ECTOPLASM HAS NO TOPOLOGY

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ABSTRACT

In a new approach to the theory of integration over Wess-Zumino supermanifolds, we suggest that a fundamental principle is their consistency with an “Ethereal Conjecture” that asserts the topology of the supermanifold must be generated essentially from its bosonic submanifold. This naturally leads to a theory of “ectoplasmic” integration based on super $p$-forms. One consequence of this approach is that the derivation of “density projection operators” becomes trivial in a number of supergravity theories.

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I. Introduction

In some of the literature on supermanifolds, the bosonic sub-manifold of the superspace is referred to as the “body” of the superspace. Similarly, the remainder of the superspace has been called its “soul”\(^3\). This set of conventions permits us the frivolity of referring to the “spiritual” part of superspace as its “ectoplasm.” Integration theory over bosonic manifolds has been defined in a number of ways, i.e. Riemann-Steljes, Lebesque, etc. However, for a long time there was an open question regarding the issue of a general theory of integration over local supersymmetry manifolds \(^1\), also called Wess-Zumino superspaces. One approach requires the existence of a local supersymmetry measure over which the integration of all the superspace coordinates can be performed. In principle, for a superspace with \(N_B\) (sometimes denoted by the symbol D in the following) bosonic and \(N_F\) (sometimes denoted by the symbol N in the following) fermionic coordinates, such measures are provided by \(\int d^{N_B+N_F}z \left[ sdet (E^A_M) \right]^{-1} \). In practice knowing this does not simply lead to an explicit expression in terms of the component fields of the supergravity multiplet and other supermultiplets to which it may couple. For this purpose, it is most convenient to define an operator known as the “local density projector.” The local density projector is the crucial ingredient in conveniently obtaining component results directly from superspace without need for explicit \(\theta\)-expansions. Alternately, the use of the local density projector is equivalent to local “ectoplasmic” integration (i.e. integration over all \(\theta\)’s in a Wess-Zumino superspace).

Until quite recently, the construction of such projectors had been done on a case-by-case basis. Only recently \(^2\) has a complete theory that extends the definition of rigid Berezinian-type Grassmannian integration to the construction of density projectors in the local case of superspace manifolds been initiated along the two distinct lines; (a.) super-differential forms \(^3\) and (b.) normal coordinate expansion \(^4\). In the latter work it was shown that a normal coordinate expansion technique is well able to compute density projectors in complete generality. The former approach, however, seems to be related to the deeper issue of topology of supersymmetry manifolds.

An outstanding question in supergravity theory (as formulated in its natural setting of curved Wess-Zumino superspace) may be cast in the form, “What is the fundamental reason why superspace supergravity theories must be formulated in terms of constraints on torsion and curvature supertensors?” A closely related question is, “How should these constraints be chosen for an arbitrary supergravity theory?”\(^4\)

\(^3\)We take our “otherworldly” sounding title in deference to this set of conventions.

\(^4\)We can expand these questions, of course, to cover all supersymmetric gauge theories.
These types of questions have plagued the superspace formulation of supergravity theory for about two decades with no satisfactory answers to date. In one of our earliest works [5], we began to grope toward answers that led to the realization of the importance of representation-preserving constraints, conventional constraints, and conformal scale-breaking constraints [6] in classifying the types of constraints imposed in supergravity theory. A definite role for the superconformal group was noted and it was found that different versions of off-shell 4D, $N = 1$ supergravity correspond to different choices of how superconformal symmetry is broken to become Poincaré supersymmetry.

This scheme represented progress but did not provide a completely satisfactory rationale for why the constraints exist. Nor does it lend great insight into how to choose complete sets of constraints. Another serious deficit of this approach is that it explains the supergravity constraints in terms of the local extension of certain matter field representations from the case of global supersymmetry to local supersymmetry. Such an approach necessarily requires the existence of matter supermultiplets. For many cases of interest, such matter multiplets do not exist. The need for a deeper understanding is clearly indicated. We now believe this issue is closely related to the first class of questions that we raised in the three paragraphs above.

One other question concerning the nature of local supersymmetry is the possible topological super-extensions of arbitrary manifolds [7]. Stated most simply, “Is it possible to have superspaces where the topological properties of the superspace are substantially different from the bosonic sub-manifold it contains?” In all known examples in supergravity theories the answer to this question is no.

This suggests a very interesting approach to solving the puzzles described above. It, perhaps, is reasonable to assert that a hitherto unrecognized guiding principle may be at work here. As a working assumption we will assert that this is the case by proposing what we will refer to in the following discussion as the “Ethereal Conjecture of Local Extended Manifolds” Our colloquial statement of this conjecture is,

*The topology of an “extended” manifold must essentially arise from its bosonic sub-manifold.*

Since it is known that superspace does not possess a de Rahm cohomology, it may well be that the Ethereal Conjecture is precisely what is needed to fill this gap. In the following, we will show evidence for the conjecture.

The organization of this paper is as follows. In section two, we introduce our approach to the formulation of topological invariants in superspace. The discussion begins by introducing the “Ectoplasmic Integration Theorem” for general curved Wess-
Zumino superspaces. This is shown to be consistent with the usual “components-
by-projection” technique for evaluating rigid superfield actions. The generalization
of this technique to curved superspace is found to lead to an expansion as first en-
visioned by Zumino. For the first time to our knowledge, a definition of a general
superspace topological index is proposed and related to the Ectoplasmic Integration
Theorem.

In sections three through five, since at present it is not possible to construct a
proof\footnote{A proof of the Ethereal Conjecture within the context of supersymmetrical theories would first
require that the off-shell constraints of all supersymmetric theories be known!} we have gathered together a survey of theories in 2D, 3D and 4D superspaces
which all demonstrate the previously unrecognized realization of the Ethereal Con-
jecture as a universal feature.

In section six, we discuss the case of the 2D, (4,4) theory. Since we have yet
to establish a complete understanding of this new approach, we use this particular
example as an illustration of some difficulties that remain. As we shall show, compli-
cations do arise here. We believe that these problems are generic when \( N_F > N_B \) (as
is the case with most of the interesting theories).

In section seven, we slightly change our perspective. In the previous sections, our
attention was directed toward the problem of defining a general theory of integration
on local supersymmetry manifolds. In the seventh section, the focus is mainly directed
to the question of how the Ethereal Conjecture may be la forza del primo behind
why constraints must be imposed on all supersymmetric theories. In the example
discussed, we shall see that the constraints may be interpreted as the topological
obstructions to the realization of the Ethereal Conjecture.

In section eight we give our prospectives on applying these ideas to find new
techniques with which to attack long unsolved problems of finding off-shell constraints
for 10D supersymmetrical field theories.

II. The Ectoplasmic Integration Theorem and Topological Invariants

The theory of integrating fermionic numbers received its beginning with the
work of Berezin \footnote{A proof of the Ethereal Conjecture within the context of supersymmetrical theories would first
require that the off-shell constraints of all supersymmetric theories be known!}, who defined properties of quantities such as \( \int d^{N_B+N_F} z \) over
“flat supermanifolds.” This was extended to the case of “curved supermanifolds”
when Arnowitt, Nath and Zumino (ANZ) \footnote{A proof of the Ethereal Conjecture within the context of supersymmetrical theories would first
require that the off-shell constraints of all supersymmetric theories be known!} showed how to formally (and explicitly)
define the super-determinant. This permits one to write \( \int d^{N_B+N_F} z \left[ sdet(\mathbf{E}_{\mathbf{M}}) \right]^{-1} \) as
a formal way to integrate over curved supermanifolds. As we pointed out above, this expression is of limited practical value. For practical (i.e. component) calculations it is required to have an equation of the form:

\[ \int d^{N_B+N_F} z \, E^{-1} \mathcal{L} = \int d^{N_B} z \, e^{-1} \left[ D^{N_F} \mathcal{L} \right], \]

\[ \rightarrow D^{N_F} \equiv e \int d^{N_F} z \, E^{-1}, \]

for an arbitrary superfunction \( \mathcal{L} \). Here \( E_{\underline{A} M} \) denotes the inverse supervielbein for the entire superspace and \( e_{\underline{a} m} \) denotes the inverse vielbein on the body of the superspace. (Also we have used the notations \( \text{sdet}(E_{\underline{A} M})^{-1} \equiv E^{-1} \text{ and } \text{det}(e_{\underline{a} m})^{-1} \equiv e^{-1}. \)) The symbols \( \int d^{N_B+N_F} z \) and \( \int d^{N_B} z \) denote integrations over the full superspace and the body of the superspace, respectively. In the equation above, \( D^{N_F} \) denotes a particular \( N_F \)-order differential operator constructed from the supergravity covariant derivative \( \nabla_\alpha \). Furthermore, \( D^{N_F} \mathcal{L} \) corresponds to the action of first applying the operator \( D^{N_F} \) to \( \mathcal{L} \) and afterward in that result setting all the fermionic coordinates to zero. The range of integration on the right hand side of the equation above is evaluated on a sub-space of the range of integration on the left hand side. The a priori derivation of \( D^{N_F} \) has until recently been an unsolved problem that will be our main concern in the following. We call the quantity \( e^{-1}[D^{N_F} \mathcal{L}] \) the “local density projector” acting on \( \mathcal{L} \).

Let us point out that it is not our goal to derive equation (1). This is the starting point of the works in [2, 4]. There it is shown that a normal coordinate expansion can be applied to the left hand side of (1) and used to rigorously derive the Ectoplasmic Integration Theorem or E.I.T. Instead we wish to take the right hand side of the equation as a starting point and attempt to formulate a logically consistent formalism that can be used to derive the operator \( D^{N_F} \) from some principle and with no reference to the superdeterminant. As we will see in the following there is no need to introduce the notion of a superdeterminant. Our operatorial oriented approach is an entirely different way to view this problem and based upon the fact that super forms have a well defined meaning. We have previously given a short introduction to this alternate approach [3].

The operator \( D^{N_F} \) must be the local extension of a well known result from rigid

\[ ^6 \text{See also DeWitt [10] who discussed related issues. Unfortunately, his discussion of integration over curved manifolds is restricted to Riemannian supermanifolds which are not relevant to supergravity theories.} \]

\[ ^7 \text{We call this the “Ectoplasmic Integration Theorem.” It vaguely resembles the Stoke’s Theorem of multi-variable calculus and allows us to completely perform the integrations over the “soul” or ectoplasm of the superspace.} \]
supersymmetry. Within rigid supersymmetry theories, it is permissible to write as an extension of Berezin’s original definition\(^8\)

\[
\int d^{N_B+N_F} z \mathcal{L} \equiv \int d^{N_B z} \left[ (D \cdot \cdot \cdot D)^{N_F} \mathcal{L} \right],
\]

(2)

where \(D\) is a symbolic representation of all of the spinorial derivatives of the super-space with \(N_F\) Grassmann coordinates. For 4D, \(N = 1\) superspace as an example, we write,

\[
\int d^4x \ d^2\theta \ d^2\bar{\theta} \mathcal{L} \equiv \frac{1}{2}\left\{ \int d^4x \left[ D^2 \mathcal{T}^2 \mathcal{L} \right] + \text{h.c.} \right\}.
\]

(3)

In the local case, there must exist the extension

\[
\int d^{N_B+N_F} z E^{-1} \mathcal{L} = \int d^{N_B z} e^{-1} \left[ \sum_{i=0}^{N_F} c_{(N_F-i)} (\nabla \cdot \cdot \cdot \nabla)^{N_F-i} \mathcal{L} \right]
\]

(4)

\[
\equiv \int d^{N_B z} e^{-1} \left[ \mathcal{D}^{N_F} \mathcal{L} \right],
\]

where \(\nabla\) represents the supergravity spinorial derivatives that describe the corresponding curved Wess-Zumino superspace. The coefficients \(c_{(N_F-i)}\) in this expansion are not constants. In general they depend on the supergravity component fields (both physical and auxiliary). The challenged raised by Zumino \(^\text{[1]}\) (which has now been definitively answered \(^\text{[4, 6]}\) was to find a method by which these coefficients might be calculated from some principle. This problem had remained unsolved (and usually unrecognized) in most of the years since it was first noted.

For a long time it has been known that the leading coefficient, \(c_{(N_F)}\), can be set equal to a constant. In the rigid limit, where all supergravity component fields vanish, this permits the result of (4) to agree with (2). Similarly, it is known that the remaining coefficients represent an expansion involving the gravitino and other fields of the supergravity multiplet. For example, in many (but not all) theories we find

\[
\left. c_{(N_F-1)} \right| \propto i \psi^\alpha \left( \gamma^\mu \right)_\beta \psi^\beta.
\]

(5)

It is also clear that the volume of the full superspace is given by

\[
\int d^{N_B+N_F} z E^{-1} = \int d^{N_B z} e^{-1} c_{(0)}
\]

(6)

so that the volume vanishes whenever \(c_{(0)} = 0\).

Since to our knowledge, the whole topic of topology of supermanifolds is not well developed mathematically, we will proceed as cautiously as possible recognizing that there may not be rigorous mathematical definitions for all of the steps we define

\(^8\)This differs from Berezin’s original definition only by total derivative terms.
operationally and heuristically. Let us set up the general situation. We use $\mathcal{L}(\tilde{I})$ to represent some superspace topological invariant (i.e. we have in mind some index associated with a field theory). If it is a topological invariant then we demand that by definition the following condition be satisfied.

$$\int d^{N_{B}} z \ e^{-1} \left[ \mathcal{D}^{N_{F}} \mathcal{L}(\tilde{I}) \right] = \int d^{N_{B}} z \ e^{-1} \left[ e_{\underline{a}}^{m} \partial_{\underline{m}} \left( J_{(I)}^{\underline{a}} + J_{(E)}^{\underline{a}} \right) \right] ,$$

for some quantities $J_{(I)}^{\underline{m}} \equiv J_{(I)}^{\underline{a}} e_{\underline{a}}^{m}$ and $J_{(E)}^{\underline{m}} \equiv J_{(E)}^{\underline{a}} e_{\underline{a}}^{m}$. As can been clearly seen, these are integrals of total divergences and hence their values can only depend on the values of fields at the boundary of the integration. The first integral on the right hand side is, in fact, the corresponding topological invariant on the body of the superspace. The integrand of the second must correspond to an exact globally well defined quantity. Roughly speaking the above equations suggest that in all local supersymmetry theories the following equation is true

$$\mathcal{H}(sM_{N_{B}+N_{F}}) \approx \mathcal{H}(M_{N_{B}}) ,$$

where $\mathcal{H}(sM_{N_{B}+N_{F}})$ denotes any homotopy group (or element thereof) of the supermanifold with $N_{B}$ bosonic coordinates and $N_{F}$ fermionic coordinates. A similar interpretation of the symbol $\mathcal{H}(M_{N_{B}})$ is to be understood for the purely bosonic sub-manifold of dimension $N_{B}$.

Another question that arises is, “Which topological invariants are to be used in calculating the local density projector?” Our answer to this is that all topological invariants which can be constructed in the class of field theories over a given manifold should satisfy this condition if that manifold is regarded as the body of a supermanifold. This statement implies that any topological invariant that can be written is a candidate to use for the derivation of the local density projector. Furthermore, given that the local density projector has been calculated using one topological invariant, our previous statement implies that the same answer will be obtained from the use of any other topological invariant that exists over the body of the superspace.

With the idea that topology lies at the heart of the problem of finding the $c(N_{F} - i)$ coefficients, a role for super p-forms can be discerned. The 4D, $N = 1$ superspace formulation of irreducible super p-forms [11] can, in principle, be extended to all values of $D$ and $N$. In a superspace of $N_{B}$ bosonic coordinates, a special role is

\footnote{Discussions of this nature can be seen in some of the recent works on D-p-branes and type IIB supergravity [12].}
accorded to super $N_B$-forms. A topological index $\Delta$ results from an integral of a closed but not exact $N_B$-form. For any closed super $N_B$-form, by a choice of Wess-Zumino gauge it follows that in the presence of supergravity, the component of the super $N_B$-form that possesses only bosonic indices satisfies:

$$\left( F_{A_1 \cdots A_N} \right) = \left[ \tilde{f}_{A_1 \cdots A_N} + \lambda^{(N_B,1)} \psi_{A_1}^{\alpha_1} (F_{A_1 A_2 \cdots A_N}) \right] + \lambda^{(N_B,2)} \psi_{A_1}^{\alpha_1} \psi_{A_2}^{\alpha_2} (F_{A_1 A_2 A_3 \cdots A_N}) \cdots + \lambda^{(N_B,N_B)} \psi_{A_1}^{\alpha_1} \psi_{A_2}^{\alpha_2} \cdots \psi_{A_N}^{\alpha_N} (F_{A_1 A_2 \cdots A_N}) \right] \quad (9)$$

where $\lambda^{(N_B,i)}$ are some normalization constants that are easy to calculate and $\psi_{A}^{\alpha}$ denotes the gravitino. The explicit value of these constants depend on $N_B$ and $N_F$.

In fact, a normal coordinate expansion should offer the simplest way to derive this equation. Above $\tilde{f}_{A_1 \cdots A_N}$ is a non-supersymmetric bosonic $N_B$-form component field. If $\tilde{f}_{A_1 \cdots A_N}$ is closed (which implies that $F_{A_1 \cdots A_N}$ is super closed), it follows that a topological index ($\Delta$) is defined through the equation,

$$\Delta \equiv (N_B!)^{-1} \int d^{N_B} z \ e^{-1} e^{A_1 \cdots A_N} \tilde{f}_{A_1 \cdots A_N} \quad (10)$$

Now using (9) we see that $\Delta = \Delta$ where

$$\Delta \equiv \int d^{N_B} z \ e^{-1} e^{A_1 \cdots A_N} \left[ (N_B!)^{-1} (F_{A_1 \cdots A_N}) - \lambda^{(N_B,1)} \psi_{A_1}^{\alpha_1} (F_{A_1 A_2 \cdots A_N}) \right. \left. - \lambda^{(N_B,2)} \psi_{A_1}^{\alpha_1} \psi_{A_2}^{\alpha_2} (F_{A_1 A_2 A_3 \cdots A_N}) \cdots - \lambda^{(N_B,N_B)} (N_B!)^{-1} \left[ \psi_{A_1}^{\alpha_1} \cdots \psi_{A_N}^{\alpha_N} (F_{A_1 A_2 \cdots A_N}) \right] \right] \quad (11)$$

is used as a definition. One might not expect the equation $\Delta = \Delta$ (which may be regarded as an expansion in terms of the gravitino) to contain any information. It seems to be a simple tautology. However, in the presence of constraints (required to define irreducible p-forms) and via the solution to the Bianchi identities on $F_{A_1 \cdots A_N}$, this equation can in many cases be used to derive the operator $D^{N_F}$. What emerges from this approach is that the gravitino expansion in (11) often directly produces the

We first derived this result for the case of the 2-form within 4D, $N = 4$ supergravity theory [13].

That derivation can easily be extended to all values $p, N_B$ and $N_F$.

We emphasize that this equation is valid independent of the constraints to which $F_{A_1 \cdots A_N}$ is subject.
coefficients $c_{(N_F-i)}$ of the local density projector. Let us further note that we may write

\[ d^{N_B} e^{-1} \varepsilon^a_1 \cdots \varepsilon^a_{N_B} = d\varepsilon^a_{m_1} \wedge \cdots \wedge d\varepsilon^a_{m_{N_B}} e_{a_1} \cdots e_{a_{N_B}} \equiv d\omega^{a_1} \cdots \omega^{a_{N_B}} . \tag{12} \]

In the paragraph above, we mentioned that in order to be irreducible, the super $p$-form $F_{A_1 \cdots A_{N_B}}$ must be subject to a set of constraints. It is our belief that even in those cases where the equation (11) does not lead to a complete determination of the density projector, it is still of importance because it seems to completely determine the gravitino field dependence of the density projector. We conjecture\(^{13}\) that it is always the case that

\[ D^{N_F} = \varepsilon^{a_1} \cdots \varepsilon^{a_{N_B}} \left[ (N_B!)^{-1} D_{a_1} \cdots \omega_{N_B} - \lambda^{(N_B,1)} \psi_{a_1}^{a_2} D_{a_2} \cdots \omega_{N_B} \right. \]

\[ - \left. \lambda^{(N_B,2)} \psi_{a_1}^{a_2} \psi_{a_2}^{a_3} D_{a_1 a_2} \cdots \omega_{N_B} \cdots \right] \]

\[ - \lambda^{(N_B,N_B)} (N_B!)^{-1} \left[ \psi_{a_1} \cdots \psi_{a_{N_B}}^{a_{N_B}} \right] D_{a_1 a_2 \cdots a_{N_B}} \right] , \tag{13} \]

where the operators $D_{a_1} \cdots \omega_{N_B}, \ldots, D_{a_1 \cdots a_{N_B}}$ are independent of the gravitino field and spacetime derivatives. Dimensional analysis implies that $D_{a_1} \cdots \omega_{N_B}$ must be of order $N_F$ in $\nabla_{\alpha}$, $D_{a_1 a_2} \cdots \omega_{N_B}$ must be of order $N_F - 1$ in $\nabla_{\alpha}$, and so forth until one gets to $D_{a_1 a_2 \cdots a_{N_B}}$ which must be of order $N_F - N_B$ in $\nabla_{\alpha}$. The coefficients of the operators $D_{a_1} \cdots \omega_{N_B}, \ldots, D_{a_1 \cdots a_{N_B}}$ can only depend on the superspace torsions and their spinorial derivatives. A notable implication of (13) is that the gravitino has a maximum power to which it is raised in the density projector, i.e. $N_B$ in the final term above. This property is not at all obvious from the ANZ local measure.

We note that any closed super $N_B$-form can be used to play the role of $F_{A_1 \cdots A_{N_B}}$. Thus, even in a theory with no matter superfields, topological indices may be constructed directly from the supergravity multiplet. In this way, we may say that the topology of the bosonic sub-manifold determines the local theory of superspace integration.

Let us discuss a bit about some additional notation. The index $\tilde{\Delta}$ is defined for $\tilde{f}_{a_1} \cdots \omega_{N_B}$, the leading term in (9). Since this quantity is closed, it is possible to add an exact $N_B$-form to it without changing the fact that $\tilde{f}_{a_1} \cdots \omega_{N_B}$ is closed. In many classes of interest, this form can be thus separated where the exact form is constructed from

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\(^{12}\) We define such density projectors as stage-II density projectors. An example of such a system will be discussed in a later section.

\(^{13}\) With the application of the normal coordinate expansion technique [2, 4], it should be possible to investigate this.
fermionic fields or other “matter” fields. This is responsible for the structure of (7). We may thus introduce another non-supersymmetric index (denoted by $\Delta$) by taking the limit of $f_{a_1 \ldots a_{N_B}}$ where all matter fields are set to zero.

Our considerations also suggest some further interesting observations. For example, we can consider a closed super $p$-form (with $p < N_B$ in a superspace of $N_B$ bosonic and $N_F$ fermionic coordinates), for which it follows there exists $\tilde{f}_{a_1 \ldots a_p}^{(p)}$ such that

$$\tilde{f}_{a_1 \ldots a_p}^{(p)} \equiv \left[ (F_{a_1 \ldots a_p})^\beta - \lambda^{(p,1)} \psi_{a_1}^{\alpha_1} (F_{a_1 a_2 \ldots a_p})^\beta - \lambda^{(p,2)} \psi_{a_2}^{\alpha_2} (F_{a_1 a_2 \ldots a_p})^\beta \ldotsight].$$

(14)

If there are defects or surfaces of interest characterized by differentials $d\omega_{a_1 \ldots a_p}$, these may be coupled to a super $p$-form $F_{A_1 \ldots A_p}$ by use of (14) and

$$S(\omega) \equiv (p!)^{-1} \int d\omega_{a_1 \ldots a_p} \tilde{f}_{a_1 \ldots a_p}^{(p)} .$$

(15)

Since defects or surfaces are used to define $S(\omega)$, this may (partially) break supersymmetry. This observation may be of use in discussing the coupling to branes. Equations (14,15) constitute the definition of a theory of integration for super $p$-forms.

The crucial features of the arguments in this section are the existence of the formulae (9) and (14). In turn the most critical features of these formulae are the coefficients $\lambda^{(N_B,\ell)}$ and $\lambda^{(p,\ell)}$ where $\ell$ is a dummy index that ranges appropriately for each of these. It turns out that their determination follows by a simple observation [4]. We may return to (12) and consider its existence in a purely bosonic space where we can use it to show

$$d\omega_{a_1 \ldots a_{N_B}} f_{a_1 \ldots a_{N_B}} = dz_{m_1} \wedge \ldots \wedge dz_{m_{N_B}} e_{m_1}^{a_1} \ldots e_{m_{N_B}}^{a_{N_B}} f_{a_1 \ldots a_{N_B}} = dz_{m_1} \wedge \ldots \wedge dz_{m_{N_B}} f_{m_1 \ldots m_{N_B}} .$$

(16)

In a purely bosonic spacetime we also have

$$f_{m_1 \ldots m_{N_B}} = e_{m_1}^{a_1} \ldots e_{m_{N_B}}^{a_{N_B}} f_{a_1 \ldots a_{N_B}} .$$

(17)

In a superspace, however, this equation must be generalized to

$$f_{m_1 \ldots m_{N_B}} = (-1)^{[\frac{N_B}{2}]} E_{m_1}^{A_1} \ldots E_{m_{N_B}}^{A_{N_B}} F_{A_1 \ldots A_{N_B}} ,$$

(18)

where $[\frac{N_B}{2}]$ denotes the greatest integer in $\frac{N_B}{2}$ and

$$E_{m}^{A} \equiv (-\psi_{m}^{\alpha}(x), e_{m}^{a}(x)) .$$

(19)
When these components for the supervielbein are substituted into (18) and it is expanded over the super indices $A_1, \ldots, A_{N_B}$, the required coefficients $(\lambda^{NB,\ell})$ appear. This same argument applies to the other set of coefficients in (14)[14]. The remarkable feature of the argument in this paragraph is that the supervielbein that appears in (19) is a superfield whereas the fields that appear on the right hand side are simple component fields. Although seemingly contradictory, this identification is correct within the context it is used.

### III. 2D Local Ectoplasmic Integration

We begin with the simplest possible 2D superspace supergravity theory that exists (i.e., (1,0) supergravity). The superspace description of this theory has been known for some time [15]. Its superspace supergravity covariant derivative $(\nabla_A \equiv (\nabla_+, \nabla_\pm, \nabla_\mp)$ satisfies

$$[\nabla_+, \nabla_+] = i2\nabla_\mp \ , \ [\nabla_+, \nabla_\pm] = 0 \ , \ \nabla_+ \Sigma^+ = \frac{1}{2} \mathcal{R} \ , \ (20)$$

$$[\nabla_+, \nabla_\mp] = -i2\Sigma^+ \mathcal{M} \ , \ [\nabla_\pm, \nabla_\mp] = -\left( \Sigma^+ \nabla_\mp + \mathcal{R} \mathcal{M} \right) \ . \ (21)$$

The quantities $\Sigma^+$ and $\mathcal{R}$ are superfield field strengths and $\mathcal{M}$ denotes the generator of $SO(1,1)$, the 2D Lorentz group. On defining $\Sigma^+|\equiv \lim_{\zeta^+ \to 0} \Sigma^+\hat{M}$ and similarly for $\mathcal{R}|$, we find

$$\Sigma^+| = -\psi_+ = -\left[ e_+ \psi_+ - e_- \psi_-^+ - c_+ \xi^+ \psi_-^+ - c_- \xi^- \psi_+ \right] \ , \ (22)$$

$$r_\pm(\omega) = \frac{1}{2} \left[ r_\pm(\omega) + i2\psi_+^\pm \psi_-^\mp \right] \ , \ (23)$$

$$\nabla_+ \Sigma^+| = -\frac{1}{2} \left[ r_\pm(\omega) + i2\psi_+^\pm \psi_-^\mp \right] \ , \ (24)$$

$$\nabla_\pm \equiv e_\pm + \omega_\pm \mathcal{M} \ , \ \nabla_\pm \equiv e_\pm + \omega_\pm \mathcal{M} \ , \ \omega_\pm = e_\pm \mp \ , \ (25)$$

The Lorentz generator $\mathcal{M}$ above is defined to act according to the rules; $[\mathcal{M}, \psi_+] = \frac{1}{2} \psi_+$, $[\mathcal{M}, \psi_-] = -\frac{1}{2} \psi_-$, $[\mathcal{M}, e_\pm] = e_\pm$ and $[\mathcal{M}, e_\pm] = -e_\pm$.

From equation (22), we see that $\psi_-^\mp = -\Sigma^+$ and upon substituting into (24) we obtain

$$-\frac{1}{2} r_\pm = (\nabla_+ - i\psi_+^\pm \Sigma^+) | \equiv \left[ \mathcal{D}_+ \Sigma^+ \right] \ . \ (27)$$

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14At least one prior effort [4] has appeared, in which some discussion of superspace integration and super p-forms was given.
Further multiplying this equation by $e^{-1}$ and integrating over the 2-manifold we find

$$\tilde{\Delta} = -\frac{1}{2} \int d^2\sigma \ e^{-1} \ r_{\pm\mp} = \int d^2\sigma \ e^{-1} \left[ D_+ \Sigma^+ \right] \equiv \tilde{\Delta} \ . \quad (28)$$

According to the Ectoplasmic Integration Theorem

$$\int d^2\sigma \ d\zeta^{-} \ E^{-1} \Sigma^+ \equiv \int d^2\sigma \ e^{-1} \left[ D_+ \Sigma^+ \right] \ , \quad (29)$$

and $\Sigma^+$ may be replaced by any general $(1,0)$ superfield Lagrangian $\mathcal{L}_+$. Notice that (28) realizes the Ethereal Conjecture precisely in the form of (1) and (7), since the the spin-connection (defined in (25) and (26)) contains a contribution from the gravitino bilinear. As shown previously [13] the density projector in (27) leads to all of the $(1,0)$ superspace results appropriate for describing the heterotic string. The index $\tilde{\Delta}$ can be related to $\Delta$ (the index of the non-supersymmetric theory) by the following arguments.

In the non-supersymmetric theory, the curvature tensor is defined as in (23). However, the connection in the non-supersymmetric theory does not contain the fermionic terms in $\omega_\pm$ as given in (26). This means that the curvature tensor in the non-supersymmetric theory (denoted by $r^{B\pm\pm\mp}$) is related to $r_{\pm\pm\mp}$ via

$$r_{\pm\pm\mp} = r^{B\pm\pm\mp} + i 2 \left[ \nabla_{\pm\pm}(e^{\pm\pm\mp} \psi^{\pm\pm\mp}) \right] \ , \quad (30)$$

so that integrating both sides of this equation yields

$$\tilde{\Delta} = \Delta - i \int d^2\sigma \left\{ \partial_m \left[ e^{m}_{\pm\pm}(\psi_{\pm\pm\mp}) \right] \right\} \ . \quad (31)$$

In equation (30) $\nabla_{\pm\pm}(e)$ refers to the 2D gravitational covariant derivative constructed solely from the zweibein fields. Equations (30,31) provide a concrete example of our general discussion surrounding equation (7). In all examples known to us, a topological index calculated from superfields possesses the structure of (7). So even when adding surface terms to supersymmetrical theories, such terms must have more than purely bosonic parts if they are to be consistent with a superfield formulation. This feature has often been ignored in the literature in various discussions at the component level of anomalies, boundary terms, duality transformations, etc.

Along the same lines, one can look at 2D, $N = 1$ supergravity. The solution to the Bianchi identities are given as,

$$[ \nabla_{\alpha} , \nabla_{\beta} ] = i 2 (\gamma^\alpha)_{\alpha\beta} \nabla_{\alpha} + 2 (\gamma^3)_{\alpha\beta} \mathcal{R} \mathcal{M} \ , \quad (32)$$

$$[ \nabla_{\alpha} , \nabla_{\beta} ] = i [ \frac{1}{2} \mathcal{R} (\gamma_b)_{\alpha\beta} \nabla_{\beta} + (\gamma^3 \gamma_b)_{\alpha\beta} (\nabla_{\beta} \mathcal{R}) \mathcal{M} ] \ , \quad (33)$$
\[ \{ \nabla_a, \nabla_b \} = -\epsilon_{ab} \left[ \frac{1}{2}(\nabla^\alpha R)(\gamma^3)_\alpha^\beta \nabla_\beta - (\nabla^2 R - R^2)\mathcal{M} \right]. \]  
(34)

(We alert the reader that our conventions are such that we write \( \nabla^2 \equiv \frac{1}{2}\nabla_\alpha \nabla_\alpha \).) In writing these results, we have also simplified them by replacing the usual Lorentz generator according to:

\[ \mathcal{M}_{bc} \rightarrow \epsilon_{bc}\mathcal{M}, \]  
(35)

so that when acting on a spinor \( \psi_\alpha \) or a vector \( v_a \) we have

\[ \mathcal{M}\psi_\alpha = \frac{1}{2}(\gamma^3)_\alpha^\beta \psi_\beta, \quad \mathcal{M}v_a = \epsilon_a^b v_b. \]  
(36)

The component fields of the supergravity multiplet are the graviton \( (e^m_a) \), gravitino \( (\psi_\alpha^a) \) and auxiliary field \( (B) \). These enter the superfield \( R \) as follows

\[ R| = B, \]  
(37)

\[ \nabla_\alpha R| = (\gamma^3)_{\alpha\beta}\epsilon^{ab}\Psi_{ab}^\beta + iB(\gamma^b)_{\alpha\beta}\psi_{b}^\beta, \]  
(38)

\[ \nabla^2 R| = -\frac{1}{2}\epsilon^{ab} r_{ab}(\omega) - 2i\psi^{a}(\gamma^b)_{\alpha\beta}\psi_{ab}^\beta + B\psi_{a}^a\psi_{a}^a + B^2, \]  
(39)

here \( \psi_{ab}^a \) denotes the “curl” of the gravitino and \( R_{ab} \) denotes the two-dimensional curvature in terms of a spin-connection defined by

\[ \omega^a = -\frac{1}{2}\epsilon^{bc}C_{bc}^a - i\epsilon^{bc}\psi_{b}^a(\gamma^a)_{\alpha\beta}\psi_{c}^\beta. \]  
(40)

Upon multiplying (38) by \( \frac{1}{2}(\gamma^3)_{\alpha\beta}\epsilon_{ab} \) and using the resultant to eliminate \( \psi_{ab}^\gamma \) from (39), we find

\[ -\frac{1}{2}\epsilon^{ab} r_{ab} = \left\{ \left[ \nabla^2 - i\psi^{a}(\gamma^b)_{\alpha}\gamma\nabla_\gamma + \epsilon^{ab}\psi_{a}^a(\gamma^3)_{\alpha\beta}\psi_{b}^\beta - R \right] R| \right\} \]  
(41)

\[ \equiv \{ \mathcal{D}^2 R| \} . \]

Thus, we may define

\[ \bar{\Delta} \equiv -\frac{1}{2}\int d^2\sigma\, e^{-1}\epsilon^{ab} r_{ab} = \int d^2\sigma\, e^{-1}\{ \mathcal{D}^2 R| \} \equiv \bar{\Delta}, \]  
(42)

\[ \rightarrow \bar{\Delta} - \Delta = \int d^2\sigma\, \partial_m \left[ \epsilon^{cde}\psi_{c}^\gamma(\gamma^3)_{\gamma\delta}\psi_{d}^\delta e_{a}^m \right], \]  
(43)

or via the Ectoplasmic Integration Theorem,

\[ \int d^2\sigma\, d^2\theta\, E^{-1}\mathcal{L} = \int d^2\sigma\, e^{-1}\left[ \mathcal{D}^2 \mathcal{L}| \right]. \]  
(44)

At this point, we have shown that both 2D (1,0) and (1,1) supergravity theories do indeed realize the Ethereal Conjecture. We next turn to the the case 2D, \( N = 2 \) theories.
There are two minimal irreducible off-shell formulations of 2D, $N = 2$ supergravity which we call the $U(1)$ and $U_A(1)$ theories, respectively. There is also a reducible formulation which we refer to as the $U(1) \otimes U_A(1)$ theory. In the following, we will restrict our consideration solely to the irreducible theories. However, one can show that exactly the same arguments apply to the $U(1) \otimes U_A(1)$ theory. The differences in the various theories has to do with the structure of the holonomy group of the superspace supergravity covariant derivative. In 2D, $N = 2$ superspace, the form of this operator (for the three respective theories mentioned above) is

$$
\nabla_A \equiv E_A^M D_M + \omega_A \mathcal{M} + \Gamma_A \mathcal{Y}, \\
\nabla_A \equiv E_A^M D_M + \omega_A \mathcal{M} + \Gamma_A' \mathcal{Y}', \\
\nabla_A \equiv E_A^M D_M + \omega_A \mathcal{M} + \Gamma_A \mathcal{Y} + \Gamma_A' \mathcal{Y}',
$$

where the $U(1)$ generator $\mathcal{Y}$ and $U_A(1)$ generator $\mathcal{Y}'$ are defined according to

$$
[\mathcal{Y}, \nabla_\pm] = i \frac{1}{2} \nabla_\pm, \\
[\mathcal{Y}, \nabla_\mp] = -i \frac{1}{2} \nabla_\pm, \\
[\mathcal{Y}', \nabla_\pm] = \pm i \frac{1}{2} \nabla_\pm, \\
[\mathcal{Y}', \nabla_\mp] = \mp i \frac{1}{2} \nabla_\pm.
$$

Since we are now considering an extended superspace, in addition to the measure over the full superspace, $\int d^2 \sigma d\zeta d\bar{\zeta} d\zeta^+ d\zeta^- E^{-1}$, there must also be measures over the chiral sub-spaces of this superspace. In the discussion to follow we use $\int d^2 \sigma d\zeta d\bar{\zeta} d\zeta^+ d\zeta^- E^{-1}$ to denote the chiral measure.

Although our previous discussion of the irreducible 2D, $N = 2$ theories utilized “covariant spinor notation,” it is also possible to formulate these theories using “light-cone spinor notation” as was done with the $(1,0)$ theory earlier in this section. Using such notation, the supergravity commutator algebra for the $U(1)$ theory takes the form below.

$$
[\nabla_+, \nabla_+] = 0, \\
[\nabla_-, \nabla_-] = 0, \\
[\nabla_+, \nabla_-] = 0,
$$

\begin{align*}
[\nabla_+, \nabla_-] &= -2 \mathcal{H}(\mathcal{M} + i\mathcal{Y}) , \\
[\nabla_+, \nabla_+] &= i2 \mathcal{H} , \\
[\nabla_-, \nabla_-] &= i2 \mathcal{H} , \\
[\nabla_+, \nabla_+] &= i2 \mathcal{H} , \\
[\nabla_+, \nabla_+] &= 0 , \\
[\nabla_-, \nabla_-] &= 0 , \\
[\nabla_+, \nabla_+] &= 0 , \\
[\nabla_-, \nabla_-] &= 0 , \\
[\nabla_+, \nabla_+] &= 0 , \\
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[\nabla_-, \nabla_-] &= 0 , \\
[\nabla_+, \nabla_+] &= 0 , \\
[\nabla_-, \nabla_-] &= 0 , \\
[\nabla_+, \nabla_+] &= 0 , \\
[\nabla_-, \nabla_-] &= 0 , \\
[\nabla_+, \nabla_+] &= 0 , \\
[\nabla_-, \nabla_-] &= 0 .
\end{align*}
\[
+ \frac{1}{2} [ \nabla_+ \nabla_- H - \nabla_+ \nabla_- H - 4H\overline{H} ]\mathcal{M}
- i\frac{1}{2} [ \nabla_+ \nabla_- H + \nabla_+ \nabla_- H ]\mathcal{Y} .
\] (47)

For the \( U_A(1) \) theory using a similar notation, we find
\[
\begin{align*}
[\nabla_+ , \nabla_+] &= 0 , \quad [\nabla_-, \nabla_-] = 0 , \quad [\nabla_+ , \nabla_-] = 0 \\
[\nabla_+ , \nabla_-] &= -2B( \mathcal{M} + i\mathcal{Y}' ) , \quad \nabla_+ B = \nabla_- B = 0 , \\
[\nabla_+ , \nabla_+] &= i2\nabla_+ , \quad [\nabla_-, \nabla_-] = i2\nabla_- , \\
[\nabla_+ , \nabla_+] &= 0 , \quad [\nabla_- , \nabla_-] = 0 , \\
[\nabla_+ , \nabla_+] &= -i[ B\nabla_- + (\nabla_- B)( \mathcal{M} + i\mathcal{Y}' ) ] , \\
[\nabla_- , \nabla_+] &= i[ B\nabla_+ - (\nabla_+ B)( \mathcal{M} + i\mathcal{Y}' ) ] , \\
[\nabla_+, \nabla_-] &= \frac{1}{2} [ (\nabla_+ B)\nabla_- + (\nabla_- B)\nabla_+ ] \\
& \quad - \frac{1}{2} [ (\nabla_+ B)\nabla_- + (\nabla_- B)\nabla_+ ] \\
& \quad + \frac{1}{2} [ \nabla_+ \nabla_- B - \nabla_+ \nabla_- B - 4B\overline{B} ]\mathcal{M} \\
& \quad - i\frac{1}{2} [ \nabla_+ \nabla_- B + \nabla_+ \nabla_- B ]\mathcal{Y}' .
\end{align*}
\] (48)

Since the \( U_A(1) \) theory more closely resembles the case of 4D, \( N = 1 \) theory (reducing the 4D, \( N = 1 \) theory to 2D gives the \( U_A(1) \) theory), we will first turn our attention to that case. By looking at the coefficients of the \( \mathcal{M} \) and \( \mathcal{Y}' \) generators in (47), we can note that the 2D, \( N = 2 \) superfield curvature (\( R_{\pm, } \)) and \( U_A(1) \) field strength (\( F_{\pm} \)) must be given respectively by
\[
\begin{align*}
R_{\pm, } &= \frac{1}{2} [ \nabla_+ \nabla_- B - \nabla_+ \nabla_- B - 4B\overline{B} ] , \\
F_{\pm, } &= -i\frac{1}{2} [ \nabla_+ \nabla_- B + \nabla_+ \nabla_- B ] .
\end{align*}
\] (49)

Accordingly it follows that a complex quantity \( \hat{R}_{\pm, } \) can be defined to satisfy
\[
\hat{R}_{\pm, } \equiv R_{\pm, } + iF_{\pm, } = [ \nabla_+ \nabla_- B - 2B\overline{B} ] .
\] (50)

Taking the \( \theta \to 0 \) limit, multiplying by \( e^{-1} \) and integrating over the two bosonic coordinates yields a complex index.
\[
\int d^2\sigma \ e^{-1} [ | \hat{R}_{\pm, } | ] = \int d^2\sigma \ e^{-1} \left[ | [ \nabla_+ \nabla_- B - 2B\overline{B} | ] \right] .
\] (51)

It remains only for us to evaluate \( | \hat{R}_{\pm, } | \) which is done by the standard method of \textit{Superspace} to yield
\[
| \hat{R}_{\pm, } | = \hat{r}_{\pm, } - i2\overline{\psi}_- \nabla_+ B + i2\overline{\psi}_+ \nabla_- B + 4(\overline{\psi}_+ \overline{\psi}_- - \overline{\psi}_- + \overline{\psi}_+ ) B .
\] (52)
This gives us the \( \lambda \)-coefficients of (9) and upon substitution of this into (51), we learn that the chiral density projector may be defined by

\[
\int d^2 \zeta \, e^{-1} \mathcal{L}_c = e^{-1} \left[ \left[ \nabla_+ \nabla_- + i 2 \bar{\psi}_- \nabla_+ - i 2 \bar{\psi}_+ \nabla_- - 2 \overline{\mathcal{B}} 
- 4 ( \bar{\psi}_+ \bar{\psi}_- - \bar{\psi}_- \bar{\psi}_+ ) \right] \mathcal{L}_c \right],
\]

\[
\equiv e^{-1} \mathcal{D}^2 \mathcal{L}_c|, \tag{53}
\]

where \( \nabla_+ \mathcal{L}_c = \nabla_- \mathcal{L}_c = 0 \). This result is in complete agreement with that derived by other means in reference [18].

The 2D (2,2) density projector in (53) depended on our examination of an index (51) associated with the supergravity covariant derivative in (48). We now wish to derive this result by considering an index that is associated with a 2D (2,2) matter system. Now let us add a twisted 2D, \( N = 2 \) vector multiplet to the supergravity covariant derivative so that the commutator algebra becomes, (where \( t \) is a U(1) generator)

\[
[\nabla_+, \nabla_+] = 0 \, , \, [\nabla_-, \nabla_-] = 0 \, , \, [\nabla_+, \nabla_-] = 0
\]

\[
[\nabla_+ , \nabla_-] = -2 \overline{\mathcal{B}}( \mathcal{M} + i \mathcal{Y} ) - i 4 g \overline{\mathcal{P}} t \, , \, \nabla_+ \mathcal{B} = \nabla_- \mathcal{B} = 0 \, ,
\]

\[
[\nabla_+ , \nabla_+] = i 2 \nabla_+ \, , \, [\nabla_- , \nabla_-] = i 2 \nabla_-
\]

\[
[\nabla_+ , \nabla_\pm] = 0 \, , \, [\nabla_- , \nabla_\pm] = 0 \, ,
\]

\[
[\nabla_+ , \nabla_\pm] = -i [ \overline{\mathcal{B}} \nabla_- + (\nabla_- \mathcal{B})( \mathcal{M} + i \mathcal{Y} ) ] + 2 g'(\nabla_- \overline{\mathcal{P}}) t \, ,
\]

\[
[\nabla_- , \nabla_\pm] = i [ \overline{\mathcal{B}} \nabla_+ - (\nabla_+ \mathcal{B})( \mathcal{M} + i \mathcal{Y} ) ] + 2 g'(\nabla_+ \overline{\mathcal{P}}) t \, ,
\]

\[
[\nabla_\pm , \nabla_\pm] = ... + i g' [ \nabla_+ \nabla_- \mathcal{P} - \nabla_+ \mathcal{P} \nabla_- - 2( B \overline{\mathcal{P}} + \mathcal{B} \overline{\mathcal{P}} ) ] t . \tag{54}
\]

On the last line above... denotes the supergravity terms that were present prior to the introduction of the twisted 2D, \( N = 2 \) vector multiplet.

The final equation above implies,

\[
\mathcal{F}_{\pm \mp}^{U(1)} = \nabla_+ \nabla_- \mathcal{P} - \nabla_+ \nabla_- \overline{\mathcal{P}} - 2( B \overline{\mathcal{P}} + \mathcal{B} \overline{\mathcal{P}} ) \, , \tag{55}
\]

and by use of a Superspace technique we find,

\[
\mathcal{F}_{\pm \mp}^{U(1)} | = f_{\pm \mp}^{U(1)} - i 2 \bar{\psi}_- \nabla_+ \mathcal{P} | + i 2 \bar{\psi}_+ \nabla_- \mathcal{P} |

- i 2 \psi_- \nabla_+ \overline{\mathcal{P}} | + i 2 \psi_+ \nabla_- \overline{\mathcal{P}} |

+ 4 ( \bar{\psi}_+ \bar{\psi}_- - \bar{\psi}_- \bar{\psi}_+ ) \mathcal{P} |

- 4 ( \psi_+ \psi_- - \psi_- \psi_+ ) \overline{\mathcal{P}} | . \tag{56}
\]
Thus, the following result is valid,
\[ \int d^2 \sigma \ e^{-1} \hat{f}^{U(1)}_{\pm} = \int d^2 \sigma \ e^{-1} \left[ D^2 \mathcal{P} | + \overline{D}^2 \mathcal{P} | \right] , \tag{57} \]
where \( D^2 \) is precisely the operator in (53). The matter superfield \( \mathcal{P} \) is chiral (i.e. \( \nabla_{\pm} \mathcal{P} = 0 \)). In the presence of \( U_A(1) \) supergravity, any chiral superfield satisfies,

A previous investigation of 2D, (2,2) supermeasures \([18]\) has also revealed another previously unknown feature. Namely, in addition to the measures associated with the full and chiral superspaces, there can also exist other local measures! In particular, these exist for twisted chiral measures. In order to derives this, once more we go back to the pure supergravity commutator algebra and add an ordinary 2D, \( N = 2 \) vector multiplet. Under these circumstances the commutators take the forms (below we use \( \nabla_+ \Psi = \nabla_- \overline{\Psi} = 0 \) as is appropriate for a twisted chiral superfield),

\[
\begin{align*}
[\nabla_+, \nabla_+] &= 0 , & [\nabla_-, \nabla_-] &= 0 , & [\nabla_+, \nabla_-] &= -i4g\overline{\Psi} t \\
[\nabla_+, \nabla_-] &= -2\overline{B} (\mathcal{M} + iY') , \\
[\nabla_+, \nabla_+] &= i2\nabla_* , & [\nabla_-, \nabla_-] &= i2\nabla_\# , \\
[\nabla_+, \nabla_\#] &= 0 , & [\nabla_-, \nabla_\#] &= 0 , \\
[\nabla_+, \nabla_\#] &= -i[\overline{B} \nabla_- + (\nabla_- \overline{B})(\mathcal{M} + iY')] + 2g(\nabla_- \overline{\Psi})t , \\
[\nabla_- , \nabla_\#] &= i[\overline{B} \nabla_+ - (\nabla_+ \overline{B})(\mathcal{M} + iY')] - 2g(\nabla_+ \overline{\Psi})t , \\
[\nabla_\# , \nabla_\#] &= ... + ig[\nabla_+ \nabla_- \Psi - \nabla_- \nabla_- \overline{\Psi}]t . \tag{58}
\end{align*}
\]

On the last line above ... once again denotes the supergravity terms that were present prior to the introduction of the 2D, \( N = 2 \) vector multiplet. Also once again, the final equation in (58) above implies,

\[ \hat{F}^{U(1)}_{\pm} = \nabla_+ \nabla_- \Psi - \nabla_+ \nabla_- \overline{\Psi} , \tag{59} \]

and by use of a Superspace technique we find,

\[
\begin{align*}
\hat{F}^{U(1)}_{\pm} &= \hat{F}^{U(1)}_{\pm} - i2\psi^* \nabla_+ \overline{\Psi} + i2\overline{\psi}^* \nabla_- \Psi \\
&\quad - i2\overline{\psi} \nabla_+ \overline{\Psi} + i2\psi^* \nabla_- \Psi \\
&\quad + 4 (\overline{\psi}^* \psi^* - \overline{\psi} \psi^* - \psi^* \overline{\psi}^* ) \Psi \\
&\quad - 4 (\psi^* \overline{\psi} - \overline{\psi}^* \psi^* ) \overline{\Psi} . \tag{60}
\end{align*}
\]

Thus, the following result is valid,
\[ \int d^2 \sigma \ e^{-1} \hat{F}^{U(1)}_{\pm} = \int d^2 \sigma \ e^{-1} \left[ \overline{D}^2 \Psi \right] + \overline{D}^2 \overline{\Psi} \right] , \tag{61} \]
where $\tilde{D}^2$ is
\[
\tilde{D}^2 \equiv \nabla_+ \nabla_- + i 2 \psi_+ - \nabla_+ - i 2 \nabla_+ \nabla_- - 4 (\overline{\psi}_+^+ \psi_- - \overline{\psi}_-^+ \psi_+) . \tag{62}
\]

We thus find the following for the twisted chiral projector,
\[
\int d\zeta^+ d\bar{\zeta}^- \tilde{E}^{-1} L_{tc} = e^{-1} \left[ \nabla_+ \nabla_- + i 2 \psi_+ - \nabla_+ - i 2 \nabla_+ \nabla_- - 4 (\overline{\psi}_+^+ \psi_- - \overline{\psi}_-^+ \psi_+) \right] L_{tc} \] , \tag{63}
\]
where $\nabla_+ L_{tc} = \nabla_- L_{tc} = 0$. Here $\tilde{E}^{-1}$ is used to denote the twisted chiral density measure.

The result of (63) raises new issues to be resolved regarding what is the complete list of density projectors for a given supergravity theory. As was first shown in [18], both chiral and twisted chiral density projection formulae exist for 2D, $N = 2$ superspace. It may well be the case that the number of such chiral-like projectors occur in theories whenever there are irreducible multiplets whose definition are totally expressed in term of first order derivative constraints acting on the superfields.

The case of 2D, $N = 2$ supergravity also gives us a chance to present another aspect of density projectors. Since the $U(1)$ theory is distinct from the $U_A(1)$ theory, it possesses a very different set of density projectors. Instead of repeating the step-by-step derivation used in the $U_A(1)$ case, here we just summarize the results.

For the 2D, $N = 2$ $U(1)$ supergravity theory, the chiral projector is given (c.f. (53)) by
\[
\int d^2 \zeta \ E^{-1} L_c = e^{-1} \left[ \nabla_+ \nabla_- + i 2 \overline{\psi}_+ - \nabla_+ - i 2 \overline{\psi}_+ \nabla_- - 4 (\overline{\psi}_+^+ \psi_- - \overline{\psi}_-^+ \psi_+) \right] L_c \] , \tag{64}
\]
where $\nabla_+ L_c = \nabla_- L_c = 0$ and for the twisted chiral density projector (c.f. (63))
\[
\int d\zeta^+ d\bar{\zeta}^- \tilde{E}^{-1} L_{tc} = e^{-1} \left[ \nabla_+ \nabla_- + i 2 \psi_+ - \nabla_+ - i 2 \overline{\psi}_+ \nabla_- - 2 \mathcal{H} - 4 (\overline{\psi}_+^+ \psi_- - \overline{\psi}_-^+ \psi_+) \right] L_{tc} \] , \tag{65}
\]
where $\nabla_+ L_{tc} = \nabla_- L_{tc} = 0$. The importance of these last two equations is that they demonstrate that for distinct superspace geometries, there correspond distinct density projection operators.
IV. 3D, \( N = 1 \) Local Ectoplasmic Integration

Applying the same considerations to 3D, \( N = 1 \) supergravity begins by once again knowing the form of the commutator algebra

\[
[\nabla_\alpha, \nabla_\beta] = i 2 (\gamma^c)_{\alpha \beta} [\nabla_c - R \mathcal{M}_c] ,
\]

\[
[\nabla_\alpha, \nabla_b] = i (\gamma_b)_\alpha \delta^\gamma \left[ \frac{1}{2} R \nabla_\gamma + (\Sigma_\delta^d + i \frac{2}{3} (\gamma^d)_\delta (\nabla_\epsilon R)) \mathcal{M}_d \right]
+ (\nabla_\alpha R) \mathcal{M}_b ,
\]

\[
[\nabla_a, \nabla_b] = -\frac{1}{2} \epsilon_{abc} \left[ \Sigma^{ac} + i \frac{2}{3} (\gamma^c)_{\alpha \beta} (\nabla_\beta R) \right] \nabla_\alpha
- \epsilon_{abc} \left[ \mathcal{R}^{cd} + \frac{2}{3} \eta^{cd} (\nabla^2 R - \frac{3}{2} R^2) \right] \mathcal{M}_d ,
\]

where \( \mathcal{R}^{ab} - \mathcal{R}^{ba} = \eta_{ab} \mathcal{R}^{ab} = (\gamma_d)_{\alpha \beta} \Sigma_\delta^d = 0 \) and

\[
\nabla_a \Sigma_\beta^c = i (\gamma_b)_\alpha \mathcal{R}^{bc} - \frac{2}{3} \left[ C_{\alpha \beta} \eta^{cd} + i \frac{1}{2} (\gamma_b)_{\alpha \beta} \epsilon_{bed} \right] (\nabla_d R) .
\]

In writing these results, we have simplified their form by replacing the usual Lorentz generator according to:

\[
\mathcal{M}_{bc} \rightarrow \epsilon_{bc}^a \mathcal{M}_a ,
\]

so that when acting on a spinor \( \psi_\alpha \) or a vector \( v_a \) we have

\[
\mathcal{M}_a \psi_\alpha = i \frac{1}{2} (\gamma_a)_\alpha \beta \psi_\beta , \quad \mathcal{M}_a v_b = \epsilon_{ab}^c v_c .
\]

Three dimensions offer us a new possibility in the class of topological invariants over the body of the supermanifold. Here we may introduce a two-form gauge field \( \mathcal{B} \) whose three-form field strength \( g = d \mathcal{B} \) can be used to construct a new class of topological invariants given by \( \int g \). There exists a super 2-form that generalizes this component theory. The field strength supertensor \( \mathcal{G}_{ABC} \) satisfies the equations

\[
\mathcal{G}_{\alpha \beta \gamma} = 0 ,
\]

\[
\mathcal{G}_{\alpha \beta c} = i 2 (\gamma^c)_{\alpha \beta} \mathcal{G} ,
\]

\[
\mathcal{G}_{abc} = i \epsilon_{abc} (\gamma^a)_\alpha \beta (\nabla_\beta G) ,
\]

\[
\mathcal{G}_{abc} = \epsilon_{abc} \left[ \nabla^2 \mathcal{G} - R \mathcal{G} \right] ,
\]

Now we may use the result of a general procedure to show that the following equation must be true

\[
\mathcal{G}_{abc} = g_{abc} + \epsilon_{abc} \left[ i \psi^\alpha_a (\gamma^a)_\alpha \beta (\nabla_\beta G) - i \epsilon^{def} \psi^\alpha_d (\gamma_e)_{\alpha \beta} \psi^\beta_f (\mathcal{G}) \right] .
\]
Combining this result with the last equation in (70) we obtain
\[ \frac{1}{6} \epsilon^{abc} g_{abc} = \nabla^2 G|_i - i \psi_a^\alpha (\gamma^\alpha)_\alpha \beta (\nabla_\beta G|_i) + i \epsilon^{abc} \psi_a^\alpha (\gamma_b)_\alpha \beta \psi_c^\beta (G|_i) - (RG|_i) \tag{72} \]
and introducing the notational device \( D^2 \) we find
\[ \frac{1}{6} \epsilon^{abc} g_{abc} = (D^2 G|_i) \tag{73} \]
Upon multiplying this by \( e^{-1} \) and integrating both sides, this becomes
\[ \hat{\Delta} \equiv \int d^3 x e^{-1} \frac{1}{6} \epsilon^{abc} g_{abc} = \int d^3 x e^{-1} (D^2 G|_i) \equiv \tilde{\Delta} \tag{74} \]
where the index on the far left is defined by the value of the integral adjacent to it. According to the Ethereal Conjecture we may define the Ectoplasmic Integration theorem in 3D, \( N = 1 \) superspace so that it reads
\[ \int d^3 x d^2 \theta E^{-1} \mathcal{L} \equiv \int d^3 x e^{-1} (D^2 \mathcal{L}|_i) \tag{75} \]
and thus the superfield topological invariant is given by,
\[ \tilde{\Delta} = \frac{1}{12} \int d^3 x d^2 \theta E^{-1} \left[ i (\gamma^c)^{\alpha \beta} \Psi_{abc} \right] \tag{76} \]
The reader who has followed our arguments thus far, might offer a challenge at this point, “What independent arguments are there to support the conclusion that we have correctly identified the density projector?” An independent derivation of this density can be obtained via the normal coordinate expansion technique[2,4]. Another simple way to support this suggestion is to calculate the component version of the superfield expression \( \int d^3 x d^2 \theta E^{-1} R \). This is known to be the 3D supergravity action. To obtain the correct component-level expression depends crucially on the form of the local density projector. By solving the 3D, \( N = 1 \) Bianchi identities we find
\[ R|_i = B \ , \ \nabla_a R|_i = - \epsilon^{abc} (\gamma^a)_\alpha \beta \Psi_{bc}^\beta + i 2 B (\gamma^a)_\alpha \beta \psi_b^\beta \tag{77} \]
\[ \nabla^2 R|_i = - \frac{1}{4} \epsilon^{abc} \mathcal{R}_{abc}(\omega) - i 2 \psi^{\alpha \alpha} (\gamma^a)_\alpha \beta \Psi_{ab}^\beta + 2 B \psi^{\alpha \alpha} \psi_{aa} + i B \epsilon^{abc} \psi_a^\alpha (\gamma_b)_\alpha \beta \psi_c^\beta \tag{78} \]
where \( \Psi_{ab}^\beta \) is the usual component level gravitino field strength. This leads to
\[ \int d^2 x d^2 \theta E^{-1} R = \int d^3 x e^{-1} \left[ - \frac{1}{2} \epsilon^{abc} (\mathcal{R}_{abc}(\omega) + \psi_{aa} \Psi_{bc}^\alpha) - B^2 \right] \tag{79} \]
where the curvature tensor \( \mathcal{R}_{abc} \) is defined in terms of a spin-connection,
\[ \omega_a^b = \frac{1}{2} \epsilon^{bcd} \left[ C_{cda} - 2 C_{acd} + i 4 \left( \psi_c^\alpha (\gamma_a)_\alpha \beta \psi_d^\beta + \psi_a^\alpha (\gamma_c)_\alpha \beta \psi_d^\beta \right) \right] - \frac{1}{2} B \delta_a^b \tag{80} \]
Again we find support for the Ethereal Conjecture. We now turn our attention to four dimensional \( N = 1 \) supersymmetric theories.

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V. 4D, $N = 1$ Local Ectoplasmic Integration

Minimal 4D, $N = 1$ supergravity is described by three fields strength tensors denoted by $W_{\alpha\beta\gamma}$, $G_{\alpha\dot{\alpha}}$, and $R$. The form of the theory is completely described by

$$\left[\nabla_\alpha, \nabla_\beta\right] = -2 \mathcal{R} \mathcal{M}_{\alpha\beta},$$
$$\left[\nabla_\alpha, \nabla_\beta\right] = i \nabla_{\alpha\beta},$$
$$\left[\nabla_\alpha, \nabla_\beta\right] = -iC_{\alpha\beta}[\mathcal{R}\nabla_\beta - G_{\gamma\beta} \nabla_\gamma] - i(\nabla_\beta \mathcal{R}) \mathcal{M}_{\alpha\beta} + iC_{\alpha\beta} \left[ \mathcal{W}_{\gamma\delta} \mathcal{M}_{\delta\gamma} - (\nabla^\gamma G_{\delta\beta}) \mathcal{M}_{\gamma\delta} \right],$$
$$\left[\nabla^{a}, \nabla^{b}\right] = \left\{ C_{\alpha\beta} W_{\alpha\beta\gamma} + C_{\alpha\beta}(\nabla_\alpha G_{\gamma\beta}) - C_{\alpha\beta}(\nabla_\alpha R) \delta_{\gamma\beta} \nabla_\gamma ight.$$
$$+ iC_{\alpha\beta} G_{\gamma\beta} \nabla_\gamma - [C_{\alpha\beta}(\nabla_\alpha \nabla_\delta G_{\gamma\beta})$$
$$- C_{\alpha\beta}(\nabla_\alpha W_{\beta\gamma\delta}) + (\nabla^2 \mathcal{R} + R \mathcal{R}) C_{\gamma\beta \delta}\delta_{\alpha} \nabla_\gamma \mathcal{M}_{\delta\gamma} \right\} + h. c. \quad (81)$$

We now apply our method by introducing a 3-form gauge field matter multiplet as first appeared very long ago [11]. In the context of 4D, $N = 1$ superspace, such a supermultiplet is described by a super 4-form field strength $F_{\alpha\beta\gamma\delta}$ that is known to satisfy the constraints,

$$F_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta} = F_{\alpha\beta\gamma\delta} = 0, \quad F_{\alpha\beta\gamma\delta} = C_{\gamma\delta} C_{\alpha(\gamma C_{\delta}\beta)} \mathcal{F}. \quad (82)$$

After the imposition of the constraints and solving the Bianchi identities on $F_{\alpha\beta\gamma\delta}$ in the presence of the old minimal supergravity derivative commutator algebra we find,

$$F_{a b c d} = - \epsilon_{a b c d} \nabla^2 \mathcal{F}, \quad F_{\dot{a} \dot{b} \dot{c} \dot{d}} = \epsilon_{a b c d} \nabla^\alpha \mathcal{F}, \quad \nabla_\alpha \mathcal{F} = 0 \quad (83)$$

$$\epsilon_{a b c d} \equiv i \frac{1}{2} \left[ C_{\alpha\beta} C_{\gamma\delta} C_{\alpha(\gamma C_{\delta}\beta)} - C_{\alpha\beta} C_{\gamma\delta} C_{\alpha(\gamma C_{\delta}\beta)} \right]. \quad (84)$$

The $\theta \to 0$ limit of all types of component level supercovariantized gauge field strengths is defined in an unambiguous manner as was discussed in Superspace. We
need only slightly generalize the formulae given there to find (c.f. (9))
\[
(F_{\alpha \beta \gamma \delta} |) = f_{\alpha \beta \gamma \delta} + \left[ \frac{1}{3!} \psi_\alpha^\alpha (F_{\alpha \beta \gamma \delta} |) + \frac{1}{4} \psi_\alpha^\alpha \psi_\beta^\beta (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right]
+ \left[ \frac{1}{6} \psi_\alpha^\alpha \psi_\beta^\beta \psi_\gamma^\gamma (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right]
- \frac{1}{2} \left[ \psi_\alpha^\alpha \psi_\beta^\beta \psi_\gamma^\gamma (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right]
- \left[ \psi_\alpha^\alpha \psi_\beta^\beta \psi_\gamma^\gamma (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right]
- \left[ \psi_\alpha^\alpha \psi_\beta^\beta \psi_\gamma^\gamma (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right]
- \left[ \psi_\alpha^\alpha \psi_\beta^\beta \psi_\gamma^\gamma (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right].
\]
(85)
Due to the constraints (83) on the 4-form supermultiplet, we can re-write this as
\[
f_{\alpha \beta \gamma \delta} = (F_{\alpha \beta \gamma \delta} |) - \left[ \frac{1}{3!} \psi_\alpha^\alpha (F_{\alpha \beta \gamma \delta} |) + \frac{1}{4} \psi_\alpha^\alpha \psi_\beta^\beta (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right].
\]
(86)
Integrating both sides of this equation leads to,
\[
\hat{\Delta} = \int d^4x \ e^{-1} f_{\alpha \beta \gamma \delta} \left[ \frac{1}{3!} (F_{\alpha \beta \gamma \delta} |) - \left[ \frac{1}{3!} \psi_\alpha^\alpha (F_{\alpha \beta \gamma \delta} |) + \frac{1}{4} \psi_\alpha^\alpha \psi_\beta^\beta (F_{\alpha \beta \gamma \delta} |) + \text{h.c.} \right] \right].
\]
(87)
where \( \hat{\Delta} \) denotes the supersymmetric version of the index described by (11). Next we use the solution to the Bianchi identities for \( F_{\alpha \beta \gamma \delta} \) and \( F_{\alpha \beta \gamma \delta} \) (from (83)), which upon substitution into (87) yields,
\[
\hat{\Delta} = \int d^4x \ e^{-1} \left[ -i (\mathcal{D}^2 \mathcal{F}) | + \text{h.c.} \right],
\]
(88)
where the operator \( \mathcal{D}^2 \) is defined by
\[
\mathcal{D}^2 \equiv \nabla^2 + i \overline{\psi}_\alpha \nabla_\alpha + 3 \overline{\mathcal{R}} + \frac{1}{2} \mathcal{C}^{\alpha \beta} \overline{\psi}_\alpha (\overline{\psi}_\beta \overline{\psi}_\beta).
\]
(89)
Now we see that a superdifferential operator appears acting on the superfield \( \mathcal{F} \). This superdifferential operator is, in fact, the chiral density projector for old minimal supergravity. We continue by noting that the chiral density projector \( \int d^2\theta \mathcal{E}^{-1} \) may be defined by the equation
\[
\int d^4x \ d^2\theta \mathcal{E}^{-1} \mathcal{L}_c \equiv \int d^4x \ e^{-1} (\mathcal{D}^2 \mathcal{L}_c |) ,
\]
(90)
for any chiral superfield, \( \mathcal{L}_c \). Finally since any chiral superfield \( \mathcal{L}_c \) in the presence of old minimal supergravity satisfies \( \mathcal{L}_c = (\nabla^2 + R) \mathcal{L} \), where \( \mathcal{L} \) is a general superfield, we also have
\[
\int d^4x \ d^2\theta \ d^2\bar{\theta} \ E^{-1} \mathcal{L} \equiv \frac{1}{2} \left[ \int d^4x \ d^2\theta \mathcal{E}^{-1} (\nabla^2 + R) \mathcal{L} \right] + \text{h.c.}
= \frac{1}{2} \int d^4x \ e^{-1} \left[ (\mathcal{D}^2 (\nabla^2 + R) \mathcal{L}) | + \text{h.c.} \right].
\]
(91)
On setting $L = 1$, we derive the action for old minimal supergravity. The fourth-order differential operator $D^4 = \underline{1}/2 D^2 (\nabla^2 + R) + \text{h.c.}$ can be expanded in the form

$$\int d^4 x \ d^2 \theta \ d^2 \bar{\theta} \ E^{-1} L = \int d^4 x \ e^{-1} \left\{ \begin{array}{c} c^{(2,2)}_{\alpha \beta \dot{\alpha} \dot{\beta}} \nabla_\alpha \nabla_\beta \nabla_{\dot{\alpha}} \nabla_{\dot{\beta}} + c^{(1,2)}_{\alpha \dot{\alpha}} \nabla_\alpha \nabla_{\dot{\alpha}} + c^{(2,0)}_{\alpha \beta} \nabla_\alpha \nabla_\beta + c^{(0,2)}_{\dot{\alpha} \dot{\beta}} \nabla_{\dot{\alpha}} \nabla_{\dot{\beta}} \\ + c^{(1,0)}_{\alpha} \nabla_\alpha + c^{(0,0)} \end{array} \right\} |L| + \text{h.c.},$$

(92)

where the coefficients are given by

$$c^{(2,2)}_{\alpha \beta \dot{\alpha} \dot{\beta}} = \frac{1}{4} C^{\alpha \beta} C^{\dot{\alpha} \dot{\beta}}, \quad c^{(1,2)}_{\alpha \dot{\alpha}} = -\frac{1}{2} \bar{\psi}^{\dot{\alpha}} \psi_\gamma C^{\dot{\alpha} \dot{\beta}},$$

$$c^{(2,0)}_{\alpha \beta} = -\frac{1}{2} C^{\alpha \beta} R, \quad c^{(0,2)}_{\dot{\alpha} \dot{\beta}} = -\frac{1}{2} C^{\dot{\alpha} \dot{\beta}} (3R + \frac{1}{2} C^{\alpha \beta} \bar{\psi}^{\dot{\alpha}} \psi_\gamma (\dot{\gamma} \bar{\psi}_{\dot{\beta}} \dot{\delta})),$$

$$c^{(1,0)}_{\alpha} = \left[ (\nabla_\alpha R) + i \bar{\psi}_{\dot{\alpha}} \alpha R \right],$$

$$c^{(0,0)} = \left[ (\nabla^2 R) + i \bar{\psi}_{\dot{\alpha}} (\nabla_\alpha R) + 3 |R|^2 + \frac{1}{2} C^{\alpha \beta} \bar{\psi}_{\dot{\alpha}} (\dot{\alpha} \bar{\psi}_{\dot{\beta}} R) \right].$$

(93)

This form of the density projector shows that we have achieved our goal of calculating, from an a priori principle, the form of the coefficients described abstractly in equation (4). We also see that the respective volumes of 4D, $N = 1$ chiral and full superspaces for old minimal supergravity are

$$\int d^4 x \ d^2 \theta \ E^{-1} = \int d^4 x \ e^{-1} \left[ - C_{\dot{\alpha} \dot{\beta}} c^{(0,2)}_{\alpha \dot{\beta}} \right],$$

$$\int d^4 x \ d^2 \theta \ d^2 \bar{\theta} \ E^{-1} = \int d^4 x \ e^{-1} \left[ c^{(0,0)} + \bar{c}^{(0,0)} \right].$$

(94)

The nonminimal 4D, $N = 1$ supergravity formulation is the oldest “off-shell” form of the theory known, having first been introduced by Breitenlohner [20] in a not quite irreducible form. Nonminimal 4D, $N = 1$ supergravity is described by three fields strength tensors just as in the minimal case. Here these are denoted by $W_{\alpha \beta \gamma}$, $G_{\alpha \dot{\alpha}}$, and $T_\alpha$. The derivation of the local density projector in the theory is much more complicated than in the case of the minimal case. So much so that to our knowledge, the explicit density projector for the nonminimal theory has never been presented in the literature previously. Using the “Ethereal Conjecture”, this calculation is rendered very simple. We will next present its explicit derivation.

Our derivation begins by giving an explicit form of the nonminimal 4D, $N = 1$
supergravity commutators. These may be written as

\[
\begin{align*}
\{ \nabla_\alpha , \nabla_\beta \} &= \frac{1}{2} T_{(\alpha} \nabla_{\beta)} - 2 \mathcal{T}_\alpha \mathcal{M}_\alpha \beta , \\
\{ \nabla_\alpha , \nabla_{\dot{\beta}} \} &= i \nabla_{\dot{\alpha} \beta} , \\
\{ \nabla_\alpha , \nabla_\gamma \} &= \frac{1}{2} T_\beta \nabla_\alpha \beta + i [ C_{\alpha \beta} G_{\gamma \beta} + \frac{1}{2} (\nabla_\beta T_\alpha) \delta_\beta \gamma ] \nabla_\gamma \\
&\quad - i [ C_{\alpha \beta} (\nabla^\gamma G_{\beta \gamma}) \mathcal{M}_\alpha \gamma + (\nabla_\beta \mathcal{R}) \mathcal{M}_\alpha \beta ] \\
&\quad + i C_{\alpha \beta} [ \nabla_{\dot{\beta}} (\mathcal{T}_\gamma \mathcal{M}_\beta \dot{\gamma}) + \frac{1}{6} ( (\nabla^\gamma + \frac{1}{2} T^\gamma ) \nabla_\gamma \mathcal{T}_\beta ) \mathcal{M}_\beta \dot{\gamma} ] .
\end{align*}
\]

The final commutator \( \{ \nabla_\alpha , \nabla_\beta \} \) can be explicitly found from the equation

\[
\begin{align*}
\{ \nabla_\alpha , \nabla_\beta \} &= - i \{ \nabla_{\dot{\beta}} , \{ \nabla_\alpha , \nabla_\beta \} \} - i \{ \nabla_\beta , \{ \nabla_\alpha , \nabla_{\dot{\beta}} \} \} .
\end{align*}
\]

In the equations above we have also made a choice of the superconformal symmetry parameter \( n = 1 \) [21], so that we have

\[
\mathcal{R} = - \frac{1}{4} \nabla^\alpha T_\alpha .
\]

Other consequences of the constraints (96) are that

\[
\begin{align*}
\nabla_\alpha [ R - \frac{1}{4} T^\alpha T_\alpha ] &= 0 , \\
\nabla_\beta (\nabla^2 + \frac{3}{4} T^\gamma \nabla_\gamma - R + \frac{1}{4} T^\gamma T_\gamma ) &= 0 .
\end{align*}
\]

Now we once again simply use the constraints of the 4-form and solve its Bianchi identities in the presence of the non-minimal supergravity commutator algebra to find,

\[
\begin{align*}
F_{a b c d} &= - \epsilon_{a b c d} [( \nabla^\dot{\alpha} - T^\dot{\alpha} ) \mathcal{F}] , \\
( \nabla^\dot{\alpha} - T^\dot{\alpha} ) \mathcal{F} &= 0 , \\
F_{\dot{a} \dot{b} c d} &= \epsilon_{a b c d} ( ( \nabla^\alpha - T^\alpha ) \mathcal{F}) , \\
F_{a b c d} &= i \epsilon_{a b c d} \left\{ \frac{1}{2} [ \nabla^\epsilon ( \nabla_\epsilon + T_\epsilon ) \mathcal{F} ] - \frac{1}{2} [ \nabla^\dot{\epsilon} ( \nabla_\epsilon + T_\epsilon ) \mathcal{F} ] \right\} .
\end{align*}
\]

So that the quantity \( \hat{\Delta} \) here takes the form,

\[
\hat{\Delta} = \int d^4 x \ e^{-1} \left[ - i ( \mathcal{D}^2 \mathcal{F} ) + \text{h.c.} \right]
\]

where now the operator \( \mathcal{D}^2 \) is defined by

\[
\mathcal{D}^2 \equiv \nabla^2 + [ i \overline{\psi}_a - \frac{1}{2} T^a ] \nabla_a + [ 2 \mathcal{R} - i \overline{\psi}_a T_a + \frac{1}{2} C^{a \beta} \overline{\psi}_a ( \dot{a} \overline{\psi}_b ) ] .
\]

(This projector provides an explicit example of the comment that was made above equation (5) since \( c_{(N^p - 1)} \) is not strictly proportional to the gravitino. This behavior generically occurs in the presence of spinorial auxiliary fields.)
If we replace $\mathcal{F}$ by a chiral Lagrangian $\tilde{\mathcal{L}}_c$ in (100) above, we seem to have reached our goal of finding a chiral-type density projector for the nonminimal theory

$$\int d^4x \, d^2\theta \, \mathcal{E}^{-1} \tilde{\mathcal{L}}_c = \int d^4x \, e^{-1} \left( D^2 \tilde{\mathcal{L}}_c \right) , \quad (\nabla_\alpha - T_\alpha) \tilde{\mathcal{L}}_c = 0 \quad . \tag{102}$$

However, the discerning reader will note a slight remaining problem. Namely the modified definition of chirality satisfied by $\tilde{\mathcal{L}}_c$ is not the usual one. One way to remedy this is to note that there exists a superfield $T_\alpha$ satisfying

$$\nabla_\alpha T_\beta + \nabla_\beta T_\alpha = 0 .$$

The solution of this implies that there exist another superfield $\tilde{T}$ such that

$$\tilde{\mathcal{L}}_c = \nabla_\alpha \tilde{T}_\alpha .$$

It therefore follows that

$$(\nabla_\alpha - T_\alpha) \tilde{\mathcal{L}}_c = 0 \quad \& \quad \tilde{\mathcal{L}}_c \equiv e^{\tilde{T}} \mathcal{L}_c \rightarrow \nabla_\alpha \mathcal{L}_c = 0 \quad . \tag{103}$$

In order to use the superfield $T$ to obtain a component level expression, we need to observe $\tilde{T}| = 0$ in a suitable Wess-Zumino gauge. Assembling all of these pieces we find for the nonminimal supergravity theory,

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \, E^{-1} \mathcal{L} \equiv \frac{1}{2} \left[ \int d^4x \, d^2\theta \, \mathcal{E}^{-1} \left( (\nabla^2 + \frac{3}{4} T^\gamma \nabla_\gamma - R + \frac{1}{4} T^\gamma T_\gamma) \mathcal{L} \right) \right. \right.$$

$$\left. + \text{h.c.} \right]$$

$$= \frac{1}{2} \int d^4x \, e^{-1} \left[ \left[ (D^2 e^{\tilde{T}} (\nabla^2 + \frac{3}{4} T^\gamma \nabla_\gamma - R + \frac{1}{4} T^\gamma T_\gamma) \mathcal{L}) \right] \right. \right.$$

$$\left. + \text{h.c.} \right] . \tag{104}$$

In this equation, the quantity $\mathcal{E}^{-1}$ is the usual density for ordinary chiral superfields. In fact it is defined by

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} E^{-1} [\bar{R} - \frac{1}{4} T^\alpha T_\alpha]^{-1} \equiv \int d^4x \, d^2\theta \, \mathcal{E}^{-1} \quad . \tag{105}$$

The appearance of $\bar{T}$ still makes for an ungainly way to proceed. So we may insert a factor of $1 = \exp[-\bar{T}] \exp[\bar{T}]$ in front of $\mathcal{L}$ and “push” the factor of $\exp[-\bar{T}]$ through the chiral projection operator until it annihilates the pre-factor of $\exp[\bar{T}]$. Now we can redefine $\mathcal{L}$ to absorb the other exponential. The net effect of these operations is to “change” the chiral projection operator according to

$$(\nabla^2 + \frac{3}{4} T^\gamma \nabla_\gamma - R + \frac{1}{4} T^\gamma T_\gamma) \rightarrow \nabla^2 - \frac{1}{4} T^\beta \nabla_\beta + R = \frac{1}{2} \nabla_\beta [\nabla_\beta - \frac{1}{2} T_\beta] \quad , \tag{106}$$

and the final form of the density projector for the nonminimal supergravity theory described by (103) can be cast into the form,

$$\int d^4x \, d^2\theta \, d^2\bar{\theta} \, E^{-1} \mathcal{L} = \frac{1}{4} \left\{ \int d^4x \, e^{-1} \left[ D^2 \nabla_\beta [\nabla_\beta - \frac{1}{2} T_\beta] \mathcal{L} \right] \right\} + \text{h.c.} \right\} . \tag{107}$$
We believe that it is appropriate to note that equation (107) immediately above marks the first time, to our knowledge, that a density projection operator for nonminimal supergravity has appeared in the physics literature.

Some time ago we proposed [22] that the 4D, $N = 1$ limit of heterotic string theory was most likely to be an unusual formulation of 4D, $N = 1$ supergravity combined with a 4D, $N = 1$ super gauge 2-form multiplet. We presently call this the 4D, $N = 1$ “βFFC supergeometry”[23]. The form of the commutator algebra for this formulation is

\[
[\nabla_\alpha, \nabla_\beta] = 0 , \\
[\nabla_\alpha, \nabla_\alpha] = i\nabla_\alpha + H_{\beta\delta} M_\alpha^\beta - H_{\alpha\beta} \nabla_{\alpha}^\beta + H_\alpha Y , \\
[\nabla_\alpha, \nabla_\beta] = i(\nabla_\beta H_{\gamma\delta}) [ M_\alpha^\gamma + \delta_\alpha^\gamma Y ] \\
\quad + i( C_{\alpha\delta} \nabla_\beta^\delta - \frac{1}{2} \delta_\delta^\beta (2\nabla_\alpha H_{\beta\gamma} + \nabla_\beta H_{\alpha\gamma} ) ) [M_\beta^\gamma , \\
[\nabla_\alpha, \nabla_\beta] = \{ \frac{1}{2} C_{\alpha\beta} [ i H^\gamma (\nabla_\gamma^\beta) - (\nabla^\gamma (\nabla_\gamma H_{\beta\gamma}) ) ] Y ] \\
\quad + [ C_{\alpha\beta} ( W_{\alpha\gamma} - \frac{1}{3} (\nabla^\gamma (H_{\alpha\gamma}) ) ) \nabla_\gamma - \frac{1}{2} C_{\alpha\beta} (\nabla_{\gamma} H_{\beta\gamma}) ) ] M^\gamma \\
\quad + \frac{1}{2} C_{\alpha\beta} [ \nabla_\gamma (\nabla_{\beta} H_{\gamma\delta}) ] M^\delta + \text{h.c.} \} .
\]

The self-consistency of the Bianchi identities of the commutator algebra above requires that the following differential equation must also be satisfied.

\[
\nabla_\alpha H_\alpha = 0 , \quad \nabla_\beta W_{\alpha\beta\gamma} = 0 , \quad \nabla_\alpha W_{\alpha\beta\gamma} = 0 , \quad \nabla_\alpha H_\alpha = 0 , \\
\nabla^\alpha W_{\alpha\beta\gamma} = - \frac{1}{4} \nabla (\nabla^\gamma H_{\gamma\delta}) - \frac{1}{4} \nabla (\nabla_{\delta} H_{\gamma\gamma}) .
\]

Finally, since the theory contains a gauge 2-form, there occurs a super 3-form field strength whose various components are given by

\[
H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma} = H_{\alpha\beta\gamma} - i\frac{1}{2} C_{\alpha\gamma} C_{\beta\gamma} = 0 , \\
H_{abc} = 0 , \quad H_{abc} = i\frac{1}{4} [ C_{\beta\gamma} C_{\alpha(\beta} H_{\alpha\gamma)} - C_{\beta\gamma} C_{\alpha(\beta} H_{\alpha\gamma)} ] .
\]

Repeating the step of solving the Bianchi identity in the presence of the constraints leads to

\[
F_{\alpha\beta\gamma\delta} = - \epsilon_{\alpha\beta\gamma\delta} \nabla^\gamma F , \quad F_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} \nabla^\alpha F , \quad \nabla_\alpha F = 0 , \\
F_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta} [ \nabla^\gamma \nabla^{\delta} F - \nabla^{\gamma\delta} F ] .
\]

\[15\] The name “βFFC” (β beta-function favored constraints) was suggested by H.Nishino.
These are substituted into (87) and as previously, we see that a superdifferential operator appears acting on the superfield $\mathcal{F}$. This time the superdifferential operator is the chiral density projector for $\beta$FFC supergravity. We continue by noting that the chiral density projector $\int d^2\theta \mathcal{E}^{-1}$ may be defined by the equation

$$\int d^4x d^2\theta d^2\bar{\theta} E^{-1} \mathcal{L}_c \equiv \int d^4x e^{-1} (D^2 \mathcal{L}_c|)$$

$$D^2 \equiv \nabla^2 + i\bar{\psi}_a \hat{\gamma}_\alpha \nabla_\alpha + \frac{1}{2} C^{\alpha\beta\gamma} \bar{\psi}_a^{(\alpha} \bar{\psi}_{\beta)}$$

for any chiral superfield, $\mathcal{L}_c$. Finally since any chiral superfield $\mathcal{L}_c$ in the presence of $\beta$FFC supergravity satisfies $\mathcal{L}_c = \nabla^2 \mathcal{L}$, where $\mathcal{L}$ is a general superfield, we also have

$$\int d^4x d^2\theta d^2\bar{\theta} E^{-1} \mathcal{L} \equiv \frac{1}{2} \left[ \int d^4x d^2\theta \left( \nabla^2 \mathcal{L} \right) + \text{h.c.} \right]$$

On setting $\mathcal{L} = 1$, we derive

$$V_{\beta FF C} = \int d^4x d^2\bar{\theta} E^{-1} = 0$$

since any purely derivative operator acting on a constant vanishes, i.e. the volume of the full $\beta$FFC superspace vanishes. Interestingly enough, however, the volume of the 4D, $N = 1$ $\beta$FFC chiral superspace is non-vanishing. If we introduce a complex parameter $\mu_c$ we find

$$V_{\beta FF C}^c = \int d^4x d^2\theta \mu_c = \int d^4x e^{-1} [D^2(\mu_c|)$$

$$= \frac{1}{2} \mu_c \int d^4x e^{-1} \left[ \bar{\psi}_{(\dot{\alpha}} \dot{\gamma}_{\alpha)} \psi^{(\beta)} \right]$$

This last expression (taken together with its conjugate) is recognizable as a mass term for the gravitino. Thus we find the very elegant result that the mass of the gravitino in 4D, $N = 1$ $\beta$FFC supergravity is proportional to the volume of chiral superspace. Since we have proposed that $\beta$FFC supergeometry is the limit of the supergravity theory associated with heterotic and superstring theory, the results in (114) and (115) must have interesting “string” implications.

We simply close this section by noting that we have presented a number of results that have not appeared previously to our knowledge. These were given in (107), (114) and (115). With this we end our discussion of examples of how to derive density projectors. The methods that we have described in the last two sections may be extended to a number of other supergravity theories with no impediments.
VI. Stage-II Density Projection Operators

In the previous sections, we have presented evidence from a wide number of examples which show that density projection operators can often be derived by looking at topological indices. One feature of all of the examples is that the construction of the density projection operators were obtained by essentially algebraic means. We shall call these Stage-I density projection operators. There are cases, however, where this is not the case and the derivation of density projection operators seem to require solving for some “prepotential” \(^{16}\). The density projection operators of this type will be called Stage-II density projection operators. Derivations for Stage-II projectors are more complicated than for Stage-I projectors. As an illustration of such a theory, we shall describe the treatment of the local 2D, \(N = 4\) theory.

A minimal off-shell 2D, \(N = 4\) supergravity theory \(^{22}\) consists of the component fields \((e_a^m, \psi_a^{\alpha i}, A_{ab}^j, B, G, H)\). These are the components of that remain after imposing the following constraints on the 2D, \(N = 4\) superspace supergravity covariant derivative, (with \(\phi_{\alpha \beta} \equiv -i[C_{\alpha \beta} G + i (\gamma^3)_{\alpha \beta} H]\))

\[
\begin{align*}
[\nabla_{\alpha i}, \nabla_{\beta j}] &= 2B[C_{\alpha \beta} C_{ij} \mathcal{M} - (\gamma^3)_{\alpha \beta} \mathcal{Y}_{ij}] , \\
[\nabla_{\alpha i}, \nabla_{\beta}^j] &= 2[ i \delta_{i}^j (\gamma^c)_{\alpha \beta} \nabla_c + \delta_{i}^j \phi_{\alpha} \gamma (\gamma^3)_{\beta} \mathcal{M} - i \phi_{\alpha \beta} \mathcal{Y}_{ij} ] , \\
[\nabla_{\alpha i}, \nabla_{b}] &= i \frac{1}{2} \phi_{\alpha} \gamma (\gamma_{b})_{\gamma} \nabla_{\gamma i} + i \frac{1}{2} (\gamma^3 \gamma_{b})_{\alpha \beta} BC_{ij} \nabla_{\beta}^j \\
&\quad - i (\gamma^3 \gamma_{b})_{\alpha \beta} \Sigma_{\beta i} \mathcal{M} + i (\gamma_{b})_{\alpha \beta} \Sigma_{\beta}^{\gamma} \mathcal{Y}_{ij} , \\
[\nabla_{a}, \nabla_{b}] &= -\frac{1}{2} \epsilon_{ab}[ (\gamma^3)_{\alpha}^{\beta} \Sigma_{\alpha i} \nabla_{\beta i} + (\gamma^3)_{\alpha}^{\beta} \Sigma_{\beta i} \nabla_{\alpha i} + \mathcal{R} \mathcal{M} + i \mathcal{F}_{i} \mathcal{Y}_{i}^{j} ] .
\end{align*}
\]

The consistency of the Bianchi identities constructed from the commutator algebra above required the conditions,

\[
\begin{align*}
\nabla_{a}^{\alpha} B &= 0 , \quad \nabla_{\alpha i} B = -2C_{ij} (\gamma^3)_{\alpha \beta} \Sigma_{\beta j} , \\
\nabla_{\alpha i} G &= \Sigma_{\alpha i} , \quad \nabla_{\alpha i} H = i(\gamma^3)_{\alpha}^{\beta} \Sigma_{\beta i} , , \\
\nabla_{\alpha i} \Sigma_{\beta j} &= iC_{ij} (\gamma^3 \gamma^a)_{\alpha}^{\beta} \nabla_{a} B , \\
\nabla_{\alpha i} \Sigma_{\beta j} &= \frac{1}{2} \delta_{\alpha}^{\beta} \delta_{i}^{j} [\mathcal{R} - 2G^2 - 2H^2 - 2B B] + i(\gamma^3)_{\alpha}^{\beta} \mathcal{F}_{i}^{j} \\
&\quad + i \frac{1}{2} \delta_{i}^{j} (\gamma^a)_{\alpha}^{\beta} (\nabla_{a} G) - \frac{1}{2} \delta_{i}^{j} (\gamma^3 \gamma^a)_{\alpha}^{\beta} (\nabla_{a} H) .
\end{align*}
\]

\(^{16}\)By prepotential, we mean in the original sense for which the word was coined \(^{23}\), not in the more recent usage as widely appears in \(N = 2\) SUSY YM theory.
The component gauge fields occur in the above supertensors in the following manner.

\[ \mathcal{R} = \epsilon^{ab} \{ \mathcal{R}_{ab}(\hat{\omega}) + [ i2(\gamma^3\gamma_a)_{\alpha\beta} \psi_b^a \bar{\Sigma}^\beta_i + \text{h.c.} ] \]

\[ + 4\phi_\alpha \gamma(\gamma^3\gamma_\beta)\psi_a^\alpha \bar{\psi}_b^\beta_i - 2\{ C_{ij} \bar{B} \psi_a^\alpha \psi_b^j + \text{h.c.} \} \} , \]

\[ \Sigma^{ai} = \epsilon^{ab} \{ \psi_b^\beta i(\gamma^3)_\beta^\alpha - i\psi_b^\beta \bar{\phi}_\beta (\gamma^3\gamma_b)\gamma^\alpha + iC^{ij} \bar{B} \bar{\psi}_a^\beta i(\gamma_b)\beta^\alpha \} , \]

\[ F_{ij} = \epsilon^{ab} \{ F_{ab}(A)_i^j - i2(\gamma_a)_{\alpha\beta} [ \psi_b^\alpha j \bar{\Sigma}^\beta_i + \bar{\psi}_b^\alpha i \Sigma^\beta j ] - \frac{1}{2} \delta_i^j (\psi_b^\alpha k \bar{\Sigma}^\beta_k + \bar{\psi}_b^\alpha k \Sigma^\beta k ) \]

\[ - 4\phi_{\alpha\beta} [ \psi_a^\alpha j \bar{\psi}_b^\beta_i - \frac{1}{2} \delta_i^j (\psi_a^{\alpha k} \bar{\psi}_b^\beta_k) \]

\[ - 2(\gamma^3)_{\alpha\beta} [ B(C_{ik}\psi_a^\alpha k \bar{\psi}_b^{\beta k} - \frac{1}{2} \delta_i^j C_{kl} \psi_a^\alpha k \bar{\psi}_b^{\beta l}) + B(C_{ik} \bar{\psi}_a^\beta k \psi_b^{\alpha i} - \frac{1}{2} \delta_i^j O^{kl} \bar{\psi}_a^\beta k \bar{\psi}_b^{\alpha l}) ] \} , \]

where \( r(\hat{\omega}) \) is the usual two-dimensional curvature in terms of \( e^m_a \) and \( \hat{\omega}_m \).

Now let us repeat the by now familiar steps which follow from our presentation thus far. This begins by totally contracting the indices on the last equation in (117) yielding

\[ \frac{1}{2} \nabla_{ai} \Sigma^{ai} = [ \mathcal{R} - 2G^2 - 2H^2 - 2BB ] . \]  

(119)

We next use the first result of (118) to write

\[ \epsilon^{ab} \mathcal{R}_{ab}(\hat{\omega}) = \left[ \frac{1}{2} \nabla_{ai} \Sigma^{ai} + 2G^2 + 2H^2 + 2BB \right. \]

\[ \left. - \epsilon^{ab} [ i2(\gamma^3\gamma_a)_{\alpha\beta} \psi_b^a \bar{\Sigma}^\beta_i + \text{h.c.} ] - 4\epsilon^{ab} \phi_\alpha \gamma(\gamma^3\gamma_\beta)\psi_a^\alpha \bar{\psi}_b^\beta_i \right. \]

\[ \left. + 2\epsilon^{ab} \{ C_{ij} \bar{B} \psi_a^\alpha \psi_b^j + \text{h.c.} \} \right] . \]

(120)

Upon multiplying both sides of this equation by \(-\frac{1}{2}\), integrating over the 2D manifold and using (11), we find

\[ \Delta = \int d^2 \sigma e^{-1} \left[ \left\{ \frac{1}{2} \nabla^{ai} \nabla_{ai} G + \epsilon^{ab} [ i(\gamma^3\gamma_a)_{\alpha\beta} \psi_b^a \nabla^\beta G + \text{h.c.} ] - G^2 - H^2 - BB + 2 \epsilon^{\alpha\beta} \psi_a^\alpha \bar{\psi}_b^\beta - \phi_\alpha \gamma(\gamma^3)_{\gamma\beta} \right. \right. \]

\[ \left. \left. - \epsilon^{\alpha\beta} \{ C_{ij} \bar{\psi}_a^\alpha \psi_b^j + \text{h.c.} \} \right] \frac{1}{2} \right] . \]

(121)

The fact that this superspace is substantially different from our previous cases can be seen by noting that this last result is not of the form of some differential operator acting on a single superfield which is characteristic of a Stage-II density projector. The basic problem is that \( G, H \) and \( B \) all simultaneously appear in this expression and there are no algebraic relations among them.

At first this seems to be an insurmountable problem. In fact, we are presently aware of three different ways to find an explicit expression for a 2D, \( N = 4 \) density
projector. Although we will not pursue these to their logical conclusion\footnote{This will be the topic of a future work.}, we deem it useful to discuss how these approaches work at least in principle. Let us call these three methods;

(a.) the 2D, \( N = 4 \) SG variational method,

(b.) the 2D, \( N = 4 \) VM-I variational method,

(c.) the 2D, \( N = 4 \) VM-II variational method.

All three of these methods can be used and remarkably enough, none of them require the process of solving the constraints in terms of prepotentials. This was the reason why we used the word “seem” in the definition of Stage-II projectors. They each rely instead on an observation first noted by Wess and Zumino \cite{24}. Namely, in a constrained supersymmetric gauge theory, the complete set of variations which preserve the constraints may be derived by a consistency method applied to the constraints. It was by use of this observation that the first proof of the correct superspace supergravity action was demonstrated in the literature.

(a.) 2D, \( N = 4 \) SG

Applying the approach of Wess and Zumino, one is ultimately led to a set of equations of the forms

\[
\delta G = D_1 \mathcal{V}, \\
\delta H = D_2 \mathcal{V}, \\
\delta B = D_3 \mathcal{V},
\]

where \( D_i \) are certain differential operators that are derived simultaneously with the derivations of the unconstrained variations here denoted by \( \mathcal{V} \). These differential operators are then substituted into (122) appropriately

\[
\tilde{\Delta} = \int d^2 \sigma e^{-1} \left\{ \left[ \frac{1}{2} \nabla^\alpha \nabla_\alpha D_1 + e^{ab} \left[ i (\gamma^3)^{\alpha_\beta} \psi^a_\alpha \psi^b_\beta \gamma_5, D_1 + h.c. \right] \\
- G D_1 - H D_2 - \frac{1}{2} B \bar{D}_3 - \frac{1}{2} B D_3 \\
+ 2 e^{ab} \psi^a_\alpha \bar{\psi}^b_\beta, \left[ C_{\alpha \beta} \bar{D}_2 - i (\gamma^3)^{\alpha \beta} D_1 \right] \\
- e^{ab} \left[ C_{ij} \psi^a_\alpha \psi^b_\beta \bar{D}_3 + h.c. \right] \mathcal{V} \right\},
\]

and lead to a density projector as described by (11).

(b.) and (c.) 2D, \( N = 4 \) VM-II, VM-II

The methods utilizing the 2D, \( N = 4 \) vector multiplets essentially work in the same manner. One first calculates \( \tilde{\Delta} \) starting from the rigid results (see appendix D) then covariantizes with respect to supergravity. After this is done, the variational
approach of Wess and Zumino is applied only to the portions of the commutator algebra describing the matter multiplets. In these cases, this leads ultimately to a set of equations analogous to (122) which are then substituted in the expressions for the respective hat Δ’s to yield an explicit expression for the projector.

VII. The Ethereal Conjecture and Superspace Constraints

So far by the survey of numbers of examples, we hope that the reader has found our arguments (which support the Ethereal Conjecture), so convincing that it may be taken as a working hypothesis to study aspects of supersymmetry theories that have been mysterious for many years. One such aspect is the matter of the constraints themselves. In the introduction we alluded to our belief that the fundamental reason that constraints are imposed in supersymmetrical field theories may also be due to the Ethereal Conjecture. In this section we wish to display some evidence for how such a vague belief can be supported by explicit calculations. In order to illustrate this, we shall discuss some of its implications within the context of 2D, (1,0) theory where all calculations are easily carried out.

Let us begin by introducing a (1,0) supervielbein denoted by E^A_M to distinguish it from the (1,0) supervielbein E^A_M of equation (13) via

\[ E^A_M = A^A_B E^B_M. \] (124)

Since the superfield quantities A^A_B are completely arbitrary, the supervielbein E^A_M correspondingly is completely arbitrary satisfying no constraints. Let us note that the decomposition in (124) is quite general and may be applied to more complicated supergravity theories. For example, the fiducial vielbein (above denoted by E^B_M) can be chosen to describe an on-shell supergravity theory.

Our strategy is quite simple. Even in a (1,0) superspace geometry defined by E^A_M, supergravity covariant derivatives

\[ \nabla_A \equiv E_A + \omega_A M , \] (125)

can be defined. After the introduction of the connections, in turn these unconstrained supergravity covariant derivatives lead to unconstrained superspace torsion and curvature superfields via the equations,

\[
\begin{align*}
\{ \nabla_+, \nabla_+ \} & = ( T_{+,+} B \nabla_B + R_{+,+} M ) , \\
\{ \nabla_+, \nabla_\pm \} & = ( T_{+,\pm} B \nabla_B + R_{+,\pm} M ) , \\
\{ \nabla_+, \nabla_\mp \} & = ( T_{+,\mp} B \nabla_B + R_{+,\mp} M ) , \\
\{ \nabla_+, \nabla_\pm \} & = - ( T^B \nabla_B + R M ) .
\end{align*}
\] (126)
In this modified theory, we have imposed no a priori constraints. An important question to ask is, “How are the topological indices of the modified theory related to the indices defined previously?”

Let us attempt to answer this by investigating a brief calculation. On the basis of dimensional analysis and Lorentz covariance, the quantity \( \hat{X} \) defined by

\[
\hat{X} = i \frac{1}{2} \int d^2 \sigma d\zeta^\gamma E^{\gamma - 1} \mathcal{K}_{+,=},
\]

is a candidate to describe an index in the (1,0) superspace. By use of the normal coordinate technique as espoused in [2, 4], this becomes

\[
\hat{X} = i \frac{1}{2} \int d^2 \sigma \left\{ \text{e}^{-1} \left[ \nabla_+ + T_{a,+}^a + \frac{1}{2} \mathcal{T}_{+,+}^a - \frac{1}{2} \overline{\psi}_{a}^+ T_{+,+}^a \right] \mathcal{K}_{+,=} \right\}. 
\]

Next we observe that one of the Bianchi identities takes the form

\[
\nabla_+ \mathcal{K}_{+,=} + \nabla_\pm \mathcal{K}_{+,=} - \mathcal{T}_{+,+}^D \mathcal{K}_{,==} - \mathcal{T}_{-,+}^D \mathcal{K}_{,+,=} = 0,
\]

which may be rewritten in the form

\[
\nabla_+ \mathcal{K}_{+,=} = \frac{1}{2} \left[ \mathcal{T}_{+,+}^a \mathcal{K}_{a,=} + \mathcal{T}_{+,+}^a \mathcal{K}_{+,=} \right] - \frac{1}{2} \left[ \nabla_+ \mathcal{K}_{+,=} \right] + \left[ \mathcal{T}_{-,+}^a \mathcal{K}_{a,=} + \mathcal{T}_{-,+}^a \mathcal{K}_{+,=} + \mathcal{T}_{-,+}^a \mathcal{K}_{+,=} \right].
\]

Substitution of this result into the leading term in the expression for \( \hat{X} \) yields,

\[
\hat{X} = i \frac{1}{2} \int d^2 \sigma \text{e}^{-1} \left\{ \frac{1}{2} \left[ \mathcal{T}_{+,+}^a \mathcal{K}_{a,=} + \mathcal{T}_{+,+}^a \mathcal{K}_{+,=} \right] - \frac{1}{2} \left[ \nabla_+ \mathcal{K}_{+,=} \right] + \left[ \mathcal{T}_{-,+}^a \mathcal{K}_{a,=} + \mathcal{T}_{-,+}^a \mathcal{K}_{+,=} + \mathcal{T}_{-,+}^a \mathcal{K}_{+,=} \right] \right\}. 
\]

Next the leading term in (131) contains a supercovariantized curvature \( \mathcal{K}_{+,=} \) that possesses an expansion in terms of the gravitino (see C.4 in an appendix),

\[
\mathcal{K}_{+,=} = \mathcal{N}_{+,=} + \mathcal{N}_{+,=}^+ - \mathcal{N}_{+,=}^+ + \mathcal{N}_{+,=}^+ \mathcal{K}_{+,=} + \mathcal{N}_{+,=}^+ \mathcal{K}_{+,=}. \]

The first term in \( \mathcal{K}_{+,=} \) allows the definition of the quantity \( \hat{X} \) via the definition

\[
\hat{X} \equiv - \frac{1}{2} \int d^2 \sigma \text{e}^{-1} \mathcal{R}_{+,=}. 
\]
Using these facts, we finally arrive at the equation \( \hat{X} - \tilde{X} = O_T \) where

\[
O_T = i \frac{1}{2} \int d^2 \sigma \ e^{-1} \left\{ \frac{1}{2} (T_{+,+}^\# - i2) \mathcal{R}_{+,+} + \left[ T_{+,+}^\# - T_{+,+}^\# \right] \right. \\
- \frac{1}{2} \overline{\psi}^\# T_{+,+}^\# - \frac{1}{2} \overline{\psi}^\# (T_{+,+}^\# - i2) \left] \mathcal{R}_{+,+} \right. \\
- \frac{1}{2} \nabla R \mathcal{R}_{+,+} + \left[ T_{+,+}^\# + i \overline{\psi}^\# \psi^\# \right] \mathcal{R}_{+,+} \\
- \left[ T_{+,+}^\# + i \overline{\psi}^\# \right] \mathcal{R}_{+,+} \left\} \right.
\]

(134)

It can be seen that when the usual (1,0) superspace constraints are imposed,

\[
\overline{T}_{+,+}^\# = i2 \quad \overline{T}_{+,+}^\# = \overline{T}_{+,+}^\# = \overline{T}_{+,+}^\# = \overline{T}_{+,+}^\# = 0
\]

(135)

all of the terms of \( O_T \), which we regard as the obstruction to the topological triviality of the ectoplasm (or “ectoplasmic obstruction”), vanish up to total derivatives. In our formulation of the Ethereal Conjecture, \( O_T \) must be trivial in order for the conjecture to be valid. So this equation shows that within the context of (1,0) supergravity, the E.C. is not satisfied without the imposition of the supergravity constraints. This strongly suggests that the reasons for the constraints in all supergravity theories have their origins in topology! However, additional study of this matter is needed in order to construct a rigorous proof of this more generally. The Ethereal Conjecture is, we believe, equivalent to the triviality of \( O_T \).

The method of carrying out the calculation of \( O_T \) above can be graphically illustrated. If we imagine that superspace is a sphere, the quantity \( \hat{X} \) corresponds to an ANZ based calculation of an index throughout the bulk of superspace (i.e. the interior of the sphere). Since purely bosonic \( p \)-forms have their support only on the boundary of the sphere, the quantity \( \tilde{X} \) corresponds to the index calculated on the purely bosonic sub-manifold of superspace (i.e. the surface of the sphere). The quantity \( O_T \) measures the difference of these two definitions.

VIII. Future Prospectives

We have seen that there is evidence from a number of supergravity theories that topology is at the heart of the process of defining the integration of Grassmann variables over local supersymmetry manifolds. If our conjecture is taken as a working assumption, future efforts may have a basis for adding to a new level of understanding of off-shell field representations of supersymmetric theories. In particular, we must begin to understand how to extend the argument of the last section to the cases of more interesting supergravity theories.
Since perhaps the most important theories which would prove of the widest interest are 10D theories, it is appropriate to review exactly what topological invariants might be available to test the Ethereal Conjecture. For the case of the coupled 10D, $N = 1$ supergravity-Yang-Mills system, the topological invariants may be denoted by

$$\Delta_{pqr}(SG + YM) = \int Tr\left[ F^r \wedge R^p \wedge H^q \wedge (d\Phi)^{10-2(p+r)-3q} \right].$$

(136)

The integers $p$, $q$ and $r$ take on the values as indicated in the following table

| $(p, r)$       | $q$  |
|----------------|------|
| $(0, 0)$       | $q = 0, 1, 2, 3$ |
| $(1, 0) (0, 1)$| $q = 0, 1, 2$   |
| $(2, 0) (1, 1) (0, 2)$ | $q = 0, 1, 2$   |
| $(3, 0) (2, 1) (1, 2) (0, 3)$ | $q = 0, 1$   |
| $(4, 0) (3, 1) (2, 2) (1, 3) (0, 4)$ | $q = 0$  |
| $(5, 0) (4, 1) (3, 2) (2, 3) (1, 4) (0, 5)$ | $q = 0$  |

Table I

The trace here is taken with respect to the matrix representation of the Lorentz generator and the Yang-Mills gauge group. The list of 38 a priori indices in Table I includes the special cases of the decoupled theories. Not all of the invariants in Table I are non-trivial. Any $(p, r)$-vector that contains a 1 as a component vanishes due to the tracing operation. This leaves only 21 non-trivial invariants. For 10D supersymmetric Yang-Mills (i.e. $\Delta_{005}$) the single invariant is

$$\Delta(YM) = \int Tr\left[ F \wedge F \wedge F \wedge F \wedge F \right] \equiv \int F^5,$$

(137)

and for decoupled 10D supergravity $\Delta_{pq0}$ the invariants are

$$\Delta_{pq}(SG) = \int Tr\left[ R^p \wedge H^q \wedge (d\Phi)^{10-2p-3q} \right].$$

(138)

Here the integers $p$ and $q$ take on the values as indicated in the following table

\footnote{There may occur multiple ways to define these traces. For example, in 4D, the Pontrjagin and Euler indices arise as two distinct ways of evaluating the trace over the Lorentz generators.}
As found by counting, there are only eleven such invariants. Although there are many additional topological invariants in the coupled case, we would expect that once the constraints are found in the decoupled cases to realize the Ethereal Conjecture separately, the additional ones would follow as consequences.

We can also foresee the possibility to use such an argument for type-II theories also. As is well-known the purely bosonic spectrum of 10D, $N = 1$ supergravity consists of $e_a^m$, $b_{ab}$ and $\Phi$. In the case of the type-IIA theory, there is a supplementary purely bosonic spectrum given by $\hat{A}_a$ and $\hat{A}_{abc}$ and hence field strengths $\hat{F}_{ab}$ and $\hat{L}_{abcd}$. In the case of the type-IIB theory, there is a supplementary purely bosonic spectrum given by $\hat{\Phi}$, $\hat{A}_{ab}$ and $\hat{A}_{abc}$ and hence field strengths $\partial_a \hat{\Phi}$, $\hat{N}_{abc}$ and $\hat{K}_{abcde}$.

In the type-IIA case the candidates for the topological invariants are

$$\Delta^{IIA}_{pqrst} = \int Tr \left[ \hat{F}^r \wedge \hat{F}^s \wedge \hat{R}^p \wedge \hat{H}^q \wedge (d\hat{\Phi})^{10-2(p+r)-3q-4s} \right].$$  \hspace{1cm} (139)

Here $p$, $q$, $r$ and $s$ denote integers similar to those discussed in the $N = 1$ case. For the type-IIB case we have

$$\Delta^{IIB}_{pqrst} = \int Tr \left[ (d\hat{\Phi})^r \wedge \hat{N}^s \wedge \hat{K}^t \wedge \hat{R}^p \wedge \hat{H}^q \wedge (d\hat{\Phi})^{10-r-2p-3(s+q)-5t} \right].$$ \hspace{1cm} (140)

Here $p$, $q$, $r$, $s$ and $t$ once again denote an appropriate set of integers 20.

We wish to observe that the ultimate formulation of covariant string field theory must ultimately confront many of the issues that we discussed in our introduction. In particular, in covariant string field theory there must also be developed a theory of local integrations. The zero-modes for covariant string field theory appear to play the role of the spacetime coordinates of superspace and the oscillator modes play the role

\[\footnote{In the enumeration above, we have not taken into account any redundancy that might result from the existence of possible distinct definitions of performing the tracing operation.}\]

\begin{table}[h]
\centering
\begin{tabular}{|c|c|}
\hline
$p$ & $q$ \\
\hline
$p = 0$ & $q = 0, 1, 2, 3$ \\
$p = 1$ & $q = 0, 1, 2$ \\
$p = 2$ & $q = 0, 1, 2$ \\
$p = 3$ & $q = 0, 1$ \\
$p = 4$ & $q = 0$ \\
$p = 5$ & $q = 0$ \\
\hline
\end{tabular}
\caption{Table II}
\end{table}
of the Grassmann coordinates. We believe that it will be the case that the concept of stringy p-forms should play an important role.

The super forms in (18) obviously belong to the general class of super forms given by

\[
 f_{\mu_1 \cdots \mu_p, \nu_1 \cdots \nu_q} = (-1)^{\frac{N_F}{2}} E_{\mu_q} C_i \cdots E_{\mu_1} C_i \frac{\delta}{\delta E_{\nu_p}} \cdots \frac{\delta}{\delta E_{\nu_1}} F_{A_1} \cdots A_q C_i \cdots C_q .
\]

(141)

It is our contention that the Ethereal Conjecture likely is equivalent to the following statement

*The topology of super manifolds with \( N_F \) and \( N_B \) fermionic and bosonic coordinates, respectively, arises solely from its purely bosonic p-forms.*

The mixed and purely fermionic super p-forms although topologically insignificant, are useful for other aspects. For example, the case of \( p = 1, q = 0 \) and \( p = 0, q = 1 \) has been used previously [11, 26] to define supersymmetric gauge phase factors.

Finally, the coefficients given in (9) are such that \( \hat{\Delta} \) defined by (11) corresponds to the integration of the super \( N_B \)-form \( f_{\nu_1 \cdots \nu_N} \) over a hypersurface in superspace. The hypersurface corresponds to the bosonic spacetime manifold. From this vantage point, it should be clear that the theory of ectoplasmic integration that we have described is based on the use of superdifferential forms and follows the path that is standard for ordinary bosonic differential forms. This observation dramatically emphasizes that the definition of ectoplasmic integration defined in the present work is *logically independent* of ANZ local superspace integration theory [9]. Dramatically, our new approach to local superspace integration is *totally independent* of the superdeterminant. More remarkably however, the local ectoplasmic integration operator \( D^N_F \) derived on the basis of super p-forms (11-13) agrees *exactly* with the local ANZ integration operator\(^{21}\) derived on the basis of a normal coordinate expansion of the superdeterminant (1). Whether this statement is necessarily true for *all* supergravity theories is an interesting question to pursue.

"I have no special talents. I am only passionately curious." – Einstein

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\(^{21}\)Whimsically, we might call this the ANZIO-i.e. the ANZ integration operator.
Research Institute where this work was initiated and which inspired him to re-consider the problem of a local superspace integration theory and the possible role of topology. He has also benefitted greatly from numerous critical comments and inputs from M. T. Grisaru, W. Siegel and especially M. E. Knutt-Wehlau for her assistance in section seven.
Appendix A: Conventions for 2D and 3D Spinors

For two dimensional superspaces and using covariant notation, we use the following conventions for the quantities associated with spinors.

\[ \eta_{ab} = (1, -1) \ , \ \epsilon_{ab} \epsilon^{cd} = - \delta_{[a} \epsilon_{d]} \delta_{b] c} \ , \ \epsilon^{01} = +1 \ , \]
\[
(\gamma^a)_{\alpha} (\gamma^b)_{\beta} = \eta^{ab} \delta_{\alpha}^\beta - \epsilon^{ab} (\gamma^3)_{\alpha}^\beta ,
\]
\[
(\gamma^3)_{\alpha} (\gamma^a)_{\beta} = - \epsilon^{ab} (\gamma^b)_{\alpha}^\beta .
\] (A.1)

Some useful Fierz identities are:

\[ C_{\alpha \beta} C^{\gamma \delta} = \delta_{[\alpha} \gamma \delta_{\beta]} \]
\[
(\gamma^a)_{\alpha \beta} (\gamma_a)_{\gamma \delta} + (\gamma^3)_{\alpha \beta} (\gamma^3)_{\gamma \delta} = - \delta_{(\alpha} \gamma \delta_{\beta)} ,
\]
\[
(\gamma^a)_{(\alpha} (\gamma_a)_{\beta)} \delta + (\gamma^3)_{(\alpha} (\gamma^3)_{\beta)} \delta = \delta_{(\alpha} \gamma \delta_{\beta)} ,
\]
\[
(\gamma^a)_{(\alpha} (\gamma_a)_{\beta)} \delta = - 2 (\gamma^3)_{\alpha \beta} (\gamma^3)_{\gamma \delta} ,
\]
\[
2 (\gamma^a)_{\alpha \beta} (\gamma_a)_{\gamma \delta} + (\gamma^3)_{\alpha \beta} (\gamma^3)_{\gamma \delta} = - \delta_{(\alpha} \gamma \delta_{\beta)} ,
\]
\[
(\gamma_a)_{\alpha} \delta_{\beta} + (\gamma^3 \gamma_a)_{\alpha} (\gamma^3)_{\beta} \delta = (\gamma^3 \gamma_a)_{\alpha \beta} (\gamma^3)_{\gamma \delta} .
\] (A.2)

For three dimensional superspaces, we use the following conventions for the quantities associated with spinors.

\[ \eta_{ab} = (1, -1, -1) \ , \ \epsilon_{abc} \epsilon^{def} = \delta_{[a} \epsilon_{d] e} \delta_{f c]} \ , \ \epsilon^{012} = +1 \ , \]
\[
(\gamma^a)_{\alpha} (\gamma^b)_{\beta} = \eta^{ab} \delta_{\alpha}^\beta + i \epsilon^{abc} (\gamma_c)_{\alpha}^\beta .
\] (A.3)

Some useful Fierz identities are:

\[ C_{\alpha \beta} C^{\gamma \delta} = \delta_{[\alpha} \gamma \delta_{\beta]} \]
\[
(\gamma^a)_{\alpha \beta} (\gamma_a)_{\gamma \delta} = - \delta_{(\alpha} \gamma \delta_{\beta)} ,
\]
\[
\epsilon^{abc} (\gamma_b)_{\alpha \beta} (\gamma_a)_{\gamma} = - i C_{\alpha \gamma} (\gamma^a)_{\beta} - i (\gamma^a)_{\alpha \gamma} \delta_{\beta} \delta .
\] (A.4)
APPENDIX B: Supercovariantized Field Strengths

In this appendix, we present some samples of component-level field strength tensors. We concentrate on the gauge $(D - 1)$-form multiplet in $D$ dimensions. The reason for looking at this particular multiplet is that the $D$-form gauge field strength is a topological invariant and provides candidates for the quantities $\tilde{\theta}$. We present these results without derivation\textsuperscript{22}. In each of the following cases, the equation is to be understood to be valid only at $\theta = 0$ order. Additionally, the first term on the rhs of each equation represents the usual component-level field strength that is present without the presence of supersymmetry.

(A.) The 2D, $N = 1, 2, 4$ Yang-Mills supercovariantized field strength takes the form,

$$F_{ab} = f_{ab} + \psi_{[a} \alpha F_{\alpha [b]} + \psi_{a}^\alpha \dot{\psi}_{b}^\beta F_{\alpha \beta} . \quad (B.1)$$

(B.) The 3D, $N = 1$ supercovariantized field antisymmetric tensor gauge field strength takes the form,

$$G_{abc} = g_{abc} + \frac{1}{2} \psi_{[a}^\alpha \theta^{\alpha} G_{\alpha [bc]} + \frac{1}{2} \psi_{[a}^\alpha \dot{\psi}_{b}^\beta G_{\alpha \beta [c]} - \dot{\psi}_{a}^\alpha \dot{\psi}_{b}^\beta \psi_{c}^\gamma G_{\alpha \beta \gamma} . \quad (B.2)$$

(C.) The 4D, $N = 1$ supercovariantized field antisymmetric rank three gauge field strength takes the form,

$$F_{abcd} = f_{abcd} + \frac{1}{3!}(\psi_{[a}^\alpha F_{\alpha [bcd]} + \tilde{\psi}_{[a}^\dot{\alpha} F_{\dot{\alpha} [bcd]})$$

$$+ \frac{1}{2} \psi_{[a}^\alpha \tilde{\psi}_{[b}^\dot{\beta} F_{\alpha \beta [cd]} + \frac{1}{2} \psi_{[a}^\alpha \dot{\psi}_{[b}^\dot{\beta} F_{\alpha \beta [cd]} + \frac{1}{4} \tilde{\psi}_{[a}^\dot{\alpha} \dot{\psi}_{[b}^\dot{\beta} \tilde{\psi}_{[c}^\dot{\gamma} F_{\dot{\alpha} \dot{\beta} \dot{\gamma} [d]}$$

$$- \frac{1}{3!} \psi_{[a}^\alpha \psi_{[b}^\dot{\beta} \psi_{[c}^\dot{\gamma} F_{\alpha \beta \gamma [d]} - \frac{1}{3!} \tilde{\psi}_{[a}^\dot{\alpha} \dot{\psi}_{[b}^\dot{\beta} \tilde{\psi}_{[c}^\dot{\gamma} F_{\dot{\alpha} \dot{\beta} \dot{\gamma} [d]}$$

$$- \frac{1}{3!} \psi_{[a}^\alpha \psi_{[b}^\dot{\beta} \psi_{[c}^\dot{\gamma} \tilde{\psi}_{[d}^\dot{\delta} F_{\alpha \beta \gamma \delta} - \frac{1}{3!} \tilde{\psi}_{[a}^\dot{\alpha} \dot{\psi}_{[b}^\dot{\beta} \tilde{\psi}_{[c}^\dot{\gamma} \tilde{\psi}_{[d}^\dot{\delta} F_{\dot{\alpha} \dot{\beta} \dot{\gamma} \dot{\delta}}$$

$$- \psi_{a}^\alpha \psi_{b}^\dot{\beta} \psi_{c}^\dot{\gamma} \psi_{d}^\dot{\delta} F_{\alpha \beta \gamma \delta} - \tilde{\psi}_{a}^\dot{\alpha} \tilde{\psi}_{b}^\dot{\beta} \tilde{\psi}_{c}^\dot{\gamma} \tilde{\psi}_{d}^\dot{\delta} F_{\dot{\alpha} \dot{\beta} \dot{\gamma} \dot{\delta}}$$

$$- \frac{1}{4} \psi_{[a}^\alpha \psi_{[b}^\dot{\beta} \tilde{\psi}_{[c}^\dot{\gamma} \tilde{\psi}_{[d}^\dot{\delta} F_{\alpha \beta \gamma \delta} . \quad (B.3)$$

\textsuperscript{22}The interested reader can find the method of derivation of this general class of results by reviewing [25].
The analogs of these identities for superspace supergravity were first given a long time ago [19] and take the form (all quantities on the rhs are to be understood as first being evaluated at $\theta = 0$)

\[
T_{ab}^c \nabla^a \nabla^b + R_{ab}^\Gamma \mathcal{M}_\Gamma = \left[ \nabla_a, \nabla_b \right] + \psi_{[a} \bar{\psi}_{|b]} \delta[\nabla_\gamma, \bar{\nabla}_\delta] + \psi_{[a} \bar{\psi}_b \delta[\nabla_\gamma, \nabla_\delta] + \text{h.c.} \right].
\]

Here the first term on the rhs is the usual set of field strengths in a non-supersymmetric theory and $\mathcal{M}_\Gamma$ denotes the generators of the tangent space.

All of the field strengths discussed in this section can be seen as special cases of the formula,

\[
f_{m_1 \ldots m_p} = (-1)^{\frac{N_p}{2}} E_{m_p, A_1} \ldots E_{m_1, A_1} F_{A_1 \ldots A_p} \right),
\]

where the leading term $f_{m_1 \ldots m_p}$ corresponds the field strength of some component level gauge field, $F_{A_1 \ldots A_p}$ corresponds to the appropriate superspace field strength and $E_{m, A}$ is identified with the component level gravitino and vielbein as described in (19).

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APPENDIX C: Some Facts About 4D, $N = 1$ Super p-forms

Although the Bianchi identities (B. I.'s) for the 4D, $N = 1$ super 4-form have been explicitly given before [13], for the convenience of the reader we will here provide explicit expressions. The equation which we schematically write as

$$0 = \nabla_{[\alpha_1} J_{\alpha_2 \ldots \alpha_5]} - T_{[\alpha_1, \alpha_2]} L J_{\alpha_3, \ldots \alpha_5]}$$  \hspace{1cm} (C.1)$$

is explicitly of the form,

$$0 = \frac{1}{2!} \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)} - \frac{1}{2 \times 3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})}$$

$$0 = \frac{1}{3!} \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}\delta \epsilon \gamma - \frac{1}{3} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})\delta \epsilon \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})\delta \epsilon \gamma}$$

$$0 = \frac{1}{2} \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\delta \epsilon \gamma} - \frac{1}{4} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma} - \frac{1}{3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma}}$$

$$0 = \frac{1}{2} \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\delta \epsilon \gamma} - \frac{1}{4} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma} - \frac{1}{3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma}}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\delta \epsilon \gamma} - \frac{1}{4} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma} - \frac{1}{3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma}}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\delta \epsilon \gamma} - \frac{1}{4} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma} - \frac{1}{3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma}}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\delta \epsilon \gamma} - \frac{1}{4} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma} - \frac{1}{3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma}}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma - \frac{1}{2} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\epsilon} \gamma}$$

$$0 = \nabla_{(\alpha J_{\beta \gamma \delta \epsilon)}_{\delta \epsilon \gamma} - \frac{1}{4} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma} - \frac{1}{3!} T_{(\alpha \beta \gamma J_{L \gamma \delta \epsilon})_{\delta \epsilon \gamma}}$$

In a similar manner we find that a super 4-form field strength $F_{ABCD}$ can be expressed as the super exterior derivative of a super 3-form gauge field $J_{ABC}$ schemati-
cally via the equation

\[ F_{\Delta_1 \ldots \Delta_4} = \nabla_{[\Delta_1} J_{\Delta_2 \ldots \Delta_4]} - T_{[\Delta_1 \Delta_2} J_L^{\mid \Delta_3] \Delta_4} \]  \hspace{1cm} \text{(C.3)}

has the explicit representation given by,

\begin{align*}
F_{\alpha \beta \gamma \delta} &= \frac{1}{3!} \nabla_{(\alpha J_{\beta \gamma \delta})} - \frac{1}{4} T_{(\alpha \beta) L \mid J_{\gamma \delta})} , \\
F_{\alpha \beta \gamma \delta} &= \frac{1}{2} \nabla_{(\alpha J_{\beta \gamma \delta})} + \nabla_{\delta} J_{\alpha \beta \gamma} - \frac{1}{2} T_{(\alpha \beta L \mid J_{\gamma \delta})} - \frac{1}{2} T_{(\alpha \beta \mid L J_{\gamma \delta})} , \\
F_{\alpha \beta \gamma \delta} &= \nabla_{(\alpha J_{\beta \gamma \delta})} + \nabla_{\delta} J_{\alpha \beta \gamma} - T_{\alpha \beta L \mid J_{\gamma \delta}} - T_{\gamma \delta L J_{\alpha \beta}} \\
&\quad - T_{(\alpha \mid (\gamma \mid \ell J_{\mid \beta \mid \delta))} , \\
F_{\alpha \beta \gamma \delta} &= \frac{1}{2} \nabla_{(\alpha J_{\beta \gamma \delta}) \delta} - \nabla_{\delta} J_{\alpha \beta \gamma} - \frac{1}{2} T_{(\alpha \beta L \mid J_{\gamma \delta})} + \frac{1}{2} T_{(\delta \mid L J_{\gamma \delta})} , \\
F_{\alpha \beta \gamma \delta} &= \nabla_{(\alpha J_{\beta \gamma \delta}) \delta} + \nabla_{\delta} J_{\alpha \beta \gamma} - \nabla_{\delta} J_{\alpha \beta \gamma} - T_{\alpha \beta L \mid J_{\gamma \delta}} \\
&\quad - T_{\gamma \delta L J_{\alpha \beta}} + T_{(\alpha \mid L \mid J_{\mid \beta \mid \delta))} , \\
F_{\alpha \beta \gamma \delta} &= \nabla_{(\alpha J_{\beta \gamma \delta}) \delta} + \nabla_{\delta} J_{\alpha \beta \gamma} - T_{\alpha \beta L \mid J_{\gamma \delta}} - T_{(\alpha \mid L \mid J_{\mid \beta \mid \delta))} \\
&\quad - T_{(\alpha \mid \beta \mid L \mid J_{\mid \delta \mid \gamma})} , \\
F_{\alpha \beta \gamma \delta} &= \nabla_{(\alpha J_{\beta \gamma \delta}) \delta} + \nabla_{\delta} J_{\alpha \beta \gamma} + \frac{1}{2} \nabla_{\delta} J_{\mid \alpha \beta \mid \delta} - T_{\alpha \beta L \mid J_{\gamma \delta}} \\
&\quad + T_{\alpha \beta L \mid J_{\gamma \delta}} + T_{\beta \gamma J_{\mid \delta \mid \alpha}} - T_{\gamma \delta L J_{\alpha \beta}} , \\
F_{\alpha \beta \gamma \delta} &= \nabla_{(\alpha J_{\beta \gamma \delta}) \delta} - \frac{1}{2} \nabla_{[b} J_{\alpha \ell \delta]} - \frac{1}{2} T_{\alpha \beta L \mid J_{\gamma \delta}} - \frac{1}{2} T_{[b} J_{\alpha \beta L \mid J_{\ell \delta]} \alpha , \\
F_{\alpha \beta \gamma \delta} &= \frac{1}{3!} \nabla_{(\alpha J_{\beta \gamma \delta})} - \frac{1}{4} T_{(\alpha \beta L \mid J_{\gamma \delta})} .
\end{align*}  \hspace{1cm} \text{(C.4)
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