On codimension two flats in Fermat-type arrangements

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Abstract

In the present note we study certain arrangements of codimension 2 flats in projective spaces, we call them Fermat arrangements. We describe algebraic properties of their defining ideals. In particular, we show that they provide counterexamples to an expected containment relation between ordinary and symbolic powers of homogeneous ideals.

1 Introduction

A Fermat-type arrangement of degree \( n \geq 1 \) of hyperplanes in projective space \( \mathbb{P}^N \) is given by linear factors of the polynomial

\[
F_{N,n}(x_0, \ldots, x_N) = \prod_{0 \leq i < j \leq N} (x_i^n - x_j^n).
\]

These arrangements sometimes appear under the name of Ceva arrangements in the literature, see e.g. [1, Section 2.3.I]. The name Fermat arrangement has been used for lines in \( \mathbb{P}^2 \) e.g. by Urzua, see [13, Example II.6]. Fermat arrangements of lines have attracted recently considerable attention, see e.g. [11], because of their appearance on the border line of the following fundamental problem.

Problem 1.1 (Containment problem). Let \( I \) be a homogeneous ideal in the polynomial ring \( \mathbb{C}[x_0, \ldots, x_N] \). Determine all pairs of integers \( m \) and \( r \) such that the containment

\[
I^{(m)} \subset I^r
\]

between the symbolic and ordinary powers of the ideal \( I \) holds.

We recall that for \( m \geq 0 \) the \( m \)-th symbolic power of \( I \) is defined as

\[
I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R),
\]

where \( \text{Ass}(I) \) is the set of associated primes of \( I \). A ground breaking result of Ein, Lazarsfeld and Smith [4] (rendered by Hochster and Huneke in positive characteristic [7]) asserts that there is always the containment in (1) for \( m \) and \( r \) subject to the inequality

\[
m \geq hr,
\]

where \( h \) is the maximum of heights of all associated primes of \( I \). The natural question: To which extend the bound in (3) is sharp has fueled a lot of research in the last 15 years. Considerable attention has been given to the following question of Huneke.

Question 1.2 (Huneke). Let \( Z \) be a finite set of points in \( \mathbb{P}^2 \) and let \( I \) be the homogeneous ideal defining \( Z \). Is then

\[
I^{(3)} \subset I^2?
\]
Note that the containment $I^{(4)} \subset I^2$ follows in this situation directly from (3). On the other hand it is very easy to find sets of points in $\mathbb{P}^2$ for which the containment $I^{(2)} \subset I^2$ fails. Question 1.2 remained open for quite a long time. It is by now known that there are sets of points in $\mathbb{P}^2$ for which the containment in (4) fails. The first counterexample has been given by Dumnicki, Szemberg and Tutaj-Gasińska in [3]. This counterexample is provided by the set of all 12 intersection points of the Fermat arrangement of 9 lines in $\mathbb{P}^2$ (i.e. $n = 3$ in this arrangement). The paper [3] suggested that arrangements with arbitrary $n \geq 3$ should provide further counterexamples. This has been worked out and verified to hold by Harbourne and Seceleanu, see [6, Proposition 2.1]. Whereas Fermat configurations of lines allow no deformations, a series of counterexamples allowing parameters has been presented recently in [8]. Apart of those series there are some sporadic counterexamples to Question 1.2 available. The nature of all these examples has not been yet fully understood.

The statement of (1) does not restrict to ideals supported on points. In particular, Huneke’s question can be reformulated for codimension 2 subvarieties in the projective space of arbitrary dimension $N$.

**Question 1.3** (Huneke-type). Let $V$ be codimension 2 subvariety in $\mathbb{P}^N$ and let $I$ be the homogeneous ideal defining $V$. Is then $I^{(3)} \subset I^2$? (5)

This question has been answered to the negative in [10]. More precisely, we showed [10, Theorem 4.3] that the containment (5) fails for $V$ consisting of all lines in $\mathbb{P}^3$ contained in at least 3 hyperplanes among those defined by linear factors of $F_{3,n}$ for all $n \geq 3$. In the present note, we investigate the ideals defining codimension 2 linear flats of multiplicity at least 3 cut out by the Fermat-type arrangement given by $F_{N,n}$ with $n \geq 3$, as well as ordinary and symbolic powers of these ideals.

## 2 Notation and basic properties

The bookkeeping of all data is quite essential for what follows. In the present section we establish the notation and prove some basics facts.

By $x_0, x_1, \ldots, x_N$ we denote the coordinates in the projective space $\mathbb{P}^N$. We fix an integer $n \geq 3$. This integer is not present in the short hand notation introduced below. We hope that it will not lead to any confusion since we work always with $n$ fixed. We introduce the following bracket symbol. For an integer $1 \leq k \leq N$ let $i_0, \ldots, i_k$ be $(k+1)$ mutually distinct elements in the set $\{0, \ldots, N\}$.

\[
[x_{i_1} \ldots x_{i_k}] := \prod_{p<q}(x_{i_p} - x_{i_q}).
\] (6)

Thus, in particular,

\[F_{N,n} = [x_0 \ldots x_N].\]

The notation in (6) fulfills the antisymmetry condition. More precisely, we have for any pair $p, q$ such that $1 \leq p < q \leq k$:

\[
[x_{i_1} \ldots x_{i_p} \ldots x_{i_q} \ldots x_{i_k}] = (-1)^{q-p}[x_{i_1} \ldots x_{i_q} \ldots x_{i_p} \ldots x_{i_k}].
\] (7)

**Lemma 2.1** (Expansion rule). For $k \geq 2$ there is

\[
[x_{i_0} \ldots x_{i_{k-1}} x_{i_k}] = [x_{i_0} \ldots x_{i_{k-1}}] \prod_{j=0}^{k-1}[x_{i_j} x_{i_k}].
\]

**Proof.** This follows straightforward from the definition in (6). \qed

For example

\[xyzw = [xyz](x^n - y^n)(y^n - w^n)(z^n - w^n).\]

We have also the following Laplace-type rule.
Lemma 2.2 (Laplace expansion). We have
\[ [x_{i_0} \ldots x_{i_k}] = \sum_{j=0}^{k} (-1)^{j+k} x_{i_0}^{n_{j}} \ldots \hat{x}_{i_j}^{n_{j}} \ldots x_{i_k}. \]
As usual \( \hat{a} \) means that the term \( a \) is omitted.

Proof. In order to alleviate notation, we drop the double index notation. It’s sufficient to show
\[ [x_0 \ldots x_k] = \sum_{j=0}^{k} (-1)^{j+k} x_{0}^{n_{j}} \ldots \hat{x}_{j}^{n_{j}} \ldots x_{k}. \]  
(8)
Both sides are polynomials of degree \( \frac{k(k+1)}{2} n \). It is enough to show that the right hand side vanishes along all hyperplanes of the form \( x_i = \delta x_j \), where \( \delta \) is some root of 1 of degree \( n \). By symmetry it is enough to check this for \( x_0 = \delta x_1 \). Then the right hand side in (8) is
\[ (-1)^{k} x_{1}^{n_{1}} \ldots \hat{x}_{1}^{n_{1}} [x_1 \ldots x_k] + (-1)^{k+1} x_{0}^{n_{2}} \ldots \hat{x}_{k}^{n_{k}} [x_0 x_2 \ldots x_k], \]
since \( x_0 = \delta x_1 \) we get 0.
Thus the right hand side of (8) is equal to \( \lambda \cdot [x_0 \ldots x_k] \) for some \( \lambda \in \mathbb{C} \). In order to establish \( \lambda \), we evaluate at \( x_0 = 0 \), which gives
\[ \lambda(-x_{1}^{n_{1}}) \ldots (-x_{k}^{n_{k}})[x_1 \ldots x_k] = (-1)^{k} x_{1}^{n_{1}} \ldots \hat{x}_{k}^{n_{k}} [x_1 \ldots x_k], \]
hence \( \lambda = 1 \).

Also the next fact is very useful.

Lemma 2.3 (Substitution rule). For any \( u \in \{0, \ldots, N\} \) and \( 1 \leq k \leq N \) there is
\[ [x_{i_0} \ldots x_{i_k}] = \sum_{j=0}^{k} [x_{i_0} \ldots \hat{x}_{i_j} x_{i_{j+1}} \ldots x_{i_k}]. \]

For example
\[ [xyz] = [wyz] + [xwz] + [xyz]. \]

Proof. In order to alleviate notation we drop the double index notation. It is clear that the statement is invariant under the symmetry group on the \( (N+1) \) variables. Also it convenient to use (7) and write the assertion in the following form
\[ [x_0 \ldots x_k] = \sum_{j=0}^{k} (-1)^{j} [x_{u} x_0 \ldots \hat{x}_{j} x_{j+1} \ldots x_k]. \]  
(9)
The argumentation is similar to that in proof of Lemma 2.2. Both sides of (9) are polynomials of degree \( \frac{k(k+1)}{2} n \). We substitute \( x_0 = \delta x_1 \). Then the right hand side is
\[ [x_{u} x_1 \ldots x_k] - [x_{u} x_0 x_2 \ldots x_k] \]
which is clearly 0. Thus we have
\[ \sum_{j=0}^{k} [x_{i_0} \ldots \hat{x}_{i_j} x_{i_{j+1}} \ldots x_{i_k}] = \lambda \cdot [x_0 \ldots x_k], \]
for some \( \lambda \in \mathbb{C} \). In order to determine \( \lambda \), we substitute \( x_u = x_0 \). Then
\[ [x_0 \ldots x_k] = \lambda \cdot [x_0 \ldots x_k], \]
which implies \( \lambda = 1 \).
\[ \square \]
We conclude these preparations by another useful rule.

**Lemma 2.4** (Useful rule). For \( k \geq 2 \) and auxiliary variables \( y_1, \ldots, y_k \) we have

\[
[x_0 \ldots x_k] = \sum_{j=0}^{k} (-1)^j [x_0 \ldots \hat{x}_j \ldots x_k][x_j y_1] \ldots [x_j y_k].
\]

**Proof.** The proof parrots that of Lemma 2.2 and Lemma 2.3 and is left to the reader. In order to determine the constant \( \lambda \) one might substitute \( y_i = x_i \) for \( i = 1, \ldots, k \).

\( \square \)

3 Fermat arrangements of codimension two flats

In this section we study, for \( n \geq 3 \), the union \( V_{N,n} \) of codimension 2 flats \( W \) in \( \mathbb{P}^N \) such that there are at least 3 hyperplanes among those defined by the linear factors of \( F_{N,n} \) vanishing along \( W \). Let \( I_{N,n} \) be the radical ideal defining \( V_{N,n} \). The set \( V_{N,n} \) is the union of \( N+1 \) cones with vertices in the coordinate points \( E_i = (0 : \ldots : 1 : \ldots : 0) \) over the sets \( V_{N-1,n}(i) \) defined in the hyperplanes \( H_i = \{ x_i = 0 \} \). Let \( I_{N-1,n}(i) \) be the ideal defining \( V_{N-1,n}(i) \) in the variables \( x_0, \ldots, \hat{x}_i, \ldots, x_N \). The geometry of the arrangement implies the following relation between the defined ideals.

**Lemma 3.1.** Keeping the notation above, we have for all \( N \geq 3 \)

\[
I_{N,n} = \bigcap_{i=0}^{N} I_{N-1,n}(i).
\]

For the proof of the main Theorem 4.1 we need a more direct description of ideals \( I_{N,n} \) in terms of generators.

**Proposition 3.2.** Consider the ideal \( I_{N,n} \) for some integers \( N \geq 2 \) and \( n \geq 3 \).

a) Let \( N = 2M \) be an even number. Let \( A = \{i_1, \ldots, i_M\} \) be a subset of \( M \) elements in the set \( \{0, 1, \ldots, N\} \) and let \( B = \{j_0, \ldots, j_M\} \) be the complimentary set. The ideal \( I_{N,n} \) is generated by all polynomials of the form

\[
g_A = x_{i_1} \ldots x_{i_M} [x_{i_1} \ldots x_{i_M}] [x_{j_0} \ldots x_{j_M}].
\]

b) Let \( N = 2M + 1 \) be an odd number. Let \( A = \{i_0, \ldots, i_M\} \) be a subset of \( M + 1 \) elements in the set \( \{0, 1, \ldots, N\} \) and let \( B = \{j_0, \ldots, j_M\} \) be the complimentary set. The ideal \( I_{N,n} \) is generated by all polynomials of the form

\[
g_A = x_{i_0} \ldots x_{i_M} [x_{i_0} \ldots x_{i_M}] [x_{j_0} \ldots x_{j_M}].
\]

**Proof.** The proof goes by induction on \( N \). The first step, \( N = 2 \) has been shown in [3, Lemma 2.1]. Using the presentation in Lemma 3.1 we will show that generators \( g_A \) are contained in each of the intersecting ideals. To this end we study first the case \( N \) is even with \( N = 2M \).

Since everything is invariant under the permutation group, it suffices to work with the set \( A = \{0, 1, \ldots, M-1\} \). Then

\[
g_A = x_0 x_1 \ldots x_{M-1} [x_0 \ldots x_{M-1}] [x_M \ldots x_{2M}].
\]

We have the following two cases. Assume that \( M \leq i \leq 2M \). Then the ideal \( I_{N-1,n}(i) \) contains as a generator

\[
h_A = x_0 \ldots x_{M-1} [x_0 \ldots x_{M-1}] [x_M \ldots \hat{x}_i \ldots x_{2M}].
\]

It is easy to see that \( g_A \) is divisible by \( h_A \), indeed

\[
g_A = \pm \prod_{j=M}^{2M} [x_j x_i] h_A.
\]
Let $I \leq i \leq M-1$. After renumbering the variables we can in fact assume that $i = 0$. Let $A_j = \{1, \ldots, M-1, M+j\}$ for $j = 0, \ldots, M$ and let $h_{A_j}$ be the corresponding generators of $I_{N-1,n}(i)$, i.e.

$$h_{A_j} = x_1 \ldots x_{M-1}x_{M+j}[x_1 \ldots x_{M-1}x_{M+j}][x_{M} \ldots \hat{x}_{M+j} \ldots x_{2M}].$$

Then

$$g_A = \sum_{j=0}^{M} (-1)^{j+M-1} x_0x_{M+j}^{n}[x_0x_1] \ldots [x_0x_{M-1}]x_1 \ldots x_{M-1}x_{M+j}[x_1 \ldots x_{M-1}x_{M+j}][x_{M} \ldots \hat{x}_{M+j} \ldots x_{2M}].$$

To see this we will alter the right hand side of the above equality. First note that by Lemma 2.1 we have

$$\sum_{j=0}^{M} (-1)^{j+M-1} x_0x_{M+j}^{n}[x_0x_1] \ldots [x_0x_{M-1}]x_1 \ldots x_{M-1}x_{M+j}x_1 \ldots x_{M-1}x_{M+j}[x_{M} \ldots \hat{x}_{M+j} \ldots x_{2M}].$$

Again by the Expansion rule it reduces to:

$$x_0 \ldots x_{M-1} \sum_{j=0}^{M} (-1)^{j+M-1} x_0x_{M+j}^{n}[x_0x_1] \ldots [x_0x_{M-1}]x_1 \ldots x_{M-1}[x_{M+j}x_1] \ldots [x_{M-1}x_{M+j}][x_{M} \ldots \hat{x}_{M+j} \ldots x_{2M}].$$

By Lemma 2.1 with $y_1 = x_1$, $y_{M-1} = x_{M-1}$, $y_M = 0$ this expression reduces to

$$x_0 \ldots x_{M-1}x_0 \ldots x_{M-1}[x_{M} \ldots x_{2M}] = g_A.$$

Now we pass to the case $N$ is odd with $N = 2M + 1$. Let $A = \{0, \ldots, M\}$ and

$$g_A = x_0x_1 \ldots x_M[x_0 \ldots x_M][x_{M+1} \ldots x_{2M+1}].$$

There are again two subcases. Assume that $0 \leq i \leq M$. Then the ideal $I_{N-1,n}(i)$ contains the generator

$$g_{A'} = x_0x_1 \ldots \hat{x}_i \ldots x_M[x_0 \ldots \hat{x}_i \ldots x_M][x_{M+1} \ldots x_{2M+1}].$$

with $A' = A \backslash \{i\}$. Then $g_A$ is divisible by $g_{A'}$, indeed

$$g_A = \pm x_1[x_0 \ldots \hat{x}_i \ldots x_M][x_{M+1} \ldots x_{2M+1}] g_{A'}.$$

For $M+1 \leq i \leq 2M+1$ it suffices, up to renumbering the variables to consider $i = 2M+1$. In the ideal $I_{N-1,n}(i)$ there are generators

$$g_{A_j} = x_0x_1 \ldots \hat{x}_j \ldots x_M[x_0 \ldots \hat{x}_j \ldots x_M][x_j x_{M+1} \ldots x_{2M}].$$

for $A_j = \{0, 1, \ldots, j, \ldots, M\}$. Then

$$g_A = \sum_{j=0}^{M} (-1)^j x_j [x_{M+1} x_{2M+1}] \ldots [x_{2M} x_{2M+1}] \cdot g_{A_j}.$$

Again, we reduce the right hand side of this equality. To begin with we have

$$\sum_{j=0}^{M} (-1)^j x_j [x_{M+1} x_{2M+1}] \ldots [x_{2M} x_{2M+1}] \cdot x_0x_1 \ldots \hat{x}_j \ldots x_M[x_0 \ldots \hat{x}_j \ldots x_M][x_j x_{M+1} \ldots x_{2M}].$$
\[ x_0 \ldots x_M \sum_{j=0}^{M} (-1)^j [x_0 \ldots \hat{x}_j \ldots x_M][x_jx_{M+1}] \ldots [x_jx_{2M}][x_{M+1} \ldots x_{2M}][x_{M+1}x_{2M+1}] \ldots [x_{2M}x_{2M+1}]. \]

Combining this with Lemma 2.1 and Lemma 2.2 we get
\[ x_0 \ldots x_M [x_0 \ldots x_M][x_{M+1} \ldots x_{2M+1}] = gA. \]

Thus we have shown that in both cases every generator \( g_A \) is indeed the whole ideal \( I_{N,n} \). We leave this to a motivated reader.

4 The non-containment result

In this section we prove our main result.

**Theorem 4.1.** For arbitrary \( N \geq 2 \) and \( n \geq 3 \) there is
\[ I_{N,n}^{(3)} \not\subset I_{N,n}^{2}. \]

**Proof.** It is convenient to abbreviate \( I = I_{N,n} \). The polynomial \( f := f_{N,n} = [x_0 \ldots x_{2M}] \) is contained in \( I^{(3)} \) by the Zariski-Nagata Theorem, see [5, Theorem 3.14] for prime ideals and [12, Corollary 2.9] for radical ideals. Let \( G \) denote the set of generators of the ideal \( I \).

The proof that it is not contained in \( I^2 \) depends on the parity of the dimension \( N \) of the ambient space.

We handle first the case \( N = 2M \). Assume to the contrary that \( f \in I^2 \). Then there are polynomials \( h_{g,g'} \) such that
\[ f = \sum_{g,g' \in G} h_{g,g'} gg'. \]  
Taking \( (\text{III}) \) modulo \( (x_0) \) we have
\[ \tilde{f} = \sum_{g,g' \in G} \tilde{h}_{g,g'} \cdot \tilde{g} \cdot \tilde{g'}, \]
where \( \tilde{q} \) denotes the residue class of \( q \in C[x_0, \ldots, x_N] \) modulo \( (x_0) \). Then
\[ \tilde{f} = x_1^n \ldots x_{2M}^n [x_1 \ldots x_{2M}]. \]

We focus now on the coefficient at the monomial
\[ m = x_1^{2Mn} x_2^{(2M-1)n} \ldots x_{2M-1}^{n} x_{2M}^{n} \]
on both sides of equation \((\text{III})\). This coefficient is 1 on the left hand side of \((\text{III})\). It is easy to see that there is exactly one way to get this monomial expanding the product defining \( \tilde{f} \).

Let \( g \in G \) be a generator of \( I \). By Proposition 3.2 \( g \) has the form
\[ g = x_{i_1} \ldots x_{i_M} [x_{i_1} \ldots x_{i_M}][x_{j_0} \ldots x_{j_M}], \]
with all indices \( i_1, \ldots, i_M, j_0, \ldots, j_M \) mutually distinct. If \( 0 \in \{i_1, \ldots, i_M\} \), then the residue class of \( g \) is zero. If it is in the second group of indices, then the residue class, after possible renumbering of indices, has the form
\[ \tilde{g} = x_{i_1} \ldots x_{i_M} x_{j_1}^n \ldots x_{j_M}^n [x_{i_1} \ldots x_{i_M}][x_{j_1} \ldots x_{j_M}]. \]  
Note that we suppress the notation and write \( x_i \) rather than \( \tilde{x}_i \).

We will now analyze how the monomial \( m \) appears on the right hand side of \((\text{III})\). To this end we run the following procedure starting with the variables with least powers in \( m \).
The variable $x_{2M}$ has to be among the variables appearing with power 1 in the product defining $\tilde{g}$ and $g'$ (variables indexed by the letter $i$) because its total power in $m$ is restricted by $n$ and this is the only possibility to fulfill this condition.

The variable $x_{2M-1}$ cannot then appear with power 1 neither in $\tilde{g}$ nor in $g'$. If it would, then it would appear with the variable $x_{2M}$ in the first bracket in the product defining $\tilde{g}$ and $g'$, hence there would be a factor

$$x_{2M-1}^2(x_{2M-1}^n - x_{2M}^n)^2$$

in the product $\tilde{g} \cdot g'$ and then the power of $x_{2M-1}$ would exceed $2n$ allowed in $m$. Hence the variable $x_{2M-1}$ appears in the second bracket in (12). Thus we have now

$$\tilde{g} = x_{2M}x_{2M-1}^n \ldots x_{2M-2}^2 \ldots [x_{2M-2}x_{2M}] \ldots x_{2M-1}$$

$$g' = x_{2M}x_{2M-1}^n \ldots x_{2M-2}^2 \ldots [x_{2M-2}x_{2M}] \ldots x_{2M-1} \ldots x_{2M-1}$$

(13)

The variable $x_{2M-2}$ in turn has to appear in the first brackets in (13). The argument is slightly more involved. In any case there is the factor $(x_{2M-2}^n - x_{2M}^n)$ in $\tilde{g}$ and $g'$ with $L$ either equal to $2M$ or $2M - 1$. From these brackets it has to be $x_{2M-2}^n$ which contributes to $m$ (otherwise the power at $xL$ would be too large). So, in any case $x_{2M-2}$ appears with power at least $2n$ in $\tilde{g} \cdot g'$. Since the total power is restricted by $3n$, the only possibility is that this variable appears with power 1 in the products in front of the brackets appearing in (13).

Hence we have

$$\tilde{g} = x_{2M}x_{2M-1}x_{2M-2} \ldots [x_{2M-2}x_{2M}] \ldots x_{2M-1}$$

$$g' = x_{2M}x_{2M-1}x_{2M-2} \ldots [x_{2M-2}x_{2M}] \ldots x_{2M-1}$$

(14)

Working down, variable by variable, in the same manner, we conclude finally that

$$\tilde{g} = g' = x_2x_4 \ldots x_{2M}x_1^n \ldots x_{2M-1}^n [x_2x_4 \ldots x_{2M}] [x_1x_3 \ldots x_{2M-1}]$$

and the only way to get the monomial $m$ from the product $\tilde{g}^2 \tilde{h}_{gg}$ comes in fact from the product

$$x_2^{(2M-2)n+2}x_4^{(2M-4)n+2} \ldots x_{2M}x_{2M-1}^{2n}x_{2M-1}^{2n} \ldots x_{2M-1}^{2n} \cdot \tilde{h}_{gg}.$$ 

This implies that $\tilde{h}_{gg}$ contains the monomial

$$p = x_2^{n-2}x_4^{n-2} \ldots x_{2M}^{n-2}$$

(15)

with coefficient 1. But this implies that the coefficient of this monomial in $h_{gg}$ is also 1 (taking modulo $(x_0)$ has no influence on this coefficient).

The next step is to take (10) modulo $(x_{2M-1})$. We will denote now the residue class of a polynomial $q$ by $\overline{q}$. Thus (10) becomes

$$\overline{f} = \sum_{g,g' \in G} \overline{g} \overline{g}' \overline{f}_{gg}.$$ 

(16)

Now we are interested in the monomial

$$m' = x_1^{2n}x_2^{(2N-1)n} \ldots x_{2M-2}x_0^{2n}x_{2N}.$$ 

Since

$$\overline{f} = -x_0^{n}x_1^{n} \ldots x_{2M-3}^{n}x_{2M-4}^{n}x_{2M}^{n} \sum_{g,g' \in G} \overline{g} \overline{g}' \overline{f}_{gg},$$

and obviously the monomial $m'$ comes up in a unique way in the above product, its coefficient in $\overline{f}$ is $-1$.

Running through an analogous procedure as in the reduction modulo $(x_0)$ step, we conclude that the monomial $m'$ appears on the right hand side of (10) only in the square of the generator

$$\overline{g} = x_2x_4 \ldots x_{2M}x_0x_3^n \ldots x_{2M-3}^{n}x_{2M-4} \ldots x_{2M-3}^{n} [x_2x_4 \ldots x_{2M}] [x_0x_1x_3 \ldots x_{2M-3}].$$
multiplied by \( \overline{h_{gg}} \). This shows that the coefficient of the monomial \( p \) defined in (15) in \( \overline{h_{gg}} \) is now \(-1\). This contradiction shows the assertion \( f \notin I^2 \).

Now, we study the case \( N = 2M + 1 \). Assume to the contrary that \( f \in I^2 \). Then there are polynomials \( h_{g,g'} \) such that
\[
f = \sum_{g,g' \in G} h_{g,g'} gg'.
\] (17)

Taking (17) modulo \( (x_0) \) we have
\[
\tilde{f} = -x_1^n \ldots x_{2M+1}^n [x_1 \ldots x_{2M+1}] = \sum_{g,g' \in G} \overline{h_{g,g'}} \cdot \tilde{g} \cdot g'.
\] (18)

Once again we focus on the coefficient at the monomial
\[
m = x_1^{(2M+1)n} x_2^{2Mn} \ldots x_{2M}^{2Mn} x_{2M+1}^{n} \]
on both sides of equation (18). This coefficient is \(-1\) on the left hand side of (18). It is easy to see that there is exactly one way to get this monomial expanding the product defining \( \tilde{f} \).

By Proposition 3.2 a generator \( g \in G \) has the form
\[
g = x_{i_0} \ldots x_{i_M} [x_{i_0} \ldots x_{i_M}] [x_{j_1} \ldots x_{j_{M+1}}],
\]
with all indices \( i_0, \ldots, i_M, j_1, \ldots, j_{M+1} \) mutually distinct. If \( 0 \in \{i_0, \ldots, i_M\} \), then the residue class of \( g \) is zero. If \( 0 \in \{j_1, \ldots, j_{M+1}\} \), then the residue class, after possible renumbering of indices \( \tilde{g} \), has the form
\[
\tilde{g} = x_{i_0} \ldots x_{i_M} x_{j_1} \ldots x_{j_{M+1}} [x_{i_0} \ldots x_{i_M}] [x_{j_1} \ldots x_{j_{M+1}}].
\]

Similarly as in the case of \( N = 2M \) one can show that there is only one possibility to get the monomial \( m \) in the right side of the equation (18).

This shows that the coefficient of
\[
x_1^{n-2} x_3^{n-2} \ldots x_{2M+1}^{n-2}
\]
in \( \overline{h_{g,g}} \) (and hence in \( h_{g,g} \)) is \(-1\), where
\[
g = x_1 x_3 \ldots x_{2M+1} [x_1 x_3 \ldots x_{2M+1}] [x_0 x_2 \ldots x_{2M}].
\] (19)

Finally, we take equation (14) modulo \( (x_{2M}) \) and look for the coefficient of
\[
m' = x_1^{(2M+1)n} x_2^{2Mn} \ldots x_0^{2Mn} x_{2M+1}^{n}.
\]

Looking at the exponents in \( m' \), we see that there is only one way to obtain this monomial in \( \overline{f} \). We present here a brief explanation how to produce such a monomial from \( \overline{f} \). We multiply the following factors
\[
x_0^n [x_0 x_1] [x_1 x_{i+1}] \ldots [x_i x_{2M+1}]
\]
and take the first element from every bracket except one bracket of the form \([x_0 x_1]\), for which we take the second element. In other words, we proceed as follows
\[
x_0^n [x_0 x_1] [x_1 x_{i+1}] \ldots [x_i x_{2M+1}] = x_0^n (-x_0^n) [x_1 x_{i+1}] \ldots [x_i x_{2M+1}] + \ldots =
\]
x_0^n (-x_0^n) (x_0^n) [x_1 x_{i+2}] \ldots [x_i x_{2M+1}] + \ldots,
\]
and so on. We do it for all possible \( i \in \{1, \ldots, 2M - 1\} \) and multiply the results by each other. Finally we multiply all by \(-x_0^n [x_0 x_{2M+1}]\). More precisely we multiply the first element in the bracket and we obtain the monomial \( m' \). Now we calculate the coefficient, which is \((-1)^{2M-1}\) from all \([x_0 x_1]\) brackets and one \((-1)\) from \(-x_0^n\).

Summing up we obtain that the coefficient of the monomial
\[
x_1^{n-2} x_3^{n-2} \ldots x_{2M+1}^{n-2}
\]
in \( \overline{h_{g,g}} \) (and hence in \( h_{g,g} \)) where \( g \) is as in (19) is 1, which gives a contradiction.
5 Concluding remarks

During preparations of this manuscript we were informed that Ben Drabkin [2] found another proof of the non-containment Theorem 4.1. Since his methods are completely different from ours we have decided to include a full proof of Theorem 4.1 also because it reveals particular symmetries of the ideals we handle here. We hope to expand this path of thoughts in our forthcoming paper [9].

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