Poisson Cloning Model for Random Graphs

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Abstract. In the random graph $G(n,p)$ with $pn$ bounded, the degrees of the vertices are almost i.i.d Poisson random variables with mean $\lambda := p(n - 1)$. Motivated by this fact, we introduce the Poisson cloning model $G_{PC}(n,p)$ for random graphs in which the degrees are i.i.d Poisson random variables with mean $\lambda$. Then, we first establish a theorem that shows the new model is equivalent to the classical model $G(n,p)$ in an asymptotic sense. Next, we introduce a useful algorithm, called the cut-off line algorithm, to generate the random graph $G_{PC}(n,p)$. The Poisson cloning model $G_{PC}(n,p)$ equipped with the cut-off line algorithm enables us to very precisely analyze the sizes of the largest component and the $t$-core of $G(n,p)$. This new approach to the problems yields not only elegant proofs but also improved bounds that are essentially best possible.

We also consider the Poisson cloning models for random hypergraphs and random $k$-SAT problems. Then, the $t$-core problem for random hypergraphs and the pure literal algorithm for random $k$-SAT problems are analyzed.

1 Introduction

The notion of a random graph was first introduced in 1947 by Erdős [26] to show the existence of a graph with a certain Ramsey property. A decade later, the theory of the random graph began with the paper entitled On Random Graphs I by Erdős and Rényi [27], and the theory had been developed by a series [28, 29, 30, 31, 32] of papers of them. Since then, the subject has become one of the most active research areas. Many researchers have devoted themselves to studying various properties of random graphs, such as the emergence of the giant component [28, 9, 52], the connectivity [27, 29, 10], the existence of perfect matching [30, 31, 32, 10], the existence of Hamiltonian cycle(s) [51, 10, 13], the $k$-core problem [10, 54, 61], and the graph invariants like the independence number [14, 56] and the chromatic number [62, 12, 53]. (The list of references here is far from being exhaustive.)

There are two canonical models for random graphs, both of which were originated in the simple model introduced in [26]. In the binomial model $G(n,p)$ on a set $V$ of $n$ vertices, each of $\binom{n}{2}$ possible edges is in the graph with probability $p$, independently of other edges. Thus, the probability of $G(n,p)$ being a fixed graph $G$ with $m$ edges is $p^m(1-p)^{\binom{n}{2}-m}$. The uniform

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model $G(n, m)$ on $V$ is a graph chosen uniformly at random from the set of all graphs on $V$ with $m$ edges. Hence, $G(n, m)$ becomes a fixed graph $G$ with probability $\binom{\binom{m}{2}}{n}^{-1}$, provided $G$ has $m$ edges. Most of asymptotic behaviors of the two models are almost identical if their expected numbers of edges are the same. (See Proposition 1.13 in [44].) The random graph process, in which random edges are added one by one, is also extensively studied. For more about models and/or basics of random graphs, we recommend two books with the identical title *Random Graphs* by Bollobás [11], and by Janson, Łuczak and Ruciński [44].

The phase transition phenomenon is among most interesting topics of random graphs. Specifically, the phase transition phenomena regarding the emergences of the giant (connected) component and the $t$-core problem have attracted much attention. In their monumental paper entitled *On the Evolution of Random Graphs* [28], Erdős and Rényi proved that, for the size $\ell_1(n, p)$ of the largest component of $G(n, p)$,

$$
\ell_1(n, p) = \begin{cases} 
O(\log n), & \text{if } \limsup_{n \to \infty} p(n-1) < 1 \\
(1 + o(1))\theta_\lambda n, & \text{if } \lim_{n \to \infty} pn = \lambda > 1,
\end{cases}
$$

where $\theta_\lambda$ is the positive solution of the equation $1 - \theta - e^{-\lambda\theta} = 0$.

Why does the size of the largest component change so dramatically around $\lambda = 1$? It was Karp [48] who nicely explained the reason. To find a component $C(v)$ of a fixed vertex $v$ of $G(n, p)$, one may first expose the vertices that are adjacent to $v$, and keep repeating the same procedure by taking each of those adjacent vertices: Initially, $v$ is active and all other vertices are neutral. At each step, we take an active vertex $w$ and expose all neutral vertices adjacent to $w$. This can be done by checking if $\{w, w'\} \in G(n, p)$ or not for all neutral vertices $w'$. Then, activate all neutral vertices that are adjacent to $w$. The vertex $w$ is no longer active, and only non-activated neutral vertices remain neutral. The process terminates when there is no more active vertex. Clearly, the process will stop after finding all the vertices in the component containing $v$. Provided the number of neutral vertices does not decrease so fast, the number of newly activated vertices has a distribution close to that of the binomial random variable $\text{Bin}(n-1, p)$, where

$$
\Pr[\text{Bin}(n-1, p) = \ell] = \binom{n-1}{\ell} p^\ell (1-p)^{n-1-\ell}.
$$

Particularly, the mean of the number is close to $pn$. If $pn \leq 1 - \delta$ for a fixed $\delta > 0$, then the process is expected to die out quickly almost every time. Thus, all $C(v)$’s are expected to be small. If $pn \geq 1 + \delta$, then the process may survive forever with positive probability. Hence, $C(v)$ can be large with positive probability; as there are many (actually $\Theta(n)$) trials, at least one of $C(v)$’s is expected to be large. Applying this approach to the random directed graph, Karp was able to prove a phase transition phenomenon for the size of the largest strong component.

Notice that, when $pn = \Theta(1)$, the distribution of $\text{Bin}(n-1, p)$ is very close to the Poisson distribution with mean $\lambda := p(n-1)$. Hence, we may further expect that the process described above could be approximated by the Galton-Watson branching process defined by a Poisson random variable $\text{Poi}(\lambda)$ with mean $\lambda$, where

$$
\Pr[\text{Poi}(\lambda) = \ell] = e^{-\lambda} \frac{\lambda^\ell}{\ell!}.
$$
Generally, the Galton-Watson branching process defined by a random variable $X$ starts with a single unisexual organism. The organism will give birth to $X_1$ children, where $X_1$ is a random variable with the same distribution as $X$. The same but independent birth process continues from each of the children and the grandchildren and so on, until no more descendant exists. (For more information regarding Galton-Watson branching processes, one may refer [8].) For simplicity, we say the Poisson$(\lambda)$ branching process for the Galton-Watson branching process defined by $\text{Poi}(\lambda)$.

**The Poisson cloning model.** To convert the above observation to a rigorous proof, it is needed to overcome or bypass two main obstacles. The first one is that the degrees of vertices of $G(n, p)$ are not exactly i.i.d Poisson random variables. Though they have the same distribution as $\text{Bin}(n-1, p)$, they are not mutually independent. For example, the sum of all degrees must be even as it is twice the number of edges. This cannot be guaranteed if the degrees are independent. The second one is that the number of neutral vertices keeps decreasing. Even if both obstacles do not cause substantial differences in many cases, one needs at least to keep tracking small differences for rigorous proofs. Since these kinds of small differences occur almost everywhere in the analysis, they sometimes make rigorous analysis significantly difficult, if not impossible. Furthermore, the fact that the number of neutral vertices decreases not only plays a crucial role but also yields a different result in the case that one wants know more precise behaviors.

As an approach to bypass the first obstacle, we introduce the Poisson cloning model $G_{PC}(n, p)$ for random graphs in which the degrees are i.i.d Poisson random variables with mean $\lambda = p(n-1)$. Moreover, the new model is equivalent to the classical model $G(n, p)$ in an asymptotic sense. Actually, defining the model is not extremely difficult: First take i.i.d Poisson $\lambda$ random variables $d(v)$ indexed by all vertices $v$ in $V$. Then take $d(v)$ copies, or clones, of each vertex $v$. If the sum of $d(v)$’s is even, then we generate a uniform random perfect matching on the set of all clones. An edge $\{v, w\}$ is in the random graph $G_{PC}(n, p)$ if a clone of $v$ is matched to a clone of $w$ in the random perfect matching. The resulting graph may or may not a simple graph. If the sum is odd, one may just take a graph with a self loop. Hence, the graph is always not simple if the sum is odd. In the next section, the Poisson cloning model is to be defined with details.

It is also possible to extend the model to uniform hypergraphs, where a $k$-uniform hypergraph on the vertex set $V$ is a collection of subsets of $V$ with size $k$. A graph is then a 2-uniform hypergraph. In the binomial model $H(n, p; k)$ for random $k$-uniform hypergraphs, each of $\binom{n}{k}$ edges is in the hypergraph with probability $p$, independently of other edges. The Poisson cloning model for random $k$-uniform hypergraphs may be similarly defined and is denoted by $H_{PC}(n, p; k)$.

The following theorem shows that the new model is essentially equivalent to the binomial model.

**Theorem 1.1** Suppose $k \geq 2$ and $p = \Theta(n^{1-k})$. Then, for any collection $\mathcal{H}$ of $k$-uniform simple hypergraphs,

$$c_1 \Pr[H_{PC}(n, p; k) \in \mathcal{H}] \leq \Pr[H(n, p; k) \in \mathcal{H}] \leq c_2 \left( \Pr[H_{PC}(n, p; k) \in \mathcal{H}] + e^{-n} \right),$$

3
where
\[ c_1 = k^{1/2}e^{\frac{k}{2}(\frac{1}{n})} + O(n^{-1/2}), \quad c_2 = \left(\frac{k}{k-1}\right) \left( c_1(k-1) \right)^{1/k} + o(1), \]
and \( o(1) \) goes to 0 as \( n \) goes to infinity.

To overcome the second obstacle, we present an algorithm, called the cut-off line algorithm, that enables us to generate the Poisson cloning model and analyze problems simultaneously. As a consequence, the size of the largest component of \( G_{PC}(n, p) \) can be described very precisely. It is also possible to analyze the size of the \( t \)-core of the random hypergraph \( H_{PC}(n, p; k) \), where the \( t \)-core of a hypergraph is the largest subhypergraph with minimum degree at least \( t \).

**The emergence of the giant component.** After the phase transition result of Erdős and Rényi, it has remained to determine the size of the largest component when \( pn \to 1 \). Though Erdős and Rényi suggested that the size \( \ell_1(n, p) \) of the largest component could be only \( O(\log n) \), \( \Theta(n^{2/3}) \), or \( \Theta(n) \), Bollobás [9] showed that \( \ell_1(n, p) \) increases rather continuously by estimating it quite accurately for \( pn - 1 \geq n^{-1/3} \sqrt{\log n} / 2 \). Later Luczak [52] was able to estimate \( \ell_1(n, p) \) for \( pn - 1 \gg n^{-1/3} \).

In statements in theorems and lemmas, etc., of this paper, we use the following convention.

**Convention:** When we say that a statement is true for all \( \alpha \) in the range \( a \ll \alpha \ll b \), it actually means that there is (small) constant \( \varepsilon > 0 \) so that the statement is true for \( \alpha \) in the range \( a/\varepsilon \leq \alpha \leq \varepsilon b \).

**Theorem 1.2** [52] (Supercritical Phase) Suppose \( \lambda = \lambda(p, n) = 1 + \varepsilon \) with \( \varepsilon \gg n^{-1/3} \). Then, for large enough \( n \), with probability at least \( 1 - 7(\varepsilon^3 n/8)^{-1/9} \),
\[ |\ell_1(n, p) - \theta \lambda n| \leq \frac{n^{2/3}}{5}, \]
and all other components are smaller than \( n^{2/3} \).

Using estimations for the number of connected graphs with certain numbers of vertices and edges, and the first and second moment methods, one may also obtain the following result for the subcritical phase.

**Theorem 1.3** (Subcritical Phase) Let \( \lambda(n, p) = 1 - \varepsilon \) with \( n^{-1/3} \ll \varepsilon \ll 1 \). Then, for any positive constant \( \delta \leq 1/3 \) and large enough \( n \), with probability at least \( 1 - \left(\frac{8}{\varepsilon^2 n}\right)^{4/3} \),
\[ |\ell_1(n, p) - \frac{2 \log(\varepsilon^3 n)}{\varepsilon^2}| \leq \frac{\delta \log(\varepsilon^3 n)}{\varepsilon^2}. \]

There have been many results regarding the structure of the largest component too, for which readers may refer [44, 52, 42, 55] and references therein.

For Poisson branching processes, a duality principle has been known. A pair \((\mu, \lambda)\) with \( \mu < 1 < \lambda \) is called a conjugate pair if \( \mu e^{-\mu} = \lambda e^{-\lambda} \). It is easy to see that \( \mu = (1 - \theta) \lambda \) for a
conjugate pair \((\mu, \lambda)\). For a conjugate pair \((\mu, \lambda)\), the distribution of the Poisson(\(\lambda\)) branching process conditioned that the process dies out is exactly the same as that of the Poisson(\(\mu\)) branching process. (See e.g. [6], p164.) A similar but a little bit coarse duality was observed for the random graph \(G(n, p)\) and \(G(n^*, p)\) with \(\lambda = \lambda(n, p) > 1\) and \(n^* = (1 - \theta_\mu)n\). Notice that \(1 - \theta_\mu\) is the extinction probability for the Poisson(\(\lambda\)) branching process. It has been known that the component sizes of \(G(n^*, p)\) and those of \(G(n, p)\) excluding the largest component are the same in an asymptotic sense (see [6]).

The Poisson cloning model \(G_{PC}(n, p)\) equipped with the cut-off line algorithm enables us to not only estimate \(\ell_1(n, p)\) more accurately but also establish a precise discrete duality principle: In the supercritical phase \(\lambda := \lambda(n, p) = 1 + \varepsilon\) with \(n^{-1/3} \ll \varepsilon \ll 1\), \(G_{PC}(n, p)\) can be decomposed into three vertex disjoint graphs \(C, S\) and \(G\) whp (with high probability), where \(C\) is a connected graph of size about \(\theta n\), \(S\) is a smaller graph of size about \(\varepsilon^{-2} \ll \theta n\), and \(G\) has the same distribution as \(G_{PC}(n^*, p^*)\) with \(n^* \approx (1 - \theta_\lambda)n\) and \(p^* \approx p\), which yields \(\lambda(n^*, p^*) \approx \mu := (1 - \theta_\mu)\lambda\). In the subcritical phase \(\lambda = 1 - \varepsilon\) with \(n^{-1/3} \ll \varepsilon \ll 1\), the largest component is of size

\[
\frac{\log(\varepsilon^3 n) - 2.5 \log \log(\varepsilon^3 n) + O(1)}{-(\varepsilon + \log(1 - \varepsilon))}
\]

whp. The precise statements are as follows. We concentrate on the cases \(\varepsilon \ll 1\) for which more careful analysis is required. It is believed that the proofs are easily modified for the cases of positive constants \(\varepsilon\).

**Theorem 1.4 Supercritical Phase:** Let \(\lambda := \lambda(n, p) = 1 + \varepsilon\) with \(n^{-1/3} \ll \varepsilon \ll 1\), \(\mu := (1 - \theta_\mu)\lambda\) and \(1 \ll \alpha \ll (\varepsilon^3 n)^{1/2}\). Then, with probability \(1 - e^{-\Omega(n^2)}\), \(G_{PC}(n, p)\) may be decomposed into three vertex disjoint graphs \(C, S\) and \(G\), where \(C\) is connected and

\[\theta_\lambda n - \alpha(n/\varepsilon)^{1/2} \leq |C| \leq \theta_\lambda n + \alpha(n/\varepsilon)^{1/2},\]

and \(|S| \leq \frac{\alpha^2}{\varepsilon^2}\), and \(G\) has the same distribution as \(G_{PC}(n^*, p^*)\) for some \(n^*\) and \(p^*\) satisfying

\[(1 - \theta_\lambda)n - \alpha(n/\varepsilon)^{1/2} \leq n^* \leq (1 - \theta_\lambda)n + \alpha(n/\varepsilon)^{1/2},\]

and

\[\mu - \alpha(\varepsilon n)^{-1/2} \leq \lambda(n^*, p^*) \leq \mu + \alpha(\varepsilon n)^{-1/2}.\]

**Subcritical Phase:** Suppose \(\lambda := \lambda(n, p) = 1 - \varepsilon\) with \(n^{-1/3} \ll \varepsilon \ll 1\). Then, the size \(\ell_1^{PC}(n, p)\) of the largest component of \(G_{PC}(n, p)\) satisfies

\[\Pr\left[\ell_1^{PC}(n, p) \geq \frac{\log(\varepsilon^3 n) - 2.5 \log \log(\varepsilon^3 n) + c}{-(\varepsilon + \log(1 - \varepsilon))}\right] \leq 2e^{-\Omega(c)},\]

and

\[\Pr\left[\ell_1^{PC}(n, p) \leq \frac{\log(\varepsilon^3 n) - 2.5 \log \log(\varepsilon^3 n) - c}{-(\varepsilon + \log(1 - \varepsilon))}\right] \leq 2e^{-\Omega(c)},\]

for any positive constant \(c > 0\).
Inside Window: Suppose \( \lambda := \lambda(n, p) = 1 + \varepsilon \) with \( |\varepsilon| = O(n^{1/3}) \). Then, whp,

\[
\ell^p_{1}(n, p) = \Theta(n^{2/3}).
\]

(All constants in \( \Omega(\cdot) \)'s do not depend on any of \( \varepsilon, \alpha \) and \( c \).)

By Theorem 1.1 a corollary regarding \( G(n, p) \) follows.

**Corollary 1.5** Supercritical region: Suppose \( \lambda = \lambda(n, p) = 1 + \varepsilon \) with \( n^{-1/3} \ll \varepsilon \ll 1 \), and \( 1 \ll \alpha \ll (\varepsilon^3 n)^{1/2} \). Then, in \( G(n, p) \),

\[
\Pr[|\ell_1(n, p) - \theta_1 n| \geq \alpha(n/\varepsilon)^{1/2}] \leq 2e^{-\Omega(\alpha^2)}.
\]

Moreover, for the size \( \ell_2(n, p) \) of the second largest component and \( \varepsilon^* = 1 - (1 - \theta_1)\lambda \),

\[
\Pr \left[ \ell_2(n, p) \geq \frac{\log((\varepsilon^*)^3 n) - 2.5 \log(2.5 \log((\varepsilon^*)^3 n) + c)}{-(\varepsilon + \log(1 - \varepsilon^*)))} \right] \leq 2e^{-\Omega(c)};
\]

and

\[
\Pr \left[ \ell_2(n, p) \leq \frac{\log((\varepsilon^*)^3 n) - 2.5 \log(2.5 \log((\varepsilon^*)^3 n) - c)}{-(\varepsilon + \log(1 - \varepsilon^*)))} \right] \leq 2e^{-\Omega(c)},
\]

for any positive constant \( c > 0 \).

Subcritical region: Suppose \( \lambda = 1 - \varepsilon \) with \( n^{-1/3} \ll \varepsilon \ll 1 \), then,

\[
\Pr \left[ \ell_1(n, p) \geq \frac{\log(\varepsilon^3 n) - 2.5 \log(2.5 \log(\varepsilon^3 n) + c)}{-(\varepsilon + \log(1 - \varepsilon))} \right] \leq 2e^{-\Omega(c)};
\]

and

\[
\Pr \left[ \ell_1(n, p) \leq \frac{\log(\varepsilon^3 n) - 2.5 \log(2.5 \log(\varepsilon^3 n) - c)}{-(\varepsilon + \log(1 - \varepsilon))} \right] \leq 2e^{-\Omega(c)},
\]

for any positive constant \( c > 0 \).

Inside Window: Suppose \( \lambda := \lambda(n, p) = 1 + \varepsilon \) with \( |\varepsilon| = O(n^{1/3}) \). Then, whp,

\[
\ell_1(n, p) = \Theta(n^{2/3}).
\]

**The emergence of the t-core.** There are at least two possible directions to extend the problem of connected components. Observing that the minimum degree in a component must be larger than or equal to 1, one may consider subgraphs with minimum degree at least \( t \geq 2 \). For a graph \( G \), the \( t \)-core is the largest subgraph with minimum degree at least \( t \). As the minimum degree of the union of two subgraphs is at least the smaller minimum degree of the two, the \( t \)-core of a graph is unique. It is also easy to see that the \( t \)-core must be an induced subgraph. For this reason, the \( t \)-core of \( G \) sometimes refers to its vertex set. Denoted by \( V_t(G) \) is (the vertex set of) the \( t \)-core of \( G \). As the 1-core \( V_1(G) \) is the set of all non-isolated vertices, we consider the cases \( t \geq 2 \) throughout this paper. If there is no subgraph with minimum degree \( t \), the \( t \)-core is defined to be empty.
Another direction is to consider the \( t \)-connectivity, where a graph is \( t \)-connected if the graph remains connected after any \( t - 1 \) vertices are removed. Higher orders of connectivity have been used to understand various structures of graphs. Clearly, if a non-empty subgraph is \( t \)-connected, then its minimum degree must be \( t \) or larger.

In 1984, Bollobás\(^1\) initiated the study of \( t \)-core, \( t \geq 2 \), and observed that, provided \( t \geq 3 \) and \( pn \) is larger than a fixed constant, the \( t \)-core of \( G(n,p) \) is non-empty and \( t \)-connected whp. Luczak\(^2\) proved that, for \( t \geq 3 \), there is an absolute constant \( c \) such that the \( t \)-core of \( G(n,p) \) is either empty, or larger than \( cn \) and \( t \)-connected, whp. In particular, as far as the random graph \( G(n,p) \) is concerned, the \( t \)-core problem is the same as the \( t \)-connectivity problem. Moreover, if \( \lambda(n,p) \leq 1 \), then the \( t \)-core of \( G(n,p) \) is empty whp as the size of the largest component is \( O(n^{2/3}) \) whp. As \( p \) increases while \( n \) is fixed, the probability of the \( t \)-core of \( G(n,p) \) being non-empty keeps increasing. Let \( p_t(n, \delta) \) be the infimum of all \( p \) that makes the probability larger than or equal to a constant \( \delta \) in the range \( 0 < \delta < 1 \). Then Bollobás’s result implies that \( np_t(n, \delta) \) is bounded from above by a constant. Though \( np_t(n, \delta) \) may still have no limit value as \( n \) goes to infinity, it seems to be more natural to expect that the limit exists. Furthermore, as it happens often in phase transition phenomena, the limit, if exists, is also expected to be independent of \( \delta \). In other words, the phase transition is expected to be sharp.

For \( t = 2 \), the 2-core of a graph \( G \) is non-empty if and only if \( G \) contains a cycle. It is easy to see by the first moment method that \( G(n,p) \) with \( p = o(1/n) \) does not contain a cycle whp. For a constant \( c \) in the range \( 0 < c < 1 \), \( G(n,p) \) with \( pn = c \) may or may not have a cycle with positive probability. Particularly, the phase transition for the existence of non-empty 2-core is not sharp. In the graph process \( (G(n,m))_{m=0,1,...} \), in which a random edge is added one by one without repetition, Janson\(^3\) found the limiting distribution for the length of the first cycle, especially he showed that the length is bounded whp. However, the expectation of the length is known to be \( \Theta(n^{1/6}) \) due to Flajolet et al.\(^4\). The two facts are not contradicting each other, since there are random variables that are bounded whp, but their expectations are not. For example, \( \Pr[X = 1] = 1 - 1/n \) and \( \Pr[X = n^2] = 1/n \).

Bollobás\(^5\) proved that, if \( t \geq 5 \) and \( \lambda(n,p) := p(n-1) \geq \max(67, 2t+6) \), then \( G(n,p) \) has a non-empty \( t \)-core. Chvátal\(^6\) introduced the notion of critical \( \lambda_t \), without proving existence, satisfying, as \( n \) goes to infinity,

\[
\Pr \left[ G(n,p) \text{ has a non-empty } t \text{-core} \right] \rightarrow \begin{cases} 0 & \text{if } \lambda(n,p) < \lambda_t - \delta \\ 1 & \text{if } \lambda(n,p) > \lambda_t + \delta, \end{cases}
\]

for any constant \( \delta > 0 \). He also proved \( \lambda_5 \geq 2.88 \), if exists, and claimed that \( \lambda_4 \geq 4.52 \) and \( \lambda_5 \geq 6.06 \) etc. could be proven by the same method. It is Pittel, Spencer and Wormald\(^6\) who proved a more general theorem that implies that \( \lambda_t \) exists for fixed \( t \geq 3 \). They identified the values too. We present a slightly weaker version of the theorem.

For a Poisson random variable \( \text{Poi}(\rho) \) with mean \( \rho \), let \( P(\rho, i) = \Pr[\text{Poi}(\rho) = i] \) and \( Q(\rho, i) = \Pr[\text{Poi}(\rho) \geq i] \), i.e.,

\[
P(\rho, i) = e^{-\rho} \frac{\rho^i}{i!}, \quad \text{and} \quad Q(\rho, i) = \sum_{j=i}^{\infty} P(\rho, j) = e^{-\rho} \sum_{j=i}^{\infty} \frac{\rho^j}{j!},
\]
and let

\[ \lambda_t = \min_{\rho > 0} \frac{\rho}{Q(\rho, t - 1)}. \]

**Theorem 1.6** Let \( t \geq 3, \lambda(n, p) = p(n - 1) \). Then

\[
\Pr \left[ G(n, p) \text{ has a non-empty } t\text{-core} \right] \rightarrow \begin{cases} 
0 & \text{if } \lambda(n, p) < \lambda_t - n^{-\delta} \\
1 & \text{if } \lambda(n, p) > \lambda_t + n^{-\delta},
\end{cases}
\]

for any \( \delta \in (0, 1/2) \), and the \( t\)-core when \( \lambda(n, p) > \lambda_t + n^{-\delta} \) has \((1 + o(1))Q(\theta, \lambda, t)n \) vertices, whp, where \( \theta_{\lambda} \) is the largest solution for the equation

\[ \theta - Q(\theta, \lambda, t - 1) = 0. \]
and, for any $\delta > 0$,
\[
\Pr[|V_t(H(n, p; k))| \geq \delta n] \leq 2e^{-\Omega(\sigma^2 n)} + 2e^{-\Omega(\beta^k/(k-1))n}.
\] (1.3)

Supercritical Phase: If $\lambda = \lambda(n, p; k) = \lambda_{\text{crt}} + \sigma$ is uniformly bounded from above, then, for all $\alpha$ in the range $1 \ll \alpha \ll \sigma n^{1/2}$,
\[
\Pr[|V_t(n, p; k)| - Q(\theta, \lambda, t)n| \geq \alpha(n/\sigma)^{1/2}] = e^{-\Omega(\alpha^2)},
\]
and, for any $i \geq t$ and the sets $V_t(i)$ (resp. $W_t(i)$) of vertices of degree $i$ (resp. larger than or equal to $i$) in the $t$-core,
\[
\Pr\left[|V_t(i)| - P(\theta, \lambda, i)n| \geq \delta n\right] \leq 2e^{-\Omega(\min\{\delta^2 \sigma n, \sigma^2 n\})},
\]
and
\[
\Pr\left[|W_t(i)| - Q(\theta, \lambda, i)n| \geq \delta n\right] \leq 2e^{-\Omega(\min\{\delta^2 \sigma n, \sigma^2 n\})}.
\]
In particular, for $\rho_{\text{crt}} := \theta_{\lambda_{\text{crt}}(k,t)} \lambda_{\text{crt}}(k, t)$,
\[
|V_t(i)| = (1 + O(\sigma^{1/2}))i^i P(\rho_{\text{crt}}, i)n + O\left((n/\sigma)^{1/2} \log n\right),
\]
with probability $1 - 2e^{-\Omega(\min\{\log^2 n, \sigma^2 n\})}$. Moreover, if all $|V_t(i)|, i \geq t$, are given, each simple graph with the degree sequence induced by $|V_t(i)|, i \geq t$, is equally likely to be the $t$-core.

As one might guess, we will prove a stronger theorem (Theorem 6.2) for the Poisson cloning model $H_{PC}(n, p; k)$, from which Theorem 1.7 easily follows.

The pure literal algorithm for the random $k$-SAT problem
Recently the satisfiability problem for Boolean formulas has played a central role in the theory of computational complexity. An instance of the problem is a formula given by a conjunctive normal form (CNF), that is, a conjunction of disjunctions. Each disjunction, or clause, is of the form $(y_1 \lor \cdots \lor y_k)$, where $y_i$’s are chosen among $2n$ literals consisting of $n$ Boolean variables and their negations. Given a formula, the problem is whether there exists an assignment of the $n$ variables satisfying the formula. When such an assignment exists, the formula is satisfiable. Otherwise, it is unsatisfiable. A pair of literals $y, z$ is strictly distinct if $y$ is neither $z$ nor the negation of $z$. When the input formula is restricted to have only clauses of $k$ pairwise strictly distinct literals, called $k$-clauses, the problem is called the $k$-SAT problem.

It is known that the satisfiability problem is NP-complete [21], so that determining whether an arbitrary formula is satisfiable or not is regarded very difficult (assuming $P \neq \text{NP}$), in the sense that it is at least as hard as any problem whose solutions can be verified in polynomial time. Cook [21] proved that even the $k$-SAT problem for $k \geq 3$ is NP-complete too, while the 2-SAT problem can be solved in polynomial time. Given these facts, researchers have tried to find heuristic algorithms that are able to determine, in polynomial time, the satisfiability of most of $k$-SAT formulas, especially 3-SAT formulas. Among others, a number of heuristic algorithms have been considered based on Davis-Putnam algorithm
Since one has to define “most” before showing that his or her algorithm works for most \( k \)-SAT formulas, random models for the \( k \)-SAT problem have been introduced.

The most common models for the random \( k \)-SAT problems is the uniform model \( F_k(n,m) \) and \( F_{PC}(n,p;k) \). Here, \( F(n, m; k) \) is sampled uniformly at random from the set of all \( k \)-SAT formulas with \( n \) variables and \( m \) clauses. The other model may be constructed by selecting each \( k \)-clause with probability \( p \) independently of all other clauses. The random formula \( F(n, p; k) \) is a conjunction of all selected clauses. Since there are \( 2^k \binom{n}{k} \) \( k \)-clauses altogether, the expected number of clauses in the formula is \( m_p := 2^k p \binom{n}{k} \).

Other models include one formed by selecting a uniform random \( k \)-clause \( m \) times with replacement. It is known that these three models are essentially equivalent if \( m_p = m \) and \( m = \Theta(n) \).

Not surprisingly, the random 2-SAT and the random 3-SAT problems have been most extensively studied and many results have been published. For \( k = 2 \), Chvátal and Reed [20], Goerdt [40] and Fernandez de la Vega [35] independently proved that the random 2-SAT problem undergoes a phase transition at \( 2pn = 1 \), that is, for \( F(n, p) = F(n, p; k) \),

\[
\lim_{n \to \infty} \Pr[F(n, p) \text{ is satisfiable}] = \begin{cases} 
1 & \text{if } \limsup_{n \to \infty} 2pn < 1 \\
0 & \text{if } \liminf_{n \to \infty} 2pn > 1.
\end{cases}
\]

Since there are \( 2n \) literals, \( 2pn \) is the right parameter. Bollobás et al. [13] took more sophisticated approaches to determine the scaling window for the problem: Let \( \rho \gg n^{-1/3} \).

Then

\[
\Pr[F(n, \frac{1-\rho}{2n}) \text{ is satisfiable}] = 1 - \Theta(1/\rho^3),
\]

and

\[
\Pr[F(n, \frac{1+\rho}{2n}) \text{ is satisfiable}] = e^{-\Theta(\rho^3)}
\]

Though it is believed that the random \( k \)-SAT problem, \( k \geq 3 \), undergoes a similar phase transition, it remains as a conjecture. Only sharp transitions are known by a seminal result of Friedgut [39].

To find a satisfying assignment for a \( k \)-SAT formula, one may apply the pure literal algorithm (PLA): A literal is pure in the formula if it belongs to one or more clauses of the formula, while its negation is in no clause. PLA keeps selecting a pure literal, setting it true, and removing clauses containing the literal as they are already satisfied. It stops when there is no more pure literal. We say that PLA succeeds if no clause remains in the formula after it stops, and it fails otherwise. Clearly, the formula is satisfiable if PLA succeeds. The converse is not true, for example, \((y \lor z) \land (\bar{y} \lor \bar{z})\) is satisfiable whereas no pure literal exists. Broder, Frieze, and Upfal [18] analyzed PLA for the random 3-SAT problem to show that, for \( F_3(n, p) \), if \( \limsup \frac{m_p}{n} < 1.63 \) then PLA succeeds whp, and if \( \liminf \frac{m_p}{n} > 1.7 \) then it fails whp. Mitzenmacher [57] used the differential equation method introduced by Wormald [63] to claim, without rigorous proof, that the threshold for PLA exists and it is the solution of certain equations, which are somewhat complicated. That is, there is a constant \( c_k \), \( k \geq 3 \), so that PLA succeeds whp if \( \limsup \frac{m_p}{n} < c_k \), and fails whp if \( \liminf \frac{m_p}{n} > c_k \). For more about upper bounds for the random satisfiability problems, readers may refer [18, 24, 25, 37, 45, 47, 46, 50, 64]. For more advanced algorithms than PLA, which give various improved lower bounds, one may refer [1, 2, 3, 19, 38]. And a variation of the random satisfiability called the \((2 + p)\)-SAT problem can be found in [4, 59, 60].
For rigorous analysis of PLA, we consider the Poisson cloning model for the random $k$-SAT problems, $k \geq 2$. Since a $k$-clause can be regarded as a (hyper)edge consisting of $k$ vertices, the Poisson cloning model $F_{PC}(n, p; k)$ can be defined as $H_{PC}(2n, p; k)$ on $V = \{x_1, \tilde{x}_1, \ldots, x_n, \tilde{x}_n\}$ without edges that contain both of a variable and its negation. Then it is not difficult to establish an asymptotic equivalence between $F_{PC}(n, p; k)$ and $F(n, p; k)$ from Theorem 1.1. The details can be found in the next section.

**Theorem 1.8** Suppose $p = \Theta(n^{1-k})$. Then, for any collection $\mathcal{F}$ of $k$-SAT formulas,

$$c_1^* \Pr[F_{PC}(n, p; k) \in \mathcal{F}] \leq \Pr[F(n, p; k) \in \mathcal{F}] \leq c_1^*(\Pr[F_{PC}(n, p; k) \in \mathcal{F}])^{1/2} + e^{-n},$$

where

$$c_1^* = k^{1/2}e^{\frac{\rho}{2n}}(k)^{2n} + \frac{3^{2n}}{2^k} + o(1), \quad c_2^* = e^{\frac{\rho(1-1/k)(2n)}{2n}}\left(\frac{k}{k-1}\right)((k-1)c_1^*)^{1/k} + o(1).$$

and $o(1)$ goes to 0 as $n$ goes infinity.

As mentioned above, PLA undergoes a sharp phase transition. Actually, for $k \geq 3$, it turns out that the phase transition is similar to that of the 2-core in the random hypergraphs. The case $k = 2$ is similar to the 2-core problem of $G(n, p)$ and will be studied in a subsequent paper. Let

$$\lambda_{\text{crit}}(k) := \min_{\rho > 0} \frac{\rho}{Q(\rho, 1)^{k-1}} = \min_{\rho > 0} \frac{\rho}{(1 - e^{-\rho})^{k-1}},$$

and, for $\lambda \leq \lambda_{\text{crit}}(k)$, let $\Theta_C$ be the largest solution of the equation $\Theta^{\frac{\lambda}{\rho}} - 1 + e^{-\Theta} = 0$. Denote by $X_R(n, p; k)$ the set of variables in $X := \{x_1, \ldots, x_n\}$ whose truth values are not determined by PLA. The remaining formula is denoted by $F_R(n, p; k)$. The residual degree $d_R(z)$ of a literal $z$ of $X_R(n, p; k)$ is the number of clauses in $F_R(n, p; k)$ containing $z$. It is easy to see that $F_R(n, p; k)$ is independent of choices of pure literals when PLA is carried out.

**Theorem 1.9** Let $\lambda(n, p; k) = p\binom{2n-1}{k-1}$, $k \geq 3$ and $\sigma \gg n^{-1/2}$. If $\lambda(n, p; k) < \lambda(k) - \sigma$ is uniformly bounded from below by 0 and $\sigma(n)$ is the minimum such that $2\sigma(n) \geq 2i/k$, then

$$\Pr[X_R(n, p; k) \neq \emptyset] \leq 2e^{-\Omega(\sigma^2 n)} + O(n^{-(1-2/k)n})
$$

**Supercritical Phase:** If $\lambda = \lambda(n, p; k) = \lambda_{\text{crit}} + \sigma$ is uniformly bounded from above, then, for all $\alpha$ in the range $1 \ll \alpha < \sigma n^{1/2}$,

$$\Pr\left[\left|X_R(n, p; k)\right| - (1 - e^{-\alpha/\sigma})^2 n \geq \alpha(n/\sigma)^{1/2}\right] = e^{-\Omega(\alpha^2)},$$

and, for any $i, j \geq 1$ and the sets $X_R(i, j)$ (resp. $Y_R(i, j)$) of variables $x \in X_R(n, p; k)$ with $d_R(x) = i, d_R(\bar{x}) = j$ (resp. $d_R(x) \geq i, d_R(\bar{x}) \geq j$),

$$\Pr\left[\left|X_R(i, j) \setminus P(\theta_{\lambda, i}) P(\theta_{\lambda, j}) n \right| \geq \delta n\right] \leq 2e^{-\Omega(\min(\delta^2 \sigma, \sigma^{2} n))},$$

and

$$\Pr\left[\left|Y_R(i, j) \setminus Q(\theta_{\lambda, i}) Q(\theta_{\lambda, j}) n \right| \geq \delta n\right] \leq 2e^{-\Omega(\min(\delta^2 \sigma, \sigma^{2} n))}.$$

Moreover, if all $|X_R(i, j)|, i, j \geq 1$, are given, each formula with the degree sequence induced by $|X_R(i, j)|, i, j \geq 1$, is equally likely to be the residual formula $F_R(n, p; k)$.
Like the t-core problem, we will prove a stronger theorem (Theorem 7.3) for the Poisson cloning model \( F_{PC}(n, p; k) \), from which Theorem 1.7 easily follows.

The rest of the paper is organized as follows. In the next section, the Poisson cloning model is defined with details. The cut-off line algorithm and the cut-off line lemma are presented in Section 3. Section 4 is for Chernoff type large derivation inequalities that will be used in most of our proofs. In Section 5, a generalized core is defined and the main lemma is presented. As the proof of Theorem 1.4 is more sophisticated, Theorems 1.7 and 1.9 are first proved in Sections 6 and 7. Section 8 is for the proof of Theorem 1.4. Closing remark follows in Section 9.

2 The Poisson Cloning Model

In this section, we define the Poisson cloning models \( G_{PC}(n, p) \) for random graphs and generally \( H_{PC}(n, p; k) \) for random hypergraphs. Then, Theorem 1.1 will be proven.

To construct \( G_{PC}(n, p) \), we first take i.i.d Poisson \( \lambda = p(n - 1) \) random variables \( d(v) \) indexed by vertices \( v \) in the set \( V \) with \( |V| = n \). Then, take \( d(v) \) copies of each vertex \( v \in V \). The copies of \( v \) are called clones of \( v \), or simply \( v \)-clones. Since the sum of Poisson random variables is also a Poisson random variable, the total number \( N_\lambda := \sum_{v \in V} d(v) \) of clones is a Poisson \( \lambda n \) random variable. It is sometimes convenient to take a reverse, but equivalent, construction. We first take a Poisson \( \lambda n = 2p \binom{n}{2} \) random variable \( N_\lambda \) and then take \( N_\lambda \) unlabelled clones. Each clone is to be independently labelled as \( v \)-clone uniformly at random, in the sense that \( v \) is chosen uniformly at random from \( V \). It is well-known that the numbers \( d(v) \) of \( v \)-clones are i.i.d Poisson random variables with mean \( \lambda \).

If \( N_\lambda \) is even, the multigraph \( G_{PC}(n, p) \) is defined by generating a (uniform) random perfect matching of those \( N_\lambda \) clones, and contracting clones of the same vertex. That is, if a \( v \)-clone and a \( w \)-clone are matched, then the edge \( \{v, w\} \) is in \( G_{PC}(n, p) \) with multiplicity. In case that \( v = w \), it produces a loop that contributes 2 in the degree of \( v \). If \( N_\lambda \) is odd, we may define \( G_{PC}(n, p) \) to be any graph with a special loop that, unlike other loops, contributes only 1 in the degree of the corresponding vertex. In particular, if \( N_\lambda \) is odd, \( G_{PC}(n, p) \) is not a simple graph.

Strictly speaking, \( G_{PC}(n, p) \) varies depending on how to define it when \( N_\lambda \) is odd. However, if only simple graphs are concerned, the case of \( N_\lambda \) being odd would not matter. For example, the probability that \( G_{PC}(n, p) \) is a simple graph with a component larger than 0.1n does not depend on how \( G_{PC}(n, p) \) is defined when \( N_\lambda \) is odd, as it is not a simple graph anyway. Generally, for any collection \( \mathcal{G} \) of simple graphs, the probability that \( G_{PC}(n, p) \) is in \( \mathcal{G} \) is totally independent of how \( G_{PC}(n, p) \) is defined when \( N_\lambda \) is odd. Notice that properties of simple graphs are actually mean collections of simple graphs. Therefore, when properties of simple graphs are concerned, it is not necessary to describe \( G_{PC}(n, p) \) for odd \( N_\lambda \).

Here are two specific ways to define \( G_{PC}(n, p) \).

Example 2.1 One may keep matching two clones chosen uniformly at random among all unmatched clones.

Example 2.2 One may keep choosing his or her favorite unmatched clone, and matching it to a clone selected uniform at random from all other unmatched clones.
If $N_\lambda$ is even, both examples would yield uniform random perfect matchings. If $N_\lambda$ odd, each of them would yield a matching and an unmatched clone. We may create the special loop consisting of the vertex for which the unmatched clone is labelled. More specific ways to choose random clones will be described in the next section.

Generally for $k \geq 3$, the Poisson cloning model $H_{PC}(n, p; k)$ for $k$-uniform hypergraphs may be defined by the same way: We take i.i.d Poisson $\lambda = p\binom{n-1}{k-1}$ random variables $d(v)$, $v \in V$, and then take $d(v)$ clones of each $v$. If $N_\lambda := \sum_{v \in V} d(v)$ is divisible by $k$, the multihypergraph $H_{PC}(n, p; k)$ is defined by generating a uniform random perfect matching consisting of $k$-tuples of clones, and contracting clones of the same vertex. That is, if $v_1$-clone, $v_2$-clone, ... , $v_k$-clone are matched in the perfect matching, then the edge $\{v_1, v_2, ..., v_k\}$ is in $H_{PC}(n, p; k)$ with multiplicity. If $N_\lambda$ is not divisible by $k$, $H_{PC}(n, p; k)$ may be any hypergraph with a special edge consisting of $N_\lambda - k\lfloor N_\lambda/k \rfloor$ vertices. In particular, $H_{PC}(n, p; k)$ is not $k$-uniform when $N_\lambda$ is not divisible by $k$. Therefore, as long as properties of simple $k$-uniform hypergraphs are concerned, we do not have to describe $H_{PC}(n, p; k)$ when $N_\lambda$ is not divisible by $k$.

We show that the Poisson cloning model $H_{PC}(n, p; k)$, $k \geq 2$, is contiguous to the classical model $H(n, p; k)$ when the expected average degree is a constant.

**Theorem 1.1 (Restated)** Suppose $k \geq 2$ and $p = \Theta(n^{1-k})$. Then, for any collection $\mathcal{H}$ of $k$-uniform simple hypergraphs,

$$c_1 \Pr[H_{PC}(n, p; k) \in \mathcal{H}] \leq \Pr[H(n, p; k) \in \mathcal{H}] \leq c_2 \left( \Pr[H_{PC}(n, p; k) \in \mathcal{H}] \right)^{\frac{1}{k}} + e^{-n},$$

where

$$c_1 = k^{1/2}e^{\frac{\pi}{2}(\frac{1}{k})^{2} + \frac{1}{2}(\frac{1}{k})^{2}} + O(n^{-1/2}), \quad c_2 = \left( \frac{k}{k-1} \right) \left( c_1 (k-1) \right)^{1/k} + o(1),$$

and $o(1)$ goes to 0 as $n$ goes to infinity.

**Proof.** We assume that the random perfect matching is generated by keeping choosing $k$ unlabelled clones and labelling them uniformly at random, as any other way to generate it is equivalent provided $N_\lambda$ is divisible by $k$. Let $H$ be a fixed simple $k$-uniform hypergraph with $m$ edges. Then $H_{PC}(n, p; k) = H$ if and only if $N_\lambda = km$ and the $km$ clones are labelled so that $H$ is yielded after contraction. The first $k$ clones are labelled to be one of the $m$ edges with probability $\frac{\binom{n}{k}}{n^k} \cdots \frac{\binom{n}{k}}{n^k}$, and the second $k$ clones are labelled to be one of the remaining $m - 1$ edges with probability $(m - 1)\frac{\binom{n}{k}}{n^k}$, and so on. That is,

$$\Pr[H_{PC}(n, p; k) = H] = \Pr[N_\lambda = km]m! \left( \frac{k!}{n^k} \right)^m.$$

As $N_\lambda$ is a Poisson random variable with mean $\lambda n = pn\binom{n-1}{k-1} = kp\binom{n}{k}$, we have

$$\Pr[N_\lambda = km] = e^{-kp\binom{n}{k}} \frac{(kp\binom{n}{k})^{km}}{(km)!} \left( 1 + O\left( \frac{1}{m} \right) \right) e^{-kp\binom{n}{k}} \frac{k^{km}p^{km}\binom{n}{k}^{km}}{(2\pi km)^{1/2}(km/e)^{km}}$$

$$= e^{O(1/m)} (2\pi km)^{-1/2} \left( e^{-p\binom{n}{k}} \frac{p^{km}\binom{n}{k}^{km}}{(2\pi/e)^m} \right)^{k},$$

$$= e^{O(1/m)} (2\pi km)^{-1/2} \left( e^{-p\binom{n}{k}} \frac{p^{km}\binom{n}{k}^{km}}{(2\pi/e)^m} \right)^{k},$$
unless \( m = 0 \). Therefore,

\[
\Pr[H(n, p; k) = H] = p^m(1 - p)^{\binom{n}{k} - m} = p^m e^{-p^{(n)} - \frac{p^2}{2}^{(n)} + pm + O(p^3 n^k + p^2 m)}
\]

implies that

\[
\frac{\Pr[H_{PC}(n, p; k) = H]}{\Pr[H(n, p; k) = H]} = e^{-pm + \frac{p^2}{2}^{(n)} + O(p^3 n^k + p^2 m + 1/m)}
\times (2\pi km)^{-1/2}m! \frac{k!}{n^k} \left( \frac{n}{k} \right)^m \left( \frac{e}{m} \right)^m \left( e^{-p^{(n)}} \frac{p^m}{(\frac{m}{e})^m} \right)^k - 1.
\]

Since

\[
(2\pi m)^{-1/2}m! \left( \frac{e}{m} \right)^m = 1 + O\left( \frac{1}{m} \right) = e^{O(1/m)}, \quad \text{and} \quad \frac{k!}{n^k} \left( \frac{n}{k} \right)^m = e^{-1 + \Theta(1/n)} \left( \frac{k}{n} \right)/n,
\]

we conclude that

\[
\frac{\Pr[H_{PC}(n, p; k) = H]}{\Pr[H(n, p; k) = H]} = k^{-1/2} e^{-p^{(n)}} \frac{p^m}{(\frac{m}{e})^m} \left( e^{-p^{(n)}} \frac{p^m}{(\frac{m}{e})^m} \right)^k e^{-1 + \Theta(1/n)} \left( \frac{k}{n} \right) + O(p^3 n^k + p^2 m + \frac{1}{m})
\]

for all \( m \) in the range \( 0 \leq m \leq \binom{n}{k} \). Let \( R_m = e^{-p^{(n)}} \frac{p^m}{(\frac{m}{e})^m} \), or equivalently \( R_m = e^{-\frac{\lambda n}{k}} \left( \frac{e \lambda m}{k m} \right)^m \) by \( \lambda n = kp^{(n)} \). Then \( \frac{R_m + 1}{R_m} = 1 + O\left( \frac{1}{m} \right) \frac{\lambda n}{km} \). This gives that \( R_m \) has its maximum \( 1 + O\left( \frac{1}{m} \right) \) when \( m = \frac{\lambda n}{k} + O(1) \), assuming \( \lambda = \Theta(1) \), or \( p = \Theta(n^{1-k}) \). Moreover, it is not difficult to show that

\[
R_m = (1 + O\left( \frac{1}{m} \right)) e^{-\frac{1 + (2m + 1)(km - \lambda n)^2}{2kn}} \text{ if } |m - \frac{\lambda n}{k}| \leq n, \quad \text{and } R_m \leq e^{-\Omega(|km - \lambda n|)} \text{ otherwise.}
\]

Hence

\[
\frac{\Pr[H_{PC}(n, p; k) = H]}{\Pr[H(n, p; k) = H]} \leq k^{-1/2} e^{-\frac{\lambda n}{2km}} + O(n^{-1/2}) = k^{-1/2} e^{- \frac{\lambda n}{k}} (\frac{n}{2})^{\frac{n}{2}} + O(n^{-1/2}) =: c_1^{-1},
\]

which yields

\[
\Pr[H(n, p; k) = H] \geq c_1 \Pr[H_{PC}(n, p; k) = H].
\]

Thus,

\[
\Pr[H(n, p; k) \in \mathcal{H}] = \sum_{H \in \mathcal{H}} \Pr[H(n, p; k) = H] \geq \sum_{H \in \mathcal{H}} c_1 \Pr[H_{PC}(n, p; k) = H] = c_1 \Pr[H_{PC}(n, p; k) \in \mathcal{H}].
\]
For the upper bound, take the minimum \( m_1 \geq p \binom{n}{k} \) such that
\[
e^{-\left(m_1 - p \binom{n}{k}\right) \left(\frac{n}{n+p}\right)} R_{m_1} \leq \left(c_1 (k - 1) \Pr[H_{PC}(n, p; k) \in \mathcal{H}]\right)^{1/k} + e^{-n}. \tag{2.2}
\]
It is routine to check that \( m_1 = \Theta(n) \). Let
\[\mathcal{H}_1 = \{ H \in \mathcal{H} : \text{the number of edges in } H \text{ is at least } p \binom{n}{k}\}.\]

Then
\[
\Pr[H(n, p; k) \in \mathcal{H}_1] = \sum_{m \geq m_1} \sum_{|H| = m} \Pr[H(n, p; k) = H] + \sum_{m_1 \leq m < 1} \sum_{|H| = m_1} \Pr[H(n, p; k) = H].
\]

For \( m > m_1 \),
\[
\frac{\Pr[\text{Bin} \left( \binom{n}{k}, p \right) = m]}{\Pr[\text{Bin} \left( \binom{n}{k}, p \right) = m - 1]} \leq (1 + O(p)) \frac{p \binom{n}{k}}{m}
\]
implies that
\[
\Pr[\text{Bin} \left( \binom{n}{k}, p \right) \geq m_1] \leq (1/2 + o(1)) (2\pi m_1)^{1/2} \Pr[\text{Bin} \left( \binom{n}{k}, p \right) = m_1] \tag{2.3}
\]
and, if \( m_1 - p \binom{n}{k} \gg n^{1/2} \),
\[
\Pr[\text{Bin} \left( \binom{n}{k}, p \right) \geq m_1] \ll m_1^{1/2} \Pr[\text{Bin} \left( \binom{n}{k}, p \right) = m_1]. \tag{2.4}
\]

Observe that
\[
\Pr[\text{Bin} \left( \binom{n}{k}, p \right) = m_1] = \binom{n}{k} m_1^{m_1} (1 - p)^{\binom{n}{k} - m_1}
\]
\[
\leq (1 + o(1)) e^{-p \binom{n}{k}} \frac{p \binom{n}{k} m_1}{(2\pi m_1)^{1/2} e} e^{-\left(\frac{m_1}{2}\right)} + \Pr[H_{PC}(n, p; k) \in \mathcal{H}],
\]
\[
(1 + o(1)) (2\pi m_1)^{-1/2} R_{m_1} e^{-\left(\frac{m_1}{2}\right)} + p m_1 - \frac{p^2}{2} \binom{n}{k} + O(1/n),
\]
as \( R_{m_1} = e^{-p \binom{n}{k}} \frac{p \binom{n}{k} m_1}{(2\pi m_1)^{1/2} e} m_1 \). Since
\[
\frac{m_1}{2} - \frac{p^2}{2} \binom{n}{k} = -\frac{1}{2}\left(m_1 - p \binom{n}{k}\right)^2 + O(1/n),
\]
(2.2) together with (2.3) and (2.4) gives
\[
\Pr[\text{Bin} \left( \binom{n}{k}, p \right) \geq m_1] \leq (1/2 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p; k) \in \mathcal{H}]\right)^{1/k} + e^{-n}. \tag{2.5}
\]
For $m$ in the range $p(n) \leq m < m_1$, 

$$e^{-(m_1 - p(n))(\xi^n/n + p)} R_m \geq \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k}.$$ 

Provided $H$ has $m$ edges, (2.1) yields 

$$\Pr[H(n, p ; k) = H] = (1 + o(1))k^{1/2} R_m^{1-k} e^{\xi(n)/m + p/n} \Pr[H_{PC}(n, p ; k) = H] 
\leq (1 + o(1))k^{1/2} e^{\xi(n)/(k)} + p/n \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} 
\times \Pr[H_{PC}(n, p ; k) = H].$$ 

Hence 

$$\sum_{m_i} \sum_{H \in \mathcal{H}} \Pr[H(n, p ; k) = H] \leq (c_1 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k - 1/k} 
\times \Pr[H_{PC}(n, p ; k) \in \mathcal{H}_1].$$ 

This together with (2.5) gives 

$$\Pr[H(n, p ; k) \in \mathcal{H}_1] \leq (1/2 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} + e^{-\xi(n)} 
+ (c_1 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} \Pr[H_{PC}(n, p ; k) \in \mathcal{H}_1].$$ 

Similarly, for the maximum $m_2 < p(n)$ such that 

$$R_{m_2} \leq \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} + e^{-n},$$ 

and 

$$\mathcal{H}_2 = \{H \in \mathcal{H} : \text{the number of edges in } H \text{ is less than } p(n)\},$$ 

we have 

$$\Pr[H(n, p ; k) \in \mathcal{H}_2] \leq (1/2 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} + e^{-n} 
+ (c_1 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} \Pr[H_{PC}(n, p ; k) \in \mathcal{H}_2].$$ 

As $\Pr[H(n, p ; k) \in \mathcal{H}] = \Pr[H(n, p ; k) \in \mathcal{H}_1] + \Pr[H(n, p ; k) \in \mathcal{H}_2]$, we finally have 

$$\Pr[H(n, p ; k) \in \mathcal{H}] \leq (1 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} + e^{-n} 
+ (c_1 + o(1)) \left( c_1 (k - 1) \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} \Pr[H_{PC}(n, p ; k) \in \mathcal{H}] 
\leq c_2 \left( \Pr[H_{PC}(n, p ; k) \in H] \right)^{1/k} + e^{-n}. $$
Theorem 2.3 may be generalized in the case that there are some small number of forbidden edges. For example, the random $k$-SAT formula $F(n, p; k)$ may be regarded as $H_{PC}(2n, p; k)$ on $V = \{x_1, \ldots, x_n, \bar{x}_1, \ldots, \bar{x}_n\}$ without edges that contain both of a variable and its negation. Suppose there is a set $B$ of forbidden edges with $|B| = \binom{n}{k}$ for $\beta = O(1)$. Each edge not in $B$ is in the random $k$-uniform hypergraph $H^{(B)}(n, p; k)$ with probability $p$ independently of all other edges.

**Theorem 2.3** Suppose $k \geq 2$, $p = \Theta(n^{1-k})$ and $B$ is a set of $\binom{n}{k}$ with $\beta = O(1)$. Then, for any collection $\mathcal{H}$ of simple $k$-uniform hypergraphs without edges in $B$,

$$c_1(\beta) \Pr[H_{PC}(n, p; k) \in \mathcal{H}] \leq \Pr[H^{(B)}(n, p; k) \in \mathcal{H}] \leq c_2(\beta) \left( \Pr[H_{PC}(n, p; k) \in \mathcal{H}] \right)^{1/2} e^{-\frac{\beta}{n}} \binom{n}{k},$$

where

$$c_1(\beta) = c_1 e^{-\frac{p(k+\beta)}{n}} \binom{n}{k}, \quad c_2(\beta) = c_2 e^{-\frac{p(k+\beta)}{n}} \binom{n}{k}.$$

**Proof.** The result follows since

$$\frac{\Pr[H^{(B)}(n, p; k) = H]}{\Pr[H(n, p; k) = H]} = (1 - p)^{-|B|} = e^{-\frac{p|B|}{n}} \binom{n}{k}$$

for $H \in \mathcal{H}$ implies that

$$\Pr[H^{(B)}(n, p; k) \in \mathcal{H}] = e^{-\frac{p|B|}{n}} \binom{n}{k} \Pr[H(n, p; k) \in \mathcal{H}].$$

For the random $k$-SAT formulas, suppose $\beta_k$ satisfies

$$\frac{\beta_k}{2n} \binom{2n}{k} = \binom{2n}{k} - 2^k \binom{n}{k}, \quad \text{or equivalently,} \quad \frac{\beta_k}{2n} = 1 - \frac{2^k \binom{n}{k}}{\binom{2n}{k}}.$$

Then

$$\frac{2^k \binom{n}{k}}{\binom{2n}{k}} = e^{-\frac{1}{n}} \binom{k}{2} + \binom{k}{2} + O(\frac{1}{n^2}) = 1 - \frac{1}{2n} \binom{k}{2} + O\left(\frac{1}{n^2}\right)$$

implies that

$$\beta_k = \binom{k}{2} + O\left(\frac{1}{n}\right).$$

**Corollary 2.4** If $k \geq 2$ and $p = \Theta(n^{1-k})$, then, for any collection $\mathcal{F}$ of $k$-SAT formulas,

$$c_1^* \Pr[F_{PC}(n, p; k) \in \mathcal{F}] \leq \Pr[F(n, p; k) \in \mathcal{F}] \leq c_2^* \left( \Pr[F_{PC}(n, p; k) \in \mathcal{F}] \right)^{1/k} e^{-n},$$

where

$$c_1^* = k^{1/2} e^{-\frac{n(1-k)}{2k}} \binom{kn}{k} + o(1), \quad c_2^* = e^{-\frac{p(k+\beta)}{2n}} \binom{n}{k} \binom{k}{k_1} \left( \frac{k}{k-1} \right)^{1/k} + o(1).$$
3 The Poisson $\lambda$-Cell and the Cut-Off Line Algorithm

To generate a uniform random perfect matching of $N$ clones, we may keep matching $k$ unmatched clones uniformly at random (cf. Example 2.1). Another way is to choose the first clone as we like and match it to $k - 1$ clones selected uniformly at random among all other unmatched clones (cf. Example 2.2). As there are many ways to choose the first clone, we may take a way that makes the given problem easier to analyze. Formally, a sequence $\mathcal{S} = (S_i)$ of choice functions determines how to choose the first clone at each step, that is, $S_i$ tells which unmatched clone is to be the first clone to form the $i$th edge in the random perfect matching. A choice function may be deterministic or random. If less than $k$ clones remain unmatched, the edge consisting of those clones will be added. The clone chosen by $S_i$ is called the $i$th chosen clone, or simply a chosen clone.

We also present a more specific way to select the $k - 1$ random clones to be matched to the chosen clone. The way presented here will be useful to solve problems mentioned in the introduction. First, independently assign to each clone a uniform random real number between 0 and $\lambda = p\left(\frac{n-1}{k-1}\right)$. For the sake of convenience, a clone is called the largest, the smallest, etc. if so is the number assigned to it. In addition, for $0 \leq \theta \leq 1$, a clone is called $\theta \lambda$-large (resp. $\theta \lambda$-small) if its assigned number is larger than or equal to (resp. smaller than) $\theta \lambda$. To visualize the labelled clones with assigned numbers, one may consider $n$ horizontal line segments from $(0,j)$ to $(\lambda,j)$, $j = 0, ..., n-1$ in the two-dimensional plane $\mathbb{R}^2$. The $v_j$-clone with assigned number $x$ can be regarded as the point $(x,j)$ in the corresponding line segment. Then, each line segment with the points corresponding to clones with assigned numbers is an independent Poisson arrival process with density 1, up to time $\lambda$. The set of these Poisson arrival processes is called a Poisson $(\lambda,n)$-cell, or simply a $\lambda$-cell.

We will consider sequences of choice functions that choose an unmatched clone without changing the joint distribution of the numbers assigned to all other unmatched clones. Such a choice function is called oblivious. A sequence of oblivious choice functions are also called oblivious. The choice function that chooses the largest unmatched clone is not oblivious, as the numbers assigned to the other clones must be smaller than the largest assigned number. For an instance of an oblivious choice function, one may consider the choice function that chooses a $v$-clone for a vertex $v$ with fewer than 3 unmatched clones. For a more general example, let a vertex $v$ and its clones be $t$-light if there are fewer than $t$ unmatched $v$-clones.

**Example 3.1** Suppose there is an order of all clones that is independent of the assigned numbers. The sequence of the choice functions that choose the first $t$-light clone is oblivious.

A cut-off line algorithm is determined by a sequence of oblivious choice functions. Once a clone is obliviously chosen, the largest $k - 1$ clones among all unmatched clones are to be matched to the chosen clone. This may be further implemented by moving the cut-off line to the left until $k - 1$ vertices are found: Initially, the cut-off line of the $\lambda$-cell is the vertical line in $\mathbb{R}^2$ containing the point $(\lambda,0)$. The initial cut-off value, or cut-off number, is $\lambda$. At the first step, once the chosen clone is given, move the cut-off line to the left until exactly $k - 1$ unmatched clones, excluding the chosen clone, are on or in the right side of the line. These $k - 1$ clones together with the chosen clone form the first edge in the random
perfect matching. The new cut-off value $\Lambda_i$ is to be the assigned number to the $(k - 1)^{th}$ largest clone. Here we assumed that no two distinct clones are assigned the same number as the probability of such an event is 0. The new cut-off line is, of course, the vertical line containing $(\Lambda_i, 0)$. Repeating this procedure, one may obtain the $i^{th}$ cut-off value $\Lambda_i$ and the corresponding cut-off line.

Notice that, after the $i^{th}$ step ends with the cut-off value $\Lambda_i$, all numbers assigned to unmatched clones are i.i.d uniform random numbers between 0 to $\Lambda_i$, as the choice functions are oblivious. Let $U_i$ be the number of unmatched clones after step $i$. That is, $U_i = N - ik$. Since the $(i + 1)^{th}$ choice function tells how to choose the first clone to form the $(i + 1)^{th}$ edge without changing the distribution of the assigned numbers, the distribution of $\Lambda_{i+1}$ is the distribution of the $(k - 1)^{th}$ largest number among $U_i - 1$ independent uniform random numbers between 0 and $\Lambda_i$. Let $1 - T_j$ be the random variable representing the largest number among $j$ independent uniform random numbers between 0 and 1. Or equivalently in distribution sense, $T_j$ is the random variable representing the smallest number among the random numbers. Then the largest number among the $U_i - 1$ random numbers has the same distribution as $\Lambda_i(1 - T_{U_{i-1}})$. Repeating this $k - 1$ times, we have

$$
\Lambda_{i+1} = \Lambda_i(1 - T_{U_{i-1}})(1 - T_{U_{i-2}}) \cdots (1 - T_{U_{i-k+1}}),
$$

and hence

$$
\Lambda_{i+1} = \Lambda_i(1 - T_{U_{i-1}}) \cdots (1 - T_{U_{i-k+1}}) = \Lambda_{i-1}(1 - T_{U_{i-1}}) \cdots (1 - T_{U_{i-2}}) \cdots (1 - T_{U_{i-k+1}}) \cdots (1 - T_{U_{i-k+1}}) = \lambda \prod_{j=N-\lambda}^{N-\lambda-(i+1)k+1} \left(1 - T_j\right).
$$

It is crucial to observe that, once $N_\lambda$ is given, all $T_i$ are mutually independent random variables. This makes the random variable $\Lambda_i$ highly concentrated near its mean, which enables us to develop theories as if $\Lambda_i$ were a constant. The cut-off value $\Lambda_i$ will provide enough information to resolve some otherwise difficult problems.

In the next section, we will prove the following slightly general lemma regarding the concentration of $\prod(1 - T_j)$.

**Lemma 3.2** For positive integer $k$, let $T_j$'s be mutually independent, $j = N, N-1, ..., N-lk$ with $N - lk \gg 1$, and let $R$ be a non-empty subset of $\{0, 1, ..., k-1\}$ with $|R| = r$. Then, denoting $\theta_i = (1 - ik/N)^{1/k}$, we have, for $\epsilon \leq 0.1$,

$$
\Pr \left[ \max_{1 \leq i \leq t} \left| \prod_{j=N}^{j=N-lk} \left(1 - T_j\right) - \theta_i^r \right| \geq \epsilon \right] \leq 10e^{-\frac{1 + o(1)}{r \log \log \log \log N} \min\left\{ \epsilon \theta_i^r N, \frac{\epsilon^2 \theta_i^r N}{2(1 - \theta_i^r)} \right\}}.
$$

In particular, if $\theta_i = \Omega(1)$, then

$$
\Pr \left[ \max_{1 \leq i \leq t} \left| \prod_{j=N}^{j=N-lk} \left(1 - T_j\right) - \theta_i^r \right| \geq \epsilon \right] \leq 10e^{-\Omega(\min\{\epsilon \theta_i, \frac{\epsilon^2 \theta_i^r}{2(1 - \theta_i^r)}\})}.
$$

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The cut-off line lemma follows from Lemma 3.2. For \( \theta \) in the range \( 0 \leq \theta \leq 1 \), let \( \Lambda(\theta) \) be the cut-off value when \((1-\theta^{\frac{k}{\theta-1}})\lambda n \) or more clones are matched for the first time. Conversely, let \( N(\theta) \) be the number of matched clones until the cut-off line reaches \( \theta \lambda \).

**Lemma 3.3 (Cut-off Line Lemma)** Let \( k \geq 2 \) and \( \lambda > 0 \) be fixed. Then, for \( \theta_i < 1 \) uniformly bounded below from 0 and \( 0 < \Delta \leq n \),

\[
\Pr \left[ \max_{\theta \leq \theta \leq 1} |\Lambda(\theta) - \theta \lambda| \geq \frac{\Delta}{n} \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n}))},
\]

and

\[
\Pr \left[ \max_{\theta \leq \theta \leq 1} |N(\theta) - (1 - \theta^{\frac{k}{\theta-1}})\lambda n| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n}))}.
\]

**Proof.** Suppose \( N_{\lambda} = \lambda n + h \) is given. As \( N_{\lambda} \) is a Poisson \( \lambda n \) random variable,

\[
\Pr \left[ |N_{\lambda} - \lambda n| \geq c \min \left\{ n, \frac{\Delta}{1 - \theta_i} \right\} \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n}))} \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n}))},
\]

where \( c \) is a (small) constant to be specified later. Hence it is enough to consider \( h \) in the range \( |h| \leq c \min \{n, \frac{\Delta}{1 - \theta_i} \} \).

For \( \theta_i \leq \theta \leq 1 \), let \( \xi_s \) be the solution of the equation \((1 - \theta^{\frac{k}{\theta-1}})\lambda n = (1 - \xi_s^{\theta^{\frac{k}{\theta-1}}})N_{\lambda} \). Then

\[
\xi_s = \left(1 - \frac{(1 - \theta^{\frac{k}{\theta-1}})\lambda n}{N_{\lambda}}\right) = \left(\theta^{\frac{k}{\theta-1}} + \frac{(1 - \theta^{\frac{k}{\theta-1}})h}{\lambda n + h}\right) = \theta + O\left(\frac{(1 - \theta)|h|}{\lambda n + h}\right)
\]

implies that \( \xi_s \) is uniformly bounded from below by 0 (for small enough \( c \)). Lemma 3.2 gives

\[
\Pr \left[ \max_{\theta \leq \theta \leq 1} |\Lambda(\theta) - \xi_s \lambda| \geq \frac{\Delta}{2n} \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n}))}.
\]

Taking small enough \( c \), we also have \( |\xi_s - \theta| \leq \frac{\Delta}{2\lambda n} \) and

\[
|\Lambda(\theta) - \theta \lambda| \leq |\Lambda(\theta) - \xi_s \lambda| + \lambda|\xi_s - \theta| \\
\leq |\Lambda(\theta) - \xi_s \lambda| + \frac{\Delta}{2n}.
\]

Therefore, if \( |\Lambda(\theta) - \theta \lambda| \geq \frac{\Delta}{n} \) then \( |\Lambda(\theta) - \xi_s \lambda| \geq \frac{\Delta}{2n} \) and hence the probability that such \( \theta \) exists in the range \( \theta_i \leq \theta \leq 1 \) is at most

\[
2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n}))} \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n})))} \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-\theta^{\frac{k}{\theta-1}})\lambda n})))}.
\]

For the second inequality, it is enough to observe that \( |N(\theta) - (1 - \theta^{\frac{k}{\theta-1}})\lambda n| \geq \Delta \) implies that \( \Lambda(\theta + \Omega(\Delta/n)) \leq \theta \lambda \) or \( \Lambda(\theta - \Omega(\Delta/n)) \geq \theta \lambda \).

□

For the Poisson \( \lambda \)-cell conditioned on \( N_{\lambda} = N \), a similar lemma may be obtained.
Lemma 3.4 (Cut-off Line Lemma for \( N \) clones) Let \( k \geq 2, \lambda > 0 \) be fixed. Then, for the Poisson \( \lambda \)-cell conditioned on \( N_{\lambda} = N \), and for \( \theta_{i} < 1 \) uniformly bounded below from 0 and \( 0 < \Delta \leq N \),

\[
\Pr\left[ \max_{\theta_{i} \leq \theta_{1}} |\Lambda(\theta) - \theta_{i} \lambda| \geq \frac{\Delta}{N} \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{1+N\lambda}))},
\]

and

\[
\Pr\left[ \max_{\theta_{i} \leq \theta_{1}} |N(\theta) - (1 - \theta_{i}^{k} N)| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1+\lambda N)^3}))}.
\]

\[
\sum_{m} \theta_{i} \leq \theta_{1} \quad \sum_{m} \theta_{i} \geq \theta_{1}
\]

4 Large Deviation Inequalities

In this section, a generalized Chernoff bound and an inequality for random processes are to be shown. Let \( X_{1}, \ldots, X_{m} \) be a sequence of random variables such that the distribution of \( X_{i} \) is determined if all the values of \( X_{1}, \ldots, X_{i-1} \) are known. For example, \( X_{i} = \Lambda(\theta_{i}) \) with \( 1 \geq \theta_{1} \geq \cdots \geq \theta_{m} \geq 0 \) in a Poisson \( \lambda \)-cell. If the upper and/or lower bounds are known for the conditional means \( E[X_{i}|X_{1}, \ldots, X_{i-1}] \) and for the conditional second and third moments, then Chernoff type large deviation inequalities may be obtained not only for \( \sum_{j=1}^{m} X_{j} \) but \( \min_{1 \leq i \leq m} \sum_{j=1}^{i} X_{j} \) and/or \( \max_{1 \leq i \leq m} \sum_{j=1}^{i} X_{j} \). Large deviation inequalities for such minimums or maximums are especially useful in various situations. Lemma 3.2 can be shown using such inequalities too.

Lemma 4.1 Let \( X_{1}, \ldots, X_{m} \) be a sequence of random variables. Suppose

\[
E[X_{i}|X_{1}, \ldots, X_{i-1}] \leq \mu_{i}, \tag{4.1}
\]

and there are positive constants \( a_{i}, b_{i}, \) and \( \xi_{0} \) so that

\[
E[(X_{i} - \mu_{i})^{2}|X_{1}, \ldots, X_{i-1}] \leq a_{i}, \tag{4.2}
\]

and

\[
E[(X_{i} - \mu_{i})^{3} e^{\xi(X_{i} - \mu_{i})}|X_{1}, \ldots, X_{i-1}] \leq b_{i} \quad \text{for all } 0 \leq \xi \leq \xi_{0}. \tag{4.3}
\]

Then for any \( \alpha \) with \( 0 < \alpha \leq \xi_{0}(\sum_{i=1}^{m} a_{i})^{1/2}, \)

\[
\Pr\left[ \sum_{i=1}^{m} X_{i} \geq \sum_{i=1}^{m} \mu_{i} + \alpha\left(\sum_{i=1}^{m} a_{i}\right)^{1/2} \right] \leq \exp\left(-\frac{\alpha^2}{2}\left(1 - \frac{\alpha \sum_{i=1}^{m} b_{i}}{3(\sum_{i=1}^{m} a_{i})^{3/2}}\right)\right).
\]

Similarly,

\[
E[X_{i}|X_{1}, \ldots, X_{i-1}] \geq \mu_{i} \tag{4.4}
\]

together with (4.2) and

\[
E[(X_{i} - \mu_{i})^{3} e^{\xi(X_{i} - \mu_{i})}|X_{1}, \ldots, X_{i-1}] \geq b_{i} \quad \text{for all } -\xi_{0} \leq \xi < 0 \tag{4.5}
\]

implies that

\[
\Pr\left[ \sum_{i=1}^{m} X_{i} \leq \sum_{i=1}^{m} \mu_{i} - \alpha\left(\sum_{i=1}^{m} a_{i}\right)^{1/2} \right] \leq \exp\left(-\frac{\alpha^2}{2}\left(1 - \frac{\alpha \sum_{i=1}^{m} b_{i}}{3(\sum_{i=1}^{m} a_{i})^{3/2}}\right)\right).
\]
Proof. We first show that
\[ E[e^{\xi \sum_{j=1}^{i}(X_j - \mu_j)}] \leq e^{\xi^2 \sum_{j=1}^{i} a_j/2 + \|\xi\|^3 \sum_{j=1}^{i} b_j/6} \quad \text{for } i = 0, \ldots, n, \]
using induction. As
\[ E[e^{\xi \sum_{j=1}^{i}(X_j - \mu_j)}] = E\left[ E[e^{\xi \sum_{j=1}^{i}(X_j - \mu_j)} | X_1, \ldots, X_{i-1}] \right] = E\left[ e^{\xi \sum_{j=1}^{i-1}(X_j - \mu_j)} E[e^{\xi(X_i - \mu_i)} | X_1, \ldots, X_{i-1}] \right], \]
it is enough to show
\[ E[e^{\xi(X_i - \mu_i)} | X_1, \ldots, X_{i-1}] \leq e^{\xi^2 a_i/2 + \xi^3 b_i/6}. \]
For \(0 < \xi \leq \xi_0\), Taylor theorem gives
\[ E[e^{\xi(X_i - \mu_i)} | X_1, \ldots, X_{i-1}] = 1 + \frac{\xi^2 E[(X_i - \mu_i)^2 | X_1, \ldots, X_{i-1}]}{2} + \frac{\xi^3 E[(X_i - \mu_i)^3 e^{\xi^*(X_i - \mu_i)} | X_1, \ldots, X_{i-1}]}{6} \leq 1 + \frac{\xi^2 a_i}{2} + \frac{\xi^3 b_i}{6} \leq e^{\xi^2 a_i/2 + \xi^3 b_i/6}, \]
for some \(\xi^*\) between 0 and \(\xi\).

Let \(\xi = \alpha(\sum_{i=1}^{m} a_i)^{-1/2} \leq \xi_0\). Then
\[ E[e^{\xi \sum_{j=1}^{i}(X_j - \mu_j)}] \leq \exp \left( \frac{\alpha^2}{2} + \frac{\alpha^3 \sum_{i=1}^{m} b_i}{6(\sum_{i=1}^{m} a_i)^{3/2}} \right), \]
and
\[ \Pr \left[ \sum_{j=1}^{m} X_j - \sum_{j=1}^{m} \mu_j \geq \alpha \left( \sum_{i=1}^{m} a_i \right)^{1/2} \right] \leq E[e^{\xi(\sum_{j=1}^{i}(X_j - \mu_j) - \alpha(\sum_{i=1}^{m} a_i)^{1/2})}] \leq \exp \left( -\frac{\alpha^2}{2} + \frac{\alpha^3 \sum_{i=1}^{m} b_i}{6(\sum_{i=1}^{m} a_i)^{3/2}} \right). \]
Similarly, \((4.4)\) together with \((4.2), (4.3)\) and \(\xi = -\alpha(\sum_{i=1}^{m} a_i)^{-1/2}\) together with \((4.3)\) gives
\[ \Pr \left[ \sum_{j=1}^{m} X_j - \sum_{j=1}^{m} \mu_j \leq -\alpha \left( \sum_{i=1}^{m} a_i \right)^{1/2} \right] \leq E[e^{\xi(\sum_{j=1}^{i}(X_j - \mu_j) + \alpha(\sum_{i=1}^{m} a_i)^{1/2})}] \leq \exp \left( -\frac{\alpha^2}{2} + \frac{\alpha^3 \sum_{i=1}^{m} b_i}{6(\sum_{i=1}^{m} a_i)^{3/2}} \right). \]
\[ \square \]

As it is sometimes tedious to point out the value of \(\alpha\) and to check the required bounds for it, the following forms of inequalities are often more convenient.

**Corollary 4.2 (Generalized Chernoff bound)** If \(\delta_0 \sum_{i=1}^{m} b_i \leq \sum_{i=1}^{m} a_i\) for some \(0 < \delta \leq 1\), then \((4.7) - (4.3)\) imply
\[ \Pr \left[ \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \mu_i + R \right] \leq e^{-\frac{1}{3} \min\{\delta_0 R, R^2 / \sum_{i=1}^{m} a_i\}}, \]
for all \(R > 0\). Similarly, If \(-\delta_0 \sum_{i=1}^{m} b_i \leq \sum_{i=1}^{m} a_i\) for some \(0 < \delta \leq 1\), then \((4.2), (4.4)\) and \((4.3)\) yield
\[ \Pr \left[ \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \mu_i - R \right] \leq e^{-\frac{1}{4} \min\{\delta_0 R, R^2 / \sum_{i=1}^{m} a_i\}}, \]
for all \(R > 0\).
Proof. For $R \leq \delta \xi_0 \sum_{i=1}^{m} a_i$, Lemma 4.1 with $\alpha = R(\sum_{i=1}^{m} a_i)^{-1/2}$ gives

$$\Pr \left[ \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \mu_i + R \right] \leq \exp \left( - \frac{R^2}{3 \sum_{i=1}^{m} a_i} \right).$$

If $R \leq \delta \xi_0 \sum_{i=1}^{m} a_i$, one may replace $a_i$ by $a_i^* \leq a_i$ satisfying $\sum a_i^* = R/(\delta \xi_0)$ and obtain

$$\Pr \left[ \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \mu_i \right] \leq \Pr \left[ \sum_{i=1}^{m} X_i - \sum_{i=1}^{m} \mu_i \geq (\delta \xi_0 R)^{1/2} \left( \frac{R}{\delta \xi_0} \right)^{1/2} \right] = \exp(-\delta \xi_0 R/3),$$

as

$$\frac{(\delta \xi_0 R)^{1/2} \sum b_i}{3(\sum a_i^*)^{3/2}} = \frac{(\delta \xi_0 R)^{1/2} \sum b_i}{3(R/(\delta \xi_0))^{3/2}} = \frac{\delta^2 \xi_0^2 \sum b_i}{3R} \leq \frac{\delta \xi_0 \sum a_i}{3R} \leq \frac{1}{3}.$$
We bound the probability $\max_{\theta_0 \leq \theta \leq \theta_1} \Gamma(\theta) \geq R$ and $\min_{\theta_0 \leq \theta \leq \theta_1} \Gamma(\theta) \leq R$ under some conditions.

**Lemma 4.5** Let $0 \leq \theta_0 < \theta_1$, $R = R_1 + R_2$, $R_1, R_2 > 0$ and $\Phi_\theta$ be events depending on $\{X_\theta\}_\theta$. If

$$\Gamma(\theta) \leq \psi + \psi_{\theta}, \quad \forall \theta_0 \leq \theta \leq \theta_1,$$

then

$$\Pr\left[ \max_{\theta_0 \leq \theta \leq \theta_1} \Gamma(\theta) \geq R \right] \leq \Pr\left[ \psi \geq R_1 \right] + \Pr\left[ \bigcup_{\theta : \theta_0 \leq \theta \leq \theta_1} \overline{\Phi_\theta} \right] + \max_{\theta : \theta_0 \leq \theta \leq \theta_1} \max_{\theta' : \theta_0 \leq \theta' \leq \theta_1} \Phi_\theta \Pr\left[ \psi \geq R_2 \bigg| \{X_\theta\}_\theta \right].$$

Similarly, if

$$\Gamma(\theta) \geq \psi + \psi_{\theta}, \quad \forall \theta_0 \leq \theta \leq \theta_1,$$

then

$$\Pr\left[ \min_{\theta_0 \leq \theta \leq \theta_1} \Gamma(\theta) \leq -R \right] \leq \Pr\left[ \psi \leq -R_1 \right] + \Pr\left[ \bigcup_{\theta : \theta_0 \leq \theta \leq \theta_1} \overline{\Phi_\theta} \right] + \max_{\theta : \theta_0 \leq \theta \leq \theta_1} \max_{\theta' : \theta_0 \leq \theta' \leq \theta_1} \Phi_\theta \Pr\left[ \psi \leq -R_2 \bigg| \{X_\theta\}_\theta \right].$$

**Example 4.3** (continued) As

$$\Pr[\psi \geq R_1] \leq e^{-\Omega(\min\{R_1, \frac{R^2}{p(1-p)n}\})}$$

and

$$\Pr[\psi_i \geq R_2 | X_1, ..., X_i] = \Pr[\psi_i \geq R_2] \leq e^{-\Omega(\min\{R_2, \frac{R^2}{p(1-p)(n-i)}\})},$$

Lemma 4.5 for $R_1 = R_2 = R/2$ and $\Phi_\theta = \emptyset$ gives

$$\Pr[\max_{i : 0 \leq i \leq n} |S_i - pi| \geq R] \leq e^{-\Omega(\min\{R, \frac{R^2}{pm}\})}.$$
Once \( \{X_{\theta'}\}_{\theta' \leq \theta} \) is given, \( \hat{V}(\theta) \) is determined and
\[
\hat{V}(\theta) = \sum_{v \in \hat{V}(\theta)} 1(\text{v has no } (1 - \theta)\lambda\text{-large clone})
\]
is a sum of i.i.d Bernoulli random variables with mean \( e^{-(\theta - \theta)\lambda} \). Thus,
\[
\Pr \left[ \psi_{\theta} \geq R/2 \left\vert \{X_{\theta'}\}_{\theta' \leq \theta} \right. \right] \leq 2e^{-\Omega(\min\{R \eta_{\alpha}^2\})} \leq 2e^{-\Omega(\min\{R \eta_{\alpha}^2/4\})},
\]
and Lemma 4.5 for \( \theta_0 = 0 \) and \( \Phi_{\theta} = \emptyset \) yields
\[
\Pr \left[ \max_{\theta_0 \leq \theta \leq \theta_1} \left\vert \hat{V}(\theta) \right\vert - e^{-\theta_{\lambda}n} \right] \geq R \right] \leq 2e^{-\Omega(\min\{R \eta_{\alpha}^2\})}.
\]

The proof of Lemma 4.5 follows.

**Proof of Lemma 4.5** Let \( \tau \) be the first time \( \theta \) in the range \( \theta_0 \leq \theta \leq \theta_1 \) when \( \Gamma(\theta) \geq R \). If no such \( \theta \) exists, \( \tau = \infty \) and \( \psi_{\tau} = -\infty \). Observe that
\[
\Pr \left[ \tau < \infty \right] \leq \Pr \left[ \tau < \infty, \psi \leq R_1, \bigcap_{\theta : \theta_0 \leq \theta \leq \theta_1} \Phi_{\theta} \right] + \Pr[\psi \geq R_1] + \Pr \left[ \bigcup_{\theta : \theta_0 \leq \theta \leq \theta_1} \Phi_{\theta} \right]
\]
\[
\leq \Pr \left[ \psi_{\tau} \geq R_2, \bigcap_{\theta : \theta_0 \leq \theta \leq \theta_1} \Phi_{\theta} \right] + \Pr[\psi \geq R_1] + \Pr \left[ \bigcup_{\theta : \theta_0 \leq \theta \leq \theta_1} \Phi_{\theta} \right].
\]

Considering the conditional probability on \( \{X_{\theta}\} \theta \leq \tau \), we have
\[
\Pr \left[ \psi_{\tau} \geq R_2, \bigcap_{\theta : \theta_0 \leq \theta \leq \theta_1} \Phi_{\theta} \right] = E \left[ \Pr \left[ \psi_{\tau} \geq R_2, \bigcap_{\theta : \theta_0 \leq \theta \leq \theta_1} \Phi_{\theta} \right| \{X_{\theta}\} \theta \leq \tau \right]
\]
\[
\leq E \left[ 1(\Phi_{\tau}) \Pr \left[ \psi_{\tau} \geq R_2 \left| \{X_{\theta}\} \theta \leq \tau \right. \right. \right] \right].
\]

As
\[
1(\Phi_{\tau}) \Pr \left[ \psi_{\tau} \geq R_2 \left| \{X_{\theta}\} \theta \leq \tau \right. \right] \leq \max_{\theta : \theta_0 \leq \theta \leq \theta_1} \max_{\theta' : \theta' \leq \theta} \Pr \left[ \psi_\theta \geq R_2 \left| \{X_{\theta'}\} \theta' \leq \theta \right. \right],
\]
the desired inequality follows.

Applying the same argument for \( -\Gamma(\theta) \), the second part also follows.

**Proof of Lemma 3.2** As
\[
\prod_{N - j \in kR}^{\theta_k N} (1 - T_j) = \exp \left( \sum_{N - j \in kR}^{\theta_k N} \log(1 - T_j) \right),
\]

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we show a high concentration for \( \log(1-T_j) \). Since \( \Pr[\exists \, j, \, T_j \geq 1/2] \leq \sum_{j=N}^{\theta_i N} 2^{-j} \leq 2^{-\theta_i N+1} \), and
\[
-x - x^2 \leq \log(1-x) \leq -x \quad \forall x : 0 \leq x \leq 1/2,
\]
with probability at least \( 1 - 2^{-\theta_i N+1} \), we have
\[
-T_j - T_j^2 \leq \log(1-T_j) \leq -T_j, \quad \text{for all } j.
\]
Thus, it is enough to show that both of \( E[S_i^*] \) and \( E[T_i^*] \) are very close to \( \theta_i^r \), and
\[
T_i^* := \sum_{j=N}^{\theta_i N} T_j, \quad \text{and} \quad S_i^* := \sum_{j=N}^{\theta_i N} S_j
\]
are highly concentrated. That is, we will show that
\[
\Pr[\max_i |S_i^* - E[S_i^*]| \geq \varepsilon] \leq 4e^{-\frac{1+o(1)}{6} \min\left\{ \frac{\epsilon^2 k N}{2(1-\theta_i)}, \epsilon \theta_i N \right\}},
\]
and
\[
\Pr[\max_i |T_i^* - E[T_i^*]| \geq \varepsilon] \leq 4e^{-\frac{1+o(1)}{6} \min\left\{ \frac{\epsilon^2 k N}{2(1-\theta_i)}, \epsilon \theta_i N \right\}},
\]
which together with
\[
E[S_i^*] = -r \log \theta_i + o\left( \left( \frac{1-\theta}{\theta_i^r N} \right)^{1/2} \right), \quad \text{and} \quad E[T_i^*] = -r \log \theta_i + o\left( \left( \frac{1-\theta}{\theta_i^r N} \right)^{1/2} \right).
\]
The \( o\left( \left( \frac{1-\theta}{\theta_i N} \right)^{1/2} \right) \) terms do not matter, since the desired inequality is trivial unless \( \varepsilon = \Omega\left( \left( \frac{1-\theta}{\theta_i N} \right)^{1/2} \right) \). If \( \varepsilon = \Omega\left( \left( \frac{1-\theta}{\theta_i N} \right)^{1/2} \right) \), then the above concentration inequalities for 0.95\( \varepsilon \) give
\[
\Pr\left[ \max_i \left| \sum_{j=N}^{\theta_i N} \log(1-T_j) - r \log \theta_i \right| \geq 0.95\varepsilon \right] \leq 8e^{-\frac{1+o(1)}{7} \min\left\{ \frac{\epsilon^2 k N}{2(1-\theta_i)}, \epsilon \theta_i N \right\}},
\]
which along with \( \varepsilon \leq 0.1 \) yields
\[
\Pr\left[ \max_{1 \leq i \leq l} \left\| \prod_{j=N}^{\theta_i N} \left( 1 - T_j \right) - \theta_i^r \right\| \geq \varepsilon \right] \leq 8e^{-\frac{1+o(1)}{7} \min\left\{ \frac{\epsilon^2 k N}{2(1-\theta_i)}, \epsilon \theta_i N \right\}} + 2^{-\theta_i N+1}\]
\[
\leq 10e^{-\frac{1+o(1)}{7} \min\left\{ \frac{\epsilon^2 k N}{2(1-\theta_i)}, \epsilon \theta_i N \right\}}.
\]
For the concentration inequalities without the maximum, it is enough to check the hypotheses of the generalized Chernoff bound. First, it is routine to check that
\[
E[T_j] = \frac{h!}{(j+1)(j+2) \cdots (j+h)} \leq \frac{h!}{j^h},
\]
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for positive integer $h$. Thus, for $j \gg 1$,

$$\text{Var}[S_j] \leq E[S_j^2] =: a_j \quad \text{and} \quad \left| \sum_{j=N}^{\theta_t N} 3 \frac{j^2}{j^2} - 3r(1 - \theta_t^k) \right| \leq \frac{4k(1 - \theta_t^k)}{\theta_t^{2k} N^2} = o\left(\frac{1 - \theta_t^k}{\theta_t^{2k} N^2}\right).$$

Furthermore, for $0 < \xi \leq \xi_0 := 0.1\theta_t^k N$,

$$E[S_j^3 e^{\xi(S_j - E[S_j])}] \leq E[S_j^3 e^{\xi S_j}] \leq \frac{10}{j^3} =: b_j,$$

and, for $-\xi_0 \leq \xi < 0$, 

$$E[S_j^3 e^{\xi(S_j - E[S_j])}] \leq E[S_j^3 e^{\xi E[S_j]]}] \leq \frac{10}{j^3}. $$

Since

$$\sum_{j=N}^{\theta_t N} \frac{10}{j^3} \leq \frac{6r(1 - \theta_t^{2k})}{k\theta_t^{2k} N^2}, \quad (4.7)$$

we have that $\xi_0 \sum b_j \leq \sum_{j=1}^i a_i$ and hence, for $\varepsilon > 0$, the generalized Chernoff bound (Corollary 4.2) gives

$$\Pr \left[ |S_i^* - E[S_i^*]| \geq \varepsilon \right] \leq 2e^{-\frac{1 + o(1)}{3} \min\{\frac{e^3 \theta_t^k N}{1 - \theta_t^k}, \varepsilon \theta_t^k N\}}$$

and

$$\Pr \left[ |S_i^* - S_i - E[S_i^*] - S_i| \geq \varepsilon \right] \leq 2e^{-\frac{1 + o(1)}{3} \min\{\frac{e^3 \theta_t^k N}{1 - \theta_t^k}, \varepsilon \theta_t^k N\}}.$$  

As

$$|S_i^* - E[S_i^*]| \leq |S_i^* - E[S_i^*]| + |S_i^* - S_i^* - E[S_i^*] - S_i^*|,$$

and $S_i^* - S_i^*$ is independent of $S_1, ..., S_i$, we may apply Lemma 4.3 with $\varepsilon_1 = \varepsilon_2 = \varepsilon/2$ to obtain

$$\Pr[\max_{1 \leq i \leq t} |S_i^* - E[S_i^*]| \geq \varepsilon] \leq 4e^{-\frac{1 + o(1)}{6} \min\{\frac{e^2 \theta_t^k N}{2(r - \theta_t^k)}, \varepsilon \theta_t^k N\}}.$$  

Similarly,

$$\Pr[\max_{1 \leq i \leq t} |T_i^* - E[T_i^*]| \geq \varepsilon] \leq 4e^{-\frac{1 + o(1)}{6} \min\{\frac{e^2 \theta_t^k N}{2(r - \theta_t^k)}, \varepsilon \theta_t^k N\}}.$$  

For the expectations, since

$$\sum_{j=N}^{\theta_t N} E[S_j] = \sum_{j=N}^{\theta_t N} \frac{1 + O(1/j)}{j + 1} \leq \frac{r}{k} \int_{\theta_t N + 1}^{N} dx + O\left(\frac{1 - \theta_t^k}{\theta_t^k N}\right) = -r \log \theta_t + o\left(\frac{1 - \theta_t^k}{\theta_t^k N}\right),$$

and

$$\sum_{j=N}^{\theta_t N} E[S_j] = \sum_{j=N}^{\theta_t N} E[T_j] = \sum_{j=N}^{\theta_t N} \frac{1}{j + 1} \geq \frac{r}{k} \int_{\theta_t N + 2}^{N+1} dx = -r \log \theta_t + o\left(\frac{1 - \theta_t^k}{\theta_t^k N}\right),$$

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we have
\[ E[S^*_t] = -r \log \theta_i + o\left(\frac{1 - \theta^k_i}{\theta^k_i N}\right)^{1/2}, \quad \text{and} \quad E[T^*_t] = -r \log \theta_i + o\left(\frac{1 - \theta^k_i}{\theta^k_i N}\right)^{1/2} \]
as desired.

\[ \square \]

5 Generalized Core-Processes and Main Lemma

In this section, we introduce generalized cores and the main lemma. The main lemma will be crucial in the proofs of theorems mentioned in the introduction.

We start with a few terminology. A \textit{generalized degree} is an ordered pair \((d_1, d_2)\) of non-negative integers. The inequality between two generalized degrees is determined by the inequality between the first coordinates and the reverse inequality between the second coordinates. That is, \((d_1, d_2) \geq (d'_1, d'_2)\) if and only if \(d_1 \geq d'_1\) and \(d_2 \leq d'_2\). A \textit{property} for generalized degrees is simply a set of generalized degrees. A property \(P\) is \textit{increasing} if generalize degrees larger than an element in \(P\) are also in \(P\). When a property \(P\) depends only on the first coordinate of generalized degrees, it is a property for degrees. For the \(t\)-core problem, we will use \(P_{t\text{-core}} = \{(d_1, d_2) : d_1 \geq t\}\). To estimate the size of the largest component, we will set \(P_{\text{comp}} = \{(d_1, d_2) : d_2 = 0\}\).

Given the Poisson \(\lambda\)-cell on the set \(V\) of \(n\) vertices and \(\theta\) in the range \(0 \leq \theta \leq 1\), let \(d_v(\theta)\) be the number of \(v\)-clones smaller than \(\theta \lambda\). Similarly, \(\bar{d}_v(\theta)\) is the number of \(v\)-clones larger than or equal to \(\theta \lambda\). Then, \(D_v(\theta) := (d_v(\theta), \bar{d}_v(\theta))\) are i.i.d random variables. In particular, for any property \(P\), the events \(D_v(\theta) \in P\) are independent and occur with the same probability, say \(p(\theta, \lambda ; P)\), or simply \(p(\theta)\).

For an increasing property \(P\), the \(P\)-process is defined as follows. Construct the Poisson \(\lambda\)-cell as described in Section 3 where \(\lambda = p^{\binom{n}{k-1}}\). The vertex set \(V = \{v_0, \ldots, v_{n-1}\}\) will be regarded as an ordered set so that the \(i\)th vertex is \(v_{i-1}\). The \(P\)-process, or generalized core-process generated by \(P\), is a generalization of Example 2.2 for which choice functions choose \(t\)-light clones.

The \(P\)-process: Initially, the cut-off value \(\Lambda = \lambda\). Activate all vertices \(v\) with \(D_v(1) \not\in P\). All clones of the activated vertices are activated too. Put activated clones in a stack in an arbitrary order. However, this does not mean that the clones are removed from the \(\lambda\)-cell.

(a) If the stack is empty, go to (b). If the stack is nonempty, choose the first clone in the stack and move the cut-off line to the left until the largest \(k-1\) unmatched clones, excluding the chosen clone, are found. (So, the cut-off value \(\Lambda\) keeps decreasing.) Then, match the \(k-1\) clones to the chosen clone. Remove all matched clones from the stack and repeat. A vertex \(v\) that has not been activated is to be activated as soon as \(D_v(\Lambda / \lambda) \not\in P\). This can be done even before all \(k-1\) clones are found. Its unmatched clones are to be activated too and put into the stack immediately. Clones found while moving the cut-off line are also in the stack until they are matched.

(b) Activate the first vertex in \(V\) that has not been activated. Its clones are activated too. Put those clones into the stack. Then, go to (a).
Clones in the stack are called active. The steps carried by the instruction described in (b) are called free steps as we are free to choose any clone.

When the cut-off line is at \( \theta \lambda \), all \( \theta \lambda \)-large clones are matched or will be matched at the end of the step and all vertices \( v \) with \( D_v(\theta) \not\in P \) have been activated. All other vertices can have been activated only by free steps. Let \( V(\theta) = V_\theta(\theta) \) be the set of vertices \( v \) with \( D_v(\theta) \in P \), and let \( M(\theta) = M_\theta(\theta) \) be the number of \( \theta \lambda \)-large clones plus the number of \( \theta \lambda \)-small clones of vertices \( v \) not in \( V(\theta) \). That is,

\[
M(\theta) = \sum_{v \in V} \tilde{d}_v(\theta) + d_v(\theta) 1(v \not\in V(\theta)) = \sum_{v \in V} \tilde{d}_v(\theta) + d_v(\theta) 1(D_v(\theta) \not\in P). \tag{5.1}
\]

Recalling that \( N(\theta) \) is the number of matched clones until the cut-off line reaches \( \theta \lambda \), the number \( A(\theta) \) of active clones (when the cut-off value \( \Lambda \) is) at \( \theta \lambda \) is at least as large as \( M(\theta) - N(\theta) \). On the other hand, the difference \( A(\theta) - (M(\theta) - N(\theta)) \) is at most the number \( F(\theta) \) of clones activated in free steps until \( \theta \lambda \), i.e.,

\[
M(\theta) - N(\theta) \leq A(\theta) \leq M(\theta) - N(\theta) + F(\theta). \tag{5.2}
\]

As the cut-off lemma gives a concentration inequality for \( N(\theta) \),

\[
\Pr \left[ \max_{\theta, \theta \leq \theta \leq 1} |N(\theta) - (1 - \theta \frac{k}{\theta})\lambda n| \geq \Delta \right] \leq 2e^{-\Omega(\min \{\Delta, \frac{\Delta^2}{(1 - \theta \frac{k}{\theta})n}\})},
\]
a concentration inequality for \( M(\theta) \) will be enough to obtain a similar inequality for \( B(\theta) := M(\theta) - N(\theta) \). More precisely, we will show that, under appropriate hypotheses,

\[
\Pr \left[ \max_{\theta, \theta \leq \theta \leq 1} |M(\theta) - (\lambda - q(\theta))n| \geq \Delta \right] \leq 2e^{-\Omega(\min \{\Delta, \frac{\Delta^2}{(1 - \theta \frac{k}{\theta})n}\})},
\]

where

\[
q(\theta) = q(\theta, \lambda; P) = E \left[ d_v(\theta) 1(D_v(\theta) \in P) \right].
\]

As \( d_v(\theta) \)'s and \( D_v(\theta) \)'s are identically distributed, \( q(\theta) \) does not depend on \( v \). Recall also \( p(\theta) = \Pr[D_v(\theta) \in P] \).

As we will see later, \( B(\theta) \) is very close to \( A(\theta) \). Hence, a concentration inequality for \( B(\theta) \) is crucial.

**Lemma 5.1 (Main lemma)** In the \( P \)-process, if \( \theta_i < 1 \) uniformly bounded from below by 0, \( 1 - p(\theta) = O(1 - \theta) \) and \( p(\theta) = \Omega(1) \), then, for all \( \Delta \) in the range \( 0 < \Delta \leq n \),

\[
\Pr \left[ \max_{\theta, \theta \leq \theta \leq 1} |V(\theta)| - p(\theta)n | \geq \Delta \right] \leq 2e^{-\Omega(\min \{\Delta, \frac{\Delta^2}{(1 - \theta \frac{k}{\theta})n}\})},
\]

and

\[
\Pr \left[ \max_{\theta, \theta \leq \theta \leq 1} |B(\theta) - (\lambda \theta \frac{k}{\theta} - q(\theta))n | \geq \Delta \right] \leq 2e^{-\Omega(\min \{\Delta, \frac{\Delta^2}{(1 - \theta \frac{k}{\theta})n}\})}.
\]
Remark. If $\Delta \gg n^{1/2} \log n$, the proof of the main lemma is much easier: Without the max, the two concentration inequalities follow from the generalized Chernoff bound. Since the bounds for the probabilities are much less than $1/n$ and there are $O(n)$ meaningful $\theta$’s, the first moment method gives the inequalities. This is already enough to prove Theorems 1.4 and 1.9 provided $|\lambda - \lambda_{\text{cr}}| \gg n^{1/2} \log n$. The full strength of the lemma is needed when the $\log n$ factor is missing.

For the proof, we first show that a concentration inequality for $|V(\theta)| = \sum_{v \in V} 1(D_v(\theta) \in P)$, which is a sum of i.i.d Bernoulli random variables with mean $p(\theta)$. More generally, we have

**Lemma 5.2** Suppose $X_i$’s are i.i.d Bernoulli random variables with mean $p(\theta)$. Then, for $\Delta > 0$,

$$ \Pr \left[ \left| \sum_{i=1}^{m} X_i - pm \right| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-p)m}))}.$$

**Proof.** Since $E[X_i] = p$, $E[(X_i - p)^2] = p(1-p)$, and for $\rho$ with $|\xi| \leq \xi_0 := \log 2$,

$$ \left| E[(X_i - p)^3 e^{\xi_1(X_i(\theta)=0)-p]} \right| \leq 2p(1-p),$$

we may set $a_i = p(1-p)$ and $b_i = 2p(1-p)$. Applying the generalized Chernoff bound, we have the desired inequality.

**Corollary 5.3** For $\theta$ in the range $\theta_i \leq \theta \leq 1$ and with the same hypotheses as in the main lemma,

$$ \Pr \left[ \left| V_P(\theta) - p(\theta)n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-p)n}))}.$$  

As in Example 4.3, Lemma 4.5 yields a concentration inequality for all of $V(\theta)$’s:

**Lemma 5.4** With the same hypotheses as in the main lemma,

$$ \Pr \left[ \max_{\theta_i \leq \theta \leq 1} \left| V(\theta) - p(\theta)n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{(1-p)n}))}.$$

**Proof.** Observing that, for $\theta_i \leq \theta \leq 1$,

$$ \frac{p(\theta_i)}{p(\theta)} \left| V(\theta) - p(\theta)n \right| \leq \left| V(\theta_i) - p(\theta_i)n \right| + \left| V(\theta) - p(\theta_i) \right| \left| V(\theta_i) \right|,$$

we set $\Gamma(\theta) = \left| V(\theta) - p(\theta)n \right|$, $\psi = \frac{1}{p(\theta_i)} \Gamma(\theta_i)$, and $\psi_\theta = \frac{p(\theta)}{p(\theta_i)} \left| V(\theta_i) - p(\theta_i) \right| V(\theta_i)$.
Clearly, $\Gamma(\theta) \leq \psi + \psi_\theta$. Corollary 5.3 gives
\[
\Pr[\psi \geq \Delta/2] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\psi)\theta}\})}.
\] (5.3)

Suppose $\{X_{\theta'} := V(\theta')\}_{\theta' \leq \theta \leq 1}$ is given, especially $V(\theta)$ is given. Then, since $P$ is increasing, we may write $|V(\theta)|$ as
\[
|V(\theta)| = \sum_{v \in V(\theta)} 1(D_v(\theta) \in P),
\]
with
\[
\Pr[D_v(\theta) \in P|\{X_{\theta'}\}_{\theta' \leq \theta}] = \Pr[D_v(\theta) \in P|v \in V(\theta)] = \frac{p(\theta)}{p(\theta')} =: p(\theta; \theta).
\]
Lemma 5.2 then gives
\[
\Pr[\psi \geq \Delta/2] \leq 2e^{-\Omega(\min\{p(\theta; \theta)\Delta, \frac{p(\theta; \theta)\Delta^2}{(1-p(\theta))\frac{\Delta^2}{\sqrt{\theta\psi}}})\})} \leq 2e^{-\Omega(\min\{p(\theta)\Delta, \frac{p(\theta)\Delta^2}{(1-p(\theta))\frac{\Delta^2}{\sqrt{\theta\psi}}})\})} \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\psi)\theta}\})}.
\] (5.4)

Lemma 4.5 together with (5.3) and (5.4) yields the desired inequality.

We now estimate $M(\theta)$. First, since $\sum_{v \in V} \delta_v(\theta)$ is a Poisson random variable with mean $(1-\theta)\lambda n$,
\[
\Pr\left[\left|\sum_{v \in V} \delta_v(\theta) - (1-\theta)\lambda n\right| \geq \Delta/2\right] \leq 2e^{-\min\{\Delta, \frac{\Delta^2}{(1-\psi)\theta}\}}.
\] (5.5)

For the second sum in (5.1), observe that
\[
\sum_{v \in V} d_v(\theta)1(\theta \notin V(\theta)) = \sum_{v \in V} d_v(\theta)1(D_v(\theta) \notin P)
\]
is a sum of i.i.d random variables with
\[
E[d_v(\theta)1(D_v(\theta) \notin P)] = E[d_v(\theta) - d_v(\theta)1(D_v(\theta) \in P)] = \theta \lambda - q(\theta).
\]
Moreover, since $P$ is an increasing property and $d_v(\theta)$ is a Poisson $\theta \lambda$ random variable, FKG inequality (see e.g. Chapter 6 of [6]) gives
\[
E[(d_v(\theta)1(D_v(\theta) \notin P)]^i \leq E[d_v(\theta)]^i \Pr[D_v(\theta) \notin P] = O(1-p(\theta)),
\]
for all fixed $i$, e.g. $i = 1, 2, 3$. Thus one may take $\xi_0 = 1$ and $a_i, b_i = \Theta(1-p(\theta))$ to satisfy all the conditions to apply the generalized Chernoff bound and to obtain
\[
\Pr\left[\left|\sum_{v \in V} d_v(\theta)1(\theta \notin V(\theta)) \geq (\theta \lambda - q(\theta)n)\right| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\psi)\theta}\})}.
\]
This together with (5.5) implies that if $1-p(\theta) = O(1-\theta)$, then
\[
\Pr\left[\left|M(\theta) - (\lambda - q(\theta)n)\right| \geq \Delta\right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\psi)\theta}\})}.
\] (5.6)

As mentioned, the following lemma is enough to prove the main lemma.
Lemma 5.5 With the same hypotheses as in the main lemma,
\[ \Pr \left[ \max_{\theta, \hat{\theta} \leq \theta \leq 1} \left| M(\theta) - (\lambda - q(\theta))n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-n)^2}\})}. \]

Proof. Clearly,
\[ |M(\theta) - (\lambda - q(\theta))n| \leq |M(\theta_i) - (\lambda - q(\theta_i))n| + |M(\theta_i) - M(\theta) - (q(\theta) - q(\theta_i))n|. \]
Let \( \Gamma(\theta) = |M(\theta) - (\lambda - q(\theta))n| \),
\[ \psi = \Gamma(\theta_i), \quad \psi = |M(\theta_i) - M(\theta) - (q(\theta) - q(\theta_i))n|, \]
and \( \Phi \) is the event \( |V(\theta) - p(\theta)n| \leq \frac{p(\theta)\Delta}{4\lambda} \). Then, (5.6) gives
\[ \Pr[\psi \geq \Delta/2] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-n)^2}\})}. \]
For \( \psi \), suppose \( \{X_{\theta'} := (V(\theta'), M(\theta'))\}_{\theta \leq \theta' \leq 1} \) is given. Using
\[ M(\theta) = \sum_{v \in V} d_v(\theta) + d_v(\theta)1(v \notin V(\theta)) = \sum_{v \in V} d_v(1) - d_v(\theta)1(v \in V(\theta)), \]
we obtain
\[ M(\theta_i) - M(\theta) = \sum_{v \in V} d_v(\theta)1(v \in V(\theta)) - d_v(\theta_i)1(v \in V(\theta_i)). \]
Once \( V(\theta) \) is given, the distributions of
\[ Y_v := d_v(\theta)1(v \in V(\theta)) - d_v(\theta_i)1(v \in V(\theta_i)) \]
depend on neither \( \{M(\theta')\}_{\theta \leq \theta' \leq 1} \) nor \( \{V(\theta')\}_{\theta \leq \theta' \leq 1} \) and hence, for \( v \in V(\theta) \),
\[ E[Y_v \mid \{X_{\theta'}\}_{\theta \leq \theta' \leq 1}] = E[d_v(\theta) - d_v(\theta_i)1(v \in V(\theta_i)) \mid v \in V(\theta)] = \frac{q(\theta) - q(\theta_i)}{p(\theta)}. \]
If \( v \notin V(\theta) \), \( Y_v = 0 \) since \( P \) is increasing.
Also, for \( v \in V(\theta) \), we may write
\[ Y_v = d_v(\theta) - d_v(\theta_i) + d_v(\theta_i)1(v \notin V(\theta_i)) \]
and
\[ E[Y_v^2 \mid v \in V(\theta)] \leq 2E[(d_v(\theta) - d_v(\theta_i))^2 \mid v \in V(\theta)] + 2E[d_v(\theta_i)^21(v \notin V(\theta_i)) \mid v \in V(\theta)]. \]
First, for \( j = 1, 2, 3 \),
\[ E[(d_v(\theta) - d_v(\theta_i))^j \mid v \in V(\theta)] \leq p(\theta)^{-1}E[(d_v(\theta) - d_v(\theta_i))^j] = O(\theta - \theta_i) = O(1 - \theta_i) \]
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for $p(\theta) \geq p(\theta_1) = \Omega(1)$ and $d_v(\theta) - d_v(\theta_1)$ is a Poisson random variable with mean $(\theta - \theta_1)\lambda = O(\theta - \theta_1)$. For the second term, FKG inequality gives

$$E \left[ \left( d_v(\theta_1)1(v \notin V(\theta_1)) \right)^j \mid v \in V(\theta) \right] \leq p(\theta)^{-1}E \left[ d_v(\theta)1(v \notin V(\theta)) \right] \leq p(\theta)^{-1}E[d_v(\theta)^j] \Pr[v \notin V(\theta)] = O(1 - p(\theta)) = O(1 - \theta),$$

for $j = 1, 2, 3$. Therefore,

$$E \left[ \left( Y_i - E[Y_i] \right)^2 \right] \{X_{\theta'} \mid \theta \leq \theta' \leq 1\} \leq E \left[ Y_i^2 \right] \{X_{\theta'} \mid \theta \leq \theta' \leq 1\} = O(1 - \theta).$$

Similarly, for $\xi$ in the range $|\xi| \leq \xi_n = 1$, it is not hard to show

$$\left| E \left[ \left( Y_i - E[Y_i] \right)^3 \varepsilon(Y_i - E[Y_i]) \right] \{X_{\theta'} \mid \theta \leq \theta' \leq 1\} \right| = O(1 - \theta).$$

Applying the generalized Chernoff bound, we have

$$\Pr \left[ \left| \sum_{v \in V} Y_v - \frac{q(\theta) - q(\theta_1)}{p(\theta)} |V(\theta)| \right| \geq \Delta/4 \right] \{X_{\theta'} \mid \theta \leq \theta' \leq 1\} \leq 2e^{-\Omega(\min(\Delta, \Delta^2))}.$$ 

Finally, as the event $\Phi_0$ guarantees

$$\frac{q(\theta) - q(\theta_1)}{p(\theta)} |V(\theta)| - p(\theta)n \leq \Delta/4,$$

for $p(\theta_1) \leq p(\theta)$ and $q(\theta) \leq \lambda$, we have

$$1(\Phi_0) \Pr \left[ \left| \sum_{v \in V} Y_v - (q(\theta) - q(\theta_1))n \right| \geq \Delta/2 \right] \{X_{\theta'} \mid \theta \leq \theta' \leq 1\} \leq \Pr \left[ \left| \sum_{v \in V} Y_v - \frac{q(\theta) - q(\theta_1)}{p(\theta)} |V(\theta)| \right| \geq \Delta/4 \right] \{X_{\theta'} \mid \theta \leq \theta' \leq 1\} \leq 2e^{-\Omega(\min(\Delta, \Delta^2))}.$$

Lemma 14.5 yields the desired inequality.

\section{Cores of Random Hypergraphs}

This section is for the proof of Theorem 1.7. Let $\lambda > 0$ and $H(\lambda) = H_{PC}(n, p)$, where $\lambda = p^\binom{n-1}{r-1}$. Let the property $P = \{(d_1, d_2) : d_1 \geq t\}$. Then

$$p(\theta) = Q(\theta\lambda, t), \quad \text{and} \quad q(\theta) = \theta\lambda Q(\theta\lambda, t - 1).$$

The main lemma gives

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Corollary 6.1 For $\theta_i \leq 1$ uniformly bounded from below by 0 and $\Delta$ in the range $0 < \Delta \leq n$,
\[ \Pr \left[ \max_{\theta_i \leq \theta \leq 1} \left| \left| V(\theta) \right| - Q(\theta\lambda, t)n \right| \geq \Delta \right] \leq 2e^{-\Omega\left(\min\left\{ \Delta, \frac{\Delta^2}{n} \right\} \right)}, \]
and
\[ \Pr \left[ \max_{\theta_i \leq \theta \leq 1} \left| B(\theta) - (\theta^{1/t} - Q(\theta\lambda, t - 1))\theta\lambda n \right| \geq \Delta \right] \leq 2e^{-\Omega\left(\min\left\{ \Delta, \frac{\Delta^2}{n} \right\} \right)}. \]

Subcritical Region: For $\lambda = \lambda_{\text{crit}} - \sigma$, $\sigma \gg n^{-1/2}$, and $\theta_i = \delta/\lambda_{\text{crit}}$ with $\delta = 0.1$, it is easy to see that there is a constant $c > 0$ such that
\[ (\theta^{1/t} - Q(\theta\lambda, t - 1))\theta\lambda n \geq c\sigma n, \quad \text{for all } \theta \text{ in the range } \theta_i \leq \theta \leq 1. \]
Let $\tau$ be the first time the number $A(\theta)$ of active clones at $\theta\lambda$ becomes 0. Then the second part of Corollary 6.1 gives
\[ \Pr[\tau \geq \theta_i] \leq \Pr[B(\theta) = 0 \text{ for some } \theta \text{ with } \theta_i \leq \theta \leq 1] \leq \Pr \left[ \max_{\theta_i \leq \theta \leq 1} \left| B(\theta) - (\theta^{1/t} - Q(\theta\lambda, t - 1))\theta\lambda n \right| \geq c\sigma n \right] \leq 2e^{-\Omega(\sigma^2 n)}. \]

As $\theta_i \lambda \leq \theta_i \lambda_{\text{crit}} = \delta$, and hence $Q(\theta_i\lambda, t) \leq \delta/2$ for $t \geq 2$, the first part of Corollary 6.1 yields
\[ \Pr[|V_i(H_{PC}(n, p; k))| \geq \delta n] \leq \Pr[\tau \geq \theta_i] + \Pr[|V(\theta_i)| \geq \delta n] \leq 2e^{-\Omega(\sigma^2 n)}. \]
Therefore, Theorem 1.1 implies that
\[ \Pr[|V_i(H(n, p; k))| \geq \delta n] \leq 2e^{-\Omega(\sigma^2 n)}. \]

To complete the proof, we observe that the $t$-core of size $i$ has at least $ti/k$ edges. Let $Z_i$ be the number of subgraphs on $i$ vertices with at least $ti/k$ edges, $i = i_0, ..., \delta n$, where $i_0 = i_0(k, t)$ is the least $i$ such that $\binom{n}{i} \geq ti/k$. Then, in $H(n, p; k)$,
\[ \mathbb{E}[Z_i] \leq \binom{n}{i} \left( \frac{\binom{i}{k}}{ti/k} \right) p^{ti/k} \leq \frac{n^i}{i!} \left( \frac{t}{i} \right)^{ti} p^{ti/k} =: L_i, \]
where $ti/k$ actually means $\lceil ti/k \rceil$. Observe that
\[ \frac{L_{i+k}}{L_i} = O\left( \frac{n^{k+1}}{i^k} \frac{i^{kt}}{t^t n^{-(k-1)t}} \right) = O\left( \left( \frac{k}{n} \right)^{(k-1)t-k} \right) = O(\delta^{(k-1)(t-1)}). \]
That is, $L_{i+k}/L_i$ exponentially decreases. Since
\[ L_i = O(n^i n^{-i(k-1)t/k}) = O(n^{-i(t-1-t/k)}), \]
for $i = i_0, ..., i_0 + k - 1$, it follows that
\[ \Pr[V_i(H(n, p; k)) \neq \emptyset] \leq 2e^{-\Omega(\sigma^2 n)} + O(n^{-i_0(t-1-t/k)}), \]
as desired. \hfill \Box

Supercritical Region: We will prove the following theorem.
Theorem 6.2 If $\lambda := p^{(n-1)}_{k-1} = \lambda_{\text{crit}} + \sigma$ with $\sigma \gg n^{-1/2}$ and $0 < \delta \leq 1$, then, with probability $1 - 2e^{-\Omega(\min\{\delta^2\sigma_n, \sigma_n^2\})}$, $V_t = V_t(H_{PC}(n, p; k))$ satisfies
\[
Q(\theta_t, t)n - \delta n \leq |V_t| \leq Q(\theta_t, t)n + \delta n,
\]
and the degrees of vertices of the $t$-core are i.i.d $t$-truncated Poisson random variables with parameter $\Lambda_t := \theta_t \lambda + \beta$ for some $\beta$ with $|\beta| \leq \delta$. Moreover, the distribution of the $t$-core is the same as that of the $t$-truncated Poisson cloning model with parameters $|V_t|$ and $\Lambda_t$.

Recall that $\theta_\lambda$ is the largest solution for the equation
\[
\theta^{k-1} - Q(\theta\lambda, t - 1) = 0.
\]

Proof. First, it is not hard to check that there are constant $c_1, c_2 > 0$ such that, for $\theta$ in the range $\theta_0 \leq \theta \leq 1$,
\[
\theta^{k-1} - Q(\theta\lambda, t - 1) \geq c_1\sigma^{1/2}(\theta - \theta_\lambda),
\]
and, for $\theta$ in the range $\theta_0 - c_2\sigma^{1/2} \leq \theta \leq \theta_0$,
\[
\theta^{k-1} - Q(\theta\lambda, t - 1) \leq -c_2\sigma^{1/2}(\theta_0 - \theta).
\]

Let $\tau$ be the largest $\theta$ with $A(\theta) = 0$. Then $V(\tau)$ is the $t$-core of $H_{PC}(n, p; k)$. For $\theta_1 = \theta_0 + \delta$ and $\theta_2 = \theta_0 - \min\{\delta, c_2\sigma^{1/2}\}$ with $0 < \delta \leq 1$, Corollary 6.1 gives
\[
\Pr[\tau \geq \theta_1] \leq \Pr[B(\theta) = 0 \text{ for some } \theta \text{ with } \theta_1 \leq \theta \leq 1]
\leq \Pr\left[\max_{\theta_0 \leq \theta \leq 1} \left|B(\theta) - (\theta^{k-1} - Q(\theta\lambda, t - 1))\theta\lambda n\right| \geq c_1\sigma^{1/2}\delta n\right]
\leq 2e^{-\Omega(\delta^2\sigma_n)},
\]
and
\[
\Pr[\tau < \theta_2] \leq \Pr[B(\theta_2) > 0]
\leq \Pr\left[\left|B(\theta_2) - (\theta_2^{k-1} - Q(\theta_2\lambda, t - 1))\theta_2\lambda n\right| \geq c_1\sigma^{1/2}\min\{\delta, c_2\sigma^{1/2}\}n\right]
\leq 2e^{-\Omega(\min\{\delta^2\sigma_n, \sigma_n^2\})}.
\]

Since $\frac{d}{d\theta}Q(\theta\lambda, t) = \lambda P(\theta\lambda, t - 1) \leq \lambda$, we have
\[
Q(\theta_{\lambda}, t) \leq Q(\theta_{\lambda}, t) + \lambda\delta, \quad \text{and} \quad Q(\theta_{\lambda}, t) \geq Q(\theta_{\lambda}, t) - \lambda\delta,
\]
and Corollary 6.1 implies that
\[
\Pr[V(\theta_{\lambda}) - Q(\theta_{\lambda}, t)n \geq 2\lambda\delta n] \leq 2e^{-\Omega(\delta^2n)},
\]
and
\[
\Pr[V(\theta_{\lambda}) - Q(\theta_{\lambda}, t)n \leq -2\lambda\delta n] \leq 2e^{-\Omega(\delta^2n)}.
\]
Therefore,

$$\Pr[|\tau - \theta_1| > \delta] \leq \Pr[\tau \geq \theta_1] + \Pr[\tau \leq \theta_2] \leq 2e^{-\Omega(\min(\delta^2 \sigma \tau, \sigma^2 n))},$$

and, replacing \( \delta \) by \( \frac{\delta}{2n} \),

$$\Pr[|V(\tau) - Q(\theta, \lambda, t)n| \geq \delta n] \leq \Pr[\tau \geq \theta_1] + \Pr[\tau \leq \theta_2] + 2e^{-\Omega(\delta^2 n)} \leq 2e^{-\Omega(\min(\delta^2 \sigma \tau, \sigma^2 n))}.$$

Clearly, once \( V(\tau) \) and \( \Lambda_t := \tau \lambda \) are given, the residual degrees \( d_v(\tau), v \in V(\tau) \), are i.i.d \( t \)-truncated Poisson random variables with parameter \( \Lambda_t \).

Once \( V_t \) and \( \Lambda_t \) are given, \( |V_t(i)|, i \geq t \), is the sum of i.i.d Bernoulli random variables with mean \( p_v(\Lambda_t) := \frac{P(\Lambda, i)}{Q(\Lambda)}, \) Similarly, the size of \( W_t(i) = \cup_{j \geq t} V_t(j) \) is the sum of i.i.d Bernoulli random variables with mean \( q_v(\Lambda_t) := \frac{Q(\Lambda, i)}{Q(\Lambda)} \). Applying the generalized Chernoff bound (Lemma \ref{lem:chernoff}), we have

$$\Pr \left[ \left| |V_t(i)| - p_v(\Lambda_t)|V_t| \right| \geq \delta|V_t|\right] \leq 2e^{-\Omega(\delta^2 |V_t|)},$$

and

$$\Pr \left[ \left| |W_t(i)| - q_v(\Lambda_t)|V_t| \right| \geq \delta|V_t|\right] \leq 2e^{-\Omega(\delta^2 |V_t|)}.$$

Combining these with Lemma \ref{lem:chernoff} and using

$$|P(\rho, i) - P(\rho', i)| \leq |\rho - \rho'|, \quad \text{and} \quad |Q(\rho, i) - Q(\rho', i)| \leq |\rho - \rho'|,$$

we obtain, for any \( i \),

$$\Pr \left[ \left| |V_t(i)| - P(\theta, \lambda, i)n \right| \geq \delta n \right] \leq 2e^{-\Omega(\min(\delta^2 \sigma \tau, \sigma^2 n))},$$

and

$$\Pr \left[ \left| |W_t(i)| - Q(\theta, \lambda, i)n \right| \geq \delta n \right] \leq 2e^{-\Omega(\min(\delta^2 \sigma \tau, \sigma^2 n))}.$$

In particular, as \( \theta_1 = \theta_{\text{crit}} + \Theta(\sigma^{1/2}) \), for uniformly bounded \( \sigma \), it follows that, for \( \lambda = \lambda_{\text{crit}} + \sigma \),

$$|V_t(i)| = (1 + O(\sigma^{1/2}))P(\theta_{\text{crit}}\lambda_{\text{crit}}, i)n + O\left(\frac{n}{\sigma^{1/2}}\log n\right),$$

with probability \( 1 - 2e^{-\Omega(\min(\log^2 n, \sigma^2 n))} \).

The last part of Theorem \ref{thm:unif_deg_seq} does not follow from Theorem \ref{thm:chernoff} as \( H(n, p; k) \) and \( H_{PC}(n, p; k) \) do not have the same distribution. We may directly prove it instead.

For a hypergraph \( H \), let \( \tilde{H} \) be the hypergraph obtained from \( H \) by removing all edges in the \( t \)-core. Then, \( \tilde{H} \) has no edge that is entirely in \( V_t(H) \), otherwise, the \( t \)-core becomes larger. Thus, the union of \( \tilde{H} \) and any simple hypergraph on \( V_t(H) \) is also simple. Therefore, conditioned on \( \tilde{H} = \tilde{H}(n, p, k) \), two hypergraphs \( H_1 \) and \( H_2 \) on \( V_t(H(n, p; k)) \) with the same degree sequence are equally likely to be the \( t \)-core of \( H(n, p; k) \): Notice that

$$\Pr[H(n, p; k) = \tilde{H} \cup H_1] = p^{\tilde{m} + m_1} (1 - p)^{\tilde{m} - m_1},$$

and

$$\Pr[H(n, p; k) = \tilde{H} \cup H_2] = p^{\tilde{m} + m_2} (1 - p)^{\tilde{m} - m_2},$$

where \( \tilde{m}, m_1 \) and \( m_2 \) are the numbers of edges in \( \tilde{H}, H_1 \) and \( H_2 \), respectively. Clearly, \( m_1 = m_2 \) as the degree sequences of \( H_1 \) and \( H_2 \) are the same.
7 The pure literal algorithm for the random $k$-SAT

To analyze the pure literal algorithm for a random $k$-SAT, $k \geq 3$, it is necessary to consider a pair of degrees. We consider this in a generalized framework. Starting with some terminology, a sequence of generalized degrees is larger than or equal to another with the same length if so is each pair of corresponding generalized degrees. A property for sequences of generalized degrees is a set of generalized degree sequences. A property $P$ is increasing if sequences of generalized degree larger than an element in $P$ are also in $P$.

Given a Poisson $\lambda$-cell on the set $V$ of $n$ vertices, let $C = \{C_i\}$ be an equipartition of the vertex set $V$. Then, $D_i(\theta) := (d_v(\theta), \bar{d}_v(\theta))_{v \in C_i}$ are i.i.d random variables. In particular, for any property $P$, the events $D_i(\theta) \in P$ are independent and occur with the same probability, say $p(\theta, \lambda; P, C)$, or simply $p(\theta)$. For the pure literal algorithm of the random $k$-SAT problem, the property is the set of pairs of generalized degrees $(d_1, d_2), (d_1', d_2')$ with $d_1, d_1' \geq 1$.

For an increasing property $P$ and an equipartition $C = \{C_i\}_{i=1}^m$, the $(P, C)$-process is defined as follows. Construct the Poisson $\lambda$-cell as described in Section 3, where $\lambda = p(\binom{n-1}{k-1})$. The $(P, C)$-process is a generalization of the $P$-process.

**The $(P, C)$-process**: Initially, the cut-off value $\Lambda = \lambda$. Activate all vertices $v \in C_i$ with $D_i(1) \not\in P$. All clones of the activated vertices are activated too. Put activated clones in a stack in an arbitrary order. However, this does not mean that the clones are removed from the $\lambda$-cell.

(a) If the stack is empty, go to (b). If the stack is nonempty, choose the first clone in the stack and move the cut-off line to the left until the largest $k-1$ unmatched clones, excluding the chosen clone, are found. (So, the cut-off value $\Lambda$ keeps decreasing.) Then, match the $k-1$ clones to the chosen clone. Remove all matched clones from the stack and repeat. A vertex in $C_i$ that has not been activated is to be activated as soon as $D_i(\Lambda/\lambda) \not\in P$. This can be done even before all $k-1$ clones are found. Its unmatched clones are to be activated too and put into the stack immediately. Clones found while moving the cut-off line are also in the stack until they are matched.

(b) Activate all vertices in the first $C_i$ no vertex of which has not been activated. Its clones are activated too. Put those clones into the stack. Then, go to (a).

Clones in the stack are called active. The steps carried by the instruction described in (b) are called free steps as it is free to artificially activate a vertex.

When the cut-off line is at $\theta \lambda$, all $\theta \lambda$-large clones are matched or will be matched at the end of the step and all vertices in $C_i$ with $D_i(\theta) \not\in P$ are activated. All other vertices can be activated only by free steps. Let $V(\theta) = V(V_{(P, C)}(\theta)$ be the union of $C_i$ with $D_i(\theta) \in P$, and let $M(\theta) = M(V_{(P, C)}(\theta)$ be the number of $\theta \lambda$-large clones plus the number of $\theta \lambda$-small clones of $v \not\in V(\theta)$. That is,

$$M(\theta) = \sum_{v \in V} \bar{d}_v(\theta) + d_v(\theta)1(v \not\in V(\theta)) = \sum_{i=1}^m \sum_{v \in C_i} \bar{d}_v(\theta) + d_v(\theta)1(D_i(\theta) \not\in P).$$

Recalling that $N(\theta)$ is the number of matched clones until the cut-off line reaches $\theta \lambda$, the number $A(\theta)$ of active clones (when the cut-off $\Lambda$ is) at $\theta \lambda$ is at least as large as $M(\theta) - N(\theta)$. 

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On the other hand, the difference \( A(\theta) - (M(\theta) - N(\theta)) \) is at most the number \( F(\theta) \) of clones activated in free steps until \( \theta \lambda \), i.e.,

\[
M(\theta) - N(\theta) \leq A(\theta) \leq M(\theta) - N(\theta) + F(\theta).
\]

As the cut-off lemma gives a concentration inequality for \( N(\theta) \),

\[
\operatorname{Pr}\left[ \max_{\theta, \theta_i \leq \theta \leq 1} |N(\theta) - (1 - \theta^{\frac{k}{1-\gamma}})\lambda n| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta^{k})n}\})},
\]

a concentration inequality for \( M(\theta) \) will be enough to obtain a similar inequality for \( B(\theta) := M(\theta) - N(\theta) \). More precisely, we will show that, under appropriate hypotheses,

\[
\operatorname{Pr}\left[ \max_{\theta, \theta_i \leq \theta \leq 1} |M(\theta) - (\lambda - q(\theta))n| \leq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta^{k})n}\})},
\]

where

\[
q(\theta) = q(\theta, \lambda; P, C) = E\left[ \left( \frac{1}{|C_i|} \sum_{v \in C_i} d_v(\theta) \right) 1(D_i(\theta) \in P) \right].
\]

As \( D_i(\theta) \) are identically distributed, \( q(\theta) \) does not depend on \( i \). Recall also \( p(\theta) = p(\theta, \lambda; P, C) = \operatorname{Pr}[D_i(\theta) \in P] \). Here is a generalization of the main lemma. Its proof is quite similar to that of the main lemma and it is presented in the Appendix.

**Lemma 7.1** (Main lemma: generalized version) In the \((P, C)\)-process described above, if \( \theta_i < 1 \) uniformly bounded from below by 0, and \( |C_1| = O(1) \), \( 1 - p(\theta_i) = O(1 - \theta_i) \) and \( p(\theta_i) = \Omega(1) \), then, for all \( \Delta \) in the range \( 0 < \Delta \leq n \),

\[
\operatorname{Pr}\left[ \max_{\theta, \theta_i \leq \theta \leq 1} |V(\theta)| - p(\theta)n| \leq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta^{k})n}\})},
\]

and

\[
\operatorname{Pr}\left[ \max_{\theta, \theta_i \leq \theta \leq 1} |B(\theta) - (\lambda \theta^{\frac{k}{1-\gamma}} - q(\theta))n| \leq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta^{k})n}\})}.
\]

Let \( \lambda > 0 \) and \( F(\lambda) = F_{PC}(n, p; k) \), where \( \lambda = p^{2n-1}_{k-1} \). As mentioned, we take the property \( P = \{(d_1, d_2), (d'_1, d'_2) : d_1, d'_1 \geq 1\} \) and \( C_i = \{x_i, \bar{x}_i\} \). Then

\[
p(\theta) = Q^2(\theta \lambda, 1) = (1 - e^{-\theta \lambda})^2, \quad q(\theta) = \theta \lambda (1 - e^{-\theta \lambda}).
\]

Let \( X(\theta) \) be the set of variables \( x \) with both of \( d_x(\theta) \) and \( d_x(\theta) \) larger than 0. Then the main lemma and \(|V(\theta)| = 2|X(\theta)| \) give

**Corollary 7.2** For \( \theta_i \leq 1 \) uniformly bounded from below by 0 and \( \Delta \) in the range \( 0 < \Delta \leq n \),

\[
\operatorname{Pr}\left[ \max_{\theta, \theta_i \leq \theta \leq 1} |X(\theta)| - (1 - e^{-\theta \lambda})^2n| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{n}\})},
\]

and

\[
\operatorname{Pr}\left[ \max_{\theta, \theta_i \leq \theta \leq 1} |B(\theta) - 2(\theta^{\frac{1}{1-\gamma}} - (1 - e^{-\theta \lambda}))\theta \lambda n| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{n}\})}.
\]
As $1 - e^{-\theta \lambda} = Q(\theta \lambda, t - 1)$ with $t = 2$, a similar argument used in the previous section may be applied to prove Theorem 1.9.

**Subcritical Region:** For $\lambda = \lambda_{\text{crit}} - \sigma$, $\sigma \gg n^{-1/2}$, let $\theta_1 = \delta / \lambda_{\text{crit}}$ with $\delta = 0.1$, and let $\tau$ be the first time the number $A(\theta)$ of active clones at $\theta \lambda$ becomes 0. Since

$$2(\theta^{\frac{1}{t-1}} - (1 - e^{\theta \lambda}))\theta \lambda n \geq c \sigma n,$$

for all $\theta$ in the range $\theta_1 \leq \theta \leq 1$, for a constant $c > 0$, the second part of Corollary 7.2 gives

$$\Pr[\tau \geq \theta_1] = \Pr[B(\theta) = 0 \text{ for some } \theta \text{ with } \theta_1 \leq \theta \leq 1] \leq \Pr \left[ \max_{\theta : \theta_1 \leq \theta \leq 1} \left| B(\theta) - 2(\theta^{\frac{1}{t-1}} - (1 - e^{\theta \lambda}))\theta \lambda n \right| \geq c \sigma n \right] \leq 2e^{-\Omega(\sigma^2 n)}.$$

As $(1 - e^{-\theta_1 \lambda})^2 \leq \delta^2$, this and the first part of Corollary 6.1 yield

$$\Pr[|X_R(F_{PC}(n, p; k))| \geq \delta n] \leq \Pr[\tau \geq \theta_1] + \Pr[|X(\theta_1)| \geq \delta n] \leq 2e^{-\Omega(\sigma^2 n)}.$$

Therefore, Theorem 1.8 implies that

$$\Pr[|X_R(F(n, p; k))| \geq \delta n] \leq 2e^{-\Omega(\sigma^2 n)}.$$

To complete the proof, we observe that the residual formula on $i$ variables has at least $2i/k$ clauses. Let $Z_i$ be number of subformulas on $i$ variables with at least $2i/k$ clauses, $i = i_0, \ldots, \delta n$, where $i_0 = i_0(k)$ is the least $i$ such that $2^k \binom{i}{k} \geq 2i/k$. Then, in $F(n, p; k)$,

$$E[Z_i] \leq \binom{n}{i} \left( \frac{2^k}{2i/k} \right)^{2i/k} \leq \frac{n^i}{i!} \left( \frac{2^k}{2i/k} \right)^{2i/k} =: L_i,$$

where $2i/k$ actually means $\lceil 2i/k \rceil$. Observe that

$$\frac{L_{i+k}}{L_i} = O\left( \frac{n^k (2i^2)2k}{i^2} n^{-2(k-1)} \right) = O\left( \left( \frac{i}{n} \right)^{k-2} \right) = O(\delta^{k-2}).$$

That is, $L_{i+k} / L_i$ exponentially decreases for $k \geq 3$. Since

$$L_i = O(n^i n^{-2(k-1)/k}) = O(n^{-i(1-2/k)}),$$

for $i = i_0, \ldots, i_0 + k - 1$, we have

$$\Pr[|X_R(F(n, p; k))| \neq \emptyset] \leq 2e^{-\Omega(\sigma^2 n)} + O(n^{-i_0(1-2/k)}),$$

as desired.

**Supercritical Region:** Applying the same argument used to prove Theorem 6.2 in the previous section, we may easily obtain
Theorem 7.3 If \( \lambda = p^{(2n-1)}_{k-1} \geq \lambda_{\text{crit}} + \sigma \) with \( \sigma \gg n^{-1/2} \) and \( 0 < \delta \leq 1 \), then, with probability \( 1 - 2e^{-\Omega(\min\{\delta^2 \sigma n, \sigma^2 n\})} \), \(|X_R(n, p; k)|\) satisfies
\[
(1 - e^{-\theta \lambda})^2 n - \delta n \leq |X_R(n, p; k)| \leq (1 - e^{-\theta \lambda})^2 n + \delta n,
\]
and the degrees of literals of \( X_R(n, p; k) \) are i.i.d 1-truncated Poisson random variables with parameter \( \Lambda_R := \theta \lambda + \beta \) for some \( \beta \) with \( |\beta| \leq \delta \). Moreover, the distribution of the residual formula on \( X_R(n, p; k) \) is the same as that of the 1-truncated Poisson cloning model with parameters \(|X_R(n, p; k)|\) and \( \Lambda_R \).

Proof. The proof is almost identical to that of Theorem 6.2 with \( t = 2 \).

Once \( X_R := X_R(n, p; k) \) and \( \Lambda_R \) are given, \(|X_R(i, j)|\), \( i, j \geq 1 \), is the sum of i.i.d Bernoulli random variables with mean \( p_{\lambda}(\Lambda_R) := \frac{P(\Lambda_R)}{(1-e^{-\Lambda_R})^2} \). Applying Lemma 5.2, the concentration inequality for a sum of i.i.d Bernoulli random variables, we now have
\[
\Pr \left[ \left| |X_R(i, j)| - p_{\lambda}(\Lambda_R)|X_R| \right| \geq \delta |X_R| \right] \leq 2e^{-\Omega(\delta^2 |X_R|)}.
\]
Combining this with Lemma 5.2 and using
\[
|P(\rho, i) - P(\rho', i)| \leq |\rho - \rho'|,
\]
we obtain, for any \( i \),
\[
\Pr \left[ \left| |X_R(i, j)| - P(\lambda, i)p(\lambda, j)\right| \geq \delta n \right] \leq 2e^{-\Omega(\min\{\delta^2 \sigma n, \sigma^2 n\})}.
\]
Similarly,
\[
\Pr \left[ \left| |Y_R(i, j)| - Q(\lambda, i)Q(\lambda, j)\right| \geq \delta n \right] \leq 2e^{-\Omega(\min\{\delta^2 \sigma n, \sigma^2 n\})}.
\]
The last statement of Theorem 1.9 follows by the same argument used in the previous section.

8 The Emergence Of the Giant Component

In this section, we prove Theorem 1.4. Let the property \( P = \{(d_1, d_2) : d_2 = 0\} \). Then
\[
p(\theta) = e^{-(1-\theta)\lambda}, \quad \text{and} \quad q(\theta) = \theta \lambda e^{-(1-\theta)\lambda},
\]
and the main lemma gives

Corollary 8.1 For \( \theta_i \leq 1 \) uniformly bounded from below by 0 and \( \Delta \) in the range \( 0 < \Delta \leq n \),
\[
\Pr \left[ \max_{\theta, \theta_i \leq \theta \leq 1} \left| V(\theta) - e^{-(1-\theta)\lambda} \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta)^2}\})},
\]
and
\[
\Pr \left[ \max_{\theta, \theta_i \leq \theta \leq 1} \left| B(\theta) - (\theta - e^{-(1-\theta)\lambda})\theta \lambda n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1-\theta)n}\})}.
\]
To estimate $A(\theta)$, it is now enough to estimate $F(\theta)$ by (5.2). Once good estimations for $F(\theta)$ are established, we may take similar (but slightly more complicated) approaches used in Section 6. We consider an (imaginary) secondary stack with parameter $\rho$, or simply $\rho$-secondary stack. Initially, the secondary stack with parameter $\rho$ consists of the first $\rho n$ vertices $v_0, ..., v_{\rho n-1}$ of $V$. The set of those $\rho n$ vertices is denoted by $V_\rho$. Whenever the primary stack is empty, the vertex in the secondary stack that has not been activated is to be activated. Its clones are activated too and put into the primary stack. If the secondary stack is empty, go back to the regular procedure. This does not change the $P$-process at all, but will be used just for the analysis. Let $\tau_\rho$ be the largest $\tau$ such that, at $\tau \rho$, the primary stack becomes empty after the secondary stack is empty. Thus, once the cut-off line reaches $\tau_\rho \rho$, no active clones are provided from the secondary stack. Denoted by $C(\rho)$ is the union of the components containing any vertex in $V_\rho$.

The following lemma is useful to predict how large $\tau_\rho$ is.

**Lemma 8.2** Suppose $0 < \delta, \rho < 1$ and $\theta_1, \theta_2 \leq 1$ are uniformly bounded from below by 0. Then

$$\Pr[\tau \geq \theta_1] \leq \Pr[\min_{\theta_1, \theta_2 \leq \theta \leq 1} B(\theta) \leq -(1 - \delta)\theta_1 \lambda e^{-(1 - \theta_1)\lambda \rho n} + 2e^{-\Omega(\delta^2 \rho n)},$$

and conversely,

$$\Pr[\tau \leq \theta_2] \leq \Pr[B(\theta_2) \geq -(1 + \delta)\theta_2 \lambda e^{-(1 - \theta_2)\lambda \rho n} + 2e^{-\Omega(\delta^2 \rho n)}].$$

**Proof.** For simplicity we will write $\tau$ and $W$ for $\tau_\rho$ and $V_\rho$, respectively. Since, at $\tau \lambda$, the primary stack is empty for the first time after no vertex is left in the secondary stack, $C(\rho)$ is exactly $W \cup V(\tau)$. And, all clones of vertices in $W \cup V(\tau)$ must have been matched. Thus,

$$M(\tau) + N(W) \geq N(\tau), \text{ or equivalently } B(\tau) \geq -N(W),$$

where, in general, $N(V')$ is the number of clones of $v \in V'$. The inequality can be strict when there are $\tau \lambda$-large clones of $v \in W$. However, clones of a vertex $v \in W$ that has no $\tau \lambda$-large clone are not counted in $M(\tau)$. Thus,

$$M(\tau) + N(W(\tau)) = N(\tau), \text{ or equivalently } B(\tau) = -N(W(\tau)),$$

where $W(\theta)$ is the set of vertices in $W$ that have no $\theta \lambda$-large clone. Thus, $\tau$ is the unique $\theta$ such that $B(\theta) = -N(W(\theta))$ and $B(\theta') > -N(W(\theta'))$ for all $\theta' > \theta$.

If $\tau \geq \theta_1$, then $B(\theta) = -N(W(\theta))$ for some $\theta$ with $\theta_1 \leq \theta \leq 1$. As $W(\theta_1) \subseteq W(\theta)$ for such $\theta$, we have

$$B(\theta) \leq -N(W(\theta_1)), \text{ for some } \theta \text{ in the range } \theta_1 \leq \theta \leq 1.$$ 

This implies that

$$\Pr[\tau \geq \theta_1] \leq \Pr[N(W(\theta_1)) < (1 - \delta)\theta_1 \lambda e^{-(1 - \theta_1)\lambda \rho n} + \Pr[\min_{\theta_1, \theta_2 \leq \theta \leq 1} B(\theta) \leq -(1 - \delta)\theta_1 \lambda e^{-(1 - \theta_1)\lambda \rho n}].$$

For $\theta$, in general,

$$N(W(\theta)) = \sum_{v \in W} d_v(\theta) (\bar{d}_v(\theta) = 0).$$
is a sum of i.i.d. random variables with mean \( \theta \lambda e^{-(1-\theta) \lambda} \), it is easy to check by the generalized Chernoff bound that

\[
\Pr[|N(W(\theta)) - \theta \lambda e^{-(1-\theta) \lambda} \rho n| \geq \delta \theta \lambda e^{-(1-\theta) \lambda} \rho n] \leq 2e^{-\Omega(\delta^2 \rho n)}.
\] (8.1)

Conversely, if \( \tau \leq \theta_2 \), then \( B(\theta_2) \geq -N(W(\theta_2)) \), which together with (8.1) yields

\[
\Pr[\tau \leq \theta_2] \leq \Pr[N(W(\theta_2)) > (1 + \delta)\theta_2 e^{-(1-\theta_2) \lambda} \rho n] + \Pr[B(\theta_2) \geq -(1 + \delta)\theta_2 e^{-(1-\theta_2) \lambda} \rho n] \\
\leq \Pr[B(\theta_2) \geq -(1 + \delta)\theta_2 e^{-(1-\theta_2) \lambda} \rho n] + 2e^{-\Omega(\delta^2 \rho n)}.
\]

\[\square\]

Since we are interested in \( \theta \) close to 1, it is convenient to define \( \bar{A}(\theta) = A(1 - \theta) \), \( \bar{V}(\theta) = V(1 - \theta) \), \( \bar{B}(\theta) = B(1 - \theta) \), \( \bar{F}(\theta) = F(1 - \theta) \) and \( \bar{\tau}_\rho = 1 - \tau_\rho \). Then Corollary 8.1 and Lemma 8.2 can be written as

**Corollary 8.3** For \( \theta_i > 0 \) uniformly bounded from above by 1 and \( \Delta \) in the range \( 0 < \Delta \leq n \),

\[
\Pr \left[ \max_{\theta: 0 \leq \theta \leq \theta_i} \left| \bar{V}(\theta) - e^{-\theta \lambda} n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta \frac{\Delta^2}{\lambda} \rho n))},
\]

and, for \( \bar{b}(\theta) = (1 - \theta)(1 - \theta - e^{-\theta \lambda}) \lambda n \),

\[
\Pr \left[ \max_{\theta: 0 \leq \theta \leq \theta_i} \left| \bar{B}(\theta) - \bar{b}(\theta) \right| \geq \Delta \right] \leq 2e^{-\Omega(\min(\Delta \frac{\Delta^2}{\lambda} \rho n))}.
\]

**Lemma 8.4** Suppose \( 0 < \delta, \rho < 1 \) and \( \theta_1, \theta_2 \geq 0 \) are uniformly bounded from above by 1. Then

\[
\Pr[\bar{\tau}_\rho \leq \theta_1] \leq \Pr[\min_{\theta: \theta_1 \leq \theta \leq \theta_2} \bar{B}(\theta) \leq -(1 - \delta)(1 - \theta) \lambda e^{-\theta \lambda} \rho n] + 2e^{-\Omega(\delta^2 \rho n)},
\]

and conversely,

\[
\Pr[\bar{\tau}_\rho \geq \theta_2] \leq \Pr[\bar{B}(\theta_2) \geq -(1 + \delta)(1 - \theta_2) \lambda e^{-\theta_2 \lambda} \rho n] + 2e^{-\Omega(\delta^2 \rho n)}.
\]

In particular, if \( 0 < \theta_1, \theta_2 \ll \delta \), then

\[
\Pr[\bar{\tau}_\rho \leq \theta_1] \leq \Pr[\min_{\theta: 0 \leq \theta \leq \theta_i} \bar{B}(\theta) \leq -(1 - \delta) \lambda \rho n] + 2e^{-\Omega(\delta^2 \rho n)},
\]

and

\[
\Pr[\bar{\tau}_\rho \geq \theta_2] \leq \Pr[\bar{B}(\theta_2) \geq -(1 + \delta) \lambda \rho n] + 2e^{-\Omega(\delta^2 \rho n)}.
\]

**Proof of Theorem 1.5**

**Supercritical Region**: Suppose \( \lambda = 1 + \varepsilon \) with \( \varepsilon \gg n^{-1/3} \) and \( 1 \ll \alpha \ll (\varepsilon^3 n)^{1/2} \). Three phases are to be considered based upon on the values of \( \theta \). Let \( \theta_1 = \frac{\alpha^2}{\lambda}, \theta_2 = \theta_1 - \alpha(\theta_1 n)^{-1/2}, \) and \( \theta_3 = \theta_1 + \alpha(\theta_1 n)^{-1/2} \). (Recall \( \theta_1 \) is the larger solution of the equation \( 1 - \theta - e^{-\theta \lambda} = 0 \).)
To bound the size of the largest component, it is enough to show that all of the following events occur with probability $1 - e^{-\Omega(\alpha^2)}$.

(i) For $\rho = \theta\theta_\lambda = \alpha^2(\theta n)^{-1}$, we have $\bar{\tau}_\rho \geq \theta_\iota$, especially $\bar{F}(\theta_\iota) \leq N(V_\rho)$.

(ii) For $\theta$ in the range $\theta_\iota \leq \theta \leq \theta_2$, all $\bar{B}(\theta)$ are positive.

(iii) For some $\theta$ between $\theta_2$ and $\theta_3$, $\bar{A}(\theta) = 0$.

Once (i) and (ii) occur, as $\bar{A}(\theta) \geq \bar{B}(\theta)$, the vertices activated between $(1 - \theta_\iota)\lambda$ and $(1 - \theta_2)\lambda$ are all in the same component, say $C_1$. Excluding the vertices in $V_\rho$, all vertices that have a $(1 - \theta_2)\lambda$-large clone but no $(1 - \theta_\iota)\lambda$-large clone belong to $C_1$. That is, $\bar{V}(\theta_\iota) \setminus \bar{V}(\theta_2) \subseteq C_1 \cup V_\rho$. Corollary 8.3 with $\Delta = 0$ yields

$$\Pr[|\bar{V}(\theta_\iota) \setminus \bar{V}(\theta_2)| \leq e^{-\theta_\iota \lambda}(1 - e^{-\theta_2\theta_\iota})n - \alpha(\theta_\iota n)^{1/2}] \leq e^{-\Omega(\alpha^2)}.$$  

As $\theta_\iota \ll \alpha(\theta n)^{-1/2} \ll \theta_\lambda \leq 1$, $\rho n \ll \alpha(\theta n)^{1/2}$, and $\theta_2 = \theta_\lambda - \alpha(\theta n)^{-1/2}$, we have

$$|C_1| \geq |\bar{V}(\theta_\iota) \setminus \bar{V}(\theta_2)| - |V_\rho| \geq (1 - e^{-\theta_\iota \lambda})n - O(\alpha(n/\theta_\iota)^{1/2}) = \theta_\iota n - O(\alpha(n/\theta_\iota)^{1/2}),$$

with probability $1 - e^{-\Omega(\alpha^2)}$.

Since (i) and (ii) imply $\bar{\tau}_\rho \geq \theta_2$, if (iii) occurs in addition, then $C_1 \subseteq (V \setminus \bar{V}(\theta_3)) \cup V_\rho$. Corollary 8.3 then yields $|V \setminus \bar{V}(\theta_3)| \leq (1 - e^{-\theta_3 \lambda})n + \alpha(\theta_3 n)^{1/2}$ with probability $1 - e^{-\Omega(\alpha^2)}$. As

$$\rho n \ll \alpha(\theta n)^{1/2}, \quad \text{and} \quad 1 - e^{-\theta_3 \lambda} = 1 - e^{-\theta_\iota \lambda} + O(\alpha(\theta n)^{-1/2}) = \theta_\iota + O(\alpha(\theta n)^{-1/2}),$$

$C_1$ is of size at most $\theta_\iota n + O((n/\theta_\iota)^{1/2})$ with the desired probability. Replacing $\alpha$ by $\delta\alpha$ for an appropriate constant $\delta > 0$, the bounds for $|C_1|$ follow as desired in Theorem 1.4.

The union $S$ of components constructed before $C_1$ is a subset of $(V \setminus \bar{V}(\theta_3)) \cup V_\rho$. Corollary 8.3 with $\Delta = 0.1\theta_\iota n$ and $1 - e^{-x} \leq x$ give

$$|V \setminus \bar{V}(\theta_\iota)| \leq \theta_\iota \lambda n + 0.1\theta_\iota n \leq 1.1\theta_\iota \lambda n,$$

with probability $1 - 2e^{-\Omega(\theta_\iota n)} \geq 1 - 2e^{-\Omega(\alpha^2)}$. Thus,

$$|S| \leq 1.1\theta_\iota \lambda n + \rho n \leq \frac{3\alpha^2 \lambda}{\theta_\iota^2} = \Theta(\frac{\alpha^2}{\epsilon^2}),$$

with probability $1 - e^{-\Omega(\alpha^2)}$.

Clearly, the vertex set $V_R$ of the residual graph without $C_1$ and $S$ satisfies

$$V(\theta_3) \setminus V_\rho \subseteq V_R \subseteq V(\theta_2),$$

and hence

$$(1 - \theta_\iota)n - \alpha(n/\epsilon)^{1/2} \leq |V_R| \leq (1 - \theta_3)n + \alpha(n/\epsilon)^{1/2},$$

by replacing $\alpha$ by $0.1\alpha$ if necessary. Furthermore, the cut-off value $\lambda^*$ when the construction of $C_1$ is concluded is between $(1 - \theta_3)\lambda$ and $(1 - \theta_2)\lambda$, as desired. (Recall, $\theta_\lambda = (2 + O(\epsilon))\epsilon.$)
For the proofs of (i), (ii) and (iii), we observe that there is a positive constant \( c < 1 \) such that
\[
\tilde{b}(\theta) = (1 - \theta)(1 - \theta - e^{-\theta \lambda}) \geq c\theta(\theta - \theta)n, \tag{8.2}
\]
for \( 0 \leq \theta \leq \theta \), and
\[
\tilde{b}(\theta) = (1 - \theta)(1 - \theta - e^{-\theta \lambda}) \leq -c\theta(\theta - \theta), \tag{8.3}
\]
for \( \theta \geq \theta \) with \( \theta \) uniformly bounded from above by 1.

**Proof of (i)** Since \( \theta_1 \leq \theta_2 \), \( \rho = \theta_1 \theta_2 \) and \( b(\theta) \geq 0 \) for all \( \theta \leq \theta \), Lemma 8.4 for \( \delta = 0.5 \) gives
\[
\Pr[\tilde{\tau}_\rho \leq \theta_1] \leq \Pr[\min_{\theta, \theta_\rho \leq \theta} \tilde{B}(\theta) \leq -0.5\rho\lambda n] + 2e^{-\Omega(\rho n)}
\leq \Pr[\min_{\theta, \theta_\rho \leq \theta} \tilde{B}(\theta) - \tilde{b}(\theta) \leq -0.5\rho\lambda n] + 2e^{-\Omega(\rho n)}.
\]
And Corollary 8.3 yields
\[
\Pr[\min_{\theta, \theta_\rho \leq \theta} \tilde{B}(\theta) - \tilde{b}(\theta) \leq -0.5\rho\lambda n] \leq 2e^{-\Omega(n^2)} \leq 2e^{-\Omega(\rho n^2)}.
\]

**Proof of (ii)** If \( \theta \) is in the range \( \theta_1^* := \alpha(\theta_1 n)^{-1/2} \leq \theta \leq \theta_2 \), then
\[
\tilde{b}(\theta) \geq c\theta(\theta - \theta)\lambda n \geq 0.9c\alpha(\theta_1 n)^{-1/2}\lambda n = 0.9c\alpha\lambda(\theta_1 n)^{1/2},
\]
(see (8.2)). Thus, Corollary 8.3 yields
\[
\Pr[\min_{\theta, \theta_1 \leq \theta \leq \theta_2} \tilde{B}(\theta) \leq 0] \leq \Pr[\min_{\theta, \theta_1 \leq \theta \leq \theta_2} \tilde{B}(\theta) - \tilde{b}(\theta) \leq -0.9c\alpha\lambda(\theta_1 n)^{1/2}] \leq 2e^{-\Omega(\rho n^2)}.
\]
For \( \theta \) between \( \theta_1 \) and \( \theta_2^* \), let \( \tilde{\theta}_i = i\theta_i \), \( i = 1, \ldots, j \), for the least integer \( i \) with \( i\theta_i \geq \theta_1 \). Then, since \( \tilde{b}(\theta) \geq \tilde{b}(\tilde{\theta}_i) \geq (1 + o(1))c\tilde{\theta}_i\tilde{\theta}_i\lambda n \) for \( \tilde{\theta}_i \leq \theta \leq \tilde{\theta}_i, \) applying Corollary 8.3 we have
\[
\Pr[\min_{\theta, \theta_1 \leq \theta \leq \theta_2} \tilde{B}(\theta) \leq 0] \leq \Pr[\min_{\theta, \theta_1 \leq \theta \leq \theta_2} \tilde{B}(\theta) - \tilde{b}(\theta) \leq -0.9c\tilde{\theta}_i\tilde{\theta}_i\lambda n]
\leq 2e^{-\Omega(\lambda\tilde{\theta}_i\tilde{\theta}_2^2)} = 2e^{-\Omega(\lambda^3)}.
\]
Thus,
\[
\Pr[\min_{\theta, \theta_1 \leq \theta \leq \theta_2} \tilde{B}(\theta) \leq 0] \leq \sum_{i=1}^\infty 2e^{-\Omega(\lambda^3)} = 2e^{-\Omega(\lambda^3)}.
\]

**Proof of (iii)** If (iii) does not occur, then (ii) implies that \( F(\theta_i) = F(\theta) \). Also,
\[
0 \leq \tilde{A}(\theta) \leq \tilde{B}(\theta) + \tilde{F}(\theta) = \tilde{B}(\theta) + \tilde{F}(\theta).
\]

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As Corollary 8.3 gives
\[ \epsilon \geq (Upper \ Bound) \ Suppose \ \text{Subcritical \ Region} \]
branching process with (independent) Poisson \( (1 - \epsilon) \)
and hence
\[ \Pr[N(V_{\rho}) \geq 1.1\lambda pn] \leq 2e^{-\Omega(\rho n)} \leq 2e^{-\Omega(\alpha^2)}, \]
and hence
\[ \Pr[\tilde{B}(\theta_3) \geq -N(V_{\rho})] \leq e^{-\Omega(\alpha^2)} + \Pr[\tilde{B}(\theta_3) \geq -1.1\lambda pn]. \]

As \( \tilde{b}(\theta_3) \leq -c\alpha(\theta n)^{-1/2}\lambda n = -c\alpha(\theta n)^{1/2} \) (see (8.3)), \( \rho n = \alpha^2/\theta_3 \ll \alpha(\theta n)^{1/2} \), and
\[ -\tilde{b}(\theta_3) - 1.1\lambda pn \geq 0.9c\alpha\lambda(\theta n)^{1/2}, \]
Corollary 8.3 gives
\[ \Pr[\tilde{B}(\theta_3) \geq -1.1\lambda pn] \leq \Pr[\tilde{B}(\theta_3) - \tilde{b}(\theta_3) \geq 0.9c\alpha\lambda(\theta n)^{1/2}] \leq e^{-\Omega(\alpha^2)}. \]

Subcritical Region: (Upper Bound) Suppose \( \lambda = 1 - \epsilon \) with \( \epsilon \gg n^{-1/3} \). We take \( \rho = \epsilon^2/\log(\epsilon^3 n) \). Until the secondary stack with \( \rho = \epsilon^2/\log(\epsilon^3 n) \) becomes empty, each free step can be regarded as the start of a branching process in which the number of children is the number of newly activated clones. The branching process ends just before the next free step. We call the \( \ell^{th} \) branching process for the branching process initiated by the \( i^{th} \) free step. The whole process ends when no vertex is left in the secondary stack at the conclusion of a branching process. As the numbers of newly activated clones are stochastically bounded by independent Poisson \( (1 - \epsilon) \) random variables, we know that the \( i^{th} \) branching process dies out, say with \( D_i \) descendants, for all \( i \). Let \( D(1 - \epsilon) \) be the number of descendants for the branching process with (independent) Poisson \( (1 - \epsilon) \) children. Then
\[ \Pr[D_i > k] \leq \Pr[D(1 - \epsilon) > k] = e^\epsilon \sum_{\ell \geq k+1} \frac{\ell^{\ell-1}e^{-\ell}}{\ell!} e^{(\epsilon + \log(1 - \epsilon))((\ell - 1)}. \]

As at most \( D_i \) clones are involved in the \( i^{th} \) branching process, the size of the component containing the vertex activated by the \( i^{th} \) free step is at most \( D_i \). Observing there are at most \( \rho n \) possible \( i \), we have the following lemma.

Lemma 8.5 Suppose \( \lambda = 1 - \epsilon \) with \( \epsilon \gg n^{-1/3} \). Then, for the secondary stack \( S_{\rho} \) with \( \rho = \epsilon^2/\log(\epsilon^3 n) \),
\[ \Pr[\exists v \in S_{\rho}, |C_v| > k] \leq \frac{\epsilon^2 e^\epsilon n}{\log(\epsilon^3 n)} \sum_{\ell \geq k+1} \frac{\ell^{\ell-1}e^{-\ell}}{\ell!} e^{(\epsilon + \log(1 - \epsilon))((\ell - 1)}. \]

In particular,
\[ \Pr[\exists v \in S_{\rho}, |C_v| > k] = O\left(\frac{e^{-\epsilon^2 k}}{k^{3/2}\log(\epsilon^3 n)}\right). \]

We will now show that the cut-off line decreases fast enough.
Lemma 8.6 Suppose \( \varepsilon \leq 0.01 \) and \( \rho = a\varepsilon^2 \) with \( a \ll 1 \). Then
\[
\Pr[(1 - \tilde{\tau}_\rho)\lambda \geq 1 - (1 + \frac{a}{2})\varepsilon] \leq 2e^{-\Omega(a\varepsilon^3n)}.
\]

Proof. For \( \delta = 0.01 \) and \( \theta_0 = 0.7a\varepsilon \), Lemma 8.4 gives
\[
\Pr[\tilde{\tau}_\rho \leq \theta_0] \leq \Pr[\min_{\theta_0 \leq \theta \leq \theta_0} \tilde{B}(\theta) \leq -(1 - \delta)(1 - \theta_0)\lambda e^{-\theta_0}\rho n] + 2e^{-\Omega(\delta^2\rho n)}.
\]

Using
\[
(1 - \delta)(1 - \theta_0)\lambda e^{-\theta_0}\rho n \geq 0.9a\varepsilon^2\lambda n,
\]
and
\[
\tilde{b}(\theta) \geq \tilde{b}(\theta_0) \geq -(1 - \theta_0)(\varepsilon\theta_0 + \theta_0^2/2)\lambda n \geq -0.8a\varepsilon^2\lambda n
\]
for \( 0 \leq \theta \leq \theta_0 \), we have, by Corollary 8.3, that
\[
\Pr[\min_{\theta_0 \leq \theta \leq \theta_0} \tilde{B}(\theta) \leq -(1 - \delta)(1 - \theta_0)\lambda e^{-\theta_0}\rho n] \leq \Pr[\min_{\theta_0 \leq \theta \leq \theta_0} \tilde{B}(\theta) - \tilde{b}(\theta) \leq -0.1a\varepsilon^2\lambda n] \leq 2e^{-\Omega(a\varepsilon^3n)}.
\]

Since \( \tilde{\tau}_\rho > \theta_0 \) implies that
\[
(1 - \tilde{\tau}_\rho)\lambda < (1 - \theta_0)(1 - \varepsilon) < 1 - (1 + \frac{a}{2})\varepsilon,
\]
the desired bound follows. \( \square \)

Applying Lemmas 8.5 and 8.6 iteratively for \( a \ll 1 \) until the cut-off value is less than 0.99, we have
\[
\Pr[\exists v, |C_v| > k] \leq \sum_{i=0}^{\infty} \varepsilon_i^2 \frac{e^{\varepsilon_i n}}{\log(\varepsilon_i^3n)} \sum_{\ell \geq k+1} \left( \frac{\ell^{\ell-1}e^{-\ell}}{\ell!} e^{(\varepsilon_i + \log(1 - \varepsilon_i))\ell - 1} \right) + 2e^{-\Omega(a\varepsilon_i^3n)} + ne^{-\Omega(k)}
\]
where \( \varepsilon_0 = \varepsilon \) and \( \varepsilon_i \geq (1 + \frac{a}{2})\varepsilon_{i-1} \), especially \( \varepsilon_i \geq (1 + \frac{a}{2})^i \varepsilon \geq (1 + \frac{ia}{2})\varepsilon \). For any \( \delta > 0 \), taking
\[
a = \frac{1}{\log(\varepsilon^3n)} \quad \text{and} \quad k = \frac{\log(\varepsilon^3n) - 2.5\log(A^3n) + c}{-(\varepsilon + \log(1 - \varepsilon))},
\]
we have
\[
\varepsilon_i^2 \sum_{\ell \geq k+1} \frac{\ell^{\ell-1}e^{-\ell}}{\ell!} e^{(\varepsilon_i + \log(1 - \varepsilon_i))\ell - 1} = O\left( \frac{e^{(\varepsilon_i + \log(1 - \varepsilon_i))k}}{k^{3/2}} \right).
\]

Using
\[
\varepsilon_i + \log(1 - \varepsilon_i) \leq (1 + \frac{ia}{2})(\varepsilon + \log(1 - \varepsilon)),
\]
we finally have
\[
\Pr[\exists v, |C_v| > k] = O\left( \frac{ne^{(\varepsilon + \log(1 - \varepsilon))k}}{k^{3/2}\log(\varepsilon^3n)} \right) + 2e^{-\Omega(\varepsilon^3n^{\log(\varepsilon^3n)})} + O\left( ne^{-\Omega(\log(\varepsilon^3n + \delta \log(\varepsilon^3n))/\varepsilon^3)} \right).
\]
and, for $\varepsilon \ll 1$,
\[
\Pr[\exists v, |C_v| > k] \leq 2e^{-\Omega(c)} + 2e^{-\Omega\left(\frac{2a\varepsilon^3}{\log(\varepsilon^3 n)}\right)} + 2e^{-\Omega(\log(\varepsilon^3 n)/\varepsilon^2)}.
\]

For the lower bound, let $a = 1/\log(\varepsilon^3 n)$ and $\rho = a\varepsilon^2$. We will approximate the size of the component $C_i$ of the vertex activated by the $i$th free step, $i = 1, \ldots, 0.9pn$. If the secondary stack becomes empty earlier, $C_i$ set to be empty. To be more precise, the following auxiliary branching process is needed. Initially, there is one organism. Generally, the number of children is given by the random variable $(1 - Y)X - Y$ where $X$ is a Poisson $1 - (1 + 2a\varepsilon)$ random variable and $Y$ is an independent Bernoulli random variable with $\Pr[Y = 1] = 2a\varepsilon^2$.

In other words, the population is given by $Z_0 = 1$, and $Z_j = Z_{j-1} + (1 - Y_j)X_j - 1 - Y_j$, where $(X_j, Y_j)$ are i.i.d random variables with the same distribution as $(X, Y)$. The branching process ends when $Z_j \leq 0$. We will couple the branching process with $C_i$ so that, under certain conditions that hold with sufficiently high probability, $|C_i|$ is at least the sum of $(1 - Y_j)$’s over $j$ with $Z_\ell > 0$ for all $\ell \leq j$.

To estimate $C_i$’s, let $\Lambda_i$ be the cut-off value at the beginning of the $i$th free step. Then the $i$th free step will generate Poisson $\Lambda_i$ active clones. In a subsequent step of the $i$th branching process, an active clone $x$ and the largest unmatched clone excluding $x$, say $y$, are to be matched. If the cut-off value $\Lambda$ were $(1 - \theta)\lambda$, $\tilde{A}(\theta) \neq 0$, and $V(\theta)$ were given at the beginning of the step, then we may lower bound the probability that $y$ has not been activated. (If $\tilde{A}(\theta) = 0$, the branching process ends.) Clearly, the set of vertices that have not been activated at the beginning of the step contains $V^* := V(\theta) \setminus V_j$. Thus, the probability is at least as large as the probability that a clone of a vertex in $V^*$ is larger than all of $\tilde{A}(\theta) - 1$ currently active clones excluding $x$. Notice that the largest number assigned to clones of vertices of $V^*$ is less than $(1 - t)\Lambda$ with probability $e^{-t\Lambda|V^*|}$ and the corresponding density function is $\Lambda|V^*|e^{-t\Lambda|V^*|}$. Using $|V^*| \geq |V(\theta)| - \rho n$, we have
\[
\Pr[y \text{ has not been activated}] \geq \int_0^1 (1 - t)^{|\tilde{A}(\theta)| - 1} \Lambda|V^*|e^{-t\Lambda|V^*|} \, dt \\
\quad \geq 1 - \Lambda|\tilde{A}(\theta)||V^*| \int_0^\infty te^{-t\Lambda|V^*|} \, dt \\
\quad = 1 - \frac{|\tilde{A}(\theta)|}{\Lambda|V^*|} \geq 1 - \frac{|\tilde{A}(\theta)|}{\Lambda(|V(\theta)| - \rho n)}.
\]
Conditioned that $y$ has not been active, the number of new active clones is a Poisson $\Lambda'$ random variable, where $\Lambda'$ is the cut-off value at the end of the step.

Provided the $i$th free step started and
\[
\frac{|\tilde{A}(\theta)|}{\Lambda(|V(\theta)| - \rho n)} \leq 2a\varepsilon^2, \quad \text{and} \quad \Lambda' \geq 1 - (1 + 2a)\varepsilon,
\]
the number of active clones after the end of the step is at least $\tilde{A}(\theta) + (1 - Y_j)X_j - 1 - Y_j$. Moreover, in each step $Y_j = 0$, a distinct vertex is added to eventually form the component $C_i$. Thus,
\[
|C_i| \geq \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j), \quad (8.4)
\]
where $Z_0 = 1$ and, for $j \geq 1$, $Z_j = Z_{j-1} + (1 - Y_j)X_j - 1 - Y_j$, or $Z_t = 1 + \sum_{j=1}^t (1 - Y_j)X_j - 1 - Y_j$. Even if any of the conditions is not satisfied, we still define the auxiliary branching process exactly the same way. However, the coupling and the inequality it implies are no longer true.

Let $\theta_i = 1.4a\varepsilon$. We will show that the following events occur with large enough probability.

(i) $\bar{\tau}_{\rho} \leq \theta_i$, (ii) $|\bar{V}_{\rho}(\theta_i)| \geq 0.9\rho n$, (iii) $|\bar{V}(\theta_i)| \geq 0.9n$, (iv) $\max_{\theta;0 \leq \theta \leq \theta_i} \bar{B}(\theta) \leq 0.1\rho n$,

and (v) the number of clones of vertices in $V_{\rho}$ is less than $1.1\rho n$. Notice that (i) $\cap$ (ii) implies that there are at least $0.9\rho n$ free steps, and (i) gives that the cut-off value never becomes less than $1 - (1 + 2a)\varepsilon$ during the entire $0.9\rho n$ branching processes. As $\bar{A}(\theta)$ with $\theta \leq \bar{\tau}_{\rho}$ is at most $\bar{B}(\theta)$ plus the number of clones of vertices in $V_{\rho}$, (i) $\cap$ (iii) $\cap$ (iv) $\cap$ (v) implies that

$$\frac{|\bar{A}(\theta)|}{\Lambda(|\bar{V}(\theta)| - \rho n)} \leq \frac{1.2\rho n}{(1 - \theta_i)\lambda(0.9n - \rho n)} \leq 2a\varepsilon^2.$$ 

Therefore, (8.4) gives

$$\Pr[|C_i| \leq k, \forall i = 1, ..., 0.9\rho n] \leq \Pr[\neg(i)] + \Pr[\neg(ii)] + \Pr[\neg(iii)] + \Pr[\neg(iv)] + \Pr[\neg(v)] + \Pr\left[\sum_{j=0}^\infty (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) \leq k\right]^{0.9\rho n}.$$ 

First, Lemma 8.4 gives

$$\Pr[\neg(i)] = \Pr[\bar{\tau}_{\rho} \geq \theta_i] \leq \Pr[\bar{B}(\theta_i) \geq -1.1(1 - \theta_i)\lambda e^{-\theta_i\lambda} \rho n] + 2e^{-\Omega(\rho n)},$$

Since $-1.1(1 - \theta_i)\lambda e^{-\theta_i\lambda} \rho n \geq -1.2\rho \lambda n = -1.2a\varepsilon^2\lambda n$, and

$$-\hat{b}(\theta_i) = -(1 - \theta_i)(1 - \theta_i - e^{-\theta_i\lambda})\lambda n \geq (1 - \theta_i)\varepsilon \theta_i \lambda n \geq 1.3a\varepsilon^2\lambda n,$$

Corollary 8.3 gives

$$\Pr[\bar{B}(\theta_i) \geq -1.1(1 - \theta_i)\lambda e^{-\theta_i\lambda} \rho n] \leq \Pr[\bar{B}(\theta_i) - \hat{b}(\theta_i) \geq 0.1a\varepsilon^2\lambda n] \leq 2e^{-\Omega(a\varepsilon^3 n)}.$$ 

Appealing Corollary 8.3 and using $e^{-\theta_i\lambda} = 1 + o(1)$, we have

$$\Pr[\neg(ii)] \leq 2e^{-\Omega(a\varepsilon^3 n)}, \text{ and } \Pr[\neg(iii)] \leq 2e^{-\Omega(n)}.$$ 

For (iv), Lemma 8.4 and $\hat{b}(\theta) \leq 0$ for $\theta \geq 0$ yield

$$\Pr[\neg(iv)] \leq \Pr[\max_{\theta;0 \leq \theta \leq \theta_i} |\bar{B}(\theta) - \hat{b}(\theta)| \geq 0.1\rho n] \leq 2e^{-\Omega(e^2\rho n)} = 2e^{-\Omega(a\varepsilon^3 n)}.$$

Clearly, $\Pr[\neg(v)] \leq 2e^{-\Omega(\rho n)} = 2e^{-\Omega(a\varepsilon^2 n)}$ as the number is a Poisson $\rho \lambda n$ random variable. Therefore, for

$$k = \frac{\log(3)}{2.5} - 2.5\log(3) - \frac{c}{\varepsilon},$$

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Using $Y$, it follows that and hence, inside a window, we have

$$\Pr[|C_i| \leq k \forall i = 1, \ldots, 0.9\rho n] \leq 2e^{-\Omega \left( \frac{3}{\log(\epsilon^3 n)} \right)} + \Pr \left[ \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) \leq k \right]^{0.9\rho n}.$$ 

Hence, it is enough to show that

$$\Pr \left[ \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) \leq k \right]^{0.9\rho n} = \left( 1 - \Pr \left[ \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) > k \right] \right)^{0.9\rho n} \leq e^{-\Omega(\epsilon^3)}.$$

Recalling $Z_\ell = 1 + \sum_{j=1}^\ell (1 - Y_j)X_j - 1 - Y_j$, we observe that, conditioned on $Y_1 = \cdots = Y_{k+1} = 0$, $Z_\ell = 1 + \sum_{j=1}^\ell X_j - 1$, $\ell = 1, \ldots, k + 1$, which is exactly the population for the Poisson branching process with mean number of children $1 - (1 + 2a)\epsilon$. As

$$\Pr[Y_1 = \cdots = Y_{k+1} = 0] = (1 - 2a\epsilon^2)^{k+1} = (1 + o(1))e^{-2ak\epsilon^2} = (1 + o(1))e^{-2},$$

it follows that

$$\Pr \left[ \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) > k \right] = \Omega \left( \frac{e^{(1+2a)\epsilon + \log(1-(1+2a)\epsilon))k}}{\epsilon^2 k^{3/2}} \right).$$

Using $k = \Theta(\epsilon^{-2}\log(\epsilon^3 n))$ and $(1 + 2a)\epsilon + \log(1 - (1 + 2a)\epsilon) \geq (1 + 3a)(\epsilon + \log(1 - \epsilon))$ for $\epsilon \leq 0.01$, we further have

$$\Pr \left[ \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) > k \right] = \Omega \left( \frac{e^{-(1+3a)\log(\epsilon^3 n) - 2.5 \log(\epsilon^3 n) - c}}{\epsilon^{-1} \log^{3/2}(\epsilon^3 n)} \right) = \Omega \left( \frac{e^c \log(\epsilon^3 n)}{\epsilon^2 n} \right),$$

and hence

$$\left( 1 - \Pr \left[ \sum_{j \geq 0} (1 - Y_j)1(Z_\ell > 0, \forall \ell \leq j) > k \right] \right)^{0.9\rho n} \leq e^{-\Omega(\epsilon^3)}.$$

Inside window: Suppose $\lambda = 1 + \epsilon$ with $|\epsilon| = O(n^{-1/3})$. For a large enough constant $K$, we may sandwich $G_{PC}(n, p)$ between $G_{PC}(n, p_1)$ and $G_{PC}(n, p_2)$, where $p_1(n-1) = 1 - Kn^{-1/3}$ and $p_2(n-1) = 1 + Kn^{-1/3}$. Thus, $G_{PC}(n, p)$ has a component of size $\Omega(K^{-2}n^{2/3} \log K)$ and no component of size $O(Kn^{2/3})$.

9 Closing Remarks

The Poisson $\lambda$-cell is introduced to analyze properties of $G_{PC}(n, p)$, in which degrees are i.i.d Poisson random variables with mean $\lambda = p(n-1)$. Then various nice properties of Poisson random variables are used to analyze sizes of the largest component and the $t$-core of $G_{PC}(n, p)$. We believe that the approaches presented in this paper are useful to analyze problems with similar flavors, especially problems related to branching processes. For example, we can easily modify the proofs of Theorem 1.7 to analyze the pure literal algorithm for the random $k$-SAT problems, $k \geq 3$. Another example may be the Karp-Sipser
algorithm to find a large matching of the random graph. (See [49,7].) In a subsequent paper, we will analyze the structures of the 2-core of $G(n, p)$ and the largest strong component of the random directed graph as well as the pure literal algorithm for the random 2-SAT problem.

For the random (hyper)graph with a given sequence $(d_i)$, we may also introduce the $(d_i)$-cell, in which the vertex $v_i$ has $d_i$ clones and each clone is assigned a uniform random real number between 0 and the average degree $\frac{1}{n} \sum_{i=0}^{n-1} d_i$. Though it is not possible to use all of the nice properties of Poisson random variables any more, we believe that the $(d_i)$-cell equipped with the cut-off line algorithm can be used to prove stronger results for the $t$-core problems considered in various papers including [22, 33, 34, 43, 58]. The case of the random $k$-SAT problem conditioned on given degree sequence can be similarly analyzed.

A better algorithm called the unit clause algorithm for random $k$-SAT problems is known. (See e.g. [1, 2, 3, 19, 38]) We believe, the unit clause algorithm and some of its variations can be analyzed using the Poisson cloning model equipped with the cut-off line algorithm.

Recall that the degrees in $G(n, p)$ has the binomial distribution with parameters $n - 1$ and $p$. By introducing the Poisson cloning model, we somehow first take the limit of the binomial distribution, which is the Poisson distribution. In general, many limiting distributions like Poisson and Gaussian ones have nice properties. In our opinion, this is because various small differences are eliminated by taking the limits, and limiting distributions have some symmetric and/or invariant properties. Thus, it may be natural to wonder if there is an infinite graph that shares most properties of the random graphs $G(n, p)$ with large enough $n$. So, in a sense, the infinite graph, if exists, can be regarded as the limit of $G(n, p)$. An infinite graph which Aldous [5] considered to solve the linear assignment problem may or may not be a (primitive) version of such an infinity graph. Though it may be impossible to construct such a graph, the approaches taken in this paper might be useful to find one, if any.

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Appendix: The proof of Lemma 7.1

For the proof of Lemma 7.1, observing that

$$|V(\theta)| = \sum_{i=1}^{m} |C_i|1(D_i(\theta) \in P) = \frac{n}{m} \sum_{i=1}^{m} 1(D_i(\theta) \in P),$$

Lemma 5.2 yields the following corollary.

**Corollary 9.1** For \( \theta \) in the range \( \theta_1 \leq \theta \leq 1 \) and with the same hypotheses as in the main lemma,

$$\Pr[|V_P(\theta)| - p(\theta)n| \geq \Delta] \leq 2e^{-\Omega(\min(\Delta, \frac{\Delta^2}{1-\theta}))}.$$  

As in Example 4.4, Lemma 4.5 yields a concentration inequality for all of \( V(\theta) \)'s:
Lemma 9.2 With the same hypotheses as in the main lemma,
\[ \Pr \left[ \max_{\theta, \theta' \leq \theta \leq 1} \left| \left| V(\theta) \right| - p(\theta)n \right| \leq \Delta \right] \leq 2e^{-\Omega(\min\{ \Delta, \frac{\Delta^2}{(1-\theta)n} \})}. \]

**Proof.** Observing that, for \( \theta_i \leq \theta \leq 1 \),
\[ \frac{p(\theta_i)}{p(\theta)} \left| |V(\theta)| - p(\theta)n \right| \leq \left| |V(\theta)| - p(\theta)n \right| + \left| |V(\theta_i)| - \frac{p(\theta_i)}{p(\theta)}|V(\theta)| \right|, \]
we set \( \Gamma(\theta) = ||V(\theta)| - p(\theta)n| \),
\[ \psi = \frac{1}{p(\theta_i)} \Gamma(\theta_i), \text{ and } \psi_\theta = \frac{p(\theta)}{p(\theta_i)} \left| |V(\theta)| - \frac{p(\theta_i)}{p(\theta)}|V(\theta)| \right|. \]
Clearly, \( \Gamma(\theta) \leq \psi + \psi_\theta \). Corollary 9.1 gives
\[ \Pr[\psi \geq \Delta/2] \leq 2e^{-\Omega(\min\{ \Delta, \frac{\Delta^2}{(1-\theta_i)n} \})}. \] (9.1)

Suppose \( \{ X_{\theta'} := V(\theta') \} \theta \leq \theta' \leq 1 \) is given, especially \( V(\theta) \) is given. Then, since \( P \) is increasing, we may write \( |V(\theta_i)| \) as
\[ |V(\theta_i)| = \sum_{i : C_i \subseteq V(\theta)} |C_i| 1(D_i(\theta_i) \in P), \]
with
\[ \Pr[D_i(\theta_i) \in P|\{ X_{\theta'} \} \theta \leq \theta' \leq 1] = \Pr[D_i(\theta_i) \in P|C_i \subseteq V(\theta)] = \frac{p(\theta_i)}{p(\theta)} = p(\theta_i, \theta), \]
for \( i \) with \( C_i \subseteq V(\theta) \). Lemma 5.2 then gives
\[ \Pr[\psi_\theta \geq \Delta/2] \leq 2e^{-\Omega(\min\{ p(\theta_i, \theta)\Delta, \frac{p(\theta_i, \theta)\Delta^2}{p(\theta_i, \theta)|V(\theta)|} \})} \leq 2e^{-\Omega(\min\{ p(\theta_i)\Delta, \frac{p(\theta_i)\Delta^2}{p(\theta_i)|V(\theta)|} \})} \leq 2e^{-\Omega(\min\{ \Delta, \frac{\Delta^2}{(1-\theta_i)n} \})}. \] (9.2)

Lemma 4.5 together with (9.1) and (9.2) yields the desired inequality. \( \square \)

We now estimate \( M(\theta) \). First, since \( \sum_{v \in V} \tilde{d}_v(\theta) \) is a Poisson random variable with mean \( (1 - \theta)\lambda n \),
\[ \Pr \left[ \left| \sum_{v \in V} \tilde{d}_v(\theta) - (1 - \theta)\lambda n \right| \geq \Delta/2 \right] \leq 2e^{-\min\{ \Delta, \frac{\Delta^2}{(1-\theta)n} \}}. \] (9.3)

For the second sum, observe that
\[ \sum_{v \in V} d_v(\theta) 1(v \notin V(\theta)) = \sum_{i=1}^{m} \left( \sum_{v \in C_i} d_v(\theta) \right) 1(D_i(\theta) \notin P) \]
is a sum of \( m \) i.i.d random variables with
\[ E \left[ \left( \sum_{v \in C_i} d_v(\theta) \right) 1(D_i(\theta) \notin P) \right] = E \left[ \sum_{v \in C_i} d_v(\theta) - \left( \sum_{v \in C_i} d_v(\theta) \right) 1(D_i(\theta) \in P) \right] = (\theta \lambda - q(\theta))|C_i|. \]
Moreover, since $P$ is an increasing property and $\sum_{v \in C_i} d_v(\theta)$ is a Poisson $\theta \lambda |C_i|$ random variable, FKG inequality (see e.g. Chapter 6 of [6]) gives

$$E \left[ \left( \sum_{v \in C_i} d_v(\theta) \right)^i \right] \leq E \left[ \left( \sum_{v \in C_i} d_v(\theta) \right)^i \right] \Pr[D_i(\theta) \notin P] = O(1 - p(\theta)),$$

for all fixed $i$, e.g. $i = 1, 2, 3$. Thus one may take $\xi_\theta = 1$ and $a_i, b_i = \Theta(1 - p(\theta))$ to satisfy all the conditions to apply the generalized Chernoff bound and to obtain

$$\Pr \left[ \left| \sum_{v \in V} d_v(\theta) 1(v \notin V(\theta)) - (\theta \lambda - q(\theta))n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1 - p(\theta))n}\})},$$

provided $|C_i| = O(1)$. This together with (9.3) implies that if $|C_i| = O(1)$ and $1 - p(\theta) = O(1 - \theta)$, then

$$\Pr \left[ \left| M(\theta) - (\lambda - q(\theta))n \right| \geq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1 - p(\theta))n}\})}. \quad (9.4)$$

We prove a concentration result for all of $M(\theta)$ that together with the cut-off lemma implies Lemma 7.1 follows.

**Lemma 9.3** With the same hypotheses as in the main lemma,

$$\Pr \left[ \max_{\theta, \theta' \leq \theta \leq 1} \left| M(\theta) - (\lambda - q(\theta))n \right| \leq \Delta \right] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1 - p(\theta))n}\})}.$$

**Proof.** Clearly,

$$\left| M(\theta) - (\lambda - q(\theta))n \right| \leq \left| M(\theta_1) - (\lambda - q(\theta_1))n \right| + \left| M(\theta_1) - M(\theta) - (q(\theta) - q(\theta_1))n \right|.$$ 

Let $\Gamma(\theta) = |M(\theta) - (\lambda - q(\theta))n|,$

$$\psi = \Gamma(\theta), \quad \psi(\theta) = |M(\theta_1) - M(\theta) - (q(\theta) - q(\theta_1))n|,$$

and $\Phi_\theta$ is the event $|V(\theta) - p(\theta)n| \leq \frac{p(\theta)\Delta}{4\lambda}$. Then, (9.4) gives

$$\Pr[\psi \geq \Delta/2] \leq 2e^{-\Omega(\min\{\Delta, \frac{\Delta^2}{(1 - p(\theta))n}\})}.$$ 

For $\psi(\theta)$, suppose $\{X_{\theta'} := (V(\theta'), M(\theta'))\}_{\theta \leq \theta' \leq 1}$ is given. Using

$$M(\theta) = \sum_{v \in V} d_v(\theta) + d_v(\theta)1(v \notin V(\theta)) = \sum_{v \in V} d_v(1) - d_v(\theta)1(v \in V(\theta)),$$

we obtain

$$M(\theta_1) - M(\theta) = \sum_{v \in V} d_v(\theta)1(v \in V(\theta)) - d_v(\theta_1)1(v \in V(\theta_1))$$

$$= \sum_{i=1}^m \left( \sum_{v \in C_i} d_v(\theta) \right)1(C_i \subseteq V(\theta)) - \left( \sum_{v \in C_i} d_v(\theta_1) \right)1(C_i \subseteq V(\theta_1)).$$
Once \( V(\theta) \) is given, the distributions of

\[
Y_i := \left( \sum_{v \in C_i} d_v(\theta) \right) 1(C_i \subseteq V(\theta)) - \left( \sum_{v \in C_i} d_v(\theta) \right) 1(C_i \subseteq V(\theta_i))
\]

depend on neither \( \{M(\theta')\}_{\theta \leq \theta'} \) nor \( \{V(\theta')\}_{\theta \leq \theta'} \) and hence, for \( C_i \subseteq V(\theta) \),

\[
E\left[ Y_i \mid \{X_{\theta'}\}_{\theta \leq \theta'} \right] = E\left[ \sum_{v \in C_i} d_v(\theta) - \left( \sum_{v \in C_i} d_v(\theta) \right) 1(v \in V(\theta_i)) \mid C_i \subseteq V(\theta) \right]
= \frac{q(\theta) - q(\theta_i)}{p(\theta)} |C_i|.
\]

If \( C_i \not\subseteq V(\theta) \), \( Y_i = 0 \) since \( P \) is increasing.

Also, for \( C_i \subseteq V(\theta) \), we may write

\[
Y_i = \sum_{v \in C_i} d_v(\theta) - d_v(\theta_i) + d_v(\theta) 1(C_i \not\subseteq V(\theta))
\]

and

\[
E\left[ Y_i^2 \mid C_i \subseteq V(\theta) \right] \leq 2E\left[ \left( \sum_{v \in C_i} d_v(\theta) - d_v(\theta_i) \right)^2 \mid C_i \subseteq V(\theta) \right] + 2E\left[ \left( \sum_{v \in C_i} d_v(\theta) 1(C_i \not\subseteq V(\theta)) \right)^2 \mid C_i \subseteq V(\theta) \right].
\]

For \( j = 1, 2, 3 \),

\[
E\left[ \left( \sum_{v \in C_i} d_v(\theta) - d_v(\theta_i) \right)^j \mid C_i \subseteq V(\theta) \right] \leq p(\theta)^{-1} E\left[ \left( \sum_{v \in C_i} d_v(\theta) - d_v(\theta_i) \right)^2 \mid C_i \subseteq V(\theta) \right] = O(\theta - \theta_i) = O(1 - \theta_i)
\]

for \( p(\theta) \geq p(\theta_i) = \Omega(1) \) and \( \sum_{v \in C_i} d_v(\theta) - d_v(\theta_i) \) is a Poisson random variable with mean \((\theta - \theta_i)\lambda |C_i| = O(\theta - \theta_i)\). For the second term, FKG inequality gives

\[
E\left[ \left( \sum_{v \in C_i} d_v(\theta) 1(C_i \not\subseteq V(\theta)) \right)^j \mid C_i \subseteq V(\theta) \right] \leq p(\theta)^{-1} E\left[ \left( \sum_{v \in C_i} d_v(\theta) 1(C_i \not\subseteq V(\theta)) \right)^j \mid C_i \subseteq V(\theta) \right]
\leq p(\theta_i)^{-1} E\left[ \left( \sum_{v \in C_i} d_v(\theta_i) \right)^j \mid C_i \not\subseteq V(\theta) \right]
= O(1 - p(\theta)) = O(1 - \theta_i),
\]

for \( j = 1, 2, 3 \). Therefore,

\[
E\left[ \left( Y_i - E[Y_i] \right)^2 \mid \{X_{\theta'}\}_{\theta \leq \theta'} \right] \leq E\left[ Y_i^2 \mid \{X_{\theta'}\}_{\theta \leq \theta'} \right] = O(1 - \theta_i).
\]

Similarly, for \( \xi \) in the range \(|\xi| \leq \xi_0 = 1\), it is not hard to show

\[
\left| E\left[ (Y_i - E[Y_i])^3 e^{\xi(Y_i - E[Y_i])} \mid \{X_{\theta'}\}_{\theta \leq \theta'} \right] \right| = O(1 - \theta_i).
\]
Applying the generalized Chernoff bound, we have

\[
\Pr \left[ \left| \sum_{j=1}^{m} Y_i - \frac{q(\theta) - q(\theta_1)}{p(\theta)} |V(\theta)| \right| \geq \Delta/4 \left| \{X_{\theta'}\}_{\theta \leq \theta'} \right| \right] \leq 2e^{-\Omega(\min\{\Delta, \Delta^2 \frac{2}{\eta} \}).}
\]

Finally, as the event \(\Phi_\theta\) guarantees

\[
\frac{q(\theta) - q(\theta_1)}{p(\theta)} |V(\theta)| - p(\theta)n \leq \Delta/4
\]

for \(p(\theta_1) \leq p(\theta)\) and \(q(\theta) \leq \lambda\), we have

\[
1(\Phi_\theta) \Pr \left[ \left| \sum_{j=1}^{m} Y_i - (q(\theta) - q(\theta_1))n \right| \geq \Delta/2 \left| \{X_{\theta'}\}_{\theta \leq \theta'} \right| \right] \\
\leq \Pr \left[ \left| \sum_{j=1}^{m} Y_i - \frac{q(\theta) - q(\theta_1)}{p(\theta)} |V(\theta)| \right| \geq \Delta/4 \left| \{X_{\theta'}\}_{\theta \leq \theta'} \right| \right] \\
\leq 2e^{-\Omega(\min\{\Delta, \Delta^2 \frac{2}{\eta} \}).}
\]

Lemma 4.5 yields the desired inequality. \(\square\)