Generalized inverses of Boolean tensors via the Einstein product

Ratikanta Behera a and Jajati Keshari Sahoo b

aDepartment of Mathematics and Statistics, Indian Institute of Science Education and Research Kolkata, Kolkata, India; bDepartment of Mathematics, BITS Pilani, K.K. Birla Goa Campus, Sancoale, India

ABSTRACT
Applications of the theory and computations of Boolean matrices are of fundamental importance to study a variety of discrete structural models. But the increasing ability of data collection systems to store huge volumes of multidimensional data, the Boolean matrix representation of data analysis is not enough to represent all the information content of the multiway data in different fields. From this perspective, it is appropriate to develop an infrastructure that supports reasoning about the theory and computations. In this paper, we discuss the generalized inverses of the Boolean tensors with the Einstein product. Further, we elaborate on this theory by producing a few characterizations of different generalized inverses and several equivalence results on Boolean tensors. We explore the space decomposition of the Boolean tensors and present reflexive generalized inverses through it. In addition to this, we address rank and the weight for the Boolean tensor.

ARTICLE HISTORY
Received 8 April 2019
Accepted 26 February 2020

COMMUNICATED BY
N.-S. Sze

KEYWORDS
Boolean tensor; generalized inverses; Moore–Penrose inverse; Space decomposition; Boolean rank.

AMS SUBJECT CLASSIFICATIONS
15A69; 15A09

1. Introduction

1.1. Background and motivation

The study of the Boolean matrices [1–4] play an important role in linear algebra [5–7], combinatorics [8], graph theory [9] and network theory [10,11]. However, this becomes particularly challenging to store huge volumes of multidimensional data. This potential difficulty can be easily overcome, thanks to tensors, which are natural multidimensional generalizations of matrices [12,13]. Here the notion of tensors is different in physics and engineering (such as stress tensors) [14], which are generally referred to as tensor fields in mathematics [15]. However, it will be more appropriate if we study the Boolean tensors and the generalized inverses of Boolean tensors. Hence the generalized inverses of Boolean tensors will encounter in many branches of mathematics, including relations theory [16], logic, graph theory, lattice theory [17] and algebraic semigroup theory.

Recently, there has been increasing interest in studying inverses [18] and different generalized inverses of tensors based on the Einstein product [19–22], and opened new perspectives for solving multilinear systems [12,23]. In [20,21], the authors have introduced...
some basic properties of the range and null space of multidimensional arrays. Further, in [21], it was discussed the adequate definition of the tensor rank, termed as reshaping rank. The weighted Moore–Penrose inverse were introduced in [24,25] via the Einstein product. Also, the authors of [26] proved some additional properties of this inverse and derived a few necessary and sufficient conditions for the reverse-order law. Within this framework, the formulation of Boolean tensors are similar to the general tensors. However, in most of the cases, the results in the general case do not immediately follow in the Boolean tensor. Hence, the purpose of this paper is to study the analogous concepts of Boolean tensors.

On the other hand, one of the most successful developments in the world of multilinear algebra is the concept of tensor decomposition [12,27,28]. This concept gives a clear and convenient way to implement all basic operations efficiently. Recently, this concept is extended in Boolean tensors [29–31]. Further, the fast and scalable distributed algorithms for Boolean tensor decompositions were discussed in [32]. In addition to that, a few applications of these decompositions are discussed in [29,33] for information extraction and clustering. At that same time, Brazell et al. in [18] discussed decomposition of tensors from the isomorphic group structure on the influence of the Einstein product and demonstrated that they are special cases of the canonical polyadic decomposition [34]. The vast work on decomposition on the tensors and its several applications in different areas of mathematics in the literature, and the recent works in [18,22], motivate us to study the generalized inverses and space decomposition in the framework of Boolean tensors. In addition to this, we introduce the rank and weight of the Boolean tensors. Further, we discuss the existence of a generalized inverse through weight of the Boolean tensor.

1.2. Organization of the paper

The rest of the paper is organized as follows. In Section 2, we present some definitions, notations, and preliminary results, which are essential in proving the main results. The main results are discussed in Section 3. It has four subparts. In the first part, we prove some identities, while the generalized inverses for Boolean tensor will be discussed in the second part. The third part mainly focuses on weighted Moore–Penrose inverses. Space decomposition and its application to generalized inverses are discussed in the last part. Finally, the results along with a few questions are concluded in Section 4.

2. Preliminaries

We first introduce some basic definitions and notations which will be used throughout the article.

2.1. Definitions and terminology

For convenience, we first briefly explain some of the terminologies which will be used here onwards. The tensor notation and definitions are followed from the article [18,22]. We refer $\mathbb{R}^{I_1 \times \cdots \times I_N}$ as the set of order $N$ real tensors. Indeed, a matrix is a second-order tensor, and a vector is a first-order tensor. Let $\mathbb{R}^{I_1 \times \cdots \times I_N}$ be the set of order $N$ and dimension $I_1 \times \cdots \times I_N$ tensors over the real field $\mathbb{R}$. $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is a tensor with $N$th order, and each entry of $\mathcal{A}$ is denoted by $a_{i_1 \cdots i_N}$. Note that throughout the paper, tensors are
represented in calligraphic letters like $\mathcal{A}$, and the notation $(\mathcal{A})_{i_1\ldots i_N} = a_{i_1\ldots i_N}$ represents the scalars. The Einstein product ([35]) $\mathcal{A} \ast_N \mathcal{B} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$ of tensors $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ and $\mathcal{B} \in \mathbb{R}^{K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$ is defined by the operation $\ast_N$ via

\[
(\mathcal{A} \ast_N \mathcal{B})_{i_1\ldots i_N j_1\ldots j_M} = \sum_{k_1\ldots k_N} a_{i_1\ldots i_N k_1\ldots k_N} b_{k_1\ldots k_N j_1\ldots j_M}.
\]

(1)

Specifically, if $\mathcal{B} \in \mathbb{R}^{K_1 \times \cdots \times K_N}$, then $\mathcal{A} \ast_N \mathcal{B} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ and

\[
(\mathcal{A} \ast_N \mathcal{B})_{i_1\ldots i_N} = \sum_{k_1\ldots k_N} a_{i_1\ldots i_N k_1\ldots k_N} b_{k_1\ldots k_N}.
\]

This product is discussed in the area of continuum mechanics [36] and the theory of relativity [35]. Further, the addition of two tensors $\mathcal{A}, \mathcal{B} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ is defined as

\[
(\mathcal{A} + \mathcal{B})_{i_1\ldots i_N k_1\ldots k_N} = a_{i_1\ldots i_N k_1\ldots k_N} + b_{i_1\ldots i_N k_1\ldots k_N}.
\]

(2)

For a tensor $\mathcal{A} = (a_{i_1\ldots i_N j_1\ldots j_M}) \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$, let $\mathcal{B} = (b_{i_1\ldots i_N j_1\ldots j_M}) \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N \times J_1 \times \cdots \times J_M}$, be the transpose of $\mathcal{A}$, where $b_{i_1\ldots i_N j_1\ldots j_M} = a_{i_1\ldots i_N j_M j_1\ldots j_M}$. The tensor $\mathcal{B}$ is denoted by $\mathcal{A}^T$. Also, we denote $\mathcal{A}^T = (a_{i_1\ldots i_N j_1\ldots j_M})$. The trace of a tensor, $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ is denoted by $tr(\mathcal{A})$ and defined as $tr(\mathcal{A}) = \sum_{i_1\ldots i_N} a_{i_1\ldots i_N i_1\ldots i_N}$. A tensor $\mathcal{O}$ denotes the zero tensor if all the entries are zero. Further, a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ is symmetric if $\mathcal{A} = \mathcal{A}^T$, and orthogonal if $\mathcal{A} \ast_N \mathcal{A} = \mathcal{A}^T \ast_N \mathcal{A} = \mathcal{I}$. Likewise, a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_N}$ is idempotent if $\mathcal{A} \ast_N \mathcal{A} = \mathcal{A}$. Furthermore, a tensor with entries $(\mathcal{D})_{i_1\ldots i_N j_1\ldots j_N}$ is called a diagonal tensor if $d_{i_1\ldots i_N j_1\ldots j_N} = 0$ for $(i_1, \ldots, i_N) \neq (j_1, \ldots, j_N)$. A few more notations and definitions are discussed below for defining generalized inverses of Boolean tensors. We first recall the definition of an identity tensor below.

**Definition 2.1 (Definition 3.13, [18]):** A tensor with entries $(\mathcal{I})_{i_1\ldots i_N j_1\ldots j_N} = \prod_{k=1}^N \delta_{i_k j_k}$, where

\[
\delta_{i_k j_k} = \begin{cases} 1, & i_k = j_k, \\ 0, & i_k \neq j_k. \end{cases}
\]

is called a unit tensor or identity tensor.

The permutation tensor is defined as follows.

**Definition 2.2:** Let $\pi$ be a permutation map on $(i_1, i_2, \ldots, i_N, j_1, j_2, \ldots, j_N)$ defined by

\[
\pi = \left( \begin{array}{cccccc} i_1 & i_2 & \cdots & i_N & j_1 & j_2 & \cdots & j_N \\ \pi(i_1) & \pi(i_2) & \cdots & \pi(i_N) & \pi(j_1) & \pi(j_2) & \cdots & \pi(j_N) \end{array} \right).
\]

A tensor $\mathcal{P}$ with entries $(\mathcal{P})_{i_1\ldots i_N j_1\ldots j_N} = \prod_{k=1}^N \epsilon_{i_k j_k}$, where

\[
\epsilon_{i_k j_k} = \begin{cases} 1, & \pi(i_k) = j_k, \\ 0, & \text{otherwise}. \end{cases}
\]

is called a permutation tensor.
Now we recall the block tensor as follows.

**Definition 2.3 ([22]):** For a tensor $\mathcal{A} = (a_{i_1 \ldots i_N j_1 \ldots j_M}) \in \mathbb{R}^I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_M$, $\mathcal{A}_{(i_1 \ldots i_N)} = (a_{i_1 \ldots i_N : :}) \in \mathbb{R}^I_1 \times \cdots \times I_M$ is a subblock of $\mathcal{A}$. $\text{Vec}(\mathcal{A})$ is obtained by lining up all the subtensors in a column, and $t$-th subblock of $\text{Vec}(\mathcal{A})$ is $\mathcal{A}_{(i_t \ldots i_N,)}$, where

$$t = i_N + \sum_{K=1}^{N-1} \left( (i_K - 1) \prod_{L=K+1}^{N} I_L \right).$$

Let $\mathcal{A} = (a_{i_1 \ldots i_N j_1 \ldots j_M}) \in \mathbb{R}^I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_M$ and $\mathcal{B} = (b_{i_1 \ldots i_N k_1 \ldots k_M}) \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_M}$. The row block tensor consisting of $\mathcal{A}$ and $\mathcal{B}$ is denoted by $[\mathcal{A} \mathcal{B}] \in \mathbb{R}^{\alpha_1 \times \beta_1 \times \cdots \times \beta_M}$, where $\alpha_1 = I_1 \times \cdots \times I_N, \beta_i = J_i + K_i, i = 1, \ldots, M$, and is defined by

$$[\mathcal{A} \mathcal{B}]_{i_1 \ldots i_N l_1 \ldots l_M} = \begin{cases} a_{i_1 \ldots i_N l_1 \ldots l_M}, & i_1 \ldots i_N \in [I_1] \times \cdots \times [I_N], l_1 \ldots l_M \in [J_1] \times \cdots \times [J_M]; \\ b_{i_1 \ldots i_N l_1 \ldots l_M}, & i_1 \ldots i_N \in [I_1] \times \cdots \times [I_N], l_1 \ldots l_M \in \Gamma_1 \times \cdots \times \Gamma_M; \\ 0, & \text{otherwise}. \end{cases}$$

where $\Gamma_i = \{J_i + 1, \ldots, J_i + K_i\}, i = 1, \ldots, M$.

Let $\mathcal{C} = (c_{j_1 \ldots j_M i_1 \ldots i_N}) \in \mathbb{R}^{I_1 \times \cdots \times J_M \times I_1 \times \cdots \times I_N}$ and $\mathcal{D} = (d_{k_1 \ldots k_M i_1 \ldots i_N}) \in \mathbb{R}^{K_1 \times \cdots \times K_M \times I_1 \times \cdots \times I_N}$. The column block tensor consisting of $\mathcal{C}$ and $\mathcal{D}$ is

$$\left[ \begin{array}{c} \mathcal{C} \\ \mathcal{D} \end{array} \right] = [\mathcal{C}^T \mathcal{D}^T]^T \in \mathbb{R}^{\beta_1 \times \cdots \times \beta_M \times \alpha_1}.$$

For $\mathcal{A}_1 \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_M}, \mathcal{B}_1 \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_M}, \mathcal{A}_2 \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_M}$ and $\mathcal{B}_2 \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_M}$, we denote $\tau_1 = [\mathcal{A}_1 \mathcal{B}_1]$ and $\tau_2 = [\mathcal{A}_2 \mathcal{B}_2]$ as the row block tensors. The column block tensor $[\tau_1 \ \tau_2]$ can be written as

$$\left[ \begin{array}{c} \mathcal{A}_1 \\ \mathcal{B}_1 \\ \mathcal{A}_2 \\ \mathcal{B}_2 \end{array} \right] \in \mathbb{R}^{\rho_1 \times \cdots \times \rho_N \times \beta_1 \times \cdots \times \beta_M},$$

where $\rho_i = I_i + L_i, i = 1, \ldots, N; \beta_j = J_j + K_j$ and $j = 1, \ldots, M$.

**Definition 2.4 (Definition 2.1, [21]):** The range space and null space of a tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N}$ are defined as per the following:

$$\mathcal{R}(\mathcal{A}) = \{ \mathcal{A} \ast_N \mathcal{X} : \mathcal{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N} \} \quad \text{and} \quad \mathcal{N}(\mathcal{A}) = \{ \mathcal{X} : \mathcal{A} \ast_N \mathcal{X} = \mathcal{O} \in \mathbb{R}^{I_1 \times \cdots \times I_M} \}.$$

The relation of range space for tensors is discussed in [21] as follows.

**Lemma 2.5 (Lemma 2.2, [21]):** Let $\mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N}, \mathcal{B} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L}$. Then $\mathcal{R}(\mathcal{B}) \subseteq \mathcal{R}(\mathcal{A})$ if and only if there exist a tensor $\mathcal{U} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_L}$ such that $\mathcal{B} = \mathcal{A} \ast_N \mathcal{U}$.

The Boolean tensor and some useful definitions are discussed in the next section.
2.2. The Boolean tensor

The binary Boolean algebra \( \mathcal{B} \) consists of the set \( \{0, 1\} \) equipped with the operations of addition and multiplication defined as follows.

\[
\begin{array}{c|cc}
+ & 0 & 1 \\
\hline
0 & 0 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \begin{array}{c|cc}
\cdot & 0 & 1 \\
\hline
0 & 0 & 0 \\
1 & 0 & 1 \\
\end{array}
\]

Definition 2.6: Let \( A = (a_{i_1\ldots i_Mj_1\ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \). If \( a_{i_1\ldots i_Mj_1\ldots j_N} \in \{0, 1\} \), then the tensor \( A \) is called Boolean tensor.

The addition and product of Boolean tensors are defined as in Equations (1) and (2) but addition and product of two entries will follow the addition and product rule of Boolean algebra. The order relation for tensors is defined as follows.

Definition 2.7: Let \( A = (a_{i_1\ldots i_Mj_1\ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( B = (b_{i_1\ldots i_Mj_1\ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \). Then \( A \leq B \) if and only if \( a_{i_1\ldots i_Mj_1\ldots j_N} \leq b_{i_1\ldots i_Mj_1\ldots j_N} \) for all \( i_s \) and \( j_t \) where \( 1 \leq s \leq M \) and \( 1 \leq t \leq N \).

We generalize the component-wise complement of the Boolean matrix [37] to Boolean tensors and defined below.

Definition 2.8: Let \( A = (a_{i_1\ldots i_Mj_1\ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) be a Boolean tensor. A tensor \( B = (b_{i_1\ldots i_Mj_1\ldots j_N}) \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) is called component-wise complement of \( A \) if

\[
b_{i_1\ldots i_Mj_1\ldots j_N} = \begin{cases} 1, & \text{when } a_{i_1\ldots i_Mj_1\ldots j_N} = 0. \\ 0, & \text{when } a_{i_1\ldots i_Mj_1\ldots j_N} = 1. \end{cases}
\]

The tensor \( B \) and its entries respectively, denoted by \( A^c \) and \( (a^c_{i_1\ldots i_Mj_1\ldots j_N}) \).

3. Main results

In this section, we prove a few exciting results on tensors which are emphasized in the binary case. We divided this section into four parts. In the first part of this section, we discuss some identities on the Boolean tensors. Then, after having introduced some necessary ingredients, we study the generalized inverses of the Boolean tensor and some equivalence results to other generalized inverses in the second part. The existence and uniqueness of weighted Moore–Penrose inverses are discussed in the third part. The space decomposition and its connection to generalized inverses are presented in the final part.

3.1. Some identities on Boolean tensors

By the definition of Boolean tensor \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_M} \), we always get \( A + A = A \). The infinite series of the Boolean tensor, \( \sum_{k=1}^{\infty} A^k \), is convergent and reduces to a finite
series since there are only finite number of Boolean tensors of the same order. Now we denote $\mathcal{A}$ for the infinite series of the Boolean tensors, i.e.

$$
\mathcal{A} = \sum_{k=1}^{\infty} A^k.
$$

Clearly $A \leq A + B$ for any two Boolean tensor (suitable order for addition) $A$ and $B$. Similarly, one can write $A = A + A \geq A + B$ for any two Boolean tensor $A \geq B$. This is stated in the next result.

**Theorem 3.1:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$. Then $A \geq B$ if and only if $A + B = A$.

If we consider $A \geq I$ in the above theorem, then it is easy to verify that $I + A + \cdots + A^n = A^n$ and obtain the following result.

**Corollary 3.1:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$ and $\mathcal{A} = \sum_{k=1}^{\infty} A^k$. If $A \geq I$, then there exist $n$, such that

(a) $\mathcal{A} = A^n$,
(b) $(\mathcal{A})^2 = \mathcal{A}$,
(c) $(\mathcal{A}) = \mathcal{A}$.

Using the above corollary, we now prove another result on the Boolean tensor.

**Theorem 3.2:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$ and $B \in \mathbb{R}^{J_1 \times \cdots \times J_N \times K_1 \times \cdots \times K_M}$, $C \in \mathbb{R}^{K_1 \times \cdots \times K_M \times J_1 \times \cdots \times J_N}$ and $D \in \mathbb{R}^{K_1 \times \cdots \times K_M \times J_1 \times \cdots \times J_N}$ be Boolean tensors with $A*NB*M = A*NC*M$. If $\mathcal{R}(B^T) = \mathcal{R}(B^T*NA^T)$, then $B*M = B*M$.

**Proof:** Since $A \geq I$ and $B \geq I$. So $\mathcal{A} \geq I$ and $\mathcal{B} \geq I$. Also we have $\mathcal{A} \geq A$ and $\mathcal{B} \geq B$. Combining these results, we get $\mathcal{A}*N\mathcal{B} \geq A$ and $\mathcal{A}*N\mathcal{B} \geq B$. Thus $\mathcal{A}*N\mathcal{B} \geq A + B$ and hence

$$
(\mathcal{A}*N\mathcal{B}) \geq A + B. \quad (3)
$$

Now $\mathcal{A} + \mathcal{B} \geq \mathcal{A}$ and $\mathcal{A} + \mathcal{B} \geq \mathcal{B}$. By using Corollary 3.1 (c), we get $\mathcal{A}*N\mathcal{B} \leq (\mathcal{A} + \mathcal{B})^2 = \mathcal{A} + \mathcal{B}$. From Corollary 3.1 (b), we have

$$
(\mathcal{A}*N\mathcal{B}) \leq (\mathcal{A} + \mathcal{B}) = \mathcal{A} + \mathcal{B}. \quad (4)
$$

From Equations (3) and (4), the proof is complete.

**Theorem 3.3:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, $B \in \mathbb{R}^{I_1 \times \cdots \times I_N \times K_1 \times \cdots \times K_M}$, $C \in \mathbb{R}^{K_1 \times \cdots \times K_M \times J_1 \times \cdots \times J_N}$ and $D \in \mathbb{R}^{K_1 \times \cdots \times K_M \times J_1 \times \cdots \times J_N}$ be Boolean tensors with $A*NB*M = A*NB*M$. If $\mathcal{R}(B^T) = \mathcal{R}(B^T*NA^T)$, then $B*M = B*M$.
**Proof:** Let $\mathcal{R}(B^T) = \mathcal{R}(B^{T*}M^TA^T)$. Then there exist a tensor $U \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_M}$ such that $B = U^{*}M^A*N^B$. Further using $A^{*}N^B*M^C = A^{*}N^B*M^D$, we obtain $B^{*}M^C = U^{*}M^A*N^B*M^D = B^{*}M^D$.

Similar way, we can prove the following corollary.

**Corollary 3.2:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$, $B \in \mathbb{R}^{K_1 \times \cdots \times K_M \times I_1 \times \cdots \times I_M}$ and $D \in \mathbb{R}^{K_1 \times \cdots \times K_M \times I_1 \times I_2 \times \cdots \times I_M}$ be Boolean tensors with $C^{*}M^A*N^B = D^{*}M^A*N^B$. If $\mathcal{R}(A) = \mathcal{R}(A^{*}N^B)$, then $C^{*}M^A = D^{*}M^A$.

Next we discuss the important result on transpose of an arbitrary-order Boolean tensor.

**Lemma 3.4:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ be any Boolean tensor. Then $A \leq A^{*}N^T*M^A$.

**Proof:** Let $B = A^{*}N^T*M^A$. We need to show that

$$a_{i_1 \cdots i_{M_1} \cdots i_{J_N}} \leq b_{i_1 \cdots i_{M_1} \cdots i_{J_N}}.$$

This inequality is trivial if $a_{i_1 \cdots i_{M_1} \cdots i_{J_N}} = 0$. Let us assume $a_{i_1 \cdots i_{M_1} \cdots i_{J_N}} = 1$. Now

$$b_{i_1 \cdots i_{M_1} \cdots i_{J_N}} = \sum_{k_1 \cdots k_N} \sum_{l_1 \cdots l_M} a_{i_1 \cdots i_{M_1} \cdots k_N} a_{l_1 \cdots l_{M_1} \cdots k_N} a_{i_1 \cdots i_{M_1} \cdots j_N}.$$

For $1 \leq s \leq N$, if $k_s = j_s$ and $l_s = i_s$, then

$$b_{i_1 \cdots i_{M_1} \cdots j_N} \geq (a_{i_1 \cdots i_{M_1} \cdots j_N})^3 = a_{i_1 \cdots i_{M_1} \cdots j_N} = 1.$$

Hence the proof is complete.

In view of the Definition 2.8, the following theorem is true for Boolean tensors.

**Proposition 3.3:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ be a Boolean tensor, then

(a) $(A^C)^C = A$.
(b) $(A^C)^T = (A^T)^C = A^{CT}$.

**Remark 3.5:** In general $B^{*}N^A \notin (A^{*}N^B)^C \notin A^{C}*N^B^C$ for any two tensor $A$, $B \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_M}$.

**Example 3.6:** Consider two Boolean tensor $A = (a_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ and $B = (b_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ such that

$$a_{ijkl} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, a_{ijkl} = a_{ijkl} = a_{ijkl} = a_{ijkl} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$b_{ijkl} = b_{ijkl} = b_{ijkl} = b_{ijkl} = b_{ijkl} = b_{ijkl} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify $B^{*}N^2A^{C} \neq (A^{*}N^B)^C \neq A^{C}*N^B^C$, where $(A^{*}N^B)^C = X = (x_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$, $A^{C}*N^B^C = Y = (y_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ and $B^{C}*N^2A^{C} = Z = (z_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$. 


$\mathbb{R}^{2 \times 3 \times 2 \times 3}$, with entries are,

$$x_{ij1} = x_{ij12} = x_{ij13} = x_{ij21} = x_{ij22} = x_{ij23} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

$$y_{ij1} = y_{ij12} = y_{ij13} = y_{ij21} = y_{ij22} = y_{ij23} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix},$$

and

$$z_{ij1} = z_{ij12} = z_{ij13} = z_{ij21} = z_{ij22} = z_{ij23} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

We now present another result based on complement of a Boolean tensor.

**Theorem 3.7:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$. Then $A * N A^C = 0$ if and only if either $A = 0$ or $A = O^C$.

**Proof:** Since the converse part is trivial, it is enough to show the sufficient part only. Let $A * N A^C = 0$. Thus $\sum_{i_1 \cdots i_N} a_{i_1 \cdots i_N k_1 \cdots k_N} a_{k_1 \cdots k_N i_1 \cdots i_N} = 0$. This implies, $a_{i_1 \cdots i_N k_1 \cdots k_N i_1 \cdots i_N} = 0$ for all $k_s$, $1 \leq s \leq N$. Which again yields either $a_{i_1 \cdots i_N k_1 \cdots k_N} = 0$ for all $i_s$, $k_s$ or $a_{k_1 \cdots k_N i_1 \cdots i_N} = 0$ for all $j_s$, $k_s$, $1 \leq s \leq N$. Therefore either $A = 0$ or $A^C = 0$. ■

Further, when $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ is a Boolean tensor, one can write

$$tr(A * N A^C) = \sum_{i_1 \cdots i_N} \sum_{k_1 \cdots k_N} a_{i_1 \cdots i_N k_1 \cdots k_N} a_{k_1 \cdots k_N i_1 \cdots i_N},$$

$$= \sum_{i_1 \cdots i_N} \sum_{k_1 \cdots k_N} a_{k_1 \cdots k_N i_1 \cdots i_N} a_{i_1 \cdots i_N k_1 \cdots k_N},$$

$$= \sum_{k_1 \cdots k_N} \sum_{i_1 \cdots i_N} a_{k_1 \cdots k_N i_1 \cdots i_N} a_{i_1 \cdots i_N k_1 \cdots k_N},$$

$$= tr(A^C * N A).$$

Hence, the tensors in the trace of a product of a tensor and its complement can be switched without changing the result. This is stated in the next result.

**Theorem 3.8:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be a Boolean tensor. Then

$$tr(A * N A^C) = tr(A^C * N A).$$

**Corollary 3.4:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N}$ be a Boolean tensor. If $A$ is symmetric, then

$$tr(A * N A^C) = tr(A^C * N A) = 0.$$

The following example is discussed in support of the Theorem 3.8.
Example 3.9: Let \( A = (a_{ijkl}) \in \mathbb{R}^{2 \times 2 \times 2 \times 2} \), where
\[
a_{ij1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_{ij2} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_{ij21} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_{ij22} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}.
\]

It is to verify that \( tr(A \ast_2 A^C) = tr(A^C \ast_2 A) = 2 \), where \( A \ast_2 A^C = (x_{ijkl}) \in \mathbb{R}^{2 \times 2 \times 2 \times 2} \) and \( A^C \ast_2 A = (y_{ijkl}) \in \mathbb{R}^{2 \times 2 \times 2 \times 2} \) with entries
\[
x_{ij1} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad x_{ij2} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad x_{ij21} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad x_{ij22} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix},
\]
\[
y_{ij1} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad y_{ij2} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad y_{ij21} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad y_{ij22} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Lemma 3.10: Let \( A \in \mathbb{R}_{1 \times \cdots \times 1 M} \times I_{1 \times \cdots \times I N} \), \( B \in \mathbb{R}_{K_1 \times \cdots \times K_L} \times I_{1 \times \cdots \times I M} \), and \( C \in \mathbb{R}_{K_1 \times \cdots \times K_L} \times I_{1 \times \cdots \times I M} \) be Boolean tensors. Then
\[
A \ast_N B \ast_L C \leq T \text{ if and only if } A^C \geq (B \ast_L C)^T.
\]

Proof: \( A \ast_N B \ast_L C \leq T \text{ if and only if } \sum_{j_1 \cdots j_N} \sum_{k_1 \cdots k_L} a_{1 \cdots i_{M_1} \cdots j_N} b_{1 \cdots i_{M_1} \cdots j_N} c_{1 \cdots k_{L_1} \cdots i_{M_1} \cdots i_{M_N}} = 0 \text{ for all } i_r, 1 \leq r \leq M. \) This is equivalent to \( a_{1 \cdots i_{M_1} \cdots j_N} b_{1 \cdots i_{M_1} \cdots j_N} c_{1 \cdots k_{L_1} \cdots i_{M_1} \cdots i_{M_N}} = 0 \text{ for all } i_r, j_s, k_t, 1 \leq r \leq M, 1 \leq s \leq N, 1 \leq t \leq L. \) This in turn is true if and only
\[
a^c_{1 \cdots i_{M_1} \cdots j_N} \geq b_{1 \cdots j_N} c_{1 \cdots k_{L_1} \cdots i_{M_1} \cdots i_{M_N}} = c^T_{1 \cdots i_{M_1} \cdots i_{M_N}} b^T_{1 \cdots i_{M_1} \cdots i_{M_N}} = (B^T \ast_L C^T)_{1 \cdots i_{M_1} \cdots j_N}.
\]

Since summation is the logical summation, we can write,
\[
a^c_{1 \cdots i_{M_1} \cdots j_N} \geq (C^T \ast_N B^T)_{1 \cdots i_{M_1} \cdots j_N} = ((B \ast_L C)^T)_{1 \cdots i_{M_1} \cdots j_N}.
\]

Thus the proof is complete.

Now we discuss the important result based on transpose and component-wise complement of an arbitrary-order Boolean tensor, as follows.

Theorem 3.11: Let \( A \in \mathbb{R}_{1 \times \cdots \times 1 M} \times I_{1 \times \cdots \times I N} \) be a Boolean tensor. Then \( A \ast_M A \leq B \) if and only if \( A \leq (B^C \ast_M A^C)^C \) and \( A^C \ast_N A \leq D \) if and only if \( A^C \leq (D^C \ast_N A^C)^C \).

Proof: Let \( A \ast_M A \leq B \). This yields \( \sum_{k_1 \cdots k_M} x_{1 \cdots i_{M_1} \cdots k_M} a_{1 \cdots k_{M_1} \cdots j_N} \leq b_{1 \cdots i_{M_1} \cdots j_N} \) for all \( i_r, 1 \leq r \leq M \) and \( j_s, 1 \leq s \leq N \). This is equivalent to \( x_{1 \cdots i_{M_1} \cdots k_M} a_{1 \cdots k_{M_1} \cdots j_N} \leq b_{1 \cdots i_{M_1} \cdots j_N} \) for all \( i_r, j_s, \) and \( k_t, 1 \leq t \leq M \). This in turns is true if and only if \( x_{1 \cdots i_{M_1} \cdots k_M} a_{1 \cdots k_{M_1} \cdots j_N} b_{1 \cdots i_{M_1} \cdots j_N} = 0 \) for all \( j_s \) and \( k_t \). Which is equivalent to
\[ x_{i_1} \ldots i_M k_1 \ldots k_M a_{k_1} \ldots k_{Mj} j_1 \ldots j_N \{ b_{j_1}^i \ldots j_N i_1 \ldots i_M \}^c = 0 \text{ for all } j_s \text{ and } k_t. \]

Summing over all \( j_s \) and \( k_t \), we get, \( \sum_{k_1} \ldots k_M \sum_{j_1} \ldots j_N x_{i_1} \ldots i_M k_1 \ldots k_M a_{k_1} \ldots k_{Mj} j_1 \ldots j_N \{ b_{j_1}^i \ldots j_N i_1 \ldots i_M \}^c = 0. \) This is true if and only if \( \mathcal{X}^* \mathcal{M}^* \mathcal{A} \mathcal{N}^* \mathcal{B}^T \mathcal{C} \leq \mathcal{T}^C. \) By Proposition 3.3 (b), this is equivalent to \( \mathcal{X}^* \mathcal{M} \mathcal{A} \mathcal{N}^* (\mathcal{B}^C)^T \leq \mathcal{T}^C. \) By Lemma 3.10, this in turns true if and only if \( \mathcal{X}^C \geq (\mathcal{A} \mathcal{N}^* (\mathcal{B}^C)^T)^T, \) that is, if and only if \( \mathcal{X} \leq (\mathcal{B}^C \mathcal{N}^* \mathcal{A}^T)^C. \)

This completes the first part of the theorem. Similar way, we can show the second part of the theorem. \( \blacksquare \)

**Corollary 3.5:** Let \( \mathcal{E} = \mathcal{O}^C \), where \( \mathcal{O} \) is the zero tensor. Then the following statements are equivalent:

(a) \( \mathcal{X}^* \mathcal{M} \mathcal{A} = \mathcal{O}. \)

(b) \( \mathcal{X} \leq ((\mathcal{A} \mathcal{N}^* \mathcal{E})^T)^C. \)

(c) \( \mathcal{E} \mathcal{N}^* \mathcal{X} \leq ((\mathcal{A} \mathcal{N}^* \mathcal{E})^T)^C. \)

The same result is also true for \( \mathcal{A}^* \mathcal{N}^* \mathcal{X} = \mathcal{O}. \) Also the following corollary easily follow from Theorem 3.11.

**Corollary 3.6:** Let \( \mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) and \( \mathcal{X} \in \mathbb{R}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M} \). Then \( \mathcal{X}^* \mathcal{M} \mathcal{A} = \mathcal{B} \) has a solution if and only if \( \mathcal{B} \leq (\mathcal{B}^C \mathcal{N}^* \mathcal{A}^T)^C \mathcal{M} \mathcal{A}. \)

### 3.2. Generalized inverses of Boolean tensors

For the generalization of the generalized inverses of Boolean matrix [4], we introduce the definition of \( [i]- \) inverses \((i = 1, 2, 3, 4)\) and the Moore–Penrose inverse of Boolean tensors via the Einstein product, as follows.

**Definition 3.12:** For any Boolean tensor \( \mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), consider the following equations in \( \mathcal{X} \in \mathbb{R}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M} : \)

\[
\begin{align*}
\mathcal{A} \mathcal{N} \mathcal{X}^* \mathcal{M} &= \mathcal{A}, \\
\mathcal{X}^* \mathcal{M} \mathcal{A} \mathcal{N} \mathcal{X} &= \mathcal{X}, \\
(\mathcal{A} \mathcal{N} \mathcal{X})^T &= \mathcal{A} \mathcal{N} \mathcal{X}, \\
(\mathcal{X}^* \mathcal{M} \mathcal{A})^T &= \mathcal{X}^* \mathcal{M} \mathcal{A}.
\end{align*}
\]

Then \( \mathcal{X} \) is called

(a) a generalized inverse of \( \mathcal{A} \) if it satisfies (1) and denoted by \( \mathcal{A}^{(1)}. \)

(b) a reflexive generalized inverse of \( \mathcal{A} \) if it satisfies (1) and (2), which is denoted by \( \mathcal{A}^{(1,2)}. \)

(c) a \([1, 3]\) inverse of \( \mathcal{A} \) if it satisfies (1) and (3), which is denoted by \( \mathcal{A}^{(1,3)}. \)

(d) a \([1, 4]\) inverse of \( \mathcal{A} \) if it satisfies (1) and (4), which is denoted by \( \mathcal{A}^{(1,4)}. \)

(e) the Moore–Penrose inverse of \( \mathcal{A} \) if it satisfies all four conditions \([1) - (4)]\), which is denoted by \( \mathcal{A}^\dagger. \)

Note that, the generalized inverse of a Boolean tensor need not be unique which explained in the next example.
Example 3.13: Consider a Boolean tensor $A = (a_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ with entries

$$
a_{ij11} = a_{ij12} = a_{ij13} = a_{ij21} = a_{ij22} = a_{ij23} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.
$$

Then it can be easily verified that both tensors $X = (x_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ and $Y = (y_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3}$ with entries

$$
x_{ij11} = x_{ij12} = x_{ij13} = x_{ij21} = x_{ij22} = x_{ij23} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},
$$

and

$$
y_{ij11} = y_{ij12} = y_{ij13} = y_{ij21} = y_{ij22} = y_{ij23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

are satisfies the required condition of the Definition 3.12.

For a Boolean tensor $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$, the number of generalized inverses are finite and the maximum number of generalized inverses is $2^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$. The next result assures the uniqueness and is true only for invertible tensors.

**Lemma 3.14:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$ be any Boolean tensor. If $A$ is invertiable then $A^{-1}$ is the only generalized inverse of $A$.

We define the maximum generalized inverse of a Boolean tensor, as follow.

**Definition 3.15:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$. A tensor $X$ is called maximum generalized inverse of $A$ if $G \leq X$ for every generalized inverse $G$ of $A$.

The following remark and corollary follows from Definition 3.12.

**Remark 3.16:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$ be a Boolean tensor. If $X$ is a generalized inverse of $A$, then $X^\ast_M A^\ast_N X$ is a reflexive generalized inverse of $A$.

**Corollary 3.7:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_N \times J_1 \times \cdots \times J_N}$ be a Boolean tensor.

(a) If $X$ is a generalized inverse of $A$, then $X^T$ is a generalized inverse of $A^T$.

(b) If $X_1$ and $X_2$ are two generalized inverse of $A$, then $(X_1 + X_2)$ is a generalized inverse of $A$.

Thus, the existence of generalized inverse of a Boolean tensor guarantees the existence of a reflexive generalized inverse. In addition to that, the Remark 3.16 and Corollary 3.7 (b) ensures that the existence of a generalized inverse implies the existence of a maximum generalized inverse of a Boolean tensor.

Next, we discus the equivalence condition for consistent system and generalized inverse.

**Theorem 3.17:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ and $X \in \mathbb{R}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M}$. Then the followings are equivalent:

(a) $A^\ast_N X^\ast_M A = A$. 

(b) \( X_M Y \) is a solution of the tensor equation \( A_N Z = Y \) whenever \( Y \in R(A) \).

(c) \( A_N X \) is idempotent and \( R(A) = R(A_N X) \).

(d) \( X_M A \) is idempotent and \( R(A^T) = R(A^T_N X^T) \).

**Proof:** First we will claim (a) if and only if (b). Let us assume (a) holds and \( Y \in R(A) \). Then there exists a Boolean tensor \( Z \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N} \) such that \( A_N Z = Y \). Now
\[
A_N X_M Y = A_N X_M A_N Z = A_N Z = Y.
\]
Therefore, \( X_M Y \) is a solution of \( A_N Z = Y \). Conversely assume (b) is true. That is \( A_N X_M Y = Y \) for all \( Y \in R(A) \). Since \( Y \in R(A) \) which implies there exists \( U \in \mathbb{R}^{I_1 \times \cdots \times I_N} \) such that \( A_N U = Y \). Thus \( A_N X_M A_N U = A_N U \) for all \( U \in \mathbb{R}^{I_1 \times \cdots \times I_N} \). Therefore, \( A_N X_M A_N = A \). Next we show the equivalence between (a) and (c). Clearly (a) implies \( A_N X \) idempotent. Since \( A = A_N X_M A \) and \( A_N X = A_N X_M A_N X \), so by Lemma 2.5 \( R(A) = R(A_N X) \). Using the same idea, we can easily show the equivalence between (a) and (d). Hence completes the proof.

**Corollary 3.8:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N} \) be a Boolean tensor.

(a) If \( R(A^T) \subseteq R(A^T_N A) \) and \( X \) is a generalized inverses of \( A^T \), then \( A_N^T A \) is a generalized inverse of \( A \).

(b) If \( R(A) \subseteq R(A_N A^T) \) and \( X \) is a generalized inverses of \( A^T \), then \( A^T \) is a generalized inverse of \( A \).

**Proof:** (a) Let \( X \) be a generalized inverses of \( A^T \). Then \( A^T_N A_N X_N A^T_N A = A^T \). Using the fact \( R(A^T) = R(A^T_N A) \), in the Theorem 3.3, we obtain \( A^T_N A_N A^T_N = A \). Thus \( X_N A^T \) is a generalized inverse of \( A \). Similarly, using Corollary 3.2, one can prove part (b).

**Corollary 3.9:** Let \( A, B, C \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N} \) be Boolean tensors with \( R(A^T) \subseteq R(B^T) \) and \( R(C) \subseteq R(B) \). If the generalized inverse of \( B \) exists, then \( A_N X_M C \) is invariant to \( X \), where \( X \) is the generalized inverse of \( B \).

**Proof:** Let \( R(A^T) \subseteq R(B^T) \) and \( R(C) \subseteq R(B) \). Then there exists \( U \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N} \) and \( V \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \) such that \( A = V*M \) and \( C = B*N \). Now \( A_N X_M C = V*M B_N X_M B*N U = V*M B*N U \) which does not rely on the tensor \( X \). Hence, it is invariant to the choice of \( X \).

To prove the next result, we define regular and singular of tensors, i.e. A tensor \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N} \), is called regular if the tensor equation \( A_N X_M A = A \) has a solution, otherwise called singular.

**Theorem 3.18:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N} \), \( S \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \), and \( T \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \). If \( S \) and \( T \) are invertible, then the following are equivalent:

(a) \( A \) is regular.

(b) \( S*M A_N T \) is regular.
(c) $A^T$ is regular.
(d) $T^T *_N A^T *_M S^T$ is regular.

Based on the block tensor [22] and their properties, we have the following lemma.

**Lemma 3.19:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$. Then $A$ is regular if and only if $[A \ O \ B]$ is regular for all regular tensors $B \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$.

**Proof:** Let $A$ and $B$ be regular tensors. Then there exist tensors $X$ and $Y$ such that $A * N X * M A = A$ and $B * N Y * M B = B$. Let $Z = [X \ O \ Y]$. Now

$$
\begin{bmatrix}
A & O \\
O & B
\end{bmatrix} *_N Z *_M \begin{bmatrix}
A & O \\
O & B
\end{bmatrix} = \begin{bmatrix}
A & O \\
O & B
\end{bmatrix} *_N \begin{bmatrix}
X & O \\
O & Y
\end{bmatrix} *_M \begin{bmatrix}
A & O \\
O & B
\end{bmatrix} = \begin{bmatrix}
A * N X *_M A & O \\
O & B * N Y * M B
\end{bmatrix} = \begin{bmatrix}
A & O \\
O & B
\end{bmatrix}.
$$

Thus $[A \ O \ B]$ is regular. The converse part can be proved in the similar way. ■

We now present another characterization of the generalized inverse of the Boolean tensor, as follows.

**Theorem 3.20:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ be a Boolean tensor. Then

$$A *_N X *_M A \leq A \text{ if and only if } X \leq \left( A *_N A^C*_M A \right)^C.$$

**Proof:** Applying Theorem 3.11 repetitively, we get $A *_N X *_M A \leq A$ if and only if $X *_M A \leq (A^T *_M A^C)^C$, which equivalently if and only if

$$X \leq \left( \left( \left( A^T *_M A^C \right)^C \right)^*_N A^T \right)^C = \left( A^T *_M A^C *_N A^T \right)^C = \left( A *_N A^C *_M A \right)^C.$$

Thus $[A \ O \ B]$ is regular. The converse part can be proved in the similar way. ■

Using the Theorem 3.20, and the fact of transpose and component-wise complement of a Boolean tensor, we obtain an important result for finding the maximum generalized inverse of a Boolean tensor.

**Corollary 3.10:** Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ be a Boolean tensor. If $A$ is regular, then the following statements hold.

(a) $A = A *_N (A *_N A^C *_M A)^C *_M A$.
(b) $(A *_N A^C *_M A)^C$ is the maximum generalized inverse of $A$.
(c) $(A *_N A^C *_M A)^C *_M A *_N (A *_N A^C *_M A)^C$ is the maximum reflexive generalized inverse of $A$. 

Next, we discuss some equivalence results between generalized and other inverses.

**Theorem 3.21:** Let \( \mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) be any Boolean tensor, then the following statements are equivalent:

(a) \( \mathcal{A}^{(1,4)} \) exists.
(b) \( \mathcal{R}(\mathcal{A}) \subseteq \mathcal{R}(\mathcal{A}^* \cdot \mathcal{A}^T) \).
(c) \( \mathcal{X}^* \cdot \mathcal{A}^* \cdot \mathcal{A}^T = \mathcal{A}^T \) for some Boolean tensor \( \mathcal{X} \).

**Proof:** Consider (a) is true and \( \mathcal{A}^{(1,4)} = \mathcal{X} \). Existence of \( \mathcal{A}^{(1)} \) is trivial and hence \( \mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^* \cdot \mathcal{A}^T) \). Now we claim (b) \( \Rightarrow \) (c). Let \( \mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^* \cdot \mathcal{A}^T) \). Then there exist a Boolean tensor \( \mathcal{U} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) such that \( \mathcal{A} = \mathcal{A}^* \cdot \mathcal{A}^T \cdot \mathcal{U} \). Which implies \( \mathcal{A}^* \cdot \mathcal{A}^T = \mathcal{A}^* \cdot \mathcal{A}^T \cdot \mathcal{U} \cdot \mathcal{U}^T \cdot \mathcal{A} \cdot \mathcal{A}^T \cdot \mathcal{U}^T \cdot \mathcal{A} \). So generalized inverse of \( \mathcal{A}^* \cdot \mathcal{A}^T \) exists. If we take \( \mathcal{X} = \mathcal{A}^T \cdot \mathcal{A} \), then

\[
\mathcal{X}^* \cdot \mathcal{A}^* \cdot \mathcal{A}^T = \mathcal{A}^T \cdot \mathcal{U}^T \cdot \mathcal{A} = \mathcal{A}^T.
\]

Finally, we claim (c) \( \Rightarrow \) (a). Let \( \mathcal{X}^* \cdot \mathcal{A}^* \cdot \mathcal{A}^T = \mathcal{A}^T \). Then

\[
(\mathcal{X}^* \cdot \mathcal{A})^T = \mathcal{A}^T \cdot \mathcal{X}^T = \mathcal{X}^* \cdot \mathcal{A}^* \cdot \mathcal{A}^T \cdot \mathcal{X}^T
\]

\[
= (\mathcal{X}^* \cdot \mathcal{A}^* \cdot \mathcal{A}^T \cdot \mathcal{X}^T)^T
\]

\[
= (\mathcal{A}^T \cdot \mathcal{X}^T)^T = \mathcal{X}^* \cdot \mathcal{A}.
\]

Using the similar way, we can show the following theorem.

**Theorem 3.22:** Let \( \mathcal{A} \) be any Boolean tensor, then the following statements are equivalent:

(a) \( \mathcal{A}^{(1,3)} \) exists.
(b) \( \mathcal{R}(\mathcal{A}^T) \subseteq \mathcal{R}(\mathcal{A}^T \cdot \mathcal{X}) \).
(c) \( \mathcal{A}^T = \mathcal{A}^T \cdot \mathcal{X} \cdot \mathcal{A} \) for some Boolean tensor \( \mathcal{X} \).

We now discuss the characterization of Moore–Penrose inverse of Boolean tensors. The similar proof of Theorem 3.2 in [22], we have the uniqueness of the Moore–Penrose inverse of a Boolean tensor in \( \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), as follows.

**Lemma 3.23:** Let \( \mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) be any Boolean tensor. If the Moore–Penrose inverse of \( \mathcal{A} \) exists then it is unique.

In the next lemma, we discuss an estimate of Moore–Penrose inverse a tensor, as follows.
Lemma 3.24: Let \( \mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_N} \) be a Boolean tensor. If \( \mathcal{A}^\dagger \) exists, then \( \mathcal{A}^* \mathcal{N} \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \leq \mathcal{A} \).

**Proof:** Let \( \mathcal{B} = \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \). Since \( \mathcal{B} \) is a Boolean tensor of even order and there are finitely many Boolean tensors of same order, so there must exist positive integers \( s, t \in \mathbb{N} \) such that \( \mathcal{B}^s = \mathcal{B}^{s+t} \). Without loss of generality, we can assume that \( s \) is the smallest positive integer for which \( \mathcal{B}^t = \mathcal{B}^{s+t} \) for some \( t \in \mathbb{N} \). Now we will show \( s = 1 \). Suppose \( s \geq 2 \). Let \( \mathcal{X} \) be the Moore–Penrose inverse of \( \mathcal{A} \). Since \( \mathcal{B} = \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^s = \mathcal{B}^{s+t} \) which implies \( \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^s = \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^{s+t} \), which implies \( \mathcal{A}^* \mathcal{N} \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^{s-2} = \mathcal{A}^* \mathcal{N} \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^{s+t-2} \). Further, pre-multiplying both side by \( \mathcal{X} \) yields \( \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^{s-2} = \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^{s+t-2} \). This implies \( \mathcal{B}^{s-1} = \mathcal{B}^{s+t-1} \).

Thus, the minimality of \( s \) is false and hence \( s = 1 \). Therefore, \( \mathcal{B} = \mathcal{B}^{t+1} \) for some \( t \in \mathbb{N} \). From the Definition 2.2, we obtain \( \mathcal{A}^* \mathcal{N} \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^{t+1} = \mathcal{A}^* \mathcal{N} \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \mathcal{B}^t \). Now pre-multiplying both side by \( \mathcal{X} \), we get \( \mathcal{A}^* \mathcal{N} \mathcal{X}^* \mathcal{M} \mathcal{A} \mathcal{B}^t = \mathcal{A}^* \mathcal{N} \mathcal{X}^* \mathcal{M} \mathcal{A} \mathcal{B}^t \mathcal{B}^t. \) Thus

\[
\mathcal{A} = \mathcal{A}^* \mathcal{N} \mathcal{B}^t = \mathcal{A}^* \mathcal{N} (\mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A})^t.
\]

Applying Lemma 3.4 repetitively and combining Equation (6), we obtain

\[
\mathcal{A}^* \mathcal{N} \mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A} \leq \mathcal{A}^* \mathcal{N} (\mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A})^2 \leq \cdots \leq \mathcal{A}^* \mathcal{N} (\mathcal{A}^T \mathcal{N}^* \mathcal{M} \mathcal{A})^t = \mathcal{A}.
\]

Using the Lemma 3.4 and 3.24 one can obtain an interesting result on invertibility of Boolean tensor as follows,

**Corollary 3.11:** A Boolean tensor \( \mathcal{A} \in \mathbb{R}^{I_1 \times \cdots \times I_N \times I_1 \times \cdots \times I_N} \) is invertible if and only if

\[
\mathcal{A}^* \mathcal{N} \mathcal{A}^T = \mathcal{A}^T \mathcal{N} \mathcal{A} = \mathcal{I}.
\]

From the Definition 2.2, we obtain

\[
(\mathcal{P}^* \mathcal{N} \mathcal{P}^T)_{i_1 \cdots i_N j_1 \cdots j_N} = \sum_{k_1 \cdots k_N} (\mathcal{P})_{i_1 \cdots i_N k_1 \cdots k_N} (\mathcal{P}^T)_{k_1 \cdots k_N j_1 \cdots j_N}
\]

\[
= \sum_{k_1 \cdots k_N} (\mathcal{P})_{i_1 \cdots i_N k_1 \cdots k_N} (\mathcal{P})_{j_1 \cdots j_N k_1 \cdots k_N}
\]

\[
= (\mathcal{P})_{i_1 \cdots i_N \pi(j_1) \cdots \pi(j_N)} (\mathcal{P})_{j_1 \cdots j_N \pi(j_1) \cdots \pi(j_N)}
\]

\[
= \begin{cases} 1 & \text{if } i_s = j_s \text{ for all } 1 \leq s \leq N, \\ 0 & \text{otherwise}. \end{cases}
\]

\[
= (\mathcal{I})_{i_1 \cdots i_N j_1 \cdots j_N}.
\]

Similar way, we can also show \( \mathcal{P}^T \mathcal{N} \mathcal{P} = \mathcal{I} \). Therefore, every permutation tensors are orthogonal and invertible. Adopting this result, we now present a characterization of the permutation tensor, as follows,

**Proposition 3.12:** A Boolean tensor \( \mathcal{A} \) has an inverse if and only if it is a permutation tensor.
Next result contains five equivalent conditions involving the existence of Moore–Penrose inverse of a Boolean tensor.

**Theorem 3.25:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times \cdots \times I_N} \) be any tensor. Then the following statements are equivalent:

(i) The Moore–Penrose inverse of \( A \) exists and unique.

(ii) \( A^{*}_{N}A^{*}_{M}A^{\top}\) \( \leq A \).

(iii) \( A^{*}_{N}A^{\top}_{M}A = A \).

(iv) The Moore–Penrose inverse of \( A \) exists and equals \( A^{\top} \).

(v) There exist a tensor \( G \) such that \( G^{*}_{N}A^{*}_{N}A^{\top} = A^{\top} \) and \( A^{T}\ast_{N}A^{*}_{N}G = A^{T} \).

**Proof:** If (i) holds then by Lemma 3.24 (ii) holds. Also (ii) \( \Rightarrow \) (iii) by Lemma 3.4. The statements (iii) \( \Rightarrow \) (iv) and (iv) \( \Rightarrow \) (i) are trivial by definition. Now we will show equivalence between (i) and (v). Suppose (i) holds. If we take \( G = A^{\top} \) then (v) hold. Conversely assume (v) is true. To prove Moore–Penrose inverse of \( A \) exists, first we show the following results:

- \( A^{*}_{N}G_{M}A = A \)
  Since \( G^{*}_{M}A^{*}_{N}A^{\top} = A^{\top} \) which implies \( A^{*}_{N}A^{*}_{M}G^{T} = A \). Pre-multiplying \( G \) and post-multiplying \( A^{\top} \) both sides, we obtain \( G^{*}_{M}A^{*}_{N}A^{*}_{M}G^{T} = A^{T} \).

- \( (G^{*}_{M}A^{T}) = (A^{T}\ast_{M}G^{T}) = G^{*}_{M}A^{*}_{N}A^{*}_{M}G^{T} = G^{*}_{M}A. \) Therefore \( G^{*}_{M}A \) is symmetric.

- \( (A^{*}_{N}G^{T}) = G^{T}\ast_{N}A^{T} = G^{T}\ast_{N}A^{T}\ast_{M}G = A^{*}_{N}G. \) Thus \( A^{*}_{N}G \) is symmetric.

Now we will show the tensor \( \chi = G^{*}_{N}A^{*}_{N}G \) is the Moore–Penrose of \( A \). Since

- \( A^{*}_{N}\chi^{*}_{M}A = A^{*}_{N}G_{M}A = A \).
- \( \chi^{*}_{M}A^{*}_{N}\chi = G^{*}_{M}A^{*}_{N}G^{*}_{M}A^{*}_{N}G = \chi \).
- \( (A^{*}_{N}\chi^{T}) = (A^{*}_{N}G^{*}_{M}A^{*}_{N}G)^{T} = (A^{*}_{N}G)^{T}\ast_{M}(A^{*}_{N}G) = A^{*}_{N}G_{M}A^{*}_{N}G = \chi \).
- \( (\chi^{*}_{M}A^{T}) = (G^{*}_{M}A^{*}_{N}G^{*}_{M}A^{T}) = (G^{*}_{M}A)^{T}\ast_{N}(G^{*}_{M}A) = G^{*}_{M}A^{*}_{N}G^{*}_{M}A = \chi^{*}_{M}A. \)

Therefore, \( \chi \) is the Moore–Penrose inverse of \( A \) and By Lemma 3.23 it is unique. \( \square \)

The reverse-order law for the Moore–Penrose inverses of tensors yields a class of challenging problems that are fundamental research in the theory of generalized inverses. Research on reverse-order law tensors has been very active recently [38,39] but as per the above theorem it is trivially true in case of Boolean tensors.

**Remark 3.26:** If Moore–Penrose inverses of \( A \in R^{I_1 \times \cdots \times I_M \times \cdots \times I_N} \), \( B \in \mathbb{R}^{I_1 \times \cdots \times K_1 \times \cdots \times I_N} \), and \( A^{*}_{N}B \) exists, then the reverse-order law for the Moore–Penrose inverse is always true, i.e.

\[ (A^{*}_{N}B)^{\dagger} = B^{\dagger}\ast_{N}A^{\dagger}. \]
3.3. Weighted Moore–Penrose Inverse

Utilizing the Einstein product, weighted Moore–Penrose inverse of even-order tensor and arbitrary-order tensor was introduced in [24,25], very recently. This work motivates us to study weighted Moore–Penrose inverse for Boolean tensors.

**Definition 3.27:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \), \( M \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M} \) and \( N \in \mathbb{R}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \) be three Boolean tensors. A Boolean tensor \( Z \in \mathbb{R}^{J_1 \times \cdots \times J_N \times I_1 \times \cdots \times I_M} \) satisfying

1. \( A \ast_N Z \ast_M A = A \),
2. \( Z \ast_M A \ast_N Z = Z \),
3. \( (M \ast_M A \ast_N Z)^T = M \ast_M A \ast_N Z \),
4. \( (Z \ast_M A \ast_N N)^T = Z \ast_M A \ast_N N \),

is called a *weighted Moore–Penrose inverse* of \( A \) and it is denoted by \( A_{M,N}^\dagger \).

Note that, the weighted Moore–Penrose inverse need not be unique in general. This can be verified by the following example.

**Example 3.28:** Consider the Boolean tensor \( A = (a_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \) as in Example 3.13. Let \( N = O \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \) and \( M = (m_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \), where

\[
m_{ij11} = m_{ij21} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad m_{ij12} = m_{ij22} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},
\]

\[
m_{ij13} = m_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\]

Then it can be easily verified that both \( X = (x_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \), \( Y = (y_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \) defined in Example 3.13 satisfies all conditions of Definition 3.27.

The uniqueness and existence of weighted Moore–Penrose inverse and some equivalent properties are discussed in the next part of this section.

**Theorem 3.29:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}, M \in \mathbb{R}^{I_1 \times \cdots \times I_M \times I_1 \times \cdots \times I_M}, \ N \in \mathbb{R}^{J_1 \times \cdots \times J_N \times J_1 \times \cdots \times J_N} \) be three Boolean tensors with \( \mathcal{R}(A) = \mathcal{R}(A_{N,N}^*N) \) and \( \mathcal{R}(A^T) = \mathcal{R}(A^T \ast_M M^T) \). If \( A_{M,N}^\dagger \) exists, then

(a) \( A_{N,N}^*N^T A_{N,N} = A_{N,N}^*N^*N A_{N,N}^T \).
(b) \( A_{M,N}^T \ast_M M \ast_M A = A_{M,N}^T \ast_M M \ast_M A \).
(c) \( A_{M,N}^\dagger \) is unique.
Proof: Let $\mathcal{X}$ be a weighted Moore–Penrose inverse of $\mathcal{A}$. Then
\[
\mathcal{A}^\ast_N \mathcal{N}^T \ast_N \mathcal{A}^T = \mathcal{A}^\ast_N \mathcal{N}^T \ast_N \mathcal{T}_M \mathcal{X}^T \ast_N \mathcal{A}^T = \mathcal{A}^\ast_N (\mathcal{X}^\ast_M \mathcal{A}^\ast_N \mathcal{N})^T \ast_N \mathcal{A}^T
\]
\[
= \mathcal{A}^\ast_N \mathcal{X}^\ast_M \mathcal{A}^\ast_N \mathcal{N} \ast_N \mathcal{A}^T = \mathcal{A}^\ast_N \mathcal{N} \ast_N \mathcal{A}^T.
\]
This completes the proof of part (a). Using the similar lines of part (a) and relation (3) of Definition 3.27, we can prove part (b). Next we will claim the uniqueness of $\mathcal{A}^\dagger_{M,N}$. Suppose there exists two weighted Moore–Penrose inverses (say $\mathcal{X}_1$ and $\mathcal{X}_2$) for $\mathcal{A}$. Then
\[
\mathcal{X}_1^\ast_M \mathcal{A}^\ast_N \mathcal{N} = \mathcal{X}_1^\ast_M \mathcal{A}^\ast_N \mathcal{X}_2^\ast_M \mathcal{A}^\ast_N \mathcal{N} = \mathcal{X}_1^\ast_M \mathcal{A}^\ast_N \mathcal{N}^T \ast_N \mathcal{A}^T \ast_M \mathcal{X}_2^T
\]
\[
= \mathcal{X}_1^\ast_M \mathcal{A}^\ast_N \mathcal{N}^T \ast_M \mathcal{A}^T \ast_M \mathcal{X}_2^T = \mathcal{N}^T \ast_M \mathcal{A}^T \ast_M \mathcal{X}_1^T \ast_M \mathcal{A}^T \ast_M \mathcal{X}_2^T
\]
\[
= \mathcal{N}^T \ast_M \mathcal{A}^T \ast_M \mathcal{X}_1^T = \mathcal{X}_2^\ast_M \mathcal{A}^\ast_N \mathcal{N}.
\]
From $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A}^\ast_N \mathcal{N})$, we obtain $\mathcal{A} = \mathcal{A}^\ast_N \mathcal{N} \ast_N \mathcal{U}$ for some Boolean tensor $\mathcal{U}$. Post-multiplying Equation (7) by $\mathcal{U}$, we get $\mathcal{X}_1^\ast_M \mathcal{A} = \mathcal{X}_2^\ast_M \mathcal{A}$. Hence
\[
\mathcal{X}_1 = \mathcal{X}_1^\ast_M \mathcal{A} \ast_N \mathcal{X}_1 = \mathcal{X}_2^\ast_M \mathcal{A} \ast_N \mathcal{X}_1.
\]
Now by using Equation (8), we get
\[
\mathcal{M}^\ast_M \mathcal{A} \ast_N \mathcal{X}_1 = \mathcal{M}^\ast_M \mathcal{A} \ast_N \mathcal{X}_2 \ast_M \mathcal{A} \ast_N \mathcal{X}_1 = \mathcal{X}_2^T \ast_N \mathcal{A}^T \ast_M \mathcal{M}^T \ast_M \mathcal{A} \ast_N \mathcal{X}_1
\]
\[
= \mathcal{X}_2^T \ast_N \mathcal{A}^T \ast_M \mathcal{M} \ast_M \mathcal{A} \ast_N \mathcal{X}_1 = \mathcal{X}_2^T \ast_N \mathcal{A}^T \ast_M \mathcal{X}_1^T \ast_N \mathcal{A}^T \ast_M \mathcal{M}^T
\]
\[
= \mathcal{X}_2^T \ast_N \mathcal{A}^T \ast_M \mathcal{M} = \mathcal{M}^\ast_M \mathcal{A} \ast_N \mathcal{X}_2.
\]
Pre-multiplying by $\mathcal{V}$ and using the fact of $\mathcal{R}(\mathcal{A}^T) = \mathcal{R}(\mathcal{A}^\ast_M \mathcal{M}^T)$, we get $\mathcal{A} \ast_N \mathcal{X}_1 = \mathcal{A} \ast_N \mathcal{X}_2$. Therefore,
\[
\mathcal{X}_2 = \mathcal{X}_2^\ast_M \mathcal{A} \ast_N \mathcal{X}_2 = \mathcal{X}_2^\ast_M \mathcal{A} \ast_N \mathcal{X}_1.
\]
Combining Equation (8) and (9), we obtain $\mathcal{X}_1 = \mathcal{X}_2$. Hence $\mathcal{A}^\dagger_{M,N}$ is unique.

The existence of weighted Moore–Penrose inverse is not trivial like other generalized inverses. The next theorem discusses the existence of weighted Moore–Penrose inverse.

Theorem 3.30: Let $\mathcal{A} \in \mathbb{R}^{l_1 \times \cdots \times l_M \times \cdots \times l_N}$, $\mathcal{M} \in \mathbb{R}^{l_1 \times \cdots \times l_M \times \cdots \times l_N}$, $\mathcal{N} \in \mathbb{R}^{l_1 \times \cdots \times l_N \times \cdots \times l_N}$ be three Boolean tensors with $\mathcal{R}(\mathcal{A}) = \mathcal{R}(\mathcal{A} \ast_N \mathcal{N})$ and $\mathcal{R}(\mathcal{A}^T) = \mathcal{R}(\mathcal{A}^\ast_M \mathcal{M}^T)$. If $\mathcal{M} \succeq \mathcal{I}$ and $\mathcal{N} \succeq \mathcal{I}$, then $\mathcal{A}^\dagger_{M,N}$ exists if and only if any one of the following conditions holds:

(a) $\mathcal{A} \ast_N \mathcal{N} \ast_N \mathcal{A}^T \ast_M \mathcal{M} \ast_M \mathcal{A} = \mathcal{A}$.
(b) $\mathcal{A} \ast_N \mathcal{N}^T \ast_N \mathcal{A}^T \ast_M \mathcal{M} \ast_M \mathcal{A} = \mathcal{A}$.
(c) $\mathcal{A} \ast_N \mathcal{N} \ast_N \mathcal{A}^T \ast_M \mathcal{M}^T \ast_M \mathcal{A} = \mathcal{A}$.
(d) $\mathcal{A} \ast_N \mathcal{N}^T \ast_N \mathcal{A}^T \ast_M \mathcal{M}^T \ast_M \mathcal{A} = \mathcal{A}$.

In particular, $\mathcal{A}^\dagger_{M,N} = \mathcal{N}^T \ast_N \mathcal{A}^T \ast_M \mathcal{M}^T$. 
\textbf{Proof:} Assume $\mathcal{A}^\dagger_{M,N}$ exists. Let $\mathcal{X} = \mathcal{A}^\dagger_{M,N}$ and $\mathcal{B} = \mathcal{A}^T * \mathcal{A}$. Since for every Boolean tensor, there are finitely many Boolean tensors of same order, so there must exist positive integers $s, t \in \mathbb{N}$ such that

$$
(A \ast_N N \ast_N A^T * M^T)^s = (A \ast_N N \ast_N A^T * M^T)^{s+t}.
$$

Without loss of generality, we can assume that $s$ is the smallest positive integer for which Equation (10) holds. Now we will claim that $s = 1$. Suppose on contradiction, assume $s > 1$. Now using Equation (10), and properties of weighted Moore–Penrose inverse, we get

\begin{align*}
\mathcal{X} * M &\ast N \ast_N A^T * M^T \ast_M (A \ast_N N \ast_N A^T * M^T)^{s-1}.
&= \mathcal{X} * M \ast N \ast_N A^T * M^T \ast_M (A \ast_N N \ast_N A^T * M^T)^{s+t-1}.
\end{align*}

This yields

\begin{align*}
N^T * N \ast_M \mathcal{X}^T * N \ast_N A^T * M^T (A \ast_N N \ast_N A^T * M^T)^{s-1}
&= N^T * N \ast_M \mathcal{X}^T * N \ast_N A^T * M^T (A \ast_N N \ast_N A^T * M^T)^{s+t-1}.
\end{align*}

From $\Re(A) = \Re(A \ast_N N)$, we obtain $A = A \ast_N N \ast_N U$ for some Boolean tensor $U$. Premultiplying $U^T$ to Equation (11), we get

\begin{align*}
A^T * M^T \ast_M (A \ast_N N \ast_N A^T * M^T)^{s-1} = A^T * M^T \ast_M (A \ast_N N \ast_N A^T * M^T)^{s+t-1}.
\end{align*}

Premultiplying $\mathcal{X}^T$ to Equation (12) and using the symmetricity of $M * M \ast_N \mathcal{X}$, we obtain

\begin{align*}
M * M \ast_N \mathcal{X} * M (A \ast_N N \ast_N A^T * M^T)^{s-1}
&= M * M \ast_N \mathcal{X} * M (A \ast_N N \ast_N A^T * M^T)^{s+t-1}.
\end{align*}

This gives

\begin{equation}
M * M (A \ast_N N \ast_N A^T * M^T)^{s-1} = M * M (A \ast_N N \ast_N A^T * M^T)^{s+t-1}.
\end{equation}

Since $\Re(A^T) = \Re(A \ast_N N \ast_M)$, so there exist a tensor $Z$ such that $Z * M * M \ast_M A = A$. Premultiplying $Z$ to Equation (13) yields

\begin{align*}
(A \ast_N N \ast_N A^T * M^T)^{s-1} = (A \ast_N N \ast_N A^T * M^T)^{s+t-1}.
\end{align*}

and contradicts the minimality of $s$. Therefore

\begin{equation}
A \ast_N N \ast_N A^T * M^T = (A \ast_N N \ast_N A^T * M^T)^{t+1}, \text{ for some } t \in \mathbb{N}.
\end{equation}

Premultiplying Equation (14) by $\mathcal{X}$, and using $(\mathcal{X} * M \ast N \ast N)^T = \mathcal{X} * M \ast N \ast N$, we obtain

\begin{align*}
N^T * N \ast_M \mathcal{X}^T * N \ast_N A^T * M^T
&= N^T * N \ast_M \mathcal{X}^T * N \ast_N A^T * M^T (A \ast_N N \ast_N A^T * M^T)^t.
\end{align*}

Since $A^T * M \ast M^T = A^T$, we get

\begin{equation}
N^T * N \ast_M A^T * M^T = N^T * N \ast_M A^T * M^T (A \ast_N N \ast_N A^T * M^T)^t.
\end{equation}

Premultiplying Equation (15) by a tensor $U^T$ and using $\Re(A) = \Re(A \ast_N N)$, we again obtain $A^T * M^T = A^T * M^T (A \ast_N N \ast_N A^T * M^T)^t$. Post-multiplying $Z^T$ and
applying $\mathcal{R}(A^T) = \mathcal{R}(A^TM^T)$, we have

$$A^T = A^T_M M^T = (A^T_N N^T A^T M^T)^t - 1 * A_N N^T A^T.
$$

Now,

$$A^T = A^T_M M^T = (A^T_N N^T A^T M^T)^t - 1 * A_N N^T A^T
\times (A^T_N N^T A^T M^T)^t - 2 * A_N N^T A^T
\times (A^T_N N^T A^T M^T)^t - 3 * A_N N^T A^T
= \cdots \cdots \cdots
\times (A^T_N N^T A^T M^T)^t - 2 * A_N N^T A^T
\times (A^T_N N^T A^T M^T)^t - 3 * A_N N^T A^T
\times (A^T_N N^T A^T M^T)^t - 3 * A_N N^T A^T
= A^T_M (A^T_N N^T A^T M^T)^t = A^T_M \left( (A^T_N N^T A^T M^T)^t \right)^T.
$$

Thus

$$\mathcal{A} = (A^T_N N^T a^T M^T)^t M^T A.
$$

As $M \geq T, N \geq T$, so by Lemma 3.4

$$A^T_N N^T a^T M^T M^T A \geq A^T_N a^T M^T A \geq \mathcal{A}.
$$

Post-multiplying $N^T a^T M^T M^T A$, we obtain

$$A^T_N N^T a^T M^T M^T A \leq (A^T_N N^T a^T M^T M^T)^2 M^T A.
$$

Combining Equations (16), (17) and (18), we have

$$\mathcal{A} \leq A^T_N N^T a^T M^T M^T A \leq (A^T_N N^T a^T M^T M^T)^2 M^T A
\leq (A^T_N N^T a^T M^T M^T)^3 M^T A \leq \cdots \leq (A^T_N N^T a^T M^T M^T)^t M^T A = \mathcal{A}.
$$

Therefore

$$\mathcal{A} = A^T_N N^T a^T M^T M^T A,
$$

and hence completes the proof of the condition (b). By using Theorem 3.29, the other conditions are holds since

$$\mathcal{A} = A^T_N N^T a^T M^T M^T A = A^T_N N^T a^T M^T M^T A
\leq (A^T_N N^T a^T M^T M^T)^2 M^T A
\leq (A^T_N N^T a^T M^T M^T)^3 M^T A \leq \cdots \leq (A^T_N N^T a^T M^T M^T)^t M^T A = \mathcal{A}.
$$

Further, we will claim not only the four conditions holds but also $\mathcal{A}_\mathcal{M}^T = N^T a^T M^T$. Let $\mathcal{X} = \mathcal{A}_\mathcal{M}^T$. From Equation (20), $\mathcal{A} = A^T_N \mathcal{X}^T M^T$ and

$$\mathcal{X}^T M^T A^T_N \mathcal{X} = N^T a^T M^T M^T \mathcal{A}^T_N N^T a^T M^T M^T = N^T a^T M^T M^T = \mathcal{X}.$$
Using Theorem 3.29, we show
\[ M^\ast M^\ast N^\ast X = M^\ast M^\ast N^T N^\ast A^T M^\ast M^T = M^\ast M^\ast N^\ast N^\ast A^T M^\ast M^T \]
\[ = (M^\ast M^\ast N^T N^\ast A^T M^\ast M^T)^T = (M^\ast M^\ast N^\ast X)^T. \]
Therefore, \( M^\ast M^\ast N^\ast X \) is symmetric. Similarly, we can show \( X^\ast M^\ast X \) is symmetric. So \( A^\dagger M^\ast N^\ast \), \( N^T N^\ast A^T M^\ast M^\ast M \) satisfies all four conditions of the weighted Moore–Penrose inverse. Thus the proof is complete.

\[ A = A^\ast N^\ast T^\ast M^\ast A = A^\ast N^\ast N^\ast T^\ast M^\ast A. \] (21)
Using the Equation (21) and symmetricity of \( A^\ast N^\ast T^\ast M^\ast A \), we obtain
\[ A^\ast N^\ast A^T = A^\ast N^\ast N^\ast A^T M^\ast M^\ast A = A^\ast N^\ast N^\ast A^T M^\ast M^\ast A. \] (22)
Similar argument yields,
\[ A^T M^\ast A = A^T M^\ast M^\ast N^\ast A^T M^\ast M^\ast A = A^T M^\ast M^\ast N^\ast A^T M^\ast M^\ast A \]
\[ = A^T M^\ast M^\ast N^\ast M^\ast A = A^T M^\ast M^\ast M^\ast A. \] (23)
Using Equations (21)–(23), it can be easily verified that \( X^\ast = N^T N^\ast A^T M^\ast M^\ast T \) satisfies all four conditions of the weighted Moore–Penrose inverse. Similarly, one can start from other conditions to verify the same. Thus the proof is complete.

Remark 3.31: The equality condition in Theorem 3.30 (a) can be replaced by \( \geq \).

### 3.4. Space decomposition

Using the theory of Einstein product, we introduce the definition of the space decomposition for Boolean tensors, which generalizes the matrix space decomposition [4].

**Definition 3.32**: A Boolean tensor \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) is said to be space decomposable if there exist two tensors \( F \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L} \) and \( R \in \mathbb{R}^{K_1 \times \cdots \times K_L \times J_1 \times \cdots \times J_N} \) such that
\[ (a) \ A = F \ast L \ R, \]
\[ (b) \ R(A) = R(F), \]
\[ (c) \ R(A^T) = R(R^T). \]
This decomposition is called a space decomposition of \( A \).

In connection with the fact of the above Definition 3.32 and Lemma 2.5, one can conclude the existence of a generalized inverse, as follows.
**Theorem 3.33:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) be a Boolean tensor and \( A = F \times_L R \) be a space decomposition of \( A \), where \( F \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L} \) and \( R \in \mathbb{R}^{K_1 \times \cdots \times K_L \times J_1 \times \cdots \times J_N} \). Then \( A^{(1)} \) exists.

We now present one of our essential results which represents not only the existence of reflexive generalized inverse but also other inverses through this decomposition.

**Theorem 3.34:** Let \( X \) be a generalized inverse of the Boolean tensor \( A \). If \( A = F \times_L R \) is a space decomposition of \( A \), where \( F \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L} \) and \( R \in \mathbb{R}^{K_1 \times \cdots \times K_L \times J_1 \times \cdots \times J_N} \), then the following statements hold.

(a) \( F^{(1)} \) and \( R^{(1)} \) exists.
(b) \( F^{(1)} \times_M F = R \times_N R^{(1)} \).
(c) \( F^{(1)} \times_M A = R \) and \( A \times_N R^{(1)} = F \).
(d) \( R^{(1)} \times_M F^{(1)} \) is a generalized inverse of \( A \).
(e) \( R \times_N X \) is a reflexive inverse of \( F \) and \( X \times_M F \) is a reflexive inverse of \( R \).

**Proof:** Let \( X \) be a generalized inverse of \( A \). Then \( A \times_N X \times_M A = A \), which implies \( F \times_L R \times_N X \times_M F \times_L R = F \times_L I \times_M F \times_L R \). Further, using Corollary 3.2, we get \( F \times_L R \times_N X \times_M F = F \). Thus \( R \times_N X \) is a generalized inverse of \( F \). Similarly, one can verify \( X \times_M F \) is a generalized inverse of \( R \). Hence completes the proof of part (a). Now using the result (a), one can prove (b) and (c). Again using (a), we obtain

\[
A \times_N R^{(1)} \times_M F^{(1)} \times_M A = A \times_N X \times_M F \times_L R \times_N X \times_M A = A \times_N X \times_M A \times_N X \times_M A = A.
\]

Hence \( R^{(1)} \times_M F^{(1)} \) is a generalized inverse of \( A \). In a similar manner, one can prove (e), by using the fact \( R \times_N X \times_M F \times_L R \times_N X \times_M F = X \times_M F \).

In view of the above theorem one can draw a conclusion, as follows.

**Remark 3.35:** Every generalized inverse of a Boolean tensor \( A \) need not of the form \( R \times_L F^{(1)} \).

We verify the Remark 3.35 with the following example.

**Example 3.36:** Let \( A = (a_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \) be a Boolean tensor with

\[
a_{ij11} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad a_{ij12} = a_{ij13} = a_{ij21} = a_{ij22} = a_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

Consider \( A^{(1)} = (x_{ijkl}) \in \mathbb{R}^{2 \times 3 \times 2 \times 3} \) is a generalized inverse of \( A \) with

\[
x_{ij11} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_{ij12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_{ij13} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

\[
x_{ij21} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad x_{ij22} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad x_{ij23} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
In light of Theorem 3.34(e), one can conclude
\[
\mathcal{R}^{(1)} *_2 \mathcal{F}^{(1)} = A^{(1)} *_2 E *_2 R *_2 A^{(1)} = A^{(1)} *_2 A *_2 A^{(1)} \neq A^{(1)}.
\]
Therefore, every generalized inverse of \( A \) need not of the form \( \mathcal{R}^{(1)}*_{I_L} \mathcal{F}^{(1)} \).

At this point, one may be interested to know when does the generalized inverse of a Boolean tensor of the form \( \mathcal{R}^{(1)}*_{I_L} \mathcal{F}^{(1)} \)? The answer to this question is given in the following theorem.

**Theorem 3.37:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) be a Boolean tensor and \( A = F*_{I_L} R \) be a space decomposition of \( A \), where \( F \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L}, \ R \in \mathbb{R}^{K_1 \times \cdots \times K_L \times J_1 \times \cdots \times J_N} \). Then \( X \) is a reflexive generalized inverse of \( A \) if and only if \( X = R^{(1)}*_{I_L} F^{(1)} \), where one of \( R^{(1)} \) and \( F^{(1)} \) is reflexive.

**Proof:** Consider the case when \( F^{(1)} \) is reflexive. Taking into account of Theorem 3.34 (d), we obtain \( X' = R^{(1)}*_{I_L} F^{(1)} \), which is a generalized inverse of \( A \). Therefore, it is enough to show \( X'*_M A*_N X' = X' \). Now using Theorem 3.34 (c), we get
\[
X'*_M A*_N X' = R^{(1)}*_{I_L} F^{(1)}*_M A*_N R^{(1)}*_{I_L} F^{(1)} = R^{(1)}*_{I_L} F^{(1)}*_M F*_L F^{(1)} = R^{(1)}*_{I_L} F^{(1)}.
\]
Conversely, let \( X' \) be a reflexive inverse of \( A \). Then by Theorem 3.34 (e),
\[
X' = X'*_M A*_N X' = X'*_M F*_L R*_N X' = R^{(1,2)}*_{I_L} F^{(1,2)} = R^{(1)}*_{I_L} F^{(1)}.
\]

**Remark 3.38:** If we drop the condition either \( F \) or \( R \) is reflexive generalized inverse of \( A \) in Theorem 3.37, then the theorem will not true in general.

In favour of Remark 3.38, we produce an example as follows.

**Example 3.39:** Let \( A \) be a Boolean tensor defined as in Example 3.36 and \( F = R = A \). Since \( A*_2 I*_2 A = I \) and \( I*_2 A*_2 I \neq I \), it follows that \( I \) is the generalized inverse for both \( F \) and \( G \) but not reflexive. In view of the Theorem 3.37, one can conclude \( R^{(1)}*_{I_L} F^{(1)} = I \) is not a reflexive generalized inverse of \( A \).

In [40] and [41], the authors have defined the rank of a Boolean matrix through space decomposition. Next, we discuss the rank and weight of a Boolean tensors.

**Definition 3.40:** Let \( A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N} \) be a Boolean tensor. If there exist a least positive integer, \( r = K_1 \times \cdots \times K_L \) such that the Boolean tensors \( B \in \mathbb{R}^{I_1 \times \cdots \times I_M \times K_1 \times \cdots \times K_L} \) and \( C \in \mathbb{R}^{K_1 \times \cdots \times K_L \times J_1 \times \cdots \times J_N} \) satisfies \( A = B*_L C \). Then \( r \) is called the Boolean rank of \( A \) and denoted by \( r_b(A) \).
Example 3.41: Consider a Boolean tensor $A = (a_{ijkl}) \in \mathbb{R}^{2 \times 2 \times 2 \times 2}$ with entries

$$a_{ij11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_{ij12} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad a_{ij21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a_{ij22} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. $$

There exist a least positive integer $r = 2$ and two tensor $B = (b_{ijk}) \in \mathbb{R}^{2 \times 2 \times 2}$ and $C = (c_{ijk}) \in \mathbb{R}^{2 \times 2 \times 2}$ with entries

$$b_{ij1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad b_{ij2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad c_{ij1} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad c_{ij2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

such that $A = B \ast_1 C$. However, if we consider $r = 1$, it is not possible to find two tensors $B \in \mathbb{R}^{2 \times 2}$ and $C \in \mathbb{R}^{2 \times 2}$ such that the product is equal to $A$.

On the other hand, the rank of the Boolean tensor is zero if it is a zero tensor. Further, we have $A = I \ast_M A = A \ast_N I$, where $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$. It is quite apparent that

$$0 \leq r_b(A) \leq \min\{I_1 \times \cdots \times I_M, J_1 \times \cdots \times J_N\}.$$

To prove the last result of this paper, we define weight of Boolean tensor as.

Definition 3.42: The weight of Boolean tensor is denoted by $w(A)$ and defined as

$$w(A) = \{ \text{Total number of non zero elements of } A \}.$$

The existence of generalized inverse can be discussed through the weight, as follows.

Theorem 3.43: Let $A \in \mathbb{R}^{I_1 \times \cdots \times I_M \times J_1 \times \cdots \times J_N}$ be any tensor with $w(A) \leq 1$. Then $A$ is regular.

Proof: It is trivial for $w(A) = 0$, as a consequence of the fact $O$ tensor is always regular. Further, consider $w(A) = 1$ and define a tensor $J$, with no zero elements. Then there exist permutation tensors $P$ and $Q$ such that $P \ast_M A \ast_N Q = [J \; O \; O \; O]$. As $J$ is regular, it implies $P \ast_M A \ast_N Q$ is regular. In view of the Lemma 3.19 and Proposition 3.12 one can conclude $A$ is regular.

4. Conclusion

In this paper, we have introduced generalized inverses ($[i]$-inverses ($i = 1, 2, 3, 4$)) with the Moore–Penrose inverse and weighted Moore–Penrose inverse for Boolean tensors via the Einstein product, which is a generalization of the generalized inverses of Boolean matrices. In addition to this, we have discussed their existence and uniqueness. This paper also provides some characterization through complement and its application to generalized inverses. Further, we explored the space decomposition for the Boolean tensors, at the same time, we have studied rank and the weight for the Boolean tensor. In particular, we limited our study for Boolean tensors with $w(A) \leq 1$. Herewith left as open problems for future studies.
**Problem:** If the Boolean rank or weight of a tensor $\mathcal{A}$ is greater than 1, then under which conditions the Boolean tensor $\mathcal{A}$ is regular?

Additionally, it would be interesting to investigate more generalized inverses on the Boolean tensors; this work is currently underway.

**Acknowledgments**

The authors would like to thank the handling editor and referees for their detailed comments and suggestions.

**Disclosure statement**

No potential conflict of interest was reported by the authors.

**Funding**

This research work was supported by Science and Engineering Research Board (SERB), Department of Science and Technology, India [grant number EEQ/2017/000747].

**ORCID**

Ratikanta Behera [http://orcid.org/0000-0002-6237-5700](http://orcid.org/0000-0002-6237-5700)

Jajati Keshari Sahoo [http://orcid.org/0000-0001-6104-5171](http://orcid.org/0000-0001-6104-5171)

**References**

[1] Bapat RB, Jain SK, Pati S. Weighted Moore-Penrose inverse of a Boolean matrix. Linear Algebra Appl. 1997;255(1-3):267–279.

[2] Bapat RB, Raghavan TES. Nonnegative matrices and applications. Vol. 64, Encyclopedia of mathematics and its applications. Cambridge: Cambridge University Press; 1997.

[3] Luce RD. A note on Boolean matrix theory. Proc Am Math Soc. 1952;3(3):382–388.

[4] Prasada Rao PSSNV, Rao KPSB. On generalized inverses of Boolean matrices. Linear Algebra Appl. 1975;11(2):135–153.

[5] Belohlavek R, Trnecka M. From-below approximations in Boolean matrix factorization: geometry and new algorithm. J Comput System Sci. 2015;81(8):1678–1697.

[6] Kim KH. Boolean matrix theory and applications. New York: Dekker; 1982.

[7] Rao CR, Mitra SK. Generalized inverse of a matrix and its applications. Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, Volume I: Theory of Statistics. California: The Regents of the University of California; 1972.

[8] Brualdi RA, Ryser HJ. Combinatorial matrix theory. Cambridge: Cambridge University Press; 1991.

[9] Berge C. The theory of graphs. New York: Dover Publications; 2001.

[10] Ledley RS. Boolean matrix equations in digital circuit design. IRE Trans Electron Comput. 1959;8(2):131–139.

[11] Li H, Wang Y. Logical matrix factorization with application to topological structure analysis of Boolean network. IEEE Trans Automat Control. 2015;60(5):1380–1385.

[12] Kolda TG, Bader BW. Tensor decompositions and applications. SIAM Rev. 2009;51(3):455–500.

[13] Ragnarsson S, Van Loan CF. Block tensor unfoldings. SIAM J Matrix Anal Appl. 2012;33(1):149–169.

[14] Narasimhan MNL. Principles of continuum mechanics. New York (NY): Wiley; 1993.

[15] De Silva V, Lek-Heng L. Tensor rank and the ill-posedness of the best low-rank approximation problem. SIAM J Matrix Anal Appl. 2008;30(3):1084–1127.
[16] Plemmons RJ. Generalized inverses of Boolean relation matrices. SIAM J Appl Math. 1971;20:426–433.
[17] Birkhoff G. Lattice theory. Rhode Island: American Mathematical Society; 1940.
[18] Brazell M, Li N, Navasca C, et al. Solving multilinear systems via tensor inversion. SIAM J Matrix Anal Appl. 2013;34(2):542–570.
[19] Behera R, Mishra D. Further results on generalized inverses of tensors via the Einstein product. Linear Multilinear Algebra. 2017;65(8):1662–1682.
[20] Ji J, Wei Y. The Drazin inverse of an even-order tensor and its application to singular tensor equations. Comput Math Appl. 2018;75(9):3402–3413.
[21] Stanimirović PS, Ciric M, Katsikis VN. Outer and (b,c) inverses of tensors. Linear Multilinear Algebra. 2018. doi:10.1080/03081087.2018.1521783.
[22] Sun L, Zheng B, Bu C, et al. Moore-Penrose inverse of tensors via Einstein product. Linear Multilinear Algebra. 2016;64(4):686–698.
[23] Liang M-L., Zheng B, Zhao R-J. Tensor inversion and its application to the tensor equations with Einstein product. Linear Multilinear Algebra. 2019;67(4):843–870.
[24] Behera R, Maji S, Mohapatra RN. Weighted Moore-Penrose inverses of arbitrary-order tensors. Submitted, arXiv:1812.03052.
[25] Ji J, Wei Y. Weighted Moore-Penrose inverses and fundamental theorem of even-order tensors with Einstein product. Front. Math. China. 2017;12(6):1319–1337.
[26] Panigrahy K, Mishra D. Reverse-order law for weighted Moore–Penrose inverse of tensors. Adv Oper Theory. 2020;5:39–63.
[27] Kolda TG. Orthogonal tensor decompositions. SIAM J Matrix Anal Appl. 2001;23(1):243–255.
[28] Lathauwer LD, De Moor B, Vandewalle J. A multilinear singular value decomposition. SIAM J Matrix Anal Appl. 2000;21(4):1253–1278.
[29] Erdos D, Miettinen P. Discovering facts with boolean tensor Tucker decomposition. Proceedings of the 22nd ACM International Conference on Information & Knowledge Management, CIKM ’13; New York (NY): ACM; 2013. p. 1569–1572.
[30] Khamis MA, Ngo HQ, Olteanu D, et al. Boolean tensor decomposition for conjunctive queries with negation. 22nd International Conference on Database Theory, Vol. 127, LIPIcs. Leibniz Int. Proc. Inform., pages Art. No. 21, 19. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2019.
[31] Rukat T, Holmes CC, Yau C. Tensormachine: probabilistic Boolean tensor decomposition. arXiv preprint arXiv:1805.04582, 2018.
[32] Miettinen P. Boolean tensor factorizations. 2011 IEEE 11th International Conference on Data Mining. IEEE; 2011. p. 447–456.
[33] Metzler S, Miettinen P. Clustering Boolean tensors. Data Min Knowl Discov. 2015;29(5):1343–1373.
[34] Carroll JD, Jih-Jie C. Analysis of individual differences in multidimensional scaling via an N-way generalization of Eckart-Young decomposition. Psychometrika. 1970;35(3):283–319.
[35] Einstein A. The foundation of the general theory of relativity. Ann Phys. 1916;49(7):769–822.
[36] Lai W, Rubin D, Kreml E. Introduction to continuum mechanics. Oxford: Butterworth Heinemann; 2009.
[37] Fitz-Gerald D. Computing the maximum generalized inverse of a Boolean matrix. Linear Algebra Appl. 1977;16(3):203–207.
[38] Mishra D, Panigrahy K. Reverse-order law of tensors revisited. 2018. arXiv:1809.07017.
[39] Panigrahy K, Behera R, Mishra D. Reverse-order law for Moore-Penrose inverses of tensors. Linear Multilinear Algebra. 2020;68(2):246–264.
[40] Beasley LB, Pullman NJ. Boolean-rank-preserving operators and Boolean-rank-1 spaces. Linear Algebra Appl. 1984;59:55–77.
[41] Song S-Z, Kang K-T, Kang M-H. Boolean regular matrices and their strongly preservers. Bull Korean Math Soc. 2009;46(2):373–385.