Abstract—This paper proposes a novel approach to resilient distributed optimization with quadratic costs in a networked control system (e.g., wireless sensor network, power grid, robotic team) prone to external attacks (e.g., hacking, power outage) that cause agents to misbehave. Departing from classical filtering strategies proposed in literature, we draw inspiration from a game-theoretic formulation of the consensus problem and argue that adding competition to the mix can enhance resilience in the presence of malicious agents. Our intuition is corroborated by analytical and numerical results showing that i) our strategy highlights the presence of a nontrivial trade-off between blind collaboration and full competition, and ii) such competition-based approach can outperform state-of-the-art algorithms based on Mean Subsequence Reduced.

I. INTRODUCTION

“With great power comes great responsibility”, and Networked Control Systems have great power indeed. From power grids regulating energy consumption, to large-scale sensor networks able to monitor vast environments, to fleets of autonomous vehicles for intelligent transportation, everyday life depends more and more on control of interacting devices.

While this brings numerous benefits, a major drawback is that malicious agent can locally intrude from any point in the system, and cause major damage at a global scale. Recently, Department of Energy secretary stated that enemies of the United States can shut down the U.S. power grid, and it is known that hacking groups around the world have high technological sophistication [1]. Cyberattacks hit Italian health care IT infrastructures during the COVID pandemic, disrupting services for weeks [2]. Another concern is represented by failure cascades generating from single nodes. Damages from cascading failures have notable examples across many domains, from electric blackouts over large areas, to denial of service or malfunctioning of web applications.

Such problems have been tackled in literature for several years, with focus on, e.g., power outage [3], cascading failure [4], [5], or denial of service [6]. Related literature in control theory has mostly focused on robustness of distributed optimization algorithms, with particular attention to so-called resilient consensus. For example, average consensus represents a fundamental tool in many applications, ranging from robot coordination to management of power grids. However, the classical consensus algorithm is fragile in the presence of misbehaving nodes, which may easily deceive the rest of the network. To contrast such a problem, the most popular approach is based on the filtering technique named “Mean Subsequence Reduced” (MSR), where each node filters out the largest and smallest incoming values. In particular, seminal work [7] improved classical MSR introducing a weighted version (W-MSR) and the notion of $r$-robustness of a graph, that was demonstrated to be a suitable measure enabling tight conditions for MSR-based resilient consensus. Among the variants of W-MSR, [8] studies resilient control for double integrators, [9] tackles mobile malicious agents, and [10] targets nonlinear systems with state constraints.

Despite the success MSR-based approaches have known in literature, and also the effectiveness showed in many applications domains, a minimal level of network robustness is required for theoretical guarantees of reaching resilient consensus. In fact, little can be said about the system behavior when robustness conditions do not hold. While sometimes algorithms still work in practice, the lack of theoretical guarantees may be undesired in certain applications, where a more conservative but safer approach may be preferred instead. Moreover, while in certain applications the agents may agree upon any common value, in other cases average consensus plays a crucial role. Thus, we depart from the classical MSR-based strategies with the aim of designing distributed algorithms that can ensure certain levels of resilience, and whose performance can be characterized, beyond the hard constraint of achieving consensus by all means, addressing the distance from the optimal solution as performance metric in a distributed optimization task.

To this aim, we draw inspiration by game theory. Distributed cooperative control and game theoretic approaches, despite their apparent contrast, are bounded from several perspectives which have been largely explored in literature [11], [12]. Our starting point is [13], where the authors discuss the connection between consensus and potential games. Stepping forward, we propose a novel approach to resilient consensus based on the celebrated Friedkin-Johnsen (FJ) dynamics [14], where an agent in the network can trade collaboration with neighbors for selfishness, which we interpret as competition against other agents. Such a mixed approach allows us to explore the performance trade-off that arises from different levels of collaboration in the network, which turns out to be crucial in the presence of attacks. In fact, we observe that the global network cost is minimized by a hybrid strategy whereby agents trust each other only partially.
Towards this goal, we first introduce and motivate average consensus for distributed quadratic optimization tasks in Section II. Then, we address the presence of outliers and misbehaving agents in the network, and propose a competition-based strategy to enhance resilience to such adversaries in Section III. In particular, we characterize the performance of our approach analytically (Section III-B) and substantiate theoretical findings with numerical tests on the cost function (Section IV). To further reinforce the validity of our approach on actual problem instances, we perform numerical simulations on large-scale networked systems and compare to W-MSR, showing that competition-based strategies can provide superior performance if robustness requirements of MSR-based algorithms are not satisfied (Section V). We conclude this paper by addressing some open questions and avenues for future research in Section VI.

II. Setup

We consider a networked control system composed of $N$ agents, collected in the set $\mathcal{V} = \{1, \ldots, N\}$. The state of agent $i \in \mathcal{V}$ is denoted as $x_i \in \mathbb{R}$, and all states are stacked in the column vector $x \in \mathbb{R}^N$. Each agent $i$ has a prior $\theta_i$, and needs to minimize the mismatches with all priors,

$$f_i(x_i) = \sum_{j \in \mathcal{V}} (x_i - \theta_j)^2.$$  

Assumption 1 (Prior distribution). The priors are distributed as i.i.d. random variables with zero mean and unit variance.

Straightforward algebraic manipulations show that the optimal values of $x_i$, $i \in \mathcal{V}$, minimizing (1) correspond to the average of the priors of all agents. Indeed,

$$f(x) = \frac{1}{N} \sum_{i \in \mathcal{V}} f_i(x_i) = \frac{1}{N} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} (x_i - \theta_j)^2$$

$$= \frac{1}{N} \sum_{i \in \mathcal{V}} \sum_{j \in \mathcal{V}} \left( x_i^2 + \sum_{j \in \mathcal{V}} \theta_j^2 - 2x_i \theta_j \right)$$

$$= \frac{1}{N} \sum_{i \in \mathcal{V}} \left( N x_i^2 + \sum_{j \in \mathcal{V}} \theta_j^2 - 2x_i \sum_{j \in \mathcal{V}} \theta_j \right)$$

$$= \sum_{i \in \mathcal{V}} \left( x_i^2 - 2x_i \bar{\theta} + \frac{1}{N} \sum_{j \in \mathcal{V}} \theta_j^2 \right)$$

$$= \sum_{i \in \mathcal{V}} \left( x_i^2 - 2x_i \bar{\theta} + \theta^2 \right) - N \bar{\theta}^2 + \sum_{j \in \mathcal{V}} \theta_j^2$$

$$= \sum_{i \in \mathcal{V}} \left( x_i - \bar{\theta} \right)^2 + \sum_{i \in \mathcal{V}} \left( \theta_i^2 - \bar{\theta}^2 \right)$$

$$= \sum_{i \in \mathcal{V}} \left( x_i - \bar{\theta} \right)^2 + \text{const},$$

where $\bar{\theta}$ is the average of all priors. In light of (2), all agents need to reach average consensus to solve the optimization task. Hence, we are interested in minimizing the expected average consensus error (or simply consensus error), defined as

$$e = \mathbb{E} \left[ \sum_{i \in \mathcal{V}} (x_i - \bar{\theta})^2 \right] = \mathbb{E} \left[ \| x - C \theta \|^2 \right],$$

where the expectation is taken with respect to the prior distribution, $\theta \in \mathbb{R}^N$ stacks all priors, $C = \frac{1}{N} \mathbb{1}_N \mathbb{1}_N^\top$, and $\mathbb{1}_N \in \mathbb{R}^N$ is the vector of all ones. According to the networked structure, agents can communicate with neighbors to decrease local costs and reach consensus dynamically.

Assumption 2 (Communication network and weights). The network topology is defined by an irreducible symmetric doubly-stochastic matrix $W$.

Assumption 3 (Local optimization). In view of the optimization tasks, we require no self-loops, i.e., $W_{ii} = 0 \forall i \in \mathcal{V}$.

It is well known that, to reach average consensus under nominal conditions and Assumption, the agents can update their states according to the so-called consensus dynamics,

$$x_i(k + 1) = \sum_{j \in \mathcal{N}_i} W_{ij} x_j(k), \quad x_i(0) = \theta_i,$$

where $x_i(k)$ is the state of agent $i$ at time $k \geq 0$ and $\mathcal{N}_i \subseteq \mathcal{V}$ is its neighborhood in the topology $W$.

III. Strategies for Resilient Consensus

A. Average Consensus in the Presence of Outliers

We now assume that some agents are outliers, i.e., their priors are corrupted by additive noise. The agents not affected by noise are called regular. We denote the subsets of outliers and regular agents as $\mathcal{M}$ and $\mathcal{R}$, respectively, $\mathcal{M} = |\mathcal{M}|$ and $\mathcal{R} = |\mathcal{R}|$. Formally speaking, the actual prior of corrupted agent $m \in \mathcal{M}$ is $\bar{\theta}_m = \theta_m + v_m$, where $v_m$ is a zero-mean random variable with variance $\sigma^2$. We assume that noises are uncorrelated with each other and with priors of all agents.

The average of priors in the presence of outliers differs from its nominal value $\bar{\theta}$, which prevents reaching average consensus via protocol (4). Indeed, the new average is

$$\bar{\theta} = \frac{1}{N} \left( \sum_{i \in \mathcal{R}} \theta_i + \sum_{m \in \mathcal{M}} \bar{\theta}_m \right) = \bar{\theta} + \frac{1}{N} \sum_{m \in \mathcal{M}} v_m,$$

and protocol (4) drives agents to the consensus error

$$e^{C} = \mathbb{E} \left[ \sum_{i \in \mathcal{V}} (\bar{\theta} - \bar{\theta})^2 \right] = \frac{dM}{N^2}.$$  

Aiming to improve the standard consensus dynamics, we draw inspiration from [13], where the authors show that (4) can be interpreted, in a game theoretic framework, as the optimal action in a quadratic game where agent $i$ repeatedly seeks to maximize the following utility function,

$$u_i(x_i) = - \sum_{j \in \mathcal{N}_i} W_{ij} (x_i - x_j)^2.$$

Note that (7) forces each agent to get as close as possible to
Assumption 4 (Malicious agent dynamics). The state of malicious agents is constant, i.e., \( x_m(k) = \hat{\theta}_m \) \( \forall m \in \mathcal{M} \). Accordingly, the rows in the matrix \( W \) corresponding to malicious agents have all off-diagonal elements equal to 0, while the elements on the main diagonal are set to 1.

Remark 2 (Malicious agents disrupt optimization). Assumption 4 is consistent with the resilient consensus literature, where often algorithms are tested against constant or drifting malicious agents that keep pulling their neighbors far off nominal average consensus [9], [16]. On the other hand, attackers may behave cleverly to not be detected, which needs to be tamed by suitable resilient strategies. However, this scenario goes beyond the scope of this paper.

In this scenario, malicious agents in the set \( \mathcal{M} \) do not collaborate to minimize cost (2). Hence, we restrict the network optimization problem to the set of regular agents, redefining their local costs as (cf. (1))

\[
  f_i(x_i) = \frac{1}{R} \sum_{j \in \mathcal{R}} (x_i - \theta_j)^2, \quad i \in \mathcal{R},
\]

and address the consensus error across regular agents (cf. (3)),

\[
  e_R = \mathbb{E} \left[ \sum_{i \in \mathcal{R}} (x_i - \hat{\theta})^2 \right] = \mathbb{E} \left[ \| x_R - \mathbb{1}_R \hat{\theta}_R \|^2 \right],
\]

where \( x_R \in \mathbb{R}^R \) and \( \theta_R \in \mathbb{R}^R \) stack respectively states and priors of regular agents.

1) Trivial FJ Dynamics: A first remarkable result is that simply letting regular agents not update their states may be sufficient to outperform the standard consensus dynamics in the presence of malicious agents.

Proposition 2 (Consensus protocol vs. FJ dynamics with malicious agents). In the presence of malicious agents, the FJ dynamics \( \{ \} \) with \( \lambda = 1 \) yields smaller error than consensus protocol \( \{ \} \) if the noise intensity \( d \) is large enough.

Proof. We first compute the error induced by standard consensus \( \{ \} \). In virtue of Assumption 4 the steady-state consensus value is the average of malicious agent’ priors,\( \hat{\theta}_M = \frac{1}{M} \sum_{m \in \mathcal{M}} \hat{\theta}_m \). The consensus error becomes

\[
  e_C^C = \mathbb{E} \left[ \sum_{i \in \mathcal{R}} \left( \frac{1}{M} \sum_{m \in \mathcal{M}} \hat{\theta}_m - \frac{1}{R} \sum_{i \in \mathcal{R}} \theta_i \right)^2 \right]
  \leq \frac{R}{M^2} \mathbb{E} \left[ \left( \sum_{m \in \mathcal{M}} \theta_m \right)^2 \right] + \frac{R}{M^2} \mathbb{E} \left[ \left( \sum_{m \in \mathcal{M}} v_m \right)^2 \right] + 1
  = \frac{R}{M} + \frac{Rd}{M} + 1.
\]

On the other hand, the FJ dynamics with \( \lambda = 1 \) simply freezes all regular agents’ priors, yielding consensus error

\[
  e^{FJ}_i = \mathbb{E} \left[ \| \theta_R - C_R \theta_R \|^2 \right] = \mathbb{E} \left[ \| (I_R - C_R) \theta_R \|^2 \right]
  = \text{tr} \left( \mathbb{E} \left[ \theta_R \theta_R^\top \right] (I - C_R) \right) = R - 1,
\]
It follows that, if
\[ d > M \left( 1 - \frac{2}{R} \right) - 1, \]  
then \( e^C > e_1^{FJ} \).

Proposition 2 states that, if outlier noises \( v_m \) are sufficiently intense, full competition where agents anchor to their priors by setting \( \lambda = 1 \) in (9) provides better performance than full collaboration with \( \lambda = 0 \), i.e., standard consensus (4).

2) Optimizing FJ Dynamics: We are now interested in choosing \( \lambda \) to reduce the consensus error. In particular, we aim to characterize the optimal parameter, which we denote as \( \lambda^* \approx \arg \min_{\lambda} e_\lambda \). Note that continuity of \( e_\lambda \) (with its continuous extension at \( \lambda = 0 \), see Remark 1 and Weierstrass theorem ensure that such \( \lambda^* \) always exists. In the following results, we use the covariance matrix of corrupted priors,
\[ \Sigma = \mathbb{E} \left[ \tilde{\theta} \tilde{\theta}^\top \right] = I_N + dW, \quad V = \begin{bmatrix} 0 & 0 \\ 0 & I_M \end{bmatrix}, \]  
where w.l.o.g. we label malicious agents as \( M = \{ N - M + 1, \ldots, N \} \), and \( V \) is the covariance matrix of noise vector \( v \).

**Lemma 1.** The optimal parameter satisfies \( \lambda^* < 1 \).

**Proof.** Let us define the following matrices: \( S_R \in \mathbb{R}^{R \times N} \) maps \( x \) to \( x_{\tilde{R}} \), and \( C_R = \frac{1}{M} I_R I_R \). According to (15), we set \( S_R = [I_R | 0] \). Then, the error (13) can be written as
\[ e_\lambda = \text{tr} \left( \Sigma E^\top E \right), \quad E = S_R L - C_R S_R \]  
(17)
and its derivative with respect to \( \lambda \) is (up to multiplicative constants)
\[ \frac{de_\lambda}{d\lambda} = \frac{1}{\lambda} \text{tr} \left( \Sigma L^\top (I - W^\top L^\top) S_R^\top E \right), \]  
(18)
which at \( \lambda = 1 \) takes value
\[ \frac{de_\lambda}{d\lambda} \bigg|_{\lambda=1} = \text{tr} \left( \Sigma (I - W^\top) S_R^\top (S_R - C_R S_R) \right). \]  
(19)
Straightforward computations show that
\[ \Sigma (I - W^\top) S_R^\top (S_R - C_R S_R) = \begin{bmatrix} A & 0 \\ \ast & 0 \end{bmatrix}, \]  
(20)
where the \( i \)th diagonal element of \( A \in \mathbb{R}^{R \times N} \), associated with regular agent \( i \in \tilde{R} \), is
\[ a_i = 1 - \frac{1}{R} \sum_{m \in M} w_{im} \geq 1 - \frac{1}{R} > 0. \]  
(21)
Being \( a_i > 0 \forall i \in \tilde{R} \), the derivative (19) is positive, hence the consensus error (13) is strictly increasing in a left neighborhood of 1. In virtue of continuity of (18) for \( \lambda > 0 \), all points of minimum of \( e_\lambda \) satisfy \( \lambda^* < 1 \). □

Interestingly, the error derivative at \( \lambda = 1 \) does not depend on the outlier noises, but only on the (weighted) topology.

**Lemma 2.** The optimal parameter satisfies \( \lambda^* > 0 \).

**Proof.** In virtue of continuity of the trace operator, we can compute the limit of the error derivative (18) as
\[ \lim_{\lambda \to 0^+} \frac{de_\lambda}{d\lambda} = \text{tr} \left( \Sigma \lim_{\lambda \to 0^+} \frac{dL^\top}{d\lambda} S_R^\top \lim_{\lambda \to 0^+} E \right) \]  
\[ = \text{tr} \left( \Sigma \Gamma S_R^\top (S_R V - C_R S_R) \right) \]  
\[ = \text{tr} \left( \Sigma \Gamma S_R^\top [-C_R | 0] \right) \]  
\[ = \text{tr} \left( \Sigma \Gamma_1 C_R + \text{tr} (\Gamma_2 C_{RM}) \right), \]  
(22)
where we have used the definitions
\[ \Sigma = \mathbb{E} \left[ \tilde{\theta} \tilde{\theta}^\top \right] = I_N + dW, \]  
(23)
and block partitions (cf. Assumption 4 for the value of \( \Gamma \))
\[ \Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ 0 & 0 \end{bmatrix}, \quad W = \begin{bmatrix} 0 & C_{RM} \\ 0 & I_M \end{bmatrix}. \]  
(24)
Matrix \( \Gamma \) can be computed exactly from the spectral decomposition of \( W \) (see Appendix for detailed derivation). In particular, its elements are finite, \( \Gamma_1 \) is positive and \( \Gamma_2 \) is negative. Hence, (22) is negative and \( e_\lambda \) is strictly decreasing in a right neighborhood of \( \lambda = 0 \). In virtue of continuity of (18), we conclude \( \lambda^* > 0 \). □

**Remark 3 (Optimal \( \lambda \) without outliers).** Lemma 2 implies that \( \lambda^* \) is always strictly positive, even with zero outlier noise, which may seem counterintuitive. This is because the malicious agents misbehave regardless of their noise level.

Lemmas 1-2 are summarized in the following proposition.

**Proposition 3.** The optimal parameter satisfies \( \lambda^* \in (0, 1) \).

3) Consensus Error vs. Noise: We now study how performance is affected by malicious agent noise variance \( d \).

We first prove an intuitive result, i.e., larger outlier noise induces larger consensus error.

**Proposition 4.** The error \( e_\lambda \) is strictly increasing with \( d \).

**Proof.** From (17) the partial derivative of \( e_\lambda \) is
\[ \frac{de_\lambda}{dd} = \text{tr} \left( V E^\top E \right). \]  
(25)
Defining the following block partition of \( L \),
\[ L = \begin{bmatrix} L_{11} & L_{12} \\ 0 & I_M \end{bmatrix}, \quad L_{11} \in \mathbb{R}^{R \times R}, \]  
(26)
it follows
\[ E^\top E = \begin{bmatrix} (L_{11} - C_R)^2 & (L_{11} - C_R) L_{12} \\ (L_{12} - C_R) L_{12}^\top & L_{12}^\top \end{bmatrix}, \]  
(27)
and hence \( \text{tr} (V E^\top E) = \text{tr} (L_{12}^\top L_{12}) > 0 \) for \( \lambda < 1 \). It follows that \( e_\lambda \) is strictly increasing with \( d \) for all \( \lambda < 1 \). □

Intuitively, the more the nominal consensus behavior is disrupted by external attacks, the more it is convenient for regular agents to stick to their own priors rather than risking to collaborate with malicious agents. Formally speaking, this would require the optimal parameter \( \lambda^* \) to increase with the noise level \( d \) and the number of malicious agents \( M \). While
the latter behavior cannot be assessed analytically because of the discrete nature of agents, the former behavior could be proven if \( \lambda^* \) was unique. Such a claim is hard to prove because of the structure of the cost function. In particular, studying the second derivative of \( e_R \) is complicated by the fact that the trace argument is not positive semidefinite, and similarly uniqueness of the root of \( [18] \) cannot be assessed in general. However, the next result contributes towards this intuition, which is confirmed numerically in Section IV.

**Proposition 5.** The critical points of \( e_R \) are strictly increasing with \( d \).

**Proof.** We start by computing partial derivatives of the error, first with respect to \( \lambda \) (cf. \( [18] \)) and then with respect to \( d \),

\[
\frac{\partial e_R}{\partial d\partial \lambda} = \frac{1}{\lambda} \text{tr} \left( LV L^T (I - W^T L^T) S_R S_R^T \right),
\]

(28)

where we use \( S_R W = 0 \). Defining the block partition

\[
W = \begin{bmatrix} W_{11} & W_{12} \\ 0 & I_M \end{bmatrix}, \quad W_{11} \in \mathbb{R}^{R \times R}, \quad (29)
\]

it follows

\[
LV = \begin{bmatrix} 0 & L_{12} \\ 0 & I_M \end{bmatrix},
\]

(30)

\[
M = I - W^T L^T = \begin{bmatrix} I_R - W_{11}^T L_{11}^T & 0 \\ -W_{12}^T L_{11}^T - L_{12}^T & 0 \end{bmatrix},
\]

(31)

\[
L^T M S_R^T S_R = \begin{bmatrix} -L_{12}^T W_{11}^T L_{11} - L_{12}^T W_{12}^T L_{11} & \star \\ \star & 0 \end{bmatrix},
\]

(32)

where the upper-left block is a negative matrix for all \( \lambda \in (0, 1) \), and is the zero matrix for \( \lambda = 1 \). Hence, the error derivative with respect to \( \lambda \) (\( [18] \)) is strictly decreasing with \( d \) for all \( \lambda \in (0, 1) \). In virtue of continuity of (\( [18] \)) in \( \lambda \), we conclude that the critical points of \( e_R \) are strictly increasing with \( d \).

**Proposition 5** implies that the points of local minimum increase strictly monotonically with \( d \). A direct consequence of this fact is that, if there is a unique critical point (and therefore a unique point of minimum) for one value of \( d \), then such a point is always unique and it is strictly increasing with \( d \). In words, this entails that higher noise intensity forces the regular agents to progressively reduce collaboration with others, trusting more their own prior instead. The next corollary refines this result by describing the limit behavior of critical points as the noise intensity grows unbounded.

**Corollary 1.** For any critical point \( \lambda_{CR} \) of \( e_R \) it holds

\[
\lim_{d \to +\infty} \lambda_{CR} = 1.
\]

**Proof.** We first expand (\( [18] \)) to highlight dependence on \( d \):

\[
\frac{de_R}{d \lambda} = \frac{d}{\lambda} \text{tr} \left( -L_{12} \left( L_{12}^T W_{11}^T + W_{12}^T L_{11}^T \right) \right) + k(\lambda).
\]

(34)

It follows that there always exists \( d > 0 \) such that the error derivative is negative, for any \( \lambda < 1 \). In fact, given \( \lambda \), the minimal such value of \( d \) is computed as

\[
d > \lambda k(\lambda) \text{tr} \left( L_{12} \left( L_{12}^T W_{11}^T L_{11}^T + W_{12}^T L_{11}^T \right) \right)^{-1} > 0.
\]

(35)

The claim follows by combining this with Proposition 5.

**IV. Numerical Experiments**

In this section, we compute numerically the consensus error function \( e_R \) in a variety of scenarios, to see how the FJ dynamics performs under different attacks.

In Fig. 1, we can see the error behavior as the noise intensity varies. In particular, we use a 3-regular graph. Further, we add one malicious agent with \( d \in \{0, 10, \ldots, 100\} \). All error curves in Fig. 1b exhibit a unique point of minimum \( \lambda^* \) and increase monotonically, according to Proposition 5. Further, \( \lambda^* \) grows with \( d \) (Fig. 1b) according to Proposition 5.
Analogous behavior is observed when varying the number of malicious agents. We next study what happens by increasing $M$ while fixing either $N$ or $R$. Figure 2 shows the error curve in a network of $N = 100$ agents when ten agents turn progressively malicious, while in Fig. 3, we fix the number of regular agents at $R = 100$ and add malicious agents, scattering them across the network. The error curves increase monotonically in Fig. 3a, which is intuitive because regular agents face progressively more attacks, while behave differently in Fig. 2a, where the error increases for small values of $\lambda$ but decreases when $\lambda$ grows. This is due to the lack of a “normalization” of the network error, which decreases with the number of regular agents (cf. [17]). This emerges for large values of $\lambda$, when the error is mainly caused by the actual values of agent’s priors, while the error curve increases when $\lambda$ is small. In both cases, we can see that $\lambda^*$ increases with $M$ (Figs. 2b,3b), which is because regular agents need to contrast larger and larger amounts of attacks.

Remark 4 (Value of optimal $\lambda$). A remarkable feature of the FJ dynamics, which emerges from the above numerical tests, is that $\lambda^*$ is relatively small, about $0.1 - 0.2$ for many relevant scenarios. In fact, $\lambda^*$ reaches 0.3 when, e.g., the malicious agent has noise variance $d = 100$, namely two order of magnitude larger than the variance of priors. This translates into the practical advantage that adding “a little” selfishness is sufficient to achieve substantial performance improvement compared to standard consensus, which may be attractive to get a good level of resilience while still letting agent states mix without forcing too conservative updates.

V. COMPARISON WITH EXISTING LITERATURE

In this section, we compare our algorithm with a state-of-the-art resilient optimization strategy, Weighted Mean Subsequence Reduced (W-MSR), originally proposed for in [7]. In words, W-MSR ensures that regular agents achieve asymptotic consensus while maintaining their states within the convex hull of their initial conditions (resilient consensus). Two main features may limit the effectiveness of W-MSR. The first is that all results rely on the notion of $r$-robustness, which expresses how effective the network is in spreading information. In particular, most results give sufficient conditions for consensus. Two practical challenges arise: on the one hand, the network may be fixed and not enough $r$-robust, possibly ruling out MSR-like strategies. On the other hand, assessing robustness in large-scale networks is computationally prohibitive [17]. Secondly, W-MSR needs to estimate the number of malicious agents to compute the minimal robustness required to succeed. This may be problematic if a reliable estimate cannot be provided, as the updates may be too conservative or misled by adversaries.

Conversely, a remarkable feature of FJ dynamics is that, even though it cannot guarantee average consensus (which, in fact, cannot be guaranteed by any resilient algorithm [17]), it always provides error bounds, which can be improved by properly tuning $\lambda$. Further, while choosing the optimal parameter requires exact knowledge of the adversary, which is not reasonable, yet our proposed approach shows good level of robustness to the choice of a specific $\lambda$, as the plots in Section IV show. Conversely, most results in literature do not characterize steady-state behavior of the system when hypotheses for resilient consensus are not satisfied. In fact, they usually either guarantee that agent’s states remain in the safety region (which in practice may be not better than using our approach with $\lambda = 1$), or let the agents reach consensus but potentially get far away from initial conditions [18]. Moreover, no extra computation or memory requirements are needed, as opposed to other algorithms proposed in literature [19]. This may be relevant to resource-constrained agents, possibly with real-time requirements.

Figure 4 shows consensus error and network cost (Fig. 4b) of the two approaches in a network with 100 agents communicating over a 3-regular graph (Fig. 4a), where the 2 malicious agents are marked as red triangles. Note that 3-regular graphs do not comply with any non-trivial level of $r$-robustness, and nothing can be said about W-MSR yielding consensus. On the other hand, we compute the theoretical error $e_\text{R}$ of FJ dynamics by sampling $\lambda$ in $(0, 1)$, and use $\lambda = \lambda^*$ computed numerically in simulation. From Fig. 4b, we can see that FJ dynamics outperforms W-MSR, proving to be a competitive approach, in particular with sparse communication networks.

VI. CONCLUSION AND FUTURE RESEARCH

In the previous sections we have analyzed the theoretical performance of a model based on FJ dynamics in the presence of malicious agents, bringing insights to algorithmic design. The proposed approach intends to motivate the general intuition that competitive approaches may make collaborative tasks more resilient to external attacks or disturbances, ideally paving the way to a new viewpoint on strategies for robust and resilient control of networked systems.

This opens several avenues of future research. First, it is desirable to address an optimal choice of the parameter $\lambda$, possibly differentiated across agents, in the realistic case where malicious agents are unknown. Further, the algorithm may be improved by letting, e.g., regular agents implement smart strategies to detect and isolate adversaries. Finally, it would be interesting to shift attention to the communication network, targeting robust design of its topology.
APPENDIX

We now show how to compute $\Gamma$ \cite{24}. First, we show how to obtain eigenvalues and eigenvectors of $\Gamma$ from $W$. Second, we show that $W$ is always diagonalizable, which implies $\Gamma$ is also diagonalizable with the same change of basis.

The derivative of $L$ is
\[
\frac{dL}{d\lambda} = \frac{dL}{d\tilde{L}} \frac{d\tilde{L}^{-1}}{d\lambda} \tilde{L} = -\lambda W \tilde{L},
\]
where $\tilde{L} = (I - (1 - \lambda)W)^{-1}$. Consider the $i$th eigenvalue of $W$, denoted as $\lambda_i$, and its associated eigenvector $v_i$, it follows that $\tilde{L}$ has $i$th eigenvalue $(1 - (1 - \lambda)\lambda_i)^{-1}$ with eigenvector $v_i$. Hence, straightforward computations yield
\[
\frac{dL}{d\lambda} v_i = \frac{1 - (1 - (1 - \lambda)\lambda_i)^{-1} \lambda_i}{(1 - (1 - \lambda)\lambda_i)} v_i = \sigma_i(\lambda) v_i.
\]
In particular, the dominant eigenvector $v_1$ (associated with $\lambda_1 = 1$) corresponds to eigenvalue $\sigma_1 = 0$ for any $\lambda > 0$. For $i > 1$, by letting $\lambda$ go to zero in (37), one gets
\[
\sigma_i = \lim_{\lambda \to 0^+} \sigma_i(\lambda) = (1 - \lambda_{i})^{-1}.
\]
Finally, the eigendecomposition of $\Gamma$ is obtained by stacking eigenvectors $\{v_i\}_{i \in V}$ in $V$ and eigenvalues $\{\sigma_i\}_{i \in V}$ in $D$.

We now show that $W$ is always diagonalizable. Considering the block decomposition \cite{20}, and recalling that $W_{11}$ is symmetric, it holds
\[
W = V \begin{bmatrix}
\Lambda_{11} & 0 \\
0 & I_M
\end{bmatrix} V^{-1},
\]
where $W_{11} = V_{11} \Lambda_{11} V_{11}$ and, for any invertible matrix $V_M$,
\[
V = \begin{bmatrix}
V_{11} & (I_R - W_{11})^{-1} W_{12} V_M \\
0 & V_M
\end{bmatrix}.
\]
In particular, $I_R - W_{11}$ is invertible because the graph is connected \cite{15}. Hence, eigendecomposition \cite{39} implies that $\Gamma$ is also diagonalizable through the change of basis $V$.

REFERENCES

[1] S. Raskin. Energy secretary says enemies are capable of shutting down us power grid. (web)
[2] L. Borghese and S. Braithwaite. Hackers block italian covid-19 vaccination booking system in ‘most serious cyberattack ever’. (web)
[3] C. Huang, R. Zhang, and S. Cui, “Optimal power allocation for wireless sensor networks with outage constraint,” IEEE Wireless Commun. Lett., vol. 3, no. 2, pp. 209–212, 2014.
[4] S. Gupta, R. Kambli, S. Wagh, and F. Kazi, “Support-vector-machine-based proactive cascade prediction in smart grid using probabilistic framework,” IEEE Trans. Ind. Electron., vol. 62, no. 4, pp. 2478–2486, 2015.
[5] J. Qi, J. Wang, and K. Sun, “Efficient estimation of component interactions for cascading failure analysis by erh algorithm,” IEEE Trans. Power Syst., vol. 33, no. 3, pp. 3153–3161, 2018.
[6] N. Chaabouni, M. Mosbah, A. Zemmari, C. Sauvignac, and P. Faruki, “Network intrusion detection for iot security based on learning techniques,” IEEE Commun. Surveys Tuts., vol. 21, no. 3, pp. 2671–2701, 2019.
[7] H. J. LeBlanc, H. Zhang, X. Koutsoukos, and S. Sundaram, “Resilient asymptotic consensus in robust networks,” IEEE J. Sel. Areas Commun., vol. 31, no. 4, pp. 766–781, 2013.
[8] S. M. Dibaji and H. Ishii, “Consensus of second-order multi-agent systems in the presence of locally bounded faults,” Systems & Control Letters, vol. 79, pp. 23–29, 2015.
[9] Y. Wang, H. Ishii, F. Bonnet, and X. Défago, “Resilient consensus against mobile malicious agents,” IFAC-PapersOnLine, vol. 53, no. 2, pp. 3409–3414, 2020, 21st IFAC World Congress.
[10] Y. Shang, “Resilient consensus in multi-agent systems with state constraints,” Automatica, vol. 122, p. 109288, 2020.
[11] J. R. Marden and J. S. Shamma, “Chapter 16 - game theory and distributed control,” in Handbook of Game Theory with Economic Applications, H. P. Young and S. Zamir, Eds. Elsevier, 2015, vol. 4, pp. 861–899.
[12] A. V. Proskurnikov and R. Tempo, “A tutorial on modeling and analysis of dynamic social networks. part i,” Annual Reviews Control, vol. 43, pp. 65–79, 2017.
[13] J. R. Marden, G. Arslan, and J. S. Shamma, “Cooperative control and potential games,” IEEE Trans. Syst., Man, Cybern. B, vol. 39, no. 6, pp. 1393–1407, 2009.
[14] N. E. Friedkin and E. C. Johnsen, “Social influence and opinions,” J. Math. Social., vol. 15, no. 3-4, pp. 193–206, 1990.
[15] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, “Novel multidimensional models of opinion dynamics in social networks,” IEEE Trans. Autom. Control, vol. 62, no. 5, pp. 2270–2285, 2017.
[16] M. Yemini, A. Nedić, A. J. Goldsmith, and S. Gil, “Characterizing trust and resilience in distributed consensus for cyberphysical systems,” IEEE Trans. Robot., vol. 38, no. 1, pp. 71–91, 2022.
[17] S. Sundaram and B. Ghasirehsafa, “Consensus-based distributed optimization with malicious nodes,” in 53rd Annu. Allerton Conf. Commun. Control Comput., Sep. 2015, pp. 244–249.
[18] J. S. Baras and X. Liu, “Trust is the cure to distributed consensus with adversaries,” in 2019 27th Mediterranean Conference on Control and Automation (MED), 2019, pp. 195–202.
[19] S. M. Dibaji, M. Safi, and H. Ishii, “Resilient distributed averaging,” in American Control Conference, 2019, pp. 96–101.