In-betweenness, a Geometrical Monotonicity Property for Operator Means

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Abstract

We introduce the notions of in-betweenness and monotonicity with respect to a metric for operator means. These notions can be seen as generalising their natural counterpart for scalar means, and as a relaxation of the notion of geodesity. We exhibit two classes of non-trivial means that are monotonic with respect to the Euclidean metric. We also show that all Kubo-Ando means are monotonic with respect to the trace metric, which is the natural metric for the geometric mean.

Key words: Power Means, Heinz Means, Kubo-Ando means, Monotonicity
1991 MSC: 15A60

1 Introduction

According to the highly respected Merriam-Webster’s dictionary, a mean is “a value that lies within a range of values and is computed according to a prescribed law.” The best-known examples of means in this sense are the arithmetic mean and the geometric mean of two real scalars $x$ and $y$, defined by the ‘prescribed laws’ $\mu(x, y) = (x + y)/2$ and $\mu(x, y) = \sqrt{xy}$, respectively. As is easily checked, these means indeed lie ‘within the range’ $[x, y]$. Many more means have been defined, like the harmonic mean $\mu(x, y) = 2(1/x + 1/y)^{-1}$ and the power means $\mu(x, y) = ((x^p + y^p)/2)^{1/p}$ (with $p \geq 1$), and they all share this property of being contained in the interval $[x, y]$. At least for real numbers, the dictionary definition appears mathematically correct. For succinctness, we will call this property that for all $x \leq y$, $x \leq \mu(x, y) \leq y$, the in-betweenness property of a mean $\mu$.

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The basic notion of mean has been extended to more general mathematical objects, like functions, vectors, matrices and operators. Because of the more complicated structure of these objects, it no longer makes sense in general to say that the mean of objects $f$ and $g$ ‘lies within a range’ defined by $f$ and $g$. The definition of in-betweenness for scalar means inherently relies on the endowment of $\mathbb{R}$ with a total ordering, $(\mathbb{R}, \leq)$. For more complicated structures a partial ordering is the best one can hope for, which in itself does not provide a solid foundation for an in-betweenness property.

In a number of cases the geometry of the space in which the mean is defined induces a total ordering; this happens when the mean can be parameterised as $t \mapsto \mu(X, Y, t)$ and the path traced out by varying $t$ is a geodesic with respect to the chosen metric of the space. A well-known example is the geometric mean, which can be parameterised as $x^t y = x^t y^{1-t}$, or as $A^t B = A^{1/2}(A^{-1/2} B A^{1/2})^t A^{1/2}$ for positive operators. It can be shown that the path $t \mapsto A^t B$ is a geodesic with respect to the trace metric [5] (see below). In-betweenness with respect to the metric then follows by definition.

In general, however, it need not be straightforward to parameterise a given mean and then find a metric such that the mean lies on a geodesic. Secondly, the context in which the mean is to be used might impose a different metric and checking in-betweenness is no longer trivial. Thirdly, it is fair to say that most means have not been defined starting from such geometric considerations. Often, the only claim that such means can lay on their name is the close resemblance between their defining prescribed law and a similar law defined for their scalar counterpart. The fact that one has proceeded with the definition of these means anyway is largely due to their applicability. Amongst the better-known means for positive operators are the arithmetic mean $(A, B) \mapsto (A + B)/2$, the geometric mean $(A, B) \mapsto A^t B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$ and the harmonic mean $(A, B) \mapsto A! B = ((A^{-1} + B^{-1})/2)^{-1}$ (for invertible $A$ and $B$, that is).

Within the last three decades, the area of operator means became largely dominated by what is now known as the class of Kubo-Ando operator means. In a beautiful and very influential paper, Kubo and Ando [9] introduced a set of axioms and showed that they were satisfied by a large number of the then known operator means, including the abovementioned arithmetic, geometric and harmonic mean. Moreover, they completely characterised the class of such means and showed that they are in one-to-one correspondence with the non-negative operator monotone functions on $(0, +\infty)$. The ensuing theory stimulated a lot of research because of its connections to Riemannian geometry, and its applications in mathematical physics. As a case in point, one should note that the Kubo-Ando axioms do not appeal to any underlying geometry, metric or geodesic.
It has to be emphasised that not all operator means in current use are Kubo-Ando means. We mention only two prominent examples here, as they are the subject of the technical part of this paper. Our first example is the class of power means, studied for operators by Bhagwat and Subramanyan [3]. They are defined as

$$(A, B) \mapsto ((A^p + B^p)/2)^{1/p},$$  

(1) with $p \in \mathbb{R}$. Clearly, this class contains the arithmetic mean ($p = 1$), and the harmonic mean ($p = -1$). These power means are Kubo-Ando means only when $-1 \leq p \leq 1$. In spite of this, the power means with $p > 1$ have many important applications, e.g. in mathematical physics and in the theory of operator spaces, where they form the basis of certain generalisations of $\ell_p$ norms to non-commutative vector-valued $L_p$ spaces [7].

Our second example of non-Kubo-Ando means is the class of Heinz means. The Heinz means for non-negative scalars are weighted versions of the geometric mean:

$$H_\nu(x, y) = x^\nu y^{1-\nu},$$  

with $0 \leq \nu \leq 1$. Sometimes another definition is adopted that is slightly more symmetrical [6]. Namely, the symmetric Heinz mean is defined as $H'_\nu(x, y) = (x^\nu y^{1-\nu} + x^{1-\nu} y^\nu)/2$, which is invariant under replacing $\nu$ by $1 - \nu$. The reason for this convention is that the symmetric Heinz mean interpolates between the arithmetic mean ($H'_0(x, y) = H'_1(x, y) = (x + y)/2$) and the geometric mean ($H'_{1/2}(x, y) = \sqrt{xy}$).

These two definitions carry over to operators in a straightforward way: one defines the symmetric Heinz mean as $H'_\nu(A, B) = (A^\nu B^{1-\nu} + A^{1-\nu} B^\nu)/2$, and the unsymmetric one as $H_\nu(A, B) = A^\nu B^{1-\nu}$. Clearly, these means cannot be Kubo-Ando means as they violate the first axiom of closure. In general, the Heinz mean of two positive operators is not even self-adjoint, let alone positive.

Nevertheless, the Heinz means have great importance. The unsymmetric Heinz mean, in particular, is a basic quantity in quantum physics. When applied to density operators, the logarithm of the trace $\log \text{Tr} H_\nu(\rho, \sigma) = \log \text{Tr} \rho^\nu \sigma^{1-\nu}$ is known as the relative Renyi entropy. The normalised mean itself, $\rho^\nu \sigma^{1-\nu}$ divided by its trace, can be considered as a quantum generalisation of the so-called Hellinger arc between two probability distributions [2]. In the present manuscript we will only consider the unsymmetric version of the Heinz mean, for reasons of simplicity.

In this paper we shall investigate one possible route towards defining an in-betweenness property for operator means, overcoming the lack of a total ordering and of the existence of a natural metric. In fact we will define two varieties of such a property. Both are based on endowing the set of positive operators with a simple Euclidean geometry; this is the topic of Section 3. For any defini-
tion to be meaningful, one would normally expect the existence of objects that satisfy it. We show that there exist indeed non-trivial operator means (apart from the arithmetic mean) that satisfy this kind of in-betweenness, namely the power means, in Section 4 and the Heinz means, in Section 5. Next, in Section 6 we will exhibit a counterexample that shows that Kubo-Ando means generally do not satisfy in-betweenness with respect to the Euclidean distance. In contrast, we will prove that they all satisfy in-betweenness with respect to the trace metric distance, the metric whose geodesics are traced out by the geometric means. In Section 7 we conclude and briefly state further research directions.

2 Kubo-Ando means

The Kubo-Ando axioms are the following, with \( \sigma \) the generic symbol for a mean in the Kubo-Ando sense, and \( A, B, C, D \) arbitrary non-negative operators:

1. **Closure:** A mean is a binary operation on and into the class of positive operators, \( A\sigma B \geq 0 \);
2. **Monotonicity:** \( A \leq C \) and \( B \leq D \) imply \( A\sigma B \leq C\sigma D \);
3. **Transformer inequality:** \( C(A\sigma B)C \leq (CAC)\sigma(CBC) \);
4. **Continuity:** \( A_n \downarrow A \) and \( B_n \downarrow B \) imply \( (A_n\sigma B_n) \downarrow (A\sigma B) \);
5. **Normalisation:** \( 11\sigma 11 = 1 \).

Here, the notation \( A_n \downarrow A \) is a shorthand for the statement that there is a sequence of positive operators \( A_1 \geq A_2 \geq \ldots \geq A_n \) with \( A_n \) converging strongly to \( A \). For further information about these axioms we refer to [9].

Dropping the normalisation condition, Kubo and Ando then showed that for any mean \( \sigma \) satisfying these axioms, the function \( f(x) := 1\sigma x \) is a non-negative operator monotone function on \((0, +\infty)\). Conversely, for any non-negative operator monotone function \( f \) on \((0, +\infty)\), there is a mean satisfying the axioms, via the construction

\[
A\sigma B = A^{1/2} f(A^{-1/2}BA^{-1/2}) A^{1/2}.
\]

Because of this correspondence, \( f \) is called the representing function of the mean.

For example, the power means \([11]\) are Kubo-Ando means only when \( -1 \leq p \leq 1 \), as this is the condition for operator monotonicity of the representing function \( f(x) = ((1 + x^p)/2)^{1/p} \).
Exploiting the theory of operator monotone functions, Kubo and Ando arrived at an integral representation of any mean satisfying their axioms (excluding the normalisation condition), see Theorem 3.4 in [9]. Given any positive Radon measure $\mu(s)$ on $[0, +\infty]$, there is a unique corresponding Kubo-Ando mean represented as

$$A\sigma B = aA + bB + \int_{(0, +\infty)} \frac{1 + s}{s} (sA) : B \, d\mu(s),$$

with $a = \mu(\{0\})$ and $b = \mu(\{+\infty\})$.

This formula can be conveniently rewritten in terms of the *weighted harmonic mean* $A!_t B$. We define this mean for $0 \leq t \leq 1$ and positive operators $A$ and $B$, as

$$A!_t B = (tA^{-1} + (1 - t)B^{-1})^{-1}.$$  

Note that Hansen also defined a weighted harmonic mean, but with a different parametrisation of $t$, ranging over the interval $[0, +\infty]$ [8]. The extremal cases are $A!_0 B = B$ and $A!_1 B = A$. In terms of the parallel sum [1]

$$A : B := (A^{-1} + B^{-1})^{-1} = B - B(A + B)^{-1}B,$$

this formula can be rewritten as

$$A!_t B = (A/t) : (B/(1 - t)) = \frac{1}{1 - t} \left( B - B \left( \frac{1 - t}{t} A + B \right)^{-1} B \right).$$

For non-invertible $A$ and/or $B$, one can replace the inverse in the latter formula by the pseudo-inverse.

Performing the substitution $s = (1 - t)/t$ (so that $t = 1/(1 + s)$), we get $A!_t B = \frac{1 + s}{s} (sA) : B$, which appears in the original integral representation [2]. Introducing the transformed Radon measure $\nu(t)$ with $d\nu(t) = -d\mu(s)$, we obtain the very simple representation of a Kubo-Ando mean

$$A\sigma B = \int_0^1 A!_t B \, d\nu(t),$$

where $aA$ and $bB$ have been absorbed into the integral. The normalisation condition $1\sigma 1 = 1$ then imposes the condition $\int_0^1 d\nu(t) = 1$, which says that $d\nu(t)$ must be a probability density. In other words, the class of Kubo-Ando
means \( A \sigma B \) are all possible convex combinations of weighted harmonic means \( A!_t B, 0 \leq t \leq 1 \).

Returning to the axioms that define the Kubo-Ando means, and comparing them to the ‘dictionary’ definition of means, none of these axioms comes very close in spirit to an in-betweenness property. The closest match, the monotonicity axiom, is not a comparison between a mean and its arguments, but a comparison between means of different pairs of arguments. The conundrum of defining a mean on a partially ordered structure has been solved here in a different way. Nevertheless, one can still ask the question whether it is not possible to reconcile the two definitions. Kubo-Ando means might still satisfy an in-betweenness property of some sorts, just like their classical scalar counterparts, not by definition but as an indirect consequence of its definition. We will answer this question affirmatively in Section 6.

### 3 Distance and Angle Monotonicity

In this paper, we shall be dealing with the space of self-adjoint trace class operators. Endowing this space with the Hilbert-Schmidt inner product \( (A, B) = \text{Tr}[A^*B] \) turns it into a real Euclidean vector space. As positive operators form a subset, \( S \), of this space, it makes perfect sense to study \( S \) from the viewpoint of Euclidean geometry too and consider Euclidean distances and angles in \( S \), the fact notwithstanding that nowadays \( S \) is usually studied from the Riemannian viewpoint, as a manifold of nonpositive curvature when endowed with the proper metric (see, e.g. [5]). One of the more obvious benefits of the Euclidean approach is that it also applies to non-positive and even non-selfadjoint operators.

In accordance with Euclidean geometry, we define Euclidean distance and angles in the usual way. These definitions apply, in particular, to positive operators:

**Definition 1** The Euclidean distance \( d \) between two trace class operators \( A, B \) is defined as

\[
d(A, B) = \sqrt{\text{Tr}[(A - B)^*(A - B)]}.
\]

**Definition 2** The angle \( \theta \) between two non-zero trace class operators \( A, B \) is defined as

\[
\cos(\theta) = \frac{\Re \text{Tr}[A^*B]}{\sqrt{\text{Tr}[A^*A]\text{Tr}[B^*B]}}.
\]
For self-adjoint operators, the $\Re$ operation can obviously be dropped.

Based on the Euclidean distance and angle, we will now define two related in-betweenness properties for means of positive operators or matrices. In the following, $A$ and $B$ will always be positive. We shall say that:

**Definition 3** An operator mean $\mu$ satisfies in-betweenness w.r.t. Euclidean distance if and only if for all positive $A$ and $B$ the distance between $A$ and $\mu(A, B)$ does not exceed the distance between $A$ and $B$.

In other words, we shall be demanding that $\mu(A, B)$ lies in the Euclidean norm ball with centre $A$ and surface containing $B$.

**Definition 4** An operator mean $\mu$ satisfies in-betweenness w.r.t. angle if and only if for all positive $A$ and $B$ the angle between $A$ and $\mu(A, B)$ does not exceed the angle between $A$ and $B$.

This condition requires that $\mu(A, B)$ lies in the cone of revolution with origin in the zero operator, central axis lying along the direction of $A$, and generated by the direction of $B$.

One can easily extend these concepts to weighted operator means $\mu(A, B, t)$, where $t$ is a real scalar in the range $[0, 1]$ that expresses how much $A$ dominates over $B$. The weighted arithmetic mean, for example, is simply defined as $\mu_{ar}(A, B, t) = tA + (1 - t)B$. In general, $\mu(A, B, 0) = B$, $\mu(A, B, 1) = A$, and the non-weighted mean is obtained as $\mu(A, B) = \mu(A, B, 1/2)$. For weighted means, the in-betweenness properties can be stated more strongly as monotonicity properties.

**Definition 5** A weighted operator mean $\mu(A, B, t)$ is distance-monotonic if and only if the Euclidean distance between $A$ and $\mu(A, B, t)$ decreases monotonically with $t \in [0, 1]$.

Thus, for a distance-monotonic mean, $\text{Tr}|A - \mu(A, B, t)|^2$ should decrease monotonically with $t \in [0, 1]$.

**Definition 6** A weighted operator mean $\mu(A, B, t)$ is angle-monotonic if and only if the angle between $A$ and $\mu(A, B, t)$ decreases monotonically with $t$.

This condition is equivalent to the monotonic increase of the function

$$t \mapsto \frac{(\Re \text{Tr} A\mu(A, B, t))^2}{\text{Tr} \mu(A, B, t)^2}.$$

In the case of scalar means, distance-monotonicity becomes monotonic decrease of $(a - \mu(a, b, t))^2$, which is the original in-betweenness property and
which should therefore hold for any reasonable definition of a weighted scalar mean. Furthermore, angle-monotonicity is trivially satisfied, as angles between real positive scalars are always 0. Finally, it goes without saying that the weighted arithmetic mean is monotonous with respect to Euclidean distance, because it is geodesic for the Euclidean metric.

4 Monotonicity of the Power Means

In this section, we will prove that the $p$-power means satisfy in-betweenness, both with respect to Euclidean distance and w.r.t. angles, whenever $1 \leq p \leq 2$. Moreover, defining a weighted $p$-power mean as

$$\mu_p(A, B, t) = (tA^p + (1-t)B^p)^{1/p},$$

we show that for $1 \leq p \leq 2$ it is both distance-monotonic and angle-monotonic.

We conjecture that these results holds more generally for larger values of $p$. The technique we use in our proofs, however, ultimately relies on the fact that in the given range of $p$, the function $x \mapsto x^p$ is convex, while the function $x \mapsto x^{p/2}$ is concave. To extend the proofs to larger values of $p$ will require a different technique.

We begin by showing that for power means the monotonicity statement is not really stronger than in-betweenness.

Lemma 1 Let $A$ and $B$ be positive operators, and let $f(t)$ be a function of $\mu_p(A, B, t)$ and $A$ (not $B$). Then $f(t)$ is monotonously increasing over $t \in [0, 1]$ if and only if $f(t) \geq f(0)$ for $t \in [0, 1]$.

Proof. Define $\tilde{B} = \mu_p(A, B, t_1) = (t_1A^p + (1-t_1)B^p)^{1/p}$ and note that if $t_2 \geq t_1$ then $\mu_p(A, B, t_2)$ can be expressed as a $(t_3A^p + (1-t_3)\tilde{B}^p)^{1/p}$ for a certain $t_3$ in the interval $[0, 1]$. Indeed, let $t_3$ be such that $t_2 = t_3 + (1-t_3)t_1$, then

$$\mu_p^p(A, B, t_2) = t_2A^p + (1-t_2)B^p$$
$$= (t_3 + (1-t_3)t_1)A^p + (1-t_3)(1-t_1)B^p$$
$$= t_3A^p + (1-t_3)\tilde{B}^p = \mu_p^p(A, \tilde{B}, t_3).$$

We also have $\mu_p(A, B, t_1) = \tilde{B} = \mu_p(A, \tilde{B}, 0)$. Thus the inequality $f(t_1) \leq f(t_2)$ reduces to $f(0) \leq f(t_3)$ when $B$ is replaced by $\tilde{B}$. □

Theorem 1 Let $A$ and $B$ be positive operators, $0 \leq t \leq 1$ and $1 \leq p \leq 2$. Then $\text{Tr}(A - \mu_p(A, B, t))^2$ decreases monotonically with $t$. 

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Proof. By the lemma it is enough to show that
\[ \text{Tr}(A - \mu_p(A, B, t))^2 \leq \text{Tr}(A - B)^2. \] (9)

Since \( \text{Tr}(A - B)^2 \geq 0, \)
\[ \text{Tr}(A^2 + B^2) \geq 2 \text{Tr}AB. \] (10)

By operator convexity of the function \( x \mapsto x^{2/p} \) for \( 1 \leq p \leq 2, \)
\[ \text{Tr} \mu_p^2(A, B, t) \leq \text{Tr} \mu_2^2(A, B, t) = \text{Tr}(tA^2 + (1 - t)B^2). \] (11)

Combining (10), multiplied with \( t, \) and (11) gives
\[
2t \text{Tr}AB \leq \text{Tr}(tA^2 + tB^2)
= \text{Tr}(2tA^2 + B^2 - tA^2 + (1 - t)B^2)))
\leq \text{Tr}(2tA^2 + B^2 - \mu_p^2(A, B, t)).
\] (12)

By operator concavity of \( x \mapsto x^{1/p} \) for \( 1 \leq p \leq 2, \)
\[ \mu_p(A, B, t) \geq \mu_1(A, B, t) = tA + (1 - t)B, \]
so that (by the fact that \( A \geq 0) \)
\[ \text{Tr}[A\mu_p(A, B, t)] \geq t \text{Tr}A^2 + (1 - t) \text{Tr}AB. \]
Combining this with (12) gives
\[
\text{Tr}[\mu_p^2(A, B, t)] - 2 \text{Tr}[A\mu_p(A, B, t)]
\leq \text{Tr}(2tA^2 + B^2 - 2tAB) - 2 \text{Tr}(tA^2 + (1 - t)AB)
= \text{Tr}B^2 - 2 \text{Tr}AB.
\]

Adding \( \text{Tr}A^2 \) to both sides finally gives [9]. \( \Box \)

Now we do the same for angle-monotonicity.

**Theorem 2** Let \( A \) and \( B \) be positive operators, \( 0 \leq t \leq 1 \) and \( 1 \leq p \leq 2. \) Then the following function of \( t \)
\[ f(t) := \frac{(\text{Tr}[A\mu_p(A, B, t)])^2}{\text{Tr}[A^2] \text{Tr}[\mu_p(A, B, t)^2]} \] (13)

is monotonously increasing with \( t. \)
Proof. Again, we can use the lemma to reduce the theorem to the statement
\( f(0) \leq f(t) \) for all \( 0 \leq t \leq 1 \). Dividing out the factor \( \text{Tr} A^2 \) and reorganising
the other factors gives:
\[
(\text{Tr}[AB])^2 \text{Tr}[\mu(A, B, t)^2] \leq \text{Tr}[B^2] (\text{Tr}[A\mu_p(A, B, t)])^2.
\]
By absorbing \( t \) in \( A \) and \( (1 - t) \) in \( B \), this is equivalent to
\[
(\text{Tr}[AB])^2 \text{Tr}[(A^p + B^p)^{2/p}] \leq \text{Tr}[B^2] (\text{Tr}[A(A^p + B^p)^{1/p}])^2.
\]
Let now \( a = \|A\|_2 = (\text{Tr} A^2)^{1/2} \) and \( b = \|B\|_2 \) and define \( G = A/a \) and
\( H = B/b \). Thus \( G \) and \( H \) both have 2-norm equal to 1. The statement then
becomes
\[
(\text{Tr}[GH])^2 \text{Tr}[(a^pG^p + b^pH^p)^{2/p}] \leq (\text{Tr}[a^pG^p + b^pH^p)^{1/p}])^2.
\]
Defining \( s = a^p/(a^p + b^p) \), which is a convex coefficient, this can be further
rewritten as
\[
(\text{Tr}[GH])^2 \text{Tr}[(sG^p + (1 - s)H^p)^{2/p}] \leq (\text{Tr}[sG^p + (1 - s)H^p)^{1/p}]]^2. \tag{14}
\]
We will prove this inequality as follows.

First note that the function \( x \mapsto x^{2/p} \) is convex, hence
\[
(\text{Tr}[GH])^2 \text{Tr}[(sG^p + (1 - s)H^p)^{2/p}] \leq (\text{Tr}[GH])^2 \text{Tr}[sG^2 + (1 - s)H^2]
= (\text{Tr}[GH])^2. \tag{15}
\]
Second, the function \( x \mapsto x^{1/p} \) is operator concave, hence
\[
sG + (1 - s)H \leq (sG^p + (1 - s)H^p)^{1/p},
\]
so that
\[
\text{Tr}[G(sG^p + (1 - s)H^p)^{1/p}] \geq \text{Tr}[G(sG + (1 - s)H)]
= s + (1 - s) \text{Tr}[GH]. \tag{16}
\]
Thirdly, by the Cauchy-Schwarz inequality
\[
\text{Tr}[GH] \leq (\text{Tr}[G^2] \text{Tr}[H^2])^{1/2} = 1,
\]
so that, for all \( 0 \leq s \leq 1 \),
\[
\text{Tr}[GH] \leq s + (1 - s) \text{Tr}[GH]. \tag{17}
\]
Combining the three inequalities \((15), (16)\) squared, and \((17)\), also squared,
gives \((14)\). □
5 Monotonicity of the Heinz Means

In this section we basically prove similar statements as in the previous section but now for the (unsymmetrised) Heinz means. As these means are not positive-operator valued, the $\Re$-operation in the definition of angle is in principle necessary. However, it can still be dropped for the Heinz means because of their special structure and the fact that $\text{Tr} XY$ is real and positive for positive $X$ and $Y$, even though $XY$ is itself not even Hermitian.

First, we need a simple lemma about convex functions.

**Lemma 2** Let $x < y$ be real scalars, and let $a, b$ be distinct real scalars in the open interval $(x, y)$. When the function $f$ is convex over the interval $[x, y]$, the following holds:

$$\frac{f(a) - f(x)}{a - x} \leq \frac{f(y) - f(b)}{y - b}. \quad (18)$$

**Proof.** Suppose first that $a < b$. By convexity of $f$ and $a < b < y$, we have $f(b) \leq (y - b)f(a)/(y - a) + (b - a)f(y)/(y - a)$, so that $(f(y) - f(b))/(y - b) \geq (f(b) - f(a))/(b - a)$. Similarly, from $x < a < b$ follows $(f(b) - f(a))/(b - a) \geq (f(a) - f(x))/(a - x)$. Combining the two inequalities yields inequality (18). For $b < a$ we proceed in a similar way by combining the inequalities $(f(b) - f(a))/(y - b) \geq (f(a) - f(b))/(a - b)$ and $(f(y) - f(b))/(y - b) \geq (f(a) - f(b))/(a - b)$. □

We start by proving angle-monotonicity for the Heinz means.

**Theorem 3** Let $A$ and $B$ be positive operators. For $0 \leq \nu \leq 1$,

$$\text{Tr} B^2(\text{Tr}[A^{1+\nu}B^{1-\nu}])^2 \geq \text{Tr}[A^{2\nu}B^{2(1-\nu)}](\text{Tr}[AB])^2. \quad (19)$$

**Proof.** Define the function $x \mapsto g(x) = \text{Tr}[A^{2x}B^{2(1-x)}]$. Inequality (19) says that $g(x)$ obeys

$$g(0)g^2(1/2 + \nu/2) \geq g(\nu)g^2(1/2).$$

Upon taking logarithms and rearranging terms, this is equivalent to

$$\log g(\nu) - \log g(0) \leq 2(\log g(1/2 + \nu/2) - \log g(1/2)).$$

This would follow from Lemma 2 with $x = 0$, $y = 1/2 + \nu/2$, $a = 1/2$ and $b = \nu$, provided $\log g(x)$ is convex. This convexity is now easily seen to be equivalent with a Cauchy-Schwarz inequality:

$$(\text{Tr}[A^{x+y}B^{2-(x+y)}])^2 \leq \text{Tr}[A^{2x}B^{2(1-x)}] \text{Tr}[A^{2y}B^{2(1-y)}].$$
Taking logarithms gives the required statement
\[
\log g((x + y)/2) \leq (\log g(x) + \log g(y))/2.
\]
\[\square\]

The corresponding result for distance-monotonicity is proven in a similar way.

**Theorem 4** Let \(A\) and \(B\) be positive semidefinite matrices. For \(0 \leq \nu \leq 1\),
\[
\text{Tr } B^2 + 2 \text{Tr}[A^{1+\nu}B^{1-\nu}] \geq \text{Tr}[A^{2\nu}B^{2(1-\nu)}] + 2 \text{Tr}[AB]. \tag{20}
\]

**Proof.** The proof proceeds in the same way as before, but now exploiting the convexity of \(g(x)\). The latter follows immediately from the convexity of \(\log g(x)\) by the fact that \(\exp(x)\) is a monotonously increasing convex function. \(\square\)

### 6 Monotonicity of the Kubo-Ando Means

Given the initial success in finding two non-trivial operator means for which in-betweenness holds, it would be very interesting if the larger class of Kubo-Ando means also satisfied it. This, however, is not the case; at least, not with the current definition of in-betweenness.

We will consider a simple counterexample for the harmonic mean. As already stated, the Kubo-Ando means are built up from the harmonic mean. We choose the following \(2 \times 2\) matrices:

\[
A = \begin{pmatrix} 5 & 7 \\ 7 & 10 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}.
\]

A simple numerical calculation reveals that, for \(t\) between 0 and about 0.32, the Euclidean distance between \(A\) and \(A_t B\) increases, rather than decreases with \(t\). Thus for \(t\) in the interval \([0, 0.32]\), \(||A - A_t B||_2 \not\leq ||A - B||_2\). By rescaling \(A\) and \(B\) one can make this happen at any value of \(t\), including \(t = 1/2\).

This appears to be very unsatisfying at first, but then one has to realise that this is all a matter of geometry. One cannot really expect that quantities that are natural in one geometry should possess properties belonging to another. The strong connections between Kubo-Ando means and hyperbolic Riemannian geometry (non-positive curvature) suggest that maybe one should modify the definition of in-betweenness to reflect this different geometry, and more
particularly the distance measure used to define in-betweenness. This is what we will attempt in the following paragraphs.

Given that the Kubo-Ando means are convex combinations of weighted harmonic means $t_1$, one might start with defining an in-betweenness that works for all the weighted harmonic means, and then take it from there. Looking at the defining formula (5), the following candidate for a distance measure comes to mind: $d_{-1}(A, B) = \|A^{-1} - B^{-1}\|_2$, which is the Euclidean distance between the inverses. It is trivial to see that, with this candidate, in-betweenness holds for all weighted harmonic means. Indeed, for invertible, positive $A$ and $B$, and $0 \leq t \leq 1$:

$$d_{-1}(A, A^{t} B) = \|A^{-1} - (A^{t} B)^{-1}\|_2 = \|A^{-1} - (tA^{-1} + (1 - t)B^{-1})\|_2 = (1 - t) \|A^{-1} - B^{-1}\|_2 = (1 - t) d_{-1}(A, B).$$

This actually shows that $t \mapsto A^{t} B$ is a minimal geodesic with respect to the inverted Euclidean metric $d_{-1}(A, B)$, just like the weighted arithmetic mean defines a geodesic w.r.t. the ordinary Euclidean metric $d$.

Now does it also work for general convex combinations of the weighted harmonic means? The answer is no – it cannot, for the following reason. Given any Kubo-Ando mean $\sigma$, we get another Kubo-Ando mean $\sigma^*$ called the adjoint via the correspondence $A \sigma^* B = (A^{-1} \sigma B^{-1})^{-1}$. Hence a Kubo-Ando mean satisfies in-betweenness w.r.t. inverted Euclidean distance $d_{-1}$ if and only if its adjoint mean satisfies in-betweenness w.r.t. the ordinary Euclidean distance $d$. The latter condition is not always satisfied, as shown by the counterexample above.

This suggests that to find a distance measure for which all Kubo-Ando means are monotonic one should be looking at a distance for which the geodesic is in the ‘middle’ of the set of means. One obvious candidate is the set of weighted geometric means $A^{#_{t}} B = A^{1/2}(A^{-1/2}BA^{-1/2})^{t}A^{1/2}$. It is well-known that these means define a geodesic $t \mapsto A^{#_{t}} B$, with respect to the so-called trace metric distance $\delta(A, B) = ||\log(A^{-1/2}BA^{-1/2})||_2$. That is, $\delta(A, A^{#_{t}} B) = t\delta(A, B)$. Hence these means trivially satisfy in-betweenness w.r.t. the trace metric distance.

It turns out that all Kubo-Ando means are monotonic w.r.t. $\delta$.

**Theorem 5** For all positive operators $A, B$, and Kubo-Ando means $\sigma$,

$$\delta(A, A \sigma B) \leq \delta(A, B). \quad (21)$$

*Proof.* We exploit the fact that $\delta$ is invariant under conjugations. That is, for
all $M$, $\delta(MAM^*, MBM^*) = \delta(A, B)$. Writing $B = A^{1/2}CA^{1/2}$, we then see that we only have to prove the statement for $A = 11$ and $B = C$.

Let $c$ be a real scalar, with $0 \leq c \leq 1$. Then we have, for $0 \leq t \leq 1$,

$$1 \leq t + (1 - t)c^{-1} \leq c^{-1}.$$  

Inverting and taking the integral over $t$ with probability measure $dp(t)$ gives

$$1 \geq \int_0^1 dp(t)(t + (1 - t)c^{-1})^{-1} \geq c.$$  

As every Kubo-Ando mean $\sigma$ can be written as a convex combination of weighted harmonic means, this shows that for every such mean, $1 \geq 1\sigma c \geq c$.

Since the function $|\log x|$ is monotonically decreasing for $x \leq 1$, this implies $|\log 1\sigma c| \leq |\log c|$.

The same inequality can be shown to hold when $0 < c \leq 1$. Passing to positive operators $C$ and taking the $|||\cdot|||_2$ norm gives the required inequality

$$\delta(11, 11\sigma C) = |||\log(11\sigma C)|||_2 \leq |||\log C|||_2 = \delta(11, C).$$

$\Box$

From the proof one sees that this theorem holds more generally for every mean $\mu$ that satisfies the closure axiom (to have positivity), achieves equality in the transformer inequality (to be able to apply the invariance of $\delta$ under conjugations) and scalar in-betweenness $x \leq \mu(x, y) \leq y$.

### 7 Conclusion

In this paper we have introduced the notion of in-betweenness for operator means, and the slightly stronger one of monotonicity with respect to a given distance measure (or metric), for those operator means that admit a simple parameterisation. These notions can be seen as a relaxation of geodesity, in the following sense. When a parameterised operator mean $\mu(x, y, t)$ traces out a geodesic $t \mapsto \mu(x, y, t)$ with respect to a given metric $d$, it satisfies $d(\mu(x, y, t), y) = (1 - t)d(x, y)$ by definition. This, however, requires a careful matching between the parameterisation of the mean and the chosen metric. This may not always be possible, be it for internal or for external reasons. In that case it might still be useful to have monotonicity, which is the *inequality* $d(\mu(x, y, t), y) \leq (1 - t)d(x, y)$. We have exhibited two non-trivial examples
of operator means that are monotonous with respect to the Euclidean metric, even though the Euclidean metric would not be considered the natural one for these means. We have also shown that all the Kubo-Ando means are monotonous w.r.t. the trace metric; in contrast only the (weighted) geometric means are geodesic in this metric.

In this work we have only scratched the surface and many questions remain. Most importantly, it would be very interesting if one could give a full characterisation of operator means that are monotonic w.r.t. a given metric, and possibly come up with an alternative axiomatic approach to operator means.

8 Acknowledgments

I am grateful for the hospitality of the Institut Mittag-Leffler, Djursholm (Sweden) where this manuscript was completed.

References

[1] W.N. Anderson, Jr. and R.J. Duffin, “Series and parallel addition of matrices”, J. Math. Anal. Appl. 26, 576–594 (1969). Lin. Alg. Appl. 422, 279–283 (2007).

[2] K.M.R. Audenaert, M. Nussbaum, A. Szkola and F. Verstraete, “Asymptotic Error Rates in Quantum Hypothesis Testing,” Comm. Math. Phys. 279, 251–283 (2008).

[3] K.V. Bhagwat and A. Subramanian, “Inequalities between means of positive operators”, Math. Proc. Camb. Phil. Soc. 83, 393–401 (1978).

[4] R. Bhatia, Matrix Analysis, Springer, Heidelberg (1997).

[5] R. Bhatia, “On the exponential metric increasing property”, Lin. Alg. Appl. 375, 211–220 (2003).

[6] R. Bhatia, “Interpolating the arithmetic-geometric mean inequality and its operator version”, Lin. Alg. Appl. 413, 355–363 (2006).

[7] E. Carlen and E. Lieb, “A Minkowski Type-Trace Inequality and Strong Subadditivity of Quantum Entropy”, in Amer. Math. Soc. Transl. (2), 189, 59–69 (1999).

[8] F. Hansen, “Means and concave products of positive semi-definite matrices”, Math. Ann. 264, 119–128 (1983).

[9] F. Kubo and T. Ando, “Means of positive linear operators”, Math. Ann. 246, 205–224 (1980).