Homogeneous Finsler spaces with only one orbit of prime closed geodesics

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Abstract

When a closed Finsler manifold admits continuous isometric actions, estimating the number of orbits of prime closed geodesics seems a more reasonable substitution for estimating the number of prime closed geodesics. To generalize the works of H. Duan, Y. Long, H.B. Rademacher, W. Wang and others on the existence of two prime closed geodesics to the equivariant situation, we propose the question if a closed Finsler manifold has only one orbit of prime closed geodesic if and only if it is a compact rank-one Riemannian symmetric space. In this paper, we study this problem in homogeneous Finsler geometry, and get a positive answer when the dimension is even or the metric is reversible. We guess the rank inequality and algebraic techniques in this paper may continue to play an important role for discussing our question in the non-homogeneous situation.

Key words: Homogeneous Finsler space, closed geodesic, compact rank-one symmetric space, connected isometry group, Killing vector field.

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1 Introduction

In Riemannian geometry, it has been conjectured for many decades that on any closed manifold $M$ with $\dim M > 1$, there exist infinitely many prime closed geodesics. In Finsler geometry, this is not true because of the Katok spheres [17] found in 1973. Katok spheres are Randers spheres of constant flag curvature which were much recently classified in [7]. Based on the Katok spheres, D.V. Anosov proposed another conjecture, claiming the existence of $2^\lceil \frac{n+1}{2} \rceil$ prime closed geodesics on the Finsler sphere $(S^n, F)$ [2]. See [6] and [27] for some recent progress on this conjecture.

Generally speaking, finding the first prime closed geodesic on a compact Finsler manifold is relatively easy (see [14] or [18]). Finding the second is already a hard problem if no topological obstacle from [15] and [25] is accessible. It was relatively recent that H. Duan and Y. Long [11] and H.B. Rademacher [23] [24] provided different proofs of the following theorem.

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**Theorem 1.1** A bumpy and irreversible Finsler metric on a sphere $S^n$ of dimension $n \geq 3$ carries two prime closed geodesics.

More generally, when $S^n$ is changed to other compact manifold, H. Duan, Y. Long and W. Wang proved the following theorem in [12].

**Theorem 1.2** There exist always at least two prime closed geodesics on every compact simply connected bumpy irreversible Finsler manifold $(M, F)$.

In this paper, we will assume the Finsler manifold $(M, F)$ admits nontrivial continuous isometries and discuss an equivalent analog of above theorems. Some thought and technique were purposed in [5] and further developed in [28] and [29], while studying the geodesics in a Finsler sphere of constant curvature.

It was suggested in [29] that, when the connected isometry group $G = I_0(M, F)$ has a positive dimension, estimating the number of prime closed geodesic seems more reasonable to be switched to estimating the number of orbits of prime closed geodesics, with respect to the action of $\hat{G} = G \times S^1$ (the precise description for this action will be explained at the end of Section 2). Though there are examples of compact Finsler manifolds with only one orbit of prime closed geodesics, they are very rare. The only known examples are compact rank-one Riemannian symmetric spaces, i.e. $S^n$, $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$ when simply connected, and $\mathbb{R}P^n$ otherwise, all endowed with their standard metrics.

Based on above observations and inspired by Theorem 1.1 and Theorem 1.2, we would like to ask

**Question 1.3** Assume $(M, F)$ is a closed connected Finsler manifold such that $G = I_0(M, F)$ has a positive dimension and $(M, F)$ has only one $\hat{G}$-orbit of prime closed geodesics. Must $M$ be one of the compact rank-one Riemannian symmetric spaces?

We will show some clue for a positive answer to Question 1.3. We assume $(M, F)$ is a Finsler manifold as described in Question 1.3 and discuss its properties. In particular, we prove that each closed geodesic on $(M, F)$ is homogeneous, i.e. the orbit of a one-parameter subgroup in $G = I_0(M, F)$ (see Lemma 3.3). It implies that the set of all prime closed geodesics on $(M, F)$ is a $G$-orbit.

The union $N$ of all the closed geodesics in $M$ is crucial for discussing Question 1.3. It is a $G$-orbit in $M$, which can be presented as $N = G/H$. Notice that the $G$-action on $N$ is almost effective. The first important theorem we prove in this paper is the rank inequality for $N$, i.e. $\operatorname{rk} G \leq \operatorname{rk} H + 1$. To be precise, $\operatorname{rk} G = \operatorname{rk} H$ when $\dim N$ is even, and $\operatorname{rk} G = \operatorname{rk} H + 1$ otherwise (see Theorem 3.5).

It is an important problem to explore if or when $N$ is totally geodesic in $(M, F)$. We will discuss this problem in subsequent works. In this paper, we only discuss the case that $N = M$, and present a classification for the compact connected homogeneous Finsler space $(M, F) = (G/H, F)$ with $G = I_0(M, F)$ and only one orbit of prime closed geodesics.

The two classification theorems, Theorem 6.1 and Theorem 7.1 can be summarized as following.
**Theorem 1.4** Assume \((M, F)\) is a compact connected homogeneous Finsler space with only one orbit of prime closed geodesics. If \(\dim M\) is odd, we further assume \(F\) is reversible. Then \((M, F)\) must be a Riemannian symmetric \(S^n, \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n\) or \(\mathbb{O}P^2\).

The proof of Theorem 1.4 (i.e. the proofs of Theorem 6.1 and Theorem 7.1) mainly uses the rank inequality in Theorem 3.5 and the discussion for compact Lie algebras. Amazingly, it rhythms with the classification of positively curved homogeneous Finsler spaces [30][31][32][34]. Many algebraic techniques are borrowed from those works.

There might be an alternative approach proving Theorem 1.4 by the following theorem of J. McCleary and W. Ziller in [20].

**Theorem 1.5** Let \(M\) be a compact connected simply connected homogeneous space which is not diffeomorphic to a symmetric space of rank 1. Then the betti numbers \(b_i(\Lambda M, \mathbb{Z}_2)\) are unbounded.

In this topological approach, we might need the non-degenerate condition for all orbits of closed geodesics to apply a generalized Gromoll-Meyer’s theorem; see Theorem 3.1 in [21]. More discussions are needed to cover the non-simply connected case and the gap from determining the manifold to determining the metric. After all, Theorem 1.5 is a very hard topological theorem. Our proof of Theorem 1.4 mainly uses Lie theory, seems much more fundamental.

Theorem 1.4 answers Question 1.3 affirmatively in homogeneous Riemannian geometry, as well as for many homogeneous Finsler spaces. Many techniques for proving Theorem 1.4 might be extended to non-homogeneous context and play an important role in future works on Question 1.3.

At the end, we discuss the special case of homogeneous Finsler spheres as the application of these techniques. We prove that a homogeneous Finsler sphere \((M, F)\) has only one orbit of prime closed geodesics only when it is a Riemannian symmetric sphere (see Proposition 8.4). In this case, we do not need to assume the reversibility of \(F\) in advance.

This paper is organized as following. In Section 2, we summarize some background knowledge in general and homogeneous Finsler geometry. In Section 3, we prove the rank inequality, i.e. Theorem 3.5. In Section 4, we discuss some examples of homogeneous Finsler spaces, with only one, or with more orbits of prime closed geodesics. In Section 5, we introduce the algebraic setup and some orthogonality lemmas in homogeneous Finsler geometry. In Section 6, we classify even dimensional homogeneous Finsler spaces with only one orbit of prime closed geodesics. In Section 7 and Section 8, we classify odd dimensional reversible homogeneous Finsler spaces with only one orbit of prime closed geodesics.

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2 Preliminaries

Here we briefly summarize some fundamental knowledge on general and homogeneous Finsler geometry. See [4] and [9] for more details.

Unless otherwise specified, we will only discuss closed connected smooth manifold in this paper.

A Finsler metric on an $n$-dimensional manifold $M$ is a continuous function $F : TM \to [0, +\infty)$, satisfying the following conditions:

1. $F$ is positive and smooth on the slit tangent bundle $TM\backslash 0$;
2. $F$ is positively homogeneous of degree one, i.e. for any $x \in M$, $y \in T_xM$ and $\lambda \geq 0$, we have $F(x, \lambda y) = \lambda F(x, y)$;
3. $F$ is strictly convex, i.e. for any standard local chart $x = (x^i) \in M$ and $y = y^i \partial_{x^i} \in T_xM$, the Hessian matrix $(g_{ij}(x, y)) = (\frac{1}{2} [F^2(x, y)]_{y^i y^j})$ is positive definite whenever $y \neq 0$.

We will also call $(M, F)$ a Finsler space or a Finsler manifold. The restriction of $F$ to each tangent space is called a Minkowski norm.

A Finsler metric $F$ is called reversible if for any $x \in M$ and $y \in T_xM$, $F(x, y) = F(x, -y)$.

The Hessian matrix $(g_{ij}(x, y))$, where $y \in T_xM$ is nonzero, defines an inner product on $T_xM$, i.e. for any $u = u^i \partial_{x^i}$ and $v = v^j \partial_{x^j}$ in $T_xM$,

$$ (u, v)^F_y = \frac{1}{2} \frac{\partial^2}{\partial s \partial t} [F^2(y + su + tv)]_{y^i y^j} = g_{ij}(x, y)u^i v^j. $$

Sometimes, we simply denote this inner product as $g^F_y$.

A geodesic $c(t)$ on $(M, F)$ is a nonconstant smooth curve satisfying the local minimizing principle for the arch length functional. Usually we parametrize it to have a constant positive speed. Then the curve $(c(t), \dot{c}(t))$ in $TM\backslash 0$ is an integration curve of the geodesic spray vector field $G = y^i \partial_{x^i} - 2G^i \partial_{y^i}$ where

$$ G_i = \frac{1}{4} g^{il} ([F^2]_{x^k y^l} y^k - [F^2]_{x^i}). $$

A geodesic is called reversible if it is still a geodesic with its direction reversed. A closed geodesic is called prime if it is not the multiple rotation of another. When we count the closed geodesics, we only count the prime ones. On the other hand, we specify the directions, i.e. a prime reversible closed geodesic is counted as two. Different closed geodesics are geometrically the same if their images are the same subsets in $M$.

The isometry group $I(M, F)$ of the compact connected Finsler space $(M, F)$ is a compact Lie group [10]. Its identity component $I_0(M, F)$ is called the connected isometry group. We call $(M, F)$ homogeneous if $I_0(M, F)$ acts transitively on $M$. A homogeneous Finsler space $(M, F)$ may have different presentations $M = G/H$ where $G$ is a closed connected subgroup of $I_0(M, F)$ which acts transitively, and $H$ is the compact isotropy subgroup at $o = eH \in G/H = M$. Denote $\mathfrak{g} = \text{Lie}(G)$, $\mathfrak{h} = \text{Lie}(H)$, and $\mathfrak{m}$ any $\text{Ad}(H)$-invariant complement of $\mathfrak{h}$ in $\mathfrak{g}$, then we call $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ a reductive
decomposition for $G/H$. The $G$-invariant Finsler metric $F$ is one-to-one determined by its restriction to $m = T_o(G/H)$, which is an $\text{Ad}(H)$-invariant Minkowski norm.

The Lie algebra $\mathfrak{g}$ of $G = I_o(M, F)$ can be identified with the linear space of all Killing vector field on $(M, F)$. For any vector $u \in \mathfrak{g}$, we denote $X^u$ the Killing vector field that $u$ defines on $(M, F)$. We denote $\text{rk} G = \text{rk} \mathfrak{g}$ the rank of the compact Lie group $G$ and its Lie algebra, which is the dimension of the maximal torus or the Cartan subalgebra.

Isometries and Killing vector fields play an important role in studying geodesics, with the following two frequently used lemmas.

**Lemma 2.1** In a closed Finsler space $(M, F)$, the common fixed point set of a family of isometries in $I_o(M, F)$, or the common zero point set of a family of Killing vector fields of $(M, F)$, is a finite disjoint union of connected imbedded totally geodesic submanifolds.

If a totally geodesic submanifold has a positive dimension, then its geodesics, with respect to the submanifold metric, are also geodesics for the ambient Finsler space.

**Lemma 2.2** Let $X$ be a Killing vector field on the Finsler space $(M, F)$, and $x \in M$ a critical point for the function $f(\cdot) = F(X(\cdot))$ with $X(x) \neq 0$. Then the integration curve of $X$ at $x$ is a geodesic. In particular, when $X$ generates an $S^1$-subgroup in $I_o(M, F)$, its integration curve at $x$ is a closed geodesic.

Lemma 2.1 is a well known easy fact in Riemannian and Finsler geometry, so we skip its proof. See [8] for the case of zero point sets of Killing vector fields. Lemma 2.2 follows immediately Lemma 3.1 in [13]. We call a geodesic homogeneous (or non-homogeneous) if and only if it is (or is not, respectively) the orbit of a one-parameter subgroup in $I_o(M, F)$. Thus Lemma 2.2 provides homogeneous geodesics from Killing vector fields which are orbits of $S^1$-subgroups in $I_o(M, F)$.

Killing vector fields which generate $S^1$-subgroups in $I_o(M, F)$ can be easily found according to the following lemma.

**Lemma 2.3** Let $G$ be a compact connected Lie group, and $\mathfrak{g}$ its Lie algebra. Then the subset $S$ of all vectors in $\mathfrak{g}$ which generate $S^1$-subgroups in $G$ is a dense subset in $\mathfrak{g}$.

**Proof.** The subset $S$ in the lemma is $\text{Ad}(G)$-invariant. Using the conjugation theorem, we only need to prove $S \cap t$ is dense in $t$, where $t$ is a Cartan subalgebra in $\mathfrak{g}$, generating a maximal torus. The statement is then obvious. ■

In [34], we have applied the following fixed point set technique based on Lemma 2.1. Let $(M, F) = (G/H, F)$ be a homogeneous Finsler space with a compact $G$. For any subset $\mathcal{L} \subset H$, we denote $\text{Fix}(\mathcal{L}, M)$ the common fixed point set of $\mathcal{L}$ on $M$. Obviously $o = eH \in \text{Fix}(\mathcal{L}, M)$, so we will denote $\text{Fix}_o(\mathcal{L}, M)$ the connected component of $\text{Fix}(\mathcal{L}, M)$ containing $o$. By Lemma 2.1 $\text{Fix}_o(\mathcal{L}, M)$ is totally geodesic in $(M, F)$. Furthermore, $\text{Fix}_o(\mathcal{L}, M) = G' \cdot o$, where $G'$ is the identity component of the centralizer $C_G(\mathcal{L})$ for $\mathcal{L}$ in $G$. Because $F' = F|_{\text{Fix}_o(\mathcal{L}, M)}$ is $G'$-invariant, so we may present $\text{Fix}(\mathcal{L}, M)$ as a homogeneous Finsler space $(G'/G' \cap H, F')$.

At the end of this section. We give the precise description for orbits of prime closed geodesics.
On the free loop space $\Lambda M$ of all the piecewise smooth curves $c(t): S^1 = \mathbb{R}/\mathbb{Z} \to M$ on the Finsler space $(M, F)$, we have the canonical action of $\hat{G} = I_0(M, F) \times S^1$ defined by

$$((\rho, t') \cdot c)(t) = \rho(c(t + t'))$$

for any $t, t' \in \mathbb{R}/\mathbb{Z}$ and $\rho \in I_0(M, F).$ (2.1)

Obviously this action preserves the subset of all the prime closed geodesics. Each $\hat{G}$-orbit of prime closed geodesics is a finite dimensional submanifold in $\Lambda M$.

3 Rank inequality

Let $(M, F)$ be a closed connected Finsler manifold such that $\dim M > 1$ and the connected isometry group $G = I_0(M, F)$ has a positive dimension. Then the existence of two prime closed geodesics follows immediately.

**Lemma 3.1** Assume $(M, F)$ is a closed connected Finsler manifold such that $G = I_0(M, F)$ has a positive dimension. Then there exists two distinct prime closed geodesics.

**Proof.** Because $G = I_0(M, F)$ is a compact connected Lie group with a positive dimension, we can find a vector $u \in \mathfrak{g} = \text{Lie}(G)$, which generates an $S^1$-subgroup. Let $x_1 \in M$ be a maximum point of the function $f_1(\cdot) = F(X^u(\cdot))$, and $x_2$ a maximum point of the function $f_2(\cdot) = F(-X^u(\cdot))$. Then the integration curve of $X^u$ at $x_1$ and the integratin curve of $-X^u$ at $x_2$ are two distinct closed geodesics of $(M, F)$.

We further assume

**Assumption (I):** The Finsler space $(M, F)$ has only one $\hat{G}$-orbit of prime closed geodesics, where $\hat{G} = G \times S^1$ acts on the closed geodesics as in (2.1).

When Assumption (I) is satisfied, by Lemma 3.1 and the connectedness of $\hat{G}$, there must exist infinitely many geometrically distinct closed geodesics, i.e. the $\hat{G}$-orbit of prime closed geodesics, as a submanifold in $\Lambda M$, has a dimension bigger than 1.

Assumption (I) implies the following immediate consequences.

**Lemma 3.2** Assume $(M, F)$ with $\dim M > 1$ and $\dim I_0(M, F) > 0$ is a closed connected Finsler space satisfying Assumption (I), then $\hat{G} = I_0(M, F)$ is semi-simple.

**Proof.** Assume conversely that $G = I_0(M, F)$ has a center of positive dimension, which corresponds to the center $\mathfrak{c}(\mathfrak{g})$ of $\mathfrak{g} = \text{Lie}(G)$. We can find a nonzero vector $u \in \mathfrak{c}(\mathfrak{g})$, which generates an $S^1$-subgroup. Let $X^u$ be the Killing vector field on $(M, F)$ defined by $u$. In the proof of Lemma 3.1, we have shown two different prime closed geodesics $\gamma_1$ and $\gamma_2$, which are integration curves of $X^u$ and $-X^u$ respectively. Because $u$ commutes with $\mathfrak{g}$, for each $g \in G = I_0(M, F)$, $g_*X = X$. So each prime closed geodesic in the $\hat{G}$-orbit of $\gamma_1$ is also an integration curve of $X^u$, i.e. $\gamma_2$ belongs to another orbit. This is a contradiction to Assumption (I), which ends the proof of the lemma.

**Lemma 3.3** Assume $(M, F)$ with $\dim M > 1$ and $\dim I_0(M, F) > 0$ is a closed connected Finsler space satisfying Assumption (I), then each closed geodesic on $(M, F)$ is homogeneous, i.e. the orbit of a one-parameter subgroup in $G$.  

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Proof. Assume conversely that there exists a non-homogeneous closed geodesic on $(M, F)$. Then Assumption (I) implies that all closed geodesics on $(M, F)$ are non-homogeneous. Because $G = I_0(M, F)$ is a compact connected Lie group with a positive dimension, we can find a nonzero vector $u \in \mathfrak{g} = \text{Lie}(G)$ which generates an $S^1$-subgroup. Denote $X^u$ the Killing vector field defined by $u$. By Lemma 2.2, at any maximum $x \in M$ for the function $f(\cdot) = F(X^u(\cdot))$, the integration curve of $X^u$ provides a homogeneous prime closed geodesic. This is a contradiction, which ends the proof of the Lemma. ■

Lemma 3.3 implies that the $S^1$-action along each prime closed geodesic $c(t)$ shifting the variable $t$ can be achieved by the $G$-action, so we have immediate that $\hat{G}$-orbit of prime closed geodesics in Assumption (I) is also a $G$-orbit. Further more, the union $N$ of all the closed geodesics is a $G$-orbit in $M$, and the induced submanifold metric $F|_N$ is $G$-invariant.

Lemma 3.4 Assume $(M, F)$ with $\dim M > 1$ and $\dim I_0(M, F) > 0$ is a closed connected Finsler space satisfying Assumption (I). Let $N$ be the union of all the closed geodesics. Then the $G$-action on $N$ is almost effective.

Proof. Assume conversely that the $G$-action on $N$ is not almost effective, i.e. there exists a closed subgroup $G'$ in $G$ with a positive dimension, which acts trivially on $N$. Then we can find a vector $u \in \mathfrak{g}' = \text{Lie}(G')$ which generates an $S^1$-subgroup, and defines a Killing vector field $X^u$ on $(M, F)$. By Lemma 2.2, the integration curve of $X^u$ at any maximum point $x \in M$ provides a prime closed geodesic of $(M, F)$ outside $N$. This is a contradiction with Assumption (I) which proves the lemma. ■

We denote $H$ the isotropy subgroup at some $o \in M$, then we can identify $N$ with $G/H$ with $o = eH \in G/H$. By Lemma 3.3, $\mathfrak{h} = \text{Lie}(H)$ contains no nonzero ideals of $\mathfrak{g} = \text{Lie}(G)$. We fix a reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ for $G/H$ which is orthogonal with respect to a chosen bi-invariant inner product on $\mathfrak{g}$. When we identify $\mathfrak{m}$ with the tangent space $T_oM$, the $\text{Ad}(H)$-action on $\mathfrak{m}$ coincides with the isotropy action. The key observation is the following rank inequality.

**Theorem 3.5** Assume $(M, F)$ with $\dim M > 1$ is a closed connected Finsler space with a connected isometry group $G = I_0(M, F)$ of positive dimension and only one $\hat{G}$-orbit of prime closed geodesics. Then $N = G/H$, the union of all prime closed geodesics on $(M, F)$, satisfies $\text{rk}\mathfrak{g} \leq \text{rk}\mathfrak{h} + 1$. To be more precise, $\text{rk}\mathfrak{g} = \text{rk}\mathfrak{h}$ when $\dim N$ is even, and $\text{rk}\mathfrak{g} = \text{rk}\mathfrak{h} + 1$ when $\dim N$ is odd.

Proof. Let $c(t) : \mathbb{R}/\mathbb{Z} \to M$ be any prime closed geodesic passing $o = eH \in M$. Define two subgroups of $G$,

$$H_1 = \{ \rho \in I_0(M, F) | \exists t_0 \in \mathbb{R}/\mathbb{Z} \text{ with } \rho(c(t)) = c(t + t_0), \forall t \},$$

$$H_2 = \{ \rho \in I_0(M, F) | \rho(c(t)) = c(t), \forall t \}.$$

Obviously, both are compact, and $H_2 = H_1 \cap H$ is a normal subgroup of $H_1$ with an $S^1$-quotient. Denote their Lie algebras as $\mathfrak{h}_i = \text{Lie}(H_i)$ for $i = 1$ and 2 respectively.

Firstly, we claim
Claim 1: the union of all the Ad($G$)-conjugations of $h_1$, i.e. $\mathcal{U} = \cup_{g \in G} \text{Ad}(g)h_1$, is a closed subset in $\mathfrak{g}$.

Proof of Claim 1. Let $\{\text{Ad}(g_i)x_i\} \subset \mathcal{U}$ be a convergent sequence in $\mathfrak{g}$, with $g_i \in G$ and $x_i \in h_1$ for each $i \in \mathbb{N}$. By the compactness of $G$, we may assume $\lim_{i \to \infty} g_i = g \in G$ by taking a subsequence. The Ad($G$)-actions preserve the chosen bi-invariant inner product on $\mathfrak{g}$, so $\{x_i\}$ is a bounded sequence in $h_1$. Taking a subsequence, we may further assume $\lim_{i \to \infty} x_i = x \in h_1$. To summarize, we have $\lim_{i \to \infty} \text{Ad}(g_i)x_i = \text{Ad}(g)x \in \mathcal{U}$, which proves the closeness of $\mathcal{U}$, i.e. Claim 1.

Secondly, we claim

Claim 2: $\mathcal{U} = \cup_{g \in G} \text{Ad}(g)h_1$ coincides with $\mathfrak{g}$.

Proof of Claim 2. Assume conversely $\mathcal{U} \neq \mathfrak{g}$. By Lemma 2.3 and Claim 1, we can find a vector $u \in \mathfrak{g}\setminus\mathcal{U}$ which generates an $S^1$-subgroup. By Assumption (I), the Killing vector field $X^u$ on $(M, F)$ is not tangent to any prime closed geodesics of $(M, F)$. By Lemma 2.2 again, its integration curve at the maximum point of $f(\cdot) = F(X(\cdot))$ provides a prime closed geodesic outside $N$. This is a contradiction which ends the proof of Claim 2.

Finally, we finish the proof of the theorem. By Claim 2, $h_1$ contains a generic vector in $\mathfrak{g}$ which generates a dense one-parameter subgroup in a maximal torus of $G$. So $H_1$ must contain a maximal torus of $G$, i.e. $rk h_1 = rk \mathfrak{g}$. Then we have

$$rk h \geq rk h_2 = rk h_1 - 1 = rk \mathfrak{g} - 1,$$

which proves the first statement in the theorem.

The other statements follow the first one, and the same argument in the classification of positively curved homogeneous spaces. See for example the proof of Theorem 4.1 in [20].

This ends the proof of the theorem. ■

4 Homogeneous examples for Assumption (I)

Since this section, we concentrate on compact connected homogeneous Finsler spaces satisfying Assumption (I), i.e. with only one orbit of prime closed geodesics.

Before the systematical discussion, we study some important examples, satisfying or not satisfying Assumption (I).

Example 1. All compact simply connected Riemannian rank-one symmetric spaces satisfy Assumption (I). They can be listed as

$$S^n = \text{SO}(n+1)/\text{SO}(n) \quad \text{with} \ n > 1, \quad \mathbb{CP}^n = \text{SU}(n+1)/\text{SU}(1)\text{U}(n), \quad \mathbb{HP}^n = \text{Sp}(n+1)/\text{Sp}(1)\text{Sp}(n), \quad \text{and} \quad \mathbb{OP}^2 = \text{F}_4/\text{Spin}(9). \quad (4.2)$$

Notice that on $S^0 = G_2/\text{SU}(3)$ and $S^7 = \text{Spin}(7)/G_2$, any invariant Finsler metrics on them must be Riemannian symmetric.

More generally, we have

Proposition 4.1 Any compact connected Riemannian manifold locally isometric to a compact rank-one symmetric space and satisfying Assumption (I) must be one of the Riemannian rank-one symmetric spaces, i.e. for all $n > 1$, $S^n$, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $\mathbb{HP}^n$, and $\mathbb{OP}^2$. 

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Proof. Assume that $M$ is a locally isometric to a compact Riemannian rank-one symmetric space, non-simply connected, and satisfies Assumption (I).

We have a locally isometric finite covering map $\pi: \tilde{M} \to M$ such that $\tilde{M}$ is one of $\{1,2\}$. All geodesics on $M$ are closed, and all prime closed geodesics on $M$ have the same length. So any geodesic passing $x \in \tilde{M}$ must contain points in $\pi^{-1}(\pi(x)) \setminus \{x\} \neq \emptyset$.

When $\tilde{M} = S^n$ with $n > 1$, $\pi^{-1}(x) \setminus \{x\}$ must coincide with the antipodal point $x'$ of $x$, otherwise $\pi^{-1}(\pi(x))$ is an infinite set, which is a contradiction. So in this case $M$ is a Riemannian symmetric $\mathbb{RP}^n$.

When $\tilde{M} = \mathbb{K}P^n$ with $\mathbb{K} = \mathbb{C}, \mathbb{H}$ and $n \geq 2$ or $\mathbb{K} = \mathbb{O}$ and $n = 2$, for any $x \in M$ and any $x'$ in the cut locus of $x$, we can find a sphere $M' = \mathbb{K}P^1$ of constant curvature, such that $M'$ is imbedded in $\tilde{M}$ as a totally geodesic submanifold and $x$ and $x'$ are contained in $M'$ as a pair of antipodal points. As for the previous case, Assumption (I) and the finiteness of $\pi^{-1}(\pi(x))$ implies $\pi^{-1}(\pi(x))$ must contain any point $x'$ in the cut locus of $x$. This is a contradiction to the finiteness of $\pi^{-1}(\pi(x))$.

This ends of proof of this proposition. 

Example 2. The flat torus $(T^n, F)$ with $n > 1$ does not satisfy Assumption (I).

We can find arbitrarily long prime closed geodesics on a flat torus. But the prime closed geodesics in the same orbit must have the same length.

Example 3. Any $SO(n_1 + 1) \times SO(n_2 + 1)$-invariant Finsler metric $F$ on $M = S^{n_1} \times S^{n_2} = SO(n_1 + 1) \times SO(n_2 + 1)/SO(n_1) \times SO(n_2)$ does not satisfy Assumption (I).

The $SO(n_1 + 1) \times SO(n_2 + 1)$-invariance of $F$ may imply more isometries. If $n_i > 1$, the term $SO(n_i + 1)$ can be changed to $O(n_i + 1)$.

For each $i = 1$ and $2$, we take $G'_i = \{e\}$ when $n_i = 1$, $G'_i = SO(n_i - 1)$ when $n_i > 3$, and $G'_i = \{g, e\} \subset O(n_i + 1)$ where $g$ is a reflection when $n_i = 3$. Then the fixed point set $\text{Fix}(G'_1 \times G'_2, M)$ is a flat 2-dimensional torus imbedded in $M$ as a totally geodesic submanifold. Our claim is then implied by Example 2.

Example 4. When $F$ is a left invariant Finsler metric on $G = SO(2) \times SU(2)$, $(G, F)$ does not satisfy Assumption (I).

This is a special case of the following proposition.

Proposition 4.2 Any left invariant Finsler metric $F$ on a compact connected Lie group $G$ with $\text{rk}G > 1$ does not satisfy Assumption (I). Further more, it has arbitrarily long prime closed geodesics.

Proof. We fix a bi-invariant inner product $| \cdot , |^2_{bi} = \langle \cdot, \cdot \rangle_{bi}$ on $g = \text{Lie}(G)$, and a Cartan subalgebra $t$ in $g$. The Killing vector fields on $(G, F)$ are right invariant vector fields. There exists a positive constant $c > 0$, such that for any nonzero vector $u \in g$, $F(X^u(g)) > c$ for all $g \in G$. The flow generated by $X^u$ has the same period (which may be infinity) everywhere on $G$.

We can find a sequence of nonzero vectors $u_n \in t$ satisfying the following:

(1) Each $u_n$ generates an $S^1$-subgroup.

(2) The period $T_n$ for the flow $\rho_{n,t}$ generated by $X^{u_n}$ diverges to infinity.
Then the integration curve of $X^{\mu \nu}$ at the maximum point of $f(\cdot) = F(X^{\nu}(\cdot))$ provides a prime closed geodesic $\gamma_n$. The length $l(\gamma_n)$ of $\gamma_n$ is at least $c \cdot T_n$ which diverges to infinity.

This ends the proof of the proposition.

**Example 5.** When $F$ is an $\text{SO}(3) \times \text{SU}(2)$-invariant Finsler metric on $M = S^2 \times S^3 = \text{SO}(3) \times \text{SU}(2)/\text{SO}(2) \times \{e\}$, it does not satisfy Assumption (I).

The metric $F$ is in fact $\text{O}(3) \times \text{SU}(2)$-invariant. The fixed point set $M' = \text{Fix}(g, M)$ of a reflection $g \in \text{O}(3)$ in $M$ can be identified as the Lie group $G = \text{SO}(2) \times \text{SU}(2)$ such that $F|_{M'}$ is left invariant. By the discussion for Example 4, $(M, F)$ does not satisfy Assumption (I).

Based on Example 2-5, we can find more examples by the following two propositions.

**Proposition 4.3** If a compact connected homogeneous Finsler space $(M, F)$ is finitely covered by and locally isometric to one of Example 2-5, then $(M, F)$ does not satisfy Assumption (I). In particular, it has arbitrarily long prime closed geodesics.

**Proof.** Assume the Finsler space $(M, F)$ has a finite cover $(\tilde{M}, \tilde{F})$ which is one of Example 2-5, such that the covering map $\pi: (\tilde{M}, \tilde{F}) \to (M, F)$ is locally isometric.

On $(\tilde{M}, \tilde{F})$, we can find a sequence of prime closed geodesics $\tilde{\gamma}_n$ such that their lengths $l(\tilde{\gamma}_n)$ diverge to infinity. Then $\pi(\tilde{\gamma}_n)$ is a closed geodesic on $(M, F)$, which corresponds to a prime closed geodesic $\gamma_n$. If $\pi$ is an $m$-fold covering, then the lengths $l(\gamma_n) \geq l(\tilde{\gamma}_n)/m$ which also diverge to infinity. So $(M, F)$ does not satisfy Assumption (I), which proves the proposition.

**Proposition 4.4** Assume $(M, F) = (G/H, F)$ is a compact connected homogeneous Finsler space such that $G$ is a compact connected Lie group and $\text{rk} G > \text{rk} H + 1$, then $(M, F)$ does not satisfy Assumption (I), and it has arbitrarily long prime closed geodesics.

**Proof.** Let $T$ be a maximal torus in $H$ and denote $M' = \text{Fix}_o(T, M)$. Then $M'$ is totally geodesic in $M$ and $(M', F'|_{M'})$ is finitely covered by and locally isometric to a compact connected Lie group $G'$ with $\text{rk} G' > 1$ and a left invariant Finsler metric $F'$. The argument in the proofs of Proposition 4.2 and Proposition 4.3 shows there exist arbitrarily long prime closed geodesics on $(G', F')$ as well as on $(M', F'|_{M'})$ and $(M, F)$.

This ends the proof of the proposition.

5 **Algebraic setup and $g^F_{\mu \nu}$-orthogonality**

Assume $(M, F) = (G/H, F)$, where $G = I_0(M, F)$, is a compact connected homogeneous Finsler space satisfying Assumption (I), i.e. it has only one orbit of prime closed geodesics.

We fix an $\text{Ad}(G)$-invariant inner product $| \cdot |^2_{bi} = \langle \cdot, \cdot \rangle_{bi}$. For simplicity, we call the orthogonality with respect to this inner product $bi$-invariant. There is a unique bi-invariant reductive decomposition $g = h + m$. We denote $\text{pr}_h$ and $\text{pr}_m$ the projection maps according to this decomposition.

Any vector $u \in m$ represents a tangent vector in $T_o M$. Meanwhile $u \in g$ also defines the Killing vector field $X^u$ which satisfies $X^u(o) = u \in T_o M = m$. Any Killing vector
field $X$ of $(M, F)$ satisfying $X(o) = u$ can be presented as $X^{u+u'}$ for some $u' \in \mathfrak{h}$. By Proposition 3.4 in [35], the integration curve of $X^{u+u'}$ at $o$ is a geodesic if and only if

$$\langle u, [u + u', m]_m \rangle_u^F = 0.$$  

In particular, when $u' = 0$ we have

**Lemma 5.1** Let $(G/H, F)$ be a compact connected homogeneous Finsler space with a bi-invariant reductive decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$. Assume $u \in \mathfrak{m}$ is a nonzero vector satisfying

$$\langle u, [u, m]_m \rangle_u^F = 0,$$

then the integration curve of $X^u$ at $o$ is a geodesic. In particular, if $u$ generates an $S^1$-subgroup in $G$, then $X$ generates a closed geodesic at $o$.

Define $\mathcal{C}$ to be the subset of all $u \in \mathfrak{m}$ such that $|u|_{bi} = 1$ and the geodesic of $(M, F)$ passing $o$ in the direction of $u$ is closed. Assumption (I) implies $\mathcal{C}$ is an Ad($H$)-orbit.

Lemma 5.1 is a key technique for us to determine $\mathcal{C}$. Vectors in $\mathfrak{m}$ which generate $S^1$-subgroups are not hard to be found, for example, from $t \cap \mathfrak{m}$ or a root plane. The equality (5.3) implies the $g_u^F$-orthogonality is the remaining issue to be considered, which will be discussed for the cases dim $M$ is even and odd separately.

**Case 1.** Assume dim $M$ is even.

By Theorem 3.5, we can find a Cartan subalgebra $t$ of $\mathfrak{g}$ which is contained in $\mathfrak{h}$. With respect to $t$, we have the root plane decomposition for $\mathfrak{g}$,

$$\mathfrak{g} = t + \sum_{\alpha \in \Delta} \mathfrak{g}_{\pm \alpha},$$

where $\Delta \subset t^*$ is the root system of $\mathfrak{g}$, and $\mathfrak{g}_{\pm \alpha}$ is the root plane.

For $\mathfrak{h}$, we have a similar root plane decomposition with respect to $t$. The root system $\Delta'$ of $\mathfrak{h}$ is a subset of $\Delta$, and $\mathfrak{h}_{\pm \alpha} = \mathfrak{g}_{\pm \alpha}$ when $\alpha \in \Delta'$. Because the reductive decomposition is bi-invariant, each root plane $\mathfrak{g}_{\pm \alpha}$ is contained in either $\mathfrak{h}$ or $\mathfrak{m}$.

For any nonzero vector $u \in \mathfrak{g}_{\pm \alpha} \subset \mathfrak{m}$, we have the following $g_u^F$-orthogonality [32].

**Lemma 5.2** Let $G/H$ be an even dimensional compact connected homogeneous Finsler space, and keep all relevant notations and assumptions. Then for any nonzero $u \in \mathfrak{g}_{\pm \alpha} \subset \mathfrak{m}$, $\mathfrak{g}_{\pm \alpha}$ is $g_u^F$-orthogonal to all other root planes in $\mathfrak{m}$.

Sketchily, Lemma 5.2 can be proved as following. Let $T' = \exp \ker \alpha$ be the subtorus in $H$ generated by the kernel of $\alpha$ in $t$. We have a Ad($T$)-invariant decomposition of $\mathfrak{m}$ such that each summand is a sum of root planes and corresponds to a different irreducible representation of $T'$. Because the $T'$-action preserves $u \in T_u M$, the Ad($T'$)-action preserves the inner product $g_u^F$ on $\mathfrak{m}$. Schur’s Lemma implies different summand in $\mathfrak{m}$ are $g_u^F$-orthogonal to each other. In particular, the summand in $\mathfrak{m}$ corresponding to the trivial representation is $\mathfrak{g}_{\pm \alpha}$, which is then $g_u^F$-orthogonal to all other root planes in $\mathfrak{m}$. See Lemma 5.3 in [32] and its proof for more details.

The same thought can be applied to the case that dim $M$ is odd, concerning different decompositions and different group actions, and providing more $g_u^F$-orthogonality.
Lemma 5.4 Assume $u \neq 0$ for $\dim \hat{m}$ of Lemma 5.3 method as for Lemma 5.2, we get $\in u$ Finsler space and keep all relevant notations and assumptions. Then for any nonzero $m$ where $\hat{g}$ subalgebra of $g$ are subsets in $t$ irreducible $T$ inner product $\langle \cdot, \cdot \rangle$ where $\Delta$ and $\Delta'$ are root systems for $G/H$, $G/H$ and $F$, respectively. Assume $T_H$ is the maximal torus in $H$, generated by $t \cap h$. With respect to different irreducible $T_H$-representations, $g$ can be decomposed as

$$g = g_0 + \sum_{0 \neq \alpha' \in t \cap h} \hat{g}_{\pm \alpha'} = c_g(t \cap h) + \sum_{0 \neq \alpha' \in t \cap g} \left( \sum_{pr_h(\alpha') = \alpha'} g_{\pm \alpha} \right).$$

(5.5)

For each $\alpha' \in t \cap h$, $\hat{g}_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap h + \hat{g}_{\pm \alpha'} \cap m$. In particular $\hat{g}_0 \cap h = t \cap h$, $\hat{g}_{\pm \alpha'} \cap h$ if and only if $\alpha' \notin \Delta'$, and $\hat{g}_{\pm \alpha'} \cap h = h_{\pm \alpha'} \neq 0$ if and only if $\alpha' \in \Delta'$.

Moreover, we have the $\text{Ad}(T_H)$-invariant decomposition

$$m = \sum_{\alpha' \in t \cap h} \hat{m}_{\pm \alpha'},$$

(5.6)

where $\hat{m}_{\pm \alpha'} = \hat{g}_{\pm \alpha'} \cap m$, and in particular, $\hat{m}_0$ is a subalgebra, the centralizer $c_m(h)$ of $h$ in $m$. Either $\hat{m}_0 = t + g_{\pm \alpha}$ when there exists a root $\alpha$ in $t \cap m$, or $\hat{m}_0 = t \cap m$ otherwise.

The $T_H$-action preserves $u \in T_o M$ for any nonzero $u \in \hat{m}_0$. So applying the similar method as for Lemma 5.2, we get

Lemma 5.3 Assume $(G/H, F)$ is an odd dimensional compact connected homogeneous Finsler space and keep all relevant notations and assumptions. Then for any nonzero $u \in \hat{m}_0$, the decomposition $u_{\hat{m}}$ is $g_u^F$-orthogonal.

When $\dim \hat{m}_0 = 3$, we need more $g_u^F$-orthogonality which requires suitable choices of $u$ as following. There exist two different vectors $u_1$ and $u_2$ in $\hat{m}_0$ such that $|u_1|_{bi} = |u_2|_{bi} = 1$, $F(u_1) = \min \{ F(u) | u \in \hat{m}_0, |u|_{bi} = 1 \}$, and $F(u_2) = \max \{ F(u) | u \in \hat{m}_0, |u|_{bi} = 1 \}$.

Then we have

Lemma 5.4 Assume $(G/H, F)$ is an odd dimensional compact connected homogeneous Finsler space with $\dim \hat{m}_0 = 3$, and keep all relevant notations and assumptions, then for $u_i$ chosen above, we have $\langle u_i, [u_i, \hat{m}_0] \rangle^F_{u_i} = 0$.  

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Assume \( \alpha \) is a root of \( \mathfrak{g} \) such that \( \text{pr}_h(\alpha) = \alpha' \neq 0 \). Denote \( t' \) the bi-invariant orthogonal complement of \( \alpha' \) in \( t \cap \mathfrak{h} \), then we have a sub-torus \( T' \subset T_H \subset H \) generated by \( t' \). Denote \( \text{pr}_t \) the bi-invariant orthogonal projection to \( t' \).

According to different irreducible \( T' \)-representations, we have the \( \text{Ad}(T') \)-invariant decomposition

\[
\mathfrak{m} = \sum_{\beta'' \in t'} \mathfrak{m}_{\pm \beta''} = \sum_{\beta'' \in t'} \left( \sum_{\beta' \in t \cap \mathfrak{h}, \text{pr}_t(\beta') = \beta''} \mathfrak{m}_{\pm \beta'} \right).
\]  

(5.7)

For any nonzero \( u \in \hat{\mathfrak{m}}_0 \), the \( T' \)-action preserves \( u \in T_oM \). Applying the similar method for Lemma 5.2 we get

**Lemma 5.5** Assume \((G/H,F)\) is an odd dimensional compact connected homogeneous Finsler space and keep all relevant notations and assumptions. Then for any nonzero \( u \in \hat{\mathfrak{m}}_0 \), the decomposition (5.7) is \( g_u^F \)-orthogonal.

We will need more \( g_u^F \)-orthogonality inside \( \hat{\mathfrak{m}}_0 = \sum_{t \geq 0} \hat{\mathfrak{m}}_{\pm ta'} \), which may be achieved with the reversibility assumption for \( F \). We can find an element \( g \in T_H \) which action on each root plane in \( \hat{\mathfrak{g}}_{\pm ta'} \) is a rotation with angle \( t\pi \). Assume the invariant metric \( F \) is reversible, then for any nonzero \( u \in \hat{\mathfrak{m}}_{\pm a'} \), \( w_1, w_2 \in \mathfrak{m} \),

\[
\langle w_1, w_2 \rangle_u^F = \langle \text{Ad}(g)w_1, \text{Ad}(g)w_2 \rangle_{\text{Ad}(g)u}^F = \langle \text{Ad}(g)w_1, \text{Ad}(g)w_2 \rangle_u^F,
\]

i.e. the \( \text{Ad}(g) \)-action preserves \( g_u^F \). Applying the similar method as for Lemma 5.2 for the action of the group generated by \( T' \) and \( g \), we get

**Lemma 5.6** Assume \((G/H,F)\) is an odd dimensional compact connected reversible homogeneous Finsler space and keep all relevant notations and assumptions. Then for any nonzero \( u \in \hat{\mathfrak{m}}_{\pm a'} \), \( \hat{\mathfrak{m}}_{\pm t_1 a'} \) and \( \hat{\mathfrak{m}}_{\pm t_2 a'} \) are \( g_u^F \)-orthogonal when \( t_1 \) and \( t_2 \) are non-negative, and \( t_1 - t_2 \in 2\mathbb{Z} \). In particular, we have \( \langle \hat{\mathfrak{m}}_{\pm a'}, \hat{\mathfrak{m}}_0 \rangle_u^F = 0 \).

For any nonzero \( u \in \hat{\mathfrak{m}}_{\pm a'} \) where \( \alpha' \in t \cap \mathfrak{h} \) is nonzero, we have a plane \( \mathbb{R}u + [t \cap \mathfrak{h}, u] \) in \( \mathfrak{m} \). The restriction of \( F \) to this plane coincides with \( |\cdot|_{\mathfrak{h}} \) up to a scalar multiplication. So we have

**Lemma 5.7** Assume \((G/H,F)\) is an odd dimensional compact connected homogeneous Finsler space and keep all relevant notations and assumptions. Then for any nonzero \( u \in \hat{\mathfrak{m}}_{\pm a'} \) such that \( \alpha' \) is a nonzero vector in \( t \cap \mathfrak{h} \), we have \( \langle u, [t \cap \mathfrak{h}, u] \rangle_u^F = 0 \). In particular, when \( u \in \mathfrak{g}_{\pm a} \subset \hat{\mathfrak{m}}_{\pm a'} \), we have \( \langle u, [t \cap m, u] \rangle_u^F = 0 \).

At the end of this section, we remark that the \( g_u^F \)-orthogonality lemmas in Case 2 are reformulations of Lemma 3.6, Lemma 3.7 and Lemma 3.8 in [31], where more details can be found.
6 Classification when \( \dim M \) is even

In this section, we assume \((M, F) = (G/H, F)\) with \(G = I_0(M, F)\) is an even dimensional compact connected homogeneous Finsler space satisfying Assumption (I), i.e. it has only one orbit of prime closed geodesics. We keep all relevant notations and assumptions in Case 1, Section 5. Recall that \(\mathcal{C}\) is the subset of all vectors \(u \in m\) satisfying \(|u|_{b1} = 1\) and the geodesic passing \(o\) in the direction of \(u \in T_oM\) is closed. Because of Assumption (I), \(\mathcal{C}\) is an \(\text{Ad}(H)\)-orbit.

Our goal is to prove the following classification theorem.

**Theorem 6.1** Any even dimensional compact connected homogeneous Finsler space with only one orbit of prime closed geodesics is a compact rank-one Riemannian symmetric space, i.e. one of the following, \(S^{2n}\), \(\mathbb{R}P^{2n}\), \(\mathbb{C}P^n\), and \(\mathbb{H}P^n\). Its proof relies on the following lemmas.

**Lemma 6.2** Assume \((G/H, F)\) is an even dimensional compact connected homogeneous Finsler space and keep all relevant notations and assumptions. Then any \(u \in g_{\pm\alpha} \subset m\) such that \(|u|_{b1} = 1\) is contained in \(\mathcal{C}\).

**Proof.** The vector \(u\) indicated in the lemma is contained in a subalgebra of type \(A_1\), so it generates an \(S^1\)-subgroup in \(G = I_0(M, F)\). By Lemma 5.2,

\[
\langle u, [u, m]_m \rangle_u^F \subset \langle g_{\pm\alpha}, \sum_{\beta \neq \pm\alpha, \beta \notin \Delta'} g_{\pm\beta} \rangle_u^F = 0.
\]

Then Lemma 5.1 indicates that \(u \in \mathcal{C}\) when \(|u|_{b1} = 1\).

As a consequence of Lemma 6.2, we have

**Lemma 6.3** Assume \((M, F) = (G/H, F)\) is an even dimensional compact connected homogeneous Finsler space satisfying Assumption (I). Then the isotropy representation of \(G/H\) is irreducible.

**Proof.** Assume conversely that the isotropy representation is not irreducible, i.e. there exists a non-trivial \(\text{Ad}(H)\)-invariant decomposition \(m = m_1 + m_2\). Because the Cartan subalgebra \(t\) is contained in \(h = \text{Lie}(H)\), each \(m_i\) is a sum of root planes. So we can find a root \(\alpha_i\) with \(g_{\pm\alpha_i} \subset m_i\), and the vector \(u_i \in g_{\pm\alpha_i} \cap \mathcal{C}\) by Lemma 6.2 for each \(i = 1\) and 2 respectively. But it is impossible because the orbits \(\mathcal{C} = \text{Ad}(H)u_i \subset m_i\) for \(i = 1\) and 2 do not intersect with itself.

**Lemma 6.4** Assume \((G/H, F)\) is an even dimensional compact connected homogeneous Finsler space satisfying Assumption (I) and keep all relevant notations and assumptions. Then there do not exist a pair of linearly independent roots \(\alpha\) and \(\beta\) such that \(g_{\pm\alpha}\) and \(g_{\pm\beta}\) are contained in \(m\), and \(\alpha \pm \beta\) are not roots of \(g\).

**Proof.** Assume conversely that there exist a pair of roots \(\alpha\) and \(\beta\) indicated in this lemma.

Inside the maximal torus \(T = \exp t\), we have a codimension two sub-torus \(T' = \exp t' = \exp(\ker \alpha \cap \ker \beta)\). The totally geodesic submanifold \((\text{Fix}_o(T', M), F|_{\text{Fix}_o(T', M)})\)
is finitely covered by and locally isometric to $M' = S^2 \times S^2 = \text{SO}(3) \times \text{SO}(3) / \text{SO}(2)$ with an $\text{SO}(3) \times \text{SO}(3)$-invariant Finsler metric.

According to Example 3 and Proposition 4.3 in Section 4, we can find a sequence of prime closed geodesics $\gamma_n$ on $(\text{Fix}_o(T', M), F|_{\text{Fix}_o(T', M)})$, as well as on $(M, F)$, which lengths $l(\gamma_n)$'s diverge to infinity. They can not belong to the same orbit, which is a contradiction to Assumption (I).

This ends the proof of this lemma. ■

Now we are ready to prove Theorem 6.1.

Proof of Theorem 6.1. Let $(M, F) = (G/H, F)$ be an even dimensional compact connected homogeneous space satisfying Assumption (I). Denote $H_0$ the identity component of the isotropy subgroup $H$.

By Lemma 6.4 and the algebraic discussion in Section 6 of [26], $(g, h)$ belongs to the Wallach's list, i.e. it is one of the following,

1. $(B_n, D_n), (A_n, \mathbb{R} \oplus A_{n-1}), (C_n, A_1 \oplus C_{n-1}), (F_4, B_4)$.
2. $(A_2, \mathbb{R} \oplus \mathbb{R}), (C_3, A_1 \oplus A_1 \oplus A_1), (F_4, D_4)$.
3. $(G_2, A_2)$.
4. $(C_n, \mathbb{R} \oplus C_{n-1})$.

When the pair $(g, h)$ belongs to (1) or (3), $(G/H, F)$ is locally isometric to one of the compact rank-one Riemannian symmetric spaces. The theorem follows Proposition 4.1 immediately.

When $(g, h)$ belongs to (4), we have a unique $\text{Ad}(H_0)$-invariant decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ where $\mathfrak{m}_i$'s are irreducible representations of $H_0$ with $\dim \mathfrak{m}_1 = 2$ and $\dim \mathfrak{m}_2 = 4$. Here $H_0$ is the identity component of $H$, which is a normal subgroup of $H$ as well.

For each $g \in H$, the $\text{Ad}(g)$-action permutes the two $\mathfrak{m}_i$-factors in $\mathfrak{m}$. Because the two factors have different dimensions, we must have $\text{Ad}(g)\mathfrak{m}_i = \mathfrak{m}_i$ for each $i \in \{1, 2\}$ and each $g \in H$. So the isotropy representation for $G/H$ is not irreducible, which contradicts Lemma 6.3.

Finally, we consider the case that $(g, h)$ belongs to (2).

In this case, $(M, F)$ is finitely covered by $G/H_0 = \text{SU}(3)/T^2, \text{Sp}(3)/\text{Sp}(1)^3$ or $F_4/\text{Spin}(8)$. We have a unique decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{m}_3$, in which all three $\mathfrak{m}_i$-factors are different irreducible representations of $H_0$ with the same dimension.

From $g \in H$ to the permutation action of $\text{Ad}(g)$ on the three $\mathfrak{m}_i$-factors defines an injection from $H/H_0$ to $S_3$, the permutation group of three elements. So the fundamental group $\pi_1(M)$ is an subgroup of $S_3$.

If $\pi_1(M) = \{e\}$ or $Z_2$, the isotropy representation is not irreducible, which contradicts Lemma 6.3. If $\pi_1(M) = S_3$, the prime closed geodesic representing a homotopy class of order 2 and another representing one of order 3 can not belong to the same orbit, which contradicts Assumption (I). To summarize, we must have $H/H_0 = \pi_1(\pi) = Z_3$.

When $G/H_0 = \text{SU}(3)/T^2$, we can find the matrix

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \in \text{SU}(3)$$ (6.8)
in $H$. The reason is following. For any element $g'$ from a suitable connected component of $H$, $\text{Ad}(gg')$ preserves each $m_i$. Direct calculation shows that $gg'$ is a diagonal matrix, i.e. it is contained in $H_0$. So we have $g \in H_0g' = g'H_0 \subset H$.

The centralizer $c(g) = \{u \in g$ with $\text{Ad}(g)u = u\}$ for $g$ in (6.8) is an Abelian subalgebra spanned by

$$u = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

and

$$v = \sqrt{-1} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

in $m$, and $c(g) \cap h = 0$. By the fixed pointed set technique in [34] (see Section 2) $\text{Fix}_o(g, M)$ is a flat torus. By Example 2 in Section 4, this is a contradiction to Assumption (I).

When $G/H_0 = \text{Sp}(3)/\text{Sp}(1)^3$, we can similarly argue that the matrix $g$ in (6.8) is contained in $H$. Then its centralizer $c(g)$ in $g = \text{sp}(3)$ can be decomposed as a direct sum of ideals,

$$c(g) = g_1 \oplus g_2 \oplus \mathbb{R}u,$$

where $u \in m$ is the same as in (6.9), and for each $i = 1$ and 2, $g_i = \text{Im} \mathbb{H} v_i$ is a subalgebra of type $A_1$, where

$$v_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and

$$v_2 = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}.$$  

(6.10)

Notice $c(g) \cap h = \text{Im} \mathbb{H} I$ is also a subalgebra of type $A_1$, diagonally imbedded in $g_1 \oplus g_2$. So $(\text{Fix}_o(g, M), F|_{\text{Fix}_o(g, M)})$ is locally isometric to and finitely covered by $(M', F')$ where $M' = S^1 \times S^3 = \text{SO}(2) \times \text{SO}(4)/e \times \text{SO}(3)$ and $F'$ is $\text{SO}(2) \times \text{SO}(4)$-invariant. By Example 3 in Section 4 and Proposition 4.3, this is a contradiction to Assumption (I).

When $G/H_0 = F_4/\text{Spin}(8)$, we view $F_4$ as the automorphism group of the exceptional Jordan algebra $\mathfrak{h}_3(\mathbb{O})$, the space of all Hermitian $3 \times 3$ Octonian matrices with the product $\circ$ defined by $A \circ B = \frac{1}{2}(AB + BA)$. Then $H_0 = \text{Spin}(8)$ can be identified as the subgroup fixing $X_1 = \text{diag}(1, 0, 0)$, $X_2 = \text{diag}(0, 1, 0)$ and $X_3 = \text{diag}(0, 0, 1)$. See Chapter 16 in [1] for more details.

The automorphism $g$ of $\mathfrak{h}_3(\mathbb{O})$ given by the conjugation by the matrix in (6.9) is an element in $F_4$ of order 3, such that $\text{Ad}(g)$ induces an outer automorphism of $h$. Because $H/H_0 = \mathbb{Z}_3 \subset S_3 = \text{Out}(h)$, we see that $g$ is contained in $H$.

For $g \in H$ described above, its centralizer $c(g)$ in $g$ is a 22-dimensional subalgebra of rank 4, and $c \cap h = G_2$. By the fixed point set technique and Proposition 4.3, there exist arbitrarily long prime closed geodesic on $(\text{Fix}_o(g, M), F|_{\text{Fix}_o(g, M)})$ as well as on $(M, F)$. This is a contradiction to Assumption (I).

In fact, Proposition 4.4 can also be applied to the cases that $G/H_0 = \text{SU}(3)/T^2$ and $\text{Sp}(3)/\text{Sp}(1)^3$. If we want to explicitly describe $\text{Fix}_o(g, M)$ when $(g, h) = (F_4, D_4)$, we will find that $\text{Fix}_o(g, M)$ is finitely covered by and locally isometric to the homogeneous Finsler space $(M', F') = (S^1 \times S^7, F') = (\text{SO}(2) \times \text{Spin}(7))/e \times G_2, F')$. The metric $F'$ is in fact $\text{SO}(2) \times \text{SO}(8)$-invariant, so we can alternatively get the contradiction by Example 3 and Proposition 4.3 in Section 4.

To summarize, by case by case discussion, we have proved Theorem 6.1.
7 Classification theorem and key lemmas when \( \dim M \) is odd

In the following two sections, we will study the odd dimensional case and prove the following theorem.

**Theorem 7.1** Any odd dimensional compact connected reversible homogeneous Finsler space with only one orbit of prime closed geodesics must be a Riemannian symmetric \( S^n \) or a Riemannian symmetric \( \mathbb{R}P^n \).

We keep all relevant notations and assumptions in Case 2, Section 5. Using the bi-invariant inner product on \( g \) and its restriction to \( h \), we identify roots of \( g \) and \( h \) as vectors in \( t \) and \( t \cap h \) respectively.

We still denote \( C \) the subset of all vectors \( u \in m \) such that \( |u|_{bi} = 1 \) and the geodesic passing \( o \) in the direction of \( u \) is closed. When Assumption (I) is satisfied, \( C \) is an \( \text{Ad}(H) \)-orbit.

For preparation, we need the following lemmas.

**Lemma 7.2** Assume \((G/H, F)\) is an odd dimensional compact connected homogeneous Finsler space satisfying \( \text{rk} G = \text{rk} H + 1 \), and keep all relevant notations and assumptions. Then we can find two different vectors \( u_1 \) and \( u_2 \) in \( \hat{m}_0 \cap C \).

**Proof.** There are two cases to be considered, \( \dim \hat{m}_0 = 1 \) or \( \dim \hat{m}_0 = 3 \).

Assume \( \dim \hat{m}_0 = 1 \), i.e. \( \hat{m}_0 = t \cap m \). We choose \( u_1 \in t \cap m \) such that \( |u_1|_{bi} = 1 \). By Lemma 5.3

\[
\langle u_1, [u_1, m]_m \rangle^F_{u_1} \subset \langle \hat{m}_0, \sum_{\alpha' \neq 0} \hat{m}_{\pm \alpha'} \rangle^F_{u_1} = 0.
\]

On the other hand, \( t \cap m \) is bi-invariant orthogonal to \( t \cap h \) which generates the torus \( T_H \). So \( u \in t \cap m \) generates an \( S^1 \)-subgroup. By Lemma 5.1, we get \( u_1 \in C \).

By the same argument, we can also get \( u_2 = -u_1 \in C \).

This proves the case when \( \dim \hat{m}_0 = 1 \).

Assume \( \dim \hat{m}_0 = 3 \). By Lemma 5.4, we can find two different vectors \( u_1 \) and \( u_2 \) in \( \hat{m}_0 \), such that \( |u_1|_{bi} = |u_2|_{bi} = 1 \), and

\[
\langle u_i, [u_i, \hat{m}_0] \rangle^F_{u_i} = 0. \tag{7.11}
\]

For each \( i \),

\[
[u_i, m]_m = [u_i, \hat{m}_0]_m + \sum_{\alpha' \in t \cap h, \alpha' \neq 0} \hat{m}_{\pm \alpha'}_m
\]

\[
\subset [u_i, \hat{m}_0]_m + \sum_{\alpha' \in t \cap h, \alpha' \neq 0} \hat{m}_{\pm \alpha'}.
\]

By Lemma 5.3 and \( (7.11) \), we have

\[
\langle u_i, [u_i, m]_m \rangle^F_{u_i} \subset \langle u_i, [u_i, \hat{m}_0] \rangle^F_{u_i} + \langle u_i, \sum_{\alpha' \in t \cap h, \alpha' \neq 0} \hat{m}_{\pm \alpha'} \rangle^F_{u_i} = 0.
\]
On the other hand, \( u_1 \) and \( u_2 \) are nonzero vectors in a compact subalgebra of type \( \mathfrak{A}_1 \), so they generate \( S^1 \)-subgroups. By Lemma 5.1 we have \( u_1 \) and \( u_2 \) are contained in \( \mathcal{C} \).

This proves the lemma when \( \dim \hat{\mathfrak{m}}_0 = 3 \). □

**Lemma 7.3** Assume \((G/H, F)\) is an odd dimensional compact connected homogeneous Finsler space satisfying Assumption (I) and keep all relevant notations and assumptions. Let \( \alpha \) be a root of \( \mathfrak{g} \) such that \( \alpha \in \mathfrak{t} \cap \mathfrak{h} \), and \( \alpha \) is the only root of \( \mathfrak{g} \) contained in \( \mathbb{R}_{>0}\alpha + \mathfrak{t} \cap \mathfrak{m} \). Then \( \alpha \) is a root of \( \mathfrak{h} \) and \( \mathfrak{h}_\pm \alpha = \mathfrak{g}_\pm \alpha = \hat{\mathfrak{g}}_\pm \alpha \).

**Proof.** We assume conversely that \( \alpha \) is not a root of \( \mathfrak{h} \), then \( \mathfrak{g}_\pm \alpha \subset \mathfrak{m} \). Denote \( t' \) the bi-invariant orthogonal complement of \( \alpha \) in \( \mathfrak{t} \cap \mathfrak{h} \), and \( T' \) the torus in \( H \) generated by \( t' \). Then \( \text{Fix}_\mathfrak{o}(T', M) \) is finitely covered by \( (M, F) \).

Assume \( \dim \hat{\mathfrak{m}}_0 = 1 \), then by the fixed point set technique, \( \text{Fix}_\mathfrak{o}(T', M) \) is a compact coset space \( G' / H' \) such that \( \mathfrak{g}' = \text{Lie}(G') = \mathbb{R} \oplus \mathfrak{A}_1 \) and \( \mathfrak{h}' = \mathbb{R} \) is contained in the \( \mathfrak{A}_1 \)-factor of \( \mathfrak{g}' \). So \((\text{Fix}_\mathfrak{o}(T', M), F |_{\text{Fix}_\mathfrak{o}(T', M)}) \) is finitely covered by and locally isometric to the homogeneous Finsler space \((M', F')\) where \( M' = S^1 \times S^2 = \text{SO}(2) \times \text{SO}(3) / e \times \text{SO}(2) \) and \( F' = \text{SO}(2) \times \text{SO}(3) \)-invariant. By Example 2 in Section 3 and Proposition 4.3 we can find a sequence of prime closed geodesics \( \gamma_n \) for \((\text{Fix}_\mathfrak{o}(T', M), F |_{\text{Fix}_\mathfrak{o}(T', M)}) \) as well as \((M, F)\), such that their lengths \( l(\gamma_n) \) diverge to infinity. They can not belong to the same orbit, which contradicts Assumption (I).

Assume \( \dim \hat{\mathfrak{m}}_0 = 3 \), then \((\text{Fix}_\mathfrak{o}(T', M), F |_{\text{Fix}_\mathfrak{o}(T', M)}) \) is finitely covered by and locally isometric to \( M' = S^3 \times S^2 = \text{SU}(2) \times \text{SO}(3) / e \times \text{SO}(2) \) with an \( \text{SU}(2) \times \text{SO}(3) \)-invariant metric. Using Example 5 in Section 3 instead, we can apply similar argument as the previous case to get a contradiction to Assumption (I).

This ends the proof of this lemma. □

**Lemma 7.4** Assume \((G/H, F)\) is an odd dimensional compact connected reversible homogeneous Finsler space and keep all relevant notations and assumptions. Let \( \alpha \) be a root of \( \mathfrak{g} \) such that \( \mathfrak{g}_\pm \alpha \subset \mathfrak{m} \), \( \mathfrak{pr}_0 \alpha \neq 0 \), and \( \alpha \) is the only root of \( \mathfrak{g} \) contained in \( (2N - 1)\alpha + \mathfrak{t} \cap \mathfrak{m} \). Then \( \mathfrak{g}_\pm \alpha \cap \mathcal{C} \neq \emptyset \).

**Proof.** Denote \( t' \) the bi-invariant orthogonal complement of \( \alpha' = \mathfrak{pr}_0 (\alpha) \) in \( \mathfrak{t} \cap \mathfrak{h} \). With respect to \( t' \), we have the decomposition \( \mathfrak{m} = \sum_{\beta'' \in \mathfrak{t}'} \hat{\mathfrak{m}}_{\pm \beta''} \) (see the detailed description after (5.7)).

Let \( u \) be any vector in \( \mathfrak{g}_\pm \alpha \) with \( |u|_{\mathfrak{t}'} = 1 \), then we also have \( u \in \hat{\mathfrak{m}}_0 \).

Direct calculation shows

\[
[u, \mathfrak{m}]_{\mathfrak{m}} \subset \hat{\mathfrak{m}}_0 + [\mathfrak{t} \cap \mathfrak{m}, u] + \sum_{\beta'' \in \mathfrak{t}', \mathfrak{pr}_0 (\beta'') \neq 0} \hat{\mathfrak{m}}_{\pm \beta''}. \tag{7.12}
\]

By Lemma 5.3 Lemma 5.6 and Lemma 5.7 each summation factor in the right side of (7.12) is \( F_u \)-orthogonal to \( u \). So we have \( \langle u, [u, \mathfrak{m}]_{\mathfrak{m}} \rangle_{F} = 0 \). Because \( u \) is contained in a subalgebra of type \( \mathfrak{A}_1 \), \( u \) generates an \( S^1 \)-subgroup.

This proves \( u \in \mathcal{C} \) and ends the proof of the lemma. □

When Lemma 7.4 is applied to find obstacle to Assumption (I), it is often accompanied with the following lemma.
Lemma 7.5 Assume \((G/H, F)\) is an odd dimensional compact connected homogeneous Finsler space such that \(G = I_0(M, F)\) is a compact simple Lie group and \((M, F)\) satisfies Assumption (I). Then we have the following:

(1) If \(\Delta \cap \mathfrak{m} = \emptyset\), i.e. there exist no roots of \(\mathfrak{g}\) contained in \(\mathfrak{t} \cap \mathfrak{m}\), then for any root plane \(\mathfrak{g}_{\pm \beta}\) of \(\mathfrak{g}\), \(\mathfrak{C} \cap \mathfrak{g}_{\pm \beta} = \emptyset\).

(2) If \(\mathfrak{t} \cap \mathfrak{m}\) contains a root \(\alpha\) of \(\mathfrak{g}\), then for any root \(\beta\) of \(\mathfrak{g}\) with \(|\alpha|_{\text{bi}} \neq |\beta|_{\text{bi}}\), we have \(\mathfrak{C} \cap \mathfrak{g}_{\pm \alpha} = \emptyset\).

Proof. (1) Assume conversely that \(\mathfrak{C}\) contains some vector \(v \in \mathfrak{g}_{\pm \beta}\) such that \(|\beta|_{\text{bi}} = 1\). By Lemma 7.2, there exists a vector \(u \in \mathfrak{t} \cap \mathfrak{m}\) which is contained in \(\mathfrak{C}\). Because \(v\) is \(\text{Ad}(G)\)-conjugation to \(\beta\), we get \(\mathfrak{C} = \text{Ad}(H)u \subset \text{Ad}(G)\beta\), i.e., the two vectors \(u\) and \(\beta\) in \(\mathfrak{t}\) are in the same \(\text{Ad}(G)\)-orbit. By Proposition 2.2, Chapter 7 in [16], \(u\) and \(\beta\) belong to the same Weyl group orbit. This is impossible because the Weyl group orbit of \(\beta\) consists of all roots of \(\mathfrak{g}\) with the same length, but \(u\) is not a root of \(\mathfrak{g}\). This proves the statement (1).

(2) Assume conversely that \(\mathfrak{C}\) contains some vector \(v \in \mathfrak{g}_{\pm \beta}\) such that \(|\alpha|_{\text{bi}} \neq |\beta|_{\text{bi}}\). For simplicity, we choose the bi-invariant inner product on \(\mathfrak{g}\) such that \(|\beta|_{\text{bi}} = 1\). By Lemma 7.2, \(\hat{\mathfrak{m}}_0\) contains a vector \(u \in \mathfrak{C}\), which is \(\text{Ad}(G)\)-conjugation to \(c^{-1}\alpha\). Meanwhile \(v\) is \(\text{Ad}(G)\)-conjugate to \(\alpha\). So \(c^{-1}\alpha\) and \(\beta\) belong to the same \(\text{Ad}(G)\)-orbit because \(\mathfrak{C} \subset \text{Ad}(G) \cdot (c^{-1}\alpha) \cap \text{Ad}(G)\beta\). By Proposition 2.2, Chapter 7 in [16], \(c^{-1}\alpha\) and \(\beta\) belong to the same Weyl group orbit. This is impossible because \(c^{-1}\alpha\) is not a root of \(\mathfrak{g}\). ■

Now we start the proof of Theorem 7.1. We follow the theme in [31], i.e. we divide the discussion into three cases.

Case I. Each root plane of \(\mathfrak{h}\) is a root plane of \(\mathfrak{g}\).

Case II. There exists a root plane \(\mathfrak{h}_{\pm \alpha}\) of \(\mathfrak{h}\) which is not a root plane of \(\mathfrak{g}\), and there are two roots \(\alpha\) and \(\beta\) of \(\mathfrak{g}\) from different simple ideals, such that \(\text{pr}_\mathfrak{h}(\alpha) = \text{pr}_\mathfrak{h}(\beta)\).

Case III. The same as Case II except that the different roots \(\alpha\) and \(\beta\) are from the same simple ideal of \(\mathfrak{g}\).

In the rest of this section, we will discuss Case I and Case II, and leave Case III to the next section.

Proof of Theorem 7.1 in Case I. Assume \((M, F) = (G/H, F)\) is an odd dimensional compact connected homogeneous Finsler space belonging to Case I and satisfying Assumption (I).

The deck transformation induces an injective group endomorphism from \(H/H_0\) to \(\pi_1(M)\). By Assumption (I), \(\pi_1(M)\) must be a cyclic group, and so does \(H/H_0\). Denote \(g \in H\) a representative for a generator of \(H/H_0\). Because \(\text{Ad}(g)H = H\), we also have \(\text{Ad}(g)\hat{\mathfrak{m}}_0 = \hat{\mathfrak{m}}_0\).

By Lemma 7.2, there exists a vector \(u_1 \in \hat{\mathfrak{m}}_0 \cap \mathfrak{C}\). Because in Case I, \([\mathfrak{h}, \hat{\mathfrak{m}}_0] = 0\), it is easy to get

\[
\mathfrak{C} = \{\text{Ad}(g^k)u_1 \text{ with all } k \in \mathbb{Z}\} \subset \hat{\mathfrak{m}}_0.
\]

By Lemma 7.2, \(\mathfrak{C}\) contains at least two points.
We claim Ad(g) fixes some nonzero vector \( u \in \mathfrak{m} \), which can be proved as follows.

Let \( t'' \) be a Cartan subalgebra of \( \mathfrak{g} \) such that \( T'' = \exp t'' \) contains \( g \). Then Ad(\( g \)) fixes each vector in \( t'' \). Obviously \( t'' \) is not contained in \( \mathfrak{h} \), i.e. \( \mathfrak{pr}_m(t'') \neq 0 \).

Because \( H_0 \) is normal in \( H \), we have Ad(\( g \))\( \mathfrak{h} = \mathfrak{h} \), and Ad(\( g \)) preserves the bi-invariant orthogonal reductive decomposition \( \mathfrak{g} = \mathfrak{h} + \mathfrak{m} \). So Ad(\( g \)) fixes each vector in \( \mathfrak{pr}_m(t'') \). Any nonzero vector \( u \in \mathfrak{pr}_m(t'') \) meets the requirement.

This ends the proof of the claim.

The previous claim implies \( M' = \text{Fix}_o(g, M) \) has a positive dimension. By Lemma 7.2, it is obvious that \( T_o M' \cap \mathcal{C} = \emptyset \). The submanifold \((M', F|_{M'})\) is a homogeneous Finsler space. By Lemma 3.1 and the homogeneity, there must exist one closed geodesic \( c(t) \) on \((M', F')\) such that \( c(0) = o, \ c'(0) = u' \in \mathfrak{m} \) and \( |u'|_{\mathfrak{m}} = 1 \). Because \( M' \) is totally geodesic in \( M \), we have \( u' \in \mathcal{C} \cap T_o M' \), which contradicts our previous observation.

This proves Theorem 7.1 in Case I, which can be summarized as

**Proposition 7.6** If \((M, F) = (G/H, F)\) with \( G = I_0(M, F) \) is an odd dimensional compact connected homogeneous Finsler space in Case I, then it has at least two \( G \)-orbits of prime closed geodesics.

**Proof of Theorem 7.1 in Case II.** Assume \((M, F) = (G/H, F)\) is an odd dimensional compact connected reversible homogeneous Finsler space belonging to Case II and satisfying Assumption (I).

We have a Lie algebra direct sum decomposition \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3 \) in which \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are simple, such that there are roots of \( \mathfrak{g} \), \( \alpha \in \mathfrak{t} \cap \mathfrak{g}_1 \) and \( \beta \in \mathfrak{t} \cap \mathfrak{g}_2 \), and \( \alpha' = \mathfrak{pr}_h(\alpha) = \mathfrak{pr}_h(\beta) \) is a root of \( \mathfrak{h} \). Obviously we have

\[
\hat{\mathfrak{m}}_0 = \mathfrak{t} \cap \mathfrak{m} = \mathbb{R}(\alpha - \beta), \quad \text{and} \quad \mathfrak{h}_{\pm \alpha'} \subset \hat{\mathfrak{m}}_{\pm \alpha'} = \mathfrak{g}_{\pm \alpha} + \mathfrak{g}_{\pm \beta}.
\]

By Lemma 7.2, we can find a vector \( u \in \mathcal{C} \cap \hat{\mathfrak{m}}_0 = \mathcal{C} \cap \mathfrak{t} \).

If there exists a root \( \delta \) of \( \mathfrak{g}_1 \), such that \( \delta \neq \pm \alpha \) and \( \delta \) is not bi-invariant orthogonal to \( \alpha \), then \( \mathfrak{g}_{\pm \delta} \subset \mathfrak{m} \), and by Lemma 7.4, there exists a vector \( v \in \mathfrak{g}_{\pm \delta} \cap \mathcal{C} \). By Assumption (I), \( \mathcal{C} = \text{Ad}(H)v \subset \text{Ad}(G)v \subset \mathfrak{g}_1 \). But \( u \in \mathcal{C} \) is not contained in \( \mathfrak{g}_1 \). This is a contradiction. To summarize, \( \mathfrak{g}_1 = A_1 \), otherwise it can not be simple. Similarly, we also have \( \mathfrak{g}_2 = A_1 \).

Next, we prove \( \mathfrak{g}_3 \) must be zero. Assume conversely it is not, by Lemma 3.2, \( \mathfrak{g}_3 \) is semi-simple. Because \( \mathfrak{t} \cap \mathfrak{m} \subset \mathfrak{t} \cap (\mathfrak{g}_1 + \mathfrak{g}_2) \), we have \( \mathfrak{t} \cap \mathfrak{g}_3 \subset \mathfrak{h} \). By Lemma 7.3, any root plane in \( \mathfrak{g}_3 \) is also contained in \( \mathfrak{h} \). So the ideal \( \mathfrak{g}_3 \) of \( \mathfrak{g} \) is contained in \( \mathfrak{h} \). But \( G = I_0(M, F) \) acts transitively on \( M = G/H \). This is a contradiction.

So in this case \( \mathfrak{h} \) is a diagonal \( A_1 \) in \( \mathfrak{g} = A_1 \oplus A_1 \), so \((G/H, F)\) is locally isometric to a Riemannian symmetric \( S^3 \). By Proposition 4.4, \((M, F)\) is either the Riemannian symmetric \( S^3 \) or the Riemannian symmetric \( \mathbb{R}P^3 \).

This proves Theorem 7.1 in Case II, which can be summarized as following.

**Proposition 7.7** If \((M, F) = (G/H, F)\) with \( G = I_0(M, F) \) is an odd dimensional compact connected reversible homogeneous Finsler space in Case II satisfying Assumption (I), then \((M, F)\) is either the Riemannian symmetric \( S^3 \) or the Riemannian symmetric \( \mathbb{R}P^3 \).
8 Proof of Theorem 7.1 in Case III

We continue the proof of Theorem 7.1 in Case III. Let \((M, F) = (G/H, F)\) with \(G = I_0(M, F)\) be an odd dimensional compact connected homogeneous Finsler space satisfying Assumption (I). We keep all relevant notations and assumptions. In particular, there exists a root plane \(g_{\pm \alpha'}\) of \(h\), which is not a root plane of \(g\), and there exists a pair of different roots \(\alpha\) and \(\beta\) from the same simple ideal in \(g\), such that \(\text{pr}_h(\alpha) = \text{pr}_h(\beta) = \alpha'\).

Recall that \(C \subset m\) is the subset of all vectors \(u\)'s such that \(|u|_b = 1\) and the geodesic of \((M, F)\) passing \(o\) in the direction of \(u\) is closed. By Assumption (I), \(C\) is an \(\text{Ad}(H)\)-orbit.

Using similar technique as in the last section, it is not hard to see that \(G\) must be simple, i.e. we have the following lemma.

**Lemma 8.1** For any odd dimensional compact connected homogeneous Finsler space \((M, F) = (G/H, F)\) in Case III, with \(G = I_0(M, F)\) and satisfying Assumption (I), the group \(G\) is simple.

**Proof.** Assume conversely that \(G\) is not simple. By Lemma 3.2, \(G\) is semi-simple. So we have a nontrivial direct sum decomposition \(g = g_1 \oplus g_2\), in which \(g_1\) is a simple ideal, from which we get the roots \(\alpha\) and \(\beta\) indicated in Case III, and \(g_2\) is a semi-simple ideal. Obviously \(t \cap m = \mathbb{R}(\alpha - \beta) \in g_1\), so \(t \cap g_2 \in h\). By Lemma 7.3, any root plane of \(g_2\) is contained in \(h\). So we have \(g_2 \subset h\), which contradicts to the fact that \(G = I_0(M, F)\) acts transitively on \(M = G/H\). This ends the proof of this lemma.

**Proof of Theorem 7.1 in Case III.** We will check case by case each possible type of \(g = \text{Lie}(G)\) from \(A_n\) to \(G_2\), and each possible unordered pair of roots \(\alpha\) and \(\beta\) of \(g\).

We follow the convention in [30][31][33] for presenting roots of the compact simple Lie algebra \(g\). We choose a suitable bi-invariant inner product on \(g\), and isometrically identify \(t\) with an Euclidean space \(V\) satisfying:

1. When \(g = A_n\), we denote \(\{e_1, \ldots, e_{n+1}\}\) the standard orthonormal basis of \(\mathbb{R}^{n+1}\), and \(V\) the orthogonal complement of \(e_1 + \cdots + e_{n+1}\);
2. When \(\text{rk}g = n\) and \(g \neq A_n\), we denote \(\{e_1, \ldots, e_n\}\) the standard orthonormal basis of \(V = \mathbb{R}^n\);
3. Under this identification, the root system \(\Delta\) of \(g\) is presented as in Table III.

Using the Weyl group action on the pair \(\alpha\) and \(\beta\), we can reduce the case number significantly. Moreover, when \(g = D_4\) or \(E_6\), we can use outer automorphism to change the pair \(\alpha = e_1 + e_2\) and \(\beta = -e_3 - e_4\) to \(\alpha = e_1 + e_2\) and \(\beta = e_2 - e_1\). But still, many subcases remain. Observing that for many subcases we can apply similar argument, we sort all the subcases into five groups, from Case III-A to Case III-E.

**Case III-A.** Assume for the root \(\alpha'\) of \(h\) described in Case III, we can find two roots \(\alpha\) and \(\beta\) of \(g\) such that \(\text{pr}_h(\alpha) = \text{pr}_h(\beta) = \alpha'\) and the angle \(\theta_{\alpha, \beta}\) between \(\alpha\) and \(\beta\) is \(\pi/3\) and \(2\pi/3\).
Lemma 8.2 Assume \((M, F) = (G/H, F)\) is an odd dimensional compact connected reversible homogeneous Finsler space in Case III such that \(G = I_0(M, F)\) and \((M, F)\) satisfies Assumption (I). Then for the roots \(\alpha\) and \(\beta\) in Case III, their angle \(\theta_{\alpha, \beta}\) can not be \(\pi/3\) or \(2\pi/3\).

Proof. Firstly, we assume that \(g \neq G_2\) and the angle between \(\alpha\) and \(\beta\) is \(\pi/3\). Then

\[
g' = R\alpha + R\beta + g_{\pm\alpha} + g_{\pm\beta} + g_{\pm(\alpha - \beta)}
\]

is a subalgebra of type \(A_2\) in \(g\), and \(g' \cap h = R\alpha' + h_{\pm \alpha'}\) is a subalgebra of type \(A_1\). As in the proof of Lemma 18 in [30], direct calculation for matrices in \(su(3)\) shows this pair \((g', h')\) can not exist. This is the contradiction.

Secondly, we assume that \(g = G_2\) and the angle between \(\alpha\) and \(\beta\) is \(2\pi/3\). In this case, \(t \cap m = \mathbb{R}(\alpha - \beta)\) does not contain roots of \(g\). On the other hand, \(2\alpha' = \alpha + \beta\) is a root of \(g\), but not a root of \(h\) because \(\alpha'\) is. So we have \(g_{\pm(\alpha + \beta)} = \bar{m}_{\pm 2\alpha'} \subset m\). By Lemma 7.3, \(C \cap g_{\pm(\alpha + \beta)} \neq \emptyset\). This is a contradiction to (1) of Lemma 7.5.

Finally, we assume that \(g = G_2\), and the angle between \(\alpha\) and \(\beta\) is \(\pi/3\) and \(2\pi/3\) respectively. We can apply (2) of Lemma 7.5 and similar argument in the previous paragraph.

This proves the lemma for all possible cases. \(\blacksquare\)

The angles are defined with respect to the chosen bi-invariant inner product on \(g\). Notice that all possible angles between the two roots \(\alpha\) and \(\beta\) in Case III are \(\pi/6, \pi/4, \pi/3, \pi/2, 2\pi/3, 3\pi/4\) and \(5\pi/6\).

The following lemma provides the contradiction.

| \(\mathfrak{g}\)   | Root system Delta of \(\mathfrak{g}\)                                                                 |
|-------------------|---------------------------------------------------------------------------------------------------------------------------------|
| \(A_n\)          | \(\pm(e_i - e_j), \forall 1 \leq i < j \leq n + 1\)                                                                             |
| \(B_n\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq n; \pm e_i, \forall 1 \leq i \leq n\)                                            |
| \(C_n\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq n; \pm 2e_i, \forall 1 \leq i \leq n\)                                       |
| \(D_n\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq n\)                                                                             |
| \(E_6\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq 6; \frac{1}{\sqrt{2}}(\pm e_1 \pm \cdots \pm \sqrt{2} e_6)\) with even plus signs |
| \(E_7\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq 7; \pm \sqrt{2} e_7; \frac{1}{\sqrt{2}} e_1 \pm \frac{\sqrt{2}}{\sqrt{3}} e_7\) with even plus signs |
| \(E_8\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq 8; \frac{1}{\sqrt{3}}(\pm e_1 \pm \cdots \pm e_8)\) with even plus signs         |
| \(F_4\)          | \(\pm e_i \pm e_j, \forall 1 \leq i < j \leq 4; \pm e_i, \forall 1 \leq i \leq 4; \frac{1}{\sqrt{2}}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\) |
| \(G_2\)          | \(\pm e_1, \pm \frac{\sqrt{2}}{\sqrt{3}} e_2, \pm \sqrt{3} e_2, \pm \frac{\sqrt{3}}{\sqrt{2}} e_1 \pm \frac{\sqrt{2}}{\sqrt{3}} e_2\) |

Table 1: Root system of compact simple Lie algebras
We can find the root $\delta$ of $\mathfrak{g}$, as listed in Table 2, such that $\mathfrak{g}_{\pm \delta} \cap \mathcal{C} \neq \emptyset$ by Lemma 7.4. Notice that $t \cap \mathfrak{m} = \mathbb{R}(\alpha - \beta)$ does not contain any root of $\mathfrak{g}$. By (1) of Lemma 7.5, $\mathfrak{g}_{\pm \delta} \cap \mathcal{C} = \emptyset$. This is a contradiction.

To summarize, any compact connected reversible homogeneous Finsler space $(M, F)$ in Case III-B can not satisfy Assumption (I).

**Case III-C.** Each row of Table 3 provides a subcase for which we can find the contradiction as following.

In the Lie algebra $\mathfrak{g}$, we can find a regular subalgebra $\mathfrak{g}' = \mathfrak{C}_2$, which roots are listed in Table 3 such that $\mathfrak{g}' \cap \mathfrak{h} = \mathfrak{A}_1$, $\alpha' = \text{pr}_\mathfrak{h}(\alpha)$ is half of a long root of $\mathfrak{g}'$, and $\mathfrak{h}_{\pm \alpha'} \subset \mathfrak{g}' \cap \mathfrak{h}$ is contained in the sum of the two root planes for short roots of $\mathfrak{g}'$. We can identify $\mathfrak{g}'$ with the matrix Lie algebra sp(2), such that $\mathfrak{g}' \cap \mathfrak{h}$ is linearly spanned by

$$H = \begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & a \\ -\bar{a} & 0 \end{pmatrix}, \quad \text{and} \quad Y = [H, X] = \begin{pmatrix} 0 & i\alpha \\ \bar{a} & 0 \end{pmatrix}, \quad (8.13)$$

where $a \in \mathbb{H}$ is nonzero. But direct calculation shows,

$$[X, Y] = \begin{pmatrix} 2|a|^2 & 0 \\ 0 & -2\bar{a}|a| \end{pmatrix} \notin \mathfrak{g}' \cap \mathfrak{h}.$$
This is a contradiction.

To summarize, Case III-C cannot happen.

**Case III-D.** Each row of Table 4 provides a subcase for which we can prove \((M, F)\) is locally isometric to a Riemannian symmetric sphere.

We take No. 2 in Table 4 as the example. The argument for the other is similar.

By Lemma 7.3, \(\pm e_i \pm e_j\) with \(1 \leq i < j \leq n\) are roots of \(\mathfrak{h}\), and \(\mathfrak{h}_{\pm (e_i \pm e_j)} = \mathfrak{g}_{\pm (e_i \pm e_j)}\).

Take any nonzero \(u \in \mathfrak{g}_{\pm (e_i \pm e_j)}\) for \(i > 2\), \(\text{ad}(u) = [u, \cdot]\) defines a linear isomorphism
\[
\text{ad}(u) : \mathfrak{g}_{\pm e_2} = \mathfrak{g}_{\pm (e_i + e_j)} + \mathfrak{g}_{\pm (e_2 - e_1)} \rightarrow \mathfrak{g}_{\pm (e_i + e_j)} + \mathfrak{g}_{\pm (e_2 - e_1)} = \mathfrak{g}_{\pm e_i},
\]
and because \(u \in \mathfrak{h}\), \(\text{ad}(u)\) maps \(\mathfrak{g}_{\pm e_2} \cap \mathfrak{h}\) and \(\mathfrak{g}_{\pm e_2} \cap \mathfrak{m}\) to \(\mathfrak{g}_{\pm e_i} \cap \mathfrak{h}\) and \(\mathfrak{g}_{\pm e_i} \cap \mathfrak{m}\) respectively. This implies that \(\pm e_i\)’s for all \(i > 1\) are roots of \(\mathfrak{h}\).

To summarize, we have found all the roots of \(\mathfrak{h}\), i.e. \(\pm e_i\) for all \(i > 1\) and \(\pm e_i \pm e_j\) for all \(1 < i < j \leq n\). So \(\mathfrak{h} = \mathfrak{b}_{n-1}\). The argument in Subcase 1, Subsection 6.3 in [30] shows this \(\mathfrak{h}\) is unique up to an \(\text{Ad}(G)\)-action. So \((G/H, F)\) is locally isometric to the Riemannian symmetric sphere \(SO(2n)/SO(2n-1)\).

In [30] and [31], there are detailed discussions for the uniqueness of \(\mathfrak{h} = \mathfrak{g}_2\) for No. 1 in Table 4.

To summarize, any compact connected homogeneous Finsler space in Case III-D is locally isometric to a Riemannian symmetric sphere. By Proposition 4.11 it satisfies Assumption (I) if and only if \(M\) is a Riemannian symmetric \(S^{2n-1}\) or a Riemannian symmetric \(\mathbb{R}P^{2n-1}\).

**Case III-E.** Each row of Table 5 provides a subcase for which we can change \(\mathfrak{h}\) by a suitable \(\text{Ad}(G)\)-action, to make it regular in \(\mathfrak{g}\). Then Proposition 7.6 provides the contradiction.

We take No. 1 in Table 5 as the example. The argument for the other subcases are similar.

By Lemma 7.3, \(\pm e_i \pm e_j\) when \(1 < i < j \leq n\) is a root of \(\mathfrak{h}\). In Subcase 1, Subsection 6.4 of [30], or Subcase 1, Subsection 4.3 of [31], it has been shown by direct calculation that, we can change \(\mathfrak{h}\) by a suitable \(\text{Ad}(G')\)-action, where \(G'\) is the subgroup generated by the subalgebra \(\mathfrak{m}_0 = \mathbb{R}e_1 + \mathfrak{g}_{\pm e_1}\), such that we have \(\mathfrak{g}_{\pm e_2} \subset \mathfrak{h}\). Because \([\mathfrak{m}_0, \mathfrak{g}_{\pm(e_1-e_j)}] = 0\) when \(1 < i < j \leq n\), we still have \(\mathfrak{g}_{\pm (e_i - e_j)} \subset \mathfrak{h}\) for all \(1 < i < j \leq n\).

| No. | \(\mathfrak{g}\) | \(\alpha\) | \(\beta\) | \(\mathfrak{h}\) | Universal cover of \(G/H\) |
|-----|------------|--------|--------|--------|----------------------------|
| 1   | \(B_n\), \(n > 2\) | \(e_1 + e_2\) | \(e_2 - e_1\) | \(G_2\) | \(S^2n-1 = SO(2n)/SO(2n-1)\) |

Table 4: Subcases in Case III-D

| No. | \(\mathfrak{g}\) | \(\alpha\) | \(\beta\) | Roots of \(\mathfrak{h}\) |
|-----|------------|--------|--------|----------------------------|
| 1   | \(B_n\), \(n > 1\) | \(e_1 + e_2\) | \(e_2 - e_1\) | \(\pm e_i, \forall i > 1; \pm e_i \pm e_j, \forall 1 < i < j \leq n\) |
| 2   | \(B_n\), \(n > 1\) | \(e_1 + e_2\) | \(e_2\) | \(\pm e_i, \forall i > 1; \pm e_i \pm e_j, \forall 1 < j \leq n\) |
| 3   | \(C_n\), \(n > 2\) | \(2e_1\) | \(e_1 + e_2\) | \(\pm (e_1 + e_2); \pm e_i \pm e_j, \forall 3 \leq i \leq n; \pm 2e_2, \forall i \geq 3\) |
| 4   | \(C_n\), \(n > 2\) | \(2e_1\) | \(2e_2\) | \(\pm (e_1 + e_2); \pm e_i \pm e_j, \forall 3 \leq i \leq n; \pm 2e_2, \forall i \geq 3\) |
| 5   | \(F_4\) | \(e_1 + e_2\) | \(e_2\) | \(\pm e_i, \forall i > 1; \pm e_i \pm e_j, \forall 1 < i < j \leq 4\) |
| 6   | \(F_4\) | \(e_1 + e_2\) | \(e_2 - e_1\) | \(\pm e_i, \forall i > 1; \pm e_i \pm e_j, \forall 1 < i < j \leq 4\) |

Table 5: Subcases in Case III-E

We take No. 1 in Table 5 as the example. The argument for the other subcases are similar.
Proof. Table 6 lists all the possible subcases of compact simple $\pi/\alpha + \pi/\beta$ in Case III (up to Weyl group action and outer automorphisms of $D_{\beta}$). If Proposition 8.3 (M, F) is a compact connected reversible homogeneous Finsler space in Case III satisfying Assumption (I), then \( G = I_0(M, F) \) or a Riemannian symmetric $SO(n)$ or a Riemannian symmetric $\mathbb{R}P^n$. 

To summarize, any odd dimensional compact connected homogeneous Finsler space $(M, F)$ in Case III-E can not satisfy Assumption (I). 

All above case by case discussions can be summarized as the following proposition.

**Proposition 8.3** If $(M, F) = (G/H, F)$ with $G = I_0(M, F)$ is an odd dimensional compact connected reversible homogeneous Finsler space in Case III satisfying Assumption (I), then $(M, F)$ is a Riemannian symmetric $S^n$ or a Riemannian symmetric $\mathbb{R}P^n$.

**Proof.** Table 6 lists all the possible subcases of compact simple $g$, the roots $\alpha$ and $\beta$ in Case III (up to Weyl group action and outer automorphisms of $D_4$ and $E_6$) with $|\alpha|_\beta > |\beta|_\alpha$ and $\theta_{\alpha, \beta} \notin \{\pi/3, 2\pi/3\}$, and where they are discussed. When $\theta_{\alpha, \beta} = \pi/3$ or $2\pi/3$, the subcase belongs to Case III-A. 

Theorem 7.1 is just a summation of Proposition 7.6, Proposition 7.7, and Proposition 8.3.

At the end, we prove the following proposition as an application.

**Proposition 8.4** Any homogeneous Finsler sphere $(M, F)$ has only one orbit of prime closed geodesics only when it is a Riemannian symmetric sphere.

**Proof.** By the classification of homogeneous spheres [19], a homogeneous Finsler sphere $(M, F) = (G/H, F)$ with dim $M > 1$, $G = I_0(M, F)$ must be one of the following:

1. $G = SO(n)$ when $M = S^{n-1} = SO(n)/SO(n-1)$ and $n > 2$;
2. $G = U(n)$ when $M = S^{2n-1} = U(n)/U(n-1)$ and $n > 1$;
3. $G = Sp(n)$ when $M = S^{4n-1} = Sp(n)/Sp(n-1)$ and $n > 0$;
4. $G = Sp(n)U(1)$ when $M = S^{4n-1} = Sp(n)U(1)/Sp(n-1)U(1)$ and $n > 1$;
5. $G = Sp(n)Sp(1)$ when $M = S^{4n-1} = Sp(n)Sp(1)/Sp(n-1)Sp(1)$ and $n > 1$;
6. $G = Spin(9)$ when $M = S^{15} = Spin(9)/Spin(7)$.

Now we assume $(M, F)$ satisfies Assumption (I). By Lemma 3.2 (2) and (4) are impossible. The homogeneous spheres in (5) and (6) only admit reversible $G$-invariant Finsler metrics. The discussion in Case III-B above proves (6) is impossible. By Proposition 7.7, (5) is impossible. The homogeneous spheres in (3) belong to Case I, i.e. $H$ is regular in $G$ for this case. It is impossible by Proposition 7.6.

The only case left is (1), in which $S^{n-1} = SO(n)/SO(n-1)$ with $n > 2$ only admits Riemannian symmetric $SO(n)$-invariant metrics.

This ends the proof of the proposition.

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| No. | Subcase in | $g$ | $\alpha$ | $\beta$ | $\theta_{\alpha,\beta}$ |
|-----|------------|-----|-----------|-----------|--------------------------|
| 1   | $A_n, n > 3$ | $e_1 - e_4$ | $e_3 - e_2$ | $\pi/2$ | III-B |
| 2   | $B_n, n > 1$ | $e_1 + e_2$ | $e_2$ | $\pi/4$ | III-E |
| 3   | $B_n, n > 1$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-E |
| 4   | $B_n, n > 3$ | $e_1 + e_2$ | $-e_3 - e_4$ | $\pi/2$ | III-B |
| 5   | $B_n, n > 1$ | $e_1 + e_2$ | $-e_3$ | $\pi/2$ | III-D |
| 6   | $B_n, n > 3$ | $e_1 + e_2$ | $-e_3$ | $\pi/2$ | III-B |
| 7   | $B_n, n > 1$ | $e_1$ | $e_2$ | $\pi/2$ | III-C |
| 8   | $B_n, n > 1$ | $e_1 + e_2$ | $-e_1$ | $3\pi/4$ | III-B |
| 9   | $C_n, n > 2$ | $2e_1$ | $e_1 + e_2$ | $\pi/4$ | III-E |
| 10  | $C_n, n > 2$ | $2e_1$ | $2e_2$ | $\pi/2$ | III-E |
| 11  | $C_n, n > 2$ | $e_1 + e_2$ | $-e_2 - e_3$ | $\pi/2$ | III-B |
| 12  | $C_n, n > 2$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-C |
| 13  | $C_n, n > 3$ | $e_1 + e_2$ | $-e_3 - e_4$ | $\pi/2$ | III-B |
| 14  | $C_n, n > 2$ | $2e_1$ | $-e_1 - e_2$ | $3\pi/4$ | III-B |
| 15  | $D_n, n > 2$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-D |
| 16  | $D_n, n > 4$ | $e_1 + e_2$ | $-e_3 - e_4$ | $\pi/2$ | III-B |
| 17  | $E_6$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-B |
| 18  | $E_7$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-B |
| 19  | $E_8$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-B |
| 20  | $E_8$ | $e_1 + e_2$ | $-e_3 - e_4$ | $\pi/2$ | III-B |
| 21  | $F_4$ | $e_1 + e_2$ | $e_2$ | $\pi/4$ | III-E |
| 22  | $F_4$ | $e_1 + e_2$ | $e_2 - e_1$ | $\pi/2$ | III-E |
| 23  | $F_4$ | $e_1 + e_2$ | $-e_3$ | $\pi/2$ | III-B |
| 24  | $F_4$ | $e_1$ | $e_2$ | $\pi/2$ | III-C |
| 25  | $F_4$ | $e_1 + e_2$ | $-e_2$ | $3\pi/4$ | III-B |
| 26  | $G_2$ | $\frac{2}{3}e_1 + \frac{\sqrt{3}}{3}e_2$ | $e_1$ | $\pi/6$ | III-A |
| 27  | $G_2$ | $\frac{2}{3}e_1 + \frac{\sqrt{3}}{3}e_2$ | $-\frac{1}{3}e_1 + \frac{\sqrt{3}}{3}e_2$ | $\pi/2$ | III-A |
| 28  | $G_2$ | $\frac{2}{3}e_1 + \frac{\sqrt{3}}{3}e_2$ | $e_1$ | $5\pi/6$ | III-B |

Table 6: Subcases in Case III with $\theta_{\alpha,\beta} \neq \pi/3$ or $2\pi/3$
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