Radiative decays of mesons in the quark model: relativistic and non-relativistic approaches

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Abstract

Different conclusions about quark nature of $f_0(980)$ based on the analysis of data on the reaction $\phi(1020) \rightarrow \gamma f_0(980)$ make it necessary to perform detailed comparison of formulae used by various groups for the calculation of partial widths of radiative decays. We carry out a comparative analysis of methods of calculation of radiative decays like $\phi \rightarrow \gamma f_0$ and $\phi \rightarrow \gamma \eta$ performed by F.E. Close et al., Phys. Rev. D 65, 092003 (2002), and A.V. Anisovich et al., Eur. Phys. J. A 12, 103 (2001); Yad. Fiz. 65, 523 (2002).

1 Introduction

At present there is a common understanding that radiative decays are a powerful tool for the study of the quark structure of mesons, and the calculation of corresponding amplitudes is subject of the increasing interest. Several groups are involved in such calculations, with different results and different conclusions about the quark structure of mesons based on these calculations. One of the main differences concerns the quark structure of $f_0(980)$ and $a_0(980)$. According to [1, 2], $f_0(980)$ and $a_0(980)$ are dominantly $qq\bar{q}\bar{q}$ states, and the authors claimed the hypothesis about a dominant $q\bar{q}$ component to be contradicting the data on radiative decays. At the same time, in the papers of our group [3, 4, 5], an opposite conclusion is made: the widths of radiative decays agree with the hypothesis about the $q\bar{q}$ origin of $f_0(980)$ and $a_0(980)$, and the $s\bar{s}$ component in $f_0(980)$ should be about 60%. The sharp discrepancy of conclusions demands to clarify the points, which resulted in such a disagreement.

The recent paper [6], where the calculation of a variety of reactions of one-photon radiative decays was carried out, provided us with an opportunity to compare the methods of calculation. Still, in [6], the calculation was performed in non-relativistic approximation, while in [3, 4, 5] we adopted a relativistic approach for the $q\bar{q}$-systems. Let us point out that in [4, 5], the formulae for the amplitudes of radiative decays were written in a form allowing a simple transformation to non-relativistic approximation for quark composite systems. The result of comparison appeared to be startling: in [6], the transition amplitudes depend explicitly on the momentum of the
decay particles, while in [4, 5] there is no such dependence. This circumstance enforces us to perform a comparative analysis of the two methods — this is the subject of the present paper.

We consider form factors of radiative decays involving vector (V), scalar (S) and pseudoscalar (P) mesons, such as $V \to \gamma P$ and $V \to \gamma S$, written in terms of Feynman integrals; then we transform them to non-relativistic approximation. In the framework of such a procedure, one can see directly that in the amplitude of radiative decays $V \to \gamma P$ and $V \to \gamma S$ the momentum dependence, like that obtained in [6], is absent. The momentum dependence of the matrix element implemented in [6] causes the violation of Lorentz-invariance: the requirement of Lorentz-invariance, due to zero photon mass, reveals itself also in a non-relativistic description of composite systems.

Let us clarify this statement in more detail. The Lorentz-invariant structure of the amplitudes of the transitions $V \to \gamma P$ and $V \to \gamma S$ have the form:

$$A_{V \to \gamma P} = -i\epsilon_{\mu\alpha\nu_1\nu_2}\epsilon_\alpha^{(V)} q_{\nu_1} p_{\nu_2} F_{V \to \gamma P}(q^2),$$

$$A_{V \to \gamma S} = \left( g_{\mu\alpha} - \frac{q_{\mu} p_{\alpha}}{(pq)} \right) \epsilon_\alpha^{(V)} \epsilon_\alpha^{(\gamma)} F_{V \to \gamma S}(q^2).$$

(1)

Here $\epsilon_{\mu\alpha\nu_1\nu_2}$ is the antisymmetrical tensor, $\epsilon^{(V)}$ and $\epsilon^{(\gamma)}$ are polarization four-vectors for the vector meson and photon, and $p$ and $q$ are their four-momenta; $g_{\mu\alpha}$ is the metric tensor. Invariant form factor $F_{V \to \gamma P}(q^2)$ (or $F_{V \to \gamma S}(q^2)$) is a function of three independent variables: $m_V^2$, $q^2$ and $m_P^2$ (or $m_S^2$). For real decays $q^2 = 0$, and this circumstance simplifies calculations of the form factors $F_{V \to \gamma P}(0)$ and $F_{V \to \gamma S}(0)$ as convolutions of quark wave functions. Form factors represented in terms of light-cone variables $k_\perp$ and $x$ read as follows [4, 5]:

$$F_{V \to \gamma P}(0) \sim \psi_V(k_\perp, x) \otimes \psi_P(k_\perp, x),$$

$$F_{V \to \gamma S}(0) \sim \psi_V(k_\perp, x) \otimes \psi_S(k_\perp, x),$$

(2)

and in these convolutions the photon momentum $q$ is not present explicitly. In the relativistic form factor representation, such a property is obvious because of the Lorentz invariance: form factors with $q^2 = 0$ allow one to choose any set of variables, $q_0 = |q| = 0$ included.

After coming to non-relativistic approach, the form factors obtained in [3, 4, 5] turn into the convolutions of quark wave functions:

$$\psi_1(|k|) \otimes \psi_2(|k|),$$

(3)

where $k$ is relative momentum of the $q\bar{q}$-system. Another representation of form factors is declared in [6]: form factors for the transitions $V \to \gamma P$ and $V \to \gamma S$ are written as convolutions

$$\psi_1(|k|) \otimes \psi_2 \left( |k + \frac{1}{2}q| \right),$$

(4)

(to avoid misunderstanding, let us note that in [6] the photon momentum was denoted as $p$, see Eq. (2) of this paper, but here we use $q$ for it).

The formulae (3) and (4) reflect the discrepancy between papers [3, 4, 5] and [6], so they are the subject of the present discussion.
By discussing non-relativistic formulae for the transitions $V \rightarrow \gamma P$ and $V \rightarrow \gamma S$, it would be more transparent not to use the trick with the choice of variables but to work directly with physical momenta $q_0 = |\mathbf{q}| \neq 0$. Such a calculation is done in the next Section by studying form factors $F_{V \rightarrow \gamma P}(0)$ and $F_{V \rightarrow \gamma S}(0)$ in the non-relativistic limit. We show that, by making a correct transformation, when in the integrand the terms of different order are not kept, the radiative decay form factors are defined by the convolution (3).

Another difference between papers [6] and [3, 4, 5] consists in a way of the application of the threshold theorem [7] ($F_{V \rightarrow \gamma S}(0) = 0$ is impossible if $V$ and $S$ are basic states ($1^3S_1q\bar{q}$ and $1^3P_0q\bar{q}$), for radial wave functions of these states have no nodes and the convolution $F_{V \rightarrow \gamma S}$ does not change sign. For basic states the requirement $F_{V \rightarrow \gamma S}(0) = 0$ is fulfilled only under the inclusion of processes beyond the additive model [8], for example, it can be the photon emission by the $t$-channel exchange current [9]. In [6] the threshold behaviour, $F_{V \rightarrow \gamma S}(0) \sim \omega$ at $\omega \to 0$ (see Eq. (10) of paper [6]), is believed to be due to the proper choice of the wave functions.

We do not investigate here the threshold behaviour of form factors in detail – special articles are devoted to this subject [8, 9], where two types of reactions have been considered: with the production of a bound system and with production of two unbound particles with the same quantum numbers, for example, $\phi(1020) \rightarrow \gamma f_0(980)$ and $e^+e^- \rightarrow \phi \rightarrow (\pi\pi)_S$. The amplitudes of these reactions are connected with each other, namely, the transition amplitude of $\phi(1020) \rightarrow \gamma f_0(980)$ is a residue in the poles of the amplitude of the process $e^+e^- \rightarrow \phi \rightarrow (\pi\pi)_S$:  

$$A_{e^+e^- \rightarrow \phi \rightarrow (\pi\pi)_S}(s_{e^+e^-}, s_{\pi\pi}, 0) = \frac{A_{\phi(1020) \rightarrow \gamma f_0(980)}(0)}{(m^2_{\phi} - s_{e^+e^-})(m^2_{f_0} - s_{\pi\pi})} + \text{smoother contributions}, \quad (5)$$

where $s_{e^+e^-}$ and $s_{\pi\pi}$ are the effective masses squared of the $e^+e^-$ and $\pi\pi$ systems. The threshold theorem is valid for the reactions with stable particles, that is for $A_{e^+e^- \rightarrow \phi \rightarrow (\pi\pi)_S}(s_{e^+e^-}, s_{\pi\pi}, 0)$ and $A_{\phi(1020) \rightarrow \gamma (\pi\pi)_S}(m^2_{\phi}, s_{\pi\pi}, 0)$:

$$A_{e^+e^- \rightarrow \phi \rightarrow (\pi\pi)_S}(s_{e^+e^-}, s_{\pi\pi}, 0) \sim (s_{e^+e^-} - s_{\pi\pi}), \quad (6)$$

$$A_{\phi(1020) \rightarrow \gamma (\pi\pi)_S}(m^2_{\phi}, s_{\pi\pi}, 0) \sim (m^2_{\phi} - s_{\pi\pi}),$$

for the $\phi(1020)$ can be considered as a stable particle with a good accuracy.

The problem of direct application of the threshold theorem to the $f_0(980)$ is not self-evident because this resonance definitely cannot be considered as a stable particle: it is characterized by two poles on different sheets related to the $K\bar{K}$ threshold, at $1020 - i40$ MeV and $960 - i200$ MeV, and the second pole is essential for the formation of the $f_0(980)$ signal, see [9] for more detail.
2 Form factors $F_{V\to\gamma S}(0)$ and $F_{V\to\gamma P}(0)$ in additive quark model

In the additive quark model, radiative decay is a three-stage process: the transition $V \to q\bar{q}$, photon emission by one of the quarks and the fusion of quarks into a final meson ($S$ or $P$), see Fig.1. The processes considered here, $V \to \gamma S$ and $V \to \gamma P$, are the transitions of both electric and magnetic types. Accordingly, the formula for the photon–quark vertex reads:

$$\frac{1}{2m} (k_{1\alpha} + k'_{1\alpha}) + \sigma_{\alpha\beta} \frac{q_\beta}{2m},$$

(7)

where $m$ is the quark mass, $\sigma_{\alpha\beta} = (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)/2$, for the notations of momenta see Fig. 1. Such a representation of the vertex being equivalent to the expression with the use of $\gamma_\alpha$ simplifies the calculations related to the transform to non-relativistic limit.

In the calculations, we work with amplitudes written as $\epsilon_\alpha^{(V)} \epsilon_\mu^{(V)} A_{\mu\alpha}^{V\to\gamma P/S}$ taking into account the requirements $\epsilon_\alpha^{(V)} q_\alpha = 0$ and $\epsilon_\mu^{(V)} p_\mu = 0$. Therefore, the calculated amplitudes obey the constraints $q_\alpha A_{\mu\alpha}^{V\to\gamma P/S} = 0$ and $p_\mu A_{\mu\alpha}^{V\to\gamma P/S} = 0$.

2.1 Feynman representation of the triangle diagram of Fig. 1

Feynman integral for the diagram of Fig. 1 reads:

$$\epsilon_\alpha^{(V)} \epsilon_\mu^{(V)} \int \frac{d^4k}{i(2\pi)^4} G_V \left[ (-) Sp \left[ \gamma_\mu (\hat{k}_1 + m) \Gamma_{\alpha} (\hat{k}_1' + m) \Gamma (-\hat{k}_2 + m) \right] \right] G,$$

(8)

where for the transition $V \to \gamma S$ the spin operators $\Gamma$ and $\Gamma_{\alpha}$ are defined as follows:

$$\Gamma = I, \quad \Gamma_{\alpha} = \frac{1}{2m} (k_{1\alpha} + k'_{1\alpha}),$$

(9)

and for $V \to \gamma P$ they are equal to

$$\Gamma = \gamma_5, \quad \Gamma_{\alpha} = \sigma_{\alpha\beta} \frac{q_\beta}{2m}.$$  

(10)

Transition vertices $V \to q\bar{q}$ and $q\bar{q} \to S$ (or $q\bar{q} \to P$) are denoted as $G_V$ and $G$.

A suitable tranformation procedure for getting non-relativistic expression is to introduce in (8) the two-component spinors for quark and antiquark, $\varphi_j$ and $\chi_j$, that is realized by substituting

$$(\hat{k}_1 + m) \to \sum_{j=1,2} u^j(k_1) \bar{u}^j(k_1), \quad (\hat{k}_1' + m) \to \sum_{j=1,2} u^j(k_1') \bar{u}^j(k_1'),$$

(11)

where

$$u^j(k) = \sqrt{k_0 + m} \left( \begin{array}{c} \varphi_j \\ \sigma_k \varphi_j \end{array} \right), \quad j = 1, 2,$$

(12)
and

\[
(\hat{k}_2 - m) \to \sum_{j=3,4} u^j(-k_2)\bar{u}^j(-k_2),
\]

where

\[
u^j(-k) = i\sqrt{k_0 + m} \left( \frac{\sigma k}{k_0 + m} \chi^j \right).
\]

After substitutions (11)–(14), we have the two-dimensional trace in the numerator of the integrand (8).

2.2 The diagram of Fig.1 in non-relativistic approximation

Now we go to non-relativistic approximation in the vector-particle rest frame. Denoting the four-momentum of vector particle as \( p = (p_0, \mathbf{p}_\perp, p_z) \), we have in this frame:

\[
p = (2m - \varepsilon_V, 0, 0),
\]

where \( \varepsilon_V \) is the binding energy of vector particle, which is supposed to be small as compared to the quark mass, \( \varepsilon_V \ll m \).

Let the photon fly along the \( z \)-axis, then

\[
q = (q_z, 0, q_z),
\]

and the polarization vector of photon lays in the \((x, y)\)-plane.

The four-momentum of scalar (pseudoscalar) particle is equal to

\[
p' = (2m - \varepsilon_V - q_z, 0, -q_z) = \left( \sqrt{(2m - \varepsilon)^2 + q_z^2}, 0, -q_z \right) \simeq \left( 2m - \varepsilon + \frac{q_z^2}{2m}, 0, -q_z \right). \]

Here \( \varepsilon \) is the binding energy of scalar (pseudoscalar) particle, which is also small compared to the mass of constituent \( \varepsilon \ll m \).

2.3 The reaction \( V \to \gamma S \)

The transition to non-relativistic approximation in the numerator of the integrand (8) provides the following formula for the reaction \( V \to \gamma S \):

\[
- \text{Sp}_2 \left[ 2m\sigma \cdot (k_2 - k'_1)\sigma \right] (k_{1\alpha} + k'_{1\alpha}).
\]

The notation \( \text{Sp}_2 \) stands for the trace of two-dimensional matrices. In (18), in the transition to non-relativism, the following terms are of the leading order.

In the photon–quark vertex (electric interaction):

\[
\bar{u}(k_1) \frac{k_{1\alpha} + k'_{1\alpha}}{2m} u(k'_1) \to \varphi^+(1)(k_{1\alpha} + k'_{1\alpha})\varphi(1'),
\]
in the vertex $V \rightarrow q\bar{q}$:

$$
\bar{u}(-k_2)\gamma_\mu u(k_1) \rightarrow \chi^+(2)2m\sigma_\mu \varphi(1) ,
$$

(20)

in the vertex $q\bar{q} \rightarrow S$:

$$
\bar{u}(k'_1)u(-k_2) \rightarrow \varphi^+(1')\sigma(k_2 - k'_1)\chi(2) .
$$

(21)

The constituent propagators in the transition to non-relativism should be replaced as follows:

$$(m^2 - k^2 - i0)^{-1} \rightarrow (-2mE + k^2 - i0)^{-1} ,
$$

(22)

where $E = k_0 - m$ and $m^2 - k_0^2 \approx -2mE$. Then the amplitude for the transition $V \rightarrow \gamma S$ is defined by the diagram of Fig. 1, it reads as follows:

$$
\epsilon^{(V)}_\mu \epsilon^{(\gamma)}_\alpha \int \frac{dEd^3k}{i(2\pi)^4} G_V \frac{-\text{Sp}_2[2m\sigma_\mu \cdot (k_2 - k'_1)\sigma](k_{1\alpha} + k'_{1\alpha})}{(-2mE_1 + k_1^2 - i0)(-2mE'_1 + k'_1^2 - i0)(-2mE_2 + k_2^2 - i0)} G .
$$

(23)

Furthermore, we denote $E_2 \equiv E$, $k_2 \equiv k$. With these notations the energy–momentum conservation laws are written as follows:

$$
E_1 = -\epsilon_V - E , \quad k_1 = -k ,
$$

$$
E'_1 = -\epsilon_V - E - q_z , \quad k'_1 = -k - q .
$$

(24)

One can integrate over $E$ by closing the integration contour in the lower half-plane of the complex-valued $E$, that is equivalent to substitution in (23):

$$
(-2mE + k^2 - i0)^{-1} \rightarrow \frac{2\pi i}{2m} \delta \left( E - \frac{k^2}{2m} \right) .
$$

(25)

By fixing $E = k^2/2m$, we can evaluate the order of value of momenta entering (23). Formula (17) gives us $2m - \epsilon_V - q_z = 2m - \epsilon + q_z^2/2m^2$, or

$$
q_z \approx \epsilon_V - \epsilon ,
$$

(26)

because $q_z^2/2m$ is the magnitude of the next-to-leading order of value. In the integrand of (23), the momentum square is of the order of

$$
k^2 \sim 2m\epsilon \sim 2m\epsilon_V ,
$$

(27)

and this means that

$$
q_z \ll |k| .
$$

(28)

Therefore, within non-relativistic approximation, the amplitude for the transition $V \rightarrow \gamma S$ reads:

$$
\epsilon^{(V)}_\mu \epsilon^{(\gamma)}_\alpha \int \frac{d^3k}{(2\pi)^3} \psi_V(k)\psi_S(k) \frac{1}{2m} (-4) \text{Sp}_2[2m\sigma_\mu \cdot 2(k\sigma)](-2k_\alpha) ,
$$

(29)

where the requirement (28) is duly taken into account and the wave functions for vector and scalar particle are introduced:

$$
\psi_V(k) = \frac{G_V}{4(m\epsilon_V + k^2)} , \quad \psi_S(k) = \frac{G}{4(m\epsilon + k^2)} .
$$

(30)
Recall, polarization vector $\epsilon^{(V)}$ do not contain the time-like component and polarization vector of the photon belongs to the $(x, y)$-plane. Accounting for $\text{Sp}_{2}\sigma_{\mu}\sigma_{\beta} = 2\delta_{\mu\beta}$, where $\delta_{\mu\beta}$ is the three-dimensional Kronecker symbol, and substituting in the integrand $k_\mu k_\alpha \to \delta_{\mu\alpha}k^2/3$, we have had final expression:

$$A_{V\to\gamma S} = \left(\epsilon^{(V)}\epsilon^{(\gamma)}\right) \int \frac{d^3k}{(2\pi)^3} \psi_V(k) \psi_S(k) \frac{32}{3}k^2 = \left(\epsilon^{(V)}\epsilon^{(\gamma)}\right) \int_0^\infty \frac{dk^2}{\pi} \psi_V(k) \psi_S(k) \frac{8}{3\pi}k^3 . \quad (31)$$

Here we re-denoted $k^2 \to k^2$. The photon polarization belongs to the plane orthogonal to $p$ and $q$, so one can re-write $\text{(31)}$ in the form of Eq. (1) by using

$$\left(g_{\mu\alpha} - \frac{q_{\mu}p_{\alpha}}{(pq)}\right)\epsilon^{(V)}_{\mu}\epsilon^{(\gamma)}_{\alpha} = \left(\epsilon^{(V)}\epsilon^{(\gamma)}\right).$$

Formula (31) coincides with that obtained in terms of the double spectral-integral representation (Eq. (32) of [4]), if in the relativistic-invariant formula (32) of [4] one makes a transform to the non-relativistic limit.

**2.4 The reaction $V \to \gamma P$**

In the reaction $V \to \gamma P$, non-relativistic spin factor (numerator of the integrand of $\text{(8)}$ has the form:

$$(-) \text{Sp}_{2}[2m\sigma_{\mu} \cdot i\varepsilon_{\alpha\beta\gamma} q_{\beta}\sigma_{\gamma} \cdot 2m] = -i8m^2\varepsilon_{\mu\alpha\beta} q_{\beta} , \quad (32)$$

where $\varepsilon_{\alpha\beta\gamma}$ is three-dimensional antisymmetric tensor. As a result, we have:

$$A_{V\to\gamma P} = -i\varepsilon_{\mu\alpha\beta\gamma} q_{\alpha} p_{\beta} F_{V\to\gamma P}(q^2) = -i\varepsilon_{\mu\alpha\beta} q_{\alpha} \int_0^\infty \frac{dk^2}{\pi} \psi_V(k) \psi_P(k) \frac{4km}{\pi} . \quad (33)$$

This expression coincides also with its relativistic-invariant counterpart obtained in [4] (Eq. (61)), if in [4] one makes a transition to non-relativistic limit.

**2.5 Normalization of wave functions $\psi_V(k)$, $\psi_S(k)$ and $\psi_P(k)$**

Here we use the wave function normalization similar to what was applied in [3, 4, 5] for the relativistic treatment of composite systems.

For scalar particle, the normalization reads:

$$\int_0^\infty \frac{dk^2}{\pi} \psi_S^2(k) \frac{2k^3}{\pi m} = 1 , \quad (34)$$

and for vector particle it is

$$\int_0^\infty \frac{dk^2}{\pi} \psi_V^2(k) \frac{2km}{\pi} = 1 . \quad (35)$$
Normalization for pseudoscalar particle is the same as for vector one: in (35) one should replace \( \psi_V(k) \to \psi_P(k) \).

Normalization condition of wave function can be re-formulated as requirement for charge form factor at \( q^2 = 0 \):

\[
F_{\text{charge}}(0) = 1 .
\]

(36)

It is easy to see that formulae (34) and (35) are actually the requirements for charge form factors. Let us explain this using scalar particle as an example.

Form factor of scalar particle is defined by triangle diagram of Fig. 1 type. Using the same calculation technique, which resulted in formula (29), we obtain:

\[
F^{(S)}_{\text{charge}}(0) = \int \frac{d^3k}{(2\pi)^3} \psi^2_S(k) \frac{2}{m} \text{Sp}_2[2(k\sigma) \cdot 2(k\sigma)] \frac{1}{2} .
\]

(37)

Here, as by derivation of Eq. (29), the factor \( 2/m \) arises due to the integration over \( E \) and wave function definition \( \psi_S(k) \); the vertex \( S \to q\bar{q} \) is equal to \( 2(k\sigma) \), and the factor \( 1/2 \) appeared because of substitution \( k_{1\alpha} + k'_{1\alpha} \to (p_{1\alpha} + p'_{1\alpha})/2 \): at \( \alpha = 0 \), that corresponds to the interaction with Coulomb field, we have \( k_{10} = k'_{10} \simeq m \) and \( p_{10} = p'_{10} \simeq 2m \). Just the condition \( F^{(S)}_{\text{charge}}(0) = 1 \) gives us the formula (34).

3 Comments on the analyticity of the amplitude \( V \to \gamma S \)

1. The threshold theorem can be reformulated as the requirement of analyticity of the field theory amplitude. The transition amplitude \( V \to \gamma S \) written for elementary particles of the field theory Lagrangian has a structure as follows:

\[
\left( g_{\mu\alpha} - \frac{2q_{\mu}p_{\alpha}}{s_V - s_S} \right) A(s_V, s_S, 0) ,
\]

(38)

where \( (pq) = (s_V - s_S)/2 \) and \( q^2 = 0 \). Then analyticity constraint reads

\[
[A(s_V, s_S, 0)]_{s_V \to s_S} \to 0 .
\]

(39)

The constraint (39) is written for the particles which form the standard sets of \( |\text{in} \rangle \) and \( \langle \text{out} | \) states, i.e. for particles which can be considered as stable ones. The problem of applying (39) to resonances is questionable because the resonance amplitude is determined by the pole residue, while the analyticity requirement (5) includes smoother terms as well.

2. The amplitude of additive quark model for the transition \( V \to \gamma S \) determined by (31) cannot be equal to zero. Indeed, to turn the right-hand side of Eq. (31) to zero one would need at least that one wave function was a sign-changing one. But wave functions of the lowest radial-excitation states, with radial quantum number \( n = 1 \), have no nodes. Therefore, the transition matrix element for basic states with \( n = 1 \) does not turn to zero, this fact not depending on what potential for \( V \) and \( S \) systems is under consideration.
Although in the additive approach $A_{V \rightarrow \gamma S}(0) \neq 0$, the factor $(m_V - m_S)$ inherent in the E1-transition can be extracted explicitly from the right-hand side of (31) for a wide class of models.

In order to clarify this point let us consider as an example the exponential approximation for the wave functions $\psi_V(k^2)$ and $\psi_S(k^2)$:

\[
\psi_V(q) \sim \exp(-b_V k^2), \quad \psi_S(q) \sim \exp(-b_S k^2).
\]  

(40)

With exponential wave functions, the matrix element for the $V \rightarrow \gamma S$ amplitude given by the additive quark model diagram, Eq. (31), up to a numerical factor is equal to:

\[
A_{V \rightarrow \gamma S} \sim \left(\epsilon(\gamma)\epsilon(V)\right) \frac{b_V^{3/4}b_S^{5/4}}{(b_V + b_S)^{5/2}} m(m_V - m_S). 
\]  

(41)

In case of one-flavour quarks in the exponential potential well one has

\[
m(m_V - m_S) = b_V^{-1},
\]  

(42)

that gives the amplitude $A_{V \rightarrow \gamma S}$ in the dipole emission representation:

\[
A_{V \rightarrow \gamma S} \sim \left(\epsilon(\gamma)\epsilon(V)\right) \frac{b_V^{7/4}b_S^{5/4}}{(b_V + b_S)^{5/2}} m(m_V - m_S). 
\]  

(43)

In case under consideration the factor $(m_V - m_S)$, or $(\epsilon_S - \epsilon_V)$, relates to the difference between the $V$ and $S$ levels and is defined by the $b_V$ value only. The amplitude $A_{V \rightarrow \gamma S}(0)$ turns to zero when binding energies of either $V$-meson or $S$-meson tend to zero: $\epsilon_V \rightarrow 0$ or $\epsilon_S \rightarrow 0$.

The considered example demonstrates that the statement of Ref. [1] that the triangle diagram contribution does not contain the factor $(\epsilon_S - \epsilon_V)$ is not generally justified. Still, the equivalence of the additive model and dipole representations for transition amplitudes depends on the type of interaction under consideration, see [9] for more detail.

3. There is one more point that is worth mentioning in connection with the present discussion, that is a representation of the spin operator. In [8], it was emphasized that the choice of the spin operator (38) is not unique as sometimes was it suggested [1]. For the spin structure of the amplitude $V \rightarrow \gamma S$ with emission of a real photon ($q^2 = 0$), one can alternatively use the metric tensor $g_{\mu \alpha}^{+\perp}(0)$ defined in the space orthogonal to the four-momenta $p$ and $q$:

\[
g_{\mu \alpha}^{+\perp}(0) = g_{\mu \alpha} + \frac{4s_V}{(s_V - s_S)^2} q_\mu q_\alpha - \frac{2}{s_V - s_S} (p_\mu q_\alpha + q_\mu p_\alpha).
\]  

(44)

The matter is that the difference

\[
g_{\mu \alpha}^{+\perp}(0) - \left(g_{\mu \alpha} - \frac{2q_\mu p_\alpha}{s_V - s_S}\right) = 4L_{\mu \alpha}(0),
\]  

(45)

where

\[
L_{\mu \alpha}(0) = \frac{s_V}{(s_V - s_S)^2} q_\mu q_\alpha - \frac{1}{2(s_V - s_S)} p_\mu q_\alpha.
\]  

(46)
is a nilpotent operator:

\[ L_{\mu\alpha}(0) L_{\mu\alpha}(0) = 0. \]  (47)

Adding a nilpotent operator to a spin operator one does not change the definition of the transverse amplitudes, see [8] for detail.

4. One can rise the question whether it is possible to re-define the radiative decay form factors (for \( V \to \gamma S \) and other similar processes) in accordance with Eq. (4). For the decay \( V \to \gamma S \), we have \(|q| = m_V - m_S\), so the re-definition does not affect the fact that \( F_{V\to\gamma S}(m_V^2, m_S^2, q^2 = 0) \) depends on \( m_V^2 \) and \( m_S^2 \) only [10]. However, it is necessary to take into consideration that the convolution (4) represents also a standard determination of the transition form factor for the space-like momenta \( q^2 = -q^2 < 0 \). So, within this re-definition and applying \( q^2 = -|q|^2 = -(m_V - m_S)^2 \) one has the equality

\[ F_{V\to\gamma S}(m_V^2, m_S^2, 0) = F_{V\to\gamma S}(m_V^2, m_S^2, -(m_V - m_S)^2), \]

which does not look reliable.

4 Conclusion

We have obtained formulae, within a non-relativistic quark model, for the transitions \( V \to \gamma S \) and \( V \to \gamma P \) starting from Feynman integrals. We have shown that, after a correct transition to the non-relativistic approximation, the decay amplitude with the emission of a real photon \((q^2 = 0)\) is determined by the convolution of wave functions (see (31) and (33)), where there is no photon-momentum dependence. This is a natural consequence of the Lorentz-invariant structure of the transition amplitudes, and it is a common property independent of whether we use relativistic or non-relativistic representations of the amplitude, see the discussion of formulae (3) and (4) here, as well as Eqs. (32) and (61) of paper [4]. In this point our formulae differ from those of the paper [6], where there is a \( q \)-dependence in the quark wave functions.

Another difference between papers [3, 4, 5] and [1, 2, 6] consists in the application of the threshold theorem [7] (see also [11, 12]) to the decay \( \phi(1020) \to \gamma f_0(980) \) – we discuss this problem in separate publication [9].

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Figure 1: Quark diagram for the transition form factor $V \to \gamma S/P$. 