On the second smallest and the largest normalized Laplacian eigenvalues of a graph *

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Abstract

Let $G$ be a simple connected graph with order $n$. Let $L(G)$ be the normalized Laplacian matrix of $G$. Let $\lambda_k(G)$ be the $k$-th smallest normalized Laplacian eigenvalue of $G$. Denote $\rho(A)$ the spectral radius of the matrix $A$. In this paper, we study the behaviors of $\lambda_2(G)$ and $\rho(L(G))$ when the graph is perturbed by three operations.

Key Words: second smallest normalized Laplacian eigenvalue, normalized Laplacian spectral radius.

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1 Introduction

Let $A$ be a matrix with order $n \times n$, and $\rho(A)$ be the spectral radius of $A$. Let $G$ be a simple connected graph. Let $V(G)$ and $E(G)$ be the vertex set and the edge set of $G$, respectively. Its order is $|V(G)|$, and its size is $|E(G)|$. For $v \in V(G)$, let $d(v)$ be the degree of $v$, $N_G(v)$ be the set of neighbours of a vertex $v$ in $G$. We use the notation $I$ for the identity matrix, $e$ for the vector consisting of all ones, $S_n$ for the star of order $n$, $C_n$ for the cycle of length $n$, $P_n$ for the path of length $n - 1$ and $Vol(G)$ for the sum of the degrees of all vertices in $G$. Meanwhile, we use the notation $S(G)$ to denote the subdivision graph of $G$, which is the graph obtained from $G$ by inserting some new vertices to some edges of $G$.

Let $A(G)$ and $D(G)$ be the adjacency matrix and the diagonal matrix of vertex degrees of $G$, respectively. The Laplacian and normalized Laplacian matrices of $G$ are defined as $L(G) = D(G) - A(G)$ and $\mathcal{L}(G) = D^{-\frac{1}{2}}(G)L(G)D^{-\frac{1}{2}}(G)$, respectively. When only one graph $G$ is under consideration, we sometimes use $A$, $D$, $L$ and $\mathcal{L}$ instead of $A(G)$, $D(G)$, $L(G)$ and $\mathcal{L}(G)$ respectively. It is easy to see that $\mathcal{L}(G)$ is a symmetric positive semidefinite matrix and $D^{\frac{1}{2}}(G)e$ is an eigenvector of $\mathcal{L}(G)$ with eigenvalue 0. Thus, the eigenvalues $\lambda_i(G)$ of $\mathcal{L}(G)$ satisfy

$$\lambda_n(G) \geq \cdots \geq \lambda_2(G) \geq \lambda_1(G) = 0.$$

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Some of them maybe repeated according to their multiplicities. \( \lambda_k(G) \) is the \( k \)-th smallest normalized Laplacian eigenvalue of \( G \). Thus \( \rho(\mathcal{L}(G)) = \lambda_2(G) \). When only one graph is under consideration, we may use \( \lambda_1 \) and \( \rho(\mathcal{L}) \) instead of \( \lambda_k(G) \) and \( \rho(\mathcal{L}(G)) \), respectively.

In terms of \( \lambda_2(G) \), Chung [1] showed that \( \lambda_2(G) \) is 0 if and only if \( G \) is disconnected. This result is closely related to the second smallest eigenvalue of its Laplacian matrix [2]. H.H. Li et al. [3, 4] studied the behavior of \( \lambda_2 \) when the graph is perturbed by grafting an edge and a pendant path, respectively. Recently, J.X. Li et al. [5] studied the behavior of \( \lambda_2 \) when the graph is perturbed by separating an edge. They determined all trees and unicyclic graphs with \( \lambda_2(G) \geq 1 - \frac{\sqrt{5}}{2} \). Guo et al. [7] studied the behavior of \( \rho(\mathcal{L}) \) when the graph is perturbed by removing pendant edges from one vertex to another. The non-bipartite unicyclic graph with fixed order and girth which has the largest \( \rho(\mathcal{L}) \) was also determined.

In this paper, we further study the behaviors of \( \lambda_2 \) and \( \rho(\mathcal{L}) \) when the graph is perturbed by three operations.

## 2 Preliminaries

In this section we recall some properties of the eigenvalues and eigenfunctions of the normalized Laplacian matrix of a graph \( G \). Let \( g \) be a vector such that \( g \neq 0 \). Then we can view \( g \) as a function which assigns to each vertex \( v \) of \( G \) a real value \( g(v) \), the coordinate of \( g \) according to \( v \) (All the vectors in this paper are dealt in this way). By letting \( g = D^{1/2} f \), we have

\[
\frac{g^T \mathcal{L} g}{g^T g} = \frac{f^T D^{1/2} \mathcal{L} D^{1/2} f}{(D^{1/2} f)^T D^{1/2} f} = \frac{f^T \mathcal{L} f}{f^T D f} = \frac{\sum_{u,v \in E(G)} (f(u) - f(v))^2}{\sum_{v \in V(G)} d(v)(f(v))^2}.
\]

Thus, we can obtain the following formulas for \( \lambda_2 \) and \( \rho(\mathcal{L}) \).

\[
\lambda_2 = \inf_{f \in \mathcal{I} \cap De} \frac{f^T \mathcal{L} f}{f^T D f} = \inf_{f \in \mathcal{I} \cap De} \frac{\sum_{u,v \in E(G)} (f(u) - f(v))^2}{\sum_{v \in V(G)} d(v)(f(v))^2}. \tag{1}
\]

\[
\rho(\mathcal{L}) = \sup_{f \in \mathcal{I} \cap De} \frac{f^T \mathcal{L} f}{f^T D f} = \sup_{f \in \mathcal{I} \cap De} \frac{\sum_{u,v \in E(G)} (f(u) - f(v))^2}{\sum_{v \in V(G)} d(v)(f(v))^2}. \tag{2}
\]

A nonzero vector that satisfies equality in (1) or (2) is called a harmonic eigenfunction associated with \( \lambda_2(G) \) or \( \rho(\mathcal{L}(G)) \).

**Lemma 2.1** [1] Let \( G \) be a simple connected graph and \( f \) be a harmonic eigenfunction associated with \( \lambda_2(G) \). Then for any \( v \in V(G) \), we have

\[
\frac{1}{d(v)} \sum_{u \in V(G)} (f(v) - f(u)) = \lambda_2(G) f(v).
\]

From Lemma 2.1, we have the following result.

**Corollary 2.2** Let \( G \) be a simple connected graph and \( f \) be a harmonic eigenfunction associated with \( \lambda_2(G) \). If \( \lambda_2(G) = 1 \), then for any \( v \in V(G) \), we have

\[
\sum_{u \in V(G)} f(u) = 0.
\]
If $f$ is a harmonic eigenfunction associated with $\rho(\mathcal{L}(G))$, the similar results about Lemma 2.1 and Corollary 2.2 are obtained.

From Corollary 2.2, we have the following result.

**Corollary 2.3** Let $v$ be the center of the star $S_n$, $f$ is a harmonic eigenfunction associated with $\lambda_2(S_n)$. Then $f(v) = 0$.

**Proof** Through a simple calculation, we can obtain $\lambda_2(S_n) = 1$. Combining the Corollary 2.2, the result is clear. □

**Lemma 2.4** For a graph which is not a complete graph, we have $\lambda_2 \leq 1$.

Next we will define three operations:

**Operation I.** $G'$ is obtained by inserting a new vertex $w$ to an edge $uv$ of $G$. That is to say $G' = G - uv + uw + wv$.

**Operation II.** Let $G_1$ and $G_2$ be two simple connected graphs, $u \in V(G_1), v \in V(G_2)$. Let $G$ be a graph obtained from $G_1$ and $G_2$ by identifying $u$ with $v$ (see Figure 1).

**Operation III.** Let $u, v$ be two vertices of the simple connected graph $G$. Suppose that $v_1, v_2, \ldots, v_s$ ($1 \leq s \leq d(v)$) are some vertices of $N_G(v) \setminus N_G(u)$ and $v_1, v_2, \ldots, v_s$ are different from $u$. Let $G'$ be the graph obtained from $G$ by deleting the edges $vv_i$ and adding the edges $uw$. That is to say $G' = G - vv_1 - vv_2 - \cdots - vv_s + uv_1 + uv_2 + \cdots + uv_s$.

![Figure 1: Operation II](image)

### 3 The effects on the $\lambda_2(G)$ of a graph by three operations

In this section we study the behavior of $\lambda_2$ when the graph is perturbed by three operations.

The following theorem studies the behavior of $\lambda_2$ when the graph is perturbed by Operation I.

**Theorem 3.1** Let $G$ be a simple connected graph of order $n$, $uv \in E(G)$ and $G' = G - uw + uv + wv$. Then $\lambda_2(G) \geq \lambda_2(G')$, and the inequality is strict if $f(u)f(v) \neq 0$, where $f$ is a harmonic eigenfunction associated with $\lambda_2(G)$.

**Proof** Let $V(G) = \{u, v, u_1, u_2, \ldots, u_{n-2}\}$ and $V(G') = \{u, v, u_1, u_2, \ldots, u_{n-2}, w\}$. Let $d(x)$ and $d'(x)$ be the degrees of $x$ in $G$ and $G'$, respectively. Let $D$ and $D'$ be the diagonal degree matrices of $G$ and $G'$, respectively. Let $L$ and $L'$ be the Laplacian matrices of $G$ and $G'$, respectively. Let $e$ and $e'$ be the vectors consisting of all ones, where $e \in \mathbb{R}^n$ and $e' \in \mathbb{R}^{n+1}$. Then $d'(w) = 2$ , $d'(x) = d(x)$, $x \in V(G)$. Since $f$ is a harmonic eigenfunction associated with $\lambda_2(G)$. Then $f \neq 0$ and $f \perp De$. Let us distinguish two cases.

**Case 1** $f(u)f(v) \leq 0$. Let $h$ be a vector such that $h(w) = 0$, $h(x) = f(x)$, where $x \in V(G)$. Then

$$h^T D' e' = \sum_{x \in V(G')} h(x) d'(x) = \sum_{x \in V(G)} h(x) d'(x) + h(w) d'(w)$$
\[
\sum_{x \in V(G)} f(x)d(x) = f^TD e = 0.
\]

Thus \( h \perp D'e' \). Note that \( h \neq 0 \). Then, we have
\[
\frac{h^T L'h}{h^TD'h} \geq \lambda_2(G').
\]

Moreover
\[
h^T D'h = \sum_{x \in V(G')} d'(x)h^2(x) = \sum_{x \in V(G)} d'(x)h^2(x) + d'(w)h^2(w) = \sum_{x \in V(G)} d(x)f^2(x) = f^TDf,
\]
and
\[
h^T L'h = \sum_{xy \in E(G')} (h(x) - h(y))^2
\]
\[
= \sum_{xy \in E(G') \setminus \{uw, wv\}} (h(x) - h(y))^2 + (h(u) - h(w))^2 + (h(w) - h(v))^2
\]
\[
= \sum_{xy \in E(G) \setminus \{uv\}} (f(x) - f(y))^2 + f^2(u) + f^2(v)
\]
\[
= \sum_{xy \in E(G)} (f(x) - f(y))^2 + 2f(u)f(v) = f^T Lf + 2f(u)f(v)
\]
\[
\leq f^T Lf.
\]

Thus, from Formula (11), we have
\[
\lambda_2(G) = \frac{f^T Lf}{f^TDf} \geq \frac{h^T L'h}{h^TD'h} \geq \lambda_2(G').
\]

If \( f(u)f(v) < 0 \), then \( f^T Lf > h^T L'h \). Thus, \( \lambda_2(G) > \lambda_2(G') \).

**Case 2** \( f(u)f(v) > 0 \). Let \( h \) be a vector such that \( h(w) = f(u), h(x) = f(x) \), where \( x \in V(G) \). Then
\[
h^T L'h = \sum_{xy \in E(G')} (h(x) - h(y))^2
\]
\[
= \sum_{xy \in E(G') \setminus \{uw, wv\}} (h(x) - h(y))^2 + (h(u) - h(w))^2 + (h(w) - h(v))^2
\]
\[
= \sum_{xy \in E(G) \setminus \{uv\}} (f(x) - f(y))^2 + (f(u) - f(v))^2
\]
\[
= \sum_{xy \in E(G)} (f(x) - f(y))^2 = f^T Lf,
\]
and
Thus, from Formula (1), we have

\[ \sum_{x \in V(G')} h(x)d'(x) = \sum_{x \in V(G)} h(x)d'(x) + h(w)d'(w) \]

\[ = \sum_{x \in V(G)} f(x)d(x) + 2f(u) = f^TDe + 2f(u) = 2f(u). \]

Let \( p = h + ce' \), where \( c = -\frac{2f(u)}{Vol(G)+2} \). Then

\[ p^T D' e' = (h + ce')^T D' e' = h^T D' e' + ce'^T D' e' = 2f(u) + c(Vol(G) + 2) = 0. \]

Thus \( p \perp D' e' \). Note that \( p \neq 0 \). Then, we have

\[ \frac{p^T L' p}{p^T D' p} \geq \lambda_2(G'). \]

It is clear that

\[ p^T L' p = h^T L' h = f^T Lf, \]

and

\[ p^T D' p = \sum_{x \in V(G')} d'(x)p^2(x) = \sum_{x \in V(G')} d'(x)(h(x) + c)^2 \]

\[ = \sum_{x \in V(G)} d(x)(f(x) + c)^2 + 2f(u)c + c^2 \]

\[ = f^T Df + 2cf^T De + c^2 Vol(G) + 2f(u)c^2 \]

\[ = f^T Df + \frac{2f^2(u)Vol(G)(2 + Vol(G))}{(2 + Vol(G))^2} \]

\[ > f^T Df. \]

Thus, from Formula (11), we have

\[ \lambda_2(G) = \frac{f^T Lf}{f^T Df} > \frac{p^T L' p}{p^T D' p} \geq \lambda_2(G'). \]

Combining Cases 1 and 2, the result follows. \( \square \)

From Theorem 3.1, we have the following result.

**Corollary 3.2** Let \( G \) be a simple connected graph and \( S(G) \) be the subdivision graph of \( G \). Then \( \lambda_2(G) \geq \lambda_2(S(G)) \).

The following theorem studies the behavior of \( \lambda_2 \) when the graph is perturbed by Operation II.

**Theorem 3.3** Let \( G_1 \) and \( G_2 \) be two simple connected graphs of orders \( m \) and \( n \), respectively. Let \( u \in V(G_1) \) and \( v \in V(G_2) \). Let \( G \) be a graph obtained from \( G_1 \) and \( G_2 \) by identifying \( u \) with \( v \). Then \( \lambda_2(G) \leq \lambda_2(G_1) \), and the inequality is strict if \( f_1(u) \neq 0 \), where \( f_1 \) is a harmonic eigenfunction associated with \( \lambda_2(G_1) \).

**Proof** Let \( V(G_1) = \{x_1, x_2, \ldots, x_{m-1}, u\} \), \( V(G_2) = \{y_1, y_2, \ldots, y_{n-1}, v\} \), and \( V(G) = \{x_1, x_2, \ldots, x_{m-1}, u, y_1, y_2, \ldots, y_{n-1}\} \). Let \( d(x), d_1(x) \) and \( d_2(x) \) be the degree of \( x \) in \( G \), the
degree of $x$ in $G_1$, and the degree of $x$ in $G_2$, respectively. Let $D$ and $D_1$ be the diagonal degree matrices of $G$ and $G_1$, respectively. Let $L$ and $L_1$ be the Laplacian matrices of $G$ and $G_1$, respectively. Let $e$ and $e_1$ be the vectors consisting of all ones, where $e \in \mathbb{R}^{m+n-1}$ and $e_1 \in \mathbb{R}^m$. Then $d(x_i) = d_1(x_i), i = 1, 2, \ldots, m - 1, d(y_j) = d_2(y_j), j = 1, 2, \ldots, n - 1$, and $d(u) = d_1(u) + d_2(v)$. Since $f_1$ is a harmonic eigenfunction associated with $\lambda_2(G_1)$. Then $f_1 \neq 0$ and $f_1 \perp D_1 e_1$.

Let $f(x) = f_1(x), \forall x \in V(G_1), f(y_j) = f_1(u), j = 1, 2, \ldots, n - 1$. Then we have

$$f^T L f = \sum_{xy \in E(G)} (f(x) - f(y))^2$$

$$= \sum_{xy \in E(G_1)} (f(x) - f(y))^2 + \sum_{xy \in E(G) \setminus E(G_1)} (f(x) - f(y))^2$$

$$= \sum_{xy \in E(G_1)} (f(x) - f(y))^2 = \sum_{xy \in E(G_1)} (f_1(x) - f_1(y))^2$$

$$= f_1^T L_1 f_1,$$

and

$$f^T D e = \sum_{x \in V(G)} d(x) f(x) = \sum_{x \in V(G_1)} d(x) f(x) + \sum_{j=1}^{n-1} d(y_j) f(y_j)$$

$$= \sum_{x \in V(G_1) \setminus \{u\}} d(x) f(x) + d(u) f(u) + \sum_{j=1}^{n-1} d(y_j) f(y_j)$$

$$= \sum_{x \in V(G_1) \setminus \{u\}} d_1(x) f_1(x) + (d_1(u) + d_2(v)) f_1(u) + \sum_{j=1}^{n-1} d_2(y_j) f_1(u)$$

$$= f_1^T D_1 e_1 + d_2(v) f_1(u) + f_1(u) \sum_{j=1}^{n-1} d_2(y_j)$$

$$= f_1(u) (d_2(v) + \sum_{j=1}^{n-1} d_2(y_j)) = f_1(u) \text{Vol}(G_2).$$

Let $h = f + c e$, where $c = -\frac{f_1(u) \text{Vol}(G_2)}{\text{Vol}(G_1) + \text{Vol}(G_2)}$, Then

$$h^T D e = (f + c e)^T D e = f_1(u) \text{Vol}(G_2) + c(\text{Vol}(G_1) + \text{Vol}(G_2)) = 0.$$ 

Thus $h \perp D e$. Note that $h \neq 0$. Then, we have

$$\frac{h^T L h}{h^T D h} \geq \lambda_2(G).$$

Moreover

$$h^T D h = (f + c e)^T D(f + c e) = f^T D f + 2c f^T D e + c^2 e^T D e$$
From the above equation, we have

\[ f^T Df = 2cf_1(u)Vol(G_2) + c^2(Vol(G_1) + Vol(G_2)) \]

\[ = f^T Df - \frac{(f_1(u)Vol(G_2))^2}{Vol(G_1) + Vol(G_2)}, \]

and

\[ f^T Df = \sum_{x \in V(G)} d(x)f^2(x) = \sum_{x \in V(G_1)} d(x)f^2(x) + \sum_{j=1}^{n-1} d(y_j)f^2(y_j) \]

\[ = \sum_{x \in V(G_1) \setminus \{u\}} d(x)f^2(x) + d(u)f^2(u) + \sum_{j=1}^{n-1} d(y_j)f^2(y_j) \]

\[ = \sum_{x \in V(G_1) \setminus \{u\}} d_1(x)f_1^2(x) + (d_1(u) + d_2(v))f_1^2(u) + \sum_{j=1}^{n-1} d(y_j)f_1^2(u) \]

\[ = f_1^T D_1 f_1 + d_2(v)f_1^2(u) + f_1^2(u) \sum_{j=1}^{n-1} d(y_j) \]

\[ = f_1^T D_1 f_1 + f_1^2(u)Vol(G_2). \]

From the above equation, we have

\[ h^T Dh = f_1^T D_1 f_1 + \frac{f_2^2(u)Vol(G_1)Vol(G_2)}{Vol(G_1) + Vol(G_2)} \]

\[ \geq f_1^T D_1 f_1 > 0. \]

Thus, from Formula (11), we have

\[ \lambda_2(G_1) = \frac{f_1^T L_1 f_1}{f_1^T D_1 f_1} = \frac{h^T L h}{h^T D h} \geq \frac{h^T L h}{h^T D h} \geq \lambda_2(G). \]

If \( f_1(u) \neq 0 \), then \( h^T D h > f_1^T D_1 f_1 \). Thus, \( \lambda_2(G_1) > \lambda_2(G) \). The result follows. \( \square \)

From Theorem 3.3, the following is easily obtained.

**Corollary 3.4** Let \( G_1 \) and \( G_2 \) be two simple connected graphs, \( u \in V(G_1), v \in V(G_2) \). Let \( G \) be a graph obtained from \( G_1 \) and \( G_2 \) by identifying \( u \) with \( v \). Then \( \lambda_2(G) \leq \min \{ \lambda_2(G_1), \lambda_2(G_2) \} \).

In particular, if \( T \) is a tree, then it is clear that \( T \) can be obtained from the subtree \( T_1 \) and \( T_2 \) by Operation II. Hence, by Corollary 3.4, the following is immediate.

**Corollary 3.5** Let \( T \) be a tree. If \( T' \) is a subtree of \( T \), then \( \lambda_2(T) \leq \lambda_2(T') \).

The following theorem studies the behavior of \( \lambda_2 \) when the graph is perturbed by Operation III.

**Theorem 3.6** Let \( u, v \) be two vertices of the simple connected graph \( G \) of order \( n \). Suppose that \( v_1, v_2, \ldots, v_s (1 \leq s \leq d(v)) \) are some vertices of \( N_G(v) \setminus N_G(u) \) and \( v_1, v_2, \ldots, v_s \) are different from \( u \). Let \( G' = G - v_1 v_2 \cdots v_s - v v_1 + w_1 + w_2 + \cdots + w_s \) and \( f \) be a harmonic eigenfunction associated with \( \lambda_2(G) \). If \( f(u) = f(v) \), Then \( \lambda_2(G) \geq \lambda_2(G') \).

**Proof** Let \( d(x) \) and \( d'(x) \) be the degree of \( x \) in \( G \) and the degree of \( x \) in \( G' \), respectively. Let \( D \) and \( D' \) be the diagonal degree matrices of \( G \) and \( G' \), respectively. Let \( L \) and \( L' \) be the Laplacian matrices of \( G \) and \( G' \), respectively. Let \( e \) be the vector consisting of all ones, where
e ∈ R^{\lvert G \rvert}. Then d'(v) = d(v) - s, d'(u) = d(u) + s, d'(x) = d(x), where x ∈ V(G) \setminus \{u, v\}. Since f is a harmonic eigenfunction associated with \lambda_2(G). Then f ≠ 0 and f ⊥ De. Let f'(u) = f(u), \forall u ∈ V(G). Then f'(u) = f(u) = f(v) = f'(v),

\[
f'^T L' f' = \sum_{xy ∈ E(G')} (f'(x) - f'(y))^2
\]

\[
= \sum_{xy ∈ E(G')} (f'(x) - f'(y))^2 + \sum_{j=1}^{s} (f'(u) - f'(v_j))^2
\]

\[
= \sum_{xy ∈ E(G')} (f(x) - f(y))^2 + \sum_{j=1}^{s} (f(v) - f(v_j))^2
\]

\[
= \sum_{xy ∈ E(G)} (f(x) - f(y))^2
\]

\[
= f'^T L f,
\]

and

\[
f'^T D' e = \sum_{x ∈ V(G')} d'(x) f'(x)
\]

\[
= \sum_{x ∈ V(G) \setminus \{u, v\}} d'(x) f'(x) + d'(u) f'(u) + d'(v) f'(v)
\]

\[
= \sum_{x ∈ V(G) \setminus \{u, v\}} d'(x) f'(x) + f'(u)(d'(u) - s) + f'(v)(d'(v) + s)
\]

\[
= \sum_{x ∈ V(G) \setminus \{u, v\}} d(x) f(x) + d(u) f(u) + d(v) f(v)
\]

\[
= \sum_{x ∈ V(G)} d(x) f(x) = f'^T D e = 0.
\]

Thus f' ⊥ D'e. Note that f' ≠ 0. Then we have

\[
\frac{f'^T L' f'}{f'^T D' f'} ≥ \lambda_2(G').
\]

It is clear that

\[
(f')^T D' f' = \sum_{x ∈ V(G')} d'(x)(f'(x))^2
\]

\[
= \sum_{x ∈ V(G') \setminus \{u, v\}} d'(x)(f'(x))^2 + d'(u)(f'(u))^2 + d'(v)(f'(v))^2
\]

\[
= \sum_{x ∈ V(G) \setminus \{u, v\}} d(x)f^2(x) + d(u)f^2(u) + d(v)f^2(v)
\]

\[
= \sum_{x ∈ V(G) \setminus \{u, v\}} d(x)f^2(x) + d(u)f^2(u) + d(v)f^2(v)
\]

\[
= \sum_{x ∈ V(G)} d(x)f^2(x) = f'^T D f.
\]
Hence, from Formula (1), we have
\[
\lambda_2(G) = \frac{f^T L f}{f^T D f} = \frac{(f')^T L' f'}{(f')^T D' f'} \geq \lambda_2(G').
\]

The result follows. □

![Graph Ga and Gb](image)

Figure 2: Graph $G_a$ and $G_b$

Figure 2 shows that the condition $f(u) = f(v)$ of Theorem 3.6 is necessary. If $f(u) \neq f(v)$, then the relation between values of $\lambda_2(G)$ and $\lambda_2(G')$ is not sure. There are the following three cases. For $G_a$ and $G_b$ in Figure 2 ($G_b$ is $G_3$ of Theorem 2.4 in [4]), the natural numbers represent the vertices and the real numbers attached to vertices in each graph are the valuations by the harmonic eigenfunction associated with $\lambda_2(G)$.

**Case 1** $\lambda_2(G) < \lambda_2(G')$. Let $G = G_a$ and 4, 3, 5 stand for $u$, $v$, $v_1$, respectively. It is clear that $f(u) > f(v)$. Denote $G' = G - vv_1 + uv_1$. By direct calculation, we obtain $\lambda_2(G) = 0.1408 < 0.1557 = \lambda_2(G')$.

**Case 2** $\lambda_2(G) = \lambda_2(G')$. Let $G = G_a$ and 1, 3, 5 stand for $u$, $v$, $v_1$, respectively. It is clear that $f(u) > f(v)$. Denote $G' = G - vv_1 + uv_1$. Because $G$ is isomorphic to $G'$, we obtain $\lambda_2(G) = \lambda_2(G') = 0.1408$.

**Case 3** $\lambda_2(G) > \lambda_2(G')$. Let $G = G_b$ and 1, 2, 8 stand for $u$, $v$, $v_1$, respectively. It is clear that $f(u) > f(v)$. Denote $G' = G - vv_1 + uv_1$. By direct calculation, we obtain $\lambda_2(G) = 0.2290 > 0.2105 = \lambda_2(G')$.

From above, we can see that if $f(u) > f(v)$, the relation between values of $\lambda_2(G)$ and $\lambda_2(G')$ is not sure. Note that if $f$ is a harmonic eigenfunction associated with $\lambda_2(G)$, $-f$ is also a harmonic eigenfunction associated with $\lambda_2(G)$. When $f(u) < f(v)$, the similar result is obtained.

4 The effects on the $\rho(\mathcal{L}(G))$ of a graph by three operations

In this section we study the behavior of $\rho(\mathcal{L})$ when the graph is perturbed by three operations.

The following theorem studies the behavior of $\rho(\mathcal{L})$ when the graph is perturbed by Operation I.

**Theorem 4.1** Let $G$ be a simple connected graph of order $n$, $uv \in E(G)$ and $G' = G - uv + uv + wv$. Let $f$ be a harmonic eigenfunction associated with $\rho(\mathcal{L}(G))$. If $f(u)f(v) \geq 0$ then $\rho(\mathcal{L}(G)) \leq \rho(\mathcal{L}(G'))$, and the inequality is strict if $f(u)f(v) > 0$. 

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Thus, from Formula (2), we have

\[ h(D) \]

\[ \rho(G) \]

\[ \rho(G') \]

Moreover, let \( e \) and \( e' \) be the vectors consisting of all ones, where \( e \in \mathbb{R}^n \) and \( e' \in \mathbb{R}^{n+1} \). Then \( d'(w) = 2, d'(x) = d(x), x \in V(G) \). Since \( f \) is a harmonic eigenfunction associated with \( \rho(\mathcal{L}(G)) \). Then \( f \neq 0 \) and \( f \perp D e \).

Let \( h \) be a vector such that \( h(w) = 0 \), \( h(x) = f(x) \), where \( x \in V(G) \). Then

\[
\begin{align*}
  h^T D' e' &= \sum_{x \in V(G')} h(x)d'(x) \\
  &= \sum_{x \in V(G)} h(x)d'(x) + h(w)d'(w) \\
  &= \sum_{x \in V(G)} f(x)d(x) = f^T D e = 0.
\end{align*}
\]

Thus \( h \perp D' e' \). Note that \( h \neq 0 \). Then, we have

\[
\frac{h^T L' h}{h^T D' h} \leq \rho(\mathcal{L}(G')).
\]

Moreover

\[
\begin{align*}
  h^T D' h &= \sum_{x \in V(G')} d'(x)h^2(x) \\
  &= \sum_{x \in V(G)} d'(x)h^2(x) + d'(w)h^2(w) \\
  &= \sum_{x \in V(G)} d(x)f^2(x) = f^T D f,
\end{align*}
\]

and

\[
\begin{align*}
  h^T L' h &= \sum_{xy \in E(G')} (h(x) - h(y))^2 \\
  &= \sum_{xy \in E(G') \setminus \{uv, uv\}} (h(x) - h(y))^2 + (h(u) - h(w))^2 + (h(w) - h(v))^2 \\
  &= \sum_{xy \in E(G) \setminus \{uv\}} (f(x) - f(y))^2 + f^2(u) + f^2(v) \\
  &= \sum_{xy \in E(G)} (f(x) - f(y))^2 + (f(u) - f(v))^2 + 2f(u)f(v) \\
  &= \sum_{xy \in E(G)} (f(x) - f(y))^2 + 2f(u)f(v) = f^T L f + f^T u f(v) \\
  &\geq f^T L f.
\end{align*}
\]

Thus, from Formula (2), we have

\[
\rho(\mathcal{L}(G)) = \frac{f^T L f}{f^T D f} \leq \frac{h^T L' h}{h^T D' h} \leq \rho(\mathcal{L}(G')).
\]
If \( f(u)f(v) > 0 \), then \( f^T L f < h^T L' h \). Thus, \( \rho(\mathcal{L}(G)) < \rho(\mathcal{L}(G')) \). □

It is clear that the proof of Theorem 4.1 is similar to the proof of Case 1 in Theorem 3.1.

Figure 3 shows that the condition \( f(u)f(v) \geq 0 \) of Theorem 4.1 is necessary. If \( f(u)f(v) < 0 \), then the relation between values of \( \rho(\mathcal{L}(G)) \) and \( \rho(\mathcal{L}(G')) \) is not sure. There are the following two cases. For \( G_c \) and \( G_d \) in Figure 3, the real numbers attached to vertices in each graph are the valuations by the harmonic eigenvector associated with \( \rho(\mathcal{L}(G)) \).

**Case 1** \( \rho(\mathcal{L}(G)) < \rho(\mathcal{L}(G')) \). Let \( G = G_c \) and 5, 6 stand for \( u, v \), respectively. It is clear that \( f(u)f(v) < 0 \). Denote \( G' = G - uv + uw + vw \). By direct calculation, we obtain \( \rho(\mathcal{L}(G)) = 1.8993 < 1.9382 = \rho(\mathcal{L}(G')) \).

**Case 2** \( \rho(\mathcal{L}(G)) \geq \rho(\mathcal{L}(G')) \). Let \( G = G_d \) and 1, 2 stand for \( u, v \), respectively. It is clear that \( f(u)f(v) < 0 \). Denote \( G' = G - uv + uw + vw \). By direct calculation, we obtain \( \rho(\mathcal{L}(G)) = \rho(\mathcal{L}(C_4)) = 2 > 1 - \cos \frac{\pi}{2} = \rho(\mathcal{L}(C_5)) = \rho(\mathcal{L}(G')) \).

The following theorem studies the behavior of \( \rho(\mathcal{L}) \) when the graph is perturbed by Operation II.

**Theorem 4.2** Let \( G_1 \) and \( G_2 \) be two simple connected graphs of orders \( m \) and \( n \), respectively. Let \( u \in V(G_1) \) and \( v \in V(G_2) \). Let \( G \) be a graph obtained from \( G_1 \) and \( G_2 \) by identifying \( u \) with \( v \). Let \( f_1 \) be a harmonic eigenvector associated with \( \rho(\mathcal{L}(G_1)) \). If \( f_1(u) = 0 \), then \( \rho(\mathcal{L}(G)) \geq \rho(\mathcal{L}(G_1)) \).

**Proof** Let \( V(G_1) = \{x_1, x_2, \ldots, x_{m-1}, u\} \), \( V(G_2) = \{y_1, y_2, \ldots, y_{n-1}, v\} \), and \( V(G) = \{x_1, x_2, \ldots, x_{m-1}, u, y_1, y_2, \ldots, y_{n-1}\} \). Let \( d(x), d_1(x) \) and \( d_2(x) \) be the degree of \( x \) in \( G \), the degree of \( x \) in \( G_1 \), and the degree of \( x \) in \( G_2 \), respectively. Let \( D \) and \( D_1 \) be the diagonal degree matrices of \( G \) and \( G_1 \), respectively. Let \( L \) and \( L_1 \) be the Laplacian matrices of \( G \) and \( G_1 \), respectively. Let \( e \) and \( e_1 \) be the vectors consisting of all ones, where \( e \in \mathbb{R}^{m+n-1} \) and \( e_1 \in \mathbb{R}^m \). Then \( d(x_i) = d_1(x_i), i = 1, 2, \ldots, m - 1, d(y_j) = d_2(y_j), j = 1, 2, \ldots, n - 1, \) and \( d(u) = d_1(u) + d_2(v) \). Since \( f_1 \) is a harmonic eigenvector associated with \( \rho(\mathcal{L}(G_1)) \). Then \( f_1 \neq 0 \) and \( f_1 \perp D_1 e_1 \).
Let \( f(x) = f_1(x), \forall x \in V(G_1),\ f(y_j) = 0, j = 1, 2, \ldots, n - 1. \) Then we have

\[
 f^T Df = \sum_{x \in V(G)} d(x) f(x)^2 = \sum_{x \in V(G_1)} d(x) f(x)^2 + \sum_{j=1}^{n-1} d(y_j) f(y_j)^2
\]

\[
 = \sum_{x \in V(G_1) \setminus \{u\}} d(x) f(x)^2 + d(u) f(u)^2
\]

\[
 = \sum_{x \in V(G_1) \setminus \{u\}} d_1(x) f_1^2(x) + (d_1(u) + d_2(v)) f_1^2(u)
\]

\[
 = f_1^T D_1 f_1 + d_2(v) f_1^2(u) = f_1^T D_1 f_1,
\]

\[
 f^T Lf = \sum_{xy \in E(G)} (f(x) - f(y))^2
\]

\[
 = \sum_{xy \in E(G_1)} (f_1(x) - f_1(y))^2 + \sum_{xy \in E(G_1) \setminus E(G_1)} (f(x) - f(y))^2
\]

\[
 = \sum_{xy \in E(G_1)} (f_1(x) - f_1(y))^2 + \sum_{uy \in E(G)} (f(u) - f(y))^2
\]

\[
 = f_1^T L_1 f_1 + \sum_{uy \in E(G)} f^2(u)
\]

\[
 = f_1^T L_1 f_1 + d_2(v) f_1^2(u) = f_1^T L_1 f_1,
\]

and

\[
 f^T De = \sum_{x \in V(G)} d(x) f(x) = \sum_{x \in V(G_1)} d(x) f(x)
\]

\[
 = \sum_{x \in V(G_1) \setminus \{u\}} d(x) f(x) + d(u) f(u)
\]

\[
 = \sum_{x \in V(G_1) \setminus \{u\}} d_1(x) f_1(x) + (d_1(u) + d_2(v)) f_1(u)
\]

\[
 = f_1^T D_1 e_1 + d_2(v) f_1(u) = f_1^T D_1 e_1 = 0.
\]

Thus \( f \perp D e. \) Note that \( f \neq 0. \) Then, we have

\[
 \frac{f^T L f}{f^T D f} \leq \rho(L(G)).
\]

Thus, from Formula 2, we have

\[
 \rho(L(G_1)) = \frac{f_1^T L_1 f_1}{f_1^T D_1 f_1} = \frac{f^T L f}{f^T D f} \leq \rho(L(G)).
\]

The result follows. □

It is clear that the proof of Theorem 4.2 is similar to the proof of Theorem 3.3.
Figure 3 shows that the condition $f_1(u) = 0$ of Theorem 4.2 is necessary. If $f_1(u) \neq 0$, then the relation between values of $\rho(L(G_1))$ and $\rho(L(G))$ is not sure. There are the following two cases.

**Case 1** $\rho(L(G_1)) < \rho(L(G))$. Let $G_1 = G_0$, 7 stand for $u$ and $G_2 = P_2$. It is clear that $f_1(u) \neq 0$. $G$ is obtained from $G_1$ and $G_2$ by Operation 2. By direct calculation, we obtain $\rho(L(G_1)) = 1.8993 < 1.9382 = \rho(L(G))$.

**Case 2** $\rho(L(G_1)) \geq \rho(L(G))$. Let $G_1 = G_d = C_4$, 4 stand for $u$ and $G_2 = C_3$. It is clear that $f_1(u) \neq 0$. $G$ is obtained from $G_1$ and $G_2$ by Operation 2. By direct calculation, we obtain $\rho(L(G_1)) = \rho(L(C_4)) = 2 > 1.9010 = \rho(L(G))$.

The following theorem studies the behavior of $\rho(L)$ when the graph is perturbed by Operation 3.

**Theorem 4.3** Let $u, v$ be two vertices of the simple connected graph $G$ of order $n$. Suppose that $v_1, v_2, \ldots, v_s$ ($1 \leq s \leq d(v)$) are some vertices of $N_G(v) \setminus N_G(u)$ and $v_1, v_2, \ldots, v_s$ are different from $u$. Let $G' = G - vv_1 - vv_2 - \cdots - vv_s + uv_1 + uv_2 + \cdots + uv_s$, and $f$ be a harmonic eigenfunction associated with $\rho(L(G))$. If $f(u) = f(v)$, then $\rho(L(G)) \leq \rho(L(G'))$.

**Proof** Let $d(x)$ and $d'(x)$ be the degree of $x$ in $G$ and the degree of $x$ in $G'$, respectively. Let $D$ and $D'$ be the diagonal degree matrices of $G$ and $G'$, respectively. Let $L$ and $L'$ be the Laplacian matrices of $G$ and $G'$, respectively. Let $e$ be the vector consisting of all ones, where $e \in R^{[G]}$. Then $d'(v) = d(v) - s$. $d'(u) = d(u) + s$, $d'(x) = d(x)$, where $x \in V(G) \setminus \{u, v\}$. Since $f$ is a harmonic eigenfunction associated with $\rho(L(G))$. Then $f \neq 0$ and $f \perp De$. Let $f'(u) = f(u), \forall u \in V(G)$. Then $f'(u) = f(u) = f(v) = f'(v)$.

Similar to the proof of Theorem 3.6, we have

$$f'^TL'f' = f^TLe, f'^TD'f' = f^TDe = 0.$$ 

Thus $f' \perp D'e$. Note that $f' \neq 0$. Then we have

$$\frac{f'^TL'f'}{f'^TD'f'} \leq \rho(L(G')).$$ 

Thus, from Formula (2), we have

$$\rho(L(G)) = \frac{f^TLf}{f^TDf} = \frac{f'^TL'f'}{f'^TD'f'} \leq \rho(L(G')).$$

The result follows. $\square$

Figure 3 shows that the condition $f(u) = f(v)$ of Theorem 4.3 is necessary. If $f(u) \neq f(v)$, then the relation between values of $\rho(L(G))$ and $\rho(L(G'))$ is not sure. There are the following three cases.

**Case 1** $\rho(L(G)) < \rho(L(G'))$. Let $G = G_c$ and 4, 3, 5 stand for $u$, $v$, $v_1$, respectively. It is clear that $f(u) > f(v)$. Denote $G' = G - vv_1 + uv_1$. By direct calculation, we obtain $\rho(L(G)) = 1.8993 < 1.9063 = \rho(L(G'))$.

**Case 2** $\rho(L(G)) = \rho(L(G'))$. Let $G = G_c$ and 1, 3, 5 stand for $u$, $v$, $v_1$, respectively. It is clear that $f(u) > f(v)$. Denote $G' = G - vv_1 + uv_1$. Because $G$ is isomorphic to $G'$, we obtain $\rho(L(G)) = \rho(L(G')) = 1.8993$.

**Case 3** $\rho(L(G)) > \rho(L(G'))$. Let $G = G_c$ and 2, 5, 6 stand for $u$, $v$, $v_1$, respectively. Considering $g = -f$ as the harmonic eigenfunction associated with $\rho(L(G))$. It is clear that $g(u) = -0.0214 > -0.3250 = g(v)$. Denote $G' = G - vv_1 + uv_1$. By direct calculation, we obtain $\rho(L(G)) = 1.8993 > 1.8243 = \rho(L(G'))$. 

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From the above, we can see that if $f(u) > f(v)$, then the relation between values of $\rho(\mathcal{L}(G))$ and $\rho(\mathcal{L}(G'))$ is not sure. Note that if $f$ is a harmonic eigenfunction associated with $\rho(\mathcal{L}(G))$, then $-f$ is also a harmonic eigenfunction associated with $\rho(\mathcal{L}(G))$. When $f(u) < f(v)$, the similar result is obtained.

References

[1] F.R.K. Chung, Spectral Graph Theory, American Math. Soc. Providence, 1997.

[2] S. Butler, Eigenvalues and structures of graphs, Ph.D. dissertation, University of California, San Diego, 2008.

[3] H.H. Li, J.S. Li, Y.-Z. Fan, The effect on the second smallest eigenvalue of the normalized Laplacian of a graph by grafting edges, Linear Multilinear Algebra 56 (2008), 627–638.

[4] H.H. Li, J.S. Li, A note on the normalized Laplacian spectra, Taiwanese J. Math. 15 (2011), 129–139.

[5] J. Li, J.-M. Guo, W.C. Shiu, A. Chang, An edge-separating theorem on the second smallest normalized Laplacian eigenvalue of a graph and its applications, Discrete Appl. Math. 171 (2014), 104–115.

[6] J.X. Li, J.-M. Guo, W.C. Shiu, A. Chang, Six classes of trees with largest normalized algebraic connectivity, Linear Algebra Appl. 452 (2014), 318–327.

[7] J.-M. Guo, J. Li, W.C. Shiu, The Largest Normalized Laplacian Spectral Radius of Non-Bipartite Graphs, Bull. Malays. Math. Sci. Soc. 2015:DOI 10.1007/s40840-015-0241-y 1–11.