Classifying unavoidable Tverberg partitions

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Abstract

Let $T(d, r) \overset{\text{def}}{=} (r - 1)(d + 1) + 1$ be the parameter in Tverberg’s theorem. We say that a partition $\mathcal{I}$ of $\{1, 2, \ldots, T(d, r)\}$ into $r$ parts occurs in an ordered point sequence $P$ if $P$ contains a subsequence $P'$ of $T(d, r)$ points such that the partition of $P'$ that is order-isomorphic to $\mathcal{I}$ is a Tverberg partition. We say that $\mathcal{I}$ is unavoidable if it occurs in every sufficiently long point sequence.

In this paper we study the problem of determining which Tverberg partitions are unavoidable. We conjecture a complete characterization of the unavoidable Tverberg partitions, and we prove some cases of our conjecture for $d \leq 4$. Along the way, we study the avoidability of many other geometric predicates, and we raise many open problems.

Our techniques also yield a large family of $T(d, r)$-point sets for which the number of Tverberg partitions is exactly $(r - 1)!^d$. This lends further support for Sierksma’s conjecture on the number of Tverberg partitions.

Keywords: Geometric predicate, Ramsey theory, stair-convexity, Tverberg’s theorem.

1 Introduction

By a (geometric) predicate of arity $k$ we mean a $k$-ary relation $\Phi$ on $\mathbb{R}^d$, i.e. a property which $k$-tuples of points $(p_1, \ldots, p_k) \in (\mathbb{R}^d)^k$ might or might not satisfy. We focus on semialgebraic predicates, which are predicates given by Boolean combinations of terms of the form $f(p_1, \ldots, p_k) > 0$, where the $f$'s are nonzero polynomials.

An example of a geometric predicate is the planar predicate “$p_1, p_2, p_3, p_4 \in \mathbb{R}^2$ are in convex position”. (We show in Section 2 that all predicates we consider are semialgebraic.) Another example
is the \((d + 1)\)-ary orientation predicate in \(\mathbb{R}^d\), \(\text{orient}(p_1, \ldots, p_{d+1})\), whose defining polynomial is
\[
\det \begin{bmatrix} 1 & \cdots & 1 \\ p_1 & \cdots & p_{d+1} \end{bmatrix}.
\]

For example, in the plane, \(\text{orient}(p_1, p_2, p_3)\) means that \(p_1, p_2, p_3\) are in counterclockwise order.

Let \(P \overset{\text{def}}{=} (p_1, \ldots, p_n) \in (\mathbb{R}^d)^n\) be an ordered sequence of \(n\) points in \(\mathbb{R}^d\). We say that a \(k\)-ary predicate \(\Phi\) occurs in \(P\) if \(P\) contains a subsequence \(p_{i_1}, \ldots, p_{i_k}\), \(i_1 < \cdots < i_k\), for which \(\Phi(p_{i_1}, \ldots, p_{i_k})\) holds. Otherwise, we say that \(P\) avoids \(\Phi\). For simplicity, we will assume that \(P\) is \(\Phi\)-generic, which means that none of the polynomials defining \(\Phi\) evaluate to 0 on any \(k\)-tuple of distinct points of \(P\). Genericity can be achieved, if necessary, by an appropriate arbitrarily small perturbation of \(P\).

We say that \(P\) is homogeneous with respect to \(\Phi\) if \(P\) avoids either \(\Phi\) or \(\neg\Phi\) (the negation of \(\Phi\)). Ramsey’s theorem implies the existence of arbitrarily-long homogeneous point sequences with respect to any predicate: Indeed, given \(k\), Ramsey’s theorem states that there exists a large enough \(n\) such that every \(n\)-point sequence \(P\) contains a \(\Phi\)-homogeneous subsequence of length \(k\).

Point sequences that are homogeneous with respect to the orientation predicate are important because they form the vertices of cyclic polytopes (see e.g. [Zie95]); these point sequences have been studied previously e.g. in [BN16, BMP14, EMRPS14, Suk14]. In the plane, \(P\) is orientation-homogeneous if and only if its points are in convex position and are listed in the order they appear along the boundary of \(\text{conv} P\).

We say that a predicate \(\Phi\) is unavoidable if it occurs in every sufficiently-long \(\Phi\)-generic point sequence. Otherwise, if there exist arbitrarily long point sequences avoiding \(\Phi\), then \(\Phi\) is avoidable.

For example, the planar convex-position predicate mentioned above is unavoidable, since every generic five-point set contains four points in convex position. This result is known as the “happy ending problem”, and it is actually one of the original results that motivated the development of Ramsey theory [GN13].

Hence, for each predicate \(\Phi\) there are three mutually exclusive possibilities: Either \(\Phi\) is unavoidable, or \(\neg\Phi\) (the negation of \(\Phi\)) is unavoidable, or both \(\Phi\) and \(\neg\Phi\) are avoidable. For us in this paper, to solve a predicate means to determine on which of these categories it falls.

This problem has been shown to be decidable for semialgebraic predicates in dimension 1 [BM14]. However, for dimensions \(d \geq 2\) the problem remains open.

In this paper we focus on Tverberg-partition predicates. We present a conjecture and prove some partial results in low dimensions. We also examine along the way some related predicates.

1.1 Tverberg partitions

For a positive integer \(n\), denote \([n] \overset{\text{def}}{=} \{1, 2, \ldots, n\}\). Define \(T(d, r) \overset{\text{def}}{=} (r - 1)(d + 1) + 1\). Tverberg’s theorem [Tve66] asserts that for every point sequence \(P \overset{\text{def}}{=} (p_1, \ldots, p_{T(d, r)})\) in \(\mathbb{R}^d\) there exists a partition \(I \overset{\text{def}}{=} \{I_1, \ldots, I_r\}\) of \([T(d, r)]\) into \(r\) parts, such that the \(r\) convex hulls \(\text{conv}\{p_i \mid i \in I_j\}\), \(1 \leq j \leq r\), intersect at a common point. Such a partition \(I\) is called a Tverberg partition for \(P\). If
the convex hulls intersect at a single point (which happens whenever \( P \) is generic), then that point is called the *Tverberg point* of the partition.

In this paper we will also call a partition of \([T(d,r)]\) into \( r \) parts a *Tverberg partition*.

Let \( \mathrm{Tv}_\mathcal{I}(P) \) be the \( T(d,r) \)-ary predicate stating that \( \mathcal{I} \) is a Tverberg partition for \( P \).

Our main objective in this paper is to classify the Tverberg partitions according to the three possibilities mentioned above.

**Definition 1.1.** Let \( \mathcal{I} \overset{\text{def}}{=} \{I_1,\ldots,I_r\} \) be a Tverberg partition. We call \( \mathcal{I} \) *colorful* if, for each \( 1 \leq i \leq d+1 \), the \( r \) consecutive integers \( \{(r-1)(i-1)+1,\ldots,(r-1)i+1\} \) belong one to each of the \( r \) parts \( I_1,\ldots,I_r \).

It is sometimes convenient to encode a Tverberg partition as a string \( \sigma \in [r]^{T(d,r)} \), indicating to which part each integer belongs. Since the order of the parts within the partition does not matter, there are \( r! \) different ways of encoding each partition. For example, for \( d = 2, r = 3 \), the partition \( \{\{1,3,6\},\{2,7\},\{4,5\}\} \) can be encoded as 1213312.

In this representation, the colorful Tverberg partitions are those \( \sigma \) for which \( \{\sigma(i+1),\ldots,\sigma(i+r)\} = [r] \) for each \( i = 0, r-1, 2(r-1), \ldots, d(r-1) \). An example of a colorful Tverberg partition for \( d = 3, r = 5 \) is 12345241351425134. The lines above and below the digits indicate the intervals in which all “colors” must show up.

It is easily seen that the number of colorful Tverberg partitions with parameters \( d, r \) is \( (r-1)!^d \). We will expound on the significance of this number below.

**Theorem 1.2.** For every Tverberg partition \( \mathcal{I} \), if \( \mathcal{I} \) is colorful, then \( \neg \mathrm{Tv}_\mathcal{I} \) is avoidable; otherwise, \( \mathrm{Tv}_\mathcal{I} \) is avoidable.

**Proof sketch.** Take \( P \) to be the *stretched diagonal* previously studied in [BMN11, BMN10, Niv09]. As we will show in Section 4, the stretched diagonal is homogeneous with respect to all Tverberg-partition predicates, and furthermore, the Tverberg partitions that occur in it are exactly the colorful ones. Since the number of points in the stretched diagonal can be made arbitrarily large, the claim follows.

**Conjecture 1.3.** For every Tverberg partition \( \mathcal{I} \), if \( \mathcal{I} \) is colorful, then \( \mathrm{Tv}_\mathcal{I} \) is unavoidable; otherwise, \( \neg \mathrm{Tv}_\mathcal{I} \) is unavoidable.

We call the colorful Tverberg partition encoded by 12\( \cdots \)r\( \cdots \)212\( \cdots \)r\( \cdots \) the *zigzag partition*.

**Theorem 1.4.** Conjecture 1.3 holds in the following cases:

- For \( d \leq 2 \) and all \( r \).

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1Our colorful partitions are unrelated to the *colored Tverberg theorem* (see [Mat02]).

2This was noticed independently by Imre Bárány and Attila Pór, as well as by Isaac Mabillard and Uli Wagner (private communication).
• For \(d = 3\), for all Tverberg partitions that have parts of sizes \(\{2, 3, 4, 4, \ldots, 4\}\).

• For \(d = r = 3\), for all the four colorful Tverberg partitions that have parts of sizes \(\{3, 3, 3\}\).

• For the zigzag partition for \(d = 4\) and all \(r\).

The results of Theorem 1.4 regarding zigzag partitions for \(d \leq 4\) were previously announced in [BMI14].

Motivation. We started studying this topic when we tried to prove that the zigzag partition is unavoidable for \(r = 3\) and all \(d\). This was the missing link in our argument that there exist one-sided epsilon-approximants of constant size with respect to convex sets [BN16]. We managed to prove the zigzag-partition claim—and hence, the one-sided epsilon-approximant corollary—only for \(d \leq 4\). However, we subsequently realized that we could do without the zigzag-partition claim, relying instead what is now Lemma 4 in [BN16], which holds for all \(d\).

1.2 Proof strategy

Our proof strategy for proving that a given predicate \(\Phi\) is unavoidable is as follows: Suppose for a contradiction that for every \(n\) there exists an \(n\)-point sequence \(P\) that avoids \(\Phi\). Let \(\Psi_1, \ldots, \Psi_k\) be other predicates. By Ramsey’s theorem, by making \(n\) large enough, we can guarantee the existence of arbitrarily large subsequences \(P'\) of \(P\) that are homogeneous with respect to \(\Psi_1, \ldots, \Psi_k\).

Hence, in our search for a contradiction, we can assume without loss of generality that our sequence \(P\) not only avoids \(\Phi\), but is also homogeneous with respect to a fixed finite family \(\Psi = (\Psi_1, \ldots, \Psi_k)\) of other predicates. Furthermore, if we previously showed that some of the \(\Psi_i\)’s are unavoidable, then we can assume that \(P\) specifically avoids their negations \(\neg \Psi_i\).

In our proofs, we will start with an orientation-homogeneous sequence (which corresponds to taking \(\Psi_1 = \text{orient}\)), and we will add additional predicates to \(\Psi\) “on-the-fly”; this is OK as long as we do it only a finite number of times.

We say that predicates \(\Psi_1\) and \(\Phi_2\) are equivalent if there is an unavoidable predicate \(\Phi\) such that \(\Phi \implies (\Phi_1 \equiv \Phi_2)\). In our proofs, \(\Phi\) will usually be the orientation predicate. We write \(\Phi_1 \equiv \Phi_2\) if \(\Phi_1\) and \(\Phi_2\) are equivalent.

1.3 Sierksma’s conjecture

Let \(P\) be a sequence of \(T(d, r)\) points in \(\mathbb{R}^d\). Tverberg’s theorem asserts the existence of at least one Tverberg partition for \(P\). However, usually there is more than one Tverberg partition. If \(r = 2\) (the particular case known as Radon’s lemma [Rad21]), and the points of \(P\) are in general position, then the Radon partition is indeed unique. However, it seems that for \(r \geq 3\) the Tverberg partition is never unique.
According to Reay [Rea82, Problem 14], Sierksma conjectured in [Sie79] that the number of Tverberg partitions is always at least \((r-1)^d\). Sierksma pointed out that there exist \(T(d,r)\)-point sets that have exactly \((r-1)^d\): Choose \(d+1\) affinely independent points \(p_1, \ldots, p_{d+1}\), and let \(q\) be a point in the interior of the simplex \(\text{conv}\{p_1, \ldots, p_{d+1}\}\). Replace each \(p_i\) by a tiny cloud \(P_i\) of \((r-1)\) points. Then, the Tverberg partitions of this point set are exactly those that have \(r-1\) parts containing exactly one point from each cloud, plus an \(r\)-th part containing only \(q\). Hence, the number of Tverberg partitions here equals exactly \((r-1)^d\). (However, for \(d \geq 2\) none of these partitions are colorful, no matter how the points are linearly ordered, since for \(d \geq 2\), colorful partitions never contain parts of size 1.)

White [Whi15] recently found a more general family of \(T(d,r)\)-point sets that have exactly \((r-1)^d\) Tverberg partitions. In fact, he constructs, for every partition \(T(d,r) = n_1 + \cdots + n_r\) of \(T(d,r)\) into \(r\) integers satisfying \(1 \leq n_i \leq d+1\), a \(T(d,r)\)-point set \(P\) that has exactly \((r-1)^d\) Tverberg partitions, all of which have parts of sizes \(n_1, \ldots, n_r\) and have the origin as their Tverberg point. Furthermore, in his construction, each point \(p_i\) is only specified by a vector of signs \(v_i \in \{+, 0, -\}^d\), which indicates the sign of each coordinate of \(p_i\); the magnitudes of the coordinates can be chosen arbitrarily.

Regarding lower bounds, Hell [Hel08] showed that the number of Tverberg partitions is always at least \((r-d)!\), and that if \(r = p^k\) is a prime power, then the number is at least

\[
\frac{1}{(r-1)!} \left( \frac{r}{k+1} \right)^{\left\lfloor T(d,r)/2 \right\rfloor}.
\]

For large \(d\) and \(r\), this number is roughly the square root of Sierksma’s conjectured bound.

**Our result.** In this paper we construct a broader family of \(T(d,r)\)-point sets that have exactly \((r-1)^d\) Tverberg partitions. Our result is a corollary of the proof of Theorem 1.2. We show that in stair-convex geometry (previously studied by the authors in [BMN11, BN12, Niv09]) every generic \(T(d,r)\)-point set has exactly \((r-1)^d\) stair-Tverberg partitions. As a consequence, in Euclidean geometry, \(T(d,r)\) randomly chosen points from the stretched grid ([BMN11, BN12, Niv09]) will almost surely have exactly \((r-1)^d\) Tverberg partitions.

### 1.4 Concurrent work

Very recently, and independently, Attila Pór announced [Pör16] that he has found a full proof of Conjecture 1.3. Pór uses a different approach from the one we use in this paper.

### 1.5 Organization of the paper

In Section 2 we briefly show that the predicates we consider in this paper are semialgebraic. In Section 3 we prove Theorem 1.4. We first show how solving Tverberg-partition predicates reduces to solving hyperplane-side predicates. Then we solve many hyperplane-side predicates, first in the plane, then in \(d = 3\), and then in \(d = 4\). We leave many open problems along the way. In Section 4 we prove...
Theorem 1.2 and our result regarding Sierksma’s conjecture. A key technical lemma that is obvious but whose proof is quite tedious is proven in Appendix A.

2 Our predicates are semialgebraic

In this section we show that all the predicates defined in the Introduction are semialgebraic. The orientation predicate is clearly semialgebraic.

**Lemma 2.1.** Given generic points \( q, p_1, \ldots, p_{d+1} \in \mathbb{R}^d \), we have \( q \in \text{conv}\{p_1, \ldots, p_{d+1}\} \) if and only if, for each \( 1 \leq i \leq d + 1 \), \( \text{orient}\{p_1, \ldots, p_{d+1}\} \) equals the orientation obtained by replacing \( p_i \) by \( q \).

**Proof.** This follows immediately from the definition of convex combination and Cramer’s rule.

**Corollary 2.2.** The \((d + 2)\)-ary predicate “\( q \in \text{conv}\{p_1, \ldots, p_{d+1}\} \)” is semialgebraic.

**Observation 2.3.** All predicates mentioned in the Introduction are semialgebraic.

**Proof.** The four-point planar convex-position predicate can be formulated by stating that none of the four given points lies in the convex hull of the other three. Hence, by Corollary 2.2, this predicate is semialgebraic.

Now, consider a Tverberg-partition predicate \( \text{Tv}_{\mathcal{I}} \) for \( \mathcal{I} \equiv \{I_1, \ldots, I_r\} \). For each \( j \), let \( x_j \) be an affine combination of the points \( p_i, i \in I_j \). Hence, there are a total of \( T(d, r) \) coefficients in the \( r \) affine combinations; these are our unknowns. For each \( j \) there is an equation requiring that the coefficients of the \( j \)-th affine combination add up to 1. Further, we express the requirement \( x_1 = x_2 = \ldots = x_r \) by \((r - 1)d\) equations. Hence, the total number of equations is also \( T(d, r) \). Therefore, we have a linear system, which has a unique solution if the given points are generic. The unique solution can be expressed using Cramer’s rule. Then, the predicate asserts that all the values in this solution are positive, so that the affine combinations are in fact convex combinations.

3 Proofs that predicates are unavoidable

In this section we prove Theorem 1.4. The case \( d = 1 \) is trivial, so let \( d \geq 2 \). Recall that a point sequence \( P \in (\mathbb{R}^d)^n \) is orientation-homogeneous if all size-\((d + 1)\) subsequences of \( P \) have the same orientation.

**Lemma 3.1** (Radon-partition lemma). Let \( P = (p_1, \ldots, p_{d+2}) \) be orientation-homogeneous. Then the Radon partition for \( P \) is the alternating one, i.e. the one encoded by \( \sigma = 12121 \ldots. \)

**Proof.** In general, the Radon partition of a set of \( d + 2 \) points is obtained as follows: We find a nontrivial solution to \( \sum \alpha_i = 0, \sum \alpha_i p_i = \vec{0} \); then, in the latter equation we move all terms with negative \( \alpha_i \)'s to the right-hand side; finally, we normalize both sides so the sum of coefficients is 1.

In our case, we add the equation \( \alpha_1 = 1 \) to ensure the linear system has a unique, nontrivial solution. Then, Cramer’s rule yields that the signs of the \( \alpha_i \)'s alternate.
3.1 Some notation

To avoid subscripts we shall use integers to denote points. So, for example, 7 will denote the 7th point of sequence $P$. Points after the 9th are denoted by the letters $A, B, C, \ldots$.

The convex hull operation will be indicated by the concatenation of the corresponding integers. So, for example 27 is the line segment going from the 2nd point to the 7th, and 245 is the triangle with vertices 2, 4 and 5.

We shall use two notations for intersection. First, we use the conventional $179 \cap 028$ to denote the intersection of triangles 179 and 028. Secondly, we denote the same by placing the two objects to be intersected one above the other, like so: $179 \ 028$.

Next, we introduce notation for hyperplane-separation statements. If $p_1, \ldots, p_d, q_1, q_2, \ldots, r_1, r_2, \ldots$ are points in $\mathbb{R}^d$, then we write $p_1 \cdots p_d(q_1 q_2 : r_1 r_2 \cdots)$ to mean that the hyperplane spanned by $p_1, \ldots, p_d$ separates $q_1, q_2, \ldots$, from $r_1, r_2, \ldots$. In other words, orient($p_1, \ldots, p_d, q_i$) has the same value for all $i$, which is the opposite of orient($p_1, \ldots, p_d, r_i$) for all $i$. For instance, $148(2 : 7 \ 37 \ 258)$ means that 7 and $37 \ 258$ are on one side of the hyperplane 148, whereas 2 is on the other. (Of course, in order for $1, 4, 8$ to span a hyperplane and for $37 \ 258$ to be a Radon point, we must be in $\mathbb{R}^3$.)

Given a hyperplane-separation statement $s \overset{\text{def}}{=} H(p : q)$ involving $k$ distinct points, we denote by $\Pi[s]$ the corresponding $k$-ary geometric predicate. For example, in $\mathbb{R}^2$, consider the two statements $s_1 \overset{\text{def}}{=} 25(1 : \frac{14}{36})$ and $s_2 \overset{\text{def}}{=} 37(1 : \frac{15}{49})$. Then $\Pi[s_1] \equiv \Pi[s_2]$; both denote the 6-ary semialgebraic predicate that asserts, given a length-6 orientation-homogeneous sequence ($p_1, \ldots, p_6$), that the line through $p_2$ and $p_5$ separates $p_1$ from $p_1p_4 \cap p_3p_6$; the latter intersection point exists if $p_1, \ldots, p_6$ are orientation-homogeneous by the Radon-partition lemma (Lemma 3.1). (If the input points are not orientation-homogeneous, then we do not care what the predicate asserts.)

**Lemma 3.2.** Let $P$ be orientation-homogeneous. Let $p_1 < p_2 < \cdots < p_d$ be points of $P$, and let $q < q'$ be two other points of $P$. Then, we have $p_1p_2 \cdots p_d(q : q')$ if and only if the number of $p_i$’s between $q$ and $q'$ is odd; otherwise, we have $p_1p_2 \cdots p_d(qq' : )$.

**Proof.** Recall every interchange between two columns of a matrix multiplies the value of its determinant by $-1$. Then the claim follows by counting the number of column interchanges. \[\]

Thus, for example, in $\mathbb{R}^4$ we have $1368(27 : 459)$.

**Observation 3.3** (Same-side rule). If a simplex $\sigma$ lies entirely on one side of a hyperplane $H$, then any intersection $\sigma \cap \tau$ lies on the same side of $H$.

For example, in $\mathbb{R}^2$ we have $14(5 : \frac{13}{25})$, since the segment 13 is entirely on one side of the line through 14, specifically, on the side opposite to 5.
3.2 The six-point lemma

**Observation 3.4.** Let points 1, ..., 6 be orientation-homogeneous in the plane. Let \( x \overset{\text{def}}{=} 25 \cap 36 \), \( y \overset{\text{def}}{=} 14 \cap 36 \), \( z \overset{\text{def}}{=} 14 \cap 25 \). Then the statements 14(3 : x), 25(1 : y), 36(4 : z) are all equivalent. (See Figure 1.)

**Proof.** Suppose 14(3 : x). Then, along the segment 36, the points 3, y, x, 6 lie in this order. Hence, 25(6 : y), which is equivalent to 25(1 : y).

Now suppose 25(1 : y). Therefore, along the segment 14 the order is 1, z, y, 4. Hence, 36(4 : z).

Finally, suppose 36(4 : z), which is equivalent to 36(5 : z). Then, along the segment 25 the order is 2, z, x, 5. Hence, 14(2 : x), which is equivalent to 14(3 : x). \( \square \)

**Lemma 3.5** (Six-point lemma). *The planar predicate*

\[
\Pi_{\text{six}} \overset{\text{def}}{=} \Pi [14(3 : \frac{25}{36})] \quad \left( \equiv \Pi [25(1 : \frac{14}{36})] \equiv \Pi [36(4 : \frac{14}{25})] \right)
\]

*of Observation 3.4* is unavoidable.

**Proof.** Let 1, ..., 7 be orientation-homogeneous. We will show that we cannot have both \( -\Pi_{\text{six}}(1, 2, 3, 4, 5, 6) \) and \( -\Pi_{\text{six}}(2, 3, 4, 5, 6, 7) \).

Let \( x \overset{\text{def}}{=} 14 \cap 25 \), \( y \overset{\text{def}}{=} 25 \cap 47 \). Suppose that \( -\Pi_{\text{six}}(1, \ldots, 6) \), or in other words, that 36( : x5). However, by the same-side rule (Observation 3.3), we have 14( : y5), so the order along the segment...
unavoidable by the same-side rule, while the remaining ones are equivalent to \( \Pi \). And the second predicate is equivalent to \( \Pi \).

\( \sigma \) (assuming the given points are orientation-homogeneous). For example, for the case \( 3213231 \), the corresponding line-separation predicates are the four colorful partitions with \( r \) (so, for example, if \( I \) is assigned the points of \( P \) which is contained in all the triangles corresponding to parts \( r \)).

\( \Pi \) asserts that the segments corresponding to parts 1 and 2 intersect at a point \( x \), which is contained in all the triangles corresponding to parts \( i \geq 3 \).

Let \( 3 \leq i \leq r \). For simplicity assume \( i = 3 \). Then, the restriction of \( \sigma \) to \( \{1, 2, 3\} \) is of the form \( \sigma|_{\{1,2,3\}} = \{2,3\}132\{1,3\} \) (where \( \{a,b\} \) means either \( ab \) or \( ba \)). These are exactly the encodings of the four colorful partitions with \( r = 3 \).

By Lemma 2.1, each corresponding predicate is a conjunction of three line-separation predicates (assuming the given points are orientation-homogeneous). For example, for the case \( \sigma|_{\{1,2,3\}} = 3213231 \), the corresponding line-separation predicates are
\[
\Pi[16( : x4)], \quad \Pi[14( : x6)], \quad \Pi[46( : x1)],
\]
where \( x \equiv 25 \cap 37 \). The first and third predicates hold in any orientation-homogeneous sequence by the same-side rule. And the second predicate is equivalent to \( \Pi_{\text{six}} \), which we showed in Lemma 3.5 to be unavoidable. See Figure 3(b).

Similarly, in the other three possible values for \( \sigma|_{\{1,2,3\}} \), there are one or two predicates that are unavoidable by the same-side rule, while the remaining ones are equivalent to \( \Pi_{\text{six}} \). See Figure 3(a,c,d).

We now proceed to show that, for all planar non-colorful Tverberg partitions, their negation is unavoidable. For this, we first prove a lemma that will be very useful in higher dimensions as well:

**Lemma 3.7** (No-consecutive-points lemma). If \( I = \{I_1, \ldots, I_r\} \) is a Tverberg partition in which some part contains two consecutive integers, then \( \neg \text{Tv}_I \) is unavoidable.

**Proof.** Suppose without loss of generality that \( \{a,a+1\} \subseteq I_1 \). Let \( k \equiv |I_1| \).

Suppose first that \( 2 \leq k \leq d \). Let \( n \equiv T(d,r) + 1 \), and suppose for a contradiction that the orientation-homogeneous point sequence \( P \equiv (p_1, \ldots, p_n) \) avoids \( \neg \text{Tv}_I \). Let
\[
P' = \{p_b : b \in I_1 \land b \leq a\} \cup \{p_{b+1} : b \in I_1 \land b \geq a\}
\]
(so, for example, if \( I_1 = \{1,3,5,6,8,11\} \) and \( a = 5 \), then \( P' = \{p_1, p_3, p_5, p_6, p_7, p_9, p_{12}\} \)). Let \( P'_i \equiv P' \setminus \{p_{a+i}\} \) for \( i = 0,1,2 \). When we evaluate \( \text{Tv}_I \) at the points \( P \setminus \{p_{a+i}\} \) for \( i = 0,1,2 \), part \( I_1 \) is assigned the points of \( P'_i \), whereas the remaining parts are assigned points independently of \( i \).
Figure 3: Case analysis for $d = 2$. In each case, the position with respect to $x$ of the solid edges is given by the same-side rule, whereas the position with respect to $x$ of the dotted edges is given by the six-point lemma.
Let \( f \) be the intersection of the affine hulls of the parts \( I_j, j \geq 2 \); hence, \( f \) is a \((d - k + 1)\)-dimensional flat. The affine hull of \( P' \) intersects \( f \) at a line \( \ell \). However, this line \( \ell \) can intersect the interior of at most two of the convex hulls of \( P'_i, i = 0, 1, 2 \), since they are three distinct faces of the simplex spanned by \( P' \). Contradiction.

If \( k = d + 1 \) we use a different argument: Recall that by Lemma \ref{lem:2.1}, \( T_{v, I} \) reduces to a Boolean combination of \( d + 1 \) predicates of the form \( \Pi(b) \overset{\text{def}}{=} \text{“the intersection point of parts } I_2, \ldots, I_r \text{ lies on the positive side of the hyperplane } I_1 \setminus \{b\},” \) for each \( b \in I_1 \). Hence, \( \Pi(a) \) and \( \Pi(a + 1) \) are equivalent predicates. We can assume that the given point sequence \( P \) is homogeneous with respect to it. But, in order for \( T_{v, I} \) to hold, \( \Pi(a) \) and \( \Pi(a + 1) \) must have opposite values. \( \square \)

**Lemma 3.8.** For every non-colorful Tverberg partition \( I \) in the plane, \( \neg T_{v, I} \) is unavoidable.

**Proof.** If one of the parts in \( I \) has size 1, or two of the parts have size 2 but they do not alternate, then, by the Radon-partition lemma (Lemma \ref{lem:3.1}), \( T_{v, I} \) does not hold in an orientation-homogeneous sequence.

Hence, suppose parts \( I_1 \) and \( I_2 \) alternate, so the encoding of \( I \) has the form \( \ldots 1 \ldots 2 \ldots 1 \ldots 2 \ldots \), partitioning the interval \([T(d, r)]\) into five gaps. Consider a part \( I_j, j \geq 3 \). Assume \( j = 3 \) for simplicity. By the no-consecutive-points lemma (Lemma \ref{lem:3.7}), we can rule out all cases in which two 3’s belong to the same gap. Hence, there are only a few cases left, which can be easily ruled out. See Figure \( 3(e-j) \). \( \square \)

### 3.4 Central projection

We now present a technique for lifting results to higher dimensions. Let \( H \) be a hyperplane in \( \mathbb{R}^d \). To define an orientation predicate within \( H \), fix a rigid motion \( \rho \) in \( \mathbb{R}^d \) that takes \( H \) to the hyperplane \( H_0 \) given by \( x_d = 0 \). Then define \( \text{orient}_H \) by \( \text{orient}_H(p_1, \ldots, p_d) \overset{\text{def}}{=} \text{orient}(\rho(p_1), \ldots, \rho(p_d)) \) (where in the last predicate we ignore the last coordinates of the points, which equal 0).

Fix a point \( q \in \mathbb{R}^d \setminus H \). Each choice of \( \rho \) produces one of two possible predicates \( \text{orient}_H \), which are negations of one another, depending on whether \( \rho \) sends \( q \) above or below \( H_0 \). Let us fix a \( \rho \) that sends \( q \) above \( H_0 \). Call the resulting predicate \( \text{orient}_{H,q} \). It is not hard to see that

\[
\text{orient}_{H,q}(p_1, \ldots, p_d) = \text{orient}(p_1, \ldots, p_d, q).
\] (1)

Given a point set \( X \subset \mathbb{R}^d \) and a point \( p \notin \text{conv} X \), fix a hyperplane \( H \) that separates \( p \) from \( X \). Then we define the central projection of \( X \) from \( p \) into \( H \) by taking each point \( q \in X \) to the intersection point \( q' \overset{\text{def}}{=} pq \cap H \); and we define the orientation within \( H \) by \( \text{orient}_{H,p} \).

**Observation 3.9.** Let \( p_1, \ldots, p_n \in \mathbb{R}^d \) be orientation-homogeneous, and let \( H \) be a hyperplane that separates \( p_1 \) (resp. \( p_n \)) from the rest of the points. Then, centrally-projecting the rest of the points from \( p_1 \) (resp. \( p_n \)) into \( H \) produces an orientation-homogeneous sequence in \( H \).

**Proof.** By (1). \( \square \)
Note that projection from an intermediate point $p_i$ does not produce an orientation-homogeneous sequence, since then the orientation of a $d$-tuple of projected points will depend on the parity of the number of points that appear after $p_i$.

**Observation 3.10.** Let $p_1, \ldots, p_n \in \mathbb{R}^d$, and let $p$ be another point not in $\text{conv}\{p_1, \ldots, p_n\}$. Let their central projection from $p$ into a hyperplane $H$ be $p'_1, \ldots, p'_n$ respectively. Then the central projection of $\text{conv}\{p_1, \ldots, p_n\}$ equals $\text{conv}\{p'_1, \ldots, p'_n\}$.

**Corollary 3.11.** Let $P = (p_0, p_1, \ldots, p_{d+1}) \in (\mathbb{R}^d)^{d+2}$, with $p_0 \notin \text{conv}\{p_1, \ldots, p_{d+1}\}$. Let $P' = (p'_1, \ldots, p'_{d+1})$ be the central projection of $\{p_1, \ldots, p_{d+1}\}$ from $p_0$ into a hyperplane $H$. Then:

- The Radon point of $P'$ is the central projection of the Radon point of $P$.
- Let $[d + 1] = I_1 \cup I_2$ be the Radon partition of $P'$. Then the Radon partition of $P$ is either $I_1 \cup \{0\}, I_2$ or $I_1, I_2 \cup \{0\}$.

Hence, we can lift up lower-dimensional results by adding a new point at the beginning or at the end. For example:

**Lemma 3.12.** In $\mathbb{R}^3$, the predicates

$$\Pi[025(1 : 14)] \quad \text{and} \quad \Pi[257(1 : 147)]$$

are unavoidable.

**Proof.** Recall that the planar predicate $\Pi_\text{six} = \Pi[25(1 : 14)]$ is unavoidable. Hence, the claim about the left predicate follows by projecting centrally from 0, while the claim about the right predicate follows by projecting centrally from 7. (Alternatively, the right predicate is the mirror image of the left one.)

**Corollary 3.13.** The predicate

$$\Pi[147(3 : 258)]$$

is unavoidable.

**Proof.** Let $P = (1, \ldots, 9)$ be a point sequence in $\mathbb{R}^3$ that is orientation-homogeneous, as well as homogeneous with respect to the two predicates in (2). Let us reformulate the left predicate in (2): This predicate asserts that the Radon points $x \overset{\text{def}}{=} 025 \cap 14$ and $y \overset{\text{def}}{=} 036 \cap 14$ lie along the segment 14 in the order $1, x, y, 4$. Hence, the predicate is equivalent to $\Pi[036(1^{025} : )]$. In particular, in $P$ we have $159(3^{148} : )$.

Similarly, the right predicate in (2) is equivalent to $\Pi[147(3 : 257)]$. Hence, in $P$ we have $159(3 : 269)$.

Hence, if we let $u \overset{\text{def}}{=} 148 \cap 37$, $v \overset{\text{def}}{=} 159 \cap 37$, $w \overset{\text{def}}{=} 269 \cap 37$, the order along 37 is $3, u, v, w, 7$. Therefore, $148(3 : w)$. This is an instance of predicate (3).
Note that there are many plane-side predicates in \( \mathbb{R}^3 \) involving a Radon point, which are not covered by the above result, nor are they trivially solved by the same-side rule; for example, \( \Pi[345( : 168)] \) and \( \Pi[368( : 147)] \). Below in Section 3.6 we will solve the latter one (and its mirror image).

3.5 A movement interpretation

We now give a useful way of reinterpreting Lemma 3.12 and Corollary 3.13.

Let \( P \in (\mathbb{R}^d)^n \) be orientation-homogeneous, and let \( a_1, a_2, a_3, a'_1, a'_2, a'_3, b_1, b_2 \in P \), such that \( b_1, b_2 \) interlace both \( a_1, a_2, a_3 \) and \( a'_1, a'_2, a'_3 \), meaning, \( a_1 < b_1 < a_2 < b_2 < a_3 \) and \( a'_1 < b_1 < a'_2 < b_2 < a'_3 \). Define the Radon points

\[
r \defeq \frac{b_1 + b_2}{a_1 a_2 a_3}, \quad r' \defeq \frac{b_1 b_2}{a'_1 a'_2 a'_3}.
\]

Hence, both \( r \) and \( r' \) lie along the segment \( b_1 b_2 \). The question is in which order.

If \( a'_1 \leq a_1 \) and \( a'_2 \geq a_2 \) and \( a'_3 \leq a_3 \), then, by the same-side rule (Observation 3.3), it is immediate that the order is \( b_1, r, r', b_2 \).

However, what happens if there are “conflicting” movements, e.g. if \( a'_1 > a_1 \) and \( a'_2 > a_2 \)? In fact, in such cases both outcomes are possible. However, one of the outcomes is unavoidable. Specifically:

**Lemma 3.14 (Movement lemma).** In the above setting, suppose \( a'_2 > a_2 \). Then the predicate “The order along \( b_1 b_2 \) is \( b_1, r, r', b_2 \)” is unavoidable. In other words, the movement of the point \( a_2 \) is the decisive one.

3.6 The case \( d = 3 \)

Unlike in the planar case, for \( d = 3 \) we have incomplete results.

Every colorful Tverberg partition with \( d = 3 \) is either of the form

\[
\pi_1 \pi_2 2 \pi_3 1 \pi_4,
\]

where \( \pi_1, \pi_4 \) are permutations of \( [r] \setminus \{1\} \) and \( \pi_2, \pi_3 \) are permutations of \( [r] \setminus \{1, 2\} \); or of the form

\[
\pi_1 \pi_2 2 \pi_3 3 \pi_4,
\]

where \( \pi_1 \) is a permutation of \( [r] \setminus \{1\} \), \( \pi_2 \) is a permutation of \( [r] \setminus \{1, 2\} \), \( \pi_3 \) is a permutation of \( [r] \setminus \{2, 3\} \), and \( \pi_4 \) is a permutation of \( [r] \setminus \{3\} \).

In the case (4), part 1 has size 2, part 2 has size 3, and all the other parts have size 4. In the case (5), parts 1, 2, and 3 have size 3, and all the other parts have size 4.

**Parts of sizes 2,3,4.** Let \( P \defeq (1, \ldots, 9, A, B) \). Assume \( P \) is orientation-homogeneous, as well as homogeneous with respect to the predicates covered by the movement lemma (Lemma 3.14). Let \( I_1 \defeq \{4, 8\}, I_2 \defeq \{2, 6, A\} \), and let their Radon point be \( x \defeq 48 \cap 26A \). In order for \( \{I_1, I_2, I_3\} \) to be a colorful Tverberg partition, \( I_3 \) must contain points 5 and 7, as well as one of the points 1, 3 and one
of the points \(9, B\). Hence, there are four colorful Tverberg partitions. Each one decomposes into four plane-separation predicates by Lemma 3.14 out of these four plane-separation predicates, two hold by the same-side rule, while the remaining two hold by the movement lemma. For example, for the case \(I_3 = \{3, 5, 7, B\}\), the corresponding plane-separation predicates are

\[\Pi[357( : xB)], \quad \Pi[35B( : x7)], \quad \Pi[37B( : x5)], \quad \Pi[57B( : x3)].\]

The first and last predicate hold by the same-side rule, while the middle two predicates hold by the movement lemma.

Next, we show that for each non-colorful Tverberg partition with parts of sizes \(\{2, 3, 4\}\), its negation is unavoidable. The Radon-partition lemma takes care of all cases in which the parts of sizes 2 and 3 do not alternate. Further, by the no-consecutive-points lemma (Lemma 3.7), it is enough to consider those cases where \(I_1 \defeq \{4, 8\}\), \(I_2 \defeq \{2, 6, A\}\), and \(I_3 \subset \{1, 3, 5, 7, 9, B\}\). Hence, the number of cases is \(\binom{9}{3} - 4 = 11\). All cases can be solved using either the same-side rule or the movement lemma. As before, let \(x \defeq 48 \cap 26A\).

If \(I_3 = \{1, 3, 5, 7\}\), then by the same-side rule, \(357(x : 1)\). If \(I_3 = \{1, 3, 5, 9\}\), then by the movement lemma, \(159(x : 3)\). If \(I_3 = \{1, 3, 7, B\}\), then, by the movement lemma, \(15B(x : 3)\). If \(I_3 = \{1, 3, 7, 9\}\), then, by the movement lemma, \(379(x : 1)\). If \(I_3 = \{1, 3, 9, B\}\), then, by the same-side rule, \(139(x : B)\). The remaining five cases are the mirror images of the first five.

**Parts of sizes 3,3,3.** We start by handling the four colorful Tverberg partitions of this form.

**Lemma 3.15.** The following four colorful Tverberg partitions in \(\mathbb{R}^3\) are unavoidable:

\[
\{1, 4, 7\}, \{2, 5, 8\}, \{3, 6, 9\}, \quad \{2, 4, 7\}, \{1, 5, 8\}, \{3, 6, 9\},
\]

\[
\{1, 4, 7\}, \{2, 5, 9\}, \{3, 6, 8\}, \quad \{2, 4, 7\}, \{1, 5, 9\}, \{3, 6, 8\}.
\]

In addition, the following two plane-side predicates are unavoidable:

\[
\Pi[368(4 : \frac{147}{25})], \quad \Pi[136(5 : \frac{258}{17})].
\] (6)

**Proof.** Consider the first partition. Denote \(T_1 \defeq 147, T_2 \defeq 258, T_3 \defeq 369\). Let \(w \defeq 147 \cap 25\), \(x \defeq 47 \cap 258, y \defeq 147 \cap 36, z \defeq 47 \cap 369\). Hence, \(T_1 \cap T_2 = wx\), and \(T_1 \cap T_3 = yz\). Let us see where the points \(w, x, y, z\) lie within \(T_1\).

The points \(x\) and \(z\) lie on the segment 47. Furthermore, by the movement lemma (Lemma 3.14), their order is \(4, x, z, 7\). The point \(w\) lies somewhere in the interior of \(T_1\). Where does \(y\) lie with respect to \(w\)? By the same-side rule, we have \(245(y7 : )\). Since the plane 245 intersects \(T_1\) along the line \(4w\), it follows that within \(T_1\) we have \(4w(7y : )\). Furthermore, by the movement lemma, we

---

\(^3\)It also takes care of all cases in which a part has size 1.
have $258(4y:17)$, so within $T_1$ we have $wx(4y:17)$. Hence, the situation is as in Figure 4(a), so the segments $wx$ and $yz$ indeed intersect.

We further see from Figure 4(a) that the plane through $T_3$ separates $w$ from 4. Hence, $369(4:147\frac{17}{25})$. Renaming the 9 to an 8 we obtain the first predicate in (6). The second predicate in (6) is its mirror image. The other partitions are handled in a similar way; see Figure 4(b–d).

Out of the 280 Tverberg partitions of $\{1,\ldots,9\}$ into three parts of size 3, a computer enumeration shows that there are 17 in which all three pairs of triangles intersect. Four of them are the colorful ones listed above in Lemma 3.15. Of the 13 remaining ones, six are solved by the no-consecutive-points lemma (Lemma 3.7). The seven remaining partitions are

$$\{1,4,7\}, \{2,5,8\}, \{3,6,9\}, \quad \{1,4,8\}, \{2,6,9\}, \{3,5,7\},$$

$$\{1,4,9\}, \{2,5,7\}, \{3,6,8\}, \quad \{1,4,9\}, \{2,6,8\}, \{3,5,7\},$$

and their mirror images. The avid reader is invited to try to solve them.

**Parts of sizes 3,3,3,4.** A computer enumeration shows that there are 144 colorful Tverberg partitions with sizes 3,3,3,4. By Lemma 2.1 each one is a conjunction of four triple-point plane-side predicates of the form “The intersection $abc\cap def \cap ghi$ is on such a side of the plane $xyz$”. The total number of distinct such plane-side predicates is 336 (according to our computer program).
Many of these predicates can be proven unavoidable by a straightforward use of the same-side rule. Consider, for example, the predicate $47A(5 : x)$ for $x \overset{\text{def}}{=} 159 \cap 26C \cap 38B$. The intersection of the triangles $159$ and $38B$ equals the segment $yz$, where $y \overset{\text{def}}{=} 159 \cap 38$ and $z \overset{\text{def}}{=} 59 \cap 38B$. By the same-side rule we have $47A(5 : y)$, and by the movement lemma, $47A(5 : z)$. Therefore, since $x$ lies along the segment $yz$, we have, again by the same-side rule, $47A(5 : x)$.

There are predicates that do not yield to such simple analysis; for example, $48B(1 : x)$ for $x \overset{\text{def}}{=} 16A \cap 259 \cap 37C$. One predicate is highly symmetric, so it can be solved with a trick similar to the one for $\Pi_{\text{six}}$:

**Lemma 3.16.** The predicate $\Pi[48C(5 : x)]$ for $x \overset{\text{def}}{=} 159 \cap 26A \cap 37B$ is unavoidable.

**Proof.** Let $P = (1, \ldots, 9, A, \ldots, D)$. Define the triangles

$$
T_1 \overset{\text{def}}{=} 159, \quad T_2 \overset{\text{def}}{=} 26A, \quad T_3 \overset{\text{def}}{=} 37B,
T_4 \overset{\text{def}}{=} 48C, \quad T_5 \overset{\text{def}}{=} 59D.
$$

Hence, $x = T_1 \cap T_2 \cap T_3$. Define the triple-intersection points

$$
w \overset{\text{def}}{=} T_2 \cap T_3 \cap T_4, \quad y \overset{\text{def}}{=} T_2 \cap T_3 \cap T_5.
$$

Define the intersection points

$$a \overset{\text{def}}{=} T_2 \cap 37, \quad b \overset{\text{def}}{=} T_3 \cap 6A.
$$

So the intersection of the triangles $T_2$ and $T_3$ equals the segment $ab$.

The points $x, w, y$ all lie within the segment $ab$. But in which order do they lie?

**Claim 1.** Along $ab$ we have the order $a, x, y, b$.

**Proof.** By the movement lemma we have $T_1(D : a)$. Furthermore, by the same-side rule, we have $T_1(: yD)$. Since $T_1$ passes through $x$, the claim follows. \qed

Now, suppose for a contradiction that $P$ avoids $\Pi[48C(5 : x)]$. This means that $T_4(x : A)$, as well as $T_5(w : B)$—here is where the symmetry of the predicate comes into play.

Now, $T_4(x : A)$ implies that, along the segment $ab$, the order is $a, w, x, b$. (Proof: By the movement lemma, we have $T_4(b : A)$, so $T_4(bx : )$. But $T_4$ passes through $w$.)

From the orders $a, w, x, b$ and $a, x, y, b$ follows the order $a, w, y, b$.

However, $T_5(w : B)$ implies the opposite order $a, y, w, b!$ (Proof: By the movement lemma, we have $T_5(b : B)$, so $T_5(wb : )$. But $T_5$ passes through $y$.)

This contradiction concludes the proof. \qed
$$x = 158 \cap 37A$$
$$y_1 = 158 \cap 269$$
$$y_2 = 158 \cap 469$$
$$y_3 = 158 \cap 047$$

Figure 5: Proof of Lemma 3.17. Point $y_3$ must lie in one of the regions I, II, III; more specifically, in region I.

### 3.7 The case $d = 4$

In this section we prove that the zigzag Tverberg partition for $d = 4$, $r = 3$ is unavoidable. Then it follows immediately that the same is true for all $r \geq 3$. Recall that this partition is encoded by 12321232123, and it is given by $I_1 \overset{\text{def}}{=} \{1, 5, 9\}$, $I_2 \overset{\text{def}}{=} \{2, 4, 6, 8, A\}$, $I_3 \overset{\text{def}}{=} \{3, 7, B\}$. Let $x \overset{\text{def}}{=} 159 \cap 37B$ be the Radon point of parts $I_1$ and $I_3$.

By Lemma 2.1, in order to show that the simplex spanned by $I_2$ contains $x$, we have to show that the following five hyperplane-side statements are unavoidable:

- $2468(Ax :)$
- $246A(8x :)$
- $248A(6x :)$
- $268A(4x :)$
- $468A(2x :)$

By symmetry, we only have to deal with the first three statements. The first statement follows immediately from the same-side rule.

**Lemma 3.17.** The predicate $\Pi[(246A(8\overset{\text{def}}{=}159}{37B} :)]$ is unavoidable.

**Proof.** Rewrite the predicate as $\Pi[(246A(7\overset{\text{def}}{=}159}{37B} :)]$ and then as $\Pi[(2469(7 : x :)]$ where $x \overset{\text{def}}{=} 158 \cap 37A$.

Suppose for a contradiction that $P \overset{\text{def}}{=} (0, 1, \ldots, 9, A, B)$ is orientation-homogeneous and satisfies $2469(7 : x)$. Let us look at the relative position of several points inside the triangle $T \overset{\text{def}}{=} 158$. The hyperplane $H_1 \overset{\text{def}}{=} 2469$ intersects $T$ along the line passing through $y_1 \overset{\text{def}}{=} 158 \cap 269$ and $y_2 \overset{\text{def}}{=} 158 \cap 469$. The assumption $H_1(7 : x)$ implies that, within $T$,

$$y_1y_2(15x : 8).$$

Now consider the hyperplane $H_2 \overset{\text{def}}{=} 137A$. It intersects $T$ along the line $1x$. Now, by the movement lemma, the predicate $\Pi[37A(269\overset{\text{def}}{=}158}{469} 5 : 8)]$ in $\mathbb{R}^3$ is unavoidable. Hence, by central projection from 1, in $\mathbb{R}^4$ the predicate

$$\Pi[137A(269\overset{\text{def}}{=}158}{469} 5 : 8)] \equiv \Pi[H_2(y_1y_25 : 8)]$$
is unavoidable. Let us assume $P$ avoids its negation. Hence, within $T$,

$$1x(y_1y_25 : 8).$$

Next, consider the hyperplane $H_3 \overset{\text{def}}{=} 2689$, which intersects $T$ along the line $8y_1$. By the same-side rule we have $H_3(5y_2 : 1)$. Therefore, within $T$ we have

$$8y_1(5y_2 : 1).$$

By (7), (8), and (9), the position of $x$, $y_1$, and $y_2$ within $T$ must be as in Figure 5. From the figure we see that, within $T$, $5x(y_1 : 1)$. Hence, $P$ satisfies the hyperplane-separation statement

$$s \overset{\text{def}}{=} 357A(1 : \frac{158}{269}).$$

Now, assume $P$ is homogeneous with respect to the predicate $\Pi[\cdot]$. Then, in particular, $P$ satisfies $2469(0 : y_3)$ for $y_3 \overset{\text{def}}{=} 047 \cap 158$. Hence, within $T$ we have

$$y_1y_2(8y_3 : 15).$$

We will now try to locate $y_3$ more precisely within $T$.

By the same-side rule, we have $137A(8 : 5_{158})$. Therefore, within $T$, we have

$$1x(8 : 5y_3).$$

By (10) and (11) it follows that $y_3$ lies in one of the regions labeled I, II, III in Figure 5.

**Claim 1.** There exists a line through $y_1$ that separates $y_3$ from 1 and 8.

**Proof.** By the movement lemma and central projection from 0, the predicate $\Pi[0269(0_{158}^{047} : 18)]$ is unavoidable. Let us assume $P$ avoids its negation. Hence, letting $y_4 \overset{\text{def}}{=} 158 \cap 026$, within $T$ we have $y_1y_4(5y_3 : 18)$. 

This rules out region II for $y_3$.

**Claim 2.** There exists a line through $y_3$ that separates $y_1$ from 1 and 8.

**Proof.** By the movement lemma and central projection from 9, the predicate $\Pi[0479(269_{158}^{18} : 18)]$ is unavoidable. So let us assume $P$ avoids its negation. Let $y_5 \overset{\text{def}}{=} 479 \cap 158$. Therefore, within $T$ we have $y_3y_5(18 : 5y_1)$.

This rules out region III for $y_3$. Hence, $y_3$ lies in region I, which implies that $y_1$ lies inside the triangle $15y_3$.

Now, by the same-side rule, we have $457A(1y_3 : )$. Therefore, again by the same-side rule (since 5, 1, and $y_3$ are all on the same side of $457A$, and since $y_1 \in 15y_3$),

$$457A(1y_1 : ).$$

This, however, is an instance of $-\Pi[\cdot]$. Contradiction.
Lemma 3.18. The predicate $\Pi[248A(6,159,37,6:B):]$ is unavoidable.

Proof. Rewrite the predicate as $\Pi[248A(5,159,37,6:B):]$ and then as $\Pi[2479(5x:)1]$ where $x \overset{\text{def}}{=} 158 \cap 36A$.

As before, let $P \overset{\text{def}}{=} (1, \ldots, 9, A)$ be orientation-homogeneous, and let us look at the relative position of several points within the triangle $T \overset{\text{def}}{=} 158$. Let $z_1 \overset{\text{def}}{=} 158 \cap 479$. By the same side rule, we have $356A(8z_1:1)$. Hence, within $T$ we have $5x(8z_1:1)$. Furthermore, by the movement lemma and central projection from 1, we have $136A(8\overset{\text{def}}{=}158\cap479:5)$. Therefore, within $T$ we have $1x(8z_1:5)$. See Figure 6.

Now, suppose for a contradiction that $2479(x:15)$. Then, within $T$ we would have $z_1z_2(x:15)$ for $z_2 \overset{\text{def}}{=} 158 \cap 279$; in other words, $x$ would be separated from 1 and 5 by a line through $z_1$. But, as we see from Figure 6, this is impossible.

4 The stretched grid and stair-convexity

In this section we recall the definition of the *stretched diagonal*, and we prove that it is homogeneous with respect to all Tverberg partitions and moreover, that the Tverberg partitions that occur in it are precisely the colorful ones. This constitutes the proof of Theorem 1.2.

The stretched diagonal is a subset of a more general construction called the *stretched grid*. The stretched grid yields our result regarding Sierksma’s conjecture mentioned in the Introduction. Hence, we start by describing the stretched grid, and then we go on to the stretched diagonal. The *stretched grid*, previously introduced in [BMN11, BN12, Niv09], is an axis-parallel grid of points where, in each direction $i$, $2 \leq i \leq d$, the spacing between consecutive “layers” increases rapidly, and furthermore, the rate of increase for direction $i$ is much larger than that for direction $i-1$. To simplify calculations, we also make the coordinates increase rapidly in the first direction.

---

\[ x = 158 \cap 36A \]
\[ z_1 = 158 \cap 479 \]

Figure 6: Proof of Lemma 3.18. As we can see, no line through $z_1$ can separate $x$ from 1 and 5.
The definition is as follows: Given \( n \), the desired number of points, let \( m \overset{\text{def}}{=} n^{1/d} \) be the side of the grid (assume for simplicity that this quantity is an integer), and let

\[
G_s \overset{\text{def}}{=} \{(K_1^{a_1}, K_2^{a_2}, \ldots, K_d^{a_d}) : a_i \in \{0, \ldots, m - 1\} \text{ for all } 1 \leq i \leq d\},
\]

for some appropriately chosen constants \( 1 < K_1 \ll K_2 \ll K_3 \ll \cdots \ll K_d \). Each constant \( K_i \) must be chosen appropriately large in terms of \( K_{i-1} \) and in terms of \( m \). Specifically:

\[
K_1 = 2; \quad K_i \geq 2d^2 K_{i-1}^m \quad \text{for } 2 \leq i \leq m.
\] (13)

We refer to the \( d \)-th coordinate as the “height”, so we call a hyperplane in \( \mathbb{R}^d \) horizontal if all its points have the same last coordinate; and we call a line in \( \mathbb{R}^d \) vertical if all its points share the first \( d - 1 \) coordinates. A vertical projection onto \( \mathbb{R}^{d-1} \) is obtained by removing the last coordinate. The \( i \)-th horizontal layer of \( G_s \) is the subset of \( G_s \) obtained by letting \( a_d = i \) in (12).

The following lemma provides the motivation for the stretched grid:

**Lemma 4.1.** Let \( a \in G_s \) be a point at horizontal layer 0, and let \( b \in G_s \) be a point at horizontal layer \( i \). Let \( c \) be the point of intersection between segment \( ab \) and the horizontal hyperplane containing layer \( i - 1 \). Then \( |c_j - a_j| \leq 1/d^2 \) for every \( 1 \leq j \leq d - 1 \).

Lemma 4.1 follows from a simple calculation (we chose the constants \( K_i \) in (13) large enough to make this and later calculations work out). The grid \( G_s \) is hard to visualize, so we apply to it a logarithmic mapping \( \pi \) that converts \( G_s \) into the uniform grid in the unit cube. Formally, let \( B \overset{\text{def}}{=} [1, K_1^{m-1}] \times \cdots \times [1, K_d^{m-1}] \) be the bounding box of the stretched grid, let \([0, 1]^d\) be the unit cube in \( \mathbb{R}^d \), and define the mapping \( \pi : B \rightarrow [0, 1]^d \) by

\[
\pi(x) \overset{\text{def}}{=} \left( \frac{\log K_1 x_1}{m - 1}, \ldots, \frac{\log K_d x_d}{m - 1} \right).
\]

Then, it is clear that \( \pi(G_s) \) is the uniform grid in \([0, 1]^d\).

Lemma 4.1 implies that the map \( \pi \) transforms straight-line segments into curves composed of almost-straight axis-parallel parts: Let \( s \) be a straight-line segment connecting two points of \( G_s \). Then \( \pi(s) \) ascends almost vertically from the lower endpoint, almost reaching the height of the higher endpoint, before moving significantly in any other direction; from there, it proceeds by induction. See Figure 7.

This observation motivates the notions of *stair-convexity*, which describe, in a sense, the limit behavior of \( \pi \) as \( m \rightarrow \infty \).

### 4.1 Stair-convexity

Given a pair of points \( a, b \in \mathbb{R}^d \), the *stair-path* \( \sigma(a, b) \) between them is a polygonal path connecting \( a \) and \( b \) and consisting of at most \( d \) closed line segments, each parallel to one of the coordinate axes. The definition goes by induction on \( d \); for \( d = 1 \), \( \sigma(a, b) \) is simply the segment \( ab \). For \( d \geq 2 \), after
possibly interchanging \( a \) and \( b \), let us assume \( a_d \leq b_d \). We set \( a' \overset{\text{def}}{=} (a_1, \ldots, a_{d-1}, b_d) \), and we let \( \sigma(a, b) \) be the union of the segment \( ab' \) and the stair-path \( \sigma(a', b) \); for the latter we use the recursive definition, ignoring the common last coordinate of \( a' \) and \( b \). Note that, if \( c \) and \( d \) are points along \( \sigma(a, b) \), then \( \sigma(c, d) \) coincides with the portion of \( \sigma(a, b) \) that lies between \( c \) and \( d \).

We call a set \( S \subseteq \mathbb{R}^d \) stair-convex if for every \( a, b \in S \) we have \( \sigma(a, b) \subseteq S \). For a real number \( y \) let \( h(y) \) denote the “horizontal” hyperplane \( \{x \in \mathbb{R}^d : x_d = y\} \). For a horizontal hyperplane \( h \overset{\text{def}}{=} h(y) \), let \( h^+ \overset{\text{def}}{=} \{x \in \mathbb{R}^d : x_d \geq y\} \) be the upper closed half-space bounded by \( h \), and let \( h^- \) be the lower closed half-space. For a set \( S \subseteq \mathbb{R}^d \) let \( S(y) \overset{\text{def}}{=} S \cap h(y) \) be the horizontal slice of \( S \) at height \( y \).

For a point \( x \overset{\text{def}}{=} (x_1, \ldots, x_d) \in \mathbb{R}^d \), let \( \overline{x} \overset{\text{def}}{=} (x_1, \ldots, x_{d-1}) \) be the projection of \( x \) into \( \mathbb{R}^{d-1} \), and define \( \overline{S} \) for \( S \subset \mathbb{R}^d \) similarly. For a point \( x \in \mathbb{R}^{d-1} \) and a real number \( x_d \), let \( x \times x_d \overset{\text{def}}{=} (x_1, \ldots, x_{d-1}, x_d) \), with a slight abuse of notation.

**Lemma 4.2** ([BMN11]). A set \( S \subseteq \mathbb{R}^d \) is stair-convex if and only if the following two conditions hold:

1. Every horizontal slice \( S(y) \) is stair-convex.

2. For every \( y_1 \leq y_2 \leq y_3 \) such that \( S(y_3) \neq \emptyset \) we have \( \overline{S(y_1)} \subseteq \overline{S(y_2)} \) (meaning, the horizontal slice can only grow with increasing height, except that it can end by disappearing abruptly).

Since the intersection of stair-convex sets is obviously stair-convex, we can define the stair-convex hull \( \text{stconv}(S) \) of a set \( S \subseteq \mathbb{R}^d \) as the intersection of all stair-convex sets containing \( S \).

**Lemma 4.3** ([BMN11]). The stair-convex hull can be characterized by induction on \( d \) in the following way: Let \( S \subseteq \mathbb{R}^d \), and let \( T \overset{\text{def}}{=} \text{stconv}(S) \). Then for every \( y \in \mathbb{R} \), if \( h(y)^+ \cap S = \emptyset \), then \( T(y) = \emptyset \); otherwise, we have \( T(y) = \text{stconv}(S \cap h^-) \). (In other words, the horizontal slice of \( T \) at height \( y \) is obtained inductively by taking the stair-convex hull in dimension \( d-1 \) of all points not above height \( y \)—unless \( h(y) \) is strictly above all of \( S \), in which case the slice is empty.)
Corollary 4.4 (Axis-parallel closedness). Let $S \subseteq \mathbb{R}^d$ be a finite point set, let $x \in \text{stconv}(S)$, and let $1 \leq i \leq d$. Let $p$ be the point of $S$ with largest $i$-coordinate that satisfies $p_i \leq x_i$, and let $q$ be the point of $S$ with smallest $i$-coordinate that satisfies $q_i \geq x_i$. Then, if we replace the $i$-th coordinate of $x$ by any real number $p_i \leq t \leq q_i$, the new point will still belong to $\text{stconv}(S)$.

Let $a \in \mathbb{R}^d$ be a fixed point, and let $b \in \mathbb{R}^d$ be another point. We say that $b$ has type 0 with respect to $a$ if $b_i \leq a_i$ for every $1 \leq i \leq d$. For $1 \leq j \leq d$ we say that $b$ has type $j$ with respect to $a$ if $b_j \geq a_j$ but $b_i \leq a_i$ for every $i$ satisfying $j + 1 \leq i \leq d$. (It might happen that $b$ has more than one type with respect to $a$, but only if some of the above inequalities are equalities.)

The following lemma is the stair-convex analogue of Carathéodory’s theorem:

Lemma 4.5 ([BMN11]). Let $S \subseteq \mathbb{R}^d$ be a point set, and let $x \in \mathbb{R}^d$ be a point. Then $x \in \text{stconv}(S)$ if and only if $S$ contains a point of type $j$ with respect to $x$ for every $j = 0, 1, \ldots, d$.

The following is a simple claim on regular convexity and its stair-convex analogue:

Lemma 4.6 ([N93]).

1. Let $p \in \mathbb{R}^d$ be a point contained in $\text{conv}(Q)$ for some $Q \subseteq \mathbb{R}^d$. Then there exists a $k \leq d + 1$ and there exist points $q_1, \ldots, q_k \in Q$ and $r_1, \ldots, r_k \in \mathbb{R}^d$ such that $r_1 = q_1$, $r_k = p$, and for every $2 \leq i \leq k$ the point $r_i$ lies in the segment $r_{i-1}q_i$. (In other words, we can get to $p$ by starting at $q_1$ and “walking” towards $q_2, q_3, \ldots, q_k$ in succession.)

2. Let $p \in \mathbb{R}^d$ be a point contained in $\text{stconv}(Q)$ for some $Q \subseteq \mathbb{R}^d$. Then there exists a $k \leq d + 1$ and there exist points $q_1, \ldots, q_k \in Q$ and $r_1, \ldots, r_k \in \mathbb{R}^d$ such that $r_1 = q_1$, $r_k = p$, and for every $2 \leq i \leq k$ we have $r_i \in \sigma(r_{i-1}, q_i)$.

Lemma 4.7 ([BMN11]). Let $S$ be a $k$-point set in $\mathbb{R}^d$ for some $k \leq d + 1$, and let $p$ be a point in $\text{stconv}(S)$. Then $p$ shares at least $d + 1 - k$ coordinates with $S$.

Definition 4.8. A set $S \subseteq \mathbb{R}^d$ is said to be in stair-general position if $p_i \neq q_i$ for every two distinct points $p, q \in S$ and every $1 \leq i \leq d$.

The multipartite Kirchberger theorem ([ABB+09], [P98]) states that if $P_1, \ldots, P_r \subseteq \mathbb{R}^d$ are $r$ point sets of total size $\sum |P_i| \geq T(d, r)$, such that their convex hulls intersect at a common point $x$, then there exist subsets $P'_1 \subseteq P_1, \ldots, P'_r \subseteq P_r$ of total size $\sum |P'_i| = T(d, r)$, whose convex hulls still intersect at a common point (not necessarily $x$).\footnote{Proof sketch: Suppose the total number of points is larger than $T(d, r)$ and that the points within each $P_i$ are affinely independent. Then the affine hulls of the $P_i$’s intersect at a flat $f$ of dimension at least 1. Starting at $x$, let us move within $f$ in a straight line, until we first hit the boundary of some $\text{conv} P_i$. This allows us to remove one point from that $P_i$.}

The following lemma is the stair-convex analogue of the multipartite Kirchberger theorem. It generalizes Lemma 5.4 of [BMN11] from 2 parts to $r$.\footnote{Proof sketch: Suppose the total number of points is larger than $T(d, r)$ and that the points within each $P_i$ are affinely independent. Then the affine hulls of the $P_i$’s intersect at a flat $f$ of dimension at least 1. Starting at $x$, let us move within $f$ in a straight line, until we first hit the boundary of some $\text{conv} P_i$. This allows us to remove one point from that $P_i$.}
Lemma 4.9. Let \( P \subset \mathbb{R}^d \) be an \( n \)-point set in stair-general position, and let \( P_1, \ldots, P_r \) be a partition of \( P \) into \( r \) parts. Let \( X = \text{stconv}(P_1) \cap \cdots \cap \text{stconv}(P_r) \). Then:

(a) If \( n < T(d,r) \) then \( X = \emptyset \).

(b) If \( n = T(d,r) \) and \( X \neq \emptyset \), then \( X \) contains a single point. Furthermore, each of the \( r \) highest points of \( P \) belongs to a different part \( P_i \).

(c) If \( n \geq T(d,r) \) and \( X \neq \emptyset \), then there exist subsets \( Q_i \subset P_i \) of total size \( \sum |Q_i| = T(d,r) \) such that \( \bigcap_{i} \text{stconv}(Q_i) \neq \emptyset \).

Proof. Suppose there exists a point \( x \in X \). By Lemma 4.7 \( x \) shares at least \( c \equiv \sum (d+1-|P_i|) = d + (T(d,r) - n) \) coordinates with the points of \( P \). Hence, \( n < T(d,r) \) would imply \( c \geq d+1 \), meaning \( x \) shares the same coordinate with two different points of \( P \), a contradiction. This proves part (a).

Now suppose \( n = T(d,r) \). Hence, \( x \) shares all \( d \) coordinates with the points of \( P \), and the same is true of every other point of \( X \). Hence, if there were another point \( y \in X \), then we would have \( \sigma(x,y) \subseteq X \), which leads to a contradiction since \( \sigma(x,y) \) contains infinitely many points.

Now let \( \{p_1, \ldots, p_r\} \) be the \( r \) highest points of \( P \). Suppose for a contradiction that two of them, say \( p_1 \) and \( p_2 \), belong to the same part \( P_i \), and hence none of them belong to \( P_j \) for some \( j \neq i \). Then \( x \) is not higher than the highest point of \( P_j \), which is lower than \( p_1 \) and \( p_2 \). Therefore, we could remove one of them, say \( p_1 \), from \( P \), and by Lemma 4.3 its stair-convex hull would still contain \( x \). This would contradict part (a). Hence, we have proven part (b).

We now prove part (c) by induction on \( d \). Suppose \( n \geq T(d,r) \). If \( d = 1 \) then each \( \text{stconv}(P_i) \) is an interval on the real line. Let \( y \) be the rightmost point of \( X \). Then \( y \in P_i \) for some \( i \), so we can take one point from \( P_i \) and the two extremal points of every other \( P_j \), for a total of \( T(1,r) = 2r - 1 \) points.

Now suppose \( d \geq 2 \). Let \( h \equiv h(x_d) \) be the horizontal hyperplane containing \( x \), and let \( P_i^\perp \equiv P_i \cap h^\perp \) for each \( i \). By Lemma 4.3 we have \( \pi \in \text{stconv}(P_i^\perp) \) for each \( i \). Hence, by induction, we can choose subsets \( Q_i^\perp \subseteq P_i^\perp \), of total size \( T(d-1,r) \), such that \( \bigcap \text{stconv}(Q_i^\perp) \) is not empty. Let \( y \) be a point in this intersection (we do not necessarily have \( y = \pi \)).

Let \( q \) be the highest point of \( \bigcup Q_i^\perp \), and say \( q \in Q_1^\perp \) for simplicity. Let \( z \equiv y \times q_d \), so by Lemma 4.3 we have \( z \in \text{stconv}(Q_1^\perp) \). For each \( 2 \leq i \leq r \), let \( Q_i^\perp \equiv Q_i^\perp \cup \{p_i\} \), where \( p_i \) is the highest point of \( P_i \). Since \( p_i \) is not lower than \( x \) and \( x \) is not lower than \( z \), it follows again by Lemma 4.3 that \( z \in \text{stconv}Q_i \). Hence, the subsets \( Q_1^\perp, Q_2, Q_3, \ldots, Q_r \) are the desired subsets of \( P_1, \ldots, P_r \), since their total size is \( T(d-1,r) + (r-1) = T(d,r) \).

Definition 4.10. If \( P \subset \mathbb{R}^d \) has size \( |P| = T(d,r) \) and is in stair-general position, then a stair-Tverberg partition of \( P \) is one that satisfies \( \bigcap_{i=1}^r \text{stconv}(P_i) \neq \emptyset \). The unique point in this intersection is called the stair-Tverberg point of this partition.

It turns out that in stair-convex geometry Siersma’s conjecture is true, and furthermore, all the \((r-1)!^d\) Tverberg partitions have the same Tverberg point:
Lemma 4.11. Let $P \subseteq \mathbb{R}^d$ be a point set of size $|P| = T(d, r)$ in stair-general position. Let $p_1, p_2, \ldots, p_{T(d, r)}$ be the points of $P$ listed by decreasing last coordinate. Let $Q \equiv P \setminus \{p_1, \ldots, p_{r-1}\}$. Then:

(a) Each stair-Tverberg partition of $P$ is obtained inductively as follows: Let $Q_1, \ldots, Q_r$ be a partition of $Q$ such that $Q_1^c, \ldots, Q_r^c$ is a stair-Tverberg partition of $Q$. Then arbitrarily assign the points $p_1, \ldots, p_{r-1}$ one to each of the $r-1$ parts that do not contain $p_r$.

(b) The stair-Tverberg point of all the above partitions is $x \equiv y \times p_r$ where $y$ is the stair-Tverberg point of $Q$.

Proof. First, let $P_1, \ldots, P_r$ be a stair-Tverberg partition of $P$, and let $x$ be the corresponding stair-Tverberg point. By Lemma 4.9, each of the points $p_1, \ldots, p_r$ belongs to a different part $P_i$. Say for simplicity that $p_i \in P_i$ for each $i$.

As pointed out in the proof of Lemma 4.9, $x$ shares each of its $d$ coordinates with some point of $P$. Therefore, it must be that $x_d = p_{rd}$. Let $Q_i \equiv P_i \setminus \{p_i\}$ for $1 \leq i \leq r-1$, and $Q_r \equiv P_r$. Hence, $Q_1, \ldots, Q_r$ is a partition of $Q$. Further, by Lemma 4.3, we have $x \in \operatorname{stconv}(Q_i)$ for all $i$, as desired.

Now let $Q_1, \ldots, Q_r$ be a partition of $Q$ such that $Q_1^c, \ldots, Q_r^c$ is a stair-Tverberg partition of $Q$ with stair-Tverberg point $y$. Say $p_r \in Q_r$ for simplicity. Let $x \equiv y \times p_{rd}$. Then $x \in Q_r$ by Lemma 4.3. Arbitrarily assign the points $p_1, \ldots, p_{r-1}$ one to each of the sets $Q_1, \ldots, Q_{r-1}$, obtaining sets $P_1, \ldots, P_{r-1}$. Then, again by Lemma 4.3, $x \in P_i$ for each $1 \leq i \leq r-1$. Hence, $P_1, \ldots, P_{r-1}, Q_r$ is a stair-Tverberg partition of $P$ and $x$ is its stair-Tverberg point.

Corollary 4.12. Let $P$ be a $T(d, r)$-point set in $\mathbb{R}^d$ in stair-general position. Then, $P$ has exactly $(r-1)!^d$ stair-Tverberg partitions.

Proof. By induction on $d$. The case $d = 1$ is straightforward. For $d \geq 2$, by Lemma 4.11, the number of stair-Tverberg partitions of $P$ equals $(r-1)!$ times the number of stair-Tverberg partitions of $Q$ for the $Q$ mentioned in the lemma.

Remark 4.13. By Corollary 4.4, if $P$ is in stair-degenerate position then the number of partitions of $P$ with intersecting stair-convex hulls can only increase.

4.2 A transference lemma

Let $p, q \in B$ be two points inside the bounding box of the stretched grid (not necessarily grid points). For $1 \leq i \leq d$, we say that the stretched distance between $p$ and $q$ in direction $i$ is $c$ if $p_i = K_i^c q_i$ or $q_i = K_i^c p_i$, or, in other words, if $|\pi(q)_i - \pi(p)_i| = c/(m-1)$.

If the stretched distance between $p$ and $q$ in direction $i$ is at most $c$, then we say that $p$ and $q$ are $c$-close in direction $i$. If this distance is at least $c$ then we say that $p$ and $q$ are $c$-far apart in direction $i$. If $p$ and $q$ are $c$-close in every direction $1, \ldots, d$ then we say that they are $c$-close. If they are $c$-far apart in every direction $1, \ldots, d$ then we say that they are $c$-far apart.
Lemma 4.14 (Transference lemma). Let $P \subset B$ be a finite point set such that every two points of $P$ are $(2d + 3)$-far apart. Let $P_1, \ldots, P_r$ be a partition of $P$ into $r$ parts. Then $\bigcap \text{conv}(P_i) \neq \emptyset$ if and only if $\bigcap \text{stconv}(P_i) \neq \emptyset$.

The lemma is intuitively obvious, given that stair-convexity is the limit behavior of regular convexity in the stretched grid under $\pi$. The reason we need the points of $P$ to be far apart enough from each other is to avoid the “rounded” parts of the $\pi(\text{conv}(P_i))$’s—the parts in which the correspondence between convexity and stair-convexity breaks down.

Unfortunately, the proof of the lemma is quite tedious. We relegate it to Appendix A.

4.3 Our results on the stretched grid

This is our result regarding Sierksma’s conjecture:

Theorem 4.15. Let $P$ be any set of $T(d, r)$ points in $B$ such that every two points of $P$ are $(2d + 3)$-far apart. Then $P$ has exactly $(r - 1)!d$ Tverberg partitions.

In particular, a randomly chosen set of $T(d, r)$ points from the stretched grid will satisfy the condition of the theorem with probability tending to 1 as $n \to \infty$.

Proof of Theorem 4.15. By the transference lemma and Corollary 4.12.

Definition 4.16. The stretched diagonal is the sequence of points obtained by taking $a_1 = a_2 = \cdots = a_d = (2d + 3)j$ for $j = 0, 1, 2, \ldots, (m - 1)/(2d + 3)$ in (12).

Lemma 4.17. The stretched diagonal is homogeneous with respect to all Tverberg partitions; moreover, the Tverberg partitions that occur in the stretched diagonal are exactly the colorful ones.

Proof. By the transference lemma and Lemma 4.11, since the stretched diagonal is monotonic with respect to all coordinates simultaneously.

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A Proof of the transference lemma (Lemma 4.14)

We start with some simple claims about axis-parallel boxes.

Recall that \([d] \overset{\text{def}}{=} \{1, \ldots, d\}\). Let \(C \subset \mathbb{R}^d\) be an axis-parallel box \(C \overset{\text{def}}{=} \{x \in \mathbb{R}^d \mid a_i \leq x_i \leq b_i \text{ for } i \in [d]\}\). For each \(i \in [d]\), call the facet of \(C\) that satisfies \(x_i = a_i\) facet \(-i\), and the facet of \(C\) that satisfies \(x_i = b_i\) facet \(i\).

**Lemma A.1.** A \(k\)-flat that intersects the interior of \(C\) must intersect a facet \(\pm i\) for at least \(k\) distinct indices \(i \in [d]\).

**Proof.** If \(k = 0\) there is nothing to prove. If \(k = 1\) the claim is simple, since a line must enter \(C\) through some facet.

For \(k \geq 2\) we proceed by induction on \(d\). Let \(f\) be a \(k\)-flat, \(k \geq 2\), that contains point \(p\) the interior of \(C\). Let \(I\) be the set of indices \(i \in [d]\) such that \(f\) does not intersect the facets \(\pm i\). We want to show that \(|I| \leq d - k\). As pointed out above, we certainly have \(I \neq [d]\). Hence, let \(i \notin I\). Let \(h\) be the hyperplane given by \(x_i = p_i\). Then \(C' \overset{\text{def}}{=} C \cap h\) is a \((d - 1)\)-dimensional axis-parallel box, whose \(d - 1\) pairs of facets can be labeled \(\pm j\) for \(j \in [d] \setminus \{i\}\) in the natural way. Further, \(f' \overset{\text{def}}{=} f \cap h\) is a \((k - 1)\)-dimensional flat, which does not intersect any of the facets \(\pm j, j \in I\). Hence, by induction on \(d\) we have \(|I| \leq (d - 1) - (k - 1) = d - k\).  

**Definition A.2.** A \(k\)-flat \(f\) \((0 \leq k \leq d)\) is said to be \(I\)-oriented with respect to \(C\), for \(I \subseteq [d]\) of size \(|I| = d - k\), if \(f\) intersects the interior of \(C\) but does not intersect any of the facets \(\pm i, i \in I\).

**Lemma A.3.** If the \(k\)-flat \(f\) is \(I\)-oriented with respect to \(C\), then \(f\) intersects the interior of every facet \(\pm i, i \notin I\).

**Proof.** If \(k = 0\) there is nothing to prove. If \(k = 1\) the claim is simple, since \(f\) is a line, which must enter and exit \(C\) through two distinct facets, and there are only two facets left.

For \(k \geq 2\) we proceed by induction on \(d\). Let \(p\) be a point of \(f\) lying in the interior of \(C\). Pick an index \(i \in [d] \setminus I\), and let \(h\) be the hyperplane given by \(x_i = p_i\). As before, let \(C' \overset{\text{def}}{=} C \cap h\), and label its facets \(\pm j\) for \(j \in [d] \setminus \{i\}\) in the natural way. Then \(f' \overset{\text{def}}{=} f \cap h\) intersects the interior of \(C'\), but
does not intersect any of its facets $\pm j$, $j \in I$. The dimension of $f'$ is either $k - 1$ or $k$, but it cannot be $k$ by Lemma A.1, so it is $k - 1$. Hence, $f'$ is $I$-oriented with respect to $C'$. Hence, by induction, $f'$ intersects the interior of all the facets of $C'$ labeled $\pm j$ for $j \notin I \cup \{i\}$. Therefore, $f$ intersects the interior of the equally-named facets of $C$.

To prove that $f$ intersects the interior of the facets $\pm i$, repeat the above argument with a different index $j \in [d] \setminus I$. Such a $j \neq i$ is guaranteed to exist since $k \geq 2$. \hfill \qed

**Lemma A.4.** Let $I_1, \ldots, I_m$ be $m$ pairwise-disjoint subsets of $[d]$ whose union equals $[d]$. Let $f_1, \ldots, f_m$ be flats that are $I_1$-, $\ldots$, $I_m$-oriented with respect to $C$, respectively. Then $f_1 \cap \cdots \cap f_m$ contains a single point, which lies in the interior of $C$.

**Proof.** Without loss of generality assume there is no $j$ for which $I_j = \emptyset$, since that implies $f_j = \mathbb{R}^d$.

We first prove the claim for the special case where each $I_j$ has size 1 (so the flats $f_j$ are hyperplanes and $m = d$). In this case we proceed by induction on $d$. Say for simplicity that $I_i = \{i\}$ for each $1 \leq i \leq d$.

For each $a_1 < z < b_1$ let $h(z)$ be the hyperplane satisfying $x_1 = z$. Consider the $(d-1)$-dimensional axis-parallel box $C'(z) = C \cap h(z)$. Let $i \geq 2$. Since $f_i$ intersects facets $\pm 1$ of $C$, by convexity it intersects $C'(z)$. Hence $f_i'(z) = f_i \cap C'(z)$ is a $(d-2)$-flat (a hyperplane within $h(z)$) that intersects the interior of $C'(z)$ but avoids its facets $\pm i$ (since they are contained in the equally-named facets of $C$). Hence, $f_i'(z)$ is $\{i\}$-oriented with respect to $C'(z)$.

Therefore, by induction on $d$, the hyperplanes $f_2'(z), \ldots, f_d'(z)$ intersect at a point $p(z)$ in the interior of $C'(z)$. The points $p(z)$, $a_1 < z < b_1$, form a line segment. This line segment goes from one side of $f_1$ to the other one, so it must intersect $f_1$. This proves the special case.

The general case can be reduced to the above special case by applying the following claim:

**Claim 1.** Let $f$ be a $k$-flat that is $I$-oriented with respect to $C$. Then $f$ can be written as the intersection of $d - k$ hyperplanes, each of which is $\{i\}$-oriented with respect to $C$ for a different $i \in I$.

**Proof.** Let $e_1, \ldots, e_d$ be the standard unit vectors in $\mathbb{R}^d$.

We first show that that if $g$ is a $J$-oriented $\ell$-flat and $j \in J$, then $g + \mathbb{R}e_j \overset{\text{def}}{=} \{x + \alpha e_j \mid x \in g, \alpha \in \mathbb{R}\}$ (the extrusion of $g$ in the direction $e_j$) is a $(J \setminus \{j\})$-oriented $(\ell + 1)$-flat. Indeed, suppose for a contradiction that $g + \mathbb{R}e_j$ intersects a facet $\pm i$ in $J \setminus \{j\}$. That means that there is some point $y$ in $g$ (but outside of $C$) such that $y + \alpha e_j$ falls on facet $\pm i$. Let $q$ be a point of $g$ in the interior of $C$. Then the segment $qy$, which lies in $g$, intersects one of the facets $\pm j$—contradiction.

Now let us come back to the given $k$-flat $f$. For each $i \in I$, let $h_i$ be the hyperplane obtained by extruding $f$ in all directions $e_j$, $j \in I \setminus \{i\}$. By the above argument, $h_i$ is $\{i\}$-oriented with respect to $C$. Hence, these are the desired $d - k$ hyperplanes. \hfill \qed

This concludes the proof of Lemma A.4. \hfill \qed

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A.1 Back to the stretched grid

The following lemma is the main motivation behind the stretched grid. It says that, under $\pi$, straight-line segments become very close to stair-paths.

**Lemma A.5.** Let $a, b \in B$. Then every point of the line segment $ab$ is $1$-close to some point of the stair-path $\sigma(a, b)$ and vice versa.

*Proof sketch.* Here we give a sketch of the proof. The full proof can be found in [Niv09]. Say without loss of generality that $a_d \leq b_d$. Let $c$ be the lowest point of the segment $ab$ that is $1$-close to $b$ in direction $d$. Let $c'$ be the point directly above $a$ at the same height as $b$. As mentioned above (Lemma 4.1), for each $1 \leq i \leq d - 1$ we have $|c_i - c'_i| \leq 1/d^2$.

Hence, let us split the segment $ab$ into segments $ac$ and $cb$, and let us split the stair-path $\sigma(a, b)$ into the vertical segment $ac'$ and the $(d - 1)$-dimensional stair-path $\sigma(c', b)$. Then every point of $ac$ is $1$-close to a point of $ac'$ and vice versa, with plenty of room to spare in the first $d - 1$ coordinates. And every point of $cb$ is $1$-close in direction $d$ to a point of $\sigma(c', b)$. For directions $1, \ldots, d - 1$ we would like to argue by induction on $d$. The problem is that $\tau$ and $\tau'$ do not exactly coincide.

The way to solve this problem is to prove by induction a stronger claim: If $a'$ is very close to $a$ and $b'$ is very close to $b$, then every point of the line segment $ab$ is $1$-close to some point of the stair-path $\sigma(a', b')$ and vice versa. We omit the details which are technical and not very hard. \hfill $\square$

**Corollary A.6.** Let $P \subseteq B$ be a point set. Then every point of $\text{conv}(P)$ is $d$-close to a point of $\text{stconv}(P)$ and vice versa.

*Proof.* By Lemma 4.6 applying Lemma A.5 $k - 1$ times for $k \leq d + 1$. \hfill $\square$

We finally get to the transference lemma:

*Proof of Lemma A.14.* By the multipartite Kirchberger theorem and its stair-convex analogue, it is enough to focus on the case $|P| = T(d, r)$, so let us assume this is the case.

For the first direction, suppose $P_1, \ldots, P_r$ is a stair-Tverberg partition of $P$, and let $p$ be its stair-Tverberg point. For each $i$, let $k_i \overset{\text{def}}{=} |P_i|$, and let $I_i \subseteq [d]$ be the set of coordinates that $p$ shares with $P_i$. By the proof of Lemma 4.9, $|I_i| = d + 1 - k_i$, and the sets $I_i$ form a partition of $[d]$. Let $f_i$ be the $(k_i - 1)$-flat satisfying the equations $x_i = p_i$ for $i \in I_i$ (so $f_i$ is “tangent” to stconv($P_i$) at point $p$).

Let $C$ be an axis-parallel box containing $p$ such that each facet of $C$ is at stretched distance $d + 1$ from $p$. Hence, each $f_i$ is $I_i$-oriented with respect to $C$.

**Claim 1.** We have $\text{stconv}(P_i) \cap C = f_i \cap C$ for each $i$.

*Proof.* Suppose for a contradiction that $\text{stconv}(P_i) \cap C$ contains a point $x \notin f_i$. Then $x$ shares with $P_i$ a *different* subset $I'_i \subset [d]$ of $d + 1 - k_i$ coordinates. Hence, $I'_i$ intersects some $I_j$, $j \neq i$. Say $\ell \in I'_i \cap I_j$. Since both $p$ and $x$ are contained in $C$, this means that a point of $P_i$ is too close to a point of $P_j$ in coordinate $\ell$—contradiction.

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Figure 8: We have \( p \in \text{stconv}(P_i) \). Suppose for a contradiction that there exists a point \( q \in \text{conv}(P_i) \) at a facet \( j \) of \( C \) (here \( j = +1 \)). Then there exists a point \( z \in \text{stconv}(P_i) \) not too far away. But then the point \( w \), which lies on face \( j \) and is aligned with \( p \), also belongs to \( \text{stconv}(P_i) \).

Similarly, suppose for a contradiction that \( \text{stconv}(P) \cap C \) is a strict subset of \( f_i \cap C \). Then the relative boundary of \( \text{stconv}(P_i) \cap f_i \) contains some point \( x \in C \). By axis-parallel closedness (Corollary 4.4), \( x \) shares with \( P_i \) an additional coordinate not in \( I_i \), yielding a similar contradiction.

We now examine how \( \text{conv}(P_i) \) intersects \( C \). We first note that \( \text{conv}(P_i) \cap C \) is not empty, by Corollary A.6 and by our choice of the size of \( C \).

By an argument similar to the one above, we have \( \text{conv}(P_i) \cap C = \text{ahull}(P_i) \cap C \), where \( \text{ahull} \) denotes the affine hull. Indeed, otherwise \( C \) would intersect a facet of \( \text{conv}(P_i) \), given by \( \text{conv}(P_i') \) for some strict subset \( P_i' \subseteq P_i \). Then, by Corollary A.6, \( \text{stconv}(P_i') \) would contain a point \( q \) that is \((2d + 1)\)-close to \( p \). This point \( q \) shares one more coordinate with \( P_i \) than \( p \) does, leading to a contradiction as before.

Finally, we claim that \( \text{ahull}(P_i) \) is \( I_i \)-oriented with respect to \( C \), just like \( f_i \). Indeed, suppose for a contradiction that \( \text{ahull}(P_i) \) intersects a facet \( j \) for \( \pm j \in I_i \), at a point \( q \). By Corollary A.6, there exists a point \( z \in \text{stconv}(P_i) \) that is \( d \)-close to \( q \); in particular, \( z_j \neq p_j \). Let \( w \) be the point on facet \( j \) of \( C \) satisfying \( w_\ell = x_\ell \) for all \( \ell \neq j \). Then, by axis-parallel closedness of \( \text{stconv}(P_i) \) (repeated application of Corollary 4.4), we have \( w \in \text{stconv}(P_i) \), contradicting the fact that \( \text{stconv}(P_i) \) is \( I_i \)-oriented. See Figure 8.

Hence, Lemma A.4 applies, so \( \bigcap \text{conv}(P_i) \) contains a point in the interior of \( C \), as desired.

For the second direction, suppose that \( P_1, \ldots, P_r \) is a Tverberg partition of \( P \), and let \( p \) be its Tverberg point. Let \( C \) be an axis-parallel box containing \( p \) such that each facet of \( C \) is at stretched distance \( d + 1 \) from \( p \). By Corollary A.6, for each \( 1 \leq i \leq r \), there is a point \( y_i \in \text{stconv}(P_i) \) in the interior of \( C \). By Lemma 4.7, \( y_i \) shares a set \( I_i \subseteq [d] \) of coordinates with \( P_i \), where \( |I_i| \geq d + 1 - k_i \). However, since every two points of \( P \) are \((2d + 3)\)-far apart, the sets \( I_i \) must be pairwise disjoint. Further, \( \sum(d + 1 - k_i) = d \), so \( |I_i| = d + 1 - k_i \), and the sets \( I_i \) form a partition of \([d]\).

As before, for each \( i \) we have \( \text{stconv}(P_i) \cap C = f_i \cap C \) for some axis-parallel \((k_i - 1)\)-flat \( f_i \) that
is $I_i$-oriented with respect to $C$. Hence, Lemma A.4 applies\footnote{Actually, Lemma A.4 is overkill in this case.} so $\bigcap \text{stconv}(P_i)$ contains a point in the interior of $C$, as desired. \hfill \Box