Developing methods for identifying the inflection point of a convex/concave curve

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We are introducing two methods for revealing the true inflection point of data that contains or not error. The starting point is a set of geometrical properties that follow the existence of an inflection point p for a smooth function. These properties connect the concept of convexity/concavity before and after p respectively with three chords defined properly. Finally a set of experiments is presented for the class of sigmoid curves and for the third order polynomials.

Keywords: Inflection point estimation, convex/concave, sigmoid curve, third order polynomial

INTRODUCING GEOMETRICAL METHODS

We are going to present some new methods for identifying the inflection point of any given convex/concave or concave/convex curve based on the definition and the geometry of it. Before starting it is necessary to give some preliminary definitions.

Preliminary Definitions

Let a function \( f : [a, b] \rightarrow R \), \( f \in C^n, n \geq 2 \) which is convex for \( x \in [a, p] \) and concave for \( x \in [p, b] \), \( p \) is the unique inflection point of \( f \) in \([a, b]\) and let an arbitrary \( x \in [a, b]\).

**Definition .1** Total, left and right chord are the lines connecting \( \{(a, f(a)), (b, f(b))\} \), \( \{(a, f(a)), (x, f(x))\} \) and \( \{(x, f(x)), (b, f(b))\} \) with Cartesian equations \( g(x) \), \( l(x) \) and \( r(x) \) respectively.

**Definition .2** Distance from total, left and right chord are the functions \( F, F_l, F_r : [a, b] \rightarrow R \) with:

\[
\begin{align*}
F(x) &= f(x) - g(x) \\
F_l(x) &= f(x) - l(x) \\
F_r(x) &= f(x) - r(x)
\end{align*}
\]

**Definition .3** \( s_l(a, x) \) and \( s_r(b, x) \) are the algebraic surfaces:

\[
\begin{align*}
s_l(a, x) &= \int_a^x F_l(t)dt \\
s_r(b, x) &= \int_x^b F_r(t)dt
\end{align*}
\]

**Definition .4** \( x_l \) and \( x_r \) are such that:

\[
\begin{align*}
x_l &= \text{argmin} \{ s_l(a, x) \} \\
x_r &= \text{argmax} \{ s_r(b, x) \}
\end{align*}
\]

with \( \delta_1, \delta_2 > 0 \) taken as small as necessary for \( x_l, x_r \) to be unique unconstrained extremums in the corresponding intervals.

We call standard partition \((SP)\) the strictly sorted grid of points, not necessary equal spaced:

\[
x_i, i = 0, 1, \ldots, n, \quad a = x_0 < x_1 < \ldots < x_n = b
\]

The corresponding \((x_i, f_i)\) data produced from \( f \) by the no error process:

\[
f_i = f(x_i), i = 0, 1, \ldots, n
\]

If errors occur then we have the \((x_i, \phi_i)\) errored data produced from \( f \) by the process:

\[
\phi_i = f(x_i) + \epsilon_i, \quad \epsilon_i \sim \text{iid}(0, \sigma^2) \quad i = 0, 1, \ldots, n
\]

Our analysis is focused on uniform distributions \((\epsilon_i \sim U(-r, r))\) but it is applicable for every distribution with zero mean, for example the normal \(N(0, \sigma^2)\). If the error distribution is not a zero mean one, then the results are ambiguous.

**Definition .5** For the errored data \( \{\{x_i, \phi_i\}\} \) we have that:

\[
\begin{align*}
\Phi(x_i) &= \phi(x_i) - g(x_i), \quad i = 0, 1, \ldots, n \\
\Phi_f(x_i) &= \phi(x_i) - l(x_i), \quad i = 0, 1, \ldots, n \\
\Phi_r(x_i) &= \phi(x_i) - r(x_i), \quad i = 0, 1, \ldots, n
\end{align*}
\]

**Definition .6** Function \( f \) is called symmetrical around inflection point or symmetry around inflection point exists when:

\[
f(p + x) - f(p) = f(p) - f(p - x), \quad \forall x \in R
\]

or

\[
f(p + x) + f(p - x) - 2f(p) = 0, \quad \forall x \in R
\]

**Definition .7** Function \( f \) is called locally \((\epsilon, \delta)\) asymptotical symmetric around inflection point or local \(\epsilon - \delta\) asymptotic symmetry exists when:

\[
|f(p + x) + f(p - x) - 2f(p)| < \epsilon, \quad \forall x \in (p - \delta, p + \delta)
\]
Definition .8 For a function $f : [a, b] \rightarrow R$ we have that:

1. Data symmetry w.r.t. inflection point exists if $p - b = a - p$
2. Data left asymmetry w.r.t. inflection point exists if $p - b < a - p$
3. Data right asymmetry w.r.t. inflection point exists if $p - b > a - p$

Definition .9 A function $f : [a, b] \rightarrow R$ is called totally symmetric or totally symmetry exist, if it is symmetric around inflection point $p$ and also exist data symmetry w.r.t. $p$.

Definition .10 For every subsequent $x_i < x_j$ the elementary trapezoidal estimation holds:

$$\int_{x_i}^{x_j} f(x)dx \approx T_{i,j}(f, x_i, x_j) = \frac{f(x_i)+f(x_j)}{2}(x_j - x_i)$$ (19)

And for every standard partition the total trapezoidal estimation holds:

$$\int_{a}^{b} f(x)dx \approx T_{n+1}(f, a, b) = \sum_{i=0}^{n-1} T_{i,i+1}(f, x_i, x_{i+1})$$ (20)

The Extremum Surface Method

We can prove that:

Lemma .1 $x$-left ($x_1$) and $x$-right ($x_r$) are the points where left and right chord respectively are tangent to the graph $G(f)$.

Proof

1. Let $x_l \leq p$ be the first point where $l(x)$ cuts $G(f)$ from left to right. The function is convex for $x \in [a, p]$, so $G(f)$ is always below the left chord, thus giving a negative value for $s_l(a, x_l)$, which is increasing in absolute value as $x_l$ departures from $a$ to the right until point $p$. After inflection point $f$ is increasing, so it is possible to continue adding negative values of surface until the point $x^* \in (p, b]$ for which $G(f)$ and $l(x)$ have one only common point. If we continue beyond this point, then $G(f)$ and $l(x)$ have again two common points $(x_1, y_1), (x_2, y_2)$, $x_1 < x_2$ such that $s_l(a, x_1) < 0$ and $s_l(x_1, x_2) > 0$, thus we have started adding positive values to the $s_l(a, x)$ and this leads to a raise of the total value $s_l(a, x) = s_l(a, x_1) + s_l(x_1, x_2)$. So, function $s_l(a, x)$ is decreasing for $x \in [a, x^*]$ and increasing for $x \in [x^*, b]$, thus $x^* > p$ is a local minimum and we call it $x_l$.

2. Let $x_r \geq p$ be the first point where $r(x)$ cuts $G(f)$ from right to left. The function is concave for $x \in [p, b]$, so $G(f)$ is always above the right chord, thus giving a positive value for $s_r(b, x_r)$, which is increasing as $x_r$ departures from $b$ to the left until inflection point $p$. After that point $f$ is still above the right chord until the point $x^* \in [a, p)$ for which $G(f)$ and $r(x)$ have only one common point. If we continue again beyond this point, then $G(f)$ and $r(x)$ have again two common points $(x_1, y_1), (x_2, y_2)$, $x_1 > x_2$ such that $s_r(b, x_1) > 0$ and $s_r(x_1, x_2) < 0$, so we have started adding negative values to the $s_r(b, x)$ and this leads to a reduction of the total value $s_r(b, x) = s_r(b, x_1) + s_r(x_1, x_2)$. So, function $s_r(b, x)$ is increasing for $x \in [a, x^*]$ and decreasing for $x \in [x^*, b]$, thus $x^* < p$ is a local minimum and we call it $x_r$.

Corollary .1

$$x_l = \arg_{x \in [a, b+\delta_1]} \left\{ f'(x) = \frac{f(x) - f(a)}{x - a} \right\}$$ (21)

$$x_r = \arg_{x \in [a - \delta_2, b]} \left\{ f'(x) = \frac{f(b) - f(x)}{b - x} \right\}$$ (22)

with $\delta_1, \delta_2 > 0$ taken as small as necessary for $x_l, x_r$ to be unique unconstrained solutions in the corresponding intervals.

Corollary .2 Let a function $f : [a, b] \rightarrow R$, $f \in C^{(n)}$, $n \geq 2$ which is convex for $x \in [a, p]$ and concave for $x \in [p, b]$. Then we have one of the following possibilities:

1. If $x_l, x_r \in [a, b]$ then $a \leq x_r < x_l \leq b$
2. If $x_l \notin [a, b]$ then $x_l > b$
3. If $x_r \notin [a, b]$ then $x_r < a$

We define the next theoretical estimator of the inflection point:

Definition .11 The theoretical extremum surface estimator (TESE) is

$$x_S = \begin{cases} \frac{x_1+x_r}{2} & , \ x_1, x_r \in [a, b] \\ \frac{x_1+x_r}{2} & , \ x_1 > b \\ \frac{x_1+x_r}{2} & , \ x_r < a \end{cases}$$ (23)

Lemma .2 If the mesh $\lambda(n)$ of the standard partition is such that:

$$\lim_{n \rightarrow \infty} n\lambda(n)^2 = 0$$

then $T_{n+1}(\phi, a, b)$ is a consistent estimator of the true value of $T_{n+1}(f, a, b)$. 

Proof
For every subsequent \( x_i < x_{i+1} \) the elementary trapezoidal estimation is:

\[
T_{i,i+1}(\phi, x_i, x_{i+1}) = \frac{x_{i+1} - x_i}{2} \phi(x_i) + \frac{x_{i+1} - x_i}{2} \phi(x_{i+1})
\]  
(24)

Taking the expected value we obtain:

\[
E(T_{i,i+1}(\phi, x_i, x_{i+1})) = \frac{x_{i+1} - x_i}{2} E(\phi(x_i)) + \frac{x_{i+1} - x_i}{2} E(\phi(x_{i+1}))
\]
(25)

so from the linearity of expected value we have also that:

\[
E(T_{n+1}(\phi, a, b)) = \sum_{i=0}^{n-1} E(T_{i,i+1}(\phi, x_i, x_{i+1})) = T_{n+1}(f, a, b)
\]  
(26)

Thus our estimator is unbiased.

We continue by computing the variance of the elementary trapezoidal estimation:

\[
V(T_{i,i+1}(\phi, x_i, x_{i+1})) = \frac{(x_{i+1} - x_i)^2}{4} V(\phi(x_i)) + \frac{(x_{i+1} - x_i)^2}{4} V(\phi(x_{i+1}))
\]
(27)

We have two cases.
If standard partition is equal spaced, then \( x_{i+1} - x_i = \frac{b-a}{n} \) and we obtain:

\[
V(T_{i,i+1}(\phi, x_i, x_{i+1})) = \frac{(b-a)^2}{2n^2} \sigma^2
\]  
(28)

Let’s compute now the variance of estimator \( T_{n+1}(\phi, a, b) \):

\[
V(T_{n+1}(\phi, a, b)) = V\left(\sum_{i=0}^{n-1} T_{i,i+1}(\phi, x_i, x_{i+1})\right)
\]
(29)

so clearly it holds:

\[
\lim_{n \to \infty} V(T_{n+1}(\phi, a, b)) = \lim_{n \to \infty} \frac{(b-a)^2}{2n^2} \sigma^2 = 0
\]

For the second case, if standard partition is not equal spaced then the mesh or norm of the partition is

\[
\lambda(n) = \max_{i=0, \ldots, n-1} (x_{i+1} - x_i)
\]

Then it is easy to show that:

\[
V(T_{i,i+1}(\phi, x_i, x_{i+1})) \leq \frac{\lambda(n)^2}{2} \sigma^2
\]  
(30)

and the total variance is:

\[
V(T_{n+1}(\phi, a, b)) \leq \frac{\sigma^2}{2} n \lambda(n)^2 \xrightarrow{n \to \infty} 0
\]  
(31)

from our hypothesis.
So the estimator is consistent.

Now we are able to compute using our trapezoidal rule \(10\) data estimations for \( s_l(x_0, x_j) \) and \( s_r(x_n, x_j) \):

Definition .12

\[
s_{l,j+1}(x_0, x_j) = T_{j+1}(\Phi, x_0, x_j)
\]  
(32)

\[
s_{r,n-j+1}(x_n, x_j) = T_{n-j+1}(\Phi_r, x_j, x_n)
\]  
(33)

(34)

It is time to define the next data estimators for \( x_l, x_r \).

Definition .13 \( \chi_l, \chi_r \) are such that:

\[
\chi_l = x_{j_l}
\]  
(35)

\[
\chi_r = x_{j_r}
\]  
(36)

We define now the errored data estimator of the inflection point:

Definition .14 The data extremum surface estimator (ESE) is

\[
\chi_S = \frac{\chi_l + \chi_r}{2}
\]  
(38)

Lemma .3 The ESE is a consistent estimator of TESE with all relevant integrals calculated via trapezoidal rule.

Proof
We have proven in Lemma[3] that trapezoidal rule for the errored data gives a consistent estimator for the trapezoidal estimation of the actual data, thus \( \chi_l, \chi_r \) are consistent estimators of the true \( x_l, x_r \) respectively, with relevant integrals trapezoidal calculated.

If the interval \([a, b]\) is such that both \( x_l, x_r \in [a, b] \) then ESE is a consistent estimator of trapezoidal calculated \( x_S = \frac{x_l+x_r}{2} \).

If \( x_l > b \) then recalling Proof of Lemma[1] \( s_l(a, x) \) is a decreasing function, so the minimum \( \chi_l \) is achieved when \( \chi_l = b \), the rightmost value of \([a, b]\).

If \( x_r < a \) then recalling Proof of Lemma[1] \( s_r(b, x) \) is an increasing function, so the maximum \( \chi_r \) is achieved when \( \chi_r = a \), the leftmost value of \([a, b]\).

So, for every possible case, ESE is a consistent estimator of the TESE given by integrals calculated via trapezoidal rule.

Because trapezoidal rule is a good estimation for the true values of integrals, we conclude that ESE should also be a good estimator of the true value of TESE.
The Extremum Distance Method

Definition 15 $x_F$-left ($x_{F,1}$) and $x_F$-right ($x_{F,2}$) are such that:

$$x_{F,1} = \min_{x \in [a-\delta_1, b]} F(x)$$  \hspace{1cm} (39)

$$x_{F,2} = \max_{x \in [a, b+\delta_2]} F(x)$$  \hspace{1cm} (40)

with $\delta_1, \delta_2 > 0$ taken as small as necessary for $x_{F,1}, x_{F,2}$ to be unique unconstrained extremums in the corresponding intervals.

We can prove that:

Lemma 4

$$x_{F,1,2} = \arg_{x \in [a-\delta_1, b+\delta_2]} \left\{ f'(x) = \frac{f(b) - f(a)}{b - a} \right\}$$  \hspace{1cm} (42)

with $\delta_1, \delta_2 > 0$ taken as small as necessary for $x_{F,1,2}$ to be unique unconstrained extremums in the corresponding intervals.

Proof

We have extended the interval $[a, b]$ such that there exist (both unconstrained) a local minimum and a local maximum inside. For our convex/concave case let $\rho \in [a-\delta_1, b+\delta_2], \rho \notin \{a, b\}$ is the internal root of $F(x)$. Then we have that $F(x) < 0, x \in [a-\delta_1, \rho]$ and $F(x) > 0, x \in [\rho, b+\delta_2]$, because function is convex near a and concave near b. Thus the local minimum exists at $[a-\delta_1, \rho]$ and the local maximum lies in $[\rho, b+\delta_2]$. If we take the first derivative we have that:

$$F'(x) = f'(x) - g'(x) = f'(x) - \lambda = 0 \Rightarrow f'(x) = \lambda$$  \hspace{1cm} (43)

where $\lambda = \frac{f(b) - f(a)}{b - a}$ is the slope of the total chord. But the above equation must hold for both local minimum/maximum $x_{F,1,2}$, so it is necessary to hold:

$$f'(x_{F,1}) = f'(x_{F,2}) = \frac{f(b) - f(a)}{b - a}$$  \hspace{1cm} (44)

We can also check the second derivative which is:

$$F''(x) = f''(x)$$  \hspace{1cm} (45)

so it holds $F''(x_{F,1}) = f''(x_{F,1}) > 0$ and $F''(x_{F,2}) = f''(x_{F,2}) < 0$, i.e. we have the correct signs for local minimum and maximum respectively.

Corollary 3 Let a function $f : [a, b] \rightarrow R, f \in C^n, n \geq 2$ which is convex for $x \in [a, p]$ and concave for $x \in [p, b]$. Then we have one of the following possibilities:

1. If $x_{F,1,2} \in [a, b]$ then $a \leq x_{F,1} < x_{F,2} \leq b$
2. If $x_{F,1} \notin [a, b]$ then $x_{F,1} < a$

3. If $x_{F,2} \notin [a, b]$ then $x_{F,2} < b$

We define the next theoretical estimator of the inflection point:

Definition 16 The theoretical extremum distance from total chord estimator (TEDE) is such that:

$$x_D = \frac{x_{F,1} + x_{F,2}}{2}$$  \hspace{1cm} (46)

Now we define the data estimators of $x_{F,1}, x_{F,2}$.

Definition 17

$$\chi_{F,1} = x_{j_1}, \hspace{1cm} j_1 = \arg \min_{j \in [0, n]} \{ \Phi(x_j) \}$$  \hspace{1cm} (47)

$$\chi_{F,2} = x_{j_2}, \hspace{1cm} j_2 = \arg \max_{j \in [0, n]} \{ \Phi(x_j) \}$$  \hspace{1cm} (48)

The data extremum distance from total chord estimator (EDE) is

Definition 18

$$\chi_D = \frac{\chi_{F,1} + \chi_{F,2}}{2} \text{ iff } \chi_{F,2} \geq \chi_{F,1}$$  \hspace{1cm} (49)

Lemma 5 The EDE is an unbiased estimator of TEDE.

Proof

For all $\Phi(x_j), j = 0, 1, \ldots, n$ it holds:

$$E(\Phi(x_j)) = F(x_j)$$  \hspace{1cm} (50)

so if we take the errored data instead of the true data it has to be also that:

$$E\left( \min_{j \in [0, n]} \{ \Phi(x_j) \} \right) = \min_{j \in [0, n]} \{ F(x_j) \}$$  \hspace{1cm} (51)

$$E\left( \max_{j \in [0, n]} \{ \Phi(x_j) \} \right) = \max_{j \in [0, n]} \{ F(x_j) \}$$

Iterative application of geometrical based methods

Another very important opportunity is the possibility of iterations like the well known bisection method in root finding. Recall that for a continuous function if $f(\alpha) f(\beta) < 0$ then exist $\xi \in (\alpha, \beta)$ such that $f(\xi) = 0$. Our ESE method always gives an interval that contains the true inflection point p or a point close to the edge a or b, if data is just convex (or just concave) and inflection point does not exist. EDE method also gives an interval, although it is more sensitive to errors, so it does not always give a point close to a or b, if simple convexity or concavity exist.
1. **ESE iterative method or Bisection-ESE or BEDE**

We apply to initial data \((x_i, \phi_i), i = 0, \ldots, n\) the ESE method and have the 0th output for ESE method:

\[
[j_r^{(0)}, j_l^{(0)}], \chi_r^{(0)} = x_{j_r^{(0)}}, \chi_l^{(0)} = x_{j_l^{(0)}}, \chi_S^{(0)} = \frac{\chi_r^{(0)} + \chi_l^{(0)}}{2}
\]

If and only if \(j_r^{(0)} > j_l^{(0)}\), then we apply again ESE for data:

\[(x_i, \phi_i), i = j_r^{(0)}, \ldots, j_l^{(0)}\]

and obtain the 1st output for ESE method:

\[
[j_r^{(1)}, j_l^{(1)}], \chi_r^{(1)} = x_{j_r^{(1)}}, \chi_l^{(1)} = x_{j_l^{(1)}}, \chi_S^{(1)} = \frac{\chi_r^{(1)} + \chi_l^{(1)}}{2}
\]

We continue until \(j_r^{(k)} < j_l^{(k)}\) or until \(|\chi_S^{(k)} - \chi_S^{(k-1)}| < \varepsilon\), with \(\varepsilon = 10^{-8}\) to be a good tolerance for all examined data.

2. **EDE iterative method or Bisection-EDE or BEDE**

We apply to initial data \((x_i, \phi_i), i = 0, \ldots, n\) all four methods and have the 0th output for EDE (iff \(x_{F,2} > x_{F,1}\)) and ESE methods:

\[
[j_r^{(0)}, j_l^{(0)}], \chi_{F,1}^{(0)} = x_{j_r^{(0)}}, \chi_{F,2}^{(0)} = x_{j_l^{(0)}}, \chi_D^{(0)} = \frac{\chi_{F,1}^{(0)} + \chi_{F,2}^{(0)}}{2}
\]

\[
[j_r^{(0)}, j_l^{(0)}], \chi_r^{(0)} = x_{j_r^{(0)}}, \chi_l^{(0)} = x_{j_l^{(0)}}, \chi_S^{(0)} = \frac{\chi_l^{(0)} + \chi_r^{(0)}}{2}
\]

If and only if \(j_2^{(0)} > j_1^{(0)}\), then we apply again all four methods for data:

\[(x_i, \phi_i), i = j_1^{(0)}, \ldots, j_2^{(0)}\]

and obtain the 1st output for EDE (iff \(\chi_{F,2}^{(1)} > \chi_{F,2}^{(1)}\)) and ESE methods:

\[
[j_1^{(1)}, j_2^{(1)}], \chi_{F,1}^{(1)} = x_{j_1^{(1)}}, \chi_{F,2}^{(1)} = x_{j_2^{(1)}}, \chi_D^{(1)} = \frac{\chi_{F,1}^{(1)} + \chi_{F,2}^{(1)}}{2}
\]

\[
[j_r^{(1)}, j_l^{(1)}], \chi_r^{(1)} = x_{j_r^{(1)}}, \chi_l^{(1)} = x_{j_l^{(1)}}, \chi_S^{(1)} = \frac{\chi_l^{(1)} + \chi_r^{(1)}}{2}
\]

**EXPERIMENTS AND RESULTS**

We design small experiments by taking a suitable smooth function of known inflection point \(p\), an interval \([a, b]\) that covers all the possible cases \(p \in [a, b]\), \(p < a\), \(p > b\) and we add a uniform error \(\epsilon_i \sim U(-r, r)\) via the process \([11]\)

| \(j_r\) | \(j_l\) | \(\chi_r\) | \(\chi_l\) | \(\chi_S\) |
|---|---|---|---|---|
| 1.570 | 3.322 | 4.028 | 5.872 | 5.0 |
| 1.567 | 3.347 | 3.884 | 6.152 | 5.0 |

**Symmetric sigmoid curves**

We find the points \(x_1, x_{99}\) that give the first 1% and the 99% of sigmoid’s capacity \(L\) because those points have economic sense. So our interval \([a, b]\) is always relevant to the interval \([x_1, x_{99}]\).

**The Fisher-Pry sigmoid curve with total symmetry**

Let’s take the function:

\[
f(x) = 5 + 5 \tanh(x - 5) \quad (52)
\]

after \([11]\), which has \(p = 5, L = 10, x_1 = 2.7024, x_{99} = 7.2976\) and examine it at the interval \([2, 8]\) in order to have data symmetry w.r.t. inflection point. The function is also symmetrical around inflection point, i.e. we have total symmetry.

From Corollary \([11]\) we compute \(x_1 = 5.970315941, x_r = 4.029684059, x_{F,1} = 3.850750196, x_{F,2} = 6.149249804\), all inside \([2, 8]\), thus all methods are theoretically applicable. We first take \(n = 500\) sub-intervals equal spaced without error just for checking our estimators. The results are presented at Table \([11]\).

We observe that \(\chi_1 = 5.9720, \chi_r = 4.0280, \chi_{F,1} = 3.8480, \chi_{F,2} = 6.1520\) are very close to the theoretical expected values, so we are on the results of Lemma \([11]\).

The absolutely accuracy from the first apply of all methods confirms our theoretical analysis. All important lines, curves and points are presented at Figure \([11]\).

**FIG. 1:** (color online, scaled) Fisher-Pry sigmoid with total symmetry and without error

We next add the error term \(\epsilon_i \sim U(-0.05, 0.05)\) via the process \([11]\) and run our algorithms again. The results
are presented at Table [III]

Again the estimations are close to the theoretically expected and both methods gave the true answer from the first apply. We present the ESE and EDE intervals and estimators together with the true function and the errored data at Figure 2 where we present all important points \(x_1, x_r, x_{F,1}, x_{F,2}\) with the relevant tangent lines and the size of the minimum/maximum of \(F\). Due to the total symmetry the (dashed) line connecting \((x_{F,1}, f(x_{F,1})), (x_{F,2}, f(x_{F,2}))\) passes from the point \((p, f(p))\).

The Fisher-Pry sigmoid curve with data left asymmetry

We continue with the same sigmoid function, but now we form proper our \([a, b]\) to show data asymmetry w.r.t. inflection point.

Let’s take for example \([4.2, 8]\). If we do our theoretical computations we find \(x_1 = 5.974322740, x_r = 4.029684059, x_{F,1} = 4.025677260, x_{F,2} = 5.974322740\). We have that \(x_r < a\), so \(x_r\) has to estimate \(a = 4.2\) and \(\chi_S\) must be close to 4.703504993. Additionally, \(x_{F,1} < a\), so \(x_{F,1}\) must be also an estimation of \(a\), thus \(\chi_D\) must lie near the value 5.087161370. It’s time to see if our theoretical predictions will be confirmed by experiment.

We use for comparability the same Standard Partition as before and have the output presented at Table [III]. We have confirmation of our theory.

It is time now to try iterations based on ESE and EDE intervals that contain inflection point and to observe remarkable convergence to the real value of \(p = 5\) for both methods. We present ESE iterations at Table [IV] and EDE iterations at Table [V].

Let’s add the same error term \(\epsilon_i \sim U(-0.05, 0.05)\) and run our algorithms. The results are at Table [VI] and clearly we are close enough to our theoretical expectations.

We observe that ESE method did not estimae the inflection point with acceptable accuracy, so it is time to run the ESE and EDE iterations. The results, Table [VII] and Table [VIII] show a clear improvement of both estimations.

All the points of interesting are presented at Figure 3 where we see that interval does not contain both \(x_1, x_r\) and \(x_{F,1}, x_{F,2}\).
TABLE VII: ESE iterations for Fisher-Pry sigmoid, p=5, data left asymmetry, \([a, b] = [4.2, 8]\), n=500, error \(r=0.05\)

| \(x_r\) | \(x_l\) | \(\chi_S\) |
|--------|--------|-----------|
| 4.8156 | 5.3248 | 5.0702    |
| 4.9144 | 5.1576 | 5.0836    |

TABLE VIII: EDE iterations for Fisher-Pry sigmoid, p=5, data left asymmetry, \([a, b] = [4.2, 8]\), n=500, error \(r=0.05\)

| \(\chi_F, 1\) | \(\chi_F, 2\) | \(\chi_D\) | \(\chi_S\) |
|----------|----------|--------|--------|
| 4.5268   | 5.5148   | 4.9258 | 4.7728 |
| 4.7244   | 5.2412   | 4.9828 | 4.9828 |

**Non symmetric sigmoid curves**

We continue our study with non symmetric sigmoid curves that appear in Economics and other disciplines.

*The Gompertz sigmoid curve*

Let’s examine the function:

\[ f(x) = 10 e^{-e^{-e^{-x}}} \]  \(\text{(53)}\)

after [2], in the interval \([3.5, 8]\). The basic properties are presented at IX.

It is easy to prove that \(f\) is \((0.224, 1.0)\)-asymptotical symmetrical around inflection point, so we can handle it similar to a symmetric sigmoid only for a distance of \(\pm 1\) from \(p = 5\).

We use, for comparison reasons, the same SP with 500 sub-intervals without error and obtain the Table X which is absolutely compatible with theoretical predictions. The ESE iterations are showed at Table XI, while EDE iterations can be found at Table XII. We observe convergence to the real \(p\) for all methods used in this Chapter.

We continue with our familiar SP by adding uniform error distributed by \(U(-0.05, 0.05)\). Results at Table XIII while the ESE iterations are shown at Table XIV and the EDE iterations at Table XV. Convergence to the true value occur. All points of interest and data are presented at Figure 4. There exists satisfactory estimation of the \(p = 5\), so there is no need for further investigation.

**Non sigmoid curves**

Our analysis is applicable also to non sigmoid curves, not necessary symmetric or with data symmetry. We
TABLE XIII: Gompertz sigmoid, p=5, asymmetry, \([a, b] = [3.5, 8]\), n=500, error \(r=0.05\)

| \(j_r\) | \(j_l\) | \(\chi_r\) | \(\chi_l\) | \(\chi_S\) |
|------|------|--------|--------|--------|
| 74   | 274  | 5.1570 | 5.0980 | 5.0670 |
| 66   | 319  | 4.0890 | 4.0020 | 4.2235 |

TABLE XIV: ESE iterations for Gompertz sigmoid, p=5, asymmetry, \([a, b] = [3.5, 8]\), n=500, error \(r=0.05\)

| \(\chi_r\) | \(\chi_l\) | \(\chi_S\) |
|--------|--------|--------|
| 4.6340 | 5.5340 | 5.0840 |
| 4.8590 | 5.1560 | 5.0075 |

TABLE XV: EDE iterations for Gompertz sigmoid, p=5, asymmetry, \([a, b] = [3.5, 8]\), n=500, error \(r=0.05\)

| \(\chi_F\) | \(\chi_S\) | \(\chi_D\) |
|---------|---------|---------|
| 4.5020 | 5.6020 | 5.0570 |
| 4.8590 | 5.1560 | 5.0075 |

shall proceed making two experiments for a symmetric polynomial of \(r^{rd}\) order.

A symmetric third order polynomial with total symmetry

Let the polynomial function:

\[
f(x) = -\frac{1}{3} x^3 + \frac{5}{2} x^2 - 4 x + \frac{1}{2}\]

We study it at \([-2, 7]\), it has inflection point at \(p = 2.5\) and we have total symmetry. The interesting points are presented here:

| \(x_r\) | \(x_l\) | \(x_S\) |
|-------|-------|-------|
| 0.25  | 4.75  | 2.50  |
| \(x_F,1\) | \(x_F,2\) | \(x_D\) |
| -0.09807621078 | 5.098076211 | 2.50 |

The SP with 500 sub-intervals without error gives the Table XVI which is absolutely compatible with theoretical predictions. There is no need for any kind of iteration, because all methods agree.

FIG. 4: (color online, scaled) Gompertz asymmetrical sigmoid with error \(r=0.05\)

TABLE XVI: 3\(^{rd}\) order polynomial, total symmetry, p=2.5, n=500, no-error

| \(j_r\) | \(j_l\) | \(\chi_r\) | \(\chi_l\) | \(\chi_S\) |
|------|------|--------|--------|--------|
| 126  | 376  | 0.25   | 4.75   | 2.50   |
| 107  | 395  | -0.092 | 4.732  | 2.302  |

TABLE XVII: Symmetric 3\(^{rd}\) order polynomial, p=2.5, n=500, error \(r=2.0\)

| \(j_r\) | \(j_l\) | \(\chi_r\) | \(\chi_l\) | \(\chi_S\) |
|------|------|--------|--------|--------|
| 115  | 375  | 0.052  | 4.732  | 2.392  |
| 105  | 375  | -0.128 | 4.732  | 2.302  |

TABLE XVIII: ESE iterations for 3\(^{rd}\) order polynomial, p=5, total symmetry, n=500, error \(r=2.0\)

| \(\chi_r\) | \(\chi_l\) | \(\chi_S\) |
|--------|--------|--------|
| 1.222 | 3.688 | 2.455 |
| 1.564 | 3.382 | 2.473 |

The same SP with uniform error distributed by \(U(-2, 2)\) gives the results of Table XVII two ESE iterations at Table XVIII. All points and data are presented at Figure 5.

A symmetric third order polynomial with data right asymmetry

For the same symmetric 3\(^{rd}\) order polynomial \(54\) we change the interval to \([-2, 8]\), thus we have data right asymmetry now. Our critical points are written here:

| \(x_r\) | \(x_l\) | \(x_S\) |
|-------|-------|-------|
| -0.25 | 4.75  | 2.50  |
| \(x_F,1\) | \(x_F,2\) | \(x_D\) |
| -0.429732639 | 5.429732639 | 2.50 |
The case of SP with 500 sub-intervals and no error gives the Table XXIX while ESE and EDE iterations are presented at Table XXIII and Table XXII respectively. First results are absolutely compatible with theoretical predictions. For example we are waiting that $\chi_S = 2.25$ and we found 2.24.

We add uniform error distributed by $U(-2, 2)$ and we have the results of Table XXII one ESE iteration at Table XXIII and one EDE iteration at Table XXIV. All points and data are presented at Figure 6.

![Figure 5](image-url)

**FIG. 5:** (color online, unscaled) Symmetric 3rd order polynomial with error $r=2.0$

| $x_r$ | $\chi_l$ | $\chi_S$ |
|------|----------|----------|
| 1.3800 | 3.8800 | 2.6300 |
| 1.8200 | 3.0600 | 2.4400 |
| 2.2200 | 2.8400 | 2.5300 |
| 2.3200 | 2.6400 | 2.4800 |
| 2.4200 | 2.5800 | 2.5000 |
| 2.4600 | 2.5400 | 2.5000 |

**TABLE XXI:** EDE iterations for symmetric 3rd order polynomial, $p=2.5$, $n=500$, $[-2,8]$, no-error

| $x_r$ | $\chi_l$ | $\chi_S$ |
|------|----------|----------|
| 0.8200 | 4.1800 | 2.5000 |
| 0.8600 | 3.8400 | 2.3500 |
| 1.4600 | 3.8400 | 2.6500 |

There exist a problem here. Although we have a symmetric polynomial, the TESE is not equal to the true inflection point. A remedy for this problem for the class of 3rd order polynomials is given with the next Lemma.

**Lemma 6** 3rd order polynomial ESE correction.

Let a 3rd order polynomial $f(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$ and let $p$ its inflection point. Then it holds exactly that:

$$ p = \frac{1}{3} x_l + \frac{1}{3} x_r + \frac{1}{6} a + \frac{1}{3} b $$

and

$$ \hat{p} = \frac{1}{3} \chi_l + \frac{1}{3} \chi_r + \frac{1}{6} a + \frac{1}{3} b $$

is a consistent estimator of trapezoidal estimated $p$.

**Proof**

The inflection point because $\alpha \neq 0$ is found from the root of the second derivative, i.e. $6 \alpha p^2 + 2 \beta = 0$ or $p = -\frac{\beta}{3\alpha}$. Due to Corollary 1 we have for the $x_l$ that:

$$ 3 \alpha x^2 + 2 \beta x + \gamma = \frac{\alpha x^3 + \beta x^2 + \gamma x - \alpha a^3 - \beta a^2 - \gamma a}{x - a} $$

or

$$(x - a)^2 (\alpha a + \beta + 2 \alpha x) = 0$$

so the internal solution $x_l$ is:

$$ x_l = -\frac{\alpha a + \beta}{2 \alpha} $$
For the $x_r$ we have similar that:

$$3\alpha x^2 + 2\beta x + \gamma = \frac{\alpha b^3 + \beta b^2 + \gamma b - \alpha x^3 - \beta x^2 - \gamma x}{b - x}$$

or

$$(b - x)^2 (\alpha b + \beta + 2\alpha x) = 0$$

so the internal solution $x_r$ is:

$$x_r = -\frac{\alpha b + \beta}{2\alpha}$$

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By adding $x_l$ and $x_r$ we obtain:

$$x_l + x_r = \frac{1}{2} a - \frac{\beta}{\alpha} - \frac{1}{2} b$$

and if we remember that $p = -\frac{\beta}{3\alpha}$ we obtain:

$$x_l + x_r = \frac{1}{2} a + 3p - \frac{1}{2} b$$

or finally

$$p = \frac{1}{3} x_l + \frac{1}{3} x_r + \frac{1}{6} a + \frac{1}{3} b$$

Since we have proven that $\chi_l, \chi_r$ are consistent estimators of trapezoidal calculated values of $x_l, x_r$ we can take a consistent estimation for trapezoidal calculated $p$ by replacing the unknown $x_l, x_r$ with the estimators $\chi_l, \chi_r$.

As an example, we come back to the case of $3^{rd}$ order symmetric polynomial with data right asymmetry. We have that $a = -2, b = 8$ and from Table XXII is $\chi_r = -0.26, \chi_l = 4.74$, so we have that:

$$\hat{p} = \frac{1}{3} \chi_l + \frac{1}{3} \chi_r + \frac{1}{6} a + \frac{1}{3} b = 2.493333333$$

which is much closer to the true value of 2.5.

The above analysis can be extended to every function, if we can find analytically a relation between inflection point and $x_l, x_r, a, b$.

**CONCLUSION**

Starting from the problem of identifying the inflection point for the data $(x_i, \phi_i) i = 0, 1, \ldots, n$ we have created two geometrical methods in order to solve this problem: the Extremum Surface Estimator (ESE method) and the Extremum Distance Estimator (EDE method).

The methods can be applied iteratively just like bisection method and converge to the true i.p. either from the first or after a few iterations only. The implementation of the methods have been done with Fortran, Maple and Matlab. It is interesting to mention that for the problem of estimating the inflection point of $n = 10000$ data pairs we needed less than 7 sec CPU time in Fortran GNU compiler and a typical Intel Core i5 CPU with 4 GB RAM memory.

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