Improved tripartite uncertainty relation with quantum memory

Fei Ming,1 Dong Wang,1,2,† Xiao-Gang Fan,1 Wei-Nan Shi,1 Liu Ye,1 and Jing-Ling Chen3,†

1School of Physics & Material Science, Anhui University, Hefei 230601, People’s Republic of China
2CAS Key Laboratory of Quantum Information, University of Science and Technology of China, Hefei 230026, People’s Republic of China
3Theoretical Physics Division, Chern Institute of Mathematics, Nankai University, Tianjin 300071, People’s Republic of China
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Uncertainty principle is a striking and fundamental feature in quantum mechanics distinguishing from classical mechanics. It offers an important lower bound to predict outcomes of two arbitrary incompatible observables measured on a particle. This principle can be depicted by taking advantage of various quantities. In quantum information theory, this uncertainty principle is popularly formulated in terms of entropy. Here, we present an improvement of tripartite quantum-memory-assisted entropic uncertainty relation. The uncertainty’s lower bound is derived by considering mutual information and Holevo quantity. It shows that the bound derived by this method will be tighter than the lower bound in [Phys. Rev. Lett. 103, 020402 (2009)]. Furthermore, regarding a pair of mutual unbiased bases as the incompatibilities, our bound will become extremely tight for the three-qubit $X$ states, completely coinciding with the entropy-based uncertainty, and can restore Renes et al.’s bound with respect to arbitrary tripartite pure states. In addition, by applying our lower bound, one can attain the tighter bound of quantum secret key rate, which is of basic importance to enhance the security of quantum key distribution protocols.

I. INTRODUCTION

Uncertainty principle proposed by Heisenberg is one of the pivotal cores in the area of quantum mechanics and exhibits basic and clear difference distinguishing from its classical counterpart [1]. The uncertainty principle specifically sets a lower bound for estimation of the measurement outcomes for two arbitrary incompatible observables on a quantum system. Kennard [2] and Robertson [3] formulated the uncertainty principle in terms of a standard deviation $\Delta X \Delta Z \geq |\langle \psi | [X, Z] | \psi \rangle | / 2$ in regard to a pair of incompatible observables $X$ and $Z$ for the systemic state $| \psi \rangle$. Note that, it can be viewed that the lower bound of the relation is not an optimal prediction result, because the bound is state-dependent, resulting in trivial result if the system is prepared in one of the eigenstates of $X$ or $Z$. Afterwards, there have been some efforts made to reform this relation uncertainty and generalize the case of multi-observable [4–12]. In 1983, Deutsch took entropy measure into account depicting the uncertainty principle, and conjectured the well-known form of entropic uncertainty relation (EUR) [13]. Further, Kraus improved Deutsch’s uncertainty relation [14], and later Maassen and Uffink proved the improvement [15]

$$H(X) + H(Z) \geq -\log_2 c(X, Z) \equiv q_{MU},$$

where $H(\tau) = -\sum_i p_i \log p_i$ is the Shannon entropy of the measured observables $\tau \in \{X, Z\}$ and $p_i = \langle \Phi_i^X | \rho | \Phi_i^X \rangle$ is the probability of obtaining the $i$-th outcome for a measurement $\tau$. $c(X, Z) = \max_{j,k} \langle \psi_j^X | \langle \phi_k^Z | \langle \phi_k^Z | \rho | \phi_k^Z \rangle \rangle $ is maximal overlap, and $| \psi_j^X \rangle$ and $| \phi_k^Z \rangle$ correspond to the eigenvectors of $X$ and $Z$, respectively. Since $c(X, Z)$ is relevant to the two observables themselves, it directly shows that the lower bound of EUR is state-independent. Compared with the standard deviation, the EUR can enable us to better predict the measured uncertainty.

The original conjecture of quantum cryptography [16] was inspired by the uncertainty relation. Yet, it is overlooked that the eavesdropper possibly has entanglement [17] with the measured system. Therefore, it cannot precisely prove the security of quantum cryptography by above-mentioned uncertainty relations. Up to 2010, considering the existing possibility of quantum memory, Berta et al. [18] filled such a blank and generalized EUR, i.e., quantum-memory-assisted entropic uncertainty relation (QMA-EUR), which reads

$$S(X|B) + S(Z|B) \geq S(A|B) - \log_2 c(X, Z),$$

where $S(X|B) = S(\rho_{XB}) - S(\rho_B)$ denotes the conditional von Neumann entropy [19] of post-measurement state with $\rho_{XB} = \sum_i (\langle \phi_i^X \rangle_A | \phi_i^X \rangle_2 \otimes \mathbb{1}_B) \rho_{AB} (| \phi_i^X \rangle_A \langle \phi_i^X | \otimes \mathbb{1}_B)$, likewise for $\rho_{ZB}$ and $\mathbb{1}_B$ is an identical operator in the Hilbert space of $B$. And $S(A|B) = S(\rho_{AB}) - S(\rho_B)$ represents the conditional von Neumann entropy of systemic density operator $\rho_{AB}$ with $S(\rho_{AB}) = -\text{tr} (\rho_{AB} \log_2 \rho_{AB})$ and $\rho_B = \text{tr} A (\rho_{AB})$. Following this novel inequality, several interesting results will be manifested: (i) if the measured particle $A$ is entangled with the memory particle $B$, the conditional von Neumann entropy $S(A|B)$ can be negative, Bob’s uncertainty for Alice’s measured outcomes will be reduced; (ii) when particle $A$ is maximally entangled with particle $B$, one can obtain $S(A|B) = -\log_2 d$ with the dimension $d$ of the measured particle and this will lead to a zero-valued bound, reflecting that Bob can perfectly predict the Alice’s measured outcomes of both observables $X$ and $Z$; (iii) when the quantum memory is absent, Eq. (2) can be reduced to

$$H(X) + H(Z) \geq S(\rho^A) - \log_2 c(X, Z),$$

which offers a tighter bound in comparison with Maassen and Uffink’s result, because of $S(\rho^A) \geq 0$ holds.
Actually, there have existed some promising improvements and generalizations related to QMA-EUR expressed by Eq. (2) [20–33]. To be explicit, Pati et al. [22] derived the uncertainty relation with a tighter lower bound including the classical correlation and the quantum correlation quantified by quantum discord. Coles and Piani [26] presented a strong bound via considering the second largest value of the overlap $c(X, Z)$. Later on, Adabi et al. [32] presented the uncertainty lower bound by adding a term about mutual information and Holevo quantity. More recently, Huang et al. [33] proposed a Holevo bound of QMA-EUR, which yields an interesting result that the difference between the entropic uncertainties and the new lower bound is always a constant value. So far, from an experimental viewpoint, ongoing progress has been made by some groups [34–40].

Typically, Renes and Boileau [41] proposed the tripartite uncertainty relation and generalized it into arbitrary two measurements as

$$S(X|B) + S(Z|C) \geq q_{MU},$$

where $q_{MU}$ is same as in Eq. (1). Technically, there is a trade-off that is quantified by the complementarity of the measurements. And this relation can be interpreted by a guessing game, so-called monogamy game. Explicitly, there are three participants, Alice, Bob and Charlie in this game. Preparing a tripartite state $\rho_{ABC}$, particle A is sent to Alice, B to Bob and C to Charlie. After receiving A, Alice randomly chooses a measurement $(X$ or $Z)$ and obtains the corresponding measured outcome $\epsilon$. Then Alice informs Bob and Charlie of her measurement choice. Bob and Charlie will win this game if and only if both predict $\epsilon$. By utilizing the monogamy of entanglement, Eq. (4) shows the uncertainty via the game: if Bob correctly produces a guess in case that Alice measured $X$ on A, as a result, Charlie cannot produce a good guess in case that Alice measured $Z$ on A, and vice versa. With respect to given observables X and Z, the bound $q_{MU}$ will become a constant, which will be independent of the characteristic of the system to be probed. In principle, the bound should be associated with the system. Although there have existed some improvement of entropic uncertainty relation, these explorations are confined to bipartite systems. Till now, there have been few improvement of tripartite quantum-memory uncertainty relations. Here we put forward a tighter bound of tripartite uncertainty relation with quantum memory, which resorts to mutual information and Holevo quantity, and thus is applicable to a lower bound of the tighter bound of quantum secret key rate to enhance the security of quantum key distribution protocols.

The outline of this article is organized as follows. In Sec. II, we derive a new tripartite uncertainty relation for arbitrary tripartite state, and manifest that the new bound is tighter than the previous bound. In Sec. III, we investigate our presented lower bound for several examples (generalized GHZ state, generalized W state, Werner-type state and random three-qubit states) to show its performance. In Sec. IV, the application of our result on quantum secret key rate is discussed. Lastly, the concise conclusions and discussions are given in Sec. V.

II. IMPROVED TRIPARTITE ENTROPIC UNCERTAINTY RELATION

Here, we present a brand-new and tighter lower bound for tripartite uncertainty relation with quantum memory, as an improvement of the existed uncertainty relation expressed as Eq. (4).

**Theorem.** By considering mutual information and Holevo quantity, a new tripartite uncertainty relation can be obtained as

$$S(X|B) + S(Z|C) \geq q_{MU} + \max \{0, \Delta\}$$

with

$$\Delta = q_{MU} + 2S(\rho^A) - [\mathcal{I}(A:B) + \mathcal{I}(A:C)]$$

$$+ [\mathcal{I}(Z:B) + \mathcal{I}(X:C)] - H(X) - H(Z),$$

where $\mathcal{I}(A:B) = S(\rho^A) + S(\rho^B) - S(\rho^{AB})$ is mutual information. The Holevo quantity $\mathcal{I}(X:B) = S(\rho^B) - \sum p_i S(\rho_i^B)$ [32] denotes the upper bound of accessible information of Bob for Alice’s measurement outcome. When Alice performs measurement $X$ on particle $A$, and obtains the $i$-th measurement outcome with probability $p_i = \text{tr}_{AB}(\Pi_i^A \rho^{AB} \Pi_i^A)$, and particle $B$ corresponds to the state $\rho_i^B = \text{tr}_{AC}(\Pi_i^A \rho^{ABC} \Pi_i^A)$. The dynamics of entropic uncertainty can be reflected by the evolution of tighter lower bound. This is a remarkable result that the new lower bound can capture the characteristic how the entropic uncertainty would behave.

**Proof.** Based on QMA-EUR in Eq. (2), with regard to the subsystems $B$ and $C$, we have

$$S(X|B) + S(Z|B) \geq S(A|B) + q_{MU},$$

$$S(X|C) + S(Z|C) \geq S(A|C) + q_{MU}.$$  

By combining Eqs. (7) and (8), a new inequality can be derived by

$$S(X|B) + S(Z|C) \geq 2q_{MU} + S(A|B) + S(A|C) - S(Z|B) - S(X|C).$$

Making use of the relations $S(\rho^A) = S(A|B) + \mathcal{I}(A:B)$, $S(\rho^A) = S(A|C) + \mathcal{I}(A:C)$, $H(Z) = S(Z|B) + \mathcal{I}(Z:B)$ and $H(X) = S(X|C) + \mathcal{I}(X:C)$, and resorting to Eqs. (4) and (9), the tripartite quantum memory uncertainty relation can be reformulated into the desired outcome, i.e., Eq. (5).

Noting that there are some special cases that $\Delta$ can be reduced. One is that if $X$ and $Z$ are complementary observables and subsystem $A$ is a maximally mixed state, we have $H(X) + H(Z) = q_{MU} + S(\rho^A)$, such as GHZ state. And, another case is that the observables are Pauli measurements $\sigma_x$ and $\sigma_z$, and the subsystem $A$ is an incoherent state, this equality mentioned above holds, such as generalized GHZ state, generalized $W$ state Werner-type state. Hence, in both cases, $\Delta = S(\rho^A) - [\mathcal{I}(A:B) + \mathcal{I}(A:C)] +$
in orthogonal basis \{\{000\}, \{001\}, \{010\}, \{011\}, \{100\}, \{101\}, \{110\}, \{111\}\}. \(\rho_{ij}(i,j = 1, 2, 3, 4, 5, 6, 7, 8)\) are all real parameters, and it satisfies the normalized condition \(\sum_{i=1}^{s} \rho_{ii} = 1\). Choosing two Pauli measurements \(\sigma_x\) and \(\sigma_z\) performed on the particle \(A\), we can derive the analytical solution of the sum of Bob’s uncertainty and Charlie’s uncertainty \(U_A = S(X|B) + S(Z|C)\) (the left-hand side of Eq. (5)) about Alice’s measurement outcome and the uncertainty lower bound \(U_R = q_{MU} + \max\{0, \Delta\}\) (the right-hand side of Eq. (5)) under the normalization condition as following

\[
U_L = U_R = 1 - S_{bin}(\rho_{11} + \rho_{33} + \rho_{55} + \rho_{77})
- (\rho_{11} + \rho_{33}) \log_2 (\rho_{11} + \rho_{33})
- (\rho_{22} + \rho_{44}) \log_2 (\rho_{22} + \rho_{44})
- (\rho_{55} + \rho_{77}) \log_2 (\rho_{55} + \rho_{77}),
\]

where \(S_{bin}(Y) = -Y \log_2 Y - (1 - Y) \log_2 (1 - Y)\) represents the binary entropy. Therefore, our lower bound is always equal to the sum of Bob’s uncertainty and Charlie’s uncertainty in the current architecture.

**Corollary 2.** If the prepared state is a tripartite pure state, owing to that \(\Delta\) is always less than or equal to zero, our lower bound will recover Renes et al.’s result.

**Proof.** For an any tripartite pure state, we have that the conditional entropies satisfy

\[
S(A|B) + S(A|C) = 0.
\]

From Eq. (4), one easily obtains

\[
q_{MU} - S(Z|B) - S(X|C) \leq 0.
\]

By linking Eqs. (5) and (9) with Eqs. (12) and (13), it is obtained that \(\Delta\) always less than or equal to zero, and our lower bound will recover Renes et al.’s lower bound. Moreover, we reveal the relationship between our lower and the purity of the systemic state (the purity \(P(\rho_{ABC}) = \text{Tr}[\rho_{ABC}^2]\)) by means of the approach of random states. That is our lower bound decreases with the increasing purity, and Renes et al.’s lower bound will be restored once the purity maximizes (i.e., the case of pure states).

### III. EXAMPLES

Considering a pair of incompatible observables such that perfect knowledge about observable \(X\) implies complete ignorance about observable \(Z\), the observables are called unbiased or mutually unbiased. For any finite-dimensional space, there are many pairs of orthogonal bases that satisfy this property. On the another hand, if two orthogonal bases \(X\) and \(Z\) are mutually unbiased bases (MUBs), they must satisfy the condition \(|\langle \psi_j^X | \varphi_k^Z \rangle|^2 := 1/d (\forall j,k)\). For example, if the measured particle is a qubit, Pauli measurements \(\sigma_x\), \(\sigma_y\) and \(\sigma_z\) can be chosen as the incompatible observables and MUBs. Generally, spin-1/2 Pauli matrices can be written as \(\sigma_x = |0\rangle \langle 1| + |1\rangle \langle 0|\), \(\sigma_y = -i |0\rangle \langle 1| + i |1\rangle \langle 0|\) and \(\sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1|\), which form a set of three MUBs. Through calculating the eigenvectors of Pauli matrices, the maximal overlap \(c(X,Z) \equiv \max_{j,k} |\langle \psi_j^X | \varphi_k^Z \rangle|^2\) is always \(\frac{1}{2}\). Thereupon, the incompatible term is \(\log_2 \frac{1}{2} = 1\). As an illustration, we herein choose Pauli measurements \(\sigma_x = X\) and \(\sigma_z = Z\) as the incompatibility, and discuss our result in different scenarios.

#### A. Generalized GHZ state

First of all, let us consider a typical pure tripartite state, generalized GHZ state, which can be written as in the Schmidt
Werner-type state under the basis \( |\psi\rangle \), the blue dashed line shows our result (the right-hand side of Eq. (5)) and the green dashed line shows Renes et al.’s result (the right-hand side of Eq. (4)).

\[ |\psi\rangle_{\text{Werner}} = (1 - p) |\Phi\rangle \langle \Phi| + \frac{p}{8} \mathbb{I}_{8 \times 8}, \]  

where the GHZ state \( |\Phi\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \), \( \mathbb{I}_{8 \times 8} \) stands for an identity \( 8 \times 8 \) matrix and \( 0 \leq p \leq 1 \). In Fig. 3, considering Werner-type state as a specific \( X \) state and two specific incompatible observables, the entropic uncertainty and our lower always are synchronized all the time. This reflects that our bound not only depends on the observables, but on the initial state. It shows that our lower bound is tighter than previous lower bound for all \( p \), which essentially is in agreement with Corollary 1, because \( \Delta \geq 0 \) holds in the case.
random numbers by this method

\[ \mathcal{P}_1 = f(0, 1), \]
\[ \mathcal{P}_2 = f(0, 1) \mathcal{P}_1, \]
\[ \mathcal{P}_3 = f(0, 1) \mathcal{P}_2, \]
\[ \mathcal{P}_4 = f(0, 1) \mathcal{P}_3, \]
\[ \mathcal{P}_5 = f(0, 1) \mathcal{P}_4, \]
\[ \mathcal{P}_6 = f(0, 1) \mathcal{P}_5, \]
\[ \mathcal{P}_7 = f(0, 1) \mathcal{P}_6, \]
\[ \mathcal{P}_8 = f(0, 1) \mathcal{P}_7. \quad (17) \]

Then a set random probabilities \( p_n \) (\( n \in \{1, 2, 3, 4, 5, 6, 7, 8\} \)) can be expressed as

\[ p_n = \frac{\mathcal{P}_n}{\sum_{m=1}^{8} \mathcal{P}_m}. \quad (18) \]

From Eqs. (17) and (18), it is straightforward to get a set of random probabilities in descending order. For the random generation of random unitary operation, we first randomly give an 8-order real matrix \( T \) by the random number function.

Using the real matrix \( T \), a random Hermitian matrix can be given as

\[
H = \begin{pmatrix}
T_{11} & T_{12} + iT_{21} & T_{13} + iT_{31} & T_{14} + iT_{41} & T_{15} + iT_{51} & T_{16} + iT_{61} & T_{17} + iT_{71} & T_{18} + iT_{81} \\
T_{12} - iT_{21} & T_{22} & T_{23} + iT_{32} & T_{24} + iT_{42} & T_{25} + iT_{52} & T_{26} + iT_{62} & T_{27} + iT_{72} & T_{28} + iT_{82} \\
T_{13} - iT_{31} & T_{23} - iT_{32} & T_{33} & T_{34} + iT_{43} & T_{35} + iT_{53} & T_{36} + iT_{63} & T_{37} + iT_{73} & T_{38} + iT_{83} \\
T_{14} - iT_{41} & T_{24} - iT_{42} & T_{34} - iT_{43} & T_{44} & T_{45} + iT_{54} & T_{46} + iT_{64} & T_{47} + iT_{74} & T_{48} + iT_{84} \\
T_{15} - iT_{51} & T_{25} - iT_{52} & T_{35} - iT_{53} & T_{45} - iT_{54} & T_{55} & T_{56} + iT_{65} & T_{57} + iT_{75} & T_{58} + iT_{85} \\
T_{16} - iT_{61} & T_{26} - iT_{62} & T_{36} - iT_{63} & T_{46} - iT_{64} & T_{56} - iT_{65} & T_{66} & T_{67} + iT_{76} & T_{68} + iT_{86} \\
T_{17} - iT_{71} & T_{27} - iT_{72} & T_{37} - iT_{73} & T_{47} - iT_{74} & T_{57} - iT_{75} & T_{67} - iT_{76} & T_{77} & T_{78} + iT_{87} \\
T_{18} - iT_{81} & T_{28} - iT_{82} & T_{38} - iT_{83} & T_{48} - iT_{84} & T_{58} - iT_{85} & T_{68} - iT_{86} & T_{78} - iT_{87} & T_{88}
\end{pmatrix}
\]

where \( T_{ij} \) are elements of the real matrix \( T \). By the calculation, we can attain eight normalized eigenvectors \( |\psi_n\rangle \) of the Hermitian matrix and a random unitary operation \( E \). Thus, we can construct the random three-qubit states. As an illustration, we take \( 10^5 \) random states to depict the corresponding uncertainty and our bounds in Fig. 4. It is apparent to show that, \( U_L \geq U_R \) (Eq. (5)) satisfies always. By means of utilizing random states, we verify that our theorem presented here is hold.

Additionally, following Fig. 5, it can be easily found that the lower bound \( U_L \) obtained by us gradually reduce with the increasing systemic state purity \( P \). For \( P = \frac{1}{8} \) corresponding to the purity of maximum mixed state, our lower bound will reach to the maximal value \( U_R = 2 \). When \( P = 1 \) corresponds to the three-qubit pure state, our lower bound will reduce to the minimal value \( U_R = 1 \). In this case, our lower bound will restore Renes et al.’s lower bound.

IV. APPLICATION

Entropic uncertainty relation not only reflects the fundamental discrepancy between quantum mechanics and classical counterpart, but also gives rise to many potential applications in the course of quantum information processing, including entanglement criterion [42, 43], quantum randomness [44], quantum steering [45–48], quantum key distribution [18, 41, 49, 50], and so on. Here we focus on the application of our finding on quantum key distribution. Specifically, we derive quantum secret key rate lower bound based on the lower bound of tripartite uncertainty relation. Technically, the entropic uncertainty relations can be applied to confirm the security of quantum key distribution protocols. To be specific, the lower bound of tripartite uncertainty relation is closely associated with the quantum secret key (QSK) rate. The key distribution protocol is that two honest part (Alice and Bob) share a key together by communicating over a public channel, and the key is secret from any eavesdropping by the third part (Eve). Devetak and Winter [49] reported that the amount of key \( K \) that can be extracted by Alice and Bob is lower bounded by

\[ K \geq S(Z|E) - S(Z|B), \quad (20) \]
Bob only needs to upper bound the additional term tended to arbitrary attacks via the post-selection technique. The argument applies only to collective attacks, it can be extended to arbitrary attacks via the post-selection technique. The communication between Alice and Bob is completely overheard by an eavesdropper Eve who try to get the key. Even in this case, Alice and Bob can still generate a secure key if their measurement outcomes are sufficiently relevant, which can be revealed by the stronger bound of quantum secret key rate $K'$.

\[ K \geq q_{MU} - S(X|B) - S(Z|B). \]  

(21)

According to the new lower bound of tripartite uncertainty relation in Eq. (5), the bound of quantum secret key rate can be rewritten as

\[ K' \geq q_{MU} + \max\{0, \Delta\} - S(X|B) - S(Z|B). \]  

(22)

Since the additional term $\max\{0, \Delta\}$ is greater than or equal to zero all along, we declare that the QSK rate obtained by us is tighter than the result obtained by Berta et al. [18]. It is well known that any measurements cannot reduce entropy. Thus, the bound of quantum secret key rate can be derived as

\[ K' \geq q_{MU} + \max\{0, \Delta\} - S(X|X') - S(Z|Z'). \]  

(23)

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure5}
\caption{(Color online) Our lower bound $U_R$ versus the systemic state purity with respect to $10^8$ randomly generated three-qubit states. X-axis denotes the systemic state purity and Y-axis represents our lower bound, respectively. The red line stands for the case of $P = \frac{1}{2}$.}
\end{figure}

When conjugate observables are applied to qubits and assumed symmetric, we have $S(X|X') = S(Z|Z')$. Although the argument applies only to collective attacks, it can be extended to arbitrary attacks via the post-selection technique [51]. The advantage of security argument is that Alice and Bob only need to upper bound the additional term $\max\{0, \Delta\}$ and the entropies $S(X|X')$ and $S(Z|Z')$, which can improve the performance of the actual quantum key distribution protocols. The statistics required to estimate states are critical for the security of protocols [52]. We can analyze the security of quantum key distribution protocols by assuming that the eavesdropper creates a quantum state $\rho_{ABE}$, and sends particles $A$ and $B$ to Alice and Bob, respectively. Although in this case, a security proof certainty means security when Alice and Bob distribute the states themselves [18]. In order to generate their key, Alice and Bob randomly choose the measurements and measure the states. $X$ and $Z$ are Alice’s choosing measurements, and $X'$ and $Z'$ are Bob’s measurements. To ensure that the same key can be generated, Alice and Bob inform each other of their measurement choices. In the worst case, the communication between Alice and Bob is completely overheard by an eavesdropper Eve who try to get the key. Even in this case, Alice and Bob can still generate a secure key if their measurement outcomes are sufficiently relevant, which can be revealed by the stronger bound of quantum secret key rate $K'$.

\section{DISCUSSIONS AND CONCLUSIONS}

We have derived a lower bound of tripartite uncertainty relation with quantum memory by adding an additional term related with mutual information and Holevo quantity. It has been demonstrated that our lower bound outperform the previous bound to some extent. We prove that our lower bound will completely coincide with the total preparation uncertainty, with regard to $\sigma_x$ and $\sigma_z$ as the incompatibility in the framework of an arbitrary three-qubit X-structure state. For an arbitrary tripartite pure state, since the quantity $\Delta$ is less than or equal to zero at all, our lower bound can recover Renes et al.’s lower bound. As illustrations, we specifically take into account the tripartite uncertainty relation for generalized GHZ state, generalized $W$ state, Werner-type state and random three-qubit states. By the analytical calculation, it testifies that our lower bound is tighter than the previous [41]. This supports that the communicators in quantum key distribution can improve the security bounds by employing our derived result. It will bring on more potential applications in the further quantum communication. For examples, in the monogamy game, our new lower bound can more precisely capture the tradeoff of entanglement monogamy during quantum information processing, which enhance the precision of prediction of measurement outcome [50]. It means that the new tripartite uncertainty relation should have an important application in precision measurements. For another application, the new tripartite uncertainty relation implies the tighter security bounds on the ability that the eavesdropper Eve predicts the measurement outcome of $Z$ measurement on subsystem $A$ in the case where the additional term $\Delta > 0$. Furthermore, the bound of quantum secret key rate can be strengthened by utilizing the new lower bound of tripartite uncertainty relation. In this sense, even though in the case that the eavesdropper Eve overhears the measurement outcomes in the communication of Alice and Bob, a secure key may still be generated between them. Thereby, our security argument can effectively improve the security of quantum key distribution schemes.

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