A NEW REALIZATION OF THE COHOMOLOGY OF SPRINGER FIBERS

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Dedicated to Professor M.S. Raghunathan

1. Introduction

Fix a positive integer \( n \) and consider the algebraic group \( G = SL_n(\mathbb{C}) \) with its Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \). For any partition \( \sigma \) of \( n \), let \( X_\sigma \subset G/B \) be the associated Springer fiber, where \( B \) is the standard Borel subgroup consisting of the upper triangular matrices. By the pioneering work of Springer, its cohomology \( H^*(X_\sigma) \) with complex coefficients admits an action of the Weyl group \( S_n \). Subsequently, the \( S_n \)-algebra \( H^*(X_\sigma) \) played a fundamental role in several diverse problems. The aim of this short note is to give a geometric realization of \( H^*(X_\sigma) \). More specifically, we prove the following theorem which is the main result of this note.

**Theorem 2.** The coordinate ring \( \mathbb{C}[(N_G(T) \cdot N_{\sigma^\vee}) \cap \mathfrak{h}] \) of the scheme theoretic intersection of \( N_G(T) \cdot N_{\sigma^\vee} \) with the Cartan subalgebra \( \mathfrak{h} \) in \( \mathfrak{sl}_n(\mathbb{C}) \) is isomorphic to \( H^*(X_\sigma) \) as a graded \( S_n \)-algebra, where \( \sigma^\vee \) is the dual partition of \( \sigma \), \( T \) is the maximal torus consisting of the diagonal matrices with \( \mathfrak{h} := \text{Lie } T \), \( N_G(T) \) is its normalizer in \( G \), and \( N_{\sigma^\vee} \) is the full nilpotent cone of the Levi component of the parabolic subalgebra of \( \mathfrak{sl}_n(\mathbb{C}) \) associated to the partition \( \sigma^\vee \).

This theorem should be contrasted with the following theorem of de Concini-Procesi.

**Theorem** ([3], Theorem 4.4). The cohomology algebra \( H^*(X_\sigma) \) is isomorphic, as a graded \( S_n \)-algebra, with the coordinate ring \( \mathbb{C}[G \cdot M_{\sigma^\vee} \cap \mathfrak{h}] \) of the scheme theoretic intersection of \( \mathfrak{h} \) with the closure of the \( G \)-orbit of \( M_{\sigma^\vee} \), where \( M_{\sigma^\vee} \) is a nilpotent matrix associated to the partition \( \sigma^\vee \).

The proof of our theorem is based on a certain characterization of the \( S_n \)-algebra \( H^*(X_\sigma) \) given in Proposition [1], which seems to be of independent interest. The proof of this proposition is based on some works of Bergeron-Garsia, Garsia-Haiman and Garsia-Procesi revolving around the so called \( n! \) conjecture.

Finally, it should be mentioned that the direct analogue of our theorem (and also the above theorem of de Concini-Procesi) for other groups does
not hold in general. However a partial generalization of the result of de Concini-Procesi is obtained by Carrell [2].

2. Notation and Preliminaries

Fix a positive integer \( n \) and consider the algebraic group \( G = \text{SL}_n(\mathbb{C}) \). By \( \mathcal{N} \) we denote the full nilpotent cone inside the Lie algebra \( \mathfrak{sl}_n(\mathbb{C}) \) of \( G \). The group \( G \) acts on \( \mathcal{N} \) by the adjoint action with finitely many orbits. An orbit is determined uniquely by the sizes of the Jordan blocks of any element in the orbit, and this sets up a one to one correspondence between the partitions of \( n \) and the \( G \)-conjugacy classes inside \( \mathcal{N} \). For each partition \( \sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0 \) of \( n \), we let \( M_\sigma \) denote the nilpotent matrix in the Jordan normal form with blocks of sizes \( \sigma_0, \sigma_1, \ldots, \sigma_m \) along the diagonal in the stated order and starting from the upper left corner.

Let \( B \) denote the Borel subgroup of \( G \) consisting of the upper triangular matrices and let \( T \) denote the group of diagonal matrices in \( G \). The Lie algebras of \( B \) and \( T \) will be denoted by \( \mathfrak{b} \) and \( \mathfrak{h} \) respectively. For any partition \( \sigma \) of \( n \) we let \( X_\sigma \) denote the closed subset (called the Springer fiber)

\[
X_\sigma := \{ gB \in G/B : \text{Ad}(g^{-1})M_\sigma \in \mathfrak{b} \}
\]

of \( G/B \). This can also be identified with the set of Borel subalgebras of \( \mathfrak{sl}_n(\mathbb{C}) \) containing \( M_\sigma \) or with certain fibers of the Springer resolution of the nilpotent cone.

The singular cohomology ring \( H^*(X_\sigma) = \text{H}^*(X_\sigma, \mathbb{C}) \) with complex coefficients has an action of the symmetric group \( S_n \) on \( n \)-letters, the well known Springer representation. It is known that \( H^*(X_\sigma, \mathbb{C}) \) is zero in odd degrees, so in the following we will consider it as a (commutative) graded algebra under rescaled grading by assigning degree \( i \) to the elements of degree \( 2i \). By the Springer correspondence, the top degree part \( H^d_\sigma(X_\sigma) \) is an irreducible \( S_n \)-module.

For the partition \( \sigma_0 : 1 \geq 1 \geq \cdots \geq 1 \) of \( n \), the variety \( X_{\sigma_0} \) coincides with \( G/B \). Thus, in this case, one may \( S_n \)-equivariantly identify \( H^*(X_{\sigma_0}) \) with the covariant ring \( \mathbb{C}[Z_1, \ldots, Z_n]/I \), where \( I \) is the ideal generated by the elementary symmetric functions in the variable \( Z_1, \ldots, Z_n \). For a general partition \( \sigma \) of \( n \), the natural map

\[
H^*(G/B) \to H^*(X_\sigma)
\]

is a surjective \( S_n \)-equivariant map [3]. This also follows from the result of de Concini-Procesi mentioned in the introduction.

2.1. The algebra \( A_\sigma \). For any partition \( \sigma : \sigma_0 \geq \sigma_1 \geq \cdots \geq \sigma_m > 0 \) of \( n \), let \( D_\sigma \) be the set of pairs of nonnegative integers \((i,j)\) satisfying \( i < \sigma_j \). Then \( D_\sigma \) consists of \( n \) elements and we fix an ordering \( \{(i_s, j_s)\}_{s=1,2,\ldots,n} \) of these. Define the polynomial

\[
\Delta_\sigma = \det[X_\sigma^{i_s}Y_\sigma^{j_s}]_{1 \leq s, t \leq n} \in R_n,
\]
where $R_n$ is the polynomial ring $\mathbb{C}[X_1, \ldots, X_n, Y_1, \ldots, Y_n]$. Observe that, up to a sign, $\Delta_\sigma$ does not depend on the choice of the ordering of the elements in $D_\sigma$.

The group $S_n$ acts on $R_n$ by acting in the natural way on the two sets of variables $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ diagonally. We think of $R_n$ as a bigraded $S_n$-module by counting the degrees in the two sets of variables separately. Define a bigraded $S_n$-equivariant ideal in $R_n$ by:

$$K_\sigma = \{ f \in R_n : f(\partial_{X_1}, \ldots, \partial_{X_n}, \partial_{Y_1}, \ldots, \partial_{Y_n})\Delta_\sigma = 0 \},$$

where $\frac{\partial}{\partial X_i}$ and $\frac{\partial}{\partial Y_j}$ are the usual differential operators on $R_n$.

Define now a bigraded $S_n$-algebra $A_\sigma := R_n/K_\sigma$. This algebra was introduced by Garsia-Haiman who conjectured that $A_\sigma$ has dimension $n!$ for any partition $\sigma$ of $n$. This conjecture, which was called the $n!$-conjecture, is now proved by Haiman.

Let $d_1$ (resp. $d_2$) be the $X$-degree (resp. $Y$-degree) of $\Delta_\sigma$. Then, it is easy to see that, the bigraded component $A_\sigma^{(d_1,d_2)}$ is one dimensional and, moreover, if $A_\sigma^{(e_1,e_2)} \neq 0$, then $e_1 \leq d_1, e_2 \leq d_2$. Clearly,

$$d_1 = \sum_{(i,j) \in D_\sigma} i, \quad d_2 = \sum_{(i,j) \in D_\sigma} j = \sum_{s=0}^\ell \left( \sigma'^\vee_s \right),$$

where $\sigma'^\vee : \sigma'^\vee_0 \geq \sigma'^\vee_1 \geq \cdots \geq \sigma'^\vee_\ell > 0$ is the dual partition.

By definition, it follows easily that $A_\sigma$ is Gorenstein (see, e.g., [4], Exercise 21.7), and hence that $A_\sigma$ has a unique minimal nonzero ideal $A_\sigma^{(d_1,d_2)}$.

As explained in [4], Section 3.1, the following theorem follows from the results in [1] and [3].

**Theorem 1.** The subalgebra of $A_\sigma$ generated by the images of $X_1, \ldots, X_n$ is $S_n$-equivariantly isomorphic to $H^*(X_\sigma)$. Similarly, the subalgebra of $A_\sigma$ generated by the images of $Y_1, \ldots, Y_n$ is $S_n$-equivariantly isomorphic to $H^*(X_\sigma)$. If we assign degree 1 to all the elements $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, then both of these isomorphisms are graded algebra isomorphisms.

3. A GEOMETRIC REALIZATION OF $H^*(X_\sigma)$

In this section we give a new geometric realization of $H^*(X_\sigma)$. Recall that the socle of a ring is defined to be the sum of all its minimal nonzero ideals.

Then, with the notation from the previous section, we have the following:

**Lemma 1.** The top degree $d_\sigma$ of $H^*(X_\sigma)$ is equal to $d_2$. Moreover, the socle of $H^*(X_\sigma)$ coincides with the top degree part $H^{d_2}(X_\sigma)$.

**Proof.** Let $z$ denote any nonzero homogeneous element in $H^*(X_\sigma)$. By Theorem 1 we may regard $z$ as the image in $A_\sigma$ of a homogeneous polynomial $f$ in the variables $Y_1, \ldots, Y_n$. As $A_\sigma$ is Gorenstein, we can find a monomial

$$g = X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_n^{\beta_n}$$

such that $z = \Delta_\sigma g$. Then $\Delta_\sigma g$ is a degree $d_\sigma$ monomial in $A_\sigma$, and $H^{d_\sigma}(X_\sigma)$ is the image of $\Delta_\sigma g$ in $H^*(X_\sigma)$.
such that the image of \( f \cdot g \) in \( A_\sigma \) is nonzero and has the maximal degree, i.e., has bidegree \((d_1, d_2)\). But then the image of \( f' := fY_1^{\beta_1} \cdots Y_n^{\beta_n} \) in \( A_\sigma \) is nonzero and, of course, lies in the subalgebra generated by the images of \( Y_1, \ldots, Y_n \). In particular, by Theorem 2, the image of \( f' \) corresponds to a nonzero element in \( H^{d_2}(X_\sigma) \) which equals the product \( z \cdot z' \) for some \( z' \) in \( H^*(X_\sigma) \).

This proves that \( d_2 \) equals the top degree of \( H^*(X_\sigma) \) and that any nonzero element of \( H^*(X_\sigma) \) can be multiplied by an element of \( H^*(X_\sigma) \) to produce a nonzero element in the top degree. This immediately implies the desired result. \( \square \)

The above lemma provides us with the following characterization of the algebra \( H^*(X_\sigma) \).

**Proposition 1.** Let \( K \) be a graded algebra with an action of \( S_n \) such that there exists a surjective \( S_n \)-equivariant graded algebra homomorphism \( \phi : H^*(X_\sigma) \to K \). Assume further that the top degree of \( K \) is \( d_\sigma \). Then \( \phi \) is an isomorphism.

*Proof.* Assume that \( \phi \) is not injective. Then \( \ker(\phi) \) will meet the socle of \( H^*(X_\sigma) \) nontrivially. Thus, by Lemma 1, the degree \( d_\sigma \) part \( \ker^{d_\sigma}(\phi) \) of \( \ker(\phi) \) is nonzero. But \( \ker^{d_\sigma}(\phi) \) is a submodule of the irreducible \( S_n \)-module \( H^{d_\sigma}(X_\sigma) \) and hence \( \ker^{d_\sigma}(\phi) \) coincides with \( H^{d_\sigma}(X_\sigma) \). This is a contradiction, since the top degree of \( K \) is equal to \( d_\sigma \) by assumption. \( \square \)

Let \( Z_T(M_\sigma) \) denote the centralizer of \( M_\sigma \) in \( T \), and let \( L_\sigma \) denote the centralizer in \( G \) of the group \( Z_T(M_\sigma) \). In other words, \( L_\sigma \) is the set of block diagonal matrices in \( G \) with blocks of sizes \( \sigma_0, \sigma_1, \ldots, \sigma_m \). Then \( L_\sigma \) is a reductive group with a Borel subgroup \( B_\sigma = B \cap L_\sigma \). Moreover, \( M_\sigma \) is a regular nilpotent element in the Lie algebra of \( L_\sigma \) and consequently the nilpotent cone \( N_\sigma \) of the Lie algebra of \( L_\sigma \) coincides with the closure \( L_\sigma M_\sigma \) of the \( L_\sigma \)-orbit of \( M_\sigma \) under the adjoint action. In the following, \( N_G(T) \) will denote the normalizer of \( T \) in \( G \).

**Lemma 2.** The coordinate ring \( \mathbb{C}[(N_G(T) \cdot N_\sigma) \cap \mathfrak{h}] \) of the scheme theoretic intersection of the \( N_G(T) \)-orbit of \( N_\sigma \) with \( \mathfrak{h} \) is a graded \( S_n \)-algebra.

*Proof.* The multiplication action of \( \mathbb{C}^* \) on the Lie algebra \( sl_n(\mathbb{C}) \) keeps \( N_G(T) \cdot N_\sigma \) and \( \mathfrak{h} \) stable. This defines the desired grading on \( \mathbb{C}[(N_G(T) \cdot N_\sigma) \cap \mathfrak{h}] \). The natural actions of \( N_G(T) \) on \( N_G(T) \cdot N_\sigma \) and \( \mathfrak{h} \) define an action of \( N_G(T) \) on the coordinate ring \( \mathbb{C}[(N_G(T) \cdot N_\sigma) \cap \mathfrak{h}] \). As \( T \) acts trivially on \( \mathfrak{h} \), this gives the desired \( S_n \)-algebra structure by identifying \( S_n \) with the Weyl group \( N_G(T)/T \). \( \square \)

**Theorem 2.** The algebra \( \mathbb{C}[(N_G(T) \cdot N_\sigma^\vee) \cap \mathfrak{h}] \) is isomorphic to \( H^*(X_\sigma) \) as a graded \( S_n \)-algebra.

*Proof.* By the result of de Concini-Procesi mentioned in the introduction, we may identify \( H^*(X_\sigma) \) as a graded \( S_n \)-algebra with the coordinate ring
\[ \mathbb{C}[G \cdot M_\sigma \cap \mathfrak{h}] \] of the scheme theoretic intersection of \( \mathfrak{h} \) with the closure of the \( G \)-orbit of \( M_\sigma \). The graded \( S_n \)-algebra structure on the latter algebra is defined similarly to the graded \( S_n \)-algebra structure on \( \mathbb{C}[\mathcal{N}_G(T) \cdot \mathcal{N}_\sigma \cap \mathfrak{h}] \) (as defined in the proof of Lemma 3). As \( N_G(T) \cdot \mathcal{N}_\sigma \cap \mathfrak{h} = N_G(T) \cdot (L_\sigma \cdot M_\sigma) \) is a closed subscheme of \( G \cdot M_\sigma \), we have a surjective morphism of graded \( S_n \)-algebras:

\[ \mathbb{C}[G \cdot M_\sigma \cap \mathfrak{h}] \to \mathbb{C}[\mathcal{N}_G(T) \cdot \mathcal{N}_\sigma \cap \mathfrak{h}] . \]

Thus, we get a surjective map of graded \( S_n \)-algebras:

\[ \phi : H^*(X_\sigma) \to \mathbb{C}[\mathcal{N}_G(T) \cdot \mathcal{N}_\sigma \cap \mathfrak{h}] . \]

To prove the theorem, in view of Proposition 1, it suffices to show that the top degree \( d \) of the graded algebra \( \mathbb{C}[\mathcal{N}_G(T) \cdot \mathcal{N}_\sigma \cap \mathfrak{h}] \) is \( d_\sigma \). As \( \phi \) is surjective, we already know that \( d \leq d_\sigma \). For the other inequality, consider the graded surjective map

\[ \mathbb{C}[\mathcal{N}_G(T) \cdot \mathcal{N}_\sigma \cap \mathfrak{h}] \to \mathbb{C}[\mathcal{N}_\sigma \cap \mathfrak{h}] , \]

obtained by the \( C^* \)-equivariant embedding \( \mathcal{N}_\sigma \subset \mathcal{N}_G(T) \cdot \mathcal{N}_\sigma \). By a classical result by Kostant [8], \( \mathbb{C}[\mathcal{N}_\sigma \cap \mathfrak{h}] \) is isomorphic with the cohomology \( H^*(L_\sigma/B_\sigma, C) \) as graded algebras (in fact, also as modules for the Weyl group \( N_L \sigma / T \)). But the top degree of \( H^*(L_\sigma/B_\sigma, C) \) coincides with the complex dimension of \( L_\sigma/B_\sigma \), which is easily seen to be equal to \( d_2 \) (use formula (1)). All together, this implies that \( d \geq d_2 \). But, by Lemma 1, \( d_2 = d_\sigma \), which ends the proof.

\[ \Box \]

References

[1] Bergeron, N. and Garsia, A. On certain spaces of harmonic polynomials. In: Hypergeometric functions on domains of positivity, Jack polynomials, and applications.
[2] Carrell, J. Orbits of the Weyl group and a theorem of de Concini and Procesi. Compositio Math. 60 (1986), 45–52.
[3] de Concini, C. and Procesi, C. Symmetric functions, conjugacy classes and the flag variety. Invent. Math. 64 (1981), 203–219.
[4] Eisenbud, D. Commutative Algebra with a View Towards Algebraic Geometry. Springer-Verlag 1995.
[5] Garsia, A. and Haiman, M. A graded representation model for Macdonald’s polynomials. Proc. Nat. Acad. Sci. 90 (1993), 3607–3610.
[6] Garsia, A. and Procesi, C. On certain graded \( S_n \)-modules and the q-Kostka polynomials. Adv. Math. 94 (1992), 82–138.
[7] Haiman, M. Hilbert schemes, polygraphs and the Macdonald positivity conjecture. J. Amer. Math. Soc. 14 (2001), 941–1006.
[8] Kostant, B. Lie groups representations on polynomial rings. Amer. J. Math. 85 (1963), 327–404.
[9] Spaltenstein, N. The fixed point set of a unipotent transformation on the flag manifold. Nederl. Akad. Wetens. Proc. Ser. A 79 (1976), 452–456.

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