The decoration of a Coxeter–Dynkin diagram and the Schläfli symbol as two methods to describe polytopes generated by finite reflection groups

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Abstract. The aim of this paper is to present two methods of describing uniform convex polytopes in three dimensions, namely, the Schläfli symbol and the method of decorating a Coxeter–Dynkin diagram. The reflection-generated polytopes are important objects, both in an abstract and in a physical context. In mathematics such polytopes are orbits of finite reflections groups, while in chemistry, biology and physics they serve as physical models for molecules, proteins, viruses and other three-dimensional structures. For this reason it is important to be able to determine the structure of polytopes and polyhedra, and to have a uniform notation that describes them. The two techniques are explained and demonstrated on the examples of polytopes generated by the finite Coxeter groups $A_3$ and $H_3$.

1. Introduction
The purpose of this paper is to present and compare two techniques which describe uniform convex polytopes. These two methods, i.e., the Schläfli symbol notation and decorating a Coxeter–Dynkin diagram, are explained in Section 3 and Section 4, and they are applied to polytopes generated as an action of a finite Coxeter group on a point in space. Although both of them are recursive methods and can be used in higher dimensions, we restrict ourselves to the three-dimensional case. Polytopes related to the reflection groups $A_3$ and $H_3$ are used to illustrate how both techniques are performed.

Before getting to the heart of the matter, let us recall some facts about finite reflection groups.

Let $E$ be a $n$-dimensional Euclidean space with a standard scalar product $(\cdot, \cdot)$. Every non-zero $\alpha \in E$ determines a hyperplane, i.e., a reflecting mirror $P_\alpha = \{ \gamma \in E : (\alpha, \gamma) = 0 \}$, and then the corresponding reflection is given by the formula

$$r_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)} \alpha = \beta - \langle \beta, \alpha \rangle \alpha,$$

(1)

where $\beta \in E$. Let $\Phi \subset E$ be a root system with a basis of simple roots $\{\alpha_1, \ldots, \alpha_n\}$. A group $G \subset GL(n)$ generated by the reflections $r_{\alpha_1}, \ldots, r_{\alpha_n}$ is called a finite Coxeter group. We adapt the notation, such that $r_{\alpha_1} = r_1$. Every root system has a basis of simple roots, called the $\alpha$-basis. We introduce its dual, i.e. the $\omega$-basis of fundamental weights. The relation between the two bases is given by

$$\langle \omega_i, \alpha_j \rangle = \delta_{ij}, \quad i, j = 1, \ldots, n.$$

(2)
The simple roots are normal vectors to reflecting mirrors, hence, to specify the Coxeter group, it suffices to provide the information about the simple roots. To do so, we use either the Cartan matrix, or the Coxeter–Dynkin diagram of a group. The Cartan matrix is defined as
\[ C = (C)_{ij} = (\langle \alpha_i, \alpha_j \rangle), \quad i, j = 1, \ldots, n, \] (3)
and provides information about the relative angles between the simple roots and their length [8]. The quadratic form matrix is defined as
\[ C^q = (C^q)_{ij} = (\omega_i, \omega_j), \] and gives the same information about the fundamental weights of the system. For the Coxeter groups related to root systems, where all the simple roots are of the same length, it is simply the inverse of the Cartan matrix. Furthermore, the following relations hold:
\[ \alpha_i = \sum_{j=1}^{n} C_{ij} \omega_j, \quad \omega_k = \sum_{j=1}^{n} C^q_{kj} \alpha_j, \quad i, j, k = 1, \ldots, n. \] (4)
Moreover, when all the entries \( \langle \alpha_i, \alpha_j \rangle \) in the Cartan matrix \( C \) are integer, the group is called crystallographic. Finite Coxeter groups of crystallographic type are also called Weyl groups.

In this paper we also consider a non-crystallographic case, namely the \( H_3 \) Coxeter group. The Cartan matrices for crystallographic root systems are well known [8], and they can be found along with their quadratic form matrices in [1]. The Cartan matrix and its inverse in the non-crystallographic case \( H_3 \) are the following:
\[ C(H_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}, \quad C^q(H_3) = \frac{1}{2} \begin{pmatrix} 2 + \tau & 2 + 2\tau & 1 + 2\tau \\ 2 + 2\tau & 4 + 4\tau & 2 + 4\tau \\ 1 + 2\tau & 2 + 4\tau & 3 + 3\tau \end{pmatrix}, \] (5)
where \( \tau = \frac{1}{2}(1 + \sqrt{5}) \) is the golden ratio.

The Coxeter–Dynkin diagram provides essentially the same information about the simple roots of a system. Namely, every node of the diagram represents a simple root and the number of lines connecting two roots specifies the angle between them. In general, the diagrams can have up to three links connecting the nodes to specify the angles. In addition, the nodes are marked differently to distinguish between long and short roots, if needed.

Finite reflection groups can be also defined in terms of their generators in the following way:
\[ A_3 = < r_1, r_2, r_3 : r_1^2 = r_2^2 = r_3^2 = 1, (r_1r_2)^3 = (r_1r_3)^2 = (r_2r_3)^3 = 1 >, \]
\[ H_3 = < r_1, r_2, r_3 : r_1^2 = r_2^2 = r_3^2 = 1, (r_1r_2)^3 = (r_1r_3)^2 = (r_2r_3)^5 = 1 >. \]

2. Polytopes
By a polytope (polyhedra) we mean a three-dimensional solid whose vertices are obtained by the action of a finite reflection group \( G \) on a point in space. More precisely, a polytope is a collection of vertices that are an orbit under the action of a given Coxeter group. In this paper we consider spherical uniform convex polytopes related to the finite Coxeter groups \( A_3 \) as an example of the
crystallographic case, and $H_3$ as an example of the non-crystallographic case. The polytopes are three-dimensional, with flat two-dimensional facets, i.e. regular planar polygons. Moreover, the vertices are equidistant from the origin, and the polytopes are vertex-transitive. Hence in each vertex, there is the same number of facets of a given type. The convexity of each polytope is ensured by checking Euler’s polyhedral formula:

$$V + F - E = 2,$$

where $V, F, E$ respectively stands for vertices, facets and edges.

As mentioned above, a reflection-generated polytope is an orbit of a finite reflection group. It is known, that each orbit has a unique dominant point. Once the vertices of polytopes are given in the $\omega$-basis, there is a unique point with non-negative coordinates. It is the dominant point of an orbit, we call it the seed point of the polytope. Furthermore, because of the duality of the bases (2), when working in the $\omega$-basis the reflection formula (1) becomes simplified:

$$r_i(\omega_j) = \omega_j - \delta_{ji} \alpha_i, \quad i = 1, \ldots, n.$$  

There are seven different seed points in three dimensions. Hence, there are seven types of polytopes that can be obtained by reflecting a point in space. That is the case with the symmetry group $H_3$. In the case of $A_3$, we have only five different polytopes, since the polytopes $(1,1,0)$ and $(0,1,1)$ have the same structure. Similarly, the polytopes $(1,0,0)$ and $(0,0,1)$ have the same structure.

The polytopes, generated by the finite Coxeter groups $A_3$ and $H_3$ are listed in Table 1 and Table 2, respectively. Each table provides information about the seed point, the number of vertices, the Schl"afli symbol and about the name of each polytope.

| $A_3$ seed point | vertices | Schl"afli symbol | name |
|----------------|----------|------------------|------|
| $(1,0,0)$      | 4        | $\{3,3\}$       | tetrahedron |
| $(0,1,0)$      | 6        | $\{3,4\}$ or $t_1\{3,3\}$ | octahedron |
| $(1,1,0)$      | 12       | $t_{0,1}\{3,3\}$ | truncated tetrahedron |
| $(1,0,1)$      | 12       | $t_1\{4,3\}$ or $t_{0,2}\{3,3\}$ | cuboctahedron |
| $(1,1,1)$      | 24       | $t_{0,1,2}\{3,3\}$ | omnitruncated tetrahedron |

| $H_3$ seed point | vertices | Schl"afli symbol | name |
|----------------|----------|------------------|------|
| $(1,0,0)$      | 12       | $\{3,5\}$       | icosahedron |
| $(0,0,1)$      | 20       | $\{5,3\}$       | dodecahedron |
| $(0,1,0)$      | 30       | $t_1\{5,3\}$    | icosidodecahedron |
| $(1,1,0)$      | 60       | $t_{0,1}\{3,5\}$ | truncated icosahedron |
| $(1,0,1)$      | 60       | $t_{0,2}\{5,3\}$ | rhombicosidodecahedron |
| $(0,1,1)$      | 60       | $t_{0,1}\{5,3\}$ | truncated dodecahedron |
| $(1,1,1)$      | 120      | $t_{0,1,2}\{5,3\}$ | truncated icosidodecahedron |
3. Schläfi symbols

The Schläfi symbol is a notation, that describes the local structure of a polytope. It is named after Ludwig Schläfi, who introduced higher-dimensional analogs of polygons [13]. This symbol is used to characterize regular polytopes (not necessarily convex) and tessellations of space, both for Euclidean and hyperbolic spaces. In this paper we focus on the Schläfi symbols describing uniform convex polytopes in three dimensions.

The simplest non-trivial Schläfi symbol is of the form \(\{p\}\), and denotes a regular \(p\)-sided polygon. For example, \(\{3\}\) refers to an equilateral triangle, \(\{4\}\) is a square, etc. The Schläfi symbol in three dimensions is of the form \(\{p,q\}\), and describes polyhedra, such that, in each vertex there are \(q\) regular facets of \(\{p\}\) form. For example, \(\{4,3\}\) specifies that there are three squares at each vertex (cube), while \(\{3,4\}\) describes four triangles in each vertex (octahedron). Therefore, the symbol denotes the Platonic solids, i.e. polyhedra with one type of facets. The notation is recursive and continues in similar form to higher dimensions.

To describe reflection-generated polytopes, that are not Platonic solids, we use the Schläfi symbol with a prefix. In this paper use the ‘t’ notation, that corresponds to a geometrical operation applied to a polytope. In this context, the Platonic solids are sometimes called the parent polytopes.

The geometrical operations in three dimensions are as follows:

- **Truncation**: cutting off a vertex of a polytope, and creating a new facet in its place, denoted as \(t_{0,1}\{p,q\}\), see Fig. 2.

![Figure 2: An illustration of a truncation operation. A tetrahedron \{3,3\} becomes a truncated tetrahedron \(t_{0,1}\{3,3\}\).](image)

- **Rectification**: truncating a polytope by marking the midpoints of all its edges, and cutting off the vertices at those points, denoted as \(t_{1}\{p,q\}\), see Fig. 3. Rectification is also called complete truncation.

![Figure 3: An illustration of a rectification operation. A dodecahedron \{5,3\} becomes an icosidodecahedron \(t_{1}\{5,3\}\).](image)

- **Cantellation**: creating a new facet in place of each edge and each vertex of a polytope, denoted as \(t_{0,2}\{p,q\}\), see Fig. 4.
Figure 4: An illustration of a cantellation operation. A dodecahedron \( \{5, 3\} \) becomes a rhombicosidodecahedron \( t_{0,2}\{5, 3\} \).

- **Omnitruncation**: creating a maximum number of facets of a polytope, by applying cantellation and truncation, denoted by \( t_{0,1,2}\{p, q\} \), see Fig. 5. Omnimtruncation is also called cantitruncation.

Figure 5: An illustration of an omnitruncation operation. A dodecahedron \( \{5, 3\} \) becomes a truncated icosidodecahedron \( t_{0,1,2}\{5, 3\} \).

4. **Decoration of the diagram**

The method of decorating a Coxeter–Dynkin diagram was introduced in [11], and applied to polyhedra in [2]. It can be applied to any connected Coxeter–Dynkin diagram of a finite Coxeter group. By labelling the nodes of a diagram in a certain way, we can easily determine a stabilizer of a face. That allows us to get the information about the number, dimension, and types of faces of polytopes generated from a chosen seed point. The method is recursive and it is a powerful tool that works in all finite dimensions.

Once a Coxeter group \( G \) and a seed point are selected, there are few rules to follow while decorating a diagram. Each node can be labelled by a \( □, ◊ \) or \( ♦ \), but two nodes that are connected (linked) cannot have a \( ◊ \) and a \( ♦ \) side by side. Recursive decoration rules are as follows:

(i) Choose the seed point. On the diagram, decorate the nodes corresponding to ‘1’ with \( □ \) and nodes corresponding to ‘0’ with \( ◊ \).

(ii) Replace one of the \( □ \) with a \( ♦ \).

(iii) Replace every \( ◊ \) linked to a \( ♦ \) by a \( □ \).

(iv) Continue as long as there are \( □ \)

Once we have obtained all the decoration steps, we can read the desired information about the structure of the considered polytope, i.e. the number and the dimension of its faces, their type (their symmetry), and the local structure.
Each step of the decoration process corresponds to one type of faces of the polytope. The number of ♦ in the decorated diagram gives the dimension \( q = 0, 1, 2 \) of a face \( f_q \). Indeed, faces \( f_0 \) are vertices, \( f_1 \) are edges, \( f_2 \) are regular polygons. The stabilizer of a face \( f_q \) is a product of two subgroups of \( G \), namely

\[
\text{Stab}_G f_q = \mathcal{W}(D) \times \mathcal{W}(V),
\]

where \( \mathcal{W}(D) \) is the symmetry group of the face \( f_q \), and \( \mathcal{W}(V) \) fixes that face pointwise. The \( \mathcal{W}(D) \) is generated by the reflections corresponding to ♦ on the decorated diagram, while \( \mathcal{W}(V) \) is generated by reflections corresponding to ◊. To determine which polygon represents a face \( f_2 \) of a polytope, we need to check if □ and ♦ coincide. Indeed, the symmetry group of a face is labelled by ♦, but the ‘active’ reflecting mirrors are labelled by □ in the initial decoration. When two reflections are labelled by □ the initial decoration, and by ♦ in the decoration of a face \( f_2 \), that face is a polygon generated by the symmetry group \( \mathcal{W}(D) \), and has as many vertices as is the order of \( \mathcal{W}(D) \). If only one of two reflections is simultaneously labelled by both ♦ and □, then the number of vertices of the polygon is equal to half of the order of \( \mathcal{W}(D) \), see Example 1.

Furthermore, the number of faces of dimension \( q \) is given by

\[
N(f_q) = \frac{|G|}{|\mathcal{W}(D)||\mathcal{W}(V)|},
\]

where \( G \) is the Coxeter group under consideration. Another information that can be read from the decorated diagram is the number of faces meeting in a face of lower dimension. Assume \( m < n \), then the number of faces \( f_n \) meeting in a face \( f_m \) is given by

\[
\frac{|\text{Stab}_W f_m|}{|\text{Stab}_W f_n \cap \text{Stab}_W f_m|}.
\]

In general, the polytopes have multiple types of faces of a given dimension, in such a case we use an upper index to denote them.

**Example 1.** Let us demonstrate the decoration method on polytopes with a seed point \((1, 1, 0)\) generated by the Coxeter group \( A_3 \) and \( H_3 \), see Fig. 6. The decoration steps are independent of the choice of the reflection group, and are presented in Fig. 7. In both cases we have vertices denoted as \( f_0 \), two types of edges denoted as \( f_1^1 \) and \( f_1^2 \), and two types of two-dimensional faces, denoted as \( f_2^1 \) and \( f_2^2 \). We examine the structure of a polytope depending on the related reflection group.

**Figure 6:** The \((1, 1, 0)\) polytopes of the \( A_3 \) and \( H_3 \) reflection groups. In the \( A_3 \) case we have a truncated tetrahedron \( t_{0,1}\{3,3\} \), and in the \( H_3 \) case we have a truncated icosahedron \( t_{0,1}\{3,5\} \).

Using the formula (9) with \( |G| = 24 \) for \( A_3 \), and \( |G| = 120 \) for \( H_3 \), we calculate how many times a given face appears in the polytope.

The active reflecting mirrors here are \( r_1 \) and \( r_2 \), labelled by □. Therefore,
| face type | decoration | number of faces | $A_3$ faces | $H_3$ faces |
|-----------|------------|----------------|------------|------------|
| $f_0$     | □ □ ♦      | 12 60 vertices | vertices   | vertices   |
| $f_1^1$   | ♦ □ ♦      | 6 30 hexagon-hexagon | hexagon-hexagon |
| $f_1^2$   | □ ♦ □      | 12 60 triangle-hexagon | hexagon-pentagon |
| $f_2^1$   | ♦ ♦ □      | 4 20 hexagon | hexagon |
| $f_2^2$   | □ □ ♦      | 4 12 triangle | pentagon |

Figure 7: The decorations of the Coxeter–Dynkin diagram of the $A_3$ and $H_3$ group for a polytope with the seed point $(1,1,0)$. The decoration method is independent of the choice of the Coxeter group, the number 5 on the diagram applies to the $H_3$ group only. Listed here are the type of a face, related decoration step, the number of times that face appears in a polytope, and what kind of a face it is.

- the symmetry group $W(D)$ of faces of type $f_1^1$, labelled by ♦, coincides with the active reflecting mirrors. It is the $A_2$ Coxeter group (of order six), hence, the $f_1^1$ faces are hexagons;
- the symmetry group $W(D)$ of faces of type $f_2^1$, labelled by ♦, does not coincide with the active reflecting mirrors. Hence, for the reflection group $A_3$ we have that $f_2^1$ faces are triangles. In the $H_3$ case, $W(D)$ is the $H_2$ group of order 10, so $f_2^1$ faces are pentagons.

To determine the local structure of the polytopes we exploit formula (10). The stabilizers are generated by the following reflections:

$\text{Stab}_{W}f_0 = <r_3>$
$\text{Stab}_{W}f_1^1 = <r_1, r_3>$
$\text{Stab}_{W}f_1^2 = <r_2>$
$\text{Stab}_{W}f_2^1 = <r_1, r_2>$
$\text{Stab}_{W}f_2^2 = <r_2, r_3>$

$\text{Stab}_{W}f_0 \cap \text{Stab}_{W}f_1^2 = \text{Stab}_{W}f_0 \cap \text{Stab}_{W}f_2^1 = <1>$
$\text{Stab}_{W}f_0 \cap \text{Stab}_{W}f_1^1 = \text{Stab}_{W}f_0 \cap \text{Stab}_{W}f_2^2 = <r_3>$

Hence, we obtain that in each vertex of the $(1,1,0)$ polytope we have

- 3 edges: one of type $f_1^1$ and two of type $f_1^2$
- 3 faces: two of type $f_2^1$ (hexagon) and one of type $f_2^2$ (triangle or pentagon)

That means edges of type $f_1^1$ are between hexagons in both cases, while edges of type $f_1^2$ are between a triangle and a hexagon for the $A_3$ polytope, and between a pentagon and a hexagon for the $H_3$ polytope.

5. Concluding remarks

The purpose of this paper is to spread awareness about the Schl"afli symbol notation and the method of decorating a Coxeter–Dynkin, as two tools that describe the structure of the reflection-generated polytopes. Brief explanations of those techniques are provided in the text and illustrated on the examples. Here, we summarize the advantages of both approaches.

The Schl"afli symbol notation is in general better known and provides a good description of the local structure of the polytopes. The notation is recursive and in higher dimension
is presented as \{p, q, r, \ldots\} along with the \( t \) prefixes, which describe geometrical operations applied to the polytopes. However, the notation is not unique, i.e., there are polytopes that can be obtained from different parent polytopes and they are described by two (or more) different Schläfli symbols.

The method of decorating a Coxeter–Dynkin diagram describes the local structure of a polytope, and provides the number of times certain faces appear in it. This technique is recursive, and can be used to describe a structure of the polytopes related to reflection groups in higher dimensions. This tool was used to explain the Platonic solids and the root polytopes in all finite dimensions [14], [15]. Furthermore, the decoration method works well when the dominant points of the polytopes have coordinates different than 1 and 0, i.e., arbitrary non-negative numbers. Such polytopes usually have non-regular faces and their edges are of different lengths, but we can still recover the form of a face and the number of times it appears in the polytope. In addition, the choice of the Coxeter group is important when reading off the information from the decoration steps, but not during the decoration process.

Both of those tools provide the information about the dual polytopes. Indeed, by inverting the order of the Schläfli symbol we obtain a description of the dual polytope, and by reversing the role of ♦ and ♦ in the decoration method, we get the decorations of the dual polytope.

In conclusion, the Schläfli symbol and the decoration method are powerful and useful techniques, which describe the polytopes and other symmetry-related structures.

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