Symbol-based Convergence Analysis in Block Multigrid Methods with Applications for Stokes Problems

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Abstract

The main focus of this paper is the study of efficient multigrid methods for large linear systems with a particular saddle-point structure. Indeed, when the system matrix is symmetric, but indefinite, the variational convergence theory that is usually used to prove multigrid convergence cannot be directly applied.

However, different algebraic approaches analyze properly preconditioned saddle-point problems, proving convergence of the Two-Grid method. In particular, this is efficient when the blocks of the coefficient matrix possess a Toeplitz or circulant structure. Indeed, it is possible to derive sufficient conditions for convergence and provide optimal parameters for the preconditioning of the saddle-point problem in terms of the associated generating symbols. In this paper, we propose a symbol-based convergence analysis for problems that have a hidden block Toeplitz structure. Then, they can be investigated focusing on the properties of the associated generating function $\mathbf{f}$, which consequently is a matrix-valued function with dimension depending on the block size of the problem.

As numerical tests we focus on the matrix sequence stemming from the finite element approximation of the Stokes problem. We show the efficiency of the methods studying the hidden 9-by-9 block multilevel structure of the obtained matrix sequence. Moreover we propose an efficient algebraic multigrid method with convergence rate independent of the matrix size. Finally, we present several numerical tests comparing the results with state-of-the-art strategies.

Keywords: Multigrid methods, block matrix sequences, spectral symbol, Stokes equation

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1. Introduction

Many works have focused on efficient iterative methods for the solution of large linear systems with coefficient matrix in saddle-point form $\mathbf{A} = (\mathbf{B} \mathbf{C}^{-1} \mathbf{B}^T)$. Indeed, many precondi-
tioned Krylov subspace methods have been developed based on the study of the spectral properties of the coefficient matrix \([3, 4, 6, 23]\). When we deal with multilevel and multilevel block systems the performance of some known preconditioners can deteriorate \([24]\). Multigrid methods (MGM) are in general a good choice in terms of computational cost and optimal convergence rate. Indeed, optimal convergence rates are obtained by constructing and employing a proper sequence of linear systems of decreasing dimensions. Moreover, when the coefficient matrix has a particular multilevel (block) Toeplitz structure, the convergence analysis for such methods can be obtained in compact form exploiting the concept of symbols \([16, 16]\).

The knowledge and the use of the symbol has been successfully employed for designing very efficient preconditioning and multigrid techniques in various settings and several applications \([11, 12, 14, 27]\). However, even in the case where the coefficient matrix is somehow related to a Toeplitz/Toeplitz-like structure, some saddle-point problems are intrinsically indefinite. This limits the possible convergence results that can be exploited.

In many works \([2, 4, 11]\) a common choice for problems with saddle-point structure is that of constructing preconditioning for two-by-two block systems involving the Schur complement. In many cases stemming from the approximation of Partial Differential Equations (PDEs), even though the whole problem is indefinite, the latter procedure is based on the solution of a linear system with a positive definite coefficient matrix.

In this paper we consider the approach suggested by \([19]\) which provides a point-wise smoother and a coarse grid correction to the linear system after preconditioning from the left and the right using lower and upper triangular matrices, respectively.

This strategy is particularly effective in the case where the blocks of the matrix possess a scalar unilevel Toeplitz and circulant structure. Indeed, in \([5]\) sufficient conditions for the Two-Grid Method (TGM) convergence and “level independency” for the W-cycle have been obtained in terms of the generating functions of the blocks. Studying the latter functions, also the optimal parameters for the preconditioning of the saddle-point problem and for the point smoother were found. Clearly, in the scalar setting, the involved generating functions were univariate or multivariate scalar trigonometric polynomials. Consequently, the exploited MGM theory is that for structured unilevel or multilevel scalar matrices \([1, 2]\). When discretizing a PDE with higher order approximation techniques, the coefficient matrix - or its blocks - possess a (multilevel) block structure.

In this paper we focus on saddle-point problems where the blocks of the coefficient matrix have a (hidden) multilevel block structure. Consequently, we combine the method given in \([19]\) with the recent results on TGM convergence analysis for block structured matrices \([8, 12]\). Moreover, we show that the structure can be preserved on the coarse level, allowing for a recursive application of the method, after proper symmetrization. The global algorithm is then a 3-step procedure whose main body is formed by a symbol-based block multigrid method.

The outline of the paper is as follows. In Subsections \([1.1, 1.4]\) we provide the notation and the necessary results on multigrid methods for both saddle-point and symmetric definite positive (SPD) matrices. Regarding SPD matrices we focus especially on the case where the multigrid is applied to Toeplitz matrices with both scalar or matrix-valued generating functions. At the end of the Section \([1]\) we report the main results of the TGM when applied to saddle-point problems. In Section \([2]\) we provide TGM convergence results for saddle-point systems with blocks that are multilevel block structured matrices. In Section \([3]\) we introduce the Stokes problem in 2D and we focus on a Finite Element
approximation in Subsection 3.1. The discrete problem involves solving a saddle-point linear system with block multilevel Toeplitz blocks. We theoretically show that such a problem is within the setting and fulfills the assumptions of the TGM convergence results described in Section 2. In Section 4, we introduce an iterative 3-step procedure that can be exploited for solving recursively linear systems with structure described as in Subsection 3.1. Precisely, we focus on the structure and on the spectral features of the involved matrix sequences in order to prove the efficiency of the multigrid strategy with the proposed projection operators given in Subsection 4.2. Finally, we conclude with Section 5 presenting selected numerical tests that confirm the efficiency of the proposed multigrid method in comparison with a state-of-the-art multigrid strategy for saddle-point structured problems.

1.1. Notation and preliminary results

This subsection is devoted to fix the notation and present the key concepts on multigrid methods and Toeplitz matrices.

We report the main definitions and properties of Toeplitz and circulant matrices generated by a function $f$. The domain of $f$ can be either one-dimensional $[-\pi, \pi]$ or $d$-dimensional $[-\pi, \pi]^d$ and this gives either unilevel or $d$-level structured matrices, respectively. The co-domain of $f$ can be either the complex field or the linear space of $d \times d$ complex matrices, generating scalar or block structured matrices, respectively. We will denote the generating function by bold $f$ when it is a matrix-valued function.

When we want to highlight that the generating function is multivariate, we specify its argument $f(\theta) = (\theta_1, \theta_2, \ldots, \theta_d) \in [-\pi, \pi]^d$. The structure and the size of the Toeplitz matrix is clearly related to its generating function.

In the whole paper we adopt the following notation on norms and relations between Hermitian Positive Definite (HPD) matrices. If $X \in \mathbb{C}^{N \times N}$ is a HPD matrix, then $\| \cdot \|_X = \|X^{1/2} \cdot \|_2$ denotes the Euclidean norm weighted by $X$ on $\mathbb{C}^N$. If $X$ and $Y$ are Hermitian matrices, then the notation $X \preceq Y$ means that $Y - X$ is a nonnegative definite matrix. Finally, given a matrix-valued function $f$ defined on the cube $[-\pi, \pi]^m$, we denote by $\|f\|_\infty = \text{ess sup}_{\theta \in [-\pi, \pi]^m} \|f(\theta)\|_2$.

In what follows, we report the formal definition of Toeplitz and circulant matrices.

**Definition 1.** Given a Lebesgue integrable function $f : [-\pi, \pi]^d \to \mathbb{C}^{s \times s}$, its Fourier coefficients are given by

$$\hat{f}_k = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(\theta)e^{-i\langle k, \theta \rangle} d\theta \in \mathbb{C}^{s \times s}, \quad k = (k_1, \ldots, k_d) \in \mathbb{Z}^d,$$

where $\theta = (\theta_1, \ldots, \theta_d)$, $(k, \theta) = \sum_{i=1}^d k_i \theta_i$, and the integrals of matrices are computed elementwise.

The $n$th Toeplitz matrix, where $n = (n_1, \ldots, n_d)$, associated with $f$ is the matrix of dimension $s \cdot n_1 \cdot \ldots \cdot n_d$ given by

$$T_n(f) = \sum_{1 \leq k \leq n-1} J^{k_1}_{n_1} \otimes \cdots \otimes J^{k_d}_{n_d} \otimes \hat{f}_k,$$

where $1$ is the vector of all ones, $J^{k_i}_{n}$ is the $n_i \times n_i$ matrix whose $(i, h)$th entry equals 1 if $(i - h) = k_i$ and 0 otherwise, and where $s \leq t$ means that $s_j \leq t_j$ for any $j = 1, \ldots, d$. 3
The $n$th circulant matrix associated with $f$ is the matrix of dimension $s_1 \times s_2$ given by

$$C_n(f) = \sum_{1 \leq i \leq s_1} Z_{n_1}^{k_i} \otimes \cdots \otimes Z_{n_d}^{k_d} \otimes \hat{f}_k,$$

where $Z_{n_\xi}^{k_\xi}$ is the $n_\xi \times n_\xi$ matrix whose $(i,h)$th entry equals 1 if $(i-h) \mod n_\xi = k_\xi$ and 0 otherwise.

**Theorem 1** (10). Let $f \in L^1([-\pi, \pi]^d, s)$ be a matrix-valued function with $d \geq 1$, $s \geq 2$. Then, the following (block-Schur) decomposition of $C_n(f)$ is valid:

$$C_n(f) = (F_n \otimes I_s)D_n(f)(F_n \otimes I_s)^H,$$

(1)

where

$$D_n(f) = \text{diag} \left( S_n(f) \left( \theta_k^{(n)} \right) \right),$$

(2)

$F_n$ is the $d$-level Fourier matrix, $F_n = F_{n_1} \otimes \cdots \otimes F_{n_d}$, $I_s$ is the identity matrix of size $s$ and

$$\theta_k^{(n)} = 2\pi k/n, \quad k = 0, \ldots, n - 1.$$

Moreover, $S_n(f)(\cdot)$ is the $n$-th Fourier sum of $f$ given by

$$S_n(f)(\theta) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_d=1-n_d}^{n_d} \hat{f}_j e^{(j, \theta)}, \quad (j, \theta) = \sum_{t=1}^d j_t \theta_t.$$

(4)

**Remark 1.** The latter results states that the eigenvalues of $C_n(f)$ are given by the evaluations of $\lambda_\xi(S_n(f)(\cdot))$, $t = 1, \ldots, s$, at the grid points $\{1\}$.

Moreover, if $f$ is a trigonometric polynomial of fixed degree (with respect to $n$), then it is worth noticing that $S_n(f)(\cdot) = \hat{f}(\cdot)$ for $n$ large enough. Therefore, in such a setting, the eigenvalues of $C_n(f)$ are the evaluations of $\lambda_\xi(f(\cdot))$, $t = 1, \ldots, s$, at the very same grid points. The latter implies that if $f \geq 0$ trigonometric polynomial and $\lambda_{\min}(f)$ does not vanish at any of the grid points $\{1\}$, then $C_n(f)$ is HPD.

**Remark 2.** In Definition 1 we considered square matrix-valued functions. In the general case where $f : [-\pi, \pi]^d \to C^{s_1 \times s_2}$, when we write $C_n(f)$, we mean that

$$C_n(f) = \Pi_{s_1n^d \times s_2n^d}^T \begin{bmatrix} C_n(f^{(1,1)}) & C_n(f^{(1,2)}) & \cdots & C_n(f^{(1,s_2)}) \\ C_n(f^{(2,1)}) & C_n(f^{(2,2)}) & \cdots & C_n(f^{(2,s_2)}) \\ \vdots & \vdots & \ddots & \vdots \\ C_n(f^{(s_1,1)}) & C_n(f^{(s_1,2)}) & \cdots & C_n(f^{(s_1,s_2)}) \end{bmatrix} \Pi_{s_2n^d \times s_1n^d},$$

(5)

where $f^{(j,k)}$ is the $(j,k)$th component of the function $f$ and, for $s \in \{s_1, s_2\}$, the permutation matrices $\Pi_{s_1n^d \times s_2n^d}$ are defined by

$$\Pi_{s_1n^d \times s_2n^d} = [\Pi_1 | \Pi_2 | \ldots | \Pi_{n^d}],$$

(6)

with

$$[\Pi_k]_{i,j} = \begin{cases} 1 & \text{if } i = (j-1)n^d + k, \\ 0 & \text{otherwise}, \end{cases} \quad i = 1, \ldots, s_1n^d, \quad j = 1, \ldots, s.$$
Remark 3. In the non-square case it is still possible to exploit symbols to perform products between circulant matrices as follows. Suppose \( f : [-\pi, \pi]^d \to \mathbb{C}^{s_1 \times s_2} \), \( g : [-\pi, \pi]^d \to \mathbb{C}^{s_2 \times s_3} \) and denote the components of the product \( fg \) with \( h^{(i,j)} \), then

\[
C_n(f)C_n(g) = \Pi^{T}_{n, s_1 \times s_1 \times s_1} \left[ C_n(f^{(i,j)}) \right]_{i=1, \ldots, s_1} \cdot \Pi^{T}_{n, s_2 \times s_2 \times s_2} \left[ C_n(g^{(i,j)}) \right]_{j=1, \ldots, s_2}.
\]

This approach is consistent with Definition 1, however, we avoided to include non-square matrices in the definition of circulants because in the next sections we will exploit the circulant algebra structure, which is natural in the square case.

Theorem 2 ([24, 25]). Let \( f \in L^1([\pi, \pi]^m) \) be a Hermitian \( d \times d \) matrix-valued function with \( m \geq 1 \), \( d \geq 2 \). Let \( m_1 \) be the essential infimum of the minimal eigenvalue of \( f \), \( M_1 \) be the essential supremum of the minimal eigenvalue of \( f \), \( m_4 \) be the essential infimum of the maximal eigenvalue of \( f \), and \( M_4 \) be the essential supremum of the maximal eigenvalue of \( f \).

1. If \( m_1 < M_4 \) then all the eigenvalues of \( T_n(f) \) belong to the open set \( (m_1, M_4) \) for every \( n \in \mathbb{N}^m \). If \( m_4 < M_4 \) then all the eigenvalues of \( T_n(f) \) belong to the open set \( [m_1, M_4) \) for every \( n \in \mathbb{N}^m \).
2. If \( m_1 = 0 \) and \( \theta_0 \) is the unique zero of \( \lambda_{\min}(f) \) such that there exist positive constants \( c, C, t \) for which

\[
c\|\theta - \theta_0\|^t \leq \lambda_{\min}(f(\theta)) \leq C\|\theta - \theta_0\|^t,
\]

then the minimal eigenvalue of \( T_n(f) \) goes to zero as \( (n_1 n_2 \ldots n_m)^{-1/m} \).

1.2. Multigrid methods

In this subsection we briefly report the relevant results concerning the convergence theory of algebraic multigrid methods [22] especially when the coefficient matrix is a block-Toeplitz matrix generated by a matrix-valued trigonometric polynomial [3].

In the general case, we are interested in solving a linear system \( A_N x_N = b_N \) where \( A_N \) is a Hermitian positive definite matrix. Assume \( N_0 > N_1 > \cdots > N_l > \ldots N_{l_{\min}} \) and define two sequences of full-rank rectangular matrices \( P_{N_t, N_{t+1}} \in \mathbb{C}^{N_t \times N_{t+1}} \) and \( R_{N_t, N_{t}} \in \mathbb{C}^{N_{t} \times N_{t+1}} \). Let us consider the stationary iterative methods \( V_{N_t, \text{pre}} \), with iteration matrix \( V_{N_t, \text{pre}} \), and \( V_{N_t, \text{post}} \), with iteration matrix \( V_{N_t, \text{post}} \).
that no pre-smoothing is applied.

Theorem 3. Let $A_{N_t}$ and $V_{N_t, \text{post}}$ be defined as in the TGM algorithm and assume that no pre-smoothing is applied ($V_{N_t, \text{pre}} = I_{N_t}$). Let us define

$$\text{TGM}(A_{N_t}, I_{N_t}, V_{N_t, \text{post}}, P_{N_t, N_{t+1}}, N_t, N_{t+1}) = V_{N_t, \text{post}}^{t+1} I_{N_t} - P_{N_t, N_{t+1}} \left(P_{N_t, N_{t+1}}^H A_{N_t} P_{N_t, N_{t+1}} \right)^{-1} P_{N_t, N_{t+1}}^H A_{N_t},$$

the iterative matrix of the TGM. Assume that

(a) the smoother fulfills the smoothing property:

$$\exists \sigma_{\text{post}} > 0 : \|V_{N_t, \text{post}} x_{N_t}\|_{A_{N_t}}^2 \leq \|x_{N_t}\|_{A_{N_t}}^2 - \sigma_{\text{post}} \|x_{N_t}\|_{A_{N_t}}^2,$$

$$\forall x_{N_t} \in \mathbb{C}^{N_t}.$$

(b) The grid transfer operator fulfills the approximation property:

$$\exists \gamma > 0 : \min_{y \in \mathbb{C}^{N_{t+1}}} \|x_{N_t} - P_{N_t, N_{t+1}} y\|_{A_{N_t}}^2 \leq \gamma \|x_{N_t}\|_{A_{N_t}}^2,$$

$$\forall x_{N_t} \in \mathbb{C}^{N_t}.$$
Then $\gamma \geq \sigma_{\text{post}}$ and

$$\|TGM(AN_{\ell}, I_{N_{\ell}}, V_{N_{\ell}, \text{post}}^{N_{\ell}, \text{post}}, P_{N_{\ell}, N_{\ell+1}} \|A_{N_{\ell}} \leq \sqrt{1 - \sigma_{\text{post}}/\gamma}.$$  

1.3. Multigrid methods for block circulant and block Toeplitz systems

If the coefficient matrix is of the form $C_n(f)$ or $T_n(f)$, with $f$ being a matrix-valued trigonometric polynomial, $f \geq 0$, sufficient conditions for the convergence and optimality of the TGM can be expressed in relation to $f$.

In particular, in [8] the authors derive sufficient conditions that should be fulfilled by the grid transfer operators in order to validate the approximation property (b) of Theorem 3. In the following we report a brief description of the result [8] in the general multilevel block Toeplitz setting.

Suppose $f$ is a $s \times s$ matrix-valued trigonometric polynomial, $f \geq 0$, such that there exist unique $\theta_0 \in [0, 2\pi)^d$ and $\bar{j} \in \{1, \ldots, s\}$ such that

$$\begin{aligned}
\lambda_j(f(\theta)) &= 0 \quad \text{for } \theta = \theta_0 \text{ and } j = \bar{j}, \\
\lambda_j(f(\theta)) &> 0 \quad \text{otherwise}.
\end{aligned}$$

(7)

The latter means that only one eigenvalue function of $f$ has exactly one zero in $\theta_0$ and $f$ is positive definite in $[0, 2\pi)^d \setminus \{\theta_0\}$. Define $q_j(\theta_0)$ as the eigenvector function of $f(\theta_0)$ associated with $\lambda_j(f(\theta_0)) = 0$. Moreover, define $\Omega(\theta) = \{\theta + \pi\eta, \eta \in \{0, 1\}^d\}$. Let us define a matrix-valued trigonometric polynomial $p(\cdot)$ such that

$$\begin{aligned}
\sum_{\xi \in \Omega(\theta)} p(\xi)^H p(\xi) > 0, \quad \forall \theta \in [0, 2\pi)^d, \\
\lambda_j(f(\theta)) - 1 - \lambda_j(s(\theta))) &= c,
\end{aligned}$$

(8)  

which implies that the trigonometric function

$$s(\theta) = p(\theta) \left( \sum_{\xi \in \Omega(\theta)} p(\xi)^H p(\xi) \right)^{-1} p(\theta)^H$$

is well-defined for all $\theta \in [0, 2\pi)^d$.

$$s(\theta_0)q_j(\theta_0) = q_j(\theta_0).$$

(9)

$$\limsup_{\theta \rightarrow \theta_0} \lambda_j(f(\theta))^{-1}(1 - \lambda_j(s(\theta))) = c,$$

(10)

where $c \in \mathbb{R}$ is a constant.

In order to have optimal convergence of the TGM it is sufficient compute a grid transfer operator as

$$P_{N_{\ell}, N_{\ell+1}} = T_{n_{\ell}}(p) \left(K_{N_{\ell}, N_{\ell+1}} \otimes I_s\right),$$

(11)

where $K_{N_{\ell}, N_{\ell+1}}$ is a $N_{\ell} \times N_{\ell+1}$ multilevel cutting matrix obtained as the Kronecker product of proper unilevel cutting matrices and $T_{n_{\ell}}(p)$ is a multilevel block-Toeplitz matrix generated by $p$.  

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1.4. TGM for saddle-point systems

In the previous section we focused on the case where the coefficient matrix is a block structured and non-negative definite matrix. For matrices in saddle-point form, Rüge-Stuben theory and Theorem 3 cannot be directly exploited. In [19] a general TGM procedure has been presented for a system of the form

\[
A = \begin{bmatrix} A & B^T \\ B & -C \end{bmatrix}
\] (12)

where \( A \) is an \( n \times n \) HPD matrix, \( C \) is an \( \hat{n} \times \hat{n} \) non-negative definite matrix and \( B \) is such that its has rank \( \hat{n} \leq n \) or that \( C \) is positive definite on the null space of \( B^T \).

Indeed, we have the following result.

**Theorem 4** ([19]). Let \( A \) be as in (12). Let \( \alpha \) be a positive number such that

\[
\alpha < \frac{2(\lambda_{\text{max}}(D^{-1}A))^{-1} - 1}{\kappa(A, \hat{P}_A)}
\]

and define

\[
\hat{A} = \mathcal{L}A\mathcal{U}, \quad \mathcal{L} = \begin{bmatrix} I_n & -I_{\hat{n}} \\ \alpha BD^{-1}A & -I_{\hat{n}} \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} I_n & -\alpha D^{-1}AB \end{bmatrix}
\] (13)

and

\[
\hat{C} = C + B(2\alpha D^{-1}A - \alpha^2 D^{-1}AD^{-1}A)B^T.
\] (14)

Let \( P_A \) and \( P_{\hat{C}} \) be, respectively, \( n \times k \) and \( \hat{n} \times \ell \) matrices of rank \( k < n \) and \( \ell < \hat{n} \), define the prolongation

\[
P = \begin{bmatrix} P_A \\ P_{\hat{C}} \end{bmatrix}
\] (15)

for the global system involving \( \hat{A} \) and suppose that the pairs \((A, P_A)\) and \((\hat{C}, P_{\hat{C}})\) fulfill the approximation property as in Theorem 3. Then, the spectral radius of the TGM iteration matrix using one iteration of the damped Jacobi method with relaxation parameter \( \omega \) as post smoother satisfies

\[
\rho(\text{TGM}(\hat{A}, P, \omega)) \leq \max \left(1 - \frac{\omega}{\kappa(A, P_A)}, 1 - \frac{\omega}{\kappa(\hat{C}, P_{\hat{C}})}, \omega \hat{\gamma}_A - 1, \omega \hat{\gamma}_{\hat{C}} - 1, \sqrt{1 - \frac{\omega(2 - \omega \hat{\gamma})}{\hat{\kappa}}}, \right)
\] (16)

where

\[
\hat{\gamma}_A = (\alpha (2 - \alpha \lambda_{\text{max}}(D^{-1}A)))^{-1}, \quad \hat{\gamma}_{\hat{C}} = \lambda_{\text{max}}(D_{\hat{C}}^{-1}(C + BA^{-1}B^T)),
\]

\[
\hat{\gamma} = \frac{2\hat{\gamma}_A \hat{\gamma}_{\hat{C}}}{\hat{\gamma}_A + \hat{\gamma}_{\hat{C}}}, \quad \hat{\kappa} = \frac{2\kappa(A, P_A)\kappa(\hat{C}, P_{\hat{C}})}{\kappa(A, P_A) + \kappa(\hat{C}, P_{\hat{C}})}.
\]

2. TGM for saddle-point systems in the multilevel block setting

The aim of the current section is to provide the theoretical background for the development and analysis of a TGM for the Stokes problem discretized with the technique of Subsection 3.1. The starting point is Theorem 3 which we want to combine with the theory for multilevel block structured matrices of Subsection 1.3. The derivations are an extension of the results in [7], where the authors consider a general unilevel circulant...
setting and a multilevel circulant setting which arises from the discretization of Stokes-like equations. The generalization that we consider in what follows consists in dealing with matrix-valued symbols instead of scalar-valued ones and this forces us to give the conditions on the symbols in a different form, owing to the non-commutativity of the matrix-matrix product.

The following results concern the properties of a matrix of the form

$$A = \begin{bmatrix} C_n(f_{A_x}) & O \\ O & C_n(f_{A_y}) \end{bmatrix} C_n^T(f_{B_x}) \quad \in \mathbb{R}^{(2s_n+s_c)n \times (2s_n+s_c)n}$$

with multilevel block circulant blocks generated by multivariate matrix-valued functions. For the sake of readability, we focus on the case \(d = 2\), \(n = (n, n)\) and, moreover, we fix \(s_n \geq s_c\). An extension to larger dimensional problems is straightforward, in this case additional blocks will be part of \(A\), e.g., for \(d = 3\) we will have an additional block \(A_z\) on the block diagonal and additionally \(B_z\) and \(B_z^T\) in the last block row and column, respectively.

**Lemma 1.** Let \(\tilde{A} = \begin{bmatrix} A_x & O \\ O & A_y \end{bmatrix} \in \mathbb{R}^{2n^2 \times 2n^2}\), \(A_x = C_n(f_{A_x})\), \(A_y = C_n(f_{A_y})\) with \(f_{A_x}\), \(f_{A_y}\) \(s \times s\) HPD matrix-valued bi-variate trigonometric polynomials.

If \(\alpha\) is a positive number such that

$$\alpha < 2 \left( \max_{z \in \{x,y\}} \left\{ \max_{j=1, \ldots, s} \left\| \left( \tilde{a}_0^{(j, j)}(f_{A_z}) \right)^{-1} \right\|_{\infty} \right\} \right)^{-1}$$

then the matrix \(2\alpha D_{\tilde{A}}^{-1} - \alpha^2 D_{\tilde{A}}^{-1} \tilde{A} D_{\tilde{A}}^{-1}\) is HPD, where \(D_{\tilde{A}} = \text{diag}(\tilde{A})\).

**Proof.** Since \(\tilde{A}\) is HPD, then \(D_{\tilde{A}}\) is HPD. Hence, writing

$$2\alpha D_{\tilde{A}}^{-1} - \alpha^2 D_{\tilde{A}}^{-1} \tilde{A} D_{\tilde{A}}^{-1} = D_{\tilde{A}}^{-\frac{1}{2}} \left( 2\alpha I_s - \alpha^2 D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}} \right) D_{\tilde{A}}^{-\frac{1}{2}}$$

we see that we need to prove that the Hermitian matrix \(2\alpha I_s - \alpha^2 D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}}\) is HPD. The eigenvalues of \(2\alpha I_s - \alpha^2 D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}}\) are of the form \(2\alpha - \alpha^2 \lambda_j \left( D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}} \right)\), where \(\lambda_j \left( D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}} \right)\) are the positive eigenvalues of \(D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}}\). Recalling that \(\alpha > 0\) by assumption, we have

$$\lambda_{\text{min}} \left( 2\alpha I_s - \alpha^2 D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}} \right) = 2\alpha - \alpha^2 \lambda_{\text{max}} \left( D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}} \right),$$

which implies that \(\alpha I_s - \alpha^2 D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}}\) is HPD if

$$0 < \alpha < \frac{2}{\lambda_{\text{max}} \left( D_{\tilde{A}}^{-\frac{1}{2}} \tilde{A} D_{\tilde{A}}^{-\frac{1}{2}} \right)}.$$ (17)

By construction, \(D_{\tilde{A}}^{-1} = \begin{bmatrix} D_{\tilde{A}_x}^{-1} & 0 \\ 0 & D_{\tilde{A}_y}^{-1} \end{bmatrix}\), with

$$D_{\tilde{A}_x}^{-1} = I_{n^2} \otimes \text{diag} \left( \tilde{a}_0^{(j, j)}(f_{A_x}) \right)^{-1}, \quad D_{\tilde{A}_y}^{-1} = I_{n^2} \otimes \text{diag} \left( \tilde{a}_0^{(j, j)}(f_{A_y}) \right)^{-1},$$

9
the parameter \( \alpha \) fulfills condition (17) and the thesis of the Lemma follows.

**Remark 4.** In order to follow the procedure analyzed by Theorem 4 we need to study the matrix \( \hat{C} \) defined in (14). Theorem 4 requires that either \( \hat{C} \) is HPD on the null space of \( B^T \). In the next theorem, we will require that \( \hat{C} \) is HPD, but this implies that the latter condition on \( B \) and \( C \) is fulfilled. Indeed, if \( \hat{C} \) is HPD and there exists \( v \in \mathbb{R}^{s_c \times s_c} \) such that \( v^T B = 0 \), then

\[
0 < v^T \hat{C} v < v^T C v + v^T B (2\alpha D^{-1}_A - \alpha^2 D^{-1}_A A D^{-1}_A) B^T v < v^T C v.
\]

Conversely, note that if \( B \) is full-rank or \( C \) is HPD on the null space of \( B^T \), then \( \hat{C} \) is HPD as long as \( \alpha \) is chosen according to Lemma 1.

We are now ready to state the main theorem concerning the convergence of a TGM of the form of Theorem 3 in the multilevel block circulant setting.

**Theorem 5.** Let us define the matrix

\[
\mathcal{A} = \begin{bmatrix} \hat{A} & B^T \\ B & -C \end{bmatrix} \in \mathbb{R}^{(2s_a + s_x) n^2 \times (2s_a + s_x) n^2}
\]  

having the blocks with the following structure:

- \( \hat{A} = \begin{bmatrix} A_x & O \\ O & A_y \end{bmatrix} \) where \( A_x = \mathcal{C}_n(f_{A_x}) \), \( A_y = \mathcal{C}_n(f_{A_y}) \) with \( f_{A_x}, f_{A_y} \) being \( s_a \times s_a \)

matrix-valued bivariate trigonometric polynomials.

- \( C = \mathcal{C}_n(f_C) \) with \( f_C \) \( s_c \times s_c \) non-negative bivariate trigonometric polynomial.
\[ B = [B_x, B_y] \text{ where } B_x = C_n(f_{B_x}), B_y = C_n(f_{B_y}) \text{ with } f_{B_x} \text{ and } f_{B_y} \text{ being } s_a \times s_c \text{ matrix-valued bivariate trigonometric polynomials such that} \]
\[ f_{B_x}(f_{A_x})^{-1}f_{B_x}^H + f_{B_y}(f_{A_y})^{-1}f_{B_y}^H \in L^\infty([-\pi, \pi]^2, s_c). \]  
\[ (19) \]

Let \( \alpha \) be a positive number such that
\[ \alpha < 2 \left( \max_{z \in \{x,y\}} \left\{ \max_{j=1,\ldots,n} \left( \frac{1}{\hat{a}_0(z)}(f_{A_x})^{-1} \right) \|f_{A_x}\|_\infty \right\} \right)^{-1} \]
\[ (20) \]

and define \( \bar{A} = \mathcal{L}A\mathcal{U} \) where
\[ \mathcal{L} = \begin{bmatrix} I_{2s_xn^2} & -\alpha BD^{-1} \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} I_{2s_xn^2} & -\alpha D^{-1}B^T \end{bmatrix}. \]
\[ (21) \]

Let \( P_{\bar{A}} \) and \( P_C \) be the full rank matrices defining the prolongation \( \mathcal{P} \) for the global system \( \bar{A} \). That is,
\[ \mathcal{P} = \begin{bmatrix} P_{\bar{A}} \\ P_C \end{bmatrix}, \]

where \( \bar{C} = C + B \left( 2\alpha D_{\bar{A}_1} - \alpha^2 D_{\bar{A}_1} \bar{A} D_{\bar{A}_1} \right) B^T = C_n(f_C), \]
\[ P_{\bar{A}} = \begin{bmatrix} C_n(p_{A_x})(K_n^T \otimes I_{s_x}) \\ C_n(p_{A_y})(K_n^T \otimes I_{s_x}) \end{bmatrix}, \quad P_C = C_n(p_C)(K_n^T \otimes I_{s_x}) \]
with \( p_{A_x}, p_{A_y}, \) and \( p_C \) bi-variate matrix-valued trigonometric polynomials. Consider a TGM associated with a basic scheme using only a single step of damped Jacobi post-smoothing with relaxation parameter \( \omega \). If

1. the trigonometric polynomials \( f_{A_x}, f_{A_y}, \) and \( f_C \) are in the setting of [7] and they are full rank at all grid points defined in [3];
2. the pairs \( (p_{A_x}, f_{A_x}), (p_{A_y}, f_{A_y}), \) and \( (p_C, f_C) \) fulfill relations [5]–[7];
3. the relaxation parameter \( \omega \) is a positive number such that

\[ \omega < 2 \min \left\{ 2\alpha - \alpha^2 \max_j \left\{ \frac{\|f_{A_x}\|_\infty}{\hat{a}_0(z)(f_{A_x})^{-1}} \|f_{A_y}\|_\infty \right\}, \quad \max_j \frac{\hat{a}_0(z)(f_C)}{\hat{a}_0(z)(f_{A_x})^{-1}} \|f_C + f_{B_x}f_{A_x}^{-1}f_{B_x}^H + f_{B_y}f_{A_y}^{-1}f_{B_y}^H \|_\infty \right\}, \]
\[ (22) \]

then \( \rho(TGM(\bar{A}, \mathcal{P}, \omega)) < 1 \).

**Proof.** As a consequence of Remark [1] the assumptions on \( f_{A_x}, f_{A_y}, \) and \( f_C \) imply that \( \bar{A} \) is HPD. Moreover, we see from Remark [4] that either \( B \) is full-rank or \( C \) is HPD on the null space of \( B^T \).

The assumptions on \( p_{A_x}, p_{A_y}, p_C \) and \( f_{A_x}, f_{A_y}, f_C \) ensure the validation of the approximation property by [3, Section 6]. That is, the pairs \( (\bar{A}, P_{\bar{A}}) \) and \( (\bar{C}, P_C) \) fulfill...
the approximation property in the 2D block setting. Therefore, we can apply Theorem 4 and by (16) we have

$$
\rho(TGM(\hat{A},\mathcal{P},\omega)) \leq \max \left( 1 - \frac{\omega}{\kappa(A,P_A^1)}, 1 - \frac{\omega}{\kappa(C,P_C^1)}, \omega\hat{\gamma}_A - 1, \omega\hat{\gamma}_C - 1, \sqrt{1 + \frac{\omega(2 - \omega\hat{\gamma})}{\kappa}} \right).
$$

In conclusion, in order to prove that \(\rho(TGM(\hat{A},\mathcal{P},\omega)) < 1\), we need to prove that each of the quantities in the maximum is bounded from above by a constant strictly smaller than 1.

The quantities \(1 - \frac{\omega}{\kappa(A,P_A^1)}\) and \(1 - \frac{\omega}{\kappa(C,P_C^1)}\) are strictly smaller than 1 because the approximation property constants \(\kappa(A,P_A^1)\) and \(\kappa(C,P_C^1)\) are finite and positive.

Concerning the two terms \(\omega\hat{\gamma}_A - 1\) and \(\omega\hat{\gamma}_C - 1\) in the maximum, we estimate

$$
\hat{\gamma}_A = \left( \frac{\alpha(2 - \alpha\lambda_{\text{max}}(D^{-1}A))}{2 - \alpha^2\lambda_{\text{max}}(D^{-1}A)} \right)^{-1} =
$$

$$
\left( 2\alpha - \alpha^2\lambda_{\text{max}} \right) \left( \frac{\left[ D^{-1}A_{ij} \ O \right]^{-1} A_{ij}^{-1}A_y}{O \ D^{-1}A_y} \right)^{-1} \leq \frac{\|a_j\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})} \left( \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})} \right),
$$

$$
\hat{\gamma}_C = \lambda_{\text{max}} \left( D^{-1}_y \left( C + B \hat{A}^{-1}B^T \right) \right) \leq \max_j \frac{1}{\tilde{D}_y^{(j)}(f_{\lambda})} \lambda_{\text{max}} \left( C + f_{\lambda} \cdot f_{\lambda}^{-1} f_{\lambda}^H f_{\lambda}^{-1} f_{\lambda}^H \right)
$$

$$
\leq \max_j \frac{1}{\tilde{D}_y^{(j)}(f_{\lambda})} \left( f_{\lambda} + f_{\lambda}^{-1} f_{\lambda}^H f_{\lambda}^{-1} f_{\lambda}^H \right) \left\| f_{\lambda} + f_{\lambda}^{-1} f_{\lambda}^H f_{\lambda}^{-1} f_{\lambda}^H \right\|_{\infty},
$$

where in the latter inequality we are using assumption 10 and the sub-additivity of norms. Hence with the choice of \(\omega\) in assumption 3., we have

$$
\omega\hat{\gamma}_A - 1 < 2 \left( 2\alpha - \alpha^2 \max_j \left\{ \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})}, \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})} \right\} \right),
$$

$$
\cdot \left( 2\alpha - \alpha^2 \max_j \left\{ \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})}, \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})} \right\} \right)^{-1} - 1
$$

$$
= 1,
$$

and

$$
\omega\hat{\gamma}_C - 1 < 2 \max_j \left\{ \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})}, \frac{\|f_{\lambda}\|_{\infty}}{\tilde{D}_y^{(j)}(f_{\lambda})} \right\} \left\| f_{\lambda} + f_{\lambda}^{-1} f_{\lambda}^H f_{\lambda}^{-1} f_{\lambda}^H \right\|_{\infty},
$$

$$
\cdot \max_j \frac{1}{\tilde{D}_y^{(j)}(f_{\lambda})} \left\| f_{\lambda} + f_{\lambda}^{-1} f_{\lambda}^H f_{\lambda}^{-1} f_{\lambda}^H \right\|_{\infty} - 1
$$

$$
= 1.
$$
Finally, in order to prove that \( \sqrt{1 - \frac{\omega(2 - \omega)}{\kappa}} < 1 \), we prove that \( 2 - \omega \tilde{\gamma} > 0 \). From hypothesis 3., we have

\[
\omega < 2\alpha - \alpha^2 \max_j \left\{ \frac{\|A_j\|_\infty}{\hat{a}_{ij}^{(j,j)}(C_{ij})}, \frac{\|B_i\|_\infty}{\hat{a}_{ij}^{(j,j)}(C_{ij})} \right\} + 
\max_j \hat{a}_{ij}^{(j,j)}(C_{ij}) \left\| f_C + \hat{f}_{B_2}^\dagger f_B^H + \hat{f}_{B_3}^\dagger f_B^H \right\|_\infty^{-1}.
\]  

(25)

Moreover, using the estimations (23) and (24), we have

\[
\tilde{\gamma} = \frac{2\hat{A}_\gamma C}{\hat{A}_\gamma + C} 
\leq 2 \left( \frac{2\alpha - \alpha^2 \max_j \left\{ \frac{\|A_j\|_\infty}{\hat{a}_{ij}^{(j,j)}(C_{ij})}, \frac{\|B_i\|_\infty}{\hat{a}_{ij}^{(j,j)}(C_{ij})} \right\} }{2\alpha - \alpha^2 \max_j \left\{ \frac{\|A_j\|_\infty}{\hat{a}_{ij}^{(j,j)}(C_{ij})}, \frac{\|B_i\|_\infty}{\hat{a}_{ij}^{(j,j)}(C_{ij})} \right\} + \max_j \hat{a}_{ij}^{(j,j)}(C_{ij})} \right)^{-1}
\]  

(26)

where we used the fact that the function \( (x, y) \mapsto 2 \frac{x}{x+y} \) is increasing in \((0, +\infty) \times (0, +\infty)\) with respect to the component-wise partial ordering in \( \mathbb{R}^2 \). Then, combining (25) and (20), we obtain \( \omega \tilde{\gamma} < 2 \).

**Remark 5.** In the previous theorem, some of the assumptions concern the pair \((\hat{C}, P_C)\).

The matrix \( \hat{C} \) is computed from \( A, B \) and \( C \), so the required assumptions can be stated involving explicitly the latter three matrices. For the sake of brevity, we discuss this point here in a particular setting, which is relevant from the application point of view. Indeed, we focus on the case where the matrix \( A \) in (18) is such that \( C \) is the null matrix and \( s_C = 1 \), so that \( \hat{f}_C \) is the scalar–value function

\[
\hat{f}_C = f_{B_2} g_z f_{B_3}^H + f_{B_3} g_y f_{B_3}^H,
\]

where we denoted by \( g_z \), the generating function of \( 2\alpha D_{A}^{-1} - \alpha^2 D_{A}^{-1} A_z D_{A}^{-1} \), for \( z \in \{x, y\} \). If we choose \( \alpha \) according to (20), then the functions \( g_z \) are HPD at all points. Assumption 1. of Theorem 5 then corresponds to requiring that there exists a \( \theta_0 \) such that \( f_{B_i}^\dagger(\theta) \) and \( f_{B_i}(\theta) \) are both the null vectors if and only if \( \theta = \theta_0 \). Regarding the choice of \( p_{\hat{C}} \), it is straightforward to see that in this scalar setting relations (8)–(10) are fulfilled by the pair \((p_{\hat{C}}, \hat{f}_C)\) if we have (8) and

\[
\lim_{\theta \to \theta_0} \frac{\sum_{\xi \in \Omega(\theta) \setminus \{\theta\}} |p(\xi)|^2}{\sum_{\xi \in \{x, y\}} f_{B_2}(\theta) g_z(\theta) |f_{B_3}^H(\theta)|} < \infty.
\]

Since the enumerator is a scalar, this second condition is the finiteness of the following limit superior:

\[
\lim_{\theta \to \theta_0} \frac{1}{\sum_{\xi \in \{x, y\}} \left( f_{B_2}(\theta) / \sum_{\xi \in \Omega(\theta) \setminus \{\theta\}} |p(\xi)| \right) g_z(\theta) \left( f_{B_3}^H(\theta) / \sum_{\xi \in \Omega(\theta) \setminus \{\theta\}} |p(\xi)| \right) 13}.
\]
Recalling again that the matrices $g_z(\theta)$ are HPD for all $\theta$, the latter condition corresponds to requiring that

$$\limsup_{\theta \to \theta_0} \frac{f_{B_z}(\theta)}{\sum_{\xi \in \Omega(\theta) \setminus \{\theta\}} |p(\xi)|} \neq 0 \quad \text{or} \quad \limsup_{\theta \to \theta_0} \frac{f_{B_z}(\theta)}{\sum_{\xi \in \Omega(\theta) \setminus \{\theta\}} |p(\xi)|} \neq 0. \quad (27)$$

**Remark 6.** We expect that the convergence result in Theorem 5 holds also for multilevel block Toeplitz matrices generated by trigonometric polynomials. Indeed, they are a low rank correction of multilevel block circulant matrices with the same generating function. The convergence could slightly deteriorate, but the combination of the proposed multigrid method with Krylov methods kills the outliers and guarantees the efficiency proven in the circulant case.

### 3. Saddle-point matrices stemming from the Stokes equation

In this section we are interested in applying the multigrid method described before to large linear systems stemming from the finite element approximation of the Stokes equation. The Stokes equation has the form

$$\begin{align*}
\Delta u + \text{grad } p &= -\Phi \quad \text{in } \Omega, \\
\text{div } u &= 0 \quad \text{in } \Omega, \\
\mathbf{u} &= g_D \quad \text{on } \Gamma_D, \\
\frac{\partial \mathbf{u}}{\partial n} &= g_N \quad \text{on } \Gamma_N,
\end{align*} \quad (28)$$

where $\mathbf{u}$ is the velocity and $p$ is the pressure, by $\Gamma_D$ we denote the Dirichlet boundary part of $\Gamma := \partial \Omega$ and by $\Gamma_N$ the Neumann part. For the construction and analysis of the system matrices we consider the domain $\Omega = (0,1)^2$, homogenous Dirichlet boundary conditions are attained for $x_1 = 0$ and $x_2 = 1$, homogenous Neumann boundary conditions for $x_1 = 1$ and $x_2 = 0$.

#### 3.1. A TGM for a Finite Element discretization of the Stokes problem

The weak form of (28) is given by

$$\begin{align*}
\mathbf{a}(\mathbf{u}, \mathbf{v}) - b(v, q) &= (\Phi, \mathbf{v}), \quad \text{for all } \mathbf{v}, \\
b(\mathbf{u}, q) &= 0, \quad \text{for all } q,
\end{align*}$$

where

$$\begin{align*}
\mathbf{a}(\mathbf{u}, \mathbf{v}) &= \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, dx \quad \text{and} \quad b(\mathbf{u}, q) = \int_\Omega \text{div } \mathbf{u} \, q \, dx.
\end{align*}$$

Higher order elements are needed for the discretization of the velocities $\mathbf{u}$ than for the discretization of the pressure $p$, e.g., Q2-Q1 Taylor-Hood-elements. Alternatively, linear elements can be used for the displacement but using a subdivision of the element, corresponding to a 2-times finer—or once refined—mesh for the displacements. We choose the latter approach that is often called Q1-iso-Q2/Q1 [9]. The properties of the system matrix are similar to those of the system matrix produced by Q2-Q1 elements. The advantage of using Q1-iso-Q2/Q1 is that the discrete operator representing $a$ is more
sparse than that for Q2-Q1 and thus cheaper to apply. The resulting algebraic system of equations given by

\[
A_N x = \begin{bmatrix}
A_x & O \\
O & A_y \\
B_x & B_y
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
\mathbf{b}_u \\
\mathbf{b}_p
\end{bmatrix} = \mathbf{b},
\] (29)

where \(A_x, A_y \in \mathbb{R}^{4n^2 \times 4n^2}\) and \(B_x, B_y \in \mathbb{R}^{4n^2 \times n^2}\), so \(N = 9n^2\). In particular, the matrices \(A_x\) and \(A_y\) are non-negative and they are a permutation of \(4 \times 4\) block bi-level Toeplitz matrices with partial dimension \(n\). Precisely, let \(e_i\) be the \(i\)th column of the identity matrix of size \(4n^2\), we can define a proper \(4n^2 \times 4n^2\) permutation matrix \(\Pi_2\) as

\[
\Pi_2 = \begin{bmatrix}
\theta & \xi \\
\xi & \theta
\end{bmatrix} \in \mathbb{R}^{4n \times 4n},
\]

such that the \(i\)th column of the identity matrix of size \(9n^2\) is

\[
\begin{bmatrix}
e^\theta & e^{i\theta} & 1 + e^{i\theta} & 1 + e^{-i\theta}
1 + e^{i\theta} & 1 + e^{-i\theta} & \varphi(-\theta_1, -\theta_2)
\varphi(\theta_1, -\theta_2) & \varphi(-\theta_1, \theta_2)
\varphi(\theta_1, \theta_2) & 1 + e^{i\theta_2} & 1 + e^{i\theta_1} & 1 + e^{-i\theta_1}
\end{bmatrix},
\] (30)

with \(\varphi(\theta_1, \theta_2) = 1 + e^{i\theta_1} + e^{i\theta_2} + e^{i(\theta_1 + \theta_2)}\). An analogous transformation can be applied to \(A_y\) and the generating function is \(f_{A_y}(\theta_1, \theta_2) = f_{A_x}(\theta_2, \theta_1)\).

We consider an analogous \(N \times N\) permutation matrix, \(\Pi_9 = [P_1|P_2|\ldots|P_9], P_l \in \mathbb{R}^{N \times n^2}, l = 1, \ldots, 9\), such that the \(l\)th column of \(P_l\) is \(e_l + e_9(k-1)\), with \(e_i\) being the \(i\)th column of the identity matrix of size \(9n^2\). We can transform the global matrix \(A_N\) in a \(9 \times 9\) block bi-level system. This property will be used later when we will exploit the information of the permuted matrix \(G_N = \Pi_9 A_N \Pi_9^T = T_N(f)\), where \(T_N(f)\) is the bi-level \(9 \times 9\) block Toeplitz \(T_N(f) = [\mathbf{f}]_{j \in [-1, \ldots, 1]^{15}} \in \mathbb{C}^{N \times N}\) generated by \(f : [-\pi, \pi]^2 \rightarrow \mathbb{C}^{9 \times 9}\). In particular,

\[
f(\theta_1, \theta_2) = \begin{bmatrix}
f_{A_x}(\theta_1, \theta_2) & O \\
O & f_{A_y}(\theta_1, \theta_2)
f_{B_x}(\theta_1, \theta_2) & f_{B_y}(\theta_1, \theta_2)
\end{bmatrix},
\] (31)

where

\[
f_{B_x}(\theta_1, \theta_2) = \begin{bmatrix}
\frac{1}{\sqrt{2}}(-\sin \theta_1)(1 + 2 \cos \theta_2) \\
\frac{1}{\sqrt{2}}(1 - e^{i\theta_1})(5 + \cos \theta_2) \\
\frac{1}{\sqrt{2}}(\sin \theta_1)(1 + e^{i\theta_2})
\end{bmatrix},
f_{B_y}(\theta_1, \theta_2) = \begin{bmatrix}
\frac{1}{\sqrt{2}}(1 + 2 \cos \theta_1)(-\sin \theta_2) \\
\frac{1}{\sqrt{2}}(1 + e^{i\theta_1})(-\sin \theta_2) \\
\frac{1}{\sqrt{2}}(5 + \cos \theta_1)(1 - e^{i\theta_2})
\end{bmatrix}.
\] (32)

To solve efficiently the system (29) with TGM satisfying the assumptions of Theorem 5 we investigate the structure of the coefficient matrix \(A_N\). It is a block matrix of the form

\[
A_N = \begin{bmatrix}
\tilde{A} & B^T \\
B & -C
\end{bmatrix},
\]
with \( C = O, B = [B_x, B_y] \) and \( \tilde{A} = \begin{bmatrix} A_x & O \\ O & A_y \end{bmatrix} \).

### 3.2. Spectral analysis of \( \tilde{A} \) and choice of \( P_A \)

Lemma \( \text{[1]} \) and Theorem \( \text{[5]} \) highlight the importance of the block \( \tilde{A} \) for the choice of the parameter \( \alpha \) and the construction of an efficient operator \( P_{3\tilde{A}} \). In practice we can just consider the matrix \( A_x \) and the related generating function \( f_{A_x} \), since \( A_y \) and \( f_{A_y} \) enjoy the same spectral properties. For this reason in the following we will avoid the specification \( x \) or \( y \). We already know that \( A \) is similar to a bi-level \( 4 \times 4 \) block Toeplitz \( T_n(f_A) \). The following proposition provides us with important spectral information on \( f_A \), and consequently \( T_n(f_A) \).

**Proposition 1.** Let \( f_A \) be the \( 4 \times 4 \) matrix-valued function defined in \( \text{[30]} \) and \( T_n(f_A) = \Pi_4 A \Pi_4^T \) the associated Toeplitz matrix. Then the four eigenvalue functions of \( f_A \) are

\[
\begin{align*}
\lambda_1(f_A) &= 3 - \frac{1}{3} e^{-(\frac{2j+1}{2})} \left( e^{2j} + e^{2j+1} \right) \left( e^{2j} + e^{2j+1} + 1 \right); \\
\lambda_2(f_A) &= 3 - \frac{1}{3} e^{-(\frac{2j+1}{2})} \left( e^{2j} + e^{2j+1} + 1 \right) \left( e^{2j} - e^{2j+1} \right); \\
\lambda_3(f_A) &= 3 + \frac{1}{3} e^{-(\frac{2j+1}{2})} \left( e^{2j} - e^{2j+1} + 1 \right) \left( e^{2j} + e^{2j+1} \right); \\
\lambda_4(f_A) &= 3 + \frac{1}{3} e^{-(\frac{2j+1}{2})} \left( e^{2j} + e^{2j+1} + 1 \right) \left( e^{2j} + e^{2j+1} \right).
\end{align*}
\]

Moreover,

1. the minimum eigenvalue function of \( \lambda_1(f_A) \) is non-negative with a zero of order 2 in \( \theta_0 = (0, 0) \). Furthermore, \( e_4 \) is the eigenvector of \( f_A(\theta_0) \) associated with \( \lambda_1(f_A(\theta_0)) = 0 \), where \( e_4 = [1, 1, 1, 1]^T \).
2. the minimal eigenvalue of \( T_n(f_A) \) (and consequently of \( A \)) goes to zero as \( (4n^2)^{-1} \);
3. the condition number \( \kappa(T_n(f_A)) \) of \( T_n(f_A) \) (and consequently of \( A \)) is proportional to \( O(n^2) \).

**Proof.** The function \( f_A \) can be written in a more compact form as

\[
\begin{align*}
f_A(\theta_1, \theta_2) &= \frac{9}{4} I_4 - \frac{1}{3} h(\theta_2) \otimes h(\theta_1) = 3I_4 - \frac{1}{3} h(\theta_2) \otimes h(\theta_1),
\end{align*}
\]

where \( h(\theta) = \begin{bmatrix} 1 & e^{-i\theta} + 1 \\ e^{i\theta} + 1 & 1 \end{bmatrix} \). This implies that \( T_n(f_A) \) is the SPD matrix

\[
T_n(f_A) = 3I_{4n^2} - \frac{1}{3} T_n(h(\theta_2)) \otimes T_n(h(\theta_1)).
\]

Moreover, formula \( \text{[34]} \) and the properties of the tensor product imply that the four eigenvalue functions \( \lambda_k(f_A), k = 1, \ldots, 4 \) of \( f_A \) are given by the following combination of the two eigenvalue functions of \( h(\theta) \)

\[
\left\{ 3 - \frac{1}{3} \lambda_i(h(\theta_2))\lambda_j(h(\theta_1)) \right\}_{i,j=1}^2.
\]
The eigenvalue functions of $h(\theta)$ can be computed analytically and they are given by

$$
\lambda_1(h(\theta)) = -e^{-\frac{\pi}{2}}(e^{\theta} - e^{\frac{\pi}{2}} + 1); \\
\lambda_2(h(\theta)) = e^{-\frac{\pi}{2}}(e^{\theta} + e^{\frac{\pi}{2}} + 1).
$$

Then, using their combination, we derive the expressions of the four eigenvalue functions of $f_A(\theta_1, \theta_2)$ which are exactly those given in formula (33). Moreover, by direct computation, we have

$$
f_A(0, 0)e_4 = \begin{bmatrix}
8/3 & -2/3 & -2/3 & -4/3 \\
-2/3 & 8/3 & -4/3 & -2/3 \\
-2/3 & -4/3 & 8/3 & -2/3 \\
-4/3 & -2/3 & -2/3 & 8/3
\end{bmatrix}e_4 = 0e_4.
$$

Then, $e_4$ is an eigenvector of $f_A(\theta_0)$ associated with 0. In addition, the minimal eigenvalue function is

$$
\lambda_1(f_A) = 3 - \frac{1}{3}e^{-\frac{(\theta_1 + \theta_2)}{2}}\left(e^{\theta_1} + e^{\frac{\theta_2}{2}} + 1\right)\left(e^{\theta_2} + e^{\frac{\theta_2}{2}} + 1\right),
$$

and it is straightforward to see that is a non-negative function over $[-\pi, \pi]$ and it is such that

$$
\frac{\partial \lambda_1(f_A)(\theta_1, \theta_2)}{\partial \theta_1}(0, 0) = 0, \\
\frac{\partial^2 \lambda_1(f_A)(\theta_1, \theta_2)}{\partial \theta_1\partial \theta_2}(0, 0) = 0,
$$

and

$$
\frac{\partial^2 \lambda_1(f_A)(\theta_1, \theta_2)}{\partial \theta_1^2}(0, 0) = \frac{1}{2}. \\
\frac{\partial^2 \lambda_1(f_A)(\theta_1, \theta_2)}{\partial \theta_2^2}(0, 0) = \frac{1}{2}.
$$

Therefore, $\lambda_1(f_A)$ has a zero of order 2 in $\theta_0 = (0, 0)$. In light of the third item of Theorem 2, we conclude that the minimal eigenvalue of $T_n(f_A)$ goes to zero as $(4n^2)^{-1}$ and, by similitude, that of $A$ as well. Furthermore,

$$
M = \max_{(\theta_1, \theta_2)\in[-\pi, \pi]} \lambda_4(f_A) = \max_{(\theta_1, \theta_2)\in[-\pi, \pi]} \lambda_3(f_A) = 4, 
$$

and from Theorem 2 we have that the spectrum of $T_n(f_A)$ is contained in $(0, 4)$. Consequently, the condition number $\kappa(T_n(f_A))$ of $T_n(f_A)$ is

$$
\kappa(T_n(f_A)) = \frac{\lambda_{\max}(T_n(f_A))}{\lambda_{\min}(T_n(f_A))} \leq \frac{4}{\lambda_{\min}(T_n(f_A))} \approx n^2.
$$

The similitude between $A$ and $T_n(f_A)$ concludes the proof of the Proposition. \qed

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A graphical confirmation of the result of the Proposition concerning the behavior of the eigenvalue functions is given in Figure 1 where the plot of \( \lambda_i(f_A) \), \( i = 1, \ldots, 4 \), is shown.

**Remark 7.** Because of the spectral properties of the symbol \( f_A \) and \( T_n(f_A) \), we need to choose

\[
\alpha < 2 \max_{j=1,\ldots,4} \left\{ \left( \frac{a_{i,j}}{\|\lambda_j(f_A)\|_{\infty}} \right)^{-1} - 1 \right\}^{-1} = 2 \left( \frac{3}{8} \right)^{-1} = 4/3
\]

Then, in the following we will fix \( \alpha \) equal to \( 4/3 \) since is the middle value in the range \((0, 4/3]\).

As suggested by Theorem 5, we need to construct

\[
P_A = \begin{bmatrix} C_n(p_{A_x})(K_n^T \otimes I_s) \\ C_n(p_{A_y})(K_n^T \otimes I_s) \end{bmatrix}, \quad P_C = C_n(p_C)(K_n^T \otimes I_s) \tag{37}
\]

with \( p_{A_x} \), \( p_{A_y} \), and \( p_C \) proper bi-variate matrix-valued trigonometric polynomials. Since \( f_{A_x} \) and \( f_{A_y} \) have the same spectral properties, we fix \( p_{A_x} = p_{A_y} = p_A \). In the following we discuss the choice of the polynomial \( p_A \).

We recall that a classical approach is that of constructing \( P_A \) as the multilevel linear interpolation projector. That is,

\[
P_A = \Pi_4^T \left( T_n(p^{(4)}) (K_{n,k} \otimes I_4) \right) \Pi_4, \tag{38}
\]

where \( K_{n,k} = K_{n,k} \otimes K_{n,k}, k = \frac{n-1}{2}, n \) odd, and
Theorem 6. Let $p^{(2)}(\theta) = p^{(2)}(\theta_1) \otimes p^{(2)}(\theta_2)$ be the 2 \times 2 trigonometric polynomial defined in (39) and $p^{(4)}(\theta_1, \theta_2) = p^{(2)}(\theta_1) \otimes p^{(2)}(\theta_2)$. Consider $f_A(\theta_1, \theta_2)$ defined in (37). Then, $p^{(4)}(\theta_1, \theta_2) = p^{(2)}(\theta_1) \otimes p^{(2)}(\theta_2)$ is such that:

\begin{itemize}
  \item \[ \sum_{\xi \in \Omega(\theta)} p^{(4)}(\xi) p^{(4)}(\xi) > 0, \quad \forall \theta \in [0, 2\pi)^2, \] \hspace{1cm} (40)

where $\Omega(\theta) = \{(\theta_1, \theta_2), (\theta_1 + \pi, \theta_2), (\theta_1, \theta_2 + \pi), (\theta_1 + \pi, \theta_2 + \pi)\}$.

  \item The function $s^{(4)}(\theta) = p^{(4)}(\theta) \left( \sum_{\xi \in \Omega(\theta)} p^{(4)}(\xi) p^{(4)}(\xi) \right)^{-1} p^{(4)}(\theta)^H$ is well-defined for all $\theta \in [0, 2\pi)^2$ and

\[ s^{(4)}(\theta_0)e_4 = e_4, \] \hspace{1cm} (41)

where $e_4 = [1, 1, 1, 1]^T$ and $\theta_0 = (0, 0)$.

  \item \[ \lim_{\theta \to \theta_0} (\lambda_1(f_A(\theta)))^{-1} (1 - \lambda_1(s^{(4)}(\theta))) = c, \] \hspace{1cm} (42)

where $c \in \mathbb{R}$ is a constant and $\lambda_1(f_A)$ and $\lambda_1(s^{(4)})$ are the minimal eigenvalue functions of $f_A$ and $s^{(4)}$, respectively.
\end{itemize}

Proof. Given $e_2 = [1, 1]^T$, the trigonometric polynomial $p^{(2)}$ by direct computation fulfills the following:

1) $p^{(2)}(0) e_2 = 4 e_2$.
2) $p^{(2)}(\pi) e_2 = 0 e_2$. 

Next theorem highlights the properties of the polynomial $p^{(2)}(\theta)$ and its multivariate version $p^{(4)}(\theta_1, \theta_2)$ in order to show that the grid transfer operator $P_A$ fulfills the conditions for the convergence and optimality of the TGM methods proposed in Section 6, here conditions \[ \text{8 - 10}. \]
3. Analysis of Item 4 clearly implies that particular to Lemma 4.3 in \[26\]. Then, we will exploit Remark 5 to theoretically validate of \(\hat{T}\) correction of the Toeplitz matrix on which our choice of sequences have been deeply investigated in \[1, 2, 15, 16, 26\], so we will first draft the idea instead of matrix-valued. Multigrid methods for scalar non-negative Toeplitz matrix \(A\) because of the structure of \(\hat{\lambda}\) we have \[\forall s \in \theta \] where \(\theta\) ensures that it verifies the analogous positivity condition given in \((40)\). Consequently, the quantity \[\hat{s}^{(2)}(0)e_2 = e_2,\]

where \(s^{(2)} = p^{(2)}(\theta) \left( p^{(2)}(\theta) H p^{(2)}(\theta) + p^{(2)}(\theta + \pi) H p^{(2)}(\theta + \pi)\right)^{-1} p^{(2)}(\theta) H\), is well-defined for all \(\theta \in [0, 2\pi]\). Because of the tensor structure of \(p^{(4)}\), the result in \[8,\] Lemma 6.2 ensures that it verifies the analogous positivity condition given in \((40)\). Consequently, the quantity

\[s^{(4)}(\theta) = p^{(4)}(\theta) \left( \sum_{\xi \in \Omega(\theta)} p^{(4)}(\xi) H p^{(4)}(\xi) \right)^{-1} p^{(4)}(\theta) H\]

is well-defined for all \(\theta = (\theta_1, \theta_2) \in [0, 2\pi]^2\). In addition, by \[8,\] Lemma 6.2, we have \(s^{(4)}(\theta_1, \theta_2) = s^{(2)}(\theta_1) \otimes s^{(2)}(\theta_2)\). Writing \(e_4\) as \(e_2 \otimes e_2\) and exploiting the properties of the Kronecker product, we have

\[s^{(4)}(\theta_0) e_4 = (s^{(2)}(\theta_0) \otimes s^{(2)}(\theta_0))(e_2 \otimes e_2) = (s^{(2)}(0)e_2) \otimes (s^{(2)}(0)e_2) = e_2 \otimes e_2 = e_4.\]

which concludes the proof of \((41)\). The last thing to be proved is that

\[\lim_{\theta \to \theta_0} \lambda_1(f_A(\theta))^{-1}(1 - \lambda_1(s^{(4)}(\theta))) < +\infty.\]

Note that \(s^{(2)}\) is an algebraic projector, that is, it can be verified by direct computation that \(s^2(\theta) - s(\theta) = 0\). Since \(\lambda_1(s(0)) = 1\) and the eigenvalue functions are continuous, we have \(\lambda_1(s^{(2)}(\theta)) \equiv 1\) and so \(\lambda_1(s^{(4)}(\theta))\) is identically 1 as well. Moreover, by \[83\], the function \(\lambda_1(f_A)\) is different from 0 in a neighbourhood of \(\theta_0\). Then, the bound in \((42)\) holds.

3.3. Analysis of \(\hat{C}\) and choice of \(P_C\)

In order to compute the sequence of grid transfer operators \(P_C\), we study the structure of \(\hat{C}\), which in the presented setting has a (bi-level) scalar nature.

Then, the associated spectral symbol will be a (bi-variate) scalar-valued function instead of matrix-valued. Multigrid methods for scalar non-negative Toeplitz matrix sequences have been deeply investigated in \[1, 2, 15, 16, 26\], so we will first draft the idea on which our choice of \(P_C\) is based to see that it fits into a classical setting, referring in particular to Lemma 4.3 in \[26\]. Then, we will exploit Remark \[5\] to theoretically validate our choice.

Firstly, we recall that \(\hat{C}\) is a non-negative matrix and \(\hat{C} = B(2\alpha D_A^{-1} - \alpha^2 D_A^{-1} \tilde{A} D_A^{-1}) B^T\). Because of the structure of \(A\), \(D_A = \frac{\alpha}{4} I_{8n^2}\). Moreover \(\alpha = \frac{2}{3}\). Then, \(\hat{C}\) is a low-rank correction of the Toeplitz matrix \(T_C(f_C)\) with

\[f_C(\theta_1, \theta_2) = \left( f_{B_F} \left( \frac{1}{2} I_4 - \frac{1}{16} f_{A_F} \right) f_{B_{F_{20}}} + f_{B_F} \left( \frac{1}{2} I_4 - \frac{1}{16} f_{A_F} \right) f_{B_{F_{20}}} \right) (\theta_1, \theta_2),\]
see Remark 3 for more details on the construction of the function $f_C$.

The function $f_C^\hat{}$ is a bivariate scalar valued function such that $\hat{a}_0(f_C^\hat{}) = \frac{11}{96}$. Moreover, it is straightforward to show that $f_C^\hat{}$ has a zero of order 2 in the origin. For example we can use the procedure analogous to the proof of Proposition 1 with the additional simplification that $f_C^\hat{}$ is scalar valued then its eigenvalue function coincides with $f_C^\hat{}$ itself. Figure 2 shows the plot of $f_C^\hat{}$ on the grid $[0, \pi]^2$.

Figure 2: Plot of the function $f_C^\hat{}$ on the grid $[0, \pi]^2$ with $n = 20$ points.

Then, we construct $P_C$ as

$$P_C = T_n((2 + 2 \cos \theta_1)(2 + 2 \cos \theta_2)) \, K_{n,k}.$$  \hfill (44)

In order to be sure that with this choice of grid transfer operator assumption 2. of Theorem 5 is fulfilled, we recall Remark 5. Positivity condition (8) is satisfied since the sum

$$|(2 + 2 \cos(\theta_1))(2 + 2 \cos(\theta_2))|^2 + |(2 + 2 \cos(\theta_1 + \pi))(2 + 2 \cos(\theta_2))|^2 + |(2 + 2 \cos(\theta_1 + \pi))(2 + 2 \cos(\theta_2 + \pi))|^2$$

is different from 0 for all $(\theta_1, \theta_2) \in \mathbb{R}^2$. Condition (27) is fulfilled since the functions in (32) vanish in $(0,0)$ with a zero of order 1.

3.4. Choice of the smoothing parameter

In order to fulfill the assumption 3. of Theorem 5 in our setting we require that

$$\omega < 2 \min \left\{ 2\alpha - \alpha^2 \max_j \left\{ \frac{\|f_{A_\theta}\|_{\infty}}{\hat{a}_0^{(j,j)}(f_{A_\theta})}, \frac{\|f_{A_\theta}\|_{\infty}}{\hat{a}_0^{(j,j)}(f_{A_\theta})} \right\}, \hat{a}_0(f_C^\hat{}) \left\| \begin{array}{c} f_C + f_{B_x} \Gamma_{A_y}^{-1} f_{B_x}^\dagger + f_{B_y} \Gamma_{A_x}^{-1} f_{B_y}^\dagger \end{array} \right\|_{\infty} \right\}.$$
That is

\[
ω < 2 \min \left\{ 2 \left( \frac{2}{3} \right)^2 - \left( \frac{2}{3} \right)^2 \frac{4 \cdot 3}{8}, \frac{11}{96} \left\| f_{B_x} f_{A_x}^{-1} f_{B_x}^H + f_{B_y} f_{A_y}^{-1} f_{B_y}^H \right\|_{\infty} \right\}
\]

\[
= 2 \min \left\{ 2 \left( \frac{2}{3} \cdot \frac{11}{24} \right) = \frac{11}{12} \right\}
\]

(45)

In the first equality above we are using the fact that the function \( f_S = f_{B_x} f_{A_x}^{-1} f_{B_x}^H + f_{B_y} f_{A_y}^{-1} f_{B_y}^H \) is a scalar valued function whose maximum is \( \frac{1}{4} \) corresponding to the point \((0, \pi)\) (equiv. \((\pi, 0)\)). Figure 3 shows the plot of \( f_S \) on a uniform grid over \([0, \pi]^2\).

Figure 3: Plot of the function \( f_S \) on a uniform grid over \([0, \pi]^2\) with \( n = 20 \) points.

Therefore, Theorem 5 ensures that the TGM applied to the system having \( \hat{A} \) as coefficient matrix converges with the choice \( \omega \in (0, 11/12) \).

4. Extension to MGM

In Section 2 we presented a TGM procedure for solving a system of the form (12) consisting in the transformation (21) and the application of the projecting strategy given by \( P \). In this section we propose a multigrid strategy with more than two grids. Moreover, in order to analyze the multigrid method for multiple grids, we need to study the problem at the coarser levels.

4.1. A 3-step procedure

Firstly, we define the CGC introducing a procedure consisting of 3 steps (3SP):

(i) transform the system into a new one by making use of the two invertible matrices \( L \) and \( U \).

(ii) Apply the projecting strategy given by \( P \).

(iii) “Symmetrize” the problem with a proper variable transformation.
First of all, note that after step one the transformed system is not symmetric. However, the block \( \tilde{C} = B(2\alpha D_A^{-1} - \alpha^2 D_A^{-1} \tilde{A} D_A^{-1}) B^T \) is a SPD matrix as long as the parameter \( \alpha \) is such that \( \alpha < 2(\lambda_{\text{max}}(D_A^{-1} A))^{-1} \). Equivalently, the matrix in block position \((1,1)\) is SPD, since it is given by \( \tilde{A} \).

After the application of the grid transfer operator

\[
P = \begin{bmatrix} P_A & P_C \end{bmatrix},
\]

the matrix at the coarse level has the form

\[
\begin{bmatrix}
P_A^T \hat{A} P_A \\
- P_C^T B (I_{(d-1)n^2} - \alpha D_A^{-1} \hat{A}) P_A \\
P_C^T C P_C + P_C^T B (2\alpha D_A^{-1} - \alpha^2 D_A^{-1} \tilde{A}) B P_C
\end{bmatrix},
\]

which is a matrix of a form which resembles that of the system (12), apart from the signs of the blocks in position \((2,1)\) and \((2,2)\). This explains why the third step of our procedure consists in the variable transformation obtained multiplying both \((41)\) and the right hand side on the left by the matrix

\[
\begin{bmatrix}
I_{2s_n n^2} \\
-I_{s_n n^2}
\end{bmatrix}.
\]

This last step restores the symmetry of the matrix and gives to \((41)\) the same block form of the original coefficient matrix so we can proceed recursively, defining a V-cycle procedure.

Precisely, the base of the recursion is the original system written as

\[
A_{N_0}(0) = \begin{bmatrix} \hat{A}(0) \\ B(0) \\ -C(0) \end{bmatrix},
\]

with \( \hat{A}(0) \in \mathbb{R}^{s_n n^2 \times s_n n^2} \) SPD, \( C(0) \in \mathbb{R}^{s_n n^2 \times s_n n^2} \) non-negative definite, and \( B(0) \in \mathbb{R}^{s_n n^2 \times 2s_n n^2} \).

At level \( \ell \), we have

\[
P_{\hat{A}(\ell)} \in \mathbb{R}^{2s_n n^2 \times 2s_n n^2} \quad \text{and} \quad P_C(\ell) \in \mathbb{R}^{s_n n^2 \times s_n n^2 + 1}
\]

the prolongation operators chosen for solving efficiently the scalar systems with coefficient matrix \( \hat{A}(\ell) \) and \( \tilde{C}(\ell) = C(\ell) + B(\ell)(2\alpha \ell D_A^{-1} \hat{A}(\ell) - \alpha^2 \ell D_A^{-1} \tilde{A}(\ell) D_A^{-1} \hat{A}(\ell) B(\ell)^T \), respectively, where \( \alpha_\ell \) is such that \( \alpha_\ell < 2\lambda_{\text{max}}(D_A^{-1} \hat{A}(\ell))^{-1} \). Then, the inductive step of our recursive definition is given by

\[
A_{N_{\ell+1}}(\ell + 1) = \begin{bmatrix} \hat{A}(\ell + 1) \\ B(\ell + 1) \\ -\tilde{C}(\ell + 1) \end{bmatrix},
\]

where

\[
\hat{A}(\ell + 1) = P_{\hat{A}(\ell)}^T \hat{A}(\ell) P_{\hat{A}(\ell)},
\]

\[
B(\ell + 1) = P_{\hat{A}(\ell)}^T B(\ell) (I_{2s_n n^2} - \alpha_\ell D_A^{-1} \hat{A}(\ell)) P_{\hat{A}(\ell)},
\]

\[
\tilde{C}(\ell + 1) = \left( P_{C(\ell)}^T \tilde{C}(\ell) P_C(\ell) + P_C^T B(\ell) (2\alpha \ell D_A^{-1} \hat{A}(\ell) - \alpha^2 \ell D_A^{-1} \tilde{A}(\ell) D_A^{-1} \hat{A}(\ell) B(\ell)^T P_C(\ell) \right).
\]
For the considerations that we made when we described our 3SP, for each \( \ell \) we have that \( \hat{A}\{\ell + 1\} \in \mathbb{R}^{2n_s^2n_{\ell+1}\times 2n_s^2n_{\ell+1}} \) is SPD, \( C\{\ell + 1\} \in \mathbb{R}^{s_s^2n_{\ell+1}\times s_s^2n_{\ell+1}} \) is non-negative definite, and \( B\{\ell + 1\} \in \mathbb{R}^{s_s^2n_{\ell+1}\times 2s_s^2n_{\ell}^2} \).

This procedure can be seen globally as one block multigrid algorithm with restriction and prolongation operators \( R\{\ell\} \) and \( P\{\ell\} \).

**Proposition 2.** Consider the three steps procedure described by items (i)-(ii). At level \( \ell \), define the prolongation and restriction operators as

\[
R\{\ell\} = \begin{bmatrix}
\alpha\{\ell\} P^T_{C(\ell)} I_{2s_s n_s^2}\{\ell\} & B\{\ell - 1\} D^{-1}_{\hat{A}(\ell)} P_{\hat{A}(\ell)} & -I_{s_s^2 n_s^2}\{\ell\}
\end{bmatrix},
\]

\[
P\{\ell\} = \begin{bmatrix}
P_{\hat{A}(\ell)}^T & P_{C(\ell)}^T \\
I_{2s_s n_s^2}\{\ell\} & -\alpha\{\ell\} P^T_{\hat{A}(\ell)} D^{-1}_{\hat{A}(\ell)} B^T\{\ell - 1\} P_{\hat{A}(\ell)}
\end{bmatrix}
\]

After \( \ell + 1 \) iterations of 3SP, the resulting coefficient matrix \( \mathcal{A}_{N_{\ell+1}}\{\ell + 1\} \) is given by

\[
\mathcal{A}_{N_{\ell+1}}\{\ell + 1\} = R\{\ell\} \begin{bmatrix}
\hat{A}\{\ell\} & B^T\{\ell\}
\end{bmatrix} P\{\ell\}. \tag{56}
\]

*Proof.* The assertion follows immediately by direct computation. \( \square \)

### 4.2. Smoother and grid transfer operator at coarser levels

In order to develop a multigrid method as described in the previous subsection, we need to choose a sequence of smoothing parameters \( \omega_\ell \) and to construct two sequences of grid transfer operators whose elements are \( P_{\hat{A}(\ell)} \) and \( P_{C(\ell)} \), associated respectively with \( \hat{A}\{\ell\} \) and \( \hat{C}\{\ell\} \).

In the circulant setting, the matrices \( \hat{A}\{\ell\} \) and \( \hat{C}\{\ell\} \) are still circulant matrices and we denote by \( f_{\hat{A}(\ell)} \) and \( f_{\hat{C}(\ell)} \) the respective generating functions. In the Toeplitz setting, the Toeplitz structure is lost at coarser levels, in particular for the block \( \hat{C}\{\ell\} \). However, \( \hat{C}\{\ell\} \) can still be associated with the symbol \( f_{\hat{C}(\ell)} \), computed like in the circulant case. This is possible because the corrections at the corners have a rank which remains bounded as \( \ell \) increases, as a consequence of the bound on the matrix bandwidth on which we will comment in the next section.

In principle, our strategy at level \( \ell \) is to choose the smoothing parameter \( \omega_\ell \) and the grid transfer operators \( P_{\hat{A}(\ell)} \) and \( P_{C(\ell)} \) such that the TGM applied to the system with coefficient matrix \( \mathcal{A}_{N_\ell}\{\ell\} \) converges. The convergence of the “coarser TGMs” is a necessary condition for the convergence of full multigrid methods such as the W-cycle and the V-cycle, see [2] for a discussion on the topic.

Combining all the previous considerations, for constructing an MGM for \( \mathcal{A} \) in [2] we aim at fulfilling the assumptions of Theorem [3] applied to the matrix \( \mathcal{A}_{N_\ell}\{\ell\} \) for all \( \ell \). A rigorous theoretical analysis of the properties of the symbols \( f_{\hat{A}(\ell)} \) and \( f_{\hat{C}(\ell)} \) in the block
multilevel setting would require a significant amount of computations, undermining the readability of the paper, so we omit it. Concerning $f_{A(t)}$, the reader can refer to \([5]\). In the applications, it is usually possible to choose at the finest level the trigonometric polynomials $p_{A_s}$, $p_{A_s}$, and $p_C$ in such a way that the same choice is suitable for all levels, that is, the properties of $f_{A(t)}$ and $f_{C(t)}$ which are relevant for the TGM convergence remain unchanged level after level. For instance, this is the case for the problem and TGM analysed in Section \(3\) as we will remark in Section \(5\).

Regarding the choice of $\omega_t$, considering equation (22) we derive the following condition:

$$\omega_t < 2 \min \left\{ 2\alpha t - \alpha^2 \max_j \frac{\|f_{A_0}(t)\|_\infty}{\hat{a}_0^{(j,j)}(f_{A_0}(t))}, \frac{\|f_{A_0}(t)\|_\infty}{\hat{a}_0^{(j,j)}(f_{A_0}(t))} \right\},$$

$$\max_j \hat{a}_0^{(j,j)}(f_{C(t)}) \left\| f_{C(t)} + f_{B_0(t)} \hat{f}_{A_0}(t) \hat{f}_{B_0}(t) + f_{B_0(t)} \hat{f}_{A_0}(t) \hat{f}_{B_0}(t) \right\|_\infty^{-1},$$

which suggests how the smoothing parameter should be chosen at each level for obtaining a convergent W-cycle.

### 4.3. Bandwidth at coarser levels

When the generating function is a (multivariate matrix-valued) trigonometric polynomial, the matrix-vector product with the associated circulant or Toeplitz matrix has a computational cost linear in the matrix size. If this is the case for the blocks $A$, $B$, and $C$, the matrix-vector product with matrix $A$ in \([15]\) is still linear in the matrix size. Therefore, in order to bound the computational cost of an iteration of the multigrid method, it is fundamental to make sure that the degree of the generating trigonometric polynomials is bounded as the level increases. In \([1]\) and \([2]\) the authors prove the latter in the unilevel scalar case, we report the results below.

**Lemma 2.** \([4]\), Proposition 2\] Let $A(t)$ be defined in \([51]\), with $P_{A(t)} = C_n(p_A)K_n^T$, where $f_{A(0)}$ and $p_A$ are trigonometric polynomials of degree $z_{A(0)}$ and $q_A$, respectively. Then $f_{A(t)}$ is a trigonometric polynomial of degree $z_{A(t)}$ such that:

1. $z_{A(t)} \leq \max(z_{A(0)}, 2q_A)$ for all $t$;
2. $z_{A(t)} \leq 2q_A$ for $t$ large enough.

**Lemma 3.** \([2]\) Consider the matrices $B(t) = C_n(f_{B(t)})$ and $C(t) = C_n(f_{C(t)})$ defined by formulas \((22)\)–\((23)\), with $P_{A(t)} = C_n(p_A)K_n^T$ and $P_{C(t)} = C_n(p_C)K_n^T$. Let $z_{B(t)}$ and $z_{C(t)}$ be the degrees of $f_{B(t)}$ and $f_{C(t)}$, respectively. Let $q = \max\{q_A, q_C\}$, where $q_A$ and $q_C$ are the polynomial degrees associated with $p_A$ and $p_C$, respectively.

Then, for $t$ large enough, it holds

1. $z_{B(t)} \leq \max(2z_{B(0)}, 4q)$,
2. $z_{C(t)} \leq \max(4z_{B(0)}, 6q, 2z_{C(0)})$.

Seeing a matrix-valued trigonometric polynomial as a matrix whose elements are trigonometric polynomials, it is straightforward to extend the latter results to block circulant matrices. Indeed, an element of a generating matrix-valued trigonometric polynomial $f_{A(t)}$, $f_{B(t)}$, or $f_{C(t)}$ at coarser levels is the sum of trigonometric polynomials whose degree is bounded by Lemmas \([2, 4, 8]\).
Table 1: Two-Grid iterations for the matrix $A_N$ with $\epsilon = 10^{-6}$ and $\omega_{\text{post}} = [2/5, 3/5, 4/5]$.

| $N = 9 \cdot (2^t + 1)^2$ | $\omega_{\text{post}} = 2/5$ | $\omega_{\text{post}} = 3/5$ | $\omega_{\text{post}} = 4/5$ |
|--------------------------|-----------------|-----------------|-----------------|
| 5 | 38 | 23 | 17 |
| 6 | 36 | 22 | 16 |
| 7 | 35 | 20 | 15 |
| 8 | 35 | 19 | 15 |

5. Numerical Examples

In this section we present the numerical confirmation of the efficiency of the proposed block multigrid method when applied on the linear system (29).

We exploit the theoretical results presented in Section 2 and Subsection 3.2 to choose the smoothing parameter $\omega$ and the grid transfer operator $P = \begin{bmatrix} P_A & P_C \end{bmatrix}$. Precisely, $P_A$ and $P_C$ are defined by formulae (37) and (44), respectively. The choice of the Jacobi smoothing parameter $\omega$, which completes the multigrid procedure, is related with the range of admissibility in (45), that is $\omega \in (0, 11/12)$.

All the tests are performed using MATLAB 2021a and the error equation at the coarsest level is solved with the MATLAB backslash function. In all the experiments we use the standard stopping criterion $\|r(k)\|_2/\|b\|_2 < \epsilon$, where $r(k) = b - Mx(k)$ and $\epsilon = 10^{-6}$.

Concerning $b$, we consider the case where the true solution $x$ of the linear system $Mx = b$ is a uniform sampling of $\sin(4t) + \cos(6t) + 1$ on $[0, \pi]$ and we compute the right-hand side $b$ as $b = Mx$.

Since the grid-transfer operator $P_A$ has a geometrical meaning—that is, it represents standard bilinear interpolation—we take the partial dimension $n$ of the form $2^t + 1$, $t = 5, \ldots, 8$.

We first consider a TGM consisting of only 1 post smoothing step of damped Jacobi. We test the efficiency of the TGM with three different values, $\omega = 2/5, 3/5, 4/5$. The iterations shown in Table 1 remain almost constant as the matrix size increases, confirming the theoretical optimal convergence rate.

As a second experiment we consider different number of steps of pre/post smoother. In order to damp the error both in the middle and in the high frequencies, we take a different parameter for the pre-smoother and the post-smoother (when both are present). In particular, we choose $\omega_{\text{post}} = 4/5$ and $\omega_{\text{pre}} = 3/5$, since they give the fastest convergence over the considered sample of admissible values. The convergence rate in all these cases is optimal (see Table 2) since the presence of the pre smoothing or of additional smoothing steps can accelerate the convergence of the TGM.

In Section 4 we described the structure of the blocks at coarser levels and how the multigrid ingredients should be chosen when considering more than two grids. In this case, our choice of trigonometric polynomials $p_{A_x}$, $p_{A_y}$, and $p_{C}$ ensures that the spectral properties of $\tilde{A}\{\ell\}$ and $\hat{C}\{\ell\}$ remain unchanged at coarser levels, as we verified numerically using the associated symbols. Regarding the choice of $\omega_\ell$, we numerically verified that choosing at all levels the smoothing parameter which is suitable for the first level is
Table 2: Two-Grid iterations for the matrix $A_N$ with $\epsilon = 10^{-6}$ and with different number of pre/post smoothing steps. If both present, $\omega_{\text{post}} = 4/5$ and $\omega_{\text{pre}} = 3/5$.

$N = 9 \cdot (2^t + 1)^2$

| $t$  | TGM(0, 1) | TGM(1, 0) | TGM(1, 1) | TGM(2, 2) |
|------|-----------|-----------|-----------|-----------|
| 5    | 17        | 20        | 16        | 15        |
| 6    | 16        | 19        | 16        | 15        |
| 7    | 15        | 18        | 15        | 14        |
| 8    | 15        | 17        | 14        | 13        |

Table 3: Comparison of TGM, W-cycle and V-cycle iterations for the matrix $A_N$ with 2 iterations of pre/post smoother of Jacobi and the choice $\omega_{\text{post}} = 4/5$ and $\omega_{\text{pre}} = 3/5$, $\epsilon = 10^{-6}$.

$N = 9 \cdot (2^t + 1)^2$

| $t$  | TGM | W-cycle | V-cycle |
|------|-----|---------|---------|
| 5    | 15  | 15      | 15      |
| 6    | 15  | 15      | 15      |
| 7    | 14  | 14      | 15      |
| 8    | 13  | 13      | 16      |

a computationally cheaper and more robust choice in order to accelerate the convergence of the V-cycle. This approach was already taken in [7], where a comparison with an adaptive choice of $\omega_{\ell}$ was provided. In Table 3 we numerically prove the independence of the convergence rate with respect to the matrix size also for the W-cycle and V-cycle methods using 2 steps of pre/post smoother, with the choice $\omega_{\text{post}} = 4/5$ and $\omega_{\text{pre}} = 3/5$ at all levels.

In many works [3, 4, 11, 13] problems with such saddle-point structure have been treated, exploiting particular preconditioned iterative strategies. In particular, a common choice is that of constructing preconditioning for two-by-two block systems involving the Schur complement, whose computation requires the solution of the linear system with coefficient matrix the positive definite part $\tilde{A}$ of $A_N$. Then, the theoretical findings of Subsection 3.2 on $\tilde{A}$ and on the grid transfer operator $P_{\tilde{A}(\ell)}$ can be exploited also for strategies different from the proposed ones. Moreover, Subsection 3.2 suggests valid tools that can be used for constructing grid transfer operators, since it shows how to proceed for verifying the conditions which lead to convergence and optimality. Furthermore, another approach to solve saddle-point problem of the form (12) consists of multigrid methods in which the smoother takes into account a special coupling, represented by the off-diagonal blocks in (12). Then, as a last test we compare our results with the standard Vanka smoother [28]. The analysis of block smoothers like the one mentioned can be found in [17, 18, 21]. In Table 4 we compare the iterations of the V-cycle method using the two different smoothers: column I shows the iterations for the system $A_N$ considering 2 pre/post smooting steps of Vanka method. The iterations of the second column are related to the V-cycle method applied to system $LAU$, whose efficiency has been already discussed. Even if the Vanka smoother performs well in combination with the linear interpolation projecting strategy, we stress that the computational cost of a
Table 4: V-cycle iterations with \( \epsilon = 10^{-6} \), 2 pre 2 post smoothing steps

| \( N = 9 \cdot (2^t + 1)^2 \) | \# Iterations of V-cycle with Vanka for \( \tilde{A}_N \) | \# Iterations of V-cycle with Jacobi for \( \hat{A}_N \) |
|-----------------|------------------------|------------------------|
| 5               | 10                     | 15                     |
| 6               | 12                     | 15                     |
| 7               | 12                     | 15                     |
| 8               | 11                     | 16                     |

A single Vanka-based MGM iteration is higher than the one of an iteration of our procedure.

6. Conclusions

In this paper we studied a symbol-based convergence analysis for problems that have a saddle-point form with block Toeplitz structure. From theoretical point of view, we extended the sufficient conditions for the TGM convergence and the choice of smoothing parameters to the case where the associated generating function \( f \) is a matrix-valued function. Moreover, we demonstrated that a recursive procedure is possible, preserving the saddle-point structure at the coarser levels, after proper symmetrization. In order to test the efficiency of the theory we considered the saddle-point system involved in the numerical solution of the Stokes problem in 2D discretized with FEM.

In particular, we exploited the properties of the generating functions of the blocks and how these can be used to develop a block multigrid method which is convergent and optimal. Numerically, the resulting methods show the expected convergence behavior, confirming the validity of our analysis.

The promising results suggest further investigation into the efficiency of the procedure for other practical applications with challenging problems. The analysis is noteworthy as it provides with reasonable effort a theoretical guarantee of convergence by solely examining the symbols of the matrix sequences involved.

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References

[1] A. Aricò and M. Donatelli. A v-cycle multigrid for multilevel matrix algebras: proof of optimality. *Numer. Math.*, 105(4):511–547, 2007.
[2] A. Aricò, M. Donatelli, and S. Serra-Capizzano. V-cycle optimal convergence for certain (multilevel) structured linear systems. *SIAM J. Matrix Anal. Appl.*, 26(1):186–214, 2004.
[3] O. Axelsson, S. Farouq, and M. Neytcheva. Comparison of preconditioned Krylov subspace iteration methods for PDE-constrained optimization problems: Poisson and convection-diffusion control. *Numer. Algorithms*, 73(3):631–663, 2016.
[4] O. Axelsson and M. Neytcheva. Eigenvalue estimates for preconditioned saddle point matrices. *Numer. Linear Algebra Appl.*, 13(4):339–360, 2006.
[5] M. Benzi, G. H. Golub, and J. Liesen. Numerical solution of saddle point problems. *Acta Numer.*, 14:1–137, 2005.
[6] M. Benzi and V. Simoncini. On the eigenvalues of a class of saddle point matrices. *Numer. Math.*, 103(2):173–196, 2006.
[7] M. Bolten, M. Donatelli, P. Ferrari, and I. Furci. Symbol based convergence analysis in multigrid methods for saddle point problems (under revision) arXiv 2203.05959.
[8] M. Bolten, M. Donatelli, P. Ferrari, and I. Furci. A symbol-based analysis for multigrid methods for block-circulant and block-Toeplitz systems. *SIAM J. Matrix Anal. Appl.*, 43(1):405–438, 2022.
[9] F. Brezzi and M. Fortin. *Mixed and hybrid finite element methods*, volume 15 of Springer Series in Computational Mathematics. Springer-Verlag, New York, 1991.
[10] P. Davis. *Circulant Matrices*. J. Wiley and Sons, New York, 1979.
[11] M. Donatelli, A. Dorostkar, M. Mazza, M. Neytcheva, and S. Serra-Capizzano. Function-based block multigrid strategy for a two-dimensional linear elasticity-type problem. *Comput. Math. Appl.*, 74(5):1015–1028, 2017.
[12] M. Donatelli, P. Ferrari, I. Furci, D. Sesana, and S. Serra-Capizzano. Multigrid methods for block-circulant and block-Toeplitz large linear systems: Algorithmic proposals and two-grid optimality analysis. *Numer. Linear Algebra Appl.*, e2356, 2020.
[13] A. Dorostkar, M. Neytcheva, and S. Serra-Capizzano. Spectral analysis of coupled PDEs and of their Schur complements via generalized locally Toeplitz sequences in 2D. *Comput. Methods Appl. Mech. Engrg.*, 309:74–105, 2016.
[14] F. Durastante and I. Furci. Spectral analysis of saddle-point matrices from optimization problems with elliptic PDE constraints. *Electron. J. Linear Algebra*, 36:773–798, 2020.
[15] G. Fiorentino and S. Serra. Multigrid methods for Toeplitz matrices. *Calcolo*, 28(3-4):283–305 (1992), 1991.
[16] G. Fiorentino and S. Serra. Multigrid methods for symmetric positive definite block Toeplitz matrices with nonnegative generating functions. *SIAM J. Sci. Comput.*, 17(5):1068–1081 (1996), 1996.
[17] Y. He and S. P. MacLachlan. Local Fourier analysis of block-structured multigrid relaxation schemes for the Stokes equations. *Numer. Linear Algebra Appl.*, 25(3):e2147, 28, 2018.
[18] S. P. MacLachlan and C. W. Oosterlee. Local Fourier analysis for multigrid with overlapping smoothers applied to systems of PDEs. *Numer. Linear Algebra Appl.*, 18(4):751–774, 2011.
[19] Y. Notay. A new algebraic multigrid approach for Stokes problems. *Numer. Math.*, 132(1):51–84, 2016.
[20] D. Noutsos, S. Serra-Capizzano, and P. Vassalos. Matrix algebra preconditioners for multilevel Toeplitz systems do not insure optimal convergence rate. *Theoret. Comput. Sci.*, 315(2):557–579, 2004.
[21] C. Rodrigo, F. J. Gaspar, and F. J. Lisbona. On a local Fourier analysis for overlapping block smoothers on triangular grids. *Appl. Numer. Math.*, 105:96–111, 2016.
[22] J. W. Ruge and K. Stüben. Algebraic multigrid. In *Multigrid methods*, volume 3 of Frontiers Appl. Math., pages 73–130. SIAM, Philadelphia, PA, 1987.
[23] S. Serra. Asymptotic results on the spectra of block Toeplitz preconditioned matrices. *SIAM J. Matrix Anal. Appl.*, 20(1):31–44, 1999.
[24] S. Serra-Capizzano. Asymptotic results on the spectra of block Toeplitz preconditioned matrices. *SIAM Journal on Matrix Analysis and Applications*, 20(1):31–44, 1998.
[25] S. Serra-Capizzano. Spectral and computational analysis of block Toeplitz matrices with nonnegative definite generating functions. *BIT*, 39:152–175, 1999.
[26] S. Serra-Capizzano and C. Tablino-Possio. Multigrid methods for multilevel circulant matrices. *SIAM J. Sci. Comput.*, 26(1):55–85, 2004.
[27] D. Sesana and V. Simoncini. Spectral analysis of inexact constraint preconditioning for symmetric saddle point matrices. *Linear Algebra Appl.*, 438(6):2683–2700, 2013.
[28] S. P. Vanka. Block-implicit multigrid solution of Navier-Stokes equations in primitive variables. *J. Comput. Phys.*, 65(1):138–158, 1986.