Constraints and Soliton Solutions for the KdV Hierarchy and AKNS Hierarchy

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Abstract: It is well-known that the finite-gap solutions of the KdV equation can be generated by its recursion operator. We generalize the result to a special form of Lax pair, from which a method to constrain the integrable system to a lower-dimensional or fewer variable integrable system is proposed. A direct result is that the \( n \)-soliton solutions of the KdV hierarchy can be completely depicted by a series of ordinary differential equations (ODEs), which may be gotten by a simple but unfamiliar Lax pair. Furthermore the AKNS hierarchy is constrained to a series of univariate integrable hierarchies. The key is a special form of Lax pair for the AKNS hierarchy. It is proved that under the constraints all equations of the AKNS hierarchy are linearizable.

1 Introduction

Much effort has been devoted to finding the exact solutions to integrable systems since Gardner, Greene, Kruskal and Miura found the inverse scattering (IST) transform method to solve the KdV. In general on the full line the reflectionless potential is solved by the IST as soliton or multi-soliton solutions. It is widely believed that the IST is inefficient to obtain the other kinds of solutions. Therefore, from this point of view it is amazing that the periodic KdV is completely solved by the algebraic-geometric solutions. The crucial fact in obtaining the algebraic-geometric solutions is that the stationary solutions of any higher-order KdV is invariant to the usual KdV. The idea that solving PDEs in its finite-dimensional invariant subspaces somehow has been developed to the method of nonlinearization of Lax pair[2] or symmetry constraint[3]. In most cases, the solutions obtained by nonlinearization of Lax pair are algebraic-geometric solutions, from which useful information is hard to get because of the complex expressions of the solutions. Furthermore, returning to get the soliton solutions some proper constraints has to be imposed for the algebraic-geometric solutions. This seems to be not straightforward. It is even more complicated to characterize the other kinds of solutions such as the elliptic solutions of the KdV. What is even worse is that there is no a rigid classification for the kinds of solutions for integrable partial differential equations (IPDEs). In this paper we will neatly characterize the soliton solutions by a less popular Lax pair for the KdV hierar-
chy without any knowledge of the algebraic-geometric solutions or any other old methods for obtaining the soliton solutions such as the Bäcklund or Darboux transformations.

For an IPDE the kind of solutions that can be gotten by only solving some linear PDE are of special interest. At first glance there seems to be little chance to realize this for an S-integrable system such as KdV or AKNS. The most desirable situation for solving a nonlinear partial differential equation is that it can be linearized by an appropriate change of variables, which is called C-integrable. The famous example of such kind is the Burgers equation. But most researchers firmly believe that a true S-integrable system such as KdV or AKNS will never be linearized by a common change of variables. So it will be very interesting to know to what an extent the S-integrable system is solvable by the change of variables. In this paper we will give a series of constraints on the AKNS hierarchy. The final result is that under the constraints the resulting equations are all linearizable by some proper transformations.

The paper is organized as follows. Section 2 introduces a special form of Lax pair, about which a basic theorem is given. The theorem states how to construct invariant manifolds corresponding to the special form of Lax pair. Section 3 introduces a special form of Lax pair generating the KdV hierarchy. By the theorem introduced in Section 2, the invariant manifolds corresponding to the Lax pair are constructed. It turns out that the invariant manifolds are just the $n$-soliton solutions of the KdV hierarchy. Section 4 deals with a special form of Lax pair of the AKNS hierarchy. We will first obtain the special kind of constraints. Then we will solve the first few invariant manifolds in detail. At last we will prove the main theorem for the constraints of the AKNS hierarchy.

## 2 Constraint for evolution equations with a special form of Lax pair

Integrable equations are consistency conditions of the Lax pair

\[
\hat{L}\varphi = \lambda \varphi, \tag{1}
\]

\[
\varphi_t = \hat{P}\varphi, \tag{2}
\]

where the eigenfunction $\varphi = \varphi(x, t)$ is $n$-dimensional vector and linear operators $\hat{L}$ and $\hat{P}$ are differential polynomials of potential $u = u(x, t)$. The method of nonlinearization of Lax pair or symmetry constraint method set up additional constraints between the potential $u$ and the eigenfunction $\varphi$. With the additional constraints Equations (1) and (2) will become ODEs in most cases. Let us still take the KdV as an example. The KdV $u_t = 6uu_x + u_{xxx}$ has a well-known Lax pair

\[
(\partial^2 + u)\varphi_i = \lambda_i \varphi_i, \tag{3}
\]

\[
\partial_t \varphi_i = (4\partial^2 + 3(u\partial + \partial u))\varphi_i \tag{4}
\]

and the well-known constraint for the KdV is

\[
u = c_0 + \sum_{i=1}^{n} c_i \varphi_i^2. \tag{5}
\]
With the constraint (5), Equation (3) and Equation (4) become two sets of ODEs. In most cases the efficient way to find a constraint for Lax equations is the symmetry constraint method. But symmetry constraints are not all constraints. It is observed in [7] that systems with the following special form of Lax pair have natural constraints.

- Operator \( \hat{L} \) has form \( \hat{L} = \hat{L}_{+} + \sum_{i=1}^{n} h_{i} \partial^{-1} g_{i} \), where \( \hat{L}_{+} \) is a differential operator and \( h_{i} \) and \( g_{i} \) are differential polynomials of potential \( u \).

- Operator \( \hat{P} \) is a differential operator.

The following theorem guarantees a natural constraint.

**Theorem 2.1** For systems with Lax pairs in the above form, there is a function \( \hat{L}_{F} = \sum_{i=1}^{n} a_{i} h_{i} \) such that a constraint 
\[ \hat{L}_{F} = \sum_{m=1}^{\infty} b_{m} \varphi_{m} \] exists, where \( \varphi_{m} \) is the eigenfunction \( \hat{L}_{m} \varphi_{m} = \lambda_{m} \varphi_{m} \), \( a_{i} \)s are some proper constants, \( b_{j} \)s are arbitrary constants and \( m \) is an arbitrary positive integer.

It is also well-known that Lax pair has a less popular form
\[ f_{j} + 1 = \hat{L} f_{j}, \tag{6} \]
\[ f_{j} = \hat{P} f_{j}. \tag{7} \]

Correspondingly Theorem 2.1 has a variant form:

**Theorem 2.2** There is a function \( \hat{L}_{F} = \sum_{i=1}^{n} a_{i} h_{i} \) such that a constraint \( \sum_{m=1}^{\infty} b_{m} f_{m} = 0 \) exists, where \( f_{1} = L_{F} \) and the requirements for \( \hat{L}, \hat{P}, a_{i}s, b_{j}s \) are the same as in Theorem 2.1.

In the following paper Theorem 2.2 will be applied more frequently.

### 3 The soliton constraint for the KdV hierarchy

The KdV hierarchy is defined by its recursion operator \( \hat{\varphi}_{1} = \partial^{2} + 4u + 2u_{x} \partial^{-1} \)

\[ u_{t} = \partial^{2} u_{x}, \quad \partial = \partial^{2} + 4u + 2u_{x} \partial^{-1} \] \tag{8}

where \( n \) is an arbitrary positive integer. The first nontrivial equation of the hierarchy is the KdV equation

\[ u_{t} = 6uu_{x} + u_{xxx}. \tag{9} \]

**Theorem 3.1** The following Lax pair [6],

\[ \hat{L}_{m} \varphi = (\partial + u \partial^{-1}) \varphi = \lambda \varphi, \quad \varphi_{m} = \hat{P}_{m} \varphi = (\hat{L}_{m})_{+} \varphi \tag{10} \]

generates the KdV hierarchy, where \( m \) is an odd integer and \( \hat{L}_{m} \) is the differential part of pseudo-differential operator \( \hat{L}_{m} \).

**Proof:** First we prove

\[ \hat{L}_{m}^{n} = \partial h, \tag{12} \]

where \( n \) is an odd positive integer and \( h \) is a differential operator. In fact for odd \( n \) we will prove

\[ (\partial + u \partial^{-1})_{m}^{n} \partial = \partial(\partial + \partial^{-1} u)_{m}^{n}. \tag{13} \]
By $\partial^{-1}(\partial + u\partial^{-1})^n \partial = (\partial + \partial^{-1}u)^n$, we obtain $(\partial + u\partial^{-1})^n \partial = \partial(\partial + \partial^{-1}u)^n$. Then we immediately get

$$[(\partial + u\partial^{-1})^n \partial]_+ = [\partial(\partial + \partial^{-1}u)^n]_+.$$  \hspace{1cm} (14)

Equation (14) is equivalent to

$$(\partial + u\partial^{-1})^n \partial + \text{res}(\partial + u\partial^{-1})^n = \partial(\partial + \partial^{-1}u)^n + \text{res}(\partial + \partial^{-1}u)^n.$$  \hspace{1cm} (15)

In fact for odd $n$ we have

$$\text{res}((\partial + u\partial^{-1})^n) = \text{res}((\partial + \partial^{-1}u)^n),$$  \hspace{1cm} (16)

because

$$\text{res}((\partial + u\partial^{-1})^n) = \text{res}((-1)^n[(\partial + \partial^{-1}u)^n]^+) = (-1)^n \text{res}((\partial + u\partial^{-1})^n).$$

Formula (13) is just a direct result of Equation (15) and Equation (16).

Secondly we prove that $\frac{d}{dt}\hat{L} = [\hat{P}_m, \hat{L}]$ is only one PDE for $u$, or in other words we will prove

$$[\hat{P}_m, \hat{L}] = f[u] \partial^{-1},$$  \hspace{1cm} (17)

where $f[u]$ denotes a differential polynomial of $u$. By (12)

$$[\hat{P}_m, \hat{L}] = \hat{G} + f[u] \partial^{-1},$$  \hspace{1cm} (18)

where $\hat{G}$ is a differential operator. But we also have [5]

$$[\hat{P}_m, \hat{L}] = [\hat{L}_m - \hat{L}_m, \hat{L}] = -[\hat{L}_m, \hat{L}].$$

So the order of $[\hat{P}_m, \hat{L}]$ is less than 0. This fact and Equation (18) imply Equation (17). So we have proved that $\frac{d}{dt}\hat{L} = [\hat{P}_m, \hat{L}]$ is only a PDE $u_t = f[u]$.

At last we prove $\frac{d}{dt}\hat{L} = [\hat{P}_m, \hat{L}]$, $m = 1, 2, 3, \cdots$, is the KdV hierarchy. This can be verified by its recursion operator $\hat{\phi} = \partial^2 + 4u + 2u_x \partial^{-1}$, which may be easily carried out by the method established by [6].

By Theorem 3.1 and Theorem 2.2 we immediately know

$$\sum_{j=1}^{m} b_j(\partial + u\partial^{-1})^{j-1}u = 0$$

is a proper constraint, which has been proved [7] to be all the soliton solutions of the KdV equation.
4 Special constraints for the AKNS hierarchy

4.1 The special type of Lax pair for the AKNS hierarchy

The AKNS hierarchy \([1, 8]\) is

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = \hat{\varphi}^n \begin{pmatrix}
-iq \\
ir
\end{pmatrix},
\]

where \(\hat{\varphi}\) is the recursion operator

\[
\hat{\varphi} = \frac{1}{i} \begin{pmatrix}
-\partial + 2q\partial^{-1}r & 2q\partial^{-1}q \\
-2r\partial^{-1}r & \partial - 2r\partial^{-1}q
\end{pmatrix}
\]

and \(n\) is an arbitrary positive integer. The first equation of the hierarchy is

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = \frac{1}{i} \begin{pmatrix}
-q_{xx} + 2q^2r \\
r_{xx} - 2qr^2
\end{pmatrix}.
\] (19)

The natural Lax pair

\[
\hat{\varphi}_t = [\hat{P}_m, \hat{\varphi}]
\]

of the AKNS hierarchy fulfills Theorem 2.2, where \(\hat{P}_m\) is the linearization operator of the AKNS. So the constraint

\[
\sum_{j=1}^{m} b_j \varphi^{j-1} \begin{pmatrix}
q \\
r
\end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

is a proper constraint. But here we will not discuss this useful symmetry constraint. We will investigate the constraints induced by the following Lax pair of the AKNS.

**Theorem 4.1** \([9]\) Let \(\hat{L} = \frac{1}{i}(-\partial + q\partial^{-1}r), \hat{P}_m = \frac{1}{i}\hat{L}_m^m\). Then Lax equation

\[
\hat{L}_t = [\hat{P}_m, \hat{L}]
\] (20)

generates the AKNS hierarchy.

**Proof:** First we must prove the Lax equation (20) is just two PDEs for \(q\) and \(r\) respectively. In fact we will prove

\[
\begin{pmatrix}
q \\
r
\end{pmatrix}_t = \frac{1}{i} \begin{pmatrix}
P_m \cdot q \\
-P_m^* \cdot r
\end{pmatrix},
\] (21)

where \(\hat{P}_m \cdot q\) denotes the differential polynomial gotten by acting the operator \(\hat{P}_m\) on \(q\) and \(\hat{P}^*\) is the conjugate operator of \(\hat{P}\). Because

\[
[\hat{P}_m, \hat{L}] = \frac{1}{i}[\hat{L}^m - \hat{L}_m^m, \hat{L}] = i[\hat{L}_m^m, \hat{L}],
\]

we know \([\hat{P}_m, \hat{L}] = 0\). So

\[
[\hat{P}_m, \hat{L}] = (P_m \cdot q)\partial^{-1}r - q\partial^{-1}(P_m^* \cdot r).
\] (22)
With Equation (22) and Equation (20) we immediately get Equation (21).
Secondly we will prove the recursion operator of the hierarchy (22) is
\[
\hat{\varphi} = \frac{1}{i} \begin{pmatrix}
-\partial + 2q\partial^{-1}r & 2q\partial^{-1}q \\
-2r\partial^{-1}r & \partial - 2r\partial^{-1}q
\end{pmatrix}.
\] (23)

With (23) Theorem 4.1 becomes obvious, while Equation (23) follows from the following theorem.

**Theorem 4.2** For any 
\[
\hat{L}_n+1 = \hat{L}_n \hat{L} + [\tilde{R}_n, \hat{L}],
\] (24)

where \(\tilde{R}_n = \frac{1}{i}(a_n + b_n\partial^{-1}r)\). Equation (24) is just the recursion equation
\[
\begin{pmatrix} q_{n+1} \\ r_{n+1} \end{pmatrix} = \hat{\varphi} \begin{pmatrix} q_n \\ r_n \end{pmatrix},
\] (25)

where \(\hat{\varphi}\) is the recursion operator (23).

**Proof:** We will first prove Equation (24). Since \(\hat{L}_n+1 = \hat{L}\hat{L}_n\), we have
\[
\hat{P}_{n+1} = \frac{1}{i}(\hat{L}^n_{n+1})_+ = \frac{1}{i}(\hat{L}^n \hat{L})_+ = \frac{1}{i}((\hat{L}^n)_+ \hat{L} + (\hat{L}^n)_- \hat{L})_+ = \frac{1}{i}(\hat{L}^n \hat{L} - (\hat{L}^n)_+ \hat{L} + (\hat{L}^n)_- \hat{L}),
\]
which leads directly to
\[
\hat{L}_n+1 = [\hat{P}_{n+1}, \hat{L}] = \hat{L}_{n+1} \hat{L} + [\tilde{R}_n, \hat{L}],
\]
where \(\tilde{R}_n\) has the expression
\[
\tilde{R}_n = \frac{1}{i}((-\hat{L}^n \hat{L})_+ + (\hat{L}^n \hat{L})_+).
\]

So \(\tilde{R}_n\) can be expressed as
\[
\tilde{R}_n = \frac{1}{i}(a_n + b_n\partial^{-1}r).
\] (26)

Then we will prove (25). Substituting (26) to (24) we get
\[
i \times (q_{n+1} \partial^{-1}r + q\partial^{-1}r_{n+1}) = (q_n \partial^{-1}r + q\partial^{-1}r_n)(-\partial + q\partial^{-1}r) + [a_n + b_n\partial^{-1}r, -\partial + q\partial^{-1}r].
\] (27)

The positive part of Equation (27) gives
\[
a_n = \int (q_n r + qr_n) dx.
\] (28)

Rearranging the negative part of Equation (27) we get
\[
(iq_{n+1} - a_nq - b_n'\partial^{-1}r + q\partial^{-1}(ir_{n+1} - r_n' + ra_n) = (q_n + b_n)\partial^{-1}(r' + rq\partial^{-1}r) + q\partial^{-1}(r_nq - rb_n)\partial^{-1}r.
\] (29)
Left multiplying (29) with $\frac{1}{r} \partial_r$ and right multiplying (29) with $\partial_q$ simultaneously, considering its negative part we get

$$b_n = -q_n. \quad (30)$$

Substituting $b_n = -q_n$ to (29) and considering $q_n r + qr_n = a'_n$, we get

$$(iq_{n+1} - 2a_n q - b'_n)\partial^{-1} r + q\partial^{-1}(ir_{n+1} - r'_n + 2ra_n) = 0.$$  

Note we have applied the well-known formula $\partial^{-1} f' \partial^{-1} = f \partial^{-1} - \partial^{-1} f$. Now it is clear

$$q_{n+1} = \frac{1}{i}(2a_n q + b'_n), \quad r_{n+1} = \frac{1}{i}(r'_n - 2ra_n).$$

By substituting (28) and (30) to the above equations we immediately get the recursion equation (25).

### 4.2 Specially constraints for AKNS hierarchy

Applying Theorem 2.2 to the Lax pair in Theorem 4.1 does not generate two systems of ODEs, because the constraint $\hat{L}^a(0) = 0$ only offers one constraint between $q$ and $r$ and still another constraint between $q$ and $r$ must be given for $q$ and $r$ being ODEs of the independent variable $x$. The situation is best explained by the case $n = 1$. When $n = 1$, by $\hat{L}(0) = 0$ we immediately get $q = 0$. So for the usual AKNS (19) only one PDE is left $r_1 = -ir_{xx}$.

The second constraint $\hat{L}^2(0) = 0$ can be simplified to

$$qr = \frac{q_x}{q}.$$  

It has been noticed [4] that with the constraint (31) the AKNS hierarchy is constrained to the Burgers hierarchy of $w$ by the transformation $w_x = qr = (\frac{q_x}{q})_x$. And it is also well-known that the Burgers hierarchy can be linearized by the famous Cole-Hopf transformation. So with the constraint (31) the AKNS hierarchy may be linearized, which is best explained by the following theorem:

**Theorem 4.3** With the constraint (31) the AKNS hierarchy can be constrained to linear equations $u_t = (-i)^{m-1}u^{(m)}$, where $u = \frac{1}{q}$.

**Proof:** Let’s prove this theorem by mathematical induction. When $k = 2$ the AKNS, which is just the NLS equation in this case, is constrained to

$$iq_t + q_{xx} - 2q(\frac{q_x}{q})_x = 0.$$  

Substituting $q$ with $\frac{1}{r}$, we get the linear equation $u_{t_2} = -iu_{xx}$. When $k = n$, we assume the equation has been transformed to $u_{t_n} = (-i)^{n-1}u^{(n)}$, i.e.,

$$\left(\begin{array}{c} q_{t_n} \\ r_{t_n} \end{array}\right) = \hat{\varphi}^n \left(\begin{array}{c} -iq \\ ir \end{array}\right) = \left(\begin{array}{c} -(\ln q)_{xx}(\frac{1}{q})_{t_n} \\ ((\ln q)_{xx}(\frac{1}{q})_{t_n} \end{array}\right).$$
When \( k = n + 1 \), by \( qr = \left( \frac{dw}{q} \right)_x \) we get \( \alpha^{-1}(qr)_{\mu} = \left( \frac{dw}{q} \right)_{\mu} \), then by the recursion operator we get

\[
q_{n+1} = \frac{1}{r} \left( -\partial + 2q\partial^{-1}r - 2q\partial^{-1}q \right) \phi^n \begin{pmatrix} -iq \\ ir \end{pmatrix}.
\]

Substituting \( q = \frac{1}{u} \), we finally we get \( u_{n+1} = (-i)^n u^{n+1} \). This finishes the proof.

Then we consider the constraint \( \hat{L}^3(0) = 0 \), which is equivalent to

\[
qr = \left( \frac{q^3 r}{x} - q q_{xxx} + q v_{q_{xx}} \right) \gamma = \left( \ln(q^3 r - q q_{xx} + q_x^2) \right)_{xx}.
\]

**Theorem 4.4** With (32) the standard AKNS is constrained to \( g_t = -ig_{xx} \) by the transformation

\[
g = \frac{q}{q^3 r - q q_{xx} + q_x^2}.
\]

**Proof:** Direct calculation shows \( g_t + ig_{xx} = 2i g \left( qr - \left( \ln(q^3 r - q q_{xx} + q_x^2) \right)_{xx} \right) \).

By (33) and (32) \( q \) can be easily obtained

\[
q = -\frac{1}{g(\ln g)_{xx}}.
\]

Let us explain how (33) is obtained. Equation \( iq_t + q_{xx} - 2q^2 r = 0 \) is transformed to

\[
\begin{align*}
-4u_x^2 u_t^2 + 7u_t u_{x} u_{xx} - 2u_t u^2 u_{xxt} + 2u_x^2 u^2 - 2u_{xx} u^2 + 11i u_x u_{xxx} - 2u_t u^2 u_{xxxx} - iu_x^3 u \\
-5u_x^3 u + 4iu^2 u_{x} u_{xxx} - 2u_t u^2 u_{xxt}u_x - 8iu_x^2 u_{ux}u_{xx} + 4u_x^2 u_{xx}^2 + 2u_{xxx} u^2 u_{xx} = 0
\end{align*}
\]

by (32) and \( q = \frac{1}{u} \). Here \( u_t = -iu_{xx} \) is still a solutions of Equation (35). Then Equation (35) is transformed to

\[
iv_{xxt} - iv_{xx} - v_{xxx} + 2v_{xxx} v_x - v_x v_x^2 + 3v_x^2 = 0
\]

by \( e^\nu = \frac{u_t + iu_{xx}}{u_x} \). Now the solution \( u_t = -iu_{xx} \) has been ruled out. Equation (36) can be written into a more compact form

\[
i(\partial^2 - v_{xx})(v_t + iv_{xx} - iv_x^2) = 0.
\]

Equation (37) suggests us to calculate \( v_t + iv_{xx} - iv_x^2 \), which turned out to be 0. But \( v_t + iv_{xx} - iv_x^2 = 0 \) is linearized by transformation \( v = -ln(g) \). Therefore we get the final transformation (33). Note that \( v_t + iv_{xx} - iv_x^2 = 0 \) can also be gotten by considering the compatibility condition between equation \( ir_t - r_{xx} + 2q r^2 = 0 \), Equation (32) and (36).

The third equation of the AKNS hierarchy is the coupled KdV

\[
\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} 6qr q_x - q_{xxx} \\ 6qr r_x - r_{xxx} \end{pmatrix}.
\]

It is easy to verify that under constraint (32) Equation (38) is simplified to \( g_t = -g_{xxx} \), where \( g \) is also defined by (33).
Then we consider the case $\hat{L}^3(0) = 0$, which is equivalent to

$$qr = (\ln p_3)_{xx},$$

where $p_3$ is defined as

$$p_3 = 5q_x^2 r^3 - q_{xxx} q_{xx} + q_{xxx} q_x^3 r - 2 q_{xxx} q^3 r - 6q_{xxx} q_x q^2 r - 2 q_{xxx} q_x q_{xx} + 7q_{xxx} q_x^2 q r + 8q_x r_x q^2 x_x + q_{xxx} q_x^2 q_x^2 + 3q_x^2 q_x^2 q r - 5 r^2 q_{xxx} q^4 - 6 q_x^3 q r_x - q_x^5 r_{xx} r + q_x^3 r_{xxx} q_x - q_x^3 q_x^3 r_x - 5 q_x^4 r_x + q_x^2 q_{xx} q_x + q_x^5 r_x + q_x^5 r^3 + q_x^3 r_{xx}.$$ (40)

Let us define $p_2$ as

$$p_2 = q_x^3 r - q_x q_{xx} + q_x^2.$$ (41)

It can be directly verified that under constraint (39) the standard AKNS is simplified to $h = -i h_{xx}$, where $h = \frac{p_2^2}{p_3}$. Then $q$ can be obtained by

$$
\frac{1}{q} = h(\ln h)_{xx} (\ln h (h(\ln h))_{xx}).
$$ (42)

Before summarizing all results above, let us first define $p_k$ and $\theta_k(g)$ recursively. Function $p_k$ is completely determined by $k$:

$$
p_{k-1} = r, \quad p_0 = 1, \quad p_1 = q, \quad \frac{p_{k+1}}{p_k} = (qr - (\ln p_k)_{xx}) \frac{p_k}{p_{k-1}}.
$$ (43)

Function $\theta_k(g)$ is determined by both $k$ and $g$:

$$
\theta_1(g) = g, \\
\theta_{k+1}(g) = (\ln(\theta_1(g)\theta_2(g)\cdots\theta_k(g)))_{xx} \theta_k(g).
$$ (44)

Now we summarize our main result as following:

**Theorem 4.5** The constraint $\hat{L}^3 q = 0$ is equivalent to $qr = (\ln p_3)_{xx}$, where $\hat{L} = \frac{1}{i}(-\partial + q \partial^{-1} r)$. With the constraint the $m$-th AKNS equation is equivalent to $\frac{\partial u_n}{\partial t} = (-i)^{m-1} u_n^{(m)}$, where $u_n = \frac{\ln p_k}{p_k}$ and $p$ is defined by (43). Given $u_n$, $q$ and $r$ are determined by $q^{-1} = (-1)^{n-1} \theta_n(u_n)$ and $r = (-1)^{n} \theta_{n+1}(u_n)$ respectively, where $\theta$ is defined by (44).

The proof of Theorem 4.5 consists of several parts. The following theorems greatly reduce the complexity of the proof of Theorem 4.5. So we will first prove the following Lemma 4.6, Theorem 4.7 and Proposition 4.8, Theorem 4.9. At last we will prove Theorem 4.5.

Let us define $(i\hat{L})^k q = f_k$, and define $\alpha_j$ by $\alpha_0 = q$ for $j = 0$ and $\alpha_j = qr - (\ln p_j)_{xx}$ for $j \neq 0$.

**Lemma 4.6** $f_k = \alpha_0 \alpha_1 \partial^{-1} \alpha_2 \cdots \alpha_{k-1} \partial^{-1} \alpha_k$.

**Proof:** Obviously the theorem is true for $k = 1$. Suppose the theorem is true for $k = s$. Then we must compute $f_{s+1} = (-\partial + q \partial^{-1} r) f_s$. Let us define $\hat{C}_j$ by $\hat{C}_j = \alpha_j \partial^{-1}$. Then we have $f_k = \hat{C}_0 \hat{C}_1 \cdots \hat{C}_k 0$, where we have set $\partial^{-1} = 1$. It can be verified $\hat{B}_k \hat{C}_k = \hat{C}_k \hat{B}_k + 1$, where $\hat{B}_k$ is defined as $\hat{B}_0 = i\hat{L}$ for $k = 0$ and $\hat{B}_k = -\partial - (\ln \frac{p_k}{p_{k-1}}) + \alpha_k \partial^{-1}$ for $k \neq 0$. Therefore, the theorem must be true for $k = s + 1$, because $\hat{B}_{s+1} 0 = \alpha_{s+1}$. 

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By Lemma 4.6, we immediately get:

**Theorem 4.7** \( \hat{L}^n q = 0 \) implies \( qr = (\ln p_n)_{xx} \).

The following proposition is also crucial for the proof of our final result.

**Proposition 4.8** For the AKNS hierarchy

\[
\frac{(p_k)_{t+1}}{p_k} - \frac{(p_{k-1})_{t+1}}{p_{k-1}} = \frac{1}{i} \left( 2\partial^{-1}(qr)_t - \frac{(p_{k-1})_{tx}}{p_{k-1}} + \frac{(p_k)_{tx}}{p_k} - \frac{(p_k)_{t}}{p_k} + \frac{(p_{k-1})_{t}}{p_{k-1}} \right) \quad (45)
\]

and

\[
(qr)_{t+1} - (\ln p_k)_{t+1,xx} = \frac{p_{k+1}p_{k-1}}{i p_k^2} \left( \frac{(p_{k-1})_{tx}}{p_{k-1}} - \frac{(p_{k+1})_{tx}}{p_{k+1}} + \frac{(p_{k+1})_{t}}{p_{k+1}p_k} - \frac{(p_{k+1})_{t}}{p_{k+1}p_k} \right) \]

\[
+ \frac{(p_k)_{t}}{p_k} - \frac{(p_{k-1})_{t}}{p_{k-1}p_k} \quad (46)
\]

**Sketch of the proof:** According to the recursion equation (25) the proposition is clearly true for \( k = 1 \). So let us suppose the proposition is true for \( k = j \). Then we must prove it is also true for \( k = j + 1 \). First we can solve \((p_{j-1})_{t+1}\) and \((p_j)_{t,xx}\) by (45) and (46) when \( k = j \). When \( k = j + 1 \), we can verify, by a lengthy but straightforward calculation, Equations (45) and (46) are simply two identities after considering (43).

**Theorem 4.9** If \( q \) and \( \theta \) are defined by (43) and (44) respectively. Given arbitrary function \( g \), set \( \theta_1 = g \), \( q^{-1} = (-1)^{n-1} \theta_2(g) \) and \( r = (-1)^n \theta_{n+1}(g) \). Then \( \theta_{n-k+1} = (-1)^{n-k} \frac{p_{n-k}}{p_k} \). As a result, \( g = \frac{p_{n-k}}{p_k} \), \( qr = (\ln p_n)_{xx} \). Conversely if \( q \) is defined by (43), \( \theta \) is defined by \( \theta_{n-k+1} = (-1)^{n-k} \frac{p_{n-k}}{p_k} \) and there is a constraint \( qr = (\ln p_n)_{xx} \) between \( q \) and \( r \), then \( \theta \) satisfies (44), \( q^{-1} = (-1)^{n-1} \theta_2(g) \) and \( r = (-1)^n \theta_{n+1}(g) \), where \( g = \frac{p_{n-k}}{p_k} \).

**Proof:** The proof is obvious by mathematical induction.

Now we will give the proof of Theorem 4.5.

**Proof of Theorem 4.5:** We only need to prove

\[
\frac{\partial u_n}{\partial t} = (-i)^{m-1} u_n^{(m)}. \quad (47)
\]

In fact (47) is a direct result of the following equation

\[
(u_n)_{t+1} = -i \times (u_n)_{x,x}, \quad (48)
\]

because (47) is obviously true for \( m = 1 \). Suppose (48) has been true for \( m = k \).

Then \( u_{t+1} \) can be directly calculated as

\[
\begin{align*}
\frac{p_{n-1}}{p_n} \left( \frac{(p_n)_{t+1}}{p_n} - \frac{(p_{n-1})_{t+1}}{p_{n-1}} \right) = \frac{i p_{n-1}}{p_n} \left( 2\partial^{-1}(qr)_t - \frac{(p_{n-1})_{tx}}{p_{n-1}} + \frac{(p_n)_{tx}}{p_n} - \frac{(p_n)_{t}}{p_n} + \frac{(p_{n-1})_{t}}{p_{n-1}} \right) \\
= \frac{i p_{n-1}}{p_n} \left( 2(\ln p_n)_{tx} - \frac{(p_{n-1})_{tx}}{p_{n-1}} + \frac{(p_n)_{t}}{p_n} - \frac{(p_{n-1})_{t}}{p_{n-1}} \right) \\
= -i \left( \frac{p_{n-1}(p_n)_t}{p_n^2} + \frac{(p_{n-1})_{t}}{p_n} \right) \\
= -i u_{t+1}.
\end{align*}
\]
So Equation (48) is proved.

Let us give a simple example to illustrate Theorem 4.5. Choose \( n = 4 \) and \( m = 3 \) in Theorem 4.5. Then \( \frac{\partial}{\partial t} \phi = -(u_4)_{xxx} \). Choose a solution of it such as
\[
 u_4 = 1 + e^{8t - 2x} + e^{x - t} + e^{4t} + e^{2x - 8t}. 
\]
Then we obtain
\[
 q = \frac{-e^{18t} - e^{6x} + 9e^{17t+x} - 36e^{10t+2x} - 9e^{16t+2x} - 65e^{9t+3x} - 9e^{2t+4x} - 36e^{8t+4x} - 9e^{t+5x}}{36e^{17t+x} + 576e^{10t+2x} + 1296e^{9t+3x} + 576e^{8t+4x} + 36e^{t+5x}},
\]
\[
 r = \frac{e^{16t} + e^{4x} + 16e^{9t+x} + 36e^{8t+2x} + 16e^{7t+3x}}{576e^{8t+2x}}. 
\]
It is easy to check that the above solution is indeed a solution of (38), the \( m \)-th equation of AKNS hierarchy.

5 Conclusions

In this work, we apply a special form of Lax pair to analyze the solutions of KdV hierarchy and AKNS hierarchy. For the KdV hierarchy, the soliton solutions are completely depicted by a new, simple and direct way. For the AKNS Hierarchy a special form of Lax pair is analyzed. With the special kind of Lax pair a wide class of solutions of the AKNS hierarchy have been obtained. At last we give the main theorem which states how to linearize all equations of the AKNS hierarchy with the special kind of constraints mentioned in this paper.

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