HOMOLOGICAL MIRROR SYMMETRY FOR CURVES OF HIGHER GENUS

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ABSTRACT. Katzarkov has proposed a generalization of Kontsevich’s mirror symmetry conjecture, covering some varieties of general type. Seidel [Se1] has proved a version of this conjecture in the simplest case of the genus two curve. In this paper we prove the conjecture (in the same version) for curves of genus $g \geq 3$, relating the Fukaya category of a genus $g$ curve to the category of Landau-Ginzburg branes on a certain singular surface.

We also prove a kind of reconstruction theorem for hypersurface singularities. Namely, formal type of hypersurface singularity (i.e. a formal power series up to a formal change of variables) can be reconstructed, with some technical assumptions, from its D $(\mathbb{Z}/2)$-G category of Landau-Ginzburg branes. The precise statement is Theorem 1.2.

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The author was partially supported by the Moebius Contest Foundation for Young Scientists, and by the NSh grant 1983.2009.1.
1. Introduction

The Homological Mirror Symmetry conjecture relates symplectic and algebraic geometry through their associated categorical structures. Kontsevich’s original version [Ko1] concerned Calabi-Yau varieties. Now, there are complete proofs of some cases [PZ, Se3] and partial results for many more [KS, F]. Soon after, Kontsevich proposed an analogue of the conjecture for Fano varieties. This was gradually extended further, and it seems that varieties with effective anticanonical divisor provide a natural context [Au]. The mirror in this case is not another variety but rather a Landau-Ginzburg theory, which means a variety together with a holomorphic function. Because of this asymmetry, the two directions of the mirror correspondence lead to substantially different mathematics. The one relevant here is where the Landau-Ginzburg theory is considered algebro-geometrically, through matrix factorizations or more generally Orlov’s Landau-Ginzburg branes [Or2].

Recently, Katzarkov [Ka, KKP] has proposed an extension of Homological Mirror Symmetry, encompassing some varieties of general type. The mirror is a Landau-Ginzburg theory. Abouzaid, Auroux, Gross, Katzarkov, and Orlov have explored both directions of the correspondence, and accumulated large amounts of evidence (K-theory computations [Ab, Or1] and more unpublished material). One direction of Katzarkov’s conjecture was proved by Seidel in the case of the genus 2 curve [Se1]. The aim of this paper is to prove it in the case of curves of genus $\geq 3$.

Let $M$ be a curve of genus $g \geq 3$, equipped with a symplectic structure. Its mirror is a three-dimensional Landau-Ginzburg theory $X \to \mathbb{C}$, whose zero fibre $H \subset X$ is the union of $(g+1)$ surfaces. Details of the construction of this mirror will be given in Section 8. Let $\mathcal{F}(M)$ be the Fukaya category of $M$, and $D^\pi(\mathcal{F}(M))$ its split-closed (Karoubi completed) derived category. On the other side, take $D_{sg}(H)$ to be the category of Landau-Ginzburg branes, and let $\overline{D_{sg}}(H)$ be the split-closure of that.

**Theorem 1.1.** There is an equivalence of triangulated categories, $D^\pi(\mathcal{F}(M)) \cong \overline{D_{sg}}(H)$.

The main ideas in the proof are the same as in [Se1]. We now sketch the steps of the proof, simultaneously fixing the notation.

Take $V = \mathbb{C}^3$. We write $\xi_k$ for the standard basis vectors of $V$, thought of as constant vector fields, and $z_k$ for the dual basis of functions. The superpotential is the polynomial

$$W = -z_1z_2z_3 + z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1} \in \mathbb{C}[V^\vee]^K,$$

where $K \cong \mathbb{Z}/(2g + 1)$ is the subgroup of $SL(V)$ generated by the diagonal matrix $\text{diag}(\zeta, \zeta, \zeta^{2g-1})$ with $\zeta = \exp(\frac{2\pi i}{2g+1})$.

An orbifold covering: We represent $M$ as a covering of a genus zero orbifold $\bar{M}$, where the covering group is $\Sigma = \text{Hom}(K, \mathbb{C}^*) \cong \mathbb{Z}/(2g + 1)$. We choose a collection of $(2g + 1)$
curves \( L_1, \ldots, L_{g+1} \) which split-generate \( D^\pi F(M) \), and which all project to the same immersed curve \( \bar{L} \subset \bar{M} \). The Floer cohomology of \( \bar{L} \) is isomorphic to the exterior algebra \( \Lambda(V) \), but with nontrivial higher order \( A_\infty \)-operations. We compute some higher products which we need using basic combinatorial techniques.

**Kontsevich formality.** The graded algebra \( \Lambda(V) \) admits a rich moduli space of \( \mathbb{Z}/2 \)-graded \( A_\infty \)-structures. The relevant deformation theory is governed by the differential graded Lie algebra of Hochschild cochains. We apply a version of the formality theorem from [Ko2], and standard tools from Maurer-Cartan theory, to reduce this to a problem about polyvector fields on \( V \). It turns out that the \( A_\infty \)-structure encountered in the Floer cohomology computation, corresponds to the (gauge equivalence class of) the superpotential \( W \).

**Koszul duality.** The cohomology level category of Landau-Ginzburg branes is known to be equivalent to that of matrix factorizations of \( W \) [Or2]. In our case, the structure sheaf of the origin \( \mathcal{O}_0 \) is a split-generator in LG branes. We take the matrix factorization corresponding to this skyscraper sheaf \( \mathcal{O}_0 \). The endomorphism DGA of this matrix factorization turns out to be quasi-isomorphic to the \( A_\infty \)-algebra computed on the Fukaya side. Namely, the cohomology super-algebra of this DGA is isomorphic to the exterior algebra \( \Lambda(V) \) and again the resulted \( A_\infty \)-structure corresponds to the superpotential \( W \) in polyvector fields.

Here we also prove the following general theorem (more precise formulation is Theorem 7.1):

**Theorem 1.2.** Let \( k \) be a field of characteristic zero, \( n \geq 1 \), and \( V = k^n \). Let \( W = \sum_{i=3}^d W_i \in k[V^\vee] \) be a non-zero polynomial, where \( W_i \in \text{Sym}^i(V^\vee) \). Then \( W \) can be reconstructed, up to a formal change of variables, from the quasi-isomorphism class of \( D_\mathbb{Z}/2(G) \)-algebra \( B_W \cong \mathbb{R} \text{Hom}_{D_{sg}(W^{-1}(0))}(\mathcal{O}_0, \mathcal{O}_0) \), the endomorphism \( DG_\mathbb{Z}/2 \)-graded algebra of the structure sheaf \( \mathcal{O}_0 \) in \( D_{sg}(W^{-1}(0)) \), together with identification \( H^*(B_W) \cong \Lambda(V) \). Moreover, formal change of variables is of the form

\[
(1.2) \quad z_i \rightarrow z_i + O(z^2).
\]

**The McKay correspondence.** Let \( X \rightarrow \bar{X} = V/A \) be the crepant resolution given by the \( A \)-Hilbert scheme [CR], and \( H \subset X \) the preimage of \( \bar{H} = W^{-1}(0)/A \subset \bar{X} \). We can explicitly determine the geometry of \( H \), which yields the description in our main theorem above. The categorical McKay correspondence [BKR] yields an equivalence \( D^b_A(V) \cong D^b(X) \). We use an analogous result for Landau-Ginzburg branes [BP]: \( D_{sg,K}(W^{-1}(0)) \cong D_{sg}(H) \).

**The sign convention.** We will treat an \( A_\infty \)-algebra as a \( \mathbb{Z} \)- (or \( \mathbb{Z}/2 \)-) graded vector space equipped with a sequence of maps \( \mu^d : A^{\otimes d} \rightarrow A \) of degree \( 2 - d \) (resp. of parity \( d \))
such that the maps \( m_d : A^\otimes d \to A \), where

\[
m_d(a_d, \ldots, a_1) = (-1)^{|a_1|+2|a_2|+\cdots+d|a_d|} \mu^d(a_d, \ldots, a_1).
\]

Acknowledgements. I am grateful to D. Kaledin, A. Kuznetsov, S. Nemirovski, D. Orlov and P. Seidel for their help and useful discussions.

2. Kontsevich formality

Let \( g \) be a DG Lie algebra over \( \mathbb{C} \). An element \( \alpha \in g^1 \) is called Maurer-Cartan (MC) element if it satisfies Maurer-Cartan (MC) equation

\[
\partial \alpha + \frac{1}{2} [\alpha, \alpha] = 0.
\]

There is a natural Lie algebra morphism from \( g^0 \) to the Lie algebra of affine vector fields on \( g^1 \); it maps \( \gamma \in g^0 \) to \( (\alpha \mapsto -\partial \gamma + [\gamma, \alpha]) \). It is easy to check that all vector fields in the image are tangent to the subscheme of solutions of (2.1). Thus, if these vector fields can be exponentiated, we obtain a group action on the set of Maurer-Cartan elements.

We will need to deal with \( L_\infty \)-morphisms between DG Lie algebras. Such a morphism \( \Phi : g \to h \) is given by a sequence of maps \( \Phi^k : g^\otimes k \to h \). These maps are required to be anti-symmetric (in super sense) and to satisfy equations of compatibility with DG Lie algebra structures on \( g \) and \( h \), see [LM]. In particular, \( \Phi^1 \) is a morphism of complexes, and induces a morphism of Lie algebras in cohomology.

Such \( \Phi \) is called a quasi-isomorphism if \( \Phi^1 \) is a quasi-isomorphism. We will need the following statement, which is implied by Homological perturbation Lemma:

Lemma 2.1. Let \( g \) be a graded Lie algebra considered as a DG Lie algebra with zero differential. Let \( h \) be a DG Lie algebra, and \( \Psi : g \to h \) an \( L_\infty \)-quasi-isomorphism. Take some morphism of complexes \( \Phi^1 : h \to g \) together with a homogeneous map \( H : h \to h \) of degree \(-1\), such that

\[
\Phi^1 \Psi^1 = \text{id}, \quad \Psi^1 \Phi^1 - \text{id} = \partial H + H \partial.
\]

Then \( \Phi^1 \) can be extended to an \( L_\infty \)-morphism \( \Phi : h \to g \), so that the higher order terms \( \Phi^k \) are given by a universal formulae, depending only on \( \Psi \), \( \Phi^1 \) and \( H \).

Moreover, one can choose \( \Phi \) in such a way that the composition \( \Phi \circ \Psi \) equals to the identity \( L_\infty \)-morphism.

Proof. For the proof of the first statement, see [Se1], Lemma 2.1. Further, for the constructed \( \Phi \), we have that the composition \( \Phi \circ \Psi \) is an \( L_\infty \)-automorphism of \( h \). Define
\( \Phi' = (\Phi \circ \Psi)^{-1} \Phi \). Then \( \Phi' \) satisfies the required property, and the higher order terms \( \Phi'^k \) are again given by a universal formulae, depending only on \( \Psi, \Phi^1 \) and \( H \).

In order to be able to exponentiate the gauge vector fields on \( g^1 \), we will deal with pro-nilpotent DG Lie algebras.

**Definition 2.2.** A DG Lie algebra \( g \) is called pro-nilpotent if it is equipped with a complete decreasing filtration \( g = L_1 g \supset L_2 g \supset \ldots \), such that

\[
\partial(L_r g) \subset L_r g, \quad [L_r g, L_s g] \subset L_{r+s} g.
\]

If \( g \) is pro-nilpotent, then Lie algebra \( g^0 \) is also such, and hence can be exponentiated to a pro-nilpotent group by Baker-Campbell-Hausdorff formula. This group then acts on MC elements \( \alpha \in g^1 \). We call two such elements equivalent if they lie in the same orbit of this action.

**Definition 2.3.** Let \( g, h \) be pro-nilpotent DG Lie algebras. An \( L_\infty \)-morphism \( \Phi : g \to h \) is called filtered if

\[
\Phi^k(L_{r_1} g \otimes \cdots \otimes L_{r_k} g) \subset L_{r_1 + \cdots + r_k} h.
\]

**Definition 2.4.** A filtered \( L_\infty \)-morphism \( \Phi : g \to h \) of filtered DG Lie algebras is called a filtered \( L_\infty \)-quasi-isomorphism if the induced morphisms of complexes \( L_r g / L_{r+1} g \to L_r h / L_{r+1} h \) are quasi-isomorphisms.

**Remark 2.5.** In Lemma 2.1 we can require \( g, h \) to be pro-nilpotent, \( \Psi \) to be filtered \( L_\infty \)-quasi-isomorphisms, and \( \Phi^1, H \) to be compatible with filtrations. Then the constructed \( L_\infty \)-morphism \( \Phi \) is also filtered.

If \( \Phi : g \to h \) is a filtered \( L_\infty \)-morphism of filtered DG Lie algebras, then we have an induced map on Maurer-Cartan elements

\[
\alpha \mapsto \sum_{k \geq 1} \frac{1}{k!} \Phi^k(\alpha, \ldots, \alpha).
\]

This map preserves equivalence relation. The following statement is an adapted version of the corresponding result in \([Ko2]\).

**Lemma 2.6.** Let \( \Phi : g \to h \) be a filtered \( L_\infty \)-quasi-isomorphism of filtered DG Lie algebras. Then the induced map on equivalence classes of MC elements is a bijection.

Lemma can be proved by standard obstruction theory, as in \([GM]\) (or \([ELO2]\) for \( A_\infty \)-algebras).
Now we summarize the result of Kontsevich formality theorem [Ko2], with some modifications. Let $V$ be a finite-dimensional $\mathbb{C}$-vector space. By definition, the space of formal polyvector fields on $V$ is

$$\mathbb{C}[[V^\vee]] \otimes \Lambda(V) = \prod_{i,j} \text{Sym}^i(V^\vee) \otimes \Lambda^j(V).$$

If we assign to the summand $\mathbb{C}[[V^\vee]] \otimes \Lambda^j(V)$ the grading $j - 1$, then the whole space becomes a graded Lie algebra with respect to the Schouten bracket

$$[f_{\xi_1} \wedge \cdots \wedge \xi_k, g_{\xi_1} \wedge \cdots \wedge \xi_l] = \sum_{q=1}^k (-1)^{k-q} (f \partial_{\xi_q} g)_{\xi_1} \wedge \cdots \wedge \widehat{\xi_q} \wedge \cdots \wedge \xi_k \wedge \xi_1 \wedge \cdots \wedge \xi_l + \sum_{p=1}^l (-1)^{l-p-1+(k-1)(l-1)} (g \partial_{\xi_p} f)_{\xi_1} \wedge \cdots \wedge \widehat{\xi_p} \wedge \cdots \wedge \xi_j \wedge \xi_1 \wedge \cdots \wedge \xi_k.$$

The MC equation on $\alpha \in \mathbb{C}[[V^\vee]] \otimes \Lambda^2(V)$ says that $\alpha$ gives rise to a formal Poisson structure. The elements $\gamma \in \mathbb{C}[[V^\vee]] \otimes V$, which are formal vector fields, act on Poisson brackets by their Lie derivatives. If the value of $\gamma$ at the origin vanishes, then it can be exponentiated to a formal diffeomorphism of $V$, and the resulting action on Poisson brackets is just the pushforward action by formal diffeomorphisms.

Now let $A$ be a graded algebra over $\mathbb{C}$. Its Hochshild cochain complex $CC^*(A, A)$ is the space of multilinear maps:

$$CC^d(A, A) = \prod_{i+j-1=d} \text{Hom}^i(A^\otimes i, A).$$

The Hochshild differential and Gerstenhaber bracket are given by the formulas.

$$\partial \phi^j(a_j, \ldots, a_1) = \sum_k (-1)^{|\phi| + |a_1| + \cdots + |a_k| + k} \phi^{j-1}(a_j, \ldots, a_{k+1}a_k, \ldots, a_1) + (-1)^{|\phi| + |a_1| + \cdots + |a_{j-1}| + j} a_j \phi^{j-1}(a_{j-1}, \ldots, a_1) + (-1)^{|\phi|-1} |a_1|-1 \phi^{j-1}(a_j, \ldots, a_2)a_1,$$
and
\[ (\phi, \psi)^j(a_j, \ldots, a_1) = \sum_{k,l} (-1)^{|\psi|(|a_1|+\cdots+|a_k|)-k} \phi^{j-l+1}(a_j, \ldots, a_{k+l+1}, \psi^l(a_{k+l}, \ldots, a_{k+1}), a_k, \ldots, a_1) - \]
\[ \sum_{k,l} (-1)^{|\phi|+|\psi|(|a_1|+\cdots+|a_k|)-k} \psi^{j-l+1}(a_j, \ldots, a_{k+l+1}, \phi^l(a_{k+l}, \ldots, a_{k+1}), a_k, \ldots, a_1). \]

Its cohomology is the Hochschild cohomology $HH \cdot (A, A)$ with grading shifted down by 1 from the standard convention. Take $\alpha \in CC^1(A, A)$, i.e. a sequence of maps $\alpha^j : A^\otimes j \to A$ of degree $2 - j$. Put
\[ (2.11) \]
\[ \begin{align*}
\mu^j &= \alpha^j \text{ for } j \neq 2; \\
\mu^2(a_2, a_1) &= \alpha^2(a_2, a_1) + (-1)^{|a_1|} a_2 a_1.
\end{align*} \]

The Maurer-Cartan equation for $\alpha$ says that the sequence of maps $\mu^j$ satisfies the equation of a curved $A_\infty$-structure.

**Remark 2.7.** As we have already mentioned in Introduction, our sign convention differs from the standard one. To obtain an $A_\infty$-structure in standard sign convention, one should put
\[ (2.12) \]
\[ m_j(a_j, \ldots, a_1) = (-1)^{|a_1|+2|a_2|+\cdots+j|a_j|} \mu^j(a_j, \ldots, a_1). \]

Suppose that $A^i$ is finite-dimensional for all $i$, and take some $\gamma \in CC^0(A, A)$, with vanishing component $\gamma^0 : \mathbb{C} \to A^1$. Put
\[ (2.13) \]
\[ \begin{align*}
\phi^1 &= \text{id} + \gamma^1 + \frac{1}{2} \gamma^1 \gamma^1 + \cdots; \\
\phi^2 &= \gamma^2 + \frac{1}{2} \gamma^1 \gamma^2 + \frac{1}{2} \gamma^2 (\gamma^1 \otimes \text{id}) + \frac{1}{2} \gamma^2 (\text{id} \otimes \gamma^1); \\
\vdots
\end{align*} \]

In general, $\phi^r$ is obtained by summing over all ways of concatenating the components of $\gamma$ to get an $r$-linear map. The associated term is taken with the coefficient $\frac{s!}{r!}$, where $s$ is the number of ways of ordering the components, compatibly with their appearance in concatenation. If two MC elements $\alpha$ and $\tilde{\alpha}$ are related by the exponential action of $\gamma$, then the associated curved $A_\infty$-structures are related by $\phi$, which is an $A_\infty$-isomorphism.

Now let again $V$ be a finite-dimensional vector space, and take $A = \Lambda(V)$. It is a classical result (see [HKR]) that $HH \cdot (A, A) \cong \mathbb{C}[[[V^*]]] \otimes \Lambda(V)$. This isomorphism is induced by Hochshild-Kostant-Rosenberg map
\[ (2.14) \]
\[ \Phi^1 : CC^*(A, A) \to \mathbb{C}[[[V^*]]] \otimes \Lambda(V). \]
If one thinks of formal polyvector fields $\mathbb{C}[[V^\vee]] \otimes \Lambda(V)$ as $\Lambda(V)$-valued formal power series, then

\[(2.15) \quad \Phi^1(\beta)(\xi) = \sum_{j \geq 1} \beta^j(\xi, \ldots, \xi).\]

**Theorem 2.8.** The map $\Phi^1$ is the first term of some $L_\infty$-morphism $\Phi$, which can be taken to be $GL(V)$-equivariant.

Theorem 2.8 is implied by Kontsevich formality Theorem using Lemma 2.1 and reductiveness of $GL(V)$, see [Se1] and Remark 2.9.

**Remark 2.9.** In contrast to our situation, Kontsevich deals with the algebra of smooth functions on smooth manifolds. He proves that for each smooth manifold $X$ the graded Lie algebra of polyvector fields $T_{\text{poly}}(X)$ is quasi-isomorphic to the DG Lie algebra of polydifferential operators $D_{\text{poly}}(X)$. In the case when $X$ is an open domain $U$ in affine space $\mathbb{R}^d$, he constructs an explicit $L_\infty$-quasi-isomorphism. However, one can replace the smooth functions by polynomials over $\mathbb{C}$, and his construction works as well. Then one exchanges even an odd variables, and obtains an $L_\infty$-quasi-isomorphism

\[(2.16) \quad \Psi : \mathbb{C}[[V^\vee]] \otimes \Lambda(V) \to \mathbb{C} \cdot (A, A).\]

This $\Psi$ is $GL(V)$-equivariant, and using Lemma 2.1 and reductiveness of $GL(V)$, one obtains the required $\Phi$, which can be taken to be left inverse to $\Psi$.

### 3. Finite determinacy

Put $V = \mathbb{C}^3$. Take the subgroup $G \subset SL(V)$ which consists of diagonal matrices with $(2g + 1)$-the roots of unity on the diagonal. Clearly, $G \cong (\mathbb{Z}/(2g + 1))^2$. Define the pronilpotent graded Lie algebra $\mathfrak{g}$ as follows:

\[(3.1) \quad \mathfrak{g}^d = \prod_{2i + j - (4g - 4)k = 3d + 3 \atop k \geq 0, i \geq d + 2} (\text{Sym}^i V^\vee \otimes \Lambda^j V)^G \hbar^k.\]

The Lie bracket comes from Schouten bracket on polyvector fields, and $L_r \mathfrak{g}^d$ is the part of the product which consists of terms with $i \geq d + 1 + r$.

We can omit $\hbar^k$ but remember that

\[(3.2) \quad 2i + j - 3d - 3 \geq 0, \quad \text{and} \quad 2i + j - 3d - 3 \equiv 0 \mod 4g - 4.\]

Let $F_r \mathbb{C}[[V^\vee]]$ be the complete decreasing filtration, s.t. $F_r \mathbb{C}[[V^\vee]]$ consists of power series with no terms of order strictly less than $r$. Any Maurer-Cartan solution $\alpha \in \mathfrak{g}^1$ can be
written as \((\alpha^0, \alpha^2)\), where \(\alpha^0 \in F_3 \mathbb{C}[[V^\vee]]\), and \(\alpha^2 \in F_2g \mathbb{C}[[V^\vee]] \otimes \Lambda^2 V\). Any element \(\gamma \in g^0\) can be written as \((\gamma^1, \gamma^3)\), where \(\gamma^1 \in F_2g_{-1} \mathbb{C}[[V^\vee]] \otimes V\), and \(\gamma^3 \in F_{2g-2} \mathbb{C}[[V^\vee]] \otimes \Lambda^3 V\). We also have \(G\)-invariance condition, as well as condition on the components of \(\alpha^i, \gamma^j\) coming from \([3.2]\).

Maurer-Cartan equation for \(\alpha = (\alpha^0, \alpha^2)\) splits into the components

\[
(3.3) \quad \frac{1}{2} [\alpha^2, \alpha^2] = 0, \quad [\alpha^0, \alpha^2] = 0.
\]

The first part says that \(\alpha^2\) gives a Poisson bracket \(\{\cdot, \cdot\}\). The second one says that the Poisson vector field associated to \(\alpha^0\) is identically zero. Equivalently, \(\alpha^2\) is a cocycle in the Koszul complex \(\mathbb{C}[[V^\vee]] \otimes \Lambda V\) with differential being contraction with \(d\alpha^0\).

The exponentiated adjoint action of \(\gamma = (\gamma^1, 0) \in g^0\) on the solutions of MC equation is the usual action by formal diffeomorphisms. For \(\gamma = (0, \gamma^3)\), this action is given by the formula

\[
(3.4) \quad (\alpha^0, \alpha^2) \mapsto (\alpha^0, \alpha^2 + t_{d\alpha^0} \gamma^3).
\]

Take

\[
(3.5) \quad W = -z_1 z_2 z_3 + z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1} \in \mathbb{C}[V^\vee]^G.
\]

Then \((W, 0) \in g^1\) is a solution of MC equation (as any other \(\alpha \in g^1\) of type \((\alpha^0, 0)\) does).

**Lemma 3.1.** Any Maurer-Cartan solution \((\alpha^0, \alpha^2) \in g^1\), such that

\[
(3.6) \quad \alpha^0 \equiv \begin{cases} W \mod F_{2g+2} \mathbb{C}[[V^\vee]] & \text{if } g \not\equiv 1 \mod 3 \\ W + \lambda(z_1 z_2 z_3)^{\frac{2g+1}{3}}, & \text{where } \lambda \in \mathbb{C} \quad \text{if } g \equiv 1 \mod 3,
\end{cases}
\]

is equivalent to \((W, 0)\).

**Proof.** First we note that in the case \((g \equiv 1 \mod 3)\) one may assume that \(\lambda = 0\). Indeed, in this case we have

\[
(3.7) \quad \exp(\lambda z_1^{\frac{2g+1}{3}} z_2^{\frac{2g-2}{3}} z_3^{\frac{2g-2}{3}} \otimes \xi_1)^* \alpha^0 \equiv \alpha^0 + \lambda z_1^{\frac{2g+1}{3}} z_2^{\frac{2g-2}{3}} z_3^{\frac{2g-2}{3}} \frac{\partial \alpha^0}{\partial z_1} \mod F_{2g+2} \mathbb{C}[[V^\vee]] \equiv W \mod F_{2g+2} \mathbb{C}[[V^\vee]].
\]

Thus, we may and will assume that \(\alpha^0 \equiv W \mod F_{2g+2} \mathbb{C}[[V^\vee]]\).

Let \(I \subset \mathbb{C}[V^\vee]\) be an ideal generated by \(\frac{\partial W}{\partial z_i}, \quad i = 1, 2, 3\). It is easy to see that

\[
(3.8) \quad z_i z_j \in I + F_{2g} \mathbb{C}[[V^\vee]] \quad \text{for } i < j, \quad z_i^{2g+2} \in I \cdot F_2 \mathbb{C}[[V^\vee]] + F_{4g} \mathbb{C}[[V^\vee]].
\]

Indeed, for example \(z_1 z_2 \equiv -\frac{\partial W}{\partial z_3} \mod F_{2g} \mathbb{C}[[V^\vee]],\) and

\[
(3.9) \quad z_1^{2g+2} \equiv \frac{1}{2g+1} z_1 \frac{\partial W}{\partial z_1} - \frac{1}{2g+1} z_1 z_2 \frac{\partial W}{\partial z_2} - z_2^2 \frac{\partial W}{\partial z_3} \mod F_{4g} \mathbb{C}[[V^\vee]].
\]
Put $W_{4g-1} = \alpha^0$. It follows from (3.2) that $\alpha^0$ contains only monomials of degree $3 + (2g - 2)k$, where $k \geq 0$. The difference $W - W_{4g-1}$ does not contain monomials $\xi_i^{4g-1}$, since they are not $G$-invariant. It follows from (3.3) that $W - W_{4g-1} \in \mathcal{I} \cdot F_{4g-3}\mathbb{C}[\lbrack V \rbrack] + F_{4g-3}\mathbb{C}[\lbrack V \rbrack]$. Therefore, there exist homogeneous polynomials $f_{4g-3,1}, f_{4g-3,2}, f_{4g-3,3}$ of degree $(4g - 3)$, such that

\begin{equation}
W_{4g-3} = \exp(f_{4g-3,1} \otimes \xi_1 + f_{4g-3,2} \otimes \xi_2 + f_{4g-3,3} \otimes \xi_3)^* W_{4g-3}
\end{equation}

\[\equiv W_{2g+1} + f_{4g-3,1} \frac{\partial W}{\partial \xi_1} + f_{4g-3,2} \frac{\partial W}{\partial \xi_2} + f_{4g-3,3} \frac{\partial W}{\partial \xi_3} \mod F_{6g-3}\mathbb{C}[\lbrack V \rbrack]\]

\[\equiv W \mod F_{6g-3}\mathbb{C}[\lbrack V \rbrack].\]

Moreover, we can take $f_{4g-3,i}$ such that $(f_{4g-3,1} \otimes \xi_1 + f_{4g-3,2} \otimes \xi_2 + f_{4g-3,3} \otimes \xi_3, 0) \in \mathfrak{g}^0$. We obtain a new formal function $W_{6g-3} \equiv W \mod F_{6g-3}\mathbb{C}[\lbrack V \rbrack]$.

Now suppose that we are given with some formal function $W_{3+(2g-2)k}$, where $k \geq 3$, such that $(W_{3+(2g-2)k}, 0) \in \mathfrak{g}^1$ and $W_{3+(2g-2)k} \equiv W \mod F_{3+(2g-2)k}\mathbb{C}[\lbrack V \rbrack]$. It follows from (3.3) that $W - W_{3+(2g-2)k} \in \mathcal{I} \cdot F_{1+(2g-2)(k-1)}\mathbb{C}[\lbrack V \rbrack] + F_{1+(2g-2)(k+1)}\mathbb{C}[\lbrack V \rbrack]$. Thus, there exist homogeneous polynomials $f_1+(2g-2)(k-1), f_1+(2g-2)(k-1), f_1+(2g-2)(k-1)$ of degree $1 + (2g - 2)(k - 1)$ such that

\begin{equation}
\exp(f_1+(2g-2)(k-1,1) \otimes \xi_1 + f_1+(2g-2)(k-1,2) \otimes \xi_2 + f_1+(2g-2)(k-1,3) \otimes \xi_3)^* W_{3+(2g-2)k} \equiv W \mod F_{3+(2g-2)(k+1)}\mathbb{C}[\lbrack V \rbrack].
\end{equation}

Again, the exponentiated formal vector field can be taken to belong to $\mathfrak{g}^0$. We obtain a new formal function $W_{3+(2g-2)(k+1)}$, such that $(W_{3+(2g-2)(k+1)}, 0) \in \mathfrak{g}^1$ and $W_{3+(2g-2)(k+1)} \equiv W \mod F_{3+(2g-2)(k+1)}\mathbb{C}[\lbrack V \rbrack]$.

Iterating, we obtain infinite sequence of formal diffeomorphisms, and their product obviously converges. As a result, our MC solution $\alpha$ is equivalent to $(W, \alpha^2)$ for some $\alpha^2 \in F_{2g}\mathbb{C}[\lbrack V \rbrack] \otimes \Lambda^2 V$. Since the quotient $\mathbb{C}[\lbrack V \rbrack]/\mathcal{I}$ is finite-dimensional, it follows that the sequence $(\frac{\partial W}{\partial \xi_1}, \frac{\partial W}{\partial \xi_2}, \frac{\partial W}{\partial \xi_3})$ is regular in $\mathbb{C}[\lbrack V \rbrack]$, and hence the Koszul complex $\mathbb{C}[\lbrack V \rbrack] \otimes \Lambda(V)$ with differential $\iota_{dW}$ is a resolution of $\mathbb{C}[\lbrack V \rbrack]/\mathcal{I}$. Since $\alpha^2$ is a cocycle in the Koszul complex, we have that there exists $\gamma^3 \in \mathbb{C}[\lbrack V \rbrack] \otimes \Lambda^3 V$ such that $\iota_{dW}\gamma^3 = -\alpha^2$. Again, $\gamma^3$ can be chosen to belong to $\mathfrak{g}^0$. By the explicit formula (3.4), the exponential of $(0, \gamma^3)$ maps $(W, \alpha^2)$ to $(W, 0)$, and we are done.

\[\square\]

4. Classification theorem

Put $A = \Lambda(V)$ with natural grading $(\deg(V) = 1)$. Consider the following DG Lie algebra $\mathfrak{h}$:
Lemma 4.1. There exists a filtered obvious $\hbar$-linear quasi-isomorphism $\Phi : \mathfrak{h} \rightarrow \mathfrak{g}$, with $\Phi^1$ being the obvious $\hbar$-linear extension of Hochschild-Kostant-Rosenberg map.

Each $\alpha \in \mathfrak{h}^1$ consists of $i$-linear components $\alpha^i$ with $i \geq 3$. Further, each $\alpha^i$ has (finite) decomposition $\alpha^i = \alpha_0^i + \alpha_1^i \hbar + \alpha_2^i \hbar^2 + \ldots$, where

$$\alpha_k^i \in \text{Hom}^{6-3i+(4g-4)k}(A^\otimes i, A)^G.$$ 

Note that if $\alpha_k^i \neq 0$, then $(6-3i+(4g-4)k) \leq 3$. It follows that $\alpha_k^i = 0$ for $3 \leq i < \frac{4g-1}{2}$. We will also need the following elementary observations:

$$L_{2g} g^1 = (\hbar^2 g)^1.$$ 

(4.4) $\Phi^1(\text{Hom}^{2g-2g}(A^\otimes 2g, A)^G) = (\text{Sym}^{2g}(V^\vee) \otimes \Lambda^2(V))^G =

\begin{cases}
C \cdot z_1^{2g} \otimes (\xi_2 \wedge \xi_3) + C \cdot z_2^{2g} \otimes (\xi_3 \wedge \xi_1) + C \cdot z_3^{2g} \otimes (\xi_1 \wedge \xi_2) & \text{if } g \neq 1 \text{ mod } 3 \\
C \cdot z_1^{2g} \otimes (\xi_2 \wedge \xi_3) + C \cdot z_2^{2g} \otimes (\xi_3 \wedge \xi_1) + C \cdot z_3^{2g} \otimes (\xi_1 \wedge \xi_2) + C \cdot z_1^{2g+1} \otimes (\xi_2 \wedge \xi_3) & \text{if } g \equiv 1 \text{ mod } 3;
\end{cases}$

(4.5) $\Phi^1(\text{Hom}^{-2g-1}(A^\otimes (2g+1), A)^G) = (\text{Sym}^{2g+1}(V^\vee))^G =

\begin{cases}
C \cdot z_1^{2g+1} + C \cdot z_2^{2g+1} + C \cdot z_3^{2g+1} & \text{if } g \neq 1 \text{ mod } 3 \\
C \cdot z_1^{2g+1} + C \cdot z_2^{2g+1} + C \cdot z_3^{2g+1} + C \cdot (z_1 z_2 z_3)^{2g+1} & \text{if } g \equiv 1 \text{ mod } 3.
\end{cases}$
Theorem 4.2. Up to equivalence, there is a unique Maurer-Cartan element $\alpha \in \mathfrak{h}^1$ such that $\Phi^1(\alpha_0^3) = -z_1 z_2 z_3$ and

(4.6) \[
\Phi^1(\alpha_1^{2g+1}) = \begin{cases} 
\frac{z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1}}{4}, & \text{if } g \not\equiv 1 \mod 3 \\
\frac{z_1^{2g+1} + z_2^{2g+1} + z_3^{2g+1} + \lambda(z_1 z_2 z_3)^{2g+1}}{3}, & \text{if } g \equiv 1 \mod 3.
\end{cases}
\]

Proof. First we will replace $\alpha$ with another $\alpha'$ satisfying the assumptions of the theorem, and such that $\alpha_i^0 = 0$ for $3 \leq i < 2g$. We will need the following

Lemma 4.3. 1) Take some $\gamma_i^j \in \mathfrak{h}^0$ lying in the component $\text{Hom}^{3-3i+(4g-4)}(A^{\otimes i}, A)$. Then for each MC element $\alpha \in \mathfrak{h}^1$ we have

(4.7) \[
\alpha' = \exp(\gamma_1^i) \cdot \alpha \equiv \alpha - \partial \gamma + \gamma, \alpha \mod (h^2\mathfrak{h})^1.
\]

2) If, moreover, $i \leq 2g - 2$, then we have that $\alpha'$ satisfies the assumptions of the theorem.

Proof. 1) This is evident.

2) According to 1) and (4.3), we only need to check that

(4.8) \[
\Phi^1([\gamma_1^i, \alpha_0^{2g+2-i}]) = 0.
\]

But for degree reasons we have that $\gamma_1^i$ vanishes when restricted to $V^{\otimes i}$, and $\alpha_1^{2g+2-i}$ vanishes when restricted to $V^{\otimes (2g+2-i)}$. Therefore, $[\gamma_1^i, \alpha_0^{2g+2-i}]$ vanishes on $V^{\otimes (2g+1)}$, hence the assertion. \qed

Take the smallest $i_0$ such that $\alpha_1^{i_0} \neq 0$. Suppose that $i_0 < 2g$. Since $\alpha$ is MC solution, we have that $\partial \alpha_1^{i_0} = 0$. Denote by $\bar{A} = \sum_{k \geq 1} \Lambda^k(V)$ the augmentation ideal of $A$. Simple degree counting shows that $\text{Hom}^{6-3i_0+4g-4}(\bar{A}^{\otimes i_0}, A) = 0$. Since the reduced Hochshild complex embeds quasi-isomorphically to the standard one, we have that there exists $\gamma_1^{i_0-1} \in \mathfrak{h}^0$ such that $\partial \gamma_1^{i_0-1} = \alpha_1^{i_0}$. Then, it follows from Lemma 4.3 that $\alpha' = \exp(\gamma_1^{i_0-1})\alpha$ satisfies the assumptions of the theorem. Moreover, $\alpha_1^i = 0$ for $3 \leq i \leq i_0$.

Iterating, we obtain some equivalent MC solution $\alpha' \in \mathfrak{h}^1$ satisfying the assumptions of the theorem and such that $\alpha_i^0 = 0$ for $3 \leq i < 2g$. Assume from this moment that $\alpha$ itself satisfies this property.

Since $\alpha$ is MC solution, we have

(4.9) \[
\partial \alpha_0^3 = 0, \quad \partial \alpha_1^{2g} = 0, \quad \partial \alpha_1^{2g+1} + [\alpha_0^3, \alpha_1^{2g}] = 0.
\]

Therefore, $\alpha_1^{2g}$ satisfies the identity

(4.10) \[
[z_1 z_2 z_3, \Phi^1(\alpha_1^{2g})] = -[\Phi^1(\alpha_0^3), \Phi^1(\alpha_1^{2g})] = -\Phi^1([\alpha_0^3, \alpha_1^{2g}]) = \Phi^1(\partial \alpha_1^{2g+1}) = 0.
\]
From (4.10) and from (4.12) we conclude that

\( \Phi^1(\alpha_1^{2g}) = \left\{ \begin{array}{ll}
0 & \text{if } g \neq 1 \text{ mod } 3 \\
\lambda'(z_1^3 z_2^3 z_3^3) \otimes (\xi_1 \wedge \xi_2) + z_1^3 z_2^3 z_3^3 \otimes (\xi_3 \wedge \xi_1) + z_1^3 z_2^3 z_3^3 \otimes (\xi_2 \wedge \xi_3), & \lambda' \in \mathbb{C} \\
\text{if } g \equiv 1 \text{ mod } 3.
\end{array} \right. \) (4.11)

Simple degree counting shows that

\( \bar{\alpha} := \sum_{n \geq 1} \frac{1}{n!} \Phi^n(\alpha, \ldots, \alpha) \equiv \Phi^1(\alpha_0^3) + h\Phi^1(\alpha_1^{2g+1}) + h\Phi^2(\alpha_0^3, \alpha_1^{2g}) \mod L_{2g} \mathfrak{g}^1 = (h^2 \mathfrak{g})^1. \) (4.12)

**Lemma 4.4.** The polynomial \( \Phi^2(\alpha_0^3, \alpha_1^{2g}) \in \text{Sym}^{2g+1}(V^\vee) \) does not contain terms \( z_i^{2g+1} \).

**Proof.** If \( \alpha_1^{2g} \in \text{Hom}^{2-2g}(A^{\otimes 2g}, A) \) is a Hochshild cocycle and \( \gamma_0^2 \in \text{Hom}^{-3}(A^{\otimes 2}, A) \), then

\( \Phi^2(\partial \gamma_0^2, \alpha_1^{2g}) = \pm \Phi^2(\gamma_0^2, \partial \alpha_1^{2g}) \pm \Phi^1([\gamma_0^2, \alpha_1^1]) \pm \partial \Phi^2(\gamma_0^2, \alpha_1^2) \pm [\Phi^1(\gamma_0^2), \Phi^1(\alpha_1^2)] = \pm \Phi^1([\gamma_0^2, \alpha_1^1]). \) (4.13)

It follows from (4.11) that the RHS of the above identity does not contain terms \( z_i^{2g+1} \). Analogously, if \( \alpha_0^{2g} \in \text{Hom}^{-3}(A^{\otimes 2}, A) \) is a Hochshild cocycle and \( \gamma_0^{2g-1} \in \text{Hom}^{2-2g}(A^{\otimes (2g-1)}, A) \), then we have that \( \Phi^2(\alpha_0^3, \partial \gamma_1^{2g-1}) \) does not contain terms \( z_i^{2g+1} \). Therefore, we may assume that

\( \alpha_0^3 = \Psi^1 \Phi^1(\alpha_0^3), \quad \alpha_1^2 = \Psi^1 \Phi^1(\alpha_1^{2g}), \) (4.14)

where \( \Psi : \mathfrak{g} \to \mathfrak{h} \) is (the obvious \( h \)-linear extension of) Kontsevich’s \( L_\infty \)-quasi-isomorphism. Further, \( L_\infty \)-morphim \( \Phi \) can be taken to be strictly left inverse to \( \Psi \), that is \( \Phi \Psi = \text{Id} \) (Remark 2.9). Under this assumptions, the coefficients of \( \Phi^2(\alpha_0^3, \alpha_1^{2g}) \) in the monomials \( z_i^{2g+1} \) equal to

\[ \pm \Psi^2(\Phi^1(\alpha_0^3), \Phi^1(\alpha_1^{2g}))(\xi_i^{\otimes (2g+1)}), \quad i = 1, 2, 3. \] (4.15)

From the precise formulas for \( \Phi^1(\alpha_0^3) = -z_1 z_2 z_3 \) and \( \Phi^1(\alpha_1^{2g}) \) (formula (4.11)), as well as for the component \( \Psi^2 \) ([Ko2], subsection 6.4, with suitable changes) one obtains that (4.15) equals to zero, as follows. In the notation of [Ko2], subsection 6.4, for each relevant admissible graph \( \Gamma \) we have \( U_{\alpha^1}(\Phi^1(\alpha_0^3), \Phi^1(\alpha_1^{2g}))(\xi_i^{\otimes (2g+1)}) = 0. \) Since \( \Psi^2 \) is a linear combination of \( U_{\Gamma} \), we obtain that (4.15) equals to zero. \( \square \)

Further, \( L_{2g} \mathfrak{g}^1 = (h^2 \mathfrak{g})^1 \) consists of pairs \( (\bar{\alpha}^0, \bar{\alpha}^2) \) such that \( \alpha^0 \in F_4g-1 \mathbb{C}[[V^\vee]] \), and \( \bar{\alpha}^2 \in F_4g-2 \mathbb{C}[[V^\vee]] \otimes \Lambda^2 V. \) From (4.12) and Lemma 4.4 it follows that \( \bar{\alpha} \) satisfies the assumptions of Lemma 3.1. Therefore, \( \bar{\alpha} \) is equivalent to \( (W, 0) \). Since \( \Phi \) induces a bijection on the equivalence classes of Maurer-Cartan solutions, it follows that \( \alpha \) with required properties is unique. \( \square \)
We are interested in the following reformulation of the above Theorem. Suppose that we are given a \((\mathbb{Z}/2)\)-graded \(A_\infty\)-structure \((\mu^1, \mu^2, \ldots)\) on \(A = \Lambda(V)\). Moreover, assume that all \(\mu^i\) are \(G\)-equivariant, \(\mu^1 = 0\), \(\mu^2\) is the usual wedge product, and for \(i \geq 3\) we have (finite) decomposition
\[
\mu^i = \mu^i_0 + \mu^i_1 + \ldots,
\]
where \(\mu^i_k\) is homogeneous of degree \(6 - 3i + (4g - 4)k\) with respect to \(\mathbb{Z}\)-gradings. Moreover, assume that for \(z \in V \subset A\) we have
\[
\mu^3_0(z, z, z) = -z_1z_2z_3,
\]
and
\[
\mu^{2g+1}_1(z, \ldots, z) = \begin{cases} 
\sum_{j=1}^{2g+1} z_j & \text{if } g \not\equiv 1 \mod 3 \\
\sum_{j=1}^{2g+1} z_j + \lambda(z_1z_2z_3) & \text{if } g \equiv 1 \mod 3,
\end{cases}
\]
where \(\lambda \in \mathbb{C}\).

Then such a structure is determined uniquely up to \(G\)-equivariant \(A_\infty\)-quasi-isomorphisms. We denote such an \(A_\infty\)-structure by \(A'\).

### 5. Matrix factorizations

Take \(V = \mathbb{C}^n\) and take some polynomial \(W \in \mathbb{C}[V]\) such that the hypersurface \(W^{-1}(0)\) has (not necessarily isolated) singularity at the origin. Following Orlov, associate to it the triangulated category of singularities:
\[
D_{sg}(W^{-1}(0)) = D^b_{coh}(W^{-1}(0))/\text{Perf}(W^{-1}(0)).
\]
Denote by \(\overline{D}_{sg}(W^{-1}(0))\) the idempotent completion of \(D_{sg}(W^{-1}(0))\). The following Lemma is clear:

**Lemma 5.1.** If \(W\) has the only singular point at the origin, then the triangulated category \(\overline{D}_{sg}(W^{-1}(0))\) is split-generated by the image of the structure sheaf \(\mathcal{O}_0\).

It turns out that the triangulated category \(D_{sg}(W^{-1}(0))\) is \((\mathbb{Z}/2)\)-graded, i.e. the shift by 2 in \(D_{sg}(W^{-1}(0))\) is canonically isomorphic to the identity (this follows from Theorem 5.2 below). Matrix factorizations give a \((\mathbb{Z}/2)\)-graded enhancement of this category. A matrix factorization for \(W\) as above is a \((\mathbb{Z}/2)\)-graded projective (and hence free) \(\mathbb{C}[V]\)-module together with an odd map \(\delta_E : E \to E\), such that \(\delta_E^2 = W \cdot \text{id}\) (in particular, \(E^0\) and \(E^1\) have the same rank). We call this map "differential", although its square does not equal to zero. Matrix factorizations form a strongly pre-triangulated \(D(\mathbb{Z}/2)\)-G category \(MF(W)\).

**Theorem 5.2.** ([Or2], Theorem 3.9) There is a natural exact equivalence of triangulated categories \(\text{Ho}(MF(W)) \sim D_{sg}(W^{-1}(0))\).
This equivalence associates to a matrix factorization \((E, \delta_E)\) a projection of \(\text{Coker}(\delta^1 : E^1 \to E^0)\) (clearly, \(W\) annihilates this \(\mathbb{C}[V^\vee]\)-module, hence it can be considered an object of \(D^b_{\text{coh}}(W^{-1}(0))\)).

Decompose the polynomial \(W\) into the sum of its graded components:

\[(5.2) \quad W = \sum_{i=2}^{k} W_i, \quad W_i \in \text{Sym}^i(V^\vee).\]

Take the one-form

\[(5.3) \quad \gamma = W = \sum_{i=2}^{k} dW_i, \quad W_i \in \text{Sym}^i(V^\vee).\]

Denote by \(\eta = \sum z_k \xi_k\) the Euler vector field on \(V\).

Now take the matrix factorization \((E, \delta_E)\) with \(E = \Omega(V)\), and \(\delta_E = \iota_\eta + \gamma \wedge \cdot\). It is easy to see that \(\delta_E^2 = \gamma(\eta) \cdot \text{id} = W \cdot \text{id}\).

**Lemma 5.3.** ([Se1], Lemma 11.3) The object \(\text{Coker}(\delta^1_E)\) is isomorphic to \(O_0\) in \(D_{sg}(W^{-1}(0))\).

Take the \(D(\mathbb{Z}/2)-G\) algebra

\[(5.4) \quad B_W := \text{End}_{MF(W)}(E).\]

By Lemma 5.3 it is quasi-isomorphic to the \(D(\mathbb{Z}/2)-G\) algebra \(R\text{Hom}_{D_{sg}(W^{-1}(0))}(O_0, O_0)\). We have the following

**Corollary 5.4.** Suppose that \(W\) has the only singular point at the origin. Then there is an equivalence \(D_{sg}(W^{-1}(0)) \cong \text{Perf}(B_W)\).

### 6. Koszul Duality

In this section we describe more explicitly the DG algebra \(B_W\) introduced in 5.4. We also prove that in the special case of our LG model, it is (equivariantly) quasi-isomorphic to the \(A_\infty\)-algebra \(\mathcal{A}'\) from the end of section 4 (Proposition 6.1).

Let \(V = \mathbb{C}^n\). Consider \(\Omega(V) = \mathbb{C}[V^\vee] \otimes \Lambda(V^\vee)\) as a complex of \(\mathbb{C}[V^\vee]\)-modules with \(\deg(\mathbb{C}[V^\vee] \otimes \Lambda^k V^\vee) = -k\) and differential \(\iota_\eta\), where \(\eta = \sum_{k=1}^{n} z_k \xi_k\) is the Euler vector field.

Consider the DG algebra \(B = \text{End}_{\mathbb{C}[V^\vee]}(\Omega(V))\). We have \(H(B) \cong \Lambda(V)\). Further, we can identify

\[(6.1) \quad B \cong \Omega(V) \otimes \Lambda(V),\]

where for \(f \in \text{Sym}(V^\vee), \beta \in \Lambda^s V^\vee, \theta \in \Lambda(V)\) the element \(f \beta \otimes \theta \in \Omega(V) \otimes \Lambda(V)\) corresponds to the endomorphism \(f \beta \wedge \iota_\theta(\cdot) \in B = \text{End}_{\mathbb{C}[V^\vee]}(\Omega(V))\).
Explicitly, the differential $d : \Omega(V) \otimes \Lambda(V) \to \Omega(V) \otimes \Lambda(V)$ is given by the formula
\begin{equation}
(6.2)
d(f \beta \otimes \theta) = \iota_{\eta}(f \beta) \otimes \theta.
\end{equation}

It is well known that DG algebra $B$ is formal. Moreover, we can write down explicitly the quasi-isomorphism of DG algebras $i : \Lambda(V) \to B$,
\begin{equation}
(6.3)
i(\theta) = 1 \otimes \theta.
\end{equation}

Also, consider the natural projection $p : B \to \Lambda(V)$. Clearly, $pi = id_{\Lambda(V)}$. Further, $ip$ differs from $id_B$ by homotopy given by the formula
\begin{equation}
(6.4)
h(f \beta \otimes \theta) = \begin{cases} 
0 & \text{if } f \beta = \lambda, \lambda \in \mathbb{C} \\
\frac{1}{w}df \wedge \beta \otimes \theta & \text{otherwise},
\end{cases}
\end{equation}

where $w = r + s$, $f \in \text{Sym}^r(V^\vee)$, $\beta \in \Lambda^s(V^\vee)$. Moreover, the maps $h$, $p$, $i$ satisfy the following identities:
\begin{equation}
(6.5)
h^2 = 0, \quad ph = 0, \quad hi = 0.
\end{equation}

Now take the polynomial $W \in \mathbb{C}[V^\vee]$ with singularity at the origin. It is clear that $B_W^{gr} \cong B_W^{gr}$, where $B_W^{gr}$ (resp. $B_W^{gr}$) is the underlying $(\mathbb{Z}/2)$-graded algebra of $B_W$, and similarly for $B^{gr}$. Denote the differential on $B_W$ by $\tilde{\partial}$. We have the following explicit formula for the difference of differentials:
\begin{equation}
(6.6)
(\tilde{\partial} - \partial)(f \beta \otimes \theta) = (-1)^{|\beta| - 1} \sum_{k=1}^{n} g_k f \beta \otimes \iota_{dz_k} \theta.
\end{equation}

From Homological Perturbation Lemma we obtain a $(\mathbb{Z}/2)$-graded $A_\infty$-structure $A$ on the graded vector space $A$ together with $A_\infty$-quasi-isomorphism $A \to \hat{B}$. Explicit computation of $\mu^k : A^{\otimes k} \to A$ goes as follows. Consider a ribbon tree with $(k + 1)$ semi-infinite edges, $k$ incoming and one outgoing, which has only bivalent and trivalent vertices. Associate with each vertex and each edge an operation as follows:
\begin{equation}
(6.7)
\begin{cases} 
\text{for a bivalent vertex} & b \mapsto (-1)^{|b|}(\tilde{\partial} - \partial)(b), B \to B; \\
\text{for a trivalent vertex} & (b_2, b_1) \mapsto (-1)^{|b_1|}b_2b_1, B^{\otimes 2} \to B; \\
\text{for a finite edge} & b \mapsto (-1)^{|b|-1}h(b), B \to B; \\
\text{for an incoming edge} & a \mapsto i(a), A \to B; \\
\text{for an outgoing edge} & b \mapsto p(b), B \to A.
\end{cases}
\end{equation}

Then each such tree gives a map $A^{\otimes k} \to A$ in an obvious way. The explicit expression is just the sum of contributions of all possible trees. The sum is actually finite because
\begin{equation}
(6.8)
(\tilde{\partial} - \partial)(C[[V^\vee]] \otimes \Lambda^k(V^\vee) \otimes \Lambda(V)) \subset C[[V^\vee]] \otimes \Lambda^k(V^\vee) \otimes \Lambda(V), \quad \text{and}
\end{equation}
The components $f_k : A \otimes A \to B$ of the $A_{\infty}$-quasi-isomorphism are defined in the same way with the only difference: to the outgoing edge one attaches the operation $b \to h(b)$. Again, the sum over trees is actually finite.

Proposition 6.1. In the above notation, the resulting $A_{\infty}$-algebra $A$ is $G$-equivariantly equivalent to $\Lambda(V)$ with the $A_{\infty}$-structure $A'$ from the end of section 4.

Proof. It is useful to take the following $\mathbb{Z}$-grading on $B = \Omega(V) \otimes \Lambda(V)$.

\begin{equation}
\deg(\text{Sym}^i(V^\vee) \otimes \Lambda^j(V^\vee) \otimes \Lambda^k(V)) = 2i - j + k.
\end{equation}

Then $\partial$ has degree 3, $h$ has degree $-3$. If we want $\tilde{\partial}$ to have degree 3, we should introduce a formal parameter $h$ with degree $(4 - 4g)$. Further, we should write $g_1 = -\frac{z_2 z_3}{3} + \frac{2g}{3}$, $g_2 = -\frac{z_1 z_3}{3} + \frac{2g}{3}$, $g_3 = -\frac{z_1 z_2}{3} + \frac{2g}{3}$.

Corollary 6.2. The category $\mathcal{D}_{sg}(W^{-1}(0))$ is equivalent to $\text{Perf}(A')$. 

Figure 1.

Figure 2.

From Corollary 5.4 and Proposition 6.1, we obtain the following

Corollary 6.2. The category $\mathcal{D}_{sg}(W^{-1}(0))$ is equivalent to $\text{Perf}(A')$.
Further, Orlov’s theorem can be extended to the equivariant setting. Let \( K \subset G \) be the cyclic subgroup of order \( 2g+1 \), generated by the diagonal matrix with diagonal entries \((\zeta, \zeta, \zeta^{2g-1})\), where \( \zeta \) is the primitive \((2g+1)\)-th root of unity. Then \( D_{sg,K}(W^{-1}(0)) \) is equivalent to \( MF_{K}(W) \). The projection of \( O_0 \otimes \mathbb{C}[K] \) split-generates \( D_{sg,K}(W^{-1}(0)) \). The projection of \( O_0 \otimes \mathbb{C}[K] \) split-generates \( D_{sg,K}(W^{-1}(0)) \).

In the \( K \)-equivariant matrix factorizations it corresponds to \((\Omega(V) \otimes \mathbb{C}[K], \iota_{\eta} + \gamma \wedge \cdot)\). Its endomorphism DG algebra is naturally isomorphic to the smash product \( \mathbb{C}[K] \# B_W \), which is further \( A_{\infty} \)-quasi-isomorphic to \( \mathbb{C}[K] \# A' \).

The result is

\[
D_{sg,K}(W^{-1}(0)) \sim \text{Perf}(\mathbb{C}[K] \# A').
\]

7. Reconstruction Theorem

In this section we show that one can recover the polynomial \( W \) (up to formal change of variables) from the \( A_{\infty} \)-structure on \( \Lambda(V) \) transferred from \( D(\mathbb{Z}/2) \)-G algebra \( B_W \), as in the previous section, for general \( W \). Our proof is based on Kontsevich formality theorem, and on Keller’s paper [Ke1].

More precisely, our setting is the following. Let \( k \) be any field of characteristic zero and \( V = k^n, \ n \geq 1 \). Consider a polynomial \( W = \sum_{r=3}^{\infty} W_r \in k[V^\vee] \), with \( W_i \in \text{Sym}^i(V^\vee) \). Take the D \((\mathbb{Z}/2)\)-G algebra \( B_W \). We have the canonical isomorphism of super-algebras

\[
\Lambda(V) \cong H(B_W).
\]

**Theorem 7.1.** Let \( W, W' \) be non-zero polynomials as above. Suppose that DG algebras \( B_W \) and \( B_{W'} \) are quasi-isomorphic, and the chain of quasi-isomorphisms connecting \( B_W \) with \( B_{W'} \) induces the identity in cohomology via identifications (7.1). Then \( W' \) can be obtained from \( W \) by a formal change of variables of the form

\[
z_i \rightarrow z_i + O(z^2).
\]

**Proof.** We introduce four pro-nilpotent DG algebras. First define the DGLA \( \tilde{g} \) by the formula

\[
\tilde{g}^d = \prod_{j+2k=d+1 \atop k \in \mathbb{Z}, i \geq d+2} (\text{Sym}^i(V^\vee) \otimes \Lambda^j(V)) \cdot h^k,
\]

and \( L_r \tilde{g}^d \) is the part of the product with \( i \geq d+1+r, \ r \geq 1 \) (the differential is zero, and the bracket is Schouten one). Further, put

\[
\tilde{b}_1^d = \prod_{i+j+2k=d+1 \atop k \in \mathbb{Z}, i \geq d+2} (\text{Hom}^j(\Lambda(V) \otimes \Lambda(V)) \cdot h^k,
\]

and \( L_r \tilde{b}_1^d \) is the part with \( i \geq d+1+r \) (the differential is Hochshild one and the bracket is Gerstenhaber one). Now, take the ”lower” grading on \( k[[V^\vee]] \), with \( k[[V^\vee]]_d = \text{Sym}^d(V^\vee) \).
Of course, $k[[V^\vee]]$ is the direct product of its graded components, but not direct sum. For the rest of this section we will denote the standard grading by upper indices, and the “lower” grading by the lower indices. Define the DGLA $\tilde{h}_2$ by the formula

$$\tilde{h}_2^d = \prod_{i=0}^{d} \prod_{k \in \mathbb{Z}, i \geq 0, j + 2k \geq 1} \left( \text{Hom}_{j'}(k[[V^\vee]]^i, k[[V^\vee]]) \cdot h^k, \right)$$

with $L_r \tilde{h}_2^d$ being the part of the product with $j' + 2k \geq r$.

Now take the Koszul DG $k[[V^\vee]]$-Λ(V)- bimodule $X = \Lambda(V) \otimes k[[V^\vee]]$ with the ”upper” and ”lower” gradings $X^j = \Lambda^{-j}(V^\vee) \otimes \text{Sym}^j(V^\vee)$, and with differential $\iota_\eta$ of degree $(1, 0)$. Define the DGLA $Q$ by the formula

$$Q^d = \tilde{h}_1^d + \tilde{h}_2^d + \prod_{i_1 + i_2 + j - 2k = d} \left( \text{Hom}_{j'}(\Lambda(V)^{i_1} \otimes X \otimes k[[V^\vee]]^{i_2}, X) \cdot h^k, \right)$$

where the differential and the bracket are induced by those in the Hochshild complex of the DG category $C$, where

- $\text{Ob}(C) = \{Y_1, Y_2\}$;
- $\text{Hom}_C(Y_1, Y_1) = k[[V^\vee]]$;
- $\text{Hom}_C(Y_2, Y_2) = \Lambda(V)$;
- $\text{Hom}_C(Y_1, Y_2) = X$;
- $\text{Hom}_C(Y_2, Y_1) = 0$.

Composition law in $C$ comes from the bimodule structure on $X$ (and from algebra structures on $k[[V^\vee]], \Lambda(V)$). Thus, the DGLA structure on $Q$ is defined. Further, define

$$L_r Q^d = L_r \tilde{h}_1^d + L_r \tilde{h}_2^d + \text{(part of the product with } 2k + j' - j \geq r \text{), } r \geq 1.$$ 

It follows from [Ke1], Lemma in Subsection 4.5, that natural projections $p_i : Q \rightarrow \tilde{h}_i$, $i = 1, 2$, are quasi-isomorphisms of DGLA’s. Moreover, both $p_1, p_2$ are filtered quasi-isomorphisms, as it is straightforward to check.

According to [Ko3], one can attach to all Kontsevich admissible graphs (relevant for the formality theorem) rational weights, in such a way that they give formality $L_\infty$-quasi-isomorphism (i.e. satisfy the relevant system of quadratic equations). In this way we obtain filtered $L_\infty$- quasi-isomorphism $\mathcal{U} : \tilde{g} \rightarrow \tilde{h}_2$.

Since $p_1, p_2, \mathcal{U}$ are filtered $L_\infty$- quasi-isomorphisms, we have by Lemma 2.6 that the composition $p_1 \circ p_2^{-1} \circ \mathcal{U} : \tilde{m}g \rightarrow \tilde{h}_1$, considered as morphism in the homotopy category of pro-nilpotent DG Lie algebras, induces a bijection between the sets of equivalence classes of MC solutions in $\tilde{g}$ and $\tilde{h}_1$.

To prove the theorem, we need to prove that, under the assumptions of the theorem, MC equations $W, W' \in \tilde{g}$ are equivalent. Indeed, this means that $W'$ is the pullback of $W$.
under the formal diffeomorphism of $V$ with zero differential at the origin. Therefore, it suffices to prove the following

**Lemma 7.2.** Under the above bijection between equivalence classes of MC solutions, the class of $W \in \tilde{g}^1$ corresponds to the class $\alpha \in \tilde{h}^1$ of the $(\mathbb{Z}/2)$-graded $A_\infty$-structure on $\Lambda(V)$ transferred from $\mathcal{B}_W$ to $H(\mathcal{B}_W) \cong \Lambda(V)$.

**Proof.** First note that $U^k(W,\ldots,W) = 0$ for $k > 1$, and $U^1(W)$ has the only constant component which is equal to $W$.

Denote by $\mu = (\mu^3, \mu^4, \ldots)$ the $A_\infty$-structure on $\Lambda(V) \cong H(\mathcal{B}_W)$ transferred from $\mathcal{B}_W$, as in the previous section. Let $A$ be the resulting $A_\infty$-algebra. Denote by $f = (f_1, f_2, \ldots)$ the $A_\infty$-quasi-isomorphism $A \to \mathcal{B}_W$. Also denote by $f_0 \in \mathcal{B}_W^1$ the multiplication by the 1-form $\gamma$. We can consider $f_i$ as maps $f_i : A^{\otimes i} \otimes X \to X$. Now define $\tilde{\alpha} \in Q^1$ with components $\mu^i$, $i \geq 3$, $f_j$, $j \geq 0$, and $W \in \tilde{h}_2^1$. Then $\tilde{\alpha}$ is MC solution,

$$p_1(\tilde{\alpha}) = \alpha, \quad p_2(\tilde{\alpha}) = U^1(W) = \sum_{k \geq 1} \frac{1}{k!} U^k(W,\ldots,W).$$

Thus, classes of MC solutions $W \in \tilde{g}^1$ and $\alpha \in \tilde{h}^1$ correspond to each other. Lemma is proved.\square

Theorem is proved.\square

It follows from the proof of the above Theorem that there exists filtered $L_\infty$-morphism $\tilde{\Phi} : \tilde{h}_1 \to \tilde{g}$ such that the polynomial $W$ can be reconstructed from $\mathcal{B}_W$ as follows. Take $\alpha \in \tilde{h}_1^1$ to be MC solution corresponding to the $A_\infty$-structure on $\Lambda(V)$ transferred from $\mathcal{B}_W$. Put

$$\beta = \sum_{k \geq 1} \frac{1}{k!} \tilde{\Phi}^k(\alpha,\ldots,\alpha).$$

Decompose $\beta$ into the sum $\beta^0 + \beta^2 + \cdots + \beta^{2[\frac{n}{2}]}$, with $\beta^{2j} \in k[[V^\vee]] \otimes \Lambda^{2j}(V)$. Then $W$ can be obtained from $\beta^0$ by a formal change of variables of type (7.2).

**Remark 7.3.** Note that in Theorem 7.1 we required our polynomials $W, W'$ not to have terms of order 2, and also required the induced isomorphism $H(\mathcal{B}_W) \to H(\mathcal{B}_{W'})$ to be compatible with identifications (7.1). The reason is that in general Maurer-Cartan theory for DGLA’s works well only in the pro-nilpotent case. However, it should be plausible that in the case $k = \mathbb{C}$ one can drop these assumptions. Of course, in this case one also should drop the requirement on the change of variables to be of type (7.2).
8. The McKay correspondence

Take $V = \mathbb{C}^3$ and $K \subset G$ as in Section 6. The quotient $V/K$ has a canonical crepant resolution

\[
X = \text{Hilb}_K(V) \to V/K.
\]

More elementary, $X$ can be given by a fan describing it, due to the paper [CR]. Take $\Gamma \subset \mathbb{R}^3$, $N = \mathbb{Z}^3 + \mathbb{Z} \cdot \frac{1}{2g+1}(1,1,2g-1)$. Now, if we take a fan $\Sigma$ consisting of a positive octant and its faces, then we have $X_\Sigma \cong V/K$. To describe $X$, we should subdivide the fan $\Sigma$. Namely, take the fan $\Sigma'$ consisting of the cones generated by

\[
\begin{align*}
(8.2) & \quad \frac{1}{2g+1}(k,k,2g+1-2k), \frac{1}{2g+1}(k+1,k+1,2g-1-2k), (1,0,0), \quad 0 \leq k \leq g-1; \\
(8.3) & \quad \frac{1}{2g+1}(k,k,2g+1-2k), \frac{1}{2g+1}(k+1,k+1,2g-1-2k), (0,1,0), \quad 0 \leq k \leq g-1; \\
(8.4) & \quad \frac{1}{2g+1}(g,g,1), (1,0,0), (0,1,0),
\end{align*}
\]

and all their faces (see Figure 3 for the case $g = 3$). Then $X \cong X_{\Sigma'}$.

The exceptional surface $Y_k \subset X$ corresponding to the vector $\frac{1}{2g+1}(k,k,2g+1-2k) \in N$ is

\[
(8.5) \begin{cases}
\text{the rational ruled surface } F_{2g+1-2k} \cong \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O} \oplus \mathcal{O}(2g+1-2k)) & \text{for } 1 \leq k \leq g-1 \\
\mathbb{C}P^2 & \text{for } k = g.
\end{cases}
\]
The surfaces $Y_i$ and $Y_j$ have empty intersection if $|i - j| \geq 2$. Further, the surfaces $Y_i$ and $Y_{i+1}$ intersect transversally along the curve $C_i \subset X$, where $1 \leq i \leq g - 1$. The curve $C_i$ is

\begin{equation}
(8.6)
\begin{cases}
\text{the } "\infty\text{-section" } & \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(2g + 1 - 2i)) \subset \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O} \oplus \mathcal{O}(2g + 1 - 2i)) \cong Y_i \text{ on } Y_i \quad \text{for } 1 \leq i \leq g - 1 \\
\text{the } "\text{zero-section" } & \mathbb{P}_{\mathbb{C}P^1}([\mathcal{O}]) \subset \mathbb{P}_{\mathbb{C}P^1}([\mathcal{O} \oplus \mathcal{O}(2g - 1 - 2i)]) \cong Y_{i+1} \text{ on } Y_{i+1} \quad \text{for } 1 \leq i \leq g - 2 \\
\text{the line on } \mathbb{C}P^2 & \cong Y_g \quad \text{for } k = g.
\end{cases}
\end{equation}

The function $W \in \mathbb{C}[V^\vee]$ is $A$-invariant, hence gives a function on $V/K$, and on $X$. The LG model $(X, W)$ is a mirror to the genus $g$ curve. The only singular fiber of $W$ on $X$ is $X_0 = H$. The surface $H$ has $(g + 1)$ irreducible components $H_1, \ldots, H_{g+1}$, where $H_i$ is $Y_i$ defined above for $1 \leq i \leq g$, and $H_{g+1}$.

The divisor $H$ has simple normal crossings. We have already described the intersections between $H_i$ for $1 \leq i \leq g$. Further, the intersection $H_i \cap H_{g+1}$ is:

\begin{equation}
(8.7)
\begin{cases}
\text{the section } & \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O} \times (y_0 y_1, y_0^{2g+1} + y_1^{2g+1})) \subset \mathbb{P}_{\mathbb{C}P^1}(\mathcal{O}(2) \oplus \mathcal{O}(2g + 1)) \cong H_1 \quad \text{for } i = 1 \\
\text{the union of two fibers } & \{y_0 y_1 = 0\} \subset \mathbb{P}_{\mathbb{C}P^1}([\mathcal{O} \oplus \mathcal{O}(2g - 1 - 2i)]) \quad \text{for } 2 \leq i \leq g - 1 \\
\text{a non-degenerate conic in } & \mathbb{C}P^2 \cong H_g \quad \text{for } i = g.
\end{cases}
\end{equation}

Here $(y_0 : y_1)$ are homogeneous coordinates on $\mathbb{C}P^1$.

The triple intersection $H_i \cap H_{i+1} \cap H_{g+1}$ consists of two points for each $1 \leq i \leq g - 1$. The corresponding dual CW complex of this configuration is homeomorphic to $S^2$.

**Theorem 8.1.** The triangulated category $D_{sg}(H)$ is equivalent to $D_{sg,K}(W^{-1}(0))$.

**Proof.** This is a special case of [BP], Theorem 1.1. \hfill \Box

Denote by $\overline{D_{sg}(H)}$ the split-closure of the triangulated category of singularities $D_{sg}(H)$.

**Corollary 8.2.** There is an equivalence $\overline{D_{sg}(H)} \cong \text{Perf}(\mathbb{C}[K] \# \mathcal{A}')$.

**Proof.** Indeed, this follows from Theorem 8.1 and the equivalence (6.12). \hfill \Box

### 9. Fukaya categories

Let $M$ be a compact oriented surface of genus $\geq 2$. Choose a symplectic for $\omega$ on $M$. Let $\pi : S(TM) \to M$ be a bundle of unit circles in the tangent bundle (it does not depend on Riemann metric). Fix a 1-form $\theta$ on $S(TM)$, such that $d\theta = \pi^* \omega$. The definition of the Fukaya category $\mathcal{F}(M)$ involves the equivalence class of $\theta$ modulo exact 1-forms.
Consider some connected Lagrangian submanifold of $M$, i.e. just simple closed curve $L \subset M$. Let $\sigma : L \to S(TM)|_L$ be the natural section for some choice of orientation on $L$. The curve $L$ is called balanced if $\int_L \sigma^* \theta = 0$. This property does not depend on the orientation on $L$. Contractible curves can not be balanced. Every other isotopy class of curves in $M$ contains a balanced representative, which is unique up to Hamiltonian isotopy.

Objects of the Fukaya category $\mathcal{F}(M)$ are balanced curves $L \subset M$ equipped with orientations and spin structures. For each two objects $L_0, L_1$ of $\mathcal{F}(M)$ we choose a family of functions $(H_t)_{t \in [0,1]}$ such that the associated Hamiltonian isotopy $(\phi_t)$ has the property that $\phi_1(L_0)$ is transverse to $L_1$. The morphism space between $L_0$ and $L_1$ in $\mathcal{F}(M)$ is the associated $(\mathbb{Z}/2)$-graded vector space

$$\text{Hom}_{\mathcal{F}(M)}(L_0, L_1) = CF^c(L_0, L_1) = \bigoplus_{x \in L_0 \cap \phi_1^{-1}(L_1)} \mathbb{C}x.$$

The generator $x$ is even if the local intersection number is $-1$, and is odd otherwise.

Take some objects $L_0, \ldots, L_d$ for some $d \geq 1$. Let $x_0 \in M$ be a point that gives rise to a generator of $CF^c(L_0, L_d)$ and analogously $x_k \in M$ for $CF^c(L_{k-1}, L_k)$, $1 \leq k \leq d$. Making some choices of auxiliary data, one obtains a moduli space of perturbed pseudo-holomorphic polygons. Points in this space are represented by maps $u : S \to M$, where

1) $S$ is a $(d + 1)$-pointed disc, i.e. a Riemann surface isomorphic to the closed disk minus $(d + 1)$ boundary points. The points at infinity are denoted by $\zeta_0, \ldots, \zeta_d$. Their ordering is fixed in a way compatible with the clockwise cyclic order. We denote by $\partial_k S$ the boundary component between $\zeta_k$ and $\zeta_{k+1}$, for $0 \leq k \leq d - 1$, and by $\partial_d S$ the boundary component between $\zeta_d$ and $\zeta_0$. Further, $u : S \to M$ is a smooth map satisfying the condition $u(\partial_k S) \subset L_k$. Near infinite points $\zeta_k$, we have convergence to $x_k$ (in some to be specified sense). Also, $u$ satisfies the inhomogeneous Cauchy-Riemann equation

$$\frac{1}{2} (du + J_z(u(z)) \circ du \circ i) = \nu_z(u(z)),$$

where $i$ is the complex structure on $S$, $J$ is a family of almost complex structures on $M$ parameterized by $S$, and $\nu$ is the inhomogeneous term.

2) $J$ and $\nu$ are the auxiliary data, which should be chosen carefully.

For details of the definition, see [Se2].

Every perturbed pseudo-holomorphic polygon $u$ has an associated virtual or expected dimension $\text{vdim}(u)$, defined using index theory. For generic choice of auxiliary data, the moduli space will be regular, hence smooth and of dimension $\text{vdim}(u)$ near any $u$. Moreover, there will be only finitely many $u \in \mathcal{M}(x_0, \ldots, x_d)$ with $\text{vdim}(u) = 0$. There is a particular signed count of such points (where the signs depend on the Spin structures,
among other things), denoted by \( m(x_0, \ldots, x_d) \in \mathbb{Z} \). One defines the \( A_\infty \)-structure on \( \mathcal{F}(M) \) by setting
\[
\mu^d(x_d, \ldots, x_1) = \sum_{x_0} m(x_0, \ldots, x_d) x_0.
\]

We will use the following Seidel’s version of the definition of morphisms and higher products \( \mu^k \) in the Fukaya category \( \mathcal{F}(M) \), which works under some assumptions, and is sufficient for our purposes.

Fix some complex structure on \( M \) and a finite collection \( \mathcal{L} \) of objects in the Fukaya category, which are in general position. From this moment, we will consider only curves from this set. For \( L_0 \neq L_1 \), let \( \mathcal{CF}(L_0, L_1) \) be the direct sum of 1-dimensional spaces \( \mathbb{C}x \), where \( x \in L_0 \cap L_1 \). The generator \( x \) is even if the local intersection number of \( L_0 \) and \( L_1 \) equals to \(-1\), and odd otherwise.

Further, for \( L_0 = L_1 = L \) fix some generic point \( * \in L \) and define \( \mathcal{CF}(L, L) := \mathbb{C}e \oplus \mathbb{C}q \), where \( e \) and \( q \) are even and odd respectively. Let \( (L_0, L_1, \ldots, L_d) \) be a collection of objects, and let \( x_0, x_1, \ldots, x_d \) be generators of Floer complexes \( \mathcal{CF}(L_0, L_d), \mathcal{CF}(L_0, L_1), \ldots, \mathcal{CF}(L_{d-1}, L_d) \) respectively. Define \( \mathcal{M}(x_0, \ldots, x_d) \) to be the moduli space of maps \( u' : S \to M \), where \( S \) is a \((d + 1)\)-pointed disk, \( u'(\partial_k S) \subset L_k \), and \( u' \) is holomorphic. If the generator \( x_k \) corresponds to transversal intersection point, then we require \( \lim_{z \to \zeta_k} u'(z) = x_k \). If \( x_k \) is of the type \( e \) or \( q \), then \( u' \) is required to extend over \( \zeta_k \). Further, in the cases \((k = 0 \text{ and } x_0 = e), (k > 0 \text{ and } x_k = q)\) we impose the condition \( u'(\zeta_k) = * \). Otherwise \( u'(\zeta_k) \in L_k \) can be arbitrary.

**Lemma 9.1.** ([Se1], Lemma 5.3) All \( u' \in \mathcal{M}(x_0, \ldots, x_d) \) which are non-constant, are regular points. The virtual dimension at such a point is at least the number of \( k \) such that no point constraint are imposed on \( u'(\zeta_k) \). Equality holds iff \( u' \) is an immersion, extends to a map with no branching at \( x_k \) which are of type \( e \) and \( q \), and is locally embedded as a convex corner at those \( x_k \) which are transverse intersection points.

We define \( m(x_0, \ldots, x_d) \) to be the signed count of points \( u' \in \mathcal{M}(x_0, \ldots, x_d) \) such that \( \mathrm{vdim}(u') = 0 \), assuming that all these \( u' \) are regular and there are only finitely many of them.

We will need another space \( \hat{\mathcal{M}}(x_0, \ldots, x_d) \) of holomorphic polygons which can break into pieces. A point in this space is the following data:

1) A planar tree \( T \) with \((d + 1)\) semi-infinite edges and with vertices of valency \( \geq 2 \). The regions of \( \mathbb{R}^2 \setminus T \) must be labelled with \( L_0, \ldots, L_d \) in the anti-clockwise order. The semi-infinite edge separating \( L_0 \) and \( L_d \) is called a root. Consider a flag in \( T \), i.e. a pair (edge, adjacent vertex). This edge separates two regions labelled by \( L_i, L_j \). We attach to
it a generator of \( \text{CF}^*(L_i, L_j) \). If the edge is semi-infinite and separates \( L_{k-1}, L_k \) (resp. \( L_d, L_0 \)) then the corresponding generator is the given \( x_k \) (resp. \( x_0 \)) If \( L_i \neq L_j \), then we require the generators associated with both flags containing this edge to coincide. If \( L_i = L_j \), then the flag closer to the root should carry \( q \) as generator, and the other one should carry \( e \) as generator.

2) For every vertex \( v \) of valency \((r + 1)\) in \( T \) we want to have \((r + 1)\)-pointed disc \( S_v \) together with a stable holomorphic map \( u'_v : S_v \to M \). The vertex is surrounded by regions labelled \((L_{i_0}, \ldots, L_{i_r})\), where \( i_0 < \cdots < i_r \). We require \( u'_v(\partial_k S_v) \subset L_{i_k} \). Further, the flags containing our vertex give generators of Floer complexes, and we impose the conditions on the behavior of \( u'_v \) near \( \zeta_k \) as above.

We still denote such an object by \( u' \). It has a virtual dimension which can be computed as the sum of virtual dimensions of holomorphic polygons attached to the vertices, and then adding 1 for each finite edge separating regions labelled by \( L_i \neq L_j \).

**Proposition 9.2.** ([Se1], Proposition 5.4) Let \((L_0, \ldots, L_d)\) and \((x_0, \ldots, x_d)\) be such that each point \( u' \in \mathcal{M}(x_0, \ldots, x_d) \) with \( \text{vdim}(u') \leq 0 \) is regular. Suppose that \( \overline{\mathcal{M}}(x_0, \ldots, x_d) \) contains no points of virtual dimension \( \leq 0 \). Then, by making appropriate choices in the definition of the Fukaya category, one can achieve that \( m(x_0, \ldots, x_d) = m(x_0, \ldots, x_d)' \).

**Corollary 9.3.** Take \((L_0, \ldots, L_d) \in \mathcal{L} \). Then, by making appropriate choices, one can achieve that \( m(x_0, \ldots, x_d) = m(x_0, \ldots, x_d)' \) in the following two situations: if the \( L_k \) are pairwise different; or only two of them coincide and these two are either \((L_{k-1}, L_k)\), or \((L_d, L_0)\).

Now we consider some examples.

**Constant triangles.** Let \( L_0 \neq L_1 \). Then the constant triangle at any point \( x \in L_0 \cap L_1 \) contributes to the products

\[ \mu^2(e, x), \mu^2(x, e) : \text{CF}^*(L_0, L_1) \to \text{CF}^*(L_0, L_1); \]

\[ \mu^2(x, x) : \text{CF}^*(L_1, L_0) \otimes \text{CF}^*(L_0, L_1) \to \text{CF}^*(L_0, L_0). \]

No non-constant triangles can contribute to these products, and taking signs into account one obtains

\[ \mu^2(x, e) = x, \quad \mu^2(e, x) = (-1)^{|x|} x, \quad \mu^2(x, x) = (-1)^{|x|} = q. \]

Further,

\[ \mu^2(e, e) = e, \quad \mu^2(e, q) = -q, \quad \mu^2(q, e) = q, \mu^2(q, q) = 0. \]
Non-constant polygons. Here the Spin structures become essential. We encode them picking a generic marked point $\circ \neq \ast$ on each $L$.

Consider pairwise distinct curves $L_0, \ldots, L_d$ and take the generators of corresponding Floer complexes $x_0, \ldots, x_d$. Further, let $u' \in \mathcal{M}(x_0, \ldots, x_d)'$ be of virtual dimension zero. Suppose that $L_1, \ldots, L_d$ are oriented compatibly with the anti-clockwise orientation on $\partial S$, and none of the points $\circ$ lies on the boundary of $u$. Reversing the orientation of $L_k$, $0 < k < d$, changes the sign by $(-1)^{|x_k|}$. Reversing the orientation of $L_d$ changes the sign by $(-1)^{|x_0| + |x_d|}$. Finally, each time when the boundary of $u'$ passes through one of the points $\circ \in L_k$, the sign changes by $(-1)$.

The other case we are interested in is when $L_0 = L_d$, and $L_0, \ldots, L_{d-1}$ are pairwise distinct. Take $u' \in \mathcal{M}(e, x_1, \ldots, x_d)$ with $vdim(u') = 0$. If $L_1, \ldots, L_d$ are oriented compatibly with the anti-clockwise orientation on $\partial S$, and the boundary of $u'$ does not meet $\circ$, then $u'$ contributes with the sign $+1$. Otherwise the sign counting is the same as in the previous case.

10. Split-generators in Fukaya categories

Suppose that $\mathcal{A}$ is some $(\mathbb{Z}/2)$-graded $A_{\infty}$-category with weak units, and $E \in \text{Perf}(\mathcal{A})$ is an object which split-generates $\text{Perf}(\mathcal{A})$. Then it is well-known that the natural $A_{\infty}$-functor $\text{Hom}(-, E) : \text{Perf}(\mathcal{A}) \to \text{Perf}(\text{End}(E))$ is a quasi-equivalence, see [Ke2].

Let $L_0, L_1$ be two objects of the Fukaya category $\mathcal{F}(M)$, and the Spin structure on $L_1$ is non-trivial. The Dehn twist $\tau_{L_1}$ is a balanced symplectic automorphism of $M$, hence $\tau_{L_1}(L_0)$ is again balanced. According to [Se1] and [Se2], we then have the following exact triangle in $D^\pi \mathcal{F}(M)$:

$$HF(L_1, L_0) \otimes L_1 \to L_0 \to \tau_{L_1}(L_0).$$

(10.1)

We will need the following two Lemmas from [Se1].

Lemma 10.1. ([Se1], Lemma 6.1) Let $L_1, \ldots, L_r$ be objects of $\mathcal{F}(M)$ whose Spin structures are non-trivial. Suppose that $L_0$ is another object, and $\tau_{L_r} \ldots \tau_{L_1}(L_0) \cong L_0[1]$. Then $L_0$ is split-generated by $L_1, \ldots, L_r$.

Lemma 10.2. ([Se1], Lemma 6.2) Let $L_1, \ldots, L_r$ be objects of $\mathcal{F}(M)$ whose Spin structures are non-trivial and which are such that $\tau_{L_r} \ldots \tau_{L_1}$ is isotopic to the identity. Then they split-generate $\mathcal{F}(M)$.
11. Gradings

Since $M$ is not Calabi-Yau, the $(\mathbb{Z}/2)$-grading on $M$ cannot be improved to $\mathbb{Z}$-gradings. However, one can improve the situation as follows. Fix a complex structure on $M$ and take a meromorphic section $\eta^r$ of the line bundle $T^*M^\otimes r$. Let $D$ be its divisor. For any oriented $L \subset M \setminus \text{Supp}(D)$ we have natural map $L \to S^1$, defined by the formula

\[(11.1) \quad x \to \frac{\eta^r(X^\otimes r)}{[\eta^r(X^\otimes r)]},\]

where $X$ is a tangent vector to $L$ at $x$, which points in the positive direction. An $\frac{1}{r}$-grading on $L$ is a real-valued lift of this map. Let $\mathcal{F}(M,D)$ be a version of Fukaya category, with the only difference that Lagrangian submanifolds $L$ should lie in $M \setminus \text{Supp}(D)$, and come equipped with $\frac{1}{r}$-grading. In particular, we have full and faithful $A_\infty$-functor $\mathcal{F}(M,D) \to \mathcal{F}(M)$.

Suppose that two objects $L_0, L_1$ of $\mathcal{F}(M,D)$, which have only transversal intersection. Then each $x \in L_0 \cap L_1$, is equipped with an integer $i^r(x)$. Namely, let $\alpha \in (0,\pi)$ be an angle counted clockwise from $TL_0,x$ to $TL_1,x$. Let $\alpha_0(x)$, $\alpha_1(x)$ be the values of $\frac{1}{r}$-gradings of $L_0$ and $L_1$ respectively. Then

\[(11.2) \quad i^r(x) = \frac{r\alpha + \alpha_1(x) - \alpha_0(x)}{\pi}.\]

If $r$ is odd, then $i^r(x) \mod 2$ coincides with the value of $(\mathbb{Z}/2)$-grading on $x$. Further, if $L_0 = L_1$, then $i^r(e) = 0$, $i^r(q) = r$.

Let $u \in \mathcal{M}(x_0,\ldots,x_d)$ be one of the perturbed pseudo-holomorphic polygons which contribute to the $A_\infty$-structure of $\mathcal{F}(M,D)$. Then it follows from the index formula that

\[(11.3) \quad i^r(x_0) - i^r(x_1) - \cdots - i^r(x_d) = r(2 - d) + \sum_{z \in D} \text{ord}(\eta^r,z) \deg(u,z).\]

Now suppose that for all points $z \in \text{Supp}(D)$ the order $\text{ord}(\eta^r,z)$ is the same positive integer $m > 0$ (in our application to genus $g$ curves, $m$ will be equal to $(2g - 2)$). With respect to our $\mathbb{Z}$-gradings $i^r(x)$, the higher operations $\mu^i$ will decompose into the sum

\[(11.4) \quad \mu^i = \mu^i_0 + \mu^i_1 + \cdots,\]

where $\mu^i_k$, $k \geq 0$, are homogeneous maps of degree $r(2 - d) + 2mk$.

12. Orbifolds

Let $\bar{M}$ be a Riemann surface with its finite set of orbifold points $\bar{D}$. Near each $z \in \bar{D}$ we have a chart in which the neighborhood of $z$ is represented as a quotient of disc by a
cyclic group $\mathbb{Z}/\text{iso}(z)$, where $\text{iso}(z) \geq 2$. We require that the orbifold Euler characteristic
\begin{equation}
\chi_{\text{orb}}(\overline{M}) = \chi_{\text{top}}(\overline{M}) - \sum_{z \in \overline{D}} \frac{\text{iso}(z) - 1}{\text{iso}(z)}
\end{equation}
is negative. Then there exists a 1-form $\tilde{\theta}$ on $S(T\overline{M})$ such that $d\tilde{\theta}$ equals to the pullback of $\tilde{\omega}$ — the orbifold symplectic form on $\overline{M}$.

Let $\overline{l}: L \to \overline{M}$ be an immersion, where $L$ is a circle, and write $\overline{L}$ for its image. As before, $\overline{L}$ is called balanced if the integral of the pullback of $\tilde{\theta}$ with respect to the natural map $L \to S(T\overline{M})$. Further, the self-intersections of $\overline{L}$ are required to be generic, and we also require absence of teardrops:

**Definition 12.1.** Let $x_-, x_+ \in L$, $x_\neq x_+$, and $\overline{l}(x_-) = \overline{l}(x_+) = x$. Denote by $H$ the closed upper half-plane. A tear-drop is a pair $(\bar{u}, w)$, where $\bar{u}: H \to \overline{M} \setminus \overline{D}$ is a smooth immersion, and $w: \partial H \to L$ is a smooth map, such that $\overline{l} \circ w = \bar{u}|_{\partial H}$, and $\lim_{x \to \pm \infty} x = x_\pm$, $\lim_{z \to \pm i \infty} \bar{u}(z) = x$.

We also put a Spin structure on $L$. One can work Floer theory for such $L$, together with Fukaya higher products, see [S1], section 8.

Assume now that our orbifold $\overline{M}$ is a quotient of some actual Riemann surface $M$ by the finite group $\Gamma$, and $\overline{L}$ comes from some embedded $L \subset M$. This implies the absence of teardrops. For each generator $x$ of $\text{CF}^{-}(L, L)$, we have the associated element $\gamma \in \Gamma$. Write the corresponding decomposition of the Floer complex as

\begin{equation}
\text{CF}^{-}(\overline{L}, \overline{L}) = \bigoplus_{\gamma \in \Gamma} \text{CF}^{-}(\overline{L}, \overline{L})\gamma
\end{equation}

Then it is clear that

\begin{equation}
\mu^d(\text{CF}^{-}(\overline{L}, \overline{L})\gamma_1 \otimes \cdots \otimes \text{CF}^{-}(\overline{L}, \overline{L})\gamma_2) \subset \text{CF}^{-}(\overline{L}, \overline{L})\gamma_1 \cdots \gamma_2.
\end{equation}

Suppose that $\Gamma$ is abelian. Then we have the action of the group of characters $G = \text{Hom}(\Gamma, \mathbb{C}^*)$ on $\text{CF}^{-}(\overline{L}, \overline{L})$: $g \in G$ acts on $\text{CF}^{-}(\overline{L}, \overline{L})\gamma$ with multiplication by $g(\gamma)$.

Now, take the collection of curves $\gamma(L)$, $\gamma \in \Gamma$. These are all non-trivial lifts of $\overline{L}$ onto $M$. Then it is elementary that we have $G$-equivariant $A_\infty$-isomorphism

\begin{equation}
\bigoplus_{\gamma_0, \gamma_1 \in \Gamma} \text{CF}^{-}(\gamma_0(L), \gamma_1(L)) \cong \mathbb{C}[G] \# \text{CF}^{-}(L, L).
\end{equation}

**13. Fukaya category of a genus $g \geq 3$ curve**

It is convenient to represent the genus $g$ curve $M$ as a 2-fold covering of $\mathbb{C}P^1$, branched in $(2g + 2)$ points: $(2g + 1)$-th roots of unity and 0. Take the curves $L_1, \ldots, L_{2g+1}$, which
are preimages of intervals $[\zeta^0, \zeta^2], [\zeta^1, \zeta^3], \ldots, [\zeta^{2g-1}, \zeta^0], [\zeta^{2g}, \zeta^1]$ respectively, where $\zeta = \exp(\frac{2\pi i}{2g+1})$. The special case $g = 3$ is shown in Figure 4.

The special case $g = 3$ is shown in Figure 4.

Figure 4.

Figure 5.

Lemma 13.1. The curves $L_1, \ldots, L_{2g+1}$, equipped with non-trivial spin structures, split-generate $D^\pi \mathcal{F}(M)$.

**Proof.** Take the curves $K_1, \ldots, K_{2g}$, which are preimages of intervals $[\zeta^0, \zeta^2], [\zeta^1, \zeta^3], \ldots, [\zeta^{2g-1}, \zeta^2]$ respectively. (The special case $g = 3$ is illustrated in Figure 5, then by [Ma] we have $(\tau_{K_2} \cdots \tau_{K_1})^4 \sim \text{id}$. From Lemma 10.2 it follows that the curves $K_1, \ldots, K_{2g}$, equipped with non-trivial spin structures, split-generate $D^\pi \mathcal{F}(M)$. Further, it is straightforward to check that $\tau_{L_{2g+1}} \cdots \tau_{L_1}(K_1)$ is isotopic to $K_1[1]$. Thus, it follows from Lemma 10.1 that $K_1$ is split-generated by $L_1, \ldots, L_{2g+1}$. Analogously, all the other $K_i$ are split-generated by $L_1, \ldots, L_{2g+1}$.

Hence, $L_1, \ldots, L_{2g+1}$ split-generate $D^\pi \mathcal{F}(M)$. □

We now compute partially the $A_\infty$-algebra $\bigoplus_{1 \leq i, j \leq 2g+1} CF(L_i, L_j)$. Our computation is in fact analogous to the computations in [Se1], Section 9.

Take a natural $\Sigma = \mathbb{Z}/(2g + 1)$-action on $M$ which lifts the rotational action on $\mathbb{CP}^1$. The quotient $M/\Sigma$ is a sphere $\bar{M}$ with 3 orbifold points. Denote the set of orbifold points by $\bar{D}$.

Explicitly, the hyperelliptic curve $M$ is given (in affine chart) by the equation

\begin{equation}
(13.1) \quad y^2 = z(z^{2g+1} - 1).
\end{equation}

The generator of $\Sigma$ acts by the formula

\begin{equation}
(13.2) \quad (y, z) \rightarrow (\zeta^{g+1}y, \zeta z).
\end{equation}

We have that $\mathbb{C}(M)^\Sigma \cong \mathbb{C}(\overline{\mathbb{C}(z^{2g+1})})$, hence $t = \frac{1}{z^{2g+1}}$ is a coordinate on an affine chart $\mathbb{C} \subset \mathbb{CP}^1 \cong \overline{\mathbb{M}}$. The set $\bar{D}$ consists of the points $t = 1$, $t = -1$, and $t = \infty$.

Each of the curves $L_i$ projects to the same curve $\bar{L} \subset \bar{M}$. It lies in $\mathbb{C} \subset \bar{M}$ and has the same isotopy type for all $g \geq 3$. The case $g = 3$ is shown in Figure 6. We have natural
$\Sigma$-equivariant $A_\infty$-isomorphism, as in (12.4):

\[(13.3) \bigoplus_{1 \leq i, j \leq 2g+1} CF^*(L_i, L_j) \cong CF^*(\bar{L}, \bar{L}) \times \Sigma_{2g+1}.\]

The super vector space $CF^*(\bar{L}, \bar{L})$ has 8 generators: two standard $e$ (even) and $q$ (odd), together with three pairs $(\bar{x}_i$ (even), $x_i$ (odd)), $1 \leq i \leq 3$, coming from each self-intersection point of $\bar{L}$ (see Figure 6). Take $\tilde{\Gamma} = \pi_{orb}^b(M)$, and put $\Gamma = [\tilde{\Gamma}, \tilde{\Gamma}]$. Then $\Gamma$ is naturally the quotient of $((\mathbb{Z}/(2g + 1))^3$ by the diagonal subgroup $\mathbb{Z}/(2g + 1)$. The class of our immersed curve $\bar{L}$ in $\Gamma$ is trivial, hence the generators of $CF^*(L, L)$ are labelled by the weights which are elements of $\Gamma$.

Further, take a meromorphic section $\eta^3$ of $(T^*\bar{M})^\otimes 3$, having double pole at each point of $\bar{D}$. Explicitly,

\[(13.4) \eta^3 = \frac{(dt)^\otimes 3}{(t - 1)^2(t + 1)^2}.\]

Each generator of $CF^*(\bar{L}, \bar{L})$ is equipped with additional integer grading, together with weight in $\Gamma$:

| generator | $e$ | $x_1$ | $x_2$ | $x_3$ |
|-----------|-----|-------|-------|-------|
| weight    | $(0, 0, 0)$ | $(1, 0, 0)$ | $(0, 1, 0)$ | $(0, 0, 1)$ |
| index     | 0   | 1     | 1     | 1     |

(13.5)

| generator | $\bar{x}_1$ | $\bar{x}_2$ | $\bar{x}_3$ | $q$ |
|-----------|-------------|-------------|-------------|-----|
| weight    | $(0, 1, 1)$ | $(1, 0, 1)$ | $(1, 1, 0)$ | $(1, 1, 1)$ |
|           | $= (-1, 0, 0)$ | $= (0, -1, 0)$ | $= (0, 0, -1)$ | $= (0, 0, 0)$ |
| index     | 2           | 2           | 2           | 3   |
Since the $A_\infty$-structure is homogeneous with respect to $\Gamma$ by (12.3) we have that $\mu^1 = 0$.

Further, the inverse image of $\eta^3$ on $M$ has three poles of order $(2g - 2)$. Therefore, according to (11.4), we have a decomposition $\mu^i = \mu^i_0 + \mu^i_1 + \ldots$, where $\mu^i_k$ has degree $6 - 3i + (4g - 4)k$.

For degree reasons, $\mu^2_k$ vanishes for $k > 0$. Further, according to (9.6), (9.7), we have

$$(13.6) \quad \mu^2(x_i, e) = x_i = -\mu^2(e, x_i), \quad \mu^2(\bar{x}_i, e) = \bar{x}_i = \mu^2(e, \bar{x}_i), \quad \mu^2(q, e) = q = -\mu^2(e, q),$$

$$\mu^2(q, q) = 0, \quad \mu^2(x_i, \bar{x}_i) = q = -\mu^2(\bar{x}_i, x_i).$$

Further, there are only six (taking into account the ordering of the vertices) non-constant triangles which avoid $\bar{D}$. To determine the sign of their contributions, choose generic points $\circ \neq \ast$ on $\bar{L}$, as in Figure 6. Then we have

$$(13.7) \quad \mu^2(x_1, x_2) = \bar{x}_3 = -\mu^2(x_2, x_1);$$

$$\mu^2(x_2, x_3) = \bar{x}_1 = -\mu^2(x_3, x_2);$$

$$\mu^2(x_3, x_1) = \bar{x}_2 = -\mu^2(x_1, x_3).$$

Further, one of the triangles (passing through $\ast$) can be thought as a four-pointed disc with one of the vertex being $\ast$. It gives contribution to

$$(13.8) \quad \mu^3_0(x_3, x_2, x_1) = -e.$$
Further, \( \mu_0^3(x_{i_1}, x_{i_2}, x_{i_3}) = 0 \) for \((i_1, i_2, i_3) \neq (3, 2, 1)\), since such an expression is a multiple of \(e\) (for degree reasons), and all the relevant spaces \( \mathcal{M}(e, x_{i_1}, x_{i_2}, x_{i_3}) \) are empty.

There are six holomorphic \((2g + 1)\)-gons in our picture. Namely, each point \( x_i \in \tilde{L} \) breaks the curve \( \tilde{L} \) into two loops \( \gamma'_i, \gamma''_i \). Choose the orientations on them in such a way that they go anti-clockwise around the corresponding orbifold point \( t_{\gamma'_i} = t_{\gamma''_i} \).

Then for each such loop \( \gamma_j \) we have a bi-holomorphic map \( v_j : S \to \tilde{M} \), where \( S \) is a 1-pointed disk. The image of \( v_j \) is the area bounded by \( \gamma_j \) and containing the orbifold point \( t_{\gamma_j} \). Also require \( v_j \) to map the center of \( S \) to \( t_{\gamma_j} \) and the marked point to the corresponding \( x_i \).

Further, each \( u_j \) hits exactly one of the points of \( \tilde{D} \) and has \((2g + 1)\)-fold ramification there, and no ramification elsewhere, which means that it lifts to a genuine immersed \((2g + 1)\)-gon in \( M \). We take the three \((2g + 1)\)-gons that go through \(*\), and determine their contributions to \( \mu_{1^{2g+1}} \), namely:

\[
\mu_{1^{2g+1}}(x_i, \ldots, x_i) = e.
\]

Now identify \( CF(\tilde{L}, \tilde{L}) \), mapping \( e \) to 1, \( x_i \) to \( \xi_i \), \( \bar{x}_1 \) to \( \xi_2 \wedge \xi_3 \) and analogously for other \( \bar{x}_i \), and \( q \) to \( -\xi_1 \wedge \xi_2 \wedge \xi_3 \). Then, it follows from the above computations and Theorem 4.2 that the resulting \( A_{\infty} \)-structure on \( \Lambda(V) \) is \( G \cong \text{Hom}(\Gamma, \mathbb{C}^*) \)-equivariantly \( A_{\infty} \)-isomorphic to \( \mathcal{A}' \) from the end of the section 4. The covering \( M \to \tilde{M} \) is classified by the surjective homomorphism \( \Gamma \to \Sigma \), which is dual to the inclusion \( K \subset G \). Combining this with Lemma 13.1 and (12.4), we obtain the following

**Corollary 13.2.** We have an equivalence \( D^\pi F(M) \cong \text{Perf}(\mathbb{C}[K]\# \mathcal{A}') \).

Corollaries 8.2 and 13.2 imply the main

**Theorem 13.3.** There is an equivalence \( \mathcal{D}_{Dg}(H) \cong D^\pi F(M) \).

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