THE YONEDA EXTENSION ALGEBRA OF $GL_2(\overline{F}_p)$

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Abstract. Let $p$ be a prime number. We compute the Yoneda extension algebra of $GL_2$ over an algebraically closed field of characteristic $p$ by developing a theory of Koszul duality for a certain class of 2-functors, one of which controls the category of rational representations of $GL_2$ over such a field.

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1. Intro

Let $F$ be an algebraically closed field of characteristic $p > 0$. Let $G = GL_2(F)$ denote the group of $2 \times 2$ invertible matrices over $F$. Let $\mathcal{L}$ denote a complete set of irreducible objects in the category $G$-mod of rational representations of $G$. The object of this paper is to give an explicit description of the Yoneda extension algebra

$$Y = \bigoplus_{L,L' \in \mathcal{L}} \text{Ext}_{G\text{-mod}}^\bullet(L, L')$$

of $G$. The best previous work in this direction was done by A. Parker, who outlined an intricate algorithm to compute the dimension of $\text{Ext}_{G\text{-mod}}^n(L, L')$ for $L, L' \in \mathcal{L}$ and $n \geq 0$. Our approach is quite different. We develop a theory of homological duality for certain algebraic operators $\mathcal{O}$ which we introduced previously in our study of the category of rational representations of $G$, and use this to give a combinatorial description of $Y$ as an algebra.
The category $G$-mod has countably many blocks, all of which are equivalent. Therefore, the algebra $Y$ is isomorphic to a direct sum of countably many copies of $y$, where $y$ is the Yoneda extension algebra of the principal block of $G$. Our problem is to compute $y$.

Suppose $\Gamma = \bigoplus_{i,j,k \in \mathbb{Z}} \Gamma_{ijk}$ is a $\mathbb{Z}$-trigraded algebra. We have a combinatorial operator $\mathcal{D}_{\Gamma}$ which acts on the collection of bigraded algebras $\Delta$ after the formula

$$\mathcal{D}_{\Gamma}^{ijk}(\Delta) = \bigoplus_{j,k_1+k_2=k} \Gamma_{ijk_1} \otimes \Delta^{jk_2},$$

where $\otimes$ denotes the super tensor product. We now define a trigraded algebra $\Lambda$.

Let $\Pi$ denote the preprojective algebra of bi-infinite type $A$: it is the path algebra of the quiver

$$\cdots \xrightarrow{x} 1 \xleftarrow{y} x \xrightarrow{x} 2 \xleftarrow{y} x \xrightarrow{x} 3 \cdots \quad \cdots \xrightarrow{x} p-1 \xleftarrow{y} x \xrightarrow{x} p \xleftarrow{y} x \xrightarrow{x} \cdots,$$

modulo relations $xy - yx$. This algebra is naturally $\mathbb{Z}_+$-graded so that paths are homogeneous with degree given by their length. We denote by $\sigma$ the involution of $\Pi$ which sends vertex $i$ to vertex $p-i$ and exchanges $x$ and $y$.

Let $\Pi_{\leq p}$ denote the subalgebra of $\Pi$ generated by arrows which begin and end at vertices indexed by $i \leq p$. Let $\Omega$ denote the quotient of $\Pi_{\leq p}$ by the ideal given by paths passing through vertices $i \leq 0$. Let $\Theta$ denote the quotient of $\Pi$ by the ideal given by paths passing through vertices $i \leq 0$ or $i \geq p$.

The algebra $\Theta$ is the quotient of $\Omega$ by the ideal generated by vertex $p$; we denote by $\pi$ the relevant quotient map. The automorphism $\sigma$ lifts to an automorphism $\Theta$. Let $\Theta^\sigma$ denote the $\Pi$-$\Pi$ bimodule given by lifting along $\pi$ the $\Theta$-$\Theta$-bimodule obtained by twisting the regular $\Theta$-bimodule on the right by $\sigma$. We define

$$\Lambda = T_{\Pi}(\Theta^\sigma) \otimes F[\zeta].$$

to be the tensor product over $F$ of the tensor algebra over $\Omega$ of $\Theta$ twisted on the right by $\sigma$, with a polynomial algebra in a single variable. We define a trigrading $\Lambda = \bigoplus_{i,j,k} \Lambda^{ijk}$ on $\Lambda$ as follows: the $i$-grading is defined by placing $\Omega$ in degree 0 and $\Theta^\sigma$ and $\zeta$ in degree 1; the $j$-grading is defined by grading elements of $\Omega$ and $\Theta^\sigma$ according to path length and placing $\zeta$ in $j$ degree $p$; the $k$-grading is defined by grading elements of $\Omega$ and $\Theta^\sigma$ according to path length and placing $\zeta$ in $k$ degree $p-1$.

Let us consider the field $F$ as a trigraded algebra concentrated in degree $(0, 0, 0)$. We have a natural embedding of bigraded algebras $F \rightarrow \Lambda$, which sends 1 to $e_1$. This embedding lifts to a morphism of operators $\mathcal{D}_F \rightarrow \mathcal{D}_{\Lambda}$. We have $\mathcal{D}_F^2 = \mathcal{D}_F$. Putting these together, we obtain a sequence of operators

$$\mathcal{D}_F \rightarrow \mathcal{D}_F \mathcal{D}_{\Lambda} \rightarrow \mathcal{D}_F \mathcal{D}_{\Lambda}^2 \rightarrow \cdots$$

which, applied to the graded algebra $F[z]$ with $z$ placed in $jk$-degree $(1, 0)$, gives a sequence of algebra embeddings

$$\lambda_1 \rightarrow \lambda_2 \rightarrow \lambda_3 \rightarrow \cdots,$$

where $\lambda_q = \mathcal{D}_F \mathcal{D}_{\Lambda}^q(F[z])$. Taking the union of the algebras in this sequence gives us an algebra $\lambda$. Our main theorem is the following:

**Theorem 1.** The algebra $y$ is isomorphic to $\lambda$. 
The algebras $\lambda_q$ are isomorphic to Yoneda extension algebras of certain Schur algebras, and are consequently finite dimensional. However, the algebra $\lambda$ is infinitely generated. In the following section we describe a monomial basis for $\lambda$.

There is a natural grading on $\gamma$, the Yoneda grading, in which extensions of length $k$ are given degree $k$. This can be identified with the $k$-grading on $\lambda$.

### 2. Monomial bases

In this paper, we describe a number of constructions yielding algebras isomorphic to the Yoneda extension algebra $\gamma$. We believe the most elegant such construction is that given in the introduction via the combinatorial operator $\Omega_\Lambda$. However, if you like algebras to be described via a product on a basis, you will not be happy with this description. With hope of bringing happiness, in this section we describe a monomial basis for $\lambda$.

When we speak of a (signed) monomial basis for an algebra, we mean a vector space basis $B$ containing a complete set of primitive idempotents, such that the product of any two elements of $B$ is either equal to (±) another basis element or equal to zero.

Our first step is to describe a monomial basis $B_\Pi$ for $\Pi$. We set

$$B_\Pi = \{(s, \alpha, \beta) \mid s \in \mathbb{Z}, \alpha, \beta \in \mathbb{N}\}.$$  

Here, basis elements of $\Pi$ correspond to paths in the quiver of $\Pi$; the integer $s$ describes the source of such a path whilst the integers $\alpha$ and $\beta$ describe the powers of $x$ and $y$ involved in the path. The target $t$ of such a path can be computed via the formula $t - s = \alpha - \beta$. The product in $\Pi$ of two such basis elements is defined by the following formula:

$$(s, \alpha, \beta) \cdot (s', \alpha', \beta') = \begin{cases} (s, \alpha + \alpha', \beta + \beta') & \text{if } s' = s + \alpha - \beta; \\ 0 & \text{otherwise.} \end{cases}$$

For $b = (s, \alpha, \beta) \in B_\Pi$, we define the degree of $b$ to be $|b| = \alpha + \beta$.

A monomial basis for $\Omega$ is given by the subset

$$B_\Omega = \{(s, \alpha, \beta) \in B_\Pi \mid 1 \leq s \leq p, \alpha \leq p - 1, \beta \leq s - 1\}$$

of $B_\Pi$. When multiplying elements of $B_\Omega$, we use the multiplication rule for $B_\Pi$, with the caveat that products of basis elements are zero if their product in $B_\Pi$ does not belong to $B_\Omega$. A monomial basis for $\Theta$ is given by the subset

$$B_\Theta = \{(s, \alpha, \beta) \in B_\Pi \mid 1 \leq s \leq p - 1, \alpha \leq p - s - 1, \beta \leq s - 1\}$$

of $B_\Pi$.

The action of $\sigma$ on a basis element $(s, \alpha, \beta) \in B_\Theta$ is given by the formula

$$\sigma(s, \alpha, \beta) = (p - s, \beta, \alpha).$$

Our next step is to describe a monomial basis $B_\Lambda$ for $\Lambda$. This is given by

$$B_\Lambda = \{(b, n, h) \mid n, h \in \mathbb{N}, b \in B_\Theta \text{ if } n = 0, b \in B_\Omega \text{ if } n > 0\}.$$  

The product is given by

$$(b, n, h)(b', n', h') = \begin{cases} (bb', n + n', h + h') & \text{if } n \text{ is even;} \\ (b\sigma(b'), n + n', h + h') & \text{if } n \text{ is odd.} \end{cases}$$
Here, basis elements in $B_\Lambda$ are indexed by the pair $(n, h)$ if they belong to the component $(\Theta^\pi)^{\otimes n} \otimes \zeta^h$ of $\Lambda$. We define the bidegree of an element $e = (b, n, h)$ of $B_\Lambda$ to be

$$|e| = (e_i, e_r) = (n + h, ph + |b|).$$

Let $\hat{1} = ((1, 0, 0), 0, 0)$ denote the element of $B_\Lambda$ of bidegree zero which corresponds to the idempotent in $\Pi$ indexed by the vertex 1, in bidegree $(0, 0)$.

We define $|b|_k = |b|$ and $|(b, n, h)|_k = |b|_k + (p - 1)h$, for $(b, n, h) \in B_\Lambda$. A signed monomial basis for the super tensor product $\Lambda^\otimes q \otimes F[z]$ is given by

$$B_q = \{[\beta_1, \ldots, \beta_q, l]|\beta_i \in B_\Lambda, l \in \mathbb{N}\}.$$  

We multiply $q + 1$ tuples in $B_q$ coordinate-wise, according to the super sign convention

$$[\beta_1, \ldots, \beta_q, l][\beta'_1, \ldots, \beta'_q, l'] = (-1)^{\sum_{r < t} |\beta_t| |\beta'_t|} [\beta_1 \beta'_1, \ldots, \beta_q \beta'_q, l + l'].$$

We define the weight of an element $[\beta_1, \ldots, \beta_q, u]$ of $B_q$ to be the element of $\mathbb{Z}^{q+1}$ given by

$$(\beta_1, \beta_2 - \beta_1, \beta_3 - \beta_2, \ldots, \beta_q - \beta_{q-1}, u - \beta_q).$$

We have an embedding of $B_q$ in $B_{q+1}$ which takes $(\beta_1, \ldots, \beta_q, l)$ to $(\hat{1}, \beta_1, \ldots, \beta_q, l)$. This embedding is multiplicative and weight preserving. We define $\mathcal{B}$ to be the union of the sequence of embeddings

$$B_1 \to B_2 \to B_3 \to \ldots$$

We define $B_\Lambda$ to be the subset of $\mathcal{B}$ consisting of elements of weight 0.

**Theorem 2.** The basis $B_\Lambda$, with product obtained by restriction from $\mathcal{B}$, forms a signed monomial basis for $\lambda$.

**Proof.** Let us work this through in terms of torus actions. To say $\Gamma$ is bigraded is to say that a torus of rank 2 acts on $\Gamma$, with $(i, j)$-weight space given by the component $\Gamma^{(i,j)}$. Likewise, to say that $\Delta$ is graded is to say that a torus of rank 1 acts on $\Delta$, with $i$-weight space given by the component $\Delta^i$. Taking the tensor product, we find a torus $G \cong F^{\times 3}$ of rank 3 acts on $\Gamma \otimes \Delta$. By definition, $\mathcal{O}_\Gamma(\Delta)$ is the set of fixed points under the action of a one-dimensional subtorus

$$F^\times \hookrightarrow G$$

$$x \mapsto (1, x^{-1}, x)$$

of $G$ on $\Gamma \otimes \Delta$, with $\mathbb{Z}$-grading given by the weight spaces of the action of another one-dimensional subtorus

$$F^\times \hookrightarrow G$$

$$x \mapsto (x, 1, 1).$$

More generally, $\mathcal{O}_\Gamma^q(\Delta)$ is the set of fixed points under the action of a $q$-dimensional torus acting on $\Gamma^{\otimes q} \otimes \Delta$; and $\mathcal{O} F^\times(\Delta)$ is the set of fixed points under the action of a $q + 1$-dimensional subtorus

$$F^\times(q+1) \hookrightarrow F^\times \times \ldots \times F^\times(q+1),$$

$$(x_1, \ldots, x_{q+1}) \mapsto (x_1, x_2^{-1}, x_2, x_3^{-1}, x_3, \ldots, x_{q+1}^{-1}, x_{q+1})$$

of the direct product of tori acting on $\Gamma^{\otimes q} \otimes \Delta$. Let us now specialise to the case $\Gamma = \Lambda$ and $\Delta = F[z]$. The weight of an element $[\beta_1, \ldots, \beta_q, u]$ of $B_q$ is precisely the weight taken with respect to the action of this $q + 1$-dimensional torus. Elements
of $B_q$ belonging to $\lambda_q$ are precisely the fixed points under the action of this torus, which is to say elements of weight zero. □

The algebra $\lambda$ has a signed monomial basis which, under the isomorphism of Theorem 1, maps to a signed monomial basis for $y$. To complete this section we make some remarks on the structure of this basis.

*Extension grading.* There is a natural grading on $y$, in which extensions of length $n$ are given degree $n$; we call the degree $n$ of such an extension its Yoneda degree. Under the isomorphism between $\lambda$ and $y$, elements of $B_\lambda$ map to elements of $y$ concentrated in a single Yoneda degree. Therefore an element of $B$ has a well defined Yoneda degree. We are left with this question: how do we compute the Yoneda degree of an element of $B_\lambda$? We define the ext-degree of an element $(\beta_1, \ldots, \beta_q, u)$ of $B_q$ to be $u$. The Yoneda degree of an element of $B_\lambda$ is then equal to the ext-degree of the corresponding element of $B_q$, for any sufficiently large $q$.

*Polytopes.* The basis $B_\lambda$ is infinite. However, it is a union of monomial bases $B_{\lambda_q}$ for $\lambda_q$, whose elements are in natural one-one correspondence with elements of finite lattice polytopes $P_q$ of dimension $4q$.

We can build this up as follows: by definition elements of $B_{\Pi}$ are indexed by lattice elements of an infinite polytope in $\mathbb{Z}^3$; elements of $B_{\Omega}$ and $B_{\Theta}$ are indexed by lattice elements of a finite polytope in $\mathbb{Z}^3$; elements of $B_\Lambda$ are indexed by lattice elements of a polytope in $\mathbb{Z}^5$; elements of $B_q$ are indexed by lattice elements of an infinite nonconvex polytope in $\mathbb{Z}^{5q+1}$; elements of $B_q$ of weight 0 are indexed by elements of a polytope in $\mathbb{Z}^{5q+1-(q+1)}$ since they are the intersection of $B_q$ with the kernel of a linear surjection from $\mathbb{Z}^{5q+1}$ to $\mathbb{Z}^{q+1}$; elements of a monomial basis $B_{\lambda_q}$ for $\lambda_q$ are therefore indexed by lattice elements of a polytope $P_q$ in $\mathbb{Z}^{5q}$; the polytope $P_q$ is finite because $\lambda_q$ is finite dimensional.

3. Example

The algebra $\lambda_q$ is isomorphic to the Yoneda extension algebra of a block of Schur algebra $S(2, r)$ with $p^q$ simple modules. Here, and throughout this paper, by the Yoneda extension algebra of an abelian category, we mean the Yoneda extension algebra of a complete set of nonisomorphic simple objects in that category; by the Yoneda extension algebra of an algebra, we mean the Yoneda extension algebra of the category of finite dimensional modules for that algebra. To place our feet on the ground, let us describe an example of such an algebra.

Let $p = 3$. Let $y_2$ denote the Yoneda extension algebra of a block of a Schur algebra $S(2, r)$ with 9 simple modules. The algebra $y_2$ is isomorphic to $FQ/J$, where $Q$ is
the quiver

\[
\begin{array}{c}
\text{(1, 1)} \\
\text{(1, 2)} \\
\text{(1, 3)} \\
\text{(2, 1)} \\
\text{(2, 2)} \\
\text{(2, 3)} \\
\text{(3, 1)} \\
\text{(3, 2)} \\
\text{(3, 3)}
\end{array}
\]

and $J$ is the ideal generated by the following relations

- $xy - yx, fg - gf, \alpha \beta - \beta \alpha$
- $x \beta - \beta x, x \alpha - \alpha x$
- $y \beta - \beta y, y \alpha - \alpha y,$
- $f \beta - \beta f, f \alpha - \alpha f,$
- $g \beta - \beta g, g \alpha - \alpha g$
- $y f - g x, y g - f x, g y - x f, f y - x g,$
- $y x(i, 1), g f(1, i), \beta \alpha(1, i), \beta f(1, i), g \alpha(1, i)$
- $(i, 1) g y(i + 1, 3), (i + 1, 3) x f(i, 1), (i, 1) f y(i - 1, 3), (i - 1, 3) x g(i, 1).$

The solid arrows have Yoneda degree 1, whilst the dotted arrows have Yoneda degree 3.

For example, the projective indecomposable module indexed by vertex $(1, 1)$ has a basis given by paths

\[
\mathcal{B} := \{ e, x e, x^2 e, \\
f f e, y f e, \\
f^2 e, x f^2 e, \\
\alpha e, x \alpha e, x^2 \alpha e, \\
f \alpha e, y f \alpha e, \\
\alpha^2 e, x \alpha^2 e, x^2 \alpha^2 e \}
\]

where $e$ is the idempotent corresponding to vertex $(1, 1)$.

We now describe the monomial basis for this projective indecomposable module. Let $\{a, b, c\}$ be a basis of the projective indecomposable $\Pi$-module indexed by vertex 1, the elements $a$, $b$ and $c$ in degrees 0, 1, 2 respectively. Let $\{\eta, \theta\}$ and $\{\xi, \zeta\}$ be bases of the indecomposable direct summands of the modules $\Theta$ and $\Theta^\sigma$ indexed by the vertex 1; we take $\eta$ and $\xi$ in degree 0 and $\theta$ and $\zeta$ in degree 1. The monomial basis $\mathcal{B}_2$ is the set

\[
\{[(\gamma, 0, h), (\gamma', 0, h')], [(\gamma, 0, h), (\delta, n, h')], l], \\
[(\delta, n, h), (\gamma, 0, h')], l], [(\delta, n, h), (\delta', n', h')], l]\}
\]
where \( h, h', l \in \mathbb{N}, n, n' \in \mathbb{N}_{>0}, \gamma \in \{a, b, c\} \) and \( \delta \in \{\eta, \theta\} \) or \( \{\xi, \zeta\} \) depending on whether \( n \) is even or odd.

The elements of weight zero in here are

\[
\{(\gamma, 0, 0), (\gamma', 0, |\gamma|), p|\gamma| + |\gamma'|, [(\gamma, 0, 0), (\delta, n, |\gamma| - n), |\delta| + p|\gamma| - pn]\}
\]

where \( \gamma, \delta \) are before and \( n = 1, \ldots, |\gamma| \). So the full set is

\[
\mathfrak{B}' := \{[(a, 0, 0), (a, 0, 0), 0], [(a, 0, 0), (b, 0, 0), 1], [(a, 0, 0), (c, 0, 0), 2],
[(b, 0, 0), (\xi, 1, 0), 1], [(b, 0, 0), (\zeta, 1, 0), 2],
[(c, 0, 0), (\eta, 2, 0), 2], [(c, 0, 0), (\theta, 2, 0), 3]
[(b, 0, 0), (a, 0, 1), 3], [(b, 0, 0), (b, 0, 1), 4], [(b, 0, 0), (c, 0, 1), 5]
[(c, 0, 0), (\xi, 1, 1), 4], [(c, 0, 1), (\xi, 1, 1), 5],
[(c, 0, 0), (a, 0, 2), 6], [(c, 0, 0), (b, 0, 2), 7], [(c, 0, 0), (c, 0, 2), 8]\}
\]

We can identify the bases \( \mathfrak{B} \) and \( \mathfrak{B}' \) for \( y_2 \), as ordered sets. Note that the last entry \( l \) in a tuple in the monomial basis gives the Yoneda degree of the corresponding element of \( \mathfrak{B} \).

4. Approach

Our proof of Theorem \([1]\) is delicate. We know of no more direct route from \( y \) to \( \lambda \) than the one we take, which passes through more than ten isomorphisms, if we include isomorphisms we invoke from work in our previous papers. We enjoy the diverse meanders of our proof, and the theory involved which is interesting in itself, but there might be a shorter route.

This section is intended as a guide through those parts of the proof which are described in this paper. In previous papers, we have given many different descriptions of the module category of a block of the Schur algebra \( S(2, r) \) with \( p^\theta \) simple modules. Here we give some further descriptions of the derived category of such a block:

Description 1. As the derived category of a dg algebra \( \mathbb{O}_{F,0}E_0\mathbb{O}_{c,\mathbb{A}}^q(F, F) \).

Description 2. As the derived category of a dg algebra \( \mathbb{O}_{F,0}E_0\mathbb{O}_{c,t}^q(F, F) \).

Description 3. As the derived category of a dg algebra \( \mathbb{O}_{F,0}E_0\mathbb{O}_{c,x}^q(F, F) \).

The algebraic operator \( \mathbb{O} \) is described in section \([4]\). The derived equivalence between 1 and 2 comes from Keller’s theory of dg algebras, as described in section \([7]\). The derived equivalence between 2 and 3 comes from fact that operator \( \mathbb{O}_\gamma(-, -) \) behaves well with respect to Koszul duality equivalence in \(?, \) as established in section \([8]\) and the fact that operator \( \mathbb{O}_\gamma(-, -) \) respects derived equivalence in \( -, \) as shown in section \([6]\). The derived equivalence between 2 and a block of \( S(2, r) \) comes from our previous paper, and is described in section \([10]\).

Let \( y_q \) denote the Yoneda extension algebra of a block of the Schur algebra \( S(2, r) \) with \( p^\theta \) simple modules. Let \( \mathbb{H} \) denote the cohomology operator. We give a number of different descriptions of \( y_q \):

Description 1. As \( \mathbb{H} \) of a dg algebra \( \mathbb{O}_{F,0}E_0\mathbb{O}_{c,\mathbb{A}}^q(F, F) \).

Description 2. As \( \mathbb{H} \) of a dg algebra \( \mathbb{O}_{c,t}^q(F, F) \).

Description 3. As \( \mathbb{O}_\lambda(F[z]) \).

Description 4. In terms of a monomial basis.
The isomorphism between 1 and 2 comes from a quasi-isomorphism between operators $E \circ c$ and $O_\circ c$ as proved in section 9. The isomorphism between 2 and 3 gives a proof of theorem 1, as described in section 12. The isomorphism between 3 and 4 we have already described, in section 2.

5. Algebraic operators

For the development of our theory, we need to discuss trigraded structures

$$S = \bigoplus_{i,j,k \in \mathbb{Z}} S^{i,j,k}.$$ 

It will be necessary to differentiate between the three $\mathbb{Z}$-gradings; we will call them the $i$-grading, the $j$-grading, and the $k$-grading. The $i$ and $j$-gradings will be algebraic: among our constructions, these get mixed up. We denote by $\langle 1 \rangle$ a shift by 1 in the $j$-grading, thus $(M(n))_j = M_{j-n}$.

The $k$-grading will always be a homological grading, representing the degree in a chain complex. Therefore, differentials have $k$-degree 1, and they will always have $(i,j)$-degree $(0,0)$. When we speak of a differential (bi-, tri-)graded algebra, we mean (bi-, tri-)graded algebra which is a differential graded algebra with respect to the $k$-grading. We denote by $\mathbb{H}$ the cohomology functor, which takes a differential $k$-graded complex $C$ to the $k$-graded vector space $\mathbb{H}C = H^\bullet C$. We denote by $[1]$ a shift by 1 in the $k$-grading. Throughout we follow the super sign convention with respect to the $k$-grading.

If we wished, we could make the statements of our paper stronger by considering $\mathbb{Z}^n$-gradings, in which $n$ grows as more operators are applied, but for simplicity, we have decided to stick to trigradings. All the gradings we consider in this paper are gradings by the group $\mathbb{Z}$ of additive integers.

The collection $\mathcal{T}$. Let $\mathcal{T}$ denote the collection of dg algebras with a dg bimodule, $$(A,M) | \ A = \bigoplus A^k \text{ a dg algebra, } M = \bigoplus M^k \text{ a dg A-A-bimodule }$$

Let $(A,M)$ and $(B,N)$ be objects of $\mathcal{T}$.

**Definition 3.** A dg equivalence between objects $(A,M)$ and $(B,N)$ of $\mathcal{T}$ is

$(i)$ A dg $A$-$B$-bimodule $X$ such that $AX$ belongs to $A$-perf, such that $AX$ generates $D_{dg}(A)$, and the natural map $$B \to \text{End}(X)$$

is a quasi-isomorphism;

$(ii)$ A quasi-isomorphism

$$X \otimes_B N \to M \otimes_A X.$$

If there is a dg equivalence between $(A,M)$ and $(B,N)$, we write $(A,M) \bowtie (B,N)$. We then have a derived equivalence

$$D_{dg}(A) \xrightarrow{\sim} D_{dg}(B),$$
and a diagram which commutes up to a natural isomorphism:

\[
\begin{array}{ccc}
D_{dg}(B) & \xrightarrow{X \otimes B} & D_{dg}(A) \\
\downarrow & & \downarrow \\
D_{dg}(B) & \xrightarrow{Y \otimes B} & D_{dg}(A).
\end{array}
\]

We define a *quasi-isomorphism* from \((A, M)\) to \((B, N)\) to be a quasi-isomorphism \(A \to B\), along with a compatible quasi-isomorphism \(_A M_A \to \_B N_B\).  

The operator \(\bigcirc\). We define a *j-graded object* of \(\mathcal{T}\) to be an object \((a, m)\) of \(\mathcal{T}\), where \(a = \bigoplus a^{jk}\) is a differential bigraded algebra, and \(m = \bigoplus m^{jk}\) a differential bigraded \(a\)-\(b\)-bimodule, and \(a^{jk} = m^{jk} = 0\) for \(j < 0\). Given a \(j\)-graded object of \(\mathcal{T}\), we have an operator

\[\bigcirc_{a,m} \circ \mathcal{T}\]

given by

\[\bigcirc_{a,m}(A, M) = (\bigoplus a^{jk} \otimes F M^{\otimes \lambda j}, \bigoplus m^{jk} \otimes F M^{\otimes \lambda j})\]

The algebra structure on \(\bigoplus a^{jk} \otimes F M^{\otimes \lambda j}\) is the restriction of the algebra structure on the super tensor product of algebras \(a \otimes T_M(A)\). The \(k\)-grading and differential on the complex \(\bigoplus a^{jk} \otimes M^{\otimes \lambda j}\) are defined to be the total \(k\)-grading and total differential on the tensor product of complexes. The bimodule structure, grading and differential on \(\bigoplus m^{jk} \otimes M^{\otimes \lambda j}\) are defined likewise. We sometimes write

\[\bigcirc_{a,m}(A, M) = (a(A, M), m(A, M))\]

If \(a\) and \(b\) are differential bigraded algebras, and \(_a x_b\) is differential bigraded \((a, b)\)-bimodule, then we have a differential graded \((a(A, M), b(A, M))\)-bimodule

\[x(A, M) := \bigoplus_{j, k} x^{jk} \otimes M^{\otimes \lambda j}\]

**Lemma 4.** Let \(a\) be a differential bigraded algebra, \(x_a\) and \(_a y\) differential bigraded modules, \((A, M)\) an object of \(\mathcal{T}\). Then

\[x(A, M) \otimes_{a(A, M)} y(A, M) \cong (x \otimes_a y)(A, M)\]

**Proof.** We define a map \(x(A, M) \otimes_{a(A, M)} y(A, M) \to (x \otimes_a y)(A, M)\) sending homogeneous elements \((\alpha \otimes m_1 \otimes \cdots \otimes m_{j_1}) \otimes (\beta \otimes n_1 \otimes \cdots \otimes n_{j_2})\) (where \(\alpha \in x^{j_1}, \beta \in y^{j_2}\), and \(m_1, \ldots, m_{j_1}, n_1, \ldots, n_{j_2} \in M\) to \((\alpha \otimes \beta) \otimes (m_1 \otimes \cdots \otimes m_{j_1} \otimes n_1 \otimes \cdots \otimes n_{j_2})\). This is an isomorphism of differential graded \((a(A, M), a(A, M))\)-bimodules. \(\square\)

We define a *Rickard object* of \(\mathcal{T}\) to be an object \((A, M)\) of \(\mathcal{T}\), where \(A\) is an algebra (aka a dg algebra concentrated in degree zero with trivial differential), and \(_A M_A\) is a two-sided tilting complex \([7]\).

Let us suppose \((A, M)\) is a Rickard object of \(\mathcal{T}\). Then \(M^{-1} = \text{Hom}(M, A)\) is a two-sided tilting complex such that \(M^{-1} \otimes_A - \cong \text{Hom}(M, -)\) induces an inverse equivalence to \(M \otimes -\). The natural map \(q : M \otimes M^{-1} \to A\) which takes \(m \otimes \phi\) to \(\phi(m)\) represents the counit of the adjunction \((M \otimes -, M^{-1} \otimes -)\), and is a quasi-isomorphism.
We previously assumed that \((a, m)\) was a graded object of \(T\) such that \(a^{jk} = m^{jk} = 0\) for \(j < 0\). If we assume rather that \(a^{jk} = m^{jk} = 0\) for \(j > 0\), we define the operator

\[
\mathcal{O}_{a, m} \circ T
\]

by the formula

\[
\mathcal{O}_{a, m}(A, M) = \bigoplus a^{jk} \otimes_F (M^{-1})^{\otimes A - j}, \bigoplus m^{jk} \otimes_F (M^{-1})^{\otimes A - j}
\]

Given a differential bigraded \(a\)-module \(x\), with components in positive and negative \(j\)-degrees, we define \(x(A, M)\) to be the \(a(A, M)\)-module given by

\[
x(A, M) = \bigoplus_{j < 0} x^{j} \otimes (M^{-1})^{\otimes A - j} \otimes A^{-j} \oplus \bigoplus_{j > 0} x^{j} \otimes M^{\otimes A j},
\]

where the action is defined via the action of \(a\) on \(x\), along with the quasi-isomorphism \(q\).

**Lemma 5.** Let \(c\) be a differential bigraded algebra, and \((A, M)\) a Rickard object of \(T\). If \(x\) and \(y\) are differential bigraded \(c\)-modules, then we have a quasi-isomorphism

\[
\text{Hom}_c(x, y)(A, M) \to \text{Hom}_c(A, M)(x(A, M), y(A, M)).
\]

**Proof.** We established this in a previous paper [5, Proof of Theorem 13]. In that paper we only consider the case where \((c, x) = y\) is a Rickard object. However, exactly the same proof works in this more general case. □

**The operator \(\mathcal{O}\).** Let \(\Gamma = \bigoplus \Gamma^{ijk}\) be a differential trigraded algebra. We have an operator

\[
\mathcal{O}_{\Gamma} \circ \{\Delta \mid \Delta = \bigoplus \Delta^{jk} \text{ a differential bigraded algebra}\}
\]

given by

\[
\mathcal{O}_{\Gamma}(\Delta)^{jk} = \bigoplus_{j, k_1 + k_2 = k} \Gamma^{ijk_1} \otimes \Delta^{jk_2}.
\]

The algebra structure and differential are obtained by restricting the algebra structure and differential from \(\Gamma \otimes \Delta\). If we forget the differential and the \(k\)-grading, the operator \(\Gamma\) is identical to the operator \(\Gamma\) defined in the introduction.

**Lemma 6.** We have

\[
\mathbb{H}\mathcal{O}_{\Gamma} \cong \mathbb{H}\mathcal{O}_{\mathbb{H}\Gamma} \cong \mathcal{O}_{\mathbb{H}\mathbb{H}},
\]

for a differential trigraded algebra \(\Gamma\).

**Proof.** The two isomorphisms are basic algebraic checks, based on the facts that the tensor products in \(\mathbb{H}\) are over \(F\), so \(\mathbb{H}(\Gamma^{jk_1} \otimes \Delta^{jk_2}) \cong \mathbb{H}(\Gamma^{jk_1}) \otimes \mathbb{H}(\Delta^{jk_2})\), and that \(\mathbb{H}\mathbb{H} = \mathbb{H}\). Using this we obtain

\[
\mathbb{H}(\mathcal{O}_{\Gamma}(\Delta))^{jk} \cong \mathbb{H}(\bigoplus_{j, k_1 + k_2 = k} \Gamma^{ijk_1} \otimes \Delta^{jk_2})
\]

\[
\cong \bigoplus_{j, k_1 + k_2 = k} \mathbb{H}(\Gamma^{ijk_1}) \otimes \mathbb{H}(\Delta^{jk_2}).
\]

Similarly, we have
Comparing $\mathcal{O}$ and $\mathcal{D}$. Here we describe relations between the operators $\mathcal{O}$ and $\mathcal{D}$. We assume $a = \bigoplus a^{ik}$ is a differential bigraded algebra and $m = \bigoplus m^{jk}$ a differential bigraded $a$-$a$-bimodule. The tensor algebra $T_a(m)$ is a differential trigraded algebra, where $a$ has $i$-degree zero and $m$ has $j$-degree 1, and the $k$-bigrading is inherited from the bigrading of $a$ and $m$. Let $A = \bigoplus A^k$ be a dg algebra, and $M = \bigoplus M^k$ a dg $A$-$A$-bimodule. The tensor algebra $T_a(A)$ is differential bigraded with $k$-degree inherited from the gradings on $A$ and $M$, with $A$ in $j$-degree zero, and $M$ in $j$-degree 1. The algebra $T_a(m)(A, M)$ formed with respect to the $j$-grading on $T_a(m)$, is a differential bigraded algebra, with

$$T_a(m)(A, M)^{ik} = \bigoplus_j T_a(m)^{ijk} \otimes M^k.$$ 

The algebra $T_{a(A, M)}(m(A, M))$ is a differential bigraded algebra, with

$$T_{a(A, M)}(m(A, M))^{ik} = (m(A, M)^{\otimes a(A, M)})^k.$$ 

Lemma 7. (i) We have an isomorphism of objects of $T$

$$\mathcal{O}_{a(m)}(A, M) = (T_a(A))^0, \mathcal{O}_{a(m)}(T_a(A))^1,$$ 

where the $k$-grading on the components of $\mathcal{O}_{a,m}(A, M)$ can be identified with the $k$-grading on $T_{\mathcal{O}_{a(m)}(T_a(A))}$.

(ii) We have an isomorphism of differential bigraded algebras

$$\mathcal{O}_{a(m)}(T_a(A)) \cong T_a(m)(A, M).$$

(iii) We have an isomorphism of differential bigraded algebras

$$T_a(m)(A, M) \cong T_{a(A, M)}(m(A, M)).$$

Proof. The isomorphisms in (i) and (iii) come by definition, the isomorphism in (iii) by Lemma 4. 

Suppose we are given $(a_i, m_i)$ for $1 \leq i \leq n$, and $(A, M)$. Let us define $(A_i, M_i)$ recursively via $(A_i, M_i) = \mathcal{O}_{a_i, m_i}(A_{i-1}, M_{i-1})$ and $(A_0, M_0) = (A, M)$.

Lemma 8. (i) We have an algebra isomorphism

$$T_{A_n}(M_n) \cong \mathcal{O}_{T_{a_n}(m_n)} \cdots \mathcal{O}_{T_{a_1}(m_1)}(T_a(A)).$$
Lemma 10. Let $\mathcal{T}$ be a $j$-graded Rickard object of $\mathcal{T}$. Let

\[ \mathcal{O}_{a,m} \otimes \mathcal{O}_{a,m}(A,M) \cong (\mathcal{O}_{a,m}(1) \otimes \mathcal{O}_{a,m}(T_A(M)))^{1\bullet}. \]

Proof. The first part follows from part (ii) of Lemma 7, and the second part follows from parts (i) and (iii) of Lemma 10. The second step then follows from part (ii) of Lemma 7 and the first part. □

The application of Lemma 8 which will be relevant for our computation of $y$ is:

Corollary 9. Let $a$ be a dg algebra and $m$ an $a$-$a$-bimodule. Then we have an isomorphism of algebras

\[ \mathcal{O}_F \circ \mathcal{O}_{a,m}(F,F) \cong \mathcal{O}_F \mathcal{O}_{T_{a,m}}(F[z]). \]

6. OPERATORS $\mathcal{O}$ RESPECT EQUIVALENCE

Lemma 10. Let $(A,M)$ and $(B,N)$ be objects of $\mathcal{T}$ such that $(A,M) \triangleright (B,N)$. Let $(a,m)$ be a $j$-graded Rickard object of $\mathcal{T}$. Then $\mathcal{O}_{a,m}(A,M) \triangleright \mathcal{O}_{a,m}(B,N)$.

Proof. Let $X$ denote an $A$-$B$-bimodule inducing the dg equivalence $(A,M) \triangleright (B,N)$.

We define

\[ x = a(A,M) \otimes_A X. \]

We claim that $x$ induces a dg equivalence $\mathcal{O}_{a,m}(A,M) \triangleright \mathcal{O}_{a,m}(B,N)$. Clearly $x$ is a left $(A,M)$-module. We have a sequence of canonical homomorphisms

\[ a(B,N) \cong B \otimes_B a(B,N) \cong \text{Hom}_A(X,X) \otimes_B a(B,N) \cong \text{Hom}_A(X,X \otimes_B a(B,N)) \rightarrow \text{Hom}_A(X,a(A,M) \otimes_A X) \cong \text{Hom}_A(X,\text{Hom}_{a(A,M)}(a(A,M),a(A,M) \otimes_A X)) \cong \text{Hom}_{a(A,M)}(a(A,M) \otimes_A X,a(A,M) \otimes_A X) = \text{Hom}_{a(A,M)}(x,x), \]

whose composition give $x$ the structure of an $(a(A,M)-a(B,N)$-bimodule. Here the third isomorphism is by virtue of $X$ being projective over $A$. Every term in this sequence is a quasi-isomorphism, which means their composite is also a quasi-isomorphism, as we require.

We have a sequence of quasi-isomorphisms.

\[ x \otimes_{a(B,N)} m(B,N) = a(A,M) \otimes_A X \otimes_{a(B,N)} m(B,N) \rightarrow (a \otimes m) (A,M) \otimes_A X \cong m(A,M) \otimes_A X \cong m(A,M) \otimes_{a(A,M)} a(A,M) \otimes_A X \cong m(A,M) \otimes_{a(A,M)} x \]

where the quasi-isomorphism comes just like the isomorphism of Lemma 8. □

Lemma 11. Let $(A,M)$ and $(B,N)$ be quasi-isomorphic objects of $\mathcal{T}$. Let $(a,m)$ be a $j$-graded Rickard object of $\mathcal{T}$. Then $\mathcal{O}_{a,m}(A,M)$ and $\mathcal{O}_{a,m}(B,N)$ are quasi-isomorphic objects of $\mathcal{T}$. 
Reversing gradings. Let us denote by $\mathbb{R}$ the sign reversing operator $\mathbb{R}$ on differential bigraded algebras, which sends a differential bigraded algebra $a = \oplus_{j,k} a^{j,k}$ to itself with reverse grading, $\mathbb{R}a^{j,k} = a^{-j,k}$. Likewise, if $m$ is a differential bigraded $a$-$a$-bimodule, we define $\mathbb{R}m$ by $\mathbb{R}m^{j,k} = m^{-j,k}$.

**Lemma 12.** Let $(a, m)$ and $(b, n)$ be graded Rickard objects of $\mathcal{T}$, and $(A, M)$ a Rickard object of $\mathcal{T}$. Then there is a quasi-isomorphism of objects of $\mathcal{T}$, 

$$O_{\mathbb{R}a, \mathbb{R}m}O_{b, n^{-1}}(A, M) \rightarrow O_{a, m}O_{b, n}(A, M).$$

**Proof.** We have an isomorphism

$$O_{a, m}(B, N) \cong O_{\mathbb{R}a, \mathbb{R}m}(B, N^{-1}),$$

since both sides are given by the object

$$(\oplus_j a^{j} \otimes_F M^{\otimes \lambda^j}, \oplus_j m^{j} \otimes_F M^{\otimes \lambda^j})$$

of $\mathcal{T}$. We have a quasi-isomorphism

$$n^{-1}(A, M) \rightarrow n(A, M)^{-1}$$

by Lemma 5, meaning we have a quasi-isomorphism

$$O_{a, n^{-1}}(A, M) \rightarrow (a(A, M), n(A, M)^{-1}).$$

The operator $O_{\mathbb{R}a, \mathbb{R}m}$ respects quasi-isomorphisms by Lemma 10. Putting this together in case $(B, N) = O_{b, n}(A, M)$, we obtain a quasi-isomorphism

$$O_{\mathbb{R}a, \mathbb{R}m}O_{b, n^{-1}}(A, M) \rightarrow O_{a, m}O_{b, n}(A, M)$$

as required. 

\[\Box\]

### 7. Homological Duality

There are a number of approaches to homological duality in representation theory. Here we describe two. One is a general approach via differential graded algebras, due to Keller. The other is a special approach for Koszul algebras, which is easier to work with explicitly.

**Keller equivalence.** Let $A$ be a finite dimensional algebra with simple modules $S_1, \ldots, S_d$. Let $P_1 = \bigoplus_i P_k^1$ be a projective resolution of $S_i$. Denote by $\mathcal{E}(A)$ the dg-algebra $\bigoplus_{k, k'} \text{Hom}_A(\bigoplus_{i=1}^d P_k^i, \bigoplus_{i=1}^d P_{k'}^i)$. Then $P = \bigoplus_i P_i$ is a differential $A$-$\mathcal{E}(A)$-bimodule. There are mutually inverse equivalences

$$D_{dg}(P) \cong D_{dg}(\mathcal{E}(A)),$$

by a theorem of Keller [4, Theorem 3.10]. Since $P$ is projective as an $A$-module, we have a natural isomorphism of functors

$$\text{Hom}_A(P, -) \cong \text{Hom}_A(P, A) \otimes_A -.$$

**The operator $\mathcal{E}$.** Here we define an operator $\mathcal{E}$ on $\mathcal{T}$. Given a Rickard tilting complex $M$ of $(A, A)$-bimodules denote by $\mathcal{E}(M)$ the dg-bimodule

$$\mathcal{E}(M) = \text{Hom}_A(P, A) \otimes_A M \otimes_A P.$$ 

We have $\mathcal{E}(M) \cong \text{Hom}_A(P, M) \otimes_A P$. Let $E(A, M) := (\mathcal{E}(A), \mathcal{E}(M))$. 

Lemma 13. The bimodule $P$ induces a dg equivalence $(A, M) \cong E(A, M)$.

Proof. We need to give a quasi-isomorphism $P \otimes_{E(A)} E(M) \to M \otimes_A P$, or in other words a quasi-isomorphism

$$P \otimes_{E(A)} \text{Hom}_A(P, A) \otimes_A M \otimes_A P \to M \otimes_A P.$$ 

We obtain this via the natural maps

$$P \otimes_{E(A)} \text{Hom}_A(P, A) \to A, \quad A \otimes_A P \to P.$$ 

The first of these is an isomorphism since $P$ is a progenerator for $A$, and the second is also an isomorphism. Since $M \otimes_A P$ is projective on the left, the entire map is a quasi-isomorphism. □

We thus have a natural isomorphism making the following diagram commute:

$$
\begin{array}{ccc}
D_{dg}(A) & \xrightarrow{\text{Hom}_A(P, -)} & D_{dg}(E(A)) \\
\downarrow M \otimes_A - & & \downarrow \text{E}M \otimes_{E(A)} - \\
D_{dg}(A) & \xrightarrow{\text{Hom}_A(P, -)} & D_{dg}(E(A))
\end{array}
$$

Lemma 14. We have an isomorphism

$$\text{E}(M)^{\otimes_{E(A)^r}} \to \text{Hom}(P_A, M^{\otimes_{A^r}} \otimes P_A).$$

Proof. If we write out $\text{E}(M)^{\otimes_{E(A)^r}}$ in full, we find that internal terms $P \otimes_{E(A)} \text{Hom}_A(P, A)$ cancel, thanks to the isomorphism $P \otimes_{E(A)} \text{Hom}_A(P, A) \to A$, leaving us with

$$\text{Hom}(P_A, A) \otimes_A M^{\otimes_{A^r}} \otimes_A P_A \cong \text{Hom}(P_A, M^{\otimes_{A^r}} \otimes_A P_A)$$

□

Koszul duality. In this subsection we summarise the results we need from [1]. We assume $a = \bigoplus a^j$ is a graded finite dimensional algebra with finite global dimension, with $a^0$ semisimple, and $a^j = 0$ for $j < 0$. Then

$$a = T_{a^0}(a^1)/R,$$

where $R$ is the kernel of the multiplication map

$$a^1 \otimes_{a^0} a^1 \to a^2.$$

We write $x^* = \text{Hom}_{(a^0, x, a^0 a^0)}$, and $^* x = \text{Hom}(x_{a^0}, a^0 a^0)$ if $x$ is a left/right $a^0$-module. If $x$ is concentrated in degree $j$, then by convention $x^*$ and $^* x$ are concentrated in degree $-j$. The quadratic dual algebra $a^!$ is given by

$$a^! = T_{a^0}(a^1*)/a^{2*},$$

where $a^{2*}$ embeds in $a^{1*} \otimes_{a^0} a^{1*}$ via the dual of the multiplication map, composed with the inverse of the natural isomorphism

$$a^{1*} \otimes_{a^0} a^{1*} \to (a^1 \otimes_{a^0} a^1)^*.$$ 

We then have $a^{1j} = 0$ for $j > 0$. The Koszul complex is defined to be the differential graded $a-a^!$-bimodule

$$K = a \otimes_{a^0} (a^1),$$
We now consider how Koszul duality behaves towards the operator \( s \).

The action of \( s \) is given by

\[
K^k = a \otimes_{a^0} (a^{1-k}) \cong \text{Hom}(a^{1-k}_{a^0}, a_{a^0}),
\]

and whose differential is given by the composition

\[
\text{Hom}(a^{1-k}_{a^0}, a_{a^0}) \to \text{Hom}(a^{1-k}_{a^0} \otimes_{a^0} a^1, a_{a^0} \otimes_{a^0} a^1) \to \text{Hom}(a^{1-k}_{a^0}, a_{a^0}),
\]

obtained from the multiplication map

\[
a \otimes_{a^0} a^1 \to a,
\]

and the natural composition

\[
a^{1-k} = a^{1-k} \otimes_{a^0} a^0 \to a^{1-k} \otimes_{a^0} a^1 \to a^{1-k-1} \otimes a^1.
\]

Note that the differential has \( j \)-degree zero.

The algebra \( a \) is said to be Koszul precisely when \( K \to a^0 \) is a projective resolution of \( a a^0 \), in which case the map \( a' \to \mathcal{E}(a) \) is a quasi-isomorphism. There are mutually inverse equivalences

\[
\text{Hom}_a(K, -) \xrightarrow{\text{D}^{b}(a, -)} \text{D}^{b}(a', -)
\]

Under Koszul duality, the irreducible \( a \)-module \( a^0 e \) corresponds to the projective \( a' \)-modules \( a' e \), for a primitive idempotent \( e \in a^0 \); a shift \( \langle j \rangle \) in \( D^{b}(a, -) \) corresponds to a shift \( \langle j \rangle [-j] \) in \( D^{b}(a', -) \).

**Warning.** Our \( j \)-grading convention does not coincide with that in the literature. We assume that \( a \) is generated in degrees 0 and 1 whilst \( a' \) is generated in degrees 0 and \(-1\). Beilinson, Ginzburg and Soergel assume that \( a \) is generated in degrees 0 and 1 but that \( a' \) is also generated in degrees 0 and 1. We have changed conventions in order to obtain gradings which behave well with respect to our operators.

**The operator \(!\).** Let us now assume \( (a, m) \) is a \( j \)-graded Rickard object of \( T \), with a Koszul. We call such an object a **Koszul object** of \( T \). Let \( K \) denote a projective resolution of the complex \( K \) of graded \( a' \)-modules. Let \( m^i \) denote the differential bigraded \( a' \)-bimodule \( \text{Hom}(K, a) \otimes_a m \otimes_a K \).

**Lemma 15.** (i) The \( \text{d} \) bimodule \( K \) induces a \( \text{d} \)g equivalence \( (a, m) \to (a', m^i) \).

(ii) We have a quasi-isomorphism from \( (a', m^i) \) to \( \mathbb{E}(a, m) \).

**Proof.** The proof of the first part goes as the proof Lemma 13.

The action of \( a' \) induces a quasi-isomorphism \( a' \to \mathcal{E}(a) \), which gives the second part. \( \square \)

**8. Koszul duality for operators \( \Box \)**

We now consider how Koszul duality behaves towards the operators \( \Box \).

**Lemma 16.** Let \( (A, M) \) be a Rickard object of \( T \). Let \( (a, m) \) be a Koszul object of \( T \), with Koszul dual \( (a', m^i) \). Then we have a \( \text{d} \)g equivalence \( \Box_{a, m}(A, M) \to \Box_{a', m^i}(A, M) \).

**Proof.** Consider \( K(A, M) \). We show that \( K(A, M) \) is a bimodule inducing a \( \text{d} \)g equivalence \( \Box_{a, m}(A, M) \sim \Box_{a', m^i}(A, M) \). By Lemma 14 we have an isomorphism

\[
K(A, M) = a(A, M) \otimes_{a^0(A, M)} (a')^*(A, M).
\]
Obviously \(a(A, M)\) acts on the left of \(a(A, M)\); since \(a^!\) acts on the right of \(* (a^!)*\), \(a^!\) acts on the right of \(* (a^!)(A, M)\); this implies that \(K(A, M)\) is a \(a(A, M)\)-\(a^! (A, M)\)-bimodule. The action of \(a^!\) on \(K\) induces a quasi-isomorphism

\[ a^! \rightarrow \text{Hom}_a(K, K). \]

Consequently, by \([5, \text{Lemma 15}]\), we have a quasi-isomorphism

\[ a^!(A, M) \rightarrow \text{Hom}_a(K, K)(A, M) \]

which, when composed with the quasi-isomorphism

\[ \text{Hom}_a(K, K)(A, M) \rightarrow \text{Hom}_{a(A, M)}(K(A, M), K(A, M)) \]

of Lemma \([5] \) gives us a quasi-isomorphism

\[ a^!(A, M) \rightarrow \text{Hom}_{a(A, M)}(K(A, M), K(A, M)) \]

which is induced by the action of \(a^!(A, M)\) on \(K(A, M)\).

The object \(K(A, M)\) generates \(D_{dg}(a(A, M)) \cong D_{dg}(a^!(A, M))\), because \(K(A, M)\) is quasi-isomorphic to \(a^!(A, M) \cong a^0 \otimes F A\), and \(A\) generates \(D_{dg}(A)\). All that remains for us to do is to establish a quasi-isomorphism of \(a(A, M)\)-\(a^!(A, M)\)-bimodules

\[ K(A, M) \otimes_{a^!(A, M)} m^!(A, M) \rightarrow m(A, M) \otimes_{a(A, M)} K(A, M) \]

However, this follows from Lemma \([4] \) and the fact that we have a quasi-isomorphism of \(a\)-\(a^!\)-bimodules

\[ K \otimes_{a^!} m^! \rightarrow m \otimes_a K. \]

\[ \square \]

9. A QUASI-ISOMORPHISM OF OPERATORS

Here we show that \(E_0 \Gamma\) is quasi-isomorphic to \(O_\Gamma E\).

**Theorem 17.** Let \((A, M)\) be a Rickard object of \( \mathcal{T} \). Let \((a, m)\) be a be a Koszul object of \( \mathcal{T} \). We have a chain of dg equivalences

\[ E(\mathcal{O}_{a, m}(A, M)) \leftrightarrow \mathcal{O}_{a, m}(A, M) \triangleright \mathcal{O}_{a^!, m^!}(A, M) \triangleright \mathcal{O}_{a^!, m^!}(E(A, M)). \]

**Proof.** This follows from Lemmas \([13, 10, 16]\) \[\square\]

Theorem \([17]\) implies we have a dg equivalence between the objects \(E(\mathcal{O}_{a, m}(A, M))\) and \(\mathcal{O}_{a^!, m^!}(E(A, M))\) of \( \mathcal{T} \). We strengthen this as follows:

**Theorem 18.** The chain of equivalences in Theorem \([17]\) lifts to a quasi-isomorphism from \( \mathcal{O}_{a^!, m^!}(E(A, M)) \) to \( E(\mathcal{O}_{a, m}(A, M)) \).

**Proof.** We need to prove that \(\mathcal{E}(a(A, M))\) is quasi-isomorphic to \(a^!(E(A, M))\) and that \(\mathcal{E}(m(A, M))\) is quasi-isomorphic to \(m^!(E(A, M))\). We start with the first assertion. By definition, \(\mathcal{E}(a(A, M))\) is obtained by the following recipe: first take a projective resolution of the simple \(a(A, M)\)-modules, then take its endomorphism ring. It is therefore enough to show that we can compute \(a^!(E(A, M))\) by taking a projective resolution of the simple \(a(A, M)\)-modules, and then taking the endomorphism ring.

We denote by \(A^0\) a direct sum of a complete set of nonisomorphic simple \(A\)-modules. Then \(a^0 \otimes_F A^0\) is a direct sum of a complete set of nonisomorphic simple \(a(A, M)\)-modules. Let \(P^*_a\) denote a minimal projective resolution of \(A^0\), and \(P^*_a\) a minimal projective resolution of \(a^0\). We have \(\mathcal{E}(A) = \text{End}(P^*_a)\). A projective resolution of
the \( a(A, M) \)-module \( a^0 \otimes A^0 \) is given by \( P_a(A, M) \otimes_A P_A \). We can therefore write \( \mathcal{E}(a(A, M)) = \text{End}(P_a(A, M) \otimes_A P_A) \). We have a sequence of quasi-isomorphisms

\[
a^1(\mathcal{E}(A), \mathcal{E}(M)) \cong \text{Hom}_A(P_A, a'(A, M) \otimes_A P_A)
\]

by Lemma [13]

\[
\rightarrow \text{Hom}_A(P_A, \mathcal{E}(a)(A, M) \otimes_A P_A)
\]

by Koszul duality, and projectivity of \( P_A \),

\[
\cong \text{Hom}_A(P_A, \text{Hom}_a(P_a, a')(A, M) \otimes_A P_A)
\]

by definition of \( \mathcal{E}(a) \),

\[
\cong \text{Hom}_A(P_A, \text{Hom}_a(A, M)(P_a(A, M), a'(A, M)) \otimes_A P_A)
\]

by Lemma [14]

\[
\cong \text{Hom}_A(P_A, \text{Hom}_a(A, M)(P_a(A, M), a'(A, M) \otimes_A P_A))
\]

by projectivity of \( P_A \),

\[
\cong \text{Hom}_a(A, M)(P_a(A, M) \otimes_A P_A, a'(A, M) \otimes_A P_A)
\]

by adjunction,

\[
\cong \text{End}(P_a(A, M) \otimes_A P_A)
\]

\[
= \mathcal{E}(a(A, M)).
\]

We likewise have a sequence of quasi-isomorphisms

\[
m^1(\mathcal{E}(A), \mathcal{E}(M)) \cong \text{Hom}_A(P_A, m^1(A, M) \otimes_A P_A)
\]

\[
\rightarrow \text{Hom}_A(P_A, \mathcal{E}(m)(A, M) \otimes_A P_A)
\]

\[
\cong \text{Hom}_A(P_A, \text{Hom}_a(P_a, m \otimes_A P_a)(A, M) \otimes_A P_A)
\]

\[
\cong \text{Hom}_A(P_A, \text{Hom}_a(A, M)(P_a(A, M), m(A, M) \otimes_A P_a(A, M)) \otimes_A P_A)
\]

\[
\cong \text{Hom}_A(P_A, \text{Hom}_a(A, M)(P_a(A, M), m(A, M) \otimes_A P_a(A, M)) \otimes_A P_A)
\]

\[
\cong \text{Hom}_a(A, M)(P_a(A, M) \otimes_A P_A, m(A, M) \otimes_A P_a(A, M))
\]

\[
= \mathcal{E}(m(A, M))
\]

The composition of these gives the quasi-isomorphisms we require. \hfill \Box

In light of our interest in the object \( \mathcal{E}(\mathcal{O}_{\ell', r'}(F, F)) \) (see Section [10]), we record the following corollary.

**Corollary 19.** Let \((a, m)\) be a be a Koszul object of \( \mathcal{T} \). Then \( \mathcal{E}(\mathcal{O}^g_{a, m}(F, F)) \) and \( \mathcal{O}^g_{a', m'}(F, F) \) are quasi-isomorphic.

**Proof.** This is proved by induction on \( q \). The base step is given by Lemma [13] (ii), so we can inductively assume \( \mathcal{E}(\mathcal{O}^{g-1}_{a, m}(F, F)) \) is quasi-isomorphic to \( \mathcal{O}^{g-1}_{a', m'}(F, F) \). By Theorem [16] and the fact that \( \mathcal{O}^{g-1}_{a, m}(F, F) \) is a Rickard object of \( \mathcal{T} \), the dg algebra \( \mathcal{E}(\mathcal{O}^g_{a', m'}(F, F)) \) is quasi-isomorphic to \( \mathcal{O}^{g-1}_{a', m'}(F, F) \). This dg algebra is in turn, by Lemma [11] and the induction hypothesis, quasi-isomorphic to \( \mathcal{O}^g_{a', m'}(F, F) \). \hfill \Box

10. \( GL_2 \)

Here we recall some facts about the rational representation theory of \( G = GL_2(F) \). The category \( G\text{-mod} \) is by definition the category of rational representations of \( G \). The category \( G\text{-mod} \) is a direct sum of countably many blocks, each of which is equivalent to the principal block, namely the block of \( G\text{-mod} \) containing the trivial module. The category of polynomial representations of \( G \) of degree \( r \) is equivalent to the category \( S(2, r)\text{-mod} \) of representations of the Schur algebra \( S(2, r) \) [23]. All blocks of \( S(2, r)\text{-mod} \) whose number of isomorphism classes of simple objects is \( p^g \)
are equivalent. We have a combinatorial way to describe these blocks, which we now describe.

Let $Z$ denote the zigzag algebra generated by the quiver

$$\cdots \xrightarrow{\xi} \xrightarrow{\eta} \xrightarrow{\xi} \xrightarrow{\eta} \xrightarrow{\xi} \xrightarrow{\eta} \cdots,$$

modulo relations $\xi^2 = \eta^2 = \xi\eta + \eta\xi = 0$. We denote by $\tau$ the algebra involution of $Z$ which sends vertex $i$ to vertex $p - i$ and exchanges $\xi$ and $\eta$. Let $e_l$ denote the idempotent of $Z$ corresponding to vertex $l \in \mathbb{Z}$. Let

$$t = \sum_{1 \leq i \leq p, 0 \leq m \leq p-1} e_l Z e_m.$$

Then $t$ admits a natural left action by the subquotient $c$ of $Z$ given by

$$c = F(\cdots \xrightarrow{\xi_1} \xrightarrow{\eta_1} \xrightarrow{\xi_2} \xrightarrow{\eta_2} \xrightarrow{\xi_3} \xrightarrow{\eta_3} \cdots \xrightarrow{\xi_{p-1}} \xrightarrow{\eta_{p-1}} \cdots) / I,$$

where $I = \langle \xi_{i+1}\xi_i, \eta_i\eta_{i+1}, \xi_i\eta_i + \eta_{i+1}\xi_i, \xi_{p-1}\eta_{p-1} \mid 1 \leq l \leq p - 2 \rangle$. By symmetry, $t$ admits a right action by $c$, if we twist the regular right action by $\tau$. In this way, $t$ is naturally a $c$-$c$-bimodule.

The algebra $c$ is a very well-known quasi-hereditary algebra, and the left restriction $c t$ of $t$ is a full tilting module for $c$. The natural homomorphism $c \rightarrow \text{Hom}(c t, c t)$ defined by the right action of $c$ on $t$ is an isomorphism, implying that $c$ is Ringel self-dual. If we let $\tilde{t}$ denote a projective resolution of $t$ as a $c$-$c$ bimodule, then $\tilde{t}$ is a two-sided tilting complex, and $\tilde{t} \otimes_c -$ induces a self-equivalence of the derived category $\text{D}^b(c)$ of $c$.

The operator $O_{c, t}$ acts on the collection of algebras with a bimodule, such as the pair $(F, F)$ whose algebra is $F$ and whose bimodule is the regular bimodule $F$. The operator $O_{F, 0} O_{c, t}$ takes an algebra with a bimodule to an algebra, along with a zero bimodule which we disregard. We define $b_q$ to be the category of modules over the algebra $O_{F, 0} O_{c, t}(F, F)$. We have an algebra homomorphism $c \rightarrow F$ which sends a path in the quiver to $1 \in F$ if it is the path of length zero based at $1$, and $0 \in F$ otherwise. This algebra homomorphism lifts to a morphism of operators $O_{c, t} \rightarrow O_{F, 0}$. We have $O_{F, 0}^2 = O_{F, 0}$: We thus have a natural sequence of operators

$$O_{F, 0} \leftarrow O_{F, 0} O_{c, t} \leftarrow O_{F, 0} O_{c, t}^2 \leftarrow \cdots,$$

which, if we apply each term to $(F, F)$ and take representations, gives us a sequence of embeddings of abelian categories

$$b_1 \rightarrow b_2 \rightarrow b_3 \rightarrow \cdots.$$

We denote by $b$ the union of these abelian categories. In a previous paper [5], we have proved the following theorem:

**Theorem 20.** [5 Corollary 27] *Every block of $G$-mod is equivalent to $b$. Every block of $S(2, r)$-mod whose number of isomorphism classes of simple objects is $p^a$ is equivalent to $b_q$.***

To compute the Yoneda extension algebra $Y$ of $G$-mod, it is enough to compute the Yoneda extension algebra $y$ of the principal block of $G$-mod which, thanks to the above theorem, we can identify with the Yoneda extension algebra of $b$. Since each
embedding $b_q \to b_{q+1}$ corresponds to taking an ideal in the poset of irreducibles for $b_{q+1}$, the theory of quasi-hereditary algebras gives us a sequence of fully faithful embeddings of derived categories $D^b(b_1) \to D^b(b_2) \to D^b(b_3) \to ...$

Therefore, if we define $y_q$ to be the Yoneda extension algebra of $b_q$, we have a sequence of algebra embeddings $y_1 \to y_2 \to y_3 \to ...$

whose union is equal to $y$.

11. **An explicit dg algebra**

Let $c$ denote the quasi-hereditary algebra with $p$ irreducible modules introduced in Section 10, and $t$ its tilting bimodule. Let us replace the $c$-$c$-bimodule $t$ by a Rickard tilting complex quasi-isomorphic to $t$, and thus define a differential bigraded $c$-$c$-bimodule $t$. In light of Theorem 18, we wish to compute iterated applications of the operator $\mathcal{O}_c$, $t$ to the pair $(F,F)$. However, $c$ is now negatively graded so to evaluate this we need to work with the adjoint of $t$, which is given by the differential bigraded $c$-$c$-bimodule $t^{-1} = \text{Hom}_c(t, c)$.

To be optimistic, we wish to describe algebras with positive gradings rather than negative gradings. Therefore rather than working with the negatively graded algebra $c$, we work with the positively graded algebra $\Omega$, which is canonically isomorphic to $c$ as an algebra, but whose $j$th homogeneous component is equal to the $-j$th homogeneous component of $c$. We define the differential bigraded $\Omega$-$\Omega$-bimodule $\Xi$ to be the differential bigraded $c$-$c$-bimodule $t^{-1}$, whose $(j,k)$th homogeneous component is equal to the $(-j,k)$th homogeneous component of $c$. Thus $\Omega = \mathbb{R}c$ and $\Xi = \mathbb{R}t^{-1}$.

Note that

$$\Omega = \mathbb{F} \left( \begin{array}{ccc} x_1 & \cdots & x_p \end{array} \right) \left( \begin{array}{ccc} y_1 & \cdots & y_p \end{array} \right) / I^\perp,$$

where $I^\perp = \langle x_ly_l - y_{l+1}x_{l+1}, y_1x_1 \mid 1 \leq l \leq p - 2 \rangle$.

Since we consider $\Omega$ as the extension algebra of $c$ (with the $j$-grading multiplied by $-1$), it is naturally a dg algebra concentrated in positive $k$-degrees with zero differential. The generators $x$ and $y$ are in $k$-degree 1.

In this section, we are concerned with establishing the following result:

**Proposition 21.** We have an isomorphism of $j$-graded algebras $\mathbb{H}(\mathcal{T}_\Omega(\Xi)) \cong \Lambda$.

We want to describe the self-equivalence of $D^b(\Omega)$ which is induced by $\Xi$. We first compute the functor $t^{-1} \otimes -$ on various $c$-modules.

The algebra $c$ has irreducible modules $L(l)$ indexed by integers $1 \leq l \leq p$. The algebra has standard modules $\Delta(l)$ which have top $L(l)$ and socle $L(l-1)$ in case
2 \leq l \leq p, and \Delta(1) = L(1). The algebra has costandard modules \nabla(l) which have socle \( L(l) \) and top \( L(l-1) \) in case 2 \( \leq l \leq p, \) and \nabla(1) = L(1). We define \( L(l), \Delta(l) \) and \nabla(l) as \( j \)-graded modules by insisting their tops are concentrated in degree 0.

**Lemma 22.** We have exact triangles in the derived category of graded \( \mathbf{c} \)-modules as follows:

\[ L(p-l) \rightarrow t^{-1} \otimes_{\mathbf{c}} L(l) \rightarrow L(p)[-l][l-1] \rightarrow, \]

for \( 1 \leq i < p, \) and

\[ 0 \rightarrow t^{-1} \otimes_{\mathbf{c}} L(p) \rightarrow L(p)[-p][p-1] \rightarrow . \]

**Proof.** We can identify \( t^{-1} \otimes - \) with \( \text{Hom}(t, -) \), which by Ringel duality takes costandard modules \( \nabla \) to standard modules \( \Delta \). Easy computations establish the following formulae:

\[ t^{-1} \otimes_{\mathbf{c}} \nabla(l) = \Delta(p-l+1), \]

for \( 2 \leq l \leq p \) and

\[ t^{-1} \otimes_{\mathbf{c}} \nabla(1) = \Delta(p)[-1]. \]

Since we have a quasi-isomorphism

\[ L(l) \cong (\nabla(l)[-1] \rightarrow \nabla(l-1)[-2] \rightarrow \cdots \rightarrow \nabla(2)\langle-(l-1)\rangle \rightarrow \nabla(1)\langle-(l-1)\rangle), \]

we obtain

\[ t^{-1} \otimes_{\mathbf{c}} L(l) = (\Delta(p-l+1)[-1] \rightarrow \Delta(p-l+2)[-2] \rightarrow \cdots \rightarrow \Delta(p)[-l]) \]

for \( 1 \leq l \leq p, \) where the leftmost term lies in \( k \)-degree 0. For \( 1 \leq l \leq p-1, \) this is exact except in \( k \)-degree 0, where the kernel is \( L(p-l) \) and \( k \)-degree \( l-1, \) where the cokernel is \( L(p)[-l]. \) For \( l = p, \) this is exact except in \( k \)-degree \( p-1, \) where the cokernel is \( L(p)[-p]. \) This completes the proof of the lemma.

We now compute a one-sided tilting complex which is quasi-isomorphic to \( \Xi. \) The advantage of this complex over \( \Xi \) is that its structure is extremely explicit, making computations possible.

Consider the differential \( jk \)-bigraded \( \Omega \)-module which, written as a two-term complex \( W = (W^1 \xrightarrow{e} W^0), \) is given by

\[ \Omega e_p \otimes_F e_p FQ e \rightarrow \Omega e, \]

where \( Q \) is the type \( A_p \), subquiver of the quiver of \( \Omega \) generated by arrows \( y, \) where \( FQ \) is the corresponding hereditary subalgebra of \( \Omega, \) where \( e = \sum_{1 \leq l \leq p-1} e_l \) is the sum over the idempotents \( e_l \) at \( l \) and where the differential on the complex is given by the algebra product. Consider also the differential \( jk \)-bigraded \( \Omega \)-module which, written as a two term complex, is given by

\[ \Omega e_p(p)[p] \rightarrow 0. \]

Let \( X \) denote the sum of these two-term complexes.

**Lemma 23.** We have a quasi-isomorphism between the differential \( jk \)-bigraded \( \Omega \)-modules \( X \) and \( \Xi. \)

**Proof.** Under Koszul duality, considered as an equivalence between derived categories of graded modules, an irreducible \( \mathbf{c} \)-module corresponds to a projective \( \mathbf{c} \)-module, hence a projective \( \Omega \)-module. A shift \( \langle j \rangle \) for \( \mathbf{c} \)-modules corresponds to a shift \( \langle j \rangle \rightarrow \langle j \rangle \) for \( \mathbf{c} \)-modules, therefore a shift \( \langle -j \rangle \rightarrow \langle -j \rangle \) for \( \Omega \)-modules. (Warning: the homological grading in the derived category of \( \Omega \)-modules does not coincide with
the $k$-grading since in the first $\Omega$ is concentrated in a single degree whilst in the second $\Omega$ is concentrated in many degrees). We therefore have exact triangles in the derived category of graded $\Omega$-modules as follows:

$$\Omega e_{p-l} \to \Xi \otimes \Omega e_l \to \Omega e_p(l)[1] \to,$$

for $1 \leq l < p$, and

$$0 \to \Xi \otimes \Omega e_p \to \Omega e_p(p)[1] \to.$$

Here $\Omega e_l$ denotes the projective indecomposable $\Omega$-module indexed by $l$. We therefore have exact triangles in the derived category of graded $\Omega$-modules

$$\Omega e_p(l) \to \Omega e_{p-l} \to \Xi \otimes \Omega e_l \to,$$

for $1 \leq l < p$. Morphisms in this derived category between projective objects lift to morphisms of chain complexes. The object $\Xi \otimes \Omega e_l$ is indecomposable. Up to scalar, there is a unique graded homomorphism $\phi$ from $\Omega e_p(l)$ to $\Omega e_{p-l}$. This implies we have an isomorphism in the derived category

$$\Xi \otimes \Omega e_l \cong (\Omega e_p(l) \xrightarrow{\phi} \Omega e_{p-l}),$$

or to put it another way, an isomorphism

$$\Xi \otimes \Omega e_l \cong (\Omega e_p \otimes_F e_p F Q e_{p-l} \to \Omega e_{p-l}),$$

where the term $\Omega e_{p-l}$ is concentrated in homological degree 0, and the differential is given by multiplication. We thus find $X$ is quasi-isomorphic to $\Xi$ as required. □

**Lemma 24.** We have an exact sequence of $j$-graded $\Omega$-modules,

$$0 \to \Omega e_l(p) \to \Omega e_p(l) \to \Omega e_{p-l} \to 0,$$

for $1 \leq l \leq p - 1$.

**Proof.** Here is a picture of the Loewy structure of the projective indecomposable $\Omega$-modules:
Here is a picture of the Loewy structure of the projective indecomposable Θ-modules:

The exact sequence is visible from these pictures.

The sum of the first two terms in the above exact sequence are isomorphic to $W^0(p)$ and $W^1$. We define the sum of the maps between these terms to be

$$\delta : W^0(p) \to W^1.$$
Let $\gamma$ denote the automorphism of $\Omega$ that sends a homogeneous element $\omega$ to $(-1)^{|\omega|} \omega$.

**Lemma 25.** $\gamma \circ \Omega$ is an inner automorphism that sends $\omega$ to $g \omega g^{-1}$, where $g = \sum (-1)^i e_i$.

**Lemma 26.**

(i) $X$ is a $j$-graded tilting complex for $\Omega$.

(ii) We have an isomorphism of $j$-graded algebras

\[ \mathbb{H} \text{End}_\Omega(X) \cong \Omega. \]

(iii) As a $jk$-graded $\Omega$-module, the kernel of the differential on $X$ is isomorphic to $\Omega(p)[p-1]$, and the cokernel of the differential on $X$ is isomorphic to $\Theta^\gamma$.

(iv) We have isomorphisms of $jk$-bigraded $\Omega$-$\Omega$-bimodules with zero differential

\[ \Lambda^{1\otimes \bullet} = \Omega(p)[p-1] \oplus \Theta^\sigma \]

\[ \cong (\Omega(p)[p-1] \oplus \Theta^\sigma)^\gamma \]

\[ \cong \mathbb{H}(X) \]

\[ \cong \mathbb{H}(\Xi). \]

Before proving the lemma, let us pause a moment on the statement of part (iv). Implicit is the existence of a $\Omega$-$\Omega$-bimodule structure on $\mathbb{H}(X)$. The reason for the existence of such a structure is the action of $\mathbb{H} \text{End}_\Omega(X)$ on $\mathbb{H}(X)$, and the isomorphism $\mathbb{H} \text{End}_\Omega(X) \cong \Omega$ of part (ii).

**Proof.**

(i) and (ii). The functor $\Xi \otimes_\Omega -$ induces an autoequivalence of $\mathbb{D}^b(\Omega)$. The image of the regular module $\Omega \Omega$ under a derived auto equivalence $\dashv \mathbb{D}^b(\Omega)$ is necessarily a tilting complex whose endomorphism ring is isomorphic to $\Omega$.

(iii) Isomorphisms as $\Omega$-modules follows from (iii) and Lemma 23. To establish isomorphisms of $\Omega$-$\Omega$-bimodules, since $\gamma$ is inner, it suffices to show the right action of $\text{End}_{\mathbb{D}^b(\Omega)}(X) \cong \Omega$ on $X$ corresponds to the action of $\Omega$ on $\Omega(p)[p-1] \oplus \Theta^\sigma$ by right multiplication, twisted by $\gamma$. To see this, we write down an action of the $j$-degree 1 generators $x$ and $y$ of $\text{End}_{\mathbb{D}^b(\Omega)}(X) \cong \Omega$ on $X$. Indeed, the generators $x$ lift to morphisms of complexes

\[ \Omega e_p(p-l) \rightarrow \Omega e_l \]

\[ \Omega e_p(p-l+1) \rightarrow \Omega e_{l-1} \]

\[ \Omega e_p(p-1) \rightarrow \Omega e_1 \]

\[ \Omega e_p(p) \rightarrow 0 \]

where the arrow 1 denotes the identity in $\text{End}(\Omega e_p)$; whilst the generators $y$ lift to morphisms of complexes

\[ \Omega e_p(p-l) \rightarrow \Omega e_l \]

\[ \Omega e_p(p-l-1) \rightarrow \Omega e_{l+1} \]
where the arrow $t$ denotes the degree 2 generator of $\text{End}(\Omega e_p) \cong F[t]/t^p$. It is not difficult to see from our diagrams of projective indecomposable $\Omega$ modules that upon taking cohomology, these generators correspond to the action of $x$ and $y$ by right multiplication on $\Omega(p)[p-1] \oplus \Theta^p$.

Since as we have constructed it the differential commutes with the action of $\Omega$ on the left and on the right, to work with dg bimodules we finally twist our right actions of $\Omega$ by the automorphism $\gamma$, so that the formulas

$$(a.m)d = a.(m)d, \quad (m.a)d = (-1)^{|a||m|}m(d).a$$

hold, for $a \in \Omega$, $m \in \mathbb{H}(X)$; these are needed for the dg formalism to make sense. □

In order to prove that $\Lambda$ is isomorphic to $\mathbb{H}(T_{\Omega}(\Xi))$, we must calculate the effect of raising the auto-equivalence of $D^b(\Omega)$ which sends $\Omega$ to a power of $X$.

Consider the complex $X_i$ of $\Omega$-modules, obtained by splicing together $n$ copies of $X$. To be more precise, let $X_i$ denote the complex

$$X^1(i-1)p \to W^1(i-2)p \to \cdots \to W^1(p) \to W^1 \to W^0$$

whose differentials are obtained via the composition

$$W^1(p) \xrightarrow{d} W^1 \xrightarrow{\delta} W^0(p)$$

We define $\Xi^i$ to be $\Xi^i \otimes \Omega^i$.

**Lemma 27.**

(i) We have a quasi-isomorphism of differential bigraded $\Omega$-modules

$$X_i \cong \Xi^i.$$  

(ii) We have isomorphisms of $j$-$k$-bigraded $\Omega$-$\Omega$-bimodules with zero differential

$$\Lambda^{i\cdot\bullet} = (\Omega[ip][i(p-1)] \oplus \Theta^p(i-1)p)[i-1](p-1) \oplus \cdots \oplus \Theta^p(0)[0] \cong (\Omega[ip][i(p-1)] \oplus \Theta^p(i-1)p)[i-1](p-1) \oplus \cdots \oplus \Theta^p(0)[0][\gamma] \cong \mathbb{H}(X_i) \cong \mathbb{H}(\Xi^i).$$

(iii) For a suitable choice of such isomorphisms, the multiplication map

$$\Lambda^i \otimes \Lambda^{i'} \to \Lambda^{i+i'}$$

corresponds to $\mathbb{H}$ applied to the multiplication map

$$\Xi^i \otimes \Xi^{i'} \to \Xi^{i+i'}.$$  

**Proof.** The auto-equivalence $\Xi \otimes \Omega$ — is a tilt of a standard kind in representation theory, sometimes called a *perverse equivalence* ([8] 2.6). Its effect is to shift the projective $\Omega e_p$ by 1 in the derived category, whilst taking $\Omega e_i$ to a complex $\Omega e_i \to \Omega e_i$ whose differential has image covering the image of all possible maps $\Omega e_p \to \Omega e_i$. 
If we iterate this construction $i$ times in a graded setting, we obtain precisely $X_i$. This establishes (i).

(ii) The first equality comes from the definition of $\Lambda$. Indeed, we have

$$\Lambda^i \cong \zeta i \Omega (i^p) [i] \oplus \zeta^{i-1} \Theta^\sigma (i^p) [i-1] \oplus \ldots \oplus \Theta^\sigma \langle 0 \rangle [0].$$

The first isomorphism comes from the fact that $\gamma$ is inner: it is given by right multiplication by $g$.

For the second isomorphism, consider the homology concentrated in the middle term of

$$W^1 ((l+1)p) \xrightarrow{d_W (l+1)p)} W^1 \langle lp \rangle \xrightarrow{\delta(lp)} W^1 ((l-1)p)$$

which is given by

$$Q = \frac{\text{Ker} d_w \langle lp \rangle}{\text{Im}(\delta \langle lp \rangle \circ d_w \langle (l+1)p \rangle)}.$$

Note that the subsequence

$$W^0 \langle (l+1)p \rangle \to W^1 \langle lp \rangle \to W^0 \langle lp \rangle$$

is exact on the left and in the middle by Lemma 24, so in fact

$$Q \cong \text{Coker} d_W \langle (l+1)p \rangle.$$

This was computed in Lemma 24, which also shows that we pick up a twist by $\sigma$ with every shift $[-1]$ in homological degree. Note that our choice of notation implies a canonical choice

$$\psi : (\Omega \langle ip \rangle [i(p-1)] \oplus \Theta^\sigma (i-1) (p-1) \oplus \ldots \oplus \Theta^\sigma \langle 0 \rangle \langle 0 \rangle \cong \mathbb{H}(X_i)$$

for the resulting isomorphism, where the twist by $\gamma$ appears as in Lemma 24 (iii). The third isomorphism follows from (i).

(iii) Here we write $\Lambda^i = \bigoplus_{j,k \in \mathbb{Z}} \Lambda^{ijk}$. The algebra $\Lambda$ is quadratic with respect to the $i$-grading, with generators $\Lambda^0 = \Omega$ in degree 0, generators $\Lambda^1 = \zeta \Omega \oplus \Theta^\sigma$ in degree 1, and relations

$$R \subset (\zeta \Omega \otimes \Omega \Theta^\sigma) \oplus (\Theta^\sigma \otimes \Omega \zeta) \subset \Lambda^1 \otimes_{\Lambda^0} \Lambda^1$$

given by the image of the composition

$$\Theta^\sigma \underset{a \to (a,-a)}{\longrightarrow} \Theta^\sigma \oplus \Theta^\sigma \underset{(a,b) \to (\zeta \otimes a,b \otimes Q)}{\longrightarrow} (\zeta \Omega \otimes \Theta^\sigma) \oplus (\Theta^\sigma \otimes \Omega \zeta).$$
It follows that to prove (iii), it is sufficient to check that the hexagon

\[
\begin{array}{ccc}
\Theta^\sigma \otimes_\Omega \Omega & \sim & \Omega \otimes_\Omega \Theta^\sigma \\
H(\Sigma^1)^0 \otimes_\Omega H(\Sigma^1)^1 & \sim & H(\Sigma^1)^1 \otimes_\Omega H(\Sigma^1)^0 \\
\end{array}
\]

commutes, where \(\mu\) denotes the multiplication map in \(T_\Omega(\Sigma)\) and our bimodule decompositions of \(H(X_i)\) for \(i = 1, 2\) are denoted \(H(X_i) = (\Omega \oplus \Theta^\sigma \otimes \Omega)\). Note the quasi-isomorphism between \(H(X_i)\) and \((\Omega \langle ip \rangle [i(p-1)] \oplus \Theta^\sigma ((i-1)p)(i-1)(p-1) \oplus \ldots \oplus \Theta^\sigma (0)(0))\) is canonically fixed. However, the quasi-isomorphism between \(X\) and \(\Xi\) of (i) is not canonically fixed: for example, if we had such a quasi-isomorphism, we could multiply by a scalar and obtain a different quasi-isomorphism with similar properties. In the remains of our proof, we fix a graded quasi-isomorphism \(\phi : X \to \Xi\) of \(\Omega\)-modules which induces an isomorphism of \(\Omega\)-\(\Omega\)-bimodules \(H(X) \cong H(\Xi)\).

Let us now explain how once we have fixed our quasi-isomorphism between \(X\) and \(\Xi\), we can define an associated fixed isomorphism \(H(X_2) \cong H(\Xi^2)\).

Indeed we have an isomorphism

\[H(\Xi \otimes \phi) : H(\Xi \otimes_\Omega X) \to H(\Xi^2)\]

and the isomorphisms

\[H(\Xi \otimes_\Omega X) = H((\Xi \otimes_\Sigma (\Omega e_p \otimes_F e_p F Q e \to \Omega e) \oplus (\Omega e_p (p) \to 0)))\]

\[H(\phi e_t) : H(X e_t) \cong H(\Xi \otimes_\Omega e_t)\]

imply an isomorphism

\[H((X e_p \otimes_F e_p F Q e \to X e) \oplus (X e_p (p) \to 0)) \cong H(\Xi \otimes_\Omega X)\]

Our explicit description of \(X\), of \(X_2\), and of the right action of \(\Omega\) on \(X\), induces an isomorphism

\[H(X_2) \to H((X e_p \otimes_F e_p F Q e \to X e) \oplus (X e_p (p) \to 0))\]

Composing these isomorphisms gives us our fixed isomorphism \(H(X_2) \cong H(\Xi^2)\).

We can now give a heuristic explanation of the commutativity of our diagram. Suppose it does not commute. Since all maps are functorial, the worst that could happen is it could fail to commute by some twist, which would be essentially an automorphism of the \(\Omega\)-\(\Omega\)-bimodule \(\Theta^\sigma\). But what could this automorphism be?
If it were not the identity it would have to come from $\gamma$ in some way. But the automorphism $\gamma$ is inner, so it can be used to construct a bimodule iso $\Theta^\sigma \cong \Theta^{\sigma\gamma}$, but cannot be used to construct a bimodule automorphism in any easy way (such would come from some invertible central element of $\Omega$). So the diagram must commute after all.

To rigorously establish our commuting hexagon we now proceed to examine in detail the effect of right multiplication by $H(\Xi)$ upon fixing the above identifications. We will write the corresponding bimodule decompositions

$$
H(X) = H(X)^1 \oplus H(X)^0 = (\Omega \oplus \Theta^\sigma)^\gamma
$$

$$
H(X_2) = H(X_2)^2 \oplus H(X)^1 \oplus H(X)^0 = (\Omega \oplus \Theta^\sigma \oplus \Theta)^\gamma
$$

The complex $X$ has homology generated in degrees 0 and 1. We consequently have a canonical exact triangle

$$
\Omega[p-1]^{\gamma} \rightarrow X \rightarrow \Theta^{\sigma\gamma} \rightarrow
$$

Via $\phi$ we obtain an exact triangle

$$
\Omega[p-1]^{\gamma} \rightarrow \Xi \rightarrow \Theta^{\sigma\gamma} \rightarrow
$$

in the derived category of $\Omega$-$\Omega$-bimodules. Right multiplication by an element of $\Omega$ defines a map $\Omega[p-1]^{\gamma} \rightarrow \Omega[p-1]^{\gamma}$; this creates a fixed recipe for lifting elements $\omega$ of the $\Omega$-component of $\Xi$ to arrows

$$
a_\omega : \Omega[p-1]^{\gamma} \rightarrow \Xi
$$

in the derived category of left $\Omega$-modules. Right multiplication by an element of $\Theta^\sigma$ defines a map $\Omega \rightarrow \Theta^\sigma$; this creates a fixed recipe for lifting elements $\theta$ of $\Theta^\sigma$ to arrows

$$
a_\theta : \Omega \rightarrow \Xi
$$

in the derived category of left $\Omega$-modules. To multiply an element of $H(\Xi)$ on the right by an element $\theta$, $\omega$ of $H(\Xi)$, it is enough to compute the effect of the functor $\Xi \otimes -$ on the relevant morphism $a_\theta, a_\omega$ in the derived category. The functor

$$
\Xi \otimes - : D^b(\Omega) \rightarrow D^b(\Omega)
$$

sends $\Omega$ to $\Xi$ and $\Xi$ to $\Xi^2$. Therefore, to describe the effect of multiplying $H(\Xi)$ on the right by $\omega$, we consider the map

$$
H(\Xi \otimes a_\omega) : H(\Xi[1]) \rightarrow H(\Xi^2);
$$

to describe the effect of multiplying $H(\Xi)$ on the right by $\theta$, we consider the map

$$
H(\Xi \otimes a_\theta) : H(\Xi) \rightarrow H(\Xi^2);
$$

We have a fixed isomorphism $H(\Xi) \cong H(X)$ and a fixed isomorphism $H(\Xi^2) \cong H(X_2)$. Thanks to the construction of the isomorphism $H(\Xi^2) \cong H(X_2)$, the map

$$
H(X)^0 \rightarrow H(X_2)^1
$$

corresponding to $\Xi \otimes a_1$ where $1 \in \Omega$ can be identified with the identity on $\Theta^{\sigma\gamma}$. Consequently the map $H(X)^0 \rightarrow H(X_2)^1$ corresponding to right multiplication by $\omega \in \Omega$ can be identified with the map

$$
\Theta^{\sigma\gamma} \rightarrow \Theta^{\sigma\gamma}
$$

that sends $\theta$ to $\theta \sigma \gamma(\omega)$. Likewise the map

$$
H(X)^1 \rightarrow H(X_2)^1
$$
corresponding to \( \Xi \otimes a_1 \) where \( 1 \in \Theta \cong \Theta^\sigma \) can be identified with the natural quotient map \( \Omega \to \Theta^\sigma \). Therefore the map \( \mathbb{H}^1(X) \to \mathbb{H}^1(X) \) corresponding to \( \theta \in \Theta^\sigma \) can be identified with the map

\[
\Theta^\sigma \to \Theta^\sigma
\]

that sends \( \omega \) to \( \omega \gamma(\theta) \). This implies that in the hexagon of fixed bimodule homomorphisms

we have correspondence of elements

\[
\gamma \sigma(\omega) = \sigma \gamma(\omega)
\]

implies the hexagon is well-defined and commutes, implying our original hexagon commutes, completing the proof of the Lemma.

\[ \square \]

**Proof of Proposition [21](#):** The isomorphism \( \mathbb{H}(T_{\Omega}(\Xi)) \cong \Lambda \) comes from Lemma 27[(ii)]. It is an algebra isomorphism, thanks to Lemma 27[(iii)].

We want to refine the statement of Proposition 21 to an isomorphism of trigraded algebras.

Take the \( i \)-grading on the tensor algebra \( T_{\Omega}(\Xi) \), with \( \Omega \) lies in \( i \)-degree 0 and \( \Xi \) in \( i \)-degree 1. The \( j \)-grading is inherited by regarding \( \Xi \) as a differential \( jk \)-bigraded \( \Omega \)-module, where the \( j \) and \( k \) gradings on \( \Omega \) are both by path length.

We have \( \Lambda = T_{\Omega}(\Theta^\sigma) \otimes F[\zeta] \); we let \( \Omega \) have \( i \)-degree 0 and let \( \zeta \) and \( \Theta^\sigma \) have \( i \)-degree 1; the \( j \)-grading on \( \Lambda \) is obtained by grading \( \Omega \) and \( \Theta^\sigma \) by path length and placing \( \zeta \) in degree \( p \); finally the \( k \)-grading is obtained by grading \( \Omega \) and \( \Theta^\sigma \) by path length and placing \( \zeta \) in degree \( p - 1 \).
With these gradings, we can say that $\Lambda \cong \mathbb{H}T_{\Omega}(\Xi)$ as trigraded algebras.

For our computation of $y$ we will use the following corollary of Lemma 12.

**Corollary 28.**

$$\mathcal{O}_{F,0}\mathcal{O}_{c,t}^\Omega(F,F) \cong \mathcal{O}_{F,0}\mathcal{O}_{\Omega,\Xi}(F,F)$$

as differential graded algebras.

**Proof.** We have $\mathbb{R}c^1 = \Omega$ and $\mathbb{R}t^{1-1} = \Xi$. Applying Lemma 12 and Lemma 10 successively gives us the advertised formula. 

12. Computing $y$

Here we apply the theory of the preceding sections to compute $y$.

**Proof of Theorem 7.** We have algebra isomorphisms

\[
\begin{align*}
y_q & \cong \mathbb{H}\mathcal{O}_{F,0}\mathcal{O}_{c,t}^\Omega(F,F) & \text{Theorem 20} \\
& \cong \mathbb{H}\mathcal{O}_{F,0}\mathcal{O}_{c,t}^\Omega(F,F) \mathcal{E}(F,F) & \text{Corollary 19} \\
& \cong \mathbb{H}\mathcal{O}_{F,0}\mathcal{O}_{c,t}^\Omega(F,F) \mathcal{E}(F,F) = (F,F) & \text{Corollary 28} \\
& \cong \mathbb{H}\mathcal{O}_{F,0}\mathcal{O}_{\Omega,\Xi}(F,F) & \text{Corollary 19} \\
& \cong \mathbb{H}\mathcal{O}_{F,0}\mathcal{O}_{\Omega,\Xi}(F,F) \mathbb{H}(F[z]) & \text{Corollary 8} \\
& \cong \mathbb{H}\mathcal{O}_{F,0}\mathcal{O}_{\Omega,\Xi}(F,F) \mathbb{H}(F[z]) & \text{Corollary 9} \\
& \cong \lambda_q & \text{Proposition 21} \\
& \cong \lambda_q & \text{definition of } \lambda_q
\end{align*}
\]

Sending 1 to $\infty$, we obtain $y \cong \lambda$ as required.

13. Categorical aesthetic

This paper gives an example of how a higher categorical sensibility can lead to the explicit computation of a classical invariant, namely the Yoneda extension algebra of $GL_2(F)$. The theory we use to make our computation unavoidably involves categories, and properly speaking much of it involves higher categories: the collection $\mathcal{T}$ is a 2-category, and the operators $\mathcal{O}$ and $\mathcal{D}$ are 2-functors, whilst dg algebras also encode some higher categorical information. The theory of higher categories poses an interesting challenge at this time. According to one puritan aesthetic, all mathematical phenomena should be presented in the highest categorical dimension possible; according to another, even plain categories are an abstract absurdity. Such postures give the impression we have a choice between staring at the sun and living in perpetual darkness. But beauty can be found in the twilight. It can feel like there are as many different shades of higher categorical light as there are shades of daylight. If this is so, the challenge we face is not to prevent the earth from turning so that we can avoid shifts in the light, but rather to choose our light carefully, according to the picture we are painting.

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