Domination versus independent domination in regular graphs

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Abstract
A set S of vertices in a graph G is a dominating set if every vertex of G is in S or is adjacent to a vertex in S. If, in addition, S is an independent set, then S is an independent dominating set. The domination number \( \gamma(G) \) of G is the minimum cardinality of a dominating set in G, while the independent domination number \( i(G) \) of G is the minimum cardinality of an independent dominating set in G. We prove that for all integers \( k \geq 3 \) it holds that if G is a connected \( k \)-regular graph, then \( \frac{i(G)}{\gamma(G)} \leq \frac{k}{2} \), with equality if and only if \( G = K_{k,k} \). The result was previously known only for \( k \leq 6 \). This affirmatively answers a question of Babikir and Henning.

Keywords
domination, extremal graph, independent domination

1 | INTRODUCTION

Given a graph \( G = (V, E) \) an independent set is a subset of vertices \( U \subseteq V \), such that no two vertices in \( U \) are adjacent. An independent set is maximal if no vertex can be added without violating independence. An independent set of maximum cardinality is called a maximum
independent set. A set $S$ of vertices in a graph $G$ is a dominating set if every vertex of $G$ is in $S$ or is adjacent to a vertex in $S$. If, in addition, $S$ is an independent set, then $S$ is an independent dominating set. The domination number of $G$, denoted $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. The independent domination number of $G$, denoted $i(G)$, is the minimum cardinality of an independent dominating set in $G$. Note that an independent set of vertices in a graph $G$ is a dominating set of $G$ if and only if it is a maximal independent set. Therefore, $i(G)$ is equal to the minimum cardinality of a maximal independent set of vertices in $G$. The object of study in this paper are $k$-regular graphs, that is, such that every vertex has degree $k$.

Dominating and independent dominating sets have been studied extensively in the literature; see, for example, the books [7,8] and a survey [5]. In early studies authors considered the difference between $\gamma(G)$ and $i(G)$ [2,3,6,9]. In [6] the authors initiated the study of the ratio $\frac{i(G)}{\gamma(G)}$. They showed that this ratio is at most $\frac{3}{2}$ for connected cubic graphs $G$, with equality if and only if $G = K_{3,3}$. Southey and Henning [11] proved that the $\frac{3}{2}$ ratio can be strengthened to a $\frac{4}{3}$ ratio if $K_{3,3}$ is excluded. Suil and West [12] constructed an infinite family of connected cubic graphs $G$ such that $\frac{i(G)}{\gamma(G)} = \frac{5}{4}$. A question of determining whether $\frac{4}{3}$ ratio from the above-mentioned result of Southey and Henning can be improved to a $\frac{5}{4}$ ratio if finitely many graphs are forbidden, remains open.

The ratio of the independent domination number to the domination number for general graphs was studied by Furuya et al. [4] who showed that for a graph $G$ this ratio is at most $\Delta(G) - 2\sqrt{\Delta(G)} + 2$, where $\Delta(G)$ denotes the maximum degree of $G$.

In this paper we give the affirmative answer to the following question from [1], with which the above sharp bound of Furuya et al. is also improved in the case of connected $k$-regular graphs.

**Question 1.** Is it true that for all integers $k \geq 3$ if $G$ is a connected $k$-regular graph, then $\frac{i(G)}{\gamma(G)} \leq \frac{k}{2}$, with equality if and only if $G = K_{k,k}$?

As mentioned above, Question 1 was already answered affirmatively for $k = 3$ in [6], and for $k \in \{4, 5, 6\}$ in [1].

## 2 PROOF OF THE MAIN THEOREM

In the proof of our theorem we use $G[A]$ to denote the subgraph of $G$ induced by a vertex set $A \subseteq V$, and we also use an old result by Rosenfeld [10].

**Proposition 2.** If $G$ is a regular graph of order $n$ with no isolated vertex, then $i(G) \leq \frac{n}{2}$.

**Theorem 3.** For $k \geq 3$, if $G$ is a connected $k$-regular graph, then

$$\frac{i(G)}{\gamma(G)} \leq \frac{k}{2},$$

with equality if and only if $G = K_{k,k}$. 
Proof. Let $G = (V, E)$ be a connected $k$-regular graph, $k \geq 3$. Let $A$ be a dominating set in $G$ with $|A| = \gamma(G)$, and $B = V \setminus A$. We distinguish two cases with respect to the number of edges in $G[A]$, which we denote by $s$.

Case 1: $s \geq \frac{\gamma(G)}{2}$. Denote by $e$ the number of edges having one end-vertex in $A$ and the other in $B$. Since $G$ is $k$-regular, we derive

$$e = k \cdot |A| - 2s = k\gamma(G) - 2s \leq k\gamma(G) - \gamma(G) = (k - 1)\gamma(G).$$

Since $A$ is a dominating set this readily implies that $|B| \leq e \leq (k - 1)\gamma(G)$. We now estimate $n = |A| + |B| \leq \gamma(G) + (k - 1)\gamma(G) = k\gamma(G)$, and using Proposition 2, we derive

$$\frac{i(G)}{\gamma(G)} \leq \frac{n/2}{n/k} = \frac{k}{2}.$$ 

Case 2: $s < \frac{\gamma(G)}{2}$. Let $A'$ denote a maximum independent set in $G[A]$. Clearly, $|A'| \geq |A| - s = \gamma(G) - s$. Let $|A'| = \gamma(G) - s + x$ for some $x \geq 0$. Then $A \setminus A'$ contains $s - x$ vertices which we denote by $b_1, b_2, ..., b_{s-x}$. Let $B_i$ be the set of neighbors of $b_i$ in $B$, $i \in \{1, 2, ..., s-x\}$, and let $B' = B_1 \cup B_2 \cup \cdots \cup B_{s-x}$. Then $|B'| \leq (s - x)(k - 1)$, and since $k \geq 3$ we derive

$$|A' \cup B'| = |A'| + |B'|$$

$$\leq \gamma(G) - s + x + (s - x)(k - 1)$$

$$= \gamma(G) - 2s + sk + x(2 - k)$$

$$\leq \gamma(G) + (k - 2)s.$$

Our next aim is to show that $A' \cup B'$ contains an independent dominating set of $G$. Let $B''$ be the set of vertices in $B'$ that have a neighbor in $A'$, and $C$ an independent dominating set in the subgraph of $G$ induced by the set $B' \setminus B''$. It is straightforward to verify that $I = A' \cup C$ is an independent dominating set in $G$. Therefore we obtain

$$i(G) \leq |I| = |A'| + |C| \leq |A'| + |B'| \leq \gamma(G) + (k - 2)s.$$ 

Recall that $s < \frac{\gamma(G)}{2}$. Now we consider the following cases with respect to the parity of $\gamma(G)$. If $\gamma(G)$ is even, then $s \leq \frac{\gamma(G)}{2} - 1$, and we have

$$i(G) \leq \gamma(G) + (k - 2)\left(\frac{\gamma(G)}{2} - 1\right) = \frac{k}{2}\gamma(G) + 2 - k$$

$$< \frac{ky(G)}{2},$$

and therefore

$$\frac{i(G)}{\gamma(G)} < \frac{ky(G)}{2} \gamma(G) = \frac{k}{2}.$$
If $\gamma(G)$ is odd, then $s \leq \frac{\gamma(G)-1}{2}$, and we obtain

$$i(G) \leq \gamma(G) + (k-2)\frac{\gamma(G)-1}{2} = \frac{k\gamma(G)}{2} - \frac{k-2}{2} < \frac{k\gamma(G)}{2},$$

which again implies the desired inequality.

Now we describe the extremal graphs, that is, graphs $G$ with $i(G) = \frac{k\gamma(G)}{2}$. Since $i(G) < \frac{k\gamma(G)}{2}$ if there are less than $\gamma(G)/2$ edges in $G[A]$ (see calculations in Case 2) the extremal graphs can be obtained only if there are at least $\gamma(G)/2$ edges in $G[A]$. In fact, $G[A]$ must have exactly $\gamma(G)/2$ edges, that is, $s = \gamma(G)/2$, since otherwise we get $e < (k-1)\gamma(G)$ which implies $n < k\gamma(G)$ and consequently $i(G) < \frac{k\gamma(G)}{2}$.

Furthermore, we will show that $G[A]$ is a collection of independent edges. Suppose that $G[A]$ has exactly $t$ components. Take one vertex from each component to form an independent set. This set can be completed with at most $(k-1)(\gamma(G) - t)$ vertices of $B$ (i.e., with at most $k-1$ vertices of $B$ for each of nonselected vertices from $A$) to a maximal independent set $M$. Recall that $M$ is an independent dominating set of $G$ and therefore $i(G) \leq |M|$. If $t > \gamma(G)/2$, then

$$i(G) \leq |M| \leq t + (k-1)(\gamma(G) - t) = k\gamma(G) - \gamma(G) - t(k-2) < k\gamma(G) - \gamma(G) - \frac{\gamma(G)}{2}(k-2) = \frac{k}{2} \gamma(G),$$

which implies $i(G) < \frac{k\gamma(G)}{2}$, a contradiction. Therefore $t \leq \gamma(G)/2$, that is, $t \leq s$. If a component of $G[A]$ contains a cycle, then $s > \gamma(G) - t$, which together with $s = \gamma(G)/2$ implies that $t > \gamma(G)/2$, a contradiction. Thus $G[A]$ is a forest. Since $t \leq s$ we have $s = \gamma(G) - t \geq \gamma(G) - s = s$, and thus $t = s$.

If there is a component of $G[A]$ which contains at least three vertices, then this component contains two independent vertices (recall that $G[A]$ is a forest). So take two independent vertices from this component and one vertex from every other component of $G[A]$. This set contains $t + 1$ independent vertices and analogously as above it can be completed with at most $(k-1)(\gamma(G) - t - 1)$ vertices of $B$ to a maximal independent set $M$. We get

$$i(G) \leq |M| \leq (t + 1) + (k-1)(\gamma(G) - t - 1) = k\gamma(G) - \gamma(G) - (t + 1)(k-2) = \frac{k}{2} \gamma(G) - (k-2) < \frac{k}{2} \gamma(G),$$

which implies $i(G) < \frac{k\gamma(G)}{2}$, a contradiction. Thus, $G[A]$ is a collection of independent edges $\{u_1v_1, u_2v_2, ..., u_kv_k\}$.

Our next aim is to show that for each $i \in \{1, 2, ..., s\}$ vertices $u_i$ and $v_i$ do not have a common neighbor. If there is $z \in N(u_i) \cap N(v_i)$, then taking $u_i$ and one vertex from every other edge of $G[A]$ to an independent set, we can complete it to a maximal
independent set $I$ with at most $k - 2$ neighbors of $v_i$ in $V \setminus A$, and at most $k - 1$ neighbors for each of nonselected vertices of $G[A]$. Therefore

$$i(G) \leq |I| \leq \frac{\gamma(G)}{2} + k - 2 + \left(\frac{\gamma(G)}{2} - 1\right)(k - 1) = \frac{\gamma(G)}{2}k - 1 \leq \frac{\gamma(G)}{2}k,$$

which again yields $\frac{i(G)}{\gamma(G)} < \frac{k}{2}$. Thus, $N(u_i) \cap N(v_i) = \emptyset$.

Now suppose that there is $z \in N(u_i)$ which has a neighbor $w$ outside $N(u_i) \cup N(v_i)$. We distinguish two cases.

Case A: $w \in A \setminus \{u_1, v_1\}$. Without loss of generality we may assume that $w = v_2$. Then put to an independent set $v_1$ and complete it to an independent dominating set $I$ analogously as above. More precisely, $I$ contains $v_i$ for every $i \in \{1, 2, ..., s\}$, at most $k - 1$ neighbors of $u_i$ in $B$ for every $i \in \{2, 3, ..., s\}$, and at most $k - 2$ neighbors of $u_1$ in $B$ since $u_2 z \in E$ and $z \in N(u_i)$. Then

$$i(G) \leq |I| \leq \frac{\gamma(G)}{2} + k - 2 + \left(\frac{\gamma(G)}{2} - 1\right)(k - 1) < \frac{\gamma(G)}{2}k,$$

which again implies $\frac{i(G)}{\gamma(G)} < \frac{k}{2}$, a contradiction.

Case B: $w \in V \setminus (A \cup N(u_i) \cup N(v_i))$. Without loss of generality we may assume that $w$ is a neighbor of $u_2$. Let $I$ be the set consisting of $v_i$ for every $i \in \{1, 2, ..., s\}$, at most $k - 1$ neighbors of $u_i$ in $B$ for every $i \in \{3, 4, ..., s\}$, and at most $(k - 1) + (k - 1) - 1$ vertices in $(N(u_i) \cup N(u_2)) \setminus \{v_1, v_2\}$ since $u_2 w, z w \in E$. Note that $I$ is an independent dominating set. We derive

$$i(G) \leq |I| \leq \frac{\gamma(G)}{2} + \left(\frac{\gamma(G)}{2} - 2\right)(k - 1) + 2k - 3 < \frac{\gamma(G)}{2}k,$$

leading to a contradiction again.

With this we have shown that no neighbor of $u_1$ in $B$ has a neighbor outside $N(u_i) \cup N(v_i)$. Proceeding analogously for neighbors of $v_1$ we see that no vertex of $N(u_i) \cup N(v_i)$ has a neighbor outside $N(u_i) \cup N(v_i)$. That is, $G[N(u_i) \cup N(v_i)]$ is a component of $G$, and since $G$ is connected, we have $\gamma(G) = 2$.

Now suppose that there are vertices $z_1, z_2 \in N(u_i) \setminus \{v_i\}$ such that $z_1 z_2 \in E$. Then put to an independent set $v_1$ and complete it to an independent dominating set of $G$ with neighbors of $u_1$ in $V \setminus A$. Since $z_1 z_2 \in E$, we get an independent dominating set of size at most $1 + (k - 2) < k = \frac{\gamma(G)}{2}k$, which gives $\frac{i(G)}{\gamma(G)} < \frac{k}{2}$ again. Thus, $N(u_i) \setminus \{v_i\}$ is an independent set in $G$. In fact, $N(u_i)$ itself is an independent set of $G$, since we have already shown that $v_1$ has no neighbors in $N(u_i) \cap V \setminus A$. Analogously it can be shown that $N(v_i)$ is an independent set, which means that $G$ is a $k$-regular graph with $2k$ vertices, and two independent sets $N(u_i)$ and $N(v_i)$ of size $k$. Consequently, $G$ is $K_{k,k}$. \hfill \qquad \square

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