WAVE EQUATION AND MULTIPLIER ESTIMATES
ON DAMEK–RICCI SPACES

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Abstract. Let $S$ be a Damek–Ricci space and $L$ be a distinguished left invariant Laplacian on $S$. We prove pointwise estimates for the convolution kernels of spectrally localized wave operators of the form
\[ e^{it\sqrt{L}} \psi(\sqrt{L}/\lambda) \]
for arbitrary time $t$ and arbitrary $\lambda > 0$, where $\psi$ is a smooth bump function supported in $[-2,2]$ if $\lambda < 1$ and supported in $[1,2]$ if $\lambda \geq 1$. This generalizes previous results in [MT]. We also prove pointwise estimates for the gradient of these convolution kernels. As a corollary, we reprove basic multiplier estimates from [HS] and [V1] and derive Sobolev estimates for the solutions to the wave equation associated to $L$.

1. Introduction

Let $n = v \oplus z$ be an $H$-type algebra and let $N$ be the connected and simply connected Lie group associated to $n$ (see Section 2 for the details). By $S$ we denote the one-dimensional solvable extension of $N$ obtained by making $A = \mathbb{R}^+$ act on $N$ by homogeneous dilations $\delta_a$. We choose $H$ in the Lie algebra $a$ of $A$ so that $\exp(tH) = e^t, t \in \mathbb{R}$, and extend the inner product on the Lie algebra $n$ of $N$ to the Lie algebra $s = n \oplus a$ of $S$, by requiring $n$ and $a$ to be orthogonal and $H$ to be a unit vector. Let $d$ be the left invariant Riemannian metric on $S$ which agrees with the inner product on $s$ at the identity.

By $\lambda$ and $\rho$ we denote the left and right invariant Haar measures on $S$, respectively. It is well known that both the right and the left Haar measures of geodesic balls are exponentially growing functions of the radius, so that $S$ is a group of exponential growth.

The space $S$ is called a Damek–Ricci space; these spaces were introduced by E. Damek and F. Ricci [D1, D2, DR1, DR2], and include all rank one symmetric spaces of the noncompact type. Most of them are nonsymmetric harmonic manifolds, and provide counterexamples to the Lichnerowicz conjecture. The geometry of these extensions was studied by M. Cowling, A. H. Dooley, A. Korányi and Ricci in [CDKR1, CDKR2].

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The simplest example of Damek–Ricci spaces is given by the so called \( ax + b \)-groups, which can be thought as the harmonic extensions of \( N = \mathbb{R}^d \).

Let \( \{X_0, \ldots, X_{n-1}\} \) be an orthonormal basis of the Lie algebra \( \mathfrak{s} \) such that \( X_0 = H \), the elements \( X_1, \ldots, X_m \) form an orthonormal basis of \( \mathfrak{v} \) and \( X_{m+1}, \ldots, X_{n-1} \) form an orthonormal basis of \( \mathfrak{z} \). As usually, we shall identify an element \( X \in \mathfrak{s} \) with the corresponding left invariant differential operator on \( S \) given by the Lie derivative \( Xf(g) = \frac{d}{dt} f(g \exp tX) \bigg|_{t=0} \).

Viewing \( X_0, X_1, \ldots, X_{n-1} \) in this way as left invariant vector fields, we define the left invariant Laplacian \( L \) by

\[
L = -\sum_{i=0}^{n-1} X_i^2.
\]

Then \( L \) is essentially selfadjoint on \( C_c^\infty(S) \subset L^2(\rho) \) and its \( L^2(\rho) \)-spectrum is given by \([0, \infty)\).

For any Borel measurable bounded multiplier \( m \) on \([0, \infty)\), we can thus define the operator \( m(L) \) on \( L^2(\rho) \) by means of spectral calculus. Then \( m(L) \) is left invariant too, so that, as a consequence of the Schwartz kernel theorem, there exists a unique distribution \( k \) on \( S \) such that

\[
m(L)f = f * k \quad \forall f \in \mathcal{D}(S).
\]

Several authors have investigated the \( L^p \)-functional calculus for the Laplacian \( L \), i.e., they studied sufficient and necessary conditions on a multiplier \( m \) such that the operator \( m(L) \) extends from \( L^2(\rho) \cap L^p(\rho) \) to an \( L^p(\rho) \)-bounded operator, for a given \( p \) in \((1, \infty)\) \[HI, CGHM, A2, HS, V1\] (in some of these papers, the authors work with a right invariant Laplacian \( L_r \), sometimes on the class of solvable groups arising in the Iwasawa decomposition of noncompact connected semisimple Lie groups of finite centre, but one can easily pass to our left-invariant Laplacian \( L \) by means of group inversion).

The main purpose of this article is to prove pointwise estimates for the convolution kernels of spectrally localized multiplier operators of the form

\[
e^{it\sqrt{L}} \psi(\sqrt{L}/\lambda),
\]

for arbitrary time \( t \) and \( \lambda > 0 \), where \( \psi \) is a bump function supported in \([-2, 2]\), if \( \lambda \leq 1 \), and in \([1, 2]\), if \( \lambda > 1 \). Such estimates were proved in \[MT\] in the case of \( ax + b \)-groups. In this paper we generalize the results in \[MT\] to Damek–Ricci spaces, and we also find pointwise estimates of the gradient of these kernels.

We shall use these estimates to give a new proof of basic multiplier results in \[HIS, V1\], which is based entirely on the wave equation.

Moreover, as a corollary we obtain \( L^p \)-estimates for Fourier multiplier operators of the form \( m(\sqrt{L}) \cos(t \sqrt{L}) \) and \( m(\sqrt{L}) \frac{\sin(t \sqrt{L})}{\sqrt{L}} \), where \( m \) is a suitable symbol, and we derive
Sobolev estimates for solutions of the wave equation associated with $L$ on Damek–Ricci spaces.

This problem was first studied in the euclidean setting in [M, P]. The problem of the regularity in space for fixed time of the wave equation associated with the Laplace–Beltrami operator on a noncompact symmetric space of arbitrary rank was studied in [GM, CGM]. In the case of noncompact symmetric spaces of rank one A. Ionescu [I] estimated the $L^p$-norm of the Fourier integral operators for variable time and derived Sobolev estimates for the solution of the wave equation associated with a shifted Laplace–Beltrami operator. Since this is the counterpart of our result in the same setting for a different Laplacian we shall discuss it in more details at the end of our paper.

Our paper is organized as follows. In Section 2 we recall the definition of $H$-type groups and Damek–Ricci spaces and we summarize some results about spherical analysis on such spaces. In Section 3 we recall some properties of the Laplacian $L$ and derive a formula for convolution kernels of multipliers of $L$. In Section 4 and 5 we prove pointwise estimates for the spectrally localized wave propagator and for its gradient, respectively. We deduce an $L^1$-estimate for these kernels; then we apply it to reprove a basic multiplier estimate from [HS] and [V1]. In Section 6 we derive Sobolev estimates for the solutions to the wave equation associated to $L$.

We shall apply the "variable constant" convention in this paper, according to which $C$ will usually denote a positive, finite constant which may vary from line to line and may depend on parameters according to the context. Given two quantities $f$ and $g$, by $f \preceq g$ we mean that there exists a constant $C$ such that $f \leq Cg$ and by $f \asymp g$ we mean that there exist constants $C_1, C_2$ such that $C_1 g \leq f \leq C_2 g$.

2. Damek–Ricci spaces

In this section we recall the definition of $H$-type groups, which had been introduced by Kaplan [K], describe their harmonic extensions and recall the main results of spherical analysis on these extensions. For the details see [ADY, CDKR1, CDKR2].

Let $\mathfrak{n}$ be a two-step nilpotent Lie algebra equipped with an inner product $\langle \cdot, \cdot \rangle$ and denote by $| \cdot |$ the corresponding norm. Let $\mathfrak{v}$ and $\mathfrak{z}$ be complementary orthogonal subspaces of $\mathfrak{n}$ such that $[\mathfrak{n}, \mathfrak{z}] = \{0\}$ and $[\mathfrak{n}, \mathfrak{n}] \subseteq \mathfrak{z}$.

The algebra $\mathfrak{n}$ is of $H$-type if for every $z$ in $\mathfrak{z}$ the map $J_z : \mathfrak{v} \to \mathfrak{v}$ defined by

$$\langle J_z v, v' \rangle = \langle z, [v, v'] \rangle \quad \forall v, v' \in \mathfrak{v}$$

satisfies the condition

$$|J_z X| = |Z| |X| \quad \forall X \in \mathfrak{v} \quad \forall Z \in \mathfrak{z}.$$
The connected and simply connected Lie group $N$ associated to $\mathfrak{n}$ is called an $H$-type group. We identify $N$ with its Lie algebra $\mathfrak{n}$ via the exponential map

$$\mathfrak{v} \times \mathfrak{z} \to N$$

$$(v, z) \mapsto \exp(v + z).$$

The product law in $N$ is given by

$$(v, z)(v', z') = (v + v', z + z' + (1/2) [v, v']) \quad \forall v, v' \in \mathfrak{v} \quad \forall z, z' \in \mathfrak{z}.$$

The group $N$ is two-step nilpotent, hence unimodular, with Haar measure $dv\,dz$. We define the following dilations on $N$:

$$\delta_a(v, z) = (a^{1/2}v, az) \quad \forall (v, z) \in N \quad \forall a \in \mathbb{R}^+.$$

These are automorphisms, so that $N$ is an homogeneous group. A homogeneous norm is given by

$$N(v, z) = \left(\frac{|v|^4}{16} + |z|^2 \right)^{1/4} \quad \forall (v, z) \in N.$$

Note that $N(\delta_a(v, z)) = a^{1/2}N(v, z)$, so that $N$ is homogeneous of degree one with respect to the modified dilation group $\{\delta_{a^{1/2}}\}_{a > 0}$. Set $Q = (m_v + 2m_z)/2$, where $m_v$ and $m_z$ denote the dimensions of $\mathfrak{v}$ and $\mathfrak{z}$, respectively.

Let $H$ be the element of $\mathfrak{a}$ such that $\text{ad}(H)v = \frac{1}{2}v$, if $v \in \mathfrak{v}$, and $\text{ad}(H)z = z$, if $z \in \mathfrak{z}$. We extend the inner product on $\mathfrak{n}$ to the algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$, by requiring $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and $H$ to be unitary. The algebra $\mathfrak{s}$ is a solvable Lie algebra. The corresponding Lie group $S$ is the semi-direct extension $S = N \rtimes A$, where $A = \mathbb{R}^+$ acts on $N$ by the above dilations.

The map

$$\mathfrak{v} \times \mathfrak{z} \times \mathbb{R}^+ \to S$$

$$(v, z, a) \mapsto \exp(v + z) \exp(aH)$$

gives global coordinates on $S$. The product in $S$ is then given by the rule

$$(v, z, a)(v', z', a') = (v + a^{1/2}v', z + a z' + (1/2) a^{1/2}[v, v'], a a')$$

for all $(v, z, a), (v', z', a')$ in $S$. Let $e$ be the identity of the group $S$. We shall denote by $n = m_v + m_z + 1$ the dimension of $S$. The group $S$ is nonunimodular: the right and left Haar measures on $S$ are given by $d\rho(v, z, a) = a^{-1}dv\,dz\,da$ and $d\lambda(v, z, a) = a^{-(Q+1)}dv\,dz\,da$, respectively. In particular,

$$d\lambda = \delta \, d\rho,$$

where the modular function is given by $\delta(v, z, a) = a^{-Q}$.
We endow $S$ with the left invariant Riemannian metric which agrees with the inner product on $s$ at the identity. Let $d$ denote the distance induced by this Riemannian structure. The Riemannian manifold $(S, d)$ is then called a Damek–Ricci space or harmonic $NA$ group.

It is well known [ADY, formula (2.18)] that

$$\cosh^2 \left( \frac{d((v, z, a), e)}{2} \right) = \left( \frac{a^{1/2} + a^{-1/2}}{2} + \frac{1}{8} a^{-1/2} |v|^2 \right)^2 + \frac{1}{4} a^{-1} |z|^2 \quad \forall (v, z, a) \in S. \quad (2.1)$$

We shall later use the notation

$$R(x) = d(x, e), \quad x \in S. \quad (2.2)$$

Let us remark at this point that it is natural to include also the "degenerate" cases where $N$ is abelian into the definition of $H$-type group respectively of Damek-Ricci space, since then all symmetric spaces of noncompact type and rank one, including real hyperbolic spaces, are special cases of these spaces, when considered as Riemannian manifolds (cf. [CDKR2]). Indeed, all of our previous definitions and subsequent arguments will apply as well when $N$ is abelian. Only when $N$ is one-dimensional, some estimates will have to be modified. However, since the case where $N$ is abelian had essentially already been dealt with in [MT], we shall restrict ourselves in this paper to the case where $N$ is non-abelian.

We denote by $B((v_0, z_0, a_0), r)$ the ball in $S$ centred at $(v_0, z_0, a_0)$ of radius $r$. In particular let $B_r$ denote the ball of center $e$ and radius $r$; note that [ADY, formula (1.18)]

$$\rho(B_r) \asymp \begin{cases} r^n & \text{if } r < 1 \\ e^{Qr} & \text{if } r \geq 1. \end{cases} \quad (2.4)$$

This shows in particular that $S$, equipped with the right Haar measure $\rho$, is a group of exponential growth.

A radial function on $S$ is a function that depends only on the distance from the identity. If $f$ is radial, then [ADY, formula (1.16)]

$$\int_S f \, d\lambda = \int_0^\infty f(r) A(r) \, dr, \quad (2.3)$$

where

$$A(r) = 2^{m_a+m_3} \sinh^{m_a+m_3} \left( \frac{r}{2} \right) \cosh^{m_3} \left( \frac{r}{2} \right) \quad \forall r \in \mathbb{R}^+. \quad (2.5)$$

One easily checks that

$$A(r) \lesssim \left( \frac{r}{1+r} \right)^{n-1} e^{Qr} \quad \forall r \in \mathbb{R}^+. \quad (2.6)$$
A radial function \( \phi \) is spherical if it is an eigenfunction of the Laplace-Beltrami operator \( \Delta \) (associated to \( d \)) and \( \phi(e) = 1 \). Let \( \phi_s \), for \( s \in \mathbb{C} \), be the spherical function with eigenvalue \( s^2 + Q^2/4 \), as in \([ADY\), formula (2.6)].

In \([A1\), Lemma 1] it is shown that
\[
\phi_0(r) \lesssim (1 + r) e^{-Qr/2} \quad \forall \ r \in \mathbb{R}^+.
\]

We shall use the following integration formula on \( S \), whose proof is reminiscent of \([CGHM,\ Lemma 1.3]\) and \([A1,\ Lemma 3]\):

**Lemma 2.1.** For every radial function \( f \) in \( C_c^\infty(S) \)
\[
\int_S \delta^{1/2} f \, d\rho = \int_0^\infty \phi_0(r) f(r) A(r) \, dr
= \int_0^\infty f(r) J(r) \, dr,
\]
where
\[
J(r) \lesssim \begin{cases} 
  r^{n-1} & \text{if } r < 1 \\
  r e^{Qr/2} & \text{if } r \geq 1
\end{cases}
\]

The spherical Fourier transform of an integrable radial function \( f \) on \( S \) is defined by the formula
\[
\mathcal{H}f(s) = \int_S \phi_s f \, d\lambda.
\]
For “nice” radial functions \( f \) on \( S \) an inversion formula and a Plancherel formula hold:
\[
f(x) = c_S \int_0^\infty \mathcal{H}f(s) \phi_s(x) |\mathcal{c}(s)|^{-2} \, ds \quad \forall x \in S,
\]
and
\[
\int_S |f|^2 \, d\lambda = c_S \int_0^\infty |\mathcal{H}f(s)|^2 |\mathcal{c}(s)|^{-2} \, ds,
\]
where the constant \( c_S \) depends only on \( m_0 \) and \( m_3 \) and \( \mathcal{c} \) denotes the Harish-Chandra function.

Let \( \mathcal{A} \) denote the Abel transform and let \( \mathcal{F} \) denote the Fourier transform on the real line, defined by \( \mathcal{F}g(s) = \int_{-\infty}^{+\infty} g(r) e^{-isr} \, dr \), for each integrable function \( g \) on \( \mathbb{R} \). It is well known that \( \mathcal{H} = \mathcal{F} \circ \mathcal{A} \), hence \( \mathcal{H}^{-1} = \mathcal{A}^{-1} \circ \mathcal{F}^{-1} \). We shall use the inversion formula for the Abel transform \([ADY,\ formula (2.24)]\), which we now recall. Let \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) be the differential operators on the real line defined by
\[
\mathcal{D}_1 = -\frac{1}{\sinh r} \frac{\partial}{\partial r}, \quad \mathcal{D}_2 = -\frac{1}{\sinh(r/2)} \frac{\partial}{\partial r}.
\]
If \( m_3 \) is even, then
\[
\mathcal{A}^{-1} f(r) = \tilde{a}_S \mathcal{D}_1^{m_3/2} \mathcal{D}_2^{m_0/2} f(r),
\]
where \( \tilde{a}_S \) is the constant determined in \([ADY,\ Lemma 2.11]\).
where \( \tilde{a}^e_S = 2^{-(2m_v + m_z)/2} \pi^{-(m_v + m_z)/2} \), while if \( m_j \) is odd, then

\[
A^{-1}f(r) = \tilde{a}^o_S \int_r^\infty D_1^{(m_j+1)/2} D_2^{m_v/2} f(s) (\cosh s - \cosh r)^{-1/2} \sinh s \, ds ,
\]

where \( \tilde{a}^o_S = 2^{-(2m_v + m_z)/2} \pi^{-n/2} \).

3. The Laplacian \( L \)

Let \( \{X_0, \ldots, X_{n-1}\} \) be an orthonormal basis of the Lie algebra \( \mathfrak{s} \) such that \( X_0 = H \), the elements \( X_1, \ldots, X_{m_v} \) form an orthonormal basis of \( \mathfrak{v} \) and \( X_{m_v+1}, \ldots, X_{n-1} \) form an orthonormal basis of \( \mathfrak{z} \). As before, we shall view the \( X_i \) as left invariant vector fields on \( S \).

In particular, if \( i = 0 \), then

\[
X_0 f(v, z, a) = a \partial_a f(v, z, a) \quad \forall f \in C^\infty(S) ,
\]

while for \( i \neq 0 \) the vector fields \( X_i \) do not involve a derivative in the variable \( a \).

These vector fields \( X_i, \, i = 0, \ldots, n - 1 \), form an orthonormal basis of the tangent space at every point of \( S \), since the Riemannian metric is left invariant. If we denote by \( \nabla \) the Riemannian gradient on \( S \), it is easy to verify that for any function \( f \) in \( C^\infty(S) \)

\[
\nabla f = \sum_i (X_i f) X_i .
\]

This implies that the Riemannian norm of the gradient, which we denote by \( \| \nabla f \| \), is given by

\[
\| \nabla f \| = (\sum_i |X_i f|^2)^{1/2}.
\]

Let \( L = -\sum_{i=0}^{n-1} X_i^2 \) be the left invariant Laplacian defined in the Introduction.

The operator \( L \) has a special relationship with the (positive definite) Laplace-Beltrami operator \( \Delta = -\text{div} \circ \text{grad} \). Indeed, let \( \Delta_Q \) denote the shifted operator \( \Delta - Q^2/4 \); it is known that [A1 Proposition 2]

\[
\delta^{-1/2} L \, \delta^{1/2} f = \Delta_Q f ,
\]

for smooth radial functions \( f \) on \( S \).

The spectra of \( \Delta_Q \) on \( L^2(\lambda) \) and \( L \) on \( L^2(\rho) \) are both \([0, +\infty)\). Let \( E_{\Delta_Q} \) and \( E_L \) be the spectral resolution of the identity for which

\[
\Delta_Q = \int_0^{+\infty} t \, dE_{\Delta_Q}(t) \quad \text{and} \quad L = \int_0^{+\infty} t \, dE_L(t) .
\]

For each bounded Borel measurable function \( \psi \) on \( \mathbb{R}^+ \) the operators \( \psi(\Delta_Q) \) and \( \psi(L) \), spectrally defined by

\[
\psi(\Delta_Q) = \int_0^{+\infty} \psi(t) \, dE_{\Delta_Q}(t) \quad \text{and} \quad \psi(L) = \int_0^{+\infty} \psi(t) \, dE_L(t) ,
\]
are bounded on $L^2(\lambda)$ and $L^2(\rho)$ respectively. By (3.1) and the spectral theorem, we see that
\[
\delta^{-1/2} \psi(L) \delta^{1/2} f = \psi(\Delta_Q) f,
\]
for smooth compactly supported radial functions $f$ on $S$.

Let $k_{\psi(L)}$ and $k_{\psi(\Delta_Q)}$ denote the convolution kernels of $\psi(L)$ and $\psi(\Delta_Q)$ respectively; we have that
\[
\psi(\Delta_Q)f = f \ast k_{\psi(\Delta_Q)} \quad \text{and} \quad \psi(L)f = f \ast k_{\psi(L)} \quad \forall f \in C_c^\infty(S),
\]
where $\ast$ denotes the convolution on $S$ defined by
\[
f \ast g(x) = \int_S f(y)g(y^{-1}x) \, d\lambda(y) = \int_S f(xy)g(y^{-1}) \, d\lambda(y) = \int_S f(xy^{-1})g(y) \, d\rho(y),
\]
for all functions $f, g$ in $C_c(S)$ and $x$ in $S$.

The integral kernel of $\psi(L)$ is the function defined on $(S, d\rho) \times (S, d\rho)$ by
\[
K_{\psi(L)}(x, y) = k_{\psi(L)}(y^{-1}x) \delta(y) \quad \forall x, y \in S.
\]

**Proposition 3.1.** Let $\psi$ be a bounded measurable function on $\mathbb{R}^+$. Then $k_{\psi(\Delta_Q)}$ is radial and $k_{\psi(L)} = \delta^{1/2} k_{\psi(\Delta_Q)}$. The spherical transform of $k_{\Delta_Q}$ is
\[
\mathcal{H}k_{\psi(\Delta_Q)}(s) = \psi(s^2) \quad \forall s \in \mathbb{R}^+.
\]

**Proof.** See [A1], [ADY]. \qed

By Proposition 3.1 it follows that if $\psi$ is a function in $C_c(\mathbb{R})$, then the convolution kernel of $\psi(L)$ is equal to
\[
k_{\psi(L)}(x) = (2\pi)^{-1} \delta^{1/2}(x) A^{-1} \left( \int_{\mathbb{R}} \psi(s^2) e^{isx} \, ds \right) (R(x)) \quad \forall x \in S,
\]
where $R(x) = d(x, e)$ is as in (2.2). By using the expression of the inverse Abel transform (2.6) and (2.7) we deduce that if $m_3$ is even, then
\[
k_{\psi(L)}(x) = a_S^e \delta^{1/2}(x) D_{1,v}^{m_3/2} D_{2,v}^{m_3/2} \left( \int_{\mathbb{R}} \psi(s^2) e^{isx} \, ds \right) (R(x)) \quad \forall x \in S,
\]
while if $m_3$ is odd, then
\[
k_{\psi(L)}(x) = a_S^o \delta^{1/2}(x) \int_{R(x)}^\infty D_{1,v}^{(m_3+1)/2} D_{2,v}^{m_3/2} \left( \int_{\mathbb{R}} \psi(s^2) e^{isx} \, ds \right) (v) \, d\nu_{R(x)}(v)
\]
\[
= a_S^o \delta^{1/2}(x) \int_{R(x)}^\infty \psi(s^2) \int_{R(x)}^\infty D_{1,v}^{(m_3+1)/2} D_{2,v}^{m_3/2} (e^{isv})(v) \, d\nu_{R(x)}(v) \, ds,
\]
Lemma 3.2. For all integers $p, q$ such that $p + q \geq 1$ and all $v \geq 1$

$$D_{p,v}^q e^{i sv}(v) = \sum_{k=1}^{p+q} s^k q_k(v) e^{-(p+q/2) v} e^{i sv},$$

where $q_k$ is in $S^0$ for all $k$.

Proof. Since $\frac{1}{\sinh v} = \frac{2}{e^v - e^{-v}}$, where $\frac{2}{e^v - e^{-v}} = \sum_{m=0}^{\infty} e^{-2mv}$ is in $S^0$ for $v > 1$, the lemma follows easily by induction on $q$ and $p$ (compare [MT] Lemma 5.3). \hfill \Box

Lemma 3.3. For all integers $p, q$ such that $p + q \geq 1$ and all $v$ in $[0,4]$

$$D_{p,v}^q e^{i sv}(v) = \sum_{k=1}^{p+q} s^k q_k(v) v^{k-2(p+q)} e^{i sv},$$

where $q_k$ is in $S^0$ for all $k$.

Proof. On the interval $[0,4]$ we have that $\frac{1}{\sinh v} = g(v) v^{-1}$, where $g$ is in $S^0$, and a similar statement holds for $\frac{1}{\sinh(v/2)}$. The lemma follows easily by induction on $q$ and $p$ (compare [MT] Lemma 5.8). \hfill \Box

In the following proposition we study the asymptotic behaviour of $F_R$ and its first derivative.

Proposition 3.4. If $m_3$ is even, then

$$F_R(s) = \begin{cases} e^{-QR/2} e^{i Rs} \sum_{k=1}^{(n-1)/2} s^k q_k(R) & \text{if } R \geq 1 \\ e^{-QR/2} e^{i Rs} \sum_{k=1}^{(n-1)/2} s^k q_k(R) R^{1-n+k} & \text{if } R < 1, \end{cases}$$
and

\[ \partial_R F_R(s) = \begin{cases} 
  e^{-QR/2} e^{iRs} \sum_{k=1}^{(n+1)/2} s^k p_k(R) & \text{if } R \geq 1 \\
  e^{-QR/2} e^{iRs} \sum_{k=1}^{(n+1)/2} s^k p_k(R) R^{-n+k} & \text{if } R < 1 ,
\end{cases} \]

where \( q_k \) and \( p_k \) are functions in \( S^0 \) uniformly in \( R \) for all \( k \).

\[ \text{If } m_3 \text{ is odd, then for every } \tau \text{ in } [0,1/2] \]

\[ F_R(s) = \begin{cases} 
  e^{-QR/2} e^{iRs} \sum_{k=1}^{n/2} s^k b_k^{-1/2}(s) & \text{if } R \geq 1 \\
  e^{-QR/2} e^{iRs} \sum_{k=1}^{n/2} s^k b_k^{-1/2}(s) R^{-n+k+\tau+1/2} & \text{if } R < 1 ,
\end{cases} \]

and

\[ \partial_R F_R(s) = \begin{cases} 
  e^{-QR/2} e^{iRs} \sum_{k=1}^{n/2+1} s^k p_k^{-1/2}(s) & \text{if } R \geq 1 \\
  e^{-QR/2} e^{iRs} \sum_{k=1}^{n/2+1} s^k p_k^{-1/2}(s) R^{-n+k+\tau-1/2} & \text{if } R < 1 ,
\end{cases} \]

where \( b_k^\beta(s) \) and \( p_k^\beta(s) \) are functions depending also on \( R \) but which are in \( S^3 \) uniformly in \( R \) for all \( k \) and \( \beta \).

**Proof.** We first study the case when \( m_3 \) is even. In this case by (3.3) we have that

\[ F_R(s) = a^s \mathcal{D}^{m_3/2}_{1,v} \mathcal{D}^{m_3/2}_{2,v} (e^{isv})(R). \]

By Lemma 3.2 it follows that if \( R \geq 1 \), then

\[ F_R(s) = e^{-QR/2} e^{iRs} \sum_{k=1}^{(n-1)/2} s^k q_k(R) , \]

while if \( R < 1 \), then

\[ F_R(s) = e^{-QR/2} e^{iRs} \sum_{k=1}^{(n-1)/2} s^k q_k(R) R^{1-n+k} , \]

where \( q_k \) is in \( S^0 \) for all \( k \).

By deriving with respect to \( R \) the expressions above we obtain that if \( R \geq 1 \), then

\[ \partial_R F_R(s) = e^{-QR/2} e^{iRs} \sum_{k=1}^{(n-1)/2} s^k (-Q/2 + is) q_k(R) + \partial_R q_k(R) ] \]

\[ = e^{-QR/2} e^{iRs} \sum_{k=1}^{(n+1)/2} s^k p_k(R) , \]

while if \( R < 1 \), then

\[ \partial_R F_R(s) = e^{-QR/2} e^{iRs} \sum_{k=1}^{(n-1)/2} s^k R^{1-n+k} ( -Q/2 + is + (1-n+k)R^{-1}) q_k(R) + \partial_R q_k(R) \]

\[ = e^{-QR/2} e^{iRs} \sum_{k=1}^{(n+1)/2} s^k R^{-n+k} p_k(R) , \]
where \( p_k \) is in \( S^0 \) for all \( k \).

This proves the proposition in the case when \( m_3 \) is even.

We now study the case when \( m_3 \) is odd. In this case by (3.4) we deduce that

\[
F_R(s) = a_s \int_R^\infty D_{1,v}^{(m_3+1)/2} D_{2,v}^{m_2/2} (e^{isv})(v) \, d\nu_R(v),
\]

where \( d\nu_R(v) = (\cosh v - \cosh R)^{-1/2} \sinh v \, dv \). By Lemma (3.2) it follows that if \( R \geq 1 \), then

\[
F_R(s) = \sum_{k=1}^{n/2} s^k \int_0^\infty q_k(v + R) e^{-(Q/2+1/2)(v+R)} e^{is(v+R)} \times
\]

\[
\left( \cosh(v + R) - \cosh R \right)^{-1/2} \sinh(v + R) \, dv
\]

\[
= \sum_{k=1}^{n/2} s^k e^{-QR/2} e^{isR} \int_0^\infty q_k(v + R) \sinh(v + R) e^{isR} \, dv
\]

\[
= \sum_{k=1}^{n/2} s^k e^{-QR/2} e^{isR} b_k^{-1/2}(s),
\]

by [MT] Lemmata 5.4, 5.5, where \( b_k^{-1/2} \) is in \( S^{-1/2} \) uniformly with respect to \( R \geq 1 \). If \( R < 1 \), then

\[
F_R(s) = F^1_R(s) + F^2_R(s)
\]

\[
= \int_R^\infty \chi(v) D_{1,v}^{(m_3+1)/2} D_{2,v}^{m_2/2} (e^{isv})(v) \, d\nu_R(v)
\]

\[
+ \int_0^\infty (1 - \chi(v)) D_{1,v}^{(m_3+1)/2} D_{2,v}^{m_2/2} (e^{isv})(v) \, d\nu_R(v)
\]

where \( \chi \) is a \( C^\infty \) function supported in \([-4, 4]\) equal to 1 in \([-2, 2]\). Let \( \tau \) be in \([0, 1/2]\). By [MT] Lemmata 5.4, 5.5 \( F^2_R \) is in \( S(\mathbb{R}) \) uniformly with respect to \( R < 1 \). We now study the behaviour of \( F^1_R \), which by Lemma (3.3) is equal to

\[
F^1_R(s) = \sum_{k=1}^{n/2} s^k \int_R^\infty \chi(v) q_k(v) v^{k-n+1} \frac{\sinh v}{v} (\cosh v - \cosh R)^{-1/2} e^{isv} \, dv.
\]
By [MT] Lemma 5.9 there exists a function $\gamma$ in $\mathcal{S}$ such that $\inf_{v \in [0,4]} \gamma(v) > 0$ and the previous sum is equal to
\[
\sum_{k=1}^{n/2} s^k \int_{R}^{\infty} \chi(v) q_k(v) v^{k-n+1} \frac{\sinh v}{v} \gamma(v)^{-1/2} (v + R)^{-1/2} (v - R)^{-1/2} e^{isv} dv
\]
\[
= \sum_{k=1}^{n/2} s^k \int_{0}^{\infty} \chi(v + R) \gamma(v + R)^{-1/2} q_k(v + R) (v + R)^{k-n+1} \frac{\sinh(v + R)}{v + R} \times
\]
\[
(2R + v)^{-1/2} v^{-1/2} e^{is(v + R)} dv
\]
\[
= e^{isR} \sum_{k=1}^{n/2} s^k \int_{0}^{\infty} \gamma_k, R(v) (v + R)^{k-n+1} (2R + v)^{-1/2} v^{-1/2} e^{isv} dv
\]
\[
= e^{isR} \sum_{k=1}^{n/2} s^k f_{k,R}(s).
\]

By [MT] Lemma 5.10 we deduce that $R^{m-k-1/2-\tau} f_{k,R}$ is in $\mathcal{S}^{-1/2}$ uniformly with respect to $R < 1$. Thus
\[
F_R^1(s) = e^{-QR/2} e^{isR} \sum_{k=1}^{n/2} s^k R^{-n+k+1/2+\tau} b_k^{-1/2}(s),
\]
where $b_k^{-1/2}(s)$, depending also on $R$, is in $\mathcal{S}^{-1/2}$ uniformly with respect to $R < 1$.

This concludes the study of the asymptotic behaviour of $F_R$ when $m_3$ is odd. To study the behaviour of its derivative, we derive with respect to $R$ the expressions above. If $R \geq 1$, then
\[
\partial_R F_R(s) = \sum_{k=1}^{n/2} s^k e^{-QR/2} e^{isR} (-Q/2 + is) b_k^{-1/2}(s)
\]
\[
= \sum_{k=1}^{n/2+1} s^k e^{-QR/2} e^{isR} p_k^{-1/2}(s),
\]
where $p_k^{-1/2}$ is in $\mathcal{S}^{-1/2}$ uniformly with respect to $R \geq 1$. If $R < 1$ and $\tau \in [0,1/2]$, then
\[
\partial_R F_R(s) = e^{-QR/2} e^{isR} \sum_{k=1}^{n/2} s^k R^{-n+k+1/2+\tau} b_k^{-1/2}(s) (-Q/2 + is + (-n + k + 1/2 + \tau) R^{-1})
\]
\[
= e^{-QR/2} e^{isR} \sum_{k=1}^{n/2+1} s^k R^{-n+k+1/2+\tau} p_k^{-1/2}(s),
\]
where $p_k^{-1/2}$ is in $\mathcal{S}^{-1/2}$ uniformly with respect to $R < 1$.

This proves the proposition also in the case when $m_3$ is odd. □
4. Spectrally localized estimates for the wave propagator

In this section we state pointwise estimates for the convolution kernel of spectrally localized wave propagators associated to the Laplacian $L$ on Damek–Ricci spaces.

Let $t \in \mathbb{R}$, $\lambda > 0$, and let $\psi$ be an even bump function in $C^\infty(\mathbb{R})$ supported in $[-2, 2]$. If $\lambda \geq 1$ we shall in addition suppose that $\psi$ vanishes on $[-1, 1]$. We denote by $k_\lambda^t$ the convolution kernel of $m_\lambda^t(L) = \psi(\frac{s}{\lambda}) \cos(t \sqrt{L})$. We know by (3.3) that

$$
(4.1) \quad k_\lambda^t(x) = \delta^{1/2}(x) \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) \cos(ts) F_{R(x)}(s) \, ds.
$$

In the following theorem we estimate the kernel $k_\lambda^t$.

**Theorem 4.1.** The kernel $k_\lambda^t$ is of the form

$$
\begin{align*}
k_\lambda^t(x) &= \delta^{1/2}(x) e^{-QR(x)/2} \left[ G_\lambda(R(x), R(x) - t) + G_\lambda(R(x), R(x) + t) \right],
\end{align*}
$$

where the function $G_\lambda$ satisfies for every $N$ in $\mathbb{N}$ the following estimates:

(i) if $m_3$ is even, then

$$
|G_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{(n-1)/2} \lambda^{k+1} & \text{if } R \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{(n-1)/2} R^{1-n+k} \lambda^{k+1} & \text{if } R < 1;
\end{cases}
$$

(ii) if $m_3$ is odd, then for every $\tau$ in $[0, 1/2]$,

$$
|G_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \lambda^{(n+1)/2} & \text{if } R \geq 1 \text{ and } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \lambda^2 & \text{if } R \geq 1 \text{ and } \lambda < 1 \\
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2} R^{-n+k+1/2+\tau} \lambda^{k+1/2+\tau} & \text{if } R < 1 \text{ and } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} R^{-n+3/2+\tau} \lambda^2 & \text{if } R < 1 \text{ and } \lambda < 1,
\end{cases}
$$

where the constants $C_N$ depend only on $\tau$ and on the $C^N$-norms of $\psi$.

**Proof.** We consider the case when $m_3$ is odd. If $R \geq 1$, then by Proposition [3.3] we deduce that

$$
k_\lambda^t(x) = \delta^{1/2}(x) e^{-QR(x)/2} \sum_{k=1}^{n/2} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) b_k^{-1/2}(s) s^k \frac{e^{i(R(x)-t)s} + e^{i(R(x)+t)s}}{2i} \, ds
$$

$$
= \delta^{1/2}(x) e^{-QR(x)/2} \left[ G_\lambda(R(x), R(x) - t) + G_\lambda(R(x), R(x) + t) \right],
$$

where $G_\lambda(R, u) = \sum_{k=1}^{n/2} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) b_k^{-1/2}(s) s^k \frac{e^{i(R(x)+s)u}}{2i} \, ds$. By [MT] Lemma 6.2 it follows that

$$
|G_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2} \lambda^{k+1/2} & \text{if } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2} \lambda^{k+1} & \text{if } \lambda < 1.
\end{cases}
$$
\[ k_\lambda^t(x) = \delta^{1/2}(x) e^{-QR(x)/2} \sum_{k=1}^{n/2} R(x)^{-n+k+\tau+1/2} \int_{\mathbb{R}} \psi \left( \frac{s}{\lambda} \right) b_k^{\tau-1/2}(s) s^k \frac{e^{i(R(x)-t)s} + e^{i(R(x)+t)s}}{2i} \, ds \]

where \( G_\lambda(R, u) = \sum_{k=1}^{n/2} R^{-n+k+\tau+1/2} \int_{\mathbb{R}} \psi \left( \frac{s}{\lambda} \right) b_k^{\tau-1/2}(s) s^k \frac{s\lambda u_s}{2\pi} \, ds \).

By [MT, Lemma 6.2] it follows that

\[
|G_\lambda(R, u)| \leq \begin{cases} 
C_N \left( 1 + |\lambda u| \right)^{-N} \sum_{k=1}^{n/2} R^{-n+k+\tau+1/2} \lambda^{k+\tau+1/2} & \text{if } \lambda \geq 1 \\
C_N \left( 1 + |\lambda u| \right)^{-N} \sum_{k=1}^{n/2} R^{-n+k+\tau+1/2} \lambda^{k+1} & \text{if } \lambda < 1 
\end{cases}
\]

This proves the theorem when \( m_j \) is odd.

The proof in the case when \( m_j \) is even is similar and easier, which is why we omit it. \( \square \)

As a consequence of Theorem 4.1 we obtain estimates of the \( L^1 \)-norms of the convolution kernel of \( \psi \left( \frac{\sqrt{t}}{\lambda} \right) \cos \left( \frac{t\sqrt{t}}{\lambda} \right) \).

**Proposition 4.2.** Let \( W_\lambda^t = k_\lambda^t \) denote the convolution kernel of \( \psi \left( \frac{\sqrt{t}}{\lambda} \right) \cos \left( \frac{t\sqrt{t}}{\lambda} \right) \) and let \( \varepsilon \geq 0 \).

(i) If \( \lambda \geq 1 \), then

\[
\int_{\mathbb{R}} |W_\lambda^t(x)| R(x)^{\varepsilon} \, d\rho(x) \lesssim \begin{cases} 
\lambda^{t-\varepsilon} (1 + t)^{(n-1)/2+\varepsilon} & \text{if } t < \lambda \\
\lambda^{-\varepsilon} \lambda^{(n-3)/2} t^{1+\varepsilon} & \text{if } t \geq \lambda 
\end{cases}
\]

(ii) If \( \lambda < 1 \), then

\[
\int_{\mathbb{R}} |W_\lambda^t(x)| R(x)^{\varepsilon} \, d\rho(x) \lesssim \lambda^{-\varepsilon} (1 + t)^{1+\varepsilon} \quad \forall t \in \mathbb{R}.
\]

In particular,

(iii) If \( \lambda \geq 1 \), then

\[
\int_{\mathbb{R}} |W_\lambda^t(x)| (1 + \lambda R(x))^{\varepsilon} \, d\rho(x) \lesssim (1 + t)^{(n-1)/2+\varepsilon} \quad \forall t \in \mathbb{R},
\]

(iv) If \( \lambda < 1 \), then

\[
\int_{\mathbb{R}} |W_\lambda^t(x)| (1 + \lambda R(x))^{\varepsilon} \, d\rho(x) \lesssim (1 + t)^{1+\varepsilon} \quad \forall t \in \mathbb{R}.
\]
The constants in these estimates depend only on the $C^N$-norms of $\psi$.

Proof. We first assume that $m_3$ is odd. Without loss of generality we shall suppose that $t \geq 0$, so that in Theorem 4.1 the dominant term is $G_\lambda(R, R-t)$. Consider first the case when $\lambda \geq 1$. By applying Lemma 2.1 and Theorem 4.1 with $\tau = 0$ we deduce that

$$I = \int_S |W_\lambda^t(x)| R(x)^\varepsilon \, d\rho(x)$$

$$\lesssim \sum_{k=1}^{n/2} \int_{R(x)<1} \delta^{1/2} R(x)^{1/2} \varepsilon (1 + |\lambda R(x) - t|)^{-N} R(x)^{-n+k+1/2} \lambda^{k+1/2} R(x)^\varepsilon \, d\rho(x)$$

$$+ \int_{R(x)\geq1} \delta^{1/2} R(x)^{1/2} \varepsilon (1 + |\lambda R(x) - t|)^{-N} \lambda^{(n+1)/2} R(x)^\varepsilon \, d\rho(x)$$

$$\lesssim \sum_{k=1}^{n/2} \int_0^1 (1 + |\lambda R - t|)^{-N} R^{-n+k+1/2} \lambda^{k+1/2} R^{n-1} \varepsilon \, dR$$

$$+ \int_1^\infty (1 + |\lambda R - t|)^{-N} \lambda^{(n+1)/2} R^{1+\varepsilon} \, dR$$

If $N$ is sufficiently large, then (i) follows by [MT] Lemma 6.4.

Assume next that $\lambda < 1$. We then apply Theorem 4.1 with $\tau = 1/2$ to obtain

$$I = \int_S |W_\lambda^t(x)| R(x)^\varepsilon \, d\rho(x)$$

$$\lesssim \int_{R(x)<1} \delta^{1/2} R(x)^{1/2} \varepsilon (1 + |\lambda R(x) - t|)^{-N} R(x)^{-n+3/2+1/2} \lambda^2 R(x)^\varepsilon \, d\rho(x)$$

$$+ \int_{R(x)\geq1} \delta^{1/2} R(x)^{1/2} \varepsilon (1 + |\lambda R(x) - t|)^{-N} \lambda^2 R(x)^\varepsilon \, d\rho(x)$$

$$\lesssim \int_0^1 (1 + |\lambda R - t|)^{-N} R^{-n+3/2+1/2} \lambda^2 R^{n-1} \varepsilon \, dR$$

$$+ \int_1^\infty (1 + |\lambda R - t|)^{-N} \lambda^2 R^{1+\varepsilon} \, dR$$

$$\lesssim \int_0^\infty (1 + |\lambda R - t|)^{-N} \lambda^2 R^{1+\varepsilon} \, dR.$$
By means of the subordination principle described e.g. in [49] we immediately obtain
the following corollary, which gives a new proof of [HS Theorem 6.1] and [47] Theorem
4.3, estimate (20)].

Corollary 4.3. Let \( \varepsilon, s_0, s_1 \) be positive constants such that \( s_0 > 3/2 + \varepsilon \) and \( s_1 > n/2 + \varepsilon \).
Then there exists a constant \( C \) such that for every continuous function \( F \) supported in \([1, 2]\)
and \( 0 < \lambda < 1 \),
\[
\int_S |k_F(\frac{x}{\lambda})| \left(1 + \lambda R(x)\right)^\varepsilon \, d\rho(x) \leq C \|F\|_{H^{s_0}(R)},
\]
while for \( \lambda \geq 1 \)
\[
\int_S |k_F(\frac{x}{\lambda})| \left(1 + \lambda R(x)\right)^\varepsilon \, d\rho(x) \leq C \|F\|_{H^{s_1}(R)}.
\]

Proof. The proof follows along the lines of the proof of [49 Corollary 6.5]. We omit the
details. \( \square \)

5. Spectrally localized estimates for the gradient of the wave propagator

In this section we prove an estimate of the gradient of the wave propagator associated to
the Laplacian \( L \).

Let \( k^i_\lambda \) denote the convolution kernel of \( m^i_\lambda(L) = \psi(\frac{\sqrt{L}}{\lambda}) \cos(t \sqrt{L}) \) and \( X_0, \ldots, X_{n-1} \) be
the left invariant vector fields which we introduced in Section 3. We have that
\[
X_0 k^i_\lambda(x) = \left( X_0 \delta^{1/2}(x) \right) \int_R \psi(\frac{s}{\lambda}) \cos(t s) F_{R(x)}(s) \, ds
+ \delta^{1/2}(x) \int_R \psi(\frac{s}{\lambda}) \cos(t s) \partial_R F_{R(x)}(s) (X_0 R)(x) \, ds
+ \delta^{1/2}(x) \int_R \psi(\frac{s}{\lambda}) \cos(t s) [-Q/2 F_{R(x)}(s) + \partial_R F_{R(x)}(s) (X_0 R)(x)] \, ds,
\]
and
\[
X_i k^i_\lambda(x) = \delta^{1/2}(x) \int_R \psi(\frac{s}{\lambda}) \cos(t s) \partial_R F_{R(x)}(s) (X_i R)(x) \, ds \quad \forall i = 1, \ldots, n-1.
\]
We recall that \( |X_i R(x)| \leq 1 \), for all \( x \in S \) and \( i = 0, \ldots, n-1 \) [47].

Theorem 5.1. For all \( i = 0, \ldots, n-1 \) we have that
\[
X_i k^i_\lambda(x) = \delta^{1/2}(x) e^{-QR(x)/2} \left[ H_\lambda(R(x), R(x) - t) + H_\lambda(R(x), R(x) + t) \right],
\]
where the function \( H_\lambda \) satisfies for every \( N \) in \( \mathbb{N} \) the following estimates:

(i) if \( m_\lambda \) is even, then
\[
|H_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{(n+1)/2} \lambda_{k+1} & \text{if } R \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{(n+1)/2} R^{-n+k} \lambda_{k+1} & \text{if } R < 1 
\end{cases}
\]
Lemma 6.2] it follows that

\[
|H_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \lambda^{(n+3)/2} & \text{if } R \geq 1 \text{ and } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \lambda^2 & \text{if } R \geq 1 \text{ and } \lambda < 1 \\
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2+1} R^{-n+k-1/2+\tau} \lambda^{k+1/2+\tau} & \text{if } R < 1 \text{ and } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} R^{-n+1/2+\tau} \lambda^2 & \text{if } R < 1 \text{ and } \lambda < 1,
\end{cases}
\]

where the constants \(C_N\) depend only on \(\tau\) and on the \(C^N\)-norms of \(\psi\).

Proof. As before, we shall only consider the case when \(m_3\) is odd; the case when \(m_3\) is even is similar and easier. Moreover, we shall only discuss the case where \(i = 0\). When \(i = 1, \ldots, n - 1\) the proof is again similar and easier, which is why we omit it.

If \(R \geq 1\), then by Proposition 5.3 we deduce that

\[
X_0 k_\lambda^I(x) = \delta^{1/2}(x) e^{-QR(x)/2} \sum_{k=1}^{n/2+1} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) \left[ -\frac{Q}{2} b_k^{-1/2}(s) + p_k^{-1/2}(s) (X_0 R)(x) \right] \times \\
\times s^k \frac{e^{i(R(x)-t)s} + e^{i(R(x)+t)s}}{2i} ds
\]

\[
= \delta^{1/2}(x) e^{-QR(x)/2} \left[ H_\lambda(R(x), R(x) - t) + H_\lambda(R(x), R(x) + t) \right],
\]

where \(H_\lambda(R, u) = \sum_{k=1}^{n/2+1} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) \left[ -\frac{Q}{2} b_k^{-1/2}(s) + p_k^{-1/2}(s) (X_0 R) \right] s^k \frac{e^{iR_u s}}{2i} ds\). By [MT] Lemma 6.2] it follows that

\[
|H_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2+1} \lambda^{k+1/2} & \text{if } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2+1} \lambda^{k+1} & \text{if } \lambda < 1,
\end{cases}
\]

which gives the required estimates in (ii) when \(R \geq 1\). If \(R < 1\), and if \(\tau \in [0, 1/2]\), then

\[
X_0 k_\lambda^I(x) = \delta^{1/2}(x) e^{-QR(x)/2} \sum_{k=1}^{n/2+1} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) \left[ -\frac{Q}{2} R(x)^{-n+k+\tau+1/2} b_k^{-1/2}(s) + R(x)^{-n+k+\tau-1/2} p_k^{-1/2}(s) (X_0 R)(x) \right] \times \\
\times s^k \frac{e^{i(R(x)-t)s} + e^{i(R(x)+t)s}}{2i} ds
\]

\[
= \delta^{1/2}(x) e^{-QR(x)/2} \left[ H_\lambda(R(x), R(x) - t) + H_\lambda(R(x), R(x) + t) \right],
\]

where

\[
H_\lambda(R, u) = \sum_{k=1}^{n/2+1} \int_{\mathbb{R}} \psi\left(\frac{s}{\lambda}\right) R^{-n+k+\tau-1/2} \left[ -\frac{Q}{2} R b_k^{-1/2}(s) + p_k^{-1/2}(s) (X_0 R) \right] s^k \frac{e^{iR_u s}}{2i} ds.
\]
By [MT, Lemma 6.2] it follows that
\[
|H_\lambda(R, u)| \leq \begin{cases} 
  C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2+1} R^{-n+k+\tau-1/2} \lambda^{k+\tau+1/2} & \text{if } \lambda \geq 1 \\
  C_N (1 + |\lambda u|)^{-N} \sum_{k=1}^{n/2+1} R^{-n+k+\tau-1/2} \lambda^{k+1} & \text{if } \lambda < 1,
\end{cases}
\]
which yields the required estimates in (ii) also when $R < 1$.
\[\square\]

As a consequence of Theorem 5.1 we obtain estimates of the $L^1$-norms of the gradient of the convolution kernel of $\psi\left(\frac{\sqrt{x}}{\lambda}\right) \cos\left(t \frac{\sqrt{x}}{\lambda}\right)$. To do it we need the following technical lemma.

**Lemma 5.2.** The following estimates hold:

(i) for all $x = (v, z, a)$ in $S$
\[
|X_0\left(s^{1/2} \left(\cosh(R/2)\right)^{-Q}\right)(v, z, a)| \leq \frac{a (a + 1 + |v|^2/4)^{-Q-1}}{[1 + (a + 1 + |v|^2/4)^2 |z|^2]^{Q/2+1}};
\]

(ii) $\int_N \frac{(a+1+|v|^2/4)^{-Q-1}}{[1+(a+1+|v|^2/4)^2 |z|^2]^{Q/2+1}} \, dv \, dz = C (a + 1)^{-1} \forall a \in \mathbb{R}^+.$

**Proof.** For a proof of (i) see [V1, Lemma 4.5]. In order to prove (ii), notice that the change variables $(a+1+|v|^2/4)^{-1} z = w$ in the integral on the left-hand side of (ii) transforms it into
\[
\int_{\mathbb{R}} (a + 1 + |v|^2/4)^{-Q-1+m_3} \, dv \int_{\mathbb{R}} \frac{dw}{(1 + |w|^2)^{Q/2+1}}
\]
\[
\leq \int_{\mathbb{R}} (a + 1 + |v|^2/4)^{-m_3/2-1} \, dv
\]
\[
\leq C (a + 1)^{-1},
\]
as required. \[\square\]

We can now prove an $L^1$-estimate for the Riemannian gradient of the convolution kernel of $\psi\left(\frac{\sqrt{x}}{\lambda}\right) \cos\left(t \frac{\sqrt{x}}{\lambda}\right)$.

**Proposition 5.3.** Let $W^t_{\lambda} = k^{1/\lambda}_r$ denote the convolution kernel of $\psi\left(\frac{\sqrt{x}}{\lambda}\right) \cos\left(t \frac{\sqrt{x}}{\lambda}\right)$.

(i) If $\lambda \geq 1$ and supp $\psi \subset [-2, -1] \cup [1, 2]$ then
\[
\int_S \|\nabla W^t_{\lambda}(x)\| \, d\rho(x) \leq \begin{cases} 
  C \lambda (1+t)^{(n-1)/2} & \text{if } t < \lambda \\
  C \lambda^{(n-1)/2} t & \text{if } t \geq \lambda.
\end{cases}
\]

(ii) If $\lambda < 1$ and supp $\psi \subset [-2, 2]$ then
\[
\int_S \|\nabla W^t_{\lambda}(x)\| \, d\rho(x) \leq C \lambda (1+t) \forall t \in \mathbb{R}.
\]
The constants in these estimates depend only on the $C^N$-norms of $\psi$.

Proof. Again, we shall only deal with the case when $m_3$ is odd, since the other case is similar and easier. Without loss of generality we may and shall assume that $t \geq 0$ so that the dominant term in Theorem 5.1 is $H_\lambda(R, R - t)$. We first observe that

$$I = \int_S \|\nabla W_\lambda^t(x)\| \, d\rho(x)$$

$$= \int_{R(x) < 1} \|\nabla W_\lambda^t(x)\| \, d\rho(x) + \int_{R(x) \geq 1} \|\nabla W_\lambda^t(x)\| \, d\rho(x)$$

$$= I_0 + I_\infty,$$

and we estimate these integrals separately.

1. Case: $\lambda \geq 1$. By applying Lemma 2.1 and Theorem 5.1 with $\tau = 0$ we obtain that

$$I_0 \lesssim \int_{R(x) < 1} \delta^{1/2}(x) e^{-QR(x)/2} |H_\lambda(R(x), R(x) - t/\lambda)| \, d\rho(x)$$

$$\lesssim \int_1^R R^{n-1} |H_\lambda(R, R - t/\lambda)| \, dR$$

$$\lesssim \sum_{k=1}^{n/2+1} \int_1^R (1 + |\lambda R - t|)^{-N} R^{-n+k-1/2} \lambda^{k+1/2} \, dR,$$

which by [MT, Lemma 6.4] satisfies (i).

We proceed in the same way for $I_\infty$. By Theorem 5.1 and Lemma 2.1

$$I_\infty \lesssim \int_{R(x) \geq 1} \delta^{1/2}(x) e^{-QR(x)/2} |H_\lambda(R(x), R(x) - t/\lambda)| \, d\rho(x)$$

$$\lesssim \int_1^\infty R |H_\lambda(R, R - t/\lambda)| \, dR$$

$$\lesssim \int_1^\infty R (1 + |\lambda R - t|)^{-N} \lambda^{(n+3)/2} \, dR.$$

By [MT, Lemma 6.4] the estimate (i) follows.

2. Case: $\lambda < 1$. In this case the estimate of $I_0$ is easy. Indeed, by applying Lemma 2.1 and Theorem 5.1 with $\tau = 1/2$ we deduce that

$$I_0 \lesssim \int_0^1 R^{n-1} R^{-n+1/2+1/2} \lambda^2 (1 + |\lambda R - t|)^{-N} \, dR$$

$$\lesssim \int_0^1 \lambda^2 (1 + |\lambda R - t|)^{-N} \, dR,$$

which is clearly bounded above by $\lambda (1 + t)$.

The estimation of the integral $I_\infty$ is more delicate. To obtain it we need a more precise study of the behaviour of $\nabla W_\lambda^t$ when $R \geq 1$ and $\lambda < 1$. We recall that by [33] and
Proposition 3.4

\[ W^i_k(x) = \delta^{1/2}(x) e^{-QR(x)/2} \sum_{k=1}^{n/2} \int_R \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) s^k b^{-1/2}_k(s) e^{i R(x) s} ds \]

\[ = \sum_{k=1}^{n/2} L_k(x) , \]

where \( L_k(x) = \delta^{1/2}(x) e^{-QR(x)/2} \int_R \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) s^k b^{-1/2}_k(s) e^{i R(x) s} ds \).

Since \( I_\infty \lesssim \sum_{k=1}^{n/2} \| \nabla L_k(x) \| d\rho(x) \) we estimate each summand separately. More precisely we distinguish the cases when \( k \geq 2 \) and \( k = 1 \) (which corresponds to the worst summand).

Let us suppose that \( k \geq 2 \) and consider the derivative of \( L_k \) along the vector field \( X_0 \):

\[ X_0 L_k(x) = (X_0 \delta^{1/2})(x) e^{-QR(x)/2} \int_R \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) s^k b^{-1/2}_k(s) e^{i R(x) s} ds \]

\[ - Q/2 (X_0 R)(x) \delta^{1/2}(x) e^{-QR(x)/2} \times \]

\[ \int \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) s^k b^{-1/2}_k(s) e^{i R(x) s} ds \]

\[ + (X_0 R)(x) \delta^{1/2}(x) e^{-QR(x)/2} \int \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) is^{k+1} b^{-1/2}_k(s) e^{i R(x) s} ds . \]

Thus

\[ |X_0 L_k(x)| \lesssim \delta^{1/2}(x) e^{-QR(x)/2} \left[ |L^k_\lambda(R(x), R(x) - t/\lambda)| + |L^{k+1}_\lambda(R(x), R(x) + t/\lambda)| \right] \]

\[ + \delta^{1/2}(x) e^{-QR(x)/2} \left[ |L_\lambda^{k+1}(R(x), R(x) - t/\lambda)| + |L_\lambda^{k+1}(R(x), R(x) + t/\lambda)| \right] , \]

where \( L^k_\lambda(R, u) = \int_R \psi \left( \frac{s}{\lambda} \right) s^k b^{-1/2}_k(s) \frac{e^{iu_s}}{2} ds \). Since \( k \geq 2 \) by [MT] Lemma 6.2 it follows that for all \( N \in \mathbb{N} \)

\[ |L^k_\lambda(R, u)| + |L^{k+1}_\lambda(R, u)| \leq C_\lambda (1 + |\lambda u|)^{-N} (\lambda^{k+1} + \lambda^{k+2}) \leq C_\lambda (1 + |\lambda u|)^{-N} \lambda^3 . \]

Thus by (5.1) and (5.2)

\[ \int_{R(x) \geq 1} |X_0 L_k(x)| d\rho(x) \lesssim \int_1^\infty (1 + |\lambda R - t|)^{-N} \lambda^3 R dR \]

\[ \leq \lambda (1 + t) . \]

For \( i = 1, \ldots, n - 1 \) the conclusion follows in the same way so that

\[ \sum_{k=2}^{n/2} \int_{R(x) \geq 1} \| \nabla L_k(x) \| d\rho(x) \lesssim \lambda (1 + t) . \]
It remains to consider the case when \( k = 1 \). We first write \( L_1 \) in the following form:

\[
L_1(x) = \delta^{1/2}(x) \left( \cosh(R(x)/2) \right)^{-Q} \left( \frac{\cosh(R(x)/2)}{e^{R(x)/2}} \right)^Q \int_{\mathbb{R}} \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) b_1^{-1/2}(s) e^{i R(x) s} \, ds
\]

\[
= \delta^{1/2}(x) \left( \cosh(R(x)/2) \right)^{-Q} g(R(x)) \, h(R(x)) ,
\]

where the function \( g \) is in \( S^0 \) for \( R \geq 1 \), and where

\[
h(R) = \int_{\mathbb{R}} \psi \left( \frac{s}{\lambda} \right) \cos \left( \frac{t s}{\lambda} \right) b_1^{-1/2}(s) e^{i R s} \, ds .
\]

By deriving along the vector field \( X_0 \) we obtain that

\[
X_0 L_1(x) = X_0 \left( \delta^{1/2}(x) \left( \cosh(R(x)/2) \right)^{-Q} \right) g(R(x)) \, h(R(x))
\]

\[
+ \delta^{1/2}(x) \left( \cosh(R(x)/2) \right)^{-Q} \left[ \partial_{Rg}(R(x)) \, h(R(x)) + g(R(x)) \partial_{Rh}(R(x)) \right] (X_0 R)(x)
\]

\[
= A(x) + B(x) .
\]

An easy application of [MT, Lemma 6.2] shows that for all \( N \) in \( \mathbb{N} \)

\[
|B(x)| \leq C_N \delta^{1/2}(x) e^{-Q R(x)/2} \left[ R(x)^{-1} \lambda^2 + \lambda^3 \right] (1 + |\lambda R(x) - t|)^{-N},
\]

and

\[
|A(x)| \leq C_N \left| X_0 \left( \delta^{1/2}(x) \left( \cosh(R(x)/2) \right)^{-Q} \right) \right| \lambda^2 (1 + |\lambda R(x) - t|)^{-N} .
\]

It suffices to prove that the integrals of \(|A|\) and \(|B|\) over the complement of the unit ball satisfy the required estimate.

By \((5.4)\) and [MT, Lemma 6.4] we deduce that

\[
\int_{R(x) \geq 1} |B(x)| \, d\rho(x) \lesssim \int_1^\infty R \left[ R^{-1} \lambda^2 + \lambda^3 \right] (1 + |\lambda R - t|)^{-N} \, dR
\]

\[
\lesssim \begin{cases} 
\lambda^2 \lambda^{-1} (1 + \lambda)^{-N+1} + \lambda^3 \lambda^{-2} (1 + \lambda)^{-N+1} & \text{if } t < \lambda/2 \\
\lambda^2 \lambda^{-1} + \lambda^3 \lambda^{-2} (1 + t) & \text{if } t \geq \lambda/2 
\end{cases}
\]

\[
\lesssim \lambda (1 + t) ,
\]

as required. We next integrate \(|A|\):

\[
\int_{R(x) \geq 1} |A(x)| \, d\rho(x) \leq \int_{R(x) < 2t/\lambda} |A(x)| \, d\rho(x) + \int_{R(x) \geq 2t/\lambda} |A(x)| \, d\rho(x)
\]

\[
= J_1 + J_2 .
\]

We estimate \( J_1 \) and \( J_2 \) separately by using Lemma [5.2] above. Note that if \( R(x) \geq 2t/\lambda \), then

\[
(1 + |\lambda R(x) - t|)^{-N} \lesssim (1 + \lambda R(x))^{-N} \lesssim (1 + \lambda \log \max(a, 1/a))^{-N} ,
\]
since by (2.1)
\[
\log \max(a, 1/a) \lesssim R(x),
\]
if \(a\) denotes the \(A\)-component of \(x \in S\). Thus, by Lemma 5.2,
\[
J_2 \lesssim \lambda^2 \int_{\mathbb{R}^+} \frac{da}{a} \left( 1 + \lambda \log \max(a, 1/a) \right)^{-N} a \int_N \frac{(a + 1 + |v|^2/4)^{-Q-1}}{\left[ 1 + (a + 1 + |v|^2/4)^{-2} |z|^2 \right]^{Q/2+1}} dv \ dz
\]
\[
\lesssim \lambda^2 \int_{\mathbb{R}^+} da \left( 1 + \lambda \log \max(a, 1/a) \right)^{-N} (a + 1)^{-1}
\]
\[
\lesssim \int_1^\infty (1 + \lambda \log a)^{-N} a^{-1} da \lesssim \lambda (t + 1).
\]
On the other hand, we note that if \(R(x) < 2t/\lambda\), then similarly
\[
|\log a| \lesssim R(x) < 2t/\lambda,
\]
so that by Lemma 5.2
\[
J_1 \lesssim \lambda^2 \int_{|\log a| \lesssim 2t/\lambda} \frac{da}{a} \int_N \frac{(a + 1 + |v|^2/4)^{-Q-1}}{\left[ 1 + (a + 1 + |v|^2/4)^{-2} |z|^2 \right]^{Q/2+1}} dv \ dz
\]
\[
\lesssim \lambda^2 \int_{|\log a| \lesssim 2t/\lambda} a^{-1} da \lesssim \lambda^2 2t/\lambda \lesssim \lambda (1 + t).
\]
It follows that
\[
(5.7) \quad \int_{R(x) \geq 1} |A(x)| \ d\rho(x) \lesssim \lambda (1 + t).
\]
By (5.3), (5.3), (5.7) we deduce that \(I_\infty \lesssim \lambda (1 + t)\).
This concludes the proof also in the case when \(\lambda < 1\).

By means of the subordination principle described e.g. in [Mu] we obtain the following corollary which gives a new proof of [V1 Theorem 4.3].

**Corollary 5.4.** Let \(s_0, s_1\) be positive constants such that \(s_0 > 3/2\) and \(s_1 > n/2\). Then there exists a constant \(C\) such that for every continuous function \(F\) supported in \([1, 2]\)
\[
\int_S \left| K_F\left( \frac{x}{\lambda^2} \right)(x, y) - K_F\left( \frac{x}{\lambda^2} \right)(x, z) \right| \ d\rho(x) \leq \begin{cases} C \lambda d(y, z) \|F\|_{H^{s_0}(\mathbb{R})} & \text{if } \lambda < 1 \\ C \lambda d(y, z) \|F\|_{H^{s_1}(\mathbb{R})} & \text{if } \lambda \geq 1. \end{cases}
\]

**Proof.** By (5.2) the integral kernel \(K_F\left( \frac{x}{\lambda^2} \right)\) is given by
\[
K_F\left( \frac{x}{\lambda^2} \right)(x, y) = \delta(y) k_F\left( \frac{x}{\lambda^2} \right)(y^{-1} x).
\]
Since $L$ is symmetric it suffices to prove the analogous estimates for \( \int_S \left| K_F (\frac{t}{\lambda^2}) (y, x) - K_F (\frac{t}{\lambda^2}) (z, x) \right| d\rho(x) \), since $K_F (\frac{t}{\lambda^2}) (x, y) = K_F (\frac{t}{\lambda^2}) (y, x)$. But

\[
\int_S \left| K_F (\frac{t}{\lambda^2}) (y, x) - K_F (\frac{t}{\lambda^2}) (z, x) \right| d\rho(x) \\
= \int_S \left| \delta(x) k_F (\frac{t}{\lambda^2}) (x^{-1} y) - \delta(x) k_F (\frac{t}{\lambda^2}) (x^{-1} z) \right| d\rho(x) \\
= \int_S \left| k_F (\frac{t}{\lambda^2}) (xy) - k_F (\frac{t}{\lambda^2}) (xz) \right| d\rho(x) \\
= \int_S \left| k_F (\frac{t}{\lambda^2}) (xz^{-1} y) - k_F (\frac{t}{\lambda^2}) (x) \right| d\rho(x) \\
\leq d(z^{-1} y, e) \int_S \| \nabla k_F (\frac{t}{\lambda^2}) (u) \| d\rho(u) \\
= d(y, z) \| \nabla k_F (\frac{t}{\lambda^2}) \|_{L^1(\rho)}.
\]

Now choose an even function $\psi$ in $C^\infty_0 (\mathbb{R})$ such that $\text{supp} \, \psi \subset [-4, -1/2] \cup [1/2, 4]$ and $\psi = 1$ on $[1, 2]$. Set $f(v) = F(v^2)$. Then $\| f \|_{H^s} \sim \| F \|_{H^s}$ and $F (\frac{t}{\lambda^2}) = f (\frac{\sqrt{2} L}{\lambda}) = \psi (\frac{\sqrt{2} L}{\lambda}) f (\frac{\sqrt{2} L}{\lambda})$. Moreover, by the Fourier inverse formula and Fubini's theorem, one easily obtains that

\[
f (\frac{\sqrt{2} L}{\lambda}) = \frac{1}{\pi} \int_0^\infty \hat{f}(t) \cos \left( t \frac{\sqrt{2} L}{\lambda} \right) dt,
\]

since $f$ is an even function. Thus

\[
F (\frac{L}{\lambda^2}) = \frac{1}{\pi} \int_0^\infty \hat{f}(t) \psi \left( \frac{\sqrt{2} L}{\lambda} \right) \cos \left( t \frac{\sqrt{2} L}{\lambda} \right) dt,
\]

which implies

\[
\| \nabla k_F (\frac{t}{\lambda^2}) \|_{L^1(\rho)} \lesssim \int_0^\infty |\hat{f}(t)| \int_S \| \nabla W_A (x) \| d\rho(x) dt.
\]

If $0 < \lambda < 1$, then by Proposition 5.3

\[
\| \nabla k_F (\frac{t}{\lambda^2}) \|_{L^1(\rho)} \lesssim \int_0^\infty |\hat{f}(t)| \lambda (1 + |t|) dt \\
\leq \lambda \left( \int_0^\infty |\hat{f}(t) (1 + |t|)^{s_0}|^2 dt \right)^{1/2} \left( \int_0^\infty (1 + |t|)^{2s_0} dt \right)^{-1/2} \\
\leq \lambda \| f \|_{H^{s_0}} \lesssim \lambda \| F \|_{H^{s_0}},
\]

if $s_0 > 3/2$. This proves the corollary when $0 < \lambda < 1$. The proof in the case when $\lambda \geq 1$ is similar and omitted. \( \square \)

**Remark 5.5.** The estimates in Theorem 5.4 are good enough for $L^1$–estimates, but not for $L^\infty$–estimates, since they exhibit singularities at $R = 0$. One knows that the singular support of the wave propagator for time $t$ is the sphere $R = t$, so that these singularities
are in fact not present. As in [MT, Section 7] we shall improve the estimates of \( k_1 \) when \( R < 1 \).

More precisely we can prove that for \( R < 1 \) the function \( G_\lambda \) which appears in Theorem 4.1 satisfies the following estimates for any \( N \) in \( \mathbb{N} \):

\[
|G_\lambda(R, u)| \leq \begin{cases} 
C_N (1 + |\lambda u|)^{-N} \lambda^n & \text{if } \lambda \geq 1 \\
C_N (1 + |\lambda u|)^{-N} \lambda^2 & \text{if } \lambda < 1.
\end{cases}
\]

The proof of (5.8) follows the same outline as the proof of [MT, Theorem 7.1].

By (5.8) and Theorem 4.1 it follows easily that for \( \lambda \geq 1 \) and \( t \geq 0 \)

\[
\|k_\lambda^t\|_{L^\infty} \leq (1 + t^{-(n-1)/2}) \lambda^{(n+1)/2}.
\]

Notice that, for small times, this estimate agrees with the one valid for the Laplacian on the Euclidean space \( \mathbb{R}^n \). However, for large times, there appears no dispersive effect.

6. Growth estimates for solutions to the wave equation in terms of spectral Sobolev norms

Given a symbol \( m \) in \( S^{-\alpha} \) with \( \alpha \geq 0 \), consider the operators \( T_1^t := m(\sqrt{L}) \cos(t \sqrt{L}) \) and \( T_2^t := m(\sqrt{L}) \sin(t \sqrt{L}) \), for \( t \in \mathbb{R} \), which are bounded on \( L^2(\rho) \) by the spectral theorem. We look for a condition on \( \alpha \) such that these operators are bounded on \( L^p(\rho) \), for some \( 1 \leq p \leq \infty \). Throughout this section we often write \( L^p \) instead of \( L^p(\rho) \).

**Theorem 6.1.** Let \( m \) be a symbol in \( S^{-\alpha} \) and \( 1 \leq p \leq \infty \). Set \( \alpha_n(p) := (n-1)|1/p - 1/2| \).

(i) If \( \alpha > \alpha_n(p) \), then \( T_1^t \) extends from \( L^p(\rho) \cap L^2(\rho) \) to a bounded operator on \( L^p(\rho) \), and

\[
\|T_1^t\|_{L^p \to L^p} \leq C_p (1 + |t|)^{1/2 - 1/2}.
\]

(ii) If \( \alpha > \alpha_n(p) - 1 \), then \( T_2^t \) extends from \( L^p(\rho) \cap L^2(\rho) \) to a bounded operator on \( L^p(\rho) \), and

\[
\|T_2^t\|_{L^p \to L^p} \leq C_p (1 + |t|).
\]

**Proof.** Without loss of generality we shall assume that \( t > 0 \). We first prove (i). Let \( \chi \in C_c^\infty(\mathbb{R}) \) be an even function such that \( \chi(s) = 1 \) if \( |s| \leq 1/2 \), and \( \chi(s) = 0 \) if \( |s| \geq 1 \). Put \( \psi_0(s) := \chi(s/2) \), and \( \psi_j(s) = \chi(2^{-j-1} s) - \chi(2^{-j} s) = \psi(2^{-j} s), j = 1, \ldots, \infty, \) where \( \psi(s) := \chi(s/2) - \chi(s) \) is supported in \( \{s : 1/2 \leq |s| \leq 2\} \). Then \( \psi_0 \) is supported in \( [-2, 2] \), \( \psi_j \) in \( \{s : 2^{-j-1} \leq |s| \leq 2^{j+1}\} \) for \( j \geq 1 \), and

\[
\sum_{j=0}^{\infty} \psi_j(s) = 1 \quad \forall s \in \mathbb{R}.
\]
We shall restrict ourselves to the case $1 \leq p < 2$, since the case $p = 2$ is trivial and the case $p > 2$ follows from the case $p > 2$ by duality. Using (6.1) we decompose the symbol $m$ as

$$m(s) = \sum_{j=0}^{\infty} m_j(2^{-j}s),$$

where $m_0 = m \chi$ and $m_j(s) := (m \psi_j)(2^j s) = m(2^j s) \psi(s)$, if $j \geq 1$. Notice that for all $N \in \mathbb{N}$

$$\parallel m_j \parallel_{C_N} \leq C 2^{-\alpha j},$$

where the constant $C$ depends on the semi-norms $\parallel m \parallel_{S_{\alpha,k}}$ only.

Then for every $f$ in $L^2(\rho)$

$$T^1 f = \sum_{j=0}^{\infty} T_j f \quad \text{in} \ L^2(\rho),$$

where $T_j := m_j \left( \frac{s}{2^j} \right) \cos \left( (2^j t) \frac{s}{2^j} \right)$. Estimating the operator norms of $T_j$ on $L^1(\rho)$ by means of Proposition 4.2 and (6.2) we obtain

$$\parallel T_j \parallel_{L^1 \rightarrow L^1} \lesssim \begin{cases} 2^{-\alpha j} (1 + 2^j t)^{(n-1)/2} & \text{if } t < 1 \\ 2^{-\alpha j} 2^j (n-3)/2 2^j t & \text{if } t \geq 1. \end{cases}$$

Interpolating the previous estimates with the trivial $L^2$ estimate $\parallel T_j \parallel_{L^2 \rightarrow L^2} \lesssim 2^{-\alpha j}$, we obtain the following inequalities:

$$\parallel T_j \parallel_{L^p \rightarrow L^p} \lesssim \begin{cases} 2^{-\alpha j} (1 + 2^j t)^{(n-1)(1/p-1/2)} & \text{if } t < 1 \\ 2^{-\alpha j} 2^{j(n-1)(1/p-1/2)} t^{(2/p-1)} & \text{if } t \geq 1. \end{cases}$$

If $\alpha > \alpha_n(p)$, then by summation over all $j \geq 0$ (i) follows immediately.

As for (ii), observe first that if we replace $m^t_{\lambda}(s) = \psi \left( \frac{s}{\lambda^2} \right) \cos(t \sqrt{s})$ in Section 4 by $\hat{m}^t_{\lambda}(s) = \psi \left( \frac{s}{\lambda} \right) \frac{\sin(ts)}{s}$, then the factor $e^{i(R-t)s} + e^{i(R+t)s}$ in the corresponding kernel has to be replaced by the factor $is^{-1} (e^{i(R-t)s} - e^{i(R+t)s})$. By [MT] Lemma 6.2] the estimates for the function $k^t_{\lambda}$ and $\hat{W}^t_{\lambda}$ are therefore the same as for $k^t_{\lambda}$ and $\hat{W}^t_{\lambda}$ in Theorem 4.1 and Theorem 4.2 except for an additional factor $\lambda^{-1}$. Moreover,

$$\sup_s \left| m_j \left( \frac{s}{2^j} \right) \frac{\sin(ts)}{s} \right| \lesssim \begin{cases} 2^{-\alpha j} 2^{-j} & \text{if } j \geq 1 \\ t & \text{if } j = 0. \end{cases}$$

Together, this implies that for $j \geq 1$, the operators $\hat{T}_j$ which appear in the dyadic decomposition of $T^t_2$ satisfy the same estimates as $T_j$, except for an additional factor $2^{-j}$,
Corollary 6.2. From Theorem 6.1 we immediately obtain adapted Sobolev norms

\[ \| \hat{T}_j \|_{L^1_t L^1_x} \lesssim \begin{cases} 2^{-\alpha j - j} (1 + 2^j t)^{(n-1)/2} & \text{if } t < 1 \\ 2^{-\alpha j} 2^j (n-3)/2 t & \text{if } t \geq 1 \end{cases} \]

By interpolating with the estimate \( \| \hat{T}_j \|_{L^2_t L^2_x} \lesssim 2^{-\alpha j - j} \), we obtain that for \( j \geq 1 \)

\[ \| \hat{T}_j \|_{L^p_t L^p_x} \lesssim \begin{cases} 2^{-\alpha j - j} 2^j (n-1)(1/p-1/2) (1 + t)^{(n-1)(1/p-1/2)} & \text{if } t < 1 \\ 2^{-\alpha j} 2^j (n-1)(1/p-1/2) t^{2/p-1} & \text{if } t \geq 1 \end{cases} \]

For \( j = 0 \) by (6.4) and Proposition 4.2 we obtain

\[ \| \hat{T}_0 \|_{L^2_t L^2_x} \lesssim t \quad \| \hat{T}_0 \|_{L^1_t L^1_x} \lesssim (1 + t), \]

hence

\[ \| \hat{T}_0 \|_{L^p_t L^p_x} \lesssim (1 + t)^{2/p-1} t^{2-2/p} \lesssim \begin{cases} t^{2-2/p} & \text{if } t < 1 \\ t & \text{if } t \geq 1 \end{cases} \]

By summation over \( j \geq 1 \) and \( j = 0 \) we obtain that if \( \alpha > \alpha_0(p) - 1 \), then

\[ \| T^f_\alpha \|_{L^p_t L^p_x} \lesssim \begin{cases} 1 & \text{if } t < 1 \\ t^{2/p-1} + t & \text{if } t \geq 1 \end{cases} \]

\[ \lesssim 1 + |t|, \]

as required. \( \square \)

Let \( u(t, x) \) be the solution of the Cauchy problem

\[ \partial_t^2 u + Lu = 0, \quad u(0, \cdot) = f \quad \partial_t u(0, \cdot) = g. \] (6.5)

For \( f, g \) in \( L^2(\rho) \) the solution \( u \) is given by \( u(t, \cdot) = \cos(t \sqrt{L}) f + \frac{\sin(t \sqrt{L})}{\sqrt{L}} g \). We define the adapted Sobolev norms

\[ \| \phi \|_{L^p_\alpha} = \| (1 + L)^{\alpha/2} \phi \|_{L^p} \quad \forall \alpha \in \mathbb{R}. \]

From Theorem 6.1 we immediately obtain

**Corollary 6.2.** If \( 1 < p < \infty \), \( \alpha_0 > (n-1) \left| 1/p - 1/2 \right| \) and \( \alpha_1 > (n-1) \left| 1/p - 1/2 \right| - 1 \), then

\[ \| u(t, \cdot) \|_{L^p} \leq C_p \left( (1 + |t|)^{1/p-1} \| f \|_{L^p_{\alpha_0}} + (1 + |t|) \| g \|_{L^p_{\alpha_1}} \right). \] (6.6)
Remark 6.3. It is likely that the estimate (6.6) even holds for $\alpha_0 = (n-1) \left|1/p - 1/2 \right| - 1$ and $\alpha_1 = (n-1) \left|1/p - 1/2 \right| - 1$. This would be the counterpart to corresponding results in the Euclidean setting by A. Miyachi and J.C. Peral \[MI\]. To prove it, we hope to prove an endpoint result in Theorem 6.1 for $p = 1$. More precisely, we expect that if $m$ is a symbol in $\mathcal{S}^{-\frac{n-1}{2}}$, then the operator $T_1^t = m(\sqrt{T}) \cos(t\sqrt{T})$ extends to a bounded operator from $H^1$ to $L^1(\rho)$, and

$$\|T_1^t\|_{H^1 \to L^1} \leq C(1 + |t|),$$

where $H^1$ is the "non-standard" Hardy space introduced in \[V2\]. A similar result should hold also for the operator $T_2^t$. This will be the object of further investigation.

Remark 6.4. We recall that the same problem on a noncompact symmetric space $X$ of rank one for the wave equation associated with the operator $\Delta_Q$ has been studied by A. Ionescu \[I\]. He proved that if $v(t, x)$ is the solution of the Cauchy problem

$$\partial_t^2 v + \Delta_Q v = 0, \quad v(0, \cdot) = f \quad \partial_t v(0, \cdot) = g,$$

for $f, g$ in $L^2(\lambda)$, then if $1 < p < \infty$, $\alpha_0 > (n-1) \left|1/p - 1/2 \right|$ and $\alpha_1 > (n-1) \left|1/p - 1/2 \right| - 1$

$$\|v(t, \cdot)\|_{L^p(\lambda)} \leq C_p e^{Q(1/2 - 1/p)t} \left[\|f\|_{L^p_{\alpha_0}(\lambda)} + (1 + |t|) \|g\|_{L^p_{\alpha_1}(\lambda)}\],$$

where we recall that $\lambda$ denotes the left Haar measure on $S$. Note that the regularity indices $\alpha_0$ and $\alpha_1$ are the same as above but there is an exponential growth with respect to the time $t$. Ionescu proved also an $L^\infty$–BMO result.

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