A New Approach to the Classical and Quantum Dynamics of Branes

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Abstract

It is shown that the Dirac-Nambu-Goto brane can be described as a point particle in an infinite dimensional brane space with a particular metric. This suggests a generalization to brane spaces with arbitrary metric, including the “flat” metric. Then quantization of such a system is straightforward: it is just like quantization of a bunch of non-interacting particles. This leads us to a system of a continuous set of scalar fields. For a particular choice of the metric in the space of fields we find that the classical Dirac-Nambu-Goto brane theory arises as an effective theory of such an underlying quantum field theory. Quantization of branes is important for the brane world scenarios, and thus for “quantum gravity”.

Keywords: Strings, Branes, Braneworld scenario, Quantization of branes, Quantum field theory, Effective theory, Position operator

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1 Introduction

Quantization of the Dirac-Nambu-Goto $p$-brane [1]–[4] is a tough problem that still awaits for a solution. The problem is rather well understood in the case of strings ($p = 1$) [5]–[7], but not of branes of arbitrary dimensionality $p$. Branes, $p$-branes and $D$-branes [8, 9] are important objects in string theory. Since according to the brane world scenarios [10]–[27], [28]–[31] a 3-brane sweeping a 4-dimensional world sheet can be associated with our spacetime, quantization of the brane would as well resolve the problem of quantum gravity.

In this paper I will show that the Dirac-Nambu-Goto action, governing the dynamics of $p$-branes, is a special case of an action in an infinite dimensional space $M$, for a special choice of the metric. This suggests that the usual string or $p$-brane theory is embedded in a more general theory in which the metric of $M$ is arbitrary [28] in the same sense as is arbitrary the metric of spacetime in general relativity. In particular, the metric of $M$ can be globally diagonal, in which case we have very special objects that I will call “flat branes”. A flat brane is like a bunch of point particles in the absence of any interaction. If we bring an interaction into the game,
then the metric of $\mathcal{M}$ is no longer globally diagonal, i.e., flat, and the space $\mathcal{M}$ has non vanishing curvature.

If such a generalized brane is compared with a bunch of interacting particles, then by analogy, the generic metric of the particle configuration space $\mathcal{C}$ should be like the generic metric of the brane space $\mathcal{M}$. Such a reasoning suggests that the usual many particle interacting theory and its quantization should be generalized so that in the limit of continuous bunch of particles it would match the theory of general branes and the curved brane space $\mathcal{M}$. In the case of flat brane space $\mathcal{M}$ and flat particle configuration space $\mathcal{C}$, the generalized brane theory that allows for the diagonal metric of $\mathcal{M}$, matches the usual theory of non interacting point particles, where the metric of $\mathcal{C}$ is also diagonal.

Quantized theory of a generalized brane should thus start just as the quantization of many particle systems: in flat brane space $\mathcal{M}$. By analogy, quantized theory of an interacting many particle system should be formulated not in 4-dimensional spacetime, but in the many dimensional configuration space $\mathcal{C}$. The corresponding quantum fields are then functions of position in $\mathcal{C}$. As a model we consider the theory of a scalar field in a multidimensional configuration space. The action for such a system can be reduced to the action for a system of scalar fields describing non interacting distinguishable particles. In the continuum limit we have a bunch of non interacting particles forming a flat brane. Such a bunch of particles—a flat brane—is described by a continuous set of scalar fields, differing in the brane parameters $\sigma^\bar{a}$, $\bar{a} = 1, 2, ..., p$, the metric $s(\sigma, \sigma')$ of the field space being diagonal ($\delta$-function like). If we replace the diagonal metric with a more general one, then we have interactions amongst the brane constituent “particles”.

We have found a particular metric $s(\sigma, \sigma') = (1 + \lambda \partial^\bar{a} \partial_\bar{b}) \delta^\bar{a}(\sigma - \sigma')$ which enables straightforward exact calculations and quantization. The inverse of such a metric is the propagator in the space $\{\sigma^\bar{a}\}$. The theory becomes a theory of the scalar field $\varphi(\sigma^\bar{a}, x^\mu)$ whose arguments are not only spacetime coordinates $x^\mu$, but also the brane coordinates $\sigma^\bar{a}$. It is straightforward to compute the exact Hamiltonian and momentum operator $\hat{p}$. We then calculated how the expectation value of $\hat{p}$ in a state which is the product of single particle wave packet profiles changes with time. We obtained the equations of motion for the wave packet centroid coordinates $X_\mu(\sigma)$ that match the classical brane equations of motion in the case when the determinant $(-\hat{\gamma})$ of the brane’s induced metric $\gamma_{\bar{a}\bar{b}}$ is equal to 1. If we generalize the field space metric according to $s(\sigma, \sigma') = \left(1 + \lambda \partial^\bar{a}(\sqrt{-\hat{\gamma}} \gamma_{\bar{a}\bar{b}} \partial_\bar{b})\right) \delta^\bar{a}(\sigma - \sigma')$, then as the expectation value we obtain exactly the classical brane equations of motion. The classical brane theory is thus obtained as an effective theory of our underlying quantum field theory of a continuous system of scalar fields for a particular choice of the field space metric $s(\sigma, \sigma')$. For a different choice of $s(\sigma, \sigma')$ we would obtain a different effective classical
brane—in agreement with our starting assumption that the brane space of a classical brane can have in principle an arbitrary metric, not necessarily the metric that gives the Dirac-Nambu-Goto brane.

2 A brane as a “point particle” in an infinite dimensional space

The Dirac-Nambu-Goto action for a $p$-brane is

$$I = \kappa \int d^{p+1}\xi \, (-\gamma)^{1/2}, \quad (1)$$

Here $\gamma \equiv \det \gamma_{ab}$, $\gamma_{ab} \equiv \partial_a X^\mu \partial_b X^\mu$, where $X^\mu(\xi^a)$, $\mu = 0, 1, 2, ..., D - 1$, $a = 0, 1, 2, ..., p$, are the embedding functions of the world volume swept by the $p$-brane, and $\kappa$ is the brane tension.

A action that is equivalent to (1) is the Schild action \[32\]

$$I_{\text{Schild}} = \frac{\kappa}{2k} \int d^{p+1}\xi \, (-\gamma), \quad (2)$$

in which the determinant of the induced metric occurs without the square root. This is a gauge fixed action. The equations of motion give $\partial_a (-\gamma) = 0$, which means that $(-\gamma) = C$, where $C$ is a constant. If we choose a gauge so that

$$(-\gamma) = C = k^2 \quad (3)$$

then the momentum $\pi^\mu = \kappa \sqrt{-\gamma} \partial^\nu X^\mu$ derived from the Dirac-Nambu-Goto action \[1\] is equal to the momentum $\kappa (-\gamma) \partial^\nu X^\mu / k$ derived from the Schild action. This is the reason why in \[2\] we included an additional constant factor $1/(2k)$.

The action \[1\] is invariant under reparametrizations of the parameters $\xi^a \equiv (\tau, \sigma^\bar{a})$, $\bar{a} = 1, 2, ..., p$. We can choose a particular gauge (choice of $\xi^a$) in which the determinant factorizes according to

$$(-\gamma) = \dot{X}^2(-\bar{\gamma}), \quad (4)$$

where $\dot{X}^2 \equiv \dot{X}^\mu \dot{X}^\mu$, $\dot{X}^\mu \equiv \partial X^\mu / \partial \tau$, and $\bar{\gamma} \equiv \det \partial_\bar{a}X^\mu \partial_\bar{b}X^\mu$, $\bar{a}, \bar{b} = 1, 2, ..., p$. The action \[1\] then becomes

$$I = \kappa \int d\tau \, d^p\sigma \, \sqrt{\dot{X}^2 \sqrt{-\bar{\gamma}}}. \quad (5)$$

The Schild action \[2\] is invariant under those coordinate transformations of $\xi^a$ which preserve the determinant $\gamma$. Also for the Schild action we can choose a gauge in which holds the factorization \[1\]. Then Eq. \[2\] becomes

$$I_{\text{Schild}} = \frac{\kappa}{2k} \int d\tau \, d^p\sigma \, \dot{X}^2(-\bar{\gamma}) \quad (6)$$
This can be written as
\[ I = \frac{\kappa}{2k} \int d\tau d^p\sigma d^p\sigma' (-\gamma) \eta_{\mu\nu} \delta(\sigma - \sigma') \dot{X}^\mu(\tau, \sigma) \dot{X}^\nu(\tau, \sigma'). \] (7)

At every \( \tau \), the integrand is a quadratic form in an infinite dimensional space with the metric
\[ \rho_{\mu\nu}(\sigma, \sigma') = (-\bar{\gamma}) \eta_{\mu\nu} \delta(\sigma - \sigma'). \] (8)

Introducing the compact notation
\[ \dot{X}^\mu(\sigma) \equiv \dot{X}^\mu(\tau, \sigma), \quad \rho_{\mu(\sigma)\nu(\sigma')} \equiv \rho_{\mu\nu}(\sigma, \sigma'), \] (9)
the action (7) reads
\[ I_{\text{Schild}} = \frac{\kappa}{2k} \int d\tau \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^\mu(\sigma)(\tau) \dot{X}^\nu(\sigma')(\tau). \] (10)

The momentum derived from the action (6) is
\[ p_\mu(\sigma) = \frac{\kappa(-\bar{\gamma})}{k} \dot{X}_\mu = \frac{\kappa\sqrt{-\bar{\gamma}}}{\sqrt{\dot{X}_\mu \dot{X}_\mu}}, \] (11)
where we have taken into account
\[ k = \sqrt{\dot{X}^2 \sqrt{-\bar{\gamma}}}, \] (12)
which follows from (3) and (4).

For the momentum belonging to the action (10) we obtain the expression which equals to (11):
\[ p_{\mu(\sigma)} = \frac{\kappa}{k} \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^\nu(\sigma') = \frac{\kappa}{k} \dot{X}_{\mu(\sigma)} = \frac{\kappa}{k} (-\bar{\gamma}) \dot{X}_\mu(\sigma), \] (13)
where \( \dot{X}_{\mu(\sigma)} = \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^\nu(\sigma') \), and \( \dot{X}_\mu(\sigma) = \eta_{\mu\nu} \dot{X}_\nu, \) \( \dot{X}^\mu(\sigma) \equiv \dot{X}^\mu(\sigma') \). We identify \( p_{\mu(\sigma)} \equiv p_{\mu(\sigma)} \), where
\[ p_{\mu(\sigma)} = \rho_{\mu(\sigma)\nu(\sigma')} p_{\nu(\sigma')} = \frac{p^\mu(\sigma)}{(-\bar{\gamma})}. \] (14)
where \( p^\mu(\sigma) = \eta^{\mu\nu} p_\nu(\sigma) \).

From the definition (11) of the momentum we obtain the following constraint:
\[ p_\mu(\sigma) p^\mu(\sigma) = \eta^{\mu\nu} p_{\mu(\sigma)} p_{\nu(\sigma')} = \kappa^2 (-\bar{\gamma}). \] (15)

\(^1\) We use the generalization of Einstein’s summation convention, so that not only summation over the repeated indices \( \mu, \nu \), but also the integration over the repeated continuous indices \( \sigma, (\sigma') \) is assumed.
We also have
\[ p_{\mu(\sigma)} p^{\mu(\sigma)} = \rho_{\mu(\sigma)\nu(\sigma')} p_{\mu(\sigma)} p_{\nu(\sigma')} = \tilde{\kappa}^2, \]  
\[ \tilde{\kappa}^2 = \int \kappa^2 d\sigma. \]  
(17)

In deriving Eq. (16) we wrote
\[ p_{\mu(\sigma)} p^{\mu(\sigma)} = \rho_{\mu(\sigma)\nu(\sigma')} p_{\mu(\sigma)} p_{\nu(\sigma')} = \eta^{\mu\nu} \delta(\sigma - \sigma') / (-\bar{\gamma}), \]  
and used (15).

Eq. (10) can then be written in the form
\[ I_{\text{Schild}} = \frac{\tilde{\kappa}}{2k} \int d\tau \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^{\mu(\sigma)}(\tau) \dot{X}^{\nu(\sigma')}(\tau), \]  
(18)

where
\[ \tilde{\kappa}^2 = \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^{\mu(\sigma)} \dot{X}^{\nu(\sigma')} = \int d\sigma (-\bar{\gamma}) \dot{X}^\mu \dot{X}^\nu = \int k^2 d\sigma. \]  
(19)

Eq. (18) is a generalization to infinite dimensions of the Schild action
\[ I_{\text{Schild}} = \frac{m}{2k} \int d\tau g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \]  
(20)

for a relativistic point particle in a curved spacetime with the metric \( g_{\mu\nu} \). The latter action is a gauge fixed action for a relativistic point particle, which is described by the reparametrization invariant action
\[ I = m \int d\tau (g_{\mu\nu} \dot{X}^\mu \dot{X}^\nu)^{1/2}. \]  
(21)

Analogously, instead of (18), we can take the action
\[ I = \tilde{\kappa} \int d\tau \left( \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^{\mu(\sigma)}(\tau) \dot{X}^{\nu(\sigma')}(\tau) \right)^{1/2}. \]  
(22)

From (17) and (19) it follows that \( \tilde{\kappa} / \tilde{k} = k / \kappa \), i.e.,
\[ \frac{\tilde{\kappa}}{\sqrt{\dot{X}^{\mu(\sigma)} \dot{X}_\mu(\sigma)}} = \frac{\kappa}{\sqrt{(-\bar{\gamma}) \dot{X}^\mu \dot{X}_\mu}}. \]  
(23)

This relation is valid not only in the gauge in which \( \sqrt{X^2 \sqrt{-\bar{\gamma}}} = k = \text{constant} \), but also in an arbitrary gauge. The same is true for the constraints (15) and (16).

Similarly to a point particle being described at every \( \tau \) by a finite set of coordinates \( x^\mu \) of a point (event) in a finite dimensional spacetime, a brane is described at every \( \tau \) by an infinite set of coordinates \( x^{\mu(\sigma)} \equiv x^\mu(\sigma) \) of a point in an infinite dimensional space, the so called brane space \( \mathcal{M} \). As \( \tau \) monotonically increases, the brane traces a worldline in \( \mathcal{M} \), described by the parametric equation
\( x^{\mu(\sigma)} = X^{\mu(\sigma)}(\tau) \), which is a mapping from the 1-dimensional space of the parameter \( \tau \) into an infinite dimensional brane space \( \mathcal{M} \) whose points are denoted by an infinite set of coordinates \( x^{\mu(\sigma)} \). The latter coordinates describe a \textit{brane event} in an analogous way as the coordinates \( x^\mu, \mu = 0, 1, 2, 3 \) described a \textit{point particle event} in spacetime. The distinction between the lower case \( x^\mu \) and the capital \( X^\mu(\sigma) \) is the distinction between coordinates of the brane space \( \mathcal{M} \) and \( \tau \)-dependent functions \( X^\mu(\sigma)(\tau) \). The derivative \( \dot{X}^{\mu(\sigma)}(\tau) \) is the velocity in \( \mathcal{M} \).

Here \( x^{\mu(\sigma)} \equiv x^\mu(\sigma) \) denotes a \textit{kinematically possible brane}. We postulate that distinct functions \( x^{\mu(\sigma)} \) can describe physically different branes, even in the case in which they are related to each other by a diffeomorphism \( \sigma^a \rightarrow \sigma'^a = f^a(\sigma) \). In such a case, a diffeomorphism is \textit{active}: it transforms one brane into another brane that is physically different in the sense that its points are tangentially displaced while the mathematical surface of both branes are the same\(^2\). On the other hand, a \textit{passive diffeomorphism} only relabels the parameters \( \sigma^a \) into new parameters \( \sigma'^a \), whereas the brane remains the same.

Consideration of the branes related by active diffeomorphisms as distinct kinematically possible objects is a crucial step that enables a formulation of the generalized brane theory. Then the tensor calculus of general relativity can be straightforwardly generalized to infinite dimensions. A point particle event with coordinates \( x^\mu \) is analogous to a brane event with coordinates \( x^{\mu(\sigma)} \). A diffeomorphism in spacetime,

\[
x^\mu \rightarrow x'^\mu = f^\mu(x'^\nu),
\]

is analogous to a diffeomorphism

\[
x^{\mu(\sigma)} \rightarrow x'^{\mu(\sigma)} = F^{\mu(\sigma)}(x^{\mu(\sigma)}),
\]

i.e.,

\[
x^\mu(\sigma) \rightarrow x'^\mu(\sigma) = F^\mu(\sigma)[x^\nu(\sigma)],
\]

where \( F^\mu(\sigma)[x^\nu(\sigma)] \) are functionals of \( x^\nu(\sigma) \). The new \( \mathcal{M} \)-space coordinates \( x'^{\mu(\sigma)} \) are functions of the old \( \mathcal{M} \)-space coordinates \( x^{\mu(\sigma)} \), i.e., the embedding functions \( x'^{\mu(\sigma)} \) are functionals of the old embedding functions \( x^\mu(\sigma) \). Both those diffeomorphisms can be either \textit{passive} or \textit{active}. If interpreted passively, then Eq. (25) means that the same brane is described either by \( \mathcal{M} \)-space coordinates \( x^{\mu(\sigma)} \) or \( x'^{\mu(\sigma)} \).

Diffeomorphisms in the brane space \( \mathcal{M} \) include the diffeomorphisms within the brane as well:

\[
\sigma^\bar{a} \rightarrow \sigma'^\bar{a} = f^\bar{a}(\sigma), \tag{27}
\]

\[
\Rightarrow \quad x^\mu(\sigma) \rightarrow x'^\mu(f(\sigma)) = x'^\mu(\sigma'). \tag{28}
\]

\(^2\) See more detailed explanation and figures of Refs. [28, 33]

\(^3\) Notice that the prime does not mean a derivative, but new quantities.
In the latter expression we can rename $\sigma'$ into $\sigma$ and write $x'^\mu(\sigma)$ instead of $X'^\mu(\sigma')$. Eq. (28) means that the brane space coordinates $x'^\mu(\sigma)$ transform into new brane space coordinates $X'^\mu(\sigma)$. Those new coordinates can be interpreted in the passive sense, namely that they describe the same brane, or in the active sense, namely that they describe a different, i.e., tangentially deformed, brane.

Tensor calculus in brane space $\mathcal{M}$ is analogous to that in spacetime. For instance, under a diffeomorphism (25) the velocity and the metric transform as

$$\dot{X}'^\mu(\sigma) = \frac{\partial x'^\mu(\sigma)}{\partial x^\nu(\sigma')} \dot{X}^\nu(\sigma'),$$

$$\rho'_{\mu(\sigma')\nu(\sigma')} = \frac{\partial x^\alpha(\sigma'')}{\partial x^\mu(\sigma')} \frac{\partial x^\beta(\sigma''')}{\partial x^\nu(\sigma''')} \rho_{\alpha(\sigma'')\beta(\sigma'''')}.$$  \hspace{1cm} (30)

Here we use the following notation for functional derivatives:

$$\frac{\partial}{\partial x^\mu(\sigma)} \equiv \frac{\partial}{\partial X^\mu(\sigma)} \equiv \frac{\delta}{\delta x^\mu(\sigma)}.$$  \hspace{1cm} (31)

In general, the metric of $\mathcal{M}$ need not be of the form (8). Moreover, it can be a metric that is not equivalent to (8) via a diffeomorphism in $\mathcal{M}$; it can be a completely different metric. In Refs. [28, 33] it was proposed that the metric of $\mathcal{M}$ is dynamical, like the metric of spacetime in general relativity.

The equations of motion derived from (22) are

$$\frac{\partial I}{\partial X^\mu(\sigma)} = \hat{\kappa} \frac{d}{d\tau} \left( \frac{\dot{X}^\mu(\sigma)}{\sqrt{\dot{X}}} \right) - \frac{\hat{\kappa}}{2} \partial_{\mu(\sigma)} \rho_{\alpha(\sigma')\beta(\sigma''')} \frac{\dot{X}^\alpha(\sigma') \dot{X}^\beta(\sigma''')}{\sqrt{\dot{X}}} = 0,$$ \hspace{1cm} (32)

where $\dot{X}^2 \equiv \dot{X}^\mu(\sigma) \dot{X}_\mu(\sigma) = \rho_{\mu(\sigma')\nu(\sigma')} \dot{X}^\mu(\sigma) \dot{X}^\nu(\sigma')$. This is the geodesic equation in the brane space $\mathcal{M}$, and, after using $\dot{X}^\mu(\sigma) = \rho_{\mu(\sigma')\nu(\sigma)} \dot{X}^\nu(\sigma)$ it can be written in the form

$$\frac{1}{\sqrt{\dot{X}}} \frac{d}{d\tau} \left( \frac{X^\mu(\sigma)}{\sqrt{\dot{X}}} \right) + \frac{\Gamma^\mu_{\alpha(\sigma')\beta(\sigma''')} \dot{X}^\alpha(\sigma') \dot{X}^\beta(\sigma''')}{\dot{X}^2} = 0,$$ \hspace{1cm} (33)

where

$$\Gamma^\mu_{\alpha(\sigma')\beta(\sigma''')} = \frac{1}{2} \rho^\mu_{\gamma(\sigma'')(\sigma''')} (\rho_{\gamma(\sigma'')\alpha(\sigma')\beta(\sigma''')} + \rho_{\gamma(\sigma'')\beta(\sigma'')\alpha(\sigma')} - \rho_{\alpha(\sigma')\beta(\sigma'')(\sigma''')})$$ \hspace{1cm} (34)

is the connection in $\mathcal{M}$, with comma denoting the functional derivative. The inverse metric $\rho'_{\mu(\sigma')\nu(\sigma'')}$ is given by $\rho'_{\mu(\sigma')\nu(\sigma'')} \rho_{\nu(\sigma'')\lambda(\sigma''')} = \delta^\nu_{\lambda(\sigma'')}$. In the usual notation this reads

$$\int d^p \sigma' \rho'_{\mu(\sigma',\sigma'')} \rho_{\nu(\sigma,\sigma')} = \delta^\nu_{\mu(\sigma' - \sigma')}.$$
Eq. (32) holds for any metric. For the particular metric (8), the action (22) reads

\[ I = \tilde{\kappa} \int d\tau \left( \int d^p \sigma \left( -\bar{\gamma} \right) \dot{X}^2 \right)^{1/2}. \] (35)

This can be written as

\[ I = \tilde{\kappa} \int d\tau L[\dot{X}^\mu(\sigma), X^\mu(\sigma)], \]

where the Lagrangian

\[ L[\dot{X}^\mu(\sigma), X^\mu(\sigma)] = \left( \int d^p \sigma \left( -\bar{\gamma} \right) \dot{X}^2 \right)^{1/2} \] (36)

is a functional of infinite dimensional velocities and coordinates.

The Euler-Lagrange equations

\[ \frac{d}{d\tau} \frac{\delta L}{\delta X^\mu(\sigma)} - \frac{\delta L}{\delta \dot{X}^\mu(\sigma)} = 0 \] (37)

give

\[ \frac{d}{d\tau} \left( \frac{\tilde{\kappa}}{\sqrt{\dot{X}^2}} \dot{X}_\mu \right) + \partial_\mu \left( \frac{\tilde{\kappa} \bar{\gamma} \dot{X}^2 \partial^\mu X_\mu}{\sqrt{\dot{X}^2}} \right) = 0, \] (38)

where \( \dot{X}^2 \equiv \dot{X}^{\mu(\sigma)} \dot{X}_{\mu(\sigma)} = \int d^p \sigma \left( -\bar{\gamma} \right) \dot{X}^2. \)

Using (23), the equation of motion (38) becomes

\[ \frac{d}{d\tau} \left( \frac{\kappa \sqrt{-\bar{\gamma}}}{\sqrt{\dot{X}^2}} \dot{X}_\mu \right) + \partial_\mu \left( \frac{\sqrt{-\bar{\gamma}} \sqrt{\dot{X}^2} \partial^\mu X_\mu}{\sqrt{X^2}} \right) = 0, \] (39)

This is the same equation as that derived from the Dirac-Nambu-Goto action (5).

We have thus verified that the action (35), which is just (22) for a particular metric (8), gives the same equations of motion as the action (5). The form of the action (22) suggests that in general the metric can be arbitrary, either a “curved” or a “flat” metric, including the brane space analog of the metric \( \eta_{\mu\nu} \).

3 Flat brane space: a brane as a bunch of non-interacting point particles

The reasoning at the end of the last section suggests that we should start formulating the brane theory with the most simple metric, i.e.,

\[ \rho_{\mu(\sigma)\nu(\sigma')} = \eta_{\mu(\sigma)\nu(\sigma')} = \eta_{\mu\nu} \delta(\sigma - \sigma'), \] (40)

which is the metric of flat brane space \( M. \) With such a metric the action (22) becomes

\[ I = \tilde{\kappa} \int d\tau \left( \int d^p \sigma \eta_{\mu\nu} \dot{X}^\mu(\tau, \sigma) \dot{X}^\nu(\tau, \sigma) \right)^{1/2}. \] (41)
This is an action for a brane in flat background space $\mathcal{M}$. Such a brane we will call flat brane. The action (41) is not invariant under general coordinate transformations (25) in $\mathcal{M}$-space. Under a diffeomorphism (25) the metric (40) occurring in the latter action transforms according to (30) into a new metric. A diffeomorphism (28) is just a particular diffeomorphism in $\mathcal{M}$-space, and the action (22) is invariant under (28), and of the same form. This shows that we need not to worry that (41) does not contain a square root of the determinant of the metric in the $\sigma^a$ space, because the action (41) is a particular case of the action (22) in which the metric is fixed according to (40), and which is invariant and covariant from the $\mathcal{M}$-space point of view.

The equations of motion derived from (41) are

$$\frac{d}{d\tau} \left( \frac{\dot{X}^\mu(\tau, \sigma)}{\tilde{X}^{1/2}} \right) = 0,$$

where now we have $\tilde{X}^2 \equiv \dot{X}^\nu(\sigma) \dot{X}^{\nu(\sigma)} = \int d\sigma \dot{X}^\mu(\sigma) \dot{X}^{\nu(\sigma)} \eta_{\mu\nu}$. In a gauge in which $\tilde{X}^2 = 1$, the equations of motion read

$$\ddot{X}^\mu(\tau, \sigma) = 0,$$

and their solution is

$$X^\mu(\tau, \sigma) = v^\mu(\sigma) \tau + X^\mu_0(\sigma).$$

This is a bunch of straight worldlines. In other words, Eq. (44) represents a continuum limit of a system of non-interacting point particles, tracing straight worldlines.

Quantization of the system described by the action (41) can be performed in analogous way as the quantization of the point particle in flat spacetime. Eq. (41) implies the constraint

$$p^{\mu(\sigma)} p_{\mu(\sigma)} - \tilde{\kappa}^2 = 0,$$

where

$$p^{\mu(\sigma)} p_{\mu(\sigma)} = \rho_{\mu(\sigma)\nu(\sigma')} \rho^{\mu(\sigma')} p^{\nu(\sigma')} = \int d\sigma \eta_{\mu\nu} p^\mu(\sigma) p^\nu(\sigma),$$

Upon quantization, Eq. (45) becomes the generalized Klein-Gordon equation,

$$\left( \hat{p}^{\mu(\sigma)} \hat{p}_{\mu(\sigma)} - \tilde{\kappa}^2 \right) \phi(x^{\nu(\sigma)}) = 0,$$

where

$$\hat{p}_{\mu(\sigma)} \equiv -i \frac{\partial}{\partial x^{\mu(\sigma)}} = -i \frac{\delta}{\delta x^{\mu(\sigma)}},$$

in which the field

$$\phi(x^{\nu(\sigma)}) \equiv \phi[x^\mu(\sigma)]$$

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is a functional of the brane’s embedding functions \( x^\mu(\sigma) \).

The corresponding action for the equation (47) is
\[
I[\phi(x^\mu(\sigma))] = \frac{1}{2} \int \mathcal{D}x^\nu(\sigma) (\partial_\mu(\sigma) \phi \partial^\mu(\sigma) \phi - \tilde{\kappa}^2 \phi^2).
\] (50)

Explicitly, Eq. (47) reads
\[
(\partial_\mu(\sigma) \partial^\mu(\sigma) + \tilde{\kappa}^2) \phi = 0,
\] (51)

which in the usual notation reads
\[
\left( \int d^p\sigma d^p\sigma' \eta^{\mu\nu} \delta(\sigma - \sigma') \frac{\delta^2}{\delta x^\mu(\sigma) \delta x^\nu(\sigma')} + \tilde{\kappa}^2 \right) \phi = 0.
\] (52)

A particular solution is
\[
\phi = e^{ip_\mu(\sigma)x^\mu(\sigma)},
\] (53)

where the momentum eigenvalue \( p_\mu(\sigma) \) satisfies the constraint (45).

A general solution of Eq. (51) is
\[
\phi(x^\mu(\sigma)) = \int \mathcal{D}p c(p) e^{ip_\mu(\sigma)x^\mu(\sigma)} \delta(p_\mu(\sigma) p^\mu(\sigma) - \tilde{\kappa}^2).
\] (54)

Upon second quantization, \( \phi(x^\mu(\sigma)) \) becomes the operator that creates or annihilates a brane with coordinates \( x^\mu(\sigma) \equiv x^\mu(\sigma) \). Because we consider the flat brane space with the metric (40), a brane with coordinates \( x^\mu(\sigma) \) is in fact a bunch of non interacting point particles, i.e., a continuous limit of a many particle system.

The coefficients \( c(p) \equiv c(p_\mu(\sigma)) \) determine the profile of the wave packet. Let us consider the case in which
\[
c(p_\mu(\sigma)) = c(p_\mu(\sigma))|_{\sigma \in (0, \Delta \sigma)} \delta(p_\mu(\sigma) - p_{0\mu(\sigma)})|_{\sigma \in (\Delta \sigma, L)}
\] (55)

This means that the brane momentum in the interval from \( \sigma = 0 \) to \( \sigma = \Delta \sigma \) is undetermined, whereas in the interval \( \sigma \in (\Delta \sigma, L) \) is sharply determined, so that it is equal to \( p_{0\mu(\sigma)} \).

If we insert (55) into the general solution (54), we obtain
\[
\phi(x^\mu(\sigma)) = A \int \mathcal{D}p c(p) e^{ip_\mu(\sigma)x^\mu(\sigma)} \delta(p_\mu(\sigma) p^\mu(\sigma) - \tilde{\kappa}^2).
\] (56)

where
\[
A = \exp \left[ i \int_{\Delta \sigma}^{L} p_{0\mu(\sigma)}x^\mu(\sigma)d\sigma \right]
\] (57)
is a phase factor. In arriving at (56) we have used
\[
\int D\mu(\sigma)\delta(p_\mu(\sigma)p^\mu(\sigma) - \tilde{\kappa}^2)\delta(p_\mu(\sigma) - p_\mu(0))
\]
\[
= \delta \left[ \int_{\sigma_0}^{\Delta\sigma} p_\mu(\sigma)p^\mu(\sigma)d\sigma + \int_{\Delta\sigma}^{L} p_\mu(\sigma)p^\mu(\sigma)d\sigma - \tilde{\kappa}^2 \right] = \delta \left( \int_{0}^{\Delta\sigma} p_\mu(\sigma)p^\mu(\sigma) - \kappa^2 \right) d\sigma,
\]
where
\[
\int_{0}^{\Delta\sigma} \kappa^2 d\sigma = \tilde{\kappa}^2 - \int_{\Delta\sigma}^{L} p_\mu(\sigma)p^\mu(\sigma)d\sigma.
\]
This is consistent with \(\tilde{\kappa}^2 = \int_{0}^{L} \kappa^2 d\sigma\) (see Eq. (17).

Eq. (56) is a solution of the generalized Klein-Gordon equation (47). But because the momenta \(p_\mu(\sigma) \equiv p_\mu(\sigma)\) for \(\sigma \in (\Delta\sigma, L)\) have been integrated out, the field (56) is a solution of the Klein-Gordon equation restricted to \(\sigma \in (0, \Delta\sigma)\) as well:
\[
\left[ \int_{0}^{\Delta\sigma} (\hat{p}_\mu(\sigma)p^\mu(\sigma) - \kappa^2) d\sigma \right] \phi(x^{\mu(\sigma)}) = 0.
\]

The \(\delta\)-function constraint in Eq. (56) can be written as
\[
\int_{0}^{\Delta\sigma} (p_\mu(\sigma)p^\mu(\sigma) - \kappa^2) d\sigma \approx (p_\mu(\sigma)p^\mu(\sigma) - \kappa^2)\Delta\sigma = 0.
\]
Multiplying the latter expression by \(\Delta\sigma\) and introducing \(p_\mu = p_\mu(\sigma)\Delta\sigma, \kappa\Delta\sigma = m\), we obtain
\[
p_\mu p^\mu - m^2 = 0,
\]
which is the constraint among the point particle momenta.

Not only in the generalized, but also in the restricted Klein-Gordon equation (60), the momentum operator is the functional derivative (48). By proceeding in the analogous way as in Eqs. (61), we obtain
\[
(\hat{p}_\mu\hat{p}^\mu - m^2)\varphi(x^{\mu}) = 0,
\]
where \(\hat{p}_\mu(\sigma_0)\Delta\sigma = \hat{p}_\mu = -i\partial/\partial x^{\mu}(\sigma_0)\) is the partial derivative with respect to the brane coordinates at \(\sigma = \sigma_0 = 0\), and \(\varphi(x^{\mu}) = \phi(x^{\mu(\sigma)})|_{\sigma_0}\).

In our setup, the segment of the brane around \(\sigma = \sigma_0(= 0)\) behaves as a point particle and satisfies the point-particle Klein-Gordon equation. The remaining segment of the brane from \(\sigma = \Delta\sigma\) to \(\sigma = L\), has definite momentum \(p_\mu(\sigma) = p_\mu(\sigma)\), and contributes only a phase factor (57). If \(p_\mu(\sigma) = 0\), this means that actually there is no brane outside the range \(\sigma \in (0, \Delta\sigma)\). Then we have only the brane within \(\sigma \in (0, \Delta\sigma)\), which in the limit \(\Delta\sigma \to 0\) behaves as a point particle. For finite, but
small $\Delta \sigma$, the brane behaves approximately as a point particle. At the end of Section 4 we further illuminate the derivation of (63) from (50).

The action for the field $\varphi(x^\mu)$, satisfying the Klein-Gordon equation (63), is
\[
I[\varphi(x^\mu)] = \frac{1}{2} \int d^D x \left( \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 \right). \tag{64}
\]

The latter action can also be straightforwardly derived from the action (50) by taking the ansatz
\[
\phi(x^\mu(\sigma)) = e^{\int_{\Delta \sigma}^L p_\mu(\sigma)x^\mu(\sigma) d\sigma} \varphi(x^\mu), \tag{65}
\]
and using (58).

In the following we will describe the flat brane by means of many particle noninteracting field theory. Different segments of the brane behave as distinguishable particles, each being described by a different scalar field $\varphi_r(x)$. The action for a system of those scalar fields is
\[
I[\varphi_r(x)] = \frac{1}{2} \int d^D x \left( \sum_{r=1}^N \partial_\mu \varphi_r \partial^\mu \varphi_r - m_r^2 \varphi_r^2 \right). \tag{66}
\]

The canonically conjugated variables are $\varphi_r(t, x)$ and $\Pi_r(t, x) = \partial \mathcal{L}/\partial \dot{\varphi}_r = \dot{\varphi}_r$, where $x \equiv x^i$, $i = 1, 2, ..., D - 1$, and the Hamiltonian is
\[
H = \frac{1}{2} \int d^D x \sum_r (\Pi_r^2 - \partial^i \varphi_r \partial_i \varphi_r + m_r^2 \varphi_r^2), \quad i = 1, 2, 3, ..., \bar{D} = D - 1. \tag{67}
\]

Upon quantization, $\varphi_r$ and $\Pi_r$ become the operators satisfying
\[
[\varphi_r(t, x), \Pi_s(t, x')] = i\delta^3(x - x')\delta_{rs},
\]
\[
[\varphi_r(t, x), \varphi_s(t, x')] = 0, \quad [\Pi_r(t, x), \Pi_s(t, x')] = 0. \tag{68}
\]

The field $\varphi_r(x), x \equiv x^\mu \equiv (t, x)$ can be expanded in terms of the creation and annihilations operators,
\[
\varphi_r(x) = \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}}(a_r(k)e^{-ikx} + a_r^\dagger(k)e^{ikx}), \tag{69}
\]
satisfying
\[
[a_r(k), a_s^\dagger(k')] = \delta^D(k - k')\delta_{rs}, \tag{70}
\]
\[
[a_r(k), a_s(k')] = 0, \quad [a_r^\dagger(k), a_s^\dagger(k')] = 0. \tag{71}
\]
We have absorbed the usual factor $(2\pi)^D 2\omega_k$, where $\omega_k = \sqrt{m^2 + k^2}$, into the definition of operators $a_r^\dagger(k), a_r^\dagger(k)$. The latter operators create and annihilate a particle with the momentum $k$. 

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Let us introduce the Fourier transformed operators

\[ a_r(x) = \frac{1}{\sqrt{(2\pi)^D}} \int d^Dk \ a_r(k)e^{ikx}, \quad a_r^\dagger(x) = \frac{1}{\sqrt{(2\pi)^D}} \int d^Dk \ a_r^\dagger(k)e^{-ikx} \] (72)

that satisfy

\[ [a_r(x), a_{r'}^\dagger(x')] = \delta^D(x-x')\delta_{rs} \] (73)
\[ [a_r(x), a_s(x')] = 0, \quad [a_r^\dagger(x), a_s^\dagger(x')] = 0. \] (74)

The operator \( a_r(x) \) annihilates the vacuum \( |0\rangle \), whereas \( a_r^\dagger(x) \) creates a particle at position \( x \):

\[ a_r^\dagger(x)|0\rangle = |x\rangle. \] (75)

In Appendix A we examine in more detail the properties of the operators \( a_r(x), a_r^\dagger(x) \) and show that in a given Lorentz frame they can indeed be interpreted, respectively, as a creation and annihilation operators for a particle at the position \( x \).

A succession of \( a_r^\dagger(x) \)'s creates a many particle state

\[ a_1^\dagger(x_1)a_2^\dagger(x_2)...a_N^\dagger(x_N)|0\rangle = |x_1x_2...x_N\rangle. \] (76)

In a more compact notation this reads

\[ \prod_r a_r^\dagger(x_r)|0\rangle \equiv A^\dagger(X_r)|0\rangle = |X_r\rangle, \quad r = 1, 2, ..., N, \] (77)

where \( X_r \) denotes a configuration of many particles, each having a different position \( x_r, r = 1, 2, ..., N \).

In the limit of infinitely many densely packed particles such a configuration can be a brane:

\[ \prod_\sigma a_\sigma^\dagger(x_\sigma)|0\rangle \equiv A^\dagger[X(\sigma)]|0\rangle = |X(\sigma)\rangle. \] (78)

The momentum operator of the \( r \)-th particle is

\[ \hat{p}_r = \int d^Dp \ a_r^\dagger(p)p \ a_r(p) = \int d^Dx \ a_r^\dagger(x)(-i) \frac{\partial}{\partial x} a_r(x). \] (79)

The latter definition is equivalent to the usual definition of momentum operator, because the factor \( 1/((2\pi)^D 2\omega_p) \) has been absorbed into the definition of the operators \( a_r^\dagger(p) \) and \( a_r(p) \).

Similarly, we can define the position operator,

\[ \hat{x}_r = \int d^Dx \ x a_r^\dagger(x) a_r(x) = \int d^Dp \ a_r^\dagger(p)i \frac{\partial}{\partial p} a_r(p). \] (80)

Notice that the position operator so defined is not equivalent to the usually defined "position operator" \[34\]–\[36\], which is then shown to be inappropriate, because it is
not self-adjoint with respect to the considered, i.e., Lorentz invariant, scalar product.

Our position operator (80) is Hermitian, because \( \hat{x}_r^\dagger = \hat{x}_r \). It is also self-adjoint with respect to the Lorentz non invariant scalar product between the wave packet states created by \( a_r^\dagger(x) \) or, equivalently, by \( a_r^\dagger(p) \).

The commutator of those operators is

\[
[\hat{x}_r, \hat{p}_s] = i\delta_{rs}1\hat{N}_r, \quad 1 \equiv \delta^i_j, \quad i, j = 1, 2, \ldots D - 1, \quad D - 1 = \bar{D},
\]

where \( \hat{N}_r = \int d^D x a_r^\dagger(x)a_r(x) \) is the number operator for an \( r \)-type particle.

Let us define the center of mass operator

\[
\hat{x}_{rT} \equiv \hat{N}^{-1}_r \hat{x}_r
\]

which satisfies

\[
\hat{x}_{rT} \left( \prod_k a_k^\dagger(x_k) \right) |0\rangle = x_T \left( \prod_k a_k^\dagger(x_k) \right) |0\rangle,
\]

\[
x_{rT} = \frac{1}{N} \sum_{k=1}^N \hat{x}_{rk}.
\]

Then Eq. (81) can be written as

\[
[\hat{x}_{rT}, \hat{p}_s] = i\delta_{rs}1.
\]

A generic many particle state is then a superposition

\[
|\psi\rangle = \int dx_1 dx_2 \ldots dx_N f(t, x_1, x_2, \ldots, x_N)a_1^\dagger(x_1)a_2^\dagger(x_2) \ldots a_N^\dagger(x_N)|0\rangle
\]

\[
= \int dp_1 dp_2 \ldots dp_N g(t, p_1, p_2, \ldots, p_N)a_1^\dagger(p_1)a_2^\dagger(p_2) \ldots a_N^\dagger(p_N)|0\rangle.
\]

Because \( a_r^\dagger(x), \ r = 1, 2, \ldots, N, \) are bosonic operators, there can be more than one operator of the same type \( r \) in the product. Thus, \( a_r^\dagger(x_r) \) can be extended to \( a_r^\dagger(x_r)a_r^\dagger(x_r') \ldots, \) and \( f(x_1, x_2, \ldots, x_N) \) into \( f(x_1, x'_1, x''_1, \ldots, x_2, x'_2, x''_2, \ldots, x_N, x'_N, x''_N, \ldots) \). A superposition then goes over all those possibilities. The wave function is symmetric with respect to the interchange of \( x_1, x'_1, x''_1, \ldots, x_2, x'_2, x''_2, \ldots, x_N, x'_N, x''_N, \ldots \). A state evolves in time according to the Schrödinger equation

\[
i \frac{d|\Psi\rangle}{dt} = H|\Psi\rangle,
\]

We perform the integration over \( d^D x \), where \( \bar{D} = D - 1 \), i.e., over a \( (D - 1) \)-dimensional hypersurface in \( D \)-dimensional spacetime. For more details about the position operator so defined, see Appendices A–C.
where the Hamiltonian operator is given in Eq. (67), which, after using (68)–(71) becomes

$$H = \int d^Dk \sum_r \omega_{rk} \left( a_r^\dagger(k) a_r(k) + \frac{\delta(0)}{2} \right).$$

(88)

Here \( \omega_{rk} = \sqrt{m_r^2 + k^2} \).

We are now interested in calculating the expectation value of the \( r \)-th particle position operator, \( \langle \hat{x}_rT \rangle \) in a state \( |\psi\rangle \). After a straightforward calculation, by using the Schrödinger equation, the commutation relations (70), (71), (73), (74), the definition (80) of the position operator, and by taking the Gaussian wave packet so that

$$g(t, p_1, \ldots, p_N) = g_1(t, p_1) g_2(t, p_2) \ldots g_N(t, p_N),$$

with

$$g_r(t, p_r) = \exp \left[-i(\omega_{rp} + E_0)t\right] \exp \left(-\frac{(p_r - p_{0r})^2}{2}\right),$$

(89)

where \( E_0 = \frac{1}{2} \int dp \sum_r \omega_{rp} \delta(0) \), we obtain

$$\langle \hat{x}_rT \rangle = \left\langle \frac{\hat{p}_r}{\omega_{rp}} \right\rangle t + x_{r0}. \tag{90}$$

Here

$$\left\langle \frac{\hat{p}_r}{\omega_{rp}} \right\rangle \equiv \langle \psi | \frac{\hat{p}_r}{\omega_{rp}} | \psi \rangle = \int dp \frac{p}{\omega_{rp}} g_r^*(p) g_r(p), \tag{91}$$

and \( x_{r0} = \langle \psi(t = 0) | \hat{x}_r | \psi(t = 0) \rangle \).

In the last equation we have an example for the expectation value of the operator \( \hat{p}_r/\omega_{rp} \) in the case when for each \( r \) we have only one particle state, \( a_r^\dagger(x) |0\rangle \), and not \( a_r^\dagger(x) a_r^\dagger(x') a_r^\dagger(x'') \ldots |0\rangle \).

In Eq. (91), \( \langle \hat{p}_r/\omega_{rp} \rangle \) is the expectation value of a particle’s velocity. Thus the expectation value of each particle’s center of mass position within our configuration traces a straight worldline. If particles are close to each other, such a configuration samples a flat brane. In the continuous limit we have a flat brane.

The position and momentum operator of the whole configuration are

$$\hat{x}^\mu = \sum_r \int d^Dx \ x^\mu a_r^\dagger(x) a_r(x) \tag{92}$$

$$\hat{p}^\mu = \sum_r \int d^Dp \ p^\mu a_r^\dagger(p) a_r(p)$$

$$= \sum_r \int d^Dx \ d^Dx' (-i) \partial_\mu \delta(x - x') a_r^\dagger(x) a_r(x')$$

$$= \sum_r \int d^Dx \ a_r^\dagger(x) (-i) \partial_\mu a_r(x) \tag{93}$$
They satisfy
\[ [\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu_\nu \hat{N}, \quad \mu, \nu = 1, 2, \ldots, \bar{D}, \quad \bar{D} = D - 1, \]  
(94)
where
\[ \hat{N} = \sum_r \int d^D x a_r^\dagger(x) a_r(x) = \sum_r \hat{N}_r \]  
(95)
is the number operator for the whole configuration.

A single state of the s-the particle is
\[ a_s^\dagger(p)|0\rangle = |p, s\rangle, \quad a_s^\dagger(x)|0\rangle = |x, s\rangle. \]  
(96)
The matrix elements are
\[ \langle p, s| \hat{p}^\dagger p', s'\rangle = p \delta_{ss'} \delta(p - p') \]  
\[ \langle x, s| \hat{x}^\dagger x', s'\rangle = x \delta_{ss'} \delta(x - x') \]  
(97)

All equations (66)–(97) can be straightforwardly generalized to a continuous set of “particles”, if instead of the discrete index \( r \) we take a continuous parameter, more precisely, a set of parameters \( \sigma \equiv \sigma^\alpha, \alpha = 1, 2, \ldots, p. \)

4 Towards curved brane space: A brane as a bunch of interacting point particles

Let us now introduce an interaction between the particles described by the fields \( \varphi_r \), and generalize the action (66) according to
\[ I[\varphi] = \frac{1}{2} \int d^D x (\partial_\mu \varphi^\sigma \partial^\mu \varphi^\sigma - m^2 \varphi^\sigma \varphi^\sigma) s_{rs}. \]  
(98)
The matrix \( s_{rs} \) has the role of a metric in the space of fields. In general, \( s_{rs} \) is a functional of \( \varphi^\sigma \). If \( s_{rs} \) is not a functional of \( \varphi^\sigma \), if it can be diagonalized, and has the inverse \( s^{rs} \), then the action (98) brings nothing new in comparison with the action (66). Interactions come into the game, if \( s_{rs} \) is a functional of \( \varphi^\sigma \), or if it cannot be diagonalized.

In the continuum limit, the discrete index \( r \) becomes the continuous index (\( \sigma \)), and \( \varphi^\sigma \) becomes \( \varphi^{(\sigma)} \). A discrete set of point particles, described by a discrete set of scalar fields \( \varphi^\sigma \), \( r = 1, 2, \ldots, N \), becomes a continuous set of point particles—a brane—described by a continuous set of scalar fields \( \varphi^{(\sigma)} \). The action (98) is then replaced by
\[ I[\varphi^{(\sigma)}] = \frac{1}{2} \int d^D x \left( \partial_\mu \varphi^{(\sigma)} \partial^\mu \varphi^{(\sigma)} - m^2 \varphi^{(\sigma)} \varphi^{(\sigma)} \right) s_{(\sigma)(\sigma')} \]  
(99)
In general, $s_{(\sigma)(\sigma')}$, is a functional of $\varphi^{(\sigma)}$, and it can thus give an interaction, provided that the space of fields has nonvanishing curvature. We will restrict our consideration to the case when $s_{(\sigma)(\sigma')} = \delta(\sigma - \sigma')$. We will assume that $s_{(\sigma)(\sigma')}\delta(\sigma - \sigma')$ has the inverse $s_{(\sigma)(\sigma')}^{-1}$, such that

$$s_{(\sigma)(\sigma')}s_{(\sigma')(\sigma)} = \delta(\sigma - \sigma').$$

(100)

The equation of motion is

$$\partial_\mu \partial^\mu \varphi^{(\sigma)} + m^2 \varphi^{(\sigma)} = 0,$$

(101)

where $\varphi^{(\sigma)} = s_{(\sigma)(\sigma')}\varphi^{(\sigma')}$. Because of (100) we also have

$$\partial_\mu \partial^\mu \varphi^{(\sigma)} + m^2 \varphi^{(\sigma)} = 0.$$  

(102)

The canonically conjugated variables $\varphi^{(\sigma)}$ and $\Pi_{(\sigma)} = \partial \mathcal{L}/\partial \dot{\varphi}^{(\sigma)} = \dot{\varphi}^{(\sigma)}$ satisfy the commutation relations

$$[\varphi^{(\sigma)}(x), \Pi_{(\sigma')}(x')]|_{x^0 - x'^0 = 0} = \delta^{(\sigma)(\sigma')}\delta^D(x - x'),$$

(103)

$$[\varphi^{(\sigma)}(x), \varphi^{(\sigma')}(x')]|_{x^0 - x'^0 = 0} = 0,$$

(104)

$$[\Pi_{(\sigma)}(x), \Pi_{(\sigma')}(x')]|_{x^0 - x'^0 = 0} = 0.$$  

The Hamiltonian is

$$H = \int d^D x (\Pi_{(\sigma)}\dot{\varphi}^{(\sigma)} - \mathcal{L}) = \frac{1}{2} \int d^D x (\Pi_{(\sigma)}\Pi^{(\sigma)} - \partial_i \varphi^{(\sigma)}\partial^i \varphi^{(\sigma)} + m^2 \varphi^{(\sigma)}\varphi^{(\sigma)}).$$

(105)

A general solution of Eq. (101) can be expanded according to

$$\varphi^{(\sigma)}(x) = \int \frac{d^D k}{\sqrt{(2\pi)^D 2\omega_k}} \left( a^{(\sigma)}(k)e^{-ikx} + a^{(\sigma)}_\dagger(k)e^{ikx} \right),$$

(106)

where $\omega_k = \sqrt{k^2 + m^2}$. Analogous expansion holds for $\varphi^{(\sigma')}$. Now we have the following commutation relations:

$$[a^{(\sigma)}(p), a^{(\sigma')}_\dagger(p')] = \delta^{(\sigma)(\sigma')}\delta^D(p - p').$$

(107)

Because $a_{(\sigma)} = s_{(\sigma)(\sigma')}a^{(\sigma')}$, and $a^{(\sigma)} = s_{(\sigma')(\sigma)}a^{(\sigma)}$, we also have

$$[a_{(\sigma)}(p), a^{(\sigma')}_\dagger(p')] = s_{(\sigma)(\sigma')}\delta^D(p - p').$$

(108)

and

$$[a^{(\sigma)}(p), a^{(\sigma')}(p')] = s_{(\sigma)(\sigma')}\delta^D(p - p').$$

(109)
Using (106) and (107), the Hamiltonian (105) becomes

\[ H = \frac{1}{2} \int d^D k \omega_k \left( a_{(\sigma)}^\dagger(k) a^{(\sigma)}(k) + a^{(\sigma)}(k) a_{(\sigma)}^\dagger(k) \right) \]

\[ = \int d^D k \omega_k a_{(\sigma)}^\dagger(k) a^{(\sigma)}(k) + H_{z.p.}, \]  

(110)

where \( H_{z.p.} \) is the "zero point" Hamiltonian, and

\[ a_{(\sigma)}^\dagger(k) a^{(\sigma)}(k) = a_{(\sigma')}^\dagger(k) a^{(\sigma')}(k) s_{(\sigma)(\sigma')} = a_{(\sigma)}^\dagger(k) a_{(\sigma')}(k) s^{(\sigma)(\sigma')} \]

(111)

For the momentum operator we obtain

\[ \hat{p} = \frac{1}{2} \int d^D k \bar{k} \left( a_{(\sigma)}^\dagger(k) a^{(\sigma)}(k) + a^{(\sigma)}(k) a_{(\sigma)}^\dagger(k) \right) \]

\[ = \int d^D k \bar{k} a_{(\sigma)}^\dagger(k) a^{(\sigma)}(k) + \hat{p}_{z.p.}. \]

(112)

We will now assume that in general \( m \) depends on position \( \sigma \) on the brane. Then also \( \omega_k = \sqrt{m^2 + k^2} \) is function of \( \sigma \). We thus have

\[ H = \int d\mathbf{k} \omega_k (\sigma) a_{(\sigma)}^\dagger(k) a^{(\sigma)}(k) s_{(\sigma)(\sigma')} + H_{z.p.}, \]

(113)

\[ H^\dagger = \int d\mathbf{k} \omega_k (\sigma') a_{(\sigma')}^\dagger(k) a^{(\sigma')}(k) s_{(\sigma)(\sigma')} + H_{z.p.}. \]

(114)

In the expression for \( H^\dagger \) we have renamed \( \sigma \rightarrow \sigma' \), \( \sigma' \rightarrow \sigma \), and used \( s_{(\sigma')(\sigma)} = s_{(\sigma)(\sigma')} \). The Hamilton operator so modified is not Hermitian. The momentum operator remains unchanged and Hermitian.

Let us now calculate the time derivative of the expectation value of the momentum operator \( \hat{p} \). We obtain:

\[ \frac{d}{dt} \langle \psi | \hat{p} | \psi \rangle = \left( \frac{d}{dt} \langle \psi | \right) \hat{p} | \psi \rangle + \langle \psi | \hat{p} \frac{d}{dt} | \psi \rangle = (-i) \langle \psi | \hat{p} H - H^\dagger \hat{p} | \psi \rangle. \]

(115)

In the last step of the above equation we have used the Schrödinger equation and its hermitian conjugate,

\[ i \frac{d}{dt} | \psi \rangle = H | \psi \rangle , \quad -i \frac{d}{dt} \langle \psi | = \langle \psi | H^\dagger. \]

(116)

---

5 From the action (69), using the standard field theoretic methods, we obtain the stress-energy tensor \( T^\mu_\nu = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi^{(\sigma)})} \partial_\nu \phi^{(\sigma)} - \mathcal{L} \delta^\mu_\nu \), and the momentum \( P_\mu = \int d\Sigma_\mu T^\mu_\nu \). Its spatial components are \( P_\mu = \int d^D x \phi^{(\sigma)} \partial_\mu \phi^{(\sigma)} \), where we have taken the reference frame in which the hypersurface has components \( d\Sigma_\mu = (d\Sigma_0, 0, 0, ..., 0) \) with \( d\Sigma_0 = d^D x \). The Fourier transform of the integrand in \( P_\mu \equiv \mathcal{P} \) gives after quantization the momentum operator (112).

6 In the discrete case this corresponds to each particle having a different mass \( m_\nu \) (see Eqs. (60), (67), (SS)).
In quantum theory we have by definition

$$\frac{d}{dt} \langle \psi | \hat{p} | \psi \rangle = \langle \psi | \frac{d}{dt} \hat{p} | \psi \rangle.$$  \tag{117}$$

Therefore, Eq. (115) gives

$$\frac{d\hat{p}}{dt} = (-i)(\hat{p}H - H^\dagger \hat{p}).$$  \tag{118}$$

The latter expression is equal to $(d\hat{p}/dt)^\dagger$; therefore $(d\hat{p}/dt)$ is Hermitian, as it should be. The zero point Hamiltonian, $H_{z.p.} = H^\dagger_{z.p.}$, cancels out in Eq. (118).

As the first step let us now consider the brane state which is the product of "single particle" wave packet profiles:

$$|\psi\rangle = \prod_\sigma \int d^Dp(\sigma) g(\sigma)(\vec{p}(\sigma)) a(\sigma) (\vec{p}(\sigma)) |0\rangle \quad \text{no integration over } (\sigma).$$  \tag{119}$$

Acting on the latter state by an annihilation operator, we obtain

$$a(\sigma')(\vec{p}'(\sigma')) |\psi\rangle = \int d\vec{p}(\sigma) d\sigma' \delta(\sigma' - \sigma) \delta(\vec{p}'(\sigma') - \vec{p}(\sigma)) g(\sigma)(\vec{p}(\sigma)) |\tilde{\psi}\rangle = g(\sigma)(\vec{p}'(\sigma')) |\tilde{\psi}\rangle,$$  \tag{120}$$

where $|\tilde{\psi}\rangle$ is the product of all the single "particle" states, except the one picked up by $a(\sigma')(\vec{p}'(\sigma')):

$$|\tilde{\psi}\rangle = \left(\prod_{\sigma \neq \sigma'} \int d\vec{p}(\sigma) g(\sigma)(\vec{p}(\sigma)) a(\sigma) (\vec{p}(\sigma))\right) |0\rangle.$$

We thus have

$$\langle \psi | a^\dagger(\sigma'')(\vec{p}''(\sigma'')) a(\sigma')(\vec{p}'(\sigma')) |\psi\rangle = g^{*\sigma''}(\vec{p}''(\sigma'')) g(\sigma')(\vec{p}'(\sigma')) \langle \tilde{\psi} | \tilde{\psi}\rangle,$$  \tag{122}$$

where normalization can be such that $\langle \tilde{\psi} | \tilde{\psi}\rangle = 1$.

Alternatively, if we take the state

$$|\psi\rangle = \int d\vec{p} g(\sigma)(\vec{p}) a(\sigma)(\vec{p}) |0\rangle,$$  \tag{123}$$

where

$$g(\sigma)(\vec{p}) a(\sigma)(\vec{p}) = g(\sigma)(\vec{p}) a^\dagger(\sigma')(\vec{p}) s(\sigma)(\sigma'),$$  \tag{124}$$

and where now we have the integration over $(\sigma)$, $(\sigma')$, then we obtain

$$a(\sigma')(\vec{p}') |\psi\rangle = g(\sigma')(\vec{p}') |0\rangle,$$  \tag{125}$$

and

$$\langle \psi | a^\dagger(\sigma'')(\vec{p}'') a(\sigma')(\vec{p}') |\psi\rangle = g^{*\sigma''}(\vec{p}'') g(\sigma')(\vec{p}') \langle 0 | 0\rangle,$$  \tag{126}$$

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where \( \langle 0|0 \rangle = 1 \).

Comparison of Eqs. (120), (122) with (125), (126) reveals us that instead of the state (119) in which we have the product of the single "particle" states, we can as well take the state (123) in which we have a superposition of the single "particles" states over \( \sigma \).

Inserting the state (119) into (115), we obtain

\[
\frac{d}{dt} \langle \psi | \hat{p} | \psi \rangle = (-i) \int d^D \mathbf{p} \mathbf{p} g^*(\sigma, \mathbf{p}) g(\sigma', \mathbf{p}) s(\sigma, \sigma')(\omega_p(\sigma) - \omega_p(\sigma')) d\sigma d\sigma',
\]

(127)

where we now write \( g(\sigma) (\mathbf{p}) \equiv g(\sigma, \mathbf{p}), \) and \( s(\sigma)(\sigma') \equiv s(\sigma, \sigma') \).

Let us now choose

\[
s(\sigma)(\sigma') = s(\sigma, \sigma') = (1 + \lambda \partial_\sigma \partial_{\sigma'} \delta^p(\sigma - \sigma')).
\]

(128)

Then Eq. (127) gives

\[
\frac{d}{dt} \langle \mathbf{p} \rangle \equiv \frac{d}{dt} \langle \psi | \hat{p} | \psi \rangle = (-i) \lambda \int d^D \mathbf{p} d^p \sigma \mathbf{p} \omega_p(\sigma)(g^* \partial_\sigma \partial_{\sigma'} g - \partial_\sigma \partial_{\sigma'} g^* g).
\]

(129)

This is the time derivative of the expectation value of the total momentum operator of the brane,

\[
\langle \psi | \hat{p} | \psi \rangle = \int d\mathbf{p} d\sigma d\sigma' \mathbf{p} g^*(\sigma, \mathbf{p}) g(\sigma', \mathbf{p}) s(\sigma, \sigma')
\]

\[
= \int d\mathbf{p} d\sigma \mathbf{p} (g^* g + g^* \partial_\sigma \partial_{\sigma'} g) = \langle \mathbf{p} \rangle = \int d\sigma \langle \mathbf{p} \rangle_\sigma
\]

(130)

where

\[
\langle \mathbf{p} \rangle_\sigma = \int d\mathbf{p} \mathbf{p} (g^* g + g^* \partial_\sigma \partial_{\sigma'} g) = \langle \psi |_\sigma \mathbf{p} | \psi \rangle_\sigma
\]

(131)

Here

\[
| \psi \rangle_\sigma = \int d\mathbf{p} g(\sigma, \mathbf{p}) a^\dagger(\sigma, \mathbf{p})|0\rangle,
\]

(132)

is the state of the brane's element at \( \sigma' \equiv \sigma'^a \). This is the state (119) in which there is no product over \( \sigma \), or equivalently, the state (123), in which there is no integration over \( \sigma \). In Eq. (131) we have thus the expectation value of the momentum of the brane's element at \( \sigma \).

Omitting the integration over \( \sigma \) in Eq. (129), we obtain

\[
\frac{d}{dt} \langle \mathbf{p} \rangle_\sigma = (-i) \lambda \int d^D \mathbf{p} \mathbf{p} \omega_p(\sigma)(g^* \partial_\sigma \partial_{\sigma'} g - \partial_\sigma \partial_{\sigma'} g^* g),
\]

(133)

which is the time derivative of the expected momentum of a brane's element, i.e., a "particle" forming the brane.
In Eq. (133) we can take $\omega_p$ that does not change with $\sigma$, and yet, in general, the expression would not vanish. For constant $\omega_p$ we obtain

$$\frac{d}{dt} \langle \hat{p} \rangle_\sigma = (-i) \lambda \partial_\sigma \int d^D p \omega_p (g^* \partial^\mu g - \partial^\mu g^* g). \quad (134)$$

This is the continuity equation for a current density on the brane. Integrating the latter equation over $\sigma$, we have $(d/dt)\langle \hat{p} \rangle = 0$. This means that for a constant $m$, and thus for constant $\omega_p = \sqrt{m^2 + p^2}$, the total momentum of the brane is constant in time, as it should be for an isolated brane\cite{Footnote}. The momentum of a brane’s element $d\sigma$ at $\sigma$ in general changes with time according to Eq. (134). With our model we have thus reproduced the well known facts about the brane’s momentum. In the following we will explore how the things look in the coordinate representation.

By taking the Fourier transform according to

$$g(\sigma, p) = \frac{1}{(2\pi)^D/2} \int e^{-i p x} f(\sigma, x) d x, \quad (135)$$

we obtain

$$\frac{d}{dt} \langle \hat{p} \rangle_\sigma = -\lambda \partial_\sigma \int d^D x \left[ f^* (\sigma, x) \left( \nabla (-i) \frac{\partial}{\partial t} \partial^\mu f (\sigma, x) \right) - \left( \nabla (-i) \frac{\partial}{\partial t} \partial^\mu f^* (\sigma, x) \right) f (\sigma, x) \right] \quad (136)$$

In the above equation, $-i \partial/\partial t$ comes from the Fourier transform of $\omega_p = \sqrt{m^2 + p^2}$, which gives

$$(m^2 + (-i)^2 \nabla^2)^{1/2} f = -i \frac{\partial}{\partial t} f. \quad (137)$$

The latter equality comes from the Schrödinger equation, as shown in Appendix B. Such equation is well known in the literature\cite{39, 40, 41}.

Eq. (136) can be written as

$$\frac{d}{dt} \langle \hat{\pi}_\mu \rangle_\sigma = -\lambda \partial_\sigma \int d^D x \left[ f^* (\sigma, x) \left( -i \frac{\partial}{\partial t} \partial^\mu \partial_\mu f \right) - \left( -i \frac{\partial}{\partial t} \partial^\mu \partial_\mu f^* \right) f \right], \quad (138)$$

where $\nabla \equiv \partial_\mu, \bar{\mu} = 1, 2, \ldots D$. The r.h.s. of Eq. (138) is the divergence of the expectation value of the operator

$$\hat{\pi}_{\bar{\mu}} = -i \lambda \frac{\partial}{\partial t} \partial_{\bar{\mu}}, \quad (139)$$

Footnote: The quantity $m$ is related to the brane tension $\kappa$ according to $m = \kappa \Delta \sigma$ (see Eq. (62) and the explanation above it). If the tension $\kappa$, and thus $m$, is not constant, this means that the brane is not isolated, i.e., it is in an interaction with other physical systems, therefore the brane’s total momentum changes with time.
which roughly corresponds to the classical quantity
\[ \pi^\bar{\mu} = \kappa \sqrt{\dot{X}^2} \partial^\alpha X_\mu = \dot{X}^2 p_0(\sigma) \partial^\alpha X_\mu, \] where \( p_0(\sigma) = \frac{\kappa}{\sqrt{\dot{X}^2}} \), \( X^0 \equiv t = \tau \), (140)
associated with a brane for which the determinant \( \bar{\gamma} \) of the spatial induced metric is constant, the constant being equal to 1 (see Eq. (39)). Such a brane can be either a flat brane, or a non flat brane, described in a gauge in which \( -\bar{\gamma} = 1 \). The correspondence is approximate (rough), because in Eq. (138) \( \partial^a \) is raised with \( \delta^{\bar{a}b} \), and not with \( \bar{\gamma}^{\bar{a}b} \).

In order to calculate the integral in Eq. (138) we need to know the wave function \( f(\sigma, x, t) \) which, in general depends on time, and must satisfy the Schrödinger equation. Its exact solution for a minimal uncertainty wave packet has been found in Ref. [42]. As a first estimation let us take the unperturbed wave function, which close to the initial time \( t = 0 \), is equal to
\[ f \approx A e^{-\frac{(x-\bar{X}(\sigma))^2}{2\sigma_0}} e^{i\bar{p}x} e^{i\bar{p}_0 t}, \] (141)
where \( \bar{x}(\sigma), \bar{p}, \bar{p}_0 \) are, respectively, the coordinates, momentum and energy of the wave packet center, and where \( A \) is the normalization constant. Because \( f \) satisfies Eq. (137), in which the square root expands to infinite order derivatives, at any \( t > 0 \), the function \( f \), even if localized at \( t = 0 \), becomes delocalized. But as it follows from Ref. [42], the deviation from the Gaussian wave packet, such as (141), is relatively small close to the initial time \( t = 0 \).

So we have
\[ -i \frac{\partial f}{\partial t} \bigg|_{t=0} = \bar{p}_0 f , \quad f(\sigma, x, 0) = A e^{-\frac{(x-\bar{X}(\sigma))^2}{2\sigma_0}} e^{i\bar{p}x} \] (142)
Introducing
\[ x - \bar{X}(\sigma) = x^\bar{a} - \bar{X}^\bar{a}(\sigma) = u^\bar{a}, \] (143)
we have
\[ \partial_\bar{a} f = \frac{u_\bar{a} \partial_{\bar{a}} \bar{X}^\nu}{\sigma_0} f , \quad \partial_{\bar{a}} f = -\frac{u_{\bar{a}}}{\sigma_0} f, \] (144)
\[ \partial_\bar{a} \partial_{\bar{b}} f = \left( \frac{\partial_\bar{a} \bar{X}^\nu}{\sigma_0} - \frac{u_{\bar{a}} u_{\bar{b}} \partial_{\bar{a}} X^\nu}{\sigma_0^2} + i \partial_\bar{a} \bar{p}_\nu u_{\bar{b}} + i \partial_{\bar{a}} p_{\bar{b}} t \right) f; \] (145)
\[ \int d^D x f^* u_{\bar{a}} u_{\bar{b}} f = A^2 (\pi \sigma_0)^{D/2} \frac{\sigma_0}{2} \delta_{\bar{a}\bar{b}} , \] (146)
\[ \int d^D x f^* u_{\bar{a}} f = 0 , \quad \int d^D x f^* u_{\bar{a}} u_{\bar{b}} u_{\bar{c}} f = 0. \] (147)
The normalization of \( f \) involves also the integration over \( d^p \sigma \), so that we have
\[ \int f^* f d^D x d^p \sigma = A^2 (\pi \sigma_0)^{D/2} S = 1 , \quad S = \int d^p \sigma. \] (148)
Using (141)–(147) in Eq. (139) and taking $t \approx 0$, we obtain

$$\frac{\langle \dot{p}_\mu \rangle}{\sigma} = -\lambda \partial_b \left( \frac{p_0}{S} \frac{\partial^a X_\mu}{\sigma_{o}} \right). \quad (149)$$

Let us compare the latter equations with the brane equation of motion (39) in which we take $(-\gamma) = 1$. The expectation value $\langle \dot{p}_\mu \rangle$ corresponds to $p_\mu(\sigma) = \frac{\kappa X_\mu}{\sqrt{x^2}}$, whereas $\frac{p_0}{S} \partial^a X_\mu$ corresponds to $\kappa \sqrt{x^2} \partial^a X_\mu = p_0(\sigma) X^2 \partial^a X_\mu$. In the gauge $\tau = t \equiv X^0$, the latter expression becomes $p_0(\sigma)(1 - v^2) \partial^a X_\mu \approx p_0(\sigma) \partial^a X_\mu$, if $v^2 \approx 0$. We have thus found that the centroid coordinates $X_\mu(\sigma)$ satisfy the equations of motion of a brane with $(-\gamma) = 1$ and $v^2 \approx 0$, up to the factor $\lambda / \sigma_{o}$. It is fascinating that such result comes from the field theory of a continuum of points particles, in which the interaction is given in terms of the metric (128) acting in the space of fields $\phi(\sigma)(t, x)$, and the wave packet profile being approximated with the expression (141) taken near $t = 0$. Therefore Eq. (149) is valid only near the initial time. For a different quantum state we would obtain an equation of motion for the expectation values that would differ from (149).

We can also consider the possibility of introducing a more general interaction than (128). First we observe that Eq. (136) can be written in the form

$$\frac{d}{dt} \langle \dot{p} \rangle = -\int d\sigma d^D x (-i) \left[ f^*(\sigma, x) (\partial_t \nabla f(\sigma', x)) - (\partial_t \nabla f^*(\sigma, x)) f \right] \lambda(\sigma, \sigma') \quad (150)$$

where

$$\lambda(\sigma, \sigma') \equiv \lambda_{(\sigma)}(\sigma') = \lambda \partial_b \partial^a \delta(\sigma - \sigma'). \quad (151)$$

If we generalize the interaction metric $\lambda(\sigma, \sigma')$ according to

$$\lambda(\sigma, \sigma') = \lambda \partial_b \left( \sqrt{-\gamma} \partial^a \right) \delta(\sigma - \sigma'), \quad (152)$$

where $\gamma = \text{det} \gamma_{\bar{a}b}$ is the determinant of the metric $\gamma_{\bar{a}b}$ in the space of parameters $\sigma \equiv \sigma^a$, and $\partial^a = \gamma^{ab} \partial_b$, then we obtain

$$\frac{d}{dt} \langle \dot{p} \rangle = -\lambda \partial_b \int d^D x \sqrt{-\gamma} (-i) \left[ f^*(\partial_t \nabla f) - (\partial_t \nabla f^*) f \right] \quad (153)$$

The latter equation corresponds to the brane equation of motion with non trivial $-\gamma \neq 1$, i.e., to the equation of motion of the Dirac-Nambu-Goto brane, provided that $\gamma_{\bar{a}b}$ is equated with the induced metric on the brane’s worldsheet, $\gamma_{ab} = \partial_a X^\mu \partial_b X^\mu$.

Taking the appropriate wave packet (141) and performing the calculations as in (142)–(148) we obtain

$$\frac{\langle \dot{p}_\mu \rangle}{\sigma} = -\lambda \partial_b \left( \frac{p_0}{S} \frac{\sqrt{-\gamma} \gamma^{\bar{a}b} \partial_b X_\mu}{\sigma_{o}} \right). \quad (154)$$

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This is indeed very close to the brane equation (38) or (39), apart from the factor $\dot{X}^2 = 1 - v^2$. In our equation (154) we have $\dot{X}^2 = 1$, which means that $v^2 = 0$. Since Eq. (154) has been calculated for the wave packet at $t \approx 0$, this is consistent with vanishing $\langle \hat{p}_\mu \rangle \propto \dot{\bar{X}}_\mu$ at $t \approx 0$. Because by our assumption $\bar{\gamma}_{\bar{a}\bar{b}}$ is the metric in the space of parameters $\sigma \equiv \sigma^a$, $a = 1, 2, ..., p$, and because $\bar{X}^{\mu}(\sigma)$ describe a brane whose induced metric is $\partial_{\bar{a}}X^{\mu}\partial_{\bar{b}}X_{\mu}$, we conclude that $\bar{\gamma}_{\bar{a}\bar{b}} = \partial_{\bar{a}}X^{\mu}\partial_{\bar{b}}X_{\mu}$.

Let us now investigate what happens if we use the metric (128) in the classical action (22) by setting

$$\rho_{\mu(\sigma)\nu(\sigma')} = \eta_{\mu\nu}(\delta(\sigma - \sigma'))$$

(155)

Because the latter metric does not functionally depend on $X^\alpha(\sigma)$, the second term in the equation of motion (32) vanishes. Therefore the equation of motion is

$$\frac{d}{dt}p_\mu(\sigma) = 0,$$

(156)

where

$$p_\mu(\sigma) = \rho_{\mu(\sigma)\nu(\sigma')}B^{\nu(\sigma')} = \frac{\tilde{\kappa}\dot{X}_\mu(\sigma)}{(\dot{X}^{\nu(\sigma')}X^{\nu(\sigma')})^{1/2}}$$

(157)

For the metric (155) we have

$$p_\mu(\sigma) = \frac{\tilde{\kappa}\dot{X}_\mu}{\sqrt{\dot{X}^2}} + \lambda\partial_{\bar{a}}\left(\frac{\tilde{\kappa}\partial_{\bar{a}}X_\mu}{\sqrt{\dot{X}^2}}\right),$$

(158)

where in the last step we have used Eq. (23) with $(-\bar{\gamma}) = 1$.

We see that our metric (155) modifies the momentum so that it contains an extra term, but otherwise the equation of motion is merely the derivative of momentum (156), with no brane-like “force” term of the form similar to the second term in Eq. (39).

For a generic metric, the constraints (16), associated with the action (22) leads to the field theory based on the action

$$I[\phi] = \frac{1}{2}\int \mathcal{D}x \left(\rho^{\mu(\sigma)\nu(\sigma')}\partial_{\mu(\sigma)}\phi\partial_{\nu(\sigma')}\phi - \tilde{\kappa}^2\phi^2\right).$$

(159)

I we take the Ansatz

$$\phi(x^{\mu(\sigma)}) = \prod_{\sigma''} \varphi(\sigma'')(x^{\mu(\sigma'')}),$$

(160)
then the functional derivative acts as a partial derivative according to
\[ \partial_{\mu(\sigma)} \phi = \lim_{\Delta \sigma \to 0} \frac{1}{\Delta \sigma} \frac{\partial \phi(\sigma)}{\partial x_{\mu}^\sigma} \prod_{\sigma' \neq \sigma} \phi(\sigma')(x^\mu_{(\sigma')}). \] (161)

The second term in (159) can be written in the form\[ \tilde{k}^2 \phi^2 = s^{(\sigma)(\sigma')} \kappa(\sigma) \kappa(\sigma') \phi^2. \] (162)

If we take the metric \[ \rho^{\mu(\sigma)\nu(\sigma')} = \eta^{\mu\nu} s^{(\sigma)(\sigma')}, \] (163)
where \( s^{(\sigma)(\sigma')} \) is the inverse of \( s_{(\sigma)(\sigma')} = (1 + \lambda \partial_{\sigma} \partial^{\sigma}) \delta(\sigma - \sigma') \), we arrive at the action
\[ I[\varphi] = \frac{1}{2} \int d^D x \ s^{(\sigma)(\sigma')} \left( \eta^{\mu\nu} \partial_{\mu} \varphi(\sigma) \partial_{\nu} \varphi(\sigma') - m^2 \varphi(\sigma) \varphi(\sigma') \right) \]
\[ = \frac{1}{2} \int d^D x \ s^{(\sigma)(\sigma')} \left( \eta^{\mu\nu} \partial_{\mu} \varphi(\sigma) \partial_{\nu} \varphi(\sigma') - m^2 \varphi(\sigma) \varphi(\sigma') \right). \] (164)

This is just the action (99) considered at the beginning of this section, by postulating the interaction metric \( s_{(\sigma)(\sigma')} \) between the continuous set of scalar fields \( \varphi^{(\sigma)}(x) \), whose quantized theory leads to the expectation value equations of motion (149), which contain the brane-like force term that is missing in the classical equations of motion (156) for the metric \( \rho^{\mu(\sigma)\nu(\sigma')} = \eta^{\mu\nu} s^{(\sigma)(\sigma')} \). The important point is that in the classical theory with the relatively simple metric (155), we have the simple equations of motion (156), whereas in the quantized theory with the same metric we also obtain the “force” term in the effective equations of motion (149) or (154). The expectation value equations of motion describe the centroid brane whose brane space metric is no longer (155), but a more general effective metric.

We will now show that to Eq. (149) corresponds the metric
\[ \rho_{\mu(\sigma)\nu(\sigma')} = \partial_{\mu} X^{\bar{\mu}} \partial^{\bar{\nu}} X_{\bar{\nu}} \delta(\sigma - \sigma'). \] (165)

If we insert this metric into (22), we obtain the following equations of motion:
\[ \frac{d}{d\tau} \left( \frac{\tilde{k} \partial_{\bar{\mu}} X^{\bar{\mu}} \partial_{\nu} X_{\nu} \dot{X}_{\bar{\mu}}}{\sqrt{\dot{x}^2}} \right) + \partial_{\bar{\mu}} \left( \frac{\tilde{k} \ddot{X}^2 \partial^{\bar{\nu}} X_{\bar{\mu}}}{\sqrt{\dot{x}^2}} \right) = 0, \] (166)

where \( \sqrt{\dot{x}^2} \equiv \dot{X}^{\mu(\sigma)} \dot{X}_{\mu(\sigma)} = \rho_{\mu(\sigma)\nu(\sigma')} \dot{X}^{\mu(\sigma)} \dot{X}^{\nu(\sigma')} \), and \( \dot{X}^2 \equiv \dot{X}^\mu \dot{X}_\mu = \eta_{\mu\nu} \dot{X}^\mu \dot{X}^\nu \).

Let us now observe that the following relation is satisfied:
\[ \frac{\kappa}{\sqrt{\dot{x}^2}} = \frac{\tilde{k}}{\sqrt{\dot{x}^2}} \sqrt{\dot{X}^2 \dot{X}_{\bar{\mu}} \partial^{\bar{\nu}} X_{\bar{\mu}}}. \] (167)

\[ \text{This comes from a massless action in higher dimensions.} \]
This relation can be easily proved by writing it in the form
\[ \tilde{k}^2 \partial_\alpha X^\mu \partial^\alpha X_\mu \dot{X}^2 = \kappa^2 \dot{X}^{\mu(\sigma)} \dot{X}_{\mu(\sigma)}, \] (168)
and integrating by \( d\sigma \). Then we obtain
\[ \tilde{k}^2 \int d\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \dot{X}^2 = \left( \int \kappa^2 d\sigma \right) \dot{X}^{\mu(\sigma)} \dot{X}_{\mu(\sigma)}, \] (169)
Since \( \int \kappa^2 d\sigma = \tilde{k}^2 \) and \( \int d\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu \dot{X}^\nu \dot{X}_\nu = \rho_{\mu(\sigma)\nu(\sigma')} X^{\mu(\sigma)} \dot{X}^{\nu(\sigma')} \), where \( \rho_{\mu(\sigma)\nu(\sigma')} \) is given by (163), we see that (169) is an identity.

Using (167) in Eq. (166), we have
\[ \frac{d}{d\tau} \left( \kappa \sqrt{\partial_\alpha X^\nu \partial^\alpha X_\mu} \dot{X}_\mu \right) + \partial_\alpha \left( \frac{\kappa \dot{X}^2 \partial^\alpha X_\mu}{\sqrt{\partial_\alpha X^\nu \partial^\alpha X_\mu}} \right) = 0. \] (170)
If we rewrite the latter equation in terms of momenta
\[ p_{\mu(\sigma)} = \frac{\tilde{k} \partial_\alpha X^\mu \partial^\alpha X_\mu \dot{X}_\mu (\sigma)}{(X^{\alpha(\sigma)} X_{\alpha(\sigma)})^{1/2}} = \frac{\kappa \sqrt{\partial_\alpha X^\nu \partial^\alpha X_\mu} \dot{X}_\mu}{\sqrt{X^2}}, \] (171)
\[ p_{\mu(\sigma)} = \frac{\tilde{k} \dot{X}_\mu}{(X^{\alpha(\sigma)} X_{\alpha(\sigma)})^{1/2}} = \frac{\kappa \dot{X}_\mu}{\sqrt{\partial_\alpha X^\nu \partial^\alpha X_\mu} \sqrt{X^2}}, \] (172)
we obtain
\[ \frac{d}{d\tau} \left( p_{\mu(\sigma)} \right) + \partial_\alpha \left( p_{\nu(\sigma')} \dot{X}^\nu \partial^\alpha X_\mu \right) = 0. \] (173)
For the spatial components \( \tilde{\mu} = 1, 2, \ldots, \tilde{D} \), in the gauge \( \tau = X^0 \equiv t \), so that \( \dot{X}^2 \equiv \dot{X}_\mu \dot{X}_\mu = 1 - v^2 \), the latter equation matches the expectation value equation (149) if \( v^2 = 0 \).

Similarly, the expectation value equation of motion (154) can be derived from the effective action (22) with the metric
\[ \rho_{\mu(\sigma)\nu(\sigma')} = \sqrt{-\gamma} (\gamma^{ab} \partial_\alpha X^\mu \partial_{\alpha} X_\mu) X^p \delta(\sigma - \sigma'). \] (174)
The equations of motion are then
\[ \frac{d}{d\tau} \left( \tilde{k} \sqrt{-\gamma} (\partial_\alpha X^\mu \partial^\alpha X_\mu) \dot{X}_\mu \right) + \partial_\alpha \left( \tilde{k} \sqrt{-\gamma} \dot{X}^2 \partial_\alpha X_\mu \right) = 0. \] (175)
Instead of (167) we have now
\[ \frac{\tilde{k}}{\sqrt{X^2}} = \frac{\kappa}{\sqrt{\partial_\alpha X^\mu \partial^\alpha X_\mu}^{p/2}}. \] (176)
Using the latter relation, Eq. (175) becomes

\[ \frac{d}{d\tau} \left( \frac{\kappa \sqrt{-\gamma (\partial_\nu X^\mu \partial_\mu X_\nu)^{\nu/2} \dot{X}_\mu)}{\sqrt{\dot{X}^2}} \right) + \partial_\mu \left( \frac{\kappa \sqrt{-\gamma \dot{X}^2 p(\partial_\nu X^\mu \partial_\mu X_\nu)^{\nu/2} \dot{X}_\mu)}{\sqrt{\dot{X}^2}} \right) = 0. \] (177)

Recall that \( \bar{\gamma}_{\bar{a}\bar{b}} \) is the metric in the space of parameters \( \sigma^{\bar{a}} \). On the other hand, \( X^\nu(\tau, \sigma^{\bar{a}}) \) describes a brane. Therefore it makes sense to equate \( \bar{\gamma}_{\bar{a}\bar{b}} \) with the induced metric on the brane:

\[ \bar{\gamma}_{\bar{a}\bar{b}} = \partial_\bar{a}X^{\mu} \partial_\bar{b}X_\mu. \] (178)

Let us now take into account that the trace of the metric is equal to the dimension of the brane:

\[ \partial_\bar{b}X^\mu \partial_\bar{b}X_\mu = p. \] (179)

Inserting this into (177) we find that \( p \) cancels out, and Eq. (177) becomes

\[ \frac{d}{d\tau} \left( \frac{\kappa \sqrt{-\gamma \dot{X}^2}}{\sqrt{\dot{X}^2}} \right) + \partial_\mu \left( \frac{\kappa \sqrt{-\gamma \dot{X}^2} \partial_\bar{a}X^{\mu}}{\sqrt{\dot{X}^2}} \right) = 0. \] (180)

This is precisely the equation of motion (39) of the Dirac-Nambu-Goto brane. Expressing it in terms of momenta, it can be written in the form

\[ \frac{dp_{\mu(\sigma)}}{d\tau} + \partial_\mu \left( p^{0(\sigma)} \dot{X}^2 \gamma^{\bar{a}\bar{b}} \partial_\bar{b}X_\mu \right) = 0. \] (181)

In the gauge \( \tau = X^0 \equiv t \), we find that for \( v^2 = 0 \) the latter equation corresponds to the expectation value equation of motion (154).

5 Further clarification of the action for a continuous set of interacting fields

We will now rewrite the action (99) into a more familiar form. Let us write \( \varphi^{(\sigma)}(x) \equiv \varphi(\sigma, x) \), and take the metric (128). Then, after performing partial integration over \( \sigma^{\bar{a}} \) and omitting the surface term, we obtain

\[ I = \frac{1}{2} \int d^Dx \, dp \sigma \left[ \partial_\mu \varphi \partial^\mu \varphi - m^2 \varphi^2 + \lambda(\partial_\mu \partial_\bar{a} \varphi \partial^\mu \partial_\bar{a} \varphi - m^2 \partial_\bar{a} \varphi \partial^\bar{a} \varphi) \right]. \] (182)

Variation with respect to \( \varphi(\sigma, x) \) gives the field equation

\[ \left( \partial_\mu \partial^\mu + m^2 \right)(1 + \lambda \partial_\bar{a} \partial^\bar{a}) \varphi = 0. \] (183)

A particular solution is

\[ \varphi = e^{-i \pi \sigma^\bar{a}} e^{-ip_\mu x^\mu}. \] (184)
subjected to the condition
\[(p_{\mu}p^{\mu} - m^2)(1 - \lambda \pi_\bar{a} \pi^a) = 0. \tag{185}\]

We can take \(\pi_\bar{a}\) arbitrary, whereas \(p_{\mu}\) satisfying the mass shell constraint \(p_{\mu}p^{\mu} - m^2 = 0\) (Case A), or vice versa, \(p_{\mu}\) arbitrary and \(\pi_\bar{a}\) satisfying \(1 - \lambda \pi^a \pi_\bar{a} = 0\) (Case B).

Eq. (183) can be written as
\[
\int d^{D}p' \sigma' (\partial_{\mu} \partial_{\mu} + m^2) s(\sigma, \sigma') \varphi(\sigma', x) = 0, \tag{186}\]
where
\[s(\sigma, \sigma') = (1 + \lambda \partial_\bar{a} \partial^a) \delta^p(\sigma - \sigma'). \tag{187}\]

The Fourier transform\(^{9}\) of \(s(\sigma, \sigma')\) is
\[s(\pi, \pi') = \delta^p(1 - \lambda \pi_\bar{a} \pi^a), \tag{188}\]
whose inverse is
\[\tilde{s}(\pi, \pi') = \frac{\delta^p(\pi - \pi')}{1 - \lambda \pi^a \pi_\bar{a}}. \tag{189}\]

Taking the Fourier transform of the latter expression, we obtain
\[
\tilde{s}(\sigma, \sigma') = \int d\pi \pi' e^{i\pi_\bar{a}(\sigma^a - \sigma'^a)} \frac{1}{1 - \lambda \pi^a \pi_\bar{a}}. \tag{190}\]

This is the propagator in the space of parameters \(\sigma^a\), and is the inverse of \(s(\sigma, \sigma')\). We thus have
\[
\int d\sigma'' \tilde{s}(\sigma, \sigma'') s(\sigma'', \sigma') = \delta^p(\sigma - \sigma'). \tag{191}\]

Multiplying Eq. (186) by \(\tilde{s}(\sigma'', \sigma)\), integrating over \(\sigma\), using (191), and renaming \(\sigma'\) into \(\sigma\), we obtain
\[(\partial_{\mu} \partial_{\mu} + m^2) \varphi(\sigma, x) = 0. \tag{192}\]

Eqs. (183) and (192), of course, correspond to (101) and (102), where
\[(1 + \lambda \partial_\bar{a} \partial^a) \varphi(\sigma, x) = \chi(\sigma, x) \equiv \varphi(\sigma)(x). \tag{193}\]

A general solution of Eq. (183) in Case A is
\[
\varphi(\sigma, x) = \int \frac{d^Dp \, d^D\pi}{\sqrt{(2\pi)^D2\omega_p}} \left[ a(\pi, \sigma) e^{-i\pi_\bar{a} \sigma^a} e^{-ip_{\mu} x^\mu} + a^\dagger(\pi, \sigma) e^{i\pi_\bar{a} \sigma^a} e^{ip_{\mu} x^\mu} \right]. \tag{194}\]

\(^{9}\) For simplicity reasons we use here the same symbol for the Fourier transformed quantity, but with different arguments.
This can be written as
\[
\varphi(\sigma, x) = \int \frac{d^Dp}{\sqrt{(2\pi)^D2\omega_p}} \left[ a(\sigma, p)e^{-ip_\mu x^\mu} + a^\dagger(\sigma, p)e^{ip_\mu x^\mu} \right].
\] (195)

where
\[
a(\sigma, p) = \int d^D\pi a(\pi, p)e^{-i\pi_\mu a^\mu}. \tag{196}
\]

Identifying \( a(\sigma, p) \equiv a(\sigma)(p) \) we find that (195) is the same equation as (106), as it should be.

We see that our continuous bunch of scalar fields whose mutual interaction is given by the field space metric \( s(\sigma, \sigma') \equiv s(\sigma') \) (given in Eq. (128) or (187)) is described by the action (182). This is an action for a field \( \varphi(\sigma^a, x^\mu) \), which depend not only on spacetime coordinates \( x^\mu \), but also on the brane parameters \( \sigma^a \). The action can be written in terms of the metric (187), whose inverse is the propagator (190) on the brane.

6 Conclusion

We have found a resolution of the problem of brane quantization, which can have important implication for the brane world scenarios that consider our spacetime as a brane living in a higher dimensional space. First we have shown that the Dirac-Nambu-Goto brane can be described as a “point particle” in an infinite dimensional brane space with a special metric. The analogy with general relativity suggests that the metric is dynamical and thus not necessarily restricted to the special form. As in general relativity the simplest metric is that of flat spacetime, so in the brane theory the brane space can have a simple “flat” metric as well. A flat brane is like a bunch of non interacting point particles. Upon quantization such a system is described by the quantum field theory of a continuous set of non interacting fields \( \varphi(\sigma) \), each one describing a different distinguishable particle.

We then considered an interacting system by introducing a coupling between the fields. We achieved this by adding an extra term to the \( \delta \)-function like metric in the field space. This extra, interacting, term was of the form \( \lambda \partial_a \partial^a \delta^p(\sigma - \sigma') \). Because of the latter term, the time derivative of the expectation value of the momentum operator, calculated for an evolving wave packet like state, does not vanish. We have found that the center of the wave packet at each \( \sigma^a \) satisfies the equations of motion of a classical brane which is nearly like the usual Dirac-Nambu-Goto brane. The difference is in the determinant \( \bar{\gamma} \equiv \det \gamma_{\bar{a}\bar{b}} \) of the induced metric on the brane being restricted to \( \bar{\gamma} = -1 \). We also showed that the interacting term \( \lambda \sqrt{-\bar{\gamma}} \partial_{\bar{a}} \gamma^{\bar{a}\bar{b}} \partial_{\bar{b}} \delta^p(\sigma - \sigma') \)
σ') for a general metric γ_{ab} and the corresponding determinant γ leads to the equation of motion of the Dirac-Nambu-Goto brane.

All this means that the special brane space metric for a Dirac-Nambu-Goto brane was induced from the underlying field theory of the continuous system of interacting scalar fields, the interaction being given by a certain coupling term. If we chose a different coupling term, we would obtain a different effective classical brane, living in a brane space whose metric were different from that of the Dirac-Nambu-Goto brane.

In this paper we concentrated on scalar field. But analogous procedure could be applied to fermion fields as well. We considered the usual canonical field quantization, which is somewhat cumbersome because of the (1 + 3) split of spacetime. How the present theory can be cast into the more elegant Fock-Schwinderg proper time formalism, or into the Stueckelberg invariant evolution parameter formalism, is beyond the scope of this paper, and will be shown elsewhere.

**Appendix A: Position operator**

The creation operator a^\dagger(x) for a particle at position x, although not Lorentz covariant, is not problematic, if it is understood that its definition is valid only in a given Lorentz reference frame. Namely, in the quantum field theory of a scalar field we have the operator a^\dagger(p) that creates a particle with momentum p. The Fourier transform of a^\dagger(p) is

\[ a^\dagger(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3k \; a^\dagger(k) e^{-ikx} \]  \hspace{1cm} (197)

The position operator is then

\[ \hat{x} = \int d^3x \; a^\dagger(x) \; x \; a(x) = \int d^3p \; a^\dagger(p) \; i \frac{\partial}{\partial p} a(p). \] \hspace{1cm} (198)

This is quite a legitimate operator in the Lorentz frame with respect to which the time t \equiv x^0 and space x \equiv x^\mu, \; \mu = 1, 2, 3 are defined. The difference with the usual treatment is that our position creation and annihilation operators a^\dagger(x), a(x) are not identified with the field operators ϕ(x) (or the positive or negative frequency part of ϕ(x)). They are just Fourier transforms of a^\dagger(p) and a(p). Since the latter operators are legitimate objects of the field theory, also a^\dagger(x), a(x) are legitimate objects in a given Lorentz frame S, although they are not covariant objects. In a different Lorentz frame one must define those operators anew.

Taking the Hamiltonian

\[ H = \int d^3p \; \omega(p) a^\dagger(p) a(p) = \hat{p}^0, \] \hspace{1cm} (199)

\footnote{We omit the zero point term.}
we have
\[ \dot{\hat{x}} = -i[\hat{x}, H] = \int d^3p \ a^\dagger(p)a(p) \frac{\partial \omega}{\partial p} = \int d^3p \ a^\dagger(p)a(p) \frac{p}{p^0} \] (200)

Since \( p/p^0 = v \), the above expression is indeed the velocity operator. Exactly the same equation (200) is satisfied by the Newton-Wigner position operator [37]. In Ref. [38] it is pointed out that the state \( |x\rangle \) created by \( a^\dagger(x) \) as defined in Eq. (197) is in fact the Newton-Wigner localized state.

From (198) we find that \( \hat{x} \) is Hermitian, \( \hat{x}^\dagger = \hat{x} \). We will show that it is self-adjoint with respect to the scalar product in the \( x \)-space.

A single particle wave packet profile is
\[ |\psi\rangle = \int d^3p \ g(p) a^\dagger(p)|0\rangle = \int d^3x \ f(x) a^\dagger(x)|0\rangle, \] (201)
where \( f(x) \) is the Fourier transform of \( g(p) \).

We define the scalar product according to
\[ \langle \psi|\psi \rangle = \int d^3p \ g^\dagger(p) g(p) = \int d^3x \ f^\dagger(x) f(x). \] (202)

The expectation value of the operator \( \hat{x} \) is
\[ \langle \psi|\hat{x}|\psi \rangle = \int d^3x \ f^\dagger(x) x f(x). \] (203)
We have
\[ \langle \psi|\hat{x}|\psi \rangle^\dagger = \langle \psi|\hat{x}|\psi \rangle, \] (204)
therefore the operator \( \hat{x} \) is self-adjoint with respect to the scalar product (202). Analogous holds for many particle wave packet profiles.

The operators \( a^\dagger(x), a(x) \), \( \hat{x} \) have been defined with respect to a particular \((1+3)\) split of spacetime, such that the simultaneity hypersurface \( \Sigma_\mu \) coincides with the space of coordinates \( x \), whereas the orthogonal to \( \Sigma_\mu \) points into the direction of the time coordinates \( x^0 = t \). In a different \((1+3)\) split, instead of \( x^\mu = (t, x) \), we have different coordinates \( x'^\mu = (t', x') \), and thus different operators \( a^\dagger(x'), a(x') \), \( \hat{x}' \). To different \((1+3)\) splits there correspond different choices of simultaneity hypersurfaces \( \Sigma_\mu \), and thus different Lorentz frames. Our operators are thus defined with respect to a given Lorentz frame \( S \), at a fixed value of time \( t \). In the frame \( S' \) one has to define different operators, which are suitable creation/annihilation and position operators in \( S' \), but not in \( S \). The theory is thus covariant in the sense that we can perform different \((1+3)\) splits and define in each of them the creation/annihilation and position

\[ ^{11} \text{For simplcity reason we take here spatial dimension } D = 3. \]
operators. But those operators themselves are not Lorentz covariant objects and cannot be transformed into another Lorentz reference frame. The transformation of \( a^\dagger(x) \) defined in Eq. (197) into another Lorentz frame makes no sense. The same is true for the localized state \( |x\rangle = a^\dagger(x)|0\rangle \). It is not correct to say that such a localized state in \( S \) when observed from \( S' \) acquires the strange properties of giving a nonzero amplitude for detection spread out all over space. To see how a localized state looks from the frame \( S' \), one must consider not only the basis state \( |x\rangle \), but also an amplitude \( f(x) \). In Appendix B we show that with the aid of the non covariant operators \( a^\dagger(x) \), we obtain the appropriate equations for amplitudes, and the corresponding 4-current which is a Lorentz covariant object.

Appendix B: Schrödinger equation and the probability current for single particle wave packet profiles

In terms of the creation operators \( a^\dagger(p) \) or \( a^\dagger(x) \), a general single particle state can be expressed as

\[
|\psi\rangle = \int d^D p \, g(p) a^\dagger(p)|0\rangle = \int d^D x \, f(x) a^\dagger(x)|0\rangle.
\]

(205)

Though we do not explicitly denote so, \( g \) and \( f \) depend on time \( t \) as well. The Schrödinger equation is

\[
i \frac{\partial |\psi\rangle}{\partial t} = H |\psi\rangle
\]

(206)

Taking the Hamiltonian (199), we obtain

\[
i \frac{\partial |\psi\rangle}{\partial t} = \int d^D p \, i \frac{\partial g(p)}{\partial t} a^\dagger(p)|0\rangle = \int d^D p \, g(p) \omega_p a^\dagger(p)|0\rangle,
\]

(207)

so that the amplitude \( g(p) \) satisfies

\[
i \frac{\partial g(p)}{\partial t} = \omega_p g(p).
\]

(208)

For the amplitude \( f(x) \) we have

\[
i \frac{\partial |\psi\rangle}{\partial t} = \int d^D x \, i \frac{\partial f(x)}{\partial t} a^\dagger(x)|0\rangle = \int d^D p \, d^D x \, d^D x' f(x) \frac{e^{ip(x' - x)}}{(2\pi)^D} \sqrt{m^2 + p^2} a^\dagger(x')|0\rangle
\]

\[
= \int d^D p \, d^D x \, d^D x' f(x) \frac{e^{ip(x' - x)}}{(2\pi)^D} m \left( 1 + \frac{p^2}{2m^2} + \ldots \right) a^\dagger(x')|0\rangle
\]

\[
= \int d^D x \, d^D x' f(x) \frac{1}{(2\pi)^D} m \left( 1 + \frac{(-i\nabla)^2}{2m^2} + \ldots \right) \delta^D(x - x') a^\dagger(x')|0\rangle
\]

(209)

\[\text{\textsuperscript{12} It is often said that those quantities are not Lorentz invariant. But this is misleading, because physical quantities need not be Lorentz invariant, they only need be Lorentz covariant; from a different Lorentz frame they may look different.}\]
After performing a partial integration, the action of the operator
\[ m \left( 1 + \frac{(-i\nabla)^2}{2m^2} + \ldots \right) = (m^2 + (-i\nabla)^2)^{1/2} \]
can be switched from \( \delta^D(x - x') \) to \( f(x) \), so that \( \delta^D(x - x') \) becomes "free", and can be integrated out. So we obtain

that the Schrödinger equation (209) for a single particle wave packet profiles is satisfied if the amplitude satisfies [39]–[41]

\[ i \frac{\partial f}{\partial t} = \left( m^2 + (-i\nabla)^2 \right)^{1/2} f. \]  

(210)

The Hamilton operator in the above equation when expanded contains derivatives up to infinite order. Therefore the function \( f \) satisfying (210), even if initially \( f(0, x) \) localized within a finite region, at any later time \( t \) has non vanishing values at all points \( x \). Therefore in the literature [39]–[41] it is usually said that such a wave function is non local. But in Ref. [42] it was shown that \( f(t, x) \) which satisfies the initial condition of a minimal uncertainty in position and momentum evolves as a wave packet whose probability density is concentrated in a finite spatial region. Using the results of Ref. [42] we have found that sufficiently close to the initial time the localization of such a wave packet is even more pronounced. The contribution of the wave packet's tail is small in comparison to the contribution of the region around the wave packet's center.

By the way, if \( f \) were a spinor, we could take the square root à la Dirac, and (210) would become the Dirac equation

\[ i \frac{\partial f}{\partial t} = (m\beta + i\alpha\nabla)f. \]  

(211)

In such a case, of course, \( a^\dagger(x) \) should be replaced by fermionic operators, and instead of the scalar field theory we would have the fermionic field theory.

What about the probability density and current? From Eq. (202) we see that

\[ f^\ast(x)f(x) = \rho(x) \]

(212)

can serve as the probability density. Differentiating \( \rho(x) \) with respect to time, and using the Schrödinger equation (210), we obtain

\[ \dot{\rho} = i \left( (Hf^\ast)f - f^\ast(Hf) \right). \]

(213)

Using the expansion

\[ Hf = \sqrt{m^2 + (-i\nabla)^2} f = m \left( 1 + \frac{1}{2} \frac{(-i\nabla)^2}{m^2} - \frac{1}{2 \cdot 4} \frac{(-i\nabla)^4}{m^4} + \ldots \right) f \]

(214)

we can rewrite Eq. (213) in the form

\[ \dot{\rho} = \nabla j, \]

(215)
where

\[ j = -im\dot{f}^* \left[ \frac{1}{2m^2}(\hat{\nabla} - \nabla) + \frac{1}{2m} \left( \hat{\nabla} - \nabla^2 + \nabla \nabla^2 - \nabla^3 \right) + \ldots \right] f \]  

(216)

Thus we can define the probability density according to (212), but for the corresponding current we then have the cumbersome expression (216). In the case of the Dirac equation (211), the current is the usual simple expression entering the 4-current

\[ j^\mu = (f^\dagger f, f^\dagger \alpha \cdot f) \]

For a scalar field, a covariant definition of the probability density can be derived from the expectation value of the operator \[ \hat{p}_0 = \left( m^2 + (-i\nabla)^2 \right)^{1/2} \]:

\[
\frac{1}{m} \langle \psi | \hat{p}_0 | \psi \rangle = \frac{1}{m} \int d^3x f^*(x) \sqrt{m^2 + (-i\nabla)^2} f(x) = \frac{1}{m} \int d^3x \ f^* i \frac{\partial}{\partial t} f
\]

\[
= \frac{i}{2m} \int d^3x \left( f^* \frac{\partial}{\partial t} f - \frac{\partial}{\partial t} f^* f \right) = \int d^3x \ j^0, \quad (217)
\]

where

\[ j^0 = \frac{i}{2m} \left( f^* \frac{\partial}{\partial t} f - \frac{\partial}{\partial t} f^* f \right) \]  

(218)

From the time derivative

\[
\frac{\partial j^0}{\partial t} = \frac{i}{2m} \left( f^* \frac{\partial}{\partial t} f - \frac{\partial}{\partial t} f^* f \right), \quad (219)
\]

by using

\[
i \dot{f} = H f, \quad -i \dot{f}^* = H f^*, \quad H = \sqrt{m^2 + (-i\nabla)^2},
\]

\[
\ddot{f} = -H^2 f, \quad \ddot{f}^* = -H^2 f^*, \quad H^2 = m^2 + (-i\nabla)^2 \]  

(220)

we obtain

\[
\frac{\partial j^0}{\partial t} = \nabla j, \quad j = \frac{i}{2m} (\nabla f^* f - f^* \nabla f). \quad (221)
\]

We see that the manipulations with position creation/annihilation operators in the \( x \)-representation are straightforward and lead to the result that make sense, and are consistent with those in the usual scalar field theory. The non covariantly defined operators \( a^\dagger (x), a(x) \) do not appear in the definition of the covariant object, the 4-current \( j^\mu = (j^0, j) \), defined in Eqs. (218), (221).

We will now complete our discussion by considering a state \( |\psi\rangle \), defined according to Eq. (205), in which \( f(x) \) represents a localized state. Let us consider the case of a state localized in the initial time \( t = 0 \) at position \( x_0 \), so that

\[ f(x) = f(0, x) = \delta^D(x - x_0). \]  

(222)

The Fourier transformed wave packet profile is

\[ g(p) = g(0, p) = \frac{1}{\sqrt{(2\pi)^D}} \int d^Dx e^{-ipx} f(x) = \frac{1}{\sqrt{(2\pi)^D}} e^{-ipx_0}. \]  

(223)
The state at any time is given in terms of \(g(t, \mathbf{p})\), which evolves according to Eq. (208), the solution being
\[
g(t, \mathbf{p}) = e^{i\omega_p t}g(0, \mathbf{p}).
\] (224)

So we have
\[
|\psi(t)\rangle = \int d^D \mathbf{p} \, g(t, \mathbf{p})a^\dagger(\mathbf{p})|0\rangle = \frac{d^D \mathbf{p}}{\sqrt{(2\pi)^D}} e^{i\omega_p t}e^{-ipx_0}a^\dagger(\mathbf{p})|0\rangle.
\] (225)

The projection of Eq. (225) into the state \(|\mathbf{p}\rangle = |0\rangle a(\mathbf{p})\) gives
\[
\langle \mathbf{p}|\psi(t)\rangle = \frac{1}{\sqrt{(2\pi)^D}} e^{i\omega_p t} = g(t, \mathbf{p}),
\] (226)

whereas the projection into the state \(|\mathbf{x}\rangle = |0\rangle a(\mathbf{x})\) gives
\[
\langle \mathbf{x}|\psi(t)\rangle = f(t, \mathbf{x}) = \frac{1}{(2\pi)^D} \int d^D \mathbf{p} e^{i\sqrt{m^2+p^2}t} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}_0)} = e^{i\sqrt{m^2+(-i\nabla)^2}t} \delta(\mathbf{x}-\mathbf{x}_0).
\] (227)

Expanding \(\sqrt{m^2+p^2} = m \left(1 + \frac{bp^2}{2m^2} + \ldots\right)\) and neglecting higher order terms, the above equation gives
\[
f(t, \mathbf{x}) = \frac{1}{(2\pi)^D} \int d^D \mathbf{p} e^{\imath mt} e^{\frac{p^2}{2m}t} e^{i\mathbf{p}(\mathbf{x}-\mathbf{x}_0)},
\] (228)

which, apart from the phase factor \(e^{i\mathbf{mt}}\), is just the Green function for a non-relativistic free particle. The full expansion gives (227), which is the Green function for a single free relativistic particle [42].

Instead of the \(\delta\)-function initial localization (222), we can take a Gaussian function
\[
f(0, \mathbf{x}) \propto \exp \left[\frac{(\mathbf{x}-\mathbf{x}_0)^2}{2\sigma_0}\right].
\] (229)

This holds in \(S\). Observed from another reference frame \(S'\), moving with respect to \(S\) with velocity \(v\) along the \(x^1 \equiv x\) direction, the wave function transform according to [13]
\[
f(t, \mathbf{x}) = f'(t', \mathbf{x}') \propto \exp \left[(1-v^2)(x'-x'_0)^2+(y'-y'_0)^2+(z'-z'_0)^2\right],
\] (230)

where \(t = 0\) and \(t' = -\frac{x'}{\sqrt{1-v^2}}\). This is also a localized wave packet, only the distance is Lorentz contracted and different points \(x'\) are not simultaneous in \(S'\). In deriving (230) we used the transformations
\[
x = \frac{x' + vt'}{\sqrt{1-v^2}}, \quad x_0 = \frac{x'_0 + vt'_0}{\sqrt{1-v^2}}, \quad y = y', \quad z = z'
\] (231)
\[
t = \frac{t' + vx'}{\sqrt{1-v^2}}, \quad t_0 = \frac{t'_0 + vx'_0}{\sqrt{1-v^2}}
\] (232)

[13] We now take the 4D spacetime, and denote \(x^1 \equiv x, x^2 \equiv y, x^3 \equiv z\).
Taking \( t = t_0 \) (which means that both points, \( x \) and \( x_0 \), are simultaneous in \( S \)), Eq. (232) gives \( t' - t'_0 = -v(x' - x'_0) \) (which means that those two points are not simultaneous in \( S' \)). From Eq. (231) we then obtain \( x - x_0 = \sqrt{1 - v^2}(x' - x'_0) \), i.e., the length contraction along the \( x \)-direction.

We see that nothing strange happens with the particle localization if we observe it from another Lorentz frame. Instead of the spherical Gaussian (229) at \( t = 0 \), we see in \( S' \) an ellipsoidal Gaussian (230) function, each \( x' \) taken at different \( t' = -vx'/\sqrt{1 - v^2} \). This reflects the fact that in \( S \) the localization is on the simultaneity hyper surface \( t = 0 \), which in \( S' \) is not a simultaneity Hyper surface. The observer in \( S' \), of course, normally does not define the spread of a localized particle’s position at different values of his time \( t' \), he defines it at the same value of \( t' \), i.e., on the simultaneity hyper surface of the Lorentz frame \( S' \). Therefore the observer in \( S' \) formulates the quantum field theory with respect to \( S' \) in the same manner as we did with respect to \( S \). In \( S' \) one then obtains results concerning wave packets and localization that are analogous to those that we obtained in the reference frame \( S \).

As already mentioned, according to Ref. [42] such an initially Gaussian wave packet evolves so that it remains localized in the sense that the contribution of its tail remains small in comparison to the contribution around the wave packet’s center. Though the tail contains superluminal propagation, it does not necessarily mean the violation of causality, if the latter is properly defined in terms of macroscopic modulated beams that can bear information. Single particle events at space-like separations can not transmit information, and therefore do not violate the properly defined causality.

**Appendix C: Comparison with the Newt-Wigner position operator**

If instead of the operators \( a(p), a^\dagger(p) \) satisfying the commutation relations

\[
[a(p), a^\dagger(p)] = \delta^3(p - p'),
\]

we use the operator \( \tilde{a}(p) = \sqrt{(2\pi)^3 2\omega_p} a(p), \tilde{a}^\dagger(p) = \sqrt{(2\pi)^3 2\omega_p} a^\dagger(p) \) satisfying

\[
[\tilde{a}(p), \tilde{a}^\dagger(p)] = (2\pi)^3 2\omega_p \delta^3(p - p'),
\]

then the position operator (198) becomes

\[
\hat{x} = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \tilde{a}^\dagger(p) i \left( \nabla_p - \frac{p}{2\omega_p^2} \right) \tilde{a}(p),
\]

where we now use \( \nabla_p \equiv \partial/\partial p \).

Let us consider the state

\[
|\psi\rangle = \int \frac{d^3p}{(2\pi)^3 2\omega_p} \hat{g}(p)\tilde{a}^\dagger(p)|0\rangle,
\]

36
act on it by the operator $\hat{x}$, and project $\hat{x}|\psi\rangle$ onto the state $|\tilde{x}\rangle$ defined as

$$|\tilde{x}\rangle = \varphi^+(0, x)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ikx} a^\dagger(k)|0\rangle,$$  \hspace{1cm} (237)

where $\varphi^+(0, x)$ is the positive energy Klein-Gordon field operator. We obtain

$$\langle \tilde{x}|\hat{x}|\psi\rangle = \int d^3p \frac{e^{-ipx}}{(2\pi)^3 2\omega_p} \left( \nabla_p - \frac{p}{2\omega_p^2} \right) g(p)$$

$$= \left( x + \frac{1}{2m^2 + (-i\nabla)^2} \right) \tilde{f}(x),$$  \hspace{1cm} (238)

where

$$\tilde{f} = \int \frac{d^3p}{(2\pi)^3 2\omega_p} e^{-ipx} g(p).$$  \hspace{1cm} (239)

In Eq. (238) we have precisely the well-known non local action of Newton-Wigner position operator on a wave function $\tilde{f}(x)$.

Why then in our paper we do not have such a non local action of the position operator? Let us rewrite the state (236) in terms of the operators $a^\dagger(p) = \tilde{a}(p) / \sqrt{(2\pi)^3 2\omega_p}$. If we introduce $g(p) = \tilde{g}(p) / \sqrt{(2\pi)^3 2\omega_p}$, then (236) becomes

$$|\psi\rangle = \int d^3p g(p) a^\dagger(p)|0\rangle,$$  \hspace{1cm} (240)

which is just our wave packet (205). But the state $|\tilde{x}\rangle = \varphi^+(0, x)|0\rangle$ defined according to Eq. (237) is not the same as the position state $|x\rangle = a^\dagger(x)|0\rangle$ used throughout this paper. If we project $\hat{x}|\psi\rangle$ onto $|x\rangle$, then

$$\langle x|\hat{x}|\psi\rangle = \int \frac{d^3pe^{ipx}}{\sqrt{(2\pi)^3}} i\nabla_p g(p) = xf(x),$$  \hspace{1cm} (241)

where

$$f(x) = \frac{1}{\sqrt{(2\pi)^3}} \int d^3pe^{ipx} g(p).$$  \hspace{1cm} (242)

It is well known that the states $|\tilde{x}\rangle$ created by the Klein-Gordon field operator are not position states. If one uses such states, then $\langle \tilde{x}|\hat{x}|\psi\rangle$ of course gives a non local result, which is indeed confirmed in Eq. (238). If, on the contrary, one uses the states $|x\rangle$ created by $a^\dagger(x)$, then $\langle x|\hat{x}|\psi\rangle$ gives the position eigenstates, as shown in Eq. (241). The states $|x\rangle = a^\dagger(x)|0\rangle$ are the basis states, defined with respect to a given Lorentz reference frame. They are not dynamical objects of the theory, they do not satisfy the Schrödinger equation. On the other hand, the state $|\psi\rangle$, whose definition (240) is equivalent to the definition (236), must satisfy the Schrödinger equation. The state $|\psi\rangle$ is a Lorentz covariant object. It is the state $|\psi\rangle$, not $|x\rangle$, which is a Lorentz covariant object.
which is physically relevant, and whose properties (e.g., localization, behaviour in different Lorentz frames, etc.) are to be considered.

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