WILSON LOOPS, $q\bar{q}$ and $3q$ POTENTIALS,

BETHE–SALPETER EQUATION

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ABSTRACT

The derivation of the $q\bar{q}$ and the $3q$ potential for two dynamical quarks in a Wilson–loop context is reviewed. Some improvements are introduced. Only the usual assumptions in the evaluation of the Wilson loop integrals and expansions in the quark velocities are required for the result. It is shown that under the same assumptions it is possible to obtain the relativistic flux–tube lagrangian and a $q\bar{q}$ Bethe–Salpeter equation with a confining kernel for spinless quarks.

1. Introduction

In this paper first we review the derivation of the $q\bar{q}$ and the $3q$ semirelativistic potentials for dynamical quarks as has been given in preceding papers\(^1\) (for a general review on the subject see\(^2\)) and introduce some significant improvements. Then we show that, under the same assumptions and in the case of spinless quarks, a Bethe–Salpeter equation with a confining kernel can be obtained.

The basic objects considered in the derivation are the appropriate Wilson loop integrals $W_{q\bar{q}}$ and $W_{3q}$ and the basic assumptions are:

i) the quantities $i \ln W$ can be expressed as the sum of a short range contribution $i \ln W^{SR}$ and a long range one $i \ln W^{LR}$;

ii) the SR–term can be obtained simply from a perturbative expansion and the LR–term from a strong coupling expansion (in practice by the area law).

The improvement consists in the fact that an ad–hoc explicit instantaneous approximation is no longer required and only expansions in the quark velocities are used. Furthermore, the $O(\alpha_s^2)$ contribution is explicitly taken into account in the static part of the potential and it is shown that a covariant Lorentz gauge as well as the Coulomb gauge can be used.

As it is well known, the arguments in favour of the two assumptions are asymptotic freedom and the observation that the SR–part of the potential vanishes for $r \to \infty$, while the LR–part vanishes for $r \to 0$. Obviously, with the simple additivity assumption i), the resulting potential or kernel is expected to be inaccurate at intermediate distances; interferences of the two mechanisms should be important there. However, no attempt is made in this paper to use a more sophisticated approximation scheme of the type proposed e.g. in Refs.\(^3\) (see also\(^4\)).

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In Sec. 2 we discuss the evaluation of the Wilson loop integrals, in Secs. 3 and 4 we derive the $q\bar{q}$ and the $3q$ potentials respectively, in Secs. 5 and 6 we sketch the derivation of the flux–tube lagrangian and of the Bethe–Salpeter equation.

2. Wilson loop integrals

For the $q\bar{q}$ case the basic object is

$$W_{q\bar{q}} = \frac{1}{3} \langle \mathrm{Tr} \ P \exp \left( ig \oint_{\Gamma} dx^\mu A_\mu(x) \right) \rangle. \quad (1)$$

Here the integration loop $\Gamma$ is assumed to be made by an arbitrary world line $\Gamma_1$ between an initial position $y_1$ at the time $t_i$ and a final one $x_1$ at the time $t_f$ for the quark ($t_i < t_f$), a similar world line $\Gamma_2$ described in the reverse direction from $x_2$ at the time $t_f$ to $y_2$ at the time $t_i$ for the antiquark and two straight lines at fixed times which connect $x_1$ to $x_2$, $y_2$ to $y_1$ and close the contour. As usual $A_\mu(x) = \frac{1}{2} \lambda a A^a_\mu(x)$, $P$ prescribes the ordering of the color matrices (from right to left) according to the direction fixed on the loop and the angular brackets denote the functional integration.

Integrating explicitly the fermion fields, for any functional of the gauge field alone one obtains

$$\langle f[A] \rangle = \frac{1}{\mathcal{D}[A]} \frac{\int f[A] e^{iS[A]} \langle \mathrm{Tr} P \rangle}{\int f[A] e^{iS[A]}}, \quad (2)$$

where $S[A]$ denotes the pure gauge action plus the gauge–fixing terms and $M_f[A]$ is the fermionic determinant

$$M_f[A] = \text{Det} \prod_j [1 + igA(i\partial - m_j)^{-1}] = \sum \left[ g \int d^4x \text{Tr} (iA(x) S_F^{(m_j)}(0)) - \frac{1}{2} g^2 \int d^4x \int d^4y \text{Tr} (A(x) iS_F^{(m_j)}(x-y) A(y) iS_F^{(m_j)}(y-x)) + \ldots \right]. \quad (3)$$

Using the above equations and writing the gauge field lagrangian as the sum of the free and the interaction parts, $\mathcal{L}(A) = \mathcal{L}_0 + \mathcal{L}_{\text{int}}$, we have the perturbative expansion

$$W^\text{pert}_{q\bar{q}} = \frac{1}{3} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{1}{n! p!} \langle \mathrm{Tr} P (ig \int dz^\mu A_\mu)^n (i \int d^4x \mathcal{L}_{\text{int}}(x))^p \rangle_0 =$$

$$= \frac{1}{3} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(ig)^n p^p}{n! p!} \int d^4x_1 \ldots \int d^4x_p \int \ldots \int_{z_1 > z_2 > \ldots > z_n} dz_1^{\mu_1} \ldots dz_n^{\mu_n} \langle \mathrm{Tr} [A_{\mu_1}(z_1) \ldots A_{\mu_n}(z_n)] \mathcal{L}_{\text{int}}(x_1) \ldots \mathcal{L}_{\text{int}}(x_p) \rangle_0. \quad (4)$$

where, due to (3), the single terms must be understood as expansions in $g$ in turn. Then, identifying $W_{SR}$ with $W^\text{pert}$ according to assumption ii), we obtain in graphical terms (we omit graphs that are obtained by permutation of other ones or completely cancelled by renormalization)
where the external circuit stands for the Wilson loop $\Gamma$, and the inserted lines for ordinary free propagators. Notice the term which includes a quark-antiquark loop, which obviously comes from (3).

The various quantities occurring in (5) have been extensively studied from the point of view of renormalization. To our knowledge however no explicit evaluation in closed form has been given other than in very special cases. For the purpose of the derivation of a semirelativistic potential, an evaluation in terms of an expansion in the quark velocities shall be sufficient.

Let $(z_0^j = t_j, \mathbf{z}_j = \mathbf{z}_j(t))$ be the equation for the world lines of the quark and the antiquark and set $\dot{z}_j^\mu = dz_j^\mu/dt = (1, \dot{\mathbf{z}}_j)$. The first–order term in $\alpha_s = g^2/4\pi$ can be written explicitly as

$$
(i \ln W_{\bar{q}q}^{\text{SR}})^{(1)} = \int_{t_i}^{t_f} dt \int_{t_i}^{t_f} dt \dot{z}_1^\mu(t_1) \dot{z}_2^\nu(t_2) D_{\mu\nu}(z_1(t_1) - z_2(t_2)) ,
$$

where the limit for large $t_f - t_i$ has been understood and the contribution from the equal–time lines are neglected. Performing the change of variables $t = \frac{t_1 + t_2}{2}$, $\tau = t_1 - t_2$, expanding $z_1$ and $z_2$ around $t$,

$$
z_1(t_1) = \mathbf{z}_1(t) + \frac{1}{2} \tau \dot{\mathbf{z}}_1(t) + \frac{1}{8} \tau^2 \ddot{\mathbf{z}}_1(t) + \ldots, \quad z_2(t_2) = \mathbf{z}_2(t) - \frac{1}{2} \dot{\mathbf{z}}_2(t) + \frac{1}{8} \tau^2 \ddot{\mathbf{z}}_2(t) - \ldots
$$

and integrating over $\tau$ (between $-\infty$ and $+\infty$), in the Coulomb gauge we obtain immediately

$$
(i \ln W_{\bar{q}q}^{\text{SR}})^{(1)} = \int_{t_i}^{t_f} dt \left\{-\frac{4 \alpha_s}{3} r [1 - \frac{1}{2} (\delta^{hk} + \hat{r}^h \hat{r}^k)] \dot{z}_1^h \dot{z}_2^k + \ldots]\right\} ,
$$

with $\mathbf{r} = \mathbf{z}_1 - \mathbf{z}_2$ and $\hat{\mathbf{r}} = \mathbf{r}/r$. If we had worked, e.g., in the Feynman gauge, we would have obtained

$$
(i \ln W_{\bar{q}q}^{\text{SR}})^{(1)} = \int_{t_i}^{t_f} dt \left\{-\frac{4 \alpha_s}{3} \frac{1}{r} [1 - \mathbf{z}_1 \cdot \dot{\mathbf{z}}_2 + \frac{1}{8} ((\dot{\mathbf{z}}_1 + \dot{\mathbf{z}}_2)^2 + \mathbf{r} \cdot \dot{\mathbf{r}}) - \frac{1}{8 r^2} ((\mathbf{r} \cdot (\dot{\mathbf{z}}_1 + \dot{\mathbf{z}}_2))^2 + \ldots]\right\} .
$$
from which (8) can be recovered by eliminating the acceleration term by partial integration. This is a consequence of the gauge invariance of the Wilson integral.

In a similar way, after renormalization, we can obtain for the \( \alpha_s^2 \) term in the static limit

\[
(i \ln W_{q\bar{q}}^{SR})^{(2)} = \int_{t_i}^{t_f} dt \left\{ -\frac{4 \alpha_s^2}{3} \frac{1}{4\pi} \left[ \left( \frac{66 - 4N_f}{3} \right) (\ln \mu r + \gamma) + A \right] + \ldots \right\}.
\]

(10)

In Eq. (10) \( \mu \) is the renormalization scale and \( A \) is a constant that depends on the renormalization convention. In the \( \overline{\text{MS}} \) scheme \( A = \frac{5}{6} \left( \frac{66 - 4N_f}{3} \right) - 8 \).

Let us come to the LR part of the Wilson integral. We shall make the assumption

\[
i \ln W_{q\bar{q}}^{LR} = \sigma S_{\text{min}} + \frac{1}{2} CP,
\]

(11)

where \( S_{\text{min}} \) denotes the minimal surface enclosed by the loop \( \Gamma \) and \( P \) its length. Eq. (11) is suggested by the pure lattice gauge theory and it is believed to be true in the so-called quenched approximation, i.e. when we replace \( M(A) \) by 1 in (2). Corrections to the pure potential theory (pair creation effects) should be introduced for this fact but they shall not be considered here.

In more explicit terms (11) can be written as

\[
i \ln W_{q\bar{q}}^{LR} = \sigma \min \int_{t_i}^{t_f} dt \int_{0}^{1} ds \left\{ -\eta (\frac{\partial \sigma}{\partial t})^2 (\frac{\partial x}{\partial s})^2 + \left( \frac{\partial x^\mu}{\partial t} \right) \left( \frac{\partial x^\mu}{\partial s} \right) \right\} + \frac{1}{2} C \sum_{j=1,2} \int_{t_i}^{t_f} dt [\hat{\sigma}^\mu \hat{\sigma}^\mu]^{1/2},
\]

(12)

where the minimum is taken over all surfaces of equation \( x^\mu = x^\mu(t, s) \) having \( \Gamma \) as contour. Obviously \( x^0 = t, x(t, 1) = z_1(t) \) and \( x(t, 0) = z_2(t) \).

By solving the appropriate Euler equations and expanding in the velocities, we obtain

\[
x_{\text{min}}(t, s) = s z_1(t) + (1 - s) z_2(t) - \frac{1}{2} s(1 - s)[\eta + \frac{1}{3}(1 + s)\zeta] + \ldots
\]

(13)

with \( \eta = (\hat{\eta} \cdot \hat{z}_2 - \hat{\eta} \cdot \hat{z}_2)\hat{r} + (\hat{r} \cdot \hat{z}_2 - 2(\hat{r} \cdot \hat{z}_2))\hat{r} + r^2 \hat{z}_2 \) and \( \zeta = -(\hat{r} \cdot \hat{r})\hat{r} + r^2 \hat{r} + (r^2 - \hat{r} \cdot \hat{r})\hat{r} \).

Actually it can be checked that the \( O(v^2) \) term in (13) does not contribute to \( S_{\text{min}} \) at order \( v^2 \) (such a term is however important in principle for the evaluation of the functional derivatives). Replacing (13) in (12), finally we have

\[
i \ln W_{q\bar{q}}^{LR} = \int_{t_i}^{t_f} dt \sigma \int_{0}^{1} ds \left[ 1 - (s \hat{z}_{1T} + (1 - s) \hat{z}_{2T})^2 \right]^{1/2} + \frac{1}{2} C \sum_{j=1}^{2} \int_{t_i}^{t_f} dt \left( 1 - \hat{z}_{jT} \cdot \hat{z}_{j} \right)^{1/2} = \int_{t_i}^{t_f} dt \sigma \left[ 1 - \frac{1}{6} \left( \hat{z}_{1T}^2 + \hat{z}_{2T}^2 + \hat{z}_{1T} \cdot \hat{z}_{2T} \right) \right] + \ldots + \frac{1}{2} C \sum_{j=1}^{2} \int_{t_i}^{t_f} dt \left( 1 - \hat{z}_{jT} \cdot \hat{z}_{j} \right) + \ldots,
\]

(14)

where \( \hat{z}_{jT} \) denotes the transversal part of \( \hat{z}_j \), \( \hat{z}_{jT} = (\hat{\gamma}^{hk} - \hat{r}^{hk} \hat{r}^k) \hat{z}_{j} \).

In conclusion, we can write

\[
i \ln W_{q\bar{q}} = (i \ln W_{q\bar{q}}^{SR})^{(1)} + (i \ln W_{q\bar{q}}^{SR})^{(2)} + \ldots + i \ln W_{q\bar{q}}^{LR},
\]

(15)
with the various terms as given by (8), (10), (14).

Let us turn to the three–quark system. In this case the basic quantity is

$$W_{3q} = \frac{1}{3!} \left\langle \varepsilon_{a_1 a_2 a_3} \varepsilon_{b_1 b_2 b_3} \left[ \mathcal{P} \exp \left( i g \int_{\Gamma_1} dx^{\mu_1} A_{\mu_1}(x) \right) \right]^{a_1 b_1} \left[ \mathcal{P} \exp \left( i g \int_{\Gamma_2} dx^{\mu_2} A_{\mu_2}(x) \right) \right]^{a_2 b_2} \left[ \mathcal{P} \exp \left( i g \int_{\Gamma_3} dx^{\mu_3} A_{\mu_3}(x) \right) \right]^{a_3 b_3} \right\rangle. \quad (16)$$

Here \(a_j, b_j\) are colour indices, \(j = 1, 2, 3\) and \(\Gamma_j\) denote the curve made by: the world lines \(\Gamma_j\) for the quark \(j\) between the times \(t_i\) and \(t_f\) \((t_i < t_f)\), a straight line on the surface \(t = t_i\) merging from an arbitrary fixed point \(I\) (which we also denote by \(y_M\)) and connected to the world line, another straight line on the surface \(t = t_f\) connecting the world line to a second fixed point \(F\) (also denoted as \(x_M\)).

Under the assumptions i) and ii) we can write in place of (8) and (11)

$$i \ln W_{3q} = \frac{2}{3} g^2 \sum_{i < j} \int_{\Gamma_i} dx^\mu_i \int_{\Gamma_j} dx^\nu_j i D_{\mu\nu}(x_i - x_j) + \sigma S_{\min} + \frac{1}{3} CP. \quad (17)$$

Here the perturbative term is taken at the lowest order in \(\alpha_s\) and \(S_{\min}\) denotes the minimum among all the surfaces made by three sheets having the curves \(\Gamma_1, \Gamma_2\) and \(\Gamma_3\) as contours and joining on a line \(\Gamma_M\) connecting \(I\) with \(F\) (the minimum is understood at fixed \(\Gamma_j\) as the surfaces and \(\Gamma_M\) change). Obviously, \(P\) denotes the total length of \(\Gamma_1, \Gamma_2\) and \(\Gamma_3\). Notice that a priori the constants \(\sigma\) and \(C\) occurring in (17) could be different from those occurring in (11); however, the fact that when two quarks coincide the potential derived from (17) must coincide with that derived from (11) (in a colour singlet state two quarks are equivalent to an antiquark) grants that they must be actually equal.

The right–hand side of (17) can be evaluated as an expansion in \(\dot{z}_j\) on the same foot used for Eqs. (8), (10) and (14). In particular up to the second order in the velocities, \(S_{\min}\) coincides with the surface described by the equations

$$x_{j,\min}^m(t, s) = s z_j(t) + (1 - s) z_M(t), \quad j = 1, 2, 3. \quad (18)$$

Here \(z_M(t)\) is constructed from the positions \(z_1(t), z_2(t)\) and \(z_3(t)\) of the three quarks according to the following rule: if no angle in the triangle made by \(z_1(t), z_2(t)\) and \(z_3(t)\) exceeds \(120^\circ\) (configuration I), \(z_M(t)\) coincides with the point inside the triangle which sees the three sides under the same angle \(120^\circ\); if one of the three angles in the triangle is \(\geq 120^\circ\) (configuration II), \(z_M(t)\) coincides with the corresponding vertex, let us say \(z_j(t)\).

In conclusion, the result is

$$i \ln W_{3q} = \int_{t_i}^{t_f} dt \left\{ \sum_{j < l} \left[ -\frac{2}{3} \frac{\alpha_s}{r_{jl}} + \frac{1}{2} \frac{\alpha_s}{r_{jl}^3} (\delta_{hl} h_{jk}^l + h_{jk}^l h_{lj}^k) \right] + \sigma \sum_{j=1}^3 r_j \left[ 1 - \frac{1}{6} (\dot{z}_{jT_j}^2 + \dot{z}_{MT_j}^2 + \dot{z}_{jT_j} \cdot \dot{z}_{MT_j}) \right] + C \frac{3}{3} \sum_{j=1}^{t_f} dt (1 - \frac{1}{2} \dot{z}_j^2 z_j^2) \right\}, \quad (19)$$
where \( r_{ji} = r_j - r_i \equiv z_j - z_i \), \( r_j = z_j - z_M \) and the transversal prescription \( T_j \) is now referred to \( r_j \). Furthermore we can notice that the quantity \( \bar{z}_M \) can be obtained by deriving the equation \( \sum_{j=1}^{3} (r_j/r_j) = 0 \). We have indeed \( \sum_{j=1}^{3} \frac{1}{r_j} (\delta_{hk} - \bar{r}_j \gamma^k) \bar{z}_j = \sum_{j=1}^{3} \frac{1}{r_j} (\delta_{hk} - \bar{r}_j \gamma^k) \bar{z}_M \). Obviously in configuration II we have \( \bar{z}_M = \bar{z}_j \).

### 3. Quark–antiquark potential

The starting point is the gauge invariant quark-antiquark \((q_1, \bar{q}_2)\) Green function (for definiteness let us assume the two particles to have different flavours)

\[
G(x_1, x_2; y_1, y_2) = \frac{1}{3} \langle 0 \mid T \bar{\psi}^c_j(x_2) U(x_2, x_1) \psi_j(x_1) \bar{\psi}^c_1(y_1) U(y_1, y_2) \psi_1(y_2) \mid 0 \rangle = \frac{1}{3} \text{Tr} \langle U(x_2, x_1) S^F_1(x_1, y_1|A) U(y_1, y_2) C^{-1} S^F_2(y_2, x_2|A) C \rangle,
\]

where \( c \) denotes the charge-conjugation matrix, \( C \) the path-ordered gauge string \( U(b, a) = \text{P} \exp \left( ig \int_a^b dx^\mu A_\mu(x) \right) \) (the integration path being the straight line joining \( a \) to \( b \)), \( S^F_1 \) and \( S^F_2 \) the quark propagators in an external gauge field \( A^\mu \).

We assume \( x_1^0 = x_2^0 = t_t, y_1^0 = y_2^0 = t_i \) (with \( t_t - t_i > 0 \) and large) and note that \( S^F_j \) are \( 4 \times 4 \) Dirac type matrices. Then, performing a Foldy–Wouthuysen transformation on \( G \), we can replace \( S^F \) with a Pauli propagator \( K_j \) (a \( 2 \times 2 \) matrix in the spin indices) and obtain a two-particle Pauli-type Green function \( K \). Solving the Schrödinger-like equation for \( K_j \) by the path-integral technique and replacing it in the expression of \( K \), we obtain even this quantity in the form of a path integral on the world lines of the two quarks (see Ref.\(^1\) for details):

\[
K(x_1, x_2, y_1, y_2; t_t - t_i) = \int_{D[z_1, p_1]}^{z_1(t_t)=x_1} D[z_1, p_1] \int_{D[z_2, p_2]}^{z_2(t_t)=x_2} D[z_2, p_2]
\exp \{ i \int_{t_i}^{t_t} dt \sum_{j=1}^{2} [p_j \cdot \dot{z}_j - m_j - \frac{p_j^2}{2m_j} + \frac{p_j^4}{8m_j^3}] \} \left( \frac{1}{3} \text{Tr} T_s \text{P} \exp \{ ig \int_{\Gamma} dx^\mu A_\mu(x) \} \right.
+ \sum_{j=1}^{2} \frac{ig}{m_j} \int_{\Gamma_j} dx^\mu (S^F_j \hat{F}_{i\mu}(x) - \frac{1}{2m_j} S^F_j \varepsilon^{ikr} p_j^k F_{i\mu}(x) - \frac{1}{8m_j} D^\nu F_{i\nu}(x)) \}.
\]

Here \( T_s \) is the time-ordering prescription for the spin matrices; \( P, \text{Tr}, \Gamma, \Gamma_1 \) and \( \Gamma_2 \) are defined as in Eq.(1). Furthermore, as usual \( F^\mu_{\nu\rho} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu] \), \( \hat{F}^\mu_{\nu\rho} = \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) and \( D^\nu F_{i\mu} = \partial^\nu F_{i\mu} + ig[A^\nu, F_{i\mu}] \), \( \varepsilon^{\mu\nu\rho\sigma} \) being the four-dimensional Ricci symbol.

In order to show that the interaction between \( q_1 \) and \( \bar{q}_2 \) can be described in terms of a semirelativistic potential we must check that at the order \( \frac{1}{m} \) we can write

\[
\left\langle \frac{1}{3} \text{Tr} T_s \text{P} \exp \ldots \right\rangle = T_s \exp \left[ -i \int_{t_i}^{t_t} dt V^{q\bar{q}}(z_1(t), z_2(t), p_1(t), p_2(t), S_1, S_2) \right],
\]

(22)
for some $V^{q\bar{q}}$. Expanding the logarithm on the left-hand side of (22), this is equivalent to state

$$i \ln W_{q\bar{q}} + i \sum_{j=1}^{2} \frac{ig}{m_j} \int_{\Gamma_{j}} dx^{\mu} \left( S'_{j} \langle \langle \hat{F}_{j\mu}(x) \rangle \rangle - \frac{1}{2m_{j}} S'_{j}\varepsilon^{lkr}_{j} p_{j}^{k} \langle \langle F_{\mu\nu}(x) \rangle \rangle - \frac{1}{8m_{j}} \langle \langle D^{\nu} F_{\nu\mu}(x) \rangle \rangle \right) - \frac{1}{2} \sum_{i,j'} \frac{ig^{2}}{m_j m_{j'}} T_{a} \int_{\Gamma_{j}} dx^{\mu} \int_{\Gamma_{j'}} dx^{\nu} S'_{i}^{l} S'_{j}^{k} \left( \langle \langle \hat{F}_{i\mu}(x) \rangle \rangle \langle \langle \hat{F}_{j\sigma}(x') \rangle \rangle - \langle \langle \hat{F}_{i\mu}(x) \rangle \rangle \langle \langle \hat{F}_{j\sigma}(x') \rangle \rangle \right) + \ldots = \left[ \int_{t_{i}}^{t_{2}} dt \right] V^{q\bar{q}} ,$$

with the notation

$$\langle \langle f[A] \rangle \rangle = \frac{1}{4} \langle \text{Tr} P [\exp(ig f_{T} dx^{\mu} A_{\mu}(x))] f[A] \rangle \left( \frac{1}{3} \langle \text{Tr} \exp(ig f_{T} dx^{\mu} A_{\mu}(x)) \rangle \right) \quad (24)$$

and $W_{q\bar{q}}$ as given by (13).

Notice that after replacing $z_{j}$ by $\frac{p_{j}}{m_{j}}$ in Eqs. (8), (10) and (14), the expression resulting for $i \ln W_{q\bar{q}}$ is already of the desired form. Concerning the spin–dependent part we observe that the occurring field expectation values can be expressed in terms of $i \ln W_{q\bar{q}}$ by the functional derivatives

$$g \langle \langle F_{\mu\nu}(z_{1}) \rangle \rangle = \frac{\delta(i \ln W_{q\bar{q}})}{\delta S^{\mu\nu}(z_{1})} ,$$

$$g^{2} \left( \langle \langle F_{\mu\nu}(z_{1}) F_{\rho\sigma}(z_{2}) \rangle \rangle - \langle \langle F_{\mu\nu}(z_{1}) \rangle \rangle \langle \langle F_{\rho\sigma}(z_{2}) \rangle \rangle \right) = \frac{\delta^{2} \ln W_{q\bar{q}}}{\delta S^{\mu\nu}(z_{1}) \delta S^{\rho\sigma}(z_{2})} = -iga \frac{\delta}{\delta S^{\rho\sigma}(z_{2})} \langle \langle F_{\mu\nu}(z_{1}) \rangle \rangle ,$$

where $\delta S^{\mu\nu}(z_{2}) = \frac{1}{2}(dz_{j}^{\nu} \delta z_{j}^{\mu} - dz_{j}^{\mu} \delta z_{j}^{\nu})$ is the element of the surface spanned by the path $z_{j}(t)$ as a consequence of the variation $z_{j}(t) \rightarrow z_{j}(t) + \delta z_{j}(t)$.

The evaluation of the right hand side of (25) and (26) requires some care, since the functional derivatives may lower the order of magnitude in the velocities. However, it can be done without any additional assumptions and the results are

$$g \langle \langle F_{0k}(z_{1}) \rangle \rangle = \left( \frac{4}{3} \alpha_{s}, \frac{1}{r_{3}} + \frac{\sigma}{r} \right) r^{k} + O(v^{2}) ,$$

$$g \langle \langle F_{hk}(z_{1}) \rangle \rangle = \left( \frac{4}{3} \alpha_{s}, \frac{1}{m_{2} r_{3}} + \frac{\sigma}{m_{2} r} \right) (r^{k} p_{2}^{h} - r^{h} p_{2}^{k}) + O(v^{3}) ,$$

$$g^{2} \left( \langle \langle F_{hk}(z_{1}) F_{lm}(z_{2}) \rangle \rangle - \langle \langle F_{hk}(z_{1}) \rangle \rangle \langle \langle F_{lm}(z_{2}) \rangle \rangle \right) = - \frac{4 ig^{2}}{38\pi} \delta(t_{1} - t_{2}) \left\{ \partial_{k} \partial_{l} \left[ \frac{1}{r} \left( \delta^{km} + r^{k} r^{m} \right) \right] - \partial_{k} \partial_{m} \left[ \frac{1}{r} \left( \delta^{kl} + r^{k} r^{l} \right) \right] - 2 \partial_{m} \partial_{k} \right\} \langle \langle F_{hk}(z_{1}) \rangle \rangle \langle \langle F_{lm}(z_{2}) \rangle \rangle \langle \langle F_{hl}(z_{2}) \rangle \rangle \langle \langle F_{mk}(z_{1}) \rangle \rangle + O(v^{2}) ,$$

$$\langle \langle F_{0k}(z_{1}) \rangle \rangle \langle \langle F_{0l}(z_{2}) \rangle \rangle = \left( \frac{4}{3} \alpha_{s}, \frac{1}{r_{3}} + \frac{\sigma}{r} \right) r^{k} r^{l} + O(v^{2}) ,$$

$$g \langle \langle F_{hk}(z_{1}) F_{0l}(z_{2}) \rangle \rangle = \left( \frac{4}{3} \alpha_{s}, \frac{1}{m_{2} r_{3}} + \frac{\sigma}{m_{2} r} \right) (r^{k} p_{2}^{l} - r^{l} p_{2}^{k}) + O(v^{3}) ,$$

$$g^{2} \left( \langle \langle F_{hk}(z_{1}) F_{0l}(z_{2}) \rangle \rangle - \langle \langle F_{hk}(z_{1}) \rangle \rangle \langle \langle F_{0l}(z_{2}) \rangle \rangle \right) = \frac{1}{2} \left( \partial_{k} \partial_{l} \right) r^{k} r^{l} + \partial_{m} \partial_{k} \left[ \frac{1}{r} \left( \delta^{km} + r^{k} r^{m} \right) \right] - \partial_{m} \partial_{k} \left[ \frac{1}{r} \left( \delta^{kl} + r^{k} r^{l} \right) \right] + O(v^{2}) ,$$

$$g \langle \langle F_{0k}(z_{1}) \rangle \rangle \langle \langle F_{0l}(z_{2}) \rangle \rangle = \left( \frac{4}{3} \alpha_{s}, \frac{1}{r_{3}} + \frac{\sigma}{r} \right) r^{k} r^{l} + O(v^{2}) ,$$

$$g \langle \langle F_{hl}(z_{1}) F_{0k}(z_{2}) \rangle \rangle = \left( \frac{4}{3} \alpha_{s}, \frac{1}{m_{2} r_{3}} + \frac{\sigma}{m_{2} r} \right) (r^{h} p_{2}^{k} - r^{k} p_{2}^{h}) + O(v^{3}) ,$$

$$g^{2} \left( \langle \langle F_{hl}(z_{1}) F_{0k}(z_{2}) \rangle \rangle - \langle \langle F_{hl}(z_{1}) \rangle \rangle \langle \langle F_{0k}(z_{2}) \rangle \rangle \right) = \frac{1}{2} \left( \partial_{h} \partial_{l} \right) r^{h} r^{l} + \partial_{m} \partial_{k} \left[ \frac{1}{r} \left( \delta^{km} + r^{k} r^{m} \right) \right] - \partial_{m} \partial_{k} \left[ \frac{1}{r} \left( \delta^{kl} + r^{k} r^{l} \right) \right] + O(v^{2}) .$$
\[
\left( \langle \langle F_{\mu\nu}(z_1)F_{\rho\sigma}(z'_1) \rangle \rangle - \langle \langle F_{\mu\nu}(z_1) \rangle \rangle \langle \langle F_{\rho\sigma}(z'_1) \rangle \rangle \right) = 0 \tag{30}
\]

and similar ones.

In the end one obtains the potential in the form of a static part, a spin–dependent part and a velocity–dependent part, \( V^\omega = V_{\text{stat}}^\omega + V_{\text{sd}}^\omega + V_{\text{vd}}^\omega \), with

\[
V_{\text{stat}}^\omega = -\frac{4}{3} \alpha_s (-\frac{\alpha_s}{3r} + \sigma r) - \frac{4}{3} \pi \left( \frac{6}{3} - \frac{\alpha_s}{2r} \right) \sum_{j=1,2} \frac{1}{m_j^2} S_j \cdot L_j + \frac{4}{6} \alpha_s \left( \frac{3}{m_1 m_2} \right) \left( S_2 \cdot L_1 + S_1 \cdot L_2 \right)
\]

\[
S = S_2 \cdot L_1 + S_1 \cdot L_2 + \frac{4}{6} \alpha_s \left[ \left( \frac{3}{r^3} (S_1 \cdot r) (S_2 \cdot r) - \frac{S_1 \cdot S_2}{r^3} \right) + \frac{8}{3} \delta^3 (r) S_1 \cdot S_2 \right] \tag{32}
\]

\[
V_{\text{vd}}^\omega = \frac{1}{m_1 m_2} \left( \frac{4}{3} \alpha_s (\delta^{kk} + \pi r^{kk}) p_i^h p_j^k \right)_{\text{weyl}} - C \sum_j \frac{p_j^2}{4 m_j^2} - \frac{1}{6} \sigma \rho (p_1 \cdot p_2)_{\text{weyl}} \tag{33}
\]

At the order \( \alpha_s \) the above potential coincides globally with that given in Ref.\(^1\). In particular \( V_{\text{sd}} \) was originally given by Eichten and Feinberg and corrected by Gromes\(^2\) (see also Ref.\(^3\)), while \( V_{\text{vd}} \) has been obtained for the first time in\(^1\). Notice that Eq.\((33)\) differs from the corresponding one proposed under the ad hoc assumption of scalar confinement and does not present the phenomenological difficulties of this\(^7\). Notice also that the terms in \( C \) can be reabsorbed in a redefinition of the masses \( m_j \rightarrow m'_j = m_j + \frac{C}{r} \). The \( O(\alpha_s^2) \) term in \( (31) \) has been obtained for the first time in\(^6\).

The \( O(\alpha_s^2) \) contributions to \( V_{\text{sd}} \) and \( V_{\text{vd}} \) have been evaluated by Gupta et al.\(^8\) in an \( S \) matrix context but they have not been included here. In fact such contributions are found to be important for an understanding of the fine and hyperfine structure of the meson spectrum. However, due to the ambiguities inherent in the derivation method, a consistent evaluation in the Wilson loop approach should be desirable. Calculations are in progress in this line.

Finally, let us come to the ordering in \( (33) \). Obviously, ordering is related to the discretization prescription in the definition of the path integral. If in the definition of the gauge field functional integration we identify the element \( U_{n'n} \) of the colour group associated to the link between the contiguous sites \( n \) and \( n' \) with \( \exp [ig(x_{n'} - x_n)A_{\mu}(\frac{z_{n'} + x_n}{\sqrt{2}})] \), we obtain the Weyl ordering

\[
\{ X^{hk}(r), p_j^h p_j^k \}_{\text{weyl}} = \frac{1}{4} \{ p_j^h, \{ X^{hk}(r), p_j^k \} \} =
\]

\[
= \frac{1}{4} (X^{hk}(r)p_j^h p_j^k + p_j^h X^{hk}(r)p_j^k + p_j^k X^{hk}(r)p_j^h + X^{hk}(r)p_j^h p_j^k). \tag{34}
\]
4. Three-quark potential

The three-quark gauge invariant Green function can be written as (again we assume the quarks to have different flavours)

\[
G(x_1, x_2, x_3, y_1, y_2, y_3) = \frac{1}{3!} \varepsilon_{a_1 a_2 a_3} \varepsilon_{b_1 b_2 b_3}
\]

\[
\langle 0 | T U^{a_3 c_3}(x_3, x_3) U^{a_2 c_2}(x_2, x_2) U^{a_1 c_1}(x_1, x_1) \psi_{1d_1}(y_1) \psi_{2d_2}(y_2) S_{3d_3}(y_3) \psi_{1d_1}(y_1, y_M) U^{d_1 b_1}(y_1, y_M) U^{d_2 b_2}(y_2, y_M) U^{d_3 b_3}(y_3, y_M) | 0 \rangle
\]

(35)

and we assume \( x_1^0 = x_2^0 = x_3^0 = t_1, y_1^0 = y_2^0 = y_3^0 = y_M^0 = t_i, t_f - t_i \) large.

The integration over the fermionic variables is again trivial and one can write

\[
G(x_1, x_2, x_3, y_1, y_2, y_3; \tau) = \frac{1}{3!} \varepsilon_{a_1 a_2 a_3} \varepsilon_{b_1 b_2 b_3} \left( \langle U(x_3, x_1) S^F_1(x_1, y_1) A(y_1, y_M) \rangle \langle U(x_2, x_3) S^F_2(x_2, y_2) A(y_2, y_M) \rangle \langle U(x_1, x_3) S^F_3(x_3, y_3) A(y_3, y_M) \rangle \right)
\]

(36)

From (36), we can proceed strictly as in Sec.3 and in conclusion we have to show that

\[
i \ln W_{3q} + i \sum_{j=1}^{3} \frac{iq}{m_j} \int_{\Gamma_j} dx^\mu \left( S^l_j \langle \hat{F}_{l\mu}(x) \rangle - \frac{1}{2m_j} S^l_j \varepsilon^{lkr} p_j^k \langle \hat{F}_{\mu}(x) \rangle - \frac{1}{8m_j} \langle \hat{D}^{\nu} F_{\mu}(x) \rangle \right) - \frac{1}{2} \sum_{j, j'} \frac{iq^2}{m_j m_{j'}} T_s \int_{\Gamma_j} dx^\mu \int_{\Gamma_{j'}} dx'^\nu \langle S^l_j S^l_{j'} \rangle.
\]

\[
\cdot \left( \langle \hat{F}_{l\mu}(x) \hat{F}_{k\sigma}(x') \rangle - \langle \hat{F}_{k\sigma}(x') \rangle \langle \hat{F}_{l\mu}(x) \rangle \right) = \left[ \int_{t_i}^{t_f} dt V^{3q}(z_j, p_j, S_j) \right]
\]

(37)

with \( W_{3q} \) given by (19) and

\[
\langle \langle f[A] \rangle \rangle = \frac{1}{3} \langle \varepsilon \varepsilon \{ \Pi_j P \exp(iq \int_{\Gamma_j} dx^\mu A_{\mu}(x)) \} f[A] \rangle \frac{1}{3} \langle \varepsilon \varepsilon \{ \Pi_j P \exp(iq \int_{\Gamma_j} dx^\mu A_{\mu}(x)) \} \rangle.
\]

(38)

Again, after the replacement \( z_j \rightarrow p_j/m_j \), the quantity \( i \ln W_{3q} \) is already of the desired form, while the field expectation values can be evaluated according to equations analogues to (23) and (26) and lead to similar expressions. The final result is again of the form \( V^{3q} = V_{3q}^{stat} + V_{3q}^{sd} + V_{3q}^{vd} \) with

\[
V_{3q}^{stat} = \sum_{j<i} \left( -\frac{2 \alpha_s}{3 \ r_{ji}} \right) + \sigma(r_1 + r_2 + r_3) + C,
\]

(39)

\[
V_{3q}^{sd} = \frac{1}{8m_1^2} \nabla_{(1)}^2 \left( \frac{2 \alpha_s}{3 \ r_{12}} - \frac{2 \alpha_s}{3 \ r_{31}} + \sigma r_1 \right) + \left\{ \frac{1}{2m_1^2} S_1 \cdot \left( r_{12} \times p_1 \right) \left( \frac{2 \alpha_s}{3 \ r_{12}} \right) + \left( r_{31} \times p_1 \right) \left( -\frac{2 \alpha_s}{3 \ r_{31}} - \sigma \frac{r_1 \times p_1}{r_1} \right) \right\} + \]

\[
+ \left\{ \frac{1}{2m_1^2} S_1 \cdot \left( r_{12} \times p_1 \right) \left( \frac{2 \ alpha_s}{3 \ r_{12}} \right) + \left( r_{31} \times p_1 \right) \left( -\frac{2 \ alpha_s}{3 \ r_{31}} - \sigma \frac{r_1 \times p_1}{r_1} \right) \right\} + \]

\[
+ \left\{ \frac{1}{2m_1^2} S_1 \cdot \left( r_{12} \times p_1 \right) \left( \frac{2 \ alpha_s}{3 \ r_{12}} \right) + \left( r_{31} \times p_1 \right) \left( -\frac{2 \ alpha_s}{3 \ r_{31}} - \sigma \frac{r_1 \times p_1}{r_1} \right) \right\} + \]

(40)
\[ + \frac{1}{m_1 m_2} \mathbf{S}_1 \cdot (\mathbf{r}_{12} \times \mathbf{p}_2) \left( -\frac{2 \alpha_s}{3 r_{12}^3} \right) + \frac{1}{m_1 m_3} \mathbf{S}_1 \cdot (\mathbf{r}_{31} \times \mathbf{p}_3) \left( \frac{2 \alpha_s}{3 r_{31}^3} \right) + \]
\[ + \frac{1}{m_1 m_2} \frac{2 \alpha_s}{3} \left\{ \frac{1}{r_{12}^3} \left[ \frac{3}{r_{12}^2} (\mathbf{S}_1 \cdot \mathbf{r}_{12})(\mathbf{S}_2 \cdot \mathbf{r}_{12}) - \mathbf{S}_1 \cdot \mathbf{S}_2 \right] + \frac{8 \pi}{3} \delta^3(\mathbf{r}_{12}) \mathbf{S}_1 \cdot \mathbf{S}_2 \right\} + \]
\[ + \text{cyclic permutations}, \quad (40) \]
\[ V_{vd}^{3q} = \sum_{j<k} \frac{1}{2m_j m_k} \left\{ \frac{2 \alpha_s}{3} \frac{r_{jl}^2}{r_{rj}^2} (\delta_{jk}^{\gamma} + \delta_{jk}^{\delta}) \right\}_{\text{Weyl}} - \sum_{j=1}^{3} \frac{1}{6m_j^2} \{ \sigma r_j \mathbf{p}_{jT_j} \}_{\text{Weyl}} - \sum_{j=1}^{3} \frac{C}{6m_j^2} p_j^2, \quad (41) \]

where the notations are the same as used in (19) and the ordering is as in (24). Notice, in particular, that the quantity \( \dot{z}_M \) in (11) is given by

\[ \dot{z}_M = \begin{cases} R^{-1} \sum_{j=1}^{3} \left( \mathbf{p}_{jT_j} / m_j r_j \right) & \text{type I configuration} \\ \mathbf{p}_j / m_j & \text{type II configuration} : (z_M \equiv z_j) \end{cases}, \]

\( R \) being the matrix with elements \( R_{hk} = \sum_{j=1}^{3} \frac{1}{r_j} (\delta_{hk} - \delta_{hk}^{jk}) \).

Notice that Eq.(10) properly refers to the configuration I case. In general one should write \( V_{sd}^{LR} = - \sum_{j=1}^{3} \frac{1}{2m_j^2} \mathbf{S}_j \cdot \nabla_j V_{\text{stat}}^{LR} \times \mathbf{p}_j \) (In comparing this with (10) one should keep in mind that the partial derivatives in \( z_M \) of \( V_{\text{stat}}^{LR} \) vanish due to the definition of \( M \)).

We observe that the short range part in Eqs.(39)–(41) is of a pure two body type: in fact it is identical to the electromagnetic potential among three equally charged particles but for the colour group factor \( 2/3 \) and it is well known. Even the static confining potential in Eq.(39) is well known (for a review see e.g.13). Furthermore the long range part in Eq.(10) coincides with the expression obtained by Ford10 starting from the assumption of a purely scalar Salpeter potential of the form

\[ \sigma (r_1 + r_2 + r_3) \beta_1 \beta_2 \beta_3, \]

but to our knowledge it was not obtained consistently in a Wilson loop context before Ref.1. Eq.(11) has been given for the first time in Ref.1. Eq. (11) differs from the corresponding equation obtained from (13). The situation for the three quarks is so similar to that occurring for the quark–antiquark system. In Eqs.(39) and (41) the terms in \( C \) can be again eliminated by the redefinition of the masses \( m_j \rightarrow m_j'' = m_j + \frac{C}{3} \). Notice however that \( m_j'' \) differs from \( m_j' \).

5. Relativistic flux tube model

Let us now neglect in Eq.(21) the spin–dependent terms and replace the \( \frac{1}{m^2} \) expansion by its exact relativistic expression

\[ K(x_1, x_2; y_1, y_2; t_f - t_i) = \]
\[
\int D[z_1, p_1] \int D[z_2, p_2] \exp \left\{ i \left[ \int_{t_i}^{t_f} dt \sum_{j=1}^{2} (p_j \cdot \dot{z}_j - \sqrt{m_j^2 + p_j^2}) \right] + \ln W_{q\bar{q}} \right\}
\] (44)

Let us further evaluate \( i \ln W_{q\bar{q}}^{SR} \) by the original Eq. (3) and assume that a sensible approximation is obtained even in the relativistic case postulating the first line of Eq. (14) in the center–of–mass system of the two particles. Then, if we expand again the exponent in (44) around the stationary values \( p_j = \frac{m z_j}{\sqrt{1 - z_j^2}} \), in the gaussian approximation we obtain the ordinary lagrangian

\[
L = -2 \sum_{j=1}^{2} m_j \sqrt{1 - \dot{z}_j^2} + \frac{4 \alpha_s}{3} \left[ 1 - \frac{1}{2} (\delta^{hk} + \hat{r}^h \hat{r}^k) \dot{z}_1^h \dot{z}_2^k \right] + \sigma r \int_0^1 ds \left[ 1 - (s \dot{z}_1 T + (1 - s) \dot{z}_2 T)^2 \right]^{1/2}.
\] (45)

This coincides with the relativistic flux–tube lagrangian

From (45) is not possible to obtain even a classical hamiltonian in a close form, due to the complicate velocity dependence. However, in terms of an expansion in \( \frac{\sigma r}{m^2} \) we have (we assume \( m_1 = m_2 = m \) for simplicity and have already eliminated the terms in \( C \))

\[
H(r, q) = 2 \sqrt{m^2 + q^2} + \frac{\sigma r}{2} \left[ \sqrt{m^2 + q^2} \arcsin \frac{q_T}{\sqrt{m^2 + q^2}} + \sqrt{m^2 + \frac{q^2}{2}} \right] + \frac{\sigma^2 r^2}{16 q_T^2} \left( \sqrt{m^2 + q^2} \arcsin \frac{q_T}{\sqrt{m^2 + q^2}} - \sqrt{m^2 + \frac{q^2}{2}} \right)^2 + \ldots (46)
\]

with \( r = z_{1\text{CM}} - z_{2\text{CM}}, q = p_{1\text{CM}} = -p_{2\text{CM}}, q_r = (\hat{r} \cdot q) / \hat{r} \) and \( q^h = (\delta^{hk} - \hat{r}^h \hat{r}^k) q^k \). From this a quantum hamiltonian can be immediately obtained by setting

\[
\langle k'| H_{FT} | k \rangle = \int \frac{dr}{(2\pi)^3} e^{i(k-k') \cdot r} H(r, \frac{k' + k}{2}),
\] (47)

in which the ordering prescription is again Weyl prescription. By an expansion in \( \frac{1}{m^2} \) a semirelativistic hamiltonian can be obviously reobtained with a potential given by (31)–(33).

6. Bethe–Salpeter equation

Let us go back to the equation analogous to (20) for spinless quarks and in it use the covariant representation for the quark propagator in an external gauge field

\[
\Delta^F(x, y|A) = -i \int_0^\infty d\tau \int_{z(0)=y}^{z(\tau)=x} D[z] \text{Pexp} i \int_0^\tau d\tau' \left\{ \frac{1}{2} \left[ \left( \frac{dz}{d\tau'} \right)^2 + m^2 \right] - g z'^\mu A_\mu(z) \right\}
\] (48)
In place of (21) we find
\[
G_4(x_1, x_2; y_1, y_2) = (-\frac{i}{2})^2 \int_0^\infty d\tau_1 \int_0^\infty d\tau_2 \int_{z_1(t_1) = x_1}^{z_1(t_1) = x_1} D[z_1] \int_{z_2(t_2) = y_2}^{z_2(t_2) = x_2} D[z_2] \exp \left\{ -\frac{i}{2} \left( \int_0^{\tau_1} d\tau_1' \left[ \left( \frac{dz_1}{d\tau_1'} \right)^2 + m_1^2 \right] + \int_0^{\tau_2} d\tau_2' \left[ \left( \frac{dz_2}{d\tau_2'} \right)^2 + m_2^2 \right] \right) \right\} \cdot \frac{1}{3} (\text{Tr} \exp \int dz^\mu A_\mu (z))
\]
(49)

where the equation of a path connecting \( y \) with \( x \) is written as \( z^\mu = z^\mu (\tau) \), in terms of an arbitrary parameter \( \tau \) (rather than the time \( t \)) and \( z' \) stands for \( z(\tau') \). Notice the occurrence again in (49) of the Wilson loop integral \( W_{q\bar{q}} \). Even in this case we can use the evaluation of \( i \ln W_{q\bar{q}}^{SR} \) as given by Eq. (4) or (3) and assume for \( i \ln W_{q\bar{q}}^{LR} \) the first line of (14) in the center–of–mass system.

Then, by appropriate manipulations of the resulting expression, one can obtain the Bethe–Salpeter equation
\[
G_4(x_1, x_2; y_1, y_2) = G_2(x_1 - y_1) G_2(x_2 - y_2) + \int d^4\xi_1 d^4\xi_2 d^4\eta_1 d^4\eta_2
\]
\[
G_2(x_1 - \xi) G_2(x_2 - \xi_2) I(\xi_1, \xi_2; \eta_1, \eta_2) G_4(\eta_1, \eta_2; y_1, y_2),
\]
(50)

with a kernel of the form \( I(\xi_1, \xi_2; \eta_1, \eta_2) = I^{SR}(\xi_1, \xi_2; \eta_1, \eta_2) + I^{LR}(\xi_1, \xi_2; \eta_1, \eta_2) \). Here \( I^{SR} \) coincides with the ordinary perturbative kernel, while in the momentum representation \( I^{LR} \) can be written as (for simplicity we have neglected the perimeter term)
\[
\tilde{I}^{LR}(p'_1, p'_2; p_1, p_2) = \frac{1}{(2\pi)^3} \int d^3r e^{i(k'-k)\cdot r} J(r, \frac{p'_1 + p_1}{2}, \frac{p'_2 + p_2}{2}) \]
(51)

\( (p'_1 + p'_2 = p_1 + p_2, \quad p_1 = -p_2 = k, \quad p'_1 = -p'_2 = k' \) with
\[
J(r, q_1, q_2) = \frac{(2\pi)^3}{2} q_{10} + q_{20} \left[ \frac{q_1^2}{q_{10}^2} + \frac{q_2^2}{q_{20}^2} - \frac{q^2}{q_{10}^2} \right] + \frac{\sqrt{q_1^2 q_2^2}}{q_{10} q_{20}} \left[ \arcsin \left( \frac{q_1}{q_{10}} \right) + \arcsin \left( \frac{q_2}{q_{20}} \right) \right] + O\left( \frac{\sigma^2}{m^4} \right)
\]
(52)

(having set \( q_1 = -q_2 = q, \quad q_{10}^2 = (\delta_{hk} - \tilde{\delta}_{hk})q^k \)).

Notice that, according to a standard procedure, the BS kernel \( \tilde{I} \) can be associated with a relativistic potential (to be used in the Salpeter equation) given by
\[
\langle k'| V | k \rangle = \frac{1}{(2\pi)^3} \frac{m_1 m_2}{\sqrt{w_1(k) w_2(k') w_1(k') w_2(k)}} \tilde{I}_{\text{inst}}(k', k)
\]
(53)

where \( w_j(k) = \sqrt{m_j^2 + k^2} \) and the instantaneous kernel \( \tilde{I}_{\text{inst}} \) is obtained from \( \tilde{I} \) by setting \( p_{10}, p_{20} \) and \( p'_{10}, p'_{20} \) equal to appropriate functions of \( k \) and \( k' \), respectively. In the present case it is convenient to choose \( p_{10} = p'_{10} = \sqrt{w_1(k) w_2(k')} \), \( p_{20} = p'_{20} = \sqrt{w_1(k) w_2(k')} \). If we do so we reobtain the Hamiltonian (17).
Going back to the expression of the kernel \( \tilde{I}^{LR} \) as given by (52)–(51), one has to notice that this is highly singular for \( k' = k \), due to the occurrence of the factor \( r \) in (52), and it must be appropriately regularized before being used in equation (50) (e.g., one can make the substitution \( r \to re^{-\varepsilon r} \)). This circumstance is related to the fact that, being confining, \( \tilde{I}^{LR} \) should admit only bound states, while the inhomogeneous BS equation provides also a continuous two–particles spectrum. Therefore one should solve (50) for the regularized kernel and only at the end take the limit for \( \varepsilon \to 0 \) (we admit in this way that resonances would evolve in bound states).

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8. References

1. A. Barchielli, E. Montaldi and G.M. Prosperi, *Nucl. Phys.* B296 (1988) 625; A. Barchielli, N. Brambilla and G.M. Prosperi, *Nuovo Cimento* 103A (1990) 59; N. Brambilla, P. Consoli and G. M. Prosperi, *Phys. Rev.* D 50 (1994).
2. W. Lucha, F.F. Schöberl and D. Gromes, *Phys. Rep.* 200 (1991) 127.
3. Y. Simonov, , HD-THEP-93-16; Y. Simonov, *Nucl. Phys.* B324 (1989) 67; H. G. Dosch, this Conference
4. J. Ball, F. Zachariasen, this conference and quoted references.
5. A.M. Polyakov, *Nucl. Phys.* B164 (1979) 171; V.S. Dotsenko and S.N. Vergeles, *Nucl. Phys.* B169 (1980) 527; R. Brandt et al. *Phys. Rev.* D24 (1981) 879; see also A. Bassetto et al., *Nucl. Phys.* B408 (1993) 62 and references therein.
6. W. Fischler, *Nucl. Phys.* B129(1977) 157; T. Appelquist et al. *Phys. Rev.* D17 (1978) 2074; A. Billoire, *Phys. Lett.* 92B (1980) 343; F. J. Yndurain and S. Titard, *Phys. Rev.* D 49 (1994) 6007.
7. A. Gara et al. *Phys. Rev.* D42 (1990) 1651; 40 (1989) 843; J.F. Lagae, *Phys. Rev.* D45 (1992) 305; 45 (1992) 317; N. Brambilla and G.M. Prosperi, *Phys. Lett.* B236 (1990) 69.
8. S.N. Gupta and S. Radford, *Phys. Rev.* D24 (1981) 2309; S.N. Gupta et al. *Phys. Rev.* D34 (1986) 201, F. Halzen et al. MAD/PH/706.
9. J.M. Richard, Phys. Rep. 212 (1992) 1.
10. J.I. Ford, *Journ. Phys.* G 15 (1989) 1641.
11. C. Olson, M.G. Olsson and K. Williams, *Phys. Rev.* D45 (1992) 4307; N. Brambilla and G.M. Prosperi, *Phys. Rev.* D47 (1993) 2107.