AN F. AND M. RIEZS THEOREM
FOR PLANAR VECTOR FIELDS

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Abstract. We prove that solutions of the homogeneous equation $Lu = 0$, where $L$ is a locally integrable vector field with smooth coefficients in two variables possess the F. and M. Riesz property. That is, if $\Omega$ is an open subset of the plane with smooth boundary, $u \in C^1(\Omega)$ satisfies $Lu = 0$ on $\Omega$, has tempered growth at the boundary, and its weak boundary value is a measure $\mu$, then $\mu$ is absolutely continuous with respect to Lebesgue measure on the noncharacteristic portion of $\partial \Omega$.

Introduction

Consider a Borel measure $\mu$ defined on the boundary $\mathbb{T}$ of the unit circle $\Delta$ of the complex plane. A classical theorem proved in 1916 by F. and M. Riesz states that if the Fourier coefficients of $\mu$ vanish for all negative integral values, i.e.,

$$(a) \quad \hat{\mu}(k) = \int_0^{2\pi} \exp(-2\pi i k \theta) d\mu(\theta) = 0, \quad k = -1, -2, \ldots,$$

then $\mu$ is absolutely continuous with respect to the Lebesgue measure $d\theta$. Condition (a) is equivalent to the existence of a holomorphic function $f(z)$ defined on $\Delta$ whose weak boundary value is $\mu$.

The F. and M. Riesz theorem has undergone an extensive generalization in the last decades, mainly in two different directions: i) generalized analytic function algebras, which has as a starting point the fact that (a) means that $\mu$ is orthogonal to the algebra of continuous functions on $\mathbb{T}$ that extend holomorphically to $\Delta$; ii) ordered groups, which emphasizes instead the role of the group structure of $\mathbb{T}$ in the classical result. Thus, although absolute continuity with respect to Lebesgue measure is a local property (i.e., if each point has a neighborhood where it holds then it holds everywhere), both directions focus on global objects. A remarkable exception is the paper [B] in which the author uses microlocal analysis to prove some generalizations of the theorem of F. and M. Riesz. Among other things, in [B] it is shown that if a CR measure on a hypersurface of $\mathbb{C}^n$ is the boundary value of a holomorphic function defined on a side, then it is absolutely continuous with respect to Lebesgue measure.

In view of Riemann’s mapping theorem and the local character of the conclusion, another way of stating the F. and M. Riesz theorem is to say that if a holomorphic
function \( f(z) \) defined on a smoothly bounded domain \( D \) of the complex plane has tempered growth at the boundary and its weak boundary value is a measure, then the measure is absolutely continuous with respect to Lebesgue measure. If we regard holomorphic functions as solutions of the homogeneous equation \( \overline{\partial} f = 0 \), it is natural to ask for which complex vector fields \( L \) it is possible to draw the same conclusion for solutions of the equation \( Lf = 0 \). In this paper we extend the F. and M. Riesz theorem to all locally integrable, smooth complex vector fields in the plane for smooth domains at the noncharacteristic part of the boundary. We recall that a nowhere vanishing smooth vector field

\[
L = a(x, y) \frac{\partial}{\partial x} + b(x, y) \frac{\partial}{\partial y}
\]

is said to be locally integrable in an open set \( \Omega \) if each \( p \in \Omega \) is contained in a neighborhood which admits a smooth function \( Z \) with the properties that \( LZ = 0 \) and the differential \( dZ \neq 0 \). Examples of locally integrable vector fields include real analytic vector fields and smooth, locally solvable vector fields. We note however that the class of locally integrable vector fields is much larger and refer to [T] on this subject.

In his work [B], the author gives a microlocal criterion for the absolute continuity of a measure analogous to (a) based on Uchiyama’s deep characterization of \( BMO(\mathbb{R}^n) \) [U]. Similarly, one of the main steps in our generalization of the F. and M. Riesz theorem is Theorem 3.1 in section 3 which concerns the location of the wave front set of the trace of a \( C^1 \) solution of a locally integrable vector field in \( \mathbb{R}^n \). Here an important tool is the use of the FBI transform in the fashion developed in [BCT] and [T]. However, since we apply this result for \( n = 2 \), in which case the trace lives in a one dimensional boundary, we do not need to rely on Uchiyama’s theorem and the classical criterion (a) suffices. On the other hand, while in the classical case and the generalizations in [B] the location of the wave front set of the measure under consideration always satisfies a restrictive hypothesis which leads to absolute continuity, this restriction is not fulfilled in general by the trace of a solution of an arbitrary locally integrable vector field even if the solution is smooth (an example concerning a vector field with real analytic coefficients is shown in Example 4.3). Thus, we need to deal as well with points where the wave front set of the measure may contain all directions; at those points the vector field \( L \) exhibits a behavior close to that of a real vector field (in a sense made precise in Lemma 3.3) and absolute continuity may be proved directly.

The paper is organized as follows. In section 1 we state our main result (Theorem 1.1) and prove a lemma on the existence of traces on a noncharacteristic boundary for continuous solutions of an arbitrary smooth complex vector field. In Section 2 we slightly extend one of the basic results in [B] with a new method of proof based on the FBI transform. In Section 3 we focus on locally integrable vector fields and give a refined description of the location of the wave front set of a solution’s trace; this is a key step in the proof of Theorem 1.1 which is also proved in this section. Finally, in section 4 we present some examples.

1. Statement of the main result

The following theorem is the main result of this article. The existence of the trace \( bf = f(x, 0) \) will be proved in Lemma 1.2 in this section.
Theorem 1.1. Suppose \( L = \frac{\partial}{\partial t} + a(x,t)\frac{\partial}{\partial x} \) is smooth in a neighborhood \( U \) of the origin in the plane. Let \( U_+ = U \cap \mathbb{R}^2_+ \), and suppose \( f \in C^1(U_+) \) satisfies \( Lf = 0 \) in \( U_+ \) and for some integer \( N \),

\[
|f(x,t)| = O(t^{-N}) \quad \text{as} \quad t \to 0.
\]

Assume that \( L \) is locally integrable in \( U \). If the trace \( \text{bf} = f(x,0) \) is a measure, then it is absolutely continuous with respect to Lebesgue measure.

We begin with a lemma on the existence of a trace that improves Theorem 3.4 in \([B]\). In the lemma, the vector field \( L \) will not be assumed to be locally integrable.

Lemma 1.2. Let \( X \subseteq \mathbb{R}^n \) be open, \( U \) an open neighborhood of \( X \times \{0\} \) in \( \mathbb{R}^{n+1} \), \( U_+ = U \cap \mathbb{R}^n_+ \). Let \( L = \frac{\partial}{\partial t} + \sum_{j=1}^{n} a_j(x,t)\frac{\partial}{\partial x_j}, a(x,t) \in C^\infty \) on \( X \cup U_+ \). Let \( f \) be a continuous function on \( U_+ \) such that \( Lf \in L^\infty(U_+) \) and for some \( N \in \mathbb{N} \),

\[
|f(x,t)| = O(t^{-N}) \quad \text{as} \quad t \to 0,
\]

uniformly on compact subsets of \( X \). Then \( \lim_{t \to 0} f(x,t) = \text{bf} \) exists in \( \mathbb{D}'(X) \). Furthermore, if \( X \times (0,T] \subseteq U_+ \), then the distributions \( \{f(\cdot,t) : 0 \leq t \leq T\} \) are uniformly bounded in \( \mathbb{D}'(X) \).

Remark. In \([B]\) this Lemma is proved under the additional assumption that \( f \) is \( C^1 \) and that

\[
|\partial_x f(x,t)| = O(t^{-N}) \quad \text{as} \quad t \to 0.
\]

Proof of Lemma 1.2. We will proceed as in \([B]\) with some modifications. Let \( \phi \in C^\infty_0(X) \), and \( T > 0 \) such that

\[
\text{supp} \phi \times [0,T] \subseteq X \cup U_+
\]

For \( \epsilon \geq 0 \) sufficiently small, set

\[
L^\epsilon = \frac{\partial}{\partial t} + \sum_{j=1}^{n} a_j(x,t+\epsilon)\frac{\partial}{\partial x_j}
\]

Let \( k \in \mathbb{N} \). We will choose \( \phi_0^\epsilon, \ldots, \phi_k^\epsilon \in C^\infty(U_+) \) such that if

\[
\Phi^{k,\epsilon}(x,t) = \sum_{j=0}^{k} \phi_j^\epsilon(x,t) \frac{t^j}{j!},
\]

then

(1) \( \Phi^{k,\epsilon}(x,0) = \phi(x) \), and (2) \( |(L^\epsilon)^k \Phi^{k,\epsilon}(x,t)| \leq Ct^k \),

where \( C \) depends only on the derivatives of \( \phi \) up to order \( k + 1 \). In particular, \( C \) will be independent of \( \epsilon \). Define \( \phi_0^\epsilon(x,t) = \phi(x) \). For \( j \geq 1 \), write

\[
L^\epsilon = \frac{\partial}{\partial t} + Q^\epsilon(x,t,\frac{\partial}{\partial x}),
\]
and define

$$\phi_j^*(x, t) = -\frac{\partial}{\partial t} \phi_{j-1}(x, t) + (Q^*)^j \phi_{j-1}$$

One easily checks that (1) and (2) above hold with these choices of the $\phi_j^*$. We will next use the integration by parts formula of the form

$$\int u(x, T)w(x, T)dx - \int u(x, 0)w(x, 0)dx = \int_0^T \int_{\mathbb{R}^n} (wPu - uP^*w)dxdt$$

which is valid for $P$ a vector field, $u$ and $w$ in $C^1(\mathbb{R}^n \times [0, T])$ and the $x$–support of $w$ contained in a compact set in $\mathbb{R}^n$. Note that the $x$-support of $\Phi^{k,\epsilon}(x, t)$ is contained in the support of $\phi(x)$. Let $\psi \in C_0^\infty(B_1(0))$, where $B_1(0)$ denotes the ball of radius 1 centered at the origin in $\mathbb{R}^{n+1}$. Assume $\int \frac{\partial}{\partial t}\psi dxdt = 1$, and for $\delta > 0$, let $\psi_\delta(x, t) = \frac{1}{\delta^{n+1}} \psi(\frac{x}{\delta}, \frac{t}{\delta})$. For $\epsilon > 0$, set $f_\epsilon(x, t) = f(x, t + \epsilon)$. Observe that if $\delta < \epsilon$, then the convolution $f_\epsilon * \psi_\delta(x, t)$ is $C^\infty$ in the region $t > 0$. In the integration by parts formula above set $u(x, t) = f_\epsilon * \psi_\delta(x, t)$, $w(x, t) = \Phi^{k,\epsilon}(x, t)$ and $P = L^\epsilon$. We get:

$$\int_X f_\epsilon * \psi_\delta(x, 0)\phi(x)dx = \int_X f_\epsilon * \psi_\delta(x, T)\Phi^{k,\epsilon}(x, T)dx$$

$$- \int_0^T \int_X L^\epsilon(f_\epsilon * \psi_\delta)\Phi^{k,\epsilon}dxdt$$

$$+ \int_0^T \int_X f_\epsilon * \psi_\delta(L^\epsilon)\Phi^{k,\epsilon}dxdt$$

Fix $\epsilon > 0$. Let $\delta \to 0^+$. Note that $f_\epsilon * \psi_\delta(x, t)$ converges uniformly to $f_\epsilon(x, t)$ on a neighborhood $W$ of $\text{supp } \phi \times [0, T]$. Hence in $\mathcal{D}'(W)$,

$$L^\epsilon(f_\epsilon * \psi_\delta) \to L^\epsilon f_\epsilon$$

as $\delta \to 0^+$. Moreover, $L^\epsilon f_\epsilon(x, t) = Lf(x, t + \epsilon) \in L^\infty$. Hence by Friederichs’ Lemma,

$$L^\epsilon(f_\epsilon * \psi_\delta) \to L^\epsilon f_\epsilon$$

in $L^2(W)$ as $\delta \to 0^+$. We thus get

$$\int_X f(x, \epsilon)\phi(x)dx = \int_X f(x, T + \epsilon)\Phi^{k,\epsilon}(x, T)dx$$

$$- \int_0^T \int_X L^\epsilon f_\epsilon(x, t)\Phi^{k,\epsilon}(x, t)dxdt$$

$$+ \int_0^T \int_X f_\epsilon(x, t)(L^\epsilon)\Phi^{k,\epsilon}(x, t)dxdt$$

In the third integral on the right, we have

$$|f_\epsilon(x, t)(L^\epsilon)\Phi^{k,\epsilon}(x, t)| \leq Ct^{k-N},$$

where $C$ depends only on the derivatives of $\phi$ upto order $k + 1$. Choose $k = N + 1$. By the dominated convergence theorem, as $\epsilon \to 0$, this third integral converges.
to $\int_0^T \int_X f L^* \Phi^k dx dt$. In the second integral on the right, note that since $Lf \in L^2(X \times (0,T))$, as $\epsilon \to 0$, the translates $L^f_\epsilon = (Lf)_\epsilon \to Lf$ in $L^2$. We thus get

$$\langle bf, \phi \rangle = \int_X f(x,T) \Phi^k(x,T) ds - \int_0^T \int_X Lf \Phi^k dx dt + \int_0^T \int_X f L^* \Phi^k dx dt,$$

where $\Phi^k = \Phi^{k,0}$. From formula (1.2), we also see that there is $C > 0$ independent of $\epsilon$ such that

$$(1.3) \quad |\langle f(.,\epsilon), \phi \rangle| \leq C \sum_{|\alpha| \leq k+1} \|\partial^\alpha \phi\|_{L^\infty}.$$  

2. The FBI approach

We will next present another proof of Theorem 3.5 in [B]. Our method of proof is based on a variant of the FBI transform developed in [BCT] and [T]; this approach will also be used in the proof of Theorem 1.1 which concerns a class of vector fields not covered in [B]. In our version, thanks to Lemma 1.2, we will not assume that $\partial_x f(x,t)$ has a tempered growth as $t \to 0^+$. Note also that in Theorem 2.1 local integrability of $L$ is not assumed.

**Theorem 2.1.** Let $X$, $U$, $U_+$ and $L = \frac{\partial}{\partial t} + \sum_{j=1}^n a_j(x,t) \frac{\partial}{\partial x_j}$ be as in Lemma 1.2. Suppose $f \in C^1(U_+)$ satisfies $|f(x,t)| = O(t^{-N})$ for some $N$ and

$$|Lf(x,t)| = O(t^k), \quad k = 1, 2, ...$$

uniformly on compact subsets of $X$. Assume

$$\partial^l_t a(x,0) = 0 \quad \forall j < l, \quad \forall x \in X$$

and that

$$\langle \partial^l_t \text{Im} a(x_0,0), \xi^0 \rangle > 0 \quad \text{for some } x_0 \in X, \quad \xi^0 \in \mathbb{R}^n.$$  

Then $(x_0, \xi^0) \notin WF(bf)$.

**Proof.** Without loss of generality, we may assume that $x_0 = 0$. Let $Z_1, \ldots, Z_n$ be a complete set of smooth approximate first integrals of $L$ near the origin in $U$ (see [T] for the existence of such). That is,

$$LZ_j(x,t) = O(t^k), \quad k = 1, 2, ... \quad \text{and} \quad Z_j(x,0) = x_j, \quad 1 \leq j \leq n.$$  

For $j = 1, \ldots, n$ let $M_j = \sum_{k=1}^n b_{jk}(x,t) \frac{\partial}{\partial x_k}$ be vector fields satisfying

$$M_j Z_k = \delta^k_j, \quad [M_j, M_k] = 0.$$  

Note that for each $j$,

$$(2.1) \quad [M_j, L] = \sum_{s=1}^n c_{js} M_s$$
where each \( c_{jk} = O(t^k), \ k = 1, 2, \ldots \) Indeed, the latter can be seen by expressing \([L_j, L]\) in terms of the basis \(\{L, M_1, \ldots, M_n\}\) and applying both sides to the \(n + 1\) functions \(\{t, Z_1, \ldots, Z_n\}\). For any \(C^1\) function \(g\), observe that the differential

\[
(2.2) \quad dg = \sum_{k=1}^{n} M_k(g) dZ_k + (Lg - \sum_{k=1}^{n} M_k(g) LZ_k) dt
\]

This is verified by evaluating each side at the basis vector fields \(\{L, M_1, \ldots, M_n\}\).

Using (2.2) we get:

\[
(2.3) \quad d(gdZ_1 \wedge \cdots \wedge dZ_n) = (Lg - \sum_{k=1}^{n} M_k(g) LZ_k) dt \wedge dZ_1 \wedge \cdots \wedge dZ_n
\]

For \(\xi \in \mathbb{R}^n, \ s \in \mathbb{R}^n\), let

\[
E(s, \xi, x, t) = i\xi \cdot (s - Z(x, t)) - |\xi|^2(s - Z(x, t))^2,
\]

where for \(w \in \mathbb{C}^n\), we write \(w^2 = \sum_{j=1}^{n} w_j^2\). Let \(B\) denote a small ball centered at 0 in \(\mathbb{R}^n\) and \(\phi \in C^\infty_0(B)\), \(\phi \equiv 1\) near the origin. We will apply (2.3) to the function

\[
g(s, \xi, x, t) = \phi(x) f(x, t) e^{E(s, \xi, x, t)}
\]

where \((s, \xi)\) are parameters. We get:

\[
(2.4) \quad d(gdZ) = \{L(\phi f) + (\phi f) LE - \sum_{k=1}^{n} (M_k(\phi f) + \phi f(M_k E)) LZ_k\} e^{E} dt \wedge dZ,
\]

where \(dZ = dZ_1 \wedge \cdots \wedge dZ_n\). Next by Stokes theorem we have, for \(t_1 > 0\) small:

\[
(2.5) \quad \int_B g(s, \xi, x, t_1) dx = \int_B g(s, \xi, x, 0) dx + \int_0^{t_1} \int_B d(gdZ)
\]

We will estimate the two integrals on the right in (2.5). Write

\[
Z = (Z_1, \ldots, Z_n) = x + t\Psi(x, t), \ \text{and} \ \Psi = \Psi_1 + i\Psi_2.
\]

Since the \(Z_j\) are approximate solutions of \(L\), we have

\[
\Psi + t\Psi_t + (I + t\Psi_x) \cdot a = O(t^k), \ k = 1, 2, \ldots
\]

and hence

\[
(2.6) \quad \partial_j^l \Psi(x, 0) = 0, \quad j < l \ \text{and} \ \langle \partial_j^l \Psi_2(x, 0), \xi^0 \rangle < 0
\]

for \(x\) in a neighborhood \(V\) of \(\overline{B}\) (after shrinking \(B\), if necessary). Observe that

\[
\text{Re} \ E(s, \xi, x, t) = t\xi \cdot \Psi_2(x, t) - |\xi|^2(s - t\Psi_1)^2 - t^2\Psi_2(x, t)^2
\]
Because of (2.6), continuity and homogeneity in \( \xi \), we can get \( c_1 > 0 \) such that
\[
\operatorname{Re} E(s, \xi, x, t) \leq -c_1|\xi|^{d+1}, \quad \text{for } x \in V, \quad 0 \leq t \leq t_1
\]  
(2.7) \( s \in \mathbb{R}^n \) and \( \xi \) in a conic nbhd \( \Gamma \) of \( \xi^0 \).

Going back to the integrals in (2.5), we clearly have
\[
\left| \int_B g(s, \xi, x, t_1) d_x Z(x, t_1) \right| \leq e^{-c_2|\xi|},
\]
for some \( c_2 > 0 \), for \( s \in \mathbb{R}^n \) and \( \xi \in \Gamma \). To estimate \( \int_0^{t_1} \int_B d(gdZ) \), we use (2.4) and look at each term that appears there. We first consider the term \( L(\phi f)e^E \). For any \( k \),
\[
|\phi(Lf)e^E| \leq C_k t^{dk} e^{-c_1 t^{d+1}|\xi|} \leq C'_{k|\xi|^{d+1}}
\]

Moreover, the \( x \)-integral
\[
\int_B (L\phi)f e^E dZ = \langle f(., t), (L\phi)e^E \rangle
\]
can be estimated using Lemma 1.2. Accordingly, after decreasing \( t_1 \), we can get \( \delta > 0 \) such that if \( |s| \leq \delta \) and \( \xi \in \Gamma \),
\[
|\langle f(., t), (L\phi)e^E \rangle| \leq C \sum_{|\alpha| \leq N+1} \left| \partial_\alpha ((L\phi)e^E) \right|_{L^\infty} \leq C e^{-c|\xi|}
\]
for some constants \( c, C > 0 \). In the latter, we have used the constancy of \( \phi \) near 0.

It follows that the integral
\[
\int_B \int_0^{t_1} L(\phi f)e^E dt \wedge dZ
\]
decays rapidly in \( \xi \). The term \( (\phi f)LEe^E \) is estimated using the fact that for any \( k \), \( |LE| \leq c_k t^k |\xi| \) for some constant \( c_k \) and that \( |e^E| \leq e^{-c_1 t^{d+1}|\xi|} \). This shows that
\[
\int_B \int_0^{t_1} (\phi f)LEe^E dt \wedge dZ
\]
decays rapidly in \( \xi \). The integral of \( \phi f(M_k E)LZ_k e^E \) is estimated likewise. To estimate the integral of \( (M_k(\phi f))LZ_k e^E \), we first integrate in \( x \) and apply Lemma 1.2 again. Indeed, the Lemma also applies to the weak derivative \( M_k(\phi f) \). Thus
\[
\int_B \int_0^{t_1} d(gdZ)
\]
has a rapid decay in \( \xi \), and going back to (2.5), we have shown:
\[
F(s, \xi) = \int_B e^{\xi(s-x)-|\xi|(s-x)^2}\phi(x)f(x,0)dx
\]  
(2.8)
decays rapidly for $|s| \leq \delta$ in $\mathbb{R}^n$ and $\xi$ in a conic neighborhood $\Gamma$ of $\xi^0$. The function $F(s, \xi)$ is the FBI transform (see [BCT]) of the distribution $\phi(x)f(x, 0)$. To prove the formula, we will exploit the inversion formula for the FBI, namely,

$$
\phi(x)f(x, 0) = \lim_{\epsilon \to 0^+} c_n \int e^{(x-s) \cdot \xi - \epsilon|x|^2} F(s, \xi) |\xi|^2 ds d\xi
$$

where $c_n$ is a dimensional constant. Assume now that $\phi(x)$ is supported in the ball centered at the origin with radius $M$. We will study the inversion integral in (2.9) by writing it as a sum of three pieces: $I_1(\epsilon)$, $I_2(\epsilon)$, and $I_3(\epsilon)$. The first piece consists of integration over the region $\{(\xi, s) : |s| \geq 2M\}$. In the second piece we integrate over $\{(\xi, s) : \delta \leq |s| < 2M\}$, and in the third piece over $\{(\xi, s) : |s| \leq \delta\}$. For the integral $I_1(\epsilon)$, after setting $s = 0$, one gets an exponential decay in $\xi$ independent of $\epsilon$, and hence $\lim_{\epsilon \to 0^+} I_1(\epsilon)$ is in fact a holomorphic function near the origin in $\mathbb{C}^n$. To study the second piece, we write it as

$$
I_2(\epsilon) = c_n \int_{\{(y, \xi, s) : \delta \leq |s| < 2M\}} e^{i(x-y) \cdot \xi - |\xi|^2 (s-y)^2 - \epsilon|\xi|^2} \phi(y)f(y, 0)|\xi|^2 dy ds d\xi
$$

We will use the holomorphic function $\zeta(\xi) = (\zeta_1^2 + \cdots + \zeta_n^2)^{\frac{1}{2}}$ where we take the principal branch of the square root in the region $|\text{Im} \zeta| < |\text{Re} \zeta|$. Observe that this function is a holomorphic extension of $|\xi|$ away from the origin. In the $\xi$ integration above, we can deform the contour to the image of

$$
\zeta(\xi) = \xi + i\beta(x - y)|\xi|
$$

where $\beta$ is chosen sufficiently small. In particular, we choose $\beta$ so that when $x$ varies near the origin and $y$ stays in the support of $\phi$, then $|\text{Im} \zeta(\xi)| < |\text{Re} \zeta(\xi)|$, away from $\xi = 0$. In the integrand of $I_2(\epsilon)$, if $|x| \leq \frac{\delta}{2}$, we get an exponential decay independent of $\epsilon$. It follows that this piece is also holomorphic near the origin in $\mathbb{C}^n$ after setting $\epsilon = 0$. Finally, for the third piece, let $\Gamma_1, \ldots, \Gamma_m$ be convex cones such that with $\Gamma_0 = \Gamma$,

$$
\mathbb{R}^n = \bigcup_{j=0}^m \Gamma_j,
$$

and for each $j \geq 1$ there exists a vector $v_j$ satisfying $v_j \cdot \Gamma_j > 0$ and $v_j \cdot \xi^0 < 0$. We now write

$$
I_3(\epsilon) = \sum_{j=0}^m K_j(\epsilon),
$$

where $K_j$ equals the integral over $\Gamma_j$. The decay in the FBI established in (2.8) shows us that $K_0$ is a smooth function even after setting $\epsilon = 0$. Each of the remaining functions $K_j$, after setting $\epsilon = 0$, is a boundary value of a tempered holomorphic function in a wedge whose inner product with $\xi^0$ is negative. Hence

$$
(0, \xi^0) \notin WF_a(K_j(0+)),
$$

where $WF_a$ denotes the analytic wave front set (see [S]). The latter implies that

$$
(0, \xi^0) \notin WF(K_j(0+)).
$$

Indeed, as is well known, a distribution $u$ is microlocally analytic (resp. smooth) at a covector $\gamma$ if there is an analytic (resp. smooth) pseudodifferential operator $P$ elliptic at $\gamma$ such that $Pu$ is analytic (resp. smooth). We have thus proved that

$$
(0, \xi^0) \notin WF(f(x, 0)).
$$
3. Proof of Theorem 1.1 and auxiliary results

We next wish to get a better description of the wave front set of the trace of a solution when the vector field in question is locally integrable. We consider a smooth vector field \( L = X + iY \) where \( X \) and \( Y \) are real vector fields defined in a neighborhood \( U \) of the origin. Let \( \Sigma \) be an embedded hypersurface through the origin in \( U \) dividing the set \( U \) into two regions, \( U^+ \) and \( U^- \) where \( U^+ \) denotes the region towards which \( X \) is pointing. We will consider a function \( f \in \mathcal{C}^1(U^+) \) that satisfies \( Lf = 0 \) on \( U^+ \) and grows in a tempered fashion as \( p \to \Sigma \).

We assume that \( L \) is noncharacteristic on \( \Sigma \) which means (after multiplying \( L \) by \( i \) if necessary) that \( X \) is noncharacteristic. Our considerations will be local and so after an appropriate choice of local coordinates \((x,t)\) and multiplication of \( L \) by a nonvanishing factor, the vector field is given by

\[
L = \frac{\partial}{\partial t} + \sum_{j=1}^{n} a_j(x,t) \frac{\partial}{\partial x_j}
\]

and \( \Sigma \) and \( U^+ \) are given by \( t = 0 \) and \( t > 0 \) respectively.

We will need to consider the integral curve \((s, \epsilon, \epsilon) \ni s \mapsto \gamma(s)\) of \( X \) that passes through the origin, i.e., \( \gamma'(0) = X \circ \gamma(0), \gamma(0) = 0 \). It is clear that for small \( \epsilon > 0 \) and \( |s| < \epsilon \), \( \gamma(s) \in U^+ \) if and only if \( s > 0 \), so \( \gamma((-\epsilon, \epsilon)) \cap U^+ = \gamma((0, \epsilon)) \). To simplify the notation we will simply write \( \gamma^+ \) to denote \( \gamma((0, \epsilon)) \).

**Theorem 3.1.** Let \( L = \frac{\partial}{\partial t} + \sum_{j=1}^{n} a_j(x,t) \frac{\partial}{\partial x_j} \) be locally integrable.

i) Suppose \( f \in \mathcal{C}^1(U^+) \) satisfies \(|f(x,t)| = O(t^{-N})\) for some \( N \) and \( Lf(x,t) = 0, (x,t) \in U^+ \).

Assume that there is a sequence \( p_k \in \gamma^+, p_k \to 0 \) such that for each \( k = 1, 2, \ldots, X(p_k) \) and \( Y(p_k) \) are linearly independent. Then there exists a unit vector \( v \) such that

\[
\xi^0 \in \mathbb{R}^n, \quad v \cdot \xi^0 > 0 \implies (0, \xi^0) \notin WF(bf).
\]

In particular, the wave front set of \( bf \) at the origin is contained in a closed half-space.

ii) Conversely, if \( X(p) \) and \( Y(p) \) are linearly dependent for all \( p \in \gamma^+ \) there exists a neighborhood \( V \subset U \) of the origin and a function \( f \in \mathcal{C}^1(V^+) \cap \mathcal{C}^0(V^-) \) such that \( Lf = 0 \) and \( (0, \xi) \in WF(bf) \) for all \( \xi \in \mathbb{R}^n \setminus \{0\} \).

**Proof.** i) Let \( Z_1, \ldots, Z_n \) be a complete set of smooth first integrals of \( L \) near the origin in \( U \) (see [T] on the subject of locally integrable structures). That is,

\[
LZ_j(x,t) = 0, \quad k = 1, \ldots, n, \quad dZ_1 \wedge \cdots \wedge dZ_n(0,0) \neq 0,
\]

and choose new local coordinates \((x,t)\) in which the \( Z_j \)'s may be written as

\[
Z_j(x,t) = x_j + i\Phi_j(x,t), \quad k = 1, \ldots, n,
\]

with \( \Phi(0,0) = 0, \Phi_x(0,0) = 0 \) and \( \Phi_{xx}(0,0) = 0 \) (\( \Phi = (\Phi_1, \ldots, \Phi_n) \)).
For \( j = 1, \ldots, n \) let \( M_j = \sum_{k=1}^{n} b_{jk}(x,t) \frac{\partial}{\partial x_k} \) be vector fields satisfying
\[
M_j Z_k = \delta_j^k, \quad [M_j, M_k] = 0.
\]
It is readily checked that for each \( j = 1, \ldots, n \),
\[
[M_j, L] = 0.
\]
(3.2)

For any \( C^1 \) function \( g \), the differential may be expressed as
\[
dg = Lg dt + \sum_{k=1}^{n} M_k g dZ_k
\]
(3.3)
Using (3.3) we get:
\[
d(gdZ_1 \wedge \cdots \wedge dZ_n) = Lg dt \wedge dZ_1 \wedge \cdots \wedge dZ_n
\]
(3.4)

For \( \zeta \in \mathbb{C}^n \), \( z \in \mathbb{C}^n \), let
\[
E(z, \zeta, x, t) = i\zeta \cdot (z - Z(x, t)) - \kappa \langle \zeta \rangle (z - Z(x, t))^2,
\]
Let \( B \) denote a small ball centered at 0 of radius \( r \) in \( \mathbb{R}^n \) and \( \phi \in C_0^\infty (B) \), \( \phi \equiv 1 \) for \( |x| \leq r/2 \), the precise value of \( r \) as well as the value of the positive parameter \( \kappa \) in the definition of \( E \) will be determined later. We will apply (3.4) to the function
\[
g(z, \zeta, x, t) = \phi(x) f(x, t) e^{E(z, \zeta, x, t)}
\]
where \( (z, \zeta) \) are parameters. We get:
\[
d(gdZ) = f L\phi e^E dt \wedge dZ
\]
(3.5)
where \( dZ = dZ_1 \wedge \cdots \wedge dZ_n \). Next by Stokes theorem we have, for \( t_1 > 0 \) small:
\[
\int_B g(z, \zeta, x, 0)d_z Z(x, 0) = \int_B g(z, \zeta, x, t_1)d_z Z(x, t_1) + \int_0^{t_1} \int_B d(gdZ)
\]
(3.6)
We will estimate the two integrals on the right in (3.6) and our aim is to show that for \( x \) and \( z \) close to the origin in real and complex space respectively, both decay exponentially as \( \zeta \to \infty \) in a conic neighborhood of \( \xi_0 \). Write
\[
Z = (Z_1, \ldots, Z_n) = x + i\Phi(x, t), \quad \Phi = (\Phi_1, \ldots, \Phi_n).
\]
Observe that, assuming without loss of generality that \( |\xi_0| = 1 \),
\[
\text{Re} E(0, \xi_0, x, t) = \Phi(x, t) \cdot \xi_0 - \kappa (|x|^2 - |\Phi(x, t)|^2).
\]
Our main task will be to determine convenient values of \( t_1, \kappa \) and \( r \) such that for some \( \gamma > 0 \)
\[
1\) \text{Re} E(0, \xi_0, x, t_1) \leq -\gamma \text{ for } |x| \leq r ;
2\) \text{Re} E(0, \xi_0, x, t) \leq -\gamma \text{ for } 0 \leq t \leq t_1 \text{ and } r/2 \leq |x| \leq r.
\]
The assumptions on \( \Phi \) allow us to write
\[
\Phi(x, t) = \Phi(0, t) + e(x, t), \quad |e(x, t)| \leq A|x| + B|x|^2
\]
(3.7)
for some positive constants \( A \) and \( B \). We may assume that \( \Phi_t(0, 0) = 0 \), otherwise the result we want to prove would follow from Theorem 2.1. Hence, we may assume that the quotient \( |\Phi(0, t)|/t^2 \leq C \) for \( (0, t) \in U^+ \). In order to find the vector \( v \) mentioned in the statement of the theorem we will need
Lemma 3.2. There exist a sequence \( t_k \searrow 0 \) such that

1. \( \Phi(0, t_k) \neq 0; \)
2. \( |\Phi(0, t)| \leq |\Phi(0, t_k)| \) for \( 0 \leq t \leq t_k; \)
3. \( \lim_{t_k \to 0} \Phi(0, t_k)/|\Phi(0, t_k)| = -v \)

We will postpone the proof of Lemma 3.2 and continue our reasoning with \( v \) given by (3). We have \( \Phi(0, t_k) + |\Phi(0, t_k)| v = o(|\Phi(0, t_k)|) \). We recall that by hypothesis \( \xi \cdot v > 0 \). Hence,

\[
\Phi(0, t_k) \cdot \xi^0 = -|\Phi(0, t_k)| v : \xi^0 + o(|\Phi(0, t_k)|) < -|\Phi(0, t_k)| v \cdot \xi^0/2 = -c|\Phi(0, t_k)|,
\]

for \( t_k \) small and \( 0 < c < 1 \). We now take \( r = \alpha|\Phi(0, t_k)|/t_k \), with \( \alpha \) and \( t_k \) small to be chosen later. Hence, for \( |x| \leq r \) and \( 0 \leq t \leq t_k \), we can choose \( \alpha \) small enough (depending on \( A, B \) and \( C \) but not on \( t_k \)) so that

\[
|c(x, t)| \leq A \alpha |\Phi(0, t_k)| \frac{t}{t_k} + B \alpha^2 \frac{|\Phi(0, t_k)|}{t_k^2} |\Phi(0, t_k)|
\]

(3.8)

\[
\leq c \frac{|\Phi(0, t_k)|}{2}.
\]

This implies that on the support of \( \phi(x) \) we have

\[-(1 + c)|\Phi(0, t_k)| \leq \Phi(x, t_k) \cdot \xi^0 \leq -\frac{c}{2} |\Phi(0, t_k)|.\]

Let \( \kappa = \epsilon/|\Phi(0, t_k)| \). A consequence of (3.7), (3.8) and the fact that \( |\Phi(0, t)| \leq |\Phi(0, t_k)| \) for \( 0 \leq t \leq t_k \) is

\[
|\Phi(x, t)| \leq (1 + c)|\Phi(0, t_k)|
\]

(3.9)

\[
|\Phi(x, t)|^2 \leq (1 + c)^2 |\Phi(0, t_k)|^2
\]

\[
\kappa|\Phi(x, t)|^2 \leq \epsilon (1 + c)^2 |\Phi(0, t_k)|^2
\]

for \( x \) in the support of \( \phi(x) \) and \( 0 \leq t \leq t_k \). Choosing \( \epsilon = c/(4(1 + c)^2) \) (thus, independent of \( t_k \)), we get, on the support of \( \phi(x) \),

\[
\Phi(x, t_k) \cdot \xi^0 + \kappa|\Phi(x, t_k)|^2 \leq -\frac{c}{2} |\Phi(0, t_k)| + c(1 + c)^2 |\Phi(0, t_k)| \leq -\frac{c}{4} |\Phi(0, t_k)|
\]

which leads to an exponential decay in the first integral on the right of (3.6) for \( z \) complex near 0 and \( \zeta \) in a complex conic neighborhood of \( \xi^0 \), as soon as we replace \( t_1 \) by \( t_k \). For the second integral, note that for \( 0 \leq t \leq t_k \) and \( x \) in the support of \( \phi \), we may invoke again (3.9) to estimate the size of \( |\Phi(x, t)| \) and \( \kappa|\Phi(x, t)|^2 \) which gives, in view of the previous choice of \( \epsilon \),

\[
|\Phi(x, t)| + \kappa|\Phi(x, t)|^2 \leq (1 + c)|\Phi(0, t_k)| + \frac{c}{4} |\Phi(0, t_k)| \leq (1 + 2c)|\Phi(0, t_k)|
\]

while on the support of \( L\phi \), \( |x| \geq r/2 = \alpha|\Phi(0, t_k)|/2t_k \) so

\[
\kappa|x|^2 \geq \frac{c\alpha^2 |\Phi(0, t_k)|}{4t_k^2}
\]
and
\[ \Phi(x, t) \cdot \xi^0 - \kappa(|x|^2 - |\Phi(x, t)|^2) \leq (1 + 2c - \frac{c\alpha^2}{4t^k})(\Phi(0, t_k)). \]

Hence, if \( t_k \) is chosen sufficiently small, we also get exponential decay for the second integral on the right hand side of (3.6) with \( t_1 \) replaced by \( t_k \).

We have thus shown that the function
\[ F(z, \zeta) = \int_B e^{\nu(z, \zeta, x, 0)} \phi(x) f(x, 0) d_z Z(x, 0) \]
satisfies an exponential decay of the form
\[ |F(z, \zeta)| \leq Ce^{-R|\zeta|} \]
for \( z \) near 0 in \( \mathbb{C}^n \) and \( \zeta \) in a complex conic neighborhood of \( \xi^0 \) in \( \mathbb{C}^n \). In particular, since \( Z(0, 0) = 0 \) and \( d_x Z(0, 0) \) is the identity matrix, the function
\[ G(x, \xi) = F(Z(x), (Z_x(x)^{-1})^t \xi) \]
has an exponential decay for \( (x, \xi) \) in a real conic neighborhood of \( (0, \xi^0) \). By Theorem 2.2 in [BC], it follows that \( (0, \xi^0) \notin WF(bf) \).

We now return to the proof of Lemma 3.2; it is here that we use the fact that \( X \) and \( Y \) are linearly independent on a sequence \( p_k \in \gamma^+ \) that approaches the origin.

We will show that \( \Phi(0, t) \) cannot vanish identically on any interval \((0, \epsilon)\). Let us write \( L = \partial_t + a \cdot \partial_x, Z = x + i\Phi, Z_x = I + i^t \Phi_x \) and recall that \( i^t \Phi_x \) has small norm for \((x, t)\) close to 0. Now \( LZ = 0 \) leads to \( a = -i(I + i^t \Phi_x)^{-1} \Phi_t \). If \( \Phi(0, t) \) vanishes identically on \((0, \epsilon)\) we will have, for those values of \( t \), that \( \Phi_t(0, t) = 0, a(0, t) = 0 \), and \( Y(0, t) = \text{Im} a(0, t) = 0 \). Furthermore, \( X(0, t) = \partial_t \) for \( 0 < t < \epsilon' \), showing that \( \gamma(s) = (0, \ldots, 0, s) \) for \( 0 < s < \epsilon' \). Thus, \( X(\gamma(s)) \) and \( Y(\gamma(s)) \) are linearly dependent for \( 0 < s < \epsilon' \), a contradiction. Therefore, there exists a sequence \( s_k \searrow 0 \) such that \( |\Phi(0, s_k)| > 0 \) and since \( \Phi(0, 0) = 0 \) there is another sequence \( t_k \searrow 0 \) satisfying (1) and (2), which in turn possesses a subsequence that satisfies (1), (2) and (3).

i) Consider as before a complete set of smooth first integrals of \( L = X + iY \) defined in neighborhood \( V \subset U \) of the origin, \( Z_1, \ldots, Z_n, LZ_j = 0, k = 1, \ldots, n, dZ_1 \wedge \cdots \wedge dZ_n(0, 0) \neq 0 \), and local coordinates \((x, t)\) in which the \( Z_j \)'s have the form \( Z_j(x, t) = x_j + \phi_j(x, t), k = 1, \ldots, n, \) with \( \Phi(0, 0) = 0, \Phi_x(0, 0) = 0 \) and \( \Phi_{xx}(0, 0) = 0 \). Since \( Y \) is proportional to \( X \) along \( \gamma^+ \) it follows that \( XZ_j = 0 \) on \( \gamma^+ \); therefore \( XRe Z_j = 0 \) on \( \gamma^+ \) which in the coordinates \((x, t)\) implies that \( x_j(\gamma^+) = 0 \), \( j = 1, \ldots, n \). Since \( Z_j \) vanishes on \( \gamma^+ \) we conclude that \( \Phi_j(0, t) = 0 \) for \( 0 \leq t < t_0 \) for some \( t_0 > 0 \) and any \( j = 1, \ldots, n \). This shows that \( Z = (Z_1, \ldots, Z_n) \) maps \( \{0\} \times [0, t_0) \subset \mathbb{R}^n \times \mathbb{R} \) into \( \{0\} \subset \mathbb{C}^n \).

Let us denote by \( F(\zeta) = \langle \zeta \rangle = (\zeta_1^2 + \cdots + \zeta_n^2)^{1/2} \) the holomorphic function \( \langle \zeta \rangle = (\zeta_1^2 + \cdots + \zeta_n^2)^{1/2} \) where we take the principal branch of the square root in the region \( |\text{Im} \zeta| < |\text{Re} \zeta| \) and set \( F(0) = 0 \). We also set \( F_c(\zeta) = (\zeta_1^2 + \cdots + \zeta_n^2 + \epsilon)^{1/2}, \epsilon > 0 \). Shrinking \( V \) if necessary we assume that \( Z(V_+) = V_+ + i\Phi(V_+) \) is contained in \( |\text{Im} \zeta| \leq |\text{Re} \zeta|/2 \) and the composition \( u(x, t) = F(Z(x, t)) \) is well defined and continuous. Approximating \( u(x, t) \) by the smooth functions \( u_c(x, t) = F_c(Z(x, t)) \), \( \epsilon \searrow 0 \) that satisfy \( Lu_c = 0 \) we see that \( u \) satisfies the homogeneous equation \( Lu = 0 \).
in the sense of distributions in $V_+$ and so does $f(x,t) = u(x,t)^3$. A moment's reflection shows that $f \in C^1(V^+) \cap C^0(V^+)$. Furthermore, $Z_1^2(x,0) + \cdots + Z_n^2(x,0) = \beta(x) |x|^2$ where $\beta$ is a smooth complex-valued function that does not vanish in a neighborhood of the origin, so $f(x,0) = \beta(x) |x|^3$ and the wave front sets of $f(x,0)$ and $x \mapsto |x|^3$ coincide for small values of $x$. The wave front set of $|x|^3$ is precisely $\{0\} \times (\mathbb{R}^n \setminus \{0\})$ because $|x|^3$ is smooth except at the origin and it is invariant under rotations.

In the proof of Theorem 1.1 we will also need the following lemma on measures which arise as traces of homogeneous solutions of vector fields.

**Lemma 3.3.** Let

$$L = \frac{\partial}{\partial t} + i \sum_{j=1}^n b_j(x,t) \frac{\partial}{\partial x_j}$$

be smooth on a neighborhood $U = B(0,a) \times (-T,T)$ of the origin in $\mathbb{R}^{n+1}$ with $B(0,a) = \{x \in \mathbb{R}^n : |x| < a\}$. We will assume that the coefficients $b_j(x,t)$, $j = 1, \ldots, n$ are real and that all of them vanish on $F \times [0,T]$, where $F \subset B(0,a)$ is a closed set. Assume that $f \in C^1(U^+)$ satisfies $Lf = 0$ on $U^+ = C^1(B(0,a) \times (0,T))$, has tempered growth as $t \searrow 0$ and its boundary value $bf(x) = f(x,0)$ is a Radon measure $\mu$. Then the restriction $\mu_F$ of $\mu$ to $F$ defined on Borel sets $X \subset B(0,a)$ by $\mu_F(X) = \mu(X \cap F)$ is absolutely continuous with respect to Lebesgue measure.

**Proof.** If $\tilde{x}$ is an arbitrary point in $F$ we may write

$$b_j(x,t) = \sum_{k=1}^n (x_k - \tilde{x}_k) \beta_{jk}(x,\tilde{x},t)$$

with $\beta_{jk}(x,\tilde{x},t)$ real and smooth. Recall that for any $\phi \in C^\infty_c(-a,a)$ we have

$$\langle \mu, \phi \rangle = \int f(x,T) \Phi^k(x,T) ds + \int_0^T \int_{-a}^a f(x,t) L^\ell \Phi^k(x,t) dx dt$$

(3.11)

$$\Phi^k(x,t) = \sum_{j=0}^k \phi_j(x,t) \frac{t^j}{j!}$$

where $\phi_0(x,t) = \phi(x)$,

$$\phi_j(x,t) = \frac{\partial}{\partial t} \phi_{j-1}(x,t) - \sum_{s,t=1}^n \frac{\partial}{\partial x_s} (x_s - \tilde{x}_s) \beta_{js}(x,\tilde{x},t) \phi_{j-1}(x,t), \quad j = 1, \ldots, k,$$

and $k$ is a convenient and fixed positive integer. We may as well write

$$\Phi^k(x,t) = A(x,t,D_x) \phi(x)$$

(3.12)

where $A(x,t,D_x) = \sum_{|\alpha| \leq k} a_\alpha(x,t) D_x^\alpha$ is a linear differential operator of order $k$ in the $x$ variables with coefficients depending smoothly on $t$. The coefficients $a_\alpha$ are obtained from the coefficients $b_j(x,t)$ of $L$ by means of algebraic operations and
Differentiations with respect to $x$ and $t$. The key observation is that (3.10) implies that, given any point $\tilde{x} \in F$, $A(x, t, D_x)$ may be written as

$$A(x, t, D_x) = \sum_{|\alpha| \leq k} \sum_{\ell=1}^{n} A_{\alpha\ell}(x, \tilde{x}, t) ((x_{\ell} - \tilde{x}_{\ell}) D_x)^{\alpha}.$$  

Notice that $|A_{\alpha\ell}(x, \tilde{x}, t)| \leq C$, for $x \in B(0, a)$, $\tilde{x} \in F$, $t \in [0, T)$, $|\alpha| \leq k$, and $\ell = 1, \ldots, n$ because the coefficients of $L$ have uniformly bounded derivatives on $B(0, a)$. Hence, we obtain from (3.11), (3.12) and (3.13) the estimate

$$\left| \int f(x, T) \Phi^k(x, T) dx \right| \leq C \sum_{|\alpha| \leq k+1} \int_{B(0, a)} d(x, F)|D_x^\alpha \phi(x)| dx,$$

where $d(x, F) = \inf_{\tilde{x} \in F} |x - \tilde{x}|$. We next consider the second integral on the right in (3.11). We recall from the proof of Lemma 1.2 that

$$|L^t \Phi^k(x, t)| \leq Ct^k$$

We need to examine this inequality more closely. We will first show that for any $j$,

$$L^t(\Phi^j) = \frac{\phi_{j+1}}{j!} t^j$$

To see this, note that (3.15) holds for $j = 0$ from the definition of $\phi_1$. To proceed by induction, assume (3.15) for $j \leq m$. Then

$$L^t(\Phi^{m+1}) = L^t(\Phi^m) + L^t\left( \frac{\phi_{m+1}}{(m+1)!} t^{m+1} \right)$$

$$= \frac{\phi_{m+1}}{m!} t^m + L^t\left( \frac{\phi_{m+1}}{(m+1)!} t^{m+1} \right)$$

$$= L^t(\phi_{m+1}) t^{m+1}$$

$$= \frac{\phi_{m+2}}{(m+1)!} t^{m+1}$$

This proves (3.15). Next we observe that since the coefficients $b_j(x, t)$ vanish on $F \times [0, T]$, each $\phi_j$ has the form

$$\phi_j(x, t) = \sum_{|\alpha| \leq j} c_{\alpha}(x, t) D_x^\alpha \phi(x)$$

where the $c_{\alpha}$ are smooth and satisfy the estimate

$$|c_{\alpha}| \leq Cd(x, F)^{|\alpha|}$$

The form (3.16) is clearly valid for $\phi_0 = \phi$. Assume it is valid for $\phi_j$. Then it will also be valid for $\phi_{j+1}$ since by definition, $\phi_{j+1} = L^t \phi_j$. If we now choose $k = N+1$,
which implies that the same conclusion holds for any continuous function which is a countable union of closed sets to which we can apply Lemma 3.3 and conclude that $\mu$ exists with respect to Lebesgue measure and the Radon-Nikodym theorem implies that there is a Borel set with $|\mu| = 0$, such that i) $\phi_\epsilon(x) = 1$ for all $x \in K$; ii) $\phi_\epsilon(x) = 0$ if $d(x, K) > \epsilon$; iii) $|D^\alpha_\epsilon \phi_\epsilon(x)| \leq C_\epsilon \epsilon^{-|\alpha|}$. Note that $\phi_\epsilon(x)$ converges pointwise to the characteristic function of $K$ as $\epsilon \to 0$ while $D^\alpha_\epsilon \phi_\epsilon(x)$ converges pointwise if $|\alpha| > 0$. Let $\psi \in C^\infty_c(B(0, a))$ and use (3.14) and (3.16) with $\phi = \phi_\epsilon \psi$ keeping in mind the trivial estimate $d(x, F) \leq d(x, K)$. By the dominated convergence theorem, $(\mu, \phi_\epsilon \psi) \to \int_K \psi \, d\mu$ while $\|d(x, K)||D^\alpha_\epsilon \phi_\epsilon(x)||_{L^1} \leq \epsilon^{1 - |\alpha|} |D^\alpha_\epsilon \phi_\epsilon(x)||_{L^1} \to 0$ as $\epsilon \to 0$ (when $\alpha = 0$ one uses the fact that $|K| = 0$).

Thus, (3.14) and (3.16) show that

$$\int_K \psi \, d\mu = 0, \quad \psi \in C^\infty_c(B(0, a)),$$

which implies that the same conclusion holds for any continuous function $\psi$ on $K$ (first extend $\psi$ to a compactly supported function on $B(0, a)$ and then approximate the extension by test functions). Thus the total variation $|\mu|(K)$ of $\mu$ on $K$ is zero and by the regularity of $\mu$ it follows that $|\mu|(F') = 0$ whenever $F' \subset F$ is a Borel set with $|F'| = 0$. This proves that $\mu_F$ is absolutely continuous with respect to Lebesgue measure.

We now consider the set

$$F_0 = \{x \in B(0, a) : \exists \epsilon > 0 : b_j(x, t) = 0, \forall t \in [0, \epsilon], j = 0, \ldots, n\}$$

which is a countable union of closed sets

$$F_\epsilon = \{x \in B(0, a) : b_j(x, t) = 0, \forall t \in [0, \epsilon], j = 0, \ldots, n\}$$

to which we can apply Lemma 3.3 and conclude that $\mu_{F_\epsilon}$ is absolutely continuous with respect to Lebesgue measure. Thus, $\mu_{F_\epsilon}$ is also absolutely continuous with respect to Lebesgue measure and the Radon-Nikodym theorem implies that there exists $g \in L^1_{loc}(B(0, a))$ such that

$$\mu_{F_\epsilon}(X) = \int_X g(x) \, dx, \quad X \subset B(0, a) \text{ a Borel set.}$$

The results proved so far immediately imply Theorem 1.1:
Proof of Theorem 1.1. We may assume that the vector field has the form

\[ L = \frac{\partial}{\partial t} + ib(x, t) \frac{\partial}{\partial x} \]

where \( b(x, t) \) is real and smooth on a neighborhood of \( U = B(0, a) \times (-T, T) \) of the origin in \( \mathbb{R}^2 \) with \( B(0, a) = \{ x \in \mathbb{R} : |x| < a \} \). Since the trace \( bf \) is a measure, by the Radon-Nikodym theorem, we may write

\[ bf = g + \mu \]

where \( g \) is a locally integrable function and \( \mu \) is a measure supported on a set \( E \) of Lebesgue measure zero. Suppose \( x_0 \) is a point for which we can find a sequence \( t_j \) converging to 0 with \( b(x_0, t_j) \neq 0 \). Let \( Z(x, t) \) be a first integral satisfying \( Z(x_0, 0) = 0 \), and \( Z_x(x_0, 0) = 1 \). If \( \text{Im} \ Z_t(x_0, 0) \neq 0 \), then \( L \) will be elliptic in a neighborhood of \( (x_0, 0) \) and so by the classical F. and M. Riesz theorem, we can conclude that \( bf \) is absolutely continuous near \( (x_0, 0) \). Otherwise, the proof of Theorem 3.1 shows that the FBI transform with this \( Z \) as a first integral and arbitrarily large \( \kappa \) decays exponentially in a complex conic neighborhood of \( (x_0, \xi_0) \), for some nonzero covector. By Theorem 2.2 in [BCT], it follows that near the point \( x_0 \), modulo a smooth nonvanishing multiple, the trace \( bf \) is the weak boundary value of a holomorphic function \( F \) defined on a side of the curve \( x \mapsto Z(x, 0) \). But then, again by the classical F. and M. Riesz theorem, \( bf \) is locally integrable near \( x_0 \), that is, \( x_0 \notin E \). Hence the set \( E \) is contained in the set \( F_0 = \{ x \in B(0, a) : \exists \epsilon > 0 : b_j(x, t) = 0, \forall t \in [0, \epsilon], j = 0, \ldots, n \} \).

But we already observed that the restriction of \( bf \) to \( F_0 \) is absolutely continuous with respect to Lebesgue measure which implies that \( \mu \) is zero.

Remarks.

1. In the preceding proof, instead of using Theorem 2.2 in [BCT], we can use Lemma 3.3 and Theorem 3.1 in this paper together with Theorem 1.4 in [B]. However, since this latter theorem in [B] uses a deep theorem of Uchiyama on the characterization of the real Hardy space, we chose to present a simpler argument.

2. In the proof of part ii) of Theorem 3.1 we showed how to construct — under the hypothesis that \( L \) is proportional to a real vector field along an integral curve of that vector — a \( C^1 \) solution such that its trace has a full wave front set at the origin. An obvious modification of the proof yields \( C^k \) solutions with the same property for any \( k = 1, 2, \ldots \) and the question arises whether it would be possible to take \( k = \infty \). Clearly, this is not true in general because singularities of the trace may propagate to the interior, as it is easy to check with the simple example \( L = \partial_t \) where the solutions \( Lf = 0 \) are functions of \( x \) alone. We will return to this matter in Example 4.3 in the next section.

4. Examples and Applications

We begin here with a lemma which shows that when the vector field is locally integrable, then a solution is determined by its trace. More precisely, we have:
Lemma 4.1. Suppose $X$, $U$, $L$ and $f$ are as in Lemma 1.2 and assume in addition that $L f = 0$ in $U_+$, $L$ is locally integrable in $U$ and that the trace $b f = 0$ in $X$. Then $f \equiv 0$ in a neighborhood of $X \times \{0\}$ in $U_+ \cup (X \times \{0\})$.

Proof. Estimate (1.2) in section 1 allows us to define a distribution $h$ in $U$ by

$$
\langle h, \psi(x, t) \rangle = \int_0^T \int_X f(x, t) \psi(x, t) dx dt
$$

We will show that $L h = 0$ in $U$. Since $h = f$ when $t > 0$ and $h = 0$ when $t < 0$, we need only show that $h$ is a solution near $t = 0$. Suppose then $\phi(x)$ and $\psi(t)$ are smooth functions of compact support and $\psi(T) = 0$. We have:

$$
\langle L h, \phi(x) \psi(t) \rangle = \langle h, L^*(\phi(x) \psi(t)) \rangle = - \lim_{\epsilon \to 0^+} \int_0^T \int_X f(x, t) L^*(\phi(x) \psi(t)) dx dt
$$

$$
= - \lim_{\epsilon \to 0^+} \int_0^T \int_X (L f(x, t)) \phi(x) \psi(t) dx dt
$$

$$
+ \left( \int_X f(x, T) \phi(x) dx \right) \psi(T) - \lim_{\epsilon \to 0^+} \int_X f(x, \epsilon) \phi(x) dx \psi(\epsilon)
$$

$$
= 0.
$$

Note that the second equality above is justified by estimate (1.3). Thus $L h = 0$ in $U$, and since the trace of $h$ on a noncharacteristic hypersurface is zero, by a well known theorem of uniqueness for locally integrable vector fields (see [BT]), it follows that $h \equiv 0$.

We will next apply the method of proof of Theorem 3.1 to present an example of a nonanalytic tube vector field in the plane which exhibits an interesting property: the trace of any $C^1$ solution of tempered growth is real analytic and extends as a smooth solution in a full neighborhood of points on the boundary.

Example 4.2. Let $e(t) = \exp(-1/t^2)$ and set

$$
\Phi(x_1, x_2, t) = (\Phi_1, \Phi_2) = e(t) \left( \cos(t^{-1}), \sin(t^{-1}) \right),
$$

$$
L = \frac{\partial}{\partial t} - i \Phi_1 \frac{\partial}{\partial x_1} - i \Phi_2 \frac{\partial}{\partial x_2},
$$

$$
Z = (x_1 + i \Phi_1, x_2 + i \Phi_2).
$$

Then, for every unit vector $v$ in $\mathbb{R}^2$ there is a sequence satisfying (1), (2) and (3) of Lemma 3.2. Since the origin can be replaced by any point in the argument, it follows from the proof of Theorem 3.1 and the fact that $Z_j(x, 0) = x_j$ that the weak trace of any $C^1$ solution of $L f = 0$, $t > 0$, with tempered growth as $t \searrow 0$, has to be real analytic. Say $f(x, 0) = \sum a_\alpha x^\alpha$, where the power series converges in some neighborhood of the origin. Let $H(x, t) = \sum a_\alpha Z(x, t)^\alpha$. Then $L H = 0$ in a neighborhood of the origin in the plane and by Lemma 4.1, $H$ agrees with $f$ in the region $t > 0$.

Example 4.3. Consider the vector field in the plane given by

$$
L = \frac{\partial}{\partial t} - \frac{2ixt}{1 + it^2} \frac{\partial}{\partial x}.
$$
The $t$-axis is an elliptic submainifold for $L$ (see [HT] for the definition) and by the main result proved in [HT] this axis propagates analyticity for solutions of the homogeneous equation. That is, if $Lu = 0$ on a neighborhood $\Omega$ of the origin and $u$ is analytic at some point $(0, t_0) \in \Omega$, then $u$ is analytic at every point $(0, t) \in \Omega$. However, Treves showed [T] that the $t$-axis does not propagate smoothness: there is a solution $u$ of $L$ which is continuous in the plane, smooth off $\{t = 0\}$, but the trace $u(x, 0)$ is not smooth at the origin. The existence of such a solution was proved in a nonconstructive fashion using a Baire category argument. Here we wish to construct a solution $h$ with the additional property that the $C^\infty$ wave front set of the trace $h(x, 0)$ at the origin contains both directions $1$ and $-1$.

Observe that the function $Z(x, t) = x(1 + it^2)$ is a first integral of $L$ in the plane.

For each $k$ a positive integer, define

$$W_k(x, t) = (Z(x, t)^2 - \frac{1}{k^2})^{\frac{1}{2}}$$

where we take the principal branch of the square root off the negative $y$ axis. Note that $W_k$ is continuous in the plane and smooth in $\mathbb{C}$ except at $(\frac{1}{k}, 0)$ and $(-\frac{1}{k}, 0)$. Let

$$h(x, t) = \sum_{k=1}^{\infty} \frac{1}{k^3} W_k(x, t)$$

The function $h$ is clearly continuous everywhere since the series converges absolutely on compact sets. We will show that $h$ is smooth when $t \neq 0$. Let

$$g_k(x, t) = Z(x, t)^2 - \frac{1}{k^2}$$

Fix two positive numbers $M > \delta$ and consider the size of $g_k$ in the region $\delta \leq |t| \leq M$. We have

$$|g_k(x, t)|^2 = (1 + 2t^4 + t^8)x^4 + (\frac{2t^4 - 2}{k^2})x^2 + \frac{1}{k^4}$$

It follows that there exists a constant $C(M, \delta) > 0$ such that

$$(4.2) \quad |g_k(x, t)| \geq \frac{C(M, \delta)}{k^2}$$

whenever $\delta \leq |t| \leq M$. We will now show that when $t \neq 0$ and for $n$ a positive integer,

$$D^n W_k(x, t) = \sum_{j=0}^{\left\lceil \frac{n}{2} \right\rceil} A_j g_k^{\frac{n}{2} - j + j(Dg_k)^n - 2j(D^2 g_k)^j}$$

where $D = \frac{\partial}{\partial x}$, $\left\lceil y \right\rceil$ denotes the greatest integer less than or equal to $y$, and the $A_j$ are constants depending only on $n$. In particular, these constants do not depend on $k$. We will prove this assertion by inducting on $n$. When $n = 1$, the formula is valid since

$$DW_k(x, t) = \frac{1}{2}(Dg_k)g_k^{-\frac{1}{2}}$$
Assume the assertion for some $n$ and apply $D$ to both sides of the formula. When $n$ is odd, since $D^3g_k = 0$, we get

$$D^{n+1}W_k = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \frac{1}{2} - n + j \right) A_j g_k^{\frac{1}{2} - n - 1 + j} (Dg_k)^{n+1-2j} (D^2g_k)^j$$

$$+ \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} A_j g_k^{\frac{1}{2} - n + j} (Dg_k)^{n-1-2j} (D^2g_k)^{j+1}$$

In the second sum replace $j$ by $j + 1$ and observe that since $n$ is odd,

$$\left\lfloor \frac{n}{2} \right\rfloor + 1 = \left\lfloor \frac{n + 1}{2} \right\rfloor$$

We are then led to

$$D^{n+1}W_k(x, t) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} B_j g_k^{\frac{1}{2} - n - 1 + j} (Dg_k)^{n+1-2j} (D^2g_k)^j$$

for some $B_j$ depending only on the $A_l$ and $n$, and hence independent of $k$. When $n$ is even, observe that in the second sum of $D^{n+1}$, the index $j$ goes only up to $\left\lfloor \frac{n}{2} \right\rfloor - 1$. Hence in this second sum if we replace $j$ by $j + 1$ and observe that

$$\left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n + 1}{2} \right\rfloor$$

we get an expression for $D^{n+1}W_k$ as required.

The expression for $D^nW_k(x, t)$ together with the lower bound (4.2) show that the series

$$\sum_{k=1}^{\infty} \frac{1}{3^k} D^n_x W_k(x, t)$$

converges absolutely for $x$ in a compact set and $0 < \delta \leq |t| \leq M$. Thus $D^n_x h(x, t)$ exists for all $n$ when $t \neq 0$. Next note that since $LW_k(x, t) = 0$ when $t \neq 0$, we have

$$D^{n+1}_x Dt W_k = \sum_{j=0}^{n+1} P_j(x, t) D^j_x W_k,$$

for some smooth $P_j$. Therefore using what was already proved, we see that for any $n$, $D^n_x D_t h(x, t)$ exists and $Lh = 0$ when $t \neq 0$. We can now iterate by differentiating the equation $Lh = 0$ to conclude that $h$ is smooth when $t \neq 0$. Finally, note that the trace

$$h(x, 0) = \frac{1}{3^k} W_k(x, 0) + E_k(x)$$

where $E_k$ is $C^1$ at the points $|x| = \frac{1}{k}$. Hence $h(x, 0)$ is not $C^1$ in any neighborhood of the origin. Moreover, since $h(x, 0) = h(-x, 0)$, both 1 and $-1$ are in the wave front set of $h(x, 0)$ at the origin.
Remarks.

(1) In the preceding example, if we restrict the solution \( h(x,t) \) to the domain \( D = \{(x,t) : \ t > x^2 \} \) bounded by the parabola \( t = x^2 \) we see that its boundary value \( bh(x) = h(x,x^2) \) is smooth except at the origin where its wave front set contains both directions.

(2) The geometric background behind Example 4.3 is as follows: the vector field \( L \) is a Mizohata vector field for \( x > 0 \) and a conjugate Mizohata vector field for \( x < 0 \). The wave front set of the trace of a smooth solution of the homogeneous Mizohata equation defined on the upper plane necessarily lacks the direction \( \xi < 0 \) while in the case of the conjugate Mizohata vector field the missing direction is \( \xi > 0 \). However, approaching the origin from both sides a singularity that contains the two microlocal directions can be produced.

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