Velocity Tails for Inelastic Maxwell Models

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We study the velocity distribution function for inelastic Maxwell models, characterized by a Boltzmann equation with constant collision rate, independent of the energy of the colliding particles. By means of a nonlinear analysis of the Boltzmann equation, we find that the velocity distribution function decays algebraically for large velocities, with exponents that are analytically calculated.

I. INTRODUCTION

Velocity distributions have over-populated high energy tails in many particle systems with in-elastic interactions, as has been discovered theoretically, and later observed in laboratory experiments with granular materials [1]. Instead of Gaussian tail distributions, many kinetic equations typically predict stretched exponentials, like \( \exp[-A|v|^{3/2}] \) in driven inelastic hard sphere systems (IHS) [2], or even tails with a higher overpopulation, like \( \exp[-A|v|] \) in the freely evolving IHS fluids [2]. These theoretical predictions have been extensively verified in Direct Monte Carlo Simulations (DSMC) of the Boltzmann equation [3], and the stretched exponentials have been observed in laboratory experiments with granular matter on vibrating plates [4]. However, the presence and also the absence of over-populated tails depends very strongly on the energy input or on how the system is thermo-statted [5,6]. Recently classes of simplified kinetic models have been studied, so-called Maxwell models, which are characterized by a Boltzmann equation with a collision rate that is independent of the relative kinetic energy of the colliding particles. Maxwell molecules are for kinetic theory, what harmonic oscillators are for quantum mechanics, and dumb-bells for polymer physics.

Ben-Naim and Krapivsky [7] have presented arguments about the non-existence of scaling solutions of the Boltzmann equation for the one-dimensional Maxwell model, and argue in favor of multi-time scales [8]. However, Puglisi et al. [2] have found an exact scaling solution \( \tilde{f}(v,t) \sim \left(1/v_0^3(t)\right)f(v/v_0(t)) \) for that equation, i.e. \( \tilde{f}(c) = (2/\pi)(1+c^2)^{3/2} \). It does have a power law tail \( (1/c^4) \). Puglisi et al. also solve the spatially homogeneous Maxwell-Boltzmann equation using MC simulations in one- and two- dimensions, and, more importantly, they show that an arbitrary initial distribution approaches this scaling solution, with power law tails at high energies. In two-dimensions the exponent \( \alpha \) of the power law tails depends on the degree of in-elasticity, i.e. on the coefficient of restitution \( \alpha \). The goal of this article is to derive these power laws from the dominant small-\( k \) singularity in the Fourier transform of the velocity distribution function.

II. DOMINANT SMALL \( k \) SINGULARITY

In order to analyze the large \( v \)-behavior of the distribution function, it is convenient to use the Fourier transform \( \phi(k,t) \) of \( f(v,t) \). It is the generating function of the moments \( \langle v^n \rangle_t \). If \( f(v,t) \) has a tail \( \sim 1/|v|^{\alpha+d} \), then the moments with \( n > \alpha \) are divergent, and so is the \( n \)-th derivative of the generating function at \( k = 0 \), i.e. \( \phi(k,t) \) is singular at \( k = 0 \). Suppose the dominant small-\( k \) singularity of \( \phi(k,t) \) is \( \sim |k|^{\alpha} \), where \( \alpha \) is different from an even integer (even powers of \( k \) represent contributions that are regular at small \( k \)), then the inverse Fourier transform scales as \( 1/|v|^{\alpha+d} \) at large \( v \).

By applying Bobylev’s Fourier transform method [9,10,11], we obtain a nonlinear equation for \( \phi(k,t) \), and we determine its dominant small-\( k \) singularity. In doing so we have derived a transcendental equation for the exponent \( \alpha \) in the power law tail \( \sim 1/|c|^{\alpha+d} \) of the scaling solutions \( \tilde{f}(v/v_0) \) of the Boltzmann equation for Maxwell molecules with in-elastic hard sphere interactions in arbitrary dimensions.

The Boltzmann equation for the \( d \)-dimensional in-elastic Maxwell model reads,
Here \(\int (\cdots) = (1/\Omega_d) \int dn(\cdots)\) is an average over a \(d\)-dimensional solid angle where \(\Omega_d = 2\pi^{d/2}/\Gamma(\frac{d}{2})\). The velocities \(v_i^{**}\) with \(i,j = \{1,2\}\) denote the \(d\)-dimensional restituting velocities, and \(v_i^\ast\) the corresponding post-collision velocities. They are defined as,

\[
\begin{align*}
\hat{v}_i^* &= v_i - \frac{1}{2}(1 + \frac{q}{K_d})v_{ij} \cdot nn \\
\hat{v}_i^\ast &= v_i - \frac{1}{2}(1 + \alpha)v_{ij} \cdot nn,
\end{align*}
\]

with \(v_{ij} = v_i - v_j\), and \(n\) is a unit vector along the line of centers of the interacting particles. In one-dimension, the tensorial product \(nn\) can be replaced by 1. From the normalization of \(f\) it follows that the loss term reduces to \(-f(v_1, t)\), i.e. the collision frequency is unity, and the dimensionless time \(t\) counts the average number of collisions per particle.

We first illustrate the method for the one-dimensional case. Fourier transformation of the Boltzmann equation yields then,

\[
\partial_t \phi(k,t) = \phi(pk,t)\phi((1-p)k,t) - \phi(k,t),
\]

where we have used that \(\phi(0,t) = 1\) and \(p = \frac{2}{3}(1 + \alpha)\). The equation for the scaling solution, \(\phi(k,t) = \Phi(v_0(t)k)\) simplifies to,

\[
- \gamma k d\Phi(k)/dk + \Phi(k) = \Phi(pk)\Phi((1-p)k),
\]

where the exponent \(\gamma\) in \(v_0(t) = v_0(0) \exp[-\gamma t]\) is still to be determined.

The requirement that the total energy be finite, imposes the the lower bound \(a > 2\) on the exponent. We therefore make the ansatz that the dominant small-k singularity has the form,

\[
\Phi(k) = 1 - \frac{1}{2\alpha}(k \cdot c)^2 + A|k|^a.
\]

Inserting this in (3), and equating the coefficient of equal powers of \(k\) yields the equation,

\[
a = \frac{1 - p^a - (1-p)^a}{p(1-p)}.
\]

The smallest root of this equation, satisfying \(a > 2\), is \(a = 3\), and \(A\) is left undetermined. Consequently the scaling solution has a power law tail, \(f(c) \sim 1/c^4\).

The same method can be applied to the \(d\)-dimensional case. Application of Bobylev’s Fourier transform method to the Boltzmann equation for this case yields the transformed equation,

\[
\partial_t \phi(k,t) = \int_n \phi(k+,t)\phi(k-,t) - \phi(k,t),
\]

where the \(n\)-average is defined below (1), and

\[
\begin{align*}
k_+ &= pk \cdot nn & |k_+|^2 &= p^2 k^2(k \cdot n)^2 \\
k_- &= k - k_+ & |k_-|^2 &= k^2[1 - q(k \cdot n)^2],
\end{align*}
\]

where \(q = p(2-p)\) is a positive number. We proceed in the same way as in the one-dimensional case, and obtain the equation for the scaling solution,

\[
- \gamma k d\Phi(k)/dk + \Phi(k) = \int_n \Phi(k_+)\Phi(k_-).
\]

Inserting the ansatz (5) into (8), and equating the coefficients of equal powers of \(k^a\) yields,

\[
a\gamma \langle |k \cdot c|^a \rangle = \int_n \langle |k \cdot c|^a - |k_+ \cdot c|^a - |k_- \cdot c|^a \rangle,
\]

for \(s = 2, a\). In order to carry out the angular \(\hat{c}\)-average in \(\langle |q \cdot c|^a \rangle\) with \(q = \{k, k_+, k_-\}\) we choose \(q\) as polar axis, and denote \(q \cdot c = q c \hat{q} \cdot \hat{c} = q c \cos \theta\), then \(\langle |q \cdot c|^a \rangle = \langle |c|^a \rangle K_a^{(d)}\), where \(K_a^{(d)}\) is the average of \(|\cos \theta|^a\) over a \(d\)-dimensional solid angle, which equals
\[ K_a^{(d)} = \Gamma\left(\frac{d}{2}(a + 1)\right) \frac{\Gamma\left(\frac{1}{2}d\right)}{\Gamma\left(\frac{1}{2}(a + d)\right)} \Gamma\left(\frac{1}{2}\right), \]  

where \( q = p(2 - p) \). Finally we carry out the angular \( n \)-averages using (8), and obtain,

\[
\int_n |k_+|^a = k^a p^a K_a^{(d)} \\
\int_n |k_-|^a = k^a \int_n [1 - q(\hat{k} \cdot n)^2]^{a/2} = k^a L_a^{(d)}(q). \tag{12}
\]

Insertion of these results in (10) for \( a=2 \), yields

\[ \gamma = \frac{1}{d} p(1 - p) = \frac{1}{4}(1 - \alpha^2). \tag{13} \]

For the exponent \( a \), featuring in the power law tail of the scaling function \( f(c) \sim 1/c^{a+d} \), we obtain the transcendental equation,

\[ a = \frac{1 - p^a K_a^{(d)} - L_a^{(d)}(q)}{\frac{1}{4} p(1 - p)}. \tag{14} \]

The two most interesting cases are \( d = 2, 3 \), where

\[
L^{(2)}_a(q) = \frac{2}{\pi} \int_0^{\pi/2} d\theta |1 - q \cos^2 \theta|^{a/2} \\
L^{(3)}_a(q) = \int_0^1 dx |1 - qx^2|^{a/2}, \tag{15}
\]

and one can verify that \( a = 2 \) is also a solution of (14). We look for the smallest solution \( a(\alpha) \) of this transcendental equation with \( a > 2 \). The numerical solutions for \( d = 2, 3 \) are shown in Figure 1 as a function of \( \alpha \). If \( p = \frac{1}{2}(1 + \alpha) \uparrow 1 \) the root \( a(\alpha) \) moves to \( \infty \), as it should, which is consistent with a Maxwellian tail distribution for the elastic case.

FIG. 1. Solution of Eq. (14) as a function of \( \alpha \) for 2 dimensions (left panel) and 3 dimensions (right panel). The solution diverges as \( \alpha \to 1 \) (elastic limit), because \( f \) becomes Maxwellian in this limit. We note that \( a = 2 \) always satisfies Eq. (14), and the solution shown here is the one different from \( a = 2 \).

The simulations in Ref. [9] of the two-dimensional Maxwell - Boltzmann equation show for the exponent \( a(\alpha = 0) + 2 \simeq 5 \), where our analytical method predicts \( a(\alpha = 0) + 2 \simeq 6.2 \). In fact, closer inspection of their two-dimensional scaling plot shows that the slope of their log-log plot of \( f(c) \) versus \( c \) increases at larger velocities, approaching the exact prediction of the Boltzmann equation. However, at these large velocities the statistical errors in their simulations are too large to make a quantitative comparison for larger \( \alpha \)-values.
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