Dynamics of Anisotropic Universes

Jérôme Perez
Laboratory of Applied Mathematics
Ecole Nationale Supérieure de Techniques Avancées
32 Bd Victor, 75739 Paris cedex 15 - France
jerome.perez@ensta.fr

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Abstract
We present a general study of the dynamical properties of Anisotropic Bianchi Universes in the context of Einstein General Relativity. Integrability results using Kovalevskaya exponents are reported and connected to general knowledge about Bianchi dynamics. Finally, dynamics toward singularity in Bianchi type VIII and IX universes are showed to be equivalent in some precise sense.

1 Homogeneous Universe and Bianchi models
Considering the usual synchronous frame of General Relativity $ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \tilde{g}_{ij} dx^i dx^j - dt^2$. A Universe is said homogeneous when there exists an isometry group which preserves the infinitesimal spatial length $dl^2 = \tilde{g}_{ij} dx^i dx^j$. A characterization of the isometry group is possible writing structure constants

$$C_c^{\, e}_{ab} = (\partial_i e^e_j - \partial_j e^e_i) e^i_a e^j_b$$

where $dx^i = e^i_j dy^j$. Constants $C_c^{\, e}_{ab}$ are tensorial, low components are antisymmetric, and follow the Jacobi rule:

$$C_c^{\, e}_{ab} C^d_{e\, c} + C_c^{\, e}_{bc} C^d_{e\, a} + C_c^{\, e}_{ca} C^d_{e\, b} = 0$$

Decomposing $C_c^{\, e}_{ab} = \varepsilon_{c\, ab} N^{de} + \delta^{c}_{a} A_{d} - \delta^{c}_{b} A_{d}$, where $N^{ab}$ is a symmetric tensor, one can show that equivalence classes of homogeneous universes are equivalence classes of $N^{ab}$ with $N^{ab} A_{b} = 0$. Without loss of generality, the symmetry of $N^{ab}$ allows us to write

$$N^{ab} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \end{bmatrix}$$

and $A_{b} = [a, 0, 0]$
Models split then into Class A with \(a = 0\) and Class B with \(a \neq 0\) and can be arranged in the well known Bianchi models

| \(n_1\) | \(n_2\) | \(n_3\) | \(a\)   | Model       |
|--------|--------|--------|--------|------------|
| 0      | 0      | 0      | 0      | \(B_I\)    |
| 0      | 0      | 0      | \(\forall\) | \(B_V\)    |
| 1      | 0      | 0      | 0      | \(B_{II}\) |
| 0      | 1      | 0      | \(\forall\) | \(B_{IV}\) |
| 1      | 1      | 0      | 0      | \(B_{VIIa}\) |
| 0      | 1      | 1      | \(\forall\) | \(B_{VIIb}\) |
| 1      | -1     | 0      | 0      | \(B_{VIIIa}\) |
| 0      | 1      | -1     | \(\neq 1\) | \(B_{VIIIb}\) |
| 1      | 1      | 1      | 0      | \(B_{IX}\) |
| 1      | 1      | -1     | 0      | \(B_{XII}\) |

2 Einstein Equations

Following [1], one writes

\[
ds^2 = \gamma(\tau) \omega \omega - N^2(\tau) d\tau^2 \text{ with } \gamma(\tau) = \text{diag}[e^{A_1(\tau)}, e^{A_2(\tau)}, e^{A_3(\tau)}]
\]

and \(dt = N(\tau) d\tau\). The so called invariant differential forms basis \(\omega^i\) are linear combinations of \(dx_i\) with exponential or trigonometric coefficient in \(x_i\) (See [1]). Finally, \(\tau\) and \(N(\tau)\) are respectively the conformal time and the lapse function.

2.1 BKL Formalism

This formalism was introduced in the 70’s by [3]. Filling Universe by a barotropic fluid with pressure \(P\) and energy density \(\rho\) such that \(P = (\Gamma - 1) \rho\), taking \(N^2(\tau) = V^2 = e^{A_1 + A_2 + A_3}\) for the lapse function, some algebra then gives from Einstein equations, the equations for the dynamics of Bianchi Universes

\[
\begin{align*}
0 &= E_c + E_p + E_m = H \\
\chi(2 - \Gamma)V^{2 - \Gamma} &= A_1'' + (n_1 e^{A_1})^2 - (n_2 e^{A_2} - n_3 e^{A_3})^2 \\
\chi(2 - \Gamma)V^{2 - \Gamma} &= A_2'' + (n_2 e^{A_2})^2 - (n_3 e^{A_3} - n_1 e^{A_1})^2 \\
\chi(2 - \Gamma)V^{2 - \Gamma} &= A_3'' + (n_3 e^{A_3})^2 - (n_1 e^{A_1} - n_2 e^{A_2})^2
\end{align*}
\]

where \(\chi = 8\pi G c^{-4}\), \(E_c = \frac{1}{2} \sum_{i \neq j=1}^3 A_i' A_j'\), \(E_p = \sum_{i \neq j=1}^3 n_i n_j e^{A_i + A_j} - \sum_{i=1}^3 n_i^2 e^{2A_i}\), \(E_m = -4\chi \rho V^2\) and \(\tau\) stands for \(d/d\tau\).

2.2 Hamiltonian formalism

This formalism was introduced by [4], at almost the same time than the precedent. It consists to diagonalize the quadratic form \(E_c\) by introducing new vari-
ables such that
\[
M := \begin{bmatrix}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\
\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\
\end{bmatrix}
\]
\[
q := [q_1, q_2, q_3]^T = M [A_1^T A_2 A_3]^T \\
p := [p_1, p_2, p_3]^T = M [A_1^T A_2 A_3']^T = q'
\]

Dynamical equations for Bianchi Universes then becomes
\[
q_{1,2}' = -\frac{\partial H}{\partial p_{1,2}}, \quad p_{1,2}' = -\frac{\partial H}{\partial q_{1,2}} \quad \text{and} \quad q_3' = \frac{\partial H}{\partial p_3}, \quad p_3' = -\frac{\partial H}{\partial q_3}
\]

where \( H = \frac{1}{2} (p, p) + \sum_{i=1}^{7} k_i e^{(\alpha_i, \alpha)} \) with the following products
\[
\forall x, y \in \mathbb{R}^3 \quad (x, y) := +x_1 y_1 + x_2 y_2 + x_3 y_3 \quad \text{and} \quad \langle x, y \rangle := -x_1 y_1 - x_2 y_2 + x_3 y_3
\]

and the constants \( k_1 := 2n_1 n_2, k_2 := 2n_1 n_3, k_3 := 2n_2 n_3, k_4 := -n_1^2, k_5 := -n_2^2, k_6 := -n_3^2, k_7 := -4\rho_o \chi \). The set of vectors \( a_i=1,\ldots,7 \) is highly symmetric (see Fig. 1): It allows to use algebraic techniques based on Lie algebra in some dynamical treatments of this problem.

3 Integrability of Bianchi models

3.1 Kovalevskaya stuff and autosimilar ODE

If an ODE
\[
\dot{x} = f(x) \quad \text{with} \quad x \in \mathbb{R}^n \quad \text{and} \quad ' \equiv d/dt
\]

admits a self-similar solution
\[
\ddot{x} = \left[ c_1 (t-t_0)^{-g_1}, \ldots, c_n (t-t_0)^{-g_n} \right]^T
\]
with the weight vector \( g \) and the constant vector \( c \) lying respectively \( \mathbb{Z}^n \) and \( \mathbb{R}^n \). One can then prove that the linearized system around \( \tilde{x} \), admits too an auto-similar solution

\[
z = \begin{bmatrix} d_1 (t - t_0)^{k_1 - g_1} & \ldots & d_n (t - t_0)^{k_n - g_n} \end{bmatrix}^T
\]

(10)

where \( d \) is a constant vector, and, this time the Kovalevskaya vector \( k \), which components are the so-called Kovalevskaya exponents, lies \( \mathbb{C}^n \). In practice, one can compute this vector: it is easy to show that Kovalevskaya exponents are eigenvalues of \( K := D [f(x)](c) + \text{diag}(g) \). A theorem by Poincaré shows that each component of the non linear general solution of equation (8) is on the form

\[
x_i(t) \propto (t - t_0)^{-g_i} \cdot S \begin{bmatrix} (t - t_0)^{k_1}, \ldots, (t - t_0)^{k_n} \end{bmatrix}
\]

(11)

where \( S[.] \) stands for a multiple series. This result is at the basis of a sufficient condition (known as Yoshida’s Theorem see [5],[6]): if all the Kovalevskaya exponents are in \( \mathbb{Q} \), then the system is algebraically integrable.

### 3.2 Kovalevskaya exponents and Bianchi Universes

Such a work was Pioneering by [7], applied for the first time to some special cases by [8], and more recently, developed with some imprecisions and incompletude by [9]. We present here results obtained in a precise way in [10].

Introducing new variables

\[
\{q,p\} \mapsto \{u,v\} \quad \text{where} \quad \begin{cases} u \in \mathbb{R}^7, u_{i=1,\ldots,7} := \langle a_i, p \rangle \\ v \in \mathbb{R}^7, v_{i=1,\ldots,7} := \exp(a_i, q) \end{cases}
\]

(12)

Einstein’s Hamiltonian equations become

\[
\forall i = 1, \ldots, 7 \quad \begin{cases} v'_i = u_i v_i \\ u'_i = \sum_{j=1}^7 W_{ij} v_j \end{cases} \quad \text{with} \quad W_{ij} := -k_j \langle a_i, a_j \rangle
\]

(13)

This dynamical system admits an auto-similar solution \( \tilde{x} = [\lambda t^{-1}, \mu t^{-2}]^T \) where the constants \( [\lambda, \mu] \in \mathbb{R}^7 \times \mathbb{R}^7 \) are solutions of the algebraic system

\[
\forall i = 1, \ldots, 7 \quad \begin{cases} \sum_{j=1}^7 W_{ij} \mu_j = -\lambda_i \\ \lambda_i \mu_i = -2 \mu_i \end{cases}
\]

(14)

Taking into account that \( \text{Rank}(W) = 3 \), it exists 45 distincts non trivial solution for system (14), and then 45 sets of 14 Kovalevskaya exponents for all Bianchi Universes. Analysis of such sets let us claim that:
• In vacuum or with stiff matter ($\Gamma = 2$), excepted $B_{IX}$ and $B_{VIII}$, all other class A Bianchi models have fractional Kovalevskaya exponents.

• With non stiff matter ($0 \leq \Gamma < 2$) class A Bianchi models have at least one real or complex Kovalevskaya exponent, except for $B_I$ with fractional values of $\Gamma$ and $B_{II}$ with $\Gamma$ fractional in $[0, \frac{11 + \sqrt{73}}{3} \approx 0.82]$.

These integrability indications are confirmed by [12] for $B_{VIII}$ using Morales-Ruiz Theory and by [11] using Painlevé’s analysis for $B_{IX}$.

4 Exact solutions

4.1 $B_I$ dynamics: The fundamental state

In vacuum, the general solution of $B_I$ model could be explicit. The line element is

$$ds^2 = t^{2p_1}dx_1^2 + t^{2p_2}dx_2^2 + t^{2p_3}dx_3^2 - dt^2$$

(15)

where

$$p_1 = -\Omega u/(1 + u + u^2)$$

$$p_2 = \Omega(1 + u)/(1 + u + u^2)$$

$$p_3 = \Omega u(1 + u)/(1 + u + u^2)$$

(16)

The real $\Omega$ is directly proportional to Universe Volume variation which is constant in $B_I$. Toward singularity ($t \to 0$), $x_1$ expands when $x_2$ and $x_3$ contracts, all with a constant exponential rate. Any couple $[u, \Omega] \in [1, +\infty[ \times \mathbb{R}$ associated to an order of axis defines a Kasner state.

4.2 $B_{II}$ dynamics: One Kasner transition

As noted by [3], $B_{II}$ dynamics corresponds to a transition between 2 asymptotic Kasner states:

\[
\begin{bmatrix}
[u, \Omega] & (\square \triangle \diamond) \\
p_1 < p_2 < p_3 & \square \triangle \diamond \\
\end{bmatrix} \quad \sim \quad \begin{bmatrix}
[u - 1, \Omega(1 - 2p_1)] & (\triangle \square \diamond) \text{ if } u \geq 2 \\
(u - 1)^{-1}, \Omega(1 - 2p_1) & (\triangle \square \diamond) \text{ if } u < 2 \\
\end{bmatrix} \quad \text{Final Kasner State : } t \to 0
\]

(17)

4.3 Conjectures (partially proven)

As indicated by the rigorous proof by [16], all classes of Bianchi models converge generically toward a simple Kasner state or a closure of Kasner states when $t \to 0$. Numerical analysis, more precisely Billiard analogy (see next section), let us think that:
• $B_{VI}$ and $B_{VII}$ dynamics correspond to a finite number of Kasner transitions

• $B_{VIII}$ and $B_{IX}$ dynamics correspond to an infinite number of Kasner transitions

5 Billiard analogy

Those works was pioneered by [4], and more recently by [13] or [15]. Introducing the super-time $\tilde{t}$ such that $d\tilde{t} = V^{1/3} dt$, a "mass" $m = V^{4/3}$ and an "energy" $E = (dV/dt)^2 / 2V^{2/3}$, dynamical equation \[6\] in vacuum becomes

\[
\begin{align*}
\frac{dq_{1,2}}{d\tilde{t}} &= \frac{p_{1,2}}{m} = \frac{\partial E}{\partial q_{1,2}}, \\
\frac{dp_{1,2}}{d\tilde{t}} &= -\frac{\partial \xi}{\partial q_{1,2}} = \frac{\partial E}{\partial p_{1,2}}
\end{align*}
\]

with $E = \frac{p_1^2 + p_2^2}{2m} - \xi(q)$, $\xi(q) = \sum_{i=1}^{6} k_i e^{i(a_i, q)}$ and $q \in \mathbb{R}^2, \pi :$ normal projector onto $(e_1, e_2)$ plane

(18)

Toward singularity, $E \to +\infty$ and $m \to 0$. When we go through time backward, equations \[18\] are ones of a ball with a decreasing mass $m$ and with an increasing energy. The exponential nature of the potential $\xi$ allows a precise description of he dynamics, namely, the billiard analogy. When $\xi$ is exponentially negligible, solutions of \[18\] are almost straight lines in phase plane $(q_1, q_2)$ : these are almost Kasner states in BKL representation. When $\xi$ cannot be neglected anymore, it is mainly due to one of it exponential term, equation of motion — for a linear combination $y$ of $(q_1, q_2)$ — takes a form equivalent to\[2^2\]

\[
\frac{d^2 y}{dx^2} = -k^2 e^y \quad \text{with } y(0) = \frac{dy}{dx} \bigg|_{x=0} = 0
\]

(19)

This equation could be solved analytically in

\[
y(x) = \ln \left[ 1 - \tanh^2 \left( \frac{kx}{\sqrt{2}} \right) \right] = -2 \ln \left[ \cosh \left( \frac{kx}{\sqrt{2}} \right) \right]
\]

(20)

It shows clearly a transition between the two Kasner states associated to the straight lines which are the asymptotes of this solution (see Fig. 2). This generalized $B_{II}$ transition corresponds to a bounce of a ball in an amazing billiard. As a matter of fact, when $p_1^2 + p_2^2 = 0$, all ball’s energy is concentrated in the potential term $\xi$, this situation corresponds to the maximum of the $y$ function plotted in Fig. 2: this is a bounce. It happens effectively against the isocontour line $\xi = \xi(q^*)$ where $dq^*/d\tilde{t} = 0$. As we go toward singularity ($\tilde{t} \to 0$) these lines move outward as one can see in Fig. 3. The Universe ball then moves in an expanding billiard. It is easy to prove that ball’s velocity norm $| dq/d\tilde{t} |$ is always greater than the one of cushions which is the energy variation

\[2^2\]The $x$ variable represents the supertime $\tilde{t}$ such that when $\tilde{t} \sim x$ one has $\xi \sim -k^2 e^y$
This property remains true when barotropic matter fills Universe provided that $\Gamma < 2$. The amplitude of the bounce depends on $k$: the largest $k$ is, the smallest is the bounce angle between the two asymptotic linear regimes. Between the two ideal cases (quasi Kasner state and bounce) the parameter $k$ is dynamical, its variations remains negligible during each phase.

6 Dynamical properties of Bianchi Billiards

In the billiard formulation, dynamics of Bianchi Universe is fully understood considering isocontours of the potential $\xi$ which are presented on Fig. 3. Excluding $B_{IX}$ and $B_{VIII}$ all isocontours are open curves. Hence after a finite number of bounces, the ball representing Universe find a hole and go away to reach the asymptotic regime solution of equation (19) which is a Kasner state. By opposite, for the two closed curves potential, there is no hole in the billiard, cushion allways expands less quickly than the ball, and there is an infinite sequence of $B_{II}$ transitions toward singularity. Although $B_{VIII}$ cushions seems open along a vertical channel, the bounce process makes the ball which enters the channel goes back. As sketched on Fig. 2 in the channel $k \sim q_2$, then as the ball sinks into this region, $q_2$ grows and bounces are more and more pinched. By consequence, as the cushion is not strictly vertical, the ball finally goes back and leaves the channel. These results are surely well known and one remaining question could be: Is $B_{VIII}$ dynamics equivalent in some precise sence to $B_{IX}$ one? In terms of complexity the two dynamics seems equivalent as they have exactly the same set of Kovalevskaya exponents. Let us show that they are also equivalent by the fact that they present the same kind of transition to chaos, and they have the same kind of attractor.
Figure 3: Isocontours of potential $\xi$ for all distincts Bianchi Universe in vacuum. Arrows indicate the increasing values of $\xi$.
6.1 $B_{IX}$ and $B_{VIII}$ Poincaré map analysis

Detailed analysis show that Bianchi Universe volume is a regular variable. It is the only variable attached to cushions expansion in the billiard formalism. Therefore, the complexity of the dynamics is contained in the cushions's shape. In order to study this complexity in the two relevant cases which are $B_{IX}$ and $B_{VIII}$ Universes, we have studied dynamical properties of temporal sections of the Hamiltonian $E$ involving in equation (18). As $V$, $t$, $\tilde{t}$ and $\tau$ are all related in a bijective way, by temporal section we mean sections of the dynamics described by equation (18) with $m = cst$ and $E = cst$. For convenience we use always a ball with unit mass, we focuss attention under Poincaré maps associated to growing values of $E$. Such maps are obtained by considering the intersection between an orbit (a solution of the 4D differential system) and a fixed plane, namely $\Pi = \{q_2 = 0, p_2 = 0\}$ for maps plotted in Fig. 4. The greater are values of the energy $E$, the closest from singularity sections are. Analysis of such maps is clear using KAM Theory: When periodic or quasiperiodic orbits correspond to Poincaré sections disposed along curves, chaotic orbits sections fill dense regions. We can then conclude that far from singularity orbits are regular in both $B_{IX}$ and $B_{VIII}$. Approaching singularity, a transition to chaos seems to appear in the dynamics of both these closed cushions expanding billiards. In order to produce a comparative measure of this chaos we propose to produce an analysis base on $B_{IX}$ and $B_{VIII}$ Universes truncated dynamics.

Figure 4: Poincaré maps toward singularity of temporal sections of $m = cst$ and $E = cst$ of $B_{IX}$ (top: $m = 1$ and $E = -2.8, -2.1, 0$ from left to right) and $B_{VIII}$ (bottom: $m = 1$ and $E = 5.0, 10.0$ from left to right) dynamics. For each map a set of 9 homogeneously distributed initial conditions have been choosen.
Figure 5: Initial conditions $\theta$ and $\omega$ for fractals maps represented on Fig. 6

6.2 $B_{IX}$ and $B_{VIII}$ fractal attractors

As suggested by all previous indicators (Kovalevskaya exponents, Poincaré maps), we suspect $B_{VIII}$ dynamics to possess the same kind of attractor exhibited initially by [17] for $B_{IX}$. In order to confirm this suspicion we have applied the truncated dynamical technique used by [17] in the context of Hamiltonian formalism described above.

Initial conditions we used are $q_o = \left[0, -\sqrt{6}\ln 2, \Omega\right]$ and $p_o = [\cos \theta, \sin \theta, \omega]$ where the constraint $H = 0$ is fulfilled provided that

$$\Omega = \frac{3}{\sqrt{6}} \ln \left[ \frac{1 - \omega^2}{2^4 (1 \pm 2)} \right]$$  \hspace{1cm} (21)

The upper and lower signs are associated respectively to $B_{IX}$ and $B_{VIII}$ Universes. The two free parameters $\omega$ and $\theta$ fixe respectively the initial variation rate of Univers volume (or the initial escape speed of cushions in the billiard formalism) and the orientation of the initial velocity vector. These initial conditions are represented on Fig. 6. The numerical integration of the dynamics from these initial conditions shows an unlimited sequence of $B_{II}$ transitions between Kasner states (see formula (17)). We stop numerical integration when the Kasnerian $u$ parameter is greater than a critical value (for maps presented on Fig. 6 we have used $u_{exit} = 8$). When the dynamic is stopped, we affect to the corresponding point in the $\theta - \omega$ plane a color which correspond to the expanding axe of stopped Kasner state (Red $\leftrightarrow A_1$, Blue $\leftrightarrow A_2$ and Green $\leftrightarrow A_3$). We have studied the $[0, \pi/2] \times [-2, -3]$ portion of the $\theta - \omega$ plane, projected on a regular $500 \times 500$ grid. The corresponding maps are presented in Fig. 6. From this map we can compute the 3 Hausdorff dimensions associated to each color set. We can resume these three numbers into one by averaging each dimension weighted by the surface proportion occupied by the corresponding color. We then obtain a mean Hausdorff dimension of each map which is $d = 1.6976 \pm 8.9 \times 10^{-3}$ for $B_{VIII}$’s map and $d = 1.7141 \pm 8.5 \times 10^{-3}$ for $B_{IX}$’s map. These two numbers are then numerically indistinguishable. In this sense we claim that $B_{VIII}$ and $B_{IX}$
Figure 6: $B_{VI}$ (left) and $B_{IX}$ (right) fractal maps in the $[0, \pi/2] \times [-2, -3]$ portion of the $\theta - \omega$ plane.

universes are dynamically equivalent

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