Trajectories of semigroups of holomorphic functions and harmonic measure

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Abstract

We give an explicit relation between the slope of the trajectory of a semigroup of holomorphic functions and the harmonic measure of the associated planar domain $\Omega$. We use this to construct a semigroup whose slope is an arbitrary interval in $[-\pi/2, \pi/2]$. The same method is used for the slope of a backward trajectory approaching a super-repulsive fixed point.

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1. Semigroups of Holomorphic Functions

A one-parameter continuous semigroup of holomorphic self-mappings of the unit disk $\mathbb{D}$ is a family $(\phi_t)_{t \in [0, \infty)}$, such that:

(i) $\phi_{t+s} = \phi_t \circ \phi_s$, for all $t, s \in [0, +\infty)$
(ii) $\phi_0(z) = z$
(iii) $\lim_{t \to s} \phi_t(z) = \phi_s(z)$, for all $s \in [0, +\infty)$.

We will simply call $(\phi_t)$ a semigroup. For general reference on semigroups we point to [1], [12] and [16].

A semigroup is called elliptic if it is not a group of hyperbolic rotations and it has an interior fixed point, which must be the same for all $\phi_t$, $t > 0$. If $(\phi_t)$ is a non-elliptic semigroup, then there exists a unique point $\xi \in \partial \mathbb{D}$, called the Denjoy-Wolff point of the semigroup [2], such that

$$\lim_{t \to \infty} \phi_t(z) = \xi, \quad \text{for every } z \in \mathbb{D}. \quad (1)$$
A semigroup with no interior fixed point is called *non-elliptic*. From now on we will only deal with non-elliptic semigroups. An important tool in the study of non-elliptic semigroups is the corresponding Koenigs function, see [1], [12], [16] and the references therein. To every non-elliptic semigroup \((\phi_t)\), corresponds a conformal mapping \(h : \mathbb{D} \to \Omega\) such that:

(i) \(h(\mathbb{D}) = \Omega\), and  
(ii) \(h(\phi_t(z)) = h(z) + t, \ z \in \mathbb{D}, \ t \geq 0\).

The domain \(\Omega\) is called the associated planar domain of \((\phi_t)\). A domain \(\Omega\) is called convex in the positive direction when \(\{z + t : z \in \Omega\} \subset \Omega\), for all \(t \in [0, \infty)\). Obviously the associated planar domain of a semigroup is convex in the positive direction. The converse is also true; for every simply connected domain \(\Omega\) convex in the positive direction, define

\[
\phi_t(z) = h^{-1}(h(z) + t),
\]

where \(h\) is the Riemann map that maps \(\mathbb{D}\) onto \(\Omega\). It is easy to verify that \((\phi_t)\), as defined above, is a semigroup.

We are interested in the boundary fixed points of \(\phi_t\). These are defined using the notion of angular limit. When \(\phi(z) \to w'\) as \(z \to w\) through any sector at \(w\) we say that \(w'\) is the angular limit of \(\phi\) as \(z\) tends to \(w\); we write

\[
\angle \lim_{z \to w} \phi(z) = w'.
\]

A point \(w \in \partial \mathbb{D}\) is called a boundary fixed point of \(\phi\), when \(\angle \lim_{z \to w} \phi(z) = w\). For a boundary fixed point \(w\), we define the angular derivative at \(w\) to be

\[
\phi'(w) = \angle \lim_{z \to w} \frac{w - \phi(z)}{w - z}.
\]

In the case when \(\phi(\mathbb{D}) \subset \mathbb{D}\), we know [14, p.82] that \(\phi'(w)\) always exists and belongs to \((0, +\infty) \cup \{\infty\}\).

Boundary fixed points in this case are divided into three categories; see [8] and references therein.

(i) When \(\phi'(w) \in (0, 1]\), \(w\) is called an attractive point,  
(ii) when \(\phi'(w) \in (1, +\infty)\), \(w\) is called a repulsive point and  
(iii) when \(\phi'(w) = \infty\), \(w\) is called a super-repulsive point.

The Denjoy-Wolff Theorem guarantees that, in the context of semigroups, the Denjoy-Wolff point \(\xi\) in relation (1), is the unique attractive boundary fixed point of \(\phi_t\), for all \(t > 0\).

Non-elliptic semigroups can be categorized according to properties of the associated planar domain \(\Omega\); see e.g. [8]. Namely:

(i) When \(\Omega\) is contained in a horizontal strip, the semigroup is called hyperbolic.
(ii) When $\Omega$ is not contained in a horizontal strip, but it is contained in a horizontal half-plane, the semigroup is called parabolic of positive hyperbolic step.

(iii) When $\Omega$ is not contained in any horizontal half-plane, the semigroup is called parabolic of zero hyperbolic step.

The trajectory of $z \in D$ of a semigroup $(\phi_t)$ is defined as the curve

$$
\gamma_z : [0, +\infty) \rightarrow \mathbb{D}, \quad \gamma_z(t) = \phi_t(z).
$$

By utilizing the associated domain $\Omega$, every trajectory can be extended as follows. Let $T$ be the infimum of $\{t : h(z) + t \in \Omega\}$. The extended trajectory of $z$ is the curve defined by

$$
\gamma_z : (T, +\infty) \rightarrow \mathbb{D}, \quad \gamma_z(t) = h^{-1}(h(z) + t).
$$

From now on $\gamma_z$ will be used for the extended trajectory. In accordance with [8], we will define the $\alpha$ and $\omega$ limits of curves. For every curve $\Gamma : (s_1, s_2) \rightarrow C$, if there exists a strictly increasing sequence $t_n \rightarrow s_2$, such that $\Gamma(t_n) \rightarrow \xi$, then $\xi$ is called an $\omega$-limit point of $\Gamma$. The set of all $\omega$-limit points of $\Gamma$ is called the $\omega$-limit set and denoted by $\omega(\Gamma)$. Replacing $s_2$ with $s_1$ and considering strictly decreasing sequences, we similarly define the $\alpha$-limit point and the $\alpha$-limit set $\alpha(\Gamma)$. From (1) it is obvious that for all $z \in D$ we have $\omega(\gamma_z) = \{\xi\}$, where $\xi$ is the Denjoy-Wolff point. The set $\alpha(\gamma_z)$ is also a single point which can be one of the following [8]:

(i) The point in $\partial D$ that corresponds to $h(z) + T \in \partial \Omega$, when $T > -\infty$.

(ii) A boundary fixed point of $(\phi_t)$, including the Denjoy-Wolff point $\xi$, when $T = -\infty$.

An interesting problem is the study of the slope of $\gamma_z$ as it approaches the boundary of $D$. For every $\gamma_z$, we consider the corresponding curve

$$
t \in (T, +\infty) \rightarrow \text{arg}(1 - \xi \gamma_z(t)) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
$$

The $\omega$-limit set of the above curve will be the set of slopes of $\gamma_z$ as it approaches the Denjoy-Wolff point $\xi$ and it will be denoted by $\text{Slope}^+(\gamma_z)$. If $\alpha(\gamma_z) = \{\chi\}$ then similarly consider the curve

$$
t \in (T, +\infty) \rightarrow \text{arg}(1 - \chi \gamma_z(t)) \in \left( -\frac{\pi}{2}, \frac{\pi}{2} \right).
$$

The $\alpha$-limit set of the above curve will be called the set of slopes of the backward trajectory $\gamma_z$ as it approaches the boundary point $\chi$ and it will be denoted by $\text{Slope}^-(\gamma_z)$. The following is already known about the $\text{Slope}^+(\gamma_z)$:

(i) When a semigroup is hyperbolic, $\text{Slope}^+(\gamma_z)$ is a singleton depending on $z$.

(ii) When a semigroup is parabolic of positive hyperbolic step, $\text{Slope}^+(\gamma_z)$ is either $\{\pi/2\}$ or $\{-\pi/2\}$ and it is independent of $z$. 

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When a semigroup is parabolic of zero hyperbolic step, it was conjectured that $\text{slope}^+(\gamma_z)$ is again a singleton. This was proven but only under some additional assumptions, see e.g. [10] and [11]. The existence of a semigroup with $\text{slope}^+(\gamma_z) = [-\pi/2, \pi/2]$ was first proven in [4] and [9]. In a more recent result, Bracci et al. [5] show that there exists a semigroup such that $\text{slope}^+(\gamma_z) \subset (-\pi/2, \pi/2)$ but it is not a singleton.

Also in [6] we find an example with $\text{slope}^+(\gamma_z) = [-\pi/2, \alpha]$, for some $-\pi/2 < \alpha < \pi/2$.

In [9] the authors posed the problem of constructing examples of one-parameter semigroups $(\phi_t)$ with $\text{slope}^+(\gamma_z) = [\theta_1, \theta_2]$ for any given $\theta_1, \theta_2 \in [-\pi/2, \pi/2]$, $\theta_1 < \theta_2$. We will construct such a semigroup.

**Theorem 1.** If $\theta_1 < \theta_2$ are real numbers with $|\theta_j| \leq \pi/2$, $j = 1, 2$, then there exists a semigroup of holomorphic functions $(\phi_t)$ such that

$$\text{slope}^+(\gamma_z) = [\theta_1, \theta_2].$$

For the $\text{slope}^-(\gamma_z)$ similar results were only known for the following cases [8]:

(i) When the $\alpha$-limit of $\gamma_z$ is the Denjoy-Wolff point $\xi$, $\text{slope}^-(\gamma_z)$ is a singleton, which is either $\{\pi/2\}$ or $\{-\pi/2\}$.

(ii) When the $\alpha$-limit of $\gamma_z$ is a repulsive point, $\text{slope}^-(\gamma_z)$ is a single point, which belongs in $(-\pi/2, \pi/2)$.

We prove that, in the case of super-repulsive points, a semigroup can have a wildly oscillating trajectory, quite similar to the case of a parabolic semigroup of zero hyperbolic step.

**Theorem 2.** If $\theta_1 \leq \theta_2$ are real numbers with $|\theta_j| \leq \pi/2$, $j = 1, 2$, then there exists a semigroup of holomorphic functions $(\phi_t)$ and a point $z \in \mathbb{D}$, such that the $\alpha$-limit of $\gamma_z$ is a super-repulsive point and

$$\text{slope}^-(\gamma_z) = [\theta_1, \theta_2].$$

2. Harmonic measure

To prove the aforementioned results we need to establish a relationship between the slope of a trajectory $\gamma_z$ and certain harmonic measures in the associated planar domain $\Omega$ of a semigroup.

The harmonic measure is the solution $u$ of the generalized Dirichlet problem for the Laplacian in a domain $D$, with boundary values equal to 1 on $E \subset \partial \Omega$ and 0 on $\partial \Omega \setminus E$. We will be using the notation $\omega(z, E, D)$.

Two basic properties of the harmonic measure that we will use are conformal invariance and domain monotonicity. When $\phi : \mathbb{D} \to \Omega$ is a conformal map, we know that, if $A$ is the set of accessible points of $\partial \Omega$, we can extend $\phi^{-1}$ to $A$. In that sense, when $E \subset A$ we have [13, p.206]

$$\omega(z, \phi^{-1}(E), \mathbb{D}) = \omega(\phi(z), E, \Omega).$$
This implies that when an arc $\hat{a}b \subset \partial \mathbb{D}$ corresponds, through $\phi$, to a boundary set $E \subset \partial \Omega$, in the sense of Caratheodory boundary correspondence, then

$$\omega(z, \hat{a}b, \mathbb{D}) = \omega(\phi(z), E, \Omega).$$

When for two domains $D_1, D_2$ in $\mathbb{C}_\infty$, with $D_1 \subset D_2$, we have a set $B \subset \partial D_1 \cap \partial D_2$, then

$$\omega(z, B, D_1) \leq \omega(z, B, D_2).$$

We also know that, if $\hat{a}b \subset \partial \mathbb{D}$ is a circular arc, then the level set

$$L_k = \{\zeta \in \mathbb{D} : \omega(\zeta, \hat{a}b, \mathbb{D}) = k\}, \ 0 < k < 1,$$

is a circular arc with endpoints $a$ and $b$ that meets the unit circle with angle $k\pi$. We will also use the notation

$$\hat{L}_k = \{\zeta \in \mathbb{D} : \omega(\zeta, \hat{a}b, \mathbb{D}) > k\}.$$

In order to establish a relation between certain harmonic measures in the case when $D$ contains, in a specific way, a rectangle, we introduce the following notation.

For any set $B$ in the complex plane $\mathbb{C}$, let $B^+ = B \cap \{z : \text{Im } z > 0\}$ and $B^- = B \cap \{z : \text{Im } z < 0\}$. Let

$$S_d = \{z : -d < \text{Im } z < d\}$$

be a horizontal strip of width $2d$,

$$S_{d,u} = \{z : -d < \text{Im } z < d, \ -u < \text{Re } z < u\}$$

be a rectangle centered at the origin with width $2d$ and length $2u$,

$$B_{d,u} = \{z : \text{Im } z = d, \ -u < \text{Re } z < u\}$$

be the upper side of $S_{d,u}$ and $B_{-d,u}$ be the lower side of $S_{d,u}$. Betsakos [4] has proven the following:

**Lemma 1.** Let $\Omega$ be a planar domain, convex in the positive direction. Assume that $\mathbb{R} \subset \Omega$ and that $(\partial \Omega)^+ \neq \emptyset$, $(\partial \Omega)^- \neq \emptyset$. Let $\epsilon > 0$ and $d > 0$. There exists a $u_0 > 0$ with the property: If $y \in (-d, d)$, $S_{d,u_0} \subset \Omega$ and $B_{d,u_0} \cup B_{-d,u_0} \subset \partial \Omega$, then

$$|\omega(ig, (\partial \Omega)^+, \Omega) - \omega(ig, (\partial S_d)^+, S_d)| < \epsilon.$$
Let also, for $A = A(w, d_1, d_2, u)$,
\[
\partial_h A = \{x + iy : |x - \text{Re } w| < u/2, \ y = \text{Im } w - d_2 \text{ or } y = \text{Im } w + d_1\},
\]
be the horizontal border of $A$. Finally for $z \in \mathbb{C}$, let
\[
\partial_z^+ \Omega = \partial \Omega \cap \{\zeta : \text{Im } \zeta > \text{Im } z\}
\]
be the part of the border of $\Omega$ that lies above $z$. Note that when $z \in \mathbb{R}$ we have $\partial_z^+ \Omega = (\partial \Omega)^+$. Note also that if the distances of $iy$ from the upper and lower parts of a strip are respectively $d_1$ and $d_2$, by applying standard conformal maps, one can see that
\[
\omega(iy,(\partial S_d)^+, S_d) = \frac{d_2}{d_1 + d_2}.
\]
By conformal invariance of the harmonic measure, Lemma 1 can be restated as follows.

**Lemma 2.** Let $d_1, d_2 > 0$. Then for every $\epsilon > 0$, there exists a $w_0 > 0$, such that for every $u > u_0$ and for all domains $\Omega$, convex in the positive direction, the following property holds: If $A = A(w, d_1, d_2, u) \subset \Omega$ and $\partial_h A \subset \partial \Omega$, then
\[
\left| \omega(w, \partial_w^+ \Omega, \Omega) - \frac{d_2}{d_1 + d_2} \right| < \epsilon.
\]
We will be working with domains convex in the positive direction but we point out that by a small modification of the proof found in [3], we can drop this requirement.

Let $z \in \mathbb{D}$. We will prove that the slope of the trajectory $\gamma_z$ of a semigroup of holomorphic functions $(\phi_t)$ is determined by certain harmonic measures. Consider the function
\[
\omega_z(t) = \omega(h(z) + t, \partial_{h(z)}^+ \Omega, \Omega), \ t \in (0, +\infty).
\]
Betsakos [4] constructed a semigroup such that for every $z \in \mathbb{D}$, $\text{Slope}^+(\gamma_z) = [-\pi/2, \pi/2]$, by considering the behavior of $\omega_0(t)$ as $t \to +\infty$. We will prove an explicit relation between the behavior of $\omega_z(t)$ and the slopes of $\gamma_z$. We will then use it to construct a semigroup such that $\text{Slope}^+(\gamma_z) = [\theta_1, \theta_2]$ with $-\pi/2 \leq \theta_1 < \theta_2 \leq \pi/2$. The same principles will be extended to an analogous result for the $\text{Slope}^-(\gamma_z)$.

**Theorem 3.** Let $(\phi_t)$ be a semigroup of holomorphic functions in $\mathbb{D}$. Denote by $h$ the corresponding Koenigs function and by $\Omega = h(\mathbb{D})$ the associated planar domain. For $z \in \mathbb{D}$, with $\partial_{h(z)}^+ \Omega \neq \emptyset$ and $\partial_{h(z)}^- \Omega \neq \partial \Omega$, let $a_1 = \limsup_{t \to -\infty} \omega_z(t)$ and $a_2 = \liminf_{t \to -\infty} \omega_z(t)$. Then
\[
\text{Slope}^+(\gamma_z) = [\pi(1/2 - a_1), \pi(1/2 - a_2)].
\]
If, in addition, for that $z$, the trajectory $\gamma_z$ is defined for all $t \in (-\infty, 0]$ and we have $b_1 = \limsup_{t \to -\infty} \omega_z(t)$ and $b_2 = \liminf_{t \to -\infty} \omega_z(t)$, then
\[
\text{Slope}^-(\gamma_z) = [\pi(1/2 - b_1), \pi(1/2 - b_2)].
\]
Using the above theorem we can argue about the slopes of the trajectories of \( \phi_t \) by focusing on the image \( h(\mathbb{D}) \) and looking at the behavior of the harmonic measure on the points of the half-line \( \{ h(z) + t : t > 0 \} \), or on \( \{ h(z) - t : t > 0 \} \) for the backward trajectories.

3. Proofs

**Proof (Theorem 3).** We assume that the Denjoy-Wolff point of \( \phi_t \) is \( \xi \) and the \( \alpha \)-limit of \( \gamma_z \) is \( \chi \). Let \( \hat{\chi} \) be the arc on \( \partial \mathbb{D} \) between \( \chi \) and \( \xi \), corresponding through \( h(z) \) to \( \partial^+_{h(z)} \Omega \). Note that \( \partial^+_{h(z)} \Omega \neq \emptyset \) and \( \partial_{h(z)}^{+} \Omega \neq \partial \Omega \) imply \( \chi \neq \xi \). Also since \( h \) is conformal we have that \( \hat{\chi} \xi \) is the arc that runs clockwise from \( \chi \) to \( \xi \). We know that the level set

\[
L_k = \{ \zeta \in \mathbb{D} : \omega(\zeta, \hat{\chi} \xi, \mathbb{D}) = k \}, \quad 0 < k < 1,
\]

is a circular arc with endpoints \( \chi \) and \( \xi \) that meets the unit circle with angle \( k \pi \). Let \( \hat{L}_k = \{ \zeta \in \mathbb{D} : \omega(\zeta, \hat{\chi} \xi, \mathbb{D}) > k \} \) and \( \Gamma_k \) be the half-line emanating from \( \xi \) that is tangent to \( L_k \) at \( \xi \). If \( \zeta \) lies on \( \Gamma_k \) then \( \arg(1 - \hat{\zeta} \xi) = \pi/2 - \pi k = \pi(1/2 - k) \).

By conformal invariance of the harmonic measure \( (7) \),

\[
\omega_z(t) = \omega(h(z) + t, \partial^+_{h(z)} \Omega, \Omega) = \omega(\phi_t(z), \hat{\chi} \xi, \mathbb{D}).
\]

Let \( a_1 = \limsup_{t \to \infty} \omega_z(t) \) and \( \theta_1 = \pi(1/2 - a_1) \) the corresponding angle.

We will prove that \( \theta_1 = \min \{ \text{Slope}^+(\gamma_z) \} \).

**Claim 1.** If \( \theta \in \text{Slope}^+(\gamma_z) \) then \( \theta_1 \leq \theta \).

If \( a_1 = 1 \) then \( \theta_1 = -\pi/2 \) and we are done. If not, since \( a_1 = \limsup_{t \to \infty} \omega_z(t) \), from (22) we must also have

\[
\limsup_{t \to \infty} \omega(\phi_t(z), \hat{\chi} \xi, \mathbb{D}) = a_1.
\]

Assume that \( \theta \in \text{Slope}(\gamma_z) \) with \( \theta_1 > \theta = \pi(1/2 - a) \). So there is an \( \varepsilon > 0 \) such that \( a_1 < a_1 + \varepsilon/2 < a_1 + \varepsilon < a \).

Then there is a sequence \( t_n \to \infty \) such that all but finite of the points \( \phi_{t_n}(z) \) lie above \( \Gamma_{a_1+\varepsilon} \) for some \( \varepsilon > 0 \). This means that \( \phi_{t_n}(z) \in \hat{L}_{a_1+\varepsilon/2} \) for almost all \( n \). This implies that \( \lim_{t_n \to \infty} \omega(\phi_{t_n}(z), \hat{\chi} \xi, \mathbb{D}) \geq a_1 + \varepsilon/2 \), a contradiction. So \( \theta_1 \leq \theta \).

**Claim 2.** \( \theta_1 \in \text{Slope}^+(\gamma_z) \).

Since there exists \( t_n \) with \( \omega(\phi_{t_n}(z), \hat{\chi} \xi, \mathbb{D}) \to a_1 \) we have that \( \arg(1 - \hat{\zeta} \phi_{t_n}(z)) \to \theta_1 \) and so \( \theta_1 \in \text{Slope}^+(\gamma_z) \).

We have shown that \( \theta_1 = \min \{ \text{Slope}^+(\gamma_z) \} \). Using the same arguments we can show that if \( a_2 = \liminf_{t \to \infty} \omega_z(t) \) and \( \theta_2 = \pi(1/2 - a_2) \) we have \( \theta_2 = \max \{ \text{Slope}^+(\gamma_z) \} \). This means that \( \text{Slope}^+(\gamma_z) = [\pi(1/2 - a_1), \pi(1/2 - a_2)] \).
In the case when the $\alpha$-limit of $\gamma_z$ is a super-repulsive point, replacing $\infty$ with $-\infty$ and $\xi$ with $\chi$, using the same arguments, we obtain relation (20) for the $\text{Slope}^- (\gamma_z)$.

**Remark 1.** The only property of the set $\partial_{h(z)}^+ \Omega$ that we use is that it corresponds, through $h^{-1}$, to an arc $\hat{\chi} \xi$ on $\partial \mathbb{D}$ with $\xi$ being the Denjoy-Wolff point, or $\chi$ being the $\alpha$-limit of $\gamma_z$, and $\chi \neq \xi$. This means that even when $\partial_{h(z)}^+ \Omega = \emptyset$ or $\partial_{h(z)}^+ \Omega = \partial \Omega$ we can use the same approach by choosing a suitable subset of $\partial \Omega$.

**Proof (Theorem 1).** We will only prove the result for $|\theta_1|, |\theta_2| < \pi / 2$ for simplicity. Small variations of the proof can also account for the cases of $\theta_1 = -\pi / 2$ or $\theta_2 = \pi / 2$. We will essentially present these variations in the proof of Theorem 2. We will modify the construction found in [4] and construct a set $\Omega$ such that for the associated semigroup we have $\text{Slope}^\pm (\gamma_0) = [\theta_1, \theta_2]$. Let $E[\zeta] = \{ \zeta + t : t \leq 0 \}$ be the half-line, parallel to the real axis, starting from $\zeta$ and extending to the left. Let $a_1 = \frac{1}{2} - \frac{\theta_1}{\pi}$ and $a_2 = \frac{1}{2} - \frac{\theta_2}{\pi}$, so that $0 < a_2 < a_1 < 1$. Let $r_n, \rho_n$ be sequences such that

\begin{equation}
\frac{1}{a_1} \rho_n = r_n
\end{equation}

and

\begin{equation}
\frac{1}{a_2} \rho_{n-1} = r_n, \quad n \geq 2.
\end{equation}

Since $a_1 > a_2$, both $r_n$ and $\rho_n$ are increasing. Note that these depend only on the choice of $a_1, a_2$ and $r_1$. For example $a_1 = \frac{3}{4}$, $a_2 = \frac{1}{3}$ and $r_1 = 6$ gives $r_n = 6^n$ and $\rho_n = 3 \cdot 6^n$.

It is easy to see that definitions (24) and (25) indeed give

\begin{equation}
\frac{\rho_n}{\rho_n + r_n} = a_1 \quad \text{and} \quad \frac{\rho_{n-1}}{\rho_{n-1} + r_n} = a_2.
\end{equation}

Note that for $w = 0$ we have $\partial_w^+ \Omega = (\partial \Omega)^+$ and choose an increasing sequence $u_n'$ from Lemma 2 such that the following hold:

When $n = 2k - 1$, for all $\Omega$ with $A = A(w, r_k, \rho_k, u_n') \subset \Omega$ and $\partial A_h \subset \partial \Omega$,

\begin{equation}
|\omega(w, (\partial \Omega)^+, \Omega) - \frac{\rho_k}{\rho_k + r_k}| < \frac{1}{n}
\end{equation}

and for all $\Omega$ with $A = A(x, r_{k+1}, \rho_k, u_n') \subset \Omega$ and $\partial A_h \subset \partial \Omega$,

\begin{equation}
|\omega(w, (\partial \Omega)^-, \Omega) - \frac{\rho_k}{\rho_k + r_{k+1}}| < \frac{1}{n}
\end{equation}

When $n = 2k$, for all $\Omega$ with $A = A(x, r_{k+1}, \rho_k, u_n') \subset \Omega$ and $\partial A_h \subset \partial \Omega$,

\begin{equation}
|\omega(w, (\partial \Omega)^-, \Omega) - \frac{\rho_k}{\rho_k + r_{k+1}}| < \frac{1}{n}
\end{equation}

and for all $\Omega$ with $A = A(x, r_{k+1}, \rho_{k+1}, u'_n) \subset \Omega$ and $\partial A_h \subset \partial \Omega$,

$$|\omega(w, (\partial \Omega)^+, \Omega) - \frac{\rho_{k+1}}{\rho_{k+1} + r_{k+1}}| < \frac{1}{n}. \quad (30)$$

Consider the partial sums $u_n = \sum_{j=1}^{n} u'_j$ and set

$$\Omega = C \setminus \bigcup_{n=1}^{\infty} (E[u_{2k-1} + ir_k] \cup E[u_{2k} - i\rho_k]). \quad (31)$$

The way $\Omega$ was constructed we have that $\Omega$ is convex in the positive direction. We also have that, for $n = 2k-1$, for the rectangles $A = A(x_n, r_k, \rho_k, u'_n)$ we have $A \subset \Omega$ and $\partial A_h \subset \partial \Omega$, where $x_n = (u_n + u_{n-1})/2$. Obviously $x_n \to \infty$. For $n = 2k$ the same holds for $A = A(x_n, r_{k+1}, \rho_k, u'_n)$.

So for $n = 2k - 1$, from relations (26) and (27), we have,

$$|\omega(x_n, (\partial \Omega)^+, \Omega) - a_1| < \frac{1}{n} \quad (32)$$

and for $n = 2k$, from relations (26) and (28),

$$|\omega(x_n, (\partial \Omega)^+, \Omega) - a_2| < \frac{1}{n}. \quad (33)$$

So we have found two sequences $x_{2k-1} \in \mathbb{R}$ and $x_{2k} \in \mathbb{R}$ with respective limits $a_2$ and $a_1$. That means

$$[a_2, a_1] \subset \left[ \lim_{t \to \infty} \omega_0(t), \lim_{t \to \infty} \omega_0(t) \right]. \quad (34)$$

We proceed to show the opposite inclusion. Consider a pair $x_{2k-1}, x_{2k}$ on the real line. Note that the rectangles $A(x_{2k-1}, r_k, \rho_k, u'_{2k-1})$ and $A(x_{2k}, r_{k+1}, \rho_k, u'_{2k})$ are both contained in $\Omega$.

Figure 1: A part of the set $\Omega$
Consider the set $\Omega_1 = \Omega \setminus E$, where $E = \{x + iy : y = r_k, u_{2k-1} < x \leq u_{2k}\}$. In Figure 2, $E$ is the dotted segment. Obviously $\Omega_1 \subset \Omega$ and $(\partial \Omega)^- = (\partial \Omega_1)^-$. Also for all $x \in [x_{2k-1}, x_{2k}]$, since $u_n'$ is increasing, we have that

$$A = A(x, r_k, \rho_k, u'_{2k-1}) \subset \Omega_1 \text{ and } \partial A_h \subset \partial \Omega_1.$$  \hfill (35)

Using the domain monotonicity of the harmonic measure and relation (27) we get

$$\omega(x, (\partial \Omega)^+, \Omega) = 1 - \omega(x, (\partial \Omega)^-, \Omega) \leq 1 - \omega(x, (\partial \Omega_1)^-, \Omega_1)$$

$$= \omega(x, (\partial \Omega_1)^+, \Omega_1) < a_1 + \frac{1}{n}.$$  \hfill (27)

Similarly consider $\Omega_2 = \Omega \cup E[u_{2k-1} + ir_k]$. Again for all $x \in [x_{2k-1}, x_{2k}]$, we have $A = A(x, r_{k+1}, \rho_k, u'_{2k-1}) \subset \Omega_2$ and $\partial A \subset \partial \Omega_2$. Since $\Omega \subset \Omega_2$, considering (28),

$$\omega(x, (\partial \Omega)^+, \Omega) > \omega(x, (\partial \Omega_2)^+, \Omega_2) > a_2 - \frac{1}{n}.$$  \hfill (28)

We can likewise treat the case where $x \in [x_{2k}, x_{2k+1}]$. These inequalities show that if there exists a sequence $t_k \to \infty$ with $\lim_{k \to \infty} \omega_0(t_k) = a$ then $a_2 \leq a \leq a_1$.

We have shown that $a_1 = \limsup_{t \to \infty} \omega_0(t)$ and $a_2 = \liminf_{t \to \infty} \omega_0(t)$. Considering the semigroup $(\phi_t)$ that corresponds to the set $\Omega$, the desired result follows from Theorem 3.

**Proof (Theorem 2).** As in the above proof let $b_1 = \frac{1}{2} - \frac{d}{\pi}$, $b_2 = \frac{1}{2} - \frac{a}{\pi}$ and $r_n, \rho_n$ be sequences such that

$$r_n = \frac{1 - b_2}{b_2} \rho_n,$$

and

$$r_n = \frac{1 - b_1}{b_1} \rho_{n-1}, \quad n \geq 2.$$  \hfill (29)

Since $b_1 > b_2$ we have that both $r_n$ and $\rho_n$ are decreasing sequences. Note that these depend only on the choice of $b_1, b_2$ and $r_1$. Similar to the above proof, if for example $b_1 = \frac{3}{4}$, $b_2 = \frac{1}{3}$ and $r_1 = \frac{1}{3}$, we get

$$r_n = \frac{1}{3} \cdot 6^{-(n-1)} \text{ and } \rho_n = 6^{-n}.$$  \hfill (30)

We define sequences $u_n, u'_n$ in the exact same way as in the proof of Theorem 1. This means that we can use relations (27) - (30). Now $\Omega$ can be defined as

$$\Omega = \mathbb{C} \setminus \bigcup_{n=1}^{\infty} (E[-u_{2k-1} + ir_k] \cup E[-u_{2k} + ir_k]).$$  \hfill (31)

Obviously $\Omega$ is convex in the positive direction and $\gamma_0$ is defined for $t \in (-\infty, +\infty)$. Similarly with before we take $x_n = -(u_n + u'_{n-1})/2$. We have that $x_n$ goes to $-\infty$ and for the subsequences $x_{2k-1}$ and $x_{2k}$ we get

$$\lim_{k \to \infty} \omega(x_{2k-1}, (\partial \Omega)^+, \Omega) = b_1$$

and

$$\lim_{k \to \infty} \omega(x_{2k}, (\partial \Omega)^+, \Omega) = b_2.$$  \hfill (32)
We can show the opposite inclusion with the same arguments as in the proof of Theorem 1. Again from Theorem 3 we get Slope\(^-\)(\(\gamma_0\)) = [\(\theta_1, \theta_2\)].

We will now consider the case when \(b_2 = 0\). We modify our sequences so that
\[
r_n = (n + m)\rho_n,
\]
and
\[
r_n = \frac{1 - b_1}{b_1}, \quad n \geq 2,
\]
where \(m\) is taken big enough, so that for all \(n\) we have \(n + m > \frac{1 - b_2}{b_2}\). We again have two decreasing sequences. The proof works out in the same way except that now, for \(n = 2k - 1\), relation (27) becomes
\[
\omega(x_n, (\partial\Omega)^+, \Omega) < \frac{1}{n + m + 1} + \frac{1}{n} < \frac{2}{n}
\]
for all \(n\). Obviously \(\omega(x_{2k-1}, (\partial\Omega)^+, \Omega) \to 0\) as \(k \to \infty\) and as before we have \(\omega(x_{2k}, (\partial\Omega)^+, \Omega) \to b_1\).

Similarly in the case when \(b_1 = 1\) we take
\[
r_n = \frac{1 - b_2}{b_2}\rho_n,
\]
and
\[
r_n = \frac{1}{n + m}, \quad n \geq 2,
\]
where \(m\) is taken big enough, so that for all \(n\) we have \(\frac{1}{n + m} < \frac{1 - b_2}{b_2}\). As before, note that, for \(n = 2k\), relation (29) becomes
\[
\omega(x_n, \partial\Omega^+, \Omega) > \frac{n + m}{n + m + 1} < \frac{1}{n
\]
\[
> \frac{n + m}{n + m + 1} - \frac{2}{n + m + 1} = 1 - \frac{3}{n + m + 1},
\]
for all \(n > m + 1\). Obviously \(\omega(x_{2k}, (\partial\Omega)^+, \Omega) \to 1\) as \(k \to \infty\), while \(\omega(x_{2k-1}, (\partial\Omega)^+, \Omega) \to b_2\).

Combining the above we can also construct an example with Slope\(^-\)(\(\gamma_z\)) = [\(-\pi/2, \pi/2\)]. Note that in this case we can simply use
\[
r_n = n\rho_n,
\]
and
\[
r_n = \frac{1}{n}\rho_{n-1}, \quad n \geq 2,
\]
which coincides with what was used in [4].

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