A note on common zeroes of Laplace–Beltrami eigenfunctions

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Abstract

Let $\Delta u + \lambda u = \Delta v + \lambda v = 0$, where $\Delta$ is the Laplace–Beltrami operator on a compact connected smooth manifold $M$ and $\lambda > 0$. If $H^1(M) = 0$ then there exists $p \in M$ such that $u(p) = v(p) = 0$. For homogeneous $M$, $H^1(M) \neq 0$ implies the existence of a pair $u, v$ as above that has no common zero.

1 Introduction

Let $M$ be a compact connected closed orientable $C^\infty$-smooth Riemannian $d$-dimensional manifold and $\Delta$ be the Laplace–Beltrami operator on it. Set

$$E_\lambda = \{ u \in C^2(M) : \Delta u + \lambda u = 0 \}.$$  

The eigenspace $E_\lambda$ can be nontrivial only for $\lambda \geq 0$. If the contrary is not stated explicitly, we assume that functions are real valued and linear spaces are finite dimensional; $H^p(M)$ denotes de Rham cohomologies.

Theorem 1. Let $M$ be as above.

(1) Suppose $H^1(M) = 0$. Then for any $\lambda \neq 0$ and each pair $u, v \in E_\lambda$ there exists $p \in M$ such that $u(p) = v(p) = 0$.

(2) If $M$ is a homogeneous space of a compact Lie group of isometries then the converse is true: $H^1(M) \neq 0$ implies the existence of $\lambda \neq 0$ and a pair $u, v \in E_\lambda$ without common zeroes.

The circle $T = \mathbb{R}/2\pi \mathbb{Z}$ and functions $u(t) = \cos t, v(t) = \sin t$ provide the simplest example for (2). Moreover, (2) is an easy consequence of this example and the following observation: for homogeneous Riemannian manifolds $M = G/H$, where $G$ is compact and connected, $H^1(M) \neq 0$ is equivalent to
the existence of $G$-equivariant mapping $M \to \mathbb{T}$ for some nontrivial action of $G$ on $\mathbb{T}$.

The corollary below gives the answer to the question in [3]: is it true that each orbit of a compact connected irreducible linear group, acting in a complex vector space, meets any hyperplane? I am grateful to P. de la Harpe for making me aware of this question which in fact was the starting point for this note.

**Corollary 1.** Let $V$ be a complex linear space, $\dim V > 1$, and $G \subset \text{GL}(V)$ be a compact connected irreducible group. Then for any $v \in V$ and every linear subspace $H \subset V$ of complex codimension 1 there exists $g \in G$ such that $gv \in H$.

There is a real version of this corollary. Let $\tau$ be a real linear irreducible representation of a compact connected Lie group $G$ in a real linear space $\tau_V$. We may assume that $\tau_V$ is endowed with the invariant inner product $\langle \cdot, \cdot \rangle$ and that $G$ is equipped with a bi-invariant Riemannian metric. Let $M_\tau$ be the space of its matrix elements; by definition, $M_\tau$ is the linear span of functions $t_{xy}(g) = \langle \tau(g)x, y \rangle$, $x, y \in \tau_V$. Then either $\tau$ admits an invariant complex structure or its complexification is irreducible. It follows from the Schur lemma that $\Delta u = \lambda_\tau u$ for each $u \in M_\tau$, where $\Delta$ is the Laplace–Beltrami operator for a bi-invariant metric on $G$. Let us fix $\Delta$ and denote by $\Lambda_\sigma$, where $\sigma$ is a finite dimensional real representation, the spectrum of $\Delta$ on $M_\sigma$; it is the union of $\lambda_\tau$ over all irreducible components $\tau$ of $\sigma$.

**Corollary 2.** Let $G$ be a compact connected semisimple Lie group, $\sigma$ be as above. Suppose that $\Lambda_\sigma$ is a single point $\lambda \neq 0$. Then the orbit of any vector in $\tau_V$ meets each linear subspace of codimension 2.

If $u$ is an eigenfunction of $\Delta$ on a Riemannian manifold $M$ then

$$N_u = \{ x \in M : u(x) = 0 \}$$

is said to be the *nodal set*, and connected components of its complement $M_u = M \setminus N_u$ are called *nodal domains*. In the following lemma, we formulate the main step in the proof of the theorem (the fact seems to be known but I failed to find a reference).

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1If $\Lambda_\sigma$ is not a single point then the assertion is not true. The simplest example is the representation of $\text{SO}(3)$ in the space of harmonic polynomials on $\mathbb{R}^3$ of the type $l(x) + q(x)$, where $l$ is linear and $q$ is quadratic. Let us fix $x_0 \neq 0$ and define a subspace $W$ of codimension 2 by equalities $l(x_0) = 0$ and $q(x_0) = 0$. If $q_0$ is nondegenerate then there exists $l_0$ such that $l_0^{-1}(0) \cap q_0^{-1}(0) = \{0\}$; the orbit of $l_0 + q_0$ does not intersect $W$. 

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Lemma 1. Let \( u, v \in E_\lambda \), \( u, v \neq 0 \), and let \( U, V \) be nodal domains for \( u \), \( v \), respectively. If \( U \subseteq V \) then \( u = cv \) for some \( c \in \mathbb{R} \).

There are many natural questions concerning the distribution of common zeroes; they seem to be difficult. We prove a very particular result for \( d = \dim M = 2 \). Note that \( M \) is diffeomorphic to the sphere \( S^2 \) if \( d = 2 \) and \( H^1(M) = 0 \).

Proposition 1. Let \( d = 2 \), \( H^1(M) = 0 \), \( \lambda \neq 0 \), \( u \in E_\lambda \). Suppose that zero is not a critical value for \( u \). Then for each \( v \in E_\lambda \) every connected component of \( N_u \) contains at least two points of \( N_v \).

In fact, each component is a Jordan contour and supports a positive measure which annihilates \( E_\lambda \).

Let \( M \) be the unit sphere \( S^2 \subset \mathbb{R}^3 \) with the standard metric. Then \( \lambda_n = n(n+1) \) is \( n \)-th eigenvalue of \( \Delta \). The corresponding eigenspace \( E_n = E_{\lambda_n} \) consists of spherical harmonics which can be defined as restrictions to \( S^2 \) of harmonic (with respect to the ordinary Laplacian in \( \mathbb{R}^3 \)) homogeneous polynomials of degree \( n \) in \( \mathbb{R}^3 \); \( \dim E_n = 2n + 1 \). The space \( E_n \) is spanned by zonal spherical harmonics \( l_{a,n}(x) = L_n((x,a))|_{S^2} \), where \( a \in S^2 \) and \( L_n \) is the Legendre polynomial. The nodal set for \( l_{a,n} \) is the union of \( n \) circles

\[
\{ x \in S^2 : \langle x, a \rangle = x_k \},
\]

where \( x_1, \ldots, x_n \in [-1,1] \) are zeroes of \( L_n \). Set \( u = l_{a,n}, v = l_{b,n} \), \( n(a,b) = \text{card}(N_u \cap N_v) \). Projections of \( N_u \) and \( N_v \) to the plane \( \pi_{ab} \) passing through \( a \) and \( b \) are families of segments in the unit disc in \( \pi_{ab} \) with endpoints in the unit circle which are orthogonal to \( a \) and \( b \), respectively. Outside the boundary circle, the preimage of each point is a pair of points. Further, \( N_u, N_v \) are symmetric with respect to \( \pi_{ab} \). Hence \( N_u \cap N_v \) corresponds to the intersection of the segments. It makes possible to calculate or estimate \( n(a,b) \).

In particular, if \( a \) and \( b \) are sufficiently close then \( n(a,b) = 2n \); if \( a \perp b \) then \( n(a,b) \approx cn^2 \), where \( c \) can be calculated explicitly since zeroes of \( L_n \) are distributed uniformly in \([-1,1]\). The set \( N_u \cap N_v \) can be infinite for independent \( u, v \in E_n \), for instance, it can be a big circle or a family of parallel circles in \( S^2 \) (this is true for suitable \( u, v \) of the type \( P(\cos \theta) \cos(k \varphi + \alpha) \), where \( P \) is a polynomial, \( \theta, \varphi \) are Euler coordinates in \( S^2 \), \( k = 1, \ldots, n \)). I do not know if there are other nontrivial examples of infinite sets \( N_u \cap N_v \) as well as examples of \( u, v \in E_n \) such that \( \text{card}(N_u \cap N_v) < 2n^2 \).

\(^2\)with multiplicities, or for generic \( u, v \). If \( n = 2 \) then \( 4 \leq \text{card}(N_u \cap N_v) \leq 8 \); for \( n = 1 \), \( \text{card}(N_u \cap N_v) = 2 \).
It is natural to ask if something like Theorem 1 is true for three or more eigenfunctions. Here is an example. Let $S^3$ be the unit sphere in $\mathbb{C}^2$ and set $u = |z_1|^2 - |z_2|^2$, $v = \text{Re} z_1 \overline{z_2}$, $w = \text{Im} z_1 \overline{z_2}$. These three Laplace–Beltrami eigenfunctions have no common zeroes in $S^3$. They are matrix elements of the three dimensional representation of $\text{SU}(2) \cong S^3$ and correspond to three linear functions on $S^2 \subset \mathbb{R}^3$; the homogeneous space $M$ admits an equivariant mapping $M \rightarrow S^2$. Perhaps, the latter property could be the right replacement of the assumption $H^1(M) \neq 0$ in a version of Theorem 1 for homogeneous spaces and three eigenfunctions.

2 Proof of results

By $\rho$ we denote the Riemannian metric in $M$, $\omega$ is the volume $n$-form. The metric $\rho$ identifies tangent and cotangent bundles, hence it extends to $T^*M$. Let $D$ be a domain in $M$, $C^2_c(D)$ be the set of all functions in $C^2(D)$ with compact support in $D$, $W_0$ be the closure of $C^2_c(D)$ in the Sobolev class $W^1_2(D)$ which consist of functions whose first derivatives (in the sense of the distribution theory) are square integrable functions. There is the natural unique up to equivalence norm making it a Banach space. For all $u, v \in C^2_c(D)$ \footnote{Thus, $\Delta = -(d\delta + \delta d)$, where $\delta$ is the adjoint operator for $d$; due to the choice of the sign, $\Delta$ is the ordinary Laplacian in the Euclidean case.}

$$\int_D \rho(du, dv) \omega = - \int_D u \Delta v \omega = - \int_D v \Delta u \omega.$$  

Hence for every $u, v, w \in C^2_c(D)$

$$\int_D u \rho dv, dw \omega = - \int_D v(\rho(du, dw) + u \Delta w) \omega. \quad (1)$$

For a domain $D \subseteq M$ and a function $u \in W^1_2(D)$, let

$$\mathcal{D}_D(u) = \int_D \rho(du, du) \omega$$

be the Dirichlet form. For most cases, we shall omit the index. For the sake of completeness, we give a proof of the classical result which states that a positive eigenfunction corresponds to the first eigenvalue which is multiplicity free. The proof follows [2, Ch. VI, §7].
Lemma 2. Let $D$ be a domain in $M$, $v \in C^2(D) \cap W_0$. Suppose $v > 0$ and $\Delta v + \lambda v = 0$ on $D$. Then for all $u \in W_0$

$$D(u) \geq \lambda \int_D u^2 \omega,$$  \hfill (2)

and the equality holds if and only if $u = cv$ in $D$ for some $c \in \mathbb{R}$.

Proof. Since $v > 0$, each $u \in C^2(D)$ admits the unique factorization $u = \eta v$, where $\eta \in C^2_c(D)$. Due to (1) and the equality $2\eta v \rho(d\eta, dv) = v \rho(d\eta^2, dv)$,

$$D(u) = \int_D \rho(d(\eta v), d(\eta v)) \omega =$$

$$\int_D \left( v^2 \rho(d\eta, d\eta) + 2\eta v \rho(d\eta, dv) + \eta^2 \rho(dv, dv) \right) \omega =$$

$$\int_D \left( v^2 \rho(d\eta, d\eta) + \eta^2 \rho(dv, dv) \right) \omega - \int_D \eta^2 \left( \rho(dv, dv) + v \Delta v \right) \omega =$$

$$\int_D \left( v^2 \rho(d\eta, d\eta) + \lambda \eta^2 v^2 \right) \omega \geq \lambda \int_D \eta^2 v^2 \omega = \lambda \int_D u^2 \omega.$$

Using the approximation, we get (2). Suppose that the equality in (2) holds for some $u \in W_0$. Let $\eta_n$ be such that $\eta_n v \to u$ in $W_0$ as $n \to \infty$. Then $D(\eta_n v) \to D(u)$. Due to the calculation above,

$$\lim_{n \to \infty} \int_D v^2 \rho(d\eta_n, d\eta_n) \omega = 0.$$

Let $D' \subset D$ be a domain whose closure is contained in $D$. Standard arguments show that any limit point of the sequence $\{\eta_n\}$ in $W^2_0(D')$ is a constant function. Hence $u = cv$ in $D$ for some $c \in \mathbb{R}$. The converse is obvious. \hfill \Box

Proof of Lemma 1. Let $D = V \supset U$; we may assume $v > 0$ in $D$ and $u > 0$ in $U$. Let $\tilde{u}$ be zero outside $U$ and coincide with $u$ in $U$. Clearly, $\tilde{u} \in W_0$. Furthermore,

$$D_D(\tilde{u}) = D_U(u) = \lambda \int_U u^2 \omega = \lambda \int_D \tilde{u}^2 \omega.$$

By Lemma 2, $u = cv$ in $D$. To conclude the proof, we refer to Aronszajn’s unique continuation theorem [1] which implies $u = cv$ on $M$. \hfill \Box

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\textsuperscript{4}$\tilde{u}$ can be approximated in $W_0$ by functions $u_n = \max\{\varepsilon_n, u\} - \varepsilon_n$ in $U$, $u_n = 0$ outside $U$, where $\varepsilon_n > 0$ are regular values for $u$ and $\varepsilon_n \to 0$ as $n \to \infty$ (note that $u \in C^1(M)$).
Proof of Theorem 1. \textbf{1)} Let \( \mathfrak{U} \) and \( \mathfrak{V} \) be families of nodal domains for \( u \) and \( v \), respectively. The assumption \( \lambda \neq 0 \) and the orthogonality relations imply \( M_u, M_v \neq M \). Obviously, \( u \) and \( v \) have no common zeroes if and only if \( \mathfrak{C} = \mathfrak{U} \cup \mathfrak{V} \) is a covering:

\[ M = \bigcup_{W \in \mathfrak{C}} W. \] (3)

It is sufficient to prove, assuming (3), that there exists a closed 1-form on \( M \) which is not exact. The covering \( \mathfrak{C} \) has following properties:

(A) sets in \( \mathfrak{U} \) are pairwise disjoint, and the same is true for \( \mathfrak{V} \);

(B) nor \( U \subseteq V \) neither \( U \supseteq V \) for every \( U \in \mathfrak{U} \), \( V \in \mathfrak{V} \).

The first is obvious, the second is a consequence of Lemma 1. Also, Lemma 1 implies that

\[ U \cap N_v \neq \emptyset, \quad V \cap N_u \neq \emptyset \quad \text{for all} \quad U \in \mathfrak{U}, \ V \in \mathfrak{V}. \] (4)

Due to (4), \( \mathfrak{C} \) is finite; \( \mathfrak{V} \) covers the compact set \( N_u \) by open disjoint sets, and the same is true for \( \mathfrak{U} \) and \( N_v \). This also means that a connected component \( X \) of \( N_u \) is contained in some nodal domain for \( v \). Further, \( u \) cannot keep its sign near \( X \). Otherwise, we get a contradiction assuming \( u > 0 \) and applying the Green formula to functions \( u, 1 \) and the component of the set \( u < \varepsilon \) which contains \( X \) (for sufficiently small regular \( \varepsilon > 0 \)). Hence \( X \) lies in the boundary of at least two domains in \( \mathfrak{U} \). Components of \( N_v \) have this property with respect to \( \mathfrak{V} \). Let \( \Gamma \) be the incidence graph for \( \mathfrak{C} \) whose family of vertices is \( \mathfrak{C} \) and edges join sets with nonempty intersection. For \( \Gamma \), the conditions above read as follows:

(a) each edge of \( \Gamma \) joins \( \mathfrak{U} \) and \( \mathfrak{V} \);

(b) any vertex is common for (at least) two different edges\(^6\).

\(^5\)Indeed, \( \int_{\partial U_\varepsilon} \delta(u \omega) = \int_{U_\varepsilon} d(\delta(u \omega)) = -\int_{U_\varepsilon} \Delta u \omega = \lambda \int_{U_\varepsilon} u \omega > 0 \), where the operator \( \delta \) is adjoint to \( d \) and \( U_\varepsilon \supset X \) is the component above. On the other hand, \( \delta(u \omega) = -\frac{\partial}{\partial n} \omega \varepsilon \), where \( \frac{\partial}{\partial n} \) is the outer normal and \( \omega \varepsilon \) is the volume form for \( \partial U_\varepsilon \). Since \( \frac{\partial}{\partial n} \geq 0 \) on \( \partial U_\varepsilon \), we get a contradiction.

\(^6\)Otherwise, there exist \( U \in \mathfrak{U}, \ V \in \mathfrak{V} \) such that \( U \subseteq \text{clos} \ V \) or \( V \subseteq \text{clos} \ U \). If \( U \subseteq \text{clos} \ V = V \cup \partial V \) then either \( U \subseteq V \) or \( U \cap \partial V \neq \emptyset \). The first contradicts to (B), the second implies the existence of a component \( X \) of \( N_v \) such that \( v \) keeps its sign near \( X \).
It follows that $\Gamma$ contains a nontrivial cycle $C$. Let $U \in \mathcal{U}$ and $V \in \mathcal{V}$ be consecutive vertices of $C$, $Q = \partial U \cap V$. Since $Q \cap \partial V = \emptyset$ due to (3), both $Q$ and $\partial U \setminus Q = \partial U \setminus V$ are compact. Hence there exists a smooth function $f$ on $M$ such that $f = 1$ in a neighbourhood of $Q$ and $f = 0$ near $\partial U \setminus Q$. Then $df = 0$ on $\partial U$ and the 1-form $\eta$ which is zero outside $U$ and coincides with $df$ on $U$ is well defined and smooth. Obviously, $\eta$ is closed; we claim that $\eta$ cannot be exact.

Suppose $\eta = dF$. Then $F = \text{const}$ on each connected set which does not intersect $\text{supp} \eta \subset U$. Let $U_1 = U, V_1 = V, \ldots, U_m, V_m$ be the cycle $C$. Then $m > 1$ and we may assume that

$$V_k \cap U = \emptyset \quad \text{for} \ 1 < k < m$$

replacing $C$ by a shorter cycle if necessary\footnote{We may assume that $C$ contains no proper subcycle.}. If a curve in $V$ starts outside $U$ and comes into $U$ then it meets $U$ at a point of $Q$. Hence there exists a curve $c_1$ in $V \setminus U$ with endpoints in $Q$ and $U_2$. Analogously, there is a curve $c_2$ inside $V_m \setminus U$ which joins a point in $U_m$ with a point in $V_m \cap \partial U$. The set

$$X = c_1 \cup U_2 \cup V_2 \cup \ldots \cup U_m \cup c_2$$

is connected; by A) and (5), $X \cap U = \emptyset$. Therefore, $F$ is constant on $X$. This contradicts to the choice of $f$ since the closure of $X$ has common points with $Q$ and $\partial U \setminus Q$ (recall that $dF = df$ on $U$ and that $f$ takes different values on these sets).

2) Let $M = G/H$, where $G$ is a compact group of isometries. Since $M$ is connected, the identity component of $G$ acts on $M$ transitively. Hence we may assume that $G$ is connected. If $H^1(M) \neq 0$ then there exists an invariant closed 1-form $\eta$ on $M$ that is not exact. It can be lifted to the left invariant closed 1-form $\tilde{\eta}$ on the universal covering group $\tilde{G}$. Since $\tilde{\eta}$ is left invariant and closed (hence exact), $\tilde{\eta} = d\chi$ for some nontrivial additive character $\chi : \tilde{G} \to \mathbb{R}$. According to structure theorems, $\tilde{G} = \tilde{S} \times \mathbb{R}^k$, where $\tilde{S}$ is compact, simply connected, and semisimple. Hence $\chi = 0$ on $\tilde{S}$. Further, $\eta$ is locally exact on $M$; thus $\chi = 0$ on the preimage $\tilde{H}$ of the group $H$ in $\tilde{G}$. Since $\tilde{S}$ is compact and normal, $\tilde{L} = \tilde{S}H$ is a closed subgroup of $\tilde{G}$. It follows that $\dim \tilde{L} < \dim G$. Let $S, L$ be subgroups of $G$ which are covered by $\tilde{S}, \tilde{L}$, respectively. Thus, $L = SH$ is a closed proper subgroup of $G$. The natural mapping $M = G/H \to G/L \cong \mathbb{T}^m$ can be continued to a nontrivial equivariant one $M \to \mathbb{T}$. Realizing $\mathbb{T}$ as the unit circle in $\mathbb{C}$ we get a nonconstant function whose real and imaginary parts satisfies the theorem. \hfill $\square$
Conditions A) and B) imply $H^1(\mathcal{C}, A) \neq 0$ for any nontrivial abelian group $A$. Indeed, every three distinct sets in $\mathcal{C}$ have empty intersection whence any function on the set of edges of $\Gamma$ is a cocycle, while for each coboundary the sum of its values along every cycle in $\Gamma$ is zero. Besides, for homogeneous spaces of connected compact Lie groups the condition $H^1(M) = 0$ is equivalent to each of following ones: $H_1(M, \mathbb{Z})$ is finite; $\pi_1(M)$ is finite; $M$ is compact; the semisimple part of $G$ acts on $M$ transitively; $g = h + [g, g]$, where $g$ and $h$ are Lie algebras of $G$ and $H$, respectively.

We omit the proof which is easy.

**Proof of Corollary 2.** Since $G$ is semisimple, $H^1(G) = 0$. Let $L \subset V_{\sigma}$ be a subspace of codimension 2, $x \in V_{\sigma}$, $y$ and $z$ be a linear base of $L^\perp$; set $u(g) = \langle \sigma(g)x, y \rangle$, $v(g) = \langle \sigma(g)x, z \rangle$. By Theorem 1, $u$ and $v$ have a common zero $g \in G$; then $\sigma(g)x \perp y, z$ but this is equivalent to $\sigma(g)x \in L$.

**Proof of Corollary 1.** Clearly, the semisimple part of $G$ is irreducible. Hence $G$ can be assumed to be semisimple. The condition $\dim V > 1$ implies that the representation is not trivial. Therefore, $\Lambda_\tau$ is a single point $\lambda_\tau \neq 0$ and the hyperplane $H$ has the real codimension 2 in $V$. Thus we may apply Corollary 2.

Note that the centre of $G$ consists of scalar matrices; hence, if $G$ is not semisimple then $H \cap Gv$ includes $T_v$ for any $v \in H$, where $T$ is the unit circle in $\mathbb{C}$. Therefore, $H \cap Gv$ is infinite for any $v \in V \setminus \{0\}$ in this case.

In what follows, we assume that $M$ is diffeomorphic to the sphere $S^2$. Let $D$ be a domain in $M$ bounded by a finite number of smooth curves. Then there exists a vector field $\frac{\partial}{\partial n}$ on $\partial D$ orthogonal to $\partial D$ such that the Green formula holds:

\[
\int_{\partial D} \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) \, ds = \int_D (u \Delta v - v \Delta u) \, dm \tag{6}
\]

for all smooth $u, v$, where $s$ and $m$ are linear and area measures defined by $\rho$ on $\partial D$ and $D$, respectively. The vector field $\frac{\partial}{\partial n}$ depends only on the local geometry of $\partial D$ and does not vanish on $\partial D$.

**Lemma 3.** Let $\lambda \neq 0$, $u \in E_\lambda$, and $C$ be a component of $N_u$. If $C$ is a Jordan contour that contains no critical points of $u$ then there exists a strictly positive continuous function $q$ on $C$ such that $\int_C v q \, ds = 0$ for all $v \in E_\lambda$. 

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Proof. Since $du \neq 0$ on $C$, it is a smooth curve. Let $D$ be one of the two domains bounded by $C$ due to Jordan Theorem. Applying (6) to it, we get

$$\int_C v \frac{\partial u}{\partial n} \, ds = 0.$$  

Clearly, if $\frac{\partial u}{\partial n}(p) = 0$ for $p \in C$ then $p$ is a critical point. Hence either $q = \frac{\partial u}{\partial n}$ or $q = -\frac{\partial u}{\partial n}$ satisfies the lemma.

Proposition 1 is an easy consequence of Lemma 3 (it remains to note that each component of $N_u$ is a Jordan contour if 0 is not a critical value).

References

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