Relativistic Corrections to the Exclusive Decays of $C$-even Bottomonia into $S$-wave Charmonium Pairs

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Abstract

Within the nonrelativistic quantum chromodynamics (NRQCD) factorization formalism, we compute the relativistic corrections to the exclusive decays of bottomonia with even charge conjugation parity into $S$-wave charmonium pairs at leading order in the strong coupling constant. Relativistic corrections are resummed for a class of color-singlet contributions to all orders in the charm-quark velocity $v_c$ in the charmonium rest frame. Almost every process that we consider in this work has negative relativistic corrections ranging from $-20$ to $-35\%$. Among the various processes, the relativistic corrections of the next-to-leading order in $v_c$ to the decay rate for $\chi_b^2 \rightarrow \eta_c(mS) + \eta_c(nS)$ with $m, n = 1$ or 2 are very large. In every case, the resummation of the relativistic corrections enhances the rate in comparison with the next-to-leading-order results. We compare our results with available predictions based on the NRQCD factorization formalism. The NRQCD predictions are significantly smaller than those based on the light-cone formalism by 1 or 2 orders of magnitude.

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I. INTRODUCTION

Among various bottomonium states $H$, the $\eta_b$ and $\chi_{bJ}$ mesons with $J^{PC} = 0^{-+}$ and $1^{++}$, respectively, have a common feature that the charge conjugation parity $C$ is even. Because the quantum chromodynamics (QCD) preserves $C$, these mesons can decay into a pair of charmonia $h_1 + h_2$ with the same $C$ parity. Therefore, possible decay modes of the mesons are

\begin{align*}
\eta_b \text{ or } \chi_{bJ} & \rightarrow \psi(mS) + \psi(nS), \quad (1a) \\
\eta_b \text{ or } \chi_{bJ} & \rightarrow \eta_c(mS) + \eta_c(nS), \quad (1b) \\
\eta_b \text{ or } \chi_{bJ} & \rightarrow \chi_{cJ} + \chi_{cJ}', \quad (1c)
\end{align*}

where the spin-triplet $S$-wave state $\psi(nS)$ is a $J^{PC} = 1^{--}$ eigenstate. However, some of these decay modes like $\eta_b \text{ or } \chi_{b1} \rightarrow \eta_c(mS) + \eta_c(nS)$ are forbidden due to the parity conservation in QCD.

Previous theoretical studies include the calculation of the decay rate for the process $\eta_b \rightarrow J/\psi + J/\psi$ \textsuperscript{[1]} within the nonrelativistic QCD (NRQCD) factorization formalism \textsuperscript{[2]}. This process was proposed as a candidate for the discovery mode of the $\eta_b$ meson \textsuperscript{[3]}. This work was followed by a study of the final-state interaction in Ref. \textsuperscript{[4]} and the next-to-leading-order (NLO) QCD corrections in Refs. \textsuperscript{[5, 6]}. In the case of the spin-triplet $P$-wave decay, the process $\chi_{bJ} \rightarrow J/\psi + J/\psi$ was investigated in Ref. \textsuperscript{[7]} within the light-cone (LC) formalism. The authors have extended their LC predictions to various channels in Ref. \textsuperscript{[8]}, where they also provided the NRQCD predictions at leading order (LO) in the charm-quark velocity $v_c$ in the charmonium rest frame. According to the results in Ref. \textsuperscript{[8]}, the LC predictions are greater than the NRQCD counterparts although they are in agreement within errors that are significant. One of the motivations of this work is to investigate if these discrepancies are reduced under relativistic corrections.\footnote{Very recently, the authors of Ref. \textsuperscript{[9]} have reported an NRQCD prediction for the decay $\chi_{bJ} \rightarrow J/\psi + J/\psi$ including the relativistic corrections of relative order $v_c^2$, where they used a velocity-expansion scheme that is different from the standard NRQCD approach employed in this work.}

In the NRQCD factorization formula, the decay rate is expanded in powers of the velocity $v_Q$ of the heavy quark $Q$ in the quarkonium rest frame. In the case of the LC formalism, the amplitude is expanded in powers of the inverse of the hard scale. In the limit of $m_b \gg m_c$...
as well as $1 \gg v_c^2 \gg v_b^2$, we can guess that the leading-twist LC prediction and the NRQCD prediction at LO in $v_Q$ are roughly consistent with each other. However, if there are large relativistic or QCD corrections, such a naive estimate may fail. For example, the cross section for the exclusive production process $e^+e^- \rightarrow J/\psi + \eta_c$ at the $B$ factories suffers large relativistic \cite{10} and QCD \cite{11} corrections. The NRQCD prediction including relativistic and QCD corrections is shown to be consistent with the empirical value within errors \cite{10}. One may guess that LC predictions as in Ref. \cite{12} provide a reasonable answer without further corrections. However, as is shown in Ref. \cite{13}, the LC calculation contains short-distance contributions, which are treated in the context of model LC distributions. These short-distance contributions appear in corrections of order $\alpha_s$ (and higher) in the NRQCD approach and can be computed from first principles in that approach. Therefore, it is very interesting to see what happens in the case of the bottomonium decay into charmonium pairs that have similar features as the exclusive process $e^+e^- \rightarrow J/\psi + \eta_c$.

In this work, we compute the decay rates for the $C$-even bottomonia $\eta_b$ and $\chi_{bJ}$ into pairs of $S$-wave charmonia within the NRQCD factorization formalism. The computation is carried out within the color-singlet mechanism of NRQCD because, in these exclusive processes, the color-octet channel enters only if at least two hadrons involve color-octet contributions that are suppressed.\footnote{See Ref. \cite{8} for further discussion regarding the suppression of the color-octet contributions.} In addition, we also neglect the electromagnetic decay mode such as $\eta_b \rightarrow \gamma^*\gamma^* \rightarrow J/\psi + J/\psi$ that is tiny compared with the QCD mode.\footnote{In the case of $e^+e^- \rightarrow J/\psi + J/\psi$ \cite{14, 15}, the LO cross section is comparable to that \cite{16, 17} of $e^+e^- \rightarrow J/\psi + \eta_c$ because of the enhancements due to the photon fragmentation and large collinear emissions from the electron line in the forward region. In the case of $\eta_b \rightarrow \gamma^*\gamma^* \rightarrow J/\psi + J/\psi$ only the photon fragmentation contributes and the collinear enhancement is missing because of the large bottom-quark mass. In addition, the fractional electric charge of the bottom quark makes the rate insignificant. See also Ref \cite{1}.} Relativistic corrections are computed by making use of the generalized Gremm-Kapustin relation \cite{18} that has been employed to resolve the $e^+e^- \rightarrow J/\psi + \eta_c$ puzzle \cite{10}. The method enables us to resum a class of color-singlet contributions to all orders in $v_Q^2$. Our calculation reveals that the discrepancies between the NRQCD prediction and the prediction based on the LC formalism become more severe, particularly in the $\chi_{b2}$ decay into an $S$-wave spin-singlet charmonium pair. This is in contrast to the case of $e^+e^- \rightarrow J/\psi + \eta_c$.

This paper is organized as follows. We first describe the NRQCD factorization formula
for the exclusive $C$-even bottomonium decay into an $S$-wave charmonium pair in Sec. [II]. In Sec. [III] we present the strategy to compute the short-distance coefficients for the NRQCD factorization formula. The analytic results for the decay rates at LO and the corrections of NLO in $v_c^2$ are given in Sec. [IV]. Our final numerical results for the decay rates, in which a class of the color-singlet contributions are resummed to all orders in $v_c^2$, are listed in Sec. [V] and compared with available predictions. Finally, we summarize the work in Sec. [VI].

II. NRQCD FACTORIZATION FORMULA

The exclusive decay of a bottomonium into a pair of charmonium states involves the annihilation of a $b\bar{b}$ pair followed by the creation of two pairs of $c\bar{c}$. One may guess that the generalization of the NRQCD factorization [2] for the electromagnetic decay or light-hadronic decay into this exclusive mode is possible. If we assume that the NRQCD factorization is valid for the exclusive decay of a bottomonium $H$ into a charmonium pair $h_1 + h_2$, then the decay rate can be expressed as a linear combination of the products of nonperturbative NRQCD matrix elements with numerous spectroscopic states. According to the velocity-scaling rules of NRQCD [2], these matrix elements are classified in powers of $v_Q$. These exclusive decay modes are dominated by the color-singlet channels as is stated in the previous section and we restrict ourselves to the color-singlet contributions. The typical velocity of the bottom quark $v_b$ in the initial state is significantly smaller than those of the final-state charmonia, $v_b^2 \sim 0.1 \ll v_c^2 \sim 0.3$ so that we neglect the relativistic effects of the bottom quark while we include the relativistic corrections of the charm quarks in the final-state charmonia.

Within the color-singlet mechanism at LO in $v_Q$, there are NRQCD matrix elements $\langle H|\mathcal{O}_1|H \rangle$ for the bottomonium decay and $\langle 0|\mathcal{O}_1^{h_i}|0 \rangle$ for the charmonium production that involve the decay rate $\Gamma[H \rightarrow h_1 + h_2]$ for $i = 1$ and 2. The spectroscopic states for the four-quark operators $\mathcal{O}_1$ and $\mathcal{O}_1^{h_i}$ are identical to those of the corresponding hadrons. In order to describe the relativistic corrections to the charmonium state, we denote $\langle 0|\mathcal{O}_1^{h_i}(m_i,n_i)|0 \rangle$.}

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4 For example, NRQCD factorization theorems for the exclusive quarkonium productions in $e^+e^-$ annihilation and $B$ decay have been proved [19] [21].

5 For the initial-state bottomonia $2S+1L_J = 1S_0$ and $3P_J$ with $J = 0$, 1, or 2 for $H = \eta_b$ and $\chi_{bJ}$, respectively. In the charmonium case, $2S+1L_J = 1S_0$ and $3S_1$ for $h_i = \eta_c$ and $J/\psi$, respectively.
as the NRQCD matrix element for the charmonium production that is of relative order \( u_c^{2(m_i+n_i)} \) in comparison with the LO matrix element \( \langle 0|\mathcal{O}_1|0 \rangle \). The spectroscopic state of the four-quark operator \( \mathcal{O}_{1, (m_i,n_i)}^{h_i} \) is again identical to that of \( h_i \). As a result, the NRQCD factorization formula for \( \Gamma[H \to h_1 + h_2] \) can be expressed as

\[
\Gamma[H \to h_1 + h_2] = \langle H|\mathcal{O}_1|H \rangle \sum_{m_i,n_i} c_{(m_1,n_1),(m_2,n_2)} \langle 0|\mathcal{O}_{1, (m_1,n_1)}^{h_1}|0 \rangle \langle 0|\mathcal{O}_{1, (m_2,n_2)}^{h_2}|0 \rangle ,
\]

(2)

where \( c_{(m_1,n_1),(m_2,n_2)} \) is the short-distance coefficient which is insensitive to the long-distance nature of the hadrons \( H, h_1, \) and \( h_2 \). These factors can be computed perturbatively in powers of the strong coupling \( \alpha_s \).

The LO color-singlet NRQCD four-quark operator \( \mathcal{O}_1 \) for the annihilation decay of the heavy quarkonium with the spectroscopic state \( ^2S^+L_J \) is of the form

\[
\mathcal{O}_1 = \psi^\dagger \mathcal{K}^{(2S^+L_J)} \chi \chi^\dagger \mathcal{K}^{(2S^+L_J)} \psi ,
\]

(3)

where \( \psi \) and \( \chi^\dagger \) are the Pauli spinor fields that annihilate \( Q \) and \( \bar{Q} \), respectively. For a bottomonium (charmonium), it is understood to be \( Q = b \) (\( c \)). Here, the operators \( \mathcal{K}^{(2S^+L_J)} \) are defined by

\[
\begin{align*}
\mathcal{K}^{(1S_0)} &= \mathbb{1}, \\
\mathcal{K}^{(3S_1)} &= \sigma^i, \\
\mathcal{K}^{(3P_0)} &= \frac{1}{\sqrt{3}}(-\frac{i}{2} \vec{D} \cdot \sigma), \\
\mathcal{K}^{(3P_1)} &= \frac{1}{\sqrt{2}}(-\frac{i}{2} \vec{D} \times \sigma)^i, \\
\mathcal{K}^{(3P_2)} &= -\frac{i}{2} \vec{D} (\sigma^i \sigma^j),
\end{align*}
\]

(4a, 4b, 4c, 4d, 4e)

where \( \mathbb{1} \) is the identity matrix for the spin and color space, \( \sigma^i \) is the Pauli matrix and \( \vec{D} \) is the gauge-covariant derivative. The notation \( A^{(ij)} \) in Eq. (4e) represents the symmetric traceless component \( \frac{1}{2}(A^{ij} + A^{ji}) - \frac{1}{3} A^{kk} \delta^{ij} \) of a Cartesian tensor \( A^{ij} \). The NRQCD matrix element \( \langle H|\mathcal{O}_1|H \rangle \) for the decay is averaged over the spin states of \( H \).

The NRQCD four-quark operators \( \mathcal{O}_{1, (m,n)}^{h_i} \) in Eq. (2) for the \( S \)-wave charmonium production can be expressed as

\[
\mathcal{O}_{1, (m,n)}^{h_i} = \frac{1}{2} \left[ \chi^\dagger \mathcal{K}_m^{(2S^+L_J)} \psi \left( \sum_{\lambda} a^\dagger_{h_\lambda} a_{h_\lambda} \right) \psi^\dagger \mathcal{K}_n^{(2S^+L_J)} \chi + \text{H.c.} \right],
\]

(5)
where $K_n$ are defined by

\begin{align}
K_n(^1S_0) &= (-\frac{i}{2} \nabla)^{2n}, \\
K_i(^3S_1) &= (-\frac{i}{2} \nabla)^{2n} \sigma^i.
\end{align}

As is explained earlier, the Pauli spinor fields are for the charm quark. We have listed only a class of the operators that contain ordinary derivatives rather than covariant derivatives. The neglect of the operators with gauge fields contributes first at relative order $v_c^4$ in the Coulomb gauge in which the matrix elements are evaluated. Applying the vacuum-saturation approximation, we can simplify these NRQCD matrix elements as

$$
\langle 0 | Q^{h}_{1,(m,n)} | 0 \rangle = (2J + 1) \langle 0 | \chi^\dagger K_m \psi | h \rangle \langle 0 | \chi^\dagger K_n \psi | h \rangle^* + O(v_c^4),
$$

where $J$ is the total angular momentum of the charmonium $h$. The spin-multiplicity factor $2J + 1$ appears because the spin states of the produced hadron $h$ are summed over in Eq. (5).

In carrying out the resummation of the relativistic corrections, it is convenient to define the ratio $\langle q^{2n}_h \rangle$ of the NRQCD matrix element of relative order $v_c^{2n}$ to the LO matrix element as

$$
\langle q^{2n}_h \rangle = \frac{\langle 0 | \chi^\dagger K_n \psi | h \rangle}{\langle 0 | \chi^\dagger K_0 \psi | h \rangle}.
$$

This ratio is independent of the polarization of $h$. In Ref. [22] a generalized version of the Gremm-Kapustin relation [24] was derived:

$$
\langle q^{2n}_h \rangle = \langle q^2 \rangle_h^n.
$$

This relation holds for the matrix elements in spin-independent-potential models. Thus this relation holds for both spin-singlet and -triplet states independently of the index $i$ up to corrections of $v_Q^2$ that break the heavy-quark spin symmetry. This relation has been applied to determine the NRQCD matrix elements for the $S$-wave quarkonium states precisely [18, 25] and to resum the relativistic corrections in various quarkonium processes [10, 23, 26].

Although there is an intrinsic limitation that the resummation of relativistic corrections with only the $Q\bar{Q}$ Fock-state contributions eventually has the predictive power up to corrections of relative order $v_Q^4$ [18], the method is still useful to improve the convergence in a process involving significant relativistic corrections. Such an example is the exclusive $J/\psi + \eta_c$.

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6 See Refs. [22, 23] for further discussion.
production in $e^+e^-$ annihilation \[10\]. We shall find in Sec. \[15\] that a large correction is indeed observed in the process $\chi b_2 \to \eta_c + \eta_c$. Because of the large relativistic corrections of relative order $v_c^4$, the theoretical prediction for the cross section can even be negative. This problem will be cured by resumming the relativistic corrections.

By employing the vacuum-saturation approximation and the generalized Gremm-Kapustin relation \[2\], we can express the NRQCD matrix element for the charmonium production as

$$\langle 0|\mathcal{O}_1|0 \rangle = (2J + 1)\langle q^2 \rangle_h^{m+n}\langle h|\mathcal{O}_1|h \rangle + O(v_c^4), \tag{10}$$

where the errors of order $v_c^4$ are from the vacuum-saturation approximation. Then the NRQCD factorization formula \[2\] for the decay is simplified into the form

$$\Gamma[H \to h_1 + h_2] = \langle H|\mathcal{O}_1|H \rangle\langle h_1|\mathcal{O}_1|h_1 \rangle\langle h_2|\mathcal{O}_1|h_2 \rangle \sum_{n_1,n_2} d_{n_1,n_2} \langle q^2 \rangle_{h_1}^{n_1}\langle q^2 \rangle_{h_2}^{n_2}, \tag{11}$$

where we have redefined the short-distance coefficient as $d_{n_1,n_2}$. The state $|H\rangle$ in the NRQCD matrix elements has the nonrelativistic normalization $\langle H(P)|H(P')\rangle = (2\pi)^3 \delta^{(3)}(P - P')$.

The short-distance coefficients $d_{n_1,n_2}$ are determined by the perturbative matching. In fact, we can construct the NRQCD factorization formula for the amplitude $A_{H\to h_1+h_2}$ under the vacuum-saturation approximation. Once we replace the hadrons $H, h_1,$ and $h_2$ with the perturbative heavy-quark-antiquark states $b\bar{b}, c\bar{c}_1,$ and $c\bar{c}_2$ with the same spectroscopic states as those of the corresponding hadrons, respectively, then we find that the $Q\bar{Q}$ counterpart $A_{b\bar{b}\to c\bar{c}_1+c\bar{c}_2}$ is calculable perturbatively as

$$A_{H\to h_1+h_2} = \sqrt{2m_H}\sqrt{2m_{h_1}}\sqrt{2m_{h_2}}\sqrt{\langle H|\mathcal{O}_1|H \rangle\langle h_1|\mathcal{O}_1|h_1 \rangle\langle h_2|\mathcal{O}_1|h_2 \rangle} \times \sum_{n_1,n_2} a_{n_1,n_2} \langle q^2 \rangle_{h_1}^{n_1}\langle q^2 \rangle_{h_2}^{n_2}, \tag{12a}$$

$$A_{b\bar{b}\to c\bar{c}_1+c\bar{c}_2} = \sqrt{\langle b\bar{b}|\mathcal{O}_1|b\bar{b} \rangle\langle c\bar{c}_1|\mathcal{O}_1|c\bar{c}_1 \rangle\langle c\bar{c}_2|\mathcal{O}_1|c\bar{c}_2 \rangle} \sum_{n_1,n_2} a_{n_1,n_2} q_1^{n_1} q_2^{n_2}, \tag{12b}$$

where $a_{n_1,n_2}$ is the short-distance coefficient at the amplitude level and $q_i$ is half the relative momentum of the $i$-th $c\bar{c}$ pair. While the hadron states like $|H\rangle$ and $|h_i\rangle$ are normalized nonrelativistically, we use the relativistic normalization for the heavy-quark state $|Q\rangle$: $\langle Q(P)|Q(P')\rangle = (2\pi)^3 2(m_Q^2 + P^2)^{1/2}\delta^{(3)}(P - P')$. The normalization factor $\sqrt{8m_Hm_{h_1}m_{h_2}}$ was introduced to make the amplitude $A_{H\to h_1+h_2}$ have the relativistic normalization like $A_{b\bar{b}\to c\bar{c}_1+c\bar{c}_2}$. Here, $m_H$ and $m_{h_i}$ are the masses of $H$ and $h_i$, respectively.
Because the short-distance coefficients are insensitive to the long-distance nature of the hadrons, the factor $a_{n_1,n_2}$ in Eq. (12) must be common to both $A_{bb \rightarrow c\bar{c}_1 + c\bar{c}_2}$ and $A_{H \rightarrow h_1 + h_2}$. The $Q\bar{Q}$ NRQCD matrix elements in Eq. (12b) are calculable perturbatively:

$$\langle \bar{b}b | O_1 | \bar{b}b \rangle = 2N_c(2E)^2,$$

(13a)

$$\langle \bar{c}c_i | O_1 | \bar{c}c_i \rangle = 2N_c(2E_i)^2,$$

(13b)

where $E$ and $E_i$ are the energies of the quark in the $Q\bar{Q}$ rest frame for the $\bar{b}b$ and $c\bar{c}_i$ pairs, respectively. In such a way, one can determine the short-distance coefficients $a_{n_1,n_2}$ and $d_{n_1,n_2}$. In the normalization of the $P$-wave matrix element (13) we have suppressed the factor $|q|^2$ that comes from the derivative operator.

III. PERTURBATIVE MATCHING

In this section, we present the method to compute the short-distance coefficients $d_{n_1,n_2}$ in the NRQCD factorization formula for the decay $H \rightarrow h_1 + h_2$ by employing the perturbative matching onto the full-QCD amplitude for $bb \rightarrow c\bar{c}_1 + c\bar{c}_2$.

A. Kinematics

The momenta for the quarkonia $H$, $h_1$, and $h_2$ are chosen to be $P$, $P_1$, and $P_2$, respectively. We list the definitions of the variables for the bottomonium $H$ or the $bb$ pair without any index. Variables with the index $i = 1$ or 2 indicate that they correspond to the charmonium $h_i$ or the $c\bar{c}_i$ pair. For convenience we list the definitions for various variables for a $Q\bar{Q}$ pair system without any index unless it is necessary.

We denote $P$ and $q$ by the total and half the relative momentum of a $Q\bar{Q}$ pair, respectively. Then the momenta for the quark ($p$) and the antiquark ($\bar{p}$) are expressed as

$$p = \frac{1}{2}P + q,$$

(14a)

$$\bar{p} = \frac{1}{2}P - q.$$

(14b)

\footnote{See, for example, Eq. (60) of Ref. 27.}
It is obvious, in the rest frame of the \(QQ\) pair, that \(P\) and \(q\) are orthogonal: \(P \cdot q = 0\). In that frame,

\[
P = (2E, 0),
\]

\[
q = (0, q),
\]

\[
p = (E, q),
\]

\[
\bar{p} = (E, -q),
\]

where \(E = \sqrt{m_Q^2 + q^2}\) is the energy of the \(Q\) or \(\bar{Q}\) in the \(QQ\) rest frame and \(m_Q\) is the mass of the heavy quark \(Q\). We assume that the \(Q\) and \(\bar{Q}\) are on their mass shells so that

\[
p^2 = \bar{p}^2 = m_Q^2,
\]

\[
P^2 = 4E^2.
\]

In addition, we define the ratio

\[
r \equiv \frac{m_c^2}{m_b^2},
\]

which is useful to simplify expressions.

**B. Spin and color projectors**

The Feynman diagrams for the process \(b\bar{b} \to c\bar{c}_1 + c\bar{c}_2\) at LO in \(\alpha_s\) are shown in Fig. 1. The corresponding perturbative amplitude for the process \(b\bar{b} \to c\bar{c}_1 + c\bar{c}_2\) is of the form

![Feynman diagrams](image-url)

**FIG. 1:** The Feynman diagrams for the exclusive decay of a \(C\)-even bottomonium decay into a pair of charmonium states at LO in \(\alpha_s\).
\[ \mathcal{A} = \frac{-ig_s^4}{(p_1 + \bar{p}_2)^2(p_2 + \bar{p}_1)^2} \bar{v}(\bar{p}, \bar{s}) B_{\mu\nu} u(p, s) \bar{u}(p_1, s_1) C^\mu v(\bar{p}_2, \bar{s}_2) \bar{u}(p_2, s_2) C^\nu v(\bar{p}_1, \bar{s}_1), \tag{18} \]

where \( g_s \) is the strong coupling, the denominator factor is from the gluon propagators and \( \bar{s} \) and \( s \) (\( \bar{s}_i \) and \( s_i \)) are the spins of \( \bar{b} \) and \( b \) (\( \bar{c} \) and \( c \) in the \( i \)-th \( c\bar{c} \) pair), respectively. \( B_{\mu\nu} \) contains the bottom-quark propagator and the gluon vertices to the bottom-quark line. \( C^\mu \) is the gluon vertex to the charm-quark pair \( c\bar{c} \) and we have suppressed the color indices. Both \( B_{\mu\nu} \) and \( C^\mu \) act on spinors with both Dirac and color indices.

In order to compute the amplitude for the perturbative process \( b\bar{b} \rightarrow c\bar{c}_1 + c\bar{c}_2 \) with the appropriate spectroscopic states, it is convenient to use the projection operators. In this work, we consider only the color-singlet contributions that can be projected out by replacing the color component of the outer product of the spinors for \( Q \) and \( \bar{Q} \) in each \( QQ \) pair with the color-singlet projector

\[ \pi_1 = \frac{1}{\sqrt{N_c}} \mathbb{1}, \tag{19} \]

where \( \mathbb{1} \) is the unit color matrix.

The spin-singlet and -triplet components of each \( QQ \) state can be projected out by making use of the spin projectors. After multiplying corresponding Clebsch-Gordan coefficients to the spin component of the outer product of the spinors for each \( QQ \) pair, one can find the spin-singlet and -triplet projectors \( \Pi_1 \) and \( \Pi_3 \) for the \( QQ \) decay and \( \bar{\Pi}_1 \) and \( \bar{\Pi}_3 \) for the \( QQ \) production, respectively. The spin projectors that are valid to all orders in the relative momentum can be found in Refs. [20, 28].

\[ \Pi_1 = -N(\hat{\mathbf{p}} + m_Q)(\mathbf{P} + 2E) \gamma_5 (\hat{\mathbf{p}} - m_Q), \tag{20a} \]
\[ \Pi_{3a}(\lambda)\epsilon^a(\lambda) = N(\hat{\mathbf{p}} + m_Q)(\mathbf{P} + 2E) \gamma_5 (\hat{\mathbf{p}} - m_Q), \tag{20b} \]
\[ \bar{\Pi}_1 = N(\hat{\mathbf{p}} - m_Q) \gamma_5 (\mathbf{P} + 2E) (\hat{\mathbf{p}} + m_Q), \tag{20c} \]
\[ \bar{\Pi}_{3a}(\lambda)\epsilon^{*a}(\lambda) = N(\hat{\mathbf{p}} - m_Q) \epsilon^*(\lambda)(\mathbf{P} + 2E) (\hat{\mathbf{p}} + m_Q), \tag{20d} \]

where the normalization factor \( N \) is

\[ N = \frac{1}{4\sqrt{2}E(E + m_Q)}, \tag{21} \]

if we choose the relativistic normalization for the spinors. \( \epsilon(\lambda) \) is the polarization four-vector of the spin-triplet state with the helicity \( \lambda \).
C. Projection of S- and P-wave contributions

We can extract the color-singlet amplitude \( A_1 \) with appropriate spin states of the three quarkonium states from the amplitude \( A \) as

\[
A_1 = \frac{-ig_s^4}{(p_1 + p_2)^2(p_2 + \bar{p}_1)^2} \text{Tr}[B_{\mu\nu}\Pi_{\bar{b}b}] \text{Tr}[C^\nu \Pi_{|c\bar{c}2} C^\mu \Pi_{|c\bar{c}1}],
\]

where \( B_{\mu\nu} \) and \( C^\mu \) are those in Eq. (18) and \( \Pi \) (\( \bar{\Pi} \)) is the direct product of the color and spin projectors defined in Eqs. (19) and (20):

\[
\Pi = \pi_1 \otimes (\Pi_1 \text{ or } \Pi_3 \epsilon^\alpha),
\]

\[
\bar{\Pi} = \pi_1 \otimes (\Pi_1 \text{ or } \bar{\Pi}_3 \epsilon^*\alpha).
\]

The trace in Eq. (22) is over the color and spin indices.

As the next step, we need to pull out the \( L \)-wave amplitude, where \( L = S \) or \( P \), from the color-singlet amplitude \( A_1 \) in Eq. (22) with correct spin states. In the case of the initial states \( H = \eta_b \) and \( \chi_{bJ} \), we need to project out the spin-singlet \( S \)-wave and spin-triplet \( P \)-wave states, respectively. As we have stated earlier, we consider only the contributions of LO in \( v_b \) in the bottomonium sector. Because the dependence of the momenta \( P \) and \( q \) for the \( b\bar{b} \) pair is isolated in the tensor \( B_{\mu\nu} \) in Eq. (22), the projection of the \( b\bar{b} \) state can be made only with this factor as

\[
B_{\mu\nu}^{\mu\nu} = \text{Tr}[B_{\mu\nu}(\pi_1 \otimes \Pi_1)|_{bb}] \bigg|_{q=0},
\]

\[
B_{\mu\nu}^{\mu\nu} = \mathcal{P}_j^{\alpha\beta} \frac{\partial}{\partial q^\alpha} \text{Tr}[B_{\mu\nu}(\pi_1 \otimes \Pi_3 \beta)|_{bb}] \bigg|_{q=0},
\]

where \( \mathcal{P}_j^{\mu\nu} \) projects the total angular momentum state \( J \) in the \( P \)-wave spin-triplet contribution that are defined by

\[
\mathcal{P}_0^{\alpha\beta} = \frac{1}{\sqrt{3}} I^{\alpha\beta},
\]

\[
\mathcal{P}_1^{\alpha\beta} = \frac{i}{2\sqrt{2}E} \epsilon^{\alpha\beta\rho\sigma} P_\rho \epsilon_\sigma(\lambda),
\]

\[
\mathcal{P}_2^{\alpha\beta} = \epsilon^{\alpha\beta}(\lambda).
\]

Here, \( \epsilon^\alpha(\lambda) \) and \( \epsilon^{\alpha\beta}(\lambda) \) are the polarization vector and tensor for the \( {}^3P_1 \) and \( {}^3P_2 \) states with the helicity \( \lambda \), respectively. In the normalization of \( B_{\mu\nu}^{\mu\nu} \) in Eq. (24), we have suppressed
the factor $|q|$ to make it to be consistent with the normalization of the matrix element \([13]\). The tensor $I^{\alpha\beta}$ is defined by
\[
I^{\alpha\beta} = -g^{\alpha\beta} + \frac{p^\alpha p^\beta}{P^2}.
\] (26)

Finally, we summarize the method to project out the S-wave component of the amplitude from $A_1$ in Eq. (22) for the two-charmonium final states that are either a spin-singlet or a triplet including relativistic corrections. Because $A_1$ in Eq. (22) is the color-singlet amplitude of the $c\bar{c}_i$ pairs with appropriate spin states, we only need to pull out the $S$-wave component that is independent of the direction of $q_i$ but may depend on $q_i^2$. We notice that the trace factor that includes $B^{\mu\nu}$ depends on both $q_1$ and $q_2$ through the bottom-quark propagator. The trace factor for the charm-quark pairs and the factor of the gluon propagator $1/[(p_1 + \bar{p}_2)^2(p_2 + \bar{p}_1)^2]$ have the dependence on both $q_1$ and $q_2$. After taking the average over the directions of $q_1$ and $q_2$, we find our final expression for the perturbative amplitude $A_{b\bar{b} \to c\bar{c}_1 + c\bar{c}_2}$ as
\[
A_{b\bar{b} \to c\bar{c}_1 + c\bar{c}_2} = -ig_5^4 \int \frac{d\Omega_1 d\Omega_2}{(4\pi)^2} B^{\mu\nu}_{b\bar{b}1} (p_1 + \bar{p}_2)^2(p_2 + \bar{p}_1)^2, 
\] (27)
where $B^{\mu\nu}_{b\bar{b}1}$ is defined in Eq. (24). Note the $b\bar{b}$ and $c\bar{c}_i$ pairs in Eq. (27) have definite spectroscopic states that are suppressed. Here, $d\Omega_i$ is the solid-angle element that represents the direction of $q_i$.

**D. Short-distance coefficients**

The perturbative amplitude $A_{b\bar{b} \to c\bar{c}_1 + c\bar{c}_2}$ depends only on $m_b$, $m_c$, $q_i^2$ and polarizations $\lambda$ and $\lambda_i$ of $b\bar{b}$ and $c\bar{c}_i$ pairs, respectively. By making use of the normalization of the NRQCD matrix elements for the $Q\bar{Q}$ states in Eq. (13), we can find the NRQCD factorization formula for the amplitude of the decay $H(\lambda) \to h_1(\lambda_1) + h_2(\lambda_2)$
\[
A_{H \to h_1 + h_2} = \sqrt{8m_H m_{h_1} m_{h_2}} \sqrt{\frac{\langle H|O_1|H\rangle\langle h_1|O_1|h_1\rangle\langle h_2|O_1|h_2\rangle}{(2E)^2(2E_1)^2(2E_2)^2(2N_c)^3}} A_{b\bar{b} \to c\bar{c}_1 + c\bar{c}_2}|_{q_i^2 \to (q^2)_{h_i}}, 
\] (28)
where we have suppressed the helicities of the hadrons that are the same as those for the $Q\bar{Q}$ pairs. Note that $q_i^2$ in the perturbative amplitude $A_{b\bar{b} \to c\bar{c}_1 + c\bar{c}_2}$ is replaced with the ratio $\langle q^2 \rangle_{h_i}$.

Now we finally find the NRQCD factorization formula for $\Gamma[H \to h_1 + h_2]$:
\[
\Gamma[H \to h_1 + h_2] = \int \frac{d\Phi_2}{(2J+1)2m_H} \sum_{\lambda,\lambda_1,\lambda_2} |A_{H(\lambda) \to h_1(\lambda_1) + h_2(\lambda_2)}|^2, 
\] (29)
where $J$ is the total angular momentum of $H$, $\int d\Phi_2$ is the phase space of the final state and the summation is over the polarizations of the initial and final states. The factor $2m_H$ in the denominator cancels that of the squared amplitude. After summing over the polarization states of the particles, the squared amplitude becomes invariant under rotation and the phase-space integral becomes trivial

$$\Phi_2 = \int d\Phi_2 = \frac{P_{\text{CM}}}{4\pi m_H S},$$

where $S$ is the symmetry factor, $S = 1$ ($S = 2$) for $h_1 \neq h_2$ ($h_1 = h_2$). $P_{\text{CM}}$ is the magnitude of the three-momentum of $h_i$ in the $H$ rest frame

$$P_{\text{CM}} = \frac{\lambda_i^{1/2}(m_H^2, m_{h_1}^2, m_{h_2}^2)}{2m_H},$$

where $\lambda(x, y, z) \equiv x^2 + y^2 + z^2 - 2(xy + yz + xz)$. Note that we use the physical masses $m_H$ and $m_{h_i}$ in evaluating the phase space $\Phi_2$. This prescription respects the physical endpoints of the phase space without spoiling the gauge invariance.

In evaluating the $S$-wave amplitude (27), we follow the strategy given in Ref. [10]. In order to investigate the convergence of the series in $v^2_{\text{c}}$, we also provide the fixed-order prediction at LO and NLO in $v^2_{\text{c}}$. To compute the fixed-order relativistic corrections, we first expand the amplitude (27) in powers of $q_i$ and then take the angle average of $q_i$-dependent tensors by making use of the following formulas:

$$\int \frac{d\Omega_i}{4\pi} q_i^\mu = 0,$$  \hspace{1cm} (32a)

$$\int \frac{d\Omega_i}{4\pi} q_i^\mu q_i^\nu = \frac{q_i^2}{3} I_i^{\mu\nu},$$  \hspace{1cm} (32b)

$$\int \frac{d\Omega_i}{4\pi} q_i^\mu q_i^\nu q_i^\alpha = 0,$$  \hspace{1cm} (32c)

$$\int \frac{d\Omega_i}{4\pi} q_i^\mu q_i^\nu q_i^\alpha q_i^\beta = \frac{q_i^4}{15} \left( I_i^{\mu\nu} I_i^{\alpha\beta} + I_i^{\mu\alpha} I_i^{\nu\beta} + I_i^{\mu\beta} I_i^{\nu\alpha} \right),$$  \hspace{1cm} (32d)

where $I_i^{\mu\nu}$ is the same as $I^{\mu\nu}$ defined in Eq. (26) except that we replace $P$ with $P_i$.

Our final expression for the short-distance coefficient $d_{n_1, n_2}$ in the NRQCD factorization formula (11) for the decay rate $\Gamma[H \rightarrow h_1 + h_2]$ is

$$d_{n_1, n_2} = \frac{4m_{h_1} m_{h_2}}{n_1! n_2!} \Phi_2 \frac{\partial^{n_1}}{\partial q_{1}^{2n_1}} \frac{\partial^{n_2}}{\partial q_{2}^{2n_2}} \left[ \sum_{\lambda_1, \lambda_2} \frac{A_{\lambda_1 \rightarrow \lambda_2} c_{\lambda_1} c_{\lambda_2}}{(2J + 1)(2N_c)^3(2m_b)^2(2E_1)^2(2E_2)^2} \right] \bigg|_{\frac{q_1^2 + q_2^2}{m_H^2} = 0}. \hspace{1cm} (33)$$

See Eq. (12a).
In the summation of the polarization states for the spin-1 and -2 quarkonia, we use the following formulas:

\[
\sum_{\lambda} \epsilon^\alpha(\lambda) \epsilon^{\mu}(\lambda) = I^{\alpha\mu}, \tag{34a}
\]

\[
\sum_{\lambda} \epsilon^{\alpha\beta}(\lambda) \epsilon^{*\mu\nu}(\lambda) = \frac{1}{2}(I^{\alpha\mu} I^{\beta\nu} + I^{\alpha\nu} I^{\beta\mu}) - \frac{1}{3} I^{\alpha\beta} I^{\mu\nu}, \tag{34b}
\]

where \(I^{\mu\nu}\) is defined in Eq. (26). For the spin-1 and -2 charmonium \(h_i\), \(I^{\mu\nu}\) must be replaced with \(I^{\mu\nu}_i\).

**IV. THE DECAY RATE UP TO NLO IN \(v_2^2\)**

Now it is straightforward to compute the decay rate based on the strategy and techniques described in Sec. III. In this section, we present the analytic expressions for the decay rates \(\Gamma[H \to h_1 + h_2]\) for \(H = \eta_b, \chi_{b1}, \chi_{b2}\) and for \(h_i = \psi(mS), \eta_c(mS)\) up to NLO in \(v_2^2\). In the case of \(\chi_{b1} \to J/\psi + J/\psi\), the LO contribution first appears at order \(v_2^4\) and the decay rate is of order \(v_2^6\). This process is so strongly suppressed that we present only the LO contribution.

We write the decay rate of the form

\[
\Gamma[H \to h_1 + h_2] = \Gamma^{LO}[H \to h_1 + h_2](1 + R), \tag{35}
\]

where \(\Gamma^{LO}\) is the LO contribution and \(R\) is the ratio of the relativistic corrections to the LO contribution. We also define \(R_2\) as the corresponding ratio at NLO in \(v_2^2\). In the following, we use the symbols \(\psi_i\) and \(\eta_i\) for \(h_i = \psi(mS)\) and \(\eta_c(mS)\), respectively.

**A. \(\eta_b \to \psi_1 + \psi_2\)**

Our analytic results for \(\Gamma^{LO}\) and \(R_2\) for \(\eta_b \to \psi_1 + \psi_2\) are given by

\[
\Gamma^{LO}[\eta_b \to \psi_1 + \psi_2] = \frac{4096\pi^3 \alpha_s^4 (1 - 4r)^{3/2}}{6561 m_b^{12} r S} \langle \eta_b | O_1 | \eta_b \rangle \langle \psi_1 | O_1 | \psi_1 \rangle \langle \psi_2 | O_1 | \psi_2 \rangle \times \left( (q^2)_{\psi_1} + (q^2)_{\psi_2} \right)^2, \tag{36a}
\]

\[
R_2[\eta_b \to \psi_1 + \psi_2] = \frac{-1}{15m_b^2 r (1 - 4r)(q^2)_{\psi_1} + (q^2)_{\psi_2}} \left[ 3((q^2)_{\psi_1} + (q^2)_{\psi_2})(3 + r + 8r^2) - 5(q^2)_{\psi_1}(q^2)_{\psi_2}(1 - 18r - 16r^2) \right], \tag{36b}
\]
where the symmetry factor $S = 2$ for $\psi_1 = \psi_2$ and, otherwise, $S = 1$. The variable $r$ is defined in Eq. (17). For simplicity, we have put $m_{h_i} = 2E_i$ in Eq. (36) including the phase-space factor $\Phi_2$. However, when we present the numerical results, we will use the expression (31) to evaluate $\Phi_2$. As shown in Eq. (36), $\Gamma^{LO}_{\eta_b \to \eta_1 + \eta_2}$ is of order $v_0^4$. In the color-singlet spin-singlet $b\bar{b}$ contribution in Eq. (24a), the Dirac trace of $\text{Tr}[\mathcal{B}_{\mu
u}\Pi_{b\bar{b}}]$ in Eq. (22) must be proportional to $e^{\mu \nu \alpha \beta} k_{1\alpha} k_{2\beta}$, where $k_1 = p_1 + \bar{p}_2$ and $k_2 = p_2 + \bar{p}_1$ are the momenta for the virtual gluons. In the limit $v_c = 0$, however, the two momenta become identical to each other $k_1 = k_2 = (P_1 + P_2)/2 = (m_b, 0, 0, 0)$, and therefore the amplitude vanishes before the vector indices are contracted to the polarization four-vectors of $\psi_1$ and $\psi_2$. As a result, the leading contribution to the amplitude begins at order $v_0^2$ and $\Gamma^{LO}_{\eta_b \to \eta_1 + \eta_2}$ is of order $v_0^4$. The expression for $\Gamma^{LO}_{\eta_b \to \eta_1 + \eta_2}$ in Eq. (36) agrees with Eq. (22) of Ref. 1.

The result for $R_2$ is new.

B. $\chi_{b0} \to \eta_1 + \eta_2$

The results for $\chi_{b0} \to \eta_1 + \eta_2$ are given by

$$\Gamma^{LO}_{\chi_{b0} \to \eta_1 + \eta_2} = \frac{2048\pi^3 \alpha_s^4 (1 + 2r)^2 \sqrt{1 - 4r}}{2187 m_b^{10} r S} \langle \chi_{b0} | O_1 | \chi_{b0} \rangle \langle \eta_1 | O_1 | \eta_1 \rangle \langle \eta_2 | O_1 | \eta_2 \rangle, \quad (37a)$$

$$R_2[\chi_{b0} \to \eta_1 + \eta_2] = -\frac{2(1 - 3r + 3r^2 + 8r^3)(\langle q^2 \rangle_{m_1} + \langle q^2 \rangle_{m_2})}{3m_b^2 r (1 - 2r - 8r^2)}. \quad (37b)$$

While the amplitude for $\eta_b \to \psi_1 + \psi_2$ begins at order $v_0^2$, the LO contribution to the amplitude $A_{\chi_{b0} \to \eta_1 + \eta_2}$ begins at relative order $v_0^0$. The analytic expression for $\Gamma^{LO}$ was also given in Ref. 8, where the authors evaluated the phase-space factor $\Phi_2$ in the limit of $m_{h_i} \to 0$. Aside from this difference in the phase-space factor, the analytic expression for $\Gamma^{LO}_{\eta_b \to \eta_1 + \eta_2}$ in Eq. (37) is smaller than that in Ref. 8 by a factor of 2.

C. $\chi_{b2} \to \eta_1 + \eta_2$

The results for $\chi_{b2} \to \eta_1 + \eta_2$ are given by

$$\Gamma^{LO}_{\chi_{b2} \to \eta_1 + \eta_2} = \frac{1024\pi^3 \alpha_s^4 (1 - 4r)^{5/2}}{10935 m_b^{10} r S} \langle \chi_{b2} | O_1 | \chi_{b2} \rangle \langle \eta_1 | O_1 | \eta_1 \rangle \langle \eta_2 | O_1 | \eta_2 \rangle, \quad (38a)$$

$$R_2[\chi_{b2} \to \eta_1 + \eta_2] = -\frac{(7 + 10r - 32r^2)(\langle q^2 \rangle_{m_1} + \langle q^2 \rangle_{m_2})}{6m_b^2 r (1 - 4r)}. \quad (38b)$$
As in the case of $\Gamma^{\text{LO}}[\eta_b \to \eta_1 + \eta_2]$, $\Gamma^{\text{LO}}[\chi_{b2} \to \eta_1 + \eta_2]$ in Eq. (38) is smaller than the corresponding result in Ref. [8] by a factor of 2. The result for $R_2$ is new.

D. $\chi_{b0} \to \psi_1 + \psi_2$

The LO decay rate and relativistic correction factor $R_2$ for $\chi_{b0} \to \psi_1 + \psi_2$ are

$$\Gamma^{\text{LO}}[\chi_{b0} \to \psi_1 + \psi_2] = \frac{2048\pi^3\alpha_s^4(1 - 4r + 12r^2)\sqrt{1 - 4r}}{2187m_b^{10}rS} \langle \chi_{b0}|O_1|\chi_{b0} \rangle \times \langle \psi_1|O_1|\psi_1 \rangle \langle \psi_1|O_1|\psi_1 \rangle \langle \psi_2|O_1|\psi_2 \rangle,$$

(39a)

$$R_2[\chi_{b0} \to \psi_1 + \psi_2] = -\frac{2(2 - 11r + 14r^2 + 16r^3)}{m_b^2(1 - 8r + 28r^2 - 48r^3)} \langle \psi_1|O_1|\psi_1 \rangle \langle \psi_2|O_1|\psi_2 \rangle.$$

(39b)

The results in Eq. (39) agree with those Ref. [9]. Like $\Gamma^{\text{LO}}[\eta_b \to \eta_1 + \eta_2]$ and $\Gamma^{\text{LO}}[\chi_{b2} \to \eta_1 + \eta_2]$, $\Gamma^{\text{LO}}[\chi_{b0} \to \psi_1 + \psi_2]$ in Eq. (39) is smaller than the corresponding result in Ref. [8] by a factor of 2. The result for $R_2$ is new.

E. $\chi_{b1} \to \psi_1 + \psi_2$

The LO amplitude for $\chi_{b1} \to \psi_1 + \psi_2$ is of order $v_c^4$ to make the decay rate of order $v_c^8$. Because the process is highly suppressed, we only present the decay rate at the LO in $v_c$:

$$\Gamma^{\text{LO}}[\chi_{b1} \to \psi_1 + \psi_2] = \frac{512\pi^3(1 - 4r)^{5/2}\alpha_s^4}{4428675m_b^{18}r^4S} \langle \chi_{b1}|O_1|\chi_{b1} \rangle \langle \psi_1|O_1|\psi_1 \rangle \langle \psi_2|O_1|\psi_2 \rangle$$

$$\langle \psi_1|O_1|\psi_1 \rangle \langle \psi_2|O_1|\psi_2 \rangle \times [9r(8r + 25)(\langle q^2 \rangle_{\psi_1}^4 + \langle q^2 \rangle_{\psi_2}^4) + 480r(4r + 2)(\langle q^2 \rangle_{\psi_1}^2 + \langle q^2 \rangle_{\psi_2}^2)] + 472r^2 + 1375r + 1600. (40)

The expression $\Gamma^{\text{LO}}[\chi_{b1} \to \psi_1 + \psi_2]$ in Eq. (40) is new.

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Since the authors in Ref. [8] adopted a different relativistic expansion, the comparison has been made after expanding $m_{J/\psi} = 2\sqrt{m_c^2 + q^2}$ in their expressions in powers of $q^2$.
F. $\chi_{b2} \rightarrow \psi_1 + \psi_2$

The LO decay rate and relativistic correction factor $R_2$ for $\chi_{b2} \rightarrow \psi_1 + \psi_2$ are

$$\Gamma^{\text{LO}}[\chi_{b2} \rightarrow \psi_1 + \psi_2] = \frac{1024\pi^3\alpha_s^4(13 + 56r + 48r^2)}{10935m_b^5 r S} \langle \chi_{b2}|O_1|\chi_{b2}\rangle \times \langle \psi_1|O_1|\psi_1\rangle \langle \psi_2|O_1|\psi_2\rangle,$$

$$R_2[\chi_{b2} \rightarrow \psi_1 + \psi_2] = -\frac{(13 + 62r - 32r^2 - 608r^3 - 512r^4)(\langle q^2\rangle_\psi_1 + \langle q^2\rangle_\psi_2)}{m_b^2r(26 + 8r - 352r^2 - 384r^3)}.$$ (41a)

While the results in Eq. (41) are in agreement with those in Ref. [9], $\Gamma^{\text{LO}}[\chi_{b2} \rightarrow \psi_1 + \psi_2]$ is smaller than the corresponding result in Ref. [8] by a factor of 2.

V. RESUMMATION OF RELATIVISTIC CORRECTIONS AND NUMERICAL RESULTS

In Sec. IV we have listed the NRQCD factorization formulas for the decay rates $\Gamma[H \rightarrow h_1 + h_2]$ for various processes, in which the LO predictions and the NLO corrections with respect to $v_c^2$ are included. In this section, we provide our predictions for these decay rates including relativistic corrections to all orders in $v_c^2$ in which a class of color-singlet contributions are resummed. Still we neglect $v_b$. The approach that was employed in Sec. IV at fixed orders in $v_c^2$ cannot be used to resum the relativistic corrections to all orders in $v_c^{2n}$. Instead of carrying out the $v_c^2$ expansion order by order, we evaluate the average of the amplitude (27) over the direction of $q_i$ numerically following a previous analysis in Sec. IV of Ref. [10]. $^{10}$ Because the calculations are carried out for the perturbative amplitude, expressions are for the parton process $b\bar{b} \rightarrow c\bar{c} + c\bar{c}$ rather than the process $H \rightarrow h_1 + h_2$.

For simplicity, we carry out the calculation in the rest frame of the initial $b\bar{b}$ pair. In this frame the explicit components of the four-momenta for the $b\bar{b}$, $c\bar{c}_1$, and $c\bar{c}_2$ are given by

$$P^* = (2m_b, 0, 0, 0),$$

$$P_1^* = (\tilde{E}_1, 0, 0, P_{\text{CM}}),$$

$$P_2^* = (\tilde{E}_2, 0, 0, -P_{\text{CM}}).$$ (42a)

$^{10}$ In some simple cases, complete analytic expressions that contain the relativistic corrections resummed to all orders in $v_c^2$ are known. See, for example, Refs. [23, 25, 26].
where the superscript in a four-vector $V^*$ indicates that the four-vector is defined in the $b\bar{b}$ rest frame, $P_{CM}$ and $\tilde{E}_i$ are the momentum and the energy of the $c\bar{c}_i$ pair:

$$P_{CM} = \frac{\lambda^{1/2} \left[(2m_b)^2, (2E_1)^2, (2E_2)^2 \right]}{4m_b},$$

(43a)

$$\tilde{E}_i = \sqrt{(2E_i)^2 + P_{CM}^2}.$$  

(43b)

We can parametrize the momentum $q_i$ in the rest frame of the $c\bar{c}_i$ rest frame as

$$q_i = |q_i|(0, \sin \theta_i \cos \phi_i, \sin \theta_i \sin \phi_i, \cos \theta_i),$$

(44)

where $\theta_i$ and $\phi_i$ are the polar and azimuthal angles of $q_i$ in the $c\bar{c}_i$ rest frame. After boosting $q_1$ and $q_2$ to the $b\bar{b}$ rest frame, we find that

$$q_1^* = |q_1|\left(+\gamma_1 \beta_1 \cos \theta_1, \sin \theta_1 \cos \phi_1, \sin \theta_1 \sin \phi_1, \gamma_1 \cos \theta_1 \right),$$

(45a)

$$q_2^* = |q_2|\left(-\gamma_2 \beta_2 \cos \theta_2, \sin \theta_2 \cos \phi_2, \sin \theta_2 \sin \phi_2, \gamma_2 \cos \theta_2 \right),$$

(45b)

where

$$\gamma_i = \frac{1}{\sqrt{1 - \beta_i^2}},$$

(46a)

$$\beta_i = \frac{P_{CM}}{E_i}.$$  

(46b)

In evaluating the angle average of the amplitude, we have to choose the numerical values for the input parameters such as the strong coupling $\alpha_s$, heavy-quark mass $m_Q$, NRQCD matrix elements and physical masses $m_H$ and $m_{h_i}$ of the hadrons. In this work, we take $\alpha_s(m_b) = 0.215$ that is, within uncertainties, consistent with previous analyses on $\chi_{bJ} \rightarrow c\bar{c} + X$ [27], $\Upsilon(1S) \rightarrow c\bar{c} + X$ [29], and $\eta_b \rightarrow \gamma\gamma$ [25]. The masses of the heavy quarks $c$ and $b$ are chosen to be the one-loop pole masses $m_c = 1.4$ GeV and $m_b = 4.6$ GeV, respectively, following previous NRQCD analyses [14, 16, 25, 27, 29]. The most recent fit for the matrix element for $\eta_b(1S)$ can be found in Ref. [25], which is consistent, within uncertainties, with the values used in Ref. [8]. That of the $P$-wave states $\chi_{bJ}$ can be found in Ref. [27]. In the case of $J/\psi$, the color-singlet NRQCD matrix elements were fit to the electromagnetic decay rate of the state in which the relativistic corrections are resummed to all orders in $v_c^{2n}$ including the order-$\alpha_s$ corrections. These values can be found in Ref. [18]. For the spin-singlet states $\eta_c(1S)$ and $\eta_c(2S)$, we quote the recent values in Ref. [30] that were fitted to the light-hadronic and electromagnetic decay rates with the NRQCD factorization formula.
TABLE I: The color-singlet NRQCD matrix elements (ME) \( \langle h | O_1 | h \rangle \) that is LO in \( v_Q \) and the ratio \( \langle q^2 \rangle_h \) for various heavy quarkonia \( h \). We keep only two digits of significant figures.

| ME \( \hbar \) | \( \eta_b \) | \( \chi_{bJ} \) | \( \eta_c \) | \( \eta_c(2S) \) | \( J/\psi \) |
|-----------------|-------------|----------------|-------------|----------------|-----------|
| \( \langle \mathcal{O}^0 \rangle_h \) | 3.1 GeV\(^3\) | 2.0 GeV\(^5\) | 0.40 GeV\(^3\) | 0.20 GeV\(^3\) | 0.44 GeV\(^3\) |
| \( \langle q^2 \rangle_h \) | – | – | 0.45 GeV\(^2\) | 0.50 GeV\(^2\) | 0.44 GeV\(^2\) |

TABLE II: The decay rate for the process \( H \to h_1 + h_2 \). \( \Gamma^{LO} \) is the NRQCD prediction at LO in \( v_b \) and \( v_c \). \( \Gamma \) is that in which the relativistic corrections of all orders in \( v_c^{2n} \) are resummed. The relativistic correction factor \( R \) defined in Eq. \( (35) \) resummed to all orders in \( v_c^{2n} \) and \( R_2 \) at NLO order in \( v_c^2 \). \( \Gamma^{LC} \) represents the LC prediction based on the formula given in Ref. [8]. \( \Gamma^{LO} \), \( \Gamma \) and \( \Gamma^{LC} \) are in units of eV.

| channel | \( \Gamma^{LO} \) | \( \Gamma \) | \( R_2 \) | \( R \) | \( \Gamma^{LC} \) |
|---------|----------------|-------------|-------------|---------|--------|
| \( \eta_b \to J/\psi + J/\psi \) | 0.58 | 0.45 | –27% | –22% | – |
| \( \chi_{b0} \to \eta_c + \eta_c \) | 28 | 21 | –31% | –25% | 51 |
| \( \chi_{b0} \to \eta_c + \eta_c(2S) \) | 28 | 23 | –33% | –17% | 76 |
| \( \chi_{b0} \to \eta_c(2S) + \eta_c(2S) \) | 7.0 | 6.4 | –34% | –10% | 30 |
| \( \chi_{b2} \to \eta_c + \eta_c \) | 0.79 | 0.32 | –93% | –59% | 16 |
| \( \chi_{b2} \to \eta_c + \eta_c(2S) \) | 0.79 | 0.35 | –98% | –56% | 21 |
| \( \chi_{b2} \to \eta_c(2S) + \eta_c(2S) \) | 0.20 | 0.09 | –103% | –54% | 8 |
| \( \chi_{b0} \to J/\psi + J/\psi \) | 18 | 15 | –20% | –16% | 79 |
| \( \chi_{b1} \to J/\psi + J/\psi \) | \( 1.0 \times 10^{-3} \) | \( 3.1 \times 10^{-4} \) | – | –70% | – |
| \( \chi_{b2} \to J/\psi + J/\psi \) | 45 | 35 | –34% | –23% | 270 |

with the accuracies of order \( \alpha_s v_c^2 \). All of these NRQCD matrix elements with corresponding references are listed in Table I. The values for the physical masses for the involving hadrons are taken from Ref. [31] as \( m_{\eta_b} = 9.3909 \text{ GeV} \), \( m_{\chi_{b0}} = 9.85944 \text{ GeV} \), \( m_{\chi_{b1}} = 9.89278 \text{ GeV} \), \( m_{\chi_{b2}} = 9.91221 \text{ GeV} \), \( m_{\eta_c} = 2.9803 \text{ GeV} \), \( m_{J/\psi} = 3.096916 \text{ GeV} \), and \( m_{\eta_c(2S)} = 3.637 \text{ GeV} \).

Our NRQCD predictions for the decay rate \( \Gamma[H \to h_1 + h_2] \), in which the relativistic corrections are resummed to all orders in \( v_c^{2n} \) keeping \( v_b = 0 \), are listed in Table II in units of eV. Here, \( \Gamma^{LO} \) is the prediction at LO in \( v_b \) and \( v_c \). In order to demonstrate the significance of the resummation, we list the values for \( R \) and \( R_2 \) that are defined in Eq. \( (35) \). \( \Gamma^{LC} \) is
the prediction based on the LC formula quoted from Ref. [8]. The relativistic corrections to all of the processes listed in Table. II are negative. According to $R_2$ in Table. II, the NLO relativistic corrections are between $-20$ and $-35\%$ of the LO prediction except for $\chi_{b2}$ decays into spin-singlet $S$-wave charmonium pairs $\eta_1 + \eta_2$ that are more than $-90\%$. Especially, the decay rate for $\chi_{b2} \to \eta_c(2S) + \eta_c(2S)$ becomes negative at NLO in $v_c^2$. The resummed results for the decay rates are greater than the NLO-corrected values. The resummation of the relativistic corrections to all orders in $v_c^{2n}$ makes all of the rates $\Gamma[\chi_{b2} \to \eta_1 + \eta_2]$ positive. The previous predictions of the decay rates $\Gamma^{LC}$ in Ref. [8] based on the LC formula are greater than our final results $\Gamma$ that include resummation of relativistic corrections to all orders in $v_c^{2n}$.

11 For $\chi_{b0} \to \eta_1 + \eta_2$, the LC results are greater than ours by factors ranging from 2 to 5. Especially, in the case of $\chi_{b2} \to \eta_1 + \eta_2$, the LC results are greater than ours by factors ranging from 45 to 90. In the case of $\chi_{bJ} \to J/\psi + J/\psi$ the factors are 5 and 8 for $J = 0$ and 2.

VI. SUMMARY

We have presented the NRQCD predictions for the decay rates of the bottomonia $\eta_b$ and $\chi_{bJ}$, that are even eigenstates of the charge conjugation parity $C$, into $S$-wave charmonium pairs. The short-distance coefficients for the NRQCD factorization formula are obtained at LO in $\alpha_s$. A class of relativistic corrections of the charm quark in the final-state charmonia is resummed to all orders in $v_c^{2n}$ by making use of the generalized Gremm-Kapustin relation in Refs. [18, 22] that is valid in spin-independent potential models and we have neglected the motion of the $b$ quark in the initial bottomonium.

The results show that the relativistic corrections to the decay rates at NLO in $v_c^2$ are all negative. Severe relativistic corrections are observed especially in $\chi_{b2} \to \eta_1 + \eta_2$ for $\eta_i = \eta_c$ or $\eta_c(2S)$. The decay rate for $\chi_{b2} \to \eta_c(2S) + \eta_c(2S)$ at NLO in $v_c^2$ is even negative. In almost every case, the resummation of relativistic corrections of a class of color-singlet contributions eventually gives sizable growth of the decay rate in comparison with the NLO predictions. Parts of our results are in agreement with previous NRQCD predictions in Refs. [1, 9].
In comparison with a previous analysis based on the LC formalism in Ref. [8], our NRQCD results severely underestimate the decay rates by 1 or 2 orders of magnitude. It is, therefore, very important to pin down the source of such significant discrepancies between the NRQCD factorization and LC formalisms. We recall previous theoretical studies on the exclusive process $e^+e^- \rightarrow J/\psi + \eta_c$ at $B$ factories. The NRQCD predictions of the cross section at LO in $\alpha_s$ and $v_c$ [16, 17] severely underestimated the empirical values [32–34]. According to Ref. [12], the LC prediction, which is much larger than that of NRQCD at LO in $\alpha_s$ and $v_c$, can explain the measured cross section. However, as is shown in Ref. [13], the LC calculation contains short-distance contributions, which are treated in the context of model LC distributions. These short-distance contributions appear in corrections of order $\alpha_s$ (and higher) in the NRQCD approach and can be computed from first principles in that approach. We know that the problem has been resolved within the NRQCD factorization formula in combination with the NLO corrections in $\alpha_s$ [11] and the resummation of relativistic corrections [10] within errors.

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