FE-Holomorphic Operator Function Method for Nonlinear Plate Vibrations with Elastically Added Masses

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Abstract

Vibrations of structures subjected to concentrated point loads have many applications in mechanical engineering. Experiments are expensive and numerical methods are often used for simulations. In this paper, we consider the plate vibration with nonlinear dependence on the eigen-parameter. The problem is formulated as the eigenvalue problem of a holomorphic Fredholm operator function. The Bogner-Fox-Schmit element is used for the discretization and the spectral indicator method is employed to compute the eigenvalues. The convergence is proved using the abstract approximation theory of Karma [11, 12]. Numerical examples are presented for validations.

Key words: plate vibration, nonlinear eigenvalue problem, holomorphic Fredholm operator function, finite element

1 Introduction

We consider the numerical computation of the natural frequencies of a mechanical structure joined elastically with discrete masses [1]. In structural engineering, the dynamic analysis of structure-spring-load systems, which describe the vibrations of structures such as shells and plates, has been

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an important research topic in the past a few decades [2, 3, 4]. Experimental studies are usually expensive and sometimes prohibitive. Instead, numerical methods are often used to compute the vibration characteristics.

In this paper, we consider the plate-spring-load systems. The contact surface between the oscillators and the plate is small enough and modeled by the Dirac δ-distribution. The dependence on the eigenparameter is nonlinear. As a consequence, numerical discretizations such as finite element methods or finite difference methods lead to nonlinear matrix eigenvalue problems (NLEVP). Many schemes have been proposed for the NLEVP. Iterative methods [5, 6, 7, 8] such as the Arnoldi method and Jacobi-Davidson method project the original problem onto some subspaces in which eigenvalues are computed as approximations. Linearization techniques transform the problem into an equivalent linear eigenvalue problem [9, 10]. These methods may increase the size of the problem, make the eigenvalues more sensitive to perturbations, or even destroy the symmetry structure. For iterative methods, it is difficult to obtain a good initial guess.

Compared to the linear cases, convergence analysis of numerical methods for the nonlinear eigenvalue problems associated with plate vibrations is much less developed. To treat the nonlinearity, most literatures construct a continuous function and a fitting function and the eigenvalues are the intersections of these two functions. In this paper, a new computational approach is proposed and its convergence is analyzed. We first convert the problem into the eigenvalue problem of a holomorphic Fredholm operator function. Then the Bogner-Fox-Schmit (BFS) finite element is used to discretize the operators. A spectral indicator method is developed to practically compute the eigenvalues. The convergence of the discrete eigenvalues is proved using the abstract approximation theory of Karma [11, 12].

The rest of the paper is organized as follows. In Section 2, we present the model problem and the abstract approximation theory for the eigenvalue problems of holomorphic Fredholm operator functions. In Section 3, the BFS element is employed to discretize the operators and the convergence of the eigenvalue problem is proved. The end of Section 3 is a description of the spectral indicator method to compute the eigenvalues. Numerical experiments are presented in Section 4. Finally, we draw some conclusions in Section 5.
2 Model Problem and Preliminaries

We consider the plate-spring-load model for the flexural vibration of a thin plate elastically combined with added masses. Assume that the masses are joined by elastic springs with stiffness coefficient $K_j$ at point $x_j \in D$, where $D \subset \mathbb{R}^2$ is a Lipschitz polygon. The $j$th pair of mass and spring forms a harmonic oscillator with the vibration frequency $v_j = \sqrt{K_j/M_j}$, where $M_j$ is the mass. The $\delta$-distribution represents the forcing term. Denote by $\omega(x, t)$ the vertical deflection of the plate at $x \in D$ at time $t$ and $\zeta_j(t)$ the vertical displacement of the $j$th mass $M_j$ at time $t$. The motion of the plate and masses satisfies the following equation [1]:

$$Lw(x, t) + \rho dw(x, t) + \sum_{j=1}^{p} M_j(\zeta_j(t)) \delta(x - x_j) = 0 \quad \text{in } D, \ x_j \in D,$$

$$Bw(x, t) = 0 \quad \text{on } \partial D,$$

$$M_j(\zeta_j(t)) + K_j(\zeta_j(t) - w(x_j, t)) = 0 \quad \text{for } t > 0, \ j = 1, ..., p,$$

where $L$ is the plate operator and $B$ is some boundary operator. The parameters $\rho(x), \nu(x), E(x), d(x)$ denote the volume mass density, Possion ratio, Young’s modulus, thickness of the plate, respectively, such that $K_j > 0, M_j > 0, 0 < \nu(x) < 1/2,$

$$|E(x)| \leq C, \ |\rho(x)| \leq C, \ |d(x)| \leq C,$$

where $C$ is a constant. The plate operator $L$ is given by

$$L(x) = \partial_{xx} R(x)(\partial_{xx} + \nu(x)\partial_{yy}) + \partial_{yy} R(x)(\partial_{yy} + \nu(x)\partial_{xx}) + 2\partial_{xy} R(x)(1 - \nu(x))\partial_{xy},$$

where $R(x) = Ed^3/12(1 - \nu^2)$ is the flexural rigidity of the material.

The eigenvibrations of this system are characterized by the ansatz:

$$w(x, t) = u(x)v(t), \ \zeta_j(t) = c_j u(x_j) v(t), \ x \in D, \ j = 1, ..., p,$$

where $v(t) = a_0 \cos(\sqrt{\lambda} t) + b_0 \sin(\sqrt{\lambda} t), t > 0,$ and $a_0, b_0, c_j$ are constants. Using (2.3) in (2.1), we obtain the nonlinear eigenvalue problem to find $\lambda \in \mathbb{C} \setminus \cup_j \{\sigma_j\}$ and nontrivial function $u(x)$ such that

$$Lu(x) = \lambda \rho du + \sum_{j=1}^{p} \frac{\lambda \sigma_j}{\sigma_j - \lambda} M_j \delta(x - x_j) u \quad \text{in } D,$$

$$Bu(x) = 0 \quad \text{on } \partial D,$$
where $\sigma_j = v_j^2 = K_j/M_j$. In the rest of the paper, we use the clamped plate condition, i.e., $u = \frac{\partial u}{\partial n} = 0$, where $n$ is the unit outward normal to $\partial D$. Without loss of generality, we set $p = 1$ and define the bilinear forms

$$a(u, v) = \int_D R(x)[(\partial_{xx} u + \partial_{yy} u)(\partial_{xx} v + \partial_{yy} v) + (1 - \nu)(2\partial_{xy} u \partial_{xy} v - \partial_{xx} u \partial_{yy} v - \partial_{yy} u \partial_{xx} v)] dx,$$

$$b(u, v; \lambda) = \lambda \int_D \rho du dv dx - \frac{\lambda \sigma}{\lambda - \sigma} M u(x_0) v(x_0), \ x_0 \in D.$$  

(2.6) (2.7)

The variational formulation for the eigenvalue problem is to find $\lambda \in \mathbb{C} \setminus \{\sigma\}$ and nontrivial $u \in H_0^2(D)$ such that

$$a(u, v) = b(u, v; \lambda), \ \forall v \in H_0^2(D).$$

(2.8)

The existence of a countable set of real eigenvalues for (2.8) is proved in [1] using an auxiliary parameter eigenvalue problem and a nonlinear algebraic equation. In this paper, we shall take a different approach by writing (2.8) as the eigenvalue problem of a holomorphic Fredholm operator function, which is then discretized by a finite element method. The convergence of the eigenvalues is then proved using the abstract approximation theory of Karma [11, 12].

Now we present some preliminaries from [11, 12]. The materials are adapted for the finite element method we shall use to discretize (2.8).

**Definition 1.** Let $V, W$ be Banach spaces. A bounded linear operator $F \in \mathcal{L}(V, W)$ is called Fredholm with index zero if

1. the range of $F$, denoted by $\mathcal{R}(F)$, is closed and $\text{codim} \mathcal{R}(F) := \dim(Y/\mathcal{R}(F))$ is finite;

2. the null space of $F$, denoted by $\mathcal{N}(F)$, is finite-dimensional; and

3. the Fredholm index, defined as $\text{ind}(F) = \dim \mathcal{N}(F) - \text{codim} \mathcal{R}(F)$, is zero.

**Definition 2.** (ref. [17]) Let $V$ and $V_n, n \in \mathbb{N}$, be Banach spaces. A sequence of linear operators $\mathcal{P} := \{p_n : V \to V_n\}_{n \in \mathbb{N}}$ connecting them is such that $\|p_n u\|_{V_n} \to \|u\|_V$ for all $u \in V$.

A sequence $\{v_n\}_{n \in \mathbb{N}}, v_n \in V_n$, is called $\mathcal{P}$-converging to an element $u \in V$, denoted by $v_n \overset{\mathcal{P}}{\to} u$, if $\|p_n u - v_n\|_{V_n} \to 0, n \to \infty$. A sequence $\{v_n\}_{n \in \mathbb{N}}, v_n \in V_n$, is called $\mathcal{P}$-compact if for every subsequence $\{v_n\}_{n \in \mathbb{N}'}$, $\mathbb{N}' \subseteq \mathbb{N}$, there exists $\mathbb{N}'' \subseteq \mathbb{N}'$ and $u \in V$ such that $v_n \overset{\mathcal{P}}{\to} u$, $n \in \mathbb{N}'' \to \infty$.

Let $F : \Omega \to \mathcal{L}(V, W)$ be a holomorphic operator function on $\Omega \subset \mathbb{C}$ and, for each $\eta \in \Omega$, $F(\eta)$ is a Fredholm operator of index zero.
Definition 3. A complex number $\lambda \in \Omega$ is called an eigenvalue of $F$ if there exists a nontrivial $x \in V$ such that $F(\lambda)x = 0$. The element $x$ is called an eigenelement associated with $\lambda$.

The resolvent set $\rho(F)$ and the spectrum $\sigma(F)$ of $F$ are respectively defined as

$$\rho(F) = \{\eta \in \Omega : F(\eta)^{-1} \text{ exists and is bounded}\}$$

and

$$\sigma(F) = \Omega \setminus \rho(F).$$

Since $F(\eta)$ is holomorphic, the spectrum $\sigma(F)$ has no cluster points in $\Omega$ and every $\lambda \in \sigma(F)$ is an eigenvalue for $F$. Furthermore, $F^{-1}(\cdot)$ is meromorphic (see Section 2.3 of [12]). The dimension of $\mathcal{N}(F(\lambda))$ is called the geometric multiplicity for an eigenvalue $\lambda$.

Definition 4. An ordered sequence of elements $x_0, x_1, \ldots, x_k$ in $V$ is called a Jordan chain of $F$ at an eigenvalue $\lambda$ if

$$F(\lambda)x_j + \frac{1}{1!}F^{(1)}(\lambda)x_{j-1} + \ldots + \frac{1}{j!}F^{(j)}(\lambda)x_0 = 0, \quad j = 0, 1, \ldots, k,$$

where $F^{(j)}(\lambda)$ denotes the $j$th derivative of $F(\lambda)$.

The length of any Jordan chain of an eigenvalue is finite. Denote by $m(F, \lambda, x_0)$ the length of a Jordan chain formed by an eigenelement $x_0$. The maximal length of all Jordan chains of the eigenvalue $\lambda$ is denoted by $\kappa(F, \lambda)$. Elements of any Jordan chain of an eigenvalue $\lambda$ are called generalized eigenelements of $\lambda$.

Definition 5. The closed linear hull of all generalized eigenelements of $F(\cdot)$ at an eigenvalue $\lambda$, denoted by $G(F, \lambda)$, is called the generalized eigenspace of $\lambda$ for $F(\cdot)$.

The following abstract convergence theorem was proved in [12].

Theorem 1. Let $\Omega \subset \mathbb{C}$ be open, bounded, and simply connected. Denoted by $V$ and $\{V_n\}_{n \in \mathbb{N}}$ a Banach space and a sequence of Banach spaces, respectively. They are connected by a sequence of linear operators $P := \{p_n : V \rightarrow V_n\}_{n \in \mathbb{N}}$ satisfying $\|p_n u\|_{V_n} \rightarrow \|u\|_V, \forall u \in V$. Let $F(\cdot) : \Omega \rightarrow \mathcal{L}(V, V)$ and $F_n(\cdot) : \Omega \rightarrow \mathcal{L}(V_n, V_n)$, $n \in \mathbb{N}$, be holomorphic Fredholm operator functions. Assume $\rho(F) \neq \emptyset$. Let the following properties be satisfied:

(A1) the operator $F_n(\lambda)$ is a Fredholm operator with index zero for any $\lambda \in \Omega$ and $n \in \mathbb{N}$.
(A2) the sequence \( \{ F_n(\cdot) \}_{n \in \mathbb{N}} \) is equibounded on every compact set \( \Omega' \subset \Omega \), namely, for a compact set \( \Omega' \subset \Omega \), there exists a constant \( C \) such that \( \| F_n(\lambda) \|_{C(V_n, V_n)} \leq C \) for all \( n \in \mathbb{N} \) and all \( \lambda \in \Omega' \).

(A3) for each \( \lambda \in \Omega \), the sequence \( \{ F_n(\lambda) \}_{n \in \mathbb{N}} \) approximates \( F(\lambda) \), i.e.,
\[
\| (F_n(\lambda)p_n - p_nF(\lambda))u \|_{V_n} \to 0, \quad \forall u \in V.
\]
(2.11)

(A4) the sequence \( \{ F_n(\lambda) \}_{n \in \mathbb{N}} \) is regular for each \( \lambda \in \Omega \), i.e.,
\[
\{ F_n(\lambda)x_n \}_{n \in \mathbb{N}} \text{ is } \mathcal{P}-\text{compact} \Rightarrow \{ x_n \}_{n \in \mathbb{N}} \text{ is } \mathcal{P}-\text{compact}, \quad \forall \lambda \in \Omega.
\]
(2.12)

If \( \lambda_0 \) be an eigenvalue of \( F(\lambda) \) and \( \Omega_0 \subset \Omega \) be a simply-connected compact set with boundary \( \partial \Omega_0 \subset \rho(F) \) and \( \Omega_0 \cap \sigma(F) = \{ \lambda_0 \} \), then the following estimation holds
\[
| \lambda_n - \lambda_0 | \leq c \epsilon_n^{1/\kappa} \to 0, \quad \forall \lambda_n \in \sigma(F_n) \cap \Omega_0,
\]
(2.13)

where \( \epsilon_n = \sup_{\eta \in \Gamma} \max_{g \in G(F, \lambda_0), \| g \|_V = 1} \| F_n(\eta)p_n g - p_nF(\eta)g \|_{V_n} \), and \( \kappa = \kappa(\lambda, \lambda_0) \).

3 FE-Holomorphic Operator Function Method

We now employ the \( C^1 \) BFS finite element \([13]\) to discretize (2.8) and analyze the convergence of the eigenvalues. For simplicity, we assume that \( D \) is a Lipschitz polygon whose boundary segments are parallel to \( x \)- or \( y \)-axis. Let \( \Omega \) be a connected compact set in \( \mathbb{C} \setminus \{ \sigma \} \). Let \( \| \cdot \| \) denote the usual \( L_2 \)-norm on a Sobolev space. Note that \( a(\cdot, \cdot) \) defines an inner product \( (w, u)_V = a(w, u) \) on \( V := H^2_0(D) \). The associated energy norm \( \| u \|_V = \sqrt{a(u, u)} \) is equivalent to the usual Sobolev norm defined on \( H^2_0(D) \) (see, e.g., \([14]\)). Let \( T_h \) be a regular rectangular mesh where \( h \) is the mesh size. Denote \( V_h \) the corresponding BFS finite element space satisfying the boundary conditions. We require that \( x_0 \) coincides with some mesh node.

The discrete problem for (2.8) is to find \( \lambda_h \in \Omega \) and nontrivial functions \( u_h \in V_h \) such that
\[
a(u_h, v_h) = b(u_h, v_h; \lambda_h), \quad \forall v_h \in V_h.
\]
(3.1)

Lemma 1. For a fixed \( \lambda \in \Omega \), \( b(\cdot, \cdot; \lambda) \) is a bounded bilinear form on \( V \times V \).
Proof. Clearly, \( b(\cdot, \cdot; \lambda) \) is a bilinear form for \( \lambda \in \Omega \). For \( u, v \in V \) and \( \lambda \in \Omega \), for \( s > 0 \), we have that

\[
|b(u, v; \lambda)| = \left| \lambda \int_D \rho u v dx - \frac{\lambda \sigma}{\lambda - \sigma} Mu(x_0) v(x_0) \right| \\
\leq \lambda \rho d \|u\| \|v\| + \left| \frac{\lambda \sigma M}{\lambda - \sigma} \right| \|u\|_0 \|v\|_0 \\
\leq C(\lambda, D, \rho, d, \sigma, M) \|u\|_{1+s} \|v\|_{1+s}
\]

(3.2)

where \( \| \cdot \|_0 \) is the norm on \( C(D) \) and the embedding \( H^{1+s}(D) \hookrightarrow C(D) \) for any small \( s > 0 \) is used.

We define the operator functions \( B(\lambda) : \mathbb{C} \setminus \{\sigma\} \to \mathcal{L}(V, V) \) such that

\[
b(u, v; \lambda) = (B(\lambda)u, v)_V, \quad \forall u, v \in V,
\]

(3.3)

and \( B_h(\lambda) : \mathbb{C} \to \mathcal{L}(V_h, V_h) \) such that

\[
b(u_h, v_h; \lambda) = (B_h(\lambda)u_h, v_h)_V, \quad \forall u_h, v_h \in V_h.
\]

(3.4)

Since \( V_h \subset V \), the following Galerkin orthogonality holds

\[
((B(\lambda) - B_h(\lambda)) u_h, v_h)_V = 0, \quad \forall u_h, v_h \in V_h.
\]

(3.5)

Now we define the nonlinear operator function

\[
F(\lambda) := I - B(\lambda)
\]

(3.6)

and its finite element approximation

\[
F_h(\lambda) := I_h - B_h(\lambda).
\]

(3.7)

The eigenvalue problems (2.8) and (3.1) can be written, respectively, as to find \( \lambda \in \mathbb{C} \setminus \{\sigma\} \) and \( u \in V \setminus \{0\} \) such that

\[
F(\lambda)u = 0
\]

(3.8)

and find \( \lambda_h \in \Omega \) and \( u_h \in V_h \setminus \{0\} \) such that

\[
F_h(\lambda_h)u_h = 0.
\]

(3.9)
Lemma 2. For \( \lambda \in \Omega \) and \( h > 0 \) small enough, \( B(\lambda) \) and \( B_h(\lambda) \) are compact. Furthermore, \( F(\lambda) \) and \( F_h(\lambda) \) are Fredholm operators with index zero.

Proof. Note that given a bounded sequence \( \{v_n\}_{n \in \mathbb{N}} \in V \), there is a convergent subsequence in \( C(D) \) and, for simplicity, we still denote it by \( \{v_n\}_{n \in \mathbb{N}} \). By the definition of \( B(\lambda) \), we have that

\[
\|B(\lambda)v_{n_1} - B(\lambda)v_{n_2}\|_V^2 = (B(\lambda)v_{n_1} - B(\lambda)v_{n_2}, B(\lambda)v_{n_1} - B(\lambda)v_{n_2})_V \\
= b(v_{n_1} - v_{n_2}, B(\lambda)v_{n_1} - B(\lambda)v_{n_2}; \lambda) \\
= \lambda \int_D \rho d(v_{n_1} - v_{n_2})(B(\lambda)v_{n_1} - B(\lambda)v_{n_2}) dx \\
- \frac{\lambda \sigma}{\lambda - \sigma} M(v_{n_1} - v_{n_2})(x_0)(B(\lambda)v_{n_1} - B(\lambda)v_{n_2})(x_0) \\
\leq C(\lambda, D, \rho, d, M)\|v_{n_1} - v_{n_2}\|_0\|B(\lambda)v_{n_1} - B(\lambda)v_{n_2}\|_0 + C(\lambda, \sigma, M)\|v_{n_1} - v_{n_2}\|_0\|v_{n_1} - v_{n_2}\|_0.
\]

(3.10)

Since \( \|v\|_0 \leq C\|v\|_V \) for \( v \in V \), it holds that

\[
\|B(\lambda)v_{n_1} - B(\lambda)v_{n_2}\|_V \leq C\|v_{n_1} - v_{n_2}\|_0 \to 0.
\]

(3.11)

Thus \( \{B(\lambda)v_n\}_{n \in \mathbb{N}} \) converges. Consequently, \( B(\lambda) \) is compact and \( F(\lambda) \) is a Fredholm operator with index zero. The same argument holds for \( B_h(\lambda) \) and \( F_h(\lambda) \).

Lemma 3. There exists a constant \( C \) such that \( \|F_h(\lambda)\|_{\mathcal{L}(V_h, V_h)} \leq C \) for \( \lambda \in \Omega, \; h > 0 \).

Proof. It can be seen from (3.2) and (3.4) that

\[
\|B_h(\lambda)\|_{\mathcal{L}(V_h, V_h)} = \sup_{u_h, v_h \in V_h} \frac{|(B_h(\lambda)u_h, v_h)_V|}{\|u_h\|_{V_h}\|v_h\|_{V_h}} = \sup_{u_h, v_h \in V_h} \frac{|b(u_h, v_h; \lambda)|}{\|u_h\|_{V_h}\|v_h\|_{V_h}} \leq C(\lambda, D, \rho, d, \sigma, M).
\]

Since \( \Omega \) is compact, for \( \lambda \in \Omega, \; v_h \in V_h \), one has that

\[
\|F_h(\lambda)v_h\|_{V_h} = \|v_h - B_h(\lambda)v_h\|_{V_h} \leq \|v_h\|_{V_h} + \|B_h(\lambda)v_h\|_{V_h} \leq C\|v_h\|_{V_h},
\]

(3.12)

where \( C \) is independent of \( \lambda \) and \( h \).

Define the linear projection operator \( p_h : V \to V_h \) such that

\[
a(u - p_h u, v_h) = 0, \; \forall v_h \in V_h.
\]

(3.13)
Remark 1. For the biharmonic equation

\[ \Delta^2 u = f \quad \text{in} \quad D, \]  
\[ u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial D, \]  
(3.14a)

if the largest interior angle of the boundary \( \partial D \) is less than 126.28° and \( f \in H^{-1}(D) \), then \( u \in H^3(D) \cap H^2_0(D) \). In fact, Dirac’s delta function \( \delta(x) \in H^{-\epsilon(D)} \) for any \( \epsilon > 0 \). Hence one can assume that \( u \in H^{3-\epsilon}(D) \), for any \( \epsilon > 0 \) small enough.

Lemma 4. Let \( \lambda \in \Omega \) and \( v \in H^2_0(D) \cap H^{3-\epsilon}(D) \) for any small \( \epsilon > 0 \). It holds that

\[ \|p_h F(\lambda)v - F_h(\lambda)p_h v\|_{V_h} \leq C h^{2-\epsilon-s} \|v\|_{3-\epsilon}, \]  
(3.15)

where \( s > 0 \) and small enough.

Proof. Using (3.3), (3.4), (3.6) and (3.7),

\[ |(p_h F(\lambda)v - F_h(\lambda)p_h v, v_h)_{V_h}| = |(p_h (I - B(\lambda))v - (I_h - B_h(\lambda))p_h v, v_h)_{V_h}| \]
\[ = |(-p_h B(\lambda)v + B_h(\lambda)p_h v, v_h)_{V_h}| \]
\[ = |- (B(\lambda)v, v_h)_{V_h} + (B_h(\lambda)p_h v, v_h)_{V_h}| \]
\[ = | - b(v, v_h; \lambda) + b(p_h v, v_h; \lambda)| \]
\[ = |b(p_h v - v, v_h; \lambda)|, \]  
(3.16)

where \( v_h \in V_h \) and \( \|v_h\|_{V_h} = 1 \). In view of (3.2) and the orthogonal projection theorem \[19, 20],

(3.10) can be written as

\[ |(p_h F(\lambda)v - F_h(\lambda)p_h v, v_h)_{V_h}| \leq C \|v_h\|_{V_h} \|p_h v - v\|_{1+s} \]
\[ \leq C \|p_h v - v\|_{1+s} \]  
(3.17)
\[ \leq C h^{2-\epsilon-s} \|v\|_{3-\epsilon}. \]  
(3.18)

Hence

\[ \|p_h F(\lambda)v - F_h(\lambda)p_h v\|_{V_h} = \sup_{v_h \in V_h, \|v_h\|_{V_h} = 1} |(p_h F(\lambda)v - F_h(\lambda)p_h v, v_h)_{V_h}| \leq C h^{2-\epsilon-s} \|v\|_{3-\epsilon} \]  
(3.19)

and the proof is complete. \( \square \)
Theorem 2. Let \( \lambda \in \Omega \) be fixed. For \( v_h \in V_h \),

\[
\| (F(\lambda) - F_h(\lambda))v_h \|_V = \| (B(\lambda) - B_h(\lambda))v_h \|_V \rightarrow 0 \quad \text{as} \quad h \rightarrow 0.
\]

Proof. From [3.3] and [3.4], we have \((B(\lambda) - B_h(\lambda))u_h, v_h\)_\(V\) = 0 for all \( u_h, v_h \in V_h \) and

\[
\| (B(\lambda) - B_h(\lambda))v_h \|_V^2 = ((B(\lambda) - B_h(\lambda))v_h, (B(\lambda) - B_h(\lambda))v_h)_V \\
\leq \| (B(\lambda) - B_h(\lambda))v_h \|_V \| (I - p_h)(B(\lambda)v_h) \|_V.
\]

Using the property of the projection operator \( p_h \), it holds that

\[
\| (B(\lambda) - B_h(\lambda))v_h \|_V \leq \| (I - p_h)(B(\lambda)v_h) \|_V \rightarrow 0. \tag{3.20}
\]

\[\square\]

Lemma 5. Let \( \lambda \in \Omega \) be fixed. For \( v_h \in V_h \),

\[
\| (F(\lambda) - F_h(\lambda))v_h \|_V = \| (B(\lambda) - B_h(\lambda))v_h \|_V \rightarrow 0 \quad \text{as} \quad h \rightarrow 0.
\]

Proof. From Lemmas 2, 3, and 4, we only need to verify (A4) in Theorem 1, namely, the sequence \( \{F_h(\lambda)\} \) is regular for every \( \lambda \in \Omega_0 \). Let \( \{h_n\} \) be a positive number sequence which goes to 0 monotonically when \( n \in \mathbb{N} \rightarrow \infty \). Consider an arbitrary subsequence \( \mathbb{N}' \subseteq \mathbb{N} \) and \( \{v_{h_n'}\} \subset \{v_{h_n}\}, n' \in \mathbb{N}' \). According to the definition of \( \mathcal{P} \)-compactness, for any \( \lambda \), there exists a subsequence \( \{F_{h_{n''}}(\lambda)v_{h_{n''}}\}, n'' \in \mathbb{N}'' \subseteq \mathbb{N}' \) and some \( y \in V \) such that \( \| F_{h_{n''}}(\lambda)v_{h_{n''}} - p_{h_{n''}}y \|_V \rightarrow 0 \). The goal is to show the existence of some \( v \in V \) such that \( \| v_{h_{n''}} - p_{h_{n''}}v \|_V \rightarrow 0 \) as \( n'' \in \mathbb{N}'' \rightarrow \infty \).
1. If $\lambda \in \rho(F)$, then $F(\lambda)^{-1}$ exists and is bounded. Letting $v = F(\lambda)^{-1}y$, due to Lemma 3 and Lemma 4, we have that

$$
\|v_{h_{n'}} - p_{h_{n'}}v\|_V
= \|F(\lambda)^{-1}[(F(\lambda) - F_{h_{n'}}(\lambda))(v_{h_{n'}} - p_{h_{n'}}v) + F_{h_{n'}}(\lambda)(v_{h_{n'}} - p_{h_{n'}}v)]\|_V
= \|F(\lambda)^{-1}[(F(\lambda) - F_{h_{n'}}(\lambda))(v_{h_{n'}} - p_{h_{n'}}v) + F_{h_{n'}}(\lambda)v_{h_{n'}} - p_{h_{n'}}y] + p_{h_{n'}}F(\lambda)v - F_{h_{n'}}(\lambda)p_{h_{n'}}v\|_V
\leq \|F(\lambda)^{-1}\|_V\|[F(\lambda) - F_{h_{n'}}(\lambda))(v_{h_{n'}} - p_{h_{n'}}v)]\|_V + \|F_{h_{n'}}(\lambda)v_{h_{n'}} - p_{h_{n'}}y\|_V + \|p_{h_{n'}}F(\lambda)v - F_{h_{n'}}(\lambda)p_{h_{n'}}v\|_V
\rightarrow 0.
$$

2. If $\lambda \in \sigma(F), N(F(\lambda))$ is finite-dimensional since $F(\lambda)$ is Fredholm. In fact,

$$
\|F(\lambda)v_{h_{n'}} - y\|_V = \|F(\lambda)v_{h_{n'}} - F_{h_{n'}}(\lambda)v_{h_{n'}} + F_{h_{n'}}(\lambda)v_{h_{n'}} - p_{h_{n'}}y + p_{h_{n'}}y - y\|_V \rightarrow 0, \quad (3.24)
$$

and thus $y \in R(F(\lambda))$ because $R(F(\lambda))$ is closed. $F(\lambda)$ is invertible as a mapping from $V/N(F(\lambda))$ to $R(F(\lambda))$ and $F(\lambda)^{-1}$ is well-defined. Let $v \in V/N(F(\lambda))$ such that $F(\lambda)v = y$. Then $\|v_{h_{n'}} - p_{h_{n'}}v\|_V \rightarrow 0$ can be proved as before.

Next we consider the consistency error $\epsilon_h$. The generalized eigenspace $G(F; \lambda_0)$ is finite-dimensional since $F(\lambda_0)$ is a Fredholm operator. If $\lambda_0$ is semi-simple, for $g \in G(F; \lambda_0)$, we have $g \in H^2_0(D) \cap H^{3-s}(D)$ and $\kappa = 1$. Combining (3.21) and Lemma 4, we obtain that

$$
\epsilon_h = \sup_{\eta \in \Gamma} \max_{g \in G(F; \lambda_0), \|g\|_V = 1} \|F_h(\eta)p_h g - p_h F(\eta)g\|_V \leq C h^{2-s-\epsilon}, \quad (3.25)
$$

where $\epsilon$ and $s$ are small enough. Especially, letting $\epsilon \rightarrow 0$, we obtain

$$
|\lambda_h - \lambda_0| \leq C h^{2-s}, \forall s \rightarrow 0_+.
$$

(3.26)

If $\lambda_0$ is not semi-simple, the convergence is given by

$$
|\lambda_h - \lambda_0| \leq C h^{(1-s)/\kappa}, \forall s \rightarrow 0_+.
$$

(3.27)

The proof is complete.
The rest of this section is devoted to the spectral indicator method (SIM) to compute the eigenvalues of $F_h(\lambda)$ [21, 22]. Without loss of generality, let $\Omega \in \mathbb{C}$ be a square and $\Gamma := \partial \Omega \subset \rho(F_h)$. Denote by $\mathbb{R}_h : V_h \to V_h$ the spectral projection
\[
\mathbb{R}_h = \frac{1}{2\pi i} \int_{\Gamma} F_h(\eta)^{-1} d\eta. \tag{3.28}
\]
The matrix form of $F_h(\eta)$ is given by
\[
F_h(\eta) = A_h - \eta B_h + \frac{\eta \sigma}{\eta - \sigma} C_h, \tag{3.29}
\]
where $A_h, B_h,$ and $C_h$ are, respectively, the matrices corresponding to
\[
\int_D R(x)[(\partial_{xx} u_h + \partial_{yy} u_h)(\partial_{xx} v_h + \partial_{yy} v_h) + (1 - \nu)(2\partial_{xy} u_h \partial_{xy} v_h - \partial_{xx} u_h \partial_{yy} v_h - \partial_{yy} u_h \partial_{xx} v_h)] dx, \]
\[
\int_D \rho du_h v_h dx, \quad \text{and} \quad Mu_h(x_0)v_h(x_0).
\]
If $\Gamma$ encloses no eigenvalues of $F_h$, the integral $\int_{\Gamma} F_h(\eta)^{-1} ds$ should be zero as well as $\mathbb{R}_h(\vec{y}_h)$ for any vector $\vec{y}_h$ in $V_h$. In practice, one selects a random vector $\vec{y}_h \in V_h$ and solves $\vec{x}_h(\eta_i) \in V_h$ for $F_h(\eta_i)\vec{x}_h(\eta_i) = \vec{y}_h$. Gaussian quadrature can be used to approximate (3.28)
\[
I_\Omega := \left| \frac{1}{2\pi i} \sum_{i=1}^{m} w_i \vec{x}_h(\eta_i) \right|, \tag{3.30}
\]
where $m$ is the number of quadrature nodes and $w_i, \eta_i$ represent the weights and corresponding nodes. One decides whether $\Omega$ contains eigenvalues using $I_\Omega$. This is done in SIM by setting a threshold $\alpha$. If $I_\Omega$ is less than $\alpha$, $\Omega$ contains no eigenvalues. Otherwise, one uniformly splits $\Omega$ into four squares $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ and compute $I_{\Omega_1}, I_{\Omega_2}, I_{\Omega_3}, I_{\Omega_4}$ correspondingly. The procedure continues until the size of the square is less than a given precision $\beta$. The center of the square is the computed eigenvalues of $F_h(\cdot)$. We refer the reader to [23, 24, 25] for the details of the algorithm.

4 Numerical Examples

We present some numerical results using a series of uniformly refined meshes $\{T_h\}$, where $h_i(= h_{i-1}/2)$ is the mesh size. To measure the convergence order, we use the relative error
\[
\text{Rel.Err} = \frac{|\lambda_{h_i} - \lambda_{h_{i-1}}|}{\lambda_{h_i}}, \tag{4.1}
\]
where \( \lambda_{h_i} \) is an eigenvalue computed on mesh \( T_{h_i} \). For all examples, \( R, \nu, \rho, d \) are constants and (2.4)-(2.5) can be written as

\[
\Delta^2 u(x) = \lambda \frac{\rho d}{R} u + \sum_{j=1}^{\sigma} \frac{\lambda \sigma_j}{R(\sigma_j - \lambda)} M_j \delta(x - x_j) u, \quad x \in D, \quad (4.2a)
\]

\[
u (x) = \frac{\partial u}{\partial n} = 0, \quad x \in \partial D. \quad (4.2b)
\]

**Example 1.** \( p = 1 \) for a rectangular domain

Let \( D = [0, 1] \times [0, 1] \), \( R = 1, \rho d = 1, M = 0.01, K = 100, \sigma = 10000, x_0 = (9/26, 19/26)^T \) as in [1], i.e.,

\[
\Delta^2 u(x) = \lambda u + \frac{100 \lambda}{10000 - \lambda} \delta(x - x_0) u, \quad x \in D, \quad (4.3a)
\]

\[
u (x) = \frac{\partial u}{\partial n} = 0, \quad x \in \partial D. \quad (4.3b)
\]

Fig. 1 shows the relative errors v.s. the degrees of freedom for the first five eigenvalues. In Table. 1 we show the first three eigenvalues and convergence orders. The convergence orders validate the theory, namely, at least 2 but no more than 4. The eigenvalues are consistent with those in [1].

![Figure 1: Relative errors of first five eigenvalues for (4.3).](image)

**Example 2.** \( p = 1 \) for an L-shaped domain
Table 1: Example 1: The first three eigenvalues and convergence orders.

| $h$  | $\lambda_h$(1st) | $\textit{order}$ | $\lambda_h$(2nd) | $\textit{order}$ | $\lambda_h$(3rd) | $\textit{order}$ |
|------|------------------|------------------|------------------|------------------|------------------|------------------|
| 1/5  | 1273.40439       | -                | 4847.23905       | -                | 5416.51831       | -                |
| 1/10 | 1271.63622       | -                | 4827.85974       | -                | 5389.11595       | -                |
| 1/20 | 1271.48701       | 3.57             | 4826.16599       | 3.52             | 5386.83431       | 3.59             |
| 1/40 | 1271.47561       | 3.71             | 4825.97011       | 3.11             | 5386.66812       | 3.78             |
| 1/80 | 1271.47475       | 3.72             | 4825.94165       | 2.78             | 5386.65729       | 3.94             |

Consider an L-shaped domain $D = [-1,1] \setminus (0,1] \times [-1,0)$, which has a reentrant corner. Let $M = 0.01$, $K = 20$, $\sigma = 2000$, $x_0 = (1/2,1/2)^T$. The other parameters are the same as the previous example. Fig. 2 shows the relative errors v.s. the degrees of freedom for the first five eigenvalues. Table 2 shows the first three eigenvalues and convergence orders. As expected, the convergence orders are lower than the previous example due to the reentrant corner.

Figure 2: Relative errors of first five eigenvalues for L-shaped domain.

Example 3. $p \neq 1$

Now we consider the case when $p = 2$, i.e., two added masses. Let $D = [0,1] \times [0,1]$ and set $pd = 1$, $M_1 = 0.01$, $K_1 = 20$, $\sigma_1 = 2000$, $x_1 = (0.4,0.2)^T$, $M_2 = 0.01$, $K_2 = 40$, $\sigma_2 = 4000$, $x_2 =$
Table 2: Example 2: The first three eigenvalues and convergence orders.

\[
\begin{array}{ccccccc}
\hline
h & \lambda_h(1st) & order & \lambda_h(2nd) & order & \lambda_h(3rd) & order \\
\hline
1/8 & 426.53874 & - & 678.31742 & - & 902.57169 & - \\
1/16 & 422.16617 & - & 677.93923 & - & 902.28136 & - \\
1/32 & 420.13893 & 1.10 & 677.85652 & 2.19 & 902.22727 & 2.42 \\
1/64 & 419.18665 & 1.09 & 677.83435 & 1.90 & 902.21013 & 1.66 \\
1/128 & 418.73687 & 1.08 & 677.82809 & 1.82 & 902.20313 & 1.29 \\
\hline
\end{array}
\]

In Fig. 3, the relative errors v.s. the degrees of freedoms are shown. Table 3 lists the first three eigenvalues and convergence orders. Again, the convergence orders are between 2 and 4.

Figure 3: Relative errors of first five eigenvalues for (4.4).

5 Conclusion

In the paper, we develop a new numerical method for the nonlinear eigenvalue problems associated to the vibrations of the plate-spring-load system. The problem is formulated as the eigenvalue
Table 3: Example 3: The first three eigenvalues and corresponding convergence orders.

| $h$ | $\lambda_h$ (1st) | order | $\lambda_h$ (2nd) | order | $\lambda_h$ (3rd) | order |
|-----|-------------------|-------|-------------------|-------|-------------------|-------|
| 1/5 | 1969.78685        | -     | 3713.12915        | -     | 5437.59842        | -     |
| 1/10| 1967.40653        | -     | 3706.51164        | -     | 5399.91834        | -     |
| 1/20| 1966.83124        | 2.05  | 3704.49126        | 1.71  | 5397.30692        | 3.85  |
| 1/40| 1966.69214        | 2.05  | 3703.99197        | 2.02  | 5397.13054        | 3.89  |
| 1/80| 1966.65774        | 2.02  | 3703.86826        | 2.01  | 5397.11834        | 3.85  |

The proposed method is effective for problems with nonlinear dependence on the eigen-parameter, which has been successfully used to compute the Dirichlet eigenvalues, the transmission eigenvalues, and the band structures of photonic crystals [23, 24, 25]. In the future, we plan to extend the method to treat damped vibrations. Extension of the method to treat 3D problems on general domains is another interesting topic.

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