COMPLETELY CONTROLLING THE DIMENSIONS OF FORMAL FIBER RINGS AT PRIME IDEALS OF SMALL HEIGHT

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ABSTRACT. Let \( T \) be a complete equicharacteristic local (Noetherian) UFD of dimension 3 or greater. Assuming that \(|T| = |T/\mathfrak{m}|\), where \( \mathfrak{m} \) is the maximal ideal of \( T \), we construct a local UFD \( A \) whose completion is \( T \) and whose formal fibers at height one prime ideals have prescribed dimension between zero and the dimension of the generic formal fiber. If, in addition, \( T \) is regular and has characteristic zero, we can construct \( A \) to be excellent.

1. Introduction

Let \( A \) be a local (Noetherian) ring with maximal ideal \( \mathfrak{m} \) and \( \mathfrak{m} \)-adic completion \( T \). To understand the relationship between \( \text{Spec} \, A \) and \( \text{Spec} \, T \) it is useful to consider a simplifying numerical quantity, the dimensions of the formal fiber rings of \( A \), which roughly measures how much “larger” \( \text{Spec} \, T \) is than \( \text{Spec} \, A \). We briefly recall the pertinent definitions. The formal fiber ring of \( A \) at the prime ideal \( \mathfrak{p} \) is \( T \otimes_A \kappa(\mathfrak{p}) \), where \( \kappa(\mathfrak{p}) \) is the residue field \( \mathfrak{m} \mathfrak{p}/\mathfrak{p} \mathfrak{m} \mathfrak{p} \); the formal fiber of \( A \) at \( \mathfrak{p} \) is the spectrum of the formal fiber ring of \( A \) at \( \mathfrak{p} \); if \( A \) is an integral domain, the generic formal fiber ring of \( A \) is the formal fiber ring of \( A \) at \( (0) \) and the generic formal fiber of \( A \) is the formal fiber of \( A \) at \( (0) \); the dimension of the formal fiber of \( A \) at \( \mathfrak{p} \), denoted by \( \alpha(A, \mathfrak{p}) \), is the dimension of the formal fiber ring of \( A \) at \( \mathfrak{p} \). The definitions are greatly clarified by identifying elements of the formal fibers of \( A \) with prime ideals of \( T \). This identification uses the natural bijection between the formal fiber of \( A \) at \( \mathfrak{p} \) and the inverse image of \( \mathfrak{p} \) under the map \( \text{Spec} \, T \rightarrow \text{Spec} \, A \) given by \( \mathfrak{q} \mapsto \mathfrak{q} \cap A \).

The first significant study of the dimension of formal fibers was Matsumura’s paper [8], which gives a clear picture of the quantity’s basic properties; we briefly outline some of the most important results from that paper. Following Matsumura, let \( \alpha(A) \) denote the supremum of the dimensions of the formal fibers of \( A \). First, the dimension of the formal fibers decreases weakly under inclusion, that is, if \( \mathfrak{p} \) and \( \mathfrak{q} \) are prime ideals of \( A \) such that \( \mathfrak{p} \subseteq \mathfrak{q} \) then \( \alpha(A, \mathfrak{p}) \geq \alpha(A, \mathfrak{q}) \). It follows that, if \( A \) is an integral domain, then \( \alpha(A) = \alpha(A, (0)) \). Second, if \( A \) has positive Krull dimension \( n \) then its formal fibers have dimension at most \( n - 1 \), that is, \( \alpha(A) \leq \dim A - 1 \). Third, for a large class of naturally occurring rings, the rings of essentially finite type over a field, the dimension of the formal fibers depends linearly on height and decreases as height increases: \( \alpha(A, \mathfrak{p}) = \dim A - \text{ht} \mathfrak{p} - 1 \) for \( \mathfrak{p} \) a nonmaximal prime ideal of \( A \).
It is natural to ask whether Matsumura’s formula for \(\alpha(A, p)\) can be extended to all rings, or at least, to a class of rings that behave nicely with respect to completion, such as excellent rings. We should immediately exclude the trivial case \(\alpha(A) = 0\), where every formal fiber has dimension zero. Besides this trivial case, even if the formula does fail we might reasonably expect that \(\alpha(A)\) would decrease strictly as \(ht p\) increases. In particular, simplifying the question by looking at prime ideals of small height, we might expect that \(\alpha(A)\) rarely equals \(\alpha(A, p)\) when \(ht p = 1\).

These considerations led Heinzer, Rotthaus, and Sally to informally pose the following question.

**Question.** Let \(A\) be an excellent local ring with \(\alpha(A) > 0\), and let \(\Delta\) be the set \(\{ p \in \text{Spec} A \mid ht p = 1 \text{ and } \alpha(A, p) = \alpha(A)\}\). Is \(\Delta\) a finite set?

Answering this question in the negative, Boocher, Daub, and Loepp constructed in [1] an excellent local unique factorization domain (UFD) with uncountably many height one prime ideals for which \(\Delta\) is countably infinite. Nevertheless, one could still reasonably conjecture that most height one prime ideals must lie outside \(\Delta\); perhaps, for instance, \(\Delta\) is at most countable. It is our goal to disprove this weakened conjecture. We will construct an excellent UFD \(A\) for which the set \(\Delta\) is uncountable and consists of every height one prime ideal of \(A\). Fleming et al. show in a forthcoming paper that there exist UFDs for which \(\Delta\) contains all height one prime ideals, but the rings they construct are not excellent.

Our first main result, the outcome of the construction, is Theorem 4.1. We start with a complete equicharacteristic local UFD \(T\) of dimension 3 or greater having the same cardinality as its residue field. The theorem then guarantees the existence of a local subring \(A\) of \(T\) which has \(T\) as its completion; whose formal fibers at prime ideals of height two or greater are singletons, and hence have dimension zero; for which we can prescribe the dimension of the generic formal fiber; and for which we can prescribe the number of formal fibers at height one prime ideals that have a given dimension. In particular, we can force \(\alpha(A, p)\) to equal \(\alpha(A, (0))\) for every height one prime ideal \(p\) of \(A\).

Our second main result is Theorem 5.1, a strengthening of Theorem 4.1 in which the constructed ring \(A\) is excellent. For this theorem, we require the additional hypothesis that \(T\) be regular and have characteristic zero.

Our method of construction is modeled on the construction in [4]: we build the ring \(A\) by adjoining deliberately chosen elements of \(T\) to the prime subring of \(T\). The key idea, Lemma 2.1, is that the ring \(A\) will be Noetherian and have \(T\) as its completion provided that the following two conditions hold: first, the natural map \(A \to T/m^2\) is onto, and second, every finitely generated ideal of \(A\) is unchanged by extension to \(T\) followed by contraction to \(A\). We can construct a ring satisfying the first condition by starting with the prime subring of \(T\) and adjoining to it a representative of every coset of \(T/m^2\). It turns out that the ring
can be made to satisfy the second condition by adjoining even more elements, although this step is rather technical; see Lemma 3.8. Since $T/m^2$ is infinite, the adjunction process must be formalized using transfinite recursion. That is, we construct an ascending chain of subrings of $T$ indexed by ordinals and terminating in $A$. The strength of this method rests in the abundance of viable elements to adjoin. Broadly speaking, we can force the final ring $A$ to have a certain property by choosing the adjoined elements so that the property is created or conserved during the construction of intermediate rings. Prime avoidance lemmas function as the main technical tool for making these choices, and to use these lemmas we must ensure that the intermediate rings are always smaller than $T/m$. We should caution that although our method is constructive, it makes uncountably many arbitrary choices and there is really no hope of explicitly describing the final ring $A$.

The main ingredient of our construction is a novel combination of properties and a series of simplifying assumptions which together yield the ring $A$ described above. Before giving the admittedly complicated definition of a $J$-subring in Section 3, which encapsulates these ingredients, let us briefly explain and motivate them.

The height one prime ideals of $A$ would be simplest to understand if they were all principal, or equivalently, if $A$ were a UFD. Since every ring whose completion is a UFD is itself a UFD, we can ensure that $A$ is a UFD, and hence that its height one prime ideals are simple, by stipulating that $T$ be a UFD. It would be even better if the height one prime ideals of $A$ were generated by elements that were prime not only in $A$, but also in $T$. We can force $A$ to have this property (which we call (J.5)), that is, that each of its prime elements be prime in $T$, by adjoining to each intermediate ring $R$ the prime elements of $T$ that appear as factors of elements of $R$. These simplifying assumptions make it much easier to control the formal fibers of $A$ in the intermediate rings because they allow us to restrict our attention to the prime elements of $T$.

To ensure that the formal fibers of $A$ at its height one prime ideals are large enough, whenever a generator $p$ of a height one prime ideal $pT$ of $T$ first appears in an intermediate ring $R$ we choose a prime ideal $q = q_{pT}$ of $T$, having prescribed height, for which $q_{pT} \cap R = pR$. We call the set of these chosen prime ideals $q$ the distinguished set for $R$, denoted by $Q_R$; the corresponding set of height one prime ideals of $T$ is denoted by $F_R$. To prescribe the height of each $q \in Q_R$, we first index the elements of $F_R$ by an ordinal number using an ordering function $\varphi_R$, and then specify the height of each $q_{pT}$ using a height indicator function $\Lambda$.

The prime ideal $q_{pT}$ will eventually become the maximal element in the formal fiber of $A$ at $pA$. To ensure that it lies in the fiber at all, we ensure that $q_{pT} \cap R = pR$ holds for every intermediate ring $R$, and hence also for $A$; this step requires prime avoidance lemmas to carefully choose elements to adjoin. In this way we get a lower bound of $\text{ht}(q_{pT}) - 1$ on the dimension of the formal fiber of $A$ at $pA$. To make this lower bound an upper bound, we adjoin generators of the prime ideals of $T$ that are not contained in any element of the
distinguished set. Consequently, the formal fibers of prime ideals of \( A \) that have height two or more will be singletons.

To control the dimension of the generic formal fiber, we fix a prime ideal \( \mathfrak{P} \) of \( T \) having prescribed height and choose the elements to adjoin so that \( \mathfrak{P} \) intersects each intermediate ring in the zero ideal. As with the prime ideals of the distinguished set, we use \( \mathfrak{P} \) to get a lower bound on the dimension of the generic formal fiber. The upper bound on dimension comes from adding nonzero elements of prime ideals of \( T \) whose height is larger than the height of \( \mathfrak{P} \). As long as every prime ideal of \( \text{ht} \mathfrak{P} + 1 \) or greater has nonzero intersection with an intermediate ring, none of those prime ideals can be in the generic formal fiber.

The previous paragraphs describe the main aspects of the definition of a \( J \)-subring. In addition to that definition, we found it convenient to define an extension of a \( J \)-subring, two \( J \)-subrings \( R \subseteq S \) satisfying certain compatibility conditions. Specifically, the rings should have the same size unless \( R \) is finite, a trivial case, and the distinguished sets and ordering functions should be compatible in the sense that \( \mathcal{Q}_R \subseteq \mathcal{Q}_S \) and \( \phi_S|_R = \phi_R \).

In this paper, all rings are commutative with unity. We say a ring is quasi-local if it has exactly one maximal ideal and local if it is both quasi-local and Noetherian. When we say \( (T, \mathfrak{m}) \) is a local ring, we mean that \( T \) is a local ring with maximal ideal \( \mathfrak{m} \). We denote the cardinality of a set \( X \) by \(|X|\), the union of the elements of \( X \) by \( \bigcup X \), the fraction field of an integral domain \( R \) by \( R_{(0)} \), and the set of prime ideals of a ring \( T \) having height \( k \) by \( \text{Spec}_k T \). Several constructions assume familiarity with the basic properties of ordinal and cardinal numbers. Section 2 collects preliminary lemmas, Section 3 builds the details of the construction, Section 4 uses the construction to prove our first main result, and Section 5 extends that result to excellent rings.

## 2. Preliminary Lemmas

The lemmas in this section fall into three broad classes. First, Lemma 2.1 is our main technical tool; second, prime avoidance lemmas for choosing the elements to adjoin; and third, a lemma concerning the cardinality of \( T \) and \( T/\mathfrak{p} \) where \( \mathfrak{p} \) is a nonmaximal prime ideal of \( T \).

**Lemma 2.1 ([5, Proposition 1]).** Let \( (A, \mathfrak{m} \cap A) \) be a quasi-local subring of a complete local ring \( (T, \mathfrak{m}) \). If \( aT \cap A = a \) for every finitely generated ideal \( a \) of \( A \) and the map \( A \to T/\mathfrak{m}^2 \) is onto then \( A \) is Noetherian and the natural homomorphism \( \widetilde{A} \to T \) is an isomorphism.

Our main technical tool for choosing elements of \( T \) is a pair of powerful prime avoidance lemmas for local rings, Lemmas 2.2 and 2.3. We will use the two lemmas to find elements satisfying transcendence conditions
related to the set of distinguished prime ideals of $T$. These transcendence conditions guarantee that the contractions of elements of the distinguished set remain principal as the intermediate ring grows.

**Lemma 2.2** ([4, Lemma 3]). Let $(T, m)$ be a local ring. Let $D \subset T$, let $C \subset \text{Spec} T$, and let $a$ be an ideal of $T$ such that $a \not\subseteq p$ for all $p \in C$. If $|C \times D| < |T/m|$ then

$$a \not\subseteq \bigcup \{ t + p \mid p \in C, t \in D \}.$$  

**Lemma 2.3** ([6, Lemma 4]). Let $(T, m)$ be a local ring, with $|T/m|$ infinite, and let $D \subset T$. Let $C \subset \text{Spec} T$ and $y, z \in T$ be such that $y, z \not\in p$ for all $p \in C$. If $|C \times D| < |T/m|$, then there is a unit $u \in T^\times$ such that

$$yu, zu^{-1} \not\in \bigcup \{ t + p \mid t \in D, p \in C \}.$$  

Because the constructed ring $A$ will have the cardinality of $T$, whereas the prime avoidance lemmas are suited to avoiding sets of cardinality $|T/m|$ or less, we will require that $|T| = |T/m|$. For the same reason it is necessary to constrain the growth of the size of the intermediate rings, and in particular, adjoining a single element will not increase the size, provided the ring is already infinite.

The final lemma of this section relates the cardinality of a complete local ring to that of its quotients, and it shows that complete local rings are “large.” Note that, a consequence of the lemma is that if $(T, m)$ is a complete local ring of dimension at least one and $|T/m| = |T|$, then $|T/m| > \aleph_0$.

**Lemma 2.4** ([3, Lemma 2.3]). Let $(T, m)$ be a complete local ring of dimension at least one and let $p \in \text{Spec} T$ be a nonmaximal prime ideal of $T$. Then $|T/p| = |T| \geq 2^{\aleph_0}$.

3. The Construction

Given a subring $R$ of a complete local UFD $T$, let

$$\mathcal{F}_R = \{ pT \in \text{Spec}_1 T \mid pu \in R \text{ for some unit } u \in T \}.$$  

For simplicity and without loss of generality we will subscribe to the notational convention that the generator $p$ of a prime ideal $pT \in \mathcal{F}_R$ is contained in $R$. That is, when we write $pT \in \mathcal{F}_R$ we are choosing $p$ so that $p \in R$. We could just as well have defined $\mathcal{F}_R$ to be a set of prime elements, but the current definition makes several arguments less verbose.

**Definition 3.1.** Let $(T, m)$ be a complete local UFD with $3 \leq \dim T$, let $(R, m \cap R)$ be a quasi-local subring of $T$ and let $\mathfrak{P}$ be a nonmaximal prime ideal of $T$. Suppose $\Lambda : |T| \to \{1, 2, \ldots, \min(\text{ht} \mathfrak{P} + 1, \dim T - 1)\}$ is a function defined on the cardinal number $|T|$, and let $\varphi_R : \mathcal{F}_R \to \delta$ be a bijection from $\mathcal{F}_R$ to an initial segment $\delta$ of $|T|$. Suppose that the following conditions are satisfied.
(J.1) \(|R| < |T/m|\);

(J.2) \(P \cap R = (0)\);

(J.3) for every \(a \in R\), \(aT \cap R = aR\);

(J.4) there exists a set \(Q_R \subset \bigcup \{\text{Spec}_n T \mid 1 \leq n \leq \min(\text{ht} P + 1, \text{dim} T - 1)\}\) such that the sets \(F_R\) and \(Q_R\) are in bijection and the ideal \(q_{pT} \in Q_R\) corresponding to \(pT \in F_R\) is such that

(J.4a) \(pT \subseteq q_{pT}\),

(J.4b) the image of \(q_{pT}\) in \(T/pT\) is in the regular locus of \(T/pT\),

(J.4c) \(q_{pT} \cap R = pT \cap R\), and

(J.4d) for every \(q_{pT} \in Q_R\), \(\text{ht}(q_{pT}) = \Lambda(\varphi_R(pT))\).

Then the 4-tuple \((R, P, \Lambda, \varphi_R)\) is called a \(J\)-subring of \(T\). Any set \(Q_R\) satisfying (J.4) is called a distinguished set for \(R\). The function \(\Lambda\) is called the height indicator and the function \(\varphi_R\) is called the ordering function for \(R\). If \((R, P, \Lambda, \varphi_R)\) is a \(J\)-subring of \(T\) and \(Q_R\) is a fixed distinguished set for \(R\), we say that \((R, P, \Lambda, \varphi_R)\) is a \(J\)-subring of \(T\) with distinguished set \(Q_R\). If, in addition, \(P, \Lambda,\) and \(\varphi_R\) are understood, we will say that \(R\) is a \(J\)-subring with distinguished set \(Q_R\). If \(Q_R\) is also understood, we will say that \(R\) is a \(J\)-subring.

Remark. It follows from conditions (J.3) and (J.4c) that \(q_{pT} \cap R = pT \cap R = pR\). Note also that no \(q_{pT} \in Q_R\) is contained in \(P\).

The cardinality condition (J.1) is needed to invoke prime-avoidance lemmas, and thus allows us to adjoin elements to \(R\) without adding unwanted elements of \(P\) or of prime ideals contained in \(Q_R\). Although our final ring \(A\) will not satisfy (J.1), we would like it to satisfy the other properties of a \(J\)-subring. (J.2) will ensure that \(P\) is in the generic formal fiber of \(A\). (J.3) is needed to show that the completion of \(A\) is \(T\). By maintaining (J.4) we will ensure that the elements of \(Q_R\) are precisely the maximal ideals of the formal fiber rings of \(A\) at its height one prime ideals; (J.4b) gives us that these are regular local rings. The ordering function \(\varphi_R\) implicitly identifies each element \(pT\) of \(F_R\) with an ordinal less than \(|T|\), and the height indicator \(\Lambda\) uses this ordinal to specify the height of \(q_{pT}\); this is the content of (J.4d). We may freely choose the heights of the distinguished prime ideals as long as each height is strictly less than that of the maximal ideal and the dimension of the generic formal fiber is no less than that of any other fiber; these two conditions explain the upper bound \(\min(\text{ht} P + 1, \text{dim} T - 1)\) on the heights of distinguished prime ideals.

Definition 3.2. Let \((T, m)\) be a complete local UFD with \(3 \leq \text{dim} T\), let \((R, P, \Lambda, \varphi_R)\) be a \(J\)-subring of \(T\) with distinguished set \(Q_R\), and let \((S, P, \Lambda, \varphi_S)\) be a \(J\)-subring of \(T\) with distinguished set \(Q_S\) and suppose that \(R \subseteq S\).

The ring \(S\) is called a \(J\)-extension of \(R\) if the following conditions are satisfied:
Lemma 3.3. Let $(T, m)$ be a complete local UFD with $3 \leq \dim T$, let $\delta$ be an ordinal number, and let 
$(R_\beta, \mathfrak{P}, \Lambda, \varphi_{R_\beta})_{\beta<\delta}$ be a sequence of $J$-subrings of $T$ indexed by $\delta$. Let $Q_{R_\beta}$ denote a distinguished set for $R_\beta$. Suppose that $R_0$ is infinite; that if $\beta = \gamma + 1$ is a successor ordinal then $R_\gamma$ is a $J$-extension of $R_\beta$; that if $\beta$ is a limit ordinal then $R_\beta = \bigcup_{\gamma<\beta} R_\gamma$ and that the distinguished set of $R_\beta$ is $Q_{R_\beta} = \bigcup_{\gamma<\beta} Q_{R_\gamma}$; and that the ordering function $\varphi_{R_\beta}$ is such that $\varphi_{R_\beta}(pT) = \varphi_{R_\gamma}(pT)$ for every $\gamma < \beta$ and every $pT \in F_{R_\gamma}$.

Define the ring $S$ by $S = \bigcup_{\beta<\delta} R_\beta$.

Then $|S| \leq \max(|R_0|, |\delta|)$; there is an ordering function $\varphi_S$ such that $(S, \mathfrak{P}, \Lambda, \varphi_S)$ satisfies every condition of a $J$-subring except perhaps the cardinality condition $(\text{J.1})$ and $S$ satisfies condition $(\text{2})$ of a $J$-extension of $R_0$.

Proof. The ring $S$ satisfies $(\text{J.2})$ since, if $r \in \mathfrak{P} \cap S$, then $r \in \mathfrak{P} \cap R_\beta$ for some $\beta$ and $\mathfrak{P} \cap R_\beta = (0)$. For $(\text{J.3})$ suppose that $a \in S$ and $x \in aT \cap S$. There is an $R_\beta$ containing $a$ and $x$, and since $R_\beta$ is a $J$-subring, $aT \cap R_\beta = aR_\beta$. So $x \in aT \cap R_\beta = aR_\beta \subseteq aS$, and $(\text{J.3})$ follows.

For $(\text{J.4})$ note that $F_S = \bigcup_{\beta<\delta} F_{R_\beta}$. We claim $Q_S = \bigcup_{\beta<\delta} Q_{R_\beta}$ is a distinguished set for $S$. Conditions $(\text{J.4a})$ and $(\text{J.4b})$ are trivially satisfied for each element of $Q_S$. Condition $(\text{J.4c})$ holds because if $q_{pT} \in Q_S$ then $q_{pT} \in Q_{R_\beta}$ for some $\beta$ and $q_{pT} \cap R_\gamma = pT \cap R_\gamma$ for every $\gamma \geq \beta$; we may then take the union over $\gamma$ of each side of these equations to conclude that $q_{pT} \cap S = pT \cap S$. Here we use the coherence condition that $Q_{R_\beta} \subseteq Q_{R_\gamma}$ whenever $\beta \leq \gamma$.

For $(\text{J.4d})$ define the ordering function $\varphi_S$ on $S$ by setting $\varphi_S(pT) = \varphi_{R_\gamma}(pT)$ for $pT \in F_{R_\gamma}$. It is then obvious that $(\text{J.4d})$ and $(\text{2})$ both hold.

To verify the cardinality inequality, we’ll prove by transfinite induction the stronger statement that for all $\beta$, $|R_\beta| \leq \max(|R_0|, |\beta|)$. The statement holds trivially for $\beta = 0$. If $\beta = \gamma + 1$ is a successor ordinal then $|R_\beta| = |R_\gamma|$ because $R_\gamma \subseteq R_\beta$ is a $J$-extension, and therefore $|R_\beta| \leq \max(|R_0|, |\gamma|) = \max(|R_0|, |\beta|)$. If $\beta$ is a limit ordinal then

$$|R_\beta| \leq \sum_{\gamma<\beta} |R_\gamma| \leq |\beta| \sup_{\gamma<\beta} |R_\gamma| \leq |\beta| \sup_{\gamma<\beta} (\max(|R_0|, |\gamma|)) \leq \max(|R_0|, |\beta|).$$
It follows that $|S| \leq \max(|R_0|, |\delta|)$. \hfill \square

Under the assumptions of Lemma 3.3, if $|R_0| < |T/m|$ and $|\delta| < |T/m|$ then $S$ is a $J$-subring of $T$. When we apply the lemma these two inequalities will always hold, except in the proof of Theorem 4.1.

Because many of our lemmas share hypotheses, we will state them as an assumption so that they can be concisely referenced.

**Assumption 3.4.** The local ring $(T, m)$ is a complete UFD with $3 \leq \dim T$ and $|T/m| > \aleph_0$. $\mathfrak{P}$ is a nonmaximal prime ideal of $T$, and $(R, \mathfrak{P}, \Lambda, \varphi_R)$ is a $J$-subring of $T$ with distinguished set $Q_R$.

Looking at the definition of $J$-subring, it is not immediately clear that finding a set $Q_R$ satisfying the conditions of (J.4) will be possible. We will employ the next lemma to select the prime ideals $q_{pT}$ for $Q_R$. Note that the conditions in Lemma 3.5 are precisely the first three conditions of (J.4).

**Lemma 3.5.** Under Assumption 3.4, let $(S, S \cap m)$ be a quasi-local subring of $T$ such that $|S| < |T/m|$. Let $n$ be an integer with $1 \leq n \leq \dim T - 1$. If $pT$ is a prime ideal of $T$ then there exists $q_{pT} \in \text{Spec}_n T$ satisfying the following:

1. $pT \subseteq q_{pT},$
2. the image of $q_{pT}$ in $T/pT$ is in the regular locus of $T/pT$, and
3. $q_{pT} \cap S = pT \cap S$.

**Proof.** We show the lemma by induction on $n$. If $n = 1$, then $q_{pT} = pT$ satisfies the required conditions.

Now let $1 < n \leq \dim T - 1$, and suppose that $q \in \text{Spec}_{n-1} T$ has the desired properties. Then $q \cap S = pT \cap S$ and so the map

$$\frac{S}{pT \cap S} \hookrightarrow \frac{T}{q}$$

is an inclusion. If $\pi \in S/pT \cap S$ is a nonzero element, then every height one prime ideal of $T/q$ containing $\pi$ is an associated prime ideal of $\tau(T/q)$. Thus, since $T/q$ is Noetherian, the set of its height one prime ideals with nonzero intersection with $S/pT \cap S$ is finite if $S/pT \cap S$ is finite, and has cardinality equal to $|S/pT \cap S| \leq |S| < |T/m|$ otherwise. Note that $T$ is excellent, and so it is catenary, and it follows that the dimension of $T/q$ is at least two. We next use Lemma 2.2 on the local ring $(T/q, m/q)$ to show that the number of height one prime ideals of $T/q$ is at least $|T/m|$. Suppose, on the contrary, that the number of height one prime ideals of $T/q$ is less than $|T/m|$, and note that $m/q \nsubseteq p/q$ for all height one prime ideals $p/q$ of $T/q$. Now let $C = \{p/q \mid p/q$ is a height one prime ideal of $T/q\}$, and let $D = \{0 + q\}$. Then we have that $|C \times D| < |T/m|$, and so by Lemma 2.2 $m/q \nsubseteq \bigcup \{p/q \mid p/q \in C\}$. But if $x + q \in m/q$, then $x + q$ is contained in some height one prime ideal of $T/q$, a contradiction. It follows that the number of height one
prime ideals of $T/q$ is at least $|T/m|$. So the set of height one prime ideals of $T/q$ whose intersection with $S/pT \cap S$ is the zero ideal is at least $|T/m|$. Since $T/pT$ is complete, it is excellent, and it follows that the singular locus of $T/pT$ is closed. As $T/pT$ is an integral domain, the singular locus is $V(J)$ for some nonzero ideal $J$ of $T/pT$. Let $J = (g_1, g_2, \ldots, g_t)$. Letting $\bar{q}$ denote the image of $q$ in $T/pT$, we have that $\bar{q}$ is in the regular locus of $T/pT$, and so $J \not\subset \bar{q}$. Hence, there is a generator of $J$ whose image in $T/q$ is not zero. Without loss of generality, suppose that the image of $q_1$, denoted by $\bar{q}_1$, is not zero in $T/q$. Now choose a height one prime ideal $\bar{q}_{pT}$ of $T/q$ such that $\bar{q}_1 \not\subset \bar{q}_{pT}$ and such that the intersection of $\bar{q}_{pT}$ with $S/pT \cap S$ is the zero ideal. As $T$ is excellent, it is catenary, and so $\bar{q}_{pT}$ lifts to a prime ideal $q_{pT}$ of $T$ of height $n$, and by the way we chose $\bar{q}_{pT}$, we have $q_{pT} \cap S = pT \cap S$. If the image of $q_{pT}$ in $T/pT$ is in the singular locus of $T/pT$, then it contains $J$, and so, in particular, it contains $g_1$. But this would imply that $\bar{q}_1 \in q_{pT}$, a contradiction. It follows that the image of $q_{pT}$ in $T/pT$ is in the regular locus of $T/pT$. 

The next lemma allows us to adjoin a transcendental element to a $J$-subring of $T$ and find a $J$-extension maintaining the properties that we desire. Because we repeatedly need to adjoin transcendental elements to satisfy different aspects of our construction, this lemma will be used often in proofs of later lemmas.

**Lemma 3.6.** Under Assumption \[\alpha\], suppose that $x \in T$ is such that $x + q \in T/q$ is transcendental over $R/q \cap R$ for all $q \in \mathcal{Q}_R \cup \{\mathfrak{P}\}$. Then there exists an infinite $J$-extension $S$ of $R$ that contains $x$.

**Proof.** Let $S = R[x]_{(0)} \cap T$. It is readily seen that $S$ is infinite, that $S$ satisfies \[\alpha\] and, if $R$ is infinite, that $|R| = |S|$. Additionally, if $c \in aT \cap S$, then $c = ab$ for some $b \in S_{(0)} \cap T = S$, and so $c \in aS$ and condition \[\beta\] also holds. For \[\gamma\] consider any element $r \in \mathfrak{P} \cap S$. Then $r = f/g$ for some $f \in \mathfrak{P} \cap R[x]$. Treating $f$ as a polynomial in $x$ over $R$, the assumption that $x + \mathfrak{P}$ is transcendental over $R/\mathfrak{P} \cap R$, gives us that each of its coefficients must be in $\mathfrak{P} \cap R = (0)$. Hence $r = 0$ and we have that $\mathfrak{P} \cap S = (0)$. Now most of the work of showing that $S$ is a $J$-extension of $R$ consists of constructing the distinguished set $\mathcal{Q}_S$ for $S$ and verifying that it satisfies the necessary properties.

Using the axiom of choice, find an arbitrary extension of the ordering function $\varphi_R$ to an ordering function $\varphi_S$. To define $\mathcal{Q}_S$, select for each $p' T \in F_S - F_R$ an ideal $q_{p'T}$ such that $\text{ht}(q_{p'T}) = \Lambda(\varphi_S(p'T))$, using Lemma 3.5 then \[\sigma\] holds and each $q_{p'T}$ satisfies condition \[\theta\]. Define $\mathcal{Q}_S$ to be the union of $\mathcal{Q}_R$ and the set of recently chosen $q_{p'T}$. To finish showing that $S$ is our desired $J$-subring, it remains to check condition \[\tau\] for $q_{p'T} \in \mathcal{Q}_R$.

We first show that $q_{p'T} \cap R[x] = pT \cap R[x] = pR[x]$ for every $q_{p'T} \in \mathcal{Q}_R$. The assumption that $x + q_{p'T}$ is transcendental over $R/(q_{p'T} \cap R)$ implies that if an element of $R[x]$ is contained in $q_{p'T}$, then each of its coefficients is in $q_{p'T} \cap R = pT \cap R = pR$. Thus $q_{p'T} \cap R[x] \subseteq (q_{p'T} \cap R)R[x] = pR[x] \subseteq pT \cap R[x]$, and since $pT \subset q_{p'T}$, we have equality throughout.
To now verify condition \([J,3c]\) for \(pT \in \mathcal{F}_R\), let \(r = f/g \in q_{pT} \cap R[x]_0\) with \(f, g \in R[x]\). If \(f\) and \(g\) are both contained in \(q_{pT} \cap R[x] = pR[x]\), then we can write \(f = p^n f'\) and \(g = p^m g'\) with \(f', g' \notin q_{pT}\) and \(r = (p^n f')/(p^m g')\). Thus, by dividing out the highest common power of \(p\) we can assume that \(p\) does not divide every coefficient of \(f\) and \(g\), and therefore that at most one of \(f\) or \(g\) lies in \(q_{pT}\). Since \(f = rg \in q_{pT}\), we have that \(g \notin q_{pT}\). Thus we can write \(f = pf'\) for \(f' \in R[x]\), and unique factorization gives that \(f'/g \in T\). Therefore \(r \in pT\). Intersecting both sides of \(pT \subseteq q_{pT}\) with \(S\) gives the opposite inequality, and so \(q_{pT} \cap S = pT \cap S\).

As noted above, there are various aspects of the construction that require us to adjoin transcendental elements to our ring \(R\). The following lemmas cover these scenarios.

Recall from Lemma 2.4 that in order for the constructed ring \(A\) to have completion \(T\), the map \(A \to T/m^2\) needs to be surjective. To guarantee surjectivity, we will use Lemma 3.7 to adjoin coset representatives of elements of \(T/m^2\). The proof of this lemma resembles the proof of Lemma 2.5 of [3]. We also need to ensure that \(aT \cap A = a\) for every finitely generated ideal \(a\) of \(A\); Lemmas 3.8 and 3.9 will help us obtain this property.

Lemma 3.7. Under Assumption 3.3, let \(t \in T\). Then there exists a \(J\)-extension \(S\) of \(R\) such that \(t + m^2\) is in the image of the map \(S \to T/m^2\).

Proof. For each \(q \in Q_R \cup \{\emptyset\}\), let \(D_q\) be a full set of representatives of the cosets \(x + q \in T/q\) such that \(t + x + q \in T/q\) is algebraic over \(R/q \cap R\); then, if \(R\) is infinite, \(|D_q| \leq |R| < |T/m|\). If \(R\) is finite, then it is clear that \(|D_q| < |T/m|\). Define \(D = \bigcup\{D_q \mid q \in Q_R \cup \{\emptyset\}\}\). As \(m^2 \notin q\) for any \(q \in Q_R \cup \{\emptyset\}\) and \(|D| < |T/m|\), we can choose an element \(x \in m^2\) using Lemma 2.2 such that \(x + t + q\) is transcendental over \(R/q \cap R\) for all \(q \in Q_R \cup \{\emptyset\}\). By Lemma 3.6 we can find a \(J\)-extension \(S\) of \(R\) containing \(t + x\). Then \(t + m^2\) is in the image of the map \(S \to T/m^2\).

Lemma 3.8. Under Assumption 3.3, let \(a\) be a finitely generated ideal of \(R\) and let \(c \in aT \cap R\). Then there exists a \(J\)-extension \(S\) of \(R\) such that \(c \in aS\).

Proof. Our proof is similar to that of Lemma 4 of [4], and we proceed by induction on the number of generators of \(a\). It will be clear from the construction that, if \(R\) is infinite, then \(|R| = |S|\). If \(a\) is the zero ideal, then \(S = R\) works. So, for the rest of the proof, assume that \(a\) is not the zero ideal. Note that, if \(n = 1\), then \(S = R\) works since \(R\) is a \(J\)-subring.

For the \(n = 2\) case, let \(a\) be generated by \(a_1\) and \(a_2\) and write \(c = a_1x_1 + a_2x_2\) for some \(x_1, x_2 \in T\). Suppose first that \(a_1, a_2 \in q_{pT}\) for some \(q_{pT} \in Q_R\) with \(p \in R\). Then \(a_1\) and \(a_2\) are both in \(q_{pT} \cap R = pR\), so by dividing out the highest power of \(p\) dividing both \(a_1\) and \(a_2\), we can replace these with \(a'_1\) and \(a'_2\) where
at least one of $a'_1$ and $a'_2$ is not in $q_{pT}$. To prove the lemma, it suffices to prove it with $c$ replaced by the element $c' = a'_1x_1 + a'_2x_2$, $a_1$ replaced by $a'_1$, and $a_2$ replaced by $a'_2$, but we now have that at least one of $a'_1$ and $a'_2$ is not contained in $q_{pT}$. By repeating this process if necessary we may assume without loss of generality that, if $q \in Q_R$, then $a_1$ and $a_2$ are not both in $q$. Note also that neither $a_1$ nor $a_2$ are in $\Psi$.

Let $x'_1 = x_1 + a_2y$ and $x'_2 = x_2 - a_1y$ for some $y \in T$ to be chosen soon using Lemma 2.2. For each $q \in Q_R \cup \{\Psi\}$ such that $a_2 \notin q$, let $D^1_q$ be a set of representatives for the cosets $y + q$ such that $x'_1 + q$ is algebraic over $R/q \cap R$. If $a_2 \in q$, let $D^1_q = \emptyset$. Define $D^2_q$ similarly with $a_2$ replaced by $a_1$ and $x'_1$ replaced by $x'_2$, and let $D = \bigcup \{D^1_q \cup D^2_q \mid q \in Q_R \cup \{\Psi\}\}$. Use Lemma 2.2 to choose an element $y \in T$ such that $y \notin \bigcup \{t + q \mid t \in D, q \in Q_R \cup \{\Psi\}\}$. Let $S = R[x'_1](0) \cap R[x'_2](0) \cap T$; we claim that $S$ is the desired $J$-extension. Note that (J.2) is satisfied, since the fact that $x'_1 + \Psi$ is transcendental over $R/\Psi \cap R$ implies that $\Psi \cap R[x'_1](0) = (0)$, as in the proof of Lemma 3.6 and so $\Psi \cap S = (0)$. Since $c \in aS$ and since $S$ satisfies (J.1) and since it is easy to verify, as in the proof of Lemma 3.6 that $S$ satisfies (J.3) it remains to extend the distinguished set of $R$ to $S$ and verify that the necessary properties hold. Let $q_{pT} \in Q_R$; at least one of $a_1$ or $a_2$ is not in $q_{pT}$, so without loss of generality we may assume $a_2 \notin q_{pT}$. Properties (J.4a) and (J.4b) still hold, and, as in the proof of Lemma 3.6 we can conclude that $q_{pT} \cap R[x'_1](0) = pT \cap R[x'_1](0)$. Intersecting this equation with $S$ gives $q_{pT} \cap S = pT \cap S$, as needed. Using the axiom of choice, find an arbitrary extension of the ordering function $\varphi_R$ to an ordering function $\varphi_S$. Finally, for elements $pT$ of $\mathcal{F}_S - T$, use Lemma 3.5 to choose $q_{pT} \in \text{Spec}_d T$ where $d = \Delta(\varphi_S(pT))$ so that these prime ideals satisfy the conditions listed in (J.4). Defining $Q_S$ to be the union of $Q_R$ with the set of these $q_{pT}$ completes the proof of the case $n = 2$.

For the general inductive step, let $a = (a_1, \ldots, a_n)$ be an $n$-generated ideal of $R$ and let $c = \sum_{i=1}^n a_ix_i \in aT \cap R$ for $x_i \in T$. As in the $n = 2$ case, we may assume without loss of generality that if $q \in Q_R$, then at least one of $a_1, a_2, \ldots, a_n$ is not contained in $q$. Let $b = (a_1, \ldots, a_{n-1})R$. The inductive step breaks into two cases, depending on whether or not there is a $q \in Q_R$ that contains all of $a_1, a_2, \ldots, a_{n-1}$.

Suppose first that there is no such $q$ in $Q_R$. Let $x'_n = x_n + a_1y_1 + \cdots + a_{n-1}y_{n-1}$ and $x'_i = x_i - a_ny_i$ $(i \neq n)$ for some $y_i \in T$ to be determined, so that $c = \sum_{i=1}^n a_ix'_i$. Using Lemma 2.2 choose $y_1$ so that $x_n + a_1y_1 + q$ is transcendental over $R/q \cap R$ for every $q \in Q_R \cup \{\Psi\}$ such that $a_1 \notin q$. Next, choose $y_2$ so that $x_n + a_1y_1 + a_2y_2 + q$ is transcendental over $R/q \cap R$ for every $q \in Q_R \cup \{\Psi\}$ such that $a_2 \notin q$. This choice will not affect the previous transcendence conditions: that is, if $x_n + a_1y_1 + q$ was transcendental over $R/q \cap R$ then $x_n + a_1y_1 + a_2y_2 + q$ will also be transcendental. Continuing to choose $y_i$’s in this way, the final element $x'_n$ will have the property that $x'_n + q$ is transcendental over $R/q \cap R$ for every $q \in Q_R \cup \{\Psi\}$ not containing each of $a_1, a_2, \ldots, a_{n-1}$. By our earlier assumption, there is no $q \in Q_R$ that contains all of these elements. We may therefore use Lemma 3.6 to procure a $J$-extension $R'$ of $R$ containing $x'_n$. Letting
Lemma 3.9. The next lemma allows us to find such subrings.

Suppose next that there is a \( q \in \mathcal{Q}_R \) such that \( a_1, a_2, \ldots, a_{n-1} \) are all contained in \( q \). This final step of the proof is the most elaborate, and will proceed in part by reduction to previous cases. The idea of the proof that follows is to replace \( b \) with an ideal \( b' \) whose generators share no common factor in \( R \).

Factoring out common divisors of the \( a_i \)'s as in the \( n = 2 \) case, we can find an element \( r \in R \) with \( a_i = ra_i' \) for each \( i < n \) and such that, for every \( q \in \mathcal{Q}_R \), at least one of \( a_1', a_2', \ldots, a_{n-1}' \) is not in \( q \). Let \( b' = (a_1', \ldots, a_{n-1}') \). Write \( c = cx + a_n x_n \), where \( x = a_1' x_1 + \cdots + a_{n-1}' x_{n-1} \). We can now apply the construction of the \( n = 2 \) case to the ideal \( (r, a_n) \) to find a \( J \)-extension \( R' \) of \( R \) containing elements \( x' \) and \( x'_{n-1} \) such that \( c = rx' + a_n x'_{n-1} \). To finish, we'll find a sufficient condition on a ring \( S \) containing \( R' \) so that \( c \in \mathfrak{a} S \), then construct \( S \) to satisfy the condition. In general, to ensure that \( c \in \mathfrak{a} S \) it is enough that \( rx' \in \mathfrak{a} S \), and to ensure that \( rx' \in \mathfrak{a} S \) it is enough that \( x' \in \mathfrak{b}' S + a_n S \). By the method of construction in the \( n = 2 \) case, the element \( x' \) has the form \( x' = x + t a_n \) for some \( t \in T \); hence \( x' \in (\mathfrak{b}' T + a_n T) \cap R' \). In this way the problem is reduced to constructing a \( J \)-extension \( S \) of \( R' \) such that \( x' \in (\mathfrak{b}' S + a_n S) \).

If, for every \( q \in \mathcal{Q}_R \), at least one of \( a_1', a_2', \ldots, a_{n-1}' \) is not contained in \( q \), then we can finish by reducing to the previous case. However, even though for every \( q \in \mathcal{Q}_R \), at least one of \( a_1', a_2', \ldots, a_{n-1}' \) is not contained in \( q \), it might happen that there is a \( q \in \mathcal{Q}_R \) containing all of \( a_1', a_2', \ldots, a_{n-1}' \), since \( R' \) might contain primes of \( T \) that divide elements of \( R \) but that were not present in \( R \). If so, repeat the argument of the previous paragraph to modify the generators of \( b' \). We know this process must end since the generators of \( b' \) have only finitely many prime factors in \( T \). When the process ends, we may reduce to the case where the generators of \( b' \) are not all contained in an element of \( \mathcal{Q}_R \).

At the end of our construction, if \( p T \in \mathcal{F}_T \) is not in the generic formal fiber of \( A \), then it must be contained in the formal fiber of some height one prime ideal of \( A \). In order for the formal fiber ring of this prime ideal to be regular it must be reduced, and so we would like \( p T \) to be the unique minimal element of that formal fiber. We can achieve this if \( A \) has the property that if \( p T \in \mathcal{F}_T \) has nonzero intersection with \( A \), then \( p T \in \mathcal{F}_A \). We therefore introduce the following property for a subring \( R \) of \( T \).

\((J.5)\) if \( p T \in \mathcal{F}_T \) has nonzero intersection with \( R \), then \( p T \in \mathcal{F}_R \).

Thus, as we continue our construction, we are particularly interested in \( J \)-subrings of \( T \) satisfying \((J.5)\). The next lemma allows us to find such subrings.

**Lemma 3.9.** Under Assumption \((3.4)\), there exists a \( J \)-extension \( S \) of \( R \) such that, for every finitely generated ideal \( \mathfrak{a} \) of \( S \), \( \mathfrak{a} T \cap S = \mathfrak{a} \) and \( S \) satisfies \((J.5)\).
Prove. First suppose that $R$ is finite. Let $C = \mathcal{Q}_R \cup \{\emptyset\}$, and for each $q \in C$, let $D_q$ be a full set of coset representatives of the cosets $x + q \in T/q$ that are algebraic over $R/R \cap q$. Now define $D = \cup \{D_q \mid q \in C\}$ and by previous lemmas, find an element $x \in T$ such that $x + q$ is transcendental over $R/R \cap q$ for all $q \in C$. By Lemma 3.3, there is a $J$-extension $R_0$ of $R$ that contains $x$. Note that $R_0$ is infinite. We will construct $S$ to be a $J$-extension of $R_0$, and so it will also be a $J$-extension of $R$. If $R$ is infinite, then let $R_0 = R$.

Now we inductively construct a countable ascending chain $R_0 \subseteq R_1 \subseteq \cdots$ of $J$-extensions and take $S = \bigcup_{i=1}^{\infty} R_i$. The rings will be constructed so that, if $a$ is a finitely generated ideal of $R_n$, then $aT \cap R_n \subseteq aR_{n+1}$ and $\{pT \in \text{Spec}_n T \mid pT \cap R_n \neq (0)\} \subseteq \mathcal{F}_{R_{n+1}}$ for each $n$. We may then take $S = \bigcup_{i=1}^{\infty} R_i$. By Lemma 3.3, the ring $S$ is a $J$-extension of $R$, and since $\mathcal{F}_S = \bigcup_{i=1}^{\infty} \mathcal{F}_{R_i}$, the ring $S$ satisfies (0.5). If $a$ is a finitely generated ideal of $S$ and $c \in aT \cap S$, then there is an $n$ such that the generators of $a$ are in $R_n$ and $c \in aT \cap R_n$, and hence $c \in aR_{n+1} \subseteq aS = a$.

It remains to construct the countable ascending chain. Having already defined $R_0$, suppose inductively that the ring $R_n$ has been constructed. To construct $R_{n+1}$, we proceed in two steps. First we construct $R_{n+0.5}$ by constructing a second ascending chain of rings as follows. Let $\Omega$ denote the set of tuples $(a, c)$, where $a$ is a finitely generated ideal of $R_n$ and $c \in aT \cap R_n$. We have that $|\Omega| = |R_n|$ because $R_n$ is infinite and so it has the same cardinality as the set of its finite subsets. Well order $\Omega$ so that it does not have a maximal element, write $\lambda = |\Omega|$, and let $(a_\beta, c_\beta)$ denote the element of $\Omega$ corresponding to the ordinal $\beta \in \lambda$.

We construct a chain $(R^\beta)_{\beta \in \lambda}$, and we let $R^0 = R_n$ to start. If $\beta = \gamma + 1$ is a successor ordinal, let $R^\beta$ be a $J$-extension of $R^\gamma$ such that $c_\gamma \in a_\gamma R^\gamma$, obtained by Lemma 3.3. If $\beta$ is a limit ordinal, let $R^\beta = \bigcup_{\gamma < \beta} R^\gamma$ and define $\mathcal{Q}_{R^\beta} = \bigcup_{\gamma < \beta} \mathcal{Q}_{R^\gamma}$. We claim that the chain $(R^\beta)_{\beta \in \lambda}$ satisfies the hypotheses of Lemma 3.3. To show this, we show that $R^\beta$ satisfies the conditions of Lemma 3.3 for all $\beta \in \lambda$. Note that $R^0 = R_n$ is a $J$-subring, and so it satisfies the conditions of Lemma 3.3. Now suppose that $R^\beta$ satisfies the conditions of Lemma 3.3 for all $\gamma < \beta$. If $\beta$ is a successor ordinal, then $R^\beta$ is a $J$-extension of $R^\gamma$ by definition. If $\beta$ is a limit ordinal, then Lemma 3.3 gives us that $R^\beta$ is a $J$-subring and in the proof of Lemma 3.3 we see that the distinguished set for $R^\beta$ is $\mathcal{Q}_{R^\beta} = \bigcup_{\gamma < \beta} \mathcal{Q}_{R^\gamma}$. We also see in the proof that the ordering function for $R^\beta$ satisfies $\varphi_{R^\beta}(pT) = \varphi_{R^\gamma}(pT)$ for every $\gamma < \beta$ and every $pT \in \mathcal{F}_{R^\gamma}$. It follows that $R^\beta$ satisfies the conditions of Lemma 3.3. Finally, let $R_{n+0.5} = \bigcup_{\beta \in \lambda} R^\beta$. By Lemma 3.3, $R_{n+0.5}$ is a $J$-extension of $R_n$. It is clear by construction that if $a$ is a finitely generated ideal of $R_n$ then $aT \cap R_n \subseteq aR_{n+0.5}$.

Next we proceed from $R_{n+0.5}$ to $R_{n+1}$ in a similar fashion, constructing another ascending chain of $J$-subrings of $T$. Let

$$\mathcal{G}_{R_n} = \{pT \in \text{Spec}_n T \mid pT \cap R_n \neq (0)\}$$
and let $\mu = |G_{R_n}|$. We will construct a chain of $J$-subrings $(R^\beta)_{\beta \in \mu}$ so that they all contain $R_n$. It follows that if $R^\gamma$ is an element of our chain, and $pT \in G_{R_n}$, then $pT \cap R^\gamma \neq (0)$. Well-order the set $G_{R_n}$ so that it does not have a maximal element and let $p_\beta$ denote a generator of an element of $G_{R_n}$ corresponding to the ordinal $\beta$. To start, set $R^0 = R_{n+0.5}$. If $\beta = \gamma + 1$ is a successor ordinal, and $p_\gamma T \in F_{R^\gamma}$, then define $R^\delta$ to be $R^\gamma$. If $p_\gamma T \notin F_{R^\gamma}$, and $p_\gamma \in q_{pT}$ for some $q_{pT} \in Q_{R^\gamma}$, then let $r \in p_\gamma T \cap R^\gamma$. So $r = p_\gamma t$ for some $t \in T$ and $r = p_\gamma t \in R^\gamma \cap q_{pT} = pR^\gamma$, and we have $r = p_\gamma t = pr_1$ for some $r_1 \in R^\gamma$. Since $p_\gamma T \notin F_{R^\gamma}$, $p_\gamma T \neq pT$. It follows that $p$ divides $t$. Dividing out by $p$, we get $p_\gamma t' = r_1 \in q_{pT} \cap R^\gamma = pR^\gamma$ for some $t' \in T$. Hence $p$ divides $t'$. Continue in this way to show that $r \in p^{i}T$ for all $i = 1, 2, \ldots$. Since $\bigcap_{i=1}^{\infty} p^{i}T = (0)$ we have that $r = 0$, contradicting that $p_\gamma T \cap R_n \neq (0)$. So if $q \in Q_{R^\gamma}$ then $p_\gamma \notin q$. Now use Lemma 2.3 to find a unit $u \in T$ such that $up_\gamma + q$ is transcendental over $R^\gamma/q \cap R^\gamma$ for every $q \in Q_{R^\gamma} \cup \{\mathfrak{p}\}$; then use Lemma 3.6 to construct $R^\delta$ to be a $J$-extension of $R^\gamma$ containing $up_\gamma$. If $\beta$ is a limit ordinal, let $R^\delta = \bigcup_{\gamma < \beta} R^\gamma$. Finally, let $R_n+1 = \bigcup_{\beta \in \mu} R^\delta$. Following the proof from the previous paragraph, we can conclude that $R_n+1$ is a $J$-extension of $R_{n+0.5}$, and so $R_n+1$ is a $J$-extension of $R_n$. It is clear from the construction that $G_{R_n} \subseteq F_{R_n+1}$. Furthermore, we note that if $a$ is a finitely generated ideal of $R_n$, $aT \cap R_n \subseteq aR_{n+0.5} \subseteq aR_{n+1}$. \hfill $\square$

Recall that the ring $A$ we construct will have the property that the formal fiber ring of $p \in \text{Spec} A$ is a field if $ht \ p > 1$. To achieve this, we would like $pT$ to be the only element in the formal fiber of $A$ at $p$; applying Lemma 3.11 to certain prime ideals of $T$ will allow us to accomplish this. The same lemma also enables us to add nonzero elements from prime ideals of $T$ with height greater than that of $\mathfrak{p}$ which will ensure that $\alpha(A, (0))$ is no greater than $ht \mathfrak{p}$.

The following lemma is used in the proof of 3.11 and comes from the proof of Lemma 3 in \[4\].

**Lemma 3.10.** Let $k$ be an infinite field, let $V$ be a finite-dimensional $k$-vector space, and let $\{V_\beta\}_{\beta \in A}$ be a collection of $k$-vector spaces indexed by a set $A$ with $|A| < |k|$. If for all $\beta \in A$ we have $V \not\subseteq V_\beta$, then $V \not\subseteq \bigcup_{\beta \in A} V_\beta$.

**Lemma 3.11.** Under Assumption 3.4 suppose that $T$ contains a field $k$ with $|k| = |T|$. Let $a$ be an ideal of $T$ such that $a \not\subseteq \mathfrak{p}$ and such that, if $q \in Q_R$, then $a \not\subseteq q$. Then there is a $J$-extension $S$ of $R$ such that either $a \subseteq p$ for some $p \in Q_S$ or $(a \cap S)T = a$.

**Proof.** Let $a_1, \ldots, a_n$ be a minimal set of generators for $a$, and let $V$ denote the $k$-vector space generated by the $a_i$. We will inductively construct a basis $\{b_1, \ldots, b_n\}$ for $V$ and a chain of $J$-extensions $R = R_0 \subset R_1 \subset \cdots \subset R_n$. If, for some $i$, we have $a_i \subseteq p$ for some $p \in Q_{R_i}$, we stop the construction of the basis and declare that $S = R_i$. If, on the other hand, this does not happen, then the $R_i$ will be chosen so that, for every $i$, there is a unit $u_i \in T$ such that $u_ib_i \in R_i$. It then follows that for $S = R_n$, we have $(a \cap S)T = a$. 


Suppose now that \( R_i \) has been constructed and assume that, if \( q \in Q_{R_i} \), then \( a \not\subseteq q \). To construct \( R_{i+1} \), note that each element of \( Q_{R_i} \cup \{ p \} \) is a \( k \)-vector space and apply Lemma 3.10 to the set \( Q_{R_i} \cup \{ p \} \cup \{ W_i \} \), where \( W_i \) is the proper subspace of \( V \) spanned by \( b_1, \ldots, b_i \). In this way we obtain an element \( b_{i+1} \in V \) that is not contained in any element of \( Q_{R_i} \cup \{ p \} \) and that is not in the span of \( b_1, \ldots, b_i \).

Next, we'll use Lemma 2.3 to choose \( u_{i+1} \). For each \( q \in Q_{R_i} \cup \{ p \} \), let \( D_q \) be a set of representatives for the cosets of \( q \) that are algebraic over \( R_i/q \cap R_i \). Let \( D = \bigcup \{ D_q \mid q \in Q_{R_i} \cup \{ p \} \} \), and choose \( u_{i+1} \) so that

\[
u_{i+1} b_{i+1} \notin \bigcup \{ t + q \mid t \in D, q \in Q_{R_i} \cup \{ p \} \}.
\]

Finally, let \( R_{i+1} \) be a \( J \)-extension of \( R_i \) containing \( u_{i+1} b_{i+1} \), obtained by Lemma 3.6. If \( a \subseteq p \) for some \( p \in Q_{R_{i+1}} \), then let \( S = R_{i+1} \). Otherwise, continue to construct \( R_{i+2} \). It follows that either \( a \subseteq p \) for some \( p \in Q_S \) or \( (a \cap S)T = a \). \( \square \)

**Lemma 3.12.** Under Assumption 3.4, let \( t \in T \), let \( a \) be an ideal of \( T \) not contained in \( p \) or in any element of \( Q_R \), and suppose that \( T \) contains a field \( k \) with \( |k| = |T| \). Then there exists a \( J \)-extension \( S \) of \( R \) such that

1. \( t + m^2 \) is in the image of the natural map \( S \rightarrow T/m^2 \),
2. Either \( a \subseteq p \) for some \( p \in Q_S \) or \( (a \cap S)T = a \),
3. for every finitely generated ideal \( b \) of \( S \) we have that \( bT \cap S = b \), and
4. \( S \) satisfies (J.5)

**Proof.** Use Lemma 3.11 to find a \( J \)-extension \( R' \) of \( R \) such that either \( a \subseteq p \) for some \( p \in Q_{R'} \) or \( (a \cap R')T = a \). Then use Lemma 3.7 to find a \( J \)-extension \( R'' \) of \( R' \) such that \( t + m^2 \) is in the image of the map \( R'' \rightarrow T/m^2 \). Finally, use Lemma 3.9 to find a \( J \)-extension \( S \) of \( R'' \) satisfying (J.5) such that if \( b \) is a finitely generated ideal of \( S \) then \( bT \cap S = bS \). \( \square \)

4. **The Main Theorem**

**Theorem 4.1.** Let \( T \) be a complete equicharacteristic local UFD with \( 3 \leq \dim T \) and \( |T| = |T/m| \). Let \( \mathfrak{P} \) be a nonmaximal prime ideal of \( T \) and let \( \{ \lambda_d \mid 0 \leq d \leq \min(\text{ht} \mathfrak{P}, \dim T - 2) \} \) be a set of cardinal numbers such that \( \sum \lambda_d = |T| \).

Then there exists a local UFD \( A \) such that \( \hat{A} = T \) and the following are satisfied:

1. \( \mathfrak{P} \cap A = (0) \) and \( \alpha(A, (0)) = \text{ht} \mathfrak{P} \),
2. for each \( d \) the set \( \Delta_d = \{ pA \in \text{Spec}_1 A \mid \alpha(A, pA) = d \} \) has cardinality \( |\Delta_d| = \lambda_d \),
3. the ring \( T \otimes_A \kappa(pA) \) is regular local for every \( pA \in \text{Spec}_1 A \), and
4. the ring \( T \otimes_A \kappa(p) \) is a field for every \( p \in \text{Spec} A \) with \( \text{ht} p \geq 2 \).
Proof. We will use transfinite recursion to construct $A$ as the union of an ascending chain $(R_\beta)_{\beta<|T|}$ of extensions of the prime subring of $T$. To make the construction easier to understand, we begin with an overview.

The main task is to ensure that $A$ is Noetherian and has the correct completion, using Lemma 2.1. To satisfy the first hypothesis of the lemma, that the map $A \to T/m^2$ is onto, we form $R_{\beta+1}$ from $R_\beta$ by adjoining a coset representative for an element of $T/m^2$, eventually adjoining a representative for every coset. To satisfy the second hypothesis of the lemma, that finitely generated ideals are unchanged by extension to $T$ and contraction to the original ring, we construct $R_{\beta+1}$ so that it satisfies this property. Ensuring that $A$ is a UFD requires no work since in general any local domain whose completion is a UFD is itself a UFD [2, VII.3.6, Proposition 4].

Four attributes of the chain $(R_\beta)_{\beta<|T|}$ govern the behavior of the formal fibers of $A$. First, the intermediate extensions $R_\beta \subseteq R_{\beta+1}$ are $J$-extensions. The ring $A$ consequently satisfies every defining condition of a $J$-subring except $[J.1]$ and therefore $\mathcal{P} \cap A = (0)$. Second, each $R_\beta$ satisfies $[J.5]$. The ring $A$ consequently satisfies $[J.5]$ which will help us show that $A$ satisfies property [3]. Third, we control the heights of new elements of the distinguished sets $\mathcal{Q}_{R_\beta}$ for each extension. The ring $A$ consequently satisfies [4]. Fourth, $R_{\beta+1}$ is formed from $R_\beta$ by adjoining generators of a given prime ideal $q_\beta$ of $T$, eventually adjoining generators for every prime ideal not contained in an element of some distinguished set or in $\mathcal{P}$. The ring $A$ consequently satisfies [4] and $\alpha(A, (0)) = \text{ht} \mathcal{P}$. This concludes the overview of the construction.

Before constructing the chain $(R_\beta)_{\beta<|T|}$ we require several set-theoretic bookkeeping preliminaries. The first of these is to choose coset representatives and generators of prime ideals to be added at each extension. The second is to explain how the heights of the elements of the distinguished set can be chosen in the correct proportions. This second preliminary will be used to ensure that $|\Delta_d| = \lambda_d$ for every $d$, since the heights effect the dimensions of the formal fibers. Specifically, the height of an element of the distinguished set is one more than the dimension of the corresponding formal fiber.

First, let $C_1$ be a set of coset representatives for $T/m^2$ and let $C_2 = \bigcup \{\text{Spec}_i T \mid 1 \leq i \leq \dim T - 1\}$. Now use Lemma 2.2 in the same way as we did in the proof of Lemma 3.5 to show that the number of height one prime ideals of $T$ is at least $|T/m|$. Since $|T| = |T/m|$, it follows that $C_1$ and $C_2$ both have cardinality $|T|$. We may therefore well-order $C_1$ and $C_2$ to be order isomorphic to $|T|$. We well-order these sets so that they do not have a maximal element. Let $t_\beta$ and $q_\beta$ denote the elements of $C_1$ and $C_2$, respectively, that correspond to $\beta$.

Controlling the heights is only slightly more complicated. The problem is to construct a height indicator $\Lambda$ that assumes the value $d+1$ exactly $\lambda_d$ times. This is basic set theory: the equation $|T| = \sum_d \lambda_d$ and the
definition of ordinal addition mean that the set $|T|$ can be partitioned into \( \min(\text{ht } \mathfrak{P} + 1, \dim T - 1) \) subsets, the \((d + 1)\text{th}\) of which has cardinality \( \lambda_d \).

Let us now construct the chain \((R_\beta)_{\beta < |T|}\) and the union \(A\) of its elements. To start, let \(R_0\) be the prime subring of \(T\), which is necessarily a field because \(T\) is equicharacteristic. Hence the set \(\mathcal{F}_{R_0}\) is empty and \(R_0\) has a \(J\)-subring structure with height indicator \(\Lambda\) and \(Q_{R_0} = \varphi_{R_0} = \emptyset\).

The nontrivial step of the recursion is the case where \(\beta = \gamma + 1\) is a successor ordinal. In this case, if \(q_\gamma\) is contained in \(\mathfrak{P}\) or in some element of \(Q_{R_\gamma}\), then define \(R_\beta\) in the following way. First let \(R'\) be the \(J\)-extension obtained from Lemma \ref{lem:extension} so that \(t_\gamma + m^2\) is in the image of the map \(R' \to T/m^2\). Then let \(R_\beta\) be the \(J\)-extension of \(R'\) obtained from Lemma \ref{lem:extension} so that \(R_\beta\) satisfies \((J.5)\) and so that for every finitely generated ideal \(a\) of \(R_\beta\) we have that \(aT \cap R_\beta = a\). If \(q_\gamma\) is not contained in \(\mathfrak{P}\) or in some element of \(Q_{R_\gamma}\), then use Lemma \ref{lem:construction} to construct a \(J\)-extension \(R_\gamma\) of \(R_\gamma\) satisfying \((J.5)\) such that \(t_\gamma + m^2\) is in the image of the map \(R_\beta \to T/m^2\), for every finitely generated ideal \(a\) of \(R_\beta\) we have that \(aT \cap R_\beta = a\), and either \((q_\gamma \cap R_\beta)T = q_\gamma\) or \(q_\gamma \subseteq q\) for some \(q \in Q_{R_\beta}\). Invoking Lemma \ref{lem:construction} requires that \(T\) contains a field of cardinality \(|T|\), but since \(T\) is equicharacteristic it contains a coefficient field isomorphic to \(T/m\), which has the same cardinality as \(T\).

If \(\beta\) is a limit ordinal, then take \(R_\beta = \bigcup_{\gamma < \beta} R_\gamma\). Finally, let \(A = \bigcup_{\beta < |T|} R_\beta\). The construction is thereby concluded.

We now catalog the properties of \(A\). It is clear that \(A\) satisfies \((J.5)\). Following the proof of Lemma \ref{lem:extension} the system \((R_\beta)_{\beta < |T|}\) satisfies the hypotheses of Lemma \ref{lem:construction}. Therefore \(A\) satisfies \((J.2)\), \((J.3)\) and \((J.4)\) with \(\mathcal{F}_A = \bigcup_{\beta < |T|} \mathcal{F}_{R_\beta}\) and \(Q_A = \bigcup_{\beta < |T|} Q_{R_\beta}\).

The fact that \(\widehat{A} = T\) is a consequence of Lemma \ref{lem:extension}. Specifically, if \(a = (a_1, \ldots, a_n)\) is a finitely generated ideal of \(A\) and \(c \in aT \cap A\), then there is some \(\beta < |T|\) where \(\beta\) is a successor ordinal and such that \(a_1, \ldots, a_n, c \in R_\beta\). Then \(c \in (a_1, \ldots, a_n)T \cap R_\beta = (a_1, \ldots, a_n)R_\beta \subseteq a\), and so we have that \(aT \cap A = a\). The map \(A \to T/m^2\) is surjective by construction, and \(A\) is quasi-local with maximal ideal \(m \cap A\) since each \(R_\beta\) is quasi-local with maximal ideal \(R_\beta \cap m\). Using Lemma \ref{lem:extension} we see that \(A\) is Noetherian and \(\widehat{A} = T\), and furthermore \(A\) is a UFD because \(T\) is.

We will now show that \(A\) satisfies the remaining four conditions.

The ring \(A\) inherits an ordering function \(\varphi_A\) in the obvious way, as in the proof of Lemma \ref{lem:construction}. Since we well-ordered our sets so that there was no maximal element, \(|\mathcal{F}_{R_\beta}| < |T|\). It follows that, for every \(\beta < |T|\), the map \(\varphi_{R_\beta}\) is not surjective. Furthermore, \(|A| = |T|\) because \(|T| = |T/m| = |T/m^2|\) and the map \(A \to T/m^2\) is surjective. An application of Lemma \ref{lem:construction} then shows that \(|\mathcal{F}_A| = |\text{Spec}_A| = |T|\). The map \(\varphi_A\) effects a bijection between \(\mathcal{F}_A\) and an initial segment of \(|T|\), and by what we just observed, this initial segment has cardinality \(|T|\). Hence the initial segment equals \(|T|\); this is the minimality property that characterizes
cardinal numbers among the ordinal numbers. Therefore, for every $\beta < |T|$, there is exactly one $pT \in F_A$ with $\varphi_A(pT) = \beta$ and $ht(q_{pT}) = \Lambda(\beta)$. So $|\Delta_d| = \lambda_d$ for each $d$.

Recall that, for each $p \in \text{Spec } A$, the elements of the formal fiber of $p$ are in one-to-one correspondence with the prime ideals of $T \otimes_A \kappa(p)$. Let $p$ be a prime ideal of $A$; since $T$ is faithfully flat over $A$, some $q \in \text{Spec } T$ lies over $p$ and $ht_q \geq ht_p$. First suppose that $ht_p = 1$, and so $p = pA$ for some nonzero prime element of $A$. It follows from conditions (J.4) and (J.5) that $q_{pT}$ is the only element of $Q_A$ contained in the formal fiber of $pA$. Furthermore, if $q \in \text{Spec } T$ is not contained in $q_{pT}$, then $q$ cannot be contained in the formal fiber of $pA$. To see this, first suppose $q \subseteq q_{pT}$ for some $q_{pT} \in Q_A$ with $q_{pT} \neq q_{pT'}$. Then $q \cap A \subseteq q_{pT} \cap A = p'A \neq pA$. If $q \subseteq \mathfrak{P}$, then $q \cap A = (0) \neq pA$. If, on the other hand, $q \not\subseteq \mathfrak{P}$ and no element of $Q_A$ contains $q$, then by our construction we have that $q = (q \cap A)T \neq pT$ and so $q \cap A \neq pA$.

Thus we have shown that the formal fiber ring $T \otimes_A \kappa(pA)$ is local with maximal ideal corresponding to $q_{pT}$. It follows that the ring $T \otimes_A \kappa(pA)$ is isomorphic to $T/pT$ localized at the image of the ideal $q_{pT}$ in the ring $T/pT$. Since we chose $q_{pT}$ so that its image in $T/pT$ is in the regular locus of $T/pT$, it follows that $T \otimes_A \kappa(pA)$ is a regular local ring. Every prime ideal of $T$ containing $(pA)T = pT$ and contained in $q_{pT}$ is the formal fiber of $pA$, so $\alpha(A, pA) = \dim(T \otimes_A \kappa(pA)) = ht_{q_{pT}} - ht(pT)$. By construction, there are $\lambda_d$ elements of $Q_A$ that have height $d + 1$. Note that, if $ht_{q_{pT}} = d + 1$, then $\alpha(A, pA) = (d + 1) - 1 = d$. It follows that there are $\lambda_d$ elements $pA$ of $Q_A$ satisfying $\alpha(A, pA) = d$.

Now let $ht_p \geq 2$. If $q \in \text{Spec } T$ lies over $p$, then $q$ is not contained in any element of $Q_A$, and $q$ is not contained in $\mathfrak{P}$, so $q = (q \cap A)T = pT$. Thus, $pT$ is the only prime ideal contained in the formal fiber of $p$, and so $T \otimes_A \kappa(p)$ is a zero-dimensional domain and hence a field.

Finally, we consider $p = (0)$. By construction, $A \cap \mathfrak{P} = (0)$ and so $\alpha(A, (0)) \geq ht \mathfrak{P}$. Recall that, for $q_{pA} \in Q_A$, we have $ht_{q_{pA}} \leq ht \mathfrak{P} + 1$. So if $q$ is a prime ideal of $T$ with height greater than that of $ht \mathfrak{P}$, then $q \not\subseteq \mathfrak{P}$ and either $q \in Q_A$ or $q$ is not contained in any distinguished prime ideal of $A$. It follows by our construction that $q \cap A \neq (0)$ and so $\alpha(A, (0)) = ht \mathfrak{P}$.

Remark. The formal fibers of the ring $A$ in Theorem 4.1 are completely understood. The generic formal fiber of the ring $A$ constructed in Theorem 4.1 consists of the prime ideals of $T$ contained in $\mathfrak{P}$ and the prime ideals contained in $q_{pT} \in Q_A$ that do not contain $p$. If $pA \in \text{Spec } A$, then the formal fiber ring of $pA$ is local with maximal ideal $q_{pT}$. In particular, the formal fiber of $pA$ is all prime ideals contained in $q_{pT}$ that contain $p$. Finally, if $p$ is a prime ideal of $A$ with height greater than one, then the formal fiber of $A$ at $p$ is $\{pT\}$.

Example 4.2. If $T = \mathbb{C}[w, x, y, z]/(x^2 + y^3 + z^5)$, then there exists a local UFD $A$ such that $\hat{A} = T$, $\alpha(A, (0)) = 1$, and $\alpha(A, pA) = 1$ for every $pA \in \text{Spec } A$. 

5. The Excellent Case

In this section we strengthen the results of the previous one to obtain excellent rings. Recall that by the Cohen Structure Theorem, a complete equicharacteristic local ring is regular if and only if it is a power series ring in finitely many variables with coefficients in a field.

**Theorem 5.1.** Let $T$ be a complete equicharacteristic regular local ring of characteristic zero with $3 \leq \dim T$ and $|T| = |T/m|$. Let $\mathfrak{P}$ be a nonmaximal prime ideal of $T$, and let $\{\lambda_d \mid 0 \leq d \leq \min(\text{ht } \mathfrak{P}, \dim T - 2)\}$ be a set of cardinal numbers such that $\sum_d \lambda_d = |T|$.

Then there exists an excellent regular local ring $A$ such that $\hat{A} = T$ and the following are satisfied:

1. $\mathfrak{P} \cap A = (0)$ and $\alpha(A, (0)) = \text{ht } \mathfrak{P}$,
2. for each $d$ the set $\Delta_d = \{pA \in \text{Spec}_1 A \mid \alpha(A, pA) = d\}$ has cardinality $|\Delta_d| = \lambda_d$,
3. $T \otimes_A \kappa(pA)$ is a regular local ring for every $pA \in \text{Spec}_1 A$, and
4. $T \otimes_A \kappa(p)$ is a field for every $p \in \text{Spec } A$ with $\text{ht } p \geq 2$.

**Proof.** First construct a subring $A$ of $T$ as in Theorem 4.1. Since $T$ is a regular local ring and $\hat{A} = T$, $A$ is a regular local ring. It remains to show is that $A$ is excellent.

We first show that the formal fibers of $A$ are geometrically regular. Let $p \in \text{Spec } A$; then we must show $T \otimes_A L$ is a regular ring for every finite field extension $L$ of $\kappa(p)$. It is enough to consider purely inseparable extensions (see Remark 1.3 of [9]), and so, since $A$ contains the rationals, $\kappa(p)$ has characteristic 0 and we need only show that $T \otimes_A \kappa(p)$ is regular. If $p$ is a nonzero prime ideal of $A$, then $T \otimes_A \kappa(p)$ is a regular local ring or a field, and so we now consider $T \otimes_A \kappa((0))$.

Let $q \otimes_A \kappa((0))$ be in the formal fiber of $(0)$. The localization of $T \otimes_A \kappa((0))$ at $q \otimes_A \kappa((0))$ is isomorphic to $T_q$, which is regular since $T$ is a regular local ring. Then the localization of $T \otimes_A \kappa((0))$ at each of its prime ideals is a regular local ring, and so $T \otimes_A \kappa((0))$ is regular. $A$ is formally equidimensional since $T$ is a domain, and so $A$ is universally catenary [7, Theorem 31.6] and hence is our desired excellent ring. \(\square\)

Using Theorem 5.1 we obtain the following example, which answers the original question of Heinzer, Rotthaus, and Sally. Since every uncountable cardinal number is realized as the cardinality of a field, it follows that the set $\Delta$ in their question can be made to have any uncountable cardinality.

**Example 5.2.** Let $k$ be an uncountable field of characteristic zero and let $T = k[[x_1, x_2, \ldots, x_n]]$ with $n = \dim T \geq 3$. Let $d$ be an integer with $0 \leq d \leq n - 2$. Then there exists an excellent regular local ring $A$ such that $\hat{A} = T$ and the set $\Delta = \{pA \in \text{Spec}_1 A \mid \alpha(A, (0)) = \alpha(A, pA) = d\}$ is equal to the set of all height one prime ideals of $A$. In particular, because the ring $A$ we constructed is such that $|A| = |T| = |\text{Spec}_1 A|$, we have that $|\Delta| = |T|$.
Finally, we provide an example illustrating that we can construct an excellent regular local ring such that the number of height one prime ideals whose formal fibers are of a specific dimension is prescribed.

**Example 5.3.** Let $T = \mathbb{C}[[x_1, x_2, x_3, x_4, x_5]]$. Then there exists an excellent regular local ring $A$ such that $\hat{A} = T$ and such that $\alpha(A, (0)) = 4$, $|\Delta_0| = |T|$, $|\Delta_1| = 1$, $|\Delta_2| = \aleph_0$, and $|\Delta_3| = |T|$ where $\Delta_d = \{pA \in \text{Spec}_1 A \mid \alpha(A, pA) = d\}$ for $0 \leq d \leq 3$.

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