Turbulent Equilibria for Charged Particles in Space

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Abstract. It is well known that the solar wind electron distribution function is apparently composed of several components, but the energetic tail population is well fitted with kappa distribution function. It is also well established that the solar wind protons possess quasi power-law tail distribution function that is well fitted with an inverse power law model. In the recent past, the present author developed a theory that describes a system of electrons and Langmuir turbulence that are in dynamical steady-state. In such a model, the kappa distribution function for the electrons emerges as a unique solution of the steady-state weak turbulence plasma kinetic equation. For the proton inverse power-law tail problem, Fisk and Gloeckler’s theory of compressional turbulence received much attention in the literature. In the present paper, their model is revisited in the light of plasma kinetic theory that involves low-frequency kinetic Alfvén wave fluctuations. It is shown that the proton kappa distribution function satisfies the steady-state proton particle kinetic equation. The steady-state wave kinetic equation for the kinetic Alfvén wave is also solved. This shows that the proton suprathermal distribution with an inverse power law velocity dependence may indeed result from a steady-state wave-particle interaction of the compressional kinetic Alfvénic fluctuations in the solar wind, thus providing support for, and also providing an alternative view of Fisk and Gloeckler’s model. However, in the absence of additional constraint that may arise from the balance of nonlinear wave-particle interaction terms within the wave kinetic equations for kinetic Alfvénic waves, the index of inverse power-law velocity tail distribution is undetermined. This calls for further investigation of nonlinear kinetic Alfvénic turbulence.

1. Introduction

It is well known that in the solar wind there exists a permanent low frequency turbulence [1, 2, 3]. While the source of such a turbulence may lie near the solar source, which may get convected outward into the interplanetary space, the role of self-generated low-frequency fluctuations by means of thermal emission is not really understood nor investigated in depth. In one of the noteworthy discoveries in recent heliospheric research, it was shown by means of various in situ spacecraft observations that the solar wind is replete with a ubiquitous distribution of energetic protons with an inverse power law suprathermal velocity distribution function, \( f(v) \propto v^{-5} \) [4, 5, 6].

Fisk and Gloeckler [7] proposed the so-called “pump acceleration mechanism” in order to explain such a feature. The pump acceleration model requires the following conditions, the essence of which can be found in Fisk and Gloeckler’s paper in this volume [8], and are paraphrased below:
• *The plasma contains an embedded mean magnetic field.* The energy in the particles is greater than that of the magnetic field, which has as its only function to couple the motion of the particles.

• *The plasma can be considered to be in a fixed volume that is thermally isolated,* which means simply that there is no net influx or outflow of particles and energy across the boundaries of the volume., i.e. no large-scale spatial gradients.

• *The plasma contains meso-scale compression and expansion regions.* These compressions and expansions occur normal to the mean magnetic field, and the regions are of size large compared to the gyro-radii of the particles that are to be accelerated in the plasma, but small compared to the total volume of the plasma.

Fisk and Gloeckler [8] also note that the above conditions “are very common conditions to occur in the solar wind.” The above stated conditions for pump acceleration mechanism may be repeated and rephrased as follows:

• The uniform magnetized plasma condition is applicable.

• The protons and compressional fluctuations maintain dynamically steady state interaction.

• Magnetic fluctuations must have a compressional component, whose characteristic wavelength can be as small as but generally longer that thermal gyro-radii of the protons.

Even though the model by Fisk and Gloeckler [7, 8] does not specify the precise plasma physical characteristics and properties of the compressional fluctuations, it is quite natural that the kinetic Alfvén waves thermally emitted by the protons are a prime candidate. Such waves and fluctuations satisfy the above re-stated conditions naturally, especially the last condition. Consequently, in the present paper, we put forth a theory in which the protons and kinetic Alfvénic fluctuations for a coupled steady-state solutions in a uniform plasma (conditions 1 and 2).

An analogous dynamical steady-state solution involving the suprathermal population of solar wind electrons, known as the *superhalo* electron component [9], was recently put forth [10, 11, 12]. The quiet time solar wind electrons featuring inverse power law tail with the index \( \sim -6 \) or so, \( f(v) \propto v^{-6} \), was recently explained in the light of the steady-state wave-particle interaction paradigm that involves high-frequency Langmuir fluctuations. It is thus quite natural to expect that the suprathermal protons in the solar wind may also be interacting with the low-frequency compressional turbulence of the Alfvénic variety in a dynamical steady-state. The present paper may help identify the detailed properties of such low frequency turbulence. It is in this context that the present work may be of relevance. It should be noted, however, that the present paper is preliminary in that our purpose is only to show that the kinetic Alfvénic fluctuations and suprathermal proton velocity distribution modeled by a kappa distribution may coexist in a steady-state fashion. The present paper does not yet show that the inver power-law index of \( v^{-5} \) naturally follows. As it will be argued later, the precise determination of the power-law index requires an additional dynamical constraint, which we do not have at this point.

The organization of the present paper is as follows: In Section 2 we discuss the general formalism of particle kinetic equation that describes the dynamical behavior of protons under the influence of kinetic Alfvénic fluctuations. We also discuss the wave kinetic equation for kinetic Alfvén waves and fluctuations. Then in Section 3 we consider the self-consistent steady-state solution of the coupled particle and wave kinetic equation for kinetic Alfvén waves and fluctuations in the limit of steady state, \( \partial/\partial t \rightarrow 0 \) in the sense of asymptotic situation, \( t \rightarrow \infty \). There it will be shown that the proton kappa distribution function is a legitimate solution of the coupled equations. We also derive the associated kinetic Alfvénic fluctuation spectra for electric and magnetic fields. Finally, Section 4 summarizes the present paper and discuss the findings.
2. General Formalism

2.1. Formal Particle Kinetic Equation

The mathematical details and physical models relevant to the present problem of steady-state coupled solutions between the protons and kinetic Alfvén fluctuations is reported elsewhere [13]. Here, we reproduce the final simplified form only. The general quasilinear kinetic equation including spontaneous emission and velocity space friction term is approximated by retaining only the Landau resonance terms since the kinetic Alfvén waves and fluctuations are of the low frequency type so that the cyclotron resonance conditions are not easily satisfied. The proton particle kinetic equation is expressed in the form of Fokker-Planck equation, which is written in velocity space cylindrical coordinate system as follows:

\[
\frac{\partial f_i}{\partial t} = \frac{\partial}{\partial v_\perp} \left( v_\perp A_{\perp} f_i + \frac{\partial}{\partial v_{\parallel}} (A_{\parallel} f_i) \right) + \frac{\partial}{\partial v_1} \left[ v_\perp \left( D_{1\perp} \frac{\partial f_i}{\partial v_{\perp}} + D_{1\parallel} \frac{\partial f_i}{\partial v_{\parallel}} \right) \right] + \frac{\partial}{\partial v_{\parallel}} \left( D_{\parallel \perp} \frac{\partial f_i}{\partial v_{\perp}} + D_{\parallel \parallel} \frac{\partial f_i}{\partial v_{\parallel}} \right),
\]

where

\[
A_{\perp} = \frac{e^2}{2\pi^2 m_i v_\perp} \int dk \int \frac{d\omega}{\omega^2} \left( \frac{\omega - k_{\parallel} v_{\parallel}}{k_{\perp} v_{\perp}} \right) \times \text{Im} \left( \frac{[v_\perp J_1(b)]^2}{\Lambda_{yy}} + \frac{\Lambda_{zz}^* [v_{\parallel} J_0(b)]^2}{(\Lambda_{zz}^2)} \right) \delta(\omega - k_{\parallel} v_{\parallel}),
\]

\[
D_{\perp \perp} = \frac{\pi e^2}{m_i^2 v_{\perp}^2} \int dk \int \frac{d\omega}{\omega^2} \left( \frac{(\omega - k_{\parallel} v_{\parallel})^2}{k_{\perp} v_{\perp}^2} \right) \times \left( [v_\perp J_0(b)]^2 \langle \delta E_\parallel^2 \rangle_{k,\omega} + [v_{\parallel} J_0(b)]^2 \langle \delta E_\parallel^2 \rangle_{k,\omega} \right) \delta(\omega - k_{\parallel} v_{\parallel}).
\]

In arriving at the above equation we assumed that the ambient magnetic field lies along z axis, \(B_0 = B_0\hat{z}\), the wave vector lies in \(xz\) plane, \(k = k_{\perp}\hat{x} + k_{\parallel}\hat{z}\). In Eq. (2), \(e\) and \(m_i\) represent the unit electric charge and proton mass, respectively, \(J_{n}(b)\) is the Bessel function of the first kind of order \(n\), and \(b = k_{\perp} v_{\perp}/\Omega_i\) is the argument, \(\Omega_i = eB_0/m_i c\) being the proton cyclotron frequency. Here, \(c\) is the speed of light in \(\text{vacuo}\).

In Eq. (2) the various quantities \(\Lambda_{ij}\) represent the components of simplified dielectric tensor under the assumption of retaining only the Landau resonance terms. In anticipation of the final results we assume isotropic proton velocity distribution function, \(f_i = f_i(v)\), and we ignore the displacement current, since the kinetic Alfvén waves and fluctuations satisfy the slow mode condition, \(\omega \ll ck\). For the \(zz\) component we retain the influence of isotropic electrons,

\[
\Lambda_{xx} = -\frac{\epsilon^2 k_{\parallel}^2}{\omega^2} + \frac{\epsilon^3}{v_A^2} \int dv \frac{1 - J_0^2(b)}{b^2} v_\perp \frac{\partial f_i}{\partial v_{\perp}},
\]

\[
\Lambda_{yy} = -\frac{\epsilon^2 k_{\parallel}^2}{\omega^2} + \frac{\epsilon^3}{v_A^2} \int dv \frac{1 - J_0^2(b)}{b^2} v_\perp \frac{\partial f_i}{\partial v_{\perp}} - i\pi \frac{\omega_p^2}{\omega} \int dv \delta(\omega - k_{\parallel} v_{\parallel}) \frac{J_1^2(b)}{b} v_\perp \frac{\partial f_i}{\partial v_{\perp}},
\]

\[
\Lambda_{zz} = \frac{\epsilon^2 k_{\parallel}^2}{\omega^2},
\]

\[
\Lambda_{zz} = -\frac{\epsilon^2 k_{\parallel}^2}{\omega^2} + \sum_{a=e,i} \frac{\omega_{p\alpha}^2}{\omega} \int dv \frac{v_{\parallel}^2}{\omega - k_{\parallel} v_{\parallel}} v_\perp \frac{\partial f_a}{\partial v_{\perp}}.
\]
In the above \( v_A^2 = B_0^2/(4\pi n_0 m_i) \) is the square of the Alfvén speed and \( \omega_{pa}^2 = 4\pi n_0 e^2/m_a \) is the square of the plasma oscillation frequency defined for species labeled \( a \).

Of the terms that appear within the definitions for \( A_i \) and \( D_{ij} \) in (2), those terms associated with \( \Lambda_{yy} \) and \( \langle \delta E^2 \rangle_{k,\omega} \) pertain to the magnetosonic mode branch, while the other terms that contain \( \Lambda_{xx}^* \) and \( \langle \delta E^2 \rangle_{k,\omega} \) are related to the kinetic Alfvén mode branch. If we are only concerned with the influence of kinetic Alfvénic fluctuations and the associated velocity space friction (or drag) term, then the particle kinetic equation (1) simplifies,

\[
\frac{\partial f_i}{\partial t} = \frac{\pi e^2}{m_i^2} \int dk \int d\omega \frac{\partial}{\partial \nu_{||}} \left[ J_0^2(b) \delta(\omega - k_{||}\nu_{||}) \right] \times \left( \frac{m_i}{2\pi^3\omega} \Im \frac{\Lambda_{xx}^*}{[\Lambda_{xx}\Lambda_{zz} - (\Lambda_{xx}^2)^*]} f_i + \langle \delta E^2 \rangle_{k,\omega} \frac{\partial f_i}{\partial \nu_{||}} \right),
\]

(4)

2.2. Formal Wave Kinetic Equation

In recent series of papers the general problem of spontaneous emission of EM fluctuations in magnetized plasmas has been formulated [14, 15, 16, 17]. In particular, Yoon et al. [17] discussed in detail, the spontaneous emission of Alfvénic fluctuations. The formalism outlined in [17] is particularly relevant for the present discussion. Even though Ref. [17] is exclusively relevant to thermal plasma, we may apply the formalism in Ref. [17] to the present case of non-thermal plasma in a straightforward manner. We start from the electric and magnetic field fluctuation spectra obtained in Ref. [17],

\[
\langle \delta E^2 \rangle_{k,\omega} = \frac{S_{yy}}{|\Lambda_{yy}|^2} + \frac{(|\Lambda_{xx}|^2 + |\Lambda_{xz}|^2)S_{zz}}{|\Lambda_{xx}\Lambda_{zz} - (\Lambda_{xx}^2)|^2},
\]

\[
\langle \delta B^2 \rangle_{k,\omega} = \frac{c^2}{\omega^2} \left( \frac{|k_{||}\Lambda_{xx}|^2 + |k_{||}\Lambda_{xz}|^2)S_{zz}}{|\Lambda_{xx}\Lambda_{zz} - (\Lambda_{xx}^2)|^2} \right) + \frac{c^2 k^2}{\omega^2} \frac{S_{yy}}{|\Lambda_{yy}|^2},
\]

(5)

where

\[
S_{yy} = \frac{2e^2 n_0}{\pi \omega^2} \int d\mathbf{v} [v_{\perp} J_1(b)]^2 \delta(\omega - k_{||}\nu_{||}) f_i,
\]

\[
S_{zz} = \frac{2e^2 n_0}{\pi \omega^2} \int d\mathbf{v} [v_{||} J_0(b)]^2 \delta(\omega - k_{||}\nu_{||}) f_i.
\]

(6)

If we are only interested in contributions from the kinetic Alfvén wave branch then we may ignore \( S_{yy} \) terms in Eq. (5). Consider the electric field spectrum for the kinetic Alfvén wave branch, which can be rewritten as

\[
\langle \Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2 \rangle \langle \delta E^2 \rangle_{k,\omega} = \frac{1}{(\Lambda_{xx}\Lambda_{zz} - \Lambda_{xx}^2)^*} \frac{2e^2 n_0}{\pi \omega^2} \int d\mathbf{v} \times \left( (|\Lambda_{xx}|^2 + |\Lambda_{xz}|^2) [v_{||} J_0(b)]^2 \delta(\omega - k_{||}\nu_{||}) f_i. \right)
\]

(7)

Upon introducing the slow time derivative associated with the angular frequency \( \omega \rightarrow \omega + i\partial/\partial t \), which acts upon both \( \delta E_{k,\omega} \) and \( \delta E^*_{k,\omega} \), within the ensemble average, \( \langle \delta E^2 \rangle_{k,\omega} = \langle \delta E_{k,\omega} \delta E^*_{k,\omega} \rangle \), we obtain

\[
\left( \Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2 \right) \langle \delta E^2 \rangle_{k,\omega} + i \frac{\partial \left( \Lambda_{xx}\Lambda_{zz} - \Lambda_{xx}^2 \right)}{\partial \omega} \frac{\partial \langle \delta E^2 \rangle_{k,\omega}}{\partial t} = \frac{1}{(\Lambda_{xx}\Lambda_{zz} - \Lambda_{xx}^2)^*} \frac{2e^2 n_0}{\pi \omega^2} \int d\mathbf{v} \left( (|\Lambda_{xx}|^2 + |\Lambda_{xz}|^2) [v_{||} J_0(b)]^2 \delta(\omega - k_{||}\nu_{||}) f_i. \right)
\]

(8)

This is the formal wave kinetic equation for the kinetic Alfvén wave mode.
2.3. Kinetic Alfvén Dispersion Relation

To discuss the dispersion relation, let us assume that the bulk of proton velocity space distribution can be approximated by the quasi-Maxwell form so that the relevant component of the dielectric tensor defined in Eq. (3) can be written as

\[
\Lambda_{xx} = -\frac{c^2 k_\parallel^2}{\omega^2} + \frac{c^2}{v_A^2} 1 - \frac{I_0(\lambda)e^{-\lambda}}{\lambda}, \quad \Lambda_{x\parallel,zz} = \frac{c^2 k_{\perp,\parallel} k_{\parallel}}{\omega^2},
\]

\[
\Lambda_{zz} = -\frac{c^2 k_\perp^2}{\omega^2} - \frac{\omega_p^2}{k_\perp^2 v_e^2} Z^\prime(\xi_e) - i\pi \frac{\omega_p^2}{\omega} \int dv \, \delta(\omega - k_{\parallel} v_{\parallel}) \frac{v_{\perp}^2}{v_{\parallel}} \partial f_i, \quad (9)
\]

where \( \lambda = k^2_{\parallel} T_i/(m_e \Omega_i^2) \), \( v_e^2 = 2 T_e/m_e \) is the square of the electron thermal speed, \( \xi_e = \omega/(k_{\parallel} v_e) \), and \( Z(\xi_e) \) is the plasma dispersion function, the prime representing the derivative with respect to the argument.

The dispersion relation is given by [18]

\[
\begin{pmatrix} \Lambda_{xx} & \Lambda_{x\parallel,zz} \\ \Lambda_{x\parallel,zz} & \Lambda_{zz} \end{pmatrix} \begin{pmatrix} \delta E_{x,\parallel}^{\omega} \\ \delta E_{x,\perp}^{\omega} \end{pmatrix} = 0, \quad (10)
\]

which leads to \( \Lambda_{xx} \Lambda_{zz} - \Lambda_{x\parallel,zz}^2 = 0 \). Let us approximate \( [1 - I_0(\lambda)e^{-\lambda}] / \lambda \approx 1 - 3\lambda/4 \). We also approximate \( 1 + \xi_e Z(\xi_e) \sim 1 \), which is equivalent to warm electrons. Under such approximations and following Ref. [18], we may obtain

\[
0 = \Lambda_{xx} \Lambda_{zz} - \Lambda_{x\parallel,zz}^2 = \frac{c^4}{v_A^4} \frac{2 T_i}{T_e} \left( 8 \frac{k_{\parallel}^2 v_e^2}{\Omega_i^2} \left[ 1 - \frac{9 k_{\parallel}^2 v_e^2}{8} \right] \right) \left( k_{\parallel}^2 v_e^2 / \Omega_i^2 \right)^2 - 1 = \frac{3 T_e}{2 T_i} \left( \frac{k_{\parallel}^2 v_e^2}{\Omega_i^2} \right)^2 \]

\[
+ i\pi \frac{c^4}{v_A^4} \frac{2 T_i}{T_e} \left( k_{\parallel}^2 v_e^2 / \Omega_i^2 \right)^2 \left( 8 \frac{k_{\parallel}^2 v_e^2}{\Omega_i^2} \right)^2 - 1 = \frac{3 T_e}{2 T_i} \left( \frac{k_{\parallel}^2 v_e^2}{\Omega_i^2} \right)^2 \]

\[
\int dv \, \delta(\omega - k_{\parallel} v_{\parallel}) \frac{v_{\perp}^2}{v_{\parallel}} \partial f_i. \quad (11)
\]

Setting the real part equal to zero we obtain the desired kinetic Alfvén wave dispersion relation,

\[
\omega^2 = \omega_k^2 = k_{\parallel}^2 v_A^2 \left[ 1 + \left( \frac{3}{4} + \frac{T_e}{T_i} \right) \frac{k_{\parallel}^2 v_e^2}{2 \Omega_i^2} \right]. \quad (12)
\]

Shown in Figure 1 is the dispersion “surface” corresponding to the kinetic Alfvén wave. The normalized real frequency \( \omega/\Omega_i \) is plotted in a surface plot format versus normalized parallel and perpendicular wave numbers, \( k_{\parallel} v_A/\Omega_i \) and \( k_{\perp} v_T/\Omega_i \), respectively, for \( T_e/T_i = 0.1 \).

2.4. Wave Kinetic Equation for Kinetic Alfvén Waves

Returning to the formal wave kinetic equation (8), taking the imaginary part, we have

\[
\frac{\partial \text{Re}(\Lambda_{xx} \Lambda_{zz} - \Lambda_{x\parallel,zz}^2)}{\partial \omega} \quad \text{Re}(\delta E_{x,\parallel}^{\omega})_{k,\omega} + 2 \text{Im}(\Lambda_{xx} \Lambda_{zz} - \Lambda_{x\parallel,zz}^2) \quad \text{Re}(\delta E_{x,\perp}^{\omega})_{k,\omega}
\]

\[
= \text{Im} \frac{1}{(\Lambda_{xx} \Lambda_{zz} - \Lambda_{x\parallel,zz}^2)^*} \frac{4 e^2 n_0}{\pi \omega^2} \int dv \left( \frac{|\Lambda_{xx}|^2 + |\Lambda_{zz}|^2}{\left( v_{\parallel} J_0(b) \right)^2} |v_{\parallel} J_0(b)|^2 \delta(\omega - k_{\parallel} v_{\parallel}) f_i. \quad (13)
\]
In what follows, we are interested in $\omega$ that satisfies the dispersion relation (12). Taking the residue contribution from the denominator, we have

$$\text{Im} \frac{1}{(\Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2)^*} = \text{Im} \frac{1}{(\omega - \omega_k - i0)[\partial(\Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2)/\partial\omega]_{\omega=\omega_k}} = \frac{\pi \delta(\omega - \omega_k)}{\partial\Re(\Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2)/\partial\omega]_{\omega=\omega_k}}. \quad (14)$$

We may also write the electric field spectrum by considering only the eigenmode contribution among all possible $\omega$'s,

$$\langle \delta E^2 \rangle_{k,\omega} = I_k \delta(\omega - \omega_k). \quad (15)$$

This leads to

$$\frac{\partial \Re \left( \Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2 \right)}{\partial \omega_k} \frac{\partial I_k}{\partial t} + 2 \text{Im} \left( \Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2 \right) I_k$$

$$= \frac{4e^2 n_0}{\omega_k^2} \left[ \partial \Re(\Lambda_{xx}\Lambda_{zz} - \Lambda_{xz}^2)/\partial\omega_k \right] (|\Lambda_{xx}|^2 + |\Lambda_{xz}|^2) \int dv [v|| J_0(v)]^2 \delta(\omega_k - k|| v||) f_i. \quad (16)$$

In the above the various dielectric tensor elements are defined by Eq. (9).

We evaluate the following various terms within the context of the dispersion relation, $\omega = \omega_k$. 

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**Figure 1.** Kinetic Alfvén wave (KAW) dispersion relation (12) for $T_e/T_i = 0.1$. 

![KAW Dispersion Relation](image.png)
defined in Eq. (12):

\[ \frac{\partial \text{Re}(\Lambda_{xx} \Lambda_{zz} - \Lambda_{xz}^2)}{\partial \omega_k} = \frac{e^4}{v_A^4} \frac{q^2}{T_i} \frac{1}{v_A^2} \left( 1 - 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right) , \]

\[ \frac{1}{\partial \text{Re}(\Lambda_{xx} \Lambda_{zz} - \Lambda_{xz}^2)/\partial \omega_k} = \frac{v_A^2}{c^4} \frac{T_e}{T_i} \frac{k^2 v_i^2}{\Omega_i^2} \left( 1 - 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right) , \]

\[ 2 \text{Im} (\Lambda_{xx} \Lambda_{zz} - \Lambda_{xz}^2) = 2 \pi \frac{c^4}{v_A^4} \frac{\Omega_i^2}{\omega_k} \left( 1 + 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right) \]

\[ \times \int dv \frac{v_i^2}{\omega_k} \delta (\omega_k - k v_i) \frac{\partial f_i}{\partial v_i} . \]

Inserting the above to Eq. (16), approximating \( J_0(b) \sim 1 \), and making use of

\[ |\Lambda_{xx}|^2 + |\Lambda_{xz}|^2 = \frac{e^4}{v_A^4} \left( \frac{k^2 v_i^2}{\omega_k} - 1 + 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right)^2 + \frac{e^4}{v_A^4} \left( \frac{k^2 v_i^2}{\omega_k} + 1 + 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right)^2 , \]

we have

\[ \frac{\partial I_K}{\partial t} = - \frac{\pi}{2} \frac{T_e}{T_i} \frac{v_i^2}{\omega_k} \left( \frac{k^2 v_i^2}{\omega_k} - 1 + 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right) \int dv \delta (\omega_k - k v_i) \frac{\partial f_n}{\partial v_i} I_K \]

\[ + \frac{v_A^4}{c^4} \left( \frac{T_e}{T_i} \frac{k^2 v_i^2}{\Omega_i^2} \right) \frac{\omega_k}{k^2 v_i^2} \left[ \frac{k^2 v_i^2}{\omega_k} - 1 + 3 \frac{k^2 v_i^2}{8 \Omega_i^2} \right]^2 . \]

Since thermal gyro-radius correction is considered small, we may further approximate the above by retaining only the leading terms in gyro-radius correction. Making use of the dispersion relation (12), it is possible to show that the final simplified form of the wave kinetic equation for kinetic Alfvén mode branch is given by

\[ \frac{\partial I_K}{\partial t} = \pi \left( \frac{T_e}{2T_i} \right)^2 \frac{v_A^2}{c^2} \frac{k^2 v_i^2}{\Omega_i^2} \left( \int dv \delta (\omega_k - k v_i) \frac{\partial f_i}{\partial v_i} \right) \frac{c^2}{v_A^2} I_K \]

\[ + \frac{m_i v_A^2}{4\pi^2 k^2 v_i^4} \int dv \delta (\omega_k - k v_i) f_i . \]

2.5. Particle Kinetic Equation for Protons Under the Influence of Kinetic Alfvénic Fluctuations

Returning to the formal particle kinetic equation (4), we make further simplification by approximating \( J_0(b) \sim 1 \), and making use of Eqs. (14), (15), and (17). We also note from Eq. (10), that

\[ \delta E_{k,\omega}^x = - \frac{\Lambda_{xx}}{\Lambda_{zz}} \delta E_{k,\omega}^z , \]

or

\[ \langle \delta E_{x}^2 \rangle_{k,\omega} = \langle \delta E_{z}^2 \rangle_{k,\omega} \frac{|\Lambda_{xx}|^2}{|\Lambda_{xx}|^2 + |\Lambda_{xz}|^2} \approx \langle \delta E_{z}^2 \rangle_{k,\omega} , \]

\[ \langle \delta E_{y}^2 \rangle_{k,\omega} = \langle \delta E_{z}^2 \rangle_{k,\omega} \frac{|\Lambda_{xx}|^2}{|\Lambda_{xx}|^2 + |\Lambda_{xz}|^2} \approx \langle \delta E_{z}^2 \rangle_{k,\omega} \frac{\Omega_i^2}{4T_i^2} \frac{k^2 v_i^2}{\Omega_i^2} . \]

From this we arrive at
\[
\left\langle \delta E^2_x \right\rangle_{k,\omega} = I_k \delta(\omega - \omega_k), \\
\left\langle \delta E^2_z \right\rangle_{k,\omega} = \frac{T^2_s}{4T_i^2} \frac{k^2_i k^4_i}{\Omega^4_i} I_k \delta(\omega - \omega_k).
\] (23)

These lead to the simplification of the formal particle kinetic equation (4),
\[
\frac{\partial f_i}{\partial t} = \frac{\pi e^2}{m_i c^2} \left( \frac{T_e}{2T_i} \right)^2 \int dk_\parallel \frac{\partial}{\partial v_\parallel} \left( \frac{m_i \omega_k}{4\pi^2} f_i + \frac{c^2}{v_A^2} I_k k_\parallel \frac{\partial f_i}{\partial v_\parallel} \right) \frac{k^2_i v^4_i}{\Omega^4_i} \delta(\omega_k - k_\parallel v_\parallel). 
\] (24)

The coupled set of equations, (20) and (24), can now be considered together in order to discuss the self-consistent steady-state solution for the proton velocity distribution function and a spectrum of kinetic Alfvénic fluctuations.

3. Asymptotic Steady State Between Kinetic Alfvén Fluctuations and Protons

Let us consider the wave and particle kinetic equations, (20) and (24),
\[
0 = \int dv \left( \frac{\partial f_i}{\partial v_\perp} \frac{c^2}{v_A^2} I_k + \frac{m_i \omega_k^2}{4\pi^2 k^4_i v_A^4} f_i \right) \delta(\omega_k - k_\parallel v_\parallel), \\
0 = \int dk_\parallel \frac{\partial}{\partial v_\perp} \left( \frac{m_i \omega_k}{4\pi^2} f_i + \frac{c^2}{v_A^2} I_k k_\parallel \frac{\partial f_i}{\partial v_\parallel} \right) \frac{k^2_i v^4_i}{\Omega^4_i} \delta(\omega_k - k_\parallel v_\parallel). 
\] (25)

Our interest as noted in the Introduction, is to consider the possibility of protons and kinetic Alfvénic fluctuations constantly undergoing wave-particle interaction while maintaining a dynamical steady state, so as to form a suprathermal population. In the literature, the kappa velocity distribution function is one of the most successful model to describe a population of particles with suprathermal tail population. Consequently, let us suppose that \( f_i \) is given by an isotropic kappa distribution,
\[
f_i(v) = \frac{1}{\pi^{3/2} v_T^3} \frac{\Gamma(\kappa + 1)}{(\kappa - \frac{3}{2})^2 \Gamma(\kappa - \frac{1}{2})} \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \frac{v^2}{v_T^2} \right)^{-\kappa - 1}, 
\] (26)

where \( v_T^2 = 2T_i/m_i \) is square of the thermal speed. Our limited aim is to show that indeed the above kappa distribution satisfies Eq. (25) in a self consistent manner. However, it is important to note that our formalism as it stands, namely Eq. (25), is not capable of determining the actual value of kappa index. From analogy with the present author’s theory of suprathermal electrons in the solar wind, the computation of kappa index requires an additional constraint. In the case of suprathermal electrons and steady-state Langmuir turbulence, the balance of nonlinear wave-particle interaction terms in the wave kinetic equation provides such an addition constraint, from which the kappa index for the electrons were determined to be \( \kappa = 2.25 \) [10, 11]. In the case of solar wind protons, the kappa index must be close to \( \kappa = 1.5 \) (the exact value of \( \kappa = 1.5 \) is, of course, not permitted within the context of Eq. (26), as such a value causes divergence). The additional dynamical constraint that will force the specific choice of kappa value will likely come from the nonlinear wave-particle interaction terms in the wave kinetic equation for the kinetic Alfvén wave turbulence, but such terms have not been derived in the literature. With this caveat in mind, let us proceed.
Making use of

\[ 2\pi \int_0^\infty dv_\perp \frac{\partial f_i}{v_\perp \partial v_\perp} = -\frac{2}{\pi^{1/2} v_i^3 T} \left( \frac{\Gamma(\kappa + 1)}{(\kappa - \frac{3}{2})} \frac{1}{\Gamma(\kappa - \frac{1}{2})} \right) \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \frac{v_i^2}{v_T^2} \right)^{-\kappa-1}, \]

\[ 2\pi \int_0^\infty dv_\perp \frac{\partial f_i}{v_\perp \partial v_\parallel} = -\frac{2}{\pi^{1/2} v_i^3 T} \left( \frac{\Gamma(\kappa + 1)}{(\kappa - \frac{3}{2})} \frac{1}{\Gamma(\kappa - \frac{1}{2})} \right) \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \frac{v_i^2}{v_T^2} \right)^{-\kappa-1}, \]

\[ 2\pi \int_0^\infty dv_\perp \frac{f_i}{v_\perp} = -\frac{1}{\pi^{1/2} v_i^3 T} \left( \frac{\Gamma(\kappa + 1)}{(\kappa - \frac{3}{2})} \frac{1}{\Gamma(\kappa - \frac{1}{2})} \right) \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \frac{v_i^2}{v_T^2} \right)^{-\kappa}, \tag{27} \]

one may show that the two equations (25) reduce to the following forms, if we integrate the second equation, that is, the steady-state wave kinetic equation, over perpendicular velocity,

\[ 0 = \int_0^\infty dv_\parallel \left[ -\frac{c^2}{v_A^2} I_k + \frac{m_i \omega_k^2}{4\pi^2 k_\parallel^2 v_A^2} \frac{v_\parallel^2}{2} \kappa - \frac{3}{2} \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \frac{\omega_k^2}{k_\parallel^2 v_T^2} \right) \right] \delta(\omega_k - k_\parallel v_\parallel), \]

\[ 0 = \int dk_\parallel \frac{\partial}{\partial v_\parallel} \left[ \frac{m_i v_\parallel^2}{4\pi^2} \frac{\kappa - \frac{3}{2}}{2} \left( 1 + \frac{1}{\kappa - \frac{3}{2}} \frac{\omega_k^2}{k_\parallel^2 v_T^2} \right) - \frac{c^2}{v_A^2} I_k \right] \delta(\omega_k - k_\parallel v_\parallel). \tag{28} \]

Making use of the dispersion relation and retaining the leading order terms in the small thermal gyro-radius expansion, we obtain from the necessary conditions for the two equalities in Eq. (28), the following form for the kinetic Alfvén wave electric field fluctuation spectrum:

\[ \langle \delta E^2 \rangle_k \equiv I_k = \frac{v_A^2}{c^2} \left\{ \frac{\kappa - \frac{3}{2}}{\kappa} T_i + \frac{1}{\kappa} \frac{B_0^2}{32\pi^2 n_0} \left[ 1 + \left( \frac{3}{4} + \frac{T_e}{T_i} \right) \frac{k_\parallel^2 \rho_i^2}{2} \right] \right\}, \tag{29} \]

where \( \rho_i^2 = v_i^2 / \Omega_i^2 \) is the thermal proton gyro-radius. The associated magnetic field fluctuation intensity may be obtained directly from the second equation in (5), by reconsidering all the analyses we have carried out thus far, or more simply, by considering from Faraday’s law that \( \delta B_{k_\parallel \omega} = (c/\omega_k)(k_\parallel \delta E_{k_\parallel \omega}^0 - k_\parallel \delta E_{k_\parallel \omega}^0) \). Upon making use of Eq. (23), it follows that the magnetic field fluctuation spectrum associated with the kinetic Alfvén wave mode is given by

\[ \langle \delta B^2 \rangle_k \equiv M_k = \frac{T_i}{4\pi^2} \left[ \frac{\kappa - \frac{3}{2}}{\kappa} + \frac{1}{\kappa \beta_i} \left[ 1 + \left( \frac{3}{4} + \frac{T_e}{T_i} \right) \frac{k_\parallel^2 \rho_i^2}{2} \right] \right]. \tag{30} \]

Note that the above magnetic field intensity spectrum reduces to the recently discussed solution in the limit of Maxwellian proton distribution (\( \kappa \to \infty \)) [17].

4. Conclusion and Discussions

To conclude, the assumed proton kappa distribution function (26) and the kinetic Alfvénic fluctuation spectra (29) and (30) satisfy the coupled particle and wave kinetic equations in a
Figure 2. Plot of the proton kappa velocity distribution function (26), in logarithmic vertical scale, versus logarithmic dimensionless speed, for \( \kappa = 1.55 \). The asymptotic behavior of the solution is \( f_i \propto v^{-5} \) in an approximate sense, which is consistent with observation [4, 5, 6].

self-consistent manner, and for a steady-state situation. We have thus proved that the kappa velocity distribution for the protons subject to dynamically steady-state wave-particle interaction with kinetic Alfvénic fluctuations, which are compressive fluctuations as envisioned by Fisk and Gloeckler [8, 7], is indeed a possible solution. What we have not yet proven is that the kappa index close to 1.5 is the unique choice. This requires an additional constraint and physical considerations such as nonlinear wave-particle interaction terms in the wave kinetic equation for kinetic Alfvénic turbulence. We reiterate that the purpose of the present theoretical analysis had been limited in the sense that, our aim was to rephrase Fisk and Gloeckler’s pump acceleration mechanism in an alternative way, with the emphasis placed on describing the self-consistent wave dynamics. However, at present, the precise value of \( \kappa \) is a free parameter. According to the similar and analogous theory of solar wind electrons and Langmuir fluctuations, the kappa index must be determined from the balance of nonlinear wave-particle interaction term in the Langmuir wave kinetic equation. In the present analysis, we do not yet have nonlinear term in the kinetic Alfvén wave equation. Consequently, the parameter \( \kappa \) must be treated as a free parameter for now.

Thus, in the present analysis, let us choose \( \kappa \) value in an arbitrary manner. According to observation [4, 5, 6] the observed suprathermal proton tail distribution \( f_i \propto v^{-5} \) corresponds to \( \kappa = 1.5 = 3/2 \), but the value of \( \kappa = 3/2 \) is the minimum critical value for which the model distribution function is not defined. We thus choose \( \kappa \) close to 1.5 but slightly higher than the exact value of 1.5 or equivalently 3/2, namely \( \kappa = 1.55 \) when plotting the proton distribution function. Figure 2 shows the solution (26) in log-log plot format. Note that the proton kappa distribution asymptotically behaves as \( f_i \propto v^{-2\kappa-2} \), which in the case of \( \kappa = 1.55 \) is \( f_i \propto v^{-5} \), in an approximate sense, which is consistent with observation [4, 5, 6].

The electric and magnetic field spectral intensities that form the other part of the self-consistent solution are given by (29) and (30). Recently, Yoon et al. [17] discussed the magnetic
Magnetic form factor $f(k_{\perp})$ [Eq. (31)] plotted against normalized perpendicular wave number $k_{\perp} \rho_i$ for various values of kappa index.

field intensity spectrum for kinetic Alfvén wave when the proton distribution is given by thermal Maxwellian form. The solution (30) is a direct generalization for kappa distribution. It is noteworthy that the kinetic Alfvénic fluctuations are characterized by the magnetic form factor,

$$f(k_{\perp}) = \frac{4\pi^2 \langle \delta B^2 \rangle}{T_i} = \frac{\kappa - \frac{3}{2}}{\kappa \beta_i} \left[ 1 + \frac{3}{4} + \frac{T_e}{T_i} \frac{k_{\perp}^2 \rho_i^2}{2} \right] \left[ 1 + \left( \frac{3}{4} + \frac{T_e}{T_i} \right) \frac{k_{\perp}^2 \rho_i^2}{2} \right].$$  \hspace{1cm} (31)

Shown in Figure 3 is the above form factor versus normalized perpendicular wave number $k_{\perp} \rho_i$ for various values of kappa index, $\kappa = 8$ (which is close to the Maxwellian limit), $\kappa = 4, 2$, and $\kappa = 1.55$.

**Acknowledgments**

The author acknowledges NSF grant AGS1550566 to the University of Maryland, and the BK21 plus program from the National Research Foundation (NRF), Korea, to Kyung Hee University. He also acknowledges the Science Award Grant from the GFT, Inc., to the University of Maryland. Part of this work was carried out while the author was visiting Ruhr University Bochum, Germany, which was made possible by the support from the Ruhr University Research School PLUS, funded by Germany’s Excellence Initiative (DFG GSC 98/3), and by a Mercator fellowship awarded by the Deutsche Forschungsgemeinschaft through the grant Schl 201/33-1.

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