LARGE DEVIATIONS AND WANDERING EXPONENT FOR RANDOM WALK IN A DYNAMIC BETA ENVIRONMENT

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Abstract. Random walk in a dynamic i.i.d. beta random environment, conditioned to escape at an atypical velocity, converges to a Doob transform of the original walk. The Doob-transformed environment is correlated in time, i.i.d. in space, and its marginal density function is a product of a beta density and a hypergeometric function. Under its averaged distribution the transformed walk obeys the wandering exponent 2/3 that agrees with Kardar-Parisi-Zhang universality. The harmonic function in the Doob transform comes from a Busemann-type limit and appears as an extremal in a variational problem for the quenched large deviation rate function.

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1. Introduction

We study an exactly solvable version of random walk in a random environment (RWRE) in one space dimension. The walk is nearest-neighbor and the environment dynamical and product-form. Our main results (i) construct a Doob transform of the RWRE that conditions the walk on an atypical velocity, (ii) establish that the transformed walk has path fluctuation exponent 2/3 of the KPZ (Kardar-Parisi-Zhang) class instead of the diffusive 1/2, and (iii) describe the quenched large deviation rate function of the walk.

The three points above are closely tied together. The harmonic functions in the Doob transform furnish extremals of a variational formula for the quenched large deviation rate function. Explicit distributional properties of these harmonic functions enable the derivation of the path exponent. The logarithm of the harmonic function itself obeys the KPZ longitudinal exponent 1/3.

This work rests on the development of analogues of percolation and polymer ideas for RWRE. The harmonic functions in the Doob transform arise through limits that correspond to Busemann functions of percolation and polymers. The quenched large deviation rate function is strictly above the averaged one except at their common minimum. For a deviation of small order $h$, the difference of the quenched and averaged rate functions is of order $h^4$. These properties are exactly as for the quenched and averaged free energy of 1+1 dimensional directed polymers [7, 13]. As is the case for the entire KPZ class, proofs of fluctuation exponents are restricted to models with special features. A natural expectation is that the picture that emerges here should be universal for 1+1 dimensional directed RWRE under some assumptions.

We turn to a detailed introduction of the model.

A dynamical environment is refreshed at each time step. On the two-dimensional space-time lattice $\mathbb{Z}^2$ we run time in the diagonal direction $(\frac{1}{2}, \frac{1}{2})$, and the admissible steps of the walk are $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The jump probabilities are independent and identically distributed at each lattice point of $\mathbb{Z}^2$. Transitions $\omega$ do not allow backward jumps. The distribution of the environment $\omega$ is $\mathbb{P}$ with expectation...
operator $\mathbb{E}$. Given a realization $\omega$ and a point $x \in \mathbb{Z}^2$, $P^\omega_x$ denotes the \textit{quenched} path measure of the Markov chain $(X_n)_{n\geq 0}$ on $\mathbb{Z}^2$ that starts at $x$ and uses transition probabilities $\omega$:

$$
P^\omega_x(X_0 = x) = 1 \quad \text{and, for } y \in \mathbb{Z}^2, \ n \geq 0, \ \text{and } i \in \{1, 2\}, 
\quad P^\omega_x(X_{n+1} = y + e_i \mid X_n = y) = \omega_{y, y+e_i}.
$$

Precisely speaking, $P^\omega_x$ is a probability measure on the path space $(\mathbb{Z}^2)^* \mathbb{Z}^+$ of the walk and $X$, is the coordinate process. This is a special case of \textit{random walk in a space-time random environment}.

This paper focuses on the \textit{beta RWRE} where $\omega_{x, x+e_1}$ is beta-distributed. Barraquand and Corwin [4] discovered that this special case is \textit{exactly solvable}. This means that fortuitous coincidences of combinatorics and probability permit derivation of explicit formulas and precise results far deeper than anything presently available for the general case. Some limit results uncovered in an exactly solvable case are expected to be universal. These then form natural conjectures to be investigated in the general case.

An earlier case of exact calculations for RWRE in a static environment appeared in a series of papers by Sabot and coauthors (see [23] and references therein). They discovered and utilized special features of the multidimensional Dirichlet RWRE to prove results currently not accessible for the general multidimensional RWRE. Section 8 of [23] discusses one-dimensional RWRE in a static beta environment.

Before specializing to the dynamic beta environment, we run through some known results for the general 1+1 dimensional RWRE (1.1) in an i.i.d. environment.

1.1. \textbf{Nearest-neighbor space-time RWRE.} Under an i.i.d. environment for the quenched model in (1.1), the \textit{averaged} path measure $P^0_\omega(\cdot) = \int P^\omega_x(\cdot) \mathbb{P}(d\omega)$ is a classical random walk with admissible steps $\{e_1, e_2\}$ and transition kernel $p(e_i) = \mathbb{E}(\omega_{0, e_i}), \ i = 1, 2$. Hence there is a law of large numbers $P^0_\omega\{X_N/N \to \xi^\ast\} = 1$ with limiting velocity $\xi^\ast = (\xi^\ast_1, \xi^\ast_2) = (p(e_1), p(e_2))$. Fubini’s theorem then gives the quenched law of large numbers

$$
P^0_\omega\left\{ \frac{X_N}{N} \to \xi^\ast \right\} = 1 \quad \text{for } \mathbb{P}-\text{a.e. } \omega.
$$

By Donsker’s invariance principle, under $P^0_\omega$ the centered and diffusively rescaled walk

$$
\left\{ W_N(t) = \frac{X_{[Nt]} - Nt\xi^\ast}{\sqrt{\xi^\ast_1\xi^\ast_2}N} : t \geq 0 \right\}
$$

converges weakly to $\{(W(t), -W(t)) : t \geq 0\}$, where $W(\cdot)$ is standard one-dimensional Brownian motion.

The same invariance principle holds for the \textit{quenched} RWRE if and only if $\mathbb{P}(\omega_{0, e_1} \in \{0, 1\}) < 1$. That is, for $\mathbb{P}$-almost every $\omega$ the distribution of $\{W_N(t) : t \geq 0\}$ under $P^\omega_0$ converges weakly to that of $\{(W(t), -W(t)) : t \geq 0\}$ (Theorem 1 of [16]). An invariance principle also holds for the quenched mean $E^0_\omega[X_N]$, but with scaling $N^{1/4}$ (Corollary 3.5 of [2]). In summary, as $N \to \infty$, the quenched mean of the walk has Gaussian fluctuations on a small scale of order $N^{1/4}$, while under a typical environment the walk itself has Gaussian fluctuations on the larger scale of order $N^{1/2}$. The fluctuations of the quenched walk dominate and hence the averaged process has Gaussian fluctuations of order $N^{1/2}$.
Let $\mathcal{U} = \{te_1 + (1-t)e_2 : 0 \leq t \leq 1\}$ denote the simplex of possible limiting velocities. For $\xi \in \mathcal{U}$ and $N \in \mathbb{N}$ let $[N\xi]$ denote a point closest to $N\xi$ on the antidiagonal $\{(x_1, x_2) \in \mathbb{Z}^2 : x_1 + x_2 = N\}$. The averaged large deviation principle (LDP) is the standard Cramér theorem and tells us that for $\xi \in \mathcal{U}$

$$\lim_{N \to \infty} N^{-1} \log P_\xi \{X_N = [N\xi]\} = -I_\alpha(\xi)$$

with rate function

$$I_\alpha(\xi) = \xi_1 \log \frac{\xi_1}{\xi_*} + \xi_2 \log \frac{\xi_2}{\xi_*} \quad \text{for } \xi = (\xi_1, \xi_2) \in \mathcal{U}.$$ (1.3)

Under the assumption

$$\mathbb{E}[|\log \omega_{0, e_i}|^{2+\varepsilon}] < \infty \quad \text{for } i \in \{1, 2\} \text{ and some } \varepsilon > 0$$ (1.4)
a quenched LDP holds as well. By Theorems 2.2, 3.1, 2.6(b), and 3.2(a) of [17], for all $\xi \in \mathcal{U}$,

$$\lim_{N \to \infty} N^{-1} \log P_\omega \{X_N = [N\xi]\} = -I_q(\xi)$$ (1.5)

exists $\mathbb{P}$-almost surely. The rate function $I_q$ does not depend on $\omega$. It is a nonnegative convex continuous function on $\mathcal{U}$ with a unique zero at $\xi_*$. By Fatou’s lemma and Jensen’s inequality, $I_q(\xi) \geq I_\alpha(\xi)$ for all $\xi \in \mathcal{U}$. It is shown in [27] that in fact $I_q(\xi) > I_\alpha(\xi)$ for all $\xi \in \mathcal{U}\{\xi_*\}$. The proof in [27] utilizes a uniform ellipticity assumption, namely that $\mathbb{P}(\delta \leq \omega_{0, e_1} \leq 1 - \delta) = 1$ for some $\delta > 0$, but their proof works more generally. Theorem 2.7 below states the strict inequality in the beta case.

In general in RWRE closed formulas for $I_q$ have not been found. Variational representations exist, for example in [6, 12, 17, 21, 26]. We state below one particular formula for the RWRE (1.1) on $\mathbb{Z}^2$. In the beta case extremals for this formula are identified in Section 2.3 below, in terms of harmonic functions constructed for the beta RWRE.

Let $\mathcal{K}$ denote the space of integrable stationary cocycles defined on the probability space $(\Omega, \mathcal{G}, \mathbb{P})$ of the environments. By this we mean stochastic processes $\{B_{x, y}(\omega) : x, y \in \mathbb{Z}^2\}$ that satisfy these conditions for all $x, y, z \in \mathbb{Z}^2$ and $\mathbb{P}$-a.e. $\omega$: $\mathbb{E}|B_{x, y}| < \infty$, $B_{x, y}(\omega) + B_{y, z}(\omega) = B_{x, z}(\omega)$, and $B_{x, y}(T_z \omega) = B_{x+z, y+z}(\omega)$ where $T_z$ is the shift $(T_z \omega)_{x, x+e_i} = \omega_{x+z, x+z+e_i}$. The rate function in (1.5) is then characterized as

$$I_q(\xi) = -\inf_{B \in \mathcal{K}} \left\{ \mathbb{E}[B_{0, e_1}] \xi_1 + \mathbb{E}[B_{0, e_2}] \xi_2 ight\}$$ (1.6)

$$+ \mathbb{P}-\text{ess sup} \log \left( \omega_{0, e_1} e^{-B_{0, e_1}(\omega)} + \omega_{0, e_2} e^{-B_{0, e_2}(\omega)} \right) \quad \text{for } \xi \in \text{ri} \mathcal{U}.$$ 

This formula for $I_q$ is valid for an i.i.d. environment $\omega$ under the same moment assumption (1.4) as the LDP.

For a nearest-neighbor RWRE on $\mathbb{Z}^d$ for which all directions $\pm e_i$ satisfy (1.4) formula (1.6) appeared in Theorem 2 on page 6 of [21]. In the directed case (1.6) is a special case of variational formula (4.7) in [9] for the point-to-point limiting free energy of a directed polymer. The RWRE with transition probability $\omega_{0, z}$ is obtained by taking the potential of the polymer to be $V_0(\omega, z) = \log \omega_{0, z} - \log p(z)$ where $p(\cdot)$ is a fixed transition probability in the background.
1.2. Beta RWRE. Let $\alpha, \beta > 0$ be positive real parameter values. The standard gamma and beta functions are given by

\begin{equation}
\Gamma(\alpha) = \int_{0}^{\infty} s^{\alpha-1}e^{-s} \, ds \quad \text{and} \quad B(\alpha, \beta) = \int_{0}^{1} s^{\alpha-1}(1-s)^{\beta-1} \, ds = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{equation}

The c.d.f. of the Beta($\alpha, \beta$) distribution is

\begin{equation}
F(t; \alpha, \beta) = B(\alpha, \beta)^{-1} \int_{0}^{t} s^{\alpha-1}(1-s)^{\beta-1} \, ds \quad \text{for} \quad 0 < t < 1.
\end{equation}

The case $\alpha = \beta = 1$ is the uniform distribution on $(0, 1)$.

For the remainder of this paper, the variables $\{\omega_{x,x+e_{i}} : x \in \mathbb{Z}^2\}$ in the RWRE (1.1) are i.i.d. Beta($\alpha, \beta$) distributed.

Barraquand and Corwin [4] showed that if $\alpha = \beta = 1$ and $\xi_1 - \xi_2 > 4/5$ then

\begin{equation}
\lim_{N \to \infty} \mathbb{P} \left\{ \frac{\log P_{0}^{\alpha}(X_N \cdot (e_1 - e_2) \geq N(\xi_1 - \xi_2)) + NI_{q}(\xi)}{c(\xi)N^{1/3}} \leq y \right\} = F_{\text{GUE}}(y)
\end{equation}

where the limit is the Tracy-Widom GUE distribution. Later, in a less rigorous paper, Thiery and Le Doussal [25] did the same for $\log P_{0}^{\alpha}(X_N = [N\xi]) + NI_{q}(\xi)$ and all $\alpha, \beta > 0$ and $\xi \neq \xi^*$.

These results revealed that this type of RWRE possesses features of the 1+1 dimensional Kardar-Parisi-Zhang (KPZ) universality class. A natural next question therefore is, where in the model do we find the KPZ wandering exponent $2/3$? It is not in the walk (1.1), because, as pointed out in Section 1.1, the walk in an i.i.d. environment satisfies a standard CLT under both its quenched and averaged distributions.

We answer the question by conditioning the walk on an atypical velocity. Under this conditioning the quenched process $X$, converges to a new walk given by a Doob transform of the original walk. The harmonic function in the transform is the exponential of an analogue of a Busemann function for RWRE. The Doob transform is a random walk in a correlated environment. When the environment is averaged out, at time $N$ this walk has fluctuations of the order $N^{2/3}$ and thus has the KPZ wandering exponent. This behavior deviates radically from that of classical random walk: standard random walk conditioned on an atypical velocity converges to another random walk with transitions altered to produce the new mean.

Conditioning on an atypical velocity is intimately tied to large deviations. The logarithm of the harmonic function in the Doob transform turns out to be an extremal in (1.6) and its expectation is the gradient of $I_{q}$.

Note: The notation conventions. We collect here some notation for easy reference. $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rationals, $\mathbb{R}$ the reals, and $\mathbb{C}$ the complex numbers. $\mathbb{Z}_{+} = \{0, 1, 2, 3, \ldots\}$, $\mathbb{N} = \{1, 2, 3, \ldots\}$, and $\mathbb{R}_{+} = [0, \infty)$. For real $a$, $[a]$ is the largest integer $\leq a$.

For $x, y \in \mathbb{R}^2$ we use the following conventions. Vector notation is $x = (x_1, x_2) = x_1e_1 + x_2e_2$, with canonical basis vectors $e_1 = (1, 0)$ and $e_2 = (0, 1)$. The scalar product is $x \cdot y$ and the $l^1$ norm $|x| = |x_1| + |x_2|$. Integer parts are taken coordinatewise: $[x] = ([x_1], [x_2])$. For $x \cdot (e_1 + e_2) \in \mathbb{Z}^2$, $[x]$ is the closest point to $x$ in $\{y \in \mathbb{Z}^2 : y_1 + y_2 = x_1 + x_2\}$. Inequality $y \geq x$ is interpreted coordinatewise: $y_1 \geq x_1$ and $y_2 \geq x_2$. 
Shifts or translations $T_z$ act on environments $\omega$ by $(T_z\omega)_{x,x+e_i} = \omega_{x+z,x+z+e_i}$ for $x, y \in \mathbb{Z}^2$. When subscripts are inconvenient, $\omega_{x,y}$ becomes $\omega(x,y)$, with the analogous convention for other quantities such as $\pi_{x,y}$, $B_{x,y}$, and $\rho_{x,y}$. A finite or infinite sequence is denoted by $x_{i:j} = (x_i, \ldots, x_j)$, for $-\infty \leq i < j \leq \infty$. The simplex of asymptotic velocities of walks is $\mathcal{U} = \{te_1 + (1-t)e_2 : 0 \leq t \leq 1\}$, with relative interior $\text{ri}\mathcal{U} = \{te_1 + (1-t)e_2 : 0 < t < 1\}$.

2. Results for beta RWRE

In Section 2.1 below we construct the Doob-transformed RWRE that is the limiting process of the quenched walk conditioned on an atypical velocity $\xi \neq \xi^*$. Section 2.2 states the KPZ fluctuation exponent of the averaged Doob-transformed walk. Finally in Section 2.3 we display the explicit quenched large deviation rate function and its connection with the harmonic functions of the Doob transform.

The standing assumptions for this section are that parameters $\alpha, \beta > 0$ are fixed, and the environment $\omega = (\omega_{x,x+1})_{x \in \mathbb{Z}^2}$ has the i.i.d. Beta($\alpha, \beta$) distribution. The probability space of the environment is $(\Omega, \mathcal{G}, \mathbb{P})$ where $\mathcal{G}$ is the Borel $\sigma$-field on the product space $\Omega = [0,1]^\mathbb{Z}^2$.

2.1. Doob transform of the quenched walk. The first main result is the existence of a family of increment-stationary harmonic functions, indexed by directions in $\text{ri}\mathcal{U} = \{te_1 + (1-t)e_2 : 0 < t < 1\}$.

**Theorem 2.1.** On $(\Omega, \mathcal{G}, \mathbb{P})$ there exists a stochastic process $\{B_{x,y}^\xi(\omega) : x, y \in \mathbb{Z}^2, \xi \in \text{ri}\mathcal{U}\}$ with the following properties.

For each $\xi \in \text{ri}\mathcal{U}$, $e^{-B_{0,x}^\xi}$ is a harmonic function: for all $x \in \mathbb{Z}^2$

$$
\omega_{x,x+e_i}e^{-B_{0,x+e_i}^\xi(\omega)} + \omega_{x,x+e_2}e^{-B_{0,x+e_2}^\xi(\omega)} = e^{-B_{0,x}^\xi(\omega)} \quad \mathbb{P}\text{-a.s.}
$$

(2.1)

For each $\xi \in \text{ri}\mathcal{U}$ there is an event $\Omega(\xi)$ such that $\mathbb{P}(\Omega(\xi)) = 1$ and for every $\omega \in \Omega(\xi)$,

$$
B_{x,y}^\xi(\omega) = \lim_{N \to \infty} \left( \log P_x^\omega\{X_{|z_N-|1} = z_N\} - \log P_y^\omega\{X_{|z_N-y-|1} = z_N\} \right)
$$

(2.2)

for all $x, y \in \mathbb{Z}^2$, and for any sequence $z_N \in \mathbb{Z}^2$ such that $|z_N|_1 \to \infty$ and $z_N/|z_N|_1 \to \xi$.

In the law of large numbers direction $\xi^* = (\alpha/\alpha+\beta, \beta/\alpha+\beta)$ we have

$$
B_{x,y}^{\xi^*}(\omega) = 0.
$$

(2.3)

By analogy with limits of increments in percolation and polymers, we could call $B^\xi$ the *Busemann function* in direction $\xi$. For $\xi \neq \xi^*$, the variables $B_{x,x+e_i}^\xi$ are marginally logarithms of beta-variables. From limit (2.2) we see that

$$
B_{x,y}^\xi(T_z\omega) = B_{x+z,y+z}^\xi(\omega) \quad \text{and} \quad B_{x,y}^\xi(\omega) + B_{y,z}^\xi(\omega) = B_{x,z}^\xi(\omega)
$$

for all $x, y, z \in \mathbb{Z}^2$ and $\mathbb{P}$-a.e. $\omega$. In other words, $B^\xi$ is a member of the space $\mathcal{K}$ of integrable stationary cocycles defined above (1.6). Harmonicity (2.1) comes from limit (2.2) and the Markov property

$$
P_x^\omega\{X_{|z_N-|1} = z_N\} = \omega_{x,x+e_1}P_{x+e_1}^\omega\{X_{|z_N-|1-1} = z_N\} + \omega_{x,x+e_2}P_{x+e_2}^\omega\{X_{|z_N-|1-1} = z_N\}.
$$

Further continuity, monotonicity, and explicit distributional properties of the process $B^\xi$ are given in Theorem 3.6.
Theorem 2.1 is proved by constructing a family of harmonic functions on quadrants and by using these to control the convergence of the differences on the right of (2.2). This approach is the RWRE counterpart of the arguments used for an exactly solvable polymer model in [11] and for the corner growth model with general i.i.d. weights in [10].

By (2.1) and (2.4),

$$\kappa^\xi_{x, x+e_i}(\omega) = \omega_{x, x+e_i} \frac{e^{-B^\xi_{x, x+e_i}(\omega)}}{e^{-B_{0,x}(\omega)}} = \omega_{x, x+e_i} e^{-B^\xi_{x, x+e_i}(\omega)}, \quad i \in \{1, 2\},$$

defines a new transition probability on $\mathbb{Z}^2$, as a Doob-transform of the original transition $\omega$. It is an RWRE transition as a function on $\Omega$ because, by (2.4), it obeys shifts: $\kappa^\xi_{x, x+e_i}(T_x \omega) = \kappa^\xi_{x+z, x+z+e_i}(\omega)$. The environment $\kappa^\xi(\omega) = \left(\kappa^\xi_{x, x+e_i}(\omega)\right)_{x \in \mathbb{Z}^2}$ is in general correlated over locations $x$, except that its restriction on antidiagonals is i.i.d. as stated in the next theorem.

Let $P_x^{\kappa^\xi}$ denote the quenched path measure of the Markov chain with transition probability $\kappa^\xi$. In other words, $P_x^{\kappa^\xi}$ satisfies (1.1) with $\kappa^\xi_{y, y+e_i}$ instead of $\omega_{y, y+e_i}$. $P_x^{\kappa^\xi} = P_x^{\kappa^\xi(\omega)}$ is a function of $\omega$ through its transition probability.

**Theorem 2.2.** Fix $\xi \in \text{ri} \mathcal{U}$. Then for any $n \in \mathbb{Z}$, the random variables $\{\kappa^\xi_{x, x+e_1}(\omega) : x_1 + x_2 = n\}$ are i.i.d. We have the law of large numbers:

$$P^{\kappa^\xi(\omega)}_0\{N^{-1} X_N \to \xi\} = 1 \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$ 

In [11] an RWRE in a correlated environment arose as a limit of the quenched log-gamma polymer. The transition probability of the log-gamma RWRE is marginally beta-distributed. The transition $\kappa^\xi$ described above is not the same. In particular, the marginal distribution of the random variable $\kappa^\xi_{x, x+e_i}$ is not beta. Its density function is a product of a beta density and a hypergeometric function, given in Theorem 3.7 below.

The next theorem records the limits of quenched processes conditioned on particular velocities.

**Theorem 2.3.** For each fixed $\xi \in \text{ri} \mathcal{U}$ there is an event $\Omega^{(\xi)}$ such that $\mathbb{P}(\Omega^{(\xi)}) = 1$ and the following holds for every $\omega \in \Omega^{(\xi)}$: if $z_N \in \mathbb{Z}^2$ is any sequence such that $|z_N|_1 = N$ and $z_N/N \to \xi$, then the conditioned quenched path distribution $P_0^\omega(\cdot | X_N = z_N)$ converges weakly on the path space $(\mathbb{Z}^2)^{\mathbb{Z}^+}$ to the Doob transformed path measure $P_0^{\kappa^\xi(\omega)}$.

The weak convergence claim in the theorem amounts to checking that for any finite path $x_{0,m}$ with $x_0 = 0$,

$$\lim_{N \to \infty} P^\omega_0( X_{0,m} = x_{0,m} | X_N = z_N) = \prod_{k=0}^{m-1} \kappa^\xi_{x_k, x_{k+1}}(\omega) \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$ 

This is an immediate consequence of limit (2.2). Combining (2.3) with the theorem above tells us that if $z_N/N \to \xi^*$, then $P^\omega_0(\cdot | X_N = z_N) \to P^\omega_0$. In other words, conditioning on the typical velocity $\xi^*$ introduces no new correlations in the limit and leads back to the original path measure. This behavior is consistent with classical random walk.

Observe that $P^{\kappa^\xi(\omega)}_0( X_{0,m} = x_{0,m} | X_N = z_N) = P^\omega_0( X_{0,m} = x_{0,m} | X_N = z_N)$ for $0 \leq m \leq N$. Consequently the family $\{P_0^{\kappa^\xi}\}$ is closed under taking limits of path distributions conditioned on velocities.
Theorems 2.1, 2.2, and 2.3 are proved after the statement of Theorem 3.6.

2.2. Fluctuation bounds. In 1+1 dimensional models in the KPZ class, the exponent $\frac{1}{3}$ appears in fluctuations of heights of growing interfaces and free energies of polymer models, while the exponent $\frac{2}{3}$ appears in spatial correlations and path fluctuations. The Barraquand-Corwin limit (1.9) indicated that logarithms of quenched probabilities obey $\frac{1}{3}$-fluctuations. Theorem below shows that the process $B^\xi$ has this same order of magnitude of fluctuations, though only in the direction $\xi$, as quantified by hypothesis (2.7) below. If the endpoint $(m,n)$ deviates from $N\xi$ by an amount of order $N^\nu$ for $\nu > \frac{2}{3}$, the fluctuations of $B^\xi_{0,(m,n)}$ become Gaussian. (This follows similar observations for directed polymers in Corollary 1.4 of [5] and Corollary 2.2 of [24].)

**Theorem 2.4.** Fix $\alpha, \beta > 0$. Fix $\xi = (\xi_1, \xi_2) \in \text{ri} \mathcal{U}\{\xi^*\}$. Given a constant $0 < \gamma < \infty$, there exist positive finite constants $c$, $C$, and $N_0$, depending only on $\alpha$, $\beta$, $\gamma$, and $\xi$, such that

$$cN^{2/3} \leq \text{Var}[B^\xi_{0,(m,n)}] \leq C N^{2/3}$$

for all $N \geq N_0$ and $(m,n) \in \mathbb{N}^2$ such that

$$|m - N\xi_1| \lor |n - N\xi_2| \leq \gamma N^{2/3}. \quad (2.7)$$

The same constants can be taken for $(\alpha, \beta, \gamma, \xi)$ varying in a compact subset of $(0, \infty)^3 \times \text{ri} \mathcal{U}\{\xi^*\}$.

Theorem 2.4 was proved independently and concurrently in the present work and as one case of a more general result for exactly solvable directed polymers by Chaumont and Noack (Theorem 1.2 of [5]). A proof appears in Section 4.1 of the first preprint version [3] of this paper. In the present version we omit the proof and cite [5] for details. The translation between the Doob-transformed walk and the beta polymer is explained in Section 4.

The second fluctuation result quantifies the deviations of the walk from its limiting velocity, under the averaged measure $P^\xi(\cdot) = \int P^{x(\omega)}(\cdot) \mathbb{P}(d\omega)$ of the Doob-transformed RWRE. Bounds (2.8) and (2.9) indicate that this walk is superdiffusive with the KPZ wandering exponent $\frac{2}{3}$ instead of the diffusive $\frac{1}{2}$ of classical random walk.

**Theorem 2.5.** Fix $\alpha, \beta > 0$. Fix $\xi \in \text{ri} \mathcal{U}\{\xi^*\}$. There exist finite positive constants $C$, $c$, $r_0$, and $\delta_0$, depending only on $\alpha$, $\beta$, and $\xi$, such that for $r \geq r_0$, $\delta \in (0, \delta_0)$, and any $N \geq 1$ we have

$$P^\xi_{0}(|X_N - N\xi| \geq rN^{2/3}) \leq Cr^{-\delta} \quad (2.8)$$

and

$$P^\xi_{0}(|X_N - N\xi| \geq \delta N^{2/3}) \geq c. \quad (2.9)$$

The same constants can be taken for $(\alpha, \beta, \xi)$ varying in a compact subset of $(0, \infty)^2 \times \text{ri} \mathcal{U}\{\xi^*\}$.

Theorem 2.5 is proved in Section 6. The bounds come from using harmonic functions to control the exit point of the walk from rectangles. For this proof also we can cite an estimate from [5].
2.3. Large deviations. This section records explicit large deviation rate functions and their
link with the process $B^\xi$ of Theorem 2.1.

We begin with a technical point that is needed for the rate function and for the entire
remainder of the paper. The next lemma establishes connections between three parameters:
$\xi \in \mathcal{U}$ is an asymptotic velocity of the walk, $t \in \mathbb{R}$ is the tilt dual to $\xi$, and $0 < \lambda < \infty$
parametrizes two families of increment-stationary harmonic functions that we construct in
Section 3.2 and use as tools to analyze the model. A slight inconvenience is that as $\xi$ ranges
across $\mathcal{U}$ from left to right (in the direction of $\xi_1$), $\lambda$ goes from $0$ to $\infty$ and back, with $\lambda = \infty$
corresponding to $\xi = \xi^*$. This is depicted in the first two plots of Figure 2.1.

Recall the polygamma functions $\psi_0(s) = \Gamma'(s)/\Gamma(s)$ and $\psi_n(s) = \psi_{n-1}(s)$ for $s > 0$ and
$n \in \mathbb{N}$. Some basic properties of these functions are given in Appendix A. Qualitatively
speaking, $\psi_0$ is strictly concave and increasing from $\psi_0(0+) = -\infty$ to $\psi_0(\infty-) = \infty$, while
$\psi_1$ is strictly convex and decreasing from $\psi_1(0+) = \infty$ to $\psi_1(\infty-) = 0$.

**Figure 2.1.** Leftmost and middle plots are of $\lambda$ as a function of $\xi_1$. The left plot
stretches the $\lambda$-axis to reveal the behavior away from $\xi^*_1$. The rightmost plot is of $\lambda$
as a function of $t$. These graphs are for $(\alpha, \beta) = (1, 2)$.

**Lemma 2.6.** Fix $\alpha, \beta > 0$.

(a) Given $\xi = (\xi_1, 1 - \xi_1) \in \mathcal{U}$ there is a unique $\lambda = \lambda(\xi) \in [0, \infty]$ such that

\[
\xi_1 = \frac{\psi_1(\lambda) - \psi_1(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)} \quad \text{for } \xi_1 \in [\xi^*_1, 1], \quad \text{and}
\]
\[
\xi_1 = 1 - \frac{\psi_1(\lambda) - \psi_1(\beta + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)} \quad \text{for } \xi_1 \in [0, \xi^*_1],
\]

with $\lambda = 0 \iff \xi \in \{e_1, e_2\}$ and $\lambda = \infty \iff \xi = \xi^* = (\frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta})$.

Furthermore, $\lambda$ is strictly increasing on $\xi_1 \in [0, \xi^*_1]$ and strictly decreasing on $\xi_1 \in
(\xi^*_1, 1]$.

(b) Given $t \in [0, \infty]$ there is a unique $\lambda = \lambda(t) \in [0, \infty]$ such that

\[
t = \psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)
\]

where $\lambda = 0 \iff t = \infty$ and $\lambda = \infty \iff t = 0$. 

The proof of Lemma 2.6 is given in Section 7. The formula for the quenched rate function $I_q$ in (1.5) in the beta environment can now be given. See Figure 2.2 for an illustration.

**Theorem 2.7.** Fix $\alpha, \beta > 0$ and let $\omega$ have i.i.d. Beta($\alpha, \beta$) distribution. Then for $\xi = (\xi_1, \xi_2) \in U$ we have $I_q(\xi^*) = 0$ and

$$I_q(\xi) = \begin{cases} 
\xi_1 \psi_0(\alpha + \beta + \lambda(\xi)) + \xi_2 \psi_0(\lambda(\xi)) - \psi_0(\alpha + \lambda(\xi)) & \text{for } \xi_1 \in (\xi_1^*, 1], \\
\xi_2 \psi_0(\alpha + \beta + \lambda(\xi)) + \xi_1 \psi_0(\lambda(\xi)) - \psi_0(\beta + \lambda(\xi)) & \text{for } \xi_1 \in [0, \xi_1^*),
\end{cases}$$

where in both cases $\lambda$ and $\xi$ determine each other uniquely via (2.10) and (2.11). $I_q$ is a strictly convex function on $[0, 1]$ and satisfies $I_q(\xi) > I_a(\xi)$ for all $\xi \in U \setminus \{\xi^*\}$.

**Example 2.8 (Case $\alpha = \beta = 1$).** In the i.i.d. uniform environment $\lambda$ and $I_q$ can be found in closed form with the help of the recurrence formulas (A.2). The rate function is

$$I_q(\xi) = 1 - 2\sqrt{\xi_1\xi_2} = \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)(-1)^{n+1}4^n(\xi_1 - \frac{1}{2})^{2n} \quad \text{for } \xi \in U.$$  

The series illustrates that this rate function is analytic on the entire open segment $\text{ri}U$, a property which is open for general $(\alpha, \beta)$.

**Remark 2.9 (Regularity of $I_q$).** Lemma B.1 in Appendix B shows that $I_q$ is analytic away from $\xi^*$. We compute derivatives of $I_q$ up to the fourth one, to verify that across $\xi^*$ we have at least four continuous derivatives. We obtain the following expansion around $\xi^*$:

$$I_q(\xi) = \frac{(\alpha + \beta)^2}{2\alpha\beta}(\xi_1 - \xi_1^*)^2 + \frac{(\alpha + \beta)^3(\alpha - \beta)}{6\alpha^2\beta^2}(\xi_1 - \xi_1^*)^3 + \frac{(\alpha + \beta)^4(2\alpha^2 - 2\alpha\beta + 2\beta^2 + 1)}{24\alpha^3\beta^3}(\xi_1 - \xi_1^*)^4 + O(|\xi_1 - \xi_1^*|^5).$$

The details appear at the end of Appendix B.

For the sake of comparison, here is the expansion around $\xi^*$ of the averaged rate function $I_a$ from (1.3):

$$I_a(\xi) = \frac{(\alpha + \beta)^2}{2\alpha\beta}(\xi_1 - \xi_1^*)^2 + \frac{(\alpha + \beta)^3(\alpha - \beta)}{6\alpha^2\beta^2}(\xi_1 - \xi_1^*)^3 + \frac{(\alpha + \beta)^3(\alpha^3 + \beta^3)}{12\alpha^3\beta^3}(\xi_1 - \xi_1^*)^4 + O(|\xi_1 - \xi_1^*|^5).$$

The expansions of $I_q$ and $I_a$ agree to third order. This explains the minute difference between the two graphs in Figure 2.2. One can check that

$$\frac{d^4}{d\xi_1^4} \left[I_q(\xi_1, 1 - \xi_1) - I_a(\xi_1, 1 - \xi_1)\right]_{\xi_1 = \xi^*} = \frac{(\alpha + \beta)^4}{\alpha^3\beta^3} > 0.$$ 

Thus the fourth-order terms differ in the two expansions.

We also record the convex conjugate

$$I_q^*(h) = \sup_{\xi \in \mathbb{U}} \{h \cdot \xi - I_q(\xi)\} = \lim_{n \to \infty} n^{-1} \log E_0^n[e^{h \cdot X_n}], \quad h \in \mathbb{R}^2.$$ 

The second equality above is an instance of Varadhan’s theorem [18, page 28]. Since $(X_n - X_0) \cdot (e_1 + e_2) = n$, we have $I_q^*(te_1 + se_2) = s + I_q^*((t-s)e_1)$ and it suffices to consider $h = te_1$ for real $t$. 


Theorem 2.10. Fix $\alpha, \beta > 0$ and let $\omega$ have i.i.d. Beta $(\alpha, \beta)$ distribution. For $t \geq 0$

\begin{align}
(2.17) \quad I_q^*(te_1) &= \psi_0(\alpha + \lambda(t)) - \psi_0(\lambda(t)) \\
(2.18) \quad \text{and} \quad I_q^*(-te_1) &= -t + \psi_0(\beta + \lambda(t)) - \psi_0(\lambda(t)),
\end{align}

where $\lambda$ and $t$ determine each other via (2.12).

Formula (2.13) for $I_q$ appeared earlier in equations (8)–(9) of [4] where it was derived by nontrivial asymptotic analysis. We derive $I_q$ through its convex conjugate $I_q^*$, which in turn is calculated with the help of harmonic functions to be constructed below.

Next we state the connections between $I_q$ and the processes $B^\xi$.

Theorem 2.11. (a) Fix $\xi \in \text{ri} U$. Then the process $B^\xi$ is an extremal for variational formula (1.6). In particular, we have

\begin{equation}
(2.19) \quad I_q(\xi) = -\mathbb{E}[B_{0,\xi_1}^\xi]_1 - \mathbb{E}[B_{0,\xi_2}^\xi]_2 \\
= -\inf_{\zeta \in \text{ri} U} \left\{ \mathbb{E}[B_{0,\xi_1}^\zeta]_1 + \mathbb{E}[B_{0,\xi_2}^\zeta]_2 \right\},
\end{equation}

where the last infimum is uniquely attained at $\zeta = \xi$. 

Figure 2.2. Top plot shows $I_q$ (higher, thicker graph) and $I_a$ (lower, thinner graph) as functions of $\xi_1$. Bottom figures zoom in on $I_q$ and $I_a$ over intervals $[\xi_1^* - 4.440004 \times 10^{-4}, \xi_1^* - 4.439998 \times 10^{-4}]$ and $[\xi_1^* + 4.442957038524 \times 10^{-4}, \xi_1^* + 4.442957038583 \times 10^{-4}]$. On the left, the vertical axis goes from $4.440484 \times 10^{-7}$ to $4.440497 \times 10^{-7}$. On the right, the vertical axis goes from $4.436541 \times 10^{-7}$ to $4.436554 \times 10^{-7}$. These graphs are for $(\alpha, \beta) = (1, 2)$. 

**Theorem 2.10.** Fix $\alpha, \beta > 0$ and let $\omega$ have i.i.d. Beta $(\alpha, \beta)$ distribution. For $t \geq 0$

\begin{align}
(2.17) \quad I_q^*(te_1) &= \psi_0(\alpha + \lambda(t)) - \psi_0(\lambda(t)) \\
(2.18) \quad \text{and} \quad I_q^*(-te_1) &= -t + \psi_0(\beta + \lambda(t)) - \psi_0(\lambda(t)),
\end{align}

where $\lambda$ and $t$ determine each other via (2.12).

Formula (2.13) for $I_q$ appeared earlier in equations (8)–(9) of [4] where it was derived by nontrivial asymptotic analysis. We derive $I_q$ through its convex conjugate $I_q^*$, which in turn is calculated with the help of harmonic functions to be constructed below.

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**Theorem 2.11.** (a) Fix $\xi \in \text{ri} U$. Then the process $B^\xi$ is an extremal for variational formula (1.6). In particular, we have

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= -\inf_{\zeta \in \text{ri} U} \left\{ \mathbb{E}[B_{0,\xi_1}^\zeta]_1 + \mathbb{E}[B_{0,\xi_2}^\zeta]_2 \right\},
\end{equation}

where the last infimum is uniquely attained at $\zeta = \xi$. 

Figure 2.2. Top plot shows $I_q$ (higher, thicker graph) and $I_a$ (lower, thinner graph) as functions of $\xi_1$. Bottom figures zoom in on $I_q$ and $I_a$ over intervals $[\xi_1^* - 4.440004 \times 10^{-4}, \xi_1^* - 4.439998 \times 10^{-4}]$ and $[\xi_1^* + 4.442957038524 \times 10^{-4}, \xi_1^* + 4.442957038583 \times 10^{-4}]$. On the left, the vertical axis goes from $4.440484 \times 10^{-7}$ to $4.440497 \times 10^{-7}$. On the right, the vertical axis goes from $4.436541 \times 10^{-7}$ to $4.436554 \times 10^{-7}$. These graphs are for $(\alpha, \beta) = (1, 2)$.
(b) Extend $I_q$ homogeneously to all of $\mathbb{R}^2_+$, that is, by $I_q(c\xi) = cI_q(\xi)$ for $c > 0$ and $\xi \in \mathcal{U}$. Then the gradient of $I_q$ satisfies

$$\nabla I_q(\xi) = -\mathbb{E}[B_{0,e_1}^\xi]c_1 - \mathbb{E}[B_{0,e_2}^\xi]c_2, \quad \xi \in \mathbb{ri}\mathcal{U}. \tag{2.20}$$

Corollary 4.5 and Remark 5.7 in [9] put equations (2.19)–(2.20) in the context of a general theory for directed walks in random potentials. Theorems 2.7, 2.10 and 2.11 are proved in Section 7.

Lastly, we record the LDP for the Doob-transformed RWRE. Definition (2.5) and the cocycle property in (2.4) imply that

$$P_0^{\kappa\xi(\omega)}(X_N = x) = P_0^{\omega}(X_N = x)e^{-B_{0,x}^\xi(\omega)}. \tag{2.21}$$

$B^\xi$ has i.i.d. increments along horizontal and vertical lines (Theorem 3.6(c)) and hence the law of large numbers applies: $\mathbb{P}$-almost surely

$$\lim_{N \to \infty} N^{-1}B_{0,[N\xi]}^\xi = \mathbb{E}[B_{0,e_1}^\xi]\zeta_1 + \mathbb{E}[B_{0,e_2}^\xi]\zeta_2 = -\zeta \cdot \nabla I_q(\xi) \quad \forall \zeta \in \mathbb{ri}\mathcal{U}. \tag{3.1}$$

The quenched LDP (1.5) of the beta walk then gives this theorem.

**Theorem 2.12.** For any fixed $\xi \in \mathbb{ri}\mathcal{U}$, the following holds $\mathbb{P}$-almost surely, simultaneously for all $\zeta \in \mathbb{ri}\mathcal{U}$,

$$\lim_{N \to \infty} N^{-1} \log P_0^{\kappa\xi}\{X_N = [N\zeta]\} = -I_q(\zeta) + \zeta \cdot \nabla I_q(\xi). \tag{3.2}$$

The rate function $I_q(\zeta) - \zeta \cdot \nabla I_q(\xi)$ is minimized at $\zeta = \xi$, by the convexity and homogeneity of $I_q$. The main results have been stated and we turn to proofs.

### 3. Increment-stationary harmonic functions

In this section we construct quenched harmonic functions whose probability distributions are suitably invariant under lattice translations. This is done first on restricted subsets of the lattice by solving a boundary value problem, and then extended to the entire lattice by taking limits. That this is possible with explicit distributions and useful independence properties is a feature of exact solvability.

The boundaries of the positive and negative quadrants $v + \mathbb{Z}_+^2$ and $v - \mathbb{Z}_+^2$ with a corner at $v \in \mathbb{Z}^2$ are denoted by

$$\mathbb{B}_v^+ = \{v + (i, 0), v + (0, j) : i, j \geq 0\} \quad \text{and} \quad \mathbb{B}_v^- = \{v - (i, 0), v - (0, j) : i, j \geq 0\}. \tag{3.1}$$

Hitting times of the boundaries follow analogous notation:

$$\tau_v^\pm = \inf\{n \geq 0 : X_n \in \mathbb{B}_v^\pm\}. \tag{3.2}$$

The separate axes of these boundaries are distinguished by the notation

$$\mathbb{B}_v^{(\pm 1)} = \{v \pm (i, 0) : i \geq 0\} \quad \text{and} \quad \mathbb{B}_v^{(\pm 2)} = \{v \pm (0, j) : j \geq 0\}. \tag{3.3}$$

In particular, $\mathbb{B}_v^\pm = \mathbb{B}_v^{(\pm 1)} \cup \mathbb{B}_v^{(\pm 2)}$. 

3.1. An involution for beta variables. This section undertakes some technical preparation for the construction of harmonic functions on quadrants of the lattice. A distribution-preserving involution of triples of beta variables is defined and its properties recorded. We begin by motivating this construction through a Dirichlet problem.

Consider backward nearest-neighbor transition probabilities $\tilde{\omega}_{x,x-e_1}$, $i \in \{1, 2\}$, on the lattice $\mathbb{Z}^2$. These transition probabilities allow two steps $-e_1$ and $-e_2$ and satisfy $\tilde{\omega}_{x,x-e_1} + \tilde{\omega}_{x,x-e_2} = 1$ at each $x \in \mathbb{Z}^2$. Suppose a function $f$ is given on the boundary $\mathbb{B}_0^+$ of the first quadrant $\mathbb{Z}^2_+$. When the backward walk starts in the first quadrant, the hitting time $\tau_0^+$ is obviously finite. Then

$$H(x) = E_x^\omega[f(X(\tau_0^+))]$$

defines an $\omega$-harmonic function on the positive first quadrant. That is, $H$ satisfies

$$H(x) = \tilde{\omega}_{x,x-e_1}H(x-e_1) + \tilde{\omega}_{x,x-e_2}H(x-e_2) \quad \text{for } x \in \mathbb{N}^2.$$  \hfill (3.5)

We solve (3.5) inductively, by beginning from the boundary values and then defining $H(x)$ once $H(x-e_1)$ and $H(x-e_2)$ have been defined. We formulate this induction in terms of ratios $\rho_{x,y} = H(x)/H(y)$. The induction assumption is that the nearest-neighbor ratios $\rho_{x-e_2, x-e_1-e_2}$ and $\rho_{x-e_1, x-e_1-e_2}$ have been defined on the south and west sides of a unit square with northeast corner at $x$. Then, by (3.5), the ratios on the north and east sides are obtained from the equations

$$\rho_{x,x-e_1} = \frac{\tilde{\omega}_{x,x-e_1} \rho_{x-e_1, x-e_1-e_2} + (1 - \tilde{\omega}_{x,x-e_1}) \rho_{x-e_2, x-e_1-e_2}}{\rho_{x-e_1, x-e_1-e_2}},$$

$$\rho_{x,x-e_2} = \frac{\tilde{\omega}_{x,x-e_1} \rho_{x-e_1, x-e_1-e_2} + (1 - \tilde{\omega}_{x,x-e_1}) \rho_{x-e_2, x-e_1-e_2}}{\rho_{x-e_2, x-e_1-e_2}},$$ \hfill (3.7)

It is useful to augment this pair of equations with a third equation

$$\omega_{x-e_1-e_2,x-e_2} = \frac{\rho_{x-e_2, x-e_1-e_2} (\rho_{x-e_1, x-e_1-e_2} - 1)}{\rho_{x-e_1, x-e_1-e_2} - \rho_{x-e_2, x-e_1-e_2}}$$ \hfill (3.8)

provided the denominator never vanishes. Together the three equations define an involution. In the case we specialize to below $\omega_{x-e_1-e_2,x-e_2}$ is a forward transition probability from $x-e_1-e_2$ to $x-e_2$. The complementary transition probability from $x-e_1-e_2$ to $x-e_1$ is of course then

$$\omega_{x-e_1-e_2,x-e_1} = 1 - \omega_{x-e_1-e_2,x-e_2}.$$ \hfill (3.9)

Equations (3.6)–(3.9) are illustrated by Figure 3.1, with $x$ in the upper right corner of the unit square and with

$$(U, V, W) = (\rho_{x-e_2, x-e_1-e_2}, \rho_{x-e_1, x-e_1-e_2}, \tilde{\omega}_{x,x-e_1})$$

and

$$(U', V', W') = (\rho_{x,x-e_1}, \rho_{x,x-e_2}, \omega_{x-e_1-e_2,x-e_2}).$$

Now assume that the transition probabilities $\tilde{\omega}$ come from a beta RWRE, in other words, that the variables $\{\tilde{\omega}_{x,x-e_1}\}_{x \in \mathbb{Z}^2}$ are i.i.d. Beta($\alpha, \beta$). The next key lemma indicates how to choose the distributions of the ratios of the boundary values $H(x)$ in order to get tractable harmonic functions. We regard the parameters $\alpha, \beta$ of the environment fixed, while $0 < \lambda < \infty$ parametrizes two different boundary conditions in cases (a) and (b) in the lemma.
Figure 3.1. Involution (3.10): Respectively, weights $U$ and $V$ on the south and west edges and west/south transition $(W, 1-W)$ become weights $U'$ and $V'$ on the north and east edges and east/north transition $(W', 1-W')$, and vice-versa.

**Lemma 3.1.** The equations

$$(3.10) \quad U' = \frac{WV + (1-W)U}{V}, \quad V' = \frac{WV + (1-W)U}{U} \quad \text{and} \quad W' = \frac{U(V-1)}{V-U}$$

define an involution $(U, V, W) \mapsto (U', V', W')$ on the product space $(0, 1) \times (1, \infty) \times (0, 1)$.

Let $0 < \alpha, \beta, \lambda < \infty$.

(a) Suppose that $(U, V, W)$ are independent variables with distributions

$$(3.11) \quad U \sim \text{Beta}(\alpha + \lambda, \beta), \quad V^{-1} \sim \text{Beta}(\lambda, \alpha), \quad \text{and} \quad W \sim \text{Beta}(\alpha, \beta).$$

Then the triples $(U', V', W')$ and $(U, V, W)$ have the same distribution.

(b) Suppose that $(U, V, W)$ are independent variables with distributions

$$(3.12) \quad U^{-1} \sim \text{Beta}(\lambda, \beta), \quad V \sim \text{Beta}(\beta + \lambda, \alpha), \quad \text{and} \quad W \sim \text{Beta}(\alpha, \beta).$$

Then again the triples $(U', V', W')$ and $(U, V, W)$ have the same distribution.

Proof. Algebra checks the involution property. We prove part (a). Part (b) follows by switching around $\alpha$ and $\beta$ and by switching around the axes.

Let $(W, \Gamma_\alpha, \Gamma_\beta, \Gamma_\lambda)$ be jointly independent with $W \sim \text{Beta}(\alpha, \beta)$ and $\Gamma_\nu \sim \text{Gamma}(\nu, 1)$. Set

$$(3.13) \quad U = \frac{\Gamma_\alpha + \Gamma_\lambda}{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda} \quad \text{and} \quad V = \frac{\Gamma_\alpha + \Gamma_\lambda}{\Gamma_\lambda}.$$ 

Then $(U, V, W)$ have the desired distribution because $V$ is independent of $\Gamma_\alpha + \Gamma_\lambda$.

Compute

$$U' = W + (1-W)\frac{U}{V} = W + (1-W)\frac{\Gamma_\lambda}{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda},$$

$$V' = W\frac{V}{U} + 1-W = W\frac{\Gamma_\alpha + \Gamma_\beta + \Gamma_\lambda}{\Gamma_\lambda} + 1-W,$$

$$(3.14) \quad W' = \frac{U(V-1)}{(V-U)} = \frac{\Gamma_\alpha}{\Gamma_\alpha + \Gamma_\beta}.$$
$W'$ is independent of the pair $(U', V')$ because it is independent of $\Gamma_{\alpha} + \Gamma_{\beta}$. It also clearly has the same distribution as $W$.

It remains to show that $(U', V')$ has the same distribution as $(U, V)$. For this set

$$Y = \frac{\Gamma_{\lambda}}{\Gamma_{\alpha} + \Gamma_{\beta} + \Gamma_{\lambda}}.$$

Observe that

$$U' = W + (1 - W)Y \quad \text{and} \quad V' = WY^{-1} + 1 - W.$$

Also

$$W' + (1 - W')Y = Y + W'(1 - Y) = \frac{\Gamma_{\alpha} + \Gamma_{\lambda}}{\Gamma_{\alpha} + \Gamma_{\beta} + \Gamma_{\lambda}} = U$$

and similarly

$$W'Y^{-1} + 1 - W' = V.$$

Furthermore, $(Y, W')$ are independent and so are $(Y, W)$. Consequently, the two pairs have the same distribution and then $(U', V')$ has the same distribution as $(U, V)$. The lemma is proved. \[ \square \]

Observe from (3.10) that

$$\frac{W'}{U} + \frac{1 - W'}{V} = 1 \quad \text{and} \quad \frac{W}{U'} + \frac{1 - W}{V'} = 1. \tag{3.15}$$

This is how the Doob transformed transition probabilities arise from a given forward transition $(W', 1 - W')$ or backward transition $(W, 1 - W)$. We derive the probability distribution of $W'/U$ (which is the same as that of $W/U'$). $2F_1$ below is the standard Gauss hypergeometric function

$$2F_1(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!} \tag{3.16}$$

where $(c)_k = c(c+1)\cdots(c+k-1)$ denotes the ascending factorial. Other examples of rational functions of beta variables whose densities involve hypergeometric functions appear in [8, 15].

PROPOSITION 3.2. The random variables $W'/U$ and $W/U'$ of Lemma 3.1 have the following density function $g_{\lambda}$ on the interval $(0, 1)$.

In case (a) under assumption (3.11),

$$g_{\lambda}(x) = \frac{B(\alpha + \lambda, \alpha + \beta)}{B(\alpha + \lambda, \beta)} \cdot \frac{x^\alpha(1 - x)^{\lambda - 1}}{B(\lambda, \alpha)} \cdot 2F_1(\alpha + \lambda, \alpha + \lambda, 2\alpha + \beta + \lambda; x). \tag{3.17}$$

In case (b) under assumption (3.12),

$$\tilde{g}_{\lambda}(x) = \frac{B(\beta + \lambda, \alpha + \beta)}{B(\beta + \lambda, \alpha)} \cdot \frac{x^{\lambda - 1}(1 - x)^{\beta - 1}}{B(\lambda, \beta)} \cdot 2F_1(\beta + \lambda, \beta + \lambda, \alpha + 2\beta + \lambda; 1 - x). \tag{3.18}$$

Neither $g_{\lambda}$ nor $\tilde{g}_{\lambda}$ is the density function of any beta distribution.
Proof. Consider case (a). Let $F_{V^{-1}}$ denote the Beta($\lambda, \alpha$) c.d.f. of $V^{-1}$. Fix $0 < x < 1$. From $W'/U = (1 - V^{-1})/(1 - U V^{-1})$,

$$g_\lambda(x) = \frac{d}{dx} \mathbb{P}\left( \frac{W'}{U} \leq x \right) = \frac{d}{dx} \mathbb{P}\left( V^{-1} \geq \frac{1 - x}{1 - xU} \right) = \frac{1}{B(\alpha + \lambda, \beta)} \int_0^1 \frac{\partial}{\partial x} \left( 1 - F_{V^{-1}} \left( \frac{1 - x}{1 - xu} \right) \right) u^{\alpha + \lambda - 1} (1 - u)^{\beta - 1} du$$

$$= \frac{x^{\alpha - 1} (1 - x)^{\lambda - 1}}{B(\alpha, \lambda)B(\alpha + \lambda, \beta)} \int_0^1 (1 - xu)^{-\alpha - \lambda} u^{\alpha + \lambda - 1} (1 - u)^{\alpha + \beta - 1} du. \tag{3.19}$$

The last integral equals $B(\alpha + \lambda, \alpha + \beta) F_1(\alpha + \lambda, \alpha + \lambda, 2\alpha + \beta + \lambda; x)$ (equation (9.09) on page 161 in [14]). This verifies (3.17).

In case (b) write $W'/U = 1 - (1 - U^{-1})/(1 - U^{-1}V)$ where the last fraction has the distribution found in case (a) but with $\alpha$ and $\beta$ interchanged. Hence we have (3.18).

We verify that $g_\lambda$ is not a beta density, by deriving a contradiction from the assumption that for some $\gamma, \delta > 0$,

$$g_\lambda(x) = \frac{1}{B(\gamma, \delta)} x^{\gamma - 1} (1 - x)^{\delta - 1} \quad \text{for } 0 < x < 1. \tag{3.20}$$

Set this equal to line (3.19) and let $x \to 0$. This forces $\gamma = \alpha$ and gives this identity:

$$1 = \frac{B(\gamma, \delta)}{B(\alpha, \lambda) B(\alpha + \lambda, \beta)} \int_0^1 u^{\alpha + \lambda - 1} (1 - u)^{\alpha + \beta - 1} du = \frac{B(\gamma, \delta)B(\alpha + \lambda, \alpha + \beta)}{B(\alpha, \lambda) B(\alpha + \lambda, \beta)}.$$ 

Using the identity above, the equality of (3.19) and (3.20) can be written as

$$(1 - x)^{\delta - \lambda} = \frac{1}{B(\alpha + \lambda, \alpha + \beta)} \int_0^1 (1 - x u)^{-\alpha - \lambda} u^{\alpha + \lambda - 1} (1 - u)^{\alpha + \beta - 1} du. \tag{3.21}$$

As $x \to 1$ the right-hand side converges to a limit that is strictly larger than 1 and possibly infinite. This forces $\lambda > \delta$ on the left. Expand both sides of (3.21) in series:

$$\sum_{k=0}^{\infty} \frac{(\lambda - \delta)_k}{k!} x^k = \frac{1}{B(\alpha + \lambda, \alpha + \beta)} \sum_{k=0}^{\infty} \frac{(\alpha + \lambda)_k}{k!} x^k \int_0^1 u^{\alpha + \lambda + k - 1} (1 - u)^{\alpha + \beta - 1} du$$

$$= \sum_{k=0}^{\infty} \frac{(\alpha + \lambda)_k (\alpha + \lambda)_k}{(2\alpha + \beta + \lambda)_k} \frac{(\lambda - \delta)_k}{k!} x^k.$$

The equality of the coefficients for $k = 1$ and $k = 2$ gives a contradiction.

\[ \Box \]

3.2. Harmonic functions on quadrants. Lemma 3.1 is applied to construct two processes: $(\omega^\lambda, \rho^\lambda)$ using case (a) of the lemma and $(\tilde{\omega}^\lambda, \tilde{\rho}^\lambda)$ using case (b). Parameters $(\alpha, \beta)$ are fixed while in both cases $0 < \lambda < \infty$. $\omega^\lambda$ and $\tilde{\omega}^\lambda$ are new i.i.d. Beta($\alpha, \beta$) environments. $\rho^\lambda$ and $\tilde{\rho}^\lambda$ are harmonic functions on $\mathbb{Z}^2_{+}$ that give rise to Doob transformed transition probabilities $\pi^\lambda$ and $\tilde{\pi}^\lambda$, respectively.

The need for two cases (a) and (b) arises from the two-to-one connection between parameters $\xi \in (\text{ri}\mathcal{U}) \setminus \{\xi^*\}$ and $0 < \lambda < \infty$, given in Lemma 2.6(a). Then to parametrize in terms
of $\xi$, let $\lambda(\xi)$ be given by Lemma 2.6(a) and define
\[
(\bar{\omega}^\xi, \bar{\rho}^\xi, \bar{\pi}^{\xi}) = \begin{cases} 
(\omega^{\lambda(\xi)}, \rho^{\lambda(\xi)}, \pi^{\lambda(\xi)}), & \xi_1 \in (\xi^*_1, 1) = (\frac{\alpha}{\alpha + \beta}, 1) \\
(\bar{\omega}^{\lambda(\xi)}, \bar{\rho}^{\lambda(\xi)}, \bar{\pi}^{\lambda(\xi)}), & \xi_1 \in (0, \xi^*_1) = (0, \frac{\beta}{\alpha + \beta}). 
\end{cases}
\]

This way we establish in Theorem 3.5 below that $\xi \in (r\mathcal{U})\setminus\{\xi^*\}$ is the limiting velocity of the Doob transformed RWRE with transition $\bar{\pi}^\xi$.

The law of large numbers velocity $\xi^* = (\frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta})$ does not arise from any transition $\pi^\lambda$ or $\bar{\pi}^\lambda$ for a finite $\lambda$. As stated in Lemma 2.6(a), $\xi^*$ corresponds to $\lambda = \infty$. In the proof of Lemma 3.1 above letting $\lambda \to \infty$ in (3.13) yields $U = V = 1$, which then gives also $U'' = V'' = 1$. We could define $\rho^\infty = \bar{\rho}^\infty = 1$ which corresponds to the constant harmonic function.

We now perform construction (3.6)–(3.9) of harmonic functions $\rho^\lambda$ and forward transition probabilities $\omega^\lambda$. The distributional properties of the construction will come from part (a) of Lemma 3.1. The given inputs of the construction are boundary variables and backward transition probabilities in the bulk. We create simultaneously infinitely many coupled systems indexed by the parameter $0 < \lambda < \infty$. Remark 3.4 below comments on the similar construction of $(\bar{\omega}^\lambda, \bar{\rho}^\lambda)$ based on case (b) of Lemma 3.1.

Let $\mathcal{P}$ denote the joint distribution of mutually independent random variables
\[
(\Delta(i,0), \Delta(0,j), \tilde{\omega}_{x,x-e_1} : i, j \in \mathbb{N}, x \in \mathbb{N}^2)
\]
with marginal distributions $\Delta(i,0), \Delta(0,j) \sim \text{Uniform}(0,1)$ and $\tilde{\omega}_{x,x-e_1} \sim \text{Beta}(\alpha, \beta)$. Set
\[
\tilde{\omega}_{x,x-e_2} = 1 - \tilde{\omega}_{x,x-e_1}.
\]

For fixed positive $a$ and $b$, let $F^{-1}(\cdot; a, b) : [0, 1] \to [0, 1]$ denote the inverse function of the Beta$(a, b)$ c.d.f. (1.8). For $0 < \lambda < \infty$ define coupled boundary variables on the coordinate axes:
\[
\rho^\lambda_{(i-1,0),(i,0)} = F^{-1}(\Delta(i,0); \alpha + \lambda, \beta) \quad \text{for } i \geq 1
\]
\[
\text{and} \quad \rho^\lambda_{(0,j-1),(0,j)} = \frac{1}{F^{-1}(\Delta(0,j); \lambda, \alpha)} \quad \text{for } j \geq 1.
\]
\[
\rho^\lambda_{(i-1,0),(i,0)} \quad \text{i.i.d. Beta}(\alpha + \lambda, \beta), \quad \{(\rho^\lambda_{(0,j-1),(0,j)})^{-1} : j \geq 1\} \quad \text{i.i.d. Beta}(\lambda, \alpha),
\]
and the two collections are independent of each other and of $\{\tilde{\omega}_{x,x-e_1} : x \in \mathbb{N}^2\}$.

For each $\lambda > 0$, apply equations (3.6)–(3.8) inductively to define random variables
\[
\{\rho^\lambda_{x,x+e_1}, \rho^\lambda_{x,x+e_2}, \omega^\lambda_{x,x+e_1} : x \in \mathbb{Z}_+^2\}
\]
indexed by the full quadrant. For $x \in \mathbb{Z}_+^2$ define additionally
\[
\omega^\lambda_{x,x+e_2} = 1 - \omega^\lambda_{x,x+e_1}.
\]
Conservation equations
\[
\rho^\lambda_{x,x+e_1} \rho^\lambda_{x+e_1,x+e_1+e_1+e_2} = \rho^\lambda_{x,x+e_2} \rho^\lambda_{x+e_2,x+e_1+e_2}
\]
are satisfied around all unit squares. Consequently we can extend the definition of $\rho^\lambda_{x,x+e_1}$ from directed nearest-neighbor edges to $\rho^\lambda_{x,y}$ for all $x, y \in \mathbb{Z}_+^2$ so that $\rho^\lambda_{x,x} = 1$ and
\[
\rho^\lambda_{x,y} \rho^\lambda_{y,z} = \rho^\lambda_{x,z} \quad \text{for all } x, y, z \in \mathbb{Z}_+^2.
\]
In the sequel we write $\rho^\lambda(x, y)$ for $\rho^\lambda_{x,y}$ when subscripts are not convenient.

A down-right lattice path $\{x_j\}_{j \in \mathbb{Z}}$ is by definition a nearest-neighbor path with increments $x_j - x_{j-1} \in \{e_1, -e_2\}$. Note that any bounded portion of a down-right path in $\mathbb{Z}_2^+$ can be obtained by finitely many corner flips starting from the path $x_j = (j^+, j^-)$ that lies on the coordinate axes. A single corner flip is the transformation of variables $\lambda$ that lies on the indicated order. Right: ratios $\rho^\lambda$ along the down-right path, transitions $\tilde{\omega}$ out of sites northeast of the path, and transitions $\omega^\lambda$ southwest of it are jointly independent.

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**Figure 3.2.** Illustration of the corner-flipping procedure. Left and center: To obtain the $\rho^\lambda$ values on the thick edges of the down-right path inside the quadrant start with the known values on the boundary edges and consecutively flip the corners of the squares, for example in the indicated order. Right: ratios $\rho^\lambda$ along the down-right path, transitions $\tilde{\omega}$ out of sites northeast of the path, and transitions $\omega^\lambda$ southwest of it are jointly independent.

**Proposition 3.3.** Let random variables (3.23) and (3.25) be given, and define the process (3.26) inductively through (3.6)–(3.8). Then for each $0 < \lambda < \infty$ we have the following distributional properties.

Random variables $\{\omega^\lambda_{x, x+e_1} : x \in \mathbb{Z}_2^+\}$ are i.i.d. Beta($\alpha, \beta$). For each $x \in \mathbb{Z}_2^+$ we have the marginal distributions

$$\rho^\lambda_{x, x+e_1} \sim \text{Beta}(\alpha + \lambda, \beta) \quad \text{and} \quad \frac{1}{\rho^\lambda_{x, x+e_2}} \sim \text{Beta}(\lambda, \alpha).$$

For any down-right path $\{x_j\}_{j \in \mathbb{Z}}$ in $\mathbb{Z}_2^+$, the following random variables are all mutually independent:

$$\{\rho^\lambda_{x_j, x_{j+1}} : j \in \mathbb{Z}\}, \quad \bigcup_{j \in \mathbb{Z}} \{\tilde{\omega}_{z, z-e_1} : z \geq x_j + (1, 1)\}, \quad \text{and} \quad \bigcup_{j \in \mathbb{Z}} \{\omega^\lambda_{x, x+e_1} : 0 \leq x \leq x_j - (1, 1)\}.$$

In particular, we have the translation invariance of the joint distribution: for any $a \in \mathbb{Z}_2^+$,

$$\begin{align*}
(\omega^\lambda_{x, x+e_1}, \rho^\lambda_{a, u,v}, \tilde{\omega}_{z, z-e_1})_{x,u,v \in \mathbb{Z}_2^+, z \in \mathbb{N}^2} & \overset{d}{=} (\omega^\lambda_{a+x, a+x+e_1}, \rho^\lambda_{a+u, a+v}, \tilde{\omega}_{a+z, a+z-e_1})_{x,u,v \in \mathbb{Z}_2^+, z \in \mathbb{N}^2}
\end{align*}$$

Translation invariance (3.30) is a consequence of the down-right path statement: with a new origin at $a$, the edge variables $\rho^\lambda_{a+(i-1)e_1, a+ie_1}$ for $i \in \mathbb{N}$ and $k \in \{1, 2\}$ and the bulk variables $\tilde{\omega}_{z, z-e_1}$ have the same joint distribution as the original ones given in (3.23) and (3.25).
Equations (3.15) give the identities
\begin{equation}
\frac{\omega^\lambda_{x,x+e_1}}{\rho^\lambda_{x,x+e_1}} + \frac{\omega^\lambda_{x,x+e_2}}{\rho^\lambda_{x,x+e_2}} = 1 \quad \text{for } x \in \mathbb{Z}_+^2
\end{equation}
and
\begin{equation}
\frac{\omega^\lambda_{x,x-e_1}}{\rho^\lambda_{x-e_1,x}} + \frac{\omega^\lambda_{x,x-e_2}}{\rho^\lambda_{x-e_2,x}} = 1 \quad \text{for } x \in \mathbb{N}_+^2.
\end{equation}

Consider the RWRE $P^\omega^\lambda$ that uses forward transitions $\omega^\lambda$. Combining (3.31) with (3.28) gives the following for any fixed $y \in \mathbb{Z}_+^2$:
\begin{equation}
\omega^\lambda_{x,x+e_1} \rho^\lambda_{x+e_1,y} + \omega^\lambda_{x,x+e_2} \rho^\lambda_{x+e_2,y} = \rho^\lambda_{x,y} \quad \text{for } x \in \mathbb{Z}_+^2.
\end{equation}
In other words, for any fixed $y$, $\rho^\lambda_{x,y}$ is a harmonic function of $x$ for transition probabilities $\omega^\lambda_{x,x+e_i}$ on $\mathbb{Z}_+^2$. In particular, for two points $u \leq y$ in $\mathbb{Z}_+^2$ we have
\begin{equation}
\rho^\lambda_{u,y} = E_u^{\omega^\lambda} [\rho^\lambda(X_{\tau^+_y}, y)].
\end{equation}
By (3.32) the same harmonic function $\rho^\lambda$ works for backward transitions $\tilde{\omega}$ and we have
\begin{equation}
\rho^\lambda_{u,y} = E_y^{\omega^\lambda} [\rho^\lambda(u, X_{\tau^+_y})].
\end{equation}

We perform a Doob transform on $P^\omega^\lambda$ by introducing transition probabilities
\begin{equation}
\pi^\lambda_{x,x+e_i} = \frac{\omega^\lambda_{x,x+e_i}}{\rho^\lambda_{x,x+e_i}}, \quad i \in \{1, 2\}.
\end{equation}
The RWRE that uses transitions $\pi^\lambda$ is the $\rho^\lambda$-tilted RWRE and its quenched path measure is denoted by $P^\pi^\lambda$. Let $x_{0,k} = (x_0, \ldots, x_k)$ be an up-right path from $x_0 = u$ that first enters the boundary $\mathbb{B}_y^-$ (recall definition (3.1)) at the endpoint $x_k$. Then
\begin{equation}
P_u^{\pi^\lambda} \{ X_{0,k} = x_0, k \} = \prod_{i=0}^{k-1} \frac{\omega^\lambda_{x_i,x_{i+1}}}{\rho^\lambda_{x_i,x_{i+1}}} = \frac{P_u^{\omega^\lambda} \{ X_{0,k} = x_0, k \}}{\rho^\lambda_{u,x_k}}
\end{equation}
\begin{equation}
= \frac{P_u^{\omega^\lambda} \{ X_{0,k} = x_0, k \} \rho^\lambda_{x_k,y}}{\rho^\lambda_{u,y}} = \frac{E_u^{\omega^\lambda} [\rho^\lambda(X_{\tau^-_y}, y), X_{0,k} = x_0, k]}{E_u^{\omega^\lambda} [\rho^\lambda(X_{\tau^-_y}, y)]}.
\end{equation}
A particular consequence that we use in subsequent sections is the following identity for the probability of hitting one of the two parts of the boundary. For fixed $u \leq y$ in $\mathbb{Z}_+^2$ and $i \in \{1, 2\}$, summing (3.37) over all paths that enter $\mathbb{B}_y^-$ at a point of $\mathbb{B}_y^{(-i)}$ gives
\begin{equation}
P_u^{\pi^\lambda} \{ X_{\tau^-_y} \in \mathbb{B}_y^{(-i)} \} = \frac{E_u^{\omega^\lambda} [\rho^\lambda(X_{\tau^-_y}, y), X_{\tau^-_y} \in \mathbb{B}_y^{(-i)}]}{E_u^{\omega^\lambda} [\rho^\lambda(X_{\tau^-_y}, y)]}.
\end{equation}

Equation (3.32) says that the same harmonic function $\rho^\lambda$ works for backward transitions $\tilde{\omega}$ as well. Hence we define also the backwards Doob transform
\begin{equation}
\tilde{\pi}^\lambda_{x,x-e_i} = \frac{\tilde{\omega}^\lambda_{x,x-e_i}}{\rho^\lambda_{x-e_i,x}}, \quad i \in \{1, 2\}.
\end{equation}
Then as above for fixed \( u \leq y \) in \( \mathbb{Z}^2_+ \) and a down-left path \( x_{0,k} \) started from \( y \) that first enters \( \mathbb{E}_u^+ \) at \( x_k \),

\[
(3.40) \quad P^\hat{\lambda}_y \{ X_{0,k} = x_k \} = \frac{P^\hat{\lambda}_y \{ X_{0,k} = x_{0,k} \} \rho^\lambda_{u,y}}{\rho^\lambda_{u,u}} = \frac{E^\hat{\lambda}_y [\rho^\lambda(u, X(\tau^+_u))] \odot E^{\hat{\lambda}}_y [\rho^\lambda(u, X(\tau^+_u))] }{\rho^\lambda_{u,u}}.
\]

**Remark 3.4.** Let us comment briefly on the version of the construction above that produces \((\hat{\omega}^\lambda, \hat{\rho}^\lambda)\) based on case (b) of Lemma 3.1. Instead of (3.25), begin with

\[
(3.41) \quad \hat{\rho}^\lambda_{(i-1,0),(i,0)} = \frac{1}{F^{-1}(\Delta_{(i,0)}; \lambda, \beta)} \quad \text{for } i \geq 1
\]

and
\[
\hat{\rho}^\lambda_{(0,j-1),(0,j)} = F^{-1}(\Delta_{(0,j)}; \beta + \lambda, \alpha) \quad \text{for } j \geq 1.
\]

Equations (3.6)–(3.9) are iterated exactly as before. Proposition (3.3) is valid word for word for \((\hat{\omega}^\lambda, \hat{\rho}^\lambda, \hat{\omega})\), except that (3.29) is replaced with

\[
(3.42) \quad \frac{1}{\hat{\rho}^\lambda_{x,x+e_1}} \sim \text{Beta}(\lambda, \beta) \quad \text{and} \quad \hat{\rho}^\lambda_{x,x+e_2} \sim \text{Beta}(\beta + \lambda, \alpha).
\]

The Doob-transformed transitions are defined again by

\[
(3.43) \quad \hat{\pi}^\lambda_{x,x+e_i} = \frac{\hat{\omega}^\lambda_{x,x+e_i}}{\hat{\rho}^\lambda_{x,x+e_i}}, \quad i \in \{1, 2\},
\]

with quenched path measure \( P^\hat{\pi}^\lambda \). Equations (3.34) and (3.37) are valid for \((\hat{\omega}^\lambda, \hat{\rho}^\lambda, \hat{\pi}^\lambda)\). \( \triangle \)

Now let \( \lambda(\xi) \) be given by Lemma 2.6(a) for \( \xi = (\xi_1, 1 - \xi_1) \in (\text{ri}\mathcal{U})\setminus\{\xi^*\} \). Combine the two constructions \((\omega^\lambda, \rho^\lambda, \pi^\lambda)\) and \((\hat{\omega}^\lambda, \hat{\rho}^\lambda, \hat{\pi}^\lambda)\) by defining \((\omega^\xi, \rho^\xi, \pi^\xi)\) by (3.22) for all \( \xi \in (\text{ri}\mathcal{U})\setminus\{\xi^*\} \). The quenched path measure of the RWRE that uses transition \( \pi^\xi \) is given by

\[
(3.44) \quad P^\pi^\xi_x = \begin{cases} P^\pi^\xi_x \quad : \xi_1 \in (\xi_1^*, 1) \allowbreak \\
\hat{P}^\pi^\xi_x \quad : \xi_1 \in (0, \xi_1^*). \end{cases}
\]

**Theorem 3.5.** We have this almost sure law of large numbers: for all \( \xi \in (\text{ri}\mathcal{U})\setminus\{\xi^*\} \),

\[
P^\pi^\xi_0 \{ n^{-1} X_n \to \xi \} = 1 \quad \mathbb{P}\text{-almost surely.}
\]

**Proof.** We give the details for the case \( \xi_1 \in (\xi_1^*, 1) = \left( \frac{\alpha}{\alpha + \beta}, 1 \right) \) with \( \lambda = \lambda(\xi) \). By translation invariance (Proposition 3.3) we can extend \( \log \rho^\lambda \) to a process \( \{ \log \rho^\lambda_{x,y} : x, y \in \mathbb{Z}^2 \} \) indexed by the entire lattice. This process has the shift-invariance and additivity properties of (2.4), in other words it is a stationary \( L^1 \) cocycle. Such processes satisfy a uniform ergodic theorem under certain regularity assumptions, as for example given in Theorem A.3 in the Appendix of [11]. Variable \( \log \rho^\lambda_{0,e_1} \) is integrable and (3.31) gives the lower bound \( \log \rho^\lambda_{x,x+e_1} \geq \log \omega^\lambda_{x,x+e_1} \) in terms of an i.i.d. process with strictly more than two moments. This is sufficient for Theorem A.3 of [11] which gives the almost sure limit

\[
(3.45) \quad \lim_{n \to \infty} n^{-1} \max_{|x| \leq n} \left| \log \rho^\lambda_{0,x} - m(\lambda) \cdot x \right| = 0,
\]

with mean vector

\[
m(\lambda) = \mathbb{E}[\log \rho^\lambda_{0,e_1}] e_1 + \mathbb{E}[\log \rho^\lambda_{0,e_2}] e_2
\]

\[
= (\psi_0(\alpha + \lambda) - \psi_0(\alpha + \beta + \lambda)) e_1 + (\psi_0(\alpha + \lambda) - \psi_0(\lambda)) e_2.
\]
Proposition 3.3 says that under $\P$ transitions $\omega^\lambda$ have the same distribution as $\omega$ does under $\P$. Then, by (1.5) and (3.45) we have $\P$-almost surely
\[
\lim_{n \to \infty} n^{-1} \log P_{0}^\lambda \{ X_n = [n\zeta] \} = \lim_{n \to \infty} (n^{-1} \log P_{0}^\lambda \{ X_n = [n\zeta] \} - n^{-1} \log \rho^\lambda_0 \{ n\zeta \} )
\]
\[
= -I_\zeta(\zeta) + \zeta_1 \psi_0(\alpha + \beta + \lambda) + \zeta_2 \psi_0(\lambda) - \psi_0(\alpha + \lambda).
\]
In other words, the distribution of $X_n/n$ under $P_{0}^\lambda$ satisfies a (quenched) large deviation principle with rate function
\[
I_\zeta^\lambda(\zeta) = I_\zeta(\zeta) - \zeta_1 \psi_0(\alpha + \beta + \lambda) - \zeta_2 \psi_0(\lambda) + \psi_0(\alpha + \lambda).
\]
By the strict convexity of $I_\zeta$ and its expression (2.13), $I_\zeta^\lambda(\zeta)$ has a unique zero at $\zeta = \xi$ with $\xi_1$ given by the right-hand side of (2.10). This proves Theorem 3.5.

As the last point, let us record monotonicity and continuity that are valid for the boundary variables by definition (3.25) and then extended by the construction. For $x,y \in \mathbb{Z}_1^2$
\[
(\gamma > \lambda > 0 \implies \rho^\gamma_{x,x+e_1} > \rho^\lambda_{x,x+e_1} \quad \text{and} \quad \rho^\gamma_{x,x+e_2} < \rho^\lambda_{x,x+e_2}
\]
and
\[
(\rho^\gamma_{x,y} \xrightarrow{\gamma \to \lambda} \rho^\lambda_{x,y} \quad \text{and} \quad \omega^\gamma_x \xrightarrow{\gamma \to \lambda} \omega^\lambda_x).
\]
The limits in (3.47) are valid as stated also for $(\tilde{\omega}, \tilde{\rho})$, but the monotonicity is reversed:
\[
(\gamma > \lambda > 0 \implies \tilde{\rho}^\gamma_{x,x+e_1} < \tilde{\rho}^\lambda_{x,x+e_1} \quad \text{and} \quad \tilde{\rho}^\gamma_{x,x+e_2} > \tilde{\rho}^\lambda_{x,x+e_2}
\]

3.3. Global harmonic functions. In this section we construct the process $B^\xi_{x,y}$ discussed in the results of Section 2.1. To have a single point of reference, we summarize the construction and its properties in the next Theorem 3.6, then derive the claims made in Section 2.1, and after that proceed to prove Theorem 3.6 piece by piece.

The probability space $(\Omega, \mathcal{G}, \P)$ in the theorem is the product space $\Omega = [0,1]^2 \times \omega$ of beta environments $\omega = (\omega_{x,x+e_1} : x \in \mathbb{Z}^2)$ where the variables $\omega_{x,x+e_1}$ are i.i.d. Beta$(\alpha, \beta)$-distributed. Shift mappings $T_z$ act by $(T_z\omega)_{x,x+e_1} = \omega_{x+z,x+z+e_1}$ for $x,z \in \mathbb{Z}^2$. Velocities $\xi \in ri\mathcal{U}$ are denoted by $\xi = (\xi_1, \xi_2) = (\xi_1, 1 - \xi_1)$, and the distinguished velocity is $\xi^* = (\frac{\alpha}{\alpha+\beta}, \frac{\beta}{\alpha+\beta})$. A down-right path $\{x_i\} \subset \mathbb{Z}^2$ in part (c) below satisfies $x_{i+1} - x_i \in \{e_1, -e_2\}$. Note that the increment distributions in part (a.1) below are those of case (a) of Lemma 3.1, while part (a.2) corresponds to case (b) of Lemma 3.1.

**Theorem 3.6.** Fix $0 < \alpha, \beta < \infty$. On the probability space $(\Omega, \mathcal{G}, \P)$ there exists a stochastic process $\{B^\xi_{x,y}(\omega) : x, y \in \mathbb{Z}^2, \xi \in ri\mathcal{U}\}$ with the following properties.

I. Distribution and expectations.

(a) For $\xi = \xi^*$ the process $B^\xi_{x,y}$ is identically zero. For $\xi \in (ri\mathcal{U}) \backslash \{\xi^*\}$, the marginal distributions and expectations are as follows, with $\lambda(\xi)$ given by (2.10)–(2.11).

(a.1) For $\xi_1 \in (\xi^*_1, 1)$,
\[
e^{B^\xi_{x,x+e_1}} \sim \text{Beta}(\alpha + \lambda(\xi), \beta) \quad \text{and} \quad e^{-B^\xi_{x,x+e_2}} \sim \text{Beta}(\lambda(\xi), \alpha),
\]
and so
\[ \mathbb{E}[B^\xi_{x,x+e_1}] = \psi_0(\alpha + \lambda(\xi)) - \psi_0(\alpha + \beta + \lambda(\xi)) \quad \text{and} \]
\[ \mathbb{E}[B^\xi_{x,x+e_2}] = \psi_0(\alpha + \lambda(\xi)) - \psi_0(\lambda(\xi)). \]
(3.49)

(a.2) For \( \xi_1 \in (0, \xi^*_i) \),
\[ e^{-B^\xi_{x,x+e_1}} \sim \text{Beta}(\lambda(\xi), \beta) \quad \text{and} \quad e^{B^\xi_{x,x+e_2}} \sim \text{Beta}(\beta + \lambda(\xi), \alpha), \]
and so
\[ \mathbb{E}[B^\xi_{x,x+e_1}] = \psi_0(\beta + \lambda(\xi)) - \psi_0(\lambda(\xi)) \quad \text{and} \]
\[ \mathbb{E}[B^\xi_{x,x+e_2}] = \psi_0(\beta + \lambda(\xi)) - \psi_0(\alpha + \beta + \lambda(\xi)). \]
(3.50)

(b) For any \( z \in \mathbb{Z}^2 \), the variables \( \{B^\xi_{x,y}(\omega) : x, y \leq z, \xi \in \ri U\} \) are independent of the variables \( \{\omega_{x+e_1} : x \leq z\} \).

(c) For a fixed \( \xi \in (\ri U) \setminus \{\xi^*_i\} \), the joint distribution of \( (\omega, B^\xi) \) is the same as that of \( (\omega^\xi, \log \rho^\xi) \) defined in (3.22). This distribution is described in Proposition 3.3 and Remark 3.4. In particular, on any down-right path \( \{x_i\}_{i \in \mathbb{Z}} \) on \( \mathbb{Z}^2 \) the variables \( \{B^\xi_{x_i, x_{i+1}}\}_{i \in \mathbb{Z}} \) are independent.

(d) The quenched large deviation rate function of (1.5) satisfies
\[ I_q(\xi) = - \inf_{\xi \in \ri U} \left\{ \mathbb{E}[B^\xi_{0,e_1}]\xi_1 + \mathbb{E}[B^\xi_{0,e_2}]\xi_2 \right\} \quad \text{for all} \ \xi \in U. \]
The infimum is uniquely attained at \( \zeta = \xi \).

II. Pointwise properties. There exists an event \( \Omega_0 \subset \Omega \) such that \( \mathbb{P}(\Omega_0) = 1 \) and the following statements hold for each \( \omega \in \Omega_0 \), \( \xi, \zeta \in \ri U \), and \( x, y, z \in \mathbb{Z}^2 \).

(e) Cocycle properties: stationarity
\[ B^\xi_{x+z,y+z}(\omega) = B^\xi_{x,y}(T_2 \omega) \]
and additivity
\[ B^\xi_{x,y}(\omega) + B^\xi_{y,z}(\omega) = B^\xi_{x,z}(\omega). \]
(3.51)
(3.52)
In particular, \( B^\xi_{x,x}(\omega) = 0 \) and \( B^\xi_{y,x}(\omega) = -B^\xi_{y,x}(\omega). \)

(f) Harmonic increments:
\[ \omega_{x,x+e_1} e^{-B^\xi_{x,x+e_1}(\omega)} + \omega_{x,x+e_2} e^{-B^\xi_{x,x+e_2}(\omega)} = 1. \]
(3.53)

(g) Monotonicity: If \( \xi \cdot e_1 < \zeta \cdot e_1 \) then
\[ B^\xi_{x,x+e_1} \geq B^\xi_{x,x+e_1} \quad \text{and} \quad B^\xi_{x,x+e_2} \leq B^\xi_{x,x+e_2}. \]
(h) \( B^\xi_{x,y}(\omega) \) is a cadlag function of \( \xi_1 \in (0, 1) \).

III. Limits. For each fixed \( \xi \in \ri U \) there exists an event \( \Omega_0^{(\xi)} \subset \Omega \) that can vary with \( \xi \), has \( \mathbb{P}(\Omega_0^{(\xi)}) = 1 \), and is such that the following statements hold for each \( \omega \in \Omega_0^{(\xi)} \) and \( x, y \in \mathbb{Z}^2 \).

(i) For any sequence \( \xi^n \in \ri U \) such that \( \xi^n \to \xi \), we have \( \lim_{n \to \infty} B^\xi_{x,y}(\omega) = B^\xi_{x,y}(\omega) \).
(j) For any sequence \( z_N \in \mathbb{Z}^2 \) with \( |z_N|_1 \to \infty \) and \( z_N/N \to \xi \) we have the limit
\[ B^\xi_{x,y}(\omega) = \lim_{N \to \infty} \left( \log P^\omega_x\{X|_{z_N-x|=z_N} = z_N\} - \log P^\omega_y\{X|_{z_N-y|=z_N} = z_N\} \right). \]
A few comments about the theorem. The translation-invariant process \((\tilde{\omega}^\xi, \log \tilde{\rho}^\xi)\) referred to in part (c) was constructed in (3.22) in Section 3.2 only on the quadrant \(\mathbb{Z}_+^2\). In order for part (c) above to make full sense, \((\tilde{\omega}^\xi, \log \tilde{\rho}^\xi)\) must be extended from \(\mathbb{Z}_+^2\) to the full lattice \(\mathbb{Z}^2\) by Kolmogorov’s extension theorem. It is also important to distinguish when \(\xi\) is fixed and when it can vary. The distributional equality of \(B^\xi\) and \(\log \tilde{\rho}^\xi\) is not valid jointly across different \(\xi\) because the joint distribution of \(\{B^\xi\}\) is not the one constructed in Section 3.2 through a coupling with uniform random variables. Note also the distinction between \((h)\) and \((i)\): at a fixed \(\xi\) there is continuity almost surely, but globally over \(\xi\) the path is cadlag.

We prove the results of Section 2.1. As given in (2.5), the transformed transition probability is defined by \(\kappa_{x,x+e_1}^\xi(\omega) = \omega_{x,x+e_1}e^{-B_{x,x+e_1}^\xi(\omega)}\).

**Proof of Theorem 2.1.** Theorem 2.1 is a subset of Theorem 3.6. \(\square\)

**Proof of Theorem 2.2.** We can express \(\kappa_{x,x+e_1}^\xi\) in terms of the increments \((B_{x,x+e_1}^\xi, B_{x,x+e_2}^\xi)\):

(3.55) \[
\kappa_{x,x+e_1}^\xi = \frac{e^{B_{x,x+e_2}^\xi} - 1}{e^{B_{x,x+e_2}^\xi} - e^{B_{x,x+e_1}^\xi}}.
\]

To prove the formula, substitute in limits (3.54) and use the Markov property. Note also that this is the analogue of \(W'/U = (V - 1)/(V - U)\) from (3.10).

Given \(n \in \mathbb{Z}\), define the down-right path \(\{x^j\}\) by

\[
x^{2k} = (n + k, -k) \quad \text{and} \quad x^{2k+1} = (n + k + 1, -k) \quad \text{for} \quad k \in \mathbb{Z}.
\]

The antidiagonal \(\{x : x_1 + x_2 = n\}\) is the subsequence \(\{x^{2k}\}\), and \((B_{x^{2k},x^{2k}+e_1}^\xi, B_{x^{2k},x^{2k}+e_2}^\xi) = (B_{x^{2k},x^{2k+1}}, -B_{x^{2k-1},x^{2k}}^\xi)\). These pairs are i.i.d. by part (c) of Theorem 3.6.

For \(\xi \in \mathcal{U}\{\xi^*\}\) the law of large numbers part (2.6) of Theorem 2.2 follows from Theorem 3.5 and the observation that \((\omega, B^\xi)\) has the same distribution as \((\tilde{\omega}^\xi, \log \tilde{\rho}^\xi)\), as stated in part (c) of Theorem 3.6.

For \(\xi = \xi^*, P_0^{\kappa_{0,e_1}^\xi}(\omega) = P_0^\omega\), the original path measure in an i.i.d. environment, and the LLN is the one in (1.2). \(\square\)

**Proof of Theorem 2.3.** Immediate from the limits (3.54). \(\square\)

**Theorem 3.7.** For \(\xi \in (\mathbb{R}\mathcal{U})(\{\xi^*\}\), random variable \(\kappa_{0,e_1}^\xi\) is not beta distributed. Let \(\lambda(\xi)\) be given by (2.10)–(2.11) and let \(g_\lambda\) and \(\tilde{g}_\lambda\) be the functions defined in (3.17)–(3.18). Then the density function \(f^\xi(x)\) of \(\kappa_{0,e_1}^\xi\) for \(0 < x < 1\) is given by

(3.56) \[
f^\xi(x) = \begin{cases} g_{\lambda(\xi)}(x), & \xi_1 \in (\xi_1^*, 1) \\ \tilde{g}_{\lambda(\xi)}(x), & \xi_1 \in (0, \xi_1^*). \end{cases}
\]

**Proof.** This comes from Proposition 3.2. Formula (3.55), the independence of \(B_{0,e_1}^\xi\) and \(B_{0,e_2}^\xi\), and their distributions given in part (a) of Theorem 3.6 imply that \(\kappa_{0,e_1}^\xi\) has exactly the distribution of \(W'/U\) in Proposition 3.2. \(\square\)
We turn to prove Theorem 3.6. In addition to the probability space \((\Omega, \mathcal{G}, \mathbb{P})\) with its beta environment \(\omega\), we use the coupled processes \(\{\xi^\omega_{x,x+e_1}, \tilde{\rho}^\omega_{x,y} : x, y \in \mathbb{Z}^2\}\) under distribution \(\mathbb{F}\), constructed in Section 3.2 with properties given in Proposition 3.3 and in the subsequent discussion. The key point is that each environment \(\xi^\omega\) has the same i.i.d. Beta(\(\alpha, \beta\)) distribution as the original environment \(\omega\). The construction of \(B_{x,y}^\omega\) is based on the limits (3.54). These limits are proved by bounding ratios of hitting probabilities with the ratio variables \(\tilde{\rho}^\omega\) from (3.22) whose distributions we control.

We begin with two lemmas that do not use the beta distributions. The setting for Lemmas 3.8 and 3.9 is the following: \(a \in \mathbb{Z}^2\) and on the quadrant \(\mathbb{S} = a + \mathbb{Z}^2_+\) we have a Markov transition probability \(p\) such that

\[
0 < p_{x,x+e_1} = 1 - p_{x,x+e_2} < 1
\]

for all \(x \in \mathbb{S}\). Let \(P_x\) with expectation \(E_x\) denote the Markov chain with transition \(p\) starting at \(x \in \mathbb{S}\). Use the standard notation for hitting probabilities:

\[
F(x, y) = P_x(\exists n \geq 0 : X_n = y).
\]

**Lemma 3.8.** The following inequalities hold for all \(y \in \mathbb{S} = a + \mathbb{Z}^2_+\):

\[
\frac{F(a + e_1, y + e_2)}{F(a, y + e_2)} \leq \frac{F(a + e_1, y)}{F(a, y)} \leq \frac{F(a + e_1, y + e_1)}{F(a, y + e_1)}.
\]  

The first two numerators can vanish but the denominators are all positive. The same inequalities hold with \(e_1\) and \(e_2\) switched around.

**Proof.** The second statement follows by applying (3.58) to the transition probability \(\tilde{p}\) obtained by reflecting \(p\) across the diagonal passing through \(a\): for \(x = (x_1, x_2) \in \mathbb{Z}^2_+\) set \(\tilde{P}_{a+x,a+x+e_1} = P_{a+x,a+x+e_1}\), where \(x = (x_2, x_1)\).

We prove claim (3.58) by induction on \(y\). It is convenient to use the ratios

\[
U_{x,y} = \frac{F(x, y)}{F(x, y - e_1)} \quad \text{and} \quad V_{x,y} = \frac{F(x, y)}{F(x, y - e_2)} \quad \text{for} \ x \leq y \ \text{in} \ \mathbb{S}.
\]

The numerator does not vanish but the denominator can vanish and then the ratio has value \(\infty\).

(3.58) holds trivially for \(y = a + \ell e_2\) with \(\ell \geq 0\) because the first two numerators vanish while the other probabilities are positive. Hence we may assume \(y \geq a + e_1\). By a shift of \(y\) (3.58) is equivalent to having

\[
U_{a,y} \leq U_{a+e_1,y} \quad \text{for} \ y \geq a + 2e_1 \quad \text{and} \quad V_{a,y} \geq V_{a+e_1,y} \quad \text{for} \ y \geq a + e_1 + e_2.
\]

We check the boundaries first. For \(y = a + ke_1\) for \(k \geq 2\), \(U_{a,y} = U_{a+e_1,y} = p_{y-e_1,y}\). For \(y = a + e_1 + \ell e_2\) for \(\ell \geq 1\),

\[
V_{a,y} = \frac{F(a, y - e_2)p_{y-e_2,y} + F(a, y - e_1)p_{y-e_1,y}}{F(a, y - e_2)} > p_{y-e_2,y} = V_{a+e_1,y}.
\]

It remains the check (3.59) for \(y = a + ke_1 + \ell e_2\) for \(k \geq 2\) and \(\ell \geq 1\). For \(y \geq x + e_1 + e_2\) the Markov property and assumption (3.57) give

\[
F(x, y) = F(x, y - e_1)p_{y-e_1,y} + F(x, y - e_2)p_{y-e_2,y}
\]
from which we derive the identities
\[ U_{x,y} = p_{y-e_1,y} + p_{y-e_2,y} U_{x,y-e_2} \]
and
\[ V_{x,y} = p_{y-e_1,y} V_{x,y-e_1} + p_{y-e_2,y} \]
also for \( y \geq x + e_1 + e_2 \).

Now proceed by induction on \( y \geq a + 2e_1 + e_2 \), beginning with \( y = a + 2e_1 + e_2 \), and then taking \( e_1 \) and \( e_2 \) steps. The boundary cases checked above together with the induction assumption give \( U_{a,y-e_2} \leq U_{a+e_1,y-e_2} \) and \( V_{a,y-e_1} \geq V_{a+e_1,y-e_1} \). Then the identities above give \( U_{a,y} \leq U_{a+e_1,y} \) and \( V_{a,y} \geq V_{a+e_1,y} \). \( \square \)

**Lemma 3.9.** Let \( v \geq a \) on \( \mathbb{Z}_+^2 \) and set \( y = v + e_1 + e_2 \). Suppose \( f(x) > 0 \) for \( x \) on the boundary \( B_y \). We have the following inequalities.

For \( a + e_1 \leq v \):

\[
\frac{E_a[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(2)}]}{E_{a+e_1}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(2)}]} \leq \frac{F(a, v)}{F(a + e_1, v)} \leq \frac{E_a[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(1)}]}{E_{a+e_1}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(1)}]}.
\]

For \( a + e_2 \leq v \):

\[
\frac{E_a[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(1)}]}{E_{a+e_2}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(1)}]} \leq \frac{F(a, v)}{F(a + e_2, v)} \leq \frac{E_a[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(2)}]}{E_{a+e_2}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(2)}]}.
\]

**Proof.** For the proof, fix \( v \) and do induction on \( |v - a|_1 \geq 1 \). Consider the case \( a = v - ke_1 \) for \( k \geq 1 \). Then

\[
\frac{E_a[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(2)}]}{E_{a+e_1}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(2)}]} = p_{a,a+e_1} = \frac{F(a, v)}{F(a + e_1, v)}.
\]

On the other hand, when the walk is required to hit \( \mathbb{B}_y^{(1)} \), both steps \( e_1 \) and \( e_2 \) are feasible from \( a \), and so

\[
E_a[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(1)}] \geq p_{a,a+e_1} E_{a+e_1}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(1)}].
\]

This establishes \( 3.60 \). \( 3.61 \) for \( a = v - ke_2 \) for \( k \geq 1 \) follows in a symmetric manner. In particular, we have the full conclusion for \( |v - a|_1 = 1 \).

Suppose \( 3.60 \)–\( 3.61 \) hold for all pairs \( a \leq v \) with \( |v - a|_1 = \ell \). Consider \( a \leq v \) with \( |v - a|_1 = \ell + 1 \). We already know the result when \( a \in \{ v - (\ell + 1)e_1, v - (\ell + 1)e_2 \} \). Thus we may assume that \( a < v \) coordinatewise. For \( i \in \{1, 2\} \) take the identity

\[
p_{a,a+e_1} E_{a+e_1}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(i)}] + p_{a,a+e_2} E_{a+e_2}[f(X_{y-y}), X_{y-y} \in \mathbb{B}_y^{(i)}] = 1
\]
and rearrange it to yield the two identities

\[
\frac{E_\alpha[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]}{E_{\alpha+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]} = p_{\alpha,a+e_1}
\]

(3.62)

\[
+ p_{\alpha,a+e_2}\frac{E_{\alpha+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]}{E_{\alpha+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]} \left( \frac{E_{\alpha+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]}{E_{\alpha+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]} \right)^{-1}
\]

and

\[
\frac{E_\alpha[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]}{E_{\alpha+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]} = p_{\alpha,a+e_2}
\]

(3.63)

\[
+ p_{\alpha,a+e_1}\frac{E_{\alpha+e_1}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]}{E_{\alpha+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]} \left( \frac{E_{\alpha+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]}{E_{\alpha+e_1+e_2}[f(X_{\tau_y^-}), X_{\tau_y^-} \in B_y^{(\cdot)}]} \right)^{-1}
\]

Derive the analogous equations for the ratios of hitting probabilities from the identity

\[
p_{\alpha,a+e_1}\frac{F(a, v)}{F(a,v)} + p_{\alpha,a+e_2}\frac{F(a + e_2, v)}{F(a, v)} = 1.
\]

Apply the induction assumption on the right-hand sides of (3.62) and (3.63) and their counterparts for the ratios of hitting probabilities. This verifies (3.60) and (3.61) for \( u \). □

The remainder of the proof relies on special properties of the beta environment. The next proposition gives control over limits of hitting probability ratios through the harmonic functions constructed for Proposition 3.3. Hitting probabilities under environment \( \omega \) are denoted by

\[
F^\omega(x, y) = F^\omega_{x,y} = P^\omega_x(\exists n \geq 0 : X_n = y).
\]

When \( x \leq y \) this is of course \( F^\omega_{x,y} = P^\omega_x(X_{\lfloor y-x\rfloor} = y) \) which we also use occasionally when the notation is not too heavy.

**Proposition 3.10.** Fix \( \xi \in (r\mathcal{U})\setminus\{\xi^*\} \). If \( \eta, \zeta \in r\mathcal{U} \) are such that

\[
\eta_1 < \xi_1 < \zeta_1
\]

then for all \( x \in \mathbb{Z}^2_+ \) and \( z \in \mathbb{Z}^2 \) we have almost surely

\[
\lim_{N \to \infty} \frac{F^\xi_{x,[N\zeta]+z}}{F^\zeta_{x+e_1,[N\zeta]+z}} = \hat{\rho}^\xi_{x,x+e_1} \leq \lim_{N \to \infty} \frac{F^\zeta_{x,[N\eta]+z}}{F^\eta_{x+e_1,[N\eta]+z}}
\]

and

\[
\lim_{N \to \infty} \frac{F^\xi_{x,[N\zeta]+z}}{F^\zeta_{x+e_2,[N\zeta]+z}} = \hat{\rho}^\xi_{x,x+e_2} \geq \lim_{N \to \infty} \frac{F^\zeta_{x,[N\eta]+z}}{F^\eta_{x+e_2,[N\eta]+z}}.
\]

Recall from (3.22) that these inequalities split into two separate results: one for \((\omega^{\lambda(\zeta)}, \rho^{\lambda(\zeta)})\) when \( \xi_1 \in (\xi^*_1, 1) \), and the other for \((\omega^{\lambda(\xi)}, \hat{\rho}^{\lambda(\xi)})\) when \( \xi_1 \in (0, \xi^*_1) \).
Proof. The inequalities claimed are all proved the same way. We illustrate with the first one. Let \( y_N = [N\zeta] + z + \epsilon_1 + \epsilon_2 \). By (3.60), then (3.38) and (3.34),

\[
\frac{F_{x,[N\zeta]+z} \leq E_x \left[ \rho(X_{\tau_N}, y_N, X_{\tau_N} \in B_{y_N}^{[-1]} \right] \right] \\
= \frac{P_x \left\{ X_{\tau_N} \in B_{y_N}^{[-1]} \right\} E_x \left[ \rho(X_{\tau_N}, y_N) \right] E_{x+e_1} \left[ \rho(X_{\tau_N}, y_N) \right] = \frac{P_x \left\{ X_{\tau_N} \in B_{y_N}^{[-1]} \right\} \rho_{x,e_1} \left( X_{\tau_N}, y_N \right) \right].
\]

The probabilities in the last expression converge to one by the law of large numbers of Theorem 3.5 because by (3.65) the \( \xi \)-ray passes \( \zeta \) on the left. \( \square \)

**Corollary 3.11.** Fix \( \xi \in \text{ri}\, \mathcal{U} \). Let \( \omega \) be an i.i.d. Beta(\( \alpha, \beta \)) environment. Then almost surely, for all \( z \in \mathbb{Z}^2 \), the limits

\[
\lim_{N \to \infty} \frac{F_0, [N\zeta]+z}{F_{e_1, [N\zeta]+z}} \quad \text{and} \quad \lim_{N \to \infty} \frac{F_{e_2, [N\zeta]+z}}{F_{0, [N\zeta]+z}}
\]

exist and are independent of \( z \).

(a) When \( \xi = \xi^* \), the limits equal 1.
(b) For \( \xi \neq \xi^* \), let \( \lambda = \lambda(\xi) \) be determined by Lemma 2.6(a).

(b.i) If \( \xi_1 \in (\xi^*, 1) \) the two limits in (3.66) are, respectively, Beta(\( \alpha+\lambda, \beta \)) and Beta(\( \lambda, \alpha \)) distributed.

(b.ii) If \( \xi_1 \in (0, \xi^*) \) the reciprocals of the two limits in (3.66) are, respectively, Beta(\( \lambda, \beta \)) and Beta(\( \beta+\lambda, \alpha \)) distributed.

**Proof.** Consider the case \( \xi_1 \in (\xi^*, 1) \) and the first limit of (3.66). Let \( \gamma < \lambda < \delta \). By Lemma 2.6(a) the velocities associated with these parameters in the range \( (\xi^*, 1) \) satisfy \( \xi_1(\beta) > \xi_1(\lambda) > \xi_1(\delta) \). Hence by Proposition 3.10

\[
\lim_{N \to \infty} \frac{F_0, [N\zeta]+z}{F_{e_1, [N\zeta]+z}} \quad \text{and} \quad \lim_{N \to \infty} \frac{F_{e_2, [N\zeta]+z}}{F_{0, [N\zeta]+z}} \leq \rho_{0,e_1}^\gamma.
\]

Since \( \rho^\gamma, \rho^\delta \to \rho^\lambda \) as \( \gamma, \delta \to \lambda \) by (3.47), and since \( \rho_{0,e_1}^\lambda \) is Beta(\( \alpha+\lambda, \beta \))-distributed, we have that

\[
\lim_{N \to \infty} \frac{F_0, [N\zeta]+z}{F_{e_1, [N\zeta]+z}} \quad \text{and} \quad \lim_{N \to \infty} \frac{F_{0, [N\zeta]+z}}{F_{e_1, [N\zeta]+z}}
\]

are both Beta(\( \alpha+\lambda, \beta \)) random variables. Since \( \lim \leq \lim \), their equality in distribution implies their \( \mathbb{P} \)-almost sure equality. Same reasoning works for the second limit of (3.66). This proves the existence of the limits and claim (b.i), for any fixed \( z \).

To argue that the limit with \( z = 0 \) equals the limit with an arbitrary \( z = (z_1, z_2) \), pick integers \( k_N \) so that \( k_N \xi_2 = [N\zeta]_2 = z_2 \). Then, depending on the relative locations of \( [k_N\xi]_1 \) and \( [N\zeta]_1 + z_1 \), by (3.58) either the inequality

\[
\frac{F_0, [k_N\zeta]}{F_{e_1, [k_N\zeta]+z}} \geq \frac{F_{e_1, [N\zeta]+z}}{F_{e_1, [N\zeta]+z}}
\]
or its opposite is valid for infinitely many $N$. In the limit we get again an almost sure inequality between two Beta($\alpha + \lambda, \beta$) random variables, which therefore must coincide almost surely.

For $\xi_1 \in (0, \xi^*_p)$ one can repeat the same steps but use instead the processes $(\tilde{\omega}, \tilde{\rho})$ that follow part (b) of Lemma 3.1.

For the case $\xi = \xi^*$, pick $\eta, \zeta \in \text{ri} \mathcal{U}$ such that $\eta_1 < \xi^*_1 < \zeta_1$. By (3.58),

$$
\frac{F^{\omega}_{0, [N\xi] + z}}{F^{\omega}_{e_1, [N\xi] + z}} \leq \frac{F^{\omega}_{0, [N\xi^*] + z}}{F^{\omega}_{e_1, [N\xi^*] + z}} \leq \frac{F^{\omega}_{0, [N\eta] + z}}{F^{\omega}_{e_1, [N\eta] + z}}.
$$

By the cases already proved, the left and right ratios converge to random variables with distributions Beta($\alpha + \lambda(\zeta), \beta$) and Beta($\lambda(\eta), \beta$)$^{-1}$, respectively. These random variables converge to 1 as we let $\zeta, \eta \to \xi^*$ which sends $\lambda(\zeta), \lambda(\eta) \to \infty$ (see Lemma 2.6(a) and the middle plot of Fig. 2.1). The second ratio in (3.66) for $\xi = \xi^*$ is handled similarly.

We can now organize all the derived properties to establish Theorem 3.6.

*Proof of Theorem 3.6.* We begin by constructing the process for a fixed $\xi \in \text{ri} \mathcal{U}$, then do it simultaneously for a dense countable subset of $\text{ri} \mathcal{U}$, and finally capture all of $\text{ri} \mathcal{U}$ with limits.

Fix $\xi \in \text{ri} \mathcal{U}$. Using Corollary 3.11 and shifts $P^{\omega}_{x+a}(X_n = v) = P^{T_x \omega}_{a}(X_n = v-x)$, we define

$$
B^e_{x,y} (\omega) = \lim_{N \to \infty} \{ \log P^{\omega}_{x}(X_{|z_N-x|_1} = z_N) - \log P^{\omega}_{y}(X_{|z_N-y|_1} = z_N) \} \quad (3.69)
$$

as an almost sure limit, for all $x, y \in \mathbb{Z}^2$, and for any sequence $z_N = [N\xi] + z$ with an arbitrary fixed $z$. (The second line is the same as the first, stated to illustrate the alternative notations we use.) The limit is independent of the choice of $z$. The marginal distributional properties (a), stationary cocycle properties (e), and harmonicity (f) stated in Theorem 3.6 follow from Corollary 3.11 and the structure of the limits.

*Proof of part (b).* The independence of the weights $\{\omega_x\}$ and construction (3.69) imply directly the first independence claim of part (b).

*Proof of part (c) for fixed $\xi$.* We write the details for the case $\xi_1 \in (\xi^*_p, 1)$. Consider the joint distribution of $m$ weights $\omega_{xh}$ for $1 \leq h \leq m$ and $k + \ell$ nearest-neighbor increments $B^e_{x_i, x_{i+1}}$ and $B^e_{y_j, y_{j+1}}$ for $1 \leq i \leq k$ and $1 \leq j \leq \ell$. By a shift, we may assume that $z_h, x_i, y_j$ all lie in $\mathbb{Z}_+^2$. Let $\gamma < \lambda(\xi) < \delta$ as in the proof of Corollary 3.11. Limit (3.69) works also in environments $\omega^\gamma$ and $\omega^\delta$ since they have the same i.i.d. beta distribution as $\omega$. Let $r_h, s_i, t_j \in \mathbb{R}$. Then inequalities (3.67) and their counterparts for $e_2$ give us these bounds:

$$
P\{\omega_{xh} \leq r_h, e^B_{x_i, x_{i+1}} \leq s_i, e^B_{y_j, y_{j+1}} \geq t_j \ \forall \ h, i, j \}$$

$$= P\{\omega_{xh}^\gamma \leq r_h, e^B_{x_i, x_{i+1}}(\omega^\gamma) \leq s_i, e^B_{y_j, y_{j+1}}(\omega^\gamma) \geq t_j \ \forall \ h, i, j \}$$

$$\leq P\{\omega_{xh}^\gamma \leq r_h, \rho_{x_i, x_{i+1}}^\gamma \leq s_i, \rho_{y_j, y_{j+1}}^\gamma \geq t_j \ \forall \ h, i, j \}$$

and similar.

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and from the other side
\[ P\{\omega_{z_h} \leq r_h, e^{B_{x,i+1}^\xi} \leq s_i, e^{B_{y,j+1}^\delta} \geq t_j \ \forall \ h, i, j\} \]
\[ = P\{\omega_{z_h} \leq r_h, e^{B_{x,i+1}^\gamma} (< \omega^\delta) \leq s_i, e^{B_{y,j+1}^\delta} \geq t_j \ \forall \ h, i, j\} \]
\[ \geq P\{\omega_{z_h} \leq r_h, \rho_{x,i+1}^{\delta} \leq s_i, \rho_{y,j+1}^{\delta} \geq t_j \ \forall \ h, i, j\}. \]

Letting \( \gamma, \delta \to \lambda(\xi) \) brings the bounds together by (3.47):
\[ P\{\omega_{z_h} \leq r_h, e^{B_{x,i+1}^\xi} \leq s_i, e^{B_{y,j+1}^\delta} \geq t_j \ \forall \ h, i, j\} \]
\[ = P\{\omega_{z_h} \leq r_h, \rho_{x,i+1}^{\lambda(\xi)} \leq s_i, \rho_{y,j+1}^{\lambda(\xi)} \geq t_j \ \forall \ h, i, j\}. \]

Thus the joint distribution of \( (\omega, B^\xi) \) is the same as that of \( (\omega^{\lambda(\xi)}, \log \rho^{\lambda(\xi)}) \) described in Proposition 3.3. The independence of nearest-neighbor \( B^\xi \)-increments along a down-right path follows.

Let \( \Omega_0 \) be the event of full \( P \)-probability on which the process \( B^\xi_{x,y} \) is defined by (3.69) for all \( \xi \) in the countable set \( \mathcal{U}_0 = (\mathfrak{riU}) \cap \mathbb{Q}^2 \).

Consider \( \xi \in \mathfrak{riU} \) and \( \zeta, \eta \in \mathcal{U}_0 \) with \( \eta_1 < \xi_1 < \zeta_1 \). Take a (possibly random) sequence \( z_N \) with \( z_N/N \to \xi \). Let \( M = |z_N| \). For large enough \( N \) we have \( [M\eta] \cdot e_1 < z_N \cdot e_1 < [M\zeta] \cdot e_1 \) and \( [M\eta] \cdot e_2 > z_N \cdot e_2 > [M\zeta] \cdot e_2 \). By Lemma 3.8 we have for such \( N \)
\[ \frac{F^\omega_{x,[M\zeta]}}{F^\omega_{x+e_1,[M\zeta]}} \leq \frac{F^\omega_{x,z_N}}{F^\omega_{x+e_1,z_N}} \leq \frac{F^\omega_{x,[M\eta]}}{F^\omega_{x+e_1,[M\eta]}}. \]

The already established limit (3.69) with sequences \( [M\zeta] \) and \( [M\eta] \) gives
\[ B^\xi_{x,x+e_1} \leq \lim_{N \to \infty} \left\{ \log P^\omega_x(X_{|z_N-x|_1} = z_N) - \log P^\omega_{x+e_1}(X_{|z_N-x|_1-1} = z_N) \right\} \]
\[ \leq \lim_{N \to \infty} \left\{ \log P^\omega_x(X_{|z_N-x|_1} = z_N) - \log P^\omega_{x+e_1}(X_{|z_N-x|_1-1} = z_N) \right\} \leq B^\eta_{x,x+e_1}. \]

The reverse inequalities hold when \( e_1 \) is replaced by \( e_2 \).

(3.71) proves that the monotonicity in part (g) holds for \( \xi, \zeta \in \mathcal{U}_0, \omega \in \Omega_0, \) and \( x \in \mathbb{Z}^2 \). Consequently, for any \( \xi \in (\mathfrak{riU}) \setminus \mathcal{U}_0 \) we can define \( B^\xi_{x,x+e_1}(\omega) \) for \( \omega \in \Omega_0 \) by the monotone limit
\[ B^\xi_{x,x+e_1}(\omega) = \lim_{\mathcal{U}_0 \ni \zeta \to \xi, \zeta_1 > \xi_1} B^\zeta_{x,x+e_1}(\omega) \]
as \( \zeta_1 \) decreases to \( \xi_1 \). By shrinking \( \Omega_0 \) we can assume that (3.72) holds also when \( \xi \in \mathcal{U}_0 \). (This is because the monotonicity gives an inequality in (3.72), but the two sides agree in distribution and hence agree almost surely.) By additivity on the right-hand side we can extend (3.72) to define
\[ B^\xi_{x,y}(\omega) = \lim_{\mathcal{U}_0 \ni \zeta \to \xi, \zeta_1 > \xi_1} B^\zeta_{x,y}(\omega) \quad \text{for all } x, y \in \mathbb{Z}^2 \text{ and } \omega \in \Omega_0. \]

This definition in terms of right limits extends the properties proved thus far to all \( \xi \). Furthermore, cadlag paths (part (h)) have also been established now.

Fix \( \xi \in \mathfrak{riU} \) and \( i \in \{1, 2\} \). The almost sure continuity of \( \zeta \to B^\xi_{0,e_i} \) at \( \zeta = \xi \) follows from monotonicity (g) and from the continuity of \( \zeta \to E[B^\xi_{0,e_i}] \), which itself is a consequence of
continuity of the polygamma functions in (3.49) and (3.50). Claim (i) follows then from the cocycle property in part (e).

Continue with a fixed $\xi \in \text{ri} \mathcal{U}$. Let $\zeta, \eta \to \xi$ in (3.71) and use the almost sure continuity we just proved. This shows that limit (3.54) holds $\mathbb{P}$-almost surely, simultaneously for all $x \in \mathbb{Z}^2$, $y \in \{x + e_1, x + e_2\}$, and any sequence $z_N$ with $z_N/N \to \xi$. The case of a general $y \in \mathbb{Z}^2$ follows from additivity. Part (j) is done.

We turn to part (d). When $\xi_1 \in [\xi_1^*, 1]$, the variational formula for $I_q$ comes from (3.49) and the explicit calculations in (7.7) in Section 7. To minimize the formula take the derivative of

$$
(\psi_0(\alpha + \lambda(\zeta)) - \psi_0(\alpha + \beta + \lambda(\zeta)))\xi_1 + (\psi_0(\alpha + \lambda(\zeta)) - \psi_0(\lambda(\zeta)))\xi_2
$$

in $\xi_1$ and set it to 0. This gives the equation

$$
\xi_1 = \frac{\psi_1(\lambda(\zeta)) - \psi_1(\alpha + \lambda(\zeta))}{\psi_1(\lambda(\zeta)) - \psi_1(\alpha + \beta + \lambda(\zeta))}
$$

which by Lemma 2.6(a) has a unique solution at $\zeta = \xi$. The case $\xi_1 \in [0, \xi_1^*]$ works similarly. □

4. STATIONARY BETA POLYMER

By looking at the random walk paths under the Doob-transformed RWRE in reverse direction, we can view this model as a stationary directed polymer model, called the beta polymer. We establish this connection in the present section, and then use it in the next two sections to rely on recently published estimates in [5] for the technical work behind our Theorems 2.4 and 2.5. The polymer model described here is case (1.4) on p. 4 of [5], with their parameter triple $(\mu, \beta, \theta)$ corresponding to our $(\alpha, \beta, \lambda)$. The notation $Z_{m,n}$ and $Q_{m,n}$ used below matches the notation of [5].

Recall the backward transition probabilities $\tilde{\omega}$, introduced in (3.23) and (3.24), and random variables $(\rho^\lambda, \omega^\lambda)$ from (3.26). The quenched stationary beta polymer is a polymer distribution on up-right paths on the nonnegative first quadrant $\mathbb{Z}^2_+$ that start at the origin. In our notation this model uses potential $V(x - e_j, e_j) = \log \tilde{\omega}_{x,x-e_j}$ across edges $(x - e_j, x)$ for $x \in \mathbb{N}$, and potential $V(x - e_j, e_j) = \log \rho^\lambda_{x-e_j,x}$ across boundary edges $(x - e_j, x)$ for $x \in \mathbb{B}_0^{(+)} \setminus \{0\}$, $j \in \{1, 2\}$. Fix a point $v = (m, n) \in \mathbb{N}^2$. The point-to-point partition function for paths from 0 to $v$ is

$$
Z_{m,n} = \sum_{y_0, m+n} e^{\sum_{i=0}^{m+n-1} V(y_i, y_{i+1} - y_i)},
$$

where the sum is over up-right paths $y_{0,m+n} = (y_0, \ldots, y_{m+n})$ from 0 to $v = (m, n)$.

If $x_i = y_{m+n-i}$ denotes the reversed path and $\ell = \min\{i : x_i \in \mathbb{B}_0^+\}$ is the time of its first entry into the boundary $\mathbb{B}_0^+$, then

$$
e^{\sum_{i=0}^{m+n-1} V(y_i, y_{i+1} - y_i)} = \rho^\lambda_{0, x_{\ell}} \prod_{i=0}^{\ell-1} \tilde{\omega}_{x_i, x_{i+1}} = \rho^\lambda_{0, x_{\ell}} P^\omega_v \{X_{0,\ell} = x_{0,\ell}\} = \rho^\lambda_{0, x_{\ell}} P^\omega_v \{X_{0,\ell} = x_{0,\ell}\}.
$$

Summing up over the paths gives the first equality below, and the second comes from (3.35):

$$
Z_{m,n} = E_v^\omega [\rho^\lambda(0, X_{\ell}^+)] = \rho^\lambda_{0,v} \quad \text{for } v = (m, n).
$$

(4.1)
The quenched polymer distribution on up-right paths $y_{0,m+n} = (y_0, \ldots, y_{m+n})$ from 0 to $v = (m,n)$ is

$$Q_{m,n}(y_{0,n}) = \frac{e^{\sum_{i=0}^{m+n-1} V(y_i, y_{i+1} - x_i)}}{Z_{m,n}}.$$ 

Letting again $x_i = y_{m+n-i}$ and $\ell = \min\{i : x_i \in \mathbb{B}_0^+\}$ and using (3.40),

$$Q_{m,n}(y_{0,n}) = \frac{\rho_{0,x_{\ell}}^{\lambda} \phi_v \{X_{0,\ell} = x_{0,\ell}\}}{\rho_{0,v}^{\lambda}} = P_{v}^{\tilde{\pi}^{0}} \{X_{0,\ell} = x_{0,\ell}\} \quad \text{for } v = (m,n).$$

Thus (the reverse of) the polymer path under $Q_{m,n}$ is obtained by running the Doob-transformed RWRE under $P_{v}^{\pi^{0}}$, until it hits the boundary $\mathbb{B}_0^+$, and then following the boundary to the origin.

5. THE VARIANCE OF THE INCREMENT-STATIONARY HARMONIC FUNCTIONS

The method for bounding the fluctuations of the walk for Theorem 2.5 is to control the exit point of the walk from rectangles. This is achieved with the help of the harmonic functions $\rho^{\lambda}$ and $\tilde{\rho}^{\lambda}$ constructed in Section 3.2. We work exclusively with $\rho^{\lambda}$ and omit the analogous statements and proofs for $\tilde{\rho}^{\lambda}$. Equivalently, we are treating explicitly only the case $\xi_1 \in (\xi_1^*, 1)$ and omitting the details for $\xi_1 \in (0, \xi_1^*)$.

This section gives the connection between the fluctuations of $\log \rho^{\lambda}$ and the entry point on the boundary, and bounds on the variance of $\log \rho^{\lambda}$. Theorem 2.4 is proved at the end of the section.

Recall the beta integral $B(a,b)$ and the c.d.f. $F(s; a, b)$ of the Beta$(a, b)$ distribution from (1.7) and (1.8). Define

$$\tilde{L}(s, a, b) = \frac{1}{s} \cdot \frac{\frac{\partial}{\partial s} F(s; a, b)}{\frac{\partial}{\partial s} F(s; a, b)}.$$ 

Note that $\frac{\partial}{\partial a} B(a, b) = (\psi_0(a) - \psi_0(a + b)) B(a, b)$. A computation then gives

$$\tilde{L}(s, a, b) = -s^{-a}(1-s)^{-b-1} \int_0^s t^{-a-1}(1-t)^{-b-1} \left[ \log t - (\psi_0(a) - \psi_0(a + b)) \right] dt.$$ 

Observe that

$$\tilde{L}(s, a, b) = s^{-a}(1-s)^{-b-1} B(a, b) \text{Cov}(-\log W, 1\{W \leq s\}),$$

where $W \sim \text{Beta}(a, b)$. Since $-\log t$ and $1\{t \leq s\}$ are decreasing functions of $t$ we see that $\tilde{L}(s, a, b) > 0$ for all $s \in (0, 1)$ and $a, b > 0$. Let $\tilde{L}(s, \lambda) = \tilde{L}(s, \alpha + \lambda, \beta)$.

Recall hitting times $\tau_v^-$ and $\tau_v^+$ defined in (3.2). Let $\nabla \text{Var}$ denote the variance under the coupling $\mathbb{P}$ of Section 3.2. Let $\mathbb{P}_v^{\pi^{0}}(\cdot) = \mathbb{E} P_{v}^{\pi^{0}}(\cdot)$ denote the averaged measure of the RWRE that utilizes the backward Doob-transformed transition probability $\pi^{0}$ of (3.39). Its expectation operation is $\mathbb{E}_v^{\pi^{0}}[\cdot] = \mathbb{E} E_v^{\pi^{0}}[\cdot]$. 

Theorem 5.1. The following holds for all $\alpha, \beta, \lambda > 0$ and all $v = (m, n) \in \mathbb{N}^2$:
\[
\mathbb{V} \text{ar}(\log \rho_{0,v}^\lambda) = n(\psi_1(\lambda) - \psi_1(\alpha + \lambda)) - m(\psi_1(\alpha + \lambda) - \psi_1(\alpha + \beta + \lambda))
\]
(5.2)
\[
+ 2\widehat{\mathbb{E}}_v^\lambda \left[ \sum_{i=0}^{X(\tau_0^\lambda) - e_1 - 1} L(\rho_{i e_1, (i+1)e_1}^\lambda, \lambda) \right]
\]
\[
= m(\psi_1(\alpha + \lambda) - \psi_1(\alpha + \beta + \lambda)) - n(\psi_1(\lambda) - \psi_1(\alpha + \lambda))
\]
(5.3)
\[
+ 2\widehat{\mathbb{E}}_v^\lambda \left[ \sum_{i=0}^{X(\tau_0^\lambda) - e_2 - 1} \tilde{L}(1/\rho_{i e_2, (i+1)e_2}^\lambda, \alpha, \lambda) \right].
\]

An empty sum (e.g. $\sum_{i=0}^{-1}$) equals 0. Thus, the $\widehat{\mathbb{E}}_v^\lambda$ expectation on the right-hand side of (5.2) is in fact over the event $\{X(\tau_0^\lambda) \in \mathbb{B}_0^{(+1)}\}$. When $v$ is chosen (approximately) in the direction $\xi(\lambda)$ so that the first two terms on the right-hand side of (5.2) (approximately) cancel, the equation expresses the KPZ relation that in $1 + 1$ dimension the wandering exponent is twice the free energy exponent.

Theorem 5.1 is the same as Proposition 1.1 in [5], via the connections (4.1) and (4.2) between the RWRE and the polymer. Theorem 5.1 is also proved in Section 4.1 of the first preprint version [3] of this paper.

Starting from the identity in Theorem 5.1, a series of coupling arguments and estimates leads to upper and lower bounds on the fluctuations of $\log \rho^\lambda$. Theorem 5.2 below follows directly from Theorem 1.2 of [5]. It is the connections (4.1) and (4.2) between the RWRE and the polymer. Theorem 5.1 is also proved in Section 4.1 of the first preprint version [3] of this paper. Here, $\xi(\lambda)$ is given by (2.10).

Theorem 5.2. Fix $\alpha, \beta > 0$. Fix $\lambda > 0$. Given a constant $0 < \gamma < \infty$, there exist positive finite constants $c, C$, and $N_0$, depending only on $\alpha, \beta, \gamma$, and $\lambda$, such that
\[
c N^{2/3} \leq \mathbb{V} \text{ar}[\rho_{0,me_1+ne_2}^\lambda] \leq CN^{2/3}
\]
(5.4)
for all $N \geq N_0$ and $(m, n) \in \mathbb{N}^2$ such that
\[
|m - N\xi_1(\lambda)| \lor |n - N\xi_2(\lambda)| \leq \gamma N^{2/3}.
\]
The same constants $c, C$, and $N_0$ can be taken for $(\alpha, \beta, \gamma, \lambda)$ varying in a compact subset of $(0, \infty)^4$.

Proof of Theorem 2.4. By virtue of Theorem 3.6(c), Theorem 5.2 implies Theorem 2.4 for the case $\xi_1 \in (\xi_1^*, 1)$. The remaining case $\xi_1 \in (0, \xi_1^*)$ follows from the (omitted) version of Theorem 5.2 for $\widehat{\rho}^\lambda$. \hfill $\square$

6. Path fluctuations

In this section we prove results about path fluctuations, from which Theorem 2.5 will follow. For an up-right path $X_{0,x}$ started at the origin and an integer $n \geq 0$ let
\[
X_n^{1,\text{min}} = \min\{m \geq 0 : X_{m+n} \cdot e_2 = n\} \quad \text{and} \quad X_n^{1,\text{max}} = \max\{m \geq 0 : X_{m+n} \cdot e_2 = n\}.
\]
Then $X_n^{1,\text{min}} e_1 + ne_2$ and $X_n^{1,\text{max}} e_1 + ne_2$ are, respectively, the leftmost and rightmost points of the path on the horizontal line $ne_2 + \mathbb{Z}_+ e_1$. See the left panel in Figure 6.1. The vertical
countercparts are given by
\[ X_{m}^{2,\min} = \min\{n \geq 0 : X_{m+n} \cdot e_1 = m\} \quad \text{and} \quad X_{m}^{2,\max} = \max\{n \geq 0 : X_{m+n} \cdot e_1 = m\}. \]

Figure 6.1. In both plots the diagonal line points in direction \( \xi(\lambda) \). Left: the definition of \( X_{1}^{\min} \) and \( X_{1}^{\max} \). Right: illustration of (6.1) and (6.4). The four arms of the cross centered at \((m, n) = [N\xi]\) are of length \( rN^{2/3}\) each. The shaded box, also centered at \((m, n)\), has sides of length \( 2\delta N^{2/3}\). For large \( r \), the path has a high probability of entering and exiting through the cross and never touching the dotted lines. For small \( \delta \), there is a positive probability, uniformly in \( N \), that the path stays left of the top edge of the shaded box, completely avoiding the box.

Again, the next result is stated and proved for the case \( \xi_1 \in (\xi_1^*, 1) \) only. The other case works similarly. Recall \( \xi(\lambda) \) from (2.10). Let \( P^\lambda_0 = E P_0^{\pi^\lambda} \), with expectation \( E^\lambda_0 = E E_0^{\pi^\lambda} \). By Theorem 3.5, \( \xi(\lambda) \) is the LLN direction for \( P^\lambda_0 \).

**Theorem 6.1.** Fix \( \alpha, \beta, \lambda > 0 \).

(a) Upper bound. There exist finite positive constants \( r_0 \) and \( C \) depending on \( \alpha, \beta, \) and \( \lambda \), such that for all \( r \geq r_0 \), integer \( N \geq 1 \), and \( (m, n) = [N\xi(\lambda)] \), we have
\[ P^\lambda_0\{X_{n}^{1,\min} < m - rN^{2/3}\} \leq Cr^{-3} \quad \text{and} \quad P^\lambda_0\{X_{n}^{1,\max} > m + rN^{2/3}\} \leq Cr^{-3}. \]  
From this it follows that
\[ E^\lambda_0[(m - X_{n}^{1,\min})^+ | \mathcal{P}]^{1/p} \leq \left(1 + \frac{Cp}{3-p}\right)^{1/p} N^{2/3} \quad \text{and} \]
\[ E^\lambda_0[\|X_{n}^{1,\max} - m\|^p]^{1/p} \leq \left(1 + \frac{Cp}{3-p}\right)^{1/p} N^{2/3}. \]

(b) Lower bound. There exist finite positive constants \( \delta \) and \( c \) depending on \( \alpha, \beta, \) and \( \lambda \), such that for any integer \( N \geq 1 \) such that \( (m, n) = [N\xi(\lambda)] \in \mathbb{N}^2 \) we have
\[ E^\lambda_0[(m - X_{n}^{1,\min})^+] \geq cN^{2/3} \quad \text{and} \]
\[ P^\lambda_0\{X_{n}^{1,\min} \leq X_{n}^{1,\max} < m - \delta N^{2/3}\} \geq c. \]

Similar bounds hold for the vertical counterparts \( X_{m}^{2,\min} \) and \( X_{m}^{2,\max} \). The same constants can be taken for all \( (\alpha, \beta, \lambda) \) varying in a compact subset of \((0, \infty)^3\).
Proof. Abbreviate \( u = (m, n) = \lfloor N\xi(\lambda) \rfloor \). Inequality (6.1) is trivial if \( rN^{1/3} \geq m \). We hence assume that \( rN^{2/3} < m \).

Note that
\[
(m - X_n^{1,\text{min}})^+ = m - X(\tau_{(m, n)}^-) \cdot e_1.
\]

Thus, the first probability in (6.1) equals
\[
\mathbf{P}_0\{ X(\tau_u^-) \cdot e_1 < m - rN^{2/3} \} = \mathbf{P}_u^\lambda\{ X(\tau_u^+) \cdot e_1 > rN^{2/3} \}.
\]

Applying Lemma 4.7 of [5] and the connection (4.2) gives
\[
\mathbf{P}_u^\lambda\{ X(\tau_0^+) \cdot e_1 > rN^{2/3} \} \leq C r^{-3}.
\]

(This is also (4.24) in [3].) This proves the first inequality in (6.1).

To prove the second inequality set \( N_0 = \lfloor \frac{m + rN^{2/3}}{\xi(\lambda)} \rfloor \) and \((m_0, n_0) = \lfloor N_0\xi(\lambda) \rfloor \). Then \( m_0 < m + rN^{2/3} \) and therefore if \( X_n^{1,\text{max}} > m + rN^{2/3} \), then \( X_{m_0}^{2,\text{min}} < n_0 \). But we also have
\[
n_0 > N_0\xi_2(\lambda) - 1 \geq \frac{m\xi_2(\lambda)}{\xi_1(\lambda)} + \frac{\xi_2(\lambda)}{\xi_1(\lambda)} rN^{2/3} - 1 - \xi_2(\lambda)
\geq n + \frac{\xi_2(\lambda)}{\xi_1(\lambda)} rN^{2/3} - 1 - \xi_2(\lambda) - \frac{\xi_2(\lambda)}{\xi_1(\lambda)} = n + \frac{\xi_2(\lambda)}{\xi_1(\lambda)} rN^{2/3},
\]

provided \( r \geq 2(1 + \xi_1(\lambda_1) + \frac{\xi_2(\lambda_1)}{\xi_2(\lambda_2)}) \). The upshot is that if \( X_n^{1,\text{max}} > m + rN^{2/3} \) then \( X_{m_0}^{2,\text{min}} < n_0 - \frac{\xi_2(\lambda_1)}{2\xi_1(\lambda_2)} rN_0^{2/3} \). The second inequality in (6.1) thus follows from the vertical version of the first inequality, but with \( N_0 \) and \( r_0 = \frac{\xi_2(\lambda_1)}{2\xi_1(\lambda_2)} r \) playing the roles of \( N \) and \( r \), respectively.

Bounds (6.2) follow from (6.1). For example, for the first bound abbreviate \( Y = (m - X_n^{1,\text{min}})^+ \) and write
\[
\mathbf{E}_0^\lambda[(N^{-2/3}Y)^p] = \int_0^\infty p r^{p-1} \mathbf{P}_0^\lambda(Y > rN^{2/3}) \, dr
\leq \int_0^1 p r^{p-1} \, dr + C \int_1^\infty p r^{p-4} \, dp = 1 + \frac{Cp}{3 - p}.
\]

Next, applying Lemma 4.2 of [5] we have
\[
\mathbf{E}_u^\lambda\left[ \sum_{i=0}^{X(\tau_0^+) \cdot e_1 - 1} L(\rho_{i+1}^\lambda, e_1, 1) \right] \leq C (\mathbf{E}_u^\lambda[X(\tau_0^+) \cdot e_1] + 1).
\]

(This is also (4.15) in [3].) Now, bound (6.3) follows from stringing together (6.7), (5.2), and the lower bound in (5.4), then reversing the picture in (6.5). To get (6.4) first write
\[
cN^{2/3} \leq \mathbf{E}_0^\lambda[Y] = \mathbf{E}_0^\lambda[Y \mathbb{1}\{Y \leq \delta N^{2/3}\}] + \mathbf{E}_0^\lambda[Y \mathbb{1}\{Y > \delta N^{2/3}\}]
\leq \delta N^{2/3} + \mathbf{E}_0^\lambda[Y^2]^{1/2} \mathbf{P}_0(Y > \delta N^{2/3})^{1/2}.
\]
Applying (6.2) with say $p = 2$ and taking $\delta \leq c/2$ we get

$$P_0^\lambda \{ X_{n, \min}^1 < m - \delta N^{2/3} \} \geq \frac{c}{2\sqrt{1 + 2C}}. \tag{6.8}$$

Now take $\delta_0 > 2\delta$, $N_0 = N + [\delta N^{2/3}]$, and $(m_0, n_0) = [N_0 \xi(\lambda)]$. Note that $m_0 \leq [N \xi_1(\lambda)] + \delta N^{2/3} = m + \delta N^{2/3}$. This forces

$$m_0 - \delta_0 N_0^{2/3} \leq m + \delta N^{2/3} - 2\delta N^{2/3} = m - \delta N^{2/3}.$$

Since $n \leq n_0$ we have that if $X_{n_0, \min}^1 < m_0 - \delta_0 N_0^{2/3}$, then

$$X_n^{1, \max} \leq X_{n_0, \min}^1 < m_0 - \delta_0 N_0^{2/3} \leq m - \delta N^{2/3}.$$

Bound (6.4) follows from the above and (6.8) with $N_0$ and $\delta_0$ playing the roles of $N$ and $\delta$, respectively.

Proof of Theorem 2.5. We only argue for the case $\xi_1 \in (\xi^*_1, 1)$, the other case being similar.

By Theorem 3.6(c) the distribution of $P_0^\xi$ under $\mathbb{P}$ is the same as that of $P_0^\lambda$ under $\mathbb{P}$, provided $\lambda$ and $\xi$ are put in duality via (2.10). Hence, $P_0^\xi = P_0^\lambda$. Now, the claims of the theorem follow from (6.1) and (6.4). See Figure 6.1. \hfill \Box

7. Proofs of the large deviation results

Proof of Lemma 2.6. Define the function

$$f(\lambda) = \frac{\psi_1(\lambda) - \psi_1(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)}.$$

We prove that $f$ is strictly decreasing in $\lambda > 0$. Its derivative is

$$f'(\lambda) = \frac{(\psi_2(\lambda) - \psi_2(\alpha + \lambda))}{(\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda))} - \frac{(\psi_1(\lambda) - \psi_1(\alpha + \lambda))(\psi_2(\lambda) - \psi_2(\alpha + \beta + \lambda))}{(\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda))^2}.$$

Since $\psi_1$ is strictly decreasing, $f'(\lambda) < 0$ is equivalent to

$$\frac{\psi_2(\lambda) - \psi_2(\alpha + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \lambda)} < \frac{\psi_2(\lambda) - \psi_2(\alpha + \beta + \lambda)}{\psi_1(\lambda) - \psi_1(\alpha + \beta + \lambda)} \tag{7.1}.$$

This in turn follows from $\psi_2 \circ \psi_1^{-1}$ being strictly concave, which is proved in Lemma A.3.

We have so far shown that $f$ is strictly decreasing. Since $\psi_1(\lambda) \to \infty$ as $\lambda \searrow 0$ we have $f(\lambda) \to 1$ as $\lambda \searrow 0$. Similarly, by Lemma A.2 we have $\lambda^2(\psi_1(\lambda) - \psi_1(\alpha + \lambda)) \to a$ as $\lambda \to \infty$ and thus $f(\lambda) \to \frac{a}{\alpha + \beta}$ as $\lambda \to \infty$. The claims in part (a) for $\xi_1 \in [\xi^*_1, 1]$ now follow. The case $\xi_1 \in [0, \xi^*_1]$ comes by interchanging the roles of $\alpha$ and $\beta$ and those of $\xi_1$ and $\xi_2$.

Define the function

$$g(\lambda) = \psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda).$$

Since $\psi_1$ is strictly decreasing we see that

$$g'(\lambda) = \psi_1(\alpha + \beta + \lambda) - \psi_1(\lambda) < 0.$$
Hence, $g$ is strictly decreasing. As $\lambda \downarrow 0$ we have $\psi_0(\lambda) \to -\infty$ and $g(\lambda) \to \infty$. Representation (A.3) gives

$$
g(\lambda) = -\frac{1}{\alpha + \beta + \lambda} + \frac{1}{\lambda} - \sum_{k=1}^{\infty} \left( \frac{1}{\alpha + \beta + \lambda + k} - \frac{1}{\lambda + k} \right).
$$

Then we see that as $\lambda \to \infty$, $g(\lambda) \to 0$. Part (b) follows and Lemma 2.6 is proved. \qed

**Proof of Theorems 2.7 and 2.10.** We utilize the stationary ratios $\rho^\lambda$ and transitions $\omega^\lambda$ from Section 3.2. By Proposition 3.3, $\omega^\lambda$ under $\overline{\mathbb{P}}$ (defined on page 17) has the same distribution as the original environment $\omega$ under $\mathbb{P}$.

By the ergodic theorem

$$
n^{-1} \log \rho^\lambda_{0,ne_2} = n^{-1} \sum_{i=0}^{n-1} \log \rho^\lambda_{i+(i+1)e_2} \xrightarrow{n \to \infty} \mathbb{E}[\log \rho^\lambda_{0,e_2}] = \psi_0(\alpha + \lambda) - \psi_0(\lambda).
$$

(Recall that the logarithm of a Gamma($\nu$, 1) has expected value $\psi_0(\nu)$ and that a Beta($a, b$) is a ratio of a Gamma($a, 1$) and a Gamma($a + b, 1$). By Proposition 3.3, $\rho^{-1}_{i+(i+1)e_2}$ are i.i.d. Beta($\lambda, \alpha$).

On the other hand, harmonicity of $\rho_{x,ne_2}$ implies, similarly to (3.34), that for any $x \in \mathbb{Z}^2_+$ with $|x|_1 \leq n$

$$
\rho^\lambda_{x,ne_2} = \sum_{j=0}^{n-1} P^\omega_j \{X_n - |x|_1 = je_1 + (n - j)e_2\} \rho^\lambda_{je_1 + (n-j)e_2,ne_2}.
$$

The sum above has nonzero terms exactly when $j$ is between $x \cdot e_1$ and $n - x \cdot e_2$. Abbreviate

$$
R_{i,n} = \rho^\lambda_{(i+1)e_1 + (n-i-1)e_2,ie_1 + (n-i)e_2}.
$$

For fixed $n$, under $\overline{\mathbb{P}}$, variables $R_{i,n}$ are i.i.d. and each distributed as $\rho^\lambda_{e_1,e_2}$. Rewrite (7.3) for $x = 0$ as

$$
n^{-1} \log \rho^\lambda_{0,ne_2} = \sum_{j=0}^{n-1} P^\omega_0 \{X_n = je_1 + (n - j)e_2\} \prod_{i=0}^{j-1} R_{i,n}.
$$

Assuming the usual asymptotics and approximations work,

$$
n^{-1} \log \rho^\lambda_{0,ne_2} \approx \max_{0 \leq j \leq n} \left\{ n^{-1} \log P^\omega_0 \{X_n = je_1 + (n - j)e_2\} + n^{-1} \sum_{i=0}^{j-1} \log R_{i,n} \right\}
$$

$$
= \sup_{\xi \in \mathbb{R}} \left\{ n^{-1} \log P^\omega_0 \{X_n = [n\xi]\} + n^{-1} \sum_{i=0}^{[n\xi]e_1-1} \log R_{i,n} \right\}
$$

$$
\xrightarrow{n \to \infty} \sup_{\xi \in \mathbb{R}} \{ -I_q(\xi) + \xi \cdot e_1(\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)) \}.
$$

We defer the detailed justification of this limit to the end of the proof.

The above and (7.2) give the equation

$$
\psi_0(\alpha + \lambda) - \psi_0(\lambda) = \sup_{\xi \in \mathbb{R}} \{ \xi_1(\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)) - I_q(\xi) \}.
$$
For $t \in \mathbb{R}$ let
\[
f(t) = I_q^*(te_1) = \sup_{s \in \mathbb{R}} \{ts - I_q(se_1 + (1 - s)e_2)\}
\]
where of course $I_q(\xi) = \infty$ for $\xi \notin \mathcal{U}$ (i.e. $s \notin [0, 1]$). For $t \geq 0$ and $\lambda(t)$ defined by Lemma 2.6(b) equation (7.5) gives
\[
f(t) = \psi_0(\alpha + \lambda(t)) - \psi_0(\lambda(t)).
\]
This proves (2.17).

We have
\[
f'(t) = \left(\psi_1(\alpha + \lambda(t)) - \psi_1(\lambda(t))\right)\lambda'(t)
\]
\[
= \frac{\psi_1(\alpha + \lambda(t)) - \psi_1(\lambda(t))}{\psi_1(\alpha + \beta + \lambda(t)) - \psi_1(\lambda(t))}
\]
\[
\xrightarrow{t \to 0} \frac{\alpha}{\alpha + \beta}
\]
where the last limit has already been shown at the end of the proof of Lemma 2.6(a). Consequently, $f'(0+) = \frac{\alpha}{\alpha + \beta}$. Since $f$ is convex, we get that
\[
f'(t \pm) \leq \frac{\alpha}{\alpha + \beta} \quad \text{for } t \leq 0.
\]

Since the RWRE under the averaged measure $\int P_{0}^{\lambda} (\cdot) \, d\tilde{\omega}$ is simple random walk, RWRE with transitions $\omega^\lambda$ satisfies an almost-sure law of large numbers with velocity given by
\[
\mathbb{E}[\omega_{0,e_1}^{\lambda} e_1 + \omega_{0,e_2}^{\lambda} e_2] = \left(\frac{\alpha}{\alpha + \beta}, \frac{\beta}{\alpha + \beta}\right) = \xi^*.
\]
This gives $I_q(\xi^*) = 0$.

Let $\xi \in \mathcal{U}$ with $\xi_1 \geq \frac{\alpha}{\alpha + \beta}$. The second equality in the next computation comes from (7.6).
\[
I_q(\xi) = \sup_{t \in \mathbb{R}} \{t \xi_1 - f(t)\} = \sup_{t > 0} \{t \xi_1 - f(t)\}
\]
\[
= \sup_{\lambda > 0} \left\{\xi_1(\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)) - \psi_0(\alpha + \lambda) + \psi_0(\lambda)\right\}
\]
\[
= \xi_1 \psi_0(\alpha + \beta + \lambda(\xi_1)) + (1 - \xi_1)\psi_0(\lambda(\xi_1)) - \psi_0(\alpha + \lambda(\xi_1))
\]
because condition (2.10) picks out the maximizer above.

To derive $I_q(\xi)$ for $\xi_1 \in [0, \frac{\alpha}{\alpha + \beta}]$, switch around $\alpha$ and $\beta$ and the axes and then apply the first formula of (2.13) already proved.

To compute $I_q^*(te_1)$ for $t < 0$ write temporarily $f_{\alpha,\beta}(t)$ and $I_{\alpha,\beta}(\xi)$ to make the dependence on the parameters $\alpha, \beta$ explicit. Then
\[
f_{\alpha,\beta}(t) = t + \sup_{0 \leq s \leq 1} \{(-t)(1 - s) - I_{\alpha,\beta}(se_1 + (1 - s)e_2)\}
\]
\[
= t + \sup_{0 \leq s \leq 1} \{(-t)(1 - s) - I_{\beta,\alpha}(1 - s)e_1 + se_2)\} = t + f_{\beta,\alpha}(-t).
\]
Formula (2.18) follows. In particular, we have $f_{\alpha,\beta}'(0-) = 1 - \frac{\beta}{\alpha + \beta} = f_{\alpha,\beta}'(0+)$ and $f_{\alpha,\beta}$ is everywhere differentiable. Thus, $I_q = I_{\alpha,\beta}$ is strictly convex on $\mathcal{U}$.

We have now verified formula (2.13) for $I_q$ and Theorem 2.10. By Lemma 8.1 of [20] the statement $I_q(\xi) > I_{\alpha}(\xi)$ $\forall \xi \in \mathcal{U}\setminus\{\xi^*\}$ is equivalent to
\[
I_q^*(t) < I_{\alpha}^*(t) \quad \text{for all } t \neq 0.
\]
(The case $t = 0$ corresponds to $\xi = \xi^*$ and thus leads to an equality.)

Substituting the above functions this becomes
\[
\psi_0(\alpha + \lambda(t)) - \psi_0(\lambda(t)) < \log(\xi_1^* e^t + \xi_2^*)
\]
and
\[
-t + \psi_0(\beta + \lambda(t)) - \psi_0(\lambda(t)) < \log(\xi_1^* e^{-t} + \xi_2^*)
\]
for all $t > 0$.

Using (2.12) and a little bit of rearrangement the above is equivalent to showing that
\[
e^{\psi_0(\alpha + \lambda)} < \xi_1^* e^{\psi_0(\alpha + \beta + \lambda)} + \xi_2^* e^{\psi_0(\lambda)}
\]
and
\[
e^{\psi_0(\beta + \lambda)} < \xi_1^* e^{\psi_0(\lambda)} + \xi_2^* e^{\psi_0(\alpha + \beta + \lambda)}
\]
for all $\lambda \geq 0$.

Since $\xi_1^*(\alpha + \beta + \lambda) + \xi_2^* \lambda = \alpha + \lambda$ and $\xi_1^* \lambda + \xi_2^*(\alpha + \beta + \lambda) = \beta + \lambda$, the above inequalities would follow if $e^{\psi_0(x)}$ were a strictly convex function. The second derivative of this function is given by $e^{\psi_0(x)}(\psi_2(x) + \psi_1(x)^2)$, which is positive by Lemma A.5. We have hence shown that $I_q(\xi) > I_q(\xi)$ for all $\xi \in U$ with $\xi \neq \xi^*$.

The proofs of Theorems 2.7 and 2.10 are complete, except that it remains to give the detailed justification of (7.5). By (7.4) for any $\xi \in U$,  

\[
n^{-1} \log \rho_{0,n e_2}^\lambda \geq n^{-1} \log P_0^\lambda \{ X_n = [n \xi] \} + n^{-1} \sum_{i=0}^{[n \xi] e_1 - 1} \log R_{i,n}.
\]

Taking $n \to \infty$ and applying (7.2), (1.5), and the ergodic theorem we get

\[
\psi_0(\alpha + \lambda) - \psi_0(\lambda) \geq -I_q(\xi) + \xi_1 \mathbb{E}[\log \rho_{e_1,e_2}^\lambda] \\
= -I_q(\xi) + \xi_1 (\mathbb{E}[\log \rho_{0,e_2}^\lambda] - \mathbb{E}[\log \rho_{0,e_1}^\lambda]) \\
= -I_q(\xi) + \xi_1 (\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)).
\]

(Since the summands $\log R_{i,n}$ shift with $n$ the limit of their average is not a.s. but rather an in probability limit.) Supremum over $\xi$ gives

\[
\psi_0(\alpha + \lambda) - \psi_0(\lambda) \geq \max_{\xi \in U} \left\{ -I_q(\xi) + \xi_1 (\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)) \right\}.
\]

For the reverse inequality go back to (7.3) and write
\[
n^{-1} \log \rho_{0,n e_2}^\lambda \leq n^{-1} \log(n + 1) + n^{-1} \max_{0 \leq j \leq n} \left\{ \log P_0^\lambda \{ X_n = je_1 + (n - j)e_2 \} \\
+ n^{-1} \log \rho_{je_1 + (n-j)e_2,n e_2}^\lambda \right\}.
\]

Now take $n \geq k \geq 2$. Let $m_n$ be the integer such that

\[
(m_n - 1)(k - 1) < n \leq m_n(k - 1).
\]

Then any $j$ with $0 \leq j \leq n$ we have $j/m_n + 1 \leq k$ and thus there exists a unique $i = i(j,n)$ such that $0 \leq i \leq k$ and $j < m_n i \leq j + m_n$. Let $x_{j,n} = ie_1 + (k - i)e_2$.

For $z \in \{e_1, e_2\}$ abbreviate

\[
A_{x,z} = \max \{ |\log \omega^\lambda(x, x + z)|, |\log \rho_{z,x+z}^\lambda| \}.
\]
For $0 \leq j \leq n$ use the Markov property to bound
\[ P_0^\omega \{ X_n = j e_1 + (n - j) e_2 \} \leq P_{j e_1 + (n - j) e_2}^\omega \{ X_{m_n k - n} = m_n x_{j,n} \} \]
On the other hand, observe that one can go from $j e_1 + (n - j) e_2$ to $m_n x_{j,n}$ by taking at most $m_n$ steps of type $e_1$ and then at most $m_n$ steps of type $e_2$. Since $m_n k \leq nk/(k - 1) + k \leq 3n$ we have
\begin{equation}
\log P_{j e_1 + (n - j) e_2}^\omega \{ X_{m_n k - n} = m_n x_{j,n} \} \geq -2 \max \left\{ \sum_{0 \leq i \leq 3n/k} A_{x + i z, e} : |x|_1 \leq 3n, z \in \{ e_1, e_2 \} \right\}.
\end{equation}
(Taking a maximum over all $x$ with $|x|_1 \leq 3n$ is an overkill, but still good enough for our purposes.) Similarly,
\[ \rho^\lambda(j e_1 + (n - j) e_2, n e_2) = \rho^\lambda(j e_1 + (n - j) e_2, m_n x_{j,n}) \rho^\lambda(m_n x_{j,n}, m_n k e_2) \rho^\lambda(m_n k e_2, n e_2) \]
and for the same reason as (7.11) we have
\[ \log \rho^\lambda(j e_1 + (n - j) e_2, m_n x_{j,n}) + \log \rho^\lambda(m_n k e_2, n e_2) \leq 3 \max \left\{ \sum_{0 \leq i \leq 3n/k} A_{x + i z, e} : |x|_1 \leq 3n, z \in \{ e_1, e_2 \} \right\} \]
(7.12)
Let $D_k = \{ x/k : x \in \mathbb{Z}_+^2, |x|_1 = k \}$. Collect the above bounds and continue from (7.10) to write
\[ n^{-1} \log \rho_{0, n e_2}^\lambda \leq \max_{\xi \in D_k} \left\{ n^{-1} \log P_0^\omega \{ X_{m_n k} = m_n k \xi \} + n^{-1} \log \rho^\lambda(m_n k \xi, m_n k e_2) \right\} + 5n^{-1} \max \left\{ \sum_{0 \leq i \leq 3n/k} A_{x + i z, e} : |x|_1 \leq 3n, z \in \{ e_1, e_2 \} \right\}. \]
Since for each $x \in \mathbb{Z}_+^2$ and $z \in \{ e_1, e_2 \}$, $\{ A_{x + i z, e} : i \in \mathbb{Z}_+ \}$ are i.i.d. and have strictly more than two moments, Lemma A.4 of [19] implies that
\[ \lim_{k \to \infty} \lim_{n \to \infty} n^{-1} \max \left\{ \sum_{i=0}^{3n/k} A_{x + i z, e} : |x|_1 \leq 3n, z \in \{ e_1, e_2 \} \right\} = 0. \]
Note that $m_n k/n \to k/(k - 1)$ as $n \to \infty$. Applying the last display, (7.2), (1.5), and the ergodic theorem (similarly to how the right-hand side of (7.9) was obtained) we get
\[ \psi_0(\alpha + \lambda) - \psi_0(\lambda) \leq \lim_{k \to \infty} \max_{\xi \in D_k} \left\{ n^{-1} \log P_0^\omega \{ X_{m_n k} = m_n k \xi \} + n^{-1} \log \rho^\lambda(m_n k \xi, m_n k e_2) \right\} \]
\[ = \lim_{k \to \infty} \max_{\xi \in D_k} \left\{ - I_q(\xi) + \xi_1(\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)) \right\} \]
\[ \leq \max_{\xi \in D_k} \left\{ - I_q(\xi) + \xi_1(\psi_0(\alpha + \beta + \lambda) - \psi_0(\lambda)) \right\}. \]
(7.5) is proved. \hfill \Box

Proof of Theorem 2.11. Equation (2.19) is the same as Theorem 3.6(d) above, where it was proved directly without recourse to the general variational formula (1.6). Substitution of $B^\xi$ on the right-hand side of (1.6) now verifies that the infimum is attained at $B = B^\xi$. After $I_q$ is extended to all of $\mathbb{R}_+$, formula (2.19) remains valid. This and calculus verify (2.20).\hfill \Box
Remark A.1. In what follows, some lengthy algebraic manipulations were performed with Maple and checked with Sage. Specifically, these were the last expansion in the proof of Lemma A.2, the expansions in the proofs of Lemmas B.2 and B.3, and the computation of $I''(\xi^*)$ at the very end of the paper.

Let us recall a few facts about polygamma functions $\psi_0(x) = \Gamma'(x)/\Gamma(x)$ and $\psi_n(x) = \psi_{n-1}'(x)$ for $x > 0$ and $n \in \mathbb{N}$.

For $n \geq 1$ we have the integral representation

\begin{equation}
\psi_n(x) = -\int_0^\infty \frac{(-t)^n e^{-xt}}{1 - e^{-t}} \, dt.
\end{equation}

See formula 6.4.1 in [1]. In particular, we see that for $n \geq 1$, $\psi_n(x)$ never vanishes, has sign $(-1)^{n-1}$ for all $x > 0$, $\psi_0$ is strictly concave and increasing, while $\psi_1$ is strictly convex and decreasing with $\psi_1(x) \to \infty$ when $x \searrow 0$ and $\psi_1(x) \to 0$ when $x \to \infty$.

Differentiating the relation $\Gamma(x+1) = x\Gamma(x)$, dividing by $\Gamma(x)$, then differentiating $n$ times gives the recurrence relation

\begin{equation}
\psi_n(x) = \psi_n(x+1) - (-1)^n n! x^{-(n+1)} \quad \text{for all } n \geq 0 \text{ and } x > 0.
\end{equation}

In particular, this shows that $\psi_0(x) \sim \log x \to \infty$ as $x \to \infty$.

Combining formulas 6.3.5 and 6.3.16 from [1] we also have the expansion

\begin{equation}
\psi_0(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^\infty \left( \frac{1}{k} - \frac{1}{x+k} \right),
\end{equation}

where $\gamma = \lim_{n \to \infty} (\log n + \sum_{k=1}^n k^{-1})$ is Euler’s constant and $x > 0$. In particular, $\psi_0(x) \to -\infty$ as $x \searrow 0$.

Lemma A.2. For $n \geq 1$ and $a > 0$ fixed we have

$\psi_n(x+a) - \psi_n(x) = \frac{(-1)^n a}{x^{n+1}} \left( n! - \frac{(a-1)(n+1)!}{2x} + \frac{(a-2)(n+2)!}{12x^2} - \frac{a(a-1)^2(n+3)!}{24x^3} + \frac{(6a^4-15a^3+10a^2-1)(n+4)!}{720x^4} + \mathcal{O} \left( \frac{1}{x^5} \right) \right)$.

Proof. Using (A.1) write

$\psi_n(x+a) - \psi_n(x) = \frac{(-1)^{n+1}}{x^{n+1}} \int_0^\infty t^n e^{-xt} \frac{e^{-at} - 1}{1 - e^{-t}} \, dt$

$= \frac{(-1)^n}{x^{n+1}} \int_0^\infty s^n e^{-s} \cdot \frac{e^{-as/x} - 1}{1 - e^{-s/x}} \, ds$

$= \frac{(-1)^n}{x^{n+1}} \int_0^\infty s^n e^{-s} \cdot \frac{s}{x} \frac{a^2 s^2}{2x^2} + \frac{a^3 s^3}{6x^3} + \cdots + \mathcal{O} \left( \frac{s^6}{x^6} \right) \, ds$

$= \frac{(-1)^n a}{x^{n+1}} \int_0^\infty s^n e^{-s} \left( 1 - \frac{(a-1)s}{2x} + \frac{(a-2)(2a-1)x^2}{12x^2} - \frac{a(a-1)^2 s^3}{24x^3} + \frac{(6a^4-15a^3+10a^2-1)s^4}{720x^4} + \mathcal{O} \left( \frac{s^5}{x^5} \right) \right) \, ds$

The claim now follows from $\int_0^\infty s^n e^{-s} \, ds = n!$.

Lemma A.3. Function $\psi_2 \circ \psi_1^{-1}$ is strictly concave;
Proof. See for example the proof of Lemma 5.3 in [4]. We give here the details for completeness.

We have
\[
(\psi_2 \circ \psi_1^{-1})'' = \left( \frac{\psi_3 \circ \psi_1^{-1}}{\psi_2 \circ \psi_1^{-1}} \right)' = \frac{\psi_4 \circ \psi_1^{-1} - (\psi_3 \circ \psi_1^{-1})^2/\psi_2 \circ \psi_1^{-1}}{(\psi_2 \circ \psi_1^{-1})^2}.
\]
Since \(\psi_2\) is negative, strict concavity of \(\psi_2 \circ \psi_1^{-1}\) would follow from showing that \(\psi_4 \psi_2 > \psi_3^2\). By the integral representation (A.1) this is equivalent to
\[
\int_0^\infty \int_0^\infty \frac{t^3 s^3 e^{-xt-xs}}{(1-e^{-t})(1-e^{-s})} dt \, ds < \int_0^\infty \int_0^\infty \frac{t^2 s^4 e^{-xt-xs}}{(1-e^{-t})(1-e^{-s})} dt \, ds.
\]
Symmetrizing the right-hand side (i.e. adding another copy with \(s\) and \(t\) interchanged) the above becomes
\[
\int_0^\infty \int_0^\infty \frac{t^2 s^2 e^{-xt-xs}}{(1-e^{-t})(1-e^{-s})} (2ts) dt \, ds < \int_0^\infty \int_0^\infty \frac{t^2 s^2 e^{-xt-xs}}{(1-e^{-t})(1-e^{-s})} (t^2 + s^2) dt \, ds,
\]
which is true for all \(x > 0\).

**Lemma A.4.** For all \(x > 0\) we have
\[
\psi_1(x) > \frac{1}{x} + \frac{1}{2x^2}.
\]

**Proof.** Write
\[
\psi_1(x) = \int_0^\infty \frac{te^{-xt}}{1-e^{-t}} dt = \frac{1}{x^2} \int_0^\infty \frac{se^{-s/x}}{1-e^{-s/x}} ds.
\]
Expand
\[
\frac{1}{1-e^{-s/x}} = \frac{s}{x} - \frac{s^2}{2x^2} + \frac{s^3}{6x^3} - \frac{s^4}{24x^4} + \mathcal{O}(x^{-5})
\]
\[
= \frac{x}{s} \left(1 + \frac{s}{2x} - \frac{s^2}{6x^2} + \frac{s^3}{24x^3} + \frac{s^4}{6x^4} + \mathcal{O}(x^{-4})\right)
\]
\[
= \frac{x}{s} \left(1 + \frac{s}{2x} - \frac{s^2}{6x^2} + \frac{s^3}{24x^3} + \frac{s^4}{4x^4} - \frac{s^5}{8x^5} + \mathcal{O}(x^{-4})\right)
\]
\[
(A.4)
\]
This gives
\[
(A.5) \quad \psi_1(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} + \mathcal{O}(x^{-5}).
\]
We thus see that the claim of the lemma holds for all large enough \(x\).

Next, assume that for some \(x > 0\) we have
\[
\psi_1(x + 1) > \frac{1}{x+1} + \frac{1}{2(x+1)^2}.
\]
Use (A.2) to write
\[
\psi_1(x) - \frac{1}{x} - \frac{1}{2x^2} = \psi_1(x + 1) - \frac{1}{x + 1} + \frac{1}{2(x+1)^2} - \frac{1}{x + 1} + \frac{1}{2x^2}
\]
\[
> \frac{1}{x + 1} + \frac{1}{2(x+1)^2} - \frac{1}{x + 1} + \frac{1}{2x^2}
\]
\[
= \frac{1}{2x^2(x+1)^2} > 0.
\]
Thus, we see that if the claim holds for \(x+1\) it holds also for \(x\). This and the fact that it holds for all large enough \(x\) implies that it holds for all \(x \geq 0\).

\[\square\]

**Lemma A.5.** We have for all \(x \geq 0\)
\[
\psi_2(x) + \psi_1(x)^2 > 0.
\]

**Proof.** We proceed similarly to the above lemma. First, we compute the expansions for large \(x\). Write
\[
\psi_2(x) = -\int_0^\infty \frac{t^2e^{-xt}}{1 - e^{-t}} \, dt = -\frac{1}{x^3} \int_0^\infty \frac{s^2e^{-s}}{1 - e^{-s/x}} \, ds.
\]
From (A.4) we get
\[
\psi_2(x) = -\frac{1}{x^2} \left( 1 + \frac{1}{x} + \frac{1}{2x^2} + O(x^{-4}) \right).
\]
And from (A.5) we have
\[
\psi_1(x)^2 = \frac{1}{x^2} \left( 1 + \frac{1}{x} + \frac{1}{3x^2} + \frac{1}{4x^2} + \frac{1}{6x^3} + O(x^{-4}) \right).
\]
Hence
\[
\psi_2(x) + \psi_1(x)^2 = \frac{1}{12x^4} + O(x^{-3}),
\]
which says that the claim of the lemma holds for all large enough \(x\).

Next, assume that for some \(x > 0\) we have
\[
\psi_2(x + 1) + \psi_1(x + 1)^2 > 0.
\]
Use (A.2) (twice for the first equality and once for the last one) to write
\[
\psi_2(x) + \psi_1^2(x) = \psi_2(x + 1) - \frac{2}{x^3} + \left( \psi_1(x + 1) + \frac{1}{x^2} \right)^2
\]
\[
= \psi_2(x + 1) + \psi_1(x + 1)^2 + \frac{2\psi_1(x + 1)}{x^2} - \frac{2}{x^3} + \frac{1}{x^4}
\]
\[
> \frac{2\psi_1(x + 1)}{x^2} - \frac{2}{x^3} + \frac{1}{x^4}
\]
\[
= \frac{2}{x^2} \left( \psi_1(x + 1) - \frac{1}{x} + \frac{1}{2x^2} \right)
\]
\[
= \frac{2}{x^2} \left( \psi_1(x) - \frac{1}{x} - \frac{1}{2x^2} \right).
\]
The last quantity is positive, by the above lemma. Hence, we see that if the inequality claimed in the lemma holds for \(x + 1\) it holds for \(x\) as well. This, and the fact the inequality holds for all large enough \(x\) implies the inequality holds for all \(x > 0\). \(\square\)
APPENDIX B. DIFFERENTIABILITY AND EXPANSION OF $I_q$

When convenient we consider $I_q$ and $\lambda$ from (2.13) and Lemma 2.6(a) as functions of $\xi$: $I_q(\xi) = I_q(\xi, 1 - \xi)$ and $\lambda(\xi) = \lambda(\xi, 1 - \xi)$.

**Lemma B.1.** There exist open sets $G_0, G_1$ in $\mathbb{C}$ such that $(0, \xi^*_1) \subset G_0$ and $(\xi^*_1, 1) \subset G_1$, and holomorphic functions $d_0$ and $f_0$ on $G_0$ and $d_1$ and $f_1$ on $G_1$ such that $\lambda(\xi) = d_0(\xi_1)$ and $I_q(\xi) = f_0(\xi_1)$ for $\xi_1 \in (0, \xi^*_1)$ and $\lambda(\xi) = d_1(\xi_1)$ and $I_q(\xi) = f_1(\xi_1)$ for $\xi_1 \in (\xi^*_1, 1)$.

**Proof.** We present the proof for the case $G_1$. Let

$$\sigma_1(z) = \frac{\psi_1(z) - \psi_1(\alpha + z)}{\psi_1(z) - \psi_1(\alpha + \beta + z)}, \quad z \in \mathbb{H}_1 = \{z : \Re z > 0\},$$

denote the function defined by (2.10), now thought of as a function on the open right half plane. The set $U_1 = \{z \in \mathbb{H}_1 : \psi_1(z) \neq \psi_1(\alpha + \beta + z)\}$ is an open subset of $\mathbb{H}_1$ that contains the half-line $(0, \infty)$. Since $\psi_1$ is holomorphic on $\mathbb{H}_1$, it follows that $\sigma_1$ is holomorphic on $U_1$.

By the proof of Lemma 2.6 in Section 7, $\sigma_1'(\lambda) < 0$ for $0 < \lambda < \infty$. By Theorem 10.30 in [22], each $\lambda \in (0, \infty)$ is the center of an open disk in $U_1$ on which $\sigma_1$ is one-to-one. By using the strict negativity and continuity of $\sigma_1'$ on $(0, \infty)$, we can take these disks small enough so that $\sigma_1$ is a one-to-one mapping of the union $V_1$ of these disks.

$G_1 = \sigma_1(V_1)$ is an open set since holomorphic functions are open mappings. $G_1$ contains the open real interval $(\xi^*_1, 1)$ by Lemma 2.6. By Theorem 10.33 in [22], the inverse function $d_1(w)$ of $\sigma_1$ is holomorphic on $G_1$. Since $d_1$ maps $G_1$ into $\mathbb{H}_1$ where $\psi_0$ is holomorphic,

$$f_1(w) = w\psi_0(\alpha + \beta + d_1(w)) + (1 - w)\psi_0(d_1(w)) - \psi_0(\alpha + d_1(w))$$

is holomorphic on $G_1$. From the definitions it now follows that for $\xi_1 \in (\xi^*_1, 1)$, $d_1(\xi_1) = \lambda(\xi)$ and $f_1(\xi_1) = I_q(\xi)$.

**Lemma B.2.** We have the following expansion for $\xi_1$ from (2.10):

$$\xi_1(\lambda) = \frac{\alpha}{\alpha + \beta} \left( 1 + \frac{\beta}{\lambda} + \frac{\beta(1 - 2\alpha)}{2\lambda^2} + \frac{\beta\alpha(\alpha - 1)}{\lambda^3} - \frac{\beta(12\alpha^3 - 18\alpha^2\alpha - 2\beta + 3)}{12\lambda^4} + \mathcal{O}(\frac{1}{\lambda^5}) \right).$$

**Proof.** Applying Lemma A.2 to (2.10) we have

$$\xi_1(\lambda) = \frac{\psi_1(\alpha + \beta + \lambda) - \psi_1(\alpha + \lambda)}{\psi_1(\alpha + \beta + \lambda) - \psi_1(\alpha + \lambda)} = \frac{\alpha}{\alpha + \beta} \cdot \frac{1 - \frac{\alpha - 1}{\lambda}}{1 - \frac{\alpha + \beta - 1}{\lambda}} + \cdots + \mathcal{O}(\lambda^{-5}).$$

The claim follows.

**Lemma B.3.** With $\xi_1$ from (2.10) we have the following expansions as $\mu \to \infty$:

$$\lambda'(\xi_1(\mu)) = -\frac{\alpha + \beta}{\alpha^2} \left( \mu^4 + (2\alpha - 1)\mu + (\alpha^2 - \alpha + 1) + \frac{2\beta}{3\mu} \right) + \mathcal{O}(\frac{1}{\mu^5}),$$

$$\lambda''(\xi_1(\mu)) = \frac{(\alpha + \beta)^2}{\alpha^2 \beta^2} \left( 2\mu^3 + 3(2\alpha - 1)\mu^2 + 3(2\alpha^2 - 2\alpha + 1)\mu + \frac{6\alpha^2 - 9\alpha^2 + 10\alpha - 3}{3} \right) + \mathcal{O}(\mu^{-1}),$$

$$\lambda'''(\xi_1(\mu)) = -\frac{(\alpha + \beta)^3}{\alpha^3 \beta^3} \left( 6\mu^4 + 12(2\alpha - 1)\mu^3 + 3(12\alpha^2 - 12\alpha + 5)\mu^2 + (24\alpha^3 - 36\alpha^2 + 32\alpha - 2\beta - 9)\mu \right) + \mathcal{O}(1).$$
Proof. Use (2.10) to write
\[
\xi_1[\psi_1(\alpha + \beta + \lambda(\xi_1)) - \psi_1(\lambda(\xi_1))] = \psi_1(\alpha + \lambda(\xi_1)) - \psi_1(\lambda(\xi_1)).
\]
Differentiating this in \(\xi_1\) and solving for \(\lambda'(\xi_1)\) we get
\[
\lambda'(\xi_1) = \frac{\psi_1(\alpha + \beta + \lambda(\xi_1)) - \psi_1(\lambda(\xi_1))}{\left(\psi_2(\alpha + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right) - \xi_1\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)}.
\]
Differentiating a second time we get
\[
\lambda''(\xi_1) = \frac{2\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)\lambda'(\xi_1)}{\psi_2(\alpha + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1)) - \xi_1\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)} + \frac{\xi_1\left(\psi_3(\alpha + \beta + \lambda(\xi_1)) - \psi_3(\lambda(\xi_1))\right) - \left(\psi_3(\alpha + \lambda(\xi_1)) - \psi_3(\lambda(\xi_1))\right)}{\psi_2(\alpha + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1)) - \xi_1\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)}\lambda'(\xi_1)^2.
\]
A third round of differentiation gives
\[
\lambda'''(\xi_1) = \frac{3\left(\psi_3(\alpha + \beta + \lambda(\xi_1)) - \psi_3(\lambda(\xi_1))\right)\left(\lambda'(\xi_1)\right)^2 + 3\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)\lambda''(\xi_1)}{\psi_2(\alpha + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1)) - \xi_1\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)} + \frac{\xi_1\left(\psi_4(\alpha + \beta + \lambda(\xi_1)) - \psi_4(\lambda(\xi_1))\right) - \left(\psi_4(\alpha + \lambda(\xi_1)) - \psi_4(\lambda(\xi_1))\right)}{\psi_2(\alpha + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1)) - \xi_1\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)}\lambda'(\xi_1)^3 + \frac{3\left[\xi_1\left(\psi_3(\alpha + \beta + \lambda(\xi_1)) - \psi_3(\lambda(\xi_1))\right) - \left(\psi_3(\alpha + \lambda(\xi_1)) - \psi_3(\lambda(\xi_1))\right)\right]}{\psi_2(\alpha + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1)) - \xi_1\left(\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right)}\lambda'(\xi_1)\lambda''(\xi_1).
\]
The claims now come by applying the expansions from Lemma A.2. \(\Box\)

Now we can prove expansion (2.15) of \(I_q\), claimed in Remark 2.9. We focus only on the case \(\xi_1 \in [\frac{\alpha}{\alpha+\beta}, 1]\), the other case being similar. Consider \(I_q\) and \(\lambda\) as functions of \(\xi_1\): \(I_q(\xi_1) = I_q(\xi_1, 1 - \xi_1)\) and \(\lambda(\xi_1) = \lambda(\xi_1, 1 - \xi_1)\).

By (2.10)
\[
I_q'(\xi_1) = \psi_0(\alpha + \beta + \lambda(\xi_1)) - \psi_0(\lambda(\xi_1)) + [\xi_1\psi_1(\alpha + \beta + \lambda(\xi_1)) + (1 - \xi_1)\psi_1(\lambda(\xi_1)) - \psi_1(\alpha + \lambda(\xi_1))]\lambda'(\xi_1) = \psi_0(\alpha + \beta + \lambda(\xi_1)) - \psi_0(\lambda(\xi_1)).
\]
Thus, \(I_q'(\xi_1) \to 0\) as \(\xi_1 \searrow \xi_1^*\). A similar argument works for \(\xi_1 \nearrow \xi_1^*\) and thus \(I_q'(\xi^*) = 0\). Taking a second derivative we have
\[
I_q''(\xi_1) = [\psi_1(\alpha + \beta + \lambda(\xi_1)) - \psi_1(\lambda(\xi_1))]\lambda'(\xi_1).
\]
Since \(\lambda\) is strictly decreasing on \([\xi_1^*, 1]\) and \(\psi_1\) is strictly decreasing on \([0, \infty)\), the above is strictly positive. This gives another verification of the strict convexity of \(I_q\).
Further differentiations give
\begin{align*}
I''_q(\xi_1) &= \left[\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right] (\lambda'(\xi_1))^2 \\
&\quad + \left[\psi_1(\alpha + \beta + \lambda(\xi_1)) - \psi_1(\lambda(\xi_1))\right] \lambda''(\xi_1)
\end{align*}
and
\begin{align*}
I'''_q(\xi_1) &= \left[\psi_3(\alpha + \beta + \lambda(\xi_1)) - \psi_3(\lambda(\xi_1))\right] (\lambda'(\xi_1))^3 \\
&\quad + 3\left[\psi_2(\alpha + \beta + \lambda(\xi_1)) - \psi_2(\lambda(\xi_1))\right] \lambda'(\xi_1) \lambda''(\xi_1) \\
&\quad + \left[\psi_1(\alpha + \beta + \lambda(\xi_1)) - \psi_1(\lambda(\xi_1))\right] \lambda'''(\xi_1).
\end{align*}
From (B.1) and the expansions in Lemmas A.2, B.2, and B.3 we have
\[ I''_q(\xi_1(\lambda)) = -(\alpha + \beta)\lambda^{-2}(1 + \mathcal{O}(\lambda^{-1})) \left(\frac{\alpha + \beta}{\alpha\beta} \lambda^2 + \mathcal{O}(\lambda)\right) \xrightarrow{\lambda \to \infty} \frac{(\alpha + \beta)^2}{\alpha\beta}. \]
In other words,
\[ \lim_{\xi_1 \searrow \xi_1^*} I''_q(\xi_1) = \frac{(\alpha + \beta)^2}{\alpha\beta} > 0. \]
Similarly,
\begin{align*}
I'''_q(\xi_1(\lambda)) &= \frac{(\alpha + \beta)^3}{\alpha^2\beta^2\lambda^3} \left(2 - 3(\alpha + \beta - 1)\lambda^{-1} + \mathcal{O}(\lambda^{-2})\right) \left(\lambda^2 + (2\alpha - 1)\lambda + \mathcal{O}(1)\right)^2 \\
&\quad - \frac{(\alpha + \beta)^3}{\alpha^2\beta^2\lambda^2} \left(1 - (\alpha + \beta - 1)\lambda^{-1} + \mathcal{O}(\lambda^{-2})\right) \left(2\lambda^3 + 3(2\alpha - 1)\lambda^2 + \mathcal{O}(\lambda)\right) \\
&= \frac{(\alpha + \beta)^3(\alpha - \beta)}{\alpha^2\beta^2} \left(1 + \mathcal{O}(\lambda^{-1})\right) \xrightarrow{\lambda \to \infty} \frac{(\alpha + \beta)^3(\alpha - \beta)}{\alpha^2\beta^2}
\end{align*}
and also
\[ \lim_{\xi_1 \searrow \xi_1^*} I'''_q(\xi_1) = \frac{(\alpha + \beta)^4(2\alpha^2 - 2\alpha\beta + 2\beta^2 + 1)}{\alpha^3\beta^3}. \]
Write \( I_q(\xi_1(\alpha, \beta), \alpha, \beta) \) to make the dependence on \( \alpha \) and \( \beta \) clear. Then symmetry gives \( \xi_1^*(\alpha, \beta) = 1 - \xi_1^*(\beta, \alpha) \) and \( I_q(s, \alpha, \beta) = I_q(1 - s, \beta, \alpha) \). Thus
\[ \lim_{s \searrow \xi_1^*(\alpha, \beta)} I^{(n)}_q(s, \alpha, \beta) = (-1)^n \lim_{t \searrow \xi_1^*(\beta, \alpha)} I^{(n)}_q(t, \beta, \alpha). \]
From this we see that \( I_q \) is \( n \) times continuously differentiable at \( \xi^* \) if and only if functions
\[ F_m(\alpha, \beta) = \lim_{\xi_1 \searrow \xi_1^*(\alpha, \beta)} I^{(m)}_q(\xi_1, \alpha, \beta) \]
are symmetric in \( \alpha, \beta \) for all even \( m \leq n \) and antisymmetric for all odd \( m \leq n \). The formulas we found above satisfy this up to \( n = 4 \), therefore \( I_q \) is at least four times continuously differentiable at \( \xi^* \) and expansion (2.15) holds. \( \square \)

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