The complete solution of the conformastat electrovacuum problem

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Abstract. We find the complete solution of the Einstein-Maxwell field equations without sources for static spacetimes in which the space of Killing trajectories is conformally flat. The result is used to present an improved local characterisation of the Majumdar-Papapetrou class of solutions.

1. Introduction

We present the complete solution of the Einstein-Maxwell field equations without sources for conformastat spacetimes, i.e. the conformastat electrovacuum spacetimes. The only assumption made is that the electromagnetic field inherits the symmetry, thus being stationary.

Taking the definition in \cite{1} as standard (see the original \cite{2}) by conformastat spacetimes we mean the static subclass of the conformastationary spacetimes, which are, in turn, those stationary spacetimes with a conformally flat space of orbits. This class of spacetimes is important by its own right, since it includes the most relevant Schwarzschild solution and the Majumdar-Papapetrou class of metrics, solution of the Einstein-Maxwell equations without sources. But more interestingly, the study of conformastat electrovacuum spacetimes eventually provides us with some characterisation (or uniqueness) results for static charged black holes \cite{3}, because the resulting classes of solutions are tremendously restricted.

Conformastat vacuum spacetimes comprise three \cite{4} out of the seven families that constitute the whole set of type D static vacuum spacetimes \cite{5, 1} (Table 18.2 in \cite{1}). These three families consist of the Schwarzschild solution together with its plane and hyperbolic counterparts.

In order to generalise this class of solutions Lukášz \textit{et al.} and Perjés tackled the problem of finding the whole set of conformastationary vacuum spacetimes in a series of three papers. Let us note that they refer to conformastationary spacetimes simply as “conformastat”. In a first paper \cite{6} the solutions possessing a functional relationship between the real and imaginary parts of the Ernst potential $E$ were found to consist of three bi-parametric families of solutions. These three families are the stationary generalisations of the three conformastat vacuum solutions by the Ehlers transformation. In particular, the Schwarzschild solution generalises to the NUT solution. In two subsequent papers \cite{7, 8}, using purposely defined “Ernst coordinates”, Perjés found that solutions with functionally independent real and imaginary parts of $E$ do not exist. The final result is thus that the set of conformastationary vacuum spacetimes correspond to the
three families presented in [6], which can be thought of being the NUT-type extensions of the conformstat vacuum solutions.

Another natural step to follow is the generalisation of the conformstat vacuum spacetimes by including a stationary electromagnetic field without sources.

To find the complete solution of the conformstat electrovacuum problem we first show how this problem and the conformastationary vacuum problem can both be treated within a common framework by using a suitable notation. This framework is then used to prove that non-trivial conformastationary electrovacuum spacetimes must contain a functional relationship between the gravitational and electromagnetic potentials. The final part, which completes the study of conformastationary electrovacuum spacetimes, consists in classifying and exploiting the necessary functional relationship.

The final result is that all conformastationary electrovacuum spacetimes either belong to the Majumdar-Papapetrou class or correspond to either the Bertotti-Robinson solution or the exterior Reissner-Nordström solution together with its plane and hyperbolic counterparts. Furthermore, the procedure used for the classification provides an improved characterisation of the Majumdar-Papapetrou class. This is known to be the class of static electrovacuum spacetimes in which the Einstein-Maxwell equations reduce to

\[ \Box \mathcal{F}_{ab} = 0, \]

where \( \mathcal{F}_{ab} \) is the electromagnetic field tensor. This equation is the analogue of the Laplace equation in electrostatics and represents the condition for the existence of a static electromagnetic field.

2. Conformastationary and conformstat spacetimes

A stationary spacetime \((\mathcal{M}, g_{\mu\nu})\) is locally defined by the existence of a timelike Killing vector field \(\xi^\mu\) whose space of orbits invariantly determines a differentiable 3-dimensional Riemannian manifold \(\Sigma_3\). Local coordinates \(\{t, x^a\}\) exist for which \(\xi^\mu = \partial_t\) and such that the line-element reads [1]

\[ ds^2 = -e^{2U}(dt + A_a dx^a)^2 + e^{-2U}h_{ab}dx^a dx^b, \tag{1} \]

where \(U, A_a\) and \(h_{ab}\) do not depend on \(t\). Applying the usual projection formalism (see e.g. [1]) we will think of \(U, A_a\) as objects living on \((\Sigma_3, \hat{h}_{ab})\). Once \(U, A_a\) and \(h_{ab}\) are given, the local geometry of the stationary spacetime \((\mathcal{M}, g_{\mu\nu})\) is fully specified by using (1). Let us use the first Latin indices \(a, b, \ldots\) for objects defined on \((\Sigma_3, \hat{h}_{ab})\). In this context a static spacetime is characterised by \(A_a = 0\).

A conformastationary spacetime is a stationary spacetime whose space of orbits \((\Sigma_3, \hat{h}_{ab})\) is conformally flat [1]. The intrinsic characterisation of a conformally flat 3-space is the vanishing of the York tensor density

\[ Y^e_a = \hat{\eta}^{be} \left( 2
abla_c \hat{R}_{ba} - \frac{1}{2} \hat{h}_{ab} \nabla_c \hat{R} \right) = 0, \tag{2} \]

where \(\hat{R}_{ab}, \hat{\eta}^{be}\) and \(\nabla\) denote the Ricci tensor, the volume form and covariant derivative relative to \(\hat{h}_{ab}\). Note that \(Y_{ae} = Y_{ea}\) and \(Y^a_a = 0\).

Conformastationary spacetimes are those conformastationary spacetimes which are, in fact, static, for which \(A_a\) can be set to zero.

3. Electrovacuum field equations

Let us first fix one basic assumption and some notation. First, we will restrict ourselves to Maxwell fields \(F_{a\beta}\) in \((\mathcal{M}, g_{\mu\nu})\) which inherit the stationary symmetry, i.e. for which \(\mathcal{L}_\xi F = 0\). The Einstein-Maxwell equations outside the sources imply (locally, at least) the existence of two complex scalars, \(\Phi(x^a)\) the electromagnetic potential, and \(\mathcal{E}(x^a)\) the Ernst potential. Then, the Einstein-Maxwell equations reduce to

\[ \hat{R}_{ab} = G_a G_b + \hat{G}_a G_b - (H_a H_b + \hat{H}_a H_b), \]
\[ \nabla^a H_a + \frac{1}{2} \mathbb{C} \cdot H - \frac{3}{2} G \cdot H = 0, \]
\[ \nabla^a G_a - \mathbb{H} \cdot H - (G - \mathbb{C}) \cdot G = 0, \]
where \( H_a \equiv (\text{Re} \mathcal{E} + \Phi \mathcal{F})^{-1/2} \Phi_a \) and \( G_a \equiv 1/2(\text{Re} \mathcal{E} + \Phi \mathcal{F})^{-1}(\mathcal{E}_a + 2 \Phi \mathcal{F}_a) \), and the dot denotes the scalar product with respect to \( \mathbb{h}_{ab} \). The integrability conditions for the two potentials are given by \( dH = H \wedge \text{Re} \mathcal{G}, \quad dG = G \wedge \mathbb{G} + \mathbb{H} \wedge H. \)

The metric is determined by the relations \( e^{2U} = \text{Re} \mathcal{E} + \Phi \mathcal{F} \), \( dA_{ab} = 2e^{-4U} \eta_{abc} \text{Im} \mathcal{G} \).

### 3.1. Vacuum stationary and electro-magnetostatic cases

The vacuum stationary case is defined by \( \Phi = 0 \Rightarrow H_a = 0 \) and hence the Einstein-Maxwell equations plus the integrability for the potentials read

\[ \hat{R}_{ab} = G_a \mathbb{G}_b + \mathbb{D}_a G_b, \quad \nabla^a G_a - (G - \mathbb{C}) \cdot G = 0, \quad dG = G \wedge \mathbb{G}. \]

The static case corresponds to \( G_a - \mathbb{G}_a = 0 \). Then \( G_a = U_a \) and also \( \mathbb{H}_a = e^{-2\theta} H_a \) for some constant \( \theta \) [4]. After defining \( X_a \equiv e^{-\theta} H_a = e^{-U} \Psi_a \) where \( \Psi \equiv e^{-\theta} \Phi \) is real, and \( \Sigma_a \equiv \frac{1}{2}(U_a + j X_a) \) in the hypercomplex plane, where \( j^2 = 1 \) with conjugation \( \bar{j} = -j \), the set of equations now read

\[ \hat{R}_{ab} = 4(\Sigma_a \bar{\Sigma}_b + \bar{\Sigma}_a \Sigma_b), \quad \nabla^a \Sigma_a - (\Sigma - \bar{\Sigma}) \cdot \Sigma = 0, \quad d\Sigma = \Sigma \wedge \bar{\Sigma}. \]

### 4. Common framework

Denote by \( \iota \) either \( i \) or \( j \), so that \( \iota^2 = \pm 1 \) accordingly, and by \( \bar{\iota} \) the general conjugation. Consider a “composed” vector field \( \mathcal{V}^a \) and a metric \( \mathbb{h}_{ab} \) that satisfy

\[ \hat{R}_{ab} = N(\mathcal{V}_a \bar{\mathcal{V}}_b + \bar{\mathcal{V}}_a \mathcal{V}_b), \quad \nabla^a \mathcal{V}_a - (\mathcal{V} - \bar{\mathcal{V}}) \cdot \mathcal{V} = 0, \quad d\mathcal{V} = \mathcal{V} \wedge \bar{\mathcal{V}}. \tag{3} \]

The vacuum (stationary) case is then recovered by using \( N = 1 \) and \( \mathcal{V}_a(= G_a) \) being complex, while the static (electrovacuum) case is recovered with \( N = 4 \) and \( \mathcal{V}_a(= \Sigma_a) \) being hypercomplex.

Regarding conformastationarity, the introduction of the 1-form \( L \equiv * (\mathcal{V} \wedge \bar{\mathcal{V}}) \) allows us to write the condition (2) as

\[ (\mathcal{V}_a - \bar{\mathcal{V}}_a) L^e + \eta^{bc} (\bar{\mathcal{V}}_b \nabla_c \mathcal{V}_a + \mathcal{V}_b \nabla_c \bar{\mathcal{V}}_a) - \frac{1}{2} \mathbb{h}_{abc} \eta^{bc} \nabla_c (\mathcal{V} \cdot \bar{\mathcal{V}}) = 0. \tag{4} \]

The proof involves solving the system (3)-(4) for \( \mathcal{V}_a \) and \( \mathbb{h}_{ab} \). In the following we present the main steps.

To solve the system one has to consider first the case \( \mathcal{V} \cdot \bar{\mathcal{V}} = 0 \). In the static case \( \mathcal{V} = \Sigma \) is \( j \)-composed and \( \Sigma \cdot \Sigma = 0 \) implies in particular \( G \cdot G + X \cdot X = 0 \), which leads to the trivial case \( G = X = 0 \). However, in the vacuum case \( \mathcal{V} = \Sigma \) is complex and one can have, in principle, fields for which \( G \cdot G = 0 \). The study of these null fields was performed in [6], where it was proven that no null conformastationary vacuum spacetimes exist apart from the trivial case of flat spacetime.

Two different situations now arise depending on whether \( L_a \) vanishes or not. In the case \( L_a \neq 0 \) one can take the basis \( \{ L_a, \mathcal{V}_a, \bar{\mathcal{V}}_a \} \) and show that one of the components of (2) reads \( L \wedge dL = 0 \), and therefore \( L = \iota \chi d\varphi \) for some real functions \( \chi \) and (a nonconstant) \( \varphi \). The integrability condition \( d\mathcal{V} = \mathcal{V} \wedge \bar{\mathcal{V}} \) implies, in turn, that a composed function \( \sigma \) exists such that \( \mathcal{V} = (\sigma + \bar{\sigma})^{-1} d\sigma \). The three potentials \( \sigma, \bar{\sigma} \) and \( \varphi \) can be used as coordinates to write down the remaining equations and study the compatibility conditions, in particular for \( N = 1 \) and \( N = 4 \). A somewhat long calculation [3] shows that there are no possible solutions in this case.

The result so far is that the case \( L_a \neq 0 \) is empty, and therefore \( L_a = 0 \) necessarily, i.e. \( \mathcal{V} \) and \( \bar{\mathcal{V}} \) must be parallel. This translates in the vacuum and static cases to:
Table 1. The conformastat electrovacuum solutions.

| $\bar{R}$ | Description | Condition |
|-----------|-------------|-----------|
| $= 0$     | Majumdar-Papapetrou | $M^2 - Q^2 > 0$ |
| $> 0$     | Plane-symmetric field (plane Reissner-Nordström) | $M^2 - Q^2 > 0$ |
|           | Bertotti-Robinson | $M^2 - Q^2 > 0$ |
|           | Reissner-Nordström exterior | $M^2 - Q^2 > 0$ |
|           | hyperbolic Reissner-Nordström | |
| $< 0$     | Bertotti-Robinson | $M^2 - Q^2 < 0$ |
|           | Reissner-Nordström exterior | $M^2 - Q^2 < 0$ |

Theorem 1 Conformastationary vacuum spacetimes are always characterised by a functional relation between the potentials $U$ and $\Omega$.

Theorem 2 Conformastat electrovacuum spacetimes are always characterised by a functional relation between the potentials $U$ and $\Phi$.

5. The complete solution

The study of the $L_a = 0$ case was presented in [9], but it turned out to be incomplete (see [3]). Indeed, the divergence equation for $\nabla_a$ first fixes, for an arbitrary constant $k$, $e^{2U} = 1 - 2k\Psi + \Psi^2$, which can be rewritten in parametric form in terms of an auxiliary function $V$ that satisfies $\nabla^2 V = 0$. Three cases arise depending on $k$, for which the corresponding Ricci equations read $k^2 = 1 : \bar{R}_{ab} = 0$, $k^2 > 1 : \bar{R}_{ab} = 2V_a V_b$, $k^2 < 1 : \bar{R}_{ab} = -2V_a V_b$. The remaining equations for $\bar{h}_{ab}$ and $V_a$ are encoded in (2). All the resulting solutions [3] are presented in Table 1. The result mentioned in the introduction thus follows. Moreover one can state the following improved local characterisation of the Majumdar-Papapetrou class, known to be the static electrovacuum spacetimes with a flat $\bar{h}_{ab}$ [1]. Here we have relaxed the requirement on $\bar{h}_{ab}$ by showing that

Corollary 1 The Majumdar-Papapetrou class of solutions are the static electrovacuum spacetimes with conformally flat $\bar{h}_{ab}$ and $\bar{R} = 0$.

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References

[1] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E (2003) Exact solutions of Einstein’s field equations. Second Edition, Cambridge University Press, Cambridge
[2] Synge J L (1960) Relativity: The General Theory, North Holland, Amsterdam
[3] González G A and Vera R 2011 Class. Quantum Grav. 28 025008
[4] Das A 1971 J. Math. Phys. 12 1136-42
[5] Ehlers J and Kundt W (1962) “Exact solutions of the gravitational fields equations”, Gravitation, ed. L. Witten, Wiley, New York, p. 49
[6] Lukács B, Perjés Z and Sebestyén Á 1983 Gen. Rel. Grav. 15 511-22
[7] Perjés Z 1986 Gen. Rel. Grav. 18 511-30
[8] Perjés Z 1986 Gen. Rel. Grav. 18 531-47
[9] González G A and Vera R 2010 J. Phys.: Conf. Ser. 229 012040