EXISTENCE AND CONVERGENCE OF THE BERIS-EDWARDS SYSTEM WITH GENERAL LANDAU-DE GENNES ENERGY

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Abstract. In this paper, we investigate the Beris-Edwards system for both biaxial and uniaxial $Q$-tensors with a general Landau-de Gennes energy density depending on four non-zero elastic constants. We prove existence of the strong solution of the Beris-Edwards system for uniaxial $Q$-tensors up to a maximal time. Furthermore, we prove that the strong solutions of the Beris-Edwards system for biaxial $Q$-tensors converge smoothly to the solution of the Beris-Edwards system for uniaxial $Q$-tensors up to its maximal existence time.

1. Introduction

The classical Ericksen-Leslie theory ([9], [19]) successfully describes the dynamic flow of uniaxial nematic liquid crystals. In [3], Beris-Edwards pointed out that the Ericksen-Leslie flow theory has a limited domain of applications to liquid crystals. Therefore, based on the celebrated Landau-de Gennes $Q$-tensor theory, Beris-Edwards [3] proposed a general hydrodynamic theory to describe flows of liquid crystals in modeling both uniaxial and biaxial nematic liquid crystals.

In 1971, de Gennes [6] introduced a $Q$-tensor order parameter to establish the Landau-de Gennes theory, which has been one of the successful continuum theories in modeling both uniaxial and biaxial nematic liquid crystals (c.f. [7], [11]). Mathematically, the Landau-de Gennes theory is described by a Landau-de Gennes functional in the space of symmetric and traceless $3 \times 3$ matrices

$$ S_0 := \{ Q \in M_{3 \times 3} : Q^T = Q, \text{tr} Q = 0 \}, $$

where $M_{3 \times 3}$ denotes the space of $3 \times 3$ matrices. Let $S_*$ be the space of all uniaxial $Q$-tensors defined by

$$ S_* := \left\{ Q \in S_0 : Q = s+ (u \otimes u - \frac{1}{3} I), \quad u \in S^2, \quad s+ := \frac{b + \sqrt{b^2 + 24ac}}{4c} \right\}, $$

where the constants $a$, $b$, $c$ correspond to a lower temperature regime in liquid crystals and we assume that $a$, $b$, $c$ are positive. Let $U$ be a domain in $\mathbb{R}^3$. For a tensor $Q \in W^{1,2}(U; S_0)$, the original Landau-de Gennes energy is defined by

$$ E_{LG}(Q; U) := \int_U f_{LG} \, dx = \int_U (\tilde{f}_E + \tilde{f}_B) \, dx. \quad (1.1) $$

Here $\tilde{f}_E$ is the elastic energy density with elastic constants $L_1, ..., L_4$ of the form

$$ \tilde{f}_E(Q, \nabla Q) := \frac{L_1}{2} |\nabla Q|^2 + \frac{L_2}{2} \frac{\partial Q_{ij}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ij}}{\partial x_k} + \frac{L_4}{2} Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k}. \quad (1.2) $$
in which and in the sequel, we adopt the Einstein summation convention for repeated indices and \( f_B(Q) \) is a bulk energy density defined by

\[
\tilde{f}_B(Q) := -\frac{a}{2} \text{tr}(Q^2) - \frac{b}{3} \text{tr}(Q^3) + \frac{c}{4} [\text{tr}(Q^2)]^2
\]  

(1.3)

with three positive material constants \( a, b, c \).

In [6], de Gennes discovered first two terms of the elastic energy density in (1.2) with \( L_3 = L_4 = 0 \). Later, combining the work of Schiele-Trimper [30] with the effect of Berreman-Meiboom [4], Dickmann [5] completed the full density (1.2) with two additional terms (c.f. [22], [1]) in which the elastic energy density is consistent with the Oseen-Frank density in for uniaxial nematic liquid crystals. However, for the case of \( L_4 \neq 0 \), Ball-Majumdar [2] found an example that the Landau-de Gennes energy density (1.2) does not satisfy the coercivity condition. In fact, Golovaty et al. [13] said that “the bending constant is much larger than others”; i.e. \( k_3 > \max\{k_1, k_2\} \) at different temperatures. For example, for p-azoxyanisole (PAA) at 134°C, \( k_1 = 4.05 \), \( k_2 = 2.1 \), \( k_3 = 5.77 \), \( k_4 = 3.08 \) (see [30]). By the physical experiments on liquid crystals, the elastic constant \( L_4 = \frac{1}{2\pi^2} (k_3 - k_1) \) is not zero in general.

To solve the above coercivity problem on the Landau-de Gennes energy density in the case of \( L_4 \neq 0 \), it was observed in [10] that for uniaxial tensors, the original third order term on \( L_4 \) in (1.2), proposed by Schiele and Trimper [30, p. 268] in physics, is a linear combination of a fourth order term and a second order term; i.e. for \( Q \in S_+ \), we have

\[
Q_{lk} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} = \frac{3}{s_+} (Q_{ln} \frac{\partial Q_{ij}}{\partial x_l}) (Q_{kn} \frac{\partial Q_{ij}}{\partial x_k}) - \frac{2s_+}{3} |\nabla Q|^2.
\]  

(1.4)

Therefore, Feng and Hong [10] introduced a new Landau-de Gennes energy given by

\[
E_{LG}(Q; U) := \int_U f(Q, \nabla Q) \, dx = \int_U \left( f_E(Q, \nabla Q) + \frac{1}{L} f_B(Q) \right) \, dx,
\]  

(1.5)

where

\[
f_E(Q, \nabla Q) = \frac{\tilde{L}_1}{2} |\nabla Q|^2 + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{L_3}{2} \frac{\partial Q_{ik}}{\partial x_j} \frac{\partial Q_{ik}}{\partial x_k} + \frac{3}{2s_+} L^{(4)}(Q, \nabla Q)
\]  

(1.6)

with \( \tilde{L}_1 = L_1 - \frac{2}{3s_+} L_4 \) for \( L_4 \geq 0 \), \( \tilde{L}_1 = L_1 + \frac{4}{3s_+} L_4 \) for \( L_4 < 0 \),

\[
L^{(4)}(Q, \nabla Q) := \begin{cases} L_4 Q_{lk} Q_{kn} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} & \text{for } L_4 \geq 0, \\ L_4 [Q_{lk} Q_{kn} \frac{\partial Q_{ij}}{\partial x_l} \frac{\partial Q_{ij}}{\partial x_k} - |Q|^2 |\nabla Q|^2] & \text{for } L_4 < 0 \end{cases}
\]

and

\[
f_B(Q) := \tilde{f}_B(Q) - \min_{Q \in S_0} \tilde{f}_B(Q) \geq 0.
\]

The new elastic energy density (1.6) keeps three physical terms of the original Landau-de Gennes density (1.2) and is equivalent to the original density (1.2) for \( Q \in S_+ \). In (1.5), the constant \( L \) is a rescaled dimensionless parameter, which drives
parts of the tensor $\nabla$ depends on the director of the molecular field. The symmetric and skew-symmetric energy (1.5) with $H$ the velocity of the fluid and let $Q, \xi$, Define $[Q, \Omega] := Q\Omega - \Omega Q$ to be the Lie bracket product and set

$$S(Q, v) = \xi \left( D(Q + \frac{1}{3}I) + (Q + \frac{1}{3}I)D - 2(Q + \frac{1}{3}I)(Q \cdot D) \right) - [Q, \Omega].$$

Then the Beris-Edwards system (c.f. [3], [28]) is given by

$$\partial_t v + v \cdot \nabla v - \nu \Delta v + \nabla P = \nabla \cdot \left( \tau(Q, \nabla Q) + \sigma(Q, \nabla Q) \right),$$
$$\nabla \cdot v = 0,$$  \hspace{1cm} (1.8)

$$\partial_t Q + v \cdot \nabla Q - S(Q, v) = \Gamma H(Q, \nabla Q),$$  \hspace{1cm} (1.9)

where $H(Q, \nabla Q)$ is the molecular field, $P$ is the pressure, the antisymmetric part of the distortion stress $\tau(Q, \nabla Q) = [Q, H]$ and $\sigma(Q, \nabla Q)$ is the distortion stress (c.f. [29]) given by

$$\sigma_{ij}(Q, \nabla Q) = -\xi(QH + HQ + \frac{2}{3}H)_{ij} + 2\xi(Q \cdot H)(Q + \frac{I}{3})_{ij} - \partial_{\mu \nu}(Q, \nabla Q)\nabla_i Q_{k\mu}.$$  

Here and in the sequel, we denote $\partial_{\mu \nu}(Q, \nabla Q) := \frac{\partial f(Q, \nabla Q)}{\partial (\nabla Q_{\mu \nu})}$ with $p = \nabla Q$.

For simplicity, we assume that $\xi = 0$, $\Gamma = \nu = 1$ in the Beris-Edwards system. For biaxial $Q$-tensors, we use the Landau-de Gennes energy density (1.5) to formulate the Beris-Edwards system with $L_4 \neq 0$. The rescaled Beris-Edwards system for $Q_L \in S_0$ and $v_L \in \mathbb{R}^3$ is:

$$\partial_t v_L + v_L \cdot \nabla v_L - \Delta v_L + \nabla P_L = \nabla \cdot \left( [Q_L, \mathcal{H}(Q_L, \nabla Q_L)] + \sigma(Q_L, \nabla Q_L) \right),$$
$$\nabla \cdot v_L = 0,$$  \hspace{1cm} (1.11)

$$\partial_t Q_L + v_L \cdot \nabla Q_L + [Q_L, \Omega_L] = \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L).$$  \hspace{1cm} (1.12)

The molecular field $H_L(Q_L, \nabla Q_L)$ is then given by

$$H_L(Q_L, \nabla Q_L) := \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L),$$
where
\[ H(Q_L, \nabla Q_L)_{ij} = \frac{1}{2} \left( \nabla_k [\partial_{p_{ik}} f_E(Q_L, \nabla Q_L)] + \nabla_k [\partial_{q_{kj}} f_E(Q_L, \nabla Q_L)] \right) \]
\[ - \frac{1}{2} \left( \partial_{q_{ij}} f_E(Q_L, \nabla Q_L) + \partial_{q_{ji}} f_E(Q_L, \nabla Q_L) \right) \]
\[ - \frac{\delta_{ij}}{3} \sum_{l=1}^{3} \left( \nabla_k [\partial_{p_{lik}} f_E(Q_L, \nabla Q_L)] - \partial_{q_{il}} f_E(Q_L, \nabla Q_L) \right), \] (1.14)

the term \( g_B(Q_L) \) is
\[ g_B(Q_L) := a Q_L + b (Q_L Q_L - \frac{1}{3} \tr(Q_L^2) I) - c Q_L \tr(Q_L^2) \] (1.15)

and \( \sigma(Q_L, \nabla Q_L) \) is the distortion stress tensor with
\[ \nabla_j \sigma_{ij}(Q_L, \nabla Q_L) = -\nabla_j \left( \nabla_i (Q_L) k_l [\partial_{p_{lik}} f_E(Q_L, \nabla Q_L)] \right). \]

Set
\[ H^2_{Q_e}(\mathbb{R}^3; S_0) = \{ Q \in S_0 : Q - Q_e \in H^2(\mathbb{R}^3) \}, \]

where \( Q_e = s_+ (e \otimes e - \frac{1}{3} I) \in S_+ \) and \( e \in S^2 \) is a constant vector.

We call \((Q_L, v_L)\) to be a strong solution to the system \((1.11)-(1.13)\) in \( \mathbb{R}^3 \times (0, T) \) for some \( T > 0 \) if it satisfies the system a.e. in \( (x, t) \in \mathbb{R}^3 \times (0, T) \) and
\[ Q_L \in L^2(0, T; H^2_{Q_e}(\mathbb{R}^3)) \cap L^\infty(0, T; H^2_{Q_e}(\mathbb{R}^3)), \quad \partial_t Q_L \in L^2(0, T; H^1(\mathbb{R}^3)), \]
\[ v_L \in L^2(0, T; H^2(\mathbb{R}^3)) \cap L^\infty(0, T; H^1(\mathbb{R}^3)). \]

Then we have

**Theorem 1** (Local Existence). Let \((Q_{L,0}, v_{L,0}) \in H^2_{Q_e}(\mathbb{R}^3; S_0) \times H^1(\mathbb{R}^3; \mathbb{R}^3) \) be the initial values satisfying \( \text{div} v_{L,0} = 0 \) and \( \|Q_{L,0}\|_{L^\infty(\mathbb{R}^3)} \leq K \) for a constant \( K > 0 \). Then there is a unique strong solution \((Q_L, v_L)\) to the system \((1.11)-(1.13)\) in \( \mathbb{R}^3 \times [0, T) \) with initial data \((Q_{L,0}, v_{L,0})\) for some \( T > 0 \).

Although Theorem 1 might be known for some experts, see [28] for the case \( L_2 = L_3 = L_4 = 0 \). However, since there exists some new difficulty on \( f_E \) with \( L_4 \neq 0 \), we give a detailed proof in Section 5 for completeness.

Next, we will formulate the Beris-Edwards system for uniaxial Q-tensors. In their book [3], Beris-Edwards suggested the hydrodynamic theory to describe flows of liquid crystals for uniaxial Q-tensors \( Q \in S_+ \), but they could not write an explicit form of molecular field \( H(Q, \nabla Q) \) for \( Q \in S_+ \) with nonzero elastic constants \( L_2, L_3, L_4 \). Recently, the explicit form of the molecular field \( H(Q, \nabla Q) \) for \( Q \in S_+ \) with general elastic constants was given in [10], so we can apply the form to formulate the Beris-Edwards system for uniaxial Q-tensors. For any two matrix \( A, B \in S_0 \), we denote the standard product by \( \langle A, B \rangle := \sum_{i,j} A_{ij} B_{ij} \). Then, the molecular
Then there is a unique strong solution $u$.

In fact, multiplying $(1.16)$ with a uniaxial Q-tensors $L_{kl}$ Beris-Edwards system for nonzero elastic constants $L_1, \cdots, L_4$ is:

$$\begin{align*}
  H(Q, \nabla Q) &= \nabla_k \left( (Q + \frac{s_k^2}{3}) (Q + \frac{s_k^2}{3}, \partial_{p_k} f_E) \right) + (Q + \frac{s_k^2}{3}, \partial_{p_k} f_E)^T \\
  - 2s_k^{-1} \nabla_k \left( (Q + \frac{s_k^2}{3}) (Q + \frac{s_k^2}{3}, \partial_{p_k} f_E) \right) - \partial_{p_k} f_E \nabla_k Q - \nabla_k (\partial_{p_k} f_E)^T \\
  + 2s_k^{-1} \left[ (\partial_{p_k} f_E, \nabla_k Q) (Q + \frac{s_k^2}{3}) + (Q + \frac{s_k^2}{3}) \nabla_k Q \right] \\
  - \partial_{Q} f_E (Q + \frac{s_k^2}{3}) - (Q + \frac{s_k^2}{3}, \partial_{p_k} f_E)^T + 2s_k^{-1} \left( \partial_{Q} f_E, Q + \frac{s_k^2}{3} \right) (Q + \frac{s_k^2}{3}).
\end{align*}$$

(1.16)

Through the new molecular field (1.16) with a uniaxial Q-tensors $Q \in S_*$, the Beris-Edwards system for nonzero elastic constants $L_1, \cdots, L_4$ is:

$$\begin{align*}
  (\partial_t + v \cdot \nabla - \Delta)v + \nabla P &= \nabla \cdot \left( [Q, H] + \sigma(Q, \nabla Q) \right), \\
  \nabla \cdot v &= 0, \\
  (\partial_t + v \cdot \nabla)Q + \left[ Q, \Omega \right] &= H(Q, \nabla Q).
\end{align*}$$

(1.17)-(1.19)

Then we prove the existence of strong solutions to (1.17)-(1.19) in the following:

**Theorem 2.** Assume that $(Q_0, v_0) \in H^2_0(\mathbb{R}^3; S_*) \times H^1(\mathbb{R}^3; \mathbb{R}^3)$ and $\text{div} \ v_0 = 0$. Then there is a unique strong solution $(Q, v)$ to the system (1.17)-(1.19) in $\mathbb{R}^3 \times [0, T^*)$ with initial data $(Q_0, v_0)$. Moreover, there are two positive constants $\varepsilon_0$ and $R_0$ such that at a singular point $x_1$, the maximal existence time $T^*$ satisfies

$$\limsup_{t \to T^*} \int_{B_R(x_1)} |\nabla Q(x, t)|^3 + |v(x, t)|^3 \ dx \geq \varepsilon_0$$

for any $R > 0$ with $R \leq R_0$.

For the proof of Theorem 2, one of the key steps is to establish Proposition 3.1 and obtain that for a short time $T_1 > 0$, the strong solution to the system (1.11)-(1.13) with initial data $(Q_0, v_0)$ satisfies the uniform estimate:

$$\begin{align*}
  \sup_{0 \leq s \leq T_1} \left( \|\nabla Q_L(s)\|^2_{H^1(\mathbb{R}^3)} + \|v_L(s)\|^2_{H^1(\mathbb{R}^3)} + \frac{1}{L}\|Q_L(s) - \pi(Q_L(s))\|^2_{H^1(\mathbb{R}^3)} \right) \\
  + \|\nabla^2 Q_L\|^2_{L^2(0, T_1; H^1(\mathbb{R}^3))} + \|\partial_t Q_L\|^2_{L^2(0, T_1; H^1(\mathbb{R}^3))} \\
  + \|\nabla v_L\|^2_{L^2(0, T_1; H^1(\mathbb{R}^3))} + \frac{1}{L}\|Q_L - \pi(Q_L)\|^2_{H^1(\mathbb{R}^3)} \leq C.
\end{align*}$$

Here $\pi(Q_L)$ is the projection of $Q_L$ defined below in the proof of Theorem 3. The proof of Proposition 3.1 is sophisticated and it will also play a crucial role in the proof of Theorem 3 below. We will outline more details about it later.

**Remark 1.** It was pointed out in [3] that (1.17)-(1.19) can be reduced to the hydrodynamic flow of the Oseen-Frank energy, known as the Ericksen-Leslie system. In fact, multiplying $u_j$ to (1.19) and employing $|u|^2 = 1$, one can check that

$$\sigma(Q, \nabla Q) = -\nabla u^T \frac{\partial W(u, \nabla u)}{\partial (\nabla u)}; \quad \partial_{p_k} f_E(Q_L, \nabla Q_L) = s_{1}^{-1} u^k \frac{\partial W(u, \nabla u)}{\partial (\nabla u)}.$$
Lin and Liu [21] introduced the Ginzburg-Landau approximation for the Ericksen-Leslie system to solve the existence problem, but they could not show that the solutions of the Ginzburg-Landau approximate systems approach the solution of the Ericksen-Leslie system. In $\mathbb{R}^2$, Hong [14] and Hong-Xin [17] proved that the solutions of the Ginzburg-Landau approximate systems with unequal Frank constants approach the solution of the Ericksen-Leslie system in a short time and showed the global existence of weak solutions to the Ericksen-Leslie system in $\mathbb{R}^2$ by using the idea of Struwe [32] on the harmonic map flow. In $\mathbb{R}^3$, Hong, Li and Xin [15] showed the strong convergence of the Ginzburg-Landau approximate system with unequal Frank constants before the blow-up time of the Ericksen-Leslie system. Recently, we [11] improved the result in [15] by proving the smooth convergence of the Ginzburg-Landau approximate systems for a general Ericksen-Leslie system with Leslie tensors before the blow-up time.

By comparing with the convergence of Ginzburg-Landau models for superconductivity theory, Gartland [12] emphasised importance of the convergence on Landau-de Gennes solutions. In physics, both the Ericksen-Leslie theory and the Beris-Edwards theory should unify in modelling uniaxial state of nematic liquid crystals, so it is very interesting to give a mathematical proof to verify that the solutions of the Beris-Edwards system (1.11)-(1.13) can approach a solution of the Ericksen-Leslie system.

For each $L > 0$, let $(Q_L,v_L)$ be the unique strong solution to the system (1.11)-(1.13) in $\mathbb{R}^3 \times [0,T_L)$ with initial data $(Q_0,v_0)$ for the maximal existence time $T_L$. Let $(Q,v)$ be the strong solution to the system (1.17)-(1.20) in $\mathbb{R}^3 \times [0,T^*)$ with the same initial data $(Q_0,v_0)$ and the maximal existence time $T^*$ in Theorem 3. Then, for any $T \in (0,T^*)$, there exists a sufficiently small $L_T > 0$ such that $T \leq T_L$ for any $L \leq L_T$. Moreover, as $L \to 0$, we have

\[
(\nabla Q_L,v_L) \to (\nabla Q,v) \quad \text{in} \quad L^\infty(0,T;L^2(\mathbb{R}^3)) \cap L^2(0,T;H^1(\mathbb{R}^3))
\]

and

\[
(\nabla Q_L,v_L) \to (\nabla Q,v) \quad \text{in} \quad C^\infty(\tau,T;C^\infty_{\text{local}}(\mathbb{R}^3)) \quad \text{for any} \ \tau > 0.
\]

For the proof of Theorem 3 the main ideas are to establish uniform estimates on higher order derivatives of $(Q_L,v_L)$ in $L$. Using similar methods in [17,11], we can handle all terms involving $f_E(Q_L)$, but the main difficulty is to obtain the uniform estimate of the terms involving $\frac{1}{2}g_B(Q_L)$ when $L \to 0$. To handle those difficult terms, we use a concept of a projection near $S_\delta$, which was first introduced on Riemannian manifolds by Schoen and Uhlenbeck [31]. Denote

\[
S_\delta := \{Q \in S_0 : \text{dist}(Q;S_\delta) \leq \delta\}.
\]

Let $\pi : S_\delta \to S_\delta$ be the smooth projection map for a small $\delta > 0$ so that $\pi(Q)$ is the nearest point; i.e. $|Q - \pi(Q)| = \text{dist}(Q;S_\delta)$ for $Q \in S_\delta$. For each smooth $Q_L(x) \in S_\delta$, there is a rotation $R(Q_L(x)) \in SO(3)$ such that $R^T(Q_L(x))Q_L(x)R(Q_L(x))$ is diagonal. However, $R(Q_L(x))$ is not always smooth; i.e. there exists a measure zero set $\Sigma_\delta$ such that $R(Q_L(x))$ is differentiable in $U$ except for the singular set $\Sigma_\delta$. To study the convergence of solutions of the Landau-de Gennes system, Feng-Hong [10] proved a geometric identity $\nabla (R^T(Q)QR(Q))_{ij} = (R^T(Q)\nabla Q R(Q))_{ij}$ with $i = 1, 2, 3$ away from the singular set for handling the difficulty on $\frac{1}{2}g_B(Q_L)$. Due
to the singular set $\Sigma_L$ of the rotation $R(Q_L(x)) \in SO(3)$, the proof in \[10\] is much complicated. In this paper, we find a new approach to avoid the main difficulty arising from the singular set $\Sigma_L$ of the above rotation $R(Q_L(x))$ in \[10\]. We outline main steps of the new approach as follows:

The first key step is to establish some new estimates to overcome the difficulty arising from the term $f_B(Q_L)$. For each smooth $Q \in S_\delta$, $\pi(Q) \in S_\delta$ has a constant number of distinct eigenvalues, so there exists a smooth rotation $R_Q := R(\pi(Q)) \in SO(3)$ such that

$$R_Q^T \pi(Q) R_Q = \begin{pmatrix} -\frac{s_+}{s} & 0 & 0 \\ 0 & -\frac{s_+}{s} & 0 \\ 0 & 0 & \frac{2s}{s_+} \end{pmatrix}=: Q^r. \quad (1.23)$$

Since $\pi(Q)$ commutes with $Q$ for any $Q \in S_\delta$ (c.f. \[25\]), one can see

$$\dot{Q} = R_Q^T Q R_Q = \begin{pmatrix} \dot{Q}_{11} & \dot{Q}_{12} & 0 \\ \dot{Q}_{21} & \dot{Q}_{22} & 0 \\ 0 & 0 & \dot{Q}_{33} \end{pmatrix}. \quad (1.24)$$

For any $Q \in S_\delta$, we can derive an estimate

$$\frac{\lambda}{2} |\nabla(Q - \pi(Q))|^2 \leq \partial_{Q_{ij}Q_{kl}} f_B(\dot{Q}) \nabla_{x_\alpha} \dot{Q}_{ij} \nabla_{x_\beta} \dot{Q}_{kl} + C |\nabla Q|^2 |Q - \pi(Q)|^2 \quad (1.25)$$

with some constant $\lambda > 0$, which improves a result of diagonal matrices in \[10\].

The second key step is to establish the uniform estimate on $(\nabla^2 Q_L, \nabla v_L)$. To overcome the difficulty arising from the term $g_B(Q_L)$, we rotate the equation (1.13) by $R_Q \in SO(3)$ such that $g_B(\dot{Q}_L)$ has the same matrix form of $\dot{Q}$ in (1.24). For any $Q \in S_\delta$, we find an extension of the geometric identity of $\dot{Q}$ in (1.24) that

$$\langle \nabla g_B(\dot{Q}), \nabla R_Q^T \pi(Q) R_Q - R_Q^T \pi(Q) \nabla R_Q \rangle = 0. \quad (1.26)$$

Combining the above identity (1.25) with (1.26), we establish the uniform estimate in Lemmas \[25\] and then prove Theorem 3, which is similar to the idea in \[11\]. Finally, by an induction method, we establish the sophisticated uniform estimate of $(\nabla^k Q_L, \nabla^k v_L)$ in $L$ for any integer $k \geq 2$ to prove Theorem \[3\]. We would point out that our proof on high order uniform estimates is new and different from one used for Ginzburg-Landau approximations in \[11\].

Remark 2. When $L_4 = 0$, Wang-Zhang-Zhang \[33\] proved some related convergence of (1.11) with smooth initial values to the Ericksen-Leslie system in $\mathbb{R}^3$, but not to the uniaxial $Q$-tensor Beris-Edwards system (1.14)-(1.15). It seems that their method only works for smooth initial values. Recently, Xin-Zhang \[33\] proved that the weak convergence also holds in $\mathbb{R}^2$ for (1.11)-(1.13) with $L_2 = L_3 = L_4 = 0$.

The paper is organized as follows. In Section 2, we derive some a-priori estimates on the strong solution $(Q_L, v_L)$ of the system (1.11)-(1.13) in $\mathbb{R}^3 \times [0, T_L]$. In Section 3, we prove Theorem \[2\]. In Section 4, we prove Theorem \[3\]. In Section 5, we prove Theorem \[1\].
2. A-priori estimates

In this section, we will derive some a-priori estimates on the strong solution \((Q_L, v_L)\) of the system (1.11)-(1.13) in \(\mathbb{R}^3 \times [0, T_L]\).

2.1. Property on the density. In order to obtain a-priori energy estimates, we need to establish some key properties on the density. Under the condition (1.7), one can verify from a result in [18] that there are two uniform constants \(\alpha > 0\) and \(\Lambda > 0\) such that for any \(Q \in M^{3 \times 3}_{\text{sym}}\) and \(p \in M^{3 \times 3}_* \times \mathbb{R}^3\), \(f_E(Q, p)\) also satisfies

\[
\frac{\alpha}{2} |p|^2 \leq f_E(Q, p) \leq \Lambda (1 + |Q|^2) |p|^2, \quad |\partial_Q f_E(Q, p)| \leq \Lambda (1 + |Q|) |p|, \quad |\partial_{pp}^2 f_E(Q, p)| \leq \Lambda (1 + |Q|^2). \tag{2.1}
\]

Noting that \(f_E(Q, p)\) is quadratic in \(p\) and satisfies (2.1), one has (c.f. [16])

\[
\frac{\alpha}{2} |\xi|^2 \leq \partial_{p_{kl}p_{lm}}^2 f_E(Q, p) \xi_k^i \xi_l^j \xi_m^i \xi_n^j \leq \Lambda (1 + |Q|^2) |\xi|^2, \quad \forall \xi \in M^{3 \times 3} \times \mathbb{R}^3. \tag{2.2}
\]

Recall that

\[ S_\delta = \{ Q \in S_0 : \text{dist}(Q; S_s) \leq \delta \}. \tag{2.3} \]

We assume that \(\delta > 0\) is sufficiently small throughout this paper. Let \(\pi(Q)\) be a smooth projection from \(S_\delta\) to \(S_s\). Then \(f_B(Q)\) satisfies (c.f. [25], [10])

\[ \frac{\lambda}{2} |Q - \pi(Q)|^2 \leq f_B(Q) \leq C |Q - \pi(Q)|^2 \tag{2.4} \]

for some \(C > 0\). Since each smooth \(\pi(Q) \in S_s\) has a constant number of distinct eigenvalues, there exists a smooth matrix \(R_Q := R(\pi(Q)) \in SO(3)\) such that \(R_Q^T \pi(Q) R_Q\) is diagonal (c.f. [24]). Since \(S_s\) has only three elements of diagonal forms, we can assume without loss of generality that

\[ R_Q^T \pi(Q) R_Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & S_1 & 0 \\ 0 & 0 & S_3 \end{pmatrix} =: Q^+. \tag{2.5} \]

Since \(\pi(Q)\) commutes with \(Q\) (c.f. [25]), we have

\[ R_Q^T \pi(Q) R_Q Q^+ = Q^+ R_Q^T \pi(Q) R_Q. \tag{2.6} \]

Then for any \(Q \in S_\delta\), it follows from using (2.5)-(2.6) that

\[ \tilde{Q} = R_Q^T \pi(Q) R_Q = \begin{pmatrix} \tilde{Q}_{11} & \tilde{Q}_{12} & 0 \\ \tilde{Q}_{21} & \tilde{Q}_{22} & 0 \\ 0 & 0 & \tilde{Q}_{33} \end{pmatrix}. \tag{2.7} \]

**Lemma 2.1.** For any \(Q \in S_\delta\), let \(\tilde{Q}\) be defined in (2.7). Then the Hessian of the bulk density \(f_B(Q)\) satisfies

\[ \frac{\lambda}{2} |\xi|^2 \leq \partial_{\tilde{Q}_{kl}}^2 f_B(\tilde{Q}) \xi_k \xi_l \tag{2.8} \]

for all \(\xi \in S_0\) of the form

\[ \xi = \begin{pmatrix} \xi_{11} & \xi_{12} & 0 \\ \xi_{21} & \xi_{22} & 0 \\ 0 & 0 & \xi_{33} \end{pmatrix} \tag{2.9} \]

where \(\lambda = \min\{s_+ b, 3a\} > 0\).
In the case of \( i \neq j = \tilde{i} = \tilde{j} \) in (2.11), we apply the relation (2.10) to obtain

\[
\partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}) = \left( b \left( \frac{1}{3} s + \delta_i \delta_j - \delta_i \tilde{Q}_{ji} + \delta_j \tilde{Q}_{ij} \right) + c(\delta_i \delta_j) \right) \tilde{Q}_{ij}.
\]

For the remaining cases of either \( i \neq j \) or \( i \neq \tilde{i} \neq \tilde{j} \) in (2.11), we compute

\[
\partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}) = 2c \tilde{Q}_{ij}^+ \tilde{Q}_{ij}^+ = \frac{2s^2}{9} c = \frac{1}{3} a + \frac{s_+ b}{9},
\]

\[
\partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}) = 2c \tilde{Q}_{ij}^+ \tilde{Q}_{ij}^+ = - \left( \frac{2}{3} a + \frac{2s_+ b}{9} \right) = \partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}).
\]

For the remaining cases of either \( i \neq j \) or \( i \neq \tilde{i} \neq \tilde{j} \) in (2.11), we find

\[
\left( \sum_{i \neq j} \sum_{i,j} + \sum_{i \neq i} \sum_{i,j} \right) \partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}) \xi_{ij} = \sum_{i \neq j} b \left( \frac{1}{3} s + Q_{ii} - Q_{jj} \right) \xi_{ij}^2 = s_+ b (\xi_{12}^2 + \xi_{21}^2),
\]

where we employed (2.10) in the last step. Using the relations (2.12)-(2.16) with the fact that \( \text{tr}(\xi) = 0 \) and (2.9), we have

\[
\partial_{\tilde{Q}_{ij}} \partial_{\tilde{Q}_{ij}} f_B(\tilde{Q}) \xi_{ij} \xi_{kl} = \left( \frac{1}{3} a + \frac{10s_+ b}{9} \right) (\xi_{11}^2 + \xi_{22}^2) + \left( \frac{2}{3} a + \frac{2s_+ b}{9} \right) (\xi_{11} \xi_{22}) + 2s_+ b \xi_{12}^2
\]

\[
+ \left( \frac{4}{3} a - \frac{5s_+ b}{9} \right) \xi_{33}^2 - \left( \frac{4}{3} a + \frac{4s_+ b}{9} \right) \xi_{33} (\xi_{11} + \xi_{22})
\]

\[
= s_+ b (\xi_{11} + \xi_{22}) + \frac{8}{3} a - \frac{s_+ b}{9} \xi_{33}^2 + 2s_+ b \xi_{12}^2 + \left( \frac{1}{3} a + \frac{s_+ b}{9} \right) (\xi_{11} + \xi_{22})^2
\]

\[
= s_+ b (\xi_{11} + \xi_{22}^2 + \xi_{12}^2 + \xi_{21}^2 + 3a \xi_{33}^2 \geq \lambda |\xi|^2,
\]

where \( \lambda = \min\{s_+ b, 3a\} > 0 \).
Due to the continuity of second derivatives of $f_B(\hat{Q})$ and the fact that $|\hat{Q} - Q^+| = \text{dist}(Q; S_\delta) \leq \delta$ with sufficiently small $\delta$, the claim (2.8) follows from using (2.17).

**Corollary 1.** For any $Q \in S_\delta$, we have
\[
\frac{\lambda}{2}|\nabla(Q - \pi(Q))|^2 \leq \partial^2_{\bar{Q}_{ij}, \bar{Q}_{kl}} f_B(\bar{Q}) \nabla \bar{Q}_{ij} \nabla \bar{Q}_{kl} + C|Q - \pi(Q)|^2|\nabla Q|^2.
\] (2.18)

Moreover, for any $k \geq 2$, we have
\[
\frac{\lambda}{2}|\nabla^k(Q - \pi(Q))|^2 \leq \partial^2_{\bar{Q}_{ij}, \bar{Q}_{kl}} f_B(\bar{Q}) \nabla^k \bar{Q}_{ij} \nabla^k \bar{Q}_{kl} + C \sum_{\mu_1 \leq \mu_2 \cdots \mu_{k+1} = k} |\nabla^{\mu_1}(Q - \pi(Q))|^2 |\nabla^{\mu_2} Q|^2 \cdots |\nabla^{\mu_{k+1}} Q|^2.
\] (2.19)

**Proof.** Taking $\xi = \nabla \hat{Q}$ for $k \geq 1$ in Lemma 2.1, we have
\[
\frac{3\lambda}{4}|\nabla^k \hat{Q}|^2 \leq \partial^2_{\hat{Q}_{ij}, \hat{Q}_{kl}} f_B(\hat{Q}) \nabla^k \hat{Q}_{ij} \nabla^k \hat{Q}_{kl}.
\] (2.20)

We note
\[
|\nabla \hat{Q}|^2 = |\nabla(\hat{Q} - Q^+)|^2 = |\nabla(R_Q^T(Q - \pi(Q)) R_Q)|^2 \\
\geq \frac{2}{3}|\nabla(Q - \pi(Q))|^2 - C|Q - \pi(Q)|^2 |\nabla R_Q|^2.
\] (2.21)

Since $R_Q$ is smooth for $Q \in S_\delta$, the inequality (2.15) follows from (2.20)-(2.21). For any $k \geq 2$, we expand the left-hand side of (2.20) to show
\[
\frac{3\lambda}{4}|\nabla^k \hat{Q}|^2 = \frac{3\lambda}{4}|\nabla^k (R_Q^T(Q - \pi(Q)) R_Q)|^2 \\
\geq \frac{\lambda}{2}|\nabla^k(Q - \pi(Q))|^2 - C \sum_{\mu_1 + \mu_2 + \mu_3 = k \atop \mu_2 < k} |\nabla^{\mu_1} R_Q^T||\nabla^{\mu_2}(Q - \pi(Q))|^2 |\nabla^{\mu_3} R_Q|^2.
\] (2.22)

Since $R_Q$ is smooth for $Q \in S_\delta$, we find
\[
|\nabla^k R_Q| \leq C \left( \sum_{\mu_1 + \cdots + \mu_k = k} |\nabla^{\mu_1} Q| \cdots |\nabla^{\mu_k} Q| \right).
\] (2.23)

Then applying (2.23) and (2.22) to (2.20), we prove (2.19). \qed

**Lemma 2.2.** For any $Q \in S_\delta$, let $\tilde{Q}$ be defined in (2.7). Then
\[
\left\langle \nabla^k g_B(\tilde{Q}), \nabla R_Q^T \pi(Q) R_Q - R_Q^T \pi(Q) R_Q \right\rangle = 0.
\] (2.24)

**Proof.** For a fixed $Q_0 \in S_\delta$, there exists $R_0 = R(\pi(Q_0)) \in SO(3)$ such that $R_0^T \pi(Q_0) R_0 = Q^+$ is a diagonal matrix. Set $\bar{R}(Q) = R_Q^T R_Q$. Since $\bar{R}(Q_0) = I$, a direct calculation shows that
\[
\nabla \bar{R}_{ij}(Q_0) + \nabla \bar{R}_{ji}(Q_0) = 0, \quad \forall i, j = 1, 2, 3.
\] (2.25)
Then we obtain
\[
\begin{aligned}
&\left\langle \nabla^{k} g_{\delta}(\tilde{Q}), \nabla R_{Q}^{\delta} \pi(Q) R_{Q} - R_{Q}^{T} \pi(Q) \nabla R_{Q} \right\rangle|_{Q=Q_{0}} \\
&= \left\langle \nabla^{k} g_{\delta}(\tilde{Q}_{0}), \nabla \tilde{R}^{T}(Q_{0})(R_{Q}^{T} \pi(Q_{0}) R_{Q} - R_{Q}^{T} \pi(Q_{0}) R_{Q}) \nabla \tilde{R}(Q_{0}) \right\rangle \\
&= \sum_{i,j=1}^{3} (\nabla^{k} g_{\delta}(\tilde{Q}_{0}))_{ij} \left( \nabla \tilde{R}_{ij}(Q_{0}) Q_{jj}^{+} + Q_{ii}^{+} \nabla \tilde{R}_{ij}(Q_{0}) \right).
\end{aligned}
\]

Using (2.25) and the fact that \(Q_{11}^{+} = Q_{22}^{+} \), we have
\[
\sum_{i,j=1}^{2} (\nabla^{k} g_{\delta}(\tilde{Q}_{0}))_{ij} \left( \nabla \tilde{R}_{ij}(Q_{0}) Q_{jj}^{+} + Q_{ii}^{+} \nabla \tilde{R}_{ij}(Q_{0}) \right) = 0. \quad (2.26)
\]

It follows from (1.15) and (2.7) that \(\tilde{Q}_{13} = \tilde{Q}_{23} = 0 \) for each \(Q \in S_{3} \). Thus
\[
(g_{\delta}(\tilde{Q}))_{13} = a(\tilde{Q})_{13} + b \sum_{k=1}^{3} \tilde{Q}_{1k} \tilde{Q}_{k3} - c(\tilde{Q})_{13} \text{tr}(\tilde{Q}^2) = 0.
\]

Then \(\nabla^{k}(g_{\delta}(\tilde{Q}))_{13} = 0 \) for any \(Q \in S_{3} \). Similarly, \(\nabla^{k}(g_{\delta}(\tilde{Q}))_{23} = 0 \). For the case of \(i = 3 \), we apply (2.25) to obtain
\[
\sum_{j=1}^{3} (\nabla^{k} g_{\delta}(\tilde{Q}_{0}))_{3j} \left( \nabla \tilde{R}_{ij}(Q_{0}) Q_{jj}^{+} + Q_{ii}^{+} \nabla \tilde{R}_{ij}(Q_{0}) \right) \\
= 2(\nabla^{k} g_{\delta}(\tilde{Q}_{0}))_{33} \nabla \tilde{R}_{33}(Q_{0}) Q_{33}^{+} = 0. \quad (2.27)
\]

Similarly, we can prove it for \(j = 3 \). In view of (2.26) and (2.27), we prove the claim (2.24) for any \(Q = Q_{0} \in S_{3} \).

\[\square\]

2.2. Some a-priori estimates. For simplicity of notations, we denote \(f_{E}(Q, \nabla Q) \) by \(f_{E} \) and only write the subscript \(L \) in the statement of each lemma, but omit it in all proofs in this section.

**Lemma 2.3.** Let \(F \) be a \(3 \times 3 \) matrix. For any symmetric \(A, B \) matrices, we have
\[
\langle [A, F], B \rangle = \langle F, [A, B] \rangle = -\langle F^{T}, [A, B] \rangle. \quad (2.28)
\]

**Proof.** Note the following identity
\[
\langle [A, F], B \rangle = \langle (AF - FA), B \rangle = \text{tr} \left( (AF)^{T} B - (FA)^{T} B \right) \\
= \text{tr} \left( F^{T} A^{T} B - A^{T} (F^{T} B) \right) = \text{tr} \left( F^{T} A^{T} B - (F^{T} B) A^{T} \right) \\
= \langle F, [A^{T}, B] \rangle = \langle F, [A, B] \rangle.
\]

For the second identity in (2.28), we observe that
\[
\langle F, [A, B] \rangle = \langle F^{T}, [A, B]^{T} \rangle = -\langle F^{T}, (A^{T} B^{T} - B^{T} A^{T}) \rangle = -\langle F^{T}, [A, B] \rangle.
\]

Here we used the fact that \(A, B \) are symmetric in the last step. \[\square\]

Now, we show the following energy identity:
Lemma 2.4. Let \((Q_L, v_L)\) be a strong solution to the system (1.11)-(1.13) in \(\mathbb{R}^3 \times (0, T_L)\) with the initial condition \((Q_{L,0}, v_{L,0}) \in H^2_{Q,0}(\mathbb{R}^3; S_\ast) \times H^1(\mathbb{R}^3; \mathbb{R}^3)\) and \(\text{div} v_{L,0} = 0\). Then, for any \(s \in (0, T_L)\), we have

\[
\int_{\mathbb{R}^3} f_E(Q_L, \nabla Q_L) + \frac{1}{L} f_B(Q_L) + \frac{|v_L|^2}{2} (x, s) \, dx
\]

\[
+ \int_0^s \int_{\mathbb{R}^3} \left| \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L) \right|^2 dx dt + \int_0^s \int_{\mathbb{R}^3} |
abla v_L|^2 dx dt
\]

\[
= \int_{\mathbb{R}^3} f_E(Q_{L,0}, \nabla Q_{L,0}) + \frac{1}{L} f_B(Q_{L,0}) + \frac{|v_{L,0}|^2}{2} \, dx.
\] (2.29)

Proof. Taking \(L^2\) inner product of (1.11) with \(v\) and using integration by part yield

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |v|^2 \, dx + \int_{\mathbb{R}^3} |
abla v|^2 \, dx
\]

\[
= \int_{\mathbb{R}^3} \partial_{p_{ki}} f_E \nabla_i Q_{kl} \nabla_j v_i \, dx - \int_{\mathbb{R}^3} [Q, \mathcal{H}(Q, \nabla Q)]_{ij} \nabla_j v_i \, dx.
\] (2.30)

Next, multiplying (1.13) with \((\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q))\) gives

\[
- \int_{\mathbb{R}^3} \left< \partial_t Q, \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right> \, dx + \int_{\mathbb{R}^3} \left| \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right|^2 \, dx
\]

\[
= \int_{\mathbb{R}^3} \left< (v \cdot \nabla)Q + [Q, \Omega], \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right> \, dx.
\] (2.31)

In view of (1.14) and the relation that \(g_B(Q) = -\nabla Q f_B(Q) - \frac{1}{L} \text{tr}(Q^2) I\), we have

\[
- \int_{\mathbb{R}^3} \left< \partial_t Q, \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right> \, dx = \frac{d}{dt} \int_{\mathbb{R}^3} (f_E(Q, \nabla Q) + \frac{1}{L} f_B(Q)) \, dx.
\] (2.32)

Utilizing (1.14), (1.15) and integrating by parts, we have

\[
\int_{\mathbb{R}^3} \left< (v \cdot \nabla)Q, \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right> \, dx
\]

\[
= \int_{\mathbb{R}^3} \left< (v \cdot \nabla)Q, \partial_{j} f_E - \partial Q f_E \right> \, dx - \int_{\mathbb{R}^3} \left< (v \cdot \nabla)Q, \frac{1}{L} \partial_i f_E \right> \, dx
\]

\[
= - \int_{\mathbb{R}^3} \nabla_j v_i \nabla_i Q_{kl} \partial_{p_{ki}} f_E + v_i \left( \nabla^2 Q_{kl} \partial_{p_{ki}} f_E - \nabla i Q_{kl} \partial Q f_E - \frac{1}{L} \nabla_i f_B(Q) \right) \, dx
\]

\[
= - \int_{\mathbb{R}^3} \partial_{p_{ki}} f_E \nabla_i Q_{kl} \nabla_j v_i \, dx - \int_{\mathbb{R}^3} v_i \nabla_i f \, dx = - \int_{\mathbb{R}^3} \partial_{p_{ki}} f_E \nabla_i Q_{kl} \nabla_j v_i \, dx, \quad (2.33)
\]

where we have used \(\text{tr} Q = 0\) in the second equality.

Choosing \(A = Q, B = \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q), F = \nabla v\) in Lemma 2.3 and using the fact that \([Q, g_B] = 0\), we have

\[
\left< [Q, \Omega], \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right> = \nabla_j v_i [Q, \mathcal{H}(Q, \nabla Q)]_{ij}.
\] (2.34)
Integrating (2.31) in $x$ and substituting (2.32) - (2.33) into (2.31) give
\[\frac{d}{dt} \int_{\mathbb{R}^3} f(Q, \nabla Q) \, dx + \int_{\mathbb{R}^3} |\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q)|^2 \, dx = \int_{\mathbb{R}^3} \nabla v_k [Q, \nabla Q]_{kl} \, dx - \int_{\mathbb{R}^3} \partial_{p_{kl}} f_E Q_{kl} \nabla_j v_i \, dx. \tag{2.35}\]

Therefore, the energy identity (2.29) follows from taking the sum of (2.30) and (2.35) and integrating over the time interval $[0, s]$. □

We rotate the equation (1.13) by $R_{QL} = R(\pi(Q_L))$; i.e.
\[R_{QL}^T (\partial_t Q_L + (v_L \cdot \nabla Q_L) + [Q_L, \Omega_L]) R_{QL} = R_{QL}^T \mathcal{H}(Q_L, \nabla Q_L) R_{QL} + \frac{1}{L} g_B(Q_L), \tag{2.36}\]
where we use the fact that $R_{QL}^T g_B(Q_L) R_{QL} = g_B(Q_L)$.

The strong solutions also admit the following local energy inequality:

**Lemma 2.5.** Let $(Q_L, v_L)$ be a strong solution to the system (1.11) - (1.13) in $\mathbb{R}^3 \times (0, T_L)$. Assume that $Q \in S_3$ for sufficiently small $\delta$ on $\mathbb{R}^3 \times (0, T_L)$. Then, for any $\phi \in C_0^\infty(\mathbb{R}^3)$ and $s \in (0, T_L)$, we have
\[\int_{\mathbb{R}^3} \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) (x, s) \phi^2 \, dx \\
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla Q_L|^2 + |v_L|^2 + |\partial_t Q_L|^2 + \frac{|\nabla (Q_L - \pi(Q_L))|^2}{L} \right) \phi^2 \, dx \, dt \\
\leq C \int_{\mathbb{R}^3} \left( |\nabla Q_{L,0}|^2 + |v_{L,0}|^2 + \frac{|Q_{L,0} - \pi(Q_{L,0})|^2}{L} \right) \phi^2 \, dx \\
+ C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_L|^2 \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) \phi^2 \, dx \, dt \\
+ C \int_0^s \int_{\mathbb{R}^3} |P_L - c_1^2(t)||v_L||\nabla \phi||\phi| + (|\nabla Q_L|^2 + |v_L|^2)|\nabla \phi|^2 \, dx \, dt. \tag{2.37}\]

**Proof.** Differentiating (2.36) and multiplying by $R_{QL}^T \nabla Q R_{QL} \phi^2$ yield
\[\int_{\mathbb{R}^3} \langle \nabla (R_{QL}^T \partial_t Q + v_L \cdot \nabla Q + [Q_L, \Omega_L]) R_{QL}, R_{QL}^T \nabla Q R_{QL} \rangle \phi^2 \, dx \\
= \int_{\mathbb{R}^3} \langle \nabla (R_{QL}^T \mathcal{H}(Q, \nabla Q) R_{QL}) + \frac{1}{L} \nabla g_B(Q), R_{QL}^T \nabla Q R_{QL} \rangle \phi^2 \, dx. \tag{2.38}\]

We observe that
\[\int_{\mathbb{R}^3} \langle \nabla (R_{QL}^T \mathcal{H}(Q, \nabla Q) R_{QL}), R_{QL}^T \nabla Q R_{QL} \rangle \phi^2 \, dx \\
\leq \int_{\mathbb{R}^3} \langle \nabla \mathcal{H}(Q, \nabla Q), \nabla \phi \rangle \phi^2 \, dx + C \int_{\mathbb{R}^3} |\nabla Q||\nabla \mathcal{H}(Q, \nabla Q)||\phi|^2 \, dx \\
\leq \int_{\mathbb{R}^3} \nabla \beta \nabla \nabla_k (\partial_k f_{ij} E) \nabla \beta Q_{ij} \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} (|\mathcal{H}(Q, \nabla Q)||\nabla Q|^2 + |\partial Q f_E(Q, \nabla Q)||\nabla^2 Q||\phi|^2 \, dx. \tag{2.39}\]
Noting the condition (2.2) on $f_E$ and integrating by parts, we have
\[
\int_{\mathbb{R}^3} \nabla_\beta \nabla_k (\partial_{P_j} f_E) \nabla_\beta Q_{ij} \phi^2 \, dx
\]
\[
= \int_{\mathbb{R}^3} \nabla_k \left( \partial_{P_j}^{2} f_E \nabla_\beta Q_{mn} + \partial_{P_j}^{2} Q_{mn} f_E \nabla_\beta Q_{mn} \right) \nabla_\beta Q_{ij} \phi^2 \, dx
\]
\[
+ \int_{\mathbb{R}^3} \nabla_k (\partial_{P_j} f_E) \nabla_\beta Q_{ij} \nabla_\beta \phi^2 \, dx
\]
\[
\leq \int_{\mathbb{R}^3} \partial_{P_j}^{2} f_E \nabla_\beta Q_{mn} \nabla_\beta Q_{ij} \phi^2 \, dx + C \int_{\mathbb{R}^3} |\partial_{P_j}^{2} f_E| |\nabla_\beta Q||\nabla_\beta \phi| \phi \, dx
\]
\[
+ C \int_{\mathbb{R}^3} \left( |\nabla (\partial_{P_j} f_E)| |\nabla_\beta Q| + |\partial_{P_j}^{2} f_E| |\nabla_\beta Q| \right) |\nabla_\beta Q| \phi^2 + |\nabla (\partial_{P_j} f_E)| |\nabla_\beta Q| |\nabla_\beta \phi| \phi \, dx
\]
\[
\leq - \int_{\mathbb{R}^3} \frac{3\alpha}{8} |\nabla_\beta Q|^2 \phi^2 + C \left( |\nabla_\beta Q|^4 \phi^2 + |\nabla_\beta Q|^2 |\nabla_\beta \phi|^2 \right) \, dx. \tag{2.40}
\]

Here we used that $|\nabla (\partial_{P_j} f_E)| \leq C (|\nabla_\beta Q| + |\nabla_\beta Q|^2)$ due to $Q \in S_6$.

In view of Corollary 11, Lemma 2.2 with $k = 1$ and $g_B(\tilde{Q}) = -\nabla_\beta f_B(\tilde{Q}) - \frac{b}{3} \text{tr}(\tilde{Q})I$, we obtain
\[
\int_{\mathbb{R}^3} \left\langle \frac{1}{L} \nabla g_B(\tilde{Q}), R_Q^T \nabla_\beta QR_Q \right\rangle \phi^2 \, dx
\]
\[
= \frac{1}{L} \int_{\mathbb{R}^3} \left\langle \nabla g_B(\tilde{Q}), \nabla_\beta \tilde{Q} - \nabla R_Q^T QR_Q - R_Q^T \nabla R_Q \right\rangle \phi^2 \, dx
\]
\[
= \frac{1}{L} \int_{\mathbb{R}^3} \left\langle -\nabla (\partial_{Q_j} f_B(\tilde{Q})), \nabla_\beta \tilde{Q} \right\rangle \phi^2 \, dx
\]
\[
+ \frac{1}{L} \int_{\mathbb{R}^3} \left\langle \nabla g_B(\tilde{Q}), -\nabla R_Q^T QR_Q - R_Q^T \nabla R_Q \right\rangle \phi^2 \, dx
\]
\[
= - \frac{1}{L} \int_{\mathbb{R}^3} \partial_{Q_j}^{2} f_B(\tilde{Q}) \nabla_\beta Q_{ij} \nabla_\beta \phi^2 \, dx
\]
\[
+ \frac{1}{L} \int_{\mathbb{R}^3} \left\langle \nabla g_B(\tilde{Q}), -\nabla R_Q^T (\pi(Q) - Q) R_Q + R_Q^T (\pi(Q) - Q) \nabla R_Q \right\rangle \phi^2 \, dx
\]
\[
\leq - \frac{\lambda}{4} \int_{\mathbb{R}^3} |\nabla_\beta (Q - \pi(Q))|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} |\nabla_\beta Q|^2 \frac{|Q - \pi(Q)|^2}{L \phi^2 \, dx}, \tag{2.41}
\]

where we also used that
\[
|\nabla g_B(\tilde{Q})| \leq C |\nabla_\beta (\tilde{Q} - Q^+)| \leq C |\nabla_\beta (Q - \pi(Q))| + C |Q - \pi(Q)||\nabla_\beta Q|.
\]

Combining (2.40), (2.39) with (2.41) yields
\[
\int_{\mathbb{R}^3} \left\langle \nabla (R_Q^T (H(Q, \nabla_\beta Q) R_Q) + \frac{1}{L} \nabla g_B(\tilde{Q}), R_Q^T \nabla_\beta QR_Q \right\rangle \phi^2 \, dx
\]
\[
\leq - \int_{\mathbb{R}^3} \left( \frac{\alpha}{4} \sqrt{2Q} + \frac{\lambda}{4} \frac{|\nabla_\beta (Q - \pi(Q))|^2}{L} \right) \phi^2 \, dx
\]
\[
+ C \int_{\mathbb{R}^3} |\nabla_\beta Q|^2 |\nabla_\beta \phi|^2 + |\nabla_\beta Q|^2 \left( |\nabla_\beta Q|^2 + |\nabla_\beta \phi|^2 + \frac{|Q - \pi(Q)|^2}{L} \right) \phi^2 \, dx. \tag{2.42}
\]
Integrating by parts, we estimate the left-hand side of (2.38) to obtain
\[
\begin{align*}
\int_{\mathbb{R}^3} \langle \nabla_\beta (R_Q^2 (\partial_t Q + v \cdot \nabla Q + [Q, \Omega]) R_Q) , R_Q^2 \nabla_\beta Q R_Q \rangle \phi^2 \, dx \\
\geq \int_{\mathbb{R}^3} \langle \nabla_\beta \partial_t Q + (v \cdot \nabla) Q, \nabla_\beta Q \rangle \phi^2 \, dx - \int_{\mathbb{R}^3} \langle [Q, \Omega], \nabla_\beta (\nabla_\beta Q \phi^2) \rangle \, dx \\
- C \int_{\mathbb{R}^3} |\nabla Q|^2 (|\partial_t Q|^2 + |v|^2 |\nabla Q|^2 + |\nabla v|^2) \phi^2 \, dx \\
\geq \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla Q|^2 \phi^2 \, dx - \int_{\mathbb{R}^3} \left( \frac{\alpha}{4} |\nabla^2 Q|^2 + \frac{1}{4} |\partial_t Q|^2 + C |\nabla Q|^2 \right) \phi^2 \, dx \\
- C \int_{\mathbb{R}^3} |\nabla Q|^2 |v|^2 \phi^2 + |\nabla Q|^2 |\nabla \phi|^2 \, dx.
\end{align*}
\]  
(2.43)

Adding (2.32) to (2.33), we have
\[
\begin{align*}
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla Q|^2 \phi^2 \, dx + \int_{\mathbb{R}^3} \left( \frac{\alpha}{4} |\nabla^2 Q|^2 + \frac{1}{2} \frac{|\nabla(Q - \pi(Q))|^2}{L} \right) \phi^2 \, dx \\
\leq \int_{\mathbb{R}^3} \frac{1}{2} |\partial_t Q|^2 + C |\nabla v|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} |\nabla Q|^2 |\nabla \phi|^2 \, dx \\
+ \int_{\mathbb{R}^3} |\nabla Q|^2 \left( |\nabla Q|^2 + |v|^2 + \frac{|Q - \pi(Q)|^2}{L} \right) \phi^2 \, dx.
\end{align*}
\]  
(2.44)

Multiplying (1.13) by $\partial_t Q \phi^2$ and using (2.32) in Lemma 2.4 yield
\[
\begin{align*}
\frac{d}{dt} \int_{\mathbb{R}^3} (f_e(Q, \nabla Q) + \frac{1}{L} f_B(Q)) \phi^2 \, dx + \int_{\mathbb{R}^3} |\partial_t Q|^2 \phi^2 \, dx \\
= -2 \int_{\mathbb{R}^3} \partial_t Q_{ij} \partial_k \phi \, dx - \int_{\mathbb{R}^3} \langle (v \cdot \nabla) Q + [Q, \Omega], \partial_t Q \rangle \phi^2 \, dx \\
\leq \int_{\mathbb{R}^3} \frac{1}{4} |\partial_t Q|^2 + C |\nabla v|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} |\nabla Q|^2 |\nabla \phi|^2 + |\nabla Q|^2 |\nabla \phi|^2 \, dx.
\end{align*}
\]  
(2.45)

Adding (2.44) to (2.45), integrating in $t$ and using (2.41), we see
\[
\begin{align*}
\int_{\mathbb{R}^3} \left( |\nabla Q|^2 + \frac{|Q - \pi(Q)|^2}{L} \right) (x, s) \phi^2 \, dx \\
+ \int_{0}^{s} \int_{\mathbb{R}^3} \left( |\nabla^2 Q|^2 + |\partial_t Q|^2 + \frac{|\nabla(Q - \pi(Q))|^2}{L} \right) \phi^2 \, dx \, dt \\
\leq C \int_{\mathbb{R}^3} \left( |\nabla Q_0|^2 + \frac{|Q_0 - \pi(Q_0)|^2}{L} \right) \phi^2 \, dx \\
+ C \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla v|^2 \phi^2 \, dx \, dt + C \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla Q|^2 |\nabla \phi|^2 \, dx \, dt \\
+ \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla Q|^2 \left( |\nabla Q|^2 + |v|^2 + \frac{|Q - \pi(Q)|^2}{L} \right) \phi^2 \, dx \, dt.
\end{align*}
\]  
(2.46)

Estimating the term $\nabla v$ on the right-hand side of (2.44), we multiply (1.11) by $v \phi^2$ and (1.13) by $\left( H(Q, \nabla Q) + \frac{1}{2} g_B(Q) \right) \phi^2$. Then it follows from using the same
Lemma 2.6. Let $\phi$ be a strong solution to the system (2.30)-(2.35) that
\[
\frac{d}{dt} \int_{\mathbb{R}^3} \left( \frac{1}{2} |v|^2 + f_E(Q, \nabla Q) + \frac{1}{L} f_B(Q) \right) \phi^2 \, dx
+ \int_{\mathbb{R}^3} \left( |H(Q, \nabla Q) + \frac{1}{L} g_B(Q)|^2 + |\nabla v|^2 \right) \phi^2 \, dx
= \int_{\mathbb{R}^3} (|v|^2 + 2 \langle P - c^*(t) \rangle) v \cdot \nabla \phi - 2 \nabla_k v_i \nabla_k \phi \, dx
\]
where
\[
= \int_{\mathbb{R}^3} \left( |v|^2 + 2 \langle P - c^*(t) \rangle \right) v \cdot \nabla \phi - 2 \nabla_k v_i \nabla_k \phi \, dx
\]
Applying (2.48) to (2.46), we prove (2.37).

Integrating (2.47) in $t$, employing (2.46) and choosing sufficiently small $\eta$, we obtain
\[
\int_{\mathbb{R}^3} |v(x, s)|^2 \phi^2 \, dx + \int_0^s \int_{\mathbb{R}^3} |\nabla v|^2 \phi^2 \, dx dt
\leq C \int_{\mathbb{R}^3} (|\nabla Q_0|^2 + |v_0|^2 + |Q_0 - \pi(Q_0)|^2) \phi^2 \, dx + C \int_0^s \int_{\mathbb{R}^3} (|\nabla Q|^4 + |v|^4) \phi^2 \, dx
dx + C \int_0^s \int_{\mathbb{R}^3} (|\nabla Q|^2 + |v|^2) |\nabla \phi|^2 \, dx + C \int_0^s \int_{\mathbb{R}^3} |P_L - c^*_L(t)||v_L||\nabla \phi||\phi| \, dx.
\]
(2.48)

Applying (2.48) to (2.46), we prove (2.37).

Through Corollary 1 and the equation (2.36), we obtain second order estimates of $(\nabla Q_L, v_L)$ in the following:

Lemma 2.6. Let $(Q_L, v_L)$ be a strong solution to the system (1.11)-(1.13) in $\mathbb{R}^3 \times (0, T_L)$. Assume that $Q \in S_\delta$ for sufficiently small $\delta$ on $\mathbb{R}^3 \times (0, T_L)$. Then for any $\phi \in C_0^\infty(\mathbb{R}^3)$ and $s \in (0, T_L)$, we have the following local estimate
\[
\int_{\mathbb{R}^3} \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \right) (x, s) \phi^2 \, dx
+ \int_0^s \int_{\mathbb{R}^3} \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\nabla Q_L|^2 + \frac{|\nabla^2(Q_L - \pi(Q_L))|^2}{L} \right) \phi^2 \, dx dt
\leq C \int_{\mathbb{R}^3} \left( |\nabla^2 Q_{L,0}|^2 + |\nabla v_{L,0}|^2 + \frac{|\nabla(Q_{L,0} - \pi(Q_{L,0}))|^2}{L} \right) \phi^2 \, dx
+ C \int_{\mathbb{R}^3} \frac{|Q_{L,0} - \pi(Q_{L,0})|^2}{L} |\nabla Q_{L,0}|^2 \phi^2 + \left( \frac{|Q - \pi(Q)|^2}{L} |\nabla Q|^2 \right) (x, s) \phi^2 \, dx
+ C \int_0^s \int_{\mathbb{R}^3} e(Q_L, v_L) \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\partial_t Q_L|^2 \right) \phi^2 \, dx dt
+ C \int_0^s \int_{\mathbb{R}^3} e(Q_L, v_L) \left( \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} + e^2(Q_L, v_L) \right) \phi^2 \, dx dt.
\]
Here we used that
\[ e(Q_L, v_L) := |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L}. \]

Proof. Differentiating (2.30) with respect to \( x_\beta \) and \( x_\gamma \), we multiply by \( \nabla_\beta (R^T_Q \nabla_\gamma Q R_Q) \phi^2 \) to obtain
\[
\int_{\mathbb{R}^3} \left( R^T_Q (\partial_\nu Q + v \cdot \nabla Q + [Q, \Omega]) R_Q, \nabla_\beta (R^T_Q \nabla_\gamma Q R_Q) \right) \phi^2 \, dx
\]
\[
= \int_{\mathbb{R}^3} \nabla^2_{\beta\gamma} \left( R^T_Q \mathcal{H}(Q, \nabla Q) R_Q + \frac{1}{T} g_B(Q) \right), \nabla_\beta (R^T_Q \nabla_\gamma Q R_Q) \right) \phi^2 \, dx. \tag{2.50}
\]

Integrating by parts twice and using (2.22), we estimate
\[
\int_{\mathbb{R}^3} \nabla^2_{\beta\gamma} \nabla_k \left( \partial^2_{\gamma\gamma} f_E \right), \nabla^2_{\beta\gamma} Q \right) \phi^2 \, dx
\]
\[
= - \int_{\mathbb{R}^3} \nabla_{\gamma} \left( \partial^2_{\gamma\gamma} f_E \nabla^2_{\beta\gamma} Q \right), \nabla_k \left( \nabla^2_{\beta\gamma} Q \right) \phi^2 \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} \nabla^2_{\beta\gamma} Q, \nabla^2_{\beta\gamma} Q \right) \phi^2 \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} \frac{3a}{8} |\nabla^2 Q|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} |\nabla Q|^2 (|\nabla^2 Q|^2 + |\nabla Q|^4) \phi^2 + |\nabla^2 Q|^2 |\nabla \phi|^2 \, dx. \tag{2.51}
\]

Then using (2.51) and integrating by parts, we find
\[
\int_{\mathbb{R}^3} \left( \nabla^2_{\beta\gamma} (R^T_Q \mathcal{H}(Q, \nabla Q) R_Q), \nabla_\beta (R^T_Q \nabla_\gamma Q R_Q) \right) \phi^2 \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} \nabla^2_{\beta\gamma} \left( \nabla_k \partial_{\gamma\gamma} f_E - \partial_{\gamma\gamma} f_E \right), \nabla^2_{\beta\gamma} Q \right) \phi^2 \, dx
\]
\[
+ C \int_{\mathbb{R}^3} \nabla^2_{\beta\gamma} \left( \nabla^2 \mathcal{H}(Q, \nabla Q), \nabla^2 Q \right) \phi^2 \, dx
\]
\[
\leq C \int_{\mathbb{R}^3} \frac{a}{4} |\nabla^3 Q|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla^2 Q|^2 + |\nabla Q|^4) (|\nabla^2 Q|^2 \phi^2 + |\nabla \phi|^2) \, dx. \tag{2.52}
\]

Here we used that \( |\nabla^2 (R^T_Q \nabla Q R_Q)| + |\nabla \mathcal{H}| \leq C (|\nabla^3 Q| + |\nabla^2 Q||\nabla Q| + |\nabla Q|^3) \).
Applying Corollary [1], Lemma 2.2 we obtain

\[
\begin{aligned}
&\int_{\mathbb{R}^3} \left\langle \nabla_{\beta\gamma}^2 \frac{g_B(\tilde{Q})}{L}, \nabla_\beta (R_Q^T \nabla_\gamma Q R_Q) \right\rangle \phi^2 \, dx \\
= &\ - \int_{\mathbb{R}^3} \nabla_\beta \left( \phi^2 \nabla_{\beta\gamma}^2 \left( \frac{g_B(\tilde{Q})}{L} \right)_{ij} \right) \nabla_\gamma \tilde{Q}_{ij} \, dx \\
&\ + \int_{\mathbb{R}^3} \nabla_\beta \left( \phi^2 \nabla_{\beta\gamma}^2 \left( \frac{g_B(\tilde{Q})}{L} \right)_{ij} \right) \left( \nabla_\gamma R_Q^T (Q - \pi(Q)) R_Q + R_Q^T (Q - \pi(Q)) \nabla_\gamma R_Q \right)_{ij} \, dx \\
\leq &\ - \frac{1}{L} \int_{\mathbb{R}^3} \partial^2_{Q_{ij}Q_{kl}} f_B(\tilde{Q}) \nabla_{\beta\gamma}^2 \tilde{Q}_{ij} \nabla_{\beta\gamma}^2 \tilde{Q}_{kl} \phi^2 \, dx \\
&\ + C \int_{\mathbb{R}^3} \frac{\partial^2 f_B(\tilde{Q}) |\nabla Q|}{L^2} \frac{\nabla^2 \tilde{Q}_I}{L^2} \phi^2 \, dx \\
&\ + C \int_{\mathbb{R}^3} \frac{\nabla^2 (g_B(\tilde{Q}))}{L} \left( (|\nabla^2 Q| + |\nabla Q|^2) |Q - \pi(Q)| + |\nabla Q| |\nabla (Q - \pi(Q))| \right) \phi^2 \, dx \\
\leq &\ - \frac{\lambda}{4} \int_{\mathbb{R}^3} \frac{\nabla^2 (Q - \pi(Q))^2}{L} \phi^2 \, dx + C \int_{\mathbb{R}^3} \frac{|\nabla (Q - \pi(Q))|^2}{L} |\nabla Q|^2 \phi^2 \, dx \\
&\ + C \int_{\mathbb{R}^3} \frac{|Q - \pi(Q)|^2}{L} (|\nabla^2 Q|^2 + |\nabla Q|^4) \phi^2 \, dx, \\
\end{aligned}
\]  

(2.53)

where we used that

\[|\nabla^2 (g_B(\tilde{Q}))| \leq C (|\nabla^2 (Q - \pi(Q))| + |\nabla Q| |\nabla (Q - \pi(Q))| + (|\nabla^2 Q| + |\nabla Q|^2) |Q - \pi(Q)|.\]

We compute the left-hand side of (2.50) to get

\[
\begin{aligned}
&\int_{\mathbb{R}^3} \left\langle \nabla_{\beta\gamma}^2 \left( R_Q^T \partial_t Q + v \cdot \nabla Q + [Q, \Omega] R_Q \right), \nabla_\beta (R_Q^T \nabla_\gamma Q R_Q) \right\rangle \phi^2 \, dx \\
\geq &\ \int_{\mathbb{R}^3} \left\langle \nabla_{\beta\gamma}^2 \partial_t Q, \nabla_{\beta\gamma}^2 Q \right\rangle \phi^2 + \left\langle \nabla_\beta \partial_t Q, \nabla_{\beta\gamma}^2 \nabla_\gamma (\phi^2) \right\rangle \, dx \\
&\ - C \int_{\mathbb{R}^3} |\nabla \partial_t Q| (|\nabla^2 Q| |\nabla R_Q| + |\nabla Q| |\nabla^2 R_Q| + |\nabla Q| |\nabla R_Q|^2) \phi^2 \, dx \\
&\ - C \int_{\mathbb{R}^3} \left( |\partial_t Q| |\nabla Q| + |v| (|\nabla^2 Q| + |\nabla Q|^2) \right) |\nabla^2 (R_Q^T \nabla Q R_Q)| \phi^2 \, dx \\
&\ - C \int_{\mathbb{R}^3} \left( |\nabla v| |\nabla Q| + |\nabla^2 v| \right) |\nabla^2 (R_Q^T \nabla Q R_Q)| \phi^2 \, dx \\
\geq &\ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla^2 Q|^2 \phi^2 \, dx - \int_{\mathbb{R}^3} \left( \frac{1}{4} |\nabla \partial_t Q|^2 + \frac{\alpha}{8} |\nabla^3 Q|^2 + C |\nabla^2 v|^2 \right) \phi^2 \, dx \\
&\ - C \int_{\mathbb{R}^3} (|\nabla^2 Q|^2 + |\nabla v|^2 + |\partial_t Q|^2 + |\nabla Q|^4) (|\nabla Q|^2 + |v|^2) \phi^2 + |\nabla^2 Q|^2 |\nabla \phi|^2 \, dx. \\
\end{aligned}
\]  

(2.54)
In view of (2.50)-(2.54), we have
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla^2 Q|^2 \phi^2 \, dx + \int_{\mathbb{R}^3} \left( \frac{\alpha}{8} |\nabla^3 Q|^2 + \frac{\lambda}{4} \frac{|\nabla^2 (Q - \pi(Q))|^2}{L} \right) \phi^2 \, dx \\
\leq \int_{\mathbb{R}^3} \left( \frac{1}{4} |\nabla \partial_t Q|^2 + C |\nabla^2 v|^2 \right) \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} e(Q, v) \left( |\nabla^2 Q|^2 + |\nabla v|^2 + |\partial_t Q|^2 + |\nabla Q|^4 + \frac{|\nabla (Q - \pi(Q))|^2}{L} \right) \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} \left( c^2(Q, v) + |\nabla^2 Q|^2 \right) |\nabla \phi|^2 \, dx. \tag{2.55}
\]

Estimating the term on \( \nabla \partial_t Q \) in (2.55), we differentiate (2.36) in \( x_{\beta} \) and multiply it by \( \nabla_\beta (R^T_{\alpha Q_\beta} \partial_t Q R Q) \phi^2 \) to obtain
\[
\int_0^a \int_{\mathbb{R}^3} \left( \nabla \beta (R^T_{\alpha Q_\beta} \partial_t Q + v \cdot \nabla Q + [Q, \Omega]) R Q \right), \nabla \beta (R^T_{\alpha Q_\beta} \partial_t Q R Q) \right) \phi^2 \, dx \, dt \\
= \int_0^a \int_{\mathbb{R}^3} \nabla \beta \left( (R^T_{\alpha Q_\beta} \mathcal{H}(Q, \nabla Q) R Q + \frac{1}{L} \mathcal{G} Q) \right), \nabla \beta (R^T_{\alpha Q_\beta} \partial_t Q R Q) \right) \phi^2 \, dx \, dt. \tag{2.56}
\]

Using (2.2), we derive
\[
\int_0^a \int_{\mathbb{R}^3} \langle \nabla \beta (R^T_{\alpha Q_\beta} \mathcal{H}(Q, \nabla Q) R Q), \nabla \beta (R^T_{\alpha Q_\beta} \partial_t Q R Q) \rangle \phi^2 \, dx \, dt \\
\leq \int_0^a \int_{\mathbb{R}^3} \langle \nabla \beta \left( \partial_{p_{ij},p_{km}} f_E - \partial_t Q f_E \right), \nabla \beta \partial_t Q \rangle \phi^2 \, dx \, dt \\
+ \int_0^a \int_{\mathbb{R}^3} \left( |\nabla \mathcal{H}||\nabla R_Q||\partial_t Q| + |\nabla R_Q||\mathcal{H}||\nabla (R^T_{\alpha Q_\beta} \partial_t Q R Q)| \right) \phi^2 \, dx \, dt \\
\leq - \int_0^a \int_{\mathbb{R}^3} \frac{1}{2} \partial_t \left( \partial_{p_{ij},p_{km}} f_E \nabla^2_{\beta l} Q_{mn} \nabla^2_{\beta k} Q_{ij} \phi^2 \, dx \, dt \\
+ \int_0^a \int_{\mathbb{R}^3} \frac{1}{2} \partial_t \partial_{p_{ij},p_{km}} f_E \nabla^2_{\beta l} Q_{mn} \nabla^2_{\beta k} Q_{ij} \phi^2 \, dx \, dt + C |\partial^2_{pp} f_E ||\nabla^2 Q||\nabla \partial_t Q||\nabla \phi|||\phi| \, dx \, dt \\
+ C \int_0^a \int_{\mathbb{R}^3} \left( |\nabla \mathcal{H}(Q, \nabla Q)||\nabla Q||\partial_t Q| \right) \phi^2 \, dx \, dt \\
+ C \int_0^a \int_{\mathbb{R}^3} \left( |\nabla \mathcal{H}(Q, \nabla Q)||\nabla Q||\partial_t Q| \right) \phi^2 \, dx \, dt \\
\leq \int_{\mathbb{R}^3} (C|\nabla^2 Q|^2 - \frac{\alpha}{4} |\nabla^2 Q(x, v)|^2) \phi^2 \, dx + \int_0^a \int_{\mathbb{R}^3} \left( \frac{\alpha}{16} |\nabla^3 Q|^2 + \frac{1}{8} |\nabla \partial_t Q|^2 \right) \phi^2 \, dx \, dt \\
+ C \int_0^a \int_{\mathbb{R}^3} |\nabla Q|^2 (|\nabla^2 Q|^2 + |\partial_t Q|^2 + |\nabla Q|^4) \phi^2 + (|\nabla^2 Q|^2 + |\nabla Q|^4) |\nabla \phi|^2 \, dx \, dt. \tag{2.57}
\]

By integrating by parts and using the fact that
\[
|\nabla (R^T_{\alpha Q_\beta} \partial_t Q R Q)| \leq C (|\nabla \partial_t Q| + |\partial_t Q||\nabla Q|),
\]
we have

\[
\int_{\mathbb{R}^3} \frac{1}{2L} \partial_t \partial_i^2 Q_{ij} f_B(\tilde{Q}) \nabla_\beta \tilde{Q}_{ij} \nabla_\beta \tilde{Q}_{ij} \phi^2 \, dx \\
\leq C \frac{1}{L} \int_{\mathbb{R}^3} |Q - \pi(Q)| \left( |\nabla \partial_i Q| |\nabla \tilde{Q}| + |\partial_i Q| |\nabla Q| |\nabla \tilde{Q}| + |\partial_i Q| |\nabla^2 \tilde{Q}| \right) \phi^2 \, dx \\
+ \frac{C}{L} \int_{\mathbb{R}^3} |Q - \pi(Q)| \left( |\partial_i Q| |\nabla \tilde{Q}| \right) \phi^2 \, dx
\]

Here in the last inequality, we have used

\[
|\nabla^2 \tilde{Q}|^2 \leq C(|\nabla^2 (Q - \pi(Q))^2 | |\nabla (Q - \pi(Q))|^2 |\nabla Q|^2 + |Q - \pi(Q)|^2 (|\nabla^2 Q|^2 + |\nabla Q|^4)).
\]

Utilizing a similar argument to one in Corollary \( \Pi \) and (2.58), we have

\[
\int_0^s \int_{\mathbb{R}^3} \left( \frac{\nabla g_B(\tilde{Q})}{L} \right) \nabla_\beta (R^x_Q \partial_t Q R_Q) \phi^2 \, dx \, dt
= \int_0^s \int_{\mathbb{R}^3} \frac{\nabla g_B(\tilde{Q})}{L} \nabla_\beta \partial_i \tilde{Q}_{ij} \phi^2 \, dx \, dt
+ C \int_0^s \int_{\mathbb{R}^3} \frac{|\nabla^2 g_B(\tilde{Q})|}{L^2} \frac{|\partial_i Q R_Q| |Q - \pi(Q)|}{L^2} \phi^2 \, dx \, dt
\]

(2.58)
Applying Young’s inequality to the left-hand side of (2.56), we obtain

\[
\int_{\mathbb{R}^3} \left( \nabla \beta \left( R^T_{\beta} \partial_t Q + v \cdot \nabla Q + |Q, \Omega| R_Q \right) \right) \phi^2 \, dx \\
\geq \int_{\mathbb{R}^3} \left( \nabla \beta \partial_t Q, \nabla \beta \partial_t Q \right) \phi^2 \, dx - \int_{\mathbb{R}^3} |\nabla \partial_t Q| |\nabla R_Q| |\partial_t Q| \phi^2 \, dx \\
- C \int_{\mathbb{R}^3} \left( |\partial_t Q| |\nabla Q| + |v| (|\nabla^2 Q| + |\nabla Q|^2) + |\nabla Q| + |\nabla^2 v| \right) |\nabla (R^T_{\beta} \partial_t Q R_Q)| \phi^2 \, dx \\
\geq \int_{\mathbb{R}^3} \frac{3}{4} |\nabla \partial_t Q|^2 \phi^2 - C |\nabla^2 v|^2 \phi^2 - C (|\nabla Q|^2 (|\partial_t Q|^2 + |\nabla v|^2) + |v|^2 |\nabla^2 Q|^2) \phi^2 \, dx.
\]

(2.60)

Substituting (2.57), (2.59) and (2.60) into (2.56) yields

\[
\int_{\mathbb{R}^3} \left( \frac{\alpha}{4} |\nabla^2 Q|^2 + \frac{\lambda}{8} \frac{|\nabla (Q - \pi(Q))|^2}{L} \right) (x, s) \phi^2 \, dx + \frac{3}{8} \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla \partial_t Q|^2 \phi^2 \, dx \, dt \\
\leq C \int_{\mathbb{R}^3} \left( \frac{|\nabla^2 Q_0|^2}{L} + \frac{|\nabla (Q_0 - \pi(Q_0))|^2}{L} + \frac{|Q_{L,0} - \pi(Q_{L,0})|^2}{L} |\nabla Q_0|^2 \right) \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} \left( \frac{|Q - \pi(Q)|^2}{L} |\nabla Q|^2 \right) (x, s) \phi^2 \, dx + C \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla^2 v|^2 \phi^2 \, dx \, dt \\
+ C \int_{0}^{s} \int_{\mathbb{R}^3} \left( \frac{|Q - \pi(Q)|^2}{L} |\nabla Q|^2 + \frac{|\nabla^2 Q|^2}{L} \phi^2 \right) \phi^2 \, dx \\
+ C \int_{0}^{s} \int_{\mathbb{R}^3} \left( |\nabla^2 Q|^2 + |\nabla v|^2 + |\partial_t Q|^2 \right) (e(Q, v) \phi^2 + |\nabla \phi|^2) \, dx \, dt \\
+ C \int_{0}^{s} \int_{\mathbb{R}^3} \left( \frac{|\nabla (Q - \pi(Q))|^2}{L} + \varepsilon^2(Q, v) \right) (e(Q, v) \phi^2 + |\nabla \phi|^2) \, dx \, dt.
\]

(2.61)

Integrating (2.55) in \( t \) then adding it to (2.61), we derive

\[
\int_{\mathbb{R}^3} \left( |\nabla^2 Q|^2 + \frac{|\nabla (Q - \pi(Q))|^2}{L} \right) (x, s) \phi^2 \, dx \\
+ \int_{0}^{s} \int_{\mathbb{R}^3} \left( |\nabla^2 Q|^2 + |\nabla \partial_t Q|^2 + \frac{|\nabla^2 (Q - \pi(Q))|^2}{L} \right) \phi^2 \, dx \, dt \\
\leq C \int_{\mathbb{R}^3} \left( |\nabla^2 Q_0|^2 + \frac{|\nabla (Q_0 - \pi(Q_0))|^2}{L} + \frac{|Q_{L,0} - \pi(Q_{L,0})|^2}{L} |\nabla Q_0|^2 \right) \phi^2 \, dx \\
+ C \int_{\mathbb{R}^3} \left( \frac{|Q - \pi(Q)|^2}{L} |\nabla Q|^2 \right) (x, s) \phi^2 \, dx + \int_{0}^{s} \int_{\mathbb{R}^3} |\nabla^2 v|^2 \phi^2 \, dx \, dt \\
+ C \int_{0}^{s} \int_{\mathbb{R}^3} \left( |\nabla^2 Q|^2 + |\nabla v|^2 + |\partial_t Q|^2 \right) (e(Q, v) \phi^2 + |\nabla \phi|^2) \, dx \, dt \\
+ C \int_{0}^{s} \int_{\mathbb{R}^3} \left( \frac{|\nabla (Q - \pi(Q))|^2}{L} + \varepsilon^2(Q, v) \right) (e(Q, v) \phi^2 + |\nabla \phi|^2) \, dx \, dt.
\]

(2.62)
To estimate the term $\nabla^2 v$ in (2.62), we take $L^2$ inner product of (1.11) with $-\Delta v \phi^2$ and calculate

\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v|^2 \phi^2 \, dx + \int_{\mathbb{R}^3} |\nabla^2 v|^2 \phi^2 \, dx
\]

\[= - \int_{\mathbb{R}^3} 2 \partial_t \nabla_i v_j \nabla_j \phi \, dx + \int_{\mathbb{R}^3} 2 (\nabla_j v_i \Delta v_i - \nabla_k v_i \nabla_k v_i) \nabla_j \phi \, dx
\]

\[\quad - \int_{\mathbb{R}^3} (P - c^*) \Delta v_i \nabla_i \phi \, dx - \int_{\mathbb{R}^3} \nabla_j \sigma_{ij} \Delta v_i \phi^2 \, dx - \int_{\mathbb{R}^3} \nabla_j [Q, \mathcal{H}(Q, \nabla Q)]_{ij} \Delta v_i \phi^2 \, dx
\]

\[\leq - \int_{\mathbb{R}^3} \nabla_j [Q, \mathcal{H}(Q, \nabla Q)]_{ij} \Delta v_i \phi^2 \, dx - \int_{\mathbb{R}^3} 2 \partial_t \nabla_i v_j \nabla_j \phi \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla^2 v|^2 \phi^2 \, dx
\]

\[+ C \int_{\mathbb{R}^3} (|\nabla v|^2 + |P - c^*|^2) |\nabla \phi|^2 \, dx + C \int_{\mathbb{R}^3} (|\nabla Q|^2 + |\nabla Q|^4) |\nabla Q|^2 \phi^2 \, dx,
\]

where we have used the fact that $|\nabla \sigma(Q, \nabla Q)| \leq C(|\nabla^2 Q| + |\nabla Q|^2)|\nabla Q|$.

By using (1.11) and integrating by parts, we have

\[\quad - 2 \int_{\mathbb{R}^3} \partial_t \nabla_i v_j \nabla_j v_i \nabla_j \phi \, dx\]

\[= 2 \int_{\mathbb{R}^3} (v_k \nabla_k v_j - \Delta v_i + \nabla_k \sigma_{ik}) \nabla_j v_i \nabla_j \phi \, dx + 2 \int_{\mathbb{R}^3} (P - c^*) \nabla_j v_i \nabla_i (\nabla_j \phi) \, dx
\]

\[\quad + 2 \int_{\mathbb{R}^3} [Q, \mathcal{H}(Q, \nabla Q)]_{ik} \left( \nabla_j v_i \nabla_j \phi + \nabla_i v_k \nabla_j \phi \right) \, dx
\]

\[\leq \frac{1}{4} \int_{\mathbb{R}^3} |\nabla^2 v|^2 \phi^2 \, dx + C \int_{\mathbb{R}^3} (|v|^2 |\nabla v|^2 + (|\nabla Q|^2 + |\nabla Q|^4) |\nabla Q|^2) \phi^2 \, dx
\]

\[+ C \int_{\mathbb{R}^3} (|P - c^*|^2 + |\nabla v|^2 + |\nabla^2 Q|^2 + |\nabla Q|^4)(|\nabla^2 \phi| |\phi| + |\nabla \phi|^2) \, dx,
\]

which, plugging into the previous inequality, yields

\[\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |\nabla v|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} |\nabla^2 v|^2 \phi^2 \, dx
\]

\[\leq - \int_{\mathbb{R}^3} \nabla_j [Q, \mathcal{H}(Q, \nabla Q)]_{ij} \Delta v_i \phi^2 \, dx + C \int_{\mathbb{R}^3} |P - c^*|^2 (|\nabla^2 \phi| |\phi| + |\nabla \phi|^2) \, dx
\]

\[+ C \int_{\mathbb{R}^3} (|\nabla Q|^2 + |\nabla v|^2 + |\nabla Q|^4)(|\nabla^2 \phi| |\phi| + |\nabla \phi|^2) \, dx
\]

\[+ C \int_{\mathbb{R}^3} (|\nabla Q|^2 + |\nabla v|^2 + |\nabla Q|^4) \phi^2 \, dx.
\]

Choosing $A = Q, B = \mathcal{H}(Q, \nabla Q) + \frac{1}{\tau} g_B(Q), F = \Delta \nabla v$ in Lemma 2.3, we observe

\[\left\langle [Q, \Delta \Omega], \mathcal{H} + \frac{1}{\tau} g_B \right\rangle = \Delta \nabla v_i [Q, \mathcal{H}]_{ij}.
\]
Then, integrating by parts and using \((2.64)\) and \((1.13)\) on the term \((\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q))\), we have

\[
\int_{\mathbb{R}^3} \left\langle \nabla_{\beta} \left( R^T [Q, \Omega] R \right), \nabla_{\beta} \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle \phi^2 \, dx 
\leq - \int_{\mathbb{R}^3} \left\langle [\Delta Q, \Omega] + 2[\nabla_{\beta} Q, \nabla_{\beta} \Omega] + [Q, \Delta \Omega], \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) \right\rangle \phi^2 \, dx
\]

\[
+ C \int_{\mathbb{R}^3} \|\nabla[Q, \Omega]\| \|\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q)\| (\|\nabla \phi\|^2 + \|\phi\|^2) \, dx
\]

\[
+ C \int_{\mathbb{R}^3} \|\nabla \phi\| \|\nabla v\| \left\langle \nabla \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle \phi^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \left( \Delta v \nabla_j [Q, \mathcal{H}(Q, \nabla Q)]_{ij} + \eta \|\nabla \phi\|^2 + \frac{1}{4} \|\nabla^2 v\|^2 \right) \phi^2 \, dx
\]

\[
+C \int_{\mathbb{R}^3} (\|\nabla^2 Q\|^2 + \|\nabla Q\|^2 + \|\partial_t Q\|^2 + \|v\|^4)((\|\nabla Q\|^2 + \|v\|^2) \phi^2 + \|\nabla \phi\|^2) \, dx
\]

\[
+ \frac{1}{4} \int_{\mathbb{R}^3} \left\langle \nabla \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle^2 \phi^2 \, dx.
\]

Here we used that \(|\mathcal{H}(Q, \nabla Q)|^2 \|\nabla \phi\|^2 \leq C(\|\nabla^2 Q\|^2 + \|\nabla Q\|^4)\|\nabla \phi\|^2.

We differentiate \((2.60)\) with respect to \(t\), multiply by \(\nabla_{\beta} \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right)\) and substitute \((2.65)\) to find

\[
\int_{\mathbb{R}^3} \left\langle \nabla \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle^2 \phi^2 \, dx
\]

\[
= \int_{\mathbb{R}^3} \left\langle \nabla_{\beta} \left( R^T \partial_t Q + v \cdot \nabla Q + [Q, \Omega] R \right), \nabla_{\beta} \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle \phi^2 \, dx
\]

\[
\leq \int_{\mathbb{R}^3} \left\langle \nabla_{\beta} \left( R^T \partial_t Q R_Q \right), \nabla_{\beta} \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle \phi^2 \, dx
\]

\[
+ \int_{\mathbb{R}^3} \left( \Delta v \nabla_j [Q, \mathcal{H}(Q, \nabla Q)]_{ij} + \eta \|\nabla \phi\|^2 + \frac{1}{4} \|\nabla^2 v\|^2 \right) \phi^2 \, dx
\]

\[
+C \int_{\mathbb{R}^3} (\|\nabla^2 Q\|^2 + \|\nabla Q\|^2 + \|\partial_t Q\|^2 + \|v\|^4)((\|\nabla Q\|^2 + \|v\|^2) \phi^2 + \|\nabla \phi\|^2) \, dx
\]

\[
+ \frac{1}{2} \int_{\mathbb{R}^3} \left\langle \nabla_{\beta} \left( R^T [Q, \Omega] R \right), \nabla_{\beta} \left( R^T \mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q) R_Q \right) \right\rangle \phi^2 \, dx.
\]

Combining \((2.63)\) with \((2.66)\) and integrating in \(t\), we then apply the arguments in \((2.57)\) - \((2.59)\) to derive

\[
\int_{\mathbb{R}^3} \left( \frac{\alpha}{4} \|\nabla^2 Q\|^2 + \frac{1}{2} \|v\|^2 + \frac{\lambda}{8} \|\nabla(\pi - \Omega)\|^2 \right) (x, s) \phi^2 \, dx \leq \int_0^s \int_{\mathbb{R}^3} \|\nabla^2 v\|^2 \phi^2 \, dx dt
\]

\[
\leq C \int_{\mathbb{R}^3} \left( \|\nabla^2 Q_0\|^2 + \|\partial_t Q_0\|^2 + \frac{\|\nabla(Q_0 - \pi(Q_0))\|^2}{L} \right) \phi^2 \, dx
\]

\[
+C \int_{\mathbb{R}^3} \left( \frac{|Q_{L,0} - \pi(Q_{L,0})|^2}{L} \|\nabla Q_{L,0}\|^2 + \frac{|Q - \pi(Q)|^2}{L} \|\nabla Q\|^2(x, s) \right) \phi^2 \, dx
\]

\[
+ \int_{\mathbb{R}^3} 2\eta \|\nabla^3 Q\|^2 + \|\nabla \partial_t Q\|^2 + \frac{\|\nabla^2 (Q - \pi(Q))\|^2}{L} \phi^2 \, dx.
\]
Then for any assumption (3.1), we have

\[ \sup_{t \in [T_0, T_L]} \int_{B_{R}(x_0)} |\nabla Q_L(x, t)|^3 + |v_L(x, t)|^3 dx \leq \varepsilon_0. \]  

(3.1)

Then for any \( t \in (T_0, T_L) \), there exists a constant \( c_L(t) \in \mathbb{R} \) such that the pressure \( P_L \) satisfies the following estimate

\[ \sup_{x_0 \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |P_L - c_L|^2 dxdt \leq C \sup_{y \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_{2R}(y)} (|\nabla^2 Q_L|^2 + |\nabla v_L|^2) + \frac{\varepsilon_0^2}{R^2} (|\nabla Q_L|^2 + |v_L|^2) dxdt. \]  

(3.2)

Proof. The proof is essentially the same as the one in [11]. For completeness, we outline an approach here. Let \( \phi \) be a cut-off function satisfying \( 0 \leq \phi \leq 1 \), \( \text{supp} \phi \subset B_{2R}(x_0) \) for some \( x_0 \in \mathbb{R}^3 \) and \( |\nabla \phi| \leq \frac{C}{R} \). Note that the pressure \( P_L \) satisfies

\[ -\Delta P_L = \nabla_i^j \left( [Q_L, H(Q_L, \nabla Q_L)]_{ij} - \sigma_{ij}(Q_L, \nabla Q_L) + v_i^j v_i^j \right) \]  

on \( \mathbb{R}^3 \times [T_0, T_L] \), which implies \( P_L = \mathcal{R}_i \mathcal{R}_j (F_i^j) \), and

\[ |F_i^j| = ||[Q_L, H(Q_L, \nabla Q_L)]_{ij} - \sigma_{ij}(Q_L, \nabla Q_L) + v_i^j v_i^j|| \leq C(|\nabla^2 Q_L| + |\nabla Q_L|^2 + |v_L|^2), \]  

where \( \mathcal{R}_i \) is the \( i \)-th Riesz transform on \( \mathbb{R}^3 \). Then we have

\[ (P_L - c_L^2) \phi = \mathcal{R}_i \mathcal{R}_j (F_i^j \phi) + [\phi, \mathcal{R}_i \mathcal{R}_j] (F_i^j) - c_L^2 \phi \]  

(3.3)

for a cut-off function \( \phi \), where the commutator \( [\phi, \mathcal{R}_i \mathcal{R}_j] \) is defined by

\[ [\phi, \mathcal{R}_i \mathcal{R}_j](\cdot) = \phi \mathcal{R}_i \mathcal{R}_j(\cdot) - \mathcal{R}_i \mathcal{R}_j(\cdot) \phi. \]

By using the Riesz operator maps \( L^q \) into \( L^q \) spaces for any \( 1 < q < +\infty \) and the assumption (3.1), we have

\[ \int_{T_0}^{T_L} \int_{\mathbb{R}^3} |\mathcal{R}_i \mathcal{R}_j (F_i^j \phi)|^2 dxdt \leq C \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 dxdt + \frac{C}{R^2} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla Q_L|^2 + |v_L|^2 dxdt. \]  

(3.4)
Since \( \text{supp} \phi \subset B_{2R(x_0)} \), the commutator can be expressed as

\[
[\phi, R_i R_j](F^{ij})(x, t) - c_L^t(t) \phi(x)
= \int_{B_{4R(x_0)}} \frac{(\phi(x) - \phi(y))(x_i - y_i)(x_j - y_j)}{|x - y|^5} F^{ij}(y, t) \, dy
+ \phi(x) \left[ \int_{B_{3} \setminus B_{4R(x_0)}} \frac{(x_i - y_i)(x_j - y_j)}{|x - y|^5} F^{ij}(y, t) \, dy - c_L(t) \right]
= : f_1(x, t) + f_2(x, t).
\]

By using the Hardy-Littlewood-Sobolev inequality (c.f. [10])

\[
\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} \, dy \right\|_{L^q(\mathbb{R}^n)} \leq C \| f \|_{L^r(\mathbb{R}^n)}, \quad \frac{1}{q} = \frac{1}{r} - \frac{\alpha}{n}
\]

and the Hölder inequality, a standard covering argument yields

\[
\int_{T_0}^{T_L} \int_{\mathbb{R}^3} |f_1(x,s)|^2 \, dx dt \leq CR^{-2} \int_{T_0}^{T_L} \| (F^{ij}) \chi_{B_{4R(x_0)}} \|_{L^q(\mathbb{R}^3)}^2 \, dt
\leq \frac{C}{R^2} \int_{T_0}^{T_L} \| (|\nabla Q_L| + |v_L|) \chi_{B_{4R(x_0)}} \|_{L^q(\mathbb{R}^3)}^2 \, dt
+ \frac{C}{R^2} \int_{T_0}^{T_L} \| \chi_{B_{4R(x_0)}} \|_{L^q(\mathbb{R}^3)}^2 \| (|\nabla^2 Q_L|) \chi_{B_{4R(x_0)}} \|_{L^q(\mathbb{R}^3)}^2 \, dt
\leq \frac{C \alpha q}{R^2} \int_{T_0}^{T_L} \int_{B_{4R(x_0)}} |\nabla Q_L|^2 + |v_L|^2 \, dx dt + C \int_{T_0}^{T_L} \int_{B_{4R(x_0)}} |\nabla^2 Q_L|^2 \, dx dt,
\]

where \( \chi_{B_{4R}(x_0)}(x) = 1 \) for \( x \in B_{4R}(x_0) \) and \( 0 \) for \( x \in \mathbb{R}^3 \setminus B_{4R}(x_0) \). Choosing

\[
c_L^t(t) = \int_{\mathbb{R}^3 \setminus B_{4R}(x_0)} \frac{(x_0_i - y_i)(x_0_j - y_j)}{|x_0 - y|^5} F^{ij}(y, t) \, dy
\]

and using the Hölder inequality, we estimate

\[
\int_{T_0}^{T_L} \int_{\mathbb{R}^3} |f_2(z,s)|^2 \, dz dt
\]

\[
\leq CR^5 \int_{T_0}^{T_L} \sum_{k=4}^{\infty} \frac{C}{(kR)^3} \int_{B(k+1)R(x_0) \setminus B_kR(x_0)} |F^{ij}(x, t)|^2 \, dx dt
\leq C \sup_{y \in \mathbb{R}^3} \int_{T_0}^{T_L} \sum_{k=4}^{\infty} k^{-4} \int_{B_k(y)} |F^{ij}|^2 \, dx dt
\leq C \sup_{y \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_k(y)} \frac{e_0^2}{R^2} (|\nabla Q_L|^2 + |v_L|^2) + (|\nabla^2 Q_L|^2 + |\nabla v_L|^2) \, dx dt.
\]

Combining (3.4), (3.6) with (3.7), we can apply a standard covering argument to complete the proof. \( \square \)

Using Lemma 2.5 and Lemma 2.6, we have:

\textbf{Lemma 3.2.} Let \( (Q_L, v_L) \) be a strong solution of (1.1) - (1.3) in \( \mathbb{R}^3 \times [T_0, T_L] \) with initial value \( (Q_{L,T_0}, v_{L,T_0}) \in H^2_0(\mathbb{R}^3; S_0) \times H^1(\mathbb{R}^3; \mathbb{R}^3) \) and \( \text{div} v = 0 \). Assume
that \( Q \in S_\delta \) for sufficiently small \( \delta \) on \( \mathbb{R}^3 \times (0, T_L) \). There exist two constants \( \varepsilon_0 \) and \( R \) that

\[
\sup_{T_0 \leq t \leq T_L, x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L^2} \, dx \leq \varepsilon_0^3. \tag{3.8}
\]

Then we have

\[
\sup_{T_0 \leq s \leq T_L, x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x_0)} \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) (x, s) \, dx
\]
\[
+ \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{B_R(x_0)} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\partial_t Q_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \, dx \, dt
\]
\[
\leq C \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |\nabla Q_L, T_0|^2 + |v_L, T_0|^2 + \frac{|Q_L, T_0 - \pi(Q_L, T_0)|^2}{L} \, dx + C\varepsilon_0^2 \frac{(T_L - T_0)}{R^2}. \tag{3.9}
\]

and

\[
\sup_{T_0 \leq s \leq T_L, x_0 \in \mathbb{R}^3} \frac{R}{\int_{B_R(x_0)}} \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \right) (x, s) \, dx
\]
\[
+ \sup_{x_0 \in \mathbb{R}^3} R \int_{B_R(x_0)} |\nabla^2 Q_L|^2 + |\nabla^2 v_L|^2 + |\nabla \partial_t Q_L|^2 + \frac{|\nabla^2(Q_L - \pi(Q_L))|^2}{L} \, dx \, dt
\]
\[
\leq CR \sup_{x_0 \in \mathbb{R}^3} \int_{B_R(x_0)} |\nabla^2 Q_L, T_0|^2 + |v_L, T_0|^2 + \frac{|Q_L, T_0 - \pi(Q_L, T_0)|^2}{L} \, dx + C\varepsilon_0^2 \frac{(T_L - T_0)}{R^2}. \tag{3.10}
\]

Proof. Let \( \{B_R(x_i)\}_{i=1}^\infty \) be a standard open cover of \( \mathbb{R}^3 \) such that at each \( x \in \mathbb{R}^3 \), there are finite intersections of open balls \( B_R(x_i) \). Let \( \phi \in C_0^\infty(B_R(x_0)) \) with \( \phi \equiv 1 \) on \( B_R(x_0) \), \( |\nabla \phi| \leq \frac{C}{\pi} \) and \( |\nabla^2 \phi| \leq \frac{C}{\pi^2} \). Recall from Lemma 2.5 that

\[
\frac{1}{R} \int_{B_R(x_0)} \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) (x, s) \, dx
\]
\[
+ \frac{1}{R} \int_0^s \int_{B_R(x_0)} |\nabla^2 Q_L|^2 + |\partial_t Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \, dx \, dt
\]
\[
\leq C \int_{B_{2R}(x_0)} |\nabla Q_L, T_0|^2 + |v_L, T_0|^2 + \frac{|Q_L, T_0 - \pi(Q_L, T_0)|^2}{L} \, dx
\]
\[
+ \frac{C}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla Q_L|^2 \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) \, dx \, dt
\]
\[
+ \frac{\eta}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |P_L - c_L(t)|^2 \, dx \, dt + \frac{C(q)}{R^3} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla Q_L|^2 + |v_L|^2 \, dx \, dt. \tag{3.11}
\]
for some small $\eta$ to be chosen later. Using Hölder’s inequality and \((3.8)\), we have
\[
\sup_{T_0 \leq s \leq T_L} \frac{1}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) (x, s) \, dx \, dt 
\leq C \varepsilon_0^2 (T_L - T_0).
\tag{3.12}
\]

Then, using the Sobolev inequality, \((3.8)\) and \((3.12)\), we find
\[
\sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla Q_L|^2 |\nabla Q_L|^2 \, dx \, dt 
\leq C \sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{T_0}^{T_L} \sum_{i=1}^{\infty} \left( \int_{B_R(x_i)} |\nabla Q_L|^3 \, dx \right) \left( \int_{B_R(x_i)} |\nabla Q_L|^6 \, dx \right)^{\frac{1}{2}} \, dt 
\leq C \varepsilon_0^2 \sup_{y \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_R(y)} |\nabla^2 Q_L|^2 \, dx \, dt + C \frac{C R}{R_0} \sup_{T_0 \leq s \leq T_L, y \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_R(y)} |\nabla Q_L|^2 \, dx \, dt 
\leq C \varepsilon_0^2 \sup_{y \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_R(y)} |\nabla^2 Q_L|^2 \, dx \, dt + C \frac{T_L - T_0}{R^2}. \tag{3.13}
\]

Similarly, we obtain
\[
\sup_{x_0 \in \mathbb{R}^3} \frac{1}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla Q_L|^2 \left( |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) \, dx \, dt 
\leq C \varepsilon_0^2 \sup_{y \in \mathbb{R}^3} \int_{T_0}^{T_L} \int_{B_R(y)} |\nabla v_L|^2 + \frac{|\nabla Q_L - \pi(Q_L)|^2}{L} \, dx \, dt + C \frac{T_L - T_0}{R^2}. \tag{3.14}
\]

Substituting \((3.12)\), \((3.13)\), \((3.14)\) into \((3.11)\), using Lemma \((3.9)\) and taking the supremum of $x_0 \in \mathbb{R}^3$, we prove \((3.9)\) by choosing $\eta$ sufficiently small.

To show \((3.10)\), recall from Lemma \((3.6)\) that
\[
R \int_{B_R(x_0)} \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \right) (x, T_L) \, dx 
+ R \int_{T_0}^{T_L} \int_{B_R(x_0)} \left( |\nabla^2 Q_L|^2 + |\nabla^2 v_L|^2 + \frac{|\nabla^2(Q_L - \pi(Q_L))|^2}{L} \right) \, dx \, dt 
\leq CR \int_{B_{2R}(x_0)} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{\left| Q_L - \pi(Q_L) \right|^2}{L} \, dx 
+ CR \int_{B_{2R}(x_0)} \left| Q_{L,0} - \pi(Q_{L,0}) \right|^2 |\nabla Q_{L,0}|^2 + \left( \frac{\left| Q - \pi(Q) \right|^2}{L} |\nabla Q|^2 \right) (x, T_L) \, dx 
+ CR \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} e(Q_L, v_L) \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\partial_t Q_L|^2 \right) \, dx \, dt 
+ CR \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} e(Q_L, v_L) \left( \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} + e^2(Q_L, v_L) \right) \, dx \, dt 
+ \frac{C}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\partial_t Q_L|^2 + \frac{|\nabla(Q - \pi(Q))|^2}{L} \, dx \, dt 
+ \frac{C}{R} \int_{T_0}^{T_L} \int_{B_{2R}(x_0)} e^2(Q_L, v_L) + |\partial_t^2 L - c_L(t)|^2 \, dx \, dt.
\[
\leq CR \int_{B_{2R}(x_0)} |\nabla^2 Q_{L,T_0}|^2 + |\nabla v_{L,T_0}|^2 + \left| \frac{\nabla(Q_{L,T_0} - \pi(Q_{L,T_0}))}{L} \right|^2 dx \\
+ C\varepsilon_0^2 R \sup_{y \in \mathbb{R}^3} \int_{B_{R}(y)} |\nabla^2 Q_{L,T_0}|^2 + |\nabla^2 Q_{L}(x,T_L)|^2 dx \\
+ C \varepsilon_0^2 R \sup_{y \in \mathbb{R}^3} \int_{B_{R}(y)} |\nabla Q_{L}(x,T_L)|^2 + |v_{L,T_0}|^2 + \left| \frac{Q_{L,T_0} - \pi(Q_{L,T_0})}{L} \right|^2 dx \\
+ C\varepsilon_0^2 R \sup_{y \in \mathbb{R}^3} \int_{B_{R}(y)} |\nabla^3 Q_L|^2 + |\nabla^2 v_L|^2 + |\nabla \partial_t Q_L|^2 + \left| \frac{\nabla^2(Q_L - \pi(Q_L))}{L} \right|^2 dxdt \\
+ \varepsilon_0^2 \frac{C(T_L - T_0)}{R^2} + CR \int_{[T_0,T_L]} \int_{B_{2R}(x_0)} e^3(Q_L,v_L) dxdt.
\]

(3.15)

Here we used the argument in \(3.13\), Lemma \(3.1\) and substituted \(3.9\). Applying the argument in \(3.14\) twice, we deduce the last term in \(3.15\) to \(3.16\).

\[
\leq CR \int_{[T_0,T_L]} \int_{B_{2R}(x_0)} e^3(Q_L,v_L) dxdt \\
\leq CR \int_{[T_0,T_L]} \int_{B_{2R}(x_0)} e(Q_L,v_L) \left( |\nabla Q_L|^4 + |v_L|^4 + \left| \frac{Q_L - \pi(Q_L)}{L^2} \right|^4 \right) dxdt \\
\leq C\varepsilon_0^2 R \sup_{y \in \mathbb{R}^3} \int_{B_{R}(y)} \int_{[T_0,T_L]} |\nabla \nabla Q_L|^2 + |\nabla v_L|^2 + \left| \frac{\nabla Q_L - \pi(Q_L)}{L^2} \right|^2 dxdt \\
+ C \varepsilon_0^2 R \sup_{y \in \mathbb{R}^3} \int_{B_{R}(y)} \int_{[T_0,T_L]} |\nabla^3 Q_L|^2 + |\nabla^2 v_L|^2 + \left| \frac{\nabla^2(Q_L - \pi(Q_L))}{L} \right|^2 dxdt \\
\leq C\varepsilon_0^2 R \sup_{y \in \mathbb{R}^3} \int_{B_{R}(y)} \int_{[T_0,T_L]} |\nabla^3 Q_L|^2 + |\nabla^2 v_L|^2 + \left| \frac{Q_{L,T_0} - \pi(Q_{L,T_0})}{L} \right|^2 dx + \varepsilon_0^2 \frac{C(T_L - T_0)}{R^2}.
\]

(3.16)

Here we also used \(3.9\). Substituting \(3.16\) into \(3.14\) and then taking the supremum of \(x_0 \in \mathbb{R}^3\) on the resulting expression, we obtain \(3.10\). \(\square\)

Using the Gagliardo-Nirenberg interpolation, we establish a uniform local existence of the strong solutions:

**Proposition 3.1.** Assume that \((Q_{L,T_0},v_{L,T_0})\) satisfies

\[
\|Q_{L,T_0}\|_{H^2(\mathbb{R}^3)}^2 + \|v_{L,T_0}\|_{L^4(\mathbb{R}^3)}^2 + \frac{\|Q_{L,T_0} - \pi(Q_{L,T_0})\|^2}{L} \leq M
\]

(3.17)

for some \(M > 0\). Then there are uniform constants \(T_M,R_0 \in L\) such that the system \(1.11\), \(1.13\) with initial data \((Q_{L,T_0},v_{L,T_0})\) has a unique strong solution \((Q_L,v_L)\) in \(\mathbb{R}^3 \times [T_0,T_M]\) satisfying

\[
\sup_{T_0 \leq t \leq T_M} \int_{B_{R_0}(x_0)} |\nabla Q_L|^3 + |v_L|^3 + \left| \frac{Q_L - \pi(Q_L)}{L^2} \right|^3 dx \leq \varepsilon_0^3 \frac{3}{2}
\]

(3.18)
and

\[
\mathop{\sup}
_{T_0 \leq s \leq T_M} \left( \|\nabla Q_L(s)\|_{L^2(\mathbb{R}^3)}^2 + \|v_L(s)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{L} \|Q_L(s) - \pi(Q_L(s))\|_{L^2(\mathbb{R}^3)}^2 \right)
+ \|\partial_t Q_L\|_{L^2(T_0; T_M; L^2(\mathbb{R}^3))}^2 + \|\nabla^2 Q_L\|_{L^2(T_0; T_M; L^2(\mathbb{R}^3))}^2
+ \|\nabla v_L\|_{L^2(T_0; T_M; H^1(\mathbb{R}^3))}^2 + \frac{1}{L} \|\nabla(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \leq C \left( 1 + \varepsilon_0^2 \frac{R_0^2}{N} \right) M. \quad (3.19)
\]

**Proof.** It follows from the Sobolev embedding theorem that for any \(0 < \varepsilon_0 < 1\), there exists a positive constant \(R_0 =: \frac{\varepsilon_0^2}{C_2 N M}\) (c.f. [11]) such that

\[
\mathop{\sup}
_{x_0 \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} |\nabla Q_L|_{T_0}^2 + |v_L|_{T_0}^2 + \frac{|Q_L - \pi(Q_L)|_{T_0}^2}{L^2} \, dx \leq \varepsilon_0^3 \frac{N}{3}, \quad (3.20)
\]

where \(N > 1\) is an absolute constant independent of \(L\) and \(M\) to be chosen later. By using the Gagliardo–Nirenberg interpolation (c.f. [11]) at \(T_0\), we have

\[
\operatorname{dist}(Q_L(T_0); S_s) = \|\operatorname{dist}(Q_L(T_0); S_s)\|_{L^\infty(\mathbb{R}^3)}
\leq C \|Q_L(T_0) - \pi(Q_L(T_0))\|_{L^2(\mathbb{R}^3)}^\frac{3}{2} \|\nabla^2(Q_L(T_0) - \pi(Q_L(T_0)))\|_{L^2(\mathbb{R}^3)}^\frac{1}{2}
\leq C(LM)^{\frac{1}{4}} \left( \int_{\mathbb{R}^3} |\nabla^2 Q_L|_{T_0}^2 + |\partial_t Q_L| |\nabla^2 Q_L|_{T_0}^2 + |\partial^2 Q_L| |\nabla Q_L|_{T_0}^2 \, dx \right)^{\frac{1}{2}}
\leq C_d L^{\frac{1}{4}} M^{\frac{1}{4}} \leq \delta \frac{1}{2},
\]

where we have used the condition (3.17) and chosen \(L \leq \left( \frac{\delta}{2C_d M} \right)^8 \).

Using Theorem 1 there is a unique local strong solution \((Q_L, v_L)\) such that \((Q_L, v_L)\) is continuous in \(t\), which follows from the Sobolev inequality (c.f. [15]). Then there is a time \(T_L^* \in (T_0, T_L]\) such that

\[
\operatorname{dist}(Q_L; S_s) \leq \delta \text{ on } \mathbb{R}^3 \times (T_0, T_L^*). \quad (3.21)
\]

and

\[
\mathop{\sup}
_{T_0 \leq t \leq T_L} \int_{B_{R_0}(x_0)} |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L^2} \, dx \leq \varepsilon_0^3. \quad (3.22)
\]

Using (3.17), (3.20), \(R_0 = \frac{\varepsilon_0^2}{C_2 N M}\) and choosing \((T_L^* - T_0) \leq \sigma R_0^2\) for some small \(\sigma\) to be chosen later, we have

\[
CR_0 \mathop{\sup}
_{x_0 \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} |\nabla^2 Q_L|_{T_0}^2 + |v_L|_{T_0}^2 + \frac{|\nabla(Q_L - \pi(Q_L))|_{T_0}^2}{L} \, dx
+ \frac{C}{R_0} \mathop{\sup}
_{x_0 \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} |\nabla Q_L|_{T_0}^2 + |v_L|_{T_0}^2 + \frac{|Q_L - \pi(Q_L)|_{T_0}^2}{L} \, dx + \varepsilon_0^2 \frac{C(T_L - T_0)}{R_0^2}
\leq CM R_0 + \frac{C}{R_0} \left( \frac{\varepsilon_0^2}{N^2} + C \varepsilon_0^2 \sigma \right) \leq \frac{C \varepsilon_0^2}{N^2} + C \varepsilon_0^2 \sigma. \quad (3.23)
\]
Similarly, we obtain

\[
\sup_{T_0 \leq t \leq T_L, x_0 \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} |\nabla Q_L|^3 + |v_L|^3 + \frac{|Q_L - \pi(Q_L)|^3}{L^2} \, dx \\
\leq C \sup_{T_0 \leq t \leq T_L, x_0 \in \mathbb{R}^3} \left( \frac{1}{R_0} \int_{B_{R_0}(x_0)} |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \, dx \right)^{3/2} \\
+ C \sup_{T_0 \leq t \leq T_L, x_0 \in \mathbb{R}^3} \left( R_0 \int_{B_{R_0}(x_0)} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \, dx \right)^{3/2} \\
\leq \left( \frac{C_1 \varepsilon_0^2}{N^2} + C_2 \sigma \varepsilon_0^2 \right)^{3/2} \leq \varepsilon_0^3 \leq \frac{3}{2},
\]

where we choose \( N \geq 2(C_1 + 1)^{\frac{1}{2}} \) and \( \sigma \leq \min\{ \frac{\varepsilon_0}{C_2}, 1 \} \). Thus we prove (3.22) up to the uniform time \( T_M = T_0 + \sigma R_0^2 \).

For (3.19), using a standard open cover \( \{ B_R(x_i) \}_{i=1}^\infty \) for \( \mathbb{R}^3 \) with finite intersections at each \( x \in \mathbb{R}^3 \), the Hölder inequality and the Sobolev inequality, we find

\[
\int_{T_0}^{T_M} \int_{\mathbb{R}^3} |\nabla Q_L|^4 \, dx \, dt \leq \int_{T_0}^{T_M} \int_{\mathbb{R}^3} \left( \int_{B_{R_0}(x_i)} |\nabla Q_L|^3 \, dx \right)^\frac{4}{3} \left( \int_{B_{R_0}(x_i)} |\nabla Q_L|^6 \, dx \right)^\frac{1}{3} \, dt \\
\leq C \varepsilon_0^2 \int_{T_0}^{T_M} \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 \, dx \, dt + C \varepsilon_0^2 \frac{T_M - T_0}{R^2} \sup_{T_0 \leq s \leq T_M} \int_{\mathbb{R}^3} |\nabla Q_L(x, s)|^2 \, dx \\
\leq \frac{1}{2} \int_{T_0}^{T_M} \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 \, dx \, dt + \frac{1}{2} \sup_{T_0 \leq s \leq T_M} \int_{\mathbb{R}^3} |\nabla Q_L(x, s)|^2 \, dx.
\]

(3.24)

Similarly, we obtain

\[
\int_{T_0}^{T_M} \int_{\mathbb{R}^3} |\nabla Q_L|^2 \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) \, dx \, dt \\
\leq \frac{1}{2} \int_{T_0}^{T_M} \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \, dx \, dt \\
+ \frac{1}{2} \sup_{T_0 \leq s \leq T_M} \int_{\mathbb{R}^3} \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) (x, s) \, dx.
\]

(3.25)

Choosing \( \phi \equiv 1 \) in Lemma 2.5 using (3.17) and (3.25), we have

\[
\sup_{T_0 \leq s \leq T_M} \int_{\mathbb{R}^3} \left( |\nabla Q_L|^2 + |v_L|^2 + \frac{|Q_L - \pi(Q_L)|^2}{L} \right) (x, s) \, dx \\
+ \int_{T_0}^{T_M} \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\partial_t Q_L|^2 + \frac{|\nabla(Q_L - \pi(Q_L))|^2}{L} \, dx \, dt \leq CM.
\]

(3.26)
Using the method in (3.25) and substituting (3.26) to Lemma 2.6 with \( \phi = 1 \), we find

\[
\int_{\mathbb{R}^3} \left( |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + \frac{|\nabla (Q_L - \pi(Q_L))|^2}{L} \right) (x, s) \, dx \\
+ \frac{1}{2} \int_{T_0}^{T_1} \int_{\mathbb{R}^3} \left| \nabla^3 Q_L \right|^2 + |\nabla^2 v_L|^2 + |\nabla \partial_t Q_L|^2 + \frac{|\nabla^2 (Q_L - \pi(Q_L))|^2}{L} \, dx \, dt
\]

\[
\leq CM + C\varepsilon_0^2 \int_{\mathbb{R}^3} |\nabla^2 Q_L(x, T_M)|^2 \, dx + \frac{C\varepsilon_0^2}{R_0^2} \int_{\mathbb{R}^3} |\nabla Q_L(x, T_M)|^2 + |\nabla Q_L|_0^2 \, dx \\
+ \frac{C\varepsilon_0^2}{R_0^2} \int_{T_0}^{T_1} \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 + |\nabla v_L|^2 + |\partial_t Q_L|^2 + |\nabla (Q_L - \pi(Q_L))|^2 \, dx \, dt \\
+ \frac{C\varepsilon_0^2}{R_0^2} \int_{T_0}^{T_1} \int_{\mathbb{R}^3} |\nabla Q_L|^4 + |v_L|^4 + \frac{|Q_L - \pi(Q_L)|^4}{L^2} \, dx \, dt \leq C \left(1 + \frac{\varepsilon_0^2}{R_0^2}\right) M.
\]  

(3.27)

Combining (3.26) with (3.27), we prove (3.19).

\[\square\]

**Proof of Theorem 2** By using Proposition 3.1 and Lemma 3.2, there exist two uniform positive constants \( T_* \) and \( L_* \) such that for any \( L \leq L_* \), the strong solution \((Q_L, v_L)\) to (1.11)-(1.13) satisfies

\[
\sup_{0 \leq t \leq T_*} \left( \|\nabla Q_L(t)\|^2_{H^1(\mathbb{R}^3)} + \|v_L(t)\|^2_{H^1(\mathbb{R}^3)} + \frac{1}{L} \|Q_L(t) - \pi(Q_L(t))\|^2_{H^1(\mathbb{R}^3)} \right) \\
+ \|\partial_t Q_L\|^2_{L^2(0, T_*; H^1(\mathbb{R}^3))} + \|\nabla^2 Q_L\|^2_{L^2(0, T_*; H^1(\mathbb{R}^3))} \\
+ \|\nabla v_L\|^2_{L^2(0, T_*; H^1(\mathbb{R}^3))} + \frac{1}{L} \|Q_L - \pi(Q_L)\|^2_{H^2(\mathbb{R}^3)} \leq C \left(1 + \frac{\varepsilon_0^2}{R_0^2}\right) M.
\]  

(3.28)

Since the pressure \( P_L \) satisfies (3.3), using (3.28), we find

\[
\int_0^{T_*} \int_{\mathbb{R}^3} |P_L|^2 \, dx \, dt \leq \int_0^{T_*} \int_{\mathbb{R}^3} |\nabla Q_L|^4 + |v_L|^4 + |\nabla^2 Q_L|^2 \, dx \, dt \leq C
\]  

(3.29)

and

\[
\int_0^{T_*} \int_{\mathbb{R}^3} |\nabla P_L|^2 \, dx \, dt \\
\leq C \int_0^{T_*} \int_{\mathbb{R}^3} \left( \|\nabla (Q_L, \mathcal{H}(Q_L, \nabla Q_L))\|^2 + |\nabla (Q_L, \nabla Q_L)|^2 + |\nabla (v_L \otimes v_L)|^2 \right) \, dx \, dt \\
\leq C \int_0^{T_*} \int_{\mathbb{R}^3} |\nabla^3 Q_L|^2 + |\nabla^2 Q_L|^2 |\nabla Q_L|^2 + |\nabla Q_L|^6 + |\nabla v_L|^2 |v_L|^2 \, dx \, dt \leq C.
\]  

(3.30)

Multiplying (1.11) with \((Q_L - Q_e)\), one can show that \((Q_L - Q_e) \in L^\infty(0, T_1; L^2(\mathbb{R}^3))\). Then, letting \( L \to 0 \) (up to a subsequence) and utilizing the Aubin-Lions Lemma,
we have

\[ Q_L \to Q \text{ in } L^2(0, T_1; H^2_0(\mathbb{R}^3)) \cap H^1(0, T_1; H^2_0(\mathbb{R}^3)), \]
\[ \partial_t Q_L \to \partial_t Q \text{ in } L^2(0, T_1; H^1(\mathbb{R}^3)), \]
\[ v_L \to v \text{ in } L^2(0, T_1; H^2(\mathbb{R}^3)) \cap H^1(\mathbb{R}^3 \times (0, T_1)), \]
\[ \partial_t v_L \to \partial_t v \text{ in } L^2(0, T_1; L^2(\mathbb{R}^3)), \]
\[ P_L \to P \text{ in } L^2(0, T_1; H^1(\mathbb{R}^3)), \]
\[ (\nabla Q_L, v_L) \to (Q, v) \text{ in } L^2(0, T_1; H^1(B_R(0))) \cap C([0, T_1]; L^2(B_R(0))) \]

for any \( R \in (0, \infty) \). We claim

\[ \lim_{L \to 0} \frac{1}{L} \int_{\mathbb{R}^3} f_B(Q_L(x, s)) \, dx = 0 \]  \hspace{1cm} (3.31)

for all \( s \in [0, T_1] \). To prove this claim, we need to estimate the \( L^2 \)-norm of \( \partial_t Q_L \).

We differentiate (2.36) in \( t \), multiply by \( R^T_{Q_L} \partial_t Q_L R_{Q_L} \). To estimate the right-hand side term on \( g_B(Q_L) \), we apply Lemma 2.1 with \( \xi = \partial_t Q_L \) and Lemma 2.2 to obtain

\[
\int_0^s \int_{\mathbb{R}^3} \left\langle \partial_t Q_L, R^T_{Q_L} \partial_t Q_L R_{Q_L} \right\rangle \, dx \, dt \\
\leq -\frac{1}{L} \int_0^s \int_{\mathbb{R}^3} \partial^2_{Q_{ij}} Q_{kl} f_B(Q_L) \partial_t (Q_L)_{ij} \partial_t (Q_L)_{kl} \, dx \, dt \\
+ C \int_0^s \int_{\mathbb{R}^3} \left| \partial_t Q_L \right| \frac{|Q_L - \pi(Q_L)|}{L^2} \, dx \, dt \\
\leq -\frac{\lambda}{4} \int_0^s \int_{\mathbb{R}^3} \left| \partial_t Q_L \right|^2 \, dx \, dt + C \int_{\mathbb{R}^3} \frac{|Q_L - \pi(Q_L)|^2}{L} \left| \partial_t Q_L \right|^2 \, dx \leq CM. \]  \hspace{1cm} (3.32)

Integrating by part and using (3.18), (3.28) and Young’s inequality, we have

\[
\int_0^s \int_{\mathbb{R}^3} \left\langle \partial_t R^T_{Q_L} H(Q_L, \nabla Q_L) R_{Q_L}, R^T_{Q_L} \partial_t Q_L R_{Q_L} \right\rangle \, dx \, dt \\
= \int_0^s \int_{\mathbb{R}^3} \left\langle \partial_t \nabla k[\partial_{\rho_{ij}} f_E(Q_L, \nabla Q_L) - \partial_{Q_{ij}} f_E(Q_L, \nabla Q_L)], \partial_t Q_L \right\rangle \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \left\langle \partial_t R^T_{Q_L} \nabla k[\partial_{\rho_{ij}} f_E(Q_L, \nabla Q_L) - \partial_{Q_{ij}} f_E(Q_L, \nabla Q_L)], \partial_t Q_L R_{Q_L} \right\rangle \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \left\langle \nabla k[\partial_{\rho_{ij}} f_E(Q_L, \nabla Q_L) - \partial_{Q_{ij}} f_E(Q_L, \nabla Q_L)], \partial_t R_{Q_L}, R^T_{Q_L} \partial_t Q_L \right\rangle \, dx \, dt \\
\leq C \int_0^s \int_{\mathbb{R}^3} \left| \partial_t Q_L \right|^2 |\nabla Q_L|^2 + |\nabla \partial_t Q_L|^2 \, dx \, dt \leq CM. \]  \hspace{1cm} (3.33)
Similarly, for the left-hand side, we find
\[
\int_0^s \int_{\mathbb{R}^3} \langle \partial_t (R_{Q_L}^T (\partial_t Q_L + (v \cdot \nabla Q_L) + [Q_L, \Omega_L]) R_{Q_L}) \partial_t Q_L R_{Q_L} \rangle \ dx \ d t \\
\geq \int_0^s \int_{\mathbb{R}^3} \frac{1}{2} |\partial_t R_{Q_L}^T \partial_t Q_L R_{Q_L}|^2 \ dx \ d t - C \int_0^s \int_{\mathbb{R}^3} |\nabla \partial_t Q_L|^2 + |\partial_t v_L|^2 \ dx \ d t \\
- C \int_0^s \int_{\mathbb{R}^3} |\partial_t Q_L|^2 (|\nabla Q_L|^2 + |v_L|^2) \ dx \ d t \\
\geq \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t Q_L(x, s)|^2 \ dx - \frac{1}{2} \int_{\mathbb{R}^3} |\partial_t Q_0|^2 \ dx - CM. \quad (3.34)
\]

Here we used $\int_0^s \int_{\mathbb{R}^3} |\partial_t v_L|^2 \ dx \ d t \leq CM$ due to (1.11).

In view of (3.32)–(3.34), we derive
\[
\sup_{0 \leq s \leq T_1} \int_{\mathbb{R}^3} |\partial_t Q_L(x, s)|^2 \ dx \leq \int_{\mathbb{R}^3} |\partial_t Q_L(x, 0)|^2 \ dx + CM \leq CM, \quad (3.35)
\]
where we used (1.13), (3.15), (3.28), (3.30) and $g_B(Q_0) = 0$ for $Q_0 \in S_\delta$.

Using (1.13), (3.35) and (3.28), we have
\[
\int_{\mathbb{R}^3} \frac{|g_B(Q_L)|^2}{L^2} \ dx = \int_{\mathbb{R}^3} \frac{|g_B(Q_L^+)|^2}{L^2} \ dx \\
\leq C \int_{\mathbb{R}^3} |\partial_t Q_L|^2 + |v_L|^2 |\nabla Q_L|^2 + |Q_L|^2 |\nabla v_L|^2 + |\nabla^2 Q_L|^2 + |\nabla Q_L|^4 \ dx \leq CM. \quad (3.36)
\]

For any $Q_L \in S_\delta$ with a sufficiently small $\delta > 0$, we take the Taylor expansion and adapt the argument in (2.17) with Corollary 1 to obtain
\[
0 = f_B(Q^+) = f_B(Q_L) + \nabla Q_{ij} f_B(Q_L)(Q_L - Q^+)_ij \\
+ \partial_Q^2 Q_{ij} f_B(Q_L)(Q_L - Q^+)_ij (\tilde{Q} - Q^+)_kl \\
\geq f_B(Q_L) - g_B(Q_L)_ij (\tilde{Q} - Q^+)_ij, \quad (3.37)
\]
where $Q^+$ is an intermediate point between $Q_L$ and $Q^+$. Then using (3.37) and (3.36), we find
\[
\frac{1}{L} \int_{\mathbb{R}^3} f_B(Q_L) \ dx = \frac{1}{L} \int_{\mathbb{R}^3} f_B(Q_L) \ dx \leq \frac{1}{L} \int_{\mathbb{R}^3} \langle g_B(Q_L), Q^+ - Q_L \rangle \ dx \\
\leq \left( \int_{\mathbb{R}^3} \frac{|g_B(Q_L)|^2}{L^2} \ dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\pi(Q_L) - Q_L|^2 \ dx \right)^{\frac{1}{2}} \leq C \left( \int_{\mathbb{R}^3} |Q_L - \pi(Q_L)|^2 \ dx \right)^{\frac{1}{2}}.
\]

Using (3.28) that $\|Q_L(s) - \pi(Q_L(s))\|^2_{L^2(\mathbb{R}^3)} \leq CL$ for $s \in [0, T_1)$, we have
\[
\lim_{L \to 0} \|Q_L(s) - \pi(Q_L(s))\|_{L^\infty(0, T_1; L^2(\mathbb{R}^3))} = 0.
\]

Then, we prove the claim (3.33).

For $\varphi \in C_0^\infty(\mathbb{R}^3, S_0)$, we define
\[
\Phi_{ij}(Q, \varphi) = s_{ij}^{-1}(Q_{ij} + \frac{s_{ij} + \delta_{ij}}{3})\varphi_{ij} + s_{ij}^{-1}(Q_{ij} + \frac{s_{ij} + \delta_{ij}}{3})\varphi_{ij} \\
- 2s_{ij}^{-2}(Q_{ij} + \frac{s_{ij} + \delta_{ij}}{3})(Q_{lm} + \frac{s_{ij} + \delta_{ij}}{3})\varphi_{lm}.
\]
As $L \to 0$, we multiply (1.13) by $\Phi_{ij}(Q, \varphi)$ and obtain
\[
((\partial_t + v \cdot \nabla)Q + [Q, \Omega], \Phi(Q, \varphi)) = (\partial_t + v \cdot \nabla)(s_+ u_i u_j)(2u_j u_i \varphi_{ij})
\]
\[
+ \left( s_+(u_i u_k - \frac{1}{3} \delta_{ik})\Omega_{kj} - s_+(u_k u_j - \frac{1}{3} \delta_{kj})\Omega_{ik} \right) \Phi_{ij}(Q, \varphi)
\]
\[
= ((\partial_t + v \cdot \nabla)Q + [Q, \Omega], \varphi),
\]
(3.38)
where we used that $|Q| = \sqrt{\frac{3}{2}} s_+$ and $|u| = 1$. Multiplying (1.13) with $\Phi_{ij}(Q, \varphi)$ and using the condition (3.31), we can apply Theorem 2 in [10] that the solution $(Q_L, v_L)$ of (1.11)-(1.13) converges to a solution $(Q, v)$ of (1.17)-(1.19) as $L \to 0$. Taking the difference between two solutions under $L^2$ estimates, it can be shown (c.f. [15] or [11]) that the strong solution $(Q, v)$ is unique. The proof on uniqueness is similar to the claim 2 in the appendix, so we omit the details here. Let $T_1$ be any time with $T_1 \leq T^*$. Under the criteria
\[
\sup_{0 \leq t \leq T_1, x \in \mathbb{R}^3} \int_{B_{R_0}(x_0)} (|\nabla Q|^2 + |v|^2) \, dx \leq \varepsilon_0^3,
\]
we can use the same methods in [15] to extend the solution passing $T_1$ up to the maximal time $T^*$.

4. Smooth convergence

In this section, we will prove Theorem 3. At first, we obtain the following higher order estimate:

Lemma 4.1. Let $(Q_L, v_L)$ be a strong solution of (1.11)-(1.13) in $\mathbb{R}^3 \times [T_0, T_M]$ with initial value $(Q_{L,T_0}, v_{L,T_0}) \in H^2_0(\mathbb{R}^3) \times H^1(\mathbb{R}^3)$ and $\text{div } v = 0$. For any $\tau > T_0$, $s \in (\tau, T_M]$ and any integer $m \geq 0$, there exists a positive constant $C_m$ independently of $Q_L$ and $L$ (but depending on $m$) such that

\[
\sup_{\tau \leq s \leq T_M} \int_{\mathbb{R}^3} \left( |\nabla^{m+1} Q_L|^2 + |\nabla^m v_L|^2 + \frac{1}{L} |\nabla^m (Q_L - \pi(Q_L))|^2 \right) (x, t) \, dx
\]
\[
+ \int_{\tau}^{T_M} \int_{\mathbb{R}^3} |\nabla^{m+2} Q_L|^2 + |\nabla^m v_L|^2 + |\nabla^m \partial_t Q_L|^2 \, dx \, dt
\]
\[
+ \int_{\tau}^{T_M} \int_{\mathbb{R}^3} \frac{1}{L} |\nabla^{m+1} (Q_L - \pi(Q_L))|^2 \, dx \, dt \leq C_m.
\]
(4.1)

Proof. We prove this lemma by induction. In view of (3.26) and (3.27), one have shown (4.1) holds for $m = 0, 1$. Assume that (4.1) holds for $m = 1, \ldots, k$ with $k \geq 1$. Then we have

\[
\sup_{\tau \leq s \leq T_M} \sum_{i=0}^{k} \int_{\mathbb{R}^3} \left( |\nabla^{i+1} Q_L|^2 + |\nabla^i v_L|^2 + \frac{1}{L} |\nabla^i (Q_L - \pi(Q_L))|^2 \right) (x, s) \, dx
\]
\[
+ \sum_{i=0}^{k} \int_{\tau}^{T_M} \int_{\mathbb{R}^3} |\nabla^{i+2} Q_L|^2 + |\nabla^{i+1} v_L|^2 + |\nabla^i \partial_t Q_L|^2 \, dx \, dt
\]
\[
+ \sum_{i=0}^{k} \int_{\tau}^{T_M} \int_{\mathbb{R}^3} \frac{1}{L} |\nabla^{i+1} (Q_L - \pi(Q_L))|^2 \, dx \, dt \leq C_k(\tau).
\]
(4.2)
For \( m = k \), it follows from using (4.2) and the mean value theorem that there exists a \( \tau_L \in (\tau/2, \tau) \) such that

\[
\int_{\mathbb{R}^3} \left( |\nabla^{k+2} Q_L|^2 + |\nabla^{k+1} v_L|^2 + \frac{1}{L} |\nabla^k (Q_L - \pi(Q_L))|^2 \right) (x, \tau_L) \, dx \leq C_k(\tau). \tag{4.3}
\]

Applying the Sobolev inequality to (4.2), we obtain

\[
\sup_{\tau_L \leq s \leq T_M} \sum_{i=0}^{k-1} \|\nabla^i (Q_L - Q_\ast)(s)\|_{L^\infty(\mathbb{R}^3)} \leq C_k(\tau). \tag{4.4}
\]

For functions \( f_1, f_2 \in H^1(\mathbb{R}^3) \), by using a similar argument to one in (5.22), we observe

\[
\int_{\mathbb{R}^3} |f_1|^2 |f_2|^2 \, dx \leq C \|f_1\|^2_{H^1(\mathbb{R}^3)} \|\nabla f_2\|^2_{L^2(\mathbb{R}^3)}, \tag{4.5}
\]

\[
\int_{\mathbb{R}^3} |f_1|^2 |f_2|^4 \, dx \leq C \|\nabla f_1\|^2_{L^2(\mathbb{R}^3)} \|\nabla f_2\|^4_{L^2(\mathbb{R}^3)}. \tag{4.6}
\]

Next, we show that (4.1) holds for \( m = k + 1 \).

In order to derive the \( L^2 \)-norm of \( \nabla^{k+3} Q_L \), we apply \( \nabla^k \nabla_\beta \) to (2.36) and multiply by \( \nabla^{k+2} (R^T_{Q_L} \nabla_\beta Q_L R_{Q_L}) \) to obtain

\[
J_1 := \int_{\mathbb{R}^3} \langle \nabla^k \nabla_\beta \left( R^T_{Q_L} (\partial_t Q_L + v_L \cdot \nabla Q_L + [Q_L, \Omega_L]) R_{Q_L} \right), \nabla^{k+2} (R^T_{Q_L} \nabla_\beta Q_L R_{Q_L}) \rangle \, dx
\]

\[
= \int_{\mathbb{R}^3} \langle \nabla^k \nabla_\beta \left( R^T_{Q_L} \mathcal{H}(Q_L, \nabla Q_L) R_{Q_L} \right), \nabla^{k+2} (R^T_{Q_L} \nabla_\beta Q_L R_{Q_L}) \rangle \, dx
\]

\[
+ \frac{1}{L} \int_{\mathbb{R}^3} \langle \nabla^k \nabla_\beta g_B(Q_L), \nabla^{k+2} (R^T_{Q_L} \nabla_\beta Q_L R_{Q_L}) \rangle \, dx =: J_2 + J_3. \tag{4.7}
\]

For the term \( J_2 \), we have

\[
J_2 \geq \int_{\mathbb{R}^3} \langle \nabla^k \nabla_\beta \nabla_\nu \left( \delta^2_{\mu \nu} f_E \right), \nabla^{k+2} \nabla_\beta Q_L \rangle \, dx - \frac{\alpha}{8} \int_{\mathbb{R}^3} |\nabla^{k+3} Q_L|^2 \, dx
\]

\[
- C \int_{\mathbb{R}^3} \nabla^{k+1} \partial Q f E \|^2 \, dx - \eta \int_{\mathbb{R}^3} |\nabla^{k+2} (R^T_{Q_L} \nabla_\beta Q_L R_{Q_L})|^2 \, dx
\]

\[
- C \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 + \mu_3 = k} |\nabla^{\mu_1} \mathcal{H}(Q_L, \nabla Q_L)|^2 |\nabla^{\mu_2} R_{Q_L}|^2 |\nabla^{\mu_3} R_{Q_L}|^2 \, dx \tag{4.8}
\]

for some small \( \eta \) to be chosen later. We deduce the first term on the right-hand side in (4.8) from (2.2) that

\[
\int_{\mathbb{R}^3} \nabla^k \nabla_\beta \left( \nabla_\nu \partial_{\mu \nu} f_E \right) \nabla^{k+2} \nabla_\beta (Q_L)_{ij} \, dx
\]

\[
= \int_{\mathbb{R}^3} \nabla^k \nabla_\nu \left( \delta^2_{\mu \nu} \partial_{\mu \nu} f_E \right) \nabla^{k+2} \nabla_\beta (Q_L)_{ij} \, dx
\]

\[
+ \int_{\mathbb{R}^3} \nabla^k \nabla_\nu \left( \delta^2_{\mu \nu} Q_{mn} f_E \nabla_\beta (Q_L)_{mn} \right) \nabla^{k+2} \nabla_\beta (Q_L)_{ij} \, dx
\]

\[
+ \int_{\mathbb{R}^3} \partial^2_{\mu \nu} Q_{mn} f_E \nabla^{k+1} \nabla_\beta (Q_L)_{mn} \nabla^{k+1} \nabla^2 \nabla_\beta (Q_L)_{ij} \, dx
\]

\[
- C \int_{\mathbb{R}^3} |\nabla^{k+3} Q_L| \sum_{\mu_1 + \mu_2 = k} |\nabla^{\mu_1} \nabla^2 \partial_{\mu \nu} f_E | |\nabla^{\mu_2} \nabla^2 Q_L| \, dx
\]
Similarly, we can estimate the remaining terms in (4.4)-(4.6), the last term in (4.8) becomes

\[ \int_{\mathbb{R}^3} |\nabla^{k+2} (R_{QL}^2 Q_\lambda R_{QL})|^2 \, dx \]

\[ \leq C \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 + k = k + 2} |\nabla^{\mu_1} \nabla Q_L|^2 |\nabla^{\mu_2} R_{QL}|^2 |\nabla^{\mu_3} R_{QL}|^2 \, dx \]

\[ \leq C \int_{\mathbb{R}^3} |\nabla^{k+3} Q_L|^2 \, dx + C \|\nabla Q_L(x)\|_{L^\infty(\mathbb{R}^3)}^2 \int_{\mathbb{R}^3} |\nabla^{k+2} Q_L|^2 \, dx \]

\[ + C \left( \int_{\mathbb{R}^3} |\nabla^{k+2} Q_L|^2 \, dx + |\nabla^{k+1} Q_L|^2 \, dx \right) \left( \int_{\mathbb{R}^3} |\nabla^{k+2} Q_L|^2 \, dx + 1 \right) \]

\[ + C \left( \int_{\mathbb{R}^3} |\nabla^{k+2} Q_L|^2 \, dx \right) \left( \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 \, dx \right)^2 \]

\[ + C \|\nabla^k Q_L(x)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |\nabla^{k+2} Q_L|^2 + |\nabla^{k+1} Q_L|^2 |\nabla^k Q_L|^2 \, dx \]

\[ + C \|\nabla^k Q_L(x)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 \, dx \left( \int_{\mathbb{R}^3} |\nabla^2 Q_L|^2 \, dx \right)^2 \]

\[ \leq C \|\nabla^{k+3} Q_L\|_{L^2(\mathbb{R}^3)}^2 + C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1)^2. \]  \( (4.10) \)

Similarly, we can estimate the remaining terms in (4.4)-(4.6) as follows:

\[ \int_{\mathbb{R}^3} |\nabla^k \nabla (\partial_q f_E)|^2 + \sum_{\mu_1 + \mu_2 + \mu_3 = k} |\nabla^{\mu_1} \mathcal{H} (Q_L, \nabla Q_L)|^2 |\nabla^{\mu_2} R_{QL}|^2 |\nabla^{\mu_3} R_{QL}|^2 \, dx \]

\[ + \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 = k} |\nabla^{\mu_1} \nabla^2 \nabla f_E|^2 |\nabla^{\mu_2} \nabla^2 Q_L|^2 \, dx \]

\[ + \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 = k+1} |\nabla^{\mu_1} \nabla^2 f_E|^2 |\nabla^{\mu_2} \nabla Q_L|^2 \, dx \leq C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1)^2. \]  \( (4.11) \)

Substituting (4.4)-(4.11) into (4.8) and choosing sufficiently small \( \eta \), we have

\[ J_2 \geq \frac{\alpha}{4} \|\nabla^{k+3} Q_L\|_{L^2(\mathbb{R}^3)}^2 - C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1)^2. \]  \( (4.12) \)
To estimate $J_3$, utilizing (4.12), the Sobolev inequality and (4.10)-(4.14), it follows from Corollary 1 that

$$
\frac{1}{L} \int_{\mathbb{R}^3} \partial_{Q_{ij}Q_{kl}} f_B(\bar{Q}_L) \nabla^{k+2}(\bar{Q}_L)_{ij} \nabla^{k+2}(\bar{Q}_L)_{kl} \, dx
\geq \frac{1}{L} \int_{\mathbb{R}^3} \frac{\lambda}{2} \nabla^{k+2}(Q_L - \pi(Q_L))^2 \, dx - \frac{C}{L} \int_{\mathbb{R}^3} \nabla^{k+1}(Q_L - \pi(Q_L))^2 |\nabla Q_L|^2 \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} |\nabla^k(Q_L - \pi(Q_L))|^2 (|\nabla^2 Q_L|^2 + |\nabla Q_L|^4) \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} |\nabla^{k-1}(Q_L - \pi(Q_L))|^2 (|\nabla^3 Q_L|^2 + |\nabla^2 Q_L|^2 |\nabla Q_L|^2 + |\nabla Q_L|^6) \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} \sum_{\max\{\mu_{k-2}, \mu_{k-1}\} \leq k-1} |\nabla^{\mu_1}(Q_L - \pi(Q_L))|^2 |\nabla^{\mu_2} Q_L|^2 \cdots |\nabla^{\mu_{k+3}} Q_L|^2 \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} |\nabla^2(Q_L - \pi(Q_L))|^2 (|\nabla^k Q_L|^2 + 1) \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} |\nabla^2(Q_L - \pi(Q_L))|^2 (|\nabla^{k+1} Q_L|^2 + |\nabla^k Q_L|^2 |\nabla Q_L|^2 + 1) \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} |Q_L - \pi(Q_L)|^2 (|\nabla^{k+2} Q_L|^2 + |\nabla^{k+1} Q_L|^2 |\nabla Q_L|^2) \, dx
- \frac{C}{L} \int_{\mathbb{R}^3} |Q_L - \pi(Q_L)|^2 |\nabla^k Q_L|^2 (|\nabla^2 Q_L|^2 + |\nabla Q_L|^4 + 1) \, dx
\geq \frac{\lambda}{2L} \|\nabla^{k+2}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2
- C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1) \left( \frac{1}{L} \|\nabla^{k+1}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + 1 \right). \quad (4.13)
$$

Repeating the argument in (4.10), one has

$$
\frac{1}{L} \|\nabla^{k+2} \tilde{Q}_L\|_{L^2(\mathbb{R}^3)}^2 \geq \frac{1}{2L} \|\nabla^{k+2}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2
- C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1) \left( \frac{1}{L} \|\nabla^{k+1}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + 1 \right). \quad (4.14)
$$

Applying Lemma 2.2 to $J_3$ and combining with (4.13)-(4.14) yields

$$
J_3 = \frac{(-1)^{k+2}}{L} \int_{\mathbb{R}^3} \left< \nabla^{2k+2} \nabla_\beta (g_B(\bar{Q}_L)), \nabla_\beta \bar{Q}_L \right> \, dx
+ \frac{(-1)^{k+2}}{L} \int_{\mathbb{R}^3} \left< \nabla^{2k+2} \nabla_\beta (g_B(\bar{Q}_L)), \nabla_\beta R^T_{Q_L}(\pi(Q_L) - Q_L) R_{Q_L} \right> \, dx
+ \frac{(-1)^{k+2}}{L} \int_{\mathbb{R}^3} \left< \nabla^{2k+2} \nabla_\beta (g_B(\bar{Q}_L)), R^T_{Q_L}(\pi(Q_L) - Q_L) \nabla_\beta R_{Q_L} \right> \, dx
\geq \frac{1}{L} \int_{\mathbb{R}^3} \partial_{Q_{ij}Q_{kl}}^2 f_B(\bar{Q}_L) \nabla^{k+2}(\bar{Q}_L)_{ij} \nabla^{k+2}(\bar{Q}_L)_{kl} \, dx
- C \int_{\mathbb{R}^3} \frac{|\nabla^{k+2} \tilde{Q}_L|}{L^2} \sum_{\mu_1 + \mu_2 = k} |\nabla^{\mu_1} \nabla_\beta^2 f_B(\bar{Q}_L)| \frac{|\nabla^{\mu_2} \bar{Q}_L|}{L^2} \, dx
\geq \frac{\lambda}{2L} \|\nabla^{k+2}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2
- C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1) \left( \frac{1}{L} \|\nabla^{k+1}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + 1 \right). \quad (4.13)
$$
\[ -C \int_{\mathbb{R}^3} \frac{|\nabla^{k+2} g_{\beta}(Q_L)|}{L^2} \sum_{\mu_1 < \mu_2 \leq \mu_3} \frac{|\nabla^{\mu_1} (Q - \pi(Q_L))|}{L^2} \frac{|\nabla^{\mu_2} R^T_{Q_L}|}{L^2} \frac{|\nabla^{\mu_3} R_{Q_L}|}{L^2} \, dx \]

\[ \geq \frac{\lambda}{4L} \|\nabla^{k+2} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \]

\[ - C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1) \left( \frac{1}{L} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + 1 \right), \quad (4.15) \]

where we used that

\[ C \int_{\mathbb{R}^3} \frac{\|\nabla^{k+2} g_{\beta}(Q_L)\|^2}{L} \, dx \leq C \|\nabla^{k+2} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \]

\[ + C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1) \left( \frac{1}{L} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + 1 \right). \]

Applying (4.12), (4.14)-(4.16) to the left-hand side of (4.17) and then substituting (4.10), we obtain

\[ J_1 \leq \int_{\mathbb{R}^3} \langle \nabla^k \nabla_{\beta} \partial_t Q_L, \nabla^k \nabla_{\beta} Q_L \rangle \, dx + \eta \int_{\mathbb{R}^3} \|\nabla^{k+2} (R^T_{Q_L} \nabla_{\beta} Q_L R_{Q_L})\|^2 \, dx \]

\[ + C(\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1) \left( \frac{1}{L} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + 1 \right). \]

(4.16)

Combining \(J_2, J_3\) with \(J_1\) and integrating in \(t\), we find

\[ \frac{1}{2} \|\nabla^{k+2} Q_L(s)\|_{L^2(\mathbb{R}^3)}^2 + \int_{T_L} \frac{\alpha}{8} \|\nabla^{k+3} Q_L\|_{L^2(\mathbb{R}^3)}^2 \, dt + \frac{\lambda}{4L} \|\nabla^{k+2} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \]

\[ \leq C \int_{T_L} \|\nabla^{k+2} v_L\|_{L^2(\mathbb{R}^3)}^2 \, dt + \int_{T_L} \|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 \, dt + \frac{1}{L} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \, dt \]

\[ + C \int_{T_L} \|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 \, dt + C. \]

(4.17)

To estimate the \(L^2\)-norm of \(\nabla^{k+1} \partial_t Q_L\), we applying \(\nabla^{k+1}\) to (2.36) and multiplying by \(\nabla^{k+1} (R^T_{Q_L} \partial_t Q_L R_{Q_L})\) to have

\[ J_4 := \int_{T_L} \int_{\mathbb{R}^3} \langle \nabla^{k+1} (R^T_{Q_L} (\partial_t Q_L + v_L \cdot \nabla Q_L) R_{Q_L}^T) \rangle \langle \nabla^{k+1} (R^T_{Q_L} \partial_t Q_L R_{Q_L}) \rangle \, dxdt \]

\[ + \int_{T_L} \int_{\mathbb{R}^3} \langle \nabla^{k+1} (R^T_{Q_L} [Q_L, \Omega_L]) R_{Q_L} \rangle \langle \nabla^{k+1} (R^T_{Q_L} \partial_t Q_L R_{Q_L}) \rangle \, dxdt \]

\[ = \int_{T_L} \int_{\mathbb{R}^3} \langle \nabla^{k+1} (R^T_{Q_L} H(Q_L, \nabla Q_L) R_{Q_L}) \rangle \langle \nabla^{k+1} (R^T_{Q_L} \partial_t Q_L R_{Q_L}) \rangle \, dxdt \]
Using (2.2), (4.11), (4.19)-(4.20) and (4.3) to \( J_5 \) and \( J_6 \).

\[
\int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left< \nabla^{k+1} \frac{1}{L} g_B(\bar{Q}_L), \nabla^{k+1} (R_{Q_L}^T \partial_t \bar{Q}_L R_{Q_L}) \right> \, dx \, dt =: J_5 + J_6. \tag{4.18}
\]

Repeating the same argument in (4.10), we observe

\[
\| \nabla^{k+1} (R_{Q_L}^T \partial_t \bar{Q}_L R_{Q_L}) \|_{L^2(\mathbb{R}^3)}^2 \leq C \| \nabla^{k+1} \partial_t \bar{Q}_L \|_{L^2(\mathbb{R}^3)}^2 + C \| \partial_t \bar{Q}_L \|_{H^k(\mathbb{R}^3)} \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 + 1 \tag{4.19}
\]

and

\[
\| \nabla^{k+1} \mathcal{H}(Q_L, \nabla Q_L) \|_{L^2(\mathbb{R}^3)}^2 \leq C \| \nabla^{k+3} Q_L \|_{L^2(\mathbb{R}^3)}^2 + C \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 + 1 \tag{4.20}
\]

Using (2.2), (4.11), (4.19)-(4.20) and (4.3) to \( J_5 \), we obtain

\[
J_5 \leq \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left< \nabla^{k+1} \partial_\beta (\nabla_\nu \partial_\rho f_E - \partial_\rho f_E), \nabla^{k+1} \partial_\beta \partial_t \bar{Q}_L \right> \, dx \, dt
+ C \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \mathcal{H}(Q_L, \nabla Q_L) \right| \sum_{\mu_1 + \mu_2 + \mu_3 = k} |\nabla^{\mu_1} \partial_t \bar{Q}_L| |\nabla^{\mu_2} \nabla R_{Q_L}| |\nabla^{\mu_3} R_{Q_L}| \, dx \, dt
+ C \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \mathcal{H}(Q_L, \nabla Q_L) \right| \sum_{\mu_1 + \mu_2 + \mu_3 = k} |\nabla^{\mu_1} \mathcal{H}| |\nabla^{\mu_2} \nabla R_{Q_L}| |\nabla^{\mu_3} R_{Q_L}| \, dx \, dt
\leq \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \partial_\beta^2 \tilde{p}_{ij} \tilde{p}_{mn} f_E \nabla^{k+1} \nabla^{3} \mathcal{H}(Q_L)_{mn} \nabla^{k+1} \partial_\beta \partial_t \bar{Q}_L_{ij} \, dx \, dt
+ C \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \partial_\beta \partial_t \bar{Q}_L \right| \sum_{\mu_1 + \mu_2 = k} |\nabla^{\mu_1} \partial_\beta \partial_\rho f_E| |\nabla^{\mu_2} \nabla^2 Q_L| \, dx \, dt
+ C \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \partial_\beta \partial_t \bar{Q}_L \right| \left( |\nabla^{k+1} \left( \partial_\beta^2 \tilde{p}_{ij} f_E \cdot \nabla Q_L \right) | + |\nabla^{k+1} \partial_\beta f_E| \right) \, dx \, dt
+ \eta \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \mathcal{H}(Q_L, \nabla Q_L) \right|^2 + \left| \nabla^{k+1} \left( R_{Q_L}^T \partial_t \bar{Q}_L R_{Q_L} \right) \right|^2 \, dx \, dt
+ C(\eta) \int_{\tau_0}^{s} \left( \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 + \| \partial_\beta \bar{Q}_L \|_{H^k(\mathbb{R}^3)} \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 \right) \, dt + C
\leq - \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \frac{1}{2} \partial_t \left( \tilde{p}_{ij} \tilde{p}_{mn} f_E \nabla^{k+1} \nabla_{ij} (Q_L)_{mn} \nabla^{k+1} \nabla_\nu (Q_L)_{ij} \right) \, dx \, dt
+ \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \frac{1}{2} \partial_t \partial_\beta^2 \tilde{p}_{ij} \tilde{p}_{mn} f_E \nabla^{k+1} \nabla_{ij} (Q_L)_{mn} \nabla^{k+1} \nabla_\nu (Q_L)_{ij} \, dx \, dt
- \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \partial_\beta^2 \tilde{p}_{ij} \tilde{p}_{mn} f_E \nabla^{k+1} \nabla_{ij} (Q_L)_{mn} \nabla^{k+1} \nabla_\nu (Q_L)_{ij} \, dx \, dt
+ \eta \int_{\tau_0}^{s} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \mathcal{H}(Q_L, \nabla Q_L) \right|^2 + \left| \nabla^{k+1} \left( R_{Q_L}^T \partial_t \bar{Q}_L R_{Q_L} \right) \right|^2 + \left| \nabla^{k+1} \partial_\beta \partial_t \bar{Q}_L \right|^2 \, dx \, dt
+ C(\eta) \int_{\tau_0}^{s} \left( \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 + \| \partial_\beta \bar{Q}_L \|_{H^k(\mathbb{R}^3)} \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 \right) \, dt + C
\leq - \frac{\alpha}{4} \| \nabla^{k+2} Q_L(s) \|_{L^2(\mathbb{R}^3)}^2 + \int_{\tau_0}^{s} \frac{\alpha}{8} \| \nabla^{k+3} Q_L \|_{L^2(\mathbb{R}^3)}^2 \, dt + \frac{1}{8} \| \nabla^{k+1} \partial_t \bar{Q}_L \|_{L^2(\mathbb{R}^3)}^2 \, dt
+ C \int_{\tau_0}^{s} \left( \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 + \| \partial_\beta \bar{Q}_L \|_{H^k(\mathbb{R}^3)} \| \nabla^{k+2} Q_L \|_{L^2(\mathbb{R}^3)}^2 \right) \, dt + C. \tag{4.21}
\]
Using (4.19), we deduce the left-hand side of (4.18) to

\[ J_6 = \frac{1}{L} \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \nabla^{k+1} \partial_t (\tilde{Q}(Q_L)) \partial_k (\nabla^2_{Q_L} f_B(\tilde{Q}_L) \nabla (\tilde{Q}_L)) \, dx \, dt \]

\[ = \frac{1}{2L} \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \partial_t (\nabla^{k+1} \partial_t (\tilde{Q}_L)) \nabla^{k+1} (\tilde{Q}_L) \partial_t (\nabla^{k+1} (\tilde{Q}_L)) \, dx \, dt \]

\[ + \frac{1}{2L} \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \partial_t (\nabla^{k+1} (\tilde{Q}_L)) \nabla^{k+1} (\tilde{Q}_L) \partial_t (\nabla^{k+1} (\tilde{Q}_L)) \, dx \, dt \]

\[ + \frac{1}{L} \int_{\tau_L}^{s} \int_{\mathbb{R}^3} |\nabla^{k+1} | \partial_t (\tilde{Q}_L) \sum_{\mu_1+\mu_2=k} |\nabla^{\mu_1} (\nabla^{\mu_2} f_B(\tilde{Q}_L))| \nabla^{\mu_2} \nabla (\tilde{Q}_L) \, dx \, dt \]

\[ \leq \frac{1}{2L} \frac{d}{dt} \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \partial_t (\nabla^{k+1} (\tilde{Q}_L)) \nabla^{k+1} (\tilde{Q}_L) \, dx \, dt \]

\[ + \eta \int_{\tau_L}^{s} \int_{\mathbb{R}^3} |\nabla^{k+1} | \partial_t (\tilde{Q}_L) \sum_{\mu_1+\mu_2=k} |\nabla^{\mu_1} (\nabla^{\mu_2} f_B(\tilde{Q}_L))| \nabla^{\mu_2} \nabla (\tilde{Q}_L) \, dx \, dt \]

\[ + C(\eta) \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \frac{1}{L} |\nabla^{k+1} | \partial_t (\tilde{Q}_L) \, dx \, dt \]

\[ + C(\eta) \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \frac{1}{L} |\nabla^{k+1} | \partial_t (\tilde{Q}_L) \, dx \, dt \]

\[ \leq - \frac{\lambda}{4L} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + C \int_{\tau_L}^{s} \frac{1}{L^2} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \, dt \]

\[ + \frac{\lambda}{8L} \|\nabla^{k+2} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{8} \|\nabla^{k+1} \partial_t Q_L\|_{L^2(\mathbb{R}^3)}^2 \, dt \]

\[ + C \int_{\tau_L}^{s} \frac{\lambda}{L} \|\nabla^{k+2} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \left( \frac{1}{L} \|\nabla^{k+1} | \partial_t Q_L\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_t Q_L\|_{H^k(\mathbb{R}^3)}^2 \right) \, dt \]

\[ + C \int_{\tau_L}^{s} \frac{1}{L} \|\nabla^{k+1} (Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \|\partial_t Q_L\|_{H^k(\mathbb{R}^3)}^2 \, dt. \] (4.22)

Using (4.19), we deduce the left-hand side of (4.18) to

\[ J_4 \geq \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \langle \nabla^{k+1} \partial_t Q_L, \nabla^{k+1} | \partial_t Q_L \rangle \, dx \, dt - \eta \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \|\nabla^{k+1} (R_{Q_L}^2 | \partial_t Q_L R_{Q_L})\|_{L^2(\mathbb{R}^3)}^2 \, dx \, dt \]

\[ + \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \sum_{\mu_1+\mu_2+\mu_3=k} |\nabla^{\mu_1} | \partial_t Q_L | \nabla^{\mu_2} | \partial_t Q_L | \nabla^{\mu_3} | \partial_t Q_L | \, dx \, dt \]

\[ + \int_{\tau_L}^{s} \int_{\mathbb{R}^3} \sum_{\mu_1+\mu_2+\mu_3=k+1} |\nabla^{\mu_1} (v_L \cdot \nabla Q_L + [Q_L, \Omega_L]) | \nabla^{\mu_2} R_{Q_L} | \nabla^{\mu_3} R_{Q_L} | \, dx \, dt \]

\[ \geq \int_{\tau_L}^{s} \frac{3}{4} \|\nabla^{k+1} | \partial_t Q_L\|_{L^2(\mathbb{R}^3)}^2 - C \|\nabla^{k+2} v_L\|_{L^2(\mathbb{R}^3)}^2 \, dt \]

\[ - C \int_{\tau_L}^{s} (\|\nabla^{k+1} | v_L\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_t Q_L\|_{H^k(\mathbb{R}^3)}^2 + 1) (\|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)} + 1) \, dt. \] (4.23)
Adding \( J_5, J_6 \) to \( J_4 \), we have

\[
\frac{\alpha}{4} \left\| \nabla^{k+2} Q_L(s) \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{\lambda}{2L} \left\| \nabla^{k+1} (Q_L - \pi(Q_L))(s) \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{2} \int_{\tau_L}^s \left\| \nabla^{k+1} \partial_t Q \right\|_{L^2(\mathbb{R}^3)}^2 dt
\]

\[
\leq C \int_{\tau_L}^s \left\| \nabla^{k+2} v_L \right\|_{L^2(\mathbb{R}^3)}^2 dt + \frac{\lambda}{8L} \int_{\tau_L}^s \left\| \nabla^{k+2} (Q_L - \pi(Q_L)) \right\|_{L^2(\mathbb{R}^3)}^2 dt + \frac{1}{8} \int_{\tau_L}^s \left\| \nabla^{k+3} Q_L \right\|_{L^2(\mathbb{R}^3)}^2 dt + \frac{1}{L} \int_{\tau_L}^s \left\| \nabla^{k+1} \partial_t Q \right\|_{L^2(\mathbb{R}^3)}^2 dt
\]

\[
+ C \int_{\tau_L}^s \left( \left\| \nabla^{k+2} Q_L \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \partial_t Q \right\|_{H^k(\mathbb{R}^3)}^2 \right) \int_{\tau_L}^s \left\| \nabla^{k+1} (Q_L - \pi(Q_L)) \right\|_{L^2(\mathbb{R}^3)}^2 dt
\]

\[
C \int_{\tau_L}^s \left( \left\| \nabla^{k+2} Q_L \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \partial_t Q \right\|_{H^k(\mathbb{R}^3)}^2 \right) \int_{\tau_L}^s \left\| \nabla^{k+1} (Q_L - \pi(Q_L)) \right\|_{L^2(\mathbb{R}^3)}^2 dt.
\]

Combining (4.24) with (4.17) yields

\[
\left\| \nabla^{k+2} Q_L(s) \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{L} \left\| \nabla^{k+1} (Q_L - \pi(Q_L))(s) \right\|_{L^2(\mathbb{R}^3)}^2
\]

\[
+ \int_{\tau_L}^s \left\| \nabla^{k+3} Q_L \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \nabla^{k+1} \partial_t Q \right\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{L} \left\| \nabla^{k+2} (Q_L - \pi(Q_L)) \right\|_{L^2(\mathbb{R}^3)}^2 dt
\]

\[
\leq C \int_{\tau_L}^s \left\| \nabla^{k+2} v_L \right\|_{L^2(\mathbb{R}^3)}^2 dt + C \int_{\tau_L}^s \left( \frac{1}{L} \left\| \nabla^{k+1} (Q_L - \pi(Q_L)) \right\|_{L^2(\mathbb{R}^3)}^2 + 1 \right)^2 dt
\]

\[
+ C \int_{\tau_L}^s \left( \left\| \nabla^{k+2} Q_L \right\|_{L^2(\mathbb{R}^3)}^2 + \left\| \partial_t Q \right\|_{H^k(\mathbb{R}^3)}^2 \right) \int_{\tau_L}^s \left\| \nabla^{k+1} (Q_L - \pi(Q_L)) \right\|_{L^2(\mathbb{R}^3)}^2 dt.
\]

To estimate \( \nabla^{k+2} v_L \) in (4.23), we apply \( \nabla^{k+1} \) to (4.11) and multiply by \( \nabla^{k+1} v_L \) to obtain

\[
\frac{1}{2} \left\| \nabla^{k+1} v_L(s) \right\|_{L^2(\mathbb{R}^3)}^2 + \int_{\tau_L}^s \left\| \nabla^{k+2} v_L \right\|_{L^2(\mathbb{R}^3)}^2 dt
\]

\[
= \int_{\tau_L}^s \int_{\mathbb{R}^3} \nabla^{k+1} \left( \partial_{\mu_1 \mu_2} f E \nabla L(Q_L)_{\mu_1 \mu_2} - [Q_L, \mathcal{H}(Q_L, \nabla Q_L)]_{ij} \right)\nabla^{k+1} \nabla_j (v_L)_i dx dt
\]

\[
\leq - \int_{\tau_L}^s \int_{\mathbb{R}^3} \nabla^{k+1} [Q_L, \mathcal{H}(Q_L, \nabla Q_L)]_{ij} \nabla^{k+1} \nabla_j (v_L)_i dx dt + \frac{1}{4} \int_{\tau_L}^s \left\| \nabla^{k+2} v_L \right\|_{L^2(\mathbb{R}^3)}^2 dt
\]

\[
+ C \int_{\tau_L}^s \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 = k+1} |\nabla^{\mu_1} \partial_{\mu_2} f E|^2 |\nabla^{\mu_2} Q_L|^2 dx.
\]

\[
\leq \frac{1}{4} \int_{\tau_L}^s \left\| \nabla^{k+2} v_L \right\|_{L^2(\mathbb{R}^3)}^2 dt + C \int_{\tau_L}^s \left( \left\| \nabla^{k+2} Q_L \right\|_{L^2(\mathbb{R}^3)}^2 + 1 \right) dt
\]

\[
- \int_{\tau_L}^s \int_{\mathbb{R}^3} \nabla^{k+1} [Q_L, \mathcal{H}]_{ij} \nabla^{k+1} \nabla_j (v_L)_i dx dt.
\]
The last step follows from the argument in (4.11).
In order to cancel the Lie bracket term in (4.26), we differentiate (2.36), multiply by $\nabla^{k+1} (R^T_{Q_L} (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L)) R_{Q_L})$ and combine with (4.31), (4.30) to obtain

$$
\int_{\tau_L}^{\infty} \int_{\mathbb{R}^3} \left| \nabla^{k+1} \left( R^T_{Q_L} (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L)) R_{Q_L} \right) \right|^2 dx dt
= \int_{\tau_L}^{\infty} \int_{\mathbb{R}^3} \left\langle \nabla^{k+1} (R^T_{Q_L} \partial_t Q_L R_{Q_L}), \nabla^{k+1} \left( R^T_{Q_L} (\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q)) R_{Q_L} \right) \right\rangle dx dt
+ \int_{\tau_L}^{\infty} \int_{\mathbb{R}^3} \left\langle \nabla^{k+1} (R^T_{Q_L} (v \cdot \nabla Q_L) R_{Q_L}), \nabla^{k+1} \left( R^T_{Q_L} (\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q)) R_{Q_L} \right) \right\rangle dx dt
+ \int_{\tau_L}^{\infty} \int_{\mathbb{R}^3} \left\langle \nabla^{k+1} (R^T_{Q_L} [Q_L, \Omega_L] R_{Q_L}), \nabla^{k+1} \left( R^T_{Q_L} (\mathcal{H}(Q, \nabla Q) + \frac{1}{L} g_B(Q)) R_{Q_L} \right) \right\rangle dx dt
=: J_7 + J_8 + J_9.
$$

(4.27)

We apply Lemma 2.3 to $\Delta^{k+1} \Omega_L$ with $A = Q_L, B = \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L), F = \Delta^{k+1} \Omega_L$ and obtain

$$
\left\langle \left\{ Q_L, \Delta^{k+1} \Omega_L \right\}, \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L) \right\rangle = \Delta^{k+1} \nabla_j v_i [Q_L, \mathcal{H}(Q_L, \nabla Q_L)]_{ij}.
$$

(4.28)

Note from (1.13) that

$$
|\nabla^k (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L))|^2 \leq C (|\nabla^k \partial_t Q_L|^2 + \sum_{\mu_1 = \mu_2 = k+1} |\nabla^{\mu_1} Q_L|^2 |\nabla^{\mu_2} v_L|^2).
$$

(4.29)

Then using (1.28), (1.29), (1.2), (4.4)–(4.6), we observe

$$
J_9 \leq (-1)^{k+1} \int_{\mathbb{R}^3} \left\langle \Delta^{k+1} [Q_L, \Omega_L], \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L) \right\rangle dx
+ \eta \int_{\mathbb{R}^3} |\nabla^{k+1} (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L))|^2 + |\nabla^{k+1} (R^T_{Q_L} [Q_L, \Omega_L] R_{Q_L})|^2 dx
+ C(\eta) \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 + \mu_3 = k} |\nabla^{\mu_1} [Q_L, \Omega_L]|^2 |\nabla^{\mu_2} \nabla Q_L|^2 |\nabla^{\mu_3} R_{Q_L}|^2 dx
+ C(\eta) \int_{\mathbb{R}^3} \sum_{\mu_1 + \mu_2 + \mu_3 = k} |\nabla^{\mu_1} (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L))|^2 |\nabla^{\mu_2} \nabla R_{Q_L}|^2 |\nabla^{\mu_3} R_{Q_L}|^2 dx
\leq (-1)^{k+1} \int_{\mathbb{R}^3} |\nabla^{2k+2} \nabla_j (v_L)| [Q_L, \mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L)]_{ij} dx
+ C \int_{\mathbb{R}^3} |\nabla^{k+1} \Omega_L| \sum_{\mu_1 + \mu_2 = k} |\nabla^{\mu_1} \nabla Q_L| |\nabla^{\mu_2} (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L))| dx
+ \eta \int_{\mathbb{R}^3} |\nabla^{k+1} (\mathcal{H}(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L))|^2 + |\nabla^{k+1} (R^T_{Q_L} [Q_L, \Omega_L] R_{Q_L})|^2 dx
+ C \left( \|
abla^{k+1} v_L\|_{L^2(\mathbb{R}^3)} + 1 \right) \|
abla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + 1)
\leq \int_{\mathbb{R}^3} |\nabla^{k+1} \nabla_j (v_L)| |\nabla^{k+1} [Q_L, \mathcal{H}(Q_L, \nabla Q_L)]_{ij} dx
\quad + \frac{1}{4} \|
abla^{k+2} v_L\|_{L^2(\mathbb{R}^3)}^2
$$

(4.30)
+ \eta_1 (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}}) \\
+ C (\| \nabla^{k+1} v_L \|^2_{L^2(\mathbb{R}^3)} + 1) (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + 1). \quad (4.30)
Repeating the same arguments in (4.21)-(4.22) and integrating the resulting expression in \(t\), we obtain
\[
J_7 \leq - \left( \frac{\alpha}{4} \| \nabla^{k+2} Q_L(s) \|_{L^2(\mathbb{R}^3)}^2 + \frac{\lambda}{2L} \| \nabla^{k+1} (Q_L - \pi(Q_L))(s) \|_{L^2(\mathbb{R}^3)}^2 \right) \\
+ \eta_1 \int_{\tau_L}^s \| \nabla^{k+3} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \nabla^{k+1} \partial_t Q_L \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{L} \| \nabla^{k+2} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} \\
+ C \int_{\tau_L}^s \| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}(\mathbb{R}^3)}) \\
+ C \int_{\tau_L}^s \frac{1}{L} \| \nabla^{k+1} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}(\mathbb{R}^3)}) \\
+ C \int_{\tau_L}^s \left( \frac{1}{L} \| \nabla^{k+1} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} + 1 \right)^2 dt. \quad (4.31)
\]
We combine \(J_7\) with \(J_3\) and apply Young’s inequality to \(J_8\). Then (4.27) reduces to
\[
\int_{\tau_L}^s \int_{\mathbb{R}^3} \| \nabla^{k+1} \left( R_{Q_L}^T (H(Q_L, \nabla Q_L) + \frac{1}{L} g_B(Q_L)) R_{Q_L} \right) \|^2 dx dt \\
\leq \int_{\tau_L}^s \int_{\mathbb{R}^3} \| \nabla^{k+1} \nabla_j (v_L, \nabla^{k+1} [Q_L, \mathcal{H}(Q_L, \nabla Q_L)])_{ij} dx + \frac{1}{4} \| \nabla^{k+2} v_L \|_{L^2(\mathbb{R}^3)}^2 dt \\
+ 2 \eta_1 \int_{\tau_L}^s \| \nabla^{k+3} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \nabla^{k+1} \partial_t Q_L \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{L} \| \nabla^{k+2} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} \\
+ C \int_{\tau_L}^s \| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \nabla^{k+1} v_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}(\mathbb{R}^3)}) \\
+ C \int_{\tau_L}^s \frac{1}{L} \| \nabla^{k+1} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}(\mathbb{R}^3)}) \\
+ C \int_{\tau_L}^s \left( \frac{1}{L} \| \nabla^{k+1} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} + 1 \right)^2 dt. \quad (4.32)
\]
By adding (4.22) to (4.20), we obtain
\[
\frac{1}{2} \| \nabla^{k+1} v_L(s) \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{4} \int_{\tau_L}^s \| \nabla^{k+2} v_L \|^2_{L^2(\mathbb{R}^3)} dt \\
\leq 2 \eta_1 \int_{\tau_L}^s \| \nabla^{k+3} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \nabla^{k+1} \partial_t Q_L \|^2_{L^2(\mathbb{R}^3)} + \frac{1}{L} \| \nabla^{k+2} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} dt \\
+ C \int_{\tau_L}^s \| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \nabla^{k+1} v_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}(\mathbb{R}^3)}) \\
+ C \int_{\tau_L}^s \frac{1}{L} \| \nabla^{k+1} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} (\| \nabla^{k+2} Q_L \|^2_{L^2(\mathbb{R}^3)} + \| \partial_t Q_L \|^2_{H^{k+1}(\mathbb{R}^3)}) \\
+ C \int_{\tau_L}^s \left( \frac{1}{L} \| \nabla^{k+1} (Q_L - \pi(Q_L)) \|^2_{L^2(\mathbb{R}^3)} + 1 \right)^2 dt. \quad (4.33)
\]
By substituting (4.25) into (4.33), choosing sufficiently small \( \eta_1 \) and combining with (4.25), we conclude
\[
\|\nabla^2 Q_L(s)\|_{L^2(\mathbb{R}^3)} + \|\nabla^{k+1} v_L(s)\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{L} \|\nabla^{k+1}(Q_L - \pi(Q_L))(s)\|_{L^2(\mathbb{R}^3)}^2
\]
\[
+ \int_{\tau_L}^s \|\nabla^{k+3} Q_L\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^{k+2} v_L\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^{k+1}\partial_t Q\|_{L^2(\mathbb{R}^3)}^2 dt
\]
\[
+ \int_{\tau_L}^s \frac{1}{L} \|\nabla^{k+2}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 dt
\]
\[
\leq C \int_{\tau_L}^s \left( \|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^{k+1} v_L\|_{L^2(\mathbb{R}^3)}^2 + \frac{1}{L} \|\nabla^{k+1}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 \right)
\]
\[
\times \left( \|\nabla^{k+2} Q_L\|_{L^2(\mathbb{R}^3)}^2 + \|\nabla^{k+1} v_L\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_t Q_L\|_{H^k(\mathbb{R}^3)}^2 \right)
\]
\[
+ \frac{1}{L} \|\nabla^{k+1}(Q_L - \pi(Q_L))\|_{L^2(\mathbb{R}^3)}^2 dt + C. 
\]  
(4.34)

We apply the Gronwall inequality to (4.34) with \( \|\cdot\| \) for \( t \in (\tau_L, s) \) and conclude that (4.1) holds for \( m = k + 1 \) on the \((\tau, s)\). Since \( \tau \geq T_0 \) is an arbitrary positive constant, we prove (4.1) for any \( s \in (\tau, T_M) \) and \( m = k + 1 \) which completes a proof of this lemma. 

**Proof of Theorem 3** Let \((Q, v)\) be the strong solution to (1.17) - (1.19) in \( \mathbb{R}^3 \times [0, T^*) \) with initial data \((Q_0, v_0) \in H^2_{loc}(\mathbb{R}^3) \times H^1(\mathbb{R}^3) \), where \( T^* \) is its maximal existence time. Given any \( T \in (0, T^*) \), set
\[
M = 2 \sup_{0 \leq t \leq T} \|\nabla Q_t\|_{L^2(\mathbb{R}^3)}. 
\]

It follows from using Theorem 2 that there exists a subsequence \((Q_L, v_L)\) and
\[
(\nabla Q_L, v_L) \to (\nabla Q, v), \quad \text{in} \quad L^\infty(0, T_M; L^2_{loc}(\mathbb{R}^3)) \cap L^2(0, T_M; H^1_{loc}(\mathbb{R}^3)).
\]

Suppose that \( T_M < T \). By Lemma 4.1 with \( m = 2 \), note that
\[
\int_{\mathbb{R}^3} |\nabla^3 Q_L(x, T_M)|^2 dx \leq C.
\]

Similarly to Lemma 2.4, one can show the energy identity of the system (1.11) - (1.13)
\[
\int_{\mathbb{R}^3} \left( f_E(Q, \nabla Q) + \frac{|v|^2}{2} \right) (x, s) dx + \int_0^s \int_{\mathbb{R}^3} |H(Q, \nabla Q)|^2 dx dt + \int_0^s \int_{\mathbb{R}^3} |\nabla v|^2 dx dt
\]
\[
= \int_{\mathbb{R}^3} \left( f_E(Q_0, \nabla Q_0) + \frac{|v_0|^2}{2} \right) dx. 
\]  
(4.35)

Then by comparing (4.35) with (2.29) (c.f. Lemma 4.3 [11]) and integrating by parts, using Hölder’s inequality, we obtain
\[
\lim_{L \to 0} \|\nabla^2 Q_L - \nabla^2 Q(T_M)\|_{L^2(\mathbb{R}^3)}^2
\]
\[
\leq \lim_{L \to 0} \left( \int_{\mathbb{R}^3} |(\nabla Q_L - \nabla Q)(T_M)|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |(\nabla^3 Q_L - \nabla^3 Q)(T_M)|^2 dx \right)^{\frac{1}{2}} = 0.
\]

Similarly, we find
\[
\lim_{L \to 0} \|\nabla v_L - \nabla v)(T_M)\|_{L^2}^2 = 0, \quad \lim_{L \to 0} \frac{1}{L} \|\nabla(Q_L - \pi(Q_L))\|_{L^2}^2 = 0.
\]
Therefore, we obtain
\[
\lim_{L \to 0} \left( \| \nabla Q_L(T_M) \|^2_{H^1(\mathbb{R}^3)} + \| v_L(T_M) \|^2_{H^1(\mathbb{R}^3)} + \frac{1}{L} \| (Q_L - \pi(Q_L))(T_M) \|^2_{H^1(\mathbb{R}^3)} \right)
\]
\[= \| \nabla Q(T_M) \|^2_{H^1(\mathbb{R}^3)} + \| v(T_M) \|^2_{H^1(\mathbb{R}^3)} \leq \frac{M}{2}.\]

Hence, for sufficiently small \( L \), one has
\[
\| \nabla Q_L(T_M) \|^2_{H^1(\mathbb{R}^3)} + \| v_L(T_M) \|^2_{H^1(\mathbb{R}^3)} + \frac{1}{L} \| (Q_L - \pi(Q_L))(T_M) \|^2_{H^1(\mathbb{R}^3)} \leq M.
\]

Utilizing Proposition 3.1 with the initial data \((Q_L(T_M), v_L(T_M))\), we extend the strong solution \((Q_L, v_L)\) to the time \( T_1 := \min\{T, 2T_M\} > T_M \). That is
\[
(\nabla Q_L, v_L) \to (\nabla Q, v), \quad \text{in } L^\infty(0, T_1; L^2(\mathbb{R}^3)) \cap L^2(0, T_1; H^1(\mathbb{R}^3)). \tag{4.36}
\]

Repeating the same argument with \( T \), we establish the convergence up to \( T \) for any \( T < T^* \) and complete the first part of Theorem 3 as any sequence \( L \to 0 \) due to the uniqueness of the solution \((Q, v)\). In the view of the Lemma 4.1, the argument in the proof of (1.20) in Theorem 3 leads to the statement (1.21).

\[
\Box
\]

5. Appendix: Local existence and proof of Theorem 1

Assume the initial data \((Q_{L,0}, v_{L,0}) \in H^2_{Q_0}(\mathbb{R}^3) \times H^1(\mathbb{R}^3)\) satisfies that \( \|Q_{L,0}\|_{L^\infty(\mathbb{R}^3)} \leq K \), \( \text{div} v_{L,0} = 0 \) and
\[
\|Q_{L,0}\|^2_{H^2_{Q_0}(\mathbb{R}^3)} + \|v_{L,0}\|^2_{H^1(\mathbb{R}^3)} = M_1. \tag{5.1}
\]

For any \( f(x) \in H^1(\mathbb{R}^3) \), it follows from the Gagliardo–Nirenberg interpolation that
\[
\int_{\mathbb{R}^3} |f(x)|^4 \, dx \leq \left( \int_{\mathbb{R}^3} |f(x)|^2 \, dx \right)^{\frac{2}{3}} \left( \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx \right)^{\frac{2}{3}}.
\]

Then, we have
\[
\left( \int_{\mathbb{R}^3} |f(x)|^4 \, dx \right)^{\frac{1}{4}} \leq \eta \int_{\mathbb{R}^3} |\nabla f(x)|^2 \, dx + \frac{C}{\eta^2} \int_{\mathbb{R}^3} |f(x)|^2 \, dx. \tag{5.2}
\]

With the aid of (4.2), we now prove the local existence of \((Q_{L,0}, v_{L,0})\) with initial data \((Q_{L,0}, v_{L,0})\).

**Proof of Theorem 1.** Without loss of generality, we assume \( L = 1 \) and omit the subscript \( L \) in the proof. Define the space
\[
\mathcal{V}(0, T) = \left\{ (Q, v) : \sup_{0 \leq t \leq T} \left( \|Q(t)\|^2_{H^2_{Q_0}(\mathbb{R}^3)} + \|v(t)\|^2_{H^1(\mathbb{R}^3)} + \|\nabla^3 Q\|^2_{L^2(0,T; L^2(\mathbb{R}^3))} \right)
\]
\[+ \|\partial_t Q\|^2_{L^2(0,T; H^1(\mathbb{R}^3))} + \|\nabla^2 v\|^2_{L^2(0,T; L^2(\mathbb{R}^3))} \leq C_1 M_1, \]
\[
\nabla \cdot v = 0, \quad \sup_{0 \leq t \leq T} \|Q(t)\|_{L^\infty(\mathbb{R}^3)} \leq 2K \right\}
\]

for some \( T \) and \( C_1 \) to be chosen later.
For a given pair \((Q_m, v_m) \in \mathcal{V}(0, T)\), there exists a unique strong solution \((Q_{m+1}, v_{m+1})\) with the initial data \((Q_0, v_0)\) of the linearized system of (1.11)-(1.13):

\[
(\partial_t - \Delta)v_{m+1} + \nabla P_{m+1} - \nabla \cdot [Q_m, h(Q_m, Q_{m+1})] = -v_m \cdot \nabla v_m - \sigma_{ij}(Q_m, \nabla Q_m), \tag{5.3}
\]

\[
\nabla \cdot v_{m+1} = 0, \tag{5.4}
\]

\[
\partial_t Q_{m+1} + [Q_m, Q_{m+1}] - h(Q_m, Q_{m+1}) = -v_m \cdot \nabla Q_m + g_B(Q_m) \tag{5.5}
\]

for some \(T_{m+1} > 0\), where

\[
h_{ij}(Q_m, Q_{m+1}) := -\frac{1}{2} \left( \nabla \beta [\partial_{\beta j} f_E(Q_m, \nabla Q_{m+1})] + \nabla \beta [\partial_{\beta j} f_E(Q_m, \nabla Q_{m+1})] \right) - \frac{1}{2} \left( \partial_{Q_{ij}} f_E(Q_m, \nabla Q_m) - \partial_{Q_{ij}} f_E(Q_m, \nabla Q_m) \right) - \frac{\delta_{ij}}{3} \sum_{l=1}^{3} \left( \nabla \beta [\partial_{\beta l} f_E(Q_m, \nabla Q_{m+1})] - \partial_{Q_{li}} f_E(Q_m, \nabla Q_m) \right). \tag{5.6}
\]

**Claim 1:** There exists a uniform \(T_{M_1}\) such that \((Q_{m+1}, v_{m+1}) \in \mathcal{V}(0, T_{M_1})\) for some \(T_{M_1} \leq T_{m+1}\) for all \(m \geq 1\). To establish the \(L^2\)-norm of \(\nabla^3 Q_{m+1}\), we multiply (5.5) with \(\Delta^2 Q_{m+1}\) and observe

\[
\langle \partial_t Q_{m+1} + [Q_m, \Omega_{m+1}] - \nabla \beta \partial_{\beta j} f_E(Q_m, \nabla Q_{m+1}), \Delta^2 Q_{m+1} \rangle = \langle \partial Q f_E(Q_m, \nabla Q_m) - v_m \cdot \nabla Q_m + g_B(Q_m), \Delta^2 Q_{m+1} \rangle. \tag{5.7}
\]

We can compute the second term in the left-hand side of (5.7)

\[
\int_0^t \int_{\mathbb{R}^3} \langle [Q_m, \Omega_{m+1}], \Delta^2 Q_{m+1} \rangle \, dx \, dt
\]

\[
\leq \frac{\alpha}{8} \int_0^t \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx \, dt + C \int_0^t \int_{\mathbb{R}^3} |Q_m|^2 |\nabla^2 v_{m+1}|^2 \, dx \, dt
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} |\nabla Q_m|^2 |\nabla v_{m+1}|^2 \, dx \, dt
\]

\[
\leq \frac{\alpha}{8} \int_0^t \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx \, dt + C \int_0^t \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \, dt
\]

\[
+ \int_0^t \int_{\mathbb{R}^3} |v_m|^2 \left( |\nabla Q_m|^2 + |\nabla^2 Q_m|^2 \right) \, dx \, dt
\]

\[
\leq \frac{\alpha}{8} \int_0^t \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx \, dt + C \int_0^t \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \, dt
\]

\[
+ C \int_0^t \int_{\mathbb{R}^3} |v_{m+1}|^2 \, dx \int_{\mathbb{R}^3} |\nabla^2 Q_m|^2 \, dx \, dt + \int_0^t \int_{\mathbb{R}^3} |v_{m+1}|^2 |\nabla^2 Q_m|^2 \, dx \, dt. \tag{5.8}
\]

Using the Sobolev inequality and (5.11), we have

\[
\sup_{0 \leq t \leq T_{m+1}} \left( \|\nabla Q_{m+1}(t)\|^2_{H^1(\mathbb{R}^3)} + \|v_{m+1}(t)\|^2_{H^1(\mathbb{R}^3)} \right) \leq CM_1. \tag{5.9}
\]
We employ the inequalities (5.2) and (5.9) to show the following:

\[
\int_0^s \int_{\mathbb{R}^3} |v_{m+1}|^2 |\nabla^2 Q_m|^2 \, dx \, dt \leq \int_0^s \left( \int_{\mathbb{R}^3} |v_{m+1}|^4 \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^3} |\nabla^2 Q_m|^4 \, dx \right)^{\frac{1}{4}} \, dt
\]

\[
\leq C \int_0^s \left( \int_{\mathbb{R}^3} |\nabla v_{m+1}|^2 + |v_{m+1}|^2 \, dx \right) \left( \int_{\mathbb{R}^3} \eta_1 |\nabla^3 Q_m|^3 + \frac{C}{\eta_1^2} |\nabla^2 Q_m|^2 \, dx \right) \, dt
\]

\[
\leq CM_1^2 (\eta_1 + s \frac{8}{\eta_1}) .
\]

(5.10)

Then we can write (5.8) as

\[
\int_0^s \int_{\mathbb{R}^3} \langle [Q_m, \Omega_{m+1}], \Delta^2 Q_{m+1} \rangle \, dx \, dt
\]

\[
\leq \frac{\alpha}{8} \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx \, dt + C \int_0^s \int_{\mathbb{R}^3} |\nabla v_{m+1}|^2 \, dx \, dt + CM_1^2 (\eta_1 + s \frac{8}{\eta_1}) .
\]

(5.11)

Integrating by parts and using (2.12), we deduce the third term in (5.7) to

\[
- \int_0^s \int_{\mathbb{R}^3} \nabla_\beta \partial_\rho \partial^{\rho} \partial_\nu \partial^{\nu} f_E(Q_m, \nabla Q_{m+1}) \Delta^2 (Q_{m+1})_{ij} \, dx \, dt
\]

\[
= - \int_0^s \int_{\mathbb{R}^3} \partial_\rho \partial^{\rho} \partial_\nu \partial^{\nu} f_E(Q_m, \nabla Q_{m+1}) \nabla^3 (Q_{m+1})_{ij} \, dx \, dt
\]

\[
+ \int_0^s \int_{\mathbb{R}^3} \nabla_\beta \partial_\rho \partial^{\rho} \partial_\nu \partial^{\nu} f_E(Q_m, \nabla Q_{m+1}) \nabla^2 (Q_{m+1})_{ij} \, dx \, dt
\]

\[
+ \int_0^s \int_{\mathbb{R}^3} \nabla_\beta \partial_\rho \partial^{\rho} \partial_\nu \partial^{\nu} f_E(Q_m, \nabla Q_{m+1}) \nabla (Q_{m+1})_{ij} \, dx \, dt
\]

\[
\leq - \frac{3\alpha}{8} \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx + C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_{m+1}|^2 (|\nabla^2 Q_m|^2 + |\nabla Q_m|^4) \, dx \, dt
\]

\[
\leq - \frac{3\alpha}{8} \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx + CM_1^2 (\eta_1 + s \frac{8}{\eta_1}) ,
\]

(5.12)

where we used the argument of (5.10) in the last calculation.

Using the argument in (5.10) again, we obtain

\[
\int_0^s \int_{\mathbb{R}^3} |\nabla (\partial_Q f_E(Q_m, \nabla Q_m) - v_m \cdot \nabla Q_m + g_B(Q_m))|^2 \, dx \, dt
\]

\[
\leq C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_m|^2 |\nabla^2 Q_m|^4 + |\nabla Q_m|^2 |\nabla^2 Q_m|^2 + |\nabla v_m|^2 |\nabla Q_m|^2 \, dx \, dt
\]

\[
+ C \int_0^s \int_{\mathbb{R}^3} |v_m|^2 |\nabla^2 Q_m|^2 + |\nabla Q_m|^2 \, dx \, dt \leq CM_1^2 (\eta_1 + s \frac{8}{\eta_1}) + CM_1 s .
\]

(5.13)

In view of (5.11)-(5.13), we deduce (5.7) to

\[
\frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |\nabla^2 Q_{m+1}(x, s)|^2 \, dx + \frac{\alpha}{4} \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx \, dt
\]

\[
\leq \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |\nabla^2 Q_0|^2 \, dx + C \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \, dt + CM_1^2 (\eta_1 + s \frac{8}{\eta_1}) + CM_1 s .
\]

(5.14)
In order to estimate the $L^2$-norm of $\nabla \partial_t Q_{m+1}$, we multiply (5.5) by $\Delta \partial_t Q_{m+1}$ and compute
\begin{align}
&\langle (\partial_t Q_{m+1} + [Q_m, \Omega_{m+1}] - \nabla_\beta \partial_\rho^\ast f_E(Q_m, \nabla Q_{m+1}), \Delta \partial_t Q_{m+1} \rangle \\
&= \langle \partial_q f_E(Q_m, \nabla Q_m) - v_m \cdot \nabla Q_m + g_B(Q_m), \Delta \partial_t Q_{m+1} \rangle.
\end{align}
(5.15)

Using a similar argument to (5.11), we have
\begin{align}
&\int_0^s \int_{\mathbb{R}^3} \langle [Q_m, \Omega_{m+1}], \Delta \partial_t Q_{m+1} \rangle \, dx dt \\
&\leq \frac{1}{8} \int_0^s \int_{\mathbb{R}^3} |\nabla \partial_t Q_{m+1}|^2 \, dx dt + C \int_0^s \int_{\mathbb{R}^3} |\nabla [Q_m, \Omega_{m+1}]|^2 \, dx dt \\
&\leq \frac{1}{8} \int_0^s \int_{\mathbb{R}^3} |\nabla \partial_t Q_{m+1}|^2 \, dx dt + CK^2 \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx dt + CM^2 (\eta_1 + s + \frac{s}{\eta_1}).
\end{align}
(5.16)

Using (2.2), we compute the third term in (5.15)
\begin{align}
&\int_0^s \int_{\mathbb{R}^3} \langle (\nabla_\beta \partial_\rho^\ast f_E(Q_m, \nabla Q_{m+1}), \Delta \partial_t Q_{m+1} \rangle \, dx dt \\
&= -\int_0^s \int_{\mathbb{R}^3} \partial_{ij}^2 f_E(Q_m, \nabla Q_{m+1}) \nabla_{ij}^2 (Q_m) + \partial_t \nabla_{ij}^2 (Q_{m+1}) \, dx dt \\
&= \int_0^s \int_{\mathbb{R}^3} \partial_{ij}^2 f_E(Q_m, \nabla Q_{m+1}) \nabla_{ij} (Q_m) + \partial_t \nabla_{ij} (Q_{m+1}) \, dx dt \\
&\leq \int_0^s \int_{\mathbb{R}^3} \partial_{ij} f_E(Q_m, \nabla Q_{m+1}) \nabla_{ij} (Q_m) + \partial_t \nabla_{ij} (Q_{m+1}) \, dx dt \\
&+ C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_{m+1}| (|\nabla \partial_t Q_m| |\nabla^2 Q_{m+1}| + |\partial_t Q_m| |\nabla Q_{m+1}|) \, dx dt \\
&+ C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_{m+1}| (|\nabla \partial_t Q_m||\nabla^2 Q_{m+1}|) \, dx dt \\
&+ C \int_0^s \int_{\mathbb{R}^3} |\partial_t \nabla Q_{m+1}| (|\nabla^2 Q_{m+1}| |\nabla Q_m| + |\nabla Q_{m+1}| |\nabla Q_m|^2) \, dx dt \\
&+ C \int_0^s \int_{\mathbb{R}^3} |\partial_t \nabla Q_{m+1}| |\nabla Q_{m+1}| |\nabla Q_m| \, dx dt \\
&\leq -\frac{\alpha}{4} \int_{\mathbb{R}^3} |\nabla^2 Q_{m+1}(x, s)|^2 \, dx + \frac{\Lambda (1 + 4K^2)}{2} \int_{\mathbb{R}^3} |\nabla^2 Q_0|^2 \, dx + CM^2 (\eta_1 + s + \frac{s}{\eta_1}) \\
&+ CM^5 s + \int_0^s \int_{\mathbb{R}^3} \frac{\alpha}{8} |\nabla^3 Q_{m+1}|^2 + \frac{1}{8} \int_{\mathbb{R}^3} |\nabla \partial_t Q_{m+1}|^2 \, dx dt,
\end{align}
(5.17)
where in the last step, we used the argument in (5.10) and the following estimate
\begin{align}
C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_{m+1}|^2 |\nabla^2 Q_{m+1}|^2 \, dx dt \\
\leq CM \int_0^s \int_{\mathbb{R}^3} \eta |\nabla^3 Q_{m+1}|^2 + \frac{1}{\eta^3} |\nabla^2 Q_{m+1}|^2 \, dx dt \leq \frac{\alpha}{16} \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx dt + CM^5 s.
\end{align}
Adding (5.18) to (5.14), we obtain
\[ \frac{\alpha}{4} \int_{\mathbb{R}^3} |\nabla^2 Q_{m+1}(x,s)|^2 \, dx + \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |\nabla \partial_t Q_{m+1}|^2 \, dx \, dt \]
\[ \leq \frac{\Lambda(1 + 4K^2)}{2} \int_{\mathbb{R}^3} |\nabla^2 Q_0|^2 \, dx + \frac{\alpha}{4} \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 \, dx \, dt \]
\[ + C \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \, dt + CM_1^2(\eta_1 + s + \frac{s}{\eta_1^3}) + CM_1^2 s. \]

(5.18)

Adding (5.18) to (5.14), we obtain
\[ \int_{\mathbb{R}^3} |\nabla^2 Q_{m+1}(x,s)|^2 \, dx + \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 + |\nabla \partial_t Q_{m+1}|^2 \, dx \, dt \]
\[ \leq C \int_{\mathbb{R}^3} |\nabla^2 Q_0|^2 \, dx + C \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx + CM_1^2(\eta_1 + s + \frac{s}{\eta_1^3}) + CM_1^2 s + CM_1 s. \]

(5.19)

Here \( C \) only depends on the following constants \( \alpha, K \) and \( \Lambda \).

To estimate \( \nabla^2 v_{m+1} \) in (5.19), we multiply (5.13) by \(-\Delta v_{m+1}\) and compute
\[ \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_{m+1}(x,s)|^2 \, dx + \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \, dt \]
\[ - \int_0^s \int_{\mathbb{R}^3} [Q_m, h(Q_m, Q_{m+1})]_{ij} \nabla_j \Delta (v_{m+1}) \, i \, dx \, dt \]
\[ = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_0|^2 \, dx - \frac{1}{4} \int_0^s \int_{\mathbb{R}^3} \left( (v_m)_j \nabla_j (v_m)_i + \nabla_j \sigma_{ij}(Q_m, \nabla Q_m) \right) (\Delta v_{m+1}) \, i \, dx \, dt \]
\[ \leq \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_0|^2 \, dx + \frac{1}{4} \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \, dt + CM_1^2(\eta_1 + s + \frac{s}{\eta_1^3}). \]

(5.20)

To cancel the term involving \( h(Q_m, Q_{m+1}) \) in (5.20), we differentiate (5.5) in \( x \), multiply by \( \nabla h(Q_m, Q_{m+1}) \) and obtain
\[ \int_{\mathbb{R}^3} \langle \nabla_\beta (\partial_t Q_{m+1} + [Q_m, \Omega_{m+1}]), \nabla_\beta h(Q_m, Q_{m+1}) \rangle \, dx + \int_{\mathbb{R}^3} |\nabla h(Q_m, Q_{m+1})|^2 \, dx \]
\[ = \int_{\mathbb{R}^3} \langle \nabla_\beta (-v_m \cdot \nabla Q_m + g_B(Q_m)), \nabla_\beta h(Q_m, Q_{m+1}) \rangle \, dx. \]

(5.21)

we choose \( A = Q, B = h(Q_m, Q_{m+1}), F = \Delta v \) in Lemma 2.3 and obtain
\[ \langle [Q_m, \Omega_{m+1}], h(Q_m, Q_{m+1}) \rangle = \langle \Delta v_{m+1}, [Q_m, h(Q_m, Q_{m+1})] \rangle. \]

(5.22)

Note that
\[ h(Q_m, Q_{m+1}) \leq C(|\nabla^2 Q_{m+1}| + |\nabla Q_{m+1}|^2 + |\nabla Q_{m+1}| |\nabla Q_m|). \]
Then using (5.22), we compute the second term in (5.21)

\[
\begin{align*}
\int_{\mathbb{R}^3} & \langle \nabla_\beta [Q_m, \Omega_{m+1}], \nabla_\beta h(Q_m, Q_{m+1}) \rangle \, dx \\
= & \int_{\mathbb{R}^3} \langle [Q_m, \Delta \Omega_{m+1}], h(Q_m, Q_{m+1}) \rangle \, dx \\
+ & \int_{\mathbb{R}^3} \langle [\Delta Q_m, \Omega_{m+1}] + 2[\nabla Q_m, \nabla \Omega_{m+1}], h(Q_m, Q_{m+1}) \rangle \, dx \\
= & \int_{\mathbb{R}^3} \langle [Q_m, \Delta \Omega_{m+1}], h(Q_m, Q_{m+1}) \rangle \, dx + \int_{\mathbb{R}^3} \langle [\nabla Q_m, \nabla \Omega_{m+1}], h(Q_m, Q_{m+1}) \rangle \, dx \\
- & \int_{\mathbb{R}^3} \langle [\nabla_\alpha Q_m, \Omega_{m+1}], \nabla_\alpha h(Q_m, Q_{m+1}) \rangle \, dx \\
\geq & \int_{\mathbb{R}^3} \langle \Delta \nabla v_{m+1}, [Q_m, h(Q_m, Q_{m+1})] \rangle \, dx - \frac{1}{4} \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dx \\
- & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla h(Q_m, Q_{m+1})|^2 \, dx - \eta_1 \int_{\mathbb{R}^3} |\nabla^3 Q_m|^2 \, dx - CM_1^2 \frac{s}{\eta_1}. 
\end{align*}
\] (5.23)

We repeat the argument in (5.17) for the first term in (5.21), apply Young’s inequality to the right-hand side of (5.21). Then integrate (5.21) in \(t\) and combine with (5.20) yield

\[
\begin{align*}
\frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_{m+1}(x, s)|^2 \, dx + \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |\nabla^2 v_{m+1}|^2 \, dxdt \\
\leq & \frac{1}{2} \int_{\mathbb{R}^3} |\nabla v_0|^2 \, dx + \frac{\Lambda(1 + 4K^2)}{2} \int_{\mathbb{R}^3} |\nabla^2 Q_0|^2 \, dx + CM_1^2 \left( \eta_1 + \frac{s}{\eta_1} \right) \\
+ & \eta_2 \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 + |\nabla \partial_t Q_{m+1}|^2 \, dxdt \\
+ & C(\eta_2)M_1^2 (\eta_1 + s + \frac{s}{\eta_1}) + C(\eta_2)M_1^5 s 
\end{align*}
\] (5.24)

for some small \(\eta_1, \eta_2\). Substituting (5.19) into (5.24) and choosing \(\eta_2\) sufficiently small, we obtain the estimates for \(v_{m+1}\). Combining the resulting expression with (5.19) yields

\[
\begin{align*}
\int_{\mathbb{R}^3} |\nabla^2 Q_{m+1}(x, s)|^2 + |\nabla v_{m+1}(x, s)|^2 \, dx \\
+ & \int_0^s \int_{\mathbb{R}^3} |\nabla^3 Q_{m+1}|^2 + |\nabla \partial_t Q_{m+1}|^2 + |\nabla^2 v_{m+1}|^2 \, dxdt \\
\leq & CM_1 + CM_1^2 \left( \eta_1 + s + \frac{s}{\eta_1} \right) + CM_1^5 s + CM_1 s. 
\end{align*}
\] (5.25)

Here \(C\) only depends on \(\alpha, K\) and \(\Lambda\).
It remains to check the $L^2$-norm of the lower order terms in $\mathcal{V}(0, T)$. We multiply (5.3) by $v_{m+1}$ to obtain
\[
\frac{1}{2} \int_{\mathbb{R}^3} |v_{m+1}(x,s)|^2 \, dx + \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |\nabla v_{m+1}|^2 \, dx \, dt \\
\leq \frac{1}{2} \int_{\mathbb{R}^3} |v_0|^2 \, dx + C \int_0^s \int_{\mathbb{R}^3} ||Q_m, h(Q_m, Q_{m+1})||^2 + |\sigma_{ij}(Q_m, \nabla Q_m)|^2 \, dx \, dt \\
\leq CM_1 + C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_{m+1}|^2 \, dx \leq CM_1 + CM_1 s.
\] (5.26)

By using mean value theorem with the fact that $g_B(Q_\varepsilon) = 0$, we find
\[
|g_B(Q_m)| \leq C(K) |Q_m - Q_\varepsilon|.
\] (5.27)

Multiplying (5.5) by $\partial_t Q_{m+1}$ and $Q_{m+1} - Q_\varepsilon$ respectively then using (5.27) yield
\[
\frac{1}{2} \int_{\mathbb{R}^3} |Q_{m+1} - Q_\varepsilon|^2 \, dx + \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |Q_{m+1}|^2 \, dx \, dt \\
\leq \frac{1}{2} \int_{\mathbb{R}^3} |Q_0 - Q_\varepsilon|^2 \, dx + \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |Q_{m+1} - Q_\varepsilon|^2 \, dx \, dt \\
+ C \int_0^s \int_{\mathbb{R}^3} |\nabla Q_m, \Omega_{m+1}|^2 + |h(Q_m, Q_{m+1})|^2 + |v_m \cdot \nabla Q_m|^2 + |g_B(Q_m)|^2 \, dx \, dt \\
\leq CM_1 + CM_1 s.
\] (5.28)

Note that the $L^2$-norm of $\nabla Q_{m+1}$ from the Sobolev inequality and (5.28). Now adding (5.26) and (5.28) to (5.24), we have
\[
\left( \|Q(x,s)\|_{H^2_{Q_\varepsilon}(\mathbb{R}^3)}^2 + \|v(x,s)\|_{H^1(\mathbb{R}^3)}^2 + \|\nabla^4 Q\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2 \right) \\
+ \|\partial_t Q\|_{L^2(0,T;H^1(\mathbb{R}^3))}^2 + \|\nabla^2 v\|_{L^2(0,T;L^2(\mathbb{R}^3))}^2 \\
\leq \frac{C_1}{4} M_1 + \frac{C_1}{4} M_1^2 (\eta_1 + s + \frac{s}{\eta_1}) + \frac{C_1}{4} M_1^3 s + \frac{C_1}{4} M_1 s \leq C_1 M_1
\] (5.29)

for some $C_1$ depending on $\alpha, K$ and $\Lambda$. Here in the last step, we set $\eta_1 = M_1^{-1}$ and $s \leq \min \left\{ \frac{1}{2} M_1^{-1}, \frac{1}{2} M_1^{-1}, \frac{1}{2} \right\}$.

It remains to verify that $\|Q_{m+1}(x,s)\|_{L^\infty(\mathbb{R}^3)} \leq 2K$. Note from (5.28) that
\[
\int_{\mathbb{R}^3} |Q_{m+1}(x,s) - Q_0|^2 \, dx = \int_{\mathbb{R}^3} \left( \int_0^s |\partial_t Q_{m+1}(x,t)|^2 \, dt \right) \, dx \leq s \int_0^s \int_{\mathbb{R}^3} |\partial_t Q_{m+1}|^2 \, dx \, dt \\
\leq s \int_0^s \int_{\mathbb{R}^3} |\partial_t Q_{m+1}|^2 \, dx \, dt \leq CM_1 s(1 + s) \leq CM_1 s
\]
for $s \leq \frac{1}{2}$. By using the Gagliardo–Nirenberg interpolation (c.f. (11)) and choosing $s \leq C_2^{-8} K^8 M_1^{-4}$, we have
\[
\|Q_{m+1}(s) - Q_0\|_{L^\infty(\mathbb{R}^3)} \leq C \|Q_{m+1}(s) - Q_0\|_{L^2(\mathbb{R}^3)} \|\nabla^2 (Q_{m+1}(s) - Q_0)\|_{L^2(\mathbb{R}^3)}^2 \\
\leq C_2(M_1 s) \frac{1}{2} M_1^2 \leq K,
\]
where $C_2$ is independent from $m$. Therefore, we prove Claim 1 by choosing
\[
T_{M_1} := \min \left\{ C_2^{-8} K^8 M_1^{-4}, \frac{1}{2} M_1^{-4}, \frac{1}{2} M_1^{-1}, \frac{1}{2} \right\}.
\]
Claim 2: There exists $T > 0$ such that

$$
\sup_{0 \leq t \leq T} \left( \|Q_{m+1} - Q_m(t)\|_{H^1_\omega(\mathbb{R}^3)} + \|v_{m+1} - v_m(t)\|_{L^2(\mathbb{R}^3)} \right)
$$

$$
+ \|\nabla^2(Q_{m+1} - Q_m)\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \|\nabla(v_{m+1} - v_m)\|_{L^2(0,T;L^2(\mathbb{R}^3))}
\leq \frac{1}{2} \sup_{0 \leq t \leq T} \left( \|Q_m - Q_{m-1}(t)\|_{H^1_\omega(\mathbb{R}^3)} + \|v_m - v_{m-1}(t)\|_{L^2(\mathbb{R}^3)} \right)
$$

$$
+ \frac{1}{2} \|\nabla^2(Q_m - Q_{m-1})\|_{L^2(0,T;L^2(\mathbb{R}^3))} + \frac{1}{2} \|\nabla(v_m - v_{m-1})\|_{L^2(0,T;L^2(\mathbb{R}^3))}
$$

For given pairs $(Q_m, v_m)$ and $(Q_{m-1}, v_{m-1}) \in \mathcal{V}$, we have

$$(\partial_t - \Delta)(v_{m+1} - v_m) + \nabla(P_{m+1} - P_m)
$$

$$= \nabla \cdot [Q_m, h(Q_m, Q_{m+1})] - \nabla \cdot [Q_{m-1}, h(Q_{m-1}, Q_m)]
$$

$$- v_m \cdot \nabla v_m + v_m \cdot \nabla v_{m-1} + \sigma(Q_m, \nabla Q_m) - \sigma(Q_{m-1}, \nabla Q_{m-1}),$$

(5.30)

$$\nabla \cdot (v_{m+1} - v_m) = 0,$$

(5.31)

$$\partial_t(Q_{m+1} - Q_m) + [Q_m, \Omega_{m+1}] - [Q_{m-1}, \Omega_m]
$$

$$= \nabla_\beta \partial_\beta f_E(Q_m, \nabla Q_m) - \nabla_\beta \partial_\beta f_E(Q_{m-1}, \nabla Q_m) - v_m \cdot \nabla Q_m + v_{m-1} \cdot \nabla Q_{m-1}
$$

$$- \partial Q f_E(Q_m, \nabla Q_m) + \partial Q f_E(Q_{m-1}, \nabla Q_{m-1}) + g_B(Q_m) - g_B(Q_{m-1}).$$

(5.32)

Multiplying $\frac{1}{2} \int_{\mathbb{R}^3} \nabla(Q_{m+1} - Q_m)(x, s)^2 \, dx$ yields

$$\int_0^s \int_{\mathbb{R}^3} \langle [Q_m, \Omega_{m+1}] - [Q_{m-1}, \Omega_m], \nabla(Q_{m+1} - Q_m) \rangle \, dx \, dt
$$

$$= \int_0^s \int_{\mathbb{R}^3} \langle \nabla_\beta \partial_\beta f_E(Q_m, \nabla Q_{m+1}) - \nabla_\beta \partial_\beta f_E(Q_{m-1}, \nabla Q_m), \nabla(Q_{m+1} - Q_m) \rangle \, dx \, dt
$$

$$- \int_0^s \int_{\mathbb{R}^3} \langle -v_m \cdot \nabla Q_m + v_{m-1} \cdot \nabla Q_{m-1}, \nabla(Q_{m+1} - Q_m) \rangle \, dx \, dt
$$

$$- \int_0^s \int_{\mathbb{R}^3} \langle -\partial Q f_E(Q_m, \nabla Q_m) + \partial Q f_E(Q_{m-1}, \nabla Q_{m-1}), \nabla(Q_{m+1} - Q_m) \rangle \, dx \, dt
$$

$$- \int_0^s \int_{\mathbb{R}^3} \langle g_B(Q_m) - g_B(Q_{m-1}), \nabla(Q_{m+1} - Q_m) \rangle \, dx \, dt.$$  

(5.33)

Using Young’s inequality and (5.24), we compute the first term in the right-hand side of (5.33)

$$\int_0^s \int_{\mathbb{R}^3} \langle [Q_m, \Omega_{m+1}] - [Q_{m-1}, \Omega_m], \nabla(Q_{m+1} - Q_m) \rangle \, dx \, dt
$$

$$\leq \eta \int_0^s \int_{\mathbb{R}^3} \nabla^2(Q_{m+1} - Q_m)^2 \, dx \, dt + C(\eta) \int_0^s \int_{\mathbb{R}^3} |\nabla(v_{m+1} - v_m)|^2 \, dx \, dt
$$

$$+ C(\eta) \int_0^s \|Q_m - Q_{m-1}\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |v_m|^2 \, dx \, dt
$$

$$\leq \eta \int_0^s \int_{\mathbb{R}^3} \nabla^2(Q_{m+1} - Q_m)^2 \, dx \, dt + C(\eta) \int_0^s \int_{\mathbb{R}^3} |\nabla(v_{m+1} - v_m)|^2 \, dx \, dt
+ C(\eta)M_1 \int_0^t \left( \int_{\mathbb{R}^3} |\nabla(Q_m - Q_{m-1})|^4 \right)^{\frac{1}{2}} dt \leq \eta \int_0^t \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 \, dx \, dt + C \int_0^t \int_{\mathbb{R}^3} |\nabla(v_{m+1} - v_m)|^2 \, dx \, dt \\
+ \eta \int_0^t \int_{\mathbb{R}^3} |\nabla^2(Q_m - Q_{m-1})|^2 \, dx \, dt + C M_1^4 \int_0^t \int_{\mathbb{R}^3} |\nabla(Q_m - Q_{m-1})|^2 \, dx \, dt, \tag{5.34}

where \eta and \eta_1 are some small constants to be chosen later.

Applying (5.3) again to the second term in (5.33) and using (2.2) yields

\begin{align*}
- \int_0^t \int_{\mathbb{R}^3} &\left\{ \nabla_\beta \partial_\rho f_E(Q_m, \nabla Q_{m+1}) - \nabla_\beta \partial_\rho f_E(Q_{m-1}, \nabla Q_m), \Delta(Q_{m+1} - Q_m) \right\} \, dx \, dt \\
= &- \int_0^t \int_{\mathbb{R}^3} \partial_\beta \delta_{ij} f_E(Q_m, \nabla Q_{m+1}) \nabla^2_{\gamma\beta}(Q_{m+1} - Q_m)_{ij} \, dx \, dt \\
&+ \int_0^t \int_{\mathbb{R}^3} \partial_\beta \delta_{ij} f_E(Q_{m-1}, \nabla Q_m) \nabla^2_{\gamma\beta}(Q_m - Q_{m+1})_{ij} \, dx \, dt \\
&- \int_0^t \int_{\mathbb{R}^3} \partial_\beta \delta_{ij} f_E(Q_m, \nabla Q_{m+1}) \nabla_\gamma(Q_m) \nabla^2_{\beta\gamma}(Q_{m+1} - Q_m)_{ij} \, dx \, dt \\
&+ \int_0^t \int_{\mathbb{R}^3} \partial_\beta \delta_{ij} f_E(Q_{m-1}, \nabla Q_m) \nabla_\gamma(Q_m) \nabla^2_{\beta\gamma}(Q_m - Q_{m+1})_{ij} \, dx \, dt \\
&\leq - \left( \frac{\alpha}{2} - 2\eta \right) \int_0^t \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 \, dx \, dt \\
&\quad - \int_0^t \int_{\mathbb{R}^3} \delta_{ij} f_E(Q_m, \nabla Q_{m+1}) \nabla^2_{\gamma\beta}(Q_{m+1} - Q_m)_{ij} \, dx \, dt \\
&\quad + \int_0^t \int_{\mathbb{R}^3} \delta_{ij} f_E(Q_{m-1}, \nabla Q_m) \nabla^2_{\gamma\beta}(Q_m - Q_{m+1})_{ij} \, dx \, dt \\
&\quad + C(\eta) \int_0^t \int_{\mathbb{R}^3} \partial_\beta \delta_{ij} f_E(Q_m, \nabla Q_{m+1}) \nabla Q_m - \partial_\beta \delta_{ij} f_E(Q_{m-1}, \nabla Q_m) \nabla Q_{m-1} \, dx \, dt \\
&\quad \leq - \left( \frac{\alpha}{2} - 2\eta \right) \int_0^t \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 \, dx \, dt \\
&\quad + C(\eta) \int_0^t \int_{\mathbb{R}^3} (|Q_m - Q_{m-1}|^2_{L^\infty(\mathbb{R}^3)}) \int_{\mathbb{R}^3} |\nabla^2 Q_m|^2 \, dx \, dt \\
&\quad + C(\eta) \int_0^t \int_{\mathbb{R}^3} (|\nabla Q_{m+1} - \nabla Q_m|^2 + |\nabla Q_m|^2 |Q_m - Q_{m-1}|^2) |\nabla Q_m|^2 \, dx \, dt \\
&\quad + C(\eta) \int_0^t \int_{\mathbb{R}^3} |\nabla Q_m|^2 |\nabla Q_m - \nabla Q_{m-1}|^2 \, dx \, dt \\
&\quad \leq - \left( \frac{\alpha}{2} - 2\eta \right) \int_0^t \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 \, dx \, dt + \eta \int_0^t \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 \, dx \, dt \\
&\quad + C(M_1 + M_2^4) \int_0^t \int_{\mathbb{R}^3} |\nabla(Q_m - Q_{m-1})|^2 \, dx \, dt. \tag{5.35}
\end{align*}

The remaining terms in (5.33) are

\begin{align*}
- \int_0^t \int_{\mathbb{R}^3} (-v_m \cdot \nabla Q_m + v_{m-1} \cdot \nabla Q_{m-1}, \Delta(Q_{m+1} - Q_m)) \, dx \, dt
\end{align*}
\[
- \int_0^s \int_{\mathbb{R}^3} \left( -\partial_Q f_E(Q_m, \nabla Q_m) + \partial_Q f_E(Q_{m-1}, \nabla Q_{m-1}), \Delta(Q_{m+1} - Q_m) \right) \, dx \, dt \\
- \int_0^s \int_{\mathbb{R}^3} \left( g_B(Q_m) - g_B(Q_{m-1}), \Delta(Q_{m+1} - Q_m) \right) \, dx \, dt \\
\leq \eta \int_0^s \int_{\mathbb{R}^3} \nabla^2(Q_{m+1} - Q_m)^2 \, dx \, dt + C \int_0^s \int_{\mathbb{R}^3} |v_m - v_{m-1}|^2 |\nabla Q_m - \nabla Q_{m-1}|^2 \, dx \, dt \\
+ C \int_0^s \int_{\mathbb{R}^3} |v_m - v_{m-1}|^2 |\nabla Q_m|^2 + |\nabla Q_m| |Q_m - Q_{m-1}|^2 \, dx \, dt \\
+ C \int_0^s \int_{\mathbb{R}^3} |(\nabla Q_m - Q_{m-1})|^2 + Q_m - Q_{m-1}|^2 \, dx \, dt \\
\leq \eta \int_0^s \int_{\mathbb{R}^3} \nabla^2(Q_{m+1} - Q_m)^2 \, dx \, dt + C \int_0^s \int_{\mathbb{R}^3} |Q_m - Q_{m-1}|^2 \, dx \, dt \\
+ \eta \int_0^s \int_{\mathbb{R}^3} |Q_m - Q_{m-1}|^2 + |v_m - v_{m-1}|^2 \, dx \, dt \\
+ C(M_1 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |(Q_m - Q_{m-1})|^2 + |v_m - v_{m-1}|^2 \, dx \, dt. 
\]

Substituting (5.34)-(5.36) into (5.33), we find
\[
\frac{1}{2} \int_{\mathbb{R}^3} |(Q_{m+1} - Q_m)(x, s)|^2 \, dx \\
\leq C \int_0^s \int_{\mathbb{R}^3} |v_m - v_{m+1}|^2 \, dx \, dt + C \int_0^s \int_{\mathbb{R}^3} |Q_m - Q_{m-1}|^2 \, dx \, dt \\
+ C \eta \int_0^s \int_{\mathbb{R}^3} |\nabla^2(Q_m - Q_{m-1})|^2 + |\nabla(v_m - v_{m-1})|^2 \, dx \, dt \\
+ C(M_1 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |(Q_m - Q_{m-1})|^2 + |v_m - v_{m-1}|^2 + |Q_m - Q_{m-1}|^2 \, dx \, dt. 
\]

Now, we compute the difference \( Q_{m+1} - Q_m \). Multiplying (5.32) by \( Q_{m+1} - Q_m \), one can show
\[
\frac{1}{2} \int_{\mathbb{R}^3} |(Q_{m+1} - Q_m)(x, s)|^2 \, dx \\
\leq CM_1 \int_0^s \int_{\mathbb{R}^3} |Q_{m+1} - Q_m|^2 \, dx \, dt + C \int_0^s \int_{\mathbb{R}^3} |\nabla(v_m + v_{m+1})|^2 \, dx \, dt \\
+ C \eta \int_0^s \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 + |\nabla(v_m - v_{m-1})|^2 \, dx \, dt \\
+ C(M_1 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |(Q_m - Q_{m-1})|^2 + |v_m - v_{m-1}|^2 + |Q_m - Q_{m-1}|^2 \, dx \, dt. 
\]

Combining (5.38) with (5.37), we find
\[
\int_{\mathbb{R}^3} |(Q_{m+1} - Q_m)(x, s)|^2 + |\nabla(Q_{m+1} - Q_m)(x, s)|^2 \, dx \\
+ \int_0^s \int_{\mathbb{R}^3} |\nabla^2(Q_{m+1} - Q_m)|^2 \, dx \, dt \\
\leq C \int_0^s \int_{\mathbb{R}^3} |v_m - v_{m+1}|^2 \, dx \, dt + C \int_0^s \int_{\mathbb{R}^3} |Q_{m+1} - Q_m|^2 \, dx \, dt 
\]
Using (5.2), we find

\[ + C(M_1 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |\nabla (Q_m - Q_{m-1})|^2 + |v_m - v_{m-1}|^2 + |Q_m - Q_{m-1}|^2 \, dx \, dt \]

\[ + C \eta_1 \int_0^s \int_{\mathbb{R}^3} |\nabla^2 (Q_{m+1} - Q_m)|^2 + |\nabla (v_m - v_{m-1})|^2 \, dx \, dt. \]  

(5.39)

Next we compute the difference involving \( v_m \). Multiplying \((5.30)\) by \((v_{m+1} - v_m)\), we have

\[ \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} (v_{m+1} - v_m)(x, s)^2 \, dx \, dt + \frac{3}{4} \int_0^s \int_{\mathbb{R}^3} |\nabla (v_{m+1} - v_m)|^2 \, dx \, dt \]

\[ \leq C \int_0^s \int_{\mathbb{R}^3} |\sigma(Q_m, \nabla Q_m) - \sigma(Q_{m-1}, \nabla Q_{m-1})|^2 \, dx \, dt \]

\[ + \int_0^s \int_{\mathbb{R}^3} (\nabla \cdot [Q_m, h(Q_m, Q_{m+1})] - \nabla \cdot [Q_{m-1}, h(Q_{m-1}, Q_m)], v_{m+1} - v_m) \, dx \]

\[ + \int_0^s \int_{\mathbb{R}^3} -(v_m \cdot \nabla v_m + v_{m-1} \cdot \nabla v_{m-1}, v_{m+1} - v_m) \, dx \, dt. \]  

(5.40)

Using \((5.2)\), we find

\[ C \int_0^s \int_{\mathbb{R}^3} |\sigma(Q_m, \nabla Q_m) - \sigma(Q_{m-1}, \nabla Q_{m-1})|^2 \, dx \, dt \]

\[ \leq C \int_0^s \int_{\mathbb{R}^3} |\nabla (Q_m - Q_{m-1})| (|\nabla Q_m|^2 + |\nabla Q_{m-1}|^2) + |Q_m - Q_{m-1}|^2 |\nabla Q_{m-1}|^4 \, dx \, dt \]

\[ \leq \eta_1 \int_0^s \int_{\mathbb{R}^3} |\nabla^2 (Q_m - Q_{m-1})|^2 \, dx \, dt + C(M_1 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |\nabla (Q_m - Q_{m-1})|^2 \, dx \, dt. \]  

(5.41)

Applying \((5.2)\) to the last term in \((5.40)\), we have

\[ \int_0^s \int_{\mathbb{R}^3} -(v_m \cdot \nabla v_m + v_{m-1} \cdot \nabla v_{m-1}, v_{m+1} - v_m) \, dx \, dt \]

\[ = \int_0^s \int_{\mathbb{R}^3} -(v_m \cdot \nabla v_m - v_{m-1} - v_m \cdot \nabla v_{m-1}, v_{m+1} - v_{m}) \, dx \, dt \]

\[ \leq C \int_0^s \int_{\mathbb{R}^3} |v_m||\nabla (v_m - v_{m-1})||v_{m+1} - v_m| \, dx \, dt \]

\[ + C \int_0^s \int_{\mathbb{R}^3} |v_{m-1}||\nabla (v_m - v_{m-1})||v_{m+1} - v_m| + |v_m - v_{m-1}||\nabla (v_{m+1} - v_m)| \, dx \, dt \]

\[ \leq \eta_1 \int_0^s \int_{\mathbb{R}^3} |\nabla (v_m - v_{m-1})|^2 \, dx \, dt + \frac{1}{4} \int_0^s \int_{\mathbb{R}^3} |\nabla (v_{m+1} - v_m)|^2 \, dx \, dt \]

\[ + C M_1^4 \int_0^s \int_{\mathbb{R}^3} |v_{m+1} - v_m|^2 + |v_m - v_{m-1}|^2 \, dx \, dt. \]  

(5.42)

Thus we can write \((5.40)\) as

\[ \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} (v_{m+1} - v_m)(x, s)^2 \, dx \, dt + \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} |\nabla (v_{m+1} - v_m)|^2 \, dx \, dt \]

\[ \leq 3 \eta_1 \int_0^s \int_{\mathbb{R}^3} |\nabla^2 (Q_m - Q_{m-1})|^2 + |\nabla (v_m - v_{m-1})|^2 \, dx \, dt \]

\[ + C M_1^4 \int_0^s \int_{\mathbb{R}^3} |\nabla (Q_m - Q_{m-1})|^2 + |v_{m+1} - v_m|^2 + |v_m - v_{m-1}|^2 \, dx \, dt \]
To cancel the Lie bracket term in (5.43), it follows from Lemma 2.8 with the substitution $A = Q_m, B = h(Q_m, Q_{m+1}), F = \Omega_{m+1}$ and the other three cases that
\[
\langle \{Q_m, \Omega_{m+1}\} - [Q_m, Q_{m+1}], h(Q_m, Q_{m+1}) - h(Q_m, Q_m) \rangle
\]
\[
= \langle \{Q_m, h(Q_m, Q_{m+1})\} - [Q_m, h(Q_m, Q_m)], \nabla (v_{m+1} - v_m) \rangle. 
\]
Multiplying (5.32) by $h(Q_m, Q_{m+1}) - h(Q_m, Q_m)$ and using (5.44), we obtain
\[
\int_0^s \int_{\mathbb{R}^3} \langle \nabla \cdot [Q_m, h(Q_m, Q_{m+1})] - \nabla \cdot [Q_m, h(Q_m, Q_m)], (v_{m+1} - v_m) \rangle \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} |h(Q_m, Q_{m+1}) - h(Q_m, Q_m)|^2 \, dx \, dt \\
= \int_0^s \int_{\mathbb{R}^3} \langle \partial_t (Q_m - Q_m), h(Q_m, Q_{m+1}) - h(Q_m, Q_m) \rangle \, dx \, dt \\
- \int_0^s \int_{\mathbb{R}^3} \langle -v_m \cdot \nabla Q_m + v_m \cdot \nabla Q_m - h(Q_m, Q_{m+1}) - h(Q_m, Q_m) \rangle \, dx \, dt \\
- \int_0^s \int_{\mathbb{R}^3} \langle g_B (Q_m) - g_B (Q_{m-1}), h(Q_m, Q_{m+1}) - h(Q_m, Q_m) \rangle \, dx \, dt \\
\leq \frac{1}{2} \int_0^s \int_{\mathbb{R}^3} \langle h(Q_m - Q_m) - h(Q_m, Q_{m+1}) \rangle \, dx \, dt + \int_0^s \int_{\mathbb{R}^3} \langle \partial_t (Q_m - Q_m) \rangle \, dx \, dt \\
+ C(\eta_3) \int_0^s \int_{\mathbb{R}^3} |\partial_\beta f E(Q_m, \nabla Q_m) - \partial_\beta f E(Q_m, \nabla Q_{m-1})|^2 \, dx \, dt \\
+ C(\eta_3) \int_0^s \int_{\mathbb{R}^3} |v_m \cdot \nabla Q_m - v_m \cdot \nabla Q_{m-1}|^2 + |g_B (Q_m) - g_B (Q_{m-1})|^2 \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \langle \partial_t (Q_m - Q_m), \nabla_\beta \partial_\rho \beta f E(Q_m, \nabla Q_m) - \nabla_\beta \partial_\rho \beta f E(Q_m, \nabla Q_{m-1}) \rangle \, dx \, dt.
\]

In a similar calculation to (5.32), using (5.24), we estimate the last term in (5.45)
\[
\int_0^s \int_{\mathbb{R}^3} \langle \partial_t (Q_m - Q_m), \nabla_\beta (\partial_\rho \beta f E(Q_m, \nabla Q_m) - \partial_\rho \beta f E(Q_m, \nabla Q_{m-1})) \rangle \, dx \, dt \\
= \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_m, \nabla Q_m) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
- \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_{m-1}, \nabla Q_m) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_m, \nabla Q_{m+1}) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
- \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_{m-1}, \nabla Q_{m+1}) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
\leq \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_m, \nabla Q_m) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_m, \nabla Q_{m+1}) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_m, \nabla Q_{m-1}) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt \\
+ \int_0^s \int_{\mathbb{R}^3} \partial_\beta \rho \beta f E(Q_{m-1}, \nabla Q_{m+1}) \nabla_\beta \partial_t (Q_m - Q_m) \, dx \, dt.
Substituting (5.46) into (5.45) and using with sufficiently small $\eta$

For the term $\partial_t(Q_m - Q_m)$, we find

Here we used the fact from (5.5) that

For the term $\partial_t(Q_m - Q_m)$, it follows from (5.32) that

Substituting (5.46) into (5.45) and using with sufficiently small $\eta$, we find

$$
\int_0^s \int_{\mathbb{R}^3} |\partial_t(Q_m + Q_m)|^2 \, dx dt \\
\leq C \int_0^s \int_{\mathbb{R}^3} |v_m \cdot Q_m - v_{m-1} \cdot Q_{m-1}|^2 + |Q_m, Q_{m+1} - Q_{m-1}, Q_m|^2 \, dx dt \\
+ C \int_0^s \int_{\mathbb{R}^3} |Q_m - Q_{m-1}|^2 + |Q_m|^4 + |g_B(Q_m)|^2 \, dx dt \\
\leq C M_1 \int_0^s \int_{\mathbb{R}^3} |Q_m - Q_{m-1}|^2 + |Q_m|^4 + |g_B(Q_m)|^2 \, dx dt
$$

(5.47)
\[ + C(M_1 + M_2^2 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |\nabla (Q_{m+1} - Q_m)|^2 + |\nabla (Q_m - Q_{m-1})|^2 \, dx \, dt \]
\[ + C(M_1 + M_2^4) \int_0^s \int_{\mathbb{R}^3} |v_m - v_{m-1}|^2 + |Q_m - Q_{m-1}|^2 \, dx \, dt. \]  \tag{5.48}

Adding (5.48) to (6.43), we have
\[ \frac{1}{2} \int_{\mathbb{R}^3} |v_{m+1} - v_m|^2(x, s) \, dx + \frac{1}{4} \int_{\mathbb{R}^3} |\nabla (v_{m+1} - v_m)|^2 \, dx \, dt \]
\[ \leq 2\eta_2 \int_0^s \int_{\mathbb{R}^3} |\nabla^2 (Q_{m+1} - Q_m)|^2 \, dx \, dt \]
\[ + C\eta_1 \int_0^s \int_{\mathbb{R}^3} |\nabla^2 (Q_m - Q_{m-1})|^2 + |\nabla (v_m - v_{m-1})|^2 \, dx \, dt \]
\[ + C(M_1 + M_2^2 + M_1^4) \int_0^s \int_{\mathbb{R}^3} |\nabla (Q_{m+1} - Q_m)|^2 + |\nabla (Q_m - Q_{m-1})|^2 \, dx \, dt \]
\[ + C(M_1 + M_2^4) \int_0^s \int_{\mathbb{R}^3} |v_m - v_{m-1}|^2 + |Q_m - Q_{m-1}|^2 \, dx \, dt. \]  \tag{5.49}

Substituting (5.49) into (6.49) and choosing suitable \( \eta_2 \), we obtain
\[ \sup_{0 \leq s \leq T} \int_{\mathbb{R}^3} |(Q_{m+1} - Q_m)|^2 + |\nabla (Q_{m+1} - Q_m)|^2 + |v_{m+1} - v_m|^2(x, s) \, dx \]
\[ + \int_0^T \int_{\mathbb{R}^3} |\nabla^2 (Q_{m+1} - Q_m)|^2 + |\nabla (v_{m+1} - v_m)|^2 \, dx \, dt \]
\[ \leq C_3 \eta_1 \int_0^T \int_{\mathbb{R}^3} |\nabla^2 (Q_{m+1} - Q_m)|^2 + |\nabla^2 (Q_m - Q_{m-1})|^2 + |\nabla (v_m - v_{m-1})|^2 \, dx \, dt \]
\[ + C_3(M_1 + M_2^2 + M_1^4) \sup_{0 \leq s \leq T} \int_{\mathbb{R}^3} |(\nabla (Q_{m+1} - Q_m)|^2 + |\nabla (Q_m - Q_{m-1})|^2) \, dx \, dx \]
\[ + C_3(M_1 + M_2^4) \sup_{0 \leq s \leq T} \int_{\mathbb{R}^3} |(v_m - v_{m-1}|^2 + |Q_m - Q_{m-1}|^2) \, dx \, dx, \]  \tag{5.50}

where \( C_3 \) is a constant independent from \( m \). Then for \( m > 1 \), choosing \( \eta_1 = \frac{1}{8C_3^{-1}} \), we prove the claim 2 with
\[ T := \frac{1}{8C_3^{-1}} \min \{M_1^{-1}, M_1^{-2}, M_1^{-4}\}. \]

Using the Claim 1, \((Q_{m+1}, v_{m+1})\) and \((Q_m, v_m)\) have two limits. By Claim 2, \((Q_{m+1}, v_{m+1})\) is a Cauchy sequence in \( L^\infty ([0, T]; H^2_{Q_m} \times L^2) \cap L^2 ([0, T]; H^2_{Q_m} \times H^1) \), so two weak limit of \((Q_{m+1}, v_{m+1})\) and \((Q_m, v_m)\) are the same. One can estimate \( P_m \) using (5.5) and the argument in (3.29), (3.30). As \( m \to \infty \), we prove Theorem 1.

**References**

[1] Ball, J. M.: Mathematics and liquid crystals. Mol. Cryst. Liq. Cryst. 647, 1–27 (2017)
[2] Ball, J. M., Majumdar, A.: Nematic liquid crystals: from Maier-Saupe to a continuum theory. Mol. Cryst. Liq. Cryst. 525, 1–11 (2010)
[3] Beris, A. N., Edwards, B. J.: Thermodynamics of flowing systems with internal micro-structure. Vol. 36, Oxford University Press, New York, 1994
[4] Berreman, D. W., Meiboom, S.: Tensor representation of Oseen-Frank strain energy in uniaxial cholesterics. Phys. Rev. A 30, 1955–1959 (1984)
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[5] Chen, Y., Struwe, M.: Existence and partial regular results for the heat flow for harmonic maps. Math. Z. 201, 83–103 (1989)
[6] de Gennes, P. G.: Short range order effects in the isotropic phase of nematics and cholesterics. Mol. Cryst. Liq. Cryst. 12, 193–214 (1971)
[7] de Gennes, P.G., Prost, J.: The physics of liquid crystals. 2nd ed. Oxford University Press, Oxford, 1993
[8] Dickmann, S.: Numerische berechnung von feld und molekülaufrichtung in flüssigkristallanzeigen. PhD thesis, University of Karlsruhe, (1995)
[9] Ericksen, J. L.: Conservation laws for liquid crystals, Trans. Soc. Rheol. 5, 23–34 (1961)
[10] Feng, Z., Hong, M.-C.: Existence of minimizers and convergence of critical points for a new Landau-de Gennes energy functional in nematic liquid crystals, Calc. Var. PDEs 61, 219 (2022)
[11] Feng, Z., Hong, M.-C., Mei, Y.: Convergence of the Ginzburg-Landau approximation for the Ericksen-Leslie system. SIAM J. Math. Anal. 52, 481–523 (2020)
[12] Gartland, Jr. E. C.: Scalings and limits of Landau-de Gennes models for liquid crystals: a comment on some recent analytical papers. Math. Model. Anal. 23, 414–432 (2018)
[13] Golovaty, D., Novack, M., Sternberg, P.: A novel Landau-de Gennes model with quartic elastic terms. Euro. Jnl. of Applied Mathematics, 1–22 (2020)
[14] Hong, M.-C.: Global existence of solutions of the simplified Ericksen-Leslie system in dimension two. Calc. Var. PDEs 40, 15–36 (2011)
[15] Hong, M.-C., Li, J., Xin, Z.: Blow-up Criteria of Strong Solutions to the Ericksen-Leslie System in $\mathbb{R}^3$, Commun. Partial Differ. Equ. 39, 1284-1328 (2014)
[16] Hong, M.-C., Mei, Y.: Well-posedness of the Ericksen-Leslie system with the Oseen-Frank energy in $L^3_{uloc}(\mathbb{R}^3)$. Calc. Var. PDEs 58, 3 (2019)
[17] Hong, M.-C., Xin, Z.: Global existence of solutions of the Liquid Crystal flow for the Oseen-Frank model in $\mathbb{R}^2$, Adv. Math. 231, 1364-1400 (2012)
[18] Kitavtsev, G., Robbins, J. M., Slastikov, V., Zarnescu, A.: Liquid crystal defects in the Landau-de Gennes theory in two dimensions - beyond the one-constant approximation. Math. Model. Methods Appl. Sci. 26, 2769–2808 (2016)
[19] Leslie, F. M.: Some constitutive equations for liquid crystals. Arch. Ration. Mech. Anal. 28, 265–283 (1968)
[20] Lin, F.-H., Liu, C.: Nonparabolic dissipative systems modeling the flow of liquid crystals. Comm. Pure Appl. Math., 48, 501–537 (1995)
[21] Lin, F.-H., Liu, C.: Existence of solutions for the Ericksen-Leslie System, Arch. Rational Mech. Anal. 154, 135–156 (2000)
[22] Mori, H., Gartland, E. C., Kelly, J. R., Bos, P. J.: Multidimensional director modeling using the $Q$ tensor representation in a liquid crystal cell and its application to the $\pi$ cell with patterned electrodes. Jpn. J. Appl. Phys. 38, 135–146 (1999)
[23] Majumdar, A., Zarnescu, A.: Landau-de Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. Arch. Ration. Mech. Anal. 196, 227–280 (2010)
[24] Mottram, N. J., Newton, C. J. P.: Introduction to $Q$-tensor theory. Preprint, arXiv 1409.3542v2, (2014)
[25] Nguyen, L., Zarnescu, A.: Refined approximation for minimizers of a Landau-de Gennes energy functional. Calc. Var. PDEs 47, 383–432 (2013)
[26] Nomizu, K.: Characteristic roots and vectors of a differentiable family of symmetric matrices. Linear Multilinear Algebra 1, 159–162 (1973)
[27] Paicu, M., Zarnescu, A.: Global existence and regularity for the full coupled Navier–Stokes and $Q$-tensor system. SIAM J. Math. Anal. 43, 2009–2049 (2011)
[28] Paicu, M., Zarnescu, A.: Energy dissipation and regularity for a coupled Navier–Stokes and $Q$-tensor system. Arch. Ration. Mech. Anal. 203, 45–67 (2012)
[29] Qian, T., Sheng, P.: Generalized hydrodynamic equations for nematic liquid crystals. Phys. Rev. E 58, 7475–7485 (1998)
[30] Schiele, K., Trimper, S.: Elastic constants of a nematic liquid crystal. Phys. Stat. Sol. (b) 118, 267–274 (1983)
[31] Schoen, R. and Uhlenbeck, K.: A regularity theory for harmonic maps, J. Diff. Geom., 17, 305-335 (1982)
[32] Struwe, M.: On the evolution of harmonic maps of Riemannian surfaces, Commun. Math. Helv. 60, 558–581 (1985)
[33] Wang, M., Wang, W., Zhang, Z.: From the Q-tensor flow for the liquid crystal to the harmonic map flow. Arch. Rational Mech. Anal. 225, 663–683 (2017)
[34] Wang, W., Zhang, P., Zhang, Z.: Rigorous derivation from Landau-de Gennes theory to Ericksen-Leslie theory. Siam J. Math. Anal. 47, 127-158 (2015)
[35] Xin, Z., Zhang, X.: From the Landau-de Gennes theory to the Ericksen-Leslie theory in dimension two. Preprint, arXiv:2105.10652v1, (2021)

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