Identification of the spacewise dependent right-hand side in one-dimensional parabolic equation

LingDe Su, V. I. Vasil'ev
Institute of Mathematics and Information Science, North-Eastern Federal University, Russia
E-mail: sulingde@gmail.com, vasvasil@mail.ru

Abstract. In this paper we consider numerical solution of the inverse problem of determining a spacewise dependent right-hand side function, which is also called source function, in parabolic equation from measured data of the solution at the final time point. Such inverse problem occurs in mathematical modeling of various physics or engineering areas. Special iterative method is constructed for the approximate solution of the inverse problem, based on the source function at the new iterative step can be identified by the time derivative and space derivative at final time moment. The capabilities of the proposed computational algorithm is confirmed by the results of several numerical experiments.

1. Introduction
Reverse problem arises in many mathematical modelings of applied problems in science and engineering applications. It has a long history, its discovery and introduction needs to be traced back to the year 1929, when Ambarzumian studied the problem of determining the equation of a vibrating string [1]. After inverse problem in the CT technology made a major breakthrough in 1979, with the rapid development of calculation methods and the extensive application of computer technology, inverse problem has attracted more and more attention and caused widespread concern. Inverse problems are often considered non-classical issues and they are classified as ill-posed or conditionally well-posed problems [2], so, stable and efficient numerical algorithms need to be developed to obtain numerical solutions of the inverse problem.

Many researchers have made a significant contribution to inverse problem research and contributed to the development of ill-posed problems. In the 1960’s Tikhonov produced an important series of papers on ill-posed problems [3–5], the most commonly used method of regularization of ill-posed problems, Tikhonov regularization method was named after Andrey Tikhonov. In 1994, Per Christian Hansen and many other collaborators developed the package of Matlab routines for easy doing experiments with analysis and solution of discrete ill-posed problems by means of regularization methods [6]. In addition, V. Isakov [7], P. N. Vabishchevich [8,9], V. L. Kamygin [10], M. Li and T. S. Jiang [11], etc., also carried out theoretical or algorithmic research on inverse problems.

Based on the unknowns in the inverse problem, it can be classified geometric inverse problems, in which the domain or a part of it’s boundary is unknown; coefficient inverse problems, in which some coefficients in the master equation are unknown; boundary inverse problems, in which boundary conditions are unknown and evolutionary inverse problems (or time inverse problem), in which initial conditions are unknown. The problem in our work, which related to identifying
the right-hand side of an equation, belongs to coefficient inverse problem. A lot of works have been done for studying such inverse problem, the existence and uniqueness of the solution and the well-posedness in various functional class were considered in [7, 12–14], also many numerical techniques for solving the inverse source problem were presented in [15–17].

In this paper we construct a special iterative method to recover the right-hand side function, which depends on spatial variables only, in one-dimensional parabolic equation. The new iterative method based on the right-hand side function can be identified with the derivative of time and space at the final time point. The paper is organized as follows, in section 2, we briefly introduce the formulation of the problem. In section 3, we introduce the algorithm and apply on the inverse problems. The results of numerical experiments are presented in section 4. Section 5 is dedicated to a brief conclusion. Finally, some references are introduced at the end.

2. The Problem Formulation

We consider the inverse problem to determine the spacewise dependent right-hand function $f(x)$ in the following parabolic equation,

$$
\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x)g(t), \quad x \in (0, l), \quad t \in (0, T],
$$

where $g(t)$ is a given function and $l$, $T$ are given positive constants. Also the initial condition is given as

$$
u(x, 0) = u^0(x), \quad x \in [0, l],
$$

the Dirichlet boundary conditions

$$
u(0, t) = u_0(t), \quad \nu(l, t) = u_l(t), \quad t \in [0, T]\n$$

for solving an inverse problem, some additional conditions are also needed, in this kind of problem, the additional conditions are usually given as the solution at the final time moment, which is also called final overdetermination

$$
u(x, T) = u^T(x), \quad x \in [0, l].
$$

In our work, we focus on numerically solving the inverse problem (1)–(4) of determining a pair of $u(x, t)$ and $f(x)$, we assume that the inverse problem (1)–(4) is well-posed. In [18, 19] provided the existence, uniqueness and stability of such source inverse problem. Also, many numerical works have been proposed for solving the inverse problem (1)–(4), in [15] boundary element method was used to determine the heat source dependent on one variable, B. T. Johansson and D. Lesnic in their work [17] provided a variational method for identifying spacewise dependent heat source, the conjugate gradient method was considered in [20], in the work of Yan with his collaborator [21] a meshless method was used to solve the inverse source problem, P. N. Vabishchevich considered iterative computational identification of spacewise dependent source in [22], and V. I. Vasil’ev and his collaborators also have given a study on numerical solution of the right hand problems in their works [23, 24], etc.

3. The Computational Algorithm

To numerically solve the source inverse problem (1)–(4) iterative algorithm is used, in the new iterative step the right-hand side function $f^{(k+1)}$ is identified for $t = T$,

$$
f^{(k+1)}(x) = \left( \frac{\partial u^{(k)}}{\partial t}(x, T) - \frac{\partial^2 u^T}{\partial x^2}(x) \right) / g(T), \quad k = 0, 1, 2, \cdots
$$
with some given assumption initial \( f^{(0)} \), the symbol \( f^{(k)} \) represents the value of \( f(x) \) at the \( k \)-th iteration step. Then, the parabolic problem (1)-(4) can be solved by the following forms combined with (5) and the evaluation of convergence can be seen in Vabishchevich’s work [22].

\[
\frac{\partial u^{(k)}_i}{\partial t} - \frac{\partial^2 u^{(k)}_i}{\partial x^2} = f^{(k)}(x)g(t), \quad t \in (0, T].
\]

(6)

In the calculation of solving equation (6), sweep method is used, we introduce the uniform grid in the computational domain \( \Omega \),

\[ \omega_h = \{ x_i | x_i = ih, \ i = 1, 2, \ldots, M - 1, \ Mh = l \}, \]

where \( M \) is a positive integer and \( x_0 = 0, x_M = l \). For the time we have,

\[ \omega_\tau = \{ t^n | t^n = n\tau, \ n = 0, 1, \ldots, N, \ N\tau = T \}. \]

Using the notation \( u^{n,(k)}_i = u^{(k)}(x_i, t^n) \) and the implicit scheme, from (6) we get discrete forms,

\[
\frac{u^{n+1,(k)}_i - u^{n,(k)}_i}{\tau} - \frac{u^{n+1,(k)}_{i+1} - 2u^{n+1,(k)}_i + u^{n+1,(k)}_{i-1}}{h^2} = f^{(k)} g^{n+1},
\]

(7)

\[
u^{0,(k)}_i = u^{(k)}(x_i, 0) = u^0(x_i), \quad u^{N,(k)}_i = u^{(k)}(x_i, T) = u^T(x_i),
\]

(8)

\[
u^{n,(k)}_0 = u^{(k)}(0, t^n) = u_0(t^n), \quad u^{n,(k)}_M = u^{(k)}(l, t^n) = u_l(t^n).
\]

(9)

where the superscript \( (k) \) represents the \( k \)-th iteration step. We convert (7) into the following form,

\[
u^{n+1,(k)}_i - (2 + \frac{h^2}{\tau})u^{n+1,(k)}_i + u^{n+1,(k)}_{i-1} + \frac{h^2}{\tau}u^{n,(k)}_i + h^2 f^{(k)}_i g^{n+1} = 0,
\]

(10)

with the conditions (8) and (9). Using the following formula for \( u^{n+1,(k)}_i \),

\[
u^{n+1,(k)}_i = \alpha^{n+1,(k)}_i u^{n,(k)}_i + \beta^{n+1,(k)}_i,
\]

(11)

where \( \alpha_0 = 0 \) and \( \beta_0 = u_0 \). Bringing (11) into equation (10), we get,

\[
\alpha^{n+1,(k)}_i = \frac{1}{2 + \kappa - \alpha^{n+1,(k)}_{i-1}}, \quad \beta^{n+1,(k)}_i = \frac{\kappa u^{n,(k)}_i + h^2 f^{(k)}_i g^{n+1} + \beta^{n+1,(k)}_{i-1}}{2 + \kappa - \alpha^{n+1,(k)}_{i-1}},
\]

(12)

where \( \kappa = \frac{h^2}{\tau} \), combining (5) and some given assumption initial \( f^{(0)} \), we can find the right-hand side function and the solution for parabolic equation (1). In our computation, we take the initial approximation \( f^{(0)}(x) \) in the form as follows,

\[
f^{(0)}(x) = -\frac{\partial^2 u^T}{\partial x^2}(x)/g(T) = -\frac{u^T(x + h_0) - 2u^T(x) + u^T(x - h_0)}{h_0^2 g(T)},
\]

(13)

and \( h_0 = 10^{-3} \).
4. Numerical Examples

To test the effectiveness of the iterative computational algorithm, which mentioned in the previous section, for solving the inverse problem to determine the spacewise dependent right-hand side function in one-dimensional parabolic equation, we present a numerical result with a discontinuous function \( f(x) \), \( x \in [0, l] \), which is given as,

\[
f(x) = \begin{cases} 
1, & x \in (0.5, 0.8) \\
\exp(-100(x - 0.25)^2), & \text{otherwise}
\end{cases}
\]

the known conditions are also given in the following,

\[
g(t) = 1, \quad u(x, 0) = 0, \quad u_0 = u_l = 0, \quad l = 1.
\]

In the computation, we use \( T = 0.2 \), the numerical and exact solution of \( f(x) \) with the assumption initial \( f(0) \) are presented in Figure 1. In this computation, we take \( M = N = 100 \) and the total iterative number \( K = 10 \). As can be seen from the figure, the numerical solution is a good coincidence of the exact solution.

![Figure 1. numerical and exact solution of \( f(x) \) and the assumption initial \( f(0) \).](image)

To evaluate the error of the approximate solution on every iterations, we adopt two errors, which are defined as follows,

\[
e = |f^{(k)}(x) - f(x)|, \quad \text{RMSE} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (f^{(k)}(x_i) - f(x_i))^2},
\]

where \( f^{(k)}(x) \) is the numerical solution of the right-hand side function on the \( k \)-th iterative step, while \( f(x) \) is the exact solution.

We consider the the errors with different \( M \) on every iterative steps, the RMSE and Max \( e \) can be obtained from Figure 2 with \( N = 100 \). In Figure 2(a) the RMSE error with different \( M \) is presented, in Figure 2(b) the figure of Max \( e \) on every iterative steps can be seen. The figures of RMSE and Max \( e \) with different time step are presented in Figure 3, in this computation we use \( M = 200 \).
Because the error is always accompanied by the actual measurement, we also consider the inverse problem with the noisy measured additional data, which is given by,

$$u_\delta^T = u^T + \delta \text{randn} \max(u^T)/2,$$

where $\delta$ is the tolerated noise level and $\text{randn}$ is a random variable uniformly distributed on the interval $[-0.5, 0.5]$. In order to identify the smooth solution, we using a smoothing method, which was given in [25] to get the smooth final solution $y$,

$$u_\delta^T = -\tau \frac{y_{xx}}{\lambda} + y,$$

where $\lambda$ is a constant parameter. The figures of the results with different noisy level $\delta$ can be seen in Figure 4 with $\lambda = 25$.

5. Conclusions
In this paper we consider the inverse problem of identifying right-hand side function. The numerical method based on the function is identified at the final time point. With some assumption initial source function, iterative method is used combined with sweep method for solving the parabolic equation on each iterative steps. Numerical example of the model problem demonstrates the efficiency of the iterative method for solving the inverse problem.
Figure 4. The figure of $f(x)$ with different $\delta$.

Acknowledgments
The authors would like to express their gratitude to professor P. N. Vabishchevich for his helpful and constructive remarks. This work was supported by RFBR (project 17-01-00689).

References
[1] Ambartsumyan V A 1929 Zeitschrift f"ur Physik 60 690–695
[2] Tikhonov A N, Goncharsky A, Stepanov V V and Yagola A G 1995 Numerical Methods for the Solution of Ill-Posed Problems 1st ed Mathematics and Its Applications 328 (Springer Netherlands)
[3] Tikhonov A N 1963 Soviet Math. Dokl. 4 1624–1627
[4] Tikhonov A N 1963 Soviet Math. Dokl. 5 1035–1038
[5] Tikhonov A N 1963 Dokl. Akad. Nauk SSSR 151 501–504
[6] Hansen P C 1994 Numer. Algorithms 6 1–35
[7] Isakov V 2017 Inverse Problems for Partial Differential Equations 3rd ed Applied Mathematical Sciences 127 (Springer International Publishing)
[8] Vabishchevich P N and Vasil’ev V I 2016 Inverse Probl. Sci. Eng. 24 42–59
[9] Samarskii A A and Vabishchevich P N 2007 Numerical Methods for Solving Inverse Problems of Mathematical Physics 1st ed Inverse and Ill-Posed Problems Series 52 (De Gruyter)
[10] Kamynin V L 2015 Math. Notes 97 349–361
[11] Jiang T S, Li M and Chen C S 2012 Numer. Heat Transfer Part A 61 338–352
[12] Isakov V 1991 Comm. Pure Appl. Math. 44 185–209
[13] Prilepko A I and Kostin A B 1993 Math. Notes 53 63–66
[14] Gol’dman N I, 2008 Dokl. Math. 77 350–355
[15] Farca A and Lesnic D 2006 J. Engrg. Math. 54 375–388
[16] Xiong X T, Yan Y M and Wang J X 2011 J. Phys. Conf. Ser. 290 12–17
[17] Johansson B T and Lesnie D 2007 IMA J. Appl. Math. 72 748–760
[18] Solov’ev V V 1990 Differ. Equ. 25 1114–1119
[19] Kamynin V L 2005 Math. Notes 77 482–493
[20] Erdem A, Lesnic D and Hasanov A 2013 Appl. Math. Model. 37 10231–10244
[21] Yan L, Yang F L and Fu C L 2009 J. Comput. Phys. 228 123–136
[22] Vabishchevich P N 2017 Inverse Probl. Sci. Eng. 25 1168–1190
[23] Vasil’ev V I, Popov V V and Kardashevsky A M 2018 Conjugate gradient method for identification of a spacewise heat source Large-Scale Scientific Computing ed Lirkov I and Margenov S (Springer International Publishing) pp 600–607
[24] Vasil’ev V I, Kardashevsky A M and Popov V V 2018 AIP Conf. Proc. 2025 100011
[25] Samarskii A A, Vabishchevich P N and Vasil’ev V I 1997 Matematicheskoe Modelirovanie 9 119–127