Multiplication Groups of Abelian Torsion-Free Groups of Finite Rank

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Abstract. For an Abelian group $G$, any homomorphism $\mu : G \otimes G \to G$ is called a multiplication on $G$. The set $\text{Mult}_G$ of all multiplications on an Abelian group $G$ itself is an Abelian group with respect to addition; the group is called the multiplication group of $G$. Let $A_0$ be the class of all reduced block-rigid almost completely decomposable groups of ring type with cyclic regulator quotient. In this paper, for groups $G \in A_0$, we describe groups $\text{Mult}_G$. We prove that for $G \in A_0$, the group $\text{Mult}_G$ also belongs to the class $A_0$. For any group $G \in A_0$, we describe the rank, the regulator, the regulator index, invariants of near-isomorphism, a main decomposition, and a standard representation of the group $\text{Mult}_G$.

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1. Introduction

For an Abelian group $G$, a multiplication on $G$ is a homomorphism $\mu : G \otimes G \to G$. The set $\text{Mult}_G$ of all multiplications on the group $G$ itself is an Abelian group with respect to addition; the group is called the multiplication group of $G$ or the group of multiplications on $G$ [11]. An Abelian group $G$ with multiplication on $G$ is called a ring on the group $G$. The problem of studying the relationship between the structure of an Abelian group and the properties of ring structures on it is very multifaceted and has a long history in algebra; see [1], [2], [8], [9], [12], [13], [15], [16].

In this paper, we consider only additively written Abelian groups and “a group” means “an Abelian group” in what follows.

In this paper, we study the group $\text{Mult}_G$ for an almost completely decomposable Abelian group $G$. A torsion-free group $G$ of finite rank is called an almost completely decomposable group (ACD-group) if $G$ contains completely decomposable subgroup of finite index. ACD-groups were studied in
Any $ACD$-group $G$ contains a special uniquely defined completely decomposable (see [11]) subgroup $\text{Reg} G$ of finite index which is a fully invariant subgroup of $G$; it is called the regulator of the group $G$. The regulator of an $ACD$-group can be defined as the intersection of all its completely decomposable subgroups of lowest index [6]. The factor group $G/\text{Reg} G$ is called the regulator quotient of the group $G$; the index of the subgroup $\text{Reg} G$ in the group $G$ is called a regulator index. It is denoted by $n(G)$. $ACD$-groups with cyclic regulator quotient are often called $CRQ$-groups.

Let $G$ be an almost completely decomposable group. Then the regulator quotient of the group $G$ can be uniquely, up to isomorphism, represented as a direct sum of torsion-free groups of rank 1 [10, Proposition 86.1]. For every type $\tau$, we denote by $\text{Reg}_\tau G$ the sum of summands of rank 1 and type $\tau$ in this decomposition of the group $\text{Reg} G$. The set of types

$$T(G) = T(\text{Reg} G) = \{ \tau \mid \text{Reg}_\tau G \neq 0 \}$$

is called the set of critical types of groups $G$ and $\text{Reg} G$. If $T(G)$ consists of pairwise incomparable types, then the groups $G$ and $\text{Reg} G$ are called block-rigid groups. If, in addition, for any $\tau \in T(G)$, the group $\text{Reg}_\tau G$ is of rank 1, then $G$ and $\text{Reg} G$ are called rigid groups. If all types in $T(G)$ are idempotent types, then $G$ is called a group of ring type.

We note that a block-rigid $ACD$-group is either divisible or reduced. For a divisible torsion-free group $G$, the group $\text{Mult} G$ is described in [10, Sect. 121]; therefore, we consider only reduced groups in what follows.

We denote by $\mathcal{A}_0$ the class of all reduced block-rigid $CRQ$-groups of ring type. In Sect. 2, we describe the group $\text{Mult} G$ for $G \in \mathcal{A}_0$ (Theorem 2.8). The aim of Sect. 3 is, for groups $G$ in the class $\mathcal{A}_0$, to study properties of $\text{Mult} G$. It is proved (Theorem 3.4) that if $G$ is a block-rigid $CRQ$-group of ring type, then $\text{Mult} G$ also is a block-rigid $CRQ$-group of ring type. We describe the rank, the regulator, the regulator index, invariants of near-isomorphism, a main decomposition and a standard representation of the group $\text{Mult} G$ for $G \in \mathcal{A}_0$.

The multiplication $\mu : G \otimes G \to G$ is often denoted by the symbol $\times$, i.e.,

$$\mu(g_1 \otimes g_2) = g_1 \times g_2$$

for all $g_1, g_2 \in G$. The multiplication $\times$ on the group $G$ induces a ring on this group which is denoted by $(G, \times)$. Let $G$ be a group and $g \in G$. The characteristic and the order of the element $g$ are denoted by $\chi(g)$ and $o(g)$, respectively. The rank and the divisible hull of the group $G$ are denoted by $r(G)$ and $\overline{G}$, respectively. If $S \subseteq G$, then $|S|$ is the cardinality of the set $S$ and $\langle S \rangle$ is the subgroup of the group $G$ generated by the set $S$. We write an element of a group direct product $\prod_{i \in I} G_i$ in the form $(g_i)_{i \in I}$, where $g_i \in G_i$. If $I_1 \subseteq I$, then for simplicity, we identify the subgroup $\{(g_i)_{i \in I} \mid g_i = 0 \text{ for all } i \notin I_1 \}$ of the group $\prod_{i \in I} G_i$ with the group $\prod_{i \in I_1} G_i$; we write elements of this group in the form $(g_i)_{i \in I_1}$.
As usual, $\mathbb{N}$ and $\mathbb{P}$ are the sets of positive integers and all prime integers, respectively, $\mathbb{Z}$ is the group (the ring) of integers, $\mathbb{Q}$ is the group (the field) of rational numbers. If $R$ is a unital ring, then $Re$ is the cyclic module over $R$ generated by the element $e$. If $S$ is a finite subset in $\mathbb{Z}$, then $\gcd(S)$ is the greatest common divisor of all integers in $S$ and $\text{lcm}(S)$ is the least common multiple of the integers in $S$. If $P_1 \subseteq \mathbb{P}$, then $P_1$-integer is an integer such that any prime divisor of it (if it exists) is contained in $P_1$. It follows from the definition that 1 is a $P_1$-integer for any $P_1 \subseteq \mathbb{P}$. For any type $\tau$, we set

$$P_\infty(\tau) = \{p \in \mathbb{P} \mid \tau(p) = \infty\}, \quad P_0(\tau) = \mathbb{P} \setminus P_\infty(\tau).$$

Unless otherwise stated, we use notation and definitions from [10,11] and [19].

2. Multiplication Groups of Block-Rigid CRQ-Groups of Ring Type

All over this section, $G$ is a reduced block-rigid CRQ-group of ring type with regulator $A$, regulator quotient $G/A = \langle d + A \rangle$ where $d \in G$, regulator index $n$ and set of critical types $T(G) = T(A)$.

By setting $\text{Reg}_\tau G = A_\tau$, we can represent the group $A$ in the form $A = \bigoplus_{\tau \in T(G)} A_\tau$. According to [21, Proposition 2.4.11], such decomposition of the completely decomposable group $A$ is unique if and only if $A$ is a block-rigid group. For divisible hulls $\tilde{G}$, $\tilde{A}$, $\tilde{A}_\tau$ of the groups $G$, $A$ and $A_\tau$, respectively, we have relations

$$\tilde{G} = \tilde{A} = \bigoplus_{\tau \in T(G)} \tilde{A}_\tau.$$

For $\tau \in T(G)$, we denote by $\pi_\tau$ the natural projection from the group $\tilde{G}$ onto $\tilde{A}_\tau$.

In [7], positive integers $m_\tau = m_\tau(G) \ (\tau \in T(G))$ are defined; these integers are invariants of near-isomorphism of the group $G$. We can define integers $m_\tau \ (\tau \in T(G))$ as follows; we take an element $d \in G/A$ such that $\langle d + A \rangle = G/A$. Let $d_\tau = \pi_\tau(d) \in \tilde{A}_\tau$ and let $m_\tau = o(d_\tau + A)$ be the order of the element $d_\tau + A$ in the torsion group $\tilde{A}/A$. In [7], it is shown that integers $m_\tau \ (\tau \in T(G))$ do not depend on the choice of the element $d$. We note that $n = o(d + A) = \text{lcm}\{m_\tau \mid \tau \in T(G)\}$.

Remark 2.1. Let $T$ be a finite set of pair-wise incomparable types and let $\{m_\tau \mid \tau \in T\}$ be some set of positive integers. We say that the set $\{m_\tau \mid \tau \in T\}$ satisfies condition $(m)$ if for any $p \in \mathbb{P}$, $k \in \mathbb{N}$, $\tau \in T$, we have that $p^k$ divides $m_\sigma$ for some $\sigma \in T \setminus \{\tau\}$ provided $p^k$ divides $m_\tau$. We note that the set $\{m_\tau \mid \tau \in T\}$ satisfies condition $(m)$ if and only if the set $\{m_\tau \mid \tau \in T, \ m_\tau > 1\}$ satisfies condition $(m)$.

According to [21, Theorem 13.1.2], the set $\{m_\tau \mid \tau \in T\}$ is a system of invariants of a near-isomorphism of some block-rigid CRQ-group $G$ with $T(G) = T$ if and only if this set satisfies condition $(m)$ and $m_\tau$ are $P_0(\tau)$-integers for all $\tau \in T$. $\triangleright$
In [5, Theorem 3.5], it is proved that for any of the group \( G \in A_0 \), there exists a direct decomposition
\[
G = G_1 \oplus C,
\]
where \( C \) is a completely decomposable group and \( G_1 \) is a rigid CRQ-group which satisfies the following conditions:
\[
\tau \in T(G_1) \text{ if and only if } m_\tau(G) > 1, \quad (1')
\]
\[
m_\tau(G_1) = m_\tau(G) \text{ for all } \tau \in T(G_1). \quad (1'')
\]
Decomposition (1), which satisfies conditions (1') and (1''), is called a main decomposition of the group \( G \). In a main decomposition of the group \( G \), the group \( G \) does not contain a completely decomposable summand; such groups are said to be clipped. We note that a main decomposition of a CRQ-group is not uniquely defined, since it depends on the choice of the element \( d \) participating in the definition of the group. In what follows, we assume that a main decomposition of the group \( G \) is fixed. We set \( T_0(G) = \{ \tau \in T(G) \mid m_\tau > 1 \} \). Then \( T_0(G) \) is the set of critical types of a clipped direct summand in any main decomposition of the group \( G \).

Let \( B \) be the regulator of the group \( G_1 \), then \( T(G_1) = T(B) = T_0(G) \) and \( \tilde{G}_1 = \tilde{B} \). There exists a system \( E_0 = \{ e^{(\tau)}_0 \in B_\tau \mid \tau \in T(B) \} \) such that
\[
B = \bigoplus_{\tau \in T(B)} R_\tau e^{(\tau)}_0. \quad (2)
\]
In (2), we assume that \( R_\tau \) is a unitary subring in \( \mathbb{Q} \), the type of the additive group of \( R_\tau \) is equal to \( \tau \), characteristics \( \chi(e^{(\tau)}_0) \in \tau \) contain only zeros and symbols \( \infty \) \( (\tau \in T(B)) \).

Let \( D = \{ d \in G_1 \mid G/A = \langle d + A \rangle \} \), it is easy to see that \( D \neq \emptyset \).

Let \( d \in D \). In the group \( \tilde{B} \), the element \( d \) can be represented in the form
\[
d = \sum_{\tau \in T(B)} \frac{s_\tau}{r_\tau} e^{(\tau)}_0, \quad \text{where } s_\tau \in \mathbb{Z}, r_\tau \in \mathbb{N}, \gcd(s_\tau, r_\tau) = 1. \]

Without loss of generality, we can assume that \( s_\tau, r_\tau \) are \( P_0(\tau) \)-integers (otherwise, we can replace the system \( E_0 \)).

Let \( \tau \in T(B) \). By the definition of the integer \( m_\tau \), the relation
\[
o \left( \frac{s_\tau}{r_\tau} e^{(\tau)}_0 + A \right) = m_\tau \text{ holds in the group } \tilde{A}/A. \]

Since \( r_\tau \) is a \( P_0(\tau) \)-integer and \( \gcd(s_\tau, r_\tau) = 1 \), we have \( r_\tau = m_\tau \). Consequently, the element \( d \) of \( \tilde{B} \) is of the form
\[
\sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} e^{(\tau)}_0, \quad (3)
\]
and the integers \( n, m_\tau \) and \( s_\tau \) satisfy the following conditions:
\[
n = \text{lcm}\{ m_\tau \mid \tau \in T(B) \}, \quad (3')
\]
\[
\gcd(s_\tau, m_\tau) = 1 \text{ for all } \tau \in T(B), \quad (3'')
\]
\[
s_\tau \text{ and } m_\tau \text{ are } P_0(\tau) \text{ numbers for any } \tau \in T(B). \quad (3''')
\]
A system \( E_0 = \{ e^{(\tau)}_0 \in B_\tau \mid \tau \in T(B) \} \) which satisfies conditions (2) and (3), is called a \( B \)-basis of the group \( G \) defined by the element \( d \). We note that the pair \( (d, E_0) \) uniquely defines the numbers \( s_\tau \ (\tau \in T(B)) \). Relation
(3) is called a **standard representation** of block-rigid CRQ-group $G$ related to the pair $(d, E_0)$.

**Remark 2.2.** We note that a $B$-basis $E_0$ can be defined by more than one element $d \in D$.

Indeed, let we have a standard representation (3) of the group $G$. Let $\gamma$ be an integer which is co-prime with the regulator index $G$. Then $G/A = \langle d + A \rangle = \langle d_1 + A \rangle$, i.e., $d_1 \in D$. In addition,

$$d_1 = \sum_{\tau \in T(B)} \frac{\gamma s_{\tau}}{m_{\tau}} e_0^{(\tau)}. \quad (4)$$

If $\gamma$ is a $P_0(\tau)$-integer, then the relation (4) is a standard representation of the group $G$. Consequently, the $B$-basis $E_0$ is defined by each of elements $d$ and $d_1$.

We note that if $\gamma$ is not a $P_0(\tau)$-integer, then the relation (4) is not a standard representation of the group $G$. ▷

For a $B$-basis $E_0$, we set

$$D(E_0) = \{ d \in D \mid B$-basis $E_0$ is determined by the element $d \}. $$

It follows from the definition of the $B$-basis that $D(E_0) \neq \emptyset$ and it follows from Remark 2.2 that $D(E_0)$ can contain more than one element.

With the right choice of elements $e_i^{(\tau)} \in C_\tau$ ($i = 0, 1, \ldots, k_\tau$), the group $C$ can be written in the form

$$C = \bigoplus_{\tau \in T(C)} C_\tau = \bigoplus_{\tau \in T(C)} \bigoplus_{i=1,\ldots,k_\tau} R_\tau e_i^{(\tau)},$$

where $R_\tau$ is a unitary subring in the field of rational numbers, the type of the additive group of $R_\tau$ is equal to $\tau$, and characteristics $\chi(e_i^{(\tau)}) \in \tau$ contain only zeros and symbols $\infty$.

For $\tau \in T(G)$, we define the following sets:

$$I_\tau(B) = \begin{cases} \{0\}, & \text{for } \tau \in T(B) \\ \emptyset, & \text{for } \tau \notin T(B), \end{cases}$$

$$I_\tau(C) = \begin{cases} \{1, \ldots, k_\tau\}, & \text{for } \tau \in T(C), \ k_\tau \in \mathbb{N} \\ \emptyset, & \text{for } \tau \notin T(C), \end{cases}$$

$$I_\tau = I_\tau(B) \cup I_\tau(C).$$

Then $A_\tau = \bigoplus_{i \in I_\tau} R_\tau e_i^{(\tau)}$ for any $\tau \in T(G)$ and

$$A = \bigoplus_{\tau \in T(G)} \bigoplus_{i \in I_\tau} R_\tau e_i^{(\tau)}. \quad (5)$$

A system $E = \{e_i^{(\tau)} \in A_\tau \mid \tau \in T(G), i \in I_\tau\}$ is called an $A$-basis of the group $G$ if $E$ satisfies (5) and the subsystem $E_0 = \{e_0^{(\tau)} \in A_\tau \mid \tau \in T(B)\}$ of $E$ is a $B$-basis.

Let $(G, \times)$ be a ring on the group $G \in A_0$. Since $A$ is a fully invariant subgroup of the group $G$, we have that $A$ is an ideal of the ring $(G, \times)$ which is a direct sum of ideals $A_\tau$ ($\tau \in T(G)$). Thus, every multiplication on $G$
induces a multiplication on $A$; therefore, $\text{Mult} \, G \subseteq \text{Mult} \, A$; however, the converse is not true.

Let $E = \{ e_i^{(\tau)} \in A_\tau \mid \tau \in T(G), i \in I_\tau \}$ be an $A$-basis of the group $G$. Then for any set $\{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), i, j \in I_\tau \}$, there exists a unique ring $(A, \times)$ such that $e_i^{(\tau)} \times e_j^{(\tau)} = u_{ij}^{(\tau)}$ for all $\tau \in T(G)$ and $i, j \in I_\tau$. The multiplication $\times$ is uniquely extended to a multiplication on $\tilde{A} = \tilde{G}$, where it is defined as follows:

$$\sum_{i \in I_\tau} r_i e_i^{(\tau)} \times \sum_{i \in I_\tau} r_i' e_i^{(\tau)} = \sum_{i, j \in I_\tau} r_i r_j' (e_i^{(\tau)} \times e_j^{(\tau)})$$

(6)

for all $\tau \in T(G)$, $r_i, r_j' \in \mathbb{Q}$; and $\tilde{A} \times \tilde{A}_\sigma = 0$ for $\tau \neq \sigma$. However, $G$ is not necessarily a subring of the ring $(\tilde{A}, \times)$. We say that the set $\{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), i, j \in I_\tau \}$ defines a multiplication on $G$ with respect to the $A$-basis $E$ if there exists a ring $(G, \times)$ such that $e_i^{(\tau)} \times e_j^{(\tau)} = u_{ij}^{(\tau)}$ for all $\tau \in T(G)$ and $i, j \in I_\tau$. We note that any set $\{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), i, j \in I_\tau \}$ defines multiplication on group $G$ with respect to the $A$-basis $E$ at most one way.

Let $G \in A_0$. To describe sets defining multiplications on the group $G$, we define the following groups.

For any $\tau \in T(G)$, let $n_\tau = |I_\tau|$ and let $M_\tau^{(0)} = M_{n_\tau} (A_\tau)$ be the additive group of square matrices of order $n_\tau$ with elements in $A_\tau$,

$$M_\tau^{(1)} = \begin{bmatrix} m_\tau A_\tau & m_\tau A_\tau & \ldots & m_\tau A_\tau \\ m_\tau A_\tau & A_\tau & \ldots & A_\tau \\ \vdots & \vdots & \ddots & \vdots \\ m_\tau A_\tau & A_\tau & \ldots & A_\tau \end{bmatrix} \subseteq M_\tau^{(1)}.$$  

where the symbol $[\ldots]$ means the set of matrices of certain form, $m_\tau$ are invariants of near-isomorphism of the group $G$,

$$M_\tau^{(2)} = \begin{bmatrix} m_\tau^2 A_\tau & m_\tau A_\tau & \ldots & m_\tau A_\tau \\ m_\tau A_\tau & A_\tau & \ldots & A_\tau \\ \vdots & \vdots & \ddots & \vdots \\ m_\tau A_\tau & A_\tau & \ldots & A_\tau \end{bmatrix} \subseteq M_\tau^{(1)}.$$  

We set

$$M^{(0)} = \prod_{\tau \in T(G)} M^{(0)}_\tau, \quad M^{(1)} = \prod_{\tau \in T(G)} M^{(1)}_\tau, \quad M^{(2)} = \prod_{\tau \in T(G)} M^{(2)}_\tau.$$  

Then $M^{(2)} \subseteq M^{(1)} \subseteq M^{(0)}$ and $M^{(2)} \cong M^{(1)} \cong M^{(0)} \cong \text{Mult} \, A$.

For the standard representation (3) of the group $G$ related to the pair $(d, E_0)$ and for every $\tau \in T(G)$, we consider elements

$$X^{(\tau)} = X^{(\tau)} (d, E_0) = \begin{pmatrix} m_\tau s_\tau^{-1} e_0^{(\tau)} & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \in M^{(1)}_\tau, \text{ if } \tau \in T(B),$$  

where \( s_\tau^{-1} \) is an integer which is inverse to \( s_\tau \) modulo \( m_\tau \),

\[
X^{(\tau)} = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 0 
\end{pmatrix} \in M_1^{(1)}, \text{ if } \tau \notin T(B).
\]

We set

\[
X = X(d, E_0) = \left( X^{(\tau)} \right)_{\tau \in T(G)} = \left( X^{(\tau)} \right)_{\tau \in T(B)} \in M^{(1)}.
\]

\[
M(d, E_0) = \langle X, M^{(2)} \rangle \subseteq M^{(1)}.
\]

We note that integral solutions of the congruence \( s_\tau x \equiv 1 \pmod{m_\tau} \) form the residue class modulo \( m_\tau \). Therefore, the set \( M(d, E_0) \) does not depend on the choice of the integers \( s_\tau^{-1} \) in the definition of \( X \).

We also note that if \( \tau \in T(C) \setminus T(B) \), then \( m_\tau = 1 \) by (1'). Therefore,

\[
M_1^{(2)} = M_1^{(1)} = M_0^{(0)}
\]

in this case. For every set \( U = \{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), \ i, j \in I_\tau \} \) and every \( \tau \in T(G) \), we consider the matrix

\[
U^{(\tau)} = \begin{pmatrix}
u_{i_1,i_1}^{(\tau)} & u_{i_1,i_2}^{(\tau)} & \ldots & u_{i_1,i_{n_\tau}}^{(\tau)} \\
u_{i_2,i_1}^{(\tau)} & u_{i_2,i_2}^{(\tau)} & \ldots & u_{i_2,i_{n_\tau}}^{(\tau)} \\
\vdots & \vdots & \ddots & \vdots \\
u_{i_{n_\tau},i_1}^{(\tau)} & u_{i_{n_\tau},i_2}^{(\tau)} & \ldots & u_{i_{n_\tau},i_{n_\tau}}^{(\tau)}
\end{pmatrix} \in M_\tau^{(0)},
\]

where \( i_k \in I_\tau, i_1 < i_2 < \ldots < i_{n_\tau} \). We set \( U = \left( U^{(\tau)} \right)_{\tau \in T(G)} \in M^{(0)} \).

**Remark 2.3.** Let \( G \in A_0 \), \( G = G_1 \oplus C \) be a main decomposition of the group \( G \), and let \( \text{Reg} G = A \), \( \text{Reg} G_1 = B \). In [14], it is proved that for any multiplication \( \times \) on group \( G \), we have

\[
B_\tau \times A \subseteq m_\tau A_\tau \quad \text{and} \quad A \times B_\tau \subseteq m_\tau A_\tau \quad \text{for all} \ \tau \in T(B).
\]

**Theorem 2.4.** Let \( G \) be a block-rigid \( CRQ \)-group of ring type with \( A \)-basis \( E \) containing an \( B \)-basis \( E_0 \). Let \( U = \{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), \ i, j \in I_\tau \} \). Then the following conditions are equivalent.

1) \( U \) defines a multiplication on \( G \) with respect to the \( A \)-basis \( E \).

2) \( U \in M(d, E_0) \) for any \( d \in D(E_0) \).

3) \( U \in M(d, E_0) \) for some \( d \in D(E_0) \).

\(< 1 \Rightarrow 2. \) Let the set \( U = \{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), \ i, j \in I_\tau \} \) induce the multiplication \( \times \) on \( G \) with respect to the \( A \)-basis \( E = \{ e_i^{(\tau)} \in A_\tau \mid \tau \in T(G), \ i \in I_\tau \} \). It follows from Remark 2.3 that \( u_{0i}^{(\tau)}, u_{i0}^{(\tau)} \in m_\tau A_\tau \) and \( u_{00}^{(\tau)} = m_\tau v_{00}^{(\tau)} \) (\( v_{00}^{(\tau)} \in A_\tau \) for all \( \tau \in T(B), \ i \in I_\tau \).

Let \( d \in D(E_0) \) and let a standard representation related to the pair \((d, E_0)\) be of the form

\[
d = \sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} e_0^{(\tau)}.
\]
Since \( d \times d \in G \), we have that \( d \times d = \alpha d + a \) for some \( \alpha \in \mathbb{Z} \) and \( a \in A \). Then
\[
d \times d = \sum_{\tau \in T(B)} \frac{\alpha s_\tau}{m_\tau} e_0^{(\tau)} + a. \tag{7}
\]
On the other hand, it follows from (6) that
\[
d \times d = \sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} v_{00}^{(\tau)} = \sum_{\tau \in T(B)} \frac{s_\tau^2}{m_\tau^2} u_{00}^{(\tau)} = \sum_{\tau \in T(B)} \frac{s_\tau^2}{m_\tau} v_{00}^{(\tau)}.
\tag{8}
\]
Let \( \tau \in T(B) \). It follows from (7) and (8) that
\[
\pi_\tau(d \times d) = \frac{s_\tau^2}{m_\tau} v_{00}^{(\tau)} = \frac{\alpha s_\tau}{m_\tau} e_0^{(\tau)} + a_\tau, \quad \text{where } a_\tau = \pi_\tau(a) \in A_\tau.
\]
Consequently, \( s_\tau v_{00}^{(\tau)} = \alpha s_\tau e_0^{(\tau)} + m_\tau a_\tau \); therefore, \( v_{00}^{(\tau)} = \alpha s_\tau^{-1} e_0^{(\tau)} + m_\tau a_\tau' \) for some \( a_\tau' \in A_\tau \), where \( s_\tau^{-1} \) is an integer which is inverse to \( s_\tau \) modulo \( m_\tau \). Therefore,
\[
u_{00}^{(\tau)} = m_\tau v_{00}^{(\tau)} = \alpha m_\tau s_\tau^{-1} e_0^{(\tau)} + m_\tau^2 a_\tau'.
\]
Consequently, \( U \in \alpha X + M(2) \subseteq M(d, E_0) \).

2) \( \Rightarrow \) 3). The implication is directly verified.

3) \( \Rightarrow \) 1). Let \( U \in M(d, E_0) \) for some \( d \in D(E_0) \) and let a standard representation related to \( (d, E_0) \) be of the form \( d = \sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} e_0^{(\tau)} \). Then \( U = \alpha X + Y \) for some \( \alpha \in \mathbb{Z} \), \( Y \in M(2) \). Therefore, for all \( \tau \in T(B) \) and \( i \in I_\tau \), we have
\[
u_{0i}^{(\tau)} = m_\tau v_{0i}^{(\tau)}, \quad u_{0i}^{(\tau)} = m_\tau u_{0i}^{(\tau)}, \quad u_{00}^{(\tau)} = \alpha m_\tau s_\tau^{-1} e_0^{(\tau)} + m_\tau^2 a_\tau,
\]
where \( v_{0i}^{(\tau)}, v_{00}^{(\tau)}, a_\tau \in A_\tau \) and the integers \( s_\tau^{-1} \) satisfy conditions \( s_\tau s_\tau^{-1} = 1 + m_\tau x_\tau \) for some \( x_\tau \in \mathbb{Z} \).

There exists a ring \((A, \times)\) such that \( e_i^{(\tau)} \times e_j^{(\tau)} = u_{ij}^{(\tau)} \) for all \( \tau \in T(G) \), \( i, j \in I_\tau \); and \( A_\tau \times A_\sigma = 0 \) for \( \tau \neq \sigma \). This multiplication is extended to a multiplication on divisible hull \( \tilde{A} = \tilde{G} \) of the group \( A \). We prove that \( G \) is a subring of the ring \((\tilde{A}, \times)\).

It follows from (6) that
\[
d \times d = \sum_{\tau \in T(B)} \frac{s_\tau^2}{m_\tau^2} u_{00}^{(\tau)} = \alpha \sum_{\tau \in T(B)} \frac{s_\tau^2 s_\tau^{-1}}{m_\tau} e_0^{(\tau)} + \sum_{\tau \in T(B)} \frac{s_\tau^2}{m_\tau} a_\tau
\]
\[
= \alpha \sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} (1 + m_\tau x_\tau) e_0^{(\tau)} + \sum_{\tau \in T(B)} s_\tau^2 a_\tau
\]
\[
= \alpha \sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} e_0^{(\tau)} + \sum_{\tau \in T(B)} (\alpha s_\tau x_\tau) e_0^{(\tau)} + \sum_{\tau \in T(B)} s_\tau^2 a_\tau
\]
\[
= \alpha d + \sum_{\tau \in T(B)} (\alpha s_\tau x_\tau e_0^{(\tau)} + s_\tau^2 a_\tau) \in G.
\]
In addition, if $\sigma \in T(B)$ and $i \in I_\sigma$, then
\[
d \times e_i^{(\sigma)} = \left( \sum_{\tau \in T(B)} s_\tau \frac{e_0^{(\tau)}}{m_\tau} \right) \times e_i^{(\sigma)} = \frac{s_\sigma}{m_\sigma} u_0^{(\sigma)} = \frac{s_\sigma}{m_\sigma} m_\sigma v_0^{(\sigma)} = s_\sigma v_0^{(\sigma)} \in A.
\]
Similarly, we have $e_i^{(\sigma)} \times d \in A$.

If $\sigma \notin T(B)$, then $d \times e_i^{(\sigma)} = e_i^{(\sigma)} \times d = 0$ for any $i \in I_\sigma$. Since $G = \langle d, A \rangle$, we have that $G$ is a subring of the ring $(\hat{A}, \times)$. Therefore, the set $U$ defines a multiplication on $G$, $\triangleright$.

It follows from Theorem 2.4 that the group $M(d, E_0)$ does not depend on the choice of the element $d \in D(E_0)$. In the following assertion, we consider relations between elements of groups $M(d_1, E_0)$ and $M(d_2, E_0)$ for $d_1, d_2 \in D(E_0)$. We note that if $d_1, d_2 \in D$, then $\langle d_1 + A \rangle = \langle d_2 + A \rangle = G/A$; therefore, $d_1 = \gamma d_2 + b$ for some $b \in B$ and some integer $\gamma$ which is co-prime with $n(G)$.

**Proposition 2.5.** Let $G$ be a group in $A_0$ with main decomposition (1) and with a $B$-basis $E_0$ and regulator index $n$. Then the following assertions hold.

1) $M(d_1, E_0) = M(d_2, E_0)$ for any $d_1, d_2 \in D(E_0)$.

2) If $d_1, d_2 \in D(E_0)$ and $d_1 = \gamma d_2 + b$, where $\gamma \in \mathbb{Z}$, $\text{lcm}(\gamma, n) = 1$, $b \in B$, $X_1 = X(d_1, E_0) \in M(d_1, E_0)$, $X_2 = X(d_2, E_0) \in M(d_2, E_0)$, then $X_1 + M^{(2)} = \gamma^{-1} X_2 + M^{(2)}$, where $\gamma^{-1}$ is an integer which is inverse to $\gamma$ modulo $n$.

\[\triangleright\text{Let } d_1, d_2 \in D(E_0), d_1 = \gamma d_2 + b, \text{ where } \gamma \in \mathbb{Z}, \text{lcm}(\gamma, n) = 1 \text{ and } b = \sum_{\tau \in T(B)} b_\tau e_0^{(\tau)} (b_\tau \in R_\tau). \text{Let}
\]
\[
d_1 = \sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} e_0^{(\tau)}, \quad d_2 = \sum_{\tau \in T(B)} \frac{t_\tau}{m_\tau} e_0^{(\tau)}
\]
be standard representations of the group $G$ related to $(d_1, E_0)$ and $(d_2, E_0)$, respectively. In the divisible hull $\hat{G}$ of the group $G$, we have relations
\[
\sum_{\tau \in T(B)} \frac{s_\tau}{m_\tau} e_0^{(\tau)} = d_1 = \gamma d_2 + b = \sum_{\tau \in T(B)} \frac{\gamma t_\tau + m_\tau b_\tau}{m_\tau} e_0^{(\tau)}.
\]
Therefore,
\[
s_\tau = \gamma t_\tau + m_\tau b_\tau \text{ for all } \tau \in T(B).
\]
Let $\tau \in T(B)$, $\gamma^{-1}$ be an integer which is inverse to $\gamma$ modulo $n$, and let $t^{-1}_\tau$ be an integer which is inverse to $t_\tau$ modulo $m_\tau$. Then number $\gamma^{-1}$ is inverse to $\gamma$ modulo $m_\tau$ by (3'); therefore, the integer $\gamma^{-1} t^{-1}_\tau$ is inverse to $s_\tau$ modulo $m_\tau$ by (9).

Let $a_\tau \in A_\tau$. Then
\[
s^{-1}_\tau m_\tau e_0^{(\tau)} + m_\tau^2 a_\tau = \gamma^{-1} t^{-1}_\tau m_\tau e_0^{(\tau)} + m_\tau^2 a'_\tau, \text{ where } a'_\tau \in A_\tau.
\]
Consequently,
\[
X_1 + M^{(2)} \subseteq \gamma^{-1} X_2 + M^{(2)}.
\]
Since \( d_2 = \gamma^{-1}d_1 + b'_1 \) (where \( b'_1 \in B \)), we have
\[
\gamma^{-1}X_2 + M^{(2)} \subseteq \gamma^{-1}(\gamma X_1 + M^{(2)}) + M^{(2)} = X_1 + M^{(2)}
\]
by (10). \( \diamond \)

It follows from Proposition 2 that we can write \( M(d, E_0) = M(E_0) \), however we will often write \( M(d, E_0) \) if we want to point which a standard representation is used in the definition of the group \( M(d, E_0) \).

**Remark 2.6.** Let \( \overline{U} \in M^{(2)} \) for the set \( U = \{ u_{ij}^{(\tau)} \in A_\tau \mid \tau \in T(G), \; i, j \in I_\tau \} \). It follows from Theorem 2.4 that, with respect to any \( A \)-basis \( E \) of the group \( G \), the set \( U \) defines a multiplication \( \times_{U,E} \) on \( G \) such that \( G \times_{U,E} G \subseteq A \). Such a multiplication is called a regulator multiplication. \( \diamond \)

**Example 2.7.** It is possible that the set \( U = \{ u_{ij}^{(\tau)} \mid \tau \in T(G), \; i, j \in I_\tau \} \) defines a multiplication on \( G \) with respect to one \( A \)-basis and does not define any multiplication on \( G \) with respect to another \( A \)-basis even for the same main decomposition. Moreover, the following situation is possible: there exist two \( A \)-bases \( E \) and \( F \) such that any set \( U \), which defines non-regulator multiplication with respect to \( E \), does not define any multiplication with respect to \( F \). This means that
\[
M(E_0) \cap M(F_0) = M^{(2)}.
\]

\( \langle \) Let \( s_1, \; s_2 \) be two co-prime integers, \( s_1 > 1, \; s_2 > 1 \), and let \( m \) be a prime integer which does not divide any of the integers \( s_1, \; s_2, \; s_1^2 - s_2^2 \).

Let \( \tau_i \) be an idempotent type such that \( P_0(\tau_i) \) is the set of all prime divisors of the integers \( m \) and \( s_i \), respectively \((i = 1, 2)\). Then types \( \tau_1 \) and \( \tau_2 \) are incomparable.

We consider a group \( B = R_1 e_1 \oplus R_2 e_2 \), where \( R_1 \) and \( R_2 \) are unital subrings in the field \( \mathbb{Q} \) whose additive groups are of types \( \tau_1 \) and \( \tau_2 \), respectively.

It follows from Remark 2.1 that there exists a CRQ-group \( G = \langle d, B \rangle \) with regulator \( B \) and quasi-isomorphism invariants \( m_{\tau_1} = m_{\tau_2} = m \). We can choose the group \( G \) in such a way that a standard representation of \( G \) is of the form
\[
d = \frac{s_1}{m} e_1 + \frac{s_2}{m} e_2.
\]
Then the system \( E_0 = \{ e_1, e_2 \} \) is a \( B \)-basis of the group \( G \) defined by the element \( d \) (in this case, the \( B \)-basis coincides with an \( A \)-basis).

We set
\[
B_1 = B_{\tau_1} = R_1 e_1, \; B_2 = B_{\tau_2} = R_2 e_2.
\]
We consider \( d_1 = d + (e_1 + e_2) \in G \). Then \( d_1 \in D \); in \( \tilde{G} \), we have
\[
d_1 = \frac{s_1 + m}{m} e_1 + \frac{s_2 + m}{m} e_2. \quad (11)
\]
Since \( s_i + m \) is co-prime with each of the integers \( s_i \) and \( m \), we have that \( s_i + m \) is a \( P_\infty(\tau_i) \)-integer for \( i = 1, 2 \). Consequently, (11) is not a standard representation of the group \( G \). We set \( f_1 = (s_1 + m)e_1 \) and \( f_2 = (s_2 + m)e_2 \). Then the system \( F_0 = \{ f_1, f_2 \} \) is a \( B \)-basis defined by the element \( d_1 \). The
standard representation of the group $G$, related to the pair $(d_1, F_0)$, is of the form $d_1 = \frac{1}{m} f_1 + \frac{1}{m} f_2$.

Let the set $U = \{u_i \in B_i \mid i = 1, 2\}$ define a non-regulator multiplication on $G$ with respect to of the $B$-basis $E_0 = \{e_1, e_2\}$. By Theorem 2.4, we have $(u_1, u_2) \in M(d, E_0) \setminus M^{(2)}$. Consequently,

$$u_1 \in \alpha ms_1^{-1} e_1 + m^2 B_1, \quad u_2 \in \alpha ms_2^{-1} e_2 + m^2 B_2,$$

for some integer $\alpha$ which is not divided by the prime integer $m$.

We assume that the set $U$ defines multiplication with respect to the $B$-basis $F_0 = \{f_1, f_2\}$. Then it follows from Theorem 2.4 that for some $\beta \in \mathbb{Z}$, we have

$$u_1 \in \beta mf_1 + m^2 B_1, \quad u_2 \in \beta mf_2 + m^2 B_2.$$  

It follows from (12) and (13) that

$$\alpha ms_1^{-1} e_1 \in \beta mf_1 + m^2 B_1 = \beta ms_1 e_1 + m^2 B_1,$$

$$\alpha ms_2^{-1} e_2 \in \beta mf_2 + m^2 B_2 = \beta ms_2 e_2 + m^2 B_2.$$  

Therefore,

$$\alpha = \beta s_1^2 + mx_1, \quad \alpha = \beta s_2^2 + mx_2$$

for some $x_1 \in R_1, \ x_2 \in R_2$. Since $x_i = \frac{\alpha - \beta s_i^2}{m} \in R_i$ and $m$ is a $P_0(\tau_i)$-integer, we have $x_i \in \mathbb{Z}$ for $i = 1, 2$. Consequently, it follows from (14) that

$$\beta(s_1^2 - s_2^2) = my$$

for $y = x_2 - x_1 \in \mathbb{Z}$. Since the prime integer $m$ does not divide $s_1^2 - s_2^2$, we have that $m$ divides $\beta$; therefore, $m$ divides $\alpha$ by (14). This contradicts to the property that $(u_1, u_2) \notin M^{(2)}$. Consequently, the set $U = \{u_1, u_2\}$ does not define any multiplication with respect to the $B$-basis $F_0$. ▷

Let $\times$ be a multiplication on a group $G \in A_0$. Let $E = \{e_i^{(\tau)} \in A_\tau \mid \tau \in T(G), \ i \in I_\tau\}$ be an $A$-basis of the group $G$ and let

$$U_\times = U_\times(E) = \{u_{ij}^{(\tau)} = e_i^{(\tau)} \times e_j^{(\tau)} \in A_\tau \mid \tau \in T(G), \ i, j \in I_\tau\}.$$  

It clearly follows from Theorem 2.4 that the correspondence $\times \mapsto U_\times$ defines an isomorphism from the group Mult $G$ onto $M(E_0)$.

**Theorem 2.8.** If $G \in A_0$ and $E_0$ is a $B$-basis of the group $G$, then Mult $G \cong M(E_0)$. ▷

We note that Theorem 2.8 implies the following property: up to isomorphism, the group $M(E_0)$ does not depend on the choice of the $B$-basis $E_0$.

**Remark 2.9.** Let $G = \langle d, A \rangle \in A_0$ and let $E$ be an $A$-basis of the group $G$ containing the $B$-basis $E_0$.

1. It follows from Theorem 2.8 that the group Mult $G$ can be identified with the group $M(E_0) = \langle X, M^{(2)} \rangle$ and the multiplication $\times$ can be identified with $\bar{U}_\times \in M(E_0)$.  

2. Let \( \overline{U}_\times = \overline{U}_\times(E) \in \text{Mult} G = M(d, E_0) = \langle \alpha \rangle \), \( \alpha \in \mathbb{Z} \). It follows from the proof of Theorem 2.4 that \( \overline{U}_\times \in \alpha X + M(2) \) if and only if \( d \times d \in \alpha d + A \).

3. It follows from 2 that \( \overline{U}_\times \in M(2) \) if and only if \( G \times G \subseteq A \). In the group \( \text{Mult} G \), this means that the subgroup \( \text{Hom}(G \otimes G, A) \) of all regulator multiplications coincides with the group \( M(2) \). \( \triangleright \)

### 3. Properties of Multiplication Groups of Block-Rigid CRQ-Groups of Ring Type

The purpose of this section is to show that for any group \( G \) in the class \( A_0 \), the group \( \text{Mult} G \) belongs to this class, as well. We will also describe the rank, the set of critical types, invariants of near-isomorphism, the regulator, a main decomposition, and a standard representation of the group \( \text{Mult} G \), where \( G \in A_0 \).

**Remark 3.1.** Let \( A \) be a completely decomposable block-rigid group of finite rank and \( G = \langle d, A \rangle \), where \( d \in \hat{A} \). Let we have \( o(d_\tau + A) = m_\tau \) in the group \( \hat{A}/A \), where \( d_\tau = \pi_\tau(d) \) for \( \tau \in T(A) \). Then the set \( \{ m_\tau | \tau \in T(A) \} \) satisfies condition (m) (see Remark 2.1) if and only if for any \( \tau \in T(A) \), the subgroup \( A_\tau \) is pure in \( G \).

Indeed, let the set \( \{ m_\tau | \tau \in T(A) \} \) satisfy condition (m), \( \sigma \in T(A) \), \( a \in A_\sigma \) and \( a = k(td + x) \) for some \( k, t \in \mathbb{Z} \) and \( x \in A \). Let \( \tau \neq \sigma \), then

\[
kt \sigma + kx_\tau = 0, \quad \text{where} \quad x_\tau = \pi_\tau(x) \in A_\tau.
\]

Therefore, \( td_\tau \in A_\tau \), whence \( m_\tau \) divides \( t \). Since the set \( \{ m_\tau | \tau \in T(A) \} \) satisfies condition (m), we have that

\[
\text{lcm}\{ m_\tau | \tau \in T(A), \tau \neq \sigma \} = \text{lcm}\{ m_\tau | \tau \in T(A) \} = n.
\]

Consequently, \( n \) divides \( t \); therefore, \( td + x \in A \). Since the type of the element \( td + x \) is equal to \( \sigma \), we have that \( td + x \in A_\sigma \).

Conversely, let the subgroup \( A_\tau \) be pure in \( G \) for any \( \tau \in T(A) \). We assume that the set \( \{ m_\tau | \tau \in T(A) \} \) does not satisfy condition (m). Then there exists a type \( \sigma \in T(A) \) such that \( m_\sigma \) does not divide \( n_1 = \text{lcm}\{ m_\tau | \tau \in T(A), \tau \neq \sigma \} \). Consequently, for \( n = \text{lcm}\{ m_\tau | \tau \in T(A) \} \), it is true that \( n = n_1 n_2 \) for some integer \( n_2 > 1 \). Consequently, \( n_1 d_\sigma \notin A_\sigma \) and \( n_2 (n_1 d_\sigma) \in A_\sigma \). Therefore, the subgroup \( A_\sigma \) is not pure in \( G \). \( \triangleright \)

**Remark 3.2.** Let \( A \) be a reduced block-rigid (resp., rigid) completely decomposable group of finite rank and let \( G = \langle d, A \rangle \), where \( d \in \hat{A} \setminus A \). Then \( A \) is a subgroup of finite index of the group \( G \), and \( T(G) = T(A) \). Therefore, \( G \) is a block-rigid (resp., rigid) ACQG-group by the definition. We note that if \( A \) is a group of ring type, then \( G \) also is a group of ring type. Since \( G/A = \langle d + A \rangle \) is a cyclic group, we have that \( G \) is a CRQ-group by [5, Sect. 2]. In addition, \( A = \text{Reg} G \) if and only if subgroup \( A_\tau \) is pure in \( G \) [5, Sect. 2] for any \( \tau \in T(G) \). \( \triangleright \)
Let $G \in \mathcal{A}_0$. The following theorem describes properties of the group \( \text{Mult} G \). In what follows, $G \in \mathcal{A}_0$, $T(G) = T$, $T_0(G) = T_0$, $m_\tau(G) = m_\tau$ ($\tau \in T$), $n(G) = n$, $G = G_1 \oplus C$ is a main decomposition of the group $G$, $\text{Reg} G_1 = B = \oplus_{\tau \in T_0} R_\tau e_0(\tau)$, $\text{Reg} G = A = B \oplus C$, $G = \langle d, A \rangle$, and a standard representation of the group $G$ is of the form

$$d = \sum_{\tau \in T_0} \frac{s_\tau e_0(\tau)}{m_\tau}.$$ 

We note that the set of integral solutions of the congruence $s_\tau x \equiv 1 \pmod{m_\tau}$ always contains a $P_0(\tau)$-integer $s_\tau^\varphi(m_\tau)-1$, where $\varphi(x)$ is the Euler function. Therefore, we always can take this $P_0(\tau)$-integer as the integer $s_\tau^{-1}$ inverse to $s_\tau$ modulo $m_\tau$.

**Theorem 3.3.** Let $G \in \mathcal{A}_0$. Then the group \( \text{Mult} G \) satisfies the following conditions.

1. The group \( \text{Mult} G \) is a block-rigid CRQ-group of ring type with regulator $M^{(2)} = \text{Hom}(G \otimes G, A)$.
2. $T(\text{Mult} G) = T(G)$ and $T_0(\text{Mult} G) = T_0(G)$, as a corollary.
3. $m_\tau(\text{Mult} G) = m_\tau(G)$ for any $\tau \in T(G)$, $n(\text{Mult} G) = n(G)$.
4. $r(\text{Reg}_\tau(\text{Mult} G)) = (r(\text{Reg}_\tau(G)))^3$ for any $\tau \in T(G)$.
5. One of main decompositions of the group $\text{Mult} G$ is of the form $\text{Mult} G = M' \oplus M''$, where

$$M' = \langle X, K \rangle, \quad K = \prod_{\tau \in T_0(G)} K_\tau, \quad K_\tau = \begin{bmatrix} m_\tau^2 B_\tau & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \subseteq M^{(2)},$$

$$X = \left( X^{(\tau)} \right)_{\tau \in T_0(G)}, \quad X^{(\tau)} = \begin{bmatrix} m_\tau s_\tau^{-1} e_0(\tau) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \in M^{(1)}_\tau \text{ for } \tau \in T_0(G),$$

$$M'' = \prod_{\tau \in T(G)} M''_\tau, \quad M''_\tau = \begin{bmatrix} m_\tau^2 C_\tau & m_\tau A_\tau \ldots m_\tau A_\tau \\ m_\tau A_\tau & A_\tau \ldots A_\tau \\ \vdots & \vdots & \ddots & \vdots \\ m_\tau A_\tau & A_\tau \ldots A_\tau \end{bmatrix} \subseteq M^{(2)}_\tau.$$ 

In addition, $T(M') = T_0(G)$, $\text{Reg} M' = K$.

6. For every $\tau \in T_0(G)$, we denote by $s_\tau^{-1}$ a $P_0(\tau)$-integer which is inverse to $s_\tau$ modulo $m_\tau$,

$$E_0^{(\tau)} = \begin{bmatrix} m_\tau^2 e_0(\tau) & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} \in K_\tau.$$ 

Then the system $\{ E_0^{(\tau)} \mid \tau \in T_0(G) \}$ is one of $B$-bases of the group $\text{Mult} G$.

One of standard representations of the group $\text{Mult} G$ is of the form

$$X = \left( \frac{s_\tau^{-1} E_0^{(\tau)}}{m_\tau} \right)_{\tau \in T_0(G)}.$$
1. It follows from Theorem 2.8 that \( \text{Mult} G = \langle X, M^{(2)} \rangle \), where

\[
X = \left( X^{(\tau)} \right)_{\tau \in T_0}, \quad X^{(\tau)} = \begin{pmatrix}
    m_\tau s_\tau^{-1} e_0^\tau & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & \ldots & 0
\end{pmatrix} \in M^{(1)} \text{ for } \tau \in T_0.
\]

Since \( \gcd(s_\tau^{-1}, m_\tau) = 1 \) for \( \tau \in T_0 \), we have \( o(X_\tau + M^{(2)}) = m_\tau \) in the group \( \hat{M}^{(2)}/M^{(2)} \).

By Remark 2.1, the set \( \{m_\tau \mid \tau \in T\} \) satisfies condition (m). Therefore, it follows from Remark 3.1 that the subgroups \( M^{(2)}_\tau \) are pure in \( \text{Mult} G \) for any \( \tau \in T \). Since \( M^{(2)} \) is a block-rigid completely decomposable group of ring type, \( \text{Mult} G \) is a block-rigid CRQ-group of ring type with regulator \( M^{(2)} \) by Remark 3.2. It follows from Remark 2.9(3) that we have \( M^{(2)} = \text{Hom}(G \otimes G, A) \).

2. It follows from 1 and the definition of the group \( M^{(2)} \) that

\[
T(\text{Mult} G) = T(M^{(2)}) = T(G).
\]

3. We have \( \text{Mult} G = \langle X, M^{(2)} \rangle \) and \( \text{Reg(Mult} G) = M^{(2)} \) by 1. Let \( \tau \in T \), then

\[
m_\tau(\text{Mult} G) = o(X_\tau + M^{(2)}) = m_\tau = m_\tau(G).
\]

Therefore, \( n(\text{Mult} G) = \text{lcm}\{m_\tau \mid \tau \in T\} = n(G) \).

4. Let \( \tau \in T \). It follows from 1 that

\[
\text{Reg}_\tau(\text{Mult} G) = M^{(2)}_\tau = \begin{bmatrix}
    m_\tau^2 A_\tau & m_\tau A_\tau & \ldots & m_\tau A_\tau \\
    m_\tau A_\tau & A_\tau & \ldots & A_\tau \\
    \ldots & \ldots & \ldots & \ldots \\
    m_\tau A_\tau & A_\tau & \ldots & A_\tau
\end{bmatrix} \cong M_{n_\tau}(A_\tau),
\]

where \( n_\tau = r(A_\tau) \). Consequently, \( r(\text{Reg}_\tau(\text{Mult} G)) = n_\tau^3 = (r(\text{Reg}_\tau G))^3 \).

5. In the decomposition \( \text{Mult} G \cong M' \oplus M'' \), the group \( M'' \) is completely decomposable and \( M' = \langle X, K \rangle \). By the definition of the group \( K \), we have \( T(K) = T(M') = T_0 \). It is easy to see that \( o(X^{(\tau)} + K) = m_\tau \) in the group \( \hat{K}/K \), for any \( \tau \in T_0 \). Since \( \{m_\tau \mid \tau \in T_0\} \) satisfies condition (m) (by Remark 2.1) and \( K \) is a rigid completely decomposable group, we have that \( M' \) is a rigid group in \( A_0 \) with \( \text{Reg} M' = K \) by Remarks 3.1 and 3.2.

Since \( T(M') = T_0 \), we have that \( \tau \in T(M') \) if and only if \( m_\tau(\text{Mult} G) = m_\tau > 1 \). In addition,

\[
m_\tau(M') = o(X^{(\tau)} + K) = m_\tau = m_\tau(\text{Mult} G),
\]

by 3. It follows from (1'), (1'') that the decomposition \( \text{Mult} G = M' \oplus M'' \) is a main decomposition of the group \( \text{Mult} G \).

6. Let \( \tau \in T_0 \), \( s_\tau^{-1} \) be a \( P_0(\tau) \)-integer which is inverse to \( s_\tau \) modulo \( m_\tau \), and let

\[
E_0^{(\tau)} = \begin{pmatrix}
    m_\tau^2 e_0^{(\tau)} & 0 & \ldots & 0 \\
    0 & 0 & \ldots & 0 \\
    \ldots & \ldots & \ldots & \ldots \\
    0 & 0 & \ldots & 0
\end{pmatrix} \in K_\tau.
\]
Then \( K = \prod_{\tau \in T_0} R_{\tau} E^{(\tau)}_0 \). In addition,

\[
X = \left( \frac{s_{\tau}^{-1}}{m_{\tau}} E^{(\tau)}_0 \right)_{\tau \in T_0}.
\]  

(15)

Since \( s_{\tau}^{-1} (\tau \in T_0) \) is a \( P_0(\tau) \)-integer, (15) is a standard representation of the group \( \text{Mult} \ G \). Therefore, \( \{ E^{(\tau)}_0 \mid \tau \in T_0(G) \} \) is a \( B \)-basis of the group \( \text{Mult} \ G \).

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