Dynamic Stability Investigation of a Sandwich Beam Tapered along width and thickness with Temperature Gradient

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Abstract. The investigation to analyze a sandwich beam's dynamic stability with asymmetric configuration, tapered along the thickness and width, and influenced by an alive axial load with temperature gradient is executed for several boundary conditions employing computational method. Use of Hamilton’s principle results in the equations of motion and related boundary conditions. Hill’s equations are achieved using non-dimensionalized equations of motion with the Galerkin’s method. Then, the influence of several parameters on the dynamic stability for different boundary conditions are attained by applying Saito-Otomi conditions. The impact of different parameters on the regions of instabilities observed and is showcased in a sequence of graphs using the appropriate MATLAB program.

Nomenclature

\(i\) = 1 for top elastic layer, \(i\) = 2 for intermediate layer and \(i\) = 3 for the elastic layer at bottom.

\(A_i\) : The cross-section area of the beam

\(E_i\) : Young’s Modulus of elasticity

\(G_2\) : Core’s complex shear modulus or \(G_2(1+j)\)

\(G_2/E_i\) : The ratio of the in-phase shear modulus of the viscoelastic core and the young’s modulus of elasticity of the elastic layers.

\(2h_i\) : Thickness of the layers of the beam

\(I_i\) : Cross-sectional moment of inertia about an appropriate axis

\(l\) : Beam length

\(\bar{t}\) : Time in dimensionless form

\(U_i(x,t)\) : Displacement axially along the centre of \(i^{th}\) layer of beam

\(Y\) : Geometric parameter

\(\overline{X}\) : \(x/l\)

\(\psi_0\) : Reference temperature

\(\alpha_b\) : Width taper parameter

1. Introduction

To attain superior characteristics like better stiffness and less weight, sandwich beams are extensively developed for different engineering structures such as aerospace, helicopter blades, etc. The blades in gas or steam turbines and aerospace applications are exposed to high temperatures, so the temperature gradient’s impact must be considered during the design of beams. The beams can be economical with the variation of cross-section configuration. Karand Sujata [1] evaluated the stability of a cantilever type sandwich beam under the effect of periodic load having symmetric configuration. They witnessed that the geometric and shear parameter enhances the system’s stability, while the taper parameter had a
detrimental effect on stability. The same authors investigated the stability of a non-uniform configuration beam subjected to temperature gradient [2]. They realized that the beam's taper profile, thermal gradient, and elastic foundation stiffness affected the stability. Ray and Kar [3] inspected a sandwich beam with 3-layers and symmetric configuration for several boundary conditions. They detected that the core's loss factor, along with the shear parameter, improved the beam's stability. Asnani and Nakra [4] established the equations of motion for a multilayered sandwich beam and acquired the vibration damping features of beams with 15 layers and simply supported at the ends for cases such as constant weight, constant size, and flexural rigidity. The dynamic along with static stability of a sandwich beamlying on a Pasternak foundation in the influence of temperature environment is investigated by Pradhan et al. [7]. Pradhan et al. [8] examined a tapered sandwich beam's stability condition, which is on a variable Pasternak foundation. Chand et al. [9] examined the stability of a rotational beam with a parabolic-tapered profile and variable temperature grade. Pradhan and Dash [10] inspected the stability of a sandwich beam with non-uniform configuration and viscoelastic support with variable temperature gradient.

The literature valuation informs that selective study has been performed for the stability of non-uniform beams with various conditions. Nevertheless, no research has been implemented before to analyze the dynamic stability investigation of an asymmetric sandwich beam tapered along width and thickness with temperature gradient. This research work investigates the above-suggested configuration.

2. Problem Formulation

A taper 3-layer sandwich beam with asymmetric configuration and length \( l \) with a pulsating load, \( P(t) = P_{0b} + P_{1b}\cos(\omega t) \) acting axially along the un-deformed axis, is presented in figure 1. \( P_{1b} \) and \( P_{0b} \) represent the amplitudes of dynamic and static load. The elastic layers at top and bottom are with young's modulus \( E_1 \) and \( E_3 \) while the thickness are \( (2h_1) \) and \( (2h_3) \) respectively. The

![Figure 1. Asymmetric Tapered Beam Configuration](image-url)
middle viscoelastic core material is having shear modulus $G_2$ and thickness $(2h_2)$. At the end, the
dimension of the layers are $(2h_1), (2h_2)$ and $(2h_3)$ respectively.
The assumptions to formulate equations of motion are same as in [7].
In this, Potential energy is expressed as
\[
V = \frac{1}{2} \int_0^1 E_i A_i U_i^{s_2} dx + \frac{1}{2} \int_0^1 E_i A_i U_i^{s_2} dx + \frac{1}{2} \int_0^1 (E_i + E_i I_3) \omega_{ss}^{s_2} dx + \frac{1}{2} G_2 \int_0^1 A_i \gamma_{2}^{s_2} dx
\]
The Kinetic energy is expressed as
\[
T = \frac{1}{2} \int_0^1 \rho \omega_{s}^{s_2} dx
\]
Work done is expressed as
\[
W_p = \int_0^1 P(t) \omega_{s}^{s_2} dx
\]
Here, $U_i$ and $U_i$ represent the displacements for the top elastic and bottom elastic layer in the axial
direction and $\gamma_2$ represents the shearing strain in the viscoelastic core, represented by
$\gamma_2 = \frac{U_1-U_3}{2h_2} - C_{0x}\varepsilon$. 

The mathematical modeling of the equations are attained by the Hamilton’s principle as presented
below.
\[
\delta \int_0^1 \left( T - V + W_p \right) dt = 0
\]
\[
\bar{m}_b \left( \bar{\omega}_{s}^{s_2} \right) + \left( \bar{P}_w \right)_{s} + (D\bar{\varepsilon}_s)_{s} + g^*Y\left[ B_1 \left( \bar{C} + \bar{\omega}_{s}^{s_2} \right) \right] = 0
\]
\[g^*YB_2\bar{\omega}_{s}^{s_2} - Y(K\bar{U}_{s})_{s} + g^*YB_2\bar{U} = 0
\]
The dependent parameters for different section are,
\[
\bar{m}_b = (h_0(1-\bar{x}\alpha_b)) \left[ (1-\bar{x}) + \bar{x} \left( \frac{(h_1)}{(h_0)} \right) + \left( \frac{\rho_2}{\rho_1} \right) \left( \begin{array}{c} h_2 \cr h_0 \end{array} \right) \left( 1-\bar{x} \right) + \left( \frac{\rho_3}{\rho_1} \right) \left( \begin{array}{c} h_3 \cr h_0 \end{array} \right) \left( 1-\bar{x} \right) \right]
\]
\[D = (1-\bar{x}\alpha_1)^3 \left( \frac{E_1}{E} \right) (1-\bar{x}\alpha_2)^3
\]
\[B = 2 \left[ 1 + \bar{x}\alpha_2 \left( \frac{(h_2)}{(h_0)} \right) \right]
\]
$\alpha_1$ and $\alpha_2$ are the thickness taper parameters for the elastic layers at the top and bottom respectively.
$\alpha_b$ is the width taper parameter.
\[
g^* = \left\{ \begin{array}{l} 2 \left( \frac{(G_{2}^*)}{(\beta)E_i} \right) \left( \frac{L}{(2h_0)} \right)^2 \\
\end{array} \right\} = g(1+j\eta_2)
\]
\[Y = \left[ \begin{array}{c} (h_1) + (2h_2) + (h_3) \\
\end{array} \right] \left( \frac{(2h_0)}{(2h_0)} \right)^2
\]
Here, $\beta$ represents the thickness parameter, $\mu_1$ and $\mu_2$ represent the density parameters.

The associated boundary conditions at the ends are

$$\left[ \frac{-E_i (I_1)_b}{I^2} \right]_0 \mathcal{D}(\bar{w}_{\tau\tau}) = 0$$

Or,

$$\bar{w}_{\tau\tau}^{'}|_0 = 0$$

$$\left[ \frac{E_i (I_1)_b}{I^2} \right]_0 YKU = 0$$

Or,

$$\bar{U}^{'}|_0 = 0$$

$$\left[ \frac{E_i (I_1)_b}{I^2} \right]_0 \left[ - (D\bar{w}_{\tau\tau}) + gYB_1(\bar{w}_{\tau\tau} + \bar{U}) - (P(w_{\tau\tau})) \right]^{'}|_0 = 0$$

Or,

$$\bar{w}_{\tau\tau}^{'}|_0 = 0$$

2.1. Approximation Solution

For (1)-(2), the approximate solutions are assumed to be as follows

$$\bar{W}(\bar{x}, \bar{t}) = \sum_{j=1}^{j=P} W_j(\bar{x}) f_j(\bar{t})$$

$$\bar{\varphi}_c(\bar{x}, \bar{t}) = \sum_{i=P+1}^{i=2P} \gamma_i(\bar{x}) f_i(\bar{t})$$

Where $f_j$ and $f_i$, $w_i$ and $\gamma_i$ are used as the time shape function and co-ordinate shape functions respectively. These are considered in such a manner so that it will satisfy maximum number of boundary conditions and the equations of motion [5]. By using Galerkin method and putting the solutions achieved from (12)-(13) in dimensionless form of equation of motion (1)-(2), the consequent equations are obtained in matrix form. As in [3], the shape function is implemented for the required boundary condition.

$$[m_{ij}] [\bar{Q}_{ij}] + [k_{ij}] [\bar{Q}_{ij}] = \{0\}$$

Or,

$$[k_{ij}] [\bar{Q}_{ij}] = \{0\}$$

Where,

$$\{Q_{ij}\} = \{f_1, \ldots, f_p\}^T$$

$$\{Q_{ij}\} = \{f_{p+1}, \ldots, f_{2p}\}^T$$

$$M_{ij} = \frac{1}{2} \bar{m}_i w_j d\bar{x}$$

$$k_{11ij} = \frac{1}{2} \mathcal{D}(w_{ij}) (w_{ij}) + g*YB_1(w_{ij}) (w_{ij}) d\bar{x}$$

$$k_{12ij} = \frac{1}{2} g*YB_1 (w_{ij}) (u_i) d\bar{x}$$
\[
k_{22bj} = \int_{0}^{l} \left[ YK(u_{1},\tau)u_{j} + g^{y}YB_{1}u_{j} \right] d\xi
\]

Equations (14)-(15) are again simplified to
\[
\left[ m_{b} \right] \{ \dddot{\phi}_{b} \} + \left[ k_{b} \right] \{ \dot{\phi}_{b} \} - P_{1b} \left( H_{1b} \right) \{ Q_{1} \} - P_{1b} \cos(\omega T) \left( H_{1b} \right) \{ Q_{1} \} = \left\{ 0 \right\}
\]

Where,
\[
H_{1b} = \int_{0}^{1} w_{j}^{'}, w_{j}^{''} d\xi
\]
and
\[
\left[ k_{b} \right] = \left[ k_{12b} \right]^{-1} \left[ k_{22b} \right] \left[ k_{12b} \right]^{T}
\]

2.2. Regions of instability
The unstable zone boundaries for main and combination type resonances are found using the conditions given by Saito and Otomi [6].

3. Result Analysis
The model of sandwich beam is explored for various boundary conditions like fixed-fixed and fixed-hinged. Numerical values are determined for various parameters like temperature grade parameter, width taper parameter, and depth taper parameters. The given values of parameters are taken for the considered problem

\[\eta = 0.01, \alpha = 1, \beta_{1} = 0.5, \beta_{2} = 0.1, \beta_{3} = 0.5\]

The temperature at any point \(\xi\) from the beam end is assumed to be \(\psi = \psi_{0}(1-\xi)\). By choosing the temperature as \(\psi_{0} = \psi_{a}\) at the end.

Taking \(\xi = 1\) to be the reference temperature, the alteration in the young’s modulus of elasticity of the beam is indicated by [2]

\[E_{b}(\xi) = E_{0b}[1-\gamma\psi_{1}(1-\xi)], 0 \leq \gamma\psi_{1} < 1\]

Here, \(\gamma\) represents the temperature expansion co-efficient of the material in the beam, \(\delta = \gamma\psi_{1}\) is the temperature gradient and \(T_{b}(\xi) = [1-\delta(1-\xi)]\).

Here we are considering,

\[\alpha = \frac{E_{1}A_{1}}{E_{2}A_{2}} = \frac{E_{1}T(\xi)}{E_{2}T(\xi)} \ast \frac{A_{1}}{A_{2}} = \frac{E_{1}[1-\delta(1-\xi)]}{E_{2}[1-\delta(1-\xi)]} \ast \frac{A_{1}}{A_{2}} = \frac{1}{\delta_{2}} \ast \frac{\delta_{1} \ast (1-\xi)}{A_{2}} \ast \frac{A_{1}}{A_{2}}\]

Where \(\delta_{1}\) and \(\delta_{2}\) represent the temperature gradient for the 1st and 3rd layers respectively.

Figure 2. to Figure 5. indicate the dependency of unstable zones of the system upon various parameters for fixed-hinged and fixed-fixed case.

3.1. Dynamic Stability Analysis
Fixed-Hinged and fixed-fixed Condition
The unstable areas for dissimilar values of $\delta_2$ are shown in figure 2, for two different end conditions like fixed-fixed and fixed-hinged. With increment in the values of $\delta_2$, the unstable areas move towards higher frequency in all the considered cases, improving the stability of the beam.

The stability plots for diverse values of $\alpha_b$ are shown in figure 3. With increment in $\alpha_b$, the unstable areas move towards higher frequency improving the stability of the beam for the above mentioned cases.

Figure 2. Regions of instability for $\delta_2 = 0.2$ (a, c) and $\delta_2 = 0.5$ (b, d)
The unstable areas for dissimilar values of $\delta_1$ are shown in Figure 3. With increment in the $\delta_1$ values, the instability zones move towards lower frequency, hence deteriorating the system stability.
Figure 4. Regions of instability for $\delta_i = 0.2$ (a, c) and $\delta_i = 0.5$ (b, d).
Figure 5. Regions of instability for $\alpha_1 = 0.2$ (a, c) and $\alpha_1 = 0.5$ (b, d)

The unstable areas for dissimilar values of $\alpha_1$ are shown in Figure 5. With rise in the $\alpha_1$ values, the unstable areas shift towards lower frequency, resulting in worsening of the system stability. $\alpha_3$, also has the same trend on the system stability, as of $\alpha_1$.

4. Conclusion

The examination of dynamic stability of an asymmetric sandwich beam, tapered along thickness and width, subjected to an axial pulsating load with temperature gradient is executed for two different end conditions employing computational method.

Results proved that the upsurge in the temperature grade for the bottom layer increases the system's stiffness, thus improving the system's dynamic stability. An increase in taper parameters' values along with thickness and thermal gradient for the top layer worsen the dynamic stability. The beam's dynamic stability improves with an increase in the values of taper parameters along the width of the beam.

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