INCREASING POSITIVE MONOIDS OF
ORDERED FIELDS ARE FF-MONOIDS

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Abstract. Given an ambient ordered field $K$, a positive monoid is a countably
generated additive submonoid of the nonnegative cone of $K$. In this paper, we first
generalize a few atomic features exhibited by Puiseux monoids of the field of rational
numbers to the more general setting of positive monoids of Archimedean fields,
accordingly arguing that such generalizations might fail if the ambient field is not
Archimedean. In particular, we show that every positive monoid of an Archimedean
field is a BF-monoid provided that it does not have zero as a limit point. Then, we
prove our main result: every increasing positive monoid of an ambient ordered field
is an FF-monoid. Finally, we deduce that every positive monoid is hereditarily atomic.

1. Introduction

The family of Puiseux monoids was introduced in [9], where the atomic structure
of its members was studied. Puiseux monoids are additive submonoids of $\mathbb{Q}_{\geq 0}$. They
exhibit a very complex atomic structure. Indeed, there are nontrivial Puiseux monoids
having no irreducible elements at all (i.e., being antimatter), while others, failing to
be atomic, contain infinitely many irreducible elements.

In this paper, we generalize the notion of Puiseux monoid of $\mathbb{Q}$ by considering certain
additive submonoids of the nonnegative rational cone of an arbitrary ordered field. In
fact, we will study the atomic structure of an even more general family of commutative
monoids.

Definition 1.1. Let $K$ be an ordered field. A positive monoid of $K$ is a countably
generated additive submonoid of the nonnegative cone of $K$.

If $P$ is a positive monoid of an ordered field $K$, we say that $K$ is an ambient field
for $P$. Every Puiseux monoid is, therefore, a positive monoid of the ambient field $\mathbb{Q}$.
In [9] and [10], many techniques were introduced to understand the atomic structure
of Puiseux monoids. Here we will modify various of these results, providing the appro-
priate conditions for them to hold in the more general context of positive monoids of
an arbitrary ordered field. Furthermore, we study the family of positive monoids that
can be generated by increasing sequences. As our main result, we prove that every
increasing positive monoid of an ordered field is an FF-monoid. After verifying that

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every submonoid of a positive BF-monoid is atomic, we obtain, as a consequence of our main result, that if a positive monoid is increasing, then all its submonoids are atomic.

Our main result, which is formally stated below, is much easier to be proved in the particular case when the ambient field is assumed to be Archimedean. We present this weaker version in Proposition 5.6 as our first approach the main theorem:

**Main Theorem.** Every increasing positive monoid of an ambient ordered field is an FF-monoid.

The outline of this paper is as follows. In Section 2 we establish the nomenclature of commutative monoids as well as the terminology for the basic notions related to their atomicity and factorization theory. Then, in Section 3 we go over the fundamental concepts of ordered fields we will be using throughout this paper. In Section 4 among other minor results, we describe those positive monoids that are isomorphic to Puiseux monoids. In the same section, we also present Proposition 4.5 and Proposition 4.7 two results on Puiseux monoids that naturally generalize to positive monoids of Archimedean ordered fields, but that fail when the Archimedean condition is dropped. In Section 5 we study the atomic structure of increasing positive monoids. We will argue that a positive monoid is a BF-monoid provided that it does not have 0 as a limit point. In addition, we present the Archimedean version of our main theorem. The last section is dedicated to the main result. We present its proof in the first part of the section. Finally, we deduce from our main theorem that every increasing positive monoid is hereditarily atomic.

## 2. Atomicity and Factorization on Commutative Monoids

This section contains basic terminology concerning the atomicity and factorization theory of commutative monoids. Here we also introduce notation for two special families of commutative monoids that will appear systematically in this sequel: numerical semigroups and Puiseux monoids. For extensive background information on commutative semigroups and non-unique factorization theory, we refer readers to the monographs [11] of Grillet and [7] of Geroldinger and Halter-Koch, respectively.

We use the double-struck symbol \( \mathbb{N} \) to denote the set of positive integers. If \( R \subseteq \mathbb{R} \) and \( r \in R \), then \( R_0 \) and \( R_{\geq r} \) denote the sets \( R \cup \{0\} \) and \( \{x \in R \mid x \geq r\} \), respectively. With a similar intension we use \( R_{\leq r} \), \( R_{> r} \), and \( R_{< r} \). If \( q \in \mathbb{Q}_{>0} \), then the unique \( a, b \in \mathbb{N} \) such that \( q = a/b \) and \( \gcd(a, b) = 1 \) are denoted by \( n(q) \) and \( d(q) \), respectively. For \( Q \subseteq \mathbb{Q}_{>0} \), the sets

\[
 n(Q) = \{ n(q) \mid q \in Q \} \quad \text{and} \quad d(Q) = \{ d(q) \mid q \in Q \}
\]

are called numerator and denominator set of \( Q \), respectively. Finally, for a set \( S \) we will write sometimes \( \{ s_n \} \in S^\infty \) when \( \{ s_n \} \) is a sequence whose terms are in \( S \).
For the sake of simplicity, we make the convention in this paper that the use of the word *monoid* by itself implies that the object in question is a commutative cancellative monoid. For the rest of this section, let $M$ be a monoid. Because every monoid here is assumed to be commutative, we will use additive notation. The set $M \setminus \{0\}$ is denoted by $M^*$ while the set of units of $M$ is denoted by $M^\times$. The monoid $M$ is said to be reduced if $M^\times$ contains only the identity element. All monoids we will be dealing with are reduced. For $a, c \in M$, we say that $a$ divides $c$ in $M$ and write $a \mid_M c$ if $c = a + b$ for some $b \in M$. A submonoid $N$ of $M$ is said to be divisor-closed if for every $a \in N$ and $d \in M$ the fact that $d \mid_M a$ implies that $d \in N$. We write $M = \langle S \rangle$ when $M$ is generated by a set $S$. The monoid $M$ is *finitely generated* if it can be generated by a finite set. If $M$ is finitely generated, then it is finitely presented; this result is known as Rédéi’s theorem. A succinct exposition of finitely generated commutative monoids can be found in [5] by García-Sánchez and Rosales.

An element $a \in M \setminus M^\times$ is irreducible or an atom if $a = x + y$ for $x, y \in M$ implies that either $x$ is a unit or $y$ is a unit. The set of atoms of $M$ is denoted by $\mathcal{A}(M)$, and $M$ is called atomic if $M = \langle \mathcal{A}(M) \rangle$. By contrast, $M$ is said to be antimatter if $\mathcal{A}(M)$ is empty. Antimatter domains and monoids were first defined in [11] and [9], respectively.

Assume that $M$ is reduced. The free abelian monoid on $\mathcal{A}(M)$ is denoted by $Z(M)$ and called factorization monoid of $M$; the elements of $Z(M)$ are called factorizations. If $z = a_1 \ldots a_n \in Z(M)$ for some $n \in \mathbb{N}_0$ and $a_1, \ldots, a_n \in \mathcal{A}(M)$, then $n$ is the length of the factorization $z$, commonly denoted by $|z|$; we say that an atom $a$ shows in $z$ if $a \in \{a_1, \ldots, a_n\}$. The unique homomorphism 

$$\phi: Z(M) \to M \text{ satisfying } \phi(a) = a \text{ for all } a \in \mathcal{A}(M)$$

is called the factorization homomorphism of $M$. For $x \in M$,

$$Z(x) = \phi^{-1}(x) \subseteq Z(M)$$

is the set of factorizations of $x$. If $x \in M$ satisfies $|Z(x)| = 1$, then we say that $x$ has unique factorization. By definition, we set $Z(0) = \{0\}$. Note that the monoid $M$ is atomic if and only if $Z(x)$ is not empty for all $x \in M$. The monoid $M$ satisfies the finite factorization property if for all $x \in M$ the set $Z(x)$ is finite; in this case we also say that $M$ is an FF-monoid. The next proposition, which we will refer recurrently here, follows from [8] Theorem 3.1.4).

**Proposition 2.1.** Every finitely generated atomic monoid is an FF-monoid.

For each $x \in M$, the set of lengths of $x$ is defined by

$$L(x) = \{|z| : z \in Z(x)\}.$$  

If $L(x)$ is a finite set for all $x \in M$, we say that $M$ satisfies the bounded factorization property, in which case, we call $M$ a BF-monoid. Proposition 2.1 says, in particular, that every finitely generated atomic monoid is a BF-monoid.
A numerical semigroup is a cofinite submonoid of the additive monoid \( \mathbb{N}_0 \). Every numerical semigroup has a unique minimal set of generators, which is finite. For a numerical semigroup \( N \) minimally generated by the positive integers \( a_1, \ldots, a_n \), we have that \( \gcd(a_1, \ldots, a_n) = 1 \) and \( \mathcal{A}(N) = \{a_1, \ldots, a_n\} \). Thus, every numerical semigroup is an atomic monoid containing finitely many atoms. A great first approach to the realm of numerical semigroups can be found in [6].

A Puiseux monoid is an additive submonoid of \( \mathbb{Q}_{\geq 0} \). Albeit a natural generalization of numerical semigroups, Puiseux monoids are not always atomic. However, a Puiseux monoid is atomic provided it does not have 0 as a limit point. We say that a Puiseux monoid is monotone if it can be generated by a monotone sequence of rationals. The atomicity of Puiseux monoids was the center of attention in [9] and [10].

3. Ordered Fields

In this section, we briefly recall some concepts related to ordered fields as a way to establish the nomenclature we will be using later. For ordered fields we mostly follow the notation in [2]. In addition, in [13, Chapters 11 and 12], readers can find the rudiments on ordered fields we will assume in this sequel.

Let \( K \) be an ordered field. Since \( K \) has characteristic zero, its prime subfield is isomorphic to \( \mathbb{Q} \). We denote by \( K^+ \) the nonnegative cone (i.e., the set of nonnegative elements) of \( K \). For each \( x \in K \) set \( |x| = x \) if \( x \in K^+ \) and \( |x| = -x \) otherwise. We write \( x = O(y) \) if \( |x| \leq n|y| \) for some \( n \in \mathbb{N} \), and \( x \sim y \) if both \( x = O(y) \) and \( y = O(x) \) hold. Clearly, \( \sim \) defines an equivalence relation on \( K^\times \). Let

\[
\alpha: K \rightarrow \Gamma_K = K^\times / \sim
\]

be the quotient map. Setting \( \alpha(x) \preceq \alpha(y) \) when \( y = O(x) \), one finds that \( (\Gamma_K, \preceq) \) is a well-defined totally ordered set. Moreover, the multiplication of \( K \) induces a group structure on \( \Gamma_K \) under which \( \Gamma_K \) is a totally ordered group. The group \( \Gamma_K \) is the value group of \( K \). The elements of \( \Gamma_K \) are called Archimedean classes, and the quotient map \( \alpha: K \rightarrow \Gamma_K \) is called Archimedean valuation.

An element \( a \in K \) is finite if \( a = O(1) \), while \( a \) is called infinitesimal (resp., infinitely large) if \( |a| < 1/n \) (resp., \( |a| > n \)) for every natural \( n \). Obviously, \( K \) contains nonzero infinitesimals if and only if it contains infinitely large elements. The set of infinitesimals of \( K \) is denoted by \( K_0 \), while the set of finite elements is denoted by \( K_{\neq} \); they are both additive subgroups of \( K \). The field \( K \) is said to be Archimedean if \( K_0 = \{0\} \). Note that \( K \) is Archimedean if and only if its value group \( \Gamma_K \) is trivial; readers can find 42 equivalent definitions of Archimedean ordered field in [3, Section 4].

The order topology on \( K \) has a basis consisting of all the intervals of the forms \((a, b)\), \((-\infty, b)\), and \((a, \infty)\), where \( a, b \in K \). The field \( K \) is Archimedean if and only if its prime subfield is order-theoretically dense in \( K \) and, in such a case, \( K \) is isomorphic as
an ordered field to a subfield of $\mathbb{R}$. Moreover, $K$ is a Hausdorff topological group (under addition) and a completely regular space; see [4, Lemma 2.1]. The field $K$ is said to be complete if every Cauchy sequence converges. Every ordered field can be densely order-embedded in a complete ordered field. There are many equivalent definitions of completeness; 72 of them are given in [3, Section 3].

**Example 3.1.** Let $K$ be an ordered field, and let $K(X)$ be the field of rational functions over $K$. If $p(X) \in K[X]$ is a nonzero polynomial, then let $\ell(p(X))$ denote its leading coefficient. Now set

$$
K(X)^+ = \{0\} \cup \left\{ \frac{p(X)}{q(X)} \bigg| p(X), q(X) \in K[X] \setminus \{0\} \text{ and } \frac{\ell(p(X))}{\ell(q(X))} > 0 \right\}
$$

and check that $K(X)^+$ is indeed a nonnegative cone making $K(X)$ an ordered field. The ordered field $K(X)$ is not Archimedean for $1/X$ (resp., $X$) is infinitesimal (resp., infinitely large). For another non-Archimedean ordering on $K(X)$, see [12, Example 2.5]. In this paper we always consider $K(X)$ as an ordered field with the nonnegative cone given in (3.1).

### 4. FROM PUISEUX MONOIDS TO POSITIVE MONOIDS

We recall that a positive monoid $P$ of an ambient ordered field $K$ is a countably generated additive submonoid of the nonnegative cone of $K$. We begin this section describing the positive monoids that are isomorphic to Puiseux monoids. Then we restate two properties of Puiseux monoids, [9, Theorem 3.10] and [10, Theorem 3.9], but in the more general context of positive monoids of an Archimedean ambient field, and we verify that these results do not hold when the ambient field fails to be Archimedean. First, let us generalize the concept of Puiseux monoid.

**Definition 4.1.** Let $K$ be an ordered field. A *Puiseux monoid* of $K$ is a positive monoid that is contained in the prime subfield of $K$.

Since we can always identify the prime subfield of an ordered field with the field of rational numbers, our definition of Puiseux monoid is consistent with that one given in Section 2. It follows immediately that a Puiseux monoid is isomorphic to a numerical semigroup if and only if it is finitely generated. It is natural to wonder when a positive monoid is isomorphic to a Puiseux monoid. Notice that if $P$ is a positive monoid of an ambient ordered field $K$, then so is $aP$ for all $a \in K^+$. The next proposition classifies those positive monoids that are isomorphic to either Puiseux monoids or numerical semigroups.
Proposition 4.2. Let $K$ be an ordered field, and let $P$ be a positive monoid of $K$. Then the following statements hold.

1. $P$ is isomorphic to a Puiseux monoid of $K$ if and only if there exists $a \in K_{>0}$ such that $aP$ is a Puiseux monoid.
2. $P$ is isomorphic to a numerical semigroup if and only if $P$ is finitely generated and $aP$ is a Puiseux monoid for some $a \in K_{>0}$.

Proof. To show (1), suppose that $P$ is isomorphic to a Puiseux monoid $Q = \langle q_n | n \in \mathbb{N} \rangle$ via the isomorphism $\varphi: Q \to P$, where $\{q_n\}$ is a sequence of positive rationals. The submonoid $N = \mathbb{N}_0 \cap Q$ is finitely generated, say $N = \langle n_1, \ldots, n_k \rangle$ for some $k \in \mathbb{N}_0$ and $n_1, \ldots, n_k \in \mathbb{N}$. For every $i \in \{1, \ldots, k\}$ we have

$$\varphi(n_i) = \frac{1}{n_1} \varphi(n_1 n_i) = \frac{n_i}{n_1} \varphi(n_1).$$

Since $\varphi$ is injective, $\varphi(n_1) \neq 0$. Set $a = n_1/\varphi(n_1)$. If $q \in Q^*$, then $n(q) \in N$. Therefore there exist coefficients $c_1, \ldots, c_k \in \mathbb{N}_0$ such that $n(q) = c_1 n_1 + \cdots + c_k n_k$. As a result, one obtains that

$$d(q)\varphi(q) = \varphi(n(q)) = \varphi \left( \sum_{i=1}^{k} c_i n_i \right) = \sum_{i=1}^{k} c_i \varphi(n_i) = \sum_{i=1}^{k} c_i n_i a^{-1} = a^{-1} n(q).$$

Thus, $\varphi(q) = a^{-1} q$ for every $q \in Q$. Since $\varphi$ is surjective $a^{-1} Q = P$, which means that $aP$ is the Puiseux monoid $Q$.

Conversely, suppose that $aP$ is a Puiseux monoid for some $a \in K_{>0}$. Since multiplication by $a$ defines an isomorphism from $P$ to $aP$, it follows immediately that $P$ is isomorphic to a Puiseux monoid.

Now let us verify (2). If $P$ is isomorphic to a numerical semigroup, then it is finitely generated. Since every numerical semigroup is in particular a Puiseux monoid, $P$ is isomorphic to a Puiseux monoid. By part (1), there exists $a \in K_{>0}$ such that $aP$ is a Puiseux monoid. Finally, let us check the reverse implication of (2). Since $aP$ is a Puiseux monoid for some $a \in K_{>0}$, the positive monoid $P$ is isomorphic to a Puiseux monoid. Because finitely generated Puiseux monoids are isomorphic to numerical semigroups, the proof is done. $\Box$

The next three propositions establish sufficient conditions for positive and Puiseux monoids to be atomic.

Proposition 4.3. Let $P$ be a positive monoid of an ordered field. Then $P$ contains a minimal generating set $A$ if and only if $P$ is atomic with $A = A(P)$; in such a case, $A$ is the unique minimal generating set of $P$.

The above proposition follows immediately from the fact that every positive monoid of an ordered field is reduced (see [8 Proposition 1.1.7]).
Let $K$ be an ordered field, and let $Q$ be the prime subfield of $K$. After identifying $Q$ with $\mathbb{Q}$, it makes sense to talk about primes, naturals, and integers in $Q$. For a prime $p$, recall that the $p$-adic valuation on $Q$ is the map defined by $\nu_p(0) = \infty$ and $\nu_p(a/b) = \nu_p(a) - \nu_p(b)$ for all nonzero integers $a$ and $b$, where $\nu_p(z)$ is the exponent of the maximal power of $p$ dividing the integer $z$. We say that a Puiseux monoid $P$ of $K$ is finite if there is a finite subset $S$ of $Q^+$ consisting of primes such that $\nu_p(x) \geq 0$ for every $x \in P^*$ and $p \notin S$. It is not hard to argue the following proposition.

**Proposition 4.4.** Let $P$ be a Puiseux monoid of an ordered field. Then $P$ is finite and $\{\nu_p(P)\}$ is bounded from below for every prime $p$ if and only if $d(P^*)$ is bounded. Moreover, if one of these conditions holds, then $P$ is atomic.

If $K$ is an Archimedean field, then it contains a natural copy of the integers, which we identify with $\mathbb{Z}$. Moreover, for every $x \in K$ there exists a unique integer $n_x$ such that $n_x \leq x < n_x + 1$. Hence the floor and ceiling functions make sense in the context of Archimedean fields. The next proposition generalizes [9, Theorem 3.10], which says that if 0 is not a limit point of a Puiseux monoid $P$ of $\mathbb{R}$, then $P$ is atomic.

**Proposition 4.5.** Let $P$ be a Puiseux monoid of an Archimedean ordered field. If 0 is not a limit point of $P$, then $P$ is a BF-monoid.

**Proof.** Let $K$ denote the ambient field of $P$. It is clear that the set $A(P)$ consists of those elements of $P^*$ that cannot be written as the sum of two positive elements of $P$. Since 0 is not a limit point of $P$ there exists $\epsilon \in K_{>0}$ such that $\epsilon < x$ for all $x \in P^*$. Now we show that $P = \langle A(P) \rangle$. Take $x \in P^*$. Since $\epsilon$ is a lower bound for $P^*$, the element $x$ can be written as the sum of at most $[x/\epsilon]$ elements of $P^*$. Take the maximum natural $m$ such that $x = a_1 + \cdots + a_m$ for some $a_1, \ldots, a_m \in P^*$. By the maximality of $m$, it follows that $a_i \in A(P)$ for each $i = 1, \ldots, m$, which means that $x \in \langle A(P) \rangle$. Hence $P$ is atomic. We have already noticed that every element $x$ in $P^*$ can be written as the sum of at most $[x/\epsilon]$ positive elements, i.e., $|L(x)| \leq [x/\epsilon]$ for all $x \in P$. Thus, $P$ is a BF-monoid.

In the next section, we will prove that every increasing positive monoid of an Archimedean field is an FF-monoid, a special version of our main theorem. However, under the hypothesis of Proposition 4.5, we cannot always guarantee that $P$ is an FF-monoid. Let us exemplify this observation.

**Example 4.6.** Let $\{p_n\}$ be an enumeration of the prime numbers. Consider the Puiseux monoid $P$ of $\mathbb{R}$ generated by the set

$$A = \left\{ \frac{p_n + |p_n/2|}{p_n}, \frac{2p_n - |p_n/2|}{p_n} \middle| n \in \mathbb{N} \right\}.$$  

Since $1 < a < 2$ for every $a \in A$, it follows that $A(P) = A$ and, therefore, $P$ is atomic. As $a > 1$ for each $a \in A(P)$, we see that 0 is not a limit point of $P$. However, for every
As we work on the more general setting of positive monoids of an arbitrary ordered field, the potential inclusion of infinitesimals might cause the failure of some properties showing in the more particular scenario of Puiseux monoids of Archimedean fields. For instance, let us see that the Archimedean condition in Proposition [14,15] is required. Let $K$ be a non-Archimedean ordered field, and let $\epsilon$ be an infinitesimal of $K$. So $\epsilon \leq r$ for all $r$ in the positive cone $Q_{>0}$ of the prime subfield. Since $Q_{>0} \cap (-\epsilon, \epsilon)$ is empty, $0$ is not a limit point of the positive monoid $Q^+$. On the other hand, $Q^+$ is not atomic; indeed, $Q^+$ is antimatter because every $q \in Q_{>0}$ is divisible by $q/2$.

The fact that a Puiseux/positive monoid is strongly increasing depends on the ambient ordered field it is embedded into. For example, if $\{p_n\}$ is an increasing enumeration of the prime numbers, then

\[(4.1) \quad P = \left\langle \frac{p_n^2 + 1}{p_n} \mid n \in \mathbb{N} \right\rangle\]

is strongly increasing as a positive monoid of $\mathbb{R}$, but it is not strongly increasing as a positive monoid of the field of rational functions $\mathbb{R}(X)$. The existence of such properties depending on the embedding also refrains some standard results for Puiseux monoids of $\mathbb{Q}$ from generalizing to positive monoids of arbitrary ambient ordered fields. It was proved in [10, Section 3] that a Puiseux monoid $P$ of $\mathbb{Q}$ is strongly increasing if and only if every submonoid of $P$ is increasing; the proof given there can be mimicked to establish the proposition below.

**Proposition 4.7.** Let $P$ be a positive monoid of an Archimedean ordered field. Then $P$ is strongly increasing if and only if every submonoid of $P$ is increasing.

Proposition 4.7 constitutes another property holding for positive monoids of an Archimedean ordered field that will no longer be true if we drop the Archimedean condition. In fact, both implications might fail if the ambient field is not Archimedean. The next two examples shed light on this observation.

**Example 4.8.** Let $\mathbb{R}(X)$ be the field of rational functions regarded as an ordered field with the nonnegative cone given in Example 3.1. Consider its positive monoids

\[ P = \langle X^n \mid n \in \mathbb{N} \rangle \quad \text{and} \quad P' = \langle X^3, X + nX^2 \mid n \in \mathbb{N} \rangle. \]

Note that $P$ is strongly increasing in $\mathbb{R}(X)$ and $P'$ is a submonoid of $P$. We verify now that $P'$ is not increasing. Since $X^3 \notin \langle X + nX^2 \mid n \in \mathbb{N} \rangle$, it is an atom of $P'$. Take
\[ X + nX^2 = \alpha X^3 + \sum_{k=1}^{n} \alpha_k X + \alpha_k kX^2 \]

(note that, in the above sum, it is enough to add just up to \( n \)). The displayed polynomial equality forces \( \alpha = 0 \) and \( \alpha_1 + \cdots + \alpha_n = 1 \). Thus, there exists \( j \in \{1, \ldots, n\} \) such that \( \alpha_j = 1 \) and \( \alpha_i = 0 \) for \( i \neq j \), which implies that \( X + nX^2 \in A(P) \). Therefore the set of atoms of \( P' \) is \( \{X^3\} \cup \{X + nX^2 \mid n \in \mathbb{N}\} \). Since there are infinitely many elements in \( A(P) \) that are less than \( X^3 \), the set of atoms of \( P' \) is not the underlying set of any increasing sequence. Hence \( P' \) cannot be generated by any increasing sequence. As a consequence, \( P' \) fails to be an increasing positive monoid.

Example 4.9. Consider again the field of rational functions \( \mathbb{R}(X) \) with the ordering given in Example 3.1. Take \( \{p_n\} \) to be an increasing enumeration of the set of primes, and take \( P \) to be the Puiseux monoid of \( \mathbb{R}(X) \) given in (4.1). As we have already mentioned, \( P \) is a strongly increasing Puiseux monoid of \( \mathbb{R} \). So it follows by [10, Theorem 3.9] that every submonoid of \( P \) is increasing in \( \mathbb{R} \) and, therefore, in \( \mathbb{R}(X) \). However, \( P \) is not strongly increasing as a positive monoid of \( \mathbb{R}(X) \); this is because \( X \) is an upper bound for \( P \). Hence a positive monoid might fail to be strongly increasing even when all its submonoids are increasing.

5. Increasing Positive Monoids

The family of monotone Puiseux monoids of the ambient field \( \mathbb{Q} \) was studied in [10]. In this section we extend several results achieved in [10] to positive monoids of more general ordered fields. As in the case of Puiseux monoids, we say that a positive monoid is monotone if it can be generated by a monotone sequence.

Definition 5.1. A positive monoid of an ordered field is increasing (resp., decreasing) if it can be generated by an increasing (resp., decreasing) sequence.

If \( P \) is an increasing positive monoid of an ordered field \( K \), then \( P \) is atomic and if \( \{a_n\} \) is an increasing sequence generating \( P \), then \( A(P) = \{a_n \mid a_n \notin \langle a_1, \ldots, a_{n-1} \rangle\} \). This was proved in [10, Proposition 3.2] for \( K = \mathbb{Q} \); the proof given there applies, mutatis mutandis, when \( K \) is an arbitrary ordered field. Let us record this observation for further references.

Proposition 5.2. Let \( K \) be an ordered field, and let \( P \) be the positive monoid of \( K \) generated by the increasing sequence \( \{a_n\} \). Then \( P \) is atomic and

\[ A(P) = \{a_n \mid a_n \notin \langle a_1, \ldots, a_{n-1} \rangle\} \]

We say that a countable subset \( S \) of an ordered field is increasing (resp., decreasing) if it is the underlying set of an increasing (resp., decreasing) sequence. If \( S \) is either increasing or decreasing, then we say that it is monotone.
Lemma 5.3. A countable subset of an ordered field is both increasing and decreasing if and only if it is finite.

Proof. Let $K$ be the ordered field, and let $S$ be a countable subset of $K$. Suppose first that $S$ is increasing and decreasing. The fact that $S$ is decreasing implies that $S$ has a maximum element, namely, the first element of any decreasing sequence of $K$ with underlying set $S$. Notice now that every increasing sequence with underlying set $S$ must stabilize at $\text{max} S$. Hence $S$ is finite. On the other hand, if $S$ is finite, then it is increasing and decreasing; this is because we can increasingly (resp., decreasingly) enumerate the elements of $S$ as the first $|S|$ elements of a sequence and then complete the rest of the sequence taking copies of $\text{max} S$ (resp., $\text{min} S$).

Let $P$ be a positive monoid of some ambient ordered field. By Lemma 5.3, if $P$ is finitely generated, then it is increasing and decreasing. On the other hand, suppose that $P$ is both increasing and decreasing. By Proposition 5.2, one finds that $P$ is atomic. Since $A(P)$ is contained in every generating set, it is increasing and decreasing. Lemma 5.3 now implies that $A(P)$ is finite and, therefore, $P$ is finitely generated. Hence the next result holds.

Proposition 5.4. A positive monoid of an ordered field is finitely generated if and only if it is increasing and decreasing.

A positive monoid of an ambient ordered field $K$ is strongly increasing (resp., weakly increasing) if it can be generated by an unbounded (resp., bounded) increasing sequence of $K$. A strongly increasing positive monoid is obviously increasing. A Puiseux monoid of $\mathbb{Q}$ is both strongly and weakly increasing if and only if it is isomorphic to a numerical semigroup (see [10, Proposition 3.7]). This fact does not extend to positive monoids of an arbitrary ordered field, as the next proposition indicates.

Proposition 5.5. Let $K$ be an ordered field, and let $P$ be a finitely generated positive monoid of $K$. Then $P$ is strongly increasing if and only if $K$ is Archimedean.

Proof. For the forward implication suppose, by way of contradiction, that $K$ is not Archimedean. Take $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in K^+$ such that $P = \langle a_1, \ldots, a_k \rangle$ and $0 < a_1 < \cdots < a_k$. If $a_k$ were finite, then $P \subseteq K^+$ and, therefore, any infinitely large element of $K$ would be an upper bound for $P$, a contradiction. Thus, assume that $a_k$ is infinitely large. In this case, for all coefficients $n_1, \ldots, n_k \in \mathbb{N}_0$,

$$\sum_{i=1}^{k} n_ia_i \leq kNa_k < a_k^2,$$

where $N = \max\{n_1, \ldots, n_k\}$. Since $a_k^2$ is an upper bound for $P$, it follows that $P$ is not strongly increasing, which is a contradiction. Hence $K$ must be Archimedean.

For the reverse implication assume that $K$ is Archimedean. Once again, take $k \in \mathbb{N}$ and $a_1, \ldots, a_k \in K^+$ such that $0 < a_1 < \cdots < a_k$ and $P = \langle a_1, \ldots, a_k \rangle$. Define the
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sequence \{u_n\} of \(K\) by setting \(u_i = a_i\) for \(i \in \{1, \ldots, k\}\) and \(u_n = na_k\) for \(n > k\). The sequence \(\{u_n\}\) is increasing and generates \(P\). Since \(K\) is Archimedean, for every \(x \in K^+\) there exists \(n \in \mathbb{N}\) such that \(u_n = na_k > x\). As \(P\) is generated by the unbounded increasing sequence \(\{u_n\}\), it is a strongly increasing positive monoid. Hence every finitely generated positive monoid of an Archimedean ambient field is strongly increasing. \(\square\)

We have seen in Proposition 4.5 that every positive monoid \(P\) of an Archimedean ambient field is a BF-monoid provided that \(P\) does not have 0 as a limit point. Strengthening the hypothesis of this result, we can guarantee that \(P\) is, in fact, an FF-monoid. We show this in the following proposition, which is a weaker version of our main theorem.

**Proposition 5.6.** Every increasing positive monoid of an Archimedean ambient field is an FF-monoid.

**Proof.** Let \(K\) be an Archimedean field, and let \(P\) be an increasing positive monoid of \(K\). Since \(P\) is increasing, 0 is not a limit point of \(P\). So Proposition 4.5 ensures that \(P\) is a BF-monoid. Suppose, by way of contradiction, that \(P\) is not an FF-monoid. Consider now the set

\[ S = \{ x \in \mathbb{P} : |Z(x)| = \infty \}. \]

Since \(P\) is not an FF-monoid, \(S\) is not empty. In addition, \(s = \inf S \neq 0\) because 0 is not a limit point of \(P\). As \(P\) is increasing, \(P^*\) contains a minimum element, namely the first nonzero element of any increasing sequence generating \(P\). Set \(m = \min P^*\), and fix \(\epsilon \in (0, m)\). Now take \(x \in S\) such that \(s \leq x < s + \epsilon\). Notice that every \(a \in \mathcal{A}(P)\) shows in only finitely many factorizations of \(x\); otherwise \(x-a\) would be an element of \(S\) satisfying that \(x-a < s\), contradicting that \(s = \inf S\). Since \(P\) is a BF-monoid, \(L(x)\) is finite. Therefore there exists \(\ell \in L(x)\) such that the set

\[ Z = \{ z \in Z(x) : |z| = \ell \} \]

contains infinitely many factorizations. Fix \(z_0 = a_1 \ldots a_\ell \in Z(x)\), where \(a_1, \ldots, a_\ell\) are atoms of \(P\). Then set \(A = \max\{a_1, \ldots, a_\ell\}\). Because every atom shows in only finitely many factorizations in \(Z\) and \(|Z| = \infty\), there exists \(z_1 = b_1 \ldots b_\ell \in Z\), where \(b_1, \ldots, b_\ell \in \mathcal{A}(P)\) and \(b_n > A\) for each \(n \in \{1, \ldots, \ell\}\) (here we are using the fact that \(\mathcal{A}(P)\) is the underlying set of an increasing sequence). But now, if \(\phi : Z(P) \to P\) is the factorization homomorphism of \(P\) (see Section 2), we obtain

\[ x = \phi(z_0) = \sum_{n=1}^{\ell} a_n \leq A\ell < \sum_{n=1}^{\ell} b_n = \phi(z_1) = x, \]

which is a contradiction. Hence \(P\) is an FF-monoid. \(\square\)
Theorem 5.6 is weaker than our main theorem because in the former the Archimedean assumption on the ambient field can be dropped, as we will finally do in Theorem 6.3. The increasing condition in Proposition 5.6 is not superfluous, as we now illustrate.

**Example 5.7.** Let \( \{p_n\} \) be an increasing enumeration of the prime numbers. In the ambient field \( \mathbb{Q} \), consider the Puiseux monoid
\[
P = \langle A \rangle, \quad \text{where} \quad A = \left\{ \frac{1}{p_n} \mid n \in \mathbb{N} \right\}.
\]
It is easy to check that \( P \) is an atomic monoid with \( A(P) = A \). Since \( A \) is infinite and \( P \) is decreasing, Proposition 5.4 implies that \( P \) is not increasing. In addition, \( P \) is not a BF-monoid as 1 is the sum of \( p_n \) copies of the atom \( 1/p_n \) for every natural \( n \). In particular, \( P \) is not an FF-monoid.

On the other hand, the converse of Proposition 5.6 is not true; the following example sheds light upon this observation.

**Example 5.8.** Let \( \{p_n\} \) be a strictly increasing sequence of primes, and consider the Puiseux monoid of \( \mathbb{Q} \) defined as follows:

\[
(5.1) \quad P = \langle A \rangle, \quad \text{where} \quad A = \left\{ \frac{p_{2n}^2 + 1}{p_{2n}}, \frac{p_{2n+1}^2 + 1}{p_{2n+1}} \mid n \in \mathbb{N} \right\}.
\]

Since \( A \) is an unbounded subset of \( \mathbb{R} \) having 1 as a limit point, it cannot be increasing. In addition, since \( d(a) \neq d(a') \) for all \( a, a' \in A \) such that \( a \neq a' \), every element of \( A \) is an atom of \( P \). Thus, every generating set of \( P \) must contain \( A \). Now the fact that \( A \) is not increasing implies that \( P \) is not an increasing positive monoid.

We verify now that \( P \) is an FF-monoid. Fix \( x \in P \) and then take \( D_x \) to be the set of primes dividing \( d(x) \). Now choose a natural \( N \) large enough such that \( N > x \) and \( D_x \subset \{1, \ldots, n\} \). For each \( a \in A \) such that \( d(a) > N \), the number of copies \( \alpha \) of the atom \( a \) showing in any \( z \in Z(x) \) must be a multiple of \( d(a) \); this follows by applying the \( d(a) \)-adic valuation map to \( x = \phi(z) \), where \( \phi \) is the factorization homomorphism. Therefore \( \alpha = 0 \); otherwise, \( x \geq \alpha a \geq d(a)a > d(a) > x \). Thus, if an atom \( a \) divides \( x \) in \( P \), then \( d(a) \leq N \). As a result, only finitely many elements of \( A(M) \) divide \( x \) in \( P \). This implies that \( Z(x) \) is finite. Hence \( P \) is an FF-monoid that fails to be increasing.

Proposition 5.6 is the first prototype of our main theorem. Let us create the building blocks we need to get rid of the unnecessary Archimedean condition.

**6. The Main Theorem**

As we mentioned before, the Archimedean condition in Proposition 5.6 is unnecessary. In this section, we prove our main theorem, which is the result of dropping the Archimedean condition in Proposition 5.6. First, let us verify two technical results that are crucial in the proof of the main theorem.
Lemma 6.1. Finitely generated positive monoids do not contain strictly decreasing sequences.

Proof. Assume, by way of contradiction, that there exists a nonempty family $\mathcal{F}$ of finitely generated positive monoids containing strictly decreasing sequences. Among the members of $\mathcal{F}$ take a positive monoid $P$ such that $|\mathcal{A}(P)| = \min\{|\mathcal{A}(F)| : F \in \mathcal{F}\}$. Let $K$ be an ambient field for $P$. By Proposition 4.3, one has that $P$ is atomic. Let $\mathcal{A}(P) = \{a_1, \ldots, a_m\}$, where $m \in \mathbb{N}$ and $a_1 < \cdots < a_m$. Also, take $\{s_n\}$ to be a strictly decreasing sequence of $K^+$ contained in the finitely generated positive monoid $\mathbb{Z}$.

Let $s_n = \alpha_{n1}a_1 + \cdots + \alpha_{nm}a_m$ for some $\alpha_{ij} \in \mathbb{N}_0$. If for $\alpha \in \mathbb{N}$, there is a strictly increasing sequence of naturals $\{k_n\}$ such that $\alpha_{kn1} = \alpha$, then taking $s_n' = s_n$ we would find that $\{s_n' - \alpha a_1\}$ is a strictly decreasing sequence contained in the finitely generated positive monoid $\langle a_2, \ldots, a_m\rangle$, contradicting the minimality of $|\mathcal{A}(P)|$. As a result, $\lim_{n \to \infty} \{\alpha_{n1}\} = \infty$. Similarly, we can argue that $\lim_{n \to \infty} \alpha_{nj} = \infty$ for each $j = 2, \ldots, m$. This implies that there exists a natural $N > 1$ such that $\alpha_{Nj} > \alpha_{ij}$ for each $j = 2, \ldots, m$. As a result, $s_N > s_1$, which contradicts the fact that $\{s_n\}$ is decreasing. \hfill $\Box$

If $M$ is an atomic monoid and $N$ is an atomic submonoid of $M$, then for $x \in N$ the set $Z(x)$ depends on whether we consider $x$ as an element in $M$ or $N$. The same is true for the set $L(x)$. When there is some risk of confusion, we write $Z_M(x)$ (resp., $Z_N(x)$) to refer the factorization set of $x$ in $M$ (resp., $N$). We use the notations $L_M(x)$ and $L_N(x)$ with the same intension.

Lemma 6.2. Let $M$ be a reduced monoid, and let $M_1, M_2, \ldots$ be a sequence of divisor-closed submonoids of $M$ such that

\[ M = \bigcup_{n \in \mathbb{N}} M_n. \]

If every $M_n$ is an FF-monoid, then $M$ is also an FF-monoid.

Proof. Let $x$ be an element of $M$. Since $M$ is the union of the $M_n$’s, we have that $x \in M_n$ for some $n \in \mathbb{N}$. We verify now that $Z_M(x) \subseteq Z_{M_n}(x)$. Take $z \in Z_M(x)$. Since $M_n$ is divisor-closed, every atom of $M$ showing in $z$ belongs to $M_n$. The fact that $\mathcal{A}(M) \cap M_n \subseteq \mathcal{A}(M_n)$ now implies that $z \in Z_{M_n}(x)$. Consequently, $Z_M(x) \subseteq Z_{M_n}(x)$. Since $M_n$ is an FF-monoid, the set of factorizations $Z_{M_n}(x)$ is finite, which implies that $|Z_M(x)| < \infty$. Since $x$ was arbitrarily taken, it follows that $M$ is an FF-monoid. \hfill $\Box$

Let $M$ be a reduced atomic monoid, and let $A$ be a subset of $\mathcal{A}(M)$. For $z \in Z(M)$, we let $|z|_A$ denote the number of atoms in $A$ showing in $z$ (counting repetition). Note that $|\cdot|$ and $|\cdot|_A$ are the same if and only if $A = \mathcal{A}(M)$. Since $Z(M)$ is free on $\mathcal{A}(M)$, there exists a unique monoid homomorphism

\[ \phi_A : Z(M) \to M \]
such that \( \phi_A(a) = a \) if \( a \in A \) and \( \phi_A(a) = 0 \) if \( a \in \mathcal{A}(M) \setminus A \); we call \( \phi_A \) the factorization homomorphism restricted to \( A \).

We are now in a position to prove our main result.

**Theorem 6.3.** Every increasing positive monoid of an ordered field is an FF-monoid.

**Proof.** Let \( K \) be an ordered field, and let \( P \) be an increasing positive monoid of \( K \). Since \( P \) is increasing, it must be atomic by Proposition 5.2. Moreover, in the case of \( P \) being finitely generated, by Proposition 2.1 we have that \( P \) is an FF-monoid. Therefore let us assume that \( P \) is not finitely generated, that is, \( |\mathcal{A}(P)| = \infty \). Take \( \{a_n\} \) to be a strictly increasing sequence of \( K \) with underlying set \( \mathcal{A}(P) \). Let \( \alpha : K \to \Gamma_K \) be the Archimedean valuation of \( K \) (see Section 3). Because \( \{a_n\} \) increases, it follows that \( \alpha(a_{n+1}) \leq \alpha(a_n) \) for every \( n \in \mathbb{N} \).

We show first that \( P \) is an FF-monoid when the set \( \{\alpha(a_n) \mid n \in \mathbb{N}\} \) of Archimedean classes is finite. Let us assume, by way of contradiction, that \( P \) is not an FF-monoid. Choose \( x \in P \) such that \( Z(x) \) contains infinitely many factorizations. Take the minimum \( N \in \mathbb{N} \) such that \( \alpha(a_n) = \alpha(a_m) \) for all \( n, m \geq N \), and set

\[ A = \{a_j \mid j \geq N\}. \]

Since \( O(y + y') = O(\max\{y, y'\}) \) for all \( y, y' \in K^+ \), it follows that \( \alpha(a_N) \leq \alpha(y) \) for all \( y \in P \). As a result, there exists a smallest positive integer \( N' \) such that \( N'a_N \geq x \).

If for some \( j \geq N \) the atom \( a_j \) shows in infinitely many factorizations of \( x \), we can replace \( x \) by \( x - a_j \) and still preserve the fact that \( |Z(x)| = \infty \). Since

\[ \sum_{j \geq N} c_j a_j \geq \sum_{j \geq N} c_j a_N \geq N'a_N \geq x \]

provided that \( \{c_j\} \in \mathbb{N}_0^\infty \) satisfies \( \sum_{j \geq N} c_j \geq N' \), the replacement mentioned above can happen at most \( N' \) times. Therefore we can assume that for every \( j \geq N \) the atom \( a_j \) shows in only finitely many factorizations of \( Z(x) \). Since \( |Z(x)| = \infty \) and every factorization in \( Z(x) \) contains at most \( N' \) copies of atoms in \( \{a_j \mid j \geq N\} \), there exists \( n_0 \leq N' \) such that the set

\[ Z = \{z \in Z(x) : |z|_A = n_0\} \]

is infinite. As for every \( j \geq N \) the atom \( a_j \) shows in only finitely many factorizations of \( x \), we can construct a sequence of factorizations \( \{z_n\} \) of \( Z \) such that \( \{\phi_A(z_n)\} \) is a strictly increasing sequence of \( P \): take \( z_1 \in Z \) arbitrarily and, once we have constructed \( \{z_1, \ldots, z_{n-1}\} \) such that \( \phi_A(z_1) < \cdots < \phi_A(z_{n-1}) \), take \( z_n \in Z \) such that every atom of \( A \) showing in \( z_n \) is strictly greater than the maximum atom showing in \( z_{n-1} \). Therefore \( \{x - \phi_A(z_n)\} \) is a strictly decreasing sequence in \( \langle a_1, \ldots, a_{N-1} \rangle \), which contradicts Lemma 6.1.

To complete the proof, let us verify that \( P \) is an FF-monoid when the set of Archimedean classes \( \{\alpha(a_n) \mid n \in \mathbb{N}\} \) contains infinitely many elements. Because \( \{a_n\} \) increases, it follows that \( \alpha(a_{n+1}) \leq \alpha(a_n) \) for every \( n \in \mathbb{N} \). Let \( \{s_n\} \) be a
increasing sequence of naturals with \( s_1 = 1 \) so that \( \alpha(a_i) = \alpha(a_j) \) if and only if \( s_n \leq i, j \leq s_{n+1} - 1 \) for some natural \( n \). Set
\[
F_n = \langle a_1, \ldots, a_{s_{n+1}-1} \rangle
\]
for every natural \( n \). By Proposition 2.1, each \( F_n \) is an FF-monoid. Now we verify that \( F_n \) is a divisor-closed submonoid of \( P \) for every \( n \in \mathbb{N} \). If \( y \in F_n \), then there are nonnegative integer coefficients \( n_1, \ldots, n_{s_{n+1}-1} \) such that
\[
y = \sum_{i=1}^{s_{n+1}-1} n_i a_i \leq (s_{n+1} - 1)Na_{s_{n+1}-1} < a_j
\]
for every \( j \geq s_{n+1} \), where \( N = \max\{n_1, \ldots, n_{s_{n+1}-1}\} \); this is because \( \alpha(a_j) \prec \alpha(a_{s_{n+1}-1}) \) when \( j \geq s_{n+1} \). Therefore no atoms contained in the complement of \( F_n \) divides \( y \) in \( P \). As a result, \( F_n \) is a divisor-closed submonoid of \( P \). Since \( P \) is the union of the \( F_n \)'s, it follows by Lemma 6.2 that \( P \) is an FF-monoid. \( \square \)

The converse of Theorem 6.3 is not true even when the ambient field is Archimedean; see Example 5.8. On the other hand, in Example 5.7 we exhibited a non-increasing Puiseux monoid of \( \mathbb{Q} \) that is not even a BF-monoid. Hence the increasing condition in Theorem 6.3 is required.

We say that a monoid \( M \) is hereditarily atomic if each submonoid of \( M \) is atomic. As the next proposition indicates, in the family of positive monoids, being hereditarily atomic is a consequence of being a BF-monoid.

**Proposition 6.4.** Every positive BF-monoid of an ordered field is hereditarily atomic.

**Proof.** Let \( K \) be an ordered field, and let \( P \) be a positive BF-monoid of \( K \). In particular, \( P \) is atomic. Let \( P' \) be a submonoid of \( P \). We will verify that \( P' \) is atomic. Observe that every element of \( P' \) that cannot be written as a sum of two elements in \( P^* \) belongs to \( \mathcal{A}(P') \). Take \( x \in P^* \). Since \( P \) is a BF-monoid, \( L_P(x) \) is finite, and so there exists \( N \in \mathbb{N} \) such that \( |z| < N \) for all \( z \in \mathbb{Z}_P(x) \). Now suppose that we can write
\[
x = x'_1 + \cdots + x'_n
\]
for some \( n \in \mathbb{N} \) and \( x'_1, \ldots, x'_n \in P^* \). Since each \( x'_i \) belongs to \( P^* \), it follows that \( x \) can be written as the sum of at least \( n \) atoms of \( P \). This implies that \( n \leq N \), and so \( x \) can be expressed as the sum of at most \( N \) elements of \( P^* \). Then we can choose \( n \) in (6.1) to be maximal. In this case, each \( x'_i \) must be an atom of \( P' \). This implies that \( P' \) is atomic. Because the submonoid \( P' \) of \( P \) was arbitrarily taken, \( P \) happens to be hereditarily atomic. \( \square \)

The converse of Proposition 6.4 does not hold. For instance, according to [10, Theorem 5.5], if \( \{p_n\} \) is an enumeration of the prime numbers, then the Puiseux monoid \( P = \langle 1/p_n \mid n \in \mathbb{N} \rangle \) is hereditarily atomic. However, for every \( n \in \mathbb{N} \), the element 1 is
the sum of \( p_n \) copies of the atom \( 1/p_n \) and, therefore, \( P \) is not a BF-monoid. On the other hand, an atomic monoid that does not satisfy the bounded factorization property might not be hereditarily atomic. The next example sheds light upon this.

**Example 6.5.** Let \( \{p_n\} \) be a strictly increasing sequence comprising the odd prime numbers. Consider the Puiseux monoid of \( \mathbb{Q} \)

\[
P = \langle A \rangle, \text{ where } A = \left\{ \frac{1}{2^n p_n} \mid n \in \mathbb{N} \right\}.
\]

Since each odd prime divides exactly one element of the set \( \mathcal{d}(A) \), it follows that \( \mathcal{A}(P) = A \). Thus, \( P \) is atomic. Moreover, the fact that 1 is the sum of \( 2^n p_n \) copies of the atom \( 1/(2^n p_n) \) for every \( n \in \mathbb{N} \) implies that \( P \) is not a BF-monoid. On the other hand, the element \( 1/2^n \) is the sum of \( p_n \) copies of the atom \( 1/(2^n p_n) \) for every \( n \in \mathbb{N} \) and, therefore, the antimatter monoid \( \langle 1/2^n \mid n \in \mathbb{N} \rangle \) is a submonoid of \( P \). Hence \( P \) fails to be hereditarily atomic.

Combining Theorem 6.3 and Proposition 6.4 we immediately obtain the next result.

**Corollary 6.6.** Every increasing positive monoid of an ambient ordered field is hereditarily atomic.

7. **Acknowledgments**

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