Analytical Framework for Optimizing Weighted Average Download Time in Peer-to-Peer Networks

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Abstract—This paper proposes an analytical framework for peer-to-peer (P2P) networks and introduces schemes for building P2P networks to approach the minimum weighted average download time (WADT). In the considered P2P framework, the server, which has the information of all the download bandwidths and upload bandwidths of the peers, minimizes the weighted average download time by determining the optimal transmission rate from the server to the peers and from the peers to the other peers. This paper first defines the static P2P network, the hierarchical P2P network and the strictly hierarchical P2P network. Any static P2P network can be decomposed into an equivalent network of sub-peers that is strictly hierarchical. This paper shows that convex optimization can minimize the WADT for P2P networks by equivalently minimizing the WADT for strictly hierarchical networks of sub-peers. This paper then gives an upper bound for minimizing WADT by constructing a hierarchical P2P network, and lower bound by weakening the constraints of the convex problem. Both the upper bound and the lower bound are very tight. This paper also provides several suboptimal solutions for minimizing the WADT for strictly hierarchical networks, in which peer selection algorithms and chunk selection algorithm can be locally designed.

Index Terms—P2P network, weighted average download time, hierarchical P2P network, strictly hierarchical P2P network

I. INTRODUCTION

P2P applications (e.g., [1], [2], [3]) are increasingly popular and represent a large majority of the traffic currently transmitted over the Internet. A unique feature of P2P networks is their flexible and distributed nature, where each peer can act as both server and client [4]. Hence, P2P networks provide a cost-effective and easily deployable framework for disseminating large files without relying on a centralized infrastructure [5]. These features of P2P networks have made them popular for a variety of broadcasting and file-distribution applications [5], [6], [7], [8], [9]. Specifically, chunk-based and data-driven P2P broadcasting systems such as CoolStreaming [6], Overcast [10] and Chainsaw [7] have been developed, which adopt pull-based techniques [6], [7]. In these P2P systems, the peers possess several chunks and these chunks are shared by peers that are interested in the same content. An important problem in such P2P systems is how to transmit the chunks to the various peers and create reliable and efficient connections between peers. For this, various approaches have been proposed including tree-based and data-driven approaches (e.g., [8], [11], [12], [13], [14], [15], [16]).

Besides these practical approaches, some research has begun to analyze P2P networks from a theoretic perspective to quantify the achievable performance. The performance, scalability and robustness of P2P networks using network coding are studied in [17], [18]. In these investigations, each peer in a P2P network randomly chooses several peers including the server as its parents, and also transmits to its children a random linear combination of all packets the peer has received. Network coding, working as a perfect chunk selection algorithm, guarantees every packet transmitted in a P2P network has new information for its receiver, which makes elegant theoretical analysis possible. Other research studies the steady-state behavior of P2P networks with homogeneous peers using fluid models [19], [20], [21]. Most papers providing theoretical analysis for P2P networks assume dynamic systems with homogeneous peers.

This paper establishes an analytical framework for optimizing weighted average download time (WADT) for P2P networks with heterogeneous peers, i.e., peers with different download bandwidths and upload bandwidths. This paper focuses on static P2P networks with a single server and a fixed number of peers. In other words, no peer leaves or joins the P2P network. In the scheme of building the P2P network, the server collects all the download bandwidths and upload bandwidths of the peers, and minimizes the WADT by determining the optimal transmission rate from the server or any peer to any other peer. A dynamic P2P system can also be modeled as a sequence of static P2P systems. Therefore, this study of static P2P networks is the first step to a complete analytical framework. We leave the study of dynamic P2P networks with heterogeneous peers for future work.

A static P2P network is a directed graph which has one root, the server, and at least one directed path from the root to each of the other nodes, which are peers. In a P2P network, peers are placed into levels according to the topological distances between these peers and the server. A peer is in level $K$ if the length of the longest directed acyclic path from the server to the peer is $K$. A hierarchical P2P network is a P2P network in which each peer can only download from peers in the lower levels and upload to peers in the higher levels. Peers in the same level cannot download or upload to each other. A strictly hierarchical P2P network is a P2P network in which each peer in level $K$ can only download from peers in level $K-1$ and upload to peers in level $K+1$.

This paper shows that any static P2P network can be de-
composed into an equivalent network of sub-peers that is strictly hierarchical. Therefore, convex optimization can minimize the WADT for P2P networks by equivalently minimizing the WADT for strictly hierarchical networks of sub-peers. This paper then gives an achievable upper bound for minimizing WADT by constructing a hierarchical P2P network, and lower bound by weakening the constraints of the convex problem. Both the upper bound and the lower bound are very tight.

The strictly hierarchical P2P network is practical for protocol design because peer selection algorithms and chunk selection algorithms can be locally designed level by level instead of globally designed. Minimizing the WADT for strictly hierarchical networks is a 0-1 convex optimization problem. However, if we have assigned all peers each to a level, then the global bandwidth allocation problem decomposes into local bandwidth allocation problems at each level, which have water-filling solutions. Several suboptimal peer assignment algorithms are provided and simulated. Some of these suboptimal but practical schemes can be used for content distribution systems, e.g. Overcast [19].

This paper is organized as follows. In Section III definitions and notation for P2P networks are introduced. Section III provides and discusses a taxonomy of overlay networks. In Section IV the problem of minimizing the weighted average download time is formulated and solved. Section V presents the simulation results. Section VI presents the conclusions.

II. SETUP AND PROBLEM DEFINITION

Consider a scenario where millions of peers would like to download content from a server in the Internet. The server has sufficient bandwidth to serve tens or hundreds of peers, but not millions. In the absence of IP multicast, one solution is to form the server and the peers into a P2P overlay network and distribute the content using application layer multicast [17] [22]. In this scenario, the content in the server is partitioned into chunks. Peers not only download chunks from the server and other peers but also upload to some other peers that are interested in the content.

This paper focuses on content distribution applications (e.g. BitTorrent, Overcast [19]) in which peers are only interested in content at full fidelity, even if it means that the content does not become available to all peers at the same time. The key issue for these P2P applications is to minimize download times for peers. Since it usually takes several hours or days for a peer to download content in full fidelity, our work is less concerned with interactive response times and transmission delays in buffers and in the network.

This paper studies a scheme to minimize the weighted average download time for a static P2P network. In this scheme, the server first collects the information of peers’ weights, download bandwidths, and upload bandwidths and then performs a centralized optimization to find the best static P2P network, i.e., the optimal transmission rates of the transmission flows from the server or any peer to any other peer to minimize the weighted average download time. The server passes the optimal solution to the peers, and the peers build the connections according to the optimal solution. The rest of the paper will focus on the centralized optimization algorithm for determining the optimal static P2P network.

In a static P2P network, the server with bandwidth $S$ has a file, whose size is 1 unit for simplicity. There are $N$ peers who want to share the file in the network. Each peer has download bandwidth $d_i$ and upload bandwidth $u_i$, for $i = 1, 2, \cdots, N$. These download bandwidths and upload bandwidths are usually determined at the application layer instead of the physical layer because an Internet user can have several downloading tasks and these tasks share the physical download and upload bandwidth of the user.

It is reasonable to assume that $d_i \geq u_i$ for each $1 \leq i \leq N$. For the case of $d_i < u_i$ for peer $i$, we just use the part of the upload bandwidth which is the same as the download bandwidth and leave the rest of the upload bandwidth.

Denote the transmission rate from the server to peer $j$ as $r_{s \rightarrow j}$ and the transmission rate from peer $i$ to peer $j$ as $r_{i \rightarrow j}$. The total download rate of peer $j$, denoted as $r_j$, is the summation of $r_{s \rightarrow j}$ and $r_{i \rightarrow j}$ for all $i \neq j$. Since the total download rate is constrained by the download bandwidth, we have $r_j = r_{s \rightarrow j} + \sum_{i \neq j} r_{i \rightarrow j} \leq d_j$ for all $j = 1, \cdots, N$. Since the total upload rate is constrained by the upload bandwidth, we also have $\sum_{i \neq j} r_{j \rightarrow i} \leq u_j$ for all $j = 1, \cdots, N$. One example of the peer model is shown in Fig. 1. The download bandwidth and upload bandwidth of Peer 1 are $d_1$ and $u_1$ respectively. Thus, the total download rate $r_{s \rightarrow 1} + \sum_{i=2}^{N} r_{i \rightarrow 1}$ is less than or equal to $d_1$. The total upload rate $\sum_{i=2}^{N} r_{1 \rightarrow i}$ is less than or equal to $u_1$.

Since our work is less concerned with interactive response times and transmission delays in buffers and in the network, the download time for each peer is dominated by $1/r_i$ and the weighted average download time is $\sum_{i=1}^{N} w_i/r_i$ where $w_i \geq 0$ is the weight of peer $i$.

In a P2P network, if peer $i$ forwards content to peer $j$, then peer $i$ is a parent of peer $j$ and peer $j$ is a child of peer $i$. Each node can have several parents and several children. A primary goal of this paper is to minimize the WADT computed as $\sum_{i=1}^{N} w_i/r_i$. In the WADT computation $r_i$ could refer to the actual download rate $r_i^{(a)}$ or the bud-
geted download rate $r_i^{(b)}$. In general, $r_i^{(a)} \leq r_i^{(b)}$ because the parents of a peer might not always have new content for sharing. The network coding strategy can be used as a perfect chunk selection algorithm to guarantee that a parent always has new content for its children. This paper uses the network coding strategy as a chunk selection algorithm in P2P networks and so $r_i^{(a)} = r_i^{(b)}$. Therefore, in the rest of the paper, we use $r_i$ for both budgeted download rate and actual download rate.

III. Taxonomy of Overlay Networks

This section studies graph structures of P2P overlay networks.

**Definition 1:** static P2P network: A static P2P network is a directed graph which has one root and at least one directed path from the root to each of the other nodes. The root node is the server which has the content to share. All the other nodes are the peers that are interested in the content. In a P2P network, peers are placed into levels according to the topological distances between these peers and the server.

**Definition 2:** level of peer: A peer/node (use peer or node?) is in level $K$ if the length of the longest directed acyclic path from the server to the peer is $K$. The server is defined in level 0.

**Definition 3:** level of P2P network: A P2P network has level $K$ if the maximum level of peers is $K$.

A. The hierarchical P2P network

**Definition 4:** hierarchical P2P network: A hierarchical P2P network is a P2P network in which each peer can only download from peers in the lower levels and upload to peers in the higher levels.

Fig. 2 shows a hierarchical P2P network with 4 peers, in which peer $i$ is in level $i$. Note that in this example, each peer downloads from the lower levels and uploads to all peers in the higher levels. However, by definition, peers are not required to download from all peers in the lower levels or upload to/from? all peers in the higher levels. For example, the network shown in Fig. 2 will still be a hierarchical P2P network if the edge $1 \rightarrow 4$ vanishes. However, peers are not allowed to download from any peer in a higher level or upload to/from? any peer in a lower level. Also peers in the same level cannot download or upload to each other.

**Lemma 1:** A hierarchical P2P network contains no directed cycle.

**Proof:** (by contraposition) Suppose there is a directed cycle $A_1 \rightarrow A_2 \rightarrow \cdots \rightarrow A_n \rightarrow A_1$, then some $A_i \rightarrow A_{i+1}$ or $A_n \rightarrow A_1$ violates the requirement that peers in a hierarchical P2P network cannot upload to a peer in a lower level. Therefore, the network containing the directed cycle cannot be a hierarchical P2P network. Q.E.D.

**Lemma 2:** A directed acyclic P2P network is a hierarchical P2P network.

**Proof:** (by contraposition) Suppose a P2P network is not hierarchical, i.e., there exits a node $A$ in level $m$ and a node $B$ in level $n$ such that $m > n$ and $A \rightarrow B$. Let $S \rightarrow A_1 \rightarrow \cdots \rightarrow A_{m-1} \rightarrow A$ be the longest directed acyclic path from $S$ to $A$ and $S \rightarrow B_1 \rightarrow \cdots \rightarrow B_{n-1} \rightarrow B$ be the longest directed acyclic path from $S$ to $B$. Since $S \rightarrow A_1 \rightarrow \cdots \rightarrow A_{m-1} \rightarrow A \rightarrow B$ is a directed path from $S$ to $B$ with length $m + 1 > n$, this path must contain a directed cycle, and hence, the P2P network is not a directed acyclic graph. Therefore, a directed acyclic P2P network is a hierarchical P2P network. Q.E.D.

**Theorem 1:** The set of all hierarchical P2P networks is the set of all directed acyclic P2P networks. It is a direct consequence of Lemma 1 and Lemma 2.

B. The strictly hierarchical P2P network

**Definition 5:** strictly hierarchical P2P network: A strictly hierarchical P2P network is a P2P network in which each peer in level $K$ can only download from peers in level $K - 1$ and upload to peers in level $K + 1$.

Fig. 3 shows a strictly hierarchical P2P network. In Fig. 3, peer 1 and 2 form the virtual server and upload to peers in level $K + 1$. In Fig. 3, peer 1 and 2 form the virtual server in level 1, denoted as $S_1$. Peer 3, 4 and 5 form the virtual server in level 2, denoted as $S_2$. Since all transmission flows are between two consecutive levels, peer selection algorithms and chunk selection algorithms can be locally designed level by level.

The relationships among the P2P network, the hierarchical P2P network and the strictly hierarchical P2P network are concluded in Fig. 4.

C. Network of sub-peers

A P2P overlay network is divisible if peers can be divided into sub-peers, and it is indivisible if peers cannot be divided. The division of peers can be performed at the application layer. Even for indivisible P2P network, peers can be conceptually divided into virtual sub-peers for theoretical analysis. A simple example of peer division is given in Fig. 5. Fig. 5(a) shows the original P2P network and Fig. 5(b) shows the network of sub-peers after the division of peer 1 into sub-peer 1A and sub-peer 1B.

A peer division is equivalent if the transmission rates from the server to each peer and from each peer to each
other peer are invariant. The network of sub-peers is equivalent to the original P2P network if the peer division is equivalent. The peer division in Fig. 5 is equivalent if

\[ r_{s-1} = r_{s-1A}, \quad r_{1-2} = r_{1A-2} \text{ and } r_{3-1} = r_{3-1B}. \]

**Theorem 2:** Any P2P network with \( N \) peers and \( K \) levels can be decomposed into an equivalent network of sub-peers that is strictly hierarchical, and has at most \( K \) levels, each of which contains at most \( N \) sub-peers.

**Proof of Theorem 2** In order to prove the theorem, it suffices to construct a strictly hierarchical network of sub-peers which is equivalent to the original P2P network. For any P2P network with \( N \) peers and \( K \) levels, denote the server \( S \) as node 0 and the peers as node 1, 2, \ldots, \( N \). Let

\[ K_i = \{k : \exists \text{ a directed acyclic path from } S \text{ to peer } i \text{ with length } k\}, \quad (1) \]

and so \(|K_i| \leq K\) is the cardinality of \( K_i\). Divide peer \( i \) into \( |K_i| \) sub-peers, which are denoted as sub-peer \((i, k)\) for \( k \in K_i\). The sub-peer \((i, k)\) is the part of peer \( i \) in level \( k \). In the network of sub-peers, there is an edge from the server \( S \) to sub-peer \((i, 1)\) if and only if \( S \rightarrow i \) in the original network. We assign the transmission rate \( r_{s-(i, 1)} = r_{s-i} \). There is an edge from sub-peer \((i, k)\) to sub-peer \((j, k+1)\) if and only if there exists a directed acyclic path with length \( k + 1 \) such that \( S \rightarrow \cdots \rightarrow i \rightarrow j \) in the original P2P network. We assign the transmission rates level by level with

\[ r_{i,k} = \sum_{i', k-1} r_{i', k-1} \rightarrow (i,k), \quad (2) \]

\[ r_{i,j}^k = r_{i-j} - \sum_{m \leq k} r_{i,m-1} \rightarrow (j,m), \quad (3) \]

\[ r_{i,k} \rightarrow (j,k+1) = \min(r_{i,k} - \sum_{j' < j} r_{i,k} \rightarrow (j', k+1), r_{i,j}^k), \quad (4) \]

where \( r_{i,k} \) is the total download rate of the sub-peer \((i, k)\), \( r_{i,j} \) is the transmission rate from peer \( i \) to peer \( j \) in the original P2P network, \( r_{i,j}^k \) is the unassigned transmission rate from peer \( i \) to peer \( j \) until level \( k \) in the network of sub-peers, and \( r_{i,k} \rightarrow (j,k+1) \) is the transmission rate from \((i, k)\) to \((j, k+1)\). This division for the network in Fig. 5(a) is shown in Fig. 6(a).

Since \(|K_i| \leq K\) for all \( i \), the constructed network of sub-peers has at most \( K \) levels. There are at most \( N \) sub-peers in each level because each level contains at most one sub-peer of any peer. It is also easy to check that the constructed network of sub-peers is strictly hierarchical.

In order to prove that the network of sub-peers is equivalent to the original P2P network, we need to show that

\[ r_{i-j} = \sum_{k \leq K} r_{i,k-1} \rightarrow (j,k). \quad (5) \]
If \( i = 0 \), i.e., node \( i \) the server \( S \), then
\[
r_{S \rightarrow j} = r_{S \rightarrow (j,1)} = \sum_{k \leq K} r_{(0,k-1)\rightarrow(j,k)}.
\]
(6)

If \( i \neq 0 \), suppose for all \( k \leq K \),
\[
r_{(i,k-1)\rightarrow(j,k)} = \min \{ r_{(i,k-1)} - \sum_{j' < j} r_{(i,k-1)\rightarrow(j',k)} + r^{-1}_{i \rightarrow j},
\]
\[
< r_{i \rightarrow j}^{k-1},
\]
then
\[
\sum_{k=2}^{K-1} \sum_{j' < j} r_{(i,k-1)\rightarrow(j',k)} < \sum_{k=1}^{K-1} r_{(i,k-1)\rightarrow(j,k)},
\]
and so
\[
r_i < \sum_{j' \leq j} r_{i \rightarrow j'} + r_{i \rightarrow j} = \sum_{j' \leq j} r_{i \rightarrow j'},
\]
which is infeasible. Hence, there always exists some \( m \leq K \) such that
\[
r_{(i,m-1)\rightarrow(j,m)} = \min \{ r_{(i,m-1)} - \sum_{j' < j} r_{(i,m-1)\rightarrow(j',m)} + r^{-1}_{i \rightarrow j},
\]
\[
= r_{i \rightarrow j}^{m-1},
\]
then \( r_{i \rightarrow j} = 0 \) for \( k \geq m \), \( r_{(i,k-1)\rightarrow(j,k)} = 0 \) for \( k > m \), and hence
\[
\sum_{k \leq K} r_{(i,k-1)\rightarrow(j,k)} = \sum_{k \leq m} r_{(i,k-1)\rightarrow(j,k)} = r_{i \rightarrow j}.
\]
(11)

Finally, we need to show that the assigned transmission rates in the network of sub-peers are feasible, i.e., the download rate of each sub-peer is greater than or equal to it upload rate. Plugging \( j = N \) into (4),
\[
r_{(i,k)\rightarrow(N,k+1)} \leq r_{(i,k)} - \sum_{j' < N} r_{(i,k)\rightarrow(j',k+1)},
\]
and so
\[
r_{(i,k)} \geq \sum_{j' \leq N} r_{(i,k)\rightarrow(j',k+1)}.
\]
Q.E.D.

If edges with zero transmission rate and sub-peers with zero download bandwidth and zero upload bandwidth are allowed, divide peer \( i \) into \( K \) sub-peers, which are denoted as sub-peer \((i,k)\) for \( k = 1, \cdots, K \). The sub-peer \((i,k)\) is the part of peer \( i \) in level \( k \). There is an edge from the server \( S \) to sub-peer \((i,1)\) for all \( i \). There is an edge from sub-peer \((i,k)\) to sub-peer \((j,k+1)\) for all \( i \neq j \). The network of sub-peers after this division is also strictly hierarchical, and keeps all the transmission paths in the original network. This peer division for the P2P network in Fig. 5(a) is shown in Fig. 5(b), in which the dotted border sub-peers have zero download bandwidth and zero upload bandwidth, and the dotted edges have zero transmission rate. By Theorem 2, there exists a transmission rate assignment such that the network of sub-peers after this peer division is equivalent to the original P2P network. Hence, minimizing the WADT for strictly hierarchical networks of sub-peers is equivalent to minimizing the WADT for P2P networks.

In a very large P2P network with \( N \) peers, the length of the longest directed acyclic path \( K = \max |K_i| \) is usually much less than \( N \). Denote \( d_{i,j}, u_{i,j}, r_{i,j} \) as the download bandwidth, upload bandwidth and the download rate of the part of peer \( i \) in level \( j \), where \( d_{i,j} \geq u_{i,j} \) and \( d_{i,j} \geq r_{i,j} \). The sub-peers in level \( j \) work together as a virtual server to upload to the sub-peers in level \( j + 1 \). Denote \( S_j \) as the upload bandwidth of the virtual server in level \( j \), \( 1 \leq j \leq K \), and so \( S_j = \sum_{i=1}^{N} \min(u_{i,j}, r_{i,j}) \). Let \( S_0 = S \).

Lemma 6: Given the server with bandwidth \( S \) and the peers with download bandwidth \( d_i \) and upload bandwidth \( u_i, 1 \leq i \leq N \), there exists an optimal strictly hierarchical network of sub-peers which achieves the minimum WADT such that \( r_{i,j} \geq u_{i,j} \) for all \( 1 \leq i \leq N, 1 \leq j \leq K - 1 \).

Proof: Suppose \( d_{i,j}, u_{i,j}, r_{i,j} \) determine an optimal strictly hierarchical network of sub-peers which connects the server and the peers, and achieves the minimum WADT. It is clear that \( \tilde{d}_{i,j}, \tilde{u}_{i,j}, \tilde{r}_{i,j} \) is also an optimal strictly hierarchical network of sub-peers which has \( \tilde{r}_{i,j} \geq \tilde{u}_{i,j} \) for any \( 1 \leq i \leq N, 1 \leq j \leq K - 1 \), where
\[
\tilde{r}_{i,j} = r_{i,j}\]
(14)
\[
\tilde{u}_{i,j} = \left\{ \begin{array}{ll}
\min(u_{i,j}, r_{i,j}) & \text{for } 1 \leq j \leq K - 1 \\
u_i - \sum_{j=1}^{K-1} \tilde{u}_{i,j} & \text{for } j = K
\end{array} \right.
\]
(15)
\[
\tilde{d}_{i,j} = \left\{ \begin{array}{ll}
r_{i,j} & \text{for } 1 \leq j \leq K - 1 \\
d_i - \sum_{j=1}^{K-1} \tilde{d}_{i,j} & \text{for } j = K
\end{array} \right.
\]
(16)
The theorem is immediately proved. Q.E.D.

Corollary 1: \( S_j = \sum_{i=1}^{N} \min(u_{i,j}, r_{i,j}) = \sum_{i=1}^{N} u_{i,j} \) for any \( 1 \leq j \leq K - 1 \).

IV. Optimizing and Bounding Weighted Average Download Time

A challenging problem for content distribution applications is how to build a P2P content distribution network to achieve the minimum weighted average download time. The WADT of a P2P network with \( N \) peers is \( \sum_{i=1}^{N} w_i \cdot r_i \), where \( w_i \) is the weight and \( r_i \) is the download rate for peer \( i \). The weights \( w_i, i = 1, \cdots, N \), depend on the application. For broadcast applications such as CoolStreaming [3] and Overcast [10], in which all peers in the P2P network are interested in the same content, all weights of peers in the content distribution system can be set to be 1. For multicast applications such as “Tribler” [24], in which some helpers, who are not interested in any particular content, store part of the content and share with other peers, weights of helpers are 0 and weights of other peers are 1. In some applications, P2P systems can also partition peers into several classes and assign different weights to peers in different classes.
A. Building Optimal P2P Networks Using Convex Programming

Theorem 2 demonstrates that minimizing the WADT for P2P networks with \( N \) peers and \( K \) levels is equivalent to minimizing the WADT for strictly hierarchical networks of sub-peers with \( K \) levels, each of which contains \( N \) sub-peers. \( K \ll N \) for large P2P networks. Recall that the bandwidth of the server is \( S = S_0 \). The download bandwidth and upload bandwidth of peer \( i \) are \( d_i \) and \( u_i \) respectively, where \( d_i \geq u_i \). \( S_j \) is the bandwidth of the virtual server in level \( j \), \( 1 \leq j \leq K \). \( d_{i,j}, u_{i,j}, r_{i,j} \) are the download bandwidth, the upload bandwidth and the download rate of the part of peer \( i \) in level \( j \). The constraints on the download bandwidths and upload bandwidths are \( d_{i,j} \geq u_{i,j} \) for any \( 1 \leq i \leq N \) and \( 1 \leq j \leq K \), \( \sum_{j=1}^{K} d_{i,j} = d_i \) and \( \sum_{j=1}^{K} u_{i,j} = u_i \) for any \( 1 \leq i \leq N \). The constraints on the download rates are \( r_{i,j} \leq d_{i,j} \) for any \( 1 \leq i \leq N \) and \( 1 \leq j \leq K - 1 \). \( \sum_{j=1}^{K} r_{i,j} = r_i \) is the total download rate for peer \( i \).

Corollary 1 demonstrates that the bandwidth of the virtual server in level \( j \) is \( S_j = \sum_{i=1}^{N} u_{i,j} \) for \( j = 1, \ldots, K - 1 \). One has \( S_j = \sum_{i=1}^{N} r_{i,j} \) since the virtual server in level \( j - 1 \) uploads content to the sub-peers in level \( j \).

All the constraints are linear and the objective function, \( \sum_{i=1}^{N} w_i/r_i \), is a convex function. Therefore, the problem of minimizing the WADT is a convex optimization program, which can be solved by convex optimization solvers such as the interior-point method. The complexity of the interior-point method to solve this problem is \( o(\sqrt{K^3N^3}) \) [25]. The solution to the convex optimization program provides the topology of the optimal P2P network which achieves the minimum WADT, however, the number of the peers is usually very large for most P2P applications and so the complexity for solving the convex program is too high. Fortunately, this convex program has an analytical achievable upper bound which is very tight to the optimal solution and its complexity is linear to the number of the peers in a P2P network.

B. Analytical Upper Bound and Lower Bound for the Convex Optimization Program

Theorem 1 shows that any hierarchical P2P network contains no directed cycle and vice versa. Constructing a hierarchical P2P network shown in Fig. 7 provides an achievable upper bound for the minimum WADT. Assume that \( S > \max_i(u_i) \), which is always true practically. Without loss of generality, assume \( u_1 \geq u_2 \geq \cdots \geq u_N \). Partition the server into two parts \( S^{(1)} = \max_i(u_i) = u_1 \) and \( S^{(2)} = S - \max_i(u_i) = S - u_1 \). The first part \( S^{(1)} \) can serve all the peers with \( r_{i}^{(1)} = u_i \). The second part \( S^{(2)} \) can then freely serve for the peers only with the constraints \( d_i \geq r_{i}^{(1)} + r_{i}^{(2)} \geq u_i \) and \( S^{(2)} \geq \sum_{i=1}^{N} r_{i}^{(2)} \). Thus, the upper bound is the solution to the optimization problem

\[
\text{Minimize} \quad \sum_{i=1}^{N} \frac{w_i}{r_i} \\
\text{Subject to} \quad d_i \geq r_i \geq u_i, \quad i = 1, 2, \ldots, N, \\
\quad \sum_{i=1}^{N} r_i - u_i \leq S - \max_i(u_i). 
\]

The Karush-Kuhn-Tucker conditions for the optimal solution are

\[
\mu_i, \nu_i, \lambda \geq 0, \\
- w_i r_i^{-2} + \mu_i - \nu_i + \lambda = 0, \\
\mu_i (r_i - d_i) = 0, \\
\nu_i (u_i - r_i) = 0, \\
\lambda \left( \sum_{i=1}^{N} (r_i - u_i) - (S - \max_i(u_i)) \right) = 0. 
\]

Combining the the Karush-Kuhn-Tucker conditions and the constraints in the problem, we obtain the upper bound which is \( \sum_{i=1}^{N} w_i/r_i \) with

\[
\lambda_i = \begin{cases} 
\sqrt{\frac{w_i}{R}} & \text{if } u_i \leq \sqrt{\frac{w_i}{R}} \leq d_i \\
\frac{w_i}{u_i} & \text{if } u_i > \sqrt{\frac{w_i}{R}} \\
\frac{w_i}{d_i} & \text{if } \sqrt{\frac{w_i}{R}} \geq \frac{w_i}{d_i} 
\end{cases} 
\]

where \( R \) is chosen appropriately such that \( \sum_{i=1}^{N} (r_i - u_i) = S - \max_i(u_i) \).

This upper bound is almost optimal. Consider the following lower bound which is very close to the upper bound. Since \( S \geq \sum_{i=1}^{N} (r_i - u_i) \) is a relaxed constraint, an lower bound is the solution to the optimization problem

\[
\text{Minimize} \quad \sum_{i=1}^{N} \frac{w_i}{r_i} \\
\text{Subject to} \quad d_i \geq r_i \geq u_i, \quad i = 1, 2, \ldots, N, \\
\quad \sum_{i=1}^{N} r_i - u_i \leq S. 
\]
Note that this optimization problem (19) differs from the problem (17) only by replacing $S$ with $S - \max_i(u_i)$. Thus, the lower bound is $\sum_{i=1}^N u_i/r_i$, with
\[
 r_i = \begin{cases} 
 \sqrt{w_i}R & \text{if } u_i \leq \sqrt{w_i}R \leq d_i \\
 u_i & \text{if } u_i > \sqrt{w_i}R \\
 d_i & \text{if } \sqrt{w_i}R > d_i
\end{cases}
\] (20)
where $R$ is chosen appropriately such that $\sum_{i=1}^N (r_i - u_i) = S$.

Since $S \gg \max(u_i)$, one has $S \simeq S - \max_i(u_i)$. The upper bound and the lower bound are almost the same. Therefore, both of these two bounds are very tight, and the upper bound is almost an analytical solution to minimizing the WADT. Hence, it is sufficient to build a hierarchical P2P network to achieve the minimum WADT.

C. Practical Solutions

Compared with building networks of sub-peers and hierarchical P2P networks, it is more practical to build a strictly hierarchical P2P network. First, in a strictly hierarchical P2P network, peers do not need to be decomposed into sub-peers. Thus, it can have a much simpler protocol in network layer than a network of sub-peers. Second, in a strictly hierarchical P2P network, each level usually contains only tens or hundreds of peers. Therefore, one can locally design peer selection algorithms and chunk selection algorithms level by level, which only depend on a small collection of peers in the P2P network. The locally designed peer and chunk selection algorithm might be much simpler than the network coding strategy and other global designed peer and chunk selection algorithms. The following theorem shows that there exists a strictly hierarchical network of sub-peers which achieves the upper bound in the previous subsection, and is very close to a strictly hierarchical P2P network.

**Theorem 3:** There exist multiple upper-bound-achieving networks of sub-peers in which at most $K - 1$ peers need to be decomposed into 2 sub-peers, where $K \leq N$ is the number of levels.

**Proof:** Equation (15) gives $r_i$, $i = 1, \cdots, N$, for the hierarchical P2P network which achieves the upper bound. Convert the hierarchical P2P network to multiple optimal strictly hierarchical networks of sub-peers by the following algorithm.

This algorithm indicates that at most $K - 1$ peers need to be divided into 2 sub-peers and each level of the constructed network of sub-peers contains at most 2 sub-peers. Since there are multiple choices in step (2), this algorithm provides multiple strictly hierarchical networks of sub-peers which are very close to a strictly hierarchical P2P network.

Q.E.D.

Another practical solution is to solve the problem of minimizing the WADT directly for strictly hierarchical P2P networks. This problem can be formulated as a 0-1 convex optimization. The complexity to solve a 0-1 convex optimization problem is exponential to the problem size.

Fortunately, this problem has an analytical suboptimal solution. Suppose there are $N$ peers and $K$ levels in a strictly hierarchical P2P network and the level location of each peer is already given. Then the global optimization problem can be decomposed into $K$ local optimization problem and all of them have an analytical “Water Filling” solution.

In particular, suppose the bandwidth of the server is $S = S_0$. The download bandwidth, upload bandwidth and the allocated download rate of Peer $i$ are $d_i$, $u_i$ and $r_i$ respectively. $S_j$, the bandwidth of the virtual server in Level $j$, is equal to the summation of the upload bandwidths of the peers in Level $j$. If $S_j$ is greater or equal to the summation of the upload bandwidth of all peers in level $j + 1$, then one has $d_i \geq r_i \geq u_i$ and $S_j = \sum_{i \in \text{level } j} u_i$. If $S_j$ is less than the summation of the upload bandwidth of all peers in level $j + 1$, one has $0 \leq r_i \leq u_i$ and $S_j = \sum_{i \in \text{level } j} r_i$. Therefore, no matter how to allocate download rates to peers in each level, the bandwidth of the virtual server in each level is fixed as long as the level positions of peers are fixed. Thus, the global minimum weighted average transmission rate problem can be decomposed to $K$ local optimization problem. Each of them has a fixed server bandwidth and fixed number of peers in the level. Thus, if the virtual server $S_j$ is greater or equal to the summation of the upload bandwidth of all peers in level $j + 1$, then the local optimization problem is

\[
\text{Minimize} \quad \sum_{i=1}^{N_j+1} \frac{w_i}{r_i}
\]

subject to
\[
d_i \geq r_i \geq u_i, \quad i = 1, 2, \cdots, N_j+1,
\]
\[
\sum_{i=1}^{N_j+1} r_i \leq S_j
\]

Algorithm 1 Peer Placement Algorithm

1: Initialize level $l = 1$;
2: Initialize the rest server bandwidth for the current level $s = S$;
3: Initialize the set of the rest of the peers $G = \{1, 2, \cdots, N\}$;
4: while $G$ is not empty do
5: Choose any peer $i$ from the set $G$;
6: if $s \geq r_i$ then
7: \quad Put Peer $i$ in the current level $l$;
8: \quad Set $G = G \setminus \{i\}$;
9: \quad Set $s = s - r_i$;
else
10: \quad Decompose Peer $i$ into 2 sub-peers with $r_i^{(1)} = s$,
11: \quad $r_i^{(2)} = r_i - s, \quad u_i^{(1)} = \min(u_i, r_i^{(1)}), \quad u_i^{(2)} = u_i - \min(u_i, r_i^{(1)})$;
12: \quad Put the sub-peer (1) in level $l$ and the sub-peer (2) in level $l + 1$;
13: \quad Set $s = (\sum_{\text{Peer } j \text{ in level } l} u_j) - r_i^{(2)}$;
14: \quad Set $l = l + 1$;
15: \quad Set $G = G \setminus \{i\}$;
end if
17: end while
where \( N_{j+1} \) is the number of peers in level \( j+1 \), for \( j = 0, \cdots, K-1 \).

By the Karush-Kuhn-Tucker conditions, the optimal solution is

\[
 r_i = \begin{cases} 
 \sqrt{w_i} R & \text{if } u_i \leq \sqrt{w_i} R \leq d_i \\
 u_i & \text{if } u_i > \sqrt{w_i} R \\
 d_i & \text{if } \sqrt{w_i} R > d_i 
\end{cases} \tag{21}
\]

where \( R \) is chosen appropriately such that \( \sum_{i=1}^{N_{j+1}} r_i = S_j \).

If the virtual server \( S_j \) for level \( j+1 \) is less than the summation of the upload bandwidth of all peers in level \( j+1 \), then the local optimization problem is

\[
\begin{align*}
\text{Minimize} & \quad \sum_{i=1}^{N_{j+1}} w_i / r_i \\
\text{Subject to} & \quad u_i \geq r_i \geq 0, i = 1, 2, \cdots, N_{j+1} \\
& \quad \sum_{i=1}^{N_{j+1}} r_i \leq S_j
\end{align*}
\]

where \( N_{j+1} \) is the number of peers in level \( j+1 \), for \( j = 0, \cdots, K-1 \).

By the Karush-Kuhn-Tucker conditions, the optimal solution is

\[
 r_i = \begin{cases} 
 \sqrt{w_i} R & \text{if } 0 \leq \sqrt{w_i} R \leq u_i \\
 u_i & \text{if } u_i < \sqrt{w_i} R 
\end{cases} \tag{22}
\]

where \( R \) is chosen appropriately such that \( \sum_{i=1}^{N_{j+1}} r_i = S_j \).

V. Results

In this section, we simulate and evaluate the performance of the methods provided in Section IV. Note that in this paper the download bandwidth \( d \) and the upload bandwidth \( u \) are decided at the application layer instead of the physical layer. Thus, the download bandwidth \( d \) can be continuously distributed in a large range such as 10 kbps to 100 Mbps. We assume that the download bandwidths \( d_i, i = 1, \cdots, N \) are independently identically distributed (i.i.d.) with uniform distribution over \([\beta, 2-\beta]\), where \( \beta \) is a very small positive number such as \(10^{-2}, 10^{-4}\). This distribution has the normalized mean value 1 and large maximum to minimum ratio \((2-\beta)/\beta\). In practice, this parameter \( \beta \) can be determined as the minimum download bandwidth which is required by the P2P system. We also assume that the upload bandwidth \( u_i \) is uniformly distributed over \([\alpha d_i, d_i]\), where \(0 < \alpha < 1\) is the minimum upload to download bandwidth required by the P2P system.

A. A small network simulation

In this small simulation, the bandwidth of the server is 100 and the number of the peers in the network is \( N = 100 \). The download bandwidths \( d_i, i = 1, \cdots, 100 \), are i.i.d. with uniform distribution over \([0.01, 1.99]\), i.e., \( \beta = 10^{-2} \). For each peer \( i \), the upload bandwidth is uniformly distributed over \([0.1d_i, d_i]\), i.e., \( \alpha = 0.1 \). The weights for all the peers are equal and normalized to be \(1/N = 1/100\), and so \( \sum_i w_i = 1 \). Thus, the WADT is \( \sum w_i / r_i \). We will compare 7 methods in this experiment. They are

1. Optimal solution to the convex optimization program with level \( K = 30 \). The convex program solver is CVX [21].
2. The upper bound [17] by constructing a hierarchical P2P network.
3. The lower bound [13] by relaxing the constraints.
4. The suboptimal solution to the 0-1 convex optimization problem for strictly hierarchical P2P networks. The number of the levels are set to be \( K = 30 \) and we randomly put around \( N/K \) peers in each level.
5. The suboptimal solution to the 0-1 convex optimization problem for strictly hierarchical P2P networks. We convert the upper-bound-achieving hierarchical P2P network (Method 2) to a network of sub-peers by Algorithm 3. The peers are placed into levels according to the constructed network of sub-peers, which is very close to a strictly hierarchical P2P network.
6. This is a trivial upper bound such that the rate of each peer is the same as the upload bandwidth.
7. This is a trivial lower bound such that the rate of each peer is the same as the download bandwidth.

The distribution of the weighted average download times of 500 experiments are shown in Fig. 8. The distribution of the difference and the relative difference between Method 3 and other methods are shown in Fig. 9 and Fig. 10. The weighted average download times of Method 1 is concentrated in the range of \([1.7, 4.1]\) with mean value 2.877. Method 2 and Method 3 provide the upper bound and lower bound for the minimum WADT. Fig. 8, 9, and 10 show that these two bounds are very tight. In this case, the lower bound is almost always the same as the optimal solution since the distribution curves of Method 1 and Method 3 perfectly match. The distribution curve of Method 2 is slightly different from that of Method 1, which means that the upper bound is only slightly larger than the optimal solution for most of the experiments. In this simulation, the bandwidth of the server is 10, which is not much larger than the maximum of the upload bandwidths. That is why the upper bound is still slightly different from the optimal solution. The distribution of Method 5 shows that in most of the experiments, the WADT of Method 5 is close to the optimal solution, however, it is sometimes much larger than the optimal solution. This is because the peer placement in Method 5 is usually very good but sometimes bad. Note that it is NP complete complex to find the best peer placement. In order to improve Method 5 without increasing the complexity, we need provide a better but still low-complex peer placement algorithm. The performance of Method 4 is much worse than the performance of Method 5 because the peer placement algorithm is Method 4 is much worse that in Method 5.

The bandwidth usage is the ratio of the total transmission rate and the total download bandwidth of all the peers in the network. For a fixed WADT, it is clear that the lower bandwidth usage the better. However, it is a trade off to...
decrease the bandwidth usage and to decrease the WADT, which is also verified in Fig. 11. Fig. 11 shows the distribution of the bandwidth usage of different methods. The distribution of Method 1, the optimal solution, is exactly the same as the distribution of Method 3, the lower bound. The bandwidth usage of Method 2, the upper bound, is slightly less than that of Method 1 and the bandwidth usage of Method 5 is slightly less than that of Method 2. It is verified that the method with smaller WADT always has higher bandwidth usage. The mean values of the weighted average download times (W.A.D.T.), the normalized weighted average download times (N.W.A.D.T.) and the bandwidth usage (B.U.) of different methods for Experiment A are listed in Table I.

| Method | W.A.D.T. | N.W.A.D.T. | B.U. |
|--------|----------|------------|------|
| 7      | 2.468    | 0.854      | 1.000|
| 3      | 2.877    | 1.000      | 0.650|
| 1      | 2.877    | 1.000      | 0.650|
| 2      | 2.947    | 1.025      | 0.633|
| 5      | 3.461    | 1.200      | 0.621|
| 4      | 5.338    | 1.849      | 0.471|
| 6      | 6.361    | 2.194      | 0.551|

For one typical experiment, the plots of the virtual server bandwidths versus the levels for different methods are shown in Fig. 12. For Method 1, the number of the levels K is manually chosen. In this experiment, K is 30. One can
see that the bandwidths of the virtual servers from level 5 to level 30 are linearly decreasing to 0 for Method 1. The bandwidth of the virtual server generated by the last level is almost 0. In other words, all the upload bandwidths of the server and the peers are fully used. This is the reason why the performance of Method 3, the tight lower bound, is almost the same as that of Method 1. For Method 2, the number of the levels needed is automatically solved. It is 15 in this experiment.

B. A large network simulation

In this simulation, the bandwidth of the server is 50 and there are 4000 peers in the network. The download bandwidths $d_i$, $i = 1, \cdots, 4000$, are i.i.d. with uniform distribution over $[0.01, 1.99]$. The upload bandwidth $u_i$ is uniformly distributed over $[0.1d_i, d_i]$. The weights for all the peers are equal and normalized to be $1/N = 1/4000$, and so $\sum_i w_i = 1$.

The distribution of the weighted average download times of 800 experiments are shown in Fig. 13. The distribution of the difference and the relative difference between Method 3 and other methods are shown in Fig. 14 and Fig. 15. These figures show that the lower bound and the upper bound provided by Method 3 and Method 2 are very close. The performance of Method 1 should be between these two bounds, although we don’t simulate Method 1 in this case. In this simulation, the bandwidth of the server is 50, which is much larger than the maximum of the upload bandwidths. That is why $S \approx S - \max_i (u_i)$ and the upper bound is almost the same as the lower bound. The performance of Method 5 is worse than that of Method 2 but still a lot better that of Method 6 and 7.

The bandwidth usages of these methods are shown in Fig. 16. The distribution of Method 2 is almost the same as the distribution of Method 3, which verifies the almost same performance of Method 2 and 3. The mean values of the weighted average download times and the bandwidth usage of different methods are listed in Table II. Combining the simulation results, the comparison of these 7 methods in different criteria are listed in Table III.

VI. Conclusions

This paper proposes an analytical framework for peer-to-peer (P2P) networks and introduces schemes for building P2P networks to approach the minimum weighted average download time (WADT). In the considered P2P framework, the server, which has the information of all the download bandwidths and upload bandwidths of the peers, minimizes the weighted average download time by determining the optimal transmission rate from the server to the peers and from the peers to the other peers.

This paper first defines the static P2P network, the hi-
erarchical P2P network and the strictly hierarchical P2P network and studies the graph structures of these P2P networks. The main result is that any static P2P network can be decomposed into an equivalent network of sub-peers that is strictly hierarchical. Therefore, convex optimization can minimize the WADT for P2P networks by equivalently minimizing the WADT for strictly hierarchical networks of sub-peers. This paper then gives an achievable upper bound for minimizing WADT by constructing a hierarchical P2P network, and lower bound by weakening the constraints of the convex problem. Both the upper bound and the lower bound are very tight.

The strictly hierarchical P2P network is practical for protocol design because peer selection algorithms and chunk selection algorithms can be locally designed level by level instead of globally designed. Minimizing the WADT for strictly hierarchical networks is a 0-1 convex optimization problem. However, if we have assigned all peers each to a level, then the global bandwidth allocation problem decomposes into local bandwidth allocation problems at each level, which have water-filling solutions. Several suboptimal peer assignment algorithms are provided and simulated.

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**TABLE III**

| Method | W.A.D.T. | B.U. | S. Comp. | P. Comp. |
|--------|----------|------|----------|----------|
| 1      | Optimal  | Highest | $O(K^*N^*)$ | High     |
| 2      | Almost Opt. | Very High | $O(N)$ | Low     |
| 5      | Good     | High    | $O(N)$ | Low     |
| 4      | Bad      | Low     | $O(N)$ | Low     |
| 6      | Bad      | Low     | $O(N)$ | Low     |

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