ON UNRAMIFIED BRAUER GROUPS OF FINITE GROUPS

SUMANA HATUI

Abstract. The Bogomolov multiplier $B_0(G)$ of a finite group $G$ is the subgroup of the Schur multiplier $H^2(G,\mathbb{Q}/\mathbb{Z})$ consisting of the cohomology classes which vanishes after restricting to any abelian subgroup of $G$. We give a proof of Hopf-type formula for $B_0(G)$ and derive an exact sequence for cohomological version of the Bogomolov multiplier. Using this exact sequence we provide necessary and sufficient conditions for the corresponding inflation homomorphism to be an epimorphism and to be zero map. We provide some conditions for the triviality of $B_0(G)$ for central product of groups $G$ and show that the Bogomolov multiplier of generalized discrete Heisenberg groups is trivial. We also give a complete characterization of groups of order $p^6$, $p > 3$ having trivial Bogomolov multiplier.

1. Introduction

The Bogomolov multiplier is a group theoretical invariant isomorphic to the unramified Brauer group of a given quotient space. This object $B_0(G)$ has a connection with Noether’s problem on rationality of fields of invariants: let $G$ be a finite group and $V$ be a faithful representation of $G$ over the field $\mathbb{C}$. Then $G$ acts naturally on the field of rational functions $\mathbb{C}(V)$. Noether’s problem [17] asks that for which groups $G$, the field of $G$-invariant functions $\mathbb{C}(V)^G$ is rational over $\mathbb{C}$? A question related to the above is whether $\mathbb{C}(V)^G$ is stably rational over $\mathbb{C}$, i.e., are there independent variables $y_1, y_2, \ldots, y_s$ such that $\mathbb{C}(V)^G(y_1, y_2, \ldots, y_s)$ is a purely transcendental extension over $\mathbb{C}$? This problem has close connection with Lüroth’s problem [21] and the inverse Galois problem [20, 22]. It is known that the stable rationality does not depend upon the choice of $V$, it depends only on the group $G$. Saltman [20] introduced the unramified Brauer group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ and using it he found examples of groups $G$ of order $p^9$ such that $\mathbb{C}(V)^G$ is not stably rational over $\mathbb{C}$. Later Bogomolov [2] explored the group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$ and proved that it is canonically isomorphic to

$$B_0(G) = \bigcap_{A \leq G; A \text{ finite abelian}} \ker \left( \text{res}_A^G : H^2(G,\mathbb{Q}/\mathbb{Z}) \to H^2(A,\mathbb{Q}/\mathbb{Z}) \right),$$

which is a subgroup of $H^2(G,\mathbb{Q}/\mathbb{Z})$. Kunyavskii [11] called $B_0(G)$, the Bogomolov multiplier of $G$. Bogomolov [2] used this description to give examples of groups $G$ of order $p^6$ having $B_0(G) \neq 0$ and this non-vanishing property of $B_0(G)$ answers the classical Noether’s problem, i.e., the invariant field $\mathbb{C}(V)^G$ is not rational over $\mathbb{C}$. So the triviality of $B_0(G)$ is an obstruction to the Noether’s problem.
Another description of $B_0(G)$ was given by Moravec [15]. Let $G \wedge G$ denote the non-abelian exterior square of group $G$ (see Section 2.1 for details). For $x, y \in G$, we write a commutator $[x, y] = x^{-1}y^{-1}xy$. Let $M(G)$ be the kernel of the commutator map $G \wedge G \to [G, G]$. Define $M_0(G) = \langle x \wedge y \mid [x, y] = 1 \rangle$. Moravec [15] defined

$$B_0(G) = M(G)/M_0(G)$$

and proved that, for finite group $G$,

$$B_0(G) \cong \text{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z}).$$

Hence for finite groups $G$, $B_0(G) \cong \tilde{B}_0(G)$. This description is very much useful and helps us to compute $B_0(G)$ explicitly; see for example [14, 15, 7]. We have also used this description in Section 5 to compute $B_0(G)$ for groups $G$ of order $p^6, p$ prime.

Here we set some notations before proceeding further. We denote the commutator subgroup and center of $G$ by $G'$ and $Z(G)$ respectively. The set of commutators $\{[x, y] \mid x, y \in G\}$ is denoted by $K(G)$. For two subgroups $H$ and $K$ of $G$, the group generated by the set $\{[h, k] \mid h \in H, k \in K\}$ is denoted by $[H, K]$.

Let $F/R$ be a free presentation of a group $G$. Moravec [15] proved a Hopf-type formula for $\tilde{B}_0(G)$ by showing that

$$\tilde{B}_0(G) \cong \frac{F' \cap R}{\langle K(F) \cap R \rangle},$$

and with the help of this, he derived a five term exact sequence for homological version of $\tilde{B}_0(G)$: see [15 Proposition 3.12]. Now we are ready to state our first main result which gives a different proof of Hopf-type formula. In the proof, we provide an isomorphism between $B_0(G)$ and $\text{Hom}\left(\frac{F' \cap R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}\right)$ which is different than given in [15]. This isomorphism will be used to prove Theorem 1.2 and Theorem 1.3.

**Theorem 1.1.** Let $G$ be a finite group with the free presentation $F/R$. Then for $A = \mathbb{Q}/\mathbb{Z}$, the transgression map

$$\text{tra} : \text{Hom}(R/\langle K(F) \cap R \rangle, A) \to B_0(G),$$

corresponding to the exact sequence

$$1 \to R/\langle K(F) \cap R \rangle \to F/\langle K(F) \cap R \rangle \to G \to 1,$$

is surjective. Consequently,

$$B_0(G) \cong \text{Hom}\left(\frac{F' \cap R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}\right).$$

We prove this result in Section 3.

In the next result, we provide an exact sequence for $B_0(G)$. Let $G = F/R$ and $N = S/R$ be a normal subgroup of $G$. Our aim is to define a homomorphism $\theta : B_0(G) \to \text{Hom}\left(\frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}\right)$ and prove the exactness of the following sequence which is dual of the exact sequence given in [15, Proposition 3.12].
Theorem 1.2. Let $G$ be a finite group with free presentation $F/R$ and $N = S/R$ be a normal subgroup of $G$. Suppose $\mathbb{Q}/\mathbb{Z}$ is a trivial $G$-module. Then we have the following exact sequence,

\[ 0 \to \text{Hom}\left(\frac{N \cap G'}{(N \cap K(G))}, \mathbb{Q}/\mathbb{Z}\right) \xrightarrow{\text{tra}} B_0(G/N) \xrightarrow{\text{inf}} B_0(G) \xrightarrow{\theta} \text{Hom}\left(\frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}\right) \to 0 \tag{1.0.1} \]

In the next result, we provide necessary and sufficient conditions for the inflation homomorphism $\text{inf}: B_0(G/N) \to B_0(G)$ to be an epimorphism and to be a zero map.

Theorem 1.3. Let $G$ be a finite group with free presentation $F/R$ and $N = S/R$ be a normal subgroup of $G$. Then

(i) The map $\text{inf}: B_0(G/N) \to B_0(G)$ is an epimorphism if and only if

\[ R \cap \langle K(F) \cap S \rangle = \langle K(F) \cap R \rangle. \]

(ii) The map $\text{inf}: B_0(G/N) \to B_0(G)$ is a zero map if and only if

\[ R \cap \langle K(F) \cap S \rangle = F' \cap R. \]

In [23], Vermani gave an exact sequence for the second cohomology group and using that exact sequence he provided necessary and sufficient conditions for inflation homomorphism to be an epimorphism and to be a zero map (for the second cohomology group case). For Bogomolov multiplier case, we use his techniques to prove Theorem 1.2 and Theorem 1.3; see Section 3.

In Section 4, we discuss about the triviality of Bogomolov multiplier for central product of groups. A group $G$ is called central product of two groups $H$ and $N$ via the map $\xi$ if there exists subgroups $H_1 \leq Z(H)$, $N_1 \leq Z(N)$ and an isomorphism $\xi: H_1 \to N_1$ such that $G \cong \frac{H \times N}{Z}$, where $Z = \{ (a, \xi(a)^{-1}) \mid a \in H_1 \}$. Kang and Kunyavski˘ı raised the following question:

Question 1: ([10, Question3.3(i)]) Let $G$ be a central product of groups $H$ and $N$ such that $B_0(H) = B_0(N) = 0$. Is it true that $B_0(G) = 0$?

Rai [18] gave a negative answer to this question by constructing an example. To answer this Question, Michailov [12, Theorem 3.1] proved that if $\eta: H \to N$ is a homomorphism such that $\xi = \eta|_{H_1}$ is an isomorphism and $B_0(H) = B_0(N) = 0$, then $B_0(H/H_1) = 0$ implies $B_0(G) = 0$. For giving an answer to Question 1, we establishes the following and generalize Michailov’s result.

Theorem 1.4. Let $G$ be a central product of two groups $H$ and $N$ via the map $\xi$ such that $B_0(H) = B_0(N) = 0$. Suppose any one of the following two conditions are satisfied.

(i) $\xi(H_1 \cap H') \subseteq N_1 \cap K(N)$.

(ii) There is a homomorphism $\eta: H \to N$ such that $\xi = \eta|_{H_1}$.

Then $B_0(G) = 0$ if and only if $B_0(H/H_1) = 0$. 
Theorem 1.5. Let \( G \) be a non-abelian group of order \( p^6, p > 3 \). Then the Hochschild-Serre spectral sequence \([5]\) for cohomology of groups yields the following exact sequence

\[ 0 \to \text{Hom}(G/N, D) \xrightarrow{\text{inf}} \text{Hom}(G, D) \xrightarrow{\text{res}} \text{Hom}(N, D) \xrightarrow{\text{tra}} H^2(G/N, D) \xrightarrow{\text{inf}} H^2(G, D), \]

where \( \text{tra} : \text{Hom}(N, D) \to H^2(G/N, D) \) is called transgression homomorphism, given by \( f \mapsto \text{tra}(f) = [\alpha] \), with

\[ \alpha(x, y) = f(xN, yN) \] for all \( x, y \in G/N \),

for a section \( \mu : G/N \to G \). The inflation homomorphism, \( \text{inf} : H^2(G/A, D) \to H^2(G, D) \) is given by \( [\alpha] \mapsto [\text{inf}([\alpha])] = [\beta] \), where \( \beta(x, y) = \alpha(xN, yN) \) for all \( x, y \in G \). The map \( \text{res} \) denotes restriction homomorphism.

Now we see that these above maps can be defined for \( B_0(G) \). We define

\[ \text{res}' : \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\frac{N}{K(G) \cap N}, \mathbb{Q}/\mathbb{Z}) \]

by \( \text{res}'(f)(\bar{n}) = f(n) \) for \( f \in \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \) and \( \bar{n} \in \frac{N}{K(G) \cap N} \). It is easy to see that \( \text{res}' \) is a well defined homomorphism.

Next we define transgression map \( \text{tra} : \text{Hom}(\frac{N}{K(G) \cap N}, \mathbb{Q}/\mathbb{Z}) \to B_0(G/N) \) given by \( \chi \mapsto \text{tra}(\chi) = [\alpha] \), with

\[ \alpha(x, y) = \chi(xN, yN) \] for all \( x, y \in G/N \),

for a section \( \mu : G/N \to G \). If \( \chi \in \text{Hom}(\frac{N}{K(G) \cap N}, \mathbb{Q}/\mathbb{Z}) \), then we show that \( \text{tra}(\chi) \in B_0(G/N) \). Let \( A/N \) be a finite abelian subgroup of \( G/N \) and \( \mu : G/N \to G \) be a section.
For \(a, b \in A/N \) we have
\[
\text{tra}(\chi)(a, b) = \chi(\mu(\bar{a})\mu(\bar{b})\mu(ab)^{-1}) = \chi([\mu(\bar{a})^{-1}, \mu(\bar{b})^{-1}], \mu(\bar{a})\mu(\bar{b})^{-1})
\]
Thus, \(\text{tra}(\chi)\) is a symmetric 2-cocycle on finite abelian groups \(A/N\). Since \(\mathbb{Q}/\mathbb{Z}\) is a divisible abelian group, \([\text{tra}(\chi)]_{A/N \times A/N} = 1\). Thus \(\text{tra}\) is a well-defined homomorphism. It is also easy to see that the inflation map \(\text{inf} : \text{B}_0(G/N) \to \text{B}_0(G)\) is a well defined homomorphism.

The following result describes the Bogomolov multiplier for direct product of groups.

**Theorem 2.1.** ([16] Theorem 1.4) Let \(G_1\) and \(G_2\) be two finite groups. Then the restriction map \(\text{res} : \text{B}_0(G_1 \times G_2) \to \text{B}_0(G_1) \times \text{B}_0(G_2)\) is an isomorphism.

2.1. Another description of \(\text{B}_0(G)\). The non-abelian exterior square of a group \(G\), denoted by \(G \wedge G\), is a group generated by the symbols \(x \wedge y, x, y \in G\) with the following relations:
\[
xy \wedge z = (y^{-1}xy \wedge y^{-1}zy)(y \wedge z),
\]
\[
x \wedge yz = (x \wedge z)(z^{-1}xz \wedge z^{-1}yz),
\]
\[
x \wedge x = 1.
\]
It follows from the definition that the map \(f : G \wedge G \to G',\) defined on the generators by \(f([g, h]) = [g, h]\), is an epimorphism. Let \(M(G) = \ker f\). Miller [13] proved that \(M(G)\) is isomorphic to the second homology group \(H_2(G, \mathbb{Z})\). We define \(M_0(G)\), a subgroup of \(M(G)\), generated by the set \(\{x \wedge y | [x, y] = 1\}\). We refer [3, 13] to the readers for more details.

A different description of \(G \wedge G\) was introduced in [19], which is more handy for evaluating exterior square of group \(G\). By \(G^\phi\) we denote the isomorphic copy of group \(G\) via the isomorphism \(\phi\). Consider the group
\[
\tau(G) := \langle G, G^\phi | R, R^\phi, [g_1, g_2]^\phi = [g_1^\phi, g_2^\phi], [g, g^\phi] = 1 \rangle
\]
where \(R, R^\phi\) are the defining relations of \(G\) and \(G^\phi\) respectively. Then it follows from [11 Proposition 16] that the map \(\Phi : G \wedge G \to [G, G^\phi]\) defined by
\[
\Phi(g \wedge h) = [g, h^\phi], \quad g, h \in G
\]
is an isomorphism. Let \(\eta := f \circ \Phi^{-1} : [G, G^\phi] \to G'\) and \(M^*(G) = \ker \eta = \Phi(M(G))\), \(M_0^*(G) = \Phi(M_0(G))\). Then it follows from [15] that
\[
\text{B}_0(G) \cong M^*(G)/M_0^*(G).
\]

**Lemma 2.2.** (See [11]) For a group \(G\), the following properties hold in \(\tau(G)\).

(i) If \(G\) is nilpotency class \(c\), then \(\tau(G)\) has nilpotency class at most \(c + 1\).
(ii) If \(G'\) is of nilpotency class \(c\), then \([G, G^\phi]\) has nilpotency class \(c\) or \(c + 1\).
(iii) If \(G\) has nilpotency class \(\leq 2\), then \([G, G^\phi]\) is abelian.
where integer. Then for ker \( \eta \) is isomorphic to \( \text{Hom}(3.0.1) \) 

\[ N ] (v) \quad [[g_1, g_2^\phi], [h_1, h_2^\phi]] = [[g_1, g_2], [h_1, h_2]] \text{ for all } g_1, g_2, h_1, h_2 \in G.
\]

(vi) \( [g_1, g_2^\phi], [g_2, g_1^\phi] = 1 \) for all \( g_1, g_2 \in G \).

(vii) \( [g_1, g_2^\phi] = [g_1^\phi, g_2] \) for all \( g_1, g_2 \in G \).

We define \([x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n], \) for \( x_1, x_2, \ldots, x_n \in G, n \geq 2 \).

**Lemma 2.3.** (\[14\] Lemma 3.7) Let \( G \) be a nilpotent group of class \( \leq 5, n \) a positive integer. Then for \( x, y \in G \),

\[ [x^n, y] = [x, y]^n [x, y, x]^{(n)} [x, y, x, x]^{(n)} [x, y, x, x, x]^{(n)} [x, y, x, x, x, x]^{(n)}, \]

where \( \sigma(n) = \frac{n(n-1)(2n-1)}{6} \).

**Lemma 2.4.** (\[14\] Lemma 2.2) Let \( G \) be a finite polycyclic group with polycyclic generating sequence \( \{g_1, g_2, \ldots, g_n\} \). Then the group \([G, G^\phi]\), a subgroup of \( \tau(G) \), is generated by the set \( \{g_i, g_j^\phi\}, i, j = 1, 2, \ldots, n, i > j \} \).

### 3. Exact sequences

In this section we prove Theorem 1.1, Theorem 1.2 and Theorem 1.3. Following the proof of [24] Theorem 11.9.1, it is easy to check that the following sequence is exact.

\[ (3.0.1) \quad 0 \to \text{Hom}(G/N, Q/Z) \stackrel{\inf}{\to} \text{Hom}(G, Q/Z) \stackrel{\text{res}'}{\to} \text{Hom}(\frac{N}{K(G) \cap N}, Q/Z) \stackrel{\text{tra}}{\to} B_0(G/N) \stackrel{\inf}{\to} B_0(G) \]

**Lemma 3.1.** Let \( 1 \to N \to G \to G/N \to 1 \) be an exact sequence. Then the image of the corresponding transgression map

\[ \text{tra} : \text{Hom}(N/(K(G) \cap N), Q/Z) \to B_0(G/N) \]

is isomorphic to \( \text{Hom}(\frac{G'}{(K(G) \cap N), Q/Z}) \).

**Proof.** We have a surjective homomorphism \( \eta : \text{Hom}(\frac{N}{K(G) \cap N}, Q/Z) \to \text{Hom}(\frac{N\cap [G, G]}{(K(G) \cap N)}, Q/Z) \) induced by the inclusion map \( N \cap [G, G] \hookrightarrow N \). Let

\[ K = \{ f \in \text{Hom}(\frac{N}{K(G) \cap N}, Q/Z) \mid f \text{ can be extended to a homomorphism } G \to Q/Z \}. \]

By the exact sequence (3.0.1), we have \( K = \text{Im} (\text{res}') = \ker (\text{tra}) \). Now we show that \( K = \ker \eta \). It will imply that

\[ \text{Im}(\text{tra}) \cong \text{Hom}(\frac{N}{K(G) \cap N}, Q/Z) \cong \text{Hom}(\frac{N \cap [G, G]}{(K(G) \cap N)}, Q/Z). \]

and we are done.

Let \( f \in \text{Hom}(\frac{N}{K(G) \cap N}, Q/Z) \) such that \( f \in K \). Then \( \frac{N\cap [G, G]}{(K(G) \cap N)} \subseteq \ker f \) ans so \( f \in \ker \eta \). Conversely, let \( g \in \ker \eta \). Then \( g \) is regarded as a homomorphism \( \frac{N}{N\cap [G, G]} \to Q/Z \). As \( \frac{N}{N\cap [G, G]} \cong \frac{[G, G]}{[G, G]} \) and \( Q/Z \) is divisible abelian group, \( g \) can be extended to a homomorphism \( G/[G, G] \to Q/Z \). Hence \( g \in K \). Hence \( K = \ker \eta \). \( \square \)
Proof of Theorem 1.1

Proof. By [8, Theorem 3.1 and Proposition 2.4] it follows that, there is a bijective correspondence between $B_0(G)$ and central CP extensions of $G$ by $A = \mathbb{Q}/\mathbb{Z}$. Let $\alpha$ be a 2-cocycle of $G$ such that $[\alpha] \in B_0(G)$. Then there is a central CP extension

$$1 \to A \to H \xrightarrow{\mu} G \to 1$$

which corresponds to $\alpha$. Since $F$ is a free group, there is a homomorphism $f : F \to H$ such that the following diagram is commutative. Here $\bar{f}$ denotes the restriction of $f$ on $R$.

$$\begin{array}{cccccc}
1 & \longrightarrow & R & \longrightarrow & F & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{\mu} & G & \longrightarrow & 1
\end{array}$$

Diagram 1

Let $[a, b] \in K(F) \cap R$, then $\pi([a, b]) = [\pi(a), \pi(b)] = \mu([f(a), f(b)]) = 1$ which says that $[f(a), f(b)] \in K(H) \cap A$. By [8, Proposition 3.3], it follows that $[f(a), f(b)] = f([a, b]) = 1$. Hence $\langle K(F) \cap R \rangle \subseteq \ker f$ and we have the following commutative diagram induced from the above one.

$$\begin{array}{cccccc}
1 & \longrightarrow & R/\langle K(F) \cap R \rangle & \longrightarrow & F/\langle K(F) \cap R \rangle & \longrightarrow & G & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{\mu} & G & \longrightarrow & 1
\end{array}$$

Now we show that, $\text{tra}(\bar{f}) = [\alpha]$. Let $s$ be a section of $\pi$. Then $f \circ s$ is a section of $\mu$. Thus $\alpha$ is cohomological to a cocycle

$$\beta(x, y) = f(s(x)s(y)s(y)^{-1}) = \bar{f}(s(x)s(y)s(y)^{-1}) = \text{tra}(\bar{f}).$$

Hence the map $\text{tra} : \text{Hom}(R/\langle K(F) \cap R \rangle, A) \to B_0(G)$ is surjective. Now by Lemma 3.1 we have

$$B_0(G) \cong \text{Hom}(F/\langle K(F) \cap R \rangle, \mathbb{Q}/\mathbb{Z}).$$

this completes our proof. \hfill \Box

Theorem 3.2. Let $F/R$ be a free presentation of a finite group $G$, $N = S/R$ be a normal subgroup of $G$ and $\mathbb{Q}/\mathbb{Z}$ is a trivial $G$-module. Then the following sequence is exact.

$$0 \to \text{Hom}(G/N, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\inf} \text{Hom}(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\res} \text{Hom}(\frac{N}{\langle K(G) \cap N \rangle}, \mathbb{Q}/\mathbb{Z})$$

$$\xrightarrow{\text{tra}} B_0(G/N) \xrightarrow{\inf} B_0(G) \xrightarrow{\theta} \text{Hom}(\frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}) \to 0.$$

Proof. Our first goal is to define the map $\theta$. Let

$$\psi : \text{Hom}(\frac{S}{\langle K(F) \cap S \rangle}, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z})$$

be the map induced by the inclusion map $R \to S$. 
Claim 1. First we prove that

$$(3.0.2) \quad \frac{\text{Hom}(R \cap \langle K(F) \rangle, \mathbb{Q}/\mathbb{Z})}{\text{Im}(\psi)} \cong \text{Hom}(\frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}).$$

Consider the restriction homomorphism

$$\delta : \text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}) \to \text{Hom}(\frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a divisible abelian group and $\frac{R}{\langle K(F) \cap R \rangle}$ is an abelian group so every homomorphism $\frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle} \to \mathbb{Q}/\mathbb{Z}$ can be extended to a homomorphism $\frac{R}{\langle K(F) \cap R \rangle} \to \mathbb{Q}/\mathbb{Z}$ and thus $\delta$ is an epimorphism. Observe that $\text{Im } \psi \subseteq \text{Ker } \delta$. Let $\delta(h) = 0$ for $h \in \text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z})$. Then $h$ induces a homomorphism

$$\tilde{h} : \frac{R}{R \cap \langle K(F) \cap S \rangle} \to \mathbb{Q}/\mathbb{Z}$$

given by $\tilde{h}(r(R \cap \langle K(F) \cap S \rangle)) = h(r(\langle K(F) \cap R \rangle))$. Since $\frac{R}{R \cap \langle K(F) \cap S \rangle} \cong \frac{R \cap \langle K(F) \cap S \rangle}{\langle K(F) \cap R \rangle}$, so $\tilde{h}$ extends to a homomorphism $f : \frac{S}{\langle K(F) \cap R \rangle} \to \mathbb{Q}/\mathbb{Z}$. As $\psi(f) = h$, therefore $\text{Im } \psi = \text{Ker } \delta$, which completes the proof of Claim 1.

Consider the following exact sequences

$$1 \to \frac{F}{\langle K(F) \cap S \rangle} \to \frac{F}{\langle K(F) \cap R \rangle} \to G/N \to 1,$$

$$1 \to \frac{S}{\langle K(F) \cap R \rangle} \to \frac{S}{\langle K(F) \cap S \rangle} \to G \to 1.$$

By Theorem [1.1], the transgression maps $\text{tra}$ corresponding to the above exact sequences are surjective. Therefore, by (3.0.1), we have the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
\text{Hom}(\frac{F}{\langle K(F) \cap S \rangle}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{res'}} \text{Hom}(\frac{S}{\langle K(F) \cap S \rangle}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{tra}} B_0(G/N) \to 0 \\
\text{Hom}(\frac{F}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{res'}} \text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{\text{tra}} B_0(G) \to 0 \\
\end{array}
\]

From this diagram, it follows that the map $\phi : \frac{\text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z})}{\text{Im}(\text{res'})} \to B_0(G)$ given by

$$\phi(f + \text{Im } (\text{res'}) = \text{tra}(f)$$

is an isomorphism.

Observe that, $\text{Im } (\text{res'}) \subseteq \text{Im } (\psi)$. Hence, we have the natural projection map

$$\tau : \frac{\text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z})}{\text{Im } (\text{res'})} \to \frac{\text{Hom}(\frac{R}{\langle K(F) \cap R \rangle}, \mathbb{Q}/\mathbb{Z})}{\text{Im } (\psi)}.$$
We define the map

\[ \theta : B_0(G) \to \frac{\text{Hom}(R \cap (K(F) \cap R), Q/Z)}{\text{Im}(\psi)} \]

by \( \theta = \tau \circ \phi^{-1} \). Since \( \tau \) is an epimorphism so \( \theta \) is also an epimorphism. By the commutativity of the diagram, it is easy to see that \( \text{Im}(\text{inf}) = \ker \theta \). Thus by Claim 1, our result follows.

**Proof of Theorem 1.2.**

*Proof.* Result follows from Lemma 3.1 and Theorem 3.2. \( \square \)

**Proof of Theorem 1.3.**

*Proof.*

(i) By the exact sequence (1.0.1), it follows that \( \text{inf} : B_0(G/N) \to B_0(G) \) is an epimorphism if and only if \( \theta : B_0(G) \to \frac{R \cap (K(F) \cap S)}{\langle K(F) \cap R \rangle}, Q/Z) \) is a zero map. Since \( \theta \) is surjective, so \( R \cap (K(F) \cap S) = \langle K(F) \cap R \rangle). \)

(ii) Let \( \alpha \in B_0(G) \). By Diagram 1, there is a \( \bar{f} \in \text{Hom}(\frac{R \cap (K(F) \cap S)}{\langle K(F) \cap R \rangle}, Q/Z) \) such that \( \text{tra} \begin{array}{c} \bar{f} \end{array} = \alpha \). Now by the proof of Theorem 1.1, the isomorphism \( \gamma : B_0(G) \to \frac{F' \cap R}{\langle K(F) \cap R \rangle}, Q/Z) \)

is given by \( \gamma(\alpha) = \bar{f} |_{\langle K(F) \cap R \rangle} \).

Hence, according to the proof of Theorem 3.2 we see that \( \phi(\bar{f} + \text{Im(res')}) = \alpha \) and \( \theta(\alpha) = \bar{f} + \text{Im}(\psi) \). Thus, considering the isomorphism in (3.0.2), we have

\[ \theta : B_0(G) \to \text{Hom}(\frac{R \cap (K(F) \cap S)}{\langle K(F) \cap R \rangle}, Q/Z) \]

given by \( \theta(\alpha) = \bar{f} |_{\langle K(F) \cap R \rangle} \).

Thus identifying \( B_0(G) \) with \( \text{Hom}(\frac{F' \cap R}{\langle K(F) \cap R \rangle}, Q/Z) \) via \( \gamma \), we see that

\[ \theta : \text{Hom}(\frac{F' \cap R}{\langle K(F) \cap R \rangle}, Q/Z) \to \text{Hom}(\frac{R \cap (K(F) \cap S)}{\langle K(F) \cap R \rangle}, Q/Z) \]

is a restriction homomorphism. Hence by the exact sequence (1.0.1), \( \text{inf} \) is zero map if and only if the restriction map \( \theta \) is an isomorphism, i.e., \( F' \cap R = R \cap (K(F) \cap S) \). \( \square \)

4. **Triviality of \( B_0(G) \) for central product of groups \( G \)**

A question was asked in [10] that if \( G \) is a central product of two groups \( H \) and \( N \), whether triviality of \( B_0(H) \) and \( B_0(N) \) implies triviality of \( B_0(G) \). Rai [18] Proof of Theorem 1.1] gave the negative answer to this question by considering the following example. We observe that, for this example, Theorem 1.4 also gives negative answer.
Example 4.1. Let $H$ be the freest special $p$-group of $G$, i.e., a special $p$-group, minimally generated by the elements $\alpha_1, \alpha_2, \ldots, \alpha_d$ such that the set $\{[\alpha_i, \alpha_j] \mid 1 \leq i < j \leq d\}$ is linearly independent and $|G'| = p^{\frac{d(d-1)}{2}}$. Suppose $Z$ is the normal subgroup of $H \times H$ generated by the element $\{[\alpha_1, \alpha_2][\alpha_3, \alpha_4], [\alpha_1, \alpha_2]\}$. Consider the group

$$G = \frac{H \times H}{Z}.$$ 

Let $H_1 = \langle [\alpha_1, \alpha_2][\alpha_3, \alpha_4] \rangle$ and $N_1 = \langle [\alpha_1, \alpha_2] \rangle$. Observe that $\xi : H_1 \to N_1$ is a homomorphism satisfies condition (i) of Theorem 1.4. Since $H_1 \cap H' \neq (H_1 \cap K(H))$, so by exact sequence (1.0.1), $B_0(H/H_1) \neq 0$. Hence by Theorem 1.4, $B_0(G) \neq 0$.

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4

Proof. Let $G = \frac{H \times N}{Z}$, where $Z = \{(a, \xi(a)^{-1}) \mid a \in H_1\}$. It follows from Theorem 2.1 that

$$B_0(H \times N) = B_0(H) \times B_0(N) = 0.$$ 

By (1.0.1), we have the following exact sequence.

$$0 \to \text{Hom}(\frac{Z \cap (H' \times N')}{\langle Z \cap (K(H) \times K(N))\rangle}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{tra}, B_0(G)} \inf \to B_0(H) \times B_0(N) = 0,$$

which says

$$B_0(G) \cong \frac{Z \cap (H' \times N')}{\langle Z \cap (K(H) \times K(N))\rangle}.$$ 

(i) Suppose $B_0(G) = 0$. Then $Z \cap (H' \times N') = \langle Z \cap (K(H) \times K(N))\rangle$. Let $a \in H_1 \cap H'$. Then $\xi(a) \in N_1 \cap K(N) \subseteq N_1 \cap N'$. Therefore,

$$(a, \xi(a)^{-1}) \in Z \cap (H' \times N') = \langle Z \cap (K(H) \times K(N))\rangle.$$ 

Hence $a \in \langle H_1 \cap K(H) \rangle$ and so $H_1 \cap H' = \langle H_1 \cap K(H) \rangle$. Thus $B_0(H/H_1) = 0$, follows from the exact sequence (1.0.1).

Conversely, suppose $B_0(H/H_1) = 0$. Then by exact sequence (1.0.1), we have $H_1 \cap H' = \langle H_1 \cap K(H) \rangle$. Let $(a, \xi(a)^{-1}) \in Z \cap (H' \times N')$. Since $a \in H_1 \cap H' = \langle H_1 \cap K(H) \rangle$, we have $a = \prod_{i=1}^{k} [h_i, h_i']$, $h_i, h_i' \in H$ and $[h_i, h_i'] \in H_1$. Then $\xi([h_i, h_i']) \in N_1 \cap K(N)$ and

$$(a, \xi(a)^{-1}) = \prod_{i=1}^{k} ([h_i, h_i'], \xi([h_i, h_i'])^{-1}) \in \langle Z \cap (K(H) \times K(N))\rangle.$$ 

Hence,

$$Z \cap (H' \times N') = \langle Z \cap (K(H) \times K(N))\rangle.$$ 

Thus our result follows.
Suppose $B_0(G) = 0$. Then $Z \cap (H' \times N') = \langle Z \cap (K(H) \times K(N)) \rangle$. Let $a \in H_1 \cap H'$. Then $a = \prod_{i=1}^k [h_i, h'_i]$, $h_i, h'_i \in H$ and
\[
\xi(a) = \prod_{i=1}^k [h_i, h'_i] = \prod_{i=1}^k \eta([h_i, h'_i]) = \prod_{i=1}^k \eta(h_i), \eta(h'_i) ] \in N_1 \cap N'.
\]
Therefore, 
\[
(a, \xi(a)^{-1}) \in Z \cap (H' \times N') = \langle Z \cap (K(H) \times K(N)) \rangle.
\]
Hence $a \in \langle H_1 \cap K(H) \rangle$ and so $H_1 \cap H' = \langle H_1 \cap K(H) \rangle$. Thus $B_0(H/H_1) = 0$, follows by the exact sequence (1.0.1).

Conversely, if $B_0(H/H_1) = 0$, then by (1.0.1), we have $H_1 \cap H' = \langle H_1 \cap K(H) \rangle$. So, for $(a, \xi(a)^{-1}) \in Z \cap (H' \times N')$, we have
\[
a = \prod_{i=1}^s [h_i, h'_i],
\]
for $h_i, h'_i \in H$ and $[h_i, h'_i] \in H_1$. Thus $\xi([h_i, h'_i]) = \eta(h_i), \eta(h'_i)] \in N_1 \cap K(N)$ and
\[
\xi(a) = \prod_{i=1}^s [h_i, h'_i] = \prod_{i=1}^s [\eta(h_i), \eta(h'_i)] \in \langle N_1 \cap K(N) \rangle.
\]
Therefore, $(a, \xi(a)^{-1}) \in \langle Z \cap (K(H) \times K(N)) \rangle$ and we are done.

\[\square\]

**Theorem 4.2.** Suppose $H, N$ are two groups such that $\xi : H \to N$ is a homomorphism and $\xi|_{H_1} : H_1 \to N_1$ is an isomorphism. Let $G$ be a central product of $H$ and $N$ via the map $\xi|_{H_1}$. Suppose $B_0(H) = B_0(N) = 0$. Then the following holds:

(i) If $H' = K(H)$, then $B_0(G) = 0$.

(ii) Suppose $\xi : H \to N$ is an isomorphism. If $N' = K(N)$ or $B_0(N/N_1) = 0$, then $B_0(G) = 0$.

**Proof.** By Theorem 4.2 we have the following exact sequence.
\[
0 \to \text{Hom}(\frac{Z \cap (H' \times N')}{Z \cap (K(H) \times K(N))}, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{tr}} B_0(G) \xrightarrow{\text{inf}} B_0(H) \times B_0(N) = 0,
\]
which says
\[
B_0(G) \cong \frac{Z \cap (H' \times N')}{Z \cap (K(H) \times K(N))}.
\]

(i) Let $(a, \xi(a)^{-1}) \in H' \times N'$ for $a \in H_1$. If $H' = K(H)$, then $a = [h_1, h_2]$, for some $h_1, h_2 \in H$. So $\xi(a) = [\xi(h_1), \xi(h_2)] \in K(N)$ implies $B_0(G) = 0$.

(ii) Let $(a, \xi(a)^{-1}) \in H' \times N'$ for $a \in H_1$. If $N' = K(N)$, then $\xi(a) = [n_1, n_2]$, for $n_1, n_2 \in N$. So, $a = \xi^{-1}([n_1, n_2]) = [\xi^{-1}(n_1), \xi^{-1}(n_2)] \in H_1 \cap K(H)$. This completes the proof.
If \( B_0(N/N_1) = 0 \), then by \((1.0.1)\), we have \( N_1 \cap N' = \langle N_1 \cap K(N) \rangle \), then
\[
\xi(a) = \prod_{i=1}^{t} [n_i, n'_i],
\]
for \( n_i, n'_i \in N \) and \([n_i, n'_i] \in N_1\), so
\[
a = \xi^{-1}(\prod_{i=1}^{t} [n_i, n'_i]) = \prod_{i=1}^{s} [\xi^{-1}(n_i), \xi^{-1}(n'_i)] \in \langle H_1 \cap K(H) \rangle.
\]
Therefore, \( (a, \xi(a)^{-1}) \in \langle Z \cap (K(H) \times K(N)) \rangle \) and we are done. \( \square \)

Now we consider generalized discrete Heisenberg groups in the following example.

**Example 4.3.** Consider generalized discrete Heisenberg groups, which are defined as follows: Let \( r \in \mathbb{N} \). For positive integers \( d_1, d_2, \ldots, d_n \) such that \( d_1|d_2| \cdots |d_n|r \), we define the group \( H_{2n+1}^{d_1|d_2| \cdots |d_n|r}(\mathbb{Z}/r\mathbb{Z}) \) by the set \((\mathbb{Z}/r\mathbb{Z})^{n+1} \times (\mathbb{Z}/r\mathbb{Z})^n \) with multiplication given by,
\[
(a, b_1, \ldots, b_n, c_1, \ldots, c_n)(a', b'_1, \ldots, b'_n, c'_1, \ldots, c'_n) = (a + a', (\sum_{i=1}^{n} d_i b'_i c_i), b_1 + b'_1, \ldots, b_n + b'_n, c_1 + c'_1, \ldots, c_n + c'_n).
\]

**Theorem 4.4.** \( B_0(H_{2n+1}^{d_1|d_2| \cdots |d_n|r}(\mathbb{Z}/r\mathbb{Z})) = 0. \)

**Proof.** Let \( G = H_{2n+1}^{d_1|d_2| \cdots |d_n|r}(\mathbb{Z}/r\mathbb{Z}) \). It is easy to observe that \( G \) is of nilpotency class 2 with the following presentation
\[
G = \langle x_i, y_i, z, 1 \leq i \leq n | [x_i, y_i] = z^{d_i}, x^r = y^r = z^r = 1 \rangle
\]
First we show that \( B_0(H_3^{d_1|r}(\mathbb{Z}/r\mathbb{Z})) = 0 \). By Lemma 2.4 it follows that, \([G, G^\phi]\) is generated by \([x, y^\phi], [z, x^\phi], [z, y^\phi]\). By Lemma 2.2 we have \([G, G^\phi]\) is abelian and so any element of \( w \in [G, G^\phi]\) can be written as
\[
w = [x_1, y_1^\phi]^{m_1}( \text{ mod } M_0^s(G)).
\]
Since \([x_1, y_1]\) has order \( r/d_1 \) and \( w \in M^s(G) \), \( r/d_1 \) divides \( m_1 \). Hence \( M^s(G) = \langle [x_1, y_1^\phi]^{r/d_1} \rangle M_0^s(G) \).
By Lemma 2.3, it follows that
\[
[x_1^{r/d_1}, y_1^\phi] = [x_1, y_1^\phi]^{r/d_1} = 1 \text{ ( mod } M_0^s(G)).
\]
Hence \( B_0(H_3^{d_1|r}(\mathbb{Z}/r\mathbb{Z})) = 0. \)

For \( n > 1 \), \( G \) is a central product of normal subgroups \( H = H_{2n-1}^{d_1|r}(\mathbb{Z}/r\mathbb{Z}) \) and \( N = H_{2n-1}^{d_2|d_3| \cdots |d_n|r}(\mathbb{Z}/r\mathbb{Z}) \), having isomorphic subgroups \( H_1 = \langle z \rangle \) and \( N_1 = \langle z \rangle \) of \( H \) and \( N \) respectively. Now we use induction on \( n \). Here \( H' = \langle z^{d_1} \rangle \) and \( K' = \langle z^{d_2} \rangle \). For \( s \in \mathbb{N} \), we have
\[
z^{s d_1} = [x_1^s, y_1], \ nz^{s d_2} = [x_2^s, y_2].
\]
Therefore \( H' = K(H), N' = K(N) \). Since \( B_0(H) = B_0(N) = 0 \), by \((1.0.1)\), we have
\[
B_0(G) \cong \frac{Z \cap (H' \times N')}{Z \cap (K(H) \times K(N))} = 0.
\]
\( \square \)
5. Triviality of $B_0(G)$ for groups $G$ of order $p^6$

The polycyclic presentation of groups of order up to $p^6$, $p$ prime, has been given in [6]. Here we recall some notations from [6]. By $\nu$ we denote the smallest positive integer which is a non-quadratic residue (mod $p$), and by $g$ we denote the smallest positive integer which is a primitive root (mod $p$). Relations of the form $[\alpha, \beta] = 1$ for generators $\alpha$ and $\beta$ are omitted in the presentations of the groups. For an element $\alpha_{i+1}$ of a finite $p$-group $G$, by $\alpha_{i+1}^{(p)}$, we mean $\alpha_{i+1}^{p} \alpha_{i+2} \cdots \alpha_{i+k} \cdots \alpha_{i+p}$, where $\alpha_{i+2}, \ldots, \alpha_{i+p}$ are suitably defined elements of $G$.

Now we are ready to prove Theorem 1.5. We use Lemma 2.2, Lemma 2.3 and Lemma 2.4 heavily for the computation in the following proof, without further reference.

Proof of Theorem 1.5

Proof. It is enough to show that $B_0(G) = 0$ for $G \in \{ \Phi_{15}, \Phi_{28}, \Phi_{29} \}$, follows from [4, Theorem 1.1]. Moravec [16] proved that $B_0(G)$ is invariant under isoclinism property of $G$, so it is enough to choose one group $G$ from each isoclinism class.

For $x, y \in G$, we have

$$[xy, z] = [x, z][x, y][y, z], \quad [x, yz] = [x, z][x, y][x, y, z],$$

$$[x^{-1}, y] = [x, y, x^{-1}]^{-1}[x, y]^{-1}. $$

We use these equalities in the following computations.

**Group $G$ in $\Phi_{15}$:** We consider

$$G = \Phi_{15}(1^6) = \langle \alpha_i, \beta_1, \beta_2 \mid 1 \leq i \leq 4 \mid [\alpha_1, \alpha_2] = [\alpha_3, \alpha_4] = \beta_1, [\alpha_1, \alpha_3] = \beta_2, [\alpha_2, \alpha_4] = \beta_3^p, \alpha_2^p = \beta_4^p = 1 \rangle.$$  

Since $G'$ is abelian, so $[G, G^0]$ is of nilpotency class at most 2. Hence, any element $w \in M^*(G)$ can be written as

$$w = [\alpha_1, \alpha_2^{p^m}] [\alpha_3, \alpha_4^{p^m}] [\alpha_1, \alpha_3^{p^m}] [\alpha_2, \alpha_4^{p^m}] \pmod{M^*_0(G)}.$$  

Now $\beta_1^{m_1 + m_2} \beta_2 = 1$ imply that $(m_1 + m_2)$ and $(m_3 + gm_4)$ is divisible by $p$. Observe that, $(\pmod{M^*_0(G)})$, we have the following.

$$[\alpha_3, \alpha_4^{p^m}] = [\alpha_3^p, \alpha_4^p] = 1, \quad [\alpha_1, \alpha_3^{p^m}] = [\alpha_1^p, \alpha_3^p] = 1.$$  

Thus, $m_2 = -m_1$ and $m_3 = -gm_4$. So

$$w = ([\alpha_1, \alpha_2^p][\alpha_3, \alpha_4^p]^{-1})^{m_1} ([\alpha_1, \alpha_3^p]^{-g} [\alpha_2, \alpha_4^p])^{m_4} \pmod{M^*_0(G)}.$$
Now our aim is to show that, $[\alpha_1, \alpha_2^3] [\alpha_3, \alpha_4^3]^{-1}$ and $[\alpha_1, \alpha_3^3]^{-g} [\alpha_2, \alpha_4^3]$ are in $M_6^0(G)$. Since $[G', G^g] \in M_6^0(G)$, we have the following identities (mod $M_6^0(G)$).

$$[\alpha_3^{-1}, \alpha_4^g] = [\alpha_3, \alpha_4^g]^{-1},$$
$$[\alpha_1^g, \alpha_3^g] = [\alpha_1, \alpha_3^g]^{-g},$$
$$[\alpha_1 \alpha_2^{-1}, (\alpha_2 \alpha_4)^g] = [\alpha_1, (\alpha_2 \alpha_4)^g] [\alpha_3^{-1}, (\alpha_2 \alpha_4)^g] = [\alpha_1, \alpha_2^g] [\alpha_3, \alpha_4^g]^{-1},$$
$$[\alpha_1^g \alpha_2, (\alpha_3 \alpha_4)^g] = [\alpha_1^g, (\alpha_3 \alpha_4)^g] [\alpha_2, (\alpha_3 \alpha_4)^g] = [\alpha_1, \alpha_2]^g [\alpha_2, \alpha_4^g],$$
$$[\alpha_1 \alpha_3^{-1}, \alpha_2 \alpha_4] = [\alpha_1, \alpha_2 \alpha_4] [\alpha_3^{-1}, \alpha_2 \alpha_4] = [\alpha_1, \alpha_2] [\alpha_3, \alpha_4]^{-1} = 1,$$
$$[\alpha_1^g \alpha_2, \alpha_3 \alpha_4] = [\alpha_1^g, \alpha_3 \alpha_4] [\alpha_2, \alpha_3 \alpha_4] = [\alpha_1, \alpha_3]^{-g} [\alpha_2, \alpha_4] = 1.$$

From the above equalities, we have

$$[\alpha_1 \alpha_3^{-1}, (\alpha_2 \alpha_4)^g], [\alpha_1^g \alpha_2, (\alpha_3 \alpha_4)^g] \in M_6^0(G).$$

Therefore $[\alpha_1, \alpha_2^3] [\alpha_3, \alpha_4^3]^{-1}$ and $[\alpha_1, \alpha_3^3]^{-g} [\alpha_2, \alpha_4^3]$ are in $M_6^0(G)$ which imply that $B_0(G) = 0$.

**Groups $G$ in $\Phi_{28}, \Phi_{29}$:**

Let $p > 3$ and We consider the following groups.

$G = \Phi_{28}(222) = \langle \alpha, \alpha_i \mid 1 \leq i \leq 4 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = [\alpha_1, \alpha_2] = \alpha_4, $
$$\alpha_3 = \alpha_1^p \alpha_2^{\frac{f(p-1)}{2}}, \alpha_4 = \alpha_2^p, \alpha_2^p = \alpha_3^p = \alpha_4^p = 1 \rangle,$$

or,

$G = \Phi_{29}(222) = \langle \alpha, \alpha_i \mid 1 \leq i \leq 4 \mid [\alpha_1, \alpha] = \alpha_2, [\alpha_2, \alpha] = \alpha_3, [\alpha_3, \alpha] = [\alpha_1, \alpha_2] = \alpha_4, $
$$\alpha_3 = \alpha_1^p \alpha_2^{\frac{f(p-1)}{2}}, \alpha_4 = \alpha_2^p, \alpha_2^p = \alpha_3^p = \alpha_4^p = 1 \rangle.$$

Since $G$ is of nilpotency class 4, $\tau(G)$ has nilpotency class at most 5. Hence, for both the groups, we have the following identities (mod $M_5^0(G)$).

(5.0.1) $[\alpha_3^p, \alpha^g] = [\alpha_3, \alpha^g]^p = 1,$
$$[\alpha_2^p, \alpha^g] = [\alpha_2, \alpha^g]^p = 1,$$
$$[\alpha_2^p, \alpha^g] = [\alpha_2, \alpha^g]^p [\alpha_2, \alpha^g] = [\alpha_2, \alpha^g]^p = 1,$$
$$[\alpha_3^{-1}, \alpha^g] = [\alpha_3, \alpha, (\alpha_3^{-1})^g]^{-1} [\alpha_3, \alpha^g]^{-1} = [\alpha_3, \alpha^g]^{-1},$$
$$[\alpha_2^{-1}, \alpha^g] = [\alpha_2, \alpha, (\alpha_2^{-1})^g]^{-1} [\alpha_2, \alpha^g]^{-1} = [\alpha_2, \alpha^g]^{-1},$$
$$[\alpha^p, \alpha_2^g] = [\alpha, \alpha_2^g] [\alpha_3^{-1}, \alpha^g] [\alpha_3^{-1}, \alpha, \alpha] = [\alpha, \alpha_2^g]^p = [\alpha_2, \alpha^g]^{-p} = 1,$$
$$[\alpha_1^2, \alpha^g] = [\alpha_1, \alpha_2^g] [\alpha_2, \alpha^g] = [\alpha_1, \alpha^g]^2 = 1,$$
$$[\alpha^p, \alpha_1^g] = [\alpha, \alpha_1^g] [\alpha_2^{-1}, \alpha^g] [\alpha_2^{-1}, \alpha, \alpha^g] = [\alpha, \alpha_1^g] [\alpha_2, \alpha^g]^2 [\alpha_3, \alpha^g]^{-2} = [\alpha_1, \alpha^g]^p,$$
$$[\alpha_1^p, \alpha^g] = [\alpha_1, \alpha^g] [\alpha_2, \alpha^g] [\alpha_2, \alpha_1^g] = [\alpha_1, \alpha^g]^p [\alpha_2^g, \alpha_1^g] [\alpha_2^g, \alpha_1] = [\alpha_1, \alpha^g]^p,$$
$$[\alpha_1, \alpha^g]^{-p} = [\alpha^g, \alpha_1]^p = [\alpha, (\alpha_1^p)^g] = [\alpha^p, \alpha_1^g].$$
Since $G'$ is abelian, $[G,G^\phi]$ has nilpotency class at most 2, By Lemma [4], Lemma 2.2, any element $w \in M^*(G)$ can be written as

$$w = [\alpha_2, \alpha^\phi]^m_1[\alpha_1, \alpha^\phi]^m_2[\alpha_3, \alpha^\phi]^m_3[\alpha_1, \alpha^\phi]^m_4 \pmod{M^*_0(G)},$$

for some integers $m_1, m_2, m_3, m_4$. It is easy to see that, $[\alpha_2^p, \alpha_1^p] = 1$, so we have $\alpha_1^m \alpha_2^{m_2} \equiv \alpha_1 \alpha_2^{m_2} \pmod{M^*_0(G)}$, for any positive integer $m$.

**Now consider** $G = \Phi_{28}(222)$, $w \in M^*(G)$ imply that

$$\alpha^m_3 \alpha^m_2 \alpha^3_4 \equiv \alpha^m_1 \alpha^m_2^{p^{m_1 + m_2 + p(m_3 + m_4)}} = 1.$$ 

Therefore, $p$ divides $m_1$ and $p^2$ divides $m_2 + p(m_3 + m_4)$. Since $[\alpha_1, \alpha^\phi]^p \equiv [\alpha_2, \alpha^\phi]^p \in M^*_0(G)$, by (5.0.2) we have ( mod $M^*_0(G)$ ),

$$w = \langle \alpha_1, \alpha^\phi \rangle^{-p(m_3 + m_4)}[\alpha_3, \alpha^\phi]^m_3[\alpha_1, \alpha^\phi]^m_4$$

$$= ([\alpha_3, \alpha^\phi][\alpha_1, \alpha^\phi]^{-p})^m_3([\alpha_1, \alpha^\phi][\alpha_1, \alpha^\phi]^{-p})^m_4$$

To prove $B_0(G) = 0$, it is enough to show that, $[\alpha_3, \alpha^\phi][\alpha, (\alpha_1^p)^\phi]$ and $[\alpha_1, \alpha_2^p][\alpha^p, \alpha_1^\phi]$ are in $M^*_0(G)$. Observe that

$$[\alpha_1^p, \alpha] = [\alpha_1, \alpha]^p_1[\alpha_2, \alpha_1]^{(\phi)} = [\alpha_1, \alpha]^p_1,$$

$$[\alpha^3_3, \alpha^\phi_3] = [\alpha_3, \alpha^\phi_3][\alpha_3, \alpha^\phi_3][\alpha, \alpha^\phi_3] = [\alpha_3, \alpha][\alpha, \alpha^\phi_3] = [\alpha_3, \alpha][\alpha_1, \alpha_2]^p_1 = 1,$$

$$[\alpha^p_3 \alpha_1, \alpha_1 \alpha_2] = [\alpha_1, \alpha_1 \alpha_2][\alpha^p_3 \alpha_1, \alpha_1 \alpha_2], [\alpha, \alpha_1 \alpha_2] = [\alpha_1, \alpha_1 \alpha_2]^p_3[\alpha_1, \alpha_1 \alpha_2] = [\alpha_1, \alpha_1 \alpha_2]^p_3[\alpha_1, \alpha_1 \alpha_2] = 1.$$

Now using the relations (5.0.1), ( mod $M^*_0(G)$ ), we have

$$[\alpha^3_3, (\alpha \alpha^\phi_3)] = [\alpha_3, (\alpha \alpha^\phi_3)][\alpha_3, \alpha^\phi_3, \alpha^\phi_3][\alpha, (\alpha \alpha^\phi_3)] = [\alpha_3, \alpha^\phi_3][\alpha, (\alpha^p_3)^\phi],$$

$$[\alpha_1 \alpha^p_3, (\alpha_1 \alpha_2)^\phi] = [\alpha_1, (\alpha_1 \alpha_2)^\phi][\alpha_1, (\alpha_1 \alpha_2)^\phi], [\alpha^p_3, (\alpha_1 \alpha_2)^\phi]$$

$$= [\alpha_1, \alpha^\phi_1][\alpha_1, \alpha_2, (\alpha^p_3)^\phi][\alpha^p_3, \alpha^\phi_1] = [\alpha_1, \alpha^\phi_1][\alpha^p_3, \alpha^\phi_1].$$

Therefore,

$$[\alpha^3_3, (\alpha \alpha^\phi_3)^\phi], [\alpha^p_3 \alpha_1, (\alpha_1 \alpha_2)^\phi] \in M^*_0(G)$$

implies that

$$[\alpha_3, \alpha^\phi_3][\alpha, (\alpha^p_1)^\phi], [\alpha_1, \alpha^\phi_1][\alpha^p_1, \alpha^\phi_1] \in M^*_0(G)$$

and so by (5.0.3), $w \in M^*_0(G)$. Thus $B_0(\Phi_{28}(222)) = 0$.

**Next consider** $G = \Phi_{29}(222)$. There exists a positive integer $s$ such that $s < p$ and $\nu s = 1$ (mod $p$). Now $w \in M^*(G)$ imply that

$$\alpha^m_3 \alpha^m_2 \alpha^3_4 ^{m_3 + m_4} = \alpha^m_3 \alpha^m_2 \alpha^s_4 \alpha^{s(m_3 + m_4)} = \alpha^m_3 \alpha^m_2 \alpha^{m_1 + m_2 + s(m_3 + m_4)} = 1.$$
Therefore, \( p \) divides \( m_1 \) and \( p^2 \) divides \( m_2 + ps(m_3 + m_4) \). Since \([\alpha, \alpha^p] = \alpha^p \), \([\alpha^p, \alpha^p] = \alpha^p \),

\[
\begin{align*}
[\alpha^p, \alpha] &= [\alpha_1, \alpha^p][\alpha_2, \alpha_1] = [\alpha_1, \alpha^p], \\
[\alpha^p, (\alpha_1^p)\phi] &= [\alpha, (\alpha_1^p)\phi][\alpha, \alpha^p] = [\alpha, (\alpha_1^p)\phi], \\
[\alpha^p, (\alpha_1^p)\phi] &= [\alpha, (\alpha_1^p)\phi]^s[\alpha, \alpha^p, \alpha^s] = [\alpha, (\alpha_1^p)\phi]^s.
\end{align*}
\]

Therefore, by \((5.0.2)\), we have \( w \equiv 0 \mod M^*_0(G) \).

\[
(5.0.4) \quad w = [\alpha^s, \alpha^p]^{-ps(m_3 + m_4)}[\alpha^s, \alpha^p][\alpha_1^p, \alpha]^{m_3} = [\alpha, \alpha^p]^{-ps(m_3 + m_4)}.
\]

We have

\[
\begin{align*}
[\alpha^s, \alpha^p] &= [\alpha, \alpha^p][\alpha_2^{-1}, \alpha^p] = [\alpha, \alpha^p], \\
[\alpha_1^p, \alpha^p] &= [\alpha_1^p, \alpha^p][\alpha_2, \alpha_1^p] = [\alpha_1^p, \alpha^p], \\
[\alpha_1^s, \alpha^s] &= [\alpha_1^s, \alpha^s][\alpha_2, \alpha_1^s] = [\alpha_1^s, \alpha^s], \\
[\alpha_3^s, \alpha_1^s, \alpha_2^s] &= [\alpha_3, \alpha_1^s][\alpha_2, \alpha_1^s, \alpha_2^s] = [\alpha_3, \alpha_1^s][\alpha_2, \alpha_1^s, \alpha_2^s], \\
[\alpha_4^s, \alpha_1^s, \alpha_2^s] &= [\alpha_4, \alpha_1^s][\alpha_2, \alpha_1^s, \alpha_2^s] = [\alpha_4, \alpha_1^s][\alpha_2, \alpha_1^s, \alpha_2^s].
\end{align*}
\]

Now \( w \equiv 0 \mod M^*_0(G) \), we have

\[
\begin{align*}
[\alpha_3^s, (\alpha_1^p)\phi] &= [\alpha_3, (\alpha_1^p)\phi][\alpha_3, \alpha_1^p, (\alpha_1^p)\phi][\alpha_3, \alpha_1^p, (\alpha_1^p)\phi] = [\alpha_3, \alpha_1^p, (\alpha_1^p)\phi], \\
[\alpha_1^p, (\alpha_1^s\alpha_2^s)] &= [\alpha_1^p, (\alpha_1^s\alpha_2^s)][\alpha_1, \alpha_1^s\alpha_2^s, \alpha_1^p] = [\alpha_1^p, (\alpha_1^s\alpha_2^s)], \\
[\alpha_3^s, (\alpha_1^s\alpha_2^s)] &= [\alpha_3, (\alpha_1^s\alpha_2^s)][\alpha_3, \alpha_1^s\alpha_2^s, \alpha_1^p] = [\alpha_3, \alpha_1^s\alpha_2^s], \\
[\alpha_4^s, (\alpha_1^s\alpha_2^s)] &= [\alpha_4, (\alpha_1^s\alpha_2^s)][\alpha_2, \alpha_1^s\alpha_2^s, \alpha_1^p] = [\alpha_4, (\alpha_1^s\alpha_2^s)].
\end{align*}
\]

Therefore, \([\alpha_3^s, (\alpha_1^p)\phi], [\alpha_1^p, (\alpha_1^s\alpha_2^s)] \in M^*_0(G) \).

Hence by \((5.0.4)\), \( w \in M^*_0(G) \) and \( B_0(\Phi(29)(222)) = 0 \).

\[ \square \]

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ON UNRAMIFIED BRAUER GROUPS OF FINITE GROUPS

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Email address: sumana.iitg@gmail.com, sumanahatui@iisc.ac.in