The Hamiltonian Dynamics of Bounded Spacetime and Black Hole Entropy: The Canonical Method

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ABSTRACT

From first principles, I present a concrete realization of Carlip’s idea on the black hole entropy from the conformal field theory on the horizon in any dimension. New formulation is free of inconsistencies encountered in Carlip’s. By considering a correct gravity action, whose variational principle is well defined at the horizon, I derive a correct classical Virasoro generator for the surface deformations at the horizon through the canonical method. The existence of the classical Virasoro algebra is crucial in obtaining an operator Virasoro algebra, through canonical quantization, which produce the right central charge and conformal weight \( \sim A_+ / \hbar G \) for the semiclassical black hole entropy. The coefficient of proportionality depends on the choice of ground state, which has to be put in by hand to obtain the correct numerical factor 1/4 of the Bekenstein-Hawking (BH) entropy. The appropriate ground state is different for the rotating and the non-rotating black holes but otherwise it has a universality for a wide variety of black holes. As a byproduct of my results, I am led to conjecture that non-commutativity of taking the limit to go to the horizon and computing variation is proportional to the Hamiltonian and momentum constraints. It is shown that almost all the known uncharged black hole solutions satisfy the conditions for the universal entropy formula.

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1 Introduction

In the recent six years, there have been several outstanding approaches toward a statistical mechanical computation of the Bekenstein-Hawking (BH) entropy [1, 2, 3, 4]. (See also Ref. [5] for an earlier work.) But there is no complete and consistent understanding about the statistical origin of black hole entropy so far. Each approach assumes a specific model in a certain regime but a universal mechanism, which can be applied to any kind of black holes, had been unclear until the recent seminal work of Carlip [6]. According to Carlip, the symmetries of a “horizon”, which is treated as a boundary, is a universal mechanism for black hole entropy. This generalizes Strominger’s approach [4], in which the statistical entropy of the BTZ black hole [7] is computed from the classical Virasoro algebra at the boundary at “infinity” [8, 9]. By looking at surface deformations [10, 8, 11] of the “$r-t$” plane that leave the horizon fixed, he has shown that the symmetry algebra contains a classical Virasoro algebra independently of spatial dimensions when appropriate boundary conditions are chosen. With the aid of Cardy’s formula for the asymptotic states, the BH entropy was derived as the leading term of a steepest descent approximation. The relevance of the $r-t$ plane at the horizon to black hole entropy, which resembles the Euclidean gravity formulation in a radial slicing [12], was a key observation: This made it possible to elevate Strominger’s idea to an arbitrary black hole in higher dimensions without requiring any microscopic model for quantum states.

There are two big differences between Carlip’s approach and Strominger’s, apart from the difference in spacetime dimension. The first is that Carlip’s Virasoro algebra is computed at the horizon, in contrast to Strominger’s, which is computed at the boundary at infinity where there is no horizon. Carlip’s approach can encode the details of the metric at the horizon unlike the Strominger’s. The second is that only “one” copy of the Virasoro algebra which lives at the horizon is involved, in contrast to the “two” copies of the Virasoro algebra which live at the boundary at infinity. Similar results have been subsequently derived in other frameworks as dimensionally reduced gravity and covariant phase space methods [13, 14].

Unfortunately it has been known that Carlip’s formulation is not complete [15, 16] and there has been no complete resolution so far [17], as far as I know, though his idea has received wide interest. This paper addresses a resolution of the problem. This provides a concrete realization of Carlip’s idea from first principles.

In section 2, a new canonical Hamiltonian which satisfies the usual variational principle in the presence of the boundary is derived from an action principle for Einstein gravity in any dimension. Full (bulk+boundary) diffeomorphism ($\text{Diff}$) generator is derived immediately from a known theorem in gravity theory. Carlip’s Hamiltonian and $\text{Diff}$ generator are also derived from an action principle with a different choice of boundary action term, but the
variational principle is not well defined in general.

In section 3, a general black hole metric in any dimension is introduced with the suitable metric fall-off conditions at the horizon. Fall-off-preserving conditions are derived for $\text{Diff}$ symmetry, which restricts the sub-leading terms as well as the leading terms for the metric and the $\text{Diff}$ parameters.

In section 4, a grand canonical ensemble, in which the horizon temperature and angular velocity as well as the horizon itself are fixed, is introduced. As an immediate consequence of the ensemble and the fall-off conditions, the new Hamiltonian satisfies the usual variational principle quite well at the black hole horizon, which is differentiable in the usual terminology.

In section 5, other consequences of the grand canonical ensemble to the $\text{Diff}$ at the horizon are investigated. First, the new $\text{Diff}$ generator satisfies all the conditions for a (differentiable) variational principle. Second, Carlip’s two main assumptions are derived which leads to a Virasoro-like algebra from the surface deformation algebra at the horizon: From the condition of fixed horizon and its temperature, I derive the equation which expresses radial $\text{Diff}$ parameter $\xi^r$ in terms of the (time or angular) derivatives of temporal or angular $\text{Diff}$ parameter $\xi^t, \xi^\alpha$; from the condition of fixed horizon angular velocity, I derive the equation of zero angular surface deformations $\hat{\xi}^\alpha = 0$, which reduces the deformation algebra to the “r-t” plane in the surface deformation space. Now, with the help of explicit spacetime dependence on $\xi^t$, which is inspired by the null surface at the horizon as well as Carlip’s two derived equations, I show that the surface deformation algebra becomes a Virasoro algebra with a classical central extension.

In deriving this result, I find, after a tedious computation, a peculiar property that first taking the limit to go to the horizon and then computing variation do not commute by the amount of the Hamiltonian constraint. This leads me to conjecture that

the non-commutativity of the two limiting process is proportional to the Hamiltonian and momentum constraints.

By restricting to the case of only one independent rotation, I obtain the usual momentum space Virasoro algebra with the classical central charge $c$ and the conformal weight $\Delta$ which are proportional to the horizon area $A_+$.

In section 6, a non-rotating black hole is analyzed with slight modifications of the formulas for a rotating black hole. The main difference is the appearance of an additional factor “1/2” in the formula compared to the rotating horizon. There is no natural connection between the temporal derivative and the angular derivative of $\text{Diff}$ parameters $\xi$. But one can still introduce an arbitrary velocity parameter, which is not related the horizon’s rotation, to obtain a Virasoro algebra. Its $c$ and $\Delta$ have additional factor “2” compared to the rotating case.
In section 7, the canonical quantization of the Virasoro algebra is considered. I find that the existence of the classical Virasoro algebra is crucial in obtaining an operator Virasoro algebra which produces the right central charge and conformal weight $\sim A_+ / \hbar G$ for the semiclassical black hole entropy. Quantum corrections of operator ordering have negligible contribution $O(1/\sqrt{\hbar})$ for a large semiclassical BH entropy. In order to obtain the correct numerical factor 1/4 of the BH entropy, the minimum conformal weight $\Delta_{\text{min}}$, which fixes the ground state of a black hole configuration, of the classical Virasoro algebra is assumed differently for rotating and non-rotating black holes: For the rotating one, it is assumed that $\Delta_{\text{min}} = -(32\pi G)^{-1} 3 A_+ \beta / T$, which has an explicit dependence on $\beta / T$ with the inverse Hawking temperature $\beta$ and a temporal-period $T$. For the non-rotating one, a $\beta, T$ independent ground state $\Delta_{\text{min}} = 0$ can be assumed. The arbitrary $\beta / T$ dependences in $c$ and $\Delta$ exactly cancel each other through effective ones $c_{\text{eff}}$ and $\Delta_{\text{eff}}$ such as the final entropy has no $\beta / T$ dependence. The appropriate ground state is different for the rotating and the non-rotating black holes but otherwise it has a universality for a wide variety of black holes.

In section 8, several applications are considered and it is found that almost all the known uncharged solutions satisfy a universal statistical entropy formula, which is the same as the BH entropy. The theory is extended to include the cosmological constant term and it is found that the BTZ solution and the rotating de-Sitter space have the universal BH entropy also. This computation is contrary to the other recent computations at the spatial infinity where there is no horizon and which, therefore, may not be relevant to a black hole. Especially for the rotating de-Sitter solution in $n = 3$ (Kerr-dS$_3$), no complex number appears at any intermediate step in contrast to the other computations at spatial infinity, which is hidden inside the horizon.

I conclude with remarks on several remaining questions which are under investigation. I shall adopt unit in which $c = 1$.

2 The variational principle and surface deformations for bounded spacetime: A general treatment

a The variational principle for bounded spacetime in general

Let me start with the spacetime split of the Einstein-Hilbert action on a $n-$dimensional manifold $\mathcal{M}$, accompanied by the extrinsic curvature terms on the boundary $\partial \mathcal{M}$

$$ S = S_{\text{EH}} + S_{\partial \mathcal{M}}, $$

(2.1)
where

\[
S_{\text{EH}} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^nx \left\{ N\sqrt{h} \left[ R + \frac{(16\pi G)^2}{h} \left( p_{ab}^{-} p_{ab}^+ - \frac{1}{n-2} p^2 \right) \right] + 2\partial_t(\sqrt{h}K) + 2\partial_a(\sqrt{h}KN^a - \sqrt{h}h^{ab}\partial_bN) \right\},
\]

\[
S_{\partial\mathcal{M}} = -\frac{1}{8\pi G} \int d^{n-1}x \sqrt{h}K \left( \left| \frac{1}{\sqrt{8\pi G}} \int_{t_i}^{t_f} dt \int_{\mathcal{B}} d^n-2x \sqrt{\sigma}n^aN_aK. \right| \right) \quad (2.2)
\]

Here, the boundaries are \( \partial \mathcal{M} = \Sigma_{t_f} \cup \Sigma_{t_i} \cup \mathcal{B} \) with the spacelike boundaries \( \Sigma_{t_f} \) and \( \Sigma_{t_i} \) at the final and initial times, respectively and the intersection \( \mathcal{B} \) of an arbitrary timelike boundary with a time slice \( \Sigma \). \( N \) and \( N^a (a = 1, 2, \ldots, n-1) \) are the lapse and shift functions, respectively. \( h_{ab} \) is the induced metric on \( \Sigma \) and \( h \) is its determinant. \( R^{ab} \) and \( K^{ab} \) are the intrinsic and extrinsic curvature tensors of the hypersurface \( \Sigma \), and \( R = h_{ab}R^{ab} \) and \( K = h_{ab}K^{ab} \) are their curvature scalars respectively. \( p_{ab} = (16\pi G)^{-1}\sqrt{h}(K^{ab} - Kh^{ab}) \) is the canonical momentum conjugate to \( h_{ab} \) and \( p = p_{ab}h_{ab} \). \( n^a \) is the unit normal \( (h_{ab}n^an^b = 1) \) to the boundary \( \mathcal{B} \) on a constant time slice \( \Sigma \), and \( \sigma_{ab} = h_{ab} - n_a n_b \) and \( \sigma \) are the induced metric on the boundary \( \mathcal{B} \) and its determinant, respectively. Then the first and the second boundary terms of \( S_{\partial\mathcal{M}} \) cancel the first and the second total derivatives terms of \( S_{\text{EH}} \), respectively, which are proportional to \( K \). The first-order total derivative action \( S \) is closely related to the so-called “gamma-gamma” action which eliminates all the second derivatives of \( g_{\mu\nu} (\mu = 0, 1, \ldots, n) \), but the advantage of the action \( S \) is that this can be written in a manifestly covariant form and moreover the first-order time derivatives of \( N, N^a \) are removed from the start.

For a general hypersurface \( \Sigma_t \), the variation of \( S \) becomes

\[
\delta S = \frac{1}{16\pi G} \int_{\mathcal{M}} G_{\mu\nu}\delta g^{\mu\nu} (n)\epsilon + \frac{1}{16\pi G} \int_{\partial\mathcal{M}} d\bar{\theta},
\]

where \( \bar{\theta} \) is the pull-back of

\[
\theta = \nabla_\mu(\gamma_\mu^\nu \delta r^\nu) (n-1)\epsilon \pm \delta r_\mu r_\nu \nabla^\nu r_\mu (n-1)\epsilon + 16\pi G \mathbf{P}^{\mu\nu} \delta \gamma_{\mu\nu}
\]

on \( \partial\mathcal{M} \). Here, the unit normal vector \( r^\mu \) of \( \partial\mathcal{M} \) is normalized as \( g_{\mu\nu}r^\mu r^\nu = \pm 1 \) (the upper sign for a timelike boundary and the lower sign for a spacelike boundary) and the induced volume element and the metric on \( \partial\mathcal{M} \) are given by

\[
\begin{align*}
(n-1)\epsilon_{\mu_1\mu_2\cdots\mu_{n-1}} = \pm r^\mu (n)\epsilon_{\mu_1\mu_2\cdots\mu_{n-1}}, & \quad \gamma_{\mu\nu} = \mp r_\mu r_\nu + g_{\mu\nu}.
\end{align*}
\]

\[
2g_{\mu\nu} = \begin{pmatrix} -N^2 + N_aN^a & N_b \\
N_a & h_{ab} \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} -N^2 & N_bN_a \\
-\frac{N_a^2}{N} & -h_{ab} + N_aN_b \end{pmatrix}, \quad h_{ac}h^{cb} = \delta^b_a, N^a = h^{ab}N_b, \quad I
\]

follow the convention of Wald with an exception of (2.6).

3Here, I assume that the spacelike hypersurface \( \Sigma_t \) intersects orthogonally the timelike boundary \( \mathcal{B} \) for each \( t \). See Hawking and Hunter’s paper for a generalization to a non-orthogonal intersection.
Here, $P^{\mu\nu} = -(\Theta^{\mu\nu} - \Theta^{\nu\mu}) (n-1)\epsilon$ is the canonical momentum $(n-1)$-form conjugate to $\gamma_{\mu\nu}$ with the extrinsic curvature $\Theta^{\mu\nu} = \gamma^{\mu\rho} \nabla_{\rho} r^\nu$. It is clear that, if the induced metric $\gamma_{\mu\nu}$ and the unit normal $r^\mu$ are fixed on $\partial M$, the equation of motion for $g^{\mu\nu}$ is the usual Einstein equation $G^{\mu\nu} = 0$ at the boundary as well as in the bulk. On the other hand, since there is no physical evidence of modification of the Einstein equation at the boundary in any case, I use the guiding principle, for the general treatment of bounded systems, that the equation of motion is the usual Einstein equation for any kinds of variations. The boundary condition, which specifies what quantities are fixed at the boundary, changes as the boundary term $S_{\partial M}$ changes. But it will be shown that my choice of $S_{\partial M}$ is the right one which is relevant to the black hole horizon when the horizon is treated as a boundary. However, in this case I do not require the fixed induced metric or the fixed unit normal at the horizon but rather some appropriate fall-off conditions for them. I am going to treat this problem within the Hamiltonian formulation, which fits my purpose well; in general, the required boundary condition in the action formulation is different from that of the Hamiltonian formulation.

The canonical Hamiltonian becomes

$$H[N, N^a] = \int_{\Sigma} d^{n-1}x \ (N H_t + N^a H_a) + \frac{1}{8\pi G} \int_{\mathcal{B}} d^{n-2}x \ (16\pi G N^a p^r_a + \sqrt{\sigma} n^a D_a N) = H_{\Sigma}[N, N^a] + H_B[N, N^a],$$

(2.7)

where $H_{\Sigma}[N, N^a]$ and $H_B[N, N^a]$ are the bulk and boundary terms on $\Sigma$ and $\mathcal{B}$, respectively. $H_t$ and $H_a$ are the "bulk" Hamiltonian and momentum constraints

$$H_t = -\frac{\sqrt{h}}{16\pi G} R + \frac{16\pi G}{\sqrt{h}} \left( p_{ab} p^{ab} - \frac{1}{n-2} p^2 \right), \quad H_a = -2 D_b p^b_a,$$

(2.8)

where $D_a$ denotes the covariant derivative with respect to the spatial metric $h_{ab}$. The variation in $H[N, N^a]$ due to arbitrary variations in $h_{ab}, n^a, N, N^a$ becomes

$$\delta H[N, N^a] = \delta H_{\Sigma}[N, N^a] + \delta H_B[N, N^a] = \text{bulk terms} + \frac{1}{8\pi G} \int_{\mathcal{B}} d^{n-2}x \ (\delta n^r \partial_r N \sqrt{\sigma}
+n^r \partial_r \delta N \sqrt{\sigma} + \frac{1}{2} n^r \sqrt{\sigma} \sigma^{\alpha\beta} D_r \delta \sigma_{\alpha\beta} + 16\pi G \delta N^a p^r_a)$$

(2.9)

with

$$\delta H_{\Sigma}[N, N^a] = \text{bulk terms} + \frac{1}{8\pi G} \int_{\mathcal{B}} d^{n-2}x \ (\frac{1}{2} n^r N \sqrt{\sigma} \sigma^{\alpha\beta} D_r \delta \sigma_{\alpha\beta}$$

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*This Hamiltonian form was first studied by Brown, Martinez and York [22] within the context of thermodynamic partition function. But the physical content is different to that of this paper because of different boundary conditions. See also [23] for another literature on the Hamiltonian.*
\[ \delta H_B[N, N^a] = \frac{1}{8\pi G} \int_B d^{n-2}x \left( \delta n^r \partial_r N \sqrt{\sigma} + n^r \partial_r \delta N \sqrt{\sigma} + \frac{1}{2} n^r D_r N \sqrt{\sigma} \sigma^{\alpha\beta} \delta \sigma_{\alpha\beta} \right) + 16\pi GN^a \delta p_a + 16\pi G \delta N^a p_a, \]

where I have chosen a coordinate system of \( n^a = (n^r, 0, \ldots, 0) \).

The bulk terms are the usual variation terms for \( \delta p^{ab}, \delta h_{ab} \), which produce the bulk equations of motion \[18\] as well as the Hamiltonian and momentum constraints

\[ H_t \approx 0, \quad H_a \approx 0. \tag{2.12} \]

The additional variations at the boundary could affect the bulk equation of motions in general. Let me first consider the first term in the boundary terms of (2.9). By using, from the definition of \( n^c \),

\[ \delta n^c = -\frac{1}{2} \delta h_{ab} h^{cb} n^a \tag{2.13} \]

the first term becomes

\[ -\frac{1}{16\pi G} \int_B d^{n-2}x \delta h_{ab} (h^{rb} n^a \partial_r N \sqrt{\sigma}). \tag{2.14} \]

So, this term would produce an additional term in the bulk equation of motion of the form

\[ \{p^{ab}(x), H[N, N^a]\}_{\text{boundary}} = \frac{1}{16\pi G} \delta(r - r_B)(n^a h^{br} \partial_r N \sqrt{\sigma}) \tag{2.15} \]

unless the quantity in the bracket ( ) vanishes on the boundary \( B \). Here \( r_B \) is the radius of the boundary \( B \). Similarly, the last boundary term in (2.9) produces an additional contribution

\[ H_a|_{\text{boundary}} = \delta(r - r_B)2p^r_a \tag{2.16} \]

to the momentum constraints \( H_a \) in general. Contrary to these two contributions, which are proportional to \( \delta(r - r_B) \), the second and third terms produce a highly singular term \( \partial_r \delta(r - r_B) \), which produces the divergence of “\( \delta(0) \)” even when I compute the commutation relations between the integrated quantities \( \{H[N, N^a], H[N', N'^a]\} \), which should be related to the measurable things. In order to avoid this problem, I need to assume the boundary conditions which restrict the radial derivatives of the variations \( \delta N \) and \( \delta \sigma_{\alpha\beta} \) on the boundary \( B \) as

\[ \sqrt{\sigma} n^r \partial_r \delta N|_B = 0, \tag{2.17} \]

\[ N \sqrt{\sigma} \sigma^{\alpha\beta} n^r D_r \delta \sigma_{\alpha\beta}|_B = 0 \tag{2.18} \]

\[ \text{Greek letters from the beginning of the alphabet are the boundary indices which do not include radial coordinate } r \text{ whence Greek letters from the middle of the alphabet are spacetime indices.} \]

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but $N$ and $\sigma_{\alpha\beta}$ can be arbitrary otherwise. One may also take, instead of (2.17) and (2.18), $\delta N_{|B} = 0$ and $\delta \sigma_{\alpha\beta}|_{B} = 0$ but this could be too strong a condition depending on the property of $B$. Hence I keep (2.17) and (2.18) for a general treatment.

Now, in summary, about the contributions of the boundary terms in (2.9), there are two possible contributions (2.15) and (2.16) to the corresponding bulk ones from the first and last terms in (2.9), respectively. The two badly-behaving terms are removed by restricting $\partial_{r}\delta N$ and $\partial_{r}\delta \sigma_{\alpha\beta}$ as in (2.17) and (2.18). In other words, the Hamiltonian (2.7) produces the Einstein equation on the boundary $B$ “as well as” in the bulk when the contributions of (2.15) and (2.16) vanish, and the boundary conditions (2.17) and (2.18) are satisfied on $B$ ($g_{ab}$ is fixed on the boundaries at infinity $\Sigma_{t_{f},t_{i}}$ as usual such that there is no contributions from $\Sigma_{t_{f},t_{i}}$).

For completeness, let me remark on another interesting type of boundary $C$ where there is no added boundary action $S_{\partial M}$ such as when the total action is just the Einstein-Hilbert action $S = S_{EH}$. In this case, $\theta$ in (2.4) becomes

$$\theta|_{C} = -\frac{2}{n-2} 16\pi G \gamma_{\mu\nu} \delta P^{\mu\nu} + (1 - \frac{2}{n-2}) 16\pi G \mathcal{P}^{\mu\nu} \delta \gamma_{\mu\nu} + \nabla_{\mu}(\gamma_{\nu}^{\mu \delta} \delta r_{\nu}) (n-1)\epsilon \pm \delta r_{\mu} r_{\nu} \nabla_{\mu} r_{\nu} (n-1)\epsilon.$$  

(2.19)

This, in general, contains $\delta P^{\mu\nu}$ term as well as $\delta \gamma_{\mu\nu}$ and $\delta r_{\mu}$ terms. Hence, for the boundary $C$, one must fix $P^{\mu\nu}$ as well as $\gamma_{\mu\nu}$ and $r_{\mu}$ on it, which may in turn over-specify the boundary degrees of freedom. The canonical Hamiltonian becomes, for a timelike boundary $C$,

$$H'[N, N^{a}] = \int_{\Sigma} d^{n-1}x \left( N \mathcal{H}_{t} + N^{a} \mathcal{H}_{a} \right) + \frac{1}{8\pi G} \int_{C} d^{n-2}x \left( \frac{16\pi G}{2 - n} \sqrt{h} n^{a} N_{a} p + 16\pi G N^{a} p_{a} + \sqrt{\sigma} n^{a} D_{a} N \right).$$  

(2.20)

This has the same form that Carlip has “postulated”, which can be seen by expressing the first $p$ term in terms of extrinsic curvature $K = 16\pi G ((2 - n)\sqrt{h})^{-1} p$ [8]. But it has a drawback that the variational principle is not well defined in general (see Appendix A for details; see also Ref. [15]), which would be related to the over-specification of boundary data in (2.19). So, I will not consider this boundary $C$ anymore hereafter and concentrate only the boundary $B$.

b **Diff generators**

Now, let me consider the generators of the spacetime $Diff$

$$\delta \xi^{\mu} = -\xi_{\mu}, \quad \delta \xi g_{\mu\nu} = \xi^{\sigma} \partial_{\sigma} g_{\mu\nu} + \partial_{\mu} \xi^{\sigma} g_{\sigma\nu} + \partial_{\nu} \xi^{\sigma} g_{\sigma\mu}. \tag{2.21}$$

The generators may be obtained directly from the usual Noether procedure [24, 25, 24, 27]. But there is a well-known theorem which identifies the generators from a well-defined Hamiltonian
If $H[N, N^a]$ is a canonical Hamiltonian of the gravity theory which does not have boundary term in the variations $\delta H[N, N^a]$ with the lapse and shift functions $N$ and $N^a$, the $Diff$ generator $L[\hat{\xi}]$ of

$$\delta_\xi h_{ab} = \{ h_{ab}, L[\hat{\xi}] \}, \quad \delta_\xi p_{ab} = \{ p_{ab}, L[\hat{\xi}] \}$$

(2.22)
is given by substituting $N, N^a$ in the Hamiltonian $H[N, N^a]$ with the so-called surface deformation parameters $[10, 8, 11]$

$$\hat{\xi}^t = N^t \xi^t, \quad \hat{\xi}^a = \xi^a + \xi^t N^a$$

(2.23)
respectively, giving

$$L[\hat{\xi}] = H[N, N^a]_{(N, N^a) \to (\hat{\xi}^t, \hat{\xi}^a)}.$$  

(2.24)

The $Diff$ of $\delta_\xi g_{\mu t}$ is instead given by the basic formula (2.21) essentially due to the absence of the canonical momentum conjugate to $g_{\mu t}$. Hence the generator becomes (I denote $L[\hat{\xi}] = H[\hat{\xi}]$)

$$H[\hat{\xi}] = \int_\Sigma d^{n-1}x \hat{\xi}^{\mu} \mathcal{H}_{\mu} + J[\hat{\xi}],$$

(2.25)
where

$$J[\hat{\xi}] = \frac{1}{8\pi G} \oint_B d^{n-2}x \left( 16\pi G \hat{\xi}^a p_{\mu a} + \sqrt{\sigma} n^a D_\mu \hat{\xi}^t \right)$$

(2.26)
for the boundary $B$, I assume the boundary conditions for the well-defined variations $\delta H[N, N^a]$ without boundary terms\footnote{Of course, this theorem will be modified if I allow the boundary terms in the variation principle \[29\].}. The usual $Diff$ at the boundary as well as in the bulk is generated by the cancellation of the boundary $Diff$ from the bulk part in (2.25) and another boundary $Diff$ from the boundary generator $J[\hat{\xi}]$. Another interesting effect, which is crucial to my analysis of black hole entropy, of $J[\hat{\xi}]$ is that it may produce the “classical” central extension in the symmetry algebra in general \[28, 8\].

Since I am interested in the symmetry algebra of $H[\hat{\xi}]$ on the physical subspace where the Hamiltonian and momentum constraints of (2.12) are imposed, I must compute the Dirac bracket in general \[28\]. The Dirac bracket algebra would show some interesting effects of the boundary but this is very complicated in my case. Rather, in this paper, I will use an effective method which gives the Dirac bracket of $H[\hat{\xi}]$’s themselves without tedious computations \[28, 31\]

$$\{ H[\hat{\xi}], H[\hat{\eta}] \}^* \approx \{ J[\hat{\xi}], J[\hat{\eta}] \}^* = \delta_\eta J[\hat{\xi}]$$

(2.27)
\footnote{This can be equivalently written as $\{ J[\hat{\xi}], J[\hat{\eta}] \}^* = (\delta_\eta J[\hat{\xi}] - \delta_\xi J[\hat{\eta}]) / 2$ in order that the antisymmetry under $\eta \leftrightarrow \xi$ is manifest. This can be realized through an explicit form of the Dirac bracket also.}
from the definition of the Dirac bracket. Here, the Dirac bracket on the right hand side has the form

\[ \{ J[\xi], J[\eta]\}^* = J[\{\xi, \eta\}_{\text{SD}}] + K[\xi, \eta] \]  

(2.28)
in general, where \( K[\xi, \eta] \) is a possible central term and \( \{\xi, \eta\}_{\text{SD}} \) is the Lie bracket for the algebra of surface deformations [10, 8, 11]:

\[ \{\xi, \eta\}_{\text{SD}} = \xi^a \partial_a \eta^t - \eta^b \partial_b \xi^t + g^{ab}(\xi^t \partial_b \eta^t - \eta^t \partial_b \xi^t). \]  

(2.29)

3 Fall-off conditions at a black hole horizon

Now, let me consider a general black-hole-like metric in \( n \) spacetime dimensions [30] with the Boyer-Lindquist coordinates \((t, r, x^\alpha)\)

\[ ds^2 = -N^2 dt^2 + f^2 (dr + N^r dt)^2 + \sigma_{\alpha\beta}(dx^\alpha + N^\alpha dt)(dx^\beta + N^\beta dt), \]  

(3.1)

where the lapse function \( N \) vanishes at the horizon and behaves as follows near the “outer-most” horizon \( r = r_+ \)

\[ N^2 = h(x^\alpha)(r - r_+) + O(r - r_+)^2, \quad \frac{2\pi}{\beta} = n^a \partial_a N|_{r_+}, \]  

(3.2)

where \( \beta \) is the inverse Hawking temperature, which is constant on the horizon \( r_+ \), and \( n^a \) is the unit normal to the horizon boundary \( r = r_+ \) on a constant time slice \( \Sigma_t \).

The suitable fall-off conditions near the horizon \( r_+ \) are

\[ f = \frac{\beta h}{4\pi} N^{-1} + O(1), \]  

(3.3)

\[ N^r = O(N^2), \quad (\partial_t - N^r \partial_r)g_{\mu\nu} = O(N)g_{\mu\nu}, \]  

(3.4)

\[ \sigma_{\alpha\beta} = O(1), \quad N^\alpha = O(1), \]  

(3.5)

\[ D_\alpha N_\beta + D_\beta N_\alpha = O(N). \]  

(3.6)

The conditions in (3.4) are the perturbations of the stationary black hole of \( N^r = 0 \); I shall keep \( N^r \) in this and the next sections for generality; but \( N^r = 0 \) shall be considered in section 5.c by computing the Virasoro algebra for a stationary horizon. (3.6) is the condition of constant angular velocity \( \Omega^\alpha = -N^\alpha \). However by comparing with (3.5) one can see that (3.6) gives a nontrivial restriction if one considers \( D_\alpha \sim \partial_\alpha \sim O(1) \). To make it more explicit, let me consider a decomposition

\[ N_\alpha = f_\alpha + K_\alpha, \]  

(3.7)
where \( f_\alpha = O(1) \), \( D_\alpha f_\beta + D_\beta f_\alpha = 0 \), and \( K_\alpha \) vanishes at the horizon \( r_+ \). Then, it is easy to see that (3.6) is satisfied if
\[
\partial_r \sigma_{\alpha\beta} \leq O(N^{-1}), \quad K^\alpha \leq O(N). \tag{3.8}
\]
[Note that \( \partial_r N = 2\pi f_{\beta}^{-1} \).] Therefore, I require the form (3.7), with the condition (3.8), for consistency with (3.3)-(3.6). These are the basic set-up of the fall-off conditions at the horizon \( r_+ \).

As another requirement of consistency, the \( Diff \) symmetry (2.21) should not accidently violate these fall-off conditions. The suitable conditions for this requirement are
\[
\xi^t \leq O(1), \quad \xi^\alpha \leq O(1), \quad \xi^r \leq O(N^2), \tag{3.9}
\]
\[
\partial_r \xi^t \leq O(N^{-2}), \quad \partial_r \hat{\xi}^\alpha \leq O(1), \tag{3.10}
\]
\[
\partial_r N^\beta \leq O(1), \quad \partial_r \sigma_{\alpha\beta} \leq O(N^{-2}). \tag{3.11}
\]
This shows that the fall-off-condition-preserving \( Diff \) requires the suitable behaviors of the sub-leading terms of \( Diff \) parameters from (3.10) as well as the behavior of the leading terms (3.3). Moreover, it depends also upon the behavior of the sub-leading term of the metric from (3.11). Then, it is straightforward to compute that
\[
K_{rr} = O(N^{-3}), \quad K_{ar} = O(N^{-1}), \quad K_{\alpha\beta} = O(1), \quad K = O(N^{-1}) \tag{3.12}
\]
and
\[
p_r^r = O(N^{-1}), \quad p_\alpha^r = O(1), \quad p_r^\alpha = O(N^{-2}), \quad p_\alpha^\beta = O(N^{-2}), \quad p = O(N^{-2}). \tag{3.13}
\]
Note that there is no inequality relation in contrast to the fall-off preserving conditions (3.8)-(3.11).

4 Grand canonical ensemble and the variational principle at the black hole horizon

I have considered the variational principle for a bounded spacetime in general in the two preceding sections. Now, let me consider the variational principle at the black hole horizon \( r_+ \) when the horizon is treated as a boundary with the fall-off boundary conditions as described in the last section. But, for discussing the statistical mechanical property of a black hole, I must specify the type of ensemble first of all. For this purpose, I take a \textit{grand canonical} ensemble, in

\textit{\footnotesize 8} Notice that the hat of \( \hat{\xi}^\alpha \) in (3.10) is not mistyping. In the unhatted expression, it becomes \( \partial_r \xi^\alpha = -N^\alpha \partial_r \xi^t + O(1) \), which is important when I prove the condition (2.18) in \( Diff \) (5.1).
which the horizon $r_+$ and its temperature and angular velocity are fixed \cite{22}. Full elaboration on the impact of this ensemble will be given in the next section, but here I only mention a direct consequence for the variational principle: When the horizon $r_+$ is treated as the boundary $B$, the boundary condition (2.17) is automatic since

$$\delta N|_{r_+} = 0 \quad (4.1)$$

in order that the boundary remains at the horizon $r = r_+$, which is the solution of $N + \delta N = 0$, and also in order that the temperature $\beta$, which is the coefficient of $N + \delta N$, is unchanged. However, the condition (2.18) is not automatic and I must impose this for consistency.

Moreover, the fixed angular velocity $\Omega^\alpha \sim N^\alpha$ in the ensemble insures that there is no boundary contributions to the momentum constraint $H_\alpha \quad (2.16)$, which is the coefficient of $\delta N^\alpha$.

One can then see that, using the fall-off conditions in (3.13), the only non-vanishing term in (2.16) is $H_{r|\text{boundary}} = O(N^{-1})$. But this is not harmful because its contribution to the bulk Hamiltonian vanishes: $\int_S d^{n-1}x \ N^r H_r \leq O(N)$. On the other hand, the boundary contributions of (2.15) vanish also since

$$\{p^{rr}(x), H[N, N^a]\}|_{\text{boundary}} \leq O(N^2), \quad (4.2)$$

and all other components in (2.15) vanish trivially from my choice of coordinates and the metric (3.1). Therefore, the new Hamiltonian (2.7) admits the variational principle even at the black hole horizon such that the usual bulk equations of motion and the bulk constraints (2.12) can be applied to the horizon as well as outside of the horizon.

Moreover, I note that since the first term in the boundary part of the Hamiltonian (2.7) is $N^a p^r_\alpha = N^\alpha p^r_\alpha + O(N) \leq O(1)$, the Hamiltonian at the black hole horizon, which will be the true Hamiltonian on the physical subspace, reduces to

$$H|_{r_+}[N, N^a] = \frac{1}{8\pi G} \oint_{r_+} d^{n-2}x \ (16\pi G N^\alpha p^r_\alpha + \sqrt{\sigma n^\prime D_r N}) \leq O(1). \quad (4.3)$$

5 **Diff and Virasoro algebra at the horizon**

In section 2.b, the general Diff generators and their symmetry algebras have been considered for an arbitrary boundary $\mathcal{B}$. But, now let me restrict to the horizon in particular, where the general Diff reduces to (3.9)-(3.11). But, to this end, I need more detailed knowledge on the Diff in the grand canonical ensemble for the black hole.
a **Diff at the horizon: Boundary conditions**

First, I note that the boundary condition (2.18) is automatic

\[ N\sqrt{\sigma}n^{\alpha\beta}D_\alpha \delta \xi \sigma_{\alpha\beta} \leq O(N) \quad (5.1) \]

by using the condition (3.8) and the fact that

\[ \delta \xi \sigma_{\alpha\beta}(x^\mu) = O(N) \quad (5.2) \]

Moreover, the general requirement (4.1) of the fixed horizon and temperature for the grand canonical ensemble in the variational principle at the black hole horizon should be applied to *Diff* transformation also. Now let me elaborate in some details what the further implications of this for *Diff*. From the fact that \( g_{tt} = -1/N^2 \) and \( \delta \xi g_{tt} = 2 \nabla_t \xi_t \), one finds

\[ \delta \xi N^2 = \frac{4Nf}{\beta}(\xi^t N^r + \xi^r) + 2N^2(\partial_t - N^a \partial_a)\xi^t + \xi^\alpha \partial_\alpha N^2 + O(N^3) \leq O(N^2). \quad (5.3) \]

Hence, an arbitrary *Diff* transformation produces change in the inverse temperature \( \beta^{-1} \) by \( (2\pi)^{-1}n^r \partial_r \delta \xi N|_{r_+} = (4\pi fN)^{-1} \partial_r \delta \xi N^2|_{r_+} \) unless the condition (4.1) is satisfied for the *Diff* transformation also, i.e.,

\[ \delta \xi N^2|_{r_+} = 0. \quad (5.4) \]

In other words, the grand canonical ensemble is satisfied if

\[ \xi^r = -\frac{N\beta}{2\pi f}(\partial_t - N^a \partial_a)\xi^t + \frac{N\beta N^r}{2\pi f}\left(\partial_t \xi^t - \frac{2\pi f}{N^r} \xi^t\right) - \frac{N^3\beta}{4\pi f^2} \xi^\alpha \partial_\alpha g_{tt}, \quad (5.5) \]

which allows to express \( \xi^r \) in terms of \( \xi^t \). Hence, another boundary condition (2.17) is also automatic in the grand canonical ensemble.

Therefore, the *Diff* generator (2.25), for the boundary \( B \) as the horizon, is well-defined without the additional contributions from the boundary when (5.5) as well as all the other conditions (2.17) and (2.18) for the metric and the conditions for the *Diff* parameters are satisfied (all the other boundary contributions vanish as the Hamiltonian did (section 4)). The existence of the well-defined *Diff* generator implies, according to the Noether theorem [25, 26, 27], that the *Diff* symmetry is not broken even with the boundary.

---

9 Notice that another term \( \delta \xi n^r \partial_r N \) from (3.2) should be dropped since this gives the temperature at the lifted position \( r_+ (n_r + \delta n^r) \) not the horizon \( r_+ n_r \).

10 Notice that here, there is no perturbation on the black hole radius \( r_+ \) since \( \delta \xi N^2 = O(N^2) \sim (r - r_+) \); this can be more directly seen from \( \delta \xi x^r|_{r_+} = -\xi^r|_{r_+} = 0 \) due to (2.21) and (3.9). But this does not imply the fixed horizon area \( A_+ = \oint_{r_+} d^{n-2}x \sqrt{\sigma} \), i.e., fixed entropy, which might cause the ensemble to be absurd: Rather, one has \( \delta \xi A_+ = \int_{r_+} d^{n-2}x \delta \xi \sqrt{\sigma} = O(1) \) according to the solution (5.2).
Furthermore, with the well-defined generator $H[\hat{\xi}]$ (2.25) all the information on the $Diff$ of the bulk part $\int \delta^{n-1}x \hat{\xi}^\mu \mathcal{H}_\mu$ is encoded into the boundary (horizon $r_+$) part $J[\hat{\xi}]$ when it is treated on the physical subspace of $\mathcal{H}_\mu \approx 0$; this is another aspect of the ’t Hooft holography principle at the black hole [33].

b  $Diff$ at the horizon: No angular surface deformations

In the last subsection I have used the condition of fixed temperature for a grand canonical ensemble. Now, let me consider the condition of fixed angular velocity, which is due to a chemical potential for the black hole system. Since $\Omega^\alpha \sim N^\alpha$, I must compute $\delta \xi^\alpha N_\beta$ to explore the rotational property of a grand canonical ensemble. On the other hand, since $g_{\alpha\beta} = \sigma_{\alpha\beta} N^\beta$, I can compute $\delta \xi^\alpha N_\beta$ (from $\sigma_{\alpha\beta} \delta \xi^\alpha = \delta \xi g_{\alpha\beta} - \delta \xi \sigma_{\alpha\beta} N^\beta$) as

$$
\delta \xi^\alpha N_\beta = \partial_\xi \hat{\xi}^\alpha - N^\beta \partial_\beta \hat{\xi}^\alpha + \xi^\delta \partial_\delta N^\alpha + \xi^t N^\beta \partial_\beta N^\alpha + O(N). \quad (5.6)
$$

Then, one finds that the condition of fixed angular velocity $\Omega^\alpha \sim N^\alpha$, i.e., $\delta \xi^\alpha N^\beta |_{r_+} = 0$ of a grand canonical ensemble is satisfied when I restrict the surface deformation space of $\hat{\xi}^\mu$ (2.23) to the “r-t” plane:

$$
\hat{\xi}^\alpha = 0; \quad (5.7)
$$

of course, this does not mean the “r-t” plane in the space of spacetime $Diff\xi^\mu$. Furthermore, the solution is unique in the grand canonical ensemble[1].

Now, equation (5.7), together with the equation (5.5), has been fixed from the conditions of the grand canonical ensemble on the black hole; these were the two main assumptions in the Carlip’s formulation. So, is this the end of story of the grand canonical ensemble? To answer this, let me elaborate what the condition (5.7) further implies. From the definition (2.21), equation (5.7), i.e., $\xi^\alpha = -N^\alpha \xi^t$ implies

$$
\delta \xi x^\alpha = -N^\alpha \delta \xi t \quad (5.8)
$$

and so,

$$
\Omega^\alpha \frac{\delta}{\delta \xi x^\alpha} = \frac{\delta}{\delta \xi t} \quad (\text{no sum}) \quad (5.9)
$$

with $\Omega^\alpha = -N^\alpha$. This does not show any definite information about arbitrary spacetime variations. But, let me introduce one assumption about $\xi^t$ inspired by these equations (5.8) and

---

[1] The question of angular $Diff$ with $\hat{\xi}^\alpha \neq 0$ has been raised by Carlip in his talk [6]. I can rule out this in a grand canonical ensemble. But it is unclear whether this can be ruled out in other ensembles also.
"ξ^t is at rest on the horizon".

This assumption determines the spacetime dependence of ξ^t as \[ \xi^t = \xi^t \left[ t - \left(1 - \frac{f}{N} N^r \right) r_* + \sum_{\hat{\alpha}} \frac{1}{\Omega^\hat{\alpha}} x^{\hat{\alpha}}, x^\alpha \right], \tag{5.10} \]

where \(t - (1 - f N^{-1} N^r) r_* = \text{constant}\) is a radial, outgoing null geodesic for a generic Kerr black hole with the Regge-Wheeler tortoise coordinate 
\[ r_* = (\beta/4\pi) \log(r - r_+) + O(r - r_+) \tag{5.11} \]

near the horizon \(r_+\); an appropriate integration constant for \(dr/dr_* = N/f\) is chosen here. The angular dependence on \(x^{\hat{\alpha}}\) reflects the assumption that \(\xi^t\) is “at rest” on the horizon which is rotating with velocity \(\Omega^{\hat{\alpha}}\); however, notice that \(x^{\hat{\alpha}}\) dependence in \(\xi^t\) is arbitrary in general. Now then, one can show that
\[ \partial_t \xi^t = -\frac{f}{N} \left(1 - \frac{f}{N} N^r \right) \partial_t \xi^t, \tag{5.12} \]
\[ \partial_t \xi^t = \Omega^\alpha \partial_{\hat{\alpha}} \xi^t \quad \text{(no sum)}, \tag{5.13} \]

which makes it possible to express all the derivatives in terms of one dimensional derivatives \(\partial_t\) or \(\partial_{\hat{\alpha}}\). Here, (5.13) shows that \(\xi^t\) respects the symmetry (5.8) exactly. Moreover, (5.12) is consistent with the boundary condition (3.10). One can also show that another non-vanishing \(\text{Diff}\) parameter \(\xi^\alpha\) satisfies the same equations as (5.12) and (5.13).

c The Virasoro algebra at a stationary horizon

So far my computation was valid for any \(N^r\), which is \(O(N^2)\). But, now let me focus on a time slice with \(N^r = 0\) such that a “stationary” black hole is considered. Then, the \(\text{Diff}\) of \(J[\hat{\xi}]\) with respect to the surface deformations becomes
\[ \delta_{\hat{\xi}_2} J[\hat{\xi}_1] = \frac{1}{8\pi G} \int_{r_*} d^{n-2}x \left[ \delta_{\hat{\xi}_2} n^r \partial_t \hat{\xi}_1 t \sqrt{\sigma} + n^r \partial_r (\delta_{\hat{\xi}_2} \hat{\xi}_1^t) \sqrt{\sigma} + \frac{1}{2} n^r \partial_t \hat{\xi}_1^t \sqrt{\sigma} \sigma^{\alpha\beta} \partial_{\hat{\xi}_2} \sigma_{\alpha\beta} + 16\pi G \delta_{\hat{\xi}_2} \hat{\xi}_1^t \hat{p}_a^t + 16\pi G \hat{\xi}_1^t \delta_{\hat{\xi}_2} \hat{p}_a \right]. \tag{5.14} \]

\(^{12}\)For \(N^r = 0\), a more generalized form \(\xi^t = \xi^t(t - \mu^{-1} r_* + \sum_{\hat{\alpha}} (\nu^\alpha \Omega^\hat{\alpha})^{-1} x^{\hat{\alpha}}, x^\alpha)\) may considered by relaxing the assumption that “\(\xi^t\) is at rest on the horizon”. But one can show that taking different \(\mu, \nu, \) and \(T/\beta\) corresponds to taking different ground state.
It is straightforward to check that the second to the fourth terms vanish based on the relations for the metric and the \textit{Diff} parameters. The only non-vanishing contributions are the fifth and the first terms.

\textbf{The fifth term}: This term needs a tedious computation of \( \{ p^{ab}, H[\hat{\xi}] \} \) from (2.22). But the final result is very simple (the derivation is sketched in Appendix B): This becomes, from \( \hat{\xi}^\alpha = 0 \),

\[
2 \oint_{r_+} d^{n-2}x \ \hat{\xi}_1^\gamma \delta_2 p_r^\gamma = \oint_{r_+} d^{n-2}x \ \hat{\xi}_1^\gamma \hat{\xi}_2^t \mathcal{H}^t + O(N), \tag{5.15}
\]

where

\[
\mathcal{H}^t = \frac{16\pi G}{\sqrt{\hbar}} \left( p_{\alpha\beta} p^{\alpha\beta} - \frac{1}{n-2} p^\alpha_{\alpha} p^\beta_{\beta} \right) + O(N^{-2}) \tag{5.16}
\]

is the Hamiltonian constraint evaluated near the horizon \( r_+ \). This result shows a very peculiar property since the \textit{Diff} of \( \delta_\xi p^r_r \) does not preserve the fall-off behavior \( p^r_r = O(N^{-1}) \) near the horizon, but rather becomes as \( \delta_\xi p^r_r = O(N^{-2}) \) such that the left hand side of (5.15) is \( O(1) \), which does not vanish in general. Notice that this is sharply contrary to the process of \textit{first taking the limit} to go to the horizon in the corresponding term \( 2 \oint_{r_+} d^{n-2}x \ \delta_\xi p^r_r = O(N) = 0 \) in \( J[\hat{\xi}] \) and then \textit{computing the functional differentiation} \( \delta_\eta 0 = 0 \); these two processes do not commute in general\footnote{This has been first pointed out by Carlip in the interpretation of a result of Ho and I \cite{15, 16}.}. However, the result (5.15) shows the interesting property that the surviving terms are nothing but the constraints of the system. Hence, the problematic situation of non-commutativity of the two limiting processes of computing the variations and the Poisson bracket is avoided by the genuine constraints of the system. This implies that there is no non-commutativity problems automatically when one consider the Dirac bracket, in which constraints can be implemented consistently through variations. Moreover, since the usual Regge-Teitelboim approach \cite{28}, which finds the appropriate \( J[\hat{\xi}] \) such that \( \delta_\xi J[\hat{\xi}] \) cancels the boundary part of the variations of the bulk symmetry generator \( \delta H_{\text{bulk}}[\hat{\xi}] \), uses the limiting process first and then the variating, this approach might not be always true. The only resolution is that \textit{the difference of the two processes is proportional to the constraints} as in my case. But since there is no other fundamental constraints besides the Hamiltonian and momentum constraints in my black hole system, I am led to a conjecture, \textit{for any black hole system}, that

\textit{Non-commutativity of taking the limit to go to the horizon and computing variation is proportional to the Hamiltonian and momentum constraints.}

I can not generally prove this conjecture since it is not clear whether the fall-off conditions that I have studied in this paper are the most general for the Virasoro algebra to exist, but this
should be the case for the consistency of functional differentiation; furthermore, it seems that this conjecture may be extended to any gauge theories, which have the Gauss’ law constraints as the fundamental constraints, if there are some appropriate boundaries where the variational principles are well-defined [26].

The first term: The first term in (5.14) is the main term. From (2.13), one has

$$\delta_{\xi} n^r = -\frac{1}{2f^2} n^r \delta n_{rr}.$$  

(5.17)

On the other hand, from (2.21) one obtains

$$\delta_{\xi} n_{rr} = -2f^2 (\partial_t - N^\alpha \partial_\alpha) \xi^t + \frac{\beta f^2}{\pi} (\partial_t - N^\alpha \partial_\alpha) \partial_\alpha \xi^t + O(N^{-1}),$$  

(5.18)

where I have used $\xi^r = -(N\beta/2\pi f) (\partial_t - N^\alpha \partial_\alpha) \xi^t$ from (5.3), $\partial_\alpha \xi^t = -N^\alpha \partial_\alpha \xi^t$ from (5.13) with $N^r = 0$, the rotational symmetry $N^\alpha \partial_\alpha N^2 = 0$, and the condition (5.11). Now then, the first term in (5.14) becomes

$$\frac{1}{16\pi G} \oint_{r_+} d^{n-1} x \frac{n^r}{f^2} \partial_r \xi^t \delta g_{rr} \sqrt{\sigma}$$

$$= -\frac{1}{16\pi G} \oint_{r_+} d^{n-1} x \sqrt{\sigma} \xi^t \left( -\frac{2\beta}{\pi} N^\alpha N^\beta \partial_\alpha \partial_\beta \xi^2 + \frac{8\pi N^\alpha}{\beta} \partial_\alpha \xi^2 \right).$$  

(5.19)

Here, I have taken a safe integration by parts for $\partial_\alpha$ [\(\bar{\alpha}\) part does not contribute] coordinate due to the rotational symmetry $\partial_\alpha N = \partial_\alpha N^2 = 0$.

By summarizing the computation, $\delta_{\xi_2} J[\hat{\xi}_1]$ of (5.14) is reduced to

$$\delta_{\xi_2} J[\hat{\xi}_1] = \frac{1}{8\pi G} \oint_{r_+} d^{n-2} x \delta_{\xi_2} n^r \partial_r \hat{\xi}_1^t \sqrt{\sigma} + 16\pi G f^2 \xi^t \delta_{\xi_2} P^r + O(N)$$

$$= \frac{1}{8\pi G} \oint_{r_+} d^{n-2} x \sqrt{\sigma} \xi^t \left( \frac{\beta}{\pi} N^\alpha N^\beta \partial_\alpha \partial_\beta \xi^2 \right) + \oint_{r_+} d^{n-2} x \hat{\xi}_1^r \xi_2^t \xi^2 + O(N).$$  

(5.20)

On the other hand, notice that

$$J[\{\hat{\xi}_1, \hat{\xi}_2\}_{\text{SD}}] = \frac{1}{8\pi G} \oint_{r_+} d^{n-2} x \left[ n^a D_a \{\hat{\xi}_1, \hat{\xi}_2\}_{\text{SD}} \sqrt{\sigma} + 16\pi G \{\hat{\xi}_1, \hat{\xi}_2\}_{\text{SD}} \delta_{\alpha} \right]$$

$$= \frac{1}{8\pi G} \oint_{r_+} d^{n-2} x \sqrt{\sigma} n^r \left( \frac{4\pi f N^\alpha}{\beta} \partial_\alpha \xi^t \xi^2 - (1 \leftrightarrow 2) \right) + O(N).$$  

(5.21)

Here I have used the fact that

$$\{\hat{\xi}_1, \hat{\xi}_2\}_{\text{SD}} = 2N N^\alpha \partial_\alpha \xi_1^t \xi_2^t - (1 \leftrightarrow 2),$$

$$\{\hat{\xi}_1, \hat{\xi}_2\}_{\text{SD}} = O(N^2), \quad \{\hat{\xi}_1, \hat{\xi}_2\}_{\text{SD}} \partial^r \partial_a = O(N).$$  

(5.22)
Fourier expansion of (5.10) as usual:
in the momentum space has no other coordinate-dependence. Then one obtains the familiar classical Virasoro algebra where \( \delta \xi_2 J[\xi_1] = J[\{\xi_1, \xi_2\}] + K[\{\xi_1, \xi_2\}] + \frac{1}{4\pi} \oint_{r+} d^{n-2}x \beta N f^{-1}\{\xi_1, \xi_2\}^t SD \mathcal{H}' + O(N) \), (5.23)

where

\[
K[\{\xi_1, \xi_2\}] = \frac{1}{8\pi G} \oint_{r+} d^{n-2}x \sqrt{\sigma} \xi_1^t \left( \frac{\beta}{\pi} N^\alpha N^\beta \partial_\alpha \partial_\beta \xi_2^t + \frac{4\pi N^\alpha}{\beta} \partial_\alpha \xi_2^t \right)
\]

From the relation (2.27), one further has a Virasoro-type algebra

\[
\{J[\hat{\xi}_1], J[\hat{\xi}_2]\}^* = \delta_{\xi_2} J[\hat{\xi}_1]
\]

\[
= J[\{\hat{\xi}_1, \hat{\xi}_2\}] + K[\{\hat{\xi}_1, \hat{\xi}_2\}] + \frac{1}{4\pi} \oint_{r+} d^{n-2}x \beta N f^{-1}\{\hat{\xi}_1, \hat{\xi}_2\}^t SD \mathcal{H}'(5.25)
\]

with a central term \( K[\{\hat{\xi}_1, \hat{\xi}_2\}] \). Notice that, the Virasoro algebra has been generalized to higher dimensions depending on the number of independent rotations \( [3] \): For uncharged black holes, the number of independent rotations is given by \( [(n-1)/2] \), which is the number of Casimir invariants of the \( SO(n-1) \) rotation group. \( [(n-1)/2] \) denotes the integer part of \( (n-1)/2 \). Moreover, since the relation (2.27) is valid with the Dirac bracket, where the constraints are imposed consistently, the last constraint term in (5.23) has no effect on the central term \( K[\{\hat{\xi}_1, \hat{\xi}_2\}] \).

Now, in order to obtain a more familiar momentum-space Virasoro algebra, I adopt the Fourier expansion of (5.10) as usual:

\[
\xi^t = \frac{T}{4\pi} \sum_n \xi_n^* \exp \left\{ \frac{2\pi i m}{T} \left( t - r^* - N^\phi |^1 r^* \phi \right) \right\}, \quad J[\hat{\xi}] = \sum_n \xi_n^* \mathcal{J}_n
\]

(5.26)

with \( \xi_n^* = \xi_{-n}^* \). The normalization of (5.26) is chosen such that one can obtain the standard factor \( i(m-n) \mathcal{J}_{m+n} \) in the Virasoro algebra below. Contrary to the lower dimensional case \( n \leq 3 \), the four and higher dimensional \( \xi_n \) have \( x^\alpha \) dependences in general, in which the momentum-space representation is different from the usual one; in this case the central term is expressed as an integral not just a number. But, for our purpose, I consider the case where \( \xi_n \) has no other coordinate-dependence. Then one obtains the familiar classical Virasoro algebra in the momentum space

\[
\{\mathcal{J}_m, \mathcal{J}_n\}^* = -i(m-n) \mathcal{J}_{m+n} - iK_{m+n},
\]

(5.27)

where

\[
\mathcal{J}_p = \frac{A_+}{16\pi G} \frac{T}{\beta} \delta_{p,0}, \quad K_{m+n} = \frac{A_+}{16\pi G} m \left( m^2 - \frac{T^2}{\beta} \right) \frac{\beta}{T} \delta_{m,-n}
\]

(5.28)
and $A_+$ is the area of the horizon $r_+$. The standard form of the central term is also obtained by a constant shift on $J_0$

$$J_0 \rightarrow J_0 = J_0 + \frac{A_+ \beta}{32\pi G T} \left( 1 - \left( \frac{T}{\beta} \right)^2 \right)$$

$$= \frac{A_+ \beta}{32\pi G T} \left( 1 + \left( \frac{T}{\beta} \right)^2 \right)$$

(5.29)

leading to

$$\{J_m, J_n\}^* = -i(m - n)J_{m+n} - i \frac{c}{12} m(m^2 - 1) \delta_{m,-n}$$

(5.30)

with a central charge

$$c = \frac{3A_+ \beta}{4\pi G T}.$$  

(5.31)

Notice the sign change of the second term in (5.29) from the shift.

6 The Virasoro algebra for a non-rotating horizon

So far, I have considered the Virasoro algebra for the surface deformation algebra on a black hole horizon when there is at least one non-zero rotation, i.e., Kerr black hole. But non-rotating ($N^\alpha = 0$) black hole, which has now a spherical symmetry, can be analyzed similarly with some modifications in the formulas for the rotating horizon.

The main differences in the basic relations are as follows (note $N^r = 0$). First, equation (5.5) becomes

$$\xi^r = -\frac{1}{2} \frac{N^\beta}{\pi f} \partial_t \xi^t \quad \text{(non-rotating horizon)}$$

(6.1)

$$\xi^r = -\frac{N^\beta}{\pi f} \partial_t \xi^t \quad \text{(rotating horizon).}$$

The second difference is that the condition of fixed angular velocity $\delta_{\xi} N^\alpha |_{r^+} = 0$ in a grand canonical ensemble produces

$$\xi^\alpha = \xi^\alpha(r, x^\beta) \quad \text{(non-rotating horizon)}$$

(6.2)

$$\xi^\alpha = -N^\alpha \xi^t \quad \text{(rotating horizon)}$$

from (5.6); even without rotation, the angular $\text{Diff}$ does not vanish, in contrast to a non-rotating limit of a rotating horizon; but now there is no connection between $\xi^\alpha$ and $\xi^t$. However,
the basic fall-off conditions (3.3)-(3.6) and their preserving conditions (3.8)-(3.11) are the same as for the rotating horizon; except that some of the conditions for the non-rotating case are milder than those of the rotating case, but in this case also, the non-rotating case can be obtained as a limit of the rotating case;

\[
\partial_r \xi^\beta \leq O(1) \quad \text{(non-rotating horizon)},
\]

\[
\partial_r \xi^\beta + N^\beta \partial_r \xi^t \leq O(1) \quad \text{(rotating horizon)}
\]

is one example.

The remaining analysis on the Virasoro algebra is straightforward and it is easily found that the coordinate-space Virasoro algebra has the form of (5.25) with

\[
J[\hat{\xi}] = \frac{1}{2} \times \frac{1}{8\pi G} \int_{r_+} d^{n-2}x \sqrt{\sigma} n^a D_\alpha \hat{\xi}^t,
\]

\[
K[\{\hat{\xi}_1, \hat{\xi}_2\}_\text{SD}] = \frac{1}{2} \times \frac{1}{8\pi G} \int_{r_+} d^{n-2}x \sqrt{\sigma} \left( \frac{4\pi}{\beta} \partial_t \xi_1^t \xi_2^t + \frac{\beta}{\pi} \partial_t \xi_1^t \partial_t \xi_2^t - 2 \partial_t \xi_1^t \partial_t \xi_2^t - 2 \partial_t \partial_t \xi_1^t \xi_2^t \right),
\]

\[
\{\hat{\xi}_1, \hat{\xi}_2\}_\text{SD}^t = \frac{1}{2} \times [-2N \partial_t \xi_1^t \xi_2^t - (1 \leftrightarrow 2)].
\]

The overall factor 1/2 is the result of the same factor in (6.1). But, due to the absence of the relation (5.13), this is not truly a Virasoro algebra because the derivative \( \partial_t \) cannot be integrated by parts and so the manifest \( 1 \leftrightarrow 2 \) antisymmetry is not attained in contrast to the generic definition (2.27). Hence, in order that \( J[\hat{\xi}] \) be a symmetry generator as (2.27), I am required to connect \( \partial_t \xi^t \) with \( \partial_\alpha \xi^t \) for some angular coordinates \( x^\hat{\alpha} \) as follows\(^{14}\):

\[
\partial_t \xi^t = v^\hat{\alpha} \partial_\alpha \xi^t.
\]

But now, the speed \( v^\hat{\alpha} \), which is arbitrary, has no connection to the horizon’s rotation. The last two terms cancel from the integration by parts and the central term (6.4) becomes the same as (5.24) with the differences in the over-all factor 1/2 and with \( v^\alpha \) instead of \( N^\alpha \):

\[
K[\{\hat{\xi}_1, \hat{\xi}_2\}_\text{SD}^a] = \frac{1}{2} \times \frac{1}{8\pi G} \int_{r_+} d^{n-2}x \sqrt{\sigma} \xi_1^t \left( \frac{\beta}{\pi} v^\beta v^\hat{\alpha} \partial_t \partial_\alpha \xi_2^t + \frac{4\pi}{\beta} v^\hat{\alpha} \partial_\alpha \xi_2^t \right)
\]

\[
= \frac{1}{2} \times \frac{1}{8\pi G} \int_{r_+} d^{n-2}x \sqrt{\sigma} \xi_1^t \left( \frac{\beta}{\pi} \partial_t \partial_t \xi_2^t - \frac{4\pi}{\beta} \partial_\alpha \xi_2^t \right).
\]

Then, for the \( Diff \xi^t \) which “lives” at the horizon

\[
\xi^t = \xi^t \left[ t - r_* + \frac{1}{v^\alpha} x^\hat{\alpha} \right]
\]

\(^{14}\)On the other hand, with no angular dependence, the orthogonality disappear, and the whole algebra contains the time-dependent factor \( \exp\{2\pi i(m + n)(t - r_*)/T\} \) as well as other unwanted terms which are proportional to \( (m^2 - n^2) \); This is the same situation as in two-dimensional gravity [34].
the momentum-space Virasoro algebra has the standard form (5.30) with

$$J_0 = 2 \times \frac{A_+}{32\pi G T} \left(1 + \left(\frac{T}{\beta}\right)^2\right), \quad c = 2 \times \frac{3A_+}{4\pi G T}. \quad (6.8)$$

The change of factor $1/2$ in (6.4) to 2 in (5.8) comes from the normalization of the Fourier expansion of $\xi^t$

$$\xi^t = 2 \times \frac{T}{4\pi} \sum_n \xi_n \exp \left\{ \frac{2\pi i n}{T} \left( t - r_* - v^{\phi-1} \phi \right) \right\} \quad (6.9)$$
in order to obtain the correct standard form factor $i(m - n)J_{m+n}$ in (5.27). Notice that furthermore, since $\phi$ should be periodic for $2\pi$ rotation, $v^\phi$ behaves as $2\pi/T$.

### 7 Canonical quantization and black hole entropy

The computation of black hole entropy from the canonical quantization of the classical Virasoro algebra is rather straightforward at this stage.

When one uses the standard canonical quantization rule

$$\left[ L_m, L_n \right] = i\hbar \{ J_m, J_n \}, \quad J_{m+n} \rightarrow L_{m+n}, \quad (7.1)$$

where $L_m$ is a quantum operator, the classical Dirac bracket algebra (5.30) becomes an operator algebra

$$\left[ L_m, L_n \right] = \hbar(m - n) L_{m+n} + \frac{\hbar c}{12} m(m^2 - 1) \delta_{m,-n}. \quad (7.2)$$

By considering the transformation

$$L_m \rightarrow \hbar (\hat{L}_m : + \hbar a \delta_{m,0}), \quad (7.3)$$

(7.2) becomes the standard operator Virasoro algebra for the normal ordered operator $\hat{L}_m :$ ($a$ is some number)

$$[\hat{L}_m : : \hat{L}_n :] = (m - n) \hat{L}_{m+n} : + \frac{c_{\text{tot}}}{12} m(m^2 - 1) \delta_{m,-n} \quad (7.4)$$

with

$$c_{\text{tot}} = \frac{c}{\hbar} + c_{\text{quant}}. \quad (7.5)$$

(See also Ref. [33] for some related discussions.) Here, $c_{\text{quant}}$, which is $O(1)$, is the quantum effect due to operator reordering.
With the Virasoro algebra of : \( \hat{L}_m \) : in the standard form, which is defined on the plane, one can use Cardy’s formula for the asymptotic states [36, 1, 37, 14]

\[
\log \rho(\Delta) \sim 2\pi \frac{1}{6} \left( c_{\text{tot}} - 24 \hat{\Delta}_{\text{min}} \right) \left( \hat{\Delta} - \frac{c_{\text{tot}}}{24} \right),
\]

(7.6)

where \( \hat{\Delta} \) is the eigenvalue, called conformal weight, of : \( \hat{L}_0 \) : for a black hole quantum state \(|\text{black hole}\rangle\) and \( \hat{\Delta}_{\text{min}} \) is its minimum value. When this is expressed in terms of the classical eigenvalue \( \Delta \equiv J_0 \) and the central charge \( c \) through

\[
\hat{\Delta} = \frac{\Delta}{\hbar} - \hbar a
\]

(7.7)

from (7.3) and (7.3), one obtains

\[
\log \rho(\Delta) \sim \frac{2\pi}{\hbar} \sqrt{\frac{1}{6} \left( c - 24 \Delta_{\text{min}} + \hbar c_{\text{quant}} + 24 \hbar^2 a \right) \left( \Delta - \frac{c}{24} - \hbar^2 a - \frac{\hbar c_{\text{quant}}}{24} \right)} = \frac{2\pi}{\hbar} \sqrt{c_{\text{eff}} \Delta_{\text{eff}} / 6 + O(\hbar)}
\]

(7.8)

with

\[
c_{\text{eff}} = c - 24 \Delta_{\text{min}}, \quad \Delta_{\text{eff}} = \Delta - \frac{c}{24}.
\]

(7.9)

This result shows explicitly how the classical Virasoro generator and central charge can give the correct order of the semiclassical BH entropy \( (c = k = 1) \)

\[
S_{\text{BH}} \sim \frac{A_+}{4\hbar G}
\]

(7.10)

since \( c \sim A_+/G \) and \( \Delta \sim A_+/G \); details on the numerical factor \( 1/4 \) of the BH entropy depends on \( \Delta_{\text{min}} \), which has to be put in by hand, and the quantum correction due to reordering gives negligible \( O(1/\sqrt{\hbar}) \) effect to the entropy when one considers the macroscopic ensemble of \( c_{\text{eff}} \Delta_{\text{eff}} \gg 1 \).

Now, let me compute the black hole entropy explicitly. For rotating horizons, (7.8) produces the entropy

\[
S = \log \rho(\Delta) \sim \frac{2\pi}{\hbar} \sqrt{\left( \frac{A_+}{16\pi G} \right)^2 - \frac{A_+}{8\pi G} \frac{T}{\beta} \Delta_{\text{min}}}
\]

(7.11)

with

\[
\Delta_{\text{eff}} = \frac{A_+}{32\pi G} \frac{T}{\beta}.
\]

(7.12)
This gives the BH entropy (7.10) if one takes
\[ \Delta_{\text{min}} = -\frac{3A_+ \beta}{32\pi GT} \] (7.13)
such as
\[ c_{\text{eff}} = \frac{3A_+ \beta}{\pi GT}. \] (7.14)

Note that \( \Delta_{\text{min}} \) has an explicit \( \beta/T \) dependence. But from the fact that this value is outside of the classical spectrum of \( \Delta \geq (16\pi G)^{-1}A_+ \), one can expect that the ground state of this black hole may be described by another class of black holes; or this might be relevant to the non-commutative spacetime near the ground state black hole which, probably, is very light [38].

On the other hand, for non-rotating horizons, there is an additional factor “2” in (6.8) and this has a remarkable consequence for the entropy computation: The entropy from the Cardy formula gives
\[ S \sim \frac{2\pi}{\hbar} \sqrt{\left( \frac{A_+}{8\pi G} \right)^2 - \frac{A_+ T}{4\pi G \beta} \Delta_{\text{min}}} \] (7.15)
with
\[ \Delta_{\text{eff}} = \frac{A_+ T}{16\pi G \beta}. \] (7.16)
This gives the BH entropy (7.10) with a \( T \)-independent ground state
\[ \Delta_{\text{min}} = 0 \] (7.17)
such as
\[ c_{\text{eff}} = \frac{3A_+ \beta}{2\pi GT}. \] (7.18)

This means that for an arbitrary choice of \( T \) and hence \( \nu^a \) the entropy is uniquely defined. Moreover, the arbitrary \( T \) dependences in \( c \) and \( \Delta \) exactly cancel each other and one obtains the \( T \)-independent correct entropy\(^{15}\).

The appropriate ground state is different for the rotating and the non-rotating black holes, but otherwise it has a *universality* for a wide variety of other black holes: The fall-off conditions described in section 3 involve only weak assumptions about the black hole, and once \( \Delta_{\text{min}} \) is

\(^{15}\)This was claimed even in the rotating case by Carlip [3] due to the additionally introduced factor 2, which is unclear in the present context. But his claim is exactly realized in the case of non-rotating case or in the \( c_{\text{eff}} \) and \( \Delta_{\text{eff}} \) even when there is rotation. The connection to the usual \( T = \beta \) relation in the path integral formulation [33, 40] needs further studies.
fixed, say for an isolated, uncharged Kerr black hole, its value is determined also for a large number of other black holes carrying electric or magnetic charge (details will appear elsewhere [41]).

8 Applications

I have shown that the statistical entropies of rotating and non-rotating stationary ($N^r = 0$) black holes through the Cardy formula have a universal form if the fall-off conditions and several fall-off-preserving conditions are satisfied. So, the problem of the entropy computation is reduced to verification of the conditions. My considerations are general enough that almost all the known solutions satisfy the required conditions. In this section I briefly discuss some of these. I shall adopt different units of $G$ depending on the usual conventions for the solutions in the literatures.

a The Kerr and Schwarzschild black holes

The Kerr black hole solution, with one-rotation, in the 4 and higher dimensions is given, in the standard form (3.1) [30, 42] as

$$ds^2 = -\frac{\rho^2 \delta}{\Sigma^2} dt^2 + \frac{\Sigma^2}{\rho^2 \sin^2 \theta} \left( d\phi - \frac{\mu a}{r^{n-5} \Sigma^2} dt \right)^2 + \frac{\rho^2}{\delta} dr^2 + \rho^2 d\theta^2 + r^2 \cos^2 \theta d\Omega_{n-4}^2,$$

(8.1)

where

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \delta = r^2 + a^2 - \frac{\mu}{r^{n-5}}, \quad \Sigma^2 = \rho^2 (r^2 + a^2) + \frac{\mu}{r^{n-5}} a^2 \sin^2 \theta$$

(8.2)

and $\Omega_{n-4}$ is the line element on a unit $n-4$ sphere; $\mu$ and $a$ are the mass and the angular momentum parameters respectively. One can easily check that all the fall-off, fall-off-preserving, and differentiability conditions are satisfied in the $n = 4$ non-extremal Kerr-solution such that it has the universal statistical entropy (7.8) and gives the BH entropy (7.10) with $A_+ = 4\pi(r_+^2 + a^2) = 8\pi Mr_+$, and

$$\Delta_{\text{min}} = -\frac{3\pi}{T} \frac{M^2(M + \sqrt{M^2 - a^2})^2}{\sqrt{M^2 - a^2}}.$$

(8.3)

Here, $M$ and $a$ are the ADM mass and the angular momentum per unit mass of the black hole, respectively and $r_\pm = MG \pm \sqrt{M^2 - a^2}$ ($G = 1$). The higher dimensional solutions have exactly the same results if only the one-rotation solution (8.1) is concerned.

Moreover, since the Schwarzschild solution can be obtained as a non-rotating limit of the Kerr solution, it is a trivial matter to check that the Schwarzschild solution also satisfies all the fall-off and other related conditions. So, in this case also the universal entropy form can
be applied. Note that, the ground state of the non-rotating black hole is contained in the full spectrum of $\Delta^{a=0}$ and obtained as $M \to 0$ limit, which is nothing but the flat spacetime: $\Delta^{a=0} = ((8\pi M^3/T) + (MT/8\pi)) \to 0$.

The lower ground state for the rotating black hole compared to the non-rotating one may be understood qualitatively as follows: Let me consider a black hole with a tiny angular velocity, which is almost static. Then, increase the angular speed adiabatically slowly such as no other physical properties of the black hole are changed. But in order that the horizon is not naked by this adiabatic process, the mass of the black hole should also be simultaneously increased by $\delta M^2 \geq \delta a^2$. In order to accommodate this increase in mass, the vacuum becomes lower without changing the identity of the black hole. But a complete understanding still needs to be discovered.

**b With a cosmological constant**

My analysis can be also generalized to include the cosmological constant (CC) term with only a small modification in the formulas of the preceding sections.

The CC term

\[
S_{CC} = -\frac{1}{16\pi G} \int d^nx N \sqrt{h}(2\Lambda)
\]

added to the action (2.2) changes the Hamiltonian constraints as

\[
\mathcal{H}_t = -\frac{\sqrt{h}}{16\pi G} (R - 2\Lambda) + \frac{16\pi G}{\sqrt{h}} \left( p_{ab} p_{ab} - \frac{1}{n-2} p^2 \right).
\]

But since this additional term does not generate any surface term in the variations, almost all the results of the boundary conditions and the surface deformations in the preceding sections are unchanged. The only exception is the computation of (5.13), which produces the constraints term in the Virasoro algebra (5.23). But in this case again, there is no effect of $\Lambda$ at the horizon: From

\[
\mathcal{H}_t^\Lambda = \mathcal{H}_t^{\Lambda=0} + \frac{\sqrt{h}}{8\pi G} \Delta \mathcal{H}_t = \Lambda \mathcal{H}_t^{\Lambda=0}, \quad \delta_\xi p^r_{\Lambda=0} = \delta_\xi p^r_{\Lambda=0} - \hat{\hat{\xi}} \frac{\sqrt{h}}{16\pi G} h^{rr} \Lambda \mathcal{H}_t \mathcal{H}_t^{\Lambda=0}.
\]

one finds that the variations of $\delta_\xi J_\Lambda[\hat{\hat{\xi}}]$ involving $\delta_\xi p^r_{\Lambda=0}$ is again the constraint term, which is the same as (6.13).

However, if the CC generates its own horizon which is not due to the black hole, it can be treated as another independent application of my original method. The interesting examples are the BTZ solution in $n = 3$ ($\Lambda < 0$) and rotating de-Sitter space solutions ($\Lambda > 0$).
b.1 The BTZ black hole

The BTZ black hole solution \[7\] in \(n = 3\) \((\Lambda < 0)\) is similar to the Kerr solution. But I consider this example since there are some points which are worthy of studying in comparison with the other entropy computations in different contexts.

The BTZ solution is given by the standard form (3.1) with \(G = 1/8\)

\[f^{-2} = N^2 = \frac{(r^2 - r_+^2)(r^2 - r_-^2)}{r^2}, \quad N^r = 0, \quad N^\phi = \frac{r_+ r_-}{r^2}, \quad N_\phi = r_+ r_-, \quad \sigma_{\phi\phi} = r^2.\]  
(8.7)

All the subsidiary conditions are satisfied the same way as in the Kerr-solution and so, the universal entropy formula (7.8) is also applicable here with \(A_+ = 2\pi r_+\) and

\[\Delta_{\text{min}} = -\frac{3\pi l^2}{T} \frac{r_+^2}{r_+^2 - r_-^2} \quad (J \neq 0), \quad \Delta_{\text{min}} = 0 \quad (J = 0).\]  
(8.8)

Here, \(r_\pm = (l/\sqrt{2}) \sqrt{M \pm \sqrt{M^2 - (J/l)^2}}\). The ground state of non-rotating solution is obtained as \(M \to 0\): \(\lim_{M \to 0} \Delta_{J=0} = -(2\pi)^{-1} |M| T = 0\). This is contrary to Strominger’s entropy computation, in which the Virasoro algebra is an algebra for an asymptotic infinity, not for the horizon and the ground state gives \(M = -1\). On the other hand, the factor “2” in the non-rotating case (6.8) corresponds to the “two” copies of the Virasoro algebra for the isometry group at spatial infinity \[3, 4, 7\].

b.2 The rotating de-Sitter space: \(n = 3\) (Kerr-dS\(_3\))

The de-Sitter space of \(\Lambda > 0\) is peculiar in that it has its own horizon, called cosmological horizon, without black holes \[13, 14, 15, 16, 17\]. Moreover, remarkably this space can have a rotating horizon with no lower bound of the mass, which is contrary to the Kerr and the BTZ solutions. The \(n = 3\) solution was studied first in Ref. \[10\] and more recently studied in other contexts in Refs. \[18, 19\]. The rotating de-Sitter solution without black holes exists also for \(n \geq 4\) and can be simply obtained from a zero-mass-black-hole limit in the usual Kerr-de Sitter black hole solution. Moreover, the computation in my method is interesting because the previous computations in the \(n = 3\) case which have used the isometry at spatial infinity, which is hidden inside the horizon, produce complex \(\Delta\) (and imaginary \(c\) for the Chern-Simons formulation) when rotations are involved \[10, 18, 19\] always. But strangely enough, the final result of the entropy is real valued and identical to the usual BH entropy \[13\]. So, it is important to check whether the complex number disappears in all the intermediate steps or not when I treat the correct boundary \(r_+\) not the suspicious boundary at infinity. In this subsection, I first consider \(n = 3\) solution and then consider \(n \geq 4\) in the next subsection.
The rotating de-Sitter in \( n = 3, \Lambda = l^{-2} > 0 \) is given by the standard metric (3.4) with \( (G = 1/8) \)

\[
f^{-2} = N^2 = M - \left( \frac{r}{l} \right)^2 + \frac{J^2}{4r^2}, \quad N^r = 0, \quad N^\phi = -\frac{J}{2r^2}, \quad N_\phi = -\frac{J}{2}, \quad \sigma_{\phi\phi} = r^2, \quad (8.9)
\]

where \( M \) and \( J \) are the mass and angular momentum parameters of the solution, respectively. The solution has only one horizon at \( r_+ = l/\sqrt{2 \sqrt{M + (J/l)^2}} \). Notice that there is no constraint on \( M^2 \) bounded by \( J^2 \) for a horizon to exit: Even a negative value of \( M \) is allowed when \( J \neq 0 \) such as the mass spectrum (ranging from \(-\infty \) to \( \infty \)) is continuous and so there is no mass gap in contrast to the BTZ solution; for \( J = 0 \) case, there is no horizon for \( M < 0 \).

The fall-off forms of the metric are the same as the BTZ solution and so the entropy also has the universal from as (7.8) with \( A_+ = 2\pi r_+ \) and

\[
\Delta_{\text{min}} = -\frac{3\pi}{2T} \quad (J \neq 0), \quad \Delta_{\text{min}} = 0 \quad (J = 0). \quad (8.10)
\]

The static de-Sitter ground state solution is obtained as \( M \to 0_- \): \( \lim_{M \to 0_-} \Delta_{J=0} = (2\pi)^{-1}Tl^2|\Delta_{J=0}| = 0 \). Note that there is no complex number at any step of computation.

### b.3 The rotating de-Sitter solution: \( n \geq 4 \) (massless Kerr-dS\(_n\))

The rotating de-Sitter solution for \( n \geq 4 \) is obtained from the \( M = 0 \) reduction of the Kerr-de Sitter solution [43, 50]. In the standard form (3.1) with \( (n = 4 \text{ for simplicity}; \text{but it is a trivial matter to generalize to higher dimensions}) \), the solution is given by \( (G = 1) \)

\[
N^2 = \frac{\Delta_r \Delta_\theta}{(r^2 + a^2) \Xi^3}, \quad f^2 = \frac{\rho^2}{\Delta_r}, \quad N^r = 0, \quad N^\theta = 0, \quad N^\phi = -\frac{1}{3} \Lambda \frac{a(r^2 + a^2)}{\Xi^2} \sin^2 \theta, \quad \sigma_{\phi\phi} = \frac{(r^2 + a^2)}{\Xi} \sin^2 \theta, \quad \sigma_{\theta\theta} = \frac{\rho^2}{\Delta_\theta}, \quad (8.11)
\]

with

\[
\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Xi = 1 + \frac{\Lambda a^2}{3}, \quad \Delta_r = (r^2 + a^2)(1 - \frac{\Lambda r^2}{3}), \quad \Delta_\theta = 1 + \frac{\Lambda a^2}{3} \cos^2 \theta, \quad (8.12)
\]

which has the cosmological horizon at \( r_+ = \sqrt{3/\Lambda} \).

This is very similar to the Kerr-solution and so one can easily check that all the conditions of the Kerr-solution are still satisfied. So the universal entropy (7.8) can be applied in this case, with \( A_+ = 4\pi r_+^2 \) and

\[
\Delta_{\text{min}} = -\frac{3\pi r_+^3}{4T} \quad (a \neq 0), \quad \Delta_{\text{min}} = 0 \quad (a = 0). \quad (8.13)
\]
The ground state of non-rotating de-Sitter space is obtained \( \Delta^{a=0} = r_+((\pi r_+^2/2T) + (T/8\pi)) \to 0 \) as \( r_+ = \sqrt{3/\Lambda} \to 0 \), which corresponds to giving an infinite moment of inertia to the space.

9 Concluding remarks

I have shown that almost all the known solutions with the horizon have the universal statistical entropy. This is identical to the Bekenstein-Hawking entropy when the appropriate ground states are chosen and the higher order quantum corrections of operator orderings are neglected. Here the existence of the classical Virasoro algebra at the horizon was crucial to the results. The remaining questions are as follows.

1. How can we understand extremal black holes in my method? Can this method explain the discrepancy between the gravity side \[51]^{16}\] and the string theory side \[6] \), which claim different entropies for the extremal black hole?

2. Can my method be generalized to non-stationary metrics such that expanding or collapsing horizons can be treated \[53]\ ?

3. Can gauge and matter fields be introduced without perturbing the universal statistical entropy formula? Can this study give another proof of the no-hair conjecture?

Some of the questions are being studied and will appear elsewhere \[41].

Finally, computing the symplectic structure on the constraint surface \( \mathcal{H}_\mu \approx 0 \) with a horizon boundary through the Dirac bracket method or the symplectic reduction will be an outstanding challenge. The higher order quantum corrections of black hole entropy can be computed by quantizing the classical symplectic structure.

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\[16\] There is a debate on the gravity side also. See Ref. \[52].
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Appendix A  Analysis on the boundary $C$

The variation in $H'[N, N^a]$ due to arbitrary variations in $h_{ab}, n^a, N, N^a$ is

$$
\delta H'[N, N^a] = \text{bulk terms} + \frac{1}{8\pi G} \int_C d^{n-2} x \left[ \frac{16\pi G}{2-n} \sqrt{\sigma} n^r N, \delta p 
+ \delta n^r \left( \partial_r N \sqrt{\sigma} + \frac{16\pi G}{2-n} \sqrt{\frac{\sigma}{h}} N_r p \right) - \frac{16\pi G}{2(2-n)} \sqrt{\frac{\sigma}{h}} n^a n^b n^r N_r p \delta h_{ab} 
+ \delta N^a \left( 16\pi G p_a^r + \frac{16\pi G}{2-n} \sqrt{\frac{\sigma}{h}} n^a p \right) + \frac{16\pi G}{2-n} \sqrt{\frac{\sigma}{h}} \delta h_{ra} n^r N^a \right] \quad (A.1)
$$

Here, I have also assumed the boundary conditions (2.17) and (2.18) in order to remove the problematic terms which persist also on $C$ as in (2.4).

The boundary contributions to the bulk equation of motion are

$$
\{h_{ab}(x), H'[N, N^a]\}_{\text{boundary}} = \delta(r - r_c) \left( \frac{2}{2-n} \sqrt{\frac{\sigma}{h}} n^r N_r h_{ab} \right),
$$

$$
\{p^{ab}(x), H'[N, N^a]\}_{\text{boundary}} = -\delta(r - r_c) \left[ \sqrt{\frac{\sigma}{h}} \frac{2}{2-n} n^r N_r p^{ab} 
+ \frac{2}{2-n} \sqrt{\frac{\sigma}{h}} \left( \delta^{(a} N^b) n^r - \frac{1}{2} n^a n^b n^r N_r - \frac{1}{2} n^{(a} h^{b) r} N_r \right) p - \frac{\sqrt{\sigma}}{16\pi G} n^{(a} h^{b) r} \partial_r N \right]. \quad (A.3)
$$

The boundary contribution to the momentum constraints is

$$
H'_a |_{\text{boundary}} = \delta(r - r_c) \left( \frac{2}{2-n} \sqrt{\frac{\sigma}{h}} n^r h_{ra} p + 2 p^r_a \right). \quad (A.4)
$$

In order to achieve the extremality under the variations with the boundary contributions, I need the boundary conditions on $C$ such that all the terms of $\delta h_{ab}, \delta p^{ab}, \delta N^a$ vanish which would be impossible in general. This can be easily checked within our black hole setup, which shows non-vanishing boundary contributions, $\{p^{\alpha\beta}(x), H'[N, N^a]\}_{\text{boundary}} \leq O(1)$, $\{p^{\alpha\beta}(x), H'[N, N^a]\}_{\text{boundary}} \leq O(1)$, $\{h_{rr}(x), H'[N, N^a]\}_{\text{boundary}} \leq O(1)$, $\{h_{rr}(x), H'[N, N^a]\}_{\text{boundary}} \leq O(1)$, $\{h_{rr}(x), H'[N, N^a]\}_{\text{boundary}} \leq O(1)$.

Finally, I note that there is a special choice of the slicing, called maximal slicing [31, 32, 28], $p = 0$, i.e., $p^{\alpha\beta} = 0$ (but $p^r_a = O(1)$ needs not be zero), in which the Hamiltonian $H'[N, N^a]$ is well defined, i.e., there is no boundary contribution, when the coordinate system $N_r = 0$ is used. But, even in this case, the $Diff$ generator $H'[\hat{\xi}^t, \hat{\xi}^a]$ is still ill-defined: $\{h_{rr}(x), H'[\hat{\xi}^t, \hat{\xi}^a]\}_{\text{boundary}} = \delta(r - r_+)((2-n)\sqrt{h})^{-1}2\sqrt{\sigma}n_r \xi^r h_{rr} \leq O(1)$ [14].

29
Appendix B  Computing $2 \int_{r_+} d^{n-2} \xi_1 \delta_2 p^r_r$

In this Appendix, I compute $2 \int_{r_+} d^{n-2} \xi_1 \delta_2 p^r_r$ of (5.15). From (2.22), one has

$$\delta_2 p^{ab} = \{ p^{ab}, H[\hat{\xi}] \} = -\hat{\xi}^i \sqrt{h} \frac{1}{16\pi G} \left( R^{ab} - \frac{1}{2} h^{ab} R \right) + \sqrt{h} \frac{1}{16\pi G} \left( D^a D^b \hat{\xi}^i - h^{ab} D_c D_c \hat{\xi}^i \right) + \frac{8\pi G \hat{\xi}^i}{\sqrt{h}} h^{ab} \left( p_{cd} p^{cd} - \frac{1}{n-2} p^2 \right) - \frac{32\pi G \hat{\xi}^i}{\sqrt{h}} \left( p^b c^{ap} - \frac{1}{n-2} p^{ab} p \right) + D_c (\hat{\xi}^i p^{ab}) - D_c (\hat{\xi}^i p^{bc}) - D_c (\hat{\xi}^i p^{ac}).$$

(B.1)

In order to determine what is involved in the computation of (5.15), let me expand the integrand of (5.13) as

$$\hat{\xi}^i \delta_2 p^r_r = \hat{\xi}^i f^2 \delta_2 p^{rr} + \hat{\xi}^i \delta_2 h_{ar} p^{ar} + \hat{\xi}^i \delta_2 h_{rr} p^{rr}$$

(B.2)

by writing $p^r_r = h_{br} p^{br}$. [Here and after, I do not restrict to the case of $N^r = 0$ for generality, though I consider $\hat{\xi}^i = 0$ to avoid unnecessary complication.] The second and the third terms are definitely $O(N^2)$ and $O(N)$, respectively. Naively, the first term is $O(N)$ if the fall-off condition (3.13) is preserved by $Diff$. But this is not a trivial matter. Let me compute this first term in detail.

With the help of (B.1), the first term of (B.2) becomes

$$\hat{\xi}^i f^2 \delta_2 p^{rr} = A + B + C + D,$$

where

$$A = -f^2 \hat{\xi}^i \hat{\xi}^j \frac{\sqrt{h}}{16\pi G} \left( R^{rr} - \frac{1}{2} h^{rr} R \right),$$

$$B = f^2 \hat{\xi}^i \hat{\xi}^j \frac{\sqrt{h}}{16\pi G} (D^r D^r \hat{\xi}^i - h^{rr} D_c D_c \hat{\xi}^i),$$

$$C = \frac{8\pi G f^2 \hat{\xi}^i \hat{\xi}^j \frac{\sqrt{h}}{16\pi G} h^{rr} \left( p_{cd} p^{cd} - \frac{1}{n-2} p^2 \right) - \frac{32\pi G f^2 \hat{\xi}^i \hat{\xi}^j \frac{\sqrt{h}}{16\pi G} \left( p^b c^{ap} - \frac{1}{n-2} p^{ab} p \right) + D_c (\hat{\xi}^i p^{br}) - f^2 \hat{\xi}^i D_c (\hat{\xi}^i p^{bc}) - f^2 \hat{\xi}^i D_c (\hat{\xi}^i p^{ac}).$$

(B.3)

Term $B$ reduces to

$$B = -(16\pi G)^{-1} f^3 \hat{\xi}^i (h^{rr})^2 \partial_r \hat{\xi}^i \partial_r \sqrt{\sigma} \leq O(N).$$

(B.4)

In order to compute the term $A$, one needs to compute the curvature tensor.

$$R^a_{\alpha \beta} = -\partial_r \Gamma^a_{\alpha \beta} + \Gamma^a_{\alpha \beta} + O(N^{-2}) = O(N^{-3}),$$

$$R^a_{\alpha \beta} = \partial_r \Gamma^a_{\alpha \beta} + \Gamma^a_{\alpha \beta} + O(1) = O(N^{-1}),$$

$$R = h^{rr} R_{rr} + h^a \beta R_{a \beta} = O(N^{-1}).$$

(B.5)
From this result, it is easy to see that

$$A = -\frac{f^3}{16\pi G \hat{\xi}_1 \hat{\xi}_2 \sqrt{\sigma}} (h_{rr})^2 R_{rr} - \frac{1}{2} h_{rr} R \leq O(N).$$ \hfill (B.6)

Term $D$ reduces to

$$D = -f^2 \hat{\xi}_1 D_c (\hat{\xi}_2 p_{rr}) + 2f^2 \hat{\xi}_1 D_\alpha (\hat{\xi}_2 p^{\alpha r}).$$ \hfill (B.7)

The first term of this equation becomes

$$- f^2 \hat{\xi}_1 \sqrt{h} \partial_r (\sqrt{h} \hat{\xi}_2) p_{rr} + f^2 \hat{\xi}_1 \hat{\xi}_2 (\partial_r p_{rr} + 2\Gamma_{rr} p_{rr} + 2\Gamma_{r\alpha} p^{\alpha r}) \leq O(N).$$ \hfill (B.8)

On the other hand, the second term of (B.7) becomes

$$2f^2 \hat{\xi}_1 h_{rr} h^{\alpha \beta} D_\alpha (\hat{\xi}_2 p_{r\beta}) = 2f^2 \hat{\xi}_1 h_{rr} h^{\alpha \beta} \left[ \partial_\alpha (\hat{\xi}_2 p_{r\beta}) + \Gamma_{\alpha r} \hat{\xi}_2 p_{\beta} - \Gamma_{\alpha \beta} \hat{\xi}_r p_{2\alpha} \right].$$ \hfill (B.9)

Now, from the computation of each term inside the bracket $[\ ]$,

$$\partial_\alpha (\hat{\xi}_2 p_{r\beta}) \leq O(1), \quad \Gamma_{\alpha r} \hat{\xi}_2 p_{\beta} \leq O(1), \quad \Gamma_{\alpha \beta} \hat{\xi}_r p_{2\alpha} \leq O(N^{-1}), \quad \Gamma_{\alpha \beta} \hat{\xi}_r p_{2\alpha} \leq O(N^{-1}).$$ \hfill (B.10)

(B.9) becomes

$$2f^2 \hat{\xi}_1 h_{rr} h^{\alpha \beta} D_\alpha (\hat{\xi}_2 p_{r\beta}) \leq O(N).$$ \hfill (B.11)

This result is contrary to the naive expectation $D_\alpha \sim \partial_\alpha$ such that this is $O(N^3)$: This implies that $\partial_r$, which is hidden in $D_\alpha$ through $\Gamma_{\alpha r}$, makes $\partial_r O(N^3) = O(N)$.

$C$ is the most important term in (B.3).

The first term of $C$ reduces to

$$\frac{8\pi G \hat{\xi}_1 \hat{\xi}_2}{\sqrt{h}} \left[ p_{\alpha \beta} p^{\alpha \beta} - \frac{1}{n-2} p_{\alpha} p^{\alpha} + O(N^{-3}) \right] \leq O(1).$$ \hfill (B.12)

The second term reduces

$$- \frac{32\pi G \hat{\xi}_1 \hat{\xi}_2}{\sqrt{h}} \left[ -\frac{1}{n-2} p_{r}^{\alpha} p^{\alpha} + O(N^{-2}) \right] \leq O(N).$$ \hfill (B.13)

[Notice that the $n = 2$ is meaningless in this formula, a separate consideration is required for that case.]

Therefore, $C$ becomes

$$C = \frac{8\pi G \hat{\xi}_1 \hat{\xi}_2}{\sqrt{h}} \left( p_{\alpha \beta} p^{\alpha \beta} - \frac{1}{n-2} p_{\alpha} p^{\alpha} \right) + O(N) \leq O(1).$$ \hfill (B.14)
On the other hand, since the Hamiltonian constraint $H_t$ becomes, near the horizon,
\[
H_t = \frac{16\pi G}{\sqrt{h}} \left( p_{\alpha\beta} p^{\alpha\beta} - \frac{1}{n-2} p^{\alpha}_{\alpha} p^{\beta}_{\beta} \right) + O(N^{-2}) \leq O(N^{-3})
\] (B.15)
from (B.3) and (B.12), $C$ is nothing but
\[
C = \frac{\hat{\xi}_1 \hat{\xi}_2}{2} H_t + O(N) \leq O(1).
\] (B.16)
In summary, one has
\[
2 \oint_{r^+} d^{n-2} \hat{\xi}_1^r \delta_2 p^r_r = 2 \oint_{r^+} d^{n-2} (A + B + C + D)
= \oint_{r^+} d^{n-2} \hat{\xi}_1^r \hat{\xi}_2^t H_t + O(N),
\] (B.17)
\[
A, B, D \leq O(N).
\] (B.18)
Notice that I have *not* used any of the constraint equations $H_t \approx 0, H_a \approx 0$.

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