I show that a generic quantum phenomenon can drive cosmic acceleration without the need for
dark energy or modified gravity. When treating the universe as a quantum system, one typically
focuses on the scale factor (of an FRW spacetime) and ignores many other degrees of freedom. How-
ever, the information capacity $S$ of the discarded variables will inevitably change as the universe
expands, generating a quantum correction [Phys. Lett. A 382, 36, 2555 (2018)] to the Friedmann
equations. If information could be stored at each Planck-volume independently, this effect would
give rise to a constant acceleration $10^{122}$ times larger than that observed, reproducing the usual
cosmological constant problem. However, once information capacity is quantified according to the
holographic principle ($S = S_h$) cosmic acceleration is far smaller, and depends on the past behaviour
of the scale factor. I calculate this holographic quantum correction, derive the semiclassical Fried-
mann equations, and obtain their general solution for a spatially-flat universe containing matter
and radiation. Comparing these $S_h$CDM solutions to those of $\Lambda$CDM, the new theory is shown
to be falsifiable, but nonetheless consistent with current observations. In general, realistic $S_h$CDM
cosmologies undergo phantom acceleration ($w_{\text{eff}} < -1$) at late times, predicting a Big Rip in the
distant future.

I. INTRODUCTION

We know the universe is expanding at an accelerating
rate [1–4], but the cause of this acceleration remains a
mystery to fundamental physics [5–7]. Current observa-
tions are broadly consistent with the simplest proposal:
acceleration driven by a cosmological constant $\Lambda > 0$ [8].
But if we are to understand $\Lambda$ as the energy-density of
empty space, we cannot explain the extremely tiny value
$\Lambda_{\text{obs}} \sim 10^{-122}/\ell_{\text{pl}}^2$ without anthropic reasoning [9–13].
Alternatively, we may hope to derive cosmic acceleration
from new dynamical fields, or modifications to Einstein’s
gravity [14, 15]. However, these models often struggle
to fit local constraints (from the solar system [16] and
gravitational wave observations [17]) and still generate
the acceleration we observe [18–20].

In this paper, I will motivate and develop a new ex-
planation for cosmic acceleration – one that does not re-
quire a cosmological constant, new dynamical fields, or
modified gravity. Instead, we will examine an overlooked
quantum phenomenon [21, 22] and show that its appli-
cation to cosmology gives rise to a new acceleration term
in the Friedmann equations. This quantum correction
depends on the maximum information the universe can
hold, which we will quantify according to the holographic
principle [23–26]. Besides this step, our approach will be
broadly independent of the details of quantum gravity at
the fundamental level.

Empirically, this new theory has many features that
distinguish it from a typical dark energy/modified gravity
model. First, it describes a purely global phenomenon: the
background undergoes accelerated expansion, but
there is no change to dynamics on sub-horizon scales.

Second, the universe can end in a Big Rip [27], with the
quantum correction resembling phantom dark energy at
late times. Third, the model has very little freedom: it
only introduces a single new parameter, has no free
functions, and cannot be tuned to mimic $\Lambda$ to arbitrary
accuracy. Nonetheless, a quick comparison with $\Lambda$CDM
will suggest the theory is consistent with current obser-
vations.

We will take a systematic approach, working all the
way from first principles to exact cosmological solutions.
(In contrast, other attempts to link holography to dark
energy have typically invoked ad hoc modifications to
the Friedmann equations, and derived only approximate
solutions, e.g. [28–38]). Before describing how the paper
will unfold, it will be helpful to first give a brief summary
of the generic quantum phenomenon [21, 22] that forms
the basis of this theory.

A. Quantum Correction from Discarded
   Degrees of Freedom

Suppose we are interested in an observable $x$ of some
physical system with many degrees of freedom. If the
classical behaviour of $x$ can be derived from an action

$$ I[x(t)] \equiv \int dt \left[ \frac{m}{2} \dot{x}^2 - V_G(x) \right], $$

without reference to the other variables, we say that the
other degrees of freedom can be discarded when predict-
ing the classical path $x(t)$.

However, once quantum effects are considered, we can-
not always continue to use the action (1) to predict
the behaviour of $x$. Indeed, if the discarded degrees of
freedom have a configuration space that varies in size
as a function of $x$, so that their information capacity

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\[ S_h \text{CDM: Cosmic acceleration from holographic information capacity} \]

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\( S(x) \neq \text{const} \), then a quantum correction will appear in the effective potential [21]:

\[
\Delta V_{\text{eff}} = \frac{\hbar^2}{8m} \left[ (1 - 4\xi \frac{d + 1}{d}) (\partial_x S)^2 + 2(1 - 4\xi) \partial_x^2 S \right],
\]

where \( \xi \in \mathbb{R} \) is a curvature coupling parameter, and \( d \in \mathbb{N} \) is the dimensionality of the discarded configuration space. (Besides these constants, the internal geometry of the discarded space is irrelevant.) The correction (2) directly affects the expected behaviour of \( x \):

\[
m \partial_t^2 (x) = - \left( \partial_x V_{\text{cl}} + \partial_x \Delta V_{\text{eff}} \right),
\]

motivating the use of a semiclassical action

\[
\mathcal{J}[x(t)] \equiv \int dt \left[ \frac{m}{2} \dot{x}^2 - V_{\text{cl}}(x) - \Delta V_{\text{eff}}(x) \right],
\]

which generates trajectories consistent with the average motion (3). Moreover, the semiclassical action (4) sets the other degrees of freedom, with information capacity \( \xi \in [\mathbb{R}] \) and check that it generates the classical Friedmann equations. In section V, having assembled the semiclassical action (4), we derive the semiclassical Friedmann equations. In section VI, we solve these equations for a spatially flat universe containing matter and radiation. Finally, in section VII, we verify the gauge of the time coordinate \( t \), and the spatial geometry is described by the function

\[
r_k(\chi) \equiv \begin{cases} 
\sin(\chi), & k = +1, \\
\chi, & k = 0, \\
\sinh(\chi), & k = -1,
\end{cases}
\]

for a closed, flat, or open universe respectively. (Note that \( \chi \) is dimensionless, and \( a \) is the radius of spatial curvature for \( k \neq 0 \).) As such, a surface of constant \( \chi \) and \( t \) is a sphere of area \( A = \mathcal{A}(\chi)[a(t)]^2 \) and volume \( V = \mathcal{V}(\chi)[a(t)]^3 \), where

\[
\mathcal{A}(\chi) \equiv 4\pi [r_k(\chi)]^2, \quad \mathcal{V}(\chi) \equiv 4\pi \int_0^\chi d\chi' [r_k(\chi')]^2.
\]

For the sake of evaluating \( \mathcal{I}_G \), we will also need the scalar curvature of the FRW spacetime (8):

\[
R = \frac{6}{a^2 N^2} \left( \frac{\dot{a}}{a} - \frac{\dot{a} N}{a N} + kN^2 \right),
\]

where dots indicate differentiation with respect to \( t \).

---

1 We set \( c = 1 \), write \( \kappa \equiv 8\pi G \), \( g \equiv \det(g_{\mu\nu}) \), \( h \equiv \det(h_{\mu\nu}) \), and adopt the sign conventions of Wald [42]: \( \eta_{\mu\nu} \equiv \text{diag}(-1, 1, 1, 1) \), \( [\nabla_\mu, \nabla_\nu]v^\alpha \equiv R^\alpha_{\beta\mu\nu}v^\beta, R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu} \). The metric \( h_{\mu\nu} \equiv g_{\mu\nu} - \epsilon_{\mu\nu} n_\nu \) and extrinsic curvature \( K_{\mu\nu} \equiv h_\mu^\alpha \nabla_\alpha n_\nu \) of the boundary \( \partial M \) are constructed from the outward unit normal \( n^\mu \), with \( \epsilon \equiv n^\alpha n_\alpha = \pm 1 \).
B. Integration Region and Boundary

Besides evaluating the action (6) on the metric (8), we must also choose a suitable region \( \mathcal{M} \) over which to integrate. Rather than attempt an integral over all space (with an infinite result for \( k \in \{0, -1\} \)), we limit ourselves to the spherically symmetric region

\[
\mathcal{M} : \quad t \in [t_-, t_+], \quad \theta \in [0, \pi], \quad \chi \in [0, \chi_*], \quad \phi \in [0, 2\pi],
\]

and promise to send \( \chi_* \rightarrow \infty \) (or \( \chi_* \rightarrow \pi \), for \( k = 1 \)) at the end of the calculation. It is easy to see that the boundary of (12) has three components: \( \partial \mathcal{M} = \partial \mathcal{M}_t \cup \partial \mathcal{M}_\chi \cup \partial \mathcal{M}_\phi \); their extrinsic scalar curvatures are

\[
K[\partial \mathcal{M}_t] = \frac{\delta^I t}{\delta a}, \quad K[\partial \mathcal{M}_\chi] = \pm \frac{3\dot{a}}{a N}, \quad K[\partial \mathcal{M}_\phi] = 0 \quad \text{at } t = t_\pm,
\]

where the prime denotes a derivative, and asterisks indicate evaluation at \( \chi = \chi_* \). With \( \mathcal{M} \) defined, we can now discuss the matter action \( \mathcal{I}_M \), and then evaluate the gravitational action \( \mathcal{I}_G \) on the FRW metric (8).

C. Matter Action

In order to provide matter terms for the Friedmann equations, we require formulae for the functional derivatives of \( \mathcal{I}_M \) with respect to variations \( \delta a(t) \), \( \delta N(t) \) in the FRW metric (8). Note that these variations cause the inverse metric to change by

\[
\delta g^{\mu \nu} = -2g^{\mu \nu} \frac{\delta a}{a} - 2g^{00} \delta_0^\mu \delta_0^\nu \frac{\delta N}{N},
\]

and hence the matter action varies according to

\[
\delta \mathcal{I}_M = \int \mathcal{M} d^4 x \frac{\delta \mathcal{I}_M}{\delta g^{\mu \nu}} \delta g^{\mu \nu}
= \int \mathcal{M} d^4 x \sqrt{-g} \left( \frac{2g^{\mu \nu} \delta a}{a} - 2g^{00} \delta_0^\mu \delta_0^\nu \frac{\delta N}{N} \right)
= \int \mathcal{M} d^4 x \sqrt{-g} \left( \frac{T \delta a}{a} + T_0 g^{00} \frac{\delta N}{N} \right),
\]

where we used (7) in the second line. Homogeneous and isotropic matter \( \Psi = \Psi(t) \) has energy-density \( \rho = \rho(t) \) and pressure \( p = p(t) \) that depend on \( t \) only, with \( T = 3p - \rho \) and \( T_0 g^{00} = -\rho \). As such, equation (15) becomes

\[
\delta \mathcal{I}_M = \mathcal{Y}_s \int_{t_-}^{t_+} dt \dot{a}^3 \left[ N(3p - \rho) \delta a - \alpha \rho \delta N \right].
\]

Consequently,

\[
\frac{\delta \mathcal{I}_M}{\delta a} = \mathcal{Y}_s a^3 N (3p - \rho), \quad \frac{\delta \mathcal{I}_M}{\delta N} = -\mathcal{Y}_s a^4 \rho,
\]

are the functional derivatives we need.

D. Gravitational Action

Finally, we assemble the gravitational part of the classical action by inserting (11) and (13) into (6). After integrating the \( \ddot{a} \) term by parts (to cancel the contributions from \( \partial \mathcal{M}_{t \pm} \)) we obtain

\[
\mathcal{I}_G = \frac{3\mathcal{Y}_s}{\kappa} \int_{t_-}^{t_+} dt \left[ -\frac{\dot{a}^2}{N} + k N a^2 \right] + \frac{\alpha' \mathcal{Y}_s}{\kappa} \int_{t_-}^{t_+} dt N a^2. \tag{18}
\]

In general, the integral proportional to \( \alpha' \mathcal{Y}_s \) can be dropped when \( \mathcal{M} \) covers the entire space. For \( k = 0 \), this happens in the obvious fashion: \( \mathcal{Y}_s = 4\pi \mathcal{Y}_s^3/3 \) and \( \alpha' = 8\pi \mathcal{Y}_s \), so the first integral dominates over the second in the limit \( \chi_* \rightarrow \infty \). For \( k = 1 \), the full space is covered by sending \( \chi_* \rightarrow \pi \), with \( \mathcal{Y}_s \rightarrow 2\pi^2 \) and \( \alpha' \rightarrow 0 \) as a result. Thus, the full-space limit gives

\[
\mathcal{I}_G[a(t), N(t)] = \frac{3\mathcal{Y}_s}{\kappa} \int_{t_-}^{t_+} dt \left[ -\frac{\dot{a}^2}{N} + k N a^2 \right], \tag{19}
\]

for \( k \in \{0, 1\} \) at least.\(^2\) This fixes the normalisation of the total action (5), being the sum of the gravitational action (19) and a matter action \( \mathcal{I}_M \) with derivatives (17). It is easy to check that this combination generates the correct Friedmann equations for the metric (8). Moreover, these equations are correct for all \( k \in \{-1, 0, 1\} \), so (19) must be the correctly normalised classical action, even for an open universe.

To complete our calculation, we express (19) in terms of the conformal time coordinate \( \eta = \eta(t) \), defined by

\[
d\eta = N dt, \quad \eta_\pm \equiv \eta(t_\pm),
\]

and find that \( N \) drops out completely:

\[
\mathcal{I}_G[a(\eta)] = \frac{3\mathcal{Y}_s}{\kappa} \int_{\eta_-}^{\eta_+} d\eta \left[ \left( \frac{da}{d\eta} \right)^2 + ka^2 \right]. \tag{21}
\]

This classical action has exactly the form (1) we require.

III. QUANTUM CORRECTION

To calculate the quantum correction, we first compare the classical action (21) to the standard form (1): formally identifying \( x \rightarrow a, t \rightarrow \eta, \) and \( m \rightarrow -6\mathcal{Y}_s/\kappa, \) the quantum correction (2) becomes

\[
\Delta V_{\text{eff}} = -\frac{4\pi^2 \xi^4}{3\mathcal{Y}_s \kappa} \left[ \left( 1 - 4\xi \frac{d + 1}{d} \right) \left( \partial_a S \right)^2 + 2(1 - 4\xi) \partial_a^2 S \right], \tag{22}
\]

\(^2\) For \( k = -1, \alpha' \sim 4\mathcal{Y}_s \) as \( \chi_* \rightarrow \infty \), so (19) cannot be obtained from the limit of (18).
where \( \ell_{\text{pl}} \equiv \sqrt{\hbar c/8\pi} \) is the Planck length.\(^3\) The correction \( \Delta V_{\text{eff}} \) arises from the many quantum degrees of freedom we have discarded by describing the universe in terms of the single observable \( a(\eta) \) – all the particles and inhomogeneities that could exist within the spatial region \( \chi \in [0, \chi_*] \). Although we would need a complete understanding of quantum gravity to describe these fundamental degrees of freedom in detail, the holographic principle will suffice to fix their maximum entropy/information \( S \); we can then treat \( \xi \) and \( d \) as unknown constants, to be determined by experiment.

I now claim that we can drop the \( \partial_d^2 S \) term in (22) and simply write

\[
\Delta V_{\text{eff}} = \frac{4\pi^2 \ell_{\text{pl}}^4}{3\sqrt{\kappa}} \left( 4 \xi \frac{d+1}{d} \right) (\partial_a S)^2 . \tag{23}
\]

There are two distinct reasons for this. The first is practical: \( (\partial_a S)^2 \sim S^2/a^2 \) is far bigger than \( \partial_d^2 S \sim S/a^2 \) whenever the information capacity is very large, i.e. \( S \gg 1 \). This will always be the case for regions \( \chi \in [0, \chi_*] \) that are much larger than the Planck length: \( a \chi_* \gg \ell_{\text{pl}} \). (We can take this for granted as \( \chi_* \rightarrow \infty \) for \( k \in (0, -1) \); for \( k = 1 \), it can only fail if the universe is Planckian \( (a \pi \sim \ell_{\text{pl}}) \) and therefore unsuitable for a semiclassical treatment anyway.)

The second reason is theoretical: even though the \( \partial_d^2 S \) contribution is tiny, it is not exactly zero, so it retains the potential to break a symmetry of the classical theory. In appendix A, I show that this is indeed the case. The classical theory has a gauge freedom \( N \), and is also invariant under a redefinition of the dynamical variable \( a \rightarrow a(\tilde{a}(t)) \); it turns out that the \( \partial_d^2 S \) term breaks this combined symmetry. Therefore, to insist that \( \Delta V_{\text{eff}} \) respect both these classical symmetries compels us to set \( \xi = 1/4 \) and banish the \( \partial_d^2 S \) term entirely. The result is equation (23) with the replacement

\[
(4 \xi \frac{d+1}{d} - 1) \rightarrow \frac{1}{d} . \tag{24}
\]

Given that we cannot properly interpret \( \xi \) or \( d \) without reference to a theory of quantum gravity, it seems wise to retain the full generality of \( \xi \in \mathbb{R} \), despite this symmetry argument. Nonetheless, this discussion motivates us to absorb the freedom in \( (\xi, d) \in \mathbb{R} \times \mathbb{N} \) into a single dimensionless parameter

\[
\bar{d} \equiv \left( 4 \xi \frac{d+1}{d} - 1 \right)^{-1} \in \mathbb{R} , \tag{25}
\]

so that (23) becomes

\[
\Delta V_{\text{eff}} = \frac{4\pi^2 \ell_{\text{pl}}^4}{3\sqrt{\kappa} \bar{d}} (\partial_a S)^2 , \tag{26}
\]

with \( \bar{d} = d \) for the symmetric case \( \xi = 1/4 \). In general, we will use an overbar to label the key dimensionless parameters of the theory.

### A. Volumetric Information Capacity

Before we invoke the holographic principle, is instructive to first consider a counterfactual argument, based on the naive idea that one should be able to store information in every Planck volume independently. This discussion will connect our work to the old cosmological constant problem, and will serve as a warm up for the holographic calculation to come.

So suppose it were possible to store exactly \( n \) qubits in every Planck volume. Then the information capacity of the region \( \chi \in [0, \chi_*] \) would be

\[
S_v = n \ln 2 \cdot V_* a^3 / \ell_{\text{pl}}^4 , \tag{27}
\]

leading to a quantum correction (26) as follows:

\[
\Delta V_{\text{eff}} = \frac{12\pi^2 n^2 (\ln 2)^2 V_* a^4}{\kappa \ell_{\text{pl}}^2 \bar{d}} . \tag{28}
\]

We would then construct the semiclassical action (4) by inserting the correction (28) into the classical action (21):

\[
\mathcal{J}_G = \mathcal{J}_G[a(\eta)] - \int_{\eta_-}^{\eta_+} d\eta \Delta V_{\text{eff}}
\]

\[
= \frac{3\mathcal{V}_*}{\kappa} \int_{\eta_-}^{\eta_+} d\eta \left[ -\left( \frac{da}{d\eta} \right)^2 + k a^2 - \frac{4\pi^2 n^2 (\ln 2)^2 a^4}{\ell_{\text{pl}}^2 \bar{d}} \right]
\]

\[
= \frac{3\mathcal{V}_*}{\kappa} \int_{t_-}^{t_+} dt \left[ \frac{\tilde{a}^2}{N} + k N a^2 - \frac{4\pi^2 n^2 (\ln 2)^2 N a^4}{\ell_{\text{pl}}^2 \bar{d}} \right] .
\]

Had we included \( \Lambda \neq 0 \) in section II, we would have subtracted

\[
\frac{1}{\kappa} \int_{\mathcal{M}} d^4 x \sqrt{-g} \Lambda = \frac{\mathcal{V}_*}{\kappa} \int_{t_-}^{t_+} dt N a^4 \Lambda \tag{30}
\]

from the classical action (19); hence (29) is equivalent to

\[
\mathcal{J}_G = \frac{3\mathcal{V}_*}{\kappa} \int_{t_-}^{t_+} dt \left[ \frac{\tilde{a}^2}{N} + k N a^2 - \frac{\Lambda_{\text{eff}} N a^4}{3} \right] , \tag{31}
\]

with an effective cosmological constant

\[
\Lambda_{\text{eff}} = \frac{12\pi^2 n^2 (\ln 2)^2}{\ell_{\text{pl}}^2 \bar{d}} . \tag{32}
\]

For \( n, \bar{d} \sim 1 \), we see that \( \Lambda_{\text{eff}} \sim 10^{124} \Lambda_{\text{obs}} \) reproduces the enormous cosmological constant that normally arises from summing zero-point energies up to the Planck scale.

It is unclear whether this resemblance is purely superficial, or evidence of some fundamental connection between vacuum energy and the quantum correction (2) as

\[^3\text{The path integral derivation of } \Delta V_{\text{eff}} \text{ ensures that (22) is valid for the general form } S = S(a(\eta), f^\nu df/\sqrt{f(a(\eta'))}) \text{, where } \partial_{\nu} \text{ derivatives acting on the first argument of } S \text{ only [22]. This includes the case } S = S(a(\eta), \eta) \text{ that will be most useful here.}\]
applied to cosmology. The second option suggests an exciting possibility: that counting degrees of freedom correctly (i.e. holographically) may not only suffice to generate the cosmic acceleration we observe, but could also explain away the large vacuum energy predicted by quantum field theory. We leave this discussion for another time, content to tackle the former problem, without a definitive answer to the latter.

B. Holographic Information Capacity

In fact, the volumetric formula (27) is wrong. As detailed in appendix B, quantum gravity considerations (the holographic principle [23–26] and black hole complementarity [43, 44]) lead us instead to the following formula for the information capacity of the region $\chi \in [0, \chi_*]$ at conformal time $\eta$:

$$S_h(\alpha, \eta) = \frac{\mathcal{A}(\bar{\eta} - \eta) a^2}{4\ell_{pl}^2} \cdot \frac{\bar{\mu} \mathcal{Y}_*}{\mathcal{V}(\bar{\eta} - \eta)} .$$

(33)

where $\bar{\eta}$ is the final conformal time, and $\bar{\mu} \lesssim 1$ is a numerical constant, to be determined elsewhere. In equation (33) the first fraction quantifies the information capacity of a sphere the size of the cosmological event horizon, and the second fraction counts the number of such spheres inside $\chi \in [0, \chi_*]$. (The filling factor $\bar{\mu}$ accounts for the organisation of holographic information in spacetime; see appendix B for details.) In section V C, we will check that $\bar{\eta}$ really does exist: i.e. that the quantum correction (26) does indeed send $a(\eta) \rightarrow \infty$ as $\eta \rightarrow \bar{\eta}$. In this sense, the final conformal time is a self-fulfilling prophesy.

The derivation of (33) assumes that the universe is expanding $\dot{a} > 0$, and that the event horizon is far smaller than the radius of spatial curvature: $|k|/(\bar{\eta} - \eta) \ll 1$. For our universe, these assumptions can only break down at very early times, either during inflation, or before a Big Bounce. Hence, equation (33) is certainly suitable for a theory of late-time cosmic acceleration. (I will revisit these assumptions in a future publication, when I examine the role of $\Delta V_{\text{eff}}$ in the very early universe.) At the very least, a reader who is sceptical of the arguments in appendix B can always take (33) to be a well-motivated holographic hypothesis, the cosmological consequences of which we will now examine in detail.

We begin, as with the volumetric case, by calculating the quantum correction (26):

$$\Delta V_{\text{eff}} = \frac{3\pi^2 \bar{\mu}^2 \mathcal{Y}_*}{\kappa d} \left( \frac{\mathcal{A}(\bar{\eta} - \eta)}{3\mathcal{V}(\bar{\eta} - \eta)} \right)^2 a^2 .$$

(34)

Once again, this combines with the classical action (21) to form the semiclassical action (4):

$$\mathcal{J}_G[a(\eta)] = \mathcal{I}_G[a(\eta)] - \int_{\eta_-}^{\eta_+} d\eta \Delta V_{\text{eff}}$$

$$= \frac{3\mathcal{Y}_*}{\kappa} \int_{\eta_-}^{\eta_+} d\eta \left[ -\left( \frac{\dot{a}}{a} \right)^2 + ka^2 - \frac{\pi^2 \bar{\mu}^2}{d} \left( \frac{\mathcal{A}(\bar{\eta} - \eta)}{3\mathcal{V}(\bar{\eta} - \eta)} \right)^2 a^2 \right] .$$

(35)

Notice that the integration limits $\eta_\pm$ determine the interval over which this action defines the dynamics of the spacetime. There is no reason to truncate our theory at late times, so we must send $\eta_+ \rightarrow \bar{\eta}$. On the other hand, we may want to keep $\eta_-$ as a cutoff at early times, for when energy-densities approach the Planck-scale and the semiclassical approximation breaks down. Unless we wish to discuss such early times in detail, it will suffice to confute this cutoff with the initial singularity: $\eta_- = 0$.

Finally, we re-express the semiclassical action (35) in terms of the generic time coordinate $t$, so that we have two dynamical variables ($a, N$) with which to derive the two semiclassical Friedmann equations. Recalling the definition of conformal time (20) the action (35) becomes

$$\mathcal{J}_G[a(t), N(t)] = \frac{3\mathcal{Y}_*}{\kappa} \int_{t_-}^{t_+} dt \left[ \frac{\dot{a}^2}{N} + kN a^2 \right.$$ 

$$- \left( \frac{\mathcal{A}(\bar{\eta} - \eta)}{3\mathcal{V}(\bar{\eta} - \eta)} \right)^2 \bar{g} N a^2 \right] ,$$

(36)

where

$$\bar{g} \equiv \frac{\pi^2 \bar{\mu}^2}{d}$$

(37)

is a convenient shorthand, and $\bar{t} \in \mathbb{R} \cup \{\infty\}$ is the final value of the $t$ coordinate:

$$\lim_{t \rightarrow \bar{t}} \eta(t) = \bar{\eta} .$$

(38)

The semiclassical action (36) is the first major result of this paper. Even though this action includes an unusual “integral inside the integral” term, it will still define well-behaved equations of motion. These are obtained in the next section, by infinitesimal variations $\delta a(t)$ and $\delta N(t)$.

IV. SEMICLASSICAL FRIEDMANN EQUATIONS

The semiclassical Friedmann equations are the equations of motion generated by the total semiclassical action, comprising both gravitational and matter parts:

$$\mathcal{J} = \mathcal{J}_G + \mathcal{I}_M .$$

(39)
which implies after relabelling the dummy variables \( t \) which becomes

\[
\int_{t_L}^{t_U} dt \left[ \frac{\dot{a}^2}{N^2} + \left( 1 + \frac{4g}{15} \right) ka^2 - \frac{\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^2} \right].
\]  

It is straightforward to take the functional derivative of this action with respect to the scale factor:

\[
\frac{\delta \mathcal{J}_G}{\delta a(t)} = \frac{6Y_*}{\kappa} \left[ \frac{d}{dt} \left( \frac{\dot{a}}{N} \right) + \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 - \frac{\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^2} \right].
\]  

However, the \( N(t) \) derivative requires a little more care. Under a variation \( \delta N(t) \), the action (41) changes by

\[
\delta \mathcal{J}_G = \frac{3Y_*}{\kappa} \int_{t_L}^{t_U} dt \left[ \delta N(t) \left( \frac{\dot{a}^2}{N^2} + \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 - \frac{\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^2} \right) \right] + \frac{2\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^3} \int_{t_L}^{t_U} \delta N(t''')dt'''
\]  

We can then swap the order of integration in the last term:

\[
\int_{t_L}^{t_U} dt \left[ \frac{N(t)[a(t)^2]}{\left( \int_t^t N(t')dt' \right)^3} \right] = \int_{t_L}^{t_U} dt \int_{t_U}^{t_U} dt' \frac{N(t)[a(t)^2]}{\left( \int_t^t N(t')dt' \right)^3} = \int_{t_L}^{t_U} dt' \int_{t_U}^{t_U} dt \frac{N(t)[a(t)^2]}{\left( \int_t^t N(t')dt' \right)^3}.
\]  

which becomes

\[
\int_{t_L}^{t_U} dt' \delta N(t''') \left[ \int_{t_U}^{t_U} dt \frac{N(t)[a(t)^2]}{\left( \int_t^t N(t')dt' \right)^3} \right] = \int_{t_L}^{t_U} dt \delta N(t) \left[ \int_{t_U}^{t_U} dt' \frac{N(t'')[a(t'')^2]}{\left( \int_{t_U}^t N(t')dt' \right)^3} \right],
\]

after relabelling the dummy variables \( t \leftrightarrow t'' \). Hence, equation (43) is equivalent to

\[
\delta \mathcal{J}_G = \frac{3Y_*}{\kappa} \int_{t_L}^{t_U} dt \delta N(t) \left[ \frac{\dot{a}^2}{N^2} + \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 - \frac{\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^2} \right] + 2\bar{g} \int_{t_N}^{t_U} dt'' \frac{N(t'')[a(t'')^2]}{\left( \int_{t_U}^t N(t')dt' \right)^3},
\]  

which implies

\[
\frac{\delta \mathcal{J}_G}{\delta N(t)} = \frac{3Y_*}{\kappa} \left[ \frac{\dot{a}^2}{N^2} + \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 - \frac{\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^2} \right],
\]

A. Functional Derivatives

Rather than proceed directly from the general formula (36) we first recall the assumption \(|k|\bar{\eta} - \eta| \ll 1\), and hence use the series expansion

\[
\left( \frac{\delta \mathcal{J}(\chi)}{3Y(\chi)} \right)^2 = \frac{1}{\chi^2} - \frac{4k}{15} + O(|k|^2)
\]  

(40)
to neglect terms \( O(|k|\bar{\eta} - \eta)^2 \) in the action (36):

\[
\int_{t_L}^{t_U} dt \left[ \frac{\dot{a}^2}{N^2} + \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 - \frac{\bar{g}a^2}{\left( \int_t^t N(t')dt' \right)^2} \right].
\]  

(41)
B. Results

We now have all we need to assemble the semiclassical Friedmann equations. Combining equations (17), (42) and (47), we see that the total semiclassical action (39) is stationary if and only if

\[ \frac{d^2}{N^2} = \frac{\kappa}{3} \rho a^4 - \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 + \frac{\bar{g}a^2}{\left( \int^t N(t')dt' \right)^2} - 2\bar{g} \int^t_{t'} \frac{N(t'')a(t'')}{(\int^t N(t')dt')^3}, \]

\[ \frac{d}{dt} \left( \frac{\dot{a}}{N} \right) = \frac{\kappa}{6} (\rho - 3p) a^3 N - \left( 1 + \frac{4\bar{g}}{15} \right) kNa + \frac{\bar{g}Na}{\left( \int^t N(t')dt' \right)^2}. \]

Note that \( \eta \) has dropped out of these equations, so we are now free to send \( \chi_\ast \to \infty \) as desired. Differentiating (48a) with respect to \( t \), and comparing the result with (48b), we see that the two equations are indeed consistent, provided matter obeys the standard continuity equation:

\[ a\dot{\rho} + 3\dot{a}(\rho + p) = 0. \]

As usual, \( N(t) \) is not determined by the dynamical equations. Instead, this function must be specified by a choice of gauge, which fixes the physical meaning of the coordinate \( t \). An intuitive representation of the dynamical equations is achieved by setting \( N(t) = 1/a(t) \), so that \( t \) is the proper time \( \tau \) of a comoving observer in the FRW spacetime (8). The semiclassical Friedmann equations (48) then become

\[ H^2 = \frac{\kappa}{3} \rho - \left( 1 + \frac{4\bar{g}}{15} \right) \frac{k}{a^2} + \frac{\bar{g}a^2}{\left( \int^\tau \frac{d\tau'}{a(\tau')} \right)^2} - 2\bar{g} \int^\tau_{\tau_0} \frac{a(\tau'')}{\left( \int^{\tau'} \frac{d\tau''}{a(\tau'')} \right)^3}, \]

\[ \frac{dH}{d\tau} + 2H^2 = \frac{\kappa}{6} (\rho - 3p) - \left( 1 + \frac{4\bar{g}}{15} \right) \frac{k}{a^2} + \frac{\bar{g}a^2}{\left( \int^\tau \frac{d\tau'}{a(\tau')} \right)^2}, \]

where \( H \equiv d\ln a/d\tau \) is the Hubble parameter. Subtracting (50a) from (50b) we can also obtain the acceleration equation:

\[ \frac{1}{a} \frac{d^2a}{d\tau^2} = 2\frac{dH}{d\tau} + H^2 = -\frac{\kappa}{6} (\rho + 3p) + 2\bar{g} \int^\tau_{\tau_0} \frac{a(\tau'')}{\left( \int^{\tau'} \frac{d\tau''}{a(\tau'')} \right)^3}. \]

This confirms our basic hypothesis – the quantum correction (34) does indeed generate cosmic acceleration, without the need for a cosmological constant, dark energy, or modified gravity. Note that \( \bar{g} > 0 \) gives the quantum correction the correct sign, producing positive cosmic acceleration. This sign is guaranteed by the symmetry-breaking argument of appendix A, which set \( \xi = 1/4, \bar{d} = 1/d \in \mathbb{N} \) and hence \( \bar{g} \equiv \pi^2 \bar{m}/\bar{d} > 0 \).

To study this new form of cosmic acceleration (51) in detail, we must of course solve the semiclassical Friedmann equations. To this end, the gauge \( N(t) = 1 \) is an extremely profitable choice: \( t \) is then equivalent to conformal time (20) and the semiclassical Friedmann equations (48) simplify to

\[ \left( \frac{da}{d\eta} \right)^2 = \frac{\kappa}{3} \rho a^4 - \left( 1 + \frac{4\bar{g}}{15} \right) ka^2 + \frac{\bar{g}a^2}{(\eta - \eta')^2} - 2\bar{g} \int_{\eta'}^{\eta'} \frac{d(\eta'')}{(\eta' - \eta'')}^2, \]

\[ \frac{d^2a}{d\eta^2} = \frac{\kappa}{6} (\rho - 3p) a^3 - \left( 1 + \frac{4\bar{g}}{15} \right) ka + \frac{\bar{g}a}{(\eta - \eta')^3}. \]

In the next section, we will find exact solutions to these equations, for \( k = 0 \).

V. SPATIALLY FLAT UNIVERSE WITH MATTER & RADIATION

We shall model our universe as a spatially flat FRW spacetime (8) containing pressure-free matter (so-called “dust”) and radiation. In other words, we set \( k = 0 \) and

\[ \rho = \frac{\rho^m a^3}{a^3} + \frac{\rho^r a^4}{a^4}, \quad p = \frac{\rho^m a^3}{3a^4}. \]
Here, \( \rho_m^\circ \) is the energy-density of matter, and \( \rho_r^\circ \) the energy-density of radiation, when the scale factor has some arbitrary reference value \( a = a_0 \). The semiclassical Friedmann equations (52) are therefore

\[
\left( \frac{da}{d\eta} \right)^2 = \frac{\kappa}{3} \left( \rho_m^\circ a^3 + \rho_r^\circ a^4 \right) + \frac{\bar{g} a^2}{(\bar{\eta} - \eta)^2} - 2\bar{g} \int_0^{\eta} d\eta' \frac{[a(\eta')]^2}{(\bar{\eta} - \eta')^3},
\]

(54a)

\[
\frac{d^2a}{d\eta^2} = \frac{\kappa}{6} \rho_m^\circ a^3 + \frac{\bar{g} a}{(\bar{\eta} - \eta)^2},
\]

(54b)

where the cutoff \( \eta_\infty \) has been placed at the initial singularity:

\[
\eta_\infty = 0, \quad \lim_{\eta \to 0} a(\eta) = 0.
\]

As with our preceding analysis, we ignore the details of the very early universe, including inflation and the possibility of a Big Bounce.

### A. Derivation

Let us first simplify our notation. We define the constants

\[
\beta_m \equiv \frac{\kappa \bar{g} \rho_m^\circ a_0^3}{3}, \quad \beta_r \equiv \frac{\kappa \bar{g} \rho_r^\circ a_0^4}{3},
\]

and express the conformal time in terms of the variable

\[
u \equiv \frac{\eta - \eta_\infty}{\bar{\eta}}.
\]

This recasts the dynamical equations (54) as

\[
\left( \frac{da}{du} \right)^2 = \beta_m a + \beta_r + \frac{\bar{g} a^2}{u^2} - 2\bar{g} \int_u^1 du' \frac{[a(u')]^2}{u'^3},
\]

(58a)

\[
\frac{d^2a}{du^2} = \frac{\beta_m}{2} + \frac{\bar{g} a}{u^2},
\]

(58b)

which we shall now proceed to solve.

To obtain the general solution of (58b), note that the homogeneous equation

\[
\frac{d^2a}{du^2} = \frac{\bar{g} a}{u^2}
\]

has general solution

\[
a = C_+ u^{(1 + \sqrt{4\bar{g} + 1})/2} + C_- u^{(1 - \sqrt{4\bar{g} + 1})/2},
\]

(60)

for arbitrary constants \( C_\pm \). Let us write this as

\[
a = C_+ u^{(1 + \bar{\gamma})/2} + C_- u^{(1 - \bar{\gamma})/2},
\]

(61)

where

\[
\bar{\gamma} \equiv \sqrt{4\bar{g} + 1} = \sqrt{4\pi^2 \bar{\mu}^2 / \bar{d} + 1}
\]

(62)

repackages the numerical constants \( \bar{\mu} \) and \( \bar{d} \) in a convenient fashion. We will generally be interested in \( \bar{\gamma} > 1 \), which corresponds to positive cosmic acceleration: \( \bar{g} > 0 \) in equation (51). Beyond this, the solutions (61) remain well-defined for all \( \bar{g} \geq -1/4 \), and we can take \( \bar{\gamma} \geq 0 \) without loss of generality. (As there are no real solutions for \( \bar{g} < -1/4 \), such values are completely untenable.)

In addition to the homogeneous solutions (61) we require a particular integral. It is easy to check that

\[
a = \frac{2\beta_m}{4 - 2\bar{g}} u^2 = \frac{2\beta_m}{9 - \bar{\gamma}^2} u^2
\]

satisfies the second semiclassical equation (58b); hence

\[
a = \frac{2\beta_m}{9 - \bar{\gamma}^2} u^2 + C_+ u^{(1 + \bar{\gamma})/2} + C_- u^{(1 - \bar{\gamma})/2}
\]

(64)

is its general solution.

We now impose the following conditions on (64):

\[
a|_{u=1} = 0, \quad \frac{da}{du} \bigg|_{u=1} = -\sqrt{\beta_r}.
\]

(65a)

(65b)

The first equation (65a) is simply the Big Bang condition (55) expressed in terms of \( u \). The second (65b) ensures that the other Friedmann equation (58a) is satisfied at \( u = 1 \), with the negative root providing an expanding universe: \( da/d\eta > 0 \). In fact, this condition guarantees that (58a) is satisfied for all \( u \). To see this clearly, move all the terms in (58a) to one side of the equation, and call this sum \( E(u) \). Differentiating with respect to \( u \), one finds that \( E'(u) \) vanishes whenever (58b) is satisfied, so our general solution (64) guarantees \( E'(u) = 0 \) \( \forall u \). Given that (65b) sets \( E(1) = 0 \), we conclude that \( E(u) = E(1) - \int_u^1 E'(u')du' = 0 \), meaning that equation (58a) is satisfied for all \( u \). Thus, the conditions (65) ensure that our solution (64) solves both semiclassical Friedmann equations (58) and has a Big Bang at \( \eta = 0 \).

Inserting (64) into (65) we obtain

\[
\frac{2\beta_m}{9 - \bar{\gamma}^2} + C_+ + C_- = 0,
\]

(66a)

\[
\frac{4\beta_m}{9 - \bar{\gamma}^2} + \frac{1 + \bar{\gamma}}{2} C_+ + \frac{1 - \bar{\gamma}}{2} C_- = -\sqrt{\beta_r}.
\]

(66b)

Hence,

\[
C_\pm = \mp \frac{1}{\bar{\gamma}} \left( \frac{2\beta_m}{3 + \bar{\gamma}} + \sqrt{\beta_r} \right)
\]

(67)

are the coefficients we require.

### B. Exact Solutions

Inserting the coefficients (67) into the solution (64), and calculating the proper time since the Big Bang

\[
\tau = \int_0^\eta d\eta' a(\eta') = \bar{\eta} \int_u^1 du' a(u'),
\]

(68)

we arrive at our main result:
This is the exact general solution to the semiclassical Friedmann equations (§IV B) for a spatially flat FRW spacetime (8) that contains matter and radiation, and begins with a Big Bang \( a = \tau = 0 \) at conformal time \( \eta = \eta_\ast = 0 \). For ease of reference, let us repeat the definitions of the various quantities used above:

\[
\begin{align*}
\beta_m & \equiv \frac{\kappa \bar{\eta}^2 \rho_m a^3_0}{3}, \\
\beta_r & \equiv \frac{\kappa \bar{\eta}^2 \rho_r a^3_0}{3}, \\
u & \equiv \frac{\bar{\eta} - \eta}{\eta}, \\
\bar{\gamma} & \equiv \sqrt{4\bar{g} + 1} \geq 0,
\end{align*}
\] (70)

where \( \bar{\eta} \) is the final conformal time, \( \rho_m/\bar{m}/\bar{d} \) is the matter/radiation energy-density when \( a = a_\ast \), and \( \bar{g} \equiv \pi^2 \bar{\mu}^2/\bar{d} \) is a dimensionless constant. (The numerical value \( \bar{\mu} \lesssim 1 \) accounts for the arrangement of holographic information in spacetime (appendix B) and will be calculated in a future paper. The parameter \( \bar{d} \) depends on unknown details of the discarded configuration space (25) but may be constrained to \( \bar{d} = d \in \mathbb{N} \) by the invariance argument of Appendix A.) These solutions are the main prediction of the theory, implicitly describing the expansion history of the universe \( a(\tau) \) for each \((\beta_m, \beta_r, \bar{\eta}, \bar{\gamma})\).

At first glance, the solutions (69) appear to break down at \( \bar{\gamma} = 0 \) and \( \bar{\gamma} = 3 \). However, we can take the limits \( \bar{\gamma} \to 0 \) and \( \bar{\gamma} \to 3 \) without issue, and let the results define the solutions at the limit points. Explicitly,

\[
\begin{align*}
\lim_{\bar{\gamma} \to 0} a & = \frac{2\beta_m}{9} \left( u^2 - u^{1/2} \left( 1 + \frac{3}{2} \ln u \right) \right) - \sqrt{\beta_r} u^{1/2} \ln u, \\
\lim_{\bar{\gamma} \to 0} \tau & = \frac{2\bar{\eta} \beta_m}{27} \left( 1 - u^3 + 3u^{3/2} \ln u \right) \\
& + \frac{2\bar{\eta} \sqrt{\beta_r}}{9} \left( u^{3/2} (3 \ln u - 2) + 2 \right),
\end{align*}
\] (71a)

gives the solutions at \( \bar{\gamma} = 0 \); while

\[
\begin{align*}
\lim_{\bar{\gamma} \to 3} a & = \frac{\beta_m}{18} \left( u^{-1} + u^3 (3 \ln u - 1) \right) + \frac{\sqrt{\beta_r}}{3} \left( u^{-1} - u^2 \right), \\
\lim_{\bar{\gamma} \to 3} \tau & = \frac{\bar{\eta} \beta_m}{18} \left( \frac{2}{3} (u^3 - 1) - (u^3 + 1) \ln u \right) \\
& + \frac{\bar{\eta} \sqrt{\beta_r}}{9} \left( u^3 - 1 - 3 \ln u \right),
\end{align*}
\] (71b)

specifies them at \( \bar{\gamma} = 3 \). The solutions (69) are therefore well-defined for all \( \bar{\gamma} \geq 0 \).

For the remainder of this section, we will describe the basic properties of the cosmologies (69) we have just derived. Then, in section VI, we will restrict our attention to the well-motivated class \( \bar{\gamma} > 1 \), and compare the expansion histories to those of the standard cosmological model, in which acceleration is driven by \( \Lambda \).

### C. Final Conformal Time

We are now in a position to check the self-consistency of the theory, confirming that \( \bar{\eta} \) really is the final conformal time (B6). Evaluating our solutions (69) in the limit \( u \to 0 \), we see that

\[
\begin{align*}
\lim_{\eta \to \bar{\eta}} a & = \begin{cases} 
0, & \bar{\gamma} \in [0, 1), \\
\sqrt{\beta_r} + \beta_m/4, & \bar{\gamma} = 1, \\
\infty, & \bar{\gamma} \in (1, \infty),
\end{cases} \\
\lim_{\eta \to \bar{\eta}} \tau & = \begin{cases} 
\frac{2\bar{\eta}}{\bar{\gamma}(3+\bar{\gamma})} (2\sqrt{\beta_r} + \beta_m/3), & \bar{\gamma} \in [0, 3), \\
\infty, & \bar{\gamma} \in [3, \infty).
\end{cases}
\end{align*}
\] (72)

For the well-motivated values \( \bar{\gamma} > 1 \), we recover exactly what we need: an accelerating expanding universe that attains infinite expansion as \( \eta \) approaches \( \bar{\eta} \). For \( \bar{\gamma} \in (1, 3) \) the universe ends in a Big Rip in finite proper time, while for \( \bar{\gamma} \in [3, \infty) \) the limit \( \eta \to \bar{\eta} \) is achieved asymptotically as \( \tau \to \infty \). In the next subsection, we will interpret these behaviours in terms of an effective equation of state \( w_{\text{eff}}(\tau) \) for the quantum correction.

Before then, let us quickly comment on the remaining (unphysical) values \( \bar{\gamma} \in [0, 1] \). For \( \bar{\gamma} \in [0, 1) \) the universe ends in a Big Crunch at \( \eta = \bar{\eta} \). These solutions pass the basic consistency check (\( \bar{\eta} \) is indeed the final conformal time) but violate the assumption of an expanding universe \( \dot{a} > 0 \). This assumption was used to derive the information capacity (33) so the self-consistency of these solutions remains dubious. The value \( \bar{\gamma} = 1 \) corresponds to the trivial case \( \bar{g} = 0 \), wherein the semiclassical Friedmann equations (54) reduce to the classical Friedmann equations. These equations make no reference to \( \eta \), so it comes as no surprise that nothing special happens at \( \eta = \bar{\eta} \) for \( \bar{\gamma} = 1 \).

### D. Effective Equation of State

It is occasionally useful to think of the quantum correction as though it were a homogeneous fluid, contributing an effective energy-density \( \rho_{\text{ef}} \) and pressure \( p_{\text{ef}} \) to the classical Friedmann equations. Consulting the semiclassical Friedmann equations (52) for \( k = 0 \), we see that this
fictitious fluid must have

\[
\kappa \rho_{\text{eff}} = \frac{3\bar{g}}{a^2 (\bar{n} - \eta)^2} - \frac{6\bar{g}}{a^2} \int_0^\eta \, \frac{d\eta'}{(\bar{n} - \eta')^2},
\]

\[
\kappa p_{\text{eff}} = -\frac{\bar{g}}{a^2 (\bar{n} - \eta)^2} - \frac{2\bar{g}}{a^2} \int_0^\eta \, \frac{d\eta'}{(\bar{n} - \eta')^2},
\]

and equation of state

\[
w_{\text{eff}} \equiv \frac{p_{\text{eff}}}{\rho_{\text{eff}}} = -\frac{a^2}{2 \int_0^\eta \, \frac{d\eta'}{(\bar{n} - \eta')^2} - \frac{2\bar{g}}{a^2} \int_0^\eta \, \frac{d\eta'}{(\bar{n} - \eta')^2}}.
\]

Of course, it is important that this description not be taken too literally: the quantum correction cannot be interpreted locally as physical fluid — this a purely global phenomenon, that only operates at the scale of cosmological event horizon.

To apply this formalism to the exact solutions (69) we first rewrite the equation of state (74) in terms of the variable \(u\):

\[
w_{\text{eff}} = \frac{a^2 u^{-2} + 2 \int_u^1 \, du'[a(u')]^2 u'^{-3}}{3 \int_0^\eta \, \frac{d\eta'}{(\bar{n} - \eta')^2} - \frac{2\bar{g}}{a^2} \int_0^\eta \, \frac{d\eta'}{(\bar{n} - \eta')^2}}.
\]

At early times, the solutions (69) become

\[
a = \epsilon \sqrt{\beta_1} + \epsilon^2 (\beta_m/4) + O(\epsilon^3)
\]

where \(\epsilon \equiv \eta/\bar{n} = 1 - u\). This result can be obtained by expanding (69) in powers of \(\epsilon\); however, it is considerably easier to insert (76) into equation (58a) and confirm that the semiclassical Friedmann equations are satisfied to the relevant order. (The Big Bang condition (65a) is also satisfied, but this is obvious.) Substituting the expansion (76) into the equation of state (75) we find

\[
w_{\text{eff}} = \frac{1}{3} - \frac{4}{9} \epsilon + O(\epsilon^2).
\]

In other words, the quantum correction scales like spatial curvature \(w_k = -1/3\), as we approach the initial singularity.

At late times \((\eta \to \bar{n}, u \to 0)\) the solutions (69) behave as follows:

\[
a \propto u^{(1-\tilde{\gamma})/2} \left(1 + O \left( u^{-\min\{\tilde{\gamma},(3+\tilde{\gamma})/2\}} \right) \right),
\]

for \(\tilde{\gamma} > 1\). Hence, the equation of state (75) tends to

\[
\lim_{\eta \to \bar{n}} w_{\text{eff}} = \frac{3 + \tilde{\gamma}}{3(1 - \tilde{\gamma})}.
\]

For \(\tilde{\gamma} \in (1, 3)\), we see that the quantum correction resembles phantom dark energy \(w_{\text{eff}} < -1\) at late times, explaining the Big Rips in equation (72). The other values \(\tilde{\gamma} \in (3, \infty)\) generate non-phantom behaviour \((-1 < w_{\text{eff}} < -1/3)\) at late times, which accelerates the universe over unbounded proper time. We also note that the special case \(\tilde{\gamma} = 3\) has \(w_{\text{eff}} \to -1\), converging on the equation of state of a cosmological constant. Hence the special solution (71b) must tend to de Sitter spacetime in the asymptotic future.

VI. COMPARISON WITH ΛCDM

Rather than attempt a full comparison with observational data here, we can assess the plausibility of the theory by comparing its predictions (69) to those of ΛCDM. This should assuage any fears that the model can be dismissed “out of hand” as inconsistent with observations.

A few notes before we start our comparison:

- For the sake of simplicity, we will ignore radiation \(\rho_r = 0\) in the following analysis. This approximation is sufficient to describe the universe as far back as recombination. Provided the quantum correction is negligible at this time, we can be confident of its irrelevance before then, by virtue of its primordial equation of state (77).

- Notation: For the sake of brevity, I shall refer to the new theory by the acronym \(S_\Lambda\)CDM, alluding to the holographic information capacity (33) that generates the quantum correction (34) and hence drives cosmic acceleration. As the acronym suggests, these cosmologies will also include the standard cold dark matter component. I will distinguish ΛCDM quantities from \(S_\Lambda\)CDM quantities with superscripts (Λ) and (S_Λ). I denote present-day values with the subscript 0, and adopt the standard convention \(a_0^{(\Lambda)} = 1\) for ΛCDM. (No generality is lost for \(k = 0\).) To calibrate \(a_0^{(S_\Lambda)}\), we will compare the two models at early times, when the cosmologies are physically identical.

- A typical observation of redshift \(z = a^{-1} - 1\) and angular diameter distance \(D = a(\eta)(\eta_0 - \eta)\) will place a model independent constraint on \(a(\eta)\), not \(a(\tau)\). It is therefore natural to compare the expansion histories \(a^{(\Lambda)}\) versus \(a^{(S_\Lambda)}\) as functions of conformal time. For this reason, I leave \(\eta\) without a superscript, as a shared coordinate for the two theories.

We now begin by quoting a standard solution of the classical Friedmann equations.

A. ΛCDM Cosmology

For \(k = 0\), \(\rho_r = 0\), \(\Lambda > 0\), and \(a_0^{(\Lambda)} = 1\), the FRW universe (8) expands according to [45]:

\[
a^{(\Lambda)} = \left[ \frac{\sinh \left( \frac{3}{2} \tau^{(\Lambda)} H_0^{(\Lambda)} \sqrt{1 - \Omega_m^{(\Lambda)}} \right)}{\left( \Omega_m^{(\Lambda)} \right)^{-1} - 1} \right]^{1/3},
\]

where \(\tau^{(\Lambda)}\) is the proper time since the Big Bang, and \(\Omega_m^{(\Lambda)} \equiv \kappa \rho_m^{(\Lambda)} / 3 (H^{(\Lambda)})^2\) is energy density as a fraction of
the critical value. The age of the universe is
\[
\tau_0^{(A)} = \frac{2 \sinh^{-1} \left( \sqrt{\left( \Omega_{m0}^{(A)} \right)^{-1} - 1} \right)}{3 H_0^{(A)} \sqrt{1 - \Omega_{m0}^{(A)}}},
\]
and conformal time can be evaluated according to
\[
\eta = \int_0^{\tau^{(A)}} \frac{d\tau'}{a^{(A)}(\tau')}.
\]
Furthermore, equations (80) and (82) imply
\[
a^{(A)} = \frac{\eta^2}{4} \left( H_0^{(A)} \right)^2 \Omega_{\Lambda m0}^{(A)} \cdots \eta^2 \frac{k \rho_{m0}^{(A)}}{12} + \cdots,
\]
at early times. These cosmologies, and the time at which the observers live, are specified by two physical parameters: \( \Omega_{m0}^{(A)} \) and \( H_0^{(A)} \).

B. \( S_0 \) CDM Cosmology

For \( k = 0 \) and \( \rho_e = 0 \), the cosmologies of the holographic theory (69) are given by
\[
a^{(S_0)} = \frac{2 \kappa \eta^2 \rho_{m0}^{(S_0)}}{3 \bar{\gamma}^3 (9 - \bar{\gamma}^2)} \left( a_0^{(S_0)} \right)^3 F_\gamma(u),
\]
\[
\tau^{(S_0)} = - \frac{2 \kappa \eta^3 \rho_{m0}^{(S_0)}}{3 \bar{\gamma}^4 (9 - \bar{\gamma}^2)} \left( a_0^{(S_0)} \right)^3 F_\gamma(u),
\]
with
\[
F_\gamma(u) \equiv \bar{\gamma} \left( u^3 - 1 \right) - u^{(3+\bar{\gamma})/2} + u^{(3-\bar{\gamma})/2}.
\]
As defined in (70), the constant \( \bar{\gamma} \) depends on unknown details of the discarded configuration space, and the coordinate \( u \in [0, 1] \) sets the conformal time
\[
\eta = \bar{\eta}(1-u)
\]
as a fraction of its final value \( \bar{\eta} \). Consulting (76) and (70), we see that
\[
a^{(S_0)} = \frac{\eta^2}{4} \frac{k \rho_{m0}^{(S_0)}}{12} \left( a_0^{(S_0)} \right)^3 + \cdots,
\]
at early times. This matter-dominated expansion is physically identical to the behaviour of \( \Lambda \)CDM in equation (83). However, the functions \( a^{(S_0)}(\eta) \) and \( a^{(A)}(\eta) \) will only agree as \( \eta \to 0 \) if we fix the arbitrary value \( a_0^{(S_0)} \) according to
\[
\rho_{m0}^{(S_0)} \left( a_0^{(S_0)} \right)^3 = \rho_{m0}^{(A)}.
\]
This calibration ensures that the comparison between \( a^{(S_0)}(\eta) \) and \( a^{(A)}(\eta) \) is physically meaningful. Once (88) has been applied, the \( S_0 \) CDM cosmologies (84) depend on two fundamental parameters \( \eta, \bar{\gamma} \), with a third value \( u_0 \) needed to specify the time at which the observers live.

C. Matching Conditions

For any given \( \bar{\gamma} \), we can think of the parameters \( (u_0, \bar{\eta}) \) as analogous to \( (\Omega_{m0}^{(A)}, H_0^{(A)}) \), defining a time-scale for the \( S_0 \)CDM universe, and placing the observers at a particular point in cosmic history. Let us postpone a full exploration of this parameter-space for a future publication (when we test \( S_0 \)CDM against actual data). Here, it suffices to choose values \( u_0 = u_0(\Omega_{m0}^{(A)}, H_0^{(A)}) \), \( \bar{\eta} = \bar{\eta}(\Omega_{m0}^{(A)}, H_0^{(A)}) \) such that the \( S_0 \)CDM universe and the \( \Lambda \)CDM universe have the same conformal age
\[
\eta_0 = \bar{\eta}(1-u_0) = \int_0^{\tau^{(A)}} \frac{d\tau'}{a^{(A)}(\tau')},
\]
and the same present-day matter-density:
\[
\rho_{m0} = \rho_{m0}^{(S_0)} = \rho_{m0}^{(A)},
\]
including dark matter. As we will soon verify, these two matching conditions are sufficient to guarantee an agreement over the angular diameter distance of any structure that occurs soon after the Big Bang, including the surface of last scattering. Given this common ground, we can then examine how the other predictions of \( a^{(S_0)} \) differ from those of \( a^{(A)} \).

D. Comparison

To compare \( a^{(S_0)}(\eta) \) with \( a^{(A)}(\eta) \), it is convenient to represent both expansion histories in terms of the dimensionless time coordinate
\[
v \equiv \tau^{(A)} / \tau_0^{(A)}.
\]
The \( \Lambda \)CDM cosmologies (80) are then
\[
a^{(A)} = \frac{W(v, \Omega)}{\Omega^{-1} - 1}^{1/3},
\]
where we have introduced \( \Omega \equiv \Omega_{m0}^{(A)} \) and
\[
W(v, \Omega) \equiv \left[ \sinh \left( v \cdot \sinh^{-1} \sqrt{\Omega^{-1} - 1} \right) \right]^{2/3},
\]
to compress notation. The conformal time (82) is then given by
\[
\frac{\eta}{\eta_0} = \frac{\int_0^v dv' / W(v', \Omega)}{\int_0^1 dv' / W(v', \Omega)},
\]
and because the cosmologies agree on the conformal age of the universe (89) we can re-express (86) as
\[
u = 1 - (1-u_0) \frac{\int_0^v dv' / W(v', \Omega)}{\int_0^1 dv' / W(v', \Omega)}.
\]}
Inserting the density condition (90) into the calibration equation (88) we see that

\[ 1 = a_0^{(S_h)} = a^{(S_h)}|_{\eta = \eta_0} = \frac{2\kappa \bar{\eta}^2 \rho_{m0}}{3 \bar{\gamma} (9 - \bar{\gamma}^2)} F^\prime_\gamma(u_0); \]  

hence the $S_h$ CDM solutions (84) are simply

\[ a^{(S_h)} = \frac{F^\prime_\gamma(u)}{F^\prime_\gamma(u_0)}, \]  

which now depend on $v$ through equation (95). Finally, we solve (89) for $\bar{\eta}$ and substitute this into (96). After simplifying, we obtain

\[ (1 - u_0)^2 \frac{\bar{\gamma} (9 - \bar{\gamma}^2)}{F^\prime_\gamma(u_0)} = 8 \left( \frac{\sinh^{-1} \sqrt{\Omega^{-1} - 1}}{9 (\Omega^{-1} - 1)^{1/3}} \right)^2 \times \left( \int_0^1 \frac{dv'}{W(v', \Omega)} \right)^2, \]  

which fixes $u_0$ as a function of $\Omega$ and $\bar{\gamma}$.

Once we have assigned values to $\Omega$ and $\bar{\gamma}$, equations (92–98) define ($a^{(\Lambda)}, a^{(S_h)}, \eta/\eta_0$) as a function of the parameteric coordinate $v$, allowing us to plot the fractional difference in the scale factor

\[ \frac{\delta a}{a} \equiv \frac{a^{(S_h)} - a^{(\Lambda)}}{a^{(\Lambda)}}, \]  

as a function of $\eta/\eta_0$. This formalism has conveniently absorbed the time-scales $\bar{\eta}$ and $F^{\prime}_{\gamma}(u)$ into the units of $v$; nonetheless we can still use the ratio

\[ \frac{\tau^{(S_h)}}{\tau^{(\Lambda)}} = -\frac{\bar{\eta}}{\tau_0^{(\Lambda)}} \frac{F^{\prime}_\gamma(u)}{F^{\prime}_\gamma(u_0)} \left( \int_0^1 \frac{dv'}{W(v', \Omega)} \right)^2, \]  

\[ = \frac{(\Omega^{-1} - 1)^{1/3} F^{\prime}_\gamma(u)}{u_0 - 1} \int_0^1 \frac{dv'}{W(v', \Omega)} \]  

as a function of $\eta/\eta_0$.

E. Results

Choosing $\Omega_{m0}^{(\Lambda)} = 0.3$ to represent the approximate state of our universe [8], the formalism above lets us generate the plots in figure 1. There are a number of details to notice:

FIG. 1. In the plots above, the cosmic expansion histories (84) of $S_h$ CDM are compared to $\Lambda CDM$ cosmologies (80) with $\Omega_{m0} = 0.3$. As described in section VI C, the parameters ($u_0, \tilde{\eta}$) have been chosen to ensure that the two models agree on the conformal age of the universe (89) and the current matter-density (90). The two topmost graphs show the fractional difference in scale factor (99) as a function of conformal time; first for a wide range of values $\bar{\gamma} > 1$; then for a small group $\bar{\gamma} \in \{1.5, 1.6, 1.7, 1.8\}$ that agree with $\Lambda CDM$ to high accuracy. The third graph depicts the fractional difference in Hubble parameter (101) as a function of conformal time. Finally, the scale factor is plotted as a function of proper time, for a wide range of $\bar{\gamma}$ that agree with $\Lambda CDM$ to high accuracy. The approximate $S_h$CDM cosmologies undergo a Big Rip.
There is no \( \tilde{\gamma} \) for which there is absolute agreement \( \delta a = 0 \) over the entire cosmic history. In general, \( S_b \) CDM cannot reproduce \( \Lambda \) CDM to arbitrary accuracy. The new theory is therefore falsifiable.

At the surface of last scattering \( (a = a_s \approx 1/1100) \) the agreement is extremely close: \( \tilde{\gamma} = 1.7 \Rightarrow \delta a_s/a_s \approx 3 \times 10^{-5} \). (This is broadly indicative of all values shown in first plot, with \( \tilde{\gamma} = 10 \) producing the worst fit: \( \delta a_s/a_s \approx 10^{-4} \).) Given that \( a \propto \eta^7 \) when matter dominates, the conformal time of last scattering \( (\eta_\gamma/\eta_0 \approx 3 \times 10^{-2}) \) will be displaced by \( \delta \eta_\gamma/\eta_0 \approx 1.5 \times 10^{-5} \), shifting the angular diameter distance of the surface of last scattering by a fraction \( \delta D_s/\eta_0 \approx 5 \times 10^{-7} \). This is well below the angular precision \( \delta \theta/\theta \sim 10^{-4} \) achieved by the Planck survey [8]. We confirm that the matching conditions (89) and (90) were more than sufficient to ensure agreement with cosmic microwave background (CMB) measurements. A more realistic treatment of observational data would presumably grant greater freedom to the value of \( \tilde{\gamma} \).

In the second plot, we see that \( \tilde{\gamma} = 1.65 \pm 0.15 \) provides a close agreement with \( \Lambda \) CDM over all cosmic history. For these values, the historic maximum \( |\delta a/a| \sim 1.5\% \) occurs at \( \eta/\eta_0 \approx 0.7 \), corresponding to redshift \( z \approx 1 \). Baryon Acoustic Oscillations (BAOs) provide the best observational constraints at this epoch, with the distance of \( z = 0.57 \) (\( \eta/\eta_0 \approx 0.85 \)) measured to a precision of roughly 1% [46]. At this level of accuracy, \( \tilde{\gamma} = 1.6 \pm 0.1 \) cannot be distinguished from \( \Lambda \) CDM.

At late times, the main constraint on \( \tilde{\gamma} \) will come from measurements of the present-day Hubble parameter \( H_0 \). As can be seen in the third plot, \( \tilde{\gamma} = 1.7 \) provides \( H_0 \) equal to that of \( \Lambda \) CDM, to within an error much less than 1%. We also notice that \( \tilde{\gamma} = 1.6 \) predicts a 5% boost in the value of \( H_0 \). This suggests that \( S_b \) CDM has the potential to resolve the well-known disagreement over the current value of the Hubble parameter: \( H_0 = (73.52 \pm 1.62)\text{km s}^{-1}\text{Mpc}^{-1} \) from standard candles in the local universe [47, 48] versus \( H_0 = (67.66 \pm 0.42)\text{km s}^{-1}\text{Mpc}^{-1} \) as inferred from the CMB using \( \Lambda \) CDM [8].

The favoured values \( \tilde{\gamma} = 1.6 \pm 0.1 \) have an effective equation of state (74) that is phantom \( w_{\text{eff}} < -1 \) at late times (79). We see the consequences (72) of this feature in the fourth plot: the \( S_b \) CDM universes end in a Big Rip at \( \tilde{\tau}/\tau_0 = 1.7 \pm 0.2 \).

This analysis indicates that current measurements cannot distinguish \( S_b \) CDM from \( \Lambda \) CDM, at least for some values of the parameters \( (u_0, \eta, \tilde{\gamma}) \). It is therefore unlikely that \( S_b \) CDM can be ruled out with present data. In a future paper, I will confront the theory with observational data directly, inferring a posterior distribution for \( (u_0, \eta, \tilde{\gamma}) \) without using \( \Lambda \) CDM as a reference model.

VII. CONCLUSIONS

We have motivated and developed a new fundamental theory of cosmic acceleration \( (S_b \) CDM) that does not require dark energy or modified gravity. Instead, the expansion of the universe is accelerated by a subtle quantum phenomenon [21, 22] that emerges in any system with information capacity \( S \) that depends on a dynamical variable. In general, a quantum correction (2) induces a bias in the behaviour of the system (3) which forces it off its classical trajectory; one accounts for this effect semiclassically by including the correction in the action (4). \( S_b \) CDM brings this formalism to bear on the universe as a whole, with the cosmological information capacity (33) quantified according to the holographic principle (appendix B). Once the quantum correction (34) has been included in the cosmological action (36), we generate semiclassical Friedmann equations (48) in which cosmic acceleration (51) arises automatically:

\[
\frac{1}{a} \frac{d^2 a}{d \tau^2} = -\frac{\kappa}{6} (\rho + 3p) + \frac{2g}{a^3} \int_\eta^{\eta_f} d\eta' \left[ \frac{[a(\eta')]}{\eta - \eta'} \right]^2, \tag{102}
\]

dependent on the past behaviour of the scale factor. We have solved the semiclassical Friedmann equations for a spatially-flat universe containing matter and radiation (69). As shown in figure 1, these solutions succeed in reproducing the predictions of \( \Lambda \) CDM to within the accuracy of current observations. We conclude that \( S_b \) CDM provides cosmic acceleration “for free”, consistent with experiment, as a natural consequence of treating the universe as a holographic quantum system.

Free Parameter: \( S_b \) CDM introduces a single unknown dimensionless constant \( \tilde{\gamma} \equiv \sqrt{9\tilde{g}} + 1 \). For no value of \( \tilde{\gamma} \) is there an exact match between the predictions of \( S_b \) CDM and \( \Lambda \) CDM, so the new theory is falsifiable. Measurements of the CMB and BAOs are expected to restrict \( \tilde{\gamma} \approx 1.6 \pm 0.1 \). Rather provocatively, \( \tilde{\gamma} \approx 1.6 \pm 0.1 \) also predicts \( H_0 \) to be \( (5 \pm 5)\% \) larger than \( \Lambda \) CDM, potentially resolving the well-known tension between local measurements [47, 48] and CMB observations [8]. The quantity \( \tilde{g} = \pi^2 \mu^2 / d \) is set by a numerical “filling factor” \( \mu \), accounting for the organisation of holographic information in spacetime (appendix B.3), and a constant \( d \), which depends on unknown details of the discarded configuration space (25). In a separate article, I will generalise the holographic covering (fig. 3) to 3+1 dimensions, and hence calculate \( \mu \). Consequently, any observational limits on \( \tilde{\gamma} \) will directly constrain \( d \).

Coincidence: The favoured values \( \tilde{\gamma} \approx 1.6 \pm 0.1 \) predict a Big Rip at \( \tilde{\tau} \approx (1.7 \pm 0.2) \times \tau_0 \). This prediction ameliorates the coincidence problem [49] because there is no longer an infinite future (with \( \Omega \) \approx 1) where we should expect to find ourselves [27, 50]. Instead, \( S_b \) CDM places us at a rather typical point in cosmological history, roughly halfway between the initial singularity \( a = 0 \), and the final singularity \( a = \infty \).

Fine Tuning: In \( S_b \) CDM, the magnitude of cosmic acceleration (102) is essentially determined by the area of
the cosmological event horizon. (This is the reverse of the usual view, wherein Λ sets the size of the horizon.) Hence, we can seek to explain the extremely small value Λ_{obs} \sim 10^{-122}/\ell^2_P as the result of some physical process that expands this area at early times. Inflation is the obvious candidate for such a mechanism, conceivably solving the fine-tuning problem in the same fashion as the flatness problem. I will investigate this possibility in a future publication, when I extend S_0CDM to the very early universe.

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Appendix A: New Variables and Gauge Invariance

In this appendix, we examine the extent to which the quantum correction ΔV_{eff} is consistent with two key symmetries of the classical theory: (i) the gauge freedom of the time coordinate, and (ii) our ability to redefine the dynamical variable a = f(\bar{a}). To keep this discussion self-contained, let us briefly summarise the process by which the semiclassical action (36) is derived.

Starting with the metric
\[ ds^2 = [a(t)]^2 \left(-[N(t)]^2 dt^2 + d\chi^2 + [rk(\chi)]^2 d\Omega^2 \right), \]
we first obtain the classical gravitational action (19):
\[ I_G[a(t), N(t)] = \frac{3\gamma_*}{\kappa} \int_{t_-}^{t_+} dt \left\{ -\frac{\dot{a}^2}{N} + kNa^2 \right\}. \]  
(A2)

The conformal time coordinate \( \eta = \eta(t) \), defined by
\[ d\eta = N dt, \quad \eta_{\pm} = \eta(t_{\pm}), \]
then allows us to write (A2) in the canonical form
\[ I_G[a(\eta)] = \frac{3\gamma_*}{\kappa} \int_{\eta_-}^{\eta_+} d\eta \left\{ -\left(\frac{da}{d\eta}\right)^2 + ka^2 \right\}. \]  
(A4)

Comparing this action with the reference (1), we formally identified \( x \rightarrow a, t \rightarrow \eta, m \rightarrow -6\gamma_*/\kappa \); hence, the quantum correction (2) becomes (22), and the semiclassical action (4) is
\[ J_G[a(\eta)] = \frac{3\gamma_*}{\kappa} \int_{\eta_-}^{\eta_+} d\eta \left\{ -\left(\frac{da}{d\eta}\right)^2 + ka^2 \right\} + Q_1 (\partial_a S)^2 + Q_2 \partial_a^2 S \]  
(A5)

where \( S = S(a, \eta) \) is the information capacity of the discarded degrees of freedom, and
\[ Q_1 = \frac{4\pi^2 \ell_P^3}{9\gamma_*^2} \left(1 - 4\xi \frac{d+1}{d}\right), \quad Q_2 = \frac{8\pi^2 \ell_P^3}{9\gamma_*^2} \left(1 - 4\xi\right) \]  
(A6)
depend on the unknown constants \( \xi \) and \( d \). Finally, we re-express the semiclassical action \( (A5) \) in terms of the generic time coordinate \( t \),
\[ J_G[a(t), N(t)] = \frac{3\gamma_*}{\kappa} \int_{t_-}^{t_+} dt \left\{ -\frac{\dot{a}^2}{N} + kNa^2 \right\} + N \left( Q_1 (\partial_a S)^2 + Q_2 \partial_a^2 S \right), \]  
(A7)

so that the semiclassical Friedmann equations (48) can be obtained by variations \( \delta a(t), \delta N(t) \)

For the present discussion, the critical step above is the selection of \( \eta \) as the time coordinate that renders \( I_G \) in the canonical form (A4). At first glance, it appears that \( \eta \) is the only such coordinate that can achieve this goal, allowing us to make contact with the quantum theory of section IA. However, suppose we define the scale factor using an invertible differentiable function \( f, \)
\[ a = f(\bar{a}(t)), \]  
(A8)
and consider \( \bar{a}(t) \) and \( N(t) \) as our new dynamical variables. Then the classical action (A2) becomes
\[ I_G[a(\bar{a}), N(t)] = I_G[f(\bar{a}), N(t)] \]  
(A9)
\[ \frac{3\gamma_*}{\kappa} \int_{t_-}^{t_+} dt \left\{ -\frac{\dot{\bar{a}}^2}{N} + [f'(\bar{a})]^2 + kN [f(\bar{a})]^2 \right\}, \]
which takes on canonical form
\[ I_G[\bar{a}(\bar{\eta})] = \frac{3\gamma_*}{\kappa} \int_{\bar{\eta}_-}^{\bar{\eta}_+} d\bar{\eta} \left\{ -\left(\frac{d\bar{a}}{d\bar{\eta}}\right)^2 + k [f'(\bar{a})f(\bar{a})]^2 \right\}, \]  
(A10)

when we use a new time coordinate \( \bar{\eta} = \bar{\eta}(t) \), with
\[ d\bar{\eta} = [f'(\bar{a})]^{-2} N dt, \quad \bar{\eta}_{\pm} = \bar{\eta}(t_{\pm}), \]  
(A11)
as its defining equations.

As far as the classical theory is concerned, the pair \( (\bar{a}, \bar{\eta}) \) stand on the same footing as \( (a, \eta) \). General covariance requires \( \eta \) and \( \bar{\eta} \) as equally valid coordinates, and there is no reason \textit{a priori} that the spacetime \( (A1) \) should be parametrised by \( a \), rather than \( \bar{a} = a/1+1/a \) or \( \bar{a} = a^2 \), say. Furthermore, since \( I_G[a(\bar{\eta})] \) has the canonical form \( (A1) \) we are free to apply the quantum theory asserted in section IA, and hence derive a new semiclassical action \( J_G[\bar{a}(\bar{\eta})] \). The question is – will this \( J_G \) agree
with the semiclassical action (A7) derived with our original variables? In other words: does the \((\bar{a}, \bar{\eta}) \leftrightarrow (a, \eta)\) equivalence survive the quantum correction?

To answer this question, we shall calculate \(\tilde{J}_G\) explicitly, and see how it differs from \(J_G\). Exactly as before, we compare the classical action (A10) to the standard (1) and see that \(m \to -6\nu_s/\kappa\). The quantum correction (2) therefore transforms the classical action (A10) into the following semiclassical action:

\[
\tilde{J}_G[\bar{a}(\bar{\eta})] = \frac{3\nu_s}{\kappa} \int_{\bar{\eta}_-}^{\bar{\eta}_+} d\eta \left[ -\left( \frac{d\bar{a}}{d\eta} \right)^2 + k \left[ f'(\bar{a}) \right]^2 + \bar{Q}_1 (\partial_a S)^2 + \bar{Q}_2 \partial_a^2 S \right], \tag{A12}
\]

with \(\bar{Q}_1\) and \(\bar{Q}_2\) defined by (A6) but allowing the unknowns to take new values \((\xi, \tilde{d})\) for the sake of generality. To evaluate the last two terms in (A12) we will need to write the discarded information capacity \(S(a, \eta)\) as a function of our new variables \((\bar{a}, \bar{\eta})\). This is achieved by noting that (A3) and (A11) imply

\[
\eta(\bar{\eta}) = \eta_- + \int_{\bar{\eta}_-}^{\bar{\eta}_+} d\eta' \left[ f'(\bar{a}(\bar{\eta})) \right]^2, \tag{A13}
\]

and hence

\[
S(a, \eta) = S \left( \bar{f}(\bar{a}), \eta_- + \int_{\bar{\eta}_-}^{\bar{\eta}_+} d\eta' \left[ f'(\bar{a}(\bar{\eta})) \right]^2 \right). \tag{A14}
\]

In terms of \((\bar{a}, \bar{\eta})\), the information capacity \(S\) has the functional form discussed in footnote 3, so the path integral construction [22] ensures the validity (A12) with the \(\partial_a\) derivatives acting on the first argument of \(S\) only. Thus, for the purposes of calculating (A12) we have

\[
\partial_a S = \bar{f}'(\bar{a}) \partial_a S, \quad \partial_a^2 S = \left[ \bar{f}'(\bar{a}) \right]^2 \partial_a^2 S + \bar{f}''(\bar{a}) \partial_a S. \tag{A15}
\]

Inserting these formulae into equation (A12) we obtain

\[
\tilde{J}_G[\bar{a}(\bar{\eta})] = \frac{3\nu_s}{\kappa} \int_{\bar{\eta}_-}^{\bar{\eta}_+} d\eta \left[ -\left( \frac{d\bar{a}}{d\eta} \right)^2 + k \left[ \bar{f}'(\bar{a}) \right]^2 + \bar{Q}_1 \left[ \bar{f}'(\bar{a}) \right]^2 (\partial_a S)^2 + \bar{Q}_2 \left( \left[ \bar{f}'(\bar{a}) \right]^2 \partial_a^2 S + \bar{f}''(\bar{a}) \partial_a S \right) \right], \tag{A16}
\]

as our new semiclassical action.

We are now in a position to “close the loop” of this calculation, and return to our original dynamical variables \(a(t)\) and \(N(t)\). We first use (A11) to write (A16) as an integral over \(t\),

\[
\tilde{J}_G[\bar{a}(t), N(t)] = \frac{3\nu_s}{\kappa} \int_{t_-}^{t_+} dt \left[ -\frac{\bar{a}^2}{N} \left[ \bar{f}'(\bar{a}) \right]^2 + k N \left[ \bar{f}(\bar{a}) \right]^2 + \bar{Q}_1 N (\partial_a S)^2 + \bar{Q}_2 N \left( \partial_a^2 S + \frac{\bar{f}''(\bar{a})}{\left[ \bar{f}'(\bar{a}) \right]^2} \partial_a S \right) \right], \tag{A17}
\]

and then invert (A8) to express everything as a function of \(a(t)\):

\[
\tilde{J}_G[\bar{f}^{-1}(a(t)), N(t)] = \frac{3\nu_s}{\kappa} \int_{t_-}^{t_+} dt \left[ -\frac{\bar{a}^2}{N} + k N a^2 + \bar{Q}_1 N (\partial_a S)^2 + \bar{Q}_2 N \left( \partial_a^2 S + \frac{\bar{f}''(f^{-1}(a))}{\left[ \bar{f}'(f^{-1}(a)) \right]^2} \partial_a S \right) \right]. \tag{A18}
\]

Comparing this with our original semiclassical action (A7) we see that the \((\bar{a}, \bar{\eta})\) approach has altered our result by

\[
\Delta J_G \equiv \tilde{J}_G - J_G = \frac{3\nu_s}{\kappa} \int_{t_-}^{t_+} dt N \left[ (\bar{Q}_1 - Q_1) (\partial_a S)^2 + (\bar{Q}_2 - Q_2) \partial_a^2 S + \bar{Q}_2 \frac{\bar{f}''(f^{-1}(a))}{\left[ \bar{f}'(f^{-1}(a)) \right]^2} \partial_a S \right]. \tag{A19}
\]

Notice that there are no \(t\)-derivatives in the integrand, so \(\Delta J_G\) contains no surface terms. Hence, \(\tilde{J}_G\) and \(J_G\) will generate identical semiclassical behaviour if and only if \(\Delta J_G = 0\). Assuming that \(\partial_a S\) and \(\partial_a^2 S\) are not identically zero, then the only way to achieve \(\Delta J_G = 0\) for all \(f\) is to set \(\bar{Q}_1 = Q_1\) and \(\bar{Q}_2 = Q_2 = 0\). Consulting (A6) we see that this is equivalent to

\[
\xi = \bar{\xi} = 1/4, \quad d = \tilde{d}. \tag{A20}
\]

We conclude that the quantum correction (22) is consistent with (i) the gauge invariance of \(t\), and (ii) arbitrary redefinitions of the dynamical variable \(a = f(\bar{a})\), if and only if \(d\) is independent of \(f\), and \(\xi = 1/4\).
Appendix B: Holographic Universe

Here we derive the holographic formula (33) that quantifies the information capacity of a comoving volume (12) of the FRW universe (8). We begin with a brief review of the holographic principle.

1. The Holographic Principle

As Bekenstein first realised [51], the maximum entropy (or information) of a system is not set by its *volume*, but by the *area* of an enclosing surface. This understanding arose from the study of black hole thermodynamics [52–57], culminating in the Bekenstein-Hawking formula

\[ S_{BH} = \frac{A}{4 \ell_{pl}^2}, \]  

for the entropy of a black hole, \( A \) being the area of its event horizon. Roughly speaking, \( S_{BH} \) is the maximum entropy that can ever be stored within a region enclosed by a surface of area \( A \). (If this upper bound were ever violated \( S > S_{BH} \), we could always send energy in through the surface until the region became a black hole. This process would lower the entropy \( S \to S_{BH} \), and hence violate the second law of thermodynamics.) This idea was given a precise and general formulation by Bousso [58] as the covariant entropy bound:

\[ S[\mathcal{L}] \leq \frac{A[\mathcal{B}]}{4 \ell_{pl}^2}. \]  

(B2)

Here, \( A[\mathcal{B}] \) is the area of an arbitrary two-dimensional spacelike surface \( \mathcal{B} \), and \( S[\mathcal{L}] \) is the entropy on a *lightsheet* \( \mathcal{L} \) (a hypersurface of null geodesics with nonpositive expansion) that originates orthogonal to \( \mathcal{B} \). Because \( \mathcal{L} \) can be past-directed or future-directed, Bousso’s bound (B2) is symmetric under time-reversal, and cannot be violated \( S > S_{BH} \). In other words, the states of \( \mathcal{L} \) live in a Hilbert space \( \mathcal{H}_L \) of dimension \( \dim[\mathcal{H}_L] \leq 2^{A[\mathcal{B}] / \delta A} \), meaning that \( \mathcal{L} \) has information capacity

\[ S[\mathcal{L}] \equiv \ln(\dim[\mathcal{H}_L]) \leq \frac{A[\mathcal{B}]}{4 \ell_{pl}^2}. \]  

(B3)

Under this premise, the entropy bound (B2) becomes trivial, because the entropy of a system can never exceed its information capacity: \( S \leq S \).

For this article, we will not need to know how the states of \( \mathcal{L} \) are encoded on \( \mathcal{B} \), nor the process by which three-dimensional physics is expected to emerge from a two-dimensional theory [59]. Nonetheless, it is sometimes useful to fix the geometry of \( \mathcal{B} \), and explore range of \( \mathcal{L} \)-states that can be encoded. For instance, let us consider the case where \( \mathcal{B} \) has the geometry of a sphere. Within a semiclassical approximation, each state encoded on \( \mathcal{B} \) should determine the geometry and matter content of a lightsheet \( \mathcal{L} \) that extends into the interior of \( \mathcal{B} \). Now, some of these states will correspond to the interior of a Schwarzschild black hole with event horizon at \( \mathcal{B} \); indeed, the Bekenstein-Hawking entropy (B1) must count all such states. Comparing this entropy to (B3), and recalling that \( S \leq S \), we conclude that the information capacity bound is saturated,

\[ S[\mathcal{L}] = \frac{A[\mathcal{B}]}{4 \ell_{pl}^2}, \]  

(B4)

whenever \( \mathcal{B} \) is spherical.4 This is the key holographic result that will allow us to quantify the information capacity of a homogenous, isotropic, expanding universe.

2. Holograms for Cosmology

To apply equation (B4) to cosmology, we require a family of (spherical) surfaces \( \mathcal{B} \), whose lightsheets \( \mathcal{L} \) cover the entire FRW spacetime (8). It is natural to insist that the “holograms” \( (\mathcal{B}, \mathcal{L}) \) respect the symmetries of the metric; hence, each surface \( \mathcal{B} \) should indeed be spherical, and must lie on some hypersurface of simultaneity \( t = \text{const} \). To complete our universal covering, we need to specify (i) the size of each \( \mathcal{B} \), (ii) whether the \( \mathcal{L} \) are directed into the past or future, and (iii) how the holograms \( (\mathcal{B}, \mathcal{L}) \) are arranged in spacetime.

Let us start by imagining we have selected a hologram \( (\mathcal{B}, \mathcal{L}) \) as a candidate for our universal covering. Now suppose we can construct a larger hologram \( (\mathcal{B}', \mathcal{L}') \) that completely engulfs our candidate: \( \mathcal{L}' \supset \mathcal{L} \). In principle, equation (B4) should apply to both holograms. However,

4 Strictly speaking, \( S[\mathcal{L}] \) must be slightly larger than \( S_{BH} \), because \( S_{BH} \) only measures the subspace of \( \mathcal{H}_L \) spanned by states that correspond to the interior of a Schwarzschild black hole with event horizon at \( \mathcal{B} \). Indeed, we should have \( S[\mathcal{L}] = S_{BH} + I_{BH} \), where \( I_{BH} > 0 \) is the amount of information conveyed by the event that \( \mathcal{B} \) is the event horizon of a Schwarzschild black hole. This information is simply the macrostate of \( \mathcal{L} \), including its total mass \( M = 2 \pi A[\mathcal{B}] / \kappa \) and angular momentum \( J = 0 \). However, (B1) and (B4) suggest that \( S[\mathcal{L}] = S_{BH} \), i.e. that \( I_{BH} \) is negligible within the semiclassical approximation, \( A[\mathcal{B}] \gg \ell_{pl}^2 \). This comes about because the smallest quantum of energy that can be confined to \( \mathcal{B} \) is a massless particle of wavelength \( \lambda \sim O(\sqrt{A[\mathcal{B}]})^{-1/2} \). Hence \( \mathcal{H}_L \) must have a discrete energy spectrum with minimum spacing \( \delta M \sim O(\hbar(\sqrt{A[\mathcal{B}]}))^{-1/2} \). The macrostate information will then be \( I_{BH} \sim O(\ln(M/\delta M)) \sim O(\ln(A[\mathcal{B}]/\ell_{pl}^2)) \ll S[\mathcal{L}] \), as claimed.
(B', L') is clearly a more fundamental description, as it contains (B, L) as a subsystem. We should therefore discard the candidate (B, L) and use the larger hologram (B', L') instead. By this logic, our universal covering must be composed of holograms that are maximal, i.e. those for which no such superset holograms exist.

As illustrated in figure 2, a superset hologram (B', L') can be constructed from a (sufficiently small) candidate (B, L) by extending the lightsheet L backwards through B. If at some point this process fails, then (B, L) will be maximal, and suitable for our universal covering. Indeed, there are two fundamental constraints that can cause backwards extension to fail:

1. **The Geometric Constraint**: By definition, L is composed of null geodesics with nonpositive expansion. This stipulation is a local representation of the notion that L should point “inwards” from B, a key property that allowed Bouso to formulate his entropy bound (B2) in the first place [58]. Backwards extension will therefore fail if we ever have \( A[B'] \leq A[B] \): the null rays from B' to B must then have positive expansion, so L' will fail to be a valid lightsheet.

2. **The Causal Constraint**: We require each hologram (B, L) to lie inside the past lightcone of some hypothetical observer. This constraint is imposed by black hole complementarity [43, 44], which prevents us from applying the laws of quantum mechanics to systems that can never be observed in their entirety.\(^5\) While it is conceivable that the entropy bound (B2) remains valid for lightsheets that break this constraint, these L cannot be be treated as quantum systems. Without a Hilbert space \( \mathcal{H}_L \) with known information capacity (B4) we cannot apply the quantum theory of section 1A.

In a universe such as ours, which is expanding \( \dot{a} > 0 \) and has low spatial curvature, holograms (B, L) with past-directed lightsheets L will always satisfy the geometric constraint. However, the causal constraint will halt backwards extension as soon as B coincides with the cosmological event horizon. In other words, a maximal past-directed hologram, centred at \( \chi = 0 \), will have its boundary at

\[
B_\chi : \quad \chi = \bar{\eta} - \eta . \tag{B5}
\]

\(^5\) Without complementarity, the unitary formation and evaporation of a black hole [60–63] would violate the no-cloning theorem [64]. Even if a firewall forms at the scrambling time [65], we still need complementarity to prevent cloning before then [66, 67]. A stricter interpretation of complementary would require (B, L) to lie inside a causal diamond, i.e. the intersection of some past lightcone and some future lightcone [68, 69]. We adopt the more tolerant version for now; in any case, this distinction would only be important in the very early universe (i.e. during inflation) when the particle horizon is closer than the event horizon.

\[
\lim_{\eta \to \bar{\eta}} a(\eta) = \infty \tag{B6}
\]

defines the final conformal time \( \bar{\eta} \). (We check that \( \bar{\eta} \) exists in section V C.) Even if spatial curvature is large, the only way (B5) will break down is if the universe is closed and the event horizon lies beyond the equator: \( \bar{\eta} - \eta > \pi/2 \). Then the geometric constraint can halt backwards extension before the event horizon is reached. However, \( \bar{\eta} - \eta > \pi/2 \) can only occur at very early times (during inflation) so we can ignore this special case for now. (We will revisit this issue in a separate publication, when we investigate \( \Delta V_{\text{eff}} \) in the very early universe.) Of course, maximal holograms need not be centred on \( \chi = 0 \); but if we place one hologram (B_\chi, L_\chi) there, then a neighbouring maximal hologram (B_{\chi+\delta\eta}, L_{\chi+\delta\eta}) will have to also be centred at \( \chi = 0 \) if the two are to be disjoint. In this fashion, maximal past-directed holograms naturally stack to form a spherically symmetric causal diamond, as depicted on the left of figure 3. We will build our universal covering from these holographic units in the next section.

Before then, we should also consider future-directed holograms. In contrast to the previous case, the causal constraint is unable to halt backwards extension, because if (B, L) fits inside the event horizon, then (B', L') will fit inside also. Instead, extension halts once B coincides with the apparent horizon,

\[
r_k(\chi_{\text{AH}}) = \left( \frac{1}{a^2} \left( \frac{da}{d\eta} \right)^2 + k \right)^{-1/2}, \tag{B7}
\]

defined by virtue of the geometric constraint. These holograms are unsuitable for our universal covering, for two distinct
reasons. Firstly, the area of the apparent horizon (B7) clearly depends on \(da/d\eta\), so we would arrive at an information capacity \(S = S(a, da/d\eta)\) that is incompatible with the formula (2) for the quantum correction. Secondly, the apparent horizon (B7) is determined by the behaviour of the scale factor, so any pattern of future-directed maximal holograms, intended to cover the universe with minimal overlap, will only succeed for a specific expansion history \(a(\eta)\). This poses a serious problem for our approach, because \(S\) must be robust to arbitrary variations \(\delta a(\eta)\) in order to be included in the semiclassical action \(\mathcal{J}[a(\eta)]\).

For the sake of practicality and generality, then, we must build our covering using the past-directed holographic units described in the previous paragraph.

3. Holographic Covering

If the classical action (21) were an integral over a single causal diamond, then the holographic unit (on the left of fig. 3) would provide all the structure we need. However, to make contact with the quantum theory of section I A, it was necessary to integrate over a region (12) of fixed comoving volume, with a view to sending \(\chi_s \to \infty\) at the end of our calculation. In order to count all the degrees of freedom in the action, we therefore need a systematic way to cover the entire FRW spacetime (8) with holographic units, such that there is minimal double counting from overlapping holograms. In 1+1 dimensions, this problem has a particularly elegant solution, shown on the right of figure 3. It is possible to generalise this self-similar pattern to 3+1 dimensions, accounting for the small gaps or overlaps that arise. For the sake of brevity, however, we leave these details for another publication. The 1+1 dimensional picture will suffice to understand the calculation below.

With a holographic covering at hand, we aim to calculate the information capacity of some spatial slice \(\eta = \text{const}\), within the integration region \(\chi \in [0, \chi_s]\). We think of the bulk spacetime as composed of holograms \((B_\eta, L_\eta)\), with the state of each lightsheet \(L_\eta\) specified by information on the boundary \(B_\eta\). Hence, the information capacity on \(\eta = \text{const}\), is simply the information

---

\[ \eta = \tilde{\eta} \]

"Holographic Unit"

Event Horizon

\[ \chi = \tilde{\eta} - \eta \]

\[ \eta = \text{const.} \]

Light Sheet

\[ \eta = \tilde{\eta} - \Delta \eta \]

\[ \chi = 0 \]

Holographic Tiling of 1+1 Dimensional Universe

FIG. 3. Holographic units are spherically symmetric causal diamonds, bounded into the future by a cosmological event horizon, and foliated by the past-directed lightsheets of the event horizon at each conformal time \(\eta\). On the right, these units are arranged into a self-similar pattern that perfectly tiles an expanding universe with one spatial dimension and final conformal time \(\tilde{\eta}\). Each holographic unit begins at \(\eta = \tilde{\eta} - 2^n \Delta \eta\) for some \(n \in \mathbb{Z}\); all reference to the arbitrary scale \(\Delta \eta\) can be removed by a natural averaging procedure described in the main text. I will extend this pattern to our 3+1 dimensional universe in a future publication, accounting for the gaps or overlaps that presumably arise. Note that on each spatial slice \(\eta = \text{const}\), the event horizon is a sphere \(B_\eta\) of area \(A[B_\eta] = \mathcal{A}(\tilde{\eta} - \eta)[a(\eta)]^2\) that encloses a volume \(V_\eta \equiv \mathcal{V}(\tilde{\eta} - \eta)[a(\eta)]^2\); each \(B_\eta\) generates a past-directed lightsheet \(L_\eta\) with information capacity set by the holographic formula (B4). Even though the pattern covers the entire 1+1 dimensional spacetime without gaps or overlap, the volumes \(V_\eta\) (shown in cyan) do not fill the entire spatial slice: some parts of the slice (magenta) are occupied by the lower half of a holographic unit (orange shaded triangle) the information capacity of which will be counted on a future slice. Hence the number of spheres \(B_\eta\) in a large volume \(V_\eta\) is \(\mathcal{N} = \mu V_\eta / V_\eta\), for some \(\mu \lesssim 1\).
capacity \( B_\eta \) of each sphere \( B_\eta \), multiplied by the number of these spheres \( N_\eta \) within \( \chi \in [0, \chi_*] \):

\[
S = N_\eta \cdot \frac{A[B_\eta]}{4 \ell^2_{pl}}. \tag{B8}
\]

If the spheres could be packed perfectly, without gap or overlap, then one might expect

\[
N_\eta = \frac{V_*}{V_\eta}, \tag{B9}
\]

where \( V_* = V(\chi_*^*)[a(\eta)]^3 \) is the volume of the integration region \( \chi \in [0, \chi_*] \), and \( V_\eta = V(\eta-\eta)[a(\eta)]^3 \) is the volume enclosed by each \( B_\eta \). However, figure 3 shows us that this is not the case. Even for the 1+1 dimensional tiling, which does indeed cover the universe without gaps or overlap, the \( B_\eta \) do not fill each spatial slice. In general, only a fraction \( \mu \lesssim 1 \) of the volume is taken up by the \( B_\eta \); the rest is occupied by the lower half of other (smaller) holographic units, foliated by holograms with their boundaries on future slices.

Consulting figure 3, it appears that \( \mu \) will oscillate — decreasing from \( \mu = 1 \), to \( \mu = 1/2 \), as the spatial slice ascends through each cycle \( \eta \in [\tilde{\eta} - 2^{n-1} \Delta \eta, \tilde{\eta} - 2^n \Delta \eta] \). However, the phase of this oscillation clearly depends on the arbitrary scale \( \Delta \eta \):

\[
\mu = \mu \left( \frac{\bar{\eta} - \eta}{\Delta \eta} \right). \tag{B10}
\]

Fortunately, there is a natural way to remove this spurious feature: a unique average over \( \Delta \eta \) that recovers the symmetry of the underlying spacetime. As we will soon show, this provides a physically well-defined constant value

\[
\bar{\mu} \equiv \frac{1}{\ln 2} \int_1^{2x} \mu \left( \frac{\bar{\eta} - \eta}{\Delta \eta} \right) d(\Delta \eta) \tag{B11}
\]

that correctly counts the spheres \( B_\eta \) in \( \chi \in [0, \chi_*] \) without reference to \( \Delta \eta \):

\[
N_\eta = \bar{\mu} V_* \frac{\bar{\eta} - \eta}{\eta - \bar{\eta}}. \tag{B12}
\]

Inserting this well-defined counting into equation (B8) we finally obtain the information capacity

\[
S = \bar{\mu} V_* \frac{\bar{\eta} - \eta}{\eta - \bar{\eta}} \frac{\mathcal{A}(\bar{\eta} - \eta)[a(\eta)]^2}{4 \ell^2_{pl}}, \tag{B13}
\]

as used in the section III.B. (We will evaluate \( \bar{\mu} \) in a future publication, using the holographic covering suitable for 3+1 dimensions.)

(All that remains is to justify the averaging procedure (B11) and show that it does not depend on the choice of \( x > 0 \). To this end, let us consider an arbitrary function \( f \) that (like \( \mu \)) depends only on the phase of a self-similar holographic covering at conformal time \( \eta \). As such, \( f \) will have the following structure:

\[
f = f \left( \frac{\bar{\eta} - \eta}{\Delta \eta} \right), \quad f(mx) = f(x), \quad \forall \, x > 0, \tag{B14}
\]

where \( m \in \mathbb{N} \) is the scaling-factor under which the pattern is self-similar. (The pattern in figure 3 has \( m = 2 \).) For a function with these properties, any arithmetic mean over \( \Delta \eta \) can be represented as an integral over a single scaling cycle:

\[
\langle f \rangle_{\Delta \eta} \equiv \int_x^{mx} f \left( \frac{\bar{\eta} - \eta}{\Delta \eta} \right) g(\Delta \eta) d(\Delta \eta), \tag{B15}
\]

with some measure \( g(\Delta \eta) \) normalised by

\[
\int_x^{mx} g(\Delta \eta) d(\Delta \eta) = 1. \tag{B16}
\]

We will seek a \( g(\Delta \eta) \) that allows \( \langle f \rangle_{\Delta \eta} \) to respect the symmetry of the underlying spacetime, for every \( f \) with the appropriate structure (B14).

Let us assume for the moment that \( k = 0 \), so that the underlying spacetime has the metric

\[
ds^2 = [a(\eta)]^2 \left( -d\eta^2 + dx^2 + \chi^2 d\Omega^2 \right). \tag{B17}
\]

Note that this spacetime is invariant under the following conformal transformation:

\[
ds^2 \rightarrow \left( \frac{a(\alpha \eta) + (1 - \alpha) \bar{\eta}}{a(\eta)} \right)^2 \alpha^2 ds^2, \tag{B18}
\]

for any constant \( \alpha > 0 \); indeed, the above transformation is equivalent to a coordinate rescaling,

\[
\eta \rightarrow \alpha \eta + (1 - \alpha) \bar{\eta}, \quad \chi \rightarrow \alpha \chi, \tag{B19}
\]

that leaves \( \bar{\eta} \) invariant. We notice, however, that the holographic covering will break this symmetry almost entirely: all that survives are transformations with \( \alpha \in \{m^n : n \in \mathbb{Z} \} \). As a case in point, consider \( f \). Because this is purely a function of the phase of the holographic covering, it will not depend on the scale factor, and so is invariant under the Weyl transformation (B18). If this function were to respect the full symmetry of the underlying spacetime, it would therefore also need to be invariant under the coordinate rescaling (B19). However, its properties (B14) only guarantee invariance for \( \alpha = m^n, \quad n \in \mathbb{Z} \).

Now, by construction, the average (B15) is also independent of \( a(\eta) \), and hence invariant under the Weyl transformation (B18). Thus, \( \langle f \rangle_{\Delta \eta} \) will recover the full symmetry of the underlying spacetime (B17) if and only if it is invariant under the coordinate rescaling (B19) for all \( \alpha > 0 \). In other words, \( \langle f \rangle_{\Delta \eta} \) cannot depend on \( \eta \) at all. Thus we seek a measure \( g(\Delta \eta) \) that ensures

\[
\langle f \rangle_{\Delta \eta} = \text{const}, \tag{B20}
\]
for all $f$ with the aforementioned properties (B14). But note that
\[ \partial_\eta \langle f \rangle_{\Delta_\eta} = \int_x^{m_\|} \partial_\eta \left( \frac{\bar{\eta} - \eta}{\Delta_\eta} \right) g(\Delta_\eta) d(\Delta_\eta) \] (B21)
\[ = \int_x^{m_\|} \left( \frac{\partial_\eta}{\partial_\eta} \right) f \frac{\bar{\eta} - \eta}{\Delta_\eta} g(\Delta_\eta) d(\Delta_\eta) \]
\[ = \frac{1}{(\bar{\eta} - \eta)} \left\{ \int_x^{m_\|} f \frac{\bar{\eta} - \eta}{\Delta_\eta} g(\Delta_\eta) d(\Delta_\eta) \right\} \]
\[ - \int_x^{m_\|} f \frac{\bar{\eta} - \eta}{\Delta_\eta} \partial_\eta \left( g(\Delta_\eta) \Delta_\eta \right) d(\Delta_\eta) \} \].

Hence the symmetry condition (B20) requires this last line to vanish for every $f$ obeying (B14). This will happen if and only if
\[ \partial_\eta \left( g(\Delta_\eta) \Delta_\eta \right) = 0, \quad \forall \Delta_\eta \in [x, m_x], \] (B22)
and recalling the normalisation (B16) we see that
\[ g(\Delta_\eta) = \frac{1}{\ln m} \cdot \frac{1}{\Delta_\eta}, \quad \forall \Delta_\eta \in [x, m_x], \] (B23)
is the only solution. Thus the unique mean (B15) that recovers the symmetry of the underlying spacetime is
\[ \langle f \rangle_{\Delta_\eta} \equiv \frac{1}{\ln m} \int_x^{m_\|} f \left( \frac{\bar{\eta} - \eta}{\Delta_\eta} \right) d(\Delta_\eta), \] (B24)
as used in equation (B11). Furthermore, it is easy to check that this construction does not depend on our choice of $x$:
\[ \partial_x \langle f \rangle_{\Delta_\eta} = \frac{1}{\ln m} \left[ m \cdot \frac{1}{m_x} f \left( \frac{\bar{\eta} - \eta}{m_x} \right) - \frac{1}{x} f \left( \frac{\bar{\eta} - \eta}{x} \right) \right] \]
\[ = 0, \] (B25)
by virtue of the second property (B14).

For $k = \pm 1$, the holographic covering will not be exactly self-similar (spatial curvature introduces a special comoving scale $\chi = 1$) and the Weyl transformation (B18) will not be an exact symmetry. Nonetheless, when the event horizon is much smaller than the radius of spatial curvature $|k|/|\bar{\eta} - \eta| \ll 1$, the $k = 0$ case will be an excellent approximation, and we can safely use the average (B24) to define $\bar{\mu}$. This approximation can only break down in the very early universe.

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