Boundary conditions in first order gravity: Hamiltonian and Ensemble

Rodrigo Aros
Departamento de Ciencias Físicas
Universidad Andrés Bello, República 252, Santiago, Chile
(Dated: March 24, 2022)

In this work two different boundary conditions for first order gravity, corresponding to a null and a negative cosmological constant respectively, are studied. Both boundary conditions allows to obtain the standard black hole thermodynamics. Furthermore both boundary conditions define a canonical ensemble. Additionally the quasilocal energy definition is obtained for the null cosmological constant case.

PACS numbers: 04.70.Dy, 04.70.Bw

I. INTRODUCTION

It is well known that the variation of the action on shell, provided the boundary conditions, must vanish in order to have a well defined variational principle, and that simultaneously the same boundary conditions must allow the existence of solutions for the equations of motion.

However there is a third role, which arises from the formal connection between quantum field theory and statistical mechanics, the boundary conditions define the ensemble in the statistical mechanical counterpart. For instance the partition function in the canonical ensemble can be written as the path ordered integral

\[ Z(\beta) = \int D\mathcal{X} e^{i\mathcal{H}_{\text{IR}} \tau - i\omega_a}, \]  

provided \( I \) effectively be such that the temperature \( \beta^{-1} \), said the inverse of the period, be fixed. In principle there is a suitable action for any other ensemble.

For gravity the same connection scenario, at least at tree level, seems to exist \[ \mathbb{I} \], justifying a general analysis. To study gravity in this direction a Hamiltonian analysis \[ 2, 3, 4 \] can be very useful since one can expect that the Hamiltonian charges be related with the mass, angular momentum and entropy in the statistical mechanics side.

Unfortunately gravity in asymptotically flat spaces is not a well defined statistical mechanics system. Actually it is necessary to put the system in a box to perform computations \[ 2, 3, 4 \]. Alternatively this can be usually regularized by introducing a background configuration.

On the other hand, gravity with a negative cosmological constant, at least formally, is a well defined statistical mechanical system \[ \mathbb{I} \]. This can be foreseen since a negative cosmological constant introduces a negative pressure which constrains the fields producing, roughly speaking, the effect of a box. Remarkably the most of results of asymptotically flat case usually can be obtained through the limit \( \Lambda \to 0 \), leading to consider \( \Lambda \) as a regulator of the theory of gravity.

In this work both \( \Lambda = 0 \) and \( \Lambda < 0 \) will be addressed, however with different approaches.

A. First Order gravity

Fermions represents a different scenario in gravity. Fermions can not be directly incorporated in metric gravity because, roughly speaking, the group of diffeomorphism does not have half integer representations. Fermions must be represented by spinors, and to incorporate spinors is necessary to introduce a local Lorentz group where they can be realized. This can be done by introducing a local orthonormal basis for the co-tangent space, called vielbein. The vielbein is usually written in terms of the set of differential forms \( e^a = e^a_\mu dx^\mu \). The metric here is the composed field \( g_{\mu\nu} = e^a_\mu e^b_\nu \epsilon_{abcd} \). In four dimensions one usually speaks of a vierbein instead.

The introduction of the vielbein motivates a reformulation of gravity \[ 5, 6 \] where the corresponding Lorentz connection is an additional independent field. The Lorentz connection is called the spin connection and also is written in terms of the differential forms \( \omega^{ab} = \omega^{ab}_\mu dx^\mu \).

Although it is direct to confirm that this reformulation is essentially different in many aspects most of the results of metric gravity in this reformulation are recovered. This work aims to analyze one those aspects. This new formulation is usually called first order gravity.

The four dimensional Einstein Hilbert action in first order formalism reads

\[ I_{EH} = \int_{\mathcal{M}} R^{ab} \wedge \epsilon^c \wedge \epsilon^d \epsilon_{abcd}. \]  

where

\[ R^{ab} = d\omega^{ab} + \omega^c \wedge \omega^{ab} = \frac{1}{2} R^{cd}_{\ \ \ \ cd} \epsilon^c \wedge \epsilon^d, \]

being \( R^{ab}_{\ cd} \) the Riemann tensor. \( \epsilon_{abcd} = \pm 1,0 \) stands for the complete antisymmetric symbol \[ \mathbb{37} \].

The variation of Eq. \[ 2 \] yields the two set of equations of motion

\[ \delta \epsilon^d \rightarrow R^{ab} \wedge \epsilon^c \epsilon_{abcd} = 0, \]  

\[ \delta \omega^{ab} \rightarrow T^c \wedge \epsilon^d \epsilon_{abcd} = 0, \]

where \( T^a = de^a + \omega^a_{\ bc} \wedge e^b = \frac{1}{2} T^a_{\ bc} e^b \wedge e^c \) corresponds to the torsion two form with \( T^a_{\ bc} \) the torsion tensor.
Note that Eq. (4) is an algebraic equation, with solution $T_{ab}^v = 0$. Once this is replaced on Eq. (3) they become the Einstein equations. Thus any solution of the metric formalism is recovered on-shell by this formulation.

If fermions are presented $T^a \neq 0$, for instance in the presence of gravitinos

$$T^a \sim \bar{\Psi} \gamma^a \Psi,$$

and thus first order in this case presents a different kind of solutions.

In general since $e^a$ and $\omega^{ab}$ are independent fields one could expect that there were an independent conjugate momentum for each one. However in four dimensions the conjugate of momentum of $e^a$ is contained in $\omega^{ab}$ or viceversa. This leads to the definitions of two equivalent phase spaces that can be mapped into each other readily. One can even confirm the equivalence of their path order integrals. These equivalent phase spaces are called $e$ and $\omega$-frames respectively.

One remarkable result of first order gravity is to reproduce the path ordered integral of metric formalism. Once its momenta are integrated out the resulting expression is the same obtained in the metric formalism once its corresponding momenta, usually denoted $\pi^{ij}$, are integrated out [2], i.e.

$$\int De^a D\pi^i_a e^{I_{Ham}} \equiv \int Dg_{ij} e^{\int R \sqrt{\gamma} dx}.$$

However both results -first order and metric - are made ignoring the boundary terms. It will be very interesting to address the same computation in first order gravity considering the presence of those boundary terms. Results in the metric formalism considering the boundary terms are very promising and for instance they are connected the entropy of black holes as observed in [10]. In this case one can expect some deviation at first loop since one should sum over also torsional degrees of freedom at the boundary.

### B. Energy

The quest for a definition of energy in gravity has been addressed by many authors (see for instance [2, 11, 12, 13, 14, 15, 16, 17, 22]). In principle a definition of energy, even for classical mechanics, relies on the boundary terms of the action, so it is in the case of gravity. In general the boundary terms fix the ground state of the system, but also the definition of a finite energy might relay on them as well (see for instance [13, 18]).

The connection between boundary conditions and a definition the energy will be explored in this work trying to shed some more light into the problem of energy in gravity. Here two definitions of energy for four dimensional first order gravity will be used, each connected boundary conditions for null and negative cosmological constants. However the two different boundary conditions will be shown to recover the (grand) canonical ensemble.

It is worth to mention another approach to this subject in [19], where another kind of first order gravity is discussed.

### C. The space

The space to be discussed in this work corresponds to a topological cylinder. One can picture it as $M = \mathbb{R} \times \Sigma$ where $\Sigma$ corresponds to a 3-dimensional spacelike hypersurface and $\mathbb{R}$ stands for the time direction and formally is a segment of the real line. In this way the boundary of the space is given by $\partial M = \mathbb{R} \times \partial \Sigma \cup \Sigma_+ \cup \Sigma_-$, where $\Sigma_+$ are the upper and lower boundaries of the topological cylinder and $\partial \Sigma$ is the boundary of $\Sigma$.

Finally $\partial \Sigma$ represents at least the asymptotical spatial region of the manifold, however in the case of a black hole be considered $\partial \Sigma = \partial \Sigma_{\infty} \cup \partial \Sigma_H$, where $\partial \Sigma_H$ stands for the horizon.

### II. $\Lambda = 0$ A DEFINITION OF ENERGY AND ENTROPY

To discuss this case one can begin by recalling that the phase space of four dimensional first order gravity can be described in either the $e$-frame or the $\omega$-frame. For $\Lambda = 0$ it seems more suitable to work in the $e$-frame.

#### A. Fixing the fields

The variation of the EH action [2] yields the boundary term

$$\delta I_{EH} \big|_{\text{on shell}} = \int_{\partial M} \delta \omega^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}, \quad (5)$$

which implies that the EH action could be a proper action principle provided $\delta \omega^{ab} = 0$ at the boundary. For reasons that would be clear later, instead of fixing the connection at $\partial M$ in this work the vierbein will be fixed. This is similar to the condition $\delta g_{ij} |_{\partial M} = 0$ discussed in [20]. This leads to modify the action by adding the boundary term

$$- \int_{\partial M} \omega^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}, \quad (6)$$

yielding

$$\tilde{I}_{EH} = \int_{M} R^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd} - \int_{\partial M} \omega^{ab} \wedge e^c \wedge e^d \varepsilon_{abcd}. \quad (7)$$

Now, the variation of $\tilde{I}_{EH}$ reads

$$\delta \tilde{I}_{EH} \big|_{\text{on shell}} = -2 \int_{\partial M} \omega^{ab} \wedge e^c \wedge \delta e^d \varepsilon_{abcd}, \quad (8)$$
confirming that $\tilde{I}_{EH}$ is a proper action principle provided $e^a$ is fixed at the boundary as expected.

The introduction of the boundary term \[ \text{(9)} \] introduces two potential problems. First, the term is not manifestly invariant under Lorentz transformations since $\omega^{ab}$ transforms as a connection. Second, it can be divergent at the spatial infinity. To solve both problems one can add another term, \[ \int_{\partial M} \omega^{ab} \land e^c \land e^d \varepsilon_{abcd}, \] where $\omega^{ab}$ transforms as a connection in the same fiber as $\omega^{ab}$ but otherwise satisfying \[ \delta \omega^{ab} |_{\partial M} = 0. \]

This condition is used since one chooses $\omega^{ab}$ to represent some particular background. Thus the role of $\omega^{ab}$ is to regularize the behavior of the action with respect to the chosen background. It is direct to prove that $(\omega^{ab} - \omega^{ab}_0)$ transforms as a tensor under Lorentz rotations.

The final expression of $\tilde{I}_{EH}$ is \[ \tilde{I}_{EH} = \int_{M} R^{ab} \land e^c \land e^d \varepsilon_{abcd} - \int_{\partial M} (\omega^{ab} - \omega^{ab}_0) \land e^c \land e^d \varepsilon_{abcd}. \] (10)

It is worth to mention that if the vierbein is properly oriented the term added to the action -on shell- can be rewritten as \[ \int_{\partial M} (K - K_0) \sqrt{\gamma} \, d^3y, \] (11) where $K$ is the trace of the extrinsic curvature of either $\Sigma_k$ or $\mathbb{R} \times \partial \Sigma$ respectively, $\gamma$ the determinant of the induced metric and $y$ an adequate coordinate system. In this view the boundary term can be considered as a generalization of the term proposed in \[ \text{(9)}. \]

To simplify the notation from now on \[ (\omega^{ab} - \omega^{ab}_0) = \tilde{\omega}^{ab}. \]

**B. First order gravity in Hamiltonian**

To proceed one needs to define an adequate vierbein and coordinate system. Here it will be used the line element \[ ds^2 = -N^2dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j). \] (12)

Now, since the coordinates are split in time and spatial, $x^\mu = (t, x^i)$, one can rewrite $\tilde{I}_{EH}$ as \[ \tilde{I}_{EH} = \int_{M} (e^a_i \Omega^{i}_{a \ bc} \omega^{bc} + \omega^{ab}_0 J_{ab} + e^a_i P_a) dt \land d^3x + B \] (13)

where \[ B = \int_{\mathbb{R} \times \partial \Sigma} 2 \epsilon^a_i (\Omega^{i}_{a \ bc} \omega^{bc} \varepsilon_{imn}) dt \land dx^m \land dx^n. \] \[ J_{ab} = 2 T^{c \ d} i_{bj} \epsilon_{abcd} \varepsilon^{ijk}, \] \[ P_0 = 2 R^{a \ bc} i_{aj} \epsilon_{abcd} \varepsilon^{ijk}, \] \[ \Omega^{i}_{a \ bc} = 2 \epsilon_{abcd} \varepsilon^{ijk} e_d^k. \]

Note that the action \[ \text{(13)} \] has only boundary terms at $\mathbb{R} \times \partial \Sigma$ but not at the lids $\Sigma_\pm$.

Recalling that the vierbein is fixed at the boundary, i.e., $\delta e^a_i |_{\partial M} = 0$, the variation of the action with respect to $e^a_i$ and $\omega^{ab}_0$ yields the constraint equations, \[ P_a = 0 \] and \[ J_{ab} = 0. \]

$J_{ab}$ is the generator of Lorentz transformations and $P_a$ is the generator of translations $\mathbb{R}$.

To continue one needs to define the vierbein. Among the different vierbeine that give rise to Eq. \[ \text{(12)} \] here, be obtained in the 18\[ \text{(13)} \] and \[ \text{(14)} \] respectively.

$\eta_a$ is the unitarian vector normal to the $t = \text{cont.}$ slices $\Sigma$. In four dimensions $\eta^a$ can be constructed as \[ \eta_a = \frac{1}{6\sqrt{g}} \epsilon_{abcd} e^a_i e^b_j e^c k \varepsilon^{ijk}. \]

where $g = \text{det} \, g_{ij}$.

Using the projection $e^a_i$ along the $N$ and $N^\perp$ the action can be rewritten as \[ \tilde{I}_{EH} = \int_{M} (e^a_i \Omega^{i}_{a \ bc} \omega^{bc} + NH_\perp + N^\perp H_i + \omega^{ab}_0 J_{ab} + B) \] (15)

where $H_\perp$ and $H_i$ are the projections of $P_a$ along the $\eta^a$ and $e^a_i$ respectively. $N$ and $N^\perp$ are Lagrange multipliers \[ \text{(14)}. \]

$B$ in Eq. \[ \text{(15)} \] stands for the boundary term \[ B = \int_{\mathbb{R} \times \partial \Sigma} (N \eta^a + N^\perp e^a_i) \left( 2 \Omega^{i}_{a \ bc} \omega^{bc} \varepsilon_{imn} \right) dt \land dx^m \land dx^n. \]

**C. Transformation**

To isolate the conjugate momenta of the 12 $e^a_i$'s, contained in the 18 $\omega^{ab}_0$'s here is introduced the following projection \[ \omega^{ab}_k = \Theta^{ab}_k \frac{\pi^a}{\pi^a} + U^{ab}_k \frac{\lambda_{mn}}{\lambda_{mn}}. \] \[ \text{(16)} \]
Note that this also gives rise to others 6 auxiliary fields $\lambda_{mn}$ (and their 6 conjugate fields $\rho^{mn}$). $\lambda_{mn}$ (and $\rho^{mn}$) is symmetric with $m, n = 1, 2, 3$. $\Theta$ and $U$ are given by

$$\Theta_{ab}^{\ i} = \frac{1}{8\sqrt{g}} \left( e^i_{\ [a} \eta^j e^c_{j]} - e^i_{\ [a} e^j_{j]} \eta^c - 2e^j_{\ [a} \eta^b e^c_{c]} \right),$$

$$U_{k\ mn} = \frac{1}{2} \delta^{(m)}_{k\ ij} \eta^a e^b e^c_i,$$  

(17)

where the square brackets indicate antisymmetrization.

In addition one can introduce

$$V_{k\ mn} = \frac{1}{g} E_a^\nu E_b^\rho \epsilon_{\nu\rho}(\delta_k^m \delta_n^i),$$

(19)

such that unveiling the relations

$$\Omega_k^{\ i} \epsilon_{ab}^\ l V_{cd}^\ k = \delta^{(l)}_{[cd]} \delta^i_{j]}$$

(20)

or equivalently

$$(\Theta U) \begin{pmatrix} \Omega \\ V \end{pmatrix} = I_{18x18}$$

in the 18 dimensional space.

Analogously at the boundary one can define

$$\omega_{k}^{\ ab} = \Theta_k^{\ ab} \eta^{j} + U_{k\ mn} V_{ij}^{\ mn} \lambda_{mn}.$$  

D. A Hamiltonian expression

Using the decomposition in Eq. (16) the action reads

$$I_{EH} = \int_{M} \left( \epsilon_a^{\ i} \sigma_{a} + N H_\perp + N^i H_i + \omega^a_b J_{ab} \right) dt \wedge d^3x$$

$$+ B.$$  

(22)

Here $\sigma_{a}^{\ i}$ is indeed the conjugate momentum of $e_a^{\ i}$. Furthermore Eq. (22) is a genuine Hamiltonian action principle provided $\delta e_a^{\ i} \big|_{\partial M} = 0$. $H_\perp$, $H_i$ and $J_{ab}$ are first class constraints. For their expressions in terms of the fields see appendix D.

The Hamiltonian of this theory reads

$$H = - \int_{\Sigma} NH_\perp + N^i H_i + \omega^a_b J_{ab} = \hat{B},$$

(23)

where

$$\hat{B} = \int_{\partial \Sigma} (N \eta^a \hat{\pi}^a + N^i \eta^i \hat{\pi}^i) \epsilon_{\imath m n} dx^m \wedge dx^n.$$  

(24)

Recalling that the constraints vanishes on shell then the Hamiltonian (23) becomes merely the boundary term, $H_{\text{on}\ shell} = -\hat{B}$. This last observation will be essential to develop an expression for the energy in the next sections.

E. Geometry and coordinates at the boundary

The boundary $\mathbb{R} \times \partial \Sigma$ has a metric of the form

$$ds^2 = -N^2 dt^2 + h_{mn}(V^m dt + d\sigma^m)(V^n dt + d\sigma^n),$$  

(25)

where $\sigma^m$, with $m = 2, 3$, are the coordinates of slice at $t = \text{const.}$ of this boundary. Since the boundary can be described as a surface $x^a(t, \sigma^m)$ one can define a set of (co-)vectors which give rise to metric (26). This set reads

$$e^a_t = N \eta^a + V^m \epsilon_m^a$$

$$e^a_m = e^a_m,$$  

(26)

where the projections are made by

$$V^m = \frac{\partial \sigma^m}{\partial x^t} \bigg|_{\mathbb{R} \times \partial \Sigma} \text{ and } e^a_m = \frac{\partial x^m}{\partial \sigma^a} \bigg|_{\mathbb{R} \times \partial \Sigma}.$$  

To complete this analysis usually is introduced the unitary vector $n^a$ which is normal to the boundary $\mathbb{R} \times \partial \Sigma$, i.e.,

$$n_a \eta^a = 0, \ n_a e^a_m = 0 \text{ and } n^a n_a = 1.$$  

This vector can written as

$$n_a = \frac{1}{2\sqrt{\gamma}} e_{abcd} \eta^b e^c d^m e^{mn},$$  

(27)

where $\gamma = N^2 h$ is the determinant of the induced metric (25) on $\mathbb{R} \times \partial \Sigma$. Note that $n^a$ is only a functional of $\eta_a$ and $e^a_m$. Note that also one can obtain

$$\eta_a = \frac{1}{2\sqrt{\hbar}} e_{abcd} n^b e^c d^m e^{mn},$$  

(28)

F. Energy and Momentum

Using the projections (26) the boundary term reads

$$B = \int_{\mathbb{R} \times \partial \Sigma} (N \eta^a + V^m e^a_m) (\hat{\pi}^a \cdot n) dt \wedge d^2 \sigma.$$  

(29)
where \( \hat{n}_a \) represents the projection \( \hat{n}_a \) at the boundary.

Following the generalization of the Hamilton Jacobi equations proposed in \( [20] \), one can define an expression for the energy. Since the fields at the boundary are \( (N, V^m, e^{a}_m) \) here it is advisable to directly variate the action with respect each of dynamical fields

\[
\frac{\delta I_{EH}}{\delta \hat{n}^a} \bigg|_{\text{onshell}} = \eta^a(\hat{n}_a \cdot \hat{n}), \quad \frac{\delta I_{EH}}{\delta V^m} \bigg|_{\text{onshell}} = e^{a}_m(\hat{n}_a \cdot \hat{n}),
\]

\[
\frac{\delta I_{EH}}{\delta e^{a}_m} \bigg|_{\text{onshell}} = \tau^a_m.
\]  

(30)

Note that \( \tau^a_m \) is not a squared matrix.

A definition of energy can be obtained by integrating Eq.\( (30) \)

\[
E = -\int_{\partial\Sigma} \eta^a(\hat{n}_a \cdot \hat{n}) d^2 \sigma.
\]  

(31)

It is straightforward to show that this expression indeed recovers the mass of Schwarzschild or Reissner Nordstrom solutions provided \( \omega^{ab}_0 \) correspond to Minkowski space.

Likewise one can define the momentum

\[
P_m = \int_{\partial\Sigma} e^{a}_m(\hat{n}_a \cdot \hat{n}) d^2 \sigma,
\]  

(32)

and an intrinsic energy momentum tensor

\[
T^a_m = \int_{\partial\Sigma} d^2 \sigma \tau^a_m.
\]  

(33)

One can define energy density \( e = -\eta^a(\hat{n}_a \cdot \hat{n}) \) and the momentum density \( P_m = e^{a}_m(\hat{n}_a \cdot \hat{n}) \).

Note that with these definitions the Hamiltonian can be written as

\[
H = H_{\text{bulk}} + \int_{\partial\Sigma} (eN - V^m p_m) d^2 \sigma.
\]  

(34)

It is interesting to compare this result with the analogous in \( [20] \), since the underlying content of fields is different. For instance after a straightforward computation the expression of the energy can be split as

\[
\eta^a(\hat{n}_a \cdot \hat{n})|_{\partial\Sigma} d^2 \sigma = [(k - k_0) + n_a K^{ab}_c h^m_b] \sqrt{\text{det} d^2 \sigma},
\]  

(35)

where \( k \) is the trace of the intrinsic curvature of \( \partial \Sigma \) immersed in \( \Sigma \) and \( h^m_b \) is the projector from \( M \) into \( \partial \Sigma \). The first part Eq.\( (35) \) recovers the expression for the energy in \( [20] \). However the second term is intrinsical to first order gravity, since it explicitly depends on the contorsion tensor \( K^{ab}_c \). In absence of fermions, since \( K^{ab}_c \) vanishes on shell, the expression in \( [20] \) is formally recovered as expected.

For the momentum \( P_m \) the expression can be written as

\[
P_m = -2 \int_{\partial\Sigma} h_{mi} \hat{n}^i j^m d^2 \sigma + F(K^{ab}_c, t^m_a)
\]  

(36)

where \( h_{mi} \) is the project from \( \Sigma \) into \( \partial \Sigma \), and \( \pi^{ij} \) is the metrical expression for the momentum. \( F(K^{ab}_c, h^m_a) \) is lineal function of the contorsion tensor, which vanishes for \( K^{ab}_c = 0 \). Therefore the first part of Eq.\( (36) \) actually recovers the metrical expression and the rest depends on the contorsion tensor, thus again in absence of fermions the expression in \( [20] \) is formally recovered. Finally one can show that the projection of \( T^a_m \) along \( e^{a}_m \) matches the metrical expressions in \( [20] \) provided \( K^{ab}_c = 0 \).

G. Canonical ensemble action

The variation of the action \( [22] \) can be cumbersome in terms of the phase space fields, however recognizing that on shell the variation is merely given by Eq.\( (8) \) one obtains that

\[
\delta I_{EH} = \int_{\Sigma_{\pm}} \pi^a \delta e^{a}_m
\]  

(37)

\[+ \int_{R \times \partial \Sigma} (e \delta N - p_m \delta V^m + \tau^a_m \delta e^{a}_m) dt \wedge d^2 \sigma.
\]

The first term basically represents the generalization of the standard \( p \delta x |_{t_f} \) in any \( 0 + 1 \) Lagrangian, in this case in the lids \( \Sigma_{\pm} \).

The next term in Eq.\( (37) \) shows that in the variational principle \( (13) \) the energy, as define in Eq.\( (31) \), is not fixed, but the lapse \( N \). The fixing of \( N \) in turn fixes the scales of time, and thus the period in the Euclidean version of the \( M, i.e., \)

\[
\beta = i \int_{\partial\Sigma} d t N.
\]  

(38)

Note that when \( \partial \Sigma \) is composed by more than a single surface, like in a black hole geometry, then one can fix \( N = N_0 \) at only one of those boundaries. In this work \( N \) will be fixed at infinity \( \partial \Sigma_\infty \) and although it is not formally necessary as \( N|_{\partial \Sigma} = N_\infty = 1 \).

The combination of an unconstrained energy in Eq.\( (37) \) and the fixing of \( \beta \) suggests that the action \( [22] \) might be suitable for the (grand)-canonical ensemble. To confirm this statement one can study the statistical mechanics framework around the charges (Eqs.\( (31) \) and \( 32 \)).

To proceed is necessary to consider a particular solution. Here the most general stationary black hole in vacuum (with \( \Lambda = 0 \)) will be considered, the Kerr solution. Since this solution has mass \( M \) and angular momentum \( J \) one must note that this solution is suitable for the grand canonical ensemble. As background configuration it has been chosen the Minkowski space.

At zero order approximation on the path order integral arises the relation for the partition function in the grand canonical ensemble

\[
\ln(Z) = \beta \hat{E} + \beta \Omega \hat{J} - S(\beta, \Omega) \approx \frac{E}{\beta} + \Omega J \ln(\Omega) + O(x^2),
\]  

(39)

where \( \Omega \) in this case corresponds to the value of the angular velocity of the horizon.
The connection between the statistical mechanics and the boundary terms becomes clearer once one notes that any stationary solution satisfies $\dot{\epsilon}^a = 0$, therefore the action merely reduces to the boundary terms

$$I_E|_{\text{on shell}} = B|_{\mathbb{R} \times \partial \Sigma}. \quad (40)$$

As mention before in this case the horizon must be considered as an internal boundary, i.e., $\partial \Sigma = \partial \Sigma_{\infty} \oplus \partial \Sigma_{\text{H}}$. Therefore,

$$I_E|_{\text{on shell}} = B|_{\mathbb{R} \times \partial \Sigma_{\infty}} - B|_{\mathbb{R} \times \partial \Sigma_{\text{H}}}. \quad (41)$$

From the definitions in the previous sections one obtains that expression at infinity give rise to the value of the charges,

$$B|_{x \rightarrow \mathbb{R} \times \partial \Sigma_{\infty}} = \beta(\bar{E} + \Omega \bar{J}).$$

If the action (10) is truly sensible for the canonical ensemble then, by connecting Eq. (39) and Eq. (40), the entropy must be given by

$$S = B|_{x \rightarrow \mathbb{R} \times \partial \Sigma_{\text{H}}}. \quad (42)$$

To compute the value of $S$ one needs to define some general properties of the horizon first. Near the horizon the Euclidean metric becomes [22]

$$ds^2|_{x \rightarrow \mathbb{R} \times \partial \Sigma} \approx N^2 dr^2 + h_{ij} dx^i dx^j, \quad (42)$$

which in terms of the vierbein reads

$$(N^i e_i^a)|_{x \rightarrow \mathbb{R} \times \partial \Sigma_{\text{H}}} \approx 0. \quad (43)$$

This general consideration permits to confirm, after computing the corresponding asymptotic limit at the horizon of $\omega^{ab}$, that the standard area law

$$S = \lim_{x \rightarrow \mathbb{R} \times \partial \Sigma_{\text{H}}} \int \omega^{ab} \wedge \epsilon^c \wedge \epsilon^d \varepsilon_{abcd} \approx \frac{A}{4} \quad (43)$$

is recovered. This confirms also that the principle action proposed in Eq. (10) indeed corresponds to the (grand) canonical action.

### III. FIRST ORDER BOUNDARY TERMS WITH $\Lambda < 0$

The boundary conditions in spaces with a negative cosmological constant within first order gravity has been observed to be fundamentally different [23]. In this case is more adequate to impose boundary conditions $\omega^{ab}$ and its derivative than on the vierbein as in $\Lambda = 0$. This leads to proceed in a generalization of $\omega$-frame to study this case.

#### A. Einstein Hilbert action with $\Lambda < 0$

To initiate the discussion one can consider the four dimensional case with a negative cosmological constant. The four dimensional Einstein Hilbert action with a negative cosmological constant in first order formalism reads

$$I_{EH} = \int_{\mathcal{M}} \left(2 R^{ab} \wedge \epsilon^c \wedge \epsilon^d + l^{-2} e^a \wedge e^b \wedge e^c \wedge e^d\right) \varepsilon_{abcd}, \quad (44)$$

where the cosmological constant has been written in terms of the AdS radius as $\Lambda = -1/(3l^2)$. The variation of Eq. (44) yields

$$\delta e^d \rightarrow \left(R^{ab} \wedge \epsilon^c + \frac{1}{l^2} e^a \wedge e^b \wedge e^c\right) \varepsilon_{abcd} = 0, \quad (45)$$

and the equation $T^a = 0$ already obtained in Eq. (43). When $T^a$ is replaced on Eq. (45) it becomes the standard Einstein equations with a negative cosmological constant.

The presence of a negative cosmological constant gives rise to several technicalities, in particular the usual expressions of the charges, as for instance the Komar’s potentials [11], become divergent. This problem has been addressed in many works (for instance [14, 18]) and is particular important in the context of the AdS/CFT conjecture (See for instance [21, 25]).

In [13, 23, 26] was discussed a set of boundary conditions that allows to transform Eq. (2) into a proper action principle. Under this boundary conditions is added to Eq. (14) the term

$$E = \int_{\mathcal{M}} \frac{l^2}{64G} R^{ab} R^{cd} \varepsilon_{abcd}, \quad (46)$$

whose variation is a total derivative and thus it does not alter the equations of motion. Eq. (46) is usually called the Euler term, but it is not the Euler number of the manifold though.

The new action principle reads

$$I_{EH} = \int_{\mathcal{M}} \frac{l^2}{64G} \bar{R}^{ab} \bar{R}^{cd} \varepsilon_{abcd} \quad (47)$$

with $\bar{R}^{ab} = R^{ab} + l^{-2} e^a e^b$. On shell the variation of Eq. (47) yields

$$\delta I_{EH} = \frac{l^2}{32G} \int_{\partial \mathcal{M}} \omega^{ab} \bar{R}^{cd} \varepsilon_{abcd} \quad (48)$$

The addition of Eq. (46) is made to obtain an action principle suitable for any asymptotically locally anti de Sitter (ALAdS) space. To confirm that one can note that generically $\bar{R}^{ab}(x)|_{x \rightarrow \partial \Sigma_{\infty}} \rightarrow 0$ for any ALAdS space, and thus Eq. (48) has no contributions from the asymptotical spatial region $\mathbb{R} \times \partial \Sigma_{\infty}$. In the other boundaries of $\mathcal{M}$ an adequate boundary condition is to fix the spin connection.
One of the surfaces in which the spin connection is to be fixed is the horizon. However the horizon requires some special attention, since to fix the spin connection at the horizon is connected with the fixing of the temperature of the black hole \textsuperscript{28}. To see that one first must recall that the horizon of a stationary black hole is the surface (in $\mathcal{M}$) where $\xi = \xi^\mu \partial_\mu$ the horizon generator, a time like Killing vector, becomes light like. Next the temperature of the black hole can be read from the relation

$$I_\xi \omega^a_b \epsilon^b_c |_{\mathbb{R} \times \partial \Sigma_H} = \kappa \xi^a,$$

(49)

where $\kappa$ is the surface gravity. The temperature is given by $T = \kappa/4\pi$. In this way the fixing of the spin connection at the horizon determines the temperature. The relation \textsuperscript{19} is the first order version of the relation

$$\xi^\mu \nabla_\mu (\xi^a |_{\mathbb{R} \times \partial \Sigma_H} = \kappa \xi^a$$

obtained in \textsuperscript{27}. By a simple translation between metric and vielbein formalism \textsuperscript{28} one can prove that the fixing of the spin connection also fixes the extrinsic curvature.

Note that here it was not necessary to explicitly require the smoothness of the Euclidean manifold at the horizon to obtain the temperature.

B. The Hamiltonian

The introduction of the coordinate system described in Eq.\textsuperscript{12} leads to rewrite the Lagrangian in Eq.\textsuperscript{17} as

$$\tilde{\mathcal{F}}^{ab}_{ij} \tilde{\mathcal{R}}^{cd}_{abcd} \epsilon^{ijk} = (2 \omega^{ab}_{,b} \tilde{\mathcal{R}}^{cd}_{abcd} \epsilon^{ijk} + \omega^{ab}_{i} J_{ab} + e^{d}_{i} P_{d} + \partial_{i} J^{d}) d^{4} x$$

(50)

with

$$J_{ab} = 4 T^{c}_{ij} T^{d}_{kj} \epsilon^{abcd} \epsilon^{ijk}$$

$$P_{d} = 4 \tilde{\mathcal{R}}^{cd}_{abcd} \epsilon^{ijk}$$

$$J^{i} = 2 \omega^{ab}_{i} \tilde{\mathcal{R}}^{cd}_{abcd} \epsilon^{ijk}$$

Remarking the Lagrangian has only a boundary term at $\mathbb{R} \times \partial \Sigma$ but not at the lids of the cylinder. The introduction of the vielbein \textsuperscript{14} yields

$$I_{EH} = \frac{l^{2}}{64 G} \int \left(2 \omega^{ab}_{i} P^{i}_{ab} + \omega^{ab}_{i} J_{ab} + N H_{\perp} + N^{i} H_{i} + \partial_{i} J^{d} \right) d^{4} x,$$

(51)

where

$$P^{i}_{ab} = \tilde{\mathcal{R}}^{cd}_{abcd} \epsilon^{ijk}$$

$$H_{\perp} = \tilde{P}^{k}_{ij} \epsilon^{d}_{i} \epsilon^{j}_{k} = 6 \sqrt{g} \tilde{\mathcal{R}}^{ij}_{ab} E^{d}_{i} E^{d}_{j}$$

$$H_{i} = \tilde{P}^{k}_{cd} \epsilon^{d}_{i} = \epsilon^{abcd} \left(\tilde{\mathcal{R}}^{ij}_{ab} \epsilon^{d}_{i} \epsilon^{jk} \right) \epsilon^{d}_{i},$$

with $E^{d}_{i}$ the inverse of the vielbein.

Eq.\textsuperscript{51} defines $P^{i}_{ab}$ as the conjugate the momentum of $\omega^{ab}_{i}$, however, the expression of the Hamiltonian is incomplete.

For $\Lambda < 0$, given that the boundary conditions depend on the spin connection, the expressions have been studied in an extension of the $\omega$ frame. In the original $\omega$ frame (i.e. $\Lambda = 0$) second class constraints arise because $P^{i}_{ab}$, which has 18 component, formally depends on $e^{i}_{a}$ with only 12 component. Here with $\Lambda < 0$, and for the same reasons, second order constraints arise as well. Thus besides the terms in Eq.\textsuperscript{51} is necessary to added these second class constraints to complete the Hamiltonian \textsuperscript{32}. Finally the Hamiltonian reads

$$H = \frac{l^{2}}{64 G} \int_{\Sigma} \left(\omega^{ab}_{i} J_{ab} + N H_{\perp} + N^{i} H_{i} + \Phi^{ij} \mu_{ij} \right) d^{3} x$$

$$+ \frac{l^{2}}{32 G} \int_{\partial \Sigma} \omega^{ab}_{i} \tilde{\mathcal{R}}^{cd}_{abcd} dx^{a} \wedge dx^{b}$$

(52)

where $\Phi^{ij}$ are 6 second order constraints, whose expression reads

$$\Phi^{ij} = \tilde{e}^{alb} b^{i} \tilde{P}^{i}_{ab} \tilde{P}^{a}_{b}$$

(53)

with

$$\tilde{P}^{i}_{ab} = P^{i}_{ab} - \epsilon^{ijk} \epsilon_{abcd} \tilde{R}_{cdk} \tilde{\epsilon}^{i}$$

This definition of momentum allows to rewrite the generator of Lorentz rotations as

$$J_{ab} = D_{i}(P^{i}_{ab}) = D_{i}(\tilde{P}^{i}_{ab}).$$

(54)

The expressions of $H_{i}$ and $H_{\perp}$ in terms of the fields are cumbersome, not very illustrative, and furthermore irrelevant for what follows.

C. Noether Charges

The vanishing of the constraints on shell implies that

$$H_{\text{on shell}} = \frac{l^{2}}{32 G} \int_{\partial \Sigma - \partial \Sigma_{H}} \omega^{ab}_{i} \tilde{R}_{ij}^{cd} \epsilon_{abcd} dx^{a} \wedge dx^{b}.$$
where $I_ξ$ stands for the projector along the vector η, in this case $I_ξω^a b = η^μω^a b_μ$. The mass or the angular momentum of the solution can be obtained from Eq. (56) as the asymptotical value at $∂Σ∞$ and provided η be the Killing vector associated with time or rotational symmetries respectively. The evaluation at the horizon for ξ, the horizon generator, leads to the entropy.

D. An expression for the Hamiltonian

To study this ideas one can choose a particular but representative solution. The most general stationary black hole in vacuum with a negative cosmological constant will be analyzed, the Kerr-AdS solution. Since this solution has angular momentum the canonical ensemble is replaced by the grand canonical ensemble.

The most general Killing vector of this solution reads

$$\eta = αξ_t + βξ_φ$$

where α and β are constant and ξ_t and ξ_φ are the Killing vectors associated with the time and rotational invariance respectively. In terms of these symmetries the Killing vector that defines the horizon, ξ, reads

$$ξ = ξ_t + Ωξ_φ,$$  \hspace{1cm} (57)

where $Ω$ is the angular velocity of the horizon \[^{29}\].

Usually the Kerr-AdS solution is presented in the Boyer-Lindquist (BL) coordinates (See appendix \[^{30}\]). The projection of the spin connection in these coordinates reads

$$I_ξω^a b = ω^a b_t + Ωω^a b_φ.$$  \hspace{1cm} (58)

Now, by noting that the expression of the Hamiltonian in Eq. (55) formally equals the expression of the Noether charge for $ξ_t = ξ_t$ in the BL coordinates one obtains

$$H|on shell = M - \frac{I^2}{2G} \int_{∂ΣH} (I_ξω^a b - Ωω^a b_φ) R^{cd}ε_{abcd}.$$  \hspace{1cm} (59)

Finally, after a long but straightforward computation, one can prove that the last term is the angular momentum of the solution. Thus,

$$H|on shell = M + ΩJ - Qξ|ΣH.$$  \hspace{1cm} (60)

Using zero order approximation on the generalization of Eq. (1) to the grand canonical ensemble one obtains,

$$ln(Z) = βE + βΩJ - S(β, Ω) ≈ -1E, Ω|on shell + O(x^2).$$

Since in this case the solution in discussion is a stationary black hole then $ω^a b_t = 0$ which in turns implies that

$$-I_ξE|on shell = -βH|on shell \approx ln(Z).$$  \hspace{1cm} (61)

Note that the right hand side also can be obtained as the zero order approximation of Eq. (59).

Combining these results is possible to recognize that the entropy is

$$S - S_0 = β \left( \frac{I^2}{2G} \int_{ΣH} I_ξω^a b R^{cd}ε_{abcd} \right).$$

where $S_0$ stands for possible higher order corrections to the value of entropy but can not depend on the values of the extensive variables. This result is equivalent to the one obtained in \[^{23}\]. Computing explicitly $S$ yields to the usual

$$S = \frac{1}{4G} A,$$

where $A$ stands for the area of the horizon.

E. Higher Dimensions

In higher dimensions besides the EH theory of gravities there are several other sensible gravitational theories, with second order equation of motion for the metric. Among them one important group are the usually called Lovelock gravities \[^{31}\]. First order formalism is particularly suitable for their study, since here the first order nature of the equation of motion for $(e^a, ω^a b)$ is manifest, and thus the second order nature of the metric ones. In absence of fermions in general these gravities have only solutions with vanishing torsion \[^{32}\].

Schematically the Lagrangian of Lovelock gravities reads

$$L = \sum_{p=0}^{[\frac{d-1}{2}]} α_p \frac{1}{d-2p} R^{a_1 a_2} ... R^{a_{2k-1} a_{2p}} e^{a_{2p+1}} ... e^{a_d} ε_{a_1 ... a_d}.$$  \hspace{1cm} (62)

where $α_p$ are arbitrary constants and $[\cdot]$ stands for function integral part of the argument.

In general this Lagrangian has solutions with multiple cosmological constants. This can be considered a problem since that produces unstable geometries that can tunnel between the different cosmological constants. To solve this one can restrict the $α_p$ coefficients such that only a single cosmological constant exist.

With a single cosmological constant the form of the equation of motion associated with $δe^a$ is

$$R^{a_1 a_2} ... R^{a_{2k-1} a_{2p}} e^{a_{2p+1}} ... e^{a_d} ε_{a_1 ... a_d} = 0$$  \hspace{1cm} (63)

which can be obtained provided

$$α_p = \begin{cases} \frac{1}{d-2p} \binom{k}{p} & p \leq k \\ 0 & p > k \end{cases}.$$  \hspace{1cm} (64)

One can prove in general that the solutions of these restricted Lovelock theories are ALAdS spaces.
Remarkably the results for the four dimensional EH gravity in the previous sections can be easily extended to these restricted Lovelock gravities in even dimensions. In odd dimensions, however, this is not direct because there is no generalization of Eq. (68), namely there is no Euler term. In general in \( d = 2n \) dimensions for an ALAdS space one can use the same boundary conditions already proposed. To do that one adds to the no restricted \( \mathbf{L} \) the term

\[
\mathbf{E}_{2n} = \frac{K}{2n} \int_{\mathcal{M}} R^{a_1 a_2 \ldots a_{2n-1} a_{2n}} \varepsilon_{a_1 \ldots a_{2n}} \tag{65}
\]

where

\[
K = - \sum_{p=0}^{n-1} (-1)^p \alpha_p = - \frac{1}{2n} \left( \frac{k - n}{n} \right).
\]

As outcome of this addition the variation of the new action principle -on shell- yields a boundary term that at the asymptotically spatial region \( (\mathbb{R} \times \partial \Sigma_\infty) \) of any ALAdS space, behaves as

\[
\Theta_{x \to \partial \Sigma_\infty} \sim \delta \omega \tilde{R}^{k-1} e^{2n-2k} \to 0, \tag{66}
\]

and thus the asymptotical spatial region \( \mathbb{R} \times \partial \Sigma_\infty \) does not contribute to the variation of the action. A proper boundary condition at the other boundaries is \( \delta \omega \omega^{ab} = 0 \). Analogous to the four dimensional case if a black hole geometry is considered the temperature is fixed by this boundary condition.

Following analogous steps as the four dimensional case one obtains that on-shell the Hamiltonian is merely the boundary term

\[
\mathbf{H}_{\text{Onshell}} = \int_{\partial \Sigma} \sum_{p=1}^{n-1} \frac{p \alpha_p}{l^{2n-2p}} \omega_t R^{p-1} e^{2n-2p} + nK \int_{\partial \Sigma} \omega_t R^{n-1}. \tag{67}
\]

To proceed one needs to consider a particular solution. Here it will be considered the topological black holes studied in Ref. [31]. They are described in Appendix C.

Recalling the expression of the Noether charge which generically reads

\[
Q_\xi = \int_{\partial \Sigma} I_\xi \omega^{ab} \frac{\partial \mathbf{L}}{\partial R^{ab}}, \tag{68}
\]

where

\[
\frac{\partial \mathbf{L}}{\partial R} = \sum_{p=1}^{n-1} \frac{p \alpha_p}{l^{2n-2p}} R^{p-1} e^{2n-2p} + nK R^{n-1},
\]

one can connect the Lagrangian and Hamiltonian results. The horizon generator \( \xi \) of these solution is merely \( \xi = \partial_t \) (See appendix C), thus it is direct to identify the part of the Hamiltonian at \( \partial \Sigma_\infty \) as the Noether charge associated with the time symmetry, and so with the mass of the solution.

Since these are static solutions is also satisfied \( \dot{\omega}^{ab} \equiv 0 \) and so

\[
-\mathbf{I}_E|_{\text{on shell}} = -\beta \mathbf{H}|_{\text{on shell}}.
\]

After analogous computations to the four dimensional case one can obtain the generic expression for the entropy,

\[
S - S_0 = \beta Q_\xi |_{\partial \Sigma_H}, \tag{69}
\]

where \( S_0 \) stands for higher order corrections to the value of entropy. The evaluation of Eq. (69) reproduces the results of [31].

\section{IV. DISCUSSION AND CONCLUSIONS}

In this work it has been recovered the basic statistical mechanical relations of black holes in the canonical ensemble using a Hamiltonian approach.

The two different boundary conditions presented, proper of \( \Lambda = 0 \) and \( \Lambda < 0 \) respectively, define a canonical ensemble. This might lead to think that somehow both boundary conditions are equivalent. It is direct to prove, however, that this is not the case. For \( \Lambda < 0 \) the boundary condition at infinity let \( \omega^{ab} \) undetermined because of Eq. (48), however since the limits \( x \to \mathbb{R} \times \partial \Sigma_H \) and \( l \to 0 \) do not commute then this boundary condition for \( \Lambda < 0 \) can not be extrapolated to the asymptotically flat case. The fundamental result of this work is that there can be more than one set of boundary conditions that lead to the canonical ensemble. The other result is to confirm that the horizon is a fundamental element in black holes thermodynamics.

The analysis in this work probably can be easily extended to higher dimensions, except for one important point, the phase variables \((\epsilon_i^a, \pi_i^a)\) are a feature proper of only four dimensions [35]. In higher dimensions there should be in principle more variables in the phase space, \((\epsilon_i^a, \omega^{ab}, \pi_i^a, \pi_{ab})\), or the reduction to a single pair of variables should be done in at least a different way.

\section{APPENDIX A: ASYMPTOTICALLY FLAT VERSUS ALADS}

To have an idea of how asymptotical flat spaces are not well behaved one can naively sketch Eq. (39) for four dimensional Schwarzschild solution. Here the entropy \( S \sim \pi r_+^2 \) and the energy \( E \sim r_+ \), therefore the partition function

\[
Z(\beta)_{\text{Sch}} \sim \int_0^\infty dr_+ e^{-\beta r_+ + \pi r_+^2} \tag{A1}
\]
which is clearly divergent.

On the other hand with a negative cosmological constant the scenario changes radically, since for Schwarzschild-AdS $E \sim r_+(1 + l^{-2} r_+^2)$, $l$ is the AdS radius, yielding the completely different $Z(\beta)$ function

$$Z(\beta)_{Sch-AdS} \sim \int_0^\infty dr_+ e^{-\beta(r_+^2 + r_+^2/|l|^2 + \pi r_+^2}$$

which trivially converges.

**APPENDIX B: KERR-ADS**

The Kerr-AdS geometry in Boyer-Lindquist-type coordinates can be expressed by the vierbein

$$e^0 = \frac{\sqrt{\Delta}}{\Xi \rho}(dt - a \sin^2 \theta d\varphi), \quad e^1 = \rho \frac{dr}{\sqrt{\Delta}},$$

$$e^2 = \rho \frac{d\theta}{\sqrt{\Delta \theta}}, \quad e^3 = \frac{\sqrt{\Delta \theta}}{\Xi \rho} \sin \omega (adt - (r^2 + a^2) d\varphi),$$

with $\Delta_r = (r^2 + a^2) \left(1 + \frac{r^2}{r^2} \right) - 2mr$, $\Delta_\theta = 1 - \frac{r^2}{r^2} \cos^2 \theta$,

$\Xi = 1 - \frac{a^2}{r^2}$ and $\rho^2 = r^2 + a^2 \cos^2 \theta$.

This vierbein indeed has the form described in Eq. (14) and the resulting metric has the required ADM form of Eq. (12) as well.

In this coordinates the horizon is define by the largest zero of $\Delta_r$, called $r_+$. The angular velocity of the horizon

$$\Omega = \frac{a}{r_+^2 + a^2}.$$ 

The mass and the angular momentum are found evaluating the charge (50) for the Killing vectors $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \varphi}$, respectively

$$Q \left( \frac{\partial}{\partial t} \right) = \frac{m a}{r^2}; \quad Q \left( \frac{\partial}{\partial \varphi} \right) = 2 \frac{a}{r^2} = J$$

in agreement with Ref. (32).

**APPENDIX C: TOPOLOGICAL BLACK HOLES**

The restricted Lovelock gravities determined by the constants $\alpha_p$, in terms of $k < n - 1$, in Eq. (64) give rise to different topological black holes solutions depending on $k$. Each one of them can be described by the vielbein

$$e^0 = f(r) dt \quad e^1 = \frac{1}{f(r)} dr \quad e^m = r e^m,$$  

and its associated torsion free connection

$$\omega^{01} = \frac{1}{2} \frac{d}{dr} f(r)^2 dt \omega^m = f(r)e^m \omega^{mn} = \bar{\omega}^{mn},$$

where

$$f^2(r) = \gamma + 2 \frac{r^2}{l^2} - \sigma \left( \frac{C_1}{r^d - 2k - 1} \right) ^{1/k},$$

$\sigma = (\pm 1)^{(k+1)}$, and the integration constant $C_1$ is identified as

$$C_1 = 2G_k(M),$$

where $M$ stands for the mass. $\bar{e}^m = \bar{e}^m(y) dy^i$ and $\bar{\omega}^{mn}$ are a vielbein and its associated torsion free connection on the transverse section with $m = 2 \ldots d - 1$. $R^{mn} = \gamma e^m e^n$. The $y^i$’s are an adequate set of coordinates.

It is straightforward to prove that the mass can be obtained by evaluating Eq. (54) for the Killing vectors $\xi$ at $\partial \Sigma^\infty$,

$$Q \left( \frac{\partial}{\partial t} \right) = M.$$ 

**APPENDIX D: EXPLICIT EXPRESSIONS**

The different constraints $H_\perp, H_1,$ and $J_{ab}$ can be written explicitly as

$$\frac{1}{2} H_\perp = \eta^a \partial_a \pi^i - \frac{1}{2} E^a_i \partial_j e^l_a \eta^b \pi^l_{\beta}$$

$$- G_{\pm}^{ij} \pi^a \pi^b - \frac{\pi^l}{l^2} G^{mn} \phi^0 \lambda^{mn}_{\pi^l} \lambda^0,$$


\[ N^m H_m = N^m \left\{ \frac{1}{2} \left( g^{-1} E_d^a \partial_e c^d c^e_m c^{ijk} e_j^b - E_d^a \partial_e c^d e_j^b \right) \pi_b^e + e_m^a \partial_a \pi_b^e + G_{mij} \pi_b^e + \frac{1}{2} N^m \epsilon_m^a \epsilon_j^b J_{ab} \right\} \]

(D1)

and

\[ J_{ab} = 2 \epsilon_{abcd} \frac{\partial e^c}{\partial x^d} c^d e^{ijk} - \frac{1}{2} (\pi^c b - \pi^c a), \]

(D2)

where

\[ \lambda^0_{pq} = \frac{1}{2g} G_{pqmn} E^{(m} \partial_{n} c^{ij)} \]

(D3)

\[ G_{a b}^{ij} = \frac{1}{16 \sqrt{g}} \left[ \epsilon_i^a \epsilon_j^a - 2 \epsilon_i^a \epsilon_j^b - g_{ij} \eta^a \eta^b \right], \]

(D4)

and

\[ G_{mij}^{ab} = \frac{1}{16 \sqrt{g}} \left[ g_{ij} \eta^a \eta^b + 2 g_{im} (\epsilon_j^a \eta^b - \epsilon_j^b \eta^a) \right]. \]

(D5)

**APPENDIX E: MICROCANONICAL BOUNDARY TERM**

To transform the action \( I_{EH} \) into the microcanonical ensemble action is necessary to add a boundary term that change the boundary conditions from \( \delta N = 0 \) to a fixed energy density \( e \) at the boundary. This is simply achieved by subtracting from Eq. (22) the term

\[ \int_{\partial \Sigma} (eN - V^m p_m) d^2 \sigma, \]

which is the boundary term \( B \). This result leads to the new the action principle

\[ \hat{I}_{EH} = \int_{\mathcal{M}} (\epsilon_i^a \pi_a^i + NH_1 + N^1 H_i + \omega_i^a J_{ab}) dt \wedge d^3 x, \]

(E1)

which should be suitable for the microcanonical ensemble. Unfortunately the analysis of the thermodynamics in this case is not straightforward, and will be discussed elsewhere.

**ACKNOWLEDGMENTS**

I would like to thank Abdus Salam International Centre for Theoretical Physics (ICTP) for the associate award granted. This work was partially funded by grants FONDECYT 1040202 and DI 06-04. (UNAB). Part of this work was written at ICTP.
[16] G. Barnich and F. Brandt, Covariant theory of asymptotic symmetries, conservation laws and central charges, Nucl. Phys. B633 (2002) 3–82, http://arxiv.org/abs/hep-th/0111246.

[17] G. Barnich, Boundary charges in gauge theories: Using stokes theorem in the bulk, hep-th/0301039.

[18] S. W. Hawking and G. T. Horowitz, The gravitational hamiltonian, action, entropy and surface terms, Class. Quant. Grav. 13 (1996) 1487–1498, gr-qc/9501014.

[19] L. Fatibene, M. Ferraris, M. Francaviglia, and M. Raiteri, Noether charges, brown-york quasilocal energy and related topics, J. Math. Phys. 42 (2001) 1173–1195, gr-qc/0003019.

[20] J. D. Brown and J. York, James W., Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D47 (1993) 1407–1419.

[21] R. Arnowitt, S. Deser, and C. W. Misner, The dynamics of general relativity, gr-qc/0405109.

[22] G. W. Gibbons and S. W. Hawking, Classification of gravitational instanton symmetries, Commun. Math. Phys. 66 (1979) 291–310.

[23] R. Aros, Analyzing charges in even dimensions, Class. Quant. Grav. 18 (2001) 5359–5369, gr-qc/0011009.

[24] R. Emparan, C. V. Johnson, and R. C. Myers, Surface terms as counterterms in the ads/cft correspondence, Phys. Rev. D60 (1999) 104001, hep-th/9903238.

[25] K. Skenderis, Lecture notes on holographic renormalization, Class. Quant. Grav. 19 (2002) 5849–5876, hep-th/0209067.

[26] R. Aros, The horizon and first order gravity, JHEP 04 (2003) 024, gr-qc/0208033.

[27] R. M. Wald, Black hole entropy in noether charge, Phys. Rev. D48 (1993) 3427–3431, gr-qc/9307038.

[28] C.-M. Y. Choquet-Bruhat and M. Dillard-Bleick, Analysis, Manifolds and Physics. Noth-Holland.

[29] S. W. Hawking, C. J. Hunter, and M. M. Taylor-Robinson, Rotation and the ads/cft correspondence, Phys. Rev. D59 (1999) 064005, hep-th/9811056.

[30] D. Lovelock, The einstein tensor and its generalizations, J. Math. Phys. 12 (1971) 498–501.

[31] R. Aros, R. Troncoso, and J. Zanelli, Black holes with topologically nontrivial ads asymptotics, Phys. Rev. D63 (2001) 084015, hep-th/0011097.

[32] M. Henneaux and C. Teitelboim, Asymptotically anti-de sitter spaces, Commun. Math. Phys. 98 (1985) 391–424.

[33] R. Aros, First order gravity: A Hamiltonian analysis in 4 and higher dimensions. Work in progress.

[34] To have an idea of the reasons see appendix A.

[35] One can confirm that the transformation of fields, i.e.,

\((e_a^i, e_i^a) \rightarrow (N, N^i, e_i^a)\),

does not change the measure of the path order integral.

[36] There is however the so called Chern Simons gravities where, even in absence of fermions, the most general solution, in first order formalism, has \(T^a \neq 0\).

[37] In three dimensions is not even necessary a projection like Eq.