A NOTE ON THE SONAR TRANSFORM AND RELATED RADON TRANSFORMS

B. RUBIN

Abstract. The sonar transform in geometric tomography maps functions on the Euclidean half-space to integrals of those functions over hemispheres centered on the boundary hyperplane. We obtain sharp $L^p$-$L^q$ estimates for this transform and new explicit inversion formulas under minimal assumptions for functions. The main results follow from intriguing connection between the sonar transform, the Radon transform over paraboloids, and the transversal Radon transform, which integrates functions on $\mathbb{R}^n$ over hyperplanes, meeting the last coordinate axis.

1. Introduction

We consider an integral transform over hemispheres in the half-space $\mathbb{R}_+^n = \{ y = (y', y_n) : y' \in \mathbb{R}^{n-1}, y_n > 0 \}$ defined by

$$ \left( H \varphi \right)(x', r) = \int_{S_+(x', r)} \varphi(y) \, d\sigma(y), \quad x' \in \mathbb{R}^{n-1}, \quad r > 0, \quad (1.1) $$

where $S_+(x', r) = \{ y \in \mathbb{R}_+^n : |y - x'| = r \}$, $n \geq 2$, and $d\sigma(y)$ denotes the induced surface area measure. This integral operator is known in tomography as the sonar transform (see, e.g., [2, 17, 26]), where the word sonar is an acronym for “sound navigation and ranging”. One of the main problems is reconstruction of $\varphi$ if $(H\varphi)(x', r)$ is known for all $(x', r)$. This problem amounts to mathematical models in acoustic imaging, seismology, thermoacoustic tomography [4, 5, 6, 21, 22] and leads to interesting developments in integral geometry and harmonic analysis. Many researchers contributed to the study of the operators (1.1) and related spherical means; see, e.g., [1, 8, 10, 18, 19, 20, 24, 25], to mention a few; see also survey articles [14, 15].

The purpose of this note is to obtain sharp weighted norm estimates and explicit inversion formulas for $H \varphi$. Unlike many other publications on this subject, in which $\varphi$ is smooth and compactly supported away
from the boundary, our consideration is focused on arbitrary Lebesgue integrable functions $\varphi$.

Our approach relies on intimate connection between $H\varphi$ and two other Radon-type transforms

$$(Pf)(x) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) \, dy', \quad (1.2)$$

$$(T\psi)(x) = \int_{\mathbb{R}^{n-1}} \psi(y', x' \cdot y' + x_n) \, dy', \quad (1.3)$$

which map functions on $\mathbb{R}^n$ to functions on $\mathbb{R}^n$. The first one can be called the parabolic Radon transform and resembles integration of $f$ over the shifted paraboloid

$$\pi_x = \pi_0 + x, \quad \pi_0 = \{y = (y', y_n) : y_n = -|y'|^2\}.$$�

However, $(Pf)(x)$ differs from the usual surface integral

$$\int_{\pi_x} f(y) \, d\sigma(y) = \int_{\mathbb{R}^{n-1}} f(x' - y', x_n - |y'|^2) (1 + 4|y'|^2)^{1/2} \, dy' \quad (1.4)$$

by the Jacobian factor, which is suppressed in our consideration. The connection between $H$ and $P$ was indicated in [3, 5, 18, 22], though the use of this connection was different from ours.

The Radon transform (1.3) differs from the most common one, as, e.g., in [13, Chapter I, Section 2]; see (2.1) below. It resembles affine Radon transforms in Gelfand’s school [11, 12] and the parametric Radon transforms in [9]. Following Strichartz [29], who used it in the context of the Heisenberg group (see also [27, 28, Section 4.13]), we call (1.3) the transversal Radon transform, taking into account that it integrates functions over only those hyperplanes which meet the last coordinate axis. Both transforms (1.2) and (1.3) were applied by Christ [7] to obtain optimal constants in the corresponding $L^p-L^q$ inequalities.

1.1. Main results. We shall prove the following statement.

**Theorem 1.1.** For the hemispherical transform (1.4), the inequality

$$\left( \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}^n_+} |(H\varphi)(x', r)|^s r^{1-s} \, dr \right]^{q/s} \, dx' \right)^{1/q} \leq c_{p,q,s} \left( \int_{\mathbb{R}^n_+} |\varphi(y)|^p \, y_n^{1-p} \, dy \right)^{1/p}$$

holds if and only if

$$1 \leq p < n/(n - 1), \quad q = p', \quad 1/s = 1 - n/p'. \quad (1.5)$$
In particular, for \( q = s = n + 1 \) and \( p = (n + 1)/n \),

\[
\left( \int_{\mathbb{R}^{n-1}} \int_0^\infty |(H\varphi)(x', r)|^{n+1} \frac{dr dx'}{r^n} \right)^{1/(n+1)} \leq c_n \left( \int_{\mathbb{R}^n_+} |\varphi(y)|^{(n+1)/n} \frac{dy}{y^{1/n}} \right)^{n/(n+1)}.
\]

Here, as usual, \( p' \) is the dual exponent, defined by \( 1/p + 1/p' = 1 \).

An important feature of this theorem is that it gives precise information about the behavior of \( H\varphi \) both globally and near the boundary, depending on the behavior of \( \varphi \).

We also obtain explicit inversion formulas for the operators \( H \) and \( P \), which follow from the known results for \( T \); see Theorems 3.4, 3.5, 4.1, 4.2.

**Plan of the Paper.** Section 2 contains notation and auxiliary facts about the transversal Radon transform \( T \). In Section 3 we establish basic connections between the operators \( H \), \( P \), and \( T \), and obtain inversion formulas for \( P \). Section 4 contains the proof of Theorem 1.1 and inversion formulas for \( H \).

### 2. Preliminaries

2.1. **Notation.** In the following, \( x = (x_1, \ldots, x_{n-1}, x_n) = (x', x_n) \in \mathbb{R}^n \); \( \Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_n^2 \) is the Laplace operator. The notation \( C(\mathbb{R}^n) \), \( C^\infty(\mathbb{R}^n) \), and \( L^p(\mathbb{R}^n) \) for function spaces is standard; \( C_0(\mathbb{R}^n) = \{ f \in C(\mathbb{R}^n) : \lim_{|x| \to \infty} f(x) = 0 \} \); \( C^\infty_c(\mathbb{R}^n) \) is the space of compactly supported infinitely differentiable functions on \( \mathbb{R}^n \). All integrals are meant as Lebesgue integrals. We say that an integral under consideration exists in the Lebesgue sense if it is finite when the integrand is replaced by its absolute value. A letter \( c \) stands for an inessential positive constant which may vary at each occurrence. Given a certain real-valued expression \( X \), and a complex number \( \lambda \), we set \( (X)_{\pm} = |X|^\lambda \) if \( \pm X > 0 \) and \( (X)_{\pm} = 0 \), otherwise.

2.2. **The transversal Radon transform.** We recall that the most familiar form of the Radon transform [13] is

\[
(Rf)(\theta, t) = \int_{\theta^\perp} f(y + t\theta) \, dy, \quad (\theta, t) \in S^{n-1} \times \mathbb{R}, \tag{2.1}
\]

where \( S^{n-1} \) is the unit sphere in \( \mathbb{R}^n \) and \( \theta^\perp \) is the hyperplane through the origin orthogonal to \( \theta \). Setting \( \theta = (\theta_1, \ldots, \theta_n) = (\theta', \theta_n) \) and

\[
(\Lambda\varphi)(\theta, t) = \varphi \left( -\frac{\theta'}{\theta_n}, \frac{t}{\theta_n} \right), \tag{2.2}
\]
we have

\[(Rf)(\theta, t) = |\theta_n|^{-1}(\Lambda Tf)(\theta, t), \quad \theta_n \neq 0. \tag{2.3}\]

This formula can be inverted and enables one to reformulate all known facts for \( R \) in terms of \( T \); see [28, Section 4.13.5] and [7, Section 5.2] for details. For example, consider the mixed norm

\[\|F\|_{q,s} \equiv \left( \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}} |F(x', x_n)|^s dx_n \right]^{q/s} dx' \right)^{1/q}. \tag{2.4}\]

Then the celebrated Oberlin-Stein theorem [23] for the operator \( R \) yields the following statement for \( T \).

**Theorem 2.1.** For \( n \geq 2 \), the inequality

\[\|T\psi\|_{q,s} \leq c_{p,q,s}\|\psi\|_p\]

holds if and only if \( p, q, \) and \( s \) satisfy (1.5). In particular, for \( q = s = n + 1 \) and \( p = (n + 1)/n \),

\[\|T\psi\|_{n+1} \leq c_n\|\psi\|_{(n+1)/n}. \tag{2.5}\]

The proof of Theorem 2.1 can be found in [27, Theorems 1.4, 8.2], [28, Theorem 4.149]; see also [7, Lemma 2.2]. The necessity of the bounds for \( p, q, \) and \( s \) are inherited from those for \( R \). They can also be checked straightforward, using the scaling argument. For example, let \( \lambda = (\lambda_1, \lambda_2), \lambda_1 > 0, \lambda_2 > 0 \). We denote

\[(A_\lambda \psi)(x) = \psi(\lambda_1 x', \lambda_2 x_n), \quad (B_\lambda \Psi)(x) = \lambda_1^{1-n} \Psi \left( \frac{\lambda_2}{\lambda_1} x', \lambda_2 x_n \right),\]

so that \( TA_\lambda \psi = B_\lambda T \psi \). Then

\[\|A_\lambda \psi\|_p = \lambda_1^{1-n/p} \lambda_2^{1-1/p}|\psi|_p, \quad \|B_\lambda \Psi\|_q = \lambda_1^{1-n+(n-1)/q} \lambda_2^{-n/q}||\Psi||_q.\]

If \( \|T\psi\|_q \leq c||\psi||_p \) is true for all \( \psi \in L^p \), then it is true for \( A_\lambda \psi \), that is, \( \|TA_\lambda \psi\|_q \leq c||A_\lambda \psi||_p \), or \( \|B_\lambda T \psi\|_q \leq c||A_\lambda \psi||_p \). The latter is equivalent to

\[\lambda_1^{1-n+(n-1)/q} \lambda_2^{-n/q} ||T\psi||_q \leq c \lambda_1^{1-n/p} \lambda_2^{-1/p} ||\psi||_p.\]

Letting \( \lambda_1 \) and \( \lambda_2 \) tend to zero and to infinity, we conclude that the last inequality is possible only if \( p = (n + 1)/n \) and \( q = n + 1 \).

A variety of inversion formulas for the transversal Radon transform can be found in [27, Section 4]; see also [28, Section 4.13]. For example, the following statement holds.
Theorem 2.2. A function \( \psi \in L^p(\mathbb{R}^n) \), \( 1 \leq p < n/(n-1) \), can be reconstructed from \( \Psi = T\psi \) by the formula

\[
\psi(x) = \frac{1}{d_n,\ell(n-1)} \int_{\mathbb{R}^n} \frac{(\Delta^\ell g)(x)}{|y|^{2n-1}} \, dy,
\]

where

\[
d_n,\ell(n-1) = \int_{\mathbb{R}^n} \frac{(1-e^{iy_1})^\ell}{|y|^{2n-1}} \, dy \quad (y_1 \text{ is the first coordinate of } y),
\]

\[
g(x) = (2\pi)^{1-n} \int_{\mathbb{R}^{n-1}} \Psi(y', x - y' \cdot x') \frac{dy'}{(1 + |y'|^2)^{(n-1)/2}},
\]

\[
(\Delta^\ell g)(x) = \sum_{j=0}^\ell \binom{\ell}{j} (-1)^j g(x - jy).
\]

Here \( \ell = n-1 \) if \( n \) is even, and \( \ell > n-1 \) is arbitrary if \( n \) is odd. The integral in (2.6) is understood as a limit \( \lim_{\varepsilon \to 0} \int_{|y|<\varepsilon} \). This limit exists in the \( L^p \)-norm and in the a.e. sense. For \( \psi \in C_0(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) \), it exists in the sup-norm.

An alternative inversion formula can be obtained in terms of powers of the minus Laplacian operator. We follow [27, Theorem 4.5] (or [28, Theorem 4.132]), assuming for simplicity \( \psi \) to be smooth and compactly supported (this assumption is weakened in [27, Subsection 4.1.2] in terms of Lipschitz functions with prescribed decay at infinity).

Theorem 2.3. Let \( \psi \in C_c^\infty(\mathbb{R}^n) \), \( \Psi = T\psi \), and let \( g \) be defined by (2.7). If \( n \) is odd, then

\[
\psi(x) = (-\Delta)^{(n-1)/2} g(x).
\]

If \( n \) is even, then

\[
\psi(x) = c_n \int_{\mathbb{R}^n} \frac{(-\Delta)^{n/2-1} g(x) - (-\Delta)^{n/2-1} g(x-y)}{|y|^{n+1}} \, dy,
\]

where \( c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2} \) and \( \int_{\mathbb{R}^n} = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \) uniformly in \( x \in \mathbb{R}^n \).

3. Basic Connections

In this section we establish basic connections between the hemispherical transform \( H \), the parabolic transform \( P \), and the transversal Radon transform \( T \).
3.1. Connection between $H$ and $P$. By the classical Calculus,

\[(H \varphi)(x', r) = r \int_{|y'|<r} \varphi(x' - y', \sqrt{r^2 - |y'|^2}) \frac{dy'}{\sqrt{r^2 - |y'|^2}}\]

\[= r \int_{|y'|<r} f(x' - y', r^2 - |y'|^2) dy', \quad f(z) = z_n^{-1/2} \varphi(z', \sqrt{z_n}); \quad z \in \mathbb{R}^n.\]

Replace $r^2$ by $x_n > 0$ to get $(H \varphi)(x', \sqrt{x_n}) = \sqrt{x_n} (P_0 f)(x)$, where

\[(P_0 f)(x) = \int_{|y'|<\sqrt{x_n}} f(x' - y', x_n - |y'|^2) dy', \quad x \in \mathbb{R}^n. \quad (3.1)\]

Setting

\[(A \varphi)(z) \equiv (A \varphi)(z', z_n) = z_n^{-1/2} \varphi(z', \sqrt{z_n}), \quad z \in \mathbb{R}^n, \quad (3.2)\]

\[(A^{-1} \Phi)(x) \equiv (A^{-1} \Phi)(x', x_n) = x_n \Phi(x', x_n^2), \quad x \in \mathbb{R}^n, \quad (3.3)\]

so that $A^{-1} A \varphi = \varphi, \quad AA^{-1} \Phi = \Phi$, we obtain

\[(H \varphi)(x', r) = (A^{-1} P_0 A \varphi)(x', r). \quad (3.4)\]

The next step is to extend (3.1) to functions on $\mathbb{R}^n$. Given a function $f^+$ on $\mathbb{R}^n_+$, we denote by $e_- f^+$ its extension by zero onto the entire space $\mathbb{R}^n$, and let $r_+$ be the restriction map, which assigns to a function on $\mathbb{R}^n$ its restriction onto $\mathbb{R}^n_+$. Clearly,

\[(P_0 f^+)(x) = (r_+ Pe_- f^+)(x), \quad x \in \mathbb{R}^n. \quad (3.5)\]

Hence (3.4) gives the following statement.

**Lemma 3.1.** The equality

\[H \varphi = A^{-1} r_+ Pe_- A \varphi \quad (3.6)\]

holds provided that either side of it exists in the Lebesgue sense.

3.2. Connection between $P$ and $T$. This connection can be found in [7, Lemma 2.3]. In view of its importance and for the sake of completeness, we present it in detail, using our notation. For $x \in \mathbb{R}^n$, let

\[(B_1 f)(x) = f(x', x_n - |x'|^2), \quad (B_2 F)(x) = F(2x', x_n - |x'|^2). \quad (3.7)\]

The corresponding inverse maps have the form

\[(B_1^{-1} u)(x) = u(x', x_n + |x'|^2), \quad (B_2^{-1} v)(x) = v\left(\frac{x'}{2}, x_n + \frac{|x'|^2}{4}\right). \quad (3.8)\]

One can readily see that

\[\|B_1 f\|_p = \|f\|_p, \quad \|B_2 F\|_{q,s} = 2^{(1-n)/q} \|F\|_{q,s}. \quad (3.9)\]
Lemma 3.2. The equality

\[ Pf = B_2 T B_1 f, \quad (3.10) \]

holds provided that either side of it exists in the Lebesgue sense.

**Proof.** We write the left-hand side as

\[ (Pf)(x) = \int_{\mathbb{R}^{n-1}} f(y', x_n - |x' - y'|^2) dy' \]

Hence

\[ (B_2^{-1} Pf)(x) = (Pf) \left( \frac{x'}{2}, x_n + \frac{|x'|^2}{4} \right) = \int_{\mathbb{R}^{n-1}} f(y', x_n - |y'|^2 + x' \cdot y') dy'. \]

On the other hand,

\[ (T B_1 f)(x) = \int_{\mathbb{R}^{n-1}} (B_1 f)(y', x' \cdot y' + x_n) dy' = \int_{\mathbb{R}^{n-1}} f(y', x_n - |y'|^2 + x' \cdot y') dy', \]

as above. This gives the result. \( \square \)

**Theorem 3.3.** For \( n \geq 2 \), the inequality

\[ \| Pf \|_{q,s} \leq c_{p,q,s} \| f \|_p \]

holds if and only if \( p, q, \) and \( s \) satisfy (1.5). In particular, for \( q = s = n + 1 \) and \( p = (n+1)/n \),

\[ \| Pf \|_{n+1} \leq c_n \| f \|_{(n+1)/n}. \quad (3.11) \]

Theorem 3.3 follows immediately from Lemma 3.2 and (3.9). The inequality (3.11) can also be found in [7, formula (1.10)].

Another important consequence of Lemma 3.2 is a series of explicit inversion formulas for the parabolic Radon transform \( P \). Specifically, let \( Pf = F \). Then, by (3.10), \( f = B_1^{-1} T^{-1} B_2^{-1} F \), or

\[ f = B_1^{-1} \psi, \quad \psi = T^{-1} \Psi, \quad \Psi = B_2^{-1} F. \]

By (3.8),

\[ (B_1^{-1} \psi)(x) = \psi(x', x_n + |x'|^2), \quad (B_2^{-1} F)(x) = F \left( \frac{x'}{2}, x_n + \frac{|x'|^2}{4} \right), \]

and \( \psi = T^{-1} \Psi \) can be computed using the results from Subsection 2.2. Hence Theorem 2.2 gives the following statement.
Theorem 3.4. A function \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p < n/(n-1) \), can be reconstructed from \( F = P f \) by the formula

\[
f(x) = \frac{1}{d_{n,t}(n-1)} \int_{\mathbb{R}^n} \frac{(\Delta_y^t g)(x', x_n + |x'|^2)}{|y|^{2n-1}} dy,
\]

where the right-hand side has the same meaning as in (2.6) but with

\[
g(x) = \pi^{1-n} \int_{\mathbb{R}^{n-1}} F(z', x_n - 2z' \cdot x' + |z'|^2) \frac{dz'}{(1 + 4|z'|^2)^{(n-1)/2}}.
\]

Another inversion result follows from Theorem 2.3

Theorem 3.5. Let \( f \in C^\infty_c(\mathbb{R}^n) \), \( F = P f \), and let \( g \) be defined by (3.13). If \( n \) is odd, then

\[
f(x) = B_1^{-1} \left[ (-\Delta)^{(n-1)/2} g(x) \right].
\]

If \( n \) is even, then

\[
f(x) = B_1^{-1} \left[ c_n \int_{\mathbb{R}^n} \frac{(-\Delta)^{n/2-1} g(x) - (-\Delta)^n g(x-y)}{|y|^{n+1}} dy \right],
\]

where \( c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2} \), \( \int_{\mathbb{R}^n} = \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \) uniformly in \( x \in \mathbb{R}^n \).

3.3. Connection between \( H \) and \( T \). Combining Lemmas 3.1 and 3.2 we obtain

\[
H \varphi = Q_2 T Q_1 \varphi, \quad Q_1 = B_1 e_{-A}, \quad Q_2 = A^{-1} r_+ B_2.
\]

Explicit expressions for \( Q_1, Q_2 \), and their inverses can be easily obtained using (3.7), (3.8), (3.2), and (3.3). We have

\[
(Q_1 \varphi)(x) = (x_n - |x'|^2)^{-1/2} \varphi(x', \sqrt{x_n - |x'|^2}), \quad x \in \mathbb{R}^n; \quad (Q_2 f)(x', r) = r f(2x', r^2 - |x'|^2), \quad x' \in \mathbb{R}^{n-1}, \quad r > 0.
\]

Similarly,

\[
(Q_1^{-1} \psi)(y) = y_n \psi(y', y_n^2 + |y'|^2), \quad y \in \mathbb{R}_n^1; \quad (Q_2^{-1} \Phi)(x) = \left( x_n + \frac{|x'|^2}{4} \right)^{-1/2} \Phi\left( \frac{x'}{2}, \sqrt{x_n + |x'|^2} \right), \quad x \in \mathbb{R}^n.
\]

Thus we get one more important connection.

Lemma 3.6. The equality

\[
H \varphi = Q_2 T Q_1 \varphi,
\]

holds provided that either side of it exists in the Lebesgue sense.
Note that
\[
\|Q_1 \varphi\|_p^p = \int_{\mathbb{R}^{n-1}} \int_{|x'|^2}^\infty (x_n - |x'|^2)^{-p/2} |\varphi(x', \sqrt{x_n - |x'|^2})|^p \, dx_n
\]
\[= 2 \int_{\mathbb{R}^n_+} |\varphi(y)|^p y_n^{1-p} \, dy. \tag{3.21}
\]
and
\[
\|Q_2^{-1} \Phi\|_{q,s}^q = \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}} \left( x_n + \frac{|x'|^2}{4} \right)^{-s/2} \left| \Phi \left( \frac{x'}{2}, \sqrt{x_n + \frac{|x'|^2}{4}} \right) \right|^s \, dx_n \right]^{q/s} \, dx'
\]
\[= 2^{n-1+q/s} \int_{\mathbb{R}^{n-1}} \left[ \int_0^\infty \left| \Phi(x', r) \right|^s r^{1-s} \, dr \right]^{q/s} \, dx'. \tag{3.22}
\]

4. PROOF OF THE MAIN RESULTS

4.1. Proof of Theorem 1.1. Denote
\[
\|\varphi\|_p^\sim = \left( \int_{\mathbb{R}^n_+} |\varphi(y)|^p y_n^{1-p} \, dy \right)^{1/p},
\]
\[
\|\Phi\|_{q,s}^\sim = \left( \int_{\mathbb{R}^{n-1}} \left[ \int_0^\infty \left| \Phi(x', r) \right|^s r^{1-s} \, dr \right]^{q/s} \, dx' \right)^{1/q}.
\]
Then (3.21) and (3.22) yield
\[
\|Q_1 \varphi\|_p = 2^{1/p} \|\varphi\|_p^\sim, \quad \|Q_2^{-1} \Phi\|_{q,s} = 2^{(n-1)/q+1/s} \|\Phi\|_{q,s}^\sim.
\]
Our aim is to show that \(|H \varphi|_{q,s}^\sim \leq c \|\varphi\|_p^\sim\) for some constant \(c\). Indeed, by Lemma 3.6 and Theorem 2.1,
\[
\|H \varphi\|_{q,s}^\sim = 2^{(1-n)/q-1/s} \|Q_2^{-1} H \varphi\|_{q,s} = 2^{(1-n)/q-1/s} \|T Q_1 \varphi\|_{q,s}
\leq c \|Q_1 \varphi\|_p = 2^{1/p} c \|\varphi\|_p^\sim,
\]
as desired.
4.2. **Inversion formulas.** Let us proceed to inversion of $H\varphi$. A formal inversion formula $\varphi = (Q_1^{-1}T^{-1}Q_2^{-1})H\varphi$ can be made precise using explicit formulas for $Q_1^{-1}$ and $Q_2^{-1}$ combined with Theorems 2.2 and 2.3. Specifically, let $H\varphi = \Phi$. Setting $\Psi = Q_2^{-1}\Phi$ in (2.7) and using (3.19), in slightly different notation we obtain

$$g(x) = \pi^{1-n} \int_{\mathbb{R}^{n-1}} (x_n - 2z' \cdot x' + |z'|^2)^{-1/2}_+$$

$$\times \Phi \left(z', \sqrt{x_n - 2z' \cdot x' + |z'|^2} \right) \frac{dz'}{(1 + 4|z'|^2)^{(n-1)/2}}. \quad (4.1)$$

Then Theorems 2.2 yields the following result.

**Theorem 4.1.** A function $\varphi$ satisfying

$$\int_{\mathbb{R}^n_+} |\varphi(y)|^p y_n^{1-p} dy < \infty, \quad 1 \leq p < n/(n-1),$$

can be reconstructed from $\Phi = H\varphi$ by the formula

$$\varphi(y) = \frac{y_n}{d_{n,\ell}(n-1)} \int_{\mathbb{R}^n} (\Delta_\ell g)(y, y_n^2 + |y'|^2) \frac{dt}{|t|^{2n-1}}. \quad (4.2)$$

where $g$ is defined by (4.1) and the right-hand side has the same meaning as in (2.6).

In a similar way, Theorem 2.3 implies the following statement.

**Theorem 4.2.** Let $H\varphi = \Phi$, where $\varphi$ is a $C^\infty$ function with compact support in $\mathbb{R}^n_+$. Suppose that $g$ is defined by (4.1). If $n$ is odd, then

$$\varphi(y) = Q_1^{-1} \left[(-\Delta)^{(n-1)/2} g(x) \right](y). \quad (4.3)$$

If $n$ is even, then

$$\varphi(y) = Q_1^{-1} \left[c_n \int_{\mathbb{R}^n} (-\Delta)^{n/2-1} g(x) - (-\Delta)^{n/2-1} g(x - z) \frac{dz}{|z|^{n+1}} \right](y), \quad (4.4)$$

where $c_n = \Gamma((n+1)/2)/\pi^{(n+1)/2}$, $\int_{\mathbb{R}^n} = \lim_{\varepsilon \to 0} \int_{|z| > \varepsilon}$ uniformly in $x \in \mathbb{R}^n$, and $Q_1^{-1}$ acts in the $x$-variable.
4.3. **Some generalizations.** It is natural to take one step further and proceed from the Radon transforms \((\text{1.2})\) and \((\text{1.3})\) to the corresponding one-sided fractional integrals

\[
(P^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (y_n - |y'|^2)^{\alpha-1} f(x - y) \, dy, \tag*{(4.5)}
\]

\[
(T^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathbb{R}^n} (x_n - y_n)^{\alpha-1} f(y', y_n + x' \cdot y') \, dy, \tag*{(4.6)}
\]

which yield \(P\) and \(T\), respectively, as \(\alpha \to 0\). These generalizations are of interest on their own right.

The integrals \((\text{4.6})\) seem to be new. The \(L^p-L^q\) estimates of the localized modifications of \((\text{4.5})\), containing a cut-off function under the sign of integration, were briefly discussed by Littman \([16]\). The non-localized integrals \((\text{4.6})\) differ from those in \([16]\), though they have some features in common. We plan to address this topic in another paper.

**References**

[1] Andersson, L.-E. “On the determination of a function from spherical averages.” *SIAM Journal on Mathematical Analysis* 19, no. 1 (1988): 214–232.

[2] Beltukov, A. *Sonar Transforms*, Thesis (Ph.D.)– Tufts University, 2004.

[3] Beltukov, A. “Inversion of the spherical mean transform with sources on a hyperplane”. [arXiv:0910.1380v1 [math.CA]], 2009.

[4] Blondel, Ph., and Murton, Br. *Handbook of Seafloor Sonar Imagery*, New York: Wiley, 1997.

[5] Buhgeim, A. L., and Kardakov, V. B. “Solution of an inverse problem for an elastic wave equation by the method of spherical means. (Russian).” *Sibirsk. Mat. Ż*. 19, no. 4 (1978): 749–758, 953.

[6] Cheney, M. “Tomography problems arising in synthetic aperture radar.” *Radon Transforms and Tomography (South Hadley, Mass, 2000)*, *Contemporary Mathematics* 278, American Mathematical Society, Rhode Island (2001): 15–27.

[7] Christ, M. “Extremizers of a Radon transform inequality”. *Advances in analysis: the legacy of Elias M. Stein*, 84–107, Princeton Math. Ser., 50, Princeton, NJ: Princeton Univ. Press, 2014.

[8] Denisjuk, A. “Integral geometry on the family of semi-spheres.” *Fractional Calculus and Applied Analysis* 2, no. 1 (1999): 31–46.

[9] Ehrenpreis, L. *The Universality of the Radon Transform*, Oxford: Clarendon Press, 2003.

[10] Fawcett, J. A. “Inversion of n-dimensional spherical averages”, *SIAM Journal on Applied Mathematics* 45, no. 2 (1985): 336–341.
[11] Gelfand, I.M., Gindikin, S.G., and Graev, M.I. *Selected Topics in Integral Geometry*, Transl. Math. Monogr., vol. 220, Providence, RI: Amer. Math. Soc., 2003.

[12] Gelfand, I.M., Graev, M.I., and Vilenkin, N.Ja. *Generalized Functions, Integral Geometry and Representation Theory*, vol. 5, Providence, RI: AMS Chelsea Publishing, 1966.

[13] Helgason, S. *Integral Geometry and Radon Transform*, New York-Dordrecht-Heidelberg-London: Springer, 2011.

[14] Kuchment, P., and Kunyansky, L. “Mathematics of thermoacoustic tomography.” *European J. Appl. Math.* 19, no. 2 (2008): 191–224.

[15] Kuchment, P., and Kunyansky, L. “Mathematics of photoacoustic and thermoacoustic tomography.” *Handbook of mathematical methods in imaging*, Vol. 1, 2, 3, 1117–1167, New York: Springer, 2015.

[16] Littman, W. “$L^p$ - $L^q$ - estimates for singular integral operators arising from hyperbolic equations.” *Partial differential equations (Proc. Sympos. Pure Math., Vol. XXIII, Univ. California, Berkeley, Calif., 1971)*, pp. 479—481, Providence, R.I.: Amer. Math. Soc., 1973.

[17] Louis, A. K., and Quinto, E. T. “Local tomographic methods in sonar (English summary).” *Surveys on solution methods for inverse problems*, 147–154, Vienna: Springer, 2000.

[18] Narayanan, E. K., and Rakesh “Spherical means with centers on a hyperplane in even dimensions.” *Inverse Problems* 26, no. 3 (2010), 035014, 12 pp.

[19] Nessibi, M. M., Rachdi, L. T., and Trimeche, K. “Ranges and inversion formulas for spherical mean operator and its dual.” *Journal of Mathematical Analysis and Applications* 196, no. 3 (1995): 861–884.

[20] Nguyen, M. K., and Truong, T. T. “Inversion of a new circular-arc Radon transform for Compton scattering tomography.” *Inverse Problems* 26, no. 6 (2010), 065005, 13 pp.

[21] Norton, S. J. *Theory of Acoustic Imaging,* Ph.D. thesis, Stanford University, Stanford Elect. Lab. Tech. Rept. No. 4956-2, Chap. 5 (1976).

[22] Norton, S. J. “Reconstruction of a reflectivity field from line integrals over circular paths,” *The Journal of the Acoustical Society of America* 67, no. 4 (1980): 853–863.

[23] Oberlin, D. M., and Stein, E. M. “Mapping properties of the Radon transform.” *Indiana Univ. Math. J.* 31, no. 5 (1982): 641–650.

[24] Palamodov, V. P. “Reconstruction from limited data of arc means.” *The Journal of Fourier Analysis and Applications* 6, no. 1 (2000): 25–42.

[25] Palamodov, V. P. *Reconstructive Integral Geometry*, Monographs in Mathematics, vol. 98, Basel: Birkhäuser, 2004.

[26] Quinto, E. T., Rieder, A., and Schuster, T. “Local inversion of the sonar transform regularized by the approximate inverse.” *Inverse Problems* 27, no. 3 (2011), 035006, 18 pp.

[27] Rubin, B. “The Radon transform on the Heisenberg group and the transversal Radon transform.” *J. of Funct. Analysis* 262, no. 1 (2012): 234–272.
[28] Rubin, B. Introduction to Radon Transforms: With Elements of Fractional Calculus and Harmonic Analysis (Encyclopedia of Mathematics and its Applications), New York: Cambridge University Press, 2015.

[29] Strichartz, R. S. “$L^p$ harmonic analysis and Radon transforms on the Heisenberg group.” J. Funct. Anal. 96, no. 2 (1991): 350–406.

DEPARTMENT OF MATHEMATICS, LOUISIANA STATE UNIVERSITY, BATON ROUGE, LOUISIANA 70803, USA

Email address: borisr@lsu.edu