Dimension Independent Generalization Error with Regularized Online Optimization

Xi Chen\textsuperscript{*} Qiang Liu\textsuperscript{†} Xin T. Tong\textsuperscript{‡}

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Abstract

One classical canon of statistics is that large models are prone to overfitting and model selection procedures are necessary for high-dimensional data. However, many overparameterized models such as neural networks, which are often trained with simple online methods and regularization, perform very well in practice. The empirical success of overparameterized models, which is often known as benign overfitting, motivates us to have a new look at the statistical generalization theory for online optimization. In particular, we present a general theory on the generalization error of stochastic gradient descent (SGD) for both convex and non-convex loss functions. We further provide the definition of “low effective dimension” so that the generalization error either does not depend on the ambient dimension \( p \) or depends on \( p \) via a poly-logarithmic factor. We also demonstrate on several widely used statistical models that the “low effect dimension” arises naturally in overparameterized settings. The studied statistical applications include both convex models such as linear regression and logistic regression, and non-convex models such as \( M \)-estimator and two-layer neural networks.

1 Introduction

The study of overfitting phenomenon has been an important topic in statistics and machine learning. From classical statistical learning theory, we understand that when the number of model parameters is large as compared to the amount of data, the generalization error can be excessively large even if the training error is small. This phenomenon is usually

\textsuperscript{*}Stern School of Business, New York University, Email: xichen@nyu.edu
\textsuperscript{†}Department of Mathematics, National University of Singapore, Email: matliuq@nus.edu.sg
\textsuperscript{‡}Department of Mathematics, National University of Singapore, Email: mattxin@nus.edu.sg
known as overfitting. For this reason, dimension reduction or feature selection mechanisms such as principle component analysis (PCA) and shrinkage methods are often required in the training phase to reduce model dimension.

In recent years, the deep neural network has achieved a great success in practical applications. Researchers have found out that overparameterized neural networks usually achieve superior performance (Golowich et al., 2018; Li and Liang, 2018; Neyshabur et al., 2019; Allen-Zhu et al., 2019; Arora et al., 2019). Moreover, these models are often trained with simple regularization and do not need dimension reduction procedures. This phenomenon is sometimes referred to as “benign overfitting” (Bartlett et al., 2019).

Although there are many practical evidence on the benefit of overparameterization, the existing theoretical study mainly focuses on linear model (see, e.g., Bartlett et al. (2019); Nakkiran (2019); Ali et al. (2019)) or neural networks with certain special data structure (see, e.g., Li and Liang (2018)). The main purpose of our paper is to systematically investigate the generalization error for a general risk minimization problem when the number of parameters is much larger than the sample size. In particular, we study the generalization error bound for the stochastic gradient descent (SGD) solution for both convex (e.g., linear regression and logistic regression) and non-convex problems (some M-estimators and neural networks). The SGD algorithm has been widely used in learning from large-scale data due to its computational and memory efficiency. Therefore it is important to study the generalization behavior for SGD solutions under the high-dimensional regime.

Let us briefly introduce our setup of the overfitting problem and stochastic gradient descent (SGD). We consider the following population risk minimization problem under the loss function $F$, which can be either convex or non-convex:

$$w^* = \arg\min_w F(w) := \mathbb{E}_\zeta f(w, \zeta).$$

In (1), $w \in \mathbb{R}^p$ is a $p$-dimensional parameter vector, $\zeta$ denotes a random sample from a certain probability distribution, and $f(\cdot, \zeta)$ is the loss function on each individual data $\zeta$. The global minimizer $w^*$ is often the true model parameter in statistical estimation problems. In practice, the distribution of $\zeta$ is usually unknown and one only has the access to $N$ i.i.d. samples $\zeta_1, \ldots, \zeta_N$ from the population. Instead of minimizing the population risk $F(w)$ in (1), the goal is to minimize the empirical loss function

$$\hat{F}(w) = \frac{1}{N} \sum_{i=1}^N f(w, \zeta_i).$$
In practice, instead of directly minimizing the empirical loss, an extra regularization term is often added to the empirical loss to avoid overfitting. In this paper, we consider the most commonly used ridge or Tikhonov regularization. The corresponding regularized empirical loss function takes the following form,

\[ \hat{F}_\lambda(w) = \hat{F}(w) + \frac{\lambda}{2} \|w\|^2 = \frac{1}{N} \sum_{i=1}^{N} f_\lambda(w, \zeta_i), \quad f_\lambda(w, \zeta) := f(w, \zeta) + \frac{\lambda}{2} \|w\|^2. \]  

(3)

The weight of regularization is controlled by \( \lambda \), which is a tuning-parameter. One popular way to optimize \( \hat{F}_\lambda \) in machine learning is via SGD. In particular, for a generic initialization parameter \( w_0 \), SGD is an iterative algorithm, where the \((n+1)\)-th iterate \( w_{n+1} \) is updated according to the following equation,

\[ w_{n+1} := w_n - \eta \nabla f_\lambda(w_n, \zeta_n) = w_n - \eta (\nabla f(w_n, \zeta_n) + \lambda w_n). \]  

(4)

By running through \( N \) samples, SGD outputs the \( N \)-th iterate \( w_N \) as the final estimator of \( w^* \). In SGD iterations (4), the hyper-parameter \( \eta \) is known as the stepsize. In our paper, we consider the constant stepsize, which is more popular in practice (Bach and Moulines, 2011). The choice of \( \eta \) will be discussed later in our theoretical results. Moreover, in (4), the gradient is taken with respect to the parameter vector \( w \). For notational simplicity, we will simply use “\( \nabla \)” as a short notation for “\( \nabla_w \)” throughout the paper.

When the sample size \( N \) is much larger as compared to the dimensionality \( p \), it is expected that \( w_N \) would be close to \( w^* \) under certain conditions. However, in an over-parameterized setting where \( N \) is less than \( p \), the solution \( w_N \) can be far away from \( w^* \). Nevertheless, instead of estimating the underlying parameter which usually requires strong assumptions, for many machine learning tasks, it is of more interest in achieving small generalization error, which is defined as follows,

\[ G(w_N) = F(w_N) - F(w^*). \]  

(5)

The main purpose of the paper is to provide a general bound of the generalization error in (5) in the overparameterized setting. We will characterize the scenarios when such a generalization error bound is independent of \( p \), or only involving in poly-logarithmic factors of \( p \).

1.1 Main results and paper organization

The main message of this paper is as follows. For a large class of statistical learning problems where the effective dimension is low (see the rigorous definition in Section 3),
the stochastic gradient descent (SGD) algorithm with proper ridge regularization will not overfit even if the ambient model dimension is much larger than the sample size. In particular, we will show that the generalization error has at most poly-logarithmic dependence on the ambient model dimension $p$.

In Section 2, we present a framework for generalization error analysis. This framework is designed to handle non-convex loss functions with high dimensional parameters. The upper bound of the generalization error is provided in Theorem 2.3. Using linear regression as an illustrative example, we show that each term in the generalization bound has a strong statistical interpretation (see Section 2.2.1). The upper bound also leads to practical guidelines on the rates of problem-related parameters given by Corollary 2.4.

While Theorem 2.3 provides an error bound on generalization, for this error bound to be almost dimension-independent, we require the effective dimension to be small. Section 3 first formally defines the general notion of low effective dimension, which can essentially be described by 1) the loss function has a fast decaying Hessian spectrum, and 2) the true parameter is either sparse or uniformly bounded along Hessian’s eigen-directions.

In Section 4, we carefully investigate generalization errors in various linear models. We consider the cases of finite projections of infinite dimensional models and linear regression with redundant features. In these scenarios, we quantify when the overparameterization does not hurt the generalization performance.

Our generalization result can be applied to a wide range of nonlinear models. In Section 5, we study both convex nonlinear models such as logistic regression and non-convex models such as $M$-estimator with the Tukey’s biweight loss function (Tukey, 1960) and two-layer neural networks. We show that the low effective dimension naturally occurs in these applications.

1.2 Related works

High dimensional statistical learning with low effective dimension is a common idea in principal component analysis (PCA) and functional data analysis applications (Jolliffe, 2002; Ramsay and Silverman, 2005). To implement PCA, we first find a $k^*$-dimensional subspace of which the truncated variability contains a significant proportion (e.g., 95%) of the full model variability. Then we project the problem onto this subspace and conduct the data analysis on it. In general, the truncated dimension $k^*$ needs to be less than the sample size $N$, and the statistical problem needs to be well conditioned in the $k^*$-dimensional subspace. While we also require the existence of a low effective dimension,
the problem setting considered in this paper is fundamentally different from the classical PCA problem. In our setting, the SGD algorithm directly runs on the original problem space of dimension $p$ and there is no projection step. Moreover, we do not need the problem to be well conditioned in the ambient space. We would argue that our setting is more useful for practical applications since it is difficult to find a proper truncation dimension $k^*$ and PCA is intractable with rank-deficient data.

In recent years, the generalization error bounds for linear models in overparameterized settings have been carefully investigated in the literature, see, e.g., Bartlett et al. (2019); Nakkiran (2019); Ali et al. (2019); Hastie et al. (2019). Our result is different from these existing results in the following perspectives:

1) Nonlinearity: Most of these works focus on linear models. For example, Bartlett et al. (2019) defined two notions of effective ranks of the data covariance matrix in linear models, and expressed the generalization error bound in terms of effective ranks. A more detailed comparison with the low effective rank condition in Bartlett et al. (2019) will be provided in Remark 3.5. In addition, Hastie et al. (2019) developed the generalization error bound for the composition of an activation function and a linear model. Our work also requires conditions similar to small effective ranks, but they are simpler in formulation, as discussed in Section 3. Moreover, our results can be applied to general non-linear and non-convex models.

2) Anisotropic spectrum and regularization: Nakkiran (2019) and Hastie et al. (2019) focus their studies on the isotropic or well-conditioned regressor covariance matrices. Our study focuses on anisotropic regressor covariance, of which the minimum eigenvalue decays to zero. Moreover, Bartlett et al. (2019), Nakkiran (2019), and Hastie et al. (2019) have already provided solid understanding of the ridgeless linear regression. Thus, this paper focuses on loss functions with a ridge penalty, which are more widely used in practice and thus deserve better understanding. Our study in Corollary 2.4 shows how the regularization parameter depends on the Hessian spectrum. Moreover, since the ridge penalty is adopted, we do not have the “double descent” phenomenon as in other literature (Belkin et al., 2018, 2019). A recent paper by Nakkiran et al. (2020) also showed that for certain linear regression models with isotropic data distribution, the ridge penalty regularized regression (in the offline setting) can avoid the “double descent” phenomenon.

3) Online optimization: The aforementioned works mainly focus on offline optimization. For example, Bartlett et al. (2019), Nakkiran (2019), and Hastie et al. (2019) show
that the generalization error of the offline linear regression solution is closely related to the spectrum of the design matrix. In particular, when the minimum eigenvalue is close to zero, the offline learning results become unstable because of singular matrix-inversions. Since the design matrix relies on data realization, such instability can only be studied through random matrix theories (RMT). While these studies of offline linear regression are interesting and technically deep, their dependence on RMT makes the extensions to nonlinear models difficult. In comparison, online learning methods process one data point at a time, and do not involve the inversion of design matrices, which facilitates the study of general nonlinear models. Moreover, in terms of practical applications, as compared to offline optimization, online optimization methods such as SGD are appealing due to their low per-iteration complexity. Therefore, this paper focuses on the generalization error for online learning in overparameterized settings.

For online optimization, the stochastic gradient descent (SGD), which dates back to Robbins and Monro (1951), is perhaps the most widely used method in practice. The convergence rates of the SGD for convex loss functions have been well studied in the literature (see, e.g., Zhang (2004), Nesterov and Vial (2008), Bach and Moulines (2011), Ghadimi and Lan (2012), Roux et al. (2012), Bach and Moulines (2013)). For the constant stepsize SGD, Bach and Moulines (2011) provided the generalization error bound in Theorem 1 of their paper. However, their bound has an explicit linear dependence on $p$, which is not applicable to the overparameterized setting. Our paper provides a more refined generalization error bound, which incorporates the Hessian spectrum to capture the “low dimension effect” in the overparameterized setting.

Over-parameterization neural network (NN) is a very active research direction. There are several existing works explaining why overfitting does not happen in large NN (Golowich et al., 2018; Li and Liang, 2018; Neyshabur et al., 2019; Allen-Zhu et al., 2019; Arora et al., 2019; E et al., 2019a,b). Interestingly, the conditions they impose are largely similar to the ones we will use. Namely, they require the high dimensional input data and the Frobenius norm of true weight matrices to be bounded by constants. For this to be true, only a small portion of the data or model components can be significantly active, which satisfies the concept of low effective dimension.

On the other hand, our study of two-layer neural network in Section 5.3 is different from existing results in the following perspectives. One popular way to analyze generalization error is by analyzing the Rademacher complexity (Golowich et al., 2018; Neyshabur et al., 2019; E et al., 2019a), which can be used to establish an upper bound of the difference
between training error and generalization error (see, e.g., Theorem 3.3 of Mohri et al. (2018)). However, these results are usually derived in an offline optimization setting, while our paper focuses on the result from online SGD. Li and Liang (2018) and Allen-Zhu et al. (2019) both studied NN generalization error with SGD iterations. But they mainly focus on classification scenarios where the loss function is bounded. Moreover, Li and Liang (2018) required the loss function to be of a logistic form, and Allen-Zhu et al. (2019) studied the running average generalization error. Our results can be applied to regression NN with unbounded loss functions. Arora et al. (2019) and E et al. (2019b) studied NN generalization bound with deterministic gradient descent. Moreover, their studies assume a certain data angle or Gram matrix to have a strictly positive minimal eigenvalue. Please see Section 5.3 on more detailed comparisons with these works.

2 A Generalization Bound

In this section, we present a general result on the generalization bound for the SGD solution from (4).

2.1 Preliminaries: non-convexity and high dimensional norms

In this paper, we consider a general setting where the population risk function $F$ in (1) can be non-convex. The non-convexity makes the statistical learning problem technically more challenging. First, the function $F$ can have multiple local minima, and each local minimum has an attraction basin, which is a “valley” in the graph of $F$. Within each valley, we assume that $F$ is “approximately locally convex” (please see the rigorous definition below). Suppose $D$ is the attraction basin of the optimal solution $w^*$ in (1), initializing SGD in $D$ will generate iterates converging to $w^*$. On the other hand, if the SGD iterates take place outside $D$, the output can be irrelevant to the properties of $w^*$.

Based on our discussion, we have the the following assumption on the population risk function,

**Assumption 2.1.** The optimal solution $w^*$ of the population risk $F$ has a neighborhood $D$, such that for some positive semidefinite (PSD) matrix $A$ and $\delta \in [0, 1/2)$,

$$-\delta A \preceq \nabla^2 F(w) \preceq A, \quad \forall w \in D. \tag{6}$$

In Assumption 2.1 and in the sequel, for two symmetric matrices $C, D$, $C \preceq D$ indicates that $D - C$ is positive semidefinite (PSD). The upper bound on the Hessian matrix
\(\nabla^2 F(w) \preceq A\) is widely assumed in statistical literature. The parameter \(\delta\) above describes the level of non-convexity. In particular, \(\delta = 0\) indicates that \(F\) is convex within \(\mathcal{D}\). However, our condition in (6) is more general since \(\delta\) can be strictly positive, which allows \(F\) to be non-convex. On the other hand, although our generalization error bound holds for \(\delta \in [0, 1/2]\), for this upper bound to be smaller than a certain threshold, \(\delta\) needs to be small. Please see Corollary 2.4 for the exact dependence of \(\delta\) in the upper bound.

In addition to the non-convexity, we define the following norms, which will be extensively used in our theoretical analysis.

**Definition 2.2.** Given a matrix \(A \in \mathbb{R}^{p \times p}\) and \(\lambda\), we decompose \(\mathbb{R}^p = S_\lambda \oplus S_\perp\), where \(S_\lambda\) consists of eigenvectors of \(A\) with eigenvalues above \(\lambda > 0\) and \(S_\perp\) is the orthogonal complement of \(S_\lambda\). Given any vector \(v\), denote its decomposition as \(v = v_\lambda + v_\perp\), where \(v_\lambda \in S_\lambda\) and \(v_\perp \in S_\perp\). Then define

\[
\|v\|_A^2 = v^T Av, \quad \|v\|_{A,\lambda}^2 := \lambda\|v_\lambda\|^2 + v_\perp^T Av_\perp. \tag{7}
\]

We introduce the norm \(\|v\|_A^2\), whose value can be independent of the ambient dimension \(p\). The second norm \(\|v\|_{A,\lambda}^2\) is a truncated version of the first norm, which essentially truncates all eigenvalues of \(A\) above \(\lambda\) to \(\lambda\). It is easy to see that \(\|v\|_{A,\lambda}^2 \leq \|v\|_A^2\). We introduce the second norm because \(\|v\|_{A,\lambda}^2\) converges to zero when the regularization parameter \(\lambda\) does, while \(\|v\|_A^2\) is independent of \(\lambda\).

### 2.2 Generalization bound

Now we are ready to give the statement of our main result:

**Theorem 2.3.** Under Assumption 2.1, suppose \(w_0 \in \mathcal{D}\) and there is a constants \(r\) and \(c_r\) such that

\[
E\|\nabla f(w, \zeta) - \nabla F(w)\|^2 \leq r^2 + c_r r^2 \min\{G(w), \|w\|^2\}, \quad \forall w \in \mathcal{D}. \tag{8}
\]

Then if the SGD hyper-parameters, the stepsize \(\eta\) and the regularization parameter \(\lambda\), satisfy

\[
\eta \leq \min \left\{1, \frac{\lambda}{12\|A\|^2 + 6\lambda^2 + 6c_r r^2}, \frac{1}{3\|A\|}, \frac{\lambda}{6c_r \|A\| r^2} \right\}, \quad 8\delta\|A\| \leq \lambda \leq 1,
\]

the generalization error has the following upper bound:

\[
E[G(w_N)1_{r \geq N}] \leq 4\|w^*\|_{A,\lambda}^2 + \frac{C_1 \eta}{\lambda} \left[ + \exp\left(-\frac{1}{4} \lambda N \eta\right)E[G(w_0) + 4N\|A\|\|w_0\|^2] + \frac{C_2 \delta}{\lambda^2}. \tag{9}\right.
\]

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In (9), the generalization error $G$ is defined in (5). The stopping time $\tau$ is the first time that an iterate $w_n$ exits region $D$. The condition $\tau \ge N$ indicates that SGD iterates have not left the region $D$. $C_1$ and $C_2$ are quantities given by,

$$C_1 = 6\|A\|(r^2 + 30\|A\||w^*|_A^2), \quad C_2 = 60\|A\|(r^2 + 5\|A\||w^*|_A^2).$$

In Section 3, we will explicitly define the low effective dimension so that $C_1$, $C_2$, and the upper bound (9) are independent of dimension $p$, or depend on $p$ only via a polynomial logarithmic factor. It is noteworthy that (9) only discusses the scenario when SGD iterates stay in the domain $D$. This is necessary since all our conditions are imposed only within $D$. Once an SGD iterate leaves $D$, there is no particular reason it can get back to $D$.

Each term in the upper bound carries a strong statistical interpretation, which will be illustrated using linear models in the next subsection (see Section 2.2.1). In particular, the term $\|w\|^2_{A,\lambda}$ in (9) can be interpreted as the bias caused by minimizing $F_\lambda$ instead of $F$, which decays with the regularization parameter $\lambda$ shrinking to zero. The term $\frac{C_2}{\lambda}$ is the variance induced by the stochastic gradient descent algorithm, which increases as $\lambda$ decreases. This reveals that under our current problem setting, $\lambda$ controls a bias-variance tradeoff. Ideally, we can choose small $\lambda$ and stepsize $\eta$ to make both the bias and variance small. However, this comes with a price. As the convergence rate scales with $\lambda \eta$, so using small $\lambda$ and $\eta$ need to be compensated with a large sample size $N$ (i.e., the number of iterations in SGD).

We can quantify these tradeoffs by considering a practical scenario where a generalization error is pre-fixed to be $\epsilon$, then our results provide guidelines on how to tune the parameters:

**Corollary 2.4.** Given any $\epsilon > 0$, if the regularization parameter $\lambda(\epsilon)$, the stepsize $\eta(\epsilon)$, the non-convexity parameter $\delta(\epsilon)$, and the sample size $N(\epsilon)$ satisfy

$$4\|w^*\|^2_{A,\lambda(\epsilon)} < \epsilon, \quad \delta(\epsilon) \le \frac{(\lambda(\epsilon))^2 \epsilon}{C_2}, \quad \eta(\epsilon) < \frac{\lambda(\epsilon) \epsilon}{C_1},$$

$$N(\epsilon) > \max \left\{ \frac{-4\log\{\epsilon/2\mathbb{E}[G(w_0)]\}}{\lambda(\epsilon) \eta(\epsilon)}, \frac{-8\log\{\epsilon \lambda(\epsilon) \eta(\epsilon)/(64\|A\|\mathbb{E}[\|w_0\|^2])\}}{\lambda(\epsilon) \eta(\epsilon)} \right\},$$

and the conditions of Theorem 2.3 hold, then $\mathbb{E}[G(w_N)1_{\tau \ge N}] \le 4\epsilon$.

To facilitate better understanding of Theorem 2.3, we will use linear models to illustrate the statistical interpretation of each term in the generalization upper bound in (9) in the next subsection.
2.2.1 Statistical interpretation in linear models and bias-variance tradeoff

In linear regression, each i.i.d. observation contains a pair of dependent and response variables, \( \zeta_i = (x_i, y_i) \in \mathbb{R}^p \times \mathbb{R} \), where \( y_i \) is generated by the following model

\[
y_i = x_i^T w^* + \xi_i. \tag{11}
\]

In (20), \( w^* \) is the true regression coefficient and the noise term \( \xi_i \) is independent of \( x_i \) with zero mean and a finite variance \( \sigma^2 \). For the ease of illustration, we assume \( x_i \sim \mathcal{N}(0, \Sigma) \).

The generalization error of this linear model takes the following form,

\[
G(w) = \mathbb{E} \left[ \frac{1}{2} (y_i - w^T x_i)^2 - \frac{1}{2} (y_i - (w^*)^T x_i)^2 \right] = \frac{1}{2} (w - w^*)^T \Sigma (w - w^*). \tag{12}
\]

From (12), we can see that \( G(w) \) has a strong dependence on the structure of \( \Sigma \). Let us denote the eigenvalues of \( \Sigma \) by \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \), and the eigenvector corresponding to \( \lambda_i \) by \( v_i \). Then the parameter error, \( v_i^T (w - w^*) \), contributes to \( G(w) \) via the factor \( \lambda_i \).

It is well known that SGD can be interpreted as a stochastic approximation of the gradient descent (Robbins and Monro, 1951), namely we can rewrite (4) as

\[
w_{n+1} = w_n - \eta \nabla F_\lambda(w_n) + \eta \xi_n, \tag{13}
\]

where \( \xi_n = -\nabla f(w_n, \zeta_n) + \nabla F(w_n) \) is the noise in stochastic gradient. For the quadratic loss with the ridge penalty \( F_\lambda \), the SGD iterates in (13) take the following form,

\[
w_{n+1} = w_n - \eta \Sigma (w_n - w^*) - \eta \lambda w_n + \eta \xi_n = (I - (\Sigma + \lambda I)\eta)(w_n - w^*_\lambda) + w^*_\lambda + \eta \xi_n, \tag{14}
\]

where \( w^*_\lambda := (\Sigma + \lambda I)^{-1} \Sigma w^* \) is the minimizer of \( F_\lambda(w) = \mathbb{E} \frac{1}{2} (y - w^T x)^2 + \frac{\lambda}{2} \|w\|^2 \).

It is easy to see that \( w_n \) then follows a vector autoregressive (VAR) model (see, e.g., Chapter 8 of Tsay (2010)). For the ease of discussion, we simply treat \( \xi_n \) as \( \mathcal{N}(0, \frac{r^2}{p} I) \), so \( \mathbb{E} \|\xi_n\|^2 = r^2 \) as we assumed in Theorem 2.3. Then the stationary distribution of \( w_n \) in (14) is a Gaussian \( \mathcal{N}(\mu, V) \). Take expectation and covariance on both sides of (14), the \( \mu \) and \( V \) take the following form (see Chapter 8.2.2 of Tsay (2010)),

\[
\mu = (I - (\Sigma + \lambda I)\eta)(\mu - w^*_\lambda) + w^*_\lambda \Rightarrow \mu = w^*_\lambda,
\]

\[
V = (I - (\Sigma + \lambda I)\eta)V(I - (\Sigma + \lambda I)\eta) + \frac{r^2 \eta^2}{p} I,
\]

which leads to

\[
V = \frac{r^2 \eta}{p} (2(\Sigma + \lambda I) - (\Sigma + \lambda I)^2 \eta)^{-1} \preceq \frac{r^2 \eta}{p} (\Sigma + \lambda I)^{-1}.
\]
This bound is sharp up to a $\frac{1}{2}$ factor, since $V \succeq \frac{\eta^2}{2p}(\Sigma + \lambda I)^{-1}$. These results give us the limiting average generalization error

$$
\lim_{n \to \infty} E G(w_n) = \frac{1}{2}(w_\lambda^* - w^*)^T \Sigma (w_\lambda^* - w^*) + \frac{1}{2} \text{tr}(V \Sigma)
\leq G(w_\lambda^*) + \frac{\eta^2}{2p} \text{tr}((\Sigma + \lambda I)^{-1} \Sigma).
$$

(15)

The first term $G(w_\lambda^*)$ is the bias caused by using regularization. Indeed, the optimizer of $F_\lambda$ is $w_\lambda^*$ rather than $w^*$. Recall that $(\lambda_i, v_i)$ are the eigenvalues and eigenvectors of $\Sigma$.

We define $a_i = \langle v_i, w^* \rangle$ and further express $G(w_\lambda^*)$ in (15) as follows,

$$
G(w_\lambda^*) = \frac{1}{2}((\Sigma + \lambda I)^{-1} \Sigma w^* - w^*)^T (\Sigma + \lambda I)^{-1} \Sigma w^* - w^*)
\leq \frac{1}{2} \lambda^2 (w^*)^T (\Sigma + \lambda I)^{-1} \Sigma (\Sigma + \lambda I)^{-1} w^* = \frac{1}{2} \sum_{i=1}^p \frac{\lambda^2 \lambda_i a_i^2}{(\lambda + \lambda_i)^2}.
$$

(16)

Note that when $\lambda_i \geq 0$, $\frac{\lambda^2 \lambda_i}{(\lambda + \lambda_i)^2} \leq \lambda$, and by Young’s inequality $\frac{\lambda^2 \lambda_i}{(\lambda + \lambda_i)^2} \leq \frac{\lambda^2 \lambda_1}{4 \lambda_1 \lambda} \leq \lambda$.

Therefore, we have the following upper bound of $G(w_\lambda^*)$

$$
G(w_\lambda^*) \leq \frac{1}{2} \sum_{i=1}^p (\lambda \land \lambda_i) a_i^2 = \frac{1}{2} \|w^*\|_{\Sigma, \lambda}.
$$

(16)

The upper bound in (16) is essentially the first term in (9) by noticing that $\Sigma = A = \nabla^2 F(w)$ in linear regression, which gives an upper bound for the bias.

For the second variance term in (15),

$$
\text{var}(\lambda) := \frac{r^2 \eta}{2p} \text{tr}((\Sigma + \lambda I)^{-1} \Sigma) = \frac{r^2 \eta}{2p} \sum_{i=1}^p \frac{\lambda_i}{\lambda_i + \lambda} \leq \frac{r^2 \eta}{2p} \sum_{i=1}^p \frac{\lambda_i}{\lambda} = \frac{\eta^2 \lambda_1}{2 \lambda}.
$$

(17)

This upper bound is essentially the second term in the generalization upper bound in (9), as it depends linearly on $\eta, \lambda^{-1}, \lambda_1 r^2$.

The first two terms in (9) are based on the limiting average generalization error. With finite SGD iterations, the iterate $w_n$ may not reach the limiting distribution. On the other hand, for VAR models, it is well known that the speed of convergence for $w_n$ is exponential and the convergence rate is closely related to the minimum eigenvalue $\lambda_{\min}(\Sigma + \lambda I) \eta = \lambda \eta$ (see, e.g., Tsay (2010) Chapter 8.2.2). The finite iterate error leads to the third term of $\exp(-\frac{1}{4} \lambda \eta N)$ in generalization error bound in (9).

Finally, we consider the scenario when $\Sigma$ is indefinite with $\delta = -\lambda_{\min}(\Sigma) > 0$. While the population loss $F$ is non-convex, when adopting $\lambda > 2 \delta$, we have that $\Sigma + \lambda I$ is positive definite and $F_\lambda$ is convex. Then the generalization upper bounds need to be updated by

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replacing \( \lambda \) with \( \lambda - \delta \), which leads to a perturbation on the order of \( \delta \). In particular, note that by Young’s inequality, the derivative of the bias term with \( \lambda \) is bounded by

\[
|\partial_\lambda G(w_\lambda^*)| = \frac{\sum_{i=1}^{p} \lambda_i^2 a_i^2}{(\lambda + \lambda_i)^2} \leq \frac{\sum_{i=1}^{p} \lambda_i a_i^2}{4(\lambda + \lambda_i)} = \frac{1}{4\lambda} \sum_{i=1}^{p} \lambda_i a_i^2 = \frac{\|w^*\|_2^2}{4\lambda}.
\]

The derivative of the variance term with \( \lambda \) in (17) is bounded by Young’s inequality,

\[
|\partial_\lambda \text{var}(\lambda)| = \frac{r^2 \eta}{2p} \sum_{i=1}^{p} \frac{\lambda_i}{(\lambda_i + \lambda)^2} \leq \frac{r^2 \eta}{2p} \sum_{i=1}^{p} \frac{\lambda_i}{\lambda^2} \leq \frac{r^2 \eta \lambda_1}{2\lambda^2}.
\]

Therefore, replacing \( \lambda \) with \( \lambda - \delta \) to handle non-convexity, we need to add the following term in the generalization error bound,

\[
\delta |\partial_\lambda F(w_\lambda^*)| + \delta |\partial_\lambda \text{var}(\lambda)| \leq \delta \left( \frac{\|w^*\|_2^2}{4\lambda} + \frac{r^2 \eta \lambda_1}{2\lambda^2} \right).
\]

This term can be further upper bounded by the last term of (9).

3 Low Effective Dimension

Given the generalization error bound in Theorem 2.3, we introduce the concept of “low effective dimension” and show that the generalization error bound in (9) can be independent (or dependent poly-logarithmically) of the ambient dimension \( p \) in an overparameterized regime. We will use the \( O \) and \( \Omega \) notations to hide constants independent of \( p \) and use the \( \tilde{O} \) and \( \tilde{\Omega} \) notations to hide constants depend poly-logarithmically on \( p \). In particular, we introduce the following standard asymptotic notations: \( A_\epsilon = O(f(\epsilon)) \), \( B_\epsilon = \tilde{O}(f(\epsilon)) \), \( C_\epsilon = \Omega(f(\epsilon)) \), \( D_\epsilon = \tilde{\Omega}(f(\epsilon)) \). These notations mean that there exist some universal constants \( C, c > 0 \) such that,

\[
A_\epsilon \leq C f(\epsilon), \quad B_\epsilon \leq C (\log p)^{-c} f(\epsilon), \quad C_\epsilon \geq C f(\epsilon), \quad D_\epsilon \geq C (\log p)^c f(\epsilon).
\]

3.1 Initialization and stochastic gradient variance

We investigate the terms appear in the generalization bound (9), whether they can be independent of \( p \), and how do they affect the necessary sample size \( N(\epsilon) \) in (10).

First, we notice that the terms related to initialization \( w_0 \), i.e., \( \mathbb{E} \|w_0\|^2 \) and \( \mathbb{E} G(w_0) \), appear in the sample size \( N(\epsilon) \) in (10). If the region \( \mathcal{D} = \mathbb{R}^p \), we can often choose appropriate \( w_0 \) so that \( \mathbb{E} \|w_0\|^2 \) and \( \mathbb{E} G(w_0) \) are independent of \( p \). For example, for linear regression loss function in (12), we can pick \( w_0 = 0 \), then \( \mathbb{E} G(w_0) = \frac{1}{2} \|w^*\|^2_A \) with \( A = \Sigma \).
which will be bounded by an $O(1)$ constant as shown below. For a restrictive region $D$, although $\mathbb{E}\|w_0\|^2$ and $\mathbb{E}G(w_0)$ may scale as a polynomial function of $p$, $N(\epsilon)$ only depends logarithmically on these two terms. Therefore, the dimension dependence of $N(\epsilon)$ is only logarithmic.

Second, we consider the stochastic gradient variance $r^2$, which contributes to the terms $C_1$ and $C_2$ in (9). In a typical setting, it scales roughly as the squared population gradient, i.e.,

$$
\mathbb{E}\|\nabla f(w, \zeta) - \nabla F(w)\|^2 \approx O(\mathbb{E}\|\nabla F(w)\|^2)
$$

$$
= O(\mathbb{E}\|\nabla F(w) - \nabla(F(w^*))\|^2)
$$

$$
= O(\mathbb{E}\|\nabla^2 F(w)(w - w^*)\|^2)
$$

Assume that $w \sim \mathcal{N}(0, I_p)$ and $\nabla^2 F \preceq A$

$$
= O(\|A\| (\|w^*\|_A^2 + \text{tr}(A))).
$$

We will see such an approximation holds for many applications of interest. Moreover, we have $\|A\| \leq \text{tr}(A)$, which can often be $p$-independent as discussed below. The scale of $\|w^*\|_A$ will also be discussed next.

From the discussion above, we only need to focus on two terms in (9), $\|w^*\|_A$ and $\|w^*\|_{A,\lambda}$. For the generalization error to be small and independent of $p$, we need to show $\|w^*\|_A$ is dimension independent, and $\|w^*\|_{A,\lambda}$ decreases as $\lambda$ decreases.

### 3.2 Low effective dimension settings

In this section, we formally define two settings of low effective dimension as Assumptions 3.1 and 3.3. In Sections 4 and 5, we will show that these assumptions easily hold for a wide range of convex and non-convex statistical models.

The first setting is characterized in the following assumption.

**Assumption 3.1.** The followings are true

1) $\|A\|$ with $A$ defined in Assumption 2.1 is bounded by an $O(1)$ constant.

2) $\|w^*\|$ is bounded by an $O(1)$ constant.

3) $r^2, c_r$ defined in Theorem 2.3 are bounded by $O(1)$ constants.

4) The initial values $\mathbb{E}\|w_0\|^2$ and $\mathbb{E}G(w_0)$ grow polynomially with $p$.  

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Under Assumption 3.1, Corollary 2.4 can be simplified as the following generalization error bound, which shows that the necessary sample size depends on $p$ in a poly-logarithmic factor.

**Proposition 3.2.** By the following inequalities,

$$
\|w^*\|^2 \leq \|A\|\|w^*\|^2, \quad \|w^*\|^2_{A,\lambda} \leq \lambda\|w^*\|^2.
$$

Assumption 3.1 implies that $\|w^*\| = O(1)$ and $\|w^*\|^2_{A,\lambda} = O(\sqrt{\lambda})$. Moreover, under the conditions in Corollary 2.4 and Assumption 3.1, given any $\epsilon > 0$, when

$$
\lambda(\epsilon) = O(\epsilon), \quad \delta(\epsilon) = O(\epsilon^3), \quad \eta(\epsilon) = O(\epsilon^2), \quad N(\epsilon) = \tilde{\Omega}\left(\frac{|\log(\epsilon)|^3}{\epsilon^3}\right),
$$

we have $\mathbb{E}[G(w_N)1_{T \geq N}] \leq 4\epsilon$. If in addition that the initial values $\mathbb{E}\|w_0\|^2$ and $\mathbb{E}G(w_0)$ are bounded by $O(1)$ constants, the sample size $N(\epsilon) = \Omega\left(\frac{|\log(\epsilon)|^3}{\epsilon^3}\right)$.

Assumption 3.1 can be interpreted as a weak sparsity condition for $w^*$, since there can only be a few significant components in $w^*$. Sparsity assumption is a very common condition in the statistical literature. However, our assumption only assumes that the $\ell_2$ norm of $w^*$, instead of the $\ell_0$ norm, is bounded. As compared to $\ell_0$-norm, the $\ell_2$-norm is rotation-free. In addition, we do not need to apply any projection or shrinkage procedures on the SGD iterates.

The second setting is technically more interesting, which assumes the data has a low effective rank in the following sense. When we say a component or a linear combination of components of $w$ is effective, it means that the loss function $F$ has a significant dependence on it. This can be analyzed through the eigen-decomposition of $\nabla^2 F(w)$ or its upper bound $A$ in (6). Let $(\lambda_i, v_i)$ be the eigenvalue-eigenvectors of $A$, where $\lambda_i$ are arranged in decreasing order. Then a small $\lambda_i$ indicates that $F$ has a weak dependence along the direction of $v_i$. For model to have a low effective dimension, there will be only constantly many $\lambda_i$ being significant, while the remaining eigenvalues in sum have negligible contribution to the overall loss function. We formally formulate this setting into the following assumption.

**Assumption 3.3.** The followings are true

1) $\text{tr}(A)$ with $A$ defined in Assumption 2.1 is bounded by an $\tilde{O}(1)$ constant.

2) the true parameter $w^*$ is bounded in each of $A$’s eigen-direction, in the sense that

$$
\|w^*\|_{A,S} := \max_i \{|\langle v_i, w^* \rangle|, i = 1, \ldots, p\} = \tilde{O}(1).
$$

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3) \( r^2, c_r \) defined in Theorem 2.3 are bounded by \( \tilde{O}(1) \) constants.

4) The initial values \( \mathbb{E}\|w_0\|^2 \) and \( \mathbb{E}G(w_0) \) grow polynomially with \( p \).

By Cauchy Schwartz, we have \( \|w^*\|_{A, S} \leq \|w^*\| \). So Assumption 3.3 condition 2) is weaker than Assumption 3.1 condition 2). In particular, it can include important cases where we can have upper and lower bound on each of \( w^* \)'s component, and \( A \) is known to be a diagonal matrix. These cases are not covered by Assumption 3.1. On the other hand, the spectrum profile of \( A \) will be required to choose the regularization parameter as shown in the following proposition.

**Proposition 3.4.** By the following inequalities,

\[
\|w^*\|_A^2 \leq \text{tr}(A)\|w^*\|_{A, S}^2, \quad \|w^*\|_{A,\lambda}^2 \leq \|w^*\|_{A, S}^2 \sum_{i=1}^p \lambda \land \lambda_i.
\]

Assumption 3.3 implies that \( \|w^*\|_A = \tilde{O}(1) \) and \( \|w^*\|_{A,\lambda}^2 = \tilde{O}(\sum_{i=1}^p \lambda \land \lambda_i) \). Moreover, under the conditions in Corollary 2.4 and Assumption 3.1, given any \( \epsilon > 0 \), if the eigenvalues of \( A \) follows,

1) Exponential decay: \( \lambda_i = e^{-ci} \) for some constant \( c > 0 \), and setting

\[
\lambda(\epsilon) = \tilde{O}\left(\frac{\epsilon}{|\log(\epsilon)|}\right), \quad \delta(\epsilon) = \tilde{O}\left(\frac{\epsilon^3}{|\log(\epsilon)|^2}\right), \quad \eta(\epsilon) = \tilde{O}\left(\frac{\epsilon^2}{|\log(\epsilon)|}\right), \quad N(\epsilon) = \tilde{\Omega}\left(\frac{|\log(\epsilon)|^3}{\epsilon^4}\right),
\]

we have \( \mathbb{E}[G(w_N)_{1\tau \geq N}] \leq 4\epsilon \).

2) Polynomial decay: \( \lambda_i = i^{-c} \) for some constant \( c > 0 \), and setting

\[
\lambda(\epsilon) = \tilde{O}\left(\frac{\epsilon^{c+1}}{\epsilon}\right), \quad \delta(\epsilon) = \tilde{O}\left(\frac{\epsilon^{3c+2}}{\epsilon}\right), \quad \eta(\epsilon) = \tilde{O}\left(\frac{\epsilon^{2c+1}}{\epsilon}\right), \quad N(\epsilon) = \tilde{\Omega}\left(\frac{|\log(\epsilon)|}{\epsilon^{3c+2}}\right),
\]

we have \( \mathbb{E}[G(w_N)_{1\tau \geq N}] \leq 4\epsilon \).

We remark that the parameter of the spectrum decay (e.g., the constant \( c \) in polynomial decay spectrum) is often assumed to be known for many functional data analysis problems (Hall and Horowitz, 2007; Cai and Hall, 2008). From Proposition 3.4, for both exponential decay and polynomial decay of the Hessian spectrum, the sample size \( N \) only depends on \( p \) in a poly-logarithmic factor.

**Remark 3.5.** It is interesting to compare our low effective dimension settings with the conditions used in Bartlett et al. (2019). For their main result, Theorem 4 in Bartlett et al.
to yield dimension-independent generalization error bound, three conditions (formulated in our notation) need to hold: 1) \( \|w^*\|^2 \) is bounded by a constant; 2) \( \text{tr}(A) \) is bounded by a constant; 3) the spectrum of \( A \) decays not so fast, so that for some \( k \), \( \sum_{i \geq k} \lambda_i \geq bN\lambda_k \) with some constant \( b \). In comparison, our Assumption 3.1 only requires conditions 1) and 2), but not the technical condition 3). Moreover, our result can also work under Assumption 3.3 where only \( \|w^*\|^2_{A,S} \) instead of \( \|w^*\|^2 \) needs to be bounded.

### 4 Overparameterization in Linear Regression

In general, overparameterization may lead to overfitting, but this sometimes can be avoided. Our main result Theorem 2.3 provides a general tool to understand why sometimes overfitting happens and sometimes does not. In this section, we will demonstrate how to apply our results on linear regression models in various high-dimensional settings. This section is technically straightforward, which is mainly used for pedagogical purpose. The discussions on more technically challenging cases for nonlinear and non-convex models are provided in the next section.

#### 4.1 Linear regression

First of all, we will find out the problem related parameters in Theorem 2.3 when applying to linear regression models. As in Section 2.2.1, we consider i.i.d. data points form \( \zeta_i = (x_i, y_i) \in \mathbb{R}^p \times \mathbb{R} \), where the response is generated by

\[
y_i = x_i^T w^* + \xi_i.
\]

In (20), \( w^* \in \mathbb{R}^p \) is the true model-parameter to be estimated. \( \xi_i \in \mathbb{R} \) are observation noise terms in the observation process, we assume they are i.i.d. with zero mean and variance \( \sigma^2 \). For simplicity, we assume that the data \( x_i \) are i.i.d. zero-mean Gaussian distributed, i.e., \( x_i \sim \mathcal{N}(0, \Sigma) \). As a remark, our proof also allows the non-Gaussian distribution with finite fourth moments.

The regression loss of parameter \( w \) on data \( \zeta_i \) is

\[
f(w, \zeta_i) = \frac{1}{2}(x_i^T w - y_i)^2.
\]

Plug (20) into (21) and taking expectation, we find the population loss function

\[
F(w) = \frac{1}{2}(w - w^*)^T \Sigma (w - w^*) + \frac{1}{2} \sigma^2.
\]

Now we show the problem related parameters in Theorem 2.3 can be set as below.
Proposition 4.1. For linear regression, Assumption 2.1 holds with \( A = \Sigma, \delta = 0, D = \mathbb{R}^p \). When \( w_0 = 0 \), \( \mathbb{E}G(w_0) = \frac{1}{2}\|w^*\|^2_\Sigma \). Moreover, the stochastic gradient variance bound in (8) holds with

\[
r^2 = 2\sigma^2 \text{tr}(\Sigma) + 12\text{tr}(\Sigma)\|w^*\|^2_\Sigma, \quad c_r = \frac{6}{\sigma^2} \max\{\|\Sigma\|, 1\}.
\]

As a consequence, by Proposition 3.2, Assumption 3.1 holds if \( \|w^*\|^2 \) and \( \text{tr}(\Sigma) \) are \( O(1) \) constants, and the necessary sample size \( N(\epsilon) \) in Corollary 2.4 is independent of \( p \). Similarly, by Proposition 3.4, Assumption 3.3 holds if \( \|w^*\|^2_{A,S} \) and \( \text{tr}(\Sigma) \) are \( \tilde{O}(1) \) constants, and the sample size \( N(\epsilon) \) depends on \( p \) only via a polynomial logarithmic factor.

### 4.2 High dimensional data with principle components

The low effective dimension settings in Section 3 naturally rise in many high dimensional problems. For example, in image processing or functional data analysis (see e.g. Ramsay and Silverman (2005); Hall and Horowitz (2007); Cai and Hall (2008); Fan et al. (2015); Xue and Yao (2018)), the data are in general assumed to take place in a Hilbert space \((H, \langle \cdot, \cdot \rangle)\) with potentially infinitely many orthonormal basis functions \( \{e_j, j = 1, 2, \ldots\} \). Each data can be written as

\[
x_i = \sum_{j=1}^\infty a_i^j e_j.
\]

Suppose \( a_i^j \) are independent Gaussian random variables with mean zero and variance \( \sigma_j^2 \). Note that \( \mathbb{E}\langle x_i, x_i \rangle = \sum_{j=1}^\infty \sigma_j^2 \). Therefore, for each data \( x_i \in H \), we assume that \( \sum_{j=1}^\infty \sigma_j^2 < \infty \) so that the norm of the data is bounded, which implicitly requires \( \sigma_j \) decaying to zero (Hall and Horowitz, 2007). Given the form of \( x_i \) in (23), the linear regression model takes the following form,

\[
y_i = \langle x_i, w^* \rangle + \xi_i,
\]

where \( w^* = \sum_{j=1}^\infty w^* e_j \). If we assume \( w^* \in H \), then \( \langle w^*, w^* \rangle = \sum_{j=1}^\infty (w^* e_j)^2 < \infty \).

When training this “infinite dimensional” linear regression model in (24), we would need a finite projection \( \mathcal{P}_p : H \mapsto \mathbb{R}^p \). When the basis functions are available, one natural choice of the projection is

\[
\mathcal{P}_p x_i = \mathcal{P}_p \left( \sum_{j=1}^\infty a_i^j e_j \right) := [a_i^1, \ldots, a_i^p]^T.
\]
Then the $p$-dimensional linear regression model is formulated as

$$y_i = (P_p x_i)^T w^*_p + \xi^p_i,$$

(25)

It is worthwhile noticing that the true infinite dimensional model in (24) is compatible with the finite dimensional model in (25), in the sense that

$$w^*_p = P_p w^* = [w^{*,1}, \ldots, w^{*,p}]^T, \quad \xi^p_i = \xi_i + \sum_{j=p+1}^{\infty} w^{*,j} a^j_i.$$

Since $a^j_i$ are independent Gaussian random variables, we have $\xi^p_i \sim N(0, \sigma^2_{\xi,p})$ with

$$\sigma^2_{\xi,p} := \sigma^2 + \sum_{j=p+1}^{\infty} \sigma^2_j (w^{*,j})^2 \leq \sigma^2 + \|w^*\|_{\Sigma,}^2, \quad \|w^*\|_{\Sigma,}^2 := \sum_{j=1}^{\infty} \sigma^2_j (w^{*,j})^2.$$

In the finite dimensional model (25), the data $P_p x_i$ has the population covariance matrix $\Sigma_p = \text{diag}(\sigma^2_1, \ldots, \sigma^2_p)$, whose trace is bounded by

$$\text{tr}(\Sigma_p) = \sum_{j=1}^{p} \sigma^2_j \leq \sum_{j=1}^{\infty} \sigma^2_j.$$

Therefore, by Proposition 4.1, the problem related parameters in Theorem 2.3 are

$$A_p = \Sigma_p, \quad \text{with} \quad \text{tr}(A_p) \leq \sum_{j=1}^{\infty} \sigma^2_j, \quad \|A_p\| = \sigma^2_1,$$

$$r^2_p = 2\sigma^2_{\xi,p} \text{tr}(\Sigma_p) + 12\text{tr}(\Sigma_p)\|w^*_p\|_{\Sigma_p}^2 \leq 2(\sigma^2 + 7\|w^*\|_{\Sigma,}^2) \sum_{j=1}^{\infty} \sigma^2_j,$$

(26)

$$c^2_{r,p} = \frac{6}{\sigma^2_{\xi,p}} \max\{1, \sigma^2_1\} \leq \frac{6}{\sigma^2} \max\{1, \sigma^2_1\}.$$

Moreover, if we use $w_0 = 0$, then $E\|w_0\|^2 = 0$ and

$$E G(w_0) = \frac{1}{2} \|w^*_p\|_{\Sigma,p}^2 \leq \|w^*\|_{\Sigma,}^2.$$

Note that

$$\|w^*\|_{\Sigma,}^2 = \sum_{j=1}^{\infty} \sigma^2_j (w^{*,j})^2 \leq \|w^*\|_{\infty}^2 \sum_{j=1}^{\infty} \sigma^2_j, \quad \|w^*\|_{\infty} := \max_{1 \leq j} |w^{*,j}|.$$

So as long as $\|w^*\|_{\infty}$ is finite, the upper bounds above are independent of dimension $p$.

When the true loading parameter $w^*$ is an element of $H$, $\|w^*_p\|^2 \leq \langle w^*, w^* \rangle < \infty$. Then we can check all items of Assumption 3.1 hold. So by Proposition 3.2, we know
the generalization error is dimension-independent. Moreover, this does not require any information of the spectrum decay profile.

More generally, we only need that each component of the true loading parameter $w^*$ is bounded, and $w^*$ does not need to be an element of $H$ itself. In particular, we note that

$$\|w_p^*\|_{\Sigma_p,S} = \max_{1 \leq j \leq p} |w^*_{j}| \leq \|w^*\|_\infty.$$  

Therefore, if $\|w^*\|_\infty$ is finite, Assumption 3.3 holds (but in general Assumption 3.1 does not). Then by Proposition 3.4, the generalization error can be dimension independent when we know the spectrum decay profile.

As a simple demonstration, we run some simulations of SGD on linear regression model (25) and present them in Figure 1. We run SGD on (25) with the sample size $N = 500$ and
the regularization parameter $\lambda = 0.01$. The covariance spectrum of predictors is set to be $\sigma^2_j = j^{-2}$ so that $\text{tr}(A_p)$ in (26) is a constant, and the true parameter is set to be $w^{*,j} = j^{-1}$ for $1 \leq j \leq p$ so that $\|w^*\|_\Sigma$ is bounded. The problem dimension ranges from $p = 100$ to $p = 1000$, which can be larger than the sample size. We use the final SGD output $w_{500}$ as the estimator, and compute the generalization error as in (12). We repeat this experiment 1000 times, and compute the mean and standard deviation. We plot the error bar plot in the upper left panel of Figure 1. As one can see, the generalization error does not increase as the dimension increases, even when $p \gg N$. As a comparison experiment, we run simulations with the same settings except for $\sigma^2_j \equiv 1$. We plot the generalization error in the upper right panel of Figure 1, which clearly shows the overfitting phenomenon.

The similar story repeats when the true parameter $w^{*,j} \equiv 1$ (i.e., the case when $\|w\|_\infty$ is bounded), where the plots are given by the lower panels in Figure 1. As one can see, the generalization error with the decaying spectrum still remains stable against the increase of dimension, and it does not change much from the previous setting when $w^*$ is an all-one vector. Meanwhile, overfitting with constant spectrum (i.e., $\sigma^2_j \equiv 1$) becomes stronger. This simple illustrative example justifies that the low effect dimension helps address the overfitting issue even when the dimension $p$ is much larger than the sample size $N$.

### 4.3 Overfitting with redundant features

Another interesting setting of overparameterization is to consider adding redundant predictors to an existing model. In this scenario, the true model is low dimensional with the true parameter $w^* \in \mathbb{R}^d$. Suppose we do not know the true model and collect additional features $z \in \mathbb{R}^{p-d}$, so that the overparameterized linear model is written as

$$y_i = x_i^T w^* + z_i^T u^* + \xi_i,$$

where $w^* \in \mathbb{R}^d$, $u^* = 0$, and $[x_i; z_i]$ is jointly Gaussian with mean zero and covariance

$$\Sigma_p = \begin{bmatrix} \Sigma_x & B \\ B^T & \Sigma_z \end{bmatrix}. $$

We assume that $\|\Sigma_z\| \leq \|\Sigma_x\|$ for the ease of discussion. Then $\Sigma_p$ being PSD implies that $\|B\| \leq \|\Sigma_x\|$. Since we do not impose any restriction on $B$ other than $\Sigma_p$ being PSD, our setting allows the possibility that some components of $z_i$ to be highly correlated or even identical with the ones of $x_i$. This in general leads to highly singular design matrices and unstable results.
We apply Proposition 4.1 and find $A_p = \Sigma_p$. By triangular inequality, for any vectors $x$ and $z$

$$\|\Sigma_p [x; z]\| = \|\Sigma_x x + B z; B^T x + \Sigma_z z\| \leq 2 \|\Sigma_x\| \|x; z\| \Rightarrow \|\Sigma_p\| \leq 2 \|\Sigma_x\|.$$

For simplicity, we also assume that we initialize with $[w_0; u_0] = 0$, so

$$\mathbb{E}G(w_0, u_0) = \frac{1}{2} \|w_0 - w^*\|_{\Sigma_x}^2 + \frac{1}{2} \|u_0\|_{\Sigma_z}^2 = \frac{1}{2} \|w^*\|_{\Sigma_x}^2.$$

Moreover, we have

$$c_{r,p} = \frac{6}{\sigma^2} \max\{\|\Sigma_p\|, 1\} \leq \frac{6}{\sigma^2} \max\{2\|\Sigma_x\|, 1\},$$

and

$$\|\frac{\Sigma_p}{\Sigma}\|_{\Sigma_p, S} = \|w^*\|_{\infty}, \quad \|\frac{\Sigma_p}{\Sigma}\|_2^2 = \|w^*\|_2^2.$$

These upper bounds are all independent of $p$, or the choice of $\Sigma_z$ and $B$.

Meanwhile,

$$r_p^2 = 2(\sigma^2 + 6\|w^*\|_{\Sigma_x}^2)(\text{tr}(\Sigma_x) + \text{tr}(\Sigma_z)), \quad \text{tr}(A_p) = \text{tr}(\Sigma_x) + \text{tr}(\Sigma_z). \quad (28)$$

Given these simple calculation, we find that the only problem related parameters that depend on $z$ are $r_p^2$ and $\text{tr}(A_p)$ in (28) through $\text{tr}(\Sigma_z)$. Therefore, our theory indicates there is a simple dichotomy on whether the model (27) will overfit.

If $\text{tr}(\Sigma_z)$ is bounded by a constant independent of $p$, Proposition 3.2 applies, which indicates the generalization error is also independent of the ambient dimension $p$. This can happen if we select data features in $z$ as PCA components. For example, suppose that the redundant data is in the form of $\sum_{j=1}^{\infty} a_j^1 e_j$ as in the setting of (23), and we collect the $p - d$ dimensional principle components as $z_i = [a_i^1, \ldots, a_i^{p-d}]^T$. Then $\text{tr}(\Sigma_z) = \sum_{j=1}^{p-d} \sigma_j^2 < \sum_{j=1}^{\infty} \sigma_j^2$, which is independent of $p$.

If $\text{tr}(\Sigma_z)$ grows with $p$, model (27) may overfit. For simplicity, we consider a special case where $\Sigma_z = I_{p-d}, B = 0$. In other words, the redundant features are independent with each other and the features of $x$. Then our derivation shows that $r_p^2 = O(p)$. This indicates the learning results may overfit.

To demonstrate this dichotomy, we simulate the SGD learning results and present their generalization error in Figure 2. In particular, we set $d = 5$ with true parameter $w^* = [1, 1, 1, 1, 1], \Sigma_x = I_5, \sigma^2 = 1$. We let $B = 0$, and choose first that $\Sigma_z$ to be diagonal with decaying entries $\frac{1}{j}$. We run SGD with 500 iterations and compute the generalization error of the final iterate. We repeat this 1000 times, and plot the error bar plot in the left
Figure 2: Generalization error bar plot with high dimensional redundant features for two different cases of $\Sigma_z$. The $x$-axis is the dimension $p$ and $y$-axis is the generalization error.

As we can see, the generalization error is stable against the increase of the dimension $p$. In comparison, if we use $\Sigma_z = I_{p-d}$, the learning results overfit, as we can see from the right panel of Figure 2.

5 Overparameterization for Nonlinear and Non-convex Models

In this section, we apply our main theorem and corollaries in Section 3 to several important nonlinear and non-convex statistical problems, such as logistic regression, M-estimator with Tukey’s biweight loss function and two-layer neural networks.

5.1 Logistic regression

We consider the logistic regression for binary classification with $N$ i.i.d. data $\zeta_i = (x_i, y_i)$. The binary response $y_i$ takes values within $\{-1, 1\}$ with probability

$$ P(y_i = y|x_i) = \frac{1}{1 + \exp(-yx_i^Tw^*)}, \quad y = \pm 1, $$

where $w^* \in \mathbb{R}^p$ is the true parameter to be estimated. We assume the predictors $x_i$ are i.i.d. with $\mathbb{E}x_ix_i^T = \Sigma$. For each data, we adopt the negative log-likelihood as the loss function

$$ f(w, \zeta_i) := \log(1 + \exp(-yx_i^Tw)), $$
and the corresponding population loss is given by
\[ F(w) = Ef(w, \zeta) = E \log(1 + \exp(-yx^Tw)). \]

The problem related parameters in Theorem 2.3 can be set by the following proposition.

**Proposition 5.1.** For logistic regression, Assumption 2.1 holds with \( A = \Sigma, \delta = 0, D = \mathbb{R}^p \). When \( w_0 = 0, EG(w_0) = \log 2 \). Moreover, the stochastic gradient variance bound in (8) holds with \( r^2 = \text{tr}(\Sigma), \ c_r = 0 \).

As a consequence, by Proposition 3.2, Assumption 3.1 holds if \( \|w^*\|^2 \) and \( \text{tr}(\Sigma) \) are \( O(1) \) constants, and the sample size \( N(\epsilon) \) in Corollary 2.4 is independent of \( p \). Similarly, by Proposition 3.4, Assumption 3.3 holds if \( \|w^*\|^2_{\bar{\Sigma},S} \) and \( \text{tr}(\Sigma) \) are \( \tilde{O}(1) \) constants, and the sample size \( N(\epsilon) \) depends on \( p \) only via a poly-logarithmic factor.

### 5.2 M-estimator with Tukey’s biweight loss function

In this example, we assume the data \( \zeta_i = (x_i, y_i) \) are generated from a linear model
\[ y_i = x_i^Tw^* + \xi_i. \tag{29} \]

We assume \( x_i \sim \mathcal{N}(0, \Sigma) \), and \( \xi_i \) are i.i.d. mean-zero noises with finite fourth moment. We adopt the non-convex Tukey’s biweight loss function as follows for the purpose of robust estimation
\[
\rho(u) = \begin{cases} 
\frac{c^2}{8} [1 - (1 - (u/c)^2)^3] & \text{if } |u| \leq c; \\
\frac{c^2}{8} & \text{if } |u| > c.
\end{cases}
\]

Then the individual data loss function and the population loss are given by,
\[ f(w, \zeta) = \rho(x^Tw - y) = \rho(x^T(w - w^*) - \xi), \quad F(w) = E\rho(x^T(w - w^*) - \xi). \]

**Proposition 5.2.** For the M-estimator with Tukey’s biweight loss in (29), the model true parameter \( w^* \) is a local minimum if and only if
\[ c_0 = E[(1 - (\xi/c)^2)(1 - 5(\xi/c)^2)1_{|\xi| \leq c}] > 0. \]

In that case, Assumption 2.1 holds with any \( \delta \geq 0, A = \Sigma \) and \( D = \{ w : \|w - w^*\|_\Sigma \leq \frac{c_0 + \delta}{16} \} \).
Moreover, the stochastic gradient variance bound in (8) holds with
\[ r^2 = \text{tr}(\Sigma), \quad c_r = 0. \]

Since \( G(w_0) \leq \frac{c^2}{6} \), by Proposition 3.2, Assumption 3.1 holds if \( \|w^*\|^2, \|w_0\|^2 \) and \( \text{tr}(\Sigma) \) are \( O(1) \) constants, and the sample size \( N(\epsilon) \) in Corollary 2.4 is independent of \( p \). Similarly, by Proposition 3.4, Assumption 3.3 holds if \( \|w_0\|_\infty, \|w^*\|^2_{\Sigma,S} \) and \( \text{tr}(\Sigma) \) are \( \tilde{O}(1) \) constants, and the sample size \( N(\epsilon) \) depends on \( p \) only via a polynomial logarithmic factor.

### 5.3 Two-layer neural network

In this example, we consider applying our result to two-layer neural networks (NN). We assume that every data point \( \zeta = (x, y) \) consists of a \( p \)-dimensional predictor \( x \in \mathcal{N}(0, \Sigma) \) and a univariate response \( y \in \mathbb{R} \). We assume that the response is generated by
\[ y = g(w, x) + \xi, \quad \mathbb{E}[\xi] = 0, \quad \mathbb{E}[\xi^2] = \sigma^2_0. \]
The function \( g \) takes the form of a two-layer NN:
\[ g(w, x) = c^T \psi(bx + a) = \sum_{i=1}^{k} c_i \psi(b_i^T x + a_i). \]  
\( (30) \)

In (30), \( a \) and \( c \) are \( k \)-dimensional vectors with \( a_i \) and \( c_i \) being their components. The notation \( b \) is a \( p \) by \( k \) matrix, and \( b_i \) denotes the \( i \)-th column of \( b \) with \( i = 1, \ldots, k \). We impose no restriction on \( k \) and it can depend on \( p \) in general. We denote all the parameters by \( w = [a; b_1, \ldots, b_k; c] \in \mathbb{R}^{(p+2)k} \). In (30), \( \psi \) denotes the activation function. Popular choices of \( \psi \) include the rectified linear unit (ReLU), sigmoid function, and the hyperbolic tangent. Here, we do not require \( \psi \) to take a specific form but only satisfy certain regularity assumptions for some constant \( C > 0 \),
\[ \psi(0) = 0, \quad |\dot{\psi}(x)| \leq C, \quad |\ddot{\psi}(x)| \leq C. \]  
\( (31) \)

It is easy to verify that hyperbolic tangent satisfies these requirements, and the sigmoid also satisfies these if we shift its center to zero. The condition \( \psi(0) = 0 \) is mainly for the ease of technical derivations. Although ReLU does not have continuous derivatives, one can find a smooth approximation to meet these requirements.

Since we consider the regression problem, the squared loss function is given by
\[ f(w, \zeta) = (y - g(w, x))^2 = (g(w^*, x) + \xi - g(w, x))^2. \]
We also introduce the following \((p + 2)k\) by \((p + 2)k\) block-diagonal matrix

\[
\Sigma^* = \text{diag}\{I_k, \Sigma, \Sigma, \ldots, \Sigma, I_k\}.
\]

This matrix introduces a high dimensional norm

\[
\|w\|_{\Sigma^*}^2 = w^T \Sigma^* w = \|a\|^2 + \sum_{i=1}^{k} \|b_i\|_{\Sigma}^2 + \|c\|^2.
\]

Recall that \(b_i\) is of dimension \(p\), its contribution to \(\|w\|_{\Sigma^*}^2\) is \(\|b_i\|_{\Sigma}^2\). By Proposition 3.4, \(\|b_i\|_{\Sigma}^2 \leq \text{tr}(\Sigma)\|b_i\|_{\Sigma,S}^2\), which can be independent of \(p\) under suitable conditions.

We are ready to show that the two-layer NN will not overfit in some overparameterized settings.

**Proposition 5.3.** Assume the activation function satisfies the condition in (31). With the two-layer NN defined in (30), Assumption 2.1 holds for any \(\delta \in (0, 1/4]\) with

\[
A = C_0(w^*)\Sigma^*, \quad \mathcal{D} = \{w : \|w - w^*\|_{\Sigma^*} \leq \delta C_1(w^*)\|w^*\|_{\Sigma^*}\}.
\]

For any \(w_0 \in \mathcal{D}\), \(G(w_0) \leq C_2(w^*)\|w^*\|_{\Sigma^*}^4\). Moreover, the stochastic gradient variance bound in (8) holds with

\[
r^2 = C_3(w^*), c_r = 0.
\]

The exact values of the problem related parameters are given by

\[
C_0(w^*) = 7C^2\|w^*\|_{\Sigma^*}^2, \quad C_1(w^*) = \frac{2}{9\sqrt{2}(2\|w^*\|_{\Sigma^*} + 1)}, \quad C_2(w^*) = C^2\|w^*\|_{\Sigma^*}^4,
\]

\[
C_3(w^*) = 8\sqrt{3}(1 + \text{tr}(\Sigma))C^2\|w^*\|_{\Sigma^*}^2, \quad C^2\|w^*\|_{\Sigma^*}^4 + \sigma_0^2).
\]

As a consequence, by Proposition 3.2, Assumption 3.1 holds if

\[
\max \left\{\|a^*\|^2 + \sum_{i=1}^{k} \|b_i^*\|^2 + \|c^*\|^2, \text{tr}(\Sigma), \|w_0\|_{\infty}\right\} = O(1),
\]

and the sample size \(N(\epsilon)\) is independent of \(p\). Similarly, by Proposition 3.4, Assumption 3.3 holds if

\[
\max \{k, \text{tr}(\Sigma), \|w_0\|_{\infty}, |a_i|, |c_i|, |v_j^T b_i|, i = 1, \ldots, k, j = 1, \ldots, p\} = \tilde{O}(1),
\]

where \(v_j\) are the eigenvectors of \(\Sigma\). Then the sample size \(N(\epsilon)\) depends on \(p\) only via a polynomial logarithmic factor.

It is worthwhile mentioning that a similar version of Condition (32) can also be found in Neyshabur et al. (2019). In particular, Neyshabur et al. (2019) assumed that
\( \|c^*\|^2, \sum_{i=1}^{k} \|b_i^*\|^2 \) to be \( O(1) \) while the parameter \( a \) is set to be 0. There is no variance assumption of \( x_i \) in Neyshabur et al. (2019), but it is assumed that \( \mathbb{E}\|x_i\|^2 \) is \( O(1) \), which is equivalent to requesting \( \text{tr}(\Sigma) = O(1) \) in our setting.

When the width of the hidden layer \( k \) is a fixed constant, the second condition (33) is in general less restrictive than the first one (32), since it allows \( \|b_i^*\| \) to grow with \( p \). When \( k \) grows with \( p \), only the first condition is applicable, and it requires that \( \|a^*\|^2 + \sum_{i=1}^{k} \|b_i^*\|^2 + \|c^*\|^2 \) is bounded by \( O(1) \). In other words, we need either \( k = O(1) \) or the true parameters to be bounded by \( O(1) \) to prevent overfitting. This can also be understood intuitively. Note that the output of the two-layer NN in (30) is a sum of \( k \) objects. Therefore, if \( k \) grows with \( p \), the output of (30) will diverge, which contradicts the common assumption that \( g \) is bounded (see, e.g., Li and Liang (2018); Neyshabur et al. (2019); Allen-Zhu et al. (2019)). Our result is consistent with the results in Allen-Zhu et al. (2019) in the sense that Allen-Zhu et al. (2019) also showed that the sample size needs to grow with \( k \). It is also possible to rescale \( g \) by multiplying (30) with a factor \( \frac{1}{k} \) or \( \frac{1}{\sqrt{k}} \), as done by Arora et al. (2019), so the generalization error is independent of the parameter \( k \).

6 Conclusions and Future Works

One classical canon of statistics is that high dimensional models are prone to overfitting when the data sample size is not sufficiently large. However, many existing models such as neural networks (NN) exhibit stable generalization error despite that they are over-parameterized. This paper developed an analysis framework of generalization errors for high dimensional regularized online learning. The error bound can be interpreted as a bias-variance tradeoff through a simplified stochastic approximation. This result indicates that over-parameterization does not lead to overfitting if the model has a low effective dimension. We demonstrated how to apply this framework on various models such as linear regression, logistic regression, \( M \)-estimator with Tukey’s biweight loss, and two-layer NN.

There are a few future directions. First, the generalization bound in Theorem 2.3 only applies when the SGD iterates stay in a local region \( D \) near the true parameter \( w^* \). Although it is a common assumption when analyzing non-convex models, this assumption might be difficult to check in practice. To address this challenge, a simple data splitting can serve as a remedy. In practice, we may split the data into two parts and run SGD through the first half. If \( w_{N/2} \) gives us some rough ideas on how to construct \( D \), we can use \( w_{N/2} \) as an initialization and run SGD on the second half of data. In the process,
we can check whether the iterates stay in $\mathcal{D}$, and then Theorem 2.3 provides us an upper bound for the generalization error of $w_N$. Of course, it would be an interesting future work on how to construct $\mathcal{D}$ based on the SGD solution from the first half of the data.

Second, our framework indicates that the ambient model dimension itself may not be a good indicator of model complexity, especially in overparameterized settings. The quantity that characterizes the data variability may lead to new information criterion for model selection in the overparameterized setting. Such results may extend the classical criteria such as the AIC and BIC.

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A Proof of the Main Results in Section 2.2

A.1 Preliminaries

Lemma A.1. For any vector $v \in \mathbb{R}^p$ and PSD matrix $A \in \mathbb{R}^{p \times p}$, the following hold

1) For any $-\delta A \preceq B \preceq A$, let $B = V \Lambda V^T$ be the eigenvalue decomposition of $B$, and denote $|\Lambda|$ as taking absolute value on each element of the diagonal matrix $\Lambda$. Denote $|B| = V |\Lambda| V^T$. Then for any vectors $v$ and $w$, $a > 0$

$$2\langle v, Bw \rangle \leq a \langle v, |B|v \rangle + \frac{1}{a} \langle w, |B|w \rangle \leq a(1 + \delta)\|v\|_A^2 + \frac{1 + \delta}{a} \|w\|_A^2.$$ 

2) For any $-\delta A \preceq B \preceq A$, and any vectors $u$ and $v$, $a > 0$

$$2|\langle u, Bv \rangle| \leq a \langle u, Bu \rangle + 2a\delta \|u\|_A^2 + \frac{1 + 2\delta}{a} \|v\|_A^2.$$ 

Proof. Claim 1). Let $(l_i, u_i)$ be the eigenvalue-eigenvectors of $B$. Assume also that $v = \sum_{i=1}^p a_i u_i$, $w = \sum_{i=1}^p b_i u_i$.

Then by Young's inequality

$$2\langle v, Bw \rangle = 2 \sum_{i=1}^p l_i a_i b_i \leq a \sum_{i=1}^p |l_i| |a_i|^2 + \frac{1}{a} \sum_{i=1}^p |l_i| |b_i|^2 = a \langle v, |B|v \rangle + \frac{1}{a} \langle w, |B|w \rangle.$$ 

Next, we denote the positive part of $\Lambda$ as $\Lambda_+$ and the negative part as $\Lambda_-$, so that

$$\Lambda = \Lambda_+ + \Lambda_- , \quad |\Lambda| = \Lambda_+ - \Lambda_-, \quad \Lambda_- \preceq 0 \preceq \Lambda_+.$$ 

Then by checking eigen-space with nonnegative eigenvalues, $B \preceq A$ indicates that $V \Lambda_+ V^T \preceq A$. Likewise, we have $-\delta A \preceq V \Lambda_- V^T$. In combination, we have

$$|B| = V \Lambda_+ V^T - V \Lambda_- V^T \preceq (1 + \delta)A.$$ 

Therefore

$$a \langle v, |B|v \rangle + \frac{1}{a} \langle w, |B|w \rangle \leq (1 + \delta)a \|v\|_A^2 + \frac{1 + \delta}{a} \|w\|_A^2.$$ 

For claim 2), denote $B_\delta = B + \delta A \succeq 0$. Then

$$2|\langle u, Bv \rangle| \leq 2|\langle u, B_\delta v \rangle| + 2\delta|\langle u, Av \rangle|$$

$$\leq a \|u\|_{B_\delta}^2 + \frac{1}{a} \|v\|_{B_\delta}^2 + a\delta \|u\|_A^2 + \frac{\delta}{a} \|v\|_A^2$$

$$= a \langle u, Bu \rangle + a\delta \|u\|_A^2 + \frac{1}{a} \langle v, Bv \rangle + \frac{\delta}{a} \|v\|_A^2 + a\delta \|u\|_A^2 + \frac{\delta}{a} \|v\|_A^2$$

$$\leq a \langle u, Bu \rangle + 2a\delta \|u\|_A^2 + \frac{1 + 2\delta}{a} \|v\|_A^2.$$ 

$\square$

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A.2 Proof of the main results

Proof of Theorem 2.3. **Step 1:** we build a bound for \(\|w_n\|^2\). We rewrite SGD update as

\[
w_{n+1} = w_n - \eta \nabla f_{\lambda}(w_n, \zeta_n) = w_n - \eta \nabla F_{\lambda}(w_n) + \eta \xi_n, \tag{34}
\]

where

\[
\xi_n = \nabla F_{\lambda}(w_n) - \nabla f_{\lambda}(w_n, \zeta_n) = \nabla F(w_n) - \nabla f(w_n, \zeta_n).
\]

Let \(\mathcal{F}_n\) be the \(\sigma\)-algebra generated by \(\{w_{i+1}, \zeta_i, i = 1, \ldots, n-1\}\). We use \(\mathbb{E}_n(\cdot)\) to denote the conditional expectation \(\mathbb{E}(\cdot|\mathcal{F}_n)\). Then \(\xi_n\) is a martingale sequence since \(\mathbb{E}_n \xi_n = 0\).

From (34), we find

\[
\|w_{n+1}\|^2 = \|w_n\|^2 - 2\eta \langle w_n, \nabla F_{\lambda}(w_n) - \xi_n \rangle + \eta^2 \|\nabla F_{\lambda}(w_n) - \xi_n\|^2. \tag{35}
\]

To continue, we try to find a bound of \(-2\eta \langle w_n, \nabla F_{\lambda}(w_n) \rangle\). We define

\[
B_n := \int_0^1 \nabla^2 F(sw_n + (1-s)w^*) ds,
\]

and apply fundamental theorem of calculus on \(\nabla F\). Note that \(\nabla F(w^*) = 0\), we obtain

\[
\nabla F(w_n) = \nabla F(w^*) + \int_0^1 \nabla^2 F(sw_n + (1-s)w^*)(w_n - w^*) ds = B_n (w_n - w^*), \tag{36}
\]

Note that \(-\delta A \preceq B_n \preceq A\). We have

\[
\langle -w_n, \nabla F(w_n) \rangle = -\langle w_n, B_n(w_n - w^*) \rangle = -\langle w_n, B_n w_n \rangle + \langle w_n, B_n w^* \rangle
\]

\[
\leq \delta \|w_n\|^2 + \frac{(1 + \delta)\lambda}{4\|A\|} \|w_n\|^2_A + \frac{(1 + \delta)\|A\|}{\lambda} \|w^*\|^2_A \text{ by Lemma A.1 1)}
\]

\[
\leq \left(\frac{1 + \delta A}{\lambda} + \delta \|A\|\right) \|w_n\|^2 + \frac{1.5\|A\|}{\lambda} \|w^*\|^2_A \text{ since } \delta \leq \frac{1}{2} \text{ and } \|w_n\|^2_A \leq \|A\| \|w_n\|^2.
\]

Recall that \(\delta \|A\| \leq \frac{A}{8}\), we have \(\left(\frac{1 + \delta A}{\lambda} + \delta \|A\|\right) \leq \frac{A}{2}\). Using these bounds, we find

\[
-2\eta \langle w_n, \nabla F_{\lambda}(w_n) \rangle = -2\eta \langle w_n, \nabla F(w_n) + \lambda w_n \rangle
\]

\[
= -2\lambda \eta \|w_n\|^2 + 2\eta \langle -w_n, \nabla F(w_n) \rangle \tag{37}
\]

\[
\leq -\lambda \eta \|w_n\|^2 + \eta \frac{3\|A\|}{\lambda} \|w^*\|^2_A.
\]

If \(B_n = QAQ^T\) is the eigendecomposition of \(B_n\), let \(|B_n| = Q|\Lambda|Q^T\), where \(|\Lambda|\) takes absolute value on each element of the diagonal matrix \(\Lambda\). From the proof of Lemma A.1
we derive how does the generalization error evolve. According to Taylor’s ex-

\[ \|\nabla F(w_n)\|^2 = \|B_n(w^* - w_n)\|^2 \leq 2\|B_nw^*\|^2 + 2\|B_nw_n\|^2 \]
\[ \leq 2(w^*)^T[B_n^{1/2}B_n]^{1/2}w^* + 2\|A\|^2\|w_n\|^2 \]
\[ \leq 2(1 + \delta)\|A\|\|B_n\|^{1/2}w^*\|^2 + 2\|A\|^2\|w_n\|^2 \]
\[ \leq 4\|A\|\|w^*\|^2 + 2\|A\|^2\|w_n\|^2. \]  

(38)

Recall that \( w_n \) is \( \mathcal{F}_n \)-measurable and \( \mathbb{E}_n\xi_n = 0 \), we have \( \mathbb{E}_n(w_n, \xi_n) = 0 \). So plugging (37) and (38) into (35), we then have

\[ \mathbb{E}_n[\|w_{n+1}\|^2] = \mathbb{E}_n[\|w_n\|^2 - 2\eta\langle w_n, \nabla F_{\lambda}(w_n) \rangle + \eta^2\|\nabla F(w_n) + \lambda w_n - \xi_n\|^2] \]
\[ \leq \mathbb{E}_n[\|w_n\|^2 - 2\eta\langle w_n, \nabla F_{\lambda}(w_n) \rangle + 3\eta^2(\|\nabla F(w_n)\|^2 + \|\xi_n\|^2 + \lambda^2\|w_n\|^2)) \]
by Cauchy Schwarz ineq.,
\[ \leq \|w_n\|^2 - \lambda\|w_n\|^2 + 6\eta^2\|A\|^2\|w_n\|^2 + 3\eta^2\lambda^2\|w_n\|^2 \]
\[ + \left( \frac{3\|A\|\eta}{\lambda} + 12\|A\|\eta^2 \right) \|w^*\|^2_A + 3\eta^2r^2(1 + c_r\|w_n\|^2) \] by (37) and (38).

Under the condition
\[ \eta \leq \frac{\lambda}{12\|A\|^2 + 6\lambda^2 + 6c_r\lambda^2}, \]
and since \( 0 \leq 1_{r\geq n+1} \leq 1_{r\geq n} \), we have
\[ \mathbb{E}[1_{r\geq n+1}\|w_{n+1}\|^2] \leq \mathbb{E}[1_{r\geq n}\|w_{n+1}\|^2] = \mathbb{E}[1_{r\geq n}\mathbb{E}_n[\|w_{n+1}\|^2]] \leq (1 - \frac{\lambda\eta}{2})\mathbb{E}[1_{r\geq n}\|w_n\|^2] + \eta M_w, \]
where
\[ M_w := \left( \frac{3\|A\|}{\lambda} + 12\|A\|\eta \right) \|w^*\|^2_A + 3\eta^2 \]
\[ = \frac{3\|A\|}{\lambda}\|w^*\|^2_A + \eta \left( 12\|A\|\|w^*\|^2_A + 3\eta^2 \right). \]

Then iterating the inequality above gives us
\[ \mathbb{E}[\|w_n\|^21_{r\geq n}] \leq \left( 1 - \frac{\lambda\eta}{2} \right)^n \mathbb{E}[\|w_0\|^2 + \frac{2}{\lambda} M_w. \]  

(39)

Step 2: we derive how does the generalization error evolve. According to Taylor’s ex-
pansion and (34), we know that there exists a $v_n$, such that

$$F(w_{n+1}) = F(w_n) - \eta \|
abla F(w_n)\|^2 - \eta \lambda \nabla F(w_n)^T w_n + \eta \xi_n^T \nabla F(w_n)$$

$$+ \frac{\eta^2}{2} (\nabla F(w_n) + \lambda w_n - \xi_n)^T \nabla^2 F(w_n)(\nabla F(w_n) + \lambda w_n - \xi_n)$$

$$\leq F(w_n) - \eta \|
abla F(w_n)\|^2 - \eta \lambda \nabla F(w_n)^T w_n + \eta \xi_n^T \nabla F(w_n)$$

$$+ \frac{\eta^2}{2} (\nabla F(w_n) + \lambda w_n - \xi_n)^T A(\nabla F(w_n) + \lambda w_n - \xi_n)$$

$$\leq F(w_n) - \eta \|
abla F(w_n)\|^2 - \eta \lambda \nabla F(w_n)^T (w_n - w^*) + \eta \lambda |\nabla F(w_n)^T w^*|$$

$$+ \eta \xi_n^T \nabla F(w_n) + \frac{3\eta^2}{2} \|A||\nabla F(w_n)\|^2 + \lambda^2 \|w_n\|^2 + \|\xi_n\|^2).$$

**Step 3:** we bound each terms in (40) through interpolations. We observe that

$$|\lambda \nabla F(w_n)^T w^*| \leq |\lambda \nabla F(w_n)^T w_\perp^*| + |\lambda \nabla F(w_n)^T w_\lambda^*|,$$  

where $w^* = w_\perp^* + w_\lambda^*$ is the decomposition introduced in Definition 2.2. Next

$$|\lambda \nabla F(w_n)^T w_\lambda^*| \leq \frac{1}{2} \|
abla F(w_n)\|^2 + \frac{1}{2} \lambda^2 \|w_\lambda^*\|^2.$$  

(42)

Recall $\nabla F(w_n) = B_n (w_n - w^*)$ in (36), and by Lemma A.1 claim 2), we further have

$$|\lambda \nabla F(w_n)^T w_\perp^*| \leq \frac{1}{2} \lambda (w_n - w^*)^T B_n (w_n - w^*)$$

$$+ \delta \lambda \|w_n - w^*\|^2_A + \frac{1 + 2\delta}{2} \lambda w_\perp^* A w_\perp^*.$$  

(43)

Plugging (42), (43) into (41), apply the result to (40)

$$F(w_{n+1}) \leq F(w_n) - \frac{1}{2} \eta \|
abla F(w_n)\|^2 - \frac{1}{2} \lambda \eta \nabla F(w_n)^T (w_n - w^*) + \frac{1 + 2\delta}{2} \lambda \eta \|w^*\|_A^2$$

$$+ \delta \lambda \eta \|w_n - w^*\|^2_A + \eta \xi_n^T \nabla F(w_n) + \frac{3\eta^2}{2} \|A||\nabla F(w_n)\|^2 + \lambda \|w_n\|^2 + \|\xi_n\|^2).$$

(Recall that $\eta \leq 1$, $\delta < \frac{1}{2}$ and $\eta/2 - 3\eta^2 \|A\|^2/2 \geq 0)$

$$\leq F(w_n) - \frac{1}{2} \lambda \eta \nabla F(w_n)^T (w_n - w^*) + \lambda \eta \|w^*\|_A^2$$

$$+ \delta \lambda \eta \|w_n - w^*\|^2_A + \eta \xi_n^T \nabla F(w_n) + \frac{3\eta^2}{2} \|A||\lambda \|w_n\|^2 + \|\xi_n\|^2).$$  

(44)

To continue, Recall $G(w_n) = F(w_n) - F(w^*)$. Apply fundamental theorem of calculus to
Namely, we have

$$G(w_n) = F(w_n) - F(w^*)$$

$$= \left[ \int_0^1 \nabla F(sw_n + (1-s)w^*) ds \right]^T(w_n - w^*)$$

$$= \left[ \int_0^1 \nabla F(w^*) + \int_0^s \nabla^2 F(tw_n + (1-t)w^*)(w_n - w^*) dt \right] ds$$

$$= (w_n - w^*)^T \left[ \int_0^1 (1-s) \nabla^2 F(sw_n + (1-s)w^*) ds \right] (w_n - w^*)$$

with

$$A_n = \int_0^1 (1-s) \nabla^2 F(sw_n + (1-s)w^*) ds.$$ 

Under Assumption 3.1, namely $0 \leq \nabla^2 F(w_n) + \delta A \leq A + \delta A$, we observe that

$$\frac{1}{2} \delta A + A_n \leq \frac{1}{2} \delta A + \int_0^1 (1-s) \nabla^2 F(sw_n + (1-s)w^*) ds$$

$$= \int_0^1 (1-s)(\nabla^2 F(sw_n + (1-s)w^*) + \delta A) ds$$

$$\leq \int_0^1 (\nabla^2 F(sw_n + (1-s)w^*) + \delta A) ds = B_n + \delta A \leq (1 + \delta)A.$$ 

Namely, we have

$$A_n \leq B_n + \frac{1}{2} \delta A \leq (1 + \frac{\delta}{2})A.$$ 

Thus

$$G(w_n) - \frac{1}{2} \delta \|w_n - w^*\|_A^2 \leq (w_n - w^*)^T B_n (w_n - w^*) = \nabla F(w_n)^T (w_n - w^*)$$

Plug this into (44), together with $\|w_n - w^*\|_A^2 \leq 2\|w_n\|_A^2 + 2\|w^*\|_A^2 \leq 2\|A\|\|w_n\|^2 + 2\|w^*\|_A^2$, we have

$$G(w_{n+1}) \leq G(w_n) - \frac{1}{2} \eta \lambda G(w_n) + \eta \xi_n^T \nabla F(w_n)$$

$$+ \frac{5}{4} \delta \eta \lambda \|w_n - w^*\|_A^2 + \lambda \eta \|w^*\|_A^2 + \lambda \eta \|w_n\|^2 + 3\eta \lambda \|A\| \|w_n\|^2 + \eta \xi_n^2.$$ 

$$\leq G(w_n) - \frac{1}{2} \eta \lambda G(w_n) + \eta \xi_n^T \nabla F(w_n)$$

$$+ \lambda \eta (\|w^*\|_A^2 + \frac{5}{2} \delta \|w^*\|_A^2) + \left( \frac{3\eta^2 \lambda^2}{2} + \frac{5}{2} \lambda \eta \delta \right) \|A\| \|w_n\|^2 + \lambda \eta \|w_n\|^2 + 3\eta \lambda \|A\| \|\xi_n\|^2.$$
Step 4: summarizing arguments. Using $0 \leq 1_{r \geq n+1} \leq 1_{r \geq n} \leq 1$ and taking conditional expectation for both sides, since $E_n \xi_n^T \nabla F(w_n) \equiv 0$, we have

$$E[G(w_{n+1})1_{r \geq n+1}] \leq E[G(w_{n+1})1_{r \geq n}] = E[1_{r \geq n} E_n[G(w_{n+1})]]$$

$$\leq (1 - \frac{1}{4} \eta^2 \lambda^2) E[G(w_n)1_{r \geq n}] + \left( \frac{3\eta^2 \lambda^2}{2} + \frac{5}{2} \lambda \eta \delta \right) \|A\| \|E[\|w_n\|^2 1_{r \geq n}]\| r^2$$

Because $\eta \leq \frac{\lambda}{\|w_n\|^2}$

$$\leq (1 - \frac{1}{4} \eta^2 \lambda^2) E[G(w_n)1_{r \geq n}] + \left( \frac{3\eta^2 \lambda^2}{2} + \frac{5}{2} \lambda \eta \delta \right) \|A\| \|E[\|w_n\|^2 1_{r \geq n}]\| r^2$$

Since $\eta \lambda \leq 1$, we have $0 \leq 1 - \frac{1}{4} \lambda \eta \leq \exp(-\frac{1}{4} \lambda \eta)$, then iterating above results gives us

$$E[G(w_{n+1})1_{r \geq n}] \leq \exp(-\frac{1}{4} \eta \lambda) E[G(w_0)] + 4 \|w^*\|^2_{A,\lambda} + 10 \delta \|w^*\|^2_A + \frac{6 \eta \|A\|}{\lambda} r^2$$

Applying (39), together with $\lambda \leq 1$, $\eta \leq 1$, $\delta \leq \frac{1}{2}$ and $1 - \frac{1}{4} \lambda \eta \leq \exp(-\frac{1}{4} \lambda \eta)$, we obtain

$$E[G(w_{n+1})1_{r \geq n}]$$

$$\leq \exp(-\frac{1}{4} \eta \lambda) E[G(w_0)] + 4 \|w^*\|^2_{A,\lambda} + 10 \delta \|w^*\|^2_A + \frac{6 \eta \|A\|}{\lambda} r^2$$

$$+ \left( \frac{3\eta^2 \lambda^2}{2} + \frac{5}{2} \lambda \eta \delta \right) \|A\| \sum_{i=0}^{n} \left( (1 - \frac{1}{4} \lambda \eta)^n - (1 - \frac{1}{4} \lambda \eta)^{n-i} \frac{2}{\lambda} M \right)$$

$$\leq \exp(-\frac{1}{4} \eta \lambda) E[G(w_0)] + 4n \|A\| \|w_0\|^2 + 4 \|w^*\|^2_{A,\lambda} + 10 \delta \|w^*\|^2_A + \frac{6 \eta \|A\|}{\lambda} r^2$$

$$+ \left( \frac{12 \lambda \eta + 20 \delta}{\lambda} \right) \|A\| \left( \frac{3 \|A\|}{\lambda} \|w^*\|^2_A + \eta \left( 12 \|A\| \|w^*\|^2_A + 3r^2 \right) \right)$$

$$\leq 4 \|w^*\|^2_{A,\lambda} + 10 \delta \|w^*\|^2_A + \frac{C_2 \delta}{\lambda^2} + \frac{C_1 \eta}{\lambda} + \exp(-\frac{1}{4} \lambda \eta) E[G(w_0)] + 4n \|A\| \|w_0\|^2,$$

with (Note $\lambda \leq 1$, $\eta \leq 1$ based on our condition)

$$C_1 = 6 \|A\| \left( r^2 + 30 \|A\| \|w^*\|^2_A \right),$$

$$C_2 = 60 \|A\| \left( r^2 + 5 \|A\| \|w^*\|^2_A \right).$$
Proof of Corollary 2.4. By Theorem 2.3, \( \mathbb{E}[G(w_N)1_{r \geq N}] \leq 4\epsilon \) holds if we choose \( \lambda, \eta, N, \delta \) such that the following hold

\[
4\|w^*\|^2_{A,\lambda} \leq \epsilon, \quad \frac{C_1 \eta}{\lambda} \leq \epsilon, \quad \frac{C_2 \delta}{\lambda^2} \leq \epsilon, \quad \exp(-\frac{1}{4} \lambda N \eta) \mathbb{E}[G(w_0)] + 4N \|A\| \|w_0\|^2 \leq \epsilon.
\]

We first choose \( \lambda(\epsilon) \) such that

\[
4\|w^*\|^2_{A,\lambda(\epsilon)} < \epsilon.
\]

The condition on \( \eta(\epsilon) \) ensures that \( \frac{C_1 \eta}{\lambda} \leq \epsilon \). For \( \delta(\epsilon) \), solving \( \frac{C_2 \delta}{\lambda^2} \leq \epsilon \) gives us its requirement. With chosen \( \lambda(\epsilon), \eta(\epsilon) \), the scale of \( N(\epsilon) \) is obtained by solving

\[
\exp(-\frac{1}{4} \lambda N \eta) \mathbb{E}[G(w_0)] + 4N \|A\| \mathbb{E}[\|w_0\|^2] \leq \epsilon.
\]

which is derived from \( \exp(-x)x \leq 2 \exp(-\frac{1}{2}x) \), since by Taylor expansion \( x \leq 2 \exp(\frac{1}{2}x) \).

\[\Box\]

B Proof for Results in Low Effective Dimension in Section 3.2

Proof of Proposition 3.2. According to the definition in (7), we have

\[
\|w^*\|^2_A = w^*^T A w^* \leq \|A\| \|w^*\|^2, \quad \|w^*\|^2_{A,\lambda} = \lambda \|w^*_\lambda\|^2 + w^*_\perp^T A w^*_\perp \leq \lambda \|w^*_\lambda\|^2 + \lambda \|w^*_\perp\|^2 = \lambda \|w^*\|^2.
\]

Since \( \|w^*\| \) is \( O(1) \), from above results, taking \( \lambda(\epsilon) = \frac{\epsilon}{4\|w^*\|^2} = O(\epsilon) \) ensures

\[
\|w^*\|^2_{A,\lambda(\epsilon)} \leq \frac{\epsilon}{4}.
\]

Moreover, the constants \( C_1 \) and \( C_2 \) are also \( O(1) \). So it suffices to apply Corollary 2.4 to obtain the remaining conclusions.

\[\Box\]

Proof of Proposition 3.4. Recall that \( (\lambda_i, v_i) \), for \( i = 1, ..., p \), are the eigenvalue-eigenvectors of \( A \) with \( \lambda_i \) decreasingly sorted. Therefore, we have

\[
\|w^*\|^2_A = w^*^T A w^* = \sum_{i=1}^p \lambda_i (w^*, v_i)^2 \leq \|w^*\|^2_{A,\text{str}}(A),
\]

\[
\|w^*\|^2_{A,\lambda} = \sum_{i=1}^p \lambda_i \wedge (w^*, v_i)^2 \leq \|w^*\|^2_{A,\text{str}} \sum_{i=1}^p \lambda_i \wedge \lambda.
\]

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For an exponential spectrum, given any $k$ and $p$,

$$\sum_{i=k+1}^{p} \lambda_i = \sum_{i=k+1}^{p} e^{-ci} = \frac{e^{-(k+1)c}(1 - e^{(k-p)c})}{1 - e^{-c}} \leq \frac{1}{e^{kc}(e-1)}.$$  

Thus, to make $\sum_{i=k+1}^{p} \lambda_i \leq \frac{8}{8} \|w^*\|_{A,S}^2$, it is sufficient for us to take $k \geq \frac{1}{e} \log \left( \frac{8}{8} \|w^*\|_{A,S}^2 \right)$. And to make $k \lambda = \frac{8}{8} \|w^*\|_{A,S}^2$, we take $\lambda = \frac{8}{8} \|w^*\|_{A,S}^2 = \widetilde{O}(\frac{c}{\log \epsilon})$. By these choices, we have

$$\|w^*\|_{A,\lambda}^2 \leq \sum_{i=1}^{p} \lambda_i \|w^*\|_{A,S}^2 \leq \frac{\epsilon}{4}.$$  

Next, we find that $\|A\| \leq \text{tr}(A) = \widetilde{O}(1)$, so $C_1 = \widetilde{O}(1), C_2 = \widetilde{O}(1)$. We implement Corollary 2.4 and find

$$\delta(\epsilon) = \widetilde{O} \left( \frac{\epsilon^3}{\log \epsilon^2} \right), \quad \eta(\epsilon) = \widetilde{O} \left( \frac{\epsilon^2}{|\log \epsilon|} \right), \quad N(\epsilon) = \widetilde{\Omega} \left( \frac{|\log \epsilon|^3}{\epsilon^3} \right).$$  

For a polynomial spectrum, the derivation is similar. Given any $k$ and $p$,

$$\sum_{i=k+1}^{p} \lambda_i = \sum_{i=k+1}^{p} i^{-(1+c)} \leq \sum_{i=k+1}^{p} \int_{i-1}^{i} \frac{1}{x^{1+c}} \, dx = \sum_{i=k+1}^{p} \frac{1}{c} x^{-c} \bigg|_{i-1}^{i} = \frac{1}{c} (k^{-c} - p^{-c}) \leq \frac{1}{ck^c}.$$  

Thus to make $\|w^*\|_{A,S}^2 \sum_{i=k+1}^{p} \lambda_i \leq \frac{1}{8} \epsilon$, that is $\sum_{i=k+1}^{p} \lambda_i = \frac{8}{8} \|w^*\|_{A,S}^2$, we take $k \geq (\frac{8}{8} \|w^*\|_{A,S}^2)^{1/c}$. Next, we take $\lambda(\epsilon) = \frac{\epsilon}{8 \|w^*\|_{A,S}^2 k} = \widetilde{O} \left( \frac{\epsilon^{1+k}}{\epsilon^k} \right)$. This leads to $\|w^*\|_{A,\lambda}^2 \leq \epsilon/4$. Again we find that $C_1 = \widetilde{O}(1), C_2 = \widetilde{O}(1)$. The order of $\delta(\epsilon), \eta(\epsilon)$ and $N(\epsilon)$ can be derived by Corollary 2.4. 

\section{Proofs of Results for Overparameterization in Statistical Models}

\subsection{Linear regression}

\textit{Proof of Proposition 4.1.} It is straightforward to find the gradient and Hessian of $F$ as:

$$\nabla F(w) = \Sigma (w - w^*), \quad \nabla^2 F(w) = \Sigma.$$  

(46)

This leads to $A = \Sigma, \delta = 0, \mathcal{D} = \mathbb{R}^p$.

Next, note that

$$\nabla f(w, \zeta) = (x^T w - y) x = (x^T (w - w^*) - \xi) x.$$  

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By Cauchy Schwarz inequality, we have
\[
\mathbb{E}\|\nabla f(w, \zeta) - \nabla F(w)\|^2 \leq \mathbb{E}[\|\nabla f(w, \zeta)\|^2] \\
\leq 2\mathbb{E}[\|xx^T(w - w^*)\|^2] + 2\mathbb{E}[\|x\xi\|^2] \\
= 2(w - w^*)^T\mathbb{E}[xx^Txx^T](w - w^*) + 2\sigma^2\text{tr}(\Sigma).
\]
(47)

Next we compute \(\mathbb{E}[xx^Txx^T]\). Since \(x \sim \mathcal{N}(0, \Sigma)\), it can be decomposed as \(x = \Sigma^{1/2}z\) with \(z \sim \mathcal{N}(0, I_d)\). Let the eigendecomposition of \(\Sigma\) be \(V^T\Lambda V\) and denote \(\Sigma^{1/2} = V^T\Lambda^{1/2}V\).

We notice that \(z' = Vz \sim \mathcal{N}(0, I_d)\), then the \((i,j)\)-th element of \(Vzz^TvV^T\) is \(\sum_{k=1}^p \lambda_k z'_i z'_j (z'_k)^2\) and taking expectation results in
\[
\mathbb{E}[Vzz^TvV^T] = \text{diag} \left[ 2\lambda_1 + \sum_{j=1}^p \lambda_j, \ldots, 2\lambda_p + \sum_{j=1}^p \lambda_j \right],
\]
thus we have
\[
\mathbb{E}[xx^Txx^T] = V^T\Lambda^{1/2}\mathbb{E}[Vzz^TvV^T]V \Lambda^{1/2}V \\
= V^T\Lambda^{1/2}\text{diag} \left[ 2\lambda_1 + \sum_{j=1}^p \lambda_j, \ldots, 2\lambda_p + \sum_{j=1}^p \lambda_j \right] \Lambda^{1/2}V \\
\leq 3\text{tr}(\Sigma)\Sigma.
\]

Plug this upper bound in (47) gives us
\[
\mathbb{E}\|\nabla f(w, \zeta) - \nabla F(w)\|^2 \leq 6\text{tr}(\Sigma)\|w - w^*\|_\Sigma^2 + 2\sigma^2\text{tr}(\Sigma).
\]

Finally, we note that \(\|w - w^*\|_\Sigma^2 = 2G(w)\) and by Young’s inequality
\[
\|w - w^*\|_\Sigma^2 \leq 2\|w\|_\Sigma^2 + 2\|w^*\|_\Sigma^2 \leq 2\|\Sigma\|\|w\|^2 + 2\|w^*\|_\Sigma^2.
\]

Therefore we conclude that
\[
\mathbb{E}\|\nabla f(w, \zeta) - \nabla F(w)\|^2 \leq 2\sigma^2\text{tr}(\Sigma) + 12\text{tr}(\Sigma)\|w^*\|_\Sigma^2 + 12\text{tr}(\Sigma)\min\{G(w), \|A\|\|w\|^2\}.
\]

\(\square\)

**Remark C.1.** In the proof above, we used the Gaussian distribution assumption only to obtain the first, second and fourth moment of \(x\). This proof can be extended to scenarios where \(x\) has a non-Gaussian distribution, as long as an upper bound of \(\mathbb{E}[xx^Txx^T]\) is available. Similar extensions can be made for other proofs in below as well.
C.2 Logistic regression

Proof for Proposition 5.1. By Fubini’s theorem,

\[ \nabla F(w) = \mathbb{E}\nabla f(w, \zeta) = \mathbb{E} \frac{-yx}{1 + \exp(yx^T w)} \]

and

\[ \nabla^2 F(w) = \mathbb{E}\nabla \nabla f(w, \zeta) = \mathbb{E} \frac{y^2 \exp(yx^T w) xx^T}{(1 + \exp(yx^T w))^2}. \]

Because \( 0 < \frac{y^2 \exp(yx^T w)}{(1 + \exp(yx^T w))^2} < 1 \) and \( 0 \preceq xx^T \), we find \( 0 \preceq \nabla^2 w F(w) \preceq \Sigma \).

Next, we observe

\[ \nabla f(w, \zeta) = \frac{-yx}{1 + \exp(yx^T w)}. \]

Then because \( y = \pm 1 \),

\[ \mathbb{E}[||\nabla f(w, \zeta)||^2] = \mathbb{E} \left[ \left( \frac{-y}{1 + \exp(yx^T w)} \right)^2 ||x||^2 \right] \leq \mathbb{E}[||x||^2] = \text{tr}(\Sigma). \]

\[ \square \]

C.3 M-estimator with Tukey’s biweight loss

Proof for Proposition 5.2. First of all, let \( v = w - w^*, u = x^T v - \xi \). We find that

\[ \nabla f(w, \zeta) = (1 - (u/c)^2)^2 ux1_{|u| \leq c} \]

Then by Fubini theorem

\[ \nabla F(w) = \nabla \rho F(w) = \mathbb{E}\nabla[(\rho(x^T v - \xi))] = \mathbb{E}[(1 - (u/c)^2)^2 ux1_{|u| \leq c}], \]

\[ \nabla^2 F(w) = \mathbb{E}[xx^T(1 - (u/c)^2)(1 - 5(u/c)^2)1_{|u| \leq c}]. \]

For the first two claims, note that

\[ \mathbb{E}[||\nabla f(w, \zeta)||^2] = \mathbb{E}[(1 - (u/c)^2)^4(u/c)^21_{|u| \leq c}||x||^2] \leq \mathbb{E}[||x||^2] = \text{tr}(\Sigma), \]

\[ \nabla^2 F(w) = \mathbb{E}[xx^T(1 - (u/c)^2)(1 - 5(u/c)^2)1_{|u| \leq c}] \preceq \mathbb{E}xx^T = \Sigma. \]

At \( w = w^* \)

\[ \nabla^2 F(w^*) = \mathbb{E}[xx^T(1 - (\xi/c)^2)(1 - 5(\xi/c)^2)1_{|\xi| \leq c}] = c_0 \Sigma. \]
We consider the directional derivative along the $v$ direction

$$\langle v, \nabla^3 F(w) \rangle := \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\nabla^2 F(w + \epsilon v) - \nabla^2 F(w))$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( E[xx^T(1 - (u/c + \epsilon x^Tv)^2)(1 - 5(u/c + \epsilon x^Tv)^2)1_{|u| \leq c}] - E[xx^T(1 - (u/c)^2)(1 - 5(u/c)^2)1_{|u| \leq c}] \right)$$

$$= E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}] - E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}] = E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}] - E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}] = E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}] - E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}] = E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}]$$

We find

$$\pm \langle v, \nabla^3 F(w) \rangle = \pm E[4xx^T x^Tv(3u/c - 5(u/c)^3)1_{|u| \leq c}]$$

$$\leq E[4xx^T x^Tv(5(u/c)^3 - 3u/c)1_{|u| \leq c}] \leq E[8xx^T x^Tv].$$

For any test vector $\psi$,

$$|\psi^T (v, \nabla^3 F(w)) \psi| \leq 8E(x^T \psi)^2 |x^Tv| \leq 8\sqrt{E(x^T \psi)^4 E|x^Tv|^2} = 8\sqrt{3(\psi^T \Sigma \psi)^2 (v^T \Sigma v)} \leq 16\|v\|_\Sigma \psi^T \Sigma \psi.$$

Therefore

$$-16||v||_\Sigma \psi \leq \langle v, \nabla^3 F(w) \rangle \leq 16||v||_\Sigma \psi.$$

Furthermore, since $w = w^* + v$, from

$$\nabla^2 F(v + w^*) = \nabla^2 F(w^*) + \int_0^1 \langle v, \nabla^3 F(w^* + sv) \rangle ds,$$

we find

$$\nabla^2 F(w) \geq -\delta \Sigma$$

if $16||v||_\Sigma \leq c_0 + \delta.$

C.4 Two layer neural network

First of all, we provide a simple upper bound when computing 4-th order moments of Gaussian random variables.

**Lemma C.2.** If $x \in \mathbb{R}^p$ is Gaussian with mean being zero, for any PSD $A \in \mathbb{R}^{p \times p}$ and $a > 0$

$$\mathbb{E}(x^T Ax + a)^2 \leq 3(\mathbb{E}(x^T Ax + a))^2.$$
Proof. Let $\Sigma$ be the covariance of $x$. Since replacing $x$ with $\Sigma^{-1/2}x$ the statement of the Lemma remains the same, therefore we can assume $x \sim \mathcal{N}(0, I_p)$. Let $A = V^T \Lambda V$ be the eigenvalue decomposition of $A$, and the eigenvalues of $A$ be $\lambda_1, \ldots, \lambda_p$. Let $z = Vx \sim \mathcal{N}(0, I_p)$. Note that

$$\mathbb{E}(x^T Ax + a)^2 = \mathbb{E}((\|z\|_{\Lambda}^4 + 2a\|z\|_{\Lambda}^2 + a^2)).$$

Note that

$$\mathbb{E}\|z\|_{\Lambda}^4 = \sum_{i,j} \lambda_i \lambda_j \mathbb{E}(z_i^2 z_j^2) \leq 3 \sum_{i,j} \lambda_i \lambda_j \mathbb{E}z_i^2 z_j^2 = 3(\mathbb{E}\|z\|_{\Lambda}^2)^2.$$

As a consequence

$$\mathbb{E}(x^T Ax + a)^2 \leq 3(\mathbb{E}\|z\|_{\Lambda}^2 + a)^2 = 3(\mathbb{E}(x^T Ax + a))^2.$$

\[ \square \]

Lemma C.3. Assume that $\psi(0) = 0$ and $|\dot{\psi}|, |\ddot{\psi}| \leq C$. Denote

$$\Sigma^* = \text{diag}\{I_k, \Sigma, \ldots, \Sigma, I_k\} \in \mathbb{R}^{(p+2)k \times (p+2)k},$$

and $\Delta w = w - w^*$. Then the following hold

1) $\mathbb{E}\|\nabla f(w)\|^2 \leq 8\sqrt{3}(1 + \text{tr}(\Sigma))(6C^2\|\Delta w\|_{\Sigma^*}^2, (\|w^*\|_{\Sigma^*}^2 + \|w\|_{\Sigma^*}^2) + \sigma_0^2)C^2\|w\|_{\Sigma^*}^2.$

2) $\mathbb{E}\nabla g(w, x)\nabla g(w, x)^T \leq 6C^2\|w\|_{\Sigma^*}^2, \Sigma^*.$

3) $-M_w \leq \mathbb{E}(g(w, x) - g(w^*, x) - \xi)\nabla^2 g \leq M_w$, where

$$M_w := 6\sqrt{2}C^2(||c||_{\infty} + 1)\|\Delta w\|_{\Sigma^*} (||w^*\|_{\Sigma^*} + \|w\|_{\Sigma^*})\Sigma^*,$$

and we define $||c||_{\infty} := \max_i \{|c_i|\}$.

4) $G(w) \leq 4C^2\|\Delta w\|_{\Sigma^*}^2(||w^*\|_{\Sigma^*}^2 + \|w\|_{\Sigma^*}^2)$.

Proof. For simplicity of discussion, we denote $z_i = b_i^T x + a_i$ and $z = bx + a$.

Proof for Claim 1) We note that $\nabla f(w) = 2(g(w, x) - g(w^*, x) - \xi)\nabla g(w, x)$,

$$\mathbb{E}\|\nabla f(w)\|^2 = 4\mathbb{E}(g(w, x) - g(w^*, x))^2\|\nabla g(w, x)\|^2 + 4\sigma_0^2\mathbb{E}\|\nabla g(w, x)\|^2$$

$$\leq 4\sqrt{\mathbb{E}(g(w, x) - g(w^*, x))^4} \sqrt{\mathbb{E}\|\nabla g(w, x)\|^4} + 4\sigma_0^2\sqrt{\mathbb{E}\|\nabla g(w, x)\|^4}. \quad (48)$$

Note that

$$\nabla g = \left[ \dot{c} \circ \dot{\psi}(z), c_1 \dot{\psi}(z_1)x, \cdots; c_k \dot{\psi}(z_k)x, \psi(z) \right]^T \in \mathbb{R}^{2k+kp}.$$
As a consequence

\[
E\|\nabla g(w, x)\|^4 = E \left( \|c \circ \psi(z)\|^2 + \sum_{i=1}^k \|c_i \psi(z_i)\|^2 + \sum_{i=1}^k \|\psi(z_i)\|^2 \right)^2 \\
\leq E \left( C^2 \|c\|^2 + \sum_{i=1}^k (c_i)^2 C^2 \|x\|^2 + 2C^2 \sum_{i=1}^k (b_i^T x)^2 + 2C^2 \|a\|^2 \right)^2 \\
\leq 3C^4 \left( \|c\|^2 + \|c\|^2 \text{tr}(\Sigma) + 2 \sum_{i=1}^k \|b_i\|_{\Sigma}^2 + 2\|a\|^2 \right)^2 \\
\leq 12C^4 (1 + \text{tr}(\Sigma))^2 \|w\|_{\Sigma}^4.
\]

Since \(x\) is mean zero Gaussian, by Lemma C.2

\[
\leq 3C^4 \left( \|c\|^2 + \|c\|^2 \text{tr}(\Sigma) + 2 \sum_{i=1}^k \|b_i\|_{\Sigma}^2 + 2\|a\|^2 \right)^2 \\
\leq 12C^4 (1 + \text{tr}(\Sigma))^2 \|w\|_{\Sigma}^4.
\]

Next we let \(w^s = sw + (1 - s)w^*\) and \(C_w^s = \|w\|_{\Sigma, w}^2 + \|w^*\|_{\Sigma, w}^2\), note that by convexity of \(\| \cdot \|_{\Sigma, w}^2\),

\[
\|w^s\|_{\Sigma, w}^4 \leq \max\{\|w\|_{\Sigma, w}^4, \|w^*\|_{\Sigma, w}^4\} \leq \|w\|_{\Sigma, w}^4 + \|w^*\|_{\Sigma, w}^4 \leq C_w^4.
\]

Then

\[
|g(w, x) - g(w^*, x)|^2 = \left( \int_0^1 \Delta w^T \nabla g(w^*, x) ds \right)^2 \\
\leq \int_0^1 \left( \Delta a^T c^s \circ \psi(z^s) + \sum_{i=1}^k c_i^s \psi(z_i^s) \Delta b_i^T x + \Delta c^T \psi(z^s) \right)^2 ds \\
\leq \int_0^1 \left( C \|\Delta a\| \|c^s\| + C \sum_{i=1}^k |c_i^s| \|\Delta b_i^T x\| + \|\Delta c\| \|\psi(z^s)\| \right)^2 ds \\
\leq \int_0^1 \left( C_w^2 \|\Delta a\|^2 + C_w^2 \sum_{i=1}^k |\Delta b_i^T x|^2 + \|\Delta c\|^2 \|\psi(z^s)\|^2 / C_w^2 \right) ds \\
\cdot \int_0^1 \left( C_w^2 \|c^s\|^2 / C_w^2 + C_w^2 \sum_{i=1}^k |c_i^s|^2 + C_w^2 \right) ds \\
\leq \int_0^1 \left( C_w^2 \|\Delta a\|^2 + C_w^2 \sum_{i=1}^k |\Delta b_i^T x|^2 + \|\Delta c\|^2 \|\psi(z^s)\|^2 / C_w^2 \right) ds \\
\cdot \int_0^1 \left( 2C_w^2 \|c^s\|^2 / C_w^2 + C_w^2 \right) ds \\
\leq 3C_w^2 \left( \int_0^1 \left( C_w^2 \|\Delta a\|^2 + C_w^2 \sum_{i=1}^k |\Delta b_i^T x|^2 + \|\Delta c\|^2 \|\psi(z^s)\|^2 / C_w^2 \right) ds \right).
\]
By Lemma C.2, and $\mathbb{E}|\Delta b_i^T x|^2 = \Delta b_i^T \Sigma \Delta b_i$,

$$
\mathbb{E} \left( C_w^2 \Delta a_i^2 + C_w^2 \sum_{i=1}^k |\Delta b_i^T x|^2 + \|\Delta c_i\|^2 \|\psi(z^s)\|^2 / C^2 \right)^2 
\leq \mathbb{E} \left( C_w^2 \Delta a_i^2 + C_w^2 \sum_{i=1}^k |\Delta b_i^T x|^2 + 2\|\Delta c_i\|^2 \left( \|a^s\|^2 + \sum_{i=1}^k |(b^s_i)^T x|^2 \right) \right)^2 
$$

Note that $\|a^s\|^2 + \sum_{i=1}^k |(b^s_i)^T x|^2 \leq \max\{\|w\|_{\Sigma^*}^2, \|w^s\|_{\Sigma^*}^2\} \leq C_w^2$

$$
\leq \left( C_w^2 \Delta a_i^2 + C_w^2 \sum_{i=1}^k \|\Delta b_i\|_{\Sigma^*}^2 + 2C_w^2 \|\Delta c_i\|^2 \right)^2 
\leq 4(\|w\|_{\Sigma^*}^2 + \|w^s\|_{\Sigma^*}^2)^2 \|\Delta w\|_{\Sigma^*}^4. 
$$

Replace these bounds into the square of (50), we find

$$
\mathbb{E}|g(w, x) - g(w^s, x)|^4 \leq 36C^4 \|\Delta w\|_{\Sigma^*}^4 (\|w^s\|_{\Sigma^*}^4 + \|w^s\|_{\Sigma^*}^2)^2. 
$$

(51)

Furthermore, we combine this with (49) into (48), we find that

$$
\mathbb{E}\|\nabla f(w)\|^2 \leq 8\sqrt{3}(1 + \text{tr}(\Sigma))(6C^2 \|\Delta w\|_{\Sigma^*}^2 (\|w^s\|_{\Sigma^*}^2 + \|w^s\|_{\Sigma^*}^2) + \sigma_0^2)C^2 \|w\|_{\Sigma^*}^2. 
$$

**Proof for Claim 2):** Recall that

$$
\nabla g = \left[ c \circ \hat{\psi}(z); c_1 \hat{\psi}(z_1) x; \ldots; c_k \hat{\psi}(z_k) x; \hat{\psi}(z) \right] \in \mathbb{R}^{2k+kp}. 
$$

With $u \in \mathbb{R}^k$, $v_1 \in \mathbb{R}^p, \ldots, v_k \in \mathbb{R}^p$, $w \in \mathbb{R}^k$, we define

$$
W = \left[ u; v_1; \ldots; v_k; w \right] \in \mathbb{R}^{2k+kp},
$$

and show that $W^T \mathbb{E}\nabla g \nabla g^T W \leq 6C^2 \|w\|_{\Sigma^*}^2 W^T \Sigma^* W$. Note that

$$
W^T \mathbb{E}\nabla g \nabla g^T W = \mathbb{E}u^T [c \circ \hat{\psi}(z)(c \circ \hat{\psi}(z))^T] u + \mathbb{E}v_i^T [c_1 \hat{\psi}(z_1) x x^T] v_i + \mathbb{E}w^T [\hat{\psi}(z)(\hat{\psi}(z))^T] w + 2u^T \mathbb{E}c \circ \hat{\psi}(z) (\hat{\psi}(z))^T w + 2 \sum_{i=1}^k u^T \mathbb{E}c \circ \hat{\psi}(z_i) c_i \hat{\psi}(z_i) x^T v_i 
$$

$$
+ 2 \sum_{i<j} v_i^T \mathbb{E}c_1 \hat{\psi}(z_i) c_j \hat{\psi}(z_j) x x^T v_j + 2 \sum_{i=1}^k v_i^T \mathbb{E}c_1 \hat{\psi}(z_i) x (\hat{\psi}(z))^T w. 
$$

For the diagonal terms, note that

$$
\mathbb{E}[c \circ \hat{\psi}(z)(c \circ \hat{\psi}(z))^T] \preceq \mathbb{E}[\|c \circ \hat{\psi}(z)\|^2 I_k] \preceq C^2 \|c\|^2 I_k,
$$

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$$\mathbb{E}[c_i^2 \hat{\psi}(z_i) \hat{\psi}(z_i)^T] \leq C^2 c_i^2 \mathbb{E}[xx^T] = C^2 c_i^2 \Sigma,$$

$$\mathbb{E}[\psi(z)(\psi(z))^T] \leq \mathbb{E}[\|\psi(z)\|^2 I_k] \leq 2C^2 \left( \|a\|^2 + \sum_{i=1}^k \|b_i\|_2^2 \right) I_k.$$

For the cross terms, note that by Cauchy Schwarz inequality

$$u^T c \circ \hat{\psi}(z)c_i \hat{\psi}(z_i)x^Tv_i \leq |u^T c \circ \hat{\psi}(z)| |c_i \hat{\psi}(z_i)x^Tv_i|$$

$$= (u^T c \circ \hat{\psi}(z)(c \circ \hat{\psi}(z))^T u)^{1/2} (v_i^T (c_i \hat{\psi}(z_i))^2 xx^Tv_i)^{1/2}$$

$$\leq \frac{c_i^2}{2\|c\|^2} (u^T c \circ \hat{\psi}(z)(c \circ \hat{\psi}(z))^T u) + \frac{\|c\|^2}{2} (v_i^T (\hat{\psi}(z_i))^2 x x^Tv_i),$$

and similarly,

$$u^T c \circ \hat{\psi}(z)(\psi(z))^T w \leq \frac{1}{2} u^T c \circ \hat{\psi}(z)(c \circ \hat{\psi}(z))^T u + \frac{1}{2} u T \psi(z)(\psi(z))^T w,$$

$$v_i^T c_i \hat{\psi}(z_i) c_j \hat{\psi}(z_j) x x^T v_j \leq \frac{c_i^2}{2} v_i^T (\hat{\psi}(z_i))^2 x x^Tv_i + \frac{c_j^2}{2} v_j^T (\hat{\psi}(z_j))^2 x x^Tv_j,$$

$$v_i^T c_i \hat{\psi}(z_i) x (\psi(z))^T w \leq \frac{\|c\|^2}{2} v_i^T (\hat{\psi}(z_i))^2 x x^Tv_i + \frac{\|c\|^2}{2} v_i^T (\hat{\psi}(z))^2 w^T (\psi(z))(\psi(z))^T w.$$

Plug the results above into (52) gives us

$$\mathbb{E}[\nabla g(w, x) \nabla g(w, x)^T] \leq C^2$$

$$\leq \frac{1}{2} \begin{bmatrix} 2\|c\|^2 I_k & 0_{k \times p} & 0_{k \times p} & 0_{k \times p} & 0_{k \times k} \\ 0_{p \times k} & 3\|c\|^2 \Sigma & 0_{p \times p} & 0_{p \times p} & 0_{p \times k} \\ 0_{p \times k} & 0_{p \times p} & \ldots & 0_{p \times p} & 0_{p \times k} \\ 0_{p \times k} & 0_{p \times p} & 0_{p \times p} & 3\|c\|^2 \Sigma & 0_{p \times k} \\ 0_{k \times k} & 0_{k \times p} & 0_{k \times p} & 0_{k \times p} & 6 \left( \|a\|^2 + \sum_{i=1}^k \|b_i\|_2^2 \right) I_k \end{bmatrix}$$

$$\leq 6C^2 \|w\|^2 \Sigma^*.$$

**Proof for Claim 3:** First of all, we find that

$$\nabla^2 g = \begin{bmatrix} D_{c \circ \hat{\psi}(z)} & c_1 \hat{\psi}(z_1) e_1 x^T & c_2 \hat{\psi}(z_2) e_2 x^T & \ldots & c_k \hat{\psi}(z_k) e_k x^T & D_{\hat{\psi}(z)} \\ c_1 \hat{\psi}(z_1) e_1^T & c_1 \hat{\psi}(z_1) x x^T & 0_{p \times p} & \ldots & 0_{p \times p} & \hat{\psi}(z_1) e_1^T \\ c_2 \hat{\psi}(z_2) e_2^T & 0_{p \times p} & c_2 \hat{\psi}(z_2) x x^T & \ldots & 0_{p \times p} & \hat{\psi}(z_2) e_2^T \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_k \hat{\psi}(z_k) e_k^T & 0_{p \times p} & \ldots & 0_{p \times p} & c_k \hat{\psi}(z_k) x x^T & \hat{\psi}(z_k) e_k^T \\ D_{\hat{\psi}(z)} & \hat{\psi}(z_1) e_1 x^T & \hat{\psi}(z_2) e_2 x^T & \ldots & \hat{\psi}(z_k) e_k x^T & 0_{k \times k} \end{bmatrix}.$$
In above, we use $D_v$ to denote the diagonal matrix with diagonal entries being components of $v$. We will first show that $\nabla^2 g \preceq Q_x \preceq (2\|c\|_\infty + 2)C\Sigma^*_x$, where
\[
Q_x := C\text{diag}\{(2\|c\|_\infty + 1)I_k, (2\|c\|_\infty + 1)xx^T, \ldots, (2\|c\|_\infty + 1)xx^T, 2I_k\},
\]
\[
\Sigma^*_x := \text{diag}\{I_k, xx^T, \ldots, xx^T, I_k\}.
\]
Recall $W = \begin{bmatrix} u; v_1; \cdots ; v_k; w \end{bmatrix} \in \mathbb{R}^{2k+kp}$. Note that
\[
W^T \nabla^2 g W = u^T D_{c^2}(z) u + \sum_{i=1}^{k} c_i \tilde{\psi}(z_i)(v_i^T x)^2 + 2w^T D_{\psi}(z) u
\]
\[
+ 2 \sum_{i=1}^{k} c_i \tilde{\psi}(z_i)(v_i^T x)(u^T e_i) + 2 \sum_{i=1}^{k} \tilde{\psi}(z_i)(v_i^T x)(w^T e_i).
\]
Note that
\[
u^T D_{c^2}(z) u \leq \| D_{c^2}(z) \| \| u \|^2 \leq C\|c\|_\infty \| u \|^2,
\]
\[
c_i \tilde{\psi}(z_i)(v_i^T x)^2 \leq C\|c\|_\infty (v_i^T x)^2,
\]
\[
2w^T D_{\psi}(z) u \leq C\|w\|^2 + C\|u\|^2,
\]
\[
2c_i \tilde{\psi}(z_i)(v_i^T x)(u^T e_i) \leq \|c\|_\infty C((v_i^T x)^2 + (u^T e_i)^2),
\]
\[
2\tilde{\psi}(z_i)(v_i^T x)(w^T e_i) \leq C((v_i^T x)^2 + (w^T e_i)^2).
\]
Replace these upperbounds with terms in (53), we find
\[
W^T \nabla^2 g W \leq W^T Q_x W
\]
because $\sum_i (u^T e_i)^2 = \|u\|^2$. Since this holds for all $W$, we have $\nabla^2 g \preceq Q_x$. Finally, we note that
\[
|W^T \mathbb{E}(g(w, x) - g(w^*, x) - \xi) \nabla^2 g W| = |W^T \mathbb{E}[(g(w, x) - g(w^*, x)) \nabla^2 g] W|
\]
\[
\leq \sqrt{\mathbb{E}(g(w, x) - g(w^*, x))^2} \sqrt{\mathbb{E}(W^T \nabla^2 g W)^2}.
\]
Recall (51), we have
\[
\sqrt{\mathbb{E}(g(w, x) - g(w^*, x))^2} \leq (\mathbb{E}(g(w, x) - g(w^*, x))^4)^{1/4} \leq \sqrt{6}C\|\Delta w\|_{\Sigma^*}(\|w^*\|_{\Sigma^*} + \|w\|_{\Sigma^*}).
\]
Then note that by Lemma C.2
\[
\mathbb{E}(W^T \nabla^2 g W)^2 \leq 4(\|c\|_\infty + 1)^2C^2\mathbb{E}(W^T \Sigma^*_x W)^2 \leq 12C^2(\|c\|_\infty + 1)^2(W^T \Sigma^* W)^2.
\]
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In combination, we find
\[ |W^T\mathbb{E}(g(w,x) - g(w^*,x) - \xi)\nabla^2 g W| \leq 6\sqrt{2}C^2(\|c\|_\infty + 1)\|\Delta w\|_{\Sigma^*} (\|w^*\|_{\Sigma^*} + \|w\|_{\Sigma^*})W^T\Sigma^* W. \]
This verifies our claim 3).

**Proof for Claim 4):** Simply note that by (51)
\[ G(w) = \mathbb{E}[g(w,x) - g(w^*,x)]^2 \leq 6C^2\|\Delta w\|^2_{\Sigma^*} (\|w^*\|^2_{\Sigma^*} + \|w\|^2_{\Sigma^*}). \]

\[ \square \]

**Proof for Proposition 5.3.** First, when \( w \in \mathcal{D} \)
\[ \|c\|^2 \leq \|w\|^2_{\Sigma^*} \leq (1 + \frac{1}{4})^2\|w^*\|^2_{\Sigma^*} \leq 2\|w^*\|^2_{\Sigma^*}. \]
Note that
\[ \nabla^2 F = \mathbb{E}[\nabla g(w,x)\nabla g(w,x)^T + \mathbb{E}(g(w,x) - g(w^*,x) - \xi)\nabla^2 g(w,x) \]
\[ = \mathbb{E}[\nabla g(w,x)\nabla g(w,x)^T + \mathbb{E}(g(w,x) - g(w^*,x))\nabla^2 g(w,x). \]
By Lemma C.3 claim 2) and claim 3)
\[ \mathbb{E}[\nabla g(w,x)\nabla g(w,x)^T] \leq 6C^2\|w^*\|^2_{\Sigma^*}, \]
\[ \mathbb{E}(g(w,x) - g(w^*,x))\nabla^2 g(w,x) \leq 6\sqrt{2}C^2(\|c\|_\infty + 1)\delta C_1(w^*)\|w^*\|_{\Sigma^*} (\|w^*\|_{\Sigma^*} + \|w\|_{\Sigma^*})\Sigma^* \]
\[ \leq 18\sqrt{2}C^2(2\|w^*\|_{\Sigma^*} + 1)\delta C_1(w^*)\|w^*\|^2_{\Sigma^*} \]
\[ \leq 4\delta C^2\|w^*\|^2_{\Sigma^*}. \]
So \( \nabla^2 F \preceq C_0(w^*)\Sigma^* \). Also note that \( \mathbb{E}[\nabla g(w,x)\nabla g(w,x)^T] \succeq 0 \), and by Lemma C.3 claim 3),
\[ \mathbb{E}(g(w,x) - g(w^*,x))\nabla^2 g(w,x) \succeq -3\delta C^2\|w^*\|^2_{\Sigma^*} \Sigma^* \preceq -\delta A. \]
Then by Lemma C.3 claim 1), we find that
\[ \mathbb{E}[\|\nabla f(w) - \nabla F(w)\|^2] \leq \mathbb{E}[\|\nabla f(w)\|^2] \]
\[ \leq 8\sqrt{3}(1 + \text{tr}(\Sigma))(6C^2\|\Delta w\|^2_{\Sigma^*}, (\|w^*\|^2_{\Sigma^*} + \|w\|^2_{\Sigma^*}) + \sigma_0^2)C^2\|w\|^2_{\Sigma^*}, \]
\[ \leq 8\sqrt{3}(1 + \text{tr}(\Sigma))(18C^2\|\Delta w\|^2_{\Sigma^*}, \|w^*\|^2_{\Sigma^*} + \sigma_0^2)C^2\|w^*\|^2_{\Sigma^*}, \]
\[ \leq 8\sqrt{3}(1 + \text{tr}(\Sigma))(18\delta C^2\|w^*\|^2_{\Sigma^*}, \|w^*\|^2_{\Sigma^*} + \sigma_0^2)C^2\|w^*\|^2_{\Sigma^*}, \]
\[ \leq 8\sqrt{3}(1 + \text{tr}(\Sigma))(2C^2\|w^*\|^2_{\Sigma^*} + \sigma_0^2)C^2\|w^*\|^2_{\Sigma^*}. \]
Finally by claim 4) of Lemma C.3, when \( w_0 \in \mathcal{D} \),
\[ G(w_0) \leq 18C^2\delta^2\|w^*\|^4_{\Sigma^*} \leq 2C^2\|w^*\|^4_{\Sigma^*}. \]
\[ \square \]
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