HOMOTOPY, HOMOLOGY AND PERSISTENT HOMOLOGY USING ČECH’S CLOSURE SPACES

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Abstract. We use Čech closure spaces, also known as pretopological spaces, to develop a uniform framework that encompasses the discrete homology of metric spaces, the singular homology of topological spaces, and the homology of (directed) clique complexes, along with their respective homotopy theories. We obtain nine homology and six homotopy theories of closure spaces. We show how metric spaces and more general structures such as weighted directed graphs produce filtered closure spaces. For filtered closure spaces, our homology theories produce persistence modules. We extend the definition of Gromov-Hausdorff distance to filtered closure spaces and use it to prove that our persistence modules and their persistence diagrams are stable. We also extend the definitions Vietoris-Rips and Čech complexes to closure spaces and prove that their persistent homology is stable.

1. Introduction

Like topological spaces, Čech closure spaces consist of a set together with a structure satisfying a few simple axioms. This structure is called a closure. If this closure operation is idempotent then one recovers the definition of a topological space. Removing the idempotent requirement gives a generalization of topological spaces that is useful in modeling proximity. For example, consider a graph, with the closure of a vertex defined to be itself together with its adjacent vertices. While not nearly as well known as topological spaces or graphs, closure space have been used in applications. Examples include shape recognition [21, 23], image analysis [29, 9], supervised learning [22, 24] and complex systems modeling [1, 30].

A major theme in applied topology is the adaptation of tools from algebraic topology to discrete settings. Here we generalize fundamental tools of algebraic topology to closure spaces. Specifically, we develop homotopy theories and homology theories for closure spaces in a uniform way. Unlike topological spaces, we have various choices of interval objects and product operations, each of which leads to a corresponding homotopy theory and homology theory. We describe the relationships between these theories and apply them to a number of examples. Closure spaces generalize topological spaces, simple graphs and simple directed graphs and our homotopy and homology theories recover previously defined theories in those settings.

Finally, we develop the persistent homology of closure spaces. We define filtered closure spaces and show how they arise from metric spaces and also from more general structures such as weighted directed graphs. Applying our homology theories to these filtered closure spaces, we obtain persistence modules. If the filtered closure spaces are obtained from sublevel sets of a function on a closure space then we prove that the resulting persistence modules are stable with respect to the supremum norm distance on the functions. To prove more general stability theorems, we extend the definition of Gromov-Hausdorff distance from metric spaces to filtered closure spaces. We prove that our persistence modules are stable with respect to
this Gromov-Hausdorff distance. We also extend the definition of Vietoris-Rips complexes and Čech complexes to closure spaces and prove that their persistent homology is stable, generalizing the stability theorem of [16].

This paper is structured as follows. In Section 2 we recall necessary background on closure spaces. In Section 3 we consider two important subcategories of closure spaces; directed and undirected graphs. In Section 4 we show how to generate closure operations from metric spaces. In Section 5 we define several homotopy theories of closure spaces and we study the relationship between these. We remark that some of these have been considered previously in applied topology when working in a full subcategory of closure spaces. In Section 6 we define several homology theories of closure spaces. We remark that some of these have been considered previously in applied topology when working in a full subcategory of closure spaces. In Section 7 we show the existence of adjoint functors between closure spaces and hypergraphs. We also define persistent homology groups of closure spaces and show stability results.

Related work. Antonio Rieser [33] and Demaria and Bogin [19, 18] used the unit interval to define a homotopy theory for closure spaces. Our work is particularly indebted to [33]. An important case of closure spaces is given by symmetric and reflexive relations. These were called physical continua by Poincaré [31], tolerance spaces by Zeeman [35], and fuzzy spaces by Poston [32]. When the underlying set is a finite subset of the lattice $\mathbb{Z}^n$, they are also called digital images and studied in digital topology [27]. (Symmetric) reflexive relations are equivalent to simple (undirected) graphs. Discrete homotopy theory, originally called $A$-theory, was developed for undirected graphs [28, 5, 6, 2]. Barcelo, Capraro and White [3] developed a discrete homology theory for metric spaces that is compatible with the discrete homotopy theory. Our work extends these discrete notions of homology and homotopy to closure spaces. A directed analogue of $A$-homotopy for simple directed graphs was also recently developed [25]. Our work also extends these ideas to closure spaces.

2. Closure spaces

In this section we provide background on Eduard Čech’s closure spaces [15, 33].

2.1. Elementary definitions. We start with the elementary definitions we need to work with closure spaces.

Definition 2.1. Let $X$ be a set and let $\mathcal{P}(X)$ denote the collection of subsets of $X$. A function $c : \mathcal{P}(X) \to \mathcal{P}(X)$ is called a closure operation (or just closure) for $X$ if the following axioms are satisfied:

1) $c(\emptyset) = \emptyset,$
2) $A \subseteq c(A)$ for all $A \subseteq X,$
3) $c(A \cup B) = c(A) \cup c(B)$ for all $A, B \subseteq X.$

An ordered pair $(X, c)$ where $X$ is a set and $c$ a closure for $X$ is called a (Čech) closure space. Elements of $X$ are called points.

Lemma 2.2. Let $(X, c)$ be a closure space. If $A \subset B$ are subsets of $X$ then $c(A) \subset c(B).$

Proof. Note that $B = A \cup B$. Then $c(A) \subset c(A) \cup c(B) = c(A \cup B) = c(B)$ by axiom 3. \qed
**Example 2.3.** Let $X$ be a set. Then the identity map $1_{\mathcal{P}(X)} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a closure operation for $X$. It is called the **discrete closure** for $X$. The closure operation defined by the map $A \mapsto X$ for $A \neq \emptyset$ and $\emptyset \mapsto \emptyset$ is called the **indiscrete closure** for $X$.

**Definition 2.4.** Let $(X,c)$ be a closure space. A subset $A \subset X$ is **closed** if $c(A) = A$. A subset $A \subset X$ is **open** if its complement $X - A$ is closed.

**Definition 2.5.** Given a set $X$ and a closure $c$ for $X$ we can associate to it an **interior operation** for $X$, $\text{int}_c : \mathcal{P}(X) \to \mathcal{P}(X)$ given by

$$\text{int}_c(A) := X - c(X - A).$$

The interior operation satisfies the following:

1. $\text{int}_c(X) = X$,
2. For all $A \subset X$, $\text{int}_c(A) \subset A$,
3. For all $A, B \subset X$, $\text{int}_c(A \cap B) = \text{int}_c(A) \cap \text{int}_c(B)$.

If $\text{int} : \mathcal{P}(X) \to \mathcal{P}(X)$ is a function satisfying the 3 conditions in Definition 2.5 and one defines $c_{\text{int}} : \mathcal{P}(X) \to \mathcal{P}(X)$ by $c_{\text{int}}(A) := X - \text{int}(X - A)$, then $c_{\text{int}}$ is a closure operation for $X$.

**Proposition 2.6.** [15, 14.A.12] A subset $A$ of a closure space $(X,c)$ is open if and only if $\text{int}_c(A) = A$.

**Definition 2.7.** [15, Definition 14.B.1] Let $(X,c)$ be a closure space. Let $A \subset X$. We say $B \subset X$ is a **neighborhood** of $A$ if $A \subset X - c(X - B) = \text{int}_c(B)$. In the case that $A = \{x\}$, we say $B$ is a neighborhood of $x$. The **neighborhood system** of $A$ is the collection of all neighborhoods of $A$.

**Definition 2.8.** [15, Definition 16.A.1] Let $(X,c_X)$ and $(Y,c_Y)$ be closure spaces. A map $f : (X,c_X) \to (Y,c_Y)$ is **continuous at $x \in X$** if for all $A \subset X$ such that $x \in c_X(A)$ it follows that $f(x) \in c_Y(f(A))$. If $f$ is continuous at every point of $X$ we say $f$ is **continuous**. Equivalently, $f$ is continuous if for every $A \subset X$, $f(c_X(A)) \subset c_Y(f(A))$. A continuous map $f$ is called a **homeomorphism** if $f$ is a bijection with a continuous inverse.

**Theorem 2.9.** [15, Theorem 16.A.4 and Corollary 16.A.5] Let $(X,c_X)$ and $(Y,c_Y)$ be closure spaces. A map $f : (X,c_X) \to (Y,c_Y)$ is continuous at $x \in X$ if and only if for every neighborhood $V \subset Y$ of $f(x)$, the inverse image $f^{-1}(V)$ is a neighborhood of $x$. Equivalently, $f$ is continuous at $x$ if and only if for each neighborhood $V \subset Y$ of $f(x)$, there exists a neighborhood $U \subset X$ of $x$ such that $f(U) \subset V$.

**Proposition 2.10.** [15, 16.A.3] Let $f : (X,c_X) \to (Y,c_Y)$ and $g : (Y,c_Y) \to (Z,c_Z)$ be continuous maps of closure spaces. Then the composition $g \circ f : (X,c_X) \to (Z,c_Z)$ is continuous.

**Definition 2.11.** Let $X$ be a set. Suppose $c_1$ and $c_2$ are two closure operations for $X$. We say $c_2$ is **coarser** than $c_1$ and $c_1$ is **finer** than $c_2$ if $c_1(A) \subset c_2(A)$ for all $A \subset X$. Equivalently, since $(\mathcal{P}(X), \subset)$ is a partial order, with the order given by inclusion of subsets, $c_2$ is coarser than $c_1$ if and only if $c_2$ dominates $c_1$ as a poset morphism. Write $c_1 \leq c_2$. Observe also that $c_2$ is coarser than $c_1$ if and only if the identity map $1_X : (X,c_1) \to (X,c_2)$ is continuous.

Under the relation $\leq$, the collection of closure operators on a set $X$ forms a poset with initial element the discrete closure and terminal element the indiscrete closure.
Definition 2.12. [15, Definition 17.A.17] Given a closure space \((X, c)\). A cover is a family of subsets of \(X\), \(\mathcal{C} = \{U_j\}_{j \in J}\), whose union is \(X\). \(\mathcal{C}\) is an interior cover of \((X, c)\) if every point \(x \in X\) has a neighborhood in \(\mathcal{C}\). We say \(\mathcal{C}\) is an open cover if every \(U_j\) is open and a closed cover if every \(U_j\) is closed. An open cover is an interior cover.

Definition 2.13. A family \(\{U_\alpha \mid \alpha \in A\}\) of subsets of a closure space \((X, c_X)\) is called locally finite of each point \(x \in X\) possesses a neighborhood intersecting only finitely many \(U_\alpha\).

Theorem 2.14. [15, 17.A.16] [Pasting Lemma] Let \((X, c_X)\) and \((Y, c_Y)\) be closure spaces and let \(\{U_\alpha \mid \alpha \in A\}\) be a locally finite cover of \((X, c_X)\). Let \(f : X \rightarrow Y\) be a map of sets. If \(f|_{c_X(U_\alpha)} : (c_X(U_\alpha), c_X) \rightarrow (Y, c_Y)\) is continuous for each \(\alpha \in A\), then \(f : (X, c_X) \rightarrow (Y, c_Y)\) is continuous.

Definition 2.15. Let \((X, c)\) be a closure space and let \(A \subset X\). For \(B \subset A\), define \(c_A(B) = c(B) \cap A\). Then \((A, c_A)\) is a closure space called a subspace of \((X, c)\).

2.2. Topological spaces. Here we consider an important class of closure spaces, namely topological spaces.

Definition 2.16. A topological closure operation for a set \(X\) is a closure \(c\) for \(X\) satisfying the following additional axiom:

\[(4)\] For all \(A \subset X\), \(c(c(A)) = c(A)\).

A closure operation satisfying this additional axiom is also called a Kuratowski closure operation (see [33, Remark 2.19]). A closure space \((X, c)\) is a topological space (Kuratowski closure space) if the closure operation \(c\) is topological.

Example 2.17. [33, Example 2.17] Let \((X, \tau)\) be a topological space. Then the operation of taking the closure of a subset \(A \subset X\) defines a closure operation for \(X\), which is topological. Definition 2.16 is equivalent to the standard definition of a topological space in the sense that the collection of open sets obtained from a topological closure form a topology and that the closure of a set is the usual closure in this topology.

Hence every topological space is a closure space. The converse is false – there are closure spaces whose closure operation does not arise from any topology on the underlying set. In these cases there are subsets whose closure is not a closed set. See Example 2.19.

We will view every topological space as a closure space. If a set has a standard topology, such as the unit interval (with the metric topology, which equals the order topology), we will use \(\tau\) to signify the corresponding topological closure operation.

Example 2.18. The discrete and indiscrete closure on a set \(X\) define the discrete and indiscrete topology on \(X\), respectively.

Example 2.19. Consider \(\mathbb{R}^n\) with the euclidean metric \(d\). Let \(r > 0\) and let \(c_r\) be the closure structure on \(\mathbb{R}^n\) defined by \(c_r(A) := \{x \in \mathbb{R}^n \mid \text{dist}(x, A) \leq r\}\) for all \(A \subset \mathbb{R}^n\), where \(\text{dist}(x, A) := \inf_{y \in A} d(x, y)\). Let \(U = \{x \in \mathbb{R}^n \mid d(x, 0) \leq 1\}\). Observe that \(c_r(c_r(U)) \neq c_r(U)\). Thus \(c_r\) is not induced by any topology on \(\mathbb{R}^n\).

Theorem 2.20. [15, Theorem 16.A.10] Let \((X, c_X)\) be a closure space and let \((Y, c_Y)\) be a topological space. A map \(f : (X, c_X) \rightarrow (Y, c_Y)\) is continuous if and only if the inverse image of every open set is open. Equivalently, \(f\) is continuous if and only if the inverse image of every closed set is closed.
**Definition 2.21.** Denote the category with objects closure spaces \((X, c)\) and morphisms continuous maps between closure spaces by \(\text{Cl}\). Denote the full subcategory of \(\text{Cl}\) with objects topological spaces by \(\text{Top}\).

**Proposition 2.22.** [15, 16.B.1-16.B.3] Let \((X, c)\) be a closure space. Define \(\tau(c) : \mathcal{P}(X) \to \mathcal{P}(X)\) by

\[
\tau(c)(A) := \bigcap \{ F \subset X \mid c(F) = F \text{ and } A \subset F \}
\]

Then \(\tau(c)\) is a topological closure operation. Furthermore, \(\tau(c)\) is the finest topological closure operation coarser than \(c\). The closure operation \(\tau(c)\) is called the topological modification of \(c\).

**Proposition 2.23.** [15, 16.B.4] Let \((X, c)\) be a closure space and let \((Y, \tau)\) be a topological space. A map of sets \(f : X \to Y\) is continuous as a closure space map \(f : (X, c) \to (Y, \tau)\) if and only if the map \(f : (X, \tau(c)) \to (Y, \tau)\) is a continuous map of topological spaces. That is, there exists a natural bijection between the sets of morphisms

\[
\text{Cl}((X, c), (Y, \tau)) \cong \text{Top}((X, \tau(c)), (Y, \tau)).
\]

Equivalently, \(\tau\) is the left adjoint to the inclusion functor \(\iota : \text{Top} \to \text{Cl}\).

### 2.3. Basic examples

We will use some of these in Section 5. Let \(m \geq 0\).

**Definition 2.24.**

1. Let \(I\) denote the unit interval \([0, 1]\) with its standard topology.
2. Let \(J_{m,\perp}\) denote the set \(\{0, \ldots, m\}\) with the discrete closure.
3. Let \(J_{m,\top}\) denote the set \(\{0, \ldots, m\}\) with the indiscrete closure.
4. Let \(J_m\) denote the set \(\{0, \ldots, m\}\) with the closure operator \(c(i) = \{ j \in \{0, \ldots, m\} \mid |i-j| \leq 1 \}\). As a special case, \(J_1 = J_{1,\top}\).
5. Let \(J_+\) denote the set \(\{0, 1\}\) with the closure operator \(c_+(0) = \{0, 1\}\), \(c_+(1) = \{1\}\).
6. Let \(J_-\) denote the set \(\{0, 1\}\) with the closure operator \(c_-(0) = \{0\}\), \(c_-(1) = \{0, 1\}\).
7. For each \(0 \leq k \leq 2^m - 1\) we define a closure operator \(c_k\) on the set \(\{0, \ldots, m\}\) as follows. Consider the binary representation of \(k\). For \(1 \leq i \leq m\), \(i - 1\) is contained in \(c_k(i)\) iff the \(i\)th rightmost bit is 0 and \(i\) is contained in \(c_k(i-1)\) iff the \(i\)th rightmost bit is 1. We denote this closure space by \(J_{m,k}\). Note that \(J_{1,1} = J_+\) and \(J_{1,0} = J_-\).
8. Let \(J_{m,\leq}\) denote the set \(\{0, 1, \ldots, m\}\) with the closure operation \(c(i) = \{ j \mid i \leq j \}\). This closure is topological and the open sets are the down-sets. As a special case, \(J_{1,\leq} = J_+\).

### 2.4. Products, inductive products, coproducts, and pushouts

Closure spaces have two canonical products, the product closure and the inductive product closure, which we describe below. Some technical details are relegated to the appendix.

**Definition 2.25.** [15, Definition 17.C.1] Let \(\{X_\alpha, c_\alpha\}_{\alpha \in \Lambda}\) be a family of closure spaces. Let \(X\) denote the set \(\prod_{\alpha \in \Lambda} X_\alpha\). For each \(\alpha \in \Lambda\), let \(\pi_\alpha : X \to X_\alpha\) denote the projection map. Consider for each \(x \in X\), the collection of sets \(U_x\) of the form

\[
\bigcap \{ \pi_\alpha^{-1}(V_\alpha) \mid \alpha \in F \},
\]

where \(F \subset \Lambda\) is finite and \(V_\alpha\) is a neighborhood of \(\pi_\alpha(x)\) in \((X_\alpha, c_\alpha)\). The collection \(U_x\) is a filter base (Definition 4.2) in \(X\) and \(x \in \bigcap U_x\). Thus by Theorem 4.6 there is a unique closure \(c\) for \(X\) such that \(U_x\) is a local base at \(x\) (Definition 4.4) in \((X, c)\). We
call $c$ the \textit{product closure} for $X$ and call the pair $(X, c)$ the \textit{product closure}. The product closure is also characterized by a universal property, given in Theorem B.1. When we have two closure spaces $(X, c_X)$ and $(Y, c_Y)$ we will denote the product closure space by $(X \times Y, c_X \times c_Y)$. In this case, the local base $U_{x,y}$ in (1) consists of sets of the form $U \times V$ where $U$ is a neighborhood of $x$ and $V$ is a neighborhood of $y$. Note that instead of using all neighborhoods $V_a$ of $\pi_a(x)$ in (1), we may restrict to a local base at $\pi_a(x)$ [15, 17.C.3].

As a special case, if $x$ and $y$ have local bases $U_x$ and $U_y$, respectively, then $U_{x,y} = U_x \times U_y$ is a local base at $(x, y)$.

\textbf{Definition 2.26.} [15, Definition 17.D.1] Given closure spaces $(X, c_X)$ and $(Y, c_Y)$, consider the product set $X \times Y$. For each $(x, y) \in X \times Y$, let $V_{x,y}$ be the collection of all sets of the form
\begin{equation}
\{ \{x\} \times V \} \cup \{ U \times \{y\}\}
\end{equation}
where $V$ and $U$ are neighborhoods of $y$ in $(Y, c_Y)$ and $x$ in $(X, c_X)$, respectively. Each $V_{x,y}$ is a filter base (Definition A.2) in $X \times Y$ and $(x, y) \in \bigcap V_{x,y}$. Thus, by Theorem A.6 there exists a unique closure operation for $X \times Y$ such that $V_{x,y}$ is a local base (Definition A.4) at $(x, y)$ for all $(x, y) \in X \times Y$. The \textit{inductive product} of two closure spaces $(X, c_X)$ and $(Y, c_Y)$, denoted by $(X \times Y, c_X \sqcup c_Y)$ is the set $X \times Y$ endowed with this closure operation. The closure operation $c_X \sqcup c_Y$ is called the \textit{inductive product closure}. The neighborhoods of the form (2) are called \textit{canonical neighborhoods for the inductive product}, or simply \textit{canonical inductive neighborhoods}. If $(X, c_X)$ and $(Y, c_Y)$ are closure spaces, we will denote their inductive product by $(X, c_X) \sqcup (Y, c_Y)$. The sets in (2) may be written as $\{(x', y') \in U \times V | x' = x \text{ or } y' = y\}$. The inductive product closure is also characterized by a universal property, given in Theorem B.2. Note that instead of using all neighborhoods of $x$ and $y$ in (2), we may restrict to local bases $U_x$ and $U_y$ of $x$ and $y$, respectively [15, 17.C.3].

\textbf{Proposition 2.27.} [15, Theorem 17.D.2] The product closure is coarser than the inductive product closure.

\textbf{Lemma 2.28.} Let $X$ be a closure space and let $Y$ be a discrete space. Then $X \times Y = X \sqcup Y$.

\textbf{Proof.} We will show that both closures share a local base (Definition A.4) at each point. It then follows by Theorem A.6 that they are equal. Let $(x, y) \in X \times Y$. Since $Y$ is discrete, $\{y\}$ is a local base at $y$. Choose a local base $U$ at $x$. Then $\{ U \times \{y\} | U \in U \}$ is a local base of $(x, y)$ for both closures. \qed

\textbf{Proposition 2.29.} [15, Proposition 17.C.11] Suppose we are given for each $a \in A$ closure spaces $(X_a, c_{X_a})$ and $(Y_a, c_{Y_a})$ and a map of sets $f_a : X_a \to Y_a$. If for all $a \in A$, $f_a$ is continuous, then the mapping $f : (\prod_{a \in A} X_a, \prod_{a \in A} c_{X_a}) \to (\prod_{a \in A} Y_a, \prod_{a \in A} c_{Y_a})$ defined by $\{x_a\}_{a \in A} \mapsto \{f_a(x_a)\}_{a \in A}$ is continuous. Conversely, if $f$ is continuous and $\prod_{a \in A} X_a \neq \emptyset$, then for all $a \in A$, $f_a$ is continuous.

\textbf{Proposition 2.30.} Let $f : (X_1, c_{X_1}) \to (Y_1, c_{Y_1})$ and $g : (X_2, c_{X_2}) \to (Y_2, c_{Y_2})$ be continuous maps of closure spaces. Then the map $f \times g : (X_1, c_{X_1}) \sqcup (X_2, c_{X_2}) \to (Y_1, c_{Y_1}) \sqcup (Y_2, c_{Y_2})$ defined by $f \times g(x_1, x_2) := (f(x_1), g(x_2))$ is continuous.

\textbf{Proof.} By Proposition B.3 it suffices to show that for every $x_2 \in X_2$ the map $f \times g(-, x_2) : (X_1, c_{X_1}) \to (Y_1, c_{Y_1}) \sqcup (Y_2, c_{Y_2})$ defined by $x_1 \mapsto (f(x_1), g(x_2))$ is continuous and that for every $x_1 \in X_1$, $f \times g(x_1, -) : (X_2, c_{X_2}) \to (Y_2, c_{Y_2})$ is continuous.
The complete above are the categorical coproduct and coequalizer in the category \([15, \text{Theorems 33.A.4 and 33.A.5}]\). The coproduct and coequalizer defined by Theorem 2.33.

The quotient set is continuous. Given continuous maps \(g \circ f \in \mathcal{C} \times \mathcal{D}\) for \(x \in X\), the 'round up' map \(f \in \mathcal{E} \times \mathcal{F}\) is continuous. Therefore \(f \times g(-, x_2)\) is continuous at \(x_1\) (Theorem 2.9). Therefore \(f \times g(-, x_2)\) is continuous. The other case is similar.

**Definition 2.31.** [15, Definition 17.B.1] Let \(\{ (X_i, c_i) \}_{i \in I}\) be a collection of closure spaces. The **coproduct** of \(\{ (X_i, c_i) \}_{i \in I}\) is the disjoint union of sets \(X = \bigsqcup_i X_i\) with the closure operator \(c\) defined by \(c(\bigsqcup_i A_i) := \bigsqcup_i c_i(A_i)\) for any subset \(\bigsqcup_i A_i\) of \(X\).

**Definition 2.32.** Given continuous maps \((X, c_X) \overset{f}{\underset{g}{\rightarrow}} (Y, c_Y)\) the **coequalizer** of \(f\) and \(g\) consists of the closure space \((Q, c_Q)\) and map \(p : Y \rightarrow Q\) defined as follows. Let \(Q\) be the quotient set \(Y/\sim\) for the equivalence relation given by \(f(x) \sim g(x)\) for all \(x \in X\). Let \(p : Y \rightarrow Q\) be the quotient map. For \(A \subseteq Q\), define \(c_Q(A) = p(c_Y(p^{-1}(A)))\).

**Theorem 2.33.** [15, Theorems 33.A.4 and 33.A.5] The coproduct and coequalizer defined above are the categorical coproduct and coequalizer in the category \(\mathcal{Cl}\) and hence \(\mathcal{Cl}\) is complete.

As an application, we have pushouts of closure spaces.

**Definition 2.34.** The **pushout** in \(\mathcal{Cl}\) is defined as follows. Given the solid arrow diagram

\[
\begin{array}{ccc}
  (A, c_A) & \xrightarrow{f} & (X, c_X) \\
  \downarrow g & & \downarrow i \\
  (Y, c_Y) & \xrightarrow{j} & (P, c_P)
\end{array}
\]

define \(P = (X \amalg Y)/\sim\) where \(f(a) \sim g(a)\) for all \(a \in A\), let \(i, j\) be the induced maps, and for \(B \subseteq P\), define \(c_B(B) = i(c_X(i^{-1}(B))) \cup j(c_Y(j^{-1}(B)))\).

**Lemma 2.35.** \(J_{m,1}, J_m\) and \(J_{m,k}\) are obtained by \(m\)-fold binary pushouts of \(J_{1,1}, J_1\), and \(J_-, J_+\) respectively under \(*\).

**Proof.** For \(m = 2\) consider the following pushouts.

\[
\begin{array}{ccccccccc}
  \ast & \xrightarrow{1} & J_{1,1} & \xrightarrow{1} & J_1 & \xrightarrow{1} & J_\ast & \xrightarrow{1} & J_+ \\
  \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow 0 \\
  J_{1,1} & \rightarrow & J_{2,1} & \rightarrow & J_1 & \rightarrow & J_{2,0} & \rightarrow & J_+
\end{array}
\]

For the general case proceed by induction.

**Lemma 2.36.** The identity maps \(J_{m,k} \xrightarrow{1} J_m\) are continuous for all \(m \geq 0\) and \(0 \leq k \leq 2^m - 1\). The 'round up' map \(f_+ : (I, \tau) \rightarrow J_+\) defined by \(f(x) = 0\) if \(x < \frac{1}{2}\) and \(f(x) = 1\) if \(x \geq \frac{1}{2}\) and the 'round down' map \(f_- : (I, \tau) \rightarrow J_-\) defined by \(f(x) = 0\) if \(x \leq \frac{1}{2}\) and \(f(x) = 1\) if \(x > \frac{1}{2}\) are continuous. These may be combined to obtain continuous maps
\( f : ([0, m], \tau) \to J_{m, k} \) for any \( m \geq 0 \) and \( 0 \leq k \leq 2^m - 1 \). Precomposing with the map \( t \mapsto nt \), we obtain a continuous map \( f : (I, \tau) \to J_{m, k} \).

**Proof.** For each \( m \geq 0 \) and \( 0 \leq k \leq 2^m - 1 \) the closure operations for \( J_{m, k} \) are finer than the one for \( J_m \). For the second statement, we need to show that for nonempty \( A \subset I \), \( f_+(\bar{A}) \subset c_+(f_+(A)) \) and \( f_-(\bar{A}) \subset c_-(f_-(A)) \). Consider the first case. If \( 0 \in f_+(A) \) then \( c_+(f_+(A)) = \{0, 1\} \) and we are done. If not, then \( A \subset f_+^{-1}(1) = \left[ \frac{1}{2}, 1 \right] \). In this case, we also have \( \bar{A} \subset \left[ \frac{1}{2}, 1 \right] \) and \( f_+(\bar{A}) = \{1\} = c_+(f_+(A)) \), as desired. The other case is similar. The third statement follows from Lemma 2.35 and Theorem 2.14. The last statement follows from Proposition 2.10. \( \square \)

Dually we have the following.

**Definition 2.37.** Given continuous maps \((X, c_X) \xrightarrow{f} (Y, c_Y)\) the **equalizer** of \( f \) and \( g \) consists of the closure space \((E, c_E)\) and map \( i : E \to X \) defined as follows. Let \( E \) be the subset \( \{ x \in X \mid f(x) = g(x) \} \) with \( i \) the inclusion map. For \( A \subset E \), define \( c_E(A) = c_X(A) \cap E \).

**Theorem 2.38.** [15, Theorems 32.A.4 and 32.A.10] The product and equalizer defined above are the categorical product and equalizer in the category \( Cl \) and hence \( Cl \) is complete.

**Proposition 2.39.** Every limit of topological spaces in \( Cl \) is a topological space. On the other hand, colimits of topological spaces in \( Cl \) are not necessarily topological spaces.

**Proof.** The inclusion functor \( \iota : Top \to Cl \) is a right-adjoint by Proposition 2.23 and thus preserves limits. For the second claim, consider the second pushout diagram in Lemma 2.35. Note that the one point space and \( J_1 \) are both topological spaces (with the indiscrete topology), however the pushout \( J_2 \), is not topological. See also [33, Example 2.52] which is a more modern take on [15, Introduction to Section 33.B]. \( \square \)

We remark that monomorphisms of closure spaces are continuous maps \( f : (X, c) \to (Y, c') \) whose underlying set maps \( f : X \to Y \) are injective.

### 3. Graphs and digraphs as closure spaces

In this section we identify categories of simple directed and undirected graphs with certain full subcategories of closure spaces. These observations go back Čech [15, Chapter 26] but we express them in a modern way. Note that we allow our graphs to have arbitrary cardinality.

**3.1. Symmetric and quasi-discrete closures.** We first define symmetric and quasi-discrete closure spaces.

**Definition 3.1.** A closure space \((X, c)\) is **symmetric** if \( y \in c(x) \) implies \( x \in c(y) \) for all \( x, y \in X \).

A closure space is symmetric if and only if it is **semi-uniformizable** (see Appendix C).

**Definition 3.2.** [15, Example 14.A.5(f)] Let \((X, c)\) be a closure space. The closure \( c \) is called **quasi-discrete** if for all \( A \subset X \), \( c(A) = \bigcup_{x \in A} c(x) \).

**Definition 3.3.** We denote by \( Cl_s \), \( Cl_{qd} \) and \( Cl_{sqd} \) the full subcategories of \( Cl \) consisting of symmetric, quasi-discrete, and symmetric quasi-discrete closure spaces, respectively.
Proposition 3.4. Suppose \((X, c_X)\) is a quasi-discrete closure space. Let \((Y, c_Y)\) be a closure space. Then \(f : (X, c_X) \to (Y, c_Y)\) is continuous if and only if \(\forall x \in X, f(c_X(x)) \subset c_Y(f(x))\).

Proof. The forward direction follows from the definition of continuity. For the reverse direction, let \(A \subset X\). Then \(f(c_X(A)) = f(\bigcup_{x \in A} c_X(x)) = \bigcup_{x \in A} f(c_X(x)) \subset \bigcup_{x \in A} c_Y(f(x)) \subset c_Y(f(A))\), since for all \(x \in A\), \(c_Y(f(x)) \subset c_Y(f(A))\). Thus \(f\) is continuous.

Definition 3.5. Let \(X\) be a set and let \(\rho : X \to \mathcal{P}(X)\) such that \(x \in \rho(x)\). Define \(c : \mathcal{P}(X) \to \mathcal{P}(X)\) by \(c(A) = \bigcup_{x \in A} \rho(x)\) for \(A \subset X\). It is easy to verify that \(c\) is a quasi-discrete closure operation, which we call the induced quasi-discrete closure operation.

Definition 3.6. Let \((X, c)\) be a closure space. Let \(qd(c)\) denote the induced quasi-discrete closure operation on the restriction of \(c\) to one-point sets. Then \(qd(c)\) is finer than \(c\) and is called the quasi-discrete modification of \(c\) ([15, Definition 26.A.1]). The mappings \((X, c) \mapsto (X, qd(c))\) and \((f : (X, c) \to (Y, d)) \mapsto (f : (X, qd(c)) \to (Y, qd(d)))\) define a functor \(qd : Cl \to Cl_{qd}\).

Proposition 3.7. Let \((X, c) \in Cl_{qd}\) and \((Y, d) \in Cl\). Given a set map \(f : X \to Y\), \(f : (X, c) \to (Y, qd(d))\) is continuous iff \(f : (X, c) \to (Y, d)\) is continuous. Thus, we have a natural bijection

\[
Cl((X, c), (Y, d)) \cong Cl_{qd}((X, c), (Y, qd(d))).
\]

That is, \(qd\) is right adjoint to the inclusion functor \(Cl_{qd} \hookrightarrow Cl\).

Proof. \((\Rightarrow)\) Let \(A \subset X\). Then \(f(c(A)) \subset qd(d)(f(A)) \subset d(f(A))\).

\((\Leftarrow)\) Let \(x \in X\). Then \(f(c(x)) \subset d(f(x)) = qd(d)(f(x))\).

Definition 3.8. Let \((X, c) \in Cl_{qd}\). Let \(s(c)\) be the quasi-discrete closure induced by \(\rho(x) = \{y \in c(x) \mid x \in c(y)\}\). If \(y \in \rho(x)\) then \(y \in c(x)\) and \(x \in c(y)\). So \(x \in \rho(y)\). That is, \(s(c)\) is symmetric. Thus \(s(c)\) is a symmetric quasi-discrete closure finer than \(c\), called the symmetrization of \(c\). Let \(f : (X, c) \to (Y, d)\). For \(x \in X\), \(f(s(c)(x)) = f\{y \in c(x) \mid x \in c(y)\}\) and \(s(d)(f(x)) = \{z \in d(f(x)) \mid f(x) \in d(z)\}\). Now \(y \in c(x)\) implies that \(f(y) \subset d(f(x))\) and \(x \in c(y)\) implies that \(f(x) \in d(f(y))\). Therefore \(f(s(c)(x)) \subset s(d)(f(x))\). That is, \(f : (X, s(c)) \to (Y, s(d))\). Let \(s : Cl_{qd} \to Cl_{sd}\) denote the functor defined by these mappings.

Proposition 3.9. Let \((X, c) \in Cl_{sd}\) and \((Y, d) \in Cl_{qd}\). Given a set map \(f : X \to Y\), \(f : (X, c) \to (Y, s(d))\) is continuous iff \(f : (X, c) \to (Y, d)\) is continuous. Thus, we have a natural isomorphism

\[
Cl_{qd}((X, c), (Y, d)) \cong Cl_{sd}((X, c), (Y, s(d))).
\]

That is, \(s\) is right adjoint to the inclusion functor \(Cl_{sd} \hookrightarrow Cl_{qd}\).

Proof. \((\Rightarrow)\) Let \(x \in X\). Then \(f(c(x)) \subset s(d)(f(x)) \subset d(f(x))\).

\((\Leftarrow)\) Let \(x \in X\). Then \(f(c(x)) \subset d(f(x))\). Let \(y \in c(x)\). Then \(f(y) \in d(f(x))\). Also, \(x \in c(y)\). Thus \(f(x) \in d(f(y))\). Hence \(f(y) \in s(d)(f(x))\) and therefore \(f(c(x)) \subset s(d)(f(x))\).

3.2. Simple directed graphs. Here we show that simple directed graphs are isomorphic to a full subcategory of closure spaces, namely the quasi-discrete closure spaces.
Definition 3.10. A simple directed graph or simple digraph is pair \((X, E)\) where \(X\) is a set and \(E\) is a relation on \(X\) such that for all \(x \in X\), \((x, x) \notin E\). Elements of \(E\) are called directed edges. For simplicity, we refer to simple digraphs as digraphs. Let \((X, E)\) and \((Y, F)\) be two digraphs. A map of sets \(f : X \to Y\) is a digraph homomorphism between \((X, E)\) and \((Y, F)\) if whenever \(xEx'\), we have \(f(x)Ff(x')\) or \(f(x) = f(x')\). We denote the category of digraphs and digraph homomorphisms by \(\text{DiGph}\). Given a digraph \((X, E)\), let \(\overline{E} = E \cup \Delta\), where \(\Delta = \{(x, x) \mid x \in X\}\). Given \(\overline{E}\), we can recover \(E\) by \(E = \overline{E} \setminus \Delta\). Furthermore, \(f : X \to Y\) is a digraph homomorphism if whenever \(xEx'\), we have \(f(x)\overline{E}f(x')\).

We remark that we have a bijection between simple digraphs \((X, E)\) and reflexive relations \((X, \overline{E})\). Note that we make no assumptions on the cardinality of \(X\).

Definition 3.11. Recall the category of quasi-discrete closure spaces, \(\text{Cl}_{\text{qd}}\) (Definition 3.3). Let \(\Psi : \text{DiGph} \to \text{Cl}_{\text{qd}}\) be the functor that assigns to each digraph \((X, E)\) the closure space \((X, c_E)\), where \(c_E\) is the induced quasi-discrete closure (Definition 3.3) determined by the map \(\rho_E : X \to \mathcal{P}(X)\) given by

\[
(3) \quad \rho_E(x) = \{y \in X \mid x\overline{E}y\}.
\]

Given a digraph homomorphism \(f : (X, E) \to (Y, F)\), the map \(\Psi(f) : (X, c_E) \to (Y, c_F)\) is given by the map of sets \(f\). By Proposition 3.4, the continuity of \(\Psi(f)\) is equivalent to the \(f\) being a digraph homomorphism.

Definition 3.12. Let \(\Phi : \text{Cl}_{\text{qd}} \to \text{DiGph}\) be the functor that assigns to each quasi-discrete closure space \((X, c)\) the digraph \((X, E_c)\) defined by

\[
x\overline{E_c}y \iff y \in c(x).
\]

Given a continuous map \(f : (X, c_X) \to (Y, c_Y)\), let \(\Psi(f) : (X, E_{c_X}) \to (Y, E_{c_Y})\) be the map of sets \(f\). By Proposition 3.4, the continuity of \(\Psi(f)\) is equivalent to the \(f\) being a digraph homomorphism.

Proposition 3.13. The functors \(\Psi\) and \(\Phi\) are inverses and thus define an isomorphism of categories \(\text{Cl}_{\text{qd}} \cong \text{DiGph}\). \(\square\)

Definition 3.14. Let \((X, E)\) be a digraph. The reverse digraph \((X, E^\top)\), is given by \(yE^\top x\) iff \(xEy\) for \(x, y \in X\). That is, it is the digraph obtained by reversing the directed edges. The complement digraph \((X, E^c)\) is given by \(xE^c y\) iff \(x \neq y\) and not \(xEy\). That is, a directed edge is in \(E^c\) iff it is not in \(E\).

Lemma 3.15. Given a digraph \((X, E)\), we have the corresponding quasi-discrete closure \(c_E\) (Definition 3.11), the reverse digraph \(E^\top\) (Definition 3.14), and the function \(\rho_{E^\top} : X \to \mathcal{P}(X)\) given by \(\overline{E}\). Let \(x \in X\). Then, the singleton \(\{\rho_{E^\top}(x)\}\) is a local base (Definition 3.4) at \(x\) in the closure space \((X, c_E)\).

Proof. By definition and because the closure operation \(c_E\) is quasi-discrete, we have

\[
\text{int}_{c_E}(\rho_{E^\top}(x)) := X - c_E(X - \rho_{E^\top}(x)) = X - c_E(X - (\{y \in X \mid y\overline{E}x\}))
\]

\[
= X - c_E(\{y \in X \mid y\overline{E^c}x\}) = X - \bigcup_{y \in X, y\overline{E}x} \rho_E(y).
\]
If \( y \bar{E} x \) then \( x \not\in \rho_E(y) \). Therefore \( x \in \text{int}_{c_E}(\rho_{E^\top}(x)) \). Thus, \( \rho_{E^\top}(x) \) is indeed neighborhood of \( \{x\} \). Now let \( U \) be a neighborhood of \( x \). Then \( x \in \text{int}_{c_E}(U) = X - c_E(X - U) \). Suppose \( y \not\in U \). Then \( c_E(y) \subset c_E(X - U) \). Thus \( x \not\in c_E(y) \) and hence \( y \not\in \rho_{E^\top}(x) \). Therefore \( \rho_{E^\top}(x) \subset U \) and thus \( \rho_{E^\top}(x) \) is a local base at \( x \).

\[ \square \]

**Definition 3.16.** Let \((X, E_X)\) and \((Y, E_Y)\) be two digraphs. Define the **digraph product** 
\[ X \times Y \] 
to be the digraph \((X \times Y, E_X \times E_Y)\), where 
\[ (x, y) \bar{E} x' \text{ x' } y' \text{ iff } x \bar{E}_X x' \text{ and } y \bar{E}_Y y'. \]

**Proposition 3.17.** Let \((X, E_X)\) and \((Y, E_Y)\) be digraphs. Then \((X \times Y, c_{E_X \times E_Y}) = (X \times Y, c_E) \) (Definition 2.25).

**Proof.** We will show that both closures share a local base (Definition A.4) at each point. It follows by Theorem A.6 that they are equal.

Let \( x \in X \) and \( y \in Y \). By Lemma 3.15, \( c_{E_X} \) has a local base \( \{\rho_{E_X^\top}(x)\} \) at \( x \), \( c_{E_Y} \) has a local base \( \{\rho_{E_Y^\top}(y)\} \) at \( y \), and \( c_{E_X \times E_Y} \) has a local base \( \{\rho_{(E_X \times E_Y)^\top}(x, y)\} \) at \( (x, y) \). By Definition 2.25, \( c_{E_X} \times c_{E_Y} \) has a local base \( \{\rho_{E_X^\top}(x) \times \rho_{E_Y^\top}(y)\} \) at \((x, y)\). By Equation (3) and Definition 3.16, \( \rho_{E_X^\top}(x) = \{x' \in X \mid x' \bar{E}_X x, y \text{ and } \rho_{E_Y^\top}(y) = \{y' \in Y \mid y' \bar{E}_Y y, \text{ and } \rho_{(E_X \times E_Y)^\top}(x, y) = \{(x', y') \in X \times Y \mid x' \bar{E}_X x \text{ and } y' \bar{E}_Y y, \text{ and either } x = x' \text{ or } y = y'. \}

Therefore \( \rho_{(E_X \times E_Y)^\top} = \rho_{E_X^\top} \times \rho_{E_Y^\top}. \)

Combining Proposition 3.13 and Proposition 3.17 we have the following.

**Corollary 3.18.** Let \((X, c_X)\) and \((Y, c_Y)\) be two quasi-discrete closure spaces. Then \((X \times Y, c_{E_X \times c_Y}) \) is also quasi-discrete.

**Definition 3.19.** [25, Definition 2.3] Let \((X, E_X)\) and \((Y, E_Y)\) be two digraphs. Define the **Cartesian product** \( X \times Y \) to be the digraph \((X \times Y, E_X \times E_Y)\), where \( (x, y)(E_X \times E_Y)(x', y') \) if and only if \( x = x' \) and \( y E_Y y' \), or \( x E_X x' \) and \( y y' \). Equivalently, \( (x, y)(E_X \times E_Y)(x', y') \) iff \( x \bar{E}_X x' \text{ and } y \bar{E}_Y y \). Combining Proposition 3.13 and Proposition 3.17 we have the following.

**Proposition 3.20.** Let \((X, E_X)\) and \((Y, E_Y)\) be digraphs. Then \((X \times Y, c_{E_X \times E_Y}) = (X \times Y, c_{E_X \times c_Y}) \) (Definition 2.20).

**Proof.** We will show that both closures share a local base (Definition A.4) at each point. It follows by Theorem A.6 that they are equal.

Let \( x \in X \) and \( y \in Y \). By Lemma 3.13, \( c_{E_X} \) has a local base \( \{\rho_{E_X^\top}(x)\} \) at \( x \), \( c_{E_Y} \) has a local base \( \{\rho_{E_Y^\top}(y)\} \) at \( y \), and \( c_{E_X \times E_Y} \) has a local base \( \{\rho_{(E_X \times E_Y)^\top}(x, y)\} \) at \((x, y)\). By Definition 2.20, \( c_{E_X} \times c_{E_Y} \) has a local base \( \{\rho_{E_X^\top}(x) \times \rho_{E_Y^\top}(y)\} \) at \((x, y)\). By Equation (3) and Definition 3.19, \( \rho_{E_X^\top}(x) = \{x' \in X \mid x' \bar{E}_X x, \text{ and } \rho_{E_Y^\top}(y) = \{y' \in Y \mid y' \bar{E}_Y y, \text{ and } \rho_{(E_X \times E_Y)^\top}(x, y) = \{(x', y') \in X \times Y \mid x' \bar{E}_X x \text{ and } y' \bar{E}_Y y, \text{ and either } x' = x \text{ or } y' = y'. \}

Therefore \( \rho_{(E_X \times E_Y)^\top} = \rho_{E_X^\top} \times \rho_{E_Y^\top}. \)

Combining Proposition 3.13 and Proposition 3.20 we have the following.

**Corollary 3.21.** Let \((X, c_X)\) and \((Y, c_Y)\) be two quasi-discrete closure spaces. Then \((X \times Y, c_{E_X \circ c_Y}) \) is also quasi-discrete.
3.3. **Simple graphs.** A simple graph is a simple digraph \((X, E)\) (Definition 3.10) in which the relation \(E\) is symmetric. That is, for all \(x, y \in X\), \(x Ey\) if and only if \(yEx\). Note that we have a bijection between simple graphs \((X, E)\) and symmetric, reflexive relations \((X, \overline{E})\). Also note that for a simple graph \((X, E)\), \(E\) corresponds a collection of subsets of \(X\) of cardinality two. We will also denote this collection by \(E\). We will use the term graph for a simple graph. A digraph homomorphism between graphs is called a graph homomorphism. The full subcategory of \(\text{DiGph}\) given by graphs is the category of graphs and graph homomorphisms, which we denote \(\text{Gph}\).

By specializing the results of the previous section we obtain the following.

**Proposition 3.22.** The functors \(\Psi\) and \(\Phi\) define an isomorphism of categories \(\text{Cl}_{\text{sqd}} \cong \text{Gph}\).

4. **Closures induced by metrics**

In this section, we consider closure operations induced by a metric. Our closure operators will be indexed by the set \([0, \infty) \times \{-1, 0, 1\}\), which we order by the lexicographic order. That is, \((\varepsilon, a) \leq (\varepsilon', a')\) if \(\varepsilon < \varepsilon'\) or \(\varepsilon = \varepsilon'\) and \(a \leq a'\). For \(\varepsilon \geq 0\) we denote \((\varepsilon, -1), (\varepsilon, 0)\), and \((\varepsilon, 1)\) by \(\varepsilon^-, \varepsilon\), and \(\varepsilon^+\), respectively.

4.1. **Basic properties.** We introduce various “thickening” closures on a metric space \((X, d)\) and examine some properties of these closure operations. In particular, we investigate how they interact with Lipschitz maps between metric spaces, and we classify the base of neighborhoods at each point.

**Definition 4.1.** Let \((X, d)\) be a metric space and let \(A \subset X\). For \(\varepsilon \geq 0\) define

\[
c_{\varepsilon^-, d}(A) = \bigcup_{x \in A} B_{\varepsilon}(x) \quad c_{\varepsilon, d}(A) = \bigcup_{x \in A} \overline{B}_{\varepsilon}(x) \quad c_{\varepsilon^+, d}(A) = \{x \in X \mid \text{dist}(x, A) \leq \varepsilon\},
\]

where \(B_{\varepsilon}(x) = \{y \in X \mid d(x, y) < \varepsilon\}\) for \(\varepsilon > 0\), \(B_0(x) = \{x\}\), \(\overline{B}_{\varepsilon}(x) = \{y \in X \mid d(x, y) \leq \varepsilon\}\), and \(\text{dist}(x, A) = \inf_{y \in A} d(x, y)\). For all \(\varepsilon \geq 0\) and \(a \in \{-1, 0, 1\}\), \(c_{(\varepsilon, a), d}\) is a closure operation on \(X\). If the metric \(d\) is clear from context then we will denote the closure spaces \((X, c_{(\varepsilon, a),})\) by \((X, c_{(\varepsilon, a),})\).

Let \((X, d)\) be a metric space. Then \((X, c_{0^+})\) is the topological space where the topology is induced by the metric \(d\). Also \((X, c_{0^-}) = (X, c_0)\) and this closure is the discrete (topological) closure.

**Example 4.2.** Consider \(([0, 1], d)\), the unit interval with \(d(x, y) = |x - y|\). For \((\varepsilon, a) \in [0, 1] \times \{-1, 0, 1\}\), we will denote the closure space \(([0, 1], c_{(\varepsilon, a)})\) by \(I_{(\varepsilon, a)}\). Note that \(I_{0^-} = I_0\) is the unit interval with the discrete (topological) closure, and \(I_{0^+} = I\), which is the unit interval with the standard topological closure. At the other extreme, note that \(I_1 = I_{1^+}\) is the unit interval with the indiscrete closure.

Let \((X, d)\) be a metric space. Observe that for \(\varepsilon \geq 0\) the closure space \((X, c_{\varepsilon^+})\) is symmetric and the closure spaces \((X, c_{\varepsilon^-})\) and \((X, c_{\varepsilon})\) are quasi-discrete and symmetric (Definition 3.2 and Theorem C.3). Furthermore, recall the functor \(\text{qd} : \text{Cl} \rightarrow \text{Cl}_{\text{sqd}}\) from Proposition 3.17. It follows from the definitions of \(c_{\varepsilon^+}\) and \(c_{\varepsilon}\) that given a metric space \((X, d)\), \(\text{qd}(X, c_{\varepsilon^+, d}) = (X, c_{\varepsilon, d})\).

From Definition 4.1 it is easy to see the following.
Lemma 4.3. Let \((X, d)\) be a metric space and let \((\varepsilon, a) \leq (\varepsilon', a') \in [0, \infty) \times \{-1, 0, 1\}\). Then \(c(\varepsilon, a) \leq c(\varepsilon', a')\).

Using the triangle inequality one obtains the following.

Lemma 4.4. Let \((X, d)\) be a metric space and let \(\varepsilon, \delta \geq 0\). Then \(c_{\varepsilon^-}(c_{\delta^-}) \leq c_{(\varepsilon+\delta)^-}, c_{\varepsilon^-}(c_{\delta^+}) \leq c_{(\varepsilon+\delta)^+}\).

Let \((X, d)\) and \((Y, e)\) be metric spaces and let \(f : (X, d) \to (Y, e)\) be a 1-Lipschitz map. That is, for \(x, x' \in X\), \(e(f(x), f(x')) \leq d(x, x')\). Such maps are sometimes called nonexpansive maps or short maps. Let \(\text{Met}\) denote the category of metric spaces and 1-Lipschitz maps. It is easy to check the following.

Lemma 4.5. Let \(\varepsilon \in [0, \infty) \times \{-1, 0, 1\}\) and \(f : (X, d) \to (Y, e)\) be a 1-Lipschitz map. Then \(f : (X, c_{\varepsilon}, d) \to (Y, c_{\varepsilon}, e)\) is a continuous map. Thus, for each each \(\varepsilon \in [0, \infty) \times \{-1, 0, 1\}\) we have a functor \(\text{Met} \to \text{Cl}_u\).

Lemma 4.6. Given \(m \geq 0\), \(J_m\) is a retract of \([0, m], c_1\).

Proof. First, the inclusion \(i : J_m \hookrightarrow [0, m]\) is continuous since the closure on \(J_m\) is the restriction of \(c_1\) to the subset \(J_m\). Second, let \(r\) by given by sending \(t \in [0, m]\) to the nearest integer, rounding up in case of ties. Then \(|s - t| \leq 1\) implies that \(|r(s) - r(t)| \leq 1\). Therefore for all \(A \subset J_m\), \(r(c_{J_m}(A)) \subset c_1(r(A))\).

As with topological spaces, closure spaces may be defined using a local basis or local base of neighborhoods at each point. For the details, see Appendix A.

Lemma 4.7. Let \((X, d)\) be a metric space, \(x \in X\), and \(\varepsilon \geq 0\).

1. The singleton \(\{B_\varepsilon(x)\}\) is local base at \(x\) in \((X, c_{\varepsilon^-})\). (Recall that \(B_0(x) = \{x\}\).)
2. The singleton \(\{\overline{B}_\varepsilon(x)\}\) is local base at \(x\) in \((X, c_\varepsilon)\).
3. The collection \(\{\overline{B}_{\varepsilon+\delta}(x)\}_{\delta > 0}\) is local base at \(x\) in \((X, c_{\varepsilon^+})\).

Proof of Lemma 4.7. We will prove the third case. The other cases are similar. Let \(\delta > 0\). First we verify that \(\overline{B}_{\varepsilon+\delta}(x)\) is a neighborhood of \(x\) in \((X, c_{\varepsilon^+})\).

\[
\text{int}_{c_{\varepsilon^+}}(\overline{B}_{\varepsilon+\delta}(x)) = X - c_{\varepsilon^+}(X - \overline{B}_{\varepsilon+\delta}(x)) = X - \{y \in X \mid \text{dist}(y, X - \overline{B}_{\varepsilon+\delta}(x)) \leq \varepsilon\} = \{y \in X \mid \text{dist}(y, X - \overline{B}_{\varepsilon+\delta}(x)) > \varepsilon\}.
\]

Now observe that \(\text{dist}(x, X - \overline{B}_{\varepsilon+\delta}(x)) = \inf_{y \in X - \overline{B}_{\varepsilon+\delta}(x)} d(x, y)\). However, note that for all \(y \in X - \overline{B}_{\varepsilon+\delta}(x)\), \(d(x, y) > \varepsilon + \delta\). Since \(\delta > 0\), it follows that \(\inf_{y \in X - \overline{B}_{\varepsilon+\delta}(x)} d(x, y) > \varepsilon\). Therefore \(x \in \text{int}_{c_{\varepsilon^+}}(\overline{B}_{\varepsilon+\delta}(x))\).

Now suppose \(A \subset X\) is a neighborhood of \(x\) in \((X, c_{\varepsilon^+})\). By definition we have that \(x \in \text{int}_{c_{\varepsilon^+}}(A) = X - c_{\varepsilon^+}(X - A) = \{y \in X \mid \text{dist}(y, X - A) > \varepsilon\}\). Thus

\[
\text{dist}(x, X - A) = \inf_{y \in X - A} d(x, y) > \varepsilon.
\]

Therefore there exists an \(\delta > 0\) such that \(\forall y \in X - A, d(x, y) \geq \varepsilon + 2\delta\). Therefore if \(y \in X\) is such that \(d(x, y) \leq \varepsilon + \delta\), the previous inequality implies that \(y \in A\). Therefore \(\overline{B}_{\varepsilon+\delta}(x) \subset A\).
Definition 4.8. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces and let \(r \geq 0\). A map of sets \(f : X \to Y\) is \(r\)-Lipschitz if for all \(x, y \in X\), \(d_Y(f(x), f(y)) \leq rd_X(x, y)\).

Lemma 4.9. Let \((X, d_X)\) and \((Y, d_Y)\) be a metric spaces. Suppose \(f : X \to Y\) is \(r\)-Lipschitz. Then \(f : (X, c_{1+r, d_Y}) \to (Y, c_{r+1})\) and \(f : (X, c_{1+r, d_X}) \to (Y, c_{r,d_Y})\) are continuous. Suppose additionally that \(\text{im} d_X \subset \mathbb{N}\) and that for all \(n \in \mathbb{N}\) we have \(c_{n+1}^{d_X} = c_{n+1}^{d_X}\). Then \(f : (X, c_{1+r, d_X}) \to (Y, c_{r+1})\) is continuous if and only if \(f : X \to Y\) is \(r\)-Lipschitz.

Proof. Suppose \(f\) is \(r\)-Lipschitz. Let \(A \subset X\) and let \(y \in f(c_{1+r, d_X}(A))\). Then there exists an \(x \in c_{1+r, d_X}(A)\) such that \(f(x) = y\). Furthermore, we have that \(\text{dist}_{d_X}(x, A) \leq 1\). Since \(f\) is \(r\)-Lipschitz, it follows that \(\text{dist}_{d_Y}(f(x), f(A)) = \text{dist}_{d_Y}(y, f(A)) \leq r\text{dist}_{d_X}(x, A) \leq r\). In other words, \(f(x) \in c_{r+1}(f(A))\). Therefore \(f(c_{1+r, d_X}(A)) \subset c_{r+1}(f(A))\). Hence, \(f : (X, c_{1+r, d_X}) \to (Y, c_{r+1})\) is continuous.

Now suppose \(f : (X, c_{1+r, d_X}) \to (Y, c_{r+1})\) is continuous and that \(\text{im} d_X \subset \mathbb{N}\) and \(c_{m+1}^{d_X} = c_{m+1}^{d_X}\). Let \(x_1, x_2 \in X\). Suppose \(d_X(x_1, x_2) = m\), for \(m \in \mathbb{N}\). Then \(x_2 \in c_{m+1}^{d_X}(x_1) = c_{m+1}^{d_X}(x_1)\). Since \(f : (X, c_{1+r, d_X}) \to (Y, c_{r+1})\) is continuous, we have \(f(x_2) \in f(c_{1+r, d_X}(c_{m+1}^{d_X}(x_1))) \subset c_{r+1}(f(c_{m+1}^{d_X}(x_1))) \subset \cdots \subset c_{r+1}(f(x_1)) \subset c_{rn+1}(f(x))\) by Lemma 4.3. Thus \(d_Y(f(x_1), f(x_2)) \leq rm = rd_X(x_1, x_2)\). Hence \(f : X \to Y\) is \(r\)-Lipschitz.

A continuous map \(f : (X, c_{1+r, d_X}) \to (Y, c_{r+1})\) need not be \(r\)-Lipschitz as the following example shows.

Example 4.10. Let \(X = \{x_1, x_2\}\) be a two point metric space with distance \(d_X(x_1, x_2) = 2\). Let \(Y = \{y_1, y_2\}\) be a two point metric space with distance \(d_Y(y_1, y_2) = 5\). Let \(f : X \to Y\) be a map of sets defined by \(f(x_i) = y_i\) for \(i = 1, 2\). Then observe that \(f : (X, c_{1+r, d_X}) \to (Y, c_{2+1})\) is continuous. However, \(f\) is not a \(2\)-Lipschitz map. Indeed, \(5 = d_Y(y_1, y_2) \geq 2d_X(x_1, x_2) = 4\).

Nevertheless, in the following examples of interest to us, the stronger statement in Lemma 4.9 holds.

Example 4.11. Let \(n \in \mathbb{N}\) and let \((X, d_X)\) be any one of the following metric spaces:

- The set of \(n + 1\) many elements with pairwise distance 1.
- \(\{0, 1, \ldots, n\}\) with the absolute value metric.
- \(\{0, 1\}^n\) with the \(\ell_1\) or \(\ell_\infty\) metric, for any \(n \in \mathbb{N}\) metric.
- \(\{0, 1, \ldots, n\}\) with either \(\ell_1\) or \(\ell_\infty\) metric, for any \(m \in \mathbb{N}\).
- The natural numbers \(\mathbb{N}\) and the integers \(\mathbb{Z}\) with the absolute value metrics.

4.2. Products of metric closures. The products and inductive products of metric closures can be to some extent recovered by closures of the \(\ell_\infty\) and \(\ell_1\) metrics, respectively.

Theorem 4.12. Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces. Let \(\varepsilon \geq 0\), \(a \in \{-1, 0, 1\}\) and consider the closure spaces \((X, c_{(\varepsilon,a), d_X})\) and \((Y, c_{(\varepsilon,a), d_Y})\). Give \(X \times Y\) the \(\ell_\infty\) metric induced by \(d_X\) and \(d_Y\). Then \((X \times Y, c_{(\varepsilon,a), d_X} \times c_{(\varepsilon,a), d_Y}) = (X \times Y, c_{(\varepsilon,a), \ell_\infty})\).
Proof. We prove the case for $a = 1$. The other cases are similar. First we show that $c_{e+} \times c_{e+}$ is coarser than $c_{e+}$. We proceed by showing the projection $\pi_X : (X \times Y, \epsilon_{e+}) \rightarrow (X, \epsilon_{e+})$ is continuous. Let $A \subset X \times Y$. Let $x \in \pi_X(A)$. Thus, there is a $y \in Y$ such that $(x, y) \in \epsilon_{e+}(A)$. Thus, by definition we have:

$$\text{dist}_{e+}((x, y), A) := \inf_{(x', y') \in A} d_{e+}((x, y), (x', y')) \leq \varepsilon$$

It follows that $\inf_{x' \in \pi_X(A)} d_X(x, x') \leq \varepsilon$. Thus $x \in \epsilon_{e+}(A)$. Therefore $\pi_X$ is continuous. Similarly, $\pi_Y : (X \times Y, \epsilon_{e+}) \rightarrow (Y, \epsilon_{e+})$ is continuous. By Theorem \[1.1\] the product closure is the coarsest closure so that each projection map is continuous. Thus, $c_{e+} \times c_{e+}$ is coarser than $c_{e+}$. 

Now we show that $c_{e+} \times c_{e+}$ is coarser than $c_{e+} \times c_{e+}$. Let $A \subset X \times Y$. Now let $(x, y) \in c_{e+}(A)$. Let $\delta > 0$. Then by Definition \[1.1\] and Lemma \[1.1\] $\pi_X^{-1}(B_{\epsilon_{e+}}(x)) \cap \pi_Y^{-1}(B_{\epsilon_{e+}}(y)) \cap A \neq \emptyset$. Observe that $\pi_X^{-1}(B_{\epsilon_{e+}}(x)) \cap \pi_Y^{-1}(B_{\epsilon_{e+}}(y)) = B_{\epsilon_{e+}}(x, y)$. Thus, $\text{dist}_{e+}((x, y), A) \leq \varepsilon$. Therefore, $(x, y) \in c_{e+}(A)$ and thus $c_{e+} \times c_{e+}$ is coarser than $c_{e+}$.

**Theorem 4.13.** Let $(X, d_X)$ and $(Y, d_Y)$ be metric spaces and let $\varepsilon \geq 0$, $a = \{-1, 0, 1\}$. Consider the closure spaces $(X, c_{(e, a),d_X})$ and $(Y, c_{(e, a),d_Y})$. Give $X \times Y$ the $\ell_1$ metric induced by $d_X$ and $d_Y$. Then the closure operation $c_{(e, a),\ell_1}$ for $X \times Y$ is coarser than $c_{(e, a),d_X, d_Y}$. 

Proof. We prove the first statement for the case $a = 1$. The other cases are similar. To show that $c_{(e, \ell_1)}$ is coarser than $c_{(e, d_X, d_Y)}$ by Theorem \[3.2\] it is sufficient to show that for each $x \in X$ and each $y \in Y$ the functions $f_y : (X, c_{(e, d_X)}) \rightarrow (X \times Y, c_{(e, \ell_1)})$ and $f_x : (Y, c_{(e, d_Y)}) \rightarrow (X \times Y, c_{(e, \ell_1)})$ defined by $f_y(x') = (x', y)$ and $f_x(y') = (x, y')$ are continuous. Let $y \in Y$ and consider the mapping $f_y$. Let $A \subset X$. Let $(x, y) \in f_y(c_{(e, d_X})(A))$. Then $x \in c_{(e, d_X)}(A)$, therefore $\text{dist}_{d_X}(x, A) \leq \varepsilon$. Thus, by definition of $\ell_1$, we have that $\text{dist}_{\ell_1}((x, y), A \times \{y\}) \leq \varepsilon$. Therefore, $(x, y) \in c_{(e, \ell_1)}(f_y(U))$. Thus $f_y(c_{(e, d_X)}(A)) \subset c_{(e, \ell_1)}(f_y(A))$ and hence $f_y$ is continuous. Similarly, given $x \in X$, $f_x$ is continuous. Therefore $c_{(e, \ell_1)}$ is coarser than $c_{(e, d_X, d_Y)}$. 

Now assume that $\text{im} d_Y \subset \mathbb{N}$. We prove the second statement for the case $a = 0$. The case $a = -1$ is similar. We have that $c_{1, \ell_1}$ is coarser than $c_{1, d_X, d_Y}$. Thus we only need to show that $c_{1, d_X} \sqcup c_{1, d_Y}$ is coarser than $c_{1, \ell_1}$. Let $A \subset X \times Y$ and let $(x, y) \in c_{1, \ell_1}(A)$. Thus, there exists a $(x', y') \in A$ such that $d_{\ell_1}((x, y), (x', y')) \leq 1$. 

By Definition \[2.20\] and Lemma \[4.7\] $(X \times Y, c_{1, d_X, d_Y})$ has a local base at $(x, y)$ consisting of the element $W = \{(x) \times B_{d_Y}(y) \cup B_{d_X}(x) \times \{y\}\}$. From the previous inequality we have $d_X(x, x') + d_Y(y, y') \leq 1$. If $y \neq y'$ then, since $\text{im} d_Y \subset \mathbb{N}$ it follows that $d_Y(y, y') = 1$ and thus $x = x'$. If $y = y'$ then $d_Y(y, y') = 0$ and thus $d_x(x, x') \leq 1$. In either case we have that $(x', y') \in W$. Therefore $W \cap A \neq \emptyset$ and thus by Theorem \[4.7\] $(x, y) \in c_{1, d_X} \sqcup c_{1, d_Y}(A)$. Hence, it follows that $c_{1, d_X} \sqcup c_{1, d_Y}$ is coarser than $c_{1, \ell_1}$. 

The next two examples show that it need not be the case that $c_{e+} \sqcup c_{e+} = c_{e+} \sqcup c_{e+}$, with $\varepsilon = 1$ and $\varepsilon = 0$, respectively.

**Example 4.14.** Let $(X, d_X) = (Y, d_Y) = (\mathbb{R}, d)$ where $d$ is the absolute value metric on $\mathbb{R}$. Let $(0, 0) \in \mathbb{R}$ and consider $c_{1, \ell_1}((0, 0))$. Note that $(\frac{1}{2}, \frac{1}{2}) \in c_{1, \ell_1}((0, 0))$. However, $(\frac{1}{2}, \frac{1}{2}) \notin (c_{1, d} \sqcup c_{1, d})((0, 0))$. However, $(\frac{1}{2}, \frac{1}{2}) \notin (c_{1, d} \sqcup c_{1, d})((0, 0))$. However,
Example 4.15. Let \((X, d_X) = (Y, d_Y) = (I, d)\) where \(d\) is the absolute value metric on the unit interval \(I\). Let \(A = (0, 1) \times (0, 1) \subset I \times I\). Note that the metric \(d\) generates the topology on \(I\) where the closure operation is \(c_0^+\). By Theorem 4.12 we have \(c_0^+ \times c_0^+ = c_0^+,\ell_\infty\) and we also know from point set topology that \(c_0^+,\ell_\infty = c_0^+,\ell_1\). Thus \(c_0^+,\ell_1(A) = I \times I\). On the other hand, the point \((0, 0) \not\in c_0^+ \square c_0^+(A)\) as no neighborhood of the form \(\overline{B}_c \times \{0\} \cup \{0\} \times \overline{B}_\delta\) for any \(\varepsilon, \delta > 0\) has a non-empty intersection with \(A\). This is necessary for \((0, 0)\) to be in \(c_0^+ \square c_0^+(A)\) by Lemma 4.7, Definition 2.26, and Theorem \(A.6\). Therefore \(c_0^+,\ell_1 \neq c_0^+,d \square c_0^+,d\).

5. Homotopy in closure spaces

In this section we define various homotopy theories for closure spaces using different intervals together with either the product or the inductive product. We will study the relationships between respective homotopy relations. These homotopy theories restrict to the full subcategories of closure spaces, \(\text{Top}, \text{Cl}_{\text{qld}}, \text{Cl}_{\text{sqd}}\), where they have been previously studied under different names.

Recall that the category of sets, \(\text{Set}\), together with the cartesian product and the one point set \(*\) forms a symmetric monoidal category. In addition, there is a functor \(U : \text{Cl} \to \text{Set}\) that forgets the closure. Also, the terminal object in \(\text{Cl}\) is the one point set \(*\) with its unique closure.

5.1. Product operations. We formalize the properties of the product and the inductive product that we will need.

Definition 5.1. A product operation is a functor \(\otimes : \text{Cl} \times \text{Cl} \to \text{Cl}\) for which \(\text{Cl}\) has a symmetric monoidal category structure that commutes with the forgetful functor \(U\) and the cartesian symmetric monoidal category structure on \(\text{Set}\). That is, for all \((X, c_X), (Y, c_Y), (X, c_X) \otimes (Y, c_Y) = (X \times Y, c_X \otimes c_Y)\) for some closure operator \(c_X \otimes c_Y\) on \(X \times Y\), the unit object is the one point space \(*\) with its unique closure, and the associator, unitors, and braiding are given by those on \(\text{Set}\). In addition we assume that this product distributes over binary coproducts. That is, \(c_X \otimes (c_Y \amalg c_Z) = c_X \otimes c_Y \amalg c_X \otimes c_Z\).

Example 5.2. The product closure and inductive product closures are examples of product operations. Indeed, by their definition, \(\square\) and \(\times\) commute with the forgetful functor and distribute over binary coproducts. The unit object for both \(\square\) and \(\times\) is the one point space \(*\).

The braiding isomorphisms \(\gamma_{XY} : X \times Y \to Y \times X\) and \(\gamma_{XY} : X \boxdot Y \to Y \boxdot X\) are both given by \(\gamma_{XY}(x,y) = (y,x)\). We will show that \(\gamma_{XY} : X \times Y \to Y \times X\) is continuous. The proof that \(\gamma_{XY} : X \boxdot Y \to Y \boxdot X\) is continuous is similar. Let \(A \subset X \times Y\) and let \((x, y) \in \gamma_{XY}((c_X \times c_Y)(A))\). Let \(U_x\) and \(V_y\) be neighborhoods of \(x\) and \(y\), respectively. Then by definition, \((U_x \times V_y) \cap A \neq \emptyset\). It follows that \((V_y \times U_x) \cap \gamma_{XY}(A) \neq \emptyset\). Thus, by definition, \((y,x) \in (c_Y \times c_X)(\gamma_{XY}(A))\). Hence \(\gamma_{XY}((c_X \times c_Y)(A)) \subset (c_Y \times c_X)(\gamma_{XY}(A))\). Therefore \(\gamma_{XY}\) is continuous.

The continuity of the associator \(a_{XYZ} : (X \times Y) \times Z \to X \times (Y \times Z)\) follows from observing that \((X \times Y) \times Z\) and \(X \times (Y \times Z)\) share a local base at each \((x, y, z)\) consisting of sets of the form \((U_x \times U_y \times U_z)\) where \(U_x, U_y, U_z\) are neighborhoods of \(X, Y\) and \(Z\) respectively.

The continuity of the associator \(a_{XYZ} : (X \boxdot Y) \boxdot Z \to X \boxdot (Y \boxdot Z)\) follows from observing that \((X \boxdot Y) \boxdot Z\) and \(X \boxdot (Y \boxdot Z)\) share a local base at each \((x, y, z)\) consisting of sets of
the form $U_x \times \{y, z\} \cup \{x\} \times U_y \times \{z\} \cup \{x, y\} \times U_z$ where $U_x, U_y, U_z$ are neighborhoods of $x, y$ and $z$ respectively.

Let $A \subset X$. Since $(c_x \times c_X)(* \times A) = * \times c_X(A)$, the left unitor $\lambda_X : * \times X \to X$ is continuous. Similarly, the left unitor $\lambda_X : * \sqcap X \to X$ is continuous and both right unitors are continuous. The triangle and pentagon identities are satisfied for both $\times$ and $\sqcap$ since they are satisfied for the underlying sets and all the maps in question are continuous.

5.2. Intervals. We formalize the structure of an interval which we will use to develop a homotopy theory and give examples.

Definition 5.3. Let $\otimes$ be a product operation. A interval for $\otimes$ is a closure space $J$ together with two maps $* \to J \leftarrow *$ for which the map $0 \coprod 1 : * \sqcap * \to J$ is one-to-one and a symmetric, associative morphism $\lor : J \otimes J \to J$ which has 0 as its neutral element and 1 as its absorbing element. That is, if we write $s \lor t$ for $\lor(s, t)$ then $0 \lor t = t$ and $1 \lor t = 1$. A morphism of intervals is a map of intervals $f : J \to K$ that preserves the distinguished points and commutes with the associative morphisms. That is, $f0_J = 0_K$, $f1_J = 1_K$, and $f(s \lor t) = f(s) \lor_K f(t)$.

Example 5.4. In each of the following, we show that the maximum map $J \times J \to J$ gives us an interval $J$ for $\times$. By Corollary 5.8 and Lemma 5.9 it follows that $J$ is also an interval for any product operation $\otimes$.

(1) $I$ with the inclusions of 0 and 1 (Definition 2.24) is an interval for $\times$. It is elementary to show that the map $I \times I \to I$ given by the maximum function is a continuous map of topological spaces.

(2) For $m \geq 1$ and $0 \leq k \leq 2^m - 1$, each of $J_{m, \perp}$, $J_{m, \top}$, $J_m$, $J_m, J_m, J_m$, and $J_{m,k}$ with the inclusions of the points 0 and $m$ (Definition 2.24) and the maximum map is an interval for $\times$. We will show that the maximum map is continuous in each case.

(a) Since $J_{m, \perp} \times J_{m, \perp}$ is a discrete space, the maximum map is continuous.

(b) Similarly, since $J_{m, \top}$ is an indiscrete space, the maximum map is continuous.

(c) Suppose $(s', t') \in c(s, t)$. Then $|s - s'| \leq 1$ and $|t - t'| \leq 1$. Therefore $|\max(s, t) - \max(s', t')| \leq 1$ and thus $\max(s', t') \in c(\max(s, t))$.

(d) Suppose that $(s', t') \in c(s, t)$. Then $s \leq s'$ and $t \leq t'$. It follows that $\max(s, t) \leq \max(s', t')$ and thus $\max(s', t') \in c(\max(s, t))$.

(e) Suppose that $(s', t') \in c(s, t)$. Then $s' \in c(s)$ and $t' \in c(t)$. If $\max(s, t) = s$ and $\max(s', t') = s'$ or $\max(s, t) = t$ and $\max(s', t') = t'$ then $\max(s', t') \in c(\max(s, t))$. If not, then $s' = t$ and $t' = s$ and we also have that $\max(s', t') \in c(\max(s, t))$.

(3) Let $\varepsilon \in [0, 1]$. Then $I_{\varepsilon-}$, $I_{\varepsilon}$, and $I_{\varepsilon+}$ with the inclusions of the points 0 and 1 (Example 4.2) and the maximum map is an interval for $\times$. We give the proof the case $I_{\varepsilon+}$. The other cases are similar. Let $A \subset I$ and suppose that $(x, y) \in (c_{\varepsilon+} \times c_{\varepsilon+})(A)$. Then by Lemma 4.7 for all $\delta_1, \delta_2 > 0$, $(\overline{B}_{\varepsilon+\delta_1}(x) \times \overline{B}_{\varepsilon+\delta_2}(y)) \cap A \neq \emptyset$. Thus for all $n \geq 1$, there exists $(x_n, y_n) \in A$ such that $|x - x_n| \leq \varepsilon + \frac{1}{n}$ and $|y - y_n| \leq \varepsilon + \frac{1}{n}$. It follows that $|\max(x, y) - \max(x_n, y_n)| \leq \varepsilon + \frac{1}{n}$. Therefore $d(\max(x, y), \max(A)) \leq \varepsilon$. Hence $\max(x, y) \in c_{\varepsilon+}(\max(A))$.

5.3. Relations between product operations and intervals. We study relations between product operations and between intervals which will later give us relations between their corresponding homotopy theories.
Definition 5.5. Define a partial order on product operations by setting \( \otimes_1 \leq \otimes_2 \) if there exists natural transformation \( \alpha \) from \( \otimes_1 \) to \( \otimes_2 \) such that for all closure spaces \( X \) and \( Y \), \( \alpha_{(X,Y)} : X \otimes_1 Y \to X \otimes_2 Y \) is the identity map.

Lemma 5.6. Let \( \otimes \) be a product operation. Then the projection maps \( \pi_X : X \otimes Y \to X \) and \( \pi_Y : X \otimes Y \to Y \) are continuous.

Proof. We show the statement for \( \pi_X \). The \( \pi_Y \) case is similar. Let \( 1_X : X \to X \) be the identity and \( a : Y \to * \) be the constant map. Then, by functoriality, we have that \( 1_X \otimes a : X \otimes Y \to X \otimes * \) is continuous. The result follows from composing with the right unitor isomorphism, \( X \otimes * \cong X \).

Lemma 5.7. Let \( \otimes \) be a product operation. For every pair of spaces \((X,Y)\), for every \( x \in X \), the map \( x : Y \to X \otimes Y \) given by \( x(y) = (x,y) \) and for every \( y \in Y \), the map \( y : X \to X \otimes Y \) given by \( y(x) = (x,y) \) is continuous.

Proof. Let \( x \in X \). Consider the continuous maps \( x : * \to X \) given by \( x(*) = x \) and \( 1_Y : Y \to Y \), the identity map. By functoriality, \( x \otimes 1_Y : * \otimes Y \to X \otimes Y \) is continuous. By precomposing with the right unitor isomorphism \( Y \otimes * \to Y \), we get that \( x : Y \to X \otimes Y \) is continuous. Similarly, for \( y \in Y \), the map \( y : X \to X \otimes Y \) is continuous.

From Theorems 3.1 and 3.2 and Lemmas 5.6 and 5.7 we get the following.

Corollary 5.8. Let \( \otimes \) be a product operation. Then \( \sqcap \leq \otimes \leq \times \).

Together with Lemma 2.28, we then also have that for a product operation \( \otimes \) and \( X \) discrete, \( X \otimes Y = X \times Y \).

Lemma 5.9. Suppose \( \otimes_1 \leq \otimes_2 \). If \( J \) is an interval for \( \otimes_2 \) then \( J \) is also an interval for \( \otimes_1 \).

Proof. Let \( \vee : J \otimes_2 J \to J \) be the associative morphism for \( J \) as an interval for \( \otimes_2 \). Precomposing \( \vee \) with the set-theoretic identity map \( J \otimes_1 J \to J \otimes_2 J \), which is continuous by the assumption that \( \otimes_1 \leq \otimes_2 \), we get an associative morphism \( \vee : J \otimes_1 J \to J \).

Definition 5.10. Define a preorder on intervals for \( \otimes \) by setting \( J \leq K \) if there exists a morphism of intervals \( f : J \to K \).

Lemma 5.11. Let \( J \) be an indiscrete space with at least two elements. Let \( 0 \coprod 1 : * \coprod * \to J \) be a one-to-one map, whose image are the points \( 0_J \) and \( 1_J \). Then \( J \) is an interval for any product operation \( \otimes \). Furthermore, for any interval \( K \) for \( \otimes \), \( K \leq J \).

Proof. By Corollary 5.8 and Lemma 5.9, it is sufficient to prove \( J \) is an interval for \( \times \). Define a symmetric associative morphism \( \vee_J : J \times J \to J \) by
\[
s \vee_J t = \begin{cases} 
0_J, & (s,t) = (0_J,0_J) \\
1_J, & \text{otherwise}
\end{cases}
\]
Note that \( \vee_J \) is continuous since \( J \) is indiscrete and \( 0_J \vee_J t = t \) and \( 1_J \vee_J t = 1_J \) for all \( t \in J \). Thus \( J \) is an interval for \( \times \). Now let \( K \) be an interval for \( \otimes \). Define a map \( f : K \to J \) by \( f(s) = 1_J \) if \( s \neq 0_K \) and \( f(0_K) = 0_J \). The map \( f \) is continuous since \( J \) is indiscrete. Furthermore, \( f(s \vee_K t) = f(s) \vee_J f(t) \) by the definition of \( f \) and thus \( f \) is a morphism of intervals.
Example 5.12.  
(1) Let $m \geq 1$ and $0 \leq k \leq 2^m - 1$. Recall (Lemma 2.36) that we have continuous maps $I \xrightarrow{f} J_{m,k} \xrightarrow{1} J_m$. These maps respect the inclusion of the two distinguished points and also commute with taking maximums. Thus they are morphisms of intervals and we write $I \leq J_{m,k} \leq J_m$. In particular, $I \leq J_+ \leq J_1$. 

(2) Let $(\varepsilon,a) \leq (\varepsilon',a')$. By Lemma 4.3, the identity map gives a morphism of intervals $I(\varepsilon,a) \to I(\varepsilon',a')$. Therefore $I(\varepsilon,a) \leq I(\varepsilon',a')$.

(3) Let $1 \leq m \leq n$. There is a morphism of intervals $J_n \to J_m$ given by $i \mapsto \min(i,m)$. Therefore $J_n \leq J_m$. The same map gives morphisms of intervals $J_{n,\bot} \to J_{m,\bot}$, $J_{n,\top} \to J_{m,\top}$, and $J_{n,\leq} \to J_{m,\leq}$. Therefore $J_{n,\bot} \leq J_{m,\bot}$, $J_{n,\top} \leq J_{m,\top}$, and $J_{n,\leq} \leq J_{m,\leq}$. Thus $J_{n,\bot} \leq J_{m,\bot}$ and $J_{n,\top} \leq J_{m,\top} \leq J_{n,\leq} \leq J_{m,\leq}$. In particular, $J_m \leq J_1$, $J_{m,\bot} \leq J_{1,\bot}$, $J_{m,\top} \leq J_{1,\top}$, and $J_{m,\leq} \leq J_{1,\leq}$.

(4) Let $1 \leq m \leq n$. Let $0 \leq k \leq 2^m - 1$ and $0 \leq \ell \leq 2^n - 1$, where the binary representation of $k$ is the first $m$ rightmost bits of the binary representation of $\ell$. Then the map $i \mapsto \min(i,m)$ gives a morphism on intervals $J_{n,\ell} \to J_{m,k}$. Therefore $J_{n,\ell} \leq J_{m,k}$. In particular, for any $m \geq 1$ and $0 \leq k \leq 2^m - 1$, either $J_{m,k} \leq J_+$ or $J_{m,k} \leq J_-$.

(5) Let $m \geq 1$. There is a morphism of intervals $J_{1,\leq} \to J_{m,\leq}$ given by $0 \mapsto 0$ and $1 \mapsto m$. The same map gives morphisms of intervals $J_{1,\bot} \to J_{m,\bot}$ and $J_{1,\top} \to J_{m,\top}$. Therefore $J_+ = J_{1,\leq} \leq J_{m,\leq}$, $J_{1,\bot} \leq J_{m,\bot}$, and $J_1 = J_{1,\top} \leq J_{m,\top}$.

(6) Let $m \geq 1$. We have morphisms of intervals $i : J_m \to ([0,m],c_1)$ and $r : ([0,m],c_1) \to J_m$ (see Lemma 4.6). Furthermore, by rescaling we have $([0,m],c_1) \cong ([0,1],c_{\frac{1}{m}})$ and this homeomorphism is given by inverse morphisms of intervals. Thus we have morphisms of intervals $J_m \to I_{\frac{1}{m}}$ and $I_{\frac{1}{m}} \to J_m$. Therefore $J_m \leq I_{\frac{1}{m}}$ and $I_{\frac{1}{m}} \leq J_m$.

(7) Let $f : I_0 \to J_{1,\bot}$ be the map of sets given by rounding up (see Lemma 2.36). This map is continuous because $I_0$ is discrete and it respects the structure of the intervals. Therefore it is a morphism of intervals and $I_0 \leq J_{1,\bot}$.

Lemma 5.13. If $J$ and $J'$ are intervals for $\otimes$, then $J \otimes J'$ is also an interval for $\otimes$. Furthermore, $J \otimes J' \leq J, J'$.

Proof. By assumption there exist symmetric, associative morphisms $\vee_J : J \otimes J \to J$, $\vee_{J'} : J' \otimes J' \to J'$ which have $0_J$, $0_{J'}$ as their neutral elements and $1_J$, $1_{J'}$ as their absorbing elements. Define $\vee_{J \otimes J'} : (J \otimes J') \otimes (J \otimes J') \to J \otimes J'$ by

$$
(q,r) \vee_{J \otimes J'} (s,t) = (q \vee_J s, r \vee_{J'} t).
$$

Then $\vee_{J \otimes J}$ is symmetric and associative and has $0_J \otimes 0_{J'}$ as its neutral element and $1_J \otimes 1_{J'}$ as its absorbing element. Also, the projection $\pi_J : J \otimes J' \to J$ is continuous by Lemma 5.6. Furthermore, $\pi_J(0_J,0_{J'}) = 0_J$ and $\pi_J(1_J,1_{J'}) = 1_J$ and $\pi_J((q,r) \vee_{J \otimes J'} (s,t)) = \pi_J((q \vee_J s, r \vee_{J'} t)) = q \vee_J s = \pi_J(q,r) \vee_J \pi_J(s,t)$ and thus $\pi_J$ is a morphism of intervals. Similarly, the projection $\pi_{J'} : J \otimes J' \to J'$ is a morphism of intervals. \hfill $\Box$

Definition 5.14. Let $J$ and $J'$ be interval objects for $\otimes$. Let $J \ast J'$ be the pushout of the diagram $\ast \xrightarrow{1} J$ and $\ast \xrightarrow{0} J'$.

$$
\begin{array}{ccc}
\ast & \xrightarrow{1} & J \\
\downarrow & & \downarrow \\
J' & \rightarrow & J \ast J'
\end{array}
$$

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We call \(J*J'\) the *concatenation of \(J\) and \(J'\). In the case that \(J = J'\), the \(m\)-fold concatenation of \(J\) with itself will be denoted by \(J^m\).

**Lemma 5.15.** If \(J\) and \(J'\) are intervals for \(\otimes\) then so is \(J*J'\).

*Proof.* By assumption there are symmetric, associative morphisms \(\vee_J : J \otimes J \to J\) and \(\vee_{J'} : J' \otimes J'\) which have \(0_J, 0_{J'}\) and \(1_J, 1_{J'}\) as their neutral elements and absorbing elements, respectively. Define \(\vee_{J*J'} : (J*J') \otimes (J*J') \to J*J'\) by

\[
\begin{cases}
    s \vee_{J*J'} t, & s, t \in J \\
    s \vee_{J'} t, & s, t \in J' \\
    t, & s \in J, t \in J' \\
    s, & s \in J', t \in J.
\end{cases}
\]

By the Pasting Lemma (Theorem 2.14), \(\vee_{J*J'}\) is continuous. One may also check that it is symmetric, associative, that \(0_{J'}\) is a neutral element, and that \(1_J\) is an absorbing element. \(\square\)

**Lemma 5.16.** Let \(J\) and \(J'\) be intervals for \(\otimes\). Then \(J*J' \leq J \otimes J'\).

*Proof.* There is a unique map \(\psi : J*J' \to J \otimes J'\) given by the pushout diagram

\[
\begin{array}{ccc}
J & \xrightarrow{f} & J \otimes J' \\
\downarrow & & \downarrow \psi \\
J*J' & \xrightarrow{\psi} & J \otimes J'
\end{array}
\]

where \(f\) and \(g\) are given by \(f(s) = (s,0_{J'})\) and \(g(s) = (1_J,s)\), and are continuous by Lemma 5.7. Then \(\psi(0_J) = f(0_J) = (0_J,0_{J'})\) and \(\psi(1_J) = g(1_J) = (1_J,1_{J'})\). Furthermore, we claim that \(\psi(s \vee_{J*J'} t) = \psi(s) \vee_{J \otimes J'} \psi(t)\). There are four cases. We will check two. The other two are similar. If \(s, t \in J\) then \(\psi(s \vee_{J*J'} t) = \psi(s \vee_J t) = f(s \vee_J t) = (s \vee_J t, 0_{J'}) = (s,0_{J'}) \vee_{J \otimes J'} (t,0_{J'}) = f(s) \vee_{J \otimes J'} f(t) = \psi(s) \vee_{J \otimes J'} \psi(t)\). If \(s \in J\) and \(t \in J'\) then \(\psi(s \vee_{J*J'} t) = \psi(t) = g(t) = (1_J,t) = (s,0_{J'}) \vee_{J \otimes J'} (1_J,t) = f(s) \vee_{J \otimes J'} g(t) = \psi(s) \vee_{J \otimes J'} \psi(t)\). Thus \(\psi\) is a morphism of intervals. \(\square\)

**Corollary 5.17.** Let \(J\) and \(J'\) be intervals for \(\otimes\). Then \(J*J' \leq J \otimes J'\).

*Proof.* Combine Lemmas 5.13 and 5.16. \(\square\)

### 5.4. Homotopy

An interval and a product operation give rise to a homotopy.

**Lemma 5.18.** For a closure space \(X\) and product operation \(\otimes\), \(X \otimes (\star \coprod \star) \cong X \coprod X\).

*Proof.* For the underlying sets, \(X \times (\star \coprod \star) \cong X \coprod X\) and \(c_X \otimes (c_x \coprod c_x) = c_X \coprod c_X\) by assumption. \(\square\)

**Lemma 5.19.** For a product operation \(\otimes\) and an interval \(J\) for \(\otimes\), we have the following natural map for a closure space \(X\):

\[
X \coprod X \cong X \otimes (\star \coprod \star) \xrightarrow{1_X \otimes ([0][1])} X \otimes J
\]
Lemma 5.23. Let \( f, g : X \to Y \in \text{Cl} \). Let \( \otimes \) be a product operation and let \( J \) be an interval for \( \otimes \). We say that \( f \) and \( g \) are one-step \((J, \otimes)\)-homotopic if there exists a morphism map \( H \) such that the following diagram commutes.

\[
\begin{array}{ccc}
X \sqcup X & \xrightarrow{f \sqcup g} & Y \\
\downarrow & \nearrow \quad H \\
X \otimes J
\end{array}
\]

Call \( H \) a one-step \((J, \otimes)\) homotopy. Let \( \sim_{(J, \otimes)} \) be the equivalence relation on the set \( \text{Cl}(X, Y) \) generated by one-step \((J, \otimes)\) homotopies. If \( f \sim_{(J, \otimes)} g \), say \( f \) and \( g \) are \((J, \otimes)\)-homotopic. Call the equivalence relation, \( \sim_{(J, \otimes)} \), \((J, \otimes)\)-homotopy.

Lemma 5.21. Let \( \otimes \) be a product operation and \( J \) an interval for \( \otimes \). Then \( \vee \) is a one-step \((J, \otimes)\) homotopy between the identity map on \( J \) and the composite map \( J \to * \xrightarrow{1} J \).

**Proof.** By definition, for all \( s \in I \), \( s \vee 0 = s \) and \( s \vee 1 = 1 \). The result follows. \( \square \)

Example 5.22. Let \( \otimes \) be a product operation. Since \( J_{1,1} \equiv \star \sqcup \star \), \( X \otimes J_{1,1} \equiv X \sqcup X \). Therefore, for any \( f, g : X \to Y \in \text{Cl} \), \( f \sim_{(J_{1,1}, \otimes)} g \).

Lemma 5.23. Let \( \otimes \) be a product operation and \( J \) be an interval for \( \otimes \). If \( f \sim_{(J, \otimes)} g \) and \( h \sim_{(J_{1}, \otimes)} k \) then \( f \otimes h \sim_{(J_{1} \otimes J_{1}, \otimes)} g \otimes k \).

**Proof.** It is sufficient to assume that \( f \) and \( g \) and \( h \) and \( k \) are one-step \((J, \otimes)\)-homotopic. Let \( H \) and \( F \) be one-step \((J, \otimes)\)-homotopies between \( f \) and \( g \) and \( h \) and \( k \), respectively. Then \( H \otimes F \) is a one-step \((J \otimes J, \otimes)\)-homotopy between \( f \otimes g \) and \( h \otimes k \). \( \square \)

Lemma 5.24. Let \( X \) be a closure space. Then the diagonal map \( \Delta : X \to X \times X \) is continuous.

**Proof.** Let \( A \subset X \). Let \( x \in c(A) \). Then, for each neighborhood \( U \) of \( x \), \( U \cap A \neq \emptyset \). It follows that \( (U \times U) \cap \Delta(A) \neq \emptyset \). Therefore \( (x, x) \in (c \times c)(\Delta(A)) \). \( \square \)

Lemma 5.25. Let \( J \) be an interval object for \( \times \). Then \( \Delta : J \to J \times J \) is a morphism of intervals. In particular, \( J \leq J \times J \).

**Proof.** By Lemma 5.24 \( \Delta \) is continuous. Furthermore \( \Delta(0) = (0, 0) \), \( \Delta(1) = (1, 1) \) and \( \Delta(s \lor t) = \Delta(s) \lor \Delta(t) \). \( \square \)

Example 5.26. Let \( X \) be a closure space. The diagonal map \( \Delta : X \to X \sqcup X \) need not be continuous. Consider the following. Let \( X \) be \( J_{1} \) or \( J_{+} \). Then \( 1 \in c(0) \) but \( (1, 1) \notin c(0, 0) \). Let \( X = I \). Consider \( A = [0, 1] \). Then \( 1 \in c(A) \) but \( (1, 1) \notin c(\Delta(A)) \).

We end this section with a characterization of one-step \((J_{1}, \times)\)-homotopy.

Lemma 5.27. Let \( f, g : (X, c) \to (Y, d) \in \text{Cl} \). Then \( f, g \) are one-step \((J_{1}, \times)\)-homotopic iff for all \( A \subset X \), \( f(c(A)) \cup g(c(A)) \subset d(f(A)) \cap d(g(A)) \).

**Proof.** Let \( e \) denote the indiscrete closure on \( J_{1} \). \( \Rightarrow \) Let \( H : X \times J_{1} \to Y \) be a one-step \((J_{1}, \times)\)-homotopy from \( f \) to \( g \). For all \( x \in X \), \( H(x, 0) = f(x) \) and \( H(x, 1) = g(x) \). Let \( A \subset X \). Then \( f(c(A)) \cup g(c(A)) = H(c(A) \times J_{1}) = H((c \times e)(A \times 0)) \subset d(H(A \times 0)) = d(f(A)) \). Similarly \( f(c(A)) \cup g(c(A)) \subset d(g(A)) \).

}\( \square \)
\((\Leftarrow)\) Define \(H : X \times J_1 \to Y\) by \(H(x,0) = f(x)\) and \(H(x,1) = g(x)\). Let \(A \subset X \times J_1\). Let \(A_0 = \{(x,0) \in A\}\) and \(A_1 = \{(x,1) \in A\}\). Then \(H((c \times e)(A)) = H((c \times e)(A_0 \cup A_1)) = H((c \times e)(A_0) \cup H((c \times e)(A_1)) = f(c(A_0) \cup g(c(A_0) \cup f(c(A_1)) \cup g(c(A_1)) \subset d(f(A_0)) \cap d(g(A_0)) \cup d(f(A_1)) \cap d(g(A_1)) \subset d(f(A_0)) \cup d(g(A_1)) = d(H(A_0)) \cup d(H(A_1)) = d(H(A_0) \cup H(A_1)) = d(H(A_0 \cup A_1)) = d(H(A))\). □

As a special case, consider \((X, E), (Y, F) \in \text{DiGph}\). Recall that \(f : X \to Y\) is a digraph homomorphism iff whenever \(xE.x'\) we have that \(fxFfx'\). Note that \(J_1\) is the complete digraph on \(\{0, 1\}\) and \((x, i)E \times J_1(x', j)\) iff \(xE.x'\).

**Lemma 5.28.** Let \(f, g : (X, E) \to (Y, F) \in \text{DiGph}\). Then \(f, g\) are one-step \((J_1, \times)\)-homotopic iff whenever \(xE.x'\) we have that \(fxFfx'\).

**Proof.** A map \(H : X \times J_1 \to Y\) with \(H(x, 0) = f(x)\) and \(H(x, 1) = g(x)\) is a digraph homomorphism iff whenever \(xE.x'\) we have that for all \(i, j \in J_1\), \(H(x, i)FH(x', j)\). □

5.5. **Relations between homotopy equivalences.** We study relations between our homotopy theories.

**Lemma 5.29.** Let \(\otimes\) be a product operation and let \(J\) be an interval for \(\otimes\). Let \(f, g : X \to Y \in \text{Cl}\). Then \(f \sim_{(J, \otimes)} g\) iff \(f\) and \(g\) are one-step \((J^m, \otimes)\) homotopic for some \(m \geq 1\).

**Proof.** Suppose that \(f, g\) are \((J, \otimes)\)-homotopic. A \((J, \otimes)\)-homotopy is obtained by the symmetric and transitive closure of the one-step \((J, \otimes)\)-homotopy. Thus for some \(m \geq 1\), there is a finite sequence \(f = f_0, f_1, \ldots, f_m = g\) of maps where consecutive maps \(f_i, f_{i+1}\) or \(f_{i+1}, f_i\) are one-step \((J, \otimes)\)-homotopic. We may concatenate the homotopies to obtain a homotopy \(H : X \otimes J^m \to Y\) between \(f\) and \(g\). In the reverse direction, we may restrict a one-step \((J^m, \otimes)\)-homotopy to obtain a sequence of one-step \((J, \otimes)\)-homotopies. □

**Corollary 5.30.** Let \(\otimes\) be a product operation. Let \(f, g : X \to Y \in \text{Cl}\).

1. \(f, g\) are \((I, \otimes)\)-homotopic iff \(f, g\) are one-step \((I, \otimes)\)-homotopic.

2. \(f, g\) are \((J_1, \otimes)\)-homotopic iff there exists \(m \geq 1\) such that \(f, g\) are one-step \((J_m, \otimes)\)-homotopic.

3. \(f, g\) are \((J_+, \otimes)\)-homotopic iff there exists \(m \geq 1\) and \(0 \leq k \leq 2^m - 1\) such that \(f, g\) are one-step \((J_{m,k}, \otimes)\)-homotopic.

**Proof.** All the cases follow from Lemma 5.29. In the special case (1) note that a sequence of \(m\) concatenations of \(I\) with itself gives us a one step \(([0, m], \otimes)\)-homotopy, which we can reparametrize to get a one-step homotopy \(H : X \otimes I \to Y\). □

**Proposition 5.31.** Let \(\otimes_1, \otimes_2\) be product operations with \(\otimes_1 \leq \otimes_2\) and let \(J, K\) be intervals for \(\otimes_2\) such that \(J \leq K\). If \(f, g : X \to Y\) are one-step \((K, \otimes_2)\)-homotopic then they are also one-step \((J, \otimes_1)\)-homotopic. (By Lemma 5.29, \(J, K\) are intervals for \(\otimes_1\).)

**Proof.** Let \(h : J \to K\) be a morphism of intervals. Consider the following diagram.

\[
\begin{array}{ccc}
X \otimes_1 \ast \biguplus_J \ast & \xrightarrow{\times_{J_1(0, J_1)}} & X \otimes_2 \ast \biguplus_J \ast \\
1 \otimes_{1(0, J_1)} & \downarrow & 1 \otimes_{2(0, J_1)} \\
X \otimes_1 J & \xrightarrow{1} & X \otimes_2 J \\
\end{array}
\]

\[
\begin{array}{ccc}
X \biguplus J & \xrightarrow{\biguplus g} & X \biguplus Y \\
1 \otimes_{2(0, K)} \biguplus 1 \otimes_{2(0, K)} & \downarrow & 1 \otimes_{2h} \\
X \otimes_2 J & \xrightarrow{1 \otimes_{2h}} & X \otimes_2 K \\
\end{array}
\]

\[
\begin{array}{ccc}
X \biguplus X & \xrightarrow{\biguplus f} & X \biguplus X
\end{array}
\]
The left square commutes by the natural transformation $\otimes_1 \Rightarrow \otimes_2$. The middle triangle commutes because $h$ is a morphism on intervals. If $f$ and $g$ are one-step $(K, \otimes_2)$-homotopic then there exists a map $H$ such that right triangle commutes. It follows that $f$ and $g$ are one-step $(J, \otimes_1)$-homotopic.

\textbf{Corollary 5.32.} Let $J, K$ be intervals for $\otimes_2$ with $J \leq K$ and let $\otimes_1, \otimes_2$ be product operations with $\otimes_1 \leq \otimes_2$. Let $f, g : X \to Y \in \text{Cl}$. If $f \sim_{(K, \otimes_2)} g$ then $f \sim_{(J, \otimes_1)} g$.

\textbf{Corollary 5.33.} Let $J$ be an interval for $\times$. If $f \sim_{(J, \times)} g$ and $h \sim_{(J, \times)} k$ then $f \times h \sim_{(J, \times)} g \times k$.

\textit{Proof.} By Lemma 5.23, $f \times h \sim_{(J \times J, \times)} g \times k$. The result follows from Lemma 5.25 and Corollary 5.32.

The proof of the general case, which we give next, is more difficult since in general we do not have a diagonal map $\Delta : X \to X \otimes X$. We thank Jamie Scott for suggesting to use concatenations of intervals instead.

\textbf{Corollary 5.34.} Let $J$ be an interval for $\otimes$. If $f \sim_{(J, \otimes)} g$ and $h \sim_{(J, \otimes)} k$ then $f \otimes h \sim_{(J, \otimes)} g \otimes k$.

\textit{Proof.} We have by Lemma 5.23 that $f \otimes h \sim_{(J \otimes J, \otimes)} g \otimes k$. By Lemma 5.16 and Proposition 5.30, this implies that $f \otimes h \sim_{(J \times J, \otimes)} g \otimes k$. The result then follows by Lemma 5.29.

\textbf{Theorem 5.35.} Let $m \geq 1$, $0^+ < (\varepsilon, a) \leq 1$ and let $\otimes$ be a product operation. Let $f, g : X \to Y$ be a continuous map of closure spaces. The following are equivalent.

\begin{enumerate}
  \item $f, g$ are $(J_1, \otimes)$-homotopic.
  \item $f, g$ are $(J_m, \otimes)$-homotopic.
  \item $f, g$ are $(J_{m, T}, \otimes)$-homotopic.
  \item $f, g$ are $(I_{(\varepsilon, a)}, \otimes)$-homotopic.
\end{enumerate}

\textit{Proof.} By Example 5.12(3), $J_m \leq J_1$. By Corollary 5.32, (1) implies (2). By Corollary 5.30(2), if $f, g$ are one-step $(J_m, \otimes)$-homotopic then $f, g$ are $(J_1, \otimes)$-homotopic. If $f, g$ are $(J_m, \otimes)$-homotopic then they are connected by a finite sequence of one-step $(J_m, \otimes)$-homotopies. So they are $(J_1, \otimes)$-homotopic. Thus (2) implies (1).

By Example 5.12(3), $J_{m, T} \leq J_1$. By Corollary 5.32, (1) implies (3). Similarly, by Example 5.12(4), $J_1 \leq J_{m, T}$ and thus (3) implies (1).

Choose integers $n$, $N$, with $1 \leq n \leq N$ such that $\frac{1}{N} \leq (\varepsilon, a) \leq \frac{1}{n}$. By Example 5.12(2) and Example 5.12(6), $J_N \leq I_{(\varepsilon, a)} \leq I_{(\varepsilon, a)} \leq J_n$. Thus (4) implies that $f, g$ are $(J_N, \otimes)$-homotopic, which we have shown implies that they are $(J_1, \otimes)$-homotopic. In addition, we have shown that (1) implies that $f, g$ are $(J_{m, T}, \otimes)$-homotopic and since $I_{(\varepsilon, a)} \leq J_n$ this implies (4).

\textbf{Theorem 5.36.} Let $m \geq 1$ and $0 \leq k \leq 2^m - 1$ and let $\otimes$ be a product operation. Let $f, g : X \to Y$ be continuous maps of closure spaces. The following are equivalent.

\begin{enumerate}
  \item $f, g$ are $(J_+ , \otimes)$-homotopic.
  \item $f, g$ are $(J_{m, k}, \otimes)$-homotopic.
  \item $f, g$ are $(J_{m, k}, \otimes)$-homotopic.
\end{enumerate}
Proof. By Example 5.12(3), \( J_{m, \leq} \leq J_+ \). By Corollary 5.32, (1) implies (2). Similarly, by Example 5.12(3), \( J_{\leq} \leq J_{m, \leq} \) and thus (2) implies (1). By Corollary 5.30(3), if \( f, g \) are one-step \((J_{m, k}, \otimes)\)-homotopic then they are \((J_+, \otimes)\)-homotopic. If \( f, g \) are \((J_{m, k}, \otimes)\)-homotopic then they are connected by a sequence of one-step \((J_{m, k}, \otimes)\)-homotopies. So they are \((J_+, \otimes)\)-homotopic. That is, (3) implies (1). Finally, note that the equivalence relation generated by one-step \((J_+, \otimes)\)-homotopy equals the equivalence relation generated by one-step \((J_-, \otimes)\)-homotopy. Since by Example 5.12(1) either \( J_{m, k} \leq J_+ \) or \( J_{m, k} \leq J_- \), it follows that (1) implies (3).

**Theorem 5.37.** Let \( m \geq 1 \) and let \( \otimes \) be a product operation. Let \( f, g : X \to Y \) be continuous maps of closure spaces. Then

- (1) \( f, g \) are \((J_{1, \perp}, \otimes)\)-homotopic,
- (2) \( f, g \) are \((J_{m, \perp}, \otimes)\)-homotopic, and
- (3) \( f, g \) are \((I_0, \otimes)\)-homotopic.

Proof. Example 5.22 implies (1). By Example 5.12(3), \( J_{m, \perp} \leq J_{1, \perp} \). So by Corollary 5.32, (1) implies (2). Similarly, by Example 5.12(7), \( I_0 \leq J_{1, \perp} \) and hence (1) implies (3).

**Theorem 5.38.** Let \( X \) and \( Y \) be closure spaces. Let \( f, g : X \to Y \) be continuous maps. Then we have the following implications

\[
\begin{align*}
\sim_{(J_0, \times)} & \implies \sim_{(J_+, \times)} & \sim_{(I_0, \times)} & \implies \sim_{(I_+, \times)} \\
\sim_{(J_0, \Box)} & \implies \sim_{(J_+, \Box)} & \sim_{(I_0, \Box)} & \implies \sim_{(I_+, \Box)} 
\end{align*}
\]

Furthermore, the relation \( \sim_{(J_0, \times)} \) is maximal; it implies any possible homotopy relation induced by any interval \( J \) for any product operation \( \otimes \). In other words, the relation \( \sim_{(J_0, \times)} \) is the “finest” homotopy relation we can have in the category of closure spaces.

Proof. Combining Corollaries 5.8 and 5.32 gives the vertical implications. The horizontal implications follow from Example 5.12(1) and Corollary 5.32. The maximality of \( \sim_{(J_0, \times)} \) follows from Corollaries 5.8 and 5.32 and Lemma 5.11.

**Example 5.39.** Let \( X \) be the two point discrete space. Then the identity map \( 1_X \) is not \((I, \Box)\) homotopic to a constant map. Indeed, if it was then we would have a homotopy \( H : X \Box I \to X \) between the two maps. However, since \( X \) is discrete, by Lemma 2.28 this would be equivalent to asking for a homotopy \( H : X \times I \to X \) between the two maps, which we know does not exist since the two point discrete space \( X \) is not \((I, \times)\)-contractible.

The following proposition is an immediate consequence of Example 5.39 and Theorem 5.38.

**Proposition 5.40.** For each of the equivalence relations in Theorem 5.38 there exist \( f, g : X \to Y \) such that \( f \) is not homotopy equivalent to \( g \).

**Example 5.41.** Consider \((\mathbb{Z}, d)\) where \( d(x, y) = |x-y| \). Then \((\mathbb{Z}, c_1)\) is \((I, \times)\)-contractible [33, Lemma 4.49]. On the other hand, the space \((\mathbb{Z}, c_1)\) is not \((J_+, \Box)\) contractible. Indeed, suppose that \( f_0, \ldots, f_m \) are a sequence of consecutive pairwise one-step \((J_+, \Box)\)-homotopic maps where \( f_0 = 1_Z \) is the identity map and \( f_m = 0 \) is the constant map to 0. Note that the definition of \( \Box \) and \( J_+ \) implies that \( f_i \) and \( f_{i+1} \) are homotopic if and only if for each \( n \in \mathbb{Z}, |f_i(n) - f_{i+1}(n)| \leq 1 \). Thus \( f_m(m+1) \geq 1 \), which gives us a contradiction.
Definition 5.42. Let $J$ be an interval and let $\otimes$ be a product operation. We say that $X$ and $Y$ are $(J, \otimes)$-homotopy equivalent if there exist morphisms $f : X \to Y$ and $g : Y \to X$ such that $gf \sim_{(J, \otimes)} 1_X$ and $fg \sim_{(J, \otimes)} 1_Y$. We say that $X$ is $(J, \otimes)$-contractible if it is $(J, \otimes)$-homotopy equivalent to the one-point space.

Example 5.43. Let $\otimes$ be a product operation and $J$ be an interval for $\otimes$. By Lemma 5.21, $J$ is $(J, \otimes)$-contractible. Furthermore, for $n \geq 0$, $J^\otimes n$ is $(J, \otimes)$-contractible, which follows inductively by Corollary 5.34.

5.6. Restrictions to full subcategories. We end this section by remarking that a number of our homotopy theories have been previously studied in various full subcategories of closure spaces.

The interval $I$ lies in the full subcategory $\text{Top}$. The intervals $J_{m,k}$, for $m \geq 1$, $0 \leq k \leq 2^{m-1}$ lie in the full subcategory $\text{Cl}_{qd} \cong \text{DiGph}$. The intervals $J_m$ for $m \geq 1$ and $I_{(\varepsilon,a)}$, for $a = -1$, $0$ and $0 \leq \varepsilon \leq 1$ lie in the full subcategory $\text{Cl}_{qd} \cong \text{Gph}$. The product closure restricts to $\text{Top}$, $\text{Gph}$ and $\text{DiGph}$ by Propositions 2.39 and 3.17. The inductive product closure restricts to $\text{Gph}$ and $\text{DiGph}$ by Proposition 3.20 and Corollary 3.21 where it is known as the Cartesian product of graphs and digraphs.

Lemma 5.44. (1) $(I, \times)$-homotopy restricts to $\text{Top}$ where it is called homotopy.
(2) $(J_1, \times)$-homotopy restricts to $\text{Gph}$ where it is called $\times$-homotopy [20].
(3) $(J_1, \square)$-homotopy restricts to $\text{Gph}$ where it is called $A$-homotopy or discrete homotopy [21, 10, 4, 34].
(4) $(J_+, \times)$-homotopy restricts to $\text{DiGph}$.
(5) $(J_+, \square)$-homotopy restricts to $\text{DiGph}$ where it is called homotopy [25, 26].
(6) $(I_{-\varepsilon}, \times)$-homotopy restricts to $\text{Gph}$ where it is called homotopy [34].

6. Homology in closure spaces

In this section we define several homology theories for closure spaces. We start by using some of our previously defined intervals and product operations to define various simplicial and cubical homology theories. For a closure space $X$, let $X^n = X^\times n$ and $X^{\square n}$ denote the $n$-fold product and $n$-fold inductive product, respectively, of $X$ with itself.

6.1. Cubical homology. We use intervals and either the product or the inductive product to define cubical singular homology theories.

Let $J$ be one of the intervals $I$, $J_1$, $J_+$, and let $\otimes$ denote either $\times$ or $\square$.

Definition 6.1. For $n \geq 1$, define the $(J, \otimes)$ $n$-cube to be $\square^n | (J, \otimes) := J^\otimes n$. By Definition 5.1 if $J = (J,c)$ then $\square^n | (J, \otimes) = (J^n, c^\otimes n)$. Define $\square^0 | (J, \otimes)$ to be the one point space. Denote $\square^n | (J, \times)$ by $\square^n | \times$. Note that $\square^n | \times$ is the set $\{0, 1\}^n$ with the indiscrete topology.

Definition 6.2. Let $X$ be a closure space. Given a $(J, \otimes)$ singular $n$-cube, $\sigma : \square^n | (J, \otimes) \to X$, for $1 \leq i \leq n$ define

(4) $A^n_i(\sigma)(a_1, \ldots, a_{n-1}) := \sigma(a_1, \ldots, a_{i-1}, 0, a_i, \ldots, a_{n-1})$
(5) $B^n_i(\sigma)(a_1, \ldots, a_{n-1}) := \sigma(a_1, \ldots, a_{i-1}, 1, a_i, \ldots, a_{n-1})$. 

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Say \( \sigma \) is degenerate if \( A_i^n \sigma = B_i^n \sigma \) for some \( i \). Let \( C_n^{(J,\otimes)}(X) \) be the quotient of the free abelian group on the \((J,\otimes)\) singular \( n \)-cubes in \( X \), which we will denote by \( Q_n^{(J,\otimes)}(X) \), by the free abelian group on the degenerate singular \( n \)-cubes. Elements of \( C_n^{(J,\otimes)}(X) \) are called \((J,\otimes)\) singular cubical \( n \)-chains in \( X \). The boundary map \( \partial_n : C_n^{(J,\otimes)}(X) \to C_{n-1}^{(J,\otimes)}(X) \) is the linear map defined by

\[
\partial_n \sigma := \sum_{i=1}^n (-1)^i (A_i^n \sigma - B_i^n \sigma).
\]

One can check that \( \partial_{n-1} \partial_n = 0 \) and thus \((C^{(J,\otimes)}(X), \partial_\bullet)\) is a chain complex of abelian groups. The cubical singular homology groups are the homology groups of this chain complex, which we denote by \( H^{(J,\otimes)}(X) \).

**Definition 6.3.** We may augment the singular chain complex with the augmentation map \( \varepsilon : C_0^{(J,\otimes)}(X) \to \mathbb{Z} \) given by \( \sum_i n_i \sigma_i = \sum_i n_i \). The homology of the augmented singular chain complex is called reduced homology and denoted \( \tilde{H}_k^{(J,\otimes)}(X) \).

**Example 6.4.** Let \(*\) denote the one point space. There is a single nondegenerate \((J,\otimes)\) \( 0 \)-cube given by the identity map and for \( k \geq 1 \) the \((J,\otimes)\) singular \( k \)-cubes are all degenerate. Therefore \( \tilde{H}_k^{(J,\otimes)}(* \equiv 0 \) for all \( k \geq 0 \).

**Definition 6.5.** Let \( f : X \to Y \) be a continuous map of closure spaces. Let \( \sigma : \square^n|^{(J,\otimes)} \to X \) be a \((J,\otimes)\) singular \( n \)-cube. Then \( f \circ \sigma : \square^n|^{(J,\otimes)} \to Y \) is a singular \((J,\otimes)\) singular \( n \)-cube in \( Y \). Furthermore \( f \) induces a group homomorphism \( f_\# : Q_n^{(J,\otimes)}(X) \to Q_n^{(J,\otimes)}(Y) \), which sends degenerate cubes to degenerate cubes. Thus it also induces a group homomorphism \( f_\# : C_n^{(J,\otimes)}(X) \to C_n^{(J,\otimes)}(Y) \). It can be checked that for all \( n \geq 0 \) these maps respect the boundary operators and thus they induce maps on homology \( f_* : H_n^{(J,\otimes)}(X) \to H_n^{(J,\otimes)}(Y) \).

In particular, for each \( n \geq 0 \), we have a functor \( H_n^{(J,\otimes)}(-) : \text{Cl} \to \text{Ab} \).

**Theorem 6.6.** Let \( f, g : (X, c_X) \to (Y, c_Y) \). If \( f \sim_{(J,\otimes)} g \), then \( f_* = g_* : H_n^{(J,\otimes)}(X, c_X) \to H_n^{(J,\otimes)}(Y, c_Y) \).

**Proof.** It is sufficient to assume that \( f \) and \( g \) are one-step \((J,\otimes)\)-homotopic. That is, there exists \( H : X \otimes J \to Y \) such that \( H(-,0) = f(-) \) and \( H(-,1) = g(-) \). Observe that \( \square^n|^{(J,\otimes)} \otimes J \) is by definition \( \square^{n+1}|^{(J,\otimes)} \).

Let \( \sigma : \square^n|^{(J,\otimes)} \to (X, c_X) \) be a singular \( n \)-cube. Consider the composition \( H \circ (\sigma \times 1) : \square^n|^{(J,\otimes)} \otimes J \to X \otimes J \to Y \). Define a map \( P : C_{n+1}^{(J,\otimes)}(X, c_X) \to C_{n+1}^{(J,\otimes)}(Y, c_Y) \) as follows. For a \((J,\otimes)\) singular \( n \)-cube \( \sigma : \square^n|^{(J,\otimes)} \to (X, c_X) \) let

\[
P(\sigma) = \tilde{\sigma},
\]

where \( \tilde{\sigma} \) is the augmentation of \( \sigma \).
where $\tilde{\sigma}$ is the $(J, \otimes)$ singular $(n+1)$-cube such that $A^n_{i+1}(\tilde{\sigma}) = H(\sigma, 0)$ and $B^n_{i+1}(\tilde{\sigma}) = H(\sigma, 1)$. We now show that $\partial P = g_\# - f_\# - P \partial$. We have

$$
\partial P(\sigma) = \partial \tilde{\sigma} = \sum_{j=1}^{n+1} (-1)^j (A^n_{j+1}(\tilde{\sigma}) - B^n_{j+1}(\tilde{\sigma})) = (-1)^1 (A^n_{1+1}(\tilde{\sigma}) - B^n_{1+1}(\tilde{\sigma}))+ \sum_{j=2}^{n+1} (-1)^j (A^n_{j+1}(\tilde{\sigma}) - B^n_{j+1}(\tilde{\sigma}))
$$

$$
= -(H(\sigma, 0) - H(\sigma, 1)) + \sum_{j=1}^{n} (-1)^j (H(A^n_{j+1}(\sigma), -) - H(B^n_{j+1}(\sigma), -)) ,
$$

where $- \text{ in } H(A^n_{j+1}(\sigma), -)$ and $H(B^n_{j+1}(\sigma), -)$ denotes either 0 or 1. The first sum in the equation above is $H(\sigma, 1) - H(\sigma, 0) = g_\#(\sigma) - f_\#(\sigma)$. The second sum is precisely $-P \partial \tilde{\sigma}$ as can be seen by unwrapping the definitions of $\partial$ and $P$. Thus $\partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma) - P \partial \sigma$. Extending linearly we get that $\partial P = g_\# - f_\# - P \partial$. Furthermore, if $\sigma$ is degenerate, $P(\sigma)$ is also degenerate. Thus, $P$ induces a homomorphism $P : C_n^{(J, \otimes)}(X, c_X) \to C_{n+1}^{(J, \otimes)}(Y, c_Y)$. Thus, $P$ is a chain homotopy between $f_\#$ and $g_\#$ and therefore $f_* = g_*$. □

By Theorem 6.6 and Example 6.4 we get the following corollary.

Corollary 6.7. Let $X$ be a $(J, \otimes)$-contractible closure space. Then $\tilde{H}^{(J, \otimes)}(X) = 0$ for all $n \geq 0$.

Example 6.8. The only continuous maps from $J_1$ to $J_+$ are the constant maps, which are degenerate $(J_1, \times)$ and $(J_1, \boxtimes)$ singular 1-cubes, and thus $H_0^{(J_1, \boxtimes)}(J_+) = H_0^{(J_1, \times)}(J_+) = \mathbb{Z}^2$.

Example 6.9. Consider the space $(\mathbb{R}, \tau)$. The only continuous maps from $|\Box|^{J_1}$, $|\Box|^{J_1, \boxtimes}$, $|\Box|^{J_1, \times}$ and $|\Box|^{J_1, \otimes}$ into $(\mathbb{R}, \tau)$ are the constant maps. Therefore

$$H_0^{(J_1, \times)}(\mathbb{R}, \tau) = H_0^{(J_1, \otimes)}(\mathbb{R}, \tau) = H_0^{(J_1, \times)}(\mathbb{R}, \tau) = \oplus_{x \in \mathbb{R}} \mathbb{Z}.$$

Example 6.10. Consider $|\Box|^2(J_1, \otimes) = J_+^{\otimes 2}$. We will show that $H_1^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) \cong \mathbb{Z}$. Let $a,b,c,d$, denote the vertices $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$ of $|\Box|^2(J_1, \otimes)$. Denote the singular $k$ cubes by the images of the vertices of $|\Box|^k(J_1)$, when those vertices are listed in lexicographic order. Then, $C_0^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) = \mathbb{Z}(a, b, c, d)$ and $C_1^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) = \mathbb{Z}(ab, ac, bd, cd)$. Furthermore, $C_2^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) = \langle aab, abb, aac, acc, bbd, ddd, ccd, cdd \rangle$. Note that, for example, $abab$ is degenerate, and the map corresponding to $abcd$ is not continuous. One can check that $\partial_2 = 0$. Since, ker $\partial_1 = \mathbb{Z} \langle \tau \rangle$, where $\tau = ab + bd - ac - cd$, the result follows.

Example 6.11. Consider $|\Box|^2(J_1, \otimes) = J_+^{\otimes 2}$. We will show that $H_1^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) \cong \mathbb{Z}$. Let $a,b,c,d$, denote the vertices $(0, 0), (0, 1), (1, 0)$ and $(1, 1)$ of $|\Box|^2(J_1, \otimes)$. Denote the singular $k$ cubes by the images of the vertices of $|\Box|^k(J_1)$, when those vertices are listed in lexicographic order. Then, $C_0^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) = \mathbb{Z}(a, b, c, d)$ and $C_1^{(J_1, \times)}(|\Box|^2(J_1, \otimes)) = \mathbb{Z}(ab, ba, ac, ca, bd, db, cd, dc)$. Furthermore, $C_2^{(J_1, \times)}(|\Box|^2(J_1, \otimes))$ is the free abelian group with generators of the form $uvw$, $uwu$, $wuu$, $vuu$, and $uvu$ where $(u, v) \in \{(a, b), (b, a), (a, c), (c, a), (b, d), (d, b), (c, d), (d, c)\}$. Note that, for example, $\partial_2(abaa) = ab + ba - 2aa = ab + ba$ since $aa$ is degenerate. No map whose image contains 3 distinct vertices of $J_1 \sqcup J_1$ is continuous, and the corresponding to $abcd$ is not continuous. The result follows from checking that ker $\partial_1 = \mathbb{Z}(ab + ba, ac + ca, bd + db, cd + dc, \tau)$, where $\tau = ab + bd - ac - cd$ and im $\partial_2 = \mathbb{Z}(ab + ba, ac + ca, bd + db, cd + dc)$. 27
6.2. Simplicial homology. In the case of the product, we define corresponding simplicial singular homology theories. Let \( J \) be one of \( I, J_1, \) or \( J_+ \). Denote \((J, \times)\) simply by \( J \).

Definition 6.12. For \( n \geq 0 \) define \( \iota_n : \{0, \ldots, n\} \to \{0, 1\}^n \) by \( \iota(k) = (1, \ldots, 1, 0, \ldots, 0) \).

Let \( |\Delta^n|^J \) denote the convex hull of \( \text{im}(\iota) \) in \( |\square^n|^J \) with the subspace closure. Let \( |\Delta^n|^J_1 \) and \( |\Delta^n|^J_+ \) denote \( \text{im}(\iota) \) in \( |\square^n|^J_1 \) and \( |\square^n|^J_+ \), respectively, with the subspace closure. Call \( |\Delta^n|^J \) the \( J \) \( n \)-simplex. For \( 0 \leq i \leq n \), the \( i \)-face of \( |\Delta^n|^J \) is the convex hull of the image of \( \iota|_{\{0, \ldots, i, \ldots, n\}} \) if \( J = I \) or the image of \( \iota|_{\{0, \ldots, i, \ldots, n\}} \) if \( J = J_+ \) or \( J = J_1 \).

Note that \( |\Delta^n|^J \) is homeomorphic to the standard \( n \)-simplex, \( |\Delta^n|^J_1 \) is homeomorphic to \( J_n \), the set \( \{0, 1, \ldots, n\} \) with the indiscrete topology, and \( |\Delta^n|^J_+ \) is homeomorphic to \( J_n, \leq \), the set \( \{0, 1, \ldots, n\} \) with the closure operation \( c(i) = \{j \mid i \leq j\} \).

Definition 6.13. Let \( X \) be a closure space. Let \( C_n^J(X) \) be the free abelian group on the \( J \) singular \( n \)-simplices, \( \sigma : |\Delta^n|^J \to X \). For \( n \geq 1 \), let \( \partial_n : C_n^J(X) \to C_{n-1}^J(X) \) be the map defined by

\[
\partial_n \sigma = \sum_{i=0}^n (-1)^i d_i \sigma,
\]

where \( d_i \sigma \) is the restriction of \( \sigma \) to the \( i \)-th face of \( |\Delta^n|^J \). Since \( \partial_{n-1} \partial_n = 0 \) we have a chain complex of free abelian groups, \( (C^J_n(X), \partial_n) \) whose homology groups we denote by \( H^J_n(X) \) and are called the simplicial singular homology groups.

Let \( f : X \to Y \) be a continuous map of closure spaces. Let \( \sigma : |\Delta^n|^J \to X \) be a \( J \) singular \( n \)-simplex. Then \( f \circ \sigma : |\Delta^n|^J \to Y \) is a singular \( J \) singular \( n \)-simplex in \( Y \). Furthermore \( f \) induces a group homomorphism \( f_* : C_n^J(X) \to C_n^J(Y) \). It can be checked that for all \( n \geq 0 \) these maps respect the boundary operators and thus they induce maps on homology \( f_* : H^J_n(X) \to H^J_n(Y) \). In particular, for each \( n \geq 0 \) we have a functor \( H^J_n(-) : \text{Cl} \to \text{Ab} \).

In [12] it will be shown that the corresponding simplicial singular homology groups and cubical singular homology groups agree.

6.3. Restrictions to full subcategories. We end this section by remarking that a number of our homology theories have been previously studied in various full subcategories of closure spaces.

Recall that \( \text{Cl}_{\text{sqd}} \cong \text{Gph} \). Under this isomorphism, \( |\Delta^n|^J_1 \) corresponds to \( K_{n+1} \), the complete graph on \( n+1 \) vertices, and \( |\square^n|^{|J_1, \square|} \) corresponds to the hypercube graph \( Q_n \).

Also recall that \( \text{Cl}_{\text{sqd}} \cong \text{DiGph} \). Under this isomorphism, \( |\Delta^n|^J_+ \) corresponds to \( K_{n+1}^+ \), the digraph of the poset \( \{(0, \ldots, n), \leq\} \) and \( |\square^n|^{|J_+, \square|} \) corresponds to the hypercube digraph \( Q_n^+ \), where the vertices are the elements of \( \{0, 1\} \) and the directed edges given by \( (a, a+e_i) \) where \( e_i \) is a standard basis vector.

Lemma 6.14. (1) \( H^J_0(I, \times)(X) \) restricts to \( \text{Top} \) where it is called singular homology.
(2) \( H^J_0(J_1, \times)(X) \) restricts to \( \text{Gph} \) where it is the homology of the clique complex of a graph.
(3) \( H^J_0(J_+, \times)(X) \) restricts to \( \text{Gph} \) where it is called discrete homology \( (\mathbb{Z}, \mathbb{Z}) \).
(4) \( H^J_0(J_+, \times)(X) \) restricts to \( \text{DiGph} \) where it is the homology of the directed clique complex.
(5) \( H^J_0(J_+, \times)(X) \) restricts to \( \text{DiGph} \).
6.4. Homology with coefficients. Let $C$ be one of the chain complexes of Section 6.1 or Section 6.2. Let $A$ be an abelian group, which we consider to be a chain complex concentrated in degree zero. Then the tensor product $C \otimes A$ is a chain complex whose homology $H_{\bullet}(C \otimes A)$ is called the homology of $C$ with coefficients in $A$. As a special case, for a field $k$, the homology groups $H_j(C \otimes k)$ are $k$-vector spaces.

7. Persistent homology

In this section we give functorial constructions of simplicial complexes and hypergraphs from closure spaces and vice-versa. As special cases, we obtain generalizations of the Vietoris-Rips complex and Čech complex constructions. We define filtered closure spaces and hypergraphs and show how they arise from metric spaces and various generalizations of metric spaces. Applying homology we obtain persistence modules which we prove to be stable with respect to a generalization of Gromov-Hausdorff distance to filtered closure spaces.

7.1. Hypergraphs and simplicial complexes. We define hypergraphs and simplicial complexes and related categories and functors. We obtain a sequence of adjunctions connecting closure spaces and hypergraphs via graphs and simplicial complexes.

Definition 7.1. A simple hypergraph $H$ is a pair $H = (X, E)$ where $X$ is a set and $E$ is a collection of non-empty subsets of $X$. We will call a simple hypergraph a hypergraph. Elements of $X$ are called vertices of the hypergraph $H$ and elements of $E$ are called hyperedges of the hypergraph $H$. A hypergraph homomorphism $f : (X, E) \to (Y, F)$ between two hypergraphs is a map $f : X \to Y$ such that for each $e \in E$, $f(e) \in F$. Let HypGph denote the category of hypergraphs and hypergraph homomorphisms. Say that a hypergraph has finite type if its hyperedges are finite sets. Say that a hypergraph is downward closed if $\tau \in E$ and $\emptyset \neq \sigma \subset \tau$ implies that $\sigma \in E$ and $x \in X$ implies that $\{x\} \in E$. An (abstract) simplicial complex is a downward-closed finite-type hypergraph. Let HypGph_{dc}, HypGph_{ft}, and Simp denote the full subcategories of HypGph consisting of finite type hypergraphs, downward closed hypergraphs, and simplicial complexes. In Simp, hyperedges and hypergraph homomorphisms are called simplices and simplicial maps, respectively.

Let $(X, E) \in$ HypGph. Define the downward closure of $E$, $dc(E)$, to be the collection of nonempty subsets $\sigma$ of $X$ such there exists $\tau \in E$ with $\sigma \subset \tau$ or $\sigma = \{x\}$ for some $x \in X$. Assume $f : (X, E) \to (Y, F) \in$ HypGph. Given $\emptyset \neq \sigma \subset \tau \in E$, $\emptyset \neq f(\sigma) \subset f(\tau) \in F$. So $f(\sigma) \in dc(F)$. Also $f(\{x\}) = \{f(x)\} \in Y$. Therefore $f : (X, dc(E)) \to (Y, dc(F)) \in$ HypGph. Thus the mappings $(X, E)$ to $(X, dc(E))$ and $f : (X, E) \to (Y, E)$ to $f : (X, dc(E)) \to (Y, dc(F))$ define a functor $dc :$ HypGph $\to$ HypGph_{dc}.

Proposition 7.2. Let $(X, E) \in$ HypGph and $(Y, E) \in$ HypGph_{dc}. Given a set map $f : X \to Y$, $f : (X, E) \to (Y, F)$ is a hypergraph homomorphism iff $f : (X, dc(E)) \to (Y, F)$ is a hypergraph homomorphism. Thus, we have a natural isomorphism

$$\text{HypGph}_{dc}((X, dc(E)), (Y, F)) \cong \text{HypGph}((X, E), (Y, F)).$$

That is, $dc$ is left adjoint to the inclusion functor $\text{HypGph}_{dc} \hookrightarrow$ HypGph.

Proof. $(\Rightarrow)$ If $\tau \in E$ and $\emptyset \neq \sigma \subset \tau$ then $\emptyset \neq f(\sigma) \subset f(\tau) \in F$ and thus $f(\sigma) \in F$. If $x \in X$ then $f(\{x\}) = \{f(x)\} \in F$. $(\Leftarrow) E \subset dc(E)$. \hfill $\Box$
Let \((X, E) \in \text{HypGph}_{dc}\). Define \(\text{tr}_\infty (E) = \{ \sigma \in E \mid |\sigma| < \infty \}\). Let \(\text{tr}_\infty : \text{HypGph}_{dc} \to \text{Simp}\) denote the functor defined by mapping \((X, E)\) to \((X, \text{tr}_\infty (E))\) and mapping \(f : (X, E) \to (Y, F)\) to \(f : (X, \text{tr}_\infty (E)) \to (Y, \text{tr}_\infty (F))\).

Let \((X, E) \in \text{Simp}\). Define \(\text{cosk}_\infty (E)\) be the collection of nonempty subsets \(\tau\) of \(X\) such that for all finite nonempty subsets \(\sigma \subset \tau, \sigma \in \text{tr}_\infty (E)\). Note that \(\sigma \in \text{cosk}_\infty (E)\) and \(|\sigma| < \infty\) implies that \(\sigma \in E\). Let \(\text{cosk}_\infty : \text{Simp} \to \text{HypGph}_{dc}\) denote the functor defined by mapping \((X, E)\) to \((X, \text{cosk}_\infty (E))\) and mapping \(f : (X, E) \to (Y, F)\) to \(f : (X, \text{cosk}_\infty (E)) \to (Y, \text{cosk}_\infty (F))\).

**Proposition 7.3.** Let \((X, E) \in \text{HypGph}_{dc}\) and \((Y, E) \in \text{Simp}\). Given a set map \(f : X \to Y, f : (X, \text{tr}_\infty (E)) \to (Y, F)\) is a simplicial map iff \(f : (X, E) \to (Y, \text{cosk}_\infty (F))\) is a hypergraph homomorphism. Thus, we have a natural isomorphism

\[
\text{Simp}((X, \text{tr}_\infty (E)), (Y, F)) \cong \text{HypGph}_{dc}((X, E), (Y, \text{cosk}_\infty (F))).
\]

That is, \(\text{tr}_\infty\) is left adjoint to \(\text{cosk}_\infty\).

**Proof.** \((\Rightarrow)\) Let \(\tau \in E\). Note that for all nonempty finite subsets of \(f(\tau)\) equal \(f(\sigma)\) for some nonempty finite \(\sigma \subset \tau\). Since \(\tau \in E\), for all nonempty, finite subsets \(\sigma \subset \tau, \sigma \in \text{tr}_\infty (E)\) and hence \(f(\sigma) \in F\). Thus for all nonempty finite subsets \(\sigma' \subset f(\tau), \sigma' \in F\). Therefore \(f(\tau) \in \text{cosk}_\infty (F)\).

\((\Leftarrow)\) If \(\sigma \in \text{tr}_\infty (E)\) then \(f(\sigma) \in \text{cosk}_\infty (F)\) and \(|f(\sigma)| < \infty\). Therefore \(f(\sigma) \in F\). \(\square\)

Let \((X, E) \in \text{Simp}\). Define \(\text{tr}_1 (E) = \{ \sigma \in E \mid |\sigma| = 2 \}\). Let \(\text{tr}_1 : \text{Simp} \to \text{Gph}\) denote the functor defined by mapping \((X, E)\) to \((X, \text{tr}_1 (E))\) and mapping \(f : (X, E) \to (Y, F)\) to \(f : (X, \text{tr}_1 (E)) \to (Y, \text{tr}_1 (F))\).

Let \((X, E) \in \text{Simp}\). Define \(\text{cosk}_1 (E)\) be the collection of nonempty finite subsets \(\tau\) of \(X\) such that for all distinct \(x, y \in \tau, \{x, y\} \in E\). Note that this includes all subsets of \(X\) of cardinality one. Let \(\text{cosk}_1 : \text{Gph} \to \text{Simp}\) denote the functor defined by mapping \((X, E)\) to \((X, \text{cosk}_1 (E))\) and mapping \(f : (X, E) \to (Y, F)\) to \(f : (X, \text{cosk}_1 (E)) \to (Y, \text{cosk}_1 (F))\).

Given a graph \((X, E)\), the simplicial complex \((X, \text{cosk}_1 (E))\) is called the clique complex of the graph. A simplicial complex in the image of \(\text{cosk}_1 : \text{Gph} \to \text{Simp}\) is called a flag complex or is said to satisfy Gromov’s no-\(\Delta\) condition. Observe that for a graph \((X, E), (X, \text{tr}_1 (\text{cosk}_1 (E)))) = (X, E)\).

**Proposition 7.4.** Let \((X, E) \in \text{Simp}\) and \((Y, E) \in \text{Gph}\). Given a set map \(f : X \to Y, f : (X, \text{tr}_1 (E)) \to (Y, F)\) is a graph homomorphism iff \(f : (X, E) \to (Y, \text{cosk}_1 (F))\) is a simplicial map. Thus, we have a natural isomorphism

\[
\text{Gph}((X, \text{tr}_1 (E)), (Y, F)) \cong \text{Simp}((X, E), (Y, \text{cosk}_1 (F))).
\]

That is, \(\text{tr}_1\) is left adjoint to \(\text{cosk}_1\).

**Proof.** \((\Rightarrow)\) Let \(\sigma \in E\). For all \(x \neq y \in \sigma, \{x, y\} \in \text{tr}_1 (E)\). Thus \(f(x) = f(y)\) or \(\{f(x), f(y)\} \in F\). Therefore \(f(\sigma) \in \text{cosk}_1 (F)\).

\((\Leftarrow)\) Let \(\sigma \in E\) and \(x \neq y \in \sigma\). Then \(f(\sigma) \in \text{cosk}_1 (F)\), which implies that either \(f(x) = f(y)\) or \(\{f(x), f(y)\} \in F\). Therefore \(f\) is a graph homomorphism. \(\square\)

Combining Propositions 3.7, 3.9 and 3.22 with Propositions 7.2 to 7.4, we have the following sequence of adjunctions. Recall that adjunctions compose to give adjoint functors.
Theorem 7.5. We have the following composite adjunction between \( \text{Cl} \) and \( \text{HypGph} \).

\[
\begin{array}{ccc}
\text{Cl} & \overset{_{\text{Cl}_{\text{qd}}}}{\underset{\text{Cl}_{\text{sqd}}} \leftarrow} & \text{Cl}_{\text{qd}} \\
& \overset{s}{\underset{\Phi} \leftarrow} & \text{Cl}_{\text{sqd}} \\
& \overset{\psi}{\underset{\cong} \rightarrow} & \text{Gph} \overset{\text{tr}_1}{\underset{\text{Simp}} \leftarrow} \\
& & \overset{\text{tr}_\infty}{\underset{\text{HypGph}_{\text{dc}}} \rightarrow} \\
& & \text{HypGph} \overset{d_{\text{dc}}}{\underset{\cong} \rightarrow}
\end{array}
\]

We end this section by defining one more functor that will be used in Section 7.2.

Definition 7.6. Let \((X, c) \in \text{Cl}_{\text{qd}}\). Define \(\Gamma(c)\) to be the downward closure of the collection of subsets of \(X\), \(\{c(x)\}_{x \in X}\). Let \(\Gamma : \text{Cl}_{\text{qd}} \to \text{HypGph}_{\text{dc}}\) be the functor defined by mapping \((X, c) \to (X, \Gamma(c))\) and mapping \(f : (X, c) \to (Y, d)\) to \(f : (X, \Gamma(c)) \to (Y, \Gamma(d))\).

7.2. Vietoris-Rips and \(\check{\text{C}}\)ech complexes. We give functorial constructions of Vietoris-Rips complexes and \(\check{\text{C}}\)ech complexes for closure spaces which send \((\mathcal{J}_1, \times)\)-one-step homotopic maps to contiguous simplicial maps. We give an adjoint functor to the Vietoris-Rips construction which sends contiguous simplicial maps to \((\mathcal{J}_1, \times)\)-one-step homotopic maps.

Let \((X, c)\) be a closure space. Let \(\text{VR}(c)\) be the collection of nonempty finite subsets \(\sigma \subset X\), such that for all \(x \in \sigma\), \(\sigma \subset c(x)\). Note that for all \(x \in X\), \(x \in c(x)\), so \(\{x\} \in \text{VR}(c)\). Also, if \(\tau \in \text{VR}(c)\) and \(\emptyset \neq \sigma \subset \tau\) then \(\sigma \in \text{VR}(c)\). Thus, \((X, \text{VR}(c))\) is a simplicial complex. Assume \(f : (X, c) \to (Y, d) \in \text{Cl}\). Let \(\sigma \in \text{VR}(X)\). Then \(f(\sigma)\) is a finite nonempty subset of \(Y\). Furthermore, all elements of \(f(\sigma)\) are of the form \(f(x)\) for some \(x \in \sigma\). As \(\sigma \subset c(x)\), it follows that \(f(\sigma) \subset f(c(x)) \subset d(f(x))\). Therefore \(f(\sigma) \in \text{VR}(d)\), which implies that \(f : (X, \text{VR}(c)) \to (Y, \text{VR}(d))\) is a simplicial map.

Definition 7.7. Define the functor \(\text{VR} : \text{Cl} \to \text{Simp}\) by mapping \((X, c)\) to \((X, \text{VR}(c))\) and \(f : (X, c) \to (Y, d)\) to \(f : (X, \text{VR}(c)) \to (Y, \text{VR}(d))\).

Let \((X, d)\) be a metric space and \(\varepsilon > 0\). Then \((X, \text{VR}(c_{\varepsilon,d}))\) consists of simplices \(\{x_0, \ldots, x_n\}\) where \(d(x_i, x_j) \leq \varepsilon\) for all \(i, j\). That is, it is the usual Vietoris-Rips complex on \((X, d))\). We also have the variant, \((X, \text{VR}(c_{\varepsilon,d}))\) consisting of simplices \(\{x_0, \ldots, x_n\}\) where \(d(x_i, x_j) < \varepsilon\) for all \(i, j\), which is sometime used.

Let \((X, c)\) be a closure space. Define \(\mathcal{C}(c)\) to be the collection of nonempty finite subsets \(\sigma \subset X\) such that there exists \(x \in X\) with \(\sigma \subset c(x)\). Note that for all \(x \in X, x \in c(x)\), so \(\{x\} \in \mathcal{C}(c)\). Furthermore, if \(\tau \in \mathcal{C}(c)\) and \(\emptyset \neq \sigma \subset \tau\) then \(\sigma \in \mathcal{C}(c)\). That is, \((X, \mathcal{C}(c))\) is a simplicial set. Assume \(f : (X, c) \to (Y, d) \in \text{Cl}\). Let \(\sigma \in \mathcal{C}(c)\). Then \(f(\sigma)\) is a nonempty finite subset of \(Y\). There exists \(x \in X\) such that \(\sigma \subset c(x)\), which implies that \(f(\sigma) \subset f(c(x)) \subset d(f(x))\). Therefore \(f(\sigma) \in \mathcal{C}(d)\) and thus \(f : (X, \mathcal{C}(c)) \to (Y, \mathcal{C}(d))\) is a simplicial map.

Definition 7.8. Define the functor \(\mathcal{C} : \text{Cl} \to \text{Simp}\) by mapping \((X, c)\) to \((X, \mathcal{C}(c))\) and \(f : (X, c) \to (Y, d)\) to \(f : (X, \mathcal{C}(c)) \to (Y, \mathcal{C}(d))\).

Let \((X, d)\) be a metric space and \(\varepsilon > 0\). Then \((X, \mathcal{C}(c_{\varepsilon,d}))\) consists of simplices \(\{x_0, \ldots, x_n\}\) such that there is a \(x \in X\) with \(d(x_i, x) \leq \varepsilon\) for all \(i\). That is, it is the usual \((\text{intrinsic})\) \(\check{\text{C}}\)ech complex on \((X, d))\). We also have the variant, \((X, \mathcal{C}(c_{\varepsilon,d}))\) consisting of simplices \(\{x_0, \ldots, x_n\}\) such that there is a \(x \in X\) with \(d(x_i, x) < \varepsilon\) for all \(i\).

It is straightforward from the definitions to check that \(\text{VR} = \text{VR} \circ \text{qd} = \text{qd}\) and \(\mathcal{C} = \mathcal{C} \circ \text{qd}\), where \(\text{qd} : \text{Cl}_{\text{qd}} \to \text{Cl}\) is the inclusion functor. That is, for a closure space \((X, c)\), we have \((X, \text{VR}(c)) = (X, \text{VR}(\text{qd}(c))), (X, \mathcal{C}(c)) = (X, \mathcal{C}(\text{qd}(c))).\)
Example 7.9. Let $X = \{x, y, z\}$ be a 3-point set with the closure operation $c$, defined by

$$c(x) = \{x, y\}, c(y) = \{x, y\}, c(z) = \{x, y, z\}$$

Note that $c$ does not arise from a metric since it is not symmetric. From the definitions,

$$VR(c) = \{\{x\}, \{y\}, \{z\}, \{x, y\}\} \quad \text{and} \quad \check{C}(c) = \{\{x\}, \{y\}, \{z\}, \{x, y\}, \{x, z\}, \{y, z\}, \{x, y, z\}\}.$$

See Figure 1.

![Figure 1. A closure space (left) and its Vietoris-Rips complex (middle) and Čech complex (right).](image)

Let $(X, E)$ be a simplicial complex. For $A \subset X$, let $G(E)(A)$ be the union of all simplices in $E$ that intersect $A$. In particular $G(E)(\emptyset) = \emptyset$. For all $a \in A \subset X$, $\{a\} \in E$, therefore $A \subset G(E)(A)$. For $A, B \subset X$, if a simplex in $E$ intersects $A \cup B$ then it intersects either $A$ or $B$. Thus $G(E)(A \cup B) \subset G(E)(A) \cup G(E)(B)$. In addition, if a simplex in $E$ intersects $A$ then it intersects $A \cup B$. So $G(E)(A) \subset G(E)(A \cup B)$. Hence, we have that $G(E)(A \cup B) = G(E)(A) \cup G(E)(B)$. Therefore $G(E)$ is a closure on $X$.

Assume $f : (X, E) \to (Y, F) \in \text{Simp}$. For $A \subset X$, $f(G(E)(A))$ is the union of $f(\sigma)$, where $\sigma$ is a simplex in $E$ that intersects $A$, which implies that $f(\sigma)$ intersects $f(A)$. Therefore $f(G(E)(A)) \subset G(F)(f(A))$. Hence $f : (X, G(E)) \to (Y, G(F))$ is a continuous map.

Definition 7.10. Define $G : \text{Simp} \to \text{Cl}$ to be the functor given by mapping $(X, E)$ to $(X, G(E))$ and mapping $f : (X, E) \to (Y, F)$ to $f : (X, G(E)) \to (Y, G(F))$.

Theorem 7.11. Let $(X, E) \in \text{Simp}$ and $(Y, c) \in \text{Cl}$. Given a set map $f : X \to Y$, $f : (X, G(E)) \to (Y, F)$ is a continuous map iff $f : (X, E) \to (Y, VR(c))$ is a simplicial map. Thus, we have a natural isomorphism

$$\text{Cl}((X, G(E)), (Y, F)) \cong \text{Simp}((X, E), (Y, VR(F))).$$

That is, VR is right adjoint to G.

Proof. ($\Rightarrow$) Let $\sigma \in E$. Let $x \in \sigma$. Then $\sigma \subset G(E)(x)$, which implies that $f(\sigma) \subset f(G(E)(x)) \subset c(f(x))$. Therefore $f(\sigma) \in VR(c)$.

($\Leftarrow$) Let $A \subset X$. Then $f(G(E)(A)) = f(\bigcup_{\sigma \in E, \sigma \cap A \neq \emptyset} \sigma) = \bigcup_{\sigma \in E, \sigma \cap A \neq \emptyset} f(\sigma)$. If $\sigma \in E$ then $f(\sigma) \in VR(c)$, which implies that for all $x \in E$, $f(\sigma) \subset c(f(x))$. So, $\bigcup_{\sigma \in E, \sigma \cap A \neq \emptyset} f(\sigma) \subset c(f(A))$. □

We observe that the functors VR, $\check{C}$ and $G$ may be written as composition of the more elementary functors in Theorem 7.3 and Definition 7.6.

Proposition 7.12. $VR = \text{cosk}_1 \circ \Phi \circ s \circ \text{sqd}$, $G = \Psi \circ \text{tr}_1$, and $\check{C} = \text{tr}_\infty \circ \Gamma \circ \text{qd}$.

It follows that restricted to $\text{Cl}_{\text{sqd}}$, $G \circ VR = 1_{\text{Cl}_{\text{sqd}}}$.

Simplicial maps $f, g : (X, E) \to (Y, F)$ are said to be contiguous if $f(\sigma) \cup g(\sigma) \in F$ for all $\sigma \in E$. 32
Theorem 7.13. Let \( f, g : (X, c) \rightarrow (Y, d) \in \text{Cl} \) be one-step \((J_1, \times)\)-homotopic. Then \( f, g : (X, \text{VR}(c)) \rightarrow (Y, \text{VR}(d)) \) are contiguous simplicial maps and so are \( f, g : (X, \hat{C}(c)) \rightarrow (Y, \hat{C}(d)) \). Conversely, let \( f, g : (X, E) \rightarrow (Y, F) \) be contiguous simplicial maps. Then \( f, g : (X, G(E)) \rightarrow (Y, G(F)) \) are one-step \((J_1, \times)\)-homotopic.

Proof. Assume that \( f, g : (X, c) \rightarrow (Y, d) \) are one-step \((J_1, \times)\)-homotopic.

Let \( \sigma \in \text{VR}(c) \). For \( x \in \sigma \), \( \sigma \subset c(x) \). Then by Lemma 5.27 \( f(\sigma) \subset f(c(x)) \subset d(f(x)) \cap d(g(x)) \). Similarly, \( g(\sigma) \subset g(c(x)) \subset d(f(x)) \cap d(g(x)) \). Therefore \( f(\sigma) \cup g(\sigma) \subset d(f(x)) \cap d(g(x)) \). Thus, for all \( y \in f(\sigma) \cup g(\sigma) \), \( f(\sigma) \cup g(\sigma) \subset d(y) \). Hence \( f(\sigma) \cup g(\sigma) \in \text{VR}(d) \).

Let \( \sigma \in \hat{C}(c) \). Then there exists \( x \in X \) such that \( \sigma \subset c(x) \). Then by Lemma 5.27 \( f(\sigma) \subset f(c(x)) \subset d(f(x)) \cap d(g(x)) \). Similarly, \( g(\sigma) \subset g(c(x)) \subset d(f(x)) \cap d(g(x)) \). Therefore \( f(\sigma) \cup g(\sigma) \subset d(f(x)) \cap d(g(x)) \subset d(f(x)) \). Hence \( f(\sigma) \cup g(\sigma) \in \hat{C}(d) \).

Assume that \( f, g : (X, E) \rightarrow (Y, F) \) are contiguous simplicial maps. Let \( A \subset X \). Then \( G(E)(A) \) is the union of all simplices in \( E \) that intersect \( A \). Hence \( f(G(E)(A)) \cup g(G(E)(A)) = \bigcup_{\sigma \in E, \sigma \cap A \neq \emptyset} f(\sigma) \cup g(\sigma) \subset \bigcup_{\tau \in F, \tau \cap f(A) \neq \emptyset} \tau = G(F)(f(A)) \). Similarly \( f(G(E)(A)) \cup g(G(E)(A)) \subset G(F)(f(A)) \). Therefore \( f(G(E)(A)) \cup g(G(E)(A)) \subset G(F)(f(A)) \). So by Lemma 5.27 \( f, g : (X, G(E)) \rightarrow (Y, G(F)) \) are one-step \((J_1, \times)\)-homotopic.

Corollary 7.14. Let \( f, g : (X, c) \rightarrow (Y, d) \in \text{Cl}_{\text{sqd}} \). Then \( f, g : (X, c) \rightarrow (Y, d) \in \text{Cl} \) are one-step \((J_1, \times)\)-homotopic iff \( f, g : (X, \text{VR}(c)) \rightarrow (Y, \text{VR}(d)) \in \text{Simp} \) are contiguous.

Proof. \((\Rightarrow)\) This implication is given by the first part of Theorem 7.13.

\((\Leftarrow)\) By the second part of Theorem 7.13 \( f, g : (X, G(\text{VR}(c))) \rightarrow (Y, G(\text{VR}(d))) \) are one-step \((J_1, \times)\)-homotopic. For \( (X, c) \in \text{Cl}_{\text{sqd}}, G(\text{VR}(c)) = c \). The result follows.

7.3. Filtered closure spaces and persistence modules. Given a metric space, we use the three closure operations of Section 4 to give functorial constructions of filtered closure spaces. Combined with our cubical and simplicial homology functors, we obtain functors from metric spaces to persistence modules. Furthermore, we extend these results to get functors from metric spaces and Lipschitz maps to persistence modules and graded maps.

7.3.1. Filtered closure spaces. We start by defining closure spaces filtered by a partially ordered set.

Let \( P \) be a partially ordered set. An example of interest to topological data analysis is \( P = (\mathbb{R}^n, \leq) \), where \( \leq \) is the product partial order on \( \mathbb{R}^n \). We will mostly use the totally ordered set \( (\mathbb{R}, \leq) \), which we will denote by \( R \), and its subset \([0, \infty)\).

Definition 7.15. Let \( P \) be a partially ordered set. We define a \( P \)-filtered closure space to be a \( P \)-indexed set of closure spaces \( \{(X_p, c_p)\}_{p \in P} \) such that for all \( p \leq q \in P \), \( X_p \subset X_q \) and the inclusion map \( (X_p, c_p) \rightarrow (X_q, c_q) \) is continuous. That is, for all \( A \subset X_p \), \( c_p(A) \subset c_q(A) \).

A morphism of \( P \)-filtered closure spaces, \( f : (X_\bullet, c_\bullet) \rightarrow (Y_\bullet, d_\bullet) \) consists of continuous maps \( f_p : (X_p, c_p) \rightarrow (Y_p, d_p) \) for all \( p \in P \) such that for all \( p \leq q \), \( f_p = f_q|_{X_p} \). Equivalently, a morphism \( f : (X_\bullet, c_\bullet) \rightarrow (Y_\bullet, d_\bullet) \) is a set map \( f : X \rightarrow Y \), where \( X = \bigcup_{p \in P} X_p \) and \( Y = \bigcup_{p \in P} Y_p \), such that for all \( p \in P \), \( f(X_p) \subset Y_p \) and \( f|_{X_p} : (X_p, c_p) \rightarrow (Y_p, d_p) \) is continuous. Let \( \mathbf{F}_P\text{Cl} \) denote the category of \( P \)-filtered closure spaces and their morphisms.

Example 7.16. Consider the following two examples. First, let \( X \) be a set together with closure operations \( \{c_p\}_{p \in P} \) such that for all \( p \leq q \), \( c_p \leq c_q \). Then \( (X, c_\bullet) \in \mathbf{F}_P\text{Cl} \). Second, consider a closure space \( (X, c) \) together with a set map \( f : X \rightarrow P \). For \( p \in P \), let
\(X_p = f^{-1}(D_p),\) where \(D_p := \{q \in P \mid q \leq p\},\) and let \(c_p\) be the subspace closure. That is, for \(A \subset X_p\) define \(c_p(A) = c(A) \cap X_p.\) Then \((X_\bullet, c_\bullet) \in F_P \text{Cl}\).

Note that a \(P\)-filtered closure space is a functor from \(P\) to \(\text{Cl}\) for which inequalities are mapped to monomorphisms and that a morphism of \(P\)-filtered closure spaces is a natural transformation of such functors.

7.3.2. Persistence modules. We recall the definitions of persistence modules, persistence diagrams, interleavings and matchings, and their corresponding distances.

A functor from \(P\) to a category \(C\) is called a persistence module indexed by \(P\) and with values in \(C\) \([10]\).

**Definition 7.17.** Let \(C\) be a category, let \(P\) be either of the totally ordered sets \([0, \infty)\) or \(R = (\mathbb{R}, \leq)\) and let \(\varepsilon \geq 0.\) Let \(M, N : P \to C.\) We say \(M\) and \(N\) are \(\varepsilon\)-interleaved if there are collections of maps \(\{\varphi_p : M_p \to N_{p+\varepsilon}\}_{p \in P}\) and \(\{\psi_p : N_p \to M_{p+\varepsilon}\}_{p \in P}\) such that:

- For all \(p, q \in P\) and \(p \leq q,\) \(\varphi_q \circ M_{p\leq q} = N_{p+\varepsilon\leq q+\varepsilon} \circ \varphi_p.\)
- For all \(p, q \in P\) and \(p \leq q,\) \(\psi_q \circ N_{p\leq q} = M_{p+\varepsilon\leq q+\varepsilon} \circ \psi_p.\)
- For all \(p \in P,\) \(\varphi_{p+\varepsilon} \circ \varphi_p = M_{p+2\varepsilon}.\)
- For all \(p \in P,\) \(\varphi_{p+\varepsilon} \circ \psi_p = N_{p+2\varepsilon}.\)

The interleaving distance between \(M\) and \(N\) is then defined to be

\[
d_I(M, N) := \inf \{\varepsilon \mid M\text{ and } N\text{ are }\varepsilon\text{-interleaved}\}
\]

For \(P = [0, \infty)\) or \(R\) as above, and \(\varepsilon \geq 0,\) let \(I_{\varepsilon+}\) denote the poset \(P \times \{0, 1\}\) where \((a, i) \leq (b, j)\) iff \(a + \varepsilon < b\) or \(a \leq b\) and \(i = j.\) Then \(d_I(M, N) = \varepsilon\) iff the functors \(M, N : P \to C\) extend to a functor \(I_{\varepsilon+} \to C\) \([11]\).

Assume that we have persistence modules indexed by \(R = (\mathbb{R}, \leq)\) and with values in \(\text{Vect},\) where \(\text{Vect}\) is the category of \(k\)-vector spaces and \(k\)-linear maps for some field \(k.\) Such a persistence module \(M\) is called \(q\)-tame if for each \(s < t,\) the linear map \(M_s : M_s \to M_t\) has finite rank. Persistence modules that are \(q\)-tame have well-defined persistence diagrams \([7, 17]\). Let \(\varepsilon \geq 0.\) An \(\varepsilon\)-matching between persistence diagrams is a partial bijection such that matched pairs are within \(\varepsilon\) of each other and unmatched elements are within \(\varepsilon\) of the diagonal. The bottleneck distance between two persistence diagrams is the infimum of all \(\varepsilon \geq 0\) such that there exists an \(\varepsilon\)-matching between them.

7.3.3. Filtered closure spaces obtained from metric spaces. Here we consider filtered closure spaces obtained from the three 1-parameter families of closure operations obtained from a metric space in Definition \([4, 11]\).

Let \(F_{p\text{Cl}}, F_{p\text{Clq}},\) and \(F_{p\text{Clq}}\) denote the corresponding full subcategories of \(F_P \text{Cl}.\) In particular, using the isomorphism \(\text{Cl}_{q} \cong \text{DiGph} (\text{Proposition 3.13}),\) we have the category \(F_P \text{DiGph},\) whose objects are \(P\)-filtered digraphs \((X_\bullet, E_\bullet),\) where for all \(p \in P, (X_p, E_p)\) is a digraph, and for all \(p \leq q, X_p \subset X_q\) and \(E_p \subset E_q,\) and whose morphisms \(f : (X_\bullet, E_\bullet) \to (Y_\bullet, F_\bullet)\) are given by set maps \(f_p : X_p \to Y_p\) such that for all \(p \in P, f(E_p) \subset F_p\) and for \(p \leq q, f_\bullet = f_q|X_p.\)

Recall the poset \([0, \infty) \times \{-1, 0, 1\}\) with the lexicographic order from Section \([4]\) and the corresponding closure operations (Definition \([4, 11]\)). Given a metric space \((X, d),\) there is a \([0, \infty) \times \{-1, 0, 1\}\)-filtered symmetric closure space \((X, \{c_\varepsilon\}_{\varepsilon \in [0, \infty) \times \{-1, 0, 1\}})\). This filtered closure space restricts to the \([0, \infty)\)-filtered symmetric quasi-discrete closure space
(X, \{c_ε\}_{ε \in [0,\infty)}), the [0,∞)-filtered symmetric quasi-discrete closure space (X, \{c_ε\}_{ε \in [0,\infty)}), the [0,∞)-filtered symmetric closure space (X, \{c_ε^+\}_{ε \in [0,\infty)}).

The latter three are persistence modules indexed by [0,∞) with values in Cl. For each of the three distinct pairs in the set \{-1,0,1\}, (X, \{c_ε\}_{ε \in [0,∞)} \times \{-1,0,1\}) restricts to a persistence module indexed by \{0\}, which shows that each of the three [0,∞)-indexed persistence modules have pairwise interleaving distance zero. Furthermore, (X, \{c_ε\}_{ε \in [0,∞)} \times \{-1,0,1\}) demonstrates that these pairwise interleavings are coherent [13]. Combining this construction with Lemma 15 and using the next definition we obtain the subsequent result.

**Definition 7.18.** Say that F,G : C → D^P are **objectwise interleaved** if for all C ∈ C, F(C) and G(C) are interleaved.

**Theorem 7.19.** Let Met → F_{[0,∞)} \times \{-1,0,1\} Cl be the functor defined by mapping (X,d) to (X,c_{d,a}) and mapping f : (X,d) → (Y,e) to f : (X,c_{d,a}) → (Y,c_{e,b}). Restricting to −1 and 0, this functor specializes to two functors Met → F_{[0,∞)} Clsqd. Restricting to 1 this functor specializes to a functor Met → F_{[0,∞)} Cl. The objectwise interleaving distance between the resulting three functors Met → F_{[0,∞)} Cl is zero and these interleavings are coherent.

Let k be a field. Let Vect denote the category of k-vector spaces and k-linear maps. Let P be a poset. Functors P → Vect are called **P-indexed persistence modules** and natural transformations between such functors are called **morphisms of persistence modules** [10]. These are the objects and maps in the category Vect^P.

Let H_{d,a} be one of the singular cubical homology functors, H_{d,a}^{I,\times}, H_{d,a}^{J_1,\times}, H_{d,a}^{J_0,\times}, H_{d,a}^{I,\otimes}, H_{d,a}^{J_1,\otimes}, H_{d,a}^{J_0,\otimes}, or H_{d,a}^{I,\otimes}, from Section 6.6 or one of the singular simplicial homology functors, H_{d,a}^I, H_{d,a}^J, H_{d,a}^{I'}, from Section 6.2, each with coefficients in the field k (Section 6.4). By composing the functors in Theorem 7.19 with one of these homology functors we obtain the following [10, 11].

**Corollary 7.20.** For each of our singular cubical and simplicial homology functors and for each j ≥ 0, there is a functor Met → Vect_{[0,∞)} \times \{-1,0,1\}, which maps (X,d) to H_j(X,c_{d,a}) and f : (X,d) → (Y,e) to f : H_j(X,c_{d,a}) → H_j(Y,c_{d,b}). Choosing one of \{-1,0,1\}, this functor specializes to three functors Met → Vect_{[0,∞)} , the objectwise interleaving distance between these functors is zero, and these interleavings are coherent.

For the remainder of this section, we restrict to the case 0 ∈ \{-1,0,1\}. That is, we consider the closures (X,c_{r,a}) for r ≥ 0, where for A ⊂ X, c_{r,a}(A) = \{x ∈ X \mid d(a,x) ≤ r for some a ∈ A\}.

7.3.4. Metric spaces, filtered closure spaces and intermediate structures. Here we consider a sequence of generalizations from metric spaces to filtered closure spaces and their corresponding persistence modules.

**Definition 7.21.** Let Lawv denote the category of small Lawvere metric spaces, i.e., extended quasi-pseudo-metric spaces and 1-Lipschitz maps.

**Definition 7.22.** Let P be a poset. Let wPDiGph be the category of P-weighted (simple) digraphs (X,E,w), i.e. digraphs (X,E) together with functions ω : E → P. The morphisms f : (X,E,w) → (Y,F,v) are digraph homomorphisms f : (X,E) → (Y,F) such that v(f(x),f(x')) ≤ w(x,x') for all xEx'.
For example, consider \( w_{[0,\infty]} \text{DiGph} \) of \([0,\infty)\)-weighted digraphs. Note that we can also view \( w_{[0,\infty]} \text{DiGph} \) as the category of \([0,\infty)\]-weighted complete digraphs. Indeed, given a \([0,\infty)\)-weighted digraph \((X,E,w)\) we can associate to it the complete digraph \((X,X \times X \setminus \Delta, w)\), where we extend \(w\) by assigning elements of \(X \times X \setminus \Delta \setminus E\) the value \(\infty\). Conversely, given an edge in a complete simple digraph with weight \(\infty\), delete it.

For each Lawvere metric space \((X,d)\), there is a corresponding \([0,\infty)\]-weighted digraph \((X,X \times X \setminus \Delta, d|_{X \times X \setminus \Delta})\). Under this mapping, 1-Lipschitz maps become morphisms of \([0,\infty)\]-weighted digraphs.

For each \([0,\infty)\)-weighted digraph \((X,E,w)\), there is a corresponding \([0,\infty)\)-filtered digraph \((X,E_\bullet)\), where for \(a \in [0,\infty)\), \(E_a = \{(x,x') \in E \mid w(x,x') \leq a\}\). Under this mapping, morphisms of weighted digraphs become morphisms of filtered digraphs. This \([0,\infty)\)-filtered weighted graph is right continuous. That is, for all \(r \geq 0\), \(X_r = \cap_{\varepsilon>0} X_{r+\varepsilon}\) and \(E_r = \cap_{\varepsilon>0} E_{r+\varepsilon}\). Denote the full subcategory of right continuous \([0,\infty)\)-filtered digraphs by \(F_{[0,\infty)} \text{DiGph}_{\text{rec}}\).

Similarly, say that a \([0,\infty)\)-filtered closure space \((X_\bullet,c_\bullet)\) is right continuous if for all \(r \geq 0\), \(X_r = \cap_{\varepsilon>0} X_{r+\varepsilon}\) and for all \(A \subset X_r\), \(c_r(A) = \cap_{\varepsilon>0} c_{r+\varepsilon}(A)\). Let \(F_{[0,\infty)} \text{Cl}_{\text{sqd,rc}}\) denote the full subcategory of \(F_{[0,\infty)} \text{Cl}\) of right continuous \([0,\infty)\)-filtered symmetric, quasi-discrete closure spaces. Then \(F_{[0,\infty)} \text{DiGph}_{\text{rec}} \cong F_{[0,\infty)} \text{Cl}_{\text{sqd,rc}}\).

Combining these observations we have the following.

**Proposition 7.23.** We have the following full embeddings of categories

\[
\text{Met} \hookrightarrow \text{Lawv} \hookrightarrow w_{[0,\infty)} \text{DiGph} \hookrightarrow F_{[0,\infty)} \text{DiGph}_{\text{rec}} \cong F_{[0,\infty)} \text{Cl}_{\text{sqd,rc}} \hookrightarrow F_{[0,\infty)} \text{Cl}
\]

whose composition sends the metric space \((X,d)\) to \((X,c_\bullet,d)\).

Composing these functors with homology we have the following.

**Corollary 7.24.** For each of our singular cubical and simplicial homology functors and for each of the categories \(\text{Met}, \text{Lawv}, w_{[0,\infty)} \text{DiGph}, F_{[0,\infty)} \text{Cl}\), we have a functor to \(\text{Vect}^{[0,\infty)}\).

Let \(R\) denote the ordered set \((\mathbb{R}, \leq)\). We may extend \([0,\infty)\)-filtered closure spaces to \(R\)-filtered closure spaces by setting \(c_a\) to be the discrete closure for \(a < 0\). In particular, \([0,\infty)\)-filtered digraphs become \(R\)-filtered digraphs by setting \(E_a\) to be empty for \(a < 0\).

**Proposition 7.25.** We have the following full embeddings of categories

\[
\text{Met} \hookrightarrow \text{Lawv} \hookrightarrow w_R \text{DiGph} \hookrightarrow F_R \text{DiGph}_{\text{rec}} \cong F_R \text{Cl}_{\text{sqd,rc}} \hookrightarrow F_R \text{Cl},
\]

whose composition sends the metric space \((X,d)\) to the \((X,c_\bullet,d)\).

Note that the image of the composition \(\text{Met} \hookrightarrow \text{Lawv} \hookrightarrow w_R \text{DiGph} \hookrightarrow F_R \text{DiGph}\) lies in \(F_R \text{Gph}\). In particular, a metric space \((X,d)\) is mapped to the \(R\)-filtered graph \((X,E_\bullet)\), where for \(x,x' \in X\), \(x \in E_{t\varepsilon t} x'\) iff \(d(x,x') \leq t\).

Again, composing the functors in Proposition 7.23 with homology we have the following.

**Corollary 7.26.** For each of our singular cubical and simplicial homology functors and for each of the categories \(\text{Met}, \text{Lawv}, w_R \text{DiGph}, F_R \text{Cl}\), we have a functor to \(\text{Vect}^R\).

### 7.3.5 Lipschitz maps

We now extend our results to the case of Lipschitz maps.

**Definition 7.27.** Let \(\text{Lip}\) denote the category of metric spaces and Lipschitz maps. Let \(\text{LipLawv}\) denote the category of small Lawvere metric spaces and Lipschitz maps.
Definition 7.28. Let \( \text{w}^x_{[0, \infty)} \text{DiGph} \) denote the category \([0, \infty)\)-weighted digraphs and maps \( f : (X, E, w) \to (Y, F, v) \) given by digraph homomorphisms \( f : (X, E) \to (Y, F) \) such that there exists a \( K \) such that for all \( xEX', v(f(x), f(x')) \leq Kw(x, x') \).

Definition 7.29. Let \( \text{F}^x_{[0, \infty)} \text{Cl} \) denote the category of \([0, \infty)\)-filtered closure spaces together with maps \( f : (X_\bullet, c_\bullet) \to (Y_\bullet, c'_\bullet) \) given by set maps \( f : X \to Y \), where \( X = \bigcup_{t \in R} X_t \) and \( Y = \bigcup_{t \in R} Y_t \), such that there exists a \( K \) such that for all \( x \geq 0 \) \( f(X_r) \subset Y_{Kr} \) and \( f|_{X_r} : (X_r, c_r) \to (Y_{Kr}, c'_{Kr}) \) is continuous. That is, for all \( A \subset X_r \), \( f(c_r(A)) \subset c'_{Kr}(f(A)) \). Similarly, define \( \text{F}^x_{[0, \infty)} \text{DiGph} \).

Following Proposition 7.23, we have the following.

Proposition 7.30. We have the following full embeddings of categories
\[
\text{Lip} \hookrightarrow \text{LipLawv} \hookrightarrow \text{w}^x_{[0, \infty)} \text{DiGph} \hookrightarrow \text{F}^x_{[0, \infty)} \text{DiGph}_{rc} \cong \text{F}^x_{[0, \infty)} \text{Cl}_{sqd, rc} \hookrightarrow \text{F}^x_{[0, \infty)} \text{Cl}
\]
whose composition sends the metric space \((X, d)\) to \((X, c_\bullet, d)\).

Definition 7.31. Let \( \text{w}^x_{[-\infty, \infty)} \text{DiGph} \) denote the category of \([-\infty, \infty)\)-weighted digraphs and maps \( f : (X, E, w) \to (Y, F, v) \) given by digraph homomorphisms \( f : (X, E) \to (Y, F) \) such that there exists an \( L \) such that for all \( xEX', \ v(f(x), f(x')) \leq w(x, x') + L \). Let \( \text{F}^x_R \text{DiGph}_{rc} \) denote the category of right-continuous \( R \)-filtered digraphs together with maps \( f : (X_\bullet, E_\bullet) \to (Y_\bullet, F_\bullet) \) consisting of set maps \( f : X \to Y \), where \( X = \bigcup_{t \in R} X_t \) and \( Y = \bigcup_{t \in R} Y_t \), such that there exists an \( L \) for which for all \( t \), \( f(X_t) \subset Y_{t+L} \) and \( f(E_t) \subset F_{t+L} \). Similarly, let \( \text{F}^x_R \text{Cl}_{sqd, rc} \) denote the category of right-continuous \( R \)-filtered symmetric, quasi-discrete closure spaces together with maps \( f : (X_\bullet, c_\bullet) \to (Y_\bullet, c'_\bullet) \) consisting of set maps \( f : X \to Y \), where \( X = \bigcup_{t \in R} X_t \) and \( Y = \bigcup_{t \in R} Y_t \), such that there exists an \( L \) for which for all \( t \) and for all \( A \subset X_t \), \( f(c_t(A)) \subset c'_{t+L}(f(A)) \).

There is an isomorphism of categories \( \text{w}^x_{[0, \infty)} \text{DiGph} \cong \text{w}^x_{[-\infty, \infty)} \text{DiGph} \) given by mapping \( (X, E, w) \) to \( (X, E, w') \) where \( w'(e) = \log w(e) \), for \( e \in E \), and \( \log(0) = -\infty \). Furthermore, there are isomorphisms of categories \( \text{F}^x_R \text{DiGph}_{rc} \cong \text{F}^x_R \text{Cl}_{sqd, rc} \). For a metric space \((X, d)\), its image in \( \text{F}^x_R \text{Cl}_{sqd, rc} \) is \((X, c_\bullet)\) where \( c_t(x) = \{ y \in X \mid \log d(x, y) \leq t \} = \{ y \in X \mid d(x, y) < \exp t \} \).

Proposition 7.32. We have the following full embeddings of categories
\[
\text{Lip} \hookrightarrow \text{LipLawv} \hookrightarrow \text{w}^x_{[-\infty, \infty)} \text{DiGph} \hookrightarrow \text{F}^x_R \text{DiGph}_{rc} \cong \text{F}^x_R \text{Cl}_{sqd, rc} \hookrightarrow \text{F}^x_R \text{Cl}_{rc}
\]
whose composition sends the metric space \((X, d)\) to \((X, c_{\exp(\bullet), d})\).

Let \( f : (X_\bullet, c_\bullet) \to (Y_\bullet, c'_\bullet) \in \text{F}^x_R \text{Cl}_{rc} \). Then there exists an \( L \) such that for all \( t \in \mathbb{R} \) and for all \( A \subset X_t \), \( f(c_t(A)) \subset c'_{t+L}(f(A)) \). Let \( \|f\| \) denote the infimum of all such \( L \). Since \( (Y_\bullet, c'_\bullet) \) is right continuous, for all \( t \in \mathbb{R} \) and all \( A \subset X_t \), \( f(c_t(A)) \subset c'_{t+\|f\|}(f(A)) \). That is, for all \( t \in \mathbb{R} \), \( f : (X, c_t) \to (Y, c_{t+\|f\|}) \) is continuous.

Definition 7.33. Let \( \text{Vect}^R \) denote the category whose objects are \( R \)-indexed persistence modules and whose morphisms are given by pairs \((f, L)\) where \( f : M_* \to N_{*+L} \).

Theorem 7.34. Let \( H_* \) be one of our singular cubical homology functors or singular simplicial homology functors and let \( j \geq 0 \). Then we have a functor \( \text{F}^x_R \text{Cl}_{rc} \to \text{Vect}^R \) given
by mapping \((X, c_{\bullet}) \) to \(H_j(X, c_{\bullet})\) and mapping \(f : (X, c_{\bullet}) \to (Y, c'_{\bullet})\) to \(f^*_j : H_j(X, c_{\bullet}) \to H_j(Y, c'_{\bullet + \|f\|})\).

Composing our functors we have the following.

**Theorem 7.35.** For each of our singular cubical or simplicial homology functors \(H_{\bullet}\) and \(j \geq 0\), we have

\[ \text{Lip} \hookrightarrow \text{LipLawv} \hookrightarrow w^+_{[-\infty, \infty)} \text{DiGph} \hookrightarrow F_R^+ \text{DiGph}_{\text{rec}} \cong F_R^+ \text{Cl}_{\text{sqd}, \text{rec}} \hookrightarrow F_R^+ \text{Cl}_{\text{rec}} \xrightarrow{H_{\bullet}} \text{Vect}_R, \]

whose composition maps \((X, d)\) to \(H_j(X, c_{\exp(\bullet), d})\) and maps \(f : (X, d) \to (Y, d')\) to \(f^*_j : H_j(X, c_{\exp(\bullet), d}) \to H_j(Y, c_{\exp(\bullet + \log\|f\|), d'})\), where \(|f|\) denotes the Lipschitz constant of \(f\).

7.4. **Stability.** Here we prove a number of stability theorems for filtered closure spaces and the persistence modules arising from them. We show that the homology of sublevel set filtrations is stable. For a pair of filtered closure spaces we define their Gromov-Hausdorff distance and we show that our singular cubical and simplicial homology persistence modules are stable with respect to this distance. We also show that the Vietoris-Rips and Čech constructions are stable with respect to Gromov-Hausdorff distance.

7.4.1. **Sublevel sets.** We start by defining sublevel set filtrations and showing that they are stable.

**Definition 7.36.** Let \((X, c) \in \text{Cl}\) and let \(f : X \to \mathbb{R}\) be a set map. Define \(\text{Sub}(f) \in F_R \text{Cl}\) to be given by \(\text{Sub}(f)_t = f^{-1}(-\infty, t]\) together with the subspace closure.

For a closure space \((X, c)\) and \(f, g : X \to \mathbb{R}\), let \(d_\infty(f, g) = \sup_{x \in X} |f(x) - g(x)|\). Then for \(\varepsilon = d_\infty(f, g)\), \(\text{Sub}(f), \text{Sub}(g) \in F_R \text{Cl}\) are \(\varepsilon\)-interleaved. It follows that for any of our singular cubical and simplicial homology theories, \(H_j(\text{Sub}(f))\) and \(H_j(\text{Sub}(g))\) are \(\varepsilon\)-interleaved \([10, 11]\). Furthermore, if coefficients are in a field and \(H_j(\text{Sub}(f))\) and \(H_j(\text{Sub}(g))\) are \(q\)-tame, then there is an \(\varepsilon\)-matching between \(D(H_j(\text{Sub}(f)))\) and \(D(H_j(\text{Sub}(g)))\) \([7, 17]\).

**Theorem 7.37** (Sublevel set Stability Theorem). Let \((X, c) \in \text{Cl}\) and \(f, g : X \to \mathbb{R}\). Let \(H\) denote one of our singular cubical or simplicial homology theories and let \(j \geq 0\). Then

\[ d_1(H_j(\text{Sub}(f)), H_j(\text{Sub}(g))) \leq d_\infty(f, g). \]

If coefficients are taken to be in a field and \(H_j(\text{Sub}(f))\) and \(H_j(\text{Sub}(g))\) are \(q\)-tame then

\[ d_B(D(H_j(\text{Sub}(f))), D(H_j(\text{Sub}(g)))) \leq d_\infty(f, g). \]

7.4.2. **Gromov-Hausdorff distance.** Next, we use multivalued maps and correspondences to define a Gromov-Hausdorff distance for filtered closure spaces.

**Definition 7.38.** A **multivalued map** \(C : X \rightrightarrows Y\) from a set \(X\) to a set \(Y\) is a subset of \(X \times Y\), also denoted \(C\), that projects surjectively onto \(X\) through the canonical projection \(\pi_X : X \times Y \to X\). The **image** \(C(A)\) of a subset \(A\) of \(X\) is the canonical projection of \(\pi_X^{-1}(A)\) onto \(Y\). That is, \(C(A) = \{ y \in Y \mid \exists a \in A, c_y a \}\).

**Definition 7.39.** A **single-valued** map \(f\) from \(X\) to \(Y\) is **subordinate** to \(C\) if we have \((x, f(x)) \in C\) for every \(x \in X\). In that case we write \(f : X \xrightarrow{C} Y\). The composition of two multivalued maps \(C : X \rightrightarrows Y\) and \(D : Y \rightrightarrows Z\) is the multivalued map \(D \circ C : X \rightrightarrows Z\) defined by:

\[ xD \circ Cz \iff \exists y \in Y, xCy, yDz \]
Note that if $f$ is subordinate to $C$ and $g$ is subordinate to $D$ then $g \circ f$ is subordinate to $D \circ C$. Also note that if $A \subseteq X$, then $(D \circ C)(A) = D(C(A))$.

**Definition 7.40.** If $C : X \rightrightarrows Y$ is a multivalued map, the transpose of $C$, denoted $C^T$, is the image of $C$ through the symmetry map $(x,y) \mapsto (y,z)$. Although $C^T$ is well-defined as a subset of $Y \times X$, it is not always a multivalued map because it may not project surjectively onto $Y$.

**Definition 7.41.** A multivalued map $C : X \rightrightarrows Y$ is a correspondence if the canonical projection $\pi_Y : C \to Y$ is surjective, or equivalently, if $C^T$ is also a multivalued map.

Note that if $C : X \rightrightarrows Y$ is a correspondence then the identity maps $1_X$ and $1_Y$ are subordinate to the compositions $C^T \circ C$ and $C \circ C^T$, respectively.

**Definition 7.42.** Let $(X,d), (Y,e) \in \text{Met}$ and let $\varepsilon \geq 0$. Say that $C : X \rightrightarrows Y$ is a metric $\varepsilon$-multivalued map if whenever $x Cy$ and $x' Cy'$ then $e(y,y') \leq d(x,x') + \varepsilon$. Say that a correspondence $C : X \rightrightarrows Y$ is a metric $\varepsilon$-correspondence if $C$ and $C^T$ are metric $\varepsilon$-multivalued maps. The metric distortion of a correspondence $C : X \rightrightarrows Y$ is given by $\text{dist}_m(C) = \inf \{\varepsilon \geq 0 \mid C \text{ is an } \varepsilon\text{-correspondence} \}$, with $\text{dist}_m(C) = \infty$ if there is no such $\varepsilon$. The Gromov-Hausdorff distance between $(X,d)$ and $(Y,e)$ is given by $d_{GH}(X,d),(Y,e)) = \frac{1}{2} \inf \text{ dist}_m(C)$, where the infimum is taken over all correspondences $C : X \rightrightarrows Y$ [14, Section 7.3].

Recall that $R$ denotes the totally ordered set $(\mathbb{R}, \leq)$ and for $(X_t, c_t) \in \text{F}_{R\text{Cl}}$, $X = \bigcup_t X_t$.

**Definition 7.43.** Let $(X_t, c_t), (Y_t, d_t) \in \text{F}_{R\text{Cl}}$. A multivalued map from $(X_t, c_t)$ to $(Y_t, d_t)$ is a multivalued map $C : X \rightrightarrows Y$. Write $C : (X_t, c_t) \rightrightarrows (Y_t, d_t)$. Similarly, a correspondence from $(X_t, c_t)$ to $(Y_t, d_t)$ is a correspondence $C : X \rightrightarrows Y$. Let $\varepsilon \geq 0$. Say that a multivalued map $C : (X_t, c_t) \rightrightarrows (Y_t, d_t)$ is an $\varepsilon$-multivalued map if for all $t$,

1. whenever $x \in X_t$ and $xCy$ then $y \in Y_{t+\varepsilon}$, and
2. whenever $A \subseteq X_t$ and $f : X \rightrightarrows Y$ then $C(c_t(A)) \subseteq d_{t+\varepsilon}(f(A))$.

Say that a correspondence $C : (X_t, c_t) \rightrightarrows (Y_t, d_t)$ is an $\varepsilon$-correspondence if $C : (X_t, c_t) \rightrightarrows (Y_t, d_t)$ is an $\varepsilon$-multivalued map and if $C^T : (Y_t, d_t) \rightrightarrows (X_t, c_t)$ is an $\varepsilon$-multivalued map.

**Lemma 7.44.** A multivalued map $C : (X_t, c_t) \rightrightarrows (Y_t, d_t)$ is an $\varepsilon$-multivalued map iff for all $t$,

1. whenever $x \in X_t$ and $xCy$ then $y \in Y_{t+\varepsilon}$, and
2. for all $S \subseteq \pi^{-1}_X(X_t)$, $C(c_t(\pi_X S)) \subseteq d_{t+\varepsilon}(\pi_Y S)$.

**Proof.** ($\Leftarrow$) Let $A \subseteq X_t$ and $f : X \rightrightarrows Y$. Let $S = \{(a, f(a)) \mid a \in A\}$. Then $\pi_X(S) = A$, $\pi_Y(S) = f(A)$. Thus $S \subseteq \pi^{-1}_X(X_t)$ and hence $C(c_t(A)) \subseteq d_{t+\varepsilon}(f(A))$.

($\Rightarrow$) Let $S \subseteq \pi^{-1}_X(X_t)$. Let $A = \pi_X(S)$. Then $A \subseteq X_t$. Let $f : X \rightrightarrows Y$. Then $f(A) \subseteq \pi_Y(\pi^{-1}_X(A)) \subseteq \pi_Y(S)$. Therefore $C(c_t(\pi_X S)) \subseteq d_{t+\varepsilon}(f(A)) \subseteq d_{t+\varepsilon}(\pi_Y S)$. □

**Lemma 7.45.** If $C : (X_t, c_t) \rightrightarrows (Y_t, d_t)$ is an $\varepsilon$-multivalued map and $D : (Y_t, d_t) \rightrightarrows (Z_t, e_t)$ is an $\varepsilon'$-multivalued map, then $D \circ C : (X_t, c_t) \rightrightarrows (Z_t, e_t)$ is an $(\varepsilon + \varepsilon')$-multivalued map.

**Proof.** (1) For all $t$ and $x \in X_t$, if $x DCz$ then there is a $y \in Y$ such that $xCy$, $yDz$. Since $x \in X_t$, $y \in Y_{t+\varepsilon}$ and thus $z \in Z_{t+\varepsilon+\varepsilon'}$. 39
(2) Let $S \subset (\pi_X^{DC})^{-1}X_t$. Let $T = (\pi_X^\circ)^{-1}(\pi_X^{DC}(S))$. Then $\pi_X^\circ(T) = \pi_X^{DC}(S)$ and $T \subset (\pi_Y^\circ)^{-1}(X_t)$. Thus $\pi_Y^\circ(T) \subset Y_{t+\epsilon}$ and $(DC)(c_t(\pi_X^\circ(T))) = D(C(c_t(\pi_X^\circ(T)))) \subset D(d_{t+\epsilon}(\pi_Y^\circ(T)))$. Let $U = (\pi_Y^\circ)^{-1}(\pi_Y^\circ(T))$. Then $\pi_Y^\circ(U) = \pi_Y^\circ(T)$ and $U \subset (\pi_Y^\circ)^{-1}(Y_{t+\epsilon})$. Thus $D(d_{t+\epsilon}(\pi_Y^\circ(T))) = D(d_{t+\epsilon}(\pi_Y^\circ(U))) \subset \epsilon_t+\epsilon'$. The first condition is trivial. For the second condition, let $x,y \in \pi_X^\circ(S)$. Since $\pi_X^\circ(S)$ is symmetric. Furthermore if $x \in \pi_X^\circ(S)$ then $d_{GH}(X_t,\pi_X^\circ(S)) = 0$.

**Definition 7.46.** Let $(X_t,c_t),(Y_t,d_t) \in \mathbf{F}_{\mathcal{R}}\mathbf{Cl}$. For a correspondence $C : (X_t,c_t) \Rightarrow (Y_t,d_t)$ define the distortion of $C$ by $\text{dist}(C) = \min\{\epsilon \geq 0 \mid C \text{ is an } \epsilon\text{-correspondence}\}$, where $\text{dist}(C) = \infty$ if there is no such $\epsilon$.

**Definition 7.47.** Let $(X_t,c_t),(Y_t,d_t) \in \mathbf{F}_{\mathcal{R}}\mathbf{Cl}$. Define the Gromov-Hausdorff distance, between $(X_t,c_t),(Y_t,d_t)$, by $d_{GH}((X_t,c_t),(Y_t,d_t)) = \frac{1}{2} \min \text{dist}(C)$, where the infimum is taken over all correspondences $C : (X_t,c_t) \Rightarrow (Y_t,d_t)$.

It follows from Lemma 7.45 that $d_{GH}$ satisfies the triangle inequality. Also note that since the definition of $\epsilon$-correspondence is symmetric, $d_{GH}$ is symmetric. Furthermore if $(X_t,c_t) \equiv (Y_t,d_t)$ then $d_{GH}((X_t,c_t),(Y_t,d_t)) = 0$.

**Theorem 7.48.** The Gromov-Hausdorff distance is an extended pseudometric on isomorphism classes of $R$-filtered closure spaces.

Recall that $\text{Met}$ is a full subcategory of $\mathbf{F}_{\mathcal{R}}\mathbf{Cl}$ by mapping a metric space $(X,d)$ to $(X,c_{*,d})$, where for $A \subset X$, $c_{*,d}(A) = \{x \in X \mid d(a,x) \leq t \text{ for some } a \in A\}$.

**Lemma 7.49.** Let $(X,d),(Y,c) \in \text{Met}$. Then $(X,c_{*,d}),(Y,c_{*,e}) \in \mathbf{F}_{\mathcal{R}}\mathbf{Cl}$. Let $C : X \Rightarrow Y$ and let $\epsilon \geq 0$. Then $C$ is a metric $\epsilon$-multivalued map iff $C$ is an $\epsilon$-multivalued map.

**Proof.** ($\Rightarrow$) The first condition is trivial. For the second condition, let $A \subset X$ and $f : X \Rightarrow Y$. Let $y \in C(c_{*,d}(A))$. Then, $\exists a \in A, \exists x \in X, d(a,x) \leq t, xCy$. Since $aCf(a)$, it follows that $e(f(a),y) \leq d(a,x)+\epsilon \leq t+\epsilon$ and thus $y \in c_{t+\epsilon,e}(f(A))$. Therefore $C(c_{*,d}(A)) \subset c_{t+\epsilon,e}(f(A))$.

($\Leftarrow$) Assume $xCy, x'Cy'$. Let $t = d(x,x')$ and set $S = \{(x,y)\} \subset C$. Since $x' \in c_{t+\epsilon,e}(\pi_X(S))$, we have that $y' \in C(c_{*,d}(\pi_X(S))) \subset c_{t+\epsilon,e}(\pi_Y(S))$. Therefore $e(y,y') \leq t+\epsilon$. 

It follows that our definition of Gromov-Hausdorff distance for $R$-filtered closure spaces Definition 7.47 agrees with the usual definition of the Gromov-Hausdorff distance of metric spaces Definition 7.42.

**Proposition 7.50.** Let $\epsilon \geq 0$. If $(X_t,c_t),(Y_t,d_t) \in \mathbf{F}_{\mathcal{R}}\mathbf{Cl}$ and $C : (X_t,c_t) \Rightarrow (Y_t,d_t)$ is an $\epsilon$-multivalued map then $C : (X_t,\text{qd}(c_t)) \Rightarrow (Y_t,\text{qd}(d_t))$ is an $\epsilon$-multivalued map. Furthermore, if $(X_t,c_t),(Y_t,d_t) \in \mathbf{F}_{\mathcal{R}}\mathbf{Cl}_{\text{qd}}$ and $C : (X_t,c_t) \Rightarrow (Y_t,d_t)$ is an $\epsilon$-multivalued map then $C : (X_t,\text{s}(c_t)) \Rightarrow (Y_t,\text{s}(d_t))$ is an $\epsilon$-multivalued map.

**Proof.** The first statement follows from specializing the second condition in Definition 7.43 to $x \in X_t$. By Lemma 7.44 we may write this specialized second condition as $x \in X_t$,
Proof. 

Corollary 7.51. Let \((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathcal{F}_{\mathcal{R}}\mathcal{C}_\mathcal{L}\), with \(d_{GH}((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet)) = d\). Then 
\[d_{GH}((X_\bullet, s(qd(c_\bullet))), (Y_\bullet, s(qd(d_\bullet)))) \leq d_{GH}((X_\bullet, qd(c_\bullet)), (Y_\bullet, qd(d_\bullet))) \leq d\]

7.4.3. Correspondences, homotopy, contiguity, and interleaving. We relate \(\varepsilon\)-correspondences to one-step \((J_1, \times)\) homotopies and to contiguous simplicial maps.

Lemma 7.52. Let \(X \Rightarrow Y\) be a multivalued map. Then there exists a single-valued map \(f : X \rightarrow Y\) subordinate to \(C\).

Proof. For each \(x \in X\), choose \((x, f(x)) \in \pi^{-1}_X(x)\).

Lemma 7.53. Let \((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathcal{F}_{\mathcal{R}}\mathcal{C}_\mathcal{L}\). Let \(C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)\) be an \(\varepsilon\)-multivalued map. Let \(f : X \mathcal{C}_\mathcal{L} Y\). Then for all \(t\), \(f|_{X_t} : (X_t, c_t) \rightarrow (Y_{t+\varepsilon}, d_{t+\varepsilon})\) is continuous.

Proof. Let \(x \in X_t\). Since \(Xc\mathcal{C}f(x)\), we have that \(f(x) \in Y_{t+\varepsilon}\). Let \(A \subseteq X_t\). Then \(f(c_t(A)) \subseteq C(c_t(A)) \subseteq d_{t+\varepsilon}(f(A))\).

Proposition 7.54. Let \((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathcal{F}_{\mathcal{R}}\mathcal{C}_\mathcal{L}\). Let \(C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)\) be an \(\varepsilon\)-multivalued map. For all \(f, g\) subordinate to \(C\), and for all \(t \in \mathbb{R}\), \(f|_{X_t} : (X_t, c_t) \rightarrow (Y_{t+\varepsilon}, d_{t+\varepsilon})\) are one-step \((J_1, \times)\)-homotopic.

Proof. By Lemma 7.53 \(f|_{X_t}, g|_{X_t} : (X_t, c_t) \rightarrow (Y_{t+\varepsilon}, d_{t+\varepsilon}) \in \mathcal{C}_\mathcal{L}\). Let \(A \subseteq X_t\). By the continuity of \(f|_{X_t}\) and \(g|_{X_t}\), we have \(f(c_t(A)) \subseteq d_{t+\varepsilon}(f(A))\) and \(g(c_t(A)) \subseteq d_{t+\varepsilon}(g(A))\). Since \(C\) is an \(\varepsilon\)-multivalued map, we also have \(g(c_t(A)) \subseteq C(c_t(A)) \subseteq d_{t+\varepsilon}(f(A))\) and similarly \(f(c_t(A)) \subseteq d_{t+\varepsilon}(g(A))\). By Lemma 5.27 \(f|_{X_t}\) and \(g|_{X_t}\) are one-step \((J_1, \times)\)-homotopic.

Combining this result with Theorem 7.13 we have the following.

Corollary 7.55. Let \((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathcal{F}_{\mathcal{R}}\mathcal{C}_\mathcal{L}\). Let \(C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)\) be an \(\varepsilon\)-multivalued map. For all \(f, g\) subordinate to \(C\), and for all \(t \in \mathbb{R}\), \(f|_{X_t}, g|_{X_t} : (X_t, c_t) \rightarrow (Y_{t+\varepsilon}, VR(d_{t+\varepsilon}))\) are contiguous and \(f|_{X_t}, g|_{X_t} : (X_t, c_t) \rightarrow (Y_{t+\varepsilon}, VR(d_{t+\varepsilon}))\) are one-step \((J_1, \times)\)-homotopic.

Definition 7.56. Let \((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathcal{F}_{\mathcal{R}}\mathcal{C}_\mathcal{L}\). Say that \((X_\bullet, c_\bullet)\) and \((Y_\bullet, d_\bullet)\) are \(\varepsilon\) one-step \((J_1, \times)\) homotopy interleaved if there exist \(f : (X_\bullet, c_\bullet) \rightarrow (Y_{t+\varepsilon}, d_{t+\varepsilon})\) and \(g : (Y_\bullet, d_\bullet) \rightarrow (X_\bullet, c_{t+\varepsilon})\) such that for all \(t\), \(g|_{Y_{t+\varepsilon}} \circ f|_{X_t}\) is one-step \((J_1, \times)\) homotopic with \(X_t \leftarrow X_{t+2\varepsilon}\) and \(f|_{X_{t+\varepsilon}} \circ g|_{Y_{t+\varepsilon}}\) is one-step \((J_1, \times)\) homotopic with \(Y_t \leftarrow Y_{t+2\varepsilon}\).

Definition 7.57. Let \((X_\bullet, E_\bullet), (Y_\bullet, F_\bullet) \in \mathcal{F}_{\mathcal{S}}\mathcal{S}_\mathcal{M}_\mathcal{P}\). Say that \((X_\bullet, E_\bullet)\) and \((Y_\bullet, F_\bullet)\) are \(\varepsilon\) contiguity interleaved if there exist \(f : (X_\bullet, E_\bullet) \rightarrow (Y_{t+\varepsilon}, F_{t+\varepsilon})\) and \(g : (Y_\bullet, F_\bullet) \rightarrow (X_\bullet, E_{t+\varepsilon})\) such that for all \(t\), \(g|_{Y_{t+\varepsilon}} \circ f|_{X_t}\) is contiguity with \(X_t \leftarrow X_{t+2\varepsilon}\) and \(f|_{X_{t+\varepsilon}} \circ g|_{Y_{t+\varepsilon}}\) is contiguity with \(Y_t \leftarrow Y_{t+2\varepsilon}\).

Theorem 7.58. Let \((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathcal{F}_{\mathcal{R}}\mathcal{C}_\mathcal{L}\). If there exists an \(\varepsilon\)-correspondence \(C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)\) then \((X_\bullet, c_\bullet)\) and \((Y_\bullet, d_\bullet)\) are \(\varepsilon\) one-step \((J_1, \times)\) homotopy interleaved, \((X_\bullet, VR(c_\bullet))\) and \((Y_\bullet, VR(d_\bullet))\) are \(\varepsilon\) contiguity interleaved, and \((X_\bullet, C(c_\bullet))\) and \((Y_\bullet, C(d_\bullet))\) are \(\varepsilon\) contiguity interleaved. Let \((X_\bullet, E_\bullet), (Y_\bullet, F_\bullet) \in \mathcal{F}_{\mathcal{S}}\mathcal{S}_\mathcal{M}_\mathcal{P}\). If \((X_\bullet, E_\bullet)\) and \((Y_\bullet, F_\bullet)\) are \(\varepsilon\) contiguity interleaved, then \((X_\bullet, G(E_\bullet))\) and \((Y_\bullet, G(F_\bullet))\) are \(\varepsilon\) one-step \((J_1, \times)\) homotopy interleaved.
Proof. Let $C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)$ be a \(\varepsilon\)-correspondence. By Lemma 7.53 there exist $f : X \xrightarrow{C} Y$ and $g : Y \xrightarrow{C^T} X$ such that $f : (X_\bullet, c_\bullet) \to (Y_{\bullet+\varepsilon}, d_{\bullet+\varepsilon})$ and $g : (Y_\bullet, d_\bullet) \to (X_{\bullet+\varepsilon}, c_{\bullet+\varepsilon})$. Since the composition $g \circ f : (X_\bullet, c_\bullet) \to (X_{\bullet+2\varepsilon}, c_{\bullet+2\varepsilon})$ given by $(g \circ f)_t = g_{t+\varepsilon} \circ f_t$ and the inclusion map $(X_\bullet, c_\bullet) \hookrightarrow (X_{\bullet+2\varepsilon}, c_{\bullet+2\varepsilon})$ are both subordinate to $C^T \circ C$ and similarly the composition $f \circ g : (Y_\bullet, d_\bullet) \to (Y_{\bullet+2\varepsilon}, d_{\bullet+2\varepsilon})$ given by $(f \circ g)_t = f_{t+\varepsilon} \circ g_t$ and the inclusion map $(Y_\bullet, d_\bullet) \hookrightarrow (Y_{\bullet+2\varepsilon}, d_{\bullet+2\varepsilon})$ are both subordinate to $C \circ C^T$, by Proposition 7.53 we have that $(X_\bullet, c_\bullet)$ and $(Y_\bullet, d_\bullet)$ are \(\varepsilon\) one-step \((J_1, \times)\) homotopy interleaved.

Applying the functors $\text{VR}$ and $\hat{\text{C}}$, we have morphisms $f : (X_\bullet, \text{VR}(c_\bullet)) \to (Y_{\bullet+\varepsilon}, \text{VR}(d_{\bullet+\varepsilon}))$, $f : (X_\bullet, \hat{\text{C}}(c_\bullet)) \to (Y_{\bullet+\varepsilon}, \hat{\text{C}}(d_{\bullet+\varepsilon}))$, $g : (Y_\bullet, \text{VR}(d_\bullet)) \to (X_{\bullet+\varepsilon}, \text{VR}(c_{\bullet+\varepsilon}))$, and $g : (Y_\bullet, \hat{\text{C}}(d_\bullet)) \to (X_{\bullet+\varepsilon}, \hat{\text{C}}(c_{\bullet+\varepsilon}))$. By Corollary 7.59 (The Rips and Čech Stability Theorem) we have the following.

Now suppose that $f : (X_\bullet, E_\bullet) \to (Y_\bullet, F_\bullet)$ and $g : (Y_\bullet, F_\bullet) \to (X_\bullet, E_\bullet)$ provide an \(\varepsilon\) contiguity interleaving. By Theorem 7.58 it follows that $f : (X_\bullet, G(E_\bullet)) \to (Y_\bullet, G(F_\bullet))$ and $g : (Y_\bullet, G(F_\bullet)) \to (X_\bullet, G(E_\bullet))$ provide an \(\varepsilon\) one-step \((J_1, \times)\) homotopy interleaving. \(\square\)

Corollary 7.59. Let $(X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathbb{F}_R \mathbb{C}$. Assume there exists an \(\varepsilon\)-correspondence $C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)$. Let $H$ denote one of our singular cubical or simplicial homology theories and let $j \geq 0$. Then $H_j(X_\bullet, c_\bullet)$ and $H_j(Y_\bullet, d_\bullet)$ are \(\varepsilon\)-interleaved. If $H_j(X_\bullet, c_\bullet)$ and $H_j(Y_\bullet, d_\bullet)$ are $q$-tame then there exists an \(\varepsilon\)-matching between $D(H_j(X_\bullet, c_\bullet))$ and $D(H_j(Y_\bullet, d_\bullet))$, where $D(\cdot)$ denotes the persistence diagram.

Proof. Given an \(\varepsilon\)-correspondence $C : (X_\bullet, c_\bullet) \Rightarrow (Y_\bullet, d_\bullet)$, by Theorem 7.58 $(X_\bullet, c_\bullet)$ and $(Y_\bullet, d_\bullet)$ are $\varepsilon$ \((J_1, \times)\) homotopy interleaved. The first statement follows from Theorem 6.6. The second statement follows from the Algebraic Stability Theorem 7.17. \(\square\)

7.4.4 Stability theorems. We now use our results on \(\varepsilon\)-correspondences to obtain our desired stability theorems.

As a direct consequence of Theorem 7.58 we have the following.

Theorem 7.60 (Stability Theorem). Let $(X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathbb{F}_R \mathbb{C}$. Let $H$ denote one of our singular cubical or simplicial homology theories and let $j \geq 0$.

$$d_I(H_j(X_\bullet, c_\bullet), H_j(Y_\bullet, d_\bullet)) \leq 2d_{GH}((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet)).$$

As a direct consequence of Corollary 7.59 we have the following.

Corollary 7.61 (Bottleneck Stability Theorem). Let $(X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathbb{F}_R \mathbb{C}$. Let $H$ denote one of our singular cubical or simplicial homology theories, where coefficients are in a field, and let $j \geq 0$. If $H_j(X_\bullet, c_\bullet)$ and $H_j(Y_\bullet, d_\bullet)$ are $q$-tame then

$$d_B(D(H_j(X_\bullet, c_\bullet)), D(H_j(Y_\bullet, d_\bullet))) \leq 2d_{GH}((X_\bullet, c_\bullet), (Y_\bullet, d_\bullet)),$$

where $D(\cdot)$ denotes the persistence diagram and $d_B$ denotes the bottleneck distance.

From Theorem 7.58 we also obtain the following stability theorem, where $d_{HI}$ denote Blumberg and Lesnick’s homotopy interleaving distance [8].

Theorem 7.62 (Rips and Čech Stability Theorem). Let $(X_\bullet, c_\bullet), (Y_\bullet, d_\bullet) \in \mathbb{F}_R \mathbb{C}$. Let $d = d_{GH}((X_\bullet, s(qd(c)_\bullet)), (Y_\bullet, s(qd(d)_\bullet))) \geq 1$ Then

$$d_{HI}((X_\bullet, \text{VR}(c_\bullet)), (Y_\bullet, \text{VR}(d_\bullet))), d_{HI}((X_\bullet, \hat{\text{C}}(c_\bullet)), (Y_\bullet, \hat{\text{C}}(d_\bullet))) \leq 2d.$$
Let $H$ denote simplicial homology and let $j \geq 0$. Then
\[ d_I(H_j(X_\bullet, \text{VR}(c_\bullet)), H_j(Y_\bullet, \text{VR}(d_\bullet))), d_I(H_j(X_\bullet, \tilde{C}(c_\bullet)), H_j(Y_\bullet, \tilde{C}(d_\bullet))) \leq 2d. \]
Assume that coefficients are taken in a field and that the persistence modules are $q$-tame. Then
\[ d_B(D(H_j(X_\bullet, \text{VR}(c_\bullet))), D(H_j(Y_\bullet, \text{VR}(d_\bullet))), d_B(D(H_j(X_\bullet, \tilde{C}(c_\bullet))), D(H_j(Y_\bullet, \tilde{C}(d_\bullet)))) \leq 2d. \]

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**Appendix A. Local bases**

Here we recall the definition of a filter. It turns out that neighborhoods of a point in a closure space form a filter. Furthermore, a basis of this filter at each point is sufficient information to reconstruct the closure operation. Thus, as for topological spaces, a point $x$ is in a closure of a set $A$ if and only if all basic neighborhoods of $x$ have a non-empty intersection with $A$ (see Theorem A.6).

**Definition A.1.** Given a set $S$, a filter on $S$ is a nonempty collection $F \subset \mathcal{P}(S)$ of subsets of $S$ such that:
1) (closed under binary intersection) $A, B \in F \implies A \cap B \in F$.
2) (upward closed) $A \in F, A \subset B \implies B \in F$.

**Definition A.2.** Given a set $S$, a filter base is a nonempty collection $B \subset \mathcal{P}(S)$ of subsets of $S$ such that:
1) (downward directed) for any $A_1, A_2 \in B$, there exists an $A_3 \in B$ such that $A_3 \subset A_1 \cap A_2$.

Given a filter base $B$, the filter generated by $B$ is obtained by taking the upward closure of the base. We also say $B$ is a base for $F$.

**Theorem A.3.** [15, Theorem 14.B.3] Let $U$ be the neighborhood system (Definition 2.7) of a subset $A$ of a closure space $(X, c)$. Then $U$ is a filter on $X$ such that $A \subset \bigcap U$.

**Definition A.4.** [15, Definition 14.B.4] Let $(X, c)$ be a closure space. By Theorem A.3 the neighborhood system of a set $A \subset X$ is a filter. A base of this filter is called a base of the neighborhood system of $A$ in $(X, c)$. We will call a local base at $x$ a base of the neighborhood system of $\{x\}$.

**Proposition A.5.** [15, 14.B.5] If $U_x$ is a local base at $x$ in a closure space $(X, c)$ then the following are true:
(nbd 1) $U_x \neq \emptyset$.
(nbd 2) For all $U \in U_x$, $x \in U$.
(nbd 3) For each $U_1, U_2 \in U_x$, there exists a $U \in U_x$ such that $U \subset U_1 \cap U_2$.

**Theorem A.6.** [15, Theorem 14.B.10] For each element $x$ of a set $X$, let $U_x$ be a collection of subsets satisfying the conditions in Proposition A.5. Then there exists a unique closure operation $c$ for $X$ such that for all $x \in X$, $U_x$ is a local base at $x$ in $(X, c)$. More specifically, $c$ can be defined as
\[ c(A) = \{ x \in X \mid U \in U_x \implies U \cap A \neq \emptyset \} \]
for all \( A \subset X \).

**APPENDIX B. CHARACTERIZATION OF PRODUCTS AND INDUCTIVE PRODUCTS**

**Theorem B.1.** [15, Theorem 17.C.6] The projections of a product space to its coordinate spaces are continuous. Moreover, the product closure is the coarsest closure for the product of underlying sets such that all projections are continuous.

**Theorem B.2.** [15, Theorem 17.D.3] Let \((X, c_X)\) and \((Y, c_Y)\) be closure spaces. A closure \( c \) for \( X \times Y \) is the closure operation \( c_X \boxplus c_Y \) if and only if the following two conditions are fulfilled:

a) Each mapping \( \{ x \mapsto (x, y) \} : (X, c_X) \to (X \times Y, c) \), \( y \in Y \), and also each mapping \( \{ y \mapsto (x, y) \} : (Y, c_Y) \to (X \times Y, c) \), \( x \in X \), is continuous.

b) If a closure \( c_1 \) for \( X \times Y \) satisfies a) with \( c \) replaced by \( c_1 \), then \( c_1 \) is coarser than \( c \). In other words, the inductive product closure is the finest closure operation for \( X \times Y \) for which all mappings in a) are continuous.

**Proposition B.3.** [15, Proposition 17.D.5] Let \((X, c_X), (Y, c_Y)\) and \((Z, c_Z)\) be closure spaces. A function \( f : (X, c_X) \boxplus (Y, c_Y) \to (Z, c_Z) \) is continuous if and only if each mapping

- \( \{ x \mapsto f(x, y) \} : (X, c_X) \to (Z, c_Z), y \in Y \) and also each mapping,
- \( \{ y \mapsto f(x, y) \} : (Y, c_Y) \to (Z, c_Z), x \in X \) is continuous.

**APPENDIX C. SEMI-UNIFORM SPACES**

**Definition C.1.** [15, Definition 23.A.3] Given a set \( X \), a collection \( \Phi \) of subsets of \( X \times X \) is called a semi-uniform structure on \( X \) if the following are satisfied:

- \( \Phi \) is a filter on \( X \times X \).
- \( \Delta_X \subset \bigcap \Phi \), where \( \Delta_X := \{(x, x) \mid x \in X\} \) is the diagonal in \( X \times X \).
- If \( U \in \Phi \), then \( U^{-1} \in \Phi \), where \( U^{-1} := \{(x, y) \mid (y, x) \in U\} \).

The pair \((X, \Phi)\) is called a semi-uniform space.

**Definition C.2.** [15, Definition 23.A.3] Let \( U \) be a relation on a set \( X \). For \( x \in X \), the \( x \) slice of \( U \) is \( U_x := \{y \in X \mid (x, y) \in U\} \). Given a semi-uniform structure \( \Phi \) on a set \( X \), let \( \Phi_x \) be the collection of all the \( x \) slices of elements of \( \Phi \), for \( x \in X \). Then \( \Phi_x \) is a base of a filter whose intersection contains \( x \). Thus by Theorem A.6 there is a unique closure operation \( c_\Phi \) for \( X \) such that each \( \Phi_x \) is a local base at \( x \), for each \( x \in X \). We say the closure \( c_\Phi \) is induced by the semi-uniform structure \( \Phi \). Given a closure space \((X, c)\), if there exists a semi-uniform structure on \( X \) that induces \( c \), we say \( c \) is semi-uniformizable and \((X, c)\) is a semi-uniformizable closure space.

**Theorem C.3** ([15, Theorem 23.B.3]). A closure space is semi-uniformizable if and only if it is symmetric (Definition 3.1).

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