Classical metric Diophantine approximation revisited: the Khintchine-Groshev theorem

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Abstract

Let $A_{n,m}(\psi)$ denote the set of $\psi$-approximable points in $\mathbb{R}^{mn}$. Under the assumption that the approximating function $\psi$ is monotonic, the classical Khintchine-Groshev theorem provides an elegant probabilistic criterion for the Lebesgue measure of $A_{n,m}(\psi)$. The famous Duffin-Schaeffer counterexample shows that the monotonicity assumption on $\psi$ is absolutely necessary when $m = n = 1$. On the other hand, it is known that monotonicity is not necessary when $n \geq 3$ (Schmidt) or when $n = 1$ and $m \geq 2$ (Gallagher). Surprisingly, when $n = 2$ the situation is unresolved. We deal with this remaining case and thereby remove all unnecessary conditions from the classical Khintchine-Groshev theorem. This settles a multi-dimensional analogue of Catlin’s Conjecture.

1 Introduction

Throughout, $n \geq 1$ and $m \geq 1$ are integers and $I_{nm}$ is the unit cube $[0, 1]^{nm}$ in $\mathbb{R}^{nm}$. Given a function $\psi : \mathbb{N} \rightarrow \mathbb{R}^+$, let $A_{n,m}(\psi)$ denote the set of $X \in I_{nm}$ such that

$$|qX + p| < \psi(|q|)$$

holds for infinitely many $(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\}$. Here $|\cdot|$ denotes the supremum norm, $X = (x_{ij})$ is regarded as an $n \times m$ matrix and $q$ is regarded as a row. Thus, $qX$ represents a point in $\mathbb{R}^m$ given by the system

$$q_1x_{1j} + \cdots + q_nx_{nj} \quad (1 \leq j \leq m)$$

of $m$ real linear forms in $n$ variables. For obvious reasons the function $\psi$ is referred to as an approximating function and points in $A_{n,m}(\psi)$ are said to be $\psi$-approximable.

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In the case that the approximating function is monotonic, the classical Khintchine-
Groshev theorem provides a beautiful and strikingly simple criterion for the ‘size’ of
\( A_{n,m}(\psi) \) expressed in terms of \( nm \)-dimensional Lebesgue measure. The following is an
improved modern version of this fundamental result – see [2] and references within. Given
a set \( X \subset \mathbb{I}^{nm} \), let \( |X| \) denote the \( nm \)-dimensional Lebesgue measure of \( X \).

**Theorem (Khintchine-Groshev)** Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \). Then

\[
|A_{n,m}(\psi)| = \begin{cases} 
0 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m < \infty, \\
1 & \text{if } \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty \text{ and } \psi \text{ is monotonic.}
\end{cases}
\]

The convergence part is reasonably straightforward to establish and is free from any as-
sumption on \( \psi \). The divergence part constitutes the main substance of the Khintchine-
Groshev theorem and involves the monotonicity assumption on the approximating function.
It is worth mentioning that in the original statement of the theorem [9, 10, 11] the stronger
hypothesis that \( q^n \psi^m(q) \) is monotonic was assumed. The goal of this article is to investigate
the role of the monotonicity hypothesis in the Khintchine-Groshev theorem.

In the one-dimensional case \( (m = n = 1) \), it is well know that the monotonicity hypo-
thesis in the Khintchine-Groshev theorem is absolutely crucial. Indeed, Duffin & Schaeffer
[7] constructed a non-monotonic function \( \psi \) for which \( \sum q \psi(q) \) diverges but \( A_{1,1}(\psi) \) is of
measure zero. In other words the Khintchine-Groshev theorem is false without the mono-
tonicity hypothesis and the conjectures of Catlin [6] and Duffin & Schaeffer [7] provide
appropriate alternative statements – see below. The Catlin and Duffin-Schaeffer conjec-
tures represent two key unsolved problems in metric number theory.

Beyond the one-dimensional case the situation is very different and surprisingly incom-
plete. If \( n = 1 \) and \( m \geq 2 \), a theorem of Gallagher [8] implies that the monotonicity
assumption in the Khintchine-Groshev theorem can be safely removed. Furthermore, the
monotonicity assumption can also be removed if \( n \geq 3 \), this time as a consequence of a
result of Schmidt [13, Theorem 2] or alternatively a more general result of Sprindžuk [14,
§ I.5, Theorem 15] – also see [11, §5]. It is worth mentioning that the results of Schmidt
and Sprindžuk are quantitative and we shall discuss this ‘stronger’ aspect of the theory at
the end of the paper in [8]. Despite the generality, the theorems of Schmidt and Sprindžuk
leave the case \( n = 2 \) unresolved and to the best of our knowledge the case is not covered
by any other known result. In this paper we show that the monotonicity assumption is
unnecessary when \( n = 2 \) and thereby establish the following clear-cut statement that is
best possible.

**Theorem 1** Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) and \( nm > 1 \). Then

\[
|A_{n,m}(\psi)| = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty.
\]
As already mentioned, Theorem 1 is false when \( mn = 1 \) and the Catlin conjecture provides the appropriate statement:

\[
|A_{1,1}(\psi)| = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \varphi(q) \max_{t \geq 1} \frac{\psi(qt)}{qt} = \infty.
\]

Here, and throughout, \( \varphi \) is the Euler function. For further details concerning the above mentioned classical results and the generalisations of the Catlin and Duffin-Schaeffer conjectures to linear forms see [1]. Indeed, Theorem 1 is formally stated as Conjecture A in [1] and is shown to be equivalent to the linear forms Catlin conjecture.

We shall prove Theorem 1 by establishing the analogous statement for an important subset of \( A_{n,m}(\psi) \). Given two integer points \( p = (p_1, \ldots, p_m) \in \mathbb{Z}^m \) and \( q = (q_1, \ldots, q_n) \in \mathbb{Z}^n \), let \( \gcd(p, q) \) denote the greatest common divisor of \( p_1, \ldots, p_m, q_1, \ldots, q_n \). We say that \( p \) and \( q \) are coprime if \( \gcd(p, q) = 1 \). Consider the set

\[
A'_{n,m}(\psi) := \{ X \in \mathbb{N}^n : |qX + p| < \psi(|q|) \quad \text{for infinitely many} \quad (p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\} \quad \text{with} \quad \gcd(p, q) = 1 \}.
\]

In view of the coprimeness condition, we clearly have that \( A'_{n,m}(\psi) \subset A_{n,m}(\psi) \) and so Theorem 1 is a consequence of the following theorem.

**Theorem 2** Let \( \psi : \mathbb{N} \to \mathbb{R}^+ \) and \( nm > 1 \). Then

\[
|A'_{n,m}(\psi)| = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} q^{n-1}\psi(q)^m = \infty.
\]

As with Theorem 1 for \( n = 1 \) the statement of Theorem 2 is due to Gallagher. For \( n \geq 3 \) it can be derived from Schmidt’s [13, Theorem 2] or Sprindžuk’s [14, §I.5, Theorem 15]. Furthermore, when \( mn = 1 \) the Duffin-Schaeffer conjecture provides the appropriate statement:

\[
|A'_{1,1}(\psi)| = 1 \quad \text{if} \quad \sum_{q=1}^{\infty} \varphi(q) \frac{\psi(q)}{q} = \infty.
\]

The proof of Theorem 2 presented in this paper is self-contained. In other words, there is little advantage in restricting the proof to the ‘unknown’ \( n = 2 \) case. The key to establishing the theorem is showing that the sets associated with the natural lim sup decomposition of \( A'_{n,m}(\psi) \) are quasi-independent on average — see Theorem 3 below. To the best of our knowledge, such an independence result is unavoidable when proving positive measure results for lim sup sets. More to the point, the analogue of Theorem 3 associated with the set \( A_{n,m}(\psi) \) is probably not in general true and it is absolutely necessary to work with the ‘thinner’ set \( A'_{n,m}(\psi) \). In particular, this would explain why Theorem 1 is not in general
covered by the result of Schmidt. Given the nature of his goal, Schmidt was essentially forced to work directly with $A_{n,m}(\psi)$.

Beyond the above statements, in §4 we discuss the generalizations of Theorems 1 and 2 within the framework of multivariable approximating functions $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$. In the final section §5, we discuss the quantitative theory and show that Theorem 1 cannot be deduced from Schmidt’s quantitative theorem.

2 Preliminaries

In this section we reduce the proof of Theorem 2 to establishing a quasi-independence on average statement – Theorem 3 below. We also state various known results that we appeal to during the course of proving Theorem 3.

We start with an almost trivial but nevertheless useful observation. In Theorem 2, there is no loss of generality in assuming that

$$\psi(h) < c$$

for all $h \in \mathbb{N}$ and $c > 0$.

Suppose for the moment that this was not the case and define

$$\Psi : h \to \Psi(h) := \min \{c, \psi(h)\} .$$

It is easily verified that if $\sum h^{n-1}\psi(h)^m$ diverges then $\sum h^{n-1}\Psi(h)^m$ diverges. Furthermore, $A'_{n,m}(\psi) \subset A'_{n,m}(\Psi)$ and so it suffices to establish Theorem 2 for $\Psi$.

The next statement is far from being trivial. It is a consequence of the main result in [5] and reduces the proof of Theorem 2 to showing that $A'_{n,m}(\psi)$ is of positive measure.

**Lemma 1** For any $n, m \geq 1$ and $\psi : \mathbb{N} \to \mathbb{R}^+$,

$$|A'_{n,m}(\psi)| > 0 \implies |A'_{n,m}(\psi)| = 1.$$

In order to prove positive measure, we make use of the following natural representation of $A'_{n,m}(\psi)$ as a lim sup set. Given $\delta > 0$ and $q \in \mathbb{Z}^n \setminus \{0\}$, let

$$B(q, \delta) := \{X \in \mathbb{P}^m : |qX + p| < \delta \text{ for some } p \in \mathbb{Z}^m\}$$

and

$$B'(q, \delta) := \{X \in \mathbb{P}^m : |qX + p| < \delta \text{ for some } p \in \mathbb{Z}^m \text{ with } \gcd(p, q) = 1\} .$$

Then, it is easily seen that

$$A'_{n,m}(\psi) = \limsup_{|q| \to \infty} B'(q, \psi(|q|)).$$
The following lemma provides a mechanism for establishing lower bounds for the measure of \( \limsup \) sets. The statement is a generalisation of the divergent part of the standard Borel-Cantelli lemma in probability theory, see [14, Lemma 5]. It is conveniently adapted for the setup above.

**Lemma 2** Let \( E_k \subset \mathbb{I}^{nm} \) be a sequence of measurable sets such that \( \sum_{k=1}^{\infty} |E_k| = \infty \). Then

\[
|\limsup_{k \to \infty} E_k| \geq \limsup_{N \to \infty} \frac{\left( \sum_{s=1}^{N} |E_s| \right)^2}{\sum_{s,t=1}^{N} |E_s \cap E_t|}.
\]  

(3)

In view of Lemma 2, the desired statement \( |A'_{n,m}(\psi)| > 0 \) will follow on showing that the sets \( B'_q(\psi) := B'(q, \psi(|q|)) \) are quasi-independent on average and that the sum of their measures diverges. Formally, we shall prove the following statement.

**Theorem 3 (quasi-independence on average)** Let \( nm > 1 \) and \( \psi : \mathbb{N} \to \mathbb{R}^+ \) satisfy \( \psi(h) < 1/2 \) for all \( h \in \mathbb{N} \) and \( \sum_{h=1}^{\infty} h^{n-1} \psi(h)^m = \infty \). Then

\[
\sum_{|q_1| \leq N, |q_2| \leq N} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)| \leq C \left( \sum_{|q_1| \leq N} |B'_{q_1}(\psi)| \right)^2.
\]  

(5)

The upshot of the above discussion is that

**Theorem 3** \( \Rightarrow \) **Theorem 2**.

In order to establish the quasi-independence on average statement, we will make use of the following results concerning the sets \( B(q, \delta) \).

**Lemma 3** Let \( n, m \geq 1 \) and let \( q_1, q_2 \in \mathbb{Z}^n \setminus \{0\} \) and \( \delta_1, \delta_2 \in (0,1/2) \). Then

\[
|B(q_1, \delta_1)| = (2\delta_1)^m
\]  

(6)

and

\[
|B(q_1, \delta_1) \cap B(q_2, \delta_2)| = |B(q_1, \delta_1)| \cdot |B(q_2, \delta_2)| \quad \text{if} \quad q_1 \parallel q_2.
\]  

(7)

The notation \( q_1 \parallel q_2 \) means that \( q_1 \) is parallel to \( q_2 \). The lemma is a consequence of Lemmas 8 and 9 in [14] and implies that the sets \( B(q_i, \delta_i), i = 1, 2 \) are pairwise independent for non-parallel vectors. The following statement is an analogue of Lemma 3 for the sets \( B'(q, \delta) \) with \( n = 1 \).

5
Lemma 4 Let \( n = 1 \) and \( m \geq 1 \). There is a constant \( C > 0 \) such that for \( \delta_1, \delta_2 \in (0, 1/2) \) and any distinct \( q_1, q_2 \in \mathbb{N} \)

\[
|B'(q_1, \delta_1)| = (2\delta_1)^m \prod_{p|q_1} (1 - p^{-m})
\]

and

\[
|B'(q_1, \delta_1) \cap B'(q_2, \delta_2)| \leq C (\delta_1 \delta_2)^m.
\]

The product in (8) is over prime divisors \( p \) of \( q_1 \) and is defined to be one if \( q_1 = 1 \).

In the case \( m = 1 \), the inequality given by (9) follows from equation (36) in [14]. In the case \( m \geq 2 \), the inequality follows from equation (10) in [8]. Finally, the equality given by (8) is established within the proof of Lemma 1 in [8]. Note that when \( m \geq 2 \), the product term in (8) is comparable to a constant and the lemma implies that the sets \( B'(q, \delta) \) are pairwise quasi-independent.

We bring this section of preliminaries to an end by stating a counting result that can be found in [14, p. 39]. Throughout, the symbols \( \ll \) and \( \gg \) will be used to indicate an inequality with an unspecified positive multiplicative constant. If \( a \ll b \) and \( a \gg b \) we write \( a \asymp b \), and say that the quantities \( a \) and \( b \) are comparable.

Lemma 5 Let \( h \) be a positive integer. Then

\[
\sum_{q \in \mathbb{Z}^n : |q| = h, \gcd(q)=1} 1 \asymp \begin{cases} 
\varphi(h) & \text{if } n = 2 \\
 h^{n-1} & \text{if } n \geq 3 ,
\end{cases}
\]

where the implied constants are independent of \( h \).

3 Quasi-independence on average

We have seen that establishing quasi-independence on average as stated in Theorem [3] lies at the heart of Theorem [2]. The proof of Theorem [3] splits naturally into establishing various key measure estimates.

3.1 Measure of \( B'(q, \delta) \) and \( B'(q_1, \delta_1) \cap B'(q_2, \delta_2) \)

The goal of this section is to extend the measure estimates of Lemma 4 beyond the \( n = 1 \) case. Given \( \delta > 0, q \in \mathbb{Z}^n \setminus \{0\} \) and \( p \in \mathbb{Z}^m \), let

\[
B(q, p, \delta) := \{ X \in \mathbb{R}^m : |qX + p| < \delta \}.
\]

The following observation will enable us to determine the precise measure of \( B'(q, \delta) \).
Lemma 6 Let $n, m \geq 1$ and let $q \in \mathbb{Z}^n \setminus \{0\}$ and $\delta \in (0, 1/2)$. Then, for any $l | \gcd(q)$

$$\sum_{p \in \mathbb{Z}^m} |B(q, lp, \delta)| = \left(\frac{2\delta}{l}\right)^m.$$  \hspace{1cm} (11)

**Proof.** Given that $\delta < 1/2$, it is easily seen that the sets $B(q, lp, \delta)$ as $p$ varies are pairwise disjoint. Therefore, in view of (6) the lemma follows on showing that

$$\bigcup_{p \in \mathbb{Z}^m} B(q, lp, \delta) = B(q/l, \delta/l).$$  \hspace{1cm} (12)

By definition, $X$ belongs to the left hand side of (12) if and only if there is a $p \in \mathbb{Z}^m$ such that $|qX + lp| < \delta$. Since $l$ divides $q$ this is equivalent to $|(q/l)X + p| < \delta/l$ which, by definition is equivalent to the statement that $X$ belongs to the right hand side of (12).

\boxdot

In the case $n = 1$, the following statement reduces to (8) of Lemma 4.

Lemma 7 Let $n, m \geq 1$ and let $q \in \mathbb{Z}^n \setminus \{0\}$ and $\delta \in (0, 1/2)$. Then,

$$|B'(q, \delta)| = (2\delta)^m \prod_{p | d} (1 - p^{-m}).$$  \hspace{1cm} (13)

The product is over prime divisors $p$ of $d := \gcd(q)$ and is defined to be one if $d = 1$.

**Proof.** Recall the following well known identities for the Möbius function $\mu$.

$$\sum_{l | d} \mu(l) = \left\{\begin{array}{cl} 0 & \text{if } d > 1 \\ 1 & \text{if } d = 1 \end{array}\right.$$

and

$$\sum_{l | d} \frac{\mu(l)}{l^m} = \prod_{p | d} (1 - p^{-m}).$$

Also, for any set $A$ we use the convention that $A \times 1 := A$ and $A \times 0 := \emptyset$. Then,

$$B'(q, \delta) = \bigcup_{p \in \mathbb{Z}^m} B(q, p, \delta) \sum_{l | d} \mu(l).$$  \hspace{1cm} (14)

Since $\delta < 1/2$, the sets in the above union do not overlap and so

$$|B'(q, \delta)| = \sum_{p \in \mathbb{Z}^m} |B(q, p, \delta)| \sum_{l | d} \mu(l)$$

$$= \sum_{l | d = \gcd(q)} \mu(l) \sum_{p \in \mathbb{Z}^m} |B(q, lp, \delta)|$$

$$= \sum_{l | d} \mu(l) \left(\frac{2\delta}{l}\right)^m = (2\delta)^m \prod_{p | d} \left(1 - \frac{1}{p^m}\right).$$  \hspace{1cm} \boxdot

The following is a consequence of examining the product term in Lemma 7.
Lemma 8 Let $n \geq 1$ and let $q \in \mathbb{Z}^n \setminus \{0\}$, $d := \gcd(q)$ and $\delta \in (0, 1/2)$. If $m = 1$, then
\[ |B'(q, \delta)| = 2\delta \frac{\varphi(d)}{d}. \] (15)

If $m > 1$, then
\[ \frac{6}{\pi^2} (2\delta)^m \leq |B'(q, \delta)| \leq (2\delta)^m. \] (16)

Proof. In the case $m > 1$, we trivially have that
\[ 1 \geq \prod_{p|d} (1 - p^{-m}) > \prod_{p} (1 - p^{-2}) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2}. \] (17)
Therefore (13) implies (16). In the case $m = 1$, we have that $\prod_{p|d} (1 - p^{-m}) = \varphi(d)/d$. Therefore (13) implies (15).

We now turn our attention to estimating the measure of the pairwise intersection between the sets $B'(q, \delta)$. In the case $n = 1$, the following statement coincides with (9) of Lemma 4.

Lemma 9 Let $n, m \geq 1$. There is a constant $C > 0$ such that for $\delta_1, \delta_2 \in (0, 1/2)$ and $q_1, q_2 \in \mathbb{Z}^n \setminus \{0\}$ satisfying $q_1 \neq \pm q_2$
\[ |B'(q_1, \delta_1) \cap B'(q_2, \delta_2)| \leq C(\delta_1 \delta_2)^m. \] (18)

Proof. In view of the fact that $B'(q, \delta) \subset B(q, \delta)$ and Lemma 3, we only need to deal with the situation when $q_1$ and $q_2$ are parallel. Then, it follows that there exists $q \in \mathbb{Z}^n$ with $\gcd(q) = 1$ and two different positive integers $k_1, k_2$ such that $q_1 = k_1 q$ and $q_2 = \pm k_2 q$. Without loss of generality, assume that $q_2 = k_2 q$.

Let $X \in B'(q_1, \delta_1) \cap B'(q_2, \delta_2)$. By definition, there are integer points $p_1, p_2 \in \mathbb{Z}^m$ such that $|q_i X + p_i| < \delta_i$ and $\gcd(p_i, q_i) = 1$ for $i = 1, 2$. Equivalently we have that
\[ \left\{ \begin{array}{ll}
|k_1 q X + p_1| < \delta_1, & \gcd(k_1, p_1) = 1, \\
|k_2 q X + p_2| < \delta_2, & \gcd(k_2, p_2) = 1.
\end{array} \right. \] (19)

Consider the transformation
\[ T_q : \mathbb{I}^m \to \mathbb{I}^m : X \mapsto q X \mod 1. \] (20)

It is readily verified that
\[ B'(q_1, \delta_1) \cap B'(q_2, \delta_2) \subseteq T_q^{-1}(B'(k_1, \delta_1) \cap B'(k_2, \delta_2)). \] (21)

The transformation $T_q$ is measure preserving; i.e. for any measurable set $A \subset \mathbb{I}^m$ we have that $|T_q^{-1}(A)| = |A|$ – see equation (48) in [14]. Therefore, by (21) we have that
\[ |B'(q_1, \delta_1) \cap B'(q_2, \delta_2)| \leq |B'(k_1, \delta_1) \cap B'(k_2, \delta_2)|. \] (22)

Applying Lemma 4 to (22) completes the proof of the lemma.

\[ \square \]
3.2 Measure of $B'_q(\psi)$ on average

Lemma 10 Let $nm > 1$ and $\psi(h) < 1/2$ for all $h \in \mathbb{N}$. Then with $q \in \mathbb{Z}^n \setminus \{0\}$ and $N \in \mathbb{N},$

\[
\sum_{|q| \leq N} |B'_q(\psi)| \asymp \sum_{h=1}^N h^{n-1}\psi(h)^m .
\] (23)

Proof. Naturally, the proof makes use of Lemma 8 and therefore splits into two cases: $m > 1$ and $m = 1$. We begin by considering the easy case $m > 1$. By (16) and the fact that the number of integer points $q \in \mathbb{Z}^n$ with $|q| = h$ is comparable to $h^{n-1}$, we have that

\[
\sum_{q \in \mathbb{Z}^n \setminus \{0\},} \sum_{|q| \leq N} |B'_q(\psi)| \asymp \sum_{q \in \mathbb{Z}^n \setminus \{0\},} \sum_{|q| \leq N} \psi(|q|)^m 
\]

\[
\asymp \sum_{h=1}^N \sum_{|q| = h} \psi(|q|)^m 
\]

\[
\asymp \sum_{h=1}^N h^{n-1}\psi(h)^m .
\]

This establishes (23) in the case $m > 1$.

We proceed with the case $m = 1$. By (15), it follows that

\[
\sum_{q \in \mathbb{Z}^n \setminus \{0\},} \sum_{|q| \leq N} |B'_q(\psi)| = \sum_{h=1}^N \sum_{q \in \mathbb{Z}^n,} |B'_q(\psi)| 
\]

\[
\asymp \sum_{h=1}^N \sum_{q \in \mathbb{Z}^n,} \frac{\varphi(d)}{d} \psi(h),
\]

\[
d := \gcd(q)
\]

\[
= \sum_{h=1}^N \psi(h) \sum_{d|h} \frac{\varphi(d)}{d} \sum_{|q'|=h/d, \gcd(q')=1} 1 .
\] (24)

To analyze (24) we consider $n > 2$ and $n = 2$ separately. Recall, that $nm > 1$ is a hypothesis within the statement of the lemma and so $n = 1$ is barred.

Subcase $n > 2$: By Lemma 5 it follows from (24) that

\[
\sum_{q \in \mathbb{Z}^n \setminus \{0\},} \sum_{|q| \leq N} |B'_q(\psi)| \asymp \sum_{h=1}^N \psi(h) \sum_{d|h} \frac{\varphi(d)(h/d)^{n-1}}{d} 
\]

\[
\asymp \sum_{h=1}^N h^{n-1}\psi(h) \sum_{d|h} \frac{\varphi(d)}{d^n} .
\]
This together with the fact that

\[ 1 \leq \sum_{d|h} \frac{\varphi(d)}{h^m} \leq \sum_{d=1}^{\infty} \frac{1}{d^2} = \frac{\pi^2}{6}, \]

yields (23).

Subcase \( n = 2 \): By Lemma \[5\] it follows from (24) that

\[ \sum_{q \in \mathbb{Z}^n \setminus \{0\}, \ |q| \leq N} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)| \asymp N \sum_{h=1}^{\infty} \psi(h) \sum_{d|h} \frac{\varphi(d) \varphi(h/d)}{d} = \sum_{h=1}^{\infty} \psi(h) f(h), \quad (25) \]

where

\[ f(h) := \sum_{d|h} \frac{\varphi(d) \varphi(h/d)}{d} = \sum_{d|h} \varphi(h/d) \sum_{l|d} \frac{\mu(l)}{l} = \sum_{l|h} \frac{\mu(l)}{l^2} \sum_{k|h/l} \varphi(k) = h \sum_{l|h} \frac{\mu(l)}{l^2} \text{ since } \sum_{k|d} \varphi(k) = d = h \prod_{p|h} (1 - p^{-2}). \]

Therefore, by (17) we have that

\[ \frac{6}{\pi^2} h \leq f(h) \leq h \quad \text{for all } h \in \mathbb{N}. \]

This combined with (25) yields (23) with \( m = 1 \) and \( n = 2 \).

\[ \checkmark \]

### 3.3 Measure of \( B'_{q_1}(\psi) \cap B'_{q_2}(\psi) \) on average

**Lemma 11.** Let \( nm > 1 \), \( \psi(h) < 1/2 \) for all \( h \in \mathbb{N} \) and \( \sum h^{n-1} \psi(h)^m = \infty \). Then with \( q_1, q_2 \in \mathbb{Z}^n \setminus \{0\} \) and \( N \) sufficiently large,

\[ \sum_{|q_1| \leq N, \ |q_2| \leq N} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)| \ll \left( \sum_{h=1}^{N} h^{n-1} \psi(h)^m \right)^2. \quad (26) \]
Proof. To begin with we separate out the diagonal term from the double sum in (26) and treat it separately as follows. Since the sum \( \sum h^{n-1} \psi(h)^m \) diverges, there exists a positive integer \( N_0 \) such that \( \sum_{h=1}^{N} h^{n-1} \psi(h)^m > 1 \) for all \( N > N_0 \). Then, by Lemma 10 it follows that for \( N > N_0 \)

\[
\sum_{|q_1| \leq N, \ |q_2| \leq N \atop q_2 \neq \pm q_1} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)| = 2 \sum_{|q_1| \leq N} |B'_{q_1}(\psi)| = N \sum_{h=1}^{N} h^{n-1} \psi(h)^m \lesssim \left( \sum_{h=1}^{N} h^{n-1} \psi(h)^m \right)^2.
\]

To complete the proof of the lemma, we obtain a similar estimate for the remaining part of the double sum. In view of Lemma 9 it follows that

\[
\sum_{|q_1| \leq N, \ |q_2| \leq N \atop q_2 \neq \pm q_1} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)| = \sum_{h=1}^{N} \sum_{l=1}^{N} \sum_{|q_1|=h, |q_2|=l \atop q_2 \neq \pm q_1} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)|
\]

\[
\lesssim \sum_{h=1}^{N} \sum_{l=1}^{N} \sum_{|q_1|=h, |q_2|=l} \psi(|q_1|)^m \cdot \psi(|q_2|)^m
\]

\[
= \sum_{h=1}^{N} \sum_{l=1}^{N} \psi(h)^m \cdot \psi(l)^m \sum_{|q_1|=h} 1 \sum_{|q_2|=l} 1
\]

\[
\lesssim \sum_{h=1}^{N} \sum_{l=1}^{N} h^{n-1} \psi(h)^m \cdot l^{n-1} \psi(l)^m
\]

\[
\lesssim \left( \sum_{h=1}^{N} h^{n-1} \psi(h)^m \right)^2.
\]

\[
\blacklozenge
\]

3.4 The finale

Let \( nm > 1 \), \( \psi(h) < 1/2 \) for all \( h \in \mathbb{N} \) and \( \sum h^{n-1} \psi(h)^m = \infty \). On combining Lemmas 10 and 11 we have that for \( q_1, q_2 \in \mathbb{Z}^n \setminus \{0\} \) and \( N \in \mathbb{N} \) sufficiently large

\[
\sum_{|q_1| \leq N, |q_2| \leq N} |B'_{q_1}(\psi) \cap B'_{q_2}(\psi)| \ll \left( \sum_{|q_1| \leq N} |B'_{q_1}(\psi)| \right)^2.
\]
Furthermore, an obvious implication of Lemma 10 is that
\[ \sum_{q \in \mathbb{Z}^n \setminus \{0\}} |B'_q(\psi)| = \infty. \]

The above are precisely the expressions given by (4) and (5) and thereby completes the proof Theorem 3.

4 The multivariable theory

Given a vector \( q \in \mathbb{Z}^n \), the approximating function \( \psi : \mathbb{N} \to \mathbb{R}^+ \) assigns a quantity \( \psi(|q|) \) that is dependent on the supremum norm of \( q \). Clearly, a natural and desirable generalisation is to consider multivariable approximating functions \( \Psi : \mathbb{Z}^n \to \mathbb{R}^+ \) and their associated sets \( \mathcal{A}_{n,m}(\Psi) \) and \( \mathcal{A}'_{n,m}(\Psi) \) of \( \Psi \)-approximable points. When the argument of \( \Psi \) is restricted to the supremum norm these sets are precisely the sets of \( \psi \)-approximable points considered above. For the sake of clarity, given \( \Psi : \mathbb{Z}^n \to \mathbb{R}^+ \) let
\[ \mathcal{A}'_{n,m}(\Psi) := \{ X \in \mathbb{Z}^m : |qX + p| < \Psi(q) \text{ for infinitely many } (p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\} \} \]
with \( \gcd(p, q) = 1 \).

Modifying the proof of Theorem 2 in the obvious manner, leads to the following statement.

**Theorem 4** Let \( \Psi : \mathbb{Z}^n \to \mathbb{R}^+ \) and \( m > 1 \). Then
\[ |\mathcal{A}'_{n,m}(\Psi)| = 1 \text{ if } \sum_{q \in \mathbb{Z}^n \setminus \{0\}} \Psi(q)^m = \infty. \]

As with Theorem 2, the proof of Theorem 4 reduces to establishing the analogue of Theorem 3— in particular, on showing that
\[ \sum_{|q_1| \leq N, |q_2| \leq N} |B'(q_1, \Psi(q_1)) \cap B'(q_2, \Psi(q_2))| \ll \left( \sum_{|q_1| \leq N} |B'(q_1, \Psi(q_1))| \right)^2. \tag{27} \]
However, since we are assuming that \( m > 1 \) the proof is much simpler. The reason for this is that (16) and (18) actually imply pairwise quasi-independence for the off-diagonal terms; i.e. \( |B'(q_1, \Psi(q_1)) \cap B'(q_2, \Psi(q_2))| \ll |B'(q_1, \Psi(q_1))| |B'(q_2, \Psi(q_2))| \) for \( q_2 \neq \pm q_1 \).

Our final result is a straightforward consequence of Theorem 4.

**Theorem 5** Let \( \Psi : \mathbb{Z}^n \to \mathbb{R}^+ \) and \( m > 1 \). Then
\[ |\mathcal{A}_{n,m}(\Psi)| = 1 \text{ if } \sum_{q \in \mathbb{Z}^n \setminus \{0\}} \Psi(q)^m = \infty. \tag{28} \]
The condition \( m > 1 \) cannot in general be removed from either Theorem 4 or Theorem 5. For a concrete counterexample see [1. §5]. Note that the statement of Theorem 5 was previously established by Sprindz̆uk [14] for approximating functions obeying additional constraints. For example, Theorem 14 in [14] is applicable to \( \Psi \) that vanish on non-primitive \( q \in \mathbb{Z}^n \). Our Theorem 5 carries no restrictions on \( \Psi \) and so is best possible.

For the sake of completeness, we mention that Theorems 4 and 5 are formally stated as Conjectures B and C in [1]. Furthermore, the Mass Transference Principle of [3] and the ‘slicing’ technique of [4] together with Theorem 5 establishes the general Hausdorff measure version of Catlin’s conjecture under the assumption that \( m \geq 2 \) – see Conjecture G in [1].

5 The quantitative theory

Let \( \Psi : \mathbb{Z}^n \to \mathbb{R}^+ \). Given \( X \in \mathbb{I}^{nm} \) and \( h \in \mathbb{N} \), let

\[
\mathcal{N}(X, h) := \# \{(p, q) \in \mathbb{Z}^m \times \mathbb{Z}^n \setminus \{0\} : |qX + p| < \Psi(q) \text{ with } |q| \leq h \}.
\]

In view of Theorem 3 if \( m > 1 \) and \( \sum \Psi(q)^m \) diverges then for almost all \( X \) we have that \( \mathcal{N}(X, h) \to \infty \) as \( h \to \infty \). An obvious question now arises: can we say anything more precise about the behavior of \( \mathcal{N}(X, h) \)? To some extent, the following remarkable statement provides the answer. Throughout, \( d(h) \) denotes the number of divisors of \( h \).

**Theorem (Schmidt)** Let \( \varepsilon > 0 \) be arbitrary. Let \( \Psi : \mathbb{Z}^n \to \mathbb{R}^+ \) and write

\[
\Phi(h) := \sum_{q \in \mathbb{Z}^n \setminus \{0\}, |q| \leq h} (2\Psi(q))^m \quad \text{and} \quad \chi(h) := \sum_{q \in \mathbb{Z}^n \setminus \{0\}, |q| \leq h} (2\Psi(q))^m d(gcd(q)).
\]

Then, for almost all \( X \in \mathbb{I}^{nm} \)

\[
\mathcal{N}(X, h) = \Phi(h) + O\left(\chi^{1/2}(h) \log^{3/2+\varepsilon} \chi(h) \right).
\]

The above form of the theorem is in line with the setup considered in this paper. Schmidt [13] actually proves a more general statement in which each of the \( m \) linear forms associated with the system \( qX \) are allowed to be approximated with different approximating functions.

Although not explicitly mentioned in the statement of Schmidt’s theorem, we may as well assume that \( \sum \Psi(q)^m \) diverges. Otherwise, a straightforward application of the Borel-Cantelli Lemma implies that \( \lim_{h \to \infty} \mathcal{N}(X, h) < \infty \) for almost all \( X \) and the theorem is of little interest. However, it is not the case that if the sum \( \sum \Psi(q)^m \) diverges then Schmidt’s theorem implies that \( \lim_{h \to \infty} \mathcal{N}(X, h) = \infty \) for almost all \( X \); that is to say that Schmidt’s theorem does not in general imply that \( |A_{n, m}(\Psi)| = 1 \). The reason for this is simple. The Duffin-Schaeffer counterexample and the counterexample alluded to in [4] above imply that the full measure statement is not in general true when \( n = m = 1 \) or when \( m > 1 \). Note
that these cases are not excluded from Schmidt’s theorem and so for the corresponding counterexamples we must have that the error term in (30) outweighs the main term. We now show that this conclusion is also true when \( n = 2 \) for certain approximating functions with argument restricted to the supremum norm. Thus, Schmidt’s theorem does not imply the theorems established in this paper.

With Theorem 1 in mind, let \( \Psi(q) = \psi(|q|) \) in the above and assume throughout that

\[
\sum_{q=1}^{\infty} q^{n-1} \psi(q)^m = \infty.
\]

(30)

Lemma 12 Let \( n = 2 \) and \( F : \mathbb{R}^+ \to \mathbb{R}^+ \) be an increasing function. Then there exists an approximating function \( \psi : \mathbb{N} \to \mathbb{R}^+ \) such that \( \psi \) is monotonic on its support and

\[
\chi(h) \geq F(\Phi(h)) \quad \text{for all sufficiently large } h.
\]

(31)

Remark. Note that for any \( \psi \) satisfying the divergent condition (30), we trivially have that the main term \( \Phi(h) \) in Schmidt’s theorem tends to infinity as \( h \to \infty \). The lemma shows that there exists \( \psi \) satisfying (30) for which the error term can be made as large as we please compared to the main term. For example, with \( F(x) := \exp(2x) \) there exists \( \psi \) for which the error term is eventually exponentially larger than the main term. Clearly, for such \( \psi \) Schmidt’s theorem does not enable us to conclude that \( \lim_{h \to \infty} N(X,h) = \infty \) for almost all \( X \) and therefore does not imply Theorem 1.

Proof. Given \( l \in \mathbb{N} \), it is easily seen that the number of points \( q \in \mathbb{Z}^2 \) such that \( |q| = l \) is equal to \( 8l - 4 \). With \( n = 2 \), it follows that

\[
\Phi(h) := \sum_{|q| \leq h} \Psi(q)^m = \sum_{l=1}^{h} \sum_{|q| = l} \psi(l)^m \leq 8 \sum_{l=1}^{h} l \psi(l)^m
\]

(32)

and

\[
\chi(h) := \sum_{|q| \leq h} \Psi(q)^m d(\gcd(q)) = \sum_{l=1}^{h} \sum_{|q| = l} \psi(l)^m d(\gcd(q))
\]

\[
= \sum_{l=1}^{h} \psi(l)^m \sum_{v|l} d(v) \sum_{|q'| = l/v, \gcd(q') = 1} 1
\]

\[
\geq \sum_{l=1}^{h} \psi(l)^m \sum_{v|l} d(v) \varphi(l/v) = \sum_{l=1}^{h} \psi(l)^m f(l),
\]

(33)

where \( f(l) := \sum_{v|l} d(v) \varphi(l/v) \). On exploiting the fact that \( f \) is a multiplicative function, it is readily verified that

\[
f(l) = l \times \prod_{p|l} \frac{p}{p-1} \times \prod_{p|l} (1 - p^{-k_p - 1}),
\]

where \( k_p \) is the \( p \)-adic valuation of \( l \).
where \( k_p \) is the largest integer \( k \) satisfying \( p^k \mid l \). Therefore, by (17) we have
\[
\frac{6}{\pi^2} l \theta(l) \leq f(l) \leq l \theta(l) \quad \text{where} \quad \theta(l) := \prod_{p \mid l} \frac{p}{p-1}.
\]
Substituting this into (33) yields that
\[
\chi(l) \geq \frac{1}{2} \sum_{l=1}^{h} l \psi(l)^m \theta(l).
\]
We will eventually define \( \psi \) to be supported on a subsequence of
\[
l_n := \prod_{i=1}^{n} p_i \quad (n \in \mathbb{N}),
\]
where \( p_i \) denotes the \( i \)-th prime. Obviously, \( \theta(l) \) will then be strictly increasing on the support of \( \psi \) and furthermore \( \lim_{n \to \infty} \theta(l_n) = \infty \).

Given an increasing function \( F \), let \( \{h_t\}_{t \in \mathbb{N}} \) be a subsequence of \( \{l_n\}_{n \in \mathbb{N}} \) such that for any \( T \in \mathbb{N} \)
\[
\frac{1}{2} \sum_{t=1}^{T} \theta(h_t) \geq F(8T + 8).
\]
The existence of such a subsequence is guaranteed by the fact that \( \theta(l_n) \to \infty \) as \( n \to \infty \). For \( t \in \mathbb{N} \), let \( s_t \) denote the number of terms \( l_n \) such that \( h_t \leq l_n \leq h_{t+1} - 1 \). Clearly, \( s_t \geq 1 \) because \( \{h_t\} \) is a subsequence of \( \{l_n\} \). Without loss of generality, we can assume that \( s_t \) is increasing since otherwise we work with an appropriate subsequence of \( \{h_t\} \). Now for any natural number \( l \) satisfying \( h_t \leq l \leq h_{t+1} - 1 \), define \( \psi(l) \) by setting
\[
l \psi(l)^m := \begin{cases} 
\frac{1}{s_t} & \text{if } l = l_n \text{ for some } n, \\
0 & \text{otherwise}.
\end{cases}
\]
Set \( \psi(l) := 0 \) for \( 1 \leq l < h_t \). It is easily seen that \( \psi \) is monotonically decreasing on its support. In view of the definition of \( \psi \), we have that for every \( t \in \mathbb{N} \)
\[
\sum_{l=h_t}^{h_{t+1}-1} l \psi(l)^m = 1.
\]
Since \( \theta(l) \) is increasing on the support of \( \psi \), we have that
\[
\sum_{l=h_t}^{h_{t+1}-1} l \psi(l)^m \theta(l) \geq \theta(h_t) \sum_{l=h_t}^{h_{t+1}-1} l \psi(l)^m \overset{\text{(36)}}{=} \theta(h_t).
\]
Now for any natural number $h \geq h_2$, there exists $T \in \mathbb{N}$ such that $h_{T+1} \leq h < h_{T+2}$ and it follows that

$$
\chi(h) \gtrsim \sum_{t=1}^{T} \sum_{l=h_t}^{h_{t+1}-1} l \psi(l)^m \theta(l) \gtrsim \sum_{t=1}^{T} \theta(h_t) \gtrsim F(8T + 8)
$$

$$
\gtrsim F \left( 8 \sum_{t=1}^{T+1} \sum_{l=h_t}^{h_{t+1}-1} l \psi(l)^m \right) \gtrsim F \left( 8 \sum_{t=1}^{T} l \psi(l)^m \right) \gtrsim F(\Phi(h)).
$$

This verifies (31) and thereby completes the proof of Lemma 12.

In view of Theorem 1, for any $\psi$ arising from Lemma 12 we still have that

$$
\lim_{h \to \infty} \mathcal{N}(X, h) = \infty \quad \text{for almost all } X.
$$

(38)

However, Schmidt’s theorem fails to describe the asymptotic behavior of $\mathcal{N}(X, h)$ and therefore the following problem remains open.

**Problem.** For $n = 2$ and $\psi$ satisfying the divergent sum condition (30), describe the asymptotic behavior of $\mathcal{N}(X, h)$.

Lemma 12 can be naturally adapted to the multivariable setup to show that there is not even a single choice of $n$ and $m$ for which Schmidt’s theorem implies Theorem 5.

**Lemma 13** Let $n \geq 2$ and $F : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function. Then there exists an approximating function $\Psi : \mathbb{Z}^n \to \mathbb{R}^+$ satisfying the divergent sum condition of (28) such that (31) holds.

**Proof.** Given $F$, let $\psi$ denote the approximating function arising from Lemma 12. The lemma now immediately follows on defining $\Psi$ by

$$
\Psi(q) := \begin{cases} 
\psi(|q|) & \text{if } q = (q_1, q_2, 0, \ldots, 0), \\
0 & \text{otherwise.}
\end{cases}
$$

In view of Theorem 5 for any $\Psi$ arising from Lemma 13 we still have (38). However, Schmidt’s theorem is vacuous for such $\Psi$ and describing the asymptotic behavior of $\mathcal{N}(X, h)$ remains an open problem.

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