Abstract
The study of solving the inverse eigenvalue problem for nonnegative matrices has been around for decades. It is clear that an inverse eigenvalue problem is trivial if the desirable matrix is not restricted to a certain structure. Provided with the real spectrum, this paper presents a fast numerical procedure, based on the induction principle, to solve two kinds of inverse eigenvalue problems, one for nonnegative matrices and another for symmetric nonnegative matrices. As an immediate application, our approach can offer not only the sufficient condition for solving inverse eigenvalue problems for nonnegative or symmetric nonnegative matrices, but also a quick numerical way to solve inverse eigenvalue problem for stochastic matrices. Numerical examples are presented for problems of relatively larger size.

Keywords: Inverse eigenvalue problem, nonnegative matrices, Perron-Frobenius theorem, stochastic matrices

1. Introduction
A real $n \times n$ matrix is said to be nonnegative if each of its entries is nonnegative. There has been considerable research interest in the properties of the eigeninformation of nonnegative matrices. Among the large number of established results, one well-known necessary, but insufficient, condition is the Perron-Frobenius theory. This theory asserts first that for a real square matrix with positive entries, there exists a unique largest real eigenvalue with the corresponding eigenvector having strictly positive components. It also asserts a similar statement for some classes of nonnegative matrices [1].

In contrast to this well-developed theory, it is known that every complex number is an eigenvalue of some nonnegative matrix [2]. Suleimanova [3] then...
generalized Kolmogorov’s result to the study of the well-known problem – the real nonnegative inverse eigenvalue problem (RNIEP).

**Problem 1.1 (RNIEP).** Let \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) be a set of \( n \) real numbers. Find necessary and sufficient conditions for \( \sigma \) to be the set of eigenvalues of some nonnegative \( n \times n \) matrix.

It is easy to see that the solution of the RNIEP may not be unique, once it exists, since there are \( n \) given numbers with respect to \( n^2 \) unknown variables, i.e., an \( n \times n \) matrix. More generally, let \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) be a set of eigenvalues of an \( n \times n \) matrix \( A \) and let the \( k \)th moment \( s_k \) of \( \sigma \) be defined by

\[
s_k = \sum_{i=1}^{n} \lambda_i^k = \text{trace}(A^k), \quad k = 1, 2, \ldots,
\]

It follows that if \( \sigma \) is a set of eigenvalues of a nonnegative matrix \( A \), then the moments of the nonnegative matrix are always nonnegative, i.e.,

\[
s_k \geq 0, \quad k = 1, 2, \ldots.
\]

Based on the notion given in (1), the following necessary condition provides the most broad-based necessary condition in the solvability of a nonnegative inverse eigenvalue problem and can be shown by simply applying the Hölder inequality [4].

**Theorem 1.1.** Suppose \( \{\lambda_1, \ldots, \lambda_n\} \) are eigenvalues of an \( n \times n \) nonnegative matrix. Then the inequalities

\[
s_m^m \leq n^{m-1} s_{km}
\]

are satisfied for all \( k, m = 1, 2, \ldots \).

It has been shown in [4] that inequalities (2) and (3) are the necessary and sufficient conditions for \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) with \( n \leq 3 \) to be a set of eigenvalues of some nonnegative matrix. However, for \( n \geq 4 \), (2) and (3) are not sufficient, and the problem is still open. If \( \sigma \) is further restricted to be real, i.e., the RNIEP, then conditions (2) and (3) are still necessary and sufficient for solving RNIEP with \( n = 4 \) [4]. In fact, the RNIEP is still open for \( n \geq 5 \). On the other hand, if consideration of nonnegative matrices is limited to positive matrices, i.e., matrices with positive entries, the most general result to fully characterize the eigenvalues is as follows, as per Boyle and Handelmann [5][p.313].

**Theorem 1.2.** The set \( \{\lambda_1, \ldots, \lambda_n\} \subset \mathbb{C} \) with \( \lambda_1 = \max_{1 \leq i \leq n} |\lambda_i| \) is the nonzero spectrum of a positive matrix of size \( m \geq n \) if and only if the following conditions are satisfied:

1. \( \lambda_1 > |\lambda_i| \) for all \( i > 1 \),
2. \( s_k > 0 \) for all \( k = 1, 2, \ldots \), and
3. All coefficients of the polynomial $\Pi_{i=1}^{n}(t-\lambda_i)$ are real in $t$.

Although Theorem 1.2 provides the necessary and sufficient conditions for the solvability of the inverse eigenvalue problem of positive matrices, these conditions are too complicated for most applications due to the examination of $s_k > 0$ for all $k = 1, 2, \ldots$. Naturally, the next challenge is to find a way of constructing nonnegative matrices from a given list of eigenvalues. This question has been widely investigated in the literature, but its answer is still open [4]. In [3], Suleimanova first came up with the sufficient condition for a list of $n$ real numbers to be the spectrum of a nonnegative matrix by means of a geometrical approach.

**Theorem 1.3.** Let $\sigma = \{\lambda_k\}_{k=1}^{n} \subset \mathbb{R}$, $\lambda_1 + \lambda_2 + \ldots + \lambda_n \geq 0$ and $\lambda_i < 0$ for $i = 2, \ldots, n$. Then there exists a nonnegative $n \times n$ matrix with spectrum $\sigma$.

Indeed, Suleimanova’s result can also be limited to the case of symmetric matrices and a simple proof for the case of symmetric matrices is given in [6][Theorem 2.4]. In this paper, a weaker condition than Suleimanova’s result is provided for solving RNIEP. We then apply this weaker condition for constructing a nonnegative matrix associated with the given real eigenvalues. There are many sufficient conditions for solving RNIEP in the literature [7, 8, 9, 6, 10, 11, 12] and the references contained therein. Instead of comparing our condition with other known results, we present here a numerical approach, based on an improvement to Suleimanova’s condition [3], to solve RNIEP of larger size.

In addition to RNIEP, we also discuss a related problem, called the symmetric nonnegative inverse eigenvalue problem (SNIEP), proposed by Fiedler [6].

**Problem 1.2 (SNIEP).** Let $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ be a set of $n$ real numbers. Find necessary and sufficient conditions for $\sigma$ to be the set of eigenvalues of some symmetric nonnegative $n \times n$ matrix.

Again, this problem is still open for $n \geq 5$ [4, 13, 14].

So far as we know, applicable numerical methods for solving SNIEPs have thus far been proposed only twice [13, 16]. In [13], the SNIEP is formulated as the following constrained optimization problem

$$
\min_{Q^\top Q=I_R=R^\top} \frac{1}{2} \|Q^\top \Lambda Q - R \circ R\|.
$$

Here, $\Lambda$ is a diagonal matrix with the desired spectrum and $\circ$ represents the Hadamard product. The idea is to parameterize any symmetric matrix with the desired spectrum equal to $\Lambda$ by $X = Q^\top \Lambda Q$ and to parameterize any symmetric nonnegative matrix $Y$ by $Y = R \circ R$ for some symmetric matrix $R$. Later, Orsi [16] utilizes alternating projection ideas for the SNIEP. This projection consists of two particular sets. One is the set of all real symmetric matrices with the desired spectrum. The other is the set of symmetric nonnegative
matrices. Instead of solving an optimization problem or applying the alternating projection method, our work constructs a symmetric nonnegative matrix based on a sequence of $2 \times 2$ matrices as a building block. This approach is guaranteed to construct a nonnegative matrix of size $n$ after $n - 1$ iterations. It should be noted that there are many other inverse eigenvalue problems involving matrices with a particular structure and a particular desired spectrum. For more on other inverse problems, see the papers [17, 18, 19, 20] and the book [21]. In this paper, we describe a numerical procedure for solving RNIEP and SNIEP, which, while admittedly quite crude, suggests the possibility of solving many structured inverse eigenvalue problems and is currently under investigation.

This paper is organized as follows. We begin Section 2 with a discussion of the condition of two eigenvalues to be a spectrum of a $2 \times 2$ nonnegative matrix and construct this $2 \times 2$ matrix explicitly, given its eigenvalues. This $2 \times 2$ construction then serves as a fundamental tool in the construction of an $n \times n$ nonnegative matrix. In Section 3, we briefly review Nazari and Sherafat’s result [22] in combining two nonnegative matrices with the desired spectrum. We point out, in particular, how to apply this result to a general $n \times n$ matrix by the splitting of this given matrix. In Section 4, we discuss how the $2 \times 2$ construction can be applied to the inverse eigenvalue problem for stochastic matrices. Concluding remarks are given in Section 5.

2. The $2 \times 2$ building block

In this section, we describe how a nonnegative matrix $A$ can be constructed. Specifically, we want to determine a $2 \times 2$ nonnegative matrix $A$ with $\sigma(A) = \{\lambda_1, \lambda_2\}$. This $2 \times 2$ construction will become a building block in our recursive algorithm. Note that for the existence of a nonnegative matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with eigenvalues $\{\lambda_1, \lambda_2\}$, it is true that

$$a + d = \lambda_1 + \lambda_2 \geq 0, \quad (5a)$$

$$ad - bc = \lambda_1 \lambda_2. \quad (5b)$$

Since $b$ and $c$ are nonnegative, it follows directly from (5a) and (5b) that

$$bc = a(\lambda_1 + \lambda_2 - a) - \lambda_1 \lambda_2$$

$$= -a \left( \frac{\lambda_1 + \lambda_2}{2} \right)^2 + \frac{(\lambda_1 - \lambda_2)^2}{4} \geq 0. \quad (6)$$

This implies that $\lambda_1 \geq a \geq \lambda_2$. If $\lambda_2 < 0$, then the entry $a$ is further limited to $\lambda_1 + \lambda_2 \geq a \geq 0$. Putting together the above results, the entries of nonnegative matrices with the set of eigenvalues $\{\lambda_1, \lambda_2\}$ can be completely characterized as follows.
Lemma 2.1. \( \{ \lambda_1, \lambda_2 \} \) are eigenvalues of a 2\( \times \)2 nonnegative matrix \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) if and only if \( (5) \) and the following conditions,

\[
\lambda_1 \geq a \geq \lambda_2, \quad \text{if } \lambda_2 \geq 0, \\
\lambda_1 + \lambda_2 \geq a \geq 0, \quad \text{if } \lambda_2 < 0,
\]

are satisfied.

PROOF. It follows from \( (5) \) and \( (6) \) that we need only prove that if \( (5) \) and \( (7) \) hold, then \( A \) is a nonnegative matrix with the desired spectrum \( \{ \lambda_1, \lambda_2 \} \). Suppose \( (7) \) holds. Then \( (5a) \) implies that

\[
\lambda_1 \geq d \geq \lambda_2, \quad \text{if } \lambda_2 \geq 0, \\
\lambda_1 + \lambda_2 \geq d \geq 0, \quad \text{if } \lambda_2 < 0,
\]

and thus \( A \) is a nonnegative matrix.

From Lemma 2.1, it is straightforward to see that the matrix

\[
A = \begin{cases} 
\begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}, & \text{if } \lambda_2 \geq 0, \\
\begin{bmatrix} 0 & -\lambda_1\lambda_2 \\ 1 & \lambda_1 + \lambda_2 \end{bmatrix}, & \text{if } \lambda_2 < 0.
\end{cases}
\]

is a nonnegative matrix with eigenvalues \( \{ \lambda_1, \lambda_2 \} \). Similarly, we can come up with different nonnegative matrices based on the conditions given in Lemma 2.1. These 2 \( \times \) 2 nonnegative matrices will play a decisive role in the solvability of the RNIEP and will be illustrated in the next section.

Now we know how to define 2 \( \times \) 2 nonnegative matrices so that the constructed matrices possess a prescribed pair of eigenvalues. Next, an interesting question to ask is whether the specified eigenvalues can be applied to construct a 2 \( \times \) 2 symmetric nonnegative matrix. The answer can be provided by the following result. We omit the proof here since it is so similar to the discussion in Lemma 2.1.

Lemma 2.2. \( \{ \lambda_1, \lambda_2 \} \) are eigenvalues of a 2\( \times \)2 symmetric nonnegative matrix \( A = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \) if and only if \( (5) \) and the following conditions,

\[
\lambda_1 \geq a \geq \lambda_2, \quad \text{if } \lambda_2 \geq 0, \\
\lambda_1 + \lambda_2 \geq a \geq 0, \quad \text{if } \lambda_2 < 0,
\]

are satisfied. In particular, the entry \( b \) is denoted by

\[
b = \sqrt{-a^2 + (\lambda_1 + \lambda_2)a - \lambda_1\lambda_2} \geq 0.
\]
From Lemma 2.2 it can be seen that the matrix

\[
A = \begin{cases} 
\begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}, & \text{if } \lambda_2 \geq 0, \\
\begin{bmatrix} 0 & \sqrt{-\lambda_1 \lambda_2} \\ \sqrt{-\lambda_1 \lambda_2} & \lambda_1 + \lambda_2 \end{bmatrix}, & \text{if } \lambda_2 < 0.
\end{cases}
\]

(10)

is a symmetric nonnegative matrix with eigenvalues \(\{\lambda_1, \lambda_2\}\). In summary, the above examples can serve as building blocks to construct general \(n \times n\) matrices with prescribed eigenvalues.

3. Divide and conquer

Let \(\rho(A)\) be the spectral radius of the nonnegative matrix \(A\). It is known that \(\rho(A)\) is an eigenvalue of \(A\), called the Perron eigenvalue, and that there is a right eigenvector with nonnegative entries corresponding to the Perron eigenvalue. In this section we want to derive a sufficient condition for the set of \(n\) real numbers \(\lambda_1, \ldots, \lambda_n\) to be a possible set of eigenvalues in an \(n \times n\) nonnegative matrix and then to come up with a numerical approach for constructing this nonnegative matrix. To begin with, we shall first present a useful result given in [22][Theorem 2.1] for combining the eigeninformation of two non-negative matrices.

**Theorem 3.1.** Suppose \(\{\lambda_k\}_{k=1}^n\) and \(\{\beta_k\}_{k=1}^n\) are eigenvalues of an \(n \times n\) nonnegative matrix \(A\) and an \(m \times m\) nonnegative matrix \(B\), respectively, with \(\lambda_1 \geq |\lambda_k|\) and \(\beta_1 \geq |\beta_k|\) for all \(k > 1\). Let \(v\) be the unit eigenvector corresponding to the eigenvalue \(\beta_1\). If the matrix \(A\) is of the form

\[
A = \begin{bmatrix} A_1 & a \\ b^T & \lambda_1 \end{bmatrix},
\]

(11)

where \(A_1\) is an \((n-1) \times (n-1)\) matrix, and \(a, b\) are two vectors in \(\mathbb{R}^{n-1}\), then the set of the eigenvalues of the matrix

\[
C = \begin{bmatrix} A_1 & av^T \\ vb^T & B \end{bmatrix},
\]

(12)

is \(\{\lambda_k\}_{k=1}^n \cup \{\beta_k\}_{k=2}^n\).

Furthermore, with Theorem 3.1 we can directly extend the result to the symmetric nonnegative matrices.

**Corollary 3.1.** Suppose \(\{\lambda_k\}_{k=1}^n\) and \(\{\beta_k\}_{k=1}^n\) are eigenvalues of a symmetric \(n \times n\) nonnegative matrix \(A\) and a symmetric \(m \times m\) nonnegative matrix \(B\), respectively, with \(\lambda_1 \geq |\lambda_k|\) and \(\beta_1 \geq |\beta_k|\) for all \(k > 1\). Let \(v\) be the unit eigenvector corresponding to the eigenvalue \(\beta_1\). If the matrix \(A\) is of the form

\[
A = \begin{bmatrix} A_1 & a \\ a^T & \beta_1 \end{bmatrix},
\]

(13)
where $A_1$ is an $(n-1) \times (n-1)$ matrix, and $a, b$ are two vectors in $\mathbb{R}^{n-1}$, then the set of the eigenvalues of the matrix

$$C = \begin{bmatrix} A_1 & av^\top \\ va^\top & B \end{bmatrix},$$

(14)

is $\{\lambda_k\}_{k=1}^n \cup \{\beta_k\}_{k=2}^n$.

Based on Theorem 3.1 or Corollary 3.1, we outline our ideas for the computation of a nonnegative matrix or symmetric nonnegative matrix, respectively, followed by a recursive algorithm. Our strategy is quite straightforward, but it offers an effective way to construct an RNIEP or SNEP. Here, we take the construction of a symmetric nonnegative matrix as an example. A similar approach can be applied to solve RNIEP with nonsymmetric cases and is demonstrated in section 4. Assume first that the set of eigenvalues $\{\lambda_1, \ldots, \lambda_n\}$ are arranged in the order $\lambda_1 \geq \ldots \geq \lambda_r \geq 0 \geq \lambda_{r+1} \geq \ldots \geq \lambda_n$ and satisfy the condition

$$\lambda_1 + \lambda_{r+1} + \ldots + \lambda_n \geq 0.$$

(15)

Note that condition (15) is weaker than Suleimanova’s result given in Theorem 1.3. To prove the existence of a nonnegative matrix of general dimensionality with eigenvalues $\{\lambda_i\}_{i=1}^n$, one can treat the iteration in terms of $2 \times 2$ matrices step by step. For example, start with a $2 \times 2$ matrix $A$ as follows.

Case 1. Suppose $\lambda_2 \geq 0$ and choose without loss of generality a $2 \times 2$ matrix

$$A = \begin{bmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{bmatrix}.$$

(16)

Case 2. Suppose $\lambda_2 < 0$ and choose without loss of generality a $2 \times 2$ matrix

$$A = \begin{bmatrix} 0 & \sqrt{-\lambda_2 \lambda_1} \\ \sqrt{-\lambda_2 \lambda_1} & \lambda_1 + \lambda_2 \end{bmatrix}.$$

(17)

Now let another eigenvalue $\lambda_3$ creep into the matrix $A$, i.e., obtain the matrix $C_1$ in (14), by suitably selecting a matrix $B_1$ from Lemma 2.2. Again, two cases are required to be considered.

Case 1. Suppose $\lambda_3 \geq 0$ and choose without loss of generality a $2 \times 2$ matrix

$$B_1 = \begin{bmatrix} \lambda_1 & 0 \\ 0 & A(2, 2) \end{bmatrix}.$$

(18)

Here, $A(i, j)$ represents the $(i, j)$ entry of $A$.

Case 2. Suppose $\lambda_3 < 0$ and choose without loss of generality a $2 \times 2$ matrix

$$B_1 = \begin{bmatrix} 0 & \sqrt{-\lambda_3 A(2, 2)} \\ \sqrt{-\lambda_3 A(2, 2)} & \lambda_3 + A(2, 2) \end{bmatrix}.$$

(19)

It follows from Corollary 3.1 that the matrix

$$C_1 = \begin{bmatrix} A(1, 1) & A(1, 2) v_1^\top \\ v_1 A(1, 2) & B_1 \end{bmatrix}$$

(20)
has a set of eigenvalues \( \{\lambda_1, \lambda_2, \lambda_3\} \), where \( \mathbf{v}_1 \) is the unit eigenvector corresponding to the Perron eigenvalue of the matrix \( B_1 \). Upon obtaining the matrix \( C_1 \) in (20), we shall replace entries of the original matrix \( A \) with those of the new matrix \( C_1 \), i.e., redefine \( A = C_1 \), and continue another construction of a \( 2 \times 2 \) matrix \( B_2 \) with eigenvalues \( \{\lambda_4, A(3, 3)\} \). From Corollary 3.1, the entry \( A(3, 3) \) in the lower right corner of the new matrix \( A \) is required to be the Perron eigenvalue of the subsequent \( 2 \times 2 \) matrix \( B_2 \). Since by condition (15),

\[
A(3, 3) + \lambda_4 \geq 0,
\]

that is, \( A(3, 3) \geq |\lambda_4| \), and \( A(3, 3) = \lambda_1 \geq \lambda_4 \), if \( \lambda_4 \geq 0 \), it follows from Lemma 2.2 that there is a nonnegative matrix \( B \) with eigenvalues \( \{\lambda_4, A(3, 3)\} \). Therefore another new matrix \( C_2 \) can be defined by

\[
C_2 = \begin{bmatrix}
A(1 : 2, 1 : 2) & A(1 : 2, 3)\mathbf{v}_2^\top \\
\mathbf{v}_2 A(1 : 2, 3) & B_2
\end{bmatrix},
\]

where \( \mathbf{v}_2 \) is the unit eigenvector corresponding to the Perron eigenvalue \( A(3, 3) \) of the matrix \( B_2 \) and \( A(i : j, k) = [A(i, k), \ldots, A(j : k)]^\top \). We then redefine the matrix \( A \) by \( A = C_2 \) with eigenvalues \( \{\lambda_5, A(4, 4)\} \). We then continue the above process for the next category, and finally obtain a constructive way for the solution of an \( n \times n \) nonnegative matrix \( A \) with eigenvalues \( \{\lambda_1, \ldots, \lambda_n\} \).

The above recursive process for obtaining a nonnegative matrix with the desired spectrum \( \{\lambda_1, \ldots, \lambda_n\} \) can be conveniently demonstrated in MATLAB expressions as in Algorithm 1. More specifically, the matrix obtained by Algorithm 1 is explicitly a symmetric matrix, but also implies the capacity of solving RNIEP. In other words, it is quite intriguing that different approaches using different sets of \( 2 \times 2 \) matrices end up with different kinds of nonnegative matrices with prescribed eigenvalues. We apply the following example to demonstrate this property more fully.

**Example 3.1.** Given eigenvalues \( \{2, \frac{1}{2}, -1\} \), this example follows from the approach given in Algorithm 1. We might select the initial matrix \( A \) as

\[
A = \begin{bmatrix}
\frac{1}{2} & 0 \\
0 & 2
\end{bmatrix}.
\]

and two kinds of matrix \( B \) as

\[
B = \begin{bmatrix}
0 & 2 \\
1 & 1
\end{bmatrix}
\text{ or } B = \begin{bmatrix}
0 & \sqrt{2} \\
\sqrt{2} & 1
\end{bmatrix}.
\]

This implies that the matrix \( C \) obtained from the combination of matrices \( A \) and \( B \) is written as

\[
C = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & 0 & 2 \\
0 & 1 & 1
\end{bmatrix}
\text{ or } C = \begin{bmatrix}
\frac{1}{2} & 0 & 0 \\
0 & 0 & \sqrt{2} \\
0 & \sqrt{2} & 1
\end{bmatrix}.
\]
with eigenvalues \( \{2, \frac{1}{2}, -1\} \).

**Algorithm 1** The RNIEP/SNIEP: \([A] = \text{RNIEP/SNIEP}(\Lambda)\)

*Given* \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \), where \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \),

*return* a \( n \times n \) symmetric nonnegative matrix \( A \) that is isospectral to \( \Lambda \).

% Set up an initial \( 2 \times 2 \) matrix

if \( \lambda_2 \geq 0 \) then

\[
A \leftarrow \begin{bmatrix}
\lambda_2 & 0 \\
0 & \lambda_1 \\
\end{bmatrix};
\]

else

\[
A \leftarrow \begin{bmatrix}
0 & \sqrt{-\lambda_2 \lambda_1} \\
\sqrt{-\lambda_2 \lambda_1} & \lambda_1 + \lambda_2 \\
\end{bmatrix};
\]

end if

% Divide and conquer

for \( i = 3 \ldots n \) do

if \( \lambda_i \geq 0 \) then

\[
B \leftarrow \begin{bmatrix}
\lambda_i & 0 \\
0 & A(i-1,i-1) \\
\end{bmatrix};
\]

else

\[
B \leftarrow \begin{bmatrix}
0 & \sqrt{-\lambda_i A(i-1,i-1)} \\
\sqrt{-\lambda_i A(i-1,i-1)} & A(i-1,i-1) + \lambda_i \\
\end{bmatrix};
\]

end if

% Compute the Perron eigenvector of \( B \).

\([v] = \text{PerronEigvector}(B)\)

% Apply Theorem 3.1/Corollary 3.1

\(\text{TEMP} \leftarrow A(1 : i - 2, i - 1) * v^\top;\)

\(A \leftarrow \begin{bmatrix}
A(1 : i - 2, 1 : i - 2) & \text{TEMP} \\
\text{TEMP}^\top & B \\
\end{bmatrix} \),

end for

Note that in our algorithm, we break down the construction of the desired matrix \( A \) to a sequence of submatrices of size 2 and then combine these submatrices together to give a nonnegative solution to the original problem. Based on this divide and conquer process, we then have the following sufficient condition for the construction of a nonnegative matrix and its proof can be directly observed from the above discussion.

**Theorem 3.2.** Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) be real numbers and let \( r \) be the greatest number with \( \lambda_r \geq 0 \). If the condition

\[
\lambda_1 + \lambda_{r+1} + \ldots + \lambda_n \geq 0
\]

is satisfied, then there exists an \( n \times n \) nonnegative matrix with \( \sigma = \{\lambda_1, \ldots, \lambda_n\} \) as its spectrum. Indeed, the nonnegative matrix can also be chosen to be a symmetric matrix.
Note that the sufficient condition in Theorem 3.2 does not require the negativity of the remaining $n - 1$ eigenvalues and is somewhat weaker than that given by Suleimanova. This simplified condition is also shown in [23] for the cases $n = 2$ and $3$ by a geometric point of view. In [24, Theorem 3.1], Marijuán et al. show that the following Soto condition includes the Suleimanova condition [2].

**Theorem 3.3.** [12] Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be real numbers and let $S_k = \lambda_k + \lambda_{n-k+1}$, $k = 2, \ldots, \lfloor n/2 \rfloor$ with $S_n + 1 = \min\{\lambda_{n+1}, 0\}$ for $n$ odd. If

$$\lambda_1 + \lambda_n \geq \sum_{S_k < 0} S_k,$$

then there exists a nonnegative stochastic matrix $A$ with $\sigma = \{\lambda_1, \ldots, \lambda_n\}$ as its spectrum and constant row sums equal to $\lambda_1$.

It can be seen that condition (22) holds, once condition (21) is satisfied. This implies that the condition in Theorem 3.2 implies the Soto condition in Theorem 3.3. Additional discussions on the relationship between the Soto condition and other sufficient conditions can be found in [25, 24].

Note that the sufficient conditions given in [7, 8, 9, 6, 10, 11, 12, 26] show that under certain conditions, there exists a particular type of nonnegative matrices. Instead of showing the sufficient conditions for the existence of a particular type of nonnegative matrices, we apply in this research a series of $2 \times 2$ nonnegative matrices to construct a nonnegative matrix of difference types. This application can then be seen in the following numerical experiments discussed below.

4. Numerical experiments

In this section, we demonstrate by numerical examples how Algorithm 1 can be applied to construct the solution of RNIIEP (or, SNIIEP) and stochastic matrices associated with some particular spectrum.

**Example 4.1.** To illustrate the feasibility of our approach again problems of relatively large size, we being with a set of eigenvalues of size larger than 5. To demonstrate the robustness of our approach, the test data is generated from a uniform distribution over the interval $[-10,0]$, say \{21.3323, 5.0851, 3.0635, -5.1077, -7.9483, -8.1763\}. It can be easily seen that this set of eigenvalues satisfies condition (21). Reported below is one typical result in our experiment.

$$A = \begin{bmatrix}
5.0851 & 0 & 0 & 0 & 0 & 0 \\
0 & 3.0635 & 0 & 0 & 0 & 0 \\
0 & 0 & 5.9856 & 6.0286 & 6.0653 \\
0 & 0 & 5.9856 & 8.0054 & 8.0542 \\
0 & 0 & 6.0286 & 8.0054 & 8.2261 \\
0 & 0 & 6.0653 & 8.0542 & 8.2261 & 0.1000
\end{bmatrix}$$
We note that the original algorithm considers a symmetric nonnegative matrix as the target. As is expected, the output result can be a general nonnegative matrix with the same spectrum. This result can be obtained by choosing the updated matrices as

$$B = \begin{bmatrix} 0 & -A(i-1,i-1)\lambda_i \\ 1 & A(i-1,i-1) + \lambda_i \end{bmatrix}, \quad \text{if } \lambda_2 < 0,$$

and the reported result is

$$A = \begin{bmatrix} 5.0851 & 0 & 0 & 0 & 0 \\ 0 & 3.0635 & 0 & 0 & 0 \\ 0 & 0 & 0.9922 & 0 & 128.9580 \\ 0 & 0 & 0.1248 & 1 & 8.2763 \end{bmatrix}.$$  

Example 4.2. In this example, the well-known Suleimanova’s result in Theorem 1.3 is given to test our approach. To begin with, we randomly generate a set of negative eigenvalues, for example \{5.4701, 2.9632, 7.4469, 1.8896\} from the uniform distribution on the interval \([-10, 0]\). We might select without loss of generality a positive eigenvalue 17.8698 so that the Suleimanova’s condition is satisfied. Using this spectrum, a desired nonnegative matrix can then be computed by applying Algorithm 1 as follows:

$$A = \begin{bmatrix} 0 & 2.2982 & 2.9032 & 3.1562 & 3.1773 \\ 2.2982 & 0 & 3.7431 & 4.0693 & 4.0966 \\ 2.9032 & 3.7431 & 0 & 5.9468 & 5.9866 \\ 3.1562 & 4.0693 & 5.9468 & 0 & 7.4967 \\ 3.1773 & 4.0966 & 5.9866 & 7.4967 & 0.1000 \end{bmatrix}.$$

Example 4.3. In this example, we illustrate the application of our approach to construct a stochastic matrix with a prescribed spectrum. This is the so-called inverse stochastic eigenvalue problem. Note that the inverse eigenvalue problem for nonnegative matrices is practically equivalent to that for stochastic matrices. For example, if \{\lambda_1, \ldots, \lambda_n\} with \lambda_1 = \max_{1 \leq i \leq n} |\lambda_i| is the set of eigenvalues of an \(n \times n\) nonnegative matrix, then it is known that \{1, \lambda_2/\lambda_1, \ldots, \lambda_n/\lambda_1\} is the spectrum of a \(n \times n\) row stochastic matrix [24] [Lemma 5.3.2]. Our approach is first to construct nonnegative matrix with the given spectrum and then transform the nonnegative matrix to a stochastic matrix based on the following theorem [25].

**Theorem 4.1.** Suppose \(A\) is a nonnegative matrix with a positive maximal eigenvalue \(\rho(A)\) and a positive eigenvector \(x = [x_i]\) such that \(Ax = \rho(A)x\). Let \(D = [d_{ij}]\) be a diagonal matrix with diagonal entries defined by \(d_{ii} = x_i\). Then \(\frac{1}{\rho(A)}D^{-1}AD\) is a stochastic matrix.

The example experimented here is taken from [13]. It is to find a stochastic matrix with (presumably randomly generated) eigenvalues \{1.0000, -0.2608,
To facilitate our illustration, assume the eigenvalues have been arranged in the decreasing order such that \( \lambda_1 = 1.0000, \lambda_2 = 0.6438, \lambda_3 = 0.5046, \lambda_4 = -0.2608 \) and \( \lambda_5 = -0.4483 \). Note that in order to apply Theorem 4.1, the constructed nonnegative matrix should have a positive eigenvector corresponding to a positive maximal eigenvalue. For this purpose, we have to fine-tune Algorithm 1, while including positive eigenvalues into a matrix. This adjustment is a simply application of Lemma 2.1 by computing \( \text{NEG} = \lambda_4 + \lambda_5 \), choosing the initial value \( A \) as

\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix},
\]

where \( a = \lambda_2 + \frac{\lambda_1 + \text{NEG}}{2}, \quad d = \frac{\lambda_1 - \text{NEG}}{2} \) and \( b = c = \sqrt{ad - \lambda_1 \lambda_2} \), and selecting the subsequent matrix \( B \) as

\[
B = \begin{bmatrix}
e & f \\
g & h
\end{bmatrix}
\]

where \( e = \lambda_3 + \frac{A(2,2) + \text{NEG}}{4}, \quad h = \frac{3A(2,2) - \text{NEG}}{4} \) and \( e = f = \sqrt{eh - \lambda_3 A(2,2)} \). It is true that by Lemma 2.1, we have many different choices for selecting matrices \( A \) and \( B \). Our methodology used here is to construct an irreducible nonnegative matrix in the end. It then follows from the Perron-Frobenius theorem [1] that for this nonnegative matrix, there is a positive eigenvalue associated with an eigenvector which can be chosen to be entry-wise positive. It follows that an example of a stochastic matrix with the desired spectrum is

\[
A = \begin{bmatrix}
0.7893 & 0.0219 & 0.0456 & 0.0638 & 0.0794 \\
0.1454 & 0.5410 & 0.0758 & 0.1060 & 0.1318 \\
0.1454 & 0.0364 & 0  & 0.3647 & 0.4535 \\
0.1454 & 0.0364 & 0.2608 & 0  & 0.5574 \\
0.1454 & 0.0364 & 0.2608 & 0.4483 & 0.1091
\end{bmatrix}
\] (23)

In [15], this example is further restricted to a structured stochastic matrix with the zero pattern given by the zeros of the following matrix:

\[
Z = \begin{bmatrix}
1 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1
\end{bmatrix}
\]

Our algorithm as it stands can not solve this problem directly though we might be able to select a sequence of particular matrices so that the obtained nonnegative matrix has structure corresponding or similar to the matrix \( Z \). However, unlike the methods proposed in [13, 16], our methodology is computed by simply combing a sequence of \( 2 \times 2 \) matrices, that is, the computed result can preserve the desired spectrum with high precision.
5. Conclusion

Determining the necessary and sufficient conditions of solving inverse eigenvalue problems for nonnegative matrices or symmetric nonnegative matrices is very challenging and the conditions for matrices of larger size remain unknown. The main thrust of this paper is to present a numerical procedure for constructing a nonnegative matrix or a symmetric nonnegative matrix provided that the desired spectrum is given. With slight modification, our method can solve inverse eigenvalue problems for stochastic matrices as well. The crux of our algorithm is the employment of Nazari and Sherafat’s result \[22\]. At each step, we look for a sequence of $2 \times 2$ matrices with the desired eigenvalues and a desired structure such as symmetry and combine them together for solving the RNIEP or SRIEP. We then propose, based on our procedure, a weaker sufficient condition for solving RNIEP than Suleimanova’s result and a condition for solving SNIEP.

From the existing structured inverse eigenvalue problems, this paper describes only a numerical procedure for symmetric nonnegative matrices and stochastic matrices. However this procedure might serve as a possible computational tool for inverse eigenvalue problems involving many other types of structured nonnegative matrices such as Toeplitz, Hankel, and others. In addition, we also propose, based on our procedure, a weaker sufficient condition for solving RNIEP than Suleimanova’s result and a condition for solving SNIEP. The application of our divide and conquer strategy for structured inverse eigenvalue problems is a subject worthy of further investigation.

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