A note on the generalized $q$-Bernoulli measures with weight $\alpha$

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Abstract In this paper we discuss new concept of the $q$-extension of Bernoulli measure. From those measures, we derive some interesting properties on the generalized $q$-Bernoulli numbers with weight $\alpha$ attached to $\chi$.

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1. Introduction

Let $p$ be a fixed prime number. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of $p$-adic rational integers, the field of $p$-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$
be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = \frac{1}{p}$ (see [1-14]).

When we talk of $q$-extension, $q$ is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$. Throughout this paper we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$ and we use the notation of $q$-number as

$$[x]_q = \frac{1 - q^x}{1 - q},$$

(see [1-14]).

Thus, we note that $\lim_{q \to 1}[x]_q = x$.

In [2], Carlitz defined a set of numbers $\xi_k = \xi_k(q)$ inductively by

$$\xi_0 = 1, \quad (q \xi + 1)^k - \xi_k = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

(1)

with the usual convention of replacing $\xi^k$ by $\xi_k$.

These numbers are $q$-extension of ordinary Bernoulli numbers $B_k$. But they do not remain finite when $q = 1$. So he modified (1) as follows:

$$\beta_{0,q} = 1, \quad q(q \beta + 1)^k - \beta_{k,q} = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

(2)

with the usual convention of replacing $\beta^k$ by $\beta_{k,q}$.

The numbers $\beta_{k,q}$ are called the $k$-th Carlitz $q$-Bernoulli numbers.

In [1], Carlitz also considered the extended Carlitz’s $q$-Bernoulli numbers as follows:

$$\beta_{0,q}^h = \frac{h}{[h]_q}, \quad q^h(q \beta + 1)^k - \beta_{k,q}^h = \begin{cases} 1, & \text{if } k = 1, \\ 0, & \text{if } k > 1, \end{cases}$$

with the usual convention of replacing $(\beta^h)^k$ by $\beta_{k,q}^h$.

Recently, Kim considered $q$-Bernoulli numbers, which are different extended Carlitz’s $q$-Bernoulli numbers, as follows: for $\alpha \in \mathbb{N}$ and $n \in \mathbb{Z}_+$,

$$\tilde{\beta}_{0,q}^{(\alpha)} = 1, \quad q^{\alpha} (\tilde{\beta}^{(\alpha)} + 1)^n - \tilde{\beta}_{n,q}^{(\alpha)} = \begin{cases} \frac{\alpha}{[\alpha]_q}, & \text{if } n = 1, \\ 0, & \text{if } n > 1, \end{cases}$$

(3)

with the usual convention of replacing $(\tilde{\beta}^{(\alpha)})^k$ by $\tilde{\beta}_{k,q}^{(\alpha)}$ (see [3]).
The numbers $\tilde{\beta}^{(\alpha)}_{k,q}$ are called the $k$-th $q$-Bernoulli numbers with weight $\alpha$. For fixed $d \in \mathbb{Z}_+$ with $(p, d) = 1$, we set
\[
X = X_d = \lim_{\longleftarrow N} (\mathbb{Z}/dp^N \mathbb{Z}), \quad X_1 = \mathbb{Z}_p, \\
X^* = \bigcup_{0 < a < dp \atop (a, p) = 1} (a + dp \mathbb{Z}_p),
\]
\[
a + dp^N \mathbb{Z}_p = \{ x \in X \mid x \equiv a \pmod{dp^N} \},
\]
where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$.

Let $UD(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, the $p$-adic $q$-integral on $\mathbb{Z}_p$ is defined by Kim as follows:
\[
I_q(f) = \int_{\mathbb{Z}_p} f(x) \, dq(x) = \lim_{N \to \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^{N-1}} f(x) q^x, \quad (\text{see [3, 4]}). \tag{4}
\]
By (3) and (4), the Witt’s formula for the $q$-Bernoulli numbers with weight $\alpha$ is given by
\[
\int_{\mathbb{Z}_p} [x]^n q^x \, dq(x) = \tilde{\beta}^{(\alpha)}_{n,q}, \quad \text{where } n \in \mathbb{Z}_+. \tag{5}
\]
The $q$-Bernoulli polynomials with weight $\alpha$ are also defined by
\[
\tilde{\beta}^{(\alpha)}_{n,q}(x) = \sum_{l=0}^{n} \binom{n}{l} [x]^{n-l} q^{\alpha l x} \tilde{\beta}_{l,q}^{(\alpha)}. \tag{6}
\]
From (4), (5) and (6), we can derive the Witt’s formula for $\tilde{\beta}^{(\alpha)}_{n,q}(x)$ as follows:
\[
\int_{\mathbb{Z}_p} [x + y]^n q^x \, dq(y) = \tilde{\beta}^{(\alpha)}_{n,q}(x), \quad \text{where } n \in \mathbb{Z}_+. \tag{7}
\]
For $n \in \mathbb{Z}_+$ and $d \in \mathbb{N}$, the distribution relation for the $q$-Bernoulli polynomials with weight $\alpha$ are known that
\[
\tilde{\beta}^{(\alpha)}_{n,q}(x) = \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} q^a \tilde{\beta}^{(\alpha)}_{n,q^d} \left( \frac{x + a}{d} \right), \quad (\text{see [3]}). \tag{8}
\]
Recently, several authors have studied the $p$-adic $q$-Euler and Bernoulli measures on $\mathbb{Z}_p$ (see [8, 9, 11]). The purpose of this paper is to construct $p$-adic
\(q\)-Bernoulli distribution with weight \(\alpha (= p\text{-adic } q\text{-Bernoulli unbounded measure with weight } \alpha)\) on \(\mathbb{Z}_p\) and to study their integral representations. Finally, we construct the generalized \(q\)-Bernoulli numbers with weight \(\alpha\) and investigate their properties related to \(p\text{-adic } q\text{-L-functions}.\)

2. \(p\text{-adic } q\text{-Bernoulli distribution with weight } \alpha\)

Let \(X\) be any compact-open subset of \(\mathbb{Q}_p\), such as \(\mathbb{Z}_p\) or \(\mathbb{Z}_p^*\). A \(p\text{-adic}\) distribution \(\mu\) on \(X\) is defined to be an additive map from the collection of compact open sets in \(X\) to \(\mathbb{Q}_p\):

\[
\mu \left( \bigcup_{k=1}^{n} U_k \right) = \sum_{k=1}^{n} \mu(U_k) \text{(additivity)},
\]

where \(\{U_1, U_2, \ldots, U_n\}\) is any collection of disjoint compact opensets in \(X\).

The set \(\mathbb{Z}_p\) has a topological basis of compact open sets of the form \(a + p^n \mathbb{Z}_p\).

Consequently, if \(U\) is any compact open subset of \(\mathbb{Z}_p\), it can be written as a finite disjoint union of sets

\[
U = \bigcup_{j=1}^{k} (a_j + p^N \mathbb{Z}_p),
\]

where \(N \in \mathbb{Z}_+\) and \(a_1, a_2, \ldots, a_k \in \mathbb{Z}\) with \(0 \leq a_i < p^N - 1\).

Indeed, the \(p\text{-adic}\) ball \(a + p^n \mathbb{Z}_p\) can be represented as the union of smaller balls

\[
a + p^n \mathbb{Z}_p = \bigcup_{b=0}^{p-1} (a + bp^n + p^{n+1} \mathbb{Z}_p).
\]

**Lemma 1.** Every map \(\mu\) from the collection of compact-open sets in \(X\) to \(\mathbb{Q}_p\) for which

\[
\mu(a + p^N \mathbb{Z}_p) = \bigcup_{b=0}^{p-1} (a + bp^N + dp^{N+1} \mathbb{Z}_p)
\]

holds whenever \(a + p^N \mathbb{Z}_p \subset X\), extends to a \(p\text{-adic}\) distribution on \(X\).
Now we define a map $\mu_{k,q}^{(\alpha)}$ on the balls in $\mathbb{Z}_p$ as follows:

$$
\mu_{k,q}^{(\alpha)}(a + p^n\mathbb{Z}_p) = \left[ \frac{[p^n]_q^k}{[p^n]_q} q^a f_{k,q,p^n}^{(\alpha)} \right] \left( \frac{\{a\}_n}{p^n} \right),
$$

(9)

where $\{a\}_n$ is the unique number in the set $\{0, 1, \cdots, p^n - 1\}$ such that $\{a\}_n \equiv a \pmod{p^n}$.

If $a \in \{0, 1, 2, \cdots, p^n - 1\}$, then

$$
\sum_{b=0}^{p-1} \mu_{k,q}^{(\alpha)}(a + bp^n + p^{n+1}\mathbb{Z}_p) = \sum_{b=0}^{p-1} \left[ \frac{[p^n+1]_q^k}{[p^n+1]_q} q^{a+bp^n} f_{k,q,p^{n+1}}^{(\alpha)} \left( \frac{a + bp^n}{p^{n+1}} \right) \right].
$$

(10)

From (10), we note that $\mu_{k,q}^{(\alpha)}$ is $p$-adic distribution on $\mathbb{Z}_p$ if and only if

$$
\frac{[p]_{q^{(p^n)_p}}^k}{[p]_{q^{p^n}_p}} \sum_{b=0}^{p-1} q^{bp^n} f_{k, q^{(p^n)_p}}^{(\alpha)} \left( \frac{a + bp^n}{p} \right) = f_{k, q^{p^n}_p}^{(\alpha)} \left( \frac{a}{p^n} \right).
$$

Theorem 2. Let $\alpha \in \mathbb{N}$ and $k \in \mathbb{Z}_+$. Then we see that $\mu_{k,q}^{(\alpha)}(a + p^n\mathbb{Z}_p)$ is $p$-adic distribution on $\mathbb{Z}_p$ if and only if

$$
\frac{[p]_{q^{(p^n)_p}}^k}{[p]_{q^{p^n}_p}} \sum_{b=0}^{p-1} q^{bp^n} f_{k, q^{(p^n)_p}}^{(\alpha)} \left( \frac{a + bp^n}{p} \right) = f_{k, q^{p^n}_p}^{(\alpha)} \left( \frac{a}{p^n} \right).
$$

We set

$$
f_{k, q^{p^n}_p}^{(\alpha)}(x) = \tilde{\beta}_{k, q^{p^n}_p}^{(\alpha)}(x).
$$

(11)

From (9) and (11), we get

$$
\mu_{k,q}^{(\alpha)}(a + p^n\mathbb{Z}_p) = \left[ \frac{[p^n]_q^k}{[p^n]_q} q^a \tilde{\beta}_{k,q,p^n}^{(\alpha)} \left( \frac{a}{p^n} \right) \right].
$$

(12)

By (8), (12) and Theorem 2, we obtain the following theorem.
Theorem 3. Let \( \mu^{(\alpha)}_k \) be given by

\[
\mu^{(\alpha)}_k(a + dp^N \mathbb{Z}_p) = \frac{[dp^N]_q^k}{[dp^N]_q^{\alpha}} q^a \tilde{\beta}^{(\alpha)}_{k,q^p \alpha} \left( \frac{a}{dp^N} \right). \tag{13}
\]

Then \( \mu^{(\alpha)}_k \) extends to a \( \mathbb{Q}(q) \)-valued distribution on the compact open sets \( U \subset X \).

From (13), we note that

\[
\int_X d\mu^{(\alpha)}_k(x) = \lim_{N \to \infty} \sum_{x=0}^{dp^N-1} \mu^{(\alpha)}_k(x + dp^N \mathbb{Z}_p)
= \lim_{N \to \infty} \frac{[dp^N]_q^k}{[dp^N]_q^{\alpha}} \sum_{a=0}^{dp^N-1} q^a \tilde{\beta}^{(\alpha)}_{k,q^p \alpha} \left( \frac{a}{dp^N} \right). \tag{14}
\]

By (8) and (14), we get

\[
\int_X d\mu^{(\alpha)}_k(x) = \tilde{\beta}^{(\alpha)}_{k,q^p \alpha}.
\]

Therefore, we obtain the following theorem.

Theorem 4. For \( \alpha \in \mathbb{N} \) and \( k \in \mathbb{Z}_+ \), we have

\[
\int_X d\mu^{(\alpha)}_k(x) = \tilde{\beta}^{(\alpha)}_{k,q^p \alpha}.
\]

Let \( \chi \) be Dirichlet character with conductor \( d \in \mathbb{N} \). Then we define the generalized \( q \)-Bernoulli numbers attached to \( \chi \) as follows:

\[
\tilde{\beta}^{(\alpha)}_{n,\chi,q} = \int_X \chi(x) [x]_q^n d\mu_q(x)
= \frac{[d]_q^n}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \tilde{\beta}^{(\alpha)}_{n,q^a} \left( \frac{a}{d} \right). \tag{15}
\]

From (13) and (15), we can derive the following equation.
\[
\int_X \chi(x) d\mu^{(\alpha)}_{k,q}(x) = \lim_{N \to \infty} \sum_{x=0}^{dp^{N-1}} \chi(x) d\mu^{(\alpha)}_{k,q}(x + dp^N z_p)
\]
\[
= \lim_{N \to \infty} \frac{[dp^N]_q^k}{[dp^N]_q} \sum_{x=0}^{dp^{N-1}} \chi(x) q^x \tilde{\beta}^{(\alpha)}_{k,q,q^{N+1}} \left( \frac{x}{dp^N} \right)
\]
\[
= \frac{[d^k_q]}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \left\{ \lim_{N \to \infty} \frac{[p^N]_q^k}{[p^N]_q^{pa}} \sum_{x=0}^{p^{N-1}} q^x \tilde{\beta}^{(\alpha)}_{k,q,q^{N}} \left( \frac{a + x}{p^{N+1}} \right) \right\}
\]
\[
= \frac{[d^k_q]}{[d]_q} \sum_{a=0}^{d-1} q^a \chi(a) \tilde{\beta}^{(\alpha)}_{k,q} \left( \frac{a}{d} \right) = \tilde{\beta}^{(\alpha)}_{k,q,\chi}
\]

and

\[
\int_{pX} \chi(x) d\mu^{(\alpha)}_{k,q}(x) = \lim_{N \to \infty} \frac{[dp^N+1]_q^k}{[dp^N+1]_q} \sum_{x=0}^{dp^{N-1}} \chi(px) q^{px} \tilde{\beta}^{(\alpha)}_{k,q,q^{N+1}} \left( \frac{px}{dp^N+1} \right)
\]
\[
= \frac{[p^k_q]}{[p]_q} \frac{[d^k_q]}{[d]_q} \sum_{a=0}^{d-1} \chi(pa) q^{pa} \lim_{N \to \infty} \frac{[p^N]_q^k}{[p^N]_q^{pa}} \sum_{x=0}^{p^{N-1}} q^x \tilde{\beta}^{(\alpha)}_{k,q,q^{N+1}} \left( \frac{a + px}{pd^N} \right)
\]
\[
= \frac{[p^k_q]}{[p]_q} \frac{[d^k_q]}{[d]_q} \sum_{a=0}^{d-1} \chi(p) \chi(a) q^{pa} \tilde{\beta}^{(\alpha)}_{k,q,q^{pa}} \left( \frac{a}{d} \right) = \chi(p) \frac{[p^k_q]}{[p]_q} \tilde{\beta}^{(\alpha)}_{k,q,\chi,\chi}
\]

For \( \beta(\neq 1) \in X^* \), we have

\[
\int_{p_X} \chi(x) d\mu^{(\alpha)}_{k,q,1/\beta}(\beta x) = \chi \left( \frac{p}{\beta} \right) \frac{[p^k_q]}{[p]_q} \tilde{\beta}^{(\alpha)}_{k,q,\chi,\chi,\beta}
\]

and

\[
\int_X \chi(x) d\mu^{(\alpha)}_{k,q,1/\beta}(\beta x) = \chi \left( \frac{1}{\beta} \right) \tilde{\beta}^{(\alpha)}_{k,q,\chi,\chi,1/\beta}
\]

Therefore, we obtain the following theorem.
Theorem 5. For $\beta(\neq 1) \in X^*$, we have

$$
\int_X \chi(x) d\mu_{k,q}^{(a)}(x) = \tilde{\beta}_{k,\chi,q}^{(a)}.
$$

$$
\int_{pX} \chi(x) d\mu_{k,q}^{(a)}(x) = \chi(p) \frac{[p]^k}{[p]_q^\alpha} \tilde{\beta}_{k,\chi,q}^{(a)},
$$

$$
\int_{pX} \chi(x) d\mu_{k,q^{1/\beta}}^{(a)}(\beta x) = \chi(p) \frac{[p]^k}{[p]_{q^{1/\beta}}} \tilde{\beta}_{k,\chi,q^{1/\beta}}^{(a)},
$$

$$
\int_X \chi(x) d\mu_{k,q^{1/\beta}}^{(a)}(\beta x) = \chi(\frac{1}{\beta}) \tilde{\beta}_{k,\chi,q^{1/\beta}}^{(a)}.
$$

Define

$$
\mu_{k,\beta,q}^{(a)}(U) = \mu_{k,q}^{(a)}(U) - \beta^{-1} \frac{[\beta^{-1}]^k_q}{[\beta^{-1}]^\alpha_q} \mu_{k,q^{1/\beta}}^{(a)}(\beta U). \quad (16)
$$

By simple calculation, we get

$$
\int_{X^*} \chi(x) d\mu_{k,\beta,q}^{(a)}(x)
$$

$$
= \int_X \chi(x) d\mu_{k,q}^{(a)}(x) - \beta^{-1} \frac{[\beta^{-1}]^k_q}{[\beta^{-1}]^\alpha_q} \int_{pX} \chi(x) d\mu_{k,q^{1/\beta}}^{(a)}(x) \quad (17)
$$

$$
= \tilde{\beta}_{k,\chi,q}^{(a)} - \chi(p) \frac{[p]^k}{[p]_q^\alpha} \tilde{\beta}_{k,\chi,q}^{(a)},
$$

and

$$
\frac{[\beta^{-1}]^k_q}{[\beta^{-1}]^\alpha_q} \int_X \chi(x) d\mu_{k,q^{1/\beta}}^{(a)}(\beta x) = \frac{[\frac{1}{\beta}]^k_q}{[\frac{1}{\beta}]^\alpha_q} \chi(1/\beta) \tilde{\beta}_{k,\chi,q^{1/\beta}}^{(a)} - \chi(p/\beta) \frac{[\frac{p}{\beta}]^k}{[\frac{p}{\beta}]_q^\alpha} \tilde{\beta}_{k,\chi,q^{1/\beta}}^{(a)}. \quad (18)
$$

By (16), (17) and (18), we get
\[ \int_{X^*} \chi(x) \, d\mu_{k,\beta,q}(\beta x) \]
\[ = \int_{X} \chi(x) \, d\mu_{k,q}(x) - \beta^{-1} \left[ \beta^{-1} \right]_{q}^{k} \int_{pX} \chi(x) \, d\mu_{k,q^{1/\beta}}(\beta x) \]
\[ = \beta^{(\alpha)}_{k,\chi,q} - \chi(p) \left[ \frac{p}{q} \right]^{k}_{q} \beta^{(\alpha)}_{k,\chi,q^{p}} - \frac{1}{\beta} \left[ \frac{1}{\beta} \right]_{q}^{k} \chi(1/\beta) \beta^{(\alpha)}_{k,\chi,q^{1/\beta}} \quad (19) \]
\[ + \chi(p/\beta) \left[ \frac{p}{q} \right]^{k}_{q} \beta^{(\alpha)}_{k,\chi,q^{p/\beta}}. \]

Now we define the operator \( \chi^{y} = \chi^{y,k,\alpha;q} \) on \( f(q) \) by
\[ \chi^{y} f(q) = \chi^{y,k,\alpha;q} f(q) = \frac{[y]^{k}_{q}}{[y]_{q}} \chi(y) f(q^{y}). \quad (20) \]

Thus, by (20), we get
\[ \chi^{x,k,\alpha;q} \circ \chi^{y,k,\alpha;q} f(q) = \chi^{x,k,\alpha;q} \left( \frac{[y]^{k}_{q}}{[y]_{q}} \chi(y) f(q^{y}) \right) \]
\[ = \frac{[y]^{k}_{q}}{[y]_{q}} \chi(y) \chi(x) \left( \frac{[y]^{k}_{q}}{[y]_{q}} \chi(y) f(q^{xy}) \right) \]
\[ = \chi^{xy,k,\alpha;q} f(q^{xy}) \quad (21) \]
\[ = \chi^{xy,k,\alpha;q} f(q) = \chi^{xy} f(q). \]

Let us define \( \chi^{x} \chi^{y} = \chi^{x,k,\alpha;q} \circ \chi^{y,k,\alpha;q} \). Then we have
\[ \chi^{x} \chi^{y} = \chi^{xy}. \]

From the definition of \( \chi^{x} \), we can easily derive the following equation.
\[ (1 - \chi^{p}) \left( 1 - \frac{1}{\beta} x^{1/\beta} \right) = 1 - \frac{1}{\beta} x^{1/\beta} - \chi^{p} + \frac{1}{\beta} x^{p/\beta}. \]
Let $f(q) = \tilde{\beta}_{k,\chi,q}^{(\alpha)}$. Then we get

$$
(1 - \chi^p) \left(1 - \frac{1}{\beta} x^{1/\beta}\right) \tilde{\beta}_{k,\chi,q}^{(\alpha)} \\
= \tilde{\beta}_{k,\chi,q}^{(\alpha)} - \frac{1}{\beta} \frac{[1/\beta]_q^k}{[1/\beta]_q} \chi(1/\beta) \tilde{\beta}_{k,\chi,q}^{(\alpha)} - [p]_q^{\alpha} \chi(p) \tilde{\beta}_{k,\chi,q}^{(\alpha)} \\
+ \frac{1}{\beta} \frac{[1/\beta]_q^k}{[1/\beta]_q} \chi(p/\beta) \tilde{\beta}_{k,\chi,q}^{(\alpha)}.
$$

(22)

By (19) and (22), we obtain the following equation:

$$
\int_{X^*} \chi(x) d\mu_{k,\beta,q}^{(\alpha)}(\beta x) = (1 - \chi^p) \left(1 - \frac{1}{\beta} x^{1/\beta}\right) \tilde{\beta}_{k,\chi,q}^{(\alpha)}
$$

where $\beta (\neq 1) \in X^*$.

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