KDV TYPE HIERARCHIES, THE STRING EQUATION AND $W_{1+\infty}$ CONSTRAINTS

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ABSTRACT. To every partition $n = n_1 + n_2 + \cdots + n_s$ one can associate a vertex operator realization of the Lie algebras $a_\infty$ and $\hat{gl}_n$. Using this construction we make reductions of the $s$–component KP hierarchy, reductions which are related to these partitions. In this way we obtain matrix KP type equations. Now assuming that (1) $\tau$ is a $\tau$–function of the $[n_1, n_2, \ldots, n_s]$–th reduced KP hierarchy and (2) $\tau$ satisfies a ‘natural’ string equation, we prove that $\tau$ also satisfies the vacuum constraints of the $W_{1+\infty}$ algebra.

§0. Introduction.

In recent years KdV type hierarchies have been related to 2D gravity. To be slightly more precise (see [Dij] for the details and references), the square root of the partition function of the Hermitian $(n-1)$–matrix model in the continuum limit is the $\tau$–function of the $n$–reduced Kadomtsev Petviashvili (KP) hierarchy. Hence, the $(n-1)$–matrix model corresponds to $n$–th Gelfand Dickey hierarchy. For $n = 2, 3$ these hierarchies are better known as the KdV– and Boussinesque hierarchy, respectively. The partition function is then characterized by the so-called string equation:

(0.1) \[ L_{-1}\tau = \frac{1}{n} \frac{\partial \tau}{\partial x_1}, \]

where $L_{-1}$ is an element of the $c = n$ Virasoro algebra, which is related to the principal realization of the affine lie algebra $\hat{sl}_n$, or rather $\hat{gl}_n$. Let $\alpha_k = -kx_{-k}$, $0$, $\frac{\partial}{\partial x_k}$ for $k < 0$, $k = 0$, $k > 0$, respectively, then

(0.2) \[ L_k = \frac{1}{2n} \sum_{\ell \in \mathbb{Z}} : \alpha_{-\ell} \alpha_{\ell+nk} : + \delta_{0k} \frac{n^2 - 1}{24n}. \]

By making the shift $x_{n+1} \mapsto x_{n+1} + \frac{n}{n+1}$, we modify the origin of the $\tau$–function and thus obtain the following form of the string equation:

(0.3) \[ L_{-1}\tau = 0. \]

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Actually, it can be shown ([FKN] and [G]) that the above conditions, $n$th reduced KP and equation (0.3) (which from now on we will call the string equation), on a $\tau$–function of the KP hierarchy imply more general constraints, viz. the vacuum constraints of the $W_{1+\infty}$ algebra. This last condition is reduced to the vacuum conditions of the $W_n$ algebra when some redundant variables are eliminated.

The $W_{1+\infty}$ algebra is the central extension of the Lie algebra of differential operators on $\mathbb{C}^\times$. This central extension was discovered by Kac and Peterson in 1981 [KP3] (see also [Ra], [KRa]). It has as basis the operators $W^{(\ell+1)}_k = -t^{k+\ell}(\frac{\partial}{\partial t})^\ell$, $\ell \in \mathbb{Z}_+$, $k \in \mathbb{Z}$, together with the central element $c$. There is a well-known way how to express these elements in the elements of the Heisenberg algebra, the $\alpha_k$’s. The $W_{1+\infty}$ constraints then are

\begin{equation}
W^{(\ell+1)}_k = \{W^{(\ell+1)}_k, \delta_{k,0} c^{\ell}\} \tau = 0 \text{ for } -k \leq \ell \geq 0.
\end{equation}

For the above $\tau$–function, $\hat{W}^{(1)}_k = -\alpha_{nk}$ and $\hat{W}^{(2)}_k = L_k - \frac{n+1}{n} \alpha_{nk}$.

It is well–known that the $n$–reduced KP hierarchy is related to the principal realization (a vertex realization) of the basic module of $sl_n$. However there are many inequivalent vertex realization. Kac and Peterson [KP1] and independently Lepowsky [L] showed that for the basic representation of a simply–laced affine Lie algebra these different realizations are parametrized by the conjugacy classes of the Weyl group of the corresponding finite dimensional Lie algebra. Hence, for the case of $sl_\ell$ they are parametrized by the partitions $n = n_1 + n_2 + \cdots + n_s$ of $n$. An explicit description of these realizations was given in [TV] (see also §2). There the construction was given in such a way that it was possible to make reductions of the KP–hierarchy. In all these constructions a ‘natural’ Virasoro algebra played an important role. A natural question now is: If $\tau$ is a $\tau$–function of this $[n_1, n_2, \ldots, n_s]$–th reduced KP hierarchy and $\tau$ satisfies the string equation (0.3), where $L_{-1}$ is an element of this new Virasoro algebra, does $\tau$ also satisfy some corresponding $W_{1+\infty}$ constraints? In this paper we give a positive answer to this question. As will be shown in §6, there exists a ‘natural’ $W_{1+\infty}$ algebra for which (0.4) holds.

This paper is organized as follows. Sections 1–3 give results which were obtained in [KV] and [TV] (see also [BT]). Its major part is an exposition of the $s$–component KP hierarchy following [KV]. In §1, we describe the semi–infinite wedge representation of the group $GL_\infty$ and the Lie algebras $gl_\infty$ and $a_\infty$. We define the KP hierarchy in the so–called fermionic picture. The loop algebra $gl_n$ is introduced in §2. We obtain it as a subalgebra of $a_\infty$. Next we construct to every partition $n = n_1 + n_2 + \cdots + n_s$ of $n$ a vertex operator realization of $a_\infty$ and $gl_n$. §3 is devoted to the description of $s$–component KP hierarchy in terms of formal pseudo–differential operators. §4 describes reductions of this $s$–component KP hierarchy related to the above partitions. In §5 we introduce the string equation and deduce its consequences in terms of the pseudo–differential operators. Using the results of §5 we deduce in §6 the $W_{1+\infty}$ constraints. §7 is devoted to a geometric interpretation of the string equation on the Sato Grassmannian, which is similar to that of [KS].

Notice that since the Toda lattice hierarchy of [UT] is related to the 2–component KP hierarchy, some results of this paper also hold for certain reductions of the Toda lattice hierarchy.

Finally, I would like to thank Victor Kac for valuable discussions, and the Mathematical Institute of the University of Utrecht for the computer and e-mail facilities. I want to dedicate this paper to the memory of my father, Noud van de Leur, who died quite suddenly in the period that I was writing this article.
§1. The semi-infinite wedge representation of the group $GL_\infty$ and the KP hierarchy in the fermionic picture.

1.1. Consider the infinite complex matrix group

$$GL_\infty = \{ A = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} \mid A \text{ is invertible and all but a finite number of } a_{ij} - \delta_{ij} \text{ are 0} \}$$

and its Lie algebra

$$gl_\infty = \{ a = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} \mid \text{all but a finite number of } a_{ij} \text{ are 0} \}$$

with bracket $[a,b] = ab - ba$. This Lie algebra has a basis consisting of matrices $E_{ij}, \ i, j \in \mathbb{Z} + \frac{1}{2}$, where $E_{ij}$ is the matrix with a 1 on the $(i,j)$-th entry and zeros elsewhere. Now $gl_\infty$ is a subalgebra of the bigger Lie algebra

$$\overline{gl}_\infty = \{ a = (a_{ij})_{i,j \in \mathbb{Z} + \frac{1}{2}} \mid a_{ij} = 0 \text{ if } |i-j| >> 0 \}.$$

This Lie algebra $\overline{gl}_\infty$ has a universal central extension $a_\infty := \overline{gl}_\infty \oplus \mathbb{C}c$ with Lie bracket-defined by

$$[a + ac, b + \beta c] = ab - ba + \mu(a,b)c,$$

for $a, b \in \overline{gl}_\infty$ and $\alpha, \beta \in \mathbb{C}$; here $\mu$ is the following 2-cocycle:

$$\mu(E_{ij}, E_{kl}) = \delta_{il} \delta_{jk}(\theta(i) - \theta(j)),$$

where the linear mapping $\theta : \mathbb{R} \to \mathbb{C}$ is defined by

$$\theta(i) := \begin{cases} 
0 & \text{if } i > 0, \\
1 & \text{if } i \leq 0.
\end{cases}$$

Let $\mathbb{C}^\infty = \bigoplus_{j \in \mathbb{Z} + \frac{1}{2}} \mathbb{C}v_j$ be an infinite dimensional complex vector space with fixed basis \{v_j\}_{j \in \mathbb{Z} + \frac{1}{2}}. Both the group $GL_\infty$ and the Lie algebras $gl_\infty$ and $a_\infty$ act linearly on $\mathbb{C}^\infty$ via the usual formula:

$$E_{ij}(v_k) = \delta_{jk}v_i.$$

We introduce, following [KP2], the corresponding semi-infinite wedge space $F = \Lambda^{\frac{1}{2}} \mathbb{C}^\infty$, this is the vector space with a basis consisting of all semi-infinite monomials of the form $v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \ldots$, where $i_1 > i_2 > i_3 > \ldots$ and $i_{\ell+1} = i_\ell - 1$ for $\ell >> 0$. We can now define representations $R$ of $GL_\infty$ on $F$ by

$$R(A)(v_{i_1} \wedge v_{i_2} \wedge v_{i_3} \wedge \cdots) = Av_{i_1} \wedge Av_{i_2} \wedge Av_{i_3} \wedge \cdots.$$

In order to describe representations of the Lie algebras we find it convenient to define wedging and contracting operators $\psi_j^-$ and $\psi_j^+$ ($j \in \mathbb{Z} + \frac{1}{2}$) on $F$ by

$$\psi_j^- (v_{i_1} \wedge v_{i_2} \wedge \cdots) = \begin{cases} 
0 & \text{if } j = i_s \text{ for some } s \\
(-1)^sv_{i_1} \wedge v_{i_2} \cdots \wedge v_{i_s} \wedge v_{-j} \wedge v_{i_{s+1}} \wedge \cdots & \text{if } i_s > -j > i_{s+1}
\end{cases}$$

$$\psi_j^+ (v_{i_1} \wedge v_{i_2} \wedge \cdots) = \begin{cases} 
0 & \text{if } j \neq i_s \text{ for all } s \\
(-1)^{s+1}v_{i_1} \wedge v_{i_2} \wedge \cdots \wedge v_{i_{s-1}} \wedge v_{i_{s+1}} \wedge \cdots & \text{if } j = i_s.
\end{cases}$$
Notice that the definition of $\psi^\pm_j$ differs from the one in [KV]. The reason for this will become clear in §7 where we describe the connection with the Sato Grassmannian. These wedging and contracting operators satisfy the following relations ($i, j \in \mathbb{Z} + \frac{1}{2}, \lambda, \mu = +, -$):

\[(1.1.5)\]

$$\psi_i^\lambda \psi_j^\mu + \psi_j^\mu \psi_i^\lambda = \delta_{\lambda,-\mu} \delta_{i,-j},$$

hence they generate a Clifford algebra, which we denote by $\mathcal{C}\ell$

Introduce the following elements of $F$ ($m \in \mathbb{Z}$):

$$|m\rangle = v_{m-\frac{1}{2}} \wedge v_{m-\frac{3}{2}} \wedge v_{m-\frac{5}{2}} \wedge \cdots.$$

It is clear that $F$ is an irreducible $\mathcal{C}\ell$-module such that

\[(1.1.6)\]

$$\psi_j^\pm |0\rangle = 0 \text{ for } j > 0.$$

We are now able to define representations $r, \hat{r}$ of $gl_\infty, a_\infty$ on $F$ by

$$r(E_{ij}) = \psi_{-i}^+ \psi_j^+, \quad \hat{r}(E_{ij}) = :\psi_{-i}^+ \psi_j^+:, \quad \hat{r}(c) = I,$$

where $:$ $:$ stands for the normal ordered product defined in the usual way ($\lambda, \mu = +$ or $-$):

\[(1.1.7)\]

$$:\psi_k^{\lambda(i)} \psi_\ell^{\mu(j)} := \begin{cases} 
\psi_k^{\lambda(i)} \psi_\ell^{\mu(j)} & \text{if } \ell \geq k \\
-\psi_\ell^{\mu(j)} \psi_k^{\lambda(i)} & \text{if } \ell < k.
\end{cases}$$

1.2. Define the charge decomposition

\[(1.2.1)\]

$$F = \bigoplus_{m \in \mathbb{Z}} F^{(m)}$$

by letting

\[(1.2.2)\]

$$\text{charge}(|0\rangle) = 0 \text{ and } \text{charge}(\psi_j^\pm) = \pm 1.$$

It is easy to see that each $F^{(m)}$ is irreducible with respect to $g\ell_\infty, a_\infty$ (and $GL_\infty$). Note that $|m\rangle$ is its highest weight vector, i.e., $\hat{r}(E_{ij}) = r(E_{ij}) - \delta_{ij} \theta(i)$ and

$$r(E_{ij}) |m\rangle = 0 \text{ for } i < j,$$

$$r(E_{ii}) |m\rangle = 0 \text{ (resp. } = |m\rangle) \text{ if } i > m \text{ (resp. if } i < m).$$

Let $\mathcal{O} = R(GL_\infty)|0\rangle \subset F^{(0)}$ be the $GL_\infty$-orbit of the vacuum vector $|0\rangle$, then one has
**Proposition 1.1 ([KP2]).** A non-zero element $\tau$ of $F^{(0)}$ lies in $\mathcal{O}$ if and only if the following equation holds in $F \otimes F$:

$$(1.2.3) \quad \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi_k^+ \tau \otimes \psi_{-k}^- \tau = 0.$$

**Proof.** For a proof see [KP2] or [KR]. □

Equation (1.2.3) is called the KP hierarchy in the fermionic picture.

§ 2. The loop algebra $\hat{gl}_n$, partitions of $n$ and vertex operator constructions.

2.1 Let $\hat{gl}_n = gl_n(C[t, t^{-1}])$ be the loop algebra associated to $gl_n(C)$. This algebra has a natural representation on the vector space $(C[t, t^{-1}])^n$. Let $\{w_i\}$ be the standard basis of $\mathbb{C}^n$, by identifying $(C[t, t^{-1}])^n$ over $\mathbb{C}$ with $\mathbb{C}^\infty$ via $v_{nk+j-\frac{1}{2}} = t^{-k}w_j$ we obtain an embedding $\phi : \hat{gl}_n \to \mathfrak{g}_\infty$:

$$\phi(t^k e_{ij}) = \sum_{\ell \in \mathbb{Z}} E_{n(\ell-k)+i-\frac{1}{2},n\ell+j-\frac{1}{2}},$$

here $e_{ij}$ is a basis of $gl_n(C)$.

A straightforward calculation shows that the restriction of the cocycle $\mu$ to $\phi(\hat{gl}_n)$ induces the following 2–cocycle on $\hat{gl}_n$:

$$\mu(x(t), y(t)) = \text{Res}_{t=0} dt \text{ tr}(dx(t) \frac{dt}{dt} y(t)).$$

Here and further $\text{Res}_{t=0} dt \sum_j f_j t^j$ stands for $f_{-1}$. This gives a central extension $\hat{gl}_n = \hat{gl}_n \oplus \mathbb{C}K$, where the bracket is defined by

$$[t^\ell x + \alpha K, t^m y + \beta K] = t^{\ell+m}(xy - yx) + \ell \delta_{\ell,-m} \text{ tr}(xy)K.$$

In this way we have an embedding $\phi : \hat{gl}_n \to a_\infty$, where $\phi(K) = c$.

Since $F$ is a module for $a_\infty$, it is clear that with this embedding we also have a representation of $\hat{gl}_n$ on this semi-infinite wedge space. It is well–known that the level one representations of the affine Kac–Moody algebra $\hat{gl}_n$ have a lot of inequivalent realizations. To be more precise, Kac and Peterson [KP1] and independently Lepowsky [L] showed that to every conjugacy class of the Weyl group of $gl_n(C)$ or rather $sl_n(C)$ there exists an inequivalent vertex operator realization of the same level one module. Hence to every partition of $n$, there exists such a construction.

We will now sketch how one can construct these vertex realizations of $\hat{gl}_n$, following [TV]. From now on let $n = n_1 + n_2 + \cdots + n_s$ be a partition of $n$ into $s$ parts, and denote by $N_a = n_1 + n_2 + \cdots + n_{a-1}$. We begin by relabeling the basis vectors $v_j$ and with them
the corresponding fermionic (wedging and contracting) operators: \((1 \leq a \leq s, 1 \leq p \leq n_a, \ j \in \mathbb{Z})\)

\[
\psi^{(a)}_{n_a-j-p+\frac{1}{2}} = \psi_{n_a-j-N_a-p+\frac{1}{2}}^{(a)}, \tag{2.1.6}
\]

\[
\psi^{\pm(a)}_{n_a+p\pm\frac{1}{2}} = \psi^{\pm}_{n_a+p\pm N_a\pm p\pm\frac{1}{2}}^{(a)} \tag{2.1.7}
\]

Notice that with this relabeling we have: \(\psi^{\pm(a)}_{k}|0\rangle = 0\) for \(k > 0\). We also rewrite the \(E_{ij}\)'s:

\[
E^{(ab)}_{n_a-j-p+\frac{1}{2}, n_b-k-q+\frac{1}{2}} = E_{n_a-j-N_a-p+\frac{1}{2}, n_b-N_b-k-q+\frac{1}{2}}^{(ab)}.
\]

The corresponding Lie bracket on \(a_\infty\) is given by

\[
[E_{jk}^{(ab)}, E_{\ell m}^{(cd)}] = \delta_{bc}\delta_{k\ell} E^{(ac)}_{jm} - \delta_{ad}\delta_{jm} E^{(db)}_{\ell k} + \delta_{ad}\delta_{bc}\delta_{jm}\delta_{k\ell}(\theta(j) - \theta(k))c,
\]

and \(\hat{r}(E_{jk}^{(ab)}) = :\psi^{-(a)}_{-j}\psi^{b}_{k}:\).

Introduce the fermionic fields \((z \in \mathbb{C}^\times):\)

\[
\psi^{\pm(a)}(z) \overset{\text{def}}{=} \sum_{k \in \mathbb{Z} + \frac{1}{2}} \psi^{\pm(a)}_{k} z^{-k-\frac{1}{2}}. \tag{2.1.2}
\]

Let \(N\) be the least common multiple of \(n_1, n_2, \ldots, n_s\). It was shown in [TV] that the modes of the fields

\[
: \psi^{+(a)}(\omega_\alpha^p z^{\frac{N}{n_a}}) \psi^{-(b)}(\omega_\beta^q z^{\frac{N}{n_b}}):,
\]

for \(1 \leq a, b \leq s, 1 \leq p \leq n_a, 1 \leq q \leq n_b\), where \(\omega_a = e^{2\pi i/n_a}\), together with the identity, generate a representation of \(\hat{gl}_n\) with \(K = 1\).

Next we introduce special bosonic fields \((1 \leq a \leq s):\)

\[
\alpha^{(a)}(z) \equiv \sum_{k \in \mathbb{Z}} \alpha^{(a)}_k z^{-k-1} \overset{\text{def}}{=} :\psi^{+(a)}(z)\psi^{-(a)}(z):. \tag{2.1.4}
\]

The operators \(\alpha_k^{(a)}\) satsify the canonical commutation relation of the associative oscillator algebra, which we denote by \(a:\)

\[
[a^{(i)}_k, a^{(j)}_\ell] = kj_{ij}\delta_{k, -\ell}, \tag{2.1.5}
\]

and one has

\[
\alpha^{(i)}_k|m\rangle = 0 \quad \text{for} \quad k > 0. \tag{2.1.6}
\]

It is easy to see that restricted to \(\hat{gl}_n\), \(F^{(0)}\) is its basic highest weight representation (see [K, Chapter 12]).

In order to express the fermionic fields \(\psi^{\pm(i)}(z)\) in terms of the bosonic fields \(\alpha^{(i)}(z)\), we need some additional operators \(Q_i, \ i = 1, \ldots, s\), on \(F\). These operators are uniquely defined by the following conditions:

\[
Q_i|0\rangle = \psi^{+(i)}_{-\frac{1}{2}}|0\rangle, \quad Q_i \psi^{\pm(j)}_k = (-1)^{\delta_{ij}+1} \psi^{\pm(j)}_{k+\delta_{ij}} Q_i. \tag{2.1.7}
\]

They satisfy the following commutation relations:

\[
Q_i Q_j = -Q_j Q_i \quad \text{if} \quad i \neq j, \quad [a^{(i)}_k, Q_j] = \delta_{ij} \delta_{k0} Q_j. \tag{2.1.8}
\]
\textbf{Theorem 2.1.} ([DJKM1], [JM])

\begin{equation}
\psi^{\pm(i)}(z) = Q^{\pm1} z^{\pm \alpha_0(i)} \exp(\mp \sum_{k<0} \frac{1}{k} \alpha_k^{(i)} z^{-k}) \exp(\mp \sum_{k>0} \frac{1}{k} \alpha_k^{(i)} z^{-k}).
\end{equation}

\textit{Proof.} See [TV].

The operators on the right-hand side of (2.1.9) are called vertex operators. They made their first appearance in string theory (cf. [FK]). If one substitutes (2.1.9) into (2.1.3), one obtains the vertex operator realization of $\hat{gl}_n$ which is related to the partition $n = n_1 + n_2 + \cdots + n_s$ (see [TV] for more details).

\section*{2.2.} The realization of $\hat{gl}_n$, described in the previous section, has a natural Virasoro algebra. In [TV], it was shown that the following two sets of operators have the same action on $F$.

\begin{align}
L_k &= \sum_{i=1}^{s} \left\{ \sum_{j \in \mathbb{Z}} \frac{1}{2n_i} : \alpha_j^{(i)} \alpha_{j+n_i,k} : + \delta_{k0} \frac{n_i^2 - 1}{24n_i} \right\}, \\
H_k &= \sum_{i=1}^{s} \left\{ \sum_{j \in \mathbb{Z} + \frac{1}{2}} \left( \frac{j}{n_i} + \frac{k}{2} \right) : \psi_j^{+(i)} \psi_{j+n_i,k}^{-(i)} : + \delta_{k0} \frac{n_i^2 - 1}{24n_i} \right\},
\end{align}

So $L_k = H_k$,

\begin{equation}
[L_k, \psi_j^{\pm(i)}] = -(\frac{j}{n_i} + \frac{k}{2}) \psi_{j+n_i,k}^{\pm(i)}
\end{equation}

and

\begin{equation}
[L_k, L_\ell] = (k - \ell)L_{k+\ell} + \delta_{k,-\ell} \frac{k^3 - k}{12} n.
\end{equation}

\section*{2.3.} We will now use the results of §2.1 to describe the $s$-component boson-fermion correspondence. Let $\mathbb{C}[x]$ be the space of polynomials in indeterminates $x = \{x_k^{(i)}\}$, $k = 1, 2, \ldots, i = 1, 2, \ldots, s$. Let $L$ be a lattice with a basis $\delta_1, \ldots, \delta_s$ over $\mathbb{Z}$ and the symmetric bilinear form $(\delta_i | \delta_j) = \delta_{ij}$, where $\delta_{ij}$ is the Kronecker symbol. Let

\begin{equation}
\varepsilon_{ij} = \begin{cases} -1 & \text{if } i > j \\ 1 & \text{if } i \leq j. \end{cases}
\end{equation}

Define a bimultiplicative function $\varepsilon : L \times L \to \{\pm1\}$ by letting

\begin{equation}
\varepsilon(\delta_i, \delta_j) = \varepsilon_{ij}.
\end{equation}

Let $\delta = \delta_1 + \cdots + \delta_s$, $Q = \{ \gamma \in L | (\delta | \gamma) = 0 \}$, $\Delta = \{ \alpha_{ij} := \delta_i - \delta_j | i, j = 1, \ldots, s, i \neq j \}$. Of course $Q$ is the root lattice of $sl_s(\mathbb{C})$, the set $\Delta$ being the root system.
Consider the vector space $\mathbb{C}[L]$ with basis $e^{\gamma}, \gamma \in L$, and the following twisted group algebra product:

\[(2.3.3)\]

\[e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}.\]

Let $B = \mathbb{C}[x] \otimes_{\mathbb{C}} \mathbb{C}[L]$ be the tensor product of algebras. Then the $s$-component boson-fermion correspondence is the vector space isomorphism

\[(2.3.4)\]

\[\sigma : F \sim \rightarrow B,\]

given by

\[(2.3.5)\]

\[\sigma(\alpha_{-m_1} \ldots \alpha_{-m_r} Q_1^{k_1} \ldots Q_s^{k_s} |0\rangle) = m_1 \ldots m_s x_{m_1}^{(i_1)} \ldots x_{m_r}^{(i_r)} \otimes e^{k_1 \delta_1 + \ldots + k_s \delta_s}.\]

The transported charge then will be as follows:

\[(2.3.6)\]

\[\text{charge}(p(x) \otimes e^{\gamma}) = (\delta | \gamma).\]

We denote the transported charge decomposition by

\[B = \bigoplus_{m \in \mathbb{Z}} B^{(m)}.\]

The transported action of the operators $\alpha_m^{(i)}$ and $Q_j$ looks as follows:

\[(2.3.7)\]

\[
\begin{cases}
\sigma \alpha_m^{(j)} \sigma^{-1}(p(x) \otimes e^{\gamma}) = m \alpha_m^{(j)} p(x) \otimes e^{\gamma}, \text{ if } m > 0, \\
\sigma \alpha_m^{(j)} \sigma^{-1}(p(x) \otimes e^{\gamma}) = \frac{\partial p(x)}{\partial x_m} \otimes e^{\gamma}, \text{ if } m > 0, \\
\sigma \alpha_0^{(j)} \sigma^{-1}(p(x) \otimes e^{\gamma}) = (\delta_j | \gamma)p(x) \otimes e^{\gamma}, \\
\sigma Q_j \sigma^{-1}(p(x) \otimes e^{\gamma}) = \varepsilon(\delta_j, \gamma)p(x) \otimes e^{\gamma+\delta}. 
\end{cases}
\]

For notational conveniences, we introduce $\delta_j = \sigma \alpha_0^{(j)} \sigma^{-1}$. Notice that $e^{\delta_j} = \sigma Q_j \sigma^{-1}$.

### 2.4. Using the isomorphism $\sigma$ we can reformulate the KP hierarchy (1.2.3) in the bosonic picture.

We start by observing that (1.2.3) can be rewritten as follows:

\[(2.4.1)\]

\[\text{Res}_{z=0} \ dz \left( \sum_{j=1}^{s} \psi^{+(j)}(z) \tau \otimes \psi^{-}(j)(z) \tau \right) = 0, \tau \in F^{(0)}.\]

Notice that for $\tau \in F^{(0)}$, $\sigma(\tau) = \sum_{\gamma \in Q} \tau_{\gamma}(x) e^{\gamma}$. Here and further we write $\tau_{\gamma}(x) e^{\gamma}$ for $\tau_{\gamma} \otimes e^{\gamma}$. Using Theorem 2.1, equation (2.4.1) turns under $\sigma \otimes \sigma : F \otimes F \sim \rightarrow \mathbb{C}[x', x''] \otimes (\mathbb{C}[L'] \otimes \mathbb{C}[L''])$ into the following set of equations; for all $\alpha, \beta \in L$ such that $(\alpha | \delta) = -(\beta | \delta) = 1$ we have:

\[\text{Res}_{z=0} (dz \sum_{j=1}^{s} \varepsilon(\delta_j, \alpha - \beta) z^{(\delta_j | \alpha - \beta - 2\delta_j)} \times \exp \left( \sum_{k=1}^{\infty} \left( x_{k}^{(j)'} - x_{k}^{(j)''} \right) z^{k} \right) \exp \left( - \sum_{k=1}^{\infty} \left( \frac{\partial}{\partial x_{k}^{(j)'}} - \frac{\partial}{\partial x_{k}^{(j)''}} \right) \frac{z^{-k}}{k} \right) \right) \tau_{\alpha-\delta_j}(x')(e^{\alpha}')^{\beta}(x''(e^{\beta}))^{\prime \prime} = 0.\]
§3. The algebra of formal pseudo-differential operators and the \(s\)-component KP hierarchy as a dynamical system.

3.0. The KP hierarchy and its \(s\)-component generalizations admit several formulations. The one we will give here was introduced by Sato [S], it is given in the language of formal pseudo-differential operators. We will show that this formulation follows from the \(\tau\)-function formulation given by equation (2.4.2).

3.1. We shall work over the algebra \(A\) of formal power series over \(\mathbb{C}\) in indeterminates \(x = (x^{(j)}_k)\), where \(k = 1, 2, \ldots\) and \(j = 1, \ldots, s\). The indeterminates \(x^{(1)}_1, \ldots, x^{(s)}_1\) will be viewed as variables and \(x^{(j)}_k\) with \(k \geq 2\) as parameters. Let

\[
\partial = \frac{\partial}{\partial x^{(1)}_1} + \cdots + \frac{\partial}{\partial x^{(s)}_1}.
\]

A formal \(s \times s\) matrix pseudo-differential operator is an expression of the form

\[
P(x, \partial) = \sum_{j \leq N} P_j(x) \partial^j,
\]

where \(P_j\) are \(s \times s\) matrices over \(A\). Let \(\Psi\) denote the vector space over \(\mathbb{C}\) of all expressions (3.1.1). We have a linear isomorphism \(S : \Psi \to \text{Mat}_s(A((z)))\) given by \(S(P(x, \partial)) = P(x, z)\). The matrix series \(P(x, z)\) in indeterminates \(x\) and \(z\) is called the symbol of \(P(x, \partial)\).

Now we may define a product \(\circ\) on \(\Psi\) making it an associative algebra:

\[
S(P \circ Q) = \sum_{n=0}^{\infty} \frac{1}{n!} \partial^n S(P) \partial^{-n} S(Q).
\]

From now on, we shall drop the multiplication sign \(\circ\) when no ambiguity may arise. One defines the differential part of \(P(x, \partial)\) by \(P_+(x, \partial) = \sum_{j=0}^{N} P_j(x) \partial^j\), and let \(P_- = P - P_+\). We have the corresponding vector space decomposition:

\[
\Psi = \Psi_- \oplus \Psi_+.
\]

One defines a linear map \(* : \Psi \to \Psi\) by the following formula:

\[
(\sum_j P_j \partial^j)^* = \sum_j (-\partial)^j \circ^t P_j.
\]

Here and further \(^t P\) stands for the transpose of the matrix \(P\). Note that \(*\) is an anti-involution of the algebra \(\Psi\).

3.2. Introduce the following notation

\[
z \cdot x^{(j)} = \sum_{k=1}^{\infty} x^{(j)}_k z^k, \quad e^{z \cdot x} = diag(e^{x^{(1)}_1}, \ldots, e^{x^{(s)}_1}).
\]
The algebra $\Psi$ acts on the space $U_+$ (resp. $U_-$) of formal oscillating matrix functions of the form
\[ \sum_{j \leq N} P_j e^{z \cdot x} \] (resp. \[ \sum_{j \leq N} P_j e^{-z \cdot x} \]), where $P_j \in \text{Mat}_s(A)$,
in the obvious way:
\[ P(x) \partial^j e^{\pm z \cdot x} = P(x)(\pm z)^j e^{\pm z \cdot x}. \]

One has the following fundamental lemma (see [KV]).

**Lemma 3.1.** If $P, Q \in \Psi$ are such that
\[ \text{Res}_{z=0}(P(x, \partial)e^{z \cdot x})^t(Q(x', \partial')e^{-z \cdot x'})dz = 0, \]
then $(P \circ Q^*)_\pm = 0$.

3.3. We proceed now to rewrite the formulation (2.4.2) of the $s$-component KP hierarchy in terms of formal pseudo-differential operators.

For each $\alpha \in \text{supp } \tau := \{ \alpha \in Q | \tau = \sum_{\alpha \in Q} \tau_{\alpha} e^\alpha, \tau_\alpha \neq 0 \}$ we define the (matrix valued) functions
\[ V^\pm(\alpha, x, z) = (V^\pm_{ij}(\alpha, x, z))_{i,j=1}^s \]
as follows:
\[ V^\pm_{ij}(\alpha, x, z) \overset{\text{def}}{=} \varepsilon(\delta_j, \alpha + \delta_i)z^{(\delta_{ij} | \pm \alpha + \alpha_{ij})} \]
\[ \times \exp(\pm \sum_{k=1}^\infty x_k(z^k \tau_{\alpha \pm \alpha_{ij}}(x))/\tau_\alpha(x)). \]

It is easy to see that equation (2.4.2) is equivalent to the following bilinear identity:
\[ \text{Res}_{z=0} V^+(\alpha, x, z)^tV^-(\beta, x', z)dz = 0 \]
for all $\alpha, \beta \in Q$.

Define $s \times s$ matrices $W_{ij}^{\pm (m)}(\alpha, x)$ by the following generating series (cf. (3.3.2)):
\[ \sum_{m=0}^\infty W_{ij}^{\pm (m)}(\alpha, x)(\pm z)^{-m} = \varepsilon z^{\delta_{ij} - 1}(\exp \sum_{k=1}^\infty \frac{\partial}{\partial x_k} z^{-k})\tau_{\alpha \pm \alpha_{ij}}(x))/\tau_\alpha(x). \]

We see from (3.3.2) that $V^\pm(\alpha, x, z)$ can be written in the following form:
\[ V^\pm(\alpha, x, z) = (\sum_{m=0}^\infty W_{ij}^{\pm (m)}(\alpha, x)R^\pm(\alpha, \pm z)(\pm z)^{-m})e^{\pm z \cdot x}, \]
where

\[(3.3.6) \quad R^\pm(\alpha, z) = \sum_{i=1}^{s} \epsilon(\delta_i, \alpha) E_{ii}(\pm z)^{\pm(\delta_i|\alpha)}.\]

Here and further $E_{ij}$ stands for the $s \times s$ matrix whose $(i, j)$ entry is 1 and all other entries are zero. Now it is clear that $V^\pm(\alpha, x, z)$ can be written in terms of formal pseudo-differential operators

\[(3.3.7) \quad P^\pm(\alpha) \equiv P^\pm(\alpha, x, \partial) = I_n + \sum_{m=1}^{\infty} W^{\pm(m)}(\alpha, x) \partial^{-m} \quad \text{and} \quad R^\pm(\alpha) = R^\pm(\alpha, \partial)\]

as follows:

\[(3.3.8) \quad V^\pm(\alpha, x, z) = P^\pm(\alpha) R^\pm(\alpha)e^{\pm z \cdot x}.\]

Since obviously $R^-(\alpha, \partial)^{-1} = R^+(\alpha, \partial)^*$, using Lemma 3.1 we deduce from the bilinear identity (3.3.3):

\[(3.3.9) \quad P^-(\alpha) = (P^+(\alpha)^*)^{-1},\]
\[(3.3.10) \quad (P^+(\alpha) R^+(\alpha - \beta) P^+(\beta)^{-1})_- = 0 \quad \text{for all} \quad \alpha, \beta \in \text{supp} \tau.\]

Victor Kac and the author showed in [KV] that given $\beta \in \text{supp} \tau$, all the pseudo-differential operators $P^+(\alpha), \alpha \in \text{supp} \tau$, are completely determined by $P^+(\beta)$ from equations (3.3.10). They also showed that $P = P^+(\alpha)$ satisfies the Sato equation:

\[(3.3.11) \quad \frac{\partial P}{\partial x^{(j)}_k} = -(PE_{jj} \circ \partial^k \circ P^{-1})_- \circ P.\]

To be more precise, one has the following

**Proposition 3.2.** Consider the formal oscillating functions $V^+(\alpha, x, z)$ and $V^-(\alpha, x, z)$, $\alpha \in Q$, of the form (3.3.8), where $R^\pm(\alpha, z)$ are given by (3.3.6) and $P^\pm(\alpha, x, \partial) \in I_s + \Psi_-$. Then the bilinear identity (3.3.3) for all $\alpha, \beta \in \text{supp} \tau$ is equivalent to the Sato equation (3.3.11) for each $P = P^+(\alpha)$ and the matching conditions (3.3.9-10) for all $\alpha, \beta \in \text{supp} \tau$.

### 3.4. Fix $\alpha \in Q$, introduce the following formal pseudo-differential operators $L(\alpha), C^{(j)}(\alpha)$, and differential operators $B^{(j)}_m(\alpha)$:

\[(3.4.1) \quad L \equiv L(\alpha) = P^+(\alpha) \circ \partial \circ P^+(\alpha)^{-1},\]
\[(3.4.1) \quad C^{(j)} \equiv C^{(j)}(\alpha) = P^+(\alpha) E_{jj} P^+(\alpha)^{-1},\]
\[(3.4.1) \quad B^{(j)}_m \equiv B^{(j)}_m(\alpha) = (P^+(\alpha) E_{jj} \circ \partial^m \circ P^+(\alpha)^{-1})_.\]
Then

\[ L = I_s \partial + \sum_{j=1}^{\infty} U^{(j)}(x) \partial^{-j}, \]

subject to the conditions

\[ \sum_{i=1}^{s} C^{(i)} = I_s, \quad C^{(i)} L = L C^{(i)}, \quad C^{(i)} C^{(j)} = \delta_{ij} C^{(i)}. \]

They satisfy the following set of equations for some \( P \in I_n + \Psi_- \):

\[ \left\{ \begin{array}{l}
LP = P \partial \\
C^{(i)} P = P E_{ii} \\
\frac{\partial P}{\partial x_k^{(i)}} = -(L^{(i)} k)_- P, \quad \text{where} \quad L^{(i)} = C^{(i)} L.
\end{array} \right. \]

**Proposition 3.3.** The system of equations (3.4.4) has a solution \( P \in I_s + \Psi_- \) if and only if we can find a formal oscillating function of the form

\[ W(x, z) = (I_s + \sum_{j=1}^{\infty} W^{(j)}(x) z^{-j}) e^{z \cdot x} \]

that satisfies the linear equations

\[ LW = zW, \quad C^{(i)} W = W E_{ii}, \quad \frac{\partial W}{\partial x_k^{(i)}} = B^{(i)}_k W. \]

And finally, one has the following

**Proposition 3.4.** If for every \( \alpha \in Q \) the formal pseudo-differential operators \( L \equiv L(\alpha) \) and \( C^{(j)} \equiv C^{(j)}(\alpha) \) of the form (3.4.2) satisfy conditions (3.4.3) and if the equations (3.4.4) have a solution \( P \equiv P(\alpha) \in I_s + \Psi_- \), then the differential operators \( B^{(j)}_k \equiv B^{(j)}_k(\alpha) \) satisfy one of the following equivalent conditions:

\[ \left\{ \begin{array}{l}
\frac{\partial L}{\partial x_k^{(j)}} = [B^{(j)}_k, L], \\
\frac{\partial C^{(i)}}{\partial x_k^{(j)}} = [B^{(j)}_k, C^{(i)}],
\end{array} \right. \]

(3.4.7)
\[
\frac{\partial L^{(i)}}{\partial x_j^{(j)}} = [B_k^{(j)}, L^{(i)}],
\]

\[
\frac{\partial B_k^{(i)}}{\partial x_j^{(j)}} - \frac{\partial B_k^{(j)}}{\partial x_i^{(i)}} = [B_k^{(j)}, B_k^{(i)}].
\]

Here \( L^{(j)} = L^{(j)}(\alpha) = C^{(j)}(\alpha) \circ L(\alpha) \).

Equations (3.4.7) and (3.4.8) are called \textit{Lax type} equations. Equations (3.4.9) are called the \textit{Zakharov-Shabat type} equations. The latter are the compatibility conditions for the linear problem (3.4.6).

§ 4. \([n_1, n_2, \ldots, n_s]\)-reductions of the \(s\)-component KP hierarchy.

4.1. Using (2.1.9), (2.1.3), (2.3.5) and (2.3.7), we obtain the to the partition \( n = n_1 + n_2 + \cdots + n_s \) related vertex operator realization of \( \hat{\mathfrak{gl}}_n \) in the vector space \( B^{(m)} \). Now, restricted to \( \hat{\mathfrak{sl}}_n \), the representation in \( F^{(m)} \) is not irreducible anymore, since \( \hat{\mathfrak{sl}}_n \) commutes with the operators

\[
\beta^{(s)}_{kn_s} = \sqrt{\frac{n_s}{N}} \sum_{i=1}^{s} \alpha_{kn_i}, \quad k \in \mathbb{Z}.
\]

In order to describe the irreducible part of the representation of \( \hat{\mathfrak{sl}}_n \) in \( B^{(0)} \) containing the vacuum vector 1, we choose the complementary generators of the oscillator algebra \( \mathfrak{a} \) contained in \( \hat{\mathfrak{sl}}_n \) (\( k \in \mathbb{Z} \)):

\[
\beta^{(j)}_k = \begin{cases} \alpha^{(j)}_k \\ \frac{N_{j+1} \alpha^{(j+1)}_{kn_{j+1}} - n_{j+1} (\alpha^{(1)}_{kn_1} + \alpha^{(2)}_{kn_2} + \cdots + \alpha^{(j)}_{kn_j})}{\sqrt{N_{j+1}(N_{j+1} - n_{j+1})}} & \text{if } k \notin n_j \mathbb{Z}, \\
\alpha^{(j)}_k - k \sqrt{N_{j+1}(N_{j+1} - n_{j+1})} & \text{if } k = \ell n_j \text{ and } 1 \leq j < s,
\end{cases}
\]

so that the operators (4.1.1 and 2) also satisfy relations (2.1.5). Hence, introducing the new indeterminates

\[
y^{(j)}_k = \begin{cases} x^{(j)}_k \\ \frac{N_{j+1} x^{(j+1)}_{kn_{j+1}} - (n_{1} x^{(1)}_{kn_1} + n_{2} x^{(2)}_{kn_2} + \cdots + n_{j} x^{(j)}_{kn_j})}{\sqrt{N_{j+1}(N_{j+1} - n_{j+1})}} & \text{if } k \notin n_j \mathbb{N}, \\
\frac{n_1 x^{(1)}_{kn_1} + n_2 x^{(2)}_{kn_2} + \cdots + n_s x^{(s)}_{kn_s}}{\sqrt{N n_s}} & \text{if } k = \ell n_s \text{ and } j = s,
\end{cases}
\]

we have: \( \mathbb{C}[x] = \mathbb{C}[y] \) and

\[
\sigma(\beta^{(j)}_k) = \frac{\partial}{\partial y^{(j)}_k} \text{ and } \sigma(\beta^{(j)}_{-k}) = k y^{(j)}_k \text{ if } k > 0.
\]
Now it is clear that the irreducible with respect to \( \hat{\mathfrak{sl}}_n \) subspace of \( B^{(0)} \) containing the vacuum \( 1 \) is the vector space

\[(4.1.5) \quad B^{(0)}_{[n_1,n_2,...,n_s]} = \mathbb{C}[y_k^{(j)}|1 \leq j < s, \ k \in \mathbb{N}, \ or \ j = s, \ k \in \mathbb{N} \setminus n_s \mathbb{Z}] \otimes \mathbb{C}[Q].\]

The vertex operator realization of \( \hat{\mathfrak{sl}}_n \) in the vector space \( B^{(0)}_{[n_1,n_2,...,n_s]} \) is then obtained by expressing the fields (2.1.3) in terms of vertex operators (2.1.9), which are expressed via (4.1.2) in the operators (4.1.4), the operators \( e^{\delta_i-\delta_j} \) and \( \delta_i-\delta_j \ (1 \leq i < j \leq s) \) (see [TV] for details).

The \( s \)-component KP hierarchy of equations (2.4.2) on \( \tau \in B^{(0)} = \mathbb{C}[y] \otimes \mathbb{C}[Q] \) when restricted to \( \tau \in B^{(0)}_{[n_1,n_2,...,n_s]} \) is called the \([n_1,n_2,...,n_s]\)-th reduced KP hierarchy. It is obtained from the \( s \)-component KP hierarchy by making the change of variables (4.1.3) and putting zero all terms containing partial derivates by \( y_{\ell n_s}^{(s)}, y_{2n_s}^{(s)}, y_{3n_s}^{(s)}, \ldots \).

The totality of solutions of the \([n_1,n_2,...,n_s]\)-th reduced KP hierarchy is given by the following

**Proposition 4.1.** Let \( \mathcal{O}_{[n_1,n_2,...,n_s]} \) be the orbit of 1 under the (projective) representation of the loop group \( SL_n(\mathbb{C}[t,t^{-1}]) \) corresponding to the representation of \( \hat{\mathfrak{sl}}_n \) in \( B^{(0)}_{[n_1,n_2,...,n_s]} \). Then

\[\mathcal{O}_{[n_1,n_2,...,n_s]} = \sigma(\mathcal{O}) \cap B^{(0)}_{[n_1,n_2,...,n_s]} .\]

In other words, the \( \tau \)-functions of the \([n_1,n_2,...,n_s]\)-th reduced KP hierarchy are precisely the \( \tau \)-functions of the KP hierarchy in the variables \( y_k^{(j)} \), which are independent of the variables \( y_{\ell n_s}^{(s)}, \ \ell \in \mathbb{N} \).

**Proof** is the same as of a similar statement in [KP2]. \( \square \)

**4.2.** It is clear from the definitions and results of \( \S 4.1 \) that the condition on the \( s \)-component KP hierarchy to be \([n_1,n_2,...,n_s]\)-th reduced is equivalent to

\[(4.2.1) \quad \sum_{j=1}^{s} \frac{\partial \tau}{\partial x_{kn_j}^{(j)}} = 0, \quad \text{for all} \ k \in \mathbb{N} \]

Using the Sato equation (3.3.11), this implies the following two equivalent conditions:

\[(4.2.2) \quad \sum_{j=1}^{s} \frac{\partial W(\alpha)}{\partial x_{kn_j}^{(j)}} = W(\alpha) \sum_{j=1}^{s} z_{kn_j}^{j} E_{jj}, \]

\[(4.2.3) \quad \sum_{j=1}^{s} L(\alpha)^{kn_j} C^{(j)} = 0 .\]
§5. The string equation.

5.1. From now on we assume that $\tau$ is any solution of the KP hierarchy. In particular, we no longer assume that $\tau_\alpha$ is a polynomial. For instance the soliton and dromion solutions of [KV, §5] are allowed. Of course this means that the corresponding wave functions $V^{\pm}(\alpha, z)$ will be of a more general nature than before.

Recall from §3 the wave function $V(\alpha, z) \equiv V^+(\alpha, z) = P(\alpha)R(\alpha)e^{z::x} = P^+(\alpha)R^+(\alpha)e^{z::x}$. It is natural to compute

$$
\frac{\partial V(\alpha, z)}{\partial z} = \frac{\partial}{\partial z}P(\alpha)R(\alpha)e^{z::x} = P(\alpha)R(\alpha)\frac{\partial}{\partial z}e^{z::x} = P(\alpha)R(\alpha)\sum_{a=1}^{s} kx_k^{(a)} \partial^{k-1} E_{aa} R(\alpha)^{-1} P(\alpha)^{-1} V(\alpha, z).
$$

Define

$$(5.1.1) \quad M(\alpha) := P(\alpha)R(\alpha)\sum_{a=1}^{s} kx_k^{(a)} \partial^{k-1} E_{aa} R(\alpha)^{-1} P(\alpha)^{-1},$$

then one easily checks that $[L(\alpha), M(\alpha)] = 1$ and

$$(5.1.2) \quad \left[ \sum_{a=1}^{s} L(\alpha)^{n_a} C^{(a)}(\alpha), M(\alpha) \sum_{a=1}^{s} \frac{1}{n_a} L(\alpha)^{1-n_a} C^{(a)}(\alpha) \right] = 1.$$
then

\[(PR)_{ij} = \epsilon(\delta_j|\alpha + \delta_i)z^{\delta_i - 1 + (\delta_j|\alpha)} \frac{\tau_{\alpha+\delta_i-\delta_j}}{\tau_{\alpha}}\]

and hence

\[
S((M(\alpha) \sum_{a=1}^{n_a} \frac{1}{n_a} L(\alpha)^{1-n_a} C^{(a)}(\alpha))_a^{} \cdot P(\alpha) R(\alpha))_{ij} = \frac{\epsilon(\delta_j|\alpha + \delta_i)}{n_j} \times
\]

\[
\left\{ \frac{1}{\tau_{\alpha}} \left( \sum_{k=1}^{\infty} \frac{\partial}{\partial x_k} z^{-k-n_j} + (\delta_{ij} - 1 + (\delta_j|\alpha))z^{-n_j} \right) \frac{\tilde{\tau}_{\alpha+\delta_i-\delta_j}}{\tau_{\alpha}} + \sum_{k=1}^{n_j} k \frac{(\tilde{\tau}_{\alpha+\delta_i-\delta_j})}{\tau_{\alpha}} z^{-n_j} \right\}
\]

\[\sum_{a=1}^{n_a} \sum_{k=1}^{n_a} (k + n_a) x_k^{(a)} \frac{\partial}{\partial x_k} \frac{\tilde{\tau}_{\alpha+\delta_i-\delta_j}}{\tau_{\alpha}} \right\} z^{\delta_{ij} - 1 + (\delta_j|\alpha)} \right\}
\]

5.2. We introduce the natural generalization of the string equation (0.3). Let $L_{-1}$ be given by (2.2.1), the string equation is the following constraint on $\tau \in F^{(0)}$:

\[(5.2.1) \quad L_{-1} \tau = 0.\]

Using (2.3.7) we rewrite $L_{-1}$ in terms of operators on $B^{(0)}$:

\[
L_{-1} = \sum_{a=1}^{n_a} \left\{ \delta_a x_{n_a}^{(a)} + \frac{1}{2n_a} \sum_{p=1}^{n_a-1} p(n_a - p) x_p^{(a)} x_{n_a-p}^{(a)} + \frac{1}{n_a} \sum_{k=1}^{\infty} (k + n_a) x_{k+n_a}^{(a)} \frac{\partial}{\partial x_k^{(a)}} \right\}.
\]

Since $\tau = \sum_{\alpha \in Q} \tau_{\alpha} e^{\alpha}$ and $L_{-1} \tau = 0$, we find that for all $\alpha \in Q$:

\[(5.2.2) \sum_{a=1}^{n_a} \left\{ (\delta_a|\alpha) x_{n_a}^{(a)} + \frac{1}{2n_a} \sum_{p=1}^{n_a-1} p(n_a - p) x_p^{(a)} x_{n_a-p}^{(a)} + \frac{1}{n_a} \sum_{k=1}^{\infty} (k + n_a) x_{k+n_a}^{(a)} \frac{\partial}{\partial x_k^{(a)}} \right\} \tau_{\alpha} = 0.
\]

Clearly, also $L_{-1} \tilde{\tau}_{\alpha+\delta_i-\delta_j} = 0$, this gives (see e.g. [D]):

\[
\sum_{a=1}^{n_a} \left\{ (\delta_a|\alpha + \delta_i - \delta_j) (x_{n_a}^{(a)} - \frac{\delta_{aj}}{n_j z^{n_j}}) + \frac{1}{2n_a} \sum_{p=1}^{n_a-1} p(n_a - p) (x_p^{(a)} - \frac{\delta_{aj}}{p z^{n_j}}) \times
\]

\[
\left\{ (x_{n_a-p}^{(a)} - \frac{\delta_{aj}}{(n_a-p) z^{n_j-p}}) + \frac{1}{n_a} \sum_{k=1}^{\infty} (k + n_a) (x_{k+n_a}^{(a)} - \frac{\delta_{aj}}{(k+n_j) z^{k+n_j}}) \frac{\partial}{\partial x_k^{(a)}} \right\} \tilde{\tau}_{\alpha+\delta_i-\delta_j} = 0.
\]

So, in a similar way as in [D], one deduces from (5.2.2 and 3) that

\[
\tilde{\tau}_{\alpha+\delta_i-\delta_j} \tau_{\alpha}^{-2} L_{-1} \tau_{\alpha} - \tau_{\alpha}^{-1} L_{-1} \tilde{\tau}_{\alpha+\delta_i-\delta_j} = 0.
\]
Hence, we find that for all $\alpha \in Q$ and $1 \leq i, j \leq s$:

$$\frac{1}{n_j}\left\{\frac{1}{\tau_\alpha}\left(\sum_{k=1}^{\infty} \frac{\partial}{\partial x_k^{(j)}} z^{-k-n_j} + (\delta_{ij} - 1) + (\delta_j | \alpha) + \frac{1}{2} - \frac{n_a}{2}\right)z^{-n_j} - n_j x_n^{(i)} \tilde{\tau}_\alpha + \delta_{ij} - \delta_j \right\} = 0.$$

(5.2.4)

$$+ \sum_{k=1}^{n_j} k x_k^{(j)} \frac{\tilde{\tau}_\alpha + \delta_j}{\tau_\alpha} z^{-k-n_j} - \sum_{a=1}^{s} n_a \sum_{k=1}^{\infty} (k + n_a) x_k^{(a)} \frac{\partial}{\partial x_k^{(a)}} \left(\frac{\tilde{\tau}_\alpha + \delta_j}{\tau_\alpha}\right) = 0.$$

Comparing this with (5.1.3), one finds

$$S\left(\sum_{a=1}^{s} \left\{\frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} C^{(a)}(\alpha)\right\} - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a} C^{(a)}(\alpha)\right) P(\alpha) R(\alpha)_{ij} = 0.$$

We thus conclude that the string equation induces for all $\alpha \in Q$:

$$\sum_{a=1}^{s} \left\{\frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} C^{(a)}(\alpha)\right\} - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a} C^{(a)}(\alpha)\right\} = 0.$$

(5.2.5)

So, if (5.2.5) holds

$$N(\alpha) := \sum_{a=1}^{s} \left\{\frac{1}{n_a} M(\alpha) L(\alpha)^{1-n_a} C^{(a)}(\alpha)\right\} - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a} C^{(a)}(\alpha)\right\}$$

is a differential operator that satisfies

$$\left[\sum_{a=1}^{s} L(\alpha)^{-n_a} C^{(a)}(\alpha), N(\alpha)\right] = 1.$$

§6. $W_{1+\infty}$ constraints.

6.1. Let $e_i$, $1 \leq i \leq s$ be a basis of $\mathbb{C}^s$. In a similar way as in §2, we identify $(\mathbb{C}[t, t^{-1}])^s$ with $\mathbb{C}^\infty$, viz., we put

$$v^{(a)}_{-k-\frac{1}{2}} = t^k e_a.$$

(6.1.1)

We can associate to $(\mathbb{C}[t, t^{-1}])^s$ $s$–copies of the Lie algebra of differential operators on $\mathbb{C}^\times$, it has as basis the operators (see [Ra] or [KRa]):

$$-t^{k+\ell} \left(\frac{\partial}{\partial t}\right)^\ell e_{ii}, \quad \text{for } k \in \mathbb{Z}, \ell \in \mathbb{Z}_+, 1 \leq i \leq s.$$
We will denote this Lie algebra by $D^s$. Via (6.1.1) we can embed this algebra into $\mathfrak{gl}_\infty$ and also into $a_\infty$, one finds

\[ -t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^\ell e_{ii} \mapsto \sum_{m \in \mathbb{Z}} -m(m-1) \cdots (m-\ell+1) E_{-m-k-\frac{1}{2},-m-\frac{1}{2}}^{(ii)}. \]

It is straightforward, but rather tedious, to calculate the corresponding 2–cocycle, the result is as follows (see also [Ra] or [KRa]). Let $f(t), g(t) \in \mathbb{C}[t, t^{-1}]$ then

\[
\mu(f(t)\left( \frac{\partial}{\partial t} \right)^{\ell} e_{aa}, g(t)\left( \frac{\partial}{\partial t} \right)^{m} e_{bb}) = \delta_{ab} \frac{\ell!m!}{(\ell + m + 1)!} \text{Res}_{t=0} \ f^{(m+1)}(t)g^{(\ell)}(t).
\]

Hence in this way we get a central extension $\hat{D}^s = D^s \oplus \mathbb{C}c$ of $D^s$ with Lie bracket

\[
[f(t)\left( \frac{\partial}{\partial t} \right)^{\ell} e_{aa} + \alpha c, g(t)\left( \frac{\partial}{\partial t} \right)^{m} e_{bb} + \beta c] =
\]

\[
\delta_{ab}\{ (f(t)\left( \frac{\partial}{\partial t} \right)^{\ell} g(t)\left( \frac{\partial}{\partial t} \right)^{m} - g(t)\left( \frac{\partial}{\partial t} \right)^{m} f(t)\left( \frac{\partial}{\partial t} \right)^{\ell}) e_{aa} + \frac{\ell!m!}{(\ell + m + 1)!} \text{Res}_{t=0} f^{(m+1)}(t)g^{(\ell)}(t)c \}.
\]

Since we have the representation $\hat{r}$ of $a_\infty$, we find that

\[
\hat{r}(-t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^{\ell} e_{aa}) = \sum_{m \in \mathbb{Z}} m(m-1) \cdots (m-\ell+1) : \psi^{+(a)}_{-m-\frac{1}{2}, m+k+\frac{1}{2}} : 
\]

In terms of the fermionic fields (2.1.2), we find

\[
\sum_{k \in \mathbb{Z}} \hat{r}(-t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^{\ell} e_{aa})z^{-k-\ell-1} =: \frac{\partial^\ell \psi^{+(a)}(z)}{z^\ell} \psi^{-(a)}(z) :.
\]

### 6.2. We will now express $-t^{k+\ell} \left( \frac{\partial}{\partial t} \right)^{\ell} e_{aa}$ in terms of the oscillators $\alpha_k^{(a)}$. For this purpose, we first calculate

\[
: (y - z) \psi^{+(a)}(y) \psi^{-(a)}(z) : = (y - z) \psi^{+(a)}(y) \psi^{-(a)}(z) - 1 = X_a(y, z) - 1,
\]

where

\[
X_a(y, z) = \left( \frac{y}{z} \right)^{\alpha_0^{(a)}} \exp(- \sum_{k < 0}^{a} \alpha_k^{(a)} (y - k - z - k)) \exp(- \sum_{k > 0}^{a} \alpha_k^{(a)} (y - k - z - k)).
\]
Then
\[ (6.2.2) : \frac{\partial^\ell \psi^{+(a)}(z)}{\partial z^\ell} \psi^{-(a)}(z) := \frac{1}{\ell + 1} \frac{\partial^{\ell+1} X_a(y, z)}{\partial y^{\ell+1}} |_{y=z}. \]

Notice that the right-hand-side of this formula is some normal ordered expression in the \(\alpha_k^{(a)}\)'s. For some explicit formulas of (6.2.2), we refer to the appendix of [AV].

6.3. In the rest of this section, we will show that \(\hat{D}^s\) has a subalgebra that will provide the extra constraints, the so called \(W\)–algebra constraints on \(\tau\).

From now on we assume that \(\tau\) is a \(\tau\)-function of the \([n_1, n_2, \ldots, n_s]\)-th reduced KP hierarchy, which satisfies the string equation. So, we assume that (4.2.3) and (5.2.1) holds. Hence, for all \(\alpha \in \text{supp} \tau\) both
\[ Q(\alpha) := \sum_{a=1}^s L(\alpha)^{n_a} C^{(a)}(\alpha) \quad \text{and} \]
\[ N(\alpha) = \sum_{a=1}^s \left\{ \frac{1}{n_a} M(\alpha)L(\alpha)^{1-n_a}C^{(a)}(\alpha) - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a} C^{(a)}(\alpha) \right\} \]
are differential operators. Thus, also \(N(\alpha)^p Q(\alpha)^q\) is a differential operator, i.e.,
\[ ((\sum_{a=1}^s \frac{1}{n_a} M(\alpha)L(\alpha)^{1-n_a} - \frac{n_a - 1}{2n_a} L(\alpha)^{-n_a})^p L(\alpha)^{qn_a} C^{(a)}(\alpha))_+ = 0 \quad \text{for} \ p, q \in \mathbb{Z}_+. \]

Using (6.3.1), we are able to prove the following

Lemma 6.1. For all \(\alpha \in Q\) and \(p, q \in \mathbb{Z}_+\)
\[ (6.3.2) \quad \text{Res}_{z=0}dz \sum_{a=1}^s z^{qn_a}(\frac{1}{n_a} z^{\frac{1-n_a}{2}} \frac{\partial}{\partial z} z^{\frac{1-n_a}{2}}) )^p V^+(\alpha, x, z) E_{aa}^m t^m V^-(\alpha, x', z) = 0. \]

Proof. Using Taylor’s formula we rewrite the right-hand-side of (6.3.2):
\[ (6.3.3) \quad \text{Res}_{z=0}dz \sum_{a=1}^s z^{qn_a}(\frac{1}{n_a} z^{\frac{1-n_a}{2}} \frac{\partial}{\partial z} z^{\frac{1-n_a}{2}} )^p V^+(\alpha, x, z) E_{aa} \exp\left(\sum_{\ell=1}^s \sum_{k=1}^\infty (x^{(\ell)}_k - x^{(\ell)}_k) \frac{\partial}{\partial x^{(\ell)}_k}\right)^m V^-(\alpha, x, z). \]

Since
\[ \frac{\partial V^-(\alpha, x, z)}{\partial x^{(\ell)}_k} = -(P^-(\alpha, x, z) E_{\ell\ell} \partial^k P^-(\alpha, x, z)^{-1})_+ V^-(\alpha, x, z), \]

it suffices to prove that for all \(m \geq 0\)
\[ (6.3.4) \quad \text{Res}_{z=0}dz \sum_{a=1}^s z^{qn_a}(\frac{1}{n_a} z^{\frac{1-n_a}{2}} \frac{\partial}{\partial z} z^{\frac{1-n_a}{2}} )^p V^+(\alpha, x, z) E_{aa}^m \partial^m V^-(\alpha, x, z) = 0. \]
Now, let \( \sum_{a=1}^{s} z^{q_n a}(\frac{1}{n_a} z^{\frac{1-n_a}{2}} \frac{\partial}{\partial z} z^{\frac{1-n_a}{2}}) p V^+ (\alpha, x, z) E_{a a} = \sum_i S_i \partial^{-i} e^{x z} \) and \( V^- (\alpha, x, z) = \sum_j T_j \partial^{-j} e^{-x z} \), then (6.3.4) is equivalent to
\[
0 = \text{Res}_{z=0} dz \sum_{i,j} S_i z^{-i} e^{x z} \partial^m (e^{-x z t} T_j (-z)^j)
\]
\[
= \text{Res}_{z=0} dz \sum_{i,j} \sum_{\ell=0}^{m} (-1)^{m-\ell+j} \binom{m}{\ell} S_i \partial^\ell (t^j T_j) z^{m-i-j-\ell}
\]
\[
(6.3.5) = \sum_{0 \leq \ell \leq m \atop i+j+\ell = m+1} (-1)^{\ell+j} \binom{m}{\ell} S_i \partial^\ell (t^j T_j).
\]

On the other hand (6.3.1) implies that
\[
0 = \sum_i S_i \partial^{-i} \sum_j (-\partial)^{-i t T_j}_-
\]
\[
= \sum_{i,j, \ell \geq 0} (-1)^j \binom{-i-j}{\ell} S_i \partial^\ell (t^j T_j) \partial^{-i-j-\ell}_-.
\]

Now let \( i + j + \ell = m + 1 \), then we obtain that for every \( m \geq 0 \)
\[
0 = \sum_{0 \leq \ell \leq m \atop i+j+\ell = m+1} (-1)^{\ell+j} \binom{m}{\ell} S_i \partial^\ell (t^j T_j)
\]
\[
= \sum_{0 \leq \ell \leq m \atop i+j+\ell = m+1} (-1)^{\ell+j} \binom{m}{\ell} S_i \partial^\ell (t^j T_j),
\]
which proves (6.3.5) \( \square \)

Taking the \((i,j)\)-th coefficient of (6.3.3) one obtains

**Corollary 6.2.** For all \( \alpha \in Q \), \( 1 \leq i, j \leq s \) and \( p, q \in \mathbb{Z}_+ \) one has
\[
(6.3.6) \ Res_{z=0} dz \sum_{a=1}^{s} z^{q_n a}(\frac{1}{n_a} z^{\frac{1-n_a}{2}} \frac{\partial}{\partial z} z^{\frac{1-n_a}{2}}) p \psi^+(a)(z) \tau_{\alpha+\delta_i-\delta_a} \otimes \psi^-(a)(z) \tau_{\alpha+\delta_a-\delta_j} = 0.
\]

Notice that (6.3.6) can be rewritten as infinitely many generating series of Hirota bilinear equations, for the case that \( p = q = 0 \) see [KV].

### 6.4

The following lemma gives a generalization of an identity of Date, Jimbo, Kashiwara and Miwa [DJKM3] (see also [G]):
Lemma 6.3. Let $X_b(y, w)$ be given by (6.2.1), then

$$Res_{z=0} dz \sum_{a=1}^s \psi^+(a)(z) X_b(y, w) \tau_{\alpha + \delta_i - \delta_a} e^{\alpha + \delta_i - \delta_a} \otimes \psi^-(a)(z) \tau_{\alpha + \delta_a - \delta_j} e^{\alpha + \delta_a - \delta_j} =$$

$$\sum_{\beta} Res_{\beta} (w - y) \psi^+(b)(y) \tau_{\alpha + \delta_i - \delta_b} e^{\alpha + \delta_i - \delta_b} \otimes \psi^-(b)(w) \tau_{\alpha + \delta_b - \delta_j} e^{\alpha + \delta_b - \delta_j}.$$  

(6.4.1)

Proof. The left–hand–side of (6.4.1) is equal to

$$Res_{z=0} dz \sum_{a=1}^s \epsilon(\delta_a, \delta_i + \delta_j) z^{(\delta_a | \delta_i + \delta_j)} - 2 y^{(\delta_b | \alpha + \delta_i)} w^{-(\delta_b | \alpha + \delta_i)} \left( \frac{z - y}{z - w} \right) \delta_{ab} \times$$

$$e^{x(b) \cdot y - x(b) \cdot w + x(a) \cdot z} \exp(- \sum_{k=1}^{\infty} \frac{1}{k} (y^{-k} - w^{-k}) \frac{\partial}{\partial x_k^{(b)}} + \frac{1}{z} \frac{\partial}{\partial x_k^{(a)}}) \tau_{\alpha + \delta_i - \delta_a} e^{\alpha + \delta_i} \otimes$$

$$e^{-x(a) \cdot z} \exp\left( \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} \frac{\partial}{\partial x_k^{(a)}} \right) \tau_{\alpha + \delta_a - \delta_j} e^{\alpha - \delta_j}.$$  

Recall the bilinear identity for $\beta = \alpha$:

$$Res_{z=0} dz \psi^+(a)(z) \tau_{\alpha + \delta_i - \delta_a} e^{\alpha + \delta_i - \delta_a} \otimes \psi^-(a)(z) \tau_{\alpha + \delta_a - \delta_j} e^{\alpha + \delta_a - \delta_j} = 0.$$  

Let $X_b(y, w) \otimes 1$ act on this identity, then

$$Res_{z=0} dz \sum_{a=1}^s \epsilon(\delta_a, \delta_i + \delta_j) z^{(\delta_a | \delta_i + \delta_j)} - 2 y^{(\delta_b | \alpha + \delta_i)} w^{-(\delta_b | \alpha + \delta_i)} \left( \frac{y - z}{w - z} \right) \delta_{ab} e^{x(b) \cdot y - x(b) \cdot w + x(a) \cdot z} \times$$

$$\exp(- \sum_{k=1}^{\infty} \frac{1}{k} (y^{-k} - w^{-k}) \frac{\partial}{\partial x_k^{(b)}} + \frac{1}{z} \frac{\partial}{\partial x_k^{(a)}}) \tau_{\alpha + \delta_i - \delta_a} e^{\alpha + \delta_i} \otimes$$

$$e^{-x(a) \cdot z} \exp\left( \sum_{k=1}^{\infty} \frac{1}{k} z^{-k} \frac{\partial}{\partial x_k^{(a)}} \right) \tau_{\alpha + \delta_a - \delta_j} e^{\alpha - \delta_j} = 0.$$  

Now, using this and the fact that $\frac{z - y}{z - w} = \frac{1 - w/z}{1 - w/y} = \frac{w - y}{z} \delta(w/z) + (\frac{w}{z}) \frac{1 - z/y}{1 - z/w}$, where $\delta(w/z) = \sum_{k \in \mathbb{Z}} (w/z)^k$ (such that $f(w, z) \delta(w/z) = f(w, w) \delta(w/z)$), we obtain that the left–hand–side of (6.4.1) is equal to

$$(w - y)^{(\delta_b | \alpha + \delta_i - \delta_b)} e^{x(b) \cdot y} \exp(- \sum_{k=1}^{\infty} \frac{1}{k} y^{-k} \frac{\partial}{\partial x_k^{(b)}} \tau_{\alpha + \delta_i - \delta_b} e^{\alpha + \delta_i} \otimes w^{-(\delta_b | \alpha + \delta_b - \delta_j)} e^{-x(b) \cdot w} \times$$

$$\exp\left( \sum_{k=1}^{\infty} \frac{1}{k} w^{-k} \frac{\partial}{\partial x_k^{(b)}} \right) \tau_{\alpha + \delta_b - \delta_j} e^{\alpha - \delta_j},$$
which is equal to the right-hand-side of (6.4.1) \(\Box\)

Define \(c_b(\ell, p)\) as follows

\[
(6.4.2) \quad (z^{\frac{1-n_b}{2}} \frac{\partial}{\partial z} z^{\frac{1-n_b}{2}})^p = \sum_{\ell=0}^{p} c_b(\ell, p) z^{-n_b p + \ell} \left( \frac{\partial}{\partial z} \right)^\ell.
\]

One also has

\[
(6.4.3) \quad (M(\alpha)L(\alpha)^{-n_b+1} - \frac{n_b-1}{2} L(\alpha)^{-n_b})^p = \sum_{\ell=0}^{p} c_b(\ell, p) M(\alpha)^\ell L(\alpha)^{-n_b p + \ell}.
\]

Then it is straightforward to show that

\[
(6.4.4) \quad c_b(\ell, p) = \sum_{0 \leq q_0 < q_1 \cdots < q_{p-\ell-1} \leq p-1} [[(q_0 + \frac{1}{2})(1-n_b)] \times
\]

\[
((q_1 + \frac{1}{2})(1-n_b) - 1) \cdots [(q_{p-\ell-1} + \frac{1}{2})(1-n_b) - (p-\ell-1)].
\]

Now using (6.4.2) and removing the tensor product symbol in (6.3.6), where we write \(x\), respectively \(x'\), for the first, respectively the second, component of the tensor product, one gets:

\[
\text{Res}_{z=0} dz \sum_{a=1}^{s} \left( \frac{1}{n_a} \right)^p z^q n_a \sum_{\ell=0}^{p} c_a(\ell, p) z^{-n_a p + \ell} \left( \frac{\partial^\ell \psi^+(a)}{\partial z^\ell} \right) \mid_{w=z}^{z^{-n_b p + \ell} \frac{\partial^\ell \psi^-(a)}{\partial z^\ell}} = 0
\]

Using Lemma 6.3, this is equivalent to

\[
\text{Res}_{z=0} dz \sum_{a=1}^{s} \psi^+(a) \sum_{b=1}^{s} \left( \frac{1}{n_b} \right)^p z^q n_b \sum_{\ell=0}^{p} c_b(\ell, p) z^{-n_b p + \ell} \frac{\partial^\ell+1 X_b(w, z)}{\partial z^\ell+1} \mid_{w=z} \times
\]

\[
\tau_{\alpha+\delta_i-\delta_a} (x) e^{\alpha+\delta_i-\delta_a} \psi^-(a) (y) \tau_{\alpha+\delta_a-\delta_j} (x') e^{\alpha+\delta_a-\delta_j}' = 0.
\]
Now, recall (6.1.4) and (6.2.2), then

\[ \text{Res}_{z=0}dz \sum_{b=1}^{s} \left( \frac{1}{n_b} \right) x^p z^{q_b} \sum_{\ell=0}^{p} \frac{c_b(\ell, p)}{\ell + 1} z^{-n_b p + \ell} \frac{\partial^{\ell+1} X_b(w, z)}{\partial w^{\ell+1}} \bigg|_{w=z} = \]

\[ \text{Res}_{z=0}dz \sum_{b=1}^{s} \left( \frac{1}{n_b} \right) x^p z^{q_b} \sum_{\ell=0}^{p} c_b(\ell, p) z^{-n_b p + \ell} : \frac{\partial^{\ell+1} \psi^{+(b)}(z)}{\partial z^{\ell+1}} \psi^{-(b)}(z) := \]

\[ \sum_{b=1}^{s} \left( \frac{1}{n_b} \right) p \sum_{\ell=0}^{p} c_b(\ell, p) \hat{r}(\psi^{+(b)}(z)) = \]

\[ - \sum_{b=1}^{s} \left( \frac{1}{n_b} \right) p \hat{r}(t^{\frac{1-n_b}{2}} z \frac{\partial}{\partial t} t^{\frac{1-n_b}{2}} e_{bb}) = \]

\[ - \sum_{b=1}^{s} \hat{r}(t^{\frac{n_b-1}{2}} t^{n_b} \frac{\partial}{\partial t} t^{\frac{1-n_b}{2}} e_{bb}) = \]

\[ \sum_{b=1}^{s} \hat{r}(t^{\frac{n_b-1}{2}} (-\lambda_b^{q} \frac{\partial}{\partial \lambda_b} t^{\frac{1-n_b}{2}} e_{bb}) \quad \text{where} \quad \lambda_b = t^{n_b} \]

\[ \text{(6.4.5)} \]

\[ \text{def} W_{q-p}^{(p+1)} \]

Hence, (6.3.6) is equivalent to

\[ \text{(6.4.6)} \]

\[ \text{Res}_{y=0}dy \sum_{a=1}^{s} \psi^{+(a)}(y) W_{q-p}^{(p+1)} \tau_{\alpha+\delta_i-\delta_j}(x) e^{\alpha+\delta_i-\delta_j}(y) \tau_{\alpha+\delta_i-\delta_j}(x')(e^{\alpha+\delta_i-\delta_j})' = 0. \]

If we ignore the cocycle term for a moment, then it is obvious from the sixth line of (6.4.5), that the elements \( W_{q-p}^{(p+1)} \) are the generators of the W–algebra \( W_{1+\infty} \) (the cocycle term, however, will be slightly different). Upto some modification of the elements \( W_0^{(p+1)} \), one gets the standard commutation relations of \( W_{1+\infty} \), where \( c = nI \).

As the next step, we take in (6.4.6) \( x^{(i,j)}_k = x^{(i,j)}_k \), for all \( k \in \mathbb{N}, 1 \leq i \leq s \), we then obtain

\[ \tau_{\alpha+\delta_i-\delta_j} W_{q-p}^{(p+1)} \tau_{\alpha} = \tau_{\alpha} W_{q-p}^{(p+1)} \tau_{\alpha+\delta_i-\delta_j} \quad \text{if} \quad i \neq j. \]

\[ \tau_{\alpha} \]

The last equation means that for all \( \alpha, \beta \in \text{supp} \tau \) one has

\[ \text{(6.4.7)} \]

\[ \frac{W_{q-p}^{(p+1)} \tau_{\alpha}}{\tau_{\alpha}} = \frac{W_{q-p}^{(p+1)} \tau_{\beta}}{\tau_{\beta}}. \]
Next we divide (6.4.6) by $\tau_\alpha(x)\tau_\alpha(x')$, of course only for $\alpha \in \text{supp } \tau$, and use (6.4.8). Then for all $\alpha, \beta \in \text{supp } \tau$ and $p, q \in \mathbb{Z}_+$ one has

$$\text{Res}_{z=0} dz \sum_{a=1}^s \exp\left( -\sum_{k=1}^\infty z^{-k} \frac{\partial}{\partial x_k^{(a)}} \right) \left( W_{q-p}^{(p+1)} \frac{\tau_\beta(x)}{\tau_\beta(x)} \right) \frac{\psi^+(a)(z)\tau_{\alpha+\delta_i-\delta_a}(x)}{\tau_\alpha(x)} e^{\alpha+\delta_i-\delta_a} \times$$

$$\frac{\psi^-(a)(z)'\tau_{\alpha+\delta_i-\delta_a}(x')}{\tau_\alpha(x')} \left( e^{\alpha+\delta_i-\delta_a} \right)' = 0.$$

Since one also has the bilinear identity (3.3.3) (see also (2.4.1-2), we can subtract that part and thus obtain the following

**Lemma 6.4.** For all $\alpha, \beta \in \text{supp } \tau$ and $p, q \in \mathbb{Z}_+$ one has

$$\text{Res}_{z=0} dz \sum_{a=1}^s \left\{ \exp\left( -\sum_{k=1}^\infty z^{-k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \left( W_{q-p}^{(p+1)} \frac{\tau_\beta(x)}{\tau_\beta(x)} \right) \times$$

$$\frac{\psi^+(a)(z)\tau_{\alpha+\delta_i-\delta_a}(x)}{\tau_\alpha(x)} e^{\alpha+\delta_i-\delta_a} \frac{\psi^-(a)(z)'\tau_{\alpha+\delta_i-\delta_a}(x')}{\tau_\alpha(x')} \left( e^{\alpha+\delta_i-\delta_a} \right)' = 0.$$

Define

$$S(\beta, p, q, x, z) := \sum_{a=1}^s \left\{ \exp\left( -\sum_{k=1}^\infty z^{-k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \left( W_{q-p}^{(p+1)} \frac{\tau_\beta(x)}{\tau_\beta(x)} \right) E_{aa}.$$

Notice that the first equation of (6.4.7) implies that $\partial \circ S(\beta, p, q, x, \partial) = S(\beta, p, q, x, \partial) \circ \partial$. Then viewing (6.4.9) as the $(i, j)$–th entry of a matrix, (6.4.9) is equivalent to

(6.4.10) $\text{Res}_{z=0} dz P^+(\alpha) R^+(\alpha) S(\beta, p, q, x, \partial) e^{x'z} t(P^-(\alpha)' R^-(\alpha)' e^{-z'x}) = 0$.

Now using Lemma 3.1, one deduces

(6.4.11) $(P^+(\alpha) R^+(\alpha) S(\beta, p, q, x, \partial) R^+(\alpha)^{-1} P^+(\alpha)^{-1})_+ = 0,$

hence

$$P^+(\alpha) S(\beta, p, q, x, \partial) P^+(\alpha)^{-1} = (P^+(\alpha) S(\beta, p, q, x, \partial) P^+(\alpha)^{-1})_+ = 0.$$ 

So $S(\beta, p, q, x, \partial) = 0$ and therefore

$$\left\{ \exp\left( -\sum_{k=1}^\infty z^{-k} \frac{\partial}{\partial x_k^{(a)}} \right) - 1 \right\} \left( W_{q-p}^{(p+1)} \frac{\tau_\beta(x)}{\tau_\beta(x)} \right) = 0.$$

From which we conclude that

(6.4.12) $W_k^{(p+1)} \tau_\beta = \text{constant } \tau_\beta$ for all $-k \leq p \geq 0$.

In order to determine the constants on the right–hand–side of (6.4.12) we calculate the Lie brackets

(6.4.13) $[W_{-1}^{(2)}, \frac{-1}{k+p+1} W_{k+1}^{(p+1)}] \tau_\beta = 0$

and thus obtain the main result.
Theorem 6.5. The following two conditions for \(\tau \in F^{(0)}\) are equivalent:

1. \(\tau\) is a \(\tau\)-function of the \([n_1, n_2, \ldots, n_s]\)-th reduced \(s\)-component KP hierarchy which satisfies the string equation (5.2.1).
2. For all \(-k \leq p \geq 0:\)

\[
(W^{(p+1)}_k + \delta_{k0}c_p)\tau = 0,
\]

where

\[
c_p = \frac{1}{2p + 2} \sum_{a=1}^{s} \left( -\frac{1}{n_a} \right)^{p+1} \sum_{\ell=0}^{p} \ell! \left( \frac{n_a + \ell}{\ell + 2} \right) \sum_{0 \leq q_1 < \cdots < q_{p-\ell-1} \leq p-1} [(q_0 + \frac{1}{2})(n_a - 1)]^\times
\]

\[
[(q_1 + \frac{1}{2})(n_a - 1) + 1] \cdots [(q_{p-\ell-1} + \frac{1}{2})(n_a - 1) + p - \ell - 1].
\]

For \(p = 0, 1\), the constants \(c_p\) are equal to 0, respectively \(\sum_{a=1}^{s} \frac{n_a^2 - 1}{24n_a}\).

Proof of Theorem 6.5. The case (2) \(\Rightarrow\) (1) is trivial. For the implication (1) \(\Rightarrow\) (2), we only have to calculate the left-hand-side of (6.4.13). It is obvious that this is equal to \((W^{(p+1)}_k + c_{p,k})\tau_{\beta}\), where \(c_{p,k} = \mu(W^{(2)}_{-1,k+1} W^{(p+1)}_{k+1})\). It is clear from (6.1.3) that \(c_{p,k} = 0\) for \(k \neq 0\). So from now on we assume that \(k = 0\) and \(c_p = c_{p,0}\). Then

\[
c_p = \frac{-1}{p+1} \mu(W^{(2)}_{-1}, W^{(p+1)}_1)
\]

\[
= \frac{-1}{p+1} \sum_{a=1}^{s} \left( \frac{1}{n_a} \right)^{p+1} \mu \left( \frac{1-n_a}{2} t^{-n_a} + t^{1-n_a} \frac{\partial}{\partial t} \right) \sum_{\ell=0}^{p} c_{a}(\ell, p) t^{n_a + \ell} \left( \frac{\partial}{\partial t} \right)^\ell
\]

\[
= \frac{1}{2p + 2} \sum_{a=1}^{s} \left( \frac{1}{n_a} \right)^{p+1} \sum_{\ell=0}^{p} (-1)^{\ell+1} \ell! \left( \frac{n_a + \ell}{\ell + 2} \right) c_{a}(\ell, p),
\]

which equals (6.4.15). ☐

§7. A geometrical interpretation of the string equation on the Sato Grassmannian.

7.1. It is well-known that every \(\tau\)-function of the 1-component KP hierarchy corresponds to a point of the Sato Grassmannian \(Gr\) (see e.g. [S]). Let \(H\) be the space of formal Laurent series \(\sum a_n t^n\) such that \(a_n = 0\) for \(n >> 0\). The points of \(Gr\) are those linear subspaces \(V \subset \tilde{H}\) for which the naturel projection \(\pi_+\) of \(V\) into \(H_+ = \{ \sum a_n t^n \in H | a_n = 0\) for all \(n < 0\}\) is a Fredholm operator. The big cell \(Gr^0\) of \(Gr\) consists of those \(V\) for which \(\pi_+\) is an isomorphism.

The connection between \(Gr\) and the semi-infinite wedge space is made as follows. Identify \(v_{-k-\frac{1}{2}} = t^k\). Let \(V\) be a point of \(Gr\) and \(w_0(t), w_{-1}(t), \ldots\) be a basis of \(V\), then we associate to \(V\) the following element in the semi-infinite wedge space

\[
w_0(t) \wedge w_{-1}(t) \wedge w_{-2}(t) \wedge \ldots.
\]
If \( \tau \) is a \( \tau \)-function of the \( n \)-th KdV hierarchy, then \( \tau \) corresponds to a point of \( Gr \) that satisfies \( t^n V \subset V \) (see e.g. [SW], [KS]).

In the case of the \( s \)-component KP hierarchy and its \([n_1, n_2, \ldots, n_s]\)-reduction we find it convenient to represent the Sato Grassmannian slightly different. Let now \( H \) be the space of formal laurent series \( \sum a_n t^n \) such that \( a_n \in \mathbb{C}^s \) and \( a_n = 0 \) for \( n \gg 0 \). The points \( Gr \) are those linear subspaces \( V \subset H \) for which the projection \( \pi_+ \) of \( V \) into \( H_+ = \{ \sum a_n t^n \in H | a_n = 0 \text{ for all } n < 0 \} \) is a Fredholm operator. Again, the big cell \( Gr^0 \) of \( Gr \) consists of those \( V \) for which \( \pi_+ \) is an isomorphism. The connection with the semi-infinite wedge space is of course given in a similar way via (2.1.1):

\[
v_{n_j - N_a - p + \frac{1}{2}} = v_{n_a j - p + \frac{1}{2}}^{(a)} = t^{-n_a j + p - 1} e_a,
\]

here \( e_a, 1 \leq a \leq s \) is an orthonormal basis of \( \mathbb{C}^s \).

It is obvious that \( \tau \)-functions of the \([n_1, n_2, \ldots, n_s]\)-th reduced \( s \)-component KP hierarchy correspond to those subspaces \( V \) for which

\[
(\sum_{a=1}^{s} t^{n_a} E_{aa}) V \subset V.
\]

7.2. The proof that there exists a \( \tau \)-function of the \([n_1, n_2, \ldots, n_s]\)-th reduced KP hierarchy that satisfies the string equation is in great details similar to the proof of Kac and Schwarz [KS] in the principal case, i.e., the \( n \)-th KdV case.

Recall the string equation \( L_{-1} \tau = H_{-1} \tau = 0 \). Now modify the origin by replacing \( x_{n_a+1} \) by \( x_{n_a+1} + 1 \) for all \( 1 \leq a \leq s \). Then the string equation transforms to

\[
(L_{-1} + \sum_{a=1}^{s} \frac{n_a + 1}{n_a} x_1^{(a)}) \tau = 0,
\]

or equivalently

\[
(H_{-1} + \sum_{a=1}^{s} \frac{n_a + 1}{n_a} x_1^{(a)}) \tau = 0.
\]

In terms of elements of \( \hat{D} \) this is

\[
(\hat{r}(-A)) \tau = 0
\]

where

\[
A = \sum_{a=1}^{s} \frac{1}{n_a} ((n_a + 1)t + t^{1-n_a} \frac{\partial}{\partial t} - \frac{n_a - 1}{2} t^{-n_a}) E_{aa}.
\]

Hence for \( V \in Gr \), this corresponds to

\[
AV \subset V.
\]
Now we will prove that there exists a subspace $V$ satisfying (7.1.1) and (7.2.3). We will first start by assuming that $m = n_1 = n_2 = \cdots = n_s$ (this is the case that $L(\alpha)^m$ is a differential operator). For this case we will show that there exists a unique point in the big cell $Gr^0$ that satisfies both (7.1.1) and (7.2.3). So assume that $V \in Gr^0$ and that $V$ satisfies these two conditions. Since the projection $\pi_+$ on $H_+$ is an isomorphism, there exist $\phi_a \in V$, $1 \leq a \leq s$, of the form $\phi_a = e_a + \sum_i c_{i,a} t^{-i}$, with $c_{i,a} = \sum_{b=1}^s c_{i,a}^{(b)} e_b \in \mathbb{C}$. Now $A^p \phi_a = t^p e_a + \text{lower degree terms}$, hence these functions for $p \geq 0$ and $1 \leq a \leq s$ form a basis of $V$. Therefore, $t^m \phi_a$ is a linear combination of $A^p \phi_b$; it is easy to observe that $A^m \phi_a = \text{constant} \ t^m \phi_a$. Using this we find a recurrent relation for the $c_{i,a}$'s:

$$
\tag{7.2.4} \left( \frac{m+1}{m}\right)^{m-1} i c_{i,a}^{(b)} = \sum_{\ell=1}^{m-1} d_{m,i,\ell} c_{i-(m+1),a},
$$

where the $d_{m,i,\ell}$ are coefficients depending on $m, i, \ell$, which can be calculated explicitly using (7.2.2). Since $c_{0,a}^{(b)} = \delta_{ab}$ and $c_{i,a}^{(b)} = 0$ for $i < 0$ one deduces from (7.2.4) that $c_{i,a}^{(b)} = 0$ if $b \neq a$, and $c_{i,a}^{(a)} = 0$ if $i \neq (m+1)k$ with $k \in \mathbb{Z}$. So the $\phi_a$ for $1 \leq a \leq s$ can be determined uniquely. More explicitly, all $\phi_a$ are of the form $\phi_a = \phi^{(m)} e_a$, with

$$
\tag{7.2.5} \phi^{(m)} = \sum_{i=1}^{\infty} b^{(m)}_i t^{-(m+1)i},
$$

where the $b_i$ do not depend on $a$ and satisfy

$$
\left( \frac{m+1}{m}\right)^{m-1} i (m+1) b^{(m)}_i = \sum_{\ell=1}^{m-1} d_{m,i,\ell} b^{(m)}_{i-\ell}.
$$

Thus the space $V \in Gr^0$ is spanned by $t^{km} A^\ell \phi_a$ with $1 \leq a \leq s$, $k \in \mathbb{Z}_+$, $0 \leq \ell < m$.

Notice that in the case that all $n_a = 1$ we find that $V = H_+$, meaning that the only solution of (7.1.1) and (7.2.3) in $Gr^0$ is $\tau = \text{constant} \ e^0$, corresponding to the vacuum vector $|0\rangle$.

If not all $n_a$ are the same, then it is obvious that there still is a $V \in Gr^0$ satisfying (7.1.1) and (7.2.3), viz., $V$ spanned by $t^{kn_a} A^\ell e^{(n_a)}_a$, with $1 \leq a \leq s$, $k \in \mathbb{Z}_+$, $0 \leq \ell < n_a$, where $\phi^{(n_a)}$ is the unique solution determined by (7.2.5). However, at the present moment we do not know if this $V \in Gr^0$ is still unique in $Gr^0$. 

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