Polylogarithms and a Zeta Function for Finite Places of a Function Field

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Abstract

We introduce and study new versions of polylogarithms and a zeta function on a completion of $\mathbf{F}_q(x)$ at a finite place. The construction is based on the use of the Carlitz differential equations for $\mathbf{F}_q$-linear functions.

Key words: $\mathbf{F}_q$-linear function; polylogarithms; zeta function
1 INTRODUCTION

It was shown in [12, 13] that the basic notions and results of analytic theory of differential equations have their natural counterparts in the setting of the function field arithmetic.

Consider the field $\mathbb{F}_q(x)$ of rational functions with coefficients from the Galois field $\mathbb{F}_q$ of characteristic $\nu > 0$, $q = \nu^v$, $v \in \mathbb{Z}_+$. Let $\pi \in \mathbb{F}_q[x]$ be a monic irreducible polynomial, $\deg \pi = \delta$. The absolute value $|t|_\pi$, $t \in \mathbb{F}_q(x)$, is defined as follows. We write $t = \pi^n \alpha/\alpha'$ where $n \in \mathbb{Z}$, and $\pi$ does not divide $\alpha, \alpha'$. Then $|t|_\pi = |\pi|^n| \pi|_\pi = q^{-\delta}$. As usual, $|0|_\pi = 0$. Let $K_\pi$ be the completion of $\mathbb{F}_q(x)$ with respect to the metric determined by this absolute value. Then the cardinality of its residue field equals $q^\delta$, and a full system of representatives of the residue classes consists of all polynomials from $\mathbb{F}_q[x]$ of degrees $< \delta$ (see Sect. 3.1 in [22]). Denote by $\Omega_\pi$ the completion, with respect to the canonical extension of the absolute value, of an algebraic closure of $K_\pi$.

A function $f$ defined on a $\mathbb{F}_q$-subspace $K_\pi'$ of $K$, with values in $\Omega_\pi$, is called $\mathbb{F}_q$-linear if $f(t_1 + t_2) = f(t_1) + f(t_2)$ and $f(\alpha t) = \alpha f(t)$ for any $t, t_1, t_2 \in K_\pi'$, $\alpha \in \mathbb{F}_q$.

The simplest example is an $\mathbb{F}_q$-linear polynomial $f(t) = \sum a_k t^{q^k}$, $a_k \in \Omega_\pi$. The set of $\mathbb{F}_q$-linear polynomials (as well as some wider classes of $\mathbb{F}_q$-linear functions) forms a ring with the usual addition and the composition as the multiplication operation. The function $f(t) = t$ is the unit element in this ring.

In the theory of differential equations over $K_\pi$ developed in [12, 13] the unknown functions are $\mathbb{F}_q$-linear, and the role of a derivative is played by Carlitz’s operator

$$d = \sqrt[n]{\Delta}, \quad (\Delta u)(t) = u(xt) - xu(t) \quad (1)$$

(in [12, 13] the case $\pi(x) = x$ was considered, but many results carry over to the general case). The meaning of a polynomial (or holomorphic) coefficient in the function field case is not a multiplication by a coefficient, but the action of a polynomial (or a power series) in the operator $\tau$, $\tau u = u^\alpha$. Such equations are known for many special functions on $K_\pi$ (like analogs of the power, exponential, Bessel, and hypergeometric functions; see [3, 4, 7, 13, 17, 18]). It appears that the Carlitz differential equations can be used for defining new special functions with interesting properties.

In this paper we consider an analog of the function $-\log(1-t)$ defined via the equation

$$(1 - \tau)du(t) = t, \quad t \in K_\pi, \quad (2)$$

a counterpart of the classical equation $(1-t)u'(t) = 1$.

Starting from a solution $l_1(t)$ of (2) defined by a $\mathbb{F}_q$-linear power series convergent for $|t|_\pi < 1$ (that is for $|t|_\pi \leq q^{-\delta}$; we consider only $t \in K_\pi$, while the functions may take their values in $\Omega_\pi$), we define a sequence of “polylogarithms” $l_k(t)$, $\Delta l_k = l_{k-1}$, $k \geq 2$, and show that all these functions can be extended to continuous non-holomorphic solutions of the same equations on the “closed” unit disk $O_\pi = \{t \in K_\pi : |t|_\pi \leq 1\}$. Their values at $t = 1$ can be seen as “special values” of a kind of a zeta function.

Note that the existing definitions of the polylogarithms and zeta for function fields (see [6, 8, 2]) are based on the use of the “infinite” $x^{-1}$-completion of $\mathbb{F}_q(x)$, though a part of the results in [6, 8, 2] is extended to finite places. Our approach leads to an apparently different zeta, but also with some interesting properties. In particular, if $\pi = x$, and if we identify the
value of a polylogarithm $l_k(1)$ not with $\zeta(k)$ but with $\zeta(x^{-k})$, we obtain a function $\zeta$ defined on a subset of $K_x$. Then we show that $\zeta$ has a natural continuous $F_q$-linear extension onto the whole field $K_x$. Thus $\zeta$ is purely an object of the characteristic $\kappa$ arithmetic, in contrast to Goss’s zeta function which is interpolated from natural numbers onto $\mathbb{Z}_\kappa$.

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2 A LOGARITHM-LIKE FUNCTION

Let us look for a $F_q$-linear holomorphic solution

$$u(t) = \sum_{n=0}^{\infty} a_n t^{q^n}$$

of the equation (2). It follows from the definition (1) that

$$du(t) = \sum_{n=1}^{\infty} a_n^{1/q} [n]^{1/q} t^{q^n-1}$$

where $[n] = x^{q^n} - x$. Substituting into (2) we find that

$$\sum_{j=0}^{\infty} \left( a_{j+1}^{1/q} [j+1]^{1/q} - a_j [j] \right) t^{q^j} = t.$$ 

We see that the equation (2) is satisfied if and only if $a_0$ is arbitrary, $a_1 = [1]^{-1}$,

$$a_{j+1} = a_j^{q} [j]^{q} / [j+1], \quad j \geq 1,$$

and we find by induction that $a_j = [j]^{-1}$.

Let $l_1(t)$ be the solution (3) of the equation (2) with $a_0 = 0$. Then

$$l_1(t) = \sum_{n=1}^{\infty} \frac{t^{q^n}}{n}.$$ 

By Lemma 2.13 from [14]

$$||n||_\pi = \begin{cases} q^{-\delta}, & \text{if } \delta \text{ divides } n; \\ 1, & \text{otherwise}. \end{cases}$$

Hence the series (4) converges for $|t|_\pi \leq q^{-\delta}$.

Note that $l_1(t)$ is different from the well-known Carlitz logarithm $\log_C$ (see [3]), the inverse function to the Carlitz exponential $e_C$. Analogies motivating the introduction of special functions are not so unambiguous, and, for instance, from the composition ring viewpoint, $\log_C$ is an analog of $e^{-t}$, though in other respects it is a valuable analog of the logarithm. By the way, another possible analog of the logarithm is a continuous function $u(t)$, $|t|_\pi \leq 1$, satisfying the
equation $\Delta u(t) = t$ (an analog of $tu'(t) = 1$) and the condition $u(1) = 0$. In fact, $u = D_1$, the first hyperdifferential operator; see [10], especially the proof of Theorem 3.5 in [10].

Now we consider continuous non-holomorphic extensions of $l_1$. We will use the following simple lemma.

**Lemma 1.** Consider the equation

$$z^q - z = \xi, \quad \xi \in \Omega_\pi.$$  (6)

If $|\xi|_\pi = 1$, then all the solutions $z_1, \ldots, z_q$ of the equation (6) are such that $|z_j|_\pi = 1$, $j = 1, \ldots, q$. If $|\xi|_\pi < 1$, then there exists a unique solution $z_1$ of the equation (6) with $|z_1|_\pi = |\xi|_\pi$. This solution can be written as

$$z_1 = - \sum_{j=0}^{\infty} \xi^j.$$  (7)

For all other solutions we have $|z_j|_\pi = 1$, $j = 2, \ldots, q$.

**Proof.** Let $|\xi|_\pi = 1$. If some solution $z_j$ of the equation (6) is such that $|z_j|_\pi < 1$, the ultra-
metric inequality would imply $|\xi|_\pi < 1$. If $|z_j|_\pi > 1$, then $|z_j|^q_\pi > |z_j|_\pi$, so that $\xi = |z_j|^q_\pi > 1$, and we again come to a contradiction.

Now suppose that $|\xi|_\pi < 1$. Then the series in (7) converges and defines a solution of

(6), such that $|z_1|_\pi = |\xi|_\pi$. All other solutions are obtained by adding elements of $F_q$ to $z_1$. Therefore $|z_2|_\pi = \ldots = |z_q|_\pi = 1$. ■

Denote by $f_i(t), i = 0, 1, 2, \ldots$, the sequence of normalized Carlitz polynomials, that is $f_i(t) = D_i^{-1}e_i(t)$,

$$D_0 = 1, \quad D_i = [i][i-1]q \cdots [1]q^{i-1}, \quad e_0(t) = t,$$

$$e_i(t) = \prod_{\omega \in F_q[x], \deg \omega < i} (t - \omega), \quad i \geq 1.$$  

It is known [20, 5] that $\{f_i\}$ is an orthonormal basis of the space of continuous $F_q$-linear functions $O_\pi \to \Omega_\pi$ (in [20, 5] the functions $O_\pi \to K_\pi$ are considered; the general case follows from Proposition 6 in [1]).

**Theorem 1.** The equation (2) has exactly $q^\delta$ continuous solutions on $O_\pi$ coinciding with (4) as $|t|_\pi \leq q^{-\delta}$. These solutions have the expansions in the Carlitz polynomials

$$u = \sum_{i=0}^{\infty} c_if_i$$  (8)

where the coefficients $c_1, \ldots, c_\delta$ are arbitrary solutions of the equations

$$c_i^q - c_i + 1 = 0,$$

$$c_i^q - c_i + [i]q c_i^q = 0, \quad 1 \leq i \leq \delta - 1,$$  (9)  (10)
higher coefficients are found from the relations
\[ c_n = \sum_{j=0}^{\infty} (c_{n-1}[n-1])^{q+1}, \quad n \geq \delta + 1, \quad (11) \]

and the coefficient \( c_0 \) is determined by the relation
\[ c_0 = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{c_i}{L_i}, \quad (12) \]

\( L_i = [i][i-1] \ldots [1] \).

**Proof.** Looking for a solution of (2) of the form (8), writing the equation (2) as \( du(t) - \Delta u(t) = t \), and using the relations
\[ df_i = f_{i-1}, \quad \Delta f_i = [i]f_i + f_{i-1}(i \geq 1), \quad df_0 = \Delta f_0 = 0 \]
(see [7][11][13]), we find that
\[ \sum_{i=0}^{\infty} \left( c_{i+1}^{1/q} - c_{i+1} - c_i[i] \right) f_i(t) = f_0(t), \quad t \in O_\pi. \]
This is equivalent to the equation (9) for \( c_1 \) and the sequence of relations (10) for \( c_i \), \( 2 \leq i < \infty \). The coefficient \( c_0 \) remains arbitrary so far.

By Lemma 1 there are \( q \) solutions of (9) and \( q \) solutions of each equation (10) with \( 1 \leq i \leq \delta - 1 \). For all of them \( |c_j|_\pi = 1, 1 \leq j \leq \delta \). Consider the equations
\[ c_n^q - c_n + [n-1]c_n^q = 0, \quad n \geq \delta + 1. \quad (13) \]
If \( n = \delta + 1 \), we use (5) and Lemma 1 to show that the corresponding equation (13) has the solution (11) with \( |c_{\delta+1}|_\pi = q^{-\delta q} \) and \( q - 1 \) other solutions with the absolute value 1. Choosing at each subsequent step the solution (11) we obtain the sequence \( c_n \), such that
\[ |c_n|_\pi \leq q^{-\delta q^n-\delta}, \quad n \geq \delta + 1, \quad (14) \]
so that \( |c_n|_\pi \to 0 \), and the series (8) indeed determines a continuous \( \mathbb{F}_q \)-linear function on \( O_\pi \).

Since
\[ f_i(t) = \sum_{j=0}^{i} (-1)^{i-j} \frac{1}{D_j L_{i-j}^q} t^{q^j} \]
(see [7]), we see that
\[ \lim_{t \to 0} \frac{f_i(t)}{t} = (-1)^i \frac{1}{L_i}. \]
Therefore, if we choose \( c_0 \) according to (12), then our solution \( u \) is such that
\[ \lim_{t \to 0} t^{-1} u(t) = 0. \quad (15) \]
Note that $|L_n|_\pi = q^{-\delta[n]}$ (where $[\cdot]$ denotes the integral part of a real number), so that the series in (12) is convergent.

By a result of Yang [23], it follows from (14) that $u$ is locally analytic; specifically, it is analytic on any ball of the radius $q^{-\delta}$. Thus it can be represented for $|t|_\pi \leq q^{-\delta}$ by the convergent power series (3), in which $a_0 = 0$ by (15). Therefore $u(t) = l_1(t)$ for $|t|_\pi \leq q^{-\delta}$, as desired.

Any other continuous solution of the equation (2) on $O_\pi$ is obtained inevitably by the same procedure, but with $|c_1|_\pi = \ldots = |c_N|_\pi = 1$, $|c_n|_\pi < 1$, if $n \geq N\delta + 1$, for some $N > 1$, and with some $c_0 \in \Omega_\pi$. In this case by Lemma 1

$$|c_{N+1}|_\pi = q^{-\delta q}, \quad |c_{N+1}|_\pi = q^{-\delta q}, \quad |c_{N+1}|_\pi = q^{-\delta(q^{l+1}+q)}$$

(here we have to proceed more accurately than in (14), in order to obtain a precise estimate).

More generally, we have

$$|c_{(N+l)}|_\pi = q^{-\delta(q^{l+1}+q)}.$$  \hspace{1cm} (16)

Indeed, this was shown above for $l = 1$. If (16) is true for some $l$, then

$$|c_{(N+l)}|_\pi = q^{-\delta\{(q^l+q^l)q+q\}},$$

and so on, so that

$$|c_{(N+l)}|_\pi = q^{-\delta\{(q^l+q^l)q+q\}q^l} = q^{-\delta(q^l+q^l+q^{2l}+q^l)},$$

and (16) is proved.

Let us consider the valuation $v_\pi(t)$, $t \in K_\pi$, connected with the absolute value by the relation $|t|_\pi = q^{-\delta v_\pi(t)}$. The equality (16) means that

$$v_\pi(c_{(N+l)}_\pi) = q^l + q^{l-1} + \ldots + q^l, \quad l = 1, 2, \ldots.$$  \hspace{1cm} (17)

Suppose that our solution coincides with the series (4) for $|t|_\pi \leq q^{-\delta}$. By $F_q$-linearity this means the analyticity of the solution on any ball of the radius $q^{-\delta}$. Then [23]

$$v_\pi(c_n) - \sum_{i=2}^{\infty} q^n - q^i \to \infty \quad \text{as} \quad n \to \infty$$

(we use the specialization of the result from [23] for the case of $F_q$-linear functions), that is

$$v_\pi(c_n) - \frac{q^n - q^{n-\delta}}{q^\delta - 1} \to \infty \quad \text{as} \quad n \to \infty.$$ 

In particular,

$$v_\pi(c_{(N+l)}_\pi) - \frac{q^{l(N+l-1)} - q^\delta}{q^\delta - 1} \to \infty \quad \text{as} \quad l \to \infty.$$  \hspace{1cm} (18)

However by (17)

$$v_\pi(c_{(N+l)}_\pi) = q^{l(N+l-1)} \frac{q^{l+1} - q^\delta}{q^\delta - 1}.$$
which contradicts (18), since \( N \geq 2 \). \( \blacksquare \)

In fact continuous solutions which satisfy (12) and have the coefficients \( c_n \) of the form (11), but starting from some larger value of \( n \), are also extensions of the functions (4), but from smaller balls.

Below we denote by \( l_1(t) \) a fixed solution of the equation (2) on \( O_\pi \) coinciding with (4) for \( |t|_\pi \leq q^{-\delta} \), as described in Theorem 1. Of course, \( l_1 \) depends on \( \pi \), but we will not indicate this dependence explicitly for the sake of brevity.

### 3 POLYLOGARITHMS

The polylogarithms \( l_n(t) \) are defined recursively by the equations

\[
\Delta l_n = l_{n-1}, \quad n \geq 2,
\]

which agree with the classical ones \( tl'_n(t) = l_{n-1}(t) \). If we look for analytic \( F_q \)-linear solutions of (19), such that \( t^{-1}l_n(t) \to 0 \) as \( t \to 0 \), we obtain easily by induction that

\[
l_n(t) = \sum_{j=1}^{\infty} \frac{t^{q^j}}{|j|!}, \quad |t|_\pi \leq q^{-\delta}.
\]

In order to find continuous extensions of \( l_n \) onto \( O_\pi \), we consider the Carlitz expansions

\[
l_n = \sum_{i=0}^{\infty} c_i^{(n)} f_i, \quad n = 2, 3, \ldots.
\]

Consider first the dilogarithm \( l_2 \). We have

\[
\Delta l_2 = \sum_{i=0}^{\infty} \left( c_{i+1}^{(2)} + [i]c_i^{(2)} \right) f_i,
\]

so that

\[
c_{i+1}^{(2)} + [i]c_i^{(2)} = c_i, \quad i = 0, 1, 2, \ldots,
\]

where \( c_i \) are the coefficients described in Theorem 1. The recursion (22) leaves \( c_0^{(2)} \) arbitrary and determines all other coefficients in a unique way:

\[
c_n^{(2)} = (-1)^n L_{n-1} \sum_{j=n}^{\infty} (-1)^j \frac{c_j}{L_j}, \quad n \geq 1,
\]

where we set \( L_0 = 1 \).

Indeed, the series in (23) is convergent, since \( c_n \) satisfies the estimate (14), while \( |L_n|_\pi = q^{-\delta[\frac{n}{\pi}]_{\text{int}}} \). For \( n = 1 \) the equality (23) means, due to (12), that \( c_1^{(2)} = c_0 \), which coincides with (22) for \( i = 0 \). If (23) is proved for some \( n \), then

\[
c_{n+1}^{(2)} = c_n - [n]c_n^{(2)} = c_n + (-1)^{n+1} L_n \sum_{j=n}^{\infty} (-1)^j \frac{c_j}{L_j} = (-1)^{n+1} L_n \sum_{j=n+1}^{\infty} (-1)^j \frac{c_j}{L_j},
\]
as desired.
We have
\[
\left| \frac{c_j}{L_j} \right| = q^\delta \left( \left\lfloor \frac{j}{\pi} \right\rfloor - q^{j-\delta} \right).
\]
Thus for \( n > \delta \)
\[
\left| \sum_{j=n}^{\infty} (-1)^j \frac{c_j}{L_j} \right| \leq \sup_{j \geq n} \left| \frac{c_j}{L_j} \right| \leq \sup_{j \geq n} q^{j-\delta q^{j-\delta}} = \delta^{-1} q^\delta \sup_{j \geq n} (\delta q^{j-\delta}) q^{-\delta q^{j-\delta}}.
\]
The function \( z \mapsto z^{-\delta} \) is monotone decreasing for \( z \geq 1 \). Therefore
\[
\left| \sum_{j=n}^{\infty} (-1)^j \frac{c_j}{L_j} \right| \leq q^n \cdot q^{-\delta q^{n-\delta}}, \quad n > \delta,
\]
so that by (23)
\[
\left| c_n^{(2)} \right| \pi \leq q^{\delta+1} \cdot q^{-\delta q^{n-\delta}}, \quad n > \delta.
\]
Using Yang’s theorem again we find that \( l_2 \) is analytic on all balls of the radius \( q^{-\delta} \). If we choose \( c_0^{(2)} \) in such a way that
\[
c_0^{(2)} = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{c_i^{(2)}}{L_i},
\]
the solution (21) of the equation (19) with \( n = 2 \) is a continuous extension of the dilogarithm \( l_2 \) given by the series (20) with \( n = 2 \).

Repeating the above reasoning for each \( n \), we come to the following result.

**Theorem 2.** For each \( n \geq 2 \), there exists a unique continuous \( \mathbf{F}_q \)-linear solution of the equation (19) coinciding for \( |t|_\pi \leq q^{-\delta} \) with the polylogarithm (20). The solution is given by the Carlitz expansion (21) with
\[
\left| c_i^{(n)} \right| \leq C_n q^{-\delta q^{i-\delta}}, \quad i > \delta, \quad C_n > 0,
\]
\[
c_0^{(n)} = \sum_{i=1}^{\infty} (-1)^{i+1} \frac{c_i^{(n)}}{L_i},
\]

### 4 Fractional Derivatives

Starting from this section and to the end of the paper we assume that \( \pi = x \).

In this section we introduce the operator \( \Delta^{(\alpha)}_\pi \), \( \alpha \in O_x \), a function field analog of the Hadamard fractional derivative \( (t^a \frac{d}{dt})^a \) from real analysis (see [12]).

Denote by \( \mathcal{D}_k(t) \), \( k \geq 0 \), \( t \in O_x \), the sequence of hyperdifferentiations defined initially on monomials by the relations \( \mathcal{D}_0(x^n) = x^n \), \( \mathcal{D}_k(1) = 0 \), \( k \geq 1 \),
\[
\mathcal{D}_k(x^n) = \binom{n}{k} x^{n-k},
\]
where it is assumed that \(^{(n)} = 0\) for \(k > n\). \(D_k\) is extended onto \(F_q[x]\) by \(F_q\)-linearity, and then onto \(O_x\) by continuity. The sequence \(\{D_k\}\) is an orthonormal basis of the space of continuous \(F_q\)-linear functions on \(O_x\) \([10]\) \([5]\).

Let \(\alpha \in O_x\), \(\alpha = \sum_{n=0}^{\infty} \alpha_n x^n\), \(\alpha_n \in F_q\). Denote \(\hat{\alpha} = \sum_{n=0}^{\infty} (-1)^n \alpha_n x^n\). The transformation \(\alpha \mapsto \hat{\alpha}\) is a \(F_q\)-linear isometry. For an arbitrary continuous \(F_q\)-linear function \(u\) on \(O_x\) we define its “fractional derivative” \(\Delta^{(\alpha)} u\) at a point \(t \in O_x\) by the formula

\[
(\Delta^{(\alpha)} u)(t) = \sum_{k=0}^{\infty} (-1)^k D_k(\hat{\alpha}) u(x^k t).
\]

The series converges for each \(t\), uniformly with respect to \(\alpha\), since \(|D_k(\hat{\alpha})|_x \leq 1\) and \(u(x^k t) \to 0\). Thus \(\Delta^{(\alpha)} u\) is, for each \(t\), a continuous \(F_q\)-linear function in \(\alpha\).

Our understanding of \(\Delta^{(\alpha)}\) as a kind of a fractional derivative is justified by the following lemma contained in \([10]\) (Corollary 3.10). We give a simple independent proof.

**Lemma 2.** \(\Delta^{(x^n)} = \Delta^n\), \(n = 1, 2, \ldots\)

**Proof.** By the definition of \(D_k\), it follows from (24) that

\[
(\Delta^{(x^n)} u)(t) = \sum_{k=0}^{n} \binom{n}{k} (-x)^{n-k} u(x^k t).
\]

If \(n = 1\), then \((\Delta^{(x)} u)(t) = u(x t) - x u(t) = (\Delta u)(t)\). Suppose we have proved that \(\Delta^{(x^{n-1})} = \Delta^{n-1}\). Then

\[
(\Delta^n u)(t) = \Delta \left( \Delta^{(x^{n-1})} u \right)(t)
= \sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^{n-1-k} u(x^{k+1} t) - x \sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^{n-1-k} u(x^k t)
= \sum_{k=1}^{n} \binom{n-1}{k-1} (-x)^{n-k} u(x^k t) + \sum_{k=0}^{n-1} \binom{n-1}{k} (-x)^{n-1-k} u(x^k t)
= u(x^n t) + \sum_{k=1}^{n-1} \left\{ \binom{n-1}{k-1} + \binom{n-1}{k} \right\} (-x)^{n-k} u(x^k t) + (-x)^n u(t) = (\Delta^{(x^n)} u)(t),
\]
as desired. \(\blacksquare\)

It follows from Lemma 2 that \(\Delta^{(x^n)} \circ \Delta^{(x^m)} = \Delta^{(x^{n+m})} = \Delta^{(x^n \cdot x^m)}\), which prompts the following composition property.

**Lemma 3.** For any \(\alpha, \beta \in O_x\)

\[
\Delta^{(\alpha)} \left( \Delta^{(\beta)} u \right)(t) = \left( \Delta^{(\alpha \beta)} u \right)(t).
\]
Proof. Using the Leibnitz rule for hyperderivatives (see [4]) we have

\[
(\Delta^{(\alpha)} \circ \Delta^{(\beta)} u)(t) = \sum_{k=0}^{\infty} (-1)^k \mathcal{D} \tilde{\beta} \sum_{l=0}^{\infty} (-1)^l \mathcal{D} \tilde{\alpha} u(x^{k+l} t)
\]

\[
= \sum_{n=0}^{\infty} (-1)^n u(x^n t) \sum_{k+l=n} \mathcal{D} \tilde{\beta} \mathcal{D} \tilde{\alpha} = \sum_{n=0}^{\infty} (-1)^n \mathcal{D} \eta (\tilde{\alpha} \tilde{\beta}) u(x^n t) = (\Delta^{(\alpha \beta)} u)(t). \quad \blacksquare
\]

5 ZETA FUNCTION

We define \( \zeta(t), t \in K_x \), setting \( \zeta(0) = 0, \zeta(x^{-n}) = \ln(1), \quad n = 1, 2, \ldots, \)

and

\( \zeta(t) = (\Delta^{(\theta_0 + \theta_1 x + \cdots)} l_n)(1), \quad n = 1, 2, \ldots, \)

if \( t = x^{-n}(\theta_0 + \theta_1 x + \cdots), \theta_j \in \mathbb{F}_q \). The correctness of this definition follows from Lemma 3. It is clear that \( \zeta \) is a continuous \( \mathbb{F}_q \)-linear function on \( K_x \) with values in \( \Omega_x \).

In particular, we have

\( \zeta(x^m) = (\Delta^{m+1} l_1)(1), \quad m = 0, 1, 2, \ldots. \)

The above definition is of course inspired by the classical polylogarithm relation

\[
\left( z \frac{d}{dz} \right) \sum_{n=1}^{\infty} \frac{z^n}{n^s} = \sum_{n=1}^{\infty} \frac{z^n}{n^{s-1}}.
\]

Let us write down some relations for “special values” \( \zeta(x^n), n \in \mathbb{N} \). Let us consider the expansion of \( l_n(t) \) in the sequence of hyperdifferentiations. We have

\( l_n(t) = \sum_{i=0}^{\infty} (\Delta^i l_n)(1) \mathcal{D}_i(t) \)

(see [10]). Therefore

\( l_n(t) = \sum_{i=0}^{\infty} \zeta(x^{-n+i}) \mathcal{D}_i(t), \quad n \in \mathbb{N}, \ t \in O_x. \quad (25)\)

In particular, combining (25) and (20) we get

\[
\sum_{j=1}^{\infty} \frac{t^j}{[j]^n} = \sum_{i=0}^{\infty} \zeta(x^{-n+i}) \mathcal{D}_i(t), \quad |t|_x \leq q^{-1}.
\]

Let us consider the double sequence \( A_{n,r} \in K_x, A_{n,1} = (-1)^{n-1} L_{n-1}, \)

\( A_{n,r} = (-1)^{n+r} L_{n-1} \sum_{0 < i_1 < \ldots < i_{r-1} < n} \frac{1}{[i_1][i_2] \ldots [i_{r-1}]}, \quad r \geq 2. \)
This sequence appears in the expansion \([19]\) of a hyperdifferentiation \(\mathcal{D}_r\) in the normalized Carlitz polynomials
\[
\mathcal{D}_r(t) = \sum_{n=0}^{\infty} A_{n,r} f_n(t), \quad t \in O_x.
\]
Its another application \([9]\) is the expression of the Carlitz difference operators \(\Delta_n, \Delta_1 = \Delta\),
\[
(\Delta_n u)(t) = (\Delta_{n-1} u)(xt) - x^{q^{n-1}} (\Delta_{n-1} u)(t),
\]
via the iterations \(\Delta^r\):
\[
\Delta_n = \sum_{r=1}^{n} A_{n,r} \Delta^r, \quad n \geq 1.
\]
For coefficients of the expansion (21) we have \(c_i^{(n)} = (\Delta_i l_n)(1), i \geq 1 \) (see \([7]\)), and by (27)
\[
c_i^{(n)} = \sum_{r=1}^{i} A_{i,r} (\Delta^r l_n)(1) = \sum_{r=1}^{i} A_{i,r} \zeta(x^{r-n}).
\]
Since \(c_0^{(n)} = \zeta(1-x^n)\), we have (see Theorems 1,2)
\[
\zeta(1-x^n) = \sum_{i=1}^{\infty} (-1)^{i+1} L_{i-1}^{-1} \sum_{r=1}^{i} A_{i,r} \zeta(x^{r-n}).
\]
The identity (29) may be seen as a distant relative of Riemann’s functional equation for the classical zeta.

Since \(\mathcal{D}_r(t)\) is not differentiable \([19]\), the interpretation of the sequence \(\{A_{i,r}\}\) given in (26) shows, by a result of Wagner \([21]\), that \(L_{i-1}^{-1} A_{i,r} \to 0\) as \(i \to \infty\). Thus it is impossible to change the order of summation in (29).

Finally, consider the coefficients of the expansion (8) for \(l_1\). As in (28), we have an expression
\[
c_i = \sum_{r=1}^{i} A_{i,r} \zeta(x^{r-1}).
\]
By Theorem 1, for \(i \geq 2\) we have
\[
c_i = \sum_{j=0}^{\infty} (z_i)^q^j, \quad z_i = c_{i-1}^q [i-1]^q \in \Omega_x.
\]
The series in (30) may be seen as an analog of \(\sum j^{-z}\). This analogy becomes clearer if, for a fixed \(z \in \Omega_x, |z|_x < 1\), we consider the set \(S\) of all convergent power series \(\sum_{n=1}^{\infty} z^n\) corresponding to sequences \(\{j_n\}\) of natural numbers. Let us introduce the multiplication \(\otimes\) in \(S\) setting \(z^n \otimes z^l = z^{n+l}\) and extending the operation distributively (for a similar construction
in the framework of $q$-analysis in characteristic 0 see \cite{15}). Denoting by $\prod_p^\otimes$ the product in $S$ of elements indexed by prime numbers we obtain in a standard way the identity

$$c_i = \prod_p^\otimes \sum_{n=0}^\infty (z_i)^{q^n}$$

(the infinite product is understood as a limit of the partial products in the topology of $\Omega_x$), an analog of the Euler product formula.
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