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Cartan-Kähler Theory and Applicationsto Local Isometric and Conformal Embedding

Nabil Kahouadji

Abstract

The goal of this lecture is to give a brief introduction to Cartan-Kähler’s theory. As examples to the application of this theory, we choose the local isometric and conformal embedding. We provide lots of details and explanations of the calculation and the tools used.

1 Cartan’s Structure Equations

Let \( \xi = (E, \pi, M) \) be a vector bundle. Denote \( (\mathcal{X}(M), [\cdot, \cdot]) \) the Lie algebra of vector fields on \( M \) and \( \Gamma(E) \) the moduli space of cross-sections of the vector bundle \( E \).

1.1 Connection on a vector bundle

A connection on a vector bundle \( E \) is a choice of complement of vertical vector fields on \( E \). A connection induces a covariant differential operator \( \nabla \) on \( E \). A covariant derivative \( \nabla \) on a vector bundle \( E \) is a way to ”differentiate” bundle sections along tangent vectors and it is sometimes called a connection.

**Definition 1.1.1.** A connection on a vector bundle \( E \) is an linear operator defined as follows:

\[
\nabla : \mathcal{X}(M) \times \Gamma(E) \rightarrow \Gamma(E)
\]

\[
(X, S) \mapsto \nabla_X S
\]

\(^1\)See the Master Thesis [7] on which the lecture is based
satisfying

\[ \nabla_{(X_1+X_2)}S = \nabla_{X_1}S + \nabla_{X_2}S, \quad \nabla_{(fX)}S = f\nabla_X S \]
\[ \nabla_X (S_1+S_2) = \nabla_X S_1 + \nabla_X S_2, \quad \nabla_X (fS) = X(f)S + f\nabla_X S \]

\( \forall X, X_1, X_2, Y, Y_1, Y_2 \in \mathfrak{X}(M) \) and \( \forall S, S_1, S_2 \in \Gamma(E) \).

1.1.1 Curvature of a Connection

Definition 1.1.2. The curvature of a connection \( \nabla \) is a vector valued 2-form

\[ R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \Gamma(E) \rightarrow \Gamma(E) \]
\[ X, Y, S \mapsto R(X, Y)S \]

defined by \( R(X, Y)S = \left( [\nabla_X, \nabla_Y] - \nabla_{[X, Y]} \right)S \).

Theorem 1.1. For any \( f, g \) and \( h \) smooth functions on \( M, S \in \Gamma(E) \) a section of \( \xi \) and \( X, Y \in \mathfrak{X}(M) \) two tangent vector fields of \( M \), we have

\[ R(fX, gY)(hS) = fghR(X, Y)S \tag{1} \]

1.1.2 Connection and Curvature Forms

Let \( \xi = (E, \pi, M) \) be a vector bundle over a smooth manifold \( M \) with an \( r \)-dimensional vector space \( E \) as a standard fiber. Let \( \nabla \) be a connection on \( \xi \) and \( R \) its curvature. We denote by \( U \) an open set of \( M \).

Definition 1.1.3. A set of \( r \) local sections \( S = (S_1, S_2, \ldots, S_r) \) of \( \xi \) is called a frame field (or a moving frame) if \( \forall p \in U, \ S(p) = \left( S_1(p), S_2(p), \ldots, S_r(p) \right) \) form a basis of the fiber \( E_p \) over \( p \).

Let \( S = (S_1, S_2, \ldots, S_r) \) be a frame field, \( \nabla \) a connection on \( \xi \) and \( X \in \mathfrak{X}(M) \) a tangent vector field on \( M \). Then \( \nabla_X S_j \) is another section of \( \xi \) and it can be expressed in the frame field \( S \) as follows:

\[ \nabla_X S_j = \sum_{i=1}^{r} \omega_{ij}(X)S_i \tag{2} \]

where \( \omega_{ij} \in A^1(M) \) are differential 1-forms\(^2\) on \( M \) and \( \omega_{ij}(X) \) are smooth functions on \( M \).

\(^2\)\( A^k(M) \) denote the set of differential \( k \)-form on \( M \) (we choose this notation instead of the standard notation \( \Omega^k(M) \) to not mix with the curvature form.)
Definition 1.1.4. The \( r \times r \) matrix \( \omega = (\omega_{ij}) \) is called the connection 1-form of \( \nabla \).

The connection \( \nabla \) is completely determined by the matrix \( \omega = (\omega_{ij}) \). Conversely, a matrix of differential 1-forms on \( M \) determines a connection (in a non-invariant way depending on the choice of the moving frame).

Let \( X, Y \in \mathfrak{X}(M) \) two tangent vector fields. Then \( \mathcal{R}(X,Y)S_j \) are sections of \( \xi \), and can be expressed on the frame field \( S \) as follows:

\[
\mathcal{R}(X,Y)S_j = \sum_{i=1}^{r} \Omega_{ij}(X,Y)S_i
\]

where \( \Omega_{ij} \in \mathcal{A}^2(M) \) are differential 2-forms on \( M \) and \( \Omega_{ij}(X,Y) \) are smooth functions on \( M \).

Definition 1.1.5. The \( r \times r \) matrix \( \Omega = (\Omega_{ij}) \) whose entries are differential 2-forms, is called the curvature 2-form of the connection \( \nabla \).

We state the following theorem\(^3\) that gives the relation between the connection 1-form \( \omega \) and the curvature 2-form \( \Omega \).

**Theorem 1.2.**

\[
d\omega + \omega \wedge \omega = \Omega \quad \text{(matrix form)}
\]

or

\[
d\omega_{ij} + \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj} = \Omega_{ij} \quad \text{(on components)}
\]

1.2 The Induced Connection

Let \( \xi = (E, \pi, M) \) and \( \xi' = (E', \pi', M) \) be two vector bundles on \( M \). Consider a map \( f : M \rightarrow M \) and denote \( \tilde{f} : E \rightarrow E' \) the associated bundle map i.e. \((f, \tilde{f})\) satisfies the following commutative diagramme:

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow{\pi} & & \downarrow{\pi'} \\
M & \xrightarrow{f} & M
\end{array}
\]

\(^3\)In the tangent bundle case, this theorem gives Cartan’s second equation, as we will see later.
If $\nabla'$ is a connection on $E'$, the vector bundle morphism induces a pull-back connection on $E$

$$\nabla = \tilde{f}^* \nabla'$$

such that for any $S' \in \Gamma(E')$ and $X \in \mathfrak{X}(M)$, $\nabla_X(\tilde{f}^* S') = (\tilde{f}^* \nabla')_X(\tilde{f}^* S') = \tilde{f}^* (\nabla_{f_* X} S')$ where $f_* : T_p M \rightarrow T_{f(p)} M$ is the linear tangent map.

We can also induce a connection on $\xi$ by another way. The connection $\nabla'$ is completely determined by the matrix of differential 1-forms $\omega' = (\omega'_{ij})$, and we define $\nabla$ by the matrix $\omega$ whose entries $\omega_{ij}$ are the pull-back of $\omega'_{ij}$ by $\tilde{f}$, i.e. $\omega = \tilde{f}^* \omega'$.

The pull back commute with the exterior differentiation and with the exterior product, so, the curvature 2-form $\Omega$ of $\nabla$ is the pull back of the curvature 2-form of $\nabla'$, i.e. $\Omega = \tilde{f}^* \Omega'$.

### 1.3 Metric Connection

Let $\xi = (E, \pi, M)$ be a vector bundle. We denote by $\nabla$ a connection on $\xi$ determined by a matrix of 1-forms $\omega$. Let $\Omega$ be the associated curvature 2-form and $g$ a Riemannian metric on $\xi$ (i.e. a positively-defined scalar product on each fiber).

**Definition 1.3.1.** $\nabla$ is a connection on $\xi$ compatible with the metric $g$ (or a metric connection) if $\nabla$ satisfies to the following property (Leibniz’s identity):

$$\nabla_X \left( g(S_1, S_2) \right) = g(\nabla_X S_1, S_2) + g(S_1, \nabla_X S_2)$$

$$\forall S_1, S_2 \in \Gamma(E), \text{ and } \forall X \in \mathfrak{X}(M)$$

**Proposition 1.3.1.** Let $S = (S_1, S_2, \ldots, S_n)$ be an orthonormal frame field with respect to $g$, i.e. $g_p(S_i, S_j) = \delta_{ij}$ for all $p \in U$, $i, j = 1, \ldots, r$, then the matrix of 1-forms $\omega$ associated to $S$ and the curvature matrix of 2-form are both skew-symmetric.

---

4 $d(f^* \alpha) = f^*(d\alpha)$ and $f^*(\alpha \wedge \beta) = f^*(\alpha) \wedge f^*(\beta)$ for all $\alpha, \beta \in \mathcal{A}(M)$.
1.4 Tangent Bundle Case

1.4.1 Torsion of a Connexion on a Tangent Bundle

Let us consider now, a local frame field $S = (S_1, \ldots, S_m)$ over $U \subset M$ where $S_i \in \mathfrak{X}(U)$ are local tangent vectors fields (i.e. local sections of the tangent bundle such that for all $p \in U$, $(S_1(p), \ldots, S_m(p))$ forms a basis of the tangent vector space of $M$).

**Definition 1.4.1.** If $S$ is a local orthonormal frame field, the associated coframe field $\eta = (\eta_1, \ldots, \eta_m)$ is a local frame field of 1-forms, such that for all $p \in U$, $\eta_i(p)(S_j) = \delta_{ij}$.

We define then a differential 2-form $\Theta$ as follows:

$$d\eta + \omega \wedge \eta = \Theta$$

(8)

**Definition 1.4.2.** $\Theta$ is called the torsion 2-form of $\nabla$.

**Proposition 1.4.1.** On a tangent bundle, the four forms $\eta, \omega, \Theta$ and $\Omega$ are connected by the following equations

$$d\Theta + \omega \wedge \Theta = \Omega \wedge \eta$$

and

$$d\Omega = \Omega \wedge \omega - \omega \wedge \Omega$$

(9)

The equation (10) is the expression of the Bianchi identity via the connection 1-form and the curvature 2-form. Equation (10) is also valid on an arbitrary vector bundle.

1.4.2 Cartan’s Structure Equations

Let $(M, g)$ be an $m$-dimensional Riemannian manifold. Let $\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ an orthonormed coframe field on $M$ ($\eta_j \in A^1(M)$). According to equations (8), (5) and the proposition 1.3.1, we establish the Cartan structure equations:

$$\begin{cases}
    d\eta_i + \sum_{j=1}^{m} \omega_{ij} \wedge \eta_j = 0 \quad \text{(torsion-free)} \\
    d\omega_{ij} + \sum_{k=1}^{m} \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}
\end{cases}$$

(11)
where the matrix \((\omega_{ij})\) is the Lévi-Civita connection 1-form (free
torsion connection which is compatible with the riemannian metric \(g\)).
Because \(\eta\) is an orthonormed coframe field, \((\omega_{ij})\) is skew-symmetric
(proposition 1.3.1.). \((\Omega_{ij})\) is the curvature 2-form matrix of the riemannian
connection \((\Omega_{ij} = \frac{1}{2} \sum_{k,l=1}^{m} R_{ijkl} \eta_{k} \wedge \eta_{l})\).

1.5 The Cartan Lemma

We end this section with a technical lemma, which is easy to establish and
at the same time rich applications . This lemma will not only be useful
for isometric embedding problem, but also for many calculus in differential
geometry.

**Lemma 1.5.1.** Let \(M\) be an \(m\)-dimensional manifold. \(\omega_{1}, \omega_{2}, \ldots, \omega_{r}\) a set of
linearly independent differential 1–forms \((r \leq n)\) and \(\theta_{1}, \theta_{2}, \ldots, \theta_{r}\) differential
1–forms such that

\[
\sum_{i=1}^{r} \theta_{i} \wedge \omega_{i} = 0 \quad (12)
\]

then there exists \(r^{2}\) functions \(h_{ij}\) in \(C^{1}(M)\) such that

\[
\theta_{i} = \sum_{j=1}^{r} h_{ij} \omega_{j} \quad \text{with} \quad h_{ij} = h_{ji}. \quad (13)
\]

2 Exterior Differential Systems and Ideals

2.1 Exterior Differential Systems

Denote \(\mathcal{A}(M)\) the space of smooth differential forms\(^{5}\) on \(M\).

**Definition 2.1.1.** An exterior differential system is a finite set of differential
forms \(I = \{\omega_{1}, \omega_{2}, \ldots, \omega_{k}\} \subset \mathcal{A}(M)\) for which there is a set of equations
\(\{\omega_{i} = 0 | \omega_{i} \in I\}\).
such that one can write the exterior differential system as follow:

\[
\begin{align*}
\omega_{1} &= 0 \\
\omega_{2} &= 0 \\
&\vdots \\
\omega_{k} &= 0
\end{align*}
\]

\(^{5}\)This is a graded algebra under the wedge product.
Definition 2.1.2. An exterior differential system \( I \subset A(M) \) is said to be Pffafian if \( I \) contains only differential 1-forms, i.e. \( I \subset A^1(M) \).

2.2 Exterior Ideals

Definition 2.2.1. Let \( \mathcal{I} \subset A(M) \) a set of differentiable forms. \( \mathcal{I} \) is an exterior ideal if:

1. The exterior product of any differential form of \( \mathcal{I} \) by a differential form of \( A(M) \) belong to \( \mathcal{I} \).

2. The sum of any two differential forms of the same degree belonging to \( \mathcal{I} \), belong also to \( \mathcal{I} \).

Definition 2.2.2. Let \( I \subset A(M) \) an exterior differential system. The exterior ideal generated by \( I \) is the smallest exterior ideal containing \( I \).

2.3 Exterior Differential Ideals

Definition 2.3.1. Let \( \mathcal{I} \subset A(M) \) a set of differential forms. \( \mathcal{I} \) is an exterior differential ideal if \( \mathcal{I} \) is an exterior ideal closed under the exterior differentiation, i.e. \( \forall \omega \in \mathcal{I}, d\omega \in \mathcal{I} \) (we can also write \( d\mathcal{I} \subset \mathcal{I} \)).

Definition 2.3.2. Let \( I \subset A(M) \) an exterior differential system. The exterior differential ideal generated by \( I \) is the smallest exterior differential ideal containing \( I \).

2.4 Closed Exterior Differential Systems

Definition 2.4.1. An exterior differential system \( I \subset A(M) \) is said closed if the exterior differentiation of any form of \( I \), belong to the exterior ideal generated by \( I \).

Proposition 2.4.1. An exterior differential system \( I \) is closed if and only if the exterior differential ideal generated by \( I \) is equal to the exterior ideal generated by \( I \). In particular, \( I \cup dI \) is closed.

2.5 Solutions of an Exterior Differential System

Definition 2.5.1. Let \( I \subset A(M) \) be an exterior differential system and \( N \) a sub-manifold of \( M \). \( N \) is an integral manifold of \( I \) if \( i^*\omega = 0, \forall \omega \in I \), where \( i \) is an embedding \( i : N \rightarrow M \).
3 Introduction to Cartan-Kähler Theory

We consider in this section, an \( m \)-dimensional real manifold \( M \) and \( I \subset \mathcal{A}(M) \), an exterior differential ideal on \( M \).

3.1 Integral Elements

Definition 3.1.1. Let \( z \in M \) and \( E \subset T_zM \) a linear subspace of \( T_zM \). \( E \) is an integral element of \( I \) if \( \varphi_E = 0 \) for all \( \varphi \in I \). We denote by \( V_p(I) \) the set of \( p \)-dimensional integral elements of \( I \).

Definition 3.1.2. \( N \) is an integral manifold of \( I \) if and only if each tangent space of \( N \) is an integral element of \( I \).

Proposition 3.1.1. If \( E \) is a \( p \)-dimensional integral element of \( I \), then every subspace of \( E \) are also integral elements of \( I \).

We denote by \( \mathcal{I}_p = I \cap \mathcal{A}^p(M) \) the set of differential \( p \)-forms of \( I \).

Proposition 3.1.2. \( V_p(I) = \{ E \in \mathcal{G}_p(TM) | \varphi_E = 0 \text{ for all } \varphi \in \mathcal{I}_p \} \)

Definition 3.1.3. Let \( E \) an integral element of \( I \). Let \( \{ e_1, e_2, \ldots, e_p \} \) a basis of \( E \subset T_zM \). The polar space of \( E \), denoted by \( H(E) \), is the vector space defined as follow:

\[
H(E) = \{ v \in T_zM | \varphi(v, e_1, e_2, \ldots, e_p) = 0 \text{ for all } \varphi \in \mathcal{I}_{p+1} \}. \tag{14}
\]

Notice that \( E \subset H(E) \). This implies that a differential form is alternate. The polar space plays an important role in exterior differential system theory as we shall see in the following proposition.

Proposition 3.1.3. Let \( E \in V_p(I) \) be an \( p \)-dimensional integral element of \( I \). A \((p+1)\)-dimensional vector space \( E^+ \subset T_zM \) which contains \( E \) is an integral element of \( I \) if and only if \( E^+ \subset H(E) \).

In order to check if a given \( p \)-dimensional integral element of an exterior differential ideal \( I \) is contained in a \((p+1)\)-dimensional integral element of \( I \), we introduce the following function \( r : V_p(I) \to \mathbb{Z}, r(E) = \dim H(E) - (p+1) \) is a relative integer, \( \forall E \in V_p(I) \).

Notice that \( r(E) \geq 1 \). If \( r(E) = -1 \), then \( E \) is contained in any \((p+1)\)-dimensional integral element of \( I \).
3.1.1 Kähler-Ordinary and Kähler-Regular Integral Elements

Let $\Delta$ a differential $n$-form on a $m$-dimensional manifold $M$. Let $G_n(TM,\Delta) = \{ E \in G_n(TM) | \Delta_E \neq 0 \}$. We denote by $\mathcal{V}_n(I,\Delta) = \mathcal{V}_n(I) \cap G_n(TM,\Delta)$ the set of integral elements of $I$ on which $\Delta_E \neq 0$.

**Definition 3.1.4.** An integral element $E \in \mathcal{V}_n(I)$ is called Kähler-ordinary if there exists a differential $n$-form $\Delta$ such that $\Delta_E \neq 0$. Moreover, if the function $r$ is locally constant in some neighborhood of $E$, then $E$ is said Kähler-regular.

3.1.2 Integral Flags, Ordinary and Regular Integral Elements

**Definition 3.1.5.** An integral flag of $I$ on $z \in M$ of length $n$ is a sequence of integral elements $E_k$ of $I$: $(0)_{z} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_z M$.

**Definition 3.1.6.** Let $I$ be an exterior differential system on $M$. An integral element $E \in \mathcal{V}(I)$ is said ordinary if its base point $z \in M$ is an ordinary zero of $I_0 = I \cap A^0(M)$ and if there exists an integral flag $(0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n = E \subset T_z M$ where the $E_k$, $k = 1, \ldots, (n-1)$ are Kähler-regular integral elements. Moreover, if $E$ is Kähler-regular, then $E$ is said regular.

3.2 Cartan’s Test

**Theorem 3.1.** (Cartan’s test)

Let $I \subset A^*(M)$ be an exterior ideal which does not contain 0-forms (functions on $M$). Let $(0)_{z} \subset E_1 \subset E_2 \subset \cdots \subset E_n \subset T_z M$ be an integral flag of $I$. For any $k < n$, we denote by $c_k$ the codimension of the polar space $H(E_k)$ in $T_z M$. Then $\mathcal{V}_n(I) \subset G_n(TM)$ is at least of $c_0 + c_1 + \cdots + c_{n-1}$ codimension at $E_n$. Moreover, $E_n$ is an ordinary integral flag if and only if $E_n$ has a neighborhood $U$ in $G_n(TM)$ such that $\mathcal{V}_n(I) \cap U$ is a manifold of $c_0 + c_1 + \cdots + c_{n-1}$ codimension in $U$.

**Proof.** See [1], page 74.

**Proposition 3.2.1.** Let $I \cap A^*(M)$ an exterior ideal which do not contains 0-forms. Let $E \in \mathcal{V}_n(I)$ be an integral element of $I$ at the point $z \in M$. Let $\omega_1, \omega_2, \ldots, \omega_n, \pi_1, \pi_2, \ldots, \pi_s$ (where $s = \dim M - n$) be a coframe in a open neighborhood of $z \in M$ such that $E = \{ v \in T_z M / \pi_a(v) = 0 \text{ for all } a = 1, \ldots, s \}$. For all $p \leq n$, we define $E_p = \{ v \in E | \omega_k(v) = 0 \text{ for all } k > p \}$. Let $\{ \varphi_1, \varphi_2, \ldots, \varphi_r \}$ be the set of differential forms which generate the exterior $G_n(TM)$ is the Grassmanian of $TM$, i.e. the set of $n$-dimensional subspace of $TM$. 

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6$G_n(TM)$ is the Grassmanian of $TM$, i.e. the set of $n$-dimensional subspace of $TM$. 

---
ideal \( I \), where \( \varphi_\rho \) is of \((d_\rho + 1)\) degree.

For all \( \rho \), there exists an expansion

\[
\varphi_\rho = \sum_{|J| = d_\rho} \pi^J_\rho \wedge \omega_J + \tilde{\varphi}_\rho
\]

where the 1-forms \( \pi^J_\rho \) are linear combinations of the forms \( \pi \) and the terms \( \tilde{\varphi}_\rho \) are, either of degree 2 or more on \( \pi \), or vanish at \( z \).

Moreover, we have

\[
H(E_p) = \{ v \in T_z M | \pi^J_\rho(v) = 0 \text{ for all } \rho \text{ and } \sup J \leq p \}
\]

In particular, for the integral flag \((0)_z \subset E_1 \subset E_2 \subset \cdots \subset E_n \cap T_z M \) de \( I \), \( c_p \) correspond to the number of linear independent forms \( \{ \pi^J_\rho \} \) such that \( \sup J \leq p \).

Proof. See [1], page 80.

\[\Box\]

### 3.3 Cartan-Kähler’s Theorem

The following theorem is a generalization of the well-known Frobenius’s theorem.

**Theorem 3.2. (Cartan-Kähler)**

Let \( I \subset \mathcal{A}^r(M) \) be a real analytic exterior differential ideal. Let \( P \subset M \) a \( p \)-dimensional connected real analytic Kähler-Regular integral manifold of \( I \).

Suppose that \( r = r(P) \geq 0 \). Let \( R \subset M \) be a real analytic submanifold of \( M \) of codimension \( r \) which contains \( P \) and such that \( T_x R \) and \( H(T_x P) \) are transversals in \( T_x M \) for all \( x \in P \subset M \).

There exists a \((p + 1)\)-dimensional connected real analytic integral manifold \( X \) of \( I \), such that \( P \subset X \subset R \). \( X \) is unique in the sense that another integral manifold of \( I \) having the stated properties, coincides with \( X \) on an open neighborhood of \( P \).

Proof. See [1], page 82.

The analicity condition of the exterior differential ideal is crucial because of the requirements in the Cauchy-Kowalewski theorem used in the proof of the Cartan-Kähler theorem.

Cartan-Kähler’s theorem has an important corollary. Actually, this corollary is often more used than the theorem and it is sometimes called the Cartan-Kähler theorem.
**Corollary 3.3.1. (Cartan-Kähler)**
Let $\mathcal{I}$ be an analytic exterior differential ideal on a manifold $M$. If $E \subset T_z M$ is an ordinary integral element of $\mathcal{I}$, there exists an integral manifold of $\mathcal{I}$ passing through $z$ and having $E$ as a tangent space at the point $z$.

## 4 Local Isometric Embedding Problem

We shall state and prove the Burstin-Cartan-Janet-Schlaefli’s theorem concerning local isometric embedding of a real analytic Riemannian manifold. The names of the mathematicians are given in alphabetic order. Schlaefli in his paper in 1871 [8] conjectured that an $m$-dimensional Riemannian manifold can always be, locally, embedded in an $N = \frac{1}{2} m(m + 1)$ dimensional Euclidean space. In 1926, Janet [6] proved the result for the dimension 2 by resolving a differential system and explain how we get the result in the general case. In 1927, Élie Cartan [3] gave the complete proof of the result. His method is based on his theory of involutive Pfaffian system. Later in 1931, Burstin [2] generalized Janet’s method and obtained the result in the general case.

The proof that we shall give is inspired by Cartan’paper [3], the Bryant, Chern, Gardner, Goldschmidt et Griffiths’s book [1] and the Griffiths et Jensen’s book [4].

### 4.1 The Burstin-Cartan-Janet-Schlaefli theorem

**Theorem 4.1.** (Burstin 1931-Cartan 1927-Janet 1926-Schlaefli 1871)
Every $m$-dimensional real analytic Riemannian manifold can be locally embedded isometrically in an $\frac{m(m + 1)}{2}$-dimensional Euclidean space.

### 4.2 Proof of Burstin-Cartan-Janet-Schlaefli’s theorem

**Steps of the proof of theorem 4.1.**

1. We shall write down the Cartan structure equations for an $m$-dimensional real analytic Riemannian manifold $M$.
2. We shall define a subbundle $\mathcal{F}_m(\mathbb{E}^N)$ of the bundle $\mathcal{F}(\mathbb{E}^N)$ of the Euclidean space $\mathbb{E}^N$, then shall write down the Cartan structure equations for the subbundle $\mathcal{F}_m(\mathbb{E}^N)$.
3. Given an exterior differential system $I_0$ on $M \times \mathcal{F}_m(\mathbb{E}^N)$, which is not close, we shall prove Claim 4.2.2, which proves that the existence of a
local isometric embedding of $M$ is the existence of an $m$-dimensional integral manifold of $I_0$.

4. We will extend this differential system to obtain a closed one. In the process of extension, we will get new equations (the Gauss equation (equ. 29)). We will also show that a closed exterior differential system $I$ with fewer 1-forms than $I$, will generate the same differential ideal that the one generated by $I$ if the Gauss’s equation is satisfied.

5. We establish the lemma 4.2.1., that ensure that the Gauss equations is a surjective submersion. We shall obtain a submanifold with a known dimension.

6. Given the closed exterior differential ideal, we shall prove the existence of an ordinary integral element by using claim 3.2.1 and the Cartan test. Finally, the Cartan-Kahler theorem ensure then the existence of an integral manifold and lead us to conclude.

Step 1:

Let $(M,g)$ be an $m$-dimensional real analytic Riemannian manifold, where $g$ is a Riemannian metric, i.e. a covariant symmetric positive defined 2-tensor, such that at a given point $p$ of $M$, $g_p$ in an orthonormal basis reduce to the identity matrix. However in a open neighborhood of $p$, the matrix of $g$ can not always be the identity but it can always be reduced to diagonal matrix:

$$g = g_{11}dx^1 \otimes dx^1 + g_{22}dx^2 \otimes dx^2 + \cdots + g_{mm}dx^m \otimes dx^m$$ (17)

where the terms $g_{ii}$ are positive functions such that $g_{ii} = 1$ at $p$. We denote than $\eta_i = \sqrt{g_{ii}} dx^i$. $g$ can be written as follows:

$$g = \eta_1 \otimes \eta_1 + \eta_2 \otimes \eta_2 + \cdots + \eta_m \otimes \eta_m$$ (18)

$\eta = (\eta_1, \eta_2, \ldots, \eta_m)$ is than an orthonormal coframe in the neighborhood of $p \in M$. We can establish the Cartan’s structure equations:

**Cartan’s structure equations on $M$:**

$$\begin{cases}
d\eta_i + \sum_{j=1}^{m} \eta_j \wedge \eta_j = 0 \quad \text{(torsion-free)} \\
d\eta_{ij} + \sum_{k=1}^{m} \eta_{ik} \wedge \eta_{kj} = \Omega_{ij}
\end{cases}$$ (19)
where \((\eta_{ij})\) is the matrix of 1-form of the Lévi-Civita’s connection on \(M\) (a torsion-free connection compatible with the metric \(g\)). \(\Omega_{ij}\) is the curvature 2-form of the connection.

**Step 2:**

Let \(\mathbb{E}^N\) be an \(N\)-dimensional Euclidean space (for the moment, \(N > m\)) endowed with the usual scalar product \(\varepsilon_N\). Let us consider \(\mathcal{F}(\mathbb{E}^N)\) a positively-oriented orthonormal frame bundle on \(\mathbb{E}^N\). In what follows, we will not work on the entire bundle \(\mathcal{F}(\mathbb{E}^N)\), but on a quotient. An element in \(\mathcal{F}_m(\mathbb{E}^N)\) has the form \((x; e_1, e_2, \ldots, e_m)\), where \(x \in \mathbb{E}^N\) and \((e_1, e_2, \ldots, e_m)\) is a positively-oriented orthonormal set of vectors in \(\mathbb{E}^N\). We can consider \(\mathcal{F}_m(\mathbb{E}^N)\) as follows: among all the positively-oriented orthonormal frames of \(\mathcal{F}(\mathbb{E}^N)\), we take the frames such that the first \(m\) elements form a positively-oriented orthonormal set of vectors, then we take the \(m\) first vectors of these frames. So, \(\mathcal{F}_m(\mathbb{E}^N)\) is diffeomorphic to \(\mathbb{E}^N \times \frac{SO(N)}{SO(N - m)}\).

**Proposition 4.2.1.**

\[
\dim \mathcal{F}_m(\mathbb{E}^N) = N(m + 1) - \frac{m(m + 1)}{2} \quad (20)
\]

We define on \(\mathcal{F}(\mathbb{E}^N)\) a set of 1-forms as follows\(^7\):

\[
\omega_\mu = e_\mu dx \quad \text{and} \quad \omega_{\mu\nu} = e_\mu de_\nu = -e_\nu de_\mu = -\omega_{\nu\mu} \quad (21)
\]

So \((\omega_1, \omega_2, \ldots, \omega_m; \omega_{m+1}, \ldots, \omega_N)\) form an orthonormal coframe of \(\mathcal{F}(\mathbb{E}^N)\). Than the Cartan structure equations on \(\mathcal{F}_m(\mathbb{E}^N)\) are:

\[
\begin{cases}
d\omega_\mu + \sum_{\nu=1}^{N} \omega_{\mu\nu} \wedge \omega_\nu = 0 \quad (\text{torsion-free}) \\
\sum_{\lambda=1}^{N} \omega_{\mu\lambda} \wedge \omega_{\lambda\nu} = 0 \quad (\text{flat curvature})
\end{cases} \quad (22)
\]

Notice that \((\omega_{\mu\nu})\) is the \(N \times N\) skew-symmetric matrix connection form of the Lévi-Civita connection on \(\mathbb{E}^N\).

\(^7\)The indices \(i, j\) and \(k\) vary from 1 to \(m\), the indexes \(a, b\) and \(c\) vary from \(m+1\) to \(N\) and the indexes \(\mu, \nu\) and \(\lambda\) vary from 1 to \(N\).
Step 3:
Let consider the product manifold $M \times \mathcal{F}_m(\mathbb{E}^N)$. Let $\mathcal{I}_0$ be the exterior ideal on $M \times \mathcal{F}_m(\mathbb{E}^N)$ generated by the Pfaffian system $I_0 = \{ (\omega_i - \eta_i), \omega_\alpha \}$.

**Proposition 4.2.2.** Every $m$-dimensional integral manifold of $\mathcal{I}_0$ on which the form $\Delta = \omega_1 \wedge \cdots \wedge \omega_m$ does not vanish locally is the graph of a function $f : M \rightarrow \mathcal{F}_m(\mathbb{E}^N)$ having the property that $u = x \circ f : M \rightarrow \mathbb{E}^N$ is a local isometric embedding.

Step 4:
According to proposition 4.2.2., the existence of an integral manifold of $\mathcal{I}_0$ for which $\Delta$ is non zero, is a necessary condition for the existence of a local isometric embedding. However, the theorems and the results that we discussed deal with closed exterior differential system. Therefore it is natural to add to the Pfaffian system $I_0$ the exterior differentiation of each 1-form. We obtain so a closed exterior differential system: $I_0 \cup dI_0$. When we compute the exterior differentiation of $(\omega_i - \eta_i)$, we remark new differential forms and an interesting result,

\[
d(\omega_i - \eta_i) = - \sum_{j=1}^m (\omega_{ij} - \eta_{ij}) \wedge \omega_i = 0
\]  

(23)

By Cartan’s lemma, $\omega_{ij} - \eta_{ij} = \sum_{k=1}^m h_{ijk}\omega_k$, with $h_{ijk} = h_{ikj} = -h_{jik}$. With the symmetry and the skew-symmetry of the functions $h_{ijk}$, we conclude that $h_{ijk}$ are zero and so, $\omega_{ij} - \eta_{ij} = 0$. This result has a geometric interpretation: $\omega_{ij} - \eta_{ij} = 0$ implies that $f^*(\omega_{ij}) = \eta_{ij}$ where $f$ is the function of proposition 4.2.2, which means that the pull-back of Lévi-Civita connection by an isometric embedding is the Lévi-Civita connection on $M$.

So, we extend the exterior differential $I_0$ and we obtain an exterior differential system on $M \times \mathcal{F}_m(\mathbb{E}^N)$ $I_1 = \{ (\omega_i - \eta_i), \}

\[8\text{Conversely, each local isometric embedding } u : M \rightarrow \mathbb{E}^N \text{ come uniquely from this construction.}\]
\( \eta_i \) \( i = 1, \ldots, m \), \((\omega_a)_{a=m+1}, \ldots, N, (\omega_{ij} - \eta_{ij})_{1 \leq i < j \leq m} \). In order to have a closed one, we add the exterior differentiation of each form and we get
\( I = I_1 \cup dI_1 \). We denote \( \mathcal{I} \) the exterior differential ideal generated by
\( I = \{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), d(\omega_i - \eta_i), d\omega_a, d(\omega_{ij} - \eta_{ij})\} \).

Instead of looking for integral manifold of \( I_0 \), we will look for the existence of an integral manifold of \( \mathcal{I} \).

From the structure equations stated earlier, we obtain the following system:

\[
\begin{align*}
    d(\omega_i - \eta_i) & \equiv 0 \mod I_1 \\
    d\omega_a & \equiv -\sum_{i=1}^{m} \omega_{ai} \wedge \omega_i \mod I_1 \\
    d(\omega_{ij} - \eta_{ij}) & \equiv \sum_{a=m+1}^{N} \omega_{ai} \wedge \omega_{aj} - \Omega_{ij} \mod I_1
\end{align*}
\]  \hspace{1cm} (24)

On \( X \), the integral manifold of \( X \), \( \omega_a = 0 \), so \( d\omega_a = 0 \) too. We conclude that \( \sum_{i=1}^{m} \omega_{ai} \wedge \omega_i = 0 \). The Cartan lemma (lemma 1.5.1., page 300) ensures the existence of \( m^2 \) functions \( h_{aij} \) such that \( \omega_{ai} = \sum_{j=1}^{m} h_{aij} \omega_j \) where \( h_{aij} = h_{aji} \).

We can write then: \( \omega_{ai} - \sum_{j=1}^{m} h_{aij} \omega_j = 0 \) on \( X \).

However, nothing lead us to think that this equality will be true outside \( X \). We define then the differential 1-form \( \pi_{ai} \) on \( M \times F_m(E^N) \) as follows

\[
\pi_{ai} = \omega_{ai} - \sum_{j=1}^{m} h_{aij} \omega_j
\]  \hspace{1cm} (25)

Consider now, the last equation of the system (24)

\[
    d(\omega_{ij} - \eta_{ij}) \equiv \sum_{a=m+1}^{N} \omega_{ai} \wedge \omega_{aj} - \Omega_{ij} \mod I
\]  \hspace{1cm} (26)

On \( X \), \( \omega_{ij} - \eta_{ij} = 0 \), so \( d(\omega_{ij} - \eta_{ij}) = 0 \). restricted to \( X \), (26) becomes

\[
    \sum_{a=m+1}^{N} \omega_{ai} \wedge \omega_{aj} = \Omega_{ij}.
\]  \hspace{1cm} (27)
Using (25), we can write (27) as follows

\[ \Omega_{ij} = \sum_{k,l=1}^{m} \left( \sum_{a=m+1}^{N} (h_{aij}h_{ajl} - h_{ail}h_{ajk}) \right) \omega_k \otimes \omega_l \]

On \( X \): \( \Omega_{ij} = \sum_{k,l=1}^{m} R_{ijkl} \eta_k \otimes \eta_l \), from \( \Omega_{ij} = \sum_{k,l=1}^{m} R_{ijkl} \eta_k \otimes \eta_l = \sum_{k,l=1}^{m} R_{ijkl} \omega_k \otimes \omega_l \), we conclude that

\[ \sum_{a=m+1}^{N} (h_{aij}h_{ajl} - h_{ail}h_{ajk}) = R_{ijkl} \]

Equation (29) is called the Gauss equation.

We see that the exterior differential system \( \tilde{I} = \{(\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai}\} \) when the Gauss’s equation is satisfied, generates the exterior differential ideal \( \tilde{I} \). Actually, the 1-forms \( (\omega_i - \eta_i) \) and \( \omega_a \) belong to \( I \) and to \( \tilde{I} \). The 1-forms \( (\omega_{ij} - \eta_{ij}) = 0 \). This implies that \( d(\omega_i - \eta_i) = 0 \). The 1-forms \( \pi_{ai} = 0 \), so \( d\omega_a = 0 \). From the Gauss equation, \( d(\omega_{ij} - \eta_{ij}) = 0 \). Looking for integral elements of \( I \) is equivalent to looking for integral elements of \( \tilde{I} \) for which the Gauss equation is satisfied. We shall proceed this in the following steps. Moreover, \( \tilde{I} \) contains less differential 1-form than the exterior differential system \( I \).

Step 5:

The functions \( h_{aij} \) are symmetric in their two last indeces. If we consider an \((N - m)\)-dimensional euclidean space \( \mathcal{W} \), then the matrix \((h_{aij})\) can be viewed as a symmetric element of \( \mathbb{R}^m(i, j = 1, \ldots, m) \) taking value in \( \mathcal{W} \), i.e. \( (h_{aij}) \in \mathcal{W} \otimes S^2(\mathbb{R}^m) \). Notice that \( \dim \mathcal{W} \otimes S^2(\mathbb{R}^m) = (N - m) \frac{m(m + 1)}{2} \).

**Proposition 4.2.3.** Let \( K_m \) the set of Riemannian curvature tensors \( \mathcal{R} \) such that:

1. \( \mathcal{R}_{ijkl} = \mathcal{R}_{klij} \).
2. \( \mathcal{R}_{ijkl} = -\mathcal{R}_{jikl} \).
3. \( \mathcal{R}_{ijkl} + \mathcal{R}_{kijl} + \mathcal{R}_{jkl} = 0 \).

where the indeces \( i, j, k \) and \( l \) vary from 1 to \( m \). Then

\[ \dim K_m = \frac{m^2(m^2 - 1)}{12} \]
Lemma 4.2.1. Suppose that \( r = N - m \geq \frac{m(m - 1)}{2} \). Let \( \mathcal{H} \subset \mathcal{W} \otimes S^2(\mathbb{R}^m) \) an open set containing the elements \( h = (h_{ij}) \) such that the vectors \( \{h_{ij}|1 \leq i \leq j \leq m - 1\} \) are linearly independents as elements of \( \mathcal{W} \). The map \( \gamma : \mathcal{H} \rightarrow \mathcal{K}_m \) that for \( h \in \mathcal{H} \) associate \( \gamma(h) \in \mathcal{K}_m \) such that

\[
\left( \gamma(h) \right)_{ijkl} = \sum_{a=m+1}^{N} h_{ai}h_{aj} - h_{aj}h_{jl},
\]

is a surjective submersion.

Step 6: The existence of an \( m \)-dimensional ordinary integral element

Let \( I \) the exterior ideal of \( M \times \mathcal{F}_m(\mathbb{E}^N) \) generated by \( s = N(m + 1) - \frac{m(m + 1)}{2} \) 1-forms:

\[
\left\{ (\omega^i_1 - \eta^i_1), \ldots, (\omega^i_m), (\omega^a_{m+1}, \ldots, N^a), (\omega_{ij} - \eta_{ij}), \ldots, (\pi_{ai}), \ldots, m, a = m + 1, \ldots, N \right\}
\]

Let \( Z = \{(x, \Upsilon, h) \in M \times \mathcal{F}_m(\mathbb{E}^N) \times \mathcal{H} | \gamma(h) = \mathcal{R}(x) \} \). \( Z \) is a submanifold (the fiber of \( \mathcal{R} \) by a submersion. The surjectivity of \( \gamma \) ensure that \( Z \neq \emptyset \)). So,

\[
\dim Z = \dim M + \dim \mathcal{F}_m(\mathbb{E}^N) + \dim \mathcal{H} - \dim \mathcal{K}_m
\]

\[
= \frac{m}{\dim M} + N(m + 1) - \frac{m(m + 1)}{2} + \frac{(N - m)m}{2} - \frac{m^2(m^2 - 1)}{12}
\]

(31)

We define the map \( \Phi : Z \rightarrow \mathcal{V}_m(I, \Delta) \) that associate to \((x, \Upsilon, h)\) the \( m \)-plane at \((x, \Upsilon)\) annihilated by the 1-forms that generate \( I \) (the exterior differential system \( \mathcal{I} \)). The map \( \Phi \) is an embedding and so \( \Phi(Z) \) is a submanifold of \( \mathcal{V}_m(I, \Delta) \). We will show that \( \Phi(Z) \) contains only ordinary integral elements. In the proof, we will use the proposition 3.2.1.

Let \((x, \Upsilon, h) \in Z\) be a point. Let denote \( E = \Phi(x, \Upsilon, h) \) the integral element defined as follows: \( E = \{v \in T_{(x, \Upsilon)}(M \times \mathcal{F}_m(\mathbb{E}^N))| (\omega^i - \eta^i)(v) = \omega^a(v) = (\omega_{ij} - \eta_{ij})(v) = \pi_{ai}(v) = 0 \}\).

\( E \) is an \( m \)-dimensional integral element. As a matter of fact, \( s \) is the number of differential forms that generate the ideal \( I \) and
\[ \dim \left( M \times \mathcal{F}_m(\mathbb{E}^N) \right) - m = N(m + 1) - \frac{m(m + 1)}{2} = s. \]

We will apply word by word the proposition 3.2.1. Let \( \mathcal{I} \) the exterior ideal of \( M \times \mathcal{F}_m(\mathbb{E}^N) \) defined above\(^9\). This ideal does not contain 0-forms. \( E \in \mathcal{V}_m(\mathcal{I}) \) at \((x, \Upsilon) \in M \times \mathcal{F}_m(\mathbb{E}^N)\). Let \( \omega_i, (\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai} \) be a coframe\(^10\) of \( M \times \mathcal{F}_m(\mathbb{E}^N) \) in the neighborhood of \((x, \Upsilon)\) such that\(^11\) \( E = \{ v \in T_{x, \Upsilon} \left( M \times \mathcal{F}_m(\mathbb{E}^N) \right) \mid (\omega_i - \eta_i)(v) = \omega_a(v) = (\omega_{ij} - \eta_{ij})(v) = \pi_{ai}(v) = 0 \} \).

For \( p \leq m \), we define the \( p \)-dimensional integral element\(^12\) \( E_p = \{ x \in E \mid \omega_k(v) = 0 \ \text{pour tout} \ k > p \} \).\(^13\). We obtain so, an integral flag \((0)_{(x, \Upsilon)} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_m \subset T_{(x, \Upsilon)} \left( M \times \mathcal{F}_m(\mathbb{E}^N) \right) \). We remind that \( I = \{ (\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), d(\omega_i - \eta_i), d\omega_a, d(\omega_{ij} - \eta_{ij}) \} \).

By computing \( d(\omega_i - \eta_i), d\omega_a \) and \( d(\omega_{ij} - \eta_{ij}) \), we shall find the differential forms that are linear combinations of the forms which generate \( \mathcal{I} \).\(^14\)

After simple calculations, we find that

\[ d\omega_a \equiv - \sum_{i=1}^{m} \pi_{ai} \wedge \omega_i \quad (32) \]

and

\[ d(\omega_{ij} - \eta_{ij}) = \sum_{a=m+1}^{N} \pi_{ai} \wedge \pi_{aj} + \sum_{k=1}^{m} \left( \sum_{a=m+1}^{N} h_{ajk} \pi_{ai} - h_{aij} \pi_{aj} \right) \wedge \omega_k \quad (33) \]

the term \( \spadesuit \) is quadratic in \( \pi_{ai} \) and vanishes on \( \mathbb{K} \).\(^15\)

\(^9\)\( M \times \mathcal{F}_m(\mathbb{E}^N) \) play the role of the manifold "M" in the proposition 3.2.1.
\(^10\)There is \( m + s = \dim \left( M \times \mathcal{F}_m(\mathbb{E}^N) \right) \) 1-forms.
\(^11\)the \( (\omega_i)_{i=1, \ldots, m} \) play the role of "\( \omega_1, \omega_2, \ldots, \omega_m \)". the \( (\omega_i - \eta_i), \omega_a, (\omega_{ij} - \eta_{ij}), \pi_{ai} \) play the role of "\( \pi_\rho \)" in the proposition 3.2.1.
\(^12\)the exterior differential system \( I \) play the role of \( \{ \varphi_1, \varphi_2, \ldots, \varphi_r \} \) in the proposition 3.2.1.
\(^13\)\( E_p \in \mathcal{V}_p(\mathcal{I}, \Delta) \) Because it is annihilated by \( s + m - p \) differential 1-forms .
\(^14\)the forms that play the role of \( \pi_\rho \) in the proposition 3.2.1.
\(^15\)\( \spadesuit \) play the role of \( \hat{\varphi}_p \) in the proposition 3.2.1.
According to proposition 3.2.1., \( c_p \) represents the number of linear independent differential 1-forms.

| The differential 1-forms | The indexes | Number of linear independent 1-forms |
|-------------------------|-------------|-------------------------------------|
| \( \omega_i - \eta_k \) | \( 1 \leq i \leq m \) | \( m \) |
| \( \omega_a \) | \( m + 1 \leq a \leq N \) | \( N - m \) |
| \( \omega_{ij} - \eta_{kj} \) | \( 1 \leq i < j \leq m \) | \( \frac{m(m - 1)}{2} \) |
| \( \pi_{ai} \) | \( 1 \leq i \leq p, \) \( m + 1 \leq a \leq N \) | \( (N - m)p \) |
| \( \sum_{a=m+1}^{N} (h_{aik}\pi_{aj} - h_{ajk}\pi_{ai}) \) | \( 1 \leq k \leq p, \) \( 1 \leq i \leq j \leq m \) | \( p\frac{(m - p)(m - p - 1)}{2} + \frac{p(p + 1)(m - p)}{2} \) |

Finally, by the sum of the number of linear independent 1-forms of the above table, \( c_p \) is the codimension of \( H(E_p) \) in \( G_m\left(T(M \times F_m\mathbb{E}^N)\right) \) defined earlier, and is equal to:

\[
c_p = N + \frac{m(m - 1)}{2} + (N - m)p + \frac{mp(m - p)}{2}
\]

(34)

so,

\[
\sum_{p=0}^{m-1} c_p = \frac{Nm(m + 1)}{2} + \frac{m^2(m^2 - 1)}{12}.
\]

(35)

To apply the proposition 3.2.1 and show that \( E \) is an \( m \)-dimensional ordinary integral element, we need just to compute the codimension of \( \Phi(Z) \) in \( G_m(\mathcal{I}, \Delta) \).

Let \( \mathcal{U} \) an open set of \( F_m(\mathbb{E}^N) \). So, \( \dim(M \times \mathcal{U}) = N(m - 1) - \frac{m(m + 1)}{2} + m \). We remind that if \( E \) is an \( n \)-dimensional euclidean space, then the space of all \( p \)-planes of \( E \) \((G_p(E))\), with \( p < n \), is of \( p(n-p) \) dimension. Let \((x, \Upsilon) \in M \times \mathcal{U}, \)}
\[
\text{dim} G_m \left( T(M \times \mathfrak{U}) \right) = \text{dim} G_m \left( T(x, \mathfrak{Y}) (M \times \mathfrak{U}) \right) + \text{dim} (M \times \mathfrak{U}) \\
= m \left( N(m + 1) - \frac{m(m + 1)}{2} + m - m \right) + N(m + 1) - \frac{m(m + 1)}{2} + m \\
= m \left( N(m + 1) - \frac{m(m + 1)}{2} \right) + N(m + 1) - \frac{m(m + 1)}{2} + m 
\]

Since that \( \Phi \) is an embedding, \( \text{dim} \Phi(\mathcal{Z}) = \text{dim} \mathcal{Z} \), so

\[
\text{dim} G_m \left( T(M \times \mathfrak{U}) \right) - \text{dim} \Phi(\mathcal{Z}) = \\
Nm(m + 1) - m - Nm(m + 1) + \frac{m(m + 1)}{2} + \frac{m(m + 1)}{2} + \frac{m^2(m^2 - 1)}{12} \\
= \frac{Nm(m + 1)}{2} + \frac{m^2(m^2 - 1)}{12} 
\]

We conclude that the codimension of \( \Phi(\mathcal{Z}) \) in \( G_m \left( T(M \times \mathfrak{U}) \right) \) is equal to \( c_0 + c_1 + \cdots + c_{m-1} \). By the Cartan’s test, \( E \in \mathcal{V}_m(\mathcal{I}, \Omega) \) is an ordinary integral element of \( \mathcal{I} \). The Cartan-Kähler theorem (Corollary 3.3.1.) ensure the existence of an integral manifold \( \mathcal{X} \) passing through \( (x, \mathfrak{Y}) \) and having \( E \) as a tangent space at \( (x, \mathfrak{Y}) \).

\( E \in \mathcal{V}_m(\mathcal{I}, \Omega) \), In particular, \( E \in \mathcal{V}_m(\mathcal{I}_0, \Omega) \). By the proposition 4.2.2. , there exists an isometric embedding of \( (M, g) \) in \( (\mathbb{R}^N, \varepsilon_N) \).

## 5 Local Conformal Embedding Problem

### Definition 5.0.1

Let \( (M, g) \) and \( (N, h) \) be two real analytic Riemannian manifolds of dimension \( m \) and \( n \) respectively. Let \( f \) be a map from \( M \) to \( N \). Then \( f \) is a conformal embedding from \( (M, g) \) to \( (N, h) \) if:

1. \( f \) is a local diffeomorphism;
2. $f^* h = Sg$, where $S : M \rightarrow \mathbb{R}^+$ is a strictly positive function on $M$.

**Theorem 5.1.** *(Jacobowitch-Moore [5])*

If $\text{dim} N = n \geq \frac{1}{2} m(m + 1) - 1$, then each point $p \in M$ admit a neighborhood on $M$ which can be conformally embedded in $N$.

Jacobowitch and Moore gave two different proofs of this result; one is based on Janet’s method and the second on Cartan’s method which is close to the proof of Burstin-Cartan-Janet-Schlaefli theorem that we gave.

Roughly speaking, we consider $\mathcal{F}(M) \times \mathcal{F}(N) \times \mathbb{R}^m \times \mathbb{R}^+$ and we look for integral manifolds of $I_0 = \{\omega_i - S\eta_i, \omega_a\}$ (the forms are defined as on the previous section). Similarly, we can extend the exterior differential system to obtain a closed one. We lead the details of the proof for the reader who should take care of $S$ when he apply the exterior differentiation cause it’s a function. When the new exterior differential system is involutive, we look for ordinary integral element and so conclude. (see [5]).

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