Characterization of Filippov representable maps and Clarke subdifferentials

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Abstract
The ordinary differential equation \( \dot{x}(t) = f(x(t)), \ t \geq 0, \) for \( f \) measurable, is not sufficiently regular to guarantee existence of solutions. To remedy this we may relax the problem by replacing the function \( f \) with its Filippov regularization \( F_f \) and consider the differential inclusion \( \dot{x}(t) \in F_f(x(t)) \) which always has a solution. It is interesting to know, inversely, when a set-valued map \( \Phi \) can be obtained as the Filippov regularization of a (single-valued, measurable) function. In this work we give a full characterization of such set-valued maps, hereby called Filippov representable. This characterization also yields an elegant description of those maps that are Clarke subdifferentials of a Lipschitz function.

Keywords Filippov regularization · Krasovskii regularization · Differential inclusion · Cusco map · Clarke subdifferential

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1 Introduction

We consider the differential equation

\[
\dot{x}(s) = f(x(s)), \ s \geq 0, \ x(0) = x_0, \tag{1}
\]

where \( f: \mathbb{R}^d \to \mathbb{R}^d \) is a bounded measurable function and \( x_0 \in \mathbb{R}^d \). The above Cauchy problem might have no solution due to the lack of regularity of \( f \). A way to overcome this difficulty is to replace (1) by a “minimal” differential inclusion which
is sufficiently regular to have a solution. A natural way to do this is to replace $f$ by its Krasovskii regularization $K_f$ given by

$$K_f(x) := \bigcap_{\delta > 0} \co f(B_\delta(x))$$

and obtain, accordingly:

$$\dot{x}(s) \in K_f(x(s)), \ x(0) = x_0, \ s \geq 0. \quad (2)$$

Another possibility is to consider, instead of $K_f$, the Filippov regularization $F_f$ of $f$ given by

$$F_f(x) := \bigcap_{L(N)=0} \bigcap_{\delta > 0} \co f((B_\delta(x)) \setminus N),$$

where the first intersection is taken over the sets $N \subset \mathbb{R}^d$ with Lebesgue measure $L(N)$ equal to zero. In this way, we obtain the so-called Filippov solutions of (1), that is, solutions of the differential inclusion

$$\dot{x}(s) \in F_f(x(s)), \ x(0) = x_0, \ s \geq 0. \quad (3)$$

The Filippov regularization is based on the idea that sets of measure zero should play no role in the relaxed dynamics.

Inclusions (2) and (3) always have a solution, since the set-valued mappings $K_f$ and $F_f$ are upper semicontinuous, with nonempty convex compact values (c.f. [1, 12]). For simplicity, borrowing terminology from [4,5], we shall refer to such set-valued mappings as cusco maps (see forthcoming Definition 2.1). If the function $f$ is continuous, then both maps $K_f$ and $F_f$ are single-valued and equal to $f$.

The techniques of Krasovskii and Filippov regularizations were introduced for obtaining solutions of discontinuous differential equations. Both regularizations have further been widely used in optimal control and differential games, see [3,8,14,17,19–21] e.g.

The main goal of this paper is to consider the inverse problem: given a cusco set-valued mapping $F$ from $\mathbb{R}^d$ to $\mathbb{R}^d$, does there exist a single-valued function $f$, such that $F$ is the Krasovskii/Filippov regularization of $f$? We shall refer to such maps as Krasovskii representable (respectively, Filippov representable). Notice that “being cusco” is clearly a necessary condition for being representable. We completely characterize Filippov representable maps, even in a slightly more general setting, namely, for maps defined in $\mathbb{R}^d$ with values in $\mathbb{R}^\ell$.

The other main contribution of this work is an equivalent characterization of the set-valued maps that are Clarke subdifferentials of a Lipschitz function in the finite-dimensional case. We show that these maps are exactly the Filippov regularizations of functions satisfying a so-called nonsmooth Poincaré condition. This condition is recently stated and used independently in [9,16] for a different purpose. We refer to
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[4] for another characterization of set-valued maps that are Clarke subdifferentials of a Lipschitz function in Banach spaces.

The manuscript is organized as follows: in Sect. 2 we introduce basic notation and background for Krasovskii and Filippov regularizations. In Sect. 3 we obtain several key results for both regularizations, while in Sect. 4 we provide the main result (characterization of Filippov representability) and use it to obtain an alternative characterization of those set-valued maps that are Clarke subdifferentials of Lipschitz functions (Sect. 5).

2 Preliminaries

Throughout the paper, we denote by $B_X$ (respectively, $\bar{B}_X$) the open (respectively, closed) unit ball, centered at the origin of the normed space $X$. The index will often be omitted if there is no ambiguity about the space. In this case, we denote by $B_\delta(x) := x + \delta B_X$ the (open) ball centered at $x$ with radius $\delta$. We also denote by $\mathcal{L}_d$ the Lebesgue measure in $\mathbb{R}^d$ and by $\mathcal{N}_d$ the set of $\mathcal{L}_d$-null subsets of $\mathbb{R}^d$, that is,

$$\mathcal{N}_d = \{N \subset \mathbb{R}^d : \mathcal{L}_d(N) = 0\}.$$

We shall also omit the index $d$ and simply write $\mathcal{L}$ for the Lebesgue measure and $\mathcal{N}$ for the family of null sets, whenever there is no ambiguity about the dimension.

For a set-valued mapping $\Phi$ from $\mathbb{R}^d$ to the subsets of $\mathbb{R}^\ell$, we will use the notation $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$, while a (single-valued) function will be denoted by $f: \mathbb{R}^d \rightarrow \mathbb{R}^\ell$. The following definition provides a convenient abbreviation for several statements in the sequel.

**Definition 2.1** *(Cusco map)* An upper semi-continuous set-valued map $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ with nonempty compact convex values will be called cusco.

Under the above terminology, the Krasovskii regularization $K_f$ is the smallest cusco map $\Phi$ satisfying $f(x) \in \Phi(x)$ for all $x \in \mathbb{R}^d$ and the Filippov regularization $F_f$ is the smallest cusco map $\Psi$ satisfying $f(x) \in \Psi(x)$ for almost all $x \in \mathbb{R}^d$. We refer the reader to [6,14,15] for more information on Filippov’s regularization and its applications. We also refer to [4] for properties of cusco maps.

We shall also need the following classical notion of a point of approximate continuity of a measurable function.

**Definition 2.2** *(Points of approximate continuity)* Let $f: \mathbb{R}^d \rightarrow \mathbb{R}^\ell$ be a measurable function. A point $x \in \mathbb{R}^d$ is called a point of approximate continuity for $f$ if for every $\varepsilon > 0$ it holds:

$$\lim_{\delta \rightarrow 0^+} \frac{\mathcal{L}\{x' \in B_\delta(x), \ |f(x') - f(x)| \geq \varepsilon\}}{\mathcal{L}(B_\delta(x))} = 0. \quad (4)$$

It is well-known that the complement $\mathcal{N}_f$ of the set of points of approximate continuity of a locally bounded measurable $f: \mathbb{R}^d \rightarrow \mathbb{R}^\ell$ is $\mathcal{L}_d$-null (c.f. [13] e.g.). Based on this result we can establish the following useful lemma.
Lemma 2.3 Let $f: \mathbb{R}^d \to \mathbb{R}^\ell$ be a (locally) bounded measurable function and $\mathbb{R}^d \setminus N_f$ be the set of points of approximate continuity. Then for every $\bar{x} \in \mathbb{R}^d$, $\delta > 0$ and $N \in \mathcal{N}$ we have:

\[
\begin{align*}
  f(B_\delta(\bar{x}) \setminus N_f) &\subset f(B_\delta(\bar{x}) \setminus (N_f \cup N)) \quad \text{and} \\
  \overline{\text{co}} f(B_\delta(\bar{x}) \setminus N_f) &\supset \overline{\text{co}} \left( f(B_\delta(\bar{x}) \setminus (N_f \cup N)) \right).
\end{align*}
\]

(5)

Consequently, for every $\bar{x} \in \mathbb{R}^d$ and $\delta > 0$ it holds:

\[
\overline{\text{co}} f(B_\delta(\bar{x}) \setminus N_f) = \bigcap_{N \in \mathcal{N}} \overline{\text{co}} f(B_\delta(\bar{x}) \setminus N).
\]

(6)

Proof Let us prove (5). Fix $\varepsilon > 0$, $N \in \mathcal{N}$ and $x \in B_\delta(\bar{x}) \setminus N_f$. Take $0 < \delta_1 < \delta$ such that $B_{\delta_1}(x) \subset B_{\delta}(\bar{x}) \setminus N_f$. By (4), there exists $\delta_2 \in (0, \delta_1)$ such that

\[
\frac{\mathcal{L}\{x' \in B_{\delta_2}(x), \ |f(x') - f(x)| \geq \varepsilon\}}{\mathcal{L}(B_{\delta_2}(x))} < 1,
\]

which yields

\[
\mathcal{L}\{x' \in B_{\delta_2}(x), \ |f(x') - f(x)| < \varepsilon\} > 0.
\]

Thus

\[
\mathcal{L}\{x' \in B_{\delta_2}(x), \ |f(x') - f(x)| < \varepsilon\} \setminus (N_f \cup N) > 0.
\]

Hence there exists $x' \in B_{\delta_2}(x) \setminus (N_f \cup N) \subset B_{\delta}(\bar{x}) \setminus (N_f \cup N)$ such that $|f(x') - f(x)| < \varepsilon$. Since $\varepsilon$ is arbitrary we deduce

\[
f(x) \in \overline{f(B_\delta(\bar{x}) \setminus (N_f \cup N))}.
\]

The right-hand side of (5) follows from the fact that for every subset $A$ of $\mathbb{R}^\ell$ we have

\[
A \subset \overline{\text{co}}(A) \quad \Rightarrow \quad \overline{A} \subset \overline{\text{co}}(A) \quad \Rightarrow \quad \overline{\text{co}}(\overline{A}) = \overline{\text{co}}(A).
\]

Assertion (6) follows directly from (5). \qed

We recall the following result due to Castaing (see [2, Theorem 8.1.4] e.g.)

Proposition 2.4 Let $\Phi: \mathbb{R}^d \rightharpoonup \mathbb{R}^\ell$ be a measurable set-valued map. Then there exists a sequence of measurable selections $\{f_n\}_{n=1}^\infty$ of $\Phi$ such that

\[
\Phi(x) = \overline{\{f_n(x) | n \in \mathbb{N}\}}, \quad \text{for all } x \in \mathbb{R}^d.
\]

Combining above proposition with Lemma 2.3, we deduce the following useful result.
Corollary 2.5 Let $\Phi: \mathbb{R}^d \to \mathbb{R}^\ell$ be cusco. Then there exists $N_\Phi \in \mathcal{N}_d$ (Lebesgue null set) such that for every $\bar{x} \in \mathbb{R}^d$, $\delta > 0$ and $N \in \mathcal{N}$ we have:

$$
\Phi(B_\delta(\bar{x}) \setminus N_\Phi) \subset \Phi(B_\delta(\bar{x}) \setminus (N_\Phi \cup N)) \quad \text{and} \quad \overline{\partial} \Phi(B_\delta(\bar{x}) \setminus N_\Phi) = \overline{\partial} \left( \Phi(B_\delta(\bar{x}) \setminus (N_\Phi \cup N)) \right). \tag{7}
$$

Consequently, for every $\bar{x} \in \mathbb{R}^d$ and $\delta > 0$ it holds:

$$
\overline{\partial} \Phi(B_\delta(\bar{x}) \setminus N_\Phi) = \bigcap_{N \in \mathcal{N}} \overline{\partial} \Phi(B_\delta(\bar{x}) \setminus N). \tag{8}
$$

Proof Let $\{f_n\}_{n \geq 1}$ be a sequence of measurable sets associated to $\Phi$ (c.f. Proposition 2.4). We set $N_\Phi := \bigcup_{k \geq 1} N_k$, where $N_k = N_{f_k}$ is the complement of the set of points of approximate continuity of $f_k$. We obviously have that $N_\Phi$ is a null set. Let us show that (7) holds.

To this end, let $N \in \mathcal{N}$, $\bar{x} \in \mathbb{R}^d$ and $\delta > 0$. Fix $\bar{x} \in B_\delta(\bar{x}) \setminus N_\Phi$ and take $\delta_1 \in (0, 1)$ such that $B_{\delta_1}(\bar{x}) \subset B_\delta(\bar{x}) \setminus (N_\Phi \cup N)$. By Lemma 2.3 we have for any $k \geq 1$,

$$
f_k(x) \in f_k(B_{\delta_1}(\bar{x}) \setminus N_k) \subset f_k(B_{\delta_1}(\bar{x}) \setminus (N_k \cup N_\Phi \cup N)) = f_k(B_{\delta_1}(\bar{x}) \setminus (N_\Phi \cup N)) \subset \Phi(B_\delta(\bar{x}) \setminus (N_\Phi \cup N)).
$$

So

$$
\Phi(x) = \{f_k(x), \ k \geq 1\} \subset \Phi(B_\delta(\bar{x}) \setminus (N_\Phi \cup N)),
$$

which established the left-hand side of (7). The remaining assertions are easily deduced in a similar manner as in Lemma 2.3. \hfill \square

Let us now recall (see [6, Proposition 2] e.g.) the following useful results. In [6], the results below have been stated and proved for the case $\ell = d$. The proofs for the general case ($\ell$ arbitrary) are identical. In what follows, $\mathcal{N}$ will always denote the class of Lebesgue null sets.

Proposition 2.6 Let $f: \mathbb{R}^d \to \mathbb{R}^\ell$ be a measurable and (locally) bounded function. Then,

(i) there exists a set $N_f \in \mathcal{N}$ such that

$$
F_f(x) := \bigcap_{\delta > 0} \overline{\partial} \Phi((B_\delta(x) \setminus N_f), \quad \text{for all } x \in \mathbb{R}^d
$$

and $f(x) \in F_f(x)$ for almost all $x \in \mathbb{R}^d$.

(ii) $F_f$ is the smallest cusco map $\Phi$ such that $f(x) \in \Phi(x)$, for almost all $x \in \mathbb{R}^d$.

(iii) $F_f$ is single-valued if and only if there exists a continuous function $g$ which coincides almost everywhere with $f$. In this case, $F_f(x) = \{g(x)\}$ for almost all $x \in \mathbb{R}^d$.

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(iv) there exists a (necessarily measurable) function $\tilde{f}$ which is equal almost everywhere to $f$ and such that

$$F_f(x) := \bigcap_{\delta > 0} \overline{co} \tilde{f}(B_\delta(x)), \text{ for all } x \in \mathbb{R}^d.$$  

(v) if a function $\tilde{f}$ coincides with $f$ for almost all $x \in \mathbb{R}^d$, then

$$F_f(x) = F_{\tilde{f}}(x), \text{ for all } x \in \mathbb{R}^d.$$  

(vi) for all $x \in \mathbb{R}^d$

$$F_f(x) = \bigcap_{\tilde{f} = f \text{ a.e. } \delta > 0} \bigcap_{\delta > 0} \overline{co} \tilde{f}(B_\delta(x)),$$

where the first intersection is taken over all functions $\tilde{f}$ equal to $f$ almost everywhere.

3 Cusco maps and Filippov representability

Before we proceed, we shall need the following classical result, whose proof is provided for completeness. According to the terminology of Kirk [18], the result asserts the existence, for every Euclidean space, of a countable partition that splits the family of open sets. For alternative proofs, or proofs of similar statements see [10,11,22].

Lemma 3.1 (Splitting partition) There exists a partition $\{A_n\}_{n=1}^\infty$ of $\mathbb{R}^d$, such that for every $n \in \mathbb{N}$ the set $A_n$ has a positive measure in every open subset of $\mathbb{R}^d$.

**Proof** Consider the countable family $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open balls with rational centers and rational radii in $\mathbb{R}^d$. Let

$$b : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$$

be a bijection such that $b(1, 1) = 1$.

Using that each nonempty open set contains a closed nowhere dense set with positive measure (e.g. a Smith–Volterra–Cantor set, also called “fat” Cantor set), we can choose $T_1 \subset \mathcal{U}_1$ to be a nowhere dense closed set with positive measure. Then, we construct a sequence $\{T_m\}_{m=2}^\infty$ of disjoint closed nowhere dense sets with positive measure such that

$$\text{if } m = b(k, j), \text{ then } T_m \subset \mathcal{U}_k \setminus \bigcup_{l < m} T_l. \quad (9)$$

This can be done since the set $\mathcal{U}_k \setminus \bigcup_{l < m} T_l$ is open.

We now set

$$A_n := \bigcup_{k=1}^{\infty} T_{b(k,n)}, \ n \geq 2$$
and

$$A_1 := \mathbb{R}^d \setminus \bigcup_{n=2}^{\infty} A_n.$$  

It is clear that \(\{A_n\}_{n=1}^{\infty}\) are measurable and disjoint. Moreover, if \(O\) be a nonempty open set, then there exists \(k\) such that \(U_k \subset O\). Using (9), we obtain that

$$A_n \cap O \supset A_n \cap U_k \supset T_{b(k,n)}, \quad n \geq 2$$

and

$$A_1 \cap O \supset \left( \mathbb{R}^d \setminus \bigcup_{n=2}^{\infty} A_n \right) \cap U_k \supset T_{b(1)}.$$  

Hence, \(\mathcal{L}(A_n \cap O) \geq \mathcal{L}(T_{b(k,n)}) > 0\) and \(\mathcal{L}(A_1 \cap O) \geq \mathcal{L}(T_{b(1)}) > 0\). This completes the proof of the lemma. \(\square\)

We are now ready to prove the following result.

**Theorem 3.2** Let \(\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell\) be a cusco map. Then there exists a measurable function \(f: \mathbb{R}^d \to \mathbb{R}^\ell\) such that \(\Phi\) is almost everywhere equal to \(F_f\) (the Filippov regularization of \(f\)), that is:

$$\Phi(x) = F_f(x), \quad \text{for almost every } x \in \mathbb{R}^d.$$  

**Proof** In view of Proposition 2.4, there exists a sequence of measurable selections \(\{f_n\}_{n=1}^{\infty}\) of \(\Phi\) such that

$$\Phi(x) = \{f_n(x) \mid n \in \mathbb{N}\}, \quad \text{for every } x \in \mathbb{R}^d.$$  

Let \(\{A_n\}_{n=1}^{\infty}\) be a splitting partition of \(\mathbb{R}^d\) given in Lemma 3.1. We define the measurable function \(f: \mathbb{R}^d \to \mathbb{R}^d\) as follows:

$$f(x) := \sum_{n=1}^{\infty} f_n(x) 1_{A_n}(x),$$

where \(1_A\) denotes the characteristic function of the set \(A\) (equal to 1 if \(x \in A\) and to 0 if \(x \notin A\)). Let

$$F_f(x) := \bigcap_{N, \mathcal{L}(N)=0} \bigcap_{\delta>0} f(B_\delta(x) \setminus N).$$
be the Filippov regularization of $f$. Since $\mathcal{L}(B_\delta(x) \cap A_n) > 0$ for all $n \in \mathbb{N}$ and for all $\delta > 0$, we obtain that

$$F_f(x) \supset \bigcap_{N, \mathcal{L}(N)=0} \bigcap_{\delta > 0} \overline{\partial} f((B_\delta(x) \cap A_n) \setminus N) = \bigcap_{N, \mathcal{L}(N)=0} \bigcap_{\delta > 0} \overline{\partial} f_n((B_\delta(x) \cap A_n) \setminus N)$$

for all $n \in \mathbb{N}, x \in \mathbb{R}^d$.

The next step in the proof consists in showing that the last expression in (10) contains $f_n(x)$ for almost all $x \in \mathbb{R}^d$. In order to do it, we will need the following assertion.

**Claim** There exists a sequence of measurable sets $\{K_m\}_{m=1}^\infty$ such that:

1. $K_1 \subset K_2 \subset \cdots K_m \subset \cdots$
2. $\mathbb{R}^d = \bigcup_{m=1}^\infty K_m \cup N_0$, where $\mathcal{L}(N_0) = 0$
3. the restrictions $f_n|_{K_m}$ are continuous for all $m, n \in \mathbb{N}$.

We postpone the proof of the claim at the end of this proof. Assuming the above claim, we deduce from Lemma 3.1 that for all $n \in \mathbb{N}, x \in \mathbb{R}^d$ and $\delta > 0$ it holds:

$$0 < \mathcal{L}(B_\delta(x) \cap A_n) = \mathcal{L}(B_\delta(x) \cap A_n \cap (\mathbb{R}^d \setminus N_0))$$

$$= \mathcal{L} \left( B_\delta(x) \cap A_n \cap \bigcup_{m=1}^\infty K_m \right) = \mathcal{L} \left( \bigcup_{m=1}^\infty (B_\delta(x) \cap A_n \cap K_m) \right)$$

$$= \lim_{m \to \infty} \mathcal{L}(B_\delta(x) \cap A_n \cap K_m),$$

since $K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$. Therefore, for some $m_0 \in \mathbb{N}$ sufficiently large we have

$$\mathcal{L}(B_\delta(x) \cap A_n \cap K_m) > 0,$$

for all $n \in \mathbb{N}, x \in \mathbb{R}^d, \delta > 0$ and $m \geq m_0$.

Let us fix an arbitrary $x \notin N_0$. Then, $x \in K_{m_1}$ for some $m_1 \in \mathbb{N}$. Let $\tilde{m} := \max(m_0, m_1)$. Since $x \in K_m$ for all $m \geq m_1$, we can continue (10) in the following way

$$F_f(x) \supset \bigcap_{N, \mathcal{L}(N)=0} \bigcap_{\delta > 0} \overline{\partial} f_n((B_\delta(x) \cap A_n \cap K_{\tilde{m}}) \setminus N) \ni f_n(x),$$

where the last inclusion is due to continuity of $f_n|_{K_{\tilde{m}}}$.

We have obtained that for all $n \in \mathbb{N}$ and for all $x \in \mathbb{R}^d \setminus N_0$

$$F_f(x) \ni f_n(x).$$
Since the Filippov regularization \( F_f \) is closed-valued, we obtain

\[
F_f(x) \supset \Phi(x) \ni f(x), \quad \text{for all } x \in \mathbb{R}^d \setminus N_0.
\]

We deduce from Proposition 2.6 (ii) that \( F_f(x) = \Phi(x) \) for almost every \( x \in \mathbb{R}^d \).

It remains to prove the claim about the existence of the sequence of sets \( \{K_m\}_{m=1}^\infty \). Since the functions \( f_n \) are measurable, due to Lusin’s theorem, for every \( m, n \in \mathbb{N} \) we can find a set \( K_{n,m} \subset \mathbb{R}^d \) such that \( f_n|_{K_{n,m}} \) is continuous and

\[
\mathcal{L}(\mathbb{R}^d \setminus K_{n,m}) < \frac{1}{2^n + m}.
\]

Let us set \( K'_m := \bigcap_{n=1}^\infty K_{n,m} \). We have that the restrictions \( f_n|_{K'_m} \) are continuous for all \( m, n \in \mathbb{N} \) and

\[
\mathcal{L}(\mathbb{R}^d \setminus K'_m) = \mathcal{L}\left(\bigcup_{n=1}^\infty (\mathbb{R}^d \setminus K_{n,m})\right) \leq \sum_{n=1}^\infty \mathcal{L}(\mathbb{R}^d \setminus K_{n,m}) < \sum_{n=1}^\infty \frac{1}{2^n + m} = \frac{1}{2m}.
\]

The inclusions \( K_1 \subset K_2 \subset \cdots K_m \subset \cdots \) are obtained by taking

\[
K_m := \bigcap_{l \geq m} K'_l, \quad m = 1, 2, \ldots.
\]

We have that

\[
\mathcal{L}(\mathbb{R}^d \setminus K_m) = \mathcal{L}\left(\bigcup_{l=m}^\infty (\mathbb{R}^d \setminus K'_l)\right) \leq \sum_{l=m}^\infty \mathcal{L}(\mathbb{R}^d \setminus K'_l) < \sum_{l=1}^\infty \frac{1}{2^l} = \frac{1}{2m-1}.
\]

Let us set \( N_0 := \mathbb{R}^d \setminus \bigcup_{m=1}^\infty K_m \). Since \( \mathbb{R}^d \setminus K_{m+1} \subset \mathbb{R}^d \setminus K_m \), we obtain that

\[
\mathcal{L}(N_0) = \mathcal{L}\left(\bigcap_{m=1}^\infty (\mathbb{R}^d \setminus K_m)\right) = \lim_{m \to \infty} \frac{1}{2^{m-1}} = 0.
\]

The proof is complete. \( \Box \)

We also obtain the following

**Proposition 3.3** Let \( \Phi: \mathbb{R}^d \to \mathbb{R}^\ell \) be a cusco map. Then, there exists a measurable selection \( f: \mathbb{R}^d \to \mathbb{R}^\ell \) of \( \Phi \) (that is, \( f(x) \in \Phi(x) \) for all \( x \in \mathbb{R}^d \)), such that

(i) \( \Phi \) is equal almost everywhere to the Filippov regularization of \( f \), that is,

\[
\Phi(x) = F_f(x), \quad \text{for almost all } x \in \mathbb{R}^d.
\]
(ii) there exists some \( \hat{f} : \mathbb{R}^d \to \mathbb{R}^\ell \) such that \( \Phi \) is equal almost everywhere to the Krasovskii regularization of \( \hat{f} \), that is,
\[
\Phi(x) = K_{\hat{f}}(x), \quad \text{for almost all } x \in \mathbb{R}^d.
\]

(iii) \( \Phi \) is equal almost everywhere to the intersection of all Filippov regularizations defined by functions \( \tilde{f} \) which are equal to \( f \) almost everywhere, that is,
\[
\Phi(x) = \bigcap_{\tilde{f} = f \text{ a.e.}} F_{\tilde{f}}(x), \quad \text{for almost all } x \in \mathbb{R}^d.
\]

**Proof** Using Theorem 3.2, we obtain a measurable function \( \bar{f} : \mathbb{R}^d \to \mathbb{R}^\ell \) such that \( \Phi_1 \) is equal almost everywhere to the Filippov regularization \( F_{\bar{f}} \) of \( \bar{f} \), that is,
\[
\Phi_1(x) = \bigcap_{\tilde{f} = f \text{ a.e.}} F_{\tilde{f}}(x), \quad \text{for almost every } x \in \mathbb{R}^d.
\]

Due to Proposition 2.6 (i) there exists a function \( \hat{f} : \mathbb{R}^d \to \mathbb{R}^\ell \) such that for all \( x \in \mathbb{R}^d \)
\[
\Phi_1(x) = \bigcap_{N, L(N) = 0} \bigcap_{\delta > 0} \hat{f}(B_{\delta}(x) \setminus N), \quad \text{for almost every } x \in \mathbb{R}^d.
\]

Clearly at every point \( x \in \mathbb{R}^d \setminus N_f \) of approximate continuity of \( \hat{f} \) we have that \( \hat{f}(x) \in \Phi(x) \). So setting \( f(x) = \hat{f}(x) \), whenever \( x \in \mathbb{R}^d \setminus N_f \) and taking \( f(x) \) to be any element of \( \Phi(x) \) if \( x \in N_f \), we obtain both claims (i) and (ii).

In order to establish (iii), we use (i) to obtain that for all \( x \in \mathbb{R}^d \setminus N_f \)
\[
\Phi(x) = \bigcap_{\delta > 0} F_{\hat{f}}(B_{\delta}(x)).
\]

At the same time we also have:
\[
\bigcap_{\tilde{f} = f \text{ a.e.}} \bigcap_{\delta > 0} \tilde{f}(B_{\delta}(x)) \supset \bigcap_{\tilde{f} = f \text{ a.e.}} \bigcap_{N, L(N) = 0} \tilde{f}(B_{\delta}(x)) \setminus N_f.
\]

The right-hand side is \( \bigcap_{\tilde{f} = f \text{ a.e.}} F_{\tilde{f}}(x) \), which by Proposition 2.6 (vi) is equal to \( F_f(x) \), for all \( x \in \mathbb{R}^d \).

The proof is complete. \( \square \)

**Remark 3.4** Notice that (completely) different functions may give rise to the same Filippov regularization: Indeed, let \( A \subset \mathbb{R} \) be a splitting set, that is, \( A \) and \( \mathbb{R} \setminus A \) have positive measure on every nontrivial interval. Then both \( f(x) := 1_{A}(x) \) and \( \tilde{f}(x) := 1_{\mathbb{R} \setminus A}(x) \) satisfy \( F_f(x) = [0, 1] \) and at the same time \( f(x) \neq \tilde{f}(x) \) for all \( x \in \mathbb{R} \).
**Definition 3.5 (The map $m(\Phi)$)** Let $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. We define the following “minimal” map:

$$m(\Phi)(x) := \bigcap_{N \in \mathcal{N}} \bigcap_{\delta > 0} \overline{\co} \Phi(B_\delta(x) \setminus N), \quad \text{for all } x \in \mathbb{R}^d. \quad (11)$$

Thanks to Corollary 2.5, we have also

$$m(\Phi)(x) = \bigcap_{\delta > 0} \overline{\co} \Phi(B_\delta(x) \setminus N_\Phi). \quad (12)$$

**Proposition 3.6** Let $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. Then the map $m(\Phi)$ is cusco and satisfies

$$m(\Phi)(\bar{x}) \subset \bigcap_{\delta > 0} \overline{\co} \Phi(B_\delta(x)) \subset \Phi(\bar{x}), \quad \text{for all } \bar{x} \in \mathbb{R}^d \quad (13)$$

$$m(\Phi)(\bar{x}) = \Phi(\bar{x}), \quad \text{for almost all } \bar{x} \in \mathbb{R}^d. \quad (14)$$

**Proof** Fix $N \in \mathcal{N}$, $x \in \mathbb{R}^d$ and set

$$G_N(x) := \bigcap_{\delta > 0} \overline{\co} \Phi(B_\delta(x) \setminus N).$$

Being a decreasing intersection of nonempty compact convex sets, $G_N(x)$ is itself a nonempty compact convex set. Notice that the family $\{G_N(x)\}_{N \in \mathcal{N}}$ has the finite intersection property. It follows from (11) that the map $m(\Phi)$ has nonempty convex compact values, while from its definition it follows easily that it is also upper semicontinuous, that is, $m(\Phi)$ is cusco.

We now fix $\varepsilon > 0$ and $\bar{x} \in \mathbb{R}^d$. Since $\Phi$ is upper semicontinuous there exists $\delta > 0$ such that

$$\forall x \in B_\delta(\bar{x}), \quad \Phi(x) \in \Phi(\bar{x}) + \varepsilon B.$$ 

So $\Phi(B_\delta(\bar{x})) \subset \Phi(\bar{x}) + \varepsilon B$ and $\overline{\co} \Phi(B_\delta(\bar{x})) \subset \Phi(\bar{x}) + 2\varepsilon B$ because $\Phi(\bar{x})$ is convex closed. Therefore

$$\bigcap_{\delta > 0} \overline{\co} \Phi(B_\delta(\bar{x})) \subset \Phi(\bar{x}) + 2\varepsilon B.$$

Taking the intersection over all $\varepsilon > 0$ we get

$$\bigcap_{\delta > 0} \overline{\co} \Phi(B_\delta(\bar{x})) \subset \bigcap_{\varepsilon > 0} (\Phi(\bar{x}) + 2\varepsilon B) = \Phi(\bar{x}).$$
This proves (13). Let us prove (14). In view of Corollary 2.5 we get from (13)
\[\forall \bar{x} \in \mathbb{R}^d, m(\Phi)(\bar{x}) = \bigcap_{\delta > 0} \overline{\partial} \Phi(B_\delta(x) \setminus N_\Phi) \subset \Phi(\bar{x}).\] (15)

If \(\bar{x} \not\in N_\Phi\) then
\[\Phi(\bar{x}) \subset \bigcap_{\delta > 0} \Phi(B_\delta(x) \setminus N_\Phi) \subset m(\Phi)(\bar{x}).\]

Consequently in view of (15) we obtain (14) for any \(\bar{x} \not\in N_\Phi\). \(\square\)

### 4 Characterization of Filippov representable maps

Let \(\hat{C}(\mathbb{R}^d, \mathbb{R}^\ell)\) be the set of all cusco maps \(\Phi: \mathbb{R}^d \to \mathbb{R}^\ell\). We now define on \(\hat{C}(\mathbb{R}^d, \mathbb{R}^\ell)\) the equivalence relation
\[\Phi_1 \sim \Phi_2 \iff \Phi_1(x) = \Phi_2(x) \text{ for almost all } x \in \mathbb{R}^d\]
and the associated quotient set
\[\hat{C}(\mathbb{R}^d, \mathbb{R}^\ell)/\sim := \{ [\Phi], \Phi \in \hat{C}(\mathbb{R}^d, \mathbb{R}^\ell) \}\]
where
\[[\Phi] := \{ \Psi \in \hat{C}(\mathbb{R}^d, \mathbb{R}^\ell), \Phi \sim \Psi \} .\]

We also define an order on \(\hat{C}(\mathbb{R}^d, \mathbb{R}^\ell)\) by
\[\Phi_1 \preceq \Phi_2 \iff \Phi_1(x) \subseteq \Phi_2(x), \text{ for all } x \in \mathbb{R}^d.\] (16)

**Lemma 4.1** (Equivalent elements in \(\hat{C}(\mathbb{R}^d, \mathbb{R}^\ell)\)) For all \(\Phi_1, \Phi_2 \in \hat{C}(\mathbb{R}^d, \mathbb{R}^\ell)\) we have:
\[\Phi_1 \sim \Phi_2 \iff m(\Phi_1) = m(\Phi_2).\]

**Proof** Let \(N \in \mathcal{N}\) be such that \(\Phi_1(x) = \Phi_2(x)\) for all \(x \in \mathbb{R}^d \setminus N\). Fix \(\bar{x} \in \mathbb{R}^d\). In view of Corollary 2.5, we deduce that for every \(\delta > 0\)
\[\overline{\partial} \Phi_1(B_\delta(\bar{x}) \setminus N_{\Phi_1}) = \overline{\partial} \Phi_1(\{B_\delta(\bar{x}) \setminus (N_{\Phi_1} \cup N_{\Phi_2} \cup N)\)
\[= \overline{\partial} \Phi_2(B_\delta(\bar{x}) \setminus (N_{\Phi_1} \cup N_{\Phi_2} \cup N)) = \overline{\partial} \Phi_2(B_\delta(\bar{x}) \setminus N_{\Phi_2})\]
because \(\Phi_1 = \Phi_2\) on the complement of \(N\). By taking intersection over all \(\delta > 0\) we obtain
\[m(\Phi_1)(\bar{x}) = \bigcap_{\delta > 0} \overline{\partial} \Phi_1(B_\delta(\bar{x}) \setminus N_{\Phi_1}) = \bigcap_{\delta > 0} \overline{\partial} \Phi_2(B_\delta(\bar{x}) \setminus N_{\Phi_2}) = m(\Phi_2)(\bar{x}).\]
The proof is complete. □

**Corollary 4.2** (minimality of $m(\Phi)$) Let $\Phi \in \hat{\mathcal{C}}(\mathbb{R}^d, \mathbb{R}^\ell)$. Then $m(\Phi) \in [\Phi]$ and $m(\Phi)$ is the minimum element in $[\Phi]$ for the order $\preceq$ defined in (16).

The fact that every cusco map $\Phi$ is equivalent to $m(\Phi)$ and that the latter is the minimum element of $[\Phi]$ under set-inclusion, has an interesting consequence, see (17) in the following remark.

**Remark 4.3** For every cusco map $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ we have:

$$m(\Phi)(x) = \bigcap_{\Phi' \sim \Phi} \Phi'(x), \text{ for all } x \in \mathbb{R}^d.$$  

This yields the following relation (which is not completely obvious at a first glance):

$$\Phi(x) = \bigcap_{\Phi' \sim \Phi} \Phi'(x), \text{ for a.e. } x \in \mathbb{R}^d. \quad (17)$$

We are now ready to establish our main result

**Theorem 4.4** (Characterization of Filippov representable maps) Let $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a cusco map. Then $\Phi$ is Filippov representable if and only if $\Phi = m(\Phi)$ (that is, $\Phi$ is the $\preceq$-minimal element in its equivalent class).

**Proof** Let $\Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell$ be a Filippov representable cusco map. Then

$$\Phi(x) = F_f(x) = \bigcap_{\delta > 0} \overline{co} f(B_\delta(x)) \setminus N_f), \text{ for all } x \in \mathbb{R}^d,$$

where $f: \mathbb{R}^d \rightarrow \mathbb{R}^\ell$ is some (bounded) measurable function. By Lemma 2.3 we deduce that

$$f(x) \in \Phi(x), \forall x \in \mathbb{R}^d \setminus N_f.$$  

This together with (12) and Lemma 2.3 yields that for any $x \in \mathbb{R}^d$

$$\Phi(x) = \bigcap_{\delta > 0} \overline{co} f(B_\delta(x)) \setminus (N_f \cup N_\Phi)) \subset \bigcap_{\delta > 0} \overline{co} \Phi(B_\delta(x)) \setminus (N_f \cup N_\Phi)).$$

In view of Corollary 2.5, we get

$$\bigcap_{\delta > 0} \overline{co} \Phi(B_\delta(x)) \setminus (N_f \cup N_\Phi)) = \bigcap_{\delta > 0} \overline{co} \Phi(B_\delta(x)) \setminus N_\Phi)$$

which is equal to $m(\Phi)(x)$ by (12). This yields $\Phi = m(\Phi)$.

To prove the opposite direction, note that by Theorem 3.2 every cusco map $\Phi$ is equivalent to a Filippov regularization $F_f$, and consequently, $F_f = m(F_f) = m(\Phi)$.

□
The following corollary follows directly.

**Corollary 4.5** The following assertions are equivalent for every cusco map \( \Phi : \mathbb{R}^d \rightrightarrows \mathbb{R}^\ell \):

(i) \( \Phi \) is a Filippov representable map;

(ii) \( \Phi = m(\Phi) \);

(iii) for every \( \bar{x} \in \mathbb{R}^d \) and \( N \in \mathcal{N} \) we have:

\[
\overline{\operatorname{co}} \left( \limsup_{x \not\in N, x \to \bar{x}} \Phi(x) \right) = \Phi(\bar{x}).
\]

Whenever \( \Phi \) is cusco, the left-hand side of (iii) above is always contained in \( \Phi(\bar{x}) \). According to (ii) above, it is very easy to obtain explicit examples of cusco maps that are not Filippov representable. Indeed, take any measurable function \( f \), consider its Filippov regularization \( F_f \) and modify it at some point \( \bar{x} \) (or at all points of a discrete set) to get an equivalent cusco map \( \Phi_1 \) different from \( F_f \). Indeed, it is sufficient to replace \( F_f(\bar{x}) \) by any convex compact strict superset \( \Phi(\bar{x}) \supset F_f(\bar{x}) \). Then \( \Phi \) is not Filippov representable, since \( \Phi \neq F_f = m(F_f) = m(\Phi) \), see forthcoming examples.

**Example 4.6** (i) We deduce easily that the following cusco maps, based on a one-point modification of the minimal map \( F_f(x) = \{0\} \), for all \( x \in \mathbb{R} \) (trivial regularization of the constant function \( f \equiv 0 \)), cannot be obtained as Filippov regularizations:

\[
\Phi_1(x) = \begin{cases} [0, 1], & \text{if } x = 0 \\ \{0\}, & \text{if } x \neq 0 \end{cases} \quad \text{and} \quad \Phi_2(x) = \begin{cases} [-1, 1], & \text{if } x = 0 \\ \{0\}, & \text{if } x \neq 0. \end{cases}
\]

It is worth noting that \( \Phi_2 \) cannot even be a Krasovskii regularization of a function, while \( \Phi_1 = K_g \), where \( g(x) = 0 \), for \( x \neq 0 \) and \( g(0) = 1 \).

(ii) A slightly more elaborated example of a function that can neither be obtained as Filippov nor as Krasovskii regularization is the following:

\[
\Phi_3(x) = \begin{cases} [-\frac{1}{m}, \frac{1}{m}], & \text{if } x = p/m \in \mathbb{Q} \setminus \{0\} \\ \{0\}, & \text{if } x \notin \mathbb{Q} \setminus \{0\}. \end{cases}
\]

where every nonzero rational number is given its irreducible form \( p/m \), where \( p, m \) are relatively prime integers.

(iii) Let us define the following measurable function:

\[
f(x) = \begin{cases} 1/m, & \text{if } x = p/m \in \mathbb{Q} \setminus \{0\} \\ \{0\}, & \text{if } x \notin \mathbb{Q} \setminus \{0\}. \end{cases}
\]

Then for every \( x \in \mathbb{R} \) we have: \( F_f(x) = \{0\} \) and \( K_f(x) = [0, f(x)] \). In particular \( F_f \sim K_f \) and consequently, the cusco map \( \Phi = K_f \) cannot be represented as a Filippov regularization.

\( \square \) Springer
5 Characterization of Clarke subdifferentials

In this section we deal with the problem of determining whether a cusco map \( \Phi \in \hat{C}(\mathbb{R}^d, \mathbb{R}^d) \) is the Clarke subdifferential of some locally Lipschitz function \( \varphi: \mathbb{R}^d \rightarrow \mathbb{R} \). A full characterization of such maps has been given in [4] and relevant results had been previously established in [5]. We shall complement the results of [4] by establishing, via our approach, another elegant characterization of Clarke subdifferentials. Our method is based on the characterization of Filippov representability (for the case \( \ell = d \)) together with a nonsmooth Poincaré condition. This latter has been recently stated and used independently in [9,16] for a different purpose (namely, to identify the free space of a finite-dimensional Euclidean space). Before we proceed, let us recall the relevant statement.

**Theorem 5.1** (nonsmooth Poincaré condition (Proposition 3.2(ii) in [9])) Let \( \mathcal{U} \neq \emptyset \) be an open convex subset of \( \mathbb{R}^d \). An essentially (locally) bounded measurable function \( f: \mathcal{U} \rightarrow \mathbb{R}^d \) is equal almost everywhere to the derivative of a (locally) Lipschitz function \( \varphi: \mathcal{U} \rightarrow \mathbb{R} \) if and only if

\[
\partial_i f_j = \partial_j f_i \quad \text{for all } i, j \in \{1, \ldots, d\},
\]

where \( \partial_i f_j \) denotes the partial derivative (in the sense of distributions) of the \( j \)th component of \( f \) with respect to \( x_i \). That is, if \( C^\infty_0(\mathcal{U}) \) is the space of compactly supported \( C^\infty \)-functions on \( \mathcal{U} \) (test functions), then (18) becomes:

\[
\int_{\mathcal{U}} f_j(x) \frac{\partial \psi}{\partial x_i}(x) dx = \int_{\mathcal{U}} f_i(x) \frac{\partial \psi}{\partial x_j}(x) dx, \quad \text{for every } \psi \in C^\infty_0(\mathcal{U}).
\]

We now give an elegant characterization of Clarke subdifferentials in the spirit of this work.

**Theorem 5.2** (Characterization of Clarke subdifferentials) Let \( \Phi: \mathbb{R}^d \rightrightarrows \mathbb{R}^d \) be a cusco map. The following are equivalent:

(i) \( \Phi = \partial \varphi \) for some locally Lipschitz function \( \varphi: \mathbb{R}^d \rightarrow \mathbb{R} \);

(ii) \( \Phi = F_f \) for some measurable selection \( f \) of \( \Phi \) that satisfies (18).

**Proof** (i) \( \Rightarrow \) (ii) Assume that \( \Phi = \partial \varphi \) for a locally Lipschitz function \( \varphi: \mathbb{R}^d \rightarrow \mathbb{R} \). Then by Rademacher’s theorem, there exists \( N_\varphi \in \mathcal{N} \) such that the derivative \( \nabla \varphi(x) \) exists for all \( x \in \mathbb{R}^d \setminus N_\varphi \). For \( x \in N_\varphi \), pick \( s(x) \in \partial \varphi(x) \) and set

\[
f(x) = \begin{cases} 
\nabla \varphi(x), & \text{if } x \in \mathbb{R}^d \setminus N_\varphi \\
s(x), & \text{if } x \in N_\varphi.
\end{cases}
\]

Then \( f: \mathbb{R}^d \rightarrow \mathbb{R}^d \) is a measurable selection of \( \partial \varphi \) and (being a.e. equal to a gradient) it satisfies (18), see [9, Proposition 3.1 (ii)]. Moreover,

\[
F_f(x) := \bigcap_{\delta > 0} \text{co} \ f(B_\delta(x) \setminus N_\varphi) = \bigcap_{\delta > 0} \text{co} \ \{ \nabla \varphi(x'): x' \in B_\delta(x) \setminus N_\varphi \}.
\]
Since \( \varphi \) is locally Lipschitz, we deduce [7, Chapter 2.6]
\[
\bigcap_{\delta > 0} \{ \nabla \varphi(x') : x' \in B_\delta(x) \setminus N\varphi \} = \{ \lim_{x_n \to x} \nabla \varphi(x_n) : \{ x_n \} \subset R^d \setminus N\varphi \} = \partial \varphi(x), \tag{20}
\]
which shows that (ii) holds for \( f \) being equal to \( \nabla \varphi \) a.e.

(ii) \( \implies \) (i) Assume that \( \Phi = F_f \), where \( f : R^d \to R^d \) is a measurable selection of \( \Phi \) that satisfies (18). Then by Theorem 5.1, there exists a locally Lipschitz function \( \varphi : R^d \to R \) such that \( f(x) = \nabla \varphi(x) \), for a.e. \( x \in R^d \). Then it follows from Proposition 2.6(v) and (19), (20) above that
\[
\partial \varphi(x) = F_{\nabla \varphi}(x) = F_f(x) = \Phi(x) \text{ for all } x \in R^d.
\]

\[\square\]

**Remark 5.3**

(i) It is possible to have \( \Phi = F_f \), without \( \Phi \) being a subdifferential; consider for instance the function \( f(x_1, x_2) = (x_2, -x_1) \), for all \( (x_1, x_2) \in R^2 \) [which obviously fails (18)]. Then \( \Phi = f \) cannot be a subdifferential.

(ii) It is possible to have infinite many measurable selections \( f(x) \in \Phi(x) \), for all \( x \in R^d \), each of which satisfies the nonsmooth Poincaré condition (18). Indeed, if we take \( \Phi \) to be identically equal to the closed ball \( \bar{B} \) for all \( x \in U \), then the set of all measurable selections that satisfy (18) contains isometrically the unit ball of the nonseparable Banach space \( \ell^\infty(N) \), see [11].

(iii) If \( \Phi = F_f \) and \( f \) is unique a.e. and satisfies (18), then by Theorem 5.2, \( \Phi = \partial \varphi \) and \( f = \nabla \varphi \) a.e. It follows that the locally Lipschitz function \( \varphi \) is unique up to a constant.

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