On generalized Coulomb–Amontons’ law in the context of rigid body dynamics

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Abstract A generalization of Coulomb–Amontons’ law of dry friction recently proposed by V. V. Kozlov is considered in the context of rigid body dynamics. Universal requirements for dry friction tensor formulated by V. V. Kozlov are complemented by a condition taking into account the contact nature of dry friction, and applied to several models. For the famous Painlevé problem, a generalized Coulomb–Amontons’ force without singularities, yet such that the dissipation takes place only at the point of contact, is found. By the example of the motion of a rigid ball on a plane with a single point of contact, it is shown that these principles are consistent with the well-known equations, studied by G.-G. Coriolis. Further, a ball simultaneously touching two perpendicular planes at two points of contact is considered. The corresponding equations of motion are derived and analyzed. An exact particular solution that describes a technique used in practice in billiards is obtained. It is shown that unlike the single contact case, the Lagrange multipliers can depend on friction coefficients. With this in mind, a generalization of the conditions on the tensor of dry friction for the case of an arbitrary number of contacts is proposed.

Keywords Dry friction · Rigid body dynamics · Coulomb–Amontons’ law · Painlevé paradox

Mathematics Subject Classification 70F40

1 Introduction

Starting from Painlevé, an approach has taken root in rigid body dynamics in which dry friction is accounted for by adding empirical terms that express the Coulomb–Amontons’ law to the right-hand side of the equations of motion. Despite the apparent simplicity of the law itself, analysis of the resulting piecewise-smooth equations turns out to be a challenging problem even in simple two-dimensional examples formulated by Painlevé in [12] 125 years ago. Not only they can exhibit rich and complex dynamics (details about bifurcations and onset of chaos in piecewise-smooth systems can be found for example in [5]), but as it was already noticed by Painlevé, such systems, in general, do not possess the properties of existence and uniqueness of solutions for all possible initial conditions. These situations known as Painlevé paradoxes have been extensively studied in the literature (a detailed overview of recent results on Painlevé paradoxes can be found in [3]).

One common workaround to the inconsistency problem is the hypothesis of “impact without collision”, also referred to as “dynamic jamming” or “tangential impact” (see, e.g. [6]). This idea has become widespread in modern literature, although there is no convincing experimental evidence that it is a real phenomenon rather than a mathematical abstraction [3]. The indeterminacy aspect in recent works is actively studied by means of regularization in the neighbour-
hood of the point of contact (see, e.g. [10,11]). However, in this direction, our knowledge is yet limited mostly to particular scenarios in certain planar problems with a single point of contact. Besides, by giving up the rigidity assumption, one cannot answer the original question stated by Painlevé, whether the law of Coulomb–Amontons is in contradiction with rigid body mechanics.

With that been said let us cite V. V. Kozlov [8]: “In our opinion, Painlevé paradoxes are also related to the inaccurate application of the laws of dry friction in describing the dynamics of systems with many degrees of freedom and unilateral constraints. (...) It is proposed to look at the laws of dry friction from a more generic point of view of Lagrangian mechanics. In this approach, the covariance property of the expression for the dry friction force, typical for the equations of motion without friction, is essential.”

In articles [8,9] V. V. Kozlov suggested a generalization of Coulomb–Amontons’ law of dry friction for constrained Lagrangian systems that eliminates singularities in friction forces and constraint reactions and applied it to an arbitrary motion of a rigid body on a fixed surface in case, where at any moment the contact between the body and the surface takes place at exactly one point. According to V. V. Kozlov, the generalized Coulomb–Amontons’ force of dry friction acting on a body moving on a surface given by an equation $f(x) = 0$, $x = \{x^i\} \in \mathbb{R}^n$, is stated as follows [8, eq. (2.6)]:

$$F = -|R| \frac{\Phi \dot{x}}{|\dot{x}|}. \tag{1}$$

Covector $R = \lambda \frac{\partial f}{\partial x}$ with $\lambda$ being the Lagrange multiplier is the constraint reaction. The quantities $|\dot{x}|$ and $|R|$ are defined via metric tensor $A = \{\partial^2 T / \partial \dot{x}^i \partial \dot{x}^j\}$ with $T$ being the kinetic energy of the system, namely,

$$|\dot{x}|^2 = (A \dot{x}, \dot{x}),$$

$$|R|^2 = \left( R, A^{-1} R \right),$$

where the parentheses denote convolution of a covector with a vector. The same expressions in terms of covariant and contravariant indices take the form

$$|\dot{x}|^2 = A_{ij} \dot{x}^i \dot{x}^j,$$

$$|R|^2 = (A^{-1})^{ij} R_i R_j,$$

where summation over repeated indices is implied. Finally, $\Phi = \{\Phi_{ij}\}$ is a matrix, for all $x$ and $t$ satisfying the following two conditions:

(i) $\Phi^T A^{-1} \frac{\partial f}{\partial x} = \rho \frac{\partial f}{\partial x}$, $\rho \in \mathbb{R}$ (see [8, eq. (2.9)]).

This restriction means mutual orthogonality of the generalized forces of dry friction and normal support reaction in the metrics, defined by matrix $A$:

$$\left( \frac{\partial f}{\partial x}, A^{-1} F \right) = -|R| \left( \frac{\partial f}{\partial x}, A^{-1} \Phi \dot{x} \right) = -|R| \left( \frac{\partial f}{\partial x}, \dot{x} \right) = -\rho |R| \left( \frac{\partial f}{\partial x}, \dot{x} \right) = 0.$$ 

The last equality is a consequence of the constraint equation. Also, as shown in [8], this condition implies that multiplier $\lambda$ does not depend on $\Phi$ and can be found from the equations of motion without friction.

(ii) $(\Phi \dot{x}, \dot{x}) \geq 0$ (see [8, eq. (2.8)]).

This condition expresses dissipativity of the friction force:

$$(F, \dot{x}) = -|R| \left( \Phi \dot{x}, \dot{x} \right) \leq 0.$$ 

From a geometric point of view, matrices $A$ and $\Phi$ are components of twice covariant tensor fields in the configuration space—tensor of inertia and tensor of dry friction accordingly, which implies invariance of conditions (i) and (ii) with respect to changes of the generalized coordinates.

Remark 1 Actually, the Coulomb–Amontons’ law (no matter, classical or generalized) can be considered as an idealization. For instance, in [5] linear and cubic perturbations in velocity are used in order to describe the effect of dynamic friction being lower than the static one. An even more sophisticated 3-parameter model also exhibiting hysteresis is proposed in [7].

In the presented work some of the results from [8] are refined under assumption of absolute rigidity of the interacting bodies. This assumption, as will be shown, leads to additional restrictions on matrix $\Phi$ and friction force $F$, and can be formulated in a simple universal form. As a consequence, for the Painlevé falling rod problem and the problem of a rigid body rolling on a fixed plane, we obtain much simpler formulas for the tensor of dry friction than in [8, §§ 5–6]. It is worth noting that rigidity of the considered objects is mentioned in the formulations of these problems in [8] as well, but it is not reflected in conditions (i)–(ii) themselves, because of what, among others, the components...
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2 Painlevé problem

Problem setting and support reaction in the absence of friction Consider a mechanical system consisting of two material points \( M_1 \) and \( M_2 \) of mass \( m_1 \) and \( m_2 \), connected to each other by a weightless rigid rod of length \( l \) (see Fig. 1). Line \( Ox \) constrains the motion of point \( M_1 \) to half-plane \( y \geq 0 \).

Lagrangian of the described system in generalized coordinates \( x, \theta, y \) has the form

\[
L = m_1 \dot{x}^2 + \dot{y}^2 + m_2 (\dot{x} - \dot{\theta} l \sin \theta)^2 + (\dot{\theta} + \dot{\theta} l \cos \theta)^2
- (m_1 + m_2) g y - m_2 g l \sin \theta + \lambda y,
\]

where \( x, y \) denote the Cartesian coordinates of point \( M_1 \), \( \theta \) is the angle between rod \( M_1 M_2 \) and axis \( Ox \) (positive direction of the reference angle is counterclockwise), \( g \) is the acceleration of gravity, and \( \lambda \) is the Lagrange multiplier. Write down the equations of motion:

\[
\begin{align*}
(m_1 + m_2) \ddot{x} - m_2 \ddot{\theta} \sin \theta &= m_2 \dot{\theta}^2 \cos \theta, \\
m_2 (\ddot{x} \sin \theta - \dot{y} \cos \theta - l \dot{\theta}) &= m_2 g \cos \theta, \\
(m_1 + m_2) \ddot{y} + m_2 \ddot{\theta} \cos \theta &= -(m_1 + m_2) g \\
+ m_2 \dot{\theta}^2 \sin \theta + \lambda.
\end{align*}
\]

Depending on the value of \( \lambda \), the motion of the system can be attributed to one of the following modes:

- \( \lambda = 0 \), there is no contact between the rod and the support, i.e. \( y > 0 \).
- \( \lambda \in (0, +\infty) \), there is a long-time contact of the rod with the support. In this case, letting \( y \) together with derivatives equal to zero, we find

\[
\lambda = \frac{2m_1 ((m_1 + m_2) g - m_2 \dot{\theta}^2 \sin \theta)}{2m_1 + m_2 (1 + \cos 2\theta)} > 0.
\]

- \( \lambda = +\infty \), there is an impact of the rod on the support, during which the generalized velocities undergo a discontinuity.

Painlevé paradox The classical Coulomb–Amontons’ law of dry friction for the considered system is defined by expression

\[
F = -\lambda x \sigma (\dot{x}), \quad \sigma (x) = \begin{cases} -1, & x < 0, \\
[-1, 1], & x = 0, \\
+1, & x > 0. \end{cases}
\]

where \( x > 0 \) is the coefficient of friction, and \([-1, 1]\) on the right-hand side means that \( \sigma (0) \) can take any value from this range. Following P. Painlevé (see \([12, p. 16, \text{formula (d')}]\)), let us add to the right-hand side of the Lagrange equation from system (2) corresponding to coordinate \( x \) force \( F \) to obtain a system with friction:

\[
\begin{align*}
(m_1 + m_2) \ddot{x} - m_2 \ddot{\theta} \sin \theta &= m_2 \dot{\theta}^2 \cos \theta + F, \\
m_2 (\ddot{x} \sin \theta - \dot{y} \cos \theta - l \dot{\theta}) &= m_2 g \cos \theta, \\
(m_1 + m_2) \ddot{y} + m_2 \ddot{\theta} \cos \theta &= -(m_1 + m_2) g \\
+ m_2 \dot{\theta}^2 \sin \theta + \lambda.
\end{align*}
\]

Putting these equations \( y \) with derivatives equal to zero, we get

\[
\lambda = \frac{2m_1 ((m_1 + m_2) g - m_2 \dot{\theta}^2 \sin \theta)}{2m_1 + m_2 (1 + \cos 2\theta) + \kappa m_2 \sigma (\dot{x}) \sin 2\theta}.
\]

As can be seen, for sufficiently large values of the friction coefficient the denominator may turn to zero, leading under a continuous contact to infinite values of the
support reaction and friction forces, which is devoid of physical sense.

V. V. Kozlov’s approach to accounting for friction As noted in [8, §5], condition (i) is violated for system (5), which leads to singularities in λ. Indeed, for the Painlevé problem

\[
A = \begin{bmatrix}
m_1 + m_2 & -lm_2 \sin \theta & 0 \\
-2m_2 \sin \theta & m_2^2 & lm_2 \cos \theta \\
0 & lm_2 \cos \theta & m_1 + m_2
\end{bmatrix}, \quad f = y, \tag{6}
\]

so for \( \Phi = \text{diag}(\kappa, 0, 0) \), we have

\[
\Phi^T A^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{m_2 \sin 2\theta}{2m_1(m_1 + m_2)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \neq \rho \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

On the contrary, if condition (i) is satisfied, namely when (see [8, formulas (5.1)–(5.2)])

\[
\Phi = A \Omega^T, \quad \Omega = \begin{bmatrix}
x_1 & \mu_1 & 0 \\
x_2 & \mu_2 & 0 \\
x_3 & \mu_3 & v
\end{bmatrix}, \tag{7}
\]

then dry friction force (1) is uniquely defined by (6), (7) and (3) at any moment in time and any point of the phase space.

As can be seen from (7), (1, 1)-tensor \( \Omega \) is obtained from tensor \( \Phi \) by transposition and lifting of one of the indices by means of contravariant metric tensor \( A^{-1} \). In the most interesting cases \( \Omega \) has simpler form than \( \Phi \). For instance, in the isotropic case \( \Omega \) is proportional to the identity matrix, while \( \Phi \) is proportional to \( A \). The components \( x_{1,2}, \mu_{1,2} \) have the meaning of coefficients of friction, and the components \( x_3, \mu_3, \) and \( v \) are associated with losses during inelastic impact with the support.

Insufficiency of conditions (i)–(ii) in case of absolutely rigid bodies Let in (7) \( \Omega = \rho E_3, \rho > 0 \), where \( E_3 \) is the 3 × 3 identity matrix, which corresponds to an isotropic tensor of dry friction. In this case both conditions (i)–(ii) are met, and friction force (1) for \( y = 0 \) and \( \lambda \) from (3) is

\[
F = -k \nu \begin{bmatrix}
(m_1 + m_2)\dot{x} - lm_2 \dot{\theta} \sin \theta \\
lm_2 (l \dot{\theta} - \dot{x} \sin \theta) \\
lm_2 \dot{\theta} \cos \theta
\end{bmatrix}, \tag{8}
\]

where

\[
k = \frac{(m_1 + m_2)g - lm_2 \dot{\theta}^2 \sin \theta}{\sqrt{2m_1}}
\]

\[
\div \sqrt{(m_1 + m_2)(2m_1 + m_2(1 + \cos 2\theta))}
\]

\[
\div \sqrt{(m_1 + m_2 \cos^2 \theta)\dot{x}^2 + m_2(\dot{x} \sin \theta - \dot{\theta})^2}.
\]

Denote the vector of generalized velocities of the body by

\[
v = (\dot{x}, \dot{\theta}, \dot{y})^T.
\]

The power dissipated by friction force (8) subject to \( y = 0 \) equals to

\[
(F, v) = -k \nu ((m_1 + m_2 \cos^2 \theta)\dot{x}^2 + m_2(\dot{x} \sin \theta - \dot{\theta})^2)
\]

\[
\leq 0.
\]

On the other hand, the power of the same force dissipated at the point of contact equals to

\[
F_c \dot{x} = -k \nu ((m_1 + m_2)\dot{x}^2 - lm_2 \dot{\theta} \sin \theta).
\]

Difference between these expressions indicates that the work of friction force (8) is not concentrated in the point of contact. Besides, the expression for the power dissipated at the point of contact is not sign-definite. Thus, conditions (i)–(ii) alone turn out to be insufficient for a correct definition of the dry friction in case of a motion of an absolutely rigid body on an undeformable surface.

Condition of contact interaction Denote by \( P \) a linear operator that maps an \( n \)-dimensional vector \( v \) of generalized velocities of the system to an \( r \)-dimensional vector \( v_c (r \leq n) \) of the velocity of the body at the point of contact with the supporting surface:

\[
 Pv = v_c. \tag{9}
\]

By definition, operator \( P \) is surjective. In particular, matrix \( PP^T \) is invertible.

Lemma 1 For an absolutely rigid body interacting with support at a single point of contact,

\[
F = PP^T F_c, \tag{10}
\]

where \( F_c \) is a Coulomb–Amontons’ friction force at the point of contact, and \( F \) is the corresponding generalized friction force.

Proof Under the lemma conditions, all power of the friction force must be dissipated at the point of contact, in other words \( (F, v) = (F_c, v_c) \). From here, taking into account (9) follows formula (10).
As a consequence from the proved lemma, we obtain a supplementary to conditions (i)–(ii) constraint on matrix $\Phi$ from (1):

(iii) $\Phi = P^T \Psi$,

where $\Psi$ is an arbitrary $r \times n$ matrix depending only on generalized coordinates.

**Lemma 2** Conditions (ii)–(iii) are equivalent to equality

$$\Phi = P^T \Phi_c P,$$

where $\Phi_c$ is a non-negative-definite $r \times r$ matrix.

**Proof** Let conditions (ii)–(iii) be met. Let us show that $\ker P \subseteq \ker \Phi \subseteq \mathbb{R}^n$. By contradiction, assume that there exists $\xi$ such that $P \xi = 0$, but $\Phi \xi \neq 0$. Then for arbitrary $u \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$

$$0 \leq (\Phi (u + \alpha \xi), u + \alpha \xi) = u^T \Phi (u + \alpha \xi) = u^T \Phi u + \alpha u^T \Phi \xi.$$

But for $u = \Phi \xi$ and sufficiently large in absolute value $\alpha$ the last expression is negative, which leads to a contradiction. Hence, $\Phi = \Xi P$, which together with condition (iii) implies $\Phi = P^T \Phi_c P$. Non-negative-definiteness of $\Phi_c$ follows from (9) and condition (ii):

$$(\Phi_c \xi_c, \xi_c) = (P^T \Phi_c P v, v) = (\Phi v, v) \geq 0.$$

Reverse implication is obvious. $\Box$

Vector of generalized support reaction $R$ is an image of $P^T$ of normal support reaction $N$ at the point of contact, i.e. $R = P^T N$. Consequently, the norm of $R$ in formula (1) can be replaced with the Euclidean length of $N$, and multiplier $|R|/|N| > 0$ that depends only on generalized coordinates can be included in not yet defined tensor $\Phi_c$. At the same time, the norm of projection of vector of generalized velocities $\dot{\xi}$ onto the point of contact coincides with the Euclidean length of the velocity of this point. The latter follows from the fact that in the metrics defined by matrix $A$, the square of length of the specified projection of generalized velocity equals to the doubled kinetic energy of a material point, proportional to the Euclidean square of velocity. With that said, holds

**Theorem 1** In case of an absolutely rigid body and a single point of contact, equality (1) can be rewritten as

$$F = -|N| \frac{P^T \Phi_c v_c}{|v_c|},$$

where $|N|$ and $|v_c|$ are the Euclidean lengths of vectors of the force of normal support reaction and the velocity of the body at the point of contact, and $\Phi_c$ is a non-negative-definite matrix such that

$$\Phi_c Q N = \rho N, \quad Q = P A^{-1} P^T, \quad \rho \in \mathbb{R}. \quad (13)$$

**Proof** Equality (12) follows from (1) by substitution (11) subject to (9). In order to prove formula (13), write down condition (i) in the form

$$\Phi_c^T A^{-1} R = \rho R.$$

By substituting here $R = P^T N$ and (11), we get equality

$$P^T \Phi_c^T P A^{-1} P^T N = \rho P^T N, \quad \rho \in \mathbb{R},$$

equivalent to (13). $\Box$

**Generalized dry friction force for the Painlevé problem**

Let us find the general view of friction force $F$ representable in the form (12) for the Painlevé problem. Herewith,

$$v_c = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}, \quad N = \begin{bmatrix} 0 \\ \lambda \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

From (6) we get

$$Q = \begin{bmatrix} 2m_1 + m_2(1 - \cos 2\theta) & -m_2 \sin 2\theta \\ -m_2 \sin 2\theta & 2m_1 + m_2(1 + \cos 2\theta) \end{bmatrix} \div 2m_1(m_1 + m_2),$$

and from (13) it follows that

$$\Phi_c = \begin{bmatrix} \lambda \frac{\kappa_1}{\kappa_1 m_1 \sin 2\theta} & \lambda \frac{\kappa_2}{2m_1 + m_2(1 + \cos 2\theta)} \\ \frac{\kappa_1}{2m_1 + m_2(1 + \cos 2\theta)} & \lambda \frac{\mu_2}{2m_1 + m_2(1 + \cos 2\theta)} \end{bmatrix} (\Phi_c \geq 0). \quad (14)$$

The corresponding Lagrange equations taking into account dry friction take the form

$$(m_1 + m_2)\ddot{x} - l m_2 \dot{\theta} \sin \theta = l m_2 \dot{\theta}^2 \cos \theta \div \lambda \lambda \frac{\kappa_1}{\sqrt{x^2 + y^2}} + \frac{\kappa_2}{\sqrt{x^2 + y^2}},$$

$$l m_2 (\ddot{x} \sin \theta - \dot{y} \cos \theta - l \dot{\theta}) = l m_2 g \cos \theta, \quad (m_1 + m_2)\ddot{y} + l m_2 \dot{\theta} \cos \theta = -(m_1 + m_2) g + l m_2 \dot{\theta}^2 \sin \theta \div \lambda \lambda \frac{\kappa_1}{\sqrt{x^2 + y^2}} + \frac{\kappa_2}{\sqrt{x^2 + y^2}},$$

$$+ \lambda \lambda \frac{\kappa_1}{\sqrt{x^2 + y^2}} + \frac{\kappa_2}{\sqrt{x^2 + y^2}} \left( \frac{\mu_2}{2m_1 + m_2(1 + \cos 2\theta)} \right), \quad (15)$$

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System (15) for different $\lambda$ describes all possible movements of the rod in the Painlevé problem. By setting in this system $\lambda = 0$, we obtain the equations of motion of the rod in the absence of contact. In turn, setting $y$ with derivatives equal to zero, we get $\lambda$ same as in (3), and the equations of motion under continuous contact:

\[
(m_1 + m_2)\ddot{x} - lm_2\dot{\theta}\sin \theta = lm_2\dot{\theta}^2 \cos \theta - \lambda x_1\dot{x},
\]

where the function $\sigma$ is defined in (4). The latter system covers two cases. The first case takes place when $\dot{x} \neq 0$ and corresponds to a motion with sliding friction. The second one, where $\dot{x} = 0$, corresponds to rest of the contact point with force of static friction

\[
F = \lambda \sigma \lambda_1 = m_2 \cos \theta (l \dot{\theta}^2 - g \sin \theta), \quad \sigma_0 \in [-1, 1].
\]

Hence the rest condition of the contact point is given by inequality

\[
|\cos \theta (l \dot{\theta}^2 - g \sin \theta)| \leq \frac{\lambda x_1}{m_2}.
\]

When equality is reached in this formula, rest turns into a slide.

Instant contact with the support (impact) corresponds to $\lambda = +\infty$. Formally, the equations describing velocity discontinuities can be obtained by integrating (15) over an infinitesimal time interval containing the moment of impact,

\[
(m_1 + m_2)\Delta \dot{x} - lm_2\Delta \dot{\theta}\sin \theta = -x_1 S_x - x_2 S_y,
\]

\[
lm_2(\Delta \dot{x} \sin \theta - \Delta \dot{\theta} \cos \theta - I \Delta \dot{\theta}) = 0,
\]

\[
(m_1 + m_2)\Delta \dot{y} + lm_2\Delta \dot{\theta} \cos \theta = I - \mu_2 S_y - x_1 S_y,
\]

\[
\mu_2 \sin \theta \frac{m_2 \sin 2\theta}{2m_1 + m_2 (1 + \cos 2\theta)},
\]

where $I$ is the impulse of the support reaction force, and $S_x$ and $S_y$ are tangent and normal components of the impulses of the impact force of friction. Note that the coefficients $x_2$ and $\mu_2$ only participate in the equations describing the friction for impact.

### 3 Rigid ball on a plane

Consider the problem of the motion of a rigid homogeneous ball of mass $m$ and radius $R$ on a fixed horizontal plane $z = 0$, and compare results with [8, § 6]. As generalized coordinates, we select coordinates of the center of the ball $x$, $y$, $z$ and Euler angles relative to the center of the ball $\alpha$, $\beta$, $\gamma$. The Lagrangian of this problem has the form

\[
L = m \left( \frac{x^2 + y^2 + z^2}{2} + \frac{m R^2}{5} (\dot{\alpha}^2 + \dot{\beta}^2 + \dot{\gamma}^2 + 2\dot{\alpha}\dot{\gamma} \cos \beta) \right) - mgz + \lambda z,
\]

where $\lambda$ is the Lagrange multiplier, and $f = z - R \geq 0$ is a unilateral constraint. Thus

\[
A = \begin{bmatrix}
    m E_3 & 0 \\
    0 & \frac{2m R^2}{5} \begin{bmatrix}
        1 & 0 & 0 \\
        0 & 1 & 0 \\
        \cos \beta & 0 & 1
    \end{bmatrix}
\end{bmatrix},
\]

Velocity of the point of the ball touching the surface and the normal support reaction are

\[
v_c = \begin{bmatrix}
    \dot{x} - \dot{\beta} \sin \alpha + \dot{\gamma} \cos \alpha \\
    \dot{y} - \dot{\beta} \cos \alpha + \dot{\gamma} \sin \alpha \\
    \dot{z}
\end{bmatrix}, \quad N = \begin{bmatrix}
    0 \\
    0 \\
    \lambda
\end{bmatrix}.
\]

from where and from (13)

\[
P = \begin{bmatrix}
    1 & 0 & 0 & -R \sin \alpha & R \cos \alpha \sin \beta \\
    0 & 1 & 0 & -R \cos \alpha & -R \sin \alpha \sin \beta \\
    0 & 0 & 1 & 0 & 0
\end{bmatrix},
\]

\[
Q = \frac{1}{m} \begin{bmatrix}
    7/2 & 0 & 0 \\
    0 & 7/2 & 0 \\
    0 & 0 & 1
\end{bmatrix}, \quad \Phi_c = \begin{bmatrix}
    x_1 & x_2 & x_3 \\
    \mu_1 & \mu_2 & \mu_3 \\
    0 & 0 & \nu
\end{bmatrix}.
\]

When written in terms of angular velocity vector $\omega = (\omega_x, \omega_y, \omega_z)^T$ defined by

\[
\dot{\alpha} = -\omega_z + \cot (\omega_x \sin \alpha + \omega_y \cos \alpha),
\]

\[
\dot{\beta} = -\omega_x \cos \alpha + \omega_y \sin \alpha,
\]

\[
\dot{\gamma} = -\csc \beta (\omega_x \sin \alpha + \omega_y \cos \alpha),
\]

instead of the Euler angles, the equations of motion look especially simple:

\[
m \ddot{x} = -\lambda (x_1 \dot{x} - R \omega_x) + x_2 (\dot{y} + R \omega_x) + x_3 \dot{z} \\
\sqrt{\dot{x}^2 - (R \omega_y)^2 + (\dot{y} + R \omega_x)^2 + \dot{z}^2},
\]

\[
m \ddot{y} = -\lambda (\mu_1 \dot{x} - R \omega_y) + \mu_2 (\dot{y} + R \omega_x) + \mu_3 \dot{z} \\
\sqrt{\dot{x}^2 - (R \omega_y)^2 + (\dot{y} + R \omega_x)^2 + \dot{z}^2},
\]

\[
m \ddot{z} = \lambda - mg - \frac{\lambda \dot{z}^2}{\sqrt{\dot{x}^2 - R \omega_y)^2 + (\dot{y} + R \omega_x)^2 + \dot{z}^2}}
\]

\[
2 \frac{m R \dot{\alpha}_x = -\lambda (\mu_1 \dot{x} - R \omega_y) + \mu_2 (\dot{y} + R \omega_x) + \mu_3 \dot{z} \\
\sqrt{\dot{x}^2 - (R \omega_y)^2 + (\dot{y} + R \omega_x)^2 + \dot{z}^2}},
\]
\[
\frac{2}{5}mR \dot{\omega}_y = \frac{\lambda (x_1 (\dot{x} - R \omega_y) + x_2 (\dot{y} + R \omega_x) + x_3 \dot{z})}{\sqrt{(\dot{x} - R \omega_y)^2 + (\dot{y} + R \omega_x)^2 + \dot{z}^2}}.
\]

\[
\frac{2}{5}mR \dot{\omega}_z = 0.
\]

Here depending on the value of \( \lambda \) the motion of the system can be attributed to one of the following modes:

- \( \lambda = 0 \), there is no contact between the ball and the surface, i.e. \( z > R \).
- \( \lambda \in (0, +\infty) \) corresponds to a motion of the ball on the surface. By setting \( z = R \), and its derivatives equal to zero, from the third equation we get \( \lambda = mg \).

The obtained system is well known and was studied among others by G.-G. Coriolis in his famous book [4]. Particularly, the quantities

\[
\dot{x} + \frac{2R}{5} \omega_y, \quad \dot{y} - \frac{2R}{5} \omega_x
\]

are first integrals of this system and represent velocity components of the so-called upper center of percussion of the ball, i.e. the point of the ball, located at distance 2/5 of the radius vertically above the center of the ball (see Fig. 2).

- \( \lambda = +\infty \) corresponds to an impact of the ball on the plane, when velocity components of the center of the ball and components of the angular velocity undergo a discontinuity. At the same time, the horizontal velocity components of the upper center of percussion are preserved.

**Remark 2** One may ask, how does the ball reach the state with \( \lambda > 0 \) if, initially, it was falling (\( \lambda = 0 \)). This is where another important phenomenon in piecewise-smooth dynamics called “chatter” comes into play. For example, if we apply a simple Newtonian restitution law with the coefficient of restitution \( \epsilon \in [0, 1] \), we will find ourselves in the situation of (complete) chatter, that is an infinite sequence of impacts (\( \lambda = +\infty \)) accumulating in finite time (see, e.g. [5]).

The problem of rolling of a heavy homogeneous ball on a horizontal plane was also considered by V. V. Kozlov in [8, § 6] subject to conditions (i) and (ii) only. The sliding friction force obtained in [8], generally speaking, depends on components of the angular velocity, that correspond to pivoting and rolling. Besides, in [8] there arise additional moments of forces of rolling and pivoting friction, which indicates the non-contact character of such a generalized force of dry friction. In Eq. (21) such terms are, obviously, absent.

4 Motion of a rigid ball in contact with two perpendicular planes

In [8, § 7] conditions (i)–(ii) are generalized to the case of an arbitrary finite number of points of contact, which is equivalent to an arbitrary number of constraints (see [8, formula (7.1)]):

\[
f_1(x) = 0, \ldots, f_p(x) = 0, \quad p < n.
\]

Covectors \( \partial f_i/\partial x \) are assumed linearly independent. Condition (ii) is transferred to this case without changes, and instead of condition (i), it is suggested (see [8, formula (7.4)])

\[
\Phi^T A^{-1} \frac{\partial f_i}{\partial x} = \sum_{j=1}^{p} c_{ij} \frac{\partial f_j}{\partial x}, \quad c_{ij} \in \mathbb{R}.
\]

If this condition is met Lagrange multipliers \( \lambda_1, \ldots, \lambda_p \) are uniquely defined, and as in the case of a single point of contact, they do not depend on \( \Phi \). The following example demonstrates that this requirement is too restrictive. In fact, in problems with several points of contacts, we should consider multiple tensors \( \Phi_1, \ldots, \Phi_p \), one for each point of contact, and the Lagrange multipliers can depend on coefficients of friction.

Equations of motion of the ball along the cushion Consider equations of motion of a ball on a horizontal half-plane \( z = 0, y \geq 0 \), bounded by a vertical plane \( y = 0 \). Further, these planes will be referred to as the table and the cushion.\(^1\) In the absence of contact between

\(^1\) In billiards, the term “cushion” means the cloth-covered piece of rubber which borders the playing surface.
the contacts with the cushion and the table by
and the surfaces of the cushion and the table by
\(\mu_c\), where for the sake of simplicity, we assume the friction of the cushion and the table to be isotropic, i.e.

\[ \Phi_\perp = \mu_c E_3, \quad \Phi_{ic} = \mu_{ic} E_3 \quad (\mu_{ic}, \mu_c > 0). \quad (25) \]

For the contact with the table, formulas (17)–(19) obtained in the previous paragraph are transferred without changes. For the contact with the cushion,

\[ v_{ic} = \begin{bmatrix} \dot{x} - \alpha_R - \gamma_R \cos \beta \\ \dot{y} \\ \dot{z} + \beta R \cos \alpha + \gamma_R \sin \alpha \sin \beta \end{bmatrix}, \quad N_{ic} = \begin{bmatrix} 0 \\ \lambda_{ic} \\ 0 \end{bmatrix}, \]

from where and from (13)

\[ P_{ic} = \begin{bmatrix} 1 & 0 & 0 & -R & 0 & -R \cos \beta \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & R \cos \alpha & R \sin \alpha \sin \beta \end{bmatrix}, \]

\[ Q_{ic} = \frac{1}{m} \begin{bmatrix} 7/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 7/2 \end{bmatrix}. \]

Now using (12), we can evaluate generalized friction force \( F \). The Lagrangian for the problem considered differs from (16) by substitution of \( \lambda \) by \( \lambda_{ic} \) and addition of term \( \lambda_{ic} \dot{y} \). Moving to the angular velocity vector in Lagrange equations using formulas (20) and substituting constraint equations (24), we obtain the system

\[ \frac{2}{5} m R \ddot{\omega}_x = -R \omega_x \left( \frac{\lambda_c \mu_c}{\sqrt{R^2 \omega_x^2 + (\dot{x} - R \omega_y)^2}} + \frac{\lambda_{ic} \mu_{ic}}{\sqrt{R^2 \omega_x^2 + (\dot{x} + R \omega_c)^2}} \right), \]

\[ \frac{2}{5} m R \ddot{\omega}_y = \frac{\lambda_c \mu_c (\dot{x} - R \omega_y)}{\sqrt{R^2 \omega_x^2 + (\dot{x} - R \omega_y)^2}}, \]

\[ \frac{2}{5} m R \ddot{\omega}_z = \frac{\lambda_{ic} \mu_{ic} (\dot{x} + R \omega_c)}{\sqrt{R^2 \omega_x^2 + (\dot{x} + R \omega_c)^2}}. \quad (26) \]

From the second and the third equations we find the Lagrange multipliers, and hence the normal reactions of the cushion and the table:

\[ \lambda_{ic} = mg \mu_{ic} R (1 + 2 \mu_{ic}) \sqrt{R^2 \omega_x^2 + (\dot{x} + R \omega_c)^2}, \]

\[ \lambda_{ic} = mg \sqrt{R^2 \omega_x^2 + (\dot{x} - R \omega_y)^2} \sqrt{R^2 \omega_y^2 + (\dot{x} + R \omega_c)^2}, \]

\[ \lambda_{ic} = mg \mu_{ic} R (1 + 2 \mu_{ic}) \sqrt{R^2 \omega_x^2 + (\dot{x} + R \omega_c)^2}. \quad (27) \]

Note that without account for friction the expression for \( \lambda_{ic} \) would turn to 0, and the expression for \( \lambda_{ic} \) would come down to mg, which is not sufficient for the description of the motions of the ball, observed in practice in billiards (see the next paragraph).

Stability of the contacts is determined by \( \lambda_{ic} \) and \( \lambda_{ic} \): the contact is stable when the corresponding Lagrange multiplier is positive. For \( \lambda_{ic} \), this is always true, i.e. continuous contact with the table while moving along the cushion is never violated. For \( \lambda_{ic} \), the requirement of positiveness is equivalent to inequality \( \lambda_{ic} > 0 \), which corresponds to a rotation pressing the ball to the cushion.

Substituting (27) into (26) and performing a change of variables

\[ \omega_x = \Omega_x, \quad \omega_y = \Omega_y + \frac{\dot{x}}{R}, \quad \omega_z = \Omega_z - \frac{\dot{x}}{R}, \]

we find the final form of the equations of motion of the ball along the cushion:

\[ \ddot{x} = -g \mu_c (\mu_{ic} \Omega_x \Omega_c - \Omega_y \sqrt{\Omega_x^2 + \Omega_c^2} \sqrt{\Omega_y^2 + \Omega_c^2}) / \mu_{ic} \mu_c \Omega_x^2 + (1 + \mu_{ic}) \sqrt{\Omega_x^2 + \Omega_c^2} \sqrt{\Omega_y^2 + \Omega_c^2}. \]
The upper side center of percussion

$$\dot{\Omega}_x = -\frac{5\mu_c \Omega_x (\mu_c \Omega_x + \sqrt{\Omega_x^2 + \Omega_z^2})}{2R(\mu_c \mu_c \Omega_x^2 + \Omega_x^2 + \Omega_z^2 \sqrt{\Omega_x^2 + \Omega_z^2})},$$

$$\dot{\Omega}_y = \frac{g\mu_c (2\mu_c \Omega_x \Omega_z - 7\Omega_y \sqrt{\Omega_x^2 + \Omega_z^2})}{2R(\mu_c \mu_c \Omega_x^2 + \Omega_x^2 + \Omega_z^2 \sqrt{\Omega_x^2 + \Omega_z^2})},$$

$$\dot{\Omega}_z = -\frac{g\mu_c (7\mu_c \Omega_x \Omega_z - 2\Omega_y \sqrt{\Omega_x^2 + \Omega_z^2})}{2R(\mu_c \mu_c \Omega_x^2 + \Omega_x^2 + \Omega_z^2 \sqrt{\Omega_x^2 + \Omega_z^2})}.$$  (28)

It can be easily seen that function

$$J = \frac{9}{5} \dot{x} + \frac{2}{5} R(\dot{x} \Omega_x - \dot{z} \Omega_z) = \dot{x} + \frac{2}{5} \omega_x - \frac{2}{5} \omega_z$$  (29)

is an integral of this system. Thus, the problem comes down to solving the system of the last three equations (28). Integral (29) has the mechanical sense of the $x$-component of velocity of the upper side point of the ball, located at distance $2\sqrt{2}/5$ of the radius from its center along the line, perpendicular to the intersection of the supporting planes and passing through the center of the ball. In this regard, this point can be referred to as the upper side center of percussion of the ball (see Fig. 3).

**Remark 3** In article [1] a mechanical model of a liquid flowmeter based on the kinematics of a two-point contact of a ball placed in an axisymmetric vessel is investigated. The authors derive the equations of motion of the ball forced to move along the edge of the vessel by swirling fluid flow from Newton’s second law in the coordinate system rotating together with the center of the ball around the axis of symmetry of the vessel. Up to forces of viscous friction and Coriolis forces system [1, eq. (26)] coincides with (28) which we obtained on the basis of the generalized Coulomb–Amontons’ law.²

The “Frenchman” stroke in billiards We will look for a solution of system (28) such that

$$\Omega_y = \eta \Omega_x, \quad \Omega_z = \zeta \Omega_x \quad (\eta, \zeta \in \mathbb{R}, \ t \in [0, T]).$$  (30)

Here $T > 0$ is a finite moment at which such a motion eventually turns into the final rolling regime

$$\dot{x} = v_{\text{fin}}, \quad \omega_x = 0, \quad \omega_y = \frac{v_{\text{fin}}}{R}, \quad \omega_z = -\frac{v_{\text{fin}}}{R}$$  (31)

for $t \in [T, \infty)$, where $v_{\text{fin}}$ is the final longitudinal velocity of the ball center.

Finding the time derivative of (30) using (28), we arrive at the equations for $\eta$ and $\zeta$:

$$(2\sqrt{1 + \zeta^2} - 5\mu_c)\eta - 2\mu_c \zeta = 0,$$

$$(5\sqrt{1 + \zeta^2} - 2\mu_c)\zeta + 2\sqrt{1 + \zeta^2} = 0.$$  

These equations have the following solutions:

1. $\eta = 0$, $\zeta = 0$. This case corresponds to uniform rolling of the ball along the cushion. Explicit equations of motion in this case take the form

$$\dot{x} = v_0, \quad \omega_x = \omega_{x,0} - \frac{5\mu_c g t (1 + \mu_c)}{2R(1 + \mu_c \mu_c)}$$

$$\omega_y = \frac{v_0}{R}, \quad \omega_z = -\frac{v_0}{R}$$  (32)

where $v_0$ and $\omega_{x,0}$ denote the initial values of the longitudinal velocity of the ball center and the spin, $t \in [0, T_1]$ and

$$T_1 = \frac{2\omega_{x,0} (1 + \mu_c \mu_c)}{5\mu_c (1 + \mu_c)} > 0.$$  

At moment $T_1$, $\omega_x$ vanishes and the motion turns into (31) with $v_{\text{fin}} = v_0$.

2. $\eta = \pm 2\sqrt{4\mu_c^2 - 1}$, $\zeta = \mp \sqrt{4\mu_c^2 - 1}$. In order for these solutions to correspond to a real motion, the friction coefficient of the ball and the cushion must satisfy inequality $\mu_c \geq 0.5$. Write down explicit dependencies of the velocities on time:

$$\dot{x} = v_0 \pm \frac{5\mu_c g t \sqrt{4\mu_c^2 - 1}}{2\sqrt{16\mu_c^2 - 3 + \mu_c}}.$$  

² In [1, eq. (26)] there is a misprint in the last row of the right-hand side, namely, the term $\omega_c A_{SS}^5 / (\omega_c jmr)$ must enter with a minus sign.
Consider a rigid body touching the support at \( p \) points of contact according to constraint equations (22). Denote by \( \mathbf{v}_i, i = 1, \ldots, p \), the velocity of the body at the point of contact with the \( i \)-th supporting surface \( f_i(x) = 0 \), by \( \mathbf{P}_i : \mathbb{R}^n \rightarrow \mathbb{R}^r, r \leq n \), the operator that maps generalized velocity \( \mathbf{v} \) to \( \mathbf{v}_i \), by \( \mathbf{N}_i \) the normal support reaction of the \( i \)-th surface, and by \( \Phi_i \) the tensor of dry friction at the \( i \)-th contact point. Also introduce covectors \( \mathbf{n}_i \) such that \( \mathbf{N}_i = \lambda_i \mathbf{n}_i \), or equivalently

\[
\mathbf{P}_i^T \mathbf{n}_i = \frac{\partial f_i}{\partial \mathbf{x}}.
\]

Then according to (12), the generalized dry friction force takes the form

\[
\mathbf{F} = -\sum_{j=1}^p \lambda_j |\mathbf{n}_j| \frac{\mathbf{P}_j^T \Phi_j \mathbf{v}_j}{|\mathbf{v}_j|}.
\]

Following V. V. Kozlov’s considerations for the case of a single point of contact (see [8, formulas (2.10)–(2.13)]), rewrite the Lagrange equations as

\[
\ddot{\mathbf{x}} = \mathbf{A}^{-1} \mathbf{X} + \sum_{j=1}^p \lambda_j \mathbf{A}^{-1} \mathbf{P}_j^T \left( \mathbf{n}_j - |\mathbf{n}_j| \frac{\Phi_j \mathbf{v}_j}{|\mathbf{v}_j|} \right),
\]

where \( \mathbf{X} \) is a known function of \( \mathbf{x} \) and \( \dot{\mathbf{x}} \). Now, multiplying scalarly this equation by \( \partial f_i/\partial \mathbf{x} = \mathbf{P}_i^T \mathbf{n}_i \), we get a system on \( \lambda_j \):

\[
\sum_{j=1}^p \lambda_j \left( \mathbf{P}_j^T \mathbf{n}_i, \mathbf{A}^{-1} \mathbf{P}_j^T \left[ \mathbf{n}_j - |\mathbf{n}_j| \frac{\Phi_j \mathbf{v}_j}{|\mathbf{v}_j|} \right] \right) = Z_i - \left( \mathbf{P}_i^T \mathbf{n}_i, \mathbf{A}^{-1} \mathbf{X} \right),
\]

where \( Z_i \) are also known functions of \( \hat{\mathbf{x}} \) and \( \mathbf{x} \),

\[
Z_i = \left( \frac{\partial f_i}{\partial \mathbf{x}} \times \dot{\mathbf{x}} \right) = -\left( \frac{d}{dt} \frac{\partial f_i}{\partial \mathbf{x}} \times \dot{\mathbf{x}} \right),
\]

and the right-hand side of (34) does not contain \( \lambda_j \). Thus, the absence of Painlevé paradoxes in the Lagrange equations is equivalent to unique solvability of (34) with respect to \( \lambda_j \).

**Definition 1** We call a set of dry friction tensors \( \Phi_i, i = 1, \ldots, p \), regular if for any \( \mathbf{v} \) satisfying constraint equations

\[
\left( \frac{\partial f_i}{\partial \mathbf{x}}, \mathbf{v} \right) = \cdots = \left( \frac{\partial f_p}{\partial \mathbf{x}}, \mathbf{v} \right) = 0,
\]

and for any \( k > 0 \)

\[
\det \left\{ \left[ \mathbf{n}_i, \mathbf{P}_i \mathbf{A}^{-1} \mathbf{P}_j^T \left[ \mathbf{n}_j - k \Phi_j \mathbf{v}_j \right] \right] \right\} > 0.
\]

5 Tensors of dry friction in multi-contact case

In this section we suggest a multi-contact generalization of condition (i) alternative to (23), that is appropriate for the description of motions like in the previous example.

Remark 4 It would be interesting to analyze the Lyapunov stability of these solutions as well as possible chattering scenarios in presence of impacts, but we leave it outside the scope of this paper. Analysis of related problems can be found in [14] and [2].
Condition (35) expresses a natural property of dry friction tensors that scale conversions of the friction coefficients should not lead to Painlevé paradoxes. In particular, assuming parameter $k$ sufficiently large and leaving the highest terms in $k$ only, we get a simple necessary condition of regularity:

$$\det \left\{ (n_i, P A^{-1} P^T \Phi_j v_j) \right\} \geq 0,$$

where in case of odd $p$ the inequality turns to equality.

**Example 1** In case of a single point of contact, (35) takes the form

$$\left(n_c, PA^{-1} P^T \Phi_c v_c \right) = 0,$$

which is equivalent to condition (i). Consequently, tensors $\Phi_c$ from (14) for the Painlevé problem and (19) for the problem of a ball on a plane are regular.

**Example 2** Let for the problem of a ball moving along the cushion

$$\Phi_{lc} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \mu_2 & 0 \\ 0 & \nu_2 & \nu_3 \end{bmatrix}, \quad \Phi_{lc} = \begin{bmatrix} \alpha_4 & \alpha_5 & \alpha_6 \\ 0 & \mu_5 & \mu_6 \\ 0 & 0 & \nu_6 \end{bmatrix},$$

where $\alpha_{1,4}, \mu_{2,5}, \nu_{3,6} \geq 0$. Then (35) turns into

$$1 + \frac{k^2 R^2 \mu_5 \nu_3 \alpha_3^2}{m^2} \geq 0,$$

which is always true. In particular, isotropic tensors $\Phi_{lc}$ and $\Phi_{lc}$ from (25) are regular.

**Additional information**

A preprint of this work is available on arXiv [13].

**Compliance with ethical standards**

**Conflict of interest** The author declares that he has no conflict of interest.

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