We investigate maxima of linear processes with i.i.d. heavy-tailed innovations and random coefficients. Using the point process approach we derive functional convergence of the partial maxima stochastic process in the space of non-decreasing càdlàg functions on \([0, 1]\) with the Skorokhod \(M_1\) topology.

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1. INTRODUCTION

Consider a strictly stationary sequence of random variables \((X_i)\) and denote by \(M_n = \max\{X_1, X_2, \ldots, X_n\}, n \geq 1\), its accompanying sequence of partial maxima. The principal concern of classical extreme value theory is with asymptotic distributional properties of the maximum \(M_n\). It is well known that in the i.i.d. case if there exist normalizing constants \(a_n > 0\) and \(b_n\), such that

\[
P\left(\frac{M_n - b_n}{a_n} \leq x\right) \rightarrow G(x) \quad \text{as } n \rightarrow \infty,
\]

where \(G\) is assumed non-degenerate, then \(G\) necessarily belongs to the class of extreme value distributions (see for instance Gnedenko, 1943; Resnick, 1987). In particular, (1.1) holds with \(G(x) = \exp(-x^{-\alpha}), x > 0\), for some \(\alpha > 0\), that is, the distribution of \(X_1\) is in the domain of attraction of the Fréchet distribution, if and only if

\[
x \mapsto P(X_1 > x)
\]

is regularly varying at infinity with index \(\alpha\), that is

\[
\lim_{t \to \infty} \frac{P(X_1 > tx)}{P(X_1 > t)} = x^{-\alpha}
\]

for every \(x > 0\) (see Resnick, 1987, Proposition 1.11). A functional version of this is known to be true as well, the limit process being an extremal process, and the convergence takes place in the space of càdlàg functions endowed with the Skorokhod \(J_1\) topology. More precisely, relation (1.1) is equivalent to

\[
M_n(\cdot) = \sqrt[n]{\frac{X_1}{a_n}} \rightarrow Y(\cdot)
\]

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in \(D([0, 1], \mathbb{R})\), the space of real-valued càdlàg functions on \([0, 1]\), with the Skorokhod \(J_1\) topology, where \(Y(\cdot)\) is an extremal process generated by \(G\). If \(G\) is the Fréchet distribution, then \(Y\) has marginal distributions

\[ P(Y(t) \leq x) = e^{-\alpha t}, \quad x \geq 0, \; t \in [0, 1]. \]

This result was first proved by Lamperti (1964) (see also Resnick, 1987, Proposition 4.20). For convenience we can put \(M_n(t) = X_t/a_n\) (or \(M_n(t) = 0\)) for \(t \in [0, 1/n]\).

In the dependent case, Adler (1983) obtained \(J_1\) extremal functional convergence with the weak dependence condition similar to ‘asymptotic independence’ condition introduced by Leadbetter (1974). For stationary sequences of jointly regularly varying random variables Basrak and Tafro (2016) showed the invariance principle for the partial maximum process \(M_n(\cdot)\) in \(D([0, 1], \mathbb{R})\) with the Skorokhod \(M_1\) topology (cf. Krizmanić, 2014).

For a special class of weakly dependent random variables, the linear processes or moving averages processes with i.i.d. heavy-tailed innovations (and deterministic coefficients), it is known that (1.2) holds, see for instance Resnick (1987, Proposition 4.28). In this article we aim to obtain the functional convergence as in (1.2) for linear processes with random coefficients. Due to possible clustering of large values, the \(J_1\) topology becomes inappropriate even for finite order moving averages with at least two non-zero coefficients (cf. Avram and Taqqu, 1992), and hence we will use the weaker Skorokhod \(M_1\) topology. In the proofs of our results we will use some methods and results which appear in Basrak and Krizmanić (2014), where they obtained functional convergence of partial sum processes with respect to Skorokhod \(M_1\) topology; Krizmanić (2018), where joint functional convergence of partial sums and maxima for linear processes was investigated and Krizmanić (2019), where a functional limit theorem for sums of linear processes with heavy-tailed innovations and random coefficients was established. For some related results on limit theory for sums of moving averages with random coefficients see Kulik (2006).

In general, functional \(M_1\) convergence of partial sum processes fails to hold. Clusters of large values in the sequence \((X_n)\) may contain positive and negative values yielding the corresponding partial sum processes having jumps of opposite signs within temporal clusters of large values, and this precludes the \(M_1\) convergence. For instance, this occurs for linear process with i.i.d. heavy-tailed innovations \(Z_i\) and deterministic coefficients \(C_0 = 1, C_1 = -1, C_2 = 1\) and \(C_i = 0\) for \(i \geq 3:\n\)

\[ X_i = Z_i - Z_{i-1} + Z_{i-2}, \quad i \in \mathbb{Z}. \]

But in this case the convergence in distribution of the partial sum processes in the weaker \(M_2\) topology can be shown to hold, see Krizmanić (2019). For partial maxima processes we do not have similar problems with positive and negative values in clusters of big values since these processes are non-decreasing and thus only jumps with positive sign appear in them, which means one can have functional \(M_1\) convergence.

The article is organized as follows. In Section 2 we introduce basic notions about linear processes, regular variation and Skorokhod topologies. In Section 3 we derive functional convergence of the partial maxima stochastic process for finite order linear processes with i.i.d. heavy-tailed innovations and random coefficients, and then we extend this result to infinite order linear processes using the uniform metric.

2. PRELIMINARIES

2.1. Linear Processes

Let \((Z_i)_{i \in \mathbb{Z}}\) be a sequence of i.i.d. random variables with regularly varying balanced tails, that is

\[ P(|Z_i| > x) = x^{-\alpha}L(x), \quad x > 0, \] \[ (2.1) \]

for some \(\alpha > 0\) and slowly varying function \(L\), and

\[ \lim_{x \to \infty} \frac{P(Z_i > x)}{P(|Z_i| > x)} = p, \quad \lim_{x \to \infty} \frac{P(Z_i < -x)}{P(|Z_i| > x)} = r, \] \[ (2.2) \]

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where \( p \in [0, 1] \) and \( p + r = 1 \). Let \((a_n)\) be a sequence of positive real numbers such that
\[
n P(|Z_1| > a_n) \to 1, \tag{2.3}
\]
as \( n \to \infty \). Then regular variation of \( Z_i \) can be expressed in terms of vague convergence of measures on \( E = \mathbb{R} \setminus \{0\} \):
\[
n P(a^{-1}_n Z_i \in \cdot) \Rightarrow \mu(\cdot) \quad \text{as} \quad n \to \infty, \tag{2.4}
\]
where \( \mu \) is a measure on \( E \) given by
\[
\mu(dx) = (p \mathbf{1}_{(0, \infty)}(x) + r \mathbf{1}_{(-\infty, 0)}(x)) \alpha |x|^{-\alpha - 1} dx. \tag{2.5}
\]

We study linear processes with random coefficients, defined by
\[
X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z}, \tag{2.6}
\]
where \((C_j)_{j \geq 0}\) is a sequence of random variables independent of \((Z_i)\) such that the above series is a.s. convergent. One sufficient condition for that, which is commonly used in the literature is
\[
\sum_{j=0}^{\infty} E|C_j|^\delta < \infty \quad \text{for some} \quad \delta < \alpha, \quad 0 < \delta \leq 1. \tag{2.7}
\]
The regular variation property and Karamata’s theorem imply \( E|Z_1|^\beta < \infty \) for every \( \beta \in (0, \alpha) \) (cf. Bingham et al., 1989, Proposition 1.5.10), which together with the moment condition (2.7) yield the a.s. convergence of the series in (2.6):
\[
E|X_i|^\delta \leq \sum_{j=0}^{\infty} E|C_j|^\delta E|Z_{i-j}|^\delta = E|Z_1|^\delta \sum_{j=0}^{\infty} E|C_j|^\delta < \infty.
\]
The same holds with the following moment conditions: if \( \alpha < 1 \) then there exists \( \delta \in (0, \alpha) \) such that \( \alpha + \delta < 1 \) and
\[
\sum_{j=0}^{\infty} E|C_j|^\alpha \leq \sum_{j=0}^{\infty} E|C_j|^\alpha < \infty, \quad \sum_{j=0}^{\infty} E|C_j|^{-\delta} < \infty,
\]
and if \( \alpha > 1 \) then there exists \( \delta > 0 \) such that
\[
\sum_{j=0}^{\infty} E(|C_j|^{\alpha + \delta})^{1/(\alpha + \delta)} < \infty, \quad \sum_{j=0}^{\infty} E(|C_j|^{\alpha - \delta})^{1/(\alpha - \delta)} < \infty,
\]
see Kulik (2006). Another condition that assures the a.s. convergence of the series in the definition of linear processes in (2.6) with
\[
E(Z_1) = 0, \quad \text{if} \quad \alpha > 1,
\]
\( Z_1 \) is symmetric, \quad \text{if} \quad \alpha = 1,

and a.s. bounded coefficients can be deduced from the results in Aastrauskas (1983) for linear processes with deterministic coefficients:
\[
\sum_{j=0}^{\infty} c^a L(c^{-1}) < \infty,
\]
and additionally $\sum c_j^2 < \infty$ for $\alpha > 2$ and $\sum c_j (1 + L(c_j^{-1/2})) < \infty$ for $\alpha = 2$, where $(c_j)$ is a sequence of positive real numbers such that $C_j \leq c_j$ a.s. for all $j$, and $L$ as in (2.1) (c.f. Balan et al., 2016).

2.2. Skorokhod Topologies

We start by considering $D([0, 1], \mathbb{R}^d)$, the space of all right-continuous $\mathbb{R}^d$–valued functions on $[0, 1]$ with left limits. For $x \in D([0, 1], \mathbb{R}^d)$ the completed (thick) graph of $x$ is the set

$$G_x = \{(t, z) \in [0, 1] \times \mathbb{R}^d : z \in [[x(t−), x(t)]]\},$$

where $x(t−)$ is the left limit of $x$ at $t$ and $[[a, b]]$ is the product segment, that is $[[a, b]] = [a_1, b_1] \times [a_2, b_2] \ldots \times [a_d, b_d]$ for $a = (a_1, a_2, \ldots), b = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$, and $[a_i, b_i]$ coincides with the closed interval $[a_i \wedge b_i, a_i \vee b_i]$, with $c \wedge d = \min\{c, d\}$ and $c \vee d = \max\{c, d\}$ for $c, d \in \mathbb{R}$. We define an order on the graph $G_x$ by saying that $(t_1, z_1) \leq (t_2, z_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and $|x(t_1−)−z_1| \leq |x(t_2−)−z_2|$ for all $j = 1, 2, \ldots, d$. A weak parametric representation of the graph $G_x$ is a continuous non-decreasing function $(r, u)$ mapping $[0, 1]$ into $G_x$, with $r$ being the time component and $u$ being the spatial component, such that $r(0) = 0$, $r(1) = 1$ and $u(1) = x(1)$.

Let $\Pi_x(x)$ denote the set of weak parametric representations of the graph $G_x$. For $x_1, x_2 \in D([0, 1], \mathbb{R}^d)$ define

$$d_w(x_1, x_2) = \inf\{\|r_1 − r_2\|_{[0, 1]} \vee \|u_1 − u_2\|_{[0, 1]} : (r_i, u_i) \in \Pi_x(x_i), i = 1, 2\},$$

where $\|x\|_{[0, 1]} = \sup\{|x(t)| : t \in [0, 1]\}$. Now we say that $x_n \to x$ in $D([0, 1], \mathbb{R}^d)$ for a sequence $(x_n)$ in the weak Skorokhod $M_1$ (or shortly WM$_1$) topology if $d_w(x_n, x) \to 0$ as $n \to \infty$.

If we replace above the graph $G_x$ with the completed (thin) graph

$$\Gamma_x = \{(t, z) \in [0, 1] \times \mathbb{R}^d : z = \lambda x(t−) + (1 − \lambda)x(t) \text{ for some } \lambda \in [0, 1]\},$$

and weak parametric representations with strong parametric representations (i.e. continuous non-decreasing functions $(r, u)$ mapping $[0, 1]$ onto $\Gamma_x$), then we obtain the standard (or strong) Skorokhod $M_1$ topology. This topology is induced by the metric

$$d_s(x_1, x_2) = \inf\{\|r_1 − r_2\|_{[0, 1]} \vee \|u_1 − u_2\|_{[0, 1]} : (r_i, u_i) \in \Pi_s(x_i), i = 1, 2\},$$

where $\Pi_s(x)$ is the set of strong parametric representations of the graph $\Gamma_x$. The standard $M_1$ topology is stronger than the weak $M_1$ topology on $D([0, 1], \mathbb{R}^d)$, but they coincide for $d = 1$.

The WM$_1$ topology coincides with the topology induced by the metric

$$d_p(x_1, x_2) = \max\{d_s(x_{ij}, x_{j2}) : j = 1, \ldots, d\} \quad (2.8)$$

for $x_j = (x_{i1}, \ldots, x_{id}) \in D([0, 1], \mathbb{R}^d)$ and $i = 1, 2$. The metric $d_p$ induces the product topology on $D([0, 1], \mathbb{R}^d)$.

Using completed graphs and their parametric representations the Skorokhod $M_2$ topology can also be defined. Here we give only its characterization by the Hausdorff metric on the space of graphs: for $x_1, x_2 \in D([0, 1], \mathbb{R})$ the $M_2$ distance between $x_1$ and $x_2$ is given by

$$d_{M_2}(x_1, x_2) = \left(\sup_{a \in G_{x_1}} \inf_{b \in G_{x_2}} d(a, b)\right) \vee \left(\sup_{b \in G_{x_2}} \inf_{a \in G_{x_1}} d(a, b)\right),$$

where $d$ is the metric on $\mathbb{R}^2$ defined by $d((x_1, y_1), (x_2, y_2)) = |x_1 − x_2| \vee |y_1 − y_2|$ for $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2$. The metric $d_{M_2}$ induces the $M_2$ topology, which is weaker than the more frequently used $M_1$ topology. For more details and discussion on the $M_1$ and $M_2$ topologies we refer to Whitt (2002, sections 12.3–12.5).
Since the sample paths of the partial maximum process $M_n(\cdot)$ that we study in this article are non-decreasing, we will restrict our attention to the subspace $D_1([0, 1], \mathbb{R}^d)$ of functions $x$ in $D([0, 1], \mathbb{R}^d)$ for which the coordinate functions $x_i$ are non-decreasing for all $i = 1, \ldots, d$.

In the next section we will use the following two lemmas about the $M_1$ continuity of multiplication and maximum of two càdlàg functions. The first one is based on Whitt (2002, Theorem 13.3.2), and the second one follows easily from the fact that for monotone functions $M_1$ convergence is equivalent to point-wise convergence in a dense subset of $[0, 1]$ including 0 and 1 (cf. Whitt, 2002, Corollary 12.5.1). Denote by Disc$(x)$ the set of discontinuity points of $x \in D([0, 1], \mathbb{R})$.

**Lemma 2.1.** Suppose that $x_n \rightarrow x$ and $y_n \rightarrow y$ in $D([0, 1], \mathbb{R})$ with the $M_1$ topology. If for each $t \in \text{Disc}(x) \cap \text{Disc}(y)$, $x(t)$, $x(t−)$, $y(t)$ and $y(t−)$ are all non-negative and $[x(t)−x(t−)][y(t)−y(t−)] \geq 0$, then $x_n y_n \rightarrow xy$ in $D([0, 1], \mathbb{R})$ with the $M_1$ topology, where $(xy)(t) = x(t)y(t)$ for $t \in [0, 1]$.

**Lemma 2.2.** The function $h : D_1([0, 1], \mathbb{R}^2) \rightarrow D_1([0, 1], \mathbb{R})$ defined by $h(x, y) = x \lor y$, where

$$(x \lor y)(t) = x(t) \lor y(t), \quad t \in [0, 1],$$

is continuous when $D_1([0, 1], \mathbb{R}^2)$ is endowed with the weak $M_1$ topology and $D_1([0, 1], \mathbb{R})$ is endowed with the standard $M_1$ topology.

### 3. FUNCTIONAL LIMIT THEOREMS

Let $(Z_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence of regularly varying random variables with index $\alpha > 0$, and $(C_i)_{i \geq 0}$ a sequence of random variables independent of $(Z_i)$ such that the series defining the linear process

$$X_t = \sum_{i=0}^{\infty} C_i Z_{t-i}, \quad t \in \mathbb{Z},$$

is a.s. convergent. Define the corresponding partial maximum process by

$$M_n(t) = \begin{cases} \frac{1}{a_n^\alpha} \sum_{i=1}^{[nt]} X_i, & t \geq \frac{1}{n}, \\ \frac{X_1}{a_n}, & t < \frac{1}{n}, \end{cases} \quad (3.1)$$

for $t \in [0, 1]$, with the normalizing sequence $(a_n)$ as in (2.3). Let

$$C_{\ast} = \max\{C_j \lor 0 : j \geq 0\} \quad \text{and} \quad C_{\ast} = \max\{-C_j \lor 0 : j \geq 0\}. \quad (3.2)$$

Before the main theorem we have two auxiliary results. Define the maximum functional $\Phi : M_n([0, 1] \times \mathbb{E}) \rightarrow D_1([0, 1], \mathbb{R}^2)$ by

$$\Phi \left( \sum_i \delta_{(t_i, x_i)} \right)(t) = \left( \bigvee_{t_i \leq t} |x_i| 1_{\{x_i > 0\}}, \bigvee_{t_i \leq t} |x_i| 1_{\{x_i < 0\}} \right)$$

for $t \in [0, 1]$ (with the convention $\lor \emptyset = 0$), where the space $M_n([0, 1] \times \mathbb{E})$ of Radon point measures on $[0, 1] \times \mathbb{E}$ is equipped with the vague topology (see Resnick, 1987, Chapter 3).
**Proposition 3.1.** The maximum functional $\Phi : M_p([0,1] \times \mathbb{E}) \to D_1([0,1], \mathbb{R}^2)$ is continuous on the set

$$\Lambda = \{ \eta \in M_p([0,1] \times \mathbb{E}) : \eta([0,1] \times \mathbb{E}) = \eta([0,1] \times [\pm \infty]) = 0 \},$$

when $D_1([0,1], \mathbb{R}^2)$ is endowed with the weak $M_1$ topology.

**Proof.** Take an arbitrary $\eta \in \Lambda$ and suppose that $\eta_n \xrightarrow{\gamma} \eta$ as $n \to \infty$ in $M_p([0,1] \times \mathbb{E})$. We need to show that $\Phi(\eta_n) \to \Phi(\eta)$ in $D_1([0,1], \mathbb{R}^2)$ according to the $WM_1$ topology. By Whitt (2002, Theorem 12.5.2), it suffices to prove that, as $n \to \infty$,

$$d_p(\Phi(\eta_n), \Phi(\eta)) = \max_{k=1,2} d_{M_k}(\Phi_1(\eta_n), \Phi_1(\eta)) \to 0.$$

Let

$$T = \{ t \in [0,1] : \eta([t] \times \mathbb{E}) = 0 \}.$$

Since $\eta$ is a Radon point measure, the set $T$ is dense in $[0,1]$. Fix $t \in T$ and take $\epsilon > 0$ such that $\eta([0,t] \times \{ \pm \epsilon \}) = 0$. Since $\eta$ is a Radon point measure, we can arrange that, letting $\epsilon \downarrow 0$, the convergence to 0 is through a sequence of values $(\epsilon_j)$ such that $\eta([0,t] \times \{ \pm \epsilon_j \}) = 0$ for all $j \in \mathbb{N}$. For $u > 0$ let $E_u = \mathbb{E} \setminus (-u, u)$. Since the set $[0,t] \times E_u$ is relatively compact in $[0,1] \times \mathbb{E}$, there exists a non-negative integer $k = k(\eta)$ such that

$$\eta([0,t] \times E_u) = k < \infty.$$

By assumption, $\eta$ does not have any atoms on the border of the set $[0,t] \times E_u$, and therefore by Resnick (2007, Lemma 7.1) there exists a positive integer $n_0$ such that

$$\eta_n([0,t] \times E_u) = k \quad \text{for all } n \geq n_0.$$

Let $(t_i, x_i)$ for $i = 1, \ldots, k$ be the atoms of $\eta$ in $[0,t] \times E_u$. By the same lemma, the $k$ atoms $(t_i^{(n)}, x_i^{(n)})$ of $\eta_n$ in $[0,t] \times E_u$ (for $n \geq n_0$) can be labeled in such a way that for every $i \in \{1, \ldots, k\}$ we have

$$(t_i^{(n)}, x_i^{(n)}) \to (t_i, x_i) \quad \text{as } n \to \infty.$$

In particular, for any $\delta > 0$ there exists a positive integer $n_\delta \geq n_0$ such that for all $n \geq n_\delta$,

$$|t_i^{(n)} - t_i| < \delta \quad \text{and} \quad |x_i^{(n)} - x_i| < \delta \quad \text{for } i = 1, \ldots, k.$$

If $k = 0$, then (for large $n$) the atoms of $\eta$ and $\eta_n$ in $[0,t] \times \mathbb{E}$ are all situated in $[0,t] \times (-\epsilon, \epsilon)$. Hence $\Phi_1(\eta_n)(t) \in [0,\epsilon)$ and $\Phi_1(\eta_n)(t) \in [0,\epsilon)$, which imply

$$|\Phi_1(\eta_n)(t) - \Phi_1(\eta)(t)| < \epsilon. \quad (3.3)$$

If $k \geq 1$, take $\delta = \epsilon$. Note that $|x_i^{(n)} - x_i| < \delta$ implies $x_i^{(n)} > 0$ iff $x_i > 0$. Hence we have

$$|\Phi_1(\eta_n)(t) - \Phi_1(\eta)(t)| = \left| \bigvee_{i=1}^k |x_i^{(n)}| \mathbf{1}_{x_i^{(n)}>0} - \bigvee_{i=1}^k |x_i| \mathbf{1}_{x_i>0} \right| \leq \bigvee_{i=1}^k \left| |x_i^{(n)}| - |x_i| \right| \mathbf{1}_{x_i>0}$$

$$\leq \bigvee_{i=1}^k |x_i^{(n)} - x_i| < \epsilon, \quad (3.4)$$
where the first inequality above follows from the elementary inequality
\[
\left| \sum_{i=1}^{k} a_i - \sum_{i=1}^{k} b_i \right| \leq \sum_{i=1}^{k} |a_i - b_i|,
\]
which holds for arbitrary real numbers \(a_1, \ldots, a_k, b_1, \ldots, b_k\). Therefore from (3.3) and (3.4) we obtain
\[
\lim_{n \to \infty} \sup \{|\Phi_1(\eta_n)(t) - \Phi_1(\eta)(t)| \leq \epsilon,
\]
and letting \(\epsilon \to 0\) it follows that \(\Phi_1(\eta_n)(t) \to \Phi_1(\eta)(t)\) as \(n \to \infty\). Since \(\Phi_1(\eta)\) and \(\Phi_1(\eta_a)\) are non-decreasing functions, and by Whitt (2002, Corollary 12.5.1) \(M_1\) convergence for monotone functions is equivalent to point-wise convergence in a dense subset of points plus convergence at the endpoints, we conclude that \(d_{M_1}(\Phi_1(\eta_n), \Phi_1(\eta)) \to 0\) as \(n \to \infty\). In the same manner we obtain \(d_{M_1}(\Phi_2(\eta_n), \Phi_2(\eta)) \to 0\), and therefore we conclude that \(\Phi\) is continuous at \(\eta\).

\[\square\]

**Proposition 3.2.** Let \((X_i)\) be a linear process defined by
\[
X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z},
\]
where \((Z_i)_{i \in \mathbb{Z}}\) is an i.i.d. sequence of random variables satisfying (2.1) and (2.2) with \(\alpha > 0\), and \((C_i)_{i \geq 0}\) is a sequence of random variables independent of \((Z_i)\) such that the series defying the above linear process is a.s. convergent. Let
\[
W_n(t) := \sum_{i=1}^{[nt]} |Z_i| a_n^{-1} C_{i} 1_{\{Z_i > 0\}} + C_{-} 1_{\{Z_i < 0\}}, \quad t \in [0, 1],
\]
with \(C_{+}\) and \(C_{-}\) defined in (3.2). Then, as \(n \to \infty\),
\[
W_n(\cdot) \Rightarrow C^{(1)} W^{(1)}(\cdot) \lor C^{(2)} W^{(2)}(\cdot)
\]
in \(D_{1}[0, 1] := D_{1}([0, 1], \mathbb{R})\) with the \(M_1\) topology, where \(W^{(1)}\) and \(W^{(2)}\) are extremal processes with exponent measures \(pax^{-a-1} 1_{(0,\infty)}(x) dx\) and \(rax^{-a-1} 1_{(0,\infty)}(x) dx\) respectively, with \(p\) and \(r\) defined in (2.2), and \((C^{(1)}, C^{(2)})\) is a random vector, independent of \((W^{(1)}, W^{(2)})\), such that \((C^{(1)}, C^{(2)}) \overset{d}{=} (C_{+}, C_{-})\).

**Remark 3.1.** In Proposition 3.2, as well as in the sequel of this article, we suppose \(W^{(1)}\) is an extremal process if \(p > 0\), and a zero process if \(p = 0\). Analogously for \(W^{(2)}\).

**Proof of Proposition 3.2.** Since the random variables \(Z_i\) are i.i.d. and regularly varying, Corollary 6.1 in Resnick (2007) yields
\[
N_n := \sum_{j=1}^{n} \delta(\frac{\epsilon_j}{\epsilon_n}) \Rightarrow N := \sum_{j} \delta(\epsilon_j) \quad \text{as} \quad n \to \infty,
\]
in \(M_{p}([0, 1] \times \mathbb{E})\), where the limiting point process \(N\) is a Poisson process with intensity measure \(Leb \times \mu\), with \(\mu\) as in (2.5). Since \(P(N \in \Lambda) = 1\) (cf. Resnick, 2007, p. 221) from (3.6) by an application of Proposition 3.1 and...
the continuous mapping theorem (see for instance Resnick, 2007, Theorem 3.1) we obtain \( \Phi(N_n)(\cdot) \overset{d}{\to} \Phi(N)(\cdot) \) as \( n \to \infty \), that is

\[
(W_n^{(1)}(\cdot), W_n^{(2)}(\cdot)) := \left( \sqrt{\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i > 0\}}}, \sqrt{\sum_{i=1}^{\lfloor n/2 \rfloor} \frac{|Z_i|}{a_n} 1_{\{Z_i < 0\}}}, \right)
\]

\[
\overset{d}{\to} (W^{(1)}(\cdot), W^{(2)}(\cdot)) := \left( \sqrt{\sum_{\tau \leq n} |U_1| 1_{\{U_1 > 0\}}}, \sqrt{\sum_{\tau \leq n} |U_1| 1_{\{U_1 < 0\}}} \right)
\]  \( (3.7) \)

in \( D_1([0, 1], \mathbb{R}^2) \) under the weak \( M_1 \) topology.

The space \( D([0, 1], \mathbb{R}) \) equipped with the Skorokhod \( J_1 \) topology is a Polish space (i.e. metrizable as a complete separable metric space), see Billingsley (1968, Section 14), and therefore the same holds for the (standard) \( M_1 \) topology, since it is topologically complete (see Whitt, 2002, Section 12.8) and separability remains preserved in the weaker topology. The space \( D_1([0, 1]) \) is a closed subspace of \( D([0, 1], \mathbb{R}) \) (cf. Whitt, 2002, Lemma 13.2.3), and hence also Polish. Furthermore, the space \( D_1([0, 1], \mathbb{R}^2) \) equipped with the weak \( M_1 \) topology is separable as a direct product of two separable topological spaces. It is also topologically complete since the product metric in (2.8) inherits the completeness of the component metrics. Thus we conclude that \( D_1([0, 1], \mathbb{R}^2) \) with the weak \( M_1 \) topology is also a Polish space, and hence by Corollary 5.18 in Kallenberg (1997), we can find a random vector \((C^{(1)}, C^{(2)})\), independent of \((W^{(1)}, W^{(2)})\), such that

\[
(C^{(1)}, C^{(2)}) \overset{d}{=} (C_+, C_-).
\]  \( (3.8) \)

This, relation (3.7) and the fact that \((C_+, C_-)\) is independent of \((W^{(1)}, W^{(2)})\), by an application of Theorem 3.29 in Kallenberg (1997), imply that, as \( n \to \infty \),

\[
(B^{(1)}(\cdot), B^{(2)}(\cdot), W^{(1)}(\cdot), W^{(2)}(\cdot)) \overset{d}{\to} (B^{(1)}(\cdot), B^{(2)}(\cdot), W^{(1)}(\cdot), W^{(2)}(\cdot))
\]  \( (3.9) \)

in \( D_{1,t}([0, 1], \mathbb{R}^4) \) with the product \( M_1 \) topology, where \( B^{(1)}(t) = C_+, B^{(2)}(t) = C_-, B^{(1)}(t) = C^{(1)} \) and \( B^{(2)}(t) = C^{(2)} \) for \( t \in [0, 1] \).

Let \( g : D_{1,t}([0, 1], \mathbb{R}^4) \to D_{1,t}([0, 1], \mathbb{R}^2) \) be a function defined by

\[
g(x) = (x_1x_3, x_2x_4), \quad x = (x_1, x_2, x_3, x_4) \in D_{1,t}([0, 1], \mathbb{R}^4).
\]

Denote by \( \tilde{D}_{1,2} \) the set of all functions in \( D_{1,t}([0, 1], \mathbb{R}^4) \) for which the first two component functions have no common discontinuity points, that is

\[
\tilde{D}_{1,2} = \{(u, v, z, w) \in D_{1,t}([0, 1], \mathbb{R}^4) : \text{Disc}(u) = \text{Disc}(v) = \emptyset\}.
\]

By Lemma 2.1 the function \( g \) is continuous on the set \( \tilde{D}_{1,2} \) in the weak \( M_1 \) topology, and hence \( \text{Disc}(g) \subseteq \tilde{D}_{1,2} \). Denoting \( \hat{D}_t = \{u \in D_{1,t}([0, 1] : \text{Disc}(u) = \emptyset\} \) we obtain

\[
P[(B^{(1)}, B^{(2)}, W^{(1)}, W^{(2)}) \in \text{Disc}(g)] \leq P[(B^{(1)}, B^{(2)}, W^{(1)}, W^{(2)}) \in \tilde{D}_{1,2}]
\]

\[
\leq P[\{B^{(1)} \in \hat{D}_t\} \cup \{B^{(2)} \in \hat{D}_t\}] = 0,
\]

where the last equality holds since \( B^{(1)} \) and \( B^{(2)} \) have no discontinuity points. This allows us to apply the continuous mapping theorem to relation (3.9) yielding \( g(B^{(1)}, B^{(2)}, W^{(1)}, W^{(2)}) \overset{d}{\to} g(B^{(1)}, B^{(2)}, W^{(1)}, W^{(2)}) \), that is

\[
(C^{(1)}_n W^{1}_n, C^{(2)}_n W^{2}_n) \overset{d}{\to} (C^{(1)} W^{1}, C^{(2)} W^{2}) \quad \text{as} \quad n \to \infty,
\]  \( (3.10) \)
in $D_1([0, 1], \mathbb{R}^2)$ with the weak $M_1$ topology. Now from (3.10) by Lemma 2.2 and the continuous mapping theorem it follows $C_+W_n^1 \vee C_-W_n^2 \Rightarrow C^{(1)}W(t) \vee C^{(2)}W(t)$ as $n \to \infty$, that is

$$
\left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_+ [Z_i]_1 \right\rfloor_{t \geq 0} \overset{d}{\to} \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_- [Z_i]_1 \right\rfloor_{t \leq 0} \vee \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_+ [Z_i]_1 \right\rfloor_{t \leq 0} \vee \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_- [Z_i]_1 \right\rfloor_{t \leq 0}
$$

in $D_1[0, 1]$ with the $M_1$ topology. This is in fact (3.5) since the process in the converging sequence in the last relation is equal to $W_n(t)$. It remains only to show that the corresponding limiting process is of the form claimed in the statement of the proposition. Denote it by $M(\cdot)$. By an application of Proposition 3.7 in Resnick (1987) we obtain that the restricted processes $\sum_i \delta_{(t_j, W_n(i))}$ and $\sum_i \delta_{(t_j, -W_n(i))}$ are independent Poisson processes with intensity measures $Leb \times \mu_+$ and $Leb \times \mu_-$ respectively, where

$$
\mu_+(dx) = p \mathbb{1}_{(0, \infty)}(x) x^{-\alpha} \, dx \quad \text{and} \quad \mu_-(dx) = r \mathbb{1}_{(0, \infty)}(x) x^{-\alpha} \, dx.
$$

(cf. Last and Penrose, 2018, Theorem 5.2). From this we conclude that the processes

$$
W^{(1)}(\cdot) = \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_+ [Z_i]_1 \right\rfloor_{t \geq 0} \quad \text{and} \quad W^{(2)}(\cdot) = \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_- [Z_i]_1 \right\rfloor_{t \leq 0}
$$

are extremal processes with exponent measures $\mu_+$ and $\mu_-$ respectively (see Resnick, 1987, Section 4.3; Resnick, 2007, p. 161), and hence $M(t) = C^{(1)}W^{(1)}(t) \vee C^{(2)}W^{(2)}(t)$ for $t \in [0, 1]$. $\square$

In deriving functional convergence of the partial maxima process we first deal with finite order linear processes. Fix $q \in \mathbb{N}$ and let

$$
X_t = \sum_{i=0}^{q} C_i Z_{t-i}, \quad t \in \mathbb{Z}.
$$

In this case $C_+$ and $C_-$ reduce to $C_+ = \max\{C_j \vee 0 : j = 0, \ldots, q\}$ and $C_- = \max\{-C_j \vee 0 : j = 0, \ldots, q\}$. Denote by $M$ the limiting process in Proposition 3.2, that is

$$
M(\cdot) = C^{(1)}W^{(1)}(\cdot) \vee C^{(2)}W^{(2)}(\cdot), \quad (3.11)
$$

where $W^{(1)}$ is an extremal process with exponent measure $\mu_+(dx) = pax^{-\alpha} \, dx$ for $x > 0$, $W^{(2)}$ is an extremal process with exponent measure $\mu_-(dx) = rax^{-\alpha} \, dx$ for $x > 0$, and $(C^{(1)}, C^{(2)})$ is a two dimensional random vector, independent of $(W^{(1)}, W^{(2)})$, such that $(C^{(1)}, C^{(2)}) \overset{d}{=} (C_+, C_-)$. Taking into account the proof of Proposition 3.2 observe that

$$
M(t) = \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_+ [Z_i]_1 \right\rfloor_{t \geq 0} \vee \left\lfloor \frac{1}{n} \sum_{i=1}^{n} C_- [Z_i]_1 \right\rfloor_{t \leq 0} + \delta_{(t_j, 0)}(C^{(1)}1_{\{t_j > 0\}} + C^{(2)}1_{\{t_j < 0\}}), \quad t \in [0, 1],
$$

where $\sum_i \delta_{(t_j, 0)}$ is a Poisson process with intensity measure $Leb \times \mu$, with $\mu$ as in (2.5).

**Theorem 3.3.** Let $(Z_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence of random variables satisfying $(2.1)$ and $(2.2)$ with $\alpha > 0$. Assume $C_0, C_1, \ldots, C_q$ are random variables independent of $(Z_i)$. Then, as $n \to \infty$,

$$
M_n(\cdot) \overset{d}{\to} M(\cdot)
$$

in $D_1[0, 1]$ endowed with the $M_1$ topology.
Proof. Our aim is to show that for every $\delta > 0$

$$\lim_{n \to \infty} P[d_{M_1}(W_n, M_n) > \delta] = 0,$$

since then from Proposition 3.2 by an application of Slutsky’s theorem (see for instance Resnick, 2007, Theorem 3.4) we will obtain $M_n(\cdot) \xrightarrow{d} M(\cdot)$ as $n \to \infty$ in $D_1[0, 1]$ endowed with the $M_1$ topology. It suffices to show that

$$\lim_{n \to \infty} P[d_{M_1}(W_n, M_n) > \delta] = 0. \quad (3.12)$$

Indeed, by Remark 12.8.1 in Whitt (2002) the following metric is a complete metric topologically equivalent to $d_{M_1}$:

$$d_{M_1}^*(x_1, x_2) = d_{M_1}(x_1, x_2) + \lambda(\hat{\omega}(x_1, \cdot), \hat{\omega}(x_2, \cdot)),$$

where $\lambda$ is the Lévy metric on a space of distributions

$$\lambda(F_1, F_2) = \inf \{ \epsilon > 0 : F_2(x - \epsilon) - \epsilon \leq F_1(x) \leq F_2(x + \epsilon) + \epsilon \ \text{for all} \ x \}$$

and

$$\hat{\omega}(x, z) = \begin{cases} \omega(x, e^z), & z < 0, \\ \omega(x, 1), & z \geq 0, \end{cases}$$

with $\omega(x, \rho) = \sup_{0 \leq t \leq 1} \omega(x, t, \rho)$ and

$$\omega(x, t, \rho) = \sup_{0 \leq t \leq 1} \| [x(t_2) - [x(t_1), x(t_3)] \|$$

where $\rho > 0$ and $\| z - A \|$ denotes the distance between a point $z$ and a subset $A \subseteq \mathbb{R}$. Since $W_n(\cdot)$ and $M_n(\cdot)$ are non-decreasing, for $t_1 < t_2 < t_3$ it holds that $\| W_n(t_2) - [W_n(t_1), W_n(t_3)] \| = 0$, which implies $\omega(W_n, \rho) = 0$ for all $\rho > 0$, and similarly $\omega(M_n, \rho) = 0$. Hence $\lambda(W_n, M_n) = 0$, and $d_{M_1}^*(W_n, M_n) = d_{M_1}(W_n, M_n)$.

To show (3.12) fix $\delta > 0$ and let $n \in \mathbb{N}$ be large enough, that is $n > \max\{2q, 2q/\delta\}$. Then by the definition of the metric $d_{M_1}$ we have

$$d_{M_1}(W_n, M_n) = \left( \sup_{v \in \Gamma_{W_n}} \inf_{z \in \Gamma_{M_n}} d(v, z) \right) \vee \left( \sup_{v \in \Gamma_{M_n}} \inf_{z \in \Gamma_{W_n}} d(v, z) \right) =: Y_n \vee T_n.$$

Hence

$$P[d_{M_1}(W_n, M_n) > \delta] \leq P(Y_n > \delta) + P(T_n > \delta). \quad (3.13)$$

Now, we estimate the first term on the right-hand side of (3.13). Let

$$D_n = \{ \exists v \in \Gamma_{W_n} \text{ such that } d(v, z) > \delta \text{ for every } z \in \Gamma_{M_n} \},$$

and note that by the definition of $Y_n$

$$\{ Y_n > \delta \} \subseteq D_n. \quad (3.14)$$

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On the event $D_n$ it holds that $d(v, \Gamma_{M_n}) > \delta$. Let $v = (t_v, x_v)$. We claim that

$$|W_n\left(\frac{i}{n}\right) - M_n\left(\frac{i}{n}\right)| > \delta,$$  \hspace{1cm} (3.15)

where $i^* = \lfloor nt_v \rfloor$ or $i^* = \lfloor nt_v \rfloor - 1$. To see this, observe that $t_v \in [i/n, (i + 1)/n)$ for some $i \in \{1, \ldots, n - 1\}$ (or $t_v = 1$). If $x_v = W_n(i/n)$ (i.e. $v$ lies on a horizontal part of the completed graph), then clearly

$$|W_n\left(\frac{i}{n}\right) - M_n\left(\frac{i}{n}\right)| \geq d(v, \Gamma_{M_n}) > \delta,$$

and we put $i^* = i$. On the other hand, if $x_v \in [W_n((i - 1)/n), W_n(i/n))$ (i.e. $v$ lies on a vertical part of the completed graph), one can similarly show that

$$|W_n\left(\frac{i - 1}{n}\right) - M_n\left(\frac{i - 1}{n}\right)| > \delta \quad \text{if} \quad M_n\left(\frac{i - 1}{n}\right) > x_v,$$

and

$$|W_n\left(\frac{i}{n}\right) - M_n\left(\frac{i}{n}\right)| > \delta \quad \text{if} \quad M_n\left(\frac{i - 1}{n}\right) < x_v.$$

In the first case put $i^* = i - 1$ and in the second $i^* = i$. Since $i = \lfloor nt_v \rfloor$ we conclude that (3.15) holds. Moreover, since $|i^*/n - (i^* + l)/n| \leq q/n < \delta$ for every $l = 1, \ldots, q$ (such that $i^* + l \leq n$), from the definition of the set $D_n$ one can similarly conclude that

$$|W_n\left(\frac{i^*}{n}\right) - M_n\left(\frac{i^* + l}{n}\right)| > \delta.$$

(3.16)

Let $C = C_+ \cup C_-$. We claim that

$$D_n \subseteq H_{n,1} \cup H_{n,2} \cup H_{n,3},$$

(3.17)

where

$$H_{n,1} = \left\{ \exists l \in \{-q, \ldots, q\} \cup \{n-q+1, \ldots, n\} \text{ such that } \frac{C|Z_l|}{a_n} > \frac{\delta}{4(q+1)} \right\},$$

$$H_{n,2} = \left\{ \exists k \in \{1, \ldots, n\} \text{ and } \exists l \in \{k-q, \ldots, k+q\} \setminus \{k\} \text{ such that } \frac{C|Z_k|}{a_n} > \frac{\delta}{4(q+1)} \right\},$$

$$H_{n,3} = \left\{ \exists k \in \{1, \ldots, n\}, \exists j \in \{1, \ldots, n\} \setminus \{k, \ldots, k+q\}, \exists l \in \{0, \ldots, q\} \text{ and } \exists l \in \{0, \ldots, q\} \setminus \{l_1\} \text{ such that } \frac{C|Z_{j-l}\setminus l_1|}{a_n} > \frac{\delta}{4(q+1)} \right\}.$$

Relation (3.17) will be proven if we show that

$$\hat{D}_n := D_n \cap (H_{n,1} \cup H_{n,2} \setminus C) \subseteq H_{n,3}.$$
Assume the event $\hat{D}_n$ occurs. Then necessarily $W_n(i^*/n) > \delta/[4(q + 1)]$. Indeed, if $W_n(i^*/n) \leq \delta/[4(q + 1)]$, that is

$$\sqrt{i^*/n}(C_{1Z > 0} + C_{1Z < 0}) = W_n\left(\frac{i^*}{n}\right) \leq \frac{\delta}{4(q + 1)},$$

then for every $s \in \{q + 1, \ldots, i^*\}$ it holds that

$$\frac{X_i}{a_n} = \sum_{j=0}^{q} \frac{C_{Z_{s-j}}}{a_n} \leq \sum_{j=0}^{q} \frac{|Z_{s-j}|}{a_n} \leq \sum_{j=0}^{q} \frac{C|Z_{s-j}|}{a_n} \leq (q + 1) \frac{\delta}{4(q + 1)} = \frac{\delta}{4} \tag{3.19}$$

Combining (3.18) and (3.19) we obtain

$$-\frac{\delta}{4} \leq \frac{X_i}{a_n} \leq \frac{\delta}{4} \tag{3.20}$$

and hence

$$\left|W_n\left(\frac{i^*}{n}\right) - M_n\left(\frac{i^*}{n}\right)\right| \leq \left|W_n\left(\frac{i^*}{n}\right)\right| + \left|M_n\left(\frac{i^*}{n}\right)\right| \leq \frac{\delta}{4(q + 1)} + \frac{\delta}{4} \leq \frac{\delta}{2},$$

which is in contradiction with (3.15).

Therefore $W_n(i^*/n) > \delta/[4(q + 1)]$. This implies the existence of some $k \in \{1, \ldots, i^*\}$ such that

$$W_n\left(\frac{i^*}{n}\right) = \frac{|Z_k|}{a_n} (C_{1Z > 0} + C_{1Z < 0}) > \frac{\delta}{4(q + 1)}, \tag{3.21}$$

and hence

$$\frac{C|Z_k|}{a_n} > \frac{\delta}{4(q + 1)}.$$

From this, since $H_{n,1}^c$ occurs, it follows that $q + 1 \leq k \leq n - q$. Since $H_{n,2}^c$ occurs, it holds that

$$\frac{C|Z_k|}{a_n} \leq \frac{\delta}{4(q + 1)} \quad \text{for all } l \in \{k - q, \ldots, k + q\} \setminus \{l\}. \tag{3.22}$$

Now we want to show that $M_n(i^*/n) = X_j/a_n$ for some $j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k + q\}$. If this is not the case, then $M_n(i^*/n) = X_j/a_n$ for some $j \in \{k, \ldots, k + q\}$ (with $j \leq i^*$). On the event $\{Z_k < 0\}$ it holds that

$$|Z_k| (C_{1Z > 0} + C_{1Z < 0}) = C_n Z_k = C_n Z_k$$
for some $j_0 \in \{0, \ldots, q\}$ (with $C_{j_0} \geq 0$). Here we distinguish two cases:

(i) $k + q \leq i^*$. Since $k + j_0 \leq i^*$, we have

$$\frac{X_j}{a_n} = M_n\left(\frac{i^*}{n}\right) \geq \frac{X_{k+j_0}}{a_n}. \quad (3.23)$$

Observe that we can write

$$\frac{X_j}{a_n} = \frac{C_{j_k}Z_k}{a_n} + \sum_{s=0}^{q} \frac{C_{s}Z_{j-s}}{a_n} = \frac{C_{j-k}Z_k}{a_n} + F_1,$$

and

$$\frac{X_{k+j_0}}{a_n} = \frac{C_{j_0}Z_k}{a_n} + \sum_{s=0}^{q} \frac{C_{s}Z_{k+j_0-s}}{a_n} = \frac{C_{j_0}Z_k}{a_n} + F_2.$$

From relation (3.22) (similarly as in (3.19)) we obtain

$$|F_1| \leq q \cdot \frac{\delta}{4(q+1)} < \frac{\delta}{4},$$

and similarly $|F_2| < \delta/4$. Since $C_{j_0} - C_{j-k} = C_+ - C_{j-k} \geq 0$, using (3.23) we obtain

$$0 \leq \frac{C_{j_0}Z_k - C_{j-k}Z_k}{a_n} \leq F_1 - F_2 \leq |F_1| + |F_2| < \frac{\delta}{2}.$$

By (3.15) we have

$$\left| \frac{C_{j_0}Z_k}{a_n} - \frac{X_j}{a_n} \right| = \left| W_n\left(\frac{i^*}{n}\right) - M_n\left(\frac{i^*}{n}\right) \right| > \delta,$$

and hence

$$\delta < \left| \frac{C_{j_0}Z_k}{a_n} - \frac{C_{j-k}Z_k}{a_n} - F_1 \right| \leq \left| \frac{C_{j_0}Z_k}{a_n} - \frac{C_{j-k}Z_k}{a_n} \right| + |F_1| < \frac{\delta}{2} + \frac{\delta}{4} = \frac{3\delta}{4},$$

which is not possible.

(ii) $k + q > i^*$. Note that in this case $k \leq j \leq i^* < k + q$. Since

$$M_n\left(\frac{k+q}{n}\right) = \sqrt[n]{\frac{X_{k+q}}{a_n}} \geq M_n\left(\frac{i^*}{n}\right) = \frac{X_{i^*}}{a_n},$$

it holds that

$$M_n\left(\frac{k+q}{n}\right) = \frac{X_{i^*}}{a_n}.$$
for some \( p \in \{j, \ldots, k + q\} \subseteq \{k, \ldots, k + q\} \). Observe that we can write

\[
\frac{X_p}{a_n} = \frac{C_{p-k}Z_k}{a_n} + \sum_{s=0}^q \frac{C_sZ_{p-s}}{a_n} =: \frac{C_{p-k}Z_k}{a_n} + F_3,
\]

with \(|F_3| < \delta/4\), which holds by relation (3.22). By relation (3.16) we have

\[
\left| \frac{C_{j_0}Z_j}{a_n} - \frac{X_p}{a_n} \right| = \left| W_n\left( \frac{r^*}{n} \right) - M_n\left( \frac{k + q}{n} \right) \right| > \delta,
\]

and repeating the arguments as in (i), but with

\[
\frac{X_p}{a_n} = M_n\left( \frac{k + q}{n} \right) \geq \frac{X_{k+j_0}}{a_n}
\]

instead of (3.23), we arrive at

\[
\delta < \left| \frac{C_{j_0}Z_j}{a_n} - \frac{C_{p-k}Z_k}{a_n} - F_3 \right| \leq \left| \frac{C_{j_0}Z_j}{a_n} - \frac{C_{p-k}Z_k}{a_n} \right| + |F_3| < 3\delta/4.
\]

Thus we conclude that this case also cannot happen.

One can similarly handle the event \( \{Z_k < 0\} \) to arrive at a contradiction. Therefore indeed \( M_n(i^*/n) = X_j/a_n \) for some \( j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k + q\} \). Now we have three cases: (A1) all random variables \( Z_{j-q}, \ldots, Z_j \) are ‘small’, (A2) exactly one is ‘large’ and (A3) at least two of them are ‘large’, where we say \( Z \) is ‘small’ if \( C|Z|/a_n \leq \delta/[4(q + 1)] \), otherwise it is ‘large’. We will show that the first two cases are not possible.

(A1) \( C|Z_j|/a_n \leq \delta/[4(q + 1)] \) for every \( l = 0, \ldots, q \). This yields (as in (3.19))

\[
\left| M_n\left( \frac{i^*}{n} \right) \right| = \frac{|X_j|}{a_n} \leq \frac{\delta}{4}.
\]

Let \( j_0 \) be as above (on the event \( \{Z_k > 0\} \)), that is

\[
|Z_k|(C_+1_{Z_k>0} + C_-1_{Z_k<0}) = C_+Z_k = C_{j_0}Z_k.
\]

If \( k + q \leq i^* \), then

\[
\frac{X_j}{a_n} \geq \frac{X_{k+j_0}}{a_n} = \frac{C_{j_0}Z_k}{a_n} + F_2,
\]

where \( F_2 \) is as in (i) above, with \(|F_2| < \delta/4\), that is

\[
F_2 = \frac{X_{k+j_0}}{a_n} - \frac{C_{j_0}Z_k}{a_n} = \sum_{s=0}^q \frac{C_sZ_{k+j_0-s}}{a_n}.
\]
There exists 
\[ \frac{C_{j_l} Z_k}{a_n} \leq \frac{X_j}{a_n} - F_2 \leq \frac{|X_j|}{a_n} + |F_2| < \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2}, \]
and
\[ \left| W_n \left( \frac{i^*}{n} \right) - M_n \left( \frac{i^*}{n} \right) \right| = \left| \frac{C_{j_l} Z_k}{a_n} - \frac{X_j}{a_n} \right| \leq \frac{C_{j_l} Z_k}{a_n} + \frac{|X_j|}{a_n} < \frac{\delta}{2} + \frac{\delta}{4} = \frac{3\delta}{4}, \]
which is in contradiction with (3.15). On the other hand, if \( k + q > i^* \), we have two possibilities: \( M_n((k+q)/n) = M_n(i^*/n) \) or \( M_n((k+q)/n) > M_n(i^*/n) \). When \( M_n((k+q)/n) = M_n(i^*/n) = X_j/a_n \), since \( k + j_0 \leq k + q \) note that relation (3.24) holds, and similarly as above we obtain
\[ \left| W_n \left( \frac{i^*}{n} \right) - M_n \left( \frac{k + q}{n} \right) \right| = \left| \frac{C_{j_l} Z_k}{a_n} - \frac{X_j}{a_n} \right| < \frac{3\delta}{4}, \]
which is in contradiction with (3.16). Alternatively, when \( M_n((k+q)/n) > M_n(i^*/n) \), it holds that \( M_n((k+q)/n) = X_j/a_n \) for some \( p \in \{ i^*, \ldots, k + q \} \). Now in the same manner as in (ii) above we get a contradiction. We handle the event \( \{ Z_k < 0 \} \) similarly to arrive at a contradiction, and therefore this case cannot happen.

(A2) There exists \( l_1 \in \{ 0, \ldots, q \} \) such that \( C|Z_{n-l_1}|/a_n > \delta/\sqrt{4(q+1)} \) and \( C|Z_{n-l_1}|/a_n < \delta/\sqrt{4(q+1)} \) for every \( l \in \{ 0, \ldots, q \} \setminus \{ l_1 \} \). Here we analyze only what happens on the event \( \{ Z_k > 0 \} \) (the event \( \{ Z_k < 0 \} \) can be treated analogously and is therefore omitted). Assume first \( k + q \leq i^* \). Then
\[ \frac{X_j}{a_n} \geq \frac{X_{j+l_0}}{a_n} = \frac{C_{j_l} Z_k}{a_n} + F_2, \quad (3.25) \]
where \( j_0 \) and \( F_2 \) are as in (i) above, with \( |F_2| < \delta/4 \). Write
\[ \frac{X_j}{a_n} = \frac{C_{j_l} Z_{j-l_1}}{a_n} + \sum_{s=0}^{q} \frac{C_s Z_{j-s}}{a_n} = : \frac{C_{j_l} Z_{j-l_1}}{a_n} + F_4. \]
Similarly as before we obtain \( |F_4| < \delta/4 \). Since \( j - l_1 \leq j \leq i^* \), by the definition of the process \( W_n(\cdot) \) we have
\[ W_n \left( \frac{i^*}{n} \right) \geq \frac{|Z_{i-l_1}|}{a_n} \left( C_1 I_{\{ Z_{i-l_1} > 0 \}} + C_{-1} I_{\{ Z_{i-l_1} < 0 \}} \right) \geq \frac{C_{i_l} Z_{j-l_1}}{a_n}. \]
Thus
\[ \frac{C_{j_l} Z_k}{a_n} = \frac{|Z_k|}{a_n} \left( C_1 I_{\{ Z_k > 0 \}} + C_{-1} I_{\{ Z_k < 0 \}} \right) = W_n \left( \frac{i^*}{n} \right) \geq \frac{C_{i_l} Z_{j-l_1}}{a_n}, \]
which yields
\[ \frac{C_{j_l} Z_k}{a_n} - \frac{X_j}{a_n} \geq \frac{C_{i_l} Z_{j-l_1}}{a_n} - \frac{X_j}{a_n} = -F_4. \quad (3.26) \]
Relations (3.25) and (3.26) yield
\[-(|F_2| + |F_4|) \leq -F_4 \leq \frac{C_{j\ast}Z_k}{a_n} - \frac{X_j}{a_n} \leq -F_2 \leq |F_2| + |F_4|,
\]
that is
\[|W_n\left(\frac{i^\ast}{n}\right) - M_n\left(\frac{i^\ast}{n}\right)| = \left|\frac{C_{j\ast}Z_k}{a_n} - \frac{X_j}{a_n}\right| \leq \frac{\delta}{4} + \frac{\delta}{4} = \frac{\delta}{2},
\]
which is in contradiction with (3.15). Assume now $k+q > i^\ast$. If $M_n((k+q)/n) = M_n(i^\ast/n) = X_j/a_n$, relation (3.25) still holds and this leads to
\[|W_n\left(\frac{i^\ast}{n}\right) - M_n\left(\frac{k+q}{n}\right)| = \left|\frac{C_{j\ast}Z_k}{a_n} - \frac{X_p}{a_n}\right| < \frac{3\delta}{4},
\]
which is in contradiction with (3.16). Hence this case also cannot happen.

(A3) There exist $l_1 \in \{0, \ldots, q\}$ and $l \in \{0, \ldots, q\} \setminus \{l_1\}$ such that $C|Z_{j-l_1}|/a_n > \delta/[4(q+1)]$ and $CZ_{j-l}/a_n > \delta/[4(q+1)]$. In this case the event $H_{n,3}$ occurs.

Therefore only case (A3) is possible, and this yields $\hat{D}_n \subseteq H_{n,3}$. Hence (3.17) holds. By stationarity we have
\[P(H_{n,1}) \leq (3q + 1) P\left(\frac{|Z|}{a_n} > \frac{\delta}{4(q+1)}\right). \tag{3.27}
\]
For an arbitrary $M > 0$ it holds that
\[P\left(\frac{|Z|}{a_n} > \frac{\delta}{4(q+1)}\right) = P\left(\frac{|Z|}{a_n} > \frac{\delta}{4(q+1)}, C > M\right) + P\left(\frac{|Z|}{a_n} > \frac{\delta}{4(q+1)}, C \leq M\right) \leq P(C > M) + P\left(\frac{|Z|}{a_n} > \frac{\delta}{4(q+1)M}\right).
\]
Using the regular variation property we obtain
\[\lim_{n \to \infty} P\left(\frac{|Z|}{a_n} > \frac{\delta}{4(q+1)M}\right) = 0,
\]
and therefore from (3.27) we get $\limsup_{n \to \infty} P(H_{n,1}) \leq (3q + 1) P(C > M)$. Letting $M \to \infty$ we conclude
\[\lim_{n \to \infty} P(H_{n,1}) = 0. \tag{3.28}
\]
Since \( Z_k \) and \( Z_l \) that appear in the formulation of \( H_{n,2} \) are independent, for an arbitrary \( M > 0 \) it holds that

\[
P(H_{n,2} \cap \{ C \leq M \}) = \sum_{k=1}^{n} \sum_{l=1}^{\min\{k+q, n\}} P \left( \frac{|Z_k|}{a_n} > \frac{\delta}{4(q+1)}, ~ \frac{|Z_l|}{a_n} > \frac{\delta}{4(q+1)}, ~ C \leq M \right)
\]

\[
\leq \sum_{k=1}^{n} \sum_{l=1}^{\min\{k+q, n\}} P \left( \frac{|Z_k|}{a_n} > \frac{\delta}{4(q+1)} \right) \cdot P \left( \frac{|Z_l|}{a_n} > \frac{\delta}{4(q+1)} \right)
\]

\[
= \frac{2q}{n} \left[ n P \left( \frac{|Z_1|}{a_n} > \frac{\delta}{4(q+1)} \right) \right]^2,
\]

and an application of the regular variation property yields \( \lim_{n \to \infty} P(H_{n,2} \cap \{ C \leq M \}) = 0 \). Hence

\[
\limsup_{n \to \infty} P(H_{n,2}) \leq \limsup_{n \to \infty} P(H_{n,2} \cap \{ C > M \}) \leq P(C > M),
\]

and letting again \( M \to \infty \) we conclude

\[
\lim_{n \to \infty} P(H_{n,2}) = 0.
\]

(3.29)

From the definition of the set \( H_{n,3} \) it follows that \( k, j \neq l, j \neq l \) are all different, which implies that the random variables \( Z_k, Z_{j-l} \) and \( Z_{j-l} \) are independent. Using this and stationarity we obtain

\[
P(H_{n,3} \cap \{ C \leq M \}) \leq \frac{q(q+1)}{n} \left[ n P \left( \frac{|Z_1|}{a_n} > \frac{\delta}{4(q+1)} \right) \right]^2
\]

for arbitrary \( M > 0 \), and hence

\[
\lim_{n \to \infty} P(H_{n,3}) = 0.
\]

(3.30)

Now from (3.17) and (3.28)–(3.30) we obtain \( \lim_{n \to \infty} P(D_n) = 0 \), and hence (3.14) yields

\[
\lim_{n \to \infty} P(Y_n > \delta) = 0.
\]

(3.31)

It remains to estimate the second term on the right-hand side of (3.13). Let

\[
E_n = \{ \exists v \in \Gamma_{M_n} \text{ such that } d(v, z) > \delta \text{ for every } z \in \Gamma_{W_n} \}.
\]

Then by the definition of \( T_n \)

\[
\{ T_n > \delta \} \subseteq E_n.
\]

(3.32)

On the event \( E_n \) it holds that \( d(v, \Gamma_{W_n}) > \delta \). Interchanging the roles of the processes \( M_n(\cdot) \) and \( W_n(\cdot) \), in the same way as before for the event \( D_n \) it can be shown that

\[
\left| W_n \left( \frac{i-v}{n} \right) - M_n \left( \frac{i}{n} \right) \right| > \delta
\]

for all \( l = 0, \ldots, q \) (such that \( i^* - l \geq 0 \)), where \( i^* = \lfloor nt_v \rfloor \) or \( i^* = \lfloor nt_v \rfloor - 1 \), and \( v = (t_v, x_v) \).
Now we want to show that $E_n \cap (H_n,1 \cup H_n,2)^c \subseteq H_n,3$, and hence assume the event $E_n \cap (H_n,1 \cup H_n,2)^c$ occurs. Since (3.33) (for $l = 0$) is in fact (3.15), repeating the arguments used for $D_n$ we conclude that (3.21) holds. Here we also claim that $M_n(i^*/n) = X_j/a_n$, for some $j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k + q\}$. Hence assume this is not the case, that is $M_n(i^*/n) = X_j/a_n$ for some $j \in \{k, \ldots, k + q\}$ (with $j \neq i^*$). We can repeat the arguments from (i) above to conclude that $k + q \leq i^*$ is not possible. It remains to see what happens when $k + q > i^*$. Let

$$W_n\left(\frac{i^* - q}{n}\right) = \frac{|Z_i|}{a_n} (C_{+1|Z_i| > 0} + C_{-1|Z_i| < 0})$$

for some $s \in \{1, \ldots, i^* - q\}$. Note that $i^* - q \geq 1$ since $q + 1 \leq k \leq i^*$. We distinguish two cases:

(a) $W_n(i^*/n) > M_n(i^*/n)$. In this case the definition of $i^*$ implies that $M_n(i^*/n) \leq x_v \leq W_n(i^*/n)$. Since $|t_v - (i^* - q)/n| < (q + 1)/n \leq \delta$, from $d(v, \Lambda_W) > \delta$ we conclude

$$\tilde{d}\left((x_v, [W_n\left(\frac{i^* - q}{n}\right), W_n\left(\frac{q}{n}\right)]\right) > \delta,$$

where $\tilde{d}$ is the Euclidean metric on $\mathbb{R}$. This yields $W_n(i^*/n) > M_n(i^*/n)$, and from (3.33) we obtain

$$W_n\left(\frac{i^*-q}{n}\right) > M_n\left(\frac{i^*}{n}\right) + \delta. \quad (3.34)$$

From this, taking into account relation (3.20), we obtain

$$\frac{C_l|Z_i|}{a_n} \geq W_n\left(\frac{i^* - q}{n}\right) > -\frac{\delta}{4} + \delta = \frac{3\delta}{4} > \frac{\delta}{4(q + 1)},$$

and since $H_n,2$ occurs it follows that

$$\frac{C_l|Z_i|}{a_n} \leq \frac{\delta}{4(q + 1)} \quad \text{for every } l \in \{s - q, \ldots, s + q\} \setminus \{s\}. \quad (3.35)$$

Let $p_0 \in \{0, \ldots, q\}$ be such that $C_{p_0}Z_i = |Z_i| (C_{+1|Z_i| > 0} + C_{-1|Z_i| < 0})$. Since $s + p_0 \leq i^*$, it holds that

$$X_j/a_n = M_n\left(\frac{i^*}{n}\right) \geq \frac{X_{s+p_0}}{a_n} = \frac{C_{p_0}Z_i}{a_n} + F_s, \quad (3.36)$$

where

$$F_s = \sum_{m=0}^{q} \frac{C_mZ_{s+p_0-m}}{a_n}.$$  

From (3.34) and (3.36) we obtain

$$\frac{C_{p_0}Z_i}{a_n} \geq \frac{X_j}{a_n} + \delta \geq \frac{C_{p_0}Z_i}{a_n} + F_s + \delta,$$

that is $F_s < -\delta$. But this is not possible since by (3.35), $|F_s| \leq \delta/4$, and we conclude that this case cannot happen.
(b) $W_n(i^*/n) \leq M_n(i^*/n)$. Then from (3.33) we get

$$M_n\left(\frac{k+q}{n}\right) \geq M_n\left(\frac{i^*}{n}\right) \geq W_n\left(\frac{i^*}{n}\right) + \delta. \quad (3.37)$$

Therefore

$$\left| W_n\left(\frac{i^*}{n}\right) - M_n\left(\frac{k+q}{n}\right) \right| > \delta,$$

and repeating the arguments from (ii) above we conclude that this case also cannot happen.

Thus we have proved that $M_n(i^*/n) = X_j/a_n$ for some $j \in \{1, \ldots, i^*\} \setminus \{k, \ldots, k+q\}$. Similar as before one can prove now that Cases (A1) and (A2) cannot happen (when $k+q > i^*$ we use also the arguments from (a) and (b)), which means that only Case (A3) is possible. In that case the event $H_{n,3}$ occurs, and thus we have proved that $E_n \cap (H_{n,1} \cup H_{n,2})^c \subseteq H_{n,3}$. Hence

$$E_n \subseteq H_{n,1} \cup H_{n,2} \cup H_{n,3},$$

and from (3.28)–(3.30) we obtain $\lim_{n \to \infty} P(E_n) = 0$. Therefore (3.32) yields

$$\lim_{n \to \infty} P(T_n > \delta) = 0. \quad (3.38)$$

Now from (3.13), (3.31) and (3.38) we obtain (3.12), which means that $M_n(\cdot) \overset{d}{\rightarrow} M(\cdot)$ in $D_1[0, 1]$ with the $M_1$ topology. This proves the theorem. $\square$

**Remark 3.2.** If the sequence of coefficients $(C_j)$ is a.s. of the same sign, the limiting process $M(\cdot)$ in Theorem 3.3 reduces to $C^{(\infty)}W^{(\infty)}(\cdot)$ with $W^{(\infty)}$ being an extremal process with exponent measure $\mu_\infty$, independent of $C^{(\infty)}$, where in the case of non-negative coefficients $C^{(\infty)} = C^{(1)} \overset{d}{=} \max\{C_j \varnothing : j = 0, \ldots, q\}$ and $\mu_\infty(dx) = \mu_\infty(dx) = \rho a x^{-\sigma-1}dx$ for $x > 0$, while in the case of non-positive coefficients $C^{(\infty)} = C^{(2)} \overset{d}{=} \max\{-C_j \varnothing : j = 0, \ldots, q\}$ and $\mu_\infty(dx) = \mu_\infty(dx) = \rho a x^{-\sigma-1}dx$ for $x > 0$. Similarly, if the innovations $(Z_i)$ are a.s. of the same sign, the limiting process is again $C^{(\infty)}W^{(\infty)}(\cdot)$, with $\mu_\infty(dx) = \alpha x^{-\alpha-1}dx$ for $x > 0$, and $C^{(\infty)} = C^{(1)}$ if the innovations are non-negative and $C^{(\infty)} = C^{(2)}$ if they are non-positive.

Now we turn our attention to infinite order linear processes. The idea is to approximate them by a sequence of finite order linear processes for which Theorem 3.3 holds, and to show that the error of approximation is negligible in the limit with respect to the uniform metric. To accomplish this in the case $\alpha \in (0, 1)$ we will use the arguments from Krizmanić (2019), and in the case $\alpha \in [1, \infty)$ the arguments from the proof of Lemma 2 in Tyran-Kamińska (2010) adapted to linear processes with random coefficients instead of deterministic.

**Theorem 3.4.** Let $(X_i)$ be a linear process defined by

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z},$$

where $(Z_i)_{i \in \mathbb{Z}}$ is an i.i.d. sequence of random variables satisfying (2.1) and (2.2) with $\alpha > 0$, and $(C_i)_{i \geq 0}$ is a sequence of random variables independent of $(Z_i)$ such that the series defying the above linear process is a.s. convergent. If $\alpha \in (0, 1)$ suppose

$$\sum_{j=0}^{\infty} E|C_j|^\delta < \infty \quad \text{for some } \delta \in (0, \alpha), \quad (3.39)$$
and
\[ \sum_{j=0}^{\infty} E[C_j]^{\gamma} < \infty \quad \text{for some } \gamma \in (\alpha, 1), \] (3.40)

while if \( \alpha = 1 \) suppose also (3.39), and if \( \alpha > 1 \) suppose
\[ \sum_{j=0}^{\infty} E|C_j| < \infty. \] (3.41)

Then \( M_n(\cdot) \overset{d}{\to} M(\cdot) \), as \( n \to \infty \), in \( D_{\alpha}[0, 1] \) with the \( M_1 \) topology, with \( M \) as defined in (3.11).

**Proof.** For \( q \in \mathbb{N}, q \geq 2 \), define
\[ X^q_i = \sum_{j=0}^{q-2} C_j Z_{i-j} + C_{q, \max} Z_{i-q+1} + C_{q, \min} Z_{i-q}, \quad i \in \mathbb{Z}, \]
and
\[ M_{n,q}(t) = \frac{[nt]}{a_n}, \quad t \in [0, 1] \]
(with \( M_{n,q}(t) = X^q_i/a_n \) if \( t \in [0, 1/n) \)), where \( C_{q, \max} = \max\{C_j : j \geq q - 1\} \) and \( C_{q, \min} = \min\{C_j : j \geq q - 1\} \).

Observe that
\[ \max\{C_j \lor 0 : j = 0, \ldots, q - 1\} \lor (C_{q, \max} \lor 0) \lor (C_{q, \min} \lor 0) = C_+, \]
\[ \max\{-C_j \lor 0 : j = 0, \ldots, q - 1\} \lor (-C_{q, \max} \lor 0) \lor (-C_{q, \min} \lor 0) = C_- \]
and therefore for the finite order moving average process \((X^q_i)\) by Theorem 3.3 we obtain
\[ M_{n,q}(\cdot) \overset{d}{\to} M(\cdot) \quad \text{as } n \to \infty, \]
in \( D_{\alpha}[0, 1] \) with the \( M_1 \) topology. If we show that for every \( \epsilon > 0 \)
\[ \lim_{q \to \infty} \limsup_{n \to \infty} P[d_{M_{\alpha}}(M_n, M_{n,q}) > \epsilon] = 0, \] (3.42)
then by a generalization of Slutsky’s theorem (see Resnick, 2007, Theorem 3.5) it will follow \( M_{\alpha}(\cdot) \overset{d}{\to} M(\cdot) \)
in \( D_{\alpha}[0, 1] \) with the \( M_1 \) topology. Since the metric \( d_{M_{\alpha}} \) on \( D_{\alpha}[0, 1] \) is bounded above by the uniform metric on \( D_{\alpha}[0, 1] \), it suffices to show that
\[ \lim_{q \to \infty} \limsup_{n \to \infty} P\left(\sup_{0 \leq t \leq 1} |M_n(t) - M_{n,q}(t)| > \epsilon \right) = 0. \]

Now we treat separately the cases \( \alpha \in (0, 1) \) and \( \alpha \in [1, \infty) \).
Case $\alpha \in (0, 1)$. Recalling the definitions, we have

$$P \left( \sup_{0 \leq t \leq 1} \left| M_n(t) - M_{n,q}(t) \right| > \epsilon \right) \leq P \left( \sqrt[n]{\sum_{i=1}^{n} \frac{|X_i - X_i^q|}{a_n}} > \epsilon \right) \leq P \left( \sum_{i=1}^{n} \frac{|X_i - X_i^q|}{a_n} > \epsilon \right).$$

Observe that

$$\sum_{i=1}^{n} |X_i - X_i^q| = \sum_{i=1}^{n} \left| \sum_{j=0}^{q-2} C_j Z_{i-j} - C_{q-1} Z_{i-q} - C_{q-max} Z_{i-q+1} - C_{q-min} Z_{i-q} \right| \leq \sum_{j=1}^{n} \left| (C_{q-1} - C_{q-max}) Z_{i-q+1} + (C_q - C_{q-min}) Z_{i-q} + \right. \left. \sum_{j=q+1}^{\infty} C_j Z_{i-j} \right| \leq \left( 5 \sum_{j=q+1}^{\infty} |C_j| \right) \left( \sum_{i=1}^{n} |Z_{i-q+1}| + |Z_{i-q}| \right) + \sum_{i=1}^{n} \sum_{j=q+1}^{\infty} |C_j| |Z_{i-j}| \leq \left( 5 \sum_{j=q+1}^{\infty} |C_j| \right) \sum_{i=1}^{n} |Z_{i-q+1}| + \sum_{i=-\infty}^{0} |Z_{i-q}| \sum_{j=1}^{n} |C_{q-i+j}|,$$

where in the second inequality above we used the simple fact that $|C_{q-1} - C_{q-max}| \leq 2 \sum_{j=q+1}^{\infty} |C_j|$ (and analogously if $C_{q,max}$ is replaced by $C_{q,min}$), and a change of variables and rearrangement of sums in the third inequality. Since conditions (3.39) and (3.40) by Lemma 3.2 in Krizmanić (2019) imply

$$\lim_{q \to \infty} \limsup_{n \to \infty} P \left( \sum_{j=1}^{n} \frac{|Z_{i-q+1}|}{a_n} + \sum_{i=-\infty}^{0} |Z_{i-q}| \sum_{j=1}^{n} |C_{q-i+j}| > \epsilon \right) = 0,$$

we obtain

$$\lim_{q \to \infty} \limsup_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} \left| M_n(t) - M_{n,q}(t) \right| > \epsilon \right) = 0,$$

which means that $M_n(\cdot) \overset{d}{\to} M(\cdot)$ as $n \to \infty$ in $(D_t[0, 1], d_{M_t})$.

Case $\alpha \in [1, \infty)$. Define $Z_{n,j}^d = a_n^{-1} Z_1 J_{\{Z_1 \leq a_n\}}$ and $Z_{n,j}^d = a_n^{-1} Z_1 J_{\{|Z_1| > a_n\}}$ for $j \in \mathcal{Z}$ and $n \in \mathbb{N}$,

$$\hat{C}_j = \begin{cases} C_j, & \text{if } j < q, \\ C_j^q - C_{q-min}, & \text{if } j = q, \\ C_j^q - C_{q-max}, & \text{if } j = q - 1, \end{cases}$$

and note that

$$\left| M_n(t) - M_{n,q}(t) \right| = \left| \sum_{i=1}^{[nt]} \frac{X_i - X_i^q}{a_n} \right| \leq \left| \sum_{i=1}^{[nt]} \frac{|X_i - X_i^q|}{a_n} \right| = \left| \sum_{j=q+1}^{\infty} C_j Z_{i-q} \right| \leq \left( \sum_{j=q+1}^{\infty} \hat{C}_j Z_{n-j}^d \right) + \left( \sum_{j=q+1}^{\infty} \hat{C}_j Z_{n,j}^d \right).$$
Using again the fact that the $M_1$ metric on $D_1[0, 1]$ is bounded above by the uniform metric we get

$$
P[d_{M_1}(M_n, M_{n,q}) > \epsilon] \leq P \left( \sup_{0 \leq t \leq 1} |M_n(t) - M_{n,q}(t)| > \epsilon \right)
= P \left( \sqrt{\sum_{j=1}^{\infty} |\tilde{C}_j Z_{n,i-j}^\leq|} > \frac{\epsilon}{2} \right) + P \left( \sqrt{\sum_{j=1}^{\infty} |\tilde{C}_j Z_{n,i-j}^\leq|} > \frac{\epsilon}{2} \right).
$$

(3.43)

To estimate $I_1$ note that

$$
I_1 \leq P \left( \sum_{j=1}^{\infty} |\tilde{C}_j| > 1 \right) + P \left( \sqrt{\sum_{j=1}^{\infty} |\tilde{C}_j Z_{n,i-j}^\leq|} > \frac{\epsilon}{2}, \sum_{j=1}^{\infty} |\tilde{C}_j| \leq 1 \right)
\leq \sum_{j=1}^{\infty} E[|\tilde{C}_j|] + \sum_{j=1}^{\infty} P \left( \sum_{j=1}^{\infty} |\tilde{C}_j Z_{n,i-j}^\leq| > \frac{\epsilon}{2}, \sum_{j=1}^{\infty} |\tilde{C}_j| \leq 1 \right).
$$

where the last inequality follows by Markov’s inequality. Take some $\varphi > \alpha$ and let $\psi$ be such that $1/\varphi + 1/\psi = 1$. Then by Hölder’s inequality we have

$$
\left( \sum_{j=1}^{\infty} |\tilde{C}_j Z_{n,i-j}^\leq| \right)^\varphi = \left( \sum_{j=1}^{\infty} |\tilde{C}_j|^{1/\psi} \cdot |\tilde{C}_j|^{1/\varphi} |Z_{n,i-j}^\leq| \right)^\varphi \leq \left( \sum_{j=1}^{\infty} |\tilde{C}_j| \right)^{\varphi/\psi} \sum_{j=1}^{\infty} |\tilde{C}_j| \cdot |Z_{n,i-j}^\leq|^{\psi/\varphi},
$$

which leads to

$$
I_1 \leq \sum_{j=1}^{\infty} E[|\tilde{C}_j|] + \sum_{j=1}^{\infty} P \left( \left( \sum_{j=1}^{\infty} |\tilde{C}_j| \right)^{\varphi/\psi} \sum_{j=1}^{\infty} |\tilde{C}_j| \cdot |Z_{n,i-j}^\leq|^{\psi/\varphi} > \left( \frac{\epsilon}{2} \right)^\varphi, \sum_{j=1}^{\infty} |\tilde{C}_j| \leq 1 \right)
\leq \sum_{j=1}^{\infty} E[|\tilde{C}_j|] + \sum_{j=1}^{\infty} P \left( \sum_{j=1}^{\infty} |\tilde{C}_j| \cdot |Z_{n,i-j}^\leq|^{\psi/\varphi} > \left( \frac{\epsilon}{2} \right)^\varphi \right).
$$

This together with the Markov’s inequality, the fact that the sequence $(C_j)$ is independent of $(Z_i)$ and stationarity of the sequence $(Z_i)$ yields

$$
I_1 \leq \sum_{j=1}^{\infty} E[|\tilde{C}_j|] + \left( \frac{\epsilon}{2} \right)^{-\varphi} \sum_{j=1}^{\infty} E \left( \sum_{j=1}^{\infty} |\tilde{C}_j| \cdot |Z_{n,i-j}^\leq|^{\psi/\varphi} \right)
= \sum_{j=1}^{\infty} E[|\tilde{C}_j|] + \frac{2^\varphi}{\epsilon^\varphi} \sum_{j=1}^{\infty} \sum_{j=1}^{\infty} E[|\tilde{C}_j|] E[|Z_{n,i-j}^\leq|]^{\psi/\varphi}
= \left( 1 + \frac{2^\varphi}{\epsilon^\varphi} n E[|Z_{n,i}|]^{\psi/\varphi} \right) \sum_{j=1}^{\infty} E[|\tilde{C}_j|].
$$

(3.44)

From the definition of $\tilde{C}_j$ it follows $\sum_{j=1}^{\infty} E[|\tilde{C}_j|] \leq 5 \sum_{j=1}^{\infty} |E[C_j]|$, and by Karamata’s theorem and (2.3), as $n \to \infty$,

$$
n E[|Z_{n,i}^\leq|]^{\psi/\varphi} = \frac{E(|Z_{1}|^{\varphi+1}[|Z_{1}| > a_n])}{a_n^\varphi P(|Z_{1}| > a_n)} \cdot n P(|Z_{1}| > a_n) \to \frac{\alpha}{\varphi - \alpha} < \infty.
$$
Hence from (3.44) we conclude that there exists a positive constant $D_1$ such that

$$\limsup_{n \to \infty} I_1 \leq D_1 \sum_{j=q}^{\infty} E|C_j|.$$  \hspace{1cm} (3.45)

To estimate $I_2$, assume first $\alpha \in (1, \infty)$. Applying again Markov’s inequality, the fact that the sequence $(C_i)$ is independent of $(Z_i)$ and stationarity of $(Z_i)$ we obtain

$$I_2 \leq \sum_{j=1}^{n} P \left( \left| \sum_{i=j+q-1}^{\infty} \tilde{C}_{Z_{n,i-j}} \right| > \frac{\epsilon}{2} \right) \leq \frac{2^\delta}{\epsilon} \sum_{j=1}^{n} E|\tilde{C}_{Z_{n,i-j}}| = \frac{2^\delta}{\epsilon} \sum_{j=q}^{\infty} E|\tilde{C}_j| \cdot nE|Z_{n,1}|.$$

By Karamata’s theorem and relation (2.3), as $n \to \infty$,

$$nE|Z_{n,1}| = \frac{E|Z_1|1_{|Z_1| > a_n}}{a_n} P(Z_1 > a_n) \to \frac{\alpha}{\alpha - 1} < \infty,$$

and hence we see that there exists a positive constant $D_2$ such that

$$\limsup_{n \to \infty} I_2 \leq D_2 \sum_{j=q}^{\infty} E|C_j|.$$  \hspace{1cm} (3.46)

In the case $\alpha = 1$ Markov’s inequality implies

$$I_2 \leq \frac{2^\delta}{\epsilon} \sum_{i=1}^{n} \sum_{j=q}^{\infty} E|\tilde{C}_{Z_{n,i-j}}| = \frac{2^\delta}{\epsilon} \sum_{j=q}^{\infty} E|\tilde{C}_j| \cdot nE|Z_{n,1}|.$$

with $\delta$ as in relation (3.39). Since $\delta < 1$, an application of the triangle inequality $|\sum_{i=1}^{\infty} a_i|^s \leq \sum_{i=1}^{\infty} |a_i|^s$ with $s \in (0,1]$ yields

$$I_2 \leq \frac{2^\delta}{\epsilon^s} \sum_{i=1}^{n} \sum_{j=q}^{\infty} E|\tilde{C}_{Z_{n,i-j}}|^s = \frac{2^\delta}{\epsilon^s} \sum_{j=q}^{\infty} E|\tilde{C}_j|^s \cdot nE|Z_{n,1}|.$$

From this, since by Karamata’s theorem

$$\lim_{n \to \infty} nE|Z_{n,1}| = \frac{n}{a_n} E\left( |Z_1|^s 1_{|Z_1| > a_n} \right) = \frac{1}{1 - \delta} < \infty,$$

and by a new application of the triangle inequality

$$\sum_{j=q}^{\infty} E|\tilde{C}_j|^s = E|\tilde{C}_{q-1}|^s + E|\tilde{C}_q|^s + \sum_{j=q+1}^{\infty} E|\tilde{C}_j|^s \leq E\left( 2 \sum_{j=q-1}^{\infty} |C_j| \right)^s + E\left( 2 \sum_{j=q-1}^{\infty} |C_j| \right)^s + \sum_{j=q+1}^{\infty} E|C_j|^s \leq 2^s \sum_{j=q-1}^{\infty} E|C_j|^s + 2^s \sum_{j=q-1}^{\infty} E|C_j|^s + \sum_{j=q+1}^{\infty} E|C_j|^s.$$
it follows that there exists a positive constant $D_3$ such that

$$ \limsup_{n \to \infty} I_2 \leq D_3 \sum_{j=q-1}^{\infty} E[C_j]^{\frac{1}{\delta}}. $$

This together with (3.43), (3.45) and (3.46) yields

$$ \limsup_{n \to \infty} P[d_{M_1}(M_n, M_{n+}) > \epsilon] \leq D_1 \sum_{j=q-1}^{\infty} E[C_j] + (D_2 + D_3) \sum_{j=q-1}^{\infty} E[C_j]^{\frac{1}{\delta}}, $$

where $s = \delta$ for $\alpha = 1$ and $s = 1$ for $\alpha > 1$. Now, conditions (3.39) and (3.41) yield (3.42), and hence we again obtain $M_n(\cdot) \overset{d}{\to} M(\cdot)$ in $(D_1[0, 1], d_{M_1})$. This concludes the proof. \hfill \Box

**Remark 3.3.** Since $(C^{(1)}, C^{(2)})$ is independent of $(W^{(1)}, W^{(2)})$, the limiting process $M$, defined in (3.11) by

$$ M(t) = C^{(1)}W^{(1)}(t) \lor C^{(2)}W^{(2)}(t) = \bigvee_{t_i \leq t} |j_i|C^{(1)}1_{(j_i > 0)} + C^{(2)}1_{(j_i < 0)}, \quad t \in [0, 1], $$

where $\sum \delta_{t_i,j_i}$ is a Poisson process with intensity measure $Leb \times \mu$, with $\mu$ as in (2.5), conditionally on $(C^{(1)}, C^{(2)}) = (a, b)$, is an extremal process with exponent measure $(pa^x + rb^x)x^{-\alpha-1}dx$ for $x > 0$ and non-negative real numbers $a$ and $b$. Indeed, for $x > 0$ we have

$$ P[M(t) \leq x | (C^{(1)}, C^{(2)}) = (a, b)] = \mathbb{P} \left( \bigvee_{t_i \leq t} |j_i|a1_{(j_i > 0)} + b1_{(j_i < 0)} \leq x \right) = \mathbb{P} \left( \bigvee_{t_i \leq t} j_iS_i \leq x \right) $$

(3.47)

with $S_i = a1_{(j_i > 0)} - b1_{(j_i < 0)}$. Propositions 3.7 and 3.8 in Resnick (1987) yield that $\sum \delta_{t_i,j_i}$ is a Poisson process with intensity measure $Leb \times \tilde{\mu}$, where $\tilde{\mu}(dx, dy) = \mu(dx)K(x, dy)$ and $K(x, dy) = \mathbb{P} \{ a1_{(j_i > 0)} - b1_{(j_i < 0)} \in dy \}$, and that $\sum \delta_{t_i,j_i}$ is a Poisson process with intensity measure $Leb \times \hat{\mu}$, where

$$ \hat{\mu}(x, \infty) = \tilde{\mu}(\{ (y, z) : yz > x \}) = \int_{y > x} \mu(dy)K(y, dz), \quad x > 0. $$

From this we conclude that the process $\bigvee_{t_i \leq t} j_iS_i$ is an extremal process with exponent measure $\hat{\mu}$ (see Resnick, 1987, Section 4.3 and Resnick, 2007, p. 161). Standard computations give

$$ \hat{\mu}(x, \infty) = \int_0^\infty \int_{x/y}^\infty K(y, dz)\mu(dy) + \int_0^\infty \int_0^{x/y} K(y, dz)\mu(dy) = px^{-\alpha}a^x + rx^{-\alpha}b^x. $$

Hence

$$ \hat{\mu}(dx) = (pa^x + rb^x)x^{-\alpha-1}1_{(0,\infty)}(x) \, dx, $$

and we conclude from (3.47) that the limiting process $M$, conditionally on $(C^{(1)}, C^{(2)}) = (a, b)$, is an extremal process with exponent measure $\hat{\mu}$.

Note that, conditionally on $\{C_j = c_j \text{ for all } j \geq 0\}$, where $\{c_j\}$ is a sequence of real numbers, the process $X_i$ in (2.6) is a linear process with deterministic coefficients $c_j$. Therefore, Proposition 4.28 in Resnick (1987) yields that the limit of $M_n$ in $D_1[0, 1]$ with the $M_1$ topology is a process which is, conditionally on $\{C_j = c_j \text{ for all } j \geq 0\}$, an extremal process with exponent measure $(pc^x + re^x)x^{-\alpha-1}dx$ for $x > 0$, provided $c_x p + c_r r > 0$, where
\[ c_+ = \max\{c_j \lor 0 : j \geq 0\} \quad \text{and} \quad c_- = \max\{-c_j \lor 0 : j \geq 0\}. \]

This corresponds to the above considerations about the structure of the limiting process for \(a = c_+\) and \(b = c_-\).

**Remark 3.4.** If the sequence \((C_j)\) is deterministic, condition (3.40) can be dropped since it is implied by (3.39). Note that condition (3.39) implies \(|C_j|^{\delta} < 1\) for large \(j\), and since \(|C_j|^{\delta\alpha} \) is decreasing in \(x\), it follows that for large \(j\)

\[ |C_j|^{\alpha} = (|C_j|^{\delta})^{\alpha/\delta} \leq |C_j|^{\delta}. \]

This suffices to conclude that (3.40) holds. In general this does not hold when the coefficients are random (see Krizmanić, 2019, p. 739).

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Data sharing is not applicable to this article as no new data were generated or analyzed in this study.

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