A Bayesian GED-Gamma stochastic volatility model for return data: a marginal likelihood approach

T. R. Santos *
Department of Statistics, Universidade Federal de Minas Gerais

Abstract

Several studies explore inferences based on stochastic volatility (SV) models, taking into account the stylized facts of return data. The common problem is that the latent parameters of many volatility models are high-dimensional and analytically intractable, which means inferences require approximations using, for example, the Markov Chain Monte Carlo or Laplace methods. Some SV models are expressed as a linear Gaussian state-space model that leads to a marginal likelihood, reducing the dimensionality of the problem. Others are not linearized, and the latent parameters are integrated out. However, these present a quite restrictive evolution equation. Thus, we propose a Bayesian GED-Gamma SV model with a direct marginal likelihood that is a product of the generalized Student’s t-distributions in which the latent states are related across time through a stationary Gaussian evolution equation. Then, an approximation is made for the prior distribution of log-precision/volatility, without the need for model linearization. This also allows for the computation of the marginal likelihood function, where the high-dimensional latent states are integrated out and easily sampled in blocks using a smoothing procedure. In addition, extensions of our GED-Gamma model are easily made to incorporate skew heavy-tailed distributions. We use the Bayesian estimator for the inference of static parameters, and perform a simulation study on several properties of the estimator. Our results show that the proposed model can be reasonably estimated. Furthermore, we provide case studies of a Brazilian asset and the pound/dollar exchange rate to show the performance of our approach in terms of fit and prediction.

Keywords: SV model, New sequential and smoothing procedures, Generalized Student’s t-distribution, Non-Gaussian errors, Heavy tails, Skewness

*T. R. Santos Universidade Federal de Minas Gerais, Brazil. E-mail: thi-agords@est.ufmg.br. The author received support from the FAPEMIG Foundation and CNPq-Brazil
1 Introduction

There is evidence of non-Gaussianity, skewness, and heavy tails in the distribution of return data. Therefore, we need to choose more flexible models in order to incorporate these stylized facts. Volatility is an important statistical measure, representing the conditional variance of an underlying asset return, and plays a key role in finance [Tsay 2010]. It is also an important component of risk management, portfolio optimization, and options trading.

Since volatility is a latent component, its estimation calls for specific techniques and suitable statistical inferences. Several models have been proposed for estimating the volatility of asset return data. For example, [Engle 1982] introduced an autoregressive conditional heteroskedasticity (ARCH) model, where volatility is a function of past time series values. The generalized ARCH (GARCH) model was proposed by [Bollerslev 1986], where volatility can depend on its own past. [Taylor 1982] proposed a stochastic volatility (SV) model with an error term in its volatility equation, capturing some of the characteristics found in financial return time series in a better way than the GARCH model.

The usual approach to using a SV model is to use linearization to convert it into a linear state-space model by a transformation in the return data. [Harvey et al. 1994], who adopted this approach, presented the quasi-maximum likelihood (QML) estimator from the classical perspective, considering the innovation distribution to be approximately Gaussian. [Danielsson 1994] proposed the simulated maximum likelihood (SML) method to estimate the SV model. Subsequently, [Sandmann and Koopman 1998] discussed the Monte Carlo likelihood estimation (MCL), and how a very efficient MCL estimator can be obtained, while keeping the linear state-space form under the classical inference. Their procedure first linearizes the SV model and then a better approximation of the observation equation error distribution is made using the MCL method. Further, [Kim et al. 1998] used a normal mixture to approximate the observation distribution once the SV model is linearized. They also proposed a different estimation method based on Gibbs sampling under the Bayesian approach.

Another interesting approach is the method of moments (MM). Several MM estimators have been introduced in the literature; for example, see [Taylor 1986] and [Melino and Turnbull 1990]. The latter used the generalized method of moments (GMM) to estimate the SV model parameters. These estimators prevent the problems related to model linearization and full likelihood function evaluation. However, they have poor finite sample properties, and do not estimate the underlying volatility directly [Broto and Ruiz 2004].

A Bayesian estimation approach to SV models using a Markov Chain Monte Carlo (MCMC) method and the full likelihood function was developed by [Jacquier et al. 1994]. Their extensive simulation experiments showed that the MCMC method performs better than the QML and MM estimation techniques. Subsequently, [Jacquier et al. 2004] introduced a new version of this model to accommodate fat tails and correlated errors and [Cappuccio et al. 2004] presented an interesting Skew-GED SV model.
The MCMC procedure requires large, computer-intensive simulations and its computational implementation is not a simple task. Another problem is the dimensionality of the parameter space, once the latent (log-volatility) and static parameters are simultaneously estimated using the full posterior distribution, which is based on a full likelihood function, although it is does not necessarily require the linearization of the SV model. Two alternatives to the MCMC methods under the Bayesian perspective are the particle filter and Laplace approximations.

Several studies have examined using SV models from a Bayesian perspective, including those of Taylor, Chib et al., Yu, Omori et al., Raggi and Bordignon, and Kastner and Fruhwirth-Schnatter. Then, Ferrante and Vidoni, Vidoni, and Davis and Yan considered nonlinear and non-Gaussian state-space models. See also Watanabe, Knight and Yu, Feunou and Tedongop, and Koopman and Bos. A detailed review of SV models can be found in Broto and Ruiz.

The family of non-Gaussian state-space models (NGSSM) was proposed by Gamerman et al. and is an attractive alternative to both the SV and GARCH models. These models have a dynamic level associated with volatility and a multiplicative Beta evolution equation. This evolution provides an exact marginal likelihood function and filtering and smoothing distributions. In spite of the analytical tractably of this family of models, the evolution equation is a random walk in log-scale, and does not include drift (a quite restrictive). Pinho et al. presented several heavy-tailed distributions representing particular cases of the NGSSM family. In this class, Shepard introduced local scale models, which were then generalized by Deschamps.

Several studies explore inferences based on stochastic volatility (SV) models, taking into account the stylized facts of return data. The general problem with these models is that the latent parameters are high-dimensional, which makes it difficult to integrate out or to use high-dimensional numerical integration. Thus, inferences using these models require approximations using, for example, Markov Chain Monte Carlo or Laplace methods. The GARCH model has been an attractive option among the users due to the difficulty in obtaining the marginal likelihood of the SV model (its computational implementation) according to Fridman and Harris. Some SV models are expressed as a linear Gaussian state-space model, leading to an approximated marginal likelihood function and a marginal posterior distribution, which reduces the dimensionality of the problem. However, the observation disturbance is either Gaussian or requires approximations. Other models are not linearized, and possess a marginal likelihood that is approximated using Monte Carlo integration/importance sampling.

Thus, the main objective of this study is to develop a Bayesian GED-Gamma SV model for return data with a new sequential analysis procedure and an approximated marginal likelihood that is a product of the generalized Student's
t-distributions and is evaluated directly, where the inferential procedure is fast and easy to implement under the Bayesian approach. The latent states in our proposed GED-Gamma model are related across time through a stationary Gaussian evolution equation, and an analytical approximation is made for the prior distribution of the log-precision/volatility, without the need for model linearization. This also allows us to approximate the marginal likelihood function. Furthermore, the high-dimensional latent states are easily integrated out and sampled in blocks using a new approximated smoothing procedure that is introduced, enabling inferences to be made for these states.

The main advantages of the employed method are its mathematical and computational simplicity, and its ability to accommodate the stylized facts of return data and a stationary Gaussian evolution equation. This circumvents the problem of high-dimensional latent states, without the need for model linearization.

Section 2 presents the GED-Gamma SV model. Then, Section 3 presents a simulation, and Section 4 provides a case study of the proposed model using real return data. Finally, Section 5 concludes the paper, including an indication of potential areas for future research.

2 GED-Gamma SV model

Because of the stylized facts common to return data, we need to choose more flexible models that allow for the use of non-Gaussian heavy-tailed skew distributions. The GED is a non-Gaussian distribution with the flexibility to capture heavy-tailed patterns, and is discussed in detail in Box and Tiao [1992] and used in Nelson [1991] and Deschamps [2011]. Another possibility is the skew-GED distribution that was used and motivated by Pinho et al. [2016] and Cappuccio et al. [2004]. However, we opt for a GED distribution that is a skew-GED distribution with the asymmetry parameter \( \kappa = 0 \), as in Deschamps [2011]. It is no difficult to extend the GED-Gamma SV model to other cases, as it will be shown in Subsection 2.3.

The GED-Gamma SV model, which is a composing of the GED distribution with precision distributed as a gamma distribution, for the return time series \( \{y_t\}_{t=1}^n \) is defined as follows:

(A1) **The observation equation** is

\[
p(y_t|\lambda_t, \varphi) = \frac{r \Gamma(3/r)^{1/2} \lambda_t^{1/r}}{2 \Gamma(1/r)^{3/2}} \exp\left(-\lambda_t \psi(r)|y_t|\right),
\]

for \( y_t \in \mathbb{R} \), where \( \psi(r) = \frac{\Gamma(3/r)}{\Gamma(1/r)} r^{1/2} \), \( \varphi \) is a static parameter vector, the latent states \( \lambda_t = h_t^{-1} \) (precision), and \( h_t \) is the volatility at time \( t \). If \( r = 1 \), it is the Laplace model, and if \( r = 2 \), it is the normal model. Deschamps [2011].

We consider a correlation structure in the mean of the returns series, such that \( y_t = (R_t - \mu_t) \), where \( R_t \) is the usual return series and \( \mu_t \) is the mean of the data.

The model is fully specified by the following remaining assumptions:
• **(A2) The prior distribution** is \( \lambda_t | Y_{t-1}, \varphi \sim \text{Gamma}(a_{t|t-1}, b_{t|t-1}) \):

• **(A3) The evolution equation** is \( \ln(\lambda_t) = -\alpha + \phi \ln(\lambda_{t-1}) + \eta_t \), where \( \eta_t \sim \text{i.i.d.} N \left(0, \sigma^2_\eta\right), \alpha \in \mathbb{R}, \phi \in [0, 1) \) and \( \sigma^2_\eta > 0 \).

• **The initial information** is \( \lambda_0 | Y_0, \varphi \sim \text{Gamma}(a_0, b_0) \), that is, \( \ln(\lambda_0) | Y_0 \sim \text{Log-Gamma}(f_0, q_0) \), where the mean is \( f_0 = \ln(a_0) - \gamma(b_0) \) and the variance \( q_0 = \gamma'(a_0) \). Then, \( \gamma(\cdot) \) and \( \gamma'(\cdot) \) are the digamma and trigamma functions, respectively.

Note that \( Y_{t-1} = (Y_0, y_1, \ldots, y_{t-1})' \) is the information available up to time \( t-1 \). Furthermore, the evolution equation (A3) in terms of the volatility \( h_t \) can be written as \( \ln(h_t) = \alpha + \phi \ln(h_{t-1}) + \eta^*_t \), where \( \eta^*_t \sim N \left(0, \sigma^2_\eta\right) \) and \( E(\varepsilon^*_t) = 0 \), \( \{\varepsilon_t\} \) is the disturbance term of the observation equation.

Instead of approximations of the observation distribution, as in the QML, MCL, and MCMC [Kim et al., 1998] methods, our approach approximates the distribution of the natural logarithm of the latent states, the log-precision, in terms of the two first moments, using an analytical approximation approach. Once the distribution of the natural logarithm of the latent states is a normal distribution or can be approximated by a normal, we can specify it in terms of its two first moments. Figure 1 shows a comparison of the log gamma and normal distributions for the states to illustrate and assess the quality of the approximation in terms of two first moments.

At the top, we have the shape parameter \( a \) at 2 and the scale parameter \( b \) assuming the values 2 and 100 and a reasonable approximation of the log gamma distribution by the normal distribution. When the shape parameter is large, the difference between the distributions become indistinguishable, because of the central limit theorem. The values of the parameters \( a \) and \( b \) were chosen based on the usual values of the shape and scale parameters of the updated distribution in our simulation experiments. This approach is similar to that adopted in the dynamic generalized linear model (DGLM) [West and Harrison, 1997].

Hereafter, we present the proposed sequential analysis (inferential) procedure of this model, that is more similar to that of the Dynamic Linear Model (DLM) than the DGLM (see Figure 2). This consists of the one-step ahead predictive and filtering (or online) distributions of the latent states \( \lambda = \{\lambda_t\}_{t=1:n} \), and the one-step ahead predictive distribution of the observations. If the model is defined as proposed in this section, we can use an approximation of the state distribution to obtain the following results:

**Proposition 1.**

1. The one-step ahead predictive (prior) distribution of the latent states at time \( t \)
   \( \lambda_t | Y_{t-1}, \varphi \sim \text{Gamma}(a_{t|t-1}, b_{t|t-1}) \), where
   \[
   a_{t|t-1} = \left(\phi^2 a_{t-1}^{-1} + \sigma^2_\eta\right)^{-1},
   \]
   \[
   b_{t|t-1} = \frac{\exp(\alpha)(a_{t-1}/b_{t-1})^{-\phi}}{\left(\phi^2 a_{t-1}^{-1} + \sigma^2_\eta\right)}. \]
2. The update or online (posterior) distribution at time $t_{\lambda|Y_t, \varphi} \sim \text{Gamma}(a_t, b_t)$, where

$$
a_t = a_{t|t-1} + 1/r,
$$

$$
b_t = b_{t|t-1} + \psi(r)|y_t|^r. \quad (4)
$$

3. The one-step ahead predictive distribution of the observations at time $t$ is given by

$$
p(y_t|Y_{t-1}, \varphi) = \frac{\Gamma(1/r + a_{t|t-1}) \Gamma(3/r + \psi(r)|y_t|^r + b_{t|t-1})}{\Gamma(a_{t|t-1}) \Gamma(1/r + a_{t|t-1}) \Gamma(3/r + \psi(r)|y_t|^r + b_{t|t-1})}, y_t \in \mathbb{R}, \quad (6)
$$

for $t = 1, \ldots, n$, where $n$ is the number of observations of the time series and $\Gamma(\cdot)$ is the gamma function. This predictive distribution is the generalized Student’s t-distribution with $2a_{t|t-1}$ degrees of freedom and if $r = 2$, then it is Student’s t-distribution [Triantafyllopoulos, 2008], an interesting feature of the proposed model.

The proof of this proposition is given in Appendix I. The important distribution of $\lambda_{t|Y_{t-1}, \varphi}$ in Part 1 of Proposition 1 preserves, in general, the mean of the distribution of $\lambda_{t-1|Y_{t-1}, \varphi}$ and increases the variance.

The approximated marginal log-likelihood function, which is a product of the generalized Student’s t-distributions, is given by

$$
\ln L(\varphi; Y_n) = \ln \prod_{t=1}^{n} p(y_t|Y_{t-1}, \varphi) = \sum_{t=1}^{n} \ln \Gamma(a_{t|t-1} + 1/r) - \ln \Gamma(a_{t|t-1}) - a_{t|t-1} \ln b_{t|t-1} + \ln \left( \frac{r(3/r)^{1/2}}{2 \Gamma(1/r)} \right) - (1/r + a_{t|t-1}) \ln \psi(r)|y_t|^r + b_{t|t-1}, \quad (7)
$$

where $\varphi$ is composed of $\alpha$, $\phi$, $\sigma^2_n$, and $r$; $Y_n = (Y_0, y_1, \ldots, y_n)'$ (when all information is available).

### 2.1 Bayesian Inference

Since the marginal posterior distribution of parameter vector $\varphi$ is not analytically tractable, a Bayesian inference for $\varphi$ can be performed using a MCMC [Gamerman and Lopes, 2006] or an Adaptive Rejection Metropolis Sampling (ARMS) [Gilks et al., 1995] algorithm. The marginal posterior distribution of $\varphi$ is given by

$$
p(\varphi|Y_n) \propto L(\varphi; Y_n)p(\varphi), \quad (8)
$$

where $L(\varphi; Y_n)$ is the likelihood function defined in (7), and $p(\varphi)$ is the prior distribution of $\varphi$. In this work, independent proper uniform priors are adopted for $\varphi$, as in Gamerman et al. 2013 and Cappuccio et al. 2004. The idea is to introduce vague uniform priors with a large variance, if we have no knowledge
about the value of the parameters. However, other priors for the components of \( \varphi \) could be \( \alpha \sim N(\mu_\alpha, \sigma^2_\alpha), \frac{\omega + 1}{2} \sim B(\alpha_\omega, \beta_\omega), \sigma^2 \sim \text{InvGamma}(a_\sigma, b_\sigma), \) and \( r \sim \text{Gamma}(a_r, b_r) \) \citep{Kastner2014}, see.

Once a sample \( \varphi^{(1)}, \ldots, \varphi^{(M)} \) is provided by the ARMS or MCMC algorithm, the approximated posterior mean, median and percentiles can be calculated. The posterior mode can be obtained by maximizing function \( \mathcal{L} \). This task is typically performed numerically using a maximization algorithm, such as the Broyden--Fletcher--Goldfarb--Shanno (BFGS) and sequential quadratic programming (SQP) algorithms \citep{Avriel2003}. In general, \( \varphi \) may be re-parameterized in order to utilize these algorithms.

An inference for the latent variables can be made using the output from the MCMC and ARMS algorithms. Once a sample \( \varphi^{(1)}, \ldots, \varphi^{(M)} \) is available, the predictive, filtering or smoothed distributions of the latent states can be calculated in the following way. Note that

\[
p(\lambda_{t+h}|\mathbf{Y}_t) = \int p(\lambda_{t+h}|\mathbf{Y}_t, \varphi) p(\varphi|\mathbf{Y}_t) \, d\varphi. \tag{9}
\]

Thus, the \( h \)-step-ahead predictive or filtering distributions can be approximated by

\[
\frac{1}{M} \sum_{j=1}^{M} p(\lambda_{t+h}|\mathbf{Y}_t, \varphi^{(j)}),
\]

from which summaries such as means, variances, and credibility intervals can be obtained. Since \( p(\lambda_{t+h}|\mathbf{Y}_t) \) is not available analytically, a draw \( \lambda_{t+h}^{(s)} \) from \( p(\lambda_{t+h}|\mathbf{Y}_t) \) can be obtained from \( \mathcal{L} \) by sampling \( \varphi^{(s)} \) from \( p(\varphi|\mathbf{Y}_n) \), and then sampling \( \lambda_{t+h}^{(s)} \) from \( p(\lambda_{t+h}|\mathbf{Y}_t, \varphi^{(s)}) \). In addition, smoothing procedures may be built, following \citep{Gamerman1991}. See also \citep{Migon2005}.

### 2.2 Smoothing

In order to infer the latent states \( \lambda = (\lambda_1, \ldots, \lambda_n)' \), we can utilize an approximated smoothed distribution for \( \ln(\lambda) \), and apply the inverse transformation. If the model is defined as proposed here, we can use the results of the sequential analysis to obtain the following smoothed distribution. The joint distribution of \( \ln(\lambda)|\mathbf{Y}_n, \varphi \) has density

\[
p(\ln(\lambda)|\varphi, \mathbf{Y}_n) = p(\varphi|\mathbf{Y}_n)p(\ln(\lambda_n)|\varphi, \mathbf{Y}_n) \prod_{t=1}^{n-1} p(\ln(\lambda_t)|\ln(\lambda_{t+1}), \varphi, \mathbf{Y}_t). \tag{10}
\]

**Proposition 2.** The distribution

\[
p(\ln(\lambda_t)|\ln(\lambda_{t+1}), \mathbf{Y}_t, \varphi) \sim N(\mu_t^*, \sigma_t^{2*}), \tag{11}
\]

where \( \sigma_t^{2*} = \left( \frac{\phi^2}{\sigma^2_t} + \frac{1}{q_t} \right)^{-1} \), \( \mu_t^* = \sigma_t^{2*} \left[ \frac{\phi(\ln(\lambda_{t+1})+\alpha)}{\sigma^2_t} + \frac{f_t}{q_t} \right] \), \( f_t = \ln(a_t) - \gamma(b_t) \) and \( q_t = \gamma'(a_t) \), which depend on the shape and scale parameters of the filtering distribution of \( \lambda_t \). The proof of Proposition 2 is given in Appendix II.
The inference for the latent variables or states can be made using the output from the MCMC and ARMS algorithms. Once a sample $\varphi^{(1)}, \ldots, \varphi^{(M)}$ is available, posterior samples $\ln(\lambda)^{(1)}, \ldots, \ln(\lambda)^{(M)}$ from the latent variables are obtained according to the following procedure.

**Smoothing procedure:**

1. set $j = 1$;
2. sample the static parameter $\varphi^{(j)}$ from the MCMC or ARMS algorithm;
3. sample the set $\ln(\lambda)^{(j)}$ of latent variables from $p(\ln(\lambda) | \varphi^{(j)}, Y_n)$ in (10);
4. set $j \rightarrow j + 1$ and return to 2, if $j \leq M$; otherwise, stop.

### 2.3 Extensions of the GED-Gamma SV model

The model, for the observations, in Equation (1) can be generalized using a scale mixture for the observation disturbance to obtain other (skew) heavy-tailed distributions directly, such as the (skew) Student’s t-distribution [Nakajima and Omori, 2009, Gamerman et al., 2013, see]. If $\epsilon_t^\star = \gamma_t^{-1/2} \epsilon_t$ is the observation disturbance of the model, where $\gamma_t \sim \text{Gamma}(\nu/2, \nu/2)$ and $\epsilon_t \sim \text{GED}(r = 2, \mu = 0, \sigma^2 = 1)$, $\epsilon_t^\star$ will have a Student’s t-distribution, with $\nu$ degrees of freedom [Gamerman et al., 2013]. Furthermore, other probability distributions may be considered for $\gamma_t$, leading to other (skew) heavy-tailed distributions for $\epsilon_t^\star$. However, the (skew) GED specification in Equation (1) leads to the one-step ahead predictive (skew) generalized Student’s t-distribution for the observations (see Equation (6)) and the marginal likelihood that is a product of the (skew) generalized Student’s t-distributions, which was also used by [Wang et al., 2013] for modelling volatility data.

### 3 A simulation

We assess the performance of the proposed model using a Monte Carlo simulation, following the design of [Sandmann and Koopman, 1998, Jacquier et al., 1994].

The values of $\varphi = (\alpha, \phi, \sigma^2, r)^T$ are chosen in the following manner. First, we set the autoregressive parameter $\phi$ to 0.90, 0.95, and 0.98. Next, we take the values of $\sigma^2$ for each value of $\phi$, so as to ensure that the coefficient of variation (CV) $\exp(\frac{\sigma^2}{\phi^2}) - 1$ takes the values 10, 1, and 0.10. Then, we determine the values of $\alpha$, such that the expected variance is equal to 0.0009. Finally, we set parameter $r$ to 2 and 1 and, thus, assume a Gaussian distribution (GED($r = 2$)) and Laplace distribution (GED($r = 1$), the heavy-tailed case), respectively, for the observation disturbance.

For each parameter setting, we generate 500 time series of length $n = 500$ with normal and Laplace errors. We then estimate the proposed model and
calculate the mean and mean squared error (MSE) of the posterior mode estimates. We adopt proper uniform priors for $\varphi$. The prior distributions are $\phi \sim \text{Unif}(0, 1)$, $\alpha \sim \text{Unif}(-10^3, 10^3)$, $\sigma^2_{\eta} \sim \text{Unif}(0, 10^3)$, $r \sim \text{Unif}(0, 10^3)$, and $\lambda_0 | Y_0 \sim \text{Gamma}(0.001, 0.001)$, as in Gamerman et al. [2013]. Using the Bayesian approach, we use the Metropolis–Hastings (MCMC) with truncated normal proposed densities and the BFGS algorithms, implemented using Ox [Doornik, 2009]. We use two chains, 5,000 iterations of the MCMC algorithm, and a burn-in of 4,000 iterations. We perform simulations on a Pentium dual-core computer, with a 2.3 GHz processor and 4GB of RAM.

The parameter estimates of the GED-Gamma and normal-Gamma models are close to the true values for several settings with different coefficient variation values (see Table 1) in the light-tailed case (normal errors). In general, the bias and MSE of the GED-Gamma model are close to the normal-Gamma model, which is the true model considered in this case. The MSE values are small, and compete with other methods [Sandmann and Koopman, 1998, Davis and Yam, 2005, see]. Among the four static parameters, $\alpha$ has the largest bias, in general. The bias for $\alpha$ and $\sigma^2_{\eta}$, with $CV = 10$, is larger than with $CV = 1$ and 0.1. For $CV = 10$, the bias of our method for $\alpha$ is slightly larger than those of the MCMC, QML [Sandmann and Koopman, 1998, see], IS, and AIS [Davis and Yam, 2005, see] methods. However, for $CV = 0.1$, the estimates are not as biased as they are in Davis and Yam [2005] and Sandmann and Koopman [1998]. For the heavy-tailed case (Laplace errors of the observation equation), clearly, the bias and MSE of the normal-Gamma model are larger than those in the GED-Gamma model. This indicates a need for more flexible heavy-tailed models, such as the proposed GED-Gamma model in this work, and that ignoring flexible tails may lead you a poor scenario in terms of estimation.

4 A case study with return data

This case study uses the daily return data a Petrobrás (a Brazilian company) asset and the pound/dollar exchange rate. The first is for the period 02/01/2001 to 06/02/2015 (3546 observations), and the second is for the period 10/01/1981 to 06/28/1985 (946 observations). The data can be found at the Yahoo finance website, and the second data set is also available in Durbin and Koopman [2001]. Here, the return series at time $t$ is defined as $y_t = R_t = 100 \ln \left( \frac{P_t}{P_{t-1}} \right)$, centered around the sample mean, where $P_t$ is the daily closing spot price. For the second data set, $P_t$ represents the daily closing exchange rate. Data irregularity due to holidays and weekends is ignored. We perform our case study using Ox [Doornik, 2009] installed on a Pentium dual-core computer, with a 2.3 GHz processor and 4GB of RAM. The codes are available upon authors request.

Figure 3 presents the time series plots of the Petrobrás and pound/dollar returns. The Pound/Dollar return data set was analyzed by Harvey et al. [1994] and then reanalyzed by Davis and Yam [2005]. A distinctive feature of financial time series is that they usually present nonconstant variance or volatility (see Figure 3). Descriptive statistics are shown in Table 2. The Petrobrás return se-
Table 1: Comparison of static parameter estimates of the proposed GED-Gamma model, with different CV values and normal and Laplace errors, based on 500 replications. For each parameter, the posterior mode estimate and mean square error are presented.

| CV=10 | GED(r=2) (Normal) Errors | CV=10 | GED(r=1) (Normal) Errors | CV=10 | GED(r=1) (Laplace) Errors |
|-------|--------------------------|-------|---------------------------|-------|--------------------------|
| σ²    | φ                        | µ     | r                         | σ²    | φ                        | µ     | r                         | σ²    | φ                        | µ     | r                         |
| True  | 0.456                    | 0.900 | -0.821                    | 1.000 | 0.234                    | 0.950 | -0.411                    | 2.000 | 0.095                    | 0.980 | -0.164                    | 2.000 |
| Normal| 0.222                    | 0.001 | 0.105                     | -0.004 | 2.938 -0.035             | -0.001 | 8.45E-5                   | 0.008 | -                     |
| GED   | 0.235                    | 0.859 | -0.876                    | 1.838 | 0.153                    | 0.945 | -0.518                    | 1.948 | 0.070                    | 0.976 | -0.226                    | 1.998 |
| MSE   | 0.001                    | 0.001 | 0.998                     | 0.010 | 3.42E-4                  | 0.044 | 0.078                     | 0.002 | 3.85E-5                  | 0.011 | 0.080                     |
| CV=1  | True  | 0.018                    | 0.900 | -0.736                    | 1.000 | 0.068                    | 0.950 | -0.368                    | 2.000 | 0.028                    | 0.980 | -0.141                    | 2.000 |
| Normal| 0.000                    | 0.001 | 0.033                     | -0.001 | 1.55E-4                 | 0.001 | 1.41E-4                   | 3.22E-5 | 0.002 |                     |
| GED   | 0.019                    | 0.896 | -0.709                    | 2.053 | 0.055                    | 0.946 | -0.424                    | 2.924 | 0.022                    | 0.978 | -0.171                    | 2.033 |
| MSE   | 0.002                    | 0.001 | 0.063                     | 0.001 | 2.65E-4                  | 0.022 | 0.081                     | 1.76E-4 | 0.003 | 0.068 |                     |
| CV=0.1| True  | 0.132                    | 0.900 | -0.710                    | 1.000 | 0.099                    | 0.950 | -0.353                    | 2.000 | 0.004                    | 0.980 | -0.141                    | 2.000 |
| Normal| 0.102                    | 0.895 | -0.792                    | 2.009 | 0.038                    | 0.949 | -0.373                    | 2.045 | 0.003                    | 0.976 | -0.148                    | 2.039 |
| GED   | 0.034                    | 0.001 | 0.085                     | 0.091 | 6.47E-5                  | 0.007 | 7.82E-5                   | 9.80E-6 | 0.001 | 0.060 |                     |
| MSE   | 0.025                    | 0.001 | 0.140                     | 0.016 | 9.07E-4                  | 0.034 | 0.013                     | 8.67E-6 | 0.003 | 0.010 |                     |
| CV=1  | True  | 0.132                    | 0.900 | -0.706                    | 1.000 | 0.009                    | 0.950 | -0.353                    | 2.000 | 0.004                    | 0.980 | -0.141                    | 2.000 |
| Normal| 0.099                    | 0.859 | -2.359                    | -0.593 | 2.44E-4                 | 0.044 | 2.44E-4                   | 0.203 | -                     |
| GED   | 0.055                    | 0.852 | -0.486                    | 1.001 | 0.052                    | 0.944 | -0.257                    | 1.008 | 0.022                    | 0.976 | -0.107                    | 1.003 |
| MSE   | 0.032                    | 0.001 | 0.140                     | 0.016 | 3.97E-4                  | 0.034 | 0.013                     | 1.07E-4 | 0.005 | 0.010 |                     |
| CV=1  | True  | 0.048                    | 0.900 | -0.704                    | 1.000 | 0.099                    | 0.950 | -0.353                    | 2.000 | 0.004                    | 0.980 | -0.141                    | 2.000 |
| Normal| 0.303                    | 0.851 | -2.349                    | -0.400 | 0.922                  | -1.359 | -0.226                    | 0.969 | -0.563 |                     |
| GED   | 0.099                    | 0.005 | 3.259                     | 0.120 | 0.001                   | 1.135 | 0.044                    | 2.44E-4 | 0.203 |                     |
| MSE   | 0.025                    | 0.850 | -0.422                    | 1.047 | 0.016                    | 0.945 | -0.210                    | 1.015 | 0.006                    | 0.977 | -0.087                    | 1.014 |
| CV=1  | True  | 0.000                    | 0.900 | -0.700                    | 1.000 | 0.000                    | 0.950 | -0.350                    | 2.000 | 0.004                    | 0.980 | -0.141                    | 2.000 |
| Normal| 0.845                    | 0.845 | -2.340                    | -0.182 | 0.929                  | -1.985 | -0.694                    | 0.976 | -0.395 |                     |
| GED   | 0.260                    | 0.005 | 3.269                     | 0.040 | 0.001                   | 0.715 | 0.012                    | 7.90E-5 | 0.079 |                     |
| MSE   | 0.037                    | 0.878 | -0.457                    | 1.021 | 0.004                    | 0.947 | -0.195                    | 1.023 | 0.001                    | 0.979 | -0.077                    | 1.032 |
| MSE   | 0.000                    | 0.011 | 0.228                     | 0.015 | 9.46E-4                 | 0.056 | 0.011                    | 6.07E-6 | 1.68E-5 | 0.004 | 0.008 |                     |

GED(r=1) (Laplace) Errors
Table 2: Descriptive statistics for the return series.

|                | Petrobrás | Pound/Dollar |
|----------------|-----------|--------------|
| No. of obs     | 3546      | 946          |
| Mean           | 0.0000    | 0.0000       |
| Median         | 0.003662  | -0.0104      |
| Std. dev.      | 3.059     | 0.711        |
| Skewness       | 0.340     | 0.604        |
| Kurtosis       | 11.54     | 7.862        |
| P-value (Normality test) | 0.000 | 0.000 |

The series presents an excess of kurtosis compared to that of the pound/dollar returns. Both series have a slight positive skewness.

The proposed GED-Gamma model is fitted to the return data of these assets using the Bayesian approach and the Metropolis–Hastings (MCMC) and BFGS algorithms, implemented in Ox [Doornik, 2009]. For our model, we adopt independent proper uniform priors for \( \varphi \). The prior distributions for the parameters are \( \phi \sim \text{Unif}(0, 1) \), \( \alpha \sim \text{Unif}(\text{−10}^3, \text{10}^3) \), \( \sigma_\eta^2 \sim \text{Unif}(0, \text{10}^3) \), \( r \sim \text{Unif}(0, \text{10}^3) \), and \( \lambda_0|Y_0 \sim \text{Gamma}(0.001, 0.001) \). We used two chains, 5,000 iterations of the MCMC algorithm, and a burn-in of 4,000 iterations.

Table 3 shows the log-likelihood value and the Bayes factor used to evaluate the model fit. For the Petrobrás returns, the results show that our GED-Gamma model outperforms the normal-Gamma model. There is strong evidence in favor of the GED-Gamma model \( BF = 0.0067 \), which allows a more flexible heavy-tailed distribution for the observation disturbance and this was also indicated for the heavy-tailed case in the simulation. For pound/dollar returns, the results are similar, with a slight, but negligible preference by the normal model, according to the Bayes factor. The parameter estimates for the GED-Gamma and normal-Gamma models are shown in Table 4. A residual analysis shows no strong violation of the proposed model assumptions. Figures 4 and 5 show the smoothed volatility for the two assets, using the procedure described in Subsection 2.2. The estimated volatility follows the volatility pattern of the return series well, and presents peaks that correspond with crisis periods.

Table 3: Values of the log-likelihood and Bayes factors for the proposed GED-Gamma and normal-Gamma models fitted to the Petrobrás and pound/dollar returns.

| Criterion | Assets         | Penny/Dollar | Petrobrás |
|-----------|----------------|--------------|-----------|
| LogLik†   | GED            | -925.74      | -890.52   |
| MLogLik** |                | -928.94      | -893.00   |
| BF        | Normal*†      | -925.97      | -896.45   |
|           |                | -928.03      | -898.00   |

Note: Bayes factors against the proposed GED-Gamma model. †at the posterior mode; *the normal model is the proposed GED-Gamma model, with \( r = 2 \); **marginal log-likelihood after integrating out the parameters.
4.1 Comparisons of the competing models

In this section, we compare the GED-Gamma model to other models, including the Kim, Shephard, and Chib SV model (SV-KSC) [Kim et al., 1998], and the Bayesian GARCH(1,1) model with Student’s t-disturbances [Ardia, 2008] (t-GARCH(1,1)). The former is implemented using Ox, by Pelagatti [2011], and latter using the R package ‘bayesGARCH’ [Ardia, 2015].

For the SV-KSC approach, the simplest SV model is linearized, so \( \log(y_t^2) \) is used. An approximation for the log \( \chi^2_1 \) distribution is performed using a seven-component Gaussian mixture. Therefore, conditional on latent indicator variables \( w_t \in \{1, \ldots, 7\} \), \( t = 1, \ldots, n \), the SV-KSC model is given by

\[
\log y_t^2 - \mu_{w_t} = h_t + \log \varepsilon_t^2, \quad \varepsilon_t \sim N(0, \sigma^2_{w_t}),
\]

(12)

\[
h_t = \alpha + \phi h_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim N(0, \sigma^2_{\varepsilon_t}),
\]

(13)

\( t = 1, \ldots, n \). We specify the independent noninformative priors for the model parameters given by \( p(\alpha, \phi) \propto p_N(\alpha, \phi) \) and \( p(\sigma^2_{\varepsilon_t}) \propto p_{IG}(\sigma^2_{\varepsilon_t}) \), where IG denotes the inverse gamma distribution. The hyperparameters of the priors are the package default values.

Another competing model is the GARCH(1,1) model with Student’s t-innovations [Ardia, 2008], given as:

\[
y_t = \left( \frac{(\nu - 2)h_t}{\nu} \right)^{-1/2} \varepsilon_t,
\]

(14)

\( t = 1, \ldots, n \), where the conditional variance equation is \( h_t = \alpha_0 + \alpha_1 y^2_{t-1} + \beta h_{t-1} \), \( \alpha_0 > 0 \), \( \alpha_1 \geq 0 \) and \( \beta \geq 0 \) to ensure a positive conditional variance and, finally, \( \varepsilon_t \sim \text{Student-t}(\nu) \). The independent prior used in the t-GARCH(1,1) model is: \( p(\alpha_0) \propto p_N(\alpha_0; 0, 1000)I_{[\alpha_0 > 0]} \), \( p(\alpha_1) \propto p_N(\alpha_1; 0, 1000)I_{[\alpha_1 > 0]} \), \( p(\beta) \propto p_N(\beta; 0, 1000)I_{[\beta > 0]} \) and \( p(\nu) = 0.01 \exp(-0.01(\nu - 2))I_{[\nu > 2]} \).

For the t-GARCH(1,1) and the SV-KSC models, we utilized one chain, 12,000 iterations of the MCMC algorithm, and a burn-in of 10,000 iterations for the Bayesian methods. The convergence of the chain was checked using methods, such as graphs.

Table 4 presents the static parameter estimates of the proposed approach, the SV-KSC, and t-GARCH(1,1) methods. The estimates of our proposed model with normal innovations are very close to those of the SV-KSC method, with normal innovations, in most of the cases for the two assets. The interval estimates (Table 4) show that the needed of a more flexible model than the normal-Gamma, as the GED-Gamma model.

For the pound/dollar returns, our parameter estimates (Table 4) are very close to those obtained by Davis and Yara [2002, p.397] and Durbin and Koopman [2001, p.236]. The QML estimates from Harvey et al. [1994] are similar to ours, but mainly in the normal-Gamma case. Furthermore, the parameter estimates of our model are very similar to those of the SV-KSC (Table 4). The parameters \( \alpha_0, \alpha_1, \beta, \) and \( \nu \) belong to the t-GARCH(1,1). The estimate of \( \nu \) indicates a heavy-tailed pattern for the two return series.
Table 4: Static parameter estimates of the models fitted to the Petrobrás and pound/dollar returns.

| Methods                      | GED       | Normal§§ | SV-KSC    | t-GARCH(1,1) |
|------------------------------|-----------|-----------|-----------|--------------|
|                              | P. Mode   | P. Mode   | P. Mean   | P. Mean      |
| Estimates Petrobrás           |           |           |           |              |
| $\hat{\sigma}^2_\eta$       | 0.011 (0.012) | 0.019 (0.021) | 0.025 | -            |
| CI                           | [0.0065;0.0226] | [0.0140;0.0310] | [0.0161;0.0375] | -            |
| $\phi$                       | 0.985 (0.983) | 0.980 (0.978) | 0.980 | -            |
| CI                           | [0.9685;0.9929] | [0.9670;0.9868] | [0.9680;0.9889] | -            |
| $\alpha$                     | 0.019 (0.022) | 0.028 (0.031) | 0.037 | -            |
| CI                           | [0.0083;0.0426] | [0.0167;0.0490] | [0.0198;0.0588] | -            |
| $\hat{\nu}$                  | 1.725 (1.744) | - | - | -            |
| CI                           | [1.5972;1.9041] | - | - | -            |
| $\hat{\alpha}_0$             | - | - | - | 0.205        |
| CI                           | - | - | - | [0.13498;0.2900] |
| $\alpha_1$                   | - | - | - | 0.082        |
| CI                           | - | - | - | [0.06468;0.1023] |
| $\beta$                      | - | - | - | 0.894        |
| CI                           | - | - | - | [0.87091;0.9154] |
| $\hat{\nu}$                  | - | - | - | 7.289        |
| CI                           | - | - | - | [5.7441;9.390] |
| Estimates Pound/Dollar        |           |           |           |              |
| $\hat{\sigma}^2_\eta$       | 0.019 (0.047) | 0.025 (0.042) | 0.036 | -            |
| CI                           | [0.0139;0.0999] | [0.0159;0.0880] | [0.0155;0.0861] | -            |
| $\phi$                       | 0.978 (0.956) | 0.974 (0.957) | 0.969 | -            |
| CI                           | [0.9107;0.9847] | [0.9054;0.9878] | [0.9376;0.9917] | -            |
| $\alpha$                     | -0.028 (-0.063) | -0.036 (-0.059) | -0.031 | -            |
| CI                           | [-0.1755;0.0193] | [-0.1288;0.0191] | [-0.0669;0.0057] | -            |
| $\hat{\nu}$                  | 1.884 (2.049) | - | - | -            |
| CI                           | [1.7097;2.4793] | - | - | -            |
| $\hat{\alpha}_0$             | - | - | - | 0.031        |
| CI                           | - | - | - | [0.0148;0.0557] |
| $\alpha_1$                   | - | - | - | 0.138        |
| CI                           | - | - | - | [0.08970;0.20394] |
| $\beta$                      | - | - | - | 0.810        |
| CI                           | - | - | - | [0.7323;0.8664] |
| $\hat{\nu}$                  | - | - | - | 8.635        |
| CI                           | - | - | - | [4.9359;13.7402] |

Note: §The proposed GED-Gamma model; §§the proposed GED-Gamma model, with $\nu = 2$ (normal case); CI: 95% percentile credibility interval.

For the in-sample analysis of the GED-Gamma model, we use the smoothed mean of volatility, calculated using the smoothing procedure of Subsection 2.2. For the SV-KSC and t-GARCH(1,1) models, we use the posterior mean of the volatility. The square of the log-return is used as a proxy for the true unobserved volatility $\sigma^2_t$ [Bauwens et al., 2012]. Thus, the square root of the mean squared error, $\text{SRMSE} = \sqrt{\frac{1}{n} \sum_{t=1}^{n} (y_t^2 - \hat{\sigma}^2_t)^2}$, and mean absolute error, $\text{MAE} = \frac{1}{n} \sum_{t=1}^{n} |y_t^2 - \hat{\sigma}^2_t|$, are used to compare the models. For the in-sample analysis of the pound/dollar returns, the GED-Gamma model has the smallest SRMSE value (Table 5). For the Petrobrás return series, the smallest MAE value. In most cases, the GED-Gamma model is the best or second best of the competing models, indicating that it performs well in terms of fit.
Table 5: The SRMSE and MAE of in-sample estimation of the volatility of the proposed GED-Gamma, SV-KSC and t-GARCH models fitted to the Petrobrás and pound/dollar returns.

| Assets       | GED Model | SV-KSC | GARCH-t |
|--------------|-----------|--------|---------|
| Petrobrás    | SRMSE     | 27.23 (2) | 25.89 (1) | 30.34 (3) |
|              | MAE       | 8.36 (1) | 8.87 (3) | 8.53 (2) |
| Pound/Dollar | SRMSE     | 1.15 (1) | 1.25 (2) | 1.34 (3) |
|              | MAE       | 0.48 (2) | 0.59 (3) | 0.47 (1) |

Note: The numbers in parentheses denote the ranking among the competing models.

4.1.1 Out-of-sample forecast comparisons

For the out-of-sample analysis, a direct comparison of volatility forecasts is adopted using the square of the log-return as a proxy for the true unobserved volatility $\sigma^2_{t+1}$ [Bauwens et al., 2012]. For the proposed GED-Gamma model, the one-step ahead forecast volatility $\hat{\sigma}^2_{t+1}$ is calculated using the distribution of Item 1 on page 5. Under the Bayesian approach, the SRMSE and MAE are computed using the one-step ahead forecast $\hat{\sigma}^2_{t+1}$ of the competing models, leaving the out last five observations, then the last four observations out, and so on, until the last observation is left out. Finally, the SRMSE and MAE are computed as $\sqrt{\frac{1}{5} \sum_{k=1}^{5} (y^2_{t+k} - \hat{\sigma}^2_{t+k})^2}$ and $\frac{1}{5} \sum_{k=1}^{5} |y^2_{t+k} - \hat{\sigma}^2_{t+k}|$, respectively, where the index $k$ varies over the last five observations.

Table 6 presents the SRMSE and MAE of one-step ahead forecasts of the proposed GED-Gamma, SV-KSC, and t-GARCH(1,1) models. For the out-of-sample analysis of the pound/dollar returns, the SRMSE and MAE values of the GED-Gamma and SV-KSC models are similar, while the MAE and MSE of the GED-Gamma model are smaller than those of the SV-KSC and t-GARCH(1,1) models for the Petrobrás return series. In most cases, the GED-Gamma model is the best or second best of the three competing models. This indicates that the proposed GED-Gamma model is also a good option in terms of prediction.

Table 6: The SRMSE and MAE of the one-step ahead forecasts for the volatility of the proposed GED-Gamma, SV-KSC and t-GARCH models fitted to the Petrobrás and pound/dollar returns.

| Assets       | GED Model | SV-KSC | t-GARCH(1,1) |
|--------------|-----------|--------|--------------|
| Petrobrás    | SRMSE     | 80.77 (1) | 86.58 (2) | 86.78 (3) |
|              | MAE       | 56.57 (1) | 57.05 (2) | 59.45 (3) |
| Pound/Dollar | SRMSE     | 1.88 (2) | 1.84 (1) | 2.06 (3) |
|              | MAE       | 1.27 (2) | 1.10 (1) | 1.29 (3) |

Note: The numbers in parentheses denote the ranking among the competing models.
5 Conclusion

In this study, we introduced a GED-Gamma SV model for return data with an approximated expression for the marginal likelihood, which can be evaluated directly, under the Bayesian approach. Using the model, we propose new sequential analysis and smoothing procedures and a marginal likelihood that is a product of the generalized Student’s t-distributions based on an analytical approximation for the distribution of the latent states. The main advantages of the proposed method are its mathematical and computational simplicity and its ability to accommodate the stylized facts of return data and a stationary Gaussian evolution equation, circumventing the problem of the high-dimensional latent states. There is no need to linearize the model; that is, the data scale is not changed and is free from approximation of the observation distribution. Non-Gaussian, heavy-tailed skew distributions for the observations are naturally accommodated. Beyond of the approximated sequential analysis procedure, the smoothing procedure is provided. Another interesting feature is the availability of the one-step ahead predictive distribution, which is the generalized Student-t distribution.

A limitation of the model is the use of approximations for the distribution of the latent states in terms of the two first moments, because it was developed as a DGLM [West and Harrison, 1997, Souza et al., 2018]. The quality of this approximation depends on the quality of the normal approximation to the log-gamma prior distribution of the latent states. The DGLM has a dynamic structure in the mean of the data, which here is volatility. Both methods preserve the sequential analysis of the data.

Our approach performed well in the parameter estimation of the GED-Gamma SV model using the posterior mode, mean and quantiles under the Bayesian perspective. The empirical results are competitive compared to other methods in the literature in terms of fit and prediction. Thus, we achieved our primary objective of introducing a Bayesian GED-Gamma SV model that can be implemented in a fast and easy way, and that is free of approximations for the observation equation. The results and the proposed procedures of this study are also useful to the closely related time series model. For example, the dynamic linear models proposed by [West and Harrison, 1997], for normal observations with time varying means and variances, allows a stationary evolution equation for the volatility. Our results can also be used in the model of [Nakajima and Omori, 2009], without the need to linearize the model for the volatility sampling.

Future works could include a study of other distributions for the observation equation (especially skew distributions), the inclusion of exogenous explanatory variables on volatility.
Appendix I

This appendix presents the proof of proposition 1 of the inferential procedure in the text.

Proposition 1.

We first provide the proofs of Parts 1 to 3 relating to the basic sequential inference of the proposed model. For ease of notation, we omit the static parameter vector \( \varphi \) from the proofs.

**Proof of Part 1:**

Assume from the hypothesis that \( \lambda_{t-1} | Y_{t-1} \sim \text{Gamma} (a_{t-1}, b_{t-1}) \); thus, according to [West and Harrison, 1997], Chapter 14,

\[
\ln(\lambda_{t-1}) | Y_{t-1} \sim \text{Log-Gamma} [f_{t-1} = \gamma(a_{t-1}) - \ln(b_{t-1}), q_{t-1} = \gamma'(a_{t-1})],
\]

where \( \gamma(b_{t-1}) \) and \( \gamma'(a_{t-1}) \) are the digamma and trigamma functions, respectively. Next, we approximate the log-gamma distribution by the normal distribution in terms of the two first moments. Then,

\[
\ln(\lambda_{t-1}) | Y_{t-1} \sim \text{Normal} (f_{t-1}, q_{t-1}).
\]

Now, we combine the above approximated distribution of \( \ln(\lambda_{t-1}) \) with the evolution equation \( \ln(\lambda_t) | \ln(\lambda_{t-1}) \sim \text{Normal} (-\alpha + \phi \ln(\lambda_{t-1}), \sigma_n^2) \) to obtain

\[
p(\ln(\lambda_t) | Y_{t-1}).
\]

Using the properties of the multivariate normal distribution of [Harvey, 1989; West and Harrison, 1997], we have

\[
p(\ln(\lambda_t) | Y_{t-1}) \equiv \int p(\ln(\lambda_{t-1}) | Y_{t-1}) p(\ln(\lambda_t) | \ln(\lambda_{t-1})) d\ln(\lambda_{t-1})
\]

\[
d = \text{Normal} (f_{t|t-1}, q_{t|t-1}),
\]

where \( f_{t|t-1} = -\alpha + \phi f_{t-1} \) and \( q_{t|t-1} = \phi^2 q_{t-1} + \sigma_n^2 \).

Since \( \lambda_t | Y_{t-1} \sim \text{Gamma} (a_{t|t-1}, b_{t|t-1}) \) and \( \ln(\lambda_t) | Y_{t-1} \sim \text{Normal} (f_{t|t-1}, q_{t|t-1}) \), the pair \((a_{t|t-1}, b_{t|t-1})\) can be elicited in terms of the two first moments \( f_{t|t-1} = \gamma(a_{t|t-1}) - \ln(b_{t|t-1}) \) and \( q_{t|t-1} = \gamma'(a_{t|t-1}) \). With suitable approximations for the log-gamma and trigamma functions [Abramovitz and Stegun, 1964], we have \( f_{t|t-1} \approx \ln(a_{t|t-1}) - \ln(b_{t|t-1}) \) and \( q_{t|t-1} \approx 1/a_{t|t-1} \), and then \( a_{t|t-1} = q_{t|t-1}^{-1} \) and \( b_{t|t-1} = \exp(-f_{t|t-1}) q_{t|t-1}^{-1} \). Now, by replacing \( f_{t|t-1} \) and \( q_{t|t-1} \) by their respective expressions, we have \( a_{t|t-1} = (\phi^2 a_{t-1}^{-1} + \sigma_n^2)^{-1} \) and \( b_{t|t-1} = \exp(\alpha)(a_{t-1}/b_{t-1})^{-\alpha} \).

Therefore,

\[
\lambda_t | Y_{t-1} \sim \text{Gamma}(a_{t|t-1}, b_{t|t-1}),
\]

\[
a_{t|t-1} = (\phi^2 a_{t-1}^{-1} + \sigma_n^2)^{-1} \text{ and } b_{t|t-1} = \frac{\exp(\alpha)(a_{t-1}/b_{t-1})^{-\alpha}}{(\phi^2 a_{t-1}^{-1} + \sigma_n^2)},
\]

to complete the proof of Part 1.
Proof of Part 2:
To calculate the on-line or update distribution of $\lambda_t$, we have

$$p(\lambda_t|Y_t) \propto p(y_t|\lambda_t)p(\lambda_t|Y_{t-1}) \propto \lambda_t^{a_{t|t-1}+1/r-1} \exp[-\lambda_t(b_{t|t-1} + \psi(r)|y_t|\gamma)].$$

Thus, it follows that $\lambda_t|Y_t \sim \text{Gamma}(a_t, b_t)$, where $a_t = a_{t|t-1} + 1/r$ and $b_t = b_{t|t-1} + \psi(r)|y_t|\gamma$, completing the proof.

\[\square\]

Proof of Part 3:

$$p(y_t|Y_{t-1}) = \int_0^\infty p(y_t|\lambda_t)p(\lambda_t|Y_{t-1})d\lambda_t$$

$$= \frac{(r\Gamma(3/r)^{1/2})}{\Gamma(a_{t|t-1})(b_{t|t-1})^{-a_{t|t-1}} \int_0^\infty [\lambda_t^{1/r+a_{t|t-1}-1} \exp(-\lambda_t(\psi(r)|y_t|\gamma + b_{t|t-1}))] d\lambda_t}$$

$$= \frac{\Gamma(1/r + a_{t|t-1}) (r\Gamma(3/r)^{1/2})^{a_{t|t-1}}}{\Gamma(a_{t|t-1})(\psi(r)|y_t|\gamma + b_{t|t-1})^{a_{t|t-1}+1/r}}, y_t \in \mathbb{R},$$

where $a_{t|t-1} = (\phi^2 a_{t-1} + \sigma^2)^{-1}$ and $b_{t|t-1} = \frac{\exp(\alpha(a_{t-1}/b_{t-1})^{1/2})}{\phi^2 a_{t-1} + \sigma^2}$

are parameters of the prior distribution of $\lambda_t$ in Part 1 of the results.

\[\square\]

Appendix II

Proposition 2.

This appendix presents the proof of Proposition 2 of the smoothing procedure in the text. We omit the static parameter vector $\varphi$ from the proof. Samples are taken from the smoothed log-precision $\ln(\lambda)$ distribution. Consequently, we obtain samples from the precision $\lambda$ and $h = \lambda^{-1}$ volatility distributions.
\[ p(\ln(\lambda_t)|\ln(\lambda_{t+1}), Y_t) = \frac{p(\ln(\lambda_{t+1})|\ln(\lambda_t), Y_t) \times p(\ln(\lambda_t)|Y_t)}{p(\ln(\lambda_{t+1})|Y_t)} \]

\[
p(\ln(\lambda_t)|\ln(\lambda_{t+1}), Y_t) \propto \exp\left[ \frac{-1}{2} \left( \frac{\phi^2}{\sigma^2 \eta} + \frac{1}{q_t} \right)^{-1} \times \left( \ln(\lambda_t)^2 - 2 \ln(\lambda_t) \times \left( \frac{\phi^2}{\sigma^2 \eta} + \frac{1}{q_t} \right)^{-1} \times \left( \frac{\phi^2 (\ln(\lambda_{t+1}) + \alpha)}{\sigma^2 \eta} + \frac{f_t}{q_t} \right) \right) \right].
\]

Therefore, \( p(\ln(\lambda_t)|\ln(\lambda_{t+1}), Y_t) \) is a normal distribution, with approximate mean \( \mu^*_t = \sigma^*_t \times \left( \frac{\phi^2 (\ln(\lambda_{t+1}) + \alpha)}{\sigma^2 \eta} + \frac{f_t}{q_t} \right) \) and variance \( \sigma^*_t = \left( \frac{\phi^2}{\sigma^2 \eta} + \frac{1}{q_t} \right)^{-1}. \)

\[ \square \]

References

R. S. Tsay. *Analysis of financial time series*. Wiley Series in Probability and Statistics, New Jersey, 2010.

R. F. Engle. Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica*, 50(4):987–1007, 1982.

T. Bollerslev. Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, 31:317–327, 1986.

S. J. Taylor. Financial returns modelled by the product of two stochastic processes, A study of daily sugar prices. In O. D. Anderson (Ed.), Time Series Analysis: Theory and Practice 1, pp. 203-226, Amsterdam: North-Holland, 1982.

A. C. Harvey, E. Ruiz, and N. Shephard. Multivariate stochastic variance models. *The Review of Economic Studies*, 61:247–264, 1994.

J. Danielsson. Stochastic volatility in asset prices: estimation with simulated maximum likelihood. *Journal of Econometrics*, 61:375–400, 1994.

G. Sandmann and S. J. Koopman. Estimation of stochastic volatility models via monte carlo maximum likelihood. *Journal of Econometrics*, 87:271–301, 1998.
S. Kim, N. Shepard, and S. Chib. Stochastic volatility: likelihood inference and comparison with arch models. *The Review of Economic Studies*, 65(3): 361–393, 1998.

S. J. Taylor. *Modelling Financial Time Series*. Wiley, Chichester, 1986.

A. Melino and S. M. Turnbull. Pricing foreign currency options with stochastic volatility. *Journal of Econometrics*, 45:239–265, 1990.

C. Broto and E. Ruiz. Estimation methods for stochastic volatility models: a survey. *Journal of Economic Surveys*, 18:613–649, 2004.

E. Jacquier, N. G. Polson, and P. E. Rossi. Bayesian analysis of stochastic volatility models. *Journal of Business and Economic Statistics*, 12(4):371–389, 1994.

E. Jacquier, N. G. Polson, and P. E. Rossi. Bayesian analysis with fat-tails and correlated errors. *Journal of Econometrics*, 122(1):185–212, 2004.

N. Cappuccio, D. Lubian, and D. Raggi. Mcmc bayesian estimation of a skew-gev stochastic volatility model. *Studies in Nonlinear Dynamics & Econometrics*, 8(2):Article 6. Available at http://www.bepress.com/snde/vol8/iss2/art6, 2004.

M. K. Pitt and N. Shephard. Filtering via simulation: Auxiliary particle filters. *Journal of the American Statistical Association*, 94(446):590–599, 1999.

H. F. Lopes and R. S. Tsay. Particle filters and bayesian inference in financial econometrics. *Journal of Forecasting*, 30(1):168–209, 2011.

S. Malik and M. K. Pitt. Particle filters for continuous likelihood evaluation and maximisation. *Journal of Econometrics*, 165:190–209, 2011.

H. Rue, S. Martino, and N. Chopin. Approximate bayesian inference for latent gaussian models by using integrated nested laplace approximations. *Journal of the royal statistical society: Series B*, 71(2):319–392, 2009.

S.J. Taylor. Modelling stochastic volatility: A review and comparative study. *Mathematical Finance*, 4:183–204, 1994.

S. Chib, F. Nardari, and N. Shephard. Markov chain monte carlo methods for stochastic volatility models. *Journal of Econometrics*, 108(2):281–316, 2002.

J. Yu. On leverage in a stochastic volatility model. *Journal of Econometrics*, 127(2):165–178, 2005.

Y. Omori, S. Chib, N. Shephard, and J. Nakajima. Stochastic volatility with leverage: Fast and efficient likelihood inference. *Journal of Econometrics*, 140 (2):425–449, 2007.
D. Raggi and S. Bordignon. Comparing stochastic volatility models through monte carlo simulations. *Computational Statistics and Data Analysis*, 50:1678–1699, 2006.

G. Kastner and S. Fruhwirth-Schnatter. Ancillarity-sufficiency interweaving strategy (asis) for boosting mcmc estimation of stochastic volatility models. *Computational Statistics and Data Analysis*, 76:408–423, 2014.

M. Ferrante and P. Vidoni. Finite dimensional filters for nonlinear stochastic difference equations with multiplicative noises. *Stochastic Processes and Their Applications*, 77:69–81, 1998.

P. Vidoni. Exponential family state space models based on conjugate latent process. *Journal of the Royal Statistical Society: Series B*, 61:213–221, 1999.

R. A. Davis and G. R. Yam. Estimation for state-space models based on a likelihood approximation. *Statistica Sinica*, 15:381–406, 2005.

T. Watanabe. A non linear filtering approach to stochastic volatility models with an application to daily stock returns. *Journal of Applied Econometrics*, 14:101–121, 1999.

J. L. Knight and J. Yu. The empirical characteristic function in time series estimation. *Econometric Theory*, 18:691–721, 2002.

B. Feunou and R. Tedongap. A stochastic volatility model with conditional skewness. *Journal of Business and Economic Statistics*, 30:576–591, 2012.

S. J. Koopman and C. S. Bos. State space models with a common stochastic variance. *Journal of Business and Economic Statistics*, 22:346–357, 2012.

D. Gamerman, T. R. Santos, and G. C. Franco. A non-gaussian family of state-space models with exact marginal likelihood. *Journal of Time Series Analysis*, 34:625–645, 2013.

F. M. Pinho, G. C. Franco, and R. S. Silva. Modelling volatility using state space models with heavy tailed distributions. *Mathematics and Computers in Simulation*, 119:108–127, 2016.

N. Shepard. Local scale models: State space alternative to integrated GARCH processes. *Journal of Econometrics*, 60(1):181–202, 1994.

P. J. Deschamps. Bayesian estimation of an extended local scale stochastic volatility model. *Journal of Econometrics*, 162(2):369–382, 2011.

M. Fridman and L. Harris. A maximum likelihood approach for non-gaussian stochastic volatility models. *Journal of Business and Economic Statistics*, 16:284291, 1998.
P. Abad, S. Benito, and C. Lopez. A comprehensive review of value at risk methodologies. *The Spanish Review of Economics*, 12:15–32, 2014.

G.E.P. Box and G.C. Tiao. *Bayesian Inference in Statistical Analysis*. Wiley, New York, 1992.

D.B. Nelson. Conditional heteroskedasticity in asset returns: a new approach. *Econometrica*, 59:347–370, 1991.

M. West and J. Harrison. *Bayesian Forecasting and Dynamic Models*. Springer, New York, 1997.

K. Triantafyllopoulos. Multivariate stochastic volatility with bayesian dynamic linear models. *Journal of Statistical Planning and Inference*, 138(4):1021–1037, 2008.

D. Gamerman and H. F. Lopes. *Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference*. Chapman & Hall, 2006.

W. R. Gilks, N. G. Best, and K. K. C. Tan. Adaptive rejection metropolis sampling within gibbs sampling. *Journal of the Royal Statistical Society. Series C (Applied Statistics)*, 44:455–472, 1995.

M. Avriel. *Nonlinear Programming: Analysis and Methods*. Courier Corporation, 2003.

D. Gamerman. Dynamic bayesian models for survival data. *Journal of the Royal Statistical Society: Series C (Applied Statistics)*, 40:63–79, 1991.

H. S. Migon, D. Gamerman, H. F. Lopes, and M. A. R. Ferreira, editors. *Dynamic models*. Elsevier, Amsterdam, 2005.

J. Nakajima and Y. Omori. Leverage, heavy-tails and correlated jumps in stochastic volatility models. *Computational Statistics & Data Analysis*, 53(6):2335–2353, 2009.

J. J. Wang, S. B. Choy, and J. S. Chan. Modelling stochastic volatility using generalized t distribution. *Journal of Statistical Computation and Simulation*, 83(2):340–354, 2013.

J.A. Doornik. *An Object-Oriented Matrix Programming Language Ox 6*. Timberlake Consultants Press, London, UK, 2009.

J. Durbin and S. J. Koopman. *Time Series Analysis by State Space Methods*. Oxford University Press, New York, 2001.

D. Ardia. *Financial Risk Management with Bayesian Estimation of GARCH Models: Theory and Applications*. Springer-Verlag, Berlin, 2008.

M. M. Pelagatti. State space methods in ox ssfpack. *Journal of Statistical Software*, 41:1–25, 2011.
D. Ardia. *Package ‘bayesGARCH’: Bayesian Estimation of the GARCH(1,1) Model with Student-t Innovations in R.* http://CRAN.R-project.org/package=bayesGARCH. R project, 2015.

L. Bauwens, C. M. Hafner, and S. Laurent. *Handbook of volatility models and their applications.* John Wiley & Sons, 2012.

M. A. D. O. Souza, H. D. S. Migon, and J. B. M. Pereira. Extended dynamic generalized linear models: the two-parameter exponential family. *Computational Statistics & Data Analysis,* 121:164–179, 2018.

A. C. Harvey. *Forecasting, structural time series models and the Kalman filter.* University Press, Cambridge, 1989.

M. Abramovitz and I. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables.* National Bureau of Standards Applied Mathematics, US Government Printing Office, Washington, DC, 1964.
Figure 1: The log-gamma and normal distributions of the states for some values of the shape (a) and scale (b) parameters.

Figure 2: The sequential analysis procedure.
Figure 3: The Petrobrás and pound/dollar return series.
Figure 4: The smoothed mean volatility obtained using the proposed GED-Gamma volatility model for the Petrobrás returns. The grey area indicates the 95% credibility intervals.
Figure 5: The smoothed mean volatility obtained using the proposed GED-Gamma volatility model for the pound/dollar exchange rate. The grey area indicates the 95% credibility intervals.