Abstract. In this paper, we extend the notions of statistically convergence of order $\beta$ and strong Cesàro summability of order $\beta$, and introduce the notions $f-$statistically convergence of order $\beta$ and strong Cesàro summability of order $\beta$ for $\beta \in (0,1]$ with respect to an unbounded modulus function $f$ for sequences of fuzzy numbers and give some inclusion theorems.

Key words and phrases: Fuzzy number, sequence of fuzzy numbers, statistical convergence, Cesàro summability, modulus function.

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1. Introduction

In order to generalize the concept of convergence of real sequences, the notion of statistical convergence was introduced by Fast [17] and Schoenberg [27], independently. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, Ergodic theory and Number theory. Later on it was further investigated from the sequence space point of view and linked with summability theory by Fridy [18], Šalát [24], Connor [13], topological groups by Çakallı [8], function spaces by Caserta et al. [10].

Matloka [20] defined the notion of fuzzy sequence and introduced bounded and convergent sequences of fuzzy real numbers and studied their some properties. After then, Nuray and Savas [23] defined the notion of statistical convergence for sequences of fuzzy numbers. Since then, there has been increasing interest in the study of statistical convergence of fuzzy sequences (see [29], [9], [21], [26], [19], [6]).

Çolak [11] generalized the statistical convergence by ordering the interval $[0,1]$ and defined the statistical convergence of order $\alpha$ and strong $p-$Cesàro summability of order $\alpha$, where $0 < \alpha \leq 1$ and $p$ is a positive real number. Altinok et al. [4] introduced the concepts of statistical convergence of order $\beta$ and strong $p-$Cesàro summability of order $\beta$ for sequences of fuzzy numbers. Aizpuru et al. [11] defined the $f-$density of the subset $A$ of $\mathbb{N}$ by using an unbounded modulus function. After then, Bhardwaj [7] introduced $f-$statistical convergence of order $\alpha$ and strong Cesaro summability of order $\alpha$ with respect to a modulus function $f$ for real sequences. The purpose of this paper is to generalize the study of Bhardwaj [7] and Çolak [11] applying to sequences of fuzzy numbers so as to fill up the existing gaps in the summability theory of fuzzy numbers. For a detailed account of many more interesting investigations concerning statistical convergence of order $\alpha$ and $\beta$, one may refer to ([12], [4], [15], [16], [2], [3], [5]).

This paper organizes as follows: In section 2, we give the basic notions which will be used throughout the paper. In section 3, we define the spaces $S^\beta (F,f)$ and $w^\beta (F,f)$, the set of all $f-$statistically convergent sequences of order $\beta$ and strong Cesàro summability of order...
\[ f \] with respect to an unbounded modulus function \( f \) for fuzzy sequences, respectively and establish inclusion relations among the spaces \( S^\beta (F, f) \) for different values of \( \beta \). Moreover, we obtain inclusion relations between the spaces \( w^\beta (F, f), w^\beta (0, F, f) \) and \( w^\beta,\infty (F, f) \) and give some conditions related to modulus function \( f \) for inclusion relation \( w^\beta (F, f) \subset S^\gamma (F, f) \).

2. Definitions and Preliminaries

In this section, we recall some basic definitions and notations that we are going to use in this paper.

A fuzzy set \( u \) on \( \mathbb{R} \) is called a fuzzy number if it has the following properties:

\( i \) \( u \) is normal, that is, there exists an \( x_0 \in \mathbb{R} \) such that \( u(x_0) = 1 \);

\( ii \) \( u \) is fuzzy convex, that is, for \( x, y \in \mathbb{R} \) and \( 0 \leq \lambda \leq 1 \), \( u(\lambda x + (1 - \lambda)y) \geq \min[u(x), u(y)] \);

\( iii \) \( u \) is upper semicontinuous;

\( iv \) \( \text{supp } u = \text{cl}\{x \in \mathbb{R} : u(x) > 0\} \), or denoted by \( [u]^0 \), is compact.

\( \alpha \)-level set \( [u]^\alpha \) of a fuzzy number \( u \) is defined by

\[
[u]^\alpha = \begin{cases} 
\{ x \in \mathbb{R} : u(x) \geq \alpha \}, & \text{if } \alpha \in (0, 1] \\
\text{supp } u, & \text{if } \alpha = 0.
\end{cases}
\]

It is clear that \( u \) is a fuzzy number if and only if \( [u]^\alpha \) is a closed interval for each \( \alpha \in [0, 1] \) and \( [u]^1 \neq \emptyset \). We denote space of all fuzzy numbers by \( L(\mathbb{R}) \).

In order to calculate the distance between two fuzzy numbers \( u \) and \( v \), we use the metric

\[
d(u, v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha)
\]

where \( d_H \) is the Hausdorff metric defined by

\[
d_H([u]^\alpha, [v]^\alpha) = \max \{ |u^\alpha - v^\alpha|, |u^\alpha - v^\alpha| \}.
\]

It is known that \( d \) is a metric on \( L(\mathbb{R}) \), and \( (L(\mathbb{R}), d) \) is a complete metric space.

A sequence \( X = (X_k) \) of fuzzy numbers is a function \( X : N \to L(\mathbb{R}) \). Let \( X = (X_k) \) be a sequence of fuzzy numbers. The sequence \( X = (X_k) \) of fuzzy numbers is said to be bounded if the set \( \{ X_k : k \in N \} \) of fuzzy numbers is bounded and convergent to the fuzzy number \( X_0 \), written as \( \lim_{k \to \infty} X_k = X_0 \), if for every \( \varepsilon > 0 \) there exists a positive integer \( k_0 \) such that

\[
d(X_k, X_0) < \varepsilon \] for \( k > k_0 \). Let \( s(F) \), \( \ell_\infty (F) \) and \( c(F) \) denote the set of all sequences, all bounded sequences and all convergent sequences of fuzzy numbers, respectively \([20]\).

The concept of modulus function was formally introduced by Nakano \([22]\). A mapping \( f : [0, \infty) \to [0, \infty) \) is said to be a modulus if

\( i \) \( f(x) = 0 \) iff \( x = 0 \),

\( ii \) \( f(x + y) \leq f(x) + f(y) \) for \( x, y \geq 0 \),

\( iii \) \( f \) is increasing,

\( iv \) \( f \) is right-continuous at 0.

The continuity of \( f \) everywhere on \( [0, \infty) \) follows from above definition. A modulus function can be bounded or unbounded. For example \( f(x) = x^p \), \( (0 < p \leq 1) \) is bounded and \( f(x) = \frac{1}{1+x^2} \) is bounded. For an extensive view on this subject we refer \([2, 25, 28, 14, 15]\).

We recall notions of statistically convergence of order \( \beta \) and \( f_\beta \)-density.

**Definition 2.1.** \([3]\) Let \( \beta \in (0, 1] \) and \( X = (X_k) \) be a sequence of fuzzy numbers. Then the sequence \( X = (X_k) \) of fuzzy numbers is said to be statistically convergent of order \( \beta \), to fuzzy number \( X_0 \) if for every \( \varepsilon > 0 \),

\[
\lim_{n \to \infty} \frac{1}{n^\beta} | \{ k \leq n : d(X_k, X_0) \geq \varepsilon \} | = 0,
\]
where the vertical bars indicate the number of elements in the enclosed set. In this case we write $S^β(F) − \lim X_k = X_0$. We denote the set of all statistically convergent sequences of order $β$ by $S^β(F)$.

We take $β$ instead of $α$ for suitability with our spaces.

**Definition 2.2.** [7] Let $f$ be an unbounded modulus function and $β \in (0, 1]$ be any real number. The $f_β$–density of the subset $A$ of $\mathbb{N}$ defined by

$$d^f_β(A) = \lim_{n \to \infty} \frac{1}{f(n^β)} f(|\{k \leq n : k \in A\}|).$$

**Remark 2.3.** [7] The $f_β$–density reduces to the natural density for $β = 1$ and $f(x) = x$, and $f_β$–density becomes the $f$–density in case $β = 1$. $f_β$–density is $β$–density when $f(x) = x$.

Generally, the equality $d^f_β(A) + d^f_β(\mathbb{N} − A) = 1$ does not hold for any unbounded modulus $f$. For example, if we take $f(x) = x^β$ for $0 < p ≤ 1$, $β \in (0, 1)$, and $A = \{2n : n \in \mathbb{N}\}$, then $d^f_β(A) = \infty = d^f_β(\mathbb{N} − A)$. Moreover, finite sets have zero $f_β$–density for any unbounded modulus function $f$.

**Remark 2.4.** [7] If $d^f_β(A) = 0$, then $d_β(A) = 0$, and so $d(A) = 0$, where $f$ is any unbounded modulus function and $β \in (0, 1]$. Conversely, a set having zero natural density may have non-zero $f_β$–density for the same $f$ and $β$. For this, we give following example.

**Example 2.5.** We consider the modulus function $f(x) = \ln(x + 1)$ and the set $A = \{1, 8, 27, 64, ...\}$. Then, it is easy to see that $d(A) = 0$ and $d_β(A) = 0$ for $β \in \left(\frac{1}{3}, 1\right]$, but $d^f_β(A) ≥ d^f(A) = \frac{1}{3}$, so $d^f_β(A) ≠ 0$.

**Lemma 2.6.** [7] Let $f$ be any unbounded modulus, $0 < β ≤ γ ≤ 1$ and $A \subset \mathbb{N}$. Then $d^f_β(A) ≤ d^f_γ(A)$.

Thus, if the set $A$ has zero $f_β$–density for any unbounded modulus $f$ and $0 < β ≤ γ ≤ 1$, then it has zero $f_γ$–density. In particular, for some $β \in (0, 1]$, a set having zero $f_β$–density has zero $f$–density. But, the converse is not true.

3. Main Results

We now introduce the concept of $f$–statistical convergence of order $β$ for sequences of fuzzy numbers as follows.

**Definition 3.1.** Let $X = (X_k)$ be a fuzzy sequence, $f$ be an unbounded modulus and $β \in (0, 1]$. A sequence $X = (X_k)$ of fuzzy numbers is said to be $f$–statistically convergent of order $β$ to fuzzy number $X_0$ or $S^β(F, f)$–convergent to $X_0$ if for each $ε > 0$,

$$\lim_{n \to \infty} \frac{1}{f(n^β)} f(|\{k \leq n : d(X_k, X_0) ≥ ε\}|) = 0.$$ 

In this case, we write $S^β(F, f) − \lim X_k = X_0$. By $S^β(F, f)$, we shall denote the set of all sequences of fuzzy numbers which are $f$–statistically convergent of order $β$ for convenience with our previous studies and by $S_{β, 0}^β(F, f)$, the set of all $f$–statistically null sequences of order $β$. For any unbounded modulus function $f$ and $β \in (0, 1]$, the inclusion relation $S_{β, 0}^β(F, f) \subset S^β(F, f)$ is clear. Furthermore, we point out $f$–statistical convergence of order $β$ reduces statistical convergence of order $β$ defined in [23] for $f(x) = x$ and $f$–statistical convergence of order $β$ reduces to statistical convergence defined in [23] for $f(x) = x$ and $β = 1$. 

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Now, we generalize the space of strongly Cesàro summable sequences of order $\beta$ using a modulus function.

**Definition 3.2.** Let $X = (X_k)$ be a sequence of fuzzy numbers, $f$ be an unbounded modulus function and $\beta \in (0, 1]$. We define following spaces:

\[
\begin{align*}
\omega^{\beta,0} (F, f) & = \left\{ X \in s (F) : \lim_{n \to \infty} \frac{1}{n^{\beta}} \sum_{k=1}^{n} f (d (X_k, 0)) = 0 \right\}, \\
\omega^{\beta} (F, f) & = \left\{ X \in s (F) : \lim_{n \to \infty} \frac{1}{n^{\beta}} \sum_{k=1}^{n} f (d (X_k, X_0)) = 0 \text{ for some } X_0 \in L (\mathbb{R}) \right\}, \\
\omega^{\beta,\infty} (F, f) & = \left\{ X \in s (F) : \sup_{n} \frac{1}{n^{\beta}} \sum_{k=1}^{n} f (d (X_k, 0)) < \infty \right\},
\end{align*}
\]

In case $f (x) = x$, the space $\omega^{\beta} (F, f)$ is the same as the space $\omega^{\beta} (F, p)$ of Altinok et al. \[1\] for $p = 1$. We shall denote the space $\omega^{\beta,\infty} (F, f)$ by $\omega^{\beta,\infty} (F)$ for $f (x) = x$. It is easy to see that $\omega^{\beta} (F, f)$, $\omega^{\beta,0} (F, f)$ and $\omega^{\beta,\infty} (F, f)$ are linear spaces over the complex field $\mathbb{C}$.

**Remark 3.3.** In Definition 3.1, although the $f$–statistical convergence of order $\beta$ is well defined for $\beta \in (0, 1]$, but not well defined for $\beta > 1$. (See Example 3.4).

**Example 3.4.** Let $f$ be an unbounded modulus function such that $\lim_{t \to \infty} \frac{f(t)}{t} > 0$ and $X = (X_k)$ be a sequence of fuzzy numbers as follows:

\[
X_k (x) = \begin{cases} 
  x + 3, & -3 \leq x \leq -2 \\
  -x - 1, & -2 \leq x \leq -1 \\
  0, & \text{otherwise} \\
  x - 1, & 1 \leq x \leq 2 \\
  -x + 3, & 2 \leq x \leq 3 \\
  0, & \text{otherwise}
\end{cases}
: X_0', \text{if } k \text{ is even}
\]

If we calculate the $\alpha$–level set of sequence $(X_k)$, then we find the set

\[
[X_k]^\alpha = \begin{cases} 
  [\alpha - 3, -1 - \alpha] : [X_0]^\alpha, & \text{if } k \text{ is odd} \\
  [\alpha + 1, 3 - \alpha] : [X_0']^\alpha, & \text{if } k \text{ is even}
\end{cases}
\]

Then we can write

\[
\frac{1}{f (n^\beta)} f (|\{k \leq n : d (X_k, X_0) \geq \varepsilon\}|) \leq \frac{f (\frac{n}{2})}{f (n^\beta)}
\]

and

\[
\frac{1}{f (n^\beta)} f (|\{k \leq n : d (X_k, X_0') \geq \varepsilon\}|) \leq \frac{f (\frac{n}{2})}{f (n^\beta)}
\]

So, we get

\[
\lim_{n \to \infty} \frac{1}{f (n^\beta)} f (|\{k \leq n : d (X_k, X_0) \geq \varepsilon\}|) = 0
\]

and

\[
\lim_{n \to \infty} \frac{1}{f (n^\beta)} f (|\{k \leq n : d (X_k, X_0') \geq \varepsilon\}|) = 0
\]
for \( \beta > 1 \) and for each \( \varepsilon > 0 \) using the property \( \lim_{t \to \infty} \frac{f(t)}{t} > 0 \). Therefore, sequence \( X = (X_k) \) of fuzzy numbers is \( f \)-statistically convergent of order \( \beta \) to both \( X_0 \) and \( X'_0 \). That is, the \( f \)-statistical limit of order \( \beta \) may not be unique for \( \beta > 1 \) (See Fig. 3.1).

Every convergent sequence of fuzzy numbers is \( f \)-statistically convergent of order \( \beta \) for any unbounded modulus function \( f \) and \( 0 < \beta \leq 1 \). But, the converse is not true, for this we can give the following example.

**Example 3.5.** Define the sequence \( X = (X_k) \) of fuzzy numbers as follows:

\[
X_k(x) = \begin{cases} 
  x + 3, & -3 \leq x \leq -2 \\
  -x - 1, & -2 \leq x \leq -1 \\
  0, & 1 \leq x \leq 2 \\
  x - 1, & 2 \leq x \leq 3 \\
  -x + 3, & 3 \leq x \leq 4 \\
  0, & \text{otherwise}
\end{cases}
\]

if \( k = n^3 \)

\[
X_k(x) = \begin{cases} 
  x + 3, & -3 \leq x \leq -2 \\
  -x - 1, & -2 \leq x \leq -1 \\
  0, & 1 \leq x \leq 2 \\
  x - 1, & 2 \leq x \leq 3 \\
  -x + 3, & 3 \leq x \leq 4 \\
  0, & \text{otherwise}
\end{cases}
\]

if \( k \neq n^3 \)

Take modulus function \( f(x) = x^p \) for \( 0 < p \leq 1 \). We can find the \( \alpha \)-level set of fuzzy sequence \( (X_k) \) as follows:

\[
[X_k]^\alpha = \begin{cases} 
  [\alpha - 3, -1 - \alpha], & \text{if } k = n^3 \\
  [\alpha + 1, 3 - \alpha], & \text{if } k \neq n^3
\end{cases}
\]

Then, the fuzzy sequence \( (X_k) \) is \( f \)-statistically convergent of order \( \beta \) for \( \beta \in \left( \frac{1}{3}, 1 \right] \), but not convergent.

**Theorem 3.6.** Let \( X = (X_k) \), \( Y = (Y_k) \) be any two fuzzy sequences and \( L_1, L_2 \) be fuzzy numbers. Also, let \( f \) be an unbounded modulus function and \( \beta \in (0, 1] \). Then

(i) If \( S^\beta(F, f) - \lim X_k = L_1 \) and \( c \in C \), then \( S^\beta(F, f) - \lim cX_k = cL_1 \).

(ii) If \( S^\beta(F, f) - \lim X_k = L_1 \) and \( S^\beta(F, f) - \lim Y_k = L_2 \), then \( S^\beta(F, f) - \lim (X_k + Y_k) = L_1 + L_2 \).

**Theorem 3.7.** Let \( f \) be unbounded modulus function and \( \beta, \gamma \) be real numbers such that \( 0 < \beta \leq \gamma \leq 1 \). Then \( S^\beta(F, f) \subset S^\gamma(F, f) \) and the inclusion is strict.

**Proof.** It can be easily shown the inclusion by using the fact that \( f \) is increasing for \( 0 < \beta \leq \gamma \leq 1 \). Now, we show that the inclusion is strict. For this, consider fuzzy sequence \( X = (X_k) \) defined by...
\[
X_k(x) = \begin{cases} 
  x + 3, & -3 \leq x \leq -2 \\
  -x - 1, & -2 \leq x \leq -1 \\
  0 & \text{otherwise} \\
  x - 1, & 1 \leq x \leq 2 \\
  -x + 3, & 2 \leq x \leq 3 \\
  0 & \text{otherwise} 
\end{cases}
\]

and take modulus function \( f(x) = x^p, 0 < p \leq 1 \). We can find the \( \alpha \)-level set of sequence \((X_k)\) as follows:

\[
[X_k]^\alpha = \begin{cases} 
  [\alpha - 3, -1 - \alpha], & \text{if } k = n^2 \\
  [\alpha + 1, 3 - \alpha], & \text{if } k \neq n^2 
\end{cases}
\]

Then, the fuzzy sequence \((X_k)\) is \( f \)-statistically convergent of order \( \gamma \) for \( \gamma \in \left(\frac{1}{2}, 1\right] \), but not \( f \)-statistically convergent of order \( \beta \) for \( \beta \in (0, \frac{1}{2}] \).

**Corollary 3.8.** Let \( X = (X_k) \) be a fuzzy sequence, \( f \) be an unbounded modulus function and \( \beta \in (0, 1] \). Then \( S^\beta(F, f) \subset S(F, f) \) and the inclusion is strict, also the limits of sequence \( X = (X_k) \) of fuzzy numbers are same.

We have the following theorem from Remark 2.4.

**Theorem 3.9.** Let \( f \) be an unbounded modulus function and \( \beta \in (0, 1] \). Then

(i) \( S^\beta(F, f) \subset S^\beta(F) \) and the inclusion is strict.

(ii) \( S^\beta(F, f) \subset S(F) \) and the inclusion is strict.

**Proof.** To show that the strictness of inclusion, consider the fuzzy sequence \( X = (X_k) \) defined as follows

\[
X_k(x) = \begin{cases} 
  x - k + 1, & k - 1 \leq x \leq k \\
  -x + k + 1, & k \leq x \leq k + 1 \\
  0 & \text{otherwise} 
\end{cases}
\]

The \( \alpha \)-level set of sequence \((X_k)\) is

\[
[X_k]^\alpha = \begin{cases} 
  [\alpha + k - 1, k + 1 - \alpha], & \text{if } k = n^3 \\
  0, & \text{otherwise} 
\end{cases}
\]

Take modulus function \( f(x) = \ln(x + 1) \). Then we see that the sequence \( X = (X_k) \) is statistically convergent of order \( \beta \) for \( \beta \in \left(\frac{1}{2}, 1\right] \) and so it is statistically convergent (See Fig. 3.2). However, \( X = (X_k) \) is not \( f \)-statistically convergent of order \( \beta \) since

\[
d^f_\beta \left( \{ k \in \mathbb{N} : d(X_k, \bar{0}) \geq \varepsilon \} \right) \geq d^f \left( \{ k \in \mathbb{N} : d(X_k, \bar{0}) \geq \varepsilon \} \right) = \frac{1}{3} \neq 0.
\]
Now, we give some results related to spaces \(w^\beta (F, f)\), \(w^{\beta, 0} (F, f)\) and \(w^{\beta, \infty} (F, f)\) introduced in Definition 3.2.

**Remark 3.10.** We didn’t allowed \(\beta\) to exceed 1 in the \(w^\beta (F,p)\) of Altinok et al. [4], but we consider \(\beta\) as any positive real number and it can exceed 1 in the spaces \(w^{\beta, 0} (F, f)\) and \(w^\beta (F, f)\).

**Theorem 3.11.** Let \(f\) be any modulus function. Then

(i) \(w^{\beta, 0} (F, f) \subset w^{\beta, \infty} (F, f)\) for \(\beta > 0\),

(ii) \(w^\beta (F, f) \subset w^{\beta, \infty} (F, f)\) for \(\beta \geq 1\).

**Proof.** The proof of (i) is trivial, so we give the proof of (ii). For this, take \(\beta \geq 1\) and any fuzzy sequence \(X = (X_k)\) in the space \(w^\beta (F, f)\). Then, we obtain

\[
\frac{1}{n^\beta} \sum_{k=1}^{n} f(d(X_k, 0)) \leq \frac{1}{n^\beta} \sum_{k=1}^{n} f(d(X_k, X_0)) + f(d(X_0, 0)) \frac{1}{n^\beta} \sum_{k=1}^{n} 1,
\]

from definition of modulus function, and so we have \(X \in w^{\beta, \infty} (F, f)\).

**Theorem 3.12.** Let \(f\) be any modulus function and \(\beta \geq 1\). Then, we have the inclusion relations \(w^\beta (F) \subset w^\beta (F, f)\), \(w^{\beta, 0} (F) \subset w^{\beta, 0} (F, f)\) and \(w^{\beta, \infty} (F) \subset w^{\beta, \infty} (F, f)\).

**Proof.** We shall prove the inclusion \(w^{\beta, \infty} (F) \subset w^{\beta, \infty} (F, f)\) since the proofs of first two inclusion relations are easy. For this, take \(\beta \geq 1\) and any fuzzy sequence \(X = (X_k)\) in the space \(w^{\beta, \infty} (F)\) so that

\[
\sup_n \frac{1}{n^\beta} \sum_{k=1}^{n} d(X_k, \bar{0}) < \infty.
\]

Given \(\varepsilon > 0\) and \(\delta \in (0, 1)\) such that \(f(t) < \varepsilon\) for \(t \in (0, \delta]\). Consider

\[
\frac{1}{n^\beta} \sum_{k=1}^{n} f(d(X_k, 0)) = \sum_1^1 + \sum_2^2,
\]

where \(\sum_1^1\) is over \(d(X_k, \bar{0}) \leq \delta\) and \(\sum_2^2\) is over \(d(X_k, \bar{0}) > \delta\). Then \(\sum_1^1 \leq \varepsilon \frac{1}{n^{\beta-1}}\) and we can write

\[
d(X_k, 0) < \frac{d(X_k, \bar{0})}{\delta} < 1 + \left\lfloor \frac{d(X_k, \bar{0})}{\delta} \right\rfloor,
\]

for \(d(X_k, \bar{0}) > \delta\), where \(\lfloor d(X_k, \bar{0}) / \delta \rfloor\) denotes the integer part of \(d(X_k, \bar{0}) / \delta\). Therefore, for \(d(X_k, \bar{0}) > \delta\), we obtain

\[
f(d(X_k, \bar{0})) \leq \left(1 + \left\lfloor \frac{d(X_k, \bar{0})}{\delta} \right\rfloor \right) f(1) \leq 2 f(1) \frac{d(X_k, \bar{0})}{\delta}
\]

from definition of modulus function. Hence we get inequality

\[
\sum_2^2 \leq 2 f(1) \delta^{-1} \frac{1}{n^\beta} \sum_{k=1}^{n} d(X_k, \bar{0})
\]

which together with \(\sum_1^1 \leq \varepsilon \frac{1}{n^{\beta-1}}\) yields

\[
\frac{1}{n^\beta} \sum_{k=1}^{n} f(d(X_k, \bar{0})) \leq \varepsilon \frac{1}{n^{\beta-1}} + 2 f(1) \delta^{-1} \frac{1}{n^\beta} \sum_{k=1}^{n} d(X_k, \bar{0})
\].

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Finally, we have \((X_k) \in w^{β,∞} (F, f)\) since \(β \geq 1\) and \((X_k) \in w^{β,∞} (F)\) which completes the proof.

**Theorem 3.13.** Let \(f\) be a modulus function such that \(\lim_{t \to ∞} \frac{f(t)}{t} > 0\) and \(β\) be a positive real number. Then \(w^β (F, f) \subset w^β (F)\).

**Proof.** Let \(X = (X_k)\) be a sequence of fuzzy numbers and \((X_k) \in w^{β} (F, f)\). It is known that \(\lim_{t \to ∞} \frac{f(t)}{t} = \inf \left\{ \frac{f(t)}{t} : t > 0 \right\}\). We denote the value of \(\lim_{t \to ∞} \frac{f(t)}{t}\) by \(ℓ\) for shortness. Thus, we can write \(f(t) ≥ ℓt\) and \(t ≤ ℓ^{-1} f(t)\) for all \(t ≥ 0\) since \(ℓ > 0\), and hence

\[
\frac{1}{n^β} \sum_{k=1}^{n} d (X_k, X_0) ≤ ℓ^{-1} \frac{1}{n^β} \sum_{k=1}^{n} f (d (X_k, X_0)) .
\]

Thus, we have \((X_k) \in w^β (F)\).

We have the following result from Theorem 3.12 and Theorem 3.13.

**Corollary 3.14.** Let \(f\) be a modulus function. If \(\lim_{t \to ∞} \frac{f(t)}{t} > 0\) and \(β \geq 1\), then \(w^β (F, f) = w^β (F)\).

**Theorem 3.15.** Let \(f\) be a modulus function and \(0 < β ≤ γ\). Then \(w^β (F, f) \subset w^γ (F, f)\) and the inclusion is strict.

**Proof.** It is easy to show the inclusion relation. For strictness of inclusion, let \(f\) be a modulus function and consider the fuzzy sequence \(X = (X_k)\) defined by

\[
X_k (x) = \begin{cases} 
    x - 1, & 1 \leq x \leq 2 \\
    -x + 3, & 2 < x \leq 3 \\
    0, & \text{otherwise}
\end{cases}
\], if \(k = n^3\)

\[
0, \quad \text{if} \quad k \neq n^3
\]

We can write

\[
\frac{1}{n^γ} \sum_{k=1}^{n} f (d (X_k, 0)) ≤ \frac{\sqrt{n}}{n^γ} f (3) = \frac{1}{n^{γ - \frac{3}{2}}} f (3)
\]

using the property \(f (0) = 0\) for every \(n ∈ \mathbb{N}\). So, we have \((X_k) \in w^γ (F, f)\) since the right side tends to zero for \(γ > \frac{1}{3}\) as \(n \to ∞\). On the other hand, we obtain

\[
\frac{1}{n^β} \sum_{k=1}^{n} f (d (X_k, 0)) ≥ \frac{\sqrt{n} - 3}{n^β} f (3)
\]

for every \(n ∈ \mathbb{N}\). Hence we have \((X_k) \notin w^β (F, f)\) since the right side tends to infinity for \(0 < γ < \frac{1}{3}\) as \(n \to ∞\).

**Theorem 3.16.** Let \(X = (X_k)\) be a sequence of fuzzy numbers and \(0 < β ≤ γ ≤ 1\). Also, let \(f\) be an unbounded modulus such that there is a positive constant \(c\) such that \(f (xy) ≥ cf (x) f (y)\) for all \(x ≥ 0, y ≥ 0\) and \(\lim_{t \to ∞} \frac{f(t)}{t} > 0\). If a sequence \(X = (X_k)\) of fuzzy numbers is strongly Cesaro summable of order \(β\) with respect to modulus function \(f\) to fuzzy number \(X_0\), then it is \(f\)–statistically convergent of order \(γ\) to fuzzy number \(X_0\).

**Proof.** Take any fuzzy sequence \(X = (X_k)\) and \(ε > 0\). Then, we have

\[
\sum_{k=1}^{n} f (d (X_k, X_0)) ≥ f \left( \sum_{k=1}^{n} d (X_k, X_0) \right)
\]

\[
≥ f \left( \left\{ \{k ≤ n : d (X_k, X_0) ≥ ε\} | ε \right\} \right)
\]

\[
≥ cf \left( \left\{ \{k ≤ n : d (X_k, X_0) ≥ ε\} \right\} f (ε)
\]

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by the definition of modulus function, and so we can write
\[
\frac{1}{n^\beta} \sum_{k=1}^{n} f\left(d\left(X_k, X_0\right)\right) \geq \frac{c f \left(\left\{k \leq n : d\left(X_k, X_0\right) \geq \varepsilon\right\}\right) f\left(\varepsilon\right)}{n^\beta} \\
\geq \frac{c f \left(\left\{k \leq n : d\left(X_k, X_0\right) \geq \varepsilon\right\}\right) f\left(\varepsilon\right)}{n^\gamma} \\
= \frac{c f \left(\left\{k \leq n : d\left(X_k, X_0\right) \geq \varepsilon\right\}\right) f\left(\varepsilon\right) f\left(n^\gamma\right)}{n^\gamma f\left(n^\gamma\right)}
\]
since $\beta \leq \gamma$. Hence, it is easy to see that $X \in S^\gamma \left(F, f\right)$ using the fact that $\lim_{t \to \infty} \frac{L\left(t\right)}{t} > 0$ and $X \in w^\beta \left(F, f\right)$.

We have the following result if we take $\beta = \gamma = 1$ in Theorem 3.16.

**Corollary 3.17.** Let $f$ satisfies conditions in Theorem 3.16. If a sequence $X = (X_k)$ of fuzzy numbers is strongly Cesaro summable with respect to modulus function $f$ to fuzzy number $X_0$, then it is $f$–statistically convergent to fuzzy number $X_0$.

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