LOG-TQFT AND TORSION

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The goal here is to put into place an algebraic theory, or rather a categorification, of logarithmic representations and their log-determinant characters which captures a class of additive invariants defined by generalised Reidemeister torsions on the bordism category.

Bordism invariants of this type may be viewed as semi-classical, positioned between classical bordism invariants (genera) and quantum bordism invariants (TQFTs); the former are homomorphisms

$$\mu : \Omega_* \to R$$

on the ring $\Omega_*$ of bordism classes of closed manifolds, such as the signature of a 4k dimensional manifold, while a topological quantum field theory (TQFT) of dimension $n$ with values in a category $\mathcal{B}$ refers to a symmetric monoidal functor

$$Z : \text{Bord}_n \to \mathcal{B}$$

from the category $\text{Bord}_n$ whose objects are smooth closed $(n-1)$-dimensional manifolds $M$ and whose morphisms are diffeomorphism classes of $n$-dimensional bordisms. The semi-classical bordism invariants we have in mind here arise as characters of log-functors from bordisms to a category of rings. A log-functor means an additive simplicial map from the nerve $\mathcal{N} \text{Bord}_n$ of the bordism category

$$\log : \mathcal{N} \text{Bord}_n \to \mathcal{M}_A$$

to a simplicial set $\mathcal{M}_A$ associated to a background additive category $\mathcal{A}$. The simplicial structure means that each bordism $W \in \text{mor}(M, M')$ has a logarithm $\log_{M \sqcup M'}(W)$ in a ring $\mathcal{F}(M \sqcup M') \in \text{mor}_A$ along with a hierarchy of inclusions $\mathcal{F}(M \sqcup M') \hookrightarrow \mathcal{F}(M \sqcup M' \sqcup M'')$ such that when two bordisms $W \in \text{mor}(M, M'), W' \in \text{mor}(M', M'')$ are sewn together there is an additive identity in $\mathcal{F}(M \sqcup M' \sqcup M'')$

$$\log_{M \sqcup M''}(W \cup M', W') \approx \log_{M \sqcup M'}(W) + \log_{M' \sqcup M''}(W'),$$

where $\approx$ indicates equality modulo finite sums of commutators. Neither commutators nor the inclusion maps are seen by trace maps $\tau_N : \mathcal{F}(N) \to R$ to a commutative ring $R$ and so, irrespective of in which ring it may be convenient to view the logarithm of a bordism $W$, the resulting log-character $\tau(\log W) \in R$ is invariantly defined.
Characters of log-TQFTs capture a class of semi-local invariants somewhat more general than the local invariants that occur as genera but which, in view of the log-additivity pasting property, are necessarily simpler and more restricted (possibly more delicate) than the globally determined invariants of a TQFT. Such trace-logs may be characterised as generalised (or exotic) torsions, and include classical Whitehead and Reidemeister torsions and the topological signature \( \sigma \) and the Euler characteristic \( \chi \) (note that \( \sigma \) is a genus while \( \chi \) is not). They arise formally in semi-classical expansions of Feynman path integrals, such as Reidemeister torsion \( T_M(a) \) in the asymptotic sum expansion of Chern-Simons TQFT \( Z_{cs}(M) \sim \sum_a c(a) \sqrt{T_M(a)} \) over irreducible flat connections \([5],[15]\).

Generalising the case of the classical topological signature \( \sigma \), it is known \([7]\) that higher Novikov signatures are additive with respect to gluing and it would be interesting to know if they may likewise be characterised as log-TQFTs. This is closely tied-in with a characterisation, observed first to me by Ryszard Nest, of higher log-functors as simplicial maps ranging in Hoschchild homology \( HH_k(A) \), the case \( k = 0 \) being the subject of this article.

1 Preliminaries: logarithms on monoids and torsion

Let \( B = (B, \cdot, +) \) be a ring, and let

\[
[B, B] = \{ \sum_{1 \leq j \leq n} [\beta_j, \beta_j'] \mid \beta_j, \beta_j' \in B \}
\]  

be the subgroup of the abelian group \( B = (B, +) \) consisting of finite sums of commutators \([\beta_j, \beta_j'] := \beta_j \cdot \beta_j' - \beta_j' \cdot \beta_j\). For \( \mu, \nu \in B \) we may use the notation

\[
\mu \approx \nu \text{ if } \mu - \nu \in [B, B], \quad \text{so } \mu = \nu \text{ in } B/[B, B].
\]

A logarithmic (or log) representation of a monoid \( Z \) with values in \( B \) means a homomorphism

\[
\log : Z \to (B, +)/[B, B], \quad \log(ba) = \log b + \log a,
\]

where \( ba = b \circ a \) is composition in \( Z \). Such a map is consequent on a map \( \log : Z \to B \) with \( \log(ba) = \log b + \log a + \sum_j [c_j, c_j'] \) for some \( c_j, c_j' \in B \), and if the exact sequence \( 0 \to [B, B] \to B \to B/[B, B] \to 0 \) of abelian groups splits then the converse holds. Sums of logs are logs and so form a homomorphic ring \( \text{Log}(Z, B) = \text{Hom}(Z, B/[B, B]) \).

A trace on \( B \) with values in a commutative unital ring \( (R, \cdot, +) \) is a homomorphism of abelian groups \( \tau : (B, +) \to (R, +) \) which vanishes on commutators \( \tau([b, b']) = 0 \), so \([B, B] \subset \text{Ker}(\tau)\). Again, sums of traces are traces forming an abelian group \( \text{Trace}(B, R) \).

A log-character is an evaluation \( \tau \circ \log : Z \to R \) in the abelian group \( \text{Hom}(Z, (R, +)) \),
defining a logarithmic representation with

\[ \tau(\log ba) = \tau(\log a) + \tau(\log b) \quad \text{in} \ R. \quad (1.4) \]

Composition with an exponential map \( e : R \to A^* \), \( e(x+y) = e(x)e(y) \), into the group \( A^* \) of units of a commutative ring \( A \) associates a multiplicative character \( \det_{t,e} := e \circ \tau \circ \log \) in the abelian group \( \text{Det}(\mathcal{Z}, A) \) of exponentiated trace-logs.

For example, let \( \mathcal{Z} = \text{Fred} \) be the monoid of Fredholm operators on a Hilbert space, and \( \mathcal{B} = \mathcal{F} \) the ideal of finite-rank operators. The logarithm \( \text{Fred} \to \mathcal{F}/[\mathcal{F}, \mathcal{F}] \) defined by \( \log a = \pi([a,p]) \), where \( p \in \text{Fred} \) is any parametrix for \( a \) and \( \pi : \mathcal{F} \to \mathcal{F}/[\mathcal{F}, \mathcal{F}] \) the quotient, is the abstract Fredholm index of \( a \). Its numeric log-character \( \text{Tr} \log a \) with respect to the canonical isomorphism \( \mathcal{F}/[\mathcal{F}, \mathcal{F}] \cong \mathbb{C}, c \mapsto \text{Tr}(c) \), defined by the classical trace is the integer valued Fredholm index \( \text{ind}_a = \dim \ker(a) - \dim \text{coker}(a) \). The log-additivity property \((1.4)\) then gives the classical property \( \text{ind}_{ba} = \text{ind}_a + \text{ind}_b \).

Matters extend immediately to log-characters \((1.4)\) on general categories, since these take values in a fixed ring \( R \). For a compact oriented manifold \( W \) of dimension \( 4k \) with boundary \( \partial W \), the topological signature \( \text{sgn}(W) \) of \( W \), defined as the signature of the quadratic form

\[ H^{2k}(W, \partial W) \times H^{2k}(W, \partial W) \to \mathbb{Z}, \quad (\xi, \xi') \mapsto \langle \xi \cup \xi', [W] \rangle, \quad (1.5) \]

arises as such a character. \( \text{sgn}(W) \) is a homotopy invariant of \( W \) and so pushes-down to a map on morphisms in the bordism category. By a result of Novikov (c 1967), proved for closed bordisms in \cite{2}, this is functorial (defining an integer valued log-TQFT)

\[ \log_{\text{sgn}} := \text{sgn} : \text{Bord}_{4k} \to (\mathbb{Z}, +), \quad \log_{\text{sgn}}(W \cup_M W') = \log_{\text{sgn}}(W) + \log_{\text{sgn}}(W'), \quad (1.6) \]

and hence a (weak) TQFT by \( Z(W) = e^{\text{sgn}(W)} \) and \( Z(\partial W) := \mathbb{Z} \). This contrasts with (Wall) non-additivity of the signature for higher codimension partitions \cite{13}.

The Fredholm index and the topological signature are better viewed here as an exotic Reidemeister torsion (R-torsion) invariants. Such torsions are secondary invariants associated to chain complexes over a commutative ring and admit a simple algebraic description in terms of log-characters. For this, on a given class of finite linear chain complexes \( \mathcal{C}_m \xrightarrow{d_m} \mathcal{C}_{m-1} \xrightarrow{d_{m-1}} \cdots \xrightarrow{d_1} \mathcal{C}_0 \to 0 \) we shall for brevity assume a monoid of cochain maps \( a = \oplus_{p \geq 0} a_p : \mathcal{C} \to \mathcal{C} := \oplus_{p \geq 0} \mathcal{C}_p \), logarithm maps \( a_p \mapsto \log a_p \in \mathcal{B}_p \) into \( R \)-modules \( \mathcal{B}_p \) and that each differential has a well-defined adjoint \( d^*_p : \mathcal{C}_p \to \mathcal{C}_{p+1} \). One may then contemplate for each \((m+1)\)-tuple \( \beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^{m+1} \) the generalized torsion logarithm

\[ T_\beta(C) := \sum_{p=0}^{m} (-1)^p \beta_p \log \Delta_p \in \bigoplus_{p=0}^{m} \mathcal{B}_p/[\mathcal{B}_p, \mathcal{B}_p] \]
for the Laplacians $\Delta_p := d_{p-1}d_{p-1}^* + d_p^*d_p : C_p \to C_p$, and, with respect to traces $\tau : B_p \to R$, its log-character

$$\tau_\beta(C) := \sum_{p=0}^{m} (-1)^p \beta_p \tau(\log \Delta_p).$$

For the lifted cellular complex $C(\hat{M})$ of the universal cover $\hat{M}$ of a closed manifold $M$ as an $\mathbb{R}\pi_1(M)$-module and $\rho : \pi_1(M) \to O(n)$ an orthogonal representation, defining a flat vector bundle $E \to M$, if $C = C(\hat{M}) \otimes_{\mathbb{R}\pi_1} \mathbb{R}^n$ is an acyclic complex then the combinatorial Laplacians $\Delta_p^c$ are invertible finite rank real matrices and topological Reidemeister torsion was identified by Ray and Singer [10] (Prop. 1.7) with the log-character

$$\tau^R(M, E) := \frac{1}{2} \sum_{p=0}^{m} (-1)^p p \text{Tr} (\log \Delta_p^c)$$

(1.7)

and shown in [10], [4], [8] to equal classical analytic torsion

$$\tau^\zeta(M, E) := \frac{1}{2} \sum_{p=0}^{m} (-1)^p p \text{TR}_\zeta(\log \Delta_p)$$

(1.8)

where $\Delta_p$ is the (invertible) Hodge Laplacian on the de Rham complex $\Omega^*(M, E)$ over $M$ coupled to $E$ and $\text{TR}_\zeta$ is the regularised $\zeta$-trace.

Here, on the monoid $\Psi_{p,a}((M, E)$ of classical pseudodifferential operators ($\psi$dos) acting on a bundle $E$ admitting a principal angle $\theta \in \mathbb{R}$, holomorphic functional calculus defines the logarithm by $\log A_\theta = \frac{d}{dz} A^z_\theta |_{z=0}$, which by a minor technical modification ([11] §2.7.1.3) takes values in the the algebra $\Psi^Z_m$ of integer order $\psi$dos

$$\log : \Psi_{p,a}((M, E) \to \Psi^Z_{p,a}(M, E)/[\Psi^Z_{p,a}(M, E), \Psi^Z(M, E)]$$

(1.10)

$\log A = \log_\theta A$ any $\theta$ then has order zero and (1.10) is well defined since the dependence on $\theta$ lies in $[\Psi^Z_{p,a}(M, E), \Psi^Z(M, E)]$; for a Laplacian there is, anyway, no ambiguity. If, instead of $\text{TR}_\zeta$, the character of the logarithm of the Hodge Laplacian is evaluated using the residue trace res : $\Psi^Z(M, E) \to C$, the unique trace on $\Psi^Z$ and defined by $\text{res}(T) = \int_{S^\gamma M} t_{-n}(x, \eta) dS_\gamma dx$ with $t_{-n}$ the component of the local asymptotic expansion of the symbol of $T$ of homogeneity degree $-n := -\dim M$ ([6], [12]), the corresponding exotic log-character of $T_\beta(M, E) := \sum_{p=0}^{m} (-1)^p \beta_p \log \Delta_p$ is the analytic residue.
torsion
\[ \tau^\text{res}_\beta (M, E) := \frac{1}{2} \sum_{p=0}^{m} (-1)^p \beta_p \text{res} (\log \Delta_p), \quad (1.11) \]
a complimentary invariant to the classical analytic torsion:

**Theorem 1.1** \( \tau^\text{res}_\beta (M, E) \) is zero if \( M \) is odd-dimensional. If \( M \) is even dimensional it is a topological invariant (independent of any Riemannian metric on \( M \)) if and only if
\[ \beta_p = p \text{ for each } p \quad \text{or} \quad \beta_p = 1 \text{ for each } p. \quad (1.12) \]
For \( \beta_p = p \) and a smooth path \( u \mapsto g_u \) of metrics one has
\[ \frac{d}{du} \tau^\text{res}_\beta (M, E) := \frac{1}{2} \sum_{p=0}^{m} (-1)^{p+1} \text{res}(\alpha_p). \quad (1.13) \]

The corresponding topological torsions are the (weighted) Euler characteristics
\[ \tau^\text{res}_p (M, E) = \chi_p (M, E) \quad \text{and} \quad \tau^\text{res}_1 (M, E) = \chi (M, E) \quad (1.14) \]
with \( \chi_p (M, E) := \sum_{p=0}^{m} (-1)^p \dim H^p (M, E) \); any analytic residue torsion is of the form
\[ \tau^\text{res} (M, E) = A \chi_p (M, E) + B \chi (M, E) = \sum_{p=0}^{m} (-1)^p (Ap + B) \dim H^p (M, E) \quad (1.15) \]
for constants \( A, B \).

Here, \( \Omega^* (M, E) \) is not assumed acyclic (the torsions are zero if it is), though the residue trace views any elliptic complex as acyclic. Taking an elliptic complex of length one gives the index of any elliptic operator as a residue torsion \( \tau^\text{res}_1 (M, E) \).

\( \chi_p (M, E) \) has a not insignificant role in constructing the Bismut-Lott higher torsion forms \[3\] (eq. 3.106). The similarity of (1.9) and (1.13) is notable.

**Proof of Theorem 1.1** To see (1.13), we have
\[ \frac{d}{du} \tau^\text{res}_\beta (M, E) := \frac{1}{2} \sum_{p=0}^{m} (-1)^p \beta_p \text{res}(\Delta_p), \]
where \( \Delta_p^{-1} \) really means a choice of global parametrix for \( \Delta_p \) — which on the quotient \( \Psi^2 / \Psi^{-\infty} \), where the residue trace lives, is an inverse. Using the identities for the de Rham operator \( d_p : \Omega^p (M, E) \to \Omega^{p+1} (M, E) \)
\[ \Delta_p := d_{p-1} d_p^* + d_p d_{p-1}, \quad d_p \Delta_p = \Delta_{p+1} d_p, \quad d_{p-1} \Delta_p = \Delta_{p-1} d_p^*, \]
\[ \Delta_p = \alpha_p d_p^* d_p - d_p^* \alpha_{p+1} d_p + d_{p-1} \alpha_{p-1} d_{p-1} - d_{p-1} d_{p-1}^* \alpha_p, \]
gives with \( \mu_p := \text{res}(\Delta_p^{-1} d_p^* d_{p+1} \alpha_p), \quad q = p+1, \quad r = p-1, \)
\[ 2 \frac{d}{du} \tau^\text{res}_\beta (M, E) = \sum_{p=0}^{m} (-1)^p \mu_{p+1} + \sum_{p=0}^{m} (-1)^p \mu_{p+1} + \sum_{p=0}^{m} (-1)^p \mu_{p+1} \]
\[- \sum_{p=0}^{m} (-1)^p \text{res}(\alpha_{p+1}) - \sum_{p=0}^{m} (-1)^p \text{res}(\alpha_p) \]
\[= \sum_{p=0}^{m} (-1)^p \cdot 2\mu_p - \sum_{q=0}^{m} (-1)^q (q-1) \mu_q - \sum_{p=0}^{m} (-1)^r (r+1) \mu_r \]
\[+ \sum_{q=0}^{m} (-1)^{q} (q-1) \text{res}(\alpha_q) - \sum_{p=0}^{m} (-1)^p \text{res}(\alpha_p) \]

which is the right-hand side of (1.13).

To see (1.14) one uses the formula \(\text{res} (\log \Delta_p) = -2(\zeta(\Delta_p, 0) + \dim \text{Ker} (\Delta_p))\) of [12] along with the Hodge theorem and, since [10] Theorem 2.3 applies verbatim to non-invertible \(\Delta_p\), that \(\sum_{p \geq 0} (-1)^p \zeta(\Delta_p, z) = 0\) for all \(z\) if \(M\) is even dimensional. The vanishing of the torsion in odd-dimensions follows from the vanishing of the residue trace on odd-class operators on such manifolds and is readily checked.

To see (1.12), a repeat of the above variation computation equated to zero with \(p\) replaced by \(\beta_p\) yields the difference equation \(2\beta_p - \beta_{p+1} - \beta_{p-1} = 0\) whose solutions are (1.12).

\[\square\]

More fundamentally, then, the invariant object here is the torsion logarithm

\[T_{\beta}(M, E) := \sum_{p=0}^{m} (-1)^p \beta_p \log \Delta_p \in \bigoplus_{p=0}^{m} \Psi_p^Z / [\Psi_p^Z, \Psi_p^Z] \]

where \(\Psi_p^Z := \Psi_p^Z(M, E \otimes \Omega^p(M))\) and the quotients are complex lines. It is this object which is independent of the metric in the cases (1.12) and the classical and residue analytic torsions are log-character evaluations of it; the \(\text{TR}_{\zeta}\) evaluation for \(\beta_p = 1\) on odd-dimensional \(M\) vanishes on an acyclic de Rham complex.

## 2 Log-determinant structures on categories

We collect together in this section some properties of logarithmic representations and their characters on general categories.

### 2.1 Monoidal product representations

Let \(C\) be a (small) category. Denote the set of morphisms between objects \(x, y \in \text{ob}(C)\) by \(\text{mor}_C(x, y)\), or \(\text{mor}(x, y)\), and \(\text{end}(x) := \text{mor}(x, x)\). \(C\) is monoidal if it has a bifunctor \(\otimes : C \times C \to C\) which is associative with identity object \(1 = 1_C\) up to coherent isomorphism. Any two coherence isomorphisms between associativity bracketings of an
The product functors of a monoidal category \( C \) are (iterations of) the functors \( C \to C \) obtained by holding fixed one of the inputs of the bifunctor \( \otimes : \) for \( y \in \text{ob}(C) \) the right-product functor \( m_y \otimes - : C \to C \) takes \( x \in \text{ob}(C) \) to \( x \otimes y \) and \( \alpha \in \text{mor}_C(x,z) \) to \( \alpha \otimes \iota \in \text{mor}_C(x \otimes y, z \otimes y) \), with \( \iota \) the identity morphism, the left-product functor \( m_w \otimes (x) = w \otimes x \) is defined symmetrically. The product functors are not monoidal.

The following construction allows the classical additivity of logarithms to be promoted to a categorical additivity on composed morphisms.

**Definition 2.2** Let \( C = (C, \otimes) \) be a symmetric monoidal category and let \( C^* = (C^*, \otimes) \) be a groupoid whose objects are those of \( C \) and whose morphisms are a specified closed subclass of the isomorphisms of \( C \) (containing the coherence and permutation isomorphisms (2.1)).

A monoidal product representation of the reduced category \( C^* \) into a target category \( M \) is a strict functor

\[
F : C^* \to M
\]

with for each \( y \in \text{ob}(C) \) an injective natural transformation of functors

\[
\eta_{\otimes y} : F \Rightarrow F_{\otimes y}
\]

from \( F : C^* \to M \) to \( F_{\otimes y} := F \circ \nu_{\otimes y} : C^* \to M \) compatible with \( \otimes \). (Neither the category \( M \) nor the functor \( F \) is assumed to be monoidal.)
If $S$ is a symmetric monoidal category, monoidal product representations pull-back with respect to symmetric monoidal functors $J : S^* \to C^*$.

$F$ represents the set of objects of $C$ with its monoidal product, and not (necessarily) its morphisms; but by Lemma 2.1 is it sensitive to permutation isomorphisms, and these intertwine with the covering maps $\eta_{\otimes y}$ as follows.

**Lemma 2.3** Let $y \in \text{ob}(C)$. For each $x \in \text{ob}(C)$ there is a morphism

$$\eta_{\otimes y}(x) \in \text{mor}_M(F(x), F(x \otimes y))$$

covering $m_{\otimes y}$ such that for $x, x_{\sigma}$ as in (2.1)

$$\eta_{\otimes y}(x_{\sigma}) \circ \mu_{\sigma}(x) = \mu_{\sigma 1}(x \otimes y) \circ \eta_{\otimes y}(x).$$

Proof: A natural transformation $\eta : G \Rightarrow H$ of functors $G, H : A \to B$ defines for $x \in \text{ob}(A)$ a morphism $\eta(x) \in \text{mor}_B(G(x), H(x))$ with $\eta(z) \circ G(\alpha) = H(\alpha) \circ \eta(x)$ for $\alpha \in \text{mor}_A(x, z)$. Applied to $G := F$ and $H := F_{\otimes y}$, (2.3) gives $\eta_{\otimes y}(x) := \eta(x)$ in (2.6). For (2.7), take $z = x_\sigma$ and $x = s_\sigma(x) \in \text{mor}(x, x_\sigma)$, so $\eta(z) \circ G(\alpha) = \eta_{\otimes y}(x_\sigma) \circ F(s_\sigma(x)) = \eta_{\otimes y}(x_\sigma) \circ \mu_{\sigma}(x)$ while $H(\alpha) \circ \eta(x) = F_{\otimes y}(s_\sigma(x)) \circ \eta_{\otimes y}(x)$ and

$$F_{\otimes y}(s_\sigma(x)) = F(m_{\otimes y}(s_\sigma(x))) = F(s_\sigma(x) \otimes \iota_y) = F(s_{\sigma 1}(x \otimes y)) = \mu_{\sigma 1}(x \otimes y).$$

In particular, since $F$ is strict there is for each $x \in \text{ob}(C)$ a canonical inclusion

$$\eta_{x}(1) : F(1) \hookrightarrow F(x).$$

*Compatibility* of the $\eta_{\otimes y}$ with $\otimes$ is the requirement $\eta_{\otimes(y \otimes z)} = \eta_{\otimes z} \circ \eta_{\otimes y}$, or more fully

$$\eta_{\otimes(y \otimes z)}(x) = \eta_{\otimes z}(x \otimes y) \circ \eta_{\otimes y}(x).$$

The assumption that $\eta_{\otimes y}$ is *injective* is that $\eta_{\otimes y}(x)$ is left-invertible for each $x \in \text{ob}(C)$: there is a $\pi_{\otimes y}(x) \in \text{mor}_M(F(x \otimes y), F(x))$ with $\pi_{\otimes y}(x) \circ \eta_{\otimes y}(x) = i$ with $i$ the identity morphism, and satisfying $\pi_{\otimes z} \circ \pi_{\otimes y} = \pi_{\otimes(z \otimes y)}$.

Somewhat more generally, it is useful to combine the above maps to define *insertion morphisms* for $x = x_0 \otimes \cdots \otimes x_n$ and $0 \leq k \leq n + 1$ and $w \in \text{ob}(C)$

$$\eta_{\otimes w}^k(x) : F(x_0 \otimes \cdots \otimes x_n) \to F(x_0 \otimes \cdots \otimes x_{k-1} \otimes w \otimes x_k \cdots \otimes x_n)$$

by

$$\eta_{\otimes w}^k(x) = \mu_{\sigma_{k,n+1}}(x \otimes w) \circ \eta_{\otimes w}(x),$$

where $\sigma_{k,n+1}$ is the permutation $(0, \ldots, n + 1) \to (0, \ldots, k - 1, n + 1, k, \ldots, n)$. In particular (by fiat), $\eta_{\otimes y} := \eta_{\otimes y}^{n + 1}(x)$ and $\eta_{\otimes y} := \eta_{\otimes y}^0(x)$. When it is clear what is meant, the superscript $k$ and the domain specifier $(x)$ may be omitted to write $\eta_w$. 8
Insertion morphisms commute (Lemma \ref{lem:insertion-morphisms-commute}) and hence for \( \underline{w} = (w_1, \ldots, w_r) \in \text{ob}(\Sigma(C)) \) the iterated insertion morphism
\[
\eta_{\underline{w}} := \eta_{w_1} \eta_{w_2} \cdots \eta_{w_r} := \eta_{w_1} \circ \cdots \circ \eta_{w_r} : F(x) \to F(x_{\underline{w}})
\] (2.12)
is unambiguously defined, independently of the ordering of the \( \eta_{w_i} \); here, \( x = x_0 \otimes \cdots \otimes x_n \)
while \( x_{\underline{w}} \) is the monoidal product of the \( x_i \) and \( w_i \) in a specified order. Evidently, each insertion morphism (2.12) is injective (left-invertible). The commutation property is:

**Lemma 2.4** One has
\[
\eta_k^n \eta_l^j = \begin{cases} \eta_k^n \eta_l^{j-1} & \text{if } k < l, \\ \eta_l^{k+1} \eta_k^j & \text{if } k \geq l. \end{cases}
\] (2.13)

**Proof:** The shorthand here is \( \eta_k^n \eta_l^j := \eta_k^n \circ (x \otimes w) \circ \eta_l^j \), where \( x = x_0 \otimes \cdots \otimes x_n \),
and so on. Noticing that the case \( \eta_k^n \eta_{l+1}^n = \eta_k^{n+2} \eta_l^{n+1} \) is (2.7) and states
\[
\eta_{\otimes z}(x \otimes w) \eta_{\otimes w}(x) = \mu_{z \otimes w}(x \otimes w \otimes z) \eta_{\otimes z}(x \otimes w) \eta_{\otimes w}(x)
\] (2.14)
the general case then follows easily using (2.11).

It follows that a monoidal product representation defines a simplicial set structure on the subcategory \( F(C^n) \) with \( p \)-simplices the sets \( F(x_1 \otimes x_2 \otimes \cdots \otimes x_p) \) and face and degeneracy maps \( s_j = \eta^j(x_j), d_j = \pi_j(x_j) \). By construction and the lemma, though, there are the additional ‘face’ and ‘degeneracy’ maps \( s_j(z) = \eta^j(z), d_j(z) = \pi_j(z) \) likewise obeying the simplicial relations (2.13). We shall refer to this as being strongly simplicial.

**Example:** The fundamental groupoid \( \Pi_{\leq 1}(X) \) of a smooth manifold \( X \) is the category whose objects are the points \( x \) of \( X \) and morphisms are homotopy classes of smooth paths with collared ends and whose monoidal product is disjoint union. A \( k \)-vector bundle \( E \to X \) with flat connection \( \nabla \) defines \( F_{\nabla} : \Pi_{\leq 1}(X) \to \text{Alg}_k \) to the category of finite-dimensional \( k \)-algebras by assigning to \( \underline{x} = x_1 \sqcup \cdots \sqcup x_n \) the algebra \( F_{\nabla}(\underline{x}) = \text{End}_k(E_{x_1} \oplus \cdots \oplus E_{x_n}) \) with \( E_x \) the fibre of \( E \) over \( x \) in \( X \) and to \( y \in \text{mor}(x, y) \) the isomorphism \( F_{\nabla}(\underline{x}) \cong F_{\nabla}(\underline{y}) \) induced by the matrix of parallel transports \( \underline{\beta}_\nabla \in \text{Hom}(E_{\underline{x}}, E_{\underline{y}}) \). Here, (2.11) is a permutation of the order of the disjoint union \( x_1 \sqcup \cdots \sqcup x_n \) and (2.2) the corresponding permutation of the rows and columns of the matrix \( \underline{\beta}_\nabla \), while 1 is the empty set and \( F_{\nabla}(1) = \{0\} \) the zero algebra and (2.3) the trivial inclusion. The \( \eta_y \) on \( F_{\nabla}(\underline{x}) \) are the canonical linear inclusions; in particular, \( \eta_{\otimes x} \) is the map \( T \to T \oplus 0 \).

Let \((A, \tau)\) be a Calabi-Yau category, so \( A \) is additive and on each ring \( \text{end}_A(x) := \text{mor}_A(x, x) \) there is a trace \( \tau = \tau_x : \text{end}_A(x) \to \text{end}_A(1) \). Let \( M_A \) be a category of rings whose objects are endomorphism sets in \( A \); thus if \( y \in \text{ob}(M_A) \) then \( y = \text{mor}_A(\chi_y, \chi_y) \) some unique \( \chi_y \in \text{ob}(A) \). Typically, \( A \) will be a category enriched over \( M_A \).
Definition 2.5 \( F : \mathcal{C}^* \to M_{(A, \tau)} \) is a tracial monoidal product representation, assigning to each \( x \in \text{ob}(\mathcal{C}) \) a trace \( \tau_x : \text{mor}_A(x, x^y) \to \text{mor}_A(1_A, 1_A) \), if \( \eta_{\otimes y}(x) \) of (2.6) and \( \pi_{\otimes y}(x) \) are ring homomorphisms and the \( \mu_{\otimes}(x) \) of (2.2) with \( x = x_1 \otimes \cdots \otimes x_n \) are ring isomorphisms, and if for all \( x, y \in \text{ob}(\mathcal{C}) \)

\[
\tau_{x \otimes y} \circ \eta_{\otimes y}(x) = \tau_x \quad \text{and} \quad \tau_{x \otimes} \circ \mu_{\otimes}(x) = \tau_x. \quad (2.15)
\]

Characters in a tracial monoidal product representation can be computed ‘anywhere’:

Lemma 2.6 For a tracial monoidal product representation \( F : \mathcal{C}^* \to M_A \), and in the notation of (2.12) with \( x = x_1 \otimes \cdots \otimes x_n \), each insertion morphism \( \eta_w : F(x) \to F(x_w) \) is a ring homomorphism and pushes-down to a homomorphism \( \eta_w : F(x)/[F(x), F(x)] \to F(x_w)/[F(x_w), F(x_w)] \) of abelian groups (indicated with the same notation). One has

\[
\tau_x = \tau_{x_w} \circ \eta_w. \quad (2.16)
\]

Proof: As \( F \) is adapted to \( A \) the first statement follows from the definition (2.11) of \( \eta_w \) implying \( \eta_w \left( [F(x \otimes z), F(x \otimes z)] \right) \subset [F(x \otimes w \otimes z), F(x \otimes w \otimes z)] \). Replacing \( \tau_{x \otimes z} \) by \( \tau_{x \otimes w \otimes z} \circ \eta_w \) defines another trace on \( F(x \otimes z) \), to see they are the same we have

\[
\tau_{x \otimes w \otimes z} \circ \eta_w = \tau_{x \otimes w \otimes z} \circ \mu_{\otimes}(x \otimes z \otimes w) \circ \eta_{\otimes w}(x \otimes z) = \tau_{x \otimes w \otimes z} \circ \eta_{\otimes w}(x \otimes z) = \tau_{x \otimes z}
\]

where the final two equalities use (2.15). (2.16) follows by iteration.

Example: An open-closed conformal field theory (CFT) is a symmetric monoidal functor \( Z : \mathbf{S} \to \text{Vect} \) from the category whose objects are compact one dimensional manifolds (possibly with boundary) and whose morphisms are conformal classes of compact Riemann surfaces with corners. \( Z \) assigns to each \( J \in \text{ob}(\mathbf{S}) \) a Frobenius algebra \( A_J := Z(J) \) as a tensor product of the component Frobenius algebras \( A_{S^1} \) and \( A_{[0,1]} \). Each such algebra comes equipped with a scalar valued trace. Setting \( F(J) \) to be the direct sum of the component Frobenius algebras defines, on the other hand, a tracial monoidal product representation of \( \mathbf{S} \).

2.2 Logarithmic functors

The nerve \( \mathcal{N} \mathcal{C} \) of a category \( \mathcal{C} \) is the simplicial set whose \( p \)-simplices are diagrams

\[
x_0 \xrightarrow{\alpha_0} x_1 \xrightarrow{\alpha_1} x_2 \to \cdots \to x_{p-1} \xrightarrow{\alpha_{p-1}} x_p \quad \in \mathcal{N} \mathcal{C} \quad (2.17)
\]
of morphisms $\alpha_j \in \text{mor}(x_j, x_{j+1})$. The $j^{th}$ face map $d_j : N_p C \to N_{p-1} C$ of the simplex deletes $x_j$, replacing when $0 < j < p$

$$\cdots \to x_{j-1} \xrightarrow{\alpha_{j-1}} x_j \xrightarrow{\alpha_i} x_{j+1} \to \cdots \quad \text{by} \quad \cdots \to x_{j-1} \xrightarrow{\alpha_i \circ \alpha_{j-1}} x_{j+1} \to \cdots$$

(2.18)

and the $j^{th}$ degeneracy map $s_j : N_p C \to N_{p+1} C$ replaces

$$\cdots \to x_j \xrightarrow{\alpha_j} x_{j+1} \to \cdots \quad \text{by} \quad \cdots \to x_j \xrightarrow{s_j} x_j \xrightarrow{\alpha_j} x_{j+1} \to \cdots.$$ 

(2.19)

$N C$ carries more data than $C$ — the objects and morphisms of $C$ are respectively identified with $N_0 C$ and $N_1 C$, while there is no right inverse to the composition face map $d_1 : \text{mor}_C(x_0, x_2) \to \text{mor}(x_0, x_2)$. The classifying space $B C$ of $C$ is the geometric realisation of $N C$.

Logarithms on a category $C$ have to be differentiated between according to the substrata of marked morphisms in $N_p C$ on which they act. To this end, one has the stratum of $\underline{z} = (x_1, \ldots, x_{p-1})$-marked $p$-simplices (2.17) between $x, y \in \text{ob}(C)$

$$\text{mor}\underline{z}(x, y) \subset N_p C$$

$$:= \text{mor}_C(x, x_1) \times \text{mor}_C(x_1, x_2) \times \cdots \times \text{mor}_C(x_{p-1}, y).$$

If $\text{mor}(x_j, x_{j+1}) = \emptyset$ some $j$ then $\text{mor}_{\underline{z}}(x, y) := \emptyset$, while $\text{mor}_{\emptyset}(x, y) := \text{mor}(x, y)$. One has the composition

$$\text{mor}_{\underline{z}}(x, w) \times \text{mor}_{\underline{z}'}(w, y) \xrightarrow{\circ} \text{mor}_{\underline{z} \bullet \underline{z}'}(x, y),$$

relative to concatenation $\bullet$, so $(x, z) \bullet y = (x, z, y)$ and so on, as a partially defined composition

$$N_p C \times N_q C \to N_{p+q-1} C$$

on compatible strata, while the face and degeneracy maps respectively restrict to simplicial maps

$$d_j : \text{mor}_{\underline{z}}(x, y) \to \text{mor}_{\delta_j(\underline{z})}(x, y), \quad s_j : \text{mor}_{\underline{z}}(x, y) \to \text{mor}_{\sigma_j(\underline{z})}(x, y)$$

with $\delta_j : C^p \to C^{p-1}$ and $\sigma_j : C^p \to C^{p+1}$ defined in the evident way.

**Definition 2.7** Let $C = (C, \otimes)$ be a symmetric monoidal category and let $F : C^* \to M_A$ be a (strict) monoidal product representation adapted to an additive monoidal category $A$. Then a log-functor (or logarithmic-functor) on $C$ taking values in $F$ is a strongly simplicial map

$$\log : N C \to F(C^*) \subset M_A.$$ 

Such a structure is said to define a logarithmic representation of $C$ in $M_A$. 

11
Recall that a simplicial map \( f : S_1 \to S_2 \) is a map which commutes with the face and degeneracy maps on simplicial sets \( S_1, S_2 \), here a strongly simplicial map means a simplicial map which furthermore commutes with the strong simplicial structure (as in the discussion following Lemma 2.12) on \( F(C^*) \).

Unwrapping the definition, a log-functor comprises the following:

1. A (strict) monoidal product representation (on the set \( \mathcal{N}_0 C \) of 0-simplices):
   \[
   F : C^* \to M_A
   \]
   associating to each \( x \in \text{ob}(C) \) a ring \( F(x) = \text{mor}_A(\chi_x, \chi_x) \).

2. A simplicial system of (strict) logarithm maps (on the set \( \mathcal{N}_1 C \) of 1-simplices) assigning to \( x, y \in \text{ob}(C) \), with \( x, y \) not both the monoidal identity \( 1 \in \text{ob}(C) \), a map
   \[
   \log_{x \otimes y} : \text{mor}(x, y) \to F(x \otimes y)/[F(x \otimes y), F(x \otimes y)],
   \]
   \[
   \alpha \mapsto \log_{x \otimes y} \alpha = \log (x \xrightarrow{\alpha} y)
   \]
   and, more generally, (on the set \( \mathcal{N}_p C \) of \( p \)-simplices) to each marking \( z = (z_1, \ldots, z_{p-1}) \)
   \[
   \log_{x \otimes z \otimes y} : \text{mor}(x, y) \to F(x \otimes z \otimes y)/[F(x \otimes z \otimes y), F(x \otimes z \otimes y)]
   \]
   where \( x \otimes z \otimes y := x \otimes z_1 \otimes \cdots \otimes z_{p-1} \otimes y \neq 1 \),
   \[
   \alpha \mapsto \log_{x \otimes y} \alpha = \log (x \xrightarrow{\alpha} z_1 \xrightarrow{\alpha_2} z_2 \to \cdots \to z_{p-1} \xrightarrow{\alpha_{p-1}} y),
   \]
   such that for \( \beta \alpha \in \text{mor}_z(x, y) \) associated to \( \alpha \in \text{mor}(x, z) \) and \( \beta \in \text{mor}(z, y) \) (p = 2) one has in
   \[
   F(x \otimes z \otimes y)/[F(x \otimes z \otimes y), F(x \otimes z \otimes y)].
   \]
   that
   \[
   \log_{x \otimes z \otimes y} \beta \alpha = \eta_{y}(\log_{x \otimes z} \alpha) + \eta_{x}(\log_{z \otimes y} \beta),
   \]
   or, equivalently,
   \[
   \eta_{z}(\log_{x \otimes y} \beta \alpha) = \eta_{y}(\log_{x \otimes z} \alpha) + \eta_{x}(\log_{z \otimes y} \beta).
   \]
   The requirement that the system of log maps (2.21), (2.22) be simplicial is, for \( p = 2 \),
   \[
   \log_{x \otimes z \otimes y}(x \xrightarrow{\alpha} z \xrightarrow{\beta} y) - \eta_{z} \log_{x \otimes y}(x \xrightarrow{\beta_{\alpha}} y) \in [F(x \otimes z \otimes y), F(x \otimes z \otimes y)]
   \]
   \[
   \log_{x \otimes z \otimes y}(x \xrightarrow{\alpha} x \xrightarrow{\beta} y) - \eta_{x} \log_{x \otimes y}(x \xrightarrow{\beta_{\alpha}} y) \in [F(x \otimes x \otimes y), F(x \otimes x \otimes y)]
   \]
   \[
   \log_{x \otimes z \otimes y}(x \xrightarrow{\alpha} y \xrightarrow{\beta} y) - \eta_{y} \log_{x \otimes y}(x \xrightarrow{\beta_{\alpha}} y) \in [F(x \otimes y \otimes y), F(x \otimes y \otimes y)]
   \]
and, more generally, for \( z = (x_1, \ldots, x_{p-1}) \) and \( \nu \in \text{mor}_z(x, y) \) that for \( j = 1, \ldots, p - 1 \)

\[
\log_z \nu - \eta_{x_j}(\log_{d_j(z)}(\nu)) \in [F(x \otimes z \otimes y), F(x \otimes z \otimes y)]
\]

(2.20)

along with the two end-point special cases corresponding to (2.27) and (2.28).

(2.26) implies (2.24) and (2.25) are equivalent.

If \( C \) is a category with no monoidal structure given, then a free log-functor (or free logarithmic-functor) on \( C \) is defined to be a log-functor on the free symmetric monoidal category \( (\Sigma(C), \bullet) \) with values in \( M_A \).

If \( A = (A, \tau) \) is a Calabi-Yau category and (2.20) is a tracial monoidal product representation \( F : C^* \rightarrow M_{(A, \tau)} \) then a log-functor is called a log-determinant functor (representation) of \( C \) in \( M_{(A, \tau)} \). The corresponding log-determinant of the logarithm \( \log_{x \otimes y} \alpha \in F/F, F \) of \( \alpha \in \text{mor}_z(x, y) \) defined by (2.22) is the tracial character

\[
\tau(\log \alpha) := \tau(\log_{x \otimes y} \alpha) \in \text{end}_A(1).
\]

(2.30)

If, further, \( \text{end}_A(1) \) is endowed with a non-zero exponential map \( e : \text{mor}_A(1, 1) \rightarrow R \) to a commutative ring (‘exponential’ meaning \( e(\xi + \eta) = e(\xi) \cdot e(\eta) \)), then this defines a categorical determinant structure on \( C \) with determinant

\[
\det \tau \alpha := e(\tau(\log_{x \otimes y} \alpha)) \in R.
\]

(2.31)

We may then write formally \( \tau(\log \alpha) = \log, \det \tau \alpha \).

A log-functor is not in general a functor of categories.

The assumption that (2.22) is strict is that \( \log_{x \otimes y} \alpha \) is independent of a choice of associativity bracketing of \( x \otimes z \otimes y \) and of the monoidal coherence isomorphisms.

(2.24) is consequent on an equivalence

\[
\log_{x \otimes z \otimes y} \beta \alpha \approx \eta_{y\otimes y}(\log_{x \otimes z} \alpha) + \eta_{x\otimes y}(\log_{z \otimes y} \beta) \quad \text{in} \ F(x \otimes z \otimes y);
\]

(2.32)

that is, on an equality

\[
\log_{x \otimes z \otimes y} \beta \alpha = \eta_{y\otimes y}(\log_{x \otimes z} \alpha) + \eta_{x\otimes y}(\log_{z \otimes y} \beta) + \sum_{1 \leq j \leq m} [\nu_j, \nu'_j]
\]

(2.33)

some \( \nu_j, \nu'_j \in F(x \otimes z \otimes y) \). Likewise, (2.25) derives from an equivalence

\[
\eta_z(\log_{x \otimes y} \beta \alpha) \approx \eta_{y\otimes y}(\log_{x \otimes z} \alpha) + \eta_{x\otimes y}(\log_{z \otimes y} \beta),
\]

(2.34)

meaning an equality for some \( \mu_j, \mu'_j \in F(x \otimes z \otimes y) \)

\[
\eta_z(\log_{x \otimes y} \beta \alpha) = \eta_{y\otimes y}(\log_{x \otimes z} \alpha) + \eta_{x\otimes y}(\log_{z \otimes y} \beta) + \sum_{1 \leq j \leq n} [\mu_j, \mu'_j].
\]

(2.35)
The log-additivity property (2.25) can be written in terms of the simplicial face maps as
\[ \eta_1 \log_{\delta_1(z)}(d_1(\beta\alpha)) \approx \eta_0 \log_{\delta_0(z)}(d_0(\beta\alpha)) + \eta_2 \log_{\delta_2(z)}((d_2(\beta\alpha)). \] (2.36)
where \( z = x \otimes y \otimes z, \eta_0 := \eta_{x\otimes}, \eta_1 := \eta_z, \eta_2 := \eta_{y\otimes}, \beta\alpha := x \rightarrow z \rightarrow y \in \text{mor}(z) \). This holds because the end-point face maps \( d_0, d_p : N_pC \rightarrow N_{p-1}C \) are defined by deleting the 0th or \( p^\text{th} \) morphism, respectively, from a simplex; this also the reason that (2.27), (2.28) have to be stated separately in the definition of simplicial compatibility.

We note that a \( \log \)-functor is effectively determined by its action on 1-simplices:

**Lemma 2.8** A simplicial system (2.32) of logarithm maps \( \log_{x\otimes y} \) is determined up to terms in \( [F,F] \) by the log maps \( \log_{x\otimes y} \) on \( \text{mor}(x,y) \) for each \( x,y \in \text{ob}(C) \) in (2.33). To define a compatible system of logarithm maps \( \log_{x\otimes y} \) it is enough to define the \( \log_{x\otimes y} \) on \( \text{mor}(x,y) \) satisfying (2.22).

**Proof:** Compatibility (2.20) gives
\[ \log_{x\otimes y}(\nu) = \eta_0(\log_{x\otimes y}(\nu)) \quad \text{in} \quad [F(x \otimes z \otimes y), F(x \otimes z \otimes y)] \] (2.37)
which is the first statement of the lemma. Given \( \log_{x\otimes y} \), the second statement is that (2.37), or indeed
\[ \log_{x\otimes y}(\nu) = \eta_0(\log_{x\otimes y}(\nu)) \quad \text{in} \quad F(x \otimes z \otimes y), \] (2.38)
defines by default a compatible system of logs (2.22).

Being defined modulo sums-of-commutators is a characteristic of logarithm maps. (2.40) in the following lemma says that if two \( p \) simplices collapse to the same \( (p-r) \) simplex, then they have the same logarithm. Likewise, inflating simplices does not change logarithms (2.43), (2.44), (2.41) and (2.42), and its \( s_j \) counterpart, state how the simplicial set structures (2.1) and (2.13), (2.19) entwine.

**Lemma 2.9** If \( d_1(x \rightarrow z \rightarrow y) = d_1(x \rightarrow z \rightarrow y) \) (that is, \( \beta\alpha = \beta'\alpha' \)) in \( \text{mor}(x,y) \) then
\[ \log_{x\otimes y}(\beta\alpha) = \log_{x\otimes y}(\beta'\alpha'). \] (2.39)
in \( F(x \otimes z \otimes y)/[F(x \otimes z \otimes y), F(x \otimes z \otimes y)] \). More generally, if for \( z = (x_1, \ldots, x_{p-1}) \) and \( \nu, \nu' \in \text{mor}_z(x,y) \) and \( j = 1, \ldots, p-1 \) one has \( d_j(\nu) = d_j(\nu') \), then
\[ \log_{x\otimes y}(\nu) = \log_{x\otimes y}(\nu') \] (2.40)
in \( F(x \otimes z \otimes y)/[F(x \otimes z \otimes y), F(x \otimes z \otimes y)] \). Iteratively, if \( d_k(d_j(\nu)) = d_k(d_j(\nu')) \) (2.41) continues to hold since
\[ \log_{x\otimes y}(\nu) \approx \eta_{x_1} \eta_{x_k} \log_{s_k(z)}(d_k(d_j(\nu))). \] (2.41)
For \( j < k \)
\[
\eta_x \eta_x \log \delta_k(\delta_j(\xi)) d_k(d_j(\nu)) = \eta_x \eta_j \log \delta_j(\delta_{k-1}(\xi)) d_j(d_{k-1}(\nu)).
\] (2.42)

Dually, for the degeneracy maps \( (2.19) \) one has
\[
\log \sigma_j(\sigma_i(\xi)) \approx \eta_x^j \log \sigma_j(\nu)
\] (2.43)
and a corresponding commutation formula to \( (2.42) \). For each of the above, the two end-point special cases corresponding to \( (2.27) \) and \( (2.28) \) also hold.

Proof: By \( (2.26) \)

\[
\log_{x \otimes y}(x \rightarrow z \rightarrow y) \approx \eta_x \log_{x \otimes y}(x \rightarrow z \rightarrow y) = \eta_x \log_{x \otimes y}(x \rightarrow z \rightarrow y)
\]
and in general \( \log_x \nu \approx \eta_x (\log \delta_j(\xi) d_j(\nu)) \approx \eta_x (\log \delta_j(\xi) \delta_j(\nu)) \approx \log_x \nu \) by \( (2.29) \). The iterated version follows by iterating these equalities given that \( (2.41) \) holds, and that holds because the \( \eta_x \) are ring homomorphisms. \( (2.42) \) and its \( s_j \) counterpart are immediate from \( (2.4) \) and the simplicial identities \( d^i d^j = d^j d^{i-1} \) and \( s^i s^j = s^j s^{i+1} \) for \( k < j \). The inflation formulae \( (2.43), (2.44) \) follow from \( (2.29) \) (resp. \( (2.42) \) by replacing \( \nu \) by \( s_j(\nu) \) (resp. \( s_k(s_j(\nu)) \)). The two end-point special cases of \( (2.40) \) hold from \( (2.27) \) and \( (2.28) \) by the same argument as the case \( 1 \leq j \leq p-1 \), while for \( (2.43) \) this is shown in Proposition \( (2.12) \).\( \square \)

Lemma 2.10 The character of the logarithm of \( \beta \alpha \in \mathrm{mor}_z(x, y) \) is invariantly defined: in \( \mathrm{mor}_A(1, 1) \)
\[
\tau_{x \otimes y}(\log_{x \otimes y} \beta \alpha) = \tau_{x \otimes y}(\log_{x \otimes y} \beta \alpha)
\] (2.45)
and, likewise, for \( \delta \in \mathrm{mor}(x, y) \)
\[
\tau_{x \otimes y}(\eta_x(\log_{x \otimes y} \delta)) = \tau_{x \otimes y}(\log_{x \otimes y} \delta).
\]
These generalise naturally to all \( p \)-simplices: for \( z = (z_1, \ldots, z_r) \) and \( x = x_1 \otimes \cdots \otimes x_n \) and the insertion morphism \( \eta_z := \eta_{z_1} \eta_{z_2} \cdots \eta_{z_r} : F(x) \rightarrow F(z) \)
\[
\tau_{x_z}(\log_{x_z} \nu) = \tau_{x_1 \otimes \cdots \otimes x_n}(\log_{x_1 \otimes \cdots \otimes x_n} \nu).
\]

Proof: For any \( w \in \mathrm{ob}(C) \)
\[
[F(w), F(w)] \subset \mathrm{Ker}(\tau_w)
\] (2.46)
and since all logs are simplicial \( (2.41) \)
\[
\log_{x_w}(\nu) - \eta_w(\log_{x_1 \otimes \cdots \otimes x_n} \nu) \in [F(x_w), F(x_w)].
\] (2.47)
So the trace of \( \log_{x_1 \otimes \cdots \otimes x_n} \alpha - \eta_1(\log_{x_1} \alpha) \) vanishes in \( \mathrm{end}_A(1) \) and then \( (2.16) \) applies to give the asserted trace formulæ.
Here, (2.45) is shorthand for \( \tau_{x \otimes z \otimes y} \left( \log_{x \otimes z \otimes y} (x \xrightarrow{\alpha} z \xrightarrow{\beta} y) \right) = \tau_{x \otimes y} \left( \log_{x \otimes y} (x \xrightarrow{\beta \alpha} y) \right) \), or \( \tau_{x \otimes z \otimes y} (\log_{x \otimes z \otimes y} \beta \alpha) = \tau_{x \otimes y} (\log_{x \otimes y} d_1(\beta \alpha)) \).

In view of Lemma 2.10 it is reasonable to refer to the logarithmic character \( \tau(\log \alpha) \in \text{mor}_A(1,1) \), as defined in (2.30), of a morphism \( \alpha \in \text{mor}_C(x,y) \) and to omit any subscript in the notation, and likewise for the determinant (2.31).

Despite Lemma 2.8 it can be natural to define simplicial logarithms directly on strata \( \text{mor}_z(x,y) \). In particular, this allows a log-functor to be extended to any \( \delta \in \text{mor}_C(1,1) = \text{end}_C(1) \) factorisable as \( \delta = \beta \alpha \) for \( \alpha \in \text{mor}_C(1,z) \) and \( \beta \in \text{mor}_C(z,1) \) with \( z \neq 1 \in \text{ob}(C) \) (this is always the case on \( \text{Bord}_n \), for example). Choosing such a factorisation, define
\[
\log_z \delta := \log_z (1 \xrightarrow{\alpha} z \xrightarrow{\beta} 1) \in F(z).
\] (2.49)

Here, \( \log_z := \log_{1 \otimes z \otimes 1} : \text{mor}_z(1,1) \to F(1 \otimes z \otimes 1) = F(z) \), as \( F \) is exact and log is strict, which depends on \( \delta \) and \( z \) but by Lemma 2.10 is independent of the particular choice of \( \alpha, \beta \). In the presence of a trace one then further has
\[
\log_1 \delta := \tau(\log_z \alpha + \tau(\log_z \beta))
\] (2.51)
as a particular case of (2.52):

**Lemma 2.11** For \( \alpha \in \text{mor}(x,z) \) and \( \beta \in \text{mor}(z,y) \)
\[
\tau(\log \beta \alpha) = \tau(\log \alpha) + \tau(\log \beta) \quad \text{in} \, \text{mor}_A(1,1),
\] (2.52)

\[
\det_\tau(\alpha \beta) = \det_\tau(\alpha) \cdot \det_\tau(\beta) \quad \text{in} \, R,
\] (2.53)

The space \( \text{Log}(C,M_A) \) of logarithmic representations is an abelian group
\[
\log_1, \log_2 \in \text{Log}(C,M_A) \Rightarrow \log_1 + \log_2 \in \text{Log}(C,M_A)
\] (2.54)

with respect to the additive structure of the category \( A \), as is the space \( \text{Log}(C,M_A) \) of logarithmic characters \( \tau(\log \alpha) \) (the range of the pairing between \( \text{Log}(C,M_A) \) and \( \text{Trace}(A) \)) with addition taking place in \( \text{mor}_A(1,1) \). Likewise, for determinant structures, the space \( \text{Det}(C,R) \) of determinants is an abelian group
\[
det_1, \det_2 \in \text{Det}(C,R) \Rightarrow \det_1 \cdot \det_2 \in \text{Det}(C,R)
\] (2.55)

with respect to the multiplication “\cdot” in the abelian group \( (R,\cdot) \).
If $C$ is an additive category then $\tau \circ \log$ is a log-representation from the maximal sub groupoid of $C$, whose morphisms are the isomorphisms of $C$, to the isomorphism torsion group $K_1^{\text{iso}}(C)$ of $\mathcal{C}$.

Log-functors transform naturally: if $J : S \to C$ is a symmetric monoidal functor, then, since $C \to \mathcal{N}C$ is functorial, a logarithmic representation of $C$ pulls-back to one of $S$. Basic properties of log-functors are listed in the following lemma.

**Proposition 2.12** 1. Let $p \in \text{mor}_C(x,x)$ be a projection morphism: $p \circ p = p$. Then in $F(x \otimes x)$

\[ \log_{x\otimes x} p \approx 0. \]  \hfill (2.56)

In particular, $\log_{x\otimes x} \iota \approx 0$, where $\iota$ is the identity morphism.

2. For $\alpha \in \text{mor}(x,y)$ and identity morphisms $\iota_x \in \text{mor}(x,x)$, $\iota_y \in \text{mor}(y,y)$

\[ \log_{x\otimes y\otimes y} (\iota_y \circ \alpha) \approx \eta_{\otimes y}(\log_{x\otimes y} \alpha) \quad \text{ in } F(x \otimes y \otimes y), \]  \hfill (2.57)

\[ \log_{x\otimes y\otimes y} (\alpha \circ \iota_x) \approx \eta_{\otimes y}(\log_{x\otimes y} \alpha) \quad \text{ in } F(x \otimes x \otimes y). \]  \hfill (2.58)

3. For $\alpha, \beta \in \text{mor}(x,x)$ one has in $F(x \otimes x)$

\[ \log_{x\otimes x} \beta \alpha \approx \log_{x\otimes x} \alpha + \log_{x\otimes x} \beta. \]  \hfill (2.59)

4. For $\alpha \in \text{mor}(x,x)$ and an isomorphism $q \in \text{mor}(w,x)$ one has in $F(w \otimes x \otimes x \otimes w)$

\[ \log_{w\otimes x\otimes x\otimes w} (q^{-1} \alpha q) \approx \eta_{w\otimes \beta}(\log_{x\otimes x} \alpha). \]  \hfill (2.60)

In the case $x = w$, considering $q^{-1} \alpha q \in \text{mor}(x,x)$, in $F(x \otimes x)$

\[ \log_{x\otimes x} (q^{-1} \alpha q) \approx \log_{x\otimes x} \alpha. \]  \hfill (2.61)

In either case, for a log-determinant structure one has in $\text{mor}_A(1,1)$

\[ \tau(\log q^{-1} \alpha q) = \tau(\log \alpha). \]  \hfill (2.62)

5. Let $w, w' \in \text{ob}(\Sigma(C))$ and let $\alpha \in \text{mor}_w(x,z), \beta \in \text{mor}_{w'}(z,y)$. Then for a logarithmic representation one has in $F(x \otimes w \otimes z \otimes w' \otimes y)$

\[ \log_{x \otimes w \otimes z \otimes w' \otimes y}(\beta \alpha) \approx \eta_{w' \otimes y}(\log_{x \otimes w \otimes z} \alpha) + \eta_{w \otimes y}(\log_{z \otimes w' \otimes y} \beta). \]  \hfill (2.63)

6. Let $w = (w_1, \ldots, w_m) \in \text{ob}(\Sigma(C))$ and let $\alpha \in \text{mor}_w(x,y)$ with given stratification $\alpha = \alpha_{m+1} \circ \alpha_m \circ \cdots \circ \alpha_1$ with $\alpha_j : w_{j-1} \to w_j$ and $w_0 := x$, $w_{m+1} := y$. Then

\[ \log_{x \otimes w \otimes y}(\alpha_{m+1} \circ \alpha_m \circ \cdots \circ \alpha_1) \approx \sum_{j=1}^{m+1} \eta_{j-1,j}(\log_{w_{j-1} \otimes w_j} \alpha_j) \]  \hfill (2.64)
in $F(x \otimes w \otimes y)$ with $\eta_{j-1,j} := \eta_{w_0} \circ \cdots \circ \eta_{w_{j-2}} \circ \eta_{w_{j-1}} \circ \cdots \circ \eta_{w_m}$. In the case $w_0 = w_1 = \cdots = w_{m+1} = x$ this reduces in $F(x \otimes x)$ to

$$\log_{x \otimes x} (\alpha_{m+1} \circ \alpha_m \circ \cdots \circ \alpha_1) \approx \sum_{j=1}^{m+1} \log_{x \otimes x} \alpha_j. \quad (2.65)$$

Proof: For 1. one has

$$\log_{x \otimes x} (x \xrightarrow{p} x \xrightarrow{p} x) \approx \eta_{x \otimes x} \log_{x \otimes x} (x \xrightarrow{p} x) + \eta_{x \otimes x} \log_{x \otimes x} (x \xrightarrow{p} x) \quad (2.27) \quad (2.28) \approx \log_{x \otimes x} (x \xrightarrow{p} x \xrightarrow{p} x) + \log_{x \otimes x} (x \xrightarrow{p} x \xrightarrow{p} x).$$

Hence $0 \approx \log_{x \otimes x} (x \xrightarrow{p} x \xrightarrow{p} x) \approx \eta_{x \otimes x}\left(\log_{x \otimes x} p \circ p\right) = \eta_{x \otimes x}\left(\log_{x \otimes x} p\right)$. Since $\eta_{x \otimes x}$ is left-invertible the conclusion follows.

For 4. we have

$$\log_{w \otimes x \otimes w} (w \xrightarrow{q} x \xrightarrow{\alpha} x \xrightarrow{q^{-1}} w) \approx \log_{w \otimes x \otimes w} (w \xrightarrow{\alpha q} x \xrightarrow{q^{-1}} w) \approx \eta_{x} \log_{w \otimes x \otimes w} (w \xrightarrow{\alpha q} x \xrightarrow{q^{-1}} w) \quad (2.32) \quad (2.34) \approx \eta_{w} \eta_{x} \log_{w \otimes x} (w \xrightarrow{\alpha q} x) + \eta_{w} \eta_{x} \log_{x \otimes w} (x \xrightarrow{q^{-1}} w)$$

By (2.57), (2.58) the final two summands are equated to

$$\eta_{w} \log_{w \otimes x \otimes w} (w \xrightarrow{q} x \xrightarrow{\alpha q} x \xrightarrow{q^{-1}} w) \approx \log_{w \otimes x \otimes x \otimes w} (w \xrightarrow{q} x \xrightarrow{\alpha q} x \xrightarrow{q^{-1}} w) \approx \eta_{x}^{2} \eta_{x} \log_{w \otimes w \otimes w} (w \xrightarrow{q q^{-1}} x) \approx 0. \quad (2.34)$$

The other statements follow similarly.

By Proposition 2.12 (4.) a log-functor on a monoid reduces to the definition of §2.

2.2.1 Example: Fredholm category

Let $C_{\text{Fred}}$ be the category whose objects are Hilbert spaces $H \in \text{ob}(C_{\text{red}})$ and whose morphisms are Fredholm maps. Thus, an element of $\text{mor}(H, H')$ is a bounded linear
operator $Z : H \to H'$ which has a bounded linear parametrix $Q : H' \to H$, so that 

$$L_Z := QZ - I \in \mathcal{F}(H) \quad \text{and} \quad R_Z := ZQ - I' \in \mathcal{F}(H')$$ \hspace{1cm} (2.66)

with $\mathcal{F}(H)$ the ideal of finite-rank operators. Define $F : C^*_\text{red} \to C_1$ by $H \to \mathcal{F}(H)$, with the canonical inclusions $\eta_H : \mathcal{F}(H) \to \mathcal{F}(H \oplus H')$. For $H_1, H_2 \in \text{ob}(C^*_\text{red})$ let $\hat{A}$ denote the image of $A \in \text{mor}(H_i, H_j)$ in bounded continuous operators on $H_1 \oplus H_2$ with respect to the canonical linear inclusion of $\text{mor}(H_i, H_j)$: thus, if $i = 1, j = 2$, then $\hat{A} = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}$, and so on. A compatible system of logarithm maps 

$$\log_{H \oplus H'} : \text{mor}(H, H') \to \mathcal{F}(H \oplus H') / [\mathcal{F}(H \oplus H'), \mathcal{F}(H \oplus H')]$$

is then defined by 

$$\log_{H \oplus H'} Z = \begin{bmatrix} \hat{Z} \\ \hat{Q} \end{bmatrix} - J$$ \hspace{1cm} (2.67)

where $J := -\hat{T} + \hat{\eta} = -I \oplus I'$; here, the right-hand side means the equivalence class in the quotient, but working will be done in $\mathcal{F}(H \oplus H')$. Here, $\begin{bmatrix} \hat{Z} \\ \hat{Q} \end{bmatrix}$ is not in $[\mathcal{F}(H \oplus H'), \mathcal{F}(H \oplus H')]$, but if $S, T$ are continuous operators on a Hilbert space $V$

$$ST \in \mathcal{F}(V) \quad \text{and} \quad TS \in \mathcal{F}(V) \quad \Rightarrow \quad [S, T] \in [\mathcal{F}(V), \mathcal{F}(V)]$$ \hspace{1cm} (2.68)

and the classical trace defines a canonical isomorphism $\mathcal{F}(V) / [\mathcal{F}(V), \mathcal{F}(V)] \overset{\text{Tr}}{\to} \mathbb{C}$ as the canonical generator of the complex line $(\mathcal{F}(V) / [\mathcal{F}(V), \mathcal{F}(V)])^*$. $\text{Tr}$ defines the unique trace on $\mathcal{F}$, which is equivalent to $A \in [\mathcal{F}(V), \mathcal{F}(V)] \iff \text{Tr}(A) = 0$.

**Lemma 2.13** The logarithm map (2.67) is well-defined: $\begin{bmatrix} \hat{Z} \\ \hat{Q} \end{bmatrix} - J$ is in $\mathcal{F}(H \oplus H')$ and is independent in the quotient $\mathcal{F} / [\mathcal{F}, \mathcal{F}]$ of the choice of parametrix $Q$. The character of the logarithm is the Fredholm index $\text{Tr}(\log_{H \oplus H'} Z) = \text{ind}(Z) \in \mathbb{Z}$.

**Proof:** 

$\begin{bmatrix} \hat{Z} \\ \hat{Q} \end{bmatrix} = \begin{pmatrix} 0 & 0 \\ Z & 0 \end{pmatrix}$ and so $\log_{H \oplus H'} Z = I - QZ \oplus QZ - I'$ which by (2.66) is in $\mathcal{F}(H \oplus H')$. $Q$ can be chosen with $L_Z$ and $R_Z$ projections onto the kernels of the operators $Z$ and $Z^*$, respectively, giving the log-character. If $P \in \text{mor}(H', H)$ is a second parametrix then the logarithm maps differ by $(\begin{bmatrix} \hat{Z}, \hat{Q} \end{bmatrix} - J) - (\begin{bmatrix} \hat{Z}, \hat{P} \end{bmatrix} - J) = [\hat{Z}, \hat{Q} - \hat{P}]$. But $\hat{Z}(\hat{Q} - \hat{P}) = 0 \oplus Z(Q - P)$ \hspace{1cm} (2.68) in $\mathcal{F}(H \oplus H')$ and $(\hat{Q} - \hat{P})\hat{Z} = (Q - P)Z \oplus 0 \in \mathcal{F}(H \oplus H')$ so $[\hat{Z}, \hat{Q} - \hat{P}] \in [\mathcal{F}(H \oplus H'), \mathcal{F}(H \oplus H')]$ by (2.68).

The log-additivity property (2.34) holds in the following way.
Lemma 2.14 Let \( Z \in \text{mor}(H, H') \) and \( Z' \in \text{mor}(H', H'') \). In \( \mathcal{F}(H \oplus H' \oplus H'') \)
\[
\eta_{H'}(\log_{H \oplus H''} Z' Z) \approx \eta_{H'}(\log_{H \oplus H'\oplus H''} Z) + \eta_{H}(\log_{H' \oplus H''} Z').
\] (2.69)

Proof: Let \( Q' \in \text{mor}(H'', H') \) be a parametrix for \( Z' \). Then
\[
\eta_{H'}(\log_{H \oplus H'} Z) = (I - QZ) \oplus (ZQ - I') \oplus 0,
\]
\[
\eta_{H}(\log_{H' \oplus H''} Z') = 0 \oplus (I' - Q'Z') \oplus (Z'Q' - I''),
\]
\[
\eta_{H'}(\log_{H \oplus H''} Z' Z) = (I - QQ'Z'Z) \oplus 0 \oplus (Z'ZQQ' - I'')
\]
in \( \mathcal{F}(H \oplus H' \oplus H'') \). The Fredholm property gives \( ZQ = I' + R_Z, QZ = I + L_Z, Z'Q' = I'' + R_{Z'}, Q'Z' = I' + L_{Z'}, \) for some \( L_Z \in \mathcal{F}(H), R_Z, L_{Z'} \in \mathcal{F}(H), R_{Z'} \in \mathcal{F}(H'') \), and hence \( Z'ZQQ' = I'' + R_{Z'} + Z'R_{Z'}Q' \) and \( QQ'Z'Z = I - L_Z - QL_Z/Z \). Changing the choice of any of the parametrices for \( Z, Z' \) or \( Z'Z \) only produces a change in \( [\mathcal{F}, \mathcal{F}] \) as accounted for in Lemma 2.13. For the given parametrices
\[
\eta_{H'}(\log_{H \oplus H''} Z' Z) - \eta_{H'}(\log_{H \oplus H'} Z) - \eta_{H}(\log_{H' \oplus H''} Z')
\]
\[
= \begin{bmatrix}
0 & 0 & 0 \\
Z & 0 & 0 \\
0 & Z' & 0
\end{bmatrix}
\begin{bmatrix}
0 & QL_{Z'} & 0 \\
0 & 0 & R_{Z'}Q' \\
0 & 0 & 0
\end{bmatrix}
+ \begin{bmatrix}
0 & 0 & 0 \\
0 & L_{Z'} & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
0 & R_Z & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

Since these matrix products are in \( \mathcal{F}(H \oplus H' \oplus H'') \) it follows from (2.68) that these commutators are sums of commutators in \([\mathcal{F}(H \oplus H' \oplus H''), \mathcal{F}(H \oplus H' \oplus H'')]\).

\( \square \)

Applying the trace to (2.69), the log-character additivity formula is
\[
\text{ind}(Z'Z) = \text{ind}(Z) + \text{ind}(Z').
\] (2.70)

A natural generalization of (2.67) defines \( \log_{H \oplus H \oplus H'} : \text{mor}(H,H') \to \mathcal{F}(H \oplus H \oplus H') \) directly on each stratified space of morphisms.

3 Log-structures on cobordism categories

When considering bordism categories it is often of interest to do so with a structure group and consider \( \text{Bord}_n^G \), the category of \( G \)-bordisms with \( G \) a (direct limit of) compact Lie group(s). The default is the category \( \text{Bord}_n^{SO} := \text{Bord}_n \) of oriented bordisms, while the category \( \text{Bord}_n^U \) is the category of unoriented bordisms; an unoriented TQFT is a symmetric monoidal functor \( Z : \text{Bord}_n^U \to C \).
A log-TQFT\(^G\) of dimension \(n\) with values in a category \(M_A\) is a log-functor log on \(\text{Bord}_n^G\) ranging in \(M_A\), meaning a (strongly) simplicial map

\[
\log : \mathcal{N}\text{Bord}_n^G \to M_A.
\]

specified by a monoidal product representation \(F : \text{Bord}_n^G \to M_A\) on objects \(\mathcal{N}_0\text{Bord}_n^G\) defining a simplicial structure on \(F(\text{ob}(\text{Bord}_n^G)) \subset M_A\), and for each \(p\)-simplex of bordisms

\[
M_0 \xrightarrow{W_0} M_1 \xrightarrow{W_1} M_2 \to \cdots \to M_{p-1} \xrightarrow{W_{p-1}} M_p \in \mathcal{N}_p\text{Bord}_n^G
\]
a logarithm

\[
\log (M_0 \xrightarrow{W_0} M_1 \xrightarrow{W_1} M_2 \to \cdots \to M_{p-1} \xrightarrow{W_{p-1}} M_p) \in F(M_0 \sqcup M_1 \sqcup \cdots \sqcup M_p).
\]

In \(F(M_0 \sqcup M_1 \sqcup M_2)\) (and likewise for higher simplices) log-additivity is an equivalence

\[
\eta_{M_1} \log (M_0 \xrightarrow{W_0 \cup M_1} W_1) \approx \eta_{M_2} \log (M_0 \xrightarrow{W_0} M_1) + \eta_{M_0} \log (M_1 \xrightarrow{W_1} M_2) \quad (3.1)
\]

while strongly simplicial is the property

\[
\log (M_0 \xrightarrow{W_0} M_1 \xrightarrow{W_1} M_2) \approx \eta_{M_1} \log (M_0 \xrightarrow{W_0 \cup M_1} W_1) \quad (3.2)
\]

The \(p\)-simplices of \(\mathcal{N}\text{Bord}_n\) form composed bordisms which retain data of how they were formed by gluing together other bordisms.

The case of a boundaryless bordism \(W \in \text{mor}_{\text{Bord}_n}(\emptyset, \emptyset)\) needs to be considered separately. We are instructed by \((2.49)\) to define the logarithm of \(W\) by viewing it as a composed bordism \(\emptyset \xrightarrow{W_0} M \xrightarrow{W_1} \emptyset\) relative to a choice of codimension 1 embedded submanifold \(M \in \text{ob}(\text{Bord}_n)\) and then define

\[
\log_M W := \log (\emptyset \xrightarrow{W_0} M \xrightarrow{W_1} \emptyset) \in F(M)/[F(M), F(M)].
\]  (3.3)

Log-additivity then gives

\[
\log_M W = \log (\emptyset \xrightarrow{W_0} M) + \log (M \xrightarrow{W_1} \emptyset) \in F(M)/[F(M), F(M)].
\]  (3.4)

and independently of the choice of \(M\)

\[
\tau(\log W) = \tau_M(\log W_0) + \tau_M(\log W_1) \in F(\emptyset).
\]  (3.5)

We note:

**Lemma 3.1** Let \(C_M \in \text{mor}_{\text{Bord}_n}(M, M)\) be the bordism class of \([0, 1] \times M\). Then for a log-TQFT\(^G\) one has

\[
\log_{M, M}(C_M) = 0 \text{ in } F(M \sqcup M)
\]

and

\[
\eta_{M,N} \log (M \xrightarrow{W} N) = \log (M \xrightarrow{W} N \xrightarrow{C_N} N) \text{ in } F(M \sqcup N \sqcup N)
\]  (3.6)
Proof: Restatements of Proposition 2.12 (1) and (2) to $\text{Bord}_n^G$.

\[\square\]

log-TQFTs may yield a TQFT, as in the following weak sense:

**Lemma 3.2** A determinant-TQFT $^G\text{Bord}_n^G$, defined by log-TQFT $^G\text{log}$ : $\mathcal{N}\text{Bord}_n^G \rightarrow M(\mathbb{A},\tau)$ with $(\mathbb{A},\tau)$ a Calabi-Yau category endowed with an exponential map $e : \text{end}\mathbb{A}(1) \rightarrow \mathbb{A}^*$ with $\mathbb{A}$ a commutative ring, defines a ‘weak TQFT $^G\text{log}$‘; that is, a scalar-valued symmetric monoidal functor $^{G\text{log},\tau,e}_Z$ : $\mathcal{N}\text{Bord}_n^G \rightarrow \mathbb{A}^*$, $^{G\text{log},\tau,e}_Z(W) = e(\tau(\text{log} W)) \in Z(\partial W) := A^*$.

Conversely, log-TQFTs may arise from TQFTs, but currently we know of this in a prescriptive sense in essentially trivial cases only. For example, any TQFT $^Z$ : $\mathcal{N}\text{Bord}_n \rightarrow \text{Vect}$ yields the log-TQFT

\[
^{\text{log}}Z_{M\sqcup M'}(W) = \text{log}^{\text{Z}}(M) \oplus \text{Z}(M')
\]

where the right-hand side is the logarithm (2.67) on the Fredholm category, here on finite-dimensional Hilbert spaces. Its character is

\[
\text{tr} (^{\text{log}}Z_{M\sqcup M'}(W)) = \dim(Z(M')) - \dim(Z(M)).
\]

Thus for $n = 2$ the character for a Riemann surface $W = \Sigma$ is

\[
\text{tr} (^{\text{log}}Z_{M\sqcup M'}(\Sigma)) = \dim(Z(S^1))^{m'} - \dim(Z(S^1))^m
\]

with $m = |\pi_0(M)|$. In particular, the character of any closed bordism is zero.

Non-trivial log-TQFTs are not hard to find, however.

**The topological signature:**

The integer-valued signature functor on $\text{Bord}_{ik}$ in (1.5) arises as the character of a log-TQFT, one consequence of which is the characteristic additivity property (1.6).

To construct the log-functor, define $\text{Bord}_{ik}^*$ by declaring its morphisms to consist solely of the coherence and permutations isomorphisms, and a tracial monoidal product representation into Fréchet algebras

\[
F_{-\infty} : \text{Bord}_{ik}^* \rightarrow \text{Alg}_F, \quad M = M_1 \sqcup \cdots \sqcup M_m \mapsto (F_{-\infty}(M), \text{Tr}) \quad (3.7)
\]

with each $M_i$ connected, where $F_{-\infty}(M) = \Psi^{-\infty}(M, \wedge T^* M)$ is the algebra of smoothing operators on the de Rham complex $\Omega(M_1) \oplus \cdots \oplus \Omega(M_n)$. An element $T \in F_{-\infty}(M)$ is an $n \times n$ block matrix $(T_{i,j})$ of smoothing operators $T_{i,j} \in \Psi^{-\infty}(M_j, M_i)$ specified by Schwartz kernels $k_{i,j} \in C^\infty(M_i \times M_j, (\Omega(M_1) \otimes |M|_{M_i}) \otimes (\Omega(M_j) \otimes |M|_{M_j}))$ in form valued
half-densities, whose rows and columns are permuted by $\mu_\sigma(M)$ relative to a reordering $\sigma$ of the $M_j$. Each coherence cobordism is mapped to the identity operator. Taking $m = 2$, the insertion map $\eta_{\partial W} : \Psi^{-\infty}(M_1) \to \Psi^{-\infty}(M_1 \sqcup M_2)$ is $S \mapsto \left( \begin{array}{cc} S & 0 \\ 0 & 0 \end{array} \right)$ with left inverse the canonical projection, and likewise for general $\eta_M$. The classical trace defines the categorical trace

$$\text{Tr} : F_{-\infty}(M_1 \sqcup \cdots \sqcup M_m) \to \mathbb{C}, \quad \text{Tr}(T) = \sum_{j=1}^{m} \int_M k_{j,j}(m,m).$$

(3.8)

More generally, define $F_Z(M)$ by taking $T_{i,j} \in \Psi^{-\infty}(M_j, M_i)$ smoothing if $i \neq j$ and each diagonal entry a classical $\psi$do of integer order $T_{i,i} \in \Psi^Z(M_i)$ on $M_i$. Then the residue trace

$$\text{res} : F_Z(M) \to \mathbb{C}, \quad \text{res}(T) = \sum_{j=1}^{m} \text{res}(T_{j,j})$$

with $\text{res}(T_{j,j})$ defined as prior to (1.11), is the unique categorical trace on $F_Z$.

Define a logarithm

$$\log : \text{mor}_{\text{Hor}, \mathbb{R}}(M, M') \to F_{-\infty}(M \sqcup M')/[F_{-\infty}(M \sqcup M'), F_{-\infty}(M \sqcup M')]$$

by

$$\log_{\partial W} W = \pi (\phi^*(\Pi^w) - \phi^*(P(\partial W))).$$

(3.9)

Here, $\pi : F_{-\infty} \to F_{-\infty}/[F_{-\infty}, F_{-\infty}]$ is the quotient map, $W$ a smooth representative for the bordism class with diffeomorphism $\phi : \overline{M} \sqcup M' \to \partial W$, $g_W$ is a choice of Riemannian metric on $W$ which in a collar neighbourhood $U_j$ of each boundary component $\partial W_j$ is a product metric $g_U = du^2 + g_{\partial W_j}$ with $u$ a normal coordinate in $(-1,0]$ if $\partial W_j$ is a component of $\overline{M}$ and in $[0,1]$ if $\partial W_j$ is a component of $M'$; all logarithms will be independent of the choice of $g_W$. Associated to $g_W$ there is a Hodge star isomorphism $\star : \Omega^p(W) \to \Omega^{4-k-p}(W)$ and a signature operator $\partial^w = d + d^* : \Omega^+(W) \to \Omega^-(W)$ between the $\pm$-eigenspaces of forms of the involution $\iota^{p(p-1)} \star$ on the de Rham complex. Then $P(\partial^w), \Pi^w \in F_Z(\partial W)$ are, respectively, the Calderón projection and the (positive) APS projection. These are order zero pseudodifferential projections with

$$\Pi^w - P(\partial^w) \in F_{-\infty}(\partial W).$$

(3.10)

More precisely, $\partial^w$ restricts on each boundary component $\partial W_j$ to a self-adjoint boundary signature operator $D_{\partial W_j} = (-1)^{k+p+1} (\ast d_j - d_j \ast)$, with $d_j$ the de Rham operator on $\partial W_j$, and $\Pi^w_j$ is the orthogonal projection onto the span of eigenforms of $D_{\partial W_j}$ with eigenvalue $\lambda > 0$. It is to be noted that

$$\Pi^w = \bigoplus_{j=1}^{r} \Pi^w_j \in \bigoplus_{j=1}^{r} F_Z(\partial W_j) = \Psi^Z(\partial W_j, \Lambda^+ T^* \partial W_j)$$

(3.11)
whilst the Calderón projection $P(\partial^W) \in F_Z(\partial W)$ is not block diagonal but involves non-zero off-diagonal smoothing terms — its range is generically identified with a graph of a smoothing operator $\text{ran}(\Pi^W) \to \text{ran}(\Pi^M)$. The Calderón $P(\partial^W)$ is the projection onto the subspace $K(\partial^W) \subset \Omega(\partial W)$ of boundary sections which are restrictions of interior solutions $\text{Ker}(\partial^W) \subset \Omega(W)$; the Poisson operator defines a canonical isomorphism

$$K(\partial^W) \xrightarrow{\cong} \text{Ker}(\partial^W)$$

(3.12)

in the appropriate Sobolev norms. Pull-back is functorial and so the pull-backs $\phi^*(\Pi^M_\geq), \phi^*(P(\partial^W)) \in F_Z(M \sqcup M')$ are likewise pseudo projectors of order zero whose difference lies in $F_{-\infty}(M \sqcup M')$, and is hence (classical) trace-class.

**Proposition 3.3** (3.9) is independent of $g_W$ with log-character

$$\text{Tr} (\log_{M \sqcup M'} W) = \text{sgn}(W).$$

(3.13)

Proof: Let $\partial^W = \partial^W$ with domain restrict to those sections $s \in \Omega^+(W)$ with $\Pi^W s|_{\partial W} = 0$. The APS signature theorem ([1], Thm 4.14) gives the first two equalities in

$$\text{sgn}(W) = \int_W L(p) - \eta_{\partial W}(0)$$

$$= \text{ind} (\partial^W_\geq) + h$$

$$= \text{Tr} (\Pi^W_\geq - P(\partial^W))$$

$$= \text{Tr} (\pi (\phi^*(\Pi^W_\geq) - \phi^*(P(\partial^W))))$$

with $h = \sum_j \dim \ker(D_{\partial W})$ and $L(p)$ the Hirzebruch $L$-polynomial in the Pontryagin forms, while the third is the equality

$$\text{Tr} (\Pi^W_\geq - P(\partial^W)) = \text{ind} (\Pi^W_\geq - P(\partial^W) : K(\partial^W) \to \text{ran}(\Pi^W_\geq))$$

with $\text{ran}(\Pi_\geq)$ the projection onto the span of the non-negative eigenvalues. Conjugation invariance of the trace means it is invariant with respect to pull-back by $\phi$ while $\text{Ker}(\text{Tr}) = [F_{-\infty}, F_{-\infty}]$ means it does not see $\pi$. This equality on the kernel of $\text{Tr}$ is the statement that the classical trace is the unique trace on $F_{-\infty}$ up to scale factor, which is in turn equivalent to $\text{Tr} : F_{-\infty}/[F_{-\infty}, F_{-\infty}] \cong \mathbb{C}$ in which, by the above, $\log_{M \sqcup M'} W$ maps to the homotopy invariant $\text{sgn}(W)$ and hence that $\log_{M \sqcup M'} W$ is (in the quotient) likewise independent of the metric.

For log-additivity (3.1), note bearing in mind (3.11) that

$$\log (M_0 \xrightarrow{W_0 \sqcup M_1} W_1) = (\Pi^W_\geq \oplus \Pi^W_\leq) - P(\partial^W_0 \sqcup M_1 \sqcup W_1) \text{ in } F_{-\infty}(M_0 \sqcup M_2),$$

(3.14)

$$\log (M_0 \xrightarrow{W_0} M_1) = (\Pi^W_\geq \oplus \Pi^W_\leq) - P(\partial^W_0) \text{ in } F_{-\infty}(M_0 \sqcup M_1),$$

(3.15)

24
\[ \log(M_1 \xrightarrow{W} M_2) = (\Pi^{M_1}_\geq \oplus \Pi^{M_2}_\geq) - P(\overline{\partial}^{W_1}) \] in \( F_{-\infty}(M_1 \sqcup M_2) \), \hspace{1cm} (3.16)

where we have omitted notation for the maps \( \pi \circ \phi^* \), and hence since \( \Pi^{M_1}_\geq = \Pi^{M_2}_\geq \) that (3.1) is consequent on

\[ (\Pi^{M_0}_\leq \oplus 0 \oplus \Pi^{M_2}_\geq) - \eta_{M_1} P(\overline{\partial}^{W_0 \cup M_1 \cup W_1}) \]

\[ - \{(\Pi^{M_0}_\leq \oplus \Pi^{M_0}_\geq \oplus 0) - \eta_{M_2} P(\overline{\partial}^{W_0})\} - \{(0 \oplus \Pi^{M_1}_\leq \oplus \Pi^{M_2}_\geq) - \eta_{M_0} P(\overline{\partial}^{W_1})\} \]

\[ \in [F_{-\infty}(M_0 \sqcup M_1 \sqcup M_2), F_{-\infty}(M_0 \sqcup M_1 \sqcup M_2)]. \] \hspace{1cm} (3.17)

That is, that \( \eta_{M_1} P(\overline{\partial}^{W_0 \cup M_1 \cup W_1}) - \eta_{M_2} P(\overline{\partial}^{W_0}) - \eta_{M_0} P(\overline{\partial}^{W_1}) \) is a commutator in \( F_{-\infty}(M_0 \sqcup M_1 \sqcup M_2) \), where \( P(\overline{\partial}^{W_1}) = (I \oplus 0) - P(\overline{\partial}^{W_1}) \in F_{-\infty}(M_1 \sqcup M_2) \); if \( W_0 \cup M_1, W_1 \) is a closed bordism \( (M_0 = M_2 = 0) \) this reduces to the usual complementary projection \( P(\overline{\partial}^{W_1}) = I - P(\overline{\partial}^{W_1}) \).

**Lemma 3.4** Let \( \Gamma_j = \Gamma_j^+ \oplus \Gamma_j^- \) be a polarised (Hilbert) space of sections of a vector bundle \( E_j \to Y_j \) over a closed manifold for \( j = 0, 1, 2 \) with \( \Gamma_j^\pm \) the APS spectral decomposition associated to a self-adjoint elliptic operator on \( \Gamma_j \). Let \( \Pi^+_j \) be the order zero \( \psi \)do projection with range \( \Gamma_j^+ \). Let \( \Psi_j^{-\infty} \) be the algebra of smoothing operators on \( \Gamma_j \) and \( \text{Tr}_j : \Psi_j^{-\infty} \to \mathbb{C} \) the classical trace. Assume a split short exact sequence \( 0 \to \Psi_0^{-\infty} \xrightarrow{s} \Psi_1^{-\infty} \xrightarrow{\iota} \Psi_2^{-\infty} \to 0 \) such that \( \text{Tr}_1 \circ s = \text{Tr}_2 \) and \( \text{Tr}_1 \circ \iota = \text{Tr}_0 \). Let \( P_j, P'_j \) be order zero \( \psi \)do projections with respective ranges closed subspaces \( V_j, V'_j \) of \( \Gamma_j \) such that \( P_j - \Pi^+_j \in \Psi_j^{-\infty}, P'_j - \Pi^+_j \in \Psi_j^{-\infty} \) are smoothing operators. If there is a diagram with exact rows and Fredholm columns

\[
\begin{array}{cccccc}
0 & \to & V_0 & \to & V_1 & \to & V_2 & \to & 0 \\
\downarrow a_0 & & \downarrow a_1 & & \downarrow a_2 & & \\
0 & \to & V'_0 & \to & V'_1 & \to & V'_2 & \to & 0
\end{array}
\]

and which is commutative (modulo smoothing operators) with \( a_j - P'_j \circ P_j \in \Psi_j^{-\infty} \), then

\[ (P_1 - P'_1) - \iota(P_0 - P'_0) - s(P_2 - P'_2) \in [\Psi_1^{-\infty}, \Psi_1^{-\infty}] \] \hspace{1cm} (3.18)

**Proof:** Since \( \text{Tr}_j \) is the projectively unique trace on \( \Psi_j^{-\infty} \) we have \( \text{Ker} (\text{Tr}_j) = [\Psi_j^{-\infty}, \Psi_j^{-\infty}] \). The result then follows from the identifications \( \text{ind} (a_j) = \text{ind} (P'_j \circ P_j) = \text{Tr}_j (P_j - P'_j) \) and \( \text{ind} (a_1) = \text{ind} (a_0) + \text{ind} (a_2) \).

\[ \square \]

For \( j = 0, 1, 2 \), let \( H_j \) be Hilbert spaces with subspaces \( W \subset H_0 \oplus H_1, W' \subset H_1 \oplus H_2 \), then the join \( W \ast W' \subset H_0 \oplus H_2 \) is the subspace, when defined, of pairs \( (u, v) \) such that there exists a \( w \in H_1 \) with \( (u, w) \in H_0 \oplus H_1 \) and \( (w, v) \in H_1 \oplus H_2 \).
To give an element of $\text{Ker}(\bar{\partial}^{w_0})$ is to give elements of $\text{Ker}(\bar{\partial}^{w_0})$ and $\text{Ker}(\bar{\partial}^{w_1})$ whose pointwise values match-up at the boundary, for in a collar neighbourhood $U$ of the boundary the product metric $g|_U = du^2 + g_{ow}(y)$ implies any solution $\psi$ to $\bar{\partial}^{w_i}$ takes the form $\psi|_U(u, y) = \sum_k e^{-\lambda_k u} \psi_k(0) \phi_k(y)$ for a chosen spectral resolution $(\lambda_k, \phi_k)$ of the boundary Hodge Laplacian $D_{ow}$, and so matching of higher normal derivatives follows from the zeroeth order matching. Hence we have a canonical isomorphism

$$\text{Ker}(\bar{\partial}^{w_0 \cup M_1}) = \text{Ker}(\bar{\partial}^{w_0}) \ast \text{Ker}(\bar{\partial}^{w_1})$$

(3.19)

and in view of (3.12) likewise for the boundary Calderón spaces $K(\bar{\partial}) := \text{Ker}(\bar{\partial})|_{\partial W}$

$$K(\bar{\partial}^{w_0 \cup M_1}) = K(\bar{\partial}^{w_0}) \ast K(\bar{\partial}^{w_1}).$$

(3.20)

Moreover, the Unique Continuation Property for the generalised Dirac operators $\bar{\partial}^w$ establishes the first inclusion in the short exact sequence in the top row of the diagram, the following map being $((u, w), (w', v)) \mapsto w - w'$,

$$0 \rightarrow K(\bar{\partial}^{w_0}) \ast K(\bar{\partial}^{w_1}) \rightarrow K(\bar{\partial}^{w_0}) \oplus K(\bar{\partial}^{w_1}) \rightarrow \Omega(M_1) \rightarrow 0$$

$$\downarrow G_0 \quad \downarrow G_1 \quad \downarrow \partial^*$$

$$0 \rightarrow H^0_\prec \oplus H^2_\succ \rightarrow H^0_\prec \oplus \Omega(M_1) \oplus H^2_\succ \rightarrow \Omega(M_1) \rightarrow 0$$

where $H^j_\prec, H^j_\succ$ are the ranges of $\Pi^M_{\prec}, \Pi^M_{\succ}$, and

$$G_0 = (\Pi^M_{\prec} \oplus \Pi^M_{\succ}) \circ P(\bar{\partial}^{w_0 \cup M_1})$$

$$G_1 = (\Pi^M_{\prec} \oplus I \oplus \Pi^M_{\succ}) \circ (\eta_{M_0} P(\bar{\partial}^{w_0}) + \eta_{M_0} P(\bar{\partial}^{w_1})).$$

The maps in the lower exact sequence are the canonical ones. Computing as $3 \times 3$ block matrices in $F_Z(M_0 \sqcup M_1 \sqcup M_2)$ the diagram is immediately seen to be commutative up to addition of smoothing operators. With respect to the $3 \times 3$ block matrix form in $F_Z(M_0 \sqcup M_1 \sqcup M_2)$ we obtain from Lemma 3.4 that (3.18) precisely gives us (3.4), as required.

Though we omit details here, for brevity and since it is immediate from established methods, let us mention that the case of a closed morphism $W \in \text{mor}_{\text{Bord}}(\emptyset, \emptyset)$ is, in fact, technically somewhat harder because there is no longer a Poisson operator and Calderón projection available. So one introduces them artificially by considering $W$ relative to a codimension 1 embedded submanifold $M$. Then (3.3) is in this case defined by

$$\log_M W := \pi\left(\phi^*(P(\bar{\partial}^{w_1}))^\perp - \phi^*(P(\bar{\partial}^{w_0}))\right) \in F_{-\infty}(M)/[F_{-\infty}(M), F_{-\infty}(M)]$$

and its character is the topological signature (1.3)

$$\text{Tr}_M(\log_M W) = sgn(W),$$

independently of $M$ as postulated in (2.50).
Whitehead and Reidemeister torsion:

All log-TQFTs known to us arise as logarithms on suitable categories of chain complexes. That is, they factorise \( \text{Bord}_n \rightarrow \text{Chain} \rightarrow M_A \) via a functor \( \text{Bord}_n \rightarrow \text{Chain} \) to chain complexes. This is so for Whitehead and Reidemeister torsion, which provide a refined set of secondary topological invariants when primary Chern character invariants vanish.

Torsions of this type arise as log-characters on groups. The general context is as follows. Log-determinant structures abelianise the product on a monoid \( \mathbb{Z} \), and in a simplicial sense on a general category. For the case of a group \( \mathbb{Z} = G \) the quotient \( \pi : G \rightarrow G/G' \), with \( G' \) the commutator subgroup, is likewise abelianising and may be viewed as a universal logarithm insofar as any \( \log : G \rightarrow B/[B,B] \) factors through it: there is a unique homomorphism of abelian groups \( \pi_{\log} : G/G' \rightarrow B/[B,B] \) such that \( \log = \pi_{\log} \circ \pi \), giving \( \text{Log}(G,B) = \text{Hom}(G/G',B/[B,B]) \). For example, if \( X \) is a path-connected space then \( \text{Log}(\pi_1(X),B) = H^1(X,B/[B,B]) \). On the other hand, \( \pi : G \rightarrow G/G' \) defines a universal determinant for \( G \) insofar as any determinant \( \det : G \rightarrow R^* \) to the subring of units \( R^* = \text{Gl}_1(R) \) of a commutative ring \( R \) factors \( \det = \pi_{\det} \circ \pi \), so that \( \text{Det}(G,R^*) \subset \text{Hom}(G/G',R^*) \). Likewise, the quotient \( \sigma : B \rightarrow (B,+)/[B,B] \) is a universal trace, any trace \( \tau : B \rightarrow R \) factors \( \tau = \sigma_{\tau} \circ \sigma \), and \( \text{Trace}(B,R) = \text{Hom}(B/[B,B],R) \). These fit into the commutative diagram with exact rows (and diagonals):

\[
\begin{array}{ccccccc}
0 & \rightarrow & \ker(\det) & \rightarrow & G \rightarrow & R^* & \\
\uparrow & & \uparrow i & & \uparrow & \pi_{\det} & \\
0 & \rightarrow & G' & \rightarrow & G \rightarrow & G/G' & \rightarrow 0 \\
\downarrow & & \downarrow i & & \downarrow & \pi_{\log} & \\
0 & \rightarrow & \ker(\log) & \rightarrow & G \rightarrow & B/[B,B] & \\
\sigma & \rightarrow & \downarrow \sigma_{\tau} & & & & \\
B & \rightarrow & R & & & & \\
\sigma & \rightarrow & \downarrow i & & \rightarrow & [B,B] & \\
[\sigma, B] & \rightarrow & B & \rightarrow & \text{ker}(\tau) & & \\
0 & \rightarrow & & & & & \\
0 & \rightarrow & & & & &
\end{array}
\]
A curved arrow from $R$ at the bottom of the final column to $R^*$ at the top might also be drawn representing an exponential $e : R \to R^*$. In the middle column ‘i’ is the identity, while 0 is the appropriate zero object. The arrows in the left-hand column are the canonical inclusions. Whether those inclusions are bijections is a question of whether the corresponding structure is, in an appropriate sense, unique. Precisely, since each of the vertical maps in the diagram have cokernels then the Snake Lemma can be applied to deduce the equivalence of statements 1. and 2. in each of the following three lemmas, whilst the other statements are straightforward:

**Lemma 3.5** Let the ring $B$ be an $R$-module and consider a trace $\tau : B \to R$ which is $R$-linear (a hom of $R$-modules). Then the following are equivalent.

1. $B/[B, B] \xrightarrow{\tau} (R, +)$ is an isomorphism of abelian groups.
2. $\ker(\tau) = [B, B]$.
3. If $\xi \in B$ with $\tau(\xi) \in R^*$, then for any $\beta \in B$ one has $\beta = \sum_{1 \leq j \leq n}[\delta_j, \delta_j'] + \tau(\xi)^{-1}\tau(\beta)\xi$ for some $\delta_j = \delta_j(\xi, \beta), \delta_j' = \delta_j'(\xi, \beta) \in B$.
4. $\tau$ is projectively unique (any other such trace $\tilde{\tau} = r \cdot \tau$ some constant $r \in R$).
5. $\text{Trace}(B, R) \cong (R, +)$.

Likewise:

**Lemma 3.6** For $\det \in \text{Det}(G, R^*)$ the following statements are equivalent.

1. $G/G' \xrightarrow{\det} (R^*, \cdot)$ is an isomorphism of abelian groups and the short exact sequence $1 \to G' \to G \xrightarrow{\pi} G/G' \to 1$ is split $j' : G/G' \to G$.
2. $\ker(\det) = G'$ and the short exact sequence $1 \to \ker(\det) \to G \xrightarrow{\det} R^* \to 1$ is split $j : R^* \to G$.
3. For a splitting $j : R^* \to G$ of $\det$, $g \in G$ can be written $g = \prod_k \{h_k, h_k^\prime\} \cdot j(\det g)$ for some $h_k = h_k(g, j), h^\prime_k = h^\prime_k(g, j) \in G$, where $\{h_1, h_2\} = h_1h_2h_1^{-1}h_2^{-1}$.

$\det$ is then the unique determinant split by $j$.

And:

**Lemma 3.7** The following are equivalent for $\log \in \text{Log}(G, B)$.

1. $G/G' \xrightarrow{\pi\log} B/[B, B]$ is an isomorphism of abelian groups and the short exact sequence $1 \to G' \to G \xrightarrow{\pi} G/G' \to 1$ is split $J' : G/G' \to G$.
2. $\ker(\log) = G'$ and the exact sequence $1 \to \ker(\log) \to G \xrightarrow{\log} B/[B, B] \to G$ is split $J : B/[B, B] :\to G$.
3. For a splitting $J : B/[B, B] \to G$ of $\log$, $g \in G$ can be written $g = \prod_k \{l_k, l_k'\} \cdot J(\log g)$ for some $l_k = l_k(g, J), l_k' = l_k'(g, J) \in G$. 

28
log is then the unique logarithm split by $J$.

(In Lemma 3.5 assuming $\tau$ to be $R$-linear splits the corresponding exact sequences.)

Consider an isomorphism $f : K \to L$ of based $A$-modules of finite rank $n$. The bases identify $f$ with $\phi_f \in \text{Gl}_n(A) \subset \text{Gl}(A) = \lim_n \text{Gl}_n(A) = \cup_{n=1}^\infty \text{Gl}_n(A)$. The torsion of $f$ in the chosen bases is defined by

$$T(f) := \pi(\phi_f) \in K_1(A) := \text{Gl}(A)/\text{Gl}(A)'.$$  

If $A = R$ is commutative one has the classical determinant

$$\det : K_1(R) \to R^*.$$

When $\text{Gl}(R)' = \text{Sl}(R)$ (e.g. for a field) then by Lemma 3.6

1. $\det : K_1(R) \xrightarrow{\cong} R^*$ is an isomorphism.
2. $\det : \text{Gl}(R) \to R^*$ is the unique determinant split by $j(\xi) = \xi \oplus I$.
3. $\ker(\det) = \text{Gl}(R)'$. 4. For $g \in \text{Gl}(R)$ one has $g = \prod_{1 \leq j \leq k} (h_j, h_j') \cdot (\det g \oplus I)$ some $h_j, h_j' \in G$. This is the context for Reidemeister torsion, while Whitehead torsion is defined for general non-commutative $A$. Topological torsions of this sort are constructed by associating to a compact manifold $X$ an isomorphism $\phi_X$ of $\mathbb{Z}\pi_1(X)$ module chain complexes.

To this end, let $C_\ast = (C_\ast, d)$ be a contractible finite based free $A$-module chain complex. Then $C_\ast$ is acyclic with a chain contraction $\gamma : C_\ast \to C_\ast$ and the chain map $d + \gamma : C_{odd} \to C_{even}$, with $C_{odd} = C_1 \oplus C_3 \oplus \cdots$, is an isomorphism of free $A$-modules and hence has a torsion. Define $T(C_\ast) := T(d + \gamma) \in K_1(A)$ any such $\gamma$. Further, for a chain homotopy equivalence $f : C_\ast \to C'_\ast$ of such chain complexes the associated algebraic mapping cone complex $C(f)_\ast$ is contractible and the torsion of $f$ defined by

$$T(f) := T(C(f)_\ast).$$

Suppose, then, that $h : X \to Y$ is a homotopy equivalence of finite CW complexes. Let $h : \tilde{X} \to \tilde{Y}$ be the unique lift of $h$ to the universal covers relative to a choice of base points. Then $\pi_1 := \pi_1(X) = \pi_1(Y)$ acts on $\tilde{X}$ and $\tilde{Y}$ and hence induces a $\mathbb{Z}\pi_1$ chain homotopy equivalence $f_h : C_\ast(\tilde{X}) \to C_\ast(\tilde{Y})$ on the cellular chain complexes. The Whitehead torsion of $h$ is defined by

$$\tau^{\text{wh}}(h) := T(C(f_h)_\ast) \in \text{Wh}(\pi_1(Y)) := K_1(\mathbb{Z}\pi_1(Y))/\{\pm\pi(\phi_g) \mid g \in \pi_1(Y)\},$$

projected into the quotient (the Whitehead group) to eliminate dependence on the choices made in defining $f_h$.

This provides a log-functor in a weak sense $\log_{\text{cw}} : \text{CW} \to \text{Ab}$ on the category $\text{CW}$ of finite CW complexes and homotopy equivalences, and ranging in the category of Abelian groups, by assigning to each such complex $Y$ its Whitehead group $F_{\text{cw}}^\ast(Y) := \text{Wh}(\pi_1(Y))$ and to a homotopy equivalence $h : X \to Y$ its torsion $\log_{\text{cw}}(h) := \tau^{\text{wh}}(h)$, known to be
formally additive for composition of homotopy equivalences. But a log-functor proper is readily constructed by considering, more concretely, the category \( h\text{-Bord}_n \) of \( h \)-bordisms (more usually called \( h \)-cobordisms).

An object of \( h\text{-Bord}_n \) is a pair \((M, \rho)\) with \( M \in \text{ob}(\text{Bord}_n)\) a smooth closed manifold of dimension \( n \) augmented with an acyclic orthogonal finite dimension representation \( \rho_M : \pi_1(M) \to O(m) \), or, equivalently, with a flat vector bundle \( E_{\rho_M} \to M \) with vanishing de-Rham cohomology. A morphism \([W] \in \text{mor}_{h\text{-Bord}_n}(M, M')\) (an \( h \)-cobordism) is a bordism \([W] \in \text{mor}_{\text{Bord}_n}(M, M')\) for which the inclusion maps \( \iota_M : M \hookrightarrow W \) and \( \iota_{M'} : M' \hookrightarrow W \) are homotopy equivalences, or, equivalently, such that \( M \) and \( M' \) are deformation retracts of \( W \), and such that \( j^*_M, M' (\rho_{M'}) = \rho_M \), or, equivalently, that \( j^*_M, M' (E_{\rho_{M'}}) = E_{\rho_M} \) with respect to the induced homotopy equivalence \( j^*_M, M' : M \to M' \). \( h\text{-Bord}_n \) inherits from \( \text{Bord}_n \) its symmetric monoidal structure of disjoint union with monoidal identity the empty manifold; note \( \text{mor}_{h\text{-Bord}_n}(\emptyset, \emptyset) = \emptyset \). Likewise, the restricted category \( h\text{-Bord}^* \) is the corresponding subcategory of \( \text{Bord}^*_n \).

We define a torsion log-functor here in two different but equivalent ways and in a third exotic and inequivalent way. First, the combinatorial: define a tracial monoidal product representation

\[
F^\circ_c : h\text{-Bord}^* \to \text{Alg}_C, \quad M = M_1 \sqcup \cdots \sqcup M_r \mapsto (F^c(M), \text{tr})
\]

with \( F(M) = \text{End}_\mathbb{R}(C_*\langle \tilde{M} \rangle \otimes_{\mathbb{Z}\pi_1(M)} \mathbb{R}^m) \) the linear endomorphism algebra of the finite acyclic complexes \( C_*\langle \tilde{M} \rangle \otimes_{\mathbb{Z}\pi_1(M)} \mathbb{R}^m := \bigoplus_j C_*(\tilde{M}_j) \otimes_{\mathbb{Z}\pi_1(M)} \mathbb{R}^m \), as in §1, and \( \text{tr} \) is the classical finite rank trace, while the covering maps \( \eta_M \) are the canonical inclusions.

Define

\[
\log^c : \text{mor}_{h\text{-Bord}_n}(M, M') \to F^c(M \sqcup M')/[F^c(M \sqcup M'), F^c(M \sqcup M')] \quad \text{(3.22)}
\]

by

\[
\log^c_{M \sqcup M'} W = \pi_c \left( \frac{1}{2} \bigoplus_{p \geq 0} (-1)^p p \log \Delta^c_{M, p} + \frac{1}{2} \bigoplus_{q \geq 0} (-1)^q q \log \Delta^c_{M', q} \right) \quad \text{(3.23)}
\]

where \( \pi_c : F^c_c \to F^c_c/[F^c_c, F^c_c] \) is the quotient map, \( W \) a smooth representative for the bordism class with diffeomorphism \( \phi : \overline{M} \sqcup M' \to \partial W \), and \( \Delta^c_{M, p} \) the combinatorial Laplacian, as in §1.

**Theorem 3.8** The assignment \( (3.22), (3.23) \) defines a log-functor which computes the Whitehead torsion of each \( h \)-bordism — the character of \( W \in \text{mor}_{h\text{-Bord}_n}(M, M') \) is

\[
\text{tr}(\log^c_{M \sqcup M'} W) = \log \det (\tau^W(W)) = \tau^R(M', E_{\rho_{M'}}) - \tau^R(M, E_{\rho_M}) \quad \text{(3.24)}
\]

30
with $\det : \text{Wh}(\pi_1(M')) \to \mathbb{R}_+$ the induced determinant from (3.21) and $\tau^R$ denotes Reidemeister torsion. As a partial converse to Lemma 3.1, one has

$$\dim W \geq 6 \quad \text{and} \quad \text{tr} (\log \Delta^M) = 0 \Rightarrow W = [0, 1] \times M. \quad (3.25)$$

Proof: The equality of the log-character $\text{tr} (\log \Delta^M)$ with the difference of $R$-torsions is from (3.23) and (1.7), while the equality with the Whitehead torsion of $W$ is a standard identification from the short exact sequence of complexes

$$0 \to C_\ast (\tilde{M}) \otimes_{\mathbb{Z}\pi_1(M)} \mathbb{R}^n \to C_\ast (jM, M') \otimes_{\mathbb{Z}\pi_1(M, M')} \mathbb{R}^n \to S(C_\ast (\tilde{M}) \otimes_{\mathbb{Z}\pi_1(M)} \mathbb{R}^n) \to 0$$

where $C_\ast (jM, M')$ is the cone complex of $jM, M'$ and $SC_\ast$ denotes the suspension complex. Log-additivity and strong simpliciality are straightforward to check. The topological invariance of Whitehead and $R$-torsion show that (3.23) is well-defined on $\text{mor}_{\text{Bord}}^h(M, M')$. The inference (3.25) is the $s$-cobordism theorem and that $\log \det : \text{Wh}(\pi_1(M')) \to \mathbb{R}_+$ is an isomorphism.

We might better, insofar as it utilises $W$ in a clearer way, but as a longer process, define $\log \Delta^M_W$ by pulling-back to $\text{F}^h(M \sqcup M')$ the logarithm of the combinatorial Laplacian on the chain complex over $W$ via the inclusions $\iota_M : M \hookrightarrow W$ and $\iota_{M'} : M' \hookrightarrow W$, and then checking that this logarithm and right-side of (3.23) have the same characters.

An equivalent construction is obtained using classical analytic torsion (1.8). The tracial monoidal product representation is then

$$\text{F}_h : h\text{-Bord}^* \to \text{Alg}_F, \quad M = M_1 \sqcup \cdots \sqcup M_r \mapsto (\text{F}^Z_{\text{diag}}(M), \text{TR}_\xi)$$

where $\text{F}^Z_{\text{diag}}(M) = \bigoplus_j \Psi^Z(M_j, \wedge^* M_j)$ is the diagonal sub algebra of $\text{F}^Z(M)$ (used above for the signature log-TQFT) of integer order $\psi$ dos on each component $M_j$ and the trace is the zeta quasi trace of $\text{F}^Z(M)$. The logarithm is the sum of Hodge Laplacians

$$\log M_{\sqcup M'} W = \pi_h \left( \frac{1}{2} \bigoplus_{p \geq 0} (-1)^p \log \Delta^M_p \right) \oplus \frac{1}{2} \bigoplus_{q \geq 0} (-1)^q \log \Delta^{M'}_q$$

with $\pi_h : \text{F}_h \to \text{F}_h/[\text{F}_h, \text{F}_h]$ the quotient map, with character

$$\text{TR}_\xi (\log M_{\sqcup M'} W) = \frac{1}{2} \sum_{p \geq 0} (-1)^p \text{TR}_\xi (\log \Delta^M_p) - \frac{1}{2} \sum_{q \geq 0} (-1)^q \text{TR}_\xi (\log \Delta^{M'}_q) \quad (3.26)$$

which by the Cheeger-Muller theorem coincides with (3.24).

In view of the considerations of §1, one then might consider the (inequivalent) character obtained by replacing $\text{TR}_\xi$ in (3.26) by the residue trace, giving the difference

$$\text{res} (\log M_{\sqcup M'} W) = \tau^{\text{res}} (M', E_{\rho_M}) - \tau^{\text{res}} (M, E_{\rho_M})$$

31
of analytic residue torsions (1.11). But by the identification (1.14) of Theorem 1.1 this
vanishes since the Betti numbers \( \dim H^p(M, E) \) are homotopy invariants. That is not the
case for the classical torsion characters (3.24) and (3.26) because although Reidemeister
torsion is a topological invariant it is not a homotopy invariant (a fact used classically
to distinguish certain homotopic but not homeomorphic spaces).

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