Corrigendum: A current driven electromagnetic mode in sheared and toroidal configurations (2014 Plasma Phys. Control. Fusion 56 035011)

István Pusztai1,2, Peter J Catto1, Felix I Parra1,3 and Michael Barnes1,4

1 Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
2 Department of Applied Physics, Chalmers University of Technology, SE-41296 Göteborg, Sweden
3 Department of Physics, University of Oxford, Oxford, OX1 3PU, UK
4 Department of Physics, University of Texas at Austin, Austin, TX 78712, USA

E-mail: pusztai@chalmers.se

Received 18 June 2014, revised 22 October 2014
Accepted for publication 6 November 2014
Published 21 November 2014

The local description adopted in the paper assumes that the parallel wave number is proportional to the distance from the resonant surface $k_{\parallel} \propto x$ over the radial extent of the mode ($x \sim k_{\perp}^{-1}$), in which the assumptions $|xq'/q| \ll 1$ and $|xq'/q| \ll 1$ are implicit. When the safety factor $q$ and the current gradient $j_0$ are related through Ampère’s law, it becomes apparent that the rapid variation in $q$, corresponding to the large current gradient needed to destabilize the ideal high-$m$ kink modes, makes the local description break down. For a more detailed discussion of the issue consult [1], while for a detailed discussion of obtaining MHD equations from gyrokinetics see [2]. However, keeping in mind the limitations of the local modeling, the non-physical high-$m$ kink modes can be used as a test of local gyrokinetic codes when a current gradient drive term is implemented.

In the light of the above considerations the following changes to the paper are necessary to avoid misinterpretation of the results.

- The last two sentences of the abstract should be replaced by: We use the non-physical high-$m$ ideal kink modes to test the current gradient drive term in the local gyrokinetic code gs2. Although the comparisons are done outside the validity of the local description the reassuring agreement with analytical results increases our confidence that the code is valid in the low current gradient limit.

- The 4th and 5th sentences in the 3rd paragraph of section 4 should be followed by: However, local codes and theory do not allow a rapid $q$ variation consistent with the strong current gradients needed to destabilize an ideal kink mode.

- The 1st sentence in the 2nd paragraph of section 6 should read: We find that in a radially local theory at sufficiently high current gradient high mode number kink modes are artificially destabilized because local theory and local codes do not allow a rapid $q$ variation consistent with the strong current gradients needed to destabilize an ideal kink mode.

- The 2nd sentence in the 4th paragraph of section 6 should be followed by: However, local codes and theory do not allow a rapid $q$ variation consistent with the strong current gradients needed to destabilize an ideal kink mode. We have demonstrated that our local code passes a test against local theory, albeit outside the range of validity of the local approximation. This increases our confidence that the code is correct in the low current gradient limit.
where the $q$ variation is moderate and the local approximation is valid.

Acknowledgments

The authors are grateful to J Connor and J Hastie for bringing this issue to their attention.

References

[1] Connor J W, Hastie R J, Pusztai I, Catto P J and Barnes M 2014 *Plasma Phys. Control. Fusion* **56** 125006

[2] Zheng L J, Kotschenreuther M T and Van Dam J W 2007 *Phys. Plasmas* **14** 072505
A current-driven electromagnetic mode in sheared and toroidal configurations

István Pusztai¹,², Peter J Catto¹, Felix I Parra¹,³ and Michael Barnes¹,⁴

¹ Plasma Science and Fusion Center, Massachusetts Institute of Technology, Cambridge, MA 02139, USA
² Department of Applied Physics, Chalmers University of Technology, SE-41296 Göteborg, Sweden
³ Department of Physics, University of Oxford, Oxford, OX1 3PU, UK
⁴ Department of Physics, University of Texas at Austin, Austin, TX 78712, USA

E-mail: pusztai@chalmers.se

Received 25 October 2013, revised 6 January 2014
Accepted for publication 17 January 2014
Published 11 February 2014

Abstract
The induced electric field in a tokamak drives a parallel electron current flow. In an inhomogeneous, finite beta plasma, when this electron flow is comparable to the ion thermal speed, the Alfvén mode wave solutions of the electromagnetic gyrokinetic equation can become nearly purely growing kink modes. Using the new ‘low-flow’ version of the gyrokinetic code GS2 developed for momentum transport studies (Barnes et al 2013 Phys. Rev. Lett. 111 055005), we are able to model the effect of the induced parallel electric field on the electron distribution to study the destabilizing influence of current on stability. We identify high mode number kink modes in GS2 simulations and make comparisons to analytical theory in sheared magnetic geometry. We demonstrate reassuring agreement with analytical results both in terms of parametric dependences of mode frequencies and growth rates, and regarding the radial mode structure.

Keywords: kink mode, gyrokinetic, current gradient, instability, simulation

(Some figures may appear in colour only in the online journal)

1. Introduction
The radial gradient of electric current represents a source of free energy in fusion plasmas which can drive or modify instabilities. For a sufficiently strong current gradient, kink modes can be destabilized. The criterion for destabilization in a screw pinch was derived in [1] with a magnetohydrodynamic formulation (there referred to as the screw-instability) for high mode numbers.

Low mode number kink modes are important for the internal stability of tokamaks. The $n = m = 1$ internal kink mode is believed to be responsible for the sawtooth instability [2]. Most of the work done on these modes uses a fluid formalism. However, accounting for kinetic effects is important to reproduce all details of the evolution of such instabilities. For instance, finite electron inertia can assist collisionless reconnection that can modify the dynamics of $m = 1$ internal kinks, as well as their coupling to ion sound waves, as discussed in [3]. In addition, considering fluid ions and kinetic electrons, collisional and diamagnetic effects on the $m = 1$ mode were studied in [4]. In spite of the recognized importance of kinetic effects on kink modes, at present there are only a limited number of numerical studies in the literature which employ gyrokinetic [5, 6] simulations. The first is a simulation of a sawtooth crash that was reported in [7], where a particle-in-cell (PIC) code neglecting ion finite Larmor radius (FLR) effects was used to model the instability for straight field lines. More recently, ideal-MHD internal kink and collisionless $m = 1$ tearing mode simulations were performed in a screw pinch geometry in [8] with the PIC code GYGLES [9].

Gyrokinetic studies on modifications to kinetic instabilities due to parallel current are also very limited. In [10] the effects of equilibrium current on reversed shear Alfvén eigenmodes is studied using the PIC code GTC [11]. Moreover, the continuum gyrokinetic code GENE [12] is used in [13] to study magnetic reconnection, where alternating current sheets are modeled in a periodic slab configuration. Linear gyrokinetic simulations of tearing modes in the collisional–collisionless transitional
regime in a slab geometry are presented in [14] using the ASTROGK code [15]. These last two references introduce the parallel current as a first-order gyrokinetic perturbation, rather than an unperturbed drive term (part of the background distribution) as we do in this article.

It is of interest to further develop our kinetic simulation capability for current driven instabilities. Using the tools available in the new version of the gyrokinetic code GS2 [16], developed for intrinsic rotation studies in tokamaks [17], we are now able to model the destabilizing effect of the modifications to the non-fluctuating electron distribution function due to an induced electric field in a tokamak. In particular, current driven modes can be studied using this continuum gyrokinetic code, as demonstrated herein through simulations of high mode number kink modes with GS2. The GS2 simulations presented here are radially local (flux tube), which inherently assumes a separation of the parallel and perpendicular scale lengths of perturbed quantities. Accordingly, in GS2, only high mode number modes can be simulated, while global modes, such as the \( n = m = 1 \) mode are beyond the region of applicability of local codes. The simulations are done in toroidal geometry and no simplifying assumptions (i.e., regarding FLR effects, kinetic treatment of different species, particle drifts, etc) are made to the Maxwell-gyrokinetic system apart from those consistent with the lowest order local gyrokinetic treatment. The new feature is the treatment of the modification to the electron distribution due to the induced electric field as an unperturbed drive term entering as a part of the non-fluctuating distribution. The code results will be shown to be in very good agreement with the analytical calculations we present.

The subsequent sections are organized as follows. First, in section 2 the electromagnetic gyrokinetic equations are derived in toroidal geometry in the presence of an induced parallel electron current. In section 3, the dispersion relation of the high mode number kink modes is derived in shearless toroidal geometry. The effects of magnetic shear and the eigenmode structure are discussed in section 4. Finally, in section 5 the analytical results are compared to GS2 simulations, before we conclude in section 6.

### 2. Electromagnetic gyrokinetic equations with induced current

The induced electric field driven part of the non-fluctuating electron distribution is similar to the solution of the Spitzer problem, \( C_i[f_{\text{Spit}}] = -(\varepsilon_e/T_e)E_1 v_i f_M \), where \( C_i \) is the linearized electron collision operator, \( f_{\text{Spit}} = n_a[m_a/(2\pi T_a)]^{3/2} \exp[-m_a v^2/(2T_a)] \) is the Maxwell distribution, with the density \( n_a \), temperature \( T_a \), mass \( m_a \) and charge \( e_a \) of species \( a \) (ions and electrons are denoted with the indices \( a = i \) and \( e \), respectively). Furthermore, \( E_1 \) denotes the induced parallel electric field, \( v_i^2 = v \cdot v \) and \( v_i = v - b \), with \( v \) the velocity and \( b \) the unit vector in the direction of the equilibrium magnetic field \( B_0 \). The Spitzer function is proportional to \( v_i \), but it may have a non-trivial speed dependence. However, as it will be shown later through simulations, the exact velocity space structure of \( f_{\text{Spit}} \) is unimportant for the instability to be investigated here. Therefore, the induced electric field effects will be modeled simply by allowing for a parallel drift velocity.

To derive the linearized gyrokinetic equation it is convenient to use the unperturbed total energy, \( E = v^2/2 + (\varepsilon_e/m_e)\phi_0 \), the magnetic moment, \( \mu = v_\parallel^2/(2B_0) \), and the canonical angular momentum \( \psi = \psi - (m_e c/\varepsilon_e)R \xi \cdot v \) as phase-space variables. Here, \( v_\parallel^2 = v^2 - v \cdot b \), \( B_0 = |B_0| \), \( \phi_0 \) is the non-fluctuating part of the electrostatic potential, \( \psi \) denotes the speed of light, \( R \) is the major radius, \( 2\pi \psi \) is the poloidal magnetic flux, and \( \xi = \nabla \xi / \nabla |\xi| \), with the toroidal angle \( \xi \) and \( R|\nabla |\xi| | = 1 \). The unperturbed Vlasov operator \( d_t = \partial_t + v \cdot \nabla + [\varepsilon_a/m_a E_0 + \Omega_a v \times b] \cdot \nabla \psi \) acting on functions of only \( E \) and \( \psi \), vanishes in a toroidally symmetric system which we shall consider. We have introduced \( \Omega_a = e_a B_0/(m_a c) \), with \( E_0 = -\nabla \phi_0 + E_1 \). The time independent piece of the distribution functions should be close to

\[
 f_{na}(\psi, E) = \eta_{na}(m_a/(2\pi T_{na}))^{3/2} \exp[-m_a E^2/(2T_{na})],
\]

where \( T_{na} = T_a(\psi \rightarrow \psi_0) \) and the pseudo-density is \( \eta_{na} = n_{a0} \exp[\varepsilon_a \phi_0/(T_{na})] \) with \( \phi_0 = \phi(\psi \rightarrow \psi_0) \) and \( n_{a0} = n_a(\psi \rightarrow \psi_0) \). Note that \( \eta_{na} \), \( n_{a0} \) and \( T_{na} \) are assumed to be flux functions. We consider \( \phi_0 = 0 \). By construction, \( f_{na} \) reduces to a Maxwellian as \( v \rightarrow \psi \).

In order to account for the electron flow due to the induced electric field, we model the non-fluctuating electron distribution by

\[
 f_{ie} = f_{ae}(\psi, E) + f_i(R_e, E, \mu),
\]

where \( f_i = -m_e v_i u j_0/M_i \), \( R_e = r + \Omega_{ae}^{-1} v_\perp \times b \) is the particle guiding center, and \( r \) is the particle position. Furthermore, \( v_\perp = v - v_b \), and the parallel electron flow velocity is \( -u \), where \( u > 0 \) is allowed to be comparable to the ion thermal speed \( v_i = (2T_i/m_i)^{1/2} \), and the sign of \( u \) is chosen so that the unperturbed current density is \( j_0 = e_n u \). For the electron flow to be divergence free, \( u \propto B_0 \).

The linearized kinetic equation for the fluctuating part of the electron distribution \( f_{ie} \) can be written as

\[
 d_t f_{ie} = -\frac{e_e}{m_e} \left( E_1 + \frac{1}{c} v \times B_1 \right) \cdot \nabla v f_{ie},
\]

where collisions are neglected since we are interested in the tokamak core, where the collision frequency is small. The fluctuating parts of the electric and magnetic fields are denoted by \( E_1 \) and \( B_1 \), respectively. We note, that the induced electric field \( E_1 \) is accounted for by retaining its effect on the non-fluctuating distribution, i.e. keeping \( f_i \) in \( f_{ie} \). The induced electric field is negligible in the \( d_t \) term of (3), since electron-ion drag requires it to be the same order as a collisional correction.

We represent the perturbed vector potential as \( A_\theta = A_{\phi} \nabla \psi + A_{\phi} \nabla \theta + A_\zeta \nabla \xi \), and work in the Coulomb gauge

\[
\]
where we define $\bar{A}_e + \nabla \cdot \bar{A}_e = d_t \bar{A}_e$. At this point we may neglect finite orbit width corrections to the kinetic equation by replacing $\partial_E f_{se}$ by $\partial_E f_{se}$ and $\partial_\phi f_{se}$ by $\partial_\phi f_{Me}$.

The preceding analysis for $f_{se}$ is essentially exact, however, we simplify the analytic treatment for $f_i$ by considering large aspect ratio $\epsilon = r/R \ll 1$ tokamak magnetic geometry with low normalized pressure $\beta = 8\pi p_i/B_0^2 \ll 1$, where $r$ is the minor radius and $p_i = n_i T_i$ is the ion pressure. We assume that the gyrokinetic ordering is satisfied by any perturbed quantity $Q$, namely $b \cdot \nabla Q \ll |Q|$, $1/L \ll |\nabla \ln Q|$, and $f_{ii}/f_{ie} \sim e_i \phi_i / T_i \ll 1$, where $L$ represents the perpendicular scale length of background plasma parameters. Additionally, we assume the $-b \cdot \nabla \phi$ and the $-A_e / c = -b \cdot \bar{A}_e / c \parallel E_1 = b \cdot E_1$ to be comparable in magnitude.

Next we consider $\nabla \cdot f_i$. Neglecting the $\nabla \cdot v|_{v_i}$ term and the poloidal variation of $u$ as small in $\epsilon$, gives

$$\nabla \cdot f_i = \frac{1}{c} \left( v \times B_1 \right) \cdot \nabla f_i = -\frac{v_i}{\Omega_e} (B_0 - R^2 B_0^2 \nabla \phi) \frac{d}{d\psi} \left( \frac{m_i u f_{Me}}{T_e} \right)$$

\begin{equation}
\times \left[ \frac{1}{\Omega_e} E_1 \left( \frac{v_i}{c} - \nabla \cdot A_1 \right) + m_i \frac{d \phi_1}{d \xi} \frac{d A_1}{d t} \right. \\
\left. - v_i \left( \nabla \psi \frac{d A_1}{d \xi} + \nabla \theta \frac{d A_0}{d t} + \nabla \xi \frac{d A_1}{d t} \right) \right] \end{equation}

$$= -\frac{m_i u f_{Me}}{T_e} \left( -v_i \frac{d A_1}{d t} + m_i \frac{d \phi_1}{d \xi} \frac{d A_1}{d t} \right)$$

\begin{equation}
= -\frac{m_i u f_{Me}}{T_e} \left( \nabla f_i + \frac{1}{c} \left( \nabla \psi \frac{d A_1}{d \xi} + \nabla \theta \frac{d A_0}{d t} + \nabla \xi \frac{d A_1}{d t} \right) \right). \end{equation}

(6)

and, combining (3), (4) and (6), we derive the kinetic equation governing this portion of the distribution function

$$\frac{d g_e}{d t} = -\frac{e_i}{T_e} \left( 1 - \frac{m_i u v_i}{T_e} \right)$$

\begin{equation}
\times \left[ \frac{\partial f_i}{\partial t} - v_i \left( \nabla \frac{\partial A_1}{\partial \xi} + \nabla \theta \frac{\partial A_0}{\partial t} + \nabla \xi \frac{\partial A_1}{\partial t} \right) \right] \frac{\partial f_{Me}}{\partial E}$$

$$= -c \left( \frac{\partial f_i}{\partial \phi} - v_i \left( \nabla \psi \frac{\partial A_1}{\partial \xi} + \nabla \theta \frac{\partial A_0}{\partial t} + \nabla \xi \frac{\partial A_1}{\partial t} \right) \right)$$

\begin{equation}
\times F_{ie} - \frac{m_i u f_{Me}}{T_e} F_{2e} + \left( E_i - \frac{v_i}{c} \right) \nabla A_1 \right). \end{equation}

(8)

To obtain this equation we made use of the fact that $d_t$ vanishes when acting on $\partial_E f_{se}$ and $\partial_\phi f_{Me}$, and thus it approximately vanishes when acting on $\partial_E f_{se}$ and $\partial_\phi f_{Me}$, as finite orbit width effects are neglected. Furthermore, we used $v \cdot \nabla \phi_i = (d_t - \bar{A}_e) \phi_i$.

Following a procedure similar to that in \cite{5} we can derive the gyrokinetic equation. After a transformation to gyro center variables, a gyro-phase average of the kinetic equation (8) is performed and finite orbit width effects are neglected where appropriate. We neglect compressional magnetic perturbations as small in the normalized pressure $b \cdot B_1 / B_0 \sim \beta e_i \phi_i / T_i$. For electrons we also neglect FLR corrections. We note that $\nabla \cdot \bar{A}_e$ vanishes upon gyro-phase averaging and the $\propto E_i / \nabla \ln Q$ term in the last line of (8) is small in the gyrokinetic ordering and therefore can be neglected. We arrive at the result

$$\frac{\partial g_e}{d t} + (v_i \parallel + v_i \perp) \cdot \nabla g_e = \frac{e_i}{T_e} \left( \frac{\partial f_i}{\partial t} = -\frac{m_i u v_i}{T_e} \right)$$

$$\times \left[ \frac{1 - m_i u v_i}{T_e} \right] f_{Me} - c f_{Me} \left( \frac{\partial f_i}{\partial \phi} = -\frac{v_i}{c} \frac{\partial A_1}{d \phi} \right)$$

\begin{equation}
\times \left( F_{ie} - \frac{m_i u f_{Me}}{T_e} F_{2e} + \frac{e_i}{T_e} f_{Me} \right). \end{equation}

(9)

where the electron drift velocity of the guiding center in the equilibrium magnetic field is $v_{de}$, and the parallel component of the fluctuating vector potential is $A_1 = I (A_e / \phi_e / R^2)$. This form of $A_1$ follows from assuming straight field line coordinates, i.e. $B_0 \nabla \theta = 1 / (q R^2)$ with $q$ a flux function. The full fluctuating electron distribution $f_{ie}$ and $g_e$ are related by

$$g_e = f_{ie} + e_i \phi_i f_{Me} \left( 1 - \frac{m_i u v_i}{T_e} \right).$$

(10)

where a term $A_e f_{Me} \left( F_{ie} - m_i u v_i / T_e \right)$ has been neglected as small in our ordering. The magnitude of this term will be quantified in the beginning of section 3 when a specific form for the perturbations will be assumed.

We keep FLR corrections when deriving the ion gyrokinetic equation to obtain the usual result

$$\frac{d \phi_i}{d t} + (v_i \parallel + v_i \perp) \cdot \nabla \phi_i = \frac{e_i}{T_i} f_{Mi} \left( \frac{\partial \phi_i}{\partial t} = -\frac{v_i}{c} \frac{\partial A_1}{\partial \phi} \right)$$

$$-c f_{Mi} \left( \frac{\partial \phi_i}{\partial \xi} = -\frac{v_i}{c} \frac{\partial A_1}{\partial \xi} \right) F_{Mi},$$

(11)
where \( \langle \cdot \rangle \) denotes a gyro-phase average at fixed guiding center position, and the relation between \( g_i \) and \( f_i \) is given by

\[
g_i = f_i + e_i \phi_i \left/ T_i \right. f_{Mi}, \tag{12}
\]

where again, a term \( A_i f_{Mi} F_i \) has been neglected for our ordering.

So far we have derived the linearized electromagnetic gyrokinetic equations, where we allow for a parallel flow of electrons. We assumed large aspect ratio and small beta, and neglected \( E_0 \) and electron FLR effects, but otherwise the equations (9)–(12) are still rather general. In the next section we shall derive a dispersion relation for the high mode number kink modes, where further approximations regarding the magnetic geometry and the mode structure will be made.

### 3. Dispersion relation of the high mode number kink mode

In this section we assume a flute like mode structure for the perturbed quantities \( \propto \exp(-\text{i}q \cdot r + \text{i} \theta - \text{i} \zeta \rho) \), where the fluctuations are elongated along magnetic field lines with \( m \approx nq \gg 1 \), where \( q \sim 1 \) is the safety factor. From \( \mathbf{v} \cdot A_1 = 0 \) we find \( A_1 \approx A_1 r^2 n/(R^2 m) \) and thus \( A_1 \approx I A_1 [1 + r^2/(qR)^2]/(BR^2) \approx A_1 /R \) and \( b \cdot \mathbf{\nabla} = \mathbf{i}(m - nq)/(qBR^2) \approx k_i \) with \( k_i \) the parallel wave number. The size \( \parallel \sim n/(R \epsilon) \approx \text{v}_e \) of the perturbed quantities is comparable to \( \text{v}_e \) as compared to the diamagnetic drifts when deriving our dispersion relation. The justification of neglecting magnetic drifts, which is a good approximation at long wavelengths, will be further discussed toward the end of this section. Finally, the drifts, which is a good approximation at long wavelengths, will be further discussed toward the end of this section. Finally, the mode frequency \( \omega \) is assumed to be comparable to or larger than the diamagnetic frequency \( \approx (n_c T_i/\epsilon) \ln(p_i)^2 \). We find that the \( A_i \) terms neglected in the derivation of (10) and (12) are smaller than the \( \phi_i \) terms by \( (k_i v_i/\omega)(\rho_i /L)(\epsilon/\epsilon') \), with \( \rho_i = v_i /\Omega_i \) the ion gyro radius.

From quasi-neutrality we have

\[
0 = \sum_a e_a \int d^3v \left[ g_a + (f_{ia} - g_a) \right], \tag{13}
\]

where \( f_{ia} - g_a \) for electrons and ions are given in (10) and (12). The velocity integrals of the \( f_{ia} - g_a \) parts of the distributions are straightforward to evaluate. Using quasi-neutrality for the unperturbed densities, equation (13) reduces to

\[
\sum_a e_a \int d^3v g_a = e_i n_a \left/ T_e \right. [1 + Z T_e / T_i] \phi_i, \tag{14}
\]

where \( Z = -e_i /e_c \) denotes the ion charge number. The fluctuating parallel current \( j_i \) is given by

\[
j_i = \sum_a e_a \int d^3v v_i [g_a + (f_{ia} - g_a)]. \tag{15}
\]

Again, the velocity integrals of \( f_{ia} - g_a \) can be readily evaluated using the relations (10) and (12), to find

\[
\sum_a e_a \int d^3v v_i g_a = j_i - e_n u \left/ T_e \right. e_i \phi_i. \tag{16}
\]

Note that the velocity integrals of (13)–(16) are taken at fixed particle position, while the \( g_a \) appearing in the gyrokinetic equations are functions of the guiding center position.

When the magnetic drifts are neglected the gyrokinetic equations are of the form

\[
\partial_t \phi_a + v_i \cdot \nabla g_a = \text{RHS}_a, \tag{17}
\]

where \( \text{RHS}_a \) represents the right-hand sides of (9) and (11) for \( a = e \) and \( i \), respectively. We integrate (17) over the velocity space at fixed particle position and sum over species to find

\[
\sum_a \int d^3v \phi_a = \int B \cdot \mathbf{\nabla} \left( \sum_a e_a \int d^3v v_i g_a \right) = \sum_a e_a \int d^3v \text{RHS}_a. \tag{18}
\]

The velocity integrals are to be performed in \( E \) and \( \mu \) variables, and the Jacobian is \( B [v_i]_l \), leading to the form of the second term of (18). For electrons the integral of \( \text{RHS}_e \) gives

\[
e_i \int d^3v \text{RHS}_e = e_i n_e \left/ T_e \right. \left[ \frac{\partial \phi_i}{\partial t} + \frac{u A_j}{c} \frac{\partial A_j}{\partial t} \right] - e_n u_c e_i \frac{\partial \ln n_e}{\partial \zeta} - e_n u_c e_i \frac{\partial \ln (n_i u)}{\partial \psi} + \left. \frac{e^2 e_i}{T_e} n_e u E_1. \right. \tag{19}
\]

For ions we need to account for FLR effects. We use

\[
\int d^3v f_{Mi}(\phi_i) J_0(k_1 v_i/\Omega_i) = \phi_i \int d^3v f_{Mi} J_0^2(k_1 v_i/\Omega_i) \approx n_i \phi_i [1 - (k_1 \rho_i)^2/2] \approx \text{v}_e \text{v}_i \text{v}_p \text{v}_n \text{v}_s \text{v}_g \text{v}_d \text{v}_m \text{v}_o \text{v}_p \text{v}_n \text{v}_s \text{v}_g \text{v}_d \text{v}_m \text{v}_o \text{v}_p \text{v}_n \text{v}_s \text{v}_g \text{v}_d \text{v}_m \text{v}_o \text{v}_p \text{v}_n \text{v}_s \text{v}_g \text{v}_d \text{v}_m \text{v}_o \text{v}_p
\]

\[
\int d^3v \text{RHS}_i = -e_i n_i \left/ T_i \right. \left[ \frac{\partial \ln p_i}{\partial \psi} - \frac{\partial \ln T_i}{\partial \zeta} \right] \frac{\partial \phi_i}{\partial t}. \tag{20}
\]

We then substitute (14), (16), (19) and (20) into (18) and use quasi-neutrality to find

\[
B \cdot \mathbf{\nabla} \left( \frac{j_i}{B} - \frac{e_n u_c}{B} \frac{e_i}{T_e} \phi_i \right) = -e_i n_e \left/ T_e \right. Z T_e \phi_i
\]

\[
+ e_i n_e u_c \frac{\partial A_1}{T_e} \frac{c}{\partial t} + e_i n_e u_c \frac{\partial \ln p_i}{\partial \psi} \frac{\partial \phi_i}{\partial \zeta}
\]

\[
- e_n u_c \frac{\partial A_1}{T_e} \frac{\partial \ln (n_i u)}{\partial \psi} + \frac{e_i n_e}{T_e} u E_1. \tag{21}
\]

Realizing that the \( b \cdot \mathbf{\nabla} \psi \) term on the left-hand side of (21) together with the \( \phi_i \) term on the right-hand side of (21) exactly cancel with the \( E_1 \) term, we can simplify to obtain

\[
ik \cdot j_i = i \omega a n_e \left/ T_e \right. \phi_i - i ne c n_i a_i \frac{\partial \ln p_i}{\partial \psi} \right.+ in e n_e A_{\parallel} \frac{\partial \ln (n_i u)}{\partial \psi}. \tag{22}
\]
where we employ the mode structure $\exp(-i\omega t + i\theta - i\kappa z)$ and define $\tau = ZT_e/T_i$. Introducing the background current gradient and the pressure gradient driven diamagnetic frequencies

$$\omega_{se} = \frac{ncT_e}{e} \frac{\partial \ln j_0}{\partial \psi}, \quad \omega_{pi} = \frac{ncT_i}{e} \frac{\partial \ln \rho_i}{\partial \psi},$$

(23)

with $j_0 = en_i u$ and $\rho_i = n_i T_i$, and multiplying (22) by $-ik_j e T_e/(e^2 n_e)$ we find

$$\frac{k_j^2 e T_e}{e^2 n_e} j_1 = \alpha_i \tau k_i c\phi_1(\omega - \omega_{pi}) + \omega_{se}' k_i u A_i.$$

(24)

Then we employ the parallel Ampère’s law $k_1^2 A_1 = (4\pi/c) j_1$, and recall $\beta_i = 8\pi p_i/B^2 = (v_i/v_A)^2$, where $v_A = [B^2/(4\pi m_i n_i)]^{1/2}$ is the Alfvén speed, to rewrite the left-hand side of (24) as $\tau (v_i k_i)^2 \alpha_i A_1/\beta_i$. We focus on the $E_j \approx 0$ limit, that is $\phi_1 \approx \omega A_1/(k_i c)$. This approximation will be justified at the end of this section. We use this relation to eliminate $\phi_1$ from (24) in favor of $A_1$, and then divide by $\tau (k_i \rho_i)^2/2$ to obtain the dispersion relation

$$\omega(\omega^2 - \omega_{pi}^2) = \beta_i \frac{(v_i k_i)^2}{\beta_i} - \frac{\omega_{se}' k_i u}{\tau \alpha_i}.$$  

(25)

The solution of (25) for the mode frequency is then

$$\omega = \frac{\omega_{pi}}{2} \pm \left( \frac{\omega_{pi}^2}{2} + \frac{(v_i k_i)^2}{\beta_i} - \frac{2\omega_{se}' k_i u}{\tau (k_i \rho_i)^2} \right)^{1/2}.$$  

(26)

This result is consistent with equation (15) of [18], which was derived in a shearless slab geometry. In the $\omega_{se}' \ll \omega$ limit (26) reduces to

$$\omega = \pm \left( \frac{(v_i k_i)^2}{\beta_i} - \frac{2\omega_{se}' k_i u}{\tau (k_i \rho_i)^2} \right)^{1/2}.$$  

(27)

For a given wave number if the electron flow speed $u$ or the normalized pressure $\beta_i$ is sufficiently small the first term dominates on the right-hand side of (27), and the solution is an Alfvén wave with purely real frequency $\omega \approx (v_A k_i)^2/\beta_i$. However, for high enough $\beta_i$, $u$ and $\omega_{se}'$ the second term might exceed the first and, depending on the relative size of $k_i$ and $u$, (27) describes either a pair of stable modes with purely real frequencies or a purely growing and a purely damped mode.

For the rest of this section we will be concerned with the purely growing mode driven by the current gradient. Clearly, decreasing the perpendicular wave number of the mode increases the growth rate of the mode. Since the first term in (27) is quadratic and the second term is linear in $k_i$, there is an optimal value of the parallel wave number, $k_{i0}$, where the mode has the highest growth rate, $\gamma$. When the plasma parameters and the perpendicular wave number are fixed the optimum is

$$k_{i0} = \frac{u\beta_i\omega_{se}'}{\omega_{pi}^2(k_i \rho_i)^2} \tau.$$  

(28)

and the growth rate corresponding to $k_{i0}$ is

$$\gamma_0 = \frac{u\sqrt{\beta_i \omega_{se}'} \omega_{pi}}{v_i(k_i \rho_i)^2} \tau.$$  

(29)

When $\omega_{se}' \sim \omega_{pi}^2$ and $\tau \sim 1$, the assumption $\omega_{pi}^2 \ll |\omega|$ used to obtain (27) is satisfied if $1 \ll u\sqrt{\beta_i}/(v_i\alpha_i)$. As long as there is a finite plasma beta and electron current, one can always find sufficiently small perpendicular wave number for which this relation is satisfied in the $\rho_i/L \rightarrow 0$ limit. In this case, neglecting magnetic drifts in the gyrokinetic equation is also justified as long as the pressure length scale is much smaller than the major radius.

It is shown at the end of the appendix that the perturbed quasineutrality equation can be written in the form $0 = [\phi_1 - \omega A_1/(k_i c)]G_1 + G_2$, where $G_1$ is a dimensionless function of order unity (as long as $\omega/(k_i v_e)$ is not too large) and $G_2$ is small in $\alpha_i$. Thus, neglecting the small correction from $G_2$, the approximate quasineutrality equation $0 = [\phi_1 - \omega A_1/(k_i c)]G_1$ is satisfied either if $G_1 = 0$, or $\phi_1 - \omega A_1/(k_i c) \approx 0$, that is if $E_j \approx 0$, which we assumed in deriving (27). The case $G_1 = 0$ includes drift wave solutions and the strongly damped modes corresponding to electrostatic roots of the uniform plasma dispersion relation in the presence of electron flow.

4. Magnetic shear effects

To obtain simple analytical results in section 3 we neglected magnetic drifts and assumed a flute like mode structure (with no radial variation). The mode tends to be more unstable at low perpendicular wave numbers, thus it is appropriate to neglect the magnetic drifts, $v_i \cdot k \perp \ll \omega$.

Due to the preceding assumptions, the result (26) is formally the same as what one would obtain solving the problem in a shearless slab geometry [18]. The only difference between a torus and a slab is that $\omega_{se} \propto n$ and $k \perp \rho_i \propto n$ have lower limits set by the lowest finite toroidal wave number $n = 1$. We note that in a shearless slab there is no such periodicity constraint, and $k \perp \rho_i$ can get arbitrarily small (thus $\gamma$ arbitrarily large) for sufficiently large perpendicular wave lengths. This unphysical behavior is partly resolved by taking finite magnetic shear into account, which is needed for the magnetic geometry to be consistent with a substantial parallel current. In this section we will study the consequences of a magnetic shear in slab geometry.

We choose a coordinate system $\{\hat{x}, \hat{y}, \hat{z}\}$ such that plasma parameters vary in the $\hat{x}$ direction, and consider a mode which is sinusoidally varying in the $\hat{y}$ direction with a corresponding wave number $k_y$, while the magnetic field has the form $B = B_0(z + \hat{y}v_x L_x)$. The magnetic shear produces an $x$ variation in $k_y$, namely $k_y(x) = k_y(0) + k_x x/L_x$, and we choose the origin so that $k_y(0) = 0$. We assume that the radial variations of the perturbed quantities are faster than those of the unperturbed ones and $\beta_i \ll 1$, thus the $\hat{y}$ component of the electron flow can be neglected ($u = u\hat{x}$) together with any change in the magnitude of $\hat{z} \cdot B$. 

---

Plasma Phys. Control. Fusion 56 (2014) 035011 I Pusztai et al
To obtain a dispersion relation in a sheared geometry we start with (24) and insert parallel Ampère’s law together with \( \alpha_i \rightarrow \rho_i^2 (k_y^2 - \partial_{x}^2) / 2 \) to find

\[
\tau \frac{\rho_i^2}{\beta_i} (k_y^2 - \partial_{x}^2) A_1 = \tau k_i c (\omega - \omega_{pl}) \rho_i^2 (k_y^2 - \partial_{x}^2) \phi_i + \omega_{pl} k_i u_i \rho_i^2 \phi_i,
\]

or equivalently

\[
\frac{\rho_i^2}{\beta_i} (k_y^2 - \partial_{x}^2) \hat{B} = \frac{k_i u_i \omega_{pl}}{\tau} \hat{B} = 0.
\]

(30)

where \( \hat{B} \) is defined by \( B_0 = \hat{B}(x) \exp(-i \omega t + ik_x y) \). The dispersion relation is essentially the same as in shearsless geometry, except for the linear \( x \)-dependence of \( k_i \), and that the replacement \( \partial_{x} \mapsto i k_x \) cannot be made.

Recalling \( k_i = k_{y} x / L_s \) and introducing the dimensionless ‘radial’ coordinate \( X = k_{y} x / L_s \), (31) can be rewritten in the form

\[
X \left( \partial_{XX}^2 - 1 \right) \hat{B} - \lambda \left( \partial_{XX}^2 - 1 \right) \left( \hat{B} / X \right) - \sigma \hat{B} = 0,
\]

(32)

where \( \lambda = \omega (\omega - \omega_{pl}) L_s^2 / \rho_i^2 \approx \sigma^2 L_s^2 / \rho_i^2 \) and \( \sigma = -2 L_s \beta_i u_i \omega_{pl} / (\tau k_i^2 \rho_i^2 \nu^2) \). The boundary conditions for this eigenvalue problem in \( \lambda \) are given by the requirement that

\[
\hat{B}(X \to \infty) \to 0.
\]

In (32) \( \sigma \) represents the drive and \( \lambda < -\beta_i [\omega_{pl} L_s / (2 \nu t)]^2 \) corresponds to an instability \( \text{Im}(\omega) > 0 \). During the analysis of the radial eigenmodes we shall neglect \( (\omega_{pl} / \omega)^2 \ll 1 \) corrections, and refer to the \( \lambda < 0 \) solutions as unstable and the \( \lambda = 0 \) solutions as marginally stable modes. We will retain \( \omega_{pl} / \omega \) corrections in section 5.

We note that reversing the sign of \( \sigma \), that is, the relative sign of \( u \) and \( k_i \), leads to the same eigenvalues, and the corresponding eigenfunctions satisfy \( B(X) \mid_{\sigma} = -\hat{B}(-X) \mid_{\sigma} \). Thus, henceforth we will analyze solutions corresponding to \( \sigma > 0 \), without loss of generality.

For small values of \( X \), (32) is dominated by the \( \lambda \) term, that is solved by \( B = C_1 X \exp(X) + C_2 X \exp(-X) \). Accordingly, the solutions are either linear or quadratic in \( X \) around \( X = 0 \). For high values of \( X \) the first term dominates (32), leading to the exponential asymptotic behavior \( B(X) \propto \exp (+X) \), consistent with the boundary conditions. To solve numerically we rewrite the eigenvalue problem (32) for \( F = B(X) / X \) as

\[
X (F'' + 2 F' - X F) - \lambda (F'' - F) - \sigma X F = 0,
\]

(33)

and discretize it using a second order finite difference scheme. To ensure appropriate asymptotic behavior for \( X \), we apply the boundary condition \( (X F')' - X F = 0 \) for \( X \) negative and \( (X F')' + X F = 0 \) for \( X \) positive. Then we numerically search for the eigenvalues \( \lambda \) and eigenfunctions \( F \) of the system for a given \( \sigma \).

Figure 1 shows solutions of the eigenvalue problem (32). We find that as the drive \( \sigma \) is increased, more and more unstable eigenfunctions appear, as illustrated in figure 1(a) showing the normalized growth rates \( -\lambda^{1/2} \) of all the unstable modes.

Figure 1. Solutions of the eigenvalue problem (32). (a) Normalized growth rates \((-\lambda)^{1/2}\) of unstable modes corresponding to \( \sigma = \{4, 5, 7.5, 10, 20, 40, 70, 100\} \). The number of unstable modes \( n_{\text{un}} \) increases with \( \sigma \). Lower growth rates correspond to more oscillatory structure. (c) The most unstable radial eigenmodes for \( \sigma = \{2.05, 3, 5, 10, 20\} \). The distance of the location of the maximum of \( |\hat{B}(X)| \), from \( X = 0 \) increases with \( \sigma \). (d) Solid curve: \(-X_o\), where \( X_o \) is the location of the maximum of \( |\hat{B}(X)| \) for the most unstable mode at a given value of \( \sigma \). Dashed–dotted curve: normalized growth rate \((-\lambda)^{1/2}\) of the most unstable mode. Dashed curve: \( \sigma / 2 \).
eigenmodes for different values of $\sigma$. On the $x$-axis of figure 1(a), $n_{\text{eig}}$ denotes the ordinal number of the unstable modes, with $n_{\text{eig}} = 1$ corresponding to the most unstable mode for each value of $\sigma$. In fact, a new unstable mode appears as $\sigma$ exceeds $2N$ for every positive integer $N$. In particular, no unstable mode exists for $|\sigma| < 2$. Note that in figure 1 the marginally stable ($\lambda = 0$) modes for even values of $\sigma$ are not shown.

For $\sigma = 2N$, the marginally stable ($\lambda = 0$) solutions of (32) are of the form $B(X) = X \exp(-X)F_1(1 + \sigma/2, 2, 2X) = X \exp(X)P_{\sigma}(X)$ for $X \leq 0$ and $B(X) = 0$ for $X > 0$. Here, $F_1$ denotes the Kummer confluent hypergeometric function, and $P_{\sigma}$ is a polynomial with only positive coefficients $(P_2 = 1, P_3 = 1 + X, P_6 = 1 + 2X + 2X^2/3, \ldots)$. The derivative of the marginally stable solutions is discontinuous at $X = 0$, however it is resolved by a boundary layer at $+0$ for $\sigma = 2N + \delta$ with an arbitrarily small $\delta > 0$ and a corresponding small eigenvalue $\lambda$. The boundary layer connects the $X \leq 0$ solution vanishing at $X = 0$, to a solution $\propto \exp(-X)(1 + 2X)\exp(2X)Ei(-2X)$ for $X > 0$, which is finite at $X = 0$. Here, $Ei(x) = \mathcal{P} \int_{-\infty}^{x} \exp(t)/t \, dt$ (for real values of $x$) denotes the exponential integral, where $\mathcal{P}$ indicates that the principal value is to be used for $x > 0$. No marginally stable solution to (32) exists if $|\sigma| \neq 2N$, since in this case $X \exp(-X)F_1(1 + \sigma/2, 2, 2X)$ becomes divergent at $X \to -\infty$, and the $B(X \to -\infty) = 0$ boundary condition cannot be met.

When more than a single unstable eigenmode exists ($\sigma > 4$), the ones with lower growth rates exhibit a more oscillatory radial structure, as illustrated in figure 1(b) showing the four unstable modes for $\sigma = 10$. In particular, the most unstable mode (corresponding to the thickest curve in figure 1(b)) does not change sign in the region $X < 0$, while all the other unstable modes do. This is consistent with the behavior of the marginally stable modes, since for increasing $N$ the number of roots of $P_{\sigma}(X)$ increases.

The amplitude $|B|$ of the most unstable eigenmode has a maximum close to the radial location where $k_{||}(X)$ would maximize the local dispersion relation (27), that is $k_{||}(X) \approx k_{||0}$ with the optimal wave number $k_{||0}$ given in (28).

In terms of $X$, the location of $k_{||}(X) = k_{||0}$ scales as $X_0 = -\sigma/2$ according to the local theory. As shown in figure 1(d), the location of the maximum amplitude (solid curve, representing $-X_0$) follows this expectation (dashed curve, $\sigma/2$) quite well. In the strongly driven ($\sigma \gg 1$) limit the normalized growth rate $(-\lambda)^{1/2}$ of the most stable eigenmode (dashed--dotted curve in figure 1(d)) approaches the optimal value, $\gamma_c$ given by (29). This value corresponds to $(-\lambda)^{1/2} \to \sigma/2$. However, $\sigma = 2$ gives $(-\lambda)^{1/2} = 0$ (that is, no unstable mode) in the sheared slab model (SSM), while the local theory would predict a finite growth rate equivalent with $(-\lambda)^{1/2} = 1$.

In conclusion, considering magnetic shear sets a stability limit in terms of the drive at $\sigma = 2$ in contrast to the shearseless model that predicts instability when $\beta_i$, the current gradient and the flow speed $u$ are finite. In the shearless case the mode is always allowed to pick the optimal parallel wave number.

The stability criterion of the mode $|\sigma| < 2$ is equivalent to that of the high mode number kink modes. Using the relations $u = j_0/(en_e)$, $L_n = qR/\rho_s$, $s = (r/q) dq/\,dr$, $k_y = nq/a$ and $n = m/q$, with $m$ the poloidal mode number, together with the definitions of $\sigma$ and $\omega_{ci}$, one can rewrite the stability criterion $|\sigma| < 2$ as

$$\frac{4\pi r}{c B_0} \left| \frac{d j_0}{dr} \right| < 2m \frac{|q'|}{|q|},$$

as obtained from the magnetohydrodynamic energy principle in [1]—see equation (2.29) therein.

5. Mode characteristics in toroidal geometry

In this section the high mode number kink mode investigated in sections 3 and 4 is studied numerically using the gyrokinetic code G32. G32 is free from the simplifying assumptions made in sections 2–4, except for the radial locality and the scale separation $k_{||} \ll k_z$. In the low-flow version of G32 extra terms related to neoclassical corrections to the non-fluctuating part of the distribution function and the electrostatic potential are implemented for momentum transport studies, as discussed in [17]. These quantities are specified as inputs, normally calculated by the neoclassical code NEO [19].

This infrastructure can in principle be used to include any modification to the non-fluctuating part of the distribution over a velocity range of a few thermal speeds. We use it to include $f_s$, as defined after (2), or more sophisticated Spitzer functions, to study the effect of the induced electric field on instabilities.

Normally, we include only a parallel flow in G32 simulations, instead of a full Spitzer function since the results are insensitive to the detailed form.

First we consider the parametric dependences of the mode frequency and the growth rate, and compare G32 simulations to predictions of the SSM (31). The SSM results are obtained by choosing the most unstable eigenmode from the numerical solution of (33). We use a 200 point radial grid, the extent of which is adapted to the expected width of the eigenfunctions depending on the value of $\sigma$.

The scans are performed about the following set of baseline parameters: $u/\nu_i = 1$, $\beta_i = 0.01$, $a/L_u = 3$, $a/L_T \approx a/L_{Te} = a/L_{ne} = 0$, $k_y \beta_i = 0.15$, $a/R \approx 0.1$, $r/a = 0.5$, $s = 1$, and $q = 10$, where $d \ln u/\,dr = -1/L_u$, $d \ln n_e/\,dr = -1/L_{ne}$, $d \ln T_e/\,dr = -1/L_{Te}$, and $d \ln T_i/\,dr = -1/L_{Ti}$. We set the density and temperature gradients to zero to avoid the appearance of the usual gradient driven modes (otherwise, for this $\beta_i$, magnetic shear and $L_n \sim a \sim L_T$ kinetic ballooning modes appear and pollute the results, as in [20]). Then the only instability drive is due to the gradient of the flow speed. The radial gradient of the flow speed in the Ohmic current is due to density and electron temperature gradients, thus our settings are not physically consistent. However, by artificially choosing the parameters we obtain a cleaner comparison between theory and simulations.

The binormal wave number and the aspect ratio are chosen to be small so that magnetic drifts are not expected to affect the results significantly. For a typical G32 simulation only an extended poloidal angle range of $\theta = (-\pi, \pi)$ is kept and 80 grid points along the field line are used, since the
eigenfunctions of strongly driven modes are highly oscillatory and very localized in \( \theta \). The simulations use 20 untrapped pitch angles—and 14 energy grid points. We neglect collisions and compressional magnetic perturbations.

Figure 2 shows various parameter scalings around the baseline parameter set. In a strongly driven situation (\(|\sigma| \gg 2\)), the growth rate is expected to be close to (29), which helps in interpreting the numerical results. Since \( \omega_{r0} \propto k_y \), and \( k_y \propto k_s \), we expect a 1/\( k_y \) dependence of the growth rate, which is observed in figure 2(a). Magnetic drifts should be more important toward higher wave numbers. The good agreement remains between GS2 and the SSM even at \( k_y \rho_i = 0.3 \) due to the very large aspect ratio \( R/r = 20 \). The growth rate is expected to increase linearly with the flow speed and the mode should be stable at \( u = 0 \) and this behavior is seen in figure 2(b). Similarly, the growth rate should exhibit the linear dependence on \( a/L_{ne} \) as shown, where the mode is unstable at \( a/L_{ne} = 0 \) due to the finite gradient in the flow speed, see figure 2(c).

To translate magnetic geometry parameters from the toroidal geometry of GS2 to a sheared slab we use \( 1/L_s = s/q \). Although the local model can be used to explain certain parametric dependences of the mode, it cannot provide predictions for the \( L_s \) dependence. However we know that as \( \sigma \propto L_s \), drops below 2 due to a decreasing \( L_s \), the mode should be completely stabilized. Thus we expect increasing \( s \) should reduce the growth rates, as seen in figure 2(d). Clearly, \( q \) should have the opposite effect as \( s \); since \( L_s \propto q/s \). Indeed, figure 2(e) shows that the mode is stabilized with decreasing \( q \). Also, when the mode is strongly driven, \( |\sigma| \gg 2 \), the growth rate should become independent of \( L_s \), since the mode approaches the local result. Hence, there is a saturation in the \( q \)-dependence of \( \gamma \) toward higher values of \( q \). When \( \beta_i \) and \( k_y \rho_i \) are held fixed the growth rate given in (29) normalized to \( v_i/a \) is independent of \( T_e/T_i \). The insensitivity of the result to the temperature ratio is demonstrated in figure 2(d).

The real part of the frequency \( \omega_r \) is proportional to the ion diamagnetic frequency \( \omega_{ri0} \), which should be zero in almost all the scalings of figure 2, since the ion pressure gradient is zero. The only exception is the density gradient scaling, figure 2(e), where \( \omega_r \) should increase linearly with \( a/L_{ne} \). Although, we find the right trend \( \omega_r \propto a/L_{ne} \), GS2 produces higher values than the slab model. The reason for this discrepancy is likely that the mode is not purely kink anymore, but instead develops some kinetic ballooning character due to the finite pressure gradient drive.

There are small deviations from the \( \omega_r = 0 \) result of the slab model in the GS2 simulations in figures 2(a), (b) and (d)–(f). These may be the result of the magnetic drift effects neglected in the slab model, but they may also represent the finite accuracy of the simulations. In certain cases, when \( \sigma \) is very high, making the parallel mode structure very oscillatory, exceptionally high parallel resolutions were necessary in GS2 to achieve the accuracy presented in figure 2 (for example 140 grid points in \( \theta \)).

The expected \( \sqrt{\beta_i} \) dependence of the growth rate of the high mode number kink modes is reproduced, as seen in figure 3. Apart from the sheared slab results (solid lines), figure 3 shows GS2 simulations of different levels of sophistication. In the simplest case the non-fluctuating electron distribution is modeled as a Maxwellian with a finite parallel flow velocity (shown with solid symbols). It is interesting to see that when the shifted Maxwellian is replaced by a Spitzer function with the same flow speed but considerably more complicated velocity structure (given by (B4) and (B8) of [21]), the results (empty symbols) remain practically unchanged, especially for the growth rates. Spot checks for

\[ Z_{1208} \]
and \( b \) to be sensitive to physics happening in small layers around the of the electron skin depth, these high-

\[ A_{\perp} \] (empty symbols), and using a Spitzer function keeping both \( \beta \)

with varying plasma beta; \( \omega_r \) and real frequency \( \sigma \), \( \epsilon \) and for the \( m \) mode 

kink modes studied here \( \propto k_{\perp} \). This drive, which determines the ideal MHD growth rate of the mode, is \( e^2 \) small in the \( m = 1 \) case (as compared to \( m \neq 1 \)) making the near marginally

scale separation \( \epsilon \) is small in the \( k < 1 \) mode \( \propto \epsilon^{-1} \), \( \epsilon > 1 \) mode is less unstable for the larger \( k < 1 \) kink modes.

In the sheared slab geometry, the parallel wave number increases away from the resonant surface (recall \( k_{\parallel} = k_{\perp} \times / L_z \)).

The Fourier transform of the sheared slab problem in the \( x \) coordinate can lead to an equation that is equivalent to the problem in ballooning representation with a coordinate along the magnetic field line [23]. More precisely, the radial eigenfunction in the sheared slab, \( \mathcal{B}(X) \), is related to the ballooning eigenfunction, \( B_{\parallel}(\theta) \), by \( \mathcal{B}(X) \propto \int_{-\infty}^{\infty} \hat{a}_j e^{i k_\perp y} \hat{F}_x(-\theta / y) \). Figure 4(c) shows that the kink modes considered here have this same property. It compares the variation of the radial mode structure (the magnitude of \( \mathcal{B}(X) \)) in sheared slab calculations (dashed lines), with the transform of the ballooning mode variation obtained from \( \text{GS2} \) (solid), for different values of \( \beta_i \). The solid line peaking the closest to (and furthest away from) the rational surface correspond to the ballooning eigenfunction in figure 4(a) and (b), respectively. The ‘ballooning’ character of the eigenfunctions, that is, their localization around \( \theta = 0 \), is simply a consequence of how a mode with a finite radial extent appears in ballooning representation, rather than as a result of a poloidal dependence in the drive of the mode. In particular it is not a magnetic drift effect. As the radial extent of \( \mathcal{B}(X) \) increases with increasing \( \sigma \propto \beta_i \), the equivalent \( B_{\parallel}(\theta) \) becomes more and more localized around \( \theta = 0 \) according to the properties of the Fourier transformation.

We note that from the sheared slab dispersion relation (30) and \( A_{\parallel} = k_{\parallel} c / \Omega_p \) the long wavelength ballooning equations solved by \( \text{GS2} \) can be recovered using the replacements \( i k_{\parallel} \rightarrow (q R)^{-1} \partial_\theta \) and \( i k_{\parallel} y + \hat{x} \partial_x \rightarrow i k_{\perp} y + i k_{\parallel} \partial_x \).

6. Discussion and conclusions

We have developed a procedure for modeling current gradient driven kink instabilities in a tokamak with \( \text{GS2} \) gyrokinetic simulations and compared the results to the analytical expressions we derived.

We find that at sufficiently high current gradient high mode number kink modes are destabilized. The properties of strongly driven kink modes can be understood from simple analytical expressions derived in a shearless magnetic geometry by assuming that the mode chooses an optimal, finite parallel wave number that maximizes its growth rate. In terms of kinetic quantities, the mode is destabilized by high \( \beta_i \), strong parallel electron flow \( u \), high values of \( \partial_\theta \ln n_i u \), and small perpendicular wave numbers.

Since the mode is more unstable for smaller values of the perpendicular wave numbers \( k_{\perp} \), magnetic drift effects \( \propto k_{\perp} \cdot \psi_{bc} \) are unimportant for describing the stability of the mode. A perhaps more important effect of toroidicity is that there is a lower limit on \( k_{\perp} \) set by the lowest finite toroidal mode number \( n = 1 \). However, both the analytical calculations and \( \text{GS2} \) assume a scale separation \( k_{\parallel} \ll k_{\perp} \) and disregard global profile and magnetic geometry variations, thus are unable to properly treat low mode number magnetohydrodynamic modes. Therefore the stability limit, which we derive based on kinetic theory, coincides with the magnetohydrodynamic stability limit for high mode number kink modes [1]. In the sheared slab magnetic geometry we find that the mode is strongly asymmetric, being localized on one side with respect to a resonant \( (k_{\parallel} = 0) \) surface. The parallel wave number corresponding to the radial location of the highest amplitude is close to the one that maximizes the growth rate in the local theory. The number of unstable radial eigenmodes increases with increasing drive.

We find good agreement between \( \text{GS2} \) simulations and analytical estimates both in terms of the parametric dependences of the growth rates and mode frequencies, and in terms of eigenmode structure. The large aspect ratio and small \( k_x \beta_i \) limit of high mode number kink modes may be used as a simple test case for linear validation of electromagnetic gyrokinetic codes when current drive is to be
modeled. By comparing kink modes assuming a Maxwellian electron distribution with a parallel flow and alternatively a Spitzer function departure from a Maxwellian as a drive we demonstrate that the exact velocity structure of the non-fluctuating electron distribution function is unimportant for the mode. Only the parallel flow speed of electrons matters.

For modes that are electrostatic in nature, an electron flow—even when comparable to the ion thermal speed—is not expected to significantly modify their stability. The circulating electrons which can flow along the field lines are not expected to significantly modify their stability. The plasma β and the electron flow speed exceeds even when the plasma β and the electron flow speed exceeds their experimentally relevant range in the simulations.

In a screw-pinch geometry it is known that, if an ideal magnetohydrodynamic mode is unstable at a given finite poloidal mode number \( m_0 \), it should be even more unstable at all mode numbers \( m \) satisfying \( 1 \leq m < m_0 \) [24]. Therefore, the trend of increasing growth rate with decreasing \( k_\perp \) is not terminated until the lowest wave number allowed in the system. Consequently, if the plasma is globally stable to low mode number kink modes, it should be stable for all mode numbers. However, since a similar theorem has not been proven in toroidal geometry, the relevance of high mode number kink modes in tokamaks is unclear, and should be the subject of future investigations. Toward this end, the research herein demonstrates that suitably modified gyrokinetic codes can be used to investigate current driven or kink instabilities in tokamaks. A local code such as GS2 permits the modeling of only high wave number modes, but it has the important advantage that it can effectively model the nonlinear evolution of these modes, which is a topic for future studies.

Acknowledgments

The authors are thankful to Jesus Ramos, Jack Connor, Jeff Freidberg and Jim Hastie for several fruitful discussions on MHD related problems, and to Choongki Sung for providing experimental parameters. This work was funded by the European Communities under Association Contract between EURATOM and Vetenskapsrådet (VR), and by the US Department of Energy grant at DE-FG02-91ER-54109 at MIT. The first author is grateful for the financial support of VR.

Appendix. Quasineutrality

To derive the explicit form of the quasineutrality equation from (14) we write it as

\[
0 = e_e \int d^3 v \ g_e - \frac{e_e^2 n_e}{T_e} \phi_1 + e_i \int d^3 v \ g_i - \frac{e_i^2 n_i}{T_i} \phi_1 \tag{A.1}
\]

where the integrals are taken at fixed particle position. First we will evaluate the electron contribution to quasineutrality, i.e. the first two terms of (A.1). We neglect the magnetic drifts in (9), replace time derivatives by \(-io\), toroidal derivatives by \(-in\), write \( E_i = -ik_u \phi_1 + io A_1/c \), and then divide the equation by \(-io + ik_u v_i\), to obtain

\[
g_e = \frac{e_e}{T_e} f_{Me} \left( 1 - \frac{m_e}{T_e} v_i \right) \omega \left( \frac{\phi_1 - \frac{v_i}{c} A_1}{\phi_1 - k_u v_i} \right) + f_{Me} \frac{e_e}{T_e} \frac{\phi_1 - \frac{v_i}{c} A_1}{\phi_1 - k_u v_i} \tag{A.2}
\]

The integral \( e_e \int d^3 v \ g_e \) in (A.1) can be directly evaluated in terms of the plasma dispersion function, using that

\[
Z(\xi) = \int_{-\infty}^{\infty} dx \ exp(-x^2) \sqrt{\pi} x - \xi \tag{A.3}
\]

where the integration is done along the Landau contour. After a straightforward calculation we find that the electron contribution to the dispersion relation is

\[
-\frac{T_e}{e_e^2 n_e} \int d^3 v \ f_{Le} = \left( \phi_1 + A_1 \right) \left[ \left( 1 + \xi_e Z(\xi_e) \right) \left( 1 - \frac{\omega_{ce}}{\omega} \right) \right.
\]

\[
- \frac{\omega_{ce} n_e}{\omega} \left( \xi_e^2 + Z(\xi_e) (\xi_e^3 - \xi_e/2) \right)
\]

\[
\left. + \left( \phi_1 + A_1 \right) \frac{U_k}{|k||} \left( Z(\xi_e) - 2 \xi_e \left( 1 + \xi_e Z(\xi_e) \right) \times \left( 1 - \frac{\omega_{ce}}{\omega} (1 + n_e - n_e) \right) \right) \frac{\omega_{ce} n_e}{\omega} \left( \xi_e^2 + Z(\xi_e) (\xi_e^3 - \xi_e/2) \right) \right] + \left( \frac{\omega_{ce}}{\omega} \right) \phi_1 \tag{A.4}
\]
where we introduced \( \xi_a = \omega / |k| v_\perp \), \( \eta_a = L_{ne} / L_{Ta} \), \( \eta_i = L_{ne} / L_a \) the normalized flow speed \( U = n / v_a \), the diamagnetic frequency \( \omega_{de} = (n c T_a / e_n) \partial_n n_a \), and \( \bar{A} = -\omega A_i / (k |c|) \).

Once the ion magnetic drifts are neglected, the gyro-averages \( \langle \cdot \rangle \) are replaced by \( J_0( k |c| \Omega_i) \), and the \( \xi \)-derivatives are written in terms of \( \omega_{ai} \), \( g_i \) from (11) can be easily expressed as the familiar form

\[
g_i = \frac{f M}{e_i} \left( \phi_i - \frac{v_i}{c} A_i \right) J_0 \left( \frac{k |c| v_\perp}{\Omega_i} \right)
	\times \frac{\omega - \omega_{ai} \left[ 1 + \eta_i \frac{m_i v_i^2}{M} - \frac{1}{2} \right]}{\omega - k |c| v_\perp}.
\]  

(A.5)

When we evaluate the velocity integral for ions in (A.1) we expand in the FLR parameter, writing \( J_0^2( k |c| \Omega_i) = J_0^2( k |c| \rho_i v_\perp / v_i ) \approx 1 - \alpha_i (v_\perp / v_i)^2 \), where we recall the definition \( \alpha_i = (k |c| \rho_i)^2 / 2 \). The ion contribution to the dispersion relation, normalized to \( -\omega_i^2 n_i / T_i \), is obtained to be

\[
-\frac{T_i}{\omega_i^2 \eta_i} \int d^3 v f_i = (1 - \alpha_i) \left( \phi_i + \bar{A} \right) \left[ (1 + \xi; Z(\xi)) \left( 1 - \frac{\omega_{ai}}{\omega} \right) + \frac{\omega_{ai} \eta_i}{\omega} \left( \xi^2 + Z(\xi)(\xi^3 - \xi^2 / 2) \right) \right] + \alpha_i \left( \frac{\omega_{ai} \eta_i}{\omega} \right) \left( \phi_i + \bar{A} \right) \left( \xi Z(\xi) + \bar{A} \right) + \alpha_i \phi_i \left( 1 - \frac{\omega_{ai}}{\omega} \right) + \phi_i \frac{\omega_{ai} \eta_i}{\omega}.
\]  

(A.6)

Note that (A.4) and (A.6) contain the contributions from the adiabatic responses. When the perturbed quasineutrality equation (A.1) is formed the contributions from \( \phi_i = \omega_{ai} / \omega \) and \( \phi_i = \omega_{ai} / \omega \) [the last terms in (A.4) and (A.6), respectively] cancel for a pure plasma, due to quasineutrality \( e z n_e + e_i n_i = 0 \) and \( (\ln n_e)' = (\ln n_i)' \).

Due to the high electron thermal speed \( \xi_e \) is typically small. As long as \( \xi_e \) is not much larger than unity there are \( \mathcal{O}(1) \) terms multiplying \( \phi_i + \bar{A} \) in the electron contribution to quasineutrality (A.4). In the ion contribution (A.6), the terms in the last line, which cannot be factorized by \( \phi_i + \bar{A} \) are multiplied by \( \alpha_i \) that is assumed to be small in our expressions.

In conclusion, the quasineutrality equation has an order unity part that can be factorized by \( \phi_i - \omega A_i / (k |c|) \), and the rest is small in \( e z \phi_i \). This means that to satisfy quasineutrality, either \( \phi_i \) and \( \omega A_i / (k |c|) \) should nearly cancel or the coefficient factorized by \( \phi_i - \omega A_i / (k |c|) \) should be close to zero.

References

[1] Kadomtsev B B and Pogutse O P 1970 Reviews of Plasma Physics vol 5, ed M A Leontovich (New York: Consultants Bureau) p 249
[2] Kadomtsev B B 1975 Sov. J. Plasma Phys. 1 389
[3] Basu B and Coppi C 1981 Phys. Fluids 24 465
[4] Drake J F 1978 Phys. Fluids 21 1777
[5] Catto P J 1978 Plasma Phys. 20 719
[6] Frieman E A and Chen L 1982 Phys. Fluids 25 502
[7] Naitou H, Tsuda K, Lee W W and Sydora R D 1995 Phys. Plasmas 2 4257
[8] Mishchenko A and Zocco A 2012 Phys. Plasmas 19 122104
[9] Mishchenko A, Könies A and Hatzky R 2009 Phys. Plasmas 16 082105
[10] Deng W, Lin Z and Holod I 2012 Nucl. Fusion 52 023005
[11] Lin Z, Hahn T S, Lee W W, Tang W M and White R B 1998 Science 281 1835
[12] http://gene.rzg.mpg.de
[13] Pueschel M J, Jenko F, Told D and Büchner J 2011 Phys. Plasmas 18 112102
[14] Numata R, Dorland W, Howes G G, Loureiro N F, Rogers B N and Tatsuno T 2011 Phys. Plasmas 18 112106
[15] Numata R, Howes G G, Tatsuno T, Barnes M and Dorland W 2010 J. Comput. Phys. 229 9347
[16] Kotschenreuther M, Rewoldt G and Tang W M 1995 Comput. Phys. Commun. 88 128
[17] Barnes M, Parra F I, Lee J P, Belli E A, Nave M F F and White A E 2013 Phys. Rev. Lett. 111 055005
[18] Sperling J L and Bhadra D K 1979 Plasma Phys. 21 225
[19] Belli E A and Candy J 2008 Plasma Phys. Control. Fusion 50 095010
[20] Pusztai I, Catto P J, Parra F I and Barnes M 2013 Proc. 40th EPS Conf. on Plasma Physics (Espoo) vol 37D (ECA) P4.156
[21] Pusztai I and Catto P J 2010 Plasma Phys. Control. Fusion 52 075016
[22] Rosenbluth M N, Dagazian R Y and Rutherford P H 1973 Phys. Fluids 16 1894
[23] Connor J W, Hastie R J and Taylor J B 1979 Proc. R. Soc. Lond. A 365 1
[24] Newcomb W A 1960 Ann. Phys. 10 232