Improvements to the Eddy Viscosity Wind Turbine Wake Model

K Gunn
E.ON Climate & Renewables, Westwood Way, Coventry, UK, CV4 8LG
E-mail: kester.gunn@eon.com

Abstract. This paper presents improvements to the eddy viscosity wake model to significantly improve its fidelity without compromising its low computational cost. The eddy viscosity wind turbine wake model, proposed by Ainslie, has been a staple component of the wind yield toolchain for decades. Its low computational cost and excellent predictive power are likely to keep it as such for years to come. However, the standard implementations of the model rely heavily on several heuristics which, although well validated, may not scale well for large turbines or large windfarms.

In this paper, some non-physical heuristics are examined, and alternatives proposed which better reflect the underlying physics the model seeks to represent, which is shown to provide much better fidelity at no additional computational cost. The proposed model allows for better predictive capabilities of the performance of the next generation of offshore wind farms. The two key areas investigated are: first the impact of the ambient condition (TI, stability, ...) on wake development; and second a re-examination of the wake combination method, leading to the inclusion of three additional processes in the model. It is shown that modifications to these areas provide a more verifiable wake model than the baseline implementations, and allow identification of the remaining physical processes to be further incorporated into future versions.

1. Introduction

The Eddy Viscosity (EV) wind turbine wake model, proposed by Ainslie [1], is a common component of commercial modelling packages [2, 3]. Many implementations rely on heuristics which, although well validated, may not scale well for large turbines or large windfarms and have been shown to produce artefacts which could lead to suboptimal windfarm designs when such models are used for layout optimisation [4].

In this paper, some of the non-physical heuristics are examined, and alternatives proposed which better reflect the underlying physics the model seeks to represent, resulting in much better fidelity at no additional computational cost. The resulting model is compared to measurements from two offshore windfarm, and is shown to improve agreement.

1.1. The Baseline and New Model

In order to illustrate the effect of non-physical heuristics on the results of wake models, two implementations of the EV mode are compared in this paper:

Baseline The “baseline” implementation is an approximation to commercially available implementations [2, 3]. It is given as a comparison to the “new model” and windfarm...
data allowing the impact of the heuristics to be understood.

**New Model** The “new model” replaces the non-physical heuristics with better representations of the physical processes. It is used to show how such improvements yield results with fewer modelling artefacts and better general agreement to measured data.

Initially, the single turbine wake model is discussed:

Section 2 presents a brief introduction to the underlying EV wake model (as defined by Ainslie [1]), and the numerical scheme used to solve it (derived by Anderson [5]). This model only represents the development of a single turbine’s wake but is the basis for both the baseline and new model implementations.

Section 3 discusses the “ambient mixing”. It is shown that the baseline formulation, taking its length scale from the turbine rotor diameter, is inappropriate and can result in unrealistic artefacts in results. A new formulation is recommended in its place.

Moving to multiple turbines, the wake combination method and its interaction with other physical processes is then discussed, yielding three corrections:

Section 4 analyses the “maximum deficit” wake combination method used in the baseline implementation. It is shown why, despite its lack of physical meaning, this method is often able to produce good results. Three improvements are derived in the following three sections. These, in combination, are required to correctly account for the physics which the “max deficit” heuristic is able to, in some cases, approximate:

Section 5 introduces the wake combination method used to replace “max deficit” in the new model which is based on conservation of kinetic energy. This modification, in isolation, yields significant over-estimations of wake loss. As such it must be used in combination with the two remaining modifications.

Section 6 introduces a heuristic to account for turbine-wake interaction, i.e. the impact of a downstream turbine on the wake of an upstream turbine. This process reduces wake losses of rows of turbines when they are aligned into the wind.

Section 7 re-introduces a physical process, meandering, which was first introduced by Ainslie [1] but does not appear to be in the commercial models on which the baseline implementation is based.

These three corrections are developed such that they need only be applied at locations where the wind speed needs to be estimated (i.e. at turbine locations). As such, they have very low impact on the computational cost of the model. Finally:

Section 8 shows results from the models and comparisons to measurements from two offshore windfarms; and

Section 9 draws conclusions and makes recommendations for further work.

Initially, though, it is worth discussing the philosophical approach which has been taken to identify these corrections: verification\(^1\).

1.2. Validation or Verification?

It is common to strive for a “verified and validated” model, despite this being an unachievable goal [6]. Verification is the process of ensuring a model has a correct interpretation of the relevant physical processes, while validation seeks to show that a model produces results within a required level of error.

\(^1\) Not to be confused with “verificationism” which, with logical positivism, has been abandoned.
Due, in part, to the commercial necessity of wake modelling (i.e. it is required to estimate the commercial value of a windfarm), much emphasis has been placed in the past on validating wake models (e.g. [2, 7, 8]). This has resulted in a lack of emphasis on verifying the models.

Within a given operational envelope (e.g. turbine spacing, farm size, control strategy, or rotor diameter), even a model with relatively poor representation of the underlying physics can be made to provide excellent results with appropriate tuning parameters. Such a model can therefore pass the required hurdles of a validation process [9]. However, without appropriate modelling of the underlying physical processes, the value of the model for extrapolating beyond the validated envelope remains unknown.

For example, the pioneering work of Jensen [10] accounts for conservation of momentum, but discards all other physical processes in favour of heuristics yielding a computationally efficient model. Many derivatives of this model have “passed” validation exercises. However, when used to extrapolate (e.g. to windfarm control), unrealistic results have been observed.

In this paper model results have been compared with measured data. Conditions where the model does not fit have been examined and this has been used to hypothesise what physical processes are not appropriately modelled. Thus, the model derived here should be considered better verified than the baseline model, in that the objective is to improve the representation of the physics. This model should not be considered validated, and any validation process must be performed with new datasets.

1.3. A Brief Note on Nomenclature
Coordinates are given either as \([x, y, z]\) being the stream-wise, cross-stream and vertical coordinates with the origin at a turbine’s hub (when using polar coordinates, \(r\) is the radial) or as global coordinates \([X, Y, Z]\), with \(Z\) origin at the ground.

To aid clarity and avoid “dimensionality errors”, parameters denoted with a tilde (\(\tilde{X}\)) are nondimensionalised by the inflow wind velocity at hub height (\(U_0\)) and rotor diameter (\(\delta\)), as is common in wake modelling:

\[
\tilde{U} = \frac{U}{U_0}, \quad \tilde{x} = \frac{x}{\delta}, \quad \tilde{\epsilon} = \frac{\epsilon}{U_0\delta}
\]

where \(\epsilon\) is the eddy viscosity. Upper case velocities \((U, V, \ldots)\) are means, while lower case \((u, v, \ldots)\) are turbulent components.

2. The Single Turbine Model
Ainslie [1] derived the EV model for the wake of individual turbines by solving (in cylindrical coordinates) the momentum equation:

\[
U_x \frac{\partial U}{\partial x} + V_r \frac{\partial U}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left( \epsilon r \frac{\partial U}{\partial r} \right)
\]

(1)

(where the “eddy viscosity”, \(\epsilon\), is used to describe the shear stress: \(-uv_r = \epsilon U_r/\partial r\) and \(V_r\) is the radial velocity) and the continuity equation:

\[
\frac{\partial U}{\partial x} + \frac{1}{r} \frac{\partial}{\partial r} (V_r r) = 0
\]

(2)

The “top hat” wake profile is an obvious example which Jensen [10] himself shows not to hold. A less obvious one is the linear rate of wake width expansion. Examination of results from more verifiable models (e.g. Larsen [11], the EV model, see Fig. 1b and CFD [4]) or measured data (e.g. measurements by Richardson given by Taylor [12]) show wake expansion will not be linear.

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Ainslie, and most commercial implementations [2, 3] solve the equations by finite difference, using the Thomas (tridiagonal matrix inversion) algorithm. As shown by Anderson [5], however, if the wake profile is assumed to be self-similar at all locations downstream (which can be shown by examining the results from the Thomson algorithm implementation), and invoking conservation of momentum, then Eqn. 1 can be reduced to:

$$\frac{d\tilde{U}_c}{d\tilde{x}} = \frac{16\epsilon}{\tilde{U}_c C_t} \left( \tilde{U}_c^3 - \tilde{U}_c^2 - \tilde{U}_c + 1 \right)$$

This can be solved using a Runge-Kutta method (here the classic 4th order variant is used) to give identical results to the Thomas algorithm implementation. The off-centre flow speed can then be calculated knowing that the wake takes a Gaussian profile:

$$\tilde{U}(\tilde{r}) = 1 - (1 - \tilde{U}_c) \exp\left(-\frac{\tilde{r}^2}{\tilde{w}^2}\right)$$ (3)

where $\tilde{w}$ is the wake (half-) width, which is calculated using conservation of momentum as:

$$\tilde{w}^2 = \frac{Ct}{4 \left(1 - \tilde{U}_c\right) \left(1 + \tilde{U}_c\right)}$$ (4)

The model is initialised at $\tilde{x} = 2$ as defined by Ainslie [1], where the flow is no longer dominated by pressure gradients.

3. Ambient Mixing
The EV wake model assumes that the rate of recovery of wind turbine wakes is governed by two mixing components: a “local” component ($K_l$) generated from the shear of the wake, and an “ambient” component ($K_a$) generated by the shear in the atmospheric boundary layer:

$$\epsilon = F(K_l + K_a)$$

where $F$ is a “filter function” accounting for the build-up of turbulence close to the turbine (and is here taken as $F = 1$ inline with WindFarmer [3]). Here $K_l = k_l w(U_0 - \tilde{U}_c)$ where $k_l = 0.015\sqrt{3.56}$ as estimated by Ainslie [1].

3.1. The Ainslie Formulation
The ambient mixing in terms of the “normal boundary layer parameters” is estimated as:

$$K_a = \kappa U_* Z/\phi_m(Z/L)$$ (5)

where $\kappa = 0.4$ is the Von Kármán constant, $U_*$ is the friction velocity and $\phi = 1$ for neutral conditions.

3 The Larsen [11] model solves the same equations analytically, making further simplifications at the expense of generality.

4 Ainslie uses $\tilde{d}$ as the measure of approximately the full wake width, while here, $w$ is used (equivalent to “2 standard deviations” of the Gaussian profile, thus approximately a half wake width) to remove the need for an arbitrary constant:

$$\tilde{d} = \sqrt{3.56} \tilde{w}$$

5 It is not clear why this function can be neglected. It is possible that the blade-induced turbulence [13], which would not have been present in many of the wind tunnel permeable disk datasets available to Ainslie, counters the lack of shear-induced turbulence. An improved “near wake” model accounting for pressure effects and blade turbulence may improve accuracy.
Invoking the “Law of the Wall” to substitute for $U_*$, Ainslie [1] then gives the ambient mixing in neutral conditions as:

$$K_a = h \kappa^2 U_0 / \ln (h/z_0)$$

where $z_0$ is the roughness length. However, he next presents the complete (including $K_l$) nondimensionalised form of the eddy viscosity, wherein $K_a$ has become:

$$\tilde{K}_a = \frac{K_a}{U_0 \delta} \approx \frac{\kappa^2}{\ln (h/z_0)}$$ (6)

where the hub height ($h$) has been cancelled by the rotor diameter ($\delta$). This approximation ($h \approx \delta$), although reasonable in 1988, is unnecessary and, with modern windfarms can lead to some unrealistic artefacts (see Section 3.4).

3.2. The Baseline Formulation
The WindFarmer [3] and OpenWind [2] Theory Manuals give the ambient mixing as:

$$\tilde{K}_a = \kappa^2 I_0$$ (7)

This formulation has the benefit that the mixing is a function of only known or measured quantities (the turbulence intensity ($I_0$) is normally measured while the roughness length ($z_0$) is harder to obtain). However, it is not clear whence this equation derives and if it has any basis in theory. As with the Ainslie formulation, this formulation is incorrectly nondimensionalised by the rotor diameter (i.e. assuming $h \approx \delta$).

3.3. An Alternative Formulation
Townsend [16] gives a turbulence model in the form:

$$\frac{\overline{u^2}}{U_*^2} = B_1 - A_1 \log(z/\delta)$$

where $u$ is the unsteady component of stream-wise flow, $\delta$ is the thickness of the shear layer, and $A_1$ and $B_1$ are non-dimensional constants. This equation is well grounded in theory (a detailed derivation can be found by Perry and Chong [17]), but in reality is only valid over a limited range of heights (see, for example, Marusic et al. [18]) and is unlikely to be satisfactory in non-neutral conditions. However, again substituting for $U_*$ in Eqn. 5, it allows the relation between $K_a$ and $I_0$ to be investigated:

$$K_a = \kappa I_0 U_0 h / \sqrt{B_1 - A_1 \log(h/\delta)}$$ (8)

Requiring, as it does, two poorly understood constants (for those wishing to test this formulation, $A_1 = 1.25$, $B_1 = 2.2$ are not unreasonable estimates [18]), and site measurements of $\delta$, it is not recommended that this formulation be used. However, there is now at least a good theoretical basis on which to place the use of $I_0$ in estimating $K_a$. Instead, the following is recommended:

$$K_a = k_a I_0 U_0.$$ (9)

There is some uncertainty here as all documents which present the mixing in this form seem to make a dimensional error, where the non-dimensional ambient mixing is added to the dimensional local mixing [2, 3, 14]. It is assumed that actual implementations are correct and this is only an error in the documentation.

7 WindFarmer [3] cites it to Ainslie [1], who presents no such relation. It is possible it can be derived based on Eqn. 5 and some derived values from wind standards such as EN1991-1-4 [15], however it is not clear if such an approach is soundly based in theory.
where \( k_a \approx 0.5 \) is a dimensionless constant, and \( I_0, U_0 \) and \( l_0 \) (the length scale) are all derived from site measurements (e.g. deriving \( l_0 \) from frequency analysis of anemometer data). For neutral conditions, \( l_0 \approx \kappa h \).

3.4. Impact of Ambient Mixing Length Scale

Both the Ainslie and baseline formulations for \( K_a \) (Equations 6 and 7 respectively) take the length scale as the rotor diameter (\( \delta \)) rather than any feature of the natural flow field. Thus, the rate of expansion of a wind turbine’s wake, kilometres downstream of the wind turbine which formed it, is dependent upon the size of the rotor.

This can be shown to be inappropriate by considering a system of two rotors, as illustrated in Fig. 1. The first rotor (\( \delta = 1, h = 0.8 \) at \( X = 0 \)) is operated at close to optimal thrust coefficient of \( C_T = 0.8 \). The second rotor is positioned at \( X = 1.5 \) but sufficiently far to the side that the turbines do not interact (illustrative map in Fig. 1a). It is shown that one can size the second rotor and operate it with a certain thrust such that the width and deficit of both wakes will, for some \( X (X^*) \), be identical. Thus both should now develop identically. However (left panels on Fig 1b), the two wakes will develop at different rates due to the different length scales the wind “remembers”\(^8\) from the rotors. Conversely, referring to the right panel, where the length scale is

\(^8\) A homeopathic interpretation.
taken as a function of the hub height (Eqn 8, neutral conditions), the two wakes now develop identically.

4. The “Maximum Deficit” Wake Combination Method

Much research has been performed on the subject of “wake combination”. Many researchers have concluded that the “maximum deficit” method (as used in OpenWind [2] and WindFarmer [3]) is optimal despite its distinct lack of physical meaning, in which:

\[(U_0 - U) = \max_i (U_0 - U_i)\]

where \(U_i\) is the speed modified by the wake of (only) turbine \(i\), and \(U\) is the speed once the wakes are “combined”. However, high resolution Computational Fluid Dynamics (CFD) analyses has failed to corroborate this approach [4].

Before selecting a more appropriate wake combination method, it is worth examining why the maximum deficit method is able to produce good results. Examination of the underlying physics shows several processes which are not modelled in the baseline implementation. Instead, two non-physical heuristics combine to approximate these actual processes in specific cases.

The two heuristics in the baseline implementation are:

(i) scaling the wake of a waked turbine so that it returns to \(U_0\) at the edges;
(ii) “max deficit” wake combination, which due to (i) now approximates the deficit from both turbines.

These two heuristics are illustrated in Fig. 2. In such models, the scaled wake becomes a representation of the total wake in the system (i.e. the energy deficit integrated over the scaled wake is closer to the total deficit than the deficit generated by one turbine). Thus, the single scaled wake (which has the maximum deficit) is selected as an approximation for the total wake.

The three physical processes which these heuristics balance to approximate are:

(i) Conservation of energy should be applied when combining wakes (Section 5)
(ii) The impact of downstream turbines on wakes will have a destructive impact on the wake (Section 6)
(iii) Wake meandering, caused by large-scale horizontal turbulence, will cause a reduction in the perceived wake (Section 7)

As shown in Fig. 2f the two heuristics above provide a reasonable approximation to reality when turbines line up. When this is not the case, however, the max deficit approach provides poor results.

5. Wake Combination Conserving Kinetic Energy

Katic et al. [19] suggests using conservation of Kinetic Energy (KE) when combining wakes, and propose:

\[(U_0 - U)^2 = \sum_i (U_0 - U_i)^2\]

While other wake combination methods such as “linear” or “sum of squares” may also lack a physical interpretation, the individual wakes are combined, whereas for “max deficit” the wakes exist independently, one of which becomes true at the point it is sampled. The Copenhagen interpretation.
T. 1 T. 2 T. 3a
T. 3b
(b) (c) (d, e) (f)
Total wake
Wake from T. 1
Wake from T. 2

(a) An illustrative windfarm layout. Letters in brackets show the X locations of sub-figures (b) to (f).

(b) The wake from the first turbine, T. 1, is initialised as prescribed by Ainslie [1].

(c) The wake is propagated to the location of T. 2. The wind speed in the wake is sampled as the input speed for T. 2 ($U_i$).

(d) T. 2's wake is initialised as prescribed by Ainslie [1], but taking $U_0 = U_i$ when calculating the profile.

(e) As the profile in Fig. 2d does not return to $U_0$ at the edges the wake is “scaled” to return to $U_0$ rather than $U_i$ [2, 3]. Note that the profile for T. 2 now more closely represents the total deficit (dashed black line) than the deficit of T. 2. Integrating the energy deficit in the re-scaled wake would yield an answer closer to the total energy deficit than that which is attributed to T. 2.

(f) At T. 3, the deficits from the two turbines are “combined”. As the scaled wake from T. 2 is an approximation to the total wake in the system the “worst” wake is taken. Any attempt to sum the two wakes would double count the deficit from T1. This results in a good approximation when the turbines are aligned (T. 3a), but is much poorer otherwise (T. 3b).

Figure 2: Illustration of the interaction of wake “scaling” during the initialisation of a wake, and the “max deficit” wake combination method as used in the baseline implementation and by several commercial implementations [2, 3]. The total wake is illustrative, based on high resolution CFD results [4].
However, this formulation does not correctly account for KE\textsuperscript{11}. The correct formulation, as presented by Kuo et al. [20] amongst others, is:

\[(U_0^2 - U^2) = \sum_i (U_0^2 - U_i^2)\]

When used in isolation (i.e. replacing “max deficit” and wake scaling) this approach will result in a significant over-estimation of wake losses (see Section 8.2.2). However, when combined with the turbine wake interaction (Section 6) and wake meandering (Section 7) the correct magnitude of loss is obtained (see Section 8).

6. Turbine-Wake Interaction

In the baseline wake model implementation (as in most engineering wake models), there is only a “one way coupling” between wakes and turbines. After a wake has been generated by a turbine, that wake will affect the operation of downstream turbines (wake-turbine interaction) but downstream turbines will have no impact on the development of that wake (turbine-wake interaction). As illustrated in Fig. 3, the expanding streamtube around a downstream turbine will result in the expansion of the wake from an upstream turbine. From conservation of momentum\textsuperscript{12} it is clear that the expansion of the wind turbine wake will result in a reduction of the deficit in the wake. Thus, the approach taken here is:

(i) Where the wake from turbine T. \(i\) reaches the \(X\) location of turbine T. \(j\), calculate the expansion of the stream-tube (\(e_j\)) around turbine T. \(j\) (Eqn. 10)
(ii) Calculate the expansion this will result in for the wake from T. \(i\). (\(e_{i,j}\), Eqn. 11)
(iii) Calculate the width of wake T. \(i\) using Eqn. 4, and add \(e_{i,j}\) (i.e. increase the width of the wake from T. \(i\))
(iv) Update the centreline deficit of the wake form T. \(i\) to comply with conservation of momentum (Eqn. 12)

From conservation of mass the expansion of the streamtube can be calculated as:

\[
\frac{e_j}{\partial} = \sqrt{1 - a_j} \left( \frac{1}{\sqrt{1 - 2a_j}} - 1 \right)
\]

where \(a_j\) is the induction factor of turbine T. \(j\) (calculated as \(a = \frac{1}{2} \left(1 - \sqrt{1 - C_t} \right)\)). It is now necessary to estimate what impact this stream tube expansion will have on the wake of T. \(i\). When the two turbines line up exactly, the wake will be expanded significantly. From CFD results it has been observed [4] that when the turbines are just off alignment, the upstream wake is actually marginally “squeezed” rather than expanded. Thus, the direct application of this expansion is complex. Here, the squeezing of the wake is neglected, and the magnitude of the wake expansion is taken as a Gaussian distribution, with maximum expansion if the two turbine line up, moving to zero when the turbines are separated in the cross stream direction:

\[
e_{i,j} = c_0 e_j \exp \left( -\frac{(Y_i - Y_j)^2}{c_j^2} \right)
\]

\textsuperscript{11} Despite no physical interpretation, this formation remains common in the literature but is refereed to as “sum of squares” rather than “conservation of kinetic energy”

\textsuperscript{12} Note that conservation of momentum does not hold within the expanding streamtube as the turbine applies a thrust to the flow. However, the momentum deficit introduced by the turbine is wholly accounted for by the introduction of the wake from that turbine. Thus, conservation of momentum does apply to the wake from the upstream turbine.
where $Y_i$ and $Y_j$ are the cross stream positions of turbines $T. i$ and $T. j$ respectively. In reality the correct function is likely to be “squarer” towards the centre, and dip below zero at the edges of the turbine. Further work is required to estimate and parametrise such a function. Two parameters in Eqn. 11 ($c_0$ and $c_1$) are tuning parameters and further work is required to fix their values appropriately. Here $c_0 = 2$ and $c_1 = 1.5$ have been found to work adequately to verify the concept if not validate the model\(^\text{13}\).

The centreline velocity for the wake from $T. i$ is then calculated as:

$$
\tilde{U}_c = \sqrt{1 - C_t/4\bar{w}^2}
$$

(12)

7. Meandering

Meandering impacts wakes by diverting them to either side through large scale fluctuations in the flow direction (i.e. with length scales larger than those driving wake recovery). The wake structure remains unchanged. Recent work has been done on the impact of wake meandering (e.g. Johnson et al. \cite{21} and Larsen et al. \cite{22}). However, the correction from Ainslie \cite{1} is used here\(^\text{14}\).

The motion of the wake centre-line can be interpreted in similar terms to particle diffusion by turbulent flow. Under this framework (and in a Lagrangian reference frame) the variance ($\sigma_{Y_c}^2$) of the cross-flow position of the wake ($Y_c$) can be estimated. General solutions under this approach (particularly in the complex non-isotropic atmospheric boundary layer) are hard to define (requiring knowledge of the turbulent Lagrangian autocorrelation, which is hard to measure), however it can be shown \cite{12} that:

$$
\sigma_{Y_c}^2 = \begin{cases} 
\sigma_v^2 t^2 & t \ll T_v^L \\
2\sigma_v^2 T_v^L t & t \gg T_v^L 
\end{cases}
$$

\(^\text{13}\)Values of $c_0 > 1/2$ may seem odd. Sadly, this is an introduction of an additional heuristic: a second physical process which is not accounted for here is the fact that the local mixing $K_l$ will be increased when the two wakes align (as the combined wake has a larger deficit). Accounting for this explicitly has been found to be extremely computationally expensive as the flow feels must be calculated more often, and the integration step size reduced. Thus, taking this effect as being linearly proportional to the effect of the expanding stream tube is a necessary evil. It is hoped that a better relation can be derived through high resolution CFD modelling in the future.

\(^\text{14}\)Although this correction is presented in the “original” EV model paper (cited to an earlier paper by the same author), it does not seem that it is implemented in the commercial models, or at least it does not appear in the theory manuals for them \cite{2, 3}.
where $T^L_v$ is the (cross-stream) Lagrangian (integral) time scale, $\sigma_v$ is the standard deviation of the cross-stream velocity and $\sigma_v$ is the standard deviation of the cross-stream velocity. These results effectively treat the turbulent velocity as constant ($t \ll T^L_v$) over the timescale of interest (thus the distance the centreline moves is simply $vt$), or as a Wiener process ($t \gg T^L_v$) where “steps” are uncorrelated (so offset grows with the square root of the time).

To identify which approximation ($t \ll T^L_v$ or $t \gg T^L_v$) is more appropriate, estimates of $t$ and $T^L_v$ must be made. The time for the wind to travel from the waking turbine to the waked turbine is $t = x/U_0$. Taking $x \approx 1$km as the order of distances of interest, and $U_0 = 10$ m/s as a speed of interest, $t = 100$ s. Appropriate values for $T^L_v$ are hard to find, but $T^L_v \approx 200$ s is not unreasonable [23].

Thus, the correlated case ($t \ll T^L_v$) is more appropriate, but this is marginal. Given this, and that closer turbine wake interactions (smaller $x$) are more important, $\sigma^2_{Y_C} = \sigma^2_v t^2$ is used here. However, with better analysis of site measurements (to better estimate the autocorrelation function), improved results are likely to be possible, especially for (spatially) large wind farms.

Taking $v \sim \mathcal{N}(0, \sigma^2_v)$, then $Y_C \sim \mathcal{N}(0, \sigma^2_{Y_C})$. From the above, $\sigma^2_{Y_C} = \sigma^2_v t^2 = \sigma^2_v x^2/U_0^2$. Having identified the Probability Density Function (PDF) of the wake centreline offset, the effective wake can now be calculated by convolving this PDF with the wake profile. To simplify this process rather than calculating this for all $Y$, it need only be calculated for the mean centreline (zero offset) and the remainder of the profile can be reconstructed by conservation of momentum using Equations 3 and 4.

Noting that the deficit is also a Gaussian form, the convolution yields:

$$U_0 - U^*_c = \frac{(U_0 - U_c)}{m}$$

$$m = \sqrt{1 + 2\sigma^2_\theta x^2/w^2}$$

where $\sigma_\theta$ is the standard deviation in wind direction which is the result presented by Ainslie [1]15.

7.1. Meandering and Histograms

It is common practice to perform yield estimates by undertaking a simulation for each “bin” of a 2D histogram of speed and direction, and multiplying the result by the probability of that “bin”. When analysing windfarm data, the inverse approach is often taken (i.e. the individual flow cases are “binned” and the mean calculated within each bin). In such cases $\theta$ will not be normally distributed, but will be the convolution of the true variability of $\theta$ ($\mathcal{N}(0, \sigma^2_\theta)$) and the distribution within the bin (for narrow bins, approximately uniform, $\mathcal{U}(\ldots)$). As such it is hard to differentiate the effects of meandering from the effects of the histogram. Results here (Section 8) are obtained using a local polynomial regression rather than “binning” and so the histogram effect should not exist.

15 Although $\sigma_v$ is not often measured, there is good evidence that $\sigma_v \approx \sigma_u \approx I_0 U$ (i.e. the turbulence is horizontally isotropic). For low levels of turbulence ($I_0 \lesssim 15\%$), the standard deviation of wind direction (in radians) is:

$$\sigma_\theta \approx \sigma_v/U \approx I_0$$

by the small angle approximation.

16 Note that the constant 7.12 from Ainslie [1] (Eqn. 10) is given as 2 here for consistency between the different definitions of wake breadth, $w$ and $b$. 

11
8. Results
Results of the baseline and improved models are illustrated in Figures 4a, 4b and 5, with some additional details shown in Fig. 6. The “Baseline” model is implemented with “max deficit”, “wake scaling” and the ambient mixing is calculated using Eqn. 7. The “KE” model is implemented with conservation of kinetic energy wake combination, no “wake scaling”, and Eqn. 9. “All Improvements” comprises “KE” with the new turbine-wake interaction heuristic, and wake meandering.

8.1. Wake Rose Figures
In Figures 4a 4b and 5, results are presented in the form of a “wake rose” wherein the radius shows the power generated by a turbine divided by the power that turbine would generate if unwaked. The angle shows the direction from which the wind blows. Green dots show the locations of other turbines in the farm, with the turbine at the centre (marked with an X) being the one for which the results are presented. Thus, for example, in Fig. 4a, at ∼ 90° the wind is flowing over one turbine before reaching the turbine of interest, while at ∼ 270° it is flowing over two. Figs. 4a and 4b are for a turbine at low and high speeds at Scroby Sands. Fig. 5 is for low speeds at Rodsand II. Fig. 6 shows an enlargement from Fig. 4a and illustrates the effects of the various modifications to the model.

Measured results are obtained from SCADA data, with the inflow estimated from the mean of the unwaked turbines (with some corrections applied). The line of best fit, 95% confidence and 95% prediction intervals are obtained using a multi-dimensional (direction and speed), robust (bisquare), local polynomial (2nd order in direction, 1st in speed) regression with Gaussian weighting (kernel widths of π/40rad and 1m/s for direction and speed respectively).

8.2. Discussion
It is clear from a review of Figs. 4a, 4b and 5 that the new model gives a better general agreement with the measured data than does the baseline (although, as already discussed, this does not comprise a validation). From the perspective of verification, it is worth examining specific features of the results, with the aid of Fig. 6.

8.2.1. Effect of Thrust Coefficient
Figures 4a and 4b show relatively low (7m/s) and high (12m/s) wind speeds, and as such the front row of turbines will be operating at very different thrust coefficients. The new model appears to fit well to both the low and high thrust cases, as does the baseline model. Although not the focus of this paper, it is worth noting that this is because both models have an appropriate physical interpretation of the thrust coefficient ($C_t$) which is not universally true amongst wake models: where coefficients which should be functions of $C_t$ have been estimated based on measured data, and thus the relationship has been lost.

8.2.2. Effect of Conservation of Energy in Wake Combination
As discussed in Section 4, the “max deficit” is often found to be the best approach for wake combination. With reference to (a) in Fig. 6 it is clear why conservation of energy is not generally used, as it results (when used in isolation) in a vast increase in wake loss, giving a deficit far larger than measured. However, when used in combination with the other corrections in this paper, this over-estimate in wake loss is reduced.

8.2.3. Off-line turbines
In Section 4 it was postulated that the “max deficit” wake combination fails when turbines do not line up. Scroby Sands is a particularly good windfarm to test this hypothesis as the array is fairly deep, but turbines are not aligned to a grid. With reference to (b) in Fig. 6, it is clear that when the wind flows through a large number of poorly aligned
Figure 4: Illustrative results for a turbine at Scroby Sands at low and high wind speeds (see Section 8.1)
turbines, “max deficit” significantly under-estimates the wake loss. This is a particular problem during array layout optimisation, as optimising using this model will result in a larger benefit from offsetting turbines. It is clear that the new model significantly reduces this problem.

8.2.4. Effect of Turbine-Wake Interaction

The over-estimate in wake loss from using conservation of energy wake combination is removed, in-part, by the turbine-wake interaction model defined in Section 6. In Fig. 6, for westerly winds (c) the turbine of interest is waked by two upstream turbines. As can be seen, the over-estimation has been significantly reduced.

8.2.5. Effect of Meandering

Meandering has two key effects on wakes. With reference to Fig. 6:

d.i with a similar effect to moving average in direction, peaks and troughs in the deficit are reduced in magnitude\(^{17}\).

d.ii it results in a reduction in directionally-narrow effects.

Both of these changes have a significant impact in closing the gap between the model and the measured data.

\(^{17}\) Note, however, that meandering cannot be applied as a moving average as a post processing step unless the wind angle is highly autocorrelated over the distances between turbines.

Figure 5: Illustrative results for a turbine at Rodsand II. (see Section 8.1)
8.2.6. “Large Array” effects
With reference to Fig. 5, for easterly winds (∼100°) the wind flows through the majority of the windfarm before reaching the turbine of interest. In this direction, the baseline model significantly under-predicts the wake losses. This could be seen as evidence for the requirement of a “large array correction”, such as WindFarmer’s “Correction for Large Wind Farms” [3] or OpenWind’s “Deep Array Wake Model” [24]. However, the new model does fit very closely to the measured data without any “large array correction”. Further research is required to test if these corrections are well justified or are merely an additional heuristic to account for errors introduced by the use of “max deficit”.

8.2.7. Increased Errors
In some directions (e.g. ∼95° in Fig. 4b) the new model seems to produce a marginally lower quality fit than the baseline model. The cause for this is not yet clear and further investigation is required. Some avenues to explore include: regularity within the eddies causing meandering at a length scale of the distance between the turbines [12], the interaction of wake meandering and the filter function, the form of the turbine-wake interaction function, or simply errors in the measured data.

9. Conclusions
This paper shows that the EV wind turbine wake model can, with a few modifications, be significantly improved in fidelity. These improvements will lead to better layout designs for windfarms, and improved yield estimates.

First, a new method of estimating the ambient mixing is proposed. Although this method
results in relatively modest changes to the overall results, it will result in more realistic simulations for large future turbines, especially when turbines of different sizes are present in the same simulation.

The remaining three improvements, when implemented in combination, replace the non-physical heuristics for wake combination in the baseline implementation ("wake scaling" and "max deficit"). These result in a significantly improved wake estimates. Furthermore, given their stronger basis in reality, the model’s ability to predict performance outside of the validated range of conditions (i.e. extrapolate) is much improved. For example, while most wake models work well for windfarms aligned on a grid (for which they are validated), it has been shown that the baseline implementation does not extrapolate well to non-gridded wind-farms (such as Scroby Sands).

The three improvements are all implemented at points when the flow-field is estimated (i.e. the locations of downstream turbines). As such, they come at no significant computational cost (furthermore, if the hypothesis in Section 8.2.6 is found to hold, then these corrections will significantly reduce computational cost, removing the need for a "large array correction"). These three corrections are:

(i) combine wakes accounting for conservation of energy;
(ii) account for the effect of the induction of downstream turbines on wakes; and
(iii) apply a wake meandering model.

Finally, it should be emphasised that this paper does not claim to produce a validated model, but rather a “better verifiable” model. The approach, rather than testing the accuracy of the model on mean power production, has been to seek conditions in which the models do not work, and review for what physical processes the model does not correctly account. This approach has yielded significant improvements in the model, and as such, a similar approach should be applied to improving wake models going forwards. For example, the error (x) in Fig. 6 has not been significantly reduced by the corrections here. It is possible that this is the result of wake-wake interaction (where the deficit from one wake recovers more slowly due to a neighbouring wake drawing on the same source of energy [4]) which will require a new way to estimate the local mixing $K_l$. Thus, further work is still required on:

- wake-wake interaction;
- global blockage effects;
- autocorrelation of turbulence for wake meandering;
- tuning the turbine-wake model (Section 6); and, finally,
- validating the model again new datasets.

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