The BRST formulation of G/H WZNW models.

Stephen Hwang\textsuperscript{1} and Henric Rhedin\textsuperscript{2}

Institute of Theoretical Physics
University of Göteborg
and
Chalmers University of Technology

Abstract

We consider a BRST approach to G/H coset WZNW models, \textit{i.e.} a formulation in which the coset is defined by a BRST condition. We will give the precise ingredients needed for this formulation. Then we will prove the equivalence of this approach to the conventional coset formulation by solving the the BRST cohomology. This will reveal a remarkable connection between integrable representations and a class of non-integrable representations for negative levels. The latter representations are also connected to string theories based on non-compact WZNW models. The partition functions of G/H cosets are also considered. The BRST approach enables a covariant construction of these, which does not rely on the decomposition of G as $G/H \times H$. We show that for the well-studied examples of $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$ and $SU(2)_k/U(1)$, we exactly reproduce the previously known results.

\textsuperscript{1}tfesh@fy.chalmers.se
\textsuperscript{2}hr@fy.chalmers.se
1 Introduction

In the study of two-dimensional conformal field theories the affine Lie algebras have played an essential rôle. One of the most important constructions, in this connection, is the so-called coset construction of which examples first appeared in [1] and then were constructed in general by Goddard, Kent and Olive (GKO) [2]. The corresponding models, known as the gauged Wess-Zumino-Novikov-Witten (WZNW) models, have a symmetry algebra which are of the affine Lie type [3] and appear to describe most of the known conformal field theories. It is, therefore, of importance to give a fundamental formulation of these models, enabling a consistent construction of tree and loop amplitudes.

A key to such a fundamental formulation was first given by Karabali and Schnitzer [4], who showed that the gauged $G/H$ WZNW models resulted in an action which was BRST invariant. This action contained three different sectors. Apart from the original $G$ WZNW model, it was described by an auxiliary non-unitary $H$ WZNW model and a Faddeev-Popov ghost sector. The nilpotent BRST symmetry results from a conversion of the original constraints of the gauged WZNW model, which are of second class classically [5], to new constraints which are classically of first class. One may also view the BRST symmetry as a consequence of a change of variables [6].

In an operator formulation, the BRST invariance implies that one should require that physical states of the gauged WZNW model should be BRST invariant. One must then analyze this equation and solve the resulting BRST cohomology. This will give the possible states of the theory. In the case of $H$ being Abelian, it was proved [4] that the BRST approach was equivalent, up to a degeneracy due to the ghost zero modes, to the usual coset conditions. It was, furthermore, shown that for arbitrary $G$ and $H$, the resulting energy-momentum tensor differed from the GKO one by a BRST exact term. It is the purpose of this paper to complete the work of Karabali and Schnitzer. We will first give the exact ingredients needed to treat general $G$ and $H$. In particular, we will specify the representations of the auxiliary $H$ WZNW model. The choice of representations will be one of the most essential parts in our work. We will then solve the BRST cohomology. The result of this analysis will show the equivalence of the BRST formulation and conventional coset construction. As a consequence of this analysis, we will discover an intriguing connection between the integrable representations of $H$ and a class of non-integrable
Section 1 Introduction

representations of $H$ with negative levels.

A BRST formulation of gauged WZNW models will be of importance in a further development of these theories. For example, we will study the partition function of the $G/H$ models in our formulation. We will derive the general contributions of the auxiliary and ghost sectors. Combined with the contribution of the $G$ WZNW model, which are given by the Kac-Weyl formula for simple $G$, they will give the resulting partition function for the $G/H$ model. This will give a covariant construction, which does not rely on a decomposition of $G$ as $G/H \times H$. We study in detail two simple examples, $SU(2)_k \times SU(2)/SU(2)_{k+1}$ and $SU(2)_k/U(1)$, for which we exactly reproduce the previously known results. The partition functions for arbitrary $G/H$ models have also been treated using a path-integral approach [7]. Their derivation does not, however, seem to respect the BRST symmetry.

We will also briefly discuss correlation functions. Using a decoupling theorem, originally proved in connection with a string theory based on the $SU(1,1)$ WZNW model [8], which is a modification of the original theorem due to Gepner and Witten [9], we can prove that our choice of representations for the auxiliary theory is consistent in correlation functions. A remaining problem, which is not addressed, is to determine fusion rules for the auxiliary theory.

The representations of the auxiliary non-unitary WZNW theory for the simplest case, $SU(2)$, are the discrete infinite dimensional non-unitary representations and they correspond to the unitary discrete ones of $SU(1,1)$. The range of highest weights $j$, $k/2 < j < 0$, $(k < 0)$ are the ones which have unitary physical states in the so-called $SU(1,1)$ string [10], as well as for the coset $SU(1,1)/U(1)$ [11] and will also contain the range of admissible representations in our BRST formulation. This connection is quite intriguing and might prove to be a way to understand the models based on non-compact groups. In particular, it may explain why these models seem to be so much simpler for integer values of $k$ [12]. Furthermore, using known fusion rules for the compact cosets one may deduce the fusion rules of the non-compact theory. The latter are at present not known.

The BRST formulation has attracted some attention in treating topological $G/G$ theories [13]. There one does not impose any restrictions on the representations of the auxiliary $G$ theory. This will lead to a much more complicated structure of the BRST cohomology. To analyze the BRST cohomology one has used a free field realization. This realization, together with another nil-potent operator, introduced by Felder for minimal models [14] and later generalized to $SU(2)_k$ [15], has provided
Section 2 The gauged WZNW model

an effective tool for analyzing the cohomology. This has e.g. been demonstrated in connection with $c \leq 1$ matter coupled to 2D gravity [16]. The problem in treating arbitrary groups $G$ is that, in general, the corresponding Felder reduction has not been rigourously proven. By assuming that a reduction exists, some general results on the cohomology [17] and the partition functions [18] can be derived. The methods that we will use in this paper are not based on a free field realization. Instead they rely on a formalism developed in [19]. We have some general results, using our techniques, for the cohomology of general $G$ and more general representations of the auxiliary $H$ theory. We intend to present these results in a future publication.

2 The gauged WZNW model

We consider a general WZNW model defined on a Riemann surface $\mathcal{M}$ with fields $g$ taking values in a compact Lie group $G$. The action is [3] [18] [20].

$$S_k(g) = \frac{1}{16\pi} \int_{\mathcal{M}} d^2\xi Tr(\partial_\mu g^{-1}\partial^\mu g) + \frac{1}{24\pi} \int_B d^3x \epsilon^{\alpha\beta\gamma} Tr(g^{-1}\partial_\alpha gg^{-1}\partial_\beta gg^{-1}\partial_\gamma g)$$ (2.1)

Here we assume that $g$ is well-defined on a three-dimensional manifold $B$, which has $\mathcal{M}$ as boundary. $k$ refers to levels $k_i$ according to the decomposition of $G$, $G = G_1 \times G_2 \times \ldots$, where $G_i$ are simple or belong to the center of $G$. We will assume that all $k_i$ are positive and integers and that we only have integrable representations. From here on we will, for simplicity, consider a simple group with a level $k$. Our analysis may be extended to the general case.

This action is invariant under the transformations [3].

$$g(\xi) \rightarrow \Omega(z) g(\xi) \bar{\Omega}^{-1}(\bar{z})$$ (2.2)

$\Omega$, and $\bar{\Omega}$ are $G$-valued matrices analytically depending on $z = \xi_1 + i\xi_2$, $\bar{z} = \xi_1 - i\xi_2$ respectively. The symmetry (2.2) implies an infinite number of conserved currents.

$$\partial_2 J = 0 \quad \partial_2 \bar{J} = 0$$ (2.3)

These currents $J = J^A t_A$ and $\bar{J} = \bar{J}^A t_A$, with $t_A$ antihermitean matrices representing the Lie algebra $g$ of the group $G$, satisfy the affine Lie algebra $\hat{g}$

$$[J^A_m, J^B_n] = i f^{A}{}_{C}^D J^C_{m+n} + \frac{k}{2} m \delta_{m+n} g^{AB},$$ (2.4)

We use here the same notation $g$ for the algebra as for the fields in the WZNW theory. It should be clear from the context what is meant.
with a corresponding algebra for $J_m^A$, $f^{AB}_C$ are structure constants of $g$ and $g^{AB}$ a non-degenerate metric on $G$.

In order to gauge an anomaly-free vector subgroup $H$ of the global $G \times G$ symmetry one introduces the gaugefields $A$, which belong to the adjoint representation of $H$. We denote the level of $H$ by $k_H$ and we assume again, for simplicity, $H$ to be simple. The corresponding action is then in a light-cone decomposition

$$S_k(g, A) = S_k(g) + \frac{1}{4\pi} \int d^2 \xi \text{Tr}(A_+ \partial_- g g^{-1} - A_- g^{-1} \partial_+ g + A_+ g A_- g^{-1} - A_- A_+) \tag{2.5}$$

By integrating out the gauge fields and using the Polyakov-Wiegmann identity, the partition function may be written in the form \cite{7,21}

$$Z = \int [dg][d\tilde{h}][db_+][db_-][dc_+][dc_-] \exp[-kS_k(g)] \exp[-(k_H - 2c_H)S_{-k_H - 2c_H}(\tilde{h})] \times \exp[-\text{Tr} \int d^2 \xi (b_+ \partial_- c_+ + b_- \partial_+ c_-)] \tag{2.6}$$

The partition function for the gauged $G/H$ model factorizes, therefore, into three different sectors: The original WZNW theory with a group $G$ and level $k$, an auxiliary WZNW theory with a group $H$ and level $-k_H - 2c_H$, where $c_H$ is the second Casimir of the adjoint representation of $H$, and a ghost sector. The total action in (2.6) is invariant under BRST transformations \cite{4,6}

$$\delta_B g = c_- g - gc_+$$
$$\delta_B h = c_- h - hc_+$$
$$\delta_B c_\pm = -\frac{1}{2} \{c_\pm, c_\mp\}$$
$$\delta_B b_+ = -k \frac{4\pi}{g} \partial_+ g + k \frac{h + c_H}{4\pi} \partial_+ h - \{b_+, c_\pm\}$$
$$\delta_B b_- = k \frac{4\pi}{g} \partial_- g - k \frac{h + c_H}{4\pi} \partial_- h - \{b_-, c_\pm\} \tag{2.7}$$

In writing the partition function (2.6) we have not been precise in defining the auxiliary WZNW theory. One must specify what representations occur for this sector. Since its level $-k_H - 2c_H$ is negative it is not, in general, unitary. As a consequence the choice of representations is not restricted in the same way as in the conventional unitary WZNW theory. A possible choice is the principal series of continuous representations \cite{7}. We will return to this issue in analyzing the BRST cohomology. In particular, we will see that choosing only the principal series is not consistent with the BRST symmetry.
The BRST symmetry found in the path-integral approach above is, however, also natural from an operator point of view. The action (2.5) implies classically that $J^a(z) \approx 0$ and $\bar{J}^a(\bar{z}) \approx 0$, where $a$ take values in $h$, the Lie algebra of $H^\mathbb{C}$. These constraints are of second class at the classical level \footnote{Our conventions are such that indices $A, B, \ldots$ take values in $g$ and $a, b, \ldots$ take values in $h$.}. In quantizing the theory canonically one should impose, in some way, such constraints on physical states. This is also implied in the Goddard-Kent-Olive (GKO) construction of coset theories \footnote{We will refer to (2.8) as the conventional coset conditions.}, where these currents commute with energy-momentum tensor of the coset. It is natural to impose on physical states

$$J^a_n|_{\text{phys}} = 0,$$

where $n > 0$ or, $n = 0$ and $a$ being a positive root. We will refer to (2.8) as the conventional coset conditions.

A more fundamental way of imposing constraints in a quantum theory is by using the BRST symmetry. In this case we have an obstruction due to the second class nature of the constraints, implying a BRST charge which is not nilpotent. To overcome this difficulty one introduces a new auxiliary set of variables, in order to convert the constraints into first class ones. A systematic construction was first discussed in \footnote{Our conventions are such that indices $A, B, \ldots$ take values in $g$ and $a, b, \ldots$ take values in $h$.}. The auxiliary theory must give rise to a change in the original constraint generators $J^a \rightarrow J^a + \tilde{J}^a$ such that the corresponding BRST charge is nilpotent at the quantum level. We are, therefore, led to the following BRST charge for the chiral part

$$Q = \oint \frac{dz}{2i\pi} \left[ c_a(z)(J^a(z) + \bar{J}^a(z)) : -\frac{i}{2} f_{a'd}^d : c_a(z)c_d(z)b^e(z) : \right]$$

The modes of the currents $J^a(z)$ and $\tilde{J}^a(z)$ satisfy an \( \hat{k} \) affine Lie algebra with levels $k_H$ and $k_{\bar{H}}$. We will assume that $J^a(z)$ will depend linearly on the currents of $G$. The ghosts $c_a(z)$ and $b^a(z)$ are conformal fields of dimension zero and one, respectively. They satisfy an operator product expansion $c_a(z)b^b(w) = \frac{\delta^b_a}{z-w}$. It is straightforward to check the nilpotency of the BRST charge. It holds provided we take $k_{\bar{H}} = -k_H - 2c_H$, which is the value found in the path-integral approach. Therefore, in a canonical quantization of a gauged WZNW theory, the auxiliary theory of level $-k_H - 2c_H$ arises from the requirement of a nilpotent BRST charge. The physical states are then found as solutions of

$$Q|_{\text{phys}} = 0$$
The purpose of this paper is to investigate what the implications of this condition are. In particular, if the naive conditions (2.8) are consistent with the BRST approach. The nilpotency of the BRST charge implies that the current

\[ J^{\text{tot}, \alpha}(z) = [Q, b^\alpha(z)] = J^\alpha(z) + \tilde{J}^\alpha(z) + J^{gh, \alpha}(z) \]  

(2.11)

satisfies an \( \hat{h} \) affine Lie algebra with vanishing central term. Here \( J^{gh, \alpha}(z) = if^{ad}: b^\alpha(z)c_d(z) : \) which satisfies an \( \hat{h} \) affine Lie algebra of level \( 2c_H \).

From the partition function (2.6) one may deduce the holomorphic energy-momentum tensor

\[ T(z) = \frac{1}{k + c_G} : J_A(z)J^A(z) : - \frac{1}{k + c_H} : \tilde{J}_\alpha(z)\tilde{J}^\alpha(z) : - : b^\alpha(z)\partial_z e_\alpha(z) : \]

\[ = T^G + T^H + T^{gh} \]  

(2.12)

which has a total conformal anomaly

\[ c^{\text{tot}} = \frac{k d_G}{k + c_G} + \frac{(-k - 2c_H)d_H}{(-k - 2c_H) + c_H} - 2d_H = \frac{k d_G}{k + c_G} - \frac{k d_H}{k + c_H} \]  

(2.13)

We recognize this charge as being equivalent to the one from the energy-momentum tensor in the GKO construction \[ T^{GKO} = T^G - T^H \]. The connection between the two is even more clear if we write \( T(z) = T^{GKO}(z) + T^H(z) + T^{\tilde{H}}(z) + T^{gh}(z) \), where \[ T^H(z) + T^{\tilde{H}}(z) + T^{gh}(z) = \frac{1}{k + c_H} [Q : b_\alpha(z) \left(J^\alpha(z) - \tilde{J}^\alpha(z)\right) :] \]  

(2.14)

The fact these two energy-momentum tensors differ only by a BRST exact term suggests that the conventional coset construction is at least contained in the BRST approach. For \( H \) being Abelian it was proved \[ T^{GKO} = T^G - T^H \]. The connection between the two is even more clear if we write \( T(z) = T^{GKO}(z) + T^H(z) + T^{\tilde{H}}(z) + T^{gh}(z) \), where \[ T^H(z) + T^{\tilde{H}}(z) + T^{gh}(z) = \frac{1}{k + c_H} [Q : b_\alpha(z) \left(J^\alpha(z) - \tilde{J}^\alpha(z)\right) :] \]  

(2.14)

The situation for more general \( H \) is, however, more complicated and this we will discuss in the following sections.

For future reference, let us introduce the Cartan-Weyl basis. In this basis the affine Lie algebra reads

\[ [J_m^i, J_n^j] = \frac{k}{2} m^{ij} \delta_{m, -n} \]

The BRST approach yields a two-fold degeneracy of physical states due to the doubling of ghost vacua.
Section 3 The BRST cohomology

\[
[J^i_m, J^\alpha_n] = \alpha^i J^\alpha_{m+n}
\]
\[
[J^\alpha_m, J^\beta_n] = \begin{cases} 
\epsilon(\alpha, \beta) J^\alpha_{m+n} & \text{if } \alpha + \beta \text{ is a root} \\
\frac{1}{\alpha^2} (\alpha_i J^i_{m+n} + \frac{k}{2} m \delta_{m,-n}) & \text{if } \alpha = -\beta \\
0 & \text{otherwise.}
\end{cases}
\]

Here \(i\) label the Cartan subalgebra and \(\alpha\) are the roots, which are normalized so that \(\alpha^2 = 1\) for the long roots. The ghosts are labelled by \(c^i_n, b^i_n, c^\alpha_n\) and \(b^\alpha_n\) with the non-zero anti-commutation relations \(\{c^i_n, b^j_n\} = \delta_{i,j} \delta_{m,n}\) and \(\{c^\alpha_n, b^\beta_n\} = \delta^\alpha_{\beta, -\delta_{m,n}}\).

The BRST charge is then

\[
Q = \sum_{i,n} :c^i_{-n} (J^i_n + \bar{J}^i_n) : + \sum_{\alpha,n} :c^\alpha_{-n} (J^\alpha_n + \bar{J}^\alpha_n) :
\]
\[- \frac{1}{2} \sum_{\alpha \neq \beta} \sum_{m,n \in \mathbb{Z}} \epsilon(\alpha, \beta) : b^{\alpha + \beta}_{-m-n} c^\alpha_{-n} c^{-\beta}_n - \frac{1}{2} \sum_{i,\alpha} \sum_{m,n \in \mathbb{Z}} \frac{1}{\alpha^2} \alpha^i : b^i_{-m-n} c^\alpha_m c^{-\alpha}_n :
\]
\[+ \sum_{i,\alpha} \sum_{m,n \in \mathbb{Z}} \alpha^i : b^\alpha_{-m-n} c^{-\alpha}_m c^\alpha_n :
\]

Furthermore

\[
J^{\text{tot},i}_m \equiv \{Q, b^i_n\} = J^i_m + \bar{J}^i_m + \sum_{\alpha,n} \alpha^i : b^\alpha_{m-n} c^{-\alpha}_n :
\]

and

\[
L^{\text{tot},m} \equiv L^H_m + \bar{L}^H_m + L^{gh}_m = \{Q, \frac{1}{k + c_H} \sum_n (J^a_{m+n} - \bar{J}^a_{m+n}) b_{-n,a} \}
\]

where

\[
L^H_m = \frac{1}{k + c_H} \left( \sum_{i,n} : j^i_n j^i_{m-n} : + \sum_{\alpha,n} \alpha^2 : j^\alpha_n j^{-\alpha}_{m-n} : \right)
\]
\[\bar{L}^H_m = -\frac{1}{k + c_H} \left( \sum_{i,n} \bar{j}^i_n \bar{j}^i_{m-n} : + \sum_{\alpha,n} \alpha^2 : \bar{j}^{-\alpha}_n \bar{j}^\alpha_{m-n} : \right)
\]
\[L^{gh}_m = \sum_{i,n} : b^i_{m-n} c^i_n : + \sum_{\alpha,n} : b^{-\alpha}_{m-n} c^\alpha_n :
\]

3 The BRST cohomology

We will now analyze the BRST equation (2.10). First we specify more exactly the state space. It decomposes into a product of three different sectors. A general state
Section 3  The BRST cohomology

is of the form $|s\rangle = |s_G\rangle \times |s_{\hat{H}}\rangle \times |s_{gh}\rangle$, where

$$|s_G\rangle = \sum_R \prod_{A,n} J_{-n}^A |0; R\rangle, \quad |s_{\hat{H}}\rangle = \sum_{\tilde{R}} \prod_{a,n} \tilde{J}_{-n}^a |0; \tilde{R}\rangle,$$

(3.1)

and $|s_{gh}\rangle$ is a sum of states of the form

$$\prod_{a_1,n_1} b_{-n_1}^{a_1} \prod_{a_2,n_2} c_{-n_2}^{a_2} |0\rangle_{gh}.$$  

(3.2)

Here the state $|0\rangle_{gh}$ is the SL(2, R) invariant ghost vacuum, which by the requirement that it is annihilated by $L_n^{gh}$ $n=0,\pm 1$ satisfies

$$c_{n}^{a} |0\rangle_{gh} = 0 \quad n \geq 1 \quad b_{n}^{a} |0\rangle_{gh} = 0 \quad n \geq 0.$$  

(3.3)

The state $|0\rangle_{G}$ is a highest weight primary state with respect to currents of $\hat{g}$, which transforms in some representation $R$ of $g$. The corresponding product in (3.1) is taken over $n > 0$ or $n = 0$ and $A$ being a negative root. Similarly, we have the state $|0\rangle_{\tilde{R}}$ which is highest weight primary with respect to $\hat{\tilde{h}}$ and $n > 0$. If there exists a highest weight state of the finite algebra then we include $n = 0$ and $a \in \{\alpha^+\}$ (i.e. the set of negative roots). The primary states in eq.(3.1) have both zero occupation level.

Among the states in eq.(3.1) there will exist, in general, states which have vanishing inner product with all states. These states are nullstates. The set of states $|s_G\rangle$ and $|s_{\hat{H}}\rangle$ in eq.(3.1), which are not null, will be denoted $S_G$ and $S_{\hat{H}}$, respectively. The full set of states $S_G \times S_{\hat{H}} \times S_{gh}$ is denoted $S$. The corresponding set of states for the subalgebra $\hat{h}$ of $\hat{g}$ will be denoted $S_{\hat{H}}$. It will be assumed throughout that any non-zero state in the space $S_G$ has a positive definite inner product.

It is convenient to rewrite the states in the $G$-sector to exhibit the explicit dependence on the currents of $\hat{h}$. We rewrite eq.(3.1) as

$$|s_G\rangle = \sum_{\Phi} \prod_{a,n} J_{-n}^a |\Phi\rangle.$$  

(3.4)

for some set of states $|\Phi\rangle \in \{\Phi\}$. If all states in this set are primary with respect to the currents of $\hat{h}$, then the state space of the $G$-sector will completely decompose into representations of $\hat{h}$. Symbolically we can write this as $G = G/H \times H$, where $G/H$ is the set of primaries with respect to $\hat{h}$. It will also imply a corresponding decomposition of the partition function. If, however, $\{\Phi\}$ is not equivalent to the
set of primary states, then this decomposition does not hold. One may establish the following criterion (*decomposition theorem*)

The decomposition eq.(3.4) of an arbitrary state in $S_G$ is possible, for all states in $\{\Phi\}$ being primary w.r.t. the currents of $\hat{h}$, if and only if all null-states w.r.t. $\hat{h}$ are also null w.r.t. $\hat{g}$.

We will not give the proof of this theorem here, but refer to a forthcoming publication. The theorem above is quite general and may be stated for other symmetry algebras than affine Lie algebras. In the present case we will not need this general result. For the theories that we are considering, namely when we restrict to compact $G$ and integrable representations, a theorem due to Kac and Peterson [23] states that it is always possible to choose $\{\Phi\}$ as the set of primary states w.r.t. the currents of $\hat{h}$. As remarked above, this implies a corresponding decomposition of the characters of $G$. They may be written in terms of characters of $H$ and so called *branching functions*.

The state space in eq.(3.1) decomposes completely in eigenstates of $L_0$ of each sector, respectively. We assume throughout that these eigenvalues are finite. In analyzing the BRST equation (2.10), one may restrict oneself to definite eigenvalues of $L_{0}^{\text{tot}}$, since this operator commutes with the BRST charge. One has

$$L_{0}^{\text{tot}}|s\rangle = \left(\frac{1}{k + c_H} (C - \tilde{C}) + N_J + \tilde{N}_J + N_{gh}\right)|s\rangle \tag{3.5}$$

where $C$ and $\tilde{C}$ are quadratic Casimirs of the finite algebra spanned by $J_0^a$, and $\tilde{J}_0^a$, respectively. $N$ is the total mode-number for each sector defined as the sum of the individual modes. By a standard argument, only the states with zero eigenvalue of $L_{0}^{\text{tot}}$ are non-trivial in the BRST cohomology. This implies that for positive values of $C - \tilde{C}$ all BRST invariant states are BRST exact, since $N_J + \tilde{N}_J + N_{gh}$ is a positive quantity. In the $H$-sector the values of $C$ that may occur are non-negative, since only unitary finite dimensional representations are possible. We can conclude, therefore, that in the $\tilde{H}$-sector all representations for which $\tilde{C}$ is strictly negative will have only trivial solutions to the BRST equation. This is the case for the principal series of continuous representations. We can, consequently, restrict our attention to representations in the $\tilde{H}$-sector for which the Casimir eigenvalues are non-negative.
Let the Cartan subalgebra of $h$ be denoted by $J^i_0$, $\tilde{J}^i_0$ and $J^{gh,i}_0$, $i = 1, \ldots, r_h$ (rank of $h$), in the respective sectors. These generators may also be diagonalized. The sum, $J^{tot,i}_0$, is a BRST exact operator (cf. eq.(2.17)) and, for non-trivial BRST invariant states, we can again restrict to states with zero eigenvalues. We write the BRST charge

$$Q = \hat{Q} + M_i b^i_0 + c_{0,i} J^{tot,i}_0.$$  

(3.6)

The operator $\hat{Q}$ is nilpotent on any state for which the condition $J^{tot,i}_0 |s\rangle = 0$ is met.

It is convenient to proceed by studying the cohomology of $\hat{Q}$ on the relative space

$$b_{0,i} |s\rangle = 0 \quad i = 1, \ldots, r_h.$$  

(3.7)

We will now prove the following result for the relative cohomology.

Let $\tilde{R}$ be a representation of highest weight $\tilde{\mu}$ such that all states $|s_{\tilde{R}}\rangle$ in eq.(3.3) belong to $S_{\tilde{H}}$, i.e. there are no null-states in this sector, then the relative cohomology is non-trivial only for states $|\phi\rangle$ which have zero ghost number, have no $\tilde{J}^a_{-n}$-excitations and satisfy

$$J^a_n |\phi\rangle = b^a_n |\phi\rangle = c^a_n |\phi\rangle = 0,$$  

(3.8)

for $n > 0$ or, $n = 0$ and $a \in \{\alpha^+\}$. In addition,

$$\mu^i + \tilde{\mu}^i + \rho^i = 0,$$  

(3.9)

where $\rho^i = \sum_{\alpha > 0} \alpha^i$ and $\mu^i$ is the weight of $|\phi\rangle$ w.r.t. $h$.

It should be remarked that an equivalent statement may be made for lowest weight representations $R$. Let us now prove the results above. Note first that eq.(3.9) follows from eq.(3.8) and the condition $J^{tot,i}_0 |\phi\rangle = 0$. It implies that the Casimirs of the representations of $h$ and $\tilde{h}$ satisfy $C - \tilde{C} = 0$. Furthermore, the equations $J^a_n |\phi\rangle = 0$ are implied by the BRST invariance of states with no ghost-excitations.

Introduce a gradation of states using the general form eq.(3.1). First we take $|0;\tilde{R}\rangle$, $|0^+\rangle_{gh} \equiv \prod_{\alpha > 0} c^i_0 |0\rangle_{gh}$ and an arbitrary state in the $G$-sector to have zero
Section 3 The BRST cohomology

degrees. Then a state is decomposed into states of definite degrees, which are determined by the operators acting on the ground-state

\[
\begin{align*}
\text{grad}(\tilde{J}_{-n}^a) & = 1, & \text{for } n > 0 \text{ or } n = 0, \ a \in \{\alpha^-\} \\
\text{grad}(b_{-n}^a) & = 1, & \text{for } n > 0 \text{ or } n = 0, \ a \in \{\alpha^-\} \\
\text{grad}(c_{-n}^a) & = -1 & \text{for } n > 0 \text{ or } n = 0, \ a \in \{\alpha^-\}
\end{align*}
\]

(3.10)

All other operators have zero degree. We note that a state with a ghost number \(N_{gh}\) will always decompose into states of degrees that are greater than or equal to \(-N_{gh}\). We have here taken the ghost number of \(|0^+\rangle_{gh}\) to be zero, a convention which we will adopt throughout this and the next section. The gradation above is not conserved by the commutators. This means that the degree will depend on the ordering of the operators which build up the states. It is, therefore, convenient to refer only to the maximum degree \(N\) of a state, *i.e.* a state which has a leading term of degree \(N\). One has the following decomposition (cf. eq.(2.16)): \(\hat{Q} = d_0 + d_{-1}\),

where

\[
d_0 = \sum_{m>0} \tilde{J}_{-m}^a c_{m,a} + \sum_{\alpha \in \{\alpha^+\}} \tilde{J}_{0}^{-\alpha} c_{0}^\alpha
\]

and \(d_{-1}\) is the remainder. The index indicates the degree of the operator, which when acting on a state of a maximum degree \(N\) will give a state of maximum degree not exceeding the sum of the degree of the operator and \(N\).

We now solve the BRST equation, which in the relative space (3.7) implies

\[
\hat{Q} |\phi\rangle = 0.
\]

(3.12)

Let \(|\phi\rangle\) be a state of maximum degree \(N > 0\), *i.e.* \(|\phi\rangle = |s; N\rangle + \ldots\). Then eq.(3.12) implies to highest order

\[
d_0 |s; N\rangle = 0
\]

(3.13)

In addition, \((d_0)^2 = 0 + O(-1)\), so that to leading order we should determine the cohomology of \(d_0\).

Consider a set of monomials

\[
|p, q\rangle = \tilde{J}_{-n_1}^{a_1} \ldots \tilde{J}_{-n_p}^{a_p} b_{-m_1}^{\varepsilon_1} \ldots b_{-m_q}^{\varepsilon_q} |s_G\rangle |\tilde{0}\rangle |\phi_{gh}\rangle.
\]

(3.14)

This set provides a basis for the full relative state space, provided the states \(|s_G\rangle \in S_G\) and \(|\phi_{gh}\rangle\) are chosen appropriately, and one defines a specific ordering among
Section 3  The BRST cohomology

the modes \( \tilde{J}_m^a \). Let us introduce a homotopy operation on this space, defined by its action on the monomials \( \kappa_0 \)

\[
\kappa_0 |p, q\rangle = \frac{1}{p+q} \sum_{i=1}^{p} \tilde{J}_{-n_1}^{a_1} \cdots \tilde{J}_{-n_p}^{a_p} b_{-n_1}^{e_1} \cdots b_{-n_q}^{e_q} \times |s_G\rangle \langle 0; \hat{R} |\phi_{gh}\rangle \quad p \neq 0
\]

where the capped terms are omitted. It is not essential to our argument whether \( \kappa_0 \) exists as an operator or not. However, it is easy to see that it does, in fact, exist. Let \( |s_0\rangle \) be a state and \( |s_1\rangle \equiv \kappa_0 |s_0\rangle \), as defined by eq.(3.15). If \( |s'_0\rangle \) is a state for which \( \langle s'_0 |s_0\rangle = 1 \), then we can realize \( \kappa_0 \) as the operator \( |s_1\rangle \langle s'_0| \).

Using the definition above, it is straightforward to verify the relation

\[
(k_0 d_0 + d_0 k_0) |p, q\rangle = \delta_{p+q,0} |p, q\rangle,
\]

which is valid to highest order. On the state \( |s; N\rangle \) this and eq.(3.13) implies

\[
|s; N\rangle = d_0 |s'; N\rangle + \mathcal{O}(N-1).
\]

This in turn implies

\[
|\phi\rangle = d_0 |s'; N\rangle + \mathcal{O}(N-1) = \hat{Q} |s'; N\rangle + \mathcal{O}(N-1).
\]

We have, therefore, that \( |\phi\rangle \) is \( \hat{Q} \)-exact to highest order in our gradation. One proceeds in a standard fashion, concluding to each highest order the exactness of the state. In this way any BRST invariant state in the relative space is shown to be cohomologically equivalent to a state with zero or negative maximum degree. A state with negative ghost number will, however, always decompose into states of positive degrees, and therefore, will be BRST trivial. This implies that the states with positive ghost number are trivial as well, which follows from the theorem by Kugo and Ojima [25].

We have finally only states of zero ghost number left to consider. These states may be decomposed into states of degrees greater than or equal to zero. Let us assume that it has a maximal degree larger than zero. We denote the highest order term of \( |\phi\rangle \) by \( |p, N\rangle \), where the degree \( N > 0 \). \( |p, N\rangle \) must then have some \( \tilde{J}_{-n}^a \)-excitations. This implies that in analyzing the BRST equation one may use the homotopy operation defined in eq.(3.15) to conclude that \( |p, N\rangle \) and hence \( |\phi\rangle \) is
BRST exact to this order. We can proceed in this fashion as long as the highest order term has a degree which exceeds zero. In this way one eliminates all $\tilde{J}_a^{-n}$-excitations. Now, if the state contains any $b_a^{-n}$-excitations, $|s\rangle = b_a^{-n}|s'\rangle + \{\text{terms with no } b_a^{-n} - \text{dependence}\}$, then by applying the BRST charge we get the highest order term $\tilde{J}_a^{-n}|s'\rangle + \{\text{terms with no } \tilde{J}_a^{-n} - \text{dependence}\} = 0$. This cannot be solved unless $|s'\rangle = 0$. We can conclude, therefore, that $|\phi\rangle$ does not contain any $b_a^{-n}$-excitations and, hence, no ghost dependence at all, and in addition, no $\tilde{J}_a^{-n}$-excitations.

This concludes our proof. Before discussing the relevance of the analysis to the coset model, we should also address the absolute cohomology. It is clear that the absolute cohomology contains a lot of more states using arguments due to [24], [14]. This follows from the fact that the ghost vacua has a $2^{r_H}$ degeneracy. In order to remove this degeneracy, we will in the next section impose that physical states are in the relative space.

The generalization to the cases where $H$ (and $G$) are not simple, but of the form $H_1 \times H_2 \times \ldots$, where $H_i$ are simple or in the center of $H$, is straightforward. Then the algebra $\hat{h}$ and $\tilde{\hat{h}}$ is a sum $\hat{h}_1 \oplus \hat{h}_2 \oplus \ldots$ and, therefore, the BRST charge decomposes correspondingly as $Q_1 + Q_2 + \ldots$. Each separate term $Q_i$ may then be analyzed as above.

4 Physical states and characters

We will now use the analysis of the preceding section to investigate the space of physical states relevant for the coset model. We will first remove the degeneracy due to the ghost zero modes corresponding to the Cartan subalgebra. We impose, therefore, the additional conditions

$$b_0^i|\text{phys}\rangle = 0 \quad \text{for } i = 1, \ldots, r_H$$

(4.1)

This implies that we only need to consider the relative cohomology. According to the results of the relative cohomology there exists a unique highest weight solution which is of zero ghost number (relative to the state $|0^+\rangle$) provided there are no null-states w.r.t. $\tilde{\hat{h}}$. It satisfies

$$J_0^a|\text{phys}\rangle = \tilde{J}_0^a|\text{phys}\rangle = b_0^a|\text{phys}\rangle = c_0^a|\text{phys}\rangle = 0,$$

(4.2)
for \( n > 0 \) or \( n = 0 \), \( a \in \{ \alpha^+ \} \). These conditions are equivalent to the conventional coset conditions \((2.3)\). In addition, there exists a corresponding lowest weight solution. One must now address the question of what representations of \( \hat{\mathfrak{h}} \) do not have null-states. We will establish the following important conclusion:

*For integrable representations of \( g \), and for representations of \( \hat{\mathfrak{h}} \) satisfying eq.\((3.9)\), the only solution to the BRST equation in the relative space are states satisfying eq.\((4.2)\).*

A different way of phrasing this result is as follows: If we only take representations of \( \hat{\mathfrak{h}} \), such that the states satisfying the coset conditions \((4.2)\) are at least contained in the set of solutions of the BRST equation, then these states are, in fact, the only possible solutions.

To prove this statement, it is sufficient to prove that, for representations satisfying eq.\((3.9)\), there are no null-states w.r.t. \( \hat{\mathfrak{h}} \) if we restrict to integrable representations of \( g \). Let us, therefore, investigate the null-states w.r.t. \( \hat{\mathfrak{h}} \). This may be done by examining the Kac-Kazhdan determinant \([26]\). We first consider the simplest case \( \hat{\mathfrak{su}}(2) \). Then the null-states are parametrized by two integers \( n \) and \( n' \). If the highest weight of the ground-state is denoted by \( j \) \((2j \in \mathbb{Z})\), then the highest weight of the primary null-state is \( j + n \) and the state occurs at occupation level \( N = nn' \) for the representations

\[
2j + 1 = -n + n'(k + 2). \tag{4.3}
\]

Here \( n, n' \geq 1 \) or \( n \leq -1, n' \leq 0 \). For \( \hat{\mathfrak{h}} \) we have \( \tilde{k} = -k - 4 \), so that for the highest weight representations \( \tilde{j} \) in this sector we have

\[
2\tilde{j} + 1 = -n - n'(k + 2), \tag{4.4}
\]

with \( n \) and \( n' \) as before. We see that for \( -k - 2 \leq 2\tilde{j} + 1 \leq 0 \) there are no null-states. Then by eq.\((3.9)\) we have \( \tilde{j} = -j - 1 \), which implies that \( -1/2 \leq j \leq (k + 1)/2 \). This range includes all the integrable representations of \( \mathfrak{su}(2) \), \( 0 \leq j \leq k/2 \). Thus, for the integrable representations of \( h = \mathfrak{su}(2) \) there are no null-states for \( \hat{\mathfrak{h}} \) and, therefore, the coset conditions eq.\((4.2)\) are the unique solutions. We note that apart from the values \( j = -1/2 \) and \( (k + 1)/2 \), the representations outside the range of integrable ones will not contain the usual coset conditions. The representations \( -k - 2 \leq 2\tilde{j} + 1 \leq 0 \), having negative highest weights, are infinite dimensional discrete representations and correspond to unitary representations of \( SU(1,1) \).
We now consider general simple algebras \( \hat{h} \). Let \( \mu^i \) denote the highest weight of the ground-state, then the primary null-state has highest weight \( \mu^i + n\alpha^i \), where \( n \) is an integer and \( \alpha^i \) a positive root, and occur at occupation level \( N = nn' \) for representations

\[
(2\mu + \rho) \cdot \alpha = -n\alpha^2 + n'(k + c_H),
\]

where \( n,n' \geq 1 \) or \( n \leq -1,n' \leq 0 \). From this expression we deduce that it is sufficient that \( (1/\alpha^2)(2\mu + \rho) \cdot \alpha \leq 0 \) for all positive roots and

\[
2\hat{\alpha} \cdot \hat{\mu} + 1 \geq \tilde{k} + 2,
\]

for \( \hat{\alpha} \) being the highest root, to have have no null-states. Thus, for these representations we will only have the physical states satisfying the coset conditions (4.2).

Then eqs. (3.9) and (4.6), using that \( \hat{\alpha} \cdot \rho = c_H - 1 \) and \( \tilde{k} = -k - 2c_H \), imply that \( (1-c_H)/2 \leq \hat{\alpha} \cdot \mu \leq (k+1)/2 \). This again includes all integrable representations, \( 0 \leq \hat{\alpha} \cdot \mu \leq k/2 \), which, consequently, proves the uniqueness of the coset conditions for arbitrary groups.

We now turn to the characters of the gauged WZNW models. Having proved the uniqueness of the BRST invariant physical states, we are assured that if we construct the characters in such a way that the BRST symmetry is respected, then only the correct physical degrees of freedom will propagate. Generally the character is defined as (we will omit the conventional factors of \( e^{-2i\pi \tau c/24} \))

\[
\chi(\tau, \theta) = \text{Tr} \left( e^{2i\pi \tau (L^G_0 + L^H_0 + L^{gh}_0)} e^{i\theta_i J_{0}^{\text{tot},i} + \theta_i' J_{0}^{I'}} (-1)^{N_{gh}} \right),
\]

where \( I' \in g/h \) \( ^6 \) In taking the trace, the projection onto the relative space, eq. (4.3), must be implemented. Therefore, the trace does not include a summation over the corresponding ghost vacua. This is consistent with the BRST symmetry only if we, in addition, require that the commutator of the conditions (4.1) with the BRST charge vanishes, i.e. a projection onto states satisfying \( J_0^{\text{tot},i}|s\rangle = 0 \) is made. We define, consequently, the BRST invariant character of the \( G/H \) WZNW model as

\[
\chi^{G/H} = \int \prod_i \frac{d\theta_i}{2\pi} \text{Tr} \left( e^{2i\pi \tau (L^G_0 + L^H_0 + L^{gh}_0)} e^{i\theta_i J_{0}^{\text{tot},i} + \theta_i' J_{0}^{I'}} (-1)^{N_{gh}} \right).
\]

The trace here contains no summation over ghost zero modes \( e^{0} \) and \( b^0 \) and the integration over \( \theta_i \) projects onto states satisfying \( J_0^{\text{tot},i}|s\rangle = 0 \).

\( ^6 \) We assume here that \( h \) is embedded in \( g \) in such a way that this decomposition is possible.
The characters (4.8) is a product of three different terms $\chi^G$, $\chi^\hat{H}$ and $\chi^{gh}$ due to the corresponding factorization of the state space. The first factor is the character of an arbitrary $G$ WZNW model and for a simple group is given by the Kac-Weyl formula [27]. The second factor is the character of $\hat{h}$. This character is straightforward to determine, since the state space is free of null-states for the representations we have selected. If the highest weight of the representation is denoted by $\tilde{\mu}$, then the character is given by

$$\chi^\hat{H}^+(\tau, \theta) = e^{i\theta \cdot \tilde{\mu}} e^{-\frac{2i\pi \tau}{k+r_{\hat{H}}} (\tilde{\mu} \cdot (\tilde{\mu} + \rho))} R^{-1}(\tau, \theta).$$

(4.9)

Here $\rho$ is the sum of positive roots and

$$R(\tau, \theta) = \prod_{n=1}^{\infty} \left( 1 - e^{2i\pi n \tau} \right)^{-1} \prod_{\alpha>0} \left( 1 - e^{-2i\pi (n-1)\tau} e^{-i\alpha \theta} \right)(1 - e^{2i\pi \tau} e^{i\alpha \theta}).$$

(4.10)

The corresponding character for a representation with the lowest weight ($-\tilde{\mu}$) is given by

$$\chi^\hat{H}^-(\tau, \theta) = -e^{(-i\theta \cdot (\tilde{\mu} + \rho))} e^{-\frac{2i\pi \tau}{k+r_{\hat{H}}} (\tilde{\mu} \cdot (\tilde{\mu} + \rho))} R^{-1}(\tau, \theta).$$

(4.11)

The character of the ghost sector is also easily found. For each ghost pair $b^a_\alpha$ and $c^a_\alpha$ we find the same contribution as for the conformal ghosts, with the exception that it is twisted according to the eigenvalue of $J^{gh,i}_3$. The trace over zero modes $b^0_\alpha$ and $c^0_\alpha$ yields $e^{i\rho \theta} \prod_{\alpha>0} (1 - e^{-i\alpha \theta})^2$. Consequently, we have

$$\chi^{gh}(\tau, \theta) = e^{i\rho \theta} R^2(\tau, \theta).$$

(4.12)

We will now consider some explicit examples to see that the characters above combine to yield results previously derived by other methods. First we take the simplest case of an Abelian group $H$, the parafermion theory $SU(2)_k/U(1)$. From the definition of the character of the coset eq. (4.8) we have ($q \equiv e^{2i\pi \tau}$)

$$\chi^{SU(2)/U(1)}(\tau) = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} Tr \left( q^{(L^0_{\theta}^{SU(2)} + L^0_{\theta}^{gh} + L^0_{\theta}^{gh})} e^{i\theta (J^3_0 + \tilde{J}^3_0)} \right).$$

(4.13)

Here $L^0_0 = -(\tilde{J}^3_0)^2/k$. The trace over excited modes of $\hat{U}(1)$ gives $\prod_{n=1}^{\infty} (1 - q^n)^{-1}$ and by integrating over $\theta$ we will impose $J^3_0 + \tilde{J}^3_0 = 0$. The ghost contribution is $\prod_{n=1}^{\infty} (1 - q^n)^2$. Consequently, we are left with a trace over $SU(2)$-sector of the form

$$\chi^{SU(2)/U(1)}(\tau) = Tr \left( q^{L^0_{\theta}^{SU(2)} - \frac{1}{2} (J^3_0)^2} \right) \prod_{n=1}^{\infty} (1 - q^n).$$

(4.14)
This expression gives the branching function of $\hat{su}(2)$ w.r.t. $\hat{u}(1)$ and is, up to a factor of $q^{-{(P+P)/2}} \prod_{n=1}^{\infty} (1 - q^n)$, a sum of string-functions. We may also obtain an explicit expression for the string-functions by inserting the expressions for the character of the $SU(2)$ and $U(1)$ theories before performing the integration over $\theta$.

The character for $SU(2)_k$ is given by

$$\chi_{j,k}(q, \theta) = \Delta_{k,j}(q, \theta) R^{-1}(q, \theta), \quad (4.15)$$

where

$$\Delta_{k,j}(q, \theta) = q^{j(j+1)/k+2} \sum_{n \in \mathbb{Z}} q^{(k+2)n^2 + (2j+1)n}(e^{i(j+(k+2)n)} - e^{-i(j+(k+2)n)}). \quad (4.16)$$

and

$$R(q, \theta) = (1 - e^{-\theta}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{i\theta})(1 - q^n e^{-i\theta}). \quad (4.17)$$

The $U(1)$ theory is equivalent to a free boson compactified on a radius $\sqrt{k/2}$. The character is

$$\chi_{\tilde{H}}(q, \theta) = \sum_{m \in \mathbb{Z}/2} q^{-\tilde{m}^2/2} e^{i\tilde{m} \tilde{\theta}} \prod_{n=1}^{\infty} (1 - q^n)^{-1}, \quad (4.18)$$

so that

$$\chi_{SU(2)/U(1)}(\tau) = \int_{-\infty}^{\infty} d\theta 2\pi \sum_{\tilde{m} \in \mathbb{Z}/2} \Delta_{k,j} q^{-\tilde{m}^2/2} e^{i\tilde{m} \tilde{\theta}} R^{-1}(\tau, \theta) \prod_{n=1}^{\infty} (1 - q^n). \quad (4.19)$$

Expanding $R^{-1}(\tau, \theta)$ \cite{28}, \cite{29},

$$R^{-1}(\tau, \theta) = \prod_{n=1}^{\infty} (1 - q^n)^{-3} \sum_{p=-\infty}^{\infty} \sum_{s=0}^{\infty} (-1)^s q^{(s-p+1/2)^2 - (p-1/2)^2} e^{ip\theta} \quad (4.20)$$

and integrating over $\theta$ gives the final result

$$\chi_{SU(2)/U(1)}(\tau) = q^{-j(j+1)/2} \prod_{n=1}^{\infty} (1 - q^n)^{-2} \sum_{p,n \in \mathbb{Z}} \sum_{s=0}^{\infty} (-1)^s q^{(k+2)n^2 + (2j+1)n}$$

$$\times q^{(s-p+1/2)^2 - (p-1/2)^2} \left( q^{-(1/2)(p+j+(k+2)n)^2} - q^{-(1/2)(p-j-(k+2)n)^2} \right). \quad (4.21)$$

This expression for the $SU(2)/U(1)$ parafermion theory was first derived in \cite{30} using a free field realization.
Our next example is $SU(2)_k \times SU(2)_1/SU(2)_{k+1}$, which is the unitary series of the minimal models. We choose the Cartan subalgebra of $G$ to be spanned by $J_0^{3(1)} + J_0^{3(2)}$ and $J_0^{3(2)}$, where the superindices $(1)$ and $(2)$ refer to the level $k$ and level one $SU(2)$ theories, respectively. We will, for simplicity, suppress the dependence on $J_0^{3(2)}$ and define the character as

$$\chi_{j_1,j_2,j}^{V ir} = \int_{-\infty}^{\infty} d\theta \overline{2\pi} \text{Tr} \left( q^{L_0} e^{i\theta J_0^3} \right)$$  \hspace{1cm} (4.22)$$

where $L_0$ and $J_0^3$ contain the sum over all sectors. The character for the $SU(2)_k$ theory is given by eq.(4.15). For $k = 1$ it simplifies to

$$\chi_{j_2,1} = \sum_{\lambda \in Z + j_2} q^{\lambda^2} e^{i\lambda \theta} \prod_{n=1}^{\infty} (1 - q^n)^{-1},$$  \hspace{1cm} (4.23)$$

where $j_2 = 0$ or $1/2$. The character for the auxiliary $SU(2)_{-k-5}$ theory is found from eq.(4.9),

$$\chi_{j,-k-5} = q^{-\frac{j(j+1)}{k+3}} e^{i\theta j} R^{-1}(\tau, \theta),$$  \hspace{1cm} (4.24)$$

and the ghost contribution from eq.(1.12), $\chi^{gh} = e^{i\theta} R^2(\tau, \theta)$. Thus,

$$\chi_{j_1,j_2,j}^{V ir} = \int_{-\infty}^{\infty} d\theta \overline{2\pi} \prod_{n=1}^{\infty} (1 - q^n)^{-1} \Delta_{k,j_1}(q, \theta) q^{-\frac{j(j+1)}{k+3}} e^{i\theta j} \sum_{\lambda \in Z + j_2} q^{\lambda^2} e^{i\lambda \theta}$$  \hspace{1cm} (4.25)$$

In performing the integration over $\theta$, we will get delta-functions, $\delta_{\lambda+j_1+j_1+1+(k+2)n,0}$ and $\delta_{\lambda+j_1-j_1-(k+2)n,0}$, from the two factors in eq.(1.10). They are zero unless $j_1 - j = j_2 \mod 1$. We can use the delta-functions to eliminate the sum over $\lambda$, so that

$$\chi_{j_1,j_2,j}^{V ir} = \prod_{n=1}^{\infty} (1 - q^n)^{-1} q^{\frac{j(j+1)}{k+3}} q^{-\frac{j(j+1)}{k+3}} \sum_{n \in \mathbb{Z}} q^{(k+2)n^2 + (2j_1+1)n}$$

$$\times \left[ q^{(j_1+(k+2)n+j_1+1)^2} - q^{(j_1+(k+2)n-j)^2} \right]$$  \hspace{1cm} (4.26)$$

If we set $p = 2j_1 + 1$ and $r = -(2\tilde{j} + 1)$, then for integrable representations, $1 \leq p \leq m - 1$ and $1 \leq r \leq m, \ (m \equiv k + 2)$. The range of $r$ is here determined by $\tilde{j} = -j - 1$ and $j$ being an integrable representation of $SU(2)_{k+1}$. Furthermore, from $j_1 - j = j_2 \mod 1$, we have $p - r$ is even or odd if $j_2$ is 0 or $1/2$, respectively. The character may then be written as

$$\chi_{(p,r)}^{V ir} = \prod_{s=1}^{\infty} (1 - q^n)^{-1} \sum_{n \in \mathbb{Z}} (q^{\alpha^r_{p,n}(n)} - q^{\beta^r_{p,n}(n)})$$  \hspace{1cm} (4.27)$$
Section 4 Physical states and characters

with

\[
\alpha_{p,r}^m(n) = \frac{[2m(m+1)n - rm + p(m+1)]^2 - 1}{4m(m+1)} \\
\beta_{p,r}^m(n) = \frac{[2m(m+1)n + rm + p(m+1)]^2 - 1}{4m(m+1)}
\]  

(4.28)

This expression is identical to the one derived by Goddard, Kent and Olive [2], obtained by factorizing the \( SU(2)_k \times SU(2)_1 \) characters, and to the one given by Rocha-Caridi [31] for the discrete series. The range of \( r \) given here is not the same as in ref. [2], but in summing over \( p \) and \( r \), it may be extended to the same range by using the symmetry properties of the characters. We see from the derivation above that we never needed to use the factorization property of \( SU(2)_k \times SU(2)_1 \) as a product \( SU(2)_{k+1} \times Vir \). Finally, if we would have used the lowest weight representations of the auxiliary \( SU(2) \) theory, we would have arrived at the same expression (4.27).

Let us end by briefly discussing correlation functions. We have selected a particular range of representations for the auxiliary theory. We must, therefore, address the important issue whether this selection is consistent in correlation functions or if the fusion rules require that we consider a larger set of representations. If we have two vertex operators corresponding to the highest weights \( \tilde{\mu}_1 \) and \( \tilde{\mu}_2 \), then these can only fuse to a vertex of highest weight starting at \( \tilde{\mu}_1 + \tilde{\mu}_2 \) and continuing downwards to lower weights. This follows from the conservation of the eigenvalues of the Cartan generators of the finite algebra. In order for our selection of representations to be consistent, there must exist a consistent truncation of this series of possible representations. We will now show that this is the case. We start again by considering \( SU(2) \). In this case we may make use of a decoupling theorem originally proved for \( SU(1,1) \) [8], modifying the original theorem of Gepner and Witten [9]. The important point is not whether we have \( SU(1,1) \) or \( SU(2) \), but if the level is positive or negative. We make use of the highest weight null-vectors

\[
(\tilde{J}^+(z))^NV_j(z) = 0.
\]  

(4.29)

Here \( \tilde{j} \) refers to the highest weight, which is a negative integer or half-integer for the infinite dimensional representations which we are considering. These null-vectors occur for \( \tilde{j} = \tilde{k}/2 - (N - 1)/2 \) for \( N = 1, 2, 3, \ldots \). Since \( \tilde{k} \) is negative, the null-vectors are present for all representations in the infinite dimensional discrete series,
except those for which $\tilde{j} > \tilde{k}/2$. They are also absent for the trivial representation. Inserting (4.29) into a three-point function will lead to restrictions on this correlation function. It will be zero whenever we mix vertices with null-vectors and vertices without. This is where the sign of the level enters. For positive levels the trivial representation is of the former type, while for negative levels it is of the latter type. Consequently, for negative levels the vanishing of the three-point function can only imply that representations with null-vectors decouple (or else there is no propagation at all) i.e. all highest weight representations for which $\tilde{j} \leq \tilde{k}/2$ decouple. This is the decoupling theorem. Note, that the fact that the coupling vertices do not contain null-vectors implies that we do not get further restrictions on the couplings, contrary to the case of positive level.

The generalization to arbitrary groups proceeds in the same way as our analysis of null-vectors in the beginning of this section. One selects an $\hat{su}(2)$ subalgebra spanned by $\tilde{J}_m^\alpha$, $\tilde{J}_{-m}^\alpha$ and $\frac{1}{\alpha} \alpha \cdot \tilde{J}_m$. Then by using the corresponding null-vectors, one reaches the conclusion that we have decoupling unless the highest weights are in the range $3/2 - c_H \leq \alpha \cdot \mu \leq (k + 1)/2$, which is consistent with the selection of weights we have considered. This concludes the decoupling for the general case.

The fact that we have a consistent truncation of representations is perhaps not so surprising from the point of view of the original GKO construction. In this formulation, the vertices of the coset have to commute with the generators of $\hat{h}$. One may then use this to show \[32\] that such vertices will close under fusion. In the present formulation we have seen that the conventional coset condition follows from the BRST condition. Therefore, it is natural to expect that the closure in the GKO construction follows from a corresponding closure in the BRST formulation.
Acknowledgments
It is a pleasure to thank Arne Kihlberg for discussions on group theory. We would also like to thank Robert Marnelius and Christian Preitschopf for discussions and Antti Kupiainen for explaining certain issues in ref. [7].

References

[1] M. B. Halpern, Phys. Rev. D4 (1971) 2398, Phys. Rev. D12 (1975) 1684
[2] P. Goddard, A. Kent, and D. Olive, Phys. Lett. 152, 88; Comm. Math. Phys. 103 (1986) 105
[3] E. Witten, Comm. Math. Phys. 92 (1984) 455
[4] D. Karabali, and H. Schnitzer, Nucl. Phys. B329 (1990) 649
[5] P. Bowcock, Nucl. Phys. B316 (1989) 80
[6] F. Bastianelli, Nucl. Phys. B361 (1991) 555
[7] K. Gawędski and A. Kupiainen, Nucl. Phys. B320 (1989) 625
[8] S. Hwang and P. Roberts, Interaction and modular invariance of strings on curved manifolds, to be published in the proceedings of the 16’th Johns Hopkins’ workshop, Göteborg 1992.
[9] D. Gepner, and E. Witten, Nucl. Phys. B278 (1986) 493
[10] S. Hwang, Nucl. Phys. B354 (1991) 100
[11] J.J. Dixon, J. Lykken and M.E. Peskin, Nucl. Phys. B325 (1989) 329
[12] M. Henningson, S. Hwang, P. Roberts and B. Sundborg, Phys. Lett. B267 (1991) 350
[13] O. Aharony, O. Ganor, J. Sonnenschein, S. Yankielowicz, and N. Sochen, Physical states in $G/H$ models, and 2d gravity, TAUP-1961-92
O. Aharony, O. Ganor, J. Sonnenschein, and S. Yankielowicz, On the twisted $G/H$ topological models., TAUP-1990-92
[14] G. Felder, Nucl. Phys. B317 (1989) 215
References

[15] D. Bernard and G. Felder, Comm. Math. Phys. 127 (1990) 145

[16] P. Bouwknegt, J. McCarthy, and K. Pilch, Comm. Math. Phys. 145 (1992) 541

[17] P. Bouwknegt, J. McCarthy, and K. Pilch Semi-infinite cohomology in conformal field theory, and 2d gravity., to be published in the proceedings 28’th Karpacz winter school of theoretical physics, Karpacz, Poland, 1992

[18] P. Bouwknegt, J. McCarthy, and K. Pilch, Nucl. Phys. B352 (1991) 139

[19] S. Hwang, and R. Marnelius, Nucl. Phys. B315 (1989) 638
S. Hwang, and R. Marnelius, Nucl. Phys. B320 (1989) 479

[20] V.G. Knizhnik, and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83

[21] D. Karabali, Q-H. Park, H. Schnitzer, and Z. Yang, Phys. Lett. B216 (1989) 307

[22] I.A. Batalin and E.S. Fradkin, Nucl. Phys. B279 (1987) 514

[23] V.G. Kac and D.H. Peterson, Adv. in Math. 53 (1984) 125

[24] I.B. Frenkel, M.Garland and G.J. Zuckermann, Proc. Natl. Acad. Sci. 83 (1986) 8442

[25] T. Kugo, and I. Ojima, Suppl. Progr. Theor. Phys. No.66, (1979)

[26] V.G. Kac, and D.A. Kazhdan, Adv. in Math. 34 (1979) 97

[27] V.G. Kac, Funct. Anal. Appl. 8 (1974) 68.

[28] C.B. Thorn, Phys. Rep. 175 (1989) 1

[29] K. Huito, D. Nemechansky and S. Yankielowicz, N=2 supersymmetry, coset models and characters, Univ. of Southern California preprint USC-90/010

[30] J. Distler and Z. Qui, Nucl. Phys. B336 (1990) 145

[31] A. Rocha-Caridi, in Vertex operators in mathematics and physics, ed. by J. Leopowsky et al. MSRI Publications No.3, p.51, Springer Verlag 1984

[32] P. Bowcock and P. Goddard, Nucl. Phys. B305 [FS23] (1988) 685