Recognizing Proper Tree-Graphs

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Abstract
We investigate the parameterized complexity of the recognition problem for the proper $H$-graphs. The $H$-graphs are the intersection graphs of connected subgraphs of a subdivision of a multigraph $H$, and the properness means that the containment relationship between the representations of the vertices is forbidden. The class of $H$-graphs was introduced as a natural (parameterized) generalization of interval and circular-arc graphs by Biró, Hujter, and Tuza in 1992, and the proper $H$-graphs were introduced by Chaplick et al. in WADS 2019 as a generalization of proper interval and circular-arc graphs. For these graph classes, $H$ may be seen as a structural parameter reflecting the distance of a graph to a (proper) interval graph, and as such gained attention as a structural parameter in the design of efficient algorithms. We show the following results.

For a tree $T$ with $t$ nodes, it can be decided in $2^{O(t^2 \log t)} \cdot n^3$ time, whether an $n$-vertex graph $G$ is a proper $T$-graph. For yes-instances, our algorithm outputs a proper $T$-representation. This proves that the recognition problem for proper $H$-graphs, where $H$ required to be a tree, is fixed-parameter tractable when parameterized by the size of $T$. Previously only $\mathsf{NP}$-completeness was known.

Contrasting to the first result, we prove that if $H$ is not constrained to be a tree, then the recognition problem becomes much harder. Namely, we show that there is a multigraph $H$ with 4 vertices and 5 edges such that it is $\mathsf{NP}$-complete to decide whether $G$ is a proper $H$-graph.

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1 Introduction

An intersection representation of a graph \( G = (V, E) \) is a collection of nonempty sets \( \{M_v \mid v \in V(G)\} \) over a given universe such that \( \{u, v\} \) is an edge of \( G \) if and only if \( M_u \cap M_v \neq \emptyset \). A large area of research in graph algorithms is the study of restricted families of graphs arising from specialized intersection representations, e.g., the interval graphs are the graphs with an intersection representation where the sets are intervals of \( \mathbb{R} \), and the circular-arc graphs are intersection graphs of families of arcs of the circle. The interval graphs and similarly defined graph classes are often motivated from application areas such as circuit layout problems [24, 4], scheduling problems [22], biological problems [19], or the study of wireless networks [16]. We refer to the books [5, 15] for an introduction and survey of the known results on the related graph classes.

A key feature of these specialized intersection representations is that they can often be used to obtain efficient algorithms for standard combinatorial optimization problems, e.g., it is well-known [5] that the Clique and Independent Set problems, as well as various coloring and Hamiltonicity problems are all efficiently solvable on interval graphs, and the algorithms often leverage on the intersection representation. This led Biró et al. [2] to introduce an elegant family of intersection graph classes, called \( H \)-graphs, over universes that may be seen as (multi) graphs. Formally, the parameter \( H \) is a multigraph, and a graph \( G \) is an \( H \)-graph when there is a subdivision \( H_{\text{sub}} \) of \( H \) and a collection \( M = \{M_v \subseteq V(H_{\text{sub}}) \mid v \in V(G)\} \) of sets, where we refer to \( M_v \) as the model of \( v \), such that

- for every \( v \in V(G) \), its model \( M_v \) induces a connected subgraph of \( H_{\text{sub}} \), and
- \( \{u, v\} \in E(G) \) if and only if \( M_u \cap M_v \neq \emptyset \).

In this context, \( H_{\text{sub}} \) represents \( G \). Observe that, for any interval graph \( G \), there is a path \( P \) (i.e., a subdivision of \( K_2 \)) such that \( P \) represents \( G \) meaning that the interval graphs are precisely the \( K_2 \)-graphs. Similarly, the circular-arc graphs are \( C \)-graphs for any cycle \( C \), and every chordal graph is a \( T \)-graph for some tree \( T \), i.e., indeed, \( H \)-graphs can be seen as a parameterized generalization of several important families of intersection graphs, where \( H \) is a parameter reflecting the distance of a graph to an interval graph. Biró et al. [2] provided polynomial-time algorithms (via treewidth-based techniques) for coloring problems on \( H \)-graphs for fixed \( H \), but left many interesting problems open.

The classes of \( H \)-graphs have seen renewed interest in recent years concerning their structure and recognition [8], relation to other graph parameters [8, 12], and primarily regarding the computational complexity of standard algorithmic problems when parameterized by the size of \( H \) [1, 7, 8, 9, 12, 17, 18]. Of particular relevance to our paper is the work on Hamiltonicity problems [7] as it introduces proper \( H \)-graphs, which are to proper interval graphs as \( H \)-graphs are to interval graphs. Namely, for a graph \( G \), a subdivision \( H_{\text{sub}} \) of \( H \) properly represents \( G \) when \( H_{\text{sub}} \) represents \( G \) using models \( \{M_v \subseteq V(H_{\text{sub}}) \mid v \in V(G)\} \) such that for each \( u, v \in V(G) \), neither \( M_u \subseteq M_v \) nor \( M_v \subseteq M_u \). In particular, on proper \( H \)-graphs polynomial size kernels (in the size of \( H \)) were developed for various Hamiltonicity problems [7], but the recognition problems were left open.

The cornerstone problem for every graph class is recognizability, and we focus on the recognition problem for proper \( H \)-graphs both when \( H \) is part of the input and when \( H \) is fixed. It is important to note that the problem of testing whether a given graph \( G \) and given tree \( T \), the graph \( G \) is a \( T \)-graph is NP-complete [20, Theorem 4]. In fact, the reduction [20, Theorem 4] also implies that testing whether \( G \) is a proper \( T \)-graph

\footnote{\( H_{\text{sub}} \) is obtained from \( H \) by iteratively replacing an edge \( \{u, v\} \) by a path \( uwv \), where \( w \) is a new vertex.}
is NP-complete. In contrast to this, it is known that when $T$ is fixed, testing whether a given graph is a $T$-graph can be done in polynomial-time [8], i.e., $\XP$ in the size of $T$; but it is not known whether the problem is $\FPT$ in the size of $T$. When going beyond trees the recognition problem becomes much harder. Namely, for each fixed non-cactus graph $H$, $H$-graph recognition is NP-complete [8]. However, for fixed $H$, it seems that only two cases of proper $H$-graph recognition have been studied: The proper interval graphs (proper $K_2$-graphs) [10, 11] and the proper circular-arc graphs (proper $C$-graphs, for any cycle $C$) [11] can each be recognized in linear-time.

Our Contribution. In our main result, we show that the recognition of proper $T$-graphs is fixed-parameter tractable ($\FPT$) with respect to the size of $T$ by proving the following.

▶ **Theorem 1.** There is an algorithm that, given an $n$-vertex graph $G$ and a tree $T$ with $t$ nodes, decides whether $G$ is a proper $T$-graph, and if yes, outputs a proper $T$-representation, in $2^{O(t \log t)} \cdot n^3$ time.

To obtain our $\FPT$ algorithm for proper $T$-graph recognition, we first observe that the problem can be reduced to the case when the input graph $G$ is connected and chordal. We proceed in the following three key steps.

In Section 3, we introduce compact representations which are an analog to the clique-trees of chordal graphs that incorporates the properness condition. We characterize the proper $T$-graphs via these compact representations. This allows us to work with maximal cliques of the input graph that can be listed in linear-time due to the chordality of $G$.

In Subsections 4.1 and 4.2, independent of the tree $T$, we partition the maximal cliques into a collection of the so-called chains each one necessarily forming a path in any proper $T$-representation, and the remaining singleton cliques that are marked and treated separately. We show that having a compact $T$-representation means there are, in terms of the size of $T$, at most quadratically many of these marked cliques and chains altogether.

In Subsections 4.3 and 4.4, we combine these ideas to form our $\FPT$ algorithm for proper $T$-graph recognition. First, our algorithm guesses a layout of the chains and the marked maximal cliques. The remaining non-trivial task is to decide whether there is a compact representation corresponding to the guessed layout. We select a root of the tree and show a combinatorial result that if any compact representation realizes some layout, it can be assumed to have some special properties concerning the usage of the nodes of degree at least three of the tree by the models with respect to the root. We call representations satisfying these properties normalized. Our algorithm follows the layout bottom-up and constructs a normalized representation if it exists.

We complement our algorithmic result from Theorem 1 by proving that if $H$ is not constrained to be a tree, the recognition problem for proper $H$-graphs becomes NP-complete even if $H$ has bounded size. This negative result employs a reduction quite similar to the one used for (non-proper) $H$-graphs in [8], and as such is discussed and proven in the full version.

▶ **Theorem 2 (⋆).** There is a 4-vertex, 5-edge multigraph $\mathcal{D}$ (defined by $V(\mathcal{D}) = \{a, b, c, d\}$ and $E(\mathcal{D}) = \{ab, bc, bc, cd\}$) such that proper $\mathcal{D}$-graph recognition is $\NP$-complete.

Note that this and further statements proven in the full version are marked with (⋆).
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2 Preliminaries

General Notation. We consider undirected graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. Usually we denote an edge as a set $\{u, v\}$. However, when needed, we also denote an edge an ordered pair $(u, v)$. For any subset $W$ of $V(G)$, we use $N(W)$ to denote the open neighborhood of $W$, i.e., $N(W) := \{u \in V(G) \mid \{u, w\} \in E(G), w \in W\}$, and for a single vertex $w \in V(G)$, $N(w) := N(\{w\})$. We denote the set of maximal cliques of a graph as $\mathcal{C}(G)$. The shorthand $[n]$ denotes the set $\{1, \ldots, n\}$ of integers.

A subdivision $H'$ of a graph $H$ at an edge $\{u, w\}$ is the graph resulting from replacing edge $\{u, w\}$ with a path $u, v, w$ where $v$ is new vertex. A contraction of a graph $H$ at an edge $\{u, v\}$ is the graph resulting from removing edge $\{u, v\}$ and identifying the two vertices $u$ and $w$. Then $H_{sub}$ is a re-subdivision of $H$ if it can be obtained by a series of contractions of $H$ (possibly none) followed by a series of subdivisions. In particular a graph $G$ is a (proper) $H$-graph if and only if there is a re-subdivision that (properly) represents $G$.

Let $T$ be a tree. For any pair $x, y$ of nodes of $T$, we denote by $T[x, y]$ the set of nodes of the unique path from $x$ to $y$ in $T$. Note that $T[x, y] = T[y, x]$. We similarly define $T(x, y) := T[x, y] \setminus \{x\}$ and $T(x, y) := T(x, y) \setminus \{x, y\}$. A tuple of nodes $(x_1, \ldots, x_s)$ is $T$-ordered if there exists a path in the graph $T$ from $x_1$ to $x_s$ where the nodes $x_1, \ldots, x_s$ occur in this order, i.e., $T[x_1, x_s]$ is the path $x_1, \ldots, x_s$.

$H$-graphs. Consider a re-subdivision $H_{sub}$ of a graph $H$ that (properly) represents a graph $G$ using models $\{M_v \subseteq V(H_{sub}) \mid v \in V(G)\}$. For clarity, we refer to each $x \in V(H_{sub})$ as a node and to each $v \in V(G)$ as a vertex. We further refer to each node $x \in V(H_{sub})$ as:
- a subdivision node when it has degree two,
- a branching node when it has degree more than two, and
- a leaf node if it has degree one.

For a set of nodes $X \subseteq V(H_{sub})$, let $V_X := \{v \in V(G) \mid M_v \cap X \neq \emptyset\}$. When $X = \{x\}$, we also write $V_x$ to mean $V_{\{x\}}$. For a subset of vertices $\Gamma \subseteq V(G)$, let $M_\Gamma := \bigcup_{v \in \Gamma} M_v$. We say that a set $\Gamma$ of vertices (or nodes) is connected if the graph induced by $\Gamma$ is connected.

Observation 3. Let $H_{sub}$ (properly) represent a graph $G$. For any connected subset $\Gamma$ of $V(G)$, the model $M_\Gamma$ of $\Gamma$ is connected in $H_{sub}$.

Chordal Graphs and Clique Trees. A graph is chordal when it does not contain an induced $k$-vertex cycle for any $k \geq 4$. The chordal graphs are well known to be characterized as the intersection graphs of subtrees of a tree, i.e., for every chordal graph $G$, there is a tree $T$ that represents $G$ ($G$ is a $T$-graph) [6, 14, 25]. In fact, $G$ is chordal if and only if there is a tree $T$ with models $\{M_v \subseteq V(T) \mid v \in V(G)\}$ where, for each node $x \in V(T)$, $V_x$ is a maximal clique of $G$ and for every node $y \in V(T)$ with $y \neq x$, $V_y \neq V_x$ [6, 14, 25]. These special representations of $G$ are called clique trees, and one can be constructed in linear-time [3, 13]. Note that chordal graphs have a simpler linear-time recognition algorithm [23]. Finally, every chordal graph $G$ has at most $n$ maximal cliques where $n = |V(G)|$ and the sum of the sizes of the maximal cliques of $G$ is $O(n + m)$ [15]. In particular, the total size of a clique tree of $G$ is $O(n + m)$. Clearly, the latter two properties of chordal graphs also apply to (proper) $T$-graphs independently of $T$, and we will use them implicitly throughout our discussions.

Each chordal graph $G$ is also a proper $T$-graph for a tree $T$. Namely, if a tree $T$ represents $G$ via models $\{M_v \mid v \in V(G)\}$, any tree $T'$ built from $T$ as follows properly represents $G$:

Extend each model $M_v$ by a new node $x_v$ and add $\{x_v, x_v\}$ to $E(T)$ for some $x \in M_v$. 

3 Compact Representations of Proper T-Graphs

In this section we introduce an analogue of clique trees for proper T-graphs. Ideally, G being a proper T-graph would imply a clique tree with the topology of T representing G which satisfies properness; in other words: a re-subdivision $T_{\text{sub}}$ of T with models satisfying properness (i.e., forbidding $M_u \subseteq M_v$ for every pair $u, v \in V(G)$) such that every node $x$ represents a unique maximal clique $V_x$. However, a proper tree-representation of a graph G may use a lot of nodes just to ensure that the models $M_u$ and $M_v$ obey properness; which is already the case for $K_2$ and its interval representation. Fortunately we may guarantee that almost all nodes represent a unique maximal clique by relaxing the properness condition. Instead of forbidding containment, we require that when $M_u$ intersects $M_v$, there is a place where $M_u$ may be extended (as needed) to break containment. That place is an edge $\{x, y\}$ in the tree $T_{\text{sub}}$ where $u$ strongly escapes $v$, that is, $u, v \in V_x$ and $v \notin V_y$. Actually, a weaker version of escape suffices. A vertex $u$ escapes $v$ if $u \in V_x$ and $v \notin V_y$.

Definition 4. Let a tree $T_{\text{sub}}$ with models $\{M_u \mid u \in V(G)\}$ represent a connected graph G. We say that $T_{\text{sub}}$ is a compact representation of G if

(C1) for every leaf node $x \in V(T_{\text{sub}})$, $V_x = \emptyset$,
(C2) there is a bijection between the non leaves of $V(T_{\text{sub}})$ and the maximal cliques $C(G)$, and
(C3) for every ordered pair $(u, v)$ with $u, v \in V(G)$, there is an edge $\{x, y\} \in E(T_{\text{sub}})$ where $u$ escapes $v$.

Observation 5 (∗). Let a tree $T_{\text{sub}}$ with models $\{M_u \mid u \in V(G)\}$ represent a connected graph G and satisfy condition (C1). For any vertices $u, v$ of G, $u$ and $v$ satisfy the condition (C3) if and only if $u$ and $v$ satisfy condition (C3′) if $M_u \cap M_v \neq \emptyset$, then $u$ strongly escapes $v$.

Note that, the non-leaves of a compact representation are in one-to-one correspondence with the maximal cliques $C(G)$. Namely, we identify the non-leaves with the maximal cliques, which implicitly defines the models. Thus, we often omit the explicit statement of the models.

Observation 6. Let G be a connected graph. For any compact representation $T_{\text{sub}}$ of G,

1. for every distinct non-leaves $x, y \in V(T_{\text{sub}})$ there is a vertex $u \in V_x \setminus V_y$, and
2. for every edge $\{x, y\} \in E(T_{\text{sub}})$ of non-leaves $x, y$, there is a vertex $u \in V_x \cap V_y$.

We (constructively) show that properness and compactness are essentially equivalent. To obtain compactness from properness, we carefully contract edges where a node was used solely to assure properness. This can involve contracting edges of T when the vertex sets of the nodes of an edge are comparable, e.g., if they are the same maximal clique. To obtain properness from compactness, we subdivide the tree and appropriately extend the models.

Theorem 7 (∗). For any connected graph G and tree $T \neq K_1$, the graph G is a proper T-graph if and only if there is re-subdivision $T_{\text{sub}}$ of T that is a compact representation of G.

Thus, instead of finding a proper representation, we search for a compact representation. The actual “properness” is hidden in the condition (C3), and we may refer to this condition as properness. See also examples in Figure 1.

Our algorithm further relies on the following property of the models $M_\Gamma$ of the (connected) components of $G - V_y$ for some non-leaf $y$; see also Figure 2(a). Let $\Gamma(y)$ (w.r.t. graph G) be the vertex sets of the components of $G - V_y$. We note that $N(\Gamma) \subseteq V_y$ for every $\Gamma \in \Gamma(y)$. Let a node $y$ be an edge if it is a neighbor of a leaf or if it is a branching node.
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Figure 1 (a) A proper $K_{1,3}$-graph. Triple $(\ell, y, r)$ is surrounding. Any representation positions $y$ between $\ell$ and $r$. Component $V_\ell \setminus V_y$ complies with condition (2B). Further, edge $\{z, y\}$ may be replaced by edge $\{z, \ell\}$ or $\{z, r\}$. (b) A proper $K_{1,3}$-graph. Triple $(\ell, y, r)$ is not surrounding. A “private” vertex in $V_\ell \setminus N(\Gamma_\ell) \cup N(\Gamma_r)$ contradicts condition (2A). Indeed, any of $\ell, y, r$ may realize the branching node. (c) Triple $(\ell, y, r)$ is surrounding. For $\{\Gamma_\ell, \Gamma_r\} = \Gamma(y)$ condition (1) allows private vertices in $V_y$; otherwise, this proper interval graph would have no surrounded nodes.

Lemma 8. Let $G$ be a connected graph. For any compact representation $T_{\text{sub}}$ of $G$ and any non-leaf node $y \in V(T_{\text{sub}})$,
1. $\{y\}$ and $M_\Gamma$ for $\Gamma \in \Gamma(y)$ partition the non-leaves of $T_{\text{sub}}$, and
2. each partition $M_\Gamma$ contains an eye, hence $|\Gamma(y)| \leq |V(T)|$.

4 Finding a Compact Representation

In this section, we prove Theorem 1; namely, we establish our FPT algorithm. Throughout the discussion, we assume $G$ is connected, and handle disconnected graphs within the final proof. From Section 3, it suffices to check for a compact representation $T_{\text{sub}}$. In Subsection 4.1, we establish the concept of surrounded nodes, which leads, in Subsection 4.2, to the chains that necessarily form paths in any compact tree representation. We establish that the chains (composed of surrounded nodes), and the remaining non-surrounded nodes are only quadratically many in the size of the desired tree $T$. In Subsection 4.3, we formalize the way these pieces fit together as templates. Finally, Subsection 4.4 contains the algorithm establishing Theorem 1. It proceeds by enumerating candidate templates and (non-trivially) testing whether a template admits a compact representation via a bottom-up procedure.

4.1 Surrounded Nodes

We establish conditions for arbitrary nodes $\ell, y, r$ that determines the relative position of $\ell, y, r$ in any representation $T_{\text{sub}}$, a relation which we denote as $(\ell, y, r)$ surrounding. Clearly, this positioning is unlikely to be possible for every triple $(\ell, y, r)$ since this would yield a polynomial-time algorithm. However, by carefully crafting our first two requirements, we may still relatively position almost all nodes $\ell, y, r$. We only fail for a few nodes $y$, at most quadratic in the size of the host tree $T$, hence our parameter.

Definition 9. Consider non-leaves $\ell, y, r$ of $T_{\text{sub}}$. There is a component $\Gamma_\ell \in \Gamma(y)$ containing $V_\ell \setminus V_y$, likewise a component $\Gamma_r \in \Gamma(y)$ containing $V_r \setminus V_y$. Then $(\ell, y, r)$ is a surrounding triple, if the following conditions are met:
1. If $\{\Gamma_\ell, \Gamma_r\} = \Gamma(y)$, then $V_y = N(\Gamma_\ell) \cup N(\Gamma_r)$ or $N(\Gamma_\ell) \cap N(\Gamma_r) = \emptyset$;
2. if $\{\Gamma_\ell, \Gamma_r\} \subset \Gamma(y)$,
   (2A) $V_y = N(\Gamma_\ell) \cup N(\Gamma_r)$, and
   (2B) for every $\Gamma \in \Gamma(y) \setminus \{\Gamma_\ell, \Gamma_r\}$ we have: $N(\Gamma) \subseteq N(\Gamma_\ell) \cap N(\Gamma_r)$; and
3. for every $\ell' \neq \ell$ such that satisfy (1), (2A), and (2B) where $\Gamma_\ell = \Gamma_{\ell'}$ and $\Gamma_r = \Gamma_{r'}$, we have $V_{\ell'} \cap V_y \subseteq V_\ell \cap V_y$ and $V_{r'} \cap V_y \subseteq V_r \cap V_y$. 

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Note, that the definition does not depend on the considered representation $T_{\text{sub}}$. Importantly, $\Gamma_\ell \neq \Gamma_r$ is (implicitly) required by condition (1). We say that $y$ is surrounded, if a triple $(\ell, y, r)$ is a surrounding triple for some nodes $\ell, r$. See Figure 1 for examples.

For each node $y$, the connected components $\Gamma_\ell$ and $\Gamma_r$ satisfy or falsify the first two conditions independently of the precise maximal cliques $V_\ell$ and $V_r$. However, condition (3) requires $V_\ell$ and $V_r$ to be the closest ones to $V_y$. In many cases condition (3) implies that $\ell$ and $r$ directly neighbor $y$. In fact, for a surrounded node $y$, there are sets of nodes $L$ and $R$ that exactly localize the nodes $\ell$ and $r$ forming a surrounding triple with $y$. Formally, $L, R$ are $y$-guards: A set of non-leaves $L \subseteq V(T_{\text{sub}})$ is a $y$-guard if $L \cup \{y\}$ is connected, and $y$ is adjacent to a node $\ell \in L$ such that $\{\ell\} = L$ or $\ell$ is a branching node of $T_{\text{sub}}$; see Figure 2(b).

**Lemma 10 (⋆).** Let a tree $T_{\text{sub}}$ be a compact representation of a connected graph $G$. Let $y$ be surrounded. There are distinct $y$-guards $L$ and $R$ such that $(\ell, y, r)$ is surrounding if and only if $(\ell, r) \in (L \times R) \cup (R \times L)$. Moreover, there is an $O(t n^2)$-time algorithm that determines sets $L, R$ for every surrounded node $y$; where $t = |V(T)|$ and $n = |V(G)|$.

The guards of $y$ are such distinct $y$-guards $L$ and $R$ that precisely characterize its surrounding triples. It is worth noting that a node $y$ that is surrounded by subdivision nodes has singleton guards $\{\ell\}$ and $\{r\}$, which then must be neighbors of $y$ in any representation.

Surprisingly there is also a quadratic bound in $|V(T)|$ on the number of not surrounded nodes. To show this, the main difficulty is that the conditions (2A) and (2B) fail for a subdivision node $y$ due to some remote component $\Gamma$, which is a connected component $\Gamma \in \Gamma(y) \setminus \{\Gamma_\ell, \Gamma_r\}$. Here, let $y \in T_{\text{sub}}(x, z)$ for some neighbors $x, z \in V(T)$. To cope with these remote components, we use three ingredients:

- A component $\Gamma \in \Gamma(y)$ that falsifies the conditions in question relates to $\Gamma(x)$ or $\Gamma(z)$: Its model $M_\Gamma$ must be outside of $T_{\text{sub}}[x, z]$. By examining the nodes $y'$ that have $\Gamma$ as a component, it follows that either $\Gamma \in \Gamma(x)$ or $\Gamma \in \Gamma(z)$.

- We use Lemma 8 on components $\Gamma \in \Gamma(x)$, likewise for $\Gamma(z)$: The models of the components $\Gamma \in \Gamma(x)$ partition the non-leaves $T_{\text{sub}}$, and each of its models $M_\Gamma$ contains an eye. That means that $\Gamma(x)$ contains at most $|E(T)|$ components.

- A component $\Gamma \in \Gamma(x)$ can be remote for at most one node $y$ on the path $T_{\text{sub}}(x, z)$, and hence falsify the surrounding conditions for at most one $y$ on that path. This yields a simple $2|E(T_{\text{sub}})|$-bound for not surrounded nodes on that path; see also Figure 2(c).

By incorporating these ideas in a more careful manner we obtain the following bound:

**Lemma 11 (⋆).** Let subdivision $T_{\text{sub}}$ of a tree $T$ be a compact representation of a connected graph $G$. There are at most $|E(T)|^2 + 1$ non-leaves of $V(T_{\text{sub}})$ that are not surrounded.
4.2 Chains: Paths in any Representation

As observed previously, singleton guards \{ℓ\} and \{r\} of a node \(y\) must neighbor \(y\). If a path of nodes \(y_1, \ldots, y_s\) is made of aligned guards, i.e., \(\{y_{i-1}\}\) and \(\{y_{i+1}\}\) are the guards of \(y_i\), then it is a path in any representation \(T_{\text{sub}}\). In this subsection we define such paths as chains. A chain captures a maximum length path \(y_1, \ldots, y_s\) with aligned guards. They also include the initial and final guard \(Y_0\) and \(Y_{s+1}\), their terminals.

\[ \mathcal{Y} = \langle Y_0, \{y_1\}, \ldots, \{y_s\}, Y_{s+1} \rangle \]
where \(y_i\) has guards \(Y_{i-1}\) and \(Y_{i+1}\) for every \(i \in [s]\); and where \(Y_i := \{y_i\}\) for \(i \in [s]\).

To avoid lengthy statements, let us implicitly use \(Y_i := \{y_i\}\) from now on. Let \(I(\mathcal{Y}) = \{y_1, \ldots, y_s\}\) be the set of inner nodes of a chain \(\mathcal{Y}\). Let \(\mathcal{H}(G)\) be the set of chains of a (connected) graph \(G\).

By Lemma 10 such a chain implies that \(y_1, \ldots, y_s\) is a path in any representation \(T_{\text{sub}}\). Also there are unique realizations of the terminals \(y_0 \in Y_0 \cap N_{T_{\text{sub}}}(y_1)\) and \(y_{s+1} \in Y_{s+1} \cap N_{T_{\text{sub}}}(y_s)\) in a given \(T_{\text{sub}}\). Let \(y_0y_1 \ldots y_Yy_{s+1}\) be the corresponding path of \(\mathcal{Y}\) in the tree \(T_{\text{sub}}\).

The terminals define how the chains may attach to each other. In the very simple case a terminal \(Y_0\) may only consist of a single non-surrounded node, and hence any other chain must attach to that node. Otherwise, as we show later, only the following option remains: Terminal \(Y_0\) contains some surrounded node \(y'_0\) which is part of another chain \(\langle Y'_0, \ldots, y''_{s+1} \rangle\). Interestingly \(Y_0\) then contains the whole path \(y'_0, \ldots, y''_s\). This allows us to freely change the attachment of the chain \(\langle Y_0, \{y_1\}, \ldots \rangle\) to the chain \(\langle \ldots, \{y'_i\}, \ldots \rangle\) without breaking the connectivity of the representation, though possibly the properness. Similarly, the intersection \(V_x \cap V_{\bar{x}}\), for an outside-of-path node \(x\) is equal for every \(i \in [s]\), and therefore may be reattached in the same sense.

\[ \text{Lemma 13 (⋆). Consider a chain } \langle Y_0, \ldots, \{y_i\}, \ldots, Y_{s+1} \rangle. \text{ A neighbor } x \in N(y_i) \setminus (Y_{i-1} \cup Y_{i+1}) \text{ has } V_x \cap V_{y_i} = V_x \cap V_{y_j}, \text{ for every } j \in [s]. \text{ Furthermore, } Y' \cap I(\mathcal{Y}) \in \{\emptyset, I(\mathcal{Y})\} \text{ for every terminal } Y' \text{ of any chain.} \]

In the following subsection we aim for a bound on the number of chains. We note here that chains behave in a reasonable way: Each surrounded node \(y\) is part of exactly one chain, because otherwise it contradicts the classification by \(y\)-guards as seen in Lemma 10. Clearly, a chain does not contain a node more than once, since \(y_i\)-guards \(Y_{i-1}, Y_{i+1}\) are in different subtrees of \(y_i\), for every \(i \in [s]\). As the next step, we observe that terminals only consist of either a not-surrounded node or a non-singleton guard, i.e., are from
\[ \mathcal{S}(G) := \{y_0 \mid y_0 \in C(G) \text{ is not surrounded}\}, \text{ or} \]
\[ \mathcal{U}(G) := \{Y_0 \mid Y_0 \text{ is guard of some } y_1 \in C(G), |Y_0| > 1\}. \]

\[ \text{Lemma 14 (⋆). Let } T_{\text{sub}} \text{ be a compact representation of a connected graph } G. \text{ Then every terminal } Y_0, Y_{s+1} \text{ of a chain of } G \text{ is part of } \mathcal{S}(G) \text{ or } \mathcal{U}(G). \]

Further, let the family of inner nodes be \(\mathcal{I}(G) := \{I(\mathcal{Y}) \mid \mathcal{Y} \in \mathcal{H}(G)\}\). Note here that \(\mathcal{I}(G) \cup \mathcal{S}(G)\) partition the maximal cliques \(C(G)\).
Thus, the chain \( Y \) is mapped to path \( \lambda_0 \lambda_1 \lambda_2 \). The chain \( Y' \) is mapped to the single edge path \( \lambda_1 \lambda \) (an edge despite the gap in the picture). We have \( \bar{t}^0(\lambda_1) = I(Y) \), and indeed any attachment of \( Y' \) to an inner node of \( Y \) is possible. For the chain \( Y' \) a mapping to single edge suffices since here it is realized by a path of subdivision nodes. c) A sample root-ordering (colors mark the mapping of \( \bar{t}^0 \)), and resulting orientation of chains \( Y' \) and \( Y \). Here \( \lambda_2 \) (mapped to \( Y_3 \)) is a tie-breaker for \( Y' \).

4.3 Template: Fixing the Topology of Chains

The set of chains \( \mathcal{H}(G) \) of a (connected) graph \( G \) already considerably prescribes many paths that are present in any proper representation \( T_{\text{sub}} \) of \( G \). What remains are two problems of a more global flavor: For a chain there may be a vast range of possible connections. Simultaneously we have to assure properness, i.e., that any vertices \( u \) and \( v \) escape each other. To cope with these tasks we define a preliminary representation, a template. A template considerably fixes the topology of a tree \( T_{\text{sub}} \) representing \( G \). It narrows down the possible representations such that we can focus on the properness. At the same time, our final algorithm has to guess a template, thus its possibilities should be bounded by our parameter, the size of \( T \).

To fix the relative positions of chains, a template locates the terminals of a chain, \( Y_0 \) and \( Y_{s+1} \), on some template tree \( T^0 \). A concrete realization \( T_{\text{sub}} \) of that template is a subdivision of \( T^0 \). It realizes a chain between its terminals as prescribed by the template. More precisely, \( t^0 \) maps the nodes \( \lambda \) of \( T^0 \) to the terminals of chains. To avoid ambiguity, let \( \bar{t}^0(\lambda) \) not map to a mere terminal \( Y_0 \), if \( Y_0 \in \overline{\mathcal{U}}(G) \) (as it may be huge), but narrow down the mapping to some set of inner nodes of \( \mathcal{I}(G) \). In other words we fix the neighborhood of a chain on the “chain-level”. Note that any \( Y_0 \in \overline{\mathcal{U}}(G) \) is a superset of some set of inner nodes, as seen in Lemma 13. For convenience, let us also fix a mapping \( h^0 \) of chains \( \{Y_0, \{y_1\}, \ldots, \{y_s\}, Y_{s+1}\} \) onto \( T^0 \): Let \( h^0 \) map to paths \( \lambda_0, \ldots, \lambda_{s+1} \) in \( T^0 \), which should be conforming with the terminals, which is \( \bar{t}^0(\lambda_0) \subseteq Y_0 \) and \( \bar{t}^0(\lambda_{s+1}) \subseteq Y_{s+1} \). If the chain does not contain any branching node, it suffices to represent the terminals. Then \( h^0 \) maps simply to the single-edge path \( \lambda_0, \lambda_{s+1} \) (for example \( Y' \) in Figure 3(a),(b)). In the other extreme, every inner node may be a branching node (respectively used as a terminal of another chain), thus possibly \( s' = s \).

In more detail, a chain \( Y \) may correspond to a path \( y_0 \ldots y_{s+1} \) with an inner branching node \( y'_0 \) which is the endpoint of a path \( y'_0 y'_1 \ldots \) corresponding to another chain \( Y' = (Y'_0, \ldots, \ldots) \). Thus, the chain \( Y \) must be mapped to a path with an inner node \( \lambda_j \) that is an endpoint of the path \( \lambda_j, \lambda'_1, \ldots, \lambda'_{s+1} \) that is the image of the other chain \( Y' \) (for example \( Y' \) in Figure 3a,b)). As seen before, then \( I(Y) \subseteq Y'_0 \in \overline{\mathcal{U}}(G) \). Hence, the mapping of that terminal is \( \bar{t}^0(\lambda_j) = I(Y) \). We require this behavior for inner nodes like \( \lambda_j \).
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Definition 15. Let $G$ be a connected chordal graph. A template of a tree (w.r.t. $G$) is a triple $(T^0, t^0, h^0)$ where

- $T^0$ is a re-subdivision of $T$,
- $t^0$ is a mapping of the non-leaves of $T^0$ to $\overline{S}(G) \cup I(G)$,
- $h^0$ is a bijection of the chains $\mathcal{H}(G)$ to an edge-disjoint set of non-trivial (i.e., containing at least one edge) paths between non-leaves of $T^0$, and
- for every chain $(Y_0, \ldots, Y_{s+1}) \in \mathcal{H}(G)$ mapped to a path $\lambda_0, \ldots, \lambda_{s'+1}$ we have that $t^0(\lambda_0) \subseteq Y_0$ and $t^0(\lambda_{s'+1}) \subseteq Y_{s+1}$ and $t^0(\lambda_i) = I(Y)$ for every $i \in [s']$.

Consider a tree $T_{\text{sub}}$ where each non-leaf $y$ is identified with a maximal clique $V_y$. Then $T_{\text{sub}}$ realizes the template $(T^0, t^0, h^0)$ if $T_{\text{sub}}$ results from subdividing $T^0$ and $V_y \in t^0(\lambda)$ for every non-leaf $\lambda$ of $T^0$.

Notably the image of $h^0$ does not necessarily cover every edge between non-leaves. Namely, non-surrounded nodes $y$ and $y'$ might be neighbors. The next lemma establishes that every tree $T_{\text{sub}}$ that is a compact representation realizes some template, as we intended.

Lemma 16 (⋆). If $T_{\text{sub}}$ is a re-subdivision of a tree $T$ and a compact representation of a connected graph $G$, then $T_{\text{sub}}$ realizes some template $(T^0, t^0, h^0)$ of $T$.

As formalized in the next lemma, we can enumerate the possible templates in FPT, since the number of chains is quadratically bounded in $|V(T)|$.

Lemma 17 (⋆). There are $2^{O(t^2 \log t)}$ possible templates of a tree $T$ w.r.t. a connected chordal graph $G$, which can be enumerated in time $2^{O(t^2 \log t)} \cdot n^3$; where $t = |V(T)|$, $n = |V(G)|$.

4.4 Normalized Representation: Achieving Properness

We now consider a fixed template and focus on the properness. The remaining leeway is to locally change the branching nodes of a particular chain. We use a construction which only fails if the considered template does not allow a compact representation. The result is a normalized representation.

Any representation $T_{\text{sub}}$ can be normalized by a bottom-up process: Move each branching node $y_i$ up as much as possible within the local subtree, i.e. as long as the subtree remains compact. By moving up, we mean replacing $y_i$ by $y_j$ as a branching node that is closer to a global root in the chain. The set of nodes that potentially replace a node $y_i$ with a linear fashion, and hence allow this greedy approach. Thus we may assume that a normalized representation exists for a yes-instance. Our algorithm though has to construct a representation from scratch. By incorporating this idea in a more careful manner we may assemble each subtree of a normalized $T_{\text{sub}}$ bottom-up. Here we attach the inductive subtrees in the most conservative way: then the same normalization step as before yields the desired new subtree. Again, the linear behavior of the potential replacements enable this greedy approach.

To start, let us define the root of a template. Since chains may not “align” towards a picked root $r_1$, we have to work with additional tie-breakers $r_2, r_3, \ldots$.

Definition 18. A root-ordering $\bar{r}$ is an ordering $\bar{r}_1, \bar{r}_2, \ldots$ of nodes $V(T^0) \cap \{\lambda | t^0(\lambda) \in \overline{S}\}$.

The specific root-ordering will not be of importance and we may pick one arbitrarily. We assume in the following that every tree and template comes with a root-ordering. See Figure 3 for an example.

Definition 19. A root-ordering $\bar{r}$ and a template $(T^0, t^0, h^0)$ define an orientation for every chain $Y = (Y_0, \ldots, Y_{s+1})$ as follows. Let $h^0(Y)$ map to a path in $T^0$ with end nodes $\lambda_0$ and $\lambda_{s+1}$ where $t^0(\lambda_0) \subseteq Y_0$ and $t^0(\lambda_{s+1}) \subseteq Y_{s+1}$. Let $k$ be the smallest index such that $(\lambda_0, \lambda_{s+1}, \bar{r}_k)$ or $(\bar{r}_k, \lambda_0, \lambda_{s+1})$ is $T^0$-ordered. If $(\lambda_0, \lambda_{s+1}, \bar{r}_k)$ is $T^0$-ordered, then $Y$ is oriented towards $Y_{s+1}$, which we denote by writing $(Y_0, \ldots, Y_{s+1})^{\bar{r}}$. 
We observe the following for a chain root-ordering $\rho$. Let $R(y_i, y_j)_{T_{sub}}$ be the tree resulting from replacing branching node $y_i$ by $y_j$. Its local version is $\rho[y_i, y_j]_{T_{sub}}$. The models living in the more restrict subtree $\rho[y_i, y_j]_{T_{sub}}$ are critical: Their properness is at stake. We define the possible replacements of a node $y_i$ resulting in a proper representation as the potential $\Phi(T_{sub}, y_i)$.

**Definition 20.** Let $\bar{\rho}$ be a root-ordering. Consider a branching node $y_i \in V(T_{sub})$ and its chain $\langle Y_0, \ldots, \{y_i\}, \ldots, \{y_j\}, \ldots, Y_{s+1} \rangle^\rho$ where $y_0$ realizes $Y_0$. For integers $i \leq j < s$, let

- $R(y_i, y_j)_{T_{sub}}$ be the tree $T_{sub}$ where $y_i$ replaces $y_j$ as a branching node, i.e., edge $\{y_j, z\}$ is replaced by a new edge $\{y_j, z\}$, for every node $z \in N_{T_{sub}}(y_i) \setminus (Y_{i-1} \cup Y_{j+1})$;
- $\rho[y_i, y_j]_{T_{sub}}$ be the tree consisting of the subtree of $R[y_i, y_j]_{T_{sub}}$ rooted at $y_0$ (w.r.t. global root $\bar{\rho}$) and path $y_0, \ldots, y_s$ where for every chain node $y_r \in \{y_1, \ldots, y_s\}$ and non-chain neighbor $z' \in N_{R[y_i, y_j]_{T_{sub}}} \setminus (Y_{i-1} \cap Y_{j+1})$ a new leaf node $y_{r'}$ added adjacent to $y_r$;
- $\rho^i[y_i, y_j]_{T_{sub}}$ be the tree consisting of the subtree of $R[y_i, y_j]_{T_{sub}}$ rooted at $y_0$ (w.r.t. global root $\bar{\rho}$) and path $y_0, \ldots, y_{j-1}$.

For convenience, let $\rho^i[y_i]_{T_{sub}} := \rho^i[y_i, y_i]_{T_{sub}}$ as well as $\rho[y_i]_{T_{sub}} := \rho[y_i, y_i]_{T_{sub}}$.

**Definition 21.** We define the potential $\Phi(T_{sub}, y_i)$ (w.r.t. a template $(T^0, t^0, h^0)$ and root-ordering $\bar{\rho}$) of non-leaf node $y_i$.

- For a not-surrounded node $y_i$, $\Phi(T_{sub}, y_i) = \{y_i\}$.
- For a surrounded branching node $y_i$, consider its chain $\langle \ldots, \{y_i\}, \ldots, \{y_s\}, \ldots \rangle^\rho$. The potential $\Phi(T_{sub}, y_i)$ contains every node $y_j \in \{y_1, \ldots, y_s\}$ where the tree $R[y_i, y_j]_{T_{sub}}$ is such that every vertex $u$ with model $M_u \subseteq V(\rho[y_i, y_j]_{T_{sub}})$ escapes every other vertex $v$.

A simple example is that $y_i \in \Phi(T_{sub}, y_i)$ for compact representations $T_{sub}$, as the considered replacement does nothing. In contrast, $\Phi(T_{sub}, y_i) = \emptyset$ indicates non-properness for the subtree of $y_i$. Indeed, the potential of $y_i$ captures exactly the possible replacements of $y_i$ as a branching node.

If some replacement $y_j$ of $y_i$ already is a branching node, the topology changes and $R[y_i, y_j]_{T_{sub}}$ does not realize the same template. To avoid such issues, we require (without loss of generality) a minimal representation: A tree $T_{sub}$ is minimal if there is no compact representation $T_{sub}'$ of $G$ that is a re-subdivision of $T_{sub}$ with fewer branching nodes. Clearly, if there is a representation $T_{sub}$ of $G$, we may also assume that it is minimal. In particular, the contraction would result in different candidate re-subdivision of $T$, which we consider separately.

We may compute it locally, meaning it suffices to consider the subtree $\rho[y_i]$. Since the potential $\Phi(T_{sub}, y_i)$ is a connected subsequence of $\langle y_i, \ldots, y_s \rangle$, we either view it as a set or as such a subsequence. Further, if the potential is $\langle y_i, \ldots, y_s \rangle$, then replacing $y_i$ with the last node $y_i$ makes the resulting potential at $y_i$ singleton. Finally, the potential is independent from later replacements, assuming a bottom-up (i.e., leaf-to-root) procedure.

**Lemma 22 (s).** Let $T_{sub}$ be a minimal compact representation of a connected graph $G$. We observe the following for a chain $\langle \ldots, \{y_i\}, \ldots, \{y_j\}, \ldots, \{y_s\} \rangle^\rho$ for $i \leq j \leq s$:
1. If $y_j \in \Phi(T_{sub}, y_i)$, then $R[y_i, y_j]_{T_{sub}}$ is a minimal compact representation of $G$.
2. locality, $\Phi(T_{sub}, y_i) = \Phi(\rho[y_i]_{T_{sub}}, y_i)$.
3. connectivity, $\Phi(T_{sub}, y_i)$ is connected in $T_{sub}$, and hence some subsequence $(y_i, \ldots, y_j)$.
4. linearity, $\Phi(T_{sub}, y_i) = (y_i, \ldots, y_j)$ if and only if $\Phi(R[y_i, y_j]_{T_{sub}}, y_j) = (y_j)$.
5. independence, $\Phi(R[y_i, y_j]_{T_{sub}}, x) \subseteq \Phi(T_{sub}, x)$ for every node $x \in V(T_{sub})$ where $(x, y_i, \bar{\rho})$ is $T_{sub}$-ordered.
Consider a tree $T_{\text{sub}}$ that realizes a template $(T^0, t^0, h^0)$, and has some root-ordering $\bar{r}$. We say $T_{\text{sub}}$ is normalized for a node $y$ (w.r.t. to $(T^0, t^0, h^0)$ and $\bar{r}$) if $\Phi(T_{\text{sub}}, y) = \langle y \rangle$. By the locality property, this is equivalent to $\Phi(p[y]T_{\text{sub}}, y) = \langle y \rangle$, hence it suffices to consider the local subtree. The whole tree $T_{\text{sub}}$ is normalized if it is normalized for every branching node. Now the independence of the potential as explored earlier allows normalizing any representation by a bottom-up procedure. Thus, in a yes-instance, we may assume a normalized representation.

**Lemma 23** (⋆). There is an $O(n^3)$ time algorithm that, given a connected chordal $n$-vertex graph $G$ and a template $(T^0, t^0, h^0)$, decides whether there is a minimal compact representation of $G$ that realizes $(T^0, t^0, h^0)$, and if one exists, it outputs one that is also normalized.

**Proof (Sketch).** Assuming a yes-instance, there is minimal compact representation $T'_{\text{sub}}$ of $G$ that realizes template $(T^0, t^0, h^0)$. We may also assume that $T'_{\text{sub}}$ is normalized (proven in the full version). Our algorithm outputs a representation isomorphic to $T'_{\text{sub}}$, thus a normalized one as desired. If, however, our construction fails at some point, we correctly conclude that no such representation exists. In the rest of the proof we fix an arbitrary root-ordering $\bar{r}$.

We fix an ordering $\sigma = \lambda_1, \lambda_2, \ldots$ of the non-leaf nodes of the template tree $T^0$, which follows the ordering within in a chain and otherwise is bottom-up. Pick a node $\lambda_k$ where every non-leaf child of $\lambda_k$ has been added before, and append it to the ordering. If there is a chain $(Y_0, \ldots, Y_{s' + 1})$ mapped by $h^0$ to a path of form $\lambda_0, \lambda_k, \lambda_{k', 1}, \ldots, \lambda_{k, s' + 1}$, append nodes $\lambda_{k, 1}, \ldots, \lambda_{k, s' + 1}$ as well. Then continue to picking a new node until all nodes are ordered.

For $k \geq 1$, let $T_k$ be the subtree of $T_{\text{sub}}$ induced by $\lambda_1, \ldots, \lambda_k$, every subdivision node between nodes from $\lambda_1, \ldots, \lambda_k$ and leaves neighboring $\lambda_1, \ldots, \lambda_k$. By induction over $k \geq 1$, we prove that a tree isomorphic to $T_k$ is polynomial time computable given $G$, $(T^0, t^0, h^0)$ and $\bar{r}$. Eventually this yields to a representation $T_{\text{sub}}$ isomorphic to $T'_{\text{sub}}$, thus normalized minimal compact and realizing $(T^0, t^0, h^0)$, as desired.

(Induction base, when $\lambda_k$ neighbors a leaf (w.r.t. to root $\bar{r}_1$)) The node $\lambda_k$ represents a not-surrounded node $t^0(\lambda_k) = \{\lambda_k\} \in \mathcal{S}(G)$. Then $T_k$ consists only of $\lambda_k$ adjacent to some leaf. Thus, this tree is prescribed by $(T^0, t^0, h^0)$ and hence no computation is required. The induction step where $\lambda_k \in V(T^0)$ is a node with $t^0(\lambda_k) = \{\lambda_k\} \in \mathcal{S}(G)$ is similar, and omitted here.

(Induction step $I(G)$) We consider the case where the template node $\lambda_k$ is a surrounded branching node. This means that $t^0(\lambda_k) = \{y_1, \ldots, y_s\} = I(\mathcal{Y})$ for some chain $\mathcal{Y}$. The template maps $\mathcal{Y}$ to a non-trivial path $\lambda_0, \ldots, \lambda_{s' + 1}$ in $T^0$ containing $\lambda_k$:

$$
\lambda_0 \ldots \lambda_c \lambda_k \lambda_{c'} \ldots \lambda_{s' + 1} = h^0(\{Y_0, \{y_1\}, \ldots, \{y_t\}, \ldots, \{y_s\}, Y_{s + 1}\}).
$$

For each of those inner template nodes $\lambda_{c'}$, we have $t^0(\lambda_{c'}) = \{y_1, \ldots, y_s\}$. Let us assume that $t^0(\lambda_0) \subseteq Y_0$ so that the directions of increasing indices match.

Note that $\lambda_{c'}$ is ordered before $\lambda_k$ because of how the ordering $\sigma$ is defined. Our algorithm may determine $\lambda_{c'}$ as the child in $T^0$ where $\lambda_0, \lambda_{c'}, \lambda_k, \lambda_{s' + 1}$ is $T^0$-ordered (possibly $\lambda_0 = \lambda_{c'}$). The tree $T_k$ realizes $\lambda_k$ with some inner node $y_j$ with $j \in [s]$. Our task is to determine $j$ without knowing $T_k$. Let $\lambda_{c_1}, \ldots, \lambda_{c_{s'}}$ be the (possibly non-existent, possibly containing $\lambda_{c'}$) remaining children of $\lambda_k$ in $T^0$. By the induction hypothesis, the subtrees $T_{c_1}, T_{c_1}, \ldots, T_{c_{s'}}$ are polynomial time computable.

The tree $T_{c_i}$ realizes $\lambda_{c_i}$ with some node $y_{i-1}$ for $i \in \{2, \ldots, s\}$ where $y_i \in Y_0$. Because $T_{c_i}$ is a subtree of $T_k$, this limits the possible realizations of $y_j$ to $\{y_1, \ldots, y_s\}$. Let $\bar{c}_j$ be the path $(y_{i-1}, y_1, \ldots, y_s)$. **8:12 Recognizing Proper Tree-Graphs**
Consider the adjacency \( \lambda_c, \lambda_k \). A simple case is that \( t^0(\lambda_c) = \{\lambda_c\} \in \mathcal{S}(G) \). Then in the tree \( T_k \) the two realizing nodes \( \lambda_c \) and \( \lambda_k \) must be adjacent, and we define \( c^0_i(\lambda_k) \) to be the path \( \lambda_c, \lambda_k \). Note that we define the path with a variable \( \lambda_k \) (as named to coincide with the template node since it shares the same variability). For example \( c^0_i(y_j) \) is the path contained in the tree \( T_k \) (which we aim to construct).

Otherwise, the node \( t^0(\lambda_c) \) is part of a chain with terminal \( Y_{s'+1} \) where \( t^0(\lambda_k) \subseteq Y_{s'+1} \). Now either \( \lambda_c \) is the other terminal of chain \( Y_1 \), or it is the set of inner nodes \( I(Y_1) \) and hence realized as one of them. Thus the format of the chain is either

\[
= \langle Y_0', \{y_{c_1}', \ldots, y_{c_{k-1}}', Y_{s'+1}' \} \rangle \text{ where } t^0(\lambda_c) \subseteq Y_0', \quad \text{or}
\]

\[
= \langle \{y_1, \ldots, y_{c_1}, \ldots, y_{c_{k-1}}, Y_{s'+1} \} \rangle, \quad \text{where } y_1 \text{ is the realization of } \lambda_c \text{ in the tree } T_c.
\]

Let \( c^0_i(\lambda_k) \) be the path \( (\lambda_c, y_{c_1}', y_{c_2}', \ldots, y_{c_{k-1}}', \lambda_k) \), similarly as before with variable \( \lambda_k \). For example \( c^0_i(y_j) \) is the path contained in the unknown tree \( T_k \). We define the paths \( c^0_2(\lambda_k), \ldots, c^0_{s}(\lambda_k) \) for the other children analogously. Clearly, the same observations apply.

We define the tree \( T(\lambda_k) \) similarly. Namely, \( T(\lambda_k) \) is the tree containing the subtrees \( T_{c_0}, T_{c_1}, \ldots, T_{c_s} \) together with paths \( c^0_i(\lambda_k), \ldots, c^0_{s}(\lambda_k) \) and path \( y_{s-1}, y_s, \ldots, y_s \). Then \( T_k \) is the subtree of \( T(y_j) \) rooted at \( y_j \). Thus it remains to determine \( y_j \) without knowing \( T_k \).

For that purpose, consider the tree \( T(y_j) \), the tree with the most conservative realization of \( \lambda_k \). Applying the reahng operation yields \( R[y_i, y_j]T(y_i) = T(y_j) \). Assume that node \( y_k \) of all the nodes of \( T_k \) has the smallest distance to the global root \( \bar{r} \) (the general case is handled by a slight modification to \( T(y_j) \), see full version). Then, since \( T'_{sub} \) is normalized and because of locality, we have \( \Phi(y_j) = \Phi(T'_{sub}, y_j) = \Phi(T(y_j), y_j) = \Phi(T(y_j), y_j) \).

Then by the linearity of the potential we have that \( \Phi(T(y_j), y_j) = (y_s, \ldots, y_1) \). This is how we algorithmically determine \( y_j \), assuming a yes-instance. Thus the desired tree \( T_k(y_j) \) is polynomial time computable given graph \( G \), template \( (T^0, t^0, h^0) \) and \( \bar{r} \). If our algorithm observes that \( \Phi(T(y_j), y_j) = \emptyset \) at some point, it contradicts the existence of a normalized representation \( T'_{sub} \), and our algorithm returns \( \emptyset \).

Now we outline our FPT algorithm for the parameter \( t = |V(T)| \). We assume without loss of generality that \( G \) is a chordal graph and \( T \neq K_1 \) as the problem is trivial otherwise. Note that chordality can be tested in linear time [23]. If \( G \) is not connected, each proper interval graph component always be represented using a subdivision of an edge incident to a leaf of \( T \). Thus, these components, which can be recognized in linear time [10, 11], can be excluded from the further consideration. Each of the remaining components is not a proper interval graph and, as such, contains a vertex whose model includes a branching node of \( T \). Thus, if these components number more than the number of branching nodes, \( G \) has no \( T \)-representation. Assume that this is not the case. We guess an assignment of the connected components of \( G \) to connected subtrees of \( T \) representing them. Two such subtrees may share an edge (which can be needed to represent both components of \( G \) using the end-nodes of this shared edge). Note that at most \( 2^{\mathcal{O}(t \log t)} \) possible mappings, and then we can deal with every component of \( G \) and the corresponding subtree of \( T \) separately.

From now on, we assume that \( G \) is connected. By Theorem 7, we may look for a compact representation \( T_{sub} \); further, it suffices that \( T_{sub} \) is minimal. Therefore, there is a template \( (T^0, t^0, h^0) \) that allows a representation of \( G \) as seen in Lemma 16. We compute the chains of \( G \) and try every template in time \( 2^{\mathcal{O}(t^2 \log t)} \cdot n^3 \) where \( n = |V(G)| \), as seen in Lemma 17.

Pick an arbitrary root-ordering \( \bar{r} \). Then test in polynomial time whether a minimal compact representation of \( G \) realizing this template by using Lemma 23. In a positive case, applying Theorem 7 leads to a proper representation. This implies our main result (restated here).

\textbf{Theorem 1.} There is an algorithm that, given an \( n \)-vertex graph \( G \) and a tree \( T \) with \( t \) nodes, decides whether \( G \) is a proper \( T \)-graph, and if yes, outputs a proper \( T \)-representation, in \( 2^{\mathcal{O}(t^2 \log t)} \cdot n^3 \) time.
5 Concluding Remarks and Open Problems

Our recognition algorithm for proper tree-graphs provides the following side result on proper leafage (introduced by Lin et al. [21] analogously to leafage): The proper leafage $\ell^\star$ of a chordal graph $G$ is the minimum number of leaf nodes of all trees $T$ that properly represent $G$. The side result, as in Corollary 24, is that computing the proper leafage is FPT. For the decision version, if $G$ is not a proper interval graph, we simply guess the host tree $T_{\text{sub}}$ of minimal leafage and verify properness with our algorithm from Theorem 1. Of course, it still remains open whether computing proper leafage is NP-hard.

▶ Corollary 24. Computing the proper leafage $\ell^\star$ of a chordal graph $G$ is FPT w.r.t. $\ell^\star$.

While we have shown that proper $T$-graph recognition is FPT, it remains open whether non-proper $T$-graph recognition is FPT. Perhaps most importantly, gaps remain concerning the precise conditions under which (proper) $H$-graph recognition is NP-complete for fixed $H$.

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