Charged, rotating black holes in Einstein-Gauss-Bonnet gravity

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Abstract

We consider the Einstein-Gauss-Bonnet equations in five dimensions including a negative cosmological constant and a Maxwell field. Using an appropriate Ansatz for the metric and for the electromagnetic fields, we construct numerically black holes with two equal angular momenta in the two orthogonal space-like planes of space-time. Families of such solutions, labeled by the angular momentum and by the electric charge are obtained for many representative intervals of the Gauss-Bonnet coupling constant $\alpha$. It is argued that, for fixed values of $\alpha$, the solutions terminate into extremal black holes at ($\alpha$-dependent) critical values of the angular momentum and/or of the electric charge. The influence of the Gauss-Bonnet coupling constant, of the charge and of the cosmological constant on the thermodynamics of the black holes and on their domain of existence is analyzed.

1 Introduction

There has been a huge activity in the study of fundamental interactions in space-times involving more than four dimensions in the last years. The standard Einstein-Hilbert action for $d$-dimensional space-times (with $d > 4$) is not the only choice leading to second order equations of motion. Suitable combinations of terms with higher powers in the curvature can be added such that second order field equations result. In the case $d = 5$, the most general theory of gravity possessing this feature is Einstein-Gauss-Bonnet (EGB) gravity. Higher derivative curvature terms occur in many contexts such as in semi-classical quantum gravity and in the effective low-energy action of superstring theories. The Gauss-Bonnet term appears as the first curvature stringy correction to general relativity [1, 2] when assuming that the tension of a string is large as compared to the energy scale of other variables. Since black holes seem to be the most fundamental solutions of the conventional four-dimensional theory of General Relativity, the construction of such solutions in theories formulated in $d$-dimensions and involving higher derivative curvature terms is a natural generalisation. The inclusion of the Gauss-Bonnet term in the theory leads to new features of the solutions (see e.g. [3, 4] for reviews and further references). Due to the non-linearity of the equations, however, it is very difficult to find non-trivial exact analytical solutions of the field equations with higher derivative terms. The main solution existing in a closed form is the Gauss-Bonnet counterpart of the Schwarzschild-Tangherlini black hole [5, 6]. Static Anti-de Sitter (AdS) black hole solutions in EGB gravity are also known in closed form presenting a number of interesting features (see e.g. [7, 8, 9, 10] and references therein). It is of interest to generalize these solutions to the rotating case. In five dimensions, rotating solutions with a negative cosmological constant have been considered in [11] within a perturbative approach. The numerical construction of rotating black holes (in 5-dimensional EGB gravity) with equal angular momentum in two orthogonal planes was addressed in [12, 13] and [14], respectively for asymptotically Anti-de Sitter (AdS), de Sitter (dS) and flat space-time. In [12, 13] the emphasis was put on the construction of the action and the counterterms necessary to make the variational principle and asymptotic charges well defined. Families of black holes were constructed and the physical parameters characterizing them were estimated numerically. A systematic study of the domain
of existence of the asymptotically flat black holes with respect to their rotational parameter and the Gauss-Bonnet coupling constant is reported in [14]. It turns out that for fixed value of the Gauss-Bonnet coupling constant, the rotating solutions exist up to a maximal value of the angular velocity at the horizon. The construction of the domain of existence leads to numerical difficulties at the approach of the limiting extremal configuration.

In this paper, we extend the study of the domain of existence of the rotating black holes in the presence of a negative cosmological constant (AdS case). For a vanishing cosmological constant, the solutions exist for arbitrary values of the Gauss-Bonnet coupling, which we call $\alpha$ in the following. When a negative cosmological constant is supplemented to EGB gravity, the solutions exist only on a limited interval of $\alpha$. This occurrence of a maximal value for $\alpha$ is examined within our approach of the field equations. In addition to the purely gravitating EGB theory, we further supplement the EGB action by a Maxwell term. Solving the corresponding field equations we are able to construct families of charged, rotating black holes of the Einstein-Gauss-Bonnet-Maxwell equations. The influence of the electromagnetic charge on the domain of existence of the black holes is emphasized.

In section 2 we give the model, we present the Ansatz and also discuss the boundary conditions. Several families of exact solutions are reviewed in section 3. Our results are described in section 4 and illustrated by several figures. Section 5 gives a summary and conclusions.

2 The Model

2.1 The action and field equations

We consider the Einstein-Gauss-Bonnet-Maxwell action with a cosmological constant $\Lambda$ given by

$$I = \frac{1}{16\pi G} \int_M d^5x \sqrt{-g} \left( R - 2\Lambda + \frac{\alpha}{4} L_{GB} - F^2 \right),$$

(2.1)

where $R$ is the Ricci scalar, $F^2$ is the standard Maxwell density and $L_{GB}$ represents the Gauss-Bonnet density:

$$L_{GB} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\sigma\tau}R^{\mu\nu\sigma\tau}.$$

(2.2)

In (2.1), $G$ and $\alpha$ denote the Newton constant and the Gauss-Bonnet coupling constant, respectively. It is usual to relate the cosmological constant to the AdS radius $\ell$ according to $\Lambda = -(d-2)(d-1)/(2\ell^2)$. The notations and conventions are the same as in [12].

The variation of the action (2.1) with respect to the metric tensor and the electromagnetic fields leads to the field equations

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} + \Lambda g_{\mu\nu} + \frac{\alpha}{4} H_{\mu\nu} = T_{\mu\nu}, \quad \partial_{\mu}(\sqrt{-g}F^{\mu\nu}) = 0$$

(2.3)

where $T_{\mu\nu}$ is the electromagnetic stress-momentum tensor while the Gauss-Bonnet tensor is given by

$$H_{\mu\nu} = 2(R_{\mu\rho\sigma\tau}R^{\rho\sigma\tau}_{\nu} - 2R_{\mu\rho}\varepsilon^{\rho\sigma\tau} - 2R_{\mu\rho\sigma\tau}\varepsilon^\rho\sigma\tau + RR_{\mu\nu}) - \frac{1}{2}L_{GB}g_{\mu\nu}. $$

(2.4)

For a well-defined variational principle, one has to supplement the action (2.1) with the Gibbons-Hawking surface term

$$I_b^{(E)} = -\frac{1}{8\pi G} \int_{\delta M} d^4x \sqrt{-\gamma} K,$$

(2.5)

and its counterpart for Gauss-Bonnet gravity [2]

$$I_b^{(GB)} = -\frac{\alpha_2}{16\pi G} \int_{\delta M} d^4x \sqrt{-\gamma} \left( J - 2G_{ab}K^{ab} \right).$$

(2.6)
where $\gamma_{\mu\nu}$ is the induced metric on the boundary, $K$ is the trace of the extrinsic curvature of the boundary, $G_{ab}$ is the Einstein tensor of the metric $\gamma_{ab}$ and $J$ is the trace of the tensor

$$J_{ab} = \frac{1}{3}(2KK_{ac}K_{b}^{c} + K_{cd}K^{cd}K_{ab} - 2K_{ac}K^{cd}K_{db} - K^{2}K_{ab}) \).$$

2.2 The Ansatz

While the most general EGB rotating black holes would possess two independent angular momenta and a more general topology of the event horizon, we restrict ourselves here to configurations with equal magnitude angular momenta and a spherical horizon topology. The suitable metric Ansatz reads \cite{16}

$$ds^2 = \frac{dr^2}{f(r)} + g(r)d\theta^2 + h(r)\sin^2\theta(d\varphi - w(r)dt)^2 + h(r)\cos^2\theta(d\psi - w(r)dt)^2$$

$$+ (g(r) - h(r))\sin^2\theta(d\varphi - d\psi)^2 - b(r)dt^2,$$

where $\theta \in [0, \pi/2]$, $(\varphi, \psi) \in [0, 2\pi]$, $r$ and $t$ being the radial and time coordinates, respectively. For such solutions, the isometry group is enhanced from $R \times U(1)^2$ to $R \times U(2)$, where $R$ refers to the time translations. This symmetry enhancement allows to transform the Einstein-Gauss-Bonnet equations into a system of ordinary differential equations (ODEs). In our construction of the solutions, we fix the residual freedom on the coordinate $r$ by taking $g(r) = r^2$. The variable $r$ defined in this way is the counterpart of the Schwarzschild radial coordinate. Note that in several papers dealing with black holes in higher dimensions (see e.g. \cite{16}) isotropic coordinates are adopted.

When the Maxwell term is involved, the ansatz \cite{2.8} has to be supplemented with a consistent parametrization of the electromagnetic field. The vector field is parametrized in terms of two functions according to

$$A_\mu dx^\mu = V(r)dt + A(r)(\sin^2\theta d\varphi + \cos^2\theta d\psi)$$

where $V(r)$, $A(r)$ represent respectively the electric and magnetic potentials. The ansatz \cite{2.8}, \cite{2.9} transforms the set of Einstein-Gauss-Bonnet-Maxwell equations into a system of six differential equations for the radial functions $f(r)$, $b(r)$, $h(r)$, $w(r)$ and $V(r)$, $A(r)$. These equations can be combined in such a way that the equation corresponding to the metric function $f(r)$ is of first order. The equation corresponding to the electric potential depends only on $dV/dr$ and can be integrated in terms of the other functions. For all the other functions the equations are of second order. The form of these equations is involved and not given here; it can be found e.g. in \cite{14} \cite{15}. In the following the prime will denote the derivative with respect to $r$.

2.3 Boundary conditions

We first discuss the regularity properties of the metric fields at the horizon (the conditions for the electromagnetic potentials will be given afterwards). We are interested in solutions possessing a regular horizon at $r = r_h$. This requires the conditions $f(r_h) = 0$, $b(r_h) = 0$. The fields can be expanded around the horizon leading to

$$f(r) = f_1(r - r_h) + O(r - r_h)^2,$$

$$b(r) = b_1(r - r_h) + O(r - r_h)^2,$$

$$h(r) = h_0 + O(r - r_h),$$

$$w(r) = w_h + w_1(r - r_h) + O(r - r_h)^2,$$

where $f_1, b_1, h_0$ are constants which are determined numerically while $w_h = w(r_h)$ (we also use the notation $\Omega \equiv w_h$) represents the angular velocity at the horizon. This parameter controls the rotation of the black hole and has to be given by hand. The regularity of the fields at the horizon leads to an extra condition, say $\Gamma_1(f, b, b', h, h', w, w')_{r=r_h} = 0$, which needs to be imposed. The polynomial $\Gamma_1$ is lengthy and is not given explicitly here.
In the asymptotic region, the metric is assumed to be asymptotically flat if \( \Lambda = 0 \) or Anti-de Sitter if \( \Lambda < 0 \). The dominant terms of the asymptotic expansions of the fields are (see [12])

\[
f(r) = \frac{r^2}{\alpha} \left( 1 - \sqrt{1 - \frac{2\alpha}{\ell^2}} \right) + 1 + \frac{f_2}{r^2} + O(1/r^4) \; , \; b(r) = \frac{r^2}{\alpha} \left( 1 - \sqrt{1 - \frac{2\alpha}{\ell^2}} \right) + 1 + \frac{b_2}{r^2} + O(1/r^4) \tag{2.12}
\]

\[
h(r) = r^2 + \ell^2 \frac{f_2 - h_2}{2r^2} \left( 1 + \sqrt{1 - \frac{2\alpha}{\ell^2}} \right) + O(1/r^6) \; , \; w(r) = \frac{w_4}{r^4} + O(1/r^8) \tag{2.13}
\]

Imposing the black hole condition, the regularity at the horizon, the asymptotic form of the metric and fixing the angular velocity at the horizon through \( \Omega \) leads to a consistent set of seven boundary conditions specifying, in principle, a solution once \( r_h \) and \( \alpha \) are chosen. The formula (2.12) reveals that the expansion holds for \( \alpha < \ell^2/2 \). In the limiting case \( \alpha = \ell^2/2 \) the asymptotic expansions are different; e.g. the dominant term of the rotating function \( w(r) \) is \( O(1/r^2) \) and for the metric function \( h \) we have \( h(r) = r^2 + H_0 + O(1/r^2) \). This contrasts with (2.12). The corresponding model, called Chern-Simons gravity, was studied in [19] and more recently in [20].

In the presence of an electromagnetic field, the boundary conditions for the functions \( V(r) \) and \( A(r) \) can be fixed as follows: at \( r = r_h \) we set \( V(r_h) = 0 \) and the condition for the fields to be regular at the horizon leads to an equation \( \Gamma_2(f,b,b',h,h',w,w',V,A,A')_{r=r_h} = 0 \). The polynomial \( \Gamma_2 \) (which comes out to be the same for the equations for \( V(r) \) and \( A(r) \) and is again too involved to be given explicitly) is independent from \( \Gamma_1 \) mentioned above. In the asymptotic region, the electromagnetic potential has the following form

\[
V(r) = V_{\text{inf}} + \frac{q}{2r^2} + O(1/r^4) \; , \; A(r) = \frac{q_m}{2r^2} + O(1/r^4) \; , \tag{2.14}
\]

where \( V_{\text{inf}} \), \( q \) and \( q_m \) are constants. In the absence of rotation (i.e. if \( w(r) \equiv 0 \)) the field equation corresponding to the magnetic potential is trivially fulfilled by \( A(r) = 0 \). Practically, we impose \( V'(r \to \infty) = -q/r^3 \) with a given value of the electric charge \( q \). The last boundary condition consists in imposing the product \( r^2A(r) \) to approach a constant. The value of this constant, interpreted as the magnetic charge (say \( q_m \)) of the solution, depends on \( \alpha, q, \Omega \) and can be extracted from the numerical data. Static solutions have \( \Omega = q_m = 0 \).

### 2.4 Physical quantities

Physical quantities characterizing the black holes can then be extracted from the knowledge of the different fields. In particular, the Hawking temperature \( T_H \), the area \( A_H \) of the horizon are given by

\[
T_H = \frac{\sqrt{f_1b_1}}{4\pi} \; , \; A_H = V_3 r_h^2 \sqrt{h(r_h)} \; . \tag{2.15}
\]

The energy \( E \) and the angular momentum \( J \) are given by

\[
E = \frac{V_3}{8\pi G} \frac{f_2 - 4b_2}{2} \sqrt{1 - \frac{2\alpha}{\ell^2}} \; , \; J = \frac{V_3}{8\pi G} w_4 \sqrt{1 - \frac{2\alpha}{\ell^2}} \; . \tag{2.16}
\]

In the figures in the next sections \( T_H \) will be given in units of \( 4\pi \) and \( E, J \) in units of \( V_3/(8\pi G) \). The electromagnetic properties of charged, rotating solutions can be characterized, namely by the electric charge \( Q \), the magnetic moment \( \mu_m \) and the gyromagnetic ratio \( g \)

\[
Q = \frac{V_3}{8\pi G} q \; , \; \mu_m = \frac{V_3}{8\pi G} q_m \; , \; g = 2\frac{E\mu_m}{QJ} \; . \tag{2.17}
\]

Another relevant quantity for the analysis of the thermodynamics is the entropy \( S \) of the black hole. It can be expressed in terms of the horizon \( r_h \) and of the value \( h_h \equiv h(r_h) \) of the metric function \( h(r) \) at the
Figure 1: Profiles of the solutions with $w_h = 0.0$ and $w_h = 0.68$ for $\alpha = 2$ (left) and corresponding profiles of $b''$ and of the Ricci scalar $R$ (right).

The study of $S$ as a function of the temperature reveals some properties about the stability through the specific heat $C_\Omega$ which is given by:

$$C_\Omega = T_H (\partial S / \partial T)_\Omega H.$$

(2.19)

3 Known solutions

In this section, we review the explicit solutions of the equations which are available in several specific limits. The fact that we recover these solutions with our numerical techniques provides a good crosscheck of the accuracy and robustness of our method.

3.1 Myers-Perry solutions

We set $\Lambda = -6/\ell^2$ where $\ell$ is the AdS radius. For pure Einstein gravity, the Myers-Perry (MP) black holes [21] have the form

$$f = 1 + \frac{r^2}{\ell^2} - \frac{2M\Xi}{r^2} + \frac{2Ma^2}{r^4}, \quad h = r^2 \left(1 + \frac{2Ma^2}{r^4}\right), \quad b = \frac{r^2 f}{h}, \quad w = \frac{2Ma}{r^2 h}$$

with $\Xi = 1 - a^2/\ell^2$. The parameter $a$ is related to the angular momentum and $M$ is related to the horizon $r_h$ with $f(r_h) = 0$. Setting for simplicity $r_h = 1$ the family of MP solutions present the following properties:

$$f'(r_h) = \frac{2(2a^2(\ell^4 + 2\ell^2 + 1) - \ell^4 - 2\ell^2)}{\ell^2(a^2(\ell^2 + 1) - \ell^2)}, \quad b'(r_h) = \frac{-2(2a^2(\ell^4 + 2\ell^2 + 1) - \ell^4 - 2\ell^2)}{\ell^4}$$
Figure 2: Some parameters characterizing the rotating solutions for $\alpha = 2.0$ as functions of $J$.

$$w(r_h) = a \frac{1 + \ell^2}{\ell^2}, \quad M = \frac{1 + \ell^2}{2(\ell^2 - a^2(1 + \ell^2))}.$$  

From now on we define $\Omega \equiv w_h \equiv w(r_h)$. In particular, the solution is singular for $a^2 = \ell^2/(1 + \ell^2)$. The rotating BH are therefore limited to $a^2 < \ell^2/(1 + \ell^2)$. Let $\Delta \equiv a^2 - (\ell^2(\ell^2 + 2)/2(\ell^2 + 1)^2)$. For $\Delta < 0$ the solutions have $f'(r_h) > 0$ and for $\Delta > 0$ they have $f'(r_h) < 0$ and admit a second horizon for $r = \tilde{r}_h > 1$.

Since we want to construct the solutions from the event horizon to infinity, it make sense to consider the solutions only for $\Delta < 0$ and for $r \in [1, \infty]$. In the limit $\Delta = 0$ the black hole is extremal. The corresponding values of the parameters $a, M$, say $a_{ex}, M_{ex}$, are then fixed in terms of the AdS radius $\ell$:

$$a_{ex}^2 = \frac{\ell^2(\ell^2 + 2)}{2(\ell^2 + 1)^2}, \quad M_{ex} = \frac{1 + \ell^2}{\ell^2}. \quad (3.1)$$

The energy and angular momentum of the extremal black holes can further be determined and read

$$E_{ex} = \frac{V_3}{8\pi G} \frac{6\ell^4 + 13\ell^2 + 8}{2\ell^4}, \quad J_{ex} = \frac{V_3}{8\pi G} \frac{\sqrt{2(\ell^2 + 2)(1 + \ell^2)}}{\ell^3}. \quad (3.2)$$

### 3.2 Anti-de Sitter-Reissner-Nordstöm solutions

Static charged black holes are known in an explicit form. In the limit $w(r) = A(r) = 0$, the field equations are solved by

$$f(r) = b(r) = 1 + \frac{r^2}{\ell^2} - \frac{2M}{r^{d-3}} + \frac{q^2}{2(d-2)(d-3)r^{2(d-3)}}, \quad h(r) = g(r) = r^2, \quad V(r) = \frac{q}{(d-3)r^{d-3}}. \quad (3.3)$$

where $q$ is a parameter related to the electric charge. Setting for definiteness $\ell^2 = \infty$ and $r_h = 1$, we find $M = (12 + q^2)/24$ and $T_H = (2 - q^2/6)/4\pi$. The black hole is extremal when the charge become large enough: i.e. for $q = \sqrt{12}$. 

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Figure 3: Dependence of the angular momentum $J$ on the temperature $T_H$ for (asymptotically flat) rotating black hole with $\alpha = 0, 0.35, 1.2$ (solid lines) and the charged counterparts (with $Q = 1$, dashed lines) (left). Dependence of $T_H$ on $w_h$ for rotating black hole with $\alpha = 0, 0.35, 1.2$ (solid lines) and the charged counterparts (with $Q = 1$, dashed lines) (right).

### 3.3 Static Einstein-Gauss-Bonnet solutions

Concentrating now on the EGB equations and abandoning rotations (i.e. with $w(r) = A(r) = 0$) a family of exact solutions [5, 6, 9, 10] exists with:

$$f = b = 1 + \frac{r^2}{\alpha}(1 - \sqrt{1 + 2\alpha\left(\frac{2Mr}{r^4} - \frac{1}{\ell^2} - \frac{q^2}{12r^6}\right)}) , \quad g = h = r^2 , \quad w = 0 . \quad (3.4)$$

Setting $q = 0$ and $\ell = \infty$ (i.e. $\Lambda = 0$) shows that these solutions exist for arbitrary values of $\alpha$. If $\Lambda < 0$, they exist for $\alpha \leq \frac{\ell^2}{2}$. Setting $r_h = 1$, the charged EGB static black holes have

$$M = \frac{12 + 6\alpha + q^2}{24} + \frac{1}{2\ell^2}, \quad b'(r_h = 1) = \frac{12 - q^2}{6(1 + \alpha)} + \frac{4}{\ell^2(1 + \alpha)} \quad (3.5)$$

Knowing that black hole solutions exist in these two particular cases, $\alpha = 0$ and $\Omega = 0$ we expect that rotating black holes of the Einstein-Gauss-Bonnet equations should exist in some domain of the $\alpha - \Omega$-plane. The purpose of the next section is to determine this domain with some accuracy.

### 4 Numerical results

To our knowledge, no explicit solutions exists in the generic case $w_h > 0$, $\alpha > 0$. The system of differential equations has to be solved numerically or perturbatively (e.g. using the angular momentum as expansion parameter, see [11]). We solved the system numerically by using a collocation method [22] for numerous values of $\alpha$ and of $w_h$. For the main part of the analysis we set $r_h = 1$ without loss of generality since this corresponds simply to fixing the scale of the radial variable $r$. The study of the quantity $b'(r_h, w_h) \equiv \frac{db}{dr} |_{r = r_h, w(w_h) = w_h}$ (i.e. the value of the derivative $db/dr$ for the solution with $w(r_h) = w_h$ at the horizon) is of some interest for the understanding of the results. Referring to the case $\alpha = 0$ and using an argument of continuity, it could be expected that a branch of rotating solutions exists for $\alpha > 0$, stopping at a maximal value $w_m(\alpha)$ such that $b'(r_h, w_m) = 0$. This implies that an extremal black hole with $T_H = 0$ is approached.
4.1 Asymptotically flat solutions.

The solutions corresponding to \( \Lambda = 0 \) have been studied in [14], but we find it useful to summarize the main features for completeness. Also, the deviation of the pattern of AdS solutions from the asymptotically flat solutions will clearly appear. Fixing \( \alpha > 0 \) and increasing \( w_h \) gradually, rotating black holes can be constructed up until a maximal value of \( w_h \), i.e. \( w_h = w_m(\alpha) \). Profiles of the solutions corresponding to \( \alpha = 2 \) are presented in Fig. 4 for the non-rotating solution \( w_h = 0 \) and \( w_h \approx 0.68 \) (for \( \alpha = 2 \), we have \( w_m \approx 0.72 \)). The study of the evolution of the parameters \( b'(r_h) \), \( f'(r_h) \) as functions of \( w_h \) suggests a natural explanation.
for this result and reveals the pattern of the solutions. For $w_h = 0$, we have $b'(r_h = 1) = 2/(1 + \alpha)$, as can be computed from the explicit solution (3.4). Then, increasing $w_h$, our numerical results show that $b'(r_h)$, $f'(r_h)$ both decrease monotonically and reach the value zero in the limit $w_h \to w_m$. At the same time, the ratio $f''(r_h)/f'(r_h)$ becomes very large. This critical phenomenon is illustrated in Fig. 2 for $\alpha = 2.0$, where some physical quantities characterizing the solutions have been plotted as function of the angular momentum.

Reiterating this construction for different values of $\alpha$ confirms that the families of rotating solutions of the Einstein-Gauss-Bonnet equations systematically approach an extremal black hole for a maximal value of the angular velocity $w_h$. It is then natural to study how the physical properties of the black holes are affected by the Gauss-Bonnet parameter $\alpha$. This is illustrated in Fig. 3: the left part represents a temperature-angular momentum plot for four values of $\alpha$ (solid lines). The right part of Fig. 3 shows the temperature as a function of $w_h$. Note that the lines $\alpha = 0$ are determined analytically by means of

$$T_H = \frac{2(1 - 2w_h^2)}{\sqrt{1 - w_h^2}} , \quad J = \frac{w_h}{1 - w_h^2} , \quad w_h \in [0, 1/\sqrt{2}] .$$

The different curves show the non-trivial dependence on the solutions on $\alpha$. In particular $w_m$ and $J_m$ are not monotonic functions of $\alpha$. For the slowly rotating black holes the Hawking temperature decreases while $\alpha$ increases. This statement clearly does not hold for the solutions with large angular momentum. The natural question is then to attempt to determine the domain of existence of rotating EGB black holes in the $\alpha - \Omega$-parameter space. This is summarized in Fig. 4 (left), where the values $w_m$ are plotted as a function of $\alpha$. The angular momentum of the extremal black hole is also given. This domain turns out to be the region below the solid black line of the figure (representing $w_h$) or equivalently below the solid red line if we consider it from the point of view of the angular momentum $J$. The corresponding energy of the limiting solution is plotted in Fig. 4 (right). The effect of the Gauss-Bonnet interaction is clearly perceptible in these plots. The analysis was limited to $\alpha \in [0, 2]$ since the Gauss-Bonnet term is supposed to come out as a correction to the Einstein gravity. However, we found families of solutions presenting the same pattern even for larger values of $\alpha$. 
Figure 7: The entropy $S$ and horizon $r_h$ as functions of the temperature for $\Lambda = 0$, $\Omega = 0.2$ and three different values of $\alpha$ (left). The mass as function of the temperature for $\Lambda = 0$, $\Omega = 0.2$ and three different values of $\alpha$ (right).

Figure 8: The entropy $S$ and horizon $r_h$ as functions of the temperature for $\ell = 2$, $\Omega = 0.2$ and three different values of $\alpha$ (left). The mass as function of the temperature for $\ell = 2$, $\Omega = 0.2$ and three different values of $\alpha$ (right).

4.2 Asymptotically Anti-De Sitter solutions

We now discuss the asymptotically AdS solutions (i.e. corresponding to $\ell < \infty$). The numerical results show that they approach continuously the asymptotically flat solutions in the limit $\ell \to \infty$. Fig. 4 contains the data for $\ell = \infty$, $\ell = 4$ and $\ell = 2$. The domain of existence of the solutions varies smoothly with the AdS radius $\ell$. The influence of the cosmological constant on the temperature and angular momentum is apparent in Fig. 5 where the curves are reported for two values of $\alpha$ and for $\ell = 2, 4$ and $\ell = \infty$. As pointed out
already, the main difference between the flat and AdS black holes resides in the fact that the asymptotically AdS black holes exist only for $\alpha < \ell^2/2$, irrespectively of their angular momentum. Constructing a branch of solutions with fixed $w_h$ while increasing $\alpha$ reveals that the asymptotic parameters $f_2, b_2$ and $w_4$ all diverge in the limit $\alpha \to \ell^2/2$ while the mass and angular momentum (2.16) remain finite. This is illustrated in Fig. 6. In fact the case $\alpha = \ell^2/2$ corresponds to Chern-Simons gravity [19]. The solutions in this case will be discussed elsewhere [23].

### 4.3 Thermodynamics

Above we have discussed families of solutions with fixed event horizon $r_h$ and examined the influence of the other parameters on it. In order to analyze the thermodynamics of the black holes, one usually fixes $\alpha$ and $\ell$ and constructs the families of solutions obtained by varying $r_h$. Here we emphasize the influence of the Gauss-Bonnet constant $\alpha$ and of the AdS radius $\ell$ on the pattern of the solutions. In this numerical construction, we keep the horizon angular velocity $\Omega$ fixed (more precisely we set $\Omega = 0.2$). The case $\Lambda = 0$ was presented in detail in [14]. In Fig. 7 we give the entropy (left) and the mass (right) of asymptotically flat black holes corresponding to three different values of $\alpha$. As pointed out in [14], the non-vanishing Gauss-Bonnet coupling constant leads to a branch of stable black holes with $C\Omega > 0$. These possess small horizons. Note that it turns out to be difficult to solve the equations for $r_h \to 0$. We hence limit our analysis to $r_h > 0.1$. Our results confirm that no stable rotating black holes exist for $\alpha = 0$ and that increasing $\alpha$ leads to the emergence of a stable branch corresponding to small horizons. The branch of stable black hole gets larger while $\alpha$ is increased. These results contrast drastically with the corresponding asymptotically AdS black holes. The entropy and horizon radius are plotted in Fig. 8 for $\ell = 2$. The presence of a non-vanishing cosmological constant reveal a quite different pattern. For $\alpha = 0$ Fig. 8 reveals that black holes with a small horizon are unstable. Increasing $r_h$, there is a phase transition and the solutions corresponding to $r_h > 1.8$ are thermodynamically stable. Increasing gradually the parameter $\alpha$ reveals that the unstable branch corresponding to small black holes merges into a branch of stable solution. Up to three solutions can be constructed which have the same angular velocity and same Hawking temperature but different entropy and energy. In particular we find two disjoint intervals of possible event horizon values with $C\Omega > 0$. The first branch occurs for small horizon. In this case we have $C\Omega < 0$ (i.e. unstable solutions) for intermediate values
of \( r_h \) and the specific heat becomes again positive for large \( r_h \). Finally when \( \alpha \) gets large enough (typically \( \alpha = 1 \)), the unstable branch disappears and we find only one branch of stable solution. Interestingly, a similar thermodynamical pattern was observed in [17] for charged, rotating solutions. Increasing the angular momentum in that case leads to the different phases. By contrast, our solutions are not charged but the different phases occur by changing the Gauss-Bonnet parameter. The values of the energy of these families

\[ \Omega, J \text{ and } E \] corresponding to the extremal black hole as functions of \( q \) for \( \alpha = 0 \) (left). Dependence of the gyromagnetic factor \( g \) on \( \Omega \) for several values of \( Q \) and of \( \alpha \) (right).

Figure 10: Estimation of the values \( \Omega, J \) and \( E \) corresponding to the extremal black hole as functions of \( q \) for \( \alpha = 0 \) (left). Dependence of the gyromagnetic factor \( g \) on \( \Omega \) for several values of \( Q \) and of \( \alpha \) (right).

of black holes are given in Fig. 8 (right). In the case when three solutions exist, it shows that the solutions with the smallest energy correspond to the smallest horizon.

### 4.4 Charged solutions

Because of the many parameters involved, we limit the construction of charged solutions to asymptotically flat solutions (i.e. \( \Lambda = 0 \)) and fix the scale such that \( r_h = 1 \). Charged rotating black holes in Einstein-Gauss-Bonnet-Maxwell theory can then be constructed for different values of the Gauss-Bonnet coupling constant \( \alpha \), of the electric charge \( q \) and of the angular velocity at the horizon \( \Omega \). The explicit solutions available in some special limits and the numerical results suggest these solutions to exist only in a limited domain of the \( \alpha, q, \Omega \) parameter-space and to approach extremal black holes on a surface

\[ \Sigma \equiv S(\alpha, q, \Omega) = 0 \quad (4.1) \]

The determination of this surface represents a formidable task but several limiting curves are available from the special cases:

- The set of points \( (\alpha, \sqrt{12}, 0) \) belong to \( \Sigma \).

- The critical line \( (0, q, \omega_{\text{max}}) \) is determined numerically and sketched in Fig. 10 (left). Note that numerous solutions of the Einstein-Maxwell equations have been produced, e.g. in [16], using isotropic coordinates. The pattern of solutions might look different from ours since we use a Schwarzschild-like radial coordinate and fix \( r_h = 1 \).

- The critical line \( (\alpha, 0, \omega_{\text{max}}) \) was obtained first in [14] and reproduced in Fig. 4 given by the black-solid line on the left.
The surface $\Sigma$ can be expected to smoothly extrapolate between these three limiting critical curves. Once the suitable boundary conditions are implemented, the equations can be solved for generic values of $\alpha, \Omega, q$. For non-rotating black holes, imposing a charge by means of $q \neq 0$ still leads to a zero magnetic potential $A(r) = 0$. Charged-rotating black holes necessarily have $A(r) \neq 0$ and are further characterized by a magnetic charge. We produced indeed several families of charged, rotating black holes. Fixing the electric charge $q$, our results show that the branch of rotating charged black holes exists and possesses the same qualitative pattern as that of the uncharged solutions. Profiles of charged-rotating black holes for $\alpha = 0$ and $\alpha = 2$ are presented in Fig. 9. Physical data characterizing families of rotating, charged solutions charge for several values of $\alpha$ are given in Fig. 8. The solid lines corresponding to $q = 0$ have been mentioned above. The effect of the charge (we set $q = 1$) is presented by the dashed lines for the cases $\alpha = 0$ and $\alpha = 2$ only.

This figure reveals that the presence of the charge changes the domain of existence only little as compared to the uncharged black holes. The numerical construction of several other branches (e.g. with $\alpha, \Omega$ fixed and varying $q$) confirms this tendency and suggests the surface $\Sigma$ to be smooth. Finally, we studied some electromagnetic properties of the solutions. In particular, we present the response of the magnetic moment and gyromagnetic factor $g$ (see (2.17)) on a change of the electric charge and of the Gauss-Bonnet coupling constant. This is summarized in Fig. 10 (right). It shows that the Gauss-Bonnet term has a tendency to increase the factor $g$. As expected [18], the gyromagnetic factor tends to $g = d - 2$ (i.e. $g = 3$) in the static limit. Note that the different curves on this figure stop at the approach of the extremal black hole.

5 Conclusions

In this note we have presented arguments supporting the fact that equal angular momentum rotating black holes of the Einstein-Gauss-Bonnet equations in five dimensions exist for generic values of the Gauss-Bonnet coupling constant. In the case of a negative cosmological constant, the asymptotically AdS solutions exist up to a maximal value of the Gauss-Bonnet coupling parameter, i.e. for $\alpha < \alpha_{cr} = \ell^2/3$. For $\alpha = \alpha_{cr}$ the model corresponds to Chern-Simons gravity [19]. The solutions obey a different asymptotic expansion than for generic values of $\alpha$. We were able to trace back this phenomenon with our numerical construction and obtain robust solutions even for $\alpha = \alpha_{cr}$ by implementing the appropriate boundary conditions.

The effect of the Gauss-Bonnet term would be worth studying for other kinds of solutions of the underlying Einstein equations with a negative cosmological constant. It would be challenging, in particular, to study the response of the NUT-charged solutions constructed in [24] to a Gauss-Bonnet term. Further applications of the Gauss-Bonnet black holes to the context of brane gravity could be looked for by following, e.g. the ideas of [25].

Supplementing the EGB Lagrangian with a Maxwell term leads naturally to charged black holes. The solutions terminate into extremal black hole at some maximal value of the angular momentum and/or of the electric charge. We believe that the numerical estimates of the domain of existence of charged-rotating black holes in EGB gravity (which we studied in the case $\Lambda = 0$ only) is correct qualitatively and quantitatively. This could be confirmed independently by using an appropriate attractor technique along the lines of [26].

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References

[1] D. J. Gross and E. Witten, Nucl. Phys. B 277 (1986) 1;
    R. R. Metsaev and A. A. Tseytlin, Phys. Lett. B 191 (1987) 354;
    C. G. Callan, R. C. Myers, and M. J. Perry, Nucl. Phys. B311, 673 (1988).
[2] R. C. Myers, Phys. Rev. D 36 (1987) 392.
[3] C. Garraffo and G. Giribet, Mod. Phys. Lett. A 23 (2008) 1801.
[4] C. Charmousis, Lect. Notes Phys. 769 (2009) 299.
[5] D. G. Boulware and S. Deser, Phys. Rev. Lett. 55, 2656 (1985).
[6] J. T. Wheeler, Nucl. Phys. B 268, 737 (1986).
    J. T. Wheeler, Nucl. Phys. B 273, 732 (1986).
[7] R-G. Cai, Phys.Rev.D 65 (2002) 084014.
[8] M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl.Phys.B 628 (2002) 295-330.
[9] Y.M. Cho and I.P. Neupane, Phys. Rev. D 66 (2002) 024044. I.P. Neupane, Phys. Rev. D 69 (2004) 084011. I.P. Neupane, Phys. Rev. D 67 (2003) 061501.
[10] T. Torii and H. Maeda, Phys. Rev. D 71 (2005) 124002.
[11] H-C. Kim and R-G. Cai, Phys. Rev. D 77 (2008) 024045.
[12] Y. Brihaye and E. Radu, Phys. Lett. B 661 (2008) 167, JHEP 09 (2008) 006.
[13] Y. Brihaye and E. Radu, Phys. Lett. B 678 (2009) 204.
[14] Y. Brihaye, B. Kleihaus, J. Kunz and E. Radu, JHEP 11(2010)098.
[15] Y. Brihaye and T. Delsate, Phys. Rev. D 79 (2009) 105013.
[16] J. Kunz, F. Navarro-Lerida and A. K. Petersen, Phys. Lett. B 614 (2005) 104
[17] J. Kunz, F. Navarro-L´erida and E. Radu, Phys. Lett. B 649 (2007) 463.
[18] A. N. Aliev, Class. Quant. Grav. 24 (2007) 4669. [arxiv: hep-th/0611205]
[19] P. Mora, R. Olea, R. Troncoso and J. Zanelli, JHEP 0406 (2004) 036.
[20] A. Anabalon, N. Deruelle, Y. Morisawa, J. Oliva, M. Sasaki, D. Tempo and R. Troncoso, Class. Quant. Grav. 26 (2009) 065002 1.3201
[21] R. C. Myers and M. J. Perry, Annals Phys. 172 (1986) 304.
[22] U. Ascher, J. Christiansen and R. D. Russell, Math. Comput. 33 (1979), 659; ACM Trans. Math. Softw. 7 (1981) 209.
[23] Y. Brihaye, R. Olea and E. Radu, in preparation.
[24] D. Astefanesei, R. B. Mann and E. Radu, JHEP 0501 (2005) 049.
[25] B. Cuadros-Melgar, E. Papantonopoulos, M. Tsoukalas and V. Zamarias, Phys.Rev.Lett. 100 (2008) 221601; ibid, Nucl.Phys.B 810 (2009) 246-265.
[26] D. Astefanesei, K. Goldstein, R.P. Jena, A.Sen and S.P. Trivedi, JHEP 10 (2006) 058.