NONCOMPACT GAUGE-INVARIANT LATTICE SIMULATIONS

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ABSTRACT

We have applied a new gauge-invariant, noncompact, Monte Carlo method to simulate $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on $8^4$ and $12^4$ lattices. The Creutz ratios of the Wilson loops agree with the exact results for $U(1)$ for $\beta \geq 1$ apart from a renormalization of the charge. The $SU(2)$ and $SU(3)$ Creutz ratios robustly display quark confinement at $\beta = 0.5$ and $\beta = 1.5$, respectively. At much weaker coupling, the $SU(2)$ and $SU(3)$ Creutz ratios agree with perturbation theory after a renormalization of the coupling constant.

1. INTRODUCTION

The first gauge-invariant noncompact simulations were carried out by Palumbo, Polikarpov, and Veselov and were based on earlier work by Palumbo et al. They saw a confinement signal. Their action contains five terms, constructed from two invariants, and involves noncompact auxiliary fields and an adjustable parameter.

The present paper implements and tests a new way of performing gauge-invariant noncompact simulations. This method is based upon a new noncompact action that is exactly invariant under lattice gauge transformations. The action is a natural discretization of the classical Yang-Mills action with auxiliary fields that are compact group elements representing gauge transformations.

We have used this method to simulate $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on $8^4$ and $12^4$ lattices. The Creutz ratios of Wilson loops agree with the exact results for $U(1)$ for $\beta \geq 1$ apart from a renormalization of the charge. The $SU(2)$ and $SU(3)$ Creutz ratios clearly show quark confinement at $\beta = 0.5$ and $\beta = 1.5$, respectively. At much weaker coupling, the $SU(2)$ and $SU(3)$ Creutz ratios agree with perturbation theory with a renormalized coupling constant.

2. THE METHOD

What constitutes a gauge transformation in this method? To find out, we look at the (massless) continuum fermionic action density $i\bar{\psi}\gamma_\mu \partial_\mu \psi$. A suitable discretization of this quantity is $i\bar{\psi}(n)\gamma_\mu [\psi(n + e_\mu) - \psi(n)]/a$ in which $n$ is a four-vector of integers

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representing an arbitrary vertex of the lattice, $e_\mu$ is a unit vector in the $\mu$th direction, and $a$ is the lattice spacing. The product of Fermi fields at the same point is gauge invariant as it stands. The other product of Fermi fields becomes gauge invariant if we insert a matrix $A_\mu(n)$ of gauge fields

$$\frac{i}{a} \bar{\psi}(n) \gamma_\mu [(1 + i g A_\mu(n)) \psi(n + e_\mu) - \psi(n)]$$

that transforms under a gauge transformation represented by the group elements $U(n)$ and $U(n + e_\mu)$ in such a way that

$$1 + i g A'_\mu(n) = U(n) [1 + i g A_\mu(n)] U^{-1}(n + e_\mu).$$

The required behavior is

$$A'_\mu(n) = U(n) A_\mu(n) U^{-1}(n + e_\mu) + \frac{i}{a g} U(n) \left[ U^{-1}(n) - U^{-1}(n + e_\mu) \right].$$

Let us define the lattice field strength $F_{\mu\nu}(n)$ as

$$F_{\mu\nu}(n) = \frac{1}{a} \left[ A_\mu(n + e_\nu) - A_\mu(n) \right] - \frac{1}{a} \left[ A_\nu(n + e_\mu) - A_\nu(n) \right] + ig \left[ A_\nu(n) A_\mu(n + e_\nu) - A_\mu(n) A_\nu(n + e_\mu) \right]$$

which reduces to the continuum Yang-Mills field strength in the limit $a \to 0$. Under the gauge transformation (3), this field strength transforms as

$$F'_{\mu\nu}(n) = U(n) F_{\mu\nu}(n) U^{-1}(n + e_\mu + e_\nu).$$

The field strength $F_{\mu\nu}(n)$ is antisymmetric in the indices $\mu$ and $\nu$, but it is not hermitian. To make a positive plaquette action density, we use the Hilbert-Schmidt norm of $F_{\mu\nu}(n)$

$$S = \frac{1}{4k} \text{Tr}[F_{\mu\nu}^\dagger(n) F_{\mu\nu}(n)],$$

in which it is assumed that the generators of the gauge group have been orthonormalized as $\text{Tr}(T_a T_b) = k \delta_{ab}$. Because $F_{\mu\nu}(n)$ transforms covariantly (3), this action density is invariant under the noncompact gauge transformation (3).

The gauge transformation (3) with group element $U(n) = \exp(-iag \omega^a T_a)$ maps the usual matrix of gauge fields $A_\mu(n) = T_a A^a_\mu(n)$ into a matrix that generally lies outside the Lie algebra of the gauge group, although it does remain in the algebra to lowest (zeroth) order in the lattice spacing $a$. In our simulation we use for the gauge fields this more-general space of matrices. We use the action (6) in which the field strength (4) is defined in terms of gauge-field matrices that are of the form

$$A_\mu(n) = V A^0_\mu(n) W^{-1} + \frac{i}{a g} V \left( V^{-1} - W^{-1} \right)$$
where $A^0_\mu(n)$ is a matrix of gauge fields defined in the usual way, $A^0_\mu(n) \equiv T_\alpha A^\alpha_\mu(n)$. Here the group elements $V$ and $W$ associated with the gauge field $A_\mu(n)$ are unrelated to those associated with the neighboring gauge fields $A_\mu(n+e_\nu), A_\nu(n)$, and $A_\nu(n+e_\mu)$.

We do not require the quantity $1 + i g a A_\mu(n)$ to be an element $L_\mu(n)$ of the gauge group. But if one did, then the matrix $A_\mu(n)$ of gauge fields would be related to the link $L_\mu(n)$ by $A_\mu(n) = (L_\mu(n) - 1)/(i g a)$, and the action (3) defined in terms of the field strength (4) and this gauge-field matrix would be, mirabile dictu, Wilson’s action:

$$S = \frac{k - \Re \text{Tr} L_\mu(n) L_\nu(n + e_\mu) L^\dagger_\nu(n + e_\mu) L^\dagger_\mu(n)}{2a^4 g^2 k}.$$  \hspace{1cm} (8)

3. RESULTS

We have tested this method by applying it to the $U(1)$, $SU(2)$, and $SU(3)$ gauge theories on $8^4$ and $12^4$ lattices. In our initial configurations, the unitary matrices $V$ and $W$ were set equal to the identity matrix and the gauge fields $A^0_\mu$ were either zero or random. We allowed at least 10,000 sweeps for thermalization.

For $U(1)$ and for $\beta \geq 1$, our measured Creutz ratios agreed with the exact ones apart from finite-size effects and a renormalization of the charge. For instance at $\beta = 1$, we found on the $8^4$ lattice $\chi(2,2) = 0.142(1), \chi(2,3) = 0.098(1), \chi(3,3) = 0.047(1), \chi(2,4) = 0.085(1), \chi(3,4) = 0.030(1)$, and $\chi(4,4) = 0.014(1)$. The first three of these $\chi$’s are equal to the exact Creutz ratios for a renormalized value of $\beta_r = 0.93$; the last three are smaller than the exact ratios for $\beta_r = 0.93$ due to finite-size effects by 6%, 17%, and 42%, respectively.

For $SU(2)$ on the $8^4$ lattice at $\beta = 0.5$, we found $\chi(2,2) = 0.835(3), \chi(2,3) = 0.85(1), \chi(3,3) = 0.9(2)$, and $\chi(2,4) = 0.9(6)$ which robustly display confinement. At $\beta = 1$, our six Creutz ratios track those of tree-level perturbation theory for a renormalized value of $\beta_r = 1.75$; the finite-size effects are hidden by incipient confinement.

For $SU(3)$ at $\beta = 1.5$, we found on the $8^4$ lattice $\chi(2,2) = 1.175(3), \chi(2,3) = 1.16(2), \chi(3,3) = 1.16(2)$, and $\chi(2,4) = 1.4(2)$, and on the $12^4$ lattice $\chi(2,2) = 1.171(4), \chi(2,3) = 1.14(3)$, and $\chi(2,4) = 1.9(7)$. At $\beta = 2$ we found on the $8^4$ lattice $\chi(2,2) = 0.839(2), \chi(2,3) = 0.837(7), \chi(3,3) = 0.76(9)$, and $\chi(2,4) = 0.87(3)$; and on the $12^4$ lattice $\chi(2,2) = 0.832(2), \chi(2,3) = 0.821(7), \chi(3,3) = 0.71(7)$, and $\chi(2,4) = 0.80(2)$. Within the limited statistics, these results exhibit confinement. At much weaker coupling, our ratios agree with perturbation theory apart from finite-size effects and after a renormalization of the coupling constant.

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