Visco-elastodynamics at large strains Eulerian

Tomáš Roubíček

Mathematical Institute, Charles University,
Sokolovská 83, CZ-186 75 Praha 8, Czech Republic,
and
Institute of Thermomechanics of the Czech Academy of Sciences,
Dolejškova 5, CZ-182 00 Praha 8, Czech Republic

Abstract

Isothermal visco-elastodynamics in the Kelvin-Voigt rheology is formulated in the spatial Eulerian coordinates in terms of velocity and deformation gradient. A generally nonconvex (possibly also frame-indifferent) stored energy is admitted. The model involves a nonlinear 2nd-grade non-simple (multipolar) viscosity so that the velocity field is well regular. To simplify analytical arguments, volume variations of the solid material are assumed to be only rather small so that the mass density is constant, exploiting the concept of semi-compressible materials. Existence of weak solutions is proved by using the Galerkin method combined with a suitable regularization, using nontrivial results about transport by smooth velocity fields.

Mathematics Subject Classification. 35K55, 35Q74, 74A30, 74B20, 74H20, 76A10.

Keywords. Elastodynamics, Kelvin-Voigt rheology, spatial coordinates, global weak solutions.

1 Introduction

Dynamics of deformable elastic or viscoelastic bodies at large strains is one of basic problems in continuum mechanics. In spite of long-lasting intensive effort, there still does not exist a reasonable analytical theory for existence of reasonably defined solutions to such problems. Actually, existence of global weak solutions was articulated as an open problem by J.M. Ball in [3, Problem 12] or [4, Sect. 1.5]. There seems to be an agreement that the basic model for simple purely elastic material at large strains is not amenable for rigorous analysis and some dissipative mechanisms and higher gradients (so-called non-simple material concept) are inevitable.

In the usual notation of deformation \( y : \mathbb{R}^d \to \mathbb{R}^d \) considered here with \( d = 2 \) or \( 3 \), one distinguishes the Lagrangian (referential) coordinates \( X \in \mathbb{R}^d \) and the Eulerian (i.e. actual space) coordinates \( x = y(t, X) \). In the Lagrangian coordinates, the deformation formulation of the elastodynamics is \( \frac{\partial^2}{\partial t^2} y = \text{div} x S \) with \( S = \varphi'(F) \) the Piola-Kirchhoff stress tensor and \( F = \nabla_X y \) is the deformation gradient. Restricting on the concept of so-called hyperelastic materials, the (generalized) Hooke’s law \( S = \varphi'(F) \) here involves the stored energy \( \varphi : \mathbb{R}^{d \times d} \to \mathbb{R} \). This energy is to be subjected to various requirements, namely frame-indifference and possibly also a blow-up to \(+\infty\) if \( \det F \to 0^+ \). The latter requirement will however not be covered by the presented analysis, cf. Remarks. The former requirement excludes in particular convexity of \( \varphi \), although it complies with polyconvexity, i.e. convexity in all subdeterminants of the deformation gradient. The polyconvexity is well applicable for static problems but, in the dynamical situations, polyconvex energies \( \varphi \) do not lead to any reasonably generalized monotonicity of \(-\text{div} \varphi'(\nabla \cdot)\) like monotonicity on curl-free tensor fields, so this concept unfortunately does not seem helpful.

1This research was partially supported also from the MŠMT ČR (Ministry of Education of the Czech Republic) project CZ.02.1.01/0.0/0.0/15-003/0000493 and the institutional support RVO: 61388998 (ČR).
We will be interested in deformations evolving in time (which are sometimes called “motions”). The formulation in terms of the velocity and the deformation gradient (briefly $v/F$ formulation) in the Lagrangian description writes as \( \frac{\partial}{\partial t} F = \nabla_x v \) and \( \frac{\partial}{\partial t} v = \text{div}_X S \) with \( S = \varphi'(F) \). It was largely scrutinized in [12, 13, 14, 27], exploiting the concept of a so-called measure-valued solution. In the mere deformation formulation \( \varphi_x \frac{\partial}{\partial t} y = \text{div}_X \varphi'(F) \), see also [32]. Mostly, the Kelvin-Voigt viscoelasticity rheology is used and always, various quite restrictive assumptions must be imposed. In contrast to conventional weak solutions, this concept may, however, be unacceptably nonselective if not accompanied by some other attributes, cf. the discussion in [15, Sect. 8.3]. Alternative results concern local-in-time solutions [10, 25]. Other attempts exploit a stress relaxation, i.e. \( \epsilon \) replaced by \( \text{div}_X \varphi'(F) \) while standardly \( \varphi_0 \), cf. [29], or a nonlocal (so-called peridynamic) variant of Hooke’s law, cf. [15] or the comparison with the conventional local approach in [16].

An alternative approach, used mostly for fluids and considered rather analytically even more difficult for solids, is to use the Eulerian description, exploiting the actual deforming configuration, i.e. the coordinate \( x = y(t,X) \). Then the velocity reads as \( v = \dot{x} \) with the dot-notation (\( \cdot \)) denoting the convective (also called material) time derivative. The chain rule gives the spatial gradient \( \nabla_x v = \nabla_x \dot{y} \nabla_x X = \dot{F} F^{-1} \), where we used \( F^{-1} = (\nabla_x X)^{-1} = \nabla_x X \). As we will focus here on this Eulerian description, we will omit the subscript \( x \). In other words, it gives the transport-and-evolution equation for the deformation-gradient tensor

\[
\dot{F} := \frac{\partial F}{\partial t} + (v \cdot \nabla) F = (\nabla v) F
\]

and the \( v/F \)-formulation of the elastodynamics turns into \( \rho \dot{v} = \text{div} \Sigma \) with the Cauchy stress \( \Sigma \). One can also use an Eulerian formulation in terms of \( v \) and \( F^{-1} \), as e.g. in [9, 47, 53]. The transport-and-evolution rule \( (1.1) \) was used in incompressible models with quadratic stored energies in [9, 23, 24, 30, 31, 32] or with convex stored energies in [6, 26], i.e. models at small strains. The understanding of \( (1.1) \) is a bit delicate because it mixes the Eulerian \( x \) and the Lagrangian \( X \); note that \( \nabla v = \nabla_x v(x) \) while standardly \( F = \nabla_x y = F(X) \). In fact, we consider \( F \circ \xi \) where \( \xi : x \mapsto y^{-1}(t,X) \) is the so-called return (sometimes called also a reference) mapping. Thus \( F \) depends on \( x \) and \( (1.1) \) is an equality for a.a. \( x \). The reference mapping \( \xi \), which is well defined through its transport equation \( \dot{\xi} = 0 \), actually does not explicitly occur in the formulation of the problem. Here we will benefit from the boundary condition \( v \cdot n = 0 \) below, which causes that the actual domain \( \Omega \) does not evolve in time. The same concerns \( T \) in (2.1a) below, which will make the problem indeed fully Eulerian, as announced in the title itself.

In general large-strain situations, usually involvement of some viscosity-like dissipative mechanisms can make analysis more promising. To this goal, the Kelvin-Voigt rheology is most efficient. The elastic stress (depending on \( F = \nabla y \)) is then enhanced by a contribution depending also on the rate \( \nabla v \). In the Lagrangian description, this “viscous” stress is of the form \( S_v = \Sigma_v(F, \nabla v) \) with a function \( \Sigma_v : (R^{d \times d})^2 \rightarrow R^{d \times d} \) which should be very nonlinear due to a frame-indifference principle, as pointed out by S.S. Antman [11, cf. also [23, Sect. 9.3] for an analysis. In the Eulerian description, incorporation of the Kelvin-Voigt viscosity is simpler by putting \( S_v = \Sigma_v(e(v)) \) with a monotone \( \Sigma_v : R^{d \times d}_{\text{sym}} \rightarrow R^{d \times d}_{\text{sym}} \) with \( e(v) = \frac{1}{2} \sqrt{v^T + \frac{1}{2} \sqrt{v}} \).

It is a general understanding that both the Lagrangian and the Eulerian descriptions of both the elastodynamics and visco-elastodynamics at large strains are not amenable to a reasonable analysis in the sense of mere global existence of weak solutions, and that involvement of some gradient theories seems inevitable. In the visco-elastic case, there are naturally two basic options how to impose some
higher-order terms: to the conservative (elastic) part or to the dissipative (viscous) part, or to both. In the Lagrangian deformation description, the higher-order deformation gradients are analyzed in [28, Sect. 9.2-3] or [11]. In the Eulerian $v/F$-formulation, there are even three options how to involve higher gradients: in addition to the two mentioned options in the Lagrangian variant, one can also regularize the transport equation (1.1) like (2.8b) or (3.11b) below.

The main philosophy of the model below is not to corrupt the ultimate geometrical relation (1.1) governing the transport of the strain variable $F$ by the velocity field $v$. To this goal, we consider a multipolarly dissipative solid without any stress-diffusion and without strain gradient. In Section 2 we formulate the initial-boundary-value for such model and identify formally its energetics. Then, in Section 3 we prove existence of global weak solutions using the Galerkin method combined with a suitable regularization. Various modifications and open problems are mentioned, too.

2 Semi-compressible mass-density-homogeneous visco-elastodynamics

We adopt two assumptions simplifying considerably the analysis and simultaneously not excluding interesting applications. In particular, they weaken the often imposed incompressibility assumptions $\text{div} \; v = 0$ or $\text{det} \; F = 1$ with constant mass density $\rho$, cf. Remarks [11] and [5]. The incompressible models, although well applicable in many situations, are physically questionable e.g. because they do not facilitate propagation of pressure (longitudinal) waves. On the other hand, the fully compressible models are devised rather for gases where e.g. pressure cannot be negative and zero pressure is related with zero mass density. In contrast to the incompressible or fully compressible situations, the semi-compressible models (as devised in [46, Sect. 5] for small-strain cases in fluids) allow for propagation and dispersion also of pressure waves (beside shear waves), and are suitable for slightly compressible solids or liquids without substantial mass density variations. This situation is related with pressures much lower than the elastic bulk modulus. This modulus is typically quite high in many solid and liquid materials (e.g. water about 2.2 GPa, magma or rocks about 10 GPa, steel more than 100 GPa), but anyhow considering such materials incompressible (i.e. bulk modulus infinity) would completely suppress effects (like the mentioned pressure-wave propagation) which sometimes are of interests. Simultaneously, the thermomechanical consistency (as energy balance or frame indifference) is kept so that this compromising simplification is an acceptable modelling short-cut in particular because many analytical technicalities are avoided, cf. also Remark [11].

We will consider the following visco-elastodynamic system in the $v/F$-formulation:

\begin{align}
\partial_t \rho \dot{v} &= \text{div} (T + D) - \frac{\rho}{2} (\text{div} \; v) \; v + f \quad \text{with} \quad T = \varphi'(F)F^\top + \varphi(F)I \\
\quad &\quad \text{and} \quad D = \mathbb{D}e(v) - \text{div} ((\nu/\sqrt{e(v)})^{p-2} \nabla e(v)) \\
\dot{F} &= (\nabla v)F \\
\quad &\quad \text{where} \quad \dot{v} = \frac{\partial v}{\partial t} + (v \cdot \nabla) v \quad \text{and} \quad \dot{F} = \frac{\partial F}{\partial t} + (v \cdot \nabla) F,
\end{align}

where $T$ is the conservative part of the Cauchy stress, while $D$ is its dissipative part which contains the viscous-moduli tensor $\mathbb{D}$ and also the contribution of the so-called hyperstress $\nu/\sqrt{e(v)}^{p-2} \nabla e(v)$ with some coefficient $\nu > 0$ assumed to be small. The term $-\frac{\rho}{2} (\text{div} \; v) \; v$ in the first equation (2.1a) is a force introduced by Temam [48] rather for numerical purposes to balance the energy. Actually, it violates the Galilean invariance, as pointed out in [49] where a certain justification can be found. Yet, this Temam’s force is presumably very small in media which are only very slightly compressible, so-called quasi-incompressible. This is the price for a simplification that the mass density $\rho$ is constant,
cf. Remark 1 for a full model. In (2.1), the notation "\cdot" denotes the scalar products of vectors and later "\cdot : \cdot" and "\cdot \cdot \cdot" will be used for the scalar products of matrices and 3rd-order tensors, respectively.

Together with the standard linear viscous stress \(\mathbb{D}\varepsilon(v)\) in (2.1a) we have employed the concept of the so-called nonsimple fluids, devised by E. Fried and M. Gurtin [18] and earlier, even more generally and nonlinearly as multipolar fluids, by J. Nečas at al. [3, 37, 38] or solids [41, 52]. More specifically, we use 2nd-grade nonsimple fluids, also called bipolar fluids, in a nonlinear variant. Here it leads to the viscous hyperstress \(v'\nabla\varepsilon(v)|^{p-2}\nabla\varepsilon(v)\); the preposition "hyper" means that it contributes to the stress through its divergence. Here, the goal is to ensure \(\nabla v \in L^1_w(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))\), cf. in particular the estimates (3.1), (3.6), or (3.8) below.

We will consider a fixed bounded Lipschitz domain \(\Omega \subset \mathbb{R}^d\) with the boundary \(\Gamma\). Then, the system of two nonlinear parabolic equations (2.1) should be accompanied by some boundary conditions, e.g.

\[
\mathbf{v} \cdot \mathbf{n} = 0, \quad [(\mathbf{T} + \mathbf{D}) \mathbf{n} + \text{div}_s (\nu |\nabla \varepsilon(v)|^{p-2}(\nabla \varepsilon(v)) \mathbf{n})]_T = \mathbf{g}, \quad \nabla \varepsilon(v) \cdot (\mathbf{n} \otimes \mathbf{n}) = 0, \tag{2.2}
\]

with \(\mathbf{n}\) denoting the unit outward normal to \(\Gamma\), \([\cdot]_T\) the tangential component of a vector, i.e. \([\sigma]_T = \sigma - (\sigma \cdot \mathbf{n}) \mathbf{n}\) for a vector \(\sigma\). In (2.2), \(\text{div}_s = \text{tr}(\nabla_s)\) denotes the \((d-1)\)-dimensional surface divergence with \(\text{tr}(\cdot)\) being the trace of a \((d-1)\times(d-1)\)-matrix and \(\nabla_s \mathbf{v} = \nabla \mathbf{v} - \frac{\partial}{\partial \mathbf{n}} \mathbf{n}\) being the surface gradient of \(\mathbf{v}\). Naturally, \(\mathbf{g} \cdot \mathbf{n} = 0\) is to be assumed if we want to recover the boundary conditions (2.2) in the classical form, otherwise the weak form does not directly need it.

To reveal at least formally the energetics behind the system (2.1), one should test (2.1a) by \(\mathbf{v}\) and (2.1b) by \(\mathbf{S}\). We use several calculations exploiting integration over \(\Omega\) and Green’s formula.

For the convective term in (2.1a) we use the calculus

\[
\int_{\Omega} \varrho (\mathbf{v} \cdot \nabla) \mathbf{v} \cdot \mathbf{v} \, dx = \int_{\Gamma} \varrho |\mathbf{v}|^2 (\mathbf{v} \cdot \mathbf{n}) \, dS - \int_{\Omega} \mathbf{v} \nabla (\varrho \mathbf{v} \otimes \mathbf{v}) \, dx
\]

\[
= \int_{\Omega} \varrho |\mathbf{v}|^2 (\mathbf{v} \cdot \mathbf{n}) \, dS - \int_{\Omega} \varrho |\mathbf{v}|^2 \text{div} \, \mathbf{v} + \varrho \mathbf{v} \cdot \nabla \mathbf{v} - |\mathbf{v}|^2 (\nabla \varrho \cdot \mathbf{v}) \, dx
\]

\[
= \int_{\Gamma} \frac{\varrho}{2} |\mathbf{v}|^2 (\mathbf{v} \cdot \mathbf{n}) \, dS - \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 (\text{div} \, \mathbf{v}) + \frac{1}{2} |\mathbf{v}|^2 (\nabla \varrho \cdot \mathbf{v}) \, dx = \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 (\text{div} \, \mathbf{v}) \, dx \tag{2.3}
\]

which also uses the first boundary condition (2.2) and the assumption that \(\varrho\) is constant so that \(\nabla \varrho = 0\). Here the role of Temam’s bulk force \(\frac{1}{2} \varrho \text{div} \mathbf{v} \mathbf{v}\) in (2.1a) is revealed.

Furthermore, we use the algebra \(\mathbf{S} : (\mathbf{V} \mathbf{F}) = (\mathbf{S} \mathbf{F})^\top : \mathbf{V}\) for \(\mathbf{V} = \nabla \mathbf{v}\) and \(\mathbf{S} = \varphi' (\mathbf{F})\) and the following calculus

\[
\int_{\Omega} \mathbf{T} : \nabla \mathbf{v} \, dx = \int_{\Omega} \left( \varphi' (\mathbf{F}) \mathbf{F}^\top + \varphi(\mathbf{F}) \mathbf{I} \right) : \nabla \mathbf{v} \, dx = \int_{\Omega} (\nabla \mathbf{F})^\top : \varphi' (\mathbf{F}) + \varphi(\mathbf{F}) \text{div} \, \mathbf{v} \, dx
\]

\[
= \int_{\Omega} \varphi(\mathbf{F}) \partial_t \mathbf{F} + \varphi(\mathbf{F}) \text{div} \, \mathbf{v} \, dx = \int_{\Omega} \varphi' (\mathbf{F}) : \frac{\partial \mathbf{F}}{\partial t} + \varphi' (\mathbf{F}) : (\mathbf{v} \cdot \nabla) \mathbf{F} + \varphi(\mathbf{F}) \text{div} \, \mathbf{v} \, dx
\]

\[
= \frac{d}{dt} \int_{\Omega} \varphi(\mathbf{F}) \, dx + \int_{\Gamma} \varphi(\mathbf{F}) \frac{\mathbf{v} \cdot \mathbf{n}}{\mathbf{n}} \, dS, \tag{2.4}
\]

where we used also the Green formula for

\[
\int_{\Omega} \nabla \varphi(\mathbf{F}) \cdot \mathbf{v} + \varphi(\mathbf{F}) \text{div} \, \mathbf{v} \, dx = \int_{\Gamma} \text{div} (\varphi(\mathbf{F}) \mathbf{v}) \, dx = \int_{\Gamma} \varphi(\mathbf{F}) \mathbf{v} \cdot \mathbf{n} \, dS = 0,
\]

exploiting the boundary condition \(\mathbf{v} \cdot \mathbf{n} = 0\). For the term \(\text{div}^2 (|\nabla \varepsilon(v)|^{p-2}\nabla \varepsilon(v))\) tested by \(\mathbf{v}\), we use twice Green’s formula together with a surface Green formula and the boundary conditions (2.2),
The relation between $\varphi$ in (2.1a) is related to that the stored energy ensures and omitting the extra compensating force because the concept of nonlinear nonsimple material equation with an initial condition $\varrho_{t=0} = \varrho_0$ or, equivalently, the algebraic relation $\varrho = \varrho_0 / \det F$ and omitting the extra compensating force because the concept of nonlinear nonsimple material ensures $\nabla \varrho$ in $L^p_w(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$ which in turn allows for estimation of $\nabla \varrho$, cf. also Remark 6 below.

Remark 1 (Varying mass density). The above model used the simplification based on the assumption of a constant mass density $\varrho$, cf. the calculus [23], and neglects its variations during volumetric deformation (which is typically indeed small in liquids and in solids, too, in contrast to gases). In fact, the initial (possibly spatially nonconstant) density should evolve in time by the continuity equation $\varrho = -\varrho \div \mathbf{v}$. In the context of solids in Eulerian description, see also [21, 24, 32, 33, 47]. Having omitted this continuity equation has been here compensated by the extra force $-\varrho (\div \mathbf{v}) \mathbf{v}/2$ in (2.1a), which is presumably very small. This simplifies a lot of calculations and analytical arguments. Here, the analysis would likely be doable for the full model involving also the continuity equation with an initial condition $\varrho_{t=0} = \varrho_0$ or, equivalently, the algebraic relation $\varrho = \varrho_0 / \det F$ and omitting the extra compensating force because the concept of nonlinear nonsimple material ensures $\nabla \varrho$ in $L^p_w(I; L^\infty(\Omega; \mathbb{R}^{d \times d}))$ which in turn allows for estimation of $\nabla \varrho$, cf. also Remark 6 below.

Remark 2 (The Cauchy stress $\mathbf{T}$ alternatively). The pressure contribution to the Cauchy stress $\mathbf{T}$ in (2.1a) is related to that the stored energy $\varphi$ is counted truly Eulerian, i.e. per actual volume. Another approach is to count the stored energy per the referential volume (let us denote it by $\varphi_r$ in (2.1a)). In terms of $\varphi_r$, the conservative part of the Cauchy stress $\mathbf{T}$ transforms to

$$\mathbf{T} = \varphi' (\mathbf{F}) \mathbf{F}^\top + \varphi (\mathbf{F}) \mathbf{I} = \left( \frac{\varphi'_r (\mathbf{F})}{\det \mathbf{F}} - \frac{\varphi_r (\mathbf{F}) \text{Cof} \mathbf{F}}{(\det \mathbf{F})^2} \right) \mathbf{F}^\top + \frac{\varphi_r (\mathbf{F})}{\det \mathbf{F}} \mathbf{I} \quad \text{(2.7)}$$

Then $\varphi (\mathbf{F})$ in (2.5) should be replaced by $\varphi_r (\mathbf{F}) / \det \mathbf{F}$, cf. e.g. [19] or [20, Ch.7]. This variant would need to control $\det \mathbf{F}$, which we avoided in this paper, cf. also Remark 6 below for a possible modification.

Remark 3 (Conservative multipolar variant with stress diffusion). The multipolar concept can alternatively use a conservative higher gradient instead of the dissipative higher-order term $\text{div}(\nu |\nabla \mathbf{v}|^{p-2} \nabla \mathbf{v})$ in (2.1a), as often occurs in literature since the work by R.A. Toupin [50] and R.D. Mindlin [36], cf. e.g. [17, 40]. Yet, this is more complicated because such term is reflected in a nonlinear (Korteweg-like) symmetric capillarity stress $\mathbf{K}$. In addition, for analytical reasons, the transport-and-evolution equation (2.1b) for $\mathbf{F}$ must be enhanced by stress diffusion. This leads to the visco-elastodynamic system

$$\varrho \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = \text{div} (\mathbf{T} + D + \mathbf{K}) - \frac{\varrho}{2} (\div \mathbf{v}) \mathbf{v} + \mathbf{f}, \quad \text{where} \quad \mathbf{T} = S \mathbf{F}^\top + \varphi (\mathbf{F}) \mathbf{I},$$

$$S = \varphi' (\mathbf{F}) - \kappa \Delta \mathbf{F}, \quad D = \mathbf{D}e (\mathbf{v}), \quad \text{and} \quad \mathbf{K} = \frac{\kappa}{2} |\nabla \mathbf{F}|^2 \mathbf{I} - \kappa \nabla \mathbf{F} \otimes \nabla \mathbf{F}, \quad \text{(2.8a)}$$

Further, we prescribe the initial conditions, i.e.

$$\mathbf{v} |_{t=0} = \mathbf{v}_0 \quad \text{and} \quad \mathbf{F} |_{t=0} = \mathbf{F}_0. \quad \text{(2.6)}$$

cf. also [44, Sect. 2.4.4] for technical details. This procedure yields (at least formally) the energy balance:

$$\frac{d}{dt} \int_{\Omega} \frac{\varrho}{2} |\mathbf{v}|^2 + \varphi (\mathbf{F}) \, d\mathbf{x} + \int_{\Omega} \mathbf{D}e (\mathbf{v}) : e (\mathbf{v}) + \nu |\nabla \mathbf{v}|^2 \, d\mathbf{x} = \int_{\Omega} \mathbf{f} : \mathbf{v} \, d\mathbf{x} + \int_{\Gamma} \mathbf{g} : \mathbf{v} \, dS. \quad \text{(2.5)}$$

5
\[ \frac{\partial F}{\partial t} + (\mathbf{v} \cdot \nabla) F = (\nabla \mathbf{v}) F + \epsilon \Delta S \]  

with the boundary conditions \( \mathbf{v} \cdot \mathbf{n} = 0 \), \((\mathbf{v} \cdot \nabla) F = 0 \), and \((\mathbf{v} \cdot \nabla) \mathbf{S} = 0 \). The capillarity coefficient \( \kappa > 0 \) and a stress-diffusion coefficient \( \epsilon > 0 \) are presumably small. Corrupting the "geometrical" transport-and-evolution rule \((1.1)\) in \((2.8b)\) might be rather controversial and purely mathematically motivated, as actually openly articulated in \([6, 26]\) for the variant of \( \Delta F \) instead of \( \Delta \mathbf{S} \). Anyhow, in fluid dynamics, the stress diffusion \( \epsilon \Delta \mathbf{S} \) was advocated in series of works by H. Brenner, cf. e.g. \([7, 8]\). Cf. also a discussion in \([39]\) and a thermodynamical justification in \([51]\). Here, \((2.8b)\) with \( \mathbf{S} = \varphi'(\mathbf{F}) - \kappa \Delta \mathbf{F} \) is formally rather Cahn-Hilliard’s diffusion of \( \mathbf{F} \). The energy balance is again obtained by the test of \((2.8a)\) by \( \mathbf{v} \) and of \((2.8b)\) by \( \mathbf{S} \). Using the calculus \([39]\) below, we obtain the energy balance

\[
\frac{d}{dt} \int_\Omega \left( \frac{\theta}{2} |\mathbf{v}|^2 + \frac{\kappa}{2} |\nabla \mathbf{F}|^2 \right) dx + \int_\Omega \mathbf{D} \epsilon(\mathbf{v}) : \epsilon(\mathbf{v}) + \epsilon |\nabla \mathbf{S}|^2 dx = \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dx + \int_\Gamma g \cdot \mathbf{v} \, dS.
\]

From this, one can read the estimate \( \mathbf{S} \in L^2(I; H^1(\Omega; \mathbb{R}^{d \times d})) \). Then, assuming \( \varphi'' \) bounded, one has \( \kappa \Delta (\nabla \mathbf{F}) = \nabla (\varphi'') (\mathbf{F} - \mathbf{S}) + \varphi'' (\nabla \mathbf{F} - \nabla \mathbf{S}) \in L^2(I \times \Omega; \mathbb{R}^{d \times d}) \) and, assuming \( \Omega \) smooth, one can further use the \( H^2 \)-regularity for \( \kappa \Delta \), so that \( \nabla \mathbf{F} \in L^2(I; H^2(\Omega; \mathbb{R}^{d \times d})) \). By comparison from \((2.8b)\) \( \frac{\partial}{\partial t} \mathbf{F} = (\nabla \mathbf{v}) \mathbf{F} - (\mathbf{v} \cdot \nabla) \mathbf{F} + \epsilon \Delta \mathbf{S} \) belongs to \( L^2(I; L^2(\Omega; \mathbb{R}^{d \times d}) + H^1(\Omega; \mathbb{R}^{d \times d})) \) so that, by Aubin-Lions’ theorem, \( \nabla \mathbf{F} \) and \( \nabla \mathbf{F} \otimes \nabla \mathbf{F} \) are “compact”, which is needed for the capillarity-type stress \( \mathbf{K} \). Such alternative gradient model in some sense improves some previous strain-diffusion models \([6, 26]\).

### 3 Analysis of the model \((2.1)\)

We now prove existence of weak solutions by a constructive method, i.e. by a suitable approximation and its convergence. This is rather technical in general. The time discretisation (Rothe’s method) standardly needs convexity of \( \varphi \) (which is not a realistic assumption here) possibly weakened if there is some viscosity in \( \mathbf{F} \) (which is not directly considered here, however). The conformal space discretisation (the Faedo-Galerkin method) cannot directly copy the energetics because the test of \((2.1)\) by \( \mathbf{S} \) is problematic in this approximation as \( \mathbf{S} = \varphi'(\mathbf{F}) \) is not in the respective finite-dimensional space in general. We facilitate the analytical issue here by imposing rather strong growth assumption \((3.1a)\) on \( \varphi \) below, which might be quite restrictive, cf. Remark \([6]\).

We will use the standard notation concerning the Lebesgue and the Sobolev spaces on the domain \( \Omega \subset \mathbb{R}^d \), as actually already employed in Remark \([3]\). Namely, for \( n \in \mathbb{N} \), \( L^p(\Omega; \mathbb{R}^n) \) denotes the Banach spaces of Lebesgue measurable functions \( \Omega \to \mathbb{R}^n \) whose Euclidean norm is integrable with \( p \)-power, and \( W^{k,p}(\Omega; \mathbb{R}^n) \) the space of functions from \( L^p(\Omega; \mathbb{R}^n) \) whose derivatives of the order \( k \) are in \( L^p(\Omega; \mathbb{R}^{n \times k d}) \). If \( n = 1 \), we will write simply \( L^p(\Omega) \) or \( W^{k,p}(\Omega) \). Moreover, \( W^{2,p}_0(\Omega; \mathbb{R}^d) := \{ \mathbf{v} \in W^{2,p}(\Omega; \mathbb{R}^d); \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \} \). We also write briefly \( H^k = W^{k,2} \). Moreover, for a Banach space \( X \) and for \( I = [0, T] \), we will use the notation \( L^p(I; X) \) for the Bochner space of Bochner measurable functions \( I \to X \) whose norm is in \( L^p(I) \), and \( H^1(I; X) \) for functions \( I \to X \) whose distributional derivative is in \( L^2(I; X) \). Moreover, \( (\cdot)^* \) will denote the dual space and \( p' = p/(p-1) \) is the conjugate exponent with the convention \( p' = \infty \) for \( p = 1 \). Occasionally, we will use \( L^p_w(I; X) \) for weakly* measurable functions \( I \to X \) for nonseparable spaces \( X \) which are duals to some other Banach spaces (specifically for \( L^\infty(\Omega) \)).
Let us first summarize the assumptions:

\[ \varphi : \mathbb{R}^{d \times d} \to [0, +\infty] \text{ continuously differentiable and } \]
\[ \exists \ell \in \mathbb{R} \forall F \in \mathbb{R}^{d \times d} : \ 0 \leq \varphi(F) \leq \ell (1 + |F|) \quad \text{and} \quad |\varphi'(F)| \leq \ell, \quad (3.1a) \]
\[ f \in L^1(I; L^2(\Omega; \mathbb{R}^d)) + L^2(I; L^1(\Omega; \mathbb{R}^d)), \quad g \in L^2(I; L^1(\Gamma; \mathbb{R}^d)), \quad (3.1b) \]
\[ v_0 \in L^2(\Omega; \mathbb{R}^d), \quad F_0 \in H^1(\Omega; \mathbb{R}^{d \times d}), \quad (3.1c) \]
\[ \rho, \nu > 0, \ D \in (\mathbb{R}^{d \times d})^d \text{ positive definite, } p > d. \quad (3.1d) \]

Beside, it is also reasonable to assume the frame indifference, i.e. \( \varphi(F) = \varphi(QF) \) for any \( Q \in SO(d) = \{ A \in \mathbb{R}^{d \times d} : A^T A = AA^T = I, \ det A = 1 \} \). It is equivalent to \( \varphi(F) = \phi(F^T F) \) for some \( \phi \) so that \( T = \varphi(F)F^T = 2F\phi(F^T F)F^T \) and thus the whole Cauchy stress \( T + D \) is symmetric. Yet, we will not explicitly need it for our analysis.

For the definition of a weak solution, we do not need to impose a priori smoothness of \( F \) when using the calculus \( \int_\Omega (v \cdot \nabla)F \cdot \tilde{F} \, dx = \int_\Omega F : (\text{div } v) \tilde{F} + (v \cdot \nabla) \tilde{F} \, dx \), although later we will prove even quite high regularity of \( F \). Like already in Section 2, for the term \( \text{div}^2(|\nabla e(v)|^{p-2} \nabla e(v)) \) tested by \( \tilde{v} \) with \( \tilde{v} \cdot n = 0 \), we will use twice the Green formula on \( \Omega \) together with a surface Green formula on \( \Gamma \) together with the boundary conditions \( (2.2) \). In addition, we use the by-part integration in time. This results to:

**Definition 1** (Weak solutions to \( (2.1) - (2.2) \)). A couple \((v, F)\) with \( v \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \cap H^1(I; L^2(\Omega; \mathbb{R}^d)) \) with \( v \cdot n = 0 \) on \( I \times \Gamma \) and \( F \in L^2(I \times \Omega; \mathbb{R}^{d \times d}) \) is called a weak solution to the initial-boundary-value problem \( (2.1) - (2.2) \) with \( (2.6) \) if the integral identity

\[ \int_0^T \int_\Omega \left( \varphi'(F) F^T : \nabla \tilde{v} + \left( \frac{\rho}{2} \text{div } v + \rho (v \cdot \nabla) v \right) \tilde{v} - \rho v \cdot \frac{\partial \tilde{v}}{\partial t} - \varphi(F) \text{div } \tilde{v} \right) + \nu |\nabla e(v)|^{p-2} \nabla e(v) \cdot \nabla e(\tilde{v}) \right) \, dx \, dt = \int_0^T \int_\Omega f \cdot \tilde{v} \, dx \, dt + \int_0^T \int_\Gamma g \cdot \tilde{v} \, dx \, dt + \int_\Omega \tilde{v}_0 \cdot \tilde{v}(0) \, dx \quad (3.2a) \]

holds for any \( \tilde{v} \in L^p(I; W^{2,p}(\Omega; \mathbb{R}^d)) \cap H^1(I; L^2(\Omega; \mathbb{R}^d)) \) with \( \tilde{v}(T) = 0 \) and \( \tilde{v} \cdot n = 0 \) on \( I \times \Gamma \), and also the integral identity

\[ \int_0^T \int_\Omega \left( \nabla v \cdot F \cdot \tilde{F} + (\text{div } v) F \cdot \tilde{F} + F : (v \cdot \nabla) \tilde{F} + F : \frac{\partial \tilde{F}}{\partial t} \right) \, dx \, dt + \int_\Omega F_0 : \tilde{F}(0) \, dx = 0 \quad (3.2b) \]

holds for any \( \tilde{F} \in W^{1,\infty}(I \times \Omega; \mathbb{R}^{d \times d}) \) with \( \tilde{F}(T) = 0 \).

**Theorem 1** (Existence of weak solutions). Let \( (3.1) \) hold. Then:

(i) There exists a weak solution \((v, F)\) according to Definition 1 such that also \( F \in L^\infty(I; H^1(\Omega; \mathbb{R}^{d \times d})) \), the transport-and-evolution rule \( (2.1b) \) holds a.e. on \( I \times \Omega \), and also the energy balance \( (2.5) \) integrated over a time interval \([0, t]\) is satisfied for all \( t \in I \).

(ii) If \( \varphi \) is also twice continuously differentiable with \( \varphi'' \) bounded, then \( S = \varphi'(F) \in L^\infty(I; H^1(\Omega; \mathbb{R}^{d \times d})) \).

**Proof.** For clarity, we divide the proof into four steps.

**Step 1: Galerkin approximation of a regularized problem.** Instead of \( (2.1b) \), we will consider in this step a regularization as in \( [6, 26] \):

\[ \frac{\partial F}{\partial t} + (v \cdot \nabla) F = (\nabla v) F + \varepsilon \Delta F \quad (3.3) \]
with the boundary conditions (2.2) enhanced also by other \(d \times d\) conditions \((\mathbf{n} \cdot \nabla) \mathbf{F} = 0\). The weak solution is defined analogously as in Definition 1 but with (3.2b) enhanced by the term \(-\varepsilon \nabla \mathbf{F} \cdot \nabla \mathbf{F}\).

We will then use the Faedo-Galerkin method exploiting a nested sequence of finite-dimensional sequences \(V_k \subset W_0^{2,\infty}(\Omega)\) such that \(\bigcup_{k\in\mathbb{N}} V_k\) is dense in \(W_0^{2,p}(\Omega)\) and composed with eigenfunctions of the self-adjoint \(\Delta\)-operator with the homogeneous Neumann boundary condition; recall the notation \(W_0^{2,p}(\Omega)\) used here for vectorial functions with normal boundary traces zero. This ensures that \(\Delta \mathbf{v} \in V_k\) provided \(\mathbf{v} \in V_k\). We also approximate the initial conditions: \(\mathbf{v}_{0,k} \in V_k^d\) and \(\mathbf{F}_{0,k} \in V_k^{d \times d}\) so that \(\{\mathbf{v}_{0,k}\}_{k \in \mathbb{N}}\) is bounded in \(L^2(\Omega; \mathbb{R}^d)\) and \(\{\mathbf{F}_{0,k}\}_{k \in \mathbb{N}}\) is bounded in \(H^1(\Omega; \mathbb{R}^{d \times d})\) and \(\mathbf{v}_{0,k} \rightarrow \mathbf{v}_0\) and \(\mathbf{F}_{0,k} \rightarrow \mathbf{F}_0\) strongly in these spaces.

Then we consider the Galerkin approximate solution \((\mathbf{v}_{ek}, \mathbf{F}_{ek}) : I \rightarrow V_k^d \times V_k^{d \times d}\) to the regularized system (2.1a) and (3.11b). Existence of such solution is based by standard ordinary-differential-equation arguments with a successive prolongation exploiting the \(L^\infty(I)\)-estimates below.

**Step 2: estimates for the approximated regularized problem.** To derive basic a-priori estimates for such Galerkin approximation, we test (3.11b) in its Galerkin approximation by \(\mathbf{F}_{ek}\). Thus we obtain the estimate

\[
\varepsilon \int_\Omega |\nabla \mathbf{F}_{ek}|^2 \, dx + \frac{d}{dt} \int_\Omega \frac{1}{2} |\mathbf{F}_{ek}|^2 \, dx = \int_\Omega \left( (\nabla \mathbf{v}_{ek}) \mathbf{F}_{ek} - (\mathbf{v}_{ek} \nabla) \mathbf{F}_{ek} \right) \cdot \mathbf{F}_{ek} \, dx
\]

\[
= \int_\Omega (\nabla \mathbf{v}_{ek}) \mathbf{F}_{ek} \cdot \mathbf{F}_{ek} + \frac{1}{2} \left( \nabla \mathbf{v}_{ek} \right) |\mathbf{F}_{ek}|^2 \, dx \leq \frac{3}{2} \|\nabla \mathbf{v}_{ek}\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|\mathbf{F}_{ek}\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2, \tag{3.4}
\]

here we used also the calculus

\[
\int_\Omega (\mathbf{v} \cdot \nabla) \mathbf{F} : \mathbf{F} \, dx = \int_I |\mathbf{F}|^2 \cdot (\mathbf{v} \cdot \mathbf{n}) \, dS - \int_\Omega \mathbf{F} : (\mathbf{v} \cdot \nabla) \mathbf{F} + (\nabla \mathbf{v}) |\mathbf{F}|^2 \, dx = -\frac{1}{2} \int_\Omega (\nabla \mathbf{v}) |\mathbf{F}|^2 \, dx \tag{3.5}
\]

together with the boundary condition \(\mathbf{v} \cdot \mathbf{n} = 0\). Moreover, testing (2.1a) in its Galerkin approximation by \(\mathbf{v}_{ek}\), we obtain

\[
\int_\Omega \mathbf{D}(\mathbf{v}_{ek}) : \mathbf{e}(\mathbf{v}_{ek}) + \nu |\nabla \mathbf{v}_{ek}|^p \, dx + \frac{d}{dt} \int_\Omega \frac{G}{2} |\mathbf{v}_{ek}|^2 \, dx
\]

\[
\leq \int_\Omega \varphi' (\mathbf{F}_{ek}) \mathbf{F}_{ek}^T : \nabla \mathbf{v}_{ek} + \varphi(\mathbf{F}_{ek}) \mathbf{D}(\mathbf{v}_{ek}) \, dx + \int_I g \cdot \mathbf{v}_{ek} \, dS
\]

\[
\leq \int_\Omega 2\ell (1 + |\mathbf{F}_{ek}|) |\nabla \mathbf{v}_{ek}| + |\mathbf{F}_{ek}| \, dx + \int_I g \cdot \mathbf{v}_{ek} \, dS
\]

\[
\leq C_{\ell, \delta} \|\mathbf{F}_{ek}\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 + \delta \|\nabla \mathbf{v}_{ek}\|_{L^2(\Omega; \mathbb{R}^{d \times d})}^2 + \|\mathbf{f}\|_{L^2(\Omega; \mathbb{R}^d)}^2 (1 + \|\mathbf{v}_{ek}\|_{L^2(\Omega; \mathbb{R}^d)})^2 + \|\mathbf{g}\|_{L^2(\Gamma; \mathbb{R}^d)}^2 (1 + \|\mathbf{v}_{ek}\|_{L^2(\Gamma; \mathbb{R}^d)})^2
\]

\[
+ (\|\mathbf{f}\|_{L^1(\Omega; \mathbb{R}^d)}^2 + \|\mathbf{g}\|_{L^1(\Gamma; \mathbb{R}^d)}^2) \delta \|\mathbf{v}_{ek}\|_{L^\infty(\Omega; \mathbb{R}^d)}^2 + \delta \|\mathbf{v}_{ek}\|_{L^\infty(\Gamma; \mathbb{R}^d)}^2 |\mathbf{r}|_{L^\infty(\Gamma; \mathbb{R}^d)}, \tag{3.6}
\]

where \(\ell\) is from (3.1a) and where \(\mathbf{f} = \mathbf{f}_1 + \mathbf{f}_2\) with some \(\mathbf{f}_1 \in L^1(I; L^2(\Omega; \mathbb{R}^d))\) and \(\mathbf{f}_2 \in L^2(I; L^1(\Omega; \mathbb{R}^d))\) referring to (3.1b). The last terms in (3.6) are still to be estimated using Korn’s inequality and the boundedness of the embedding/trace operators for \(\max(\|\mathbf{v}\|_{L^\infty(\Omega; \mathbb{R}^d)}, \|\mathbf{v}\|_{L^\infty(\Gamma; \mathbb{R}^d)}) \leq \|\mathbf{v}\|_{L^2(\Omega; \mathbb{R}^d)} + \|\nabla \mathbf{v}\|_{L^p(\Omega; \mathbb{R}^{d \times d})}\). Then, summing (3.4) and (3.6) and using Gronwall’s inequality, we obtain the a-priori estimates

\[
\|\mathbf{v}_{ek}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^d))} \cap L^p(I; W_0^{2,p}(\Omega; \mathbb{R}^d)) \leq C, \tag{3.7a}
\]

\[
\|\mathbf{F}_{ek}\|_{L^\infty(I; L^2(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{and} \quad \|\nabla \mathbf{F}_{ek}\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d \times d})} \leq \frac{C}{\sqrt{\varepsilon}}. \tag{3.7b}
\]
Although these estimates would already allow for passage with $k \to \infty$, we will still make another a-priori estimate by testing (3.11b) in its Galerkin approximation by $\Delta F_{\varepsilon k}$. Using the Green formula with the boundary condition $(n \cdot \nabla) F_{\varepsilon k} = 0$, we obtain
\[
\varepsilon \int_{\Omega} |\Delta F_{\varepsilon k}|^2 \, dx + \frac{1}{2} \int_{\Omega} \frac{d}{dt} \int_{\Omega} \nabla F_{\varepsilon k}^2 \, dx = \int_{\Omega} \nabla \left( (\nabla \varepsilon_k \cdot \nabla) F_{\varepsilon k} - (\varepsilon_k \cdot \nabla) F_{\varepsilon k} \right) \cdot \nabla F_{\varepsilon k} \, dx
\]
\[
= \int_{\Omega} (\nabla F_{\varepsilon k} \otimes \nabla F_{\varepsilon k}) : e(v_{\varepsilon k}) - \frac{1}{2} |\nabla F_{\varepsilon k}|^2 \, div \, \varepsilon_k - (\nabla \varepsilon_k) \nabla F_{\varepsilon k} : \nabla F_{\varepsilon k} - (\nabla^2 \varepsilon_k) F_{\varepsilon k} : \nabla F_{\varepsilon k} \, dx
\]
\[
\leq \frac{5}{2} \|\nabla \varepsilon_k\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|F_{\varepsilon k}\|^2_{L^2(\Omega; \mathbb{R}^{d \times d})} + \|\nabla^2 \varepsilon_k\|_{L^p(\Omega; \mathbb{R}^{d \times d \times d})} \|F_{\varepsilon k}\|_{L^2(\Omega; \mathbb{R}^{d \times d \times d})}(1 + \|\nabla F_{\varepsilon k}\|^2_{L^2(\Omega; \mathbb{R}^{d \times d})}) ; \tag{3.8}
\]
here $\nabla F \otimes \nabla F$ denotes the symmetric matrix $[\nabla F \otimes \nabla F]_{ij} = \sum_{k,l=1}^d \frac{\partial}{\partial x_i} F_{kl} \frac{\partial}{\partial x_j} F_{kl}$. In (3.8), we used both $p \geq d$ so that $p^{-1} + (2^*)^{-1} + 2^{-1} \leq 1$ and also the calculus
\[
\int_{\Omega} \nabla ((v \cdot \nabla) F) : \nabla F \, dx = \int_{\Omega} (\nabla F \otimes \nabla F) : e(v) + (v \cdot \nabla) \nabla F : \nabla F \, dx
\]
\[
= \int_{\Gamma} |\nabla F|^2 v \cdot n \, dS + \int_{\Omega} (\nabla F \otimes \nabla F) : e(v) - (div \, v)|\nabla F|^2 - \nabla F : (v \cdot \nabla) \nabla F \, dx
\]
\[
= \int_{\Gamma} \frac{|\nabla F|^2}{2} v \cdot n \, dS + \int_{\Omega} (\nabla F \otimes \nabla F) : e(v) - (div \, v)|\nabla F|^2 - \frac{2}{2} \, dx . \tag{3.9}
\]
By Gronwall’s inequality applied to (3.8) together with the qualification of the approximate initial condition $F_{0,k}$ bounded in $H^1(\Omega; \mathbb{R}^d)$, we obtain still the a-priori estimates
\[
\|F_{\varepsilon k}\|_{L^\infty(I; H^1(\Omega; \mathbb{R}^{d \times d}))} \leq C \quad \text{and} \quad \|\Delta F_{\varepsilon k}\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d \times d})} \leq \frac{C}{\sqrt{\varepsilon}}. \tag{3.10}
\]

Step 3: limit passage with $k \to \infty$ and with $\varepsilon \to 0$. Exploiting the estimates (3.7a) and (3.10), we can select a subsequence converging in the weak* topologies $(L^\infty(I; L^2(\Omega; \mathbb{R}^d))) \cap L^p(I; W_0^{2,p}(\Omega; \mathbb{R}^d)) \times L^\infty(I; H^1(\Omega; \mathbb{R}^{d \times d}))$. The convergence in the Galerkin approximation is then routine. The highest-order nonlinear term $div^2(\nu|\nabla e(v)|^{p-2} \nabla e(v))$ is monotone and can be handled by Minty’s trick, while the other terms are linear or, in the case of $(v \cdot \nabla)v$, $(div \, v)v$, $(v \cdot \nabla)F$, $\varphi'(F)F^T$ and $\varphi(F)$, nonlinear but of lower order and thus to be handled by compactness. Here the Aubin-Lions compact-embedding theorem can be used when employing estimates of $\frac{\partial}{\partial t} \varepsilon_k$ and $\frac{\partial}{\partial t} F_{\varepsilon k}$ by comparison; cf. [44, Sect. 8.4] for adaptations of that theorem for Galerkin method. In fact, the uniform monotonicity of $v \mapsto div^2(\nu|\nabla e(v)|^{p-2} \nabla e(v))$, the weak* convergence of $\varepsilon_k$ can be improved to the strong convergence, so that the Minty trick and the Aubin-Lions theorem and the estimate of $\frac{\partial}{\partial t} \varepsilon_k$ are actually not needed. Also $\frac{\partial}{\partial t} F_{\varepsilon k} = (\nabla \varepsilon_k) F_{\varepsilon k} - (\varepsilon_k \cdot \nabla) F_{\varepsilon k}$ can be estimated bounded in $L^p(I; L^2(\Omega; \mathbb{R}^{d \times d}))$ due to (3.7a) and the first estimate in (3.10), so that the conventional version of the Aubin-Lions theorem can be used.

The regularizing term $\varepsilon \Delta F_{\varepsilon k}$ vanishes in the limit for $\varepsilon \to 0$ because $\|\varepsilon \Delta F_{\varepsilon k}\|_{L^2(I \times \Omega; \mathbb{R}^{d \times d})} = O(\sqrt{\varepsilon}) \to 0$; or, in the weak formulation even $\|\varepsilon \nabla F_{\varepsilon k}: \nabla F\|_{L^1(I \times \Omega)} = O(\varepsilon) \to 0$. Actually, we can pass to the limit simultaneously with $k \to \infty$ and with $\varepsilon \to 0$ since we do not rely on the latter estimate in (3.7a). Thus, we obtain a weak solution to the original problem (2.1).

Step 4: test by $S = \varphi'(F)$. For the continuous problem (2.1a), we can perform the physically relevant test by $v$ and $\varphi'(F)$; recall that now we have already $\varepsilon = 0$. This gives the energy balance
Here it is important that \( \partial \varphi \mathbf{v} \in L^p(I; W_0^{2,p} (\Omega; \mathbb{R}^d)^* \) + \( L^1(I; L^2(\Omega; \mathbb{R}^d)) \) is in duality with \( \mathbf{v} \in L^p(I; W_0^{2,p}(\Omega; \mathbb{R}^d)) \cap L^\infty(I; L^2(\Omega; \mathbb{R}^d)) \), so that the test of (2.1a) by \( \mathbf{v} \) is legitimate. Also, as in Step 3, we have \( \frac{\partial}{\partial t} \mathbf{F} = (\nabla \mathbf{v}) \mathbf{F} - (\mathbf{v} \nabla) \mathbf{F} \in L^p(I; L^2(\Omega; \mathbb{R}^{d \times d})) \) is surely in duality with \( \mathbf{S} = \varphi'(\mathbf{F}) \in L^\infty(I \times \Omega; \mathbb{R}^{d \times d}) \), so that the physical test of (2.1b) by \( \mathbf{S} \) is legitimate. In particular, all the integrands in (2.4) are integrable (some of them even bounded). Thus all the calculations leading to the energy balance (2.5) are not formal.

Obviously, \( \nabla \mathbf{S} = \nabla \varphi'(\mathbf{F}) = \varphi''(\mathbf{F}) \nabla \mathbf{F} \in L^\infty(I; L^2(\Omega; \mathbb{R}^{d \times d})) \) provided \( \varphi'' \) is bounded.

Remark 4 (Stored energy compliance with (3.1a)). An example of a mechanically relevant frame-indifferent stored energy is

\[
\varphi(\mathbf{F}) = \phi(E) \quad \text{with} \quad \phi(E) = \frac{dK|\text{sph} \, E|^2}{2 + \eta|E|^{3/2}} + G|\text{dev} \, E|^2 \quad \text{where} \quad E = E(\mathbf{F}) = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbb{I})
\]

with the spherical and the deviatoric parts \( \text{sph} \, E = (\text{tr} \, E)\mathbb{I}/d \) and \( \text{dev} \, E = E - (\text{tr} \, E)\mathbb{I}/d \), and with \( K \) and \( G \) the bulk and the shear elastic moduli, respectively. The philosophy of this model is a quadratic function (here up to higher-order terms in the neighbourhood of 0) of the Green-Lagrange (sometimes called Green-St. Venant) strain tensor \( E \). The regularization parameter \( \eta > 0 \) is expectedly small just to ensure that \( \varphi(\mathbf{F}) = \Theta(|\mathbf{F}|) \) and \( \varphi'(\mathbf{F}) = \phi'(F^T \mathbf{F} - \mathbb{I})E' \) is \( \Theta(1) \) for \( |\mathbf{F}| \to \infty \), so that this \( \varphi \) complies with (3.1a). For \( \eta = 0 \) we obtain the isotropic St. Venant-Kirchhoff material for which \( \varphi \) has, however, the growth \( \Theta(|\mathbf{F}|^4) \) and thus does not comply with (3.1a).

Remark 5 (An incompressible limit). In literature, the model of the type (2.1) is sometimes interpreted rather as a viscoelastic fluid than solid, and then considered as incompressible. This is motivated by a qualitative difference of bulk and shear elastic moduli in fluids (the latter one being zero) in contrast to solids where these moduli are mostly of the same order (except rubber-like materials). The incompressible model thus modifies (2.1) as

\[
\begin{align*}
\varphi(F) &= \phi(E) \quad \text{with} \quad \phi(E) = \frac{dK|\text{sph} \, E|^2}{2 + \eta|E|^{3/2}} + G|\text{dev} \, E|^2 \quad \text{where} \quad E = E(\mathbf{F}) = \frac{1}{2}(\mathbf{F}^\top \mathbf{F} - \mathbb{I}) \\
\mathbf{F} &= (\nabla \mathbf{v}) \mathbf{F} \\
\mathbf{v} &= \text{div} \left( \mathbf{T} + \mathbf{D} \right) + \mathbf{f} \quad \text{and} \quad \text{div} \, \mathbf{v} = 0, \quad \text{where} \quad \mathbf{T} = \varphi'(\mathbf{F}) \mathbf{F}^\top + \pi \mathbb{I} \\
\mathbf{D} &= \mathbb{D} \mathbf{e}(\mathbf{v}) - \text{div} (\mathbf{v} \nabla \mathbf{e}(\mathbf{v})^p - 2 \nabla \mathbf{e}(\mathbf{v}))
\end{align*}
\]

with \( \pi \) a pressure. Sometimes, (3.11b) is considered with the nonlinear holonomic constraint \( \det \mathbf{F} = 1 \) while the linear constraint \( \text{div} \, \mathbf{v} = 0 \) is possibly omitted, relying on that \( \text{div} \, \mathbf{v} = 0 \) is equivalent with \( \det \mathbf{F} = 1 \) if \( \det \mathbf{F}_0 = 1 \). The weak formulation modifies Definition 1 by imposing \( \text{div} \, \mathbf{v} = 0 \) and \( \text{div} \, \mathbf{v} = 0 \), i.e. in particular (3.2a) omits the terms \( \frac{\partial}{\partial t}(\text{div} \, \mathbf{v}) \mathbf{v} \) and \( \varphi(\mathbf{F}) \mathbf{v} \) and (3.2b) omits \( \text{div} \, \mathbf{v} \mathbf{F} \). This model naturally arises as the limit of (2.1) by two ways. For the isotropic viscosity tensor \( \mathbb{D} \mathbf{e}(\mathbf{v}) = K \text{div} \, \mathbf{v} + 2G \text{dev} \, \mathbf{e}(\mathbf{v}) \) with \( \text{dev} \, \mathbf{e} = \mathbf{e} - (\text{tr} \, \mathbf{e})\mathbb{I}/d \) denoting the deviatoric strain rate, \( K \) the bulk modulus, and \( G \) the shear modulus, the incompressible limit can arise when sending \( K \to \infty \). Indeed, one can estimate \( \|\text{div} \, \mathbf{v}\|_{L^p(I \times \Omega; \mathbb{R}^d)} \leq \Theta(1/\sqrt{K}) \). The other way is to assume \( \varphi(\mathbf{F}) \leq K(1 - \det \mathbf{F})^2 \), which leads to \( \|1 - \det \mathbf{F}\|_{L^\infty(I; L^2(\Omega))} \leq \Theta(1/\sqrt{K}) \). In both cases, the limit passage to the weak solutions of the incompressible system (3.11) is quite straightforward.

Remark 6 (Local non-interpenetration). Another physically relevant assumption beside frame indifference is that \( \mathbf{F} \) ranges only \( \text{GL}^+(d) = \{ \mathbf{F} \in \mathbb{R}^{d \times d}; \det \mathbf{F} > 0 \} \), i.e. the subgroup of the general linear group of matrices with positive determinants. In particular, one should impose the blow-up
\( \varphi(F) \rightarrow \infty \) for \( \det F \rightarrow 0^+ \), which is however not compatible with \((3.1a)\). In some context, one can assume \( \varphi : \mathbb{R}^{d \times d} \rightarrow [0, +\infty) \) continuously differentiable on \( GL^+(d) \) and

\[
\exists \epsilon > 0, \forall F \in \mathbb{R}^{d \times d}: \quad \varphi(F) \geq \begin{cases} 
\epsilon / (\det F)^r & \text{if } \det F > 0, \\
+\infty & \text{if } \det F \leq 0,
\end{cases}
\]  

(3.12)

for \( r > qd/(q-d) \) if one could ensure that \( F(t) \) ranges a bounded set in \( W^{1,q}(\Omega; \mathbb{R}^{d \times d}) \) for some \( q > d \). Then one could use Healey-Krömer’s arguments \cite{22} to ensure \( \det F \) away from zero, like it is possible in Lagrangian formulation in \cite{28, 34, 35}. Here, formally one could strengthen \((3.1c)\) for \( F_0 \in W^{1,q}(\Omega; \mathbb{R}^{d \times d}) \) and then test \((2.1b)\) by \( \text{div}(\nabla F)^{q-2} \nabla F) \), which gives an estimate of \( F \) in \( L^\infty(I; W^{1,q}(\Omega; \mathbb{R}^{d \times d})) \). Assuming \( p \geq 2q/(2^*-q) \) so that \( p^{-1} + (2^*)^{-1} + (q')^{-1} \leq 1 \), by the Hölder and Young inequalities, this test modifies \((3.8)\) as

\[
\frac{d}{dt} \int_\Omega \frac{1}{q} |\nabla F|^q \ dx = \int_\Omega \nabla((v \cdot \nabla) F - (\nabla v) F) : |\nabla F|^{q-2} \nabla F \ dx
\]

\[
= \int_\Omega \nabla F|^{q-2} (\nabla F \otimes \nabla F) : e(v) - \frac{1}{q} |\nabla F|^q \text{div} v - (\nabla v) \nabla F : |\nabla F|^{q-2} \nabla F - (\nabla^2 v) F : |\nabla F|^{q-2} \nabla F \ dx
\]

\[
\leq \frac{2q+1}{q} \|\nabla v\|_{L^\infty(\Omega; \mathbb{R}^{d \times d})} \|\nabla F\|_q^q + \|\nabla^2 v\|_L^p(\Omega; \mathbb{R}^{d \times d}) \|F\|_L^{p*}(\Omega; \mathbb{R}^{d \times d}) \left(1 + \|\nabla F\|_q^q\right),
\]

where we used a modification of the calculus \((3.9)\):

\[
\int_\Omega \nabla((v \cdot \nabla) F) : |\nabla F|^{q-2} \nabla F \ dx = \int_\Omega |\nabla F|^{q-2} (\nabla F \otimes \nabla F) : e(v) + (v \cdot \nabla) F : |\nabla F|^{q-2} \nabla F \ dx
\]

\[
= \int_\Gamma |\nabla F|^{q-2} (\nabla F \otimes \nabla F) : e(v) - (\text{div} v) |\nabla F|^{q-2} |\nabla F| - (q-1) |\nabla F|^{q-2} (v \cdot \nabla) F \ dx
\]

\[
= \int_\Gamma |\nabla F|^{q-2} (\nabla F \otimes \nabla F) : e(v) - (\text{div} v) \frac{|\nabla F|^{q-2}}{q} \ dx.
\]

The boundary integral vanishes if \( v \cdot n = 0 \). Then, one formally obtains the estimate \( \|\nabla F\|_{L^\infty(I; L^q(\Omega; \mathbb{R}^{d \times d}))} \leq C \). Actually, to execute this strategy legitimately would be quite technical. First, a suitable cut-off of the stresses in the momentum equation is to be done together with strengthening of the \( \varepsilon \)-regularization of \((3.1b)\) by considering an \( q \)-Laplacian \( \text{div}(|\nabla F|^{q-2} \nabla F) \). The Galerkin discretization of such a system must be done separately for the momentum and for the (modified) transport-and-evolution equation \((3.1b)\), the limit passage of the latter one being performed first. Also the a-priori estimates are to be extended, in particular by proving that \( \text{div}(|\nabla F|^{q-2} \nabla F) \) is in \( L^2(I \times \Omega; \mathbb{R}^{d \times d}) \); cf. \cite{43} for such a strategy in the context of plastic enhancement of the model. Alternatively, without relying on \((3.12)\), one can employ directly the transport equation of \( 1/\det F \), namely \( 1/\det F = - (\text{div} v)/\det F \). To avoid all these very technical arguments, we confined ourselves to the simpler although less physically relevant model with \((3.1a)\) instead of \((3.12)\).

**Remark 7** (Reconstruction of an underlying deformation). Implicitly, we have in mind the situation when \( F_0 = \nabla_X y_0 \) for some initial deformation \( y_0 \in H^2(\Omega; \mathbb{R}^d) \). Although we have not explicitly needed this additional qualification of \( F_0 \), in the context of the original motivation of the model, a natural question is whether one can reconstruct the deformation \( y(t) \) such that \( v = \dot{y} \) and \( F = \nabla_X y \), and also \( y(0) = y_0 \). This seems a nontrivial question, however. We can always construct the return mapping \( \xi \) mentioned above by solving the simple transport equation \( \dot{\xi} = 0 \) with the initial condition
\(\xi(0)\) = identity. Then \(F = (\nabla_x \xi)^{-1}\) and, if \(\xi(t) : \Omega \to \Omega\) is surjective, \(y(t) = \xi^{-1}(t)\). This global surjectivity seems not automatic, however. An example for such global surjectivity, indicating the complexity of this problem, is for a completely fixed boundary deformation (not considered in this paper), i.e., in addition to \(v \cdot n = 0\), also the tangential velocity on \(\Gamma\) would be prescribed zero so that \(v = 0\) on \(\Gamma\). Then also \(\xi|_\Gamma(t)\) = identity on \(\Gamma\). As the velocity field \(v\) is enough regular, the regularity of the initial condition \(\xi_0\) = identity and local surjectivity in the sense of invertibility of \(\nabla \xi_0\), i.e. here \(\det(\nabla \xi_0) = \det(I) = 1 > 0\), is copied for all \(t > 0\). Note that we have the evolution-and-transport equation \(\det \nabla \xi = -(\text{div} \, v) \det \nabla \xi\), as actually mentioned in Remark 6 since \(\det \nabla \xi = 1/\det F\). Then the classical result of J.M. Ball [2] shows global injectivity of \(\xi(t)\), i.e. \(y(t) = \xi^{-1}(t)\) exists.

Acknowledgments

The author is thankful to Ulisse Stefanelli and Giuseppe Tomassetti for valuable discussions about the Eulerian continuum mechanics.

References

[1] S.S. Antman. Physically unacceptable viscous stresses. Zeitschrift f. angew. Math. Phys., 49:980–988, 1998.
[2] J.M. Ball. Global invertibility of Sobolev functions and the interpenetration of matter. Proc. R. Soc. Edinb., Sect.A, 88:315–328, 1981.
[3] J.M. Ball. Some open problems in elasticity. In Geometry, Mechanics, and Dynamics (Eds.: P. Newton, P. Holmes, and A. Weinstein), pages 3–59. Springer, New York, 2002.
[4] J.M. Ball. Progress and puzzles in nonlinear elasticity. In Poly-, Quasi- and Rank-One Convexity in Applied Mechanics (Eds.: J. Schröder and P. Neff), CISM Intl. Centre for Mech. Sci. 516, pages 1–15. Springer, Wien, 2010.
[5] H. Bellout, F. Bloom, and J. Nečas. Phenomenological behavior of multipolar viscous fluids. Quarterly Appl. Math., 1:559–583, 1992.
[6] B. Benešová, J. Forster, C. Liu, and A. Schlömerkemper. Existence of weak solutions to an evolutionary model for magnetoelasticity. SIAM J. Math. Anal., 50:1200–1236, 2018.
[7] H. Brenner. Kinematics of volume transport. Physica A, 349:11–59, 2005.
[8] H. Brenner. Fluid mechanics revisited. Physica A, 349:190–224, 2006.
[9] Y. Chen and P. Zhang. The global existence of small solutions to the incompressible viscoelastic fluid system in 2 and 3 space dimensions. Comm. Partial Diff. Eqs., 31:1793–1810, 2006.
[10] C. Dafermos and W. Hrusa. Energy methods for quasilinear hyperbolic initial-boundary value problems. Applications to elastodynamics. Archive Rational Mech. Anal., 87:267–292, 1985.
[11] E. Davoli, T. Roubiček, and U. Stefanelli. A note about hardening-free viscoelastic models in Maxwellian-type rheologies. Math. Mech. Solids, 26:1483–1497, 2021.
[12] S. Demoulini. Weak solutions for a class of nonlinear systems of viscoelasticity. Archive Rational Mech. Anal., 155:299–334, 2000.
[13] S. Demoulini, D.M.A. Stuart, and A.E. Tzavaras. A variational approximation scheme for three-dimensional elastodynamics with polyconvex energy. Archive Rational Mech. Anal., 157:325–344, 2001.
[14] S. Demoulini, D.M.A. Stuart, and A.E. Tzavaras. Weak-strong uniqueness of dissipative measure-valued solutions for polyconvex elastodynamics. *Archive Rational Mech. Anal.*, 205:927–961, 2012.

[15] E. Emmrich and D. Puhst. Measure-valued and weak solutions to the nonlinear peridynamic model in nonlocal elastodynamics. *Nonlinearity*, 28:285–307, 2015.

[16] E. Emmrich and D. Puhst. Survey of existence results in nonlinear peridynamics in comparison with local elastodynamics. *Comput. Methods Appl. Math.*, 15:483–496, 2015.

[17] R. Fosdick and G. Royer-Carfagni. The Lagrange multipliers and hyperstress constraint reactions in incompressible multipolar elasticity theory. *J. Mech. Phys. Solids*, 50:1627–1647, 2002.

[18] E. Fried and M.E. Gurtin. Tractions, balances, and boundary conditions for nonsimple materials with application to liquid flow at small-length scales. *Archive Rational Mech. Anal.*, 182:513–554, 2006.

[19] M.-H. Giga, A. Kirshtein, and C. Liu. Variational modeling and complex fluids. In *Handbook of mathematical analysis in mechanics of viscous fluids* (Eds.: A. Novotný and Y. Giga), pages 1–41. Springer, Cham, 2017.

[20] M.E. Gurtin. *Topics in Finite Elasticity*. SIAM, Philadelphia, 1983.

[21] M.E. Gurtin, E. Fried, and L. Anand. *The Mechanics and Thermodynamics of Continua*. Cambridge Univ. Press, New York, 2010.

[22] T. J. Healey and S. Krömer. Injective weak solutions in second-gradient nonlinear elasticity. *ESAIM: Control, Optim. & Cal. Var.*, 15:863–871, 2009.

[23] X. Hu and F. Lin. Global solutions of two-dimensional incompressible viscoelastic flows with discontinuous initial data. *Comm. Pure Appl. Math.*, 69:372–404, 2016.

[24] X. Hu and D. Wang. Global existence for the multi-dimensional compressible viscoelastic flows. *J. Differential Equations*, 250:1200–1231, 2011.

[25] T.J.R. Hughes, T. Kato, and J.E. Marsden. Well-posed quasi-linear second-order hyperbolic systems with applications to nonlinear elastodynamics and general relativity. *Archive Rational Mech. Anal.*, 63:273–294, 1977.

[26] M. Kalousek, J. Kortum, and A. Schlömerkemper. Mathematical analysis of weak and strong solutions to an evolutionary model for magnetoviscoelasticity. *Discr. Cont. Dyn. Sys. S*, 14:17–39, 2021.

[27] K. Koumatos and S. Špirito. Quasiconvex elastodynamics: Weak-strong uniqueness for measure-valued solutions. *Comm. Pure Appl. Math.*, 72:1288–1320, 2019.

[28] M. Kružík and T. Roubíček. *Mathematical Methods in Continuum Mechanics of Solids*. Springer, Switzerland, 2019.

[29] C. Lattanzio and A.E. Tzavaras. Structural properties of stress relaxation and convergence from viscoelasticity to polyconvex elastodynamics. *Archive Rational Mech. Anal.*, 180:449–492, 2006.

[30] Z. Lei, C. Liu, and Y. Zhou. Global existence for a 2D incompressible viscoelastic model with small strain. *Commun. Math. Sci.*, 5:595–616, 2007.

[31] F.-H. Lin, C. Liu, and P. Zhang. On hydrodynamics of viscoelastic fluids. *Commun. Pure Appl. Math.*, 58:1437–1471, 2005.
[32] C. Liu and N.J. Walkington. An Eulerian description of fluids containing visco-elastic particles. *Archive Rational Mech. Anal.*, 159:229–252, 2001.

[33] Z. Martinec. *Principles of Continuum Mechanics*. Birkhäuser/Springer, Switzerland, 2019.

[34] A. Mielke and T. Roubíček. Rate-independent elastoplasticity at finite strains and its numerical approximation. *Math. Models Meth. Appl. Sci.*, 6:2203–2236, 2016.

[35] A. Mielke and T. Roubíček. Thermoviscoelasticity in Kelvin-Voigt rheology at large strains. *Archive Rational Mech. Anal.*, 238:1–45, 2020.

[36] R.D. Mindlin. Micro-structure in linear elasticity. *Archive Rational Mech. Anal.*, 16:51–78, 1964.

[37] J. Nečas, A. Novotný, and M. Šilhavý. Global solution to the ideal compressible heat conductive multipolar fluid. *Comment. Math. Univ. Carolinae*, 30:551–564, 1989.

[38] J. Nečas and M. Růžička. Global solution to the incompressible viscous-multipolar material problem. *J. Elasticity*, 29:175–202, 1992.

[39] H.C. Öttinger, H. Struchtrup, and M. Liu. Inconsistency of a dissipative contribution to the mass flux in hydrodynamics. *Phys. Rev. E*, 80:Art.no. 056303, 2009.

[40] P. Podio-Guidugli and M. Vianello. Hypertractions and hyperstresses convey the same mechanical information. *Continuum Mech. Thermodynam.*, 22:163–176, 2010.

[41] M. Růžička. Mathematical and physical theory of multipolar viscoelasticity. Bonner Mathematische Schriften 233, Bonn, 1992.

[42] M.O. Rieger. Young measure solutions for nonconvex elastodynamics. *SIAM J. Math. Anal.*, 34:1380–1398, 2003.

[43] T. Roubíček. Quasistatic hypoplasticity at large strains Eulerian. *to appear*. (Preprint arXiv no.2108.12718, 2021.)

[44] T. Roubíček. *Nonlinear Partial Differential Equations with Applications*. Birkhäuser, Basel, 2nd edition, 2013.

[45] T. Roubíček. *Relaxation in Optimization Theory and Variational Calculus*. W.de Gruyter, Berlin, 2nd edition, 2020.

[46] T. Roubíček. From quasi-incompressible to semi-compressible fluids. *Disc. Cont. Dynam. Syst. S*, 14:4069–4092, 2021.

[47] T.C. Sideris and B. Thomases. Global existence for three-dimensional incompressible isotropic elastodynamics via the incompressible limit. *Comm. Pure Appl. Math.*, 58:750–788, 2005.

[48] R. Temam. Sur l’approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires (I). *Archive Rational Mech. Anal.*, 32:135–153, 1969.

[49] G. Tomassetti. An interpretation of Temam’s stabilization term in the quasi-incompressible Navier-Stokes system. *Applications in Engr. Sci.*, 5:Art.no. 100028, 2021.

[50] R.A. Toupin. Elastic materials with couple stresses. *Archive Rational Mech. Anal.*, 11:385–414, 1962.

[51] P. Van, M. Pavelka, and M. Grmela. Extra mass flux in fluid mechanics. *J. Non-Equlib. Thermodyn.*, 42:133–152, 2017.

[52] M. Šilhavý. Multipolar viscoelastic materials and the symmetry of the coefficient of viscosity. *Appl. Math.*, 37:383–400, 1992.

[53] D.H. Wagner. Symmetric-hyperbolic equations of motion for a hyperelastic material. *J. Hyperbolic Diff. Eqs.*, 6:615–630, 2009.