Abstract

We give a new and short proof of the Mallows-Sloane upper bound for self-dual codes. We formulate a version of Greene’s theorem for normalized weight enumerators. We relate normalized rank-generating polynomials to two-variable zeta functions. And we show that a self-dual code has the Clifford property, but that the same property does not hold in general for formally self-dual codes.

1 Introduction

In [3] we introduced, for an arbitrary linear code, its zeta function, as a different way to describe the weight distribution of the code. The definition is motivated by properties of algebraic curves and of codes constructed with those curves. After analyzing the definition more carefully for its coding theoretic meaning, we formulated in [4] an equivalent definition in terms of puncturing and shortening operations. Both definitions are recalled in this paper together with some basic properties of zeta functions for codes.

We introduce a polynomial $g(w)$ of small degree that interpolates the normalized differences

$$
\left( \frac{A_w}{(w)} - (q-1) \frac{A_{w-1}}{(w-1)} \right) (-1)^{w-d},
$$
for \( w = 1, 2, \ldots, n \) (Lemma 1). The fact that the polynomial is both of small degree and has many zeros when the minimum distance \( d \) is large leads us to an alternative proof for the Mallows-Sloane upper bounds (as a special case of the bounds in Theorem 3). The polynomial \( g(w) \) determines the zeta polynomial of a linear code and vice versa (Proposition 1).

Pellikaan defined a two-variable zeta function for curves \( \mathbb{P}^1 \). For codes, we can consider a similar two-variable zeta function. In the approach that we take here, we first formulate a version of Greene’s Theorem for normalized rank-generating polynomials (4). Then we define the two-variable zeta function in terms of the normalized rank-generating polynomial (7). And we show that this is compatible with Definition 2 for the one-variable zeta function.

Clifford’s theorem on the dimension of special divisors has an analogue for codes. We use an argument from [10] to show that the corresponding result holds for self-dual codes, but in general not for formally self-dual codes.

## 2 Weight enumerators and zeta functions

Let \( C \) be a linear code of length \( n \) and minimum distance \( d \) over the finite field of \( q \) elements. Let \( A_i \) be the number of words of weight \( i \) in \( C \). The weight enumerator of the code \( C \) is defined as

\[
A(x, y) = x^n + \sum_{i=d}^{n} A_i x^{n-i} y^i
\]

**Definition 1** (3) For a given weight enumerator \( A(x, y) \), of a \( q \)-ary linear code of length \( n \) and minimum distance \( d \), define \( P(T) \) as the unique polynomial of degree at most \( n - d \) such that

\[
[T^{n-d}] \frac{P(T)}{(1 - T)(1 - qT)} (y + (x - y)T)^n = \frac{A(x, y) - x^n}{q - 1}
\]

Let \( a_w = A_w / \binom{n}{w} \), for \( w = 0, 1, \ldots, n \). Define the normalized weight enumerator as

\[
a(t) = \frac{1}{q - 1} (a_d + a_{d+1} t + \cdots + a_n t^{n-d})
\]
Definition 2 ([4]) For a given normalized weight enumerator $a(t)$, of a $q$-ary linear code of length $n$ and minimum distance $d$, define $P(T)$ as the unique polynomial of degree at most $n - d$ such that

$$\frac{P(T)}{(1 - T)(1 - qT)(1 - T)^{d+1}} \equiv a\left(\frac{T}{1-T}\right) \pmod{T^{n-d+1}}$$

As a brief motivation for Definition 1 consider the special case $P(T) = 1$, and recall that

$$\frac{1}{(1 - T)(1 - qT)}$$

is a generating function for the number of monic polynomials of degree at most a given degree $a$, say. To interpret the modified generating function in the definition, we use

$$\frac{(y(1 - T) + xT)}{1 - T} = y + xT + xT^2 + \cdots$$

It is then clear that the coefficient at $x^{n-i}y^iT^a$ gives the number of those monic polynomials of degree at most $a$ that have precisely $n - i$ zeros in a given subset $\{x_1, \ldots, x_n\} \subset F_q$. Thus the weight enumerator $A(x, y)$ that corresponds to $P(T) = 1$ is realized by the linear code

$$C = \{(f(x_1), \ldots, f(x_n) : f \in F_q[x]_{\leq a}\}$$

The code $C$ has $k = a + 1$, $d = n - a$ and meets the Singleton bound $d \leq n - (k - 1)$. Definition 2 is motivated by the following property of the normalized weight enumerator.

Theorem 1 ([4]) The expression

$$a(t)(1 + t)^d \pmod{t^{n-d+1}}$$

is invariant under puncturing or shortening.

To have well-defined puncturing (projection) and shortening (restriction) operations on a weight enumerator, independent of the choice of a coordinate, we average over all coordinates, so that the effect on the weight enumerator $A(x, y)$ is given by

$$\frac{1}{n} \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right) \text{ (puncturing)} \quad \frac{1}{n} \left(\frac{\partial}{\partial x}\right) \text{ (shortening)}$$
The following properties are derived in [3]. For nondegenerate linear codes, with both \( d \geq 2 \) and \( d^\perp \geq 2 \),
\[
\deg P(T) = n + 2 - d - d^\perp \quad \text{and} \quad P(1) = 1
\]

Duality, as contained in the MacWilliams identities
\[
A_C^\perp(x, y) = \frac{1}{|C|} A_C(x + (q - 1)y, x - y),
\]
becomes
\[
P^\perp(T) = P(1/qT) q^g T^{g^\perp},
\]
where \( g = n + 1 - k - d \) and \( g^\perp = n + 1 - k^\perp - d^\perp \).

The zeros of the zeta polynomial play a rôle in the following upper bound for the minimum distance. Writing \( P(T) = a_d/(q - 1)(1 + aT + \cdots) \), Definition 2 yields
\[
a_d(a - d + q) = a_{d+1}
\]
or
\[
d + 1 = q + 1 + a - a_{d+1}/a_d \leq q + 1 + a
\]
Thus, estimates for the reciprocal zeros of \( P(T) \), and in particular for their sum \(-a\), yield upper bounds for the minimum distance of a linear code.

The following theorem describes the zeros of \( P(T) \) for an interesting infinite family of weight enumerators. For a self-dual code of type (IV), that is to say defined over \( F_4 \) with only words of even weight, \( 3d \leq n + 6 \) ([9]). When the bound is met the weight enumerator of the code is uniquely determined.

**Theorem 2 ([6])** Let \( A(x, y) \) be the unique weight enumerator of type (IV) with \( d = m + 3 \) and \( n = 3m + 3 \), for \( m \) odd. Let \( P(T) \) be the associated zeta polynomial and let \( Q(T) = P(T)(1 + 2T) \). Then
\[
Q\left(\frac{T^2}{2}\right) = \lambda_m C_m^{m+1}(\frac{T^{-1} + T}{2})T^m
\]
where \( C_m^{m+1} \) is an ultraspherical polynomial of degree \( m \) with \( m \) real zeros on \([-1, 1]\), and \( \lambda_m \) a constant depending on \( m \). In particular \( Q(e^{i\theta}/2) = 0 \) if and only if \( C_m^{m+1}(\cos \theta) = 0 \).

Details for this section and related results can be found in [3], [4], [5], [6].
3 The Mallows-Sloane bounds

As in the previous section, let \( a_w = A_w / \binom{n}{w} \).

**Lemma 1** For a linear code of length \( n \), minimum distance \( d \) and dual minimum distance \( d^\perp \), there exists a unique polynomial \( g(w) \) of degree \( n - d^\perp \) such that
\[
g(w) = (a_w - (q - 1)a_{w-1})(-1)^{w-d}, \quad \text{for } w = 1, 2, \ldots, n.
\]

**Proof.** We choose \( g(w) \) of degree at most \( n - d^\perp \) such that it interpolates the right hand side correctly for \( w = 1, 2, \ldots, n - d^\perp + 1 \). It remains to show
(1) \( g(w) \) interpolates correctly in \( n - d^\perp + 2, \ldots, n \), and
(2) \( \deg g(w) = n - d^\perp \).

Let \( S \subset \{1, 2, \ldots, n\} \) be a subset of size \( s \) and consider the subcode of \( C \) of words with support on \( S \). Averaging over all \( S \) of size \( s \) gives for the average size of such a subcode
\[
\sum_{w=0}^{n} a_w \binom{s}{w}
\]
On the other hand for \( s > n - d^\perp \), the size of each such subcode equals
\[
q^{k-(n-s)}
\]
Thus, for \( s > n - d^\perp \),
\[
\sum_{w=0}^{s+1} a_w \binom{s+1}{w} = q \sum_{w=0}^{s} a_w \binom{s}{w},
\]
\[
\sum_{w=1}^{s+1} a_w \binom{s}{w-1} = (q - 1) \sum_{w=0}^{s} a_w \binom{s}{w},
\]
\[
\sum_{w=0}^{s} \binom{s}{w}(a_{w+1} - (q - 1)a_w) = 0.
\]

With elementary calculus, this says that the value for
\[
(-1)^w(a_w - (q - 1)a_{w-1})
\]
at \( w = s + 1 \) is the polynomial extrapolation of the values at \( w = 1, \ldots, s \). This proves claim (1). For \( s = n - d^\perp \), the average size of a subcode exceeds \( q^{k-d^\perp} \), and the extrapolation relation cannot be used to obtain the value at \( w = n - d^\perp + 1 \) from the values at \( w = 1, \ldots, n - d^\perp \). This clearly implies claim (2). \( \square \)
Since \( g(w) \) has zeros at 2, 3, \ldots, \( d-1 \), we obtain \( d-2 \leq n - d^\perp \). And when equality holds,

\[
g(w) = (q-1) \left( \frac{w-2}{d-2} \right) (-1)^{w-d}
\]

For a general weight enumerator, let \( P(T) = p_0 + p_1 T + \cdots + P_r T^r \) be the zeta polynomial, with \( r = n + 2 - d - d^\perp \). Then

**Proposition 1**

\[
g(w) = (q-1) \left( p_0 \left( \frac{w-2}{d-2} \right) - p_1 \left( \frac{w-2}{d-1} \right) + \cdots + (-1)^r p_r \left( \frac{w-2}{d+r-2} \right) \right)
\]

**Theorem 3** ([6]) \( \) Let the code \( C \) have all weights divisible by \( c \). Then

\[
d + cd^\perp \leq n + c(c+1)
\]

If moreover the code is binary, even, and contains the allone word, then

\[
2d + cd^\perp \leq n + c(c+2)
\]

**Proof.** We give a proof based on Lemma [11]. From \( g(w) \) we can obtain a polynomial \( h(w) \) of same degree such that

\[
h(w) = (a_w - (q-1)^c a_{w-c}) (-1)^{w-d}, \quad \text{for } w = c, c+1, \ldots, n.
\]

It has at least \( (c-1)/c \cdot (n-c) + 1/c \cdot (d-2c) \) zeros, and

\[
(c-1)n - (c-1)c + d - 2c \leq cn - cd^\perp \iff d + cd^\perp \leq n + c(c+1)
\]

For the second claim, the degree of \( h(w) \) drops to at most \( n - d^\perp - 1 \). It has at least \( (c-1)/c \cdot (n-c) + 2/c \cdot (d-2c) \) zeros, and

\[
(c-1)n - (c-1)c + 2d - 4c \leq cn - cd^\perp - c \iff 2d + cd^\perp \leq n + c(c+2)
\]

\( \square \)

When applied to self-dual codes, with \( d = d^\perp \), we recover the Mallows-Sloane upper bounds ([9]).

| Type I \( (q = 2, c = 2) \) | \( d \leq 2 \lfloor n/8 \rfloor + 2. \) |
| Type II \( (q = 2, c = 4) \) | \( d \leq 4 \lfloor n/24 \rfloor + 4. \) |
| Type III \( (q = 3, c = 3) \) | \( d \leq 3 \lfloor n/12 \rfloor + 3. \) |
| Type IV \( (q = 4, c = 2) \) | \( d \leq 2 \lfloor n/6 \rfloor + 2. \) |
4 Two-variable zeta functions

Pellikaan defined, for an algebraic curve over a finite field, the two-variable zeta function as the convergent power series

\[ Z(T, u) = \sum_{[D]} \frac{u^{l(D)} - 1}{u - 1} T^{\deg D} \]

The summation is over divisor classes \([D]\). For a finite field of size \(q\) and for \(u = q\), it agrees with the Hasse-Weil zeta function: \(Z(T, q) = Z(T)\). Some familiar properties of the Hasse-Weil zeta function generalize to the Pellikaan zeta function \([11]\). Thus \(Z(T, u)\) is a rational function in the variables \(T\) and \(u\), with functional equation

\[ Z(T, u) = Z(\frac{1}{uT}, u) u^{g-1} T^{2g-2} \]

The vanderGeer-Schoof two-variable zeta function gives a generalization to number fields. In the version for curves it is defined as

\[ \zeta^{GS}(s, t) = \sum_{[D]} q^{s h_0 + t h_1} \]

where \(h_0 = \dim L(D)\) and \(h_1 = \dim \Omega(D) = \dim L(K-D)\). We use it in the form

\[ Z^{GS}(x, y) = \sum_{[D]} x^{h_0} y^{h_1} \]

so that \(\zeta^{GS}(s, t) = Z^{GS}(q^s, q^t)\). Deninger gives the relation between \(Z(T, u)\) and \(\zeta^{GS}(s, t)\) (Proposition 2.1 \([2]\)). For \(Z^{GS}\) it becomes,

\[ Z(T, u)(u - 1)T^{1-g} = Z^{GS}(uT, T^{-1}) \]

Two-variable rank-generating polynomials for matroids go back to Whitney and to important papers in graph theory by Tutte. The columns in the generating matrix of a code form a set \(G\). For each subset \(A\) of columns, let

\[ r(A) = \text{rank}(A) \] (rank)

\[ \rho(A) = |A| \] (degree)

\[ n(A) = \rho(A) - r(A) \] (nullity)
The rank-generating polynomial (or Whitney polynomial, or corank-nullity polynomial, e.g. [1]) is defined as
\[
W_G(x, y) = \sum_{A \in G} x^{r(G)-r(A)} y^{|A|-r(A)}
\]

For a code with column set \(G\), the weight enumerator is given by Greene’s Theorem [8], which can be written in the form
\[
\frac{A(x, y)}{(x-y)^k y^{n-k}} = W_G\left(\frac{qy}{x+y}, \frac{x-y}{y}\right)
\]

The rank-generating polynomial of a code depends only on the generators of the code. To compute the weight enumerator of a code after taking coefficients in an extension field, only \(q\) needs to be replaced. We give a version of Greene’s theorem for the normalized rank-generating polynomial. Let
\[
W_n(x, y) = \sum_{i=0}^{n} \frac{1}{\binom{n}{i}} \sum_{A \in G, |A|=i} x^{r(G)-r(A)} y^{|A|-r(A)}.
\]

\[
A_n(x, y) = \sum_{i=0}^{n} \frac{1}{\binom{n}{i}} A_i x^{n-i} y^i
\]

Then
\[
A_n(s, t)(s+t)^{n+1} = W_n\left(\frac{qt}{s+t}, \frac{s+t}{t}\right)(s+t)^{k_t n-k_s} s^{n+1}
\]
\[
+\tilde{W}_n\left(\frac{qs}{s+t}, \frac{s+t}{s}\right)(s+t)^{k_s n-k_t} t^{n+1}
\]

The relation is written as a polynomial identity but the polynomial \(\tilde{W}_n\) has a priori no particular meaning, so the relation could as well be used with \(s = 1\) as a congruence relation modulo \(t^{n+1}\),
\[
A_n(1, t)(1+t)^{n+1} \equiv W_n\left(\frac{qt}{1+t}, \frac{1+t}{t}\right)(1+t)^{k_t n-k_s} \pmod{t^{n+1}}
\]

An exception is for binary self-complementary codes that have \(A_n(s, t) = A_n(t, s)\) and \(\tilde{W}_n = W_n\). We want to show that there is a natural definition of a two-variable zeta function for codes that is compatible with our earlier definitions for the one-variable case.
To relate the two-variable zeta function of Pellikaan and the rank-generating polynomial, let, for a special divisor \( E \) on the curve,

\[
\begin{align*}
    r(E) &= l(K) - l(K - E) \\
    \rho(E) &= \deg(E) \\
    n(E) &= \deg(E) - (l(K) - l(K - E)) = l(E) - 1
\end{align*}
\]

These definitions do not make the set of special divisors into a representable matroid (unless we allow a somewhat wider definition) but they seem perfectly natural and give a satisfactory correspondence. The canonical divisor \( K \) has rank \( l(K) - 1 \) and the rank-generating polynomial for special divisors becomes

\[
W(x, y) = \sum_{[E]} x^{l(K-E)-1} y^{l(E)-1}
\]

which is similar to the vanderGeer-Schoof two-variable zeta function.

To define a two-variable zeta function for codes, we use two properties of the two-variable zeta function for curves: (1) the number of divisor classes of given degree is constant and equal to \( h \). (2) the zeta function consists of a finite contribution and an infinite tail that only depends on \( h \). The first property holds with \( h = 1 \) for the normalized rank-generating polynomial \( W_n(x, y) \). For the second property we add an infinite tail to \( W_n \).

\[
W_n^+(x, y) = W_n(x, y) + \frac{x^{k+1}}{1-x} + \frac{y^{n-k+1}}{1-y}.
\]

For a normalized rank-generating function \( W_n^+ \), define a two-variable zeta function, in analogy with (1), via

\[
Z(T, u)(u - 1)T^{1-g} = W_n^+(uT, T^{-1})
\]

We show that this definition is compatible with the one-variable zeta function in Definition 2 such that \( Z(T, q) = Z(T) \). We modify (5) to include contributions of the infinite tail that was added to \( W_n \) in (6). Let \( x = qt/(1+t) \).

\[
\left(\frac{qt}{1+t}\right)^{k+1} \frac{1+t}{1-(q-1)t} (1+t)^k t^{n-k} \equiv 0 \pmod{t^{n+1}}
\]

Let \( y = (1+t)/t \).

\[
\left(\frac{1+t}{t}\right)^{n-k+1} \frac{t}{-1} (1+t)^k t^{n-k} \equiv -(1+t)^n \pmod{t^{n+1}}
\]
Combined with (5) and (6) this gives
\[(A_n(1, t) - 1) (1 + t)^{n+1} \equiv W_n^+ \left( \frac{qt}{1+t}, \frac{1+t}{t} \right) (1 + t)^k t^{n-k} \pmod{t^{n+1}}.\]

Or, using \(A_n(1, t) - 1 = (q - 1) a(t) t^d\) and (7),
\[a(t) t^d (1 + t)^{n+1} \equiv Z \left( \frac{t}{1+t}, q \right) (1 + t)^{n-k} (1 + t)^{1-g} \\pmod{t^{n+1}}.\]

Finally, with \(d = n + 1 - k - g\) this reduces to
\[a(t) (1 + t)^{d+1} \equiv Z \left( \frac{t}{1+t}, q \right) \\pmod{t^{n+1-d}}\]
which agrees with Definition 2 after the substitution \(t = T/(1-T)\). Compare also with Theorem 1.

As an example, an MDS code of length \(n\) and dimension \(k\) has
\[W_n(x, y) = x^k + \cdots + x + 1 + y + \cdots + y^{n-k},\]
\[W_n^+(x, y) = \frac{1 - xy}{(1 - x) (1 - y)},\]
and
\[Z(T, u) = \frac{-T^{-1}}{(1 - uT)(1 - T^{-1})} = \frac{1}{(1 - T)(1 - uT)}\]
The passage from \(W_n\) to \(W_n^+\) to define the two-variable zeta function of a code is in line with Theorem 1. The effect of puncturing on the polynomial \(W_n\) is \(W_n \mapsto W_n - y^{n-k}\), and the effect of shortening is \(W_n \mapsto W_n - x^k\). Thus by adding an infinite tail, \(W_n^+\) has become invariant under puncturing or shortening.

5 A Clifford type theorem for self-dual codes

We give an interpretation of Clifford’s theorem for self-dual codes. In [12], [13], Clifford’s theorem is used to give estimates for the weight distributions of geometric Goppa codes. Clifford’s theorem says
\[l(E) - 1 + l(K - E) - 1 \leq l(K) - 1\]
which corresponds to an inequality
\[ n(A) + r(G) - r(A) \leq r(G) \iff 2r(A) \geq |A| \]
for representable matroids.

**Proposition 2** The inequality \( 2r(A) \geq |A| \) holds for any code that contains its dual and for any choice of columns \( A \). In the special case of a self-dual code \( C \), equality holds if and only if \( C = C_1 \oplus C_2 \) for self-dual codes \( C_1 \) and \( C_2 \) that are supported on \( A \) and the complement of \( A \), respectively.

**Proof.** Corank and nullity are dual notions and \( k^\perp - r^\perp(\bar{A}) = |A| - r(A) \). The subcode of the dual code with support on \( A \) therefore has dimension \( |A| - r(A) \). The subcode is self-orthogonal and thus \( 2|A| - 2r(A) \leq |A| \), with equality if and only if it is self-dual. Using duality twice, we have \( k^\perp + |\bar{A}| - 2r^\perp(\bar{A}) = k + |A| - 2r(A) \). And for a self-dual code, \( 2r(A) = |A| \) if and only if \( 2r(\bar{A}) = |\bar{A}| \) if and only if both \( A \) and \( \bar{A} \) support self-dual codes, in which case clearly \( C = C_1 \oplus C_2 \) as required. \( \Box \)

A short argument to prove the Clifford inequality for self-dual codes is provided by [10, Theorem 3.9]. The \( n = 2k \) columns in a self-dual code divide in at least one way into two independent subsets of size \( k \) each. Let the subset \( A \) have \( a_1 \) columns in the first subset and \( a_2 \) columns in the second subset. Then \( 2r(A) \geq 2 \max\{a_1, a_2\} \geq a_1 + a_2 = |A| \).

The inequality \( 2r(A) \geq |A| \) for self-dual codes, does in general not hold for formally self-dual codes. It is easy to find a formally self-dual code for which the inequality fails. We may take (10) with dual code (01).

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