Synthetic Hall tube of interacting fermions

Xiaofan Zhou,1 Gang Chen,1,2,3,* and Suotang Jia1,2

1State Key Laboratory of Quantum Optics and Quantum Optics Devices, Institute of Laser spectroscopy, Shanxi University, Taiyuan 030006, China
2Collaborative Innovation Center of Extreme Optics, Shanxi University, Taiyuan, Shanxi 030006, China
3Collaborative Innovation Center of Light Manipulations and Applications, Shandong Normal University, Jinan 250358, China

Motivated by a recent experiment [J. H. Han, et. al., Phys. Rev. Lett. 122, 065303 (2019)], we investigate many-body physics of interacting fermions in a synthetic Hall tube, using state-of-the-art density-matrix renormalization-group numerical method. Since the inter-leg couplings of this synthetic Hall tube generate an interesting spin-tensor Zeeman field, exotic topological and magnetic properties occur. Especially, four new quantum phases, such as nontopological spin-vector and -tensor paramagnetic insulators, and topological and nontopological spin-mixed paramagnetic insulators, are predicted by calculating entanglement spectrum, entanglement entropies, energy gaps, and local magnetic orders with 3 spin-vectors and 5 spin-tensors. Moreover, the topologically magnetic phase transitions induced by the interaction as well as the inter-leg couplings are also revealed. Our results pave a new way to explore many-body (topological) states induced by both the spiral spin-vector and -tensor Zeeman fields.

I. INTRODUCTION

Since the discovery of the quantum Hall effect [1], the exploration of novel topological states of matter has been attracted great attention in both theory and experiment, since they provide important applications in designing novel quantum devices and processing quantum information. The Hofstadter-Harper Hamiltonian is one of the fundamental models that are used to investigate topological states [2]. The experimental realization of such Hamiltonian in cold atomic gases opens up the avenue of simulating topological states [3–5]. In cold atoms system, the internal degrees of freedom of atoms, such as the hyperfine spins [6, 7] and clock states [8, 9], can be treated as synthetic dimension to simulate the $D+1$ dimensional quantum physics using $D$ dimensional lattices [10], e.g., four dimensional quantum Hall effect [11] and chiral edge current of Hall ribbons [6, 7].

Using three hyperfine states as synthetic lattice dimension and coupling them through synthetic gauge fields by two-photon Raman process [12], spin-1 spin-orbit coupling [13–17] and spin-tensor-momentum coupling [18, 19] have also been implemented. When the links between the hyperfine states are cyclic with a gauge flux $\phi = 2\pi/3$, the optical lattice can form a synthetic Hall tube [20, 21], which is a simple Hofstadter-Harper Hamiltonian [2]. The synthetic Hall tube supports a generalized inversion symmetry-protected topological insulator [22], which is similar as the integer quantum Hall state. Since the time-reversal, particle-hole, and chiral symmetries are broken, this topological insulator belongs to the symmetry class A (unitary) of the Altland-Zirnbauer classification [23–26]. As varying one of the inter-leg coupling strength, there exists a topological phase transition with a closing band gap at critical point [20]. In a recent experiment, this interesting synthetic Hall tube has been realized successfully in the alkaline-earth fermions [27].

Apart from the single-particle quantum regulation in cold atoms experiments, the interactions between the internal states can be controlled via Feshbach resonances [28], and more importantly, generate rich many-body phenomena [29–35]. However, the interacting synthetic Hall tube has not been fully investigated. In this paper, we investigate many-body properties of such system, based on state-of-the-art density-matrix renormalization-group (DMRG) numerical method [36, 37]. Since the inter-leg couplings of this synthetic Hall tube generate an interesting spin-tensor Zeeman field, it is very necessary to explore magnetic properties of the system, apart from the interaction-driven topological transition. Due to the coexistence of the spiral spin-vector and -tensor Zeeman fields in the synthetic Hall tube, local magnetic orders with 3 spin-vectors and 5 spin-tensors should be introduced [38]. In terms of the calculated entanglement spectrum, entanglement entropies, energy gaps, and local magnetic orders, we find four new quantum phases such as nontopological spin-vector and -tensor paramagnetic insulators, and topological and nontopological spin-mixed paramagnetic insulators. Moreover, the topologically magnetic phase transitions induced by the interaction as well as the inter-leg couplings are also revealed. Our results pave a new way to explore many-body (topological) states induced by both the spiral spin-vector and -tensor Zeeman fields.
FIG. 1: (a) Schematics of the system setup with three Raman lasers $R_{1,2,3}^\sigma$, which are represented respectively by the yellow, blue, and red arrows. $R_1^\sigma$ has an angle $\eta$ from $x$-axis. A magnetic field $B$ along $z$-axis is applied to lift the spin degeneracy of the ground state $|1\rangle_S$. (b) Three hyperfine spin states in ground state $|1\rangle_S$ of alkaline-earth-like atoms $^{173}$Yb are coupled by three two-photon Raman transitions. (c) Synthetic Hall tube with a uniform flux $\phi$ on each side plaquette and interaction $U$ between these hyperfine spin states.

II. MODEL AND HAMILTONIAN

Similar as Ref. [27], here we consider the alkaline-earth fermions $^{173}$Yb trapped in an effective one-dimensional optical lattice (along $x$-direction), as shown in Fig. 1(a). Three hyperfine spin states of the ground state $|1\rangle_S$, $|1\rangle = |F = 5/2, m_F = -5/2\rangle$, $|2\rangle = |F = 5/2, m_F = -3/2\rangle$ and $|3\rangle = |F = 5/2, m_F = -1/2\rangle$, are chosen as three legs, as shown in Fig. 1(b). Three linearly-polarized Raman laser beams $R_{1,2,3}^\sigma$ are used to make three two-photon Raman transitions between the states $|1\rangle_S$, $F = 5/2\rangle$ and $|3\rangle_P$, $F = 7/2\rangle$. The couplings $|1\rangle \leftrightarrow |2\rangle$ and $|2\rangle \leftrightarrow |3\rangle$ are $\pi - \sigma$ transitions ($\Delta m_F = 1$), while the coupling $|1\rangle \leftrightarrow |3\rangle$ is the $\sigma - \sigma$ transition ($\Delta m_F = 2$), as shown in Fig. 1(b). Thus, the three-component atomic tunneling along the lattice and three-leg couplings with a complex phase factor form a synthetic tube with a uniform flux per plaquette, as shown in Fig. 1(c).

When the effective one-dimensional optical lattice is deep enough and the Rabi frequency of the two-photon Raman transitions is not too large, we use the single-band approximation to derive the tight-binding model Hamiltonian [27]

$$\hat{H} = \hat{H}_{\text{hop}} + \hat{H}_\Omega + \hat{H}_{\text{int}},$$  

(1)

where the tunneling Hamiltonian

$$\hat{H}_{\text{hop}} = \sum_{j,\sigma} (-t \hat{c}_{j+1,\sigma} \hat{c}_{j,\sigma} + \text{H.c.}),$$  

(2)

the inter-leg coupling Hamiltonian

$$\hat{H}_\Omega = \frac{1}{2} \sum_{j,\sigma \neq \sigma'} (\Omega_{\sigma\sigma'} e^{i\phi_j} \hat{c}_{j,\sigma} ^\dagger \hat{c}_{j,\sigma'} + \text{H.c.}),$$  

(3)

and the interaction Hamiltonian

$$\hat{H}_{\text{int}} = U \sum_{j,\sigma \neq \sigma'} \hat{n}_{j,\sigma} \hat{n}_{j,\sigma'}.$$  

(4)

In Eq. (1), $\hat{c}_{j,\sigma}$ is the annihilation (creation) operator for a fermion at the real lattice site $j = 1, \cdots , L$ with spin $\sigma = (1, 2, 3)$ and the lattice length $L$. $\hat{n}_{j,\sigma} = \hat{c}_{j,\sigma} ^\dagger \hat{c}_{j,\sigma}$ is the number operator. $t$ is the tunneling rate, $\Omega_{\sigma\sigma'}$ is the Rabi frequency of the two-photon Raman transition between the spin states $|\sigma\rangle$ and $|\sigma'\rangle$ and is set to $\Omega_{12} = \Omega_{23}$ for simplicity, the $j$-dependent complex phase factor $e^{i\phi_j}$ results from the momentum imparted by the two-photon Raman transitions, the flux $\phi = k_R d_x(1 - \cos \eta)$ with $k_R$ being the recoil momentum of the Raman lasers and $d_x$ being the lattice constant, $U$ is the interaction strength, and H.c. is the Hermitian conjugate.

The Hamiltonian (1) has a distinct advantage that all parameters can be tuned independently. For example, $t$ can be tuned by varying the depth of the optical lattice, $\Omega_{\sigma\sigma'}$ can be controlled by adjusting the magnitudes of the Raman laser beams, $\phi$ can be manipulated by controlling the angle $\eta$, and $U$ can be tuned via the external magnetic field through orbital Feshbach resonance [39–41] or via the transverse trapping frequencies through the confinement induced resonance [42, 43]. In the following, we mainly consider the case of the half filling, i.e., $n = N/L = 1$ with $N$ being the total number of atoms, since the system exhibits a synthetic Hall tube in such condition. We also address the repulsive interaction $U > 0$ and set $t = 1$ as a unit.

In the absence of interaction $(U = 0)$, when $\phi = 2\pi/3$ and $\Omega_\sigma < \Omega_{31} < \Omega_\pi$ with $\Omega_\pm = \pm 3t + \sqrt{\Omega_{12}^2 + 9t^2}$, this synthetic Hall tube supports a topological insulator protected by generalized inversion symmetry [20, 22, 27]. Since the time-reversal, particle-hole, and chiral symmetries are broken, the topological insulator belongs to the unitary symmetry class A (unitary) of the Altland-Zirnbauer classification and is characterized by a $\tilde{Z}$ invariant [23–26]. More interestingly, the inter-leg couplings generate spatially periodic spin-vector and -tensor Zee-man fields with the following Hamiltonian

$$\hat{H}_\Omega = \sum_j \Omega_{12} \left[ \cos(\phi_j) S_j^x - \sin(\phi_j) S_j^y \right] + \Omega_{31} \left[ \cos(\phi_j) (N_j^{xx} - N_j^{yy}) + \sin(\phi_j) N_j^{xy} \right],$$  

(5)
where $S_j = \sum_{\sigma \sigma'} \hat{b}^\dagger_{j \sigma} \mathbf{F}_{\sigma \sigma'} \hat{b}_{j \sigma'}$ with $\mathbf{F}_{\sigma \sigma'}$ being the spin operators of the total angular momentum $F = 1$, $N^{\alpha \beta} = \{S^\alpha, S^\beta\} / 2 - \delta_{\alpha \beta} S^2 / 3$ with being the anticommutation relation and $\alpha(\beta) = (x, y, z)$, $\Omega_{12}$ and $\Omega_{31}$ are called the spin-vector and -tensor Zeeman fields respectively, and $2\pi / \phi$ is the spiral period of the Zeeman field. When $\Omega_{31} = 0$, the synthetic Hall tube reduces to the spin-1 spin-orbit coupled optical lattice only with the spin-vector Zeeman field [13–16], which has a trivial topology. Notice that $\Omega_{12}$ can also be treated as the spin-tensor Zeeman field since the Hamiltonian (5) has the rotational symmetry.

In the presence of weak interaction, the topological insulator with the $Z$ invariant still exists since the generalized inversion symmetry remains [44]. For further increasing the interaction strength, the topology of the system becomes trivial. On the other hand, the interaction Hamiltonian (4) can also be rewritten as a magnetic form

$$\hat{H}_{\text{int}} = \frac{U}{2} \sum_j S_j^2 (\hat{n}_j - 1). \quad (6)$$

It shows clearly that at the half filling ($n = 1$), the interaction has a little contribution on the magnetism of the system, i.e., the magnetic properties of the synthetic Hall tube are mainly determined by $\hat{H}_\Omega$.

Based on above qualitative analysis, it can be found that the synthetic Hall tube exhibits exotic topological and magnetic properties arising from the competition between the tunneling, spin-vector and -tensor Zeeman fields, and interaction. In order to quantitatively reveal them, we will perform state-of-the-art DMRG numerical method, for which we retain 400 truncated states per DMRG block and perform 30 sweeps with a maximum truncation error $\sim 10^{-10}$.

### III. ORDER PARAMETERS

The many-body topological properties can be well described by the degeneracy in entanglement spectrum, entanglement entropy, chemical potential spectrum, and excited energy gap. The entanglement spectrum is defined as [45]

$$\xi_i = -\ln(\rho_i), \quad (7)$$

with $\rho_i$ being the eigenvalue of the reduced density matrix $\hat{\rho}_A = \text{Tr}_B \{ \psi \} \langle \psi \}$, where $\{ \psi \}$ is the ground-state wavefunction and $A, B$ correspond to the left or the right half of the one-dimensional chain. The system is topological if the entanglement spectrum is degenerate since the entanglement spectrum resembles the energy spectrum of edge excitations and vice versa [45–51]. The quantum criticality of the interaction-driven topological phase transition can be governed by the von Neumann entropy [51–56]

$$S_{\text{vN}} = -\text{Tr}_A [\hat{\rho}_A \log \hat{\rho}_A] \quad (8)$$

The divergence of the von Neumann entropy at the critical point not only indicates a continuous transition but also yields a central charge, which reflects the universal class of phase transition. The von Neumann entropy of a subchain of length $l$ is given by

$$S_{\text{vN}} = C \frac{1}{l} \ln \left[ \frac{\pi l}{L} \right] + \text{const}, \quad (9)$$

in which the slope at large distance gives the central charge $C$ of the conformal field theory underlying the critical behavior [57, 58].

The appearance of edge states is usually considered to be a hallmark of topological properties for the bulk system. The topological ground state of the synthetic Hall tube has two gapless edge states inside the gap between the lowest and the upper branches in the chemical potential spectrum [20], which is essentially the energy required to add an atom to a system of $N$ atoms and can be defined as

$$\mu = E^e_\alpha(N) - E^e_\alpha(N - 1). \quad (10)$$

Here, $E^e_\alpha(N)$ is the ground-state energy of $N$ atoms under open boundary condition. The topological ground state of the synthetic Hall tube is nondegenerate and separated from the first excited state by a finite gap, which closes and reopens in the process of topological phase transition [27]. The excited energy gap is defined as

$$\Delta_e = E^e_\alpha(N) - E^e_\alpha(N), \quad (11)$$

where $E^e_\alpha(N)$ is the first-excited (ground) state energy of $N$ atoms under periodic boundary condition.

Due to the coexistence of the spin-vector and -tensor Zeeman fields, the magnetism of the synthetic Hall tube should be described by whole spin-1 local magnetic or- ders (8 spin-moments with 3 spin-vectors and 5 spin-tensors) and their correlations [38]. The local spin-vector

$$\hat{S}_j = \langle S_j^x \rangle \langle S_j^y \rangle \langle S_j^z \rangle^T, \quad (12)$$

while the local spin-tensor fluctuation matrix $T_j$ has tensor moments

$$T_j^{\alpha \beta} = \langle \{ S_j^\alpha, S_j^\beta \} \rangle / 2 - \langle S_j^\alpha \rangle \langle S_j^\beta \rangle. \quad (13)$$

Geometrically, $\hat{S}_j$ is characterized by an arrow and $T_j$ is governed by an ellipsoid (with principle axis lengths $l_{T}^a(j)$ ($n = a, b, c$) and orientations $\mathbf{e}_T^a(j)$ given by the square-roots of the eigenvalues and eigenvectors of $T_j^{\alpha \beta}$ [59]). Since the magnetic properties are mainly determined by $\hat{H}_\Omega$, all the insulators are spiral paramagnetic phases without any long-range correlations. As a result, eight independent geometric parameters, including the length $l_S$ and spherical coordinates $\theta_S, \phi_S$ of the arrow, the two axis lengths $l_T^{a,b}$ with the third axis length $l_T^c = \sqrt{2 - (l_S)^2 - (l_T^a)^2 - (l_T^b)^2}$, and the orientational Euler angles $\theta_T, \phi_T, \phi_T'$ of the ellipsoid, are chosen to quantitatively characterize and geometrically visualize the magnetic orders.
a plate with large Figs. 2(a1) and 2(a2). The spin-tensor ellipsoid almost is the θ field [13–16]. In this case, the spin-vector arrow has a coupled optical lattice only with the spin-vector Zeeman cal insulators as varying the spin-tensor Zeeman field \( \Omega \). We cuss the magnetisms of the topological and nontopological insulator and vice versa. As a result, we can dis-

\[ (\text{a1}) \]

\[ (\text{a2}) \]

\[ (\text{a3}) \]

\[ (\text{a4}) \]

\[ (\text{b1}) \]

\[ (\text{b2}) \]

\[ (\text{b3}) \]

\[ (\text{b4}) \]

\[ (\text{c1}) \]

\[ (\text{c2}) \]

\[ (\text{c3}) \]

\[ (\text{c4}) \]

\[ \phi \]

\[ \theta \]

\[ l \]

\[ T \]

\[ \Omega \]

\[ S \]

\[ j \]

\[ n \]

\[ a \]

\[ b \]

\[ c \]

\[ \Omega_{31}/t = 12.3 \]

\[ \Omega_{31}/t = 19 \]

\[ \Omega_{31}/t = 0 \]

\[ \Omega_{31}/t = 12.3, U/t = 0, L = 32, \text{ and } n = 1. \]

FIG. 2: (a1, b1, c1) Schematic diagrams of the spin-vector density arrows \( \vec{S}_j \) and the spin-tensor density ellipsoids \( T_j \). The blue arrow denotes the spin-vector \( \vec{S} \), while the red ellipsoid reflects the spin-tensor \( T \), in which the black arrows are the ellipsoid’s axis orientations. (a2, b2, c2) Spatial distributions of [\( l_S(j), \theta_S(j), \phi_S(j) \)] for the vector-density arrows \( \vec{S}_j \). (a3, b3, c3) Distributions of the axis lengths \( l_T^j(n) \) \( (n = a, b, c) \) of the spin-tensor density ellipsoids \( T_j \). (a4, b4, c4) Distributions of the orientational Euler angles \( \theta_T, \phi_T, \phi_T' \) of the spin-tensor density ellipsoids \( T_j \). We set \( \Omega_{31}/t = 0 \) for (a1)-(a4), \( \Omega_{31}/t = 12.3 \) for (b1)-(b4), and \( \Omega_{31}/t = 19 \) for (c1)-(c4). In all subfigures, we have \( \Omega_{12}/t = 12.3, U/t = 0, L = 32, \text{ and } n = 1. \)

IV. QUANTUM PHASES

A. Noninteracting case \((U = 0)\)

We first address the case of the noninteracting case \((U = 0)\). For \( \Omega_- < \Omega_{31} < \Omega_+ \), the ground state is a topological insulator and vice versa. As a result, we can discuss the magnetisms of the topological and nontopological insulators as varying the spin-tensor Zeeman field \( \Omega_{31} \) for a fixed spin-vector Zeeman field \( \Omega_{12}/t = 12.3 \). When \( \Omega_{31} = 0 \), the system is the same as the spin-1 spin-orbit coupled optical lattice only with the spin-vector Zeeman field [13–16]. In this case, the spin-vector arrow has a unit length \( l_S = 1 \) and spirals in the \( x-y \) plane \( (i.e., \theta_S \text{ is a constant and } \phi_S \text{ changes cyclically}) \), as shown in Figs. 2(a1) and 2(a2). The spin-tensor ellipsoid almost is a plate with large \( l_T^{a,b,c} \) and small \( l_T^T \) [see Fig. 2(a3)], and also spirals with a cyclical variation \( \phi_T \) and constants \( \theta_T, \phi_T' \) [see Fig. 2(a4)], since the spin-tensor ellipsoid depends crucially on the three spin-vector operators \( S^a \) [see Eq. (13)]. This paramagnetic insulator dominated only by the spin-vector is called nontopological spin-vector paramagnetic insulator (NTSV). For \( \Omega_{31} < \Omega_- \) \( (i.e., \text{a small spin-tensor Zeeman field}) \), the ground state is still the NTSV.

For the topological regime with \( \Omega_- < \Omega_{31} < \Omega_+ \), the spin-vector arrows also spiral in the \( x-y \) plane but have short lengths \( l_S < 1 \), as shown in Figs. 2(b1) and 2(b2). In this case, the spin-vector cannot fully de-

be considered. The spin-tensor ellipsoids have finite \( l_T^{a,b,c} \) [see Fig. 2(b3)], and also spiral in the \( x-y \) plane with a cyclical variation \( \phi_T \) and constants \( \theta_T, \phi_T' \) [see Fig. 2(b4)]. Different from the NTSV, this spiral spin-tensor ellipsoids only depend on the spin-tensor Zeeman field. This topological insulator is called topological spin-mixed paramagnetic insulator (TSM). For \( \Omega_{31} > \Omega_+ \) \( (i.e., \text{a large spin-tensor Zeeman field}) \), the spin-vector arrow vanishes \( (i.e., l_S = 0) \), as shown in Figs. 2(c1) and 2(c2). In this case, the magnetic orders are fully dominated by the spin-tensor ellipsoid. The ellipsoids have \( l_T^{a,b,c} \sim 1 \text{ and } l_T^T \to 0 \) [see Fig. 2(c3)], and also spiral in the \( x-y \) plane with a cyclical variation \( \phi_T \) and constants \( \theta_T, \phi_T' \) [see Fig. 2(c4)]. This paramagnetic insulator without the spin-vector is called nontopological spin-tensor paramagnetic insulator (NTST).

The above analysis of Fig. 2 shows that there exist two topologically magnetic phase transitions as increasing the spin-tensor Zeeman field \( \Omega_{31}/t \). One is the transition from the NTSV to the TSM at \( \Omega_{31} = \Omega_- \). At this critical point, the spin-vector arrow length \( l_S \) drops rapidly, but the ellipsoid’s axis lengths \( l_T^{a,b,c} \) increase rapidly [see Fig. 3(a)]. The other is the transition from the TSM to the NTST at \( \Omega_{31} = \Omega_+ \). At this critical point, the spin-vector arrow length \( l_S \) suddenly becomes zero, and the ellipsoid’s axis lengths \( l_T^a, l_T^b \) increase (decrease) abruptly [see Fig. 3(a)]. Figure 3(b) shows the phase diagram in the \( \Omega_{31}/t - \Omega_{12}/t \) plane. Both the phase transitions of NTSV→TSM and TSM→NTST [see blue lines in Fig. 3(b)] are of second-order with a closing excited
energy gap $\Delta_\epsilon$ at the critical points $[20, 27]$.

### B. Interacting case ($U > 0$)

We now explore many-body properties induced by the repulsive interaction ($U > 0$). We first address the topological properties driven by interaction, when $\Omega_{b2}/t = 12.3$ and $\Omega_{33}/t = 11$. For a weak interaction, the entanglement spectrum $\xi_i$ is two-fold degeneracy, and no longer degenerate beyond a critical interaction strength $U_c/t \sim 1.85$, as shown in Fig. 4(a). Without any symmetry breaking in this processing, it is a typical topological phase transition from a topological insulator to a nontopological insulator. As demonstrated in Fig. 4(b), sharp features of the von Neumann entropy $S_{VN}$ emerge at the critical point. From inset of Fig. 4(b), we estimate $C \sim 1.97$, which is close to the universality class of the Luttinger liquid ($C = 2$) and shows the continuous of the topological phase transition. On the other hand, in the absence of interaction, there are two gapless edge states inside the bulk band gap in the chemical potential spectrum $\mu$. As increasing the interaction strength beyond a critical value $U_c/t$, these edge states merge into the bulk band and the system becomes a nontopological insulator, as shown in Fig. 4(c). Moreover, the excited energy gap $\Delta_\epsilon$ closes at the same critical value and then reopens, as shown in Fig. 4(d). This critical point is well consistent with that derived from entanglements in Figs. 4(a) and 4(b).

We now explore the magnetic orders in the presence of interaction. Based on the above graphics of Fig. 2, it can be found that the spin-vector arrows and the spin-tensor ellipsoids exhibit the same spiral features, but show the distinct lengths $l_S$ and $l^{a,b,c}_T$. This means that these lengths are adequate to describe the magnetic properties. As a result, we only calculate the lengths $l_S$ and $l^{a,b,c}_T$ and ignore the angles $\theta_S$, $\phi_S$, $\theta_T$, $\phi_T$, and $\phi_T'$ hereafter.

In Fig. 4(e), we plot the lengths $l_S$ and $l^{a,b,c}_T$ as functions of the interaction strength $U/t$. This figure shows that at the critical point $U_c/t$, the spin-vector arrow length $l_S$ suddenly increases to $l_S \sim 1$, and the spin-tensor ellipsoid’s axis lengths $l^{a,b}_T$ drop rapidly, which indicate that this topologically magnetic phase transition from the TSM to the NTSV occurs. In Fig. 4(f), we plot the lengths $l_S$ and $l^{a,b,c}_T$ as functions of the interaction Zeeman field $\Omega_{33}/t$ for a large interaction strength $U/t = 6$. In this case, all the insulators are nontopological since the entanglement spectrum $\xi_i$ is nondegenerate. Interestingly, as increasing the spin-tensor Zeeman field $\Omega_{33}$, the vector length $l_S$ and the ellipsoid’s axis lengths $l^{a,b}_T$ firstly remain, then $l_S$ rapidly decreases to $l_S < 1$ and $l^{a,b}_T$ (or $l^c_T$) increase (decrease) rapidly. The corresponding phase is called the nontopological spin-mixed paramagnetic insulator (NTSM). Further increasing the spin-tensor Zeeman field $\Omega_{33}$, $l_S$ suddenly drops to $l_S \sim 0$,  

![Image](image-url)
observe these quantum phases and phase transitions in energy gap $\Delta$. Moreover, all the phase transitions with a closing excited tensor Zeeman fields as well as the repulsive interaction, which are well controlled by both the spin-vector and -tensor Zeeman fields $\Omega$. The climbing spin-vector and -tensor Zeeman fields $\Omega^{a,b}$ rapidly increase to $l_T^{a,b} \sim 1$, i.e., the system enters into the NTST. Note that for a small interaction, the fundamental properties are similar to that in Fig. 3(a) and thus not plotted here.

Finally, with the help of the calculated entanglement spectrum, entanglement entropies, energy gaps, and local magnetic orders, in Fig. 5 we map out phase diagrams in the $\Omega_{31} - U$ plane for the different spin-vector Zeeman fields $\Omega_{12}/t = 12.3$ (a) and $\Omega_{12}/t = 6.0$ (b). This figure shows clearly four different phases such as the TSM, the NTSM, the NTSV, and the NTST, which are well controlled by both the spin-vector and -tensor Zeeman fields as well as the repulsive interaction. Moreover, all the phase transitions with a closing excited energy gap $\Delta_+ \sim a$ are of second order.

\[ \Omega_{31} - U \]

FIG. 5: Phase diagrams in the $\Omega_{31} - U$ plane for the different spin-vector Zeeman fields $\Omega_{12}/t = 12.3$ (a) and $\Omega_{12}/t = 6.0$ (b). The blue and black lines show the continuous phase transitions. The blue lines denote the liquids with $C = 2$. In all subfigure, we have $L = 32$ and $n = 1$.

\[ l_T^{a,b} \]

\[ l_T^{a,b} \sim 1 \]

\[ l_T^{a,b} \sim \Omega_{31} - U \]

\[ \Delta_+ \sim a \]

\[ \Delta_+ \sim \Omega_{31} - U \]

V. CONCLUSIONS

Before ending up this paper, we briefly discuss how to observe these quantum phases and phase transitions in cold atom experiments. The entanglement entropy can be measured using quantum interference of many-body twins of ultracold atoms in optical lattices [56]. The excited energy gap closing in the processing of topological phase transition can be observed via momentum-resolved analysis of the quench dynamics [27]. The local magnetic orders can be measured by isolating the sites of interest using additional site-resolved potentials [60–63]. Thus, all the quantum phases and phase transitions can be observed in current experimental setups.

In conclusion, we have studied the many-body physics of interacting synthetic Hall tube by the state-of-the-art DMRG numerical method. We have found four quantum phases, including the TSM, the NTSM, the NTSV, and the NTST, by means of the calculated entanglement spectrum, entanglement entropies, energy gaps, and local magnetic orders. These quantum phases depend crucially on the interaction and the spiral spin-vector and -tensor Zeeman fields induced by the inter-leg couplings. Our work paves a new way to explore many-body (topological) states induced by both the spiral spin-vector and -tensor Zeeman fields.

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