The finite size effects on nucleon masses are calculated in relativistic chiral perturbation theory. Results are compared with two-flavor lattice results.

1. Introduction

Finite size effects, in particular on dynamical configurations, can be a serious impediment to precision lattice calculations of hadron masses and matrix elements. On the lattice sizes used in production runs, the nucleons and pions are the relevant degrees of freedom for understanding the finite size effects on the nucleon mass. We calculate these effects here using two-flavor relativistic baryon chiral perturbation theory $(\chi PT)$ at $O(q^3)$ (one loop).

This study of nucleon masses is based on QCDSF and UKQCD unquenched ($N_f = 2$) lattice data using a Wilson gauge action and two flavors
of non-perturbatively $O(a)$-improved Wilson fermions ($a$ denotes the lattice spacing). The corresponding pion masses are in the interval $0.45 - 1$ GeV. Valence and sea quark masses are equal. Lattice sizes are $1 - 2$ fm. The scale is set with $r_0$, using the physical value $r_0 = 0.5$ fm $\approx 1/(395\text{MeV})$. JLQCD has undertaken a study with the same lattice actions and range of simulation parameters on varying lattice sizes. In the comparison shown in Fig. 1 one can observe that masses from lattices smaller than 1.8 fm are consistently larger than the results from larger lattices.

The $a \to 0$ limit was not performed. Since discretization effects in unquenched simulations could be quite large we have to keep an eye on the discretization errors in our results. Studying the variation of nucleon masses with $a$ at a fixed lattice size we find that between two data points on approximately the same volume and at the same pion mass, but with $a$ varying by $\sim 30\%$, the nucleon mass remains unchanged within the statistical errors. Moreover, we compare our results from large lattices ($L \geq 1.8$ fm) with two CP-PACS data sets with renormalization group improved gauge fields and tree-level tadpole-improved clover quarks, $a^{-1} \approx 1.5$ and 2 GeV respectively and $L \geq 2.5$ fm, and find that they agree.

Figure 1. Nucleon masses from QCDSF and UKQCD (white symbols) and JLQCD (gray symbols).
Figure 2. Nucleon masses on large lattices. The solid curve represents a fit with relativistic $\chi PT$, the dashed curve the NR limit using the same values of $m_0$ and $c_1$, and the dot-dashed curve the NR limit using $m_0 = 0.81$ GeV and $c_1 = -1.1$ GeV$^{-1}$.

2. $\chi PT$ formalism

The one-loop contribution at $O(q^3)$ is generated by the $O(q^1)$ Lagrangian $L^{(1)}_N$:

$$L^{(1)}_N = \bar{\Psi} (i\gamma_\mu D^\mu - m_0) \Psi + \frac{1}{2} g_A \bar{\Psi} \gamma_\mu \gamma_5 u^\mu \Psi,$$

(1)

with $D_\mu = \partial_\mu + \frac{1}{2}[u^\dagger, \partial_\mu u]$, $u_\mu = iu^\dagger \partial_\mu U u^\dagger$ and $u^2 = U$. We use the infrared regularization scheme $^4$. To compute the renormalized nucleon mass $m_N$, we include the tree-level $O(q^2)$ term $-4c_1 m^2_{PS}$ and an additional contribution of the form $e_1 m^4_{PS}$, which is needed for renormalization of the relativistic $O(q^3)$ result, although formally only entering at $O(q^4)$. The one-loop correction to $m_0$ is $^5$:

$$m_N = m_0 - 4c_1 m^2_{PS} + \left[ e_1^r(\lambda) + \frac{3g_A^2}{64\pi^2 F^2_\pi m_0} \left( 1 - 2 \ln \frac{m_{PS}}{\lambda} \right) \right] m^4_{PS}$$

$$- \frac{3g_A^2}{16\pi^2 F^2_\pi m^3_{PS}} \sqrt{1 - \frac{m^2_{PS}}{4m_0^2}} \left[ \frac{\pi}{2} + \arctan \frac{\sqrt{4m^2_{PS}m_0^2 - m^4_{PS}}}{m^2_{PS}} \right].$$

(2)

e_1^r(\lambda)$ denotes the renormalized $e_1$. The renormalization procedure is detailed in $^5$. We employ $\lambda = 1$ GeV, $g_A = 1.2$ $^6$, and $F_\pi = 92.4$ MeV.

The nucleon mass in the chiral limit, $m_0$, and the values of $c_1$ and $e_1^r$ are determined by a fit to six lattice data points at the smallest masses. The fit is shown in Fig. 2. Relativistic $\chi PT$ describes the data correctly for pion
masses up to \( \sim 700 \) MeV. For the values of the parameters, we find \( m_0 = 0.85(14) \) GeV, \( c_1 = -0.80(18) \) GeV\(^{-1}\) and \( c_1'(1 \) GeV\) = 2.8(1.1) GeV\(^{-3}\). These results differ slightly from the ones quoted in 5 due to their fitting to a larger set of lattice points and using the value \( g_A = 1.267 \).

In the limit \( m_{PS} \ll m_N \), keeping only terms up to \( O(m_{PS}^3) \) of Eq. (2), one obtains non-relativistic (NR) \( \chi PT \). In Fig. 2, we also plot NR results. We find that they approximate the data and the relativistic curve only for rather small pion masses. A method to extend the validity of the NR approximation to higher momentum scales within a cutoff scheme is described in 7. The breakdown of the non-relativistic theory around \( m_{PS} = 400 \) MeV as shown in the plots can be avoided if momentum modes above a cutoff \( \Lambda \chi \) are absorbed in local counterterms (e.g. see the discussion in 7). Then the nonrelativistic \( O(q^3) \) calculation actually agrees quite well with the corresponding relativistic result up to \( m_{PS} = 600 \) MeV.

3. Finite Size Effects

We determine the finite size effect from the one-loop \( O(q^3) \) contribution to the self-energy. Putting the external nucleon line on-shell, the self-energy is given by

\[
\Sigma(q' = m_0) = \frac{3g_A^2m_0m_{PS}^2}{2F^2_\pi} \int_0^\infty dx \int \frac{d^4p}{(2\pi)^4} \left[ p^2 + m_0^2x^2 + m_{PS}^2(1-x) \right]^{-2}
\]

in Euclidean space. The temporal extent of the lattice is taken to be infinite.

We define \( \delta = \frac{1}{m_0} \left( \Sigma(q' = m_0, L) - \Sigma(q' = m_0, \infty) \right) \), and using 8 express it as an integral over Bessel functions:

\[
\delta = \frac{3g_A^2m_{PS}^2}{16\pi^2F^2_\pi} \int_0^\infty dx \sum_{\vec{n} \neq 0} K_0 \left( L|\vec{n}| \sqrt{m_0^2x^2 + m_{PS}^2(1-x)} \right).
\]

Then we calculate the finite size behavior of the nucleon mass:

\[
m_N(L) = (1 + \delta)m_N(\infty).
\]

We first extrapolate to \( m_N(\infty) \) by using the lattice result from the largest available lattice as input to Eq.(5), and then determine \( m_N(L) \) at finite lattice extent using the formula Eq.(5). In Fig. 3 we compare lattice nucleon masses from various volumes with both relativistic and NR \( \chi PT \) at \( O(q^3) \). Relativistic \( \chi PT \) describes the lattice data very well up to pion masses \( \sim 700 \) MeV. We find that for the finite size effect to be < 1%, \( L \) should be \( \geq 1.9 \) fm at the smallest pion mass shown in the plot.
Figure 3. \( m_N \) as a function of size on the lattice and in \( \chi PT \). Data sets correspond to a fixed \((\beta, \kappa)\) value and are labeled by the pion mass on the largest volume. The dashed lines show the results from NR \( \chi PT \).

The finite size effect calculated in the non-relativistic formalism (for discussion see also \(^9\)) is considerably smaller compared to the relativistic result and the lattice data. On lattices of 1 – 2 fm size it is \( \sim 30 – 40\% \) of the relativistic effect.

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