An explicit formula for a star product with separation of variables

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Abstract

For a star product with separation of variables $\ast$ on a pseudo-Kähler manifold we give a simple closed formula of the total symbol of the left star multiplication operator $L_f$ by a given function $f$. The formula for the star product $f \ast g$ can be immediately recovered from the total symbol of $L_f$.

(Dedicated to the memory of Nikolai Neumaier)

1 Introduction

Given a vector space $W$ and a formal parameter $\nu$, we denote by $W[[\nu]]$ the space of formal vectors $w = w_0 + \nu w_1 + \nu^2 w_2 + \ldots$, $w_r \in W$. One can also consider formal vectors that are formal Laurent series in $\nu$ with a finite polar part,

$$w = \sum_{r \geq k} \nu^r w_r$$

with $k \in \mathbb{Z}$.

Let $M$ be a Poisson manifold endowed with a Poisson bracket $\{\cdot, \cdot\}$. A star product $\ast$ on $M$ is an associative product on the space $C^\infty(M)[[\nu]]$ of formal functions on $M$ given by a $\nu$-adically convergent series

$$f \ast g = \sum_{r=0}^{\infty} \nu^r C_r(f, g),$$

where $C_r$ are bidifferential operators, $C_0(f, g) = fg$, and $C_1(f, g) - C_1(g, f) = i\{f, g\}$ (see [1]). We also assume that the unit constant is the unity of the star-product $\ast$. A star product can be restricted to an open subset of $M$ and recovered from its restrictions to subsets forming an open covering of $M$. Given functions $f, g \in C^\infty(M)[[\nu]]$, denote by $L_f$ and $R_g$ the left star multiplication operator by $f$ and the right star multiplication by $g$, respectively. Then $L_f g = f \ast g = R_g f$ and the associativity of $\ast$ is equivalent to the property that $[L_f, R_g] = 0$ for any $f, g$. 
The operators $L_f$ and $R_g$ are formal differential operators on $M$. It was proved by Kontsevich in [9] that deformation quantizations exist on arbitrary Poisson manifolds.

A star product is called natural if, for each $r$, the bidifferential operator $C_r$ is of order not greater than $r$ in each of its arguments (see [6]). We call a formal differential operator $A = A_0 + \nu A_1 + \nu^2 A_2 + \ldots$ natural if the order of $A_r$ is not greater than $r$. If a star product is natural, the operators $L_f$ and $R_f$ for any $f \in C^\infty(M)[[\nu]]$ are natural. The star products of Fedosov [4] and Kontsevich [9] are natural.

Now let $M$ be a pseudo-Kähler manifold of complex dimension $m$ endowed with a pseudo-Kähler form $\omega_{-1}$ and the corresponding Poisson bracket $\{\cdot, \cdot\}$. A star product with separation of variables $*$ on $M$ is a star product such that the bidifferential operators $C_r$ differentiate the first argument in antiholomorphic directions and the second argument in holomorphic ones (see [7], [3]). Star products with separation of variables appear naturally in the context of Berezin quantization (see [2]). It was proved in [3] and [8] that the star products with separation of variables are natural in the sense of [6].

A star product on a pseudo-Kähler manifold $M$ is a star product with separation of variables if and only if for any local holomorphic function $a$ and a local antiholomorphic function $b$ on $M$ the operators $L_a$ and $R_b$ are pointwise multiplication operators by the functions $a$ and $b$, respectively,

$$L_a = a, \quad R_b = b.$$ 

Otherwise speaking, if $f$ is a local holomorphic or $g$ is a local antiholomorphic function, then $f * g = fg$.

A formal form $\omega = \frac{1}{\nu} \omega_{-1} + \omega_0 + \nu \omega_1 + \ldots$ such that the forms $\omega_r$, $r \geq 1$, are of type $(1,1)$ with respect to the complex structure on $M$ and may be degenerate is called a formal deformation of the pseudo-Kähler form $\omega_{-1}$. It was proved in [7] that the star products with separation of variables on a pseudo-Kähler manifold $(M, \omega_{-1})$ are bijectively parametrized by the formal deformations of the form $\omega_{-1}$ (see also [10]).

A star product with separation of variables $*$ on $(M, \omega_{-1})$ corresponds to a formal deformation $\omega$ of the form $\omega_{-1}$ if for any contractible holomorphic chart $(U, \{z^k, \bar{z}^l\})$, where $1 \leq k, l \leq m$, and a formal potential $\Phi = \frac{1}{\nu} \Phi_{-1} + \Phi_0 + \nu \Phi_1 + \ldots$ of $\omega$ (i.e., $\omega = i \partial \bar{\partial} \Phi$) one has

$$R_{\nu \frac{\partial}{\partial z^l}} = \nu \left( \frac{\partial \Phi}{\partial \bar{z}^l} + \frac{\partial}{\partial \bar{z}^l} \right).$$

The star product with separation of variables $*$ parametrized by a given deformation $\omega$ of $\omega_{-1}$ can be constructed as follows. As shown in [7], for any formal function $f$ on $U$ one can find a unique formal differential operator $A$ on $U$ commuting with the operators $R_{\bar{z}^l} = \bar{z}^l$ and $R_{\nu \frac{\partial}{\partial z^l}}$ and such that $A1 = f$. This is the
left multiplication operator by \(f\) with respect to \(*\), \(A = L_f\). In particular, one can immediately check that

\[
L_\nu \frac{\partial \Phi}{\partial z_k} = \nu \left( \frac{\partial \Phi}{\partial z^k} + \frac{\partial}{\partial z^k} \right).
\]

Now, for any formal function \(g\) on \(U\) we recover the product of \(f\) and \(g\) as \(f \ast g = L_f g\). The local star products parametrized by \(\omega\) agree on the intersections of coordinate charts and define a global star product on \(M\).

We call the star product with separation of variables parametrized by the trivial deformation \(\omega = \frac{1}{\nu} \omega^{-1}\) of \(\omega^{-1}\) standard.

Explicit formulas for star products with separation of variables on pseudo-Kähler manifolds can be given in terms of graphs encoding the bidifferential operators \(C_r\) (see [11], [5], [12]).

In this paper we give a closed formula expressing the total symbol of the left star multiplication operator \(L_f\) of the standard star product with separation of variables \(*\) on a coordinate chart \(U\) of a pseudo-Kähler manifold \(M\) in terms of a family of differential operators on the cotangent bundle \(T^*U\) acting on symbols of differential operators on \(U\). One can immediately recover a formula for the star product \(f \ast g\) on \(U\) from the total symbol of the operator \(L_f\).

### 2 A recursive formula for the symbol of the left multiplication operator

A differential operator \(A\) on a real \(n\)-dimensional manifold \(M\) can be written in local coordinates \(\{x^i\}\) on a chart \(U \subset M\) in the normal form,

\[
A = p_{i_1 i_2 \ldots i_n}(x) \left( \frac{\partial}{\partial x^1} \right)^{i_1} \ldots \left( \frac{\partial}{\partial x^n} \right)^{i_n},
\]

where summation over repeated indices is assumed. Denote by \(\{\xi_i\}\) the dual fibre coordinates on \(T^*U\). Then the total symbol of \(A\) is given by the fibrewise polynomial function

\[
\tau(A)(x, \xi) = p_{i_1 i_2 \ldots i_n}(x)(\xi_1)^{i_1} \ldots (\xi_n)^{i_n}
\]

on \(T^*U\). The mapping \(A \mapsto \tau(A)\) is a bijection of the space of differential operators on \(U\) onto the space of fibrewise polynomial functions on the cotangent space \(T^*U\). The composition of differential operators induces via this bijection an associative operation \(\circ\) on the fibrewise polynomial functions on \(T^*U\). The composition \(\circ\) of fibrewise polynomial functions \(p(x, \xi)\) and \(q(x, \xi)\) is given by the formula
\[(p \circ q)(x, \xi) = \exp \left( \frac{\partial}{\partial \eta} \frac{\partial}{\partial y} \right) p(x, \eta)q(y, \xi) \bigg|_{y=x, \eta=\xi} = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\partial^r p}{\partial \xi_{i_1} \cdots \partial \xi_{i_r}} \frac{\partial^r q}{\partial x^{i_1} \cdots \partial x^{i_r}}, \right.\]

where the sum has a finite number of nonzero terms. If \(p = p(x)\) or \(q = q(\xi)\), then \(p \circ q = pq\), which means that the operation \(\circ\) has the separation of variables property with respect to the variables \(x\) and \(\xi\). Formula (2.1) is valid for complex coordinates as well.

Now let \(\ast\) be the standard star product with separation of variables on a pseudo-Kähler manifold \((M, \omega_{-1})\) of complex dimension \(m\). Choose a contractible coordinate chart \((U, \{z^k, \bar{z}^l\})\) on \(M\) and let \(\Phi_{-1}\) be a potential of \(\omega_{-1}\) on \(U\). Given a formal function \(f = f_0 + \nu f_1 + \ldots\) on \(U\), the left star multiplication operator \(L_f\) is the formal differential operator on \(U\) determined by the conditions that (i) \(L_f 1 = f \ast 1 = f\), (ii) it commutes with the pointwise multiplication operators \(R_{\bar{z}^l} = \bar{z}^l\), and (iii) it commutes with the operators

\[R_{\ast z^l} = \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \frac{\partial}{\partial \bar{z}^l}\]

for \(1 \leq l \leq m\). Also, the operator \(L_f\) is natural, i.e., \(L_f = A_0 + \nu A_1 + \ldots\), where \(A_r\) is a differential operator on \(U\) of order not greater than \(r\).

Denote by \(\{\zeta_k, \bar{\zeta}_l\}\) the dual fibre coordinates on \(T^*U\). We want to describe conditions (i) - (iii) on the operator \(L_f\) in terms of its total symbol \(F = \tau(L_f) = F_0 + \nu F_1 + \ldots\), where \(F_r = \tau(A_r)\). Condition (ii) means that \(F\) does not depend on the antiholomorphic fibre variables \(\bar{\zeta}_l\), \(F = F(\nu, z, \bar{z}, \zeta)\). Condition (i) means that \(F|_{\zeta=0} = f\) and \(F_r|_{\zeta=0} = f_r\). Condition (iii) is expressed as follows:

\[F \circ \left( \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \bar{\zeta}_l \right) = \left( \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \bar{\zeta}_l \right) \circ F.\]

Using the definition (2.1) of the operation \(\circ\) and its separation of variables property we simplify (2.2):

\[F \circ \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} + \nu \bar{\zeta}_l F = \frac{\partial \Phi_{-1}}{\partial \bar{z}^l} F + \nu \bar{\zeta}_l F + \nu \frac{\partial F}{\partial \bar{z}^l}.\]

We will use the conventional notation,

\[g_{k_1 \ldots k_r l} = \frac{\partial^{r+1} \Phi_{-1}}{\partial z^{k_1} \cdots \partial z^{k_r} \partial \bar{z}^l}.\]

Using (2.1) we simplify (2.3) further:

\[\sum_{r=1}^{\infty} \frac{1}{r!} g_{k_1 \ldots k_r l} \frac{\partial^r F}{\partial \zeta_{k_1} \cdots \partial \zeta_{k_r}} = \nu \frac{\partial F}{\partial \bar{z}^l}.\]
In particular, \( g_{kl} \) is the metric tensor corresponding to \( \omega_{-1} \). We denote its inverse by \( g^{lk} \) and introduce the following operators:

\[
\Gamma_r = g_{k_1...k_r l} g^{lk} \frac{\partial^r}{\partial \zeta_{k_1} \cdots \partial \zeta_{k_r}} \quad \text{and} \quad D = \nu g^{lk} \frac{\partial}{\partial \zbar{z}}.
\]

In particular,

\[
\Gamma_1 = \zeta_k \frac{\partial}{\partial \zeta_k}
\]

is the Euler operator for the holomorphic fibre variables. Multiplying both sides of (2.4) by \( g^{lk} \zeta_k \) and summing over the index \( l \), we obtain the formula

\[
(2.5) \quad \sum_{r=1}^{\infty} \frac{1}{r!} \Gamma_r F = DF.
\]

We want to assign a grading to the variables \( \nu \) and \( \zeta_k \) such that \(|\nu| = 1\) and \(|\zeta_k| = -1\). Denote by \( \mathcal{E}_p \) the space of formal series in the variables \( \nu \) and \( \zeta_k \) with coefficients in \( C^\infty(U) \) such that the grading of each monomial \( f(z, \zbar{z})\nu^r \zeta_{k_1} \cdots \zeta_{k_s} \) in such a series satisfies \( r - s \geq p \). The spaces \( \mathcal{E}_p \) form a descending filtration on the space \( \mathcal{E} := \mathcal{E}_0 \):

\[
\mathcal{E} = \mathcal{E}_0 \supset \mathcal{E}_1 \supset \ldots.
\]

Since \( L_f \) is a natural operator, its total symbol \( F = \tau(L_f) \) is an element of \( \mathcal{E} \). The operator \( \Gamma_r \) acts on \( \mathcal{E} \) and raises the filtration by \( r - 1 \). The operator \( D \) acts on \( \mathcal{E} \) and respects the filtration. Observe that the series on the left-hand side of (2.5) converges in the topology induced by the filtration on \( \mathcal{E} \). The space \( \mathcal{E} \) breaks into the direct sum of subspaces, \( \mathcal{E} = \mathcal{E}' \oplus \mathcal{E}'' \), where \( \mathcal{E}' \) consists of the elements of \( \mathcal{E} \) that do not depend on the fibre variables \( \zeta_k \), i.e., \( \mathcal{E}' = C^\infty(U)[[\nu]] \), and \( \mathcal{E}'' \) is the kernel of the mapping \( \mathcal{E} \ni H \mapsto H|_{\zeta=0} \). Observe that the Euler operator \( \Gamma_1 : \mathcal{E} \to \mathcal{E} \) respects the decomposition \( \mathcal{E} = \mathcal{E}' \oplus \mathcal{E}'' \), \( \mathcal{E}' \) is its kernel, and \( \mathcal{E}'' \) is its image. Moreover, the operator \( \Gamma_1 \) is invertible on \( \mathcal{E}'' \). Every operator \( \Gamma_k : \mathcal{E} \to \mathcal{E} \) maps \( \mathcal{E} \) to \( \mathcal{E}'' \) and has \( \mathcal{E}' \) in its kernel.

The following lemma is straightforward.

**Lemma 2.1.** The operator \( \exp D = \sum_{r=0}^{\infty} \frac{1}{r!} D^r \) acts on \( \mathcal{E} \) and \( \exp(-D) \) is its inverse operator on \( \mathcal{E} \). The operator \( \exp D \) leaves invariant the subspace \( \mathcal{E}'' \) and the operator \( \exp D - 1 \) maps \( \mathcal{E} \) to \( \mathcal{E}'' \).

**Lemma 2.2.** We have the following identity,

\[
\Gamma_1 - D = e^D \Gamma_1 e^{-D}.
\]

**Proof.** The lemma follows from the fact that \([\Gamma_1, D] = D\) and the calculation

\[
e^D \Gamma_1 e^{-D} = \sum_{r=0}^{\infty} \frac{1}{r!} (\text{ad} D)^r \Gamma_1 = \Gamma_1 - D.
\]
Using Lemma 2.2, we rewrite formula (2.5) as follows:

\[(2.6) \quad (e^D \Gamma_1 e^{-D} + \sum_{r=2}^{\infty} \frac{1}{r!} \Gamma_r) F = 0.\]

Introduce the operator

\[(2.7) \quad Q = -e^{-D} \left( \sum_{r=2}^{\infty} \frac{1}{r!} \Gamma_r \right) e^D\]

on $\mathcal{E}$. It raises the filtration on $\mathcal{E}$ by one and maps $\mathcal{E}$ to $\mathcal{E}''$. Applying the operator $\exp(-D)$ on both sides of (2.6) we obtain that

\[(2.8) \quad (\Gamma_1 - Q) e^{-D} F = 0.\]

Using the decomposition $\mathcal{E} = \mathcal{E}' \oplus \mathcal{E}''$ and the last statement of Lemma 2.1, we observe that $\exp(-D)F = f + H$ for some $H \in \mathcal{E}''$. We can rewrite formula (2.8) as follows:

\[(2.9) \quad (\Gamma_1 - Q) H = Q f.\]

Since the operator $Q$ maps $\mathcal{E}$ to $\mathcal{E}''$ and $\Gamma_1$ is invertible on $\mathcal{E}''$, the operator $\Gamma_1^{-1} Q$ is well defined on $\mathcal{E}$ and raises the filtration by one, we obtain from (2.9) that

\[(2.10) \quad (1 - \Gamma_1^{-1} Q) H = \Gamma_1^{-1} Q f.\]

The operator $1 - \Gamma_1^{-1} Q$ is invertible and its inverse is given by the convergent series

\[ (1 - \Gamma_1^{-1} Q)^{-1} = \sum_{r=0}^{\infty} (\Gamma_1^{-1} Q)^r. \]

We have

\[ F = e^D (f + H) = e^D \left( f + \left( \sum_{r=0}^{\infty} (\Gamma_1^{-1} Q)^r \right) \Gamma_1^{-1} Q f \right) = e^D \left( \sum_{r=0}^{\infty} (\Gamma_1^{-1} Q)^r \right) f = e^D \left( 1 - \Gamma_1^{-1} Q \right)^{-1} f. \]

Combining these arguments we arrive at the following theorem.

**Theorem 2.3.** Given the standard star product with separation of variables on a pseudo-Kähler manifold $(M, \omega_{-1})$, a coordinate chart $U$ on $M$, and a function $f \in C^\infty(U)[[\nu]]$, then the total symbol $F = \tau(L_f)$ of the left star multiplication operator by $f$ is given by the following explicit formula,

\[(2.11) \quad F = e^D \left( 1 - \Gamma_1^{-1} Q \right)^{-1} f.\]
Now, to find the star product \( f \ast g \), one has to calculate the total symbol \( F \) of the operator \( L_f \) using formula (2.11), recover \( L_f \) from \( F \), and apply it to \( g \), \( f \ast g = L_f g \).

One can use the same formula (2.11) to express the total symbol of the left multiplication operator \( L_f \) of the star product with separation of variables \( \ast_\omega \) corresponding to an arbitrary formal deformation \( \omega \) of the pseudo-Kähler form \( \omega_{-1} \). To this end one has to modify the operators \( \Gamma_r \) and \( D \) as follows. On a contractible coordinate chart \( U \) find a formal potential \( \Phi = \frac{1}{p} \Phi_{-1} + \Phi_0 + \ldots \) of the form \( \omega \) and set

\[
G_{k_1 \ldots k_r \bar{l}} := \frac{\partial^{r+1} \Phi}{\partial z_{k_1} \ldots \partial z_{k_r} \partial \bar{z}_l}.
\]

Then \( G_{k_1 \ldots k_r \bar{l}} = \frac{1}{p} g_{k_1 \ldots k_r \bar{l}} + \ldots \). Denote the inverse of \( G_{k \bar{l}} \) by \( G^{\bar{l}k} = \nu g^{\bar{l}k} + \ldots \). Now modify \( \Gamma_r \) and \( D \) (retaining the same notations) as follows:

\[
\Gamma_r = G_{k_1 \ldots k_r \bar{l}} G^{\bar{l}k} \zeta_k \frac{\partial^r}{\partial \zeta_{k_1} \ldots \partial \zeta_{k_r}} \quad \text{and} \quad D = G^{\bar{l}k} \zeta_k \frac{\partial}{\partial \bar{z}_l}.
\]

The Euler operator \( \Gamma_1 \) will not change. Define the operator \( Q \) by the same formula (2.7) with the modified \( \Gamma_r \) and \( D \). Observe that we get the old operators \( \Gamma_r \), \( D \), and \( Q \) for the trivial deformation \( \omega = \frac{1}{p} \omega_{-1} \). One can show along the same lines that formula (2.11) with the modified operators \( D \) and \( Q \) will be given by a convergent series in the topology induced by the filtration on \( E \) and will define the total symbol of the left star multiplication operator \( L_f \) with respect to the star product \( \ast_\omega \).

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