A large deviations principle for the Maki–Thompson rumour model

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Abstract

We consider the stochastic model for the propagation of a rumour within a population which was formulated by Maki and Thompson. Sudbury has established that, as the population size tends to infinity, the proportion of the population never hearing the rumour converges in probability to 0.2032. We prove a corresponding large deviations principle, with an explicit formula for the rate function.

Keywords: Large deviations, Maki–Thompson, Markov process, Rumour.

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1 Introduction

Rumours play an important role in various aspects of the human life: in social relationships, politics, economy, diplomacy, marketing. A classical model of the mathematical literature for the spread of a rumour within a population was introduced by Maki and Thompson [14] and discussed later by Frauenthal [6]. In this model, a closed population is subdivided into three classes: ignorants, spreaders and stiflers. The rumour is propagated by directed contact of the spreaders with other individuals in the population. When a spreader meets an ignorant, the rumour is told and the ignorant becomes a spreader. If a spreader contacts another spreader or a stiffer, the initiating spreader turns into a stiffer.

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The process eventually terminates (when there are no more spreaders in the population), so it is of interest to study the proportion of remaining ignorants.

We adopt the usual notation, denoting respectively by \(X(t), Y(t)\) and \(Z(t)\) the number of ignorants, spreaders and stiflers at time \(t\). Initially, \(X(0) = N\), \(Y(0) = 1\) and \(Z(0) = 0\), while \(X(t) + Y(t) + Z(t) = N + 1\) for all \(t \geq 0\). The process \(\{(X(t), Y(t))\}_{t \geq 0}\) is a continuous-time Markov chain which proceeds according to the following transition scheme:

- \((X(t), Y(t)) \rightarrow (X(t) - 1, Y(t) + 1)\) at rate \(X(t) Y(t)\),
- \((X(t), Y(t)) \rightarrow (X(t), Y(t) - 1)\) at rate \(Y(t) (N - X(t))\).

The first transition corresponds to a spreader telling the rumour to an ignorant, who becomes a spreader. The second transition corresponds to a spreader meeting another spreader or a stifler, in which case he loses the interest in propagating the rumour and becomes a stifler.

Let \(\tau = \inf \{t : Y(t) = 0\}\) be the time until the rumour process ceases, thus \(X(\tau)\) is the final number of ignorants in the population. It is convenient to write down explicitly the dependence of this quantity on \(N\), so we denote it by \(X_F^{(N)}\). The first theorems established for the Maki–Thompson model deal with the asymptotic behaviour as \(N \to \infty\) of \(N^{-1} X_F^{(N)}\) (i.e. the proportion of the originally ignorant individuals who remained ignorant at the end of the process). By using a martingale technique, Sudbury [16] proved that

\[
\lim_{N \to \infty} \frac{X_F^{(N)}}{N} = x_\infty \approx 0.2032 \quad \text{in probability.}
\]

Therefore, for large \(N\), approximately a fifth of the people are not aware of the rumour at the moment that the spreading process stops, with high probability. The limiting proportion of ignorants \(x_\infty\) can be expressed in terms of the so-called Lambert \(W\) function. This is the multivalued inverse of the function \(x \mapsto xe^x\); see Corless et al. [2] for more details. Denoting by \(W_0\) the principal branch of the Lambert \(W\) function, we have that

\[
x_\infty = -\frac{W_0(-2 e^{-2})}{2} \tag{1.1}
\]

Watson [17] later proved the corresponding Central Limit Theorem, which states that

\[
\sqrt{N} \left( \frac{X_F^{(N)}}{N} - x_\infty \right) \overset{D}{\to} \mathcal{N}(0, \sigma^2) \quad \text{as } N \to \infty,
\]

where \(\overset{D}{\to}\) denotes convergence in distribution, and \(\mathcal{N}(0, \sigma^2)\) is the Gaussian distribution with mean zero and variance given by

\[
\sigma^2 = \frac{x_\infty (1 - x_\infty)}{1 - 2 x_\infty} \approx 0.2727.
\]
Lefevre and Picard [12] characterized in terms of Gontcharoff polynomials the joint distribution of the number of individuals who ultimately heard the rumour and the total personal time units during which the rumour spread. The limit theorems proved by Sudbury [16] and Watson [17] were generalized by Lebensztayn et al. [11] for a Maki–Thompson rumour model with general initial configuration and in which a spreader becomes a stifler only after being involved in a random number of unsuccessful telling meetings. We refer the interested reader to Daley and Gani [3, Chapter 5] for an excellent presentation on the mathematical modelling of rumours.

The main results of the paper are stated in Section 2. Our main theorem is that a full large deviations principle holds for the proportion \( \frac{N-1}{N} X_F^{(N)} \). To prove this result, we first derive a closed formula for the probability mass function of \( X_F^{(N)} \). Then, we obtain the asymptotic behaviour of normalized logarithms of probabilities of certain events. To the best of our knowledge, large deviations for the final outcome of stochastic rumour models have never been investigated in the literature.

2 Main results

In our first result, we give a closed formula for the absorption probabilities of the Maki–Thompson model. For \( n \geq 1 \), let \( d_n \) denote the number of nonisomorphic unlabelled initially connected complete and deterministic automata with \( n \) states over a 2-letter alphabet. Liskovets [13] proves the following recursive formula:

\[
d_1 = 1, \quad \text{and} \quad d_n = \frac{n^{2n}}{(n-1)!} - \sum_{i=1}^{n-1} \frac{n^{2(n-i)}}{(n-i)!} d_i \quad \text{for } n \geq 2.
\]

This is the sequence A006689 in Sloane’s Encyclopedia [15], where more details can be found.

**Theorem 2.1.** For each \( i = 0, \ldots, N-1 \),

\[
P(X_F^{(N)} = i) = \frac{(N-1)!}{i!} \frac{d_{N-i}}{N^{2(N-i)}}.
\]

We need some definitions to state the full large deviations principle for the ultimate proportion of ignorants. We define the constants

\[
v_\infty = 1 - x_\infty \approx 0.7968, \quad \text{and} \quad \rho = 2 + \log x_\infty + \log(1 - x_\infty) \approx 0.1792,
\]
where \( x_\infty \) is given in (1.1). We also define the function \( h : [0, 1) \to \mathbb{R} \) given by

\[
h(x) = x \log x + (1 - x)[\rho - \log(1 - x)],
\]

with the usual convention that \( 0 \log 0 = 0 \), and let \( H : [0, \infty) \to [0, \infty] \) be given by

\[
H(x) = \begin{cases} 
  h(x) & \text{if } 0 \leq x < 1, \\
  \infty & \text{if } x \geq 1.
\end{cases}
\]

We note that

(i) \( h(x_\infty) = 0 \).
(ii) \( h \) is decreasing on the intervals \([0, x_\infty]\) and \([v_\infty, 1)\), and is increasing on \([x_\infty, v_\infty]\).
(iii) \( h \) is strictly convex on \([0, 1/2]\), and is strictly concave on \([1/2, 1)\).

The graph of \( h \) is presented in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{h_graph.png}
\caption{Graph of the function \( h \).}
\end{figure}

**Theorem 2.2.** Let \( \nu_N \) be the probability distribution of the random variable \( N^{-1} X_F^{(N)} \) on \([0, \infty)\). Then the following conclusions hold.

(a) For each closed set \( F \subset [0, \infty) \),

\[
\limsup_{N \to \infty} \frac{1}{N} \log \nu_N(F) \leq -\inf_{x \in F} H(x).
\]

(b) For each open set \( G \subset [0, \infty) \),

\[
\liminf_{N \to \infty} \frac{1}{N} \log \nu_N(G) \geq -\inf_{x \in G} H(x).
\]
To finish the section, we present two results which concern the asymptotic behaviour of normalized logarithms of probabilities of certain events. These results are useful in the proof of Theorem 2.2.

**Proposition 2.3.** For every \( x \in [0, 1) \), we have that
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(X_F^{(N)} = \lfloor Nx \rfloor) = \lim_{N \to \infty} -\frac{1}{N} \log P(X_F^{(N)} = \lfloor Nx \rfloor) = h(x).
\]

**Proposition 2.4.**
(a) If \( 0 \leq x < x_\infty \), then
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(X_F^{(N)} \leq Nx) = h(x).
\]
(b) If \( x_\infty < x < y \leq v_\infty \), then
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(Nx \leq X_F^{(N)} \leq Ny) = h(x).
\]
(c) If \( v_\infty \leq x < y < 1 \), then
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(Nx \leq X_F^{(N)} \leq Ny) = h(y).
\]

3 Proofs

Theorem 2.2 follows from Proposition 2.4 by using standard arguments of the large deviations theory (see for instance the proof of Theorem 2.2.3 in Dembo and Zeitouni [4]). We present the proof at Subsection 3.5 for the sake of completeness.

To facilitate the proofs of Theorem 2.1 and Propositions 2.3 and 2.4 we restate them in terms of the random variable \( V_F^{(N)} = N - X_F^{(N)} \), which represents the number of the initially ignorant individuals who heard the rumour at the end of the process. We also define \( V(t) = N - X(t) \) for \( t \geq 0 \), and let \( \psi : (0, 1] \to \mathbb{R} \) be the function given by
\[
\psi(v) = (1 - v) \log(1 - v) + v[\rho - \log v].
\]
Since \( \psi(v) = h(1 - v) \) for every \( v \in (0, 1] \), our task is done once we prove the following results.

**Theorem 3.1.** For each \( j = 1, \ldots, N \),
\[
P(V_F^{(N)} = j) = \frac{(N - 1)!}{(N - j)! N^{2j}} d_j.
\]
Proposition 3.2. For every $v \in (0, 1]$, we have that
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(V_F^{(N)} = \lfloor Nv \rfloor) = \lim_{N \to \infty} -\frac{1}{N} \log P(V_F^{(N)} = \lceil Nv \rceil) = \psi(v).
\]

Proposition 3.3. (a) If $0 < u < v \leq x_\infty$, then
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(Nu \leq V_F^{(N)} \leq Nv) = \psi(u).
\]
(b) If $x_\infty \leq u < v < v_\infty$, then
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(Nu \leq V_F^{(N)} \leq Nv) = \psi(v).
\]
(c) If $v_\infty < u \leq 1$, then
\[
\lim_{N \to \infty} -\frac{1}{N} \log P(V_F^{(N)} \geq Nu) = \psi(u).
\]

3.1 Proof of Theorem 3.1

We first observe that the continuous-time Markov chain $\{(V(t), Y(t))\}_{t \geq 0}$ proceeds according to the following transition scheme:

\[
(V(t), Y(t)) \to (V(t) + 1, Y(t) + 1) \quad \text{at rate } Y(t)(N - V(t)) ,
\]
\[
(V(t), Y(t)) \to (V(t), Y(t) - 1) \quad \text{at rate } Y(t)V(t).
\]

The distribution of $V_F^{(N)}$ depends on $(V(t), Y(t))$ only through the embedded Markov chain, whose state space is the set of points $(r, s)$ of the two-dimensional integer lattice for which $0 \leq r \leq N$ and $0 \leq s \leq r + 1$. The evolution of the embedded chain can be viewed as the motion of a particle through these lattice points, starting at the point $(0, 1)$ of the $xy$ plane. (See Figure 2.) From a point $(r, s)$ with $s > 0$, the particle moves either one step vertically downwards to $(r, s - 1)$ or one step diagonally northeast (i.e., up and right) to $(r + 1, s + 1)$. The probabilities of these two types of transitions are $r/N$ and $(1 - r/N)$, respectively. Following Daley and Gani [3], we say that this process is strictly evolutionary: once a state is visited and left, it is never visited again, so each state is entered either once or not at all. The states $\{(r, 0) : r = 1, \ldots, N\}$ are absorbing, thus the particle halts once it hits the $x$-axis. If the particle hits the line $r = N$, then only the vertically downward transitions are allowed, until it reaches the point $(N, 0)$. For each $j = 1, \ldots, N$, the event $\{V_F^{(N)} = j\}$ means that the particle hits the $x$-axis precisely at the point $(j, 0)$. 
Now we fix \( j \in \{1, \ldots, N\} \), and define \( \Gamma \) to be the set of all lattice paths running from \((0,1)\) to \((j,0)\) that use the steps in \(\{(0,-1),(1,1)\}\) and that hit the \(x\)-axis for the first time at the point \((j,0)\). Given any path \( \gamma \in \Gamma \), we note that, for each \( \ell \in \{0, \ldots, j-1\} \), there is exactly one northeast transition in \( \gamma \), which corresponds to the change of the value of the first coordinate from \( \ell \) to \( \ell + 1 \). Therefore, \( \gamma \) has \( j \) northeast transitions. As each of these transitions results in an increase by 1 of the second coordinate, we conclude that the number of downward transitions in \( \gamma \) equals \( j+1 \). For each \( i = 1, \ldots, j \), let \( m_i = m_i(\gamma) \) denote the number of downward transitions of the path \( \gamma \) that are made when the first coordinate equals \( i \). Then,

\[
P(V_F^{(N)} = j) = \sum_{\gamma \in \Gamma} \prod_{\ell=0}^{j-1} \left( \frac{N-\ell}{N} \right) \prod_{i=1}^{j} \left( \frac{i}{N} \right)^{m_i}.
\]

Defining \( a_j(\gamma) = \prod_{i=1}^{j} i^{m_i} \) (which does not depend on \( N \)) and \( b_j = \sum_{\gamma \in \Gamma} a_j(\gamma) \), we obtain that

\[
P(V_F^{(N)} = j) = \frac{(N-1)!}{(N-j)!} \frac{b_j}{N^{2j}}.
\]

Clearly \( b_1 = 1 \). Using that \( \sum_{j=1}^{N} P(V_F^{(N)} = j) = 1 \), we see that \( \{b_j\} \) satisfies the same recursive formula as \( \{d_j\} \), whence the result follows.

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**Figure 2:** Transitions of the embedded Markov chain.
3.2 Auxiliary results

We present here some definitions and theorems that will be used in the proofs of Propositions 3.2 and 3.3. We start off with Stirling’s asymptotic estimate and bounds for factorials.

Lemma 3.4. (a) \( n! \sim \sqrt{2\pi} n^{n+1/2} e^{-n} \) as \( n \to \infty \).

(b) \( \sqrt{2\pi} n^{n+1/2} e^{-n} \leq n! \leq \sqrt{2\pi} n^{n+1/2} e^{-n} e^{1/12} \) for all \( n \geq 1 \).

The proof of this result can be found in Feller [5, Section II.9]; formula in part (b) is an immediate consequence of Equation (9.15) therein.

In the sequel, we will show that there is a strong relation between the numbers \( d_n \) and the Stirling numbers of the second kind. Recall that, for \( m, n \) nonnegative integers, the Stirling number of the second kind, denoted by \( \{n \atop m\} \), is the number of ways of partitioning a set of \( n \) elements into \( m \) nonempty subsets. We refer to Graham et al. [8, Section 6.1] for more details on this subject. The following result provides the asymptotic estimate of \( \{2n \atop n\} \) for large values of \( n \), which can be obtained by means of general techniques of analytic combinatorics (see Good [7]).

Lemma 3.5. Define the constants 
\[
\alpha = \sqrt{\frac{1}{2\pi (2v_\infty - 1)}} \quad \text{and} \quad \beta = \frac{1}{e v_\infty (1 - v_\infty)}.
\]

Then, as \( n \to \infty \),
\[
\{2n \atop n\} = \alpha^n \beta^{n-1/2} \left(1 + O\left(\frac{1}{n}\right)\right).
\]

We observe that this lemma is a restatement of Lemma 6 in Bassino and Nicaud [1] with \( k = 2 \) and constants \( \alpha \) and \( \beta \) expressed in terms of \( v_\infty \).

Next, we present an asymptotic approximation and an upper bound for \( d_n \), showing how this number is related to the Stirling number \( \{2n \atop n\} \).

Lemma 3.6. (a) As \( n \to \infty \), one has
\[
d_n \sim E n \{2n \atop n\} \quad \text{where} \quad E = 2 - \frac{1}{v_\infty}.
\]

(b) \( d_n \leq n \{2n \atop n\} \) for all \( n \geq 1 \).
Proof. This basically follows from results stated in Bassino and Nicaud [1]. First, we note that, by Lemma 1 therein, the number $|A_n|$ of nonisomorphic initially connected, complete and deterministic automata of size $n$ on a 2-letters alphabet is equal to $2^n d_n$. Hence, part (a) is a consequence of the fact that

$$|A_n| \sim E n 2^n \binom{2n}{n}.$$  

This asymptotic estimate for $|A_n|$ was originally established by Korshunov [9], and reformulated in terms of the Stirling numbers by Bassino and Nicaud [1, Theorem 18]; the simplified expression for the constant $E$ was obtained by Lebensztayn [10]. The inequality in part (b) follows directly from Theorem 2, Equation (1) and Lemma 9 in Bassino and Nicaud [1]. \hfill \Box

Now we introduce some definitions that will be used in the sequel. We define the constants

$$A = \frac{\sqrt{2 v_\infty - 1}}{v_\infty} \approx 0.9669 \quad \text{and} \quad B = e^{-\rho} = \frac{1}{e^2 v_\infty (1 - v_\infty)} \approx 0.8359.$$  

We observe that

$$A = \sqrt{2 \pi \alpha E} \quad \text{and} \quad B = \beta/e. \quad (3.1)$$  

For $v \in (0, 1)$, we define

$$f(v) = \sqrt{\frac{v}{1 - v}} \quad \text{and} \quad g(v) = \exp\{-\psi(v)\} = \frac{B^v v^v}{(1 - v)^{1-v}}. \quad (3.2)$$

### 3.3 Proof of Proposition 3.2

Proposition 3.2 is an immediate consequence of the following result.

**Lemma 3.7.** (a) As $N \to \infty$, we have that

$$P(V_F^{(N)} = N) \sim A B^N.$$  

(b) For $v \in (0, 1)$, let $j^* = j^*(v)$ be either $\lfloor Nv \rfloor$ or $\lceil Nv \rceil$. Then, as $N \to \infty$,

$$P(V_F^{(N)} = j^*) \sim \frac{A}{\sqrt{2\pi N}} f(j^*/N) [g(j^*/N)]^N.$$  

9
Proof of Lemma 3.7. Using Theorem 3.1, Lemmas 3.4, 3.5 and 3.6 and Equation (3.1), we obtain that, as \(N \to \infty\),

\[
P(V_F^{(N)} = N) = \frac{N!}{N^{2N+1}} d_N \sim \frac{\sqrt{2\pi} N^{N+.5} e^{-N}}{N^{2N+1}} E N \left\{ \frac{2N}{N} \right\}
\]

\[
\sim \frac{\sqrt{2\pi} N e^{-N}}{N^{N-.5}} \alpha \beta N^{N-.5} = A B^N.
\]

This shows part (a). Part (b) can be proved similarly (note that, since \(v \in (0,1)\), both \(j^*\) and \(N - j^*\) tend to infinity as \(N \to \infty\)).

3.4 Proof of Proposition 3.3

To prove Proposition 3.3 we establish suitable asymptotic upper bounds for the absorption probabilities. We define

\[
\phi^{(N)}(j) = \begin{cases} f(j/N) [g(j/N)]^N & \text{for } 1 \leq j \leq N-1, \\ B^N & \text{for } j = N, \end{cases}
\]

where \(f\) and \(g\) are given in (3.2). We will use the following results.

Lemma 3.8. For every \(u \in (0,1)\), there exist positive constants \(C_0\) and \(C_1\) such that, for all sufficiently large \(N\),

\[
P(V_F^{(N)} = j) \leq C_0 \phi^{(N)}(j) \left( 1 + \frac{C_1}{Nu} \right) \text{ for each } j = [Nu], \ldots, N.
\]

Lemma 3.9. (a) If \(u \in (0, x_\infty)\), then for all sufficiently large \(N\),

\[
\phi^{(N)}(j) \leq \phi^{(N)}([Nu]) \text{ for every } j = [Nu], \ldots, [Nx_\infty].
\]

(b) If \(v \in (x_\infty, v_\infty)\), then for all sufficiently large \(N\),

\[
\phi^{(N)}(j) \leq \phi^{(N)}([Nv]) \text{ for every } j = [Nv], \ldots, [Nv].
\]

(c) If \(u \in (v_\infty, 1)\), then for all sufficiently large \(N\),

\[
\phi^{(N)}(j) \leq \phi^{(N)}([Nu]) \text{ for every } j = [Nu], \ldots, N.
\]

Proof of Proposition 3.3. To prove the assertion in part (a), we consider \(0 < u < v \leq x_\infty\). Since \(P(Nu \leq V_F^{(N)} \leq Nv) \geq P(V_F^{(N)} = [Nu])\), Proposition 3.2 implies

\[
\limsup_{N \to \infty} \frac{1}{N} \log P(Nu \leq V_F^{(N)} \leq Nv) \leq \psi(u).
\]
On the other hand, from Lemmas 3.8 and 3.9, it follows that for all sufficiently large \( N \),

\[
P(Nu \leq V_F^{(N)} \leq Nv) = \sum_{j=\lfloor Nu \rfloor}^{\lceil Nu \rceil} P(V_F^{(N)} = j) \leq C_0 \left( 1 + \frac{C_1}{Nu} \right) N \phi^{(N)}([Nu]).
\]

Consequently,

\[
\liminf_{N \to \infty} -\frac{1}{N} \log P(Nu \leq V_F^{(N)} \leq Nv) \geq \psi(u),
\]

which yields the result in part (a). The remaining statements are proved in a similar way. \( \square \)

Proof of Lemma 3.8

By Lemma 3.5, there exists a positive constant \( K \) such that, for all sufficiently large \( n \),

\[
\left\{ 2n \atop n \right\} \leq \alpha^n n^{-1/2} \left( 1 + \frac{K}{n} \right).
\]

Therefore, using Theorem 3.1 and Lemmas 3.4 and 3.6, we conclude that, for all sufficiently large \( N \),

\[
P(V_F^{(N)} = N) = \frac{N!}{N^{2N+1}} d_N \leq \frac{\sqrt{2\pi} N^{N+1/2} e^{-N} e^{1/12}}{N^{2N+1}} N \left\{ 2N \atop N \right\}
\leq \frac{\sqrt{2\pi} e^{1/12} e^{-N}}{N^{N-1/2}} \alpha^N N^{N-1/2} \left( 1 + \frac{K}{N} \right)
= \alpha \sqrt{2\pi} e^{1/12} B^N \left( 1 + \frac{K}{N} \right).
\]

Analogously, if \( u \in (0,1) \), then for all sufficiently large \( N \) and each \( j = \lfloor Nu \rfloor, \ldots, N-1 \),

\[
P(V_F^{(N)} = j) = \frac{N!}{(N-j)!} \frac{d_j}{N^{2j+1}} \leq \frac{N^{N+1/2} e^{-N} e^{1/12}}{(N-j)^{N-j+1/2} e^{-(N-j)}} \frac{1}{N^{2j+1}} \left\{ 2j \atop j \right\}
\leq \frac{e^{1/12}}{N^{1/2}} \frac{j N^N}{e^j N^{2j}} \frac{1}{N^{2j+1}} \frac{\alpha \beta^j j^{-1/2} \left( 1 + \frac{K}{Nu} \right)}{(N-j)^{N-j+1/2} e^{-(N-j)}}
= \frac{\alpha e^{1/12}}{N^{1/2}} \frac{j^{1/2} B^j \phi^{(N)}(j) \left( 1 + \frac{K}{Nu} \right)}{(N-j)^{N-j} e^{j}},
\]

whence the result follows. \( \square \)

Proof of Lemma 3.9

(a) We consider \( N \geq 10 \) (so that \( \lfloor N x_\infty \rfloor > 1 \) and \( \lceil N v_\infty \rceil < N - 1 \),
and note that

\[
\frac{\partial \phi^{(N)}(j)}{\partial j} = \phi^{(N)}(j) \Delta^{(N)}(j),
\]
where
\[
\Delta^{(N)}(j) = \frac{N}{2j(N-j)} + \log \left[ \frac{j/N}{v_\infty(1-v_\infty)} \right].
\]
Also, let \(j_1 < j_2 < j_3\) be given by
\[
j_1 = \frac{N}{2} \left( 1 - \sqrt{1 - 2/N} \right), \quad j_2 = \frac{N}{2}, \quad j_3 = \frac{N}{2} \left( 1 + \sqrt{1 - 2/N} \right).
\]
Then,
\[
\Xi^{(N)}(j) = \frac{\partial \Delta^{(N)}(j)}{\partial j} = \frac{(N - 2j)[2j(N - j) - N]}{2j^2(N - j)^2} = \frac{2(j - j_1)(j - j_2)(j - j_3)}{j^2(N - j)^2}.
\]
Since \(j_1 < 1\), we have that \(\Xi^{(N)}(j) > 0\) for every \(j \in (1,j_2)\). In addition, for all sufficiently large \(N\),
\[
\Delta^{(N)}(1) = \frac{N}{2(N-1)} + \log \left[ \frac{1/N}{v_\infty(1-v_\infty)} \right] < 0, \quad \text{and} \quad \Delta^{(N)}(j_2) = \frac{2}{N} + \log \left[ \frac{1}{4v_\infty(1-v_\infty)} \right] > 0.
\]
Now fix \(u \in (0,x_\infty)\). The assertion in part (a) is proved once we show that for all sufficiently large \(N\),
\[
\phi^{(N)}([Nx_\infty]) \leq \phi^{(N)}([Nu]). \tag{3.3}
\]
To prove this, we define for \(z \in (0,1)\),
\[
\Gamma^{(N)}(z) = \left[ f(z) \right]^{1/N} g(z) = \left( \frac{z}{1-z} \right)^{1/(2N)} \exp\{-\psi(z)\}.
\]
By Dini’s Theorem, as \(N \to \infty\), the sequence \(\{\Gamma^{(N)}\}\) converges to \(g\) uniformly on each closed interval contained in the interval \((0,1/2)\). Consequently,
\[
\lim_{N \to \infty} \left[ \phi^{(N)}([N x_\infty]) \right]^{1/N} = g(x_\infty) < g(u) = \lim_{N \to \infty} \left[ \phi^{(N)}([Nu]) \right]^{1/N}.
\]
This implies that (3.3) holds true for all sufficiently large \(N\).

(b) Fixed \(v \in (x_\infty,v_\infty)\), the statement in (b) follows from the facts that the function \(\phi^{(N)}\) is continuous on the interval \([\lfloor Nx_\infty \rfloor, \lfloor Nu \rfloor]\), and that
\[
\Delta^{(N)}(j) > \log \left[ \frac{j/N}{v_\infty(1-v_\infty)} \right] > 0
\]
for every \(j \in ([Nx_\infty], [Nu])\).
We first observe that for every $N \geq 2$,
\[ B < 1 \leq \frac{(N - 1)^{N-1/2}}{N^{N-2}}, \]
which implies that $\phi^{(N)}(N) \leq \phi^{(N)}(N - 1)$. Now fixed $u \in (v, 1)$, we prove that for all sufficiently large $N$, the function $\phi^{(N)}$ is decreasing on the interval $[\lfloor Nu \rfloor, N - 1]$. As $j_3 > N - 1$, we conclude that $\Xi^{(N)}(j) < 0$ for every $j \in (j_2, N - 1)$. Thus, it is enough to show that for all sufficiently large $N$,
\[ \Delta^{(N)}(\lfloor Nu \rfloor) < 0. \] (3.4)
To accomplish this, we define for $z \in (0, 1)$,
\[ \Lambda^{(N)}(z) = \frac{1}{2Nz(1-z)} + \log \left[ \frac{z(1-z)}{v(1-v)} \right], \]
and
\[ \Lambda(z) = \log \left[ \frac{z(1-z)}{v(1-v)} \right]. \]
By Dini's Theorem, the sequence $\{\Lambda^{(N)}\}$ converges to $\Lambda$ as $N \to \infty$, uniformly on each closed interval contained in the interval $(0, 1)$. Hence,
\[ \lim_{N \to \infty} \Delta^{(N)}(\lfloor Nu \rfloor) = \lim_{N \to \infty} \Lambda^{(N)}\left(\frac{\lfloor Nu \rfloor}{N}\right) = \Lambda(u) < 0. \]
From this, it follows that (3.4) is valid for all sufficiently large $N$. 

3.5 Proof of Theorem 2.2

For a set $K \subset [0, \infty)$, we denote by $I(K)$ the infimum of $H(x)$ over $K$. To prove part (a), let $F$ be a nonempty closed subset of $[0, \infty)$. If $I(F) = 0$ or $I(F) = \infty$, there is nothing to prove. Assume that $0 < I(F) < \infty$, and define
\[ x_1 = \sup (F \cap [0, x_\infty]), \quad x_2 = \inf (F \cap [x_\infty, v_\infty]), \quad x_3 = \sup (F \cap [v_\infty, \infty)). \] (3.5)
By the monotonicity properties of $H$, we have that $I(F) = H(x_1) \land H(x_2) \land H(x_3)$ (we suppose that none of the intersections in (3.5) is empty; otherwise, the corresponding term is missing). Using Proposition 2.4, we get
\[
\limsup_{N \to \infty} \frac{1}{N} \log \nu_N(F) \leq \limsup_{N \to \infty} \frac{1}{N} \log (\nu_N([0,x_1]) + \nu_N([x_2,v_\infty]) + \nu_N([v_\infty,x_3]))
\]
\[ = \max \left\{ \limsup_{N \to \infty} \frac{1}{N} \log \nu_N([0,x_1]), \limsup_{N \to \infty} \frac{1}{N} \log \nu_N([x_2,v_\infty]), \limsup_{N \to \infty} \frac{1}{N} \log \nu_N([v_\infty,x_3]) \right\}
\]
\[ = \max \{-H(x_1), -H(x_2), -H(x_3)\} = -I(F). \]
This establishes part (a).

Regarding part (b), let $G$ be a nonempty open subset of $[0, \infty)$. We will show that for each $x \in G$,

$$\liminf_{N \to \infty} \frac{1}{N} \log \nu_N(G) \geq -H(x). \quad \text{(3.6)}$$

As (3.6) trivially holds for $x \in G \cap [1, \infty)$, it is enough to prove it for $x \in G \cap [0,1)$.

First, assume that $x \in G \cap [0,x_\infty)$. Then, there exists $y < x$ such that $(y, x] \subset G \cap [0, x_\infty)$. Using that $P(X^{(N)}_F \leq Nx) = P(X^{(N)}_F \leq Ny) + \nu_N((y, x])$ and Proposition 2.4, we obtain

$$-H(x) \leq \max \left\{ -H(y), \liminf_{N \to \infty} \frac{1}{N} \log \nu_N((y, x]) \right\}.$$ 

Since $H(x) < H(y)$, it follows that (3.6) holds true.

Now if $x \in G \cap (x_\infty, v_\infty]$, then there exists $y < x$ such that $[y, x] \subset G \cap (x_\infty, v_\infty]$. Consequently,

$$\liminf_{N \to \infty} \frac{1}{N} \log \nu_N(G) \geq \liminf_{N \to \infty} \frac{1}{N} \log \nu_N([y, x]) = -H(y) \geq -H(x).$$

A similar argument shows that (3.6) also holds for each $x \in G \cap (v_\infty, 1)$. Since $G$ is open and $H$ is continuous on $x_\infty$, we conclude that (3.6) is valid for every $x \in G \cap [0,1)$. This completes the proof of part (b).

\[ \square \]

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