Sojourn Times of Brownian Sheet

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September 26, 2000

1 Introduction

Let $B$ denote the standard Brownian sheet. That is, $B$ is a centered Gaussian process indexed by $\mathbb{R}^2_+$ with continuous trajectories and covariance structure

$$E\{B_sB_t\} = \min\{s_1,t_1\} \times \min\{s_2,t_2\}, \quad s = (s_1,s_2), \quad t = (t_1,t_2) \in \mathbb{R}^2_+.$$ 

In a canonical way, one can think of $B$ as “two-parameter Brownian motion”.

In this article, we address the following question: “Given a measurable function $\upsilon : \mathbb{R} \to \mathbb{R}_+$, what can be said about the distribution of $\int_0^1 \upsilon(B_s) \, ds$?”

The one-parameter variant of this question is both easy-to-state and well understood. Indeed, if $b$ designates standard Brownian motion, the Laplace transform of $\int_0^1 \upsilon(b_s+x) \, ds$ often solves a Dirichlet eigenvalue problem (in $x$), as prescribed by the Feynman–Kac formula; cf. Revuz and Yor [6], for example. While analogues of Feynman-Kac for $B$ are not yet known to hold, the following highlights some of the unusual behavior of $\int_{[0,1]^2} \upsilon(B_s) \, ds$ in case $\upsilon = 1_{[0,\infty)}$ and, anecdotally, implies that finding explicit formulæ may present a challenging task.

**Theorem 1.1**

There exists a $c_0 \in (0,1)$, such that for all $0 < \varepsilon < \frac{1}{8}$,

$$\exp\left\{-\frac{1}{c_0} \log^2(1/\varepsilon)\right\} \leq P\{\int_{[0,1]}^1 \mathbf{1}_{\{B_s > 0\}} \, ds < \varepsilon\} \leq \exp\left\{-c_0 \log^2(1/\varepsilon)\right\}. $$

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*Research supported in part by grants from NSF and NATO
†Research supported in part by NSF grant 98-03249
Remark 1.2
By the arcsine law, the one-parameter version of the above has the following simple form: given a linear Brownian motion $b$,
\[
\lim_{\varepsilon \to 0^+} \varepsilon^{-1/2} \mathbb{P}\{\int_0^1 1\{b_s > 0\} \, ds < \varepsilon\} = \frac{2}{\pi};
\]
see [6, Theorem 2.7, Ch. 6]. \hfill \Box

Remark 1.3
R. Pyke (personal communication) has asked whether \( \int_{[0,1]^2} 1\{B_s > 0\} \, ds \) has an arcsine-type law; see [5, Section 4.3.2] for a variant of this question in discrete time. According to Theorem 1.1, as $\varepsilon \to 0$, the cumulative distribution function of $\int_{[0,1]^2} 1\{B_s > 0\} \, ds$ goes to zero faster than any power of $\varepsilon$. In particular, the distribution of time (in $[0,1]^2$) spent positive does not have any simple extension of the arcsine law. \hfill \Box

Theorem 1.4
Let $v(x) := 1_{[-1,1]}(x)$, or $v(x) := 1_{(-\infty,1)}(x)$. Then, there exists a $c_1 \in (0,1)$, such that for all $\varepsilon \in (0,\frac{1}{8})$,
\[
\exp\left\{ -\frac{\log^2(1/\varepsilon)}{c_1 \varepsilon} \right\} \leq \mathbb{P}\{\int_{[0,1]^2} v(B_s) \, ds < \varepsilon\} \leq \exp\left\{ -c_1 \frac{\log(1/\varepsilon)}{\varepsilon} \right\}.
\]
For a refinement, see Theorem 2.2 below.

Remark 1.5
The one-parameter version of Theorem 1.4 is quite simple. For example, let $\Gamma = \int_0^1 1_{[-1,1]}(b_s) \, ds$, where $b$ is linear Brownian motion. In principle, one can compute the Laplace transform of $\Gamma$ by means of Kac's formula and invert it to calculate its distribution function. However, direct arguments suffice to show that the two-parameter Theorem 1.4 is more subtle than its one-parameter counterpart:
\[
-\infty < \liminf_{\varepsilon \to 0^+} \varepsilon \ln \mathbb{P}\{\Gamma < \varepsilon\} \leq \limsup_{\varepsilon \to 0^+} \varepsilon \ln \mathbb{P}\{\Gamma < \varepsilon\} < 0,
\]
where $\ln$ denotes the natural logarithm function. We will verify this later on in the Appendix. \hfill \Box

Remark 1.6
The arguments used to demonstrate Theorem 1.4 can be used to also estimate the distribution function of additive functionals of form, e.g., $\int_{[0,1]^2} v(B_s) \, ds$, as long as $\alpha 1_{[-r,r]} \leq v \leq \beta 1_{[-R,R]}$, where $0 < r \leq R$ and $0 < \alpha \leq \beta$. Other formulations are also possible. For instance, when $\alpha 1_{(-\infty,r)} \leq v \leq \beta 1_{(-\infty,R)}$. \hfill \Box
2 Proof of Theorems 1.1 and 1.4

Our proof of Theorem 1.1 rests on a lemma that is close in spirit to a Feynman–Kac formula of the theory of one-parameter Markov processes.

**Proposition 2.1**
There exists a finite and positive constant $c_2$, such that for all measurable $D \subset \mathbb{R}$ and all $0 < \eta, \varepsilon < \frac{1}{8}$,

$$
\mathbb{P}\left\{ \int_{[0,1]^2} 1_{\{B \notin D\}} \, ds < \varepsilon \right\} \leq \mathbb{P}\left\{ \forall s \in [0,1]^2 : B_s \in D, \exp(-c_2 \varepsilon^{-\eta}) \right\},
$$

where $D_\delta$ denotes the $\delta$-enlargement of $D$ for any $\delta > 0$. That is,

$$
D_\delta := \{ x \in \mathbb{R} : \operatorname{dist}(x; D) \leq \delta \},
$$

where ‘dist’ denotes Hausdorff distance.

**Proof**

For all $t \in [0,1]^2$, let $|t| := \max\{t_1, t_2\}$. Then, it is clear that for any $\varepsilon, \delta > 0$, whenever there exists some $s_0 \in [0,1]^2$ for which $B_{s_0} \notin D_\delta$, either

1. $\sup_{|t-s| \leq \varepsilon^{1/2}} |B_t - B_s| > \delta$, where the supremum is taken over all such choices of $s$ and $t$ in $[0,1]^2$; or
2. for all $t \in [0,1]^2$ with $|t-s_0| \leq \varepsilon^{1/2}$, $B_t \notin D$, in which case, we can certainly deduce that $\int_{[0,1]^2} 1_{D^c}(B_t) \, dt > \varepsilon$.

Thus,

$$
\mathbb{P}\left\{ \exists s_0 \in [0,1]^2 : B_{s_0} \notin D_\delta \right\} \leq \mathbb{P}\left\{ \sup_{|t-s| \leq \varepsilon^{1/2}} |B_t - B_s| > \delta \right\} + \mathbb{P}\left\{ \int_{[0,1]^2} 1_{D^c}(B_t) \, dt > \varepsilon \right\}.
$$

By the general theory of Gaussian processes, there exists a universal positive and finite constant $c_2$ such that

$$
\mathbb{P}\left\{ \sup_{|t-s| \leq \varepsilon^{1/2}} |B_t - B_s| > \delta \right\} \leq \exp\left\{-c_2 \delta^2 \varepsilon^{-1/2}\right\}. \tag{2.1}
$$

Although it is well known, we include a brief derivation of this inequality for completeness. Indeed, we recall C. Borell’s inequality from Adler [1, Theorem 2.1]: if $\{g_t : t \in T\}$ is a centered Gaussian process such that $\|g\|_T = \mathbb{E}\{\sup_{t \in T} |g_t|\} < \infty$ and whenever $T$ is totally bounded in the metric $d(s,t) = \sqrt{\mathbb{E}\{(g_t - g_s)^2\}}$ $(s,t \in T)$,

$$
\mathbb{P}\left\{ \sup_{t \in T} |g_t| > \lambda + \|g\|_T \right\} \leq 2 \exp\left\{-\frac{\lambda^2}{2\sigma_T^2}\right\},
$$

where $\sigma_T^2 = \sup_{t \in T} \mathbb{E}\{g_t^2\}$. Eq. (2.1) follows from this by letting $T = \{(s,t) \in (0,1)^2 \times (0,1)^2 : |s-t| \leq \varepsilon^{1/2}\}$, $g_{t,s} = B_t - B_s$ and by making a few lines of standard calculations. Having derived (2.1), we can let $\delta := \varepsilon^{\frac{1}{1-2\varepsilon}}$ to obtain the proposition.  \hfill \Box
Proof of Theorem 1.1. Let $D = (-\infty, 0)$ and use Proposition 2.1 to see that
\[
P\{\int_{[0,1]} 1_{\{B_s > 0\}} < \varepsilon\} \leq P\{\sup_{s \in [0,1]^2} B_s \leq \varepsilon^{1/2-\eta}\} + \exp\{-c_2\varepsilon^{-\eta}\}.\]
Thus, the upper bound of Theorem 1.1 follows from Li and Shao [4], which states that
\[
\limsup_{\varepsilon \to 0^+} \frac{1}{\log^2(1/\varepsilon)} \log P\{\sup_{s \in [0,1]^2} B_s \leq \varepsilon\} < -\infty.
\]
(An earlier, less refined version, of this estimate can be found in Csáki et al. [2].) To prove the lower bound, we note that
\[
P\{\int_{[0,1]^2} 1_{\{B_s > 0\}} ds < 2\varepsilon - \varepsilon^2\}
\geq P\{\sup_{s \in [\varepsilon,1]^2} B_s < 0\}
= P\{\forall (u,v) \in [0, \ln(1/\varepsilon)]^2 : B(e^{-u}, e^{-v}) < 0\},
\]
and observe that the stochastic process $(u,v) \mapsto B(e^{-u}, e^{-v})/e^{-(u+v)/2}$ is the 2-parameter Ornstein–Uhlenbeck sheet. All that we need to know about the latter process is that it is a stationary, positively correlated Gaussian process whose law is supported on the space of continuous functions on $[0,1]^2$. We define $c_3 > 0$ via the equation
\[
e^{-c_3} := P\{\forall (u,v) \in [0,1]^2 : \frac{B(e^{-u}, e^{-v})}{e^{-(u+v)/2}} < 0\}.\]
By the support theorem, $0 < c_3 < \infty$: this is a consequence of the Cameron-Martin theorem on Gauss space; cf. Janson [3, Theorem 14.1]. Moreover, by stationarity and by Slepian’s inequality (cf. [1, Corollary 2.4]),
\[
P\{\int_{[0,1]^2} 1_{\{B_s < 0\}} ds < \varepsilon\}
\geq \prod_{0 \leq i,j \leq \ln(1/\varepsilon)+1} P\{\forall (u,v) \in [i,i+1] \times [j,j+1] : \frac{B(e^{-u}, e^{-v})}{e^{-(u+v)/2}} < 0\}
= \exp\{-c_3 \ln^2(\varepsilon/\varepsilon)\}.
\]
This proves the theorem. \(\Box\)

Next, we prove Theorem 1.4.

Proof of Theorem 1.4. Let $D_\varepsilon$ denote the collection of all points $(s,t) \in [0,1]^2$, such that $st \leq \varepsilon$. Note that
\begin{enumerate}
\item Lebesgue’s measure of $D_\varepsilon$ is at least $\varepsilon \ln(1/\varepsilon)$; and
\item if $\sup_{s \in D_\varepsilon} |B_s| \leq 1$, then $\int_{[0,1]^2} 1_{(-1,1)}(B_s) \ ds > \varepsilon \ln(1/\varepsilon)$.
\end{enumerate}
Thus,
\[ \mathbb{P}\left\{ \int_{[0,1]^2} \mathbf{1}_{(-1,1)}(B_s) \, ds < \varepsilon \ln(1/\varepsilon) \right\} \leq \mathbb{P}\left\{ \sup_{s \in \mathcal{D}_\varepsilon} |B_s| > 1 \right\}. \]

A basic feature of the set \( \mathcal{D}_\varepsilon \) is that whenever \( s \in \mathcal{D}_\varepsilon \), then \( E\{B_s^2\} \leq \varepsilon \). Since \( E\{\sup_{s \in \mathcal{D}_\varepsilon} |B_s|\} \leq E\{\sup_{s \in [0,1]^2} |B_s|\} < \infty \), we can apply Borell’s inequality to deduce the existence of a finite, positive constant \( c_4 < 1 \), such that for all \( \varepsilon > 0 \),
\[ \mathbb{P}\{\sup_{s \in \mathcal{D}_\varepsilon} |B_s| > 1/c_4\} \leq \exp\{-c_4/\varepsilon\}. \]

We apply Brownian scaling and possibly adjust \( c_4 \) to conclude that
\[ \mathbb{P}\left\{ \sup_{s \in \mathcal{D}_\varepsilon} |B_s| > 1 \right\} \leq e^{-c_4/\varepsilon}. \]

Consequently, we can find a positive, finite constant \( c_5 \), such that for all \( \varepsilon \in (0, \frac{1}{8}) \),
\[ \mathbb{P}\{\Gamma < \varepsilon\} \leq \exp\left\{-c_5 \ln(1/\varepsilon) / \varepsilon\right\}. \] (2.2)

This implies the upper bound in the conclusion of Theorem 1.4. For the lower bound, we note that for all \( \varepsilon \in (0, \frac{1}{8}) \), Lebesgue’s measure of \( \mathcal{D}_\varepsilon \) is bounded above by \( c_6 \varepsilon \log(1/\varepsilon) \). Thus,
\[ \mathbb{P}\left\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s) \, ds < c_6 \varepsilon \ln(1/\varepsilon) \right\} \geq \mathbb{P}\left\{ \inf_{s \in [0,1]^2 \setminus \mathcal{D}_\varepsilon} B_s > 1 \right\}. \]

On the other hand, whenever \( s \in [0,1]^2 \setminus \mathcal{D}_\varepsilon \), \( s_1 s_2 \geq \varepsilon \). Thus,
\[ \mathbb{P}\left\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s) \, ds < c_6 \varepsilon \log(1/\varepsilon) \right\} \geq \mathbb{P}\left\{ \inf_{s \in [0,1]^2 \setminus \mathcal{D}_\varepsilon} \frac{B_s}{\sqrt{s_1 s_2}} > \frac{1}{\sqrt{\varepsilon}} \right\} \]
\[ = \mathbb{P}\left\{ \inf_{u,v \geq 0 : u+v \leq \ln(1/\varepsilon)} O_{u,v} > \varepsilon^{-1/2} \right\}, \]

where \( O_{u,v} := B(e^{-u}, e^{-v})/e^{-(u+v)/2} \) is an Ornstein–Uhlenbeck sheet. Consequently,
\[ \mathbb{P}\left\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s) \, ds < c_3 \varepsilon \log(1/\varepsilon) \right\} \geq \mathbb{P}\left\{ \inf_{0 \leq u,v \leq \ln(1/\varepsilon)} O_{u,v} > \varepsilon^{-1/2} \right\}. \]

By appealing to Slepian’s inequality and to the stationarity of \( O \), we can deduce that
\[ \mathbb{P}\left\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s) \, ds < c_3 \varepsilon \log(1/\varepsilon) \right\} \]
\[ \geq \prod_{0 \leq i,j \leq \ln(1/\varepsilon)} \mathbb{P}\left\{ \inf_{i \leq u \leq i+1} \inf_{j \leq v \leq j+1} O_{u,v} > \varepsilon^{-1/2} \right\} \]
\[ = \left[ \mathbb{P}\left\{ \inf_{0 \leq u,v \leq 1} O_{u,v} > \varepsilon^{-1/2} \right\} \right]^{\ln^2(\varepsilon/\varepsilon)}. \] (2.3)
On the other hand, recalling the construction of $O$, we have

$$
\mathbb{P}\left\{ \inf_{0 \leq u, v \leq 1} O_{u,v} > \varepsilon^{-1/2} \right\} \\
\geq \mathbb{P}\left\{ \inf_{0 \leq s, t \leq 1/2} B_{s,t} \geq 2\varepsilon^{-1/2} \right\} \\
\geq \mathbb{P}\left\{ B_{1,1} \geq 2\varepsilon^{-1/2} \right\} \cdot \mathbb{P}\left\{ \sup_{0 \leq s_1, s_2 \leq 1/2} |B_s - B_{1,1}| \leq \varepsilon \varepsilon^{-1/2} \right\} \\
\geq c_7 \mathbb{P}\left\{ B_{1,1} \geq 2\varepsilon^{-1/2} \right\},
$$

for some absolute constant $c_7$ that is chosen independently of all $\varepsilon \in (0, \frac{1}{8})$. Therefore, by picking $c_8$ large enough, we can insure that for all $\varepsilon \in (0, \frac{1}{8})$,

$$
\mathbb{P}\left\{ \inf_{0 \leq u, v \leq 1} O_{u,v} > \varepsilon^{-1/2} \right\} \geq \exp\left\{ -c_8 \varepsilon^{-1} \right\}.
$$

Plugging this in to Eq. (2.3), we obtain

$$
\mathbb{P}\left\{ \int_{[0,1]^2} 1_{(-\infty,1)}(B_s) \, ds < c_6 \varepsilon \log(1/\varepsilon) \right\} \geq \exp\left\{ -c_8 \frac{n^2(1/\varepsilon) \log(1/\varepsilon)}{4\varepsilon} \right\}. \quad (2.4)
$$

The lower bound of Theorem 1.4 follows from replacing $\varepsilon$ by $\varepsilon / \log(1/\varepsilon)$.

The methods of this proof go through with few changes to derive the following extension of Theorem 1.4.

**Theorem 2.2**

Suppose $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a measurable function such that (a) as $r \downarrow 0$, $\varphi(r) \uparrow \infty$; and (b) there exists a finite constant $\gamma > 0$, such that for all $r \in (0, \frac{1}{2})$, $\varphi(2r) \geq \gamma \varphi(r)$. Define $J_{\varphi} = \int_{[0,1]^2} 1_{\{ |B_s| \leq \sqrt{\varphi(s_1, s_2)} \}} \, ds$. Then, there exist a finite constant $c_9 > 1$, such that for all $\varepsilon \in (0, \frac{1}{2})$,

$$
\mathbb{P}\left\{ J_{\varphi} < \varepsilon \right\} \leq \exp\left\{ -c_9 \frac{\varepsilon^2}{\log(1/\varepsilon)^2} \right\}. \quad (2.5)
$$

**Appendix: On Remark 1.5**

In this appendix, we include a brief verification of the exponential form of the distribution function of $\Gamma$; cf. Eq. (1.1). Given any $\lambda > \frac{1}{2}$ and for $\zeta = (2\lambda)^{-1/2}$, we have

$$
\mathbb{E}\{ e^{-\lambda T} \} \leq \mathbb{E}\left\{ \exp\left\{ -\lambda \int_0^\tau \varphi(b_s) \, ds \right\} \right\} \\
\leq e^{-\lambda \zeta} + \mathbb{P}\{ \sup_{0 \leq s \leq \zeta} |b_s| > 1 \} \\
\leq e^{-\lambda \zeta} + e^{-1/(2\zeta)} \\
= 2e^{-\sqrt{\lambda/2}}. \quad (2.6)
$$
By Chebyshev’s inequality, \( P\{\int_0^1 \upsilon(b_s) \, ds < \varepsilon\} \leq 2 \inf_{\lambda > 1} e^{-\sqrt{\lambda/2 + \lambda \varepsilon}} \). Choose \( \lambda = \frac{1}{8} \varepsilon^{-2} \) to obtain the following for all \( \varepsilon \in (0, \frac{1}{2}) \):
\[
P\{\Gamma < \varepsilon\} \leq 2 e^{-1/(8\varepsilon)}.
\tag{2.7}
\]
Conversely, we can choose \( \delta = (2\lambda)^{-1/2} \) and \( \eta \in (0, \frac{1}{100}) \) to see that
\[
E\{e^{-\lambda \Gamma}\} \geq E\left\{ \exp\left( -\lambda \int_0^\delta \upsilon(b_s) \, ds \right); \inf_{\delta \leq s \leq 1} |b_s| > 1 \right\}
\geq e^{-\lambda \delta} P\{ |b_\delta| > 1 + \eta, \sup_{\delta < s < 1 + \delta} |b_s - b_\delta| < \eta \}.
\]
Thus, we can always find a positive, finite constant \( c_{10} \) that only depends on \( \eta \) and such that
\[
E\{e^{-\lambda \Gamma}\} \geq c_{10} \exp\left\{ -\sqrt{\frac{\lambda}{2}} \left[ 1 + (1 + \eta)^2 (1 + \psi_\delta) \right] \right\},
\]
where \( \lim_{\delta \to 0^+} \psi_\delta = 0 \), uniformly in \( \eta \in (0, \frac{1}{100}) \). In particular, after negotiating the constants, we obtain
\[
\liminf_{\lambda \to \infty} \lambda^{-1/2} \ln E\{e^{-\lambda \Gamma}\} \geq -2^{1/2}.
\tag{2.8}
\]
Thus, for any \( \varepsilon \in (0, \frac{1}{100}) \),
\[
e^{-\sqrt{\lambda} (1 + o_1(1))} \leq E\{e^{-\lambda \Gamma}\} \leq P\{\Gamma < \varepsilon\} + e^{-\lambda \varepsilon},
\]
where \( o_1(1) \to 0 \), as \( \lambda \to \infty \), uniformly in \( \varepsilon \in (0, \frac{1}{100}) \). In particular, if we choose \( \varepsilon = (1 + \eta) \sqrt{2/\lambda} \), where \( \eta > 0 \), we obtain
\[
P\{\Gamma < (1 + \eta) \sqrt{2/\lambda}\} \geq e^{-\sqrt{\lambda} (1 + o_2(1))},
\]
where \( o_2(1) \to 0 \), as \( \lambda \to \infty \). This, Eq. (2.7) and a few lines of calculations, together imply Eq. (1.1). \( \square \)

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