CONSTANT TERM OF EISENSTEIN INTEGRALS

JACQUES CARMONA, PATRICK DELORME

Abstract. Let $H$ be the fixed point group of a rational involution $\sigma$ of a reductive $p$-adic group on a field of characteristic different from 2. Let $P$ be a $\sigma$-parabolic subgroup of $G$ i.e. such that $\sigma(P)$ is opposite to $P$. We denote by $M$ the intersection with $\sigma(P)$. Kato and Takano on one hand, Lagier on the other hand associated canonically to an $H$-form, i.e. an $H$-fixed linear form, $\xi$, on a smooth admissible $G$-module, $V$, a linear form on the Jacquet module $j_P(V)$ of $V$ along $P$ which is fixed by $M \cap H$. We call this operation constant term of $H$-fixed linear forms. This constant term is linked to the asymptotic behaviour of the generalized coefficients with respect to $\xi$.

P. Blanc and the second author defined a family of $H$-fixed linear forms on certain parabolically induced representations, associated to an $M \cap H$-fixed linear form, $\eta$, on the space of the inducing representation. The purpose of this article is to describe the constant term of these $H$-fixed linear forms.

Also it is shown that when $\eta$ is square integrable, i.e. when the generalized coefficients of $\eta$ are square integrable, the corresponding family of $H$-fixed linear forms on the induced representations is a family of tempered, in a suitable sense, of $H$-fixed linear forms. A formula for the constant term of Eisenstein integrals is given.

1. Introduction

Let $F$ be a non archimedean local field of characteristic different from two. The hypothesis on the characteristic of the residue field in the first version of the article has been removed (see Proposition 2.3 and section 10).

1991 Mathematics Subject Classification. 22E50.

Key words and phrases. Reductive group, nonarchimedean local field, symmetric space.

The second author thanks warmly Joseph Bernstein for enlighting conversations during the elaboration of this work. We thank also very much Jean-Pierre Labesse and Bertrand Lemaire for their help on algebraic groups.

The second author has been supported by the program ANR-BLAN08-2-326851 during the elaboration of this work.
Let $G$ be of $\mathbf{F}$-points of a connected reductive algebraic group defined over $\mathbf{F}$. Let $X(G)$ be the group of unramified characters of $G$ and let $A_G$ be the maximal split torus of the center of $G$.

Let $\sigma$ be a rational involution of $G$ defined over $\mathbf{F}$ and let $H$ be equal to the fixed point group of $G$ (a slightly weaker assumption is made in the main body of the article). Let $X(G)_{\sigma}$ be the connected component of 1 in the group of elements $\chi$ of $X(G)$ such that $\chi \circ \sigma = \chi^{-1}$.

If $(\pi, V)$ is a smooth representation of $G$, an $H$-form, $\xi$, on $V$ is an $H$-fixed element of the algebraic dual $V'$ of $V$ i.e. $\xi \in V'^H$. We know (cf. [D3], Theorem 4.5) that if $V$ is of finite length the vector space of $H$-forms on $V$ is finite dimensional. We will denote by $\tilde{V}$ the smooth dual of $V$.

A parabolic subgroup of $G$ is called a $\sigma$-parabolic subgroup if and only if $P$ and $P^- := \sigma(P)$ are opposite (studied by A. Helminck and Wang, A. Helminck and G. Helminck, cf. [HW] and [HH]). Then $M := P \cap \sigma(P)$ is the $\sigma$-stable Levi subgroup of $P$ and $PH$ is open in $G$. Let $U$ be the unipotent radical of $P$. If $(\pi, V)$ is a smooth representation of $G$, $(\pi_P, V_P)$ will denote the normalized Jacquet module along $P$ and $j_P : V \to V_P, v \mapsto v_P$ will denote the canonical projection.

By the Second adjointness Theorem (Casselman, Bernstein, cf. [C], [Be], [R]) $(\tilde{V})_{P^-}$ is canonically isomorphic to $(V_P')$, i.e. there exists a canonical non degenerate pairing between $(\tilde{V})_{P^-}$ and $V_P$, denoted by $\langle \cdot, \cdot \rangle_{P^-}$. Let us assume moreover that $(\pi, V)$ is of finite length. Let $\pi'$ be the contragredient representation of $\pi$ on $V'$.

Independently Kato-Takano [KT1] and Lagier [L] have associated to each $H$-form $\xi$ on $V$ an $M \cap H$-form $j_{P^-}\xi$, denoted also $\xi_{P^-}$, on $V_P$ such that, for all $v \in V$, there exists $\varepsilon$ satisfying:

$$\delta_P^{1/2}(a)\langle \xi, \pi(a)v \rangle = \langle (\xi_{P^-}, \pi_P(a)v_P) , a \in A_M^{-1}(\varepsilon)$$

where $A_M^{-1}(\varepsilon)$ is some translate of the negative Weyl chamber in the maximal split torus, $A_M$, of the center of $M$ and $\delta_P$ is the modulus function of $P$. This is a characteristic property of $\xi_{P^-}$ which describes the asymptotics of generalized coefficients. The linear form $\xi_{P^-}$ will be called the constant term of $\xi$ along $P$.

For $K$ a “$(P, H)$-good” compact open subgroup of $G$, one has :

$$\langle \xi_{P^-}, v_P \rangle = \langle (e_K \xi)_{P^-}, v_P \rangle, v \in V^K.$$

There exists arbitrary small “$(P, H)$-good” compact open subgroups of $G$ (Kato-Takano [KT1], Lemma 4.6 and Lagier [L], Theorem 1 (ii)).
Hence $\xi_{P^{-}}$ is determined by the various $(e_{K}\xi)_{P^{-}}$ where $e_{K}$ is the normalized Haar measure on $K$ and $e_{K}\xi$ is the smooth linear form defined by:

$$\langle e_{K}\xi, v \rangle = \langle \xi, e_{K}v \rangle, \quad v \in V.$$  

P. Blanc and the second author defined families of $H$-forms on parabolically induced representations (cf. [BD]).

The purpose of this article is to describe the constant term of these $H$-forms. For this we prepare several tools.

First we introduce two operations on $H$-forms on induced representations.

The first one is denoted $\tilde{j}_{Q^{-}\circ}$.

Let $P = MU$ be a parabolic subgroup of $G$, as above. Let $(\delta, E)$ be a smooth representation of finite length of $M$ and let $i_{P}^{G}E$ be the normalized parabolically induced representation. Our choice is such that the elements of $i_{P}^{G}E$ are left covariant $E$-valued functions and $G$ acts by the right regular representation. Let $(Q, Q^{-})$ be a pair of opposite parabolic subgroups of $M$. Let $P_{Q} = QU \subset P, P_{Q^{-}} = Q^{-}U \subset P$.

We define an homomorphism of $G$-modules \( \tilde{j}_{Q^{-}\circ} : (i_{P}^{G}E)^{\gamma} \rightarrow (i_{P_{Q^{-}}^{G}E_{Q})^{\gamma}, \)

as the composition of four maps:

\[
(i_{P}^{G}E)^{\gamma} \rightarrow i_{P_{Q^{-}}}^{G}((E)_{Q^{-}})^{\gamma} \rightarrow i_{P_{Q^{-}}}^{G}((E_{Q})^{\gamma}) \rightarrow (i_{P_{Q^{-}}^{G}E_{Q})^{\gamma},
\]

where the first and the last maps are deduced from the equivalence between the smooth dual of a parabolically induced representation and the parabolically induced representation of the smooth dual, where $f$ is given by the composition with $j_{Q^{-}}$ and $g$ is the induced map of the isomorphism between $(\tilde{E}_{Q^{-}}^{\gamma}$ and $(E_{Q})^{\gamma}$ given by the Second adjointness Theorem.

This operation on smooth linear forms easily extends to $H$-form: one associates to an $H$-form, $\xi$, on $i_{P}^{G}E$, an $H$-form $\tilde{j}_{Q^{-}\circ} \xi$ on $i_{P_{Q^{-}}}^{G}E_{Q}).$ Let $P_{1} = M_{1}U_{1}$ be a parabolic subgroup of $G$ such that $P \subset P_{1}, M \subset M_{1}$ and such that $P_{1}H$ is open. The second operation is denoted $\tilde{r}_{M_{1}}.$

Let $\xi$ be an $H$-form on $i_{P}^{G}E$. Inducing in stage, $\xi$ is an $H$-form on $i_{P_{1}}^{G}E_{1}$ where $E_{1} := i_{P_{1}}^{M_{1}}E$. The linear form $\xi$ might be viewed as an $E_{1}$-distribution which is right invariant by $H$ and left covariant under $P_{1}$. The restriction of this distribution to the open set $P_{1}H$ is simply a function. Its value at 1 is denoted $\tilde{r}_{M_{1}}\xi \in (E_{1})^{\gamma}_{M_{1}H}.$

The next tool is what we call the Generic basic geometric Lemma.
Let $P = MU$ be a $\sigma$-parabolic subgroup of $G$. Let $(\delta, E)$ be a finite length smooth representation of $M$. Let $X(M)_\sigma$ be the neutral component of $\sigma$-antiinvariant elements of $X(M)$. If $\chi \in X(M)_\sigma$, we denote by $E_\chi$ the space of the representation $\delta_\chi := \delta \otimes \chi$. If $x \in G$ and $X$ is a subset of $G$, $x.X$ will denote $xXx^{-1}$. Then $x\delta$ is the representation of $x.M$ on $xE := E$, such that, if $m \in M$, $x\delta(xmx^{-1}) = \delta(m)$. We denote by $\chi$ the left regular representation of $G$ acting on functions on $G$.

Let $P' = M'U'$ be another $\sigma$-parabolic subgroup of $G$. Then we have the following result that we call the Generic basic geometric Lemma (or rather its dual version).

**Lemma** For a good choice $W(M'\backslash G/M)$ of a set of representatives of $P'\backslash G/P$, one defines successively for $w \in W(M'\backslash G/M)$:

$$X_{\chi,w} := \iota_{M'\cap w,P}^M((E_\chi)_{M\cap w^{-1},P'})$$

$$\tilde{P}_w = (M \cap w^{-1}.P')U, \tilde{P}_w' = (M' \cap w.P)U',$$

and for $\chi$ in a suitable open dense subset of $X(M)_\sigma$:

the transpose of the intertwining integral $^tA(w, \tilde{P}_w, \tilde{P}_w', w j_{M\cap w^{-1},P'} \delta_\chi)$ and $\beta_{\chi,w} : (i_{P'}^E_\chi)^\vee \rightarrow (X_{\chi,w})^\vee$ by:

$$\beta_{\chi,w} = \tilde{r}_{M'} \circ ^tA(w, \tilde{P}_w, \tilde{P}_w', w j_{M\cap w^{-1},P'} \delta_\chi) \circ \lambda(w) \circ (j_{M\cap w^{-1},P'} \circ)$$

so that $\beta := \bigoplus_{w \in W(M'\backslash G/M)} \beta_{\chi,w} : (i_{P'}^E_\chi)^\vee \rightarrow \bigoplus_{w \in P'\backslash G/P} X_{\chi,w}$ goes through the quotient to an isomorphism:

$$\beta_{\chi} : ((i_{P'}^E_\chi)^\vee)_{P'} \rightarrow \tilde{X}_{\chi},$$

where $X_{\chi} = \bigoplus_{w \in W(M'\backslash G/M)} X_{\chi,w}$.

The proof of the Generic basic geometric Lemma requires the study of bijectivity of intertwining integrals and the notion of infinitesimal character linked to the Bernstein center. This notion allows to show that, for $\chi$ in an open dense subset of $X(M)_\sigma$, two distincts $X_{\chi,w}$ do not have irreducible subquotients in common. The end of the proof uses the Basic geometric Lemma which describes the graded of a natural filtration of the Jacquet modules of parabolically induced representations.

From our isomorphism $\beta_{\chi} : ((i_{P'}^E_\chi)^\vee)_{P'} \rightarrow \tilde{X}_{\chi}$, $X_{\chi}$ identifies with $(i_{P'}^E_\chi)_{P'}$, by the second adjointness theorem. Hence, if $\xi$ is an $H$-form on $i_{P'}^E_\chi$, $\xi_{P'}$ identifies with a linear form denoted again $\xi_{P'}$ on $X_{\chi}$, with components $\xi_{P',w} \in (X_{\chi,w})'$.

By reduction to the $e_K \xi$, and by unwinding the definitions, one proves a preliminary result:
Theorem 1
For $\chi$ in an open dense subset of $X(M)_\sigma$:

$$\xi_{P',w} = \tilde{r}_{M'} \circ t A(w,P_w,P_{w'},w j_{M \cap w^{-1}, P} \delta) \circ \lambda(w) \circ j_{M \cap w^{-1}, P'} \circ \xi.$$  

To simplify the exposition, we assume, only in the introduction, that there is only one open double $(P,H)$-coset, $PH$. Due to Blanc and the second author (cf. [BD], Theorem 2.8, whose proof adapts to the weaker hypothesis on $F$ of this article), for $\eta \in E(M) \cap H$, there exists a unique rational map on $X(M)_\sigma$, $\chi \mapsto \xi(P,\delta,\eta) \in \text{End}(E(M) \cap H)$ such that:

$$t A(P,Q,\delta) \xi(P,\delta,\eta) = \xi(P,\delta, B(Q,P,\delta \chi) \eta).$$

An $H$-form on a finite length smooth representation of $G$, $(\pi,V)$, is said cuspidal or $H$-cuspidal if one of the equivalent statements holds (Kato-Takano [KT1]):

1) $\xi _{P} = 0$ for every proper $\sigma$-parabolic subgroup, $P$, of $G$.
2) For all $v \in V$, $c_{\xi,v} \in C^\infty(H \backslash G)$ is compactly supported mod-center where:

$$c_{\xi,v}(Hg) = \langle \xi, \pi(g)v \rangle.$$  

Assuming that $(\pi,V)$ is irreducible and unitary, an $H$-form is said square integrable if the $c_{\xi,v}$ are square integrable on $H \backslash G/A_G$.

Kato and Takano (cf. [KT2]) have characterized a square integrable $H$-form $\xi$ by a property, for each $\sigma$-parabolic subgroup of $G$, $P = MU$, of the eigenvalues of $A_M$ acting on the linear span of the translates of $j_P^{-1} \xi$ by elements of $A_M$. These eigenvalues are called the exponents of $\xi$ along $P$.

We define similarly the tempered $H$-forms.

Theorem 2
Let $\eta \in E(M) \cap H$. If $\chi \in X(M)_\sigma$ we denote by $\xi_{\chi}$ the $H$-form $\xi(P,\delta,\eta)$, when it is defined.

(i) Let us assume that $(\xi_{\chi})_{P',w}$ is non identically zero when $\chi$ varies in $X(M)_\sigma$, one may change $w$ in the same double $(P',P)$-coset such that:

a) $M \cap w.P$ is a $\sigma$-parabolic subgroup of $M'$, $M \cap w^{-1}.P'$ is a $\sigma$-parabolic subgroup of $M$, $P'_w, P_w$ and $w.P'_w$ are $\sigma$-parabolic subgroups of $G$.

b) The exponents of $\xi_{\chi}$ along $P'$ are explicitly controlled by the exponents of $\eta$ along certain $\sigma$-parabolic subgroups of $M$.

(ii) With these choices of $w$, if $\eta$ is cuspidal, one has $w.M \subset M'$,
\[ M \cap w^{-1}.P' = M, \text{ so that } \tilde{P}_w = P \text{ and} \]
\[ (\xi_x)_{P' \cap w} = \xi(M' \cap w.P, w\delta_x, B(\tilde{P}_w, w.P, w\delta_x)\eta). \]

(iii) If \( \eta \) is square integrable and \( \chi \) is unitary, then \( \xi_x \) is tempered. The notion of weak constant term of tempered \( H \)-forms is introduced and computed for \( \xi_x \).

(iv) The \( B \)-matrices preserves cuspidal (resp., square integrable) \( M \cap H \)-forms.

**Remark**

(a) As expected, when \( \eta \) is cuspidal and \( P' \) is too small, \( \xi_{P' \cap w} \) vanishes, by (ii).

b) The analogous of part (iii) for real groups is the main result of [D1]. The long proof used the description of relative discrete series by Oshima and Matsuki. Our proof here does not need such a knowledge. It would be interesting if one could find a proof of this result of [D1] avoiding the description of the relative discrete series.

Let us state the two main key lemmas of the article and let us explain how they lead to (i) a) and to (ii) of the Theorem above.

**First key lemma**

Let \( P \) be a parabolic, not necessarily a \( \sigma \)-parabolic, subgroup of \( G \). Let \( A_0 \) be a \( \sigma \)-stable maximal split torus of \( P \), which exists, and let \( M \) be a Levi subgroup of \( P \) with \( A_0 \subset M \). Let \( \delta \) be a smooth representation of \( M \) of finite length.

Let \( (\xi_x) \) be a smooth family of \( H \)-forms on \( i_P(\delta_x) \), with \( \chi \) in a neighborhood of 1 in a complex subtorus of \( X(M), X \).

Let us assume that \( PH \) is contained and open in the support of the family \((\xi_x)\).

Then the elements of \( X \) are anti-invariant by \( \sigma \).

Moreover, if there exists \( \chi \) strictly \( P \)-dominant in \( X \), then \( P \) is a \( \sigma \)-parabolic subgroup of \( G \).

Applying this to the family \( \xi_{P' \cap w} \) (with \( G \) replaced by \( M' \), and \( P \) by a suitable conjugate in \( M' \) of \( M' \cap w.P \), one sees why, in Theorem 2 (i) a), one can take \( M' \cap w.M \) to be a \( \sigma \)-parabolic subgroup of \( M' \). Then it is possible to refine the choice of \( w \) to get (i) a).

Let us assume now that \( \eta \) is cuspidal. If \( M \cap w^{-1}.P' \) is different from \( M \), it is not possible to see directly that \( \xi_{P' \cap w} \) is zero. What is easily seen in that case, from the cuspidality of \( \eta \), is that \( \tilde{j}_{M \cap w^{-1}.P'} \circ \xi \) vanishes on the open \((\tilde{P}_w, H)\)-double cosets, hence its support has an empty interior.

**Second key lemma:** The intertwining integrals in the definition of
\( \xi_{P',w} \) preserve this property.

This was suggested by a Lemma of Matsuki on orbit closures of orbits of parabolic subgroups on a real reductive symmetric space.

Applying \( \hat{r}_{M'} \) to something which vanishes on all open \((\tilde{P}' \cap w.P, M' \cap H)\)-double cosets, you get something which vanishes on all open \((M' \cap w.P, M' \cap H)\)-double cosets. The last ingredient is (cf. [BD]) that for \( \chi \)-generic, an \( H \)-form on \( i_G^\sigma E_\chi \) is determined by its restriction to open \((P, H)\)-double cosets. This is applied to \( M' \) instead of \( G \).

The computation of \( \xi_{P',w} \), when \( M \cap w^{-1}P' = M \) follows from Theorem 1. The last statement, which says that \( B \)-matrices preserve cuspidal \( M \cap H \)-forms, comes from the hereditary properties of \( j_{P'-w} \), and the fact observed in the remark, that for \( P' \) small enough \( j_{P'-w} \xi = 0 \).

Theorem 2, together with the determination of part of the Casselmann pairing for parabolically induced representations, leads to the determination of the constant term and the weak constant term of Eisenstein integrals for \( p \)-adic reductive symmetric spaces in terms of the corresponding \( C \)-functions.

This work is an important step towards the Plancherel formula and the Paley-Wiener theorem for reductive \( p \)-adic symmetric spaces.

The role of this type of results might be seen in the \( p \)-adic case in [Wal] for the work of Harish-Chandra on the Plancherel formula for \( p \)-adic groups, and [D4], [D5] for Whittaker functions.

For real reductive symmetric spaces, see also [D1], [CarD] for results analogous to the ones of this article and [D2] for their use for the Plancherel formula.

2. Notations

2.1. Reductive \( p \)-adic groups. If \( E \) is a vector space, \( E' \) will denote its dual. If \( E \) is real, \( E_C \) will denote its complexification.

If \( G \) is a group, \( g \in G \) and \( X \) is a subset of \( G \), \( g.X \) will denote \( gXg^{-1} \). If \( J \) is a subgroup of \( G \), \( g \in G \) and \((\pi, V)\) is a representation of \( J \), \( V^J \) will denote the space of invariant elements of \( V \) under \( J \) and \((g\pi, gE)\) will denote the representation of \( g.J \) on \( gE := E \) defined by:

\[
(g\pi)(gxg^{-1}) := \pi(x), \ x \in J.
\]

We will denote by \((\pi', V')\) the contragredient representation of \( G \) in the algebraic dual vector space \( V' \) of \( V \).

If \( V \) is a vector space of vector valued functions on \( G \) which is invariant by right (resp., left ) translations, we will denote by \( \rho \) (resp., \( \lambda \)) the right (resp., left) regular representation of \( G \) in \( V \).
If $G$ is locally compact, $d_{lg}$ will denote a left invariant Haar measure on $G$ and $\delta_G$ will denote the modulus function.

Let $F$ be a non archimedean local field. Unless specified we assume:

(2.1) The characteristic of $F$ and of its residue field are different from 2.

Let $|.|_F$ be the absolute value of $F$.

We will use conventions like in [Wal]. One considers various algebraic groups defined over $F$, and a sentence like:

(2.1) "let $A$ be a split torus" will mean "let $A$ be the group of $F$-points of a torus, $A$, defined and split over $F"."

With these conventions, let $G$ be a connected reductive linear algebraic group.

Let $A$ be a split torus of $G$. Let $X_*(A)$ be the group of one-parameter subgroups of $A$. This is a free abelian group of finite type. Such a group will be called a lattice. One fixes a uniformizing element $\varpi$ of $F$. One denotes by $\Lambda(A)$ the image of $X_*(A)$ in $A$ by the morphism of groups $\lambda \mapsto \lambda(\varpi)$. By this morphism $\Lambda(A)$ is isomorphic to $X_*(A)$.

If $(P,P^-)$ are two opposite parabolic subgroups of $G$, we will denote by $M$ their common Levi subgroup and by $AM$ or $A$ the maximal split torus of its center. We denote by $U$ (resp., $U^-$) the unipotent radical of $P$ (resp., $P^-$).

Let $A^-$ (resp., $A^-$) be the set of $P$ antidominant (resp., strictly antidominant) elements in $A$. More precisely, if $\Sigma(P,A)$ is the set of roots of $A$ in the Lie algebra of $P$, and $\Delta(P,A)$ is the set of simple roots, one has:

$$A^- \text{ (resp., } A^-) = \{ a \in A | |\alpha(a)|_F \leq 1, \text{ (resp., } < 1) \alpha \in \Delta(P,A) \}.$$

We define similarly $A^+$ and $A^{++}$ by reversing the inequalities. One defines also for $\varepsilon > 0$:

$$A^-(\varepsilon) = \{ a \in A | |\alpha(a)|_F \leq \varepsilon, \alpha \in \Delta(P,A) \}.$$

If $A_0$ is a maximal split torus of $G$, a maximal compact subgroup of $G$, $K_0$, will be said $A_0$-good if it is the stabilizer of a special point of the apartment associated to $A_0$ of the Bruhat-Tits building of $G$. We choose such a $K_0$. Let $M_0$ be the the centralizer in $G$ of $A_0$ and let $P_0$ be a minimal parabolic subgroup of $G$ with Levi subgroup $M_0$. If $P$ is a parabolic subgroup of $G$ which contains $A_0$, we denote, if there is no ambiguity, by $P^-$ the opposite parabolic subgroup of $G$ to $P$ which contains $A_0$ and by $M$ the intersection of $P$ and $P^-$. 
For the following result, see [C], Prop. 1.4.4:

There exists a decreasing sequence of compact open subgroups of $G$, $K_n$, $n \in \mathbb{N}$, which forms a basis of neighborhoods of the identity in $G$, such that for all $n \in \mathbb{N}^*$, $J = K_n$ is normal in $K_0$ and such that or every parabolic subgroup, $P$, which contains $P_0$, one has:

1) The product map $J_{U^-} \times J_M \times J_U \to J$ is bijective, where $J_{U^-} = J \cap U^-$, $J_M = J \cap M$, $J_U = J \cap U$.
2) For all $a \in A^-$, $aJ_Ua^{-1} \subset J_U$, $a^{-1}J_Ua \subset J_{U^-}$.
3) $J_M$ satisfies the same properties for $M$, $A_0$ and $P_0 \cap M$.

One says that a compact open subgroup of $G$, $J$, has an Iwahori factorization with respect to $(P,P^-)$ if 1) and 2) are satisfied.

If $J$ is an algebraic group, one denotes by Rat$(J)$ the group of its rational characters defined over $F$. Let us define:

$$a_G = \text{Hom}_{\mathbb{Z}}(\text{Rat}(G), \mathbb{R}).$$

The restriction of rational characters from $G$ to $A_G$ induces an isomorphism:

$$\text{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq \text{Rat}(A_G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.3}$$

Notice that Rat$(A_G)$ appears as a generating lattice in the dual space $a'_G$ of $a_G$ and:

$$a'_G \simeq \text{Rat}(G) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{2.4}$$

One has the canonical map $H_G : G \to a_G$ which is defined by:

$$e^{(H_G(x),\chi)} = |\chi(x)|_F, \ x \in G, \ \chi \in \text{Rat}(G). \tag{2.5}$$

The kernel of $H_G$, which is denoted by $G^1$, is the intersection of the kernels of the characters of $G$, $|\chi|_F, \ \chi \in \text{Rat}(G)$. One defines $X(G) = \text{Hom}(G/G^1, \mathbb{C}^*)$, which is of unramified characters of $G$. One will use similar notations for Levi subgroups of $G$.

One denotes by $a_{G,F}$, resp., $\tilde{a}_{G,F}$ the image of $G$, resp., $A_G$, by $H_G$. Then $G/G^1$ is isomorphic to the lattice $a_{G,F}$.

There is a surjective map:

$$\tilde{a}'_G \to X(G) \to 1. \tag{2.6}$$

denoted by $\nu \mapsto \chi_\nu$ which associates to $\chi \otimes s$, with $\chi \in \text{Rat}(G)$, $s \in \mathbb{C}$, the character $g \mapsto |\chi(g)|_s^s$ (cf. [Wal], I.1.1). In other words:

$$\chi_\nu(g) = e^{(\nu,H_G(g))}, \ g \in G, \ \nu \in (\tilde{a}'_G)_\mathbb{C}. \tag{2.7}$$
The kernel is a lattice and it defines a structure of a complex algebraic variety on $X(G)$ of dimension $\dim_{\mathbb{R}} a_G$. Moreover $X(G)$ is an abelian complex Lie group whose Lie algebra is equal to $(a_G)_{\mathbb{C}}$.

If $\chi$ is an element of $X(G)$, let $\nu$ be an element of $a'_{G,\mathbb{C}}$ such that $\chi \nu = \chi$. The real part $\text{Re}(\nu) \in a'_{G}$ is independent from the choice of $\nu$. We will denote it by $\text{Re}(\chi)$. If $\chi \in \text{Hom}(G, \mathbb{C}^*)$ is continuous, the character of $G$, $|\chi|$, is an element of $X(G)$. One sets $\text{Re}(\chi) = |\chi|$. Similarly, if $\chi \in \text{Hom}(A_G, \mathbb{C}^*)$ is continuous, the character $|\chi|$ of $A_G$ extends uniquely to an element of $X(G)$ with values in $\mathbb{R}^+$, that we will denote again by $|\chi|$ and one sets $\text{Re}(\chi) = |\chi|$.

If $P$ is a parabolic subgroup of $G$ with Levi subgroup $M$ we set:

$$a_P = a_M, H_P = H_M.$$ 

The inclusions $A_G \subset A_M \subset M \subset G$ determine a surjective morphism $a_{M,F} \to a_{G,F}$, resp., an injective morphisme $\hat{a}_{G,F} \to \hat{a}_{M,F}$, which extends uniquely to a surjective linear map between $a_M$ and $a_G$, resp., injective map between $a_G$ and $a_M$. The second map allows to identify $a_G$ with a subspace of $a_M$ and the kernel of the first one, $a^G_{M}$, satisfies:

$$a_M = a^G_{M} \oplus a_G.$$ 

Let us denote by $X(G|M)$ the set of restrictions to $M$ of elements of $X(G)$. Then $X(G|M)$ is the analytic subgroup of $X(M)$ with Lie algebra $(a'_G)_{\mathbb{C}} \subset (a'_M)_{\mathbb{C}}$. This follows easily from (2.7) and (2.8). Moreover:

$$X(G|M)$$

is a closed subgroup of $X(M)$.

This will be seen by writing $X(G) = X(G)_u X(G)^+$ where $X(G)_u$ is the compact group of unitary characters in $X(G)$ and $X(G)^+$ is of elements in $X(G)$ which are strictly positive. The group $X(G)_u$ is compact and has compact image. The group $X(G)^+$ is isomorphic to a vector subgroup and the restriction, which is a morphism of Lie groups, determines an isomorphism of $X(G)^+$ to a connected Lie subgroup of $X(M)^+$: the restriction of an element of $X(G)^+$ trivial on $A_G$ is trivial by what has been said above. Hence the image of $X(G)^+$ is closed. This implies our claim on $X(G|M)$.

One has (cf. [D3], (4.5)),

$$\Lambda(A_G) \to G/G^1$$

is injective and allows to identify $\Lambda(A_G)$ to a subset of $a_G$.

Let $A$ be a maximal split torus of $G$ and let $M_0$ be the centralizer of $A$ in $G$. We fix a $W$-invariant scalar product on $a$, where $W$ is the Weyl group of $(G,A)$. Then $a_G$ identifies with the fixed point set of
a under $W$ and $a^G$ is an invariant subspace of $a$ under $W$ which is supplementary to $a_G$. Hence it is the orthogonal subspace to $a_G$ in $a$. The space $a'_G$, might be viewed as a subspace of $a'$ by (2.8). More generally let $M$ be a Levi subgroup of $G$ which contains $A$. From (2.8) applied to $M$ instead of $G$ and to $M_0$, instead of $M$, one gets a decomposition $a = a^M_{M_0} \oplus a_M$. From the $W$-invariance of the scalar product one gets:

The decomposition $a = a^M_{M_0} \oplus a_M$ is an orthogonal decomposition.

(2.11) The space $a^M_M$ appears as a subspace of $a'$ and, in the identification of $a$ with $a'$ given by the scalar product, $a^M_M$ identifies with $a_M$.

If $\nu \in a'$, we denote by $P_\nu$ the parabolic subgroup of $G$ whose Lie algebra is equal to the sum of the $A$-weights spaces for the weights $\alpha$ which are either equal to zero or to a root $\alpha$ such that $(\nu, \alpha) \geq 0$. Then one has:

(2.12) The parabolic subgroups of $G$, $P_\nu$ and $P_{-\nu}$ are opposite.

If $\rho_P \in a'$ is the half sum of the $A$-roots of $A$ in the Lie algebra of $P$, then the following is clear:

(2.13) $P = P_{\rho_P}$.

Let $G$ be the algebraic group defined over $F$ whose group of $F$-points is $G$. Let $\sigma$ be a rational involution of $G$, defined over $F$. Let $H$ be the group of $F$-points of an open $F$-subgroup of the fixed point set of $\sigma$. We will also denote by $\sigma$ the restriction of $\sigma$ to $G$.

A split torus of $G$, $A$, is said $\sigma$-split if $A$ is contained in the set of elements of $G$ which are antiinvariant by $\sigma$. Let $A$ be a $\sigma$-invariant maximal split torus of $G$. We fix a scalar product on $a$ which is invariant by $\sigma$ and the action of the Weyl group of $G$ with respect to $A$: this is possible because $\sigma$ normalizes $A$, hence its normalizer and the Weyl group of $G$ with respect to $A$. Let $A^\sigma$ (resp., $A_\sigma$) be the maximal split torus in the group of elements of $A$ which are invariant (resp., antiinvariant by $\sigma$). Then $a^\sigma$ (resp., $a_\sigma$) identifies with the set of invariant of (resp., antiinvariant) of $\sigma$ in $a$ and $A_\sigma$ is the maximal
\(\sigma\)-split torus of \(A\).

If \(M\) is a \(\sigma\)-invariant Levi subgroup of \(G\) which contains \(A\), \(a_M\) is a \(\sigma\)-invariant subspace of \(a\) and \(a_M = a_M^\sigma \oplus a_M^{-\sigma}\) where \(a_M^\sigma = a_M \cap a^\sigma, a_M^{-\sigma} = a_M \cap a^{-\sigma}\). This is an orthogonal decomposition and, in the identification of \(a\) to \(a'\), \((a_M^\sigma)^{-\sigma}\) identifies with the space \((a'_M)^{-\sigma}\) of antiinvariant elements of \(a'_M\).

Moreover the Lie algebra of the connected component, \(X(M)_\sigma\), of the group of antinvariant elements of \(X(M)\), with the identification of \(a_M\) and \(a'_M\), is equal to \(a_M^{-\sigma}\).

A parabolic subgroup of \(G\), \(P\), is called a \(\sigma\)-parabolic subgroup if \(P\) and \(\sigma(P)\) are opposite parabolic subgroups. Then \(M := P \cap \sigma(P)\) is the \(\sigma\)-stable Levi subgroup of \(P\). If \(P\) is such a parabolic subgroup, \(P^-\) will denote \(\sigma(P)\). Then the maximal split torus of the center of \(M\), \(A_M\) is \(\sigma\)-stable.

One deduces from [HW], Proposition 13.4:

\[\text{(2.15)}\]

\[PH\text{ and }P^-H\text{ are open in }G.\]

Let \(M\) be equal to \(P \cap \sigma(P)\).

The restriction to \(H \cap M\) of the modulus function of \(P\), \(\delta_P\), is trivial as it is positive and it is equal to its inverse on \(M \cap H\).

We recall also that (cf. l.c. Corollary 6.16):

\[\text{(2.16)}\]

There are only a finite number of \(H\)-conjugacy classes of parabolic subgroups of \(G\).

Let \(M_\emptyset\) be the \(\sigma\)-stable Levi subgroup of a minimal \(\sigma\)-parabolic subgroup of \(G\). Let \(A_\emptyset\) be the maximal \(\sigma\)-split torus of the center of \(M_\emptyset\).

**Definition 2.1.** An element \(x\) of \(G\) is said \(A_\emptyset\)-good if and only if \(x^{-1}.A_\emptyset\) is a \(\sigma\)-split torus.

From [BD], Lemma 2.4 one sees:

There exists a finite set of representatives of the \((P_\emptyset, H)\)-double cosets open in \(G\), \(W^{G}_{M_\emptyset}\), whose elements are \(A_\emptyset\)-good.

Moreover if \(M\) is the \(\sigma\)-stable Levi subgroup of a \(\sigma\)-parabolic subgroup of \(G\), with \(M_\emptyset\) (or \(A_\emptyset\)) contained in \(M\), there exists a subset, \(W^{G}_{M}\), of \(W^{G}_{M_\emptyset}\) such that for all \(\sigma\)-parabolic subgroup \(P\) of \(G\) with Levi subgroup \(M\), \(W^{G}_{M}\) is a set of representatives of the \((P, H)\)-double open cosets.

**Lemma 2.2.** Let the notations and the hypothesis be as above and let \(x \in G\) be \(A_\emptyset\)-good. Then one has:

(i) The group \(x^{-1}.P\) is a \(\sigma\)-parabolic subgroup of \(G\) with \(\sigma\)-stable Levi subgroup \(x^{-1}.M\). Moreover \(\sigma(x^{-1}.P) = x^{-1}.P^-\), where \(P^- = \sigma(P)\).
(ii) The set $PxH$ is open in $G$.
(iii) One defines an involution of $G$, $\sigma_x$, by:
$$\sigma_x(g) := x\sigma(x^{-1}gx)x^{-1}, \ g \in G$$
whose fixed point set contains $x.H$. Then $P$ is a $\sigma_x$-parabolic subgroup of $G$, $\sigma_x(M) = M$ and $\sigma_x(P) = P$. (iv) For all $y \in PxH$, $y^{-1}.P$ is a $\sigma$-parabolic subgroup of $G$ and $P$ is a $\sigma_y$-parabolic subgroup of $G$. Let $M_y$ be the $\sigma_y$-stable Levi subgroup of $P$. Then $M_y \cap y.H = P \cap y.H$. Moreover if $y = px$ with $p \in P$, one has $M_y = p.M$.

Proof. (i) and (ii) follows from [BD], Lemme 2.4. (iii) follows immediately from (i). (iv) If $y = pxh$ with $h \in H, p \in P$. Then $y^{-1}.P = h.(x^{-1}.P)$. Then the first part of (iv) follows from (i). Also one has $\sigma_y = \sigma_{px}$. Hence one is reduced to prove the second part of (iv) for $h = 1$ and $y = px$. A simple computation shows that $\sigma_y(P) = p.P^-$. Hence $\sigma_y(P) \cap P = p.M$, which proves (iv).

The sentence: “Let $P = MU$ be a parabolic subgroup of $G$” will mean that $U$ is the unipotent radical of $P$ and $M$ a Levi subgroup of $G$. If moreover $P$ is a $\sigma$-parabolic subgroup of $G$, $M$ will denote its $\sigma$-stable Levi subgroup. Let $A_0$ be a maximal $\sigma$-split torus of $G$ and let $A_0$ be a $\sigma$-stable maximal split torus of $G$ which contains $A_0$. Let $P_0 = M_0U_0$ be a minimal $\sigma$-parabolic subgroup of $G$, whose $\sigma$-stable Levi subgroup is equal to the centralizer, $M_0$, of $A_0$ in $G$.

**Proposition 2.3.** There exists a decreasing sequence of $\sigma$-stable compact open subgroups of $G$, $(J_n)_{n \in \mathbb{N}}$, which forms a basis of neighborhoods of 1 in $G$ and such that for each $n \geq 1$, $J := J_n$ satisfies:
(i) For every $\sigma$-parabolic subgroup of $G$, $P = MU$ which contains $P_0$, the product map $J_{U^-} \times J_M \times J_U \to J$ is bijective, where $J_{U^-} = J \cap U^-$, $J_M = J \cap M$, $J_U = J \cap U$.
(ii) Let $A$ be the maximal split torus of the center of $M$. For all $a \in A^-$, $aJ_Ua^{-1} \subset J_U$, $a^{-1}J_Ua \subset J_U^-$. (iii) One has $J = J_HJ_P$, where $J_H = J \cap H, J_P = J \cap P$.
(iv) For each $\sigma$-parabolic subgroup of $G$ which contains $P_0$, $P = MU$, the sequence $(J_n \cap M)$ enjoys the same properties that $(J_n)$ for $M$, $P_0 \cap M$.

Proof. We fix a basis of the Lie algebra of $G$, $g$, $(X_j)$, which is the union of a basis $(U_k)$ of $u_0$ made of weight vectors for $a_0$, of a basis of $m_{\bar{a}} \cap h$, a basis of the space of antiinvariant elements by $\sigma$ of
and of the basis \((\sigma(U_k))\) of \(\sigma(u_0)\). We use Lemma 10.1 (iv) for \(G_1 = U_0, G_2 = M_0, G_3 = \sigma(U_0)\) to prove (i).

Let \(A^0\) be the maximal compact open subgroup. We remark that the \(A^0\)-module \(A^-\) is finitely generated. Let \((\lambda_l)\) be a finite family of generators. We apply Lemma 10.1 (iii) to the family of automorphisms of \(G\) induced by the conjugacy by elements of \(A^0\lambda_l\). The fact that the \(J_n\) can be choosen \(\sigma\)-stable is proven similarly.

(iii) We apply Lemma 10.1 to \(G_1 = U, G_2 = M, G_3 = H\). Here \(g'_1 = u, g'_2 = m\) and \(g'_3\) is the subspace generated by the \(U_i + \sigma(U_i)\), where the \(U_i\) are those which belongs to \(u\).

(iv) results from the fact that \(J_M\) is defined like \(J\), in view of Lemma 10.1 (iv).

\[\square\]

Remark 2.4. If \(F\) is of characteristic different from 2 and the characteristic of its residue field is different from 2, the proposition is due to Katano and Takano (cf. [KT] Lemma 4.3). They added the fact that the \(J_n\) might be choosen normal in some maximal compact open subgroup of \(G\).

In [D3], Remark 3.2, it was stated uncorrectly, although not used, that the \(J_n\) have also an Iwahori factorization

If a compact open subgroup of \(G\) satisfies the properties of the Proposition, it will be said to have a \(\sigma\)-factorization. These are the \((P,H)\)-good groups from the introduction.

3. Two operations on \(H\)-forms on induced representations

3.1. Second adjointness Theorem and \(H\)-fixed linear forms. In the sequel, the smooth representations of \(G\) and of its closed subgroups will be with complex coefficients.

Let \((\pi, V)\) be a smooth representation of \(G\) and let \(P = MU\) be a parabolic subgroup of \(G\). One denotes by \((\pi_P, V_P)\) the tensor product of the quotient of \(V\) by the \(M\)-submodule generated by the \(\pi(u)v - v, u \in U, v \in V\), with the representation of \(M\) on \(\mathbb{C}\) given by \(\delta_P^{-1/2}\). We call it the normalized Jacquet module of \(V\) along \(P\). We denote by \(j_P\) the natural projection map from \(V\) to \(V_P\) and sometimes \(\pi_P\) will be denoted \(j_P(\pi)\).

The following result is due to J. Bernstein (cf. [Be], [R], Chapter V.9). We present here a slight reformulation of his result (see [D3], Lemma 2.1). This is a generalization of a result of W. Casselman [C]. Let \((P, P^-)\) be a pair of opposite parabolic subgroups of \(G\) with common Levi subgroup \(M\). Let \(A_0\) be a maximal split torus of \(M\) and let \(P_0\) be
a minimal parabolic subgroup $P_0$ such that $A_0 \subset P_0 \subset P$.

We define:

$$\Theta_P := \Delta(P_0 \cap M, A_0)$$  \hspace{1cm} (3.1)

and, for $\varepsilon > 0$, we set:

$$A_\varepsilon^-(P, < \varepsilon) := \{ a \in A^-_0 \mid |a(a)|_F < \varepsilon, a \in \Delta(P_0, A_0) \setminus \Theta_P \}.$$  \hspace{1cm} (3.2)

**Second adjointness theorem**

Let $(\pi, V)$ be a smooth representation of $G$. Let $j_P$ (resp., $j_{P-}$) denote the canonical projection of $V$ (resp., of the smooth dual $\tilde{V}$ of $V$) onto $V_P$ (resp., $(\tilde{V})_{P-}$).

Then there exists a unique non degenerate $M$-invariant bilinear form on $(V)_{P-} \times V_P$, $\langle ., . \rangle_p$ such that for all compact open subgroups, $J$, of $G$, there exists $\varepsilon_J < 1$ such that:

$$\delta_P^{1/2}(a)\langle j_P-(\tilde{\nu}), \pi_P(a)j_P(v) \rangle_p = \langle \tilde{\nu}, \pi(a)v \rangle,$$

for $a \in A_{\varepsilon^-(P, < \varepsilon)}$, $\nu \in V^J$, $\tilde{\nu} \in \tilde{V}^J$. It is part of the statement that $\varepsilon_J$ does not depend on $V$. In particular there is a canonical isomorphism between $(V_P)^*$, $(\tilde{V})_{P-}$.

One denotes by $e_J$ the normalized Haar measure on $J$, that we view as a compactly supported distribution on $G$.

An $H$-**form** on a smooth module of $G$ is an $H$-fixed linear form on $V$.

Using the same argument as in [L], Lemma 2, one sees:

Let $P = MU$ be a $\sigma$-parabolic subgroup of $G$. Let $A$ be the maximal split torus of the center of $M$. Let $A_\sigma$ be the maximal $\sigma$-split torus of the maximal split torus, $A$, of the center of $M$. Let $J$ be a compact open subgroup of $G$ with a $\sigma$-factorization with respect to $(P, P^-)$. Then for every smooth module $(\pi, V)$, $\xi \in V^{H}$:

$$\langle \xi, \pi(a)v \rangle = \langle e_J \xi, \pi(a)v \rangle, \; v \in V^J, \; a \in A^- \cap A_\sigma,$$

where $e_J \xi$ is the element of $\tilde{V}$ defined by:

$$\langle e_J \xi, v \rangle = \langle \xi, \pi(e_J)v \rangle, \; v \in V.$$

Let $A_M$, or simply $A$, be the maximal split torus of the center of $M$. For every smooth module $(\pi, V)$ of $G$ and $\xi$ an $H$-form on $V$, there exists a unique $M \cap H$-form on $V_P$, $j_P-\xi$, such that for each compact open subgroup, $J$, of $G$, there exists $\varepsilon'_J > 0$, such that $\varepsilon'_J \leq \varepsilon_J$, depending only on $J$ and not on $V$ and $\xi$, such that one has:

$$\langle \xi, \pi(a)v \rangle = \delta^{1/2}_P(a)\langle j_{P-}-(\tilde{\xi}), \pi_P(a)j_P(v) \rangle, \; v \in V^J, \; a \in A^-(\varepsilon'_J).$$  \hspace{1cm} (3.5)
From (3.4), one deduces from the above that if \( J \) is a compact open subgroup of \( G \) with a \( \sigma \)-factorization, one has

\[
\langle e_J \xi, \pi(a)v \rangle = \delta_p^{1/2}(a) \langle j_{p \cdot \xi}, \pi_P(a) j_P(v) \rangle, \quad v \in V^J, \; a \in A_\sigma \cap A^-(\varepsilon'_J).
\]

From the Second adjointness Theorem, one deduces from this, that for \( a \in A_\sigma \cap A^-(\varepsilon'_J), v \in V^J :\)

\[
\langle j_{p \cdot \xi}, \pi_P(a) j_P(v) \rangle = \langle j_{p \cdot e_J \xi}, \pi_P(a) j_P(v) \rangle.
\]

If \( \pi \) is admissible, the two sides of this equality are \( A_\sigma \)-finite functions on \( A_\sigma \), hence they are equal. In particular in \( a = 1 \), one gets:

Let \( J \) be a compact open subgroup of \( G \) with a \( \sigma \)-factorization with respect to \((P, P^-)\). Then, if \( \pi \) is admissible, (3.6) one has:

\[
\langle j_{p \cdot \xi}, v \rangle = \langle j_{p \cdot (e_J \xi)}, v \rangle_P, v \in V^{J^M}_P.
\]

3.2. \textbf{Induced representations.} Let \( K \) be a maximal compact open subgroup of \( G \) which is the fixator of a special point of an apartment of the building of \( G \). It will be fixed for the rest of the article. Let \( dk \) be the Haar measure on \( K \) of volume 1. Let \( dg \) be the Haar measure on \( G \) such that \( K \) has volume 1.

One has \( G = PK \) for all parabolic subgroups of \( G \) as it is true for at least a minimal parabolic subgroup of \( G \).

Let \( P = MU \) be a parabolic subgroup of \( G \) with Levi subgroup \( M \) and unipotent radical \( U \). If \( (\delta, E) \) is a smooth representation of \( M \), one extends it to a representation of \( P \) trivial on \( U \). Let \( \chi \) be an element of \( X(M) \). One denotes by \( E_\chi \) the space of the representation \( \delta_\chi := \delta \otimes \chi \). Let \( i^G_\chi E_\chi \) be the space of maps \( v \) from \( G \) to \( E \), right invariant by a compact open subgroup of \( G \) and such that \( v(mug) = \delta_P(m)^{1/2} \delta_\chi(m) f(g) \) for all \( m \in M, u \in U, g \in G \). Let \( i^G_P \delta_\chi \) be the representation of \( G \) in \( i^G_P E_\chi \) by right translations.

One denotes by \( i^K_{P \cap K} E \) the space of maps \( v \) from \( K \) in \( E \), which are right invariant by a compact open subgroup of \( K \) and such that \( v(pk) = \delta(p)v(k) \) for all \( k \in K \) and \( p \in P \cap K \). The restriction of functions to \( K \) determines an isomorphism from \( i^G_P E_\chi \) to \( i^K_{P \cap K} E \). We will denote by \( i^G_P(\delta_\chi) \) the representation of \( G \) in \( i^K_{P \cap K} E \) deduced from \( i^G_P \delta_\chi \) “par transport de structure”. This representation will be called the compact realization of \( i^G_P \delta_\chi \) in this space independent from \( \chi \). If \( v \in i^K_{P \cap K} E \), one denotes by \( v_\chi \) the element of \( i^G_P E_\chi \) whose restriction to \( K \) is equal to \( v \).
If $\delta$ is unitary, one defines a scalar product on $i^K_{P\cap K} E$ by:

$$
(3.7) \quad (v, v') = \int_K (v(k), v'(k)) \, dk, \, v, v' \in i^K_{P\cap K} E.
$$

The representation $i^G_P \delta_X$ is unitary for this scalar product when $\chi$ is unitary. Consequently, “par transport de structure”, $i^G_P \delta_X$ is also unitary.

If $g \in G$, one chooses $u_P(g) \in U, m_P(g) \in M$ and $k_P(g) \in K$ such that $g = u_P(g) m_P(g) k_P(g)$. Then $\delta_p(m_P(g))$ does not depend on the choice of $m_P(g)$. Let $P^- = MU^-$ be the opposite parabolic subgroup of $P$ with respect to $M$. We choose a Haar measure on $U^-$, $du^-$ such that:

$$
(3.8) \quad \int_{U^-} \delta_P(m_P(u^-)) \, du^- = 1.
$$

One has, (cf. e.g. [D4], equation (2.10)):

$$
(3.9) \quad (v_\chi, v'_\chi) = \int_{U^-} (v_\chi(u^-), v'_\chi(u^-)) \, du^-.
$$

Let $e$ be an element of $E$ and let $J$ be a compact open subgroup of $G$ such that $e$ is invariant by $J \cap P$ under $\delta$. One defines a map $v_{e,\delta}^P J$ from $G$ to $E$ by:

$$
(3.10) \quad v_{e,\delta}^P J (pj) = \delta_{p1/2}(p) \delta(p)e, \, j \in J, \, p \in P
$$

$$
\quad v_{e,\delta}^P J (g) = 0, \, g \notin PJ,
$$

the definition making sense due to our hypothesis on $J$ and $e$.

Notice that this hypothesis is satisfied if $J$ has an Iwahori factorization with respect to $(P, P^-)$ (resp., if $P$ is a $\sigma$-parabolic subgroup of $G$ and $J$ has a $\sigma$-factorization for $(P, P^-)$) and $e$ is $J_M$-invariant.

Then $v_{e,\delta}^P J$ is invariant on the right by $J$ and defines an element of $i^G_P E$.

3.3. The operation $r_M$. Let $P = MU$ be a parabolic subgroup of $G$ and $\delta, E$ a smooth representation of $M$.

Let $d_P$ be the left invariant Haar measure on $P$ such that:

$$
(3.11) \quad \int_{P} f(g) \, dg = \int_{P \times K} f(pk) \, d_P \, dk, \, f \in C_c^\infty (G).
$$

Let $d_r P = \delta_P d_P$ which is a right invariant measure on $P$. One defines a linear map $M_{\delta, P}$ from $C_c^\infty (G) \otimes E$ to $i^G_P E$ by:

$$
(3.12) \quad (M_{\delta, P}(f))(g) = \int_{P} \delta_{p1/2}(p^{-1}) \delta(p^{-1}) f(pg) \, d_r P, \, f \in C_c^\infty (G) \otimes E, \, g \in G.
$$

This map goes through the quotient to an isomorphism between $H_0(P, C_c^\infty (G) \otimes E))$ and $i^G_P E$ ([BD], Prop. 1.13 (iv)), where $H_0$ stands
for the 0-homology.
A linear form \( \xi \) on \( i_P^G E \) determines an \( E \)-distribution on \( G \), \( \tilde{\xi} \), which is \( P \)-covariant for the representation \( \pi = \delta \otimes \delta_P^{-1/2} \) (cf. subsection 9.1) for the definitions) defined by:

\[
\tilde{\xi}(f) = \xi(M_{\delta,P}(f)), f \in C_c^{\infty}(G) \otimes E.
\]

This follows from the obvious equality:

\[
\langle \tilde{\xi}, \lambda(p)(\pi(p)f) \rangle = \langle \xi, M_{\delta,P}(\lambda(p)(\pi(p)f)) \rangle, f \in C_c^{\infty}(G) \otimes E
\]

and from the equality:

\[
M_{\delta,P}(\lambda(p)(\pi(p)f)) = M_{\delta,P}(f), f \in C_c^{\infty}(G) \otimes E,
\]

that we are going to prove. Let \( p_0 \) be an element of \( P \). Taking into account the equality \( \delta_P(p_0)^{-1/2} \) with \( \delta_P(p_0)^{1/2} \delta_P^{-1}(p_0) \), one has, for \( g \in G \):

\[
[M_{\delta,P}(\lambda(p_0)(\pi(p_0)f))](g) = \int_P \delta_P^{1/2}(p^{-1} p_0) \delta(p^{-1} p_0) f(p_0^{-1} pg) \delta_P(p_0)^{-1} d_r p
\]

Using the definition of \( d_r p \), the change of variables \( p' = p_0^{-1} p \), leads to the required identity.

By definition, the support of \( \xi \) is the support of \( \tilde{\xi} \). As \( \tilde{\xi} \) is left \( P \)-

covariant, one has:

\[
\text{Supp}(\xi) \text{ is left } P \text{-invariant and is equal to the complementary subset of the largest left } P \text{-invariant open subset of } G,
\]

\[
O, \text{ such that } \langle \xi, v \rangle = 0 \text{ if the support of } v \in i_P^G E \text{ is contained in } O.
\]

If moreover \( \xi \) is invariant by the right action of \( H \) then the same is true for the \( E \)-distribution \( \tilde{\xi} \).

Let us assume that \( PH \) is open in \( G \) and that \( \xi \) is right invariant by \( H \). One remarks that the left Haar measure on \( G \) restricted to \( PH \) is left invariant by \( P \) and right invariant by \( H \). One applies Lemma 9.1 to \( P \times H \) modulo the diagonal of \( H \cap P \). From the above, it has a \( P \times H \)-invariant measure. The contragredient representation of \( \delta_P \) is \( \delta_P^{-1} \). Hence there exists a unique \( \eta \in E', H \cap P \)-invariant, such that one can define:

\[
f_\xi(g) := (\delta_P)(p)^{1/2} \delta'(p) \eta, g = ph, p \in P, h \in H
\]

which verifies:

\[
\langle \tilde{\xi}, f \rangle = \int_{PH} \langle f_\xi(g), f(g) \rangle dg \text{ if } f \in C_c^{\infty}(G) \otimes E \text{ has its support contained in } PH.
\]
Let us assume that moreover \( f \) has its support contained in \( P(K \cap H) \). The set of \((p, k) \in P \times K\) such that \( pk \in P(K \cap H) \) is equal to \( P \times (K \cap (K \cap H)) \). Thus, from (3.11), one gets:

\[
\int_{P(K \cap H)} \langle f_\xi(g), f(g) \rangle \, dg = \int_{P \times (K \cap (K \cap H))} \langle f_\xi(pk), f(pk) \rangle \, dp \, dk.
\]

Then one integrates over \( P \), taking into account the covariance property of \( \xi \). Then, as \( \delta_p dp = d_p \), one gets:

\[
\langle \xi, v \rangle = \int_{(K \cap (K \cap H))} \langle f_\xi(k), v(k) \rangle \, dk,
\]

where \( v = M_{\delta, p}(f) \).

One remarks that if \( e \) fixed by \( J \cap P \), using the notations of (3.10), one has:

\[
\langle \xi, v^{P,J} \rangle = \text{Vol}(J \cap P)^{-1} M_{\delta, p}(1 \otimes e),
\]

where \( \text{Vol}(J \cap P) \) denotes the volume of \( J \cap P \) for the measure \( dp \).

Let \( x \in G \) such that \( P x H \) is open in \( G \). Applying (3.15) and (3.16) to each \( \rho(x)f \), which is \( xHx^{-1} \)-invariant and one can define a function on the union \( \Omega \) of all \((P, H) \) open double cosets, \( \xi \), with values in \( E' \), right invariant under \( P \), left covariant under \( H \), with \( \delta' \otimes \delta^{{-1}/2} \), such that:

If \( f \in C_c^\infty(G) \otimes E \) has its support contained in \( \Omega \),

\[
\langle \xi, f \rangle = \int_{\Omega} \langle f_\xi(g), f(g) \rangle \, dg.
\]

Moreover if \( x \in G \) is such that \( P x H \) is open in \( G \):

\[
f_\xi(x) \in E'^{P \cap x, H}.
\]

We assume moreover that \( P \) is \( \sigma \)-parabolic subgroup of \( G \). Let \( J \) be a compact open subgroup of \( K \) which has a \( \sigma \)-factorization for \((P, P^-)\).

Let \( e \in E \) be fixed by \( J_M \). Let us prove:

\[
\langle \xi, v^{P,J}_{e, \delta} \rangle = \text{Vol}((K \cap P)J_H) \langle \eta, e \rangle.
\]

One can apply (3.17) with \( v = v^{P,J}_{e, \delta} \), by taking into account (3.18). As the support of \( v \) is contained \( PJ = PJ_H \) where \( J_H = J \cap H \), one gets from (3.17):

\[
\langle \xi, v \rangle = \int_{K \cap (P J_H)} \langle f_\xi(k), v(k) \rangle \, dk.
\]

But \( K \cap (P J_H) = (K \cap P)J_H \). So, one has:

\[
\langle \xi, v \rangle = \int_{(K \cap P)J_H} \langle f_\xi(k), v(k) \rangle \, dk.
\]
The function under the integral sign is left invariant by $K \cap P$, due to (3.15) and to the properties of the induced representation. It is right invariant by $J_H$ due to the fact that $\xi$ is $H$-invariant and that $v$ is $J$-invariant. Hence (3.20) follows.

**Proposition 3.1.** (i) Let $\tilde{v}$ be a smooth (resp., $\xi$ be an $H$-fixed) linear form on $i_P^G E$, where $P = MU$ is a parabolic subgroup of $G$ (resp., a parabolic subgroup of $G$ such that $PH$ is open in $G$), and $(\delta, E)$ is a smooth representation of $M$. We can identify canonically $\tilde{v}$ with an element of $i_P^G E$ and we will denote its value at $1$ by $\tilde{r}_M \tilde{v} \in \hat{E}$. Similarly $f_\xi(1) \in E^{M \cap H}$ is well defined. We will denote it $\tilde{r}_M \xi$. Then if moreover $P$ is a $\sigma$-parabolic subgroup of $G$, for all compact open subgroup of $K, J$, with a $\sigma$-factorization with respect to $(P, P^-)$,

one has:

$\tilde{r}_M (e_J \xi) = e_J \tilde{r}_M (\tilde{r}_M \xi)$.

(ii) Let $Q = LV$ be a a parabolic subgroup of $G$ such that $QH$ is open and let $P = MU$ be a parabolic subgroup, such that $P \subset Q$ and $M \subset L$. Let $(\delta, E), \tilde{v}, \xi$ be as above. Applying induction in stage, $i_P^G E$ is isomorphic to $i_Q^E(i_P^L E)$. From (i), one gets an element $\tilde{r}_L \tilde{v}$ of $(i_P^L E)^{-\sigma}$; ( resp. $\tilde{r}_L \xi$ of $(i_P^L E)_{\ell \cap H}$).

(iii) One assumes that $P$ and $Q$ are $\sigma$-parabolic subgroups of $G$. Then one has:

$\tilde{r}_M \xi = \tilde{r}_M (\tilde{r}_L \xi)$.

**Proof.** (i) One reduces easily to compare the evaluation of both sides of the equality to prove on any element $e$ of $E^{J_M}$. One introduces $v := v_{e, \delta}^P$. From (3.20), one gets, on one hand:

$\langle \xi, v \rangle = \text{Vol}((K \cap P)J_H) \langle f_\xi(1), e \rangle$.

As $v$ is $J$-invariant this implies:

(3.21) $\langle e_J \xi, v \rangle = \text{Vol}((K \cap P)J_H) \langle f_\xi(1), e \rangle$.

On the other hand:

$\langle e_J \xi, v \rangle = \int_K \langle (e_J \xi)(k), v(k) \rangle \, dk$.

Again, we use that the support of $v$ is contained in $PJ = PJ_H$:

$\langle e_J \xi, v \rangle = \int_{(K \cap P)J_H} \langle (e_J \xi)(k), v(k) \rangle \, dk$.

The function under the integral sign is left invariant by $K \cap P$, by the properties of the induced representations, and right invariant by $J_H$ as $v$ and $e_J \xi$ are invariant by $J$. So one gets:

$\langle e_J \xi, v \rangle = \text{Vol}((K \cap P)J_H) \langle (e_J \xi)(1), e \rangle$. 
The equality needed to prove (i) follows from this and from (3.21). (ii) is a simple consequence of (i).

(iii) If $\hat{v} \in (i_P^G E)^\sim$, the equality $\tilde{r}_M \hat{v} = \tilde{r}_M (\check{r}_L \hat{v})$ is clear. Then (iii) follows from the last assertion of (i). \hfill $\Box$

3.4. The operation $j_{Q-} \circ$. Our second operation needs some preparation.

Let $P = MU$ be a parabolic subgroup of $G$. Let $(Q, Q^-)$ be a pair of opposite parabolic subgroups of $M$ with $L := Q \cap Q^-$. Let $(\delta, E)$ be a smooth representation of $M$. We denote by $P_Q$ the parabolic subgroup of $G$ equals to $QU$. We define similarly $P_{Q^-}$. We define a $G$-homomorphism $j_{Q-} \circ$ from $(i_P^G E)^\sim$ to $(i_{P_Q}^G (E_Q)_{\sim})$ as follows. Let $\hat{v} \in (i_P^G E)^\sim$. We identify canonically $(i_P^G E)^\sim$ to $i_P^G \tilde{E}$ and we denote by $j_{Q-} \circ \hat{v}$ the element of $i_{P_Q}^G \tilde{(E)_Q}$ obtained by composition of $\hat{v}$ with the projection $j_{Q^-}$. By the second adjointness Theorem (cf. (3.2)), $(\tilde{E})_{Q^-}$ is canonically isomorphic to $(E_Q)_{\sim}$. Let us denote by $j_{Q^-} \circ \hat{v}$ the image of $j_{Q^-} \circ \hat{v}$ by the induced isomorphism. This is an element of $i_{P_Q}^G (E_Q)^\sim$.

Then one defines $\tilde{j}_{Q^-} \circ \hat{v}$ as the image of $j_{Q^-} \circ \hat{v}$ by the canonical isomorphism of $i_{P_Q}^G (E_Q)^\sim$ with $(i_{P_Q}^G (E_Q))_{\sim}$. Summarizing $\tilde{j}_{Q^-} \circ$ appears as the composition of the homomorphisms of $G$-modules:

$$(i_P^G E)^\sim \rightarrow i_P^G \tilde{E} \rightarrow i_{P_Q}^G (\tilde{E})_{Q^-} \rightarrow i_{P_Q}^G (E_Q)^\sim \rightarrow (i_{P_Q}^G E_Q)^\sim$$

where the first arrow is the canonical isomorphism between $(i_P^G E)^\sim$ and $i_P^G \tilde{E}$, the last arrow is the canonical isomorphism between $i_{P_Q}^G (E_Q)^\sim$ and $(i_{P_Q}^G E_Q)^\sim$, the arrow $f$ is given by composition of functions with the projection $j_{Q^-}$, and the arrow $g$ is the induced morphism from the canonical isomorphism between $(\tilde{E})_{Q^-}$ and $(E_Q)^\sim$ given by the second adjointness Theorem. Hence $\tilde{j}_{Q^-} \circ$ is a $G$-module homomorphism. One sees easily that:

For $\hat{v} \in (i_P^G E)^\sim$, one has:

$$(\tilde{j}_{Q^-} \circ \hat{v})(1) = j_{Q^-}(\hat{v}(1)),$$

where in the left hand side of the equality $\tilde{j}_{Q^-} \circ \hat{v}$ is viewed as an element of $i_{P_Q}^G (E_Q)_{\sim}$ and where in the right side $\hat{v}$ is viewed as an element of $i_P^G \tilde{E}$ and $j_{Q^-}(\hat{v}(1))$ is viewed as an element of $(E_Q)_{\sim}$ by the second adjointness Theorem. In other words:

$$\check{r}_L (\tilde{j}_{Q^-} \circ \hat{v}) = j_{Q^-}(\tilde{r}_M (\hat{v})).$$
Proposition 3.2. Let $P = MU$ be a parabolic subgroup of $G$. Let $(Q, Q^-)$ be a pair of opposite parabolic subgroups of $M$. Let $(\delta, E)$ be a smooth representation of $M$ and let $\xi$ be an $H$-form on $V = i_Q^* E$.

(i) Let $v_1 \in V_1 := i_Q^* E_Q$. Let $\pi$ (resp., $\pi_1$) the induced representation of $G$ on $V$ (resp., $V_1$). The number $\langle \hat{\jmath}_{Q^-} \circ (e_J \xi), v_1 \rangle$ does not depend on the compact open subgroup of $G$, $J$, such that $v_1$ is fixed by $J$.

(ii) This allows to define a linear form on $V_1$, denoted by $\hat{\jmath}_{Q^-} \circ \xi$, as follows.

If $v_1 \in V_1$ is fixed by the compact open subgroup of $G$, $J$, one defines

$$\langle \hat{\jmath}_{Q^-} \circ \xi, v_1 \rangle := \langle \hat{\jmath}_{Q^-} \circ (e_J \xi), v_1 \rangle.$$ 

Then $\hat{\jmath}_{Q^-} \circ \xi$ is an $H$-form on $V_1$.

(iii) For every compact open subgroup of $G$, $J$, one has:

$$e_J(\hat{\jmath}_{Q^-} \circ \xi) = \hat{\jmath}_{Q^-} \circ (e_J \xi).$$

(iv) The support of $\hat{\jmath}_{Q^-} \circ \xi$ is contained in the support of $\xi$.

(v) Let $x \in G$. The space $(x E)_x Q$ is equal to $E_Q$. Then $\rho(x)\xi$ is fixed by $x.H$ and $\hat{\jmath}_{x, Q} \circ (\rho(x)\xi)$ is equal to $\rho(x)(\hat{\jmath}_Q \circ \xi)$.

Proof. (i) Let $v_1 \in V_1$. It is enough to prove that if $J' \subset J$ are two compact open subgroups of $G$ which leave $v_1$ invariant, one has:

$$\langle \hat{\jmath}_{Q^-} \circ (e_{J'} \xi), v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (e_J \xi), v_1 \rangle.$$ 

As $e_J v_1 = v_1$, one has:

$$\langle \hat{\jmath}_{Q^-} \circ (e_{J'} \xi), v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (e_J \xi), v_1 \rangle = \langle e_J \hat{\jmath}_{Q^-} \circ (e_{J'} \xi), v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (e_J \xi), v_1 \rangle,$$ 

which proves (i).

(ii) Let $v_1$ be an element of $V_1$ and let $h$ be an element of $H$. One may choose a small enough compact open subgroup, $J$, of $G$, such that $h.J$ and $h^{-1}.J$ leave also invariant $v_1$. This implies that $\pi_1(h) v_1$ is also fixed by $J$. Then, one has, from the definition of $\hat{\jmath}_{Q^-} \circ \xi$:

$$\langle \hat{\jmath}_{Q^-} \circ \xi, \pi_1(h) v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (e_J \xi), \pi_1(h) v_1 \rangle.$$ 

By elementary operations one sees that

$$\langle \hat{\jmath}_{Q^-} \circ \xi, \pi_1(h) v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (\pi'(h^{-1}) e_J \xi), v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (\pi'(h^{-1}) e_J \pi'(h) \xi), v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ (e_{J^{-1}}J \xi), v_1 \rangle.$$ 

Hence one gets, from the definition of $\hat{\jmath}_{Q^-} \circ \xi$ and the fact that $v_1$ is $h^{-1}.J$ fixed, the equality:

$$\langle \hat{\jmath}_{Q^-} \circ \xi, \pi_1(h) v_1 \rangle = \langle \hat{\jmath}_{Q^-} \circ \xi, v_1 \rangle.$$
which proves the $H$-invariance of $\tilde{j}_Q \circ \xi$.
The linearity is proved in the same way. This proves (ii).
(iii) is an immediate corollary of (ii).
(iv) Let $F$ be the support of $\xi$ and let $v_1 \in V_1$ whose support, $F_1$, is contained in the complementary subset of $F$ in $G$. Choose a compact open subgroup, $J$, of $G$ which fixes $v_1$ and such that $FJ$ is disjoint from $F_1$, which might be achieved from the compacity of $P_Q \setminus G$.
Then the support of $e_J \xi$ is disjoint from the support of $v_1$. As the composition by $j_Q^{-}$ do not increase supports, one concludes, from the definition in (ii), that:
$$\langle \tilde{j}_Q^{-} \circ \xi, v_1 \rangle = 0.$$ This implies (iv).
(v) is simply a “transport de structure”. □

**Lemma 3.3.** We keep the notations of the preceeding Lemma, but we assume that $P$ is a $\sigma$-parabolic subgroup with $\sigma$-stable Levi subgroup $M$ and that $Q$ is a $\sigma$-parabolic subgroup of $M$ with $\sigma$-stable Levi subgroup $L$. Let $A_0$ be a maximal $\sigma$-split torus of $L$. Let $x$ be an element of $G$ which is $A_0$-good. Then $\rho(x)\xi$ is an $x.H$-form, $P, Q$ are $\sigma_x$-parabolic subgroups (see Lemma 2.2) and one has:
$$\tilde{r}_L(\rho(x)(\tilde{j}_Q^{-} \circ \xi)) = j_Q^{-}(\tilde{r}_M(\rho(x)\xi)).$$
Proof. We first treat the case where $x = 1$.
Notice that $\sigma(P_Q^{-}) = (P^{-})_Q$ is opposite to $P_Q^{-}$. Hence $P_Q^{-}$ is a $\sigma$-parabolic subgroup of $G$.
Let $e$ be an element of $E_Q$. From [KT], Lemma 4.3, (see Proposition 2.3), one can choose a compact open subgroup of $K, J$, arbitrary small, which has a $\sigma$-factorization with respect to $(P, P^{-})$ and $(P_Q^{-}, (P^{-})_Q)$, such that $J_M$ has a $\sigma$-factorization with respect to $(Q, Q^{-})$. So we can assume that $J_L$ fixes $e$. One has:
\begin{equation}
(3.23) \quad J = (J \cap P_Q^{-})(J \cap H),
\end{equation}
\begin{equation}
(3.24) \quad J_M = (J_M \cap Q^{-})(J_M \cap H).
\end{equation}

Let us prove
\begin{equation}
(3.24) \quad \langle \tilde{r}_L(e_J(\tilde{j}_Q^{-} \circ \xi)), e \rangle = \langle \tilde{r}_L(\tilde{j}_Q^{-} \circ \xi), e \rangle.
\end{equation}

Let $P'$ denote $P_Q^{-}$. As $e$ is fixed by $J_L$, $v := v_{e_j}^{P', J}$ is well defined.
One will compute in two ways $\langle e_J \xi', v \rangle$ where $\xi' = \tilde{j}_Q^{-} \circ \xi$. First $v$ is invariant by $J$ so that one has:
$$\langle e_J \xi', v \rangle = \langle \xi', v \rangle.$$
Using (3.17) and (3.18), one deduces from the preceding equality, as in the proof of (3.21), that:
\[ \langle e_J \xi', v \rangle = \int_{(P' \cap K)J_H} \langle f_{\xi'}(k), v(k) \rangle \, dk. \]

The function under the integral sign is left invariant by \( K \cap P' \). Moreover, if \( j \in J_H \), \( f_{\xi'}(j) = f_{\xi'}(1) \), by the right \( H \)-invariance of \( f_{\xi'} \) and \( v(j) = e \) by the right invariance by \( J \) of \( v \). So one gets:
\[ (3.25) \quad \langle e_J \xi', v \rangle = \text{Vol}((P' \cap K)J_H) \langle f_{\xi'}(1), e \rangle. \]

Our second computation of \( \langle e_J \xi', v \rangle \) starts with:
\[ \langle e_J \xi', v \rangle = \int_K \langle (e_J \xi')(k), v(k) \rangle \, dk. \]

As \( v \) is supported on \( P'J \), one gets:
\[ \langle e_J \xi', v \rangle = \int_{(K \cap P')J} \langle (e_J \xi')(k), v(k) \rangle \, dk. \]

As the function to integrate is invariant by \( P' \cap K \) on the left and by \( J \) on the right, one has:
\[ (3.26) \quad \langle e_J \xi', v \rangle = \text{Vol}((P' \cap K)J) \langle (e_J \xi')(1), e \rangle. \]

Notice that:
\[ f_{\xi'}(1) = \tilde{r}_L(\tilde{j}_{Q-} \circ \xi), (e_J \xi')(1) = \tilde{r}_L((e_J \xi')). \]

Then, taking into account the equality \( J = (J \cap P')J_H \), one sees that \( (P' \cap K)J = (P' \cap K)J_H \). Then (3.24) follows from these two computations of \( \langle e_J \xi', v \rangle \) (cf. (3.25) and (3.26)).

From the fact that the composition by \( j_{Q-} \) commutes with right translations by elements of \( G \), one sees:
\[ e_J(\tilde{j}_{Q-} \circ \xi) = \tilde{j}_{Q-} \circ (e_J \xi) \]

hence:
\[ \langle \tilde{r}_L(e_J(\tilde{j}_{Q-} \circ \xi)), e \rangle = \langle (\tilde{r}_L(\tilde{j}_{Q-} \circ (e_J \xi)), e \rangle. \]

From this and (3.22), one deduces:
\[ \langle \tilde{r}_L(e_J(\tilde{j}_{Q-} \circ \xi)), e \rangle = \langle j_{Q-}(\tilde{r}_M(e_J \xi)), e \rangle. \]

As \( J \) has a \( \sigma \)-factorization for \((P, P^-)\), one deduces from Proposition 3.1 (i), that:
\[ \tilde{r}_M(e_J(\xi)) = e_{JM} \tilde{r}_M(\xi). \]

Replacing in the above equality, one gets:
\[ \langle \tilde{r}_M(e_J(\tilde{j}_{Q-} \circ \xi)), e \rangle = \langle j_{Q-}(e_{JM} \tilde{r}_M \xi), e \rangle. \]
From (3.6), and using the fact that \( J_M \) has a \( \sigma \)-factorization for \( (Q, Q^-) \), this implies:

\[
\langle \tilde{r}_M(e, j_Q(\cdot \circ \xi)), e \rangle = \langle j_Q(\cdot \circ \xi), e \rangle.
\]

Together with (3.24), this shows that:

\[
\langle \tilde{r}_L(\tilde{j}_Q(\cdot \circ \xi)), e \rangle = \langle j_Q(\cdot \circ \xi), e \rangle,
\]

which proves the assertion for \( x = 1 \).

Let us treat the general case. Then (see Lemma 2.2), \( P \) is a \( \sigma_x \)-parabolic subgroup of \( G \), \( M \) is \( \sigma_x \)-stable and \( Q \) is a \( \sigma_x \)-parabolic subgroup of \( M \). One may apply the first part of the proof to \( \xi_x := \rho(x)\xi \) which is fixed by \( x.H \). The result follows from the fact that \( \tilde{j}_Q \circ \cdot \circ \xi \) commutes with right translations by elements of \( G \).

\[\square\]

4. Generic basic geometric Lemma

In the two next subsections, we make no assumptions on the characteristic of the residue field of \( F \)

4.1. Intertwining integrals. Let \( P = MU \) be a parabolic subgroup of \( G \) and let \((\delta, E)\) be a smooth representation of \( M \) with finite length. Let \( Q = MV \) be another parabolic subgroup of \( G \) with \( M \) as Levi subgroup. We denote by \( \Sigma(Q, P) \) the set of roots of \( A_M \) in the Lie algebra of \( Q \) which are not roots of \( A_M \) in the Lie algebra of \( P \). We imbed \( A_M \) in a maximal split torus of \( M \), \( A \) and we fix a scalar product on \( a' \) which is invariant by the Weyl group of the pair \((G, A)\). It induces a scalar product on \( a'_M \) (cf. (2.11)). One has (cf. [Wal], Theorem IV.1.1 and Proposition IV.2.1):

There exists \( R > 0 \) such that, for all \( \chi \in X(M) \) which satisfies

\[
(\Re(\chi), \alpha) > R, \alpha \in \Sigma(Q, P),
\]

there exists an intertwining operator, \( A(Q, P, \delta_\chi) \), between \( i_P^G \delta_\chi \) and \( i_P^G \delta_\chi \) satisfying:

\[
(A(Q, P, \delta_\chi)f)(g) = \int_{V \cap U \setminus V} f(vg) \, dv, f \in i_P^G E_\chi,
\]

(4.1)

the integral being absolutely convergent. This family of operators viewed in the compact realization admits an extension to a rational family in \( \chi \in X(M) \) denoted in the same way. More precisely, there is a non zero polynomial function on \( X(M) \), \( b \), such that for all \( f \) in \( i_K^G \cap P E_\chi \), the family \( b(\chi)(A(Q, P, \delta_\chi)f) \) is polynomial, in the compact realization.
From this characterization, one deduces:

\[(4.2)\] The intertwining integrals commute with induced operators from intertwining operators between smooth representations of finite length of \(M\).

Let us show that:

\[(4.3)\] When \(A(Q, P, \delta_\chi)\) is defined, this operator is non zero.

Let \(P^-\) be the opposite parabolic subgroup to \(P\) with Levi subgroup \(M\). Let \(e \in E\) and let \(J\) be a compact open subgroup with Iwahori factorization with respect to \((P, P^-)\) and such that \(e\) is invariant by \(J_M\). Let \(f = v^{P, J}_{e, \delta_\chi}\) whose support is \(P_J\). As \((V \cap U^-) \cap P_J = V \cap (U^- \cap J)\) one sees that

\[
(A(Q, P, \delta_\chi) f)(1) = \text{Vol}(V \cap U^- \cap J) e
\]

which proves our claim.

The following Lemma is an immediate consequence of the induction by stage and of the definitions.

**Lemma 4.1.** (i) Let \(P = M U\) be a parabolic subgroup of \(G\) and let \(Q_1 = LV_1, Q_2 = LV_2\) be two parabolic subgroups of \(M\). Let us define \(P_{Q_i} := Q_i U\), \(P_{Q^2} := Q_2 U\). Let \((\omega, F)\) be a finite length smooth representation of \(L\) and let \(\delta_i\) be the representation \(i_{Q_i}^M \omega\), with space \(E_{Q_i}\), \(i = 1, 2\). The representation \(i_{P_{Q_i}}^G \omega\) is canonically equivalent to \(i_{P_{Q_i}}^G i_{Q_i}^M \omega\).

Let \(v \in i_{P_{Q_1}}^G E_1\). Then, one has the equality of rational functions in \(\chi \in X(M)\):

\[
A(Q_1, Q_2, \omega_\chi) \circ v = A(P_{Q_1}, P_{Q_2}, \omega_\chi) v.
\]

From [Wal], IV.1 (11), one has the relation:

\[(4.4)\] \(\langle A(Q, P, \delta)f, \tilde{f}\rangle = \langle f, A(P, Q, \tilde{\delta})\tilde{f}\rangle, f \in i_{P}^G E, \tilde{f} \in i_{P}^G \tilde{E} \).

Let us prove:

**Lemma 4.2.** Let \(\chi \in X(M)\) such that \(A(Q, P, \delta_\chi)\) and \(A(P, Q, \delta_\chi)\) are defined.

(i) Let \(f \in i_{P}^G E_\chi\). Then one has:

\[
\text{Supp}(A(Q, P, \delta_\chi) f) \subset \text{cl}(V(\text{Supp}(f))),
\]

where \(\text{cl}\) denotes the closure in \(G\) and \(\text{Supp}\) the support.

(ii) Let \(T\) be a linear form on \(i_{P}^G E_\chi\). Let \(T' = T \circ A(Q, P, \delta_\chi)\). Then one has:

\[
\text{Supp}(T') \subset \text{cl}(U(\text{Supp}(T))).
\]
Proof. (i) Let \( g \notin \text{cl}(V(\text{Supp}(f))) \). Let us show that \( g \) is not element of the support of \((A(Q, P, \delta_\chi)f)\). One reduces immediately, by holomorphy, to the case where \( A(Q, P, \delta_\chi) \) is defined by a converging integral. If there exists \( v \in i_\nu^\circ E_\chi \) such that \( f(vg) \) does not vanish, \( g \) has to be an element of \( V\text{Supp}(f) \). As this is not true, this implies that \((A(Q, P, \delta_\chi)f)(g) = 0 \). This proves (i).

(ii) Let \( A := A(Q, P, \delta_\chi) \). Let \( g \) be an element of \( \text{Supp}(T') \). Then for any compact open neighborhood \( \Omega \) of \( g \) in \( G \), there exists \( f \in i_\nu^\circ E_\chi \) with support in \( P\Omega \), such that \( \langle T', f \rangle \neq 0 \). Then \( \langle T, Af \rangle \neq 0 \), so that \( \text{Supp} T \cap \text{Supp}(Af) \) is non empty. By (i), \( \text{Supp}(Af) \subset \text{cl}(V\text{Supp}(f)) \). So one has:

\[
\text{cl}(VP\Omega) \cap \text{Supp}(T) \neq \emptyset.
\]

Let us show that if \( X \) is a subset of \( G \) and \((\Omega_p)\) is a decreasing sequence of compact open neighborhoods of \( g \) in \( G \), whose intersection is reduced to \( g \),

\[
\bigcap_{p \in \mathbb{N}} \text{cl}(X\Omega_p) = \text{cl}(Xg).
\]

In order to see this, one can reduce to the case where \( g = e \). If \( y \in \bigcap_{p \in \mathbb{N}} \text{cl}(X\Omega_p) \), for all \( p \), \( y = \lim x_{n,p} \omega_{n,p} \) where \( \omega_{n,p} \in \Omega_p \) and \( x_{n,p} \in X \). Let \( V, V' \) be symmetric neighborhoods of \( e \) in \( G \) with \( V'^2 \subset V' \). Let \( p \in \mathbb{N} \) such that \( \Omega_p \subset V' \) and let \( n \in \mathbb{N} \) such that \( y^{-1}x_{n,p} \omega_{n,p} \in V' \). Then \( y^{-1}x_{n,p} \) is an element of \( V \). Hence \( y \) is an element of \( \text{cl}(X) \), which proves our claim. But, as \( VP = QU \), \( VP\Omega_p \) is left \( Q \)-invariant and the image of \( \text{cl}(VP\Omega_p) \) in \( Q \setminus G \) is closed as the projection is open. Hence this projection is compact. It is the same for the projection of \( \text{Supp}(T) \). Together with (4.5), an argument of compacity shows that the intersection \( \bigcap_{p \in \mathbb{N}} \text{cl}(VP\Omega_p) \cap \text{Supp}(T) \) is non empty. Together with (4.6), this implies:

\[
\text{cl}(VPg) \cap \text{Supp}(T) \neq \emptyset.
\]

Then, one sees that \( g \in \text{cl}(PV\text{Supp}(T)) \): if \((v_n p_n g)\) admits \( t \in \text{Supp}(T) \) as a limit, then \((v_n p_n)\) has \( tg^{-1} \) as a limit and \((v_n p_n)^{-1} t \) has \( g \) as a limit.

But, by the \( Q \)-invariance of \( \text{Supp}(T) \) and the equality \( PV = UQ \), one has:

\[
PV\text{Supp}(T) = U\text{Supp}(T).
\]

Hence \( g \) is an element of \( \text{cl}(U\text{Supp}(T)) \), which proves (ii). \( \square \)

**Definition 4.3.** A \((Q, P)\)-subset of \( X(M) \) is the complementary set in \( X(M) \) of a finite union of sets of the form \( \{ \chi_\nu | \nu \in (a'_M)_\mathcal{C}, (\nu, \alpha) = c \} \), where \( \alpha \) describes the set \( \Sigma(Q, P) \). Such a set is open and dense in \( X(M) \).
Lemma 4.4. There exists a \((Q, P)\)-subset of \(X(M)\), such that for \(\chi\) element of this set, \(A(Q, P, \delta_\chi)\) is holomorphic and invertible.

Proof. From [Wal] IV.1.1(12) and (14), it suffices to prove the statement assuming that \(P\) and \(Q\) are adjacent and opposite, hence maximal. Let \(\alpha\) be the single element of \(\Sigma(Q, P)\). Recall that \(a_M = a^G_M \oplus a_G\) (cf. (2.8)). Here \(a^G_M\) is one dimensional. Hence the image in \(X(M)\) of \((a^G_M)_C, X(M)^G\), by the map \(\lambda \mapsto \chi_\lambda\), is a one dimensional torus. Thus the family \(\chi \mapsto A(Q, P, \delta_\chi)\), depending rationally on \(\chi \in X(M)^G\), it has a finite number of poles \(\chi_i\). One remarks that \(A(Q, P, \delta_\chi)\) does not change if \(\chi\) is multiplied by an element of \(X(G|M)\), so that \(A(Q, P, \delta_\chi)\) has poles only along the sets \(\chi_i X(G|M)\). This implies the holomorphy statement.

From [Wal]. IV.3, there exists a rational function on \(X(M)\) with values in \(\mathbb{C}\), \(j\), such that \(A(P, Q, \delta_\chi)A(Q, P, \delta_\chi)\) is equal to the multiplication by \(j(\chi)\). Moreover (cf. [S], Theorem 3.2), \(i_p^G \delta_\chi\) is irreducible for \(\chi\) in an open dense subset of \(X(M)\). From (4.3), one deduces that \(j\) is not identically zero. Also it is invariant by \(X(G|M)\), by the remark above. Hence, again, its poles and zeros are along a finite number of subtori of \(X(M)\) of the form \(\chi'_i X(G|M)\) where \(\chi_i' \in X(M)^G\). The second part of the lemma follows. \(\square\)

It follows from (4.2) and the proof of the preceeding Lemma that:

The intertwining integrals and their inverses commute

\[(4.7) \quad \text{with induced operators from intertwining operators between smooth representations of finite lenght of } M.\]

Let \(P' = M'U'\) be a parabolic subgroup of \(G\) with \(M \subset M', P \subset P'\). Let \((\delta, E)\) be a finite length smooth representation of \(M\). Let \(v\) be an element of \(i_p^G E\). We denote by \(r_{M'} v\) the value at 1 of the element of \(i_{p, (i_{M' \cap P})}^G (M' \cap P)\) associated to \(v\) by the induction by stage. Thus for \(\chi \in X(M)\), it defines a map denoted again by \(r_{M'}:\)

\[(4.8) \quad r_{M'} : i_p^G E_\chi \rightarrow i_{p, M' \cap M}^G E_\chi.\]

We will identify \(i_p^M E\) with \(E\).

Lemma 4.5. There exists a \((Q, P)\)-subset of \(X(M)\) such for every \(\chi\) in this subset and for every \(G\)-submodule, \(V\), of \(i_p^G E_\chi\) such that \(r_M(V) = E\), then \(r_M(A(Q, P, \delta_\chi) V) = E\).

Proof. Let us take a \((Q, P)\)-set as in the previous Lemma and let \(\chi\) be an element of this \((Q, P)\)-set. If \(r_M(A(Q, P, \delta_\chi) V)\) is equal to a strict \(M\)-submodule, \(E_1\), of \(E\), this implies that \(A(Q, P, \delta_\chi) V\) is a submodule of \(i_p^G E_1\). By (4.7) one would have \(r_M V \subset E_1\). A contradiction which proves the Lemma. \(\square\)
4.2. Generic basic geometric Lemma. Let $P$ (resp., $P'$) be a parabolic subgroup of $G$ with Levi subgroup $M$ (resp., $M'$). Let $A$ (resp., $A'$) be a maximal split torus of $M$ (resp., $M'$). Let us show:

There exists a set of representatives of $P' \backslash G/P$ such that for each of its elements, $w$, one has $w.A = A'$

By considering a minimal parabolic subgroup of $G$, contained in $P$ (resp., $P'$) and containing $A$ (resp., $A'$), one can reduce to the case where $P$ and $P'$ are minimal parabolic subgroups of $G$. Then $P' = x.P$ for some element $x$ of $G$. As all maximal split tori in a minimal parabolic subgroup are conjugate by an element of this parabolic subgroup (cf. [BT], Theorem 11.6), one can choose $x$ such that $x.P = P'$ and $x.A = A'$. On the other hand, by the Bruhat decomposition $G = \cup_w P w P$ where the $w$ normalize $A$, Hence, $G = \cup_w P' x w P$. Then (4.9) follows from the fact that $xw.A = A'$.

We will denote by $W(M' \backslash G/M)$ (although this set is not unique) a set of representatives of $P' \backslash G/P$ such that for each $w \in W(M' \backslash G/M)$, $w.A = A'$. Then $M' \cap w.M$ (resp., $M \cap w^{-1}.M'$) is the Levi subgroup containing $A'$ (resp., $A$) of the parabolic subgroup of $M'$ (resp., $M$), $M' \cap w.P$ (resp., $M \cap w^{-1}.P'$).

If $P = MU$, $P' = M'U'$ are $\sigma$-parabolic subgroups of $G$, one will assume that $A$ (resp., $A'$) is a maximally $\sigma$-split $\sigma$-stable maximal split torus of $M$ (resp., $M'$).

Proposition 4.6. Let $P = MU$ and $P' = M'U'$ be parabolic subgroups of $G$. We denote by $\Sigma(P)$ the set of $A_M$-roots in the Lie algebra of $P$. Let $(\delta, E)$ be a smooth representation of finite length of $M$. Let $w, w'$ be two distinct elements of $W(M' \backslash G/M)$. Let $X$ be a complex subtorus of $X(M)$ stable by complex conjugacy. We assume that the Lie algebra of $X$, denoted by $b \subset (a_M)'$, contains at least an element $\nu$ such that $(\nu, \alpha)$ is strictly positive for each element $\alpha$ of the set $\Sigma(P)$. Then the following holds:

(i) The set $O_{w, w'}$ of elements $\chi$ of $X$ such that:

\[ i_{M' \cap \chi}^{M'}(w j_{M \cap \chi^{-1}, P'} E_\chi) \]

is open and dense in $X$. If $(\delta, E)$ is unitary, $O_{w, w'} \cap X(M)_{\text{ad}}$ is dense in $X_{\text{ad}} := X \cap X(M)_{\text{ad}}$.

(ii) If $\chi$ is an element of $O = \cap_{w, w'} W(M' \backslash G/M), w \neq w'O_{w, w'}$, the Jacquet module $j_{P'}(i_{P}^{\text{G}} E_\chi)$ is isomorphic to the direct sum:

\[ \bigoplus_{w} W(M' \backslash G/M) i_{M' \cap \chi}^{M'}(w j_{M \cap \chi^{-1}, P'} E_\chi). \]
Proof. Let \( \{\Lambda_1, \ldots, \Lambda_p\} \) be the set of Bernstein’s parameters of the representation \( (j_{M \cap w^{-1} \cdot P_E}) \) of \( M \cap w^{-1} \cdot M' \), where for every \( i \), \( \Lambda_i = (L_i, \omega_i)_{M \cap w^{-1} \cdot M} \), \( L_i \) is a Levi subgroup of \( M \cap w^{-1} \cdot M' \) which contains \( A \) and \( \omega_i \) is a cuspidal representation of \( L_i \) i.e. whose smooth coefficients have a support which is compact modulo the center of \( L_i \).

We introduce similar data related to \( w', L'_j, \omega'_j \). Then, using (9.3) and (9.4), one sees:

The set of Bernstein’s parameters of the finite length \( M' \)-smooth module \( V_{\chi,w} \) is equal to

\[
\{ (w.L_1, w(\omega_1 \otimes \chi_{|L_1}))), \ldots, (w.L_p, w(\omega_p \otimes \chi_{|L_p})))_{M'} \}.
\]

Let us prove that the set \( Y = X \setminus O_{w,w'} \), is closed in \( X \) and has an empty interior. From (4.11), one sees that \( \chi \in Y \), if and only if for some \( i, j \), one has:

\[
\begin{align*}
(w.L_i, w. (\omega_i \otimes \chi_{|L_i}))) & \text{ is } M' \text{-conjugate to } \\
(w'_{},L'_j, w'. (\omega_j \otimes \chi_{|L'_j})) & \text{ in } M'.
\end{align*}
\]

Let \( Y_{i,j} \) be the subset of elements of \( Y \) satisfying (4.12). Let us assume that \( Y_{i,j} \) is non empty. In particular \( w.L_i \) is conjugate in \( M' \) to \( w'.L'_j \). These are two Levi subgroups of \( M' \) which contain \( A' \) and which are conjugate under \( M' \). As two maximal split tori in \( w'.L'_j \) are conjugate, these two Levi subgroups of \( M' \) are conjugate by an element of the normalizer in \( M' \) of \( A' \). By multiplying \( w \) by this element of the normalizer in \( M' \) of \( A' \), one reduces to the case where these two Levi subgroups of \( M', w.L_i, w'L'_j \) are equal. Let us denote by \( L'' \) this Levi subgroup of \( M' \). Two cuspidal representations of \( L'' \), \( \omega \), \( \omega' \) define the same infinitesimal character for \( M' \) if for some \( x \) in the normalizer of \( L'' \) in \( M' \), \( x \omega \) is equivalent to \( \omega' \). Hence \( \chi \in Y_{i,j} \) if and only if for some \( x \in N_{M'}(L'') \), that might be chosen to normalize \( A' \),

\[
xw \omega_i \otimes xw \chi_{|L''} \text{ is equivalent to } w' \omega'_j \otimes w' \chi_{|L''}.
\]

For \( x \) given, the set \( Y_{i,j,x} \) of such \( \chi \) is easily seen to be closed because

\[
\text{The characters of these two families of irreducible representations of } L'' \text{ vary weakly holomorphically in } \chi.
\]

As \( Y_{i,j,x} \) depends only on the right coset \( xL'' \) and as \( N_{M'}(L'') \) is finite, this implies that \( Y_{i,j} \) is closed. Hence \( Y \) is closed in \( X(M) \) and \( O_{w,w'} \) is open in \( X \).

Let us assume that \( O \) is not dense. This implies that \( Y \) has a non empty interior, hence by Baire’s Theorem, there exists \( w, w', i, j, x \) as above such that \( Y_{i,j,x} \) has a non empty interior.

By multiplying \( w' \) by \( x^{-1} \) one may and one will reduce to the case where
$x = 1$. From (4.14), one deduces that for all $\chi \in X$, (4.13) holds. In particular it is true for $\chi = 1$. Denote by $\omega''$ the representation $w\omega_i$ of $L''$. Then one concludes also that for all $\chi \in X$, $(w'\chi|_{L''})(w\chi|_{L''})$ belongs to the finite set of elements $\chi''$ of $X(L'')$ such that $\omega'' \otimes \chi''$ is equivalent to $\omega''$. Hence, by connexity,

$$(4.15) \quad \text{for all } \chi \in X, \ w\chi|_{L''} = w'\chi|_{L''}. $$

By differentiation, it implies

$$w\nu = w'\nu, \nu \in b.$$ 

This might be written:

$$w''\nu = \nu, \nu \in b,$$

where $w'' = w'^{-1}w$ is an element of the normalizer of $A$ in $G$. From our hypothesis on $b$, one sees that $w''\nu = \nu$ for a strictly $P$-dominant element of $b, \nu$. But $w''$ acts on $a_C$ as an element of the Weyl group of $A$, which, by the above, is a product of symmetries with respect to roots orthogonal to $\nu$. The corresponding roots have to be roots of $A$ in the Lie algebra of $M$, by our hypothesis on $\nu$. This implies that $w''$ fixes pointwise $(a_M)'_C$. Hence $w''$ is an element of the normalizer of $A$ which fixes pointwise $a_M$. This implies that it is an element of $M$. As $w = w'^{-1}w''$, this implies that $w$ and $w'$ represents the same element of $P^\circ \backslash G/P$. A contradiction with our hypothesis. Hence $Y$ has an empty interior and $O$ is dense in $X$. This proves the first statement on $O_{w,w'}$.

The proof of the statement for $O_{w,w'} \cap X_u$ is similar.

(ii) By the basic geometric Lemma (cf. [R] VI.5.1) the Jacquet module $j_P(i_P^E\chi)$ has a filtration whose associated graded is the direct sum of the statement. (ii) is an immediate consequence of the definition of $O$ and (9.5).

**Lemma 4.7.** We keep the notations and the assumptions of the previous Lemma.

(i) Let us assume that $(\delta, E)$ is irreducible (resp. irreducible and unitary). There exists an open dense subset of $X$ (resp., $X_u$) such that for every $\chi$ in this subset, $i_P^\delta\delta_\chi$ is irreducible.

(ii) Let us assume that $(\delta, E)$ is a finite length smooth representation of $M$. There exists an open dense subset of $X, X'$, such that for every $\chi \in X'$ and for every $G$-submodule $V$ of $i_P^\delta E_\chi$ such that $r_MV = E_\chi$, then $V = i_P^E E_\chi$.

**Proof.** (i) follows easily from [S], Theorem 3.2, where no assumption of unitarity on the inducing representation is made, and our hypothesis on $X$.

(ii) Let $X'$ be an open and dense subset of $X$ such that:
1) for every irreducible subquotient of \((\delta, E)\), \((\omega, F)\) and \(\chi \in X', i_P^E \omega_\chi\) is irreducible.

2) \(X'\) is a subset of the set \(O\) of the preceeding Lemma, where we take \(P = P', M = M'\).

The existence of \(X'\) follows from (i) and from the preceeding Lemma. We proceed by induction on the length of \(E\) to prove that:

\[(4.16)\quad \text{An open dense subset, } X', \text{ of } X \text{ satisfying 1) and 2) above has the properties required by the Lemma.}\]

If \(E\) is of length one and \(\chi \in X', i_P^E \omega_\chi\) is irreducible. As \(V\) is non zero, one sees that the claim is true in that case.

Now assume that \((4.16)\) is true if \(E\) is of length \(p \geq 1\). Let \(E\) be a smooth \(M\)-module of length \(p + 1\). Let \(\chi\) be an element of \(X'\). Let \((\pi_1, V_1)\) be an irreducible \(G\)-submodule of \(V\). As \(\chi \in X', \pi_1\) is isomorphic to \(i_P^E \omega_\chi\) for some irreducible subquotient \((\omega, F)\) of \((\delta, E)\). This determines a non zero element, \(T\), of \(\text{Hom}_G(i_P^E \omega_\chi, V) \subset \text{Hom}_G(i_P^E \omega_\chi, i_P^E \omega_\chi)\). The latter space is isomorphic to \(\text{Hom}_M(j_P(i_P^E \omega_\chi), E_\chi)\). But from our hypothesis on \(\chi\) and the properties of \(X'\), \(j_P(i_P^E \omega_\chi)\) splits as a direct sum \(\oplus_{w \in W(M/G)_{w \neq \omega} P^w j_M_{\chi w^{-1}, \chi P} F_\chi\). As \(\chi\) is an element of \(O\), for \(w \notin P\), the set of Bernstein parameters of \(i_M^{\chi w} P^w j_M_{\chi w^{-1}, \chi P} F_\chi\) is disjoint from the set of Bernstein parameters of \(F_\chi\). Hence, one has:

\[\text{Hom}_M(j_P(i_P^E \omega_\chi), E_\chi) \approx \text{Hom}_M(F_\chi, E_\chi)\]

From this and from the fact that \(T\) is nonzero, it follows that \(\text{Hom}_M(F_\chi, E_\chi)\) is non reduced to zero. This proves that \(F\) appears as a submodule of \(E\), that we still denote by \(F\). Moreover \(T\) is the induced map from an element of \(\text{Hom}_M(F_\chi, E_\chi)\) and \(V_1\) is equal to \(i_P^E \omega_\chi\). Going through the quotient of \(V\) by \(i_P^E \omega_\chi\) and applying the induction hypothesis, one gets the result. \(\square\)

**Proposition 4.8.** Generic basic geometric Lemma.

One keeps the notations of Lemma 4.6, namely \(P = MU\), \(P' = M'U'\) are parabolic subgroups of \(G\), \(\Sigma(P)\) is the set of \(A_M\)-roots in the Lie algebra of \(P\), \((\delta, E)\) is a smooth representation of finite length of \(M\) and \(X\) is a complex subtorus of \(X(M)\) stable by complex conjugacy. We assume that the Lie algebra of \(X\), denoted by \(\mathfrak{b} \subset (\mathfrak{a}_M')\), contains at least an element \(\nu\) such that \((\nu, \alpha)\) is strictly positive for each \(\alpha\) element of the set \(\Sigma(P)\).

If \(w \in W(M'/G/M)\), let us denote by \(P_w\) (resp., \(P'_w\)) the parabolic subgroup of \(G\) such that \(P_w \subset P\) (resp., \(P'_w \subset P'\)) and defined by:

\[P_w = (M \cap w^{-1} P')U, P'_w = (M' \cap w P)U'.\]
There exists a dense open subset $O$ of $X$, whose intersection with $X_u$ is dense, such that:

(i) For $\chi \in O$ and $w \in W(M'/G/M)$, the map $\alpha_{\chi,w}$ is well defined from $i_P^E \chi$ to $V_{X,w} := i_{M'w,Mw}^{M}(j_{M\cap w^{-1}.P^{w}}E_\chi)$ by:

$$\alpha_{\chi,w}(v) = r_{M'}[A(P'_w, w.P_w, w.j_{M\cap w^{-1}.P^{w}}\delta(\lambda(w) \circ j_{M\cap w^{-1}.P^{w}} v)],$$

for $v \in i_P^E \chi$. Moreover it goes through the quotient to a surjective morphism of $M'$-modules from $j_P i_P^E \chi$ to $V_{X,w}$, that we will denote in the same way.

(ii) For $\chi \in O$, the map

$$\alpha_{\chi} : j_P^E(i_P^E \chi) \to \oplus_{w \in W(M\cap G/M)} V_{X,w},$$

whose components are the $\alpha_{\chi,w}$, is an isomorphism of $M'$-modules.

Proof. Let us denote by $M'_w$ the Levi subgroup of $P'_w$ which contains $A' = w.A$. Then $M'_w = M' \cap w.M$. From the properties of intertwining integrals (cf. Lemma 4.4), $\alpha_{\chi,w}$ is well defined for $\chi$ element of $X$ and such that $w\chi | M'_w$ is element of some $(P'_w, w.P_w)$-subset of $X(M'_w)$.

We denote by $X_w$ the set of such $\chi$. Such a set is open in $X$. Let us show that it is dense in $X$. If it was false, the complementary set of some $(P'_w, w.P_w)$-subset of $X(M)$ would contain the set of $w\chi | M'_w$ when $\chi$ varies in a nonempty open subset of $X$. Thus, by looking to tangent spaces, one would see that $w\mathfrak{b}$ should be contained in the orthogonal subspace to some non empty collection of roots, $\alpha$, of the maximal split torus of the center of $M'_w$, $A_{M'_w}$, in the Lie algebra of $P'_w$ and which are not root in the Lie algebra of $w.P_w$. But, by the hypothesis on $X$ in Lemma 4.6, such a root would be trivial on $w\mathfrak{a}_M$, as the roots which are orthogonal to $\mathfrak{b}$ are trivial on $\mathfrak{a}_M$. Hence it would be a root of $A_{M'_w}$ in the Lie algebra of the intersection of $P'$ with $w.M$. On the other hand, from the definition of $P_w$ one sees that:

$$w.M \cap P' \subset w.P_w.$$ Moreover one has:

$$w.M \cap P' = (w.M \cap M')(w.M \cap U').$$

From the definition of $P'_w$ one concludes:

$$w.M \cap P' \subset P'_w.$$ Hence one sees that $w.M \cap P'$ is a subset of $w.P_w \cap P'_w$ and there is no root having the required property. This achieves to prove that $X_w$ is dense in $X$. One sees similarly that $X'_w \cap X_u$ is dense in $X_u$.

Let us denote by $V$ the $G$-submodule $\{\lambda(w)(j_{M\cap w^{-1}.P^{w}} v) | v \in i_P^E \chi \}$.
of $i_{w,P}^G(j_{M\cap w^{-1},P'}E_\chi)$. From the surjectivity of $j_{M\cap w^{-1},P'}$ and the surjectivity of $r_M$ from $i_P^G E_\chi$ to $E$, one concludes, by “transport de structure”, that $r_{M'\cap w,M'}V = w j_{M\cap w^{-1},P'}E_\chi$. By Lemma 4.5, one concludes that for $\chi$ element of an open dense subset of $X_w$ one has:
$$r_{M'\cap w,M}(A(P',P_w,w j_{M\cap w^{-1},M'}E_\chi)V) = w j_{M\cap w^{-1},P'}E_\chi.$$ 
As $r_{M'\cap w,M} = r_{M'\cap w,M} \circ r_{M'}$, one concludes that for $\chi$ element of this open dense subset of $X_w$, the image of $\alpha_{X,w}$, $\alpha_{X,w}(i_{P}^G E_\chi)$ satisfies:
$$r_{M'\cap w,M}\alpha_{X,w}(i_{P}^G E_\chi) = w j_{M\cap w^{-1},P'}E_\chi.$$ 
Then from Lemma 4.7, one deduces that for $\chi$ element of an open dense subset of $X_w$, the image of $\alpha_{X,w}$ is equal to $i_{M'\cap w,P}^M(\chi,w j_{M\cap w^{-1},P'}E_\chi)$. Hence the image of $\alpha$ admits $V_{X,w} = i_{M'\cap w,P}^M(\chi,w j_{M\cap w^{-1},P'}E_\chi)$ as a quotient. The fact that $\alpha$ goes through the quotient to $j_{P'}(i_{P}^G E_\chi)$ follows from the fact that, in the definition of $\alpha_{X,w}$, all maps are $G$-morphisms except $r_{M'}$ which goes through the quotient to an $M'$-module map on the Jacquet module. This achieves to prove (i).

(ii) Using (i) and Lemma 4.6, one sees that for $\chi$ in a dense open subset of $X$, $O$, whose intersection with $X_u$ is dense in $X_u$, $\alpha_{X,w}$ is surjective for every $w \in W(M'\backslash G/M)$ and that the various $V_{X,w}$ have disjoint sets of Bernstein’s parameters. This implies (cf. (9.5)) that the image is equal to the direct sum of the $V_{X,w}$. So $\alpha$ is a surjective $M'$-module map from $j_{P'}(i_{P}^G E_\chi)$ to $\bigoplus_{w \in W(M'\backslash G/M)} V_{X,w}$. On the other hand, by Proposition 4.6, $j_{P'}(i_{P}^G E_\chi)$ is an $M'$-module isomorphic to $\bigoplus_{w \in W(M'\backslash G/M)} V_{X,w}$. By looking to the length of modules, one concludes from this that $\alpha$ is bijective. 

Let $P$ and $Q$ be two parabolic subgroups of $G$, with common Levi subgroup $M$. Let $(\delta, E)$ be a smooth representation of finite length of $M$ such that the operators $A(Q, P, \delta)$ and $A(P, Q, \delta)$ are well defined. Then the restriction of the transposed operator of $A(Q, P, \delta)$, ${}^t A(Q, P, \delta)$, to the space of smooth vectors intertwines $(i_Q^G \delta)^\vee$ with $(i_Q^P \delta)^\vee$. Using the canonical isomorphism of $i_Q^P E$ with $(i_Q^G E)^\vee$ and $i_Q^G E$ with $(i_Q^P E)^\vee$, the restriction of ${}^t A(Q, P, \delta)$ to the space of smooth vectors defines an intertwining operator between $i_Q^G E$ and $i_Q^P E$ which, by (4.4), is equal to $A(P, Q, \delta)$.

**Proposition 4.9.** One keeps the notations of the preceding Proposition. If $w \in W(M'\backslash G/M)$, let us denote by $\hat{P}_w$ (resp., $\hat{P}_w'$) the parabolic subgroup of $G$ such that $\hat{P}_w \subset P$ (resp., $\hat{P}_w' \subset P'$) and defined by:
$$\hat{P}_w = (M \cap w^{-1}.P')U, \hat{P}_w' = (M' \cap w.P)U'^{-}.$$
There exists a dense open subset $O'$ of $X$, whose intersection with $X_u$ is dense in $X_u$, such that:

(i) For $\chi \in O'$ and $w \in W(M'\backslash G/M)$ and $\tilde{v} \in (i^G_P E_\chi)^\vee$:

$$\beta_{\chi,w}(\tilde{v}) := \tilde{r}_{M'} \circ \tilde{A}(w, \tilde{P}_w, \tilde{P}'_w, w \cdot j_{M'\backslash w^{-1},P'} \delta_{\chi}) \circ \lambda(w) \circ \tilde{j}_{M'\backslash w^{-1},P'} \circ \tilde{v}$$

is a well defined element of $(V_{\chi,w})^\vee$ where $V_{\chi,w} := i^\prime_{M'\backslash w,P'}(w j_{M'\backslash w^{-1},P'} E_\chi)$. Moreover the map $\beta_{\chi,w}$ goes through the quotient to a surjective morphism of $M'$-modules from $j_{P^-}((i^G_P E_\chi)^\vee)$ to $(V_{\chi,w})^\vee$ that we will denote in the same way.

(ii) For $\chi \in O'$, the map

$$\beta_\chi : j_{P^-}((i^G_P E_\chi)^\vee) \to \oplus_{w \in W(M'\backslash G/M)} (V_{\chi,w})^\vee$$

whose components are the $\beta_{\chi,w}$ is an isomorphism of $M'$-modules.

Proof. The fact that $\beta_{\chi,w}(\tilde{v})$ is a well defined element of $(V_{\chi,w})^\vee$ follows from the definitions. The rest of the proof is similar to the proof of the preceding Proposition, using the isomorphism of the smooth dual of a parabolically induced representation with the parabolically induced representation of the smooth dual of the inducing representation. \(\square\)

4.3. Generic basic geometric Lemma and $H$-forms. We come back to our assumption that the characteristic of the residue field of $F$ is different from 2. Let us keep the notations of the preceding Proposition. Let $\chi \in O'$. We set $V = i^G_P E_\chi$. The Second adjointness Theorem shows that $(i^G_P E_\chi)_{P'}$ is canonically isomorphic to $(((i^G_P E_\chi)^\vee)_{P'^-})^\vee$. From the preceding Proposition, the isomorphism $\beta_\chi$, determines an isomorphism

$$\gamma_\chi : j_{P'}(i^G_P E_\chi) \to V_{P'^{-1},1} := \oplus_{w \in W(M'\backslash G/M)} V_{\chi,w}.$$

We denote by $\gamma_{\chi,w}$ the composition of $\gamma_{\chi}$ with the projection onto $V_{\chi,w}$. For a smooth $G$-module $V$, we recall that $\langle \cdot, \cdot \rangle_{P'}$ is the canonical pairing between $V_{P'}$ and $(\tilde{V})_{P'^-}$. In other words, the isomorphism $\gamma_{\chi}$ is characterized by

$$\langle \tilde{v} P'^-, v P' \rangle_{P'} = \langle \beta_\chi(\tilde{v}), \gamma_\chi(v) \rangle, \quad v \in V, \tilde{v} \in \tilde{V},$$

where, in the second member of the equality, the pairing is the natural pairing between $(V_{P'^{-1},1})^\vee$ and $V_{P'^{-1},1}$.

**Theorem 4.10.** We keep the notations of the preceding Proposition. We assume moreover that $P$ and $P'$ are $\sigma$-parabolic subgroups of $G$. Let $\chi \in O'$ and let $\xi$ be an $H$-form on $V := i^G_P E_\chi$. We define:

$$\xi_{P'^{-1}} := j_{P'^{-1}} \xi.$$

We will denote the components of the linear form on $V_{P'^{-1},1} = \oplus_{w \in W(M'\backslash G/M)} V_{\chi,w}$, $\xi^1 := \xi_{P'^{-1}} \circ \gamma_{\chi}^{-1}$ by $\xi_{P'^{-1},w} \in (V_{\chi,w})^\vee$, where $V_{\chi,w} := \oplus_{w \in W(M'\backslash G/M)} V_{\chi,w}.$
Using Proposition 3.2, Proposition 3.1, and the definition of $\beta$ from (4.17), one sees:

$$\langle \xi_{P''}, v_{P''} \rangle = \langle \beta(e_j \xi), \gamma(v_{P''}) \rangle.$$  

From the definition of $\xi_{P''}$ and of the $\sigma$-factorization (cf. (3.6), Proposition 2.3), one has:

$$\langle \xi_{P''}, v_{P''} \rangle = \langle (e_j \xi)_{P''}, v_{P''} \rangle_{P''}.$$  

From (4.17), one sees:

$$\langle \xi_{P''}, v_{P''} \rangle = \langle \beta(e_j \xi), \gamma(v_{P''}) \rangle.$$  

Let us denote by $\xi^2$ the element of the dual of $V_{P''}$ whose components are

$$\hat{r}_{M'} \circ A(w, \tilde{P}_w, \tilde{P}'_w, w, p_{M\cap w^{-1}, p''} \delta_\chi) \circ \lambda(w) \circ \tilde{j}_{M\cap w^{-1}, p''} \circ \xi.$$  

Using Proposition 3.2, Proposition 3.1, and the definition of $\beta$ in Proposition 4.9, one sees easily that

$$\beta(e_j \xi) = e_{J_{M'}} \xi^2.$$  

Hence we get a second expression for $\langle \xi_{P''}, v_{P''} \rangle$:

$$\langle \xi_{P''}, v_{P''} \rangle = \langle e_{J_{M'}} \xi^2, \gamma(v_{P''}) \rangle, v \in V^J.$$  

Together with (4.18), this implies that $\xi^1$ and $\xi^2$ are equal on $V_{P''}.M'$. As there are arbitrary small open compact subgroups of $G$ with a $\sigma$-factorization with respect to $(P', P''')$, this implies that $\xi^1 = \xi^2$. This finishes the proof of the Theorem. \hfill \square

5. Two key Lemmas and some of their consequences

5.1. Families of distributions on $PH$, where $P$ is a parabolic subgroup of $G$. We keep the notations of the preceding subsection. Let $O$ be a non empty open subset of $X$. A map $\chi \mapsto \xi_\chi \in (i_{E}\xi)^\prime$ defined on $O$ is said to be weakly holomorphic if for all $v \in i_{K}^{\prime} E$, the map $\chi \mapsto \langle \xi_\chi, v \rangle_\chi$ is holomorphic on $O$.

We will denote by $M_\chi$ the map $C^\infty_c(G) \otimes E \rightarrow i_{E}^{\prime} E$ denoted by $M_{\chi, P}$ in equation (3.12) and we set $\xi_\chi := \xi_\chi \circ M_\chi$. Let us prove:

Let $f \in C^\infty_c(G) \otimes E$. Then $\chi \mapsto \langle \tilde{\xi}_\chi, f \rangle$ is holomorphic on $O$, in other words $\chi \mapsto \tilde{\xi}_\chi$ is a weakly holomorphic family of $E$-distributions on $G$. 

(5.1)
Let $J$ be a compact open subgroup of $K$ such that $f$ is right and left invariant by $J$. Then $v(\chi) := (M_\chi f)_{|K}$ has its values in the finite dimensional space $(i^K_{K\cap P}E)^J$. To see that $\chi \to v(\chi)$ is holomorphic, it is enough to check that for every $k \in K$, $\chi \to (v(\chi))(k)$ is holomorphic. By using left translates by elements of $K$, one can reduce to $k = 1$. But $f \in C_c^\infty(G) \otimes E$. Hence its restriction to $P$ is invariant by a compact open subgroup of $P$, $J'$, and is supported by a finite number of right cosets of $J'$ in $P$, $x_iJ'$. Hence, using the definition of $M_\chi$ and the fact that the unramified characters are trivial on compact open subgroups, one sees that:

$$(v(\chi))(1) = \sum_i \chi(x_i^{-1})(\int_{J'} \delta^{1/2}_{J'}((x_i^{-1})p) f(x_i) \, dp).$$

Hence $\chi \mapsto v(\chi)$ is holomorphic. Then (5.1) follows from the finite dimension of $(i^K_{K\cap P}E)^J$ and from our hypothesis on the family $(\xi_\chi)$.

Let $(\xi_\chi)$ be as above and assume that every $\xi_\chi$ has a zero restriction to the complementary set of a closed set $F$ of $X$, which is left $P$-invariant. Then (cf. subsection 9.1) $\xi_\chi$ induces on $F$ an $E$-distribution denoted by $\tilde{\xi}_{F,\chi} \in (C_c^\infty(F) \otimes E)^J$. If $f \in C_c^\infty(F) \otimes E$, let $f_1 \in C_c^\infty(G) \otimes E$ be such that its restriction to $F$ is equal to $f$. Then

$$(5.2) \quad \tilde{\xi}_{F,\chi}(f) = \tilde{\xi}_\chi(f_1).$$

From this one concludes that: $\chi \to \tilde{\xi}_{F,\chi}$ is a weakly holomorphic family of $E$-distributions on $F$. Similarly, if $\Omega$ is a left $P$-invariant open set of $F$, the restriction of $\tilde{\xi}_{F,\chi}$ to $\Omega$, $(\tilde{\xi}_{F,\chi})_{|\Omega}$, is a weakly holomorphic family of $E$-distributions on $\Omega$.

The following Lemma is one of the two key Lemmas in the article.

**Lemma 5.1.** Let $P$ be a parabolic subgroup of $G$. Let $A_0$ be a $\sigma$-stable maximal split torus of $G$ contained in $P$, which exists by [HH], Lemma 2.4. Let $M$ be the Levi subgroup of $P$ which contains $A_0$ and let $U$ be the unipotent radical of $P$. Let $(\delta, E)$ be a finite length smooth representation of $M$ and let $X$ be a complex subtorus of $X(M)$. Let $O$ be a nonempty open subset of $X$.

Let $\xi \mapsto \xi_\chi$ be a weakly holomorphic family, depending on $\chi \in O$, of $E$-distribution on $PH$. We assume moreover that the family $\xi_\chi$ is non identically zero and that for every $\chi \in O$, $\xi_\chi$ is $H$-invariant on the right and $\delta_\chi$-covariant under $P$ (cf. subsection 9.1).

The morphism of Lie groups from $X(M)$ to $X(A_0)$ given by the restriction has finite kernel. This allows to view the Lie algebra $b$ of $X$, which is a subspace of the Lie algebra $(a_M)'_C$ of $X(M)$, as a subspace of Lie algebra $(a_0)'_C$ of $X(A_0)$ (cf. (2.11)).
(i) Then it follows from our assumptions that this subspace is made of antiinvariant elements by $\sigma$.

(ii) Moreover if $\mathfrak{b}$ contains a strictly $P$-dominant element, $\nu$ (i.e. such that $\langle \nu, \alpha \rangle > 0$ for every root, $\alpha$, of $A_M$ in the Lie algebra of $P$), then $P$ is a $\sigma$-parabolic subgroup of $G$.

**Proof.** There is no restriction to assume that $O$ is connected and $\xi_\chi$ is never equals to zero. Also, by translation by an element of $O$, one can assume that $O$ contains 1. The group $P \times H$ acts on $PH$ by

$$(p, h)g = pxh^{-1}, g \in PH, p \in P, h \in H.$$ 

So $PH$ is an homogeneous space under $P \times H$ homeomorphic to $(P \times H)/\text{Diag}(P \cap H)$ by the map $(p, h) \mapsto ph\varepsilon_H$ (cf. BD Lemma 3.1 (iii)). Let us denote by $\varepsilon_H$ the trivial character of $H$. Let us define the mean value operation $M_{P \cap H}$ which sends $C_c(\mathbb{R}^\infty \times H) \otimes E$ to $C_c(\mathbb{R}^\infty(PH) \otimes E$:

$$(M_{P \cap H}f)(ph) := \int_{P \cap H} f(px, x^{-1}h) \, dx, f \in C_c(\mathbb{R}^\infty \times H) \otimes E,$$

where $dx$ is a left invariant Haar measure on $P \cap H$. We define $\xi'_\chi$ by:

$$\xi'_\chi := \xi_\chi \circ M_{P \cap H}.$$ 

It is a weakly holomorphic family of $E$-distributions on $P \times H$ which are $(\delta_{P}^{1/2} \otimes \delta_\chi) \otimes \varepsilon_H$-left covariant under $P \times H$.

From Lemma 9.1, there exists $\eta_\chi \in E'$, such that for all $e \in E$ and for all compact open subgroup, $J$, of $P$ which fixes $e$ under $\delta$, one has:

$$\langle \xi'_\chi, f \rangle = \int_{P \times H} \langle \eta_\chi, \delta(p)f(p, h)e \rangle \, dp \, dh = \text{vol}(J)\langle \eta_\chi, e \rangle,$$

where $f = 1_J \otimes f_1 \otimes e \in C_c(\mathbb{R}^\infty(P \times H) \otimes E$ and $f_1$ is a smooth function on $H$ with compact support and with integral equals to 1.

As the family $(\xi'_\chi)$ is weakly holomorphic, this implies that:

(5.3) $(\eta_\chi)$ is a weakly holomorphic family of linear forms on $E$.

Also from Lemma 9.1, one deduces:

(5.4) $(\delta_\chi)'(p)\eta_\chi = \delta_{P \cap H}^{-1}(p)\delta_P(p)\eta_\chi, p \in P \cap H$.

Let $a \in A_M$ and let $b$ be equal to $aa\sigma(a) \in A_0 \cap H \subset M \cap H$. Then, one has:

$$(\delta_\chi)'(b)\eta_\chi = \delta_{P \cap H}^{-1}(b)\delta_P(b)\eta_\chi.$$ 

But $(\delta_\chi)'(b) = \chi^{-1}(b)\delta'(b)$ so that one has:

(5.5) $\delta'(b)\eta_\chi = \delta_{P \cap H}^{-1}(b)\delta_P(b)\chi(b)\eta_\chi$. 
Let us consider the parabolic subgroup of $M$, $M \cap \sigma(P)$. Its Levi subgroup containing $A_0$ is equal to $M \cap \sigma(M)$. Let us prove that:
\[(5.6)\quad \eta_\chi \in (E_{M \cap \sigma(P)})',\]
where $E_{M \cap \sigma(P)}$ is the Jacquet module of $(\delta, E)$ with respect to the parabolic subgroup of $M$, $M \cap \sigma(P)$.

Let $x \in M \cap \sigma(U)$. From [BD] Proposition 2.1, there exists a unipotent subgroup $V'$ of $P \cap \sigma(P)$, $h \in H \cap V'$ and $y \in U \cap \sigma(P)$ such that $x = yh$.
Then, as $\delta_\chi$ is trivial on $U$, one has:
\[
(\delta_\chi)'(x)\eta_\chi = (\delta_\chi)'(yh)\eta_\chi = (\delta_\chi)'(h)\eta_\chi.
\]
As $h \in V' \cap H$ and $V'$ is a unipotent subgroup of $P$, $h$ is an element of a union of compact subgroups of $P \cap H$. From the fact that a continuous positive character on a topological group is trivial on compact subgroups, one deduces:
\[
\delta_{P \cap H}^{-1}(h)\delta_P(h) = 1.
\]
Similarly, as $\chi$ is unramified, one has
\[
\chi(h) = 1.
\]
Hence, from (5.4), one gets:
\[
(\delta_\chi)'(h)\eta_\chi = \eta_\chi.
\]
As $\chi$ is an unramified character of $M$ and $x$ is an element of the unipotent subgroup of $M$, $M \cap \sigma(U)$, one has $\chi(x) = 1$. From the previous discussion, one sees:
\[
\delta'(x)\eta_\chi = \eta_\chi, x \in M \cap \sigma(U),
\]
so that one has:
\[
\eta_\chi \in (E_{\delta}^{M \cap \sigma(U)}),
\]
which proves (5.6).

The Jacquet module $E_{M \cap \sigma(P)}$ being an $M \cap \sigma(M)$-module of finite length and $b$ being an element of the center of $M \cap \sigma(M)$, the number of generalized eigenvalues of $\delta'(b)$ on $(E_{M \cap \sigma(U)})'$ is finite. From (5.5), one deduces that the map $\chi \mapsto \chi(b)$ is constant on $O$, and equals to 1 as $O$ is connected and contains 1. In other words we have proved:
\[(5.7)\quad \chi(a\sigma(a)) = 1, \chi \in X, a \in A_M.
\]
As $X \subset X(M)$, one views an element $\nu$ of $b$ as an element of $(a_0)'_\mathbb{C}$, which vanishes on $a_0^M$. One deduces from (5.7) that:
\[(5.8)\quad \nu(X + \sigma(X)) = 0, X \in a_M, \nu \in b.
\]
Hence the restriction of $\nu$ to $a_M + \sigma(a_M)$ is $\sigma$-antiinvariant.
Recall that we have chosen a scalar product on $a_0$ which is invariant by the Weyl group of $(G, A_0)$ and by $\sigma$. Then $\nu \in b \subset (a'_M)_C$ is zero on the orthogonal to $a_M$, hence also on the orthogonal to $a_M + \sigma(a_M)$. Hence $\nu$ is $\sigma$-antiinvariant. This proves (i).
Now let us assume that $\nu$ is an element of $b$ which is strictly $P$-dominant. Then, with the notations of (2.12), one has $P = P_\nu$. One sees, from the antiinvariance of $\nu$ that $\sigma(P) = P - \nu$ which is clearly opposite to $P$. Hence $P$ is a $\sigma$-parabolic subgroup of $G$. □

**Lemma 5.2.** Let $P = MU$ be a parabolic subgroup of $G$ and let $(\delta, E)$ a smooth representation of finite length of $M$. Let $X$ be a complex subtorus of $X(M)$ and let $O$ be a non empty subset of $X$. Let $\chi \mapsto \xi_\chi$ be a weakly holomorphic family of $H$-fixed linear forms on $i_G^E \mathcal{E}_\chi$ defined for $\chi \in O$.
Let $F$ be the union of the supports of the $\xi_\chi$, $\chi \in O$. As these supports are left invariant by $P$ and right invariant by $H$ and as there are only a finite number of $(P, H)$-double cosets, $F$ is closed. We call $F$ the support of the family.
Let $A$ be a maximal split torus in $M$. Let $\Omega$ be a $(P, H)$-double coset of $G$ open in $F$. Then one can choose $x \in \Omega$ such that $A_x := x^{-1}.A$ is a $\sigma$-stable maximal split torus contained in $x^{-1}.P$. For such an $x$, $x^{-1}.M$ is the Levi subgroup of $x^{-1}.P$ which contains $A_x$. The conjugacy under $x^{-1}$ induces a map $\chi \rightarrow x^{-1}\chi$ from $X$ to a subtorus $x^{-1}.X$ of $X(x^{-1}.M)$.

(i) Then the Lie algebra of $x^{-1}.X$ appears as a subspace of $(a_x)_C'$ made of antinvariant elements by $\sigma$.
(ii) Moreover if $X$ contains a strictly $P$-dominant element, $x^{-1}.P$ is a $\sigma$-parabolic subgroup.
(iii) With the assumption of (ii), one can choose $x$ such that $A_x$ is a maximally $\sigma$-split $\sigma$-invariant maximal split torus in the $\sigma$-stable Levi subgroup of $x^{-1}.P$.

**Proof.** Let $x$ be an element of $\Omega$. First $x^{-1}.P$ contains a $\sigma$-invariant maximal split torus of $G$ (cf. [HH] Lemma 2.4). Two maximal split tori of a parabolic subgroup are conjugate by an element of this subgroup (see [BT], Proposition 4.7 and Theorem 4.21). Hence changing $x$ into $px$, for a suitable $p \in P$, one can assume that $A_x$ is $\sigma$-invariant. We denote by $\xi_\chi$ the restriction to $\Omega$ of the induced $E$-distribution $\xi_{F, \chi}$ by $\xi_\chi$ on $F$ (see subsection 9.1). Then, from what follows (5.2), $\lambda(x^{-1})\xi_\chi$ satisfies the hypothesis of the preceeding Lemma with $P$ changed in $x^{-1}.P$, $A_0$ in $A_x$ and $X$ in $x^{-1}.X$. Then (i) and (ii) are an immediate consequence of the preceeding Lemma.
Let us prove (iii). We are in the case where $x^{-1}.P$ is a $\sigma$-parabolic subgroup whose $\sigma$-stable Levi subgroup contains a maximally $\sigma$-split $\sigma$-invariant maximal split torus of $G$. This implies (iii). \hfill \Box

5.2. Generic basic generic Lemma for $H$-forms and the role of $\sigma$-parabolic subgroups.

**Proposition 5.3.** Let $P = MU$ (resp., $P' = M'U'$) be a $\sigma$-parabolic subgroup and let $A$ (resp., $A'$) be a maximally $\sigma$-split maximal split torus of $M$ (resp., $M'$). Let $A_\sigma$ (resp., $A'_\sigma$) be the maximal $\sigma$-split torus of $A$ (resp., $A'$). We denote by $X := X(M)_{\sigma}$ the neutral component of elements of $X(M)$ which are antiinvariant by $\sigma$. Let $(\xi_\chi)$ be a weakly holomorphic family of $H$-forms on $i_p^G E_\chi$ defined for $\chi$ in an open subset $O$ of $X$. With the notations of Theorem 4.10, let $w$ be an element of $W(M' \setminus G / M)$. Let us assume that $\xi_{p' - w} \neq 0$ where $\xi = \xi_{\chi_0}$ for some element $\chi_0$ of $O \cap O'$, where $O'$ is as in Theorem 4.10. Then one may change our choice of $w$ in its class in $P' \setminus G / P$ in such a way that:

(i) One has $A' := w . A$.

(ii) The group $w . P$ is a $\sigma$-parabolic subgroup of $G$ with $\sigma$-stable Levi subgroup $w . M$ and $M' \cap w . P$ is a $\sigma$-parabolic subgroup of $M'$.

(iii) The groups $P'_w$ and $w . P_w$ are $\sigma$-parabolic subgroups of $G$.

(iv) One has the equality $w^{-1}.A'_\sigma = A_\sigma$.

(v) The group $w^{-1}.P'$ (resp., $w^{-1}.P'^-\sigma$) is a $\sigma$-parabolic subgroup of $G$ with $\sigma$-stable Levi subgroup $w^{-1}.M'$ and $M \cap w^{-1}.P'$ (resp., $M \cap w^{-1}.P'$) is a $\sigma$-parabolic subgroup of $M$.

The groups $P'_w, P'_w, P'_w, P'_w$ are $\sigma$-parabolic subgroups of $G$.

**Proof.** It will be more convenient for the proof of this Proposition to denote $A'$ by $A_1$ in order to avoid too many $'$. First, as $w \in W(M' \setminus G / M)$, one has $w . A = A_1$.

Moreover :

The Lie algebra of $X$ is equal to the space $(a_M)_C^{t-\sigma}$ of $\sigma$-antiinvariant elements of $(a_M)_C$ that one can view as a subspace of $a_C^{t}$ (cf. (2.11) and (2.14)).

We denote by $X' := \{ w_\chi | M' \cap w . M | \chi \in X \}$ which is closed in $X(M' \cap w . M)$ (cf., (2.9)). By looking to differential, one sees that on an open neighborhood of $\chi_0$ in $O \cap O'$, the map $\chi \mapsto w_\chi | M' \cap w . M$ from $X$ to $X'$, is an isomorphism whose inverse, defined on $O'' \subset X'$, will be be denoted $\chi' \mapsto w^{-1}\chi'$. We will see that the family $\chi' \mapsto (\xi_{w^{-1}\chi'})_{P'^-w}$ defined on $O''$ satisfies the hypothesis of the preceding Lemma with $G$ replaced by $M'$. From Theorem 4.10, only the weak holomorphy condition has to be proven. Let us study $\langle (\xi_\chi)_{P'^-w}, e_\chi \rangle$ where $e$ is an element of
the compact realization of $j_{M'\cap w,P}^M w j_{M'\cap w^{-1},P'} E_X$. We choose a compact open subgroup of $G$, $J$, with a $\sigma$-factorization for $(P',P'^\sigma)$ such that $e$ is fixed by $J_{M'}$. One starts by using (3.6) and then one uses the fact that $(e,J_\xi)$ is an holomorphic family of $J$-fixed vectors. One deduces from Proposition 4.9, that $j_{M'\cap w^{-1},P'}^M \circ \xi_\chi$ is weakly holomorphic if for all $v$ in the compact realization of $i_P^G E$, the map $\chi \mapsto ((\beta_{\chi,w}(\tilde v),e_{w,\chi})$ is holomorphic on $O'$, where $\beta_{\chi,w}(\tilde v) = (\tilde v_\chi)_{P'^\sigma,w}$. This follows easily from the definition of $\beta_{\chi,w}(\tilde v)$ (cf. Proposition 4.9) and from the holomorphy properties of the intertwining integrals.

As $P$ is a $\sigma$-parabolic subgroup, $X$ contains strictly $P$-dominant elements and $X'$ contains a strictly $M' \cap w.P$-dominant element. The conclusion of the preceeding Lemma, applied to $M'$ instead of $G$, asserts that there exists $m' \in M'$ such that $A_2 := m'.A = m'.A_1$ is a maximally $\sigma$-split, $\sigma$-stable, maximal split torus of $M'$, $Q := m'.(M' \cap w.P)$ is a $\sigma$-parabolic subgroup of $M'$, $m'.X'$ and hence $m'.w(a_M)^{-\sigma} \subset (a_2)'$ is made of $\sigma$-antiinvariant elements, where $(a_M)^{-\sigma}$ is the space of antiinvariant elements of $(a_M)'$. Then the half sum of the roots of $A_2$ in the Lie algebra of $Q$, $\rho_Q$, satisfies:

$$\rho_Q \in (a_2)^{-\sigma}. \quad (5.10)$$

Two maximally $\sigma$-split $\sigma$-stable maximal split tori of $M'$ are conjugate by an element of $M'$ which conjugates their maximal $\sigma$-split tori, because two maximal $\sigma$-split tori are conjugate, and two maximal split tori in the centralizer of a maximal $\sigma$-split torus are conjugate. So one can choose $m''$ in $M'$ such that $m''.A_2 = A'$ and $m''.(a_2)^{-\sigma} = (a_1)^{-\sigma}$. Then from (5.10), one sees that the parabolic subgroup of $M$, $Q := (m''.m'.w.M) \cap P'$, contains $A_1$ and the half sum of the roots of $A_1$ in its Lie algebra $\rho_{Q'}$ is an element of $(a_1)^{-}$. Using (2.12), one sees that it is a $\sigma$-parabolic subgroup of $M$. One will change $w$ to $m''m'.w$. Hence one has:

$$w.A = A_1, w(a_M)^{-\sigma} \subset (a_1)^{-\sigma}. \quad (5.11)$$

Let us show that $w$ satisfies (ii) and (iii). Let $\rho_P \in a'_M \subset a'$ be the half sum of roots of $A$ in the Lie algebra of $P$. We define similarly $\rho_{w.P_w}$, $\rho_{P_w}$ and $\rho_{P'^\sigma \cap w,M}$ with respect to $A_1$. These are elements of $(a_1)'$. One has:

$$\rho_{w.P_w} = \rho_{P'^\sigma \cap w,M} + w.\rho_P. \quad (5.12)$$

As $P$ is a $\sigma$-parabolic subgroup of $G$, one has $\rho_P \in (a_M)^{-\sigma}$. From (5.11), one deduces that:

$$w.\rho_P \in (a_1)'^{-} \text{ is } \sigma\text{-antiinvariant.} \quad (5.13)$$
It follows that \( w.P \) is a \( \sigma \)-parabolic subgroup of \( G \). The group \( w.M \) is the Levi subgroup of \( w.P \) which contains \( A_1 = w.A \), whose Lie algebra is the sum of the \( A_1 \)-weight spaces for weights \( \alpha \) which are equal to zero or to an \( A_1 \)-root \( \alpha \) such that \( (w\rho_p, \alpha) = 0 \). As \( w\rho_p \) is \( \sigma \)-antiinvariant, one sees that \( w.M \) is \( \sigma \)-invariant. As \( P' \) is a \( \sigma \)-parabolic subgroup of \( G \), \( \sigma(P') \cap P' = M' \), which implies:

\[
\sigma(P' \cap w.M) \cap P' \cap w.M = M' \cap w.M.
\]

As \( M' \cap w.M \) is the Levi subgroup of the parabolic subgroup \( P' \cap w.M \) of \( w.M \) which contains \( A' \), this implies that \( P' \cap w.M \) is a \( \sigma \)-parabolic subgroup of \( w.M \) with \( \sigma \)-stable Levi subgroup equals to \( w.M \cap M' \).

Hence

\[
\rho_{P' \cap w.M} \text{ is antiinvariant by } \sigma.
\]

From (5.12), (5.13), (5.14), one sees that \( \rho_{w.P} \in (a_1)' \) is \( \sigma \)-antiinvariant. It follows from (2.12) that \( w.P \) is a \( \sigma \)-parabolic subgroup of \( G \). One sees easily that its \( \sigma \)-stable Levi subgroup is \( M' \cap w.M \).

Similarly one proves that \( P'w \) is a \( \sigma \)-parabolic subgroup of \( G \), by using the equality:

\[
\rho_{P'w} = \rho_{M' \cap w.P} + \rho_{P'}
\]

and that \( P' \) (resp., \( M' \cap w_P \)) is a \( \sigma \)-parabolic subgroup of \( G \) (resp., \( M' \)), as \( w.P \) is a \( \sigma \)-parabolic subgroup of \( G \) from (5.13) and \( w.M \) is its \( \sigma \)-stable Levi subgroup.

Altogether we have found a choice of \( w \) which satisfies (i), (ii) and (iii).

We will modify our preceeding choice of \( w \) to get one which will satisfy also (iv).

Let \( w \) be as above. Then \( (w.P)wH = wPH \) is open in \( G \). Moreover \( w.P \) is a \( \sigma \)-parabolic subgroup of \( G \), \( (A_1)_\sigma \) is a maximal \( \sigma \)-split torus contained in \( w.P \), hence contained in its \( \sigma \)-stable Levi subgroup \( w.M \). As any \( (w.P,H) \)-open orbit has a representative which is \( (A_1)_\sigma \)-good (cf. 2.17), there exists \( p' = w.p \) with \( p \in P \) such that \( w' := (w.p).w = wp \) satisfies \( w'^{-1}.(A_1)_\sigma \) is \( \sigma \)-split. But \( w'^{-1}.(A_1)_\sigma \subset p^{-1}.A \subset P \). As \( w'^{-1}.(A_1)_\sigma \) is \( \sigma \)-split, it is \( \sigma \)-stable hence contained in \( M = P \cap \sigma(P) \). Hence \( w'^{-1}.(A_1)_\sigma \) is a maximal \( \sigma \)-split torus in \( M \). Then, as all maximal \( \sigma \)-split tori in \( M \) are conjugate (cf. [HH], Proposition 1.16), we can choose an element \( m \) of \( M \), such that \( w'' = w'm \) satisfies \( w''^{-1}.(A_1)_\sigma = A_\sigma \). Then \( w''^{-1}.A \) is contained in the centralizer of \( A_\sigma \) which is contained in \( M \). So one can choose \( m_1 \) element of this centralizer such that \( w_1 := w''m_1 \) is such that \( w_1^{-1}.A_1 = A \) and \( w_1^{-1}.(A_1)_\sigma = A_\sigma \). Hence, as \( \rho_P \in a'^{-\sigma} \), \( w_1\rho_P \) is \( \sigma \)-antiinvariant and \( w_1.P \) is a \( \sigma \)-parabolic subgroup of \( G \). As above, one sees also that it implies that \( w_1.M \) is its \( \sigma \)-stable Levi subgroup. Then, as above, one
sees that $w_1$ satisfies (ii) and (iii). As $P'$ is a $\sigma$-parabolic subgroup of $G$, $\rho_{P'}$ is an element of $(a_1)^{1-\sigma}$ and one gets $w_1^{-1}\rho_{P'} \in a^{1-\sigma}$. One sees, as above, that $w_1^{-1}.P'$ is a $\sigma$-parabolic subgroup of $G$ with $\sigma$-stable Levi subgroup equals to $w_1^{-1}.M'$. This implies easily that $M \cap w_1^{-1}.P'$ is a $\sigma$-parabolic subgroup of $M$ with $\sigma$-stable Levi subgroup equals to $M \cap w_1^{-1}.P'$.

Then the assertions on $P_{w_1}, P'_{w_1}, \tilde{P}_{w_1}, \tilde{P}'_{w_1}$ follow easily. Thus $w_1$ has the required properties. \hfill \Box

5.3. Intertwining integrals and support of families of $H$-forms.
An ordered pair $(P = MU, P' = MU')$ of parabolic (resp., $\sigma$-parabolic subgroups) of $G$, is said adjacent (resp., $\sigma$-adjacent) if there is a unique reduced $A_M$-root (resp., reduced $(A_M)_{\sigma}$-root), $\alpha$, which is positive for $P$ and negative for $P'$. We denote by $A^\alpha$ the group of $F$-points of the neutral component of the kernel of $\alpha$ in $A_M$ (resp., $(A_M)_{\sigma}$) and by $M^\alpha$ the centralizer of $A^\alpha$ in $G$. The group $P^\alpha$ generated by $P$ and $P'$ is a parabolic (resp., $\sigma$-parabolic) subgroup of $G$ with Levi (resp., $\sigma$-stable Levi subgroup ) $M^\alpha$ and unipotent radical $U^\alpha$ contained in $U \cap U'$.

It is easy to see that $\alpha$ is $P$-simple.

A minimal string of parabolic (resp., $\sigma$-parabolic) subgroups of $G$ between two parabolic (resp., $\sigma$-parabolic) subgroups of $G$, $P = MU$, $P' = MU'$, is a sequence $(P_i)_{i=0, \ldots, r}$ of parabolic (resp., $\sigma$-parabolic subgroups) of $G$, such that $P_0 = P, P_r = P'$ and $(P_i, P_{i+1})$ is adjacent (resp., $\sigma$-adjacent) for $i = 0, \ldots, r - 1$. Such a string always exists (cf. [KnSt], before Theorem 4.2 for parabolic subgroups and it works in a same manner for $\sigma$-parabolic subgroups).

The next Lemma is the second key Lemma of the article. It was suggested by a geometric result of Matsuki (cf. [M], Lemma 3).

**Lemma 5.4.** Let $P = MU$ and $Q = LV$ be two $\sigma$-parabolic subgroups of $G$, with $P \subset Q$. Let $(\delta, E)$ be a smooth irreducible representation of $\mathfrak{L}$. Let $P' = MU'$ be an other $\sigma$-parabolic subgroup of $G$ such that $(P, P')$ is $\sigma$-adjacent, and let $\alpha$ be the unique reduced $(A_M)_{\sigma}$-root which is positive for $P$ and negative for $P'$. One assumes that the restriction $\alpha|_{a^\sigma_L}$ of $\alpha$, to $a_{\sigma}^{-\sigma}$ is non zero, where $a_{\sigma}^{-\sigma}$ is the subspace of elements of $a_L$ antiinvariant by $\sigma$.

We denote by $X(\mathfrak{L})_{\sigma}$ the neutral component of $\sigma$-antiinvariant elements of $X(\mathfrak{L})$.

Let $\chi \mapsto \xi_\chi$ be a weakly holomorphic family of $H$-forms on $i_E^*E_X$ defined for $\chi$ in an open subset, $\mathcal{O}$, of $X(\mathfrak{L}|M)_{\sigma} := \{\chi|_M \mid \chi \in X(\mathfrak{L})_{\sigma}\}$. Let us assume that the support of every $\xi_\chi$ has an empty interior in $G$.

One has the following:
such that
defined over $U$ (resp., $Y$). We fix $(i)$ The set $O'$ of $\chi \in O$ such that $A(P, P', \delta_\chi)$ has no pole is an open and dense subset of $O$.

(ii) If $\chi \in O'$, the support of $\xi_\chi \circ A(P, P', \delta_\chi)$, which is an $H$-form on $\mathfrak{p}'_E$, has an empty interior in $G$.

(iii) Let $Q'$ be the $\sigma$-adjacent $\sigma$-parabolic subgroup of $G$ determined by the $Q$-simple $(A_L)_\sigma$-root $\alpha_{|L}$. Then $P' \subset Q'$ and $(Q, Q')$ are $\sigma$-adjacent.

Proof. The fact that $O'$ is dense follows from Lemma 4.4. This proves (i).

We want to prove (ii) for a given $\chi \in O'$. Changing $\delta$ to $\delta_\chi$, one may assume that $\chi = 1 \in O'$. We set $\xi := \xi_1$.

The union of the $(P, H)$ double cosets $PxH$ which are open in the support of $\xi$ is dense in this support. From Lemma 4.2, the support of $\xi \circ A(P, P', \delta)$ is contained in $\text{cl}(P\text{Sup}(T))$, hence in the union of $\text{cl}(P'xH)$ as for $A, B$, subsets of $G$, $\text{cl}(\text{Acl}(B)) = \text{cl}(AB)$ and $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$. Hence it suffices to show that:

(5.15) For all $x = x_i$ one has $P^a x H$ with empty interior.

Let $A$ be a maximally $\sigma$-split torus $\sigma$-stable maximal split torus in $M$, which is automatically maximally $\sigma$-split $\sigma$-stable maximal split torus in $G$ as $P$ is a $\sigma$-parabolic subgroup. We may and will choose $x$ in its double $(P, H)$-coset such that $x^{-1}.A$ is $\sigma$-stable (cf. Lemma 5.2). The Lie algebra of $X := X(L|M)_\sigma$ identifies with $(\mathfrak{a}'_L)^{\sigma}$. By a left translation by $x^{-1}$, we can apply Lemma 5.2 and one sees, using (2.14), that:

(5.16) $x^{-1}(\mathfrak{a}'_L) \subset (x^{-1}\mathfrak{a})^{\sigma}$.

One sets: $P_x = x^{-1}P, P'_x = x^{-1}P'$ etc..

We denote by $\mathfrak{g}$ the Lie algebra of the algebraic group $G$ such that $G = \mathfrak{G}(F)$. By abuse of terminology we will say that $\mathfrak{g}$ is the Lie algebra of $G$. We use a similar terminology for the subgroups of $G$ which are the group of $\mathfrak{F}$-points of a subgroup of $G$ defined over $F$.

Let $\mathfrak{g}(\alpha)$ (resp., $\mathfrak{g}(-\alpha)$) be the Lie algebra of $U \cap M^\alpha$ (resp., $U' \cap M^\alpha$). Similarly let $\mathfrak{g}(x^{-1}\alpha)$ (resp., $\mathfrak{g}(-x^{-1}\alpha)$) be the Lie algebra of $U_x \cap M^{\alpha}_x$ (resp., $U'_x \cap M^{\alpha}_x$). One has:

$p^\alpha = \mathfrak{g}(-x^{-1}\alpha) + p_x$.

We fix $Y_x = x^{-1}Y \in x^{-1}(\mathfrak{a}'_L)^\sigma$, where $Y$ is a strictly $Q$-dominant element in $(\mathfrak{a}'_L)^\sigma$ which is canonically identified to $\mathfrak{a}'_L$ (cf. (2.14)). Then an $A_\sigma$-root $\beta$ is such that $\beta(Y_x) > 0$ if and only if the corresponding root space is contained in $\mathfrak{u}_x$. But $\mathfrak{g}(\alpha) \subset \mathfrak{u}$, as $P \subset Q$ and $\alpha_{|\mathfrak{a}_L} \neq 0$. 
So one has \((x^{-1}\alpha)(Y_x) > 0\). Let us prove that:

\[(5.17)\quad h + p_x = \mathfrak{h} + p_x^\sigma.\]

The only thing to prove is that \(g(-x^{-1}\alpha) \subset h + p_x\). Let \(\beta\) be a root of \(A_x\) in \(g(-x^{-1}\alpha)\), so that \(\beta(Y_x) < 0\) by what have been said above. Let \(Z\) be an element of the corresponding weight space in \(g\). Let \(\beta\) be a root of \(A_x\) in \(g(-x^{-1}\alpha)\), so that \(\beta\) is strictly positive. Hence \(\sigma(Z)\) is an element of \(p_x\). Consequently one has:

\[Z = (Z + \sigma(Z)) - \sigma(Z) \in h + p_x,\]

which achieves to prove \((5.17)\).

Let us assume that \(P_x^\sigma H\) has a non empty interior. We will prove that it leads to a contradiction. From (2.17) and Lemma 2.2 (iv), one deduces that \(P_x^\sigma\) is a \(\sigma\)-parabolic subgroup of \(G\). Hence (9.6) implies that \(\sigma\) is a \(\sigma\)-parabolic of \(G\) and (9.7) implies

\[\mathfrak{h} + p_x^\sigma = g.\]

Then, together with (5.17), it implies that \(\mathfrak{h} + p_x = g\). Hence (cf. Lemma 9.3), \(HP_x\) is open. From Lemma 9.4, one sees that \(P_x H\) would be open in \(G\). This is a contradiction with our hypothesis on the support of \(\xi\), which implies (5.15). We have thus proved (ii).

The assertion on \(Q'\) in (iii) being clear, this achieves the proof of the Lemma.

\[\square\]

Lemma 5.5. We keep the notations of the previous Lemma, except that \(P'\) is not necessarily adjacent to \(P\). We assume that every \((A_M)_\sigma\)-root which is positive for \(P'\) and negative for \(P\) satisfies \(\alpha_{|a_L^\sigma} \neq 0\). Then the same conclusion as in (i) and (ii) of the previous Lemma is valid.

Proof. Let \(P_0, \ldots P_r\) be a minimal string of \(\sigma\)-parabolic subgroups between \(P\) and \(P'\). We will prove by induction on \(i\) that for \(\chi \in O'\), the support of \(A(P_i, P, \delta_{\chi})\xi_{\chi}\) has an empty interior in \(G\) and we will define \(\sigma\)-parabolic subgroups of \(G\), \(Q_0 = Q, \ldots, Q_r\), with \(\sigma\)-stable Levi subgroup \(L\), such that:

The family \((\xi^i_{\chi}) := (\xi_{\chi} \circ A(P_i, P, \delta_{\chi}))\) is such that \((P_i, P_{i+1}), Q_i\) satisfies the hypothesis of the previous Lemma, by the induction hypothesis.

The Lemma follows by using this previous Lemma. \[\square\]
6. Families of $H$-forms on induced representations from $\sigma$-parabolic subgroups and $B$-matrices

6.1. Families of $H$-forms on induced representations from $\sigma$-parabolic subgroups. Let $P = MU$ be a $\sigma$-parabolic subgroup of $G$. Let $(\delta, E)$ be a smooth representation of $M$ with finite length. Let $W_M^G$ be a set of representatives of the open $(P, H)$-double cosets as in (2.17): here $A_\sigma$ is a maximal $\sigma$-split torus of $M$. As remarked in Lemma 2.2, if $x \in W_M^G$, $x^{-1}P$ is a $\sigma$-parabolic subgroup of $G$. We define

$$E'(\delta, H) = \bigoplus_{x \in W_M^G} E^H_{M \cap xH}.$$

We have the following mild generalization of [BD], Theorem 2.8, that one gets in an entirely similar way (see the comments following the statement). Notice that this statement is true assuming only that the characteristic of $F$ is different from 2, as it is shown by the examination of the proof.

Proposition 6.1. Let us assume that we are given a $\sigma$-split torus $B \subset A_M$, and a complex subtorus of $X(M)_\sigma$, $X$. We use the identification $X(M)_\sigma$ and $a_M^\sigma$ given by a suitable scalar product on $a_M$ (see (2.11) and (2.14)). We assume that the Lie algebra of $X$ identifies to $b_C$ and that $\Lambda(B) \subset b$ contains strictly $P$-dominant elements.

(i) Let $\chi \in X(M)_\sigma$. Let us denote by $J_\chi$ the subspace of elements of $i_P^G E_X$ whose support is contained in the union of the open $(P, H)$-double cosets.

There is a canonical linear isomorphism between $E'(\delta, H)$ and $J^H_\chi$ which associates to $\eta \in E^H_{M \cap xH}$, the element $\xi'(P, \delta, \chi, \eta)$ of $J^H_\chi$ defined by:

$$\xi'(P, \delta, \chi, \eta)(\varphi) = \int_{(H \cap x^{-1}M) \setminus H} \langle \varphi(xh), \eta \rangle \, dh, \varphi \in J_\chi.$$  

(ii) There exists an open dense subset, $O_0$, of $X$ such that for $\chi \in O_0$, $\xi'(P, \delta, \chi, \eta)$ extends uniquely to an $H$-form, $\xi(P, \delta, \chi, \eta)$, on $i_P^G E_X$. In particular for $\chi \in O_0$, every $H$-form on $i_P^G E_X$ whose restriction to the open $(P, H)$ double cosets is equal to zero is itself equals to zero.

(iii) Moreover there exists a polynomial function, $b$, on $X$, such that for every $v \in i_K \cap P$ and $\eta \in E'(\delta, H)$, $\chi \mapsto b(\chi) \langle \xi(P, \delta, \chi, \eta), v_\chi \rangle$ extends to a polynomial function on $X$.

The main point for this generalization of [BD], Theorem 2.8, is the generalization of [BD], Lemma 2.5 with our $X$. But it is straightforward.
Then, we will see that one has:

\[ (6.1) \quad \text{For all } \eta = (\eta_x)_{x \in W_M^G}, \text{ for all } x \in W_M^G, \ r_M(\rho(x)\xi(P, \delta_\chi, \eta)) = \eta_x. \]

For \( x = 1 \), it follows from section 9.1 and [BD], Equation (2.33). Now the assertion reduces to the one for \( x = 1 \) applied to the \( x.H \)-fixed linear form \( \rho(x)\xi(P, \delta_\chi, \eta) \).

6.2. \( B \)-matrices.

**Proposition 6.2.** We keep the notations of the preceding Proposition. Let \( Q \) be a \( \sigma \)-parabolic subgroup of \( G \) with \( \sigma \)-stable Levi subgroup equals to \( M \). There exists a rational function on \( X(M)_\sigma, \chi \mapsto B(P, Q, \delta_\chi) \) with values in \( \text{End}_C(E'(\delta, H)) \) such that one has the equality of rational maps on \( X(M)_\sigma \):

\[ \xi(Q, \delta_\chi, \eta) \circ A(Q, P, \delta_\chi) = \xi(P, \sigma, B(P, Q, \delta_\chi)\eta), \eta \in E'(\delta, H). \]

More precisely, let \( b_A \) (resp., \( b_\xi \)) be a non zero polynomial function on \( X(M)_\sigma \), such that for all \( v \) in \( iK_nPE \), \( \chi \mapsto b_A(\chi)(A(Q, P, \delta_\chi)v) \) (resp., and for all \( \eta \in E'(\delta, H) \), \( \chi \mapsto b_\xi(\chi)(\xi(Q, \delta_\chi, \eta), v_\chi) \) is polynomial on \( X(M) \). Then for all \( \eta \in E'(\delta, H) \), the map \( \chi \mapsto (b_A b_\xi(\chi))B(P, Q, \sigma_\chi)\eta \) is a polynomial map on \( X(M)_\sigma \) with values in \( E'(\delta, H) \).

**Proof.** From Proposition 6.1, for \( \chi \) element of a dense open subset, \( O \), of \( X(M)_\sigma \), there is a unique \( \theta(\chi) \in E'(\delta, H) \), such that

\[ b_\xi(\chi)\xi(Q, \delta_\chi, \eta) \circ (b_A(\chi)A(Q, P, \delta_\chi)) = \xi(P, \delta_\chi, \theta(\chi)). \]

Let us show that the map \( \chi \mapsto \theta(\chi) \) is polynomial in \( \chi \in X(M)_\sigma \). Let \( \theta(\chi)_x, x \in W_M^G \) be the components of \( \theta(\chi) \). First, let us prove:

\[ (6.2) \quad \text{For all } e \in E, \langle \theta(\chi)_1, e \rangle \text{ is polynomial in } \chi \in X(M)_\sigma. \]

Let \( J \) be a compact open subgroup of \( K \) which has a \( \sigma \)-factorization for \((P, P^-)\) and such that \( e \) is fixed by \( J_M \). Then one has from (3.20):

\[ \langle \theta(\chi)_1, e \rangle = \Vol((K \cap P)J_H)^{-1}\langle \xi(Q, \delta_\chi, \theta(\chi)), v_{e, \delta_\chi}^{P, J} \rangle = \Vol((K \cap P)J_H)^{-1}b_\xi(\chi)\langle \xi(Q, \delta_\chi, \eta), b_A(\chi)A(Q, P, \delta_\chi)v_{e, \delta_\chi}^{P, J} \rangle. \]

It follows from (3.10), that the restriction to \( K \) of \( v_{e, \delta_\chi}^{P, J} \) is independent of \( \chi \). Hence, from the properties of \( b_A \), one sees that \( \chi \mapsto v(\chi) := b_A(\chi)A(Q, P, \delta_\chi)v_{e, \delta_\chi}^{P, J} \) is polynomial in the compact realization. Hence (6.2) follows.

Let \( x \in W_M^G \). One applies (6.2) to \( \rho(x)\xi(P, \delta_\chi, \eta) \) by changing \( \sigma \) to \( \sigma_x \).
One concludes that the map $\chi \to \theta(\chi)$ is polynomial in $\chi \in X(M)_{\sigma}$. Then
\[ B(Q, P, \sigma_\chi) \eta := (b_\chi b_A)^{-1}(\chi) \theta(\chi) \]
satisfies the required properties. \hfill \Box

7. Main Theorems

7.1. Let us prove

Let $P = M U$ be a $\sigma$-parabolic subgroup of $G$ and let $A_\sigma$ be

(7.1) a maximal $\sigma$-split torus of $M$. If $x, x' \in G$ are $A_\sigma$-good and

$P H x H = P x' H$ then $x' = mxh$ with $m \in M$.

One has $x' = mu xh$ with $m \in M$, $u \in U$, $h \in H$. As $x'$ is $A_\sigma$-good, $x'^{-1}.M$ is $\sigma$-stable, which implies that $x^{-1}.u^{-1}.M$ is $\sigma$-stable. Let $M' = x^{-1}.M$ which contains $x^{-1}A_\sigma$. Then $M'$ is the $\sigma$-stable Levi subgroup of the $\sigma$-parabolic subgroup $P' := x^{-1}.P$ (cf. Lemma 2.2). Hence, as $x^{-1}.(u^{-1}.M) \subset P'$, one has $x^{-1}.(u^{-1}.M) = x^{-1}.M$, which implies that $u^{-1}.M = M$. Hence, for all $m \in M$, $u^{-1}mu \in M$ which implies $m^{-1}u^{-1}mu \in M$. Then $m^{-1}u^{-1}mu$ is element of $U \cap M$ hence is equal to 1. Hence $u$ commutes to $M$. This is possible only if $u = 1$. Hence $x' = mxh$, which proves (7.1)

Lemma 7.1. (i) Let $P = M U$, $R = L V \subset P$ be two $\sigma$-parabolic subgroups of $G$. Let $Q$ be equal to $R \cap M$, which is a $\sigma$-parabolic subgroup of $M$ with $\sigma$-stable Levi subgroup equal to $L$. Let $A_\sigma$ be a maximal $\sigma$-split torus of $M$. If $x \in M$ and $\Omega = Q x (M \cap H)$ is open in $M$ then $R x H$ is open in $G$.

(ii) Let $x, x' \in M$ which are $A_\sigma$-good. If $R x H = R x' H$, one has $Q x (M \cap H) = Q x' (M \cap H)$.

(iii) Let $(\delta, E)$ be a smooth representation of $L$ and let $\xi$ be an $H$-form on $i^M E$. If $\text{Supp}(\xi) \subset G$ has an empty interior, the same is true for the support of $\tilde{r}_M \xi \in (i^M E)'$.

Proof. (i) The first claim on $Q$ is clear as $Q \cap \sigma(Q) = R \cap \sigma(R) \cap M \cap \sigma(M) = L$.

The map $P \times H \to PH$, $(p, h) \mapsto ph$, is open (cf. [BD], Lemma 3.1) and $U \Omega$ is open in $P$. Hence $U \Omega H$ is open in $PH$ and also in $G$, as $PH$ is open in $G$. But $U \Omega H \subset R x H$. It implies that $R x H$ has a non empty interior, hence is open.

(ii) By (7.1) applied to $R$, one has $x' = lxh$ with $l \in L, h \in H$. Hence $h$ is element of $M \cap H$ and (ii) follows.

(iii) Let us prove that with our hypothesis, 1 does not belong to the support of $\tilde{r}_M \xi$. The hypothesis implies that $\tilde{r}_L \xi = 0$. From Proposition 3.1, one deduces that $\tilde{r}_L \xi = \tilde{r}_L(\tilde{r}_M \xi) = 0$, which is equivalent to
1 is not in the support of $\tilde{r}_M \xi$. This proves our claim.

Now let $x$ as in (i). Changing of representative in $Qx(H \cap H)$, we may assume (cf (2.17)) that $x$ is $A_\sigma$-good. Then $P$ is a $\sigma_x$-parabolic subgroup of $G$ (cf. Lemma 2.2). Then one applies the first part of the proof to $\rho(x) \xi$, which is fixed by $x.H$, replacing $\sigma$ by $\sigma_x$. This implies that $x$ is not element of the support of $\tilde{r}_M \xi$. □

**Lemma 7.2.** Let $P = MU$, $R = LV$ be two $\sigma$-parabolic subgroups of $G$ with $R \subset P$. Let $Q = M \cap R$, which is a $\sigma$-parabolic subgroup of $M$. Let $A$ be a $\sigma$-stable maximally $\sigma$-split maximal split torus of $L$. We choose the set $W_M^G$ such that its elements are $A_\sigma$-good. Then for all $y \in W_M^G$, $y^{-1}.M$ is $\sigma$-stable, $y^{-1}.Q$ is a $\sigma$-parabolic subgroup of $y^{-1}.M$ with $\sigma$-stable Levi subgroup equals to $y^{-1}.L$. For all $y \in W_M^G$, we choose $W_{y^{-1}.L}^{y^{-1}.M}$ such that its elements are $y^{-1}.A_\sigma$-good. Then $W := \bigcup_{y \in W_M^G} y W_{y^{-1}.L}^{y^{-1}.M}$ is made of $A_\sigma$-good elements, the union being disjoint, and may be taken as $W_L^G$.

**Proof.** We will show that $W$ is a set of representatives of the open $(R, H)$-double cosets in $G$.

Let $x = yz, x' = y'z' \in W$ with $y, y' \in W_M^G, z \in W_{y^{-1}.M}^{y^{-1}.L}, z' \in W_{y'^{-1}.M}^{y'^{-1}.L}$. Then $x, x'$ are $A_\sigma$-good. Hence $RxH$ and $Rx'H$ are open. Moreover $PxH = PyH, Px'H = Py'H$ as $z \in y^{-1}.M, z' \in y'^{-1}.M$. Let us show that:

(7.2) The equality $RxH = Rx'H$ implies $y = y'$ and $z = z'$.

Our hypothesis implies $PxH = Px'H$. By what we have just seen, it implies $PyH = Py'H$. Hence one has $y = y'$.

First, let us assume that $y = y' = 1$.

Then $z, z' \in M$ are $A_\sigma$-good, $RzH = Rz'H$, and, by (7.1), $z' = lzh$ where $l \in L$ and $h \in H$. Hence one has $h \in M \cap H$. If $Q = R \cap M$, $z, z'$ determine the same $(Q, M \cap H)$-double coset. This implies that $z' = z$. Hence this proves (7.2) if $y = y' = 1$.

For the general case, we apply this to $y^{-1}.L$ and $y^{-1}.R$ in order to achieve to prove (7.2). So $W$ is a set of representatives of certain open $(R, H)$-cosets.

Reciprocally, let $RxH$ be an open $(R, H)$-double coset in $G$. We may assume that $x$ is $A_\sigma$-good. Let $Q = R \cap M$. Let us show that there is $x' \in W$ with $RxH = Rx'H$. First there exists $y \in W_M^G$ such that $PxH = PyH$. Let us assume that $y = 1$. Then, by (7.1), one has $x = mh$ with $m \in M, h \in H$. Hence one can assume $x \in M$. As $x \in M$ is $A_\sigma$-good, $Qx(H \cap M)$ is open in $M$ and $Qx(H \cap M) = Qx'(H \cap M)$ with $x' \in W_L^M$. Then $RxH = Rx'H$ as wanted. In general one changes
P to \( P' = y^{-1}P, \) \( R \) to \( R' = y^{-1}R. \) Hence one has \( P'y^{-1}xH = P'H, \) as \( PxH = PyH. \) Then one uses our last result. \( \square \)

7.2. \( \sigma \)-exponents of \( j_{P'}-\xi. \)

**Definition 7.3.** Let \((\pi, V)\) be a smooth representation of \( G \) of finite type. Then it is a finite direct sum of generalized eigenspaces under \( A_{G, \sigma} := (A_G)_{\sigma}. \) If \( \nu \) is a character of \( A_{G, \sigma}, \) let us denote by \( V(\nu) \) the corresponding generalized eigenspace of \( V \) and by \( \xi(\nu) \) the restriction to \( V(\nu) \) of any element \( \xi \) of \( \sigma \), which might be extended to an element of \( V' \), which is zero on the other generalized eigenspaces denoted also \( \xi(\nu). \) If \( \xi \in V''H, \) \( \text{Exp}(\xi) \) will denote the subset of \( \nu \) such that \( \xi(\nu) \) is non zero. The elements of \( \text{Exp}(\xi) \) are called the \( A_{G, \sigma} \)-exponents or \( \sigma \)-exponents of \( \xi. \)

**Theorem 7.4.** Let \( P = MU \) (resp., \( P' = M'U' \)) be a \( \sigma \)-parabolic subgroup of \( G \) and let \( A \) (resp., \( A' \)) be a maximally \( \sigma \)-stable \( \sigma \)-maximal split torus of \( M \) (resp., \( M' \)). Let \((P'\setminus G/P)_{\sigma} \) be the set of \((P', P)\)-double cosets in \( G \) having a representative \( w \) such that \( w.A = A', w.A_{\sigma} = A'_{\sigma}. \) We denote by \( W(M'\setminus G/M)_{\sigma} \) a set of representatives of \((P'\setminus G/P)_{\sigma} \) with this property and we assume that \( W(M'\setminus G/M)_{\sigma} \subset W(P'\setminus G/M). \)

Let \((\delta, E)\) be an irreducible smooth representation of \( M, \eta \in E'(\delta, H). \) Let \( O' \) be as in Theorem 4.10, with \( X = X(M)_{\sigma}. \) Let \( O_0 \) be the open dense subset of \( X(M)_{\sigma} \) from Proposition 6.1 (ii). Let \( \chi \in O' \cap O_0 \) and let \( \xi = \xi(P, \delta, \chi, \eta). \)

(i) Let \( w \in W(M'\setminus G/M). \) Then if \( (\xi)_{P'-w} \) is not zero, one has \( w \in W(M'\setminus G/M)_{\sigma}. \)

(ii) If \( w \in W(M'\setminus G/M)_{\sigma}, Q = M \cap w^{-1}P' \) is a \( \sigma \)-parabolic subgroup of \( M' \) and \( L = M \cap w^{-1}M' \) is its \( \sigma \)-stable Levi subgroup. We introduce \( \mathcal{W}^G_L \) as in the preceding Lemma.

If \( y \in \mathcal{W}^G_M \) and \( z \in \mathcal{W}^{y^{-1}.M}_y, \) we define \( z' := yzy^{-1} \in M \) and \( x = yz. \) Then \( \delta'(z')\eta_y \) is \( M \cap x.H \)-invariant, \( Q \) is a \( \sigma_x \)-parabolic subgroup of \( M. \) Hence \( j_Q-\delta'(z')\eta_y \) is defined.

(iii) One writes \( E_Q = E^+_Q \oplus E^0_Q, \)

where \( E^+_Q \) is the sum over \( y \in \mathcal{W}^G_M \) and \( z \in \mathcal{W}^{y^{-1}.M}_y \) of the \( A_{L, \sigma} \)-weights space corresponding to the set of exponents of \( j_Q-\delta'(z')\eta_y, \) where \( z' = yzy^{-1} \) and \( E^0_Q \) is the sum over the other weights.

Then, for \( \chi \) element of an open dense subset, \( O'' \) of \( O_0 \cap O', \) the \( A_{M', \sigma} \)-exponents of \( \xi_{P'-w} \) are of the form \( (w\chi^+)_{A_{M', \sigma}} \) where \( \chi^+ \) is an \( A_{M', \sigma} \)-eigenvalue of \((E^+_Q)_{\chi}. \)
Proof. (i) If $\xi_{P^-w}$ is non zero, one has $w \in W(M'\backslash G/M)_w$ by Proposition 5.3. This proves (i).

Let us prove (ii). From (i), $w^{-1}P'$ is a $\sigma$-parabolic subgroup of $G$, as $w$ is $A'_\sigma$-good, which contains $A$. Hence $Q$ is a $\sigma$-parabolic subgroup of $M$.

Let $x,y,z$ as in the statement of (ii). The linear form on $E$, $\delta'(z')\eta_y$ is $M \cap x.H$-invariant as $\eta_y$ is $M \cap y.H$-invariant and $z'y = yz = x$.

By construction of $W_M$, $x$ is $A_\sigma$-good. Hence $Q$, $Q^-$ are opposite $\sigma_x$-parabolic subgroup of $M$ (cf. Lemma 2.2). This achieves to prove (ii).

One defines projector $p_Q^+$ and $p_Q^0$ of $E_Q$ onto $E_Q^+$ and $E_Q^0$ corresponding to the decomposition $E_Q = E_Q^+ \oplus E_Q^-$. This defines, by induction, projectors on the space $(i_P^Q((E_\chi)_Q)'$ that we will denote in the same way.

Notice that $P_{Q^-}$ is equal to $\tilde{P}_w$. With these conventions, we define for $\chi \in O_0 \cap O'$:

$$\xi_Q^+ = p_Q^+(\tilde{j}_Q^- \circ \xi).$$

One defines similarly $\xi_Q^0$. Then

$$\xi_{P^-w}^+ := \tilde{r}_M' \circ {^tA}(w,\tilde{P}_w, \tilde{P}'_w, w,j_Q\delta_\chi) \circ \lambda(w)\xi_Q^-. $$

is a well defined element of $V'_w$ where $V_w := v_{M'\cap w.P}(wE^+_\chi)$. We define similarly $\xi_{P^-w}^0$ so that one has:

$$\xi_{P^-w} = \xi_{P^-w}^+ + \xi_{P^-w}^0.$$ We will prove that $\xi_{P^-w}^0 = 0$. We first study the restriction of $\xi_Q^0$ on the open $(M' \cap w.P, M' \cap H)$-orbits. From the definition from Lemma 3.3, one sees that if $x = yz$ with $y \in W_M^G$, $z \in W_{y^{-1}}^{y^{-1}M}$,

$$\tilde{r}_L(\rho(x)\tilde{j}_Q^- \circ \xi) = j_Q^-(\tilde{r}_M(\rho(x)\xi)).$$

But $x = yz = z'y$ with $z' = yzy^{-1} \in M$ and, with the notations of (3.19), one has:

$$\tilde{r}_M(\rho(x)\xi) = f_\xi(x) = \delta'_\chi(z')f_\xi(y) = \chi(z')^{-1}\delta'(z')\eta_y$$

Hence, one has:

$$\tilde{r}_L(\rho(x)\tilde{j}_Q^- \circ \xi) = \chi(z')^{-1}j_Q^-(\delta'(z')\eta_y),$$

and:

$$\tilde{r}_L(\rho(x)\xi^0_Q) = 0.$$

From the preceeding Lemma, one sees that the support of $\xi_Q^0$ has an empty interior. By structural transport the same is true for $\lambda(w) \circ \xi^0_w$. Hence by Lemma 5.5 and Lemma 7.1, one sees that the support of $\xi_{P^-w}^0$ has an empty interior. By Proposition 6.1, applied to $M'$ and
\[
X' = \{(w\chi)|_{M'\cap w.M}\} \text{ instead of } G \text{ and } X, \xi^0_{P',-w} \text{ is equal to zero for } \chi \text{ element of an open dense subset, } O'' \text{ of } O_0 \cap O'. \text{ Hence one has: }
\]
\[
\xi_{P',-w} = \xi^+_{P',-w}.
\]
As \(\xi^+_{P',-w}\) is a linear form on \(i_{M'\cap w.P}^w(E^+_Q)^{\chi^0_{M'\cap w-1M'}}\), (iii) follows. \(\square\)

An \(H\)-form \(\xi\) on a smooth representation of \(G\), \((\pi,V)\) is said \(H\)-cuspidal if \(j_Q\xi = 0\) for all proper \(\sigma\)-parabolic subgroups of \(G\). We denote by \(V_{\text{cusp}}^\prime\) the space of cuspidal \(H\)-forms on \(V\).

We define
\[
E'(\delta,H)_{\text{cusp}} := \bigoplus_{x \in \mathcal{W}^G_{wM}} (E'_{M'\cap x,H})_{\text{cusp}}.
\]

**Theorem 7.5.** Let \(\eta \in E'(\delta,H)_{\text{cusp}}\). Let \(w \in \mathcal{W}(M'\setminus G/M)_{\sigma}\). With the notations of the preceding Theorem, let \(\chi \in O''\) and \(\xi = \xi(P,\delta,\chi,\eta)\).

(i) If \(\xi_{P',-w} \neq 0\), one has \(M \cap w^{-1}P' = M\).

(ii) If \(M \cap w^{-1}P' = M\), one defines \(\mathcal{W}^G_{w,M} := w\mathcal{W}^G_M\) whose elements are \(w.A_{\sigma} = A'_{\sigma}\)-good. One can choose \(\mathcal{W}^G_{w,M}\) such that for \(y' \in \mathcal{W}^G_{w,M}\), there exists a unique \(y \in \mathcal{W}^G_{w,M}\) and \(h \in M' \cap H\) with \(y' = yh\).

If \(\eta \in E'(w\delta,H), \rho_{M'\eta} := (\eta_{y'})_{y' \in \mathcal{W}^G_{w,M}}\), is an element of \((wE')'(w\delta,M' \cap H)\). With these notations
\[
\xi_{P',-w} = \xi(M' \cap w.P, w\delta,\chi,\rho_{M'\eta}B(\tilde{P}'_w, w.P, w\delta,\eta)),
\]
where the \(B\)-matrices are defined with \(\mathcal{W}^G_{w,M}\).

(iii) Let us assume that \(M' = M\) and \(A' = A\). Then \(B(P', P, \delta, \chi)\) is an element of \((E'(\delta,H))_{\text{cusp}}\). Hence \(B(P', P, \delta, \chi)\) restricts to an endomorphism of \(E'(\delta,H)_{\text{cusp}}\).

**Proof.** (i) Let us assume that \(M \cap w^{-1}P' \neq M\). From the definition of \(E'(\delta,H)_{\text{cusp}}\) and of \(E^+_Q\) one will see that \(E^+_Q\) is zero. In fact, as \(\eta_y\) is \(M \cap y.H\) cuspidal, one sees by a direct computation that \(\delta'(z')\eta_y\) is \(M \cap x.H\)-cuspidal and \(j_Q \delta'(z')\eta_y = 0\). This implies that \(E^+_Q = \{0\}\).

From the preceding Theorem, \(\xi_{P',-w} = 0\).

(ii) Let us assume \(M \cap w^{-1}P' = M\). Then the \(\sigma\)-stable Levi subgroup of \(P'_w\) is equal to \(w.M\).

If \(y' \in M'\) is \(A'_{\sigma}\)-good, \(P'_w y'H\) is open and \(P'_w y'H = P'_w yH\) for some element \(y\) of \(\mathcal{W}^G_{w,M}\). In particular \(y\) is \(A'_{\sigma}\)-good. From (7.1), one sees that \(y' = lyh\) with \(l \in M' \cap w.M, h \in H\). Changing \(y'\) to \(l^{-1}y'\), one may assume that \(y' = yh\) for some \(h \in M' \cap H\). This allows to choose \(\mathcal{W}^G_{w,M}\) as in (ii). From Theorem 4.10
\[
\xi_{P',-w} = \tilde{r}_{M'}^t A(w.P, \tilde{P}'_w, w(\delta,\chi)) \lambda(w) \xi.
\]
One has chosen $\mathcal{W}_w^G := w\mathcal{W}_M^G$ whose elements are $w.A_\sigma$-good. Then, one sees that $(wE)^{\prime}(w\delta, H)_{\text{cusp}} = E^{\prime}(\delta, H)_{\text{cusp}}$, as $(E^{\prime}M\cap H = ((wE)^{\prime})(wM)\cap (wx.H))$. Also one has $\lambda(w)\xi(P, \delta, \chi, \eta) = \xi(w.P, w(\delta), \eta)$.

Hence from the definition of $B$-matrices one has:

$$\xi_{P^{\prime}, w^{-1}} = \hat{r}_{M^{\prime}} \xi(P^{\prime}_w, w(\delta), B(P^{\prime}_w, w.P, w(\delta, \chi)), \eta)$$

In order to prove the equality of the Theorem for $\chi$ in an open dense subset of $X(M)$, it is enough, due to Proposition 6.1, to prove the equality of the values at every element, $y'$, of $\mathcal{W}_w^{M^{\prime}}$ of $\hat{r}_{M^{\prime}} \xi(P^{\prime}_w, w(\delta, \chi), B(P^{\prime}_w, w.P, w(\delta, \chi), \eta))$ and $\xi(M^{\prime} \cap w.P, w(\delta, \chi, \eta), \eta)$. This is easily seen from the definitions and from the fact that $y' = yh$ for an element $y$ of $\mathcal{W}_w^G$ and $h \in H \cap M^{\prime}$.

(iii) We take $w = 1$ in (ii). Hence, one has:

$$\xi_{P^{\prime}, 1} = B(P^{\prime}, P, \delta, \chi) \eta \in V(\delta, H)$$

From (i) and the transitivity of the constant term, one sees that $\xi_{P^{\prime}, 1} \in E^{\prime}(\delta, H)_{\text{cusp}}$ is $M \cap H$-cuspidal.

If $P = MU$ is a $\sigma$-parabolic subgroup of $G$, let us recall that $A_{M, \sigma}$ denotes $(A_M)_{\sigma}$. Let $\dot{a}_M^{\prime \prime}$ (resp. $\ddot{a}_M^{\prime \prime}$) be the set of $\lambda \in a_M^{\prime \prime}$ which are linear combinations of roots of $A_{M, \sigma}$ in the Lie algebra of $U$ with coefficients greater or equal to zero (resp., greater than zero).

**Definition 7.6.** Let $(\pi, V)$ be a finite length smooth representation of $G$ and $\xi$ an $H$-form on $V$. Then $\xi$ is said tempered (resp., square integrable) if and only for every $\sigma$-parabolic subgroup of $G$, $P = MU$, every exponent, $\chi$, of $\xi_{P^{-}}$ is such that $\text{Re}(\chi)$ is element of $\dot{a}_M^{\prime \prime}$ (resp. $\ddot{a}_M^{\prime \prime}$).

If $\xi$ is a tempered $H$-form we define its weak constant term

$$(7.3) \quad \xi_{P^{-}}^{w}(\chi) = \sum_{\chi \in \exp(\xi_{P^{-}}), \text{Re}(\chi) = 0} \xi_{P^{-}}(\chi)$$

Hence a square integrable $H$-form is a tempered $H$-form such that its weak constant term is zero for all proper $\sigma$-parabolic subgroups of $G$. Notice that Kato-Takano (cf. [KT2]) showed that, if $\pi$ is irreducible and has a unitary central character, an $H$-form is square integrable if and only its generalized coefficients are square integrable modulo the center.

**Lemma 7.7.** (i) If $\xi$ is a tempered $H$-form on $V$, $\xi_{P^{-}}^{w}$ is a tempered $M \cap H$-form on $V_P$. 
(ii) If $Q$ is $\sigma$-parabolic subgroups of $M$ and $R = QU$, one has:

$$\xi_R^w = (\xi_{P^-}^w)_Q$$

Proof. (i) and (ii) follow from the transitivity of the constant term (cf. [D3]).

If $(\delta, E)$ is a smooth unitary irreducible representation of $M$, let $E'_2(M \cap H)$ be the space of square integrable $M \cap H$-forms on $E$ and let:

$$E'(\delta, H)_2 = \oplus_{x \in W^G/M} E'_2(M \cap x.H).$$

Theorem 7.8. With the notations of the Theorem 7.4, let us assume $\delta$ is unitary, $\eta \in E'(\delta, H)_2$ and $\chi \in O'' \cap X(M)_u$. Then one has:

(i) The $H$-form $\xi(P, \delta \chi, \eta)$ is tempered.

(ii) If $w \in W(M' \setminus G/M)$ is not in $W(M' \setminus G/M)_\sigma$ or if $M \cap w^{-1}P'$ is distinct from $M$, then $\xi_{P'^{-1}, w} = 0$. Otherwise, with the notations of the preceeding Theorem (ii):

$$\xi_{P'^{-1}, w} = \xi(M' \cap w.P, w\delta \chi, p_M.B(\tilde{P}'_w, w.P, w\delta \chi)\eta),$$

where the $B$-matrices are defined with $W^G_w.M$.

(iii) Let us assume that $M' = M$ and $A' = A$. Then $B(P', P, \delta \chi)$ restricts to an endomorphism of $E'(\delta, H)_2$.

Proof. Let us use the notations of Theorem 7.4 (ii). Using the criteria of square integrability of Kato and Takano (see above), one sees by “transport de structure” that $\delta'(z')\eta_y$ is square integrable. Moreover $Q$ is a $\sigma_x$-parabolic subgroup of $M$. Then (i) follows from Theorem 7.4 (iii) and from our definition of square integrable forms.

The proof of (ii)(resp., (iii)) is analogous to the proof of (i) and (ii) (resp., (iii) ) of the preceeding theorem.

8. Constant term of Eisenstein integrals

Let $P = MU$ be a $\sigma$-parabolic subgroup of $G$. If $f$ is a smooth function on $H \setminus G$ the constant term of $f$ along $P$, $f_P$, has been defined in [D3], section 3.3. It is a smooth function on $M \cap H \setminus M$.

If $(\pi, V)$ is a smooth representation of $G$, $\xi$ is an $H$-form on $V$ and $v$ is an element of $V$, we denote by $c_{\xi, v}$ the generalized coefficient defined by:

$$c_{\xi, v}(Hg) = \langle \xi, \pi(g)v \rangle$$

Then (cf. [D3], Proposition 3.13) one has:

$$c_{\xi, v}(P) = c_{\xi_{P^-}, v_P},$$

where $\xi_{P^-} = j_Q \cdot \xi, v_P = j_P v$.
The space of smooth functions on \( f \) deduced from the constant term \( \xi_{P-} \).

**Definition 8.1.** We define \( \mathcal{A}_{temp}(H \backslash G) \) to be the set of functions of the type \( c_{\xi,v} \) for a finite length smooth representation \( (\pi, V) \) of \( G \) and a tempered \( H \)-form on \( V \). It is easily seen to be a vector subspace of the space of smooth functions on \( H \backslash G \). If \( f \) is such a generalized coefficient \( f_{P}^{w} \), then \( f_{P}^{w} \) will denote the generalized coefficient \( c_{\xi,v_{-},v_{-}} \). It is naturally deduced from the constant term \( f_{P} \) like in the definition of \( \xi_{P-} \), hence it does not depend of the presentation of \( f \) as a generalized coefficient.

**Definition 8.2.** Let \( P = MU \) be a \( \sigma \)-parabolic subgroup of \( G \) and let \( (\delta, E) \) be an irreducible smooth representation of \( M \). Let \( \eta \in E'(\delta, H) \) such that \( \xi(P, \delta, \eta) \) is defined. Then if \( v \in i_{P}^{E}E_{P} \), one defines an element of \( C^{\infty}(H \backslash G), E_{P}^{G}(\eta \otimes v) \) by:

\[
E_{P}^{G}(\eta \otimes v)(g) = \langle \xi(P, \delta, \eta), i_{P}^{E} \sigma(g)v \rangle, g \in G.
\]

Then, from Proposition 6.1, there exists a non zero polynomial function on \( X(M)_{\sigma}, b \), such that for all \( v \in i_{K \cap P}^{K}E \), \( \eta \in E'(\delta, H) \), the map \( \chi \mapsto b(\chi)E_{P}^{G}(\eta \otimes v_{\chi})(g) \) is polynomial in \( \chi \in X(M)_{\sigma} \). By bilinearity, we define similarly \( E_{P}^{G}(\phi) \) where \( \phi \) is element of \( E'(\delta, H) \otimes i_{P}^{E}E \).

**Lemma 8.3.** Let \( W(M'|G|M)_{\sigma} \) the set of elements of \( W(M'|G|M)_{\sigma} \) such that \( w.M \subset M' \). Let \( w \) be an element of \( W(M'|G|M)_{\sigma} \). For \( \chi \) element of an open dense subset of \( X(M)_{\sigma} \) one has, with the notations of (4.17):

\[
\gamma_{\chi,w} = \alpha_{\chi,w}.
\]

**Proof.** If \( W(M'|G|M)_{\sigma} \) is empty, there is nothing to prove. Otherwise let \( s \in W(M'|G|M)_{\sigma} \). It is enough, by “transport de structure”, to prove the result for \( s^{-1}.P' \), as the Jacquet modules for \( P' \) and \( s^{-1}.P' \) of \( i_{P}^{E} \delta_{\chi} \) are canonically isomorphic. So we may assume that \( 1 \in W(M'|G|M)_{\sigma} \) and \( A' = A \). By Lemma 4.7 applied to \( M' \) instead of \( G \), one sees that for \( \chi \) element of a dense subset of \( X(M)_{\sigma}, V_{\chi,w} \) is irreducible for all \( w \in W(M'|G|M)_{\sigma} \). Hence \( \gamma_{\chi,w} = \gamma(P, w, \chi)\alpha_{\chi,w} \) for an element \( \gamma(P, w, \chi) \) of \( C^{*} \). The proof of the Lemma is then identical to the proof of Proposition V.1.1 in [Wal] (see Equations V.1 (2), (3) and (4)), where parabolic subgroups have to be replaced by \( \sigma \)-parabolic subgroups.

**Theorem 8.4.** We keep the notations of Theorem 7.4. Let \( \chi \) be an element of the dense open subset of \( X(M)_{\sigma}, O' \). If \( w \in W(M'|G|M)_{\sigma} \), one defines a linear map \( C(w, P', P, \delta_{\chi}) \) from \( E'(\delta, H) \otimes i_{P}^{E}E_{\chi} \) in \((wE)'(w\delta, M' \cap H) \otimes i_{P}^{E}E_{\chi} \) by:

\[
C(w, P', P, \delta_{\chi}) = p_{M'}B(P'_{w}, w.P, \delta_{\chi}) \otimes (A(P'_{w}, w.P, w(\delta_{\chi})\lambda(w))).
\]
Then, if $\phi \in E'(\delta, H)_{\text{cusp}} \otimes i_P^* E_X$ (resp., $E'(\delta, H)_2 \otimes i_P^* E_X$ and $\delta$ and $\chi$ are unitary), one has:

$$E_P^G(\phi)_{P'} = 0 \quad (\text{resp., } E_P^G(\phi)_{w_{P'}} = 0)$$

if $W(M'|G|M)_{\sigma}$ is empty. Otherwise $E_P^G(\phi)_{P'}$ (resp., $E_P^G(\phi)_{w_{P'}}$) is equal to:

$$\sum_{w \in W(M'|G|M)_{\sigma}} E_{M'\cap w.P}^{M'(C(w, P', P, \delta_P \chi))}(\phi).$$

Proof. By $M'$-equivariance, it is enough to prove the equalities of the Proposition evaluated at $m' = 1$. Then the result follows from the property (8.1) (resp., from the definition 8.1) together with Theorem 7.4 (resp., Theorem 7.8).

\section{Appendix}

\subsection{Covariant distributions on an homogeneous space}

Let $X$ be a totally disconnected topological space and let $V$ be a complex vector space. We denote by $C_c^\infty(X)$ (resp., $C_c^\infty(X, V)$) the space of locally constant functions on $X$ with compact support and with values in $\mathbb{C}$ (resp., $V$). Notice that $C_c^\infty(X, V)$ identifies with $C_c^\infty(X) \otimes V$. We denote by $\mathcal{D}'(X, V)$ the space of linear forms on $C_c^\infty(X, V)$ which are called $V$-distributions on $X$. The support of a $V$-distribution on $X$ is the complementary subset of the largest open subset $O$ such that $T$ restricted to $C_c^\infty(O) \otimes V$ is equal to zero.

Let $F$ be a closed subset of $X$ and let $O$ denote $X \setminus F$. From the exact sequence:

$$0 \rightarrow C_c^\infty(O) \otimes V \rightarrow C_c^\infty(X) \otimes V \rightarrow C_c^\infty(F) \otimes V \rightarrow 0,$$

one sees that if $T$ has support contained in $F$, then $T$ defines a $V$-distribution on $F$ which is called the distribution induced on $F$ by $T$.

Let $(\rho, V)$ be a representation of a group $J$. Recall that $H_0(J, V)$ denotes the quotient of $V$ by the subspace generated by the elements of the form $\rho(j)v - v, j \in J, v \in V$. The dual of $H_0(J, V)$ identifies with the space of $J$-fixed linear forms on $V$.

Let $G$ be a totally disconnected locally compact group and let $H$ be a closed subgroup of $G$. Let $(\pi, E)$ be a smooth representation of $G$. One denotes by $\text{ind}_H^G \pi$ the right regular representation of $G$ in the space $\text{ind}_H^G E$ of functions, $f$, from $G$ to $V$, left invariant by a compact open subgroup of $G$, with compact support modulo $H$ and such that:

$$f(hg) = \pi(h)f(g), h \in H, g \in G.$$

Let $X$ be a totally disconnected space on which $G$ acts continuously. A $V$-distribution on $X$, $T$, is said $\pi$-covariant if:

$$T(f - \pi(g)L_g f) = 0, f \in C_c^\infty(X) \otimes V.$$
Lemma 9.1. Let \( T \) be a \( \pi \)-covariant distribution on \( G/H \). Let us denote by \( d_l g \) (resp., \( d_l h \)) a left invariant Haar measure on \( G \) (resp \( H \)). Let us denote by \( \delta_G \) the modulus function of \( G \), which satisfies:
\[
\int_G f(gg_0) d_l g = \delta_G(g_0) \int_G f(g) d_l g, \quad f \in C_c(G), \quad g_0 \in G.
\]
We define a linear map from \( C_c^\infty(G) \otimes V \) to \( C_c^\infty(G/H) \otimes V \) by:
\[
M_H f(gH) := \int_H f(gh) d_l h, \quad f \in C_c^\infty(G) \otimes V.
\]
Then there exists a unique \( \eta \in V' \) such that
\[
\langle T, M_H f \rangle = \int_G \langle \pi'(g) \eta, f(g) \rangle d_l g, \quad f \in C_c^\infty(G) \otimes V.
\]
The linear form \( \eta \) will be called the value at \( 1 \) of \( T \) and denoted \( \text{ev}_1 T \) or \( T(1) \). Moreover:
\[
\pi'(h) \eta = \delta_H^{-1}(h) \delta_G(h) \eta, \quad h \in H.
\]
If \( G/H \) has a left invariant measure by \( G \), one has
\[
\langle T, f \rangle = \int_{G/H} \langle \pi'(g) \eta, f(gH) \rangle d gH, \quad f \in C_c^\infty(G/H) \otimes V.
\]

Proof. Let us assume first that \( H = \{1\} \). We remark that \( T \in (H_0(G, C_c^\infty(G) \otimes V))' \) where \( G \) acts on \( C_c^\infty(G) \otimes V \) by the tensor product \( L \otimes \pi \) of the left regular representation with \( \pi \). From [BD] Prop. 1.13 (iv), one sees:
\[
\text{(9.1) The map } f \in C_c^\infty(G) \otimes V \mapsto \int_G \pi(g^{-1}) f(g) d_l g \text{ goes through the quotient to an isomorphism of } H_0(G, C_c^\infty(G) \otimes V) \text{ with } V.
\]
Hence \( T \) defines \( \eta \in V' \) by “transport de structure”. One sees that \( T \) verifies:
\[
\text{(9.2) } \langle T, f \rangle = \int_G \langle \pi'(g) \eta, f(g) \rangle d_l g, \quad f \in C_c^\infty(G, V).
\]
which proves our claim when \( H = \{1\} \).

In general, we introduce a \( V \)-distribution on \( G \), \( \tilde{T} \), by
\[
\langle \tilde{T}, f \rangle = \langle T, M_H f \rangle, \quad f \in C_c^\infty(G) \otimes V.
\]
We can apply to \( \tilde{T} \), which is \( \pi \)-covariant, the first part of the proof. Now one has
\[
M_H R_h f = \delta_H(h) M_H f \in C_c^\infty(G) \otimes E,
\]
which implies
\[ \langle \tilde{T}, R_h f \rangle = \delta_H(h) \langle \tilde{T}, f \rangle, \quad f \in C_c^\infty(G) \otimes V. \]

Hence, it follows from (9.2) applied to \( \tilde{T} \) that
\[ \langle \tilde{T}, R_h f \rangle = \delta_H(h) \int_G \langle \eta, \pi(g)^{-1} f(g) \rangle \, d_l g. \]

But, using again (9.2), one has:
\[ \langle \tilde{T}, R_h f \rangle = \int_G \langle \pi'(g) \eta, f(gh) \rangle \, d_l g = \delta_G(h) \int_G \langle \pi'(g) \pi'(h^{-1}) \eta, f(g) \rangle \, d_l g. \]

From the preceding equalities one deduces
\[ \delta_H(h) \int_G \langle \eta, \pi(g)^{-1} f(g) \rangle \, d_l g = \delta_G(h) \int_G \langle \pi'(h^{-1}) \eta, \pi(g)^{-1} f(g) \rangle \, d_l g. \]

Then (9.1) implies:
\[ \pi'(h) \eta = \delta_H^{-1}(h) \delta_G(h) \eta, \quad h \in H. \]

\[ \square \]

**9.2. Bernstein’s parameters of finite length smooth modules.**

The Bernstein’s center \([\text{DeliBe}], ZB(G)\), identifies with an algebra of functions on the set, \( \Omega(G) \), of \( G \)-conjugacy classes of cuspidal pairs i.e. pairs \((L, \omega)\), where \( L \) is a Levi-subgroup of \( G \) and \( \omega \) is a smooth, irreducible cuspidal representation of \( L \). Here cuspidal means that the smooth coefficients of the representation are compactly supported modulo the center.

If \((L, \omega)\) is such a pair, we denote by \((L, \omega)_G\) its conjugacy class under \( G \). If \( \Lambda \in \Omega(G) \), we denote by \( \chi_\Lambda \) the character of \( ZB(G) \) given by the evaluation at \( \Lambda \) and \( I_\Lambda \) the kernel of \( \chi_\Lambda \). It is a maximal ideal of \( ZB(G) \). We say that \( \chi_\Lambda \) has Bernstein parameter \( \Lambda \).

Let \((\pi, V)\) be smooth \( G \)-module of finite length. We say that \( \{\Lambda_1, \ldots, \Lambda_p\} \subset \Omega(G) \) is the set of Bernstein’s parameters of \((\pi, V)\), if \( V \) splits as direct sum of \( G \)-modules \( V_1 \oplus \cdots \oplus V_p \) such that \( V_i \) is non reduced to \( \{0\} \) and is annihilated by a power of the ideal \( I_{\Lambda_i} \) of \( ZB(G) \). Then one sees easily:

If \( \chi \) is an unramified character of \( G \) and if the set of Bernstein’s parameter of \((\pi, V)\) is equal to \( \{(L_1, \omega_1)_G, \ldots, (L_p, \omega_p)_G\} \), the set of Bernstein’s parameter of \((\pi \otimes \chi, V)\) is equal to \( \{(L_1, \omega_1 \otimes \chi|_{L_1})_G, \ldots, (L_p, \omega_p \otimes \chi|_{L_p})_G\} \).

Let \( P \) be a parabolic subgroup of \( G \) with Levi subgroup \( M \) and let \((\delta, E)\) be a smooth representation of \( M \) with set of Bernstein’s parameters \( \{L_1, \omega_1\}_M, \ldots, (L_p, \omega_p)_M \} \), where \((L_i, \omega_i)\) is a cuspidal pair for \( M \).
Then, one has:

\[(9.4) \text{ The set of Bernstein's parameters of } (i^* E, i^*_P \delta) \text{ is } \{ (L_1, \omega_1)_{G}, \ldots, (L_p, \omega_p)_{G} \}. \]

The following is an immediate consequence of the splitting of the category of smooth modules (cf. [Be], [R])

\[(9.5) \text{ If one has a short exact sequence } 0 \to V_1 \to V_2 \to V_3 \to 0 \text{ of finite length smooth } M'-\text{modules such that } V_1 \text{ and } V_3 \text{ have disjoint sets of Bernstein's parameters, then } V_2 \text{ is isomorphic to the direct sum of } V_1 \oplus V_3. \]

9.3. Some results on \(\sigma\)-parabolic subgroups. In this subsection, we change slightly the notations of the main body of the article.

**Lemma 9.2.** Let \( G \) be a connected algebraic group acting over the non empty variety \( X \).

(i) Let \( x \in X. \) Then \( Y = Gx \) is a smooth locally closed subset of \( X. \)

(ii) There exists at least one closed orbit.

(iii) Let \( G_x \) be the stabilizer of \( x \) in \( G. \) Then \( \dim Y = \dim G - \dim G_x. \)

**Proof.** (i) and (ii) follows from [Hu], Proposition 8.3.

(iii) The morphism \( G \to G.x \) is dominant: this morphism is surjective and \( G \) is irreducible, hence \( G.x \) is also irreducible and our claim follows from the discussion in the middle of [Bo], Ch. AG. 8.2. Then, the assertion on dimensions follows from [Bo], Ch. AG. 10.1, with \( X = G, Y = G.x, W = x, Z \) the neutral component of \( G_x. \) \( \square \)

**Lemma 9.3.** Let \( G \) be a connected algebraic group acting over an irreducible nonsingular variety \( X, \) with a finite number of orbits.

If an orbit, \( X', \) of \( G \) in \( X \) has the same dimension than \( X, \) then, \( X' \) is open in \( X. \)

**Proof.** We use induction on the number of orbits. If this number is 1, our statement is clear. Otherwise, if this number is greater than 1, let \( Y \) be a closed orbit in \( X. \) Then \( Y \) is not equal to \( X. \) As \( G \) is connected, \( Y \) is irreducible. It follows from [Hu], Proposition 3.2, that \( \dim Y < \dim X. \) Then \( X \setminus Y \) contains \( X' \) and is irreducible as \( Y \) is closed and \( X \) is irreducible. Moreover the action of \( G \) on \( X \) induces an action on \( X \setminus Y. \) One applies the induction hypothesis. \( \square \)

Let \( G \) be a connected reductive group defined over a local field \( k \) of characteristic different from 2. Let \( P, Q \) be two parabolic subgroups of \( G \) defined over \( k. \) Let \( G_k \) be the set of \( k \)-points of \( G. \) We have similar
notations for subgroups of $G$.

Let us show:

The $k$-parabolic subgroups of $G$, $P$ and $Q$ are opposed if and only if $P_k \cap Q_k$ is equal to $M_k$ where $M$ is a common Levi to $P$ and $Q$.

If $P$ and $Q$ are opposed $P \cap Q$ is equal to their common Levi subgroup, $M$, and it is clear that $P_k \cap Q_k = M_k$. Reciprocally if $P_k \cap Q_k = M_k$, then $P \cap Q$ contains the Zariski closure of $M_k$ which is equal to $M$ by [Bo], Corollary 18.3. By looking to the $k$-parabolic subgroups of $G$ with Levi subgroup $M$, one sees that only the $k$-parabolic subgroup opposed to $P$ satisfies $P_k \cap Q_k = M_k$. This proves our claim.

Hence if $\sigma$ is an involution of $G$ defined over $k$ and $P$ is a parabolic subgroup of $G$ defined over $k$, $P$ is a $\sigma$-parabolic of $G$ if and only $P_k$ is a $\sigma$-parabolic subgroup of $G_k$.

Let us show:

Let $P$ a $\sigma$-parabolic subgroup of $G$, $p + h = g$, where $g$ (resp., $p$) is the Lie algebra of $G$ (resp., $P$) and $h$ is the Lie algebra of the fixed point group of $\sigma$.

As $P$ is a $\sigma$-parabolic subgroup of $G$, $p + \sigma(p) = g$. Hence any $X \in g$ is of the form to $Y + \sigma(Z)$ with $Y, Z \in p$. Hence, one has:

$$X = Y - Z + (Z + \sigma(Z)).$$

The result follows from the fact that $h$ is equal to the fixed point set of $\sigma$ in $g$ (cf. [Ri], proof of Lemma 2.4). Let $H$ be an open subgroup, defined over $k$, of the fixed points group of $\sigma$. We will show:

Let $P$ be a $\sigma$-parabolic subgroup of $G$ defined over $k$.

Then:

9.8 a) $HP$ is open in $G$.

b) $H_kP_k$ is open in $G_k$

The assertion a) follows from [HW] Lemma 4.8. The assertion b) reduces to the case where $P$ is a minimal $\sigma$-parabolic subgroup of $G$ defined over $k$. In that case it follows from [HW], Definition 13.1 and Proposition 13.4.

Lemma 9.4. Let $P$ be a $\sigma$-parabolic subgroup of $G$ defined over $k$.

Let $x \in G_k$.

The following conditions are equivalent:

(i) $H_kxP_k$ is open in $G_k$.

(ii) $HxP$ is open in $G$.

(iii) $x \in HP$.

(iv) $xPx^{-1}$ is a $\sigma$-parabolic subgroup of $G$ defined over $k$. 
Proof. (i) implies (ii): Let $P'$ be a minimal $k$-parabolic subgroup of $G$ contained in $P$. There are finitely many $(H_k, P'_k)$-double cosets in $G_k$ (cf. [HW], Corollary 6.16). Hence $H_k x P_k$ contains an open $(H_k, P'_k)$-double coset, $H_k x' P_k$ with $x' = x p$ and $p \in P_k$. From [HH], Proposition 3.5, one sees that $x'.P'$ is contained in a minimal $\sigma$-parabolic subgroup of $G$, hence $H x' P'$ is open in $G$ by [HW], Lemma 4.8. Hence (i) implies (ii).

(ii) implies (iii): as the complementary set of an open $(H, P)$-double coset in $G$ is closed and as $G$ is connected, hence irreducible, there is only one open $(H, P)$-double coset in $G$. From (9.8) a), one knows that $H x' P'$ is open in $G$ by [HW], Lemma 4.8. Hence (ii) implies (iii).

(iii) implies (iv) because the conjugacy by an element of $H$ of preserves the set of $\sigma$-parabolic subgroups of $G$.

(iv) implies (i) follows from (9.8) b). □

10. COMPACT OPEN SUBGROUPS OF LIE GROUPS OVER $F$

Let $G$ be a Lie group over $F$ in the sense of [Bou] Ch.III.1, Definition 1..and let $g$ be its Lie algebra.

We will use an idea given by Deligne in [DeliBe], top of p.16. We fix an analytic bijective map $\psi : V \to W$ between an open neighborhood $V$ of 0 in $g$ to an open neighborhood, $W$, of 1 in $G$. We assume that its differential at 0 is the identity. Such a map will be called a good chart at 1 for $G$.

Lemma 10.1. Let $\mathfrak{P}$ be the the maximal ideal of the ring of integers, $\mathcal{O}$ of $F$. We fix a basis $(X_i)$ of $g$.

(i) Let $\Lambda_n g$ (resp. $\Lambda g$) be the $\mathfrak{P}^n$-module ( resp $\mathcal{O}$-module) generated by the $X_i$. It is a basis of neighborhoods of 0 in $g$. Let $J_n$ be the image of $\Lambda_n g$ by $\psi$, which is defined for $n$ large enough.

For $n$ large enough, $J_n$ is a compact open subgroup of $G$.

(ii) If $J'_n$ is defined with another good chart $\psi'$, one has $J_n = J'_n$ for $n$ large enough.

(iii) Let $\Theta$ be a family of automorphisms of the Lie group G whose differential preserves $\Lambda g$. Then it preserves $\Lambda_n g$ for all $n \in \mathbb{N}^*$. We assume moreover that $\Theta$ has the structure of compact analytic manifold over $F$ and that the map $\Theta \times G \to G$, $(\theta, g) \mapsto \theta(g)$ is analytic. Then for $n$ large enough, $J_n$ is invariant by every $\theta \in \Theta$.

(iv) Let us assume that we are given three closed Lie subgroups of $G$, $G_1$, $G_2$, $G_3$ and vector subspaces of $g$, $g_i \subset g_i$, $i = 1, 2, 3$, such that $\Lambda g = \Lambda g \cap g'_1 \oplus \Lambda g \cap g'_2 \oplus \Lambda g \cap g'_3$. Then for $n$ large enough one has:

$$J_n = (J_n \cap G_1)(J_n \cap G_2)(J_n \cap G_3).$$
Proof. (i) The fact that $J_n$ is compact and open follows from the fact that $\Lambda_n g$ is compact and open. We choose $n$ large enough so that $J_n \subset \overline{W}$.

Denote by $x_j$ the $j$-th coordinate map on $g$. We define:

$$|X| = \text{Sup}_j |x_j(X)|_{\mathfrak{F}}.$$

Let $X, X' \in \Lambda_n g$, and let us study $x_j(\psi^{-1}(\psi(X)\psi(X')))$. By our hypothesis on $\psi$ and from the formula of the differential of the product in $G$, one sees that the differential of this map of $(X, X')$ is simply $(X, X') \mapsto x_j(X) + x_j(X')$. The definition of the differential shows that:

$$x_j(\psi^{-1}(\psi(X)\psi(X'))) = x_j(X) + x_j(X') + \text{Sup}(|X|, |X'|)\varepsilon(X, X'),$$

where $\varepsilon(X, X')$ tends to zero if $(X, X')$ tends to $(0, 0)$. Let $n_0$ be large enough such that $|\varepsilon(X, X')| < 1$ for $X, X'$ in $\Lambda_n g$. One deduces from the above equality that for $n \in \mathbb{N}$ larger than $n_0$, $jj' \in K_n$ for all $j, j' \in J_n$. One sees similarly that $j'^{-1} \in J_n$ if $j \in J_n$ and $n$ is larger than $n_0$. Then (i) follows.

(ii) One proceeds as in (i), by considering the map $X \mapsto x_j(\psi^{-1}(\psi(\varepsilon(X))^{-1}))$, whose differential at 0 is equal to zero. Arguing as in (i), for $n$ large one sees that $\psi(X)(\psi'(X))^{-1} \in J_{n+1}$. Hence if $j \in J_n$, we have found $j' \in J_n$ such that $j(j')^{-1} \in J_{n+1}$. Using that $J_n'$ is a group, and proceeding inductively, we find a sequence $(j'_p)$ in $J_n'$ such that $j(j'_p)^{-1} \in J_{n+p}$. Hence $j'_p$ converges to $j$. But $J_n'$ is compact, hence $j \in J_n'$ and $J_n \subset J_n'$. The reverse inclusion is proved similarly. This proves (ii).

(iii) We denote also by $\theta$ the differential at 1 of an element $\theta$ of $\Theta$. Shrinking $V$ if necessary, let us consider the analytic map from $V \times \Omega$ to $g$, $(\theta, X) \mapsto \psi^{-1}[\theta^{-1}(\psi(\theta(X)))])$, whose partial derivative in $X$ at any element $(0, \theta)$ is equal to zero. One fixes $\theta_0 \in \Theta$. Proceeding as above, i.e. using the differential of our map at $(0, \theta_0)$, we find a neighborhood of $\theta_0$, $V_{\theta_0}$, such for $n$ large enough, all the elements of $V_{\theta_0}$ preserve $J_n$. Then (iii) follows from the compactness of $\Theta$.

(iii) Let $\psi_i$ be a good chart at zero of $G_i$, $i = 1, 2, 3$. From the first part of the Lemma, one can use the following function $\psi$ to study $J_n$:

$$\psi(Y_1 + Y_2 + Y_3) = \psi_{g_1}(Y_1)\psi_{g_2}(Y_2)\psi_{g_3}(Y_3), Y_i \in g_i', i = 1, 2, 3.$$

From the definition of $J_n$ it is clear that

$$J_n \subset (J_n \cap G_1)(J_n \cap G_2)(J_n \cap G_3).$$

The reverse inclusion being clear this proves the Lemma. \qed
Lemma 10.2. If $G \subset GL(n)$ is a linear algebraic group defined over $\mathbb{F}$, the group $G = G(\mathbb{F})$ has a structure of analytic Lie group, whose Lie algebra $\mathfrak{g}$ is the Lie algebra of $\mathbb{F}$-points of the Lie algebra of $G$.

Proof. Let $f_1 = 0, \ldots, f_p = 0$ be a set of equations, with coefficients in $\mathbb{F}$ defining $G$. One defines the analytic structure on $G$ by applying the constant rank theorem to the map $GL(n, \mathbb{F}) \to \mathbb{F}^p, g \mapsto (f_1(g), \ldots, f_p(g))$. It has the required property. \hfill \square

11. References

[Ba], van den Ban E.P., The principal series for a reductive symmetric space. I. $H$-fixed distribution vectors. Ann. Sci. Ecole Norm. Sup. (4) 21 (1988), 359–412.

[Be] Bernstein J., Second adjointness Theorem for representations of reductive p-adic groups, unpublished manuscript.

[BD] Blanc P., Delorme P., Vecteurs distributions $H$-invariants de représentations induites pour un espace symétrique réductif p-adique $G/H$, Ann. Inst. Fourier, 58 (2008), 213–261.

[Bo] Borel A. Linear algebraic groups. Second edition. Graduate Texts in Mathematics, 126. Springer-Verlag, New York, 1991.

[BoTi] Borel A., Tits J., Groupes réductifs. Inst. Hautes Etudes Sci. Publ. Math. 27 (1965) 55–150.

[Bou] Bourbaki, Groupes et algèbres de Lie. Chapitres 2 et 3, Eléments de Mathématique XXXVII, Hermán, Paris 1972.

[CarD] Carmona J., Delorme P., Transformation de Fourier sur l’espace de Schwartz d’un espace symétrique réductif, Invent. Math. 134 (1998), 59-99.

[C] Casselman W., Introduction to the theory of admissible representations of p-adic reductive groups, http://www.math.ubc.ca/~cass/research.html.

[DeliBe] Deligne P., Le “centre” de Bernstein rédigé par Pierre Deligne. Travaux en Cours, Representations of reductive groups over a local field, 1–32, Hermann, Paris, 1984.

[D1] Delorme P., Intégrales d’Eisenstein pour les espaces symétriques réductifs: tempérance, majorations. Petite matrice $B$, J. Funct. Anal. 136 (1996), 422–509.

[D2] Delorme P., The Plancherel formula on reductive symmetric spaces from the point of view of the Schwartz space. in Lie theory, AA, 135–175, Progr. Math., 230, Birkhauser Boston, Boston, MA, 2005.

[D3] Delorme P., Constant term of smooth $H \psi$-invariant functions, Trans. Amer. Math. Soc. 362 (2010), 933–955.
[D4] Delorme P., Théorème de Paley-Wiener pour les fonctions de Whittaker sur un groupe réductif $p$-adique, to appear in J. Inst. Math. Jussieu.

[D5] Delorme P., Formule de Plancherel pour les fonctions de Whittaker sur un groupe réductif $p$-adique, to appear in Ann. Inst. Fourier.

[Hei] Heiermann, V., Une formule de Plancherel pour l’algèbre de Hecke d’un groupe réductif $p$-adique. Comment. Math. Helv. 76 (2001), 388–415.

[HH] Helminck A.G., Helminck G.F., A class of parabolic $k$-subgroups associated with symmetric $k$-varieties. Trans. Amer. Math. Soc. 350 (1998) 4669–4691.

[HW] Helminck A. G., Wang S. P., On rationality properties of involutions of reductive groups. Adv. Math. 99 (1993) 26–96.

[KT1] Kato S., Takano K., Subrepresentation theorem for $p$-adic symmetric spaces, Int. Math. Res. Not. IMRN 2008, no. 11

[KT2] Kato S., Takano K., Square integrability of representations on $p$-adic symmetric spaces. J. Funct. Anal. 258 (2010)1427-1451.

[Hu] Humphreys J. E, Linear algebraic groups, Graduate Text In Math. 21, Springer, 1981.

[KnSt] Knapp A., Stein E., Intertwining operators for semisimple groups. II. Invent. Math. 60 (1980) 984

[L], Lagier N., Terme constant de fonctions sur un espace symétrique réductif $p$-adique, J. of Funct. An., 254 (2008) 1088–1145.

[M] Matsuki, Closure relations for orbits on affine symmetric spaces under the action of minimal parabolic subgroups. Representations of Lie groups, Kyoto, Hiroshima, 1986, 541–559, Adv. Stud. Pure Math., 14, Academic Press, Boston, MA, 1988.

[R] Renard, Représentations des groupes réductifs $p$-adiques. Cours Spécialisés, 17. Société Mathématique de France, Paris, 2010.

[Ri] Richardson, R. W. Orbits, invariants, and representations associated to involutions of reductive groups. Invent. Math. 66 (1982), 287-312.

[S] Sauvageot, Principe de densité pour les groupes réductifs. Compositio Math. 108 (1997) 151–184.

[Wal] Waldspurger J.-L., La formule de Plancherel pour les groupes $p$-adiques (d’après Harish-Chandra), J. Inst. Math. Jussieu 2 (2003), 235–333.
Institut de Mathématiques de Luminy, UMR 6206 CNRS, Université de la Méditerranée, 163 Avenue de Luminy, 13288 Marseille Cedex 09, France

E-mail address: carmona@iml.univ-mrs.fr, delorme@iml.univ-mrs.fr