Where the monotone pattern (mostly) rules

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Abstract

We consider pattern containment and avoidance with a very tight definition that was used first by Riordan more than 60 years ago. Using this definition, we prove the monotone pattern is easier to avoid than almost any other pattern of the same length. We also show that with this definition, almost all patterns of length $k$ are avoided by the same number of permutations of length $n$. The corresponding statements are not known to be true for more relaxed definitions of pattern containment. This is the first time we know of that expectations are used to compare numbers of permutations avoiding certain patterns.

1 Introduction

The classic definition of pattern avoidance on permutations is as follows. Let $p = p_1p_2 \cdots p_n$ be a permutation, let $k < n$, and let $q = q_1q_2 \cdots q_k$ be another permutation. We say that $p$ contains $q$ as a pattern if there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ so that for all indices $j$ and $r$, the inequality $q_j < q_r$ holds if and only if the inequality $p_{i_j} < p_{i_r}$ holds. If $p$ does not contain $q$, then we say that $p$ avoids $q$. In other words, $p$ contains $q$ if $p$ has a subsequence of entries, not necessarily in consecutive positions, which relate to each other the same way as the entries of $q$ do.

Classic pattern avoidance has been a rapidly developing field for the last decade. One of the most fascinating subjects in this field was the enumeration of permutations avoiding a given pattern. Let $S_n(q)$ denote the number of permutations of length $n$ (or in what follows, $n$-permutations) that avoid

*Partially supported by an NSA Young Investigator Award.
the pattern \( q \), and let us consider the numbers \( S_n(q) \) for each pattern \( q \) of length \( k \). A very interesting and counter-intuitive phenomenon is that in this multiset of \( k! \) numbers, the number \( S_n(q) \) will, in general, not be the largest or the smallest number. There are several results on this fact (see [1], [2] or [3]), but the phenomenon is still not perfectly well understood.

In 2001, Elizalde and Noy [4] proposed another definition of pattern containment. We will say that the permutation \( p = p_1p_2 \cdots p_n \) tightly contains the permutation \( q = q_1q_2 \cdots q_k \) if there exists an index \( 0 \leq i \leq n - k \) so that \( q_i < q_r \) if and only if \( p_i + j < p_i + r \). In other words, for \( p \) to contain \( q \), we require that \( p \) has a consecutive string of entries that relate to each other the same way the entries of \( q \) do. For instance, 246351 contains 132 (take the second, third, and fifth entries, for instance), but it does not tightly contain 132 since there are no three entries in consecutive positions in 246351 that would form a 132-pattern. If \( p \) does not tightly contain \( q \), then we say that \( p \) tightly avoids \( q \). Let \( T_n(q) \) denote the number of \( n \)-permutations that tightly avoid \( q \). Elizalde and Noy conjectured in [4] that no pattern of length \( k \) is tightly avoided by more \( n \)-permutations than the monotone pattern. In other words, if \( q \) is a pattern of length \( k \), then

\[
T_n(q) \leq T_n(12 \cdots k). \tag{1}
\]

This conjecture is still open. (In the special case of \( k = 3 \), it was proved in [4].) Still, it is worth pointing out that changing the definition of pattern avoidance changed the status of the monotone pattern among all patterns of the same length. With this definition, it is believed that the monotone pattern is the easiest pattern to avoid.

This perceived change in the status of the monotone pattern led us to the following direction of research. Let us take the idea of Elizalde and Noy one step further, by restricting the notion of pattern containment further as follows. Let \( p = p_1p_2 \cdots p_n \) be a permutation, let \( k < n \), and let \( q = q_1q_2 \cdots q_k \) be another permutation. We say that \( p \) very tightly contains \( q \) if there is an index \( 0 \leq i \leq n - k \) and an integer \( 0 \leq a \leq n - k \) so that \( q_j < q_r \) if and only if \( p_i + j < p_i + r \), and,

\[
\{p_{i+1}, p_{i+2}, \ldots, p_{i+k}\} = \{a + 1, a + 2, \ldots, a + k\}.
\]

That is, \( p \) very tightly contains \( q \) if \( p \) tightly contains \( q \) and the entries of \( p \) that form a copy of \( q \) are not just in consecutive positions, but they are also consecutive as integers (in the sense that their set is an interval).

For example, 15324 tightly contains 132 (consider the first three entries), but does not very tightly contain 132. On the other hand, 15324 very tightly
contains 213, as can be seen by considering the last three entries. If \( p \) does not very tightly contain \( q \), then we will say that \( p \) very tightly avoids \( q \). Note that in the special case when \( q \) is the monotone pattern, this notion was studied before pattern avoidance became widely known. The literature of permutations very tightly avoiding monotone patterns goes back at least to [7]. More recent examples include [6] and [5]. However, we did not find any examples where the notion was used in connection with any other pattern.

Let \( V_n(q) \) denote the number of \( n \)-permutations that very tightly avoid \( q \). While we cannot prove that \( V_n(q) \leq V_n(12 \cdots k) \) for all patterns \( q \) of length \( k \), we will be able to prove that this inequality holds for most patterns \( q \) of length \( k \). As a byproduct, we will prove that for all \( k \), there exists a set \( W_k \) of patterns of length \( k \) so that \( \lim_{k \rightarrow \infty} \frac{|W_k|}{k!} = 1 \), and \( V_n(q) \) is identical for all patterns \( q \in W_k(q) \). In other words, almost all patterns of length \( k \) are equally difficult to very tightly avoid. There are no comparable statements known for the other two discussed notions of pattern avoidance.

Our argument will be a probabilistic one. Once the framework is set up, the computation will be elementary. However, this is the first time we know of that expectations are successfully used to compare the number of permutations avoiding a given pattern (admittedly, with a very restrictive definition of pattern avoidance). We wonder whether more sophisticated methods of enumeration could extend the reach of this technique to less restrictive definitions of pattern avoidance.

## 2 A Probabilistic Argument

### 2.1 The outline of the argument

For the rest of this section, let \( k \geq 3 \) be a fixed positive integer. Let \( \alpha = 12 \cdots k \), the monotone pattern of length \( k \). Recall that \( V_n(\alpha) \) is the number of \( n \)-permutations very tightly avoiding \( \alpha \). Our goal is true prove that

\[
V_n(q) \leq V_n(\alpha)
\]

for any pattern \( q \) of length \( k \).

Let \( q \) be any pattern of length \( k \). For a fixed positive integer \( n \), let \( X_{n,q} \) be the random variable counting the occurrences of \( q \) in a randomly selected \( n \)-permutation. As the following straightforward proposition shows, the expectation of \( X_{n,q} \) does not depend on \( q \); it only depends on \( n \), and the length \( k \) of \( q \).
Proposition 1 For any fixed $n$, and $q \in S_k$, we have

$$E(X_{n,q}) = \frac{(n - k + 1)^2}{\binom{n}{k} k!}.$$ 

Proof: Let $X_i$ be the indicator random variable of the event that the string $p_{i+1} \cdots p_{i+k}$ is a $q$-pattern in the very tight sense. Then $E(X_i) = P(p_{i+1} \cdots p_{i+k} \simeq q) = \frac{n-k+1}{\binom{n}{k} \cdot k!}$, since there are $n-k+1$ favorable choices for the set of the entries $p_{i+1}, \cdots, p_{i+k}$, and there is $1/k!$ chance that their pattern is $q$. Now note that $E(X_{n,q}) = \sum_{i=0}^{n-k} X_i$, and the statement is proved by the linearity of expectation. \(\diamondsuit\)

Let $p_{n,i,q}$ be the probability that a randomly selected $n$-permutation contains exactly $i$ copies of $q$, and let $P_{n,i,q}$ be the probability that a randomly selected $n$-permutation contains at least $i$ copies of $q$.

Set $m = n - k + 1$, and observe that no $n$-permutation can very tightly contain more than $m$ copies of any given pattern $q$ of length $k$. Now note that by the definition of expectation

$$E(X_{n,q}) = \sum_{i=1}^{m} ip_{n,i,q}$$

$$= \sum_{j=0}^{m-1} \sum_{i=0}^{j} p_{n,m-i,q}$$

$$= p_{n,m,q} + (p_{n,m,q} + p_{n,m-1,q}) + \cdots + (p_{n,m,q} + \cdots + p_{n,1,q})$$

$$= \sum_{i=1}^{m} P(n, i, q).$$

By Proposition 1 we know that $E(X_{n,q}) = E(X_{n,\alpha})$, and then previous displayed equation implies that

$$\sum_{i=1}^{m} P(n, i, q) = \sum_{i=1}^{m} P(n, i, \alpha). \quad (2)$$

So if we could show that for $i \geq 2$, the inequality

$$P(n, i, q) \leq P(n, i, \alpha) \quad (3)$$

holds, then (2) would imply that $P(n, 1, q) \geq P(n, 1, \alpha)$, which is just what we set out to prove.

The simple counting argument that we present in this paper will not prove (2) for every pattern $q$. However, it will prove (2) for most patterns $q$. We describe these patterns in the next subsection.
2.2 Condensible and Non-condensible Patterns

Let us assume that the permutation $p = p_1p_2 \cdots p_n$ very tightly contains two non-disjoint copies of the pattern $q = q_1q_2 \cdots q_k$. Let these two copies be $q^{(1)}$ and $q^{(2)}$, so that $q^{(1)} = p_{i+1}p_{i+2} \cdots p_{i+k}$ and $q^{(2)} = p_{i+j+1}p_{i+j+2} \cdots p_{i+j+k}$ for some $j \in [1, k - 1]$. Then $|q^{(1)} \cap q^{(2)}| = k - j + 1 = s$. Furthermore, since the set of entries of $q^{(1)}$ is an interval, and the set of entries of $q^{(2)}$ is an interval, it follows that the set of entries of $q^{(1)} \cap q^{(2)}$ is also an interval. So the rightmost $s$ entries of $q$, and the leftmost $s$ entries of $q$ must form identical patterns, and the respective sets of these entries must both be intervals.

For obvious symmetry reasons, we can assume that $q_1 < q_k$. We claim that then the rightmost $s$ entries of $q$ must also be the largest $s$ entries of $q$. This can be seen by considering $q^{(1)}$. Indeed, the set of these entries of $q^{(1)}$ is the intersection of two intervals of the same length, and therefore, must be an ending segment of the interval that starts on the left of the other. An analogous argument, applied for $q^{(2)}$, shows that the leftmost $s$ entries of $q$ must also be the smallest $s$ entries of $q$.

The following Proposition collects the observations made in this subsection.

**Proposition 2** Let $p$ be a permutation that very tightly contains copies $q^{(1)}$ and $q^{(2)}$ of the pattern $q = q_1q_2 \cdots q_k$. Let us assume that $q_1 < q_k$. Then $q^{(1)}$ and $q^{(2)}$ are disjoint unless all of the following hold.

There exists a positive integer $s \leq k - 1$ so that

1. the rightmost $s$ entries of $q$ are also the largest $s$ entries of $q$, and the leftmost $s$ entries of $q$ are also the smallest $s$ entries of $q$, and

2. the pattern of the leftmost $s$ entries of $q$ is identical to the pattern of the rightmost $s$ entries of $q$.

It is easy to see that if $q$ satisfies both of these criteria, then two very tightly contained copies of $q$ in $p$ may indeed intersect. For example, the pattern $q = 2143$ satisfies both of the above criteria with $s = 2$, and indeed, 214365 very tightly contains two intersecting copies of $q$, namely 2143 and 4365.

**Definition 1** Let $q$ be a pattern that satisfies both conditions of Proposition 2. Then we say that $q$ is condensible.
It is not difficult to prove that almost all patterns of length \( k \) are non-condensible. We do not want to break the course of our proof with this, so we postpone this computation until the Appendix.

### 2.3 The Computational Part of the Proof

The following Lemma is the heart of our main result.

**Lemma 1** Let \( q \) be a non-condensible pattern, and let \( i > 1 \). Then

\[
P(n, i, q) \leq P(n, i, \alpha).
\]

**Proof:** We point out that if \( n < ik \), then the statement is clearly true. Indeed, \( P(n, i, q) = 0 \) since any two copies of \( q \) in an \( n \)-permutation \( p \) would have to be disjoint, and \( n \) is too small for that. Therefore, in the rest of the proof, we can assume that \( n \geq ik \). For \( k \leq 2 \), the statement is trivial, so we assume \( k \geq 3 \) as well.

First, we prove a lower bound on \( P(n, i, \alpha) \). The number of \( n \)-permutations very tightly containing \( i \) copies of \( \alpha \) is at least as large as the number of \( n \)-permutations very tightly containing the pattern \( 12 \cdots (i + k - 1) \). The latter is at least as large as the number of \( n \)-permutations that very tightly contain a \( 12 \cdots (i + k - 1) \)-pattern in their first \( i + k - 1 \) positions, that is, \((n - k - i + 2) \cdot (n - k - i + 1)! = (n - k - i + 2)!\). Therefore,

\[
\frac{(n - k - i + 2)!}{n!} \leq P(n, i, \alpha).
\] (4)

We are now going to find an upper bound for \( P(n, i, q) \). Let \( S \) be an \( i \)-element subset of \([n]\) so that the elements of \( S \) can be the starting positions of \( i \) (necessarily disjoint) very tight copies of \( q \) in an \( n \)-permutation. If \( S = \{s_1, s_2, \ldots, s_i\} \), then this is equivalent to saying that

\[
1 \leq s_1 < s_2 - k + 1 \leq s_3 - 2k + 2 \leq \cdots \leq s_i - (i - 1)(k - 1) \leq n - i(k - 1).
\]

Therefore, there are \( \binom{n - i(k - 1)}{i} \) possibilities for \( S \). Now let \( A_S \) be the event that in a random permutation \( p = p_1 \cdots p_n \), the subsequence \( p_jp_{j+1} \cdots p_{j+k-1} \) is a very tight \( q \)-subsequence for all \( j \in S \). Let \( A_{i,q} \) be the event that \( p \) contains at least \( i \) very tight copies of \( q \). Then \( P(A_{i,q}) = P(n, i, q) \). Furthermore,

\[
A_{i,q} = \cup_S A_S,
\]

where the union is taken over all \( \binom{n - i(k - 1)}{i} \) possible subsets for \( S \). Therefore,

\[
P(n, i, q) = P(A_{i,q}) \leq \sum_S P(A_S). \] (5)
Let us now compute $P(A_S)$. We will see that this probability does not depend on the choice of $S$. Indeed, just as there are $\binom{n-i(k-1)}{i}$ possibilities for $S$, there are $\binom{n-i(k-1)}{i}$ possibilities for the entries in the positions belonging to $S$. Once those entries are known, the rest of the $q$-patterns starting in those entries are determined, and there are $(n-ik)!$ possibilities for the rest of the permutation. This shows that 

$$P(A_S) = \frac{\binom{n-i(k-1)}{i}(n-ik)!}{n!}$$

for all $S$. Therefore, (5) implies

$$P(n, i, q) \leq \binom{n-i(k-1)}{i}^2 \frac{1}{(n-ik)!} \frac{1}{n!}.$$ (6)

Comparing (6) and (4), we see that our lemma will be proved if we show that for $i > 1$, the inequality

$$\binom{n-i(k-1)}{i}^2 (n-ik)! \leq (n-i-k+2)!,$$

or, equivalently,

$$(n-i(k-1))_i \leq i!^2(n-k-i+2)(n-k-i+1) \cdots (n-i(k-1)+1)$$ (7)

holds. Where $(z)_j = z(z-1) \cdots (z-j+1)$. Note that the left-hand side has $i$ factors, while the right-hand side, not counting $i!^2$, has $(k-1)(i-1) > i$ factors, each of which are larger than the factors of the left-hand side. Therefore, (7) holds, and the Lemma is proved.

The proof of our main result is now immediate.

**Theorem 1** Let $q$ be any pattern of length $k$. Then

$$V_n(q) \leq V_n(\alpha).$$

**Proof:** Lemma 1 and formula (2) together imply that $P(n, 1, q) \geq P(n, 1, \alpha)$, which means that there are at least as many $n$-permutations that very tightly contain $q$ as $n$-permutations that very tightly contain $\alpha$.

### 2.4 A Result on Non-condensible Patterns

We have seen in Proposition 2 that if $q$ is non-condensible and $q_1 < q_k$, then any two copies of $q$ contained in a given permutation $p$ are disjoint. Therefore, the number of $n$-permutations that very tightly avoid $q$ can be computed by the Principle of Inclusion-Exclusion. Indeed, in this case, the following holds.
Proposition 3 Let $q$ be a non-condensible pattern. Then

$$V_n(q) = n! - \sum_{i=1}^{\lfloor n/k \rfloor} \binom{n-i(k-1)}{i}^2 (n-ik)!.$$ 

In particular, $V_n(q)$ does not depend on the choice of $q$.

Proof: In the proof of Lemma 1 more precisely, in our argument showing that (6) holds, we showed that there are \( \binom{n-i(k-1)}{i} \) ways to choose an $i$-element set of positions that can be the starting positions of $i$ disjoint very tight copies of $q$, and there are \( \binom{n-i(k-1)}{i} \) ways to choose the sets of entries forming these same copies. Once these choices are made, the rest of the permutation can be chosen in \( (n-ik)! \) ways. The statement now follows by the Principle of Inclusion-Exclusion. ♦

As we said, we will prove in Proposition 4 that almost all patterns are non-condensible. Note that nothing comparable is known for the other two notions of pattern avoidance. Numerical evidence suggests that similarly strong results will probably not hold if the traditional definition or the tight definition is used.

3 Further Directions

The novelty of this paper, beside the notion of very tight containment, was the application of expectations to compare the numbers of permutations avoiding two patterns. The computations themselves were elementary. This leads to the following question.

Question 1 Is it possible to apply our method to compare the numbers $T_n(q)$ and $T_n(\alpha)$, or the numbers $S_n(q)$ and $S_n(\alpha)$, for at least some patterns $q$?

As we used a very simple estimate in our proof of Lemma 1, there may be room for improvement at that point.

There are several natural questions that can be raised about the enumeration of permutations that very tightly avoid a pattern. Let us recall that we proved a formula for $V_n(q)$ for the overwhelming majority of patterns $q$, namely for non-condensible patterns. A formula for $V_n(\alpha)$, where $\alpha$ is the monotone pattern, can be found in [6] and [5]. This raises the following question.
Question 2 Are there other patterns $q$ for which $V_n(q)$ can be explicitly determined?

Let us call patterns $q$ and $q'$ very tightly equivalent if $V_n(q) = V_n(q')$ for all $n$. We have seen that almost all patterns of length $k$ are very tightly equivalent. This raises the following questions.

Question 3 How many equivalence classes are there for very tight patterns of length $k$?

Question 4 Can we say anything about the same topic for tight pattern containment, or traditional pattern containment? If equivalence is defined for them in an analogous way, how many equivalence classes will be formed, (for patterns of length $k$) and how large will the largest one be?

4 Appendix

In this section we prove the following simple fact.

Proposition 4 Let $h_n$ be the number of condensible permutations of length $n$. Then

$$\lim_{n \to \infty} \frac{h_n}{n!} = 0.$$ 

Proof: We prove that another class of permutations, one that contains all condensible permutations, is also very small. Let $a_n$ be the number of decomposable permutations, that is, permutations $p_1p_2\cdots p_n$ for which is there is an index $i$ so that $p_j < p_m$ if $j \leq i < m$. In other words, $p$ can be cut into two parts so that everything before the cut is less than everything after the cut. (Note that if a permutation is not decomposable, then it is called indecomposable, and Exercise 1.32 of [8] contains more information about these permutations.)

Counting according to the index $i$ of the above definition, we see that

$$\frac{a_n}{n!} \leq \sum_{i=1}^{n-1} \frac{i!(n-i)!}{n!}$$

$$\leq \sum_{i=1}^{n-1} \frac{n!}{n!}$$

$$\leq \frac{2}{n} + \frac{n-3}{n^2},$$

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where in the last step we used the well-known unimodal property of binomial coefficient, in particular the inequality that \( \binom{n}{i} \geq \binom{n}{2} \) if \( 2 \leq i \leq n - 2 \).

Therefore, \( a_n/n! \) converges to 0 as \( n \) goes to infinity. Clearly, all condensible patterns are decomposable since their first \( i \) entries are also their \( i \) smallest entries, so our claim follows. ◊

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