1. Introduction

Let $E$ be a holomorphic vector bundle over a compact Kähler manifold $M$, such that $E$ admits a holomorphic connection compatible with its holomorphic structure. The following question is perhaps due to S. Murakami [M1], [M2], [M3]: does there exists a flat connection on $E$ compatible with the holomorphic structure?

The same question can be posed for a more general principal $G$ bundle where $G$ is a complex Lie group. Murakami constructed examples of torus bundles over torus which admit holomorphic connection but do not admit compatible flat connection; thus providing a negative answer to the general question.

Let $M$ be a compact connected Kähler manifold of complex dimension $d$, and let $\omega$ be a Kähler form on $M$. The degree of a torsion-free $\mathcal{O}_M$-coherent sheaf $F$ is defined as [Ko, Ch. V, (7.1)]

\begin{equation}
\deg F := \int_M c_1(F) \wedge \omega^{d-1}.
\end{equation}

Consider the Harder-Narasimhan filtration of the holomorphic tangent bundle of $M$ [Ko, page 174, Ch. V, Theorem 7.15]:

\begin{equation}
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \mathcal{E}_3 \subset \cdots \subset \mathcal{E}_{l-1} \subset \mathcal{E}_l = T
\end{equation}

Let $G$ be a connected affine algebraic reductive group over $\mathbb{C}$. Let $P$ be a holomorphic principal $G$ bundle on $M$ (i.e. the transition functions of $P$ are holomorphic). Assume that $P$ admits a holomorphic connection $D$ compatible with the holomorphic structure. This means the following: $D$ is a holomorphic 1-form on $P$ with values in the Lie algebra, $\mathfrak{g}$, of $G$, such that $D$ is invariant for the action of $G$ on $P$, and, when restricted to the fibers of $P$, this form coincides with the holomorphic Maurer-Cartan form. Using the natural identification of the holomorphic tangent space of $P$ with its real tangent space, the holomorphic connection $D$ gives a $G$ connection on $P$.

A flat $G$ connection on $P$ is called compatible with respect to the holomorphic structure if the $(1,0)$ component of the connection form (which is a $\mathfrak{g}$ valued 1-form on $P$) is actually a holomorphic 1-form. This is equivalent to the following: $M$ can be covered by open set $\{U_i\}$ such that over each $U_i$ the principal bundle $P$ admits a holomorphic trivialization which is also constant (with respect to the connection). Clearly the $(1,0)$ component of such a flat connection compatible with the holomorphic structure gives a holomorphic connection compatible with the holomorphic structure.

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Our aim here is to prove the following theorem [Theorem 3.1]:

**Theorem A.** If the Kähler manifold $M$ satisfies the condition that $\deg(T/E) \geq 0$, then a holomorphic principal $G$-bundle $P$ on $M$ admitting a compatible holomorphic connection is semistable. Moreover, if $\deg(T/E) > 0$, then such a bundle $P$ actually admits a compatible flat $G$-connection.

So, in particular, if $T$ is semistable with $\deg T > 0$, then a $G$-bundle $P$ on $M$ with a holomorphic connection admits a compatible flat connection. In the case where $G = GL(n, \mathbb{C})$, the above result was proved in [3].

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2. Preliminaries

We continue with the notation of Section 1.

Let $D$ be a holomorphic connection on the holomorphic principal $G$ bundle $P$ compatible with its holomorphic structure.

Let $Ad(P)$ denote the vector bundle on $M$ associated to $P$ for the adjoint action of $G$ on its Lie algebra $\mathfrak{g}$. The holomorphic structure of $P$ will induce a holomorphic structure on the vector bundle $Ad(P)$. Let $\overline{\partial}_P$ denote the differential operator of order one defining the holomorphic structure of $Ad(P)$.

Let $\Omega^1_M$ denote the holomorphic vector bundle on $M$ given by the holomorphic $i$-forms.

The holomorphic connection $D$ on $P$ induces a holomorphic connection on $Ad(P)$, which is again denoted by $D$. In other words,

$$D : Ad(P) \rightarrow Ad(P) \otimes \Omega^1_M$$

is a first order operator satisfying the Leibniz condition $D(f.s) = \partial f.s + fD(s)$, where $f$ is a smooth function, such that the curvature, $(D + \overline{\partial}_P)^2$, of the connection $D + \overline{\partial}_P$ is a holomorphic section of $\Omega^2_M \otimes Ad(P)$. This condition implies that the operator $D$ maps a holomorphic sections of $Ad(P)$ to a holomorphic section of $Ad(P) \otimes \Omega^1_M$.

A vector bundle $E$ on $M$ is called *stable* (resp. *semistable*) if for any $O_M$-coherent proper subsheaf $0 \neq F \subset E$, with $E/F$ torsion-free, the following condition holds [Ko, Ch. V, §7]:

$$\mu(F) := \deg F / \text{rank} F < \mu(E) := \deg E / \text{rank} E \quad (\text{resp. } \mu(F) \leq \mu(E))$$

A vector bundle is called *quasistable* if it is a direct sum of stable bundles of same $\mu$ (slope).

**Lemma 2.1.** If $\deg(T/E) \geq 0$ then the vector bundle $Ad(P)$ on $M$ is semistable. Moreover, if $\deg(T/E) > 0$, then $Ad(P)$ is quasistable.

The proof of this lemma is actually contained in [3]. However, to be somewhat self-contained, we will give some details of the proof.
Proof of Lemma 2.1. We will first prove that if \( \deg(T/E_{l-1}) \geq 0 \) then the bundle \( Ad(P) \) is semistable.

Suppose \( Ad(P) \) is not semistable. In that case it has a nontrivial Harder-Narasimhan filtration. Let

\[
0 = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V_n = Ad(P)
\]

be the Harder-Narasimhan filtration of the vector bundle \( Ad(P) \).

We restrict the domain of the operator \( D \) (which gives the holomorphic connection on \( Ad(P) \)) to the subsheaf \( V_1 \), and consider the induced operator

\[
D_1 : V_1 \rightarrow (Ad(P)/V_1) \otimes \Omega^1_M
\]

The Leibniz identity for \( D \) implies that \( D_1(f.s) = f.D_1(s) \), i.e. \( D_1 \) is \( \mathcal{O}_M \)-linear.

The largest semistable subsheaf (i.e. the first nonzero term in the Harder-Narasimhan filtration) of \( \Omega^1_M \) is the kernel of the surjective homomorphism

\[
q_{l-1} : \Omega^1_M \rightarrow E^*_{l-1}
\]

obtained by taking the dual of the inclusion in (1.2). Since the tensor product of two semistable sheaves is again semistable [MR, Remark 6.6 iii], the largest semistable subsheaf of \( (Ad(P)/V_1) \otimes \Omega^1_M \) is

\[
W := (V_2/V_1) \otimes \ker(q_{l-1})
\]

The general formula for the degree of a tensor product gives \( \mu(W) = \mu(V_2/V_1) + \mu(\ker(q_{l-1})) \). From the property of the Harder-Narasimhan filtration we have that \( \mu(V_2/V_1) < \mu(V_1) \). The assumption that \( \deg(T/E_{l-1}) \geq 0 \) implies that

\[
\mu(\ker(q_{l-1})) = -\mu(T/E_{l-1}) \leq 0
\]

From these we conclude that

\[
\mu(V_1) > \mu(W)
\]

Let \( E \) denote the image of \( D_1 \) defined in (2.3). Assume that \( E \) is not the zero sheaf. Since \( V_1 \) is semistable and \( E \) is a quotient of \( V \), we have \( \mu(E) \geq \mu(V_1) \). On the other hand, since \( W \) is the largest semistable subsheaf of \( (Ad(P)/V_1) \otimes \Omega^1_M \), we have \( \mu(E) \leq \mu(W) \). But this contradicts (2.5). So \( E \) must be the zero sheaf, i.e. the homomorphism \( D_1 \) in (2.3) must be the zero homomorphism. Thus we obtain that the subsheaf \( V_1 \) is invariant under the connection \( D \) on \( Ad(P) \).

Any \( \mathcal{O}_M \) coherent sheaf with a holomorphic connection is a locally free \( \mathcal{O}_M \) module [B, p. 211, Proposition 1.7]. (This proposition in [3] is stated for integrable connections (\( D \)-modules), but the proof uses only the Leibniz rule which is valid for a holomorphic connection.)

Using the Chern-Weil construction of characteristic classes it is easy to show that for a vector bundle \( E \) on \( M \), equipped with a holomorphic connection, any Chern class \( c_i(E) \in H^{2i}(M, \mathbb{Q}), i \geq 1 \), vanishes.

So we have \( \deg V_1 = 0 = \deg Ad(P) \). If \( V_1 \neq Ad(P) \) then \( \mu(V_1) > \mu(Ad(P)) \). So \( Ad(P) = V_1 \), i.e. the bundle \( Ad(P) \) is semistable.
If $Ad(P)$ is stable then obviously it is quasistable. Suppose that $Ad(P)$ is not stable. Then there is a filtration \cite[Ch. V, §7, Theorem 7.18]{K} \[\begin{align*}
0 = W_0 \subset W_1 \subset W_2 \subset \ldots \subset W_{n-1} \subset W_n = Ad(P)\end{align*}\] such that $W_i/W_{i-1}, 1 \leq i \leq n$, is a stable sheaf with $\mu(W_i/W_{i-1}) = \mu(Ad(P))$.

Now, as done in the proof of Proposition 3.4 of \cite{Bi}, using the given condition that $\deg(T/E_{l-1})$ is strictly positive, it is possible to show that the filtration (2.6) splits, i.e. \[W_i = W_{i-1} \oplus (W_i/W_{i-1})\] where $W_i/W_{i-1}$ is a locally free sheaf. The equality (2.7) proves that \[Ad(P) = \sum_{i=1}^n W_i/W_{i-1}\] This completes the proof of the lemma. \qed

In the next section we will use Lemma 2.1 to construct a flat connection on $P$.

3. Existence of a Flat Connection

Let $G$ be a connected affine algebraic reductive group over $\mathbb{C}$. Let $P$ be a holomorphic principal $G$ bundle over $M$.

Let $i : U \rightarrow M$ be the inclusion of an open subset such that the complement $X - i(U)$ is an analytic subset of $X$ of codimension at least two. For a $\mathcal{O}_U$ coherent sheaf $F$ on $U$, the direct image $i_*F$ is a $\mathcal{O}_M$ coherent sheaf. The degree of $F$ is defined to be the degree of $i_*F$.

We will recall the definition of (semi)stability of $P$ \cite[Definition 4.7]{RR}. Let $U \subset X$ be an open subset with $X - U$ being an analytic set of codimension at least two. Let $Q$ be a parabolic subgroup of $G$, and let $P'$ be a reduction of the structure group to $Q$ of the restriction of the principal $P$ to the open set $U$. The principal bundle $P$ is said to be stable (resp. semistable) if for any such $P'$, the degree of a line bundle on $U$ associated to $P$ for any character $\chi$ on $Q$ dominant with respect to a Borel subgroup contained in $Q$, is strictly negative (resp. nonpositive).

Let $D$ be a holomorphic connection on $P$ (defined in Section 1). The following theorem is a generalization of Lemma 2.1.

**Theorem 3.1.** If $\deg(T/E_{l-1}) \geq 0$ then $P$ is semistable. Moreover, if $\deg(T/E_{l-1}) > 0$, then $P$ admits a flat $G$-connection compatible with the holomorphic structure.

**Proof.** Let $P' \subset P$ be a reduction of structure group of $P$ to a maximal parabolic subgroup $Q \subset G$. This reduction is given by a section, $\sigma$, of the fiber bundle \[\rho : P/Q \rightarrow U\]

Let $T_{rel}$ denote the relative tangent bundle for the map $\rho$.

From Lemma 2.1 of \cite{R} it follows that in order to check that $P$ is semistable it is enough to show that $\deg(\sigma^*T_{rel}) \geq 0$. 
The reduction $P' \subset P$ gives an injective homomorphism $Ad(P') \rightarrow Ad(P)$ of adjoint bundles on $M$. The bundle $\sigma^*T_{rel}$ on $M$ is the quotient bundle $Ad(P)/Ad(P')$.

From Lemma 2.1 we know that if $\deg(T/E_{l-1}) \geq 0$, the adjoint bundle $Ad(P)$ is semistable. Since $G$ is reductive, the Lie algebra $\mathfrak{g}$ admits a nondegenerate $G$ invariant bilinear form. This implies that $Ad(P) = Ad(P)^\ast$. Hence $\deg Ad(P) = 0$. Now the semistability of $Ad(P)$ implies that $\deg(Ad(P)/Ad(P')) \geq 0$. This proves that the principal bundle $P$ is semistable.

If $\deg(T/E_{l-1}) > 0$ then from Lemma 2.1 we know that $Ad(P)$ is quasistable. So from the main theorem of [UY] it follows that the vector bundle $Ad(P)$ admits a Hermitian-Yang-Mills connection. We will denote this Hermitian-Yang-Mills connection by $\nabla$. This connection $\nabla$ is unique (though the Hermitian-Yang-Mills metric is not unique), and it is irreducible if and only if $Ad(P)$ is stable.

We want to show that this connection $\nabla$ induces a connection on the principal $G$ bundle $P$.

Let $Z_0$ denote the connected component of the center of $G$ containing the identity element (the center has finitely many components). Define

$$G_0 = G/Z_0$$

which is a semisimple group. The group $G_0$ acts on the Lie algebra $\mathfrak{g}$ (of $G$) by conjugation and gives an homomorphism

$$(3.2) \quad \theta : G_0 \rightarrow GL(\mathfrak{g})$$

which has a finite group as the kernel.

From a theorem of Chevalley [H, Theorem 11.2] we know that there is a linear representation of the group $GL(\mathfrak{g})$

$$(3.3) \quad \phi : GL(\mathfrak{g}) \rightarrow GL(V)$$

in a vector space $V$ over $\mathbb{C}$ and a line $L$ in $V$ such that

$$\phi \circ \theta(G_0) = \{ g \in GL(\mathfrak{g}) \mid \phi(g)(L) = L \}$$

Since $G_0$ is semisimple, it does not have any nontrivial character. This implies that $\phi \circ \theta(G_0)$ fixes the line $L$ point-wise. Let $0 \neq v \in L$ be a nonzero vector. So the isotropy subgroup of $v$ for the action of $GL(\mathfrak{g})$ on $V$ is precisely $\phi \circ \theta(G_0)$. It is not difficult to see that the homomorphism $\phi$ can be so chosen that it maps the center of the Lie algebra of $GL(\mathfrak{g})$ into the center of the Lie algebra of $GL(V)$. We will choose $\phi$ such that it satisfies this condition.

Let $q : G \rightarrow G_0$ denote the obvious projection. Let $P(G_0)$ denote the principal $G_0$ bundle on $M$ obtained by extending the structure group of $P$ to $G_0$ using the homomorphism $q$. The vector bundle $Ad(P)$, which can be identified with a principal $GL(\mathfrak{g})$ bundle, is obtained by extending the structure group of $P(G_0)$ to $GL(\mathfrak{g})$ using the homomorphism $\theta$ defined in (3.2).

Using the homomorphism $\phi$ we may extend the structure group of $Ad(P)$ to $GL(V)$. The vector bundle on $M$ associated to this principal $GL(V)$ bundle for the natural action of $GL(V)$ on the vector space $V$ will be denoted by $E$. 
The Hermitian-Yang-Mills metric on the vector bundle $Ad(P)$ gives a reduction of the structure group of $Ad(P)$ to $U(\mathfrak{g})$, a maximal compact subgroup of $GL(\mathfrak{g})$. Since the image $\phi(U(\mathfrak{g}))$ is a compact subgroup of $GL(V)$, it is contained in some maximal compact subgroup of $GL(V)$. So the reduction of $Ad(P)$ to $U(\mathfrak{g})$ will induce a reduction of $E$ to a maximal compact subgroup of $GL(V)$ (i.e. the vector bundle $E$ will be equipped with a hermitian metric) such that the connection on $E$ obtained by extending the connection $\nabla$ (on $P(\mathfrak{g})$) is the hermitian connection (for the hermitian metric on $E$). Since the metric on $Ad(P)$ is a Hermitian-Yang-Mills metric, the metric on $E$ obtained above is also a Hermitian-Yang-Mills metric. Indeed, the Hermitian-Yang-Mills condition of the connection on $Ad(P)$ implies that the curvature is a 2-form on $M$ with values in the center of the endomorphism bundle $Ad(P)^* \otimes Ad(P)$. Since $\phi$ maps the center of the Lie algebra of $GL(\mathfrak{g})$ into the the center of the Lie algebra of $GL(V)$, the induced connection on $E$ is a Hermitian-Yang-Mills connection. Let $\nabla'$ denote the Hermitian-Yang-Mills connection on $E$ obtained this way.

Since $\theta(G_0)$ fixes the vector $v$, this vector $v$ will give a nowhere zero section of $E$ (since $E$ is obtained by extending the structure group of $P$ to $GL(V)$); let $s$ denote this section of $E$.

The holomorphic connection $D$ on $P$ will induce a holomorphic connection on $E$. As we noted in Section 2, using the Chern Weil construction it is easy to see that the existence of a holomorphic connection on $E$ implies that any Chern class, $c_i(E)$, $i \geq 1$, vanishes.

From [Ko, Ch. IV, Corollary 4.13] it follows that Hermitian-Yang-Mills connection $\nabla'$ is a flat connection. Since $\nabla'$ is a flat unitary connection, any holomorphic section of $E$ must be a flat section for the connection $\nabla'$. Indeed, the Laplacian of a flat unitary connection operator is twice the Laplacian of the Dolbeault operator. In particular, the space of harmonic sections for these two Laplacians coincide. Since any holomorphic section is a harmonic section for the Dolbeault Laplacian, it must be a flat section. So, in particular, the section $s$ is flat.

Let $P(\mathfrak{g})$ denote the principal $GL(\mathfrak{g})$ obtained by extending the structure group of $P$ to $GL(\mathfrak{g})$ using the homomorphism $\theta \circ q$. Since $P(\mathfrak{g})$ is an extension, the fiber bundle, $P(\mathfrak{g})/G$, with fiber $GL(\mathfrak{g})/G$ has a natural section (which gives the reduction of the structure group of $P(\mathfrak{g})$ to $G$). Let $\alpha$ denote this section of $P(\mathfrak{g})/G$.

Since $\theta(G_0)$ fixes $v$ for the homomorphism $\phi$ in (3.3), we have an embedding of the fiber bundle $P(\mathfrak{g})/G$ in the total space of $E$ (given by the orbit of $v$ for the action of $GL(\mathfrak{g})$ using $\phi$).

It is easy to see that the image of the section $\alpha$ by the above embedding of $P(\mathfrak{g})/G$ in $E$ is precisely the section $s$.

Now, since $s$ is a flat section for the connection $\nabla'$, the connection $\nabla$ induces a $G_0$ connection on the principal $G_0$ bundle $P(G_0)$ on $M$ as follows: Let

$$p : P(G_0) \rightarrow P(\mathfrak{g})$$

denote the holomorphic map induced by $\theta$ in (3.2). Take $x \in p(P(G_0))$, and let $v \in T_xP(\mathfrak{g})$ be a horizontal vector for the connection $\nabla$ on $P(\mathfrak{g})$. Let $w$ be the image
of $v$ by the differential of the map

$$P(g) \rightarrow E$$

induced by the homomorphism $\phi$ in (3.3). Since $s$ is flat, the tangent vector $w$ lies in the submanifold of the total space of $E$ given by the section $s$. But this implies that the tangent vector $v$ lies in the image of $TP(G_0)$ under the map given by the differential of $p$. Thus the connection $\nabla$ on $P(g)$ induces a connection on the principal bundle $P(G_0)$ with structure group $G_0$. We will call this connection on $P(G_0)$ as $\nabla_0$. Since $\nabla$ is a flat connection, $\nabla_0$ is also flat.

The commutator subgroup $G' := [G, G] \subset G$ is a semisimple group, and the restriction of $q$ to $G'$ is a surjective homomorphism with a finite kernel. So their Lie algebras are isomorphic. So

$$G = Z_0 . G'$$

with a finite intersection $\Gamma := Z_0 \cap G'$.

The abelian Lie group $Z_0/\Gamma$ is a product of copies of $\mathbb{C}^*$, since $G$ is assumed to be affine. Let

$$f : G \rightarrow Z_0/\Gamma$$

denote the obvious projection (obtained from (3.4)).

Let $P(f)$ denote the principal $Z_0/\Gamma$ bundle on $M$ obtained by extending the structure group of $P$ using the homomorphism $f$.

The holomorphic connection $D$ on $P$ induces a holomorphic connection on $P(f)$, which we will denote by $D(f)$.

Any holomorphic line bundle on $M$ admitting a holomorphic connection actually admits a compatible flat connection. Indeed, if $\partial$ is a holomorphic connection on a holomorphic line bundle $L$ whose holomorphic structure is given by the operator $\overline{\partial}$, then the curvature $(\partial + \overline{\partial})^2$ is a holomorphic 2-form which is exact (since the cohomology class represented by it is of the type $(1, 1)$). So it is of the form $\partial \beta$, where $\beta$ is a $(1, 0)$-form. The new connection

$$\partial - \beta + \overline{\partial}$$

on $L$ is a flat connection compatible with the holomorphic structure.

Recall that $Z_0/\Gamma$ is a product of copies of $\mathbb{C}^*$. In view of the above remark, the the existence of the holomorphic connection $D(f)$ implies that the principal $Z_0/\Gamma$ bundle $P(f)$ admits a flat connection. Let $\nabla_1$ be a flat connection on $P(f)$.

Since the exact sequence of the Lie algebras

$$0 \rightarrow \mathfrak{i}_0 \rightarrow \mathfrak{g} \rightarrow \mathfrak{g}_0 \rightarrow 0$$

has a natural splitting (given by the Lie algebra of $G''$), the two flat connections $\nabla_0$ and $\nabla_1$ combine together to induce a flat $G$ connection on $P$ as follows: The horizontal subspace of the tangent space at a point $p \in P$ is defined to be the intersection of the inverse images of the horizontal subspaces of $P(G_0)$ and $P(f)$ (horizontal subspaces for the flat connections $\nabla_0$ and $\nabla_1$ respectively) for the obvious projections of $P$ onto $P(G_0)$ and $P(f)$ respectively. The integrability of $\nabla_0$ and $\nabla_1$ will imply that the
connection on $P$ obtained above is actually flat. This completes the proof of the theorem.

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School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, INDIA

Current address: Université de Grenoble 1, Institut Fourier, 100 rue des Maths, B.P. 74, 38402 Saint-Martin-d’Hères, FRANCE

E-mail address: indranil@math.tifr.res.in and biswas@puccini.ujf-grenoble.fr