ON SOME GENERALIZATIONS OF THE PROPERTY B PROBLEM OF AN $n$-UNIFORM HYPERGRAPH

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Abstract. The extremal problem of hypergraph colorings related to the Erdős–Hajnal property $B$-problem is considered. Let $k$ be a natural number. The problem is to find the value of $m_k(n)$ equal to the minimal number of edges in an $n$-uniform hypergraph that does not admit 2-colorings of the vertex set such that every edge of the hypergraph contains at least $k$ vertices of each color. In this paper, we obtain new lower bounds for $m_k(n)$.

1. Introduction and History

In this work, we consider the famous problem concerning vertex colorings of uniform hypergraphs. First, we will give some basic definitions. A hypergraph is a pair $H = (V, E)$, where $V$ is a finite set, whose elements are called vertices, and $E$ is a family of subsets of $V$, called the edges. A hypergraph is said to be $n$-uniform if each of its edges contains exactly $n$ vertices.

One of the classical extremal problems of the hypergraph theory is the property B problem. We say that a hypergraph has property B if there is a two-coloring of its vertex set such that no edge is monochromatic. The problem is to find the quantity $m(n)$ that is the minimum possible number of edges of an $n$-uniform hypergraph that does not have property B. This question was first posed by Erdős and Hajnal [6]. Erdős himself (see [4,5]) obtained the following asymptotic bounds for $m(n)$:

$$2^{n-1} \leq m(n) \leq \frac{e \ln 2}{4} n^2 2^n (1 + o(1)).$$  (1)

The upper bound remains the best. The lower bound was refined in several works (see surveys [10,16]). The best-known result is due to Radhakrishnan and Srinivasan (see [15]). They proved that

$$m(n) \geq (0.1) 2^n \sqrt{\frac{n}{\ln n}}.$$

There exist a lot of different generalizations of the Erdős–Hajnal problem (see [10]). One of them was suggested by Shabanov [19]. Let $k$ be a natural number. We say that a hypergraph $H = (V, E)$ has property $B_k$ if there exists a two-coloring of $V$ such that every edge contains at least $k$ vertices of each color. Similarly to $m(n)$ we define the quantity $m_k(n)$ equal to the minimum number of edges of an $n$-uniform hypergraph that does not have property $B_k$. Clearly property $B_1$ is property B and $m_1(n)$ is $m(n)$. By means of the probabilistic method one can obtain the following lower bound for $m_k(n)$, which coincides with (1) in the case $k = 1$:

$$m_k(n) \geq \frac{2^{n-1}}{k-1} \sum_{j=0}^{\binom{n}{j}}.$$

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In [19, 20], Shabanov proved that if \( k = k(n) = o(n/\ln n) \), then there exists a function \( \psi(n) \) that depends only on the type of the function \( k(n) \) and tends to 1 as \( n \to \infty \), such that
\[
m_k(n) \leq \frac{\text{e} \ln 2}{4} n^2 \left( \frac{2^n}{\sum_{i=0}^{n-k} (n)} \right) \psi(n).
\] (2)

Moreover, Teplyakov in [26] proved that if \( k = o(n) \), then \( m_k(n) \) satisfies (2).

Also in [21] Shabanov obtained the following lower bound for \( m_k(n) \):
\[
m_k(n) = \Omega \left( \left( \frac{n}{\ln n} \right)^{1/2} \frac{e^{-k/2}}{\sqrt{2k-1}} \frac{2^{n-k}}{\binom{n}{k-1}} \right)
\] (3)

when \( k = k(n) = O(\ln n) \). In the case where the growth of \( k \) is between \( \ln n \) and \( \sqrt{n} \), the best lower bound is due to Rozovskaya (see [17,18]):
\[
m_k(n) \geq 0.19 \cdot n^{1/4} \frac{2^{n-1}}{\binom{n-1}{k-1}}.
\] (4)

Recent results and surveys on this topic can be found in [1, 8, 9, 12, 13, 16, 22–24].

2. Main Result

The main result of this paper is a new lower bound for \( m_k(n) \).

**Theorem 1.** Let \( n \geq 30, k \geq 2, \) and
\[
k \leq \sqrt{\frac{n}{\ln n}}.
\] (5)

Then
\[
m_k(n) \geq \frac{12}{\text{e}^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{2^{n-1}}{\binom{n-1}{k-1}}.
\] (6)

The estimate (6) refines the previous results (3) and (4) when the growth of \( k \) does not exceed \( n^{1/2-\delta}/\sqrt{\ln n} \), where \( 0 < \delta < 1/2 \).

3. Proof of Theorem 1

In order to prove this theorem we need to show that any \( n \)-uniform hypergraph with at least \( 12/\text{e}^{26} \times (n/k \ln n)^{1/2} \times 2^{n-1}/\binom{n-1}{k-1} \) edges has property \( B_k \). Consider an \( n \)-uniform hypergraph \( H = (V, E) \) with
\[
|E| = m < \frac{12}{\text{e}^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{2^{n-1}}{\binom{n-1}{k-1}}.
\] (7)

3.1. Criterion of Having Property \( B_k \). Let \( G = (W, U) \) be an arbitrary hypergraph and let \( \sigma \) be a numeration of its vertices. More precisely, let \( \sigma \) be some bijective mapping from \( W \) to \( \{1, 2, \ldots, |W|\} \). We say that a pair of edges \( (A_1, A_2) \) of \( G \) forms an ordered 2-chain with respect to numeration \( \sigma \) if \( |A_1 \cap A_2| = 1 \) and \( \sigma(v) \leq \sigma(u) \) for all \( v \in A_1, u \in A_2 \). Pluhár [14] suggested a lemma that connects together the existence of ordered 2-chains in a hypergraph and property \( B \).

**Lemma 1.** Let \( G = (W, U) \) be an arbitrary \( n \)-uniform hypergraph. Then \( G \) has property \( B \) if and only if there is a numeration \( \sigma \) of \( W \) without ordered 2-chains in \( G \).

Further we consider the criterion for having property \( B_k \), which generalizes a property \( B \)-criterion and was formulated by Rozovskaya [18].

For each edge \( f \in U \), let \( F_{\sigma}(f) \) denote the set of the first \( k \) vertices of \( f \) in the numeration \( \sigma \) and let \( L_{\sigma}(f) \) denote the set of the last \( k \) vertices of \( f \).
Lemma 2. Let $G = (W, U)$ be an arbitrary $n$-uniform hypergraph every edge of which contains at least $2k$ vertices. Then $G$ has property $B_k$ if and only if there is a numeration of its vertices $\sigma$ such that for each two edges $f$ and $s$, the following relation holds:

$$L_\sigma(f) \cap F_\sigma(s) = \emptyset. \quad (8)$$

Proof. Necessity. Suppose that $G$ has property $B_k$, i.e., there exists a coloring of its vertices with two colors such that each edge contains at least $k$ vertices of each color. We enumerate the vertices of a hypergraph in the following way: we assign first numbers to the vertices of the first color and after that we enumerate the vertices of the second color. It is easy to see that the condition $L_\sigma(f) \cap F_\sigma(s) = \emptyset$ holds true.

Sufficiency. Consider the numeration $\sigma$ given by the theorem. We color the first $k$ vertices of every edge in the numeration $\sigma$ with the first color and all the remaining vertices with the second one. Since the condition (8) holds, every edge contains at least $k$ vertices of each color. Therefore, $G$ has property $B_k$. $\square$

3.2. General Idea of Proof. Here we generalize the idea of Cherkashin and Kozik [2]. They suggested to enumerate $V$ in the following way. For every vertex $v$, let $X_v$ be an random variable with a uniform distribution on $(0, 1)$. The value $\sigma(v)$ is defined as follows:

$$\sigma(v) := \sum_{w \in V} I\{X_w \leq X_v\}.$$ 

For any edge $A \in E$, we define

$$\max(A) = \max_{v \in A} X_v, \quad \min(A) = \min_{v \in A} X_v.$$ 

We say that $A$ is dense if

$$\max(A) - \min(A) \leq \frac{1-p}{2},$$

where $p = \ln n/n$.

According to the criterion, a hypergraph has property $B$ if there are no pairs of edges with one common vertex in the intersection that is the first in one edge and the last in the other one. If we show that the sum of probabilities of the event that there are dense edges and the event that there exists a pair of edges intersecting in the above way (assuming that no edge is dense) is strictly less than 1, then with positive probability there exists such a numeration that the graph has property $B$. It is easy to see that

$$|E|\left[\left(\frac{1-p}{2}\right)^n + n\left(\frac{1-p}{2}\right)^{n-1}\left(1 - \frac{1-p}{2}\right)\right] + |E|^2\int_{(1-p)/2}^{(1+p)/2} x^{n-1}(1-x)^{n-1} dx < 1$$

if

$$|E| \geq c \cdot 2^n \cdot \sqrt{n \ln n},$$

where $c$ is some absolute constant. Thus, we obtain the best-known lower bound for $m(n)$.

We generalize this idea.

Let $f \in E$ be an edge of $H$. For each vertex $v \in f$, we introduce the events $F_f(v)$ and $L_f(v)$. The event $F_f(v)$ occurs if the vertex $v$ is one of the first $k$ vertices of the edge $f$, and the event $L_f(v)$ occurs if $v$ is one of the last $k$ vertices of $f$. For any two edges $f$ and $s$ such that $f \cap s \neq \emptyset$ and for any vertex $v_0 \in f \cap s$, denote

$$M(f, s, v_0) = L_f(v_0) \cap F_s(v_0). \quad (9)$$

Let $M(f, s)$ be the union of the events $M(f, s, v_0)$ over all vertices $v_0$ from the intersection of $f$ and $s$:

$$M(f, s) = \bigcup_{v_0 \in f \cap s} M(f, s, v_0).$$
Consider a numeration $\sigma$ of $V$. Let $A \in E$ be an edge of $H$. Without loss of generality, we assume that $A = \{1, 2, \ldots, n\}$. For any $v \in A$, we consider the random variable $X_v$ and arrange $X_v$, $v \in A$, in the ascending order: $\{X_{(1)}, \ldots, X_{(n)}\}$. Denote

$$l(A) = X_{(n-k+1)}, \quad f(A) = X_{(k)}.$$  

We call an edge $A$ dense if

$$l(A) - f(A) \leq \frac{1-p}{2},$$

where $p = 2k \ln n/n$. For each edge $A$, let $\mathcal{N}(A)$ denote the event that the edge $A$ is dense. Put

$$\mathcal{R} = \bigcup_A \mathcal{N}(A).$$

If we prove that the sum of probabilities of the events $\mathcal{R}$ and $\mathcal{F} := \bigcup_{f,s} \mathcal{M}(f, s)$ is strictly less than 1, assuming that no edge in $\mathcal{F}$ is dense, then with positive probability there exists such a numeration $\sigma$ of $V$ that for each $f$ and $s$,

$$L_\sigma(f) \cap F_\sigma(s) = \emptyset,$$

which implies that $H$ has property $B_k$ according to the criterion.

Our next step is to estimate the probabilities of the events $\mathcal{R}$ and $\mathcal{M}(f, s)$.

3.3. The Estimate for the Probability of $\mathcal{R}$.

$$\mathbb{P}(\mathcal{R}) = \mathbb{P}\left(\bigcup_A \mathcal{N}(A)\right) \leq |E| \max_A \mathbb{P}(\mathcal{N}(A)) = |E| \mathbb{P}\left(l(A) - f(A) \leq \frac{1-p}{2}\right)$$

$$= |E| \left[ \mathbb{P}\left(l(A) \leq \frac{1-p}{2}\right) + \mathbb{P}\left(l(A) - f(A) \leq \frac{1-p}{2}, l(A) > \frac{1-p}{2}\right) \right]$$

$$= |E| \mathbb{P}\left(l(A) \leq \frac{1-p}{2}\right) + |E| \mathbb{P}\left(l(A) - f(A) \leq \frac{1-p}{2}, l(A) > \frac{1-p}{2}\right). \tag{10}$$

Then we estimate both summands of (10) separately.

$$|E| \mathbb{P}\left(l(A) \leq \frac{1-p}{2}\right) = |E| n \binom{n-1}{k-1} \int_0^{(1-p)/2} x^{n-k+1}(1-x)^{k-1} \, dx$$

$$= |E| n \binom{n-1}{k-1} \left. \frac{x^{n-k+1}(1-x)^{k-1}}{n-k+1} \right|_0^{(1-p)/2} + \frac{k-1}{n-k+1} \int_0^{(1-p)/2} x^{n-k+1}(1-x)^{k-2} \, dx$$

$$= |E| n \binom{n-1}{k-1} \left. \frac{x^{n-k+1}(1-x)^{k-1}}{n-k+1} \right|_0^{(1-p)/2} + \frac{k-1}{n-k+1} \left. \frac{x^{n-k+2}(1-x)^{k-2}}{n-k+2} \right|_0^{(1-p)/2}$$

$$+ \frac{k-1}{n-k+1} \frac{k-2}{n-k+2} \int_0^{(1-p)/2} x^{n-k+2}(1-x)^{k-3} \, dx$$

$$= \cdots = |E| n \binom{n-1}{k-1} \left. \frac{x^{n-k+1}(1-x)^{k-1}}{n-k+1} \right|_0^{(1-p)/2} + \frac{k-1}{n-k+1} \left. \frac{x^{n-k+2}(1-x)^{k-2}}{n-k+2} \right|_0^{(1-p)/2}$$

$$+ \cdots + \frac{k-1}{n-k+1} \frac{k-2}{n-k+2} \cdots \frac{1}{n-1} \frac{x^n}{n} \left|_0^{(1-p)/2} \right.$$
Then (11) can be estimated in the following way:

\[ |E|n \left( \frac{n-1}{k-1} \right) \left( \frac{(1-p)/2}{n-k+1} \right)^{n-k+1} \left( \frac{(1-(1-p)/2)/2}{n-k+2} \right)^{k-1} + \frac{k-1}{n-k+1} \frac{(1-p)/2}{n-k+2} \]

\[ + \ldots + \frac{k-1}{n-k+1} \frac{k-2}{n-k+2} \ldots \frac{1}{n-1} \frac{(1-p)/2}{n} \] \quad \text{(11)}

Note that

\[ \frac{(1-p)/2}{n-k+1} \left( 1 - \frac{1}{n-k+2} \right)^{n-k+1} \left( 1 - \frac{1}{n-k+2} \right)^{k-2} \]

\[ \geq \ldots \geq \frac{k-1}{n-k+1} \frac{k-2}{n-k+2} \ldots \frac{1}{n-1} \frac{(1-p)/2}{n} \]

Then (11) can be estimated in the following way:

\[ |E|n \left( \frac{n-1}{k-1} \right) k \frac{(1-p)/2}{n-k+1} \left( 1 - \frac{1}{n-k+2} \right)^{n-k+1} \left( 1 - \frac{1}{n-k+2} \right)^{k-1} \]

\[ \leq \frac{6}{e^{26}} \sqrt{\frac{n}{k \ln n}} \frac{nk}{n-k+1} \left( 1 - \frac{1}{n-k+2} \right)^{n} \left( 1 + \frac{2p}{1-p} \right)^{k-1} = S_1. \] \quad \text{(12)}

Here we used the inequalities \((1 - x/n)^n \leq \exp(-x)\) and \(1 + x \leq \exp(x)\) for \(x > -1\).

Since

\[ \exp \left( \frac{2p(k-1)}{1-p} \right) \leq \exp \left( \frac{2pk}{1-p} \right) = \exp \left( \frac{2(2k \ln n/n)k}{1-2k \ln n/n} \right) = \exp \left( \frac{4k^2 \ln n}{n-2k \ln n} \right) \]

\[ \leq \exp \left( \frac{4}{n-2\sqrt{n} \ln n} \right) = \exp \left( \frac{4n}{n-2\sqrt{n} \ln n} \right) = \exp \left( \frac{4}{1-2\sqrt{n} \ln n} \right) \]

we obtain

\[ S_1 < \frac{6}{e^{13}} \sqrt{\frac{n}{k \ln n}} \frac{nk}{n-k+1} \frac{1}{n^{2k}} \leq \frac{6}{e^{13}} \sqrt{\frac{n^{3k^2}}{k \ln n(n-k+1)^{2n^{4k-3}}}} \leq \frac{6}{e^{13}} \sqrt{\frac{\sqrt{n} \ln n}{\ln n(n-\sqrt{n} \ln n)^2n^{4k-3}}} \]

\[ \leq \frac{6}{e^{13}} \sqrt{\frac{1}{27^2(n/n)3/2n^{4k-7/2}}} \leq \frac{6}{27e^{13}} \sqrt{\frac{1}{n \ln n}} < \frac{1}{45e^{13}} \text{ for all } n \geq 30. \] \quad \text{(14)}

Let us estimate the second summand in (10).

\[ |E|P \left( l(A) - f(A) \leq \frac{1-p}{2}, \ l(A) > \frac{1-p}{2} \right) \]

\[ = |E|n \left( \frac{n-1}{k-1} \right) \int_{(1-p)/2}^{1} \sum_{i=0}^{k-1} \binom{n-k}{i} \left( x - \frac{1-p}{2} \right)^i \left( \frac{1-p}{2} \right)^{n-k-i} (1-x)^{k-1} dx \]

\[ = |E|n \left( \frac{n-1}{k-1} \right) \sum_{i=0}^{k-1} \binom{n-k}{i} \left( \frac{1-p}{2} \right)^{n-k-i} \int_{(1-p)/2}^{1} \left( x - \frac{1-p}{2} \right)^i (1-x)^{k-1} dx \]
\[
= |E|n(n-1) \sum_{i=0}^{k-1} \frac{(n-k)}{i} \left( 1 - p \right)^{n-k-i} \frac{k-1}{i+1} \int_{(1-p)/2}^{1} \left( x - \frac{1-p}{2} \right)^{i+1} (1-x)^{k-2} \, dx
\]

\[
= \cdots = |E|n(n-1) \sum_{i=0}^{k-1} \frac{(n-k)}{i} \left( 1 - p \right)^{n-k-i} \frac{(k-1)!}{(i+1) \ldots (i+k-1)} \int_{(1-p)/2}^{1} \left( x - \frac{1-p}{2} \right)^{i+k-1} \, dx
\]

\[
= |E|n(n-1) \sum_{i=0}^{k-1} \frac{(n-k)}{i} \left( 1 - p \right)^{n-k-i} \frac{(k-1)!}{(i+k)!} \left( x - \frac{1-p}{2} \right)^{i+k} \bigg|_{(1-p)/2}^{1} = S_2.
\]

We will estimate the inner sum in (15). For this purpose, we will find the maximum term in the sum. The ratio of \(i\)th summand and \((i+1)\)th summand is equal to

\[
\frac{n-k}{(i+1)!} \frac{(1+p)^i}{(1-p)^{i-1}} = \frac{(i+k+1)(1-p)}{(n-k-i)(1+p)} < \frac{i+k+1}{n-k-i} \leq \frac{2k}{n-2k} = \varepsilon(n,k) = \varepsilon < 1.
\]

We used the condition \(i \leq k-1\). Hence, the greatest summand in the sum has the greatest number \(i\), which is \(k-1\). Therefore, the sum can be estimated by the greatest summand multiplied by \((1-\varepsilon)^{-1}\). Then

\[
S_2 < (1-\varepsilon)^{-1} \frac{12}{2e^{26}} \sqrt{\frac{n}{k \ln n}} \frac{n-k}{k-1} \frac{(k-1)!}{(2k-1)!} (1-p)^{n} \frac{(1+p)^{2k-1}}{1-p}
\]

\[
= (1-\varepsilon)^{-1} \frac{6}{e^{26}} \sqrt{\frac{n}{k \ln n}} \frac{n(n-k)!}{(n-2k+1)!} \frac{(k-1)!}{(2k-1)!} \left( 1 - \frac{\ln n^{2k}}{n} \right)^{n} \frac{1}{1-p} (1+2p)^{2k-1}
\]

\[
\leq (1-\varepsilon)^{-1} \frac{6}{e^{26}} \sqrt{\frac{n}{k \ln n}} \frac{n(n-k)^{k-1}}{(2k-1)!} \frac{(k-1)!}{n^{2k}} \exp \left( \frac{2p(2k-1)}{1-p} \right)
\]

\[
< (1-\varepsilon)^{-1} \frac{6}{e^{26}} \sqrt{\frac{n}{k \ln n}} \frac{n^k}{(2k-1)!} \frac{1}{n^{2k}} \frac{e^{26}}{5(1-\varepsilon)^{-1}} = 6(1-\varepsilon)^{-1} \sqrt{\frac{n}{k \ln n}} \frac{(k-1)!}{(2k-1)!n^k} < 6(1-\varepsilon)^{-1} \frac{1}{n \ln n} < \frac{3}{5}(1-\varepsilon)^{-1} \text{ for all } n \geq 30.
\]

Hence, taking into account (14) and (16), we derive the final estimate for the probability of the event \(\mathcal{R}\):

\[
P(\mathcal{R}) < \frac{1}{45e^{13}} + \frac{3}{5}(1-\varepsilon)^{-1}.
\]

3.4. The Estimate for the Probability of \(\mathcal{M}(f,s)\). Consider two arbitrary edges \(f\) and \(s\). Recall that we assume that no edge is dense. Let \(h\) be the number of their common vertices and let \(v_0\) be some vertex from \(f \cap s\). Denote

\[
A_1 = \{ v \in f \cap s : \sigma(v) < \sigma(v_0) \}, \quad B_1 = \{ v \in f \cap s : \sigma(v) > \sigma(v_0) \},
\]

\[
A_2 = \{ v \in f \setminus s : \sigma(v) > \sigma(v_0) \}, \quad B_2 = \{ v \in s \setminus f : \sigma(v) < \sigma(v_0) \}.
\]

The event \(\mathcal{M}(f,s,v_0)\) implies that

\[
|A_2| + |B_1| \leq k-1, \quad |A_1| + |B_2| \leq k-1.
\]

By the definition (9) the event \(\mathcal{M}(f,s,v_0)\) is equivalent to the intersection of the following events:

\[
\mathcal{M}(f,s,v_0) = \{ |A_2| + |B_1| \leq k-1 \} \cap \{ |A_1| + |B_2| \leq k-1 \}.
\]
We know that \(|A_1| + |B_1| = h - 1\). Consequently, it follows from \(\mathcal{M}(f, s, v_0)\) that \(h + |A_2| + |B_2| \leq 2k - 1\). If \(h \geq 2k\), then the latter inequality is impossible as well as the event \(\mathcal{M}(f, s, v_0)\). Below we consider only the case \(h \leq 2k - 1\).

It follows from (18) that

\[
\mathcal{M}(f, s, v_0) = \bigcup_{i,j,t \in \mathbb{Z}_+} \{ |A_2| = i, |B_2| = j, |A_1| = t \}.
\]  

(19)

Recall that

\[
\mathcal{M}(f, s) = \bigcup_{v_0 \in f \cap s} \mathcal{M}(f, s, v_0).
\]

Then by (19) we derive the estimate for the probability of \(\mathcal{M}(f, s)\):

\[
\mathbb{P}(\mathcal{M}(f, s)) \leq h \sum_{t = \max(0, h-k)}^{\min(k-1, h-1)} \binom{h-1}{t} \sum_{i=0}^{k-h+t} \binom{n-h}{i} \sum_{j=0}^{k-1-t} \binom{n-h}{j} \times \int_{(1-p)/2}^{(1+p)/2} x^{n-h-i+t+j} (1-x)^{n-h-j+(h-1-t)+i} \, dx.
\]

(20)

The limits of the integration in the right-hand side of (20) are from \((1-p)/2\) to \((1+p)/2\). Suppose the opposite: \(X_{v_0} < (1-p)/2\) or \(X_{v_0} > (1+p)/2\). Consider the case of \(X_{v_0} > (1+p)/2\). It follows from \(v_0 \in F_\sigma(s)\) that \(f(s) \geq X_{v_0} > (1+p)/2\). Then

\[
l(s) - f(s) < l(s) - \frac{1+p}{2} < \frac{1-p}{2},
\]

which contradicts the condition that \(s\) is not dense. The case of \(X_{v_0} < (1-p)/2\) is similar to the considered one.

Further, we estimate the expression in the right-hand side of (20). First, we estimate each integral in (20):

\[
\int_{(1-p)/2}^{(1+p)/2} x^{n-h-i+t+j} (1-x)^{n-h-j+(h-1-t)+i} \, dx
\]

\[
= \int_{-p/2}^{p/2} \left( \frac{1}{2} + y \right)^{n-h-i+t+j} \left( \frac{1}{2} - y \right)^{n-h-j+(h-1-t)+i} \, dy
\]

\[
= \left( \frac{1}{2} \right)^{2n-h-1} \int_{-p/2}^{p/2} (1+2y)^{n-h-i+t+j} (1-2y)^{n-h-j+(h-1-t)+i} \, dy
\]

\[
= \left( \frac{1}{2} \right)^{2n-h-1} \int_{-p/2}^{p/2} (1-4y^2)^{-h} (1+2y)^{-i+t+j} (1-2y)^{-j+(h-1-t)+i} \, dy
\]

\[
\leq \left( \frac{1}{2} \right)^{2n-h-1} \int_{-p/2}^{p/2} (1+2y)^{-i+t+j} (1-2y)^{-j+(h-1-t)+i} \, dy
\]
The latter inequality requires clarification. Denote $x = h - t$, then

\[
\frac{(n-h)^2(h-t)}{2(h+1)(k-h+t)(n-h-k+t+1)} = \frac{x}{x+t+1} \geq \frac{(n-h)^2}{2(k-x)(n-x-k+1)}.
\]
is an increasing function of \( x \). From the conditions
\[
x = h - t \geq 1, \quad t \leq k - 1, \quad h < 2k
\]
we obtain (24).

It follows from the condition (5) of the theorem that \( \beta \) is strictly greater than 1. We have
\[
(n - 2k)^2 - 2k^2(n - k) > n^2 - 4nk - 2k^2 n \geq n^2 - 4\sqrt{n} \ln n \quad 2n^2 \ln n = n^2 \left( \ln n - 2 - 4\sqrt{\frac{\ln n}{n}} \right) > 0,
\]

since
\[
\ln n - 2 - 4\sqrt{\frac{\ln n}{n}} \geq \ln 30 - 2 - 4\sqrt{\frac{\ln 30}{30}} > 0 \quad \text{for all} \quad n \geq 30.
\]

Hence, every summand of the right-hand side of (23) is a decreasing function of \( h \). So for all \( t \), due to the restriction \( h \geq t + 1 \) we have
\[
h\left( \frac{h - 1}{t} \right) \left( \frac{n - h}{k - 1 - t} \right) \left( \frac{n - h}{k - h + t} \right)^{2h+1} \leq (t + 1) \left( \frac{n - t - 1}{k - 1 - t} \right) \left( \frac{n - t - 1}{k - 1} \right)^{2t+2}.
\]

The relation (23) implies that
\[
\Pr(\mathcal{M}(f, s)) \leq \frac{e^{30}}{(1 - \alpha)^2} \sum_{t = \max(0, h - k)}^{\min(k - h, 1)} (t + 1) \left( \frac{n - t - 1}{k - 1 - t} \right) \left( \frac{n - t - 1}{k - 1} \right)^{2t+2}, \quad (25)
\]

Clearly the greater number of positive summand makes the sum greater, thus
\[
\Pr(\mathcal{M}(f, s)) \leq e^{30} (1 - \alpha)^{-2} \sum_{t = 0}^{\min(k - h, 1)} (t + 1) \left( \frac{n - t - 1}{k - 1 - t} \right) \left( \frac{n - t - 1}{k - 1} \right)^{2t+2}.
\]

We will find the maximal summand in the sum over \( t \) by considering the ratio of neighboring summand as before. The ratio of the summand \( t \) and the summand \( t + 1 \) is equal to
\[
\frac{(t + 1) \left( \frac{n - t - 1}{k - 1 - t} \right)^{2t+2}}{(t + 2) \left( \frac{n - t - 2}{k - 2 - t} \right)^{2t+3}} = \frac{(t + 1)(n - t - 1)^2}{2(t + 2)(k - 1 - t)(n - t - k)} \geq \frac{(n - k)^2}{4k(n - k)} = \frac{n - k}{4k} = \gamma^{-1} > 1.
\]

Here we used the inequalities \( t \geq 0 \) and \( t + 1 \leq k \).

Thus, the maximum summand corresponds to the case \( t = 0 \). The sum can be estimated by the maximal summand multiplied by \( (1 - \gamma)^{-1} \). Finally, we obtain this estimate for the probability of the event \( \mathcal{M}(f, s) \):
\[
\Pr(\mathcal{M}(f, s)) \leq e^{30} (1 - \alpha)^{-2} (1 - \gamma)^{-1} \left( \frac{n - 1}{k - 1} \right)^2 p \left( \frac{1}{2} \right)^{2n-2}, \quad (26)
\]

3.5. Auxiliary Analysis. In this section, we estimate the following factors:
\[
R(n, k) = (1 - \alpha)^{-2} (1 - \gamma)^{-1}, \quad T(n, k) = (1 - \varepsilon)^{-1},
\]

which appear in the right-hand sides of (17) and (26), respectively. We appeal to the condition (5) of the theorem.

The following relations hold true:
\[
\alpha = \frac{k}{n - 3k} \leq \frac{\sqrt{n/\ln n}}{n - 3\sqrt{n/\ln n}} = \frac{1}{\sqrt{n \ln n} - 3} \leq \frac{1}{\sqrt{30} \ln 30 - 3} \leq 0.15,
\]
\[
\gamma = \frac{4k}{n - k} \leq \frac{4\sqrt{n/\ln n}}{n - \sqrt{n/\ln n}} = \frac{4}{\sqrt{n \ln n} - 1} \leq \frac{1}{4\sqrt{30} \ln 30 - 1} \leq 0.5,
\]
\[
\varepsilon = \frac{2k}{n - 2k} \leq \frac{2\sqrt{n/\ln n}}{n - 2\sqrt{n/\ln n}} = \frac{2}{\sqrt{n \ln n} - 2} \leq \frac{2}{\sqrt{30} \ln 30 - 2} \leq 0.25,
\]

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(1 - α)^{-2} \leq (1 - 0.15)^{-2} \leq 1.5,
(1 - γ)^{-1} \leq (1 - 0.5)^{-1} = 2,
(1 - ε)^{-1} \leq (1 - 0.25)^{-1} = \frac{4}{3}.

The above inequalities yield the final upper bounds for the factors $R(n, k)$ and $T(n, k)$:

$$R(n, k) = (1 - α)^{-2}(1 - γ)^{-1} \leq 1.5 \times 2 = 3,$$

$$T(n, k) = (1 - ε)^{-1} \leq \frac{4}{3}.$$ (27) (28)

3.6. Completion of the Proof. We complete the proof of the theorem. It follows from (26) and (27) that

$$\mathbb{P}(\mathcal{M}(f, s)) \leq 3e^{30} \left(\binom{n-1}{k-1}\right)^2 2^{2-2n} \frac{2k \ln n}{n}.$$ (29)

Then the probability of $F$ can be estimated as follows:

$$\mathbb{P}(\mathcal{F}) = \mathbb{P} \left( \bigcup_{f,s} \mathcal{M}(f, s) \right) \leq |E|^2 \max_{f,s} \mathbb{P}(\mathcal{M}(f, s)) \leq |E|^2 3e^{30} \left(\binom{n-1}{k-1}\right)^2 2^{2-2n} \frac{2k \ln n}{n} < \frac{864}{e^{22}}.$$ (30)

The latter inequality follows from the condition (7) on the hypergraph $H$.

It follows from (17) and (28) that

$$\mathbb{P}(\mathcal{R}) < \frac{1}{45e^{13}} + \frac{4}{5}.$$ (31)

Thus, from (30) and (31) we obtain the final estimate for the sum of the probabilities of the events $\mathcal{F}$ and $\mathcal{R}$:

$$\mathbb{P}(\mathcal{F}) + \mathbb{P}(\mathcal{R}) < \frac{864}{e^{22}} + \frac{1}{45e^{13}} + \frac{4}{5} < 1.$$ 

Let us sum up. We have shown that the sum of the probabilities of the events $\mathcal{R}$ and $\mathcal{F}$ is strictly less than 1, assuming that none of the edges in $\mathcal{F}$ is dense. Hence with positive probability in a random numeration $σ$ of $V$ for each pair of edges $f$ and $s$, we have

$$L_σ(f) \cap F_σ(s) = \emptyset.$$ 

Therefore, $H$ has the property $B_k$ according to the criterion. We have obtained the desired inequality, since the hypergraph $H$ with the condition (7) was chosen arbitrarily:

$$m_k(n) \geq \frac{12}{e^{26}} \left(\frac{n}{k \ln n}\right)^{1/2} \frac{2^{n-1}}{(n-1)}.$$ 

Theorem 1 is proved.

3.7. Corollary: Property $B_{k,ε}$. We say that a subhypergraph $H'$ of a hypergraph $H = (V, E)$ is spanning if the set of its vertices is $V$ and the set of its edges $E'$ is a subset of $E$. We say that a hypergraph $H = (V, E)$ has property $B_{k,ε}$ if there is a spanning subhypergraph $H' = (V, E')$ with property $B_k$ and with $|E'| \geq (1 - ε)|E|$. It is easy to see that $ε$ is from 0 to 1. Property $B_{k,ε}$ is equivalent to property $B_k$ when $ε = 0$. Shabanov [20, 21] showed that if

$$ε \geq \left(\sum_{j=0}^{k-1} \binom{n}{j}\right)2^{1-n},$$

then the property $B_{k,ε}$ is trivial that is an arbitrary $n$-uniform hypergraph has property $B_{k,ε}$.

The quantity $m_{k,ε}(n)$ is the minimum number of edges of an $n$-uniform hypergraph that does not have property $B_{k,ε}$. We have $m_{k,ε}(n) = m_k(n)$ for $ε = 0$. It is easy to show that if $ε < 1/m_k(n)$, then $m_{k,ε}(n) = m_k(n)$.

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It turns out that the quantity $m_{k,\varepsilon}(n)$ makes sense only when
\[ \varepsilon \in \left( \frac{1}{m_k(n)}, \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{1-n} \right). \]

Consider inequality (2). Since $\psi(n)$ converges, it is bounded. Hence, we have
\[ \varepsilon \in \left( R \cdot \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{1-n} \cdot n^{-2}, \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{1-n} \right), \tag{32} \]
where $R$ is some constant. Shabanov [20,21] proved that under these conditions there exist constants $c$ and $C$ such that for $k \leq C \ln n$, the following inequality holds:
\[ m_{k,\varepsilon}(n) \geq c \cdot \varepsilon \cdot \frac{n}{\ln n} \cdot \frac{2^{2n}}{\left( \sum_{j=0}^{k-1} \binom{n}{j} \right)^2} \cdot \frac{2^{-2k}e^{-k}}{2k - 1}. \tag{33} \]
Clearly, (33) is similar to (3): it was produced by squaring (3) and multiplying it by $\varepsilon$. In [3] the following result for this quantity was obtained.

**Theorem 2.** Let $n \geq 14$, $k \geq 2$, and let
\[ 2k^2(n - k) \leq (n - 2k)^2. \]
Then
\[ m_{k,\varepsilon}(n) \geq 0.0361 \cdot \varepsilon \cdot \frac{n}{\ln n} \cdot \frac{2^{2n-2}}{\binom{n}{k-1}^2}. \]
Evidently, this inequality is similar to (4) as the inequality (33) was similar to (3). Its advantage is the same: the range of values of $k$ extends from $\ln n$ to the square root of $n$.

**Theorem 3.** Let $k \geq 2$ and suppose that for all $n \geq n_1 = 30$,
\[ k \leq \sqrt{\frac{n}{\ln n}}. \]
Put
\[ I = R \cdot \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{-n-1} \cdot n^{-2}, \quad J = \frac{2e^{30}}{9} \cdot \frac{1}{2^n n \ln n}. \]
Let $N_1$ be a natural number such that for all $n \geq N_1$,
\[ I \geq J. \]
Then for all $n \geq \max(n_1, N_1)$,
\[ m_{k,\varepsilon}(n) \geq \frac{1}{3e^{35}} \cdot \frac{2^{2n-2}}{\binom{n}{k-1}^2} \cdot \frac{n}{\ln(n^{2k} \ln n)} \cdot \varepsilon. \]
The estimate for $m_{k,\varepsilon}(n)$ obtained in Theorem 3 improves the result of Theorem 2 and (33) when the growth of $k$ does not exceed $n^{1/2-\delta}/\sqrt{\ln n}$, where $0 < \delta < 1/2$.

**Proof.** Let $X$ denote a random variable equal to the number of bad pairs of edges in $H$ in the random numeration $\sigma$ and let $Y$ be a random variable equal to the number of dense edges in $H$ in the random numeration $\sigma$. If we show that the sum of expected values of $X$ and $Y$ is less than $\varepsilon |E|$, then we prove that $H$ has property $B_{k,\varepsilon}$. Indeed, if $\mathbb{E}(X + Y) < \varepsilon |E|$, then there exists a numeration $\sigma$ such that $X(\sigma) + Y(\sigma) < \varepsilon |E|$. We take such $\sigma$ and remove an edge from every bad pair and remove all dense edges.
from the hypergraph as well. The obtained spanning hypergraph \( H' \) has at least \( |E|(1 - \varepsilon) \) edges. It has property \( B_k \) according to the criterion.

Set \( p = \ln(n^{2k} \ln n)/n \). We need the following auxiliary results:

\[
(1 + p)^{2k} \leq \exp\left(\frac{2k \ln(n^{2k} \ln n)}{n}\right) = \exp\left(\frac{4k^2 \ln n + 2k \ln n}{n}\right) = e^4 \cdot \left(\frac{2 \ln n}{\sqrt{n} \ln n}\right) \leq e^4 \cdot \left(\frac{2 \ln 30}{\sqrt{30} \ln 30}\right) < e^5,
\]

\[
\left(\frac{1 + p}{1 - p}\right)^k = \left(1 + \frac{2p}{1 - p}\right)^k \leq \exp\left(\frac{2k(2k \ln n + \ln n)/n}{1 - (2k \ln n + \ln n)/n}\right) \leq \exp\left(\frac{4 + 2 \ln 30/\sqrt{30} \ln 30}{1 - 2/\sqrt{30} \ln 30 - \ln 30/\sqrt{30} \ln 30}\right) < e^{15}. \quad (34)
\]

For the expected value of the random variable \( Y \), we have

\[
\mathbb{E}(Y) \leq |E| \max_A \mathbb{P}(\mathcal{N}(A)) = |E| \mathbb{P}\left(l(A) - f(A) \leq \frac{1 - p}{2}\right) = |E| \left[ \mathbb{P}\left(l(A) \leq \frac{1 - p}{2}\right) + \mathbb{P}\left(l(A) - f(A) \leq \frac{1 - p}{2}, l(A) > \frac{1 - p}{2}\right) \right] = |E| \mathbb{P}\left(l(A) \leq \frac{1 - p}{2}\right) + |E| \mathbb{P}\left(l(A) - f(A) \leq \frac{1 - p}{2}, l(A) > \frac{1 - p}{2}\right). \quad (35)
\]

We estimate separately each of two terms in the right-hand side of (35) as before:

\[
|E| \mathbb{P}\left(l(A) \leq \frac{1 - p}{2}\right) = |E| n^{\left(\frac{1 - p}{2}\right)} \int_0 \cdots = |E| n^{\left(\frac{1 - p}{2}\right)} \frac{\left(\frac{(1 - p)/2}{n - k + 1}\right)^k}{n - k + 1} 
\]

\[
|E| \mathbb{P}\left(l(A) - f(A) \leq \frac{1 - p}{2}, l(A) > \frac{1 - p}{2}\right) = |E| n^{\left(\frac{1 - p}{2}\right)} \int_0 \cdots |E| n^{\left(\frac{1 - p}{2}\right)} \frac{\left(\frac{(1 - p)/2}{n - k + 1}\right)^k}{n - k + 1} 
\]

The estimates in (36) can be obtained in the same way as in (11) and (12). Further, we estimate the second summand in the right-hand side of (35):

\[
|E| \mathbb{P}\left(l(A) - f(A) \leq \frac{1 - p}{2}, l(A) > \frac{1 - p}{2}\right) 
\]

\[
= |E| n^{\left(\frac{1 - p}{2}\right)} \int_0 \cdots |E| n^{\left(\frac{1 - p}{2}\right)} \frac{\left(\frac{(1 - p)/2}{n - k + 1}\right)^k}{n - k + 1} 
\]

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< |E| \frac{n-1}{k-1} \binom{n}{k-1}^2 \frac{(k-1)!}{(2k-1)!} \frac{1}{2^n} \binom{1+p}{1-p}^n (1-\varepsilon)^{-1}

< |E| \left( \frac{2^{(k-1)+1} n^2}{3 2^n n^2 k \ln n (2k-1)!} \right)^{2k} \left( \frac{1+p}{1-p} \right)^{2k} \leq |E| \frac{2e^{30} 1}{9 2^n n \ln n}.

(37)

From (32) we have

\varepsilon \leq \left( R \cdot \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{-n-1} \cdot n^{-2} \right) \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{-n-1}.

It follows from this relation and from (36) and (37) that if \( n > \max(n_1, N) \), then

|E| \frac{2e^{30} 1}{9 2^n n \ln n} < \varepsilon |E|,

|E| \frac{2e^{30} 1}{9 2^n n \ln n} < \varepsilon |E|.

Recall that \( p = \ln(n^{2k} \ln n)/n \). Then similarly to (29) the inequalities (34) imply that

\mathbb{P}(\mathcal{M}(f, s)) \leq 3e^{35} \frac{(n-1)^2}{n} 2^{2-2n} |\ln(n^{2k} \ln n)| n.

Let us estimate the expected value of the random variable \( X \):

\mathbb{E}(X) \leq |E|^2 \max_{f, s} \mathbb{P}(\mathcal{M}(f, s))

\leq \left( \frac{1}{3e^{35} \frac{(n-1)^2}{n} \ln(n^{2k} \ln n)} \varepsilon \right) \left( 3e^{35} \frac{(n-1)^2}{n} 2^{2-2n} |\ln(n^{2k} \ln n)| n \right) = \frac{\varepsilon}{2} |E|.

So, we derive that

\mathbb{E}(X + Y) < \left( \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} \right) |E| = \varepsilon |E|.

Theorem 3 is proved. \( \square \)

**Remark.** Since \( m_{k, \varepsilon}(n) \geq m_k(n) \), it is easy to see that if

\varepsilon \in \left( R \cdot \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{1-n} \cdot n^{-2} \cdot S \cdot \left( \sum_{j=0}^{k-1} \binom{n}{j} \right) \cdot 2^{1-n} \cdot n^{-1/2} \right),

where \( S \) is some constant, the estimate (6) is the best for \( m_{k, \varepsilon}(n) \).

### 4. Hypergraphs with Restrictions on an Intersection of Edges

Consider the following generalization of property B\(_k\)-problem. We say that a hypergraph has property \( A_h \) if any two edges of the hypergraph do not intersect or have at least \( h \) common vertices. We define \( m_{k, h}(n) \) as the minimum possible number of edges of a hypergraph that has property \( A_h \) but does not have property \( B_k \). For \( h \geq 2k \), this problem does not make sense. Evidently any hypergraph that has property \( A_{2k} \) has property \( B_k \), hence quantity \( m_{k, h}(n) \) does not exist when \( h \geq 2k \). In nontrivial case the lower bound for this quantity was obtained by Shabanov [20]:

\[ m_{k, h}(n) \geq c \left( \frac{3n}{2h n \ln n} \right)^{h/3} \frac{2^{n-1}}{\left( \frac{n}{(n-1)} \right)}, \]

(38)

where \( k = O(h \ln n) \) and \( h < 2k \).

Further, there holds true

\[ m_{k, h}(n) \geq m_k(n). \]
That is all the lower bounds for \( m_k(n) \) hold for \( m_{k,h}(n) \) as well as the lower bound (6). Rozovskaya [17] obtained the following result:

\[
m_{k,h}(n) \geq \frac{0.19n^{1/4}2^{n-1}}{\sqrt{2^{h-k}(h-k+1)(n-1)(2k-1-h)}}
\]  

(39)

provided \( 2k^2(n-k) < (n-2k)^2 \). In comparison to the previous result (38), the lower bound of Rozovskaya is valid for a larger range of values of \( k \). The next theorem yields a new lower bound for \( m_{k,h}(n) \).

**Theorem 4.** Let \( n \geq 30 \), \( k > 2 \), \( k < h < 2k \) and suppose \( (5) \) holds. Then

\[
m_{k,h}(n) \geq \frac{12}{e^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{2^{n-1}}{\sqrt{2^{h-k}(h-k+1)(n-1)(2k-1-h)}}.
\]

(40)

It is easy to check that the ratio of the right-hand sides of (40) and (6) under the conditions of Theorem 4 is not less than \((c_0n/k)^{(h-k)/2}\), where \( c_0 \) is some absolute constant. Thus, (40) estimates quantity \( m_{k,h}(n) \) better than (6). Moreover, the estimate of \( m_{k,h}(n) \) from Theorem 4 refines (39) when the growth of \( k \) does not exceed \( n^{1/2-\delta}/\sqrt{\ln n} \), where \( 0 < \delta < 1/2 \).

**Proof.** The proof is similar to the proof of Theorem 1. It suffices to show that an arbitrary \( n \)-uniform hypergraph \( H = (V, E) \) that has property \( A_h \) and with

\[
|E| < \frac{12}{e^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{2^{n-1}}{\sqrt{2^{h-k}(h-k+1)(n-1)(2k-1-h)}},
\]

(41)

has property \( B_k \) as well.

Let \( H = (V, E) \) be such a hypergraph. Consider the random numeration \( \sigma \) that was constructed in the proof of Theorem 1 (see Sec. 3.2). We introduced the event \( R \) in the section where we proved Theorem 1. It implies that there exist dense edges in the hypergraph \( H \) in the random numeration \( \sigma \). According to (12), (13), (15), and (16) under the conditions of Theorem 1 (that is the condition (5)) the following inequality holds:

\[
\mathbb{P}(R) < e^{13}|E| \left( \frac{n-1}{k-1} \right) 2^{-n} - \frac{nk}{n-k+1} \frac{1}{n^{2k}} + e^{26}(1-\varepsilon)^{-1}|E| \left( \frac{n-1}{k-1} \right) 2^{-n} n^k \frac{(k-1)!}{(2k-1)!} \frac{1}{n^{2k}}.
\]

(42)

From the condition (41) on the number of edges we have

\[
|E| \left( \frac{n-1}{k-1} \right) 2^{-n} < \frac{6}{e^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \left( \frac{n}{k \ln n} \right)^{1/2} \sqrt{2^{h-k}(h-k+1)(2k-1-h)}
\]

\[
< \frac{6}{e^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \sqrt{(2k-1-h)!/(k-1)!} (n-2k+1)! (n-k)! \frac{1}{n^{2k}} < \frac{6}{e^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \sqrt{(2k-1-h)!/(2k-h)!}.
\]

We used the inequality \( k < h < 2k \). From here we have the final estimate for (42):

\[
\mathbb{P}(R) < \frac{6}{e^{13}} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{n^k}{n-k+1} \frac{1}{n^{3k/2}} + 6(1-\varepsilon)^{-1} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{1}{n^{k/2}}
\]

\[
< \frac{6}{e^{13}} \frac{1}{n - \sqrt{n \ln n}} \frac{1}{n^{3k/2}} + 6(1-\varepsilon)^{-1} \frac{1}{n \ln n} < \frac{6}{e^{13}} \frac{1}{27} \frac{1}{10} + 6(1-\varepsilon)^{-1} \frac{1}{10}
\]

\[
= \frac{1}{45e^{13}} + \frac{1}{5} (1-\varepsilon)^{-1} < \frac{1}{45e^{13}} + \frac{4}{5} \text{ for all } n \geq 30.
\]

(43)
According to (25) under the conditions of Theorem 1 (that is the condition (5)) the following inequality holds:

\[
P(M(f, s)) \leq e^{30}(1 - \alpha)^{-2\lambda_m} \sum_{t=\max(0, l-k)}^{\min(k-1, l-1)} (t+1)(n-t-1) \binom{n-t-1}{k-1} 2^{t+2} p \left( \frac{1}{2} \right)^{2n},
\]

where \( l = f \cap s \). Under the conditions of Theorem 2, \( \max(0, l-k) = l-k \geq h-k \). Then we can increase the sum if we consider \( t \) from \( h-k \) to \( k-1 \). We find the maximal summand in the sum by considering the ratio of neighboring summands. Similarly to Theorem 1, we deduce that the maximal summand has the number \( t = h-k \) and the sum can be estimated by the maximal summand multiplied by \( (1 - \gamma)^{-1} \). Therefore,

\[
P(M(f, s)) \leq e^{30}(1 - \alpha)^{-2\lambda_m}(h-k+1) \binom{n-h+k-1}{2k-1-h} \binom{n-h+k-1}{k-1} 2^{h-k+2} p \left( \frac{1}{2} \right)^{2n}.
\]

Further, according to (27) we derive the following estimate for the probability of the event \( M(f, s) \):

\[
P(M(f, s)) \leq 6e^{30}2^{2-2n}(h-k+1) \binom{n-1}{2k-1-h} \binom{n-1}{k-1} 2^{h-k} \frac{k \ln n}{n}. \]

Finally, the probability of the event \( \mathcal{F} \) can be estimated in the following way:

\[
P(\mathcal{F}) \leq \sum_{f,s} P(M(f, s)) < |E|^2 6e^{30}2^{-2n}(h-k+1) \binom{n-1}{2k-1-h} \binom{n-1}{k-1} 2^{h-k} \frac{k \ln n}{n} < \frac{864}{e^{22}}.
\]

It follows from the latter inequality and (43) that

\[
P(\mathcal{R}) + P(\mathcal{F}) < \frac{1}{45e^{13}} + \frac{4}{5} + \frac{864}{e^{22}} < 1.
\]

We have proven that with positive probability in the random numeration \( \sigma \) for any edges \( f \) and \( s \),

\[
L_\sigma(f) \cap L_\sigma(s) = \emptyset.
\]

Therefore, according to the criterion \( H \) has property \( \text{B}_k \) and since the choice of \( H \) is arbitrary, we obtain the desired inequality

\[
m_{k,h}(n) \geq \frac{12}{e^{2n}} \left( \frac{n}{k \ln n} \right)^{1/2} 2^{n-1} \sqrt{2^{h-k}(h-k+1) \binom{n-1}{k-1} \binom{n-1}{2k-1-h}}.
\]

Theorem 4 is proved.

5. Simple Hypergraphs

Another generalization of the question of Erdős–Hajnal is connected with simple hypergraphs. We say a hypergraph is simple if any two edges of it have no more than one vertex in the intersection. Let the quantity \( m^*(n) \) denote the minimum possible number of edges in a simple \( n \)-uniform hypergraph that does not have property \( \text{B} \). Initially the problem of finding \( m^*(n) \) was posed by P. Erdős and L. Lovász [7]. They obtained the following results:

\[
c_1 \frac{4^nn}{n^3} \leq m^*(n) \leq c_2 n^4 4^n.
\]

Further the lower bound was refined by Z. Szabó [25] and by A. V. Kostochka and M. Kumbhat [11]. They proved that for any \( \varepsilon > 0 \), there exists an integer \( n_0(\varepsilon) \) such that for all \( n > n_0(\varepsilon) \),

\[
m^*(n) \geq 4^n n^{-\varepsilon}.
\]
We consider the natural generalization of this problem. Define a quantity $m^*_k(n)$ which is equal to minimum possible number of edges in a simple $n$-uniform hypergraph that does not have property $B_k$. The best lower bound for this quantity is due to Rozovskaya [18]:

$$m^*_k(n) \geq \frac{(0.19)^2}{8e} \frac{2^{2n-2}}{(n-1)^{3/2}(n-2k-1)^2},$$  \hspace{1cm} (44)

where $2k^2(n - k - 1) < (n - 2k - 1)^2$. The following theorem yields a new lower bound for $m^*_k(n)$.

**Theorem 5.** Let $k \geq 2$ and suppose that (5) hold. Then for all $n \geq 30$,

$$m^*_k(n) \geq \frac{25}{18e^2} \frac{2^{2(n-1)}}{k(n-1)\ln(n-1)(n-2k-1)^2}. \hspace{1cm} (45)$$

Clearly the obtained estimate improves the previous result (44) when the growth of $k$ does not exceed $n^{1/2-\delta}/\sqrt{\ln n}$, where $0 < \delta < 1/2$. In order to prove Theorem 5 we need the following theorem, which produces a sufficient condition for a hypergraph to have property $B_k$.

**Theorem 6.** Let $H = (V, E)$ be an $n$-uniform hypergraph any edge of which does not intersect more than $D$ other edges. Let $n \geq 30$, $k \geq 2$, and suppose that (5) hold. If

$$D \leq \frac{5}{3e^{26}} \left( \frac{n}{k \ln n} \right)^{1/2} \frac{2^{n-1}}{(n-2k-1)^2} - 1,$$

then $H$ has property $B_k$.

5.1. **Proof of Theorem 6.** Let $H = (V, E)$ be an $n$-uniform hypergraph such that every of its edges intersects no more than $D$ other edges.

Let $\sigma$ be a random numeration as in the proof of Theorem 1 (see Sec. 3.2). Recall that we introduced the events

$$\{M(f, s): f, s \in E, f \cap s \neq \emptyset, f \text{ and } s \text{ are not dense}\}$$

and

$$\{N(A): A \in E, A \text{ is dense}\}.$$

If we prove that

$$\mathbb{P}\left\{ \left( \bigcap_{f,s} M(f, s) \right) \cap \left( \bigcap_A N(A) \right) \right\} > 0,$$

then with positive probability for any two edges $f$ and $s$,

$$L_\sigma(f) \cap F_\sigma(s) = \emptyset,$$

which means that $H$ has property $B_k$.

Further we need a supplementary theorem, whose proof can be found in [16].

**Theorem 7.** Let $A_1, \ldots, A_N$ be events in an arbitrary probabilistic space, let $S_1, \ldots, S_N$ be the subsets of $\{1, \ldots, N\}$. If the conditions

(1) $A_i$ is independent of the algebra generated by the events $\{A_j, j \notin S_i\}$,

(2) for all $i \in \{1, \ldots, N\}$, holds $\sum_{j \in S_i \cup S_i} \mathbb{P}(A_j) \leq 1/4$

hold, then

$$\mathbb{P}\left( \bigcap_{i=1}^N A_i \right) > 0.$$
Due to the construction of the random numeration $\sigma$, the event $\mathcal{M}(f,s)$ is independent of the algebra generated by events

$$\{\mathcal{M}(g,u): g, u \in E, \ g \cap u \neq \emptyset, \ (g \cup u) \cap (f \cup s) = \emptyset\}$$

and

$$\{\mathcal{N}(Q): Q \in E, \ Q \text{ is dense}\}$$

and the event $\mathcal{N}(A)$ is independent of the algebra generated by the events

$$\{\mathcal{M}(g,u): g, u \in E, \ g \cap u \neq \emptyset, \ g \text{ and } u \text{ are not dense}\}$$

and

$$\{\mathcal{N}(Q): Q \in E, \ Q \cap A = \emptyset, \ Q \text{ is dense}\}.$$

For the event $\mathcal{M}(f,s)$, consider

$$S(f,s) = \{\mathcal{M}(g,u): g, u \in E, \ g \cap u \neq \emptyset, \ (g \cup u) \cap (f \cup s) \neq \emptyset, \ g \text{ and } u \text{ are not dense}\},$$

and for the event $\mathcal{N}(A)$, consider

$$Z(A) = \{\mathcal{N}(Q): Q \in E, \ Q \cap A \neq \emptyset, \ Q \text{ is dense}\}.$$

Under the conditions of Theorem 6, the number of elements in $S(f,s)$ does not exceed $4(D+1)^2$ and the number of elements in $Z(A)$ does not exceed $D + 1$. From (29) and (42) we have

$$4(D+1)^2 \mathbb{P}(\mathcal{M}(f,s)) + (D+1)\mathbb{P}(\mathcal{N}(A)) \leq 4(D+1)^2 \left(3e^{30} \left(\frac{n-1}{k-1}\right)^2 \frac{2^{2-2n}2k \ln n}{n}\right) + (D+1) \left(\left(\frac{n-1}{k-1}\right)^2 - \frac{1}{32 e^{13}} + \frac{1}{9} \left(\frac{1}{4}\right)^2\right)$$

Here we used the condition (45) on $D$.

Thus, Theorem 7 implies that

$$\mathbb{P}\left\{\left(\bigcap_{f,s} \mathcal{M}(f,s)\right) \cap \left(\bigcap_A \mathcal{N}(A)\right)\right\} > 0.$$

Theorem 6 is proved.

5.2. Proof of Theorem 5. From Theorem 6 we know that if any edge of an $n$-uniform hypergraph intersects no more than $D$ other edges, where

$$D \leq \frac{5}{3e^{26}} \left(\frac{n}{k \ln n}\right)^{1/2} 2^{n-1} \frac{2^{n-1}}{n^{n-1} (n-1)} - 1,$$

then it has property $B_k$. Put

$$X(n) = \frac{5}{3e^{26}} \left(\frac{n}{k \ln n}\right)^{1/2} 2^{n-1} \frac{2^{n-1}}{n^{n-1} (n-1)} - 1.$$

Then if $n$-uniform hypergraph does not have property $B_k$, then it has an edge $f_0$ that intersects more than $X(n)$ other edges. Hence there exists a vertex $v_0 \in f_0$ such that

$$\deg v_0 \geq \frac{X(n)}{n} + 1.$$

Let $H = (V,E)$ be an $n$-uniform hypergraph that does not have property $B_k$. Consider an $(n-1)$-uniform hypergraph $H_1 = (V,E)$ that is obtained from $H$ by deleting a vertex of a maximum degree from each edge. Clearly $H_1$ does not have property $B_k$ as $H$ does not have this property. Then there is a vertex $v_1 \in V$ such that

$$\deg_{H_1} v_1 \geq \frac{X(n-1)}{n-1} + 1.$$
Consider all the edges that contain $v_1$. We removed the vertex with the maximum degree from each such edge. Now restore them. The initial hypergraph $H$ is simple hence all restored vertices are different. Let $u_1, \ldots, u_m$ be all these vertices and
\[ m \geq \deg_{H_1} v_1 \geq \frac{X(n-1)}{n-1} + 1. \]
The degree of every $u_j$ in $H$ is not smaller than $\deg_{H_1} v_1$ and every $u_i$ and $u_j$ have no more than one common edge.

Each restored vertex is contained in at least $Y = \frac{X(n-1)}{(n-1)+1}$ edges. Then the total number of edges in the hypergraph can be estimated as follows:
\[ Y + (Y-1) + (Y-2) + \ldots = \frac{Y(Y+1)}{2}. \]
We substitute $Y$ in this expression and obtain
\[ |E| \geq \frac{Y(Y+1)}{2} \geq \frac{1}{2} \left( \frac{X(n-1)}{n-1} + 1 \right)^2 \geq \frac{25}{18e^{52}} \frac{2^{2(n-1)}}{k(n-1)\ln(n-1)(\frac{n-2}{k-1})^2}. \]

Theorem 5 is proved.

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