The new exact solutions for the deterministic and stochastic (2+1)-dimensional equations in natural sciences

Mahmoud A. E. Abdelrahman a,b, M. A. Sohaly b and Abdulghani Alharbi a

aDepartment of Mathematics, College of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia; bDepartment of Mathematics, Faculty of Science, Mansoura University, Mansoura, Egypt

ABSTRACT

This paper poses the Riccati–Bernoulli sub-ODE method in order to find the exact (random) travelling wave solutions for the (2+1)-dimensional cubic nonlinear Klein–Gordon (cKG) equation and the (2+1)-dimensional nonlinear Zakharov–Kuznetsov modified equal width (ZK-MEW) equation. The obtained travelling wave solutions are expressed by the hyperbolic, trigonometric and rational functions. Indeed, these solutions reflect some interesting physical interpretation for nonlinear phenomena. We discuss our method in deterministic case and in a random case. Additionally, we can show and discuss this method under some random distributions. Finally, some three-dimensional graphics of some solutions have been illustrated.

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1. Introduction

Nonlinear phenomena arise in many areas of applied science, such as fluid mechanics, biology, optical fibres, plasma physics and so on. Actually, these phenomena can be described by nonlinear partial differential equations (NPDEs), see [1–13]. Due to the importance of these NPDEs, searching for exact solutions for these equations has become more and more attractive field in different branches of physics and applied mathematics. Therefore investigating new methods to solve more complicated problems. Thus, many new methods have been proposed, such as the tanh-sech method [14–16], Jacobi elliptic function method [17–19], exp-function method [20,21], sine-cosine method [22–24], homogeneous balance method [25,26], F-expansion method [27,28], extended tanh-method [29,30], (G'/G)–expansion method [31,32]. Indeed, there are recent development in analytical and numerical methods for finding solutions for NPDEs, see [33–44] and references therein.

In this paper, we use the Riccati–Bernoulli sub-ODE method [11,12,45–47], to construct exact solutions, solitary wave solutions for the (2+1)-dimensional cubic nonlinear Klein–Gordon (cKG) equation and the (2+1)-dimensional nonlinear Zakharov–Kuznetsov modified equal width (ZK-MEW) equation. If we get a solution of NPDEs, we obtain new infinite sequence of solutions of these equations by using a Bäcklund transformation. Actually, we give new solutions and show that this method is efficient, powerful and vital for solving other type of NPDEs. Moreover the Riccati–Bernoulli sub-ODE method has an interesting feature, namely it is give infinite solutions. Indeed all presented solutions have so important contribution for the explanation of some practical physical phenomena and further nonlinear problems. To the best of our knowledge, no previous research work has been done using Riccati–Bernoulli sub-ODE method for solving the cKG equation and the ZK-MEW equation.

In many recent models, studying has ensued about the role of random or noise variables that end up as basic variables in predictive models. Therefore, one might think that the randomness part or the randomness effectiveness might be a normal outcome in many models. There are several difficulties in the study of stochastic models which may be as stochastic differential equations (SDEs) [48,49], or stochastic partial differential equations (SPDEs) [50,51]. As result, we implemented the Riccati–Bernoulli sub-ODE method for finding the exact stochastic solutions of the (2+1)-dimensional stochastic cKG equation and the (2+1)-dimensional nonlinear stochastic ZK-MEW equation, when the parameters are assigned random variables.
In order to find the conditions for our method in random case we will state the stability and convergence theorem.

The rest of the paper is given as follows. Section 2 describes the Riccati–Bernoulli sub-ODE method and a Bäcklund transformation of the Riccati–Bernoulli equation. In Sections 3, we apply the Riccati–Bernoulli sub-ODE method to solve the $(2+1)$-dimensional cGK equation and the $(2+1)$-dimensional nonlinear ZK-MEW equation. Additionally, in Section 4 we can discuss the Random Riccati–Bernoulli sub-RODE method. Finally, in Section 6 we give the conclusions.

2. The Riccati–Bernoulli sub-ODE method

Consider the following nonlinear evolution equation

$$P(\phi, \phi_x, \phi_t, \phi_{xx}, \phi_{xt}, \phi_{yy}, \phi_{xy}, \ldots) = 0,$$  \hspace{1em} (1)

where $P$ is a polynomial in $\phi(x, y, t)$ and its partial derivatives with even highest order derivatives and nonlinear terms.

Step 1. Let

$$\phi(x, y, t) = \phi(\xi), \quad \xi = kx + \beta y - \zeta t,$$  \hspace{1em} (2)

then Equation (1) reduces to a nonlinear ordinary differential equation ODE:

$$H(\phi, \phi', \phi'', \phi''', \ldots) = 0.$$  \hspace{1em} (3)

Step 2. Assume the solution of Equation (3) takes the form

$$\phi' = a\phi^{2-n} + b\phi + c\phi^n,$$  \hspace{1em} (4)

where $a, b, c$ and $n$ are constants to be determined. From Equation (4), we get

$$\phi'' = ab(3 - n)\phi^{2-n} + a^2(2-n)\phi^{3-2n} + nc^2\phi^{2n-1} + bc(n+1)\phi^n + (2ac + b^2)\phi,$$  \hspace{1em} (5)

$$\phi''' = (ab(3-n)(2-n)\phi^{1-n} + a^2(2-n)(3-2n)\phi^{2-2n} + n(2n-1)c^2\phi^{3n-2} + bcn(n+1)\phi^{n-1} + (2ac + b^2)\phi'),$$  \hspace{1em} (6)

2.1. Classification of the solutions

The solutions for the Riccati–Bernoulli Equation (4) are:

Case 1: When $n = 1$, then

$$\phi(\xi) = \mu e^{(a+b+c)\xi},$$  \hspace{1em} (7)

Case 2: When $n \neq 1, b = 0$ and $c = 0$, then

$$\phi(\xi) = (\sigma(n-1)(\xi + \mu))^{1/(n-1)},$$  \hspace{1em} (8)

Case 3: When $n \neq 1, b \neq 0$ and $c = 0$, then

$$\phi(\xi) = \left(-\frac{a}{b} + \mu e^{(b(n-1))\xi}\right)^{1/(n-1)},$$  \hspace{1em} (9)

Case 4: When $n \neq 1, a \neq 0$ and $b^2 - 4ac < 0$, then

$$\phi(\xi) = \left(-\frac{b}{2a} + \sqrt{\frac{4ac - b^2}{2a}} \tan \left(\frac{(1-n)\sqrt{4ac - b^2}}{2} (\xi + \mu)\right)\right)^{1/(1-n)}$$  \hspace{1em} (10)

and

$$\phi(\xi) = \left(-\frac{b}{2a} - \sqrt{\frac{4ac - b^2}{2a}} \cot \left(\frac{(1-n)\sqrt{4ac - b^2}}{2} (\xi + \mu)\right)\right)^{1/(1-n)}$$  \hspace{1em} (11)

Case 5: When $n \neq 1, a \neq 0$ and $b^2 - 4ac > 0$, then

$$\phi(\xi) = \left(-\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{2a}} \coth \left(\frac{(1-n)\sqrt{b^2 - 4ac}}{2} (\xi + \mu)\right)\right)^{1/(1-n)}$$  \hspace{1em} (12)

and

$$\phi(\xi) = \left(-\frac{b}{2a} - \sqrt{\frac{b^2 - 4ac}{2a}} \tanh \left(\frac{(1-n)\sqrt{b^2 - 4ac}}{2} (\xi + \mu)\right)\right)^{1/(1-n)}$$  \hspace{1em} (13)

Case 6: When $n \neq 1, a \neq 0$ and $b^2 - 4ac = 0$, then

$$\phi(\xi) = \left(\frac{1}{\sigma(n-1)(\xi + \mu)} - \frac{b}{2a}\right)^{1/(1-n)}.$$  \hspace{1em} (14)

Here $\mu$ is an arbitrary constant.

Step 3. Substituting the derivatives of $u$ into Equation (3) gives an algebraic equation of $u$. We can determine $n$, by using the symmetry of the right-hand item of Equation (4) and setting the highest power exponents of $u$ to equivalence in Equation (3). We compare the coefficients of $u^i$ gives a set of algebraic equations for $a, b, c, v$, and $v$. Solving these equations and superseding $n, a, b, c, v$, and $\xi = kx + \beta y - \zeta t$ into Equations (7)–(14), then the solutions of Equation (1) is obtained.
2.2. Bäcklund transformation

When \( \phi_{m-1}(\xi) \) and \( \phi_m(\xi)(\phi_m(\xi) = \phi_m(\phi_{m-1}(\xi))) \) are the solutions of Equation (4), we get

\[
\frac{d\phi_m(\xi)}{d\xi} = \frac{d\phi_m(\xi)}{d\phi_{m-1}(\xi)} \frac{d\phi_{m-1}(\xi)}{d\xi} = \frac{d\phi_m(\xi)}{d\phi_{m-1}(\xi)} (a\phi_m^{2-n} + b\phi_m + c\phi_m^n),
\]

namely

\[
\frac{d\phi_m(\xi)}{a\phi_m^{2-n} + b\phi_m + c\phi_m^n} = \frac{d\phi_m(\xi)}{a\phi_{m-1}^{2-n} + b\phi_{m-1} + c\phi_{m-1}^n}.
\]

Integrating Equation (15) once with respect to \( \xi \), we get a Bäcklund transformation of Equation (4) as follows:

\[
\phi_m(\xi) = \left( \frac{-cA_1 + aA_2 (\phi_{m-1}(\xi))^{1-n}}{bA_1 + aA_2 + aA_1 (\phi_{m-1}(\xi))^{1-n}} \right)^{1/(1-n)},
\]

where \( A_1 \) and \( A_2 \) are arbitrary constants.

The Equation (16) is used to get infinite sequence of solutions for Equation (4) and consequently Equation (1).

3. Applications

3.1. The (2+1)-dimensional cKG equation

Here we solve the (2+1)-dimensional cKG equation, see [52], which given as follows:

\[
\phi_{xx} + \phi_{yy} - \phi_{tt} + \alpha \phi + \beta \phi^3 = 0,
\]

where \( \alpha \) and \( \beta \) are non zero constants. This equation prescribes many problems in classical (quantum) mechanics, solitons and condensed matter physics. For example, it models the dislocations in crystals and the motion of rigid pendula attached to a stretched wire [53]. Using the travelling wave transformation

\[
\phi(x, y, t) = \phi(\xi), \quad \xi = x + y - \lambda t,
\]

where \( \alpha \) and \( \beta \) are real constants.

Equation (17) transforms into the following ODEs, using (18):

\[
(2 - \lambda^2)\phi'' + \alpha \phi + \beta \phi^3 = 0.
\]

Substituting Equations (5) into Equation (19), we obtain

\[
(2 - \lambda^2) (ab(3-n)\phi^{2-n} + a^2(2-n)\phi^{3-2n} + n^2\phi^{2n-1} + b(n+1)\phi^n + (2ac + b^2)\phi) + \alpha \phi + \beta \phi^3 = 0.
\]

Setting \( n = 0 \), Equation (20) is reduced to

\[
(2 - \lambda^2) (3ab\phi^2 + 2a^2\phi^3 + bc + (2ac + b^2)\phi) + \alpha \phi + \beta \phi^3 = 0.
\]

Setting each coefficient of \( \phi^i (i = 0, 1, 2, 3) \) to zero, we get

\[
\begin{align*}
(2 - \lambda^2)bc &= 0, \\
(2 - \lambda^2)(2ac + b^2) + \alpha &= 0, \\
3(2 - \lambda^2)ab &= 0, \\
2(2 - \lambda^2)a^2 + \beta &= 0.
\end{align*}
\]

Solving Equations (22)–(25), we get

\[
\begin{align*}
b &= 0, \\
ac &= \frac{\alpha}{2(\lambda^2 - 2)}, \\
a &= \pm \sqrt{\frac{\beta}{2(\lambda^2 - 2)}},
\end{align*}
\]

Case I. When \( \alpha/(\lambda^2 - 2) > 0 \), superseding Equations (26)–(28) and (18) into Equations (10) and (11), then the solutions of Equation (17) is,

\[
\phi_{1,2}(x, y, t) = \pm \sqrt{\frac{\alpha}{\beta}} \tan \left( \sqrt{\frac{\alpha}{2(\lambda^2 - 2)}}(x + y - \lambda t + \mu) \right),
\]

and

\[
\phi_{3,4}(x, y, t) = \pm \sqrt{\frac{\alpha}{\beta}} \cot \left( \sqrt{\frac{\alpha}{2(\lambda^2 - 2)}}(x + y - \lambda t + \mu) \right),
\]

where \( \alpha, \beta \), and \( \mu \) are arbitrary constants. The solution \( \phi_1 \) is depicted in Figure 1.

Case II. When \( \alpha/(\lambda^2 - 2) < 0 \), superseding Equations (26)–(28) and (18) into Equations (12) and (13),

Figure 1. The solution \( \phi = \phi_1(x, y, t) \) with \( \alpha = 1, \beta = 2, \lambda = 2, y = n = 0 \) and \(-5 \leq t, x \leq 5\).
where \( \lambda \) is a parameter. The wave speed and periodic travelling wave solution of Equation (17) is obtained using the sub-ODE method for seeking exact solitary wave solutions. Here, physical interpretations of the solutions are illustrated.

3.1.1. Physical interpretation

The solution \( \phi_5(x,y,t) \) is given by:

\[
\phi_{5,6}(x,y,t) = \pm \sqrt{\frac{2}{\alpha}} \tanh \left( \frac{1}{\sqrt{2(\lambda^2 - 2)}} (x + y - \lambda t + \mu) \right),
\]

and

\[
\phi_{7,8}(x,y,t) = \pm \sqrt{\frac{2}{\alpha}} \coth \left( \frac{1}{\sqrt{2(\lambda^2 - 2)}} (x + y - \lambda t + \mu) \right),
\]

where \( \alpha, \beta \) and \( \mu \) are arbitrary constants. The solution \( \phi_5 \) is depicted in Figure 2.

Case III. When \( b = 0 \) and \( c = 0 \), the solution of Equation (17) is

\[
\phi_9(x,y,t) = (-\alpha (x + y - \lambda t + \mu))^{-1},
\]

where \( \lambda \) and \( \mu \) are arbitrary constants.

3.2. The (2+1)-dimensional nonlinear ZK-MEW equation

Here we solve the (2+1)-dimensional nonlinear ZK-MEW equation, \([54]\). This equation given as follows:

\[
\phi_t + \alpha (\phi^3)_{x} + (\beta \phi_{xt} + \gamma \phi_{yy})_x = 0,
\]

where \( \alpha, \beta \) and \( \gamma \) are non zero constants. Here, \( x \) is the spatial domain and \( t \) is the time. \( \phi_t \) is the evolution term, \( \alpha (\phi^3)_x \) is the nonlinear term, and \( (\beta \phi_{xt} + \gamma \phi_{yy})_x \) is the dispersion term. Using the travelling wave transformation

\[
\phi(x,y,t) = \phi(\xi), \quad \xi = x + \epsilon y - \lambda t.
\]

where \( \alpha \) and \( \beta \) are real constants.

Equation (34) transforms into the following ODEs, using (35):

\[
(\gamma \epsilon^2 - \beta \lambda)\phi'' - \lambda \phi + \alpha \phi^3 = 0.
\]

with zero constant of integration. Substituting Equations (5) into Equation (36), we obtain

\[
(\gamma \epsilon^2 - \beta \lambda)(ab(3 - n)\phi^{2-n} + a^2(2 - n)\phi^{3-2n} + nc^2\phi^{2n-1} + bc(n + 1)\phi^n + (2ac + b^2)\phi) - \lambda \phi + \alpha \phi^3 = 0.
\]

Setting \( n = 0 \), Equation (37) is reduced to

\[
(\gamma \epsilon^2 - \beta \lambda) (3ab\phi^2 + 2a^2\phi^3 + bc + (2ac + b^2)\phi) - \lambda \phi + \alpha \phi^3 = 0.
\]
Putting each coefficient of \( \phi^i (i = 0, 1, 2, 3) \) to zero, we have

\[
\begin{align*}
(\gamma e^2 - \beta \lambda)bc &= 0, \\
(y e^2 - \beta \lambda)(2ac + b^2) - \lambda &= 0, \\
3(\gamma e^2 - \beta \lambda)ab &= 0, \\
2(y e^2 - \beta \lambda)\alpha^2 + \alpha &= 0.
\end{align*}
\]

Solving Equations (39)–(42), we get

\begin{align*}
b &= 0, \\
ac &= \frac{\lambda}{2(y e^2 - \beta \lambda)}, \\
\lambda &= \pm \sqrt{\frac{\alpha}{2(\beta \lambda - y e^2)}},
\end{align*}

Case I. When \( \lambda / ((y e^2 - \beta \lambda)) > 0 \), substituting Equa-
tions (43)–(45) and (35) into Equations (10) and (11), we obtain the exact wave solutions of Equation (34),

\[
\begin{align*}
\tilde{\phi}_{1,2}(x, y, t) &= \pm \sqrt{\frac{\lambda}{\alpha}} \tan \left( \sqrt{\frac{\lambda}{2(\beta \lambda - y e^2)}} (x + ey - \lambda t + \mu) \right), \\
\tilde{\phi}_{3,4}(x, y, t) &= \pm \sqrt{\frac{\lambda}{\alpha}} \cot \left( \sqrt{\frac{\lambda}{2(\beta \lambda - y e^2)}} (x + ey - \lambda t + \mu) \right),
\end{align*}
\]

where \( \lambda, \alpha, \beta, \gamma, \epsilon, \) and \( \mu \) are arbitrary constants. The solution \( \tilde{\phi}_{1,2} \) is depicted in Figure 3.

Case II. When \( \lambda / ((y e^2 - \beta \lambda)) < 0 \), substituting Equations (43)–(45) and (35) into Equations (12) and (13), we get exact travelling wave solutions of

\[
\begin{align*}
\tilde{\phi}_{5,6}(x, y, t) &= \pm \sqrt{-\frac{\lambda}{\alpha}} \tanh \left( \sqrt{-\frac{\lambda}{2(\beta \lambda - y e^2)}} (x + ey - \lambda t + \mu) \right), \\
\tilde{\phi}_{7,8}(x, y, t) &= \pm \sqrt{-\frac{\lambda}{\alpha}} \coth \left( \sqrt{-\frac{\lambda}{2(\beta \lambda - y e^2)}} (x + ey - \lambda t + \mu) \right),
\end{align*}
\]

where \( \lambda, \alpha, \beta, \gamma, \epsilon, \) and \( \mu \) are arbitrary constants. The solution \( \tilde{\phi}_{5,6} \) is depicted in Figure 4.

Case III. When \( b = 0 \) and \( c = 0 \), the solution of Equation (34) is

\[
\phi_{9}(x, y, t) = \frac{1}{a (x + ey - \lambda t + \mu)}^{-1},
\]

where \( \epsilon, \lambda, \) and \( \mu \) are arbitrary constants.

Remark 3.1: Applying Equation (16) to \( \phi_{i}(x, y, t) \) and \( \phi_{i}(x, y, t) \) \( (i = 1, 2, \ldots, 9) \) once, we can obtain an infinite solutions of Equations (17) and (34), respectively.

4. Stochastic models

In fact, the stochastic models has many applications in our life and it can be as stochastic dynamical system so, it is important for us to discuss what is the effect when dealing with these stochastic models and how to find the control of stability or the constraints on the randomness part in order to find the solution for these models. In this section, we investigate stochastic process solution of the (2+1)-dimensional Stochastic cubic nonlinear Klein–Gordon model with two random variables input and (2+1)-dimensional nonlinear stochastic...
4.1. The (2+1)-dimensional cubic stochastic nonlinear Klein–Gordon model

The nonlinear stochastic Klein–Gordon equation is used to model many nonlinear phenomena. In this part of the paper, a new Riccati–Bernoulli sub-RODE method for solving the (2+1)-dimensional cubic stochastic nonlinear Klein–Gordon model is proposed as follow:

\[ \phi_{xx} + \phi_{yy} - \phi_{tt} + \alpha \phi + \beta \phi^3 = 0, \quad (51) \]

where \( \alpha \) and \( \beta \) are non-zero random variables.

As the in deterministic case we can find the solutions as follow:

When \( \alpha, \beta \) are positive bounded random variables i.e. \( 0 < \alpha(\omega) \leq \alpha_1, 0 < \beta(\omega) \leq \beta_1, \alpha/(\lambda^2 - 2) > 0 \), we get stochastic exact travelling wave solutions as follow,

\[
\hat{\phi}_{1,2}(x,y,t) = \pm \sqrt{\frac{\alpha}{\beta}} \tan \left( \sqrt{\frac{\alpha}{2(\lambda^2 - 2)}}(x + y - \lambda t + \mu) \right), \quad (52)
\]

and

\[
\hat{\phi}_{3,4}(x,y,t) = \pm \sqrt{\frac{\alpha}{\beta}} \cot \left( \sqrt{\frac{\alpha}{2(\lambda^2 - 2)}}(x + y - \lambda t + \mu) \right). \quad (53)
\]

where \( \alpha, \beta \) are random variables and \( \zeta, \mu \) are arbitrary constants. The expected value operator of the stochastic solution \( \tilde{\phi}_1, \tilde{\phi}_3 \) is depicted in Figure 5. The variance of the stochastic solution \( \tilde{\phi}_1, \tilde{\phi}_3 \) is depicted in Figure 6, respectively.

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**Figure 5.** Graph of expectation of stochastic process solutions \( \hat{\phi} = \phi_1(x,y,t) \) and \( \hat{\phi} = \phi_3(x,y,t) \) on the left and right, respectively with \( \alpha \) and \( \beta \) have Beta distribution \( \beta(0.5, 0.5) \), \( \lambda = 2, y = n = 0 \) and \(-5 \leq t, x \leq 5 \) for Equation (51).

**Figure 6.** Graph of variance of stochastic process solutions \( \hat{\phi} = \phi_1(x,y,t) \), \( \hat{\phi} = \phi_3(x,y,t) \) on the left and right, respectively with \( \alpha \) and \( \beta \) have Beta distribution \( \beta(0.5, 0.5) \), \( \lambda = 2, y = n = 0 \) and \(-5 \leq t, x \leq 5 \) for Equation (51).
4.1.1. Theory of stability

The stochastic process solution for our problem, the (2+1)-dimensional stochastic cubic nonlinear Klein–Gordon equation is will be stable under the conditions:

1. $\alpha$ is bounded and second order random variable ($E[\alpha^2] < \infty$).
2. $\beta$ is bounded and second order random variable ($E[\beta^2] < \infty$).
3. $\gamma$ is bounded and second order random variable ($E[\gamma^2] < \infty$).

Also, for the Riccati–Bernoulli sub-ODE method that we used must be with second order the travelling wave transformation since ($E[\lambda^2] < \infty$).

4.2. The (2+1)-dimensional stochastic nonlinear ZK-MEW equation

Stochastic nonlinear ZK-MEW equation in many areas play an important role. Therefore, solving stochastic or deterministic nonlinear ZK-MEW equation or generally, the nonlinear evolution equations has become a valuable task. For this purpose, we will try to deal with the stochastic case as follow. This equation given as follows:

$$\phi_t + \alpha \phi^3_x + (\beta \phi_t + \gamma \phi_{yy})_x = 0,$$  \hspace{1cm} (54)

where $\alpha, \beta$ and $\gamma$ are non zero random variables.

As the same as in deterministic case we can find the stochastic solution relation as follow. When $\alpha, \beta$ and $\gamma$ are positive bounded random variables i.e. $0 < \alpha(\omega) \leq \alpha_1$, $0 < \beta(\omega) \leq \beta_1$, $\lambda/((\gamma \epsilon^2 - \beta \lambda)) > 0$ and $0 < \gamma(\omega) \leq \gamma_1$, we get stochastic exact travelling wave solutions as follow,

$$\hat{\phi}_{1,2}(x, y, t) = \pm \sqrt{\frac{\lambda}{\alpha}} \tan \left( \sqrt{\frac{\lambda}{2(\beta \lambda - \gamma \epsilon^2)}} (x + \epsilon y - \lambda t + \mu) \right),$$  \hspace{1cm} (55)

$$\hat{\phi}_{3,4}(x, y, t) = \pm \sqrt{\frac{\lambda}{\alpha}} \cot \left( \sqrt{\frac{\lambda}{2(\beta \lambda - \gamma \epsilon^2)}} (x + \epsilon y - \lambda t + \mu) \right),$$  \hspace{1cm} (56)

where $\alpha, \beta, \gamma$ are random variables, $\lambda$, $\epsilon$, and $\mu$ are arbitrary constants. The expected value operator of the stochastic solution $\hat{\phi}_1, \hat{\phi}_3$ is depicted in Figure 7. The variance of the stochastic solution $\hat{\phi}_1, \hat{\phi}_3$ is depicted in Figure 8.

4.2.1. Theory of stability

The stochastic process solution for our problem, the (2+1)-dimensional nonlinear stochastic Zakharov–Kuznetsov modified equal width (SZK-MEW) equation is will be stable under the conditions:

1. $\alpha$ is bounded and second order random variable ($E[\alpha^2] < \infty$).
2. $\beta$ is bounded and second order random variable ($E[\beta^2] < \infty$).
3. $\gamma$ is bounded and second order random variable ($E[\gamma^2] < \infty$).

Also, for the Riccati–Bernoulli sub-ODE method that we used must be with second order the travelling wave transformation since ($E[\lambda^2] < \infty$).

Remark 4.1: The Riccati–Bernoulli sub-ODE method has been successfully applied to find new travelling wave solutions for the (2+1)-dimensional cubic nonlinear Klein–Gordon equation and the (2+1)-dimensional nonlinear Zakharov–Kuznetsov modified equal width (ZK-MEW) models. As a result, we obtained many new exact solutions including rational function, hyperbolic function and trigonometric function solutions. It can be

Figure 7. Graph of expectation of stochastic process solutions $\hat{\phi} = \phi_1(x, y, t), \hat{\phi} = \phi_3(x, y, t)$ on the left and right, respectively with $\alpha$ and $\beta$ have Beta distribution $B(0.5, 0.5)$, $\lambda = 1, y = n = 0$ and $-5 \leq t, x \leq 5$ for Equation (54).
concluded that this method is a very robust and efficient technique to find the exact solutions for a large class of NPDEs. Moreover, from Remark 3.1 we find that the Riccati–Bernoulli sub-ODE method gives an infinite sequence of solutions. Moreover, we implemented the Riccati–Bernoulli sub-ODE method for finding the exact stochastic solutions of the proposed models, when the parameters are assigned random variables. We also state the stability and convergence theorem to find the conditions for the proposed method in random case.

5. Comparisons

We compare the results presented in this paper with other results in order to show that the Riccati–Bernoulli sub-ODE is powerful, efficient and adequate.

(1) Wang et al. [52] have presented only five solutions for the cKG equation, using the multi-function expansion method. Whereas Khan et al. [55] given eight solutions, using the modified simple equation (MSE) method. Comparing these results with presented result in this paper, we deduce that the Riccati–Bernoulli sub-ODE method gives many new exact travelling wave solutions along with additional free parameters. Thus, the Riccati–Bernoulli sub-ODE method is more effective in providing many new solutions than these two methods.

(2) Wazwaz [54] has presented only four solutions for the ZK-MEW equation, using the sine-cosine method and has introduced four solutions, using the tanh-method. On the other hand, we give many new exact travelling wave solutions with numerous free parameters.

Thus, the Riccati–Bernoulli sub-ODE method superior to other methods.

6. Conclusions

We have introduced the (2+1)-dimensional cubic nonlinear Klein–Gordon equation and the (2+1)-dimensional nonlinear Zakharov–Kuznetsov modified equal width (ZK-MEW) models in deterministic case and also, if we have some disturbance in their coefficients. We have proposed Riccati–Bernoulli sub-ODE method in order to find the exact travelling wave solutions for these models, additionally, to find the stochastic process solution. The physical cases for the solution of the (2+1)-dimensional cKG equation is discussed. The stochastic process solutions are studied for our models by stability control on the randomness part. The statistical moments are computed.

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ORCID

Mahmoud A.E. Abdelrahman http://orcid.org/0000-0002-7351-2088
M. A. Sohaly http://orcid.org/0000-0001-5971-0048
Abdulghani Alharbi http://orcid.org/0000-0002-1430-4684

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