HAHN POLYNOMIALS FOR HYPERGEOMETRIC DISTRIBUTION

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Abstract. Orthogonal polynomials for the multivariate hypergeometric distribution are defined on lattices in polyhedral domains in $\mathbb{R}^d$. Their structures are studied through a detailed analysis of classical Hahn polynomials with negative integer parameters. Factorization of the Hahn polynomials is explored and used to explain the relation between the index set of orthogonal polynomials and the lattice set in polyhedral domain. In the multivariate case, these constructions lead to nontrivial families of hypergeometric polynomials vanishing on lattice polyhedra. The generating functions and bispectral properties of the orthogonal polynomials are also discussed.

1. Introduction

We study orthogonal polynomials with respect to the hypergeometric distribution in several variables. Let $N$ be a positive integer and let $\ell_i$, $1 \leq i \leq d+1$, be nonnegative integers, such that $\ell_i \leq N$ and $\ell_i + \ell_j \geq N$ for $i \neq j$. The hypergeometric distribution in $d$ variables is defined by

$$H_{\ell,N}(x) := \frac{N!}{(-|\ell|)_N} \prod_{i=1}^d \frac{(-\ell_i)x_i}{x_i!} \frac{(-\ell_{d+1}) N-|x|}{(N-|x|)!},$$

where $|\ell| = \ell_1 + \cdots + \ell_{d+1}$, $|x| = x_1 + \cdots + x_d$ and $(a)_k = a(a+1)\cdots(a+k-1)$ is the Pochhammer symbol; it is a probability distribution on the set $V_{\ell,N}$ of discrete polyhedral domain defined by

$$V_{\ell,N} := \{ x \in \mathbb{N}_0^d : 0 \leq x_i \leq \ell_i, 1 \leq i \leq d, \text{ and } N-\ell_{d+1} \leq |x| \leq N \}.$$

We studied orthogonal polynomials with respect to this distribution in [11], which are called the Hahn polynomials on polyhedra, since an orthogonal basis can be explicitly given in terms of the Hahn polynomials. These polynomials are closely related to the classical Hahn polynomials of several variables [14] that are orthogonal with respect to the weight function

$$W_{\kappa,N}(x) = \frac{N!}{(|\kappa|+d+1)_N} \prod_{i=1}^d \frac{(\kappa_i+1)x_i}{x_i!} \frac{(\kappa_{d+1}+1) N-|x|}{(N-|x|)!},$$

where $\kappa_i > -1$, $1 \leq i \leq d+1$, defined on the lattice points within the simplex $V_{\kappa,N}^d$ defined by $V_{\kappa,N}^d = \{ x \in \mathbb{N}_0^d : |x| \leq N \}$.

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In [11] a family of Hahn polynomials on polyhedra, denoted by $Q_{\nu, \ell}(x; \ell, N)$, are explicitly given with $\nu$ belonging to an index set $H_{\ell,N}^d$, which is a subset of $V_{\ell,N}^d$, but different from $V_{\ell,N}^d$. The set $H_{\ell,N}^d$ has a complicated structure and one of the main results of [11] is to show, with a strenuous combinatorial proof, that $H_{\ell,N}^d$ and $V_{\ell,N}^d$ have the same cardinality, so that $\{Q_{\nu, \ell}(x; \ell, N) : \nu \in H_{\ell,N}^d\}$ is an orthogonal basis. The definition of $H_{\ell,N}^d$ comes from setting $\kappa_i$ as negative integers $-\ell_i - 1$ in a basis of classical Hahn polynomials and collecting those polynomials whose norms are finite and non-zero. Since the basis used in [11] is specifically normalized, one may ask if those classical Hahn polynomials whose indices lie outside of $H_{\ell,N}^d$ when setting $\kappa_i$ to be $-\ell_i - 1$ are trivial or undefined.

The purpose of this paper is to answer these questions and to study the structures of the Hahn polynomials for hypergeometric distribution. More specially, we want to understand the formal process of setting $\kappa_i$ as negative integers $-\ell_i - 1$ rigorously and, in particular, to determine if and how the formal process leads to structural relations, such as generating functions, recurrence relations and difference equations, for the Hahn polynomials on the polyhedra. The question has interesting implications even for the Hahn polynomials of one variable, which we briefly describe to facilitate the discussion.

The classical Hahn polynomials $Q_m(x; a, b, N)$ are orthogonal with respect to a discrete inner product defined on the set $V_N = \{x \in \mathbb{N}_0 : 0 \leq x \leq N\}$ and $\{Q_m(x; a, b, N) : 0 \leq m \leq N\}$ is an orthogonal basis. Assume, for example, $\ell_1, \ell_2 \leq N$ and $\ell_1 + \ell_2 \geq N$. Then, for $d = 1$, a basis of the orthogonal polynomials for the hypergeometric distribution is given by $\{Q_m(x; -\ell_1 - 1, -\ell_2 - 1, N) : 0 \leq m \leq \ell_1\}$ and the orthogonality is defined on the set $V_{l,N} = \{x \in \mathbb{N} : N - \ell_2 \leq x \leq \ell_1\}$. We shall show that all polynomials $Q_m(x; -\ell_1 - 1, -\ell_2 - 1, N)$ with $m > \ell_1$ contain the factor $\prod_{j=N-\ell_2}^{\ell_1} (x-j)$. In particular, this shows that all polynomials with index outside $H_{\ell,N} = \{0, 1, \ldots, \ell_1\}$ vanishes on the set $V_{l,N}$.

The situation for $d > 1$, however, is much more complicated. When we set $\kappa_i = -\ell_i - 1$, those classical Hahn polynomials with indices outside $H_{\ell,N}^d$ still vanish on the polyhedral lattice set $V_{l,N}^d$; however, they do not always contain linear factors that vanish trivially on $V_{l,N}^d$. In fact, they lead to non-trivial, likely irreducible, polynomials that vanish on large subsets of lattice points. The complication for $d > 1$ requires a careful consideration when deriving structural properties for the Hahn polynomials on polyhedra from those of classical Hahn polynomials.

The paper is organized as follows. In the next section we consider the case $d = 1$ and study Hahn polynomials with negative integer parameters. The definition of the Hahn polynomials on the polyhedra is discussed in Section 3, which ends with examples of nontrivial Hahn polynomials that vanish on a large set of lattice points. The examples lead to the study of factorization of the Hahn polynomials of two variables in Section 4. The generating functions of the Hahn polynomials on the polyhedra are discussed in Section 5, whereas their bispectral properties are described in Section 6, which contains, in particular, explicit recurrence relations and difference equations satisfied by these polynomials.
2. HAHN POLYNOMIALS WITH NEGATIVE INTEGER PARAMETERS

We start with a short review of the classical Hahn polynomials that depend on two real parameters $a, b > -1$. The main goal of this section is to study Hahn polynomials when $a$ and $b$ become negative integers, for which some of the properties of the Hahn polynomials remain valid whereas others become more subtle.

2.1. Classical Hahn polynomials. For $a, b > -1$, the classical Hahn polynomials are $3F_2$ hypergeometric functions given by

$$Q_n(x; a, b, N) := 3F_2\left(-n, \frac{a + b + 1, -x}{a + 1, -N}; 1\right), \quad n = 0, 1, \ldots, N.$$  

They are discrete orthogonal polynomials with respect to the weight function

$$w_{a,b}(x; N) = \frac{N!}{(a + b + 2)_N} \frac{(a + 1)_x(b + 1)_N}{x!(N - x)!}, \quad x = 0, 1, \ldots, N,$$

over the set of integers $\{0, 1, \ldots, N\}$. More precisely, they satisfy

$$\sum_{x=0}^{N} Q_n(x; a, b, N) Q_m(x; a, b, N) w_{a,b}(x; N) = (-1)^n n!(b + 1)_n(a + b + N + 2)_n (n + a + b + 1)_n \delta_{n,m}, \quad n, m \leq N.$$  

Furthermore, these polynomials can also be defined via a generating function

$$\left(1 + t\right)^N \frac{P_n(a,b)(1 - t)}{P_n(a,b)(1)} = \sum_{x=0}^{N} \binom{N}{x} Q_n(x; a, b, N) t^x,$$

where $P_n(a,b)$ is the Jacobi polynomial defined by

$$P_n(a,b)(t) = \frac{(a + 1)_n}{n!} F_2\left(-n, \frac{a + b + 1}{a + 1}; 1 - t\right).$$

They also satisfy the relation

$$Q_n(x; a, b, N) = (-1)^n \frac{(b + 1)_n}{(a + 1)_n} Q_n(N - x; b,a,N).$$

2.2. Hahn polynomials with negative integer parameters. Let $N$ be a positive integer. Let $\ell_1$ and $\ell_2$ be two positive integers that satisfy

$$\ell_1 + \ell_2 \geq N.$$  

We consider the hypergeometric distribution or the weight function

$$H_{\ell_1,N}(x) = \frac{\binom{\ell_1}{\ell_2}(N-x)}{(\ell_1 + \ell_2)_N} = \frac{N!}{(-\ell_1 - \ell_2)_N} \frac{(-\ell_1)_x(-\ell_2)_N}{x!(N - x)!}, \quad x \in \mathbb{N}_0.$$  

Throughout this paper, we shall adopt the following notation,

$$a \wedge b = \min\{a, b\} \quad \text{and} \quad a \vee b = \max\{a, b\}, \quad a, b \in \mathbb{R}.$$

**Lemma 2.1.** Let $N, \ell_1$ and $\ell_2$ be positive integers satisfying (2.5). Then

$$H_{\ell_1,N}(x) = \frac{(\ell_1 \lor N)!(\ell_2 \lor N)!}{N!(-\ell_1 - \ell_2)_N} \frac{(-\ell_1 \land N)_x(-\ell_2 \land N)_N}{(x - N + \ell_2 \lor N)!((\ell_1 \lor N - x)!}. $$
In particular, the function $H_{\ell,N}$ is positive and supported on the set
\[ V_{\ell,N} := \{ x \in \mathbb{N}_0 : N - \ell_2 \wedge N \leq x \leq \ell_1 \wedge N \}. \]

**Proof.** If $\ell_1, \ell_2 \leq N$, then $H_{\ell_1,N}$ and $H_{\ell_2,N}$ coincide. In all other cases, we can rewrite $H_{\ell,N}$ given in (2.6) to the formula in (2.7) by using the identity $(-m)_n = (-1)^m n!/(m-n)!$ for $m, n \in \mathbb{N}_0$. The claim on the support of $H_{\ell,N}$ follows readily from $(-m)_n > 0$ if $n \leq m$ and $(-m)_n = 0$ if $n > m$. □

We consider orthogonal polynomials with respect to the inner product
\[ (f,g)_{\ell,N} = \sum_{x=\ell_2 \wedge N}^{\ell_1 \wedge N} f(x)g(x)H_{\ell,N}(x), \]
which satisfies $(1,1)_{\ell,N} = 1$. Applying the Gram-Schmidt process, we can identify a family of orthogonal polynomials \{\(Q_n : 0 \leq n \leq \deg_{\ell,N}\), where
\[ \deg_{\ell,N} := \ell_1 \wedge N + \ell_2 \wedge N - N. \]
Evidently, $H_{\ell,N}(x) = w_{-\ell_1-1,-\ell_2-1}(x;N)$. We make the following definition.

**Definition 2.2.** Let $\ell_1, \ell_2$ and $N$ be positive integers such that $\ell_1 + \ell_2 \geq N$. For $0 \leq n \leq \ell_1 \wedge N$, define Hahn polynomials with negative integer parameters by
\[ Q_n(x;\ell_1,\ell_2,N) = Q_n(x;\ell_1,\ell_2,N) = \binom{-n - \ell_1 - \ell_2}{\ell_1, N} (x; 1), \]
where $Q_n$ is the classical Hahn polynomial (2.1).

For $0 \leq n \leq \deg_{\ell,N}$, the orthogonality of the polynomials $Q_n(x;\ell_1,\ell_2,N)$ follows from that of classical Hahn polynomials. Indeed, the identity (2.2) involves only rational functions in $a, b$, so that we can apply analytic continuation to extend it to $a, b$ being negative integers, while the support set of $H_{\ell,N}$ shows that the inner product becomes (2.8).

**Theorem 2.3.** For $0 \leq m, n \leq \deg_{\ell,N}$,
\[ \langle Q_n(x;\ell_1,\ell_2,N), Q_m(x;\ell_1,\ell_2,N) \rangle_{\ell,N} = \delta_{m,n} B_n(\ell,N), \]
where
\[ B_n(\ell,N) = \frac{(-1)^n n!(-\ell_2)_n(-\ell_1 - \ell_2 + N)_n(-n + \ell_1 + \ell_2 + 1)}{(-\ell_1)_n(-\ell_2)_n(-N)_n(-2n + \ell_1 + \ell_2 + 1)}. \]

From (2.9), the Hahn polynomials with negative integer parameters are well defined if $0 \leq n \leq \ell_1 \wedge N$. If $\ell_2 \geq N$, then $\deg_{\ell,N} = \ell_1 \wedge N$. If $\ell_2 < N$, however, $\ell_1 \wedge N > \deg_{\ell,N}$, we have more polynomials than needed. It turns out that the extra polynomials are entirely zero when restricted on $V_{\ell,N}$.

**Theorem 2.4.** Assume $\ell_2 \leq N$. Let $n = \deg_{\ell,N} + m + 1$ for $m = 0, 1, \ldots, N - \ell_2 - 1$. Then
\[ Q_n(x;\ell_1,\ell_2,N) = \binom{-N + \ell_2 + 1}{\ell_1 \wedge N} \prod_{j=N-\ell_2}^{\ell_1 \wedge N} (x-j) \times Q_m(x,N - \ell_2 - 1,|N - \ell_1| - 1,\ell_1 \wedge N). \]
In particular, $Q_n(x;\ell_1,\ell_2,N)$ vanishes on $V_{\ell,N}$ if $\deg_{\ell,N} < n \leq \ell_1 \wedge N$. 

Proof. The assumption on $m$ implies $n \leq \ell_1 \land N$, which implies that the constant in front of $Q_m$ in the righthand side of \((2.10)\) is nonzero.

First assume $\ell_1 \leq N$. We need the following identity in \cite[Entry (7.4.4.83)]{15},
\[
\begin{aligned}
3F_2 \left( \begin{array}{c}
-n, a, b \\
c, d \\
1 
\end{array} ; 1 \right) &= \frac{(c + d - a - b)_n}{(c)_n} 3F_2 \left( \begin{array}{c}
-n, d - a, d - b \\
c + d - a - b \\
1 
\end{array} ; 1 \right),
\end{aligned}
\]
where $n$ is a positive integer and the identity holds when both sides are finite. Choose $a = n - \ell_1 - \ell_2 - 1$, $b = -x$, $c = -\ell_1$ and $d = -N$. Then for our choice of $n = \ell_1 + \ell_2 - N + m + 1$ we obtain
\[
Q_n(x; \ell_1, \ell_2, N) = \frac{(-\ell_1 - m + x)_n}{(-\ell_1)_n} 3F_2 \left( \begin{array}{c}
-n, m, -N + x \\
-\ell_1 - m + x, -N \\
1 
\end{array} ; 1 \right).
\]

We write $(-\ell_1 - m + x)\ell_1 + \ell_2 - N + m + 1 = (-\ell_1 - m + x)\ell_1 + \ell_2 - N + 1$. While the second term is $(-\ell_1 + x)\ell_1 + \ell_2 - N + 1 = \prod_{j=1}^{1} \ell_j (x - j)$, the first term combining with the $3F_2$ function gives, using the identity \cite[Entry (7.4.4.86)]{15}
\[
\begin{aligned}
\frac{(c)_m}{(c - a)_m} 3F_2 \left( \begin{array}{c}
-m, a, b \\
c, d \\
1 
\end{array} ; 1 \right) &= \frac{(-m, a, b)_m}{(-m, a - c - m + 1)_m} 3F_2 \left( \begin{array}{c}
-n, m, -N + x \\
-\ell_1 - m + x, -N \\
1 
\end{array} ; 1 \right),
\end{aligned}
\]
with $a = -N + x$, $b = -n$, $c = -\ell_1 - m + x$, and $d = -N$, that
\[
\frac{(-\ell_1 - m + x)_m}{(-N - \ell_1 - m)_m} 3F_2 \left( \begin{array}{c}
-n, m, -N + x \\
-\ell_1 - m + x, -N \\
1 
\end{array} ; 1 \right) = \frac{(-m, -N + n, -N + x)_m}{(-N + \ell_1 + 1, -N)_m} 3F_2 \left( \begin{array}{c}
-n, m, -N + x \\
-\ell_1 - m + x, -N \\
1 
\end{array} ; 1 \right).
\]
This last function can be identified with $Q_m(N - x; N - \ell_1 - 1, N - \ell_2 - 1, N)$, which we further write, using the identity \cite[2.4]{2.10}, as
\[
(-1)^m \frac{(-N + \ell_2 + 1)_m}{(-N + \ell_1 + 1)_m} Q_m(x; N - \ell_2 - 1, N - \ell_1 - 1, N).
\]
Putting these together proves the identity \((2.10)\) when $\ell_1 \leq N$.

Next we assume $\ell_1 > N$. Exchanging the role of $\ell_1$ and $N$ in the $3F_2$ of \((2.9)\), we see that the Hahn polynomials satisfy
\[
(2.11)
Q_n(x; \ell_1, \ell_2, N) = Q_n(x; N, \ell_1 + \ell_2 - N, \ell_1).
\]
With $\tilde{\ell}_1 = N$, $\tilde{\ell}_2 = \ell_1 + \ell_2 - N$ and $\tilde{N} = \ell_1$, the polynomial in the righthand side is $Q_n(x; \tilde{\ell}_1, \tilde{\ell}_2, \tilde{N})$ with $\tilde{\ell}_1 \leq \tilde{N}$ and $\tilde{\ell}_1 + \tilde{\ell}_2 \geq \tilde{N}$, which is factorable by what we proved in the previous paragraph. Hence, \((2.10)\) for $\ell_1 > N$ follows from that for $\ell_1 \leq N$. The proof is complete. \qed

By symmetry, one may expect a factorization of $Q_n$ when $\ell_2 > N \geq \ell_1$. Indeed, this holds for the polynomial $Q_n(N - x; \ell_2, \ell_1, N)$, which is well defined for $n \leq \ell_2$ and the factorization holds for $n > \ell_1 + \ell_2 - N$. However, by \cite[2.4]{2.10},
\[
Q_n(x; \ell_1, \ell_2, N) = (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} Q_n(x; N - x; \ell_2, \ell_1, N).
\]
The constant in the right-hand side makes sense only if $n \leq \ell_1 < \ell_2$. Thus, it may seem that applying first \cite[2.4]{2.10} and then Theorem \cite[2.4]{2.10} can lead to interesting new factorizations, but Theorem \cite[2.4]{2.10} cannot be applied.

The Hahn polynomial $Q_n(x, N - \ell_2 - 1, |N - \ell_1| - 1, \ell_1 \lor N)$ in the righthand side of \((2.10)\) still has negative integer parameters. Experiment with small $\ell_1$ and $N$ shows that many such polynomials are irreducible. One may ask if these polynomials are irreducible or reducible after further factoring out the linear terms. The answer to both questions, however, is negative.
Example 2.5. If \((\ell_1, \ell_2, N) = (6, 8, 12)\), then (2.10) becomes
\[
Q_6(x; 6, 8, 12) = -\frac{1}{51}(x - 4)(x - 5)(x - 6)Q_3(x; 3, 5, 12)
\]
and the Hahn polynomial in the righthand side contains a further linear factor,
\[
Q_3(x; 3, 5, 12) = -\frac{1}{132}(x - 4)(2x^2 - 13x + 33).
\]
Furthermore, if \((\ell_1, \ell_2, N) = (8, 9, 16)\), then (2.10) becomes
\[
Q_7(x; 8, 9, 16) = \frac{1}{56}(x - 7)(x - 8)Q_5(x; 6, 7, 16)
\]
and the Hahn polynomial in the righthand side is not irreducible,
\[
Q_5(x; 6, 7, 16) = -\frac{1}{24960}(52 - 14x + x^2)(-480 + 159x - 20x^2 + x^3).
\]

2.3. Generating function. The generating function (2.3) can also be adopted for the Hahn polynomials of negative integer parameters. When \(a\) and \(b\) are negative integers, the Jacobi polynomial \(P_n^{(a,b)}\) are known to have a degree reduction for some \(n\). Indeed, if \(n\) is a positive integer and \(n + m + a + b = 0\), \(m\) is an integer, \(1 \leq m \leq n\), then [16] (4.22.3)
\[
\binom{n}{m-1} P_n^{(a,b)}(t) = \binom{n+a}{n-m+1} P_{n-1}^{(a,b)}(t).
\]
However, we claim that this degree reduction will be irrelevant in our setting. Indeed, for \(\ell_1\) and \(\ell_2\) satisfy (2.5), we are interested in \(n\) that satisfies \(0 \leq n \leq \deg_{\ell_1, N}\). With \(a = -\ell_1 - 1\), \(b = -\ell_2 - 1\), we have \(m = -n - a - b = \ell_1 + \ell_2 + 2 - n \geq \ell_1 + \ell_2 + 2 - \deg_{\ell_1, N}\). If \(N \leq \ell_2\), then \(\deg_{\ell_1, N} = \ell_1 \land N\), so that \(m \geq \ell_1 + N + 2 - \ell_1 \land N \geq N + 2 > n\), whereas if \(\ell_2 \leq N\), then \(\deg_{\ell_1, N} = \ell_1 \land N - (\ell_1 \land \ell_2)\), so that \(m \geq \ell_1 + N + 2 - \ell_1 \land N \geq N + 2 > n\). This verifies the claim. To simplify notation, we make the following definition.

Definition 2.6. Let \(\ell_1, \ell_2\) be positive integers. For \(n = 0, 1, \ldots, \ell_1 \land \ell_2\), we define
\[
G_n^{(\ell_1, \ell_2)}(t) = \frac{P_n^{(-\ell_1,-\ell_2-1)}(t)}{P_n^{(-\ell_1-1,-\ell_2-1)}(1)} = 2F1 \left(-n, -n - \ell_1 - \ell_2 - 1, 1 - t; \frac{1}{2}\right).
\]
Since \(P_n^{(-\ell_1-1,-\ell_2-1)}(1) = \frac{(-\ell_1)_n}{n!}\) and \(P_n^{(a,b)}(-t) = (-1)^nP_n^{(b,a)}(t)\), we have
\[
G_n^{(\ell_1, \ell_2)}(t) = (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} G_n^{(\ell_2, \ell_1)}(-t).
\]

By analytic continuation, the identity (2.3) remains valid when \(a = -\ell_1 - 1\) and \(b = -\ell_2 - 1\), which gives
\[
(1 + t)^N G_n^{(\ell_1, \ell_2)} \left(1 - t \over 1 + t\right) = \sum_{x=0}^{N} \binom{N}{x} Q_n(x; \ell_1, \ell_2, N) t^x.
\]

This works for all \(\ell_1 + \ell_2 \geq N\). Its right-hand side, however, sums over all integers in \([0, N]\) instead of over integers in \([N - \ell_2 \land N, \ell_1 \land N]\), on which the orthogonality of \(Q_n(x; \ell_1, \ell_2, N)\) is defined. A more general generating function is given below.
Proposition 2.7. Let $\alpha = \ell_1 + \ell_2 \wedge N - \ell_1 \wedge N$ and $\beta = \ell_2 + \ell_1 \wedge N - \ell_2 \wedge N$. Then for $0 \leq n \leq \deg_{\ell,N}$,

\begin{equation}
\label{eq:2.14}
 b_{\ell,N} t^{N-\ell_2 \wedge N} (1 + t)^{\ell_1 \wedge N + \ell_2 \wedge N - N} G_n^{(\alpha,\beta)} \left( \frac{1 - t}{1 + t} \right)
= \sum_{x=0}^{\ell_2 \wedge N} \binom{\ell_1 + \ell_2 \wedge N - N}{\ell_1 \wedge N - x} Q_n(x; \ell_1, \ell_2, N) t^x,
\end{equation}

where $b_{\ell,N}$ is the constant given by

$$b_{\ell,N} = \frac{(-\ell_2)_n (-\ell_1 - \ell_2 \wedge N + N)_n}{(-\ell_1)_n (-\ell_2 + \ell_2 \wedge N - N)_n}.$$ 

In particular, $b_{\ell,N} = 1$ if $\ell_2 \geq N$.

Proof. The identity \eqref{eq:2.13} gives \eqref{eq:2.14} when $\ell_1 \geq N$ and $\ell_2 \geq N$. In all other cases, we need to modify the right-hand side so that the summation is over the integers in the interval $[N - \ell_2 \wedge N, \ell_1 \wedge N]$. Assume $\ell_2 \geq N$. The identity \eqref{eq:2.13} for $Q_n$ in the right-hand side of \eqref{eq:2.11} gives \eqref{eq:2.14} for $\ell_1 < N$.

Next we assume $\ell_2 \leq N$. By \eqref{eq:2.4},

\begin{equation}
\label{eq:2.15}
 Q_n(x; \ell_1, \ell_2, N) = (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} Q_n(N - x, \ell_2, \ell_1, N).
\end{equation}

Applying \eqref{eq:2.11} on the Hahn polynomial in the right hand side, we deduce

$$Q_n(x; \ell_1, \ell_2, N) = (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} Q_n(N - x, N, \ell_1 + \ell_2 - N, \ell_2).$$

Using this identity, the right-hand side of \eqref{eq:2.14} for $\ell_1 \geq N$ becomes

\begin{align*}
\sum_{x=0}^{\ell_2 \wedge N} \binom{\ell_2}{N} Q_n(x; \ell_1, \ell_2, N) t^x &= \sum_{x=0}^{\ell_2 \wedge N} \binom{\ell_2}{N} Q_n(N - x; \ell_1, \ell_2, N) t^{N-x} \\
&= (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} \sum_{x=0}^{\ell_2 \wedge N} \binom{\ell_2}{x} Q_n(N - x; N, \ell_1 + \ell_2 - N, \ell_2) t^{N-x} \\
&= (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} t^{N-\ell_2} (1 + \frac{1}{t})^{\ell_2} G_n^{(N,\ell_1 + \ell_2 - N)} \left( \frac{1 - \frac{1}{t}}{1 + \frac{1}{t}} \right) \\
&= (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} t^{N-\ell_2} (1 + t)^{\ell_2} G_n^{(N,\ell_1 + \ell_2 - N)} \left( \frac{1 - t}{1 + t} \right),
\end{align*}

where the third equality follows from \eqref{eq:2.13}. Finally, applying \eqref{eq:2.12}, we have established \eqref{eq:2.14} for $\ell_1 \geq N$ and $\ell_2 \leq N$. In the last case, when $\ell_1 \leq N$ and $\ell_2 \leq N$, we rewrite the Hahn polynomial, starting from \eqref{eq:2.11}, using \eqref{eq:2.15} and then \eqref{eq:2.11} one more time, to obtain

$$Q_n(x; \ell_1, \ell_2, N) = (-1)^n \frac{(-\ell_1 - \ell_2)_n}{(-N)_n} Q_n(N - x; \ell_1 + \ell_2 - N, N, \ell_1)$$

$$= (-1)^n \frac{(-\ell_1 - \ell_2)_n}{(-N)_n} Q_n(N - x; \ell_1, \ell_2 + \ell_2 - N).$$
With this identity, the righthand side of (2.14) for $\ell_1 \leq N$ becomes

$$\sum_{x=N-\ell_2}^{\ell_1} \binom{\ell_1 + \ell_2 - N}{\ell_1 - x} Q_n(x; \ell_1, \ell_2, N) t^x$$

$$= \sum_{x=0}^{\ell_1+\ell_2-N} \binom{\ell_1 + \ell_2 - N}{x} Q_n(\ell_1 - x; \ell_1, \ell_2, N) t^{\ell_1-x}$$

$$= (-1)^n Q_n(\ell_1 - \ell_2, N) \sum_{x=0}^{\ell_1+\ell_2-N} \binom{\ell_1 + \ell_2 - N}{x} Q_n(x; \ell_1, \ell_2, \ell_1 + \ell_2 - N) t^{\ell_1-x},$$

from which the proof can be completed as in the case $\ell_1 \geq N$ and $\ell_2 \leq N$. This completes the proof. □

The usual orthogonality of the Jacobi polynomials, however, no longer holds when the parameters are negative integers, since $(1-x)^a(1+x)^b$ is not integrable on $[-1,1]$ if $a$ and/or $b$ are negative integers. It is possible, however, to define a linear functional $L$, so that the polynomials $G_{n}^{(\ell_1, \ell_2)}$ are orthogonal in the sense that $L(G_{n}^{(\ell_1, \ell_2)} G_{n'}^{(\ell_1, \ell_2)}) = 0$ for $m \neq n$, although this linear function is no longer positive definite.

Let $\ell_1$ and $\ell_2$ be positive integers. We define the linear functional $L$ on the space $\Pi_{\ell_1+\ell_2}$ of polynomials of degree at most $\ell_1 + \ell_2$ so that its moments are given by

$$L(x^k) = \binom{-k}{-\ell_1-\ell_2} 2F_1(-k,-\ell_1;2,0) \leq \ell_1 + \ell_2.$$

**Proposition 2.8.** Let $\ell_1$ and $\ell_2$ be two positive integer. Then the polynomials $G_{n}^{(\ell_1, \ell_2)}$ satisfy the orthogonal relation

$$L \left( G_{m}^{(\ell_1, \ell_2)} G_{n}^{(\ell_1, \ell_2)} \right) = h_{n}^{(\ell_1, \ell_2)} \delta_{m,n}, \quad 0 \leq m, n \leq \ell_1 \land \ell_2,$$

where the constants $h_{n}^{(\ell_1, \ell_2)}$ are given by

$$h_{n}^{(\ell_1, \ell_2)} = \frac{n!(-\ell_2) n(1+\ell_1+\ell_2-n)}{(-\ell_1) n(-\ell_1-\ell_2) n(1+\ell_1+\ell_2-2n)}.$$

**Proof.** First we claim that the following relation holds

$$L \left( \left( \frac{1-x}{2} \right)^m \right) = \frac{(-\ell_1) m}{(-\ell_1-\ell_2)m}, \quad m = 0, 1, \ldots, \ell_1 + \ell_2.$$

Indeed, by the binomial formula and the definition of the $2F_1$, we have

$$L \left( \left( \frac{1-x}{2} \right)^m \right) = \frac{1}{2^m} \sum_{k=0}^{m} \binom{m}{k} (-1)^k L(x^k)$$

$$= \frac{1}{2^m} \sum_{k=0}^{m} (-1)^k \frac{k!}{k!} \sum_{j=0}^{k} (-k)_{j} (-\ell_1)_{j} 2^j$$

$$= \frac{1}{2^m} \sum_{j=0}^{m} \frac{(-\ell_1)_{j} 2^j}{j!} \sum_{k=j}^{m} (-1)^k \frac{k!}{k!}.$$
Changing summation index and rewriting, it is easy to see that
\[
\sum_{k=j}^{m} \frac{(-m)_k (-k)_j}{k!} = (-1)^j (-m)_j \sum_{k=0}^{m-j} \frac{(-m+j)_k}{k!} = m! \delta_{j,m},
\]
from which the claimed formula (2.16) follows immediately.

Now let \( n \leq \ell_1 \land \ell_2 \). For \( 0 \leq m \leq n \), we use (2.16) and the identity \( (a)_{m+k} = (a)_m (a + m)_k \) to obtain
\[
\mathcal{L}(G^{(\ell_1, \ell_2)}_n(x) \left(\frac{1-x}{2}\right)^m) = \sum_{k=0}^{n} \frac{(-\ell_1)_m (-\ell_2)_m}{(-\ell_1 - \ell_2)_m} \mathcal{L}(\left(\frac{1-x}{2}\right)^{m+k})
\]
\[
= \frac{(-\ell_1)_m (-\ell_2)_m}{(-\ell_1 - \ell_2)_m} \sum_{k=0}^{n} \frac{(-n)_k (-n - n - 1)_k (-\ell_1 - m)_k}{k! (-\ell_1 - \ell_2 - m)_k}
\]
\[
= \frac{(-\ell_1)_m (-\ell_2)_m}{(-\ell_1 - \ell_2)_m} \, {}_3F_2 \left( -n, n - \ell_1 - \ell_2 - 1, -\ell_1 + m ; 1 \right).
\]
This hypergeometric function is a balanced terminating \( {}_3F_2 \) and, by Saalschütz summation formula, we conclude that
\[
\mathcal{L}(G^{(\ell_1, \ell_2)}_n(x) \left(\frac{1-x}{2}\right)^m) = \frac{(-\ell_1)_m (-\ell_2)_m}{(-\ell_1 - \ell_2)_m} \frac{(1 + \ell_2 - n)_n (-m)_n}{(-\ell_1)_n (-\ell_1 + \ell_2 - n - m)_n}
\]
\[
= (-1)^n n! (-\ell_2)_n g_{m,n},
\]
where we have used \((-m)_n = 0\) for \( m < n \) and \((-n)_n = (-1)^n n!\). This proves that \( G_n^{(\ell_1, \ell_2)} \) is orthogonal to \( (1 - x)^m \) for \( m < n \) and hence, by linearity, to \( G_{m}^{(\ell_1, \ell_2)} \). Furthermore, multiplying the above identity by the coefficient \((-1)^n (n - \ell_1 - \ell_2 - 1)_n / (-\ell_1)_n \) of \( \left(\frac{1-x}{2}\right)^n \) in \( G_n^{(\ell_1, \ell_2)} \) verifies the formula for \( h_{n}^{(\ell_1, \ell_2)} \). □

Since \( h_{n}^{(\ell_1, \ell_2)} \) has the sign \((-1)^n\), the moment functional \( \mathcal{L} \) is not positive definite. This can also be seen in the coefficients of the three-term relations satisfied by \( tG^{(\ell_1, \ell_2)}_m(t) \), where the coefficients of \( G_{n+1}^{(\ell_1, \ell_2)} \) and \( G_{n-1}^{(\ell_1, \ell_2)} \) have opposite signs.

The generating function (2.14) of the Hahn polynomials of negative parameters requires \( 0 \leq n \leq \deg_{\ell,N} = \ell_1 \land N + \ell_2 \land N - N \). Since \( \deg_{\ell,N} \leq \ell_1 \land \ell_2 \), we see that all \( G_{n}^{(\ell_1, \ell_2)} \) in the generating function (2.14) are orthogonal polynomials.

Finally, let us mention that the three-term relation satisfied by \( Q_n(\cdot, \ell_1, \ell_2, N) \) can be deduced from that of the classical Hahn polynomials, which shows
\[
(2.17) \quad x \phi_n = -A_n \phi_{n+1} + (A_n + C_n) \phi_n - C_n \phi_{n-1}, \quad 0 \leq n \leq \deg_{\ell,N} - 1,
\]
where \( \phi_n = Q_n(\cdot, \ell_1, \ell_2, N) \) and the coefficients \( A_n \) and \( C_n \) are given by
\[
A_n = \frac{(n - \ell_1 - \ell_2 - 1)(n - \ell_1)(N-n)}{(2n - \ell_1 - \ell_2 - 1)(2n - \ell_1 - \ell_2)},
\]
\[
C_n = \frac{n(n + N - \ell_1 - \ell_2 - 1)(n - \ell_2 - 1)}{(2n - \ell_1 - \ell_2 - 1)(2n - \ell_1 - \ell_2 - 2)}.
\]
In our statement of (2.17), we assume \( n \leq \deg_{\ell,N} - 1 \). For \( \ell_2 \leq N \), the identity also holds for \( n = \deg_{\ell,N} \) by Theorem 2.4 whenever \( A_n \) and \( C_n \) are finite. Since \( \deg_{\ell,N} \leq \frac{\ell_1 + \ell_2}{2} \), the coefficients are well defined unless \( n = \deg_{\ell,N} = \frac{\ell_1 + \ell_2}{2} \). This
last equation is attained if \( \ell_1 = \ell_2 = N \) and \( n = \deg_{\ell,N} = N \), which leads to a pole in \( A_n \) so that (2.17) fails for \( n = \deg_{\ell,N} \) in this particular case.

3. HAHN POLYNOMIALS FOR HYPERGEOMETRIC DISTRIBUTION

Classical Hahn polynomials in several variables are those on lattice points inside a simplex. A brief review of these polynomials will be given in the first subsection. When their parameters become negative integers, these polynomials become orthogonal polynomials for hypergeometric distribution, which will be discussed in the second subsection.

3.1. Classical Hahn polynomials of several variables. Let \( N \) be a positive integer. Recall that \( V^d_N \) is the set of lattice points in a discrete simplex

\[ V^d_N := \{ \nu \in \mathbb{N}_0^d : |\nu| \leq N \}, \]

and, for \( \kappa \in \mathbb{R}^{d+1} \) with \( \kappa_i > -1, 1 \leq i \leq d+1 \), the function \( W_{\kappa,N} \) defined in (1.3) is the normalized Hahn weigh function. The Hahn polynomials are orthogonal with respect to the discrete inner product

\[ \langle f, g \rangle_{W_{\kappa,N}} = \sum_{x \in V^d_N} f(x)g(x)W_{\kappa,N}(x). \]

For \( 0 \leq n \leq N \), let \( V^d_n(W_{\kappa,N}) \) denote the space of orthogonal polynomials of degree \( n \) with respect to this inner product. Then

\[ \dim V^d_n(W_{\kappa,N}) = \binom{n+d-1}{n}, \quad n = 0, 1, 2, \ldots \]

An orthogonal basis of \( V^d_n(W_{\kappa,N}) \) can be given in terms of classical Hahn polynomials of one variable, for which we need the following notation:

For \( y = (y_1, \ldots, y_d) \in \mathbb{R}^d \) and \( 1 \leq j \leq d \), we define

\[ y_j := (y_1, \ldots, \hat{y}_j, \ldots, y_d) \quad \text{and} \quad y^j := (y_1, \ldots, y_d), \]

and also define \( y_0 := 0 \) and \( y^{d+1} := 0 \). It follows that \( y_d = y^1 = y \), and

\[ |y_j| = y_1 + \cdots + y_{j-1}, \quad |y^j| = y_1 + \cdots + y_d, \quad \text{and} \quad |y_0| = |y^{d+1}| = 0. \]

For \( \kappa = (\kappa_1, \ldots, \kappa_{d+1}) \), we have \( \kappa^j := (\kappa_j, \ldots, \kappa_{d+1}) \) for \( 1 \leq j \leq d+1 \). Furthermore, for \( \nu \in \mathbb{N}_0^d \) and \( \kappa \in \mathbb{R}^{d+1} \), we define

\[ a_j := a_j(\kappa, \nu) := |\kappa^j| + 2|\nu^{j+1}| + d - j, \quad 1 \leq j \leq d. \]

Notice that \( a_d = \kappa_{d+1} \) since \( |\nu^{d+1}| = 0 \) by definition.

**Proposition 3.1.** For \( x \in \mathbb{Z}^{d+1}_N \) and \( \nu \in \mathbb{N}_0^d, |\nu| \leq N \), define

\[ Q_{\nu}(x;\kappa,N) = \prod_{j=1}^d (-N + |x_{j-1}| + |\nu^{j+1}|)_{\nu_j} \times Q_{\nu_j}(x_j;\kappa_j, a_j, N - |x_{j-1}| - |\nu^{j+1}|). \]

The polynomials in \( \{Q_{\nu}(x;\kappa,N) : |\nu| = n\} \) form a mutually orthogonal basis of \( V^d_n(W_{\kappa,N}) \) and \( B_{\nu} := (Q_{\nu}(\cdot;\kappa,N), Q_{\nu}(\cdot;\kappa,N))_{W_{\kappa,N}} \) is given by, setting \( \lambda_{\kappa} := \)
form an orthogonal basis of $V$ where 

$$a \mathcal{H}(3.7)$$

denoted by 

which it is more convenient to use a different normalization of the Hahn polynomials, 

$$\ell (3.9)$$

where we use homogeneous coordinates 

$$\alpha$$

where 

3.2. Hahn polynomials with negative integer parameters. Let $N \in \mathbb{N}_0$ and 

$$\ell_i \in \mathbb{N} \text{ for } 1 \leq i \leq d + 1.$$ 

We assume that they satisfy 

$$\ell_i \leq N \text{ and } \ell_i + \ell_j \geq N, \quad i \neq j, \quad 1 \leq i, j \leq d + 1.$$ 

Recall that $V_{\ell,N}^d$, defined in (4.2), denotes the discrete polyhedral domain, which we restate below,

$$V_{\ell,N}^d := \{ x \in \mathbb{N}_0^d : 0 \leq x_i \leq \ell_i, 1 \leq i \leq d, \text{ and } N - \ell_{d+1} \leq |x| \leq N \},$$

where $|x| = x_1 + \cdots + x_d$. Evidently, $V_{\ell,N}^d$ is the simplex $V_N^d$ if all $\ell_i = N$. The polyhedral is the simplex $V_N^d$ with its corners sliced off. In particular,

$$|V_{\ell,N}^d| = \binom{N + d}{d} - \sum_{k=1}^{d+1} \binom{N - \ell_k + d - 1}{d},$$
where we denote by \(|E|\) the cardinality of the discrete set \(E\). For \(\ell\) and \(N\) satisfying (3.9), the hypergeometric distribution in \(d\) variables is defined by (1.11), which is restated below,
\[
H_{\ell,N}(x) = \frac{1}{\binom{d}{N}} \prod_{i=1}^{d} \binom{\ell_i}{x_i} \binom{\ell_{d+1}}{N - |x|} = \frac{N!}{(-\ell)_N} \prod_{i=1}^{d} \frac{(-\ell_i)_N (-\ell_{d+1})_N - |x|}{x_i! (N - |x|)!}.
\]
This defines a probability measure on \(V_{\ell,N}^d\). The first identity is used more commonly in the probability theory; see, for example, [12]. The Hahn polynomials for this distribution are orthogonal polynomials with respect to the inner product
\[
(f, g)_{\ell,N} = \sum_{x \in V_{\ell,N}^d} f(x)g(x)H_{\ell,N}(x).
\]

Let \(\Pi_{\ell,N}^d\) denote the space of polynomials of total degree at most \(N\) in \(d\) variables. Let \(I(V_{\ell,N}^d)\) denote the ideal of polynomials that vanish on \(V_{\ell,N}^d\). It is known that the space of orthogonal polynomials, denoted by \(\Pi_{\ell,N}^d\), with respect to \((\cdot, \cdot)_{\ell,N}\) satisfies
\[
\Pi_{\ell,N}^d = \mathbb{R}[x_1, \ldots, x_d]/I(V_{\ell,N}^d).
\]

Since \(H_{\ell,N}(x) = W_{-\ell-1,N}(x)\) where \(1 = (1, 1, \ldots, 1)\) and \(H_{\ell,N}(x) = 0\) if \(x \in V_N^d \setminus V_{\ell,N}^d\), orthogonal polynomials with respect to the inner product \((\cdot, \cdot)_{\ell,N}\) can be deduced from \(Q_\nu(x; \kappa, N)\) in (3.4) by setting \(\kappa = -\ell - 1\) for \(\nu\) in an appropriate subset of \(\{\nu : |\nu| \leq N\}\). This narrative was carried out in [11]. Let us now rewrite the polynomials \(Q_\nu(x; -\ell - 1, N)\) in terms of the Hahn polynomials with negative integer parameters.

With the notation \(\ell^j := (\ell_j, \ldots, \ell_{d+1})\) for \(1 \leq j \leq d + 1\), we define
\[
a_j := -a_j(-\ell - 1, \nu) - 1 = |\ell^j| - 2|\nu^j|, \quad 1 \leq j \leq d.
\]

**Definition 3.4.** Let \(\ell \in \mathbb{N}_{d+1}^d\) and \(N \in \mathbb{N}\). Assume \(\ell_i \leq N\) and (3.9). We define Hahn polynomials with negative integer parameters on the polyhedron \(V_{\ell,N}^d\) by
\[
Q_\nu(x; \ell, N) = Q_\nu(x; -\ell - 1, N)
\]
\[
= \prod_{j=1}^{d} (-N + |x_{j-1}| + |\nu^{j+1}|)_{\nu_j} (x_j; \ell_j, a_j, N - |x_{j-1}| - |\nu^{j+1}|).
\]

In [11], we considered these polynomials for \(\nu\) in the index set \(H_{\ell,N}^d\) defined by
\[
H_{\ell,N}^d := \{\nu \in \mathbb{N}_0^d : |\nu| \leq N, |\ell| - N, \nu_j \leq \ell_j, \nu_j \leq a_j, 1 \leq j \leq d\}.
\]
The main result of [11] states that they form an orthogonal basis of the space \(\Pi_{\ell,N}^d\).

**Theorem 3.5.** Let \(N \in \mathbb{N}\) and let \(\ell_i \in \mathbb{N}\) satisfy (3.9). Then
(i) The polynomials \(Q_\nu(\cdot; \ell, N)\) are orthogonal and satisfy
\[
\langle Q_\nu(\cdot; \ell, N), Q_\mu(\cdot; \ell, N) \rangle_{\ell,N} = B_\nu(\ell, N)\delta_{\nu,\mu}
\]
for all \(\nu, \mu \in H_{\ell,N}^d\), where
\[
B_\nu(\ell, N) := (-1)^{|\nu|}(-N)|\nu|(-|\ell|)_{N+|\nu|} \prod_{j=1}^{d} \frac{(-a_j)_{\nu_j} (-\ell_j - a_j - 1)_{2\nu_j}!}{(-|\ell|)_N(-|\ell|)_{2|\nu|}}.
\]
(ii) The set \( \{ Q_\nu(\cdot;\ell, N) : \nu \in H^d_{\ell,N} \} \) is a basis of \( H^d_{\ell,N} \). In particular,

\[ |H^d_{\ell,N}| = |V^d_{\ell,N}| = \dim \Pi^d_{\ell,N}. \]

The part (i) follows from the classical Hahn polynomials on \( V^d_N \) as we indicated above. The proof of part (ii) is highly non-trivial and requires a rather involved combinatorial proof. The norm \( B_\nu(\ell, N) \) is nonzero, in fact positive, for \( \nu \in H^d_{\ell,N} \), which is how the set \( H^d_{\ell,N} \) is conceived and defined.

The definition of the polynomials \( Q_\nu(\cdot;\ell, N) \) shows that these polynomials are well defined for \( \nu \) in a set \( H^d_{\ell,N} \) that contains \( H^d_{\ell,N} \) as a subset. The next theorem is a complementary result of Theorem 3.5.

**Theorem 3.6.** Let \( N \in \mathbb{N} \) and \( \ell_i \in \mathbb{N} \) satisfying (3.9). The polynomials \( Q_\nu(\cdot;\ell, N) \) are well defined on \( V^d_{\ell,N} \) if \( \nu \in H^d_{\ell,N} \), where

\[ H^d_{\ell,N} = \{ \nu \in \mathbb{N}^d_0 : \nu_i \leq \ell_i, 1 \leq i \leq d, |\nu| \leq N \}. \]

Furthermore, if \( \nu \in H^d_{\ell,N} \setminus H^d_{\ell,N} \), then the polynomials \( Q_\nu(\cdot;\ell, N) \) vanishes on \( V_{\ell,d} \).

**Proof.** Let \( \nu \in H^d_{\ell,N} \). For \( 0 \leq k \leq \nu_j \), expanding \( Q_{\nu_j} \) in \( Q_\nu \) using its \( 3F2 \) definition, and using the identity

\[ \frac{(-N + |x_j-1| + |\nu^{j+1}|)_j}{(-N + |x_j-1| + |\nu^{j+1}|)_k} = (-N + |x_j-1| + |\nu^{j+1}| + k)_{\nu_j-k}, \]

it is easy to see that \( Q_\nu \) is well defined on \( H^d_{\ell,N} \). Since the orthogonal relation in Theorem 3.5 is derived by setting \( \kappa = -\ell - 1 \), it holds for \( \nu \in H^d_{\ell,N} \). In particular, the expression of \( B_\nu(\ell, N) \) in (3.12) shows that the norm of \( Q_\nu(\cdot;\ell, N) \) is zero if \( |\nu| > |\ell| - N \) or if \( \nu_j > a_j \) for some \( j \), so that the polynomial is entirely zero on \( V^d_{\ell,N} \) if \( \nu \in H^d_{\ell,N} \) but \( \nu \notin H^d_{\ell,N} \). \( \square \)

This theorem can be regarded as an extension of Theorem 2.4 in one variable. In that theorem, the polynomial \( Q_n(\cdot, \ell, \ell_2, N) \) in one variable is factored into two lower degree polynomials, one of which vanishes on \( V_{\ell,N} \) so that the vanishing of \( Q_n(\cdot, \ell, \ell_2, N) \) becomes obvious from the factorization. An analogous factorization, however, no longer holds for \( Q_\nu(\cdot;\ell, N) \) in higher dimension. In fact, what makes this theorem interesting lies in the existence of non-trivial polynomials that vanish on the set \( V^d_{\ell,N} \), as illustrated by the following example.

**Example 3.7.** Let \( d = 2, \ell_1 = 6, \ell_2 = 4, \ell_3 = 4, N = 7 \). Then \( V^2_{\ell,N} \) and \( H^2_{\ell,N} \) each contains 23 points; see Figure 1. The indices \( \nu = (0, 5) \) and \( \nu = (3, 3) \) are outside of \( H^2_{\ell,N} \) and their corresponding polynomials

\[ Q_{0,5}(x; \ell, N) = (x_1 - 3)(840 - 638x_1 - 910x_2 + 179x^2_1 + 480x_1x_2 + 375x^2_2) \]
\[ -22x^3_1 - 85x^2_1x_2 - 120x_1x^2_2 - 70x^3_2 \]
\[ + x^4_1 + 5x^3_1x_2 + 10x^2_1x^2_2 + 10x_1x^3_2 + 5x^4_2) \];

\[ Q_{3,3}(x; \ell, N) = -\frac{1}{48}(x_1 - 4)(x_1 - 3)(x_1 - 2)(x_1 + 2x_2 - 7) \]
\[ \times (60 - 22x_1 - 35x_2 + 2x^3_1 + 5x_1x_2 + 5x^2_2), \]

of degree 5 and 6, respectively, vanish on \( V^2_{\ell,N} \); see Figure 2.
Figure 1. \((\ell_1, \ell_2, \ell_3, N) = (6, 4, 4, 7)\). Left: \(V^2_{\ell,N}\). Right: \(H^2_{\ell,N}\)

Figure 2. \((\ell_1, \ell_2, \ell_3, N) = (6, 4, 4, 7)\). Left: \(Q_{0,5}(x; \ell, N)\). Right: \(Q_{3,3}(x; \ell, N)\)

In the case of \(\nu = (0, 5)\), the polynomial \(Q_{0,5}(x; \ell, N)\) contains an irreducible polynomial of degree 4, thus its vanishing on \(V^2_{\ell,N}\) is no longer obvious from the factorization, in contrast to Theorem 2.4 in the one variable case. Observe that the polynomial of degree 4 in \(Q_{0,5}(x; \ell, N)\) and the irreducible polynomial of degree 2 in \(Q_{3,3}(x; \ell, N)\) have common zeros at 8 lattice points, which is the maximum possible predicted by the Bezout theorem.

It is possible to be more specific about the factors in \(Q_{\nu}(x; \ell, N)\) when \(\nu\) is outside of \(H^d_{\ell,N}\). Let \(N_j := N - |x_{j-1} - |\nu^{j+1}|\) so that \(Q_{\nu_j}\) can be written as \(Q_{\nu_j}(x_j; \ell_j, a_j, N_j)\). From \(\nu_j \leq \ell_j, 1 \leq j \leq d\), we obtain \(|\nu^{j+1}| \leq |\ell^{j+1}| - \ell_{d+1}\), which implies, by (3.9), that

\[
\ell_j + a_j = \ell_j + |\ell^{j+1}| - 2|\nu^{j+1}| \geq \ell_j + \ell_{d+1} - |\nu^{j+1}| \geq N - |\nu^{j+1}|.
\]
Hence, for \( x \in V^d_{\ell,N} \), \( \ell_j + a_j \geq N - |x| \) for \( j = 1, \ldots, d \), which shows that the parameters of \( Q_{\nu_j} \) in \( Q_{\nu} \) satisfy (2.5). Thus, if \( N_j \) is a positive integer, then \( Q_{\nu_j} \) is a Hahn polynomial with negative integer parameters, so that Theorem 2.4 may apply if \( a_j < N_j \).

4. Factorization of Hahn Polynomials of Two Variables with Negative Integer Parameters

We examine the factorization of the Hahn polynomials of two variables more closely in this section. For \( d = 2 \), the polynomial \( Q_{\nu}(\cdot; \ell, N) \) is given by

\[
Q_{\nu}(x; \ell, N) = (-N)_{\nu_1} Q_{\nu_1}(x_1; \ell_1, \ell_2 + \ell_3 - 2\nu_2, N - \nu_2) \times (-N + x_1)_{\nu_2} Q_{\nu_2}(x_2; \ell_2, \ell_3, N - x_1)
\]

and the index set \( \nu \in H^2_{\ell,N} \) is given by

\[
H^2_{\ell,N} = \{(\nu_1, \nu_2) : \nu_1 + \nu_2 \leq N, \nu_1 + \nu_2 \leq |\ell| - N, \\
\nu_1 \leq \ell_1, \nu_2 \leq \ell_2, \nu_3 \leq \ell_3, \nu_1 + 2\nu_2 \leq \ell_2 + \ell_3 \}.
\]

It is easy to see that both \( Q_{\nu_i}, i = 1, 2 \), are Hahn polynomial with negative integer parameters. To determine their factorization for \( \nu \) is outside of \( H^2_{\ell,N} \), we need a precise definition of \( H^2_{\ell,N} \) using the height function \( h_{\ell,N} \) introduced in \[\text{(4.1)}\].

The height function measures the number of integer points on the vertical line \( x_1 = \nu_1 \) in \( H^2_{\ell,N} \). For \( d = 2 \), it is defined by

\[
h_{\ell,N}(\nu_1) := \min \left( \ell_2, \ell_3, \frac{\ell_2 + \ell_3 - \nu_1}{2}, \ell_1 + \ell_2 + \ell_3 - N - \nu_1, N - \nu_1 \right) + 1.
\]

It follows immediately that the index set \( H^2_{\ell,N} \) can be written as

\[
H^2_{\ell,N} := \{(\nu_1, \nu_2) : 0 \leq \nu_1 \leq \ell_1, 0 \leq \nu_2 \leq h_{\ell,N}(\nu_1) - 1 \}.
\]

The value of the height function can be described more explicitly.

**Lemma 4.1.** The height function \( h_{\ell,N} \) satisfies

(i) if \( 0 \leq \nu_1 \leq |\ell_3 - \ell_2| \), then \( h_{\ell,N}(\nu_1) = \ell_2 \land \ell_3 + 1 \);  
(ii) if \(|\ell_3 - \ell_2| \leq \nu_1 \leq \ell_1 - |2N - |\ell|| \), then

\[
h_{\ell,N}(\nu_1) = \ell_2 \land \ell_3 - \left[ \frac{\nu_1 - |\ell_3 - \ell_2| - 1}{2} \right] + 1;
\]

(iii) if \( \ell_1 - |2N - |\ell|| \leq \nu_1 \leq \ell_1 \), then

\[
h_{\ell,N}(\nu_1) = \left\{ \begin{align*}
|\ell| - N - \nu_1 + 1, & \text{ if } |\ell| \leq 2N, \\
N - \nu_1 + 1, & \text{ if } |\ell| > 2N.
\end{align*} \right.
\]

**Proof.** We can assume \( \ell_3 \geq \ell_2 \). The proof is a simple but tedious verification. The first two items have already been used in the proof of Lemma 4.3 in \[\text{[13]}\]. We omit the details. \( \square \)

To consider polynomials \( Q_{\nu}(\cdot; \ell, N) \) with \( \nu = (\nu_1, \nu_2) \) outside the domain \( H^2_{\ell,N} \), we then assume \( \nu_2 \geq h_{\ell,N}(\nu_1) \). Below we study \( Q_{\nu}(\cdot; \ell, N) \) with \( \nu_2 = h_{\ell,N}(\nu_1) \), that is, with index just outside \( H^2_{\ell,N} \). For convenience, we shall write

\[
Q_{\nu}(x; \ell, N) = (-N)_{\nu_1} Q_{\nu_1}(x_1; \ell_1, \ell_2, N) R_{\nu_2}(x; \ell_2, \ell_3, N),
\]
Moreover, \( \hat{\ell} = (\ell_1, \ell_2 + \ell_3 - 2\nu_2), \hat{N} = N - \nu_2, \) and
\[
(4.3) \quad R_{\nu_2}(x; \ell_2, \ell_3, N) := (-N + x_1)_{\nu_2} Q_{\nu_2}(x_2, \ell_2, \ell_3, N - x_1).
\]

We say that a polynomial of a variable \( x \) splits if it splits in \( \mathbb{Q} \), i.e. if it can be written as \( \prod_i (x - a_i), a_i \in \mathbb{Q} \). We examine \( Q_{\nu_1}(x_1; \hat{\ell}, \hat{N}) \) first. Applying Theorem 2.4 this polynomial has linear factors if \( \ell_2 \leq \hat{N} \) and \( \deg_{\mathbb{F}_N} < \nu_1 \leq \min\{\ell_1, \hat{N}\} \), or more specifically, if
\[
(4.4) \quad \ell_2 + \ell_3 - \nu_2 \leq N, \quad \nu_1 < N - \ell_2 - \ell_3 + \nu_2.
\]

**Proposition 4.2.** Assume \( \ell_3 \geq \ell_2 \). Let \( \nu_2 = h_{\ell, N}(\nu_1) \) and set \( Q_{\nu_1}(x_1) = Q_{\nu_1}(x_1; \hat{\ell}, \hat{N}). \) Then

1. \( Q_{\nu_1} \) is undefined if \( |\ell| > 2N \) and \( \ell_1 - (|\ell| - 2N) + 1 \leq \nu_1 \leq \ell_1; \)
2. \( Q_{\nu_1} \) splits if \( |\ell| \leq 2N \) and \( 1 \leq \ell_3 - \ell_2 - 1 \leq \nu_1 \leq \ell_1 \), or if \( |\ell| > 2N \) and \( 1 \leq \nu_1 \leq \ell_1 - (|\ell| - 2N) \).

More precisely, for \( j = -1, 0, 1, \ldots \) and \( \nu_1 = \ell_3 - \ell_2 + j \leq \ell_1 - |2N - |\ell||, \)
\[
(4.5) \quad Q_{\nu_1}(x_1) = \frac{1}{(-N + \ell_2 - i + 1)_{\nu_1}} \prod_{k=N-\ell_3-i+1}^{N-\ell_2+i-1} (x_1 - k), \quad j = 2i - 1,
\]
\[
(4.6) \quad Q_{\nu_1}(x_1) = \frac{-N + \ell_3 + i}{(-N + \ell_2 - i + 1)_{\nu_1}} \prod_{k=N-\ell_3-i+1}^{N-\ell_2+i-1} (1 - a_i x_1) (x_1 - k), \quad j = 2i,
\]
where \( a_i = ((\ell_1 + \ell_2 - \ell_3 - 2i)/(N - \ell_3 - i)(N - \ell_2 + i - 1)) \). Furthermore, if \( |\ell| \leq 2N \), then for \( \nu_1 = \ell_1 + |\ell| - 2N + j \) with \( j = 1, 2, \ldots, 2N - |\ell| \),
\[
(4.7) \quad Q_{\nu_1}(x_1) = \frac{1}{(-\ell_1)_{\nu_1}} \prod_{k=2N-|\ell|-j+1}^{\ell_1} (x_1 - k).
\]

**Proof.** If \( 0 \leq \nu_1 \leq \ell_3 - \ell_2 \), then \( h_{\ell, N}(\nu_1) = \ell_2 + 1 \). Hence, with \( \nu_2 = \ell_2 + 1 \), it is easy to see that \( \ell_2 + \ell_3 - \nu_2 = \ell_3 - 1 \leq N \) and \( \min\{\ell_1, N - \nu_2\} = N - \ell_2 - 1 \) and \( \deg_{\mathbb{F}_N} = \ell_3 - \ell_2 - 2 \). Thus, \((4.4)\) holds if \( \nu_1 > \ell_3 - \ell_2 - 2 \). Furthermore, if \( \nu_1 = \ell_3 - \ell_2 - 1 \), then \( \nu_1 = \deg_{\mathbb{F}_N} + 1 \), so that we can apply Theorem 2.4 with \( m = 0 \) to factor \( Q_{\nu_1} \), which gives \((4.5)\) for \( i = 0 \), whereas if \( \nu_1 = \ell_3 - \ell_2 \), then we can apply Theorem 2.4 with \( m = 1 \) to factor \( Q_{\nu_1} \) and the factorization contains also \( Q_1(x_1, \hat{N} - \hat{\ell} - \ell_1, |N - \ell_3| - 1, \hat{N}) \), a polynomial of degree 1 that gives the factor \( 1 - a_0 x_1 \), which gives \((4.6)\) for \( i = 0 \).

If \( \ell_3 - \ell_2 < \nu_1 \leq \ell_1 - |2N - |\ell|| \), we write \( \nu_1 = \ell_3 - \ell_2 + j \). By Lemma 4.1
\[
h_{\ell, N}(\ell_3 - \ell_2 + j) = \ell_2 - \left\lfloor \frac{j + 1}{2} \right\rfloor + 1, \quad 1 \leq \left\lfloor \frac{j + 1}{2} \right\rfloor \leq \min\{N - \ell_3, \ell_1 + 2 - N\}.
\]

With \( \nu_2 = h_{\ell, N}(\ell_3 - \ell_2 + j) \), we have \( \ell_2 + \ell_3 - \nu_2 \leq \ell_3 + \left\lfloor \frac{j + 1}{2} \right\rfloor - 1 \leq N - 1 < N \). Moreover, \( \min\{\ell_1, N - \nu_2\} = N - \ell_2 + \left\lfloor \frac{j + 1}{2} \right\rfloor - 1 \), so that \( \deg_{\mathbb{F}_N} = \ell_3 - \ell_2 + 2 \left\lfloor \frac{j + 1}{2} \right\rfloor - 2 \). Hence, \((4.4)\) holds for all \( \ell_3 - \ell_2 < \nu_1 \leq \ell_1 - |2N - |\ell|| \). Furthermore, if \( j = 2i - 1 \), then \( \nu_1 = \deg_{\mathbb{F}_N} + 1 \), so that we can apply Theorem 2.4 with \( m = 0 \) to factor \( Q_{\nu_1} \), which gives \((4.5)\) for \( i > 0 \), whereas if \( j = 2i \), then \( \nu_1 = \deg_{\mathbb{F}_N} + 2 \), we can then apply Theorem 2.4 with \( m = 1 \) to factor \( Q_{\nu_1} \) and the factorization contains also \( Q_1(x_1, \hat{N} - \hat{\ell} - \ell_1, |\hat{N} - \ell_1| - 1, \hat{N}) \), which gives \((4.6)\) for \( i > 0 \).
If $\ell_1 - |2N - |\ell|| + 1 \leq \nu_1 \leq \ell_1$, we write $\nu_1 = \ell_1 - |2N - |\ell|| + j$ for $j = 1, 2, \ldots, |2N - |\ell||$. Here we need to consider two cases. First, assume $|\ell| \leq 2N$. Then, by Lemma 4.1

$$h_{\ell,N}(\ell_1 - |2N - |\ell|| + j) = N - \ell_1 - j + 1, \quad 1 \leq j \leq 2N - |\ell|.$$ 

With $\nu_2 = h_{\ell,N}(\ell_1 - |2N - |\ell|| + j)$, we have $\ell_2 + \ell_3 - \nu_2 = |\ell| - N + j - 1 \leq N - 1 < N$ and $\deg_{\ell,N} = \ell_1 - (2N - |\ell|| + j - 1$, so that $\nu_1 = \ell_1 - |2N - |\ell|| + j = \deg_{\ell,N} + 1$ for all $j$. Thus, we can apply Theorem 2.4 with $m = 0$ to factor $Q_{\nu_1}$, which gives (4.7). Next we assume $|\ell| > 2N$. Then, by Lemma 4.1

$$h_{\ell,N}(\ell_1 - |2N - |\ell|| + j) = \ell_2 + \ell_3 - N - j + 1, \quad 1 \leq j \leq 2N - |\ell|.$$ 

With $\nu_2 = h_{\ell,N}(\ell_1 - |2N - |\ell|| + j)$, a quick verification shows that $\nu_1 + 2\nu_2 - |\ell| - 1 = -\ell_1 - j + 1$ and $N - \nu_2 = \nu_1 - 1$, which by (2.9) leads to

$$Q_{\nu_1}(x_1) = \frac{(-\nu_1 - \ell_1 - j + 1, -x, 1)}{(-\ell_1, -\nu_1 + 1)}.$$ 

Since $(-\nu_1 + 1)\nu_1 = 0$, this function is infinite for all $j \geq 1$. This completes the proof. \qed

This proposition shows that the polynomials $Q_{\nu_1}$ is either undefined or splits with only possible exception when $2 \leq \nu_1 \leq |\ell_3 - \ell_2| - 2$, and $|\ell_3 - \ell_2| \geq 4$, since the factorizations are trivial for polynomials of degree 0 or 1.

**Proposition 4.3.** For $\nu_2 = h_{\ell,N}(\nu_1)$, the polynomial $Q_\nu(x; \ell, N)$ is not well-defined if and only if

(i) $\ell_3 \geq \ell_2$ and $\nu_1 \leq \ell_3 - \ell_2$;

(ii) $|\ell| > 2N$ and $2N + 1 - \ell_2 - \ell_3 \leq \nu_1$.

**Proof.** We first assume $\ell_3 \geq \ell_2$. If $0 \leq \nu_1 \leq \ell_3 - \ell_2$, then $\nu_2 = \ell_2 + 1$ according to the proof of Proposition 4.2. For this $\nu_2$, it follows by (2.9) that

$$R_{\ell_2+1}(x; \ell_2, \ell_3, N) = \frac{(-N + x_1)_{\ell_2+1} \ldots F_2 \left( -\ell_2 - 1, -\ell_3, -x_2; 1 \right)}{-\ell_2, -N + x_1},$$

which shows that $R_{\ell_2+1}$ is undefined, since $(-\ell_2)_{\ell_2+1} = 0$, if $\ell_3 \geq \ell_2$. The case $\nu_1 > \ell_3 - \ell_2$ follows readily from Proposition 4.2. Assume now $\ell_3 < \ell_2$ and $\nu_1 \leq \ell_2 - \ell_3$. Then $\nu_2 = h_{\ell,N}(\nu_1) = \ell_3 + 1$ and

$$R_{\ell_3+1}(x; \ell_2, \ell_3, N) = \frac{(-N + x_1)\ell_{\ell_2+1} \ldots F_2 \left( -\ell_3 - 1, -\ell_2, -x_2; 1 \right)}{-\ell_2, -N + x_1},$$

so that $R_{\ell_3+1}(x; \ell_2, \ell_3, N)$ is well defined and splits. Thus, the statement follows from Proposition 4.2. \qed

This shows that the non-trivial polynomials that vanishes on a larger set of lattice points, as seen in Example 3.7, are $R_{\nu_2}(x; \ell, N)$. For $\nu_2 = h_{\ell,N}(\nu_1)$ and $\nu_1$ small, it is possible to determine $R_{\nu_2}$ more explicitly.

**Proposition 4.4.** If $\ell_2 = \ell_3$, then

$$R_{\ell_2}(x, \ell_2, \ell_2, N) = \frac{(-N + x_1 + x_2)\ell_{\ell_2+1} \ldots (-1)^{\ell_2+1}(-x_2)\ell_{\ell_2+1}}{-N + \ell_2 + x_1},$$

where $\ell_{\ell_2+1} \ldots F_2 \ldots F_2 \ldots F_2 \left( -\ell_2, -N + x_1 \right)$.
which contains a linear factor $N - x_1 - 2x_2$ if $\ell_2$ is odd. Furthermore,
\[
R_{\ell_2+1}(x; \ell_2, \ell_2, N) = (-N + \ell_2 + x_1)R_{\ell_2}(x; \ell_2, \ell_2, N).
\]

Proof. We consider the case $R_{\ell_2+1}$ first, which corresponds to $\nu_1 = 0$ and $\nu_2 = h_{\ell,N}(0) = \ell_2 + 1$. Thus,
\[
R_{\nu_2}(x; \ell_2, \ell_2, N) = (-N + x_1)_{\ell_2+1} \, _3F_2 \left( \begin{array}{c} -\ell_2 - 1, -\ell_2, -x_2 \nonumber \end{array} \right| _{-\ell_2, -N + x_1}; 1 \right).
\]

Taking as a limiting case, the $\, _3F_2$ function is given by
\[
\lim_{\ell_2 \to \ell_2} \, _3F_2 \left( \begin{array}{c} -\ell_2 - 1, -\ell_2, -x_2 \nonumber \end{array} \right| _{-\ell_2, -N + x_1}; 1 \right) = \sum_{k=0}^{\ell_2} \frac{(-\ell_2 - 1)_k (-x_2)_k}{k!(-N + x_1)_k}
\]
\[
= \left. _2F_1 \left( \begin{array}{c} -\ell_2 - 1, -x_2 \nonumber \end{array} \right| _{-N + x_1}; 1 \right) \right] \frac{(-1)^{\ell_2+1} (-x_2)_{\ell_2+1}}{(-N + x_1)_{\ell_2+1}}
\]
\[
= \frac{(-N + x_1 + x_2)_{\ell_2+1}}{(-N + x_1)_{\ell_2+1}} \frac{(-1)^{\ell_2+1} (-x_2)_{\ell_2+1}}{(-N + x_1)_{\ell_2+1}},
\]

where the last step follows from Chu-Vandermonde identity. Consequently,
\[
R_{\ell_2+1}(x; \ell_2, \ell_2, N) = (-N + x_1 + x_2)_{\ell_2+1} - (-1)^{\ell_2+1} (-x_2)_{\ell_2+1}.
\]

If $\ell_2$ is odd, then the righthand side becomes zero if $x_1 = N - 2x_2$, which shows that this polynomial contains a factor $N - x_1 - 2x_2$. Furthermore, rewriting the first Pochhammer symbol in the righthand side, we obtain
\[
R_{\ell_2+1}(x; \ell_2, \ell_2, N) = (-N + x_1 + x_2)_{\ell_2+1} - (-1)^{\ell_2+1} (-x_2)_{\ell_2+1}
\]
which is zero when $x_1 = N - \ell_2$, so that it also contains a factor $N - \ell_2 - x_1$.

The case $R_{\ell_2}$ corresponds to $\nu_1 = 1$ and $\nu_2 = \ell_2$. We end up with the same $\, _3F_2$ function, so that, using $(-N + x_1)_{\ell_2+1} = (-N + \ell_2 + x_1)(-N + x_1)_{\ell_2}$, we obtain
\[
R_{\ell_2+1}(x; \ell_2, \ell_2, N) = (-N + \ell_2 + x_1)R_{\ell_2}(x; \ell_2, \ell_2, N).
\]

Since $R_{\ell_2+1}(x; \ell_2, \ell_2, N)$ contains a factor $-N + \ell_2 + x_1$, this completes the proof. \qed

Since the first factor of $Q_{\nu}(x; \ell, N)$ when $\nu_1 = 1$, $\nu_2 = \ell_2$ and $\ell_3 = \ell_2$ is
\[
Q_{\ell_2}(x; \ell_3, \ell_3, 0, N - \ell_2) = \frac{-N + \ell_2 + x_1}{-N + \ell_2},
\]
we conclude that $R_{\ell_2}(x; \ell_2, \ell_2, N)$ is a polynomial of degree $\ell_2$ that vanishes on $V_{\ell,N}^2 \setminus \{(N - \ell_2, j) : 0 \leq j \leq \ell_2\}$. This is the polynomial of degree 4 in $Q_{0,5}( :, \ell, N)$ in Example 3.7.

If $\ell_3 > \ell_2$, then the smallest suitable $\nu_1$ is $\ell_3 - \ell_2 + 1$, for which $\nu_2 = \ell_2$. Consequently, we obtain
\[
R_{\ell_2}(x; \ell_2, \ell_3, N) = (-N + x_1)_{\ell_2} \, _3F_2 \left( \begin{array}{c} -\ell_2, -\ell_3 - 1, -x_2 \nonumber \end{array} \right| _{-\ell_2, -N + x_1}; 1 \right)
\]
\[
= (-N + x_1)_{\ell_2} \sum_{k=0}^{\ell_2} \frac{(-\ell_3 - 1)_k (-x_2)_k}{k!(-N + x_1)_k}.
\]
If $\ell_3 - \ell_2$ is small, then the last sum can be made more explicit by writing it in terms of $\binom{x}{n}$, by adding and subtracting a few terms, and then using the Chu-Vandermonde identity, which gives

$$
R_{\ell_2}(x; \ell_2, \ell_3, N) = \frac{(-N + x_1 + x_2)_{\ell_3+1}}{(-N + x_1)_{\ell_3+1}} - \sum_{k=\ell_2+1}^{\ell+1} \frac{(-\ell_3 - 1)_k (-x_2)_k}{k! (-N + x_1)_k}.
$$

By (4.5) and (4.2), the polynomial $R_{\ell_2}$ vanishes if $x_1 > N - \ell_2$ or $x_1 < N - \ell_3$, which however is not obvious from the formula.

The simplest case is when $\ell_3 = \ell_2 + 1$, for which $\nu_1 = 2$. We then obtain

$$
R_{\ell_2}(x; \ell_2, \ell_2 + 1, N) = \frac{(-N + x_1 + x_2)_{\ell_2+2}}{(-N + \ell_2 + x_1)_2} - \frac{(-1)^{\ell_2+1}(\ell_2 + 2)(-x_2)_{\ell_2+1}}{-N + \ell_2 + x_1} - \frac{(-1)^{\ell_2+2}(-x_2)_{\ell_2+2}}{(-N + \ell_2 + x_1)_2}.
$$

Since the first factor of $Q_n(x; \ell, N)$ in this case is a quadratic polynomial

$$
Q_2(x_1; \ell_1, 1, N - \ell_2) = \frac{(N - \ell_2 - x_1)(N - \ell_2 - 1 - x_1)}{(N - \ell_2)(N - \ell_2 - 1)}
$$

that vanishes when $x_1 = N - \ell_2$ and $N - \ell_2 - 1$, it follows that the polynomial $R_{\ell_2}(x; \ell_2, \ell_2 + 1, N)$ is a polynomial of degree $\ell_2$ that vanishes on $V_{\ell_2,N}^2 \setminus \{(N - \ell_2, j) : 0 \leq j \leq \ell_2\} \cup \{(N - \ell_2 - 1, j) : 0 \leq j \leq \ell_2\}$.

This is not, however, obvious from the explicit formula of $R_{\ell_2}(x; \ell_2, \ell_2 + 1, N)$ given above.

We also observe that $V_{\ell_2,N}^2$ depends on $\ell_1$, whereas $R_{\ell_2}(x; \ell_2, \ell_2 + 1, N)$ is independent of $\ell_1$. Consequently, the polynomial $R_{\ell_2}(x; \ell_2, \ell_2 + 1, N)$ vanishes on a large number of lattice points.

We end this subsection by making a conjecture on the irreducibility of the polynomial $R_{\ell_2}$, which we state more generally for the polynomial

$$
R_n(x; \ell_1, \ell_2, y) := (-y)_n Q_n(x; \ell_1, \ell_2, y)
$$

$$
= (-y)_n \sum_{k=0}^{n} \frac{(-n)_k (n - \ell_1 - \ell_2 - 1)_k (-x)_k}{(-\ell_1)_k k! (-y)_k}
$$

$$
= \sum_{k=0}^{n} \frac{(-n)_k (n - \ell_1 - \ell_2 - 1)_k (-x)_k (-y + k)_{n-k}}{(-\ell_1)_k k!}
$$

This is a well-defined polynomial in $\mathbb{Q}[x, y]$ when $n \leq \ell_1 \land \ell_2$.

**Conjecture 4.5.** For $n \in \mathbb{N}$ and $\ell_1, \ell_2 \in \mathbb{N}$ such that $n \leq \ell_1 \land \ell_2$, the polynomial $R_n(x; \ell_1, \ell_2, y)$ is irreducible unless $\ell_1 = \ell_2$ and $n$ is odd, in which case $R_n(x; \ell_1, \ell_2, y)$ is the product of $(y - 2x)$ and an irreducible polynomial of degree $n - 1$.

For all $n \leq \ell_1 \land \ell_2$, the identity

$$
R_n(x; \ell_1, \ell_2, y) = (-1)^n \frac{(-\ell_2)_n}{(-\ell_1)_n} R_n(y - x; \ell_2, \ell_1, y)
$$

holds, which shows that that $R_n(x; \ell_1, \ell_2, y)$ has a factor $y - 2x$ if $\ell_1 = \ell_2$ and $n$ is odd. The conjecture states that the polynomial is irreducible apart from this trivial factor.
The conjecture can be naturally viewed as a two-dimensional analog of irreducibility results for classical orthogonal polynomials, which have a long history. Indeed, a famous result by Hilbert asserts that there exist irreducible polynomials of every degree \( n \) over \( \mathbb{Q} \) having the largest possible Galois group \( S_n \). While Hilbert’s proof was nonconstructive, Schur provided a rather explicit example by proving that the \( n \)-th Laguerre polynomial is irreducible and has Galois group \( S_n \) over \( \mathbb{Q} \). In the early 50’s, Grosswald conjectured the irreducibility of the Bessel polynomials, which was proved in [6].

It is worth noting that, in general, irreducibility results for polynomials of two variables are easier to prove, since \( \mathbb{Q}[x,y] = \mathbb{Q}[x][y] \) and we can try to use the Eisenstein criterion for the unique factorization domain \( \mathbb{Q}[x] \). This is sometimes the “easy step” in the classical one-dimensional results, see for instance Proposition 3.1 in [2]. However, these arguments do not work in our case, since the expansion is in Pochhammer terms, and not in ordinary powers of \( x \) or \( y \).

5. Generating function

For Hahn polynomials on lattice points in the simplex, the generating function is a useful tool and plays an essential role in [10, 17]. In view of the generating function (2.14), one would expect that there is a generating function for \( Q_\nu(\cdot; \ell, N) \) given in terms of the Jacobi polynomial on the simplex with negative integer parameters. Using the normalized Hahn polynomials

\[
H_\nu(\alpha; \ell, N) = H_\nu(\alpha; -\ell - 1, N), \quad \alpha \in \mathbb{N}_0^{d+1},
\]
see (3.7), one such extension holds straightforwardly.

**Proposition 5.1.** Let \( \ell_i \) and \( N \) be positive integers, so that \( \ell_i \leq N \) and (3.9) holds. For \( \nu \in \mathbb{N}_0^d \), define

\[
G_\nu(x) := G_\nu^d(x) = \prod_{j=1}^d (1 - |x_{j-1}|)^{\nu_j} G_{\nu_j}^{(a_j, \ell_j)} \left( \frac{2x_j}{1 - |x_{j-1}|} - 1 \right),
\]
where \( a_j \) is defined in (3.11). Then, for \( \nu \in H_{\ell,N}^d \), the Hahn polynomials \( H_\nu(\cdot; \ell, N) \) satisfy

\[
G_{\nu,N}(y) = |y|^N G_e \left( \frac{y}{|y|} \right) = \sum_{|\alpha|=N} \frac{N!}{\alpha!} H_\nu(\alpha; \ell, N) y^\alpha.
\]

This follows from the identity (3.8), which is a finite sum for \( \nu \in H_{\ell,N}^d \), by setting \( \kappa_i = -\ell_i - 1 \). The sum in the right-hand side is over all lattice points in \( V_{\ell,N}^d \), which contains \( V_{\ell,N}^d \) as a subset. One may ask if this is a multidimensional analog of the generating function in Proposition 2.7 so that the right-hand side of (5.1) is summed over lattice points in \( V_{\ell,N}^d \). However, this does not look to be possible, since for the factor \( Q_{\nu_j}(x_j; \ell_j, a_j, \widetilde{N}) \) of \( H_\nu(\cdot; \ell, N) \), where \( \widetilde{N} = N - |x_{j-1}| - |\nu|^{j+1} \) and \( j \geq 2 \), both \( \ell_j \wedge \widetilde{N} \) and \( a_j \wedge \widetilde{N} \) depend on \( |x_{j-1}| \) for \( x \in V_{\ell,N}^d \).

The generating function (3.8) is used to derive properties of the Hahn polynomials on \( V_{\ell,N}^d \) in [17]. Some of those properties remain valid when the parameters are negative integers. Of particular interests are those on the reproducing kernel of
the Hahn polynomials of degree \( n \), defined by
\[
P_n(W_{\kappa,N};x,y) = \sum_{|\nu|=n} \frac{Q_\nu(x;\kappa,N)Q_\nu(y;\kappa,N)}{B_\nu(\kappa,N)}, \quad 0 \leq n \leq N.
\]

For the Hahn polynomials with negative integer parameters, the corresponding kernel is defined by
\[
\mathcal{P}_n(H_{\ell,N};x,y) = \sum_{|\nu|=n, \nu \in H_{\ell,N}^d} \frac{Q_\nu(x;\ell,N)Q_\nu(y;\ell,N)}{B_\nu(\ell,N)}, \quad 0 \leq n \leq N.
\]
The space \( \mathcal{V}_n^d(H_{\ell,N}) \) is of the dimension \( \#\{\nu \in H_{\ell,N}^d : |\nu| = n\} \), which is less than \( \#\{\nu \in \mathbb{N}_0^d : |\nu| = n\} \) for \( n \) large. However, let
\[
\ell_{\min} := \min\{\ell_i : 1 \leq i \leq d + 1\}.
\]

**Lemma 5.2.** The set \( \{\nu \in H_{\ell,N}^d : |\nu| = n\} \) coincides with \( \{\nu \in \mathbb{N}_0^d : |\nu| = n\} \) if and only if \( 0 \leq n \leq \ell_{\min} \).

**Proof.** If \( 0 \leq n \leq \ell_{\min} \) and \( |\nu| = n \), then \( \nu_j \leq n \leq \ell_{\min} \) so that \( \nu_j \leq \ell_j \) follows trivially. Moreover, \( |\nu^2| + |\nu^{j+1}| \leq 2n \leq |\ell^{j+1}| \) holds if \( j \leq d - 1 \), which implies \( \nu_j \leq a_j \) for \( 1 \leq j \leq d - 1 \), whereas \( a_d = \ell_{d+1} \), so that \( \nu_d \leq a_d \) holds as well. This shows that \( \{\nu \in \mathbb{N}_0^d : |\nu| = n\} \subset \{\nu \in H_{\ell,N}^d : |\nu| = n\} \), while the other direction of inclusion is trivial. Hence, the two sets are equal. If \( n > \ell_{\min} \) and assume, say, \( \ell_{\min} = \ell_1 \), then the element \( \nu = (n,0,\ldots,0) \) does not belong to \( \{\nu \in H_{\ell,N}^d : |\nu| = n\} \) as \( \nu_1 = n > \ell_1 \). This completes the proof. \( \square \)

For \( 0 \leq n \leq \ell_{\min} \), we can then write the sum of \( \mathcal{P}_n(H_{\ell,N};\cdot,\cdot) \) as over \( \{\nu : |\nu| = n\} \). In this way, it is easy to see that the kernel agrees with \( \mathcal{P}_n(W_{\kappa,N};\cdot,\cdot) \) when \( \kappa_j = -\ell_j - 1 \). In particular, by [17 Theorem 4.3], we can rewrite \( \mathcal{P}_n(H_{\ell,N}) \) in terms of elementary function
\[
\mathcal{E}_k(x,y;\ell) := \sum_{|\gamma| = k, \gamma_\ell \leq \ell} \frac{(-X)_\gamma (-Y)^\gamma}{(-\ell)^\gamma!}, \quad x,y \in V_{\ell,N}^d,
\]
where \( X = (x,N - |x|) \), \( Y = (y,N - |y|) \) and \( \gamma \in \mathbb{N}_0^{d+1} \), and we have used the notation \( (a)_\gamma = (a_1)_{\gamma_1} \cdots (a_{d+1})_{\gamma_{d+1}} \) for \( a \in \mathbb{R}^{d+1} \).

**Proposition 5.3.** Let \( \ell_i \) be nonnegative integers, \( \ell_i \leq N \) and \( \ell_i + \ell_j \geq N \). Then, for \( x,y \in V_{\ell,N}^d \) and \( 0 \leq n \leq \ell_{\min} \),
\[
(5.2) \quad \mathcal{P}_n(H_{\ell,N};x,y) = \frac{(-N)_n(-|\ell|)_n(-|\ell|)_n(|\ell| + 1 - 2n)}{n!(-|\ell|)_N+n(|\ell|+1-n)}
\times \sum_{k=0}^{\min\{n,\ell\}} \frac{(-n)_k(n-|\ell|)_k}{(-N)_k(-N)_k} \mathcal{E}_k(x,y;\ell).
\]

If \( n > \ell_{\min} \), then \( \{\nu : |\nu| = n\} \) contains \( \nu \) outside of \( H_{\ell,N}^d \), so that some of \( Q_\nu(\cdot;\ell,N) \) with \( |\nu| = n \) vanishes on \( V_{\ell,N}^d \) by Theorem 5.6. One may ask if it is possible to include such polynomials in the summation of \( \mathcal{P}_n(H_{\ell,N};\cdot,\cdot) \) so that some version of (5.2) can be deduced from that of \( \mathcal{P}_n(W_{\kappa,N};\cdot,\cdot) \). The answer, however, is negative since the norm of such polynomials, \( B_\nu(\ell,N) \), is necessarily zero and, as a result, no such terms can be included in the sum of \( \mathcal{P}_n(H_{\ell,N};\cdot,\cdot) \).
There is an exceptional case when \( d = 2 \), for which Hahn polynomials of negative integer parameters of degree \( n \) have full range for all suitable \( n \). This is the case when \( \ell_1 = \ell_2 = \ell_3 = \ell \) and \( N = 2\ell \), so that the set \( V_{\ell,N}^2 \) consists of lattice points in a regular triangle, whereas the set \( H_{\ell,N}^2 \) is given below.

**Proposition 5.4.** Let \( \ell \) be a positive integer. If \( d = 2, \ell_i = \ell \) for \( 1 \leq i \leq 3 \) and \( N = 2\ell \), then

\[
H_{\ell,N}^2 = \{ \nu \in \mathbb{N}_0^3 : |\nu| \leq \ell \}.
\]

**Proof.** Since \( N = 2\ell \) and \( |\ell| = 3\ell \), we also have \( a_1 = 2\ell - 2\nu_2 \) and \( a_2 = \ell \). Hence,

\[
H_{\ell,N}^2 = \{ (\nu_1, \nu_2) : |\nu| \leq 2\ell, |\nu| \leq \ell, \nu_1 \leq \ell, \nu_2 \leq \ell, \nu_1 + 2\nu_2 \leq 2\ell \}
\]

by the definition of \( H_{\ell,N}^2 \). Clearly, \( |\nu| \leq \ell \) implies \( \nu_1 \leq \ell \) and it also implies \( \nu_1 + 2\nu_2 = \nu_1 + |\nu| \leq 2\ell \), which proves the statement. \( \square \)

For the Hahn polynomials \( Q_{\nu}(x; \kappa, N) \) on \( V_N^d \), the closed form expression of the reproducing kernel is used to prove that the Poisson kernel

\[
\Phi_r(W_{\kappa,N}; x, y) := \sum_{n=0}^{N} P_n(W_{\kappa,N}; x, y) r^n, \quad 0 \leq r \leq 1
\]

is nonnegative for all \( x, y \in V_N^{d} \) and \( 0 \leq r \leq 1 \), see \( \{17\} \). The proof relies on the fact that the corresponding function

\[
E_k(x, y; \kappa) := \sum_{|\gamma| = k} \frac{(-X)_\gamma (-Y)_\gamma}{(\kappa + 1)_\gamma \gamma!},
\]

for \( \kappa_i > -1 \) is nonnegative for \( x, y \in V_N^{d} \).

For the Hahn polynomials with negative integer parameters, we could define the Poisson kernel analogously as

\[
\Phi_r(H_{\ell,N}; x, y) := \sum_{n=0}^{M} P_n(H_{\ell,N}; x, y) r^n, \quad 0 \leq r \leq 1,
\]

where \( M \leq N \) is the highest degree \( M = \max\{|\nu| : \nu \in H_{\ell,N}^2\} \). This kernel, however, is no longer nonnegative, as shown by examples of small \( \ell \) and \( N \). One of the reasons that the proof in \( \{17\} \) fails is that \( E_k(x, y; \ell) \) is of the sign \( (-1)^\ell \) instead of nonnegative. Furthermore, for \( M > \ell_{\min} \), we could question if the definition \( \Phi_r(H_{\ell,N}; \cdot, \cdot) \) reflects the symmetry of \( H_{\ell,N}^2 \). In this regard, we may examine the special case when \( d = 2, \ell_i = \ell \) and \( N = 2\ell \) in Proposition 5.4 closely.

**Proposition 5.5.** Let \( d = 2, \ell_i = \ell, 1 \leq i \leq 3 \), and \( N = 2\ell \). Then, for all \( x, y, z \),

\[
\Phi_r(H_{\ell,N}; ((x_1, x_2), (0, 0))) = \sum_{n=0}^{\ell} \frac{(-1)^n (-3\ell - 1)_n (-3\ell + 1)_{2n} Q_n(x_1; \ell, 2\ell, 2\ell) r^n}{n! (-3\ell - 1)_{2n}}
\]

**Proof.** Using the explicit formula of \( Q_{\nu}(x; \ell, N) \) in \( \{4.1\} \), it is easy to verify that

\[
Q_{\nu_1, \nu_2}(0, 0; \ell, 2\ell) = (-2\ell)_{\nu_1} \delta_{\nu_2, 0}
\]

by the Chu-Vandermonde identity. For \( \nu_2 = 0 \), we have

\[
Q_{\nu_1, 0}(x; \ell, 2\ell) = (-2\ell)_{\nu_1} Q_{\nu_1}(x; \ell, 2\ell, 2\ell).
\]
Furthermore, by (3.12), the norm of $Q_{\nu,0}$ is equal to
\begin{equation}
B_{\nu,0}(\ell, 2\ell) = \frac{(-1)^{\nu_1} \nu_1! (-2\ell)^{\nu_1} (-3\ell - 1)^{2\nu_1}}{(-3\ell - 1)^{\nu_2} (-3\ell)^{2\nu_1}}.
\end{equation}
Together, these identities lead to
\begin{equation}
P_n(H_{\ell,2\ell}; (\ell, x_2), (0, \ell)) = \sum_{\nu_1=0}^{n} \frac{Q_{\nu_2,n-\nu_1}((0, \ell); \ell, 2\ell) Q_{\nu_1,n-\nu_1}(x; \ell, 2\ell)}{B_{\nu_1,n-\nu_1}(\ell, 2\ell)}
\end{equation}
\begin{equation}
= \frac{(-1)^n (-3\ell - 1)^n (-3\ell)^{2n}}{n! (-3\ell - 1)^{2n}} Q_n(x; \ell, 2\ell, 2\ell),
\end{equation}
from which the formula for $\Phi_r(H_{\ell,N}; (\ell, x_2), (0, \ell))$ follows immediately. \(\Box\)

This gives an explicit expression for the Poisson kernel $\Phi_r(H_{\ell,N}; \cdot, \cdot)$, which is a polynomial of degree $\ell$ in the variable $r$, at the points $(x_1, x_2)$ and $(0, \ell)$ in $V_{\ell,2\ell}$, which depends on $x_1$ but not $x_2$. Numerical experiment shows that this polynomial changes sign once on $[0,1]$ when $\ell$ is even and $x = (\ell, x_2)$ and, in some cases, when $\ell$ is odd and $x$ is an even integer.

### 6. Bispectral properties

The Hahn polynomials for the hypergeometric distribution can be characterized as common eigenfunctions of two families of commutative algebras of difference operators: one acting on the variables $x_1, \ldots, x_d$, and another one acting on the indices $\nu_1, \ldots, \nu_d$. These operators can be linked to mutually commuting symmetries of a discrete extension of the generic quantum superintegrable system on the sphere \([8][11][13]\). We discuss these families of operators in the next subsections.

In this section, let $\tilde{Q}_\nu(x; \ell, N)$ denote the Hahn polynomials normalized as follows
\begin{equation}
\tilde{Q}_\nu(x; \ell, N) = \frac{1}{(-N)^{\nu_1}} \prod_{j=1}^{d} (-N + |x_{j-1}| + |\nu^{j+1}|)^{\nu_j}
\end{equation}
\begin{equation}
\times Q_{\nu_1} \left( x_j; \ell_j, a_j, N - |x_{j-1}| - |\nu^{j+1}| \right).
\end{equation}

The polynomial $\tilde{Q}_\nu$ differs from $Q_\nu$ by an an extra factor $(-N)^{\nu_1}$ in the denominator. This factor makes it possible to write the recurrence operators $\mathcal{L}_d$, defined in (6.4), in its relatively simple form.

#### 6.1. Spectral equations in the variables

We denote by $\{e_1, e_2, \ldots, e_d\}$ the standard basis for $\mathbb{R}^d$, and by $E_{x_i}$ and $E^{-1}_{x_i}$ the shift operators acting on a function $f(x)$ as follows
\begin{equation}
E_{x_i} f(x) = f(x + e_i) \quad \text{and} \quad E^{-1}_{x_i} f(x) = f(x - e_i).
\end{equation}

The operator
\begin{equation}
\mathcal{L}_d := \mathcal{L}_d^\ell(x; \ell; N) = \sum_{1 \leq i \neq j \leq d} x_j (x_i - \ell_i)(E_{x_i} E^{-1}_{x_j} - 1)
\end{equation}
\begin{equation}
+ \sum_{i=1}^{d} (x_i - \ell_i)(N - |x_i|)(E_{x_i} - 1) + \sum_{i=1}^{d} x_i (N - |x_i| - \ell_{d+1})(E^{-1}_{x_i} - 1),
\end{equation}
is self-adjoint with respect to the hypergeometric distribution and acts diagonally on the basis of polynomials in (6.1) with eigenvalue $-|\nu|(|\nu| - |\ell| - 1)$ which depends only
on the total degree $|\nu|$ of the polynomial $\hat{Q}_\nu(x; \ell, N)$, see [10] Section 5. Fix now $k < d$ and note now that, up to a factor independent of $x_1, \ldots, x_{d-k}$, the product of the last $k$-terms in (6.1) can be regarded as a Hahn polynomial in the variables $\hat{x} = x^{d-k+1} = (x_{d-k+1}, \ldots, x_d)$, with indices $\nu = \nu^{d-k+1} = (\nu_{d-k+1}, \ldots, \nu_d)$, and parameters $\ell = \ell^{d-k+1} = (\ell_{d-k+1}, \ldots, \ell_{d+1})$, $N = N - |x_{d-k}|$. Therefore, if we set

$$L_k^\ell := L_k^\ell(x^{d-k+1}; \ell^{d-k+1}, N - |x_{d-k}|), \quad \text{for } k = 1, \ldots, d-1,$$

we see that polynomials in (6.1) will be eigenfunctions of the operators $L_1^\ell, \ldots, L_d^\ell$ and satisfy the spectral equations, for $k = 1, \ldots, d$,

$$L_k^\ell \hat{Q}_\nu(x; \ell, N) = -|\nu^{d-k+1}|(\nu^{d-k+1}| - |\ell^{d-k+1}| - 1)\hat{Q}_\nu(x; \ell, N).$$

From these equations it follows easily that the operators $L_1^\ell, \ldots, L_d^\ell$ commute with each other. They generate a Gaudin subalgebra for a representation of the Kohno-Drinfeld algebra associated with the hypergeometric distribution. The operators in the larger algebra can be regarded as symmetries, or integrals of motion, for a discrete extension of the generic quantum superintegrable system on the sphere, see [11].

6.2. Spectral equations in the indices. In this section, we will use the shift operators $E_{\nu_i}$ and $E_{\nu_i}^{-1}$ acting on a function $f_\nu$ as follows

$$E_{\nu_i} f_\nu = f_{\nu + e_i}, \quad E_{\nu_i}^{-1} f_\nu = f_{\nu - e_i}.$$

For $j, k \in \{0, \pm 1\}^2$ and $i \in \{1, \ldots, d\}$, we define $B_j^{i,k}$ as follows

$$B_{i}^{0,0} = |\nu^j|(|\nu^j| - |\ell^i| - 1) + |\nu^{i+1}|(|\nu^{i+1}| - |\ell^{i+1}| - 1) + \frac{|\ell^{i+1}|(|\ell^i| + 2)}{2},$$

$$B_{i}^{0,1} = -\nu_i(\nu_i + 2|\nu^{i+1}| - |\ell^i| - 1),$$

$$B_{i}^{0,-1} = (\ell_i - \nu_i)(\nu_i + 2|\nu^{i+1}| - |\ell^{i+1}| - 1),$$

$$B_{i}^{1,0} = (\ell_i - \nu_i)(\nu_i + 2|\nu^{i+1}| - |\ell^i| - 1),$$

$$B_{i}^{-1,0} = -\nu_i(\nu_i + 2|\nu^{i+1}| - |\ell^{i+1}| - 1),$$

$$B_{i}^{1,1} = (\nu_i + 2|\nu^{i+1}| - |\ell^i| - 1)(\nu_i + 2|\nu^{i+1}| - |\ell^{i+1}|),$$

$$B_{i}^{-1,1} = \nu_i(\nu_i - 1),$$

$$B_{i}^{1,-1} = (\ell_i - \nu_i)(\ell_i - \nu_i - 1),$$

$$B_{i}^{-1,-1} = (\nu_i + 2|\nu^{i+1}| - |\ell^{i+1}| - 1)(\nu_i + 2|\nu^{i+1}| - |\ell^{i+1}| - 2).$$

We extend the formulas above and define $B_j^{0,k}$ for $k \in \{0, \pm 1\}$ as follows

$$B_0^{0,0} = -N + |\ell|/2, \quad B_0^{0,1} = N - |\nu|, \quad B_0^{0,-1} = |\nu| - |\ell| + N - 1.$$

Next, for $i \in \{1, \ldots, d\}$ we define

$$b_i^0 = \frac{(2|\nu^j| - |\ell^i|)(2|\nu^j| - |\ell^i| - 2)}{2},$$

$$b_i^1 = (2|\nu^j| - |\ell^i|)(2|\nu^j| - |\ell^i| - 1),$$

$$b_i^{-1} = (2|\nu^j| - |\ell^i| - 2)(2|\nu^j| - |\ell^i| - 1).$$
and for $\mu = (\mu_1, \ldots, \mu_d) \in \{0, \pm 1\}^d$ we set

$$C_\mu = \frac{\prod_{k=0}^{d} B_{k}^{\mu_k, \mu_{k+1}}}{\prod_{k=1}^{d} \nu_k},$$

with the convention that $\mu_0 = \mu_{d+1} = 0$. With the above notations, we define an operator acting on the indices as follows

$$(6.4) \quad L'_d := L'_d(\nu; \ell; N) = \sum_{0 \neq \mu \in \{-1, 0, 1\}^d} C_\mu \left( \prod_{k=1}^{d} E_{\nu_k}^{\mu_k - \mu_{k+1}} - 1 \right).$$

Note that this operator is significantly more complicated than the difference operator $L^x_d$ in (6.2). It can be obtained as a limit of the image of the $d$-dimensional Racah operator under the bispectral involution associated with the Racah polynomials, see [7, Section 5.2]. In particular, this shows that

$$(6.5) \quad L'_d(\nu; \ell; N) \hat{Q}_\nu(x; \ell, N) = |x| \hat{Q}_\nu(x; \ell, N).$$

Remark 6.1 (Boundary conditions). Note that the operator $L'_d$ contains backward shift operators, and the polynomials $\hat{Q}_\nu(x; \ell, N)$ are defined for $\nu \in \mathbb{N}_0^d$. However, one can easily see that the coefficients of the operator $L'_d$ that multiply the polynomials containing negative indices are zero, so we can simply ignore these terms. Indeed, the operator $L'_d$ will contain a negative power of $E_{\nu_k}$ in one of the following cases:

Case 1: $\mu_k - \mu_{k+1} = -1$, and the corresponding term in the sum in (6.4) contains $E_{\nu_k}^{-1}$. Note that $\mu_k - \mu_{k+1} = -1$ is possible for two sets of indices

- when $(\mu_k, \mu_{k+1}) = (-1, 0)$, in which case $C_\mu$ contains the term $B_{k}^{-1,0}$, which is 0 when $\nu_k = 0$,
- or when $(\mu_k, \mu_{k+1}) = (0, 1)$, in which case $C_\mu$ contains the term $B_{k}^{0,1}$, which is 0 when $\nu_k = 0$.

Case 2: $\mu_k - \mu_{k+1} = -2$, and the corresponding term in the sum in (6.4) contains $E_{\nu_k}^{-2}$. This happens only when $\mu_k = -1$ and $\mu_{k+1} = 1$, in which case $C_\mu$ contains the term $B_{k}^{-1,1}$, which is 0 when $\nu_k = 0$ or $\nu_k = 1$.

This shows that we can ignore all terms containing negative indices. With this convention, the results in [7, Section 5.2] imply that (6.5) holds for all $\nu \in \mathbb{N}_0^d$ when $\ell_j$ and $N$ are generic (non-integer) parameters. For parameters $\ell_j \in \mathbb{N}$ and $N \in \mathbb{N}$, equation (6.5) holds when the indices on the left-hand side belong to the set $H^2_{\ell, N}$. We will use the same convention for the other operators we construct in this section.

Similarly to the difference operators in $x$, we can construct a family of commuting difference operators acting on the indices $\nu_1, \ldots, \nu_d$, which represent the multiplication by the variables $x_1, \ldots, x_d$ in the basis (6.1).

Theorem 6.2. If we set

$$L'_k := L'_k(\nu; \ell_1, \ldots, \ell_k, |\ell^{k+1}| - 2|\nu^{k+1}|; N - |\nu^{k+1}|), \quad \text{for } k = 1, \ldots, d - 1,$$

the polynomials in (6.1) will satisfy also the following spectral equations

$$(6.6) \quad L'_k \hat{Q}_\nu(x; \ell, N) = |x_k| \hat{Q}_\nu(x; \ell, N), \quad \text{for } k = 1, \ldots, d.$$
Remark 6.3. In view of [6.3], we refer to (6.3) and (6.6) as bispectral equations for the Hahn polynomials.

Remark 6.4. The equations (6.6) also lead to three-term relation for \( \hat{Q}_\nu(\cdot, \ell, N) \) since we can write them as

\[
(\mathcal{L}_k^\nu - \mathcal{L}_k^{\nu-1}) \hat{Q}_\nu(x; \ell, N) = x_k \hat{Q}_\nu(x; \ell, N), \quad \text{for } k = 1, \ldots, d,
\]

where \( \mathcal{L}_k^\nu \) denotes the zero operator.

6.3. Explicit formulas in dimension two. In this subsection, we write the operators and the difference equations in Theorem 6.2 in dimension two. Note that when \( d = 2 \), we have \(|\nu| = \nu_1 + \nu_2\) and \(|\ell| = \ell_1 + \ell_2 + \ell_3\). The operator in (6.4) can be written as

\[
\mathcal{L}_2^\nu = C_{1,0}E_{\nu_1} + C_{1,1}E_{\nu_2} + C_{1,-1}E_{\nu_1}^{-1}E_{\nu_2}^{-1} + C_{0,1}E_{\nu_1}^{-1}E_{\nu_2} + C_{0,-1}E_{\nu_1}E_{\nu_2}^{-1} + C_{-1,0}E_{\nu_1}^{-1} + C_{-1,-1}E_{\nu_1}^{-2}E_{\nu_2},
\]

where

\[
C_{1,0} = \frac{(N - |\nu|)(\ell_1 - \nu_1)(\nu_1 + 2\nu_2 - |\ell| - 1)}{(2|\nu| - |\ell|)(2|\nu| - |\ell| - 1)(2\nu_2 - \ell_2 - \ell_3)(2\nu_2 - \ell_2 - \ell_3 - 2)} \times (2\nu_2(\nu_2 - \ell_2 - \ell_3 - 1) + \ell_3(\ell_2 + \ell_3 + 2))
\]

\[
C_{-1,0} = -\frac{(|\nu| - |\ell| + N - 1)\nu_1(\nu_1 + 2\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell| - 2)(2|\nu| - |\ell| - 1)(2\nu_2 - \ell_2 - \ell_3)(2\nu_2 - \ell_2 - \ell_3 - 2)} \times (2\nu_2(\nu_2 - \ell_2 - \ell_3 - 1) + \ell_3(\ell_2 + \ell_3 + 2))
\]

\[
C_{0,1} = \frac{(2N - |\ell|)\nu_1(\nu_1 + 2\nu_2 - |\ell| - 1)(\ell_2 - \nu_2)(\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell|)(2|\nu| - |\ell| - 2)(2\nu_2 - \ell_2 - \ell_3)(2\nu_2 - \ell_2 - \ell_3 - 1)}
\]

\[
C_{0,-1} = \frac{(2N - |\ell|)(\ell_1 - \nu_1)(\nu_1 + 2\nu_2 - \ell_2 - \ell_3 - 1)\nu_2(\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell|)(2|\nu| - |\ell| - 2)(2\nu_2 - \ell_2 - \ell_3 - 2)(2\nu_2 - \ell_2 - \ell_3 - 1)}
\]

\[
C_{1,1} = \frac{(N - |\nu|)(\nu_1 + 2\nu_2 - |\ell| - 1)(\nu_1 + 2\nu_2 - |\ell|)(\ell_2 - \nu_2)(\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell|)(2|\nu| - |\ell| - 1)(2\nu_2 - \ell_2 - \ell_3)(2\nu_2 - \ell_2 - \ell_3 - 1)}
\]

\[
C_{-1,1} = -\frac{((|\nu| - |\ell| + N - 1)\nu_1(\nu_1 + 2\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell| - 2)(2|\nu| - |\ell| - 1)(2\nu_2 - \ell_2 - \ell_3)(2\nu_2 - \ell_2 - \ell_3 - 1)}
\]

\[
C_{1,-1} = -\frac{(N - |\nu|)(\ell_1 - \nu_1)(\ell_1 - \nu_1 - 1)\nu_2(\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell|)(2|\nu| - |\ell| - 1)(2\nu_2 - \ell_2 - \ell_3 - 2)(2\nu_2 - \ell_2 - \ell_3 - 1)}
\]

\[
C_{-1,-1} = -\frac{(|\nu| - |\ell| + N - 1)(\nu_1 + 2\nu_2 - \ell_2 - \ell_3 - 1)(\nu_1 + 2\nu_2 - \ell_2 - \ell_3 - 2)}{(2|\nu| - |\ell| - 2)(2|\nu| - |\ell| - 1)(2\nu_2 - \ell_2 - \ell_3)(2\nu_2 - \ell_2 - \ell_3 - 1)}
\]

\[
\times \nu_2(\nu_2 - \ell_2 - \ell_3 - 1),
\]
and
\[ \hat{C}_{0,0} = - \sum_{\mu \in \{-1,0,1\} \setminus \{0,0\}} C_\mu. \]

The operator \( L_1^\nu \) can be written as
\[ L_1^\nu = C'_1 (E_{\nu_1} - 1) + C'_{-1} (E_{\nu_1}^{-1} - 1), \]
where
\[ C'_1 = \frac{(N - |\nu|)(\ell_1 - \nu_1)(\nu_1 + 2\nu_2 - |\ell| - 1)}{(2|\nu| - |\ell|)(2|\nu| - |\ell| - 1)}, \]
\[ C'_{-1} = -\frac{(|\nu| - |\ell| + N - 1)(\nu_1 + 2\nu_2 - \ell_2 - \ell_3 - 1)}{(2|\nu| - |\ell| - 2)(2|\nu| - |\ell| - 1)}. \]

With the above formulas, equations (6.6) take the form
\[
\begin{align*}
L_1^\nu \hat{Q}_\nu(x; \ell, N) &= x_1 \hat{Q}_\nu(x; \ell, N), \\
L_2^\nu \hat{Q}_\nu(x; \ell, N) &= (x_1 + x_2) \hat{Q}_\nu(x; \ell, N).
\end{align*}
\]

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