THE CHARGE QUANTUM NUMBERS OF GAUGE INVARIANT QUASI-FREE ENDO MORPHISMS

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Abstract. The representations of a group of gauge automorphisms of the
canonical commutation or anticommutation relations which appear on the
Hilbert spaces of isometries \(H_{\varrho}\) implementing quasi-free endomorphisms \(\varrho\)
on Fock space are studied. Such a representation, which characterizes the
"charge" of \(\varrho\) in local quantum field theory, is determined by the Fock space
structure of \(H_{\varrho}\) itself: Together with a "basic" representation of the group, all
higher symmetric or antisymmetric tensor powers thereof also appear. Hence \(\varrho\)
is reducible unless it is an automorphism. It is further shown by the example of
the massless Dirac field in two dimensions that localization and implementabil-
ity of quasi-free endomorphisms are compatible with each other.

1. Introduction

In quantum field theory the structure of superselection sectors is entirely encoded
in the set of localized endomorphisms of the algebra of local observables \([DHR71, Ha96]\). In the case of main physical interest, viz. in four dimensional Minkowski
space, the set of (equivalence classes of) localized endomorphisms can be identified
with the representation category of a unique compact group, the global gauge group
of the theory \([DR89a, DR90]\). This gauge group acts on a larger field algebra
containing besides the observables charge carrying fields with normal commutation
relations which reach all superselection sectors from the vacuum \([DR89a, DR90]\).
Gauge group and field algebra are intrinsically determined by the observable data.

The relation between localized endomorphisms and representations of the gauge
group is made concrete in the following way \([DR72]\). There is a functor which
assigns to a localized endomorphism \(\varrho\) the Hilbert space of isometries \(H_{\varrho}\) consisting
of all local fields \(\Psi\) which induce \(\varrho:\)

\[ H_{\varrho} \equiv \{ \Psi \mid \Psi a = \varrho(a)\Psi, \text{ for all local observables } a \}. \]

The action of the gauge group on the field algebra restricts to a unitary representa-
tion \(D_{\varrho}\) on \(H_{\varrho}\) relative to the inner product \(\langle \Psi, \Psi' \rangle_1 \equiv \Psi^* \Psi'\). This representation
of the gauge group determines the charge of the endomorphism \(\varrho\); it is customary to
refer to any label which characterizes the representation \(D_{\varrho}\) as the charge quantum
numbers of \(\varrho\). It will be used below that the representation \(D_{\varrho}\) is in a canonical
way unitarily equivalent to the representation on the Hilbert space \(H_{\varrho}\Omega\) generated
by applying the field operators in \(H_{\varrho}\) to the vacuum vector \(\Omega\).

Any orthonormal basis \(\Psi_1, \ldots, \Psi_d\) in \(H_{\varrho}\) generates a representation of the Cuntz
algebra \(O_d\) and implements the endomorphism \(\varrho\) as follows:

\[ \varrho(a) = \sum_{j=1}^d \Psi_j a \Psi_j^*, \quad \text{(1.1)} \]

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Using this formula, \( \varrho \) can be canonically extended to an endomorphism of the field algebra. This extension is \textit{gauge invariant}, i.e. commutes with all gauge automorphisms:

**Proposition 1.1.** Let \( \varrho \) be an endomorphism of the field algebra which is implemented by a Hilbert space of isometries \( H_\varrho \) as in (1.1). Then \( H_\varrho \) is gauge invariant if and only if \( \varrho \) is gauge invariant.

**Proof.** Assume first that \( H_\varrho \) is invariant under gauge automorphisms \( \gamma \). Since the representation \( D_\varrho \) of the gauge group on \( H_\varrho \), given by \( D_\varrho(\gamma) \equiv \gamma | H_\varrho \), is unitary, the \( \gamma(\Psi_j) \) also form an orthonormal basis in \( H_\varrho \). Since the endomorphism associated with a Hilbert space of isometries as in (1.1) is independent of the choice of an orthonormal basis in \( H_\varrho \), it follows that, for any field operator \( f \),

\[
\gamma(\varrho(f)) = \sum_j \gamma(\Psi_j) \gamma(f) \gamma(\Psi_j)^* = \varrho(\gamma(f)).
\]

Conversely, assume that \( \varrho \) is gauge invariant. Let \( \Psi \in H_\varrho \) and let \( \gamma \) be a gauge transformation. Then one has for any field operator \( f \)

\[
\gamma(\varrho(f)) = \gamma(\varrho^{-1}(f)) = \gamma(\varrho^{-1}(f))\Psi = \varrho(\gamma^{-1}(f))\Psi = \varrho(f)\gamma(\Psi)
\]

so that \( \gamma(\Psi) \in H_\varrho \).

The existence of localized endomorphisms and associated Hilbert spaces of isometries follows from first principles of local quantum field theory. But it is by no means obvious how to obtain them explicitly in concrete models. In previous work we have developed a general theory of quasi-free endomorphisms of the CAR and CCR algebras which can be implemented by Hilbert spaces of isometries on Fock space \([\text{Bin}95, \text{Bin}98b]\). Among the results are implementability conditions for endomorphisms, which generalize the well-known criteria of Shale and Stinespring for automorphisms \([\text{Sha}62, \text{SS}65]\), and detailed constructions of field operators which implement endomorphisms according to (1.1).

In the present paper we are interested in the possible charge quantum numbers of such endomorphisms. The CAR resp. CCR algebra will play the role of the field algebra. Therefore quasi-free endomorphisms have to be viewed as endomorphisms of the field algebra and, by Proposition 1.1, we have to restrict attention to endomorphisms which are gauge invariant under an appropriate group action. We will consider quasi-free actions of arbitrary groups which leave the Fock vacuum invariant. We show that the charge quantum numbers are then determined by the natural Fock space structure found in \([\text{Bin}95, \text{Bin}98b]\) of the Hilbert spaces of isometries \( H_\varrho \) implementing gauge invariant quasi-free endomorphisms \( \varrho \). The representation \( D_\varrho \) is unitarily equivalent to the representation \( \Lambda_{\varrho} \) on the antisymmetric Fock space over an auxiliary unitary \( G \)-module \( \mathfrak{r}_\varrho \), tensored with a certain character \( \det_{\mathfrak{b}} \):

\[
D_\varrho \simeq \det_{\mathfrak{b}} \otimes \Lambda_{\varrho} \quad (\text{CAR}),
\]

resp. to the representation \( \mathfrak{S}_{\varrho} \) on the symmetric Fock space over \( \mathfrak{r}_\varrho \):

\[
D_\varrho \simeq \mathfrak{S}_{\varrho} \quad (\text{CCR}).
\]

This is our main result, contained in Theorems 2.3 and 3.2. It follows that \( D_\varrho \) is reducible if \( \varrho \) is non-surjective. Any \( \varrho \) has a quasi-free conjugate \( \varrho^c \) such that \( D_{\varrho^c} \) is equivalent to the complex conjugate of \( D_\varrho \), provided that the single-particle space has a particle-antiparticle symmetry.

The analysis of the representations \( D_\varrho \) is completely independent of localization properties of endomorphisms. In order to show that localization and implementability are not in conflict, we give in Section 2.3 an explicit example of a \textit{localized} implementable gauge invariant endomorphism \( \varrho \), with \( \dim H_\varrho = 2^N \), of
the free massless Dirac field with $U(N)$ gauge symmetry in two dimensions. The construction rests on the use of “local” Fourier bases for the chiral components, and is in this respect similar to the known examples [Böc94, Böc96] of localized endomorphisms in conformal field theory.

The present investigations are taken from the author’s Ph.D. thesis in physics [Bin98a], to which we refer for further results and discussions.

2. The Fermionic Case

First of all we need some formalism and some results from [Bin95, Bin98a].

2.1. Preliminaries on the implementation of quasi-free endomorphisms (CAR).

Recall Araki’s approach to the canonical anticommutation relations [Ara71, Ara87]: Let $K$ be an infinite dimensional separable complex Hilbert space, endowed with a complex conjugation $f \mapsto f^\ast$. The (selfdual) CAR algebra $C(K)$ over $K$ is the unique (simple) $C^*$-algebra generated by 1 and the elements of $K$, subject to the anticommutation relation

$$f^* g + g f^* = \langle f, g \rangle 1,$$

$f, g \in K$.

Let $P_1$ be a fixed basis projection on $K$, i.e. an orthogonal projection such that $P_1^* = 1 - P_1$.

Here the bar denotes the complex conjugate of an operator $A$:

$$A(f) = A(f^\ast)^*, \quad f \in K.$$

Let $P_2$ be the complementary (basis) projection:

$$P_2 = 1 - P_1.$$

The components of an operator $A$ on $K$ with respect to the decomposition $K = K_1 \oplus K_2$ given by $P_1$ and $P_2$ will be denoted by

$$A_{mn} = P_m A P_n, \quad m, n = 1, 2$$

and will be regarded as operators from $K_n$ to $K_m$.

To the basis projection $P_1$ there corresponds a unique (pure, quasi-free) Fock state $\omega_{P_1}$ which is completely determined by the condition that

$$\omega_{P_1}(f^* f) = 0 \quad \text{if} \quad P_1 f = 0. \quad (2.1)$$

The GNS representation associated with $\omega_{P_1}$ will be denoted by $(\mathfrak{F}_{P_1}, \pi_{P_1}, \Omega_{P_1})$. The Hilbert space $\mathfrak{F}_{P_1}$ can be identified with the antisymmetric Fock space over $K_1$. The elements of $K$ are then represented by sums of creation and annihilation operators:

$$\pi_{P_1}(f) = a^*(P_1 f) + a(P_1 f^*), \quad f \in K, \quad (2.2)$$

and the cyclic vector $\Omega_{P_1}$ is the Fock vacuum vector.

Every isometry $V$ on $K$ which commutes with complex conjugation extends to a unique quasi-free endomorphism $\varrho_V$ of $\mathfrak{C}(K)$:

$$\varrho_V(f) = V(f), \quad f \in K.$$

As shown in [Bin95], an endomorphism $\varrho_V$ can be implemented by a Hilbert space of isometries $H_{\varrho_V}$ (cf. [1.1]) in the Fock state $\omega_{P_1}$ if and only if

$$[P_1, V] \text{ is Hilbert–Schmidt (HS)}. \quad (2.3)$$

These isometries form a semigroup

$$\text{End}_{P_1}(K) \equiv \{ V \in \mathfrak{B}(K) \mid V^* V = 1, \quad \nabla V = V, \quad [P_1, V] \text{ is HS} \}$$
isomorphic to the semigroup of all implementable quasi-free endomorphisms of \( \mathfrak{C}(\mathcal{X}) \). The statistics dimension \( d_{\mathcal{O}_V} \) of \( \mathcal{O}_V \) is given by
\[
d_{\mathcal{O}_V} \equiv \dim H_{\mathcal{O}_V} = 2^{4 \text{ind } V},
\]
where \( \text{ind } V \) is, up to the sign, the Fredholm index of \( V \):
\[
\text{ind } V = \dim \ker V^* \in \{0, 2, 4, \ldots, \infty \}, \quad V \in \text{End}_{P_1}(\mathcal{X}).
\]

The grading automorphism of \( \mathfrak{C}(\mathcal{X}) \) is equal to the quasi-free automorphism \( \mathcal{O}_{-1} \). Let \( \Gamma(-1) \) be the (self-adjoint, unitary) second quantization of \( \mathcal{O}_{-1} \), given by
\[
\Gamma(-1)\pi_{P_1}(a)\Omega_{P_1} = \pi_{P_1}(\mathcal{O}_{-1}(a))\Omega_{P_1}, \quad a \in \mathfrak{C}(\mathcal{X}),
\]
and let \( \theta(-1) \) be the unitary operator
\[
\theta(-1) \equiv \frac{1}{\sqrt{2}}(1 - i\Gamma(-1)).
\]
Then the twisted Fock representation \( \psi_{P_1} \) induced by \( P_1 \) is defined by
\[
\psi_{P_1}(a) \equiv \theta(-1)\pi_{P_1}(a)\theta(-1)^*. \tag{2.4}
\]
It can be used to describe the commutants of “local” subalgebras: If \( \mathcal{H} \subset \mathcal{K} \) is a subspace invariant under complex conjugation, and \( \mathfrak{C}(\mathcal{H}) \) the \( C^* \)-subalgebra of \( \mathfrak{C}(\mathcal{X}) \) generated by \( \mathcal{H} \), then
\[
\pi_{P_1}(\mathfrak{C}(\mathcal{H}))' = \psi_{P_1}(\mathfrak{C}(\mathcal{H}^+))'' \quad (\text{twisted duality}).
\]

Let \( V \in \text{End}_{P_1}(\mathcal{X}) \) be given. As mentioned in the introduction, it suffices for our purposes to consider group actions on the Hilbert space \( H_{\mathcal{O}_V}\Omega_{P_1} \). An orthonormal basis in this space can be obtained as follows. Define a finite dimensional subspace \( \mathfrak{h} \subset \mathcal{X}_1 \) by
\[
\mathfrak{h} \equiv V_{12}(\ker V_{22}), \tag{2.6}
\]
and an antisymmetric Hilbert–Schmidt operator \( T \) from \( \mathcal{X}_1 \) to \( \mathcal{X}_2 \) by
\[
T \equiv V_{21}V_{11}^{-1} - V_{22}^{-1}V_{12}^*[\ker V_{11}^*]. \tag{2.7}
\]
Here the bounded operator \( V_{11}^{-1} \) is defined to be zero on \( \ker V_{22}^* \), and \( [\mathcal{H}] \) denotes the orthogonal projection onto a closed subspace \( \mathcal{H} \subset \mathcal{X} \). Then one has \( T(\mathfrak{h}) = 0 \); such pairs \( (\mathfrak{h}, T) \) parameterize the class of all Fock states which are unitarily equivalent to the given Fock state \( \omega_{P_1} \). The basis projection \( P \) corresponding to the pair \( (\mathfrak{h}, T) \) is explicitly given by
\[
P \equiv (P_1 + T)(P_1 + T^*T)^{-1}(P_1 + T^*) - [\mathfrak{h}] + [\mathfrak{h}^*], \tag{2.8}
\]
and \( \mathfrak{h} \) and \( T \) can be recovered from \( P \) by
\[
\mathfrak{h} = \ker P_{11}, \tag{2.9}
\]
\[
T = P_{21}P_{11}^{-1}. \tag{2.10}
\]
\( (P_{11}^{-1} \) is defined in a similar way as \( V_{11}^{-1} \) above). The Fock state \( \omega_P \) associated with \( P \) is an extension of the partial Fock state \( \omega_{P_1} \circ \mathcal{O}_V^{-1}|_{\text{ran } \mathcal{O}_V} \), and is induced by the cyclic vector
\[
\Omega_P = (\det(P_1 + T^*T))^{-1/4} \psi_{P_1}(e_1 \cdots e_L) \exp(\frac{1}{2} T a^* a^*)\Omega_{P_1} \in \mathfrak{F}_{P_1}
\]
where the determinant has to be computed on \( \mathcal{X}_1 \), \( \{e_1, \ldots, e_L\} \) is an orthonormal basis in \( \mathfrak{h} \), and the exponential term has a well-defined meaning as a strongly convergent series on a dense domain containing \( \Omega_{P_1} \). The vector \( \Omega_P \) belongs to \( H_{\mathcal{O}_V}\Omega_{P_1} \); in fact, the latter Hilbert space consists precisely of the vectors in \( \mathfrak{F}_{P_1} \) which induce extensions of the partial Fock state \( \omega_{P_1} \circ \mathcal{O}_V^{-1}|_{\text{ran } \mathcal{O}_V} \).

\footnote{An operator \( A \) on \( \mathcal{X} \) is antisymmetric if \( A^* = -A \).}
A complete orthonormal basis in $H_{\vartheta \nu} \Omega_{P_1}$ can be obtained by applying suitable partial isometries from the commutant of $\text{ran} \vartheta \nu$ to $\Omega_P$. The basis projection $P$ leaves $\ker V^*$ invariant. Let

$$\mathfrak{t} \equiv P(\ker V^*), \quad \text{with} \quad \dim \mathfrak{t} = \frac{1}{2} \text{ind} V, \quad (2.11)$$

and let $\{g_j\}$ be an orthonormal basis in $\mathfrak{t}$. For any multi-index $\alpha = (\alpha_1, \ldots, \alpha_l)$, $1 \leq \alpha_1 < \cdots < \alpha_l \leq \frac{1}{2} \text{ind} V$ (resp. $\alpha = 0$ if $l = 0$), set

$$\Omega_\alpha \equiv \psi_{P_1}(g_{\alpha_1} \cdots g_{\alpha_l}) \Omega_P. \quad (2.12)$$

Then the $\Omega_\alpha$ constitute an orthonormal basis in $H_{\vartheta \nu} \Omega_{P_1}$, and they determine an orthonormal basis $\{\Psi_\alpha\}$ in $H_{\vartheta \nu}$ via

$$\Psi_\alpha \pi_{P_1}(a) \Omega_{P_1} = \pi_{P_1}(\vartheta \nu(a)) \Omega_\alpha. \quad (2.13)$$

Since $\psi_{P_1}$ is a representation of the canonical anticommutation relations and since $\Omega_P$ is annihilated by the operators $\psi_{P_1}(g_j)^*$, the spaces $H_{\vartheta \nu}$ and $H_{\vartheta \nu} \Omega_{P_1}$ can both be identified with the antisymmetric Fock space over $\mathfrak{t}$. \cite{Bin95, Bin98a}.

### 2.2. Gauge invariant quasi-free endomorphisms (CAR)

Let $G \subset \text{End}_{P_1}(\mathfrak{K})$ be a group consisting of unitary operators which commute with $P_1$. The usual second quantization of $U \in G$ (or, more precisely, of $U_{11}$) will be denoted by $\Gamma(U)$; the map $U \mapsto \Gamma(U)$ is strongly continuous. The corresponding gauge automorphisms, which leave $\omega_{P_1}$ invariant, will be denoted by $\gamma_U$.

We are interested in the representation $D_{\vartheta \nu}$ of $G$ on the Hilbert space $H_{\vartheta \nu}$ which implements a gauge invariant quasi-free endomorphism $\vartheta \nu$. **Gauge invariant implementable quasi-free endomorphisms are given by the elements of the semigroup**

$$\text{End}_{P_1}(\mathfrak{K})^G \equiv \{V \in \text{End}_{P_1}(\mathfrak{K}) \mid [V, G] = 0\}.$$  

To determine $D_{\vartheta \nu}$ up to unitary equivalence, it suffices to calculate the transformed vectors $\Gamma(U) \Omega_\alpha$, $U \in G$, where the $\Omega_\alpha$ are the basis elements in $H_{\vartheta \nu} \Omega_{P_1}$ defined in (2.12). The basic observation is that the objects entering the construction of the $\Omega_\alpha$, namely the spaces $\mathfrak{t}$ and $\mathfrak{t}$ and the operator $T$, are all gauge invariant.

**Lemma 2.1.** Let $U \in G$. Then $\Gamma(U)$ implements the gauge automorphism $\gamma_U$ in the twisted Fock representation $\psi_{P_1}$:

$$\Gamma(U) \psi_{P_1}(a) = \psi_{P_1}(\gamma_U(a)) \Gamma(U), \quad a \in \mathfrak{C}(\mathfrak{K}).$$

**Proof.** Since $U$ commutes with $P_1$, the implementer $\Gamma(U)$ is even:

$$[\Gamma(U), \Gamma(-1)] = 0.$$

This implies that $[\Gamma(U), \theta(-1)] = 0$ (see (2.4)) so that, by (2.5),

$$\Gamma(U) \psi_{P_1}(a) \Gamma(U)^* = \theta(-1) \Gamma(U) \pi_{P_1}(a) \Gamma(U)^* \theta(-1)^* = \psi_{P_1}(\gamma_U(a)).$$

**Lemma 2.2.** Let $V \in \text{End}_{P_1}(\mathfrak{K})^G$. Then $\exp(\frac{1}{2} T a^* a) \Omega_{P_1}$, with $T$ defined by (2.7), is invariant under all gauge transformations $\Gamma(U)$, $U \in G$.

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\* The results in this section hold also for non-compact groups (relative to the strong topology). However, the close relationship between representations of $G$ and superselection sectors is then lost. We also do not require that $-1 \in G$, which would be necessary if the $G$-invariant elements of $\mathfrak{C}(\mathfrak{K})$ were to be interpreted as physical observables.
Proof. If $S$ is an antisymmetric operator from $\mathcal{H}_1$ to $\mathcal{H}_2$ of finite rank, then one readily verifies that
\[ \Gamma(U)(\frac{1}{2}S^{\ast}a^{\ast}a^{\ast})\Gamma(U)^{\ast} = \frac{1}{2}(USU^{\ast})a^{\ast}a^{\ast} \]
(here the expressions of the form $S^{\ast}a^{\ast}$ are defined by expanding $S = \sum f_j \langle g_j, \cdot \rangle$ with $f_j, g_j \in \mathcal{H}_1$, and by setting $S^{\ast}a^{\ast} = \sum a^{\ast}(f_j)a^{\ast}(g_j)$). Approximating $T$ by such finite rank operators in the Hilbert–Schmidt norm (cf. [CR87]), one finds that
\[ \Gamma(U)(\frac{1}{2}T^{\ast}a^{\ast}a^{\ast})\Omega_{P_1} = \left(\frac{1}{2}(UTU^{\ast})a^{\ast}a^{\ast}\right)^{\ast}\Omega_{P_1}, \quad n \in \mathbb{N}, \]
because $\Gamma(U)\Omega_{P_1} = \Omega_{P_1}$. It follows that
\[ \Gamma(U)\exp\left(\frac{1}{2}T^{\ast}a^{\ast}a^{\ast}\right)\Omega_{P_1} = \sum_{n=0}^{\infty} \frac{1}{n!}\Gamma(U)(\frac{1}{2}T^{\ast}a^{\ast}a^{\ast})^{n}\Omega_{P_1} \]
\[ = \exp\left(\frac{1}{2}(UTU^{\ast})a^{\ast}a^{\ast}\right)\Omega_{P_1}. \]
Since $U$ commutes with $P_1, P_2$ and $V$, it also commutes with all components of $V$ and $V^{\ast}$, including the operators $V_{11}^{-1}, [\ker V_{11}^{\ast}]$ etc. The operator $T$ is by (2.7) a bounded function of these components, so that
\[ [U, T] = 0, \quad U \in G. \quad (2.14) \]
Hence we get $\Gamma(U)\exp\left(\frac{1}{2}T^{\ast}a^{\ast}a^{\ast}\right)\Omega_{P_1} = \exp\left(\frac{1}{2}T^{\ast}a^{\ast}a^{\ast}\right)\Omega_{P_1}$ as claimed. \hfill \Box

Setting $\delta \equiv (\det(P_1 + T^{\ast}T))^{-1/4}$, we thus arrive at the following formula:
\[ \Gamma(U)\Omega_{\alpha} = \delta \cdot \psi_{P_1}(U(g_{\alpha_1}) \cdots U(g_{\alpha_1})U(e_1) \cdots U(e_L)) \exp\left(\frac{1}{2}T^{\ast}a^{\ast}a^{\ast}\right)\Omega_{P_1}. \quad (2.15) \]

Theorem 2.3. Let $P_1$ be a basis projection of $\mathcal{H}$, let $G$ be a group of unitary operators on $\mathcal{H}$ commuting with $P_1$ and with complex conjugation, and let $V \in \text{End}_{P_1}(\mathcal{H})^G$. Then the finite dimensional subspace $\mathfrak{h} \subset \mathcal{H}$ and the $\frac{1}{\sqrt{2}}\text{ind} V$ dimensional subspace $\mathfrak{t} \subset \mathcal{H}$ associated with $V$ by (2.9) and (2.11) are both invariant under $G$. Let $\Lambda_{\mathfrak{h}}$ be the unitary representation of $G$ on the antisymmetric Fock space over $\mathfrak{h}$ that is obtained by taking antisymmetric tensor powers of the representation on $\mathfrak{h}$. Then the unitary representation $D_{\mathfrak{h}V}$ of $G$ on the Hilbert space of isometries $H_{\mathfrak{h}V}$ which implements $\varphi_{\mathfrak{h}V}$ in the Fock state $\omega_{P_1}$ is unitarily equivalent to $\Lambda_{\mathfrak{t}}$, tensored with the one dimensional representation $\text{det}_{\mathfrak{h}}(U) \equiv \det(U|_{\mathfrak{h}})$:
\[ D_{\mathfrak{h}V} \simeq \text{det}_{\mathfrak{h}} \otimes \Lambda_{\mathfrak{t}}. \quad (2.16) \]

Proof. The subspace $\mathfrak{h} = V_{12}(\ker V_{22})$ is invariant under $G$ because $G$ commutes with the components of $V$ (cf. the proof of Lemma 2.2). Since $\{e_1, \ldots, e_L\}$ is an orthonormal basis in $\mathfrak{h}$, it follows from the canonical anticommutation relations that
\[ U(e_1) \cdots U(e_L) = \det(U|_{\mathfrak{h}}) \cdot e_1 \cdots e_L, \quad U \in G. \]

Similarly, $G$ commutes with the basis projection $P$ (cf. (2.8) and (2.14)) and leaves $\ker V^{\ast}$ invariant, so that $\mathfrak{t} = P(\ker V^{\ast})$ is also left invariant. It then follows from the canonical anticommutation relations that $g_{\alpha_1} \cdots g_{\alpha_1}$ transforms like the $l$-fold antisymmetric tensor product of $g_{\alpha_1}, \ldots, g_{\alpha_1}$ under $G$.

Thus we see from (2.17) that the representation of $G$ on $H_{\mathfrak{h}V}|_{\Omega_{P_1}}$ is unitarily equivalent to $\text{det}_{\mathfrak{h}} \otimes \Lambda_{\mathfrak{t}}$, and the same holds true for the representation $D_{\mathfrak{h}V}$. \hfill \Box

Remarks. (i) Theorem 2.3 shows that non-surjective quasi-free endomorphisms $\varphi_{\mathfrak{h}V}$ are always reducible in the sense that the representation $D_{\mathfrak{h}V}$ (or, if $G$ is compact, the representation induced by $g_{\mathfrak{h}V}$ of the pointwise gauge invariant “observable” algebra $\mathcal{C}(\mathcal{H})^G$ on the subspace of $\Gamma(G)$-invariant vectors in $\mathfrak{F}_{P_1}$) is reducible. In fact, each “$n$-particle” subspace of $H_{\mathfrak{h}V}$, i.e. the closed linear span of all $\Psi_\alpha$ with $\alpha$ of length $n$, is invariant under $G$, and may decompose further. Let $D^{(n)}_{\mathfrak{h}V}$ be
the restriction of $D_{\nu}$ to this subspace. Closest to irreducibility is the case that at least $D_{\nu}^{(1)}$ is irreducible. In typical situations, the remaining representations $D_{\nu}^{(n)}$ will then also be irreducible. This happens for instance if $G \cong U(N)$ or $G \cong SU(N)$, and $\mathfrak{t}$ carries the defining representation of $G$. In the $U(N)$ case, the $D_{\nu}^{(n)}$ are not only irreducible, but also mutually inequivalent. In the $SU(N)$ case, the representations $D_{\nu}^{(0)}, \ldots, D_{\nu}^{(N-1)}$ are mutually inequivalent, but $D_{\nu}^{(N)}$ is equivalent to $D_{\nu}^{(0)}$. In general, it can Nevertheless happen that $D_{\nu}^{(1)}$ is irreducible but some $D_{\nu}^{(n)}$ are not, as is the case if $G \cong SO(N)$ ($N > 2$ even) and $\mathfrak{t}$ carries the defining representation of $G$ (cf. [Wey46, Boe70]). If already $D_{\nu}^{(1)}$ is reducible, then one has an additional Clebsch–Gordan type splitting.

(ii) Theorem 2.3 characterizes the representation $D_{\nu}$ associated with a fixed gauge invariant endomorphism $\nu$ in terms of the representations on $\mathfrak{h}$ and $\mathfrak{t}$. The question which representations of $G$ can be realized on the spaces $\mathfrak{h}$ and $\mathfrak{t}$ by letting $V$ vary through $\text{End}_{P_1} (\mathcal{K})^G$ is studied in [Bin98a]. In typical field theoretic situations, where the single-particle space $\mathcal{K}_1$ carries an irreducible representation of a compact group together with its complex conjugate, both with infinite multiplicity, as e.g. in Section 2.3 below, one finds that any irreducible representation of $G$ realized on Fock space $\mathcal{F}$ is equivalent to a subrepresentation of some $D_{\nu}$.

(iii) A special case worth mentioning is the case $G \cong U(1)$ and $\text{ind } V = 0$, i.e. the case of the restricted unitary group. It is well-known from the work on the external field problem (see e.g. [CHOS82]) that the charge of elements of the restricted unitary group is given by a certain Fredholm index $\text{ind } V_{++}$ (which has nothing to do with the index of $V$, but refers to a finer decomposition $V_{11} = V_{++} \oplus V_{--}$). This fact can be easily derived from our general result: The factor $\Lambda_\mathfrak{r}$ in (2.10) becomes trivial, whereas

$$\det_{\mathfrak{h}}(U_\lambda) = \exp(i \lambda \text{ ind } V_{++})$$

if $U_\lambda \in G$ corresponds to $e^{i \lambda} \in U(1)$ [Bin98a].

Similarly, in the case $G = \{ \pm 1 \} \cong \mathbb{Z}_2$ and $\text{ind } V = 0$, the factor $\Lambda_\mathfrak{r}$ is trivial, but

$$\det_{\mathfrak{h}}(-1) = (-1)^{\dim \mathfrak{h}} = (-1)^{\dim \ker V_{11}}$$

yields the $\mathbb{Z}_2$-index of Araki and Evans [AE83, Ara87, EK98].

(iv) If the single-particle space $\mathcal{K}_1$ decomposes into the direct sum of two unitarily equivalent $G$-modules (“particle-antiparticle symmetry”), then there exists an involutive automorphism $V \mapsto V^c$ of $\text{End}_{P_1} (\mathcal{K})^G$ such that the spaces $\mathfrak{h}^c$ and $\mathfrak{t}^c$ corresponding by (2.6) and (2.11) to $V^c$ are, as $G$-modules, antimitarily equivalent to the spaces $\mathfrak{h}$ and $\mathfrak{t}$ corresponding to $V$. That is, the representation $D_{\nu}^c$ is unitarily equivalent to the complex conjugate of $D_{\nu}$ (charge conjugation).

2.3. An example: Localized endomorphisms of the chiral Dirac field. In Section 2.2 we have analyzed the charge quantum numbers of gauge invariant implementable quasi-free endomorphisms in complete generality. In particular, and in sharp contrast to the field theoretic situation, it was not necessary to assume any localization properties of endomorphisms. If one could find, in a specific model, a localized implementable quasi-free endomorphism, then our methods would apply and could be used to determine its charge and to construct the corresponding local fields. It is however not clear from the outset whether localization and implementability are compatible with each other. To show that this is in fact the

*Known results concerning this question are restricted to the case of automorphisms. Building on the work of Carey and Ruijsenaars [CR87] and others, we constructed in [Bin93] a family of (implementable and transportable) localized endomorphisms, carrying arbitrary $U(1)$-charges, of the free Dirac field in two spacetime dimensions with arbitrary mass. The operators $V \in \text{End}_{P_1} (\mathcal{K})^U(1)$ belonging to these automorphisms are given by two $U(1)$-valued functions which
case, we will present below an explicit example of a non-surjective implementable localized quasi-free endomorphism of the free massless Dirac field in two spacetime dimensions.

Let us first introduce the free Dirac field with global $U(N)$ symmetry. Let
\[ \mathcal{H} \equiv L^2(\mathbb{R}^{2n-1}, \mathbb{C}^{2^n}) \]  
be the single-particle space of the time-zero Dirac field in $2n$ spacetime dimensions. Let $H = -i\alpha \nabla + \beta m$ be the free Dirac Hamiltonian, with spectral projections $E_\pm$ corresponding to the positive resp. negative part of the spectrum of $H$. Tensored with $1_N$, these operators act on the space
\[ \mathcal{H}' \equiv \mathcal{H} \otimes \mathbb{C}^N. \]  
The gauge group $U(N)$ also acts naturally on $\mathcal{H}'$. In the selfdual CAR formalism, one sets
\[ \mathcal{X} \equiv \mathcal{H}' \oplus \mathcal{H}'^*, \]  
where $\mathcal{H}'^*$ is the Hilbert space conjugate to $\mathcal{H}'$. There is a natural conjugation $f \mapsto f^*$ on $\mathcal{X}$ which is inherited from the antiunitary identification map $\mathcal{H}' \to \mathcal{H}'^*$. The basis projection $P_1$ corresponding to the vacuum representation of the field is given by
\[ P_1 \equiv E'_+ \oplus \overline{E'_-} \]  
with $E'_\pm = E_\pm \otimes 1_N$. Gauge transformations act like $U = (1_\mathcal{X} \otimes u) \oplus (1_{\mathcal{X}'^*} \otimes \overline{\mathcal{U}})$, $u \in U(N)$, on $\mathcal{X}$. They commute with $P_1$. The field operators $\varphi_t$ at time $t$ are given by
\[ \varphi_t(f) \equiv \pi_{P_1}(e^{itH'} f) = a(E'_+ e^{itH'} f)^+ + a(\overline{E'_-} e^{-itH'} f^*) \]  
with $H' \equiv H \otimes 1_N$, $f \in \mathcal{H}'$. They are solutions of the Dirac–Schrödinger equation
\[ -i \frac{d}{dt} \varphi_t(f) = \varphi_t(H' f), \quad f \in D(H'). \]

If $O$ is a double cone in Minkowski space with base $B \subset \mathbb{R}^{2n-1}$ at time $t$, then the local field algebra associated with $O$ is the von Neumann algebra generated by all $\varphi_t(f)$ with supp $f \subset B$. The local observable algebras are the fixed point subalgebras of the local field algebras under the gauge action. A whole net of local algebras is generated from these special ones by applying Lorentz transformations.

Gauge invariant implementable localized endomorphisms of the $N$-component Dirac field can be characterized as follows. A quasi-free endomorphism $\varrho_V$ is gauge invariant if and only if $V$ has the form
\[ V = (v \otimes 1_N) \oplus (\overline{v} \otimes 1_N) \]  
with respect to the decomposition (2.13), where $v$ is an isometry of $\mathcal{H}$. This follows from the fact that the defining representation of $U(N)$ and its complex conjugate are disjoint, so that the commutant of $G$ on $\mathcal{X}$ is given by
\[ G' = (\mathfrak{B}(\mathcal{H}) \otimes 1_N) \oplus (\mathfrak{B}(\mathcal{H}^*) \otimes 1_N). \]

For $V$ of the form (2.20) one has
\[ [P_1, V] = ([E_+, v] \otimes 1_N) \oplus ([E_-, v] \otimes 1_N) \]  
so that the implementability condition (2.3) holds if and only if
\[ [E_+, v] \text{ and } [E_-, v] \text{ are Hilbert–Schmidt.} \]  

are equal to 1 at spacelike infinity, and the charge ind $V_{+\pm}$ (cf. Remark (iii) in Section 2.2) of $\varrho_V$ is equal to the difference of the winding numbers of these functions. However, unlike in two dimensions, there seem to be no known examples of implementable charge-carrying automorphisms in the case $G \cong U(1)$ in four spacetime dimensions.
Therefore $\text{End}_P(\mathcal{X})^{U(N)}$ is isomorphic to the semigroup of all isometries $v$ of $\mathcal{H}$ which fulfill \eqref{eq:2.21}.

An endomorphism of the algebra of all local observables is \textit{localized} in a bounded region $O$ in Minkowski space if it acts like the identity on observables which are localized in bounded regions contained in the spacelike complement $O'$ of $O$.\cite{Haa96}. Localized elements of $\text{End}_P(\mathcal{X})^{U(N)}$ (at time zero) can be characterized as follows.

**Proposition 2.4.** Let $O$ be a double cone with base $B \subset \mathbb{R}^{2n-1}$ at time zero. Let $V$ be an element of $\text{End}_P(\mathcal{X})^{U(N)}$, and let $v$ be the isometry of $\mathcal{H}$ associated with $V$ by \eqref{eq:2.20}. Then $\varrho_V$ is localized in $O$ if and only if there exists, for each connected component $\Delta$ of $\mathbb{R}^{2n-1} \setminus B$, a phase factor $\tau_\Delta \in U(1)$ such that

$$v(f) = \tau_\Delta f \quad \text{for all } f \in \mathcal{H} \text{ with } \text{supp} f \subset \Delta.$$  \hfill (2.22)

**Proof.** Assume that $\varrho_V$ is localized in $O$. Let $b_1, \ldots, b_N$ be the standard basis in $\mathbb{C}^N$, let $\Delta$ be a component of the complement of $B$, and let $f, g \in \mathcal{H}$ with $\text{supp} f, g \subset \Delta$. Then

$$a(f, g) \equiv \sum_{j=1}^N (f \otimes b_j)(g \otimes b_j)^*$$

is gauge invariant, and $\pi_P(a(f, g))$ is an observable localized in $O'$. Since $\varrho_V$ is localized in $O$, one has $a(f, g) = \varrho_V(a(f, g)) = \sum_j (v(f) \otimes b_j)(v(g) \otimes b_j)^*$. Since the $b_j$ are linearly independent, it follows that

$$(f \otimes b_j)(g \otimes b_j)^* = (v(f) \otimes b_j)(v(g) \otimes b_j)^*, \quad j = 1, \ldots, N.$$ \hfill (2.23)

Now let $P'$ be the (basis) projection onto $\mathcal{H}' \subset \mathcal{K}$, and let $\omega_{P'}$ be the corresponding Fock state. One has

$$\omega_{P'}((f \otimes b_j)^*(f \otimes b_j)(f \otimes b_j)^*(f \otimes b_j)) = \|f\|^4,$$

and, since $(v(f) \otimes b_j)^*$ belongs to the annihilator ideal of $\omega_{P'}$,

$$\omega_{P'}((f \otimes b_j)^*(v(f) \otimes b_j)(v(g) \otimes b_j)^*(f \otimes b_j)) = |\langle v(f), f \rangle|^2.$$  \hfill (2.24)

Therefore one gets from \eqref{eq:2.23}, in the special case $f = g$, that $\|f\|^1 = |\langle v(f), f \rangle|$. It follows that there exists $\tau_f \in U(1)$ such that $v(f) = \tau_f f$. By the same argument, $v(g) = \tau_g g$ for some $\tau_g \in U(1)$. Then \eqref{eq:2.23} yields that $\tau_f = \tau_g$. Therefore these phase factors depend only on $\Delta$ and not on the functions.

Conversely, assume that \eqref{eq:2.22} holds. Then $\varrho_V$ acts on fields localized in bounded regions in $O'$ like a gauge transformation, and therefore like the identity on observables localized in $O'$. It follows that $\varrho_V$ is localized in $O$. \hfill \Box

Of course, $\mathbb{R}^{2n-1} \setminus B$ is connected if $n > 1$, but it has two connected components if $n = 1$. This is the basic reason for the possible occurrence of braid group statistics and soliton sectors in two dimensional Minkowski space.

Next let us demonstrate that at least the free massless Dirac field in two space-time dimensions possesses non-surjective implementable localized quasi-free endomorphisms. It suffices to consider one chiral component of the field. Thus consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ with Dirac Hamiltonian $-i\frac{d}{dx}$. It is convenient to transform to the Hilbert space $L^2(S^1)$ via the Cayley transform $\vartheta$  

$$\vartheta : \mathbb{R} \cup \{\infty\} \to S^1, \quad x \mapsto -e^{2i \arctan x} = \frac{x - i}{x + i} \quad (\text{cf. \cite{CR87}}).$$

$\vartheta$ induces a unitary transformation $\tilde{\vartheta}$  

$$\tilde{\vartheta} : L^2(S^1) \to L^2(\mathbb{R}), \quad (\tilde{\vartheta} f)(x) = \pi^{-\frac{1}{2}} f(\vartheta(x)) \frac{x - i}{x + i}. \quad (\text{More precisely, the normal extension of } \varrho_V \text{ in the representation } \pi_P \text{ is localized in } O).$$
The important point is that the spectral projections $E_{\pm}$ of $-i\frac{d}{dz}$ are transformed into the Hardy space projections: Set $\tilde{E}_{\pm} = \vartheta^{-1}E_{\pm}\vartheta$, then

$$\tilde{E}_+ = \sum_{n\geq 0} e_n(e_n, \cdot), \quad \tilde{E}_- = \sum_{n<0} e_n(e_n, \cdot), \quad e_n(z) \equiv z^n \ (z \in S^1, n \in \mathbb{Z}).$$

We want to construct an isometry $v$ of $L^2(S^1)$ with $[\tilde{E}_+, v]$ Hilbert–Schmidt (implementability), with $\text{ind } v = 1$ (close to irreducibility, cf. Rem. (i) in Section 2.4), and such that $v(f) = f$ for all $f \in L^2(S^1)$ with supp $f \subset S^1 \setminus I$, where $I \subset S^1$ is a fixed interval (localization). As localization region we shall choose the interval

$$I \equiv \{ e^{i\lambda} \mid \frac{\pi}{2} \leq \lambda \leq \frac{3\pi}{2} \}$$

which corresponds, by the inverse Cayley transform, to the interval $\vartheta^{-1}(I) = [-1, 1]$ in $\mathbb{R}$. We need the following orthonormal basis $(f_m)_{m \in \mathbb{Z}}$ in $L^2(I) \subset L^2(S^1)$

$$f_m(z) \equiv \sqrt{2}(-1)^m z^{2m} \chi_I(z), \quad z \in S^1,$$

where $\chi_I$ is the characteristic function of $I$. We now define the isometry $v$ by

$$v \equiv 1 + \sum_{m \geq 0} (f_{m+1} - f_m)(f_m, \cdot). \quad (2.24)$$

Note that $v$ acts like the identity on functions with support in $S^1 \setminus I$, that $v(f_m) = f_m$ if $m < 0$, and that $v$ acts like the unilateral shift on the remaining $f_m$: $v(f_m) = f_{m+1}$ if $m \geq 0$.

**Lemma 2.5.** The commutators $[\tilde{E}_+, v]$ and $[\tilde{E}_-, v]$ are Hilbert–Schmidt.

**Proof.** The rather lengthy estimates of the Hilbert–Schmidt norms of these commutators, which are essentially due to P. Grinevich, can be found in [Bin98b]. \(\square\)

Thus the operator $V \in \text{End}_{P_1}(\mathcal{K})^U(N)$ induced by $v$ via (2.20) yields a localized endomorphism $\varrho_V$ of the chiral Dirac field. Since by construction

$$\frac{1}{2} \text{ind } V = N,$$

it is clear that the space $\mathfrak{f}$ associated with $V$ by (2.11) carries either the defining representation of $U(N)$ or its complex conjugate. By Remark (i) in Section 2.2 the irreducible constituents of $\varrho_V$ correspond to the irreducible, mutually inequivalent representations $D_{\nu}^N$, $n = 0, \ldots, N$. Note that the same isometry $v$ gives rise to localized gauge invariant implementable endomorphisms for arbitrary symmetry groups $G$, by replacing the defining representation of $U(N)$ in Eq. (2.18) with a suitable finite dimensional representation of $G$.

3. The Bosonic Case

We need some preparations from [Bin98b]. The exposition will closely follow the lines of the Fermionic case considered in Section 2.

3.1. Preliminaries on the implementation of quasi-free endomorphisms (CCR). We start with a Fock representation of the canonical commutation relations. Thus we may assume as above that $\mathcal{K}$ is an infinite dimensional separable complex Hilbert space with a complex conjugation $f \mapsto f^*$, and that $P_1$ is a fixed basis projection. Let $P_2$ be the complementary projection. Define a self-adjoint unitary operator

$$C \equiv P_1 - P_2$$

so that $C^* = -C$, and a nondegenerate hermitian sesquilinear form

$$\kappa(f, g) \equiv \langle f, Cg \rangle$$
so that
\[ \kappa(f^*, g^*) = -\kappa(g, f), \quad f, g \in \mathcal{K}. \]

It must be emphasized that the basic form on \( \mathcal{K} \) which determines the canonical commutation relations is \( \kappa \) and not the Hilbert space inner product. In fact, the latter depends on the choice of the Fock state, i.e. on the choice of \( P_1 \).

The \textit{(selfdual) CCR algebra} \( \mathfrak{C}(\mathcal{K}, \kappa) \) over \( (\mathcal{K}, \kappa) \) is the simple \(*\)-algebra which is generated by 1 and elements \( f \in \mathcal{K} \), subject to the commutation relation \[ \kappa(f^*, g^*) = -\kappa(g, f), \quad f, g \in \mathcal{K}. \]

The \textit{Weyl algebra} \( \mathfrak{M}(\mathcal{K}, \kappa) \) over \( (\mathcal{K}, \kappa) \) is the simple \( C^* \)-algebra generated by unitary operators \( w(f) \), \( f \in \mathcal{K} \) with \( f = f^* \), subject to the relations
\[ w(f)^* = w(-f), \quad w(f)w(g) = e^{-\frac{i}{4}\kappa(f, g)}w(f + g). \]

The \textit{Fock state} \( \omega_{P_1} \) over \( \mathfrak{C}(\mathcal{K}, \kappa) \) induced by \( P_1 \) is again determined by condition \( (2.1) \). The GNS representation \( (\mathfrak{F}_{P_1}, \pi_{P_1}, \Omega_{P_1}) \) of \( \omega_{P_1} \) can be identified with the representation \( \pi_{P_1} \) given by formula \( (2.2) \), where \( a^*(f) \) and \( a(f) \), \( f \in \mathcal{K}_1 \), now are Bosonic creation and annihilation operators, acting on the symmetric Fock space \( \mathfrak{F}_{P_1} \) over \( \mathcal{K}_1 \) with Fock vacuum vector \( \Omega_{P_1} \). All operators \( \pi_{P_1}(a) \), \( a \in \mathfrak{C}(\mathcal{K}, \kappa) \), are defined on the invariant dense domain \( \mathcal{D} \subset \mathfrak{F}_{P_1} \) of algebraic tensors, are closable, and fulfill \( \pi_{P_1}(a^*) \subset \pi_{P_1}(a)^* \).

The irreducible Fock representation of the Weyl algebra \( \mathfrak{M}(\mathcal{K}, \kappa) \) induced by \( P_1 \) is obtained by identifying the Weyl operator \( w(f) \), \( f = f^* \in \mathcal{K} \), with the exponential of the closure of \( i\pi_{P_1}(f) \). The vacuum expectation value of \( w(f) \) is
\[ \langle \Omega_{P_1}, w(f)\Omega_{P_1} \rangle = e^{-\frac{i}{4}\|f\|^2}, \quad f = f^*. \]

Let \( \mathcal{H} \) be a subspace of \( \mathcal{K} \) with \( \mathcal{H} = \mathcal{H}^* \), and let \( \mathfrak{M}(\mathcal{H}) \) be the \( C^* \)-algebra generated by all \( w(f) \) with \( f = f^* \in \mathcal{H} \). If \( \mathcal{H}^2 \) is the orthogonal complement of \( \mathcal{H} \) with respect to \( \kappa \), then
\[ \mathfrak{M}(\mathcal{H})' = \mathfrak{M}(\mathcal{H}^2)' \quad (\text{duality}). \]

Every operator \( V \) on \( \mathcal{K} \) which preserves the form \( \kappa \) and which commutes with complex conjugation extends to a unique \textit{quasi-free endomorphism} \( \varrho_V \) of \( \mathfrak{C}(\mathcal{K}, \kappa) \): \[ \varrho_V(f) = V(f), \quad f \in \mathcal{K}, \]
and to a unique \(*\)-endomorphism, denoted by the same symbol, of \( \mathfrak{M}(\mathcal{K}, \kappa) \): \[ \varrho_V(w(f)) = w(V(f)), \quad f = f^*. \]

As shown in \[ \text{Bin98b} \], an endomorphism \( \varrho_V \) of \( \mathfrak{M}(\mathcal{K}, \kappa) \) can be implemented by a Hilbert space of isometries \( H_{\varrho_V} \) on \( \mathfrak{F}_{P_1} \) as in \[ \text{(2.3)} \] if and only if the Hilbert–Schmidt condition \( (2.3) \) holds. Such operators \( V \) form a semigroup
\[ \text{End}_{P_1}(\mathcal{K}, \kappa) \equiv \{ V \in \mathfrak{B}(\mathcal{K}) \mid V^\dagger V = 1, \quad V = V, \quad [P_1, V] \text{ is HS} \} \]

isomorphic to the semigroup of all implementable quasi-free endomorphisms of \( \mathfrak{M}(\mathcal{K}, \kappa) \). Here we use the notation
\[ A^\dagger \equiv CA^*C \]
for the adjoint of an operator \( A \) on \( \mathcal{K} \) relative to the form \( \kappa \). Every \( V \in \text{End}_{P_1}(\mathcal{K}, \kappa) \) has a well-defined Fredholm index:
\[ \text{ind } V = \dim \ker V^\dagger \in \{ 0, 2, 4, \ldots, \infty \}. \]

The \textit{statistics dimension} \( d_{\varrho_V} \) of \( \varrho_V \) is in the Bosonic case given by
\[ d_{\varrho_V} \equiv \dim H_{\varrho_V} = \begin{cases} 1, & \text{ind } V = 0, \\ \infty, & \text{ind } V \neq 0. \end{cases} \quad (3.1) \]
In order to obtain an orthonormal basis in the Hilbert space \( H_{\mathcal{H}} \Omega_{P_1} \) associated with a fixed \( V \in \text{End}_{P_1}(\mathcal{H}) \), it is again convenient to extend the partial Fock state \( \omega_{P_1} \circ \theta_{V}^{-1} \) to a proper Fock state \( \omega_P \) which is unitarily equivalent to \( \omega_{P_1} \). The basis projection \( P \) corresponding to this new Fock state has the form

\[
P \equiv V P_1 V^\dagger + p
\]

where the basis projection \( p \) of \( \ker V^\dagger \) is defined as follows. Let \( E \) be the orthogonal projection onto \( \ker V^\dagger \), and let \( A \equiv ECE \) be the operator describing the restriction of \( \kappa \) to \( \ker V^\dagger \) with respect to the scalar product. Let \( A_+ \) be the positive part of \( A \), i.e. the unique positive operator such that \( A = A_+ - A_- \) and \( A_+ A_- = 0 \). Let \( A_+^{-1} \) be defined on \( \text{ran} A_+ \) as the inverse of \( A_+ \), and on \( (\text{ran} A_+)^\perp \) as zero. Then \( p \) is defined as

\[
p \equiv A_+^{-1} C.
\]

The class of all Fock states over \( \mathcal{M} \) which are unitarily equivalent to \( \omega_{P_1} \) is parameterized by symmetrized Hilbert–Schmidt operators \( T \) from \( \mathcal{K}_1 \) to \( \mathcal{K}_2 \) with \( ||T|| < 1 \). The operator \( T \) corresponding to \( P \) is again given by (2.10):

\[
T \equiv P_{21} P_{11}^{-1}, \tag{3.2}
\]

whereas \( P \) can be recovered from \( T \) by

\[
P = (P_1 + T)(P_1 + T^\dagger T)^{-1}(P_1 + T^\dagger).
\]

The cyclic vector \( \Omega_P \) in \( \mathcal{F}_{P_1} \), unique up to a phase, which induces the state \( \omega_P \), is given by

\[
\Omega_P = (\det(P_1 + T^\dagger T))^{1/4} \exp(-\frac{1}{2} T^\dagger a^* a \dagger T) \Omega_{P_1}.
\]

It belongs to \( H_{\mathcal{H}} \Omega_{P_1} \). A complete orthonormal basis in \( H_{\mathcal{H}} \Omega_{P_1} \) is obtained by applying certain isometries from the commutant of \( \text{ran} g_V \) to \( \Omega_P \). The basis projection \( P \) leaves \( \ker V^\dagger \) invariant, and \( \kappa \) is positive definite on \( \text{ran} P \). Let

\[
\xi \equiv P(\ker V^\dagger), \quad \text{with dim} \xi = \frac{1}{2} \text{ind} V, \tag{3.3}
\]

and let \( g_1, g_2, \ldots \) be a basis in \( \xi \) such that \( \kappa(g_j, g_k) = \delta_{jk} \). Let \( \psi_j \) be the isometry obtained by polar decomposition of the closure of \( \pi_{P_1}(g_j) \). For any multi-index \( \alpha = (\alpha_1, \ldots, \alpha_l) \) with \( 1 \leq \alpha_j \leq \alpha_{j+1} \leq \frac{1}{2} \text{ind} V \) (\( \alpha = 0 \) if \( l = 0 \)), set

\[
\Omega_{\alpha} \equiv \psi_{\alpha_1} \cdots \psi_{\alpha_l} \Omega_P. \tag{3.4}
\]

Then the \( \Omega_{\alpha} \) form an orthonormal basis in \( H_{\mathcal{H}} \Omega_{P_1} \) and, by (2.13), induce an orthonormal basis in \( H_{\mathcal{H}} \mathcal{F}_{P_1} \). The spaces \( H_{\mathcal{H}} \mathcal{F}_{P_1} \) and \( H_{\mathcal{H}} \Omega_{P_1} \) can both be identified with the symmetric Fock space over \( \xi \).

### 3.2. Gauge invariant quasi-free endomorphisms (CCR)

We assume again that a group \( G \subset \text{End}_{P_1}(\mathcal{H}, \kappa) \) consisting of unitary operators \( U \) which commute with \( P_1 \) (so that \( G \) can be identified with a subgroup of \( U(\mathcal{K}_1) \)) acts by second quantization \( \Gamma(U) \) on \( \mathcal{F}_{P_1} \). Gauge invariant implementable quasi-free endomorphisms correspond to the elements of the semigroup

\[
\text{End}_{P_1}(\mathcal{H}, \kappa)^G \equiv \{ V \in \text{End}_{P_1}(\mathcal{H}, \kappa) \mid [V, G] = 0 \}.
\]

To determine the representation \( D_{\mathcal{H}} \) of \( G \) on the Hilbert space \( H_{\mathcal{H}} \mathcal{F}_{P_1} \) associated with \( V \in \text{End}_{P_1}(\mathcal{H}, \kappa)^G \), it suffices to consider the action of \( \Gamma(U) \) on the vectors \( \Omega_{\alpha} \) defined in (3.4). In contrast to the Fermionic case (cf. Lemma 2.1), there is no simple

\[\text{In the Bosonic case, a basis projection } P \text{ is an operator on } \mathcal{H} \text{ such that } P = P^2 = P^\dagger = 1 - P, \text{ and such that } CP \text{ is positive definite on } \text{ran } P.\]

\[\text{I.e. } T = \overline{P}.\]
transformation law for the $\psi_j$ under $G$. They obey however a linear transformation law when applied to $\Omega_P$; in fact, one can show that $\Omega_\alpha$ is proportional to
\[ \pi_P(g_{\alpha_1}) \cdots \pi_P(g_{\alpha_i}) \exp\left(-\frac{1}{2}T^*a^*\right)\Omega_{P_i}, \]  
(cf. [Bin98a]; taking the closures of the $\pi_P(g_{\alpha_j})$ is tacitly assumed here). The behavior of the $\pi_P(g_{\alpha_j})$ under gauge transformations is obvious.

**Lemma 3.1.** Let $V \in \text{End}_{P_1}(\mathcal{K}, \kappa)^G$ be given, and let $T$ be defined by (3.2). Then $\exp(-\frac{1}{2}T^*a^*)\Omega_{P_1}$ is invariant under all gauge transformations $\Gamma(U), U \in G$.

**Proof.** Let $E$ be the orthogonal projection onto $\ker V^\dagger = C\ker V^*$, and let $A \equiv ECE$ as in Section 3.1. Then $E$ and $A$ commute with $G$ because $V$ and $P_1$ do so. Therefore the positive part $A_+$ of $A$ and the operator $A_+^{-1}$ defined in Section 3.1 also commute with $G$. It follows that $P = VP_1V^\dagger + A_+^{-1}C$ and $T \equiv P_1P_1^{-1}$ commute with $G$ as well.

Arguing as in the proof of Lemma 2.2 one finds for $U \in G$
\[ \Gamma(U)(-\frac{1}{2}T^*a^*)^n\Omega_{P_i} = (-\frac{1}{2}(TU)^\dagger a^*)^n\Omega_{P_i}, \]  
and finally
\[ \Gamma(U)\exp(-\frac{1}{2}T^*a^*)\Omega_{P_i} = \exp(-\frac{1}{2}(TU)^\dagger a^*)\Omega_{P_i} = \exp(-\frac{1}{2}T^*a^*)\Omega_{P_i}. \]

**Theorem 3.2.** Let $P_1$ be a basis projection of $(\mathcal{K}, \kappa)$, let $G$ be a group of unitary operators on $\mathcal{K}$ commuting with $P_1$ and with complex conjugation, and let $V \in \text{End}_{P_1}(\mathcal{K}, \kappa)^G$. Then the subspace $\mathfrak{k}$ defined in (3.3) is invariant under $G$, and the unitary representation $D_{\mathfrak{k}^v}$ of $G$ on the Hilbert space of isometries $H_{\mathfrak{k}^v}$ which implements $g_V$ in the Fock representation determined by $P_1$ is unitarily equivalent to the representation $\mathfrak{S}_\mathfrak{k}$ on the symmetric Fock space over $\mathfrak{k}$ that is obtained by taking symmetric tensor powers of the representation on $\mathfrak{k}$:
\[ D_{\mathfrak{k}^v} \simeq \mathfrak{S}_\mathfrak{k}. \]

**Proof.** $\mathfrak{k}$ is invariant under $G$ because $\ker V^\dagger$ is invariant and because $P$ commutes with $G$ (see the proof of Lemma 3.1). The assertion hence follows from (3.3) and Lemma 3.1. 

**Remarks.**
(i) Theorem 3.2 shows that non-surjective quasi-free endomorphisms of the CCR algebra are even “more reducible” than endomorphisms of the CAR algebra in that they are always infinite direct sums, a fact which explains the generic occurrence of infinite statistics in the CCR case (cf. (3.1)). Again, each closed subspace of $H_{\mathfrak{k}^v}, \Omega_{P_1}$ spanned by the $\Omega_\alpha$ with length of $\alpha$ fixed is invariant under $G$.

(ii) Any representation of $G$ which is contained in $\mathcal{K}$ with infinite multiplicity is realized on some space $\mathfrak{k}$ belonging to a $V \in \text{End}_{P_1}(\mathcal{K}, \kappa)^G$.

(iii) Quasi-free automorphisms are less interesting in the CCR case because they are all neutral: $D_{\mathfrak{k}^v}$ is the trivial representation of $G$ if $\text{ind } V = 0$.

(iv) If the single-particle space $\mathcal{K}_1$ has a particle-antiparticle symmetry, then every $V \in \text{End}_{P_1}(\mathcal{K}, \kappa)^G$ has a conjugate in $\text{End}_{P_1}(\mathcal{K}, \kappa)^G$, just as in the Fermionic case.

4. **Concluding Remarks**

As we have seen, gauge invariant implementable quasi-free endomorphisms of the CAR and CCR algebras with statistics dimension $d \neq 1$ restrict to reducible endomorphisms of the observable algebra. In typical cases, e.g. if $G$ is isomorphic to one of the classical compact Lie groups, any irreducible representation of the group
is equivalent to a subrepresentation of some tensor power of the defining representation. In such cases there will exist quasi-free endomorphisms, behaving like “master endomorphisms”, which contain each superselection sector as a subrepresentation.

It is an interesting problem how to obtain the irreducible “subobjects” of a quasi-free endomorphism $\varrho$. Suppose that $\{\Psi_j\}$ is an (incomplete) orthonormal set in $H_{\varrho}$ which transforms irreducibly under $G$. According to the general theory [DHR71], there should exist a gauge invariant isometry $\Phi$ on Fock space with $\text{ran } \Phi = \bigoplus_j \text{ran } \Psi_j$. The corresponding irreducible endomorphism $\varrho_\Phi$ (which is not quasi-free) would then be given by

$$\varrho_\Phi(a) \equiv \Phi^* \left( \sum_j \Psi_j a \Psi_j^* \right) \Phi.$$

Collections of gauge invariant isometries $\{\Phi_j\}$ fulfilling the Cuntz relations would permit to define direct sums of quasi-free endomorphisms $\{\varrho_j\}$:

$$\left( \bigoplus_j \varrho_j \right)(a) \equiv \sum_j \Phi_j \varrho_j(a) \Phi_j^*,$$

so that one would get the whole Doplicher–Roberts category generated by quasi-free endomorphisms.

Another important question is how to find basis-independent examples of, say, localized isometries $v$ with index one on the single-particle space $H$ (see (2.17)) of the time-zero Dirac field, such that the implementability condition (2.21) holds. Recall that our construction of such an operator in Eq. (2.24) made essential use of the existence of local Fourier bases on the circle. Of particular interest would be the massive case in two dimensions, where one might hope to find localized quasi-free endomorphisms obeying non-Abelian braid group statistics. However, preliminary calculations based on the explicit formulas in [Bin93, Bin98b] indicate that the commutation relations of implementers corresponding to irreducible subobjects of quasi-free endomorphisms only admit Abelian braid group statistics.

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