Lorentz-invariant membranes
and finite matrix approximations

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Abstract
The question of Lorentz invariance for finite $N$ approximations of relativistic membranes is addressed. We find that one of the classical manifestations of Lorentz-invariance is \textit{not} possible for $N \times N$ matrices (at least when $N = 2$ or 3). How the symmetry is restored in the large $N$ limit is studied numerically.

1 Motivation

A crucial manifestation of Lorentz invariance in the light-cone description of the relativistic dynamics of $M$-dimensional extended objects is the existence of a field $\zeta$ (the longitudinal embedding coordinate) constructed out of the purely transverse fields $\vec{x}$ and $\vec{p}$ (and two additional discrete degrees of freedom, $\zeta_0 = \int \zeta d^M \varphi$ and $\eta = p_+$) that

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satisfies the same non-linear wave equation then the components of \( \vec{x} \), namely

\[
\eta^2 \dddot{\zeta} = \Delta \zeta := \frac{1}{\rho} \partial_a \left( \frac{gg^{ab}}{\rho} \partial_b \zeta \right),
\]

where \( g \) is the determinant of \( g_{ab} = \partial_a \vec{x} \partial_b \vec{x} \), \( \rho \) is a certain non-
dynamical density of unit weight (\( \int \rho d^M \varphi = 1 \)), and the Hamiltonian
for the phase-space variables (\( \vec{x}, \vec{p}; \eta, \zeta_0 \)), constrained by

\[
\int f^a \vec{p} \partial_a \vec{x} = 0 \quad \text{whenever} \quad \partial_a (\rho f^a) = 0,
\]

is given by (cp. \[1, 2\])

\[
H = \frac{1}{2\eta} \int \frac{\vec{p}^2 + g}{\rho} d^M \varphi.
\]

The equations of motion,

\[
\eta \ddot{x} = \frac{\vec{p}}{\rho}, \quad \eta \ddot{p} = \frac{1}{\rho} \partial_a \left( \frac{gg^{ab}}{\rho} \partial_b \vec{x} \right) = \Delta \vec{x}
\]

\[
\dot{\eta} = 0, \quad \dot{\zeta}_0 = \frac{H}{\eta},
\]

and \([2]\), imply that \( \zeta \) can, via

\[
\eta^2 \dddot{\zeta} = \frac{\vec{p}^2 + g}{2\rho^2}, \quad \eta \partial_a \zeta = \frac{\vec{p}}{\rho} \partial_a \vec{x},
\]

consistently be reconstructed; and \([1]\) easily follows from \([5]\):

\[
\eta^2 \dddot{\zeta} = \frac{\vec{p} \dot{\vec{p}}}{\rho} + \frac{g}{\rho^2} g^{ab} \partial_a \vec{x} \partial_b \vec{x} = \frac{\vec{p} \dot{\vec{p}}}{\eta \rho} \Delta \vec{x} + \frac{g}{\rho^2} g^{ab} \partial_a \vec{x} \partial_b \vec{p} \frac{\vec{p}}{\eta \rho}
\]

\[
\Delta \zeta = \frac{1}{\rho} \partial_a \left( \frac{gg^{ab}}{\rho} \partial_b \zeta \right) = \frac{\vec{p} \dot{\vec{p}}}{\eta \rho} \Delta \vec{x} + \frac{g}{\rho^2} g^{ab} \partial_a \vec{x} \partial_b \vec{p} \frac{\vec{p}}{\eta \rho}.
\]

Note that the original, manifestly Lorentz-invariant, formulation, namely

\[
\Delta x^\mu = 0 \quad (x^\mu, \mu = 0, 1, \ldots, D - 1 \text{ being the embedding coordinates of the } M + 1 \text{ dimensional manifold } \mathcal{M} \text{ swept out in space-time}),
\]

directly implies \([1]\), as \( \zeta = x^0 - x^{D-1} \) (and time \( \tau = \frac{x^0 + x^{D-1}}{2} \)), while

the chosen light-cone gauge (cp. \([1, 2]\) with \( G_{0a} = 0 \) (\( a = 1, \ldots, M \)),

\[
G_{ab} = -g_{ab} \text{ reduces } \Delta = \frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta, \text{ the Laplacian on } \mathcal{M}, \text{ to a}
non-linear wave operator proportional to $\eta^2 \partial_t^2 - \Delta$. Also note that an explicit formula for $\zeta$ was given by Goldstone in the mid-eighties \cite{2},

$$
\zeta = \zeta_0 + \frac{1}{\eta} \int G(\varphi, \tilde{\varphi}) \nabla^a \left( \frac{\tilde{\varphi}}{\rho} \partial_a \tilde{x} \right) \rho(\tilde{\varphi}) d^M \tilde{\varphi}
$$

(7)

and recently \cite{3} rewritten as

$$
\zeta_0 + \frac{\vec{x} \cdot \vec{p}}{2\eta \rho} - \int \frac{\vec{x} \cdot \vec{p}}{2\eta \rho} \rho d^M \varphi
$$

+ \frac{1}{2} \int G(\varphi, \tilde{\varphi}) \left( \frac{\tilde{\varphi}}{\eta \rho} \Delta \tilde{x} - \frac{\tilde{\varphi}}{\eta \rho} \right) \rho d^M \tilde{\varphi}
$$

(8)

-implying that $\tilde{\zeta}$, defined as the part of $2\eta(\zeta - \zeta_0)$ not containing $\vec{P} = \int \vec{p} d^M \varphi$, can be rewritten as

$$
\tilde{\zeta} = (d_{\alpha\beta\gamma} + e_{\alpha\beta\gamma}) \vec{x}_{\beta} \cdot \vec{p} Y_{\alpha}(\varphi)
$$

(9)

where

$$
d_{\alpha\beta\gamma} = \int Y_{\alpha} Y_{\beta} Y_{\gamma} \rho d^M \varphi, \quad e_{\alpha\beta\gamma} := \frac{\mu_{\beta} - \mu_{\gamma}}{\mu_{\alpha}} d_{\alpha\beta\gamma},
$$

(10)

with $Y_{\alpha}$ and $-\mu_{\alpha}$ being the (non-constant) eigenfunctions, resp. eigenvalues, of a Laplacian on the parameter space (with a metric whose determinant is $\rho^2$). Note that (7) satisfies the first equation in (5) without having to use (2) (i.e. "strongly")

$$
2\eta^2 \tilde{\zeta} = 2\eta H + 2 \int G(\varphi, \tilde{\varphi}) \tilde{\nabla}^a \left( \frac{\tilde{\varphi}}{\rho} \partial_a \tilde{\varphi} + \Delta \tilde{\varphi} \partial_a \tilde{x} \right) \rho d^M \varphi
$$

$$
= 2\eta H + \int G \Delta \left( \frac{\tilde{p}^2}{\rho^2} \right) \rho d^M \varphi + 2 \int G(\varphi, \tilde{\varphi}) \tilde{\nabla}^a \left[ \frac{1}{\rho} \partial_b \left( \frac{g_{b\alpha}}{\rho} \partial_c \tilde{x} \right) \partial_a \tilde{x} \right] \rho d^M \tilde{\varphi}
$$

$$
= 2\eta H + \int (\tilde{\Delta} G) \left( \frac{\tilde{p}^2}{\rho^2} + \frac{g}{\rho^2} \right) (\tilde{\varphi}) \rho d^M \tilde{\varphi} = \frac{\tilde{p}^2}{\rho^2} + \frac{g}{\rho^2}
$$

(11)

- having used that

$$
2\tilde{\nabla}^a \left[ \frac{1}{\rho} \partial_b \left( \frac{g}{\rho} \right) \delta^b_{\alpha} - \frac{1}{2 \rho^2} \partial_a g \right] = \Delta \left( \frac{g}{\rho^2} \right).
$$

(12)

3
On the other hand,
\[ \eta \partial_\rho \zeta - \frac{\bar{p}}{\rho} \partial_\rho \vec{x} = - \int \left( \partial_\rho \tilde{\partial}^b G(\varphi, \tilde{\varphi}) + \frac{\delta(\varphi, \tilde{\varphi})}{\rho} \delta_a^b \right) \bar{p} \partial_\rho \bar{x}(\rho) d^M \tilde{\varphi} \] (13)
is of the form (2) with
\[ \tilde{\partial}_b \left[ \rho \left( \partial_\rho \tilde{\partial}^b G(\varphi, \tilde{\varphi}) + \frac{\delta(\varphi, \tilde{\varphi})}{\rho} \delta_a^b \right) \right] = \rho \partial_\rho \tilde{\Delta} G(\varphi, \tilde{\varphi}) - \partial_\rho \delta(\varphi, \tilde{\varphi}) = 0, \] (14)
implying that (13) vanishes on the constrained phase space, hence the second part of (5) holds (weakly) too. These considerations will later when we have to guess/know which part of \( \eta \partial_\rho \zeta - \Delta \zeta \) is only weakly zero, of some relevance.

(1) (together with the first equation in (5)), immediately implies that the Lorentz-generator
\[ M_{i-} = \int (x_i \mathcal{H} - p_i \zeta) d^M \varphi \] (15)
Poission-commutes with \( \mathcal{H} \), as
\[ \eta \dot{M}_{i-} = \int p_i \left( \frac{\mathcal{H}}{\rho} - \eta \dot{\zeta} \right) d^M \varphi + \int x_i \left( \frac{\dot{\mathcal{H}}}{\rho} \eta - \Delta \zeta \right) \rho d^M \varphi. \] (16)

2 Matrix approximation, \( M = 2 \)

In [4] the question of Lorentz-invariance of Matrix Membranes was discussed and a discrete analogue of \( \tilde{\zeta} \) proposed,
\[ \zeta_N := \sum_{a,b,c=1}^{N^2-1} \frac{\mu_a + \mu_b - \mu_c}{\mu_a} d_{abc}^{(N)} \vec{x}_b \cdot \vec{p}_c T_a^{(N)} \] (17)
where the \( T_a^{(N)} \) are hermitean \( N \times N \) matrices and \( d_{abc}^{(N)} \) is proportional to \( Tr(T_a^{(N)} \{ T_b^{(N)}, T_c^{(N)} \}) \) (the normalizations suited for \( N \to \infty \) will be discussed below).

In this paper we would like to address the question of Lorentz-invariance for the Matrix theory, focusing on numerical computations, that will tell us that

- at least for low \( N \) (probably all finite \( N \)) there are no Matrix solutions for the natural analogue of (1)
• show how exactly (and how not) the finite $N$ analogue (17) will approach "solving (1)."

Before going into the details, let us note that, on general grounds (cp. [5, 6]) one is guaranteed that for $M = 2$ and any genus, a sequence $T_\alpha$, $\alpha = 1, 2, \ldots$ of linear maps (from real functions to hermitean matrices)

$$T_\alpha : f \to F^{(N_\alpha)} = T_\alpha(f)$$

exists, as well as a sequence of increasing positive integers $N_\alpha$, and decreasing positive real numbers $h_\alpha$ (with $\lim_{\alpha \to \infty} h_\alpha N_\alpha$ finite), with the following properties (for arbitrary smooth functions $f, g, \ldots$):

$$\lim_{\alpha \to \infty} ||T_\alpha(f)|| < \infty,$$

$$||T_\alpha(f)T_\alpha(g) - T_\alpha(f \cdot g)|| \to 0,$$

$$||\frac{1}{i h_\alpha}[T_\alpha(f), T_\alpha(g)] - T_\alpha(\{f, g\})|| \to 0,$$

$$2\pi h_\alpha Tr(T_\alpha(f)) \to \int f \rho d^2 \varphi.$$  (19)

Here $||T_\alpha(f)||$ can be taken as the largest eigenvalue of the hermitean matrix $T_\alpha(f)$, and

$$\{f, h\} := \frac{\epsilon^{rs}}{\rho} \partial_r f \partial_s h.$$  (20)

For functions on a sphere this map exists for all integers $N > 1$ (hence one can drop the index $\alpha$ and simply write $N$ and $h_N$ instead of $N_\alpha$ and $h_\alpha$) and, up to normalisation is given [1] via replacing in

$$Y_{lm}(\theta, \varphi) = \sum_{A_k=1, 2, 3} c_{A_1 \ldots A_l}^{(lm)} x_{A_1} \ldots x_{A_l} |x^2 = 1, \quad \text{the commuting variables}$$

$$x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta$$

by three $N \times N$ matrices $X_1, X_2, X_3$ satisfying

$$[X_A^{(N)}, X_B^{(N)}] = i \frac{2}{\sqrt{N^2 - 1}} \epsilon_{ABC} X_C^{(N)},$$

$$\vec{X}^2 := X_1^2 + X_2^2 + X_3^2 = 1_{N \times N}. \quad \text{(22)}$$
The resulting "Matrix harmonics" $T_{lm} = T_N(Y_{lm})$ (linear independent for $l = 0, 1, \ldots, N-1, m = -l, \ldots, +l$, and identically zero for $l \geq N$) are known \[1, 2, 7\] to have many special properties. In particular, they are eigenfunctions of the discrete Laplacian with eigenvalues $-\mu_{lm}$ being equal to the infinite $N$ eigenvalues $-l(l+1)$ of the parameter space Laplace

$$\Delta = \frac{1}{\sin \theta} \partial_\theta \sin \theta \partial_\theta + \frac{1}{\sin^2 \theta} \partial_\varphi^2.$$ \hspace{1cm} (23)

Apart from the fact that those $T_{lm}^{(N)}$ are not hermitean (because of the $Y_{lm}$ being complex), they are ideally suited for testing (17). Note that for an arbitrary function $f$ the map $T_N$ can be explicitly given as

$$f = \sum_{l=0,|m|\leq l}^{\infty} f_{lm} Y_{lm}(\theta, \varphi) \rightarrow F^{(N)} := \sum_{l=0}^{N-1} f_{lm} T_{lm}^{(N)}.$$ \hspace{1cm} (24)

To get the normalisation factors right is more difficult for a variety of reasons: from a practical/computational point of view it is easiest to take $\hat{T}_{lm}^{(N)}$ as given in \[7\], satisfying $Tr(\hat{T}_{lm}^{(N)} \hat{T}_{lm'}^{(N)}) = \delta_{ll'} \delta_{mm'}$, and in particular equation (6) in \[7\] as well as

$$\hat{T}_{lm}^{(N)} = \sqrt{4\pi} \sqrt{(N^2 - 1)! (N-1-l)!} \sum_{A_k=1,2,3} c_{A_1 \ldots A_l}^{(lm)} X_{A_1}^{(N)} \ldots X_{A_l}^{(N)}.$$ \hspace{1cm} (25)

Apart from further "hermiteanization" note that the usual spherical harmonics are normalized according to $\int Y_{lm}^* Y_{l'm'} \sin \theta d\theta d\varphi = \delta_{ll'} \delta_{mm'}$ whereas $\rho$ should satisfy $\int \rho d\theta d\varphi = 1$. Hence with $\rho \rightarrow \frac{\rho}{4\pi}, Y_{lm} \rightarrow \sqrt{4\pi} Y_{lm}$, explaining the factor $\sqrt{4\pi}$ in (25).

To conform with \[19\] we need $N2\pi \hbar_N \rightarrow 1$; we choose\footnote{One could also take $2\pi N \hbar = 1$ (for all $N$) and(or) multiply $\hat{T}$ by $\sqrt{N}$, rather then (cp. \[28\]) by $(N^2 - 1)^\frac{1}{2}$. This would have the advantage of having $T(1) = 1$ and $T(x_A) = X_A$ hold exactly, for any finite $N$, and also simplify (29).}

$$\hbar_N = \frac{1}{2\pi \sqrt{N^2 - 1}}.$$ \hspace{1cm} (26)

In order to have

$$2\pi \hbar_N Tr(T_{lm}^{(N)} T_{l'm'}^{(N)}) \rightarrow \delta_{ll'} \delta_{mm'}$$ \hspace{1cm} (27)
we multiply (see previous footnote) (25) by \((N^2 - 1)^{\frac{1}{4}}\) i.e. take

\[
\tilde{T}_{lm}^{(N)} = \frac{(N^2 - 1)^{\frac{1}{4}} T_{lm}^{(N)}}{\sqrt{\pi N}} \sum_{A_k=1,2,3} c_{A_1...A_l}^{(lm)} X_{A_1} \cdots X_{A_l}
\]

(28)

with \(\gamma_{Nl} \to 1\) as \(N \to \infty\).

Due to \(\mu_{lm} = \mu_{l-m}\), and \(T_{lm}^{\dagger} \sim (-1)^m T_{l-m}\) (as a consequence of \(Y_{lm}^* = (-1)^m Y_{lm}\)) one can at the end easily get rid of the non-hermicity problem and consider hermitean matrices \(T_a^{(N)}\). Forming linear combinations \(\frac{T_{lm}^{(N)} + T_{lm}^{\dagger}}{\sqrt{2}}\) and \(\frac{T_{lm}^{(N)} - T_{lm}^{\dagger}}{\sqrt{2i}}\) one obtains the desired hermitean basis \(\{T_a^{(N)}\}_{a=1}^{N^2-1}\) satisfying

\[
\frac{1}{\sqrt{N^2 - 1}} \text{Tr} \left( T_a^{(N)} T_b^{(N)} \right) = \delta_{ab}.
\]

(29)

The matrix approximation of (3) is then given by (leaving out \(\eta\) from now on, which - just as is done in string theory - can, for most purposes, be absorbed in a redefinition of "time")

\[
H_N = \frac{1}{2\sqrt{N^2 - 1}} \text{Tr} \left( \vec{P}^2 - \frac{1}{2}(2\pi)^2 (N^2 - 1)[X_i, X_j]^2 \right)
= \frac{1}{2} \left( p_{ia} p_{ia} + \frac{1}{2} f_{abc} f_{abc} \vec{x}_b \cdot \vec{x}_c \cdot \vec{x}_c \right)
\]

(30)

and the normalisations, and conventions,

\[
d_{abc}^{(N)} := \frac{1}{2\sqrt{N^2 - 1}} \text{Tr} \left( T_a^{(N)} \{T_b^{(N)}, T_c^{(N)}\} \right)
\]

(31)

(the symbol \(\{\cdot, \cdot\}\) here denotes the anticommutator of matrices) and

\[
f_{abc}^{(N)} := \frac{2\pi}{i} \text{Tr} \left( T_a^{(N)} [T_b^{(N)}, T_c^{(N)}] \right),
\]

(32)

are such that, as \(N \to \infty\), (31) and (32) approach, respectively,

\[
d_{abc} = \int Y_a Y_b Y_c \rho d\theta d\varphi, \quad g_{abc} = \int Y_a \{Y_b, Y_c\} \rho d\theta d\varphi, \quad \rho = \frac{\sin \theta}{4\pi}.
\]

(33)
3 Lorentz symmetry at finite $N$?

We now focus on calculating $\zeta_N - \Delta^{(N)} \zeta_N$ where we take $\zeta_N$ to be given by

$$\zeta_N = \sum_{a,b,c=1}^{N^2-1} L^{(N)}_{abc} \bar{x}_a \gamma_\mu T^a$$

with for the moment arbitrary coefficients $L^{(N)}_{abc}$. Using the discrete equations of motion $\dot{x}_{ia} = p_{ia}$, $\dot{p}_{ia} = f^{(N)}_{abc} \bar{x}_c \cdot \bar{x}_{ia}$, we find that

$$\ddot{\zeta}_N - \Delta^{(N)} \zeta_N = \bar{x}_m \cdot \bar{x}_n \cdot \bar{x}_\mu \cdot \bar{p}_\nu R^{(N)}_{amn\mu\nu} T^a$$

where

$$R^{(N)}_{amn\mu\nu} = L^{(N)}_{auc} f_{cdm} f^{(N)}_{nd\nu} + L^{(N)}_{a\mu c} f_{cdm} f^{(N)}_{nd\nu} + 2L^{(N)}_{a\nu c} f_{cdm} f^{(N)}_{nd\mu}$$

$$+ L^{(N)}_{amc} (f_{d\mu} f_{v\nu} + f_{d\nu} f_{v\mu}) - L^{(N)}_{c\mu\nu} f_{adm} f^{(N)}_{ndc}.$$  \hfill (35)

The question that now arises is whether there exist nontrivial coefficients $L^{(N)}_{abc}$ (i.e. which cannot be written as $M^{(N)}_{ab} f_{bc}$) such that the r.h.s of equation (34) is weakly zero. The corresponding equation for $R^{(N)}_{amn\mu\nu}$ is

$$R^{(N)}_{amn\mu\nu} + R^{(N)}_{anm\mu\nu} = \bar{G}^{(N)}_{amnk} f^{(N)}_{k\mu\nu}$$

where $\bar{G}^{(N)}_{amnk}$ are unknown coefficients. Note that $M^{(N)}_{ab} f^{(N)}_{bc}$ is a solution of (36) for arbitrary $M_{ab}$, i.e. satisfies (36) with

$$\bar{G}^{(N)}_{amnk} = -M^{(N)}_{ck} (f^{(N)}_{amn} f^{(N)}_{dnc} + f^{(N)}_{amn} f^{(N)}_{dnc}).$$

In order to see that there are no other solutions it is best to first symmetrize (36) over $\mu$ and $\nu$

$$R^{(N)}_{amn\mu\nu} + R^{(N)}_{anm\mu\nu} + R^{(N)}_{amn\mu\nu} + R^{(N)}_{anm\mu\nu} = 0$$

and solve the resulting equation with respect to $L^{(N)}_{abc}$. By explicit calculation for $N = 2$ and $N = 3$ we found the general solution to be of the form $L^{(N)}_{abc} = M^{(N)}_{ak} f^{(N)}_{kbc}$, proving that nontrivial solutions do not exist (for $N = 2, 3$, possibly for all $N$).

On the other hand, in the large $N$ limit the theory is relativistically invariant \[2\]; therefore there should exist a choice of $L^{(N)}_{abc}$ such that the
r.h.s. of (34) converges to zero when \( N \to \infty \) for \( \vec{x}_a \) and \( \vec{p}_a \) satisfying the Gauss constraint \( G_a := f^{(N)}_{abc} \vec{x}_b \vec{p}_c = 0 \). In the following we will consider \( \zeta_N \) given by (17) i.e. we take

\[
L^{(N)}_{abc} = \frac{\mu_a + \mu_b - \mu_c}{\mu_a} d^{(N)}_{abc} .
\] (37)

Inserting (37) into (34) however does not immediately yield a r.h.s. converging to zero (as we found numerically); hence it is necessary to explicitly determine \( G_a^{(N)} \). To derive the exact form of the subtraction that one has to make to render convergence (to zero), as \( N \to \infty \), is non-trivial: due to

\[
\Delta \zeta := \frac{1}{\rho} \partial_b \left( \frac{gg^{ab}}{\rho} \partial_a \zeta \right) ,
\]

the term involving constraints is (leaving out the \( \rho \)-factors and \( \eta \) for simplicity), cp. (13):

\[
- \partial_b \left( gg^{ab} \int F^c_a(\varphi, \tilde{\varphi})(\vec{p} \partial_c \vec{x})(\tilde{\varphi})d^M \varphi \right) ;
\]

for \( M = 2 \), the \( \alpha \)-component of that is

\[
= - \sum_{\gamma=1}^{\infty} \frac{1}{\mu_\gamma} \left( \partial_a Y_\gamma(\varphi) \partial_a Y_\gamma(\tilde{\varphi}) + \delta^c_{\alpha} \delta(\varphi, \tilde{\varphi}) = \sum_{\gamma=1}^{\infty} \frac{\epsilon_{\alpha a'' b''}}{\mu_\gamma} \partial_a Y_\gamma(\varphi) \epsilon^{a'' b''} \partial_c Y_\gamma(\tilde{\varphi}) .
\]

Hence (note the factor of 2 involved in the relation between \( \zeta \) and \( \tilde{\zeta} \)) the matrix

\[
U := \vec{x}_m \cdot \vec{x}_n \vec{x}_\mu \cdot \vec{p}_\nu (R^{(N)}_{amn\mu \nu} - S^{(N)}_{amn\mu \nu}) T^{(N)}_a
\] (38)

with

\[
S^{(N)}_{amn\mu \nu} := - L^{(N)}_{cnd} f^{(N)}_{admn} f^{(N)}_{e\mu \nu}
\]

should not contain any terms proportional to the \( G_a \)'s and therefore should converge strongly to 0 in the large \( N \) limit.
Numerical investigation

In order to verify that the matrix $U$ indeed converges to 0 we performed a numerical analysis for matrices with $N = 3, \ldots, 11$ using the conventions described in section 2 ($N = 2$ is trivial, $U = 0$). The elements of the matrix $U$ are polynomials of the form

$$U_{ij}^{(N)} := \tilde{x}_m \cdot \tilde{x}_n \cdot \tilde{p}_{\mu} \tilde{R}_{amn\mu\nu}^{(N)}[T_{a}^{(N)}]_{ij}$$

where $\tilde{R}_{amn\mu\nu}^{(N)} := R_{amn\mu\nu}^{(N)} - S_{amn\mu\nu}^{(N)}$. We restrict the analysis to $i, j = 1, 2, 3$, i.e. we analyze what is the $N$ dependence of the $SU(3)$ corner of matrix $U$, and to $1 \leq a, m, n, \mu, \nu \leq 8$, i.e. we consider only the range of the $SU(3)$ adjoint index.

A typical polynomial $U_{ij}^{(N)}$ consists of about 700 terms satisfying these restrictions. We found numerically that they all behave like $1/N$ (see Fig. 1).

![Figure 1: N dependence of coefficients from $U_{ij}^{(N)}$ polynomials](image)

We would like to make several comments concerning this result. First, the fact that we subtracted the Gauss constraint by considering $\tilde{R}$ instead of $R$ is necessary to see the convergence. If the Gauss
constraint is not subtracted then the corresponding polynomials $U_{ij}^{(N)}$ diverge - their coefficients behave like $N^1$. Second, the combinations of terms in (35) is of course very special i.e. crucial for the convergence. If for instance we consider only the first term in (35), i.e. $L_{acv}^{(N)} f_{cdm}^{(N)} f_{ndp}^{(N)}$ then the coefficients of the resulting polynomial $U_{ij}^{(N)}$ are divergent, behaving like $N^1$. Third, the restrictions ($1 \leq i, j \leq 3$ and $1 \leq a, m, n, \mu, \nu \leq 8$) we used are certainly minimal. The question remains to what extent one can relax these restrictions still having the convergence. It is reasonable to conjecture that for any fixed $n < N$ (i.e. $n$ independent of $N$) the elements of the matrix $U_{ij}^{(N)}$ satisfying the restrictions $1 \leq i, j \leq n, \quad 1 \leq a, m, n, \mu, \nu \leq n^2 - 1$ still converge to 0.

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