Approximating Gromov-Hausdorff Distance in Euclidean Space

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Abstract

The Gromov-Hausdorff distance \(d_{GH}\) proves to be a useful distance measure between shapes. In order to approximate \(d_{GH}\) for compact subsets \(X, Y \subset \mathbb{R}^d\), we look into its relationship with \(d_{H,iso}\), the infimum Hausdorff distance under Euclidean isometries. As already known for dimension \(d \geq 2\), the \(d_{H,iso}\) cannot be bounded above by a constant factor times \(d_{GH}\). For \(d = 1\), however, we prove that \(d_{H,iso} \leq \frac{5}{4} d_{GH}\). We also show that the bound is tight. In effect, this gives rise to an \(O(n \log n)\)-time algorithm to approximate \(d_{GH}\) with an approximation factor of \((1 + \frac{1}{4})\).

Keywords: Gromov-Hausdorff distance, approximation algorithm, abstract distance measures, shape comparison

1. Introduction

This paper grew out of our effort to compute the Gromov-Hausdorff distance between Euclidean subsets. The Gromov-Hausdorff distance between two abstract metric spaces was first introduced by M. Gromov in ICM 1979 (see Berger [1]). The notion, although it emerged in the context of Riemannian metrics, proves to be a natural distance measure between any two (compact) metric spaces. Only in the last decade the Gromov-Hausdorff distance has received much attention from the researchers in the more applied fields. In shape recognition and comparison, shapes are regarded as metric spaces that are deformable under a class of transformations. Depending on the application in question, a suitable class of transformations is chosen, then the dissimilarity between the shapes are defined by a suitable notion of distance measure or error that is invariant under the desired class of transformations. For comparing Euclidean shapes under Euclidean isometry, the use of Gromov-Hausdorff distance is proposed and discussed in [2,3,4,5].

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In this paper, we are primarily motivated by the questions pertaining to the computation of the Gromov-Hausdorff distance, particularly between Euclidean subsets. Although the distance measure puts the Euclidean shape matching on a robust theoretical foundation [2, 3], the question of computing the Gromov-Hausdorff distance, or even an approximation thereof, still remains elusive. In the recent years, some efforts have been made to address such computational aspects. Most notably, the authors of [6] show an NP-hardness result for approximating the Gromov-Hausdorff distance between metric trees. For Euclidean subsets, however, the question of a polynomial-time algorithm is still open. In [5], the authors show that computing the Gromov-Hausdorff distance is related to various NP-hard problems and study a variant of the distance measure.

The authors of [7] introduce the additive distortion—the one used in Gromov-Hausdorff distance—and consider the problem of minimizing the distortion of bijective functions between sets of the same cardinality. For the real-line case ($d = 1$), the authors demonstrate a polynomial-time 2-approximation algorithm [7, Theorem 6]. An open problem is also posed in [7]: are there polynomial-time approximations to find the distortion-minimizing bijection within a factor less than 2? Although the context is similar, we consider minimizing the distortion of correspondences between sets of possibly different cardinalities. As we see in Definition 1.5, a correspondence between two sets is a more general relation than a function.

All these computational concerns provoke the natural curiosity about the genuine computational hardness of the Gromov-Hausdorff distance between Euclidean subsets. Let $d \geq 1$ and $X, Y \subseteq \mathbb{R}^d$ be compact sets equipped with the standard Euclidean metric, and let $d_{GH}(X, Y)$ denote their Gromov-Hausdorff distance. One then wonders:

(i) Is there an algorithm to compute $d_{GH}(X, Y)$ exactly in polynomial-time?

(ii) If not, can we find a polynomial-time approximation algorithm for $d_{GH}(X, Y)$, possibly with a reasonably small approximation factor?

(iii) If not, is it NP-hard to approximate $d_{GH}(X, Y)$, like the metric graph case?

The above questions motivate our investigation into the computation of $d_{GH}$ in Euclidean spaces.

**Background and Related Work.** The notion of Gromov-Hausdorff distance is closely related to the notion of Hausdorff distance. Let $(Z, d_Z)$ be any metric space. We first give a formal definition of the directed Hausdorff distance between any two subsets of $Z$.

**Definition 1.1 (Directed Hausdorff Distance).** For any two compact subsets $X, Y$ of a metric space $(Z, d_Z)$, the directed Hausdorff distance from $X$ to $Y$, denoted $d^H(X, Y)$, is defined by

$$\max_{x \in X} \min_{y \in Y} d_Z(x, y).$$
Unfortunately, the directed Hausdorff distance is not symmetric. To retain symmetry, the *Hausdorff distance* is defined in the following way:

**Definition 1.2 (Hausdorff Distance).** For any two compact subsets $X, Y$ of a metric space $(Z, d_Z)$, their *Hausdorff distance*, denoted by $d^Z_H(X, Y)$, is defined by

$$\max \left\{ -d^Z_H(X, Y), -d^Z_H(Y, X) \right\}.$$  

To keep our notations simple, we drop the superscript when it is understood that $Z$ is taken to be $\mathbb{R}^d$ and $X, Y$ are Euclidean subsets equipped with the standard Euclidean metric $|\cdot|$. The $d^Z_H$ can be computed in $O(n \log n)$-time for finite point sets with at most $n$ points or $n$ line segments; see [8].

We follow Gromov’s book ([9]) to define the Gromov-Hausdorff distance. The primary definition uses the concept of an isometry or distance-preserving map between metric spaces, which we define first.

**Definition 1.3 (Isometry).** A map $f : (X, d_X) \to (Y, d_Y)$ is called an *isometry* if

$$d_X(x_1, x_2) = d_Y(f(x_1), f(x_2)).$$

We immediately note that an isometry $f$ is injective, and that $f : X \to f(X)$ is a homeomorphism.

We are now in a place to define the Gromov-Hausdorff distance formally. Unlike the Hausdorff distance, the Gromov-Hausdorff distance is defined between two abstract metric spaces $(X, d_X)$ and $(Y, d_Y)$ that may not share a common ambient space. We start with the following formal definition:

**Definition 1.4 (Gromov-Hausdorff Distance [9]).** The *Gromov-Hausdorff distance*, denoted by $d_{GH}(X, Y)$, between two metric spaces $(X, d_X)$ and $(Y, d_Y)$ is defined to be

$$d_{GH}(X, Y) = \inf_{f : X \to Z, g : Y \to Z} d^Z_H(f(X), g(Y)),$$

where the infimum is taken over all isometries $f : X \to Z, g : Y \to Z$ and metric spaces $(Z, d_Z)$.

The definition of Gromov-Hausdorff distance may not seem very natural at first glance—it deserves a bit of explanation. As mentioned earlier, the definition works for abstract metric spaces $X$ and $Y$, without requiring them to be embedded in a common ambient metric space. In order to anatomize Definition 1.4 we first observe that the maps $f, g$ are embedding $X$ and $Y$, respectively, into a common metric space $(Z, d_Z)$. Since $f, g$ are isometries, the subsets $f(X), g(Y)$ of $Z$ are isometric to $X$ and $Y$, respectively. As $f(X)$ and $g(Y)$ are subsets of $Z$, their Hausdorff distance $d^Z_H(f(X), g(Y))$ can now be considered. The Gromov-Hausdorff distance is defined to minimize (if the minimum exists) this $d^Z_H(f(X), g(Y))$, subject to all isometries $f, g$ and ambient metric space $(Z, d_Z)$. As a consequence, Gromov-Hausdorff distance is a
distance measure between abstract metric spaces $X$ and $Y$ that is also invariant under any isometric transformations of $X$ or $Y$. A detour to [5, 9, 10] is suggested for a detailed treatment of the definitions and properties of Gromov-Hausdorff distance.

In order to present an equivalent definition of the Gromov-Hausdorff distance that is computationally viable, we first define the notion of a correspondence.

**Definition 1.5** (Correspondence). A **correspondence** $C$ between any two (non-empty) sets $X$ and $Y$ is defined to be a subset $C \subseteq X \times Y$ with the following two properties:

i) for any $x \in X$, there exists a $y \in Y$ such that $(x, y) \in C$, and

ii) for any $y \in Y$, there exists an $x \in X$ such that $(x, y) \in C$.

A correspondence $C$ is a special relation that assigns all points of both $X$ and $Y$ a corresponding point. If the sets $X$ and $Y$ in the Definition 1.5 are equipped with metrics $d_X$ and $d_Y$, respectively, we can also define the distortion of the correspondence $C$.

**Definition 1.6** ((Additive) Distortion of Correspondence). Let $C$ be a correspondence between two metric spaces $(X, d_X)$ and $(Y, d_Y)$, then its **distortion**, denoted $\text{Dist}(C)$, is defined to be

$$\sup_{(x_1, y_1), (x_2, y_2) \in C} |d_X(x_1, x_2) - d_Y(y_1, y_2)|$$

The distortion $\text{Dist}(C)$ is sometimes called the additive distortion as opposed to the multiplicative distortion; see [11] for a definition. In the context of the Gromov-Hausdorff distance, the distortion of a correspondence measures how much the two metrics are distorted by the $C$. In the extreme case, when $C$ is an isometry, its distortion becomes zero. For non-empty sets $X, Y$, we denote by $\mathcal{C}(X, Y)$ the set of all correspondences between $X$ and $Y$. An alternative definition of the Gromov-Hausdorff distance is given in Lemma 1.7; see [10] for a proof.

**Lemma 1.7.** For any two compact metric spaces $(X, d_X)$ and $(Y, d_Y)$, the following holds:

$$d_{GH}(X, Y) = \frac{1}{2} \inf_{C \in \mathcal{C}(X, Y)} \text{Dist}(C).$$

This combinatorial formulation unveils the genuine complexity entailed in the computation of Gromov-Hausdorff distance. For two finite metric spaces $X, Y$ containing at most $n$ points, the computation takes $O(2^n)$-time by enumerating all possible correspondences between the points of $X$ and $Y$.

**Our Contribution.** Our main contribution in this work is to provide a satisfactory answer to the quest of understanding the relation between $d_{H, \text{iso}}$...
(see Definition 2.1) and \( d_{GH} \) when \( X, Y \) are compact subsets of \( \mathbb{R}^1 \) equipped with the standard Euclidean metric.

In Theorem 3.2, we show that
\[
d_{H,iso}(X,Y) \leq \frac{5}{4}d_{GH}(X,Y)
\]
for any compact \( X, Y \subset \mathbb{R}^1 \). For subsets of the real line, it is tempting to believe that \( d_{GH} = d_{H,iso} \). We show in Theorem 3.10 that this is, in fact, not true by showing that the bound \( \frac{5}{4} \) in Theorem 3.2 is tight. Since \( d_{H,iso}(X,Y) \) can be computed in \( O(n \log n) \)-time (12), we provide an \( O(n \log n) \)-time algorithm to approximate \( d_{GH}(X,Y) \) for finite \( X, Y \subset \mathbb{R}^1 \) with an approximation factor of \((1 + \frac{1}{4})\).

2. Gromov-Hausdorff vs Hausdorff Distance under Isometry

With the basic definitions now at our disposal, we make our readers acquainted with a related notion \( d_{H,iso}(X,Y) \) here, and list a few of its relevant consequences. In this section, we also introduce the concept of nearest neighbor correspondences, present some of their properties, and then illuminate the trail that has led us to the pinnacle of our findings of Section 3.

2.1. Comparing \( d_{GH} \) with \( d_{H,iso} \)

For any dimension \( d \geq 1 \), a Euclidean isometry \( T : \mathbb{R}^d \rightarrow \mathbb{R}^d \) is defined to be a map that preserves the distance, i.e.,
\[
|T(x_1) - T(x_2)| = |x_1 - x_2| \quad \text{for all } x_1, x_2 \in \mathbb{R}^d.
\]

When \( d = 1 \), the map \( T \) can only afford to be a translation or a reflection (flip). In \( d = 2 \), a Euclidean isometry is characterized by a combination of a translation, a rotation by an angle, and a reflection about the origin. For more about Euclidean isometries, see [13]. We denote by \( \mathcal{E} (\mathbb{R}^d) \) the set of all isometries of \( \mathbb{R}^d \).

**Definition 2.1 (Hausdorff under Isometry).** For any two compact subsets \( X, Y \) of \( \mathbb{R}^d \), we define
\[
d_{H,iso}(X,Y) = \inf_{T \in \mathcal{E} (\mathbb{R}^d)} d_H(X, T(Y)).
\]

We immediately note that \( d_{H,iso} \) induces a pseudo-metric on the set of compact subsets of \( \mathbb{R}^d \); if \( d_{H,iso}(X,Y) = 0 \), then \( X \) is congruent to \( Y \).

**Remark 2.2.** If \( X, Y \) are subsets of \( \mathbb{R}^1 \) with at most \( n \) points, the authors of [12] prove that their \( d_{H,iso}(X,Y) \) can be computed in \( O(n \log n) \)-time.
The \( d_{H,iso}(X,Y) \) minimizes the Hausdorff distance over only Euclidean isometries; whereas \( d_{GH}(X,Y) \) considers minimizing over all isometries and all embeddings for \( X \) and \( Y \)—not just Euclidean. The observation quickly yields the following inequality:

\[
d_{GH}(X,Y) \leq d_{H,iso}(X,Y).
\]

It is most natural to wonder if they are, in fact, equal. To our disappointment, we can contrive the following configuration in \( \mathbb{R}^2 \) to show the contrary in Example 2.3.

**Example 2.3** (\( d_{GH} < d_{H,iso} \) in \( \mathbb{R}^2 \)). Let us consider two finite sets \( X,Y \subset \mathbb{R}^2 \) as shown in Figure 1, where \( \alpha, h, \epsilon > 0 \) are constants. We take \( X = \{x_1, x_2, x_3, x_4, x_5\} \), \( Y = \{y_1, y_2, y_3, y_4, y_5\} \), and \( x_1 = y_1, x_2 = y_2 \). In a moment’s reflection, we see that the blue edges give us the correspondence with minimum distortion:

\[
C_{opt} = \{(x_1, y_1), (x_2, y_2), (x_4, y_3), (x_5, y_3), (x_3, y_4), (x_3, y_5)\}.
\]

with \( \text{dist}(C_{opt}) = h + \epsilon \). Consequently, \( d_{GH}(X,Y) = \frac{h+\epsilon}{2} \). On the other hand, \( d_{H,iso}(X,Y) = h \). So, for \( \epsilon < h \), we have \( d_{GH}(X,Y) < d_{H,iso}(X,Y) \).

In [4], Mémoli shows the following bounds, relating \( d_{H,iso} \) and \( d_{GH} \) between two compact subsets \( X,Y \) of \( \mathbb{R}^d \).

\[
d_{GH}(X,Y) \leq d_{H,iso}(X,Y) \leq c'_d(M)^{\frac{1}{2}} \sqrt{d_{GH}(X,Y)},
\]

where \( M = \max \{\text{diameter}(X), \text{diameter}(Y)\} \) and \( c'_d \) is a constant that depends only on the dimension \( d \). In the inequality (1), note the upper bound depends on the diameter of the input sets \( X \) and \( Y \). For \( d \geq 2 \), such a dependence is unavoidable. See Figure 1. This leaves us with \( d = 1 \), the compact subsets of the real line.

In \( \mathbb{R}^1 \) it often helps to visualize \( X \times Y \) on the disjoint union of two real lines in \( \mathbb{R}^2 \) and a correspondence \( C \in C(X,Y) \) by edges between the corresponding points; see Figure 2. Such a two dimensional visualization comes in handy for the proofs.
(a) An example correspondence

(b) The standard configuration

Figure 2: On the left, the (sorted) $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, y_3\}$ are identified as subsets of the top and the bottom lines respectively. The points of $X$ are shown in green, and the points of $Y$ are shown in yellow. We visualize the correspondence $C = \{(x_1, y_1), (x_2, y_2), (x_2, y_3)\}$ by the red edges between the respective points. Also, the edges $(x_1, y_1)$ and $(x_2, y_2)$ are crossing.

On the right, the distortion $D$ of a correspondence is attained by the pairs $(x', y')$ and $(x, y)$.

It is tempting to believe, due to the deceptively simple structure of the real line, that $d_{GH} = d_{H, iso}$. If it was true, we could compute $d_{GH}$ in near-linear time; see Remark 2.2. However, the following sophisticated construction shows that the conjecture is false.

**Example 2.4** ($d_{GH} < d_{H, iso}$ in $\mathbb{R}^1$). In this example, we show that for any given $\delta > 0$, there exist compact $X, Y \subset \mathbb{R}^1$ such that $d_{GH}(X, Y) = \delta$ and $d_{H, iso}(X, Y) = \delta + \frac{\delta}{8}$. As a consequence, $d_{GH}(X, Y) < d_{H, iso}(X, Y)$.

Figure 3: The points of $X$ and $Y$ are shown in green and yellow, respectively, on two copies of the real line. The optimal correspondence is shown by the red edges. The distortion for the correspondence is $2\delta$, consequently $d_{GH}(X, Y) = \delta$. We also note that the optimal correspondence is not crossing free.

The subsets $X, Y$ are taken as shown in Figure 3 and $\delta > 0$. We note that $d_{H}(X, Y) = \delta + \frac{\delta}{8}$. Now, we claim that $d_{H, iso}(X, Y) = \delta + \frac{\delta}{8}$. For a proof of our claim, we present in Table 1 the summary of $d_{H}(X, Y + \Delta)$, which considers translations of $Y$ by an amount $\Delta \in \mathbb{R}^1$. We also note that a translation of $-Y$ does not help to reduce the Hausdorff distance.

By our observation in the above example, we are intrigued by the quest of bounding $d_{H, iso}$ from above by a constant multiple of $d_{GH}$. A constant upper bound is presented in Section 3 for $d = 1$, along with the proof that the bound is tight.
\[ \Delta \quad \overrightarrow{d}_H(X, Y + \Delta) \quad \overrightarrow{d}_H(Y + \Delta, X) \quad d_H(X, Y + \Delta) \]

\begin{array}{|c|c|c|c|}
\hline
\Delta & \overrightarrow{d}_H(X, Y + \Delta) & \overrightarrow{d}_H(Y + \Delta, X) & d_H(X, Y + \Delta) \\
\hline
(-\infty, 0) & - & (y_0, x') & > \delta + \frac{\delta}{8} \\
0 & (x, y') & (y_0, x') & \delta + \frac{\delta}{8} \\
(0, \frac{\delta}{8}) & - & (y_k, x), (y_k, x_k) & \in (\delta + \frac{\delta}{8}, \delta + \frac{\delta}{4}) \\
\frac{\delta}{8} & (x, y), (x, y') & - & \delta + \frac{\delta}{4} \\
(\frac{\delta}{8}, \frac{\delta}{4}) & (x, y) & - & \delta + \frac{\delta}{8} \\
\frac{\delta}{4} & (x, y) & (y', x_0) & \delta + \frac{\delta}{8} \\
(\frac{\delta}{4}, \infty) & - & (y', x_0) & > \delta + \frac{\delta}{8} \\
\hline
\end{array}

Table 1: A summary of \( d_H(X, Y + \Delta) \) is recorded for \( \Delta \in \mathbb{R} \) for Example 2.4 shown in Figure 3. In the second and third columns, the directed Hausdorff distances are achieved for the shown pairs of points. The other columns are self-explanatory.

2.2. Nearest Neighbor Correspondence

To conclude our discussion of this section, we lastly present our line of investigation into Hausdorff correspondences. As noted previously, in Example 2.3 and Example 2.4, \( d_{GH}(X, Y) \neq d_{H,iso}(X, Y) \) in general. However, we take the analysis one step further, and explore in Theorem 2.8 that there does not exist a Euclidean isometry \( T \) such that the Hausdorff correspondence between \( X \) and \( T(Y) \) is an optimal correspondence that minimizes the distortion.

Among many possible correspondences, the following correspondence is particularly interesting when considering two Euclidean subsets.

**Definition 2.5** (Nearest Neighbor Correspondence). For any two compact subsets \( X, Y \subset \mathbb{R}^N \), we define the nearest neighbor correspondence \( C_{NN} \) to be the relation defined by the nearest neighbors of points of \( X \) and \( Y \). More precisely,

\[ C_{NN} = \{(x, y) \in X \times Y \mid (x \text{ is a nearest neighbor of } y \text{ in } X) \] or \((y \text{ is a nearest neighbor of } x \text{ in } Y)\}. \]

Since \( X \) and \( Y \) are compact, the nearest neighbors exist. Hence, the induced relation is a correspondence. Now follows an important structural property concerning crossings of a correspondence.

**Definition 2.6** (Crossing). For a correspondence \( C \in \mathcal{C}(X, Y) \), we say a pair of edges \((x_1, y_1)\) and \((x_2, y_2)\) is a **crossing** if they cross in the usual sense, i.e., either of the following happens: \( x_1 < x_2 \) but \( y_1 > y_2 \) or \( x_1 > x_2 \) but \( y_1 < y_2 \); see Figure 2.

**Lemma 2.7** (Crossing). For any two compact \( X, Y \subset \mathbb{R}^1 \), the nearest neighbor correspondence \( C_{NN}(X, Y) \) is free of crossings.
Proof. Let us consider two edges \( e = (x, y) \) and \( e' = (x', y') \) in \( \mathcal{C}_{NN}(X, Y) \) with \( x < x' \). Without loss of generality, we assume that \( y \) is a nearest neighbor of \( x \). In order to show that \( e \) cannot cross \( e' \), we assume the contrary: \( y' < y \).

Now, we consider the following cases based on the position of \( y \) with respect to \( x \). In each case we show that neither \( y' \) is a nearest neighbor of \( x' \) nor \( x' \) is a nearest neighbor of \( y' \). This would contradict the fact that \( (x', y') \in \mathcal{C}(X, Y) \).

**Case 1** \((y \leq x)\): In this case, a nearest neighbor of \( x' \) cannot be smaller than \( y \). Hence, \( y' \) cannot be its nearest neighbor. Also for \( y' < y \), its nearest neighbor has to be also smaller than \( x \), hence \( x' \) cannot be its nearest neighbor either.

**Case 2** \((x < y)\): A nearest neighbor of \( x' \) cannot be smaller than \( y \). Hence, \( y' \) cannot be its nearest neighbor. Since \( y' < y \), we have \( y' \leq x \) in this case. Therefore, a nearest neighbor \( y' \) has to be smaller than \( x \). So, \( x' \) cannot be its nearest neighbor.

We wrap up this section with our final result of this section in the following theorem.

**Theorem 2.8.** For \( d \geq 1 \), there exist compact subsets \( X, Y \subset \mathbb{R}^d \) such that \( \mathcal{C}_{NN}(X, T(Y)) \) is not an optimal correspondence for any Euclidean isometry \( T \in E(\mathbb{R}^d) \).

Proof. For \( d \geq 2 \), we refer the readers to Example 2.3 and Figure 1. We also note here that there does not exist any non-trivial Euclidean isometry \( T \) such that \( \mathcal{C}_{opt} \) becomes the nearest neighbor correspondence.

In the case \( d = 1 \), we use \( X, Y \) from Example 2.4 and Figure 3. Here, \( \mathcal{C}_{opt} \) must have crossings—even when one considers the reflection of \( Y \). By Lemma 2.7, \( \mathcal{C}_{opt} \) cannot be produced by any nearest neighbor correspondence. 

3. Approximating Gromov-Hausdorff Distance in \( \mathbb{R}^1 \)

This section is devoted to our main result (Theorem 3.2) on approximating the Gromov-Hausdorff distance between subsets of the real line. Unless stated otherwise, in this section we always assume that \( X, Y \) are compact subsets of \( \mathbb{R}^1 \) and both are equipped with the standard Euclidean metric denoted by \( |\cdot| \).

**Definition 3.1** (Standard Configuration). Let \( \mathcal{C} \) be any correspondence between two compact subsets \( X, Y \) of \( \mathbb{R}^1 \). There exist \((x, y), (x', y') \in \mathcal{C} \) such that \( |x - x'| - |y - y'| = D \), where \( D \) is the distortion of \( \mathcal{C} \). Without loss of generality, we assume that \( x \leq x' \) and \( |x - x'| \leq |y - y'| \). Then, there exists an \( \mathbb{R}^1 \)-isometry such that, when applied on \( Y \), \((x, y)\) and \((x', y')\) are arranged as in Figure 2(b). For the proofs that follow, we assume this standard configuration for any given correspondence \( \mathcal{C} \), and denote by \( \{(x, y), (x', y')\} \) a fixed pair that has this property.
We have already noted that \( d_{GH}(X,Y) \leq d_{H,iso}(X,Y) \) for any compact \( X,Y \subset \mathbb{R}^d \); see [4]. Together with this, Theorem 3.2 thus gives us the approximation algorithm for \( d_{GH} \) with an approximation factor of \((1 + \frac{1}{4})\). Later in Theorem 3.10 we also show that the upper bound of Theorem 3.2 is tight.

**Theorem 3.2** (Approximation of Gromov-Hausdorff Distance). For any two compact \( X,Y \subset \mathbb{R}^1 \) we have

\[
d_{H,iso}(X,Y) \leq \frac{5}{4} d_{GH}(X,Y).
\]

**Proof.** In order to prove the result, it suffices to show that for any correspondence \( C \in \mathcal{C}(X,Y) \) with \( \text{Dist}(C) = D \), there exists a Euclidean isometry \( T \in \mathcal{E}(\mathbb{R}^1) \) such that

\[
d_H(X,T(Y)) \leq \frac{5D}{8}.
\]

Depending on the crossing (Definition 2.6) behavior, we classify a given correspondence into three main types: no double crossing (Definition 3.3), wide crossing (Definition 3.6), and no wide crossing, and we divide the proof for each type into Theorem 3.4, Theorem 3.8, and Theorem 3.9, respectively. \( \square \)

### 3.1. No Double Crossings

We start with the definition of a double crossing edge.

**Definition 3.3.** An edge in a correspondence \( C \in \mathcal{C}(X,Y) \) is said to be a double crossing if it crosses both the edges \((x',y')\) and \((x,y)\) as defined in Definition 3.1, see Figure 8.

In the following theorem, we consider the case when there is no double crossing edge in \( C \).

**Theorem 3.4** (No Double Crossing). For a correspondence \( C \in \mathcal{C}(X,Y) \) without any double crossing, there exists a value \( \Delta \in \mathbb{R} \) such that

\[
d_{H}(X,Y + \Delta) \leq \frac{5D}{8},
\]

where \( D = \text{Dist}(C) \).

**Proof.** In the trivial case, when \( d_{H}(X,Y) \leq \frac{5D}{8} \), we take \( \Delta = 0 \). We now assume the non-trivial case that \( d_{H}(X,Y) > \frac{5D}{8} \). As shown in Figure 4 we define the following two subsets of \( X \):

\[
A = \{ a \in X \cap (x + D, \infty) \mid \exists b \in Y \cap [y',y] \text{ with } (a,b) \in C \},
A' = \{ a \in X \cap (-\infty, x' - D) \mid \exists b \in Y \cap [y',y] \text{ with } (a,b) \in C \}.
\]

We also define the following two subsets of \( Y \):

\[
B = \{ b \in Y \cap (y' + D, y - D) \mid \exists a \in X \cap (x, \infty) \text{ with } (a,b) \in C \},
B' = \{ b \in Y \cap (y' + D, y - D) \mid \exists a \in X \cap (-\infty, x') \text{ with } (a,b) \in C \}.
\]

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As shown in Lemma A.3, not all pairs of the above defined sets can be non-empty together. The assumption that \( d_H(X, Y) > \frac{5D}{8} \) implies, again from Lemma A.3, that one of the above sets must be non-empty. Without any loss of generality, it then suffices to study only the following three unique cases:

1. \( A \neq \emptyset, B = \emptyset \)
2. \( A = \emptyset, B \neq \emptyset \)
3. \( A \neq \emptyset, B \neq \emptyset \).

For each of the cases, we show that a positive number \( \Delta \) can be chosen such that \( d_H(X, Y + \Delta) \leq \frac{5D}{8} \).

**Case 1** \((A \neq \emptyset, B = \emptyset)\): We denote \( p_0 = \max A \) and \( \varepsilon = p_0 - x - D \). We also let \( q_0 \in Y \cap [y', y] \) such that \((p_0, q_0) \in C\); let \( \varepsilon' = (y - q_0) \); see Figure 5. From the assumption that \( d_H(X, Y) > \frac{5D}{8} \), we first note that \( \varepsilon > \frac{D}{8} \). We also argue that

\[
\varepsilon \leq \varepsilon' \leq D - \varepsilon, \quad \text{and} \quad \varepsilon \leq \frac{D}{2}. \tag{4}
\]

To see (4), we start by observing that \( (p_0 - x') \geq (q_0 - y') \). From the observation together with the distortion of the pair \((x', y')\) and \((p_0, q_0)\), we get

\[
D \geq |(p_0 - x') - (q_0 - y')| = (p_0 - x') - (q_0 - y') = (p_0 - q_0) - (x' - y') = \varepsilon + \varepsilon'.
\]
So, \( \varepsilon' \leq D - \varepsilon \). In particular, \( \varepsilon' \leq D \). So, from the distortion of the pair \((x, y)\) and \((p_0, q_0)\), we also get

\[
D \geq |(p_0 - x) - (y - q_0)| = |(D + \varepsilon) - \varepsilon'| = D + \varepsilon - \varepsilon'.
\]

This implies that \( \varepsilon \leq \varepsilon' \). As a result, we also have

\[
\varepsilon \leq \frac{1}{2}(\varepsilon + \varepsilon) \leq \frac{1}{2}(\varepsilon' + \varepsilon) \leq \frac{D}{2}.
\]

We now show that \( d_H(X, Y + \Delta) \leq \frac{5D}{8} \), for any translation amount \( \Delta \) such that

\[
\begin{cases}
\Delta = \varepsilon - \frac{D}{8}, & \text{when } \frac{D}{8} < \varepsilon \leq \frac{D}{4} \\
\Delta \in [\varepsilon - \frac{D}{8}, \varepsilon + \varepsilon' - \frac{3D}{8}], & \text{when } \frac{D}{4} < \varepsilon \leq \frac{D}{2}.
\end{cases}
\]

(5)

Due to (4) and the fact that \( \varepsilon > \frac{D}{8} \), we immediately note from (5) that

\[
0 < \Delta \leq \frac{5D}{8}.
\]

In order to cover \( X \), we consider the following (possibly overlapping) intervals to cover the real line:

\[
\mathcal{I}_1 = \left(-\infty, y' - \frac{5D}{8} + \Delta \right], \mathcal{I}_2 = \left[y' - \frac{5D}{8} + \Delta, y' + \frac{5D}{8} + \Delta \right],
\]

\[
\mathcal{I}_3 = \left(y' + \frac{5D}{8} + \Delta, q_0 - \frac{5D}{8} + \Delta \right], \mathcal{I}_4 = \left[q_0 - \frac{5D}{8} + \Delta, q_0 + \frac{5D}{8} + \Delta \right],
\]

\[
\mathcal{I}_5 = \left[y - \frac{5D}{8} + \Delta, y + \frac{5D}{8} + \Delta \right], \text{ and } \mathcal{I}_6 = \left(y + \frac{5D}{8} + \Delta, \infty \right).
\]

Since \( y - q_0 = \varepsilon' \leq D \) from (4), the intervals \( \mathcal{I}_4 \) and \( \mathcal{I}_5 \) intersect. So, the union of the intervals is the entire real line. For an arbitrary point \( a \in X \) from any of the above intervals, we show that there exists a point \( b \in Y \) such that \( |a - (b + \Delta)| \leq \frac{5D}{8} \). The intervals \( \mathcal{I}_2, \mathcal{I}_4, \) and \( \mathcal{I}_5 \) are covered by \( y', q_0, \) and \( y \), respectively. So, we present our argument only for \( \mathcal{I}_1, \mathcal{I}_3, \) and \( \mathcal{I}_6 \).

(\( \mathcal{I}_1 \)). Let \( a \in \mathcal{I}_1 \cap X \) and \( b \in Y \) such that \((a, b) \in C \). From \( \Delta \leq \frac{5D}{8} \), it follows that \( a < y' \). We first note that \((a, b)\) cannot cross \((x, y)\), since there is no double crossing. We next argue that \((a, b)\) does not cross \((p_0, q_0)\), either. We assume the contrary that \((a, b)\) crosses \((p_0, q_0)\), consequently \( b \in [q_0, y] \). So, we get \( b - q_0 \leq \varepsilon' \). And, the distortion of the pair \((a, b)\) and \((p_0, q_0)\) yields the following contradiction:

\[
D \geq |p_0 - a| - |q_0 - b| = |x + D + \varepsilon - a| - (b - q_0)|
\]

\[
= |x - a| + D + \varepsilon - (b - q_0)|
\]

\[
= |x - a| + D + \varepsilon - (b - q_0), \text{ since } (b - q_0) \leq \varepsilon' \leq D \text{ from (4)}
\]

\[
= |(x - x') + (x' - y') + (y' - a)| + D + \varepsilon - \varepsilon'
\]
\[
\geq (x' - y') + (y' - a) + D + \varepsilon - \varepsilon', \text{ since } (x - x') \geq 0
\]
\[
= \frac{D}{2} + (y' - a) + D + \varepsilon - \varepsilon'
\]
\[
\geq \frac{D}{2} + \left(\frac{5D}{8} - \Delta\right) + D + \varepsilon - \varepsilon', \text{ since } a \in I_1
\]
\[
= D + \left(\varepsilon - \varepsilon' + \frac{9D}{8} - \Delta\right)
\]
\[
> D, \text{ since the second term is positive by Lemma A.4}
\]

Now that we have \((a, b)\) not crossing \((p_0, q_0)\), we show that \(|a - (b + \Delta)| \leq \frac{5D}{8}\). If \(b \geq a\), then from the distortion bound of \(D\) for the pair of (non-crossing) edges \((a, b)\) and \((p_0, q_0)\), we get \(b - a \leq \frac{D}{2} - (\varepsilon + \varepsilon')\). From (5), it is evident that \(\Delta \leq \varepsilon + \varepsilon'\). So,
\[
|a - (b + \Delta)| = \Delta + (b - a) \leq \Delta + \left[\frac{D}{2} - (\varepsilon + \varepsilon')\right] \leq \frac{D}{2}.
\]

If \(b \leq a\), then from the distortion bound of \(D\) for the pair of (non-crossing) edges \((a, b)\) and \((x, y)\), we have \(a - b \leq \frac{D}{2}\). Since we have already observed that \(\Delta \leq \frac{5D}{8}\), we also have \(|a - (b + \Delta)| \leq \frac{5D}{8}\). In either case, we conclude that \(|a - (b + \Delta)| \leq \frac{5D}{8}\).

\((I_3)\). We note here that the distance between the endpoints of \(I_3\) is
\[
\left(\frac{q_0 - \frac{5D}{8} + \Delta}{x'}\right) - \left(\frac{y' + \frac{5D}{8} + \Delta}{x'}\right)
\]
\[
= (q_0 - y') - \frac{10D}{8}
\]
\[
= (y - \varepsilon') - y' - \frac{10D}{8}
\]
\[
= (y - y') - \varepsilon' - \frac{10D}{8}
\]
\[
= (x - x') + D - \varepsilon' - \frac{10D}{8}
\]
\[
= (x - x') - \left(\varepsilon' + \frac{D}{4}\right)
\]

So, \(I_3 \neq \emptyset\) if and only if \(x - x' > \varepsilon' + \frac{D}{4}\). We also note similarly that \(I_3 \subseteq [x', x]\).

If \(I_3\) is empty, there is nothing show. Let us assume that it is non-empty.

Let \(a \in I_3 \cap X \text{ and } b \in Y\) such that \((a, b) \in C\). Since \(a \in [x', x]\), the edge \((a, b)\) cannot cross \((x, y)\) or \((x', y')\) because of Lemma A.1. Moreover, \(|b - a| \leq \frac{D}{2}\). We now argue that \((a, b)\) does not cross \((p_0, q_0)\) either. We assume the contrary that the edge \((a, b)\) crosses \((p_0, q_0)\), i.e., \(q_0 < b \leq y\). So, we get the following contradiction:
\[
D \geq \left|\left|p_0 - a\right| - \left|q_0 - b\right|\right| = \left|[x + D + \varepsilon - a] - (b - q_0)\right|
\]

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\begin{align*}
&= |(x - a) + D + \varepsilon - (b - q_0)| \\
&= (x - a) + D + \varepsilon - (b - q_0), \text{ since } (b - q_0) \leq \varepsilon' \leq D \text{ from } (4) \\
&= \left( y - \frac{D}{2} \right) - a + D + \varepsilon - (b - q_0) \\
&= (q_0 + \varepsilon') - \frac{D}{2} - a + D + \varepsilon - (b - q_0) \\
&= 2q_0 + \varepsilon' + \frac{D}{2} - a + \varepsilon - b \\
&\geq 2q_0 + \varepsilon' + (b - a) - a + \varepsilon - b, \text{ since } \frac{D}{2} \geq (b - a) \\
&= 2(q_0 - a) + \varepsilon + \varepsilon' \\
&> 2\left( \frac{5D}{8} - \Delta \right) + \varepsilon + \varepsilon', \text{ since } a \in I_3 \\
&= D + \left[ \frac{D}{4} - 2\Delta + \varepsilon + \varepsilon' \right] \\
&\geq D, \text{ since the second term is non-negative by Lemma A.5}
\end{align*}

Hence, \((a, b)\) does not cross \((p_0, q_0)\). Therefore, by arguing similar to \(I_1\) we conclude \(|a - (b + \Delta)| \leq \frac{5D}{8}\).

\((I_6)\). For \(a \in I_6 \cap X\), we get
\begin{align*}
a > y + \frac{5D}{8} + \Delta &= \left( p_0 - \varepsilon - \frac{D}{2} \right) + \frac{5D}{8} + \Delta \\
&= p_0 + \frac{D}{8} - \varepsilon + \Delta \\
&\geq p_0, \text{ since } \Delta \geq \frac{D}{8} \text{ by } (5) \\
&= \max A.
\end{align*}

By the definition of the set \(A\), an edge \((a, b) \in C\) cannot cross \((x, y)\). For this reason, it cannot cross \((p_0, q_0)\) either. The rest of the argument presented for \(I_1\) then goes through. Therefore, \(|a - (b + \Delta)| \leq \frac{5D}{8}\) by arguing similar to \(I_1\).

In order to show the other direction, i.e., \(d_H(Y + \Delta, X) \leq \frac{5D}{8}\), we argue that there exists a translation amount \(\Delta\) satisfying (5). To that end, we consider the following (possibly overlapping) partition of the real line into intervals:

\(J_1 = (-\infty, x - \frac{5D}{8} - \Delta)\), \(J_2 = \left[ x - \frac{5D}{8} - \Delta, x + \frac{5D}{8} - \Delta \right]\), \(J_3 = \left( x + \frac{5D}{8} - \Delta, p_0 - \frac{5D}{8} - \Delta \right)\), \(J_4 = \left[ p_0 - \frac{5D}{8} - \Delta, p_0 + \frac{5D}{8} - \Delta \right]\), and \(J_5 = \left( p_0 + \frac{5D}{8} - \Delta, \infty \right)\).
For an arbitrary point \( b \in Y \) from any of the above intervals, we now show that there exists a point \( a \) in \( X \) such that \( |a - (b + \Delta)| \leq \frac{5D}{8} \). The intervals \( J_2 \) and \( J_4 \) are covered by \( x \) and \( p_0 \), respectively. We present our argument only for \( J_1, J_3, \) and \( J_5 \). While the argument for \( J_1 \) and \( J_5 \) goes through for any \( \Delta \) satisfying (5), the interval \( J_3 \) may require a more particular choice of \( \Delta \).

\( (J_1) \). For any \( b \in J_1 \cap Y \) with an edge \((a, b) \in C \), we argue that the edge cannot cross \((p_0, q_0)\). Since \( \Delta > 0 \), we note that either \( b \in [y', y - D] \) or \( a < y' \). If \( b \in [y', y - D] \), the edge \((a, b)\) cannot cross \((x, y)\), since \( B = 0 \). If \( a < y' \), the edge \((a, b)\) cannot cross \((x, y)\), as no double crossing is allowed. Now, \( b < y - D \leq q_0 \) implies that the edge \((a, b)\) cannot cross \((p_0, q_0)\) without first crossing \((x, y)\). The rest of the argument presented for \( I_1 \) then goes through. Therefore, \( |a - (b + \Delta)| \leq \frac{5D}{8} \) by arguing similar to \( I_1 \).

\( (J_3) \). We observe that \( J_3 \neq \emptyset \) if and only if \( \varepsilon > \frac{D}{4} \). There is nothing to show if \( J_3 \) is empty. So, we assume that \( \varepsilon > \frac{D}{4} \), and claim that there must exist a \( \Delta \) in \([\varepsilon - \frac{D}{8}, \varepsilon + \varepsilon' - \frac{3D}{8}]\), hence satisfying (5), such that for any \( b \in Y \cap J_3 \) there exists \( a \in X \) with \( |a - (b + \Delta)| \leq \frac{5D}{8} \).

We argue by contradiction. Let us assume that no such \( \Delta \) exists in given the interval, i.e., for all \( \Delta \in [\varepsilon - \frac{D}{8}, \varepsilon + \varepsilon' - \frac{3D}{8}] \) the following subset of \( J_3 \) is non-empty:

\[
E(\Delta) = \left\{ b \in J_3 \mid X \cap \left[ b - \frac{5D}{8} + \Delta, b + \frac{5D}{8} + \Delta \right] = \emptyset \right\}.
\]

Let us also define the following subsets of \( Y \):

\[
Y_L = \{ b \in Y \mid \text{there exists an edge} \ (a, b) \in C \ \text{with} \ a \leq b \}
\]

and

\[
Y_R = \{ b \in Y \mid \text{there exists an edge} \ (a, b) \in C \ \text{with} \ a \geq b \}.\]

We note from Lemma 2.6 that \( E(\Delta) \cap Y_L \) and \( E(\Delta) \cap Y_R \) cannot be both non-empty for any given \( \Delta \). Therefore, \( E(\Delta) \neq \emptyset \) for all \( \Delta \) implies

i) either \( E(\Delta) \cap Y_L \) is empty and \( E(\Delta) \cap Y_R \) non-empty for all \( \Delta \), or

ii) \( E(\Delta) \cap Y_L \) is non-empty and \( E(\Delta) \cap Y_R \) empty for all \( \Delta \).

In order to arrive at a contradiction, we now show, however, that \( E(\Delta) \cap Y_R \) is empty when \( \Delta = \varepsilon - \frac{D}{8} \), whereas \( E(\Delta) \cap Y_L \) is empty when \( \Delta = \varepsilon + \varepsilon' - \frac{3D}{8} \).

When \( \Delta = \varepsilon - \frac{D}{8} \), we show that \( E(\Delta) \cap Y_R = \emptyset \). Consider any \( b \in J_3 \cap Y_R \) and an edge \((a, b) \in C \) such that \( a \geq b \). Due to the distortion bound of \( D \) for the pair of edges \((x, y)\) and \((a, b)\), we have \( (a - x) \leq D + (y - b) \). So,

\[
\begin{align*}
(a - (b + \Delta)) & \leq (x + D + y - b) - (b + \Delta) \\
& \leq x + D + y - 2b - \Delta
\end{align*}
\]
\[< x + D + y - 2 \left( x + \frac{5D}{8} - \Delta \right) - \Delta, \text{ since } b \in \mathcal{J}_3\]
\[= D + (y - x) - \frac{10D}{8} + \Delta\]
\[= D + \frac{D}{2} - \frac{10D}{8} + \Delta\]
\[= \frac{D}{4} + \left( \varepsilon - \frac{D}{8} \right)\]
\[= \varepsilon + \frac{D}{8} \leq \frac{5D}{8}, \text{ since } \varepsilon \leq \frac{D}{2} \text{ from } \{1\}.
\]
So, \(b \notin E(\Delta)\). Therefore, \(E(\Delta) \cap Y \subseteq \emptyset\) for \(\Delta = \varepsilon - \frac{D}{8}\).

We show now that \(E(\Delta) \cap Y \leq \emptyset\) when \(\Delta = \varepsilon + \varepsilon' - \frac{3D}{8}\). To see this, consider \(b \in \mathcal{J}_3 \cap Y \leq \) and an edge \((a, b) \in \mathcal{C}\) such that \(a \leq b\). Due to the distortion bound of \(D\) for the pair of edges \((p_0, q_0)\) and \((a, b)\), we have \((p_0 - a) \leq D + (b - q_0)\).

So,
\[
(b + \Delta) - a \leq (b + \Delta) + D + (b - q_0) - p_0
\]
\[
\leq \Delta + D + 2b - q_0 - p_0
\]
\[
< \Delta + D + 2 \left( \frac{p_0 - 5D}{8} - \Delta \right) - q_0 - p_0, \text{ since } b \in \mathcal{J}_3
\]
\[
= (p_0 - q_0) - \frac{2D}{8} - \Delta
\]
\[
= \left( \varepsilon + \frac{D}{2} + \varepsilon' \right) - \frac{2D}{8} - \left( \varepsilon + \varepsilon' - \frac{3D}{8} \right)
\]
\[
= \frac{5D}{8}.
\]
So, \(b \notin E(\Delta)\). Therefore, \(E(\Delta) \cap Y \leq \emptyset\) for \(\Delta = \varepsilon + \varepsilon' - \frac{3D}{8}\). This is a contradiction. Hence, there must exist \(\Delta\) satisfying \(\{5\}\).

\((\mathcal{J}_5)\). Similarly for \(b \in \mathcal{J}_5 \cap Y\), an edge \((a, b) \in \mathcal{C}\) cannot cross \((x, y)\) due to Lemma \(A.2\). Moreover, it cannot cross \((p_0, q_0)\). If it did, i.e., \(a \in [x, p_0]\), the distortion of the pair would exceed \(D\):
\[
|(b - q_0) - (p_0 - a)|
\]
\[
\geq (b - q_0) - p_0 + \left( b - \frac{D}{2} \right), \text{ as } (b - a) \leq \frac{D}{2}, \text{ from Lemma } A.2
\]
\[
= (b - q_0) + (b - p_0) - \frac{D}{2} = [\left( b - p_0 \right) + (p_0 - q_0)] + (b - p_0) - \frac{D}{2}
\]
\[
= (p_0 - q_0) + 2(b - p_0) - \frac{D}{2} = \left( \varepsilon + \varepsilon' + \frac{D}{2} \right) + 2(b - p_0) - \frac{D}{2}
\]
\[
> \left( \varepsilon + \varepsilon' + \frac{D}{2} \right) + 2 \left( \frac{5D}{8} - \Delta \right) - \frac{D}{2}, \text{ as } b \in \mathcal{J}_5
\]
\[
\begin{align*}
= \frac{5D}{4} + \varepsilon + \varepsilon' - 2\Delta & \geq \frac{5D}{4} + \varepsilon + \varepsilon' - 2\left(\varepsilon + \varepsilon' - \frac{3D}{8}\right), \text{ from (5)} \nonumber \\
= 2D - (\varepsilon + \varepsilon') & \geq D, \text{ as } \varepsilon + \varepsilon' \leq D \text{ from (4)}. 
\end{align*}
\]

So, \((a, b)\) does not cross \((p_0, q_0)\). Hence, following the argument presented in \(I_1\), we conclude \(|a - (b + \Delta)| \leq \frac{5D}{8}\).

**Case 2** \((A = \emptyset, B \neq \emptyset)\): We denote \(q_1 = \min B\) and \(\eta = y - D - q_1\). We also let \(p_1 \in X \cap (x, \infty)\) such that \((p_1, q_1) \in C\) and \(\eta' = p_1 - x\). From the assumption that \(d_H(X, Y) > \frac{5D}{8}\), we first note that \(\eta > \frac{D}{8}\). We also argue that

\[
\eta' \leq \eta + \eta' \leq D - \eta, \text{ and } \eta \leq \frac{D}{2}. \tag{6}
\]

To see (6), we first observe that \((p_1 - x') \geq (q_1 - y')\). From the observation together with the distortion of the pair \((x', y')\) and \((p_1, q_1)\), we get

\[
D \geq |(p_1 - x') - (q_1 - y')| = (p_1 - x') - (q_1 - y'), \text{ since } (p_1 - x') \geq (q_1 - y') \\
= [(x - x') + \eta'] - \left[\frac{D}{2} + (x - x') - \frac{D}{2} - \eta\right] = \eta' + \eta.
\]

In particular, \(\eta' \leq D\). Now from the distortion of the pair \((x, y)\) and \((p_1, q_1)\), we also get

\[
D \geq |(y - q_1) - (x - p_1)| = |D - \eta + \eta'| = D - \eta - \eta'.
\]

This implies that \(\eta' \geq \eta\). Combining this with \(\eta + \eta' \leq D\), we get \(\eta \leq \frac{D}{2}\). Comparing (4) and (6), we note that \(\eta\) and \(\eta'\) are analogous to \(\varepsilon\) and \(\varepsilon'\) from Case (1), respectively. Also, the edge \((p_1, q_1)\) is analogous to the edge \((p_0, q_0)\) in Case (1). Using this analogy, we can reuse the arguments presented in Case (1) for some of the intervals here.

We now show that \(\overrightarrow{d_H}(Y + \Delta, X) \leq \frac{5D}{8}\) for any translation amount \(\Delta\) such that

\[
\begin{cases}
\Delta = \eta - \frac{D}{8}, \text{ when } \frac{D}{8} < \eta \leq \frac{D}{4} \\
\Delta \in \left[\eta - \frac{D}{8}, \eta + \eta' - \frac{3D}{8}\right], \text{ when } \frac{D}{4} < \eta \leq \frac{D}{2}.
\end{cases}
\tag{7}
\]

![Figure 6: No double crossing Case (2) is shown. The set B is a subset of the thick, blue interval in the bottom.](image)
Due to (6) and the fact that $\eta > \frac{D}{8}$, we immediately note that

$$0 < \Delta \leq \frac{5D}{8}.$$  

We consider the following intervals:

- $J_1 = \left( -\infty, x - \frac{5D}{8} - \Delta \right)$,
- $J_2 = \left[ x - \frac{5D}{8} - \Delta, x + \frac{5D}{8} - \Delta \right]$,
- $J_3 = \left[ p_1 - \frac{5D}{8} - \Delta, p_1 + \frac{5D}{8} - \Delta \right]$, and
- $J_4 = \left( p_1 + \frac{5D}{8} - \Delta, \infty \right)$.

Since $p_1 - x = \eta' \leq D$ from (6), the intervals $J_2$ and $J_3$ intersect. So, the union of the intervals is the entire real line. For an arbitrary $b \in Y$ from any of the above intervals, we show there exists a point $a \in X$ such that $|a - (b + \Delta)| \leq \frac{5D}{8}$ for any translation $\Delta$ satisfying (7). The intervals $J_2$ and $J_3$ are covered by $x$ and $p_1$, respectively. The nearest neighbor argument for the intervals $J_1$ and $J_4$ are analogous to the intervals $I_6, I_3$, as presented in Case (1), respectively.

Now, in order to show $\delta_H(X, Y + \Delta) \leq \frac{5D}{8}$, we consider the following (possibly overlapping) partition of the real line into intervals:

- $I_1 = \left( -\infty, q_1 - \frac{5D}{8} + \Delta \right)$,
- $I_2 = \left[ q_1 - \frac{5D}{8} + \Delta, q_1 + \frac{5D}{8} + \Delta \right]$,
- $I_3 = \left( q_1 + \frac{5D}{8} + \Delta, y - \frac{5D}{8} + \Delta \right)$,
- $I_4 = \left[ y - \frac{5D}{8} + \Delta, y + \frac{5D}{8} + \Delta \right]$,
- $I_5 = \left( y + \frac{5D}{8} + \Delta, \infty \right)$.

Using the analogy we mentioned, the intervals $I_1, I_2, I_3, I_4$, and $I_5$ can be reasoned about analogous to $J_5, J_4, J_3, J_2$, and $J_1$ from Case (1), respectively. Also, we note that $B = \emptyset$ in Case (1) is analogous to having $A = \emptyset$ in this case; the argument for $I_5$ here uses the fact that $A = \emptyset$.

**Case 3** ($A \neq \emptyset, B \neq \emptyset$): The case is depicted in Figure 7. The choice of $\Delta$ in this case depends on how $\epsilon$ compares to $\eta$. If $\epsilon \geq \eta$, we choose $\Delta$ according to Case (1). In Case (1), the argument for the interval $J_1$ relies on the fact
that $B = \emptyset$. Since $\varepsilon \geq \eta$, we have

$$\Delta \geq \varepsilon - \frac{D}{8} \geq \eta - \frac{D}{8}.$$  

Consequently, for any $b \in J_1 \cap Y$, we have $b < q_1$. By the definition of the set $B$ and the point $q_1$, any edge $(a, b)$ then cannot cross $(x, y)$, which is analogous to having $B = \emptyset$ for Case (1).

Similarly, if $\eta \geq \varepsilon$, we choose $\Delta$ according to Case (2). In Case (2), the argument for the interval $I_5$ relies on the fact that $A = \emptyset$. Since $\eta \geq \varepsilon$, we have

$$\Delta \geq \eta - \frac{D}{8} \geq \varepsilon - \frac{D}{8}.$$  

Consequently, for any $a \in I_5 \cap Y$, we have $a > p_0$. By the definition of the set $A$ and the point $p_0$, any edge $(a, b)$ then cannot cross $(x, y)$, which is analogous to having $A = \emptyset$ for Case (2).

3.2. Double Crossings

Now, we undertake the task of finding a suitable isometry/alignment when there is a double crossing in $C$. In this case, we may have to consider flipping $Y$ to construct such an isometry. We always flip $Y$ about the midpoint of $x$ and $x'$. After flipping, the image of $Y$ is denoted by $\tilde{Y}$ and the image of any $b \in Y$ by $\tilde{b}$.

We first present two technical lemmas.

**Lemma 3.5.** Let $(p, q) \in C$ be a double crossing; see Figure 8. If we denote $h = (x - x')$, $\varepsilon_1 = (p - x)$, and $\varepsilon_2 = (y' - q)$, then we have the following:

i) $\varepsilon_1 - \varepsilon_2 \geq h$,

ii) $\varepsilon_1 - \varepsilon_2 \leq D - h$,

iii) $h \leq \frac{D}{2}$, and

iv) $|p - \tilde{q}| \leq \frac{D}{2} - h$, where $\tilde{q}$ denotes the reflection of $q$ about the midpoint of $x$ and $x'$.

Figure 8: A double crossing $(p, q)$ is shown.
Proof. i) Let us assume the contrary, i.e., $\varepsilon_1 < \varepsilon_2 + h$. Then, the distortion for the pairs $(x, y)$ and $(p, q)$ becomes

$$|\varepsilon_2 + D + h - \varepsilon_1| = \varepsilon_2 + h + D - \varepsilon_1 > D.$$  

This contradicts the fact that the distortion of $C$ is $D$. Therefore, we conclude that $\varepsilon_1 - \varepsilon_2 \geq h$.

ii) Since from (i) we have $\varepsilon_1 \geq \varepsilon_2$, from the distortion for the pairs $(p, q)$ and $(x', y')$, we have

$$h + \varepsilon_1 - \varepsilon_2 \leq D. \quad (8)$$

So, $\varepsilon_1 - \varepsilon_2 \leq D - h$.

iii) From (ii) we have $\varepsilon_2 + D \geq \varepsilon_1$. Hence, the distortion for the pairs $(p, q)$ and $(x, y)$ implies

$$\varepsilon_2 + D + h - \varepsilon_1 \leq D.$$  

Adding (8) and (9), we get $2h \leq D$. Hence, $h \leq \frac{D}{2}$.

iv) If $p > \tilde{q}$, then

$$p - \tilde{q} = \varepsilon_1 - D \geq (D - h) - \frac{D}{2} - \varepsilon_2 \leq \frac{D}{2} - h.$$  

Otherwise,

$$\tilde{q} - p = \frac{D}{2} + \varepsilon_2 - \varepsilon_1 \leq \frac{D}{2} - (\varepsilon_1 - \varepsilon_2) \leq \frac{D}{2} - h.$$  

Therefore, $|p - \tilde{q}| \leq \frac{D}{2} - h$. \hfill $\Box$

In our pursuit of constructing the right isometry, we first define a wide (double) crossing. We show in Theorem 3.8 that we need to flip $Y$ in the presence of such a wide crossing.

Definition 3.6 (Wide Crossing). A double crossing edge $(p, q) \in C$ is called a wide crossing if either

$$p < \begin{cases} \min A', & \text{if } A' \neq \emptyset \\ x' - \frac{D}{2}, & \text{if } A \neq \emptyset \text{ and } \max A > y + \frac{3D}{4} \\ x' - D, & \text{otherwise} \end{cases} \quad (10)$$

or

$$p > \begin{cases} \max A, & \text{if } A \neq \emptyset \\ x + \frac{D}{2}, & \text{if } A' \neq \emptyset \text{ and } \min A < y' - \frac{3D}{4} \\ x + D, & \text{otherwise.} \end{cases}$$

Before presenting Theorem 3.8, we make an important observation first in the following technical lemma.
Lemma 3.7 (Wide Crossing). Let there be a wide crossing \((p, q) \in C\) and an edge \((p_0, q_0) \in C\) such that \(p_0 \in A\) and \(y' \leq q_0 \leq y\). If we denote \(\varepsilon = p_0 - x - D\), \(\varepsilon' = y - q_0\) and \(h = x - x'\), then we have

\[
\varepsilon' \geq 2h + \varepsilon. \tag{11}
\]

Proof. Since \(A \neq \emptyset\), we have from Lemma A.3 that \(A' = \emptyset\). So, for the wide edge \((p, q)\), we must have either \(p < x'\) or \(p > \max A\). We, therefore, consider the following two possible positions of \(p\).

\[
\begin{array}{c}
\text{Case 1} (p < x'): \text{From the distortion of the pair } (p, q) \text{ and } (p_0, q_0), \text{ we have} \\
D \geq |(p_0 - p) - (q - q_0)| \\
= |(\varepsilon_1 + h + D + \varepsilon) - (\varepsilon' + \varepsilon_2)| \\
= (\varepsilon_1 + h + D + \varepsilon) - (\varepsilon' + \varepsilon_2),
\end{array}
\]

since \(\varepsilon_1 \geq \varepsilon_2\) from Lemma 3.5 and \(\varepsilon' \leq D\) from (4)

\[
\geq \varepsilon_1 - \varepsilon_2 + h + \varepsilon + D - \varepsilon' \\
\geq 2h + \varepsilon + D - \varepsilon', \text{ since } \varepsilon_1 \geq \varepsilon_2 + h \text{ from Lemma 3.5}
\]

So, \(\varepsilon' \geq 2h + \varepsilon\).

\[
\begin{array}{c}
\text{Case 2} (p > p_0): \text{From the distortion of the pair } (p, q) \text{ and } (p_0, q_0), \text{ we get} \\
D \geq |(q_0 - q) - (p - p_0)| \\
= |(\varepsilon_2 + D + h - \varepsilon') - (\varepsilon_1 - D - \varepsilon)| \\
= |(D + \varepsilon_2 - \varepsilon_1) + (h + \varepsilon) + (D - \varepsilon')| \\
= (D + \varepsilon_2 - \varepsilon_1) + (h + \varepsilon) + (D - \varepsilon'),
\end{array}
\]

since \(D \geq \varepsilon'\) from (4) and \(D + \varepsilon_2 - \varepsilon_1 \geq 0\) from Lemma 3.5

\[
\geq 2h + \varepsilon + (D - \varepsilon'), \text{ since } D + \varepsilon_2 - \varepsilon_1 \geq h \text{ from Lemma 3.5}
\]

So, \(\varepsilon' \geq 2h + \varepsilon\).
Theorem 3.8 (Wide Crossing). Let $C$ be a correspondence between two compact sets $X, Y \subseteq \mathbb{R}$ with distortion $D$. If there is a wide crossing $(p, q) \in C$, then there exists a value $\Delta \in \mathbb{R}$ such that
\[
d_H(X, \tilde{Y} + \Delta) \leq \frac{5D}{8},
\]
where $\tilde{Y}$ denotes the reflection of $Y$ about the midpoint of $x$ and $x'$.

Proof. We consider the subsets $A, A'$ of $X$ as defined in (2). As already argued in Lemma A.3, the subsets $A$ and $A'$ cannot be both non-empty. Without any loss of generality, it then suffices to study only the following two unique cases:

1. $A \neq \emptyset, A' = \emptyset$,
2. $A = \emptyset, A' = \emptyset$.

For each of the cases, we show that a number $\Delta$ can be chosen such that
\[
d_H(X, \tilde{Y} + \Delta) \leq \frac{5D}{8}.
\]

Case 1 ($A \neq \emptyset, A' = \emptyset$): We denote $p_0 = \max A$ and $\varepsilon = p_0 - x - D$. We also let $q_0 \in Y \cap [y', y]$ such that $(p_0, q_0) \in C$, and $\varepsilon' = (y - q_0)$; see Figure 10. As already shown in (4), we still have
\[
\varepsilon \leq \varepsilon' \leq D - \varepsilon, \quad \text{and} \quad \varepsilon \leq \frac{D}{2}.
\]

We now define the following subset of $Y$:

\[
B_1 = \{ b \in Y \cap [y, \infty) \mid \exists a \in X \cap [x, \infty) \text{ with } (a, b) \in C \}.
\]
Let $q_1 = \max B_1$, $\eta_1 = q_1 - y$, and let there exist an edge $(p_1, q_1) \in C$ with $\eta'_1 = p_1 - x$. We note from Lemma A.10 that

$$\eta_1 \leq \eta'_1 \leq D \text{ and } \eta_1 \leq \frac{D}{2} - h. \quad (14)$$

If we already have $d_H(X, \tilde{Y}) \leq \frac{5D}{8}$, we take $\Delta = 0$. Let us, therefore, assume the non-trivial case that $d_H(X, \tilde{Y}) > \frac{5D}{8}$. As a result, we must have

$$\max\{\varepsilon, \eta_1\} > \frac{D}{8}.$$ 

This is because all points of $X$ outside $A$ and points of $\tilde{Y}$ outside $B_1$ are already covered, as we note

i) $|a - \tilde{b}| \leq \frac{D}{2}$ for a double crossing edge $(a, b)$ from Lemma 3.5

ii) $|a - \tilde{q}_0| \leq \frac{D}{2}$ for any $a \in [x', x]$ due to the fact that $\varepsilon' \geq 2h + \varepsilon$ from Lemma 3.7, and

iii) $|b - x'| \leq \frac{D}{2}$ or $|b - x| \leq \frac{D}{2}$ for any $b \in [y', y]$ due to the fact that $h \leq \frac{D}{2}$ from Lemma 3.5.

We now show that $d_H(X, \tilde{Y} + \Delta) \leq \frac{5D}{8}$ for some translation amount $\Delta$ such that

$$\left\{ \begin{array}{l}
\Delta = \max\{\varepsilon, \eta_1\} - \frac{D}{8}, \text{ when } \varepsilon \leq \eta_1 + \frac{D}{4} \\
\Delta \in \left[ \varepsilon - \frac{D}{2}, \frac{5D}{8} + h + \varepsilon - \varepsilon' \right], \text{ when } \eta_1 + \frac{D}{4} < \varepsilon \leq \frac{D}{2}. 
\end{array} \right. \quad (15)$$

While $d_H(X, \tilde{Y} + \Delta) \leq \frac{5D}{8}$ for any $\Delta$ satisfying (15), we may require a more specific choice of $\Delta$ in order to show $d_H(\tilde{Y} + \Delta, X) \leq \frac{5D}{8}$.

As already argued $\max\{\varepsilon, \eta_1\} > \frac{D}{2}$, we have $\Delta > 0$. We also note from (12), (14) that $\varepsilon, \eta_1 \leq \frac{D}{2}$ and from (11) that the upper bound of $\Delta$:

$$\frac{5D}{8} + h + \varepsilon - \varepsilon' \leq \frac{5D}{8} + h + \varepsilon - (2h + \varepsilon) \leq \frac{5D}{8} - h \leq \frac{5D}{8}.$$ 

As a result, we note for later that

$$0 < \Delta \leq \frac{5D}{8}.$$ 

In order to show $d_H(X, \tilde{Y} + \Delta) \leq \frac{5D}{8}$ for any $\Delta$ satisfying (15), we consider the following (possibly overlapping) intervals to cover the real line:

$$I_1 = (-\infty, \tilde{q}_1 - \frac{5D}{8} + \Delta), I_2 = \left[ \tilde{q}_1 - \frac{5D}{8} + \Delta, \tilde{q}_1 + \frac{5D}{8} + \Delta \right],$$

$$I_3 = \left[ y - \frac{5D}{8} + \Delta, y + \frac{5D}{8} + \Delta \right], I_4 = \left[ \tilde{q}_0 - \frac{5D}{8} + \Delta, \tilde{q}_0 + \frac{5D}{8} + \Delta \right],$$

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\[ I_6 = \left[ \tilde{y}' - \frac{5D}{8} + \Delta, \tilde{y}' + \frac{5D}{8} + \Delta \right], \quad \text{and} \quad I_8 = \left( \tilde{y}' + \frac{5D}{8} + \Delta, \infty \right). \]

Since \( \tilde{y} - \tilde{q}_1 = \eta_1 \leq D \) from (14), the intervals \( I_2 \) and \( I_3 \) intersect. Since \( \tilde{q}_0 - \tilde{y} = \varepsilon' \leq D \) from (12), the intervals \( I_3 \) and \( I_4 \) intersect. Since \( \varepsilon' \geq h \) from Lemma 3.7, we have \( \tilde{y}' - \tilde{q}_0 = (D + h) - \varepsilon' \leq D \). So, the intervals \( I_4 \) and \( I_5 \) intersect. As a result, the union of the intervals is the entire real line. For an arbitrary point \( a \in X \) from any of the above intervals, we show that there exists a point \( b \in Y \) such that \( \|a - (\tilde{b} + \Delta)\| \leq \frac{5D}{8} \). The intervals \( I_2, I_3, I_4, \) and \( I_5 \) are covered by \( q_1, y, q_0, \) and \( y' \), respectively. So, we present our argument only for \( I_1 \) and \( I_6 \).

\((I_1)\). Let \( a \in I_1 \cap X \) with an edge \((a, b) \in \mathcal{C}\). As we note now, \( a \) satisfies the condition (A.2) of Lemma A.8. When \( \varepsilon \leq \eta_1 + \frac{D}{4} \), we have

\[
x' - a > \frac{D}{2} + \eta_1 + \left( \frac{5D}{8} - \Delta \right) = D + \eta_1 - \Delta + \frac{D}{8} = D + \eta_1 - \left( \max\{\varepsilon, \eta_1\} - \frac{D}{8} \right) + \frac{D}{8}, \quad \text{from (15)}
\]

\[
= D + (\eta_1 - \max\{\varepsilon, \eta_1\}) + \frac{D}{4} = \begin{cases} D + \frac{D}{4}, & \text{if } \eta_1 \geq \varepsilon \\ D + \frac{D}{4} + \eta_1 - \varepsilon & \text{if } \eta_1 \leq \varepsilon \end{cases}
\]

\[
\geq D, \quad \text{since } \varepsilon \leq \eta_1 + \frac{D}{4}.
\]

When \( \varepsilon > \eta_1 + \frac{D}{4} \), we have

\[
x' - a > \frac{D}{2} + \eta_1 + \left( \frac{5D}{8} - \Delta \right) \geq \frac{D}{2}, \quad \text{as } \Delta \leq \frac{5D}{8}.
\]

Therefore, Lemma A.8 implies that \((a, b)\) is a double crossing. As a result, after flipping \( Y \), the edge \((a, \tilde{b})\) does not cross \((p_0, \tilde{q}_0)\). Consequently, when \( \tilde{b} \geq a \), due to the distortion bound of \( D \) of the edges \((a, \tilde{b})\) and \((p_0, \tilde{q}_0)\), we must have

\[
D \geq (p_0 - a) - (\tilde{q}_0 - \tilde{b}) = (\tilde{b} - a) + (p_0 - \tilde{q}_0).
\]

So,

\[
\Delta + (\tilde{b} - a) \leq \Delta + [D - (p_0 - \tilde{q}_0)]
\]

\[
= \Delta + D - \left( \left( \varepsilon + \frac{D}{2} \right) + (D + h - \varepsilon') \right)
\]

\[
= \Delta + \varepsilon' - \varepsilon - h - \frac{D}{2}
\]

\[
\leq \left( \frac{5D}{8} + h + \varepsilon' \right) + \varepsilon' - \varepsilon - h - \frac{D}{2}, \quad \text{from (15)}
\]

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Also, when \( \tilde{b} \leq a \), we must have \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \), since \( \Delta \leq \frac{5D}{8} \). Therefore, in either case, we have \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \).

**\( I_6 \).** Let \( a \in I_6 \cap X \) and \((a, b) \in C\) an edge. We note that \( a > p_0 \), as we have \( \Delta \geq \varepsilon - \frac{D}{8} \) from \([15]\). Lemma \([A, 8]\) implies that \((a, b)\) is a double crossing edge. Therefore, after flipping \( Y \), the edge \((a, \tilde{b})\) cannot cross \((p_0, \tilde{q}_0)\). The rest of the argument presented for \( I_1 \) then goes through. Hence arguing similar to \( I_1 \), we have \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \).

In order to show that \( \delta_H(\tilde{Y} + \Delta, X) \leq \frac{5D}{8} \), we define the following intervals:

\[
\begin{align*}
J_1 &= \left(-\infty, x' - \frac{5D}{8} - \Delta\right], \quad J_2 = \left[x' - \frac{5D}{8} - \Delta, x' + \frac{5D}{8} - \Delta\right], \\
J_3 &= \left[x - \frac{5D}{8} - \Delta, x + \frac{5D}{8} - \Delta\right], \quad J_4 = \left[p_1 - \frac{5D}{8} - \Delta, p_1 + \frac{5D}{8} - \Delta\right], \\
J_5 &= \left(p_1 + \frac{5D}{8} - \Delta, p_0 - \frac{5D}{8} - \Delta\right], \quad J_6 = \left[p_0 - \frac{5D}{8} - \Delta, p_0 + \frac{5D}{8} - \Delta\right], \\
J_7 &= \left(p_0 + \frac{5D}{8} - \Delta, \infty\right].
\end{align*}
\]

Since \( x - x' = h \leq \frac{D}{8} \) from Lemma \([3, 5]\), the intervals \( J_2 \) and \( J_3 \) intersect. Since \( p_1 - x = \eta_1' \leq \frac{D}{8} \) from \([14]\), the intervals \( J_3 \) and \( J_4 \) intersect. As a result, the union of the intervals is the entire real line. For an arbitrary point \( \tilde{b} \in \tilde{Y} \) from any of the above intervals, we show that there exists a point \( a \in X \) such that \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \). The intervals \( J_2, J_3, J_4, \) and \( J_6 \) are covered by \( x', x, p_1, \) and \( p_0 \), respectively. So, we present the argument only for \( J_1, J_5, \) and \( J_7 \). While the argument for \( J_1, J_7 \) goes through for any \( \Delta \) satisfying \([15]\), the interval \( J_5 \) may require a more particular choice of \( \Delta \).

**\( J_1 \).** Let \( \tilde{b} \in \tilde{Y} \cap J_1 \) and \((a, b) \in C\) an edge. From \([15]\), we get \( \Delta \geq \eta_1 - \frac{D}{8} \), so \( \tilde{b} < \tilde{q}_1 \), i.e., \( b > q_1 \). Since \( q_1 = \max B_1 \), the edge \((a, b)\) has to cross \((p_0, \tilde{q}_0)\). So after flipping \( Y \), the edge \((a, \tilde{b})\) does not cross \((p_0, \tilde{q}_0)\). The rest of the argument presented for the interval \( I_1 \) then goes through. Hence arguing similar to \( I_1 \), we have \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \).

**\( J_5 \).** We observe that \( J_5 \neq \emptyset \) if and only if \( \varepsilon > \frac{D}{4} + \eta_1' \). There is nothing to show if \( J_5 \) is empty. So, we assume that \( \varepsilon > \frac{D}{4} + \eta_1' \), and claim that there must exist a number \( \Delta \in \left[\varepsilon - \frac{D}{8}, \frac{5D}{8} + h + \varepsilon - \varepsilon'\right] \) such that for any \( \tilde{b} \in \tilde{Y} \cap J_5 \) there exists \( a \in X \) with \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \). We also observe that the choice of
\(\Delta\) here satisfies \([15]\), because \(\varepsilon > \frac{D}{4} + \eta'_1\) implies \(\varepsilon > \frac{D}{4} + \eta_1\) due to \([14]\).

We argue by contradiction. Let us assume that no such \(\Delta\) exists in the given interval, i.e., for all \(\Delta \in [\varepsilon - \frac{D}{8}, \frac{5D}{8} + h + \varepsilon - \varepsilon']\), the following subset of \(\mathcal{J}_5\) is non-empty:

\[
E(\Delta) = \left\{ b \in \bar{Y} \cap \mathcal{J}_5 \mid X \cap \left[ b - \frac{5D}{8} + \Delta, b + \frac{5D}{8} + \Delta \right] = \emptyset \right\}
\]

Let us also define

\[\bar{Y}_\leq = \left\{ b \in Y \mid \text{there exists an edge } (a, b) \in \mathcal{C} \text{ with } a \leq b \right\}\]

and

\[\bar{Y}_\geq = \left\{ b \in Y \mid \text{there exists an edge } (a, b) \in \mathcal{C} \text{ with } a \geq b \right\}\]

We note from Lemma \[\text{A.6}\] that \(E(\Delta) \cap \bar{Y}_\leq\) and \(E(\Delta) \cap \bar{Y}_\geq\) cannot be both non-empty for any given \(\Delta\). Therefore, \(E(\Delta) \neq \emptyset\) for all \(\Delta\) implies

i) either \(E(\Delta) \cap \bar{Y}_\leq\) is empty and \(E(\Delta) \cap \bar{Y}_\geq\) non-empty for all \(\Delta\), or

ii) \(E(\Delta) \cap \bar{Y}_\leq\) is non-empty and \(E(\Delta) \cap \bar{Y}_\geq\) empty for all \(\Delta\).

In order to arrive at a contradiction, we now show, however, that \(E(\Delta) \cap \bar{Y}_\geq\) is empty when \(\Delta = \varepsilon - \frac{D}{8}\), whereas \(E(\Delta) \cap \bar{Y}_\leq\) is empty when \(\Delta = \frac{5D}{8} + h + \varepsilon - \varepsilon'\).

When \(\Delta = \varepsilon - \frac{D}{8}\), we show that \(E(\Delta) \cap \bar{Y}_\geq\) = \(\emptyset\). Consider any \(\tilde{b} \in \mathcal{J}_5 \cap \bar{Y}_\geq\) with an edge \((a, b) \in \mathcal{C}\) such that \(a \geq \tilde{b}\). Due to the distortion bound of \(D\) for the pair of edges \((x', y')\) and \((a, b)\), we have \((a - x') \leq D + (y' - b)\). So,

\[
a - (\tilde{b} + \Delta) \leq (x' + D + y' - \tilde{b}) - (\tilde{b} + \Delta) \\
\leq x' + D + y' - 2\tilde{b} - \Delta \\
< x' + D + y' - 2 \left( p_1 + \frac{5D}{8} - \Delta \right) - \Delta, \text{ since } \tilde{b} \in \mathcal{J}_5 \\
\leq x + D + y' - 2 \left( x + \frac{5D}{8} - \Delta \right) - \Delta, \text{ since } p_1 \geq x \geq x' \\
= (y' - x) - \frac{D}{4} + \Delta \\
= \frac{D}{2} - \frac{D}{4} + \Delta \\
= \frac{D}{4} + \left( \varepsilon - \frac{D}{8} \right) \\
= \varepsilon + \frac{D}{8} \leq \frac{5D}{8}, \text{ since } \varepsilon \leq \frac{D}{2} \text{ from } [12].
\]

So, \(\tilde{b} \not\in E(\Delta)\). Therefore, \(E(\Delta) \cap \bar{Y}_\geq = \emptyset\) for \(\Delta = \varepsilon - \frac{D}{8}\).
We now show that $E(\Delta) \cap \tilde{Y}_\leq = \emptyset$ when $\Delta = \frac{5D}{8} + h + \varepsilon - \varepsilon'$. To see this, consider $\tilde{b} \in \tilde{Y} \cap J_5$ and an edge $(a, \tilde{b}) \in C$ such that $a \leq \tilde{b}$. Due to the distortion bound of $D$ for the pair of edges $(p_0, \tilde{q}_0)$ and $(a, \tilde{b})$, we have $(p_0 - a) \leq D + (\tilde{b} - \tilde{q}_0)$.

So,
\[
(\tilde{b} + \Delta) - a \leq (\tilde{b} + \Delta) + D + (\tilde{b} - \tilde{q}_0) - p_0 \\
\leq \Delta + D + 2\tilde{b} - \tilde{q}_0 - p_0 \\
< \Delta + D + 2\left( p_0 - \frac{5D}{8} - \Delta \right) - \tilde{q}_0 - p_0, \text{ since } \tilde{b} \in J_5 \\
= (p_0 - \tilde{q}_0) - \frac{D}{4} - \Delta \\
= \left(D + h - \varepsilon' + \frac{D}{2} + \varepsilon\right) - \frac{D}{4} - \Delta \\
= \left(D + h - \varepsilon' + \frac{D}{2} + \varepsilon\right) - \frac{D}{4} - \left(\frac{5D}{8} + h + \varepsilon - \varepsilon'\right) \\
= \frac{5D}{8}.
\]

So, $\tilde{b} \not\in E(\Delta)$. Therefore, $E(\Delta) \cap \tilde{Y}_\leq = \emptyset$ for $\Delta = \frac{5D}{8} + h + \varepsilon - \varepsilon'$. This is a contradiction. Hence, there must exist a number $\Delta$ satisfying (15) as claimed.

$(J_7)$. Let $\tilde{b} \in \tilde{Y} \cap J_7$ and $(a, b) \in C$ an edge. Since $\Delta \leq \frac{5D}{8}$, we have $\tilde{b} > p_0 \geq y + \frac{D}{2}$, i.e., $b < y' - \frac{D}{2}$. So, Lemma A.9 implies that $(a, b)$ is a double crossing, i.e., $a > x$. We further argue that we must also have $a > p_0$.

If not, i.e., $x < a \leq p_0$, then $(p_0, \tilde{q}_0)$ crosses $(a, \tilde{b})$. Consequently, the distortion of the pair
\[
(\tilde{b} - \tilde{q}_0) - (p_0 - a) \\
\geq (\tilde{b} - \tilde{q}_0) - (\tilde{b} - a), \text{ as } \tilde{b} \in J_7, \text{ we have } p_0 \leq \tilde{b} \\
= (\tilde{b} - p_0) + (p_0 - y') + (y' - \tilde{q}_0) - (\tilde{b} - a) \\
\geq (\tilde{b} - p_0) + (p_0 - y') + (y' - \tilde{q}_0) - \frac{D}{2}, \text{ from Lemma 3.5} \\
= (\tilde{b} - p_0) + \left(\frac{D}{2} + \varepsilon\right) + (D + h - \varepsilon') - \frac{D}{2} \\
> \left(\frac{5D}{8} - \Delta\right) + \left(\frac{D}{2} + \varepsilon\right) + (D + h - \varepsilon') - \frac{D}{2}, \text{ as } \tilde{b} \in J_7 \\
= \frac{13D}{8} + \varepsilon - \varepsilon' + h - \Delta \\
\geq \frac{13D}{8} + \varepsilon - \varepsilon' + h - \left(\frac{5D}{8} + \varepsilon - \varepsilon' + h\right), \text{ from (15)} \\
= D
\]
This is a contradiction, so \( a > p_0 \). Therefore, the edge \((a, \tilde{b})\) cannot cross \((p_0, \tilde{q}_0)\). Hence arguing similar to \( \mathcal{I}_1 \), we have \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \).

**Case 2** \((A = \emptyset, A' = \emptyset)\): We define

\[
B_2 = \{ b \in Y \cap (-\infty, y'] \mid \exists a \in X \cap (-\infty, x'] \text{ with } (a, b) \in C \}.
\]

Let \( q_2 = \min B_2 \), \( \eta_2 = y' - q_2 \), and let there exist an edge \((p_2, q_2) \in C\) with \( \eta_2' = x' - p_2' \); see Figure 11. Following the same argument presented for (14), we have

\[
\eta_2 \leq \eta_2' \leq D \text{ and } \eta_2 \leq \frac{D}{2} - h.
\]

(17)

If we already have \( d_H(X, \tilde{Y}) \leq \frac{5D}{8} \), we take \( \Delta = 0 \). Let us, therefore, assume the non-trivial case that \( d_H(X, \tilde{Y}) > \frac{5D}{8} \). We have observed in Lemma 3.5 that for a double crossing edge \((a, b)\), we already have \( |a - \tilde{b}| \leq \frac{D}{2} \), after flipping \( Y \). This together with the fact that \( A, A' = \emptyset \) implies we must have either

i) \( a_0 \in (x' + \frac{D}{8}, x - \frac{D}{8}) \cap X \) such that \( \min_{y \in Y} |a_0 - y| > \frac{5D}{8} \), or

ii) \( b_0 \in B_1 \cup B_2 \) such that \( \min_{x \in X} |\tilde{b}_0 - x| > \frac{5D}{8} \).

When \( h \leq \frac{3D}{8} \), such \( a_0 \) cannot exist. Because, we have that \( |a_0 - \tilde{b}| \leq \frac{5D}{8} \) for any edge \((a_0, b)\), due to the fact that \( |a - b| \leq \frac{D}{2} \). Consequently, \( b_0 \) must exist implying that either \( \eta_1 > \frac{D}{8} \) or \( \eta_2 > \frac{D}{8} \). Without any loss of generality, we assume \( \eta_1 \geq \eta_2 > \frac{D}{8} \) so that the direction of the translation of \( \tilde{Y} \) is from left to right.

When \( h > \frac{3D}{8} \), on the other hand, from (14), (17) we have \( \eta_1, \eta_2 \leq \frac{D}{8} \). Consequently, \( b_0 \) cannot exist, so \( a_0 \) must exist. We also note from the existence of such \( a_0 \) that the set

\[
H := \left( y' + h - \frac{D}{4}, y - h + \frac{D}{4} \right) \cap Y = \emptyset.
\]

(18)

The set \( H \) is indicated in Figure 11. This is because for any \( b \in H \) and \( a \in (x' + \frac{D}{8}, x - \frac{D}{8}) \), we have

\[
|a - \tilde{b}| \leq \left| x - \frac{D}{8} \right| + \left| y' + h - \frac{D}{4} \right| = \frac{5D}{8}.
\]

Furthermore, \( a_0 \) belongs to either \( C_L := (x' + \frac{D}{8}, x' + h - \frac{D}{4}) \cap X \) or (its mirror image) \( C_R := (x' + h + \frac{D}{4}, x - \frac{D}{8}) \cap X \). This is because for any \( a \in [x' + h - \frac{D}{4}, x' + h + \frac{D}{4}] \) with edge \((a, b)\), we would have \( b \in H \), due to the fact that \( |a - b| \leq \frac{D}{2} \). The sets \( C_L \) and \( C_R \) are also depicted in Figure 11.

Without any loss of generality, we then assume that

\[
a_0 \in \left( x' + \frac{D}{8}, x' + h - \frac{D}{4} \right) \cap X.
\]

(19)
The choice is justified, because one can alternatively consider $-X, -Y$ without altering the distortion of the correspondence. The assumption again retains the left-to-right direction of the translation of $\tilde{Y}$.

Now, we take our

$$
\Delta = \begin{cases} 
\eta_1 - \eta_2 - \frac{D}{8}, & \text{when } h \leq \frac{3D}{8} \\
\tilde{h} - \frac{3D}{8}, & \text{when } \frac{3D}{8} < h \leq \frac{D}{2}.
\end{cases}
$$

(20)

We immediately note that $0 < \Delta \leq \frac{D}{2}$, as $\eta_1 \leq \frac{D}{2}$.

In order to show that $\tilde{d}_H(X, \tilde{Y} + \Delta) \leq \frac{5D}{8}$, we define the following intervals:

$$
\mathcal{I}_1 = \left(-\infty, \tilde{q}_1 - \frac{5D}{8} + \Delta\right), \mathcal{I}_2 = \left[\tilde{q}_1 - \frac{5D}{8} + \Delta, \tilde{q}_1 + \frac{5D}{8} + \Delta\right],
$$

$$
\mathcal{I}_3 = \left[\tilde{y} - \frac{5D}{8} + \Delta, \tilde{y} + \frac{5D}{8} + \Delta\right], \mathcal{I}_4 = \left[x' + \frac{D}{8} + \Delta, x - \frac{D}{2} + \frac{D}{16} + \frac{\Delta}{2}\right],
$$

$$
\mathcal{I}_5 = \left(x - \frac{h}{2} + \frac{D}{16} + \frac{\Delta}{2}, x - \frac{D}{8} + \Delta\right), \mathcal{I}_6 = \left[\tilde{y}' - \frac{5D}{8} + \Delta, \tilde{y}' + \frac{5D}{8} + \Delta\right],
$$

$$
\mathcal{I}_7 = \left[\tilde{q}_2 - \frac{5D}{8} + \Delta, \tilde{q}_2 + \frac{5D}{8} + \Delta\right], \text{ and } \mathcal{I}_8 = \left(\tilde{q}_2 + \frac{5D}{8} + \Delta, \infty\right).
$$

Since $\tilde{y} - \tilde{q}_1 = \eta_1 \leq D$ from (14), the intervals $\mathcal{I}_2$ and $\mathcal{I}_3$ intersect. Since $\tilde{q}_2 - \tilde{y}' = \eta_2 \leq D$ from (17), the intervals $\mathcal{I}_6$ and $\mathcal{I}_7$ intersect. As a result, the union of the intervals is the entire real line. For an arbitrary point $a \in X$ from any of the above intervals, we show that there exists a point $b \in Y$ such that $|a - (b + \Delta)| \leq \frac{5D}{8}$. The intervals $\mathcal{I}_2, \mathcal{I}_3, \mathcal{I}_6,$ and $\mathcal{I}_7$ are covered by $q_1, y, y'$, and $q_2$, respectively. So, we present the argument only for $\mathcal{I}_1, \mathcal{I}_4, \mathcal{I}_5,$ and $\mathcal{I}_8$.  

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Let $a \in I_1 \cap X$ and $(a, b) \in C$ an edge. We argue that $b > q_1$. We first observe that
\[
x' - a = (x' - \tilde{q}_1) + (\tilde{q}_1 - a) = \left(\frac{D}{2} + \eta_1\right) + (\tilde{q}_1 - a) > \left(\frac{D}{2} + \eta_1\right) + \left(\frac{5D}{8} - \Delta\right) = D + \left(\frac{D}{8} + \eta_1 - \Delta\right) \geq D,
\]
as last term is non-negative for any $\Delta$ satisfying (20).

So, Lemma [A.8] implies that $b > y$. If we assume $b \in [y, q_1]$, then the distortion of the edges $(a, b)$ and $(p_1, q_1)$ exceeds $D$:
\[
(p_1 - a) - (q_1 - b) > (D + h + \eta'_1) - \eta_1 \geq D \quad \text{as} \quad \eta'_1 \geq \eta_1, \text{ from (14)}.
\]

Therefore, $b > q_1$ as claimed. As a result, after flipping $Y$ the edge $(a, \tilde{b})$ does not cross $(p_1, \tilde{q}_1)$. Consequently, when $\tilde{b} \geq a$, due to the distortion bound of $(a, b)$ and $(p_1, \tilde{q}_1)$, we must have
\[
(\tilde{b} - a) + (p_1 - \tilde{q}_1) \leq D.
\]
So,
\[
\Delta + (\tilde{b} - a) \leq \Delta + D - (p_1 - \tilde{q}_1)
= \Delta + D - \left(\eta_1 + \frac{D}{2} + h + \eta'_1\right)
\leq \Delta + \frac{D}{2} - \eta_1 - h.
\]

When $h \leq \frac{3D}{8}$, from (20) we have
\[
\left|a - (\tilde{b} + \Delta)\right| \leq \Delta + \frac{D}{2} - \eta_1 - h = \left(\eta_1 - \eta_2 - \frac{D}{8}\right) + \frac{D}{2} - \eta_1 - h \leq \frac{5D}{8}.
\]

When $h > \frac{3D}{8}$, again from (20) we have
\[
\left|a - (\tilde{b} + \Delta)\right| \leq \Delta + \frac{D}{2} - \eta_1 - h = \left(h - \frac{3D}{8}\right) + \frac{D}{2} - \eta_1 - h \leq \frac{5D}{8}.
\]

When $a \geq \tilde{b}$, from Lemma [3.5] we have $a - \tilde{b} \leq \frac{D}{2}$. Since $\Delta \leq \frac{D}{2}$ as already noted, we must have
\[
\left|a - (\tilde{b} + \Delta)\right| \leq \frac{5D}{8}.
\]

Hence, we have $\left|a - (\tilde{b} + \Delta)\right| \leq \frac{5D}{8}$.

We note that $I_4 \neq \emptyset$ if and only if $h \geq \frac{D}{4}$. There is nothing to show if $I_4$ is empty. So, we assume that $h \geq \frac{D}{4}$. Also, the choice of $\Delta$ in (20) satisfies (A.1). Therefore, Lemma [A.7] implies that $\left|a - (\tilde{b} + \Delta)\right| \leq \frac{5D}{8}$. 

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(I₅). We note that I₅ ≠ ∅ if and only if \( h > \frac{3D}{8} \). There is nothing to show if I₅ is empty. So, we assume that \( h > \frac{3D}{8} \) and \( a ∈ I₅ ∩ X \). From (20), we have \( \Delta = h - \frac{3D}{8} \). As already discussed there is a₀ satisfying (19). Let (a₀, b) be an edge. We argue that \( |a - (b + \Delta)| ≤ \frac{5D}{8} \).

We note from Lemma A.1 that \( |a₀ - b| ≤ \frac{D}{2} \). Consequently,

\[
b ∈ \left( [x' + \frac{D}{8}] - \frac{D}{2}, \left[ x' + h - \frac{D}{4} \right] + \frac{D}{2} \right) = \left( y' + \frac{D}{8}, x' + h + \frac{D}{4} \right).
\]

Hence, \( \bar{b} ∈ (x - h - \frac{D}{4}, y - \frac{D}{8}) \). Considering the opposite endpoints of this interval and I₅, we can write

\[
|\bar{b} + \Delta - a| ≤ \max\left\{ \left| (x - h - \frac{D}{4}) + \Delta - \left( x - \frac{D}{8} + \Delta \right) \right|, \left| (y - \frac{D}{8}) + \Delta - \left( x - \frac{h}{2} + \frac{D}{16} + \frac{\Delta}{2} \right) \right| \right\},
\]

\[
= \max\left\{ |\Delta - h|, \frac{\Delta}{2} + \frac{h}{2} + \frac{5D}{16} \right\}
\]

\[
= \max\left\{ \left| h - \frac{3D}{8} \right| - h, \frac{1}{2} \left( h - \frac{3D}{8} \right) + \frac{h}{2} + \frac{5D}{16} \right\}
\]

\[
= \max\left\{ \frac{3D}{8}, \frac{5D}{8} \right\}
\]

\[
= \frac{3D}{8} \text{ since } h ≤ \frac{D}{2} \text{ from Lemma 3.5}
\]

\[
= \frac{5D}{8}.
\]

(II₅). For \( a ∈ I₅ ∩ X \), we have \( a > x + D \). By Lemma A.8 we then have \( b < y' \). Consequently, after flipping \( Y \), the edge \((a, \bar{b})\) cannot cross \((p₁, \bar{q₁})\). The rest of the argument presented for I₁ then goes through. Hence arguing similar to I₁, we have \( |a - (\bar{b} + \Delta)| ≤ \frac{5D}{8} \).

In order to show that \( d_H(\bar{Y} + \Delta, X) ≤ \frac{5D}{8} \), we define the following intervals:

\[
\mathcal{J₁} = \left( -\infty, p_2 - \frac{5D}{8} - \Delta \right), \mathcal{J₂} = \left[ p_2 - \frac{5D}{8} - \Delta, p_2 + \frac{5D}{8} - \Delta \right],
\]

\[
\mathcal{J₃} = \left[ x' - \frac{5D}{8} - \Delta, x' + \frac{5D}{8} - \Delta \right], \mathcal{J₄} = \left[ x - \frac{5D}{8} - \Delta, x + \frac{5D}{8} - \Delta \right],
\]

\[
\mathcal{J₅} = \left[ p_1 - \frac{5D}{8} - \Delta, p_1 + \frac{5D}{8} - \Delta \right], \mathcal{J₆} = \left( p_1 + \frac{5D}{8} - \Delta, \infty \right).
\]

Since \( x' - p_2 = \eta' ≤ D \) from (17), the intervals \( \mathcal{J₂} \) and \( \mathcal{J₃} \) intersect. Since \( x - x' = h ≤ D \) from Lemma 3.5, the intervals \( \mathcal{J₃} \) and \( \mathcal{J₄} \) intersect. Since
\(p_1 - x = \eta'_1 \leq D\) from (14), the intervals \(\mathcal{J}_4\) and \(\mathcal{J}_5\) intersect. As a result, the union of the intervals is the entire real line. For an arbitrary point \(\tilde{b} \in \tilde{Y}\) from any of the above intervals, we show that there exists a point \(a \in X\) such that \(|a - (\tilde{b} + \Delta)| \leq \frac{5D}{8}\). We present the argument only for \(\mathcal{J}_1\) and \(\mathcal{J}_6\).

**(\mathcal{J}_1).** For any \(\tilde{b} \in \tilde{Y} \cap \mathcal{J}_1\), we first argue that \(\tilde{b} < \tilde{q}_1\). When \(\eta_1 < \frac{D}{8}\), this is definitely true. When \(\eta_1 \geq \frac{D}{8}\), then (14) yields \(h \leq \frac{3D}{8}\). So,

\[
\tilde{b} - \tilde{q}_1 < \left(p_2 - \frac{5D}{8} - \Delta\right) - \tilde{q}_1 = (x' - \eta'_2) - \frac{5D}{8} - \Delta - (\tilde{y} - \eta_1)
\]

\[
= (x' - \tilde{y}) - \frac{5D}{8} + (\eta_1 - \eta'_2 - \Delta) = \frac{D}{2} - \frac{5D}{8} + (\eta_1 - \eta'_2 - \Delta)
\]

\[
= \eta_1 - \eta'_2 - \frac{D}{8} - \Delta = \eta_1 - \eta'_2 - \frac{D}{8} - \left(\eta_1 - \eta_2 - \frac{D}{8}\right), \text{ from (20)}
\]

\[
= \eta_2 - \eta'_2 \leq 0, \text{ from (17)}.
\]

So, \(\tilde{b} < \tilde{q}_1\). Consequently, \(b > q_1\). Since \(q_1 = \max B_1\), for any edge \((a, b)\), we must have \(a < x\) by the definition of the set \(B_1\); see (13). Therefore, \((a, b)\) crosses \((p_1, q_1)\). So after flipping \(Y\), the edge \((a, \tilde{b})\) does not cross \((p_1, \tilde{q}_1)\). The rest of the argument presented for \(\mathcal{I}_1\) then goes through. Hence arguing similar to \(\mathcal{I}_1\), we have \(|a - (\tilde{b} + \Delta)| \leq \frac{5D}{8}\).

**(\mathcal{J}_6).** For \(\tilde{b} \in \tilde{Y} \cap \mathcal{J}_6\), we argue that \(\tilde{b} > \tilde{q}_2\). We have

\[
\tilde{b} - \tilde{q}_2 > \left(p_1 + \frac{5D}{8} - \Delta\right) - \tilde{q}_2 = (x + \eta'_1) + \frac{5D}{8} - \Delta - (\tilde{y} + \eta_2)
\]

\[
= (x - \tilde{y}) + \frac{5D}{8} + (\eta'_1 - \eta_2 - \Delta) = -\frac{D}{2} + \frac{5D}{8} + (\eta'_1 - \eta_2 - \Delta)
\]

\[
= \eta'_1 - \eta_2 + \frac{D}{8} - \Delta \geq \eta_1 - \eta_2 + \frac{D}{8} - \Delta, \text{ as } \eta'_1 \geq \eta_1
\]

\[
\geq \min \left\{\eta_1 - \eta_2 + \frac{D}{8} - \left(\eta_1 - \eta_2 - \frac{D}{8}\right), \eta_1 - \eta_2 + \frac{D}{8} - \left(h - \frac{3D}{8}\right)\right\}
\]

\[
= \min \left\{\frac{D}{4} \cdot \eta_1 - \eta_2 + \frac{D}{2} - h\right\}
\]

\[
\geq 0, \text{ since } \eta_1 \geq \eta_2 \text{ as assumed, and } h \leq \frac{D}{2}.
\]

So, \(\tilde{b} > \tilde{q}_2\). Consequently, \(b < q_2\). Since \(q_2 = \min B_2\), for edge \((a, b)\), we have \(a > x\) by the definition of the set \(B_2\); see (16). On the other hand, we have \(a \notin [x', x]\) due to Lemma A.1. As a result, \(a > x\). Furthermore, we must have \(a > p_1\). If not, i.e., \(a \in [x, p_1]\), then the distortion of the pair of edges \((p_1, \tilde{q}_1)\) and \((a, \tilde{b})\) exceeds \(D\):

\[
(\tilde{b} - \tilde{q}_1) - (p_1 - a) = [(\tilde{b} - p_1) + (p_1 - \tilde{q}_1)] - (p_1 - a)
\]

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Therefore, \( a > p_1 \) and the edge \((a, \tilde{b})\) does not cross \((p_1, q_1)\). Hence arguing similar to \( I_1 \), we have \( |a - (\tilde{b} + \Delta)| \leq \frac{5D}{8} \).

This completes the proof for wide crossing.

In order to complete our analysis of various types of correspondences, we show now that a flip is not required if there is no wide crossing in \( C \).

**Theorem 3.9 (No Wide Crossing).** Let \( C \) be a correspondence between two compact sets \( X,Y \subset \mathbb{R} \) with distortion \( D \). If there are double crossings but not wide, then there exists a value \( \Delta \in \mathbb{R} \) such that

\[
\text{d}_H(X,Y + \Delta) \leq \frac{5D}{8}.
\]

**Proof.** We assume that there are double crossings in \( C \), but none of them are wide; see Figure 12. Since there are double crossings, at least one of the following two subsets of \( Y \) is non-empty:

\[
C_1 = \{ b \in Y \cap (-\infty, y') \ | \ \exists a \in X \cap (x, \infty) \ \text{with} \ \langle a, b \rangle \in C \} ,
C_2 = \{ b \in Y \cap (y, \infty) \ | \ \exists a \in X \cap (-\infty, x') \ \text{with} \ \langle a, b \rangle \in C \} .
\]

We also define \( q_1 = \min C_1, \ \text{and} \ q_2 = \max C_2, \) and edges \((p_1, q_1), (p_2, q_2) \in C\). Let us now denote \( \xi_1 = y' - q_1 \) and \( \xi_2 = q_2 - y \). If \( \text{d}_H(X,Y) \leq \frac{5D}{8} \), we take \( \Delta = 0 \). Let us assume the non-trivial case that \( \text{d}_H(X,Y) > \frac{5D}{8} \). Since wide crossings are not allowed, we have from Definition 3.6 that \( p_2 \geq x' - D, \ p_1 \leq p_0 \) and from

---

Figure 12: No wide crossing exists, but there are double crossings.
Lemma 3.5 that $h \leq \frac{D}{2}$. Consequently, the assumption that $d_H(X, Y) > \frac{5D}{8}$ implies either $\varepsilon > \frac{D}{4}, \xi_1 > \frac{D}{8}$, or $\xi_2 > \frac{D}{8}$. Without any loss of generality, we also assume that $\xi_1 \geq \xi_2$ so that the translation of $Y$ occurs from left to right.

We now consider the following two cases.

**Case 1 ($\varepsilon \geq \xi_1$):** This case is equivalent to Case (1) of Theorem 3.4. In Case (1) of Theorem 3.4, the intervals $I_1, I_6$ and $J_1, J_5$ use the fact that there are no double crossings. We show here that arguments presented for the intervals still work in the presence of (non-wide) double crossings. For any $a \in I_1 \cap X$ with an edge $(a, b)$, we show that $(a, b)$ cannot cross $(x, y)$. And, the rest of the argument for $I_1$ goes through.

For any $a \in I_1 \cap X$, we note that $a$ satisfies (10), the condition of a wide crossing. When $\varepsilon \leq \frac{D}{4}$, we have

$$x' - a > \frac{D}{2} + \left(\frac{5D}{8} - \Delta\right) = D - \Delta + \frac{D}{8} = D - \left(\frac{\varepsilon}{2} - \frac{D}{8}\right) + \frac{D}{8},$$

from (5).

When $\varepsilon > \frac{D}{4}$, we have

$$x' - a > \frac{D}{2} + \left(\frac{5D}{8} - \Delta\right) \geq \frac{D}{2},$$

as $\Delta \leq \frac{5D}{8}$.

Consequently, an edge $(a, b) \in C$ cannot be a double crossing, otherwise it would become a wide crossing. Hence, $(a, b)$ cannot cross $(x, y)$.

**Case 2 ($\varepsilon < \xi_1$):** This case is equivalent to Case (2) of Theorem 3.4. In Case (2) of Theorem 3.4, the intervals $I_1, I_6$ and $J_1, J_5$ use the fact that there are no double crossings. We show here that arguments presented for the intervals still work in the presence of (non-wide) double crossings. For any $b \in J_5 \cap Y$ with an edge $(a, b)$, we argue that $(a, b)$ cannot cross $(p_0, q_0)$, i.e., $a > p_0$. And, the rest of the argument for $J_5$ goes through. Due to Lemma A.2 we can have either $a < x', a \in [x, p_0)$, or $a > p_0$. We arrive at a contradiction when assumed the first two.
We first assume that $a < x'$, i.e., $(a, b)$ is a double crossing. This yields a contradiction, as $(a, b)$ becomes a wide crossing. When $\varepsilon \leq D_4$, we have
\[
x' - a \geq b - y, \text{ from Lemma 3.5}
\]
\[
\frac{D}{2} + \varepsilon + \left(\frac{5D}{8} - \Delta\right) \geq \frac{D}{2} + \varepsilon + \frac{5D}{8} - \left(\varepsilon - \frac{D}{8}\right) \geq D.
\]
When $\varepsilon > \frac{D}{4}$, we have
\[
x' - a \geq b - y > \frac{D}{2} + \varepsilon + \left(\frac{5D}{8} - \Delta\right)
\]
\[
\geq \frac{D}{2} + \varepsilon + \frac{5D}{8} - \left(\varepsilon + \varepsilon' - \frac{3D}{8}\right), \text{ from [5]}
\]
\[
= \frac{3D}{2} - \varepsilon' \geq \frac{D}{2}, \text{ since } \varepsilon' \leq D \text{ from [4]}
\]
So, $(a, b)$ cannot be a double crossing.

We now assume that $a \in [x, p_0]$ to arrive at the contradiction that the distortion of $(a, b)$ and $(p_0, q_0)$ exceeds $D$:
\[
(b_0) - (p_0 - a) = [(b - p_0) + (p_0 - q_0)] - (p_0 - a)
\]
\[
> \left[\left(\frac{5D}{8} - \Delta\right) + \left(\varepsilon + \frac{D}{2} + \varepsilon'\right)\right] - (p_0 - a)
\]
\[
\geq \left[\frac{9D}{8} + \varepsilon + \varepsilon' - \left(\varepsilon + \varepsilon' - \frac{3D}{8}\right)\right] - (p_0 - a), \text{ from [5]}
\]
\[
= \frac{12D}{8} - (p_0 - a) \geq \frac{12D}{8} - (b - a), \text{ as } b \geq p_0
\]
\[
\geq \frac{12D}{8} - \frac{D}{2}, \text{ from Lemma A.2}
\]
\[
\geq D.
\]
Hence, $a > p_0$, i.e., $(a, b)$ cannot cross $(p_0, q_0)$.

**Case 2 ($\varepsilon \geq \xi_1$):** This case is equivalent to Case (2) of Theorem 3.4 and the argument is analogous.

This concludes the proof. 

We conclude this section by showing in Theorem 3.10 that the bound of Theorem 3.2 is a tight upper bound in the following sense:

**Theorem 3.10 (Tightness of the Bound).** For any $0 < \varepsilon < \frac{1}{4}$ and $\delta > 0$, there exist compact $X, Y \subset \mathbb{R}$ with $d_{GH}(X, Y) = \delta$ and
\[d_{H, iso}(X, Y) = \left(\frac{5}{4} - \varepsilon\right)\delta.\]
Proof. It suffices to assume that \( \varepsilon = \frac{1}{4(2k+1)} \) for some \( k \in \mathbb{N} \). We now take (sorted)

\[
X = \{ x', x, x_k, x_{k-1}, \ldots, x_1, x_0 \} \quad \text{and} \quad Y = \{ y', y_0, y_1, \ldots, y_{k-1}, y_k, y \},
\]

with distances as shown in Figure 13. As a result, we also have \( (y_i - x) = 4i\varepsilon \delta \)
and \( (x_k - y_i) = 2\delta + 4(k - i + 1)\varepsilon \delta, \forall i \in \{ 0, 1, 2, \ldots, k \} \). To prove our claim that \( d_{H,iso}(X,Y) = \left( \frac{5}{4} - \varepsilon \right) \delta \), we consider translating both \( Y \) and \( \tilde{Y} \), the reflection of \( Y \) about the midpoint of \( x \) and \( x' \).

When translating \( \tilde{Y} \), we note that the smallest Hausdorff distance of \( \frac{3\delta}{2} \) is achieved for a translation of \( \tilde{Y} \) by an amount of \( \frac{\delta}{2} \) to the right. For this amount of translation, \( \tilde{y}' \) becomes the midpoint of \( x \) and \( x_0 \), where \( \tilde{y}' \) is the reflection of \( y' \) about the midpoint of \( x \) and \( x' \). And all the other points of \( \tilde{Y} \) are at distance at least \( (50\delta - \frac{\delta}{2}) \) from \( x \).

Now, we consider translating \( Y \) by an amount \( \Delta \in \mathbb{R} \). We first observe that \( d_{H}(X,Y) = 2\delta \), and the distance is attained by \( x_0 \) and \( y \). Now, a translation of \( Y \) to the left is only going to increase the Hausdorff distance \( d_{H}(X,Y + \Delta) \). Taking this argument one step further we get the following analysis as we vary \( \Delta \):

If \( \Delta \in (-\infty, \frac{3\delta}{4} + \varepsilon \delta) \), then the pair \( (x_0, y) \) gives

\[
d_{H}(X,Y + \Delta) = 2\delta - \Delta > 2\delta - \frac{3\delta}{4} - \varepsilon \delta = \left( \frac{5}{4} - \varepsilon \right) \delta.
\]

For \( \Delta = \frac{3\delta}{4} + \varepsilon \delta \), we get \( x_0 - (y + \Delta) = \left( \frac{5}{4} - \varepsilon \right) \delta \). Also, \( y_k + \Delta - x = \left( \frac{5}{4} - \varepsilon \right) \delta \) and \( x_k - (y_k + \Delta) = \left( \frac{5}{4} - \varepsilon \right) \delta + 4\varepsilon \delta \). So, \( d_{H}(X,Y + \Delta) = \left( \frac{5}{4} - \varepsilon \right) \delta \), which is attained by \( |x_0 - y_k| \).

Following this pattern, we conclude that \( d_{H}(X,Y + \Delta) > \left( \frac{5}{4} - \varepsilon \right) \delta \), except for \( \Delta = \frac{3\delta}{4} + \varepsilon \delta + 4i\varepsilon \delta \) for \( i \in \{ 0, 1, 2, \ldots, k \} \). Therefore, \( d_{H,iso}(X,Y) = \left( \frac{5}{4} - \varepsilon \right) \delta \). We summarize our analysis in Table 2.

With the \( d_{H,iso}(X,Y) \) computed, we now define the following correspondence \( C \) between \( X \) and \( Y \):

\[
C = \{ (x_i, y_i) \mid i \in \{ 0, 1, \ldots, k \} \} \cup \{ (x', y'), (x, y) \}.
\]

The distortion of \( C \) is evidently \( 2\delta \). Moreover, we observe that \( C \) is an optimal correspondence. Therefore, \( d_{GH}(X,Y) = \delta \).

\[\square\]

4. Conclusions and Future Work

In our investigation, we focus on approximating Gromov-Hausdorff distance by the Hausdorff distance for subsets of \( \mathbb{R}^d \). The use of \( d_{H,iso} \) yields an approximation algorithm with a tight factor. We do not know, however, if other algorithms can be devised with a better approximation factor. We believe that the problem of computing the Gromov-Hausdorff distance or even approximating it by a factor less than \( \frac{3}{4} \) in \( \mathbb{R}^d \) is NP-hard. The question of a polynomial-time approximation algorithm for subsets of \( \mathbb{R}^d \) is still open for \( d \geq 2 \).
Figure 13: The top picture demonstrates the configuration of \( X \) and \( Y \). The correspondence \( C \) is shown using the red edges. In the bottom, \( X \) and \( \tilde{Y} \), the reflection of \( Y \) about the midpoint of \( x \) and \( x' \), are shown, along with the correspondence \( C \) by the red edges.
\[
\begin{array}{cccc}
\Delta & d_H(X, Y + \Delta) & d_H(Y + \Delta, X) & d_H(X, Y + \Delta) \\
(-\infty, \frac{3\delta}{4} + \varepsilon \delta) & (x_0, y) & - & > (\frac{7}{4} - \varepsilon) \delta \\
\frac{3\delta}{4} + \varepsilon \delta & (x_0, y) & (y_k, x), (y, x_k) & (\frac{7}{4} - \varepsilon) \delta \\
\left(\frac{3\delta}{4} + \varepsilon \delta, \frac{3\delta}{4} + \varepsilon \delta + 4\varepsilon \delta\right) & - & (y_k, x), (y_k, x_k) & > (\frac{7}{4} - \varepsilon) \delta \\
\frac{3\delta}{4} + 4\varepsilon \delta & - & (y_{k-1}, x), (y_k, x_k) & (\frac{7}{4} - \varepsilon) \delta \\
\ldots & \ldots & \ldots & \ldots \\
\left(\frac{3\delta}{4} + \varepsilon \delta + 4i\varepsilon \delta, \frac{3\delta}{4} + \varepsilon \delta + 4(i + 1)\varepsilon \delta\right) & - & (y_{k-i}, x), (y_{k-i}, x_k) & > (\frac{7}{4} - \varepsilon) \delta \\
\frac{3\delta}{4} + 4(i + 1)\varepsilon \delta & - & (y_{k-i-1}, x), (y_{k-i}, x_k) & (\frac{7}{4} - \varepsilon) \delta \\
\ldots & \ldots & \ldots & \ldots \\
\frac{3\delta}{4} + 4k\varepsilon \delta & (x, y_0) & (y_0, x) & (\frac{7}{4} - \varepsilon) \delta \\
\left(\frac{3\delta}{4} + \varepsilon \delta + 4k\varepsilon \delta, \infty\right) & (x, y_0) & - & > (\frac{7}{4} - \varepsilon) \delta \\
\end{array}
\]

Table 2: A summary of \( d_H(X, Y + \Delta) \) is recorded for \( \Delta \in \mathbb{R} \). In the second and third columns, the directed Hausdorff distances are achieved for the shown pairs of points. The other columns are self-explanatory.

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A. Additional Lemmas

Lemma A.1. Let \( C \) be any correspondence between \( X, Y \subset \mathbb{R}^1 \) in its standard configuration (see Definition 3.1) with \( \text{Dist}(C) = D \). For \( s \in [x', x] \) and \( (s, t) \in C \), we have \( |s-t| \leq \frac{D}{2} \). Consequently, \( (s, t) \) does not cross \( (x, y) \) or \( (x', y') \).

Proof. We prove by contradiction. Without loss of generality, let us assume that \( t > s + \frac{D}{2} \). This implies that

\[
| |x' - s| - |y' - t| | = |x' - y'| + |t - s| > \frac{D}{2} + \frac{D}{2} = D
\]

This contradicts the assumption that \( \text{Dist}(C) = D \). So, \( t - s \leq \frac{D}{2} \).

Lemma A.2. Let \( C \) be any correspondence between \( X, Y \subset \mathbb{R}^1 \) in its standard configuration (see Definition 3.1) with \( \text{Dist}(C) = D \). If \( (s, t) \in C \) such that \( s > x \) and \( t > y \), then \( |s-t| \leq \frac{D}{2} \).

Proof. We prove by contradiction. Without loss of generality, let us assume that \( t > s + \frac{D}{2} \). This implies that

\[
| |x' - s| - |y' - t| | = |x' - y'| + |t - s| > \frac{D}{2} + \frac{D}{2} = D
\]

This contradicts the assumption that \( \text{Dist}(C) = D \). So, \( t - s \leq \frac{D}{2} \).

Lemma A.3. Let \( C \in \mathcal{C}(X, Y) \) be a correspondence between \( X, Y \subset \mathbb{R} \) that does not have any double crossings. The sets \( A, A' \) and \( B, B' \) are as defined in (2) and (3), respectively. Then,

(a) \( d_H(X, Y) > \frac{5D}{8} \) implies at least one of the above sets is non-empty.
(b) \( A, A' \) cannot both be non-empty.
(c) \( B, B' \) cannot both be non-empty.
(d) the pairs \( A, B' \) and \( A', B \) cannot both be non-empty.

Proof. (a) In this case, there must exist either

(i) \( a_0 \in X \) with \( \min_{b \in Y} |a_0 - b| > \frac{5D}{8} \), or

(ii) \( b_0 \in Y \) with \( \min_{a \in X} |a - b_0| > \frac{5D}{8} \).

If such an \( a_0 \) exists, we first observe that \( a_0 \) cannot belong to \( [x', x] \) due to Lemma A.1 and the definition of \( a_0 \). From the definition of \( a_0 \), we further note that either \( a_0 > y + \frac{5D}{8} \) or \( a_0 < y - \frac{5D}{8} \); see Figure 4 for reference. In either case, for an edge \( (a_0, t) \in C \), we now argue that \( t \in [y', y] \). To see this, consider the first case: \( a_0 > y + \frac{5D}{8} \). If we assume the contrary, i.e., \( y \notin [y', y] \), then we must have \( t > y \), since no double crossing is allowed. However, Lemma A.2
would then imply that \(|a_0 - t| \leq \frac{D}{2}\) — a contradiction to the definition of \(a_0\). For \(a_0 < y' - \frac{5D}{8}\), we can similarly arrive at a contradiction. Therefore, \(a_0\) belongs to either \(A\) or \(A'\). So, either \(A\) or \(A'\) is non-empty.

If such a \(b_0\) exists, we observe that \(b_0 \in [y', y]\). We assume the contrary, for example, that \(b_0 > y\) and \((s, b_0) \in C\) for some \(s \in X\). Lemma \[A.1\] implies that \(s \notin [x', x]\). From the assumption of no double crossing, then we must have \(s > x\). However, Lemma \[A.2\] then contradicts the definition of \(b_0\). We can arrive at a similar contradiction assuming \(b_0 < y'\). From the definition of \(b_0\), we further note that \(b_0 \in (x' + \frac{5D}{8}, x - \frac{5D}{8})\). Now, we observe that an edge \((s, b_0) \in C\) has to cross either \((x, y)\) or \((x', y')\). Otherwise, Lemma \[A.1\] again contradicts the definition of \(b_0\). Therefore, \(b_0\) belongs to either \(B\) or \(B'\). So, either \(A\) or \(A'\) is non-empty.

(b) We now note that \(A\) and \(A'\) cannot be both non-empty; see Figure 4. To see this, we let \(a_1 \in A\), \(a_2 \in A'\) with \(b_1, b_2 \in Y \cap [y', y]\) so that \((a_1, b_2), (a_2, b_2) \in C\). Since \(a_1 \geq x + D\) and \(a_2 < x' - D\), we arrive at the following contradiction to the fact that \(D = \text{Dist}(C)\):

\[
| a_1 - a_2 | = | a_1 - a_2 | - | b_1 - b_2 | > D.
\]

(c) We also note that \(B\) and \(B'\) cannot be both non-empty. To see this, we let \(b_1 \in B\), \(b_2 \in B'\) with \(a_1 > x\) and \(a_2 < x'\) so that \((a_1, b_2), (a_2, b_2) \in C\). Since \(b_1, b_2 \in (y' + D, y - D)\), we arrive at the following contradiction:

\[
| a_1 - a_2 | = | a_1 - a_2 | - | b_1 - b_2 | > D.
\]

(d) By the similar argument, each of the pairs \(A, B'\) and \(A', B\) cannot be both non-empty.

\[ \Box \]

**Lemma A.4.** Let \(\varepsilon, \varepsilon', D\) be non-negative numbers satisfying \([4]\). Then, for any \(\Delta\) following \([1]\), we have \(\varepsilon - \varepsilon' + \frac{9D}{8} - \Delta > 0\).

**Proof.** The proof considers two cases: \(\varepsilon \leq \frac{D}{4}\) and \(\varepsilon > \frac{D}{4}\).

If \(\varepsilon \leq \frac{D}{4}\), then \(\Delta = \varepsilon - \frac{D}{8}\). Since \(\varepsilon' \leq D\), we get

\[
\varepsilon - \varepsilon' + \frac{9D}{8} - \Delta = \varepsilon - \varepsilon' + \frac{9D}{8} - \left(\varepsilon - \frac{D}{8}\right) = \frac{10D}{8} - \varepsilon' > D - \varepsilon' \geq 0.
\]

If \(\varepsilon > \frac{D}{4}\), then \(\Delta \leq \varepsilon + \varepsilon' - \frac{3D}{8}\). In this case, we have

\[
\varepsilon - \varepsilon' + \frac{9D}{8} - \Delta
\]

\[
\geq \varepsilon - \varepsilon' + \frac{9D}{8} - \left(\varepsilon + \varepsilon' - \frac{3D}{8}\right)
\]

\[
= \frac{12D}{8} - 2\varepsilon'
\]

\[
= \frac{12D}{8} - 2\varepsilon' - 2\varepsilon + 2\varepsilon
\]
\[
\frac{12D}{8} - 2(\varepsilon' + \varepsilon) + 2\varepsilon
\]
\[
> \frac{12D}{8} - 2(\varepsilon' + \varepsilon) + 2 \times \frac{D}{4}, \text{ since in this case } \varepsilon > \frac{D}{4}
\]
\[
= \frac{16D}{8} - 2(\varepsilon' + \varepsilon)
\]
\[
\geq 0, \text{ since } \varepsilon + \varepsilon' \leq D.
\]

In either case, we conclude that \(\varepsilon - \varepsilon' + \frac{9D}{8} - \Delta > 0\). \(\square\)

**Lemma A.5.** Let \(\varepsilon, \varepsilon', \) and \(D\) be non-negative numbers satisfying (4). Then, for any \(\Delta\) following (5), we have \(\frac{D}{4} + \varepsilon + \varepsilon' - 2\Delta \geq 0\).

**Proof.** The proof considers two cases: \(\varepsilon \leq \frac{D}{4}\) and \(\varepsilon > \frac{D}{4}\).

If \(\varepsilon \leq \frac{D}{4}\), then \(\Delta = \varepsilon - \frac{D}{8}\). Since \(\varepsilon \leq \frac{D}{4}\), we get
\[
\frac{D}{4} + \varepsilon + \varepsilon' - 2\Delta = \frac{D}{4} + \varepsilon + \varepsilon' - 2 \left(\varepsilon - \frac{D}{8}\right) = \frac{D}{2} + \varepsilon' - \varepsilon \geq \frac{D}{2} - \varepsilon \geq 0.
\]

If \(\varepsilon > \frac{D}{4}\), then \(\Delta \leq \varepsilon + \varepsilon' - \frac{3D}{8}\). In this case, we have
\[
\frac{D}{4} + \varepsilon + \varepsilon' - 2\Delta \geq \frac{D}{4} + \varepsilon + \varepsilon' - 2 \left(\varepsilon + \varepsilon' - \frac{3D}{8}\right) = D - (\varepsilon + \varepsilon')
\]
\[
\geq 0, \text{ since } \varepsilon + \varepsilon' \leq D.
\]

In either case, we conclude that \(\frac{D}{4} + \varepsilon + \varepsilon' - 2\Delta \geq 0\). \(\square\)

**Lemma A.6.** Let \(\Delta \geq 0\) be a number, and \(C\) a correspondence between \(X,Y \subset \mathbb{R}\) with distortion \(D\). Let
\[
E(\Delta) = \left\{ b \in Y \mid X \cap \left[ b - \frac{5D}{8} + \Delta, b + \frac{5D}{8} + \Delta \right] = \emptyset \right\},
\]
\[
Y_\leq = \{ b \in Y \mid \text{there exists an edge } (a,b) \in C \text{ with } a \leq b \},
\]
and
\[
Y_\geq = \{ b \in Y \mid \text{there exists an edge } (a,b) \in C \text{ with } a \geq b \}.
\]
Then \(E(\Delta) \cap Y_\leq\) and \(E(\Delta) \cap Y_\geq\) cannot be both non-empty.

**Proof.** We prove by contradiction. Let \(b_1 \in E(\Delta) \cap Y_\leq\) and \(b_2 \in E(\Delta) \cap Y_\geq\). And, \(a_1, a_2 \in X\) are such that \((a_1, b_1), (a_2, b_2) \in C\) with \(a_1 \leq b_1\) and \(a_2 \geq b_2\).

The distortion of the edges is
\[
||a_1 - a_2| - |b_1 - b_2||
\]
\[
\geq (\max\{a_1, a_2\} - \max\{b_1, b_2\}) + (\min\{b_1, b_2\} - \min\{a_1, a_2\})
\]
\[
> 2 \left(\frac{5D}{8} + \Delta\right), \text{ since } b_1, b_2 \in E(\Delta)
\]
\[
>D.
\]

This is a contradiction. \(\square\)
Lemma A.7. Let $C$ be a correspondence with distortion $D$ between $X, Y \subset \mathbb{R}$. Let $\frac{D}{4} \leq h \leq \frac{D}{2}$, where $h = x - x'$, and $\Delta$ a number such that

\[
\begin{cases}
\Delta \in [0, \frac{D}{8}], & \text{when } \frac{D}{4} \leq h \leq \frac{3D}{8} \\
\Delta = h - \frac{3D}{8}, & \text{when } \frac{3D}{8} < h \leq \frac{D}{2}.
\end{cases}
\tag{A.1}
\]

Then, for any $a \in \left[ x' + \frac{D}{8} + \Delta, x - \frac{h}{2} + \frac{D}{16} + \frac{\Delta}{2} \right]$, with an edge $(a, b)$, we have $|a - (\bar{b} + \Delta)| \leq \frac{5D}{8}$.

Proof. Without any loss of generality, let us assume that the mid-point of $x'$ and $x$ is the origin, i.e., $\bar{b} = -b$. Since $0 \leq \Delta \leq \frac{D}{8}$ and $h = x - x' > \frac{D}{4}$, we have

\[
x' \leq x' + \frac{D}{8} + \Delta \leq x - \frac{h}{2} + \frac{D}{16} + \frac{\Delta}{2} \leq x.
\]

As a result, $a \in [x', x]$. We then note from Lemma [A.1] that $b \in [y', y]$ and $|a - b| \leq \frac{D}{2}$. We consider the following two cases:

Case 1 ($a \leq 0$): Since $\Delta \geq 0$, the distance $|a - (\bar{b} + \Delta)|$ is maximum when $b = a - \frac{D}{2}$. So, the maximum value is

\[
|b + \Delta - a| = -b + \Delta - a, \text{ as we flip about the origin, } \bar{b} = -b
\]

\[
= -2a + \frac{D}{2} + \Delta
\]

\[
\leq -2 \left( x' + \frac{D}{8} + \Delta \right) + \frac{D}{2} + \Delta
\]

\[
= 2(-x') + \frac{D}{4} - \Delta = 2x + \frac{D}{4} - \Delta
\]

\[
= 2 \times \frac{h}{2} + \frac{D}{4} - \Delta, \text{ since the mid-point of } x' \text{ and } x \text{ is assumed to be 0}
\]

\[
= h + \frac{D}{4} - \Delta \leq \frac{5D}{8}, \text{ considering both cases of (A.1)}.
\]

Case 2 ($a > 0$): The distance $|a - (\bar{b} + \Delta)|$ is maximum when $b = a - \frac{D}{2}$. The maximum value is,

\[
|a - (\bar{b} + \Delta)| = 2a + \frac{D}{2} - \Delta
\]

\[
\leq 2 \left( x - \frac{h}{2} + \frac{D}{16} + \frac{\Delta}{2} \right) + \frac{D}{2} - \Delta
\]

\[
= \frac{5D}{8}, \text{ since } x = \frac{h}{2}.
\]
Therefore, in either case, we must have \(|a - (b + \Delta)| \leq \frac{5D}{8}\). □

**Lemma A.8.** Let C be a correspondence between \(X, Y \subseteq \mathbb{R}\) with distortion D and a wide crossing edge \((p, q)\). Let \((a, b) \in C\) be an edge such that

\[
a < \begin{cases}
\min A', & \text{if } A' \neq \emptyset \\
x' - \frac{D}{2}, & \text{if } \max A > y + \frac{3D}{4} \\
x' - D, & \text{otherwise}
\end{cases}
\tag{A.2}
\]

or

\[
a > \begin{cases}
\max A, & \text{if } A \neq \emptyset \\
x + \frac{D}{2}, & \text{if } \min A' < y' - \frac{3D}{4} \\
x + D, & \text{otherwise}.
\end{cases}
\tag{A.3}
\]

Then, \((a, b)\) has to be a double crossing.

**Proof.** We prove the result when \(a < x'\) and satisfies (A.2). For \(a > x\) satisfying (A.3), the argument is exactly the same due to symmetry. We consider the following three cases depending on the position of \(p\).

**Case 1** \((a \leq p < x')\): If we assume the contrary that \((a, b)\) is not a double crossing, then \((a, b)\) and \((p, q)\) cannot cross. We arrive at the contradiction in each of the following two sub-cases.

\textbf{(max }A \leq y + \frac{3D}{4}\text{).} \quad \text{We have from Definition 3.6 that } p < \min A' \text{ if } A' \neq \emptyset \text{ and } p < x' - D \text{ if } A' = \emptyset. \text{ In either case, we have } (x' - p) > D. \text{ In either case, we also have from } a \leq p \text{ and the definition of } A' \text{ that } b < y'. \text{ Now, the distortion of the pair } (a, b) \text{ and } (p, q) \text{ is}

\[
(q - b) - (p - a) = (q - p) + (a - b) \geq (q - p) - |a - b|
\]

\[
\geq (q - p) - \frac{D}{2}, \text{ from Lemma A.2}
\]

\[
= [(q - y) + (y - x') + (x' - p)] - \frac{D}{2} \geq [(y - x') + (x' - p)] - \frac{D}{2}
\]

\[
= \left(\frac{D}{2} + h\right) + (x' - p) - \frac{D}{2} \geq \left(\frac{D}{2} + h\right) + D - \frac{D}{2} \geq D.
\]

This is a contradiction.

\textbf{(max }A > y + \frac{3D}{4}\text{).} \quad \text{We have } a < x' - \frac{D}{2}. \text{ Let } (p_0, q_0) \text{ be an edge with } p_0 = \max A \text{ and } q_0 \in [y', y]. \text{ We denote } \varepsilon = p_0 - x - D \text{ and } \varepsilon' = y - q_0 \text{ so that we have } \varepsilon > \frac{D}{4}. \text{ Also, } \max A > y + \frac{3D}{4} \text{ is equivalent to } \varepsilon > \frac{D}{4}.

If we assume \(b \in [y', q_0]\), then the distortion of the pair \((a, b)\) and \((p_0, q_0)\) clearly exceeds D. If we assume \(b \in [q_0, y]\), then the distortion of the pair \((a, b)\) exceeds D again:

\[
(p_0 - a) - (b - q_0) = [(p_0 - x') + (x' - a)] - (b - q_0)
\]

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≥ [(p₀ - x') + (x' - a)] - (y - q₀), since q₀ ≤ b ≤ y
= (h + D + ε) + (x' - a) - ε' > (h + D + ε) + \frac{D}{2} - ε'
≥ (h + D + ε) + \frac{D}{2} - (D - ε), from [4]
= h + 2ε + \frac{D}{2} > D, as ε > \frac{D}{4}.

Now, we assume that b < y'. We first note that from the distortion of the pair (a, b) and (p₀, q₀) that

\[ D \geq (p₀ - a) - (q₀ - b) = (p₀ - x') + (x' - a) - [(q₀ - y') + (y' - b)] \]
\[ = (D + ε + h) + (x' - a) - [(D + h - ε') + (y' - b)] \]
\[ = [(x' - a) - (y' - b)] + (ε + ε'). \]

So, (x' - a) - (y' - b) ≤ D - (ε + ε'). From the definition of wide crossing, we have p < x' - \frac{D}{2}. So, the distortion of the pair (a, b) and (p, q) exceeds D:

\[ (q - b) - (p - a) = [(q - y') + (y' - b)] - [(x' - a) - (x' - p)] \]
\[ = (q - y') + (x' - p) - [(x' - a) - (y' - b)] \]
\[ ≥ (q - y') + (x' - p) - [D - (ε + ε')], as noted above \]
\[ ≥ (y - y') + (x' - p) - [D - (ε + ε')] \]
\[ > (D + h) + \frac{D}{2} - [D - (ε + ε')] = h + \frac{D}{2} + ε + ε' \]
\[ ≥ h + \frac{D}{2} + 2ε, from [4] \]
\[ > D, since ε > \frac{D}{4}. \]

This is a contradiction.

**Case 2 (p < a < x'):** If we assume the contrary that (a, b) is not a double crossing, then (a, b) and (p, q) cross, i.e., (a, b) and (p, q) do not cross after flipping Y. We arrive at the a contradiction in each of the following two subcases.

**Case 2 (max A ≤ y + \frac{3D}{4}).** We have a < min A'. The definition of A' implies that b < y'. We also have (x' - a) > D. So, the distortion of the pair (a, b) and (p, q) exceeds D:

\[ (\tilde{b} - \tilde{q}) - (a - p) = (\tilde{b} - a) + (p - \tilde{q}) ≥ (\tilde{b} - a) - |p - \tilde{q}| \]
\[ ≥ (\tilde{b} - a) - \frac{D}{2}, from Lemma 3.5 \]
\[ ≥ [(y - x') + (x' - a)] - \frac{D}{2} = \left( \frac{D}{2} + h \right) + (x' - a) - \frac{D}{2} \]

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This is a contradiction.

\((\max A > y + \frac{3D}{4})\). When \(a\) and \(p\) are interchanged, the configuration becomes exactly the same as considered in the second sub-case of Case (1).

**Case 3** \((p > x):\) If we assume the contrary that \((a, b)\) is not a double crossing, we arrive at the a contradiction in each of the following two sub-cases.

\((A \neq \emptyset)\). From Lemma [A.3], we then have \(A' = \emptyset\). As a result, \(b < y'\). Since \(p > x + D\) and \(a < x' - \frac{D}{2}\), the distortion of \((a, b)\) and \((p, q)\) exceeds \(D\).

\((A' \neq \emptyset)\). Since \(a < \min A'\), we have from the definition of \(A'\) that \(b < y'\). Since \(p > x + \frac{D}{2}\) and \(a < x' - D\), the distortion of \((a, b)\) and \((p, q)\) exceeds \(D\).

**Lemma A.9.** Let \(C\) be a correspondence between \(X, Y \subset \mathbb{R}\) with distortion \(D\) and a wide crossing edge \((p, q)\). Let \((a, b) \in C\) be an edge such that \(b < y' - \frac{D}{2}\) (equivalently \(b > y + \frac{D}{2}\)). If \(A \neq \emptyset\) (equivalently \(A' \neq \emptyset\)), then \((a, b)\) has to be a double crossing, i.e., \(a > x\) (equivalently \(a < x'\)).

**Proof.** We prove the result for \(b < y' - \frac{D}{2}\). For \(b > y + \frac{D}{2}\), the argument is exactly the same due to symmetry. We assume the contrary that \((a, b)\) is not a double crossing. Due to Lemma [A.2], we then have \(a < x'\). Depending on the position of \(p\), we consider the following sub-cases to arrive at a contradiction in each of them.

**Case 1** \((a \leq p < x')\): Since \((a, b)\) does not cross \((x', y')\), we have from Lemma [A.2] that \(|a - b| \leq \frac{D}{2}\). So, \(b < y' - \frac{D}{2}\) implies that \(a < x' - \frac{D}{2}\). We can use the argument presented in Case (1) of Lemma [A.8] to arrive at the desired contradiction.

**Case 2** \((p < a < x')\): We note that

\[
\begin{align*}
\bar{b} - a &= \bar{b} - \bar{y} + (\bar{y} - x') + (x' - a) \\
&= (y' - b) + \left(\frac{D}{2} + h\right) + (x' - a) \\
&= 2(y' - b) + \left(\frac{D}{2} + h\right), \text{ since } (x' - a) \geq (y' - b) \text{ from Lemma [A.2]} \\
&> \frac{3D}{2}, \text{ as } b < y' - \frac{D}{2}.
\end{align*}
\]

Therefore, we can use Case (2) of Lemma [A.8] to arrive at a contradiction.

**Case 3** \((p > x)\): Since we still have \(a < x' - \frac{D}{2}\), we can use the sub-case \((A \neq \emptyset)\) of Case (3) from Lemma [A.8] to arrive at a contradiction.
So, we arrive at a contradiction in each of the above cases. Therefore, \((a, b)\) must be a double crossing.

**Lemma A.10.** If \((p, q)\) is a wide crossing and \(\eta_1, \eta'_1\) are as defined in Theorem A.8 then

\[
\eta_1 \leq \eta'_1 \leq D \text{ and } \eta_1 \leq D - h.
\]

**Proof.** The reader may use Figure 10 for reference. We first observe that \(\eta_1 \leq \eta'_1\), otherwise the distortion of \((p_1, q_1)\) and \((x', y')\) exceeds \(D\). We now consider the following two cases depending on the position of \(p\).

**Case 1** \((p < x')\): Similarly, due to the distortion of the wide crossing edge \((p, q)\) and \((p_1, q_1)\), we must have

\[
D \geq |(p_1 - p) - (q - q_1)| = |[(x' - p) + h + \eta'_1] - [(q - y) - \eta_1]|
\]

\[
= (x' - p) + h + \eta'_1 - [(q - y) - \eta_1], \quad \text{as } (x - p') \geq (q - y) + h \text{ by Lemma 3.5}
\]

\[
\geq 2h + \eta'_1 + \eta_1, \text{ since } (x - p') \geq h + (q - y) \text{ by Lemma 3.5}
\]

\[
\geq 2h + 2\eta_1, \text{ since } \eta'_1 \geq \eta_1 \text{ as already noted.}
\]

It implies that \(\eta_1 \leq \frac{D}{2} - h\). Moreover, we have \(\eta'_1 \leq D\) from the third line of the above equation.

**Case 2** \((p > x)\): If we assumed \(p_1 > p\), then Lemma A.8 would imply \((p_1, q_1)\) is a double crossing. This is a contradiction. For this reason, we assume that \(x \leq p_1 \leq p\), i.e., the edges \((p, q)\) and \((p_1, q_1)\) cross. As a result, the edges \((p, \tilde{q})\) and \((p_1, q_1)\) do not cross after flipping \(Y\). From the distortion bound of the pair, we get

\[
D \geq (\tilde{q} - \tilde{q}_1) - (p - p_1) = (p_1 - q_1) - (p - \tilde{q})
\]

\[
\geq (p_1 - q_1) - |p - \tilde{q}| \geq (p_1 - q_1) - \left(\frac{D}{2} - h\right), \quad \text{from Lemma 3.5}
\]

\[
= [(p_1 - x) + (x - \tilde{y}) + (\tilde{y} - q_1)] - \frac{D}{2} + h
\]

\[
= \left[\eta'_1 + \left(\frac{D}{2} + h\right) + \eta_1\right] - \frac{D}{2} + h
\]

\[
\geq 2\eta_1 + 2h, \text{ since } \eta'_1 \geq \eta_1 \text{ as already noted.}
\]

So, \(\eta_1 \leq \frac{D}{2} - h\). Moreover, we have \(\eta'_1 \leq D\) from the third line of the above equation.

\[\square\]