The bipolar filtration of topologically slice knots

Jae Choon Cha and Min Hoon Kim

Abstract. The bipolar filtration of Cochran, Harvey and Horn presents a framework of the study of deeper structures in the smooth concordance group of topologically slice knots. We show that the graded quotient of the bipolar filtration of topologically slice knots has infinite rank at each stage greater than one. To detect nontrivial elements in the quotient, the proof simultaneously uses higher order amenable Cheeger-Gromov $L^2$-invariants and infinitely many Heegaard Floer correction term $d$-invariants.

1. Introduction

Understanding the difference of the topological and smooth categories is among the main objectives of the topological study of dimension 4. Knot concordance, which may be viewed as the local case of the general disk embedding problem in dimension 4, has been studied extensively from this viewpoint. Indeed it is known that several questions on 4-manifolds and smoothings can be investigated via concordance. In the study of smooth concordance, an ultimate goal is to understand the structure of the smooth concordance group of topologically slice knots, which we denote by $\mathcal{T}$. The group $\mathcal{T}$ measures the gap between the smooth and topological categories.

In the literature, there are remarkable advances in the study of $\mathcal{T}$, which are achieved by modern 4-manifold technologies. The first examples of topologically slice knots which are not smoothly slice, due to Akbulut and Casson, are established as a consequence of the results of Freedman [Fre82, Fre84] and Donaldson [Don83]. As an abelian group, $\mathcal{T}$ is infinitely generated [End95], and the 2-torsion subgroup of $\mathcal{T}$ is infinitely generated [HKL16]. We do not attempt to provide a complete list of known results, but as further investigation of the structure of $\mathcal{T}$, we note that results on summands of $\mathcal{T}$ [Liv04, Liv08, MO07, Hom15], and the study of the quotient of $\mathcal{T}$ modulo the subgroup of Alexander polynomial one knots [HLR12, HKL16] are especially significant.

Nonetheless, our understanding is still far from obtaining a classification of $\mathcal{T}$. In fact we believe that known smooth invariants and obstructions, including those from various versions of gauge theory, Heegaard Floer homology and Khovanov homology, are short of classifying $\mathcal{T}$; for instance [CHH13, Proposition 1.2] suggests that a majority of invariants would not be able to see structures in deep part of $\mathcal{T}$ unless they were coupled better with the fundamental group. We will discuss this in more details below.

Regarding the above, we remark that the role of the fundamental group is relatively better understood for topological concordance, to provide a framework toward a classification, especially in developments initiated by work of Cochran, Orr and Teichner [COT03]. It may be viewed as an obstruction theoretic approach via a filtration and exact sequences involving iterated quotients. $L^2$-signatures associated with the $n$th derived subgroup of the fundamental group give obstructions at the $n$th stage. We also remark that obstruction theoretic approaches give beautiful classifications for link homotopy [HL90] and Whitney tower concordance [CST11] where Milnor invariants extracted from the fundamental group are key obstructions.

2020 Mathematics Subject Classification. 57N13, 57M27, 57N70, 57M25.
The main result of this paper shows that modern smooth techniques can be combined with fundamental group information more strongly, to detect nontrivial elements in $\mathcal{T}$ for which the majority of known smooth obstructions vanish. Our result also provides, from an obstruction theoretic viewpoint for $\mathcal{T}$, information on what the iterated quotients may look like.

To discuss our result explicitly, we consider the bipolar filtration

$$\mathcal{T} \supset \mathcal{T}_0 \supset \mathcal{T}_1 \supset \cdots \supset \mathcal{T}_n \supset \cdots \supset \{0\}$$

introduced by Cochran, Harvey and Horn [CHH13]. Briefly, the filtration reflects definiteness of the intersection form motivated from Donaldson’s work, together with fundamental group information related to derived subgroups and the tower techniques of Casson and Freedman for 4-manifolds and work of Cochran, Orr, and Teichner on knot concordance. The definition of the subgroup $\mathcal{T}_n \subset \mathcal{T}$, which is associated with the $n$th derived subgroup, is recalled in Section 2.1.

It is noteworthy that the $\tau$-invariant [OS03b] and the $\epsilon$-invariant [Hom14] are trivial on $\mathcal{T}_0$, and the slice obstructions from the Heegaard Floer correction term invariants [MO07, OS06, JN07, GJ11] vanish on $\mathcal{T}_1$ [CHH13]. Also, from results in [CHH13, OSS17, NW15, HW16], it follows that the $\nu^+$-invariant [HW16] and the $\Upsilon$-invariant [OSS17] vanish on $\mathcal{T}_0$. This is a rigorous description of the aforementioned claim that a majority of known smooth invariants does not see deep part of $\mathcal{T}$. To understand structures in $\mathcal{T}_n$ ($n \geq 1$), it seems necessary to combine existing smooth techniques with more information associated with the fundamental group.

Toward a classification of $\mathcal{T}$, two questions on the bipolar filtration are fundamental:

(i) What is the quotient $\mathcal{T}_n/\mathcal{T}_{n+1}$ for each $n$? Especially, is it nontrivial?

(ii) What is the transfinite term $\mathcal{T}_\omega = \bigcap_{n \geq 0} \mathcal{T}_n$? Especially, is it nontrivial?

The main focus of this paper is on the first question. The case of $n = 0, 1$ were the only cases for which the nontriviality of $\mathcal{T}_n/\mathcal{T}_{n+1}$ was previously known [CHH13, CH15].

Theorem A. For each $n \geq 2$, the quotient $\mathcal{T}_n/\mathcal{T}_{n+1}$ has infinite rank.

In what follows we discuss some aspects of Theorem A and the techniques of the proof.

Combining smooth invariants with fundamental group information. To detect structures unknown to be detected by existing smooth obstructions, our approach simultaneously uses Heegaard Floer correction term $d$-invariants and amenable Cheeger-Gromov $L^2$ $p$-invariants associated with higher derived series quotients of the fundamental group. We remark that a combination of the Cheeger-Gromov invariant over a certain torsion-free solvable group and the $d$-invariant of the 3-fold cyclic branched cover was used earlier in [CHH13], in order to obtain a weaker result under an additional not-yet-proven hypothesis, which is implied by the homotopy ribbon slice conjecture. Our improved method can be carried out without the homotopy ribbon type hypothesis.

A key technique we use is to consider, even for a single knot, an infinite family of $d$-invariants associated to branched covers of various degrees, together with the Cheeger-Gromov $p$-invariants. We show that these infinitely many $d$-invariants are all nonzero for our examples, and derive the desired result using this. Another key ingredient we employ is the amenable signature theorem in [CO12, Cha14] for the Cheeger-Gromov invariants over locally $p$-indicable amenable groups.
Given that our approach successfully provides new information on the deep part of the smooth concordance group $T$, it seems natural to ask how other various modern smooth techniques can be improved to be coupled more tightly with topological information associated with the fundamental group. We believe that further investigations along this direction will be intriguing.

Satellite constructions of knots. We explicitly construct knots which are independent (and so generate a free abelian subgroup of infinite rank) in $T_n/T_{n+1}$ by using iterated satellite constructions. The knots are of the form $R(J, D)$ shown in Figure 1, where $D$ means the positive Whitehead double of the right handed trefoil, and $J$ is a pattern knot.

Figure 1. A satellite knot $R(J, D)$.

The pattern knots $J$ are given as follows. For brevity, denote by $P(K)$ a satellite knot with pattern $P$ and companion $K$. Using a fixed pattern $P$ which is defined in Figure 3 in Section 2.2 and using an infinite family $\{J_i\}_{i=0}^\infty$ of knots described in Section 4, the knots $J$ are defined to be $P^{n-1}(J_0) = P(P(\cdots P(J_0)\cdots))$, obtained by the $(n-1)$st iterated satellite construction.

Satellite constructions are used as a standard tool in many papers on concordance and related subjects. For instance, see [Gil83, Liv83, GL92, COT04, CT07, FT05, CHL09, FP14, CDR14, CHP17, Hed07, HLR12, HK12, Lev12, Hom15, HKL16]. Especially, interesting new information has been discovered via satellite operators that produce knots generating subgroups of concordance groups which are not straightforward to detect but actually large. For topological concordance, work of Cochran, Harvey and Leidy on the fractal nature [CHL11] is notable. Among remarkable results for the smooth case, Hedden and Kirk showed that Whitehead doubling has infinite rank image in $T$ [HK12], and proposed a conjecture that Whitehead doubling preserves independence in the smooth knot concordance group.

From this viewpoint, we note that the proof of Theorem A shows that satellite constructions produce a large subgroup even in the deep part of $\mathcal{T}$, where the notion of depth is rigorously given by the bipolar filtration. More precisely, the satellite operator $Q_n = R(P^{n-1}(-), D)$ injects the infinite set of knots $\{J_i\}_{i=0}^\infty$ to a linearly independent subset in $T_n/T_{n+1}$ for $n \geq 2$ (and consequently in $T_n \subset T$). It appears to be an interesting direction to enrich currently available various smooth invariants, to detect large images even in the deep part of $\mathcal{T}$ for sophisticated satellite operators such as $Q_n$ above. Our method supports that it would not be completely impossible.
Also, note that the statement of Theorem A tells that the quotients \( T_n/T_{n+1} \) (\( n \geq 2 \)) have a similarity in that they are of the same rank. The above discussion exhibits more about the similarity from a geometric viewpoint: for all \( n \geq 2 \), our \( \mathbb{Z}^\infty \) subgroup in \( T_n/T_{n+1} \) is generated by the image of the \textit{same} knots \( J_n \) under the satellite operator \( Q_n \). This shows that the fractal nature of the topological knot concordance group proposed and investigated in [CHH13] is also found, at least partially, in the smooth concordance group \( T \) of topologically slice knots.

\textit{Strategy of the proof and organization of the paper.} The construction of \( Q_n(J_n) = R(P^{n-1}(J_n), D) \) is also closely related to the strategy of the proof of Theorem A. Recall that if one attempted to prove a knot is not slice, it would be natural to start by investigating metabolizers of the Blanchfield pairing. For the knot \( R(P^{n-1}(J_n), D) \), it turns out to have exactly two metabolizers: submodules \( \langle \alpha_D \rangle \) and \( \langle \alpha_J \rangle \) generated by linking circles \( \alpha_D \) and \( \alpha_J \) of the handles along which \( D \) and \( J = P^{n-1}(J_n) \) are tied in. (See Figure 1 above, and also Figure 2 in Section 2.) Indeed, we begin similarly to prove Theorem A by considering the two possible metabolizers. In Section 2, we recall the definition of the bipolar filtration, describe the construction of the above satellite knots in detail, and divide the proof into the two cases. In Sections 3 and 4, we present the construction of \( Q_n(J_n) \), using infinitely many Heegaard Floer \( d \)-invariants. In Sections 5 and 6, we treat the case of \( \langle \alpha_J \rangle \), using infinitely many Heegaard Floer \( d \)-invariants.

\textit{Comparison with the link case.} We remark that for the multi-component link case, the nontriviality of \( T_n/T_{n+1} \) was proven earlier by the first named author and Powell [CP14]. They built a geometric operation which systematically pushes certain links nontrivial in \( T_{n-1}/T_n \) to links nontrivial in the next stage \( T_n/T_{n+1} \), using covering link calculus. This works only for links, since the covering link technique requires multi-components. The approach used in this paper for knots is of a completely different nature. The main results (Theorems 1.1 and 1.2) of [CP14] for \( n \geq 2 \) can be obtained as immediate consequences of our Theorem A.

\textit{Acknowledgements.} We would like to thank anonymous referees for comments which were very helpful in improving the exposition of this paper. Part of this work was done during the authors’ visit to the Max Planck Institute for Mathematics in Bonn. JCC and MHK were partly supported by NRF grant 2019R1A3B2067839. MHK was partly supported by POSCO TJ Park Science Fellowship and NRF grant 2021R1C1C1012939.

2. Examples and the first step of the proof

2.1. Definition of the bipolar filtration

We begin by recalling the definition of the bipolar filtration \( \{T_n\} \). In this paper, manifolds and submanifolds are always assumed to be compact, oriented and smooth. For a knot \( K \), denote by \( M(K) \) the zero-surgery manifold. Denote by \( G^{(n)} \) the \( n \)th derived subgroup of a group \( G \), which is defined by \( G^{(0)} = G \) and \( G^{(n+1)} = [G^{(n)}, G^{(n)}] \).

\textbf{Definition 2.1} ([CHH13 Definition 5.1]). A knot \( K \) in \( S^3 \) is \textit{n-negative} if \( M(K) \) bounds a connected 4-manifold \( V \) satisfying the following.

1. The inclusion induces an isomorphism \( H_1(M(K)) \rightarrow H_1(V) \) and a meridian of \( K \) normally generates \( \tau_1(V) \).
2. There is a basis for \( H_2(V) \) which consists of the classes of closed connected surfaces \( \{S_i\} \), disjointly embedded in \( V \), with self-intersection number \( S_i \cdot S_i = -1 \) (or equivalently, with normal bundle with Euler class \(-1\)).
(3) For each $i$, the image of $\pi_1(S_i)$ lies in $\pi_1(V)^{(n)}$.

The above 4-manifold $V$ is called an $n$-negaton bounded by $M(K)$.

An $n$-positive knot and an $n$-positon are defined by replacing the self-intersection condition by $S_i \cdot S_i = +1$. A knot is $n$-bipolar if it is $n$-positive and $n$-negative.

Recall that $\mathcal{T}$ denotes the smooth knot concordance group of topologically slice knots. The group operation is connected sum. For an integer $n \geq 0$, let $\mathcal{T}_n$ be the subset of $\mathcal{T}$ consisting of the concordance classes of $n$-bipolar knots. In [CHH13], it was shown that $\mathcal{T}_n$ is a subgroup of $\mathcal{T}$. It is straightforward that $\mathcal{T}_{n+1} \subset \mathcal{T}_n$.

**Definition 2.2** ([CHH13, Definition 2.6]). The descending filtration $\mathcal{T} \supset \mathcal{T}_0 \supset \mathcal{T}_1 \supset \cdots \supset \mathcal{T}_n \supset \cdots \supset \{0\}$ is called the bipolar filtration of $\mathcal{T}$.

**2.2. Construction of examples**

Fix an integer $n \geq 2$. In this section, we construct a sequence of topologically slice $n$-bipolar knots $K_i$ ($i = 1, 2, \ldots$) by using iterated satellite operations. They will be proven to be linearly independent in the quotient $\mathcal{T}_n/\mathcal{T}_{n+1}$ in later sections.

We use the following notations for satellite operations. For a knot $J$ in $S^3$, denote the exterior by $E_J$. Suppose $J$ and $P$ are knots in $S^3$ and $\eta$ is a knot in $S^3 \setminus P$ which is unknotted in $S^3$. Take the union $E_\eta \cup_\partial E_J$, where the boundaries are attached along an orientation reversing homeomorphism identifying a zero linking longitude of $\eta$ with a meridian of $J$ and a meridian of $\eta$ with a zero linking longitude of $J$. Let $P(\eta, J)$ be the image of $P$ in $E_\eta \subset E_\eta \cup_\partial E_J \cong S^3$. This is a satellite knot with pattern $P$ and companion $J$.

Our examples are of the following form. Let $R$ be the knot $9_{46}$ and $\alpha_J$, $\alpha_D$ be the curves shown in Figure 2. Denote by $R(J, D)$ the satellite knot $(R(\alpha_J, J))(\alpha_D, D)$, which is shown in Figure 1 in the introduction. The following observation will be useful in later parts: for the trivial knot $U$, both $R(U, D)$ and $R(J, U)$ are slice. A movie picture of a slice disk in $D^4$, for instance for $R(U, D)$, is obtained by cutting the 1-handle of the obvious genus one Seifert surface along which $D$ is tied.

![Figure 2. The knot $R = 9_{46}$ and the curves $\alpha_J$ and $\alpha_D$](image)

Let $D$ be the untwisted positive Whitehead double of the right-handed trefoil. This choice will be fixed throughout this paper.
On the other hand, in place of $J$, we will use knots $J_{n-1}^i$ ($i = 1, 2, \ldots$) described below. (Recall that $n \geq 2$ is fixed.) We will use the following notation. For a knot $J$, let $\sigma_J(\omega)$ be the Levine-Tristram function defined for $\omega \in S^1$. For a positive integer $d$, denote the average of the evaluations of $\sigma_J$ at the $d$th roots of unity by

$$\rho(J, \mathbb{Z}_d) := \frac{1}{d} \sum_{k=0}^{d-1} \sigma_J(e^{2\pi i k/d}).$$

We start by choosing a knot $J_0^i$ and a prime $p_i$ for each $i = 1, 2, \ldots$ satisfying the following:

(J1) For each $i$, $J_0^i$ is 0-negative.

(J2) For each $i$, $|\rho(J_0^i, \mathbb{Z}_{p_i})| > 69713280 \cdot (6n + 90)$.

(J3) For $i < j$, $\rho(J_0^i, \mathbb{Z}_{p_i}) = 0$.

An explicit construction of a sequence $\{(J_0^i, p_i)\}$ satisfying (J1) (J2) and (J3) will be given in Section 3. In this section, we will use (J1) only. The other conditions (J2) and (J3) will be used in Section 3.

For $k = 0, 1, \ldots, n-2$, define $J_{k+1}^i := P_k(\eta_k, J_0^i)$, where $P_k$ is the stevedore knot and $\eta_k$ is the curve shown in Figure 3. Although $(P_k, \eta_k)$ remains the same as $k$ varies, we will keep the index $k$ in the notation since it will be useful to distinguish the occurrences in distinct stages.

Let $K_i$ be the satellite knot $R(J_{n-1}^i, D)$. Each $K_i$ is topologically slice since $D$ is topologically slice by the work of Freedman [Fre84]. By the following lemma, $K_i$ lies in $\mathcal{T}_n$.

**Lemma 2.3.** Under the assumption that $J_0^i$ is 0-negative, the knot $K_i$ is $n$-negative. Also, $K_i$ is $k$-positive for every $k$.

**Proof.** We will use the following two facts. (i) If $P$ is slice, $J$ is $n$-negative and $[\eta] \in \pi_1(S^3 \setminus P)^{(k)}$, then $P(\eta, J)$ is $(n + k)$-negative [CHH13 Proposition 3.3]. This holds when “negative” is replaced by “positive” or by “bipolar.” (ii) A knot is 0-positive if it can be changed to a trivial knot by changing positive crossings to negative crossings [CHH13 Proposition 3.1], [CL86 Lemma 3.4].

In our case, observe that the stevedore knot $P_k$ is slice and $[\eta_k]$ lies in $\pi_1(S^3 \setminus P_k)^{(1)}$. Since $J_0^i$ is 0-negative, $J_{k}^i$ is $k$-negative for $k = 0, 1, \ldots$ by induction using (i). Since $R(U, D)$ is slice and $[\alpha_j] \in \pi_1(S^3 \setminus R(U, D))^{(1)}$, it follows that $K_i = (R(U, D))(\alpha_j, J_{n-1}^i)$ is $n$-negative once again by (i).

Let $T$ be the right-handed trefoil. It is 0-positive by (ii). The knot $D$ can be viewed as a satellite knot $Wh(\eta, T)$ where $(Wh, \eta)$ is the pattern shown in Figure 4. Since $[\eta]$ is trivial in $\pi_1(S^3 \setminus Wh) = \mathbb{Z}$, it follows that $D$ is $k$-positive for all $k$ by (i). Therefore, by (i), $K_i = (R(J_{n-1}^i, U))(\alpha_D, D)$ is $k$-positive for all $k$, since $R(J_{n-1}^i, U)$ is slice. \qed
2.3. A negaton from a linear combination and metabolizers

Suppose that a nontrivial finite linear combination $K = \sum_{i=1}^{r} a_i K_i$ ($a_i \in \mathbb{Z}$) of the knots $K_i$ is $(n+1)$-bipolar. By eliminating terms with $a_i = 0$ and by replacing $K$ by $-K$ if necessary, we may assume that $a_1 \geq 1$ and $a_i \neq 0$ for each $i$. Our strategy is to derive a contradiction by investigating consequences on the first knot $K_1$ which are implied by the hypothesis on the linear combination $K$.

For this purpose, we will construct a specific $n$-negaton for $K_1$, from a given $(n+1)$-negation for $K$, by attaching additional “negative” pieces. (Principle: negative + negative = negative.) It will be guided by the observation that $K_1$ is concordant to the connected sum of $K_1 - (a_1 - 1)K_1$ and $-a_i K_i$ ($i > 1$), where the summands added to $K$ are $n$-negative regardless of the sign of $a_i$, by Lemma 2.3.

The actual construction proceeds as follows. Let $V^-$ be an $(n+1)$-negaton bounded by $M(K)$. For each $i$, if $a_i > 0$, choose an $n$-negaton bounded by $-M(K_i)$ by invoking Lemma 2.3 and call it $Z^-_i$. If $a_i < 0$, choose an $n$-negaton bounded by $M(K_i)$ again by using Lemma 2.3 and call it $Z^+_i$. Indeed, we will use a specific choice of $Z^-_i$ later in Section 3.2 (see Lemma 3.3), but for now it suffices to assume that $Z^-_i$ is just an $n$-negaton for $\pm K_i$. Recall that there is a standard cobordism, which we call $C$, bounded by the union of $\partial_- C := -M(K)$ and $\partial_+ C := \bigsqcup_{i=1}^{r} a_i M(K_i)$. It is obtained by attaching, to $\bigsqcup_{i=1}^{r} a_i M(K_i) \times I$, $(N-1)$ 1-handles that makes it connected and attaching $(N-1)$ 2-handles that makes meridians of the involved knots parallel, where $N = \sum_{i=1}^{r} |a_i|$. See, for instance, [COT04, p. 113] for a detailed discussion on $C$. Define

$$X^- := V^- \cup_{\partial_- C} C \cup_{\partial_+ C} \left( (a_1 - 1)Z^-_1 \cup \bigsqcup_{i>1} |a_i|Z^-_i \right).$$

See the schematic diagram in Figure 5.
Lemma 2.4. The 4-manifold $X^-$ is an $n$-negaton bounded by $M(K_1)$.

Proof. It is known that $H_1(C) \cong \mathbb{Z}$ and $H_2(C) \cong \bigoplus H_2(M(K_i))^{[n_i]}$ where the second isomorphism is induced by the inclusions (for instance see [COT04, p. 113]). Also, a meridian of any one of $K, K_1, \ldots, K_r$ normally generates $\pi_1(C)$ (and hence generates $H_1(C)$), because of the 2-handle attachments in the construction of $C$. Using this and Definition 2.1 (1) for the negatons $Z^n_i$ and $V^n_i$, the following is shown by a Mayer-Vietoris argument:

$$H_1(X^-) \cong H_1(M(K_1)), \\
H_2(X^-) \cong H_2(V^-) \oplus H_2(Z^-)^{n_i-1} \oplus \left( \bigoplus_{i>1} H_2(Z^-)^{[n_i]} \right)$$

where the isomorphisms are inclusion-induced. From the $H_2$ computation, it follows that Definition 2.1 (2) and (3) are satisfied for $X^-$. Since $\pi_1(C), \pi_1(V^-)$ and $\pi_1(Z^-)$ are normally generated by meridians of $K_1, K$ and $K_i$ respectively, it follows that $\pi_1(X^-)$ is normally generated by a meridian of $K_1$. By this and the above $H_1$ computation, Definition 2.1 (1) for the negatons $Z^n_i$ and $V^n_i$ is satisfied. $\square$

Recall that the Blanchfield form

$$B:\ H_1(M(J); \mathbb{Q}[t^\pm]) \times H_1(M(J); \mathbb{Q}[t^\pm]) \longrightarrow \mathbb{Q}(t)/\mathbb{Q}[t^\pm]$$

defined on the (rational) Alexander module $H_1(M(J); \mathbb{Q}[t^\pm])$ of a knot $J$, and that a submodule $P$ of $H_1(M(J); \mathbb{Q}[t^\pm])$ is called a metabolizer if $P = P^\perp$, where

$$P^\perp := \{ x \in H_1(M(J); \mathbb{Q}[t^\pm]) \mid Bt(x, P) = 0 \}.$$

Lemma 2.5 (A special case of [CHH13, Theorem 5.8]). Let $V$ be a either 1-negaton or 1-position bounded by $M(J)$, and let $P$ be the kernel of the inclusion-induced homomorphism $H_1(M(J); \mathbb{Q}[t^\pm]) \rightarrow H_1(V; \mathbb{Q}[t^\pm])$ on the Alexander modules. Then $P$ is a metabolizer of the Blanchfield form.

Returning to our case, let

$$P := \text{Ker} \{ H_1(M(K_1); \mathbb{Q}[t^\pm]) \longrightarrow H_1(X^-; \mathbb{Q}[t^\pm]) \}.$$

We need the following facts, which can be verified by a routine computation, for instance using the Seifert matrix $\left[ \begin{array}{cc} a & 0 \\ 0 & b \end{array} \right]$ of $K_1$. Regard (zero linking parallels of) the curves $\alpha_J$ and $\alpha_D$ in Figure 2 as curves in the zero surgery manifold of $K_1 = \overline{B(J_{n-1}, D)}$, and denote them by $\alpha_J, \alpha_D \subset M(K_1)$, for brevity. Then, $H_1(M(K_1); \mathbb{Q}[t^\pm])$ is the internal direct sum of two cyclic submodules $\langle \alpha_J \rangle$ and $\langle \alpha_D \rangle$ generated by the classes of $\alpha_J$ and $\alpha_D$ respectively, and in addition, $\langle \alpha_J \rangle \cong \mathbb{Q}[t^\pm]/(l - 2)$ and $\langle \alpha_D \rangle \cong \mathbb{Q}[t^\pm]/(2t - 1)$. The Blanchfield form of $K_1$ is given by $Bt(\alpha_J, \alpha_J) = Bt(\alpha_D, \alpha_D) = 0, Bt(\alpha_J, \alpha_D) = \frac{t-2}{2t}$. For later use in Section 3, we remark that the same conclusion holds when $\mathbb{Z}$ is used as coefficients in place of $\mathbb{Q}$.

From the above paragraph, it follows that $P$ is equal to either $\langle \alpha_D \rangle$ or $\langle \alpha_J \rangle$, since $P$ is a metabolizer by Lemma 2.5.

Case 1: $P = \langle \alpha_D \rangle$. In this case, we will use the Cheeger-Gromov $L^2$- $\rho$-invariants to derive a contradiction. The proof is given in Section 3.

Case 2: $P = \langle \alpha_J \rangle$. In this case, to derive a contradiction, we will use the Heegaard Floer correction term $d$-invariants of infinitely many branched covers of $K_1$. The proof is given in Section 5.

The proof of Theorem A will be finished by completing the above two cases.
3. Case 1: Use of Cheeger-Gromov invariants

The goal of this section is to reach a contradiction in Case 1 described above. Suppose $P = (\alpha_D)$ throughout this section.

Our key ingredient is the amenable signature theorem developed in [CO12, Cha14]. We use it to extract obstructions, from Cheeger-Gromov invariants over locally p-indicable amenable groups. For this purpose, we begin by converting the negations described in Section 2.3 to 4-manifolds called integral solutions in [Cha14]. After that, we analyze the behavior of a commutator series of the fundamental group, and investigate Cheeger-Gromov invariants over the associated quotient, by applying the ideas and methods used in [Cha14] Sections 4 and 5. We remark that this type of technique is strongly influenced by earlier work of Cochran, Harvey and Leidy [CHL99].

3.1. A 4-manifold and analysis of mixed-type commutator series

Recall from (2.1) that $X^0$ is the union of $V^−, C, (a_1 − 1)Z^−_1$ and $|a_i|Z^−_i (i > 1)$. Let $V^0 = V^− \# \mathbb{CP}^2$ and $Z^0_i = Z^−_i \# (\mathbb{CP}^2)$. Then $V^0$ is an integral $(n+1)$-solution in the sense of [Cha14] Definition 3.1, and each $Z^0_i$ is an integral n-solution too, by [CHH13] Proposition 5.5. Since we do not directly use the definition of an integral n-solution, we do not spell it out but we will state some properties when we need to use them. We remark that the definition of an integral n-solution does not require a spin structure (and so there is no condition on the self-intersection $\mu$), in contrast to the definition of an n-solution defined in [COT03]. Let

$$X^0 := V^0 \cup_{\partial C} C \cup_{\partial C} \left((a_1 − 1)Z^0_1 \cup \bigcup_{i > 1} |a_i|Z^0_i\right).$$

The manifold $X^0$ is bounded by $M(K_1)$. From the hypothesis that $P = (\alpha_D)$, it follows that the kernel of $H_1(M(K_1); \mathbb{Z}[t^\pm]) \to H_1(X^0[t^\pm])$ is still equal to $(\alpha_D)$, since $\pi_1(X^0) \cong \pi_1(X^−)$. In fact, as done in [CHH13], this leads us to a proof of Lemma 2.5 since it is known that if W is an integral n-solution with $n \geq 1$ for a knot $J$, then the kernel of $H_1(M(J); \mathbb{Z}[t^\pm]) \to H_1(W; \mathbb{Z}[t^\pm])$ is a metabolizer [COT03 Theorem 4.4].

Attach more pieces to $X^0$ as follows. For the satellite knot $J_{k+1}^1 = P_k(\eta_k, J_k^1)$, there is a standard cobordism, say $E_k$, from $M(J_{k+1}^1)$ to $M(J_k^1) \cup \partial(M(P_k))$ for $k = 0, \ldots, n−2$: take the union of $M(J_k^1) \times I$ and $M(P_k) \times I$, and identify the solid torus $M(J_k^1) \times E(J_k^1) \times I$ with a tubular neighborhood of $\eta_k \subset P_k \times 1$ [CHL99, p. 1429]. The same construction gives a standard cobordism $E_{n−1}$ from $M(K_1)$ to $M(J_{n−1}^1) \cup \partial(M(R(U, D)))$. Define

$$X := X^0 \cup_{M(K_1)} E_{n−1} \cup_{M(J_{n−1}^1)} E_{n−2} \cup_{M(J_{n−2}^1)} \cdots \cup_{M(J_1)} E_0.$$

See the schematic diagram in Figure 6.

We will compute the $\rho^{(2)}$-invariant of $\partial X$, which is associated to groups obtained from a certain mixed-coefficient commutator series construction, as first done in [Cha14]. The series is defined as follows. Let $p = p_1$, which is the prime associated to $J_1^1$ (see the conditions [J1], [J2] and [J3] in Section 2.2). Let $R_i = \mathbb{Q}$ for $i = 0, \ldots, n−1$, and let $R_n = \mathbb{Z}_p$. For a group $G$, define $P^0G := G$ and

$$P^{k+1}G = \text{Ker}\left\{P^kG \to \frac{P^kG}{[P^kG, P^kG]} \to \frac{P^kG}{[P^kG, P^kG]} \otimes \mathbb{Z}_p\right\}$$

for $k = 0, \ldots, n$ inductively. For the use in (3.4) (see Section 3.2), we note that the quotient $\Gamma = G/\rho^{n+1}G$ satisfies $\Gamma^{(n+1)} = \{1\}$, and that $\Gamma$ is amenable and locally $p$-indicable by [CO12] Lemma 6.8. Here, a group is amenable if it admits a finitely additive
invariant mean, and is \textit{locally p-indicable} if each nontrivial finitely generated subgroup admits an epimorphism onto \( \mathbb{Z}_p \). (In this paper we will not use these definitions directly.)

**Remark 3.1.** In place of local \( p \)-indicability, Strebel’s class \( D(\mathbb{Z}_p) \) [Str74] was used in statements in [CO12] and subsequent papers. Indeed, a group is locally \( p \)-indicable if and only if it lies in \( D(\mathbb{Z}_p) \), due to [HS83].

Let \( \phi : \pi_1(X) \to \pi_1(X)/\mathcal{P}^{n+1}\pi_1(X) \) be the quotient homomorphism. Let \( \mu_{J_0} \) be the meridian of \( J_{10} \), and regard it as a curve in \( M(J_{10}) \subset \partial X \subset X \).

The following is the key part which makes use of the defining condition of Case 1.

**Lemma 3.2.** Under the assumption of Case 1 that \( \mathcal{P} = \langle \alpha_D \rangle \), the element \( \phi(\mu_{J_0}) \) is nontrivial and lies in the subgroup \( \mathcal{P}^n\pi_1(X)/\mathcal{P}^{n+1}\pi_1(X) \). In addition, \( \phi(\mu_{J_0}) \) has order \( p \).

**Outline of the proof.** Except the nontriviality of \( \phi(\mu_{J_1}) \), the assertions in Lemma 3.2 are shown straightforwardly. Indeed, reverse induction on \( k = n - 1, n - 2, \ldots, 0 \) shows that \( \mu_{J_1} \) lies in \( \pi_1(X)^{(n-k)} \subset \mathcal{P}^{n-k}\pi_1(X) \); use that \( \mu_{J_1} \) is parallel to the satellite curve \( \eta_k \) which lies in \( \pi_1(M(P_k))^{(1)} \), and that the meridian of \( P_k \), which normally generates \( \pi_1(M(P_k)) \), is homotopic to \( \mu_{J_{11}} \) in \( X \). Also, the order assertion in Lemma 3.2 follows from the nontriviality, since \( \mathcal{P}^n\pi_1(X)/\mathcal{P}^{n+1}\pi_1(X) \) is isomorphic to a direct sum of (possibly infinitely many) copies of \( \mathbb{Z}_p \), by the definition of the mixed coefficient commutator series.

The nontriviality of \( \phi(\mu_{J_1}) \) in Lemma 3.2 is proven by exactly the same argument as the proof of Theorem 4.14 in [Cha14], which is given in Section 5 of [Cha14]. So, instead of presenting full details, we will discuss the key difference in our case, focusing on the role of the hypothesis \( \mathcal{P} = \langle \alpha_D \rangle \) in Case 1.

Theorem 4.14 in [Cha14] gives the desired nontriviality for another 4-manifold, denoted by \( W_0 \) in [Cha14], instead of our \( X \). The manifold \( W_0 \) in [Cha14] is constructed in the exactly same way as \( X \), but using a different knot (indeed the stevedore knot) in place...
of our \( R(U,D) \), which is the pattern used to produce \( K_1 \) from the companion \( J^1_{n-1} \). This different choice in \cite{Cha14} automatically provides the property that the satellite curve used to produce \( K_1 \), which is the analogue of \( \alpha_j \) in Figure 2 in our case, does not lie in the kernel \( P \) of the homomorphism \( H_1(M(K_1);\mathbb{Q}[t^\pm 1]) \to H_1(X^0;\mathbb{Q}[t^\pm 1]) \). (This property is used in lines 1–5 on page 4801 in \cite{Cha14}.) In our case, \( \alpha_j \notin P \) is guaranteed by the hypothesis \( P = \langle \alpha_D \rangle \) of Case 1. This enables us to carry out all the arguments as in the proof of Theorem 4.14 given in \cite{Cha14} Section 5], to prove the nontriviality of \( \phi(\mu_{1,2}) \).

\[ \Box \]

3.2. Estimating Cheeger-Gromov invariants

Now we estimate the Cheeger-Gromov invariant over the mixed-type commutator series quotient \( \pi_1(X)/\mathcal{P}^{n+1}\pi_1(X) \). For the reader’s convenience, we describe some known key results on the Cheeger-Gromov invariants employed in our argument.

\( L^2 \)-signature defect interpretation. Suppose \( M \) is a 3-manifold bounding a 4-manifold \( W \) and \( \phi: \pi_1(M) \to G \) is a homomorphism factoring through \( \pi_1(W) \). Then the Cheeger-Gromov invariant \( \rho^{(2)}(M, \phi) \) is equal to the \( L^2 \)-signature defect of \( W \) over \( G \), which we denote by \( \bar{\sigma}^{(2)}_G(W) \). That is,

\[ (3.2) \quad \rho^{(2)}(M, \phi) = \bar{\sigma}^{(2)}_G(W) := \text{sign}_G^{(2)}(W) - \text{sign}(W) \]

where \( \text{sign}_G^{(2)}(W) \) is the \( L^2 \)-signature of the intersection form

\[ H_2(W;NG) \times H_2(W;NG) \to NG \]

with the group von Neumann algebra \( NG \) as coefficients, and \( \text{sign}(W) \) is the ordinary signature of \( W \). For more about this, see, for instance, \cite{CW03}, \cite{CT07} Section 2], \cite{Har08} Section 3], \cite{Cha16} Section 2.1].

Quantitative universal bound (a special case). If \( K \) has a planar diagram with \( c \) crossings, then for every homomorphism \( \phi \) of \( \pi_1(M(K)) \),

\[ (3.3) \quad |\rho^{(2)}(M(K), \phi)| \leq 69713280 \cdot c. \]

This is a special case of \cite{Cha16} Theorem 1.9].

Amenable signature theorem (a special case). Suppose \( W \) is an integral \((n,5)\)-solution bounded by \( M(J) \), \( G \) is a locally \( p \)-indicable amenable group satisfying \( G^{(n+1)} = \{1\} \), and \( \phi: \pi_1(M(J)) \to G \) is a homomorphism which factors through \( \pi_1(W) \) and takes a meridian to an infinite order element. Then

\[ (3.4) \quad \rho^{(2)}(M(J), \phi) = \bar{\sigma}^{(2)}_G(W) = 0. \]

This is a special case of \cite{Cha14} Theorem 3.2], whose proof relies on \cite{CO12}. (See also Remark 3.1.) We remark that the amenable signature theorem is a generalization of a major result in \cite{COT03}.

Also, the following explicit computation is useful for our purpose.

Computation for knots over a finite cyclic group. If \( \phi: \pi_1(M(J)) \to G \) is a homomorphism with finite cyclic image of order \( d \), then

\[ (3.5) \quad \rho^{(2)}(M(J), \phi) = \rho(J, \mathbb{Z}_d) := \frac{1}{d} \sum_{k=0}^{d-1} \sigma_J(e^{2\pi i k / d}). \]

A proof can be found in \cite{Fri05} Corollary 4.3], \cite{CO12} Lemma 8.7].
Returning to our case, let \( \phi : \pi_1(X) \to G := \pi_1(X)/\mathbb{P}^{n+1}\pi_1(X) \) be the projection. For brevity, for a subspace \( A \) of \( X \), denote the restriction of \( \phi \) on \( \pi_1(A) \) by \( \phi \). By applying the \( L^2 \)-signature defect interpretation \cite{3.2} and the Novikov additivity to the 4-manifold \( X \), we obtain the following:

\[
\rho^{(2)}(M(J^1_0), \phi) + \sum_{k=0}^{n-2} \rho^{(2)}(M(P_k), \phi) + \rho^{(2)}(M(R(U, D)), \phi)
\]

\[
\quad = \sigma^{(2)}_G(X) = \sigma^{(2)}_G(V^0) + \sum_{k=0}^{n-1} \sigma^{(2)}_G(E_k) + \sum_{i,j} \sigma^{(2)}_G(Z^0_{i,j}),
\]

where \( Z^0_{i,j} \) designates the \( j \)th copy of \( Z^0_i \) used in the construction of \( X \). (We use this notation just because each copy of \( Z^0_i \) may contribute different \( L^2 \)-signature defect.)

Since \( \phi \) restricted on \( \pi_1(M(J^1_0)) \) is onto a subgroup isomorphic to \( \mathbb{Z}_p \) (see Lemma 3.2),

\[
\rho^{(2)}(M(J^1_0), \phi) = \rho(J^1_0, \mathbb{Z}_p)
\]

by \cite{3.5}. By the quantitative universal bound \cite{3.3},

\[
|\rho^{(2)}(M(P_k), \phi)| \leq 6 \cdot 69\,713\,280
\]

since the stevedore knot \( P_k \) has 6 crossings. Similarly, by \cite{3.3},

\[
|\rho^{(2)}(M(R(U, D)), \phi)| \leq 96 \cdot 69\,713\,280
\]

since \( R(U, D) \) has a diagram with 96 crossings. Recall that \( V^0 \) is an integral \((n+1)\)-solution, \( G \) is locally \( p \)-indicable and \( G^{(n+1)} = \{1\} \). Since the meridian of \( K \) generates \( H_1(X) \cong \mathbb{Z} \) onto which \( G \) surjects, the meridian of \( K \) has infinite order in \( G \). So we have

\[
\sigma^{(2)}_G(V^0) = \rho^{(2)}(M(K), \phi) = 0
\]

by the amenable signature theorem \cite{3.4}.

Recall that \( Z^0_{i,j} = Z^{-1}_{i,j} \# (b_2(Z^{-1}_{i,j})\mathbb{C}P^2) \), where \( Z^{-1}_{i,j} \) has been assumed to be an arbitrary \( n \)-negaton for \( \pm K_i \). To control \( \sigma^{(2)}_G(Z^0_{i,j}) \), we use a specific choice of \( Z^0_{i,j} \) described below.

**Lemma 3.3.** Under our assumption that \( J^1_0 \) is a 0-negative knot, there exists an \( n \)-negaton \( Z^0_{i,j} \) for \( \pm K_i \) satisfying that \( \sigma^{(2)}_G(Z^0_{i,j}) \) is equal to either zero or \( \rho(J^1_0, \mathbb{Z}_p) \).

**Proof.** Recall from the definition that

\[
K_i = R(P_{n-2}(\eta_0, \ldots, P_0(\eta_0, J^1_0) \cdots, D), U, V, W, Z).
\]

Let \( Q \) be the knot obtained by replacing \( J^1_0 \) in this expression by the trivial knot \( U \). Since \( U \) and all the \( P_k \) are slice, \( Q \) is slice. We can view \( \eta_0 \) as a curve in \( S^3 \setminus Q \), and \( K_i \) can be written as \( K_i = Q(\eta_0, J^1_0) \). Since \( [\eta_0] \in \pi_1(S^3 \setminus P_k)^{(1)} \) for each \( k \), \( [\eta_0] \in \pi_1(S^3 \setminus Q)^{(n)} \) by induction.

Choose a slice disk exterior, say \( N \), for the slice knot \( Q \), and choose a 0-negaton, say \( N^- \), for the 0-negative knot \( J^1_0 \). Now, as our \( Z^{-1}_{i,j} \), take the union of \( N \) and \( N^- \), with a tubular neighborhood of \( \eta_0 \subset M(Q) = \partial N \) and the solid torus \( M(J^1_0) \setminus \partial \eta_0 \subset \partial N^- \) identified. In \( Z^{-1}_{i,j} \), \( \eta_0 \) is isotopic to a meridian of \( J^1_0 \), which normally generates \( \pi_1(N^-) \) by Definition 2.1(1) since \( N^- \) is a 0-negaton. Since \( [\eta_0] \in \pi_1(S^3 \setminus Q)^{(n)} \), \( \pi_1(N^-) \) maps to \( \pi_1(Z^{-1}_{i,j})^{(n)} \), and from this it follows that \( Z^{-1}_{i,j} \) is an \( n \)-negaton.

To obtain the integral \( n \)-solution \( Z^0_{i,j} \), first let \( N^0 \) be the connected sum of \( N^- \) with \( b_2(N^-) \) copies of \( \mathbb{C}P^2 \). Then \( N^0 \) is an integral \( 0 \)-solution for \( J^1_0 \), and \( Z^0_{i,j} = N \sqcup S^1 \times \mathbb{D}^2 \). \( N^0 \). The following additivity is known (e.g., see \cite[Proposition 3.2]{COT04}, \cite[Proposition 4.4]{Cha14} and their proofs): \( \sigma^{(2)}_G(Z^0_{i,j}) = \sigma^{(2)}_G(N) + \sigma^{(2)}_G(N^0) \). By the amenable
signature theorem \((5.4)\), \(\sigma_G^{(2)}(N) = 0\) since \(N\) is a slice disk exterior. By the \(L^2\)-signature defect interpretation \((5.2)\), \(\sigma_G^{(2)}(N^0) = \rho^{(2)}(M(J_0), \phi)\) since \(\partial N^0 = M(J_0)\). Since \([\eta_0] \in \pi_1(S^3 \setminus Q)^{(n)}\), \(\pi_1(M(J_0))\) maps into \(\pi_1(X)^{(n)} \subset \mathcal{P}^{n} \pi_1(X)\). So the conclusion follows by applying \((3.5)\).

By the property \([J3]\) in Section 2.2 we have \(\rho(J_0^i, \mathbb{Z}_p) = 0\) for \(i > 1\). By Lemma 3.3 it follows that
\[(3.11) \quad \sigma_G^{(2)}(Z_{i,j}^0) = 0.\]
for all \(i > 1\) and for all \(j\).

Finally, \(\sigma_C^{(2)}(C_0) = 0\) and \(\sigma_C^{(2)}(E_k) = 0\) for each \(k\), by \([\text{CHL09}]\) Lemma 2.4. From this and \((3.6)\), \((3.7)\), \((3.8)\), \((3.9)\), \((3.10)\) and \((3.11)\), we obtain the following:
\[r \cdot |\rho(J_0^i, \mathbb{Z}_p)| \leq \sum_{k=0}^{n-2} \rho^{(2)}(M(P_k), \phi) + |\rho^{(2)}(M(R(U, D)), \phi)| \leq 69713280 \cdot (6n + 90)\]
where \(r \geq 1\) is an integer. But it contradicts the property \([J2]\) of \(J_0^i\). This completes the proof for Case 1.

4. Realization of signature functions by negative knots

In the arguments in Section 3, the properties \([J1]\), \([J2]\) and \([J3]\) stated in Section 2.2 were among the essential ingredients used to realize a nontrivial value of the amenable \(\rho\)-invariant obstruction. In this section we describe a construction of an infinite sequence of knots \(J_0^i\) with primes \(p_i\) \((i = 1, 2, \ldots)\) satisfying \([J1]\), \([J2]\) and \([J3]\) For the reader's convenience, we recall them below.

(J1) For each \(i\), \(J_0^i\) is 0-negative.

(J2) For each \(i\), \(|\rho(J_0^i, \mathbb{Z}_p)| > 69713280 \cdot (6n + 90)\).

(J3) For \(i < j\), \(\rho(J_0^i, \mathbb{Z}_p) = 0\).

Here, \(\rho(J, \mathbb{Z}_q)\) is the average of the values of the Levine-Tristram signature function of \(J\) at the \(d\)th roots of unity (see Equation \((3.5)\)).

We begin with a realization of an arbitrary Alexander polynomial by a 0-negative knot.

Lemma 4.1. For every Alexander polynomial \(\Delta(t)\) over \(\mathbb{Z}\), there is a 0-negative knot whose Alexander polynomial is equal to \(\Delta(t)\) up to multiplication by \(\pm t^k\).

Proof. Write the given Alexander polynomial as
\[\Delta(t) = a_2(t^9 + t^{-9}) + \ldots + a_1(t + t^{-1}) + a_0,\]
with \(a_i \in \mathbb{Z}, a_0 + \sum_{i=1}^{d} 2a_i = \Delta(1) = -1\). Invoke a classical realization method of Levine \([\text{Lev66}]\) as follows. Perform \(-1\) surgery along the unknotted circle \(\alpha\) in Figure 7 and regard the other unknotted circle as a knot \(K\) in the result of surgery, which is \(S^3\). Then \(K\) has Alexander polynomial \(\Delta(t)\), due to \([\text{Lev66}]\) Proof of Theorem 2].

Now, regard Figure 7 as a 2-component link \(\alpha \cup K\) in \(S^3\), and let \(D\) be the standard slicing disk in \(D^4\) for the unknotted component \(K\). Attach a 2-handle to the exterior of \(D \subset D^4\) along the \(-1\) framing of \(\alpha\), and call the result \(V\). After the \(-1\) surgery, the zero framing on \(K\) in \(S^3\) remains the zero framing, since \(\text{lk}(K, \alpha) = 0\). So \(\partial V = M(K)\). Also, since \(\pi_1(D^4 \setminus D) = \mathbb{Z}\) and \(\text{lk}(K, \alpha) = 0\), we have \(\pi_1(V) = \pi_1(D^4 \setminus D)/(\alpha) \cong \mathbb{Z}\). Choose a surface in the exterior of \(D \subset D^4\) bounded by \(\alpha\), and take the union with the surgery core disk. This gives us a closed surface in \(V\), say \(S\), which generates \(H_2(V) \cong \mathbb{Z}\). Since the surgery framing is \(-1\), the self intersection of \(S\) is \(-1\). It follows that \(V\) is a 0-negaton bounded by \(M(K)\). \(\square\)
Also, we will use the following result of the first named author and Livingston [CL04].

**Lemma 4.2 (CL04 Proof of Theorem 1).** Suppose $0 < \theta_0 < \pi$ and $\epsilon > 0$. If $\frac{\pi}{b}$ is a rational number sufficiently close to $\cos \theta_0$ with sufficiently large $b > 0$, then for some $\theta_1 \in (\theta_0 - \epsilon, \theta_0 + \epsilon)$, every knot $K$ with Alexander polynomial

$$\Delta_K(t) = bt^2 - (2b + 2a)t + (4a + 2b) - 1 - (2b + 2a)t^{-1} + bt^{-2}$$

has the property that $\sigma_K(e^{i\sqrt{-1}}) = 0$ for $0 \leq \theta < \theta_1$, and $|\sigma_K(e^{i\sqrt{-1}})| \leq 2$ for $\theta_1 \leq \theta \leq \pi$.

Now, choose increasing odd primes $p_1 < p_2 < \cdots$. For each $i = 1, 2, \ldots$, apply Lemma 4.2 and then Lemma 4.1 to choose a 0-negative knot $J_0$ and a real number $d_i \in (\pi - \frac{\pi}{p_i - 1}, \pi - \frac{\pi}{p_i})$ such that $|\sigma_{J_i}(e^{i\theta})| \geq 2$ for $d_i < \theta \leq \pi$, and $|\sigma_{J_i}(e^{i\theta})| = 0$ for $0 \leq \theta < d_i$. (Here, for brevity, let $p_0 = 2$ so that $\pi - \frac{\pi}{p_i}$ is understood as $\frac{\pi}{2}$ for $i = 1$.) Since $\sigma_{J_i}(\omega) = \sigma_{J_i}(\bar{\omega})$, it follows that

\begin{equation}
\rho(J_0, \mathbb{Z}_{p_i}) \geq \frac{1}{p_i} |\sigma_{J_i}(e^{(\pi - \frac{\pi}{p_i})\sqrt{-1}}) + \sigma_{J_i}(e^{(\pi + \frac{\pi}{p_i})\sqrt{-1}})| \geq \frac{4}{p_i}.
\end{equation}

Furthermore, for $i < j$,

$$\rho(J_0, \mathbb{Z}_{p_i}) = \frac{2^{k-1}}{p_i} \sum_{k=0}^{p_j-1} \sigma_{J_i}(e^{2\pi k\sqrt{-1}/p_i}) = 0$$

since $\frac{2\pi k}{p_i} \leq \pi = \frac{\pi}{p_{j-1}} < d_j$ for $k = 0, \ldots, \frac{p_{j-1}}{2}$. It follows that (J1) and (J3) are satisfied. Finally, replace each $J_i$ with the connected sum of $N_i$ copies of $J_0$ for some $N_i > \frac{p_i}{2} \cdot 69713280 \cdot (6n + 90)$. Then, from (4.1), the property (J2) follows too.

**Remark 4.3.** The above argument works for any given increasing sequence $\{p_i\}$ of odd positive integers, without assuming that $p_i$ is prime.

### 5. Case 2: Use of Heegaard-Floer $d$-invariants

Recall that Case 2 assumes that there is an $n$-negaton $X^-$ bounded by $M(K_1)$ for which the kernel $P$ of $H_1(M(K_1); \mathbb{Q}[t^{\pm 1}]) \to H_1(X^-; \mathbb{Q}[t^{\pm 1}])$ is equal to the subgroup $(\alpha_J)$. We will reach a contradiction under a weaker hypothesis that there is a 1-negaton $X^-$ satisfying $P = (\alpha_J)$.

Recall $K_1 = R(J_1 \cup D)$. First, we claim that $J_1 \cup D$ may be replaced by the trivial knot, that is, we may assume that the knot $K_0 := R(\cup D)$ admits a 1-negaton with the same kernel property. This is due to [CHH13 Lemma 8.2], which essentially gives the following general statement:
Lemma 5.1 ([CHH13 Lemma 8.2]). Suppose $K_1 = K_0(\alpha, J)$ where $(K_0, \alpha)$ is a winding number zero pattern. Note that the Alexander modules $H_1(M(K_0); \mathbb{Q}[t^{\pm 1}])$ and $H_1(M(K_1); \mathbb{Q}[t^{\pm 1}])$ are isomorphic, via the standard degree one map associated to a satellite construction. If $J$ is unknoted by changing only positive crossings, then for every 1-negaton $X_1$ bounded by $M(K_1)$, there is an 1-negaton $X_0$ bounded by $M(K_0)$ such that the two maps $H_1(M(K_1); \mathbb{Q}[t^{\pm 1}]) \rightarrow H_1(X_i; \mathbb{Q}[t^{\pm 1}])$, $i = 0, 1$, have the same kernel under the identification of the Alexander modules.

In our case, $K_1 = R(J_n^1, D) = K_0(\alpha, J_n^1)$ and $J_n^1$ is unknoted by changing one positive crossing (the topmost crossing in Figure 3). Thus the claim follows by applying Lemma 5.1. Note that we use the assumption $n \geq 2$.

In what follows, we assume that $X^-$ is a 1-negaton bounded by $M(K_0)$ such that $P := \text{Ker}(H_1(M(K_0); \mathbb{Q}[t^{\pm 1}]) \rightarrow H_1(X^-; \mathbb{Q}[t^{\pm 1}]))$ is equal to $\langle \alpha, J \rangle$, where $K_0 = R(U, D)$ as above.

5.1. Metabolizer of finite cyclic branched covers and $d$-invariants

For a positive integer $m$, let $\Sigma_m$ be the $m$-fold cyclic cover of $S^3$ branched along $K_0$. Let $X_m$ be the 4-manifold obtained by attaching a 2-handle, to the $m$-fold cyclic cover of the given $X^-$, along the pre-image of the zero framed meridian of $K_0$. We have $\partial X_m = \Sigma_m$.

In case of $\partial X_1 = S^0$ and the core of the 2-handle is a slice disk in $X_1$ bounded by $K_0$. (Indeed, our $X_1$ is the 4-manifold used to give an alternative definition of negative knots in [CHH13, Definition 2.2].) The manifold $X_m$ is the $m$-fold branched cyclic cover of $X_1$ along the slice disk.

To state the $d$-invariant obstruction of [CHH13], we use the following notations. For a rational homology 3-sphere $Y$ and a spin$^c$ structure $t$ on $Y$, let $d(Y, t)$ be the associated correction term invariant of Ozsváth and Szabó [OS03a]. When $Y$ is a $\mathbb{Z}_2$-homology sphere (for instance it is the case for $Y = \Sigma_m$ if $m$ is an odd prime power), denote by $s_Y$ the spin$^c$ structure induced by the unique spin structure on $Y$. Every spin$^c$ structure on $Y$ is of the form $s_Y + c$ for some $c \in H^2(Y)$, where $+ \text{designates the action of } H^2(Y)$ on the spin$^c$ structures. A subgroup $G$ in $H_1(Y)$ is called a metabolizer if $G = G^\perp := \{ x \in H_1(Y) \mid \lambda(x, G) = 0 \}$, where $\lambda: H_1(Y) \times H_1(Y) \rightarrow \mathbb{Q}/\mathbb{Z}$ is the linking form.

Correction term $d$-invariant obstruction ([CHH13 Theorem 6.5]). If $X^-$ is a 1-negaton bounded by $M(K_0)$ and $m$ is an odd prime power, then $G := \text{Ker}(H_1(\Sigma_m) \rightarrow H_1(X_m))$ is a metabolizer, and

\begin{equation}
\label{eq:5.1}
d(\Sigma_m, s_{\Sigma_m} + \hat{x}) \geq 0
\end{equation}

for the Poincaré dual $\hat{x} \in H^2(\Sigma_m)$ of every $x \in G$.

Understanding the metabolizer in the $d$-invariant obstruction is essential for our purpose. We will relate the above metabolizer $G$ of the $m$-fold branched cover to the metabolizer $P \subset H_1(M(K_0); \mathbb{Z}[t^{\pm 1}])$ of the infinite cyclic cover. Recall that $H_1(\Sigma_m)$ is isomorphic to $H_1(M(K_0); \mathbb{Z}[t^{\pm 1}])/(t^m - 1)$ (for instance see [Mil68], or use a Wang sequence argument). Let $x_1, x_2 \in H_1(\Sigma_m)$ be the images of $[\alpha], [\alpha_2] \in H_1(M(K_0); \mathbb{Z}[t^{\pm 1}])$ respectively. First, we claim that a metabolizer in $H_1(\Sigma_m)$ is either $\langle x_1 \rangle$ or $\langle x_2 \rangle$. In fact, from the previous computation of $H_1(M(K_0); \mathbb{Z}[t^{\pm 1}])$ and the Blanchfield form in Section 2, it follows that each of $x_1, x_2$ generates a cyclic subgroup $\langle x_i \rangle \subset H_1(\Sigma_m)$ of order $2^m - 1$. $H_1(\Sigma_m) = \langle x_1 \rangle \oplus \langle x_2 \rangle$, and the linking form satisfies $\lambda(x_1, x_1) = \lambda(x_2, x_2) = 0$ and $\lambda(x_1, x_2) \neq 0$. (A computation confirming this can also be found in [GL92 Proposition 2].) The claim follows from this.

Lemma 5.2. Under the hypothesis of Case 2 that $P = \langle \alpha, J \rangle$, $G = \text{Ker}(H_1(\Sigma_m) \rightarrow H_1(X_m))$ is equal to $\langle x_1 \rangle$ for every sufficiently large prime $m$. 
We do not know any estimate for how large \( m \) should be. In our argument below, it depends on the \( 1 \)-negaton \( X^- \). This is the reason that we need to consider an infinite family of the \( d \)-invariants.

**Proof of Lemma 5.2.** Consider the following commutative diagram:

\[
\begin{array}{ccc}
H_1(M(K_0); \mathbb{Q}[t^{\pm 1}]) & \rightarrow & H_1(M(K_0); \mathbb{Z}[t^{\pm 1}]) \\
\downarrow & & \downarrow \\
H_1(X^-; \mathbb{Q}[t^{\pm 1}]) & \rightarrow & H_1(X^-; \mathbb{Z}[t^{\pm 1}]) \\
\end{array}
\]

Here, the vertical maps are induced by inclusions, the left horizontal maps are tensoring by \( \mathbb{Q} \), and the right horizontal maps are the quotient maps. The left and right vertical maps have kernels \( P \) and \( G \) respectively.

The image of \( [\alpha_j] \in H_1(M(K_0); \mathbb{Z}[t^{\pm 1}]) \) in \( H_1(X^-; \mathbb{Z}[t^{\pm 1}]) \) has finite order, say \( a \), since \( [\alpha_j] \) is sent to \( 0 \) in \( H_1(X^-; \mathbb{Q}[t^{\pm 1}]) \). Choose an odd prime \( m \) not smaller than each prime factor of \( a \). We need the following elementary fact.

**Lemma 5.3.** If \( p \) and \( m \) are primes and \( p \leq m \), then \( p \) and \( 2^m - 1 \) are coprime.

**Proof.** If \( p = 2 \), the conclusion is straightforward. Suppose \( p \) is odd and \( p \mid 2^m - 1 \). Let \( d \) be the multiplicative order of \( 2 \) in \( \mathbb{Z}_p^\times \). Since \( 2^m \equiv 1 \pmod{p} \), \( d \) divides \( m \). Since \( m \) is a prime, \( d = m \). Since \( 2^m - 1 \equiv 1 \pmod{p} \) by Fermat’s little theorem, it follows that \( p - 1 \) is a multiple of \( m \). This contradicts \( p \leq m \). \( \square \)

Returning to the proof of Lemma 5.2, it follows that every prime factor of \( a \) is coprime to \( 2^m - 1 \), by Lemma 5.3. So \( a \) is coprime to \( 2^m - 1 \). Choose \( b \) such that \( ab \equiv 1 \pmod{2^m - 1} \). Then the image of \( [\alpha_j] \in H_1(M(K_0); \mathbb{Z}[t^{\pm 1}]) \) in \( H_1(X^-; \mathbb{Z}[t^{\pm 1}]) \) is equal to \( [x_1] \), since \( [x_1] \) has order \( 2^m - 1 \). Also, by the choice of \( a \), the image of \( ab \cdot [\alpha_j] \) in \( H_1(X^-; \mathbb{Z}[t^{\pm 1}]) \) is zero. It follows that \( [x_1] \) lies in \( G \), and thus \( G = \langle x_1 \rangle \). This completes the proof of Lemma 5.2. \( \square \)

Now, our argument for Case 2 proceeds as follows. For an odd prime \( m \), recall that \( s_{\Sigma_m} \) is the spin\(^c\) structure of the \( \mathbb{Z}_2\)-homology 3-sphere \( \Sigma_m \) induced by the unique spin structure. Theorem 5.4, which is stated below, implies that \( d(\Sigma_m, s_{\Sigma_m} + 2^{m-1} \hat{x}_1) \) is negative for every odd prime \( m \). Since \( G = \langle x_1 \rangle \) for sufficiently large \( m \) by Lemma 5.2, this contradicts the \( d \)-invariant obstruction (5.1). This completes the proof for Case 2, modulo the proof of Theorem 5.4.

### 5.2. Computation of the \( d \)-invariants

The remaining part of this paper is devoted to prove the following:

**Theorem 5.4.** For every odd prime power \( m \), \( d(\Sigma_m, s_{\Sigma_m} + 2^m \hat{x}_1) \leq -\frac{3}{2} \).

We remark that the new contribution is the case of \( m > 3 \); for \( m = 3 \), Theorem 5.4 was shown in the earlier work of Cochran, Harvey and Horn [CHH13]. In addition, part of our proof which is given in Section 6.1 essentially follows the arguments in [CHH13].

Our arguments in Section 6.2, which prove a key lemma (see Lemma 5.6) for general \( m > 3 \), are new and use a different approach.

**Remark 5.5.** In the proof of Theorem 5.4, we will use various standard facts on spin\(^c\) structures, their Chern class, and cohomology classes. We briefly recall them below, for the reader’s convenience, and to fix notation. Let Spin\(^c\)(\( M \)) be the set of spin\(^c\) structures on a manifold \( M \). As in Section 5.1, denote the action of \( x \in H^2(\hat{M}) \) on Spin\(^c\)(\( M \)) by \( t \mapsto t + x \) for \( t \in \text{Spin}^c(\hat{M}) \). When Spin\(^c\)(\( M \)) is nonempty, \( x \mapsto t + x \) is a bijection \( H^2(\hat{M}) \rightarrow \text{Spin}^c(\hat{M}) \) for each \( t \in \text{Spin}^c(\hat{M}) \).
(1) For $t \in \text{Spin}^c(M)$, denote by $c_1(t) \in H^2(M)$ the first Chern class of the associated determinant line bundle. It satisfies $c_1(t + x) = c_1(t) + 2x$. 

(2) For a $\mathbb{Z}_2$-homology 3-sphere $Y$, $c_1 : \text{Spin}^c(Y) \to H^2(Y)$ is bijective since $H^2(Y)$ does not have 2-torsion. For the spin$^c$ structure $\mathfrak{s}_Y$ induced by the unique spin structure of $Y$, $c_1(\mathfrak{s}_Y) = 0$. So, $c_1(\mathfrak{s}) = 0$ if and only if $\mathfrak{s} = \mathfrak{s}_Y$.

(3) Let $X$ be a 4-manifold. Then the image of $c_1 : \text{Spin}^c(X) \to H^2(X)$ is equal to the set of characteristic classes. Here a cohomology class $c \in H^2(X)$ is characteristic if $c(x) \equiv x \cdot x \pmod{2}$ for all $x \in H_2(X)$, where $c(x)$ is the evaluation of $c$ on the homology class $x$ and $\cdot$ is the intersection pairing. This definition is equivalent to that $c(x) \equiv x \cdot x \pmod{2}$ holds for generators $x$ of $H_2(X)$.

(4) In addition, suppose that each component of $\partial X$ is a rational homology 3-spheres. Then for $c \in H^2(X)$, $c^2 \in \mathbb{Q}$ is defined as follows. Let $j : H_2(X) \to H_2(X, \partial X)$ be inclusion-induced. Since $H_1(\partial X)$ is torsion, $nc$ is the Poincaré dual of $j(x)$ for some nonzero $n \in \mathbb{Z}$ and $x \in H_2(X)$. We define $c^2 = (x \cdot x)/n^2$.

To prove Theorem 5.4, we will use a cobordism given in the following lemma. Let

$$Y_m = ((m - 3)L(3, 1))#S^2_{\tau}(D# - T)#S^2_4(-(T)),$$

where $L(3, 1)$ is the lens space, $-T$ is the left handed trefoil knot, and $S^2_{\tau}(K)$ designates the $\tau$-framed surgery manifold of $K \subset S^3$.

**Lemma 5.6.** For every odd prime power $m$, there exists a cobordism $W$ bounded by $(-\Sigma_m) \sqcup Y_m$ and a spin$^c$ structure $t$ on $W$ satisfying the following:

(W1) $W$ is negative definite and $\beta_2(W) = 3m - 3$.

(W2) $c_1(t|_{\partial W}) = (\hat{x}_1, 0) \in H^2(\Sigma_m) \oplus H^2(Y_m) = H^2(\partial W)$ and $c_1(t)^2 = -m$.

**Proof of Theorem 5.4.** Let $W$ and $t$ be those given by Lemma 5.6. We will first prove that $t|_{\Sigma_m} = \mathfrak{s}_{\Sigma_m} + 2^{m-1}\hat{x}_1^2$ and will obtain the desired conclusion by applying the $d$-invariant inequality to $(W, t)$. We have $c_1(t|_{\Sigma_m}) = \hat{x}_1$ and $c_1(t|_{Y_m}) = 0$ by (W2). By Remark 5.5 (1) and (2), and since $x_1$ and therefore $\hat{x}_1$ has order $2m - 1$, we have

$$c_1(\mathfrak{s}_{\Sigma_m} + 2^{m-1}\hat{x}_1^2) = 2^m \cdot \hat{x}_1 = \hat{x}_1 = c_1(t|_{\Sigma_m}).$$

It follows that $t|_{\Sigma_m} = \mathfrak{s}_{\Sigma_m} + 2^{m-1}\hat{x}_1^2$, since $c_1$ is 1-1 as mentioned in Remark 5.5 (2). Since $W$ is negative definite, the $d$-invariant inequality of Ozsváth–Szabó [OS03a, Theorem 9.6] gives

$$d(Y_m, t|_{Y_m}) - d(\Sigma_m, \mathfrak{s}_{\Sigma_m} + 2^{m-1}\hat{x}_1^2) \geq \frac{c_1(t)^2 + \beta_2(W)}{4} = \frac{2m - 3}{4},$$

where the equality follows from Lemma 5.6 (W1) and (W2).

The value of $d(Y_m, t|_{Y_m})$ is computed by using known results, as described below. For brevity, denote $A := S^3_{\tau}(D# - T)$ and $B := S^2_4(-(T))$ temporarily, so that $Y_m = ((m - 3)L(3, 1))#A#B$. Since $c_1(t|_{Y_m}) = 0$, $t$ restricts to a spin$^c$ structure whose first Chern class is trivial on each summand. By Remark 5.5 (2), $t|_{L(3, 1)} = \mathfrak{s}_{L(3, 1)}$ and $t|_A = \mathfrak{s}_A$ since $L(3, 1)$ and $A$ are $\mathbb{Z}_2$-homology spheres. By the recursive $d$-invariant formula for lens spaces given in [OS03a, Proposition 4.8], $d(L(3, 1), \mathfrak{s}_{L(3, 1)}) = \frac{1}{2}$. Due to Cochran, Harvey and Horn [CHH13, p. 2151], $d(A, \mathfrak{s}_A) = -\frac{1}{2}$. Using the formula of Owens and Strle for $L$-space knots [OST12, Theorem 6.1], we have $d(B, t|_B) \leq \frac{3}{4}$. It follows that

$$d(Y_m, t|_{Y_m}) = (m - 3) \cdot d(L(3, 1), \mathfrak{s}_{L(3, 1)}) + d(A, \mathfrak{s}_A) + d(B, t|_B) \leq \frac{2m - 9}{4}.$$

Combining the above, we obtain

$$d(\Sigma_m, \mathfrak{s}_{\Sigma_m} + 2^{m-1}\hat{x}_1^2) \leq \frac{2m - 9}{4} - \frac{2m - 3}{4} = \frac{3}{2}.$$ 

This completes the proof of Theorem 5.4 modulo the proof of Lemma 5.6. \qed
6. The cobordism $W$ from $\Sigma_m$ to $Y_m$

In this section, we describe the construction of $W$ and perform some computation on $W$ to prove Lemma 5.6.

6.1. Construction of $W$

Consider the diagram in Figure 8, which consists of $(2m-2)$ 0-framed curves and additional curves $v_1, \ldots, v_{3m-5}$. The labels $v_{3m-4}, \ldots, v_{4m-6}$ will be explained in the next paragraph. For now, ignore the arrows and labels $x_i$; they will be used in Section 6.2. The 0-framed curves form a standard surgery diagram of the $m$-fold cyclic branched cover $\Sigma_m$. 
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of $K_0$, which is obtained from Figure 2 using the Akbulut-Kirby method [AK80]. For later use, note that $v_{3m-5}$ is (isotopic to) a lift of the curve $\alpha_J$ in Figure 2. So, choosing appropriate basepoints, we may assume that the surjection $H_1(M(K_0); \mathbb{Z}[t^{\pm 1}]) \to H_1(\Sigma_m)$ takes $[\alpha_J]$ to $[v_{3m-5}]$, that is, $[v_{3m-5}]$ is equal to $x_1 \in H_1(\Sigma_m)$ used in Section 3.

Regard $v_1, \ldots, v_{3m-5}$ as curves in $\Sigma_m$. The following observation will be useful: $(-1)$-surgery along $v_1, \ldots, v_{3m-6}$ changes the enclosed crossings (at the cost of framing changes of the 0-framed components), and after the crossing changes, we would be able to isotopize the resulting $2m - 2$ curves into $m - 1$ split 2-component links, if $D$ were trivial. Furthermore, using the fact that the Whitehead double $D$ is unknotted by changing a positive crossing, we could also do additional $(-1)$ surgeries to remove the $D$ boxes. More precisely, for each of the $D$ boxes except the leftmost one, let $v_i (i = 3m-4, \ldots, 4m-6)$ be the curve inside the box shown in Figure 9. (For the rightmost $D$ box, replace the double strands in Figure 9 with a single strand.) Enumerate them from right to left, as indicated by the labels $v_{3m-4}, \ldots, v_{4m-6}$ in Figure 8. The $(-1)$-surgery along $v_i (i = 3m-4, \ldots, 4m-6)$ would eliminate the enclosing $D$ box.

Now, take $\Sigma_m \times [0, 1]$, and attach $(4m - 6)$ 2-handles to $\Sigma_m \times 1$ along the $(-1)$-framing of the curves $v_1, \ldots, v_{4m-6}$, to obtain a 4-manifold which we temporarily call $N$. The promised 4-manifold $W$ will be obtained by attaching $m - 3$ additional 3-handles to $N$. The attaching 2-spheres are described as follows. Regard Figure 8 with each $v_i$ ($-1$)-framed, as a surgery diagram of the upper boundary of $N (= \partial N \setminus \Sigma_m)$. Use the effect of the $(-1)$-surgery discussed above, and perform isotopy, to obtain a simplified surgery diagram of the upper boundary shown in Figure 10. For each of the middle $m - 3$ sublinks with two 3-framed components, perform handle slide of one of the components over the other. This gives the surgery diagram in Figure 11. The 0-framed unknotted circles in this diagram give $(m - 3) S^1 \times S^2$ summands of the upper boundary of $N$. Attach, to $N$, $(m - 3)$ 3-handles along the $S^2$ factors of these summands. The result is our 4-manifold $W$.

![Figure 10](image1.png)

**Figure 10.** The result of $(-1)$-surgery (or blowing down) along the $v_i$.

![Figure 11](image2.png)

**Figure 11.** The result of handle slides.
We need to verify that the upper boundary of $W$ is $Y_m$. Since the 3-handle attachments eliminate the $(m - 3)$ 0-framed unknotted circles, Figure 11 with the 0-framed circles removed is a surgery description of the upper boundary of $W$. By the two (+1)-surgeries in the diagram, it becomes the connected sum of $(-7)$-surgery on $D\#_{1} - T$, $(-6)$-surgery on $-T$, and $m - 3$ copies of $L(3, 1)$. This is the 3-manifold $Y_m$, as desired.

6.2. Definiteness and spin$^c$ structure computation for $W$

This subsection is devoted to the proof of Lemma 5.6. Recall that Lemma 5.6 (W1) asserts that $W$ is negative definite and $b_2(W) = 3m - 3$. Since an explicit handle decomposition of $W$ is given, one may try to compute directly the second homology and intersection form to prove the assertions. Though, after several attempts, we learned that proving the desired definiteness for all $m$ by direct computation was difficult, if feasible, because of the growth of the size and sophistication of the intersection matrix.

In what follows, we present a different approach. The idea is to obtain an “easier” 4-manifold (which we call $\tilde{W}$ below) by attaching another “easier” 4-manifold (which we call $W_0$ below) to $W$, motivated from the above surgery diagram calculus of 3-manifolds. Then we investigate $W$ as the difference of the two easier ones.

Proof of Lemma 5.6 (W1). Let $W_0$ be the 4-manifold with one 0-handle and $(2m - 2)$ 2-handles attached along the 0-framed curves shown in Figure 8. That is, Figure 8 with the $v_i$ forgotten is now viewed as a Kirby diagram of $W_0$. It is straightforward that $\partial W_0 = \Sigma_m$. Let $\bar{W} = W_0 \cup_{\Sigma_m} W$. See the schematic diagram in Figure 12 (for now, ignore the thick arcs and labels $a_i \sigma'_i$, $a_j \sigma'_j$, $z'_i$, $z'_j$ and $a_j v'_j$, which will be used later).

By a Mayer-Vietoris argument, using that $\Sigma_m$ is a rational homology sphere, we obtain

$$b_2(\bar{W}) = b_2(W) + b_2(W_0).$$
Also, by Novikov additivity, the signatures are related as follows:

\[(6.2) \quad \text{sign}(\bar{W}) = \text{sign}(W) + \text{sign}(W_0).\]

First, we compute \(b_2(W_0)\) and \(\text{sign}(W_0)\). Since \(W_0\) consists of \((2m - 2)\) 2-handles and one 0-handle, \(b_2(W_0) = 2m - 2\). By Akbulut-Kirby [AK80], \(W_0\) is the \(m\)-fold cyclic cover of \(D^4\) branched along the surface obtained by pushing into \(D^4\) the interior of the obvious Seifert surface of genus one for \(K_0 = R(U, D)\). Therefore, by a well known fact (see, for instance, [VST73] and [Kor78, Section 12]), \(\text{sign}(W_0)\) is determined by the Levine-Tristram signature function \(\sigma_{K_0}\), as follows:

\[\text{sign}(W_0) = \sum_{k=0}^{m-1} \sigma_{K_0}(e^{2\pi k\sqrt{-1}/m}).\]

Since \(K_0\) is slice, \(\sigma_{K_0}(\omega) = 0\) when \(\omega\) is a root of unity of prime power order. It follows that \(\text{sign}(W_0) = 0\).

To compute \(\beta_2(\bar{W})\) and \(\text{sign}(\bar{W})\), we use an alternative description of \(\bar{W} = W_0 \cup_{\Sigma_m} W\). By definition, \(\bar{W}\) consists of one 0-handle, \((2m - 2)\) 2-handles attached along the 0-framed curves in Figure 8, \((4m - 6)\) 2-handles attached along the \((-1)\)-framing of the curves \(v_1, \ldots, v_{m-6}\) in Figure 8, and additional \((m - 3)\) 3-handles. Blow down \(\bar{W}\) \((4m - 6)\) times, to realize the effect of the \((-1)\) surgery. This transforms Figure 8 (with each \(v_i\) \((-1)\)-framed) to Figure 10. That is, we have

\[(6.3) \quad \bar{W} = W' \# ((4m - 6)\mathbb{CP}^2)\]

where the result \(W'\) of blowing down is given by Figure 10 viewed as a 4-manifold Kirby diagram now, with additional \((m - 3)\) 3-handles. By handle slides and cancellations of 3-handles with 2-handles, it follows that Figure 11 with the \((m - 3)\) 0-framed circles removed is a Kirby diagram for \(W'\) (without 3-handles).

Now, since our final Kirby diagram consists of \((m + 1)\) 2-handles without any 1- and 3-handles, \(b_2(W') = m + 1\). Also, the intersection matrix for \(W'\) is the direct sum of \([\frac{1}{2} \frac{3}{2}], [\frac{3}{2} \frac{1}{2}]\) and \((m - 3)\) copies of the \([1 \ 1]\) matrix \([3]\). Since the two \(2 \times 2\) submatrices have vanishing signatures, \(\text{sign}(W') = m - 3\). From this and (6.3), it follows that \(b_2(\bar{W}) = 5m - 5\) and \(\text{sign}(\bar{W}) = -3m + 3\).

By substituting the above values into (6.1) and (6.2), it follows that \(b_2(W) = 3m - 3\) and \(\text{sign}(W) = -3m + 3\). This completes the proof of Lemma 5.6 (W1). 

In the proof of Lemma 5.6 [W2], the following lemma is essential. Recall that \(W\) has \((4m - 6)\) 2-handles attached to \(\Sigma_m \times [0, 1]\) along the \((-1)\)-framed curves \(v_1, \ldots, v_1\) (see Figures 8 and 12). For \(1 \leq i \leq 4m - 6\), let \(\sigma_i\) be the union of \(v_i \times [0, 1] \subset \Sigma_m \times [0, 1]\) and the core of the 2-handle attached to \(v_i \times 1\). Orient \(v_i\) in Figure 8 counterclockwise, and orient the 2-disk \(\sigma_i\) in such a way that \(\partial \sigma_i = v_i \times 0\). That is, homologically, the image of the relative class \(\sigma_i\) under the boundary map

\[\partial: H_2(W, \partial W) \longrightarrow H_1(\partial W) = H_1(\Sigma_m) \oplus H_1(Y_m)\]

is \(([v_i], 0)\). Let \(w = [\sigma_{3m-5}] + \cdots + [\sigma_{4m-6}] \in H_2(W, \partial W)\).

**Lemma 6.1.** The Poincaré dual \(\hat{w} \in H^2(W)\) is characteristic and satisfies \(\hat{w}^2 = -m\).

**Proof of Lemma 6.1 [W2].** Since \(\hat{w}\) in Lemma 6.1 is characteristic, there is a spin" structure \(t\) on \(W\) such that \(c_1(t) = \hat{w}\) by Remark 5.5 (3). We will verify that \(t\) satisfies the desired properties \(c_1(t)^2 = -m\) and \(c_1(t|_{\partial W}) = (x_1, 0) \in H^2(\partial W) = H^2(\Sigma_m) \oplus H^2(Y_m)\).

First, \(c_1(t)^2 = \hat{w}^2 = -m\). By naturality, we have

\[c_1(t|_{\partial W}) = c_1(t)|_{\partial W} = \hat{w}|_{\partial W} = \hat{w}\]
where \( \hat{w} \in H^2(\partial W) \) is the Poincaré dual of \( \partial w \in H_1(\partial W) \). By the first paragraph of Section 6.1, \( [\sigma_{2m-5}] = x_1 \) and thus \( \partial [\sigma_{2m-5}] = (x_1, 0) \). For \( i \geq 3m - 4 \), we have \( \partial [\sigma_i] = [(v_i, 0)] = 0 \), since \( v_i \) has linking number zero with other surgery curves in Figure 9. It follows that \( \partial w = (x_1, 0) \). Therefore \( c_1(t|_{\partial W}) = (\tilde{x}_1, 0) \). \( \square \)

The remaining part of this section is devoted to proving Lemma 6.1, which is largely computations of intersection data in terms of rational linking numbers.

**Proof of Lemma 6.1.** Let \( w_0 = (2^m - 1)w \in H_2(W, \partial W) \). Since \( 2^m - 1 \) is odd, \( \hat{w} \) is characteristic if and only if so is \( \tilde{w}_2 \), by the mod 2 defining property of a characteristic class in Remark 5.5(3). Also, \( \tilde{w}_0^2 = (2^m - 1)\hat{w}^2 \), by the definition of \((-1)^2\) in Remark 5.5(4). So, Lemma 6.1 is equivalent to that \( \tilde{w}_0 \) is characteristic and \( \tilde{w}_0^2 = -(2^m - 1)^2m \). We will prove this.

First, we construct a (non-relative) cycle representative \( E_0 \) of the class \( w_0 \), that is, \( j[E_0] = w_0 \) where \( j : H_2(W) \to H_2(W, \partial W) \) is the inclusion-induced map. Let

\[
\alpha_i = \begin{cases} 2^m - 1 & \text{for } i = 1, \ldots, 3m - 5, \\ 1 & \text{for } i = 3m - 4, \ldots, 4m - 6. \end{cases}
\]

Then \( a_i v_i \) is null-homologous in \( \Sigma_m \) for all \( i \). Indeed, for \( i \leq 3m - 5 \), it is straightforward since \( H_1(\Sigma_m) \cong \mathbb{Z}_{2^{m-1}} \oplus \mathbb{Z}_{2^{m-1}} \), and for \( i \geq 3m - 4 \), \( v_i \) shown in Figure 9 is null-homologous since \( v_i \) has linking number zero with other surgery curves in Figure 8.

Choose a 2-chain \( z_i \) in \( \Sigma_m \subset W \) whose oriented boundary is \( a_i v_i \). Let \( E_i \) be the 2-cycle \( a_i \sigma_i - z_i \) in \( W \).

Recall that \( W \) is obtained by attaching 2-handles to \( \Sigma_m \times [0, 1] \) along the \( v_i \). It can be seen that the classes \( [E_i] \) (for \( i = 1, \ldots, 4m - 6 \)) generate a subgroup \( \langle [E_i] \rangle \) with odd index in \( H_2(W) \). In fact, since \( H_2(\Sigma_m) = H_1(W) = 0 \), a Mayer-Vietoris argument gives us

\[
0 \to H_2(W) \to \mathbb{Z}^{4m - 6} \to H_1(\Sigma_m) \to 0.
\]

So \( H_2(W) \) is a subgroup of \( \mathbb{Z}^{4m - 6} \) with index equal to \( |H_1(\Sigma_m)| = (2^m - 1)^2 \). By the definition of \( E_i \) and \( \alpha_i \), we have \( \mathbb{Z}^{4m - 6} : \langle [E_i] \rangle = (2^m - 1)^{3m - 5} \). It follows that \( |H_2(W) : \langle [E_i] \rangle| = (2^m - 1)^{3m - 3} \), as claimed.

Let

\[
(6.4) \quad E_0 = E_{3m-5} + (2^m - 1)(E_{3m-4} + \cdots + E_{4m-6}).
\]

Since \( j[E_i] = \alpha_i [\sigma_i] \), we have \( j[E_0] = (2^m - 1)([\sigma_{3m-5}] + \cdots + [\sigma_{4m-6}]) = w_0 \). It follows that \( \tilde{w}_0^2 = E_0 \cdot E_0 \) and \( \tilde{w}_0(E_i) = E_i \cdot E_0 \), where \( \cdot \) denotes the intersection in \( W \). (As usual, the intersection of two chains is computed by taking a pushoff of one of them which is transverse to another.) We will verify that

\[
(6.5) \quad E_0 \cdot E_0 \equiv - (2^m - 1)^2m, \quad E_i \cdot E_0 \equiv E_i \cdot E_i \pmod{2}
\]

for \( i = 1, \ldots, 4m - 6 \). Since \( \langle [E_i] \rangle \) has odd index in \( H_2(W) \), the second identity in (6.5) implies that \( \tilde{w}_0 \) is characteristic. See Remark 5.5(3). So the verification of (6.5) completes the proof of Lemma 6.1.

Recall that for every pair of two disjoint 1-cycles \( (\alpha, \beta) \) in a rational homology 3-sphere \( \Sigma \), the linking number \( \text{lk}_\Sigma(\alpha, \beta) \in \mathbb{Q} \) is defined as follows: if \( u \) is a 2-chain bounded by \( r \alpha \sigma \) in \( \Sigma \), for some nonzero integer \( r \), then \( \text{lk}_\Sigma(\alpha, \beta) = -1/2(\alpha \sigma \beta) \). Note that \( \partial \sigma_i = v_i \times 0 \), and \( z_i \) is a 2-chain in \( \Sigma_m = \Sigma_m \times 0 \) such that \( \partial z_i = a_i v_i \). Fix \( i \) and \( j \) (1 \( \leq i, \leq j \)).
\[ j \leq 4m - 6. \] One can push \( E_i \) and \( E_j \) slightly, to obtain transverse 2-cycles \( E'_i \) and \( E'_j \) depicted as thick arcs in Figure 12 for which we have
\[
E_i \cdot E_j = E'_i \cdot E'_j = z_i \circ (a_j v'_j) = a_i a_j \operatorname{lk}_{\Sigma_m} (v_i, v'_j),
\]
where \( v'_j \) denotes a pushoff of \( v_j \) in \( \Sigma_m \) taken along the \((-1)\)-framing. More details are as follows. Deform \( E_i \) to get a homologous cycle \( E'_i = a_i \sigma'_i - z'_i \), where \( z'_i \) is obtained by pushing \( z_i \subset \Sigma_m = \Sigma_m \times 0 \subset \partial W \) slightly into the interior of \( W \), and \( \sigma'_i \) is obtained by removing a collar of \( \partial \sigma_i = v_i \times 0 \) from \( \sigma_i \). Also deform \( E_j \) to get a homologous cycle \( E'_j = a_j \sigma'_j - z'_j \), where \( \sigma'_j \) is obtained by pushing the 2-disk \( \sigma_j \) along its unique framing in \( W \), and \( z'_j \) is obtained from \( z_j \) by attaching annuli cobounded by \( -\partial z_j \) and \( a_j (\partial \sigma'_j) \). See Figure 12. Note that \( -\partial \sigma'_j \equiv v'_j \times 0 \), since the 2-handles are attached along the \((-1)\)-framing. Now, \( \sigma'_i \cdot \sigma'_j = \sigma'_i \cdot z'_j = z'_i \cdot z'_j = 0 \), and so \( E'_i \cdot E'_j = (z'_i \cdot (a_j \sigma'_j) \) in \( W \). This intersection is equal to the intersection \( -(z_i) \circ (a_j v'_j) \) in \( \Sigma_m = \Sigma_m \times 0 \) (see Figure 12 again), where the additional \(-\) sign is needed since the orientation of \( \Sigma_m \) is opposite to the boundary orientation on \( \partial W \). From this and the definition of \( \operatorname{lk}_{\Sigma_m} \), (6.6) is obtained immediately.

Using (6.4) and (6.6), it follows that (6.5) is equivalent to the following linking number conditions:
\[
\begin{align*}
\sum_{i=3m-5}^{4m-6} \sum_{j=3m-5}^{4m-6} \operatorname{lk}_{\Sigma_m} (v_i, v'_j) &= -m, \\
\sum_{j=3m-5}^{4m-6} a_i a_j \operatorname{lk}_{\Sigma_m} (v_i, v'_j) &\equiv a_i^2 \operatorname{lk}_{\Sigma_m} (v_i, v'_i) \pmod{2} \quad \text{for all } i.
\end{align*}
\]

This reduces the proof to a purely 3-dimensional computation. It is known that the linking number in a rational homology 3-sphere can be explicitly computed by using the linking matrix. (See, for instance, [CK02 Theorem 3.1].) To apply this to our case, let \( L \) be the link consisting of the 0-framed curves in Figure 8 from which \( \Sigma_m \) is obtained by surgery. Orient components of \( L \) along the arrows in Figure 8 and let \( x_i \) be the positively oriented meridian of the \( i \)th component (the one with label \( x_i \) in Figure 8).

Let \( P \) be the linking matrix for \( L \). That is, the \((i, j)\)-entry is the linking number, in \( S^3 \), of the \( i \)th component and a pushoff of the \( j \)th component taken along the framing. Then \( P \) is invertible over \( \mathbb{Q} \) since \( \Sigma_m \) is a rational homology sphere. With respect to the basis \( \{x_i\} \), \( P^{-1} \) gives rise to a well-defined symmetric \( \mathbb{Q} \)-valued \( 2 \times 2 \) bilinear pairing \( R \) that is, define \( R(x_i, x_j) \) to be the \((i, j)\)-entry of \( P^{-1} \), and expand it bilinearly. Then, for disjoint 1-cycles \( \alpha, \beta \) in \( S^3 \setminus L \), we have
\[
\operatorname{lk}_{\Sigma_m} (\alpha, \beta) = \operatorname{lk}_{S^3} (\alpha, \beta) - R(\alpha, \beta).
\]

From Figure 8 it is seen that \( P \) is given as the following matrix, which consists of \((m - 1) \times (m - 1)\) blocks of size \( 2 \times 2 \):
\[
P = \begin{bmatrix}
3A_0 & -A_1 \\
-A_1^T & 3A_0 & -A_1 \\
\vdots & \ddots & \ddots & \ddots & \ddots
\end{bmatrix},
\]

\[
A_0 = \begin{bmatrix}
3 & -1 \\
-1 & 3
\end{bmatrix},
\]
where $A_r := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. The inverse of $P$ is the following symmetric matrix consisting of $(m-1) \times (m-1)$ blocks:

$$P^{-1} = \frac{1}{2^m - 1} \begin{bmatrix} c_1 c_{m-1} A_0 & c_1 c_{m-2} A_1^T & \cdots & c_1 c_1 A_{m-2}^T \\ c_1 c_{m-2} A_1 & c_2 c_{m-2} A_0 & \cdots & c_2 c_1 A_{m-3}^T \\ \vdots & \vdots & \ddots & \vdots \\ c_1 c_1 A_{m-2} & c_2 c_{m-3} A_1 & \cdots & c_m c_1 A_0 \end{bmatrix},$$

where $c_r := 2^r - 1$. That is, for $k \geq \ell$, the $(k, \ell)$-block of $P^{-1}$ is $\frac{c_{\ell} c_{m-k}}{2^{m-1}} A_{k-\ell}$. A straightforward matrix multiplication confirms $P^{-1} P = I$. We omit the details since it is routine. (We remark that the identities $A_0^2 = 2^r I$, $A_{k-\ell+1} A_1 = 2 A_{k-\ell} A_0 = 2 A_{k-\ell-1} A_1^T$ and $A_{k-\ell+1} A_0 = 2 A_{k-\ell-1} A_1$ are useful in the verification of $P^{-1} P = I$, for the diagonal, below-diagonal and above-diagonal entries respectively.)

Now, we are ready to verify (6.7) and (6.8), using (6.9) and (6.10). First we prove (6.7).

Since each $v_i$ is $(1)$-framed and since $v_i$ and $v_j$ are split for $i \neq j$, we have

$$\text{lk}_{S^3}(v_i, v_j) = -\delta_{ij} \quad \text{for all } i \text{ and } j.$$  

If $i \geq 3m - 4$, $R(v_i, v_j) = 0$ for all $j$, since $v_i$ is null-homologous in $S^3 \setminus L$. So, by (6.9) and (6.11), we have

$$\text{lk}_{\Sigma_m}(v_i, v_j) = -\delta_{ij} \quad \text{for } i \geq 3m - 4.$$  

From Figure 8, $v_{3m-5} = x_1$ in $H_1(S^3 \setminus L)$. By the definition, $R(x_1, x_1)$ is equal to the $(1, 1)$-entry of $P^{-1}$ in (6.10), which is zero. Thus, by (6.9) and (6.11), we have

$$\text{lk}_{\Sigma_m}(v_{3m-5}, v'_{3m-5}) = -1.$$  

From this, (6.12) and the fact that $\text{lk}_{\Sigma_m}$ is symmetric, it follows that (6.7) holds.

It remains to verify (6.8). First, for $i \geq 3m - 4$, the left and right hand sides of (6.8) are equal to $1 - 2^m$ and $-1$, respectively, by (6.12). It follows that (6.8) holds in this case. Also, for $i = 3m - 5$, both sides of (6.8) are equal to $-(2^m - 1)^2$, by (6.12) and (6.13). So (6.8) holds in this case.

Suppose $i \leq 3m - 6$. In this case, $a_i = 2^m - 1$. From Figure 8, it is seen that $v_i$, oriented counterclockwise, is always of the form $v_i = x_o + x_e$, in $H_1(S^3 \setminus L)$, with $o$ odd and $e$ even. (In fact, $(o, e)$ is of the form $(2k - 1, 2k + 2)$ or $(2k + 1, 2k)$, depending on the choice of $i$, but we do not need this information.) Using (6.9), (6.11) and $v_{3m-5} = x_1$, we have

$$(2^m - 1) \text{lk}_{\Sigma_m}(v_i, v_{3m-5}) = -(2^m - 1) R(x_o, x_1) - (2^m - 1) R(x_e, x_1).$$

Inspecting (6.10) together with the matrix $A_r$, entries in the first column of the integer matrix $(2^m - 1) P^{-1}$ has alternating parity, starting from zero, since $c_r$ is always odd. That is, $(2^m - 1) R(x_o, x_1)$ is even and $(2^m - 1) R(x_e, x_1)$ is odd. Therefore, from (6.14), it follows that the left hand side of (6.8) is odd.

On the other hand, since the matrix $A_r$ has zero diagonals, $P^{-1}$ in (6.10) does too, and thus $R(x_o, x_o) = R(x_e, x_e) = 0$. So, using (6.9) and (6.11), we have

$$(2^m - 1) \text{lk}_{\Sigma_m}(v_i, v'_j) = -(2^m - 1) - (2^m - 1) R(x_o + x_e, x_o + x_e) \equiv 1 \pmod{2}.$$  

It follows that the right hand side of (6.8) is odd. Hence, (6.8) holds. This completes the proof of Lemma 6.1.
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Center for Research in Topology, POSTECH, Pohang Gyeongbuk 37673, Republic of Korea
School of Mathematics, Korea Institute for Advanced Study, Seoul 02455, Republic of Korea
E-mail address: jccha@postech.ac.kr

Department of Mathematics, Chonnam National University, Gwangju 61186, Republic of Korea
E-mail address: minhoonkim@jnu.ac.kr