On the Problem of Spin Diffusion in 1D Antiferromagnets

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Abstract

We study the problem of spin diffusion in magnetic systems without long-range order. We discuss the example of the 1D spin chain. For the system described by the Heisenberg Hamiltonian we show that there are no diffusive excitations. However, the addition of an arbitrarily small dissipation term, such as the spin-phonon interaction leads to diffusive excitations in the long time limit. For those excitations we estimate the spin-diffusion coefficient by means of the renormalisation group analysis.
I. INTRODUCTION

The spin dynamics in magnetic systems without long-range order is a long-standing problem. It is generally believed [1,3] that in the high temperature limit, where no long-range order is present, the microscopic spin fluctuations are governed by the diffusion equation, i.e. that at small frequencies and momenta the spin correlation function has a diffusive pole

\[ < SS > = g(\omega, k) = \frac{Z}{i\omega + Dk^2} \]

where \( D \) is the diffusion constant and \( Z \) is the residue.

Although this idea of spin diffusion is quite common, we are not aware of any theoretical approach, within which one could ”derive” the correlation function Eq. (1), starting from the nearest neighbour Heisenberg Hamiltonian

\[ H_H = \sum_i J_i \vec{S}_i \cdot \vec{S}_{i+1}. \]

In this paper we are presenting such an approach for 1D spin-1/2 Heisenberg chains of the infinite length. At any non-zero temperature the chain is in disordered state with exponentially decaying spin-spin correlations and even at \( T = 0 \) the correlations decay as a power law, so there is no true long-range order. We show that in one dimension it is impossible to derive the diffusive form of the spin-spin correlation function Eq. (1) from the Heisenberg Hamiltonian Eq. (2), without any kind of additional dissipation mechanism. We also present a general argument supporting this result. We show then, that if one takes into account an additional dissipation, for example due to spin-phonon interaction (which is always present in any real system at finite temperature), then the renormalisation group approach leads to the correlation function Eq. (1) in the long-time asymptotic.
Our results could be applied to the materials like $KCuF_3$, $CuSO_4 \cdot 5H_2O$, $Sr_2CuO_3$. In a broad temperature range they are nearly ideal 1D antiferromagnetic chains with the coupling constant $J$ ranging from 1.45 K in $CuSO_4 \cdot 5H_2O$ [1] to 190 K in $KCuF_3$ [2,3] and 1300 K in the novel $Sr_2CuO_3$ [4]. The diffusion equation was successfully used in a number of experimental papers to fit the data and explain the results of the experiments [1,5]. This approach was also confirmed by computer simulations [6].

The rest of this paper is organized as follows. In Section II we briefly review the mapping of the spin system onto 1D fermions. In the Section III we map the fermions onto a boson system, justifying bosonisation procedure at finite temperatures. Section IV gives the calculation, which shows the absence of spin diffusion in the Heisenberg model Eq. (2) and presents some general qualitative arguments, supporting this conclusion, based on the theory of the Sine-Gordon equation. Section V discusses the spin-phonon interaction and its renormalisation. Section VI is conclusions and discussion of results. The details of the calculations are presented in Appendices.

II. FERMION MODEL

In this section we review the well-known procedure of mapping the Heisenberg model Eq. (2) on a system of spinless fermions [7] and establish the notations.

The spin model Eq. (2) can be transformed into a model of spinless fermions, noting that operators $S^+_i$ and $S^-_i$ anticommute. The Jordan-Wigner transformation then relates spin to fermion operators ($a_i, a^\dagger_i$) via

$$S^+_i = a^\dagger_i \exp \left( i \pi \sum_{j=1}^{i-1} a^\dagger_j a_j \right),$$

$$S^z_i = a^\dagger_i a_i - \frac{1}{2}$$

(3)
When transforming the Hamiltonian Eq. (2) the spin-flip terms give rise to the motion of the fermions (kinetic energy in the fermion Hamiltonian) and \( S_z^i \cdot S_z^{i+1} \) interaction leads to a fermion-fermion interaction between adjacent cites. Since \( S_z^i \) is quadratic in fermion operators, the interaction between fermions is the four-particle interaction. Since the original spin model was formulated on a lattice, all possible types of four-fermion interaction are present, including the umklapp term.

The 1D fermion models are often treated using bosonisation [9]. In the case of massless fermions with the four-fermion interaction with small momentum transfer it allows exact solution. It is shown, that in the thermodynamic limit that system can be mapped onto a system of free bosons. The propagator of a free boson has a pole at \( \omega = ck \) and therefore corresponds to the particle, propagating without dissipation, so that this interaction can not lead to any significant change in the long-time dynamics of the system. Therefore the only source of dissipation might be interaction with large momentum transfer, such as the umklapp term.

The treatment of the umklapp term is much more complicated because in the boson language it corresponds to the highly non-linear term, namely \( \cos(2\beta\phi) \) (where \( \phi(x, t) \) is the boson field and the number \( \beta \) is a parameter of the transformation, depending on the fermion interaction with small momentum transfer). This term does not conserve momentum and is only marginally irrelevant, so at \( T > 0 \) it might lead to some new dynamics. Thus we should investigate the impact of that term on the long-time fermion (and thus spin) dynamics, disregarding all other possible four-fermion interaction terms. The next two Sections present the results of this investigation.

Since we are interested only in the low-energy behaviour of the system, we can linearize the fermion kinetic energy, thus allowing exact bosonisation. Therefore our model Hamiltonian is

\[
H_m = v_F \sum_k \left[ (k - k_F)\psi_R^\dagger \psi_R + (-k - k_F)\psi_L^\dagger \psi_L \right] + V \sum \left( \psi_R \psi_R \psi_L \psi_L + h.c. \right),
\]

(4)
where $\psi_{R(L)}$ is the operator of a "right" ("left") mover. The quantity we are interested in here is the density-density correlation function, which corresponds to the $\langle S_i^z \cdot S_j^z \rangle$ correlator in the spin problem. We shall now investigate, whether the long-time asymptotic of this correlator has the diffusive form, Eq. (1).

III. BOSONISATION

As we mentioned above our main technical tool will be bosonisation. The procedure is well established in 1D at zero temperature. We consider finite temperatures and we are looking for the long-time asymptotic of the spin-spin correlator. It is quite difficult to obtain the results in that limit in the Matsubara technique, since the analytic continuation from the Matsubara frequencies $\omega_n$ to real frequencies much less that the inverse temperature would require a precise knowledge of the Green’s functions on the infinite range of $\omega_n$, which is usually not the case in perturbation theory. Therefore we have to resort to the Keldysh technique [10], which incorporates finite temperatures and real-time representation. In this Section we construct the bosonisation procedure for the Keldysh technique and then confirm it’s correctness by comparing the results of the perturbation theory for bosons and fermions. The fermion and boson Green’s functions in space-time representation, which we are using in our calculations, are presented in Appendix A.

Thirty years ago Keldysh has presented a field theoretical technique to calculate the real-time correlation functions of a quantum system. To allow the treatment of advanced and retarded correlators, Keldysh introduced the time contour $C$ (Fig. 1) with the upper branch going in positive direction from $-\infty$ to $+\infty$ and the backward lower branch. All the operator products are now time-ordered along the contour $C$. To distinguish particles
on the upper and the lower branches of the contour the fermion field operator $\psi_\gamma$ is given an index $\gamma$, which equals to 1 on the upper branch and 2 on the lower. Green’s functions become $2 \times 2$ matrices with respect that index.

In a one dimensional fermion problem we have four different operators $\psi_\gamma^{L(R)}$ - left and right movers on both branches of the contour. Since operators on each separate branch are completely analogues to the zero-temperature operators, we can proceed with the bosonisation separately on each branch in exactly the same way as at $T = 0$. Thus we introduce two boson fields, $\phi_\gamma$, (one for each branch), which we shall treat as two components of the Keldysh field. The resulting bosonised Hamiltonian will thus be formulated in the Keldysh technique also.

The left $\phi_\gamma^L$ and right $\phi_\gamma^R$ moving bose fields expressed via $\phi_\gamma$ and its canonically conjugate $P_\gamma$.

$$
\phi_\gamma^{L(R)}(x) = \frac{1}{2} \left[ \phi_\gamma(x) \mp \int_{-\infty}^{x} P_\gamma(x') dx' \right] \\
= \pm \int_{0}^{\infty} \frac{dp}{2\pi \sqrt{2|p|}} e^{-\alpha |p|} \left[ \phi_\gamma(p) e^{ipx} + h.c. \right].
$$

(5)

As in the usual procedure $\phi_\gamma^{L(R)}$ are functions of only $(x \mp t)$.

The fermion operators are constructed in analogy with the zero temperature case

$$
\psi_\gamma^{L(R)} \sim \frac{1}{\sqrt{\alpha}} \exp \left( \pm i\beta \phi_\gamma^{L(R)} \right)
$$

(6)

where $\beta^2 = 4\pi$ and the upper sign corresponds to the left mover.

The commutation relations between fermion fields hold by exactly the same reason as at $T = 0$. The fact that we have a different time contour (the Keldysh contour $C$ as opposed
to the usual time axis) does not change the calculation, for the integrals involved in Eq. (5) are over space coordinate and the bose fields commute no matter which branch of the time contour they are on. Another way of saying this is that the Keldysh operators on different branches still correspond to the same particle. Dividing the time contour in two parts is a matter of mathematical convenience rather than physical distinction.

The cut-off \( \alpha \) is a lattice spacing and so should be the same for both bosons and fermions. The operator equality Eq. (6) means that any correlation function (in the limit \( \alpha \to 0 \)), calculated in the fermi theory with the cut-off \( \alpha \), is reproduced in the bosonic theory with the same cut-off if the fermion operator in the left-hand side is replaced by the bosonic operator in the right-hand side of Eq. (3). Using this operator equivalence we can construct the boson Hamiltonian from the fermion Hamiltonian Eq. (4). The fermion kinetic energy corresponds to that of bosons. The umklapp interaction term gives rise to the cosine interaction of the boson field. The conjugate operators \( P_+ \) and \( P_- \) cancel out, so the interaction is a function only of the boson fields \( \phi_+ \) and \( \phi_- \) itself, as it is in the zero-temperature bosonisation. The interaction constant \( V \) now acquires the factor \( \frac{1}{\alpha^2} \) from the prefactor in Eq. (6). The boson Hamiltonian therefore is

\[
H_B = (\partial_\mu \phi)^2 + V' \cos 2\beta \phi. \tag{7}
\]

where \( V' = \frac{V}{\alpha^2} \).

Due to the special form of the interaction (\( \cos 2\beta \phi \)) it is most convenient to formulate the Keldysh technique in the path integral representation, developed by A. Schmid [11]. The boson action for our model is

\[
A = \int dx dt \left[ \phi_+^\dagger B_{\gamma\nu} \phi_\nu - V'(\cos 2\beta \phi_+ - \cos 2\beta \phi_-) \right], \tag{8}
\]
where $\phi_+$ and $\phi_-$ are the two components of the Keldysh boson field, $\gamma$ and $\nu$ denote Keldysh indices. $B_{\mu\nu}$ is the kinetic operator, so that the inverse $B_{\mu\nu}^{-1}$ is proportional to the boson Green’s function in the Keldysh basis

$$B_{\mu\nu}^{-1} = -i \begin{pmatrix} D^{++} & D^{+-} \\ D^{-+} & D^{--} \end{pmatrix}$$

We shall later use these functions in the space-time representation (see Appendix A).

The fermion density in the conventional bosonisation is represented by the spatial derivative of the boson field $\varphi$

$$\rho = \frac{1}{\sqrt{\pi}} \partial_1 \varphi. \tag{10}$$

Here in the same manner we write the fermion density operators. The density-density correlators are represented by functional integrals in which the preexponential is some linear combination of the spatial derivatives of the bose fields corresponding to that particular correlator. The advanced correlator is

$$\langle \rho \rho \rangle_A = \Pi(x_1 - x_2) = \frac{1}{Z} \int \mathcal{D}[\phi_1, \phi_2] \ \partial_1 \phi^*_\alpha(x_1) \tau_{\alpha\beta} \partial_1 \phi_\beta(x_2) \ \exp(-A), \tag{11}$$

where the density vertex $\tau_{\alpha\beta}$ is

$$\tau_{\alpha\beta} = \frac{1}{\pi} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{12}$$

We now check, following the approach of R. Shankar [9], that the above construction yields correct results in perturbation theory. The density-density correlation function for
non-interacting fermions should be calculated separately for the "left" and "right" movers and the results should be added. That gives for the advanced correlator

$$\Pi_0 = \frac{2}{\pi} \frac{q^2}{\omega^2 - q^2 - 2i\delta\omega}, \quad (13)$$

where $\delta$ is infinitesimally small. In the next order we have two topologically nontrivial diagrams given on Fig. 2 Again we have to repeat the calculation for the "left" movers and add the results with proper combinatorial factors. For simplicity we give here only the imaginary part of the first-order correction, which for us is most important

$$Im \Pi_1 = \frac{V^2}{2\pi^2} \frac{q^2}{\omega^2 - q^2} \frac{\omega T}{\omega^2 - q^2}, \quad (14)$$

The next order of perturbation is discussed in the next Section.

We now turn to the calculation of the same perturbation series on the boson language. In the non-interacting case we have to calculate a simple Gaussian integral which in momentum space gives

$$\Pi_0(\omega, q) = \frac{2}{\pi} q^2 D_A(\omega, q) = \frac{2}{\pi} \frac{q^2}{\omega^2 - q^2 - 2i\delta\omega}, \quad (15)$$

which is the same as Eq. (13). $D_A(\omega, q)$ is the advanced boson Green’s function, connected to the original Keldysh basis via
\[ D_A = D^{--} - D^{+-}. \] (16)

We now expand the exponent in the integral Eq. (11) in the series in \( V' \). In the first non-trivial order we get

\[
\Pi_1(x_1 - x_2) = V'2 \int d^2y_1d^2y_2 \frac{1}{Z} \int \mathcal{D} [\phi_1, \phi_2] \partial_1 \phi_\alpha^*(x_1) \tau_{\alpha\beta} \partial_1 \phi_\beta(x_2) \exp \left( - \int dxdt \phi_\mu^* B_{\mu\nu} \phi_\nu \right) \\
\left[ \cos 2\beta \phi_+ (y_1) - \cos 2\beta \phi_- (y_1) \right] \left[ \cos 2\beta \phi_+ (y_2) - \cos 2\beta \phi_- (y_2) \right]. \] (17)

Since the cosines contribute the linear terms in the exponent, the integral is still Gaussian with the same prefactor of the exponent. This feature will remain in all higher orders of the perturbation series. The calculation of that integral we shall discuss in more detail due to its importance for the later arguments. In the first-order integral Eq. (17) we have four different terms of the same type, arising from the cosines. In the exponent they have combinations like \( 2\beta (\phi_1(y_2) - \phi_2(y_1)) \) with all possible permutations of indices. To calculate the functional integral we make the Fourier transform of the bose fields. We perform the transform in the most general way, since the same expressions will appear later. As we show below in any order of the perturbation theory we shall need to calculate averages of the form

\[
\langle \partial_1 \phi_\alpha^*(x_1) \tau_{\alpha\beta} \partial_1 \phi_\beta(x_2) \exp \left( \int \frac{d^2k}{(2\pi)^2} \langle^{(r)} I_\mu^*(k, y_1) \phi_\mu(k) + h.c. \rangle \right) \rangle. \] (18)

For the first-order integral Eq. (17) we four different factors \( \langle^{(r)} I_\mu^*(k) \rangle \):
Completing the square in the exponent we calculate the functional integral in Eq. (17) and apart from the numerical factor we get in the momentum space

\[
\Pi_1(\omega, q) \sim \int d^2k \ kq \prod_\{\{r\} \sum_{\{r\}}^{(r)} I_\gamma^*(k, y) B_{\alpha\gamma}^{-1}(q) \tau_{\alpha\beta} B_{\beta\nu}^{-1}(k) \ (r) I_\nu(k, x) K_1(y),
\]

(20)

where

\[
K_1(y_i) = \exp\left(\int \frac{d^2k'}{(2\pi)^2} \ (r) I_\gamma^*(k', y) B_{\gamma\delta}^{-1}(k') \ (r) I_\delta^*(k', y_i) \right),
\]

(21)

and the sum is over four different factors \((r) I_\nu(k, y_i)\) corresponding to the four different terms of the type (18) in the first order integral Eq. (17).

Here the integration measure \(\prod d^2y_i\) is equal \(dy_1dt_1dy_2dt_2\). All dependence on these variables is contained in the factors \((r) I_\nu(k, y_i)\). These factors are nothing but exponentials \(\beta e^{iky}\) with different signs. Therefore the integral in the exponent in Eq. (21) is just the Fourier transform of the boson Green’s function back from momentum space to the real space. The boson functions in real space are logarithms, so all four exponents \(K_1(x_i)\) can be evaluated. All four have the same structure

\[
K_1(y = y_1 - y_2) = \frac{f(y, t)}{S(y, t)},
\]

(22)

where
\[ S^{-1}(y, t) = \frac{(\pi T \alpha)^4}{\sinh^2 \pi T(y - t) \sinh^2 \pi T(y + t)}, \quad (23) \]

while \( f(y, t) \) is some algebraic function, which has no poles. In the long-time asymptotic \( K_1(y, t) \) acts like a derivative of a delta-function. It has (due to the denominator Eq. (23)) a sharp singularity at \( y, t = 0 \) and decays very fast as the variables go away from that point. Therefore the integral over \( y, t \) is dominated by the small region of the size \( \alpha \) around the origin. The remaining spatial integration over the sum \( y_1 + y_2 \) involves only functions \( ^{(r)}I^*_\mu(k, y_i) \) from the prefactor since \( K_1(y_i) \) depends only on the difference of the variables \( y_1 \) and \( y_2 \). Since \( ^{(r)}I^*_\mu(k, y_i) \) are exponentials the integral is easily evaluated to yield \( \delta(k - q) \), which solves the momentum integration. So after the summation over all four terms we get the result Eq. (14), as expected.

The next orders of the perturbation series, which we discuss in the following Section, could be calculated in the same manner and are also the same for the boson and the fermion formulations of the theory. The bosonisation procedure is thus justified.

IV. ABSENCE OF DIFFUSION FOR THE HEISENBERG MODEL

Although the boson and the fermion versions of the theory are completely equivalent and should give the same results in the perturbation theory, the bosonised version allows easier evaluation or the higher orders of the perturbation theory. Indeed, all the higher order corrections differ from the first order Eq. (17) only by appearance of additional cosine brackets (and corresponding spatial integrations). That means that in any order of perturbation we have to calculate averages of the form Eq. (18). The functions \( ^{(r)}I^*_\mu(k, x_i) \) will be now sums of exponentials, for example in the next order of perturbation we will have terms like
\[(r) I_\mu(k, x_i) = \beta \left[ e^{ikx_1} - e^{ikx_2} + e^{ikx_3} - e^{ikx_4} \right], \quad (24)\]

similar to the first order factors Eq. (19), but constructed from four different exponentials. We can still perform the functional integration and get the same general formula Eq. (20). Since functions \[(r) I_\mu(k, x_i)\] are exponentials the integral in the exponent in Eq. (20) still yields the boson Green’s functions in the space-time representation, but now instead of one such a function as in the first order we have a sum of them. In the second order we shall get

\[K_2(x_1, \ldots x_4) = \frac{S(x_2-x_3)S(x_1-x_4)}{S(x_1-x_2)S(x_1-x_3)S(x_2-x_4)S(x_3-x_4)} f_2(x_1, \ldots x_4), \quad (25)\]

where \(S(x)\) is the "singular" denominator, defined in Eq. (23).

In higher orders functions \(K\) have the same structure, but with more factors \(S(x)\) in the numerator and denominator. These factors have the same singular behaviour as described in the previous Section. Therefore the integration over the multidimensional space will be dominated by the regions where two pairs of variables are almost equal to each other, for example \(x_1 = x_2\) and \(x_3 = x_4\). The integration over one such pair effectively reduces the function Eq. (25) to the form of the previous order Eq. (22) (in the long-time asymptotic). One can see it by inspecting Eq. (25). Consider, for instance, the contribution from the region \(x_1 = x_2\). In this region \(K_2 \sim S^{-1}(x_1-x_2)S^{-1}(x_3-x_4)\), so the integral over \(x_2\) leads to the same form (\(\sim 1/S(x_3-x_4)\)), which is the main part of the first order function \(K_1\) Eq. (22). The integration over \(x_3\) and \(x_4\) is then the same as in the first order. Therefore the spatial integration in in the higher order terms does not yield any additional singularity to that produced by the prefactor of the exponent, which is the same as in the first order Eq. (20) and contains just two boson Green’s functions. So the structure of any term in the perturbation series is
\[ \text{Im} \Pi_1(\omega, q) \sim P \frac{\omega q^2}{(\omega^2 - q^2)^2} g(\omega, q). \]  

(26)

Here \( g(\omega, q) \) is some function, which has no poles at small \( \omega, q \). We note that \( \text{Im} \Pi_1(\omega, q) \) differs from the first order Eq. (17) only by \( g(\omega, q) \), and \( P \) denotes principal value.

Combining our conclusions we can write down the structure of the density-density correlation function in our model.

\[
\Pi(\omega, q) = \frac{2}{\pi} \frac{q^2}{\omega^2 - q^2 + 2i\delta \omega} + i \frac{\omega T q^2}{(\omega^2 - q^2)^2} g(\omega, q, V) \]  

(27)

The significance of this result for us is in the fact, that the higher order corrections do not acquire additional poles in the imaginary part, which could sum up in a diffusive pole in \( \Pi(\omega, q) \).

The fermion perturbation theory gives the same results in the low orders of the perturbation. In the second non-trivial order we have six topologically different diagrams, presented on Fig. 3. Our purpose is to show that the correction to the density-density correlation function, which is the sum of these diagrams, is not qualitatively different from the first order result. The only diagrams that produce the extra pole are the diagrams \( a \) and \( b \) on Fig. 3. They cancel each other exactly, so that the second order correction has the pole of the same order as the first order one. This cancellation seems to be an accidental property of the problem in the fermion representation, but the bosonisation approach shows that it happens in all orders of perturbation theory.

Our results clearly show the absence of spin diffusion in the Heisenberg model. Instead of the diffusive pole in the density-density correlation function we found some kind of a propagating behaviour. Note that the second order pole in Eq. (27) should be regarded as a principal value, so the imaginary part of general susceptibilities does not contain unphysical
contributions proportional to $\delta^2(\omega^2 - q^2)$. This result should have been expected. In the boson representation our problem is essentially the Sine-Gordon model. It is known in the theory of the Sine-Gordon equation that due to the infinite number of conserved charges the excitations of the model are propagating and not diffusive. So by showing the mapping of the Heisenberg model onto the Sine-Gordon model we have showed the absence of spin diffusion in the model Eq. (2).

V. SPIN-PHONON INTERACTION

The Heisenberg model Eq. (2) does not exactly describes the physics of a real material. We now try to make it a bit more realistic. Even in the absence of disorder, at finite temperatures there always are phonons in the system. The exchange integral depends in general on the instantaneous separation of magnetic ions. We consider the linear approximation for spin-phonon coupling. That is we expand the separation-dependent Heisenberg coupling constant to the first order in ionic displacement. $u(R_i)$

$$J(r_i - r_j) = J_0(R_i - R_j) + \chi (u(R_i) - u(R_j)),$$  \hspace{1cm} (28)

where $R_i$ is the equilibrium position of the ion and $r_i = R_i + u(R_i)$. That leads to the simplest form of the spin-phonon interaction Hamiltonian

$$\mathcal{H}_{sp-ph} = J' \sum_{<ij>} \vec{S}_i \cdot \vec{S}_j \left( b_i^\dagger b_i + b_i b_i \right).$$  \hspace{1cm} (29)
We shall now try, treating interaction Eq. (29) as a small perturbation of the original Hamiltonian Eq. (2), to examine it’s impact on the long-time spin dynamics. To do that we first bosonise Eq. (29). We get

$$\mathcal{H}_{b-ph} = J' \sum_k |\phi_k|^2 \left( b_k^\dagger + b_k \right).$$

The boson self-energy in the first non-trivial order in $\mathcal{H}_{b-ph}$ is presented by the diagram on Fig. 4. The solid line represents a boson and the dashed - a phonon. For the imaginary part we are interested in we get

$$Im \Sigma(\omega, q) = \begin{cases} \frac{\eta \omega q v_F}{2}, & if \ q > T \\ \frac{\eta \omega T}{2}, & if \ q < T \end{cases}$$

where restoring the units $\eta \sim (J'/c)^2$ and $c$ is the speed of sound. The momentum dependence of $Im \Sigma(\omega, q)$ arises due to the momentum in the numerator of the phonon Green’s function

$$D_{ph}(\omega, q) = \frac{c^2 q^2}{\omega^2 - c^2 q^2 + i\delta \omega}.$$ 

where $\delta$ is infinitesimally small. Note that although we consider 1D spin chain, the phonons in a real material are three-dimensional, so when evaluating the self-energy Eq. (31) one must integrate out the two other components of the phonon momentum.

We shall now investigate, how the umklapp term renormalizes this self-energy. We now divide the boson field into "slow" and "fast" parts, separated by some cut-off $k_0$. We integrate out all "fast" degrees of freedom (those with momentum bigger the cut-off) and see how the imaginary part of the boson self-energy (namely the coefficient $\eta$) changes with
the cut-off. The result is that when the cut-off is bigger than the temperature, \( T \), \( \eta \) rises as some power of the cut-off. But after the cut-off becomes smaller than the temperature, the imaginary part grows exponentially. That means, that on the large distances the motion becomes diffusive. We notice that it should happen in any experimental situation, because it is just the existence of the spin-phonon interaction (no matter how small) is responsible for the diffusion.

After we introduce the "slow" and "fast" variables as

\[
\phi_\mu = \phi_\mu^s + \phi_\mu^f.
\]  

(33)

The integral Eq. (11) then is

\[
\Pi(x_1 - x_2) = \frac{1}{Z} \int \mathcal{D} [\phi_1^s, \phi_2^s] \mathcal{D} [\phi_1^f, \phi_2^f] \partial_1 \phi_\alpha^{**}(x_1) \tau_{\alpha\beta} \partial_1 \phi_\beta^s(x_2) \exp(-A),
\]  

(34)

where the prefactor contains only slow degrees of freedom since we are looking for the infrared asymptotic. We can separate the integral over fast variables and the density integral now becomes

\[
\Pi(x_1 - x_2) = \frac{1}{Z'} \int \mathcal{D} [\phi_1^s, \phi_2^s] \partial_1 \phi_\alpha^{**}(x_1) \tau_{\alpha\beta} \partial_1 \phi_\beta^s(x_2)
\]

\[
\exp \left( \int \frac{d^2 k'}{(2\pi)^2} \left[ \phi_\mu^{**} B_{\mu\nu} \phi_\nu^s \right] \right) \exp (\ln M),
\]  

(35)

where \( M \) is the integral over the fast variables, which in the first non-trivial order in perturbation is a Gaussian integral without any prefactor.
\[ M = \frac{V'^2}{2} \int D[\phi_1^f, \phi_2^f] \exp \left( \int \frac{d^2k'}{(2\pi)^2} \left[ \phi_{\mu}^f B_{\mu\nu} \phi_{\nu}^f \right] \right) \int d^2x_1 d^2x_2 \]

\[
\left[ \cos 2\beta \left( \phi_s^s(x_1) + \phi_1^f(x_1) \right) - \cos 2\beta \left( \phi_2^s(x_1) + \phi_1^f(x_1) \right) \right] \\
\left[ \cos 2\beta \left( \phi_1^s(x_2) + \phi_2^f(x_2) \right) - \cos 2\beta \left( \phi_2^s(x_2) + \phi_2^f(x_2) \right) \right]. \tag{36}
\]

In functional integral Eq. (35) \( \ln M \) plays a role of renormalisation of the imaginary part of the kinetic operator. Now we calculate the functional integral and expand the result in the boson fields \( \phi^s_1 \) to get their bilinear combination, which gives the renormalisation of \( \eta \). We get

\[ M = \left( \frac{V'\beta}{2} \right)^2 \int d^2x_1 d^2x_2 \left[ \phi_s^s(x_1) R_{\alpha\beta}(x_{12}) \phi^s_\beta(x_2) \right], \tag{37} \]

where as usual \( x_{12} = x_1 - x_2 \). The elements of the matrix \( R_{\alpha\beta}(x) \) are the exponentials of the boson Green’s functions in the space-time representation. They are presented in Appendix B, where we give the detailed renormalisation calculation. The boson Green’s functions entering the matrix \( R_{\alpha\beta}(x_{12}) \) are now different from those in Appendix A, due to the self-energy Eq. (31). The retarded function in the momentum space is now

\[ D_R(\omega, q) = \frac{1}{\omega^2 - q^2 + i\text{Im} \Sigma(\omega, q)} \tag{38} \]

and the corresponding functions \( D_A(\omega, q) \) and \( D_F(\omega, q) \) acquire the same self-energy. Note, that \( R_{\alpha\beta}(x_{12}) \) is expressed via boson Green’s functions in the not rotated Keldysh basis. However, the coefficient \( \eta \) is easier to extract from the Green’s functions in the rotated
Keldysh basis Eq. (52). The element $R_R$ of the matrix $R_{\alpha\beta}$ in the rotated basis which corresponds to the renormalisation of the retarded function is

$$R_R = \frac{1}{2} \left( R_{11}(x) - R_{22}(x) - R_{12}(x) + R_{21}(x) \right). \tag{39}$$

If the cut-off is large, $v_F k_0 > T$, this yields after the Fourier transform

$$R_R = i \omega \eta \frac{k^3}{k_0^3} \exp(-\frac{\beta^2}{2\pi} \ln k_0 \alpha), \tag{40}$$

so that the change in $\eta$ is

$$\Delta \eta = \eta \frac{V^2}{4(k_0 \alpha)^4}, \tag{41}$$

since $\beta = \sqrt{4\pi}$. Here as always, $\alpha$ is the lattice spacing which cuts-off the large momentum integration. So here we have a power-law rise in $\eta$. When the cut-off is less that the temperature we get exponential renormalisation:

$$\Delta \eta = \eta_T \exp \left( 4 \frac{T}{v_F k_0} \right) \frac{V^2}{4(k_0 \alpha)^4}. \tag{42}$$

Here $\eta_T$ is the effective damping at the scales $v_F k_0 \sim T$. That means, that on the scales of momenta less that the temperature, the imaginary part coefficient $\eta$ grows very rapidly, and we immediately get the diffusive dynamics. The renormalisation group treatment breaks down when the correction Eq. (42) becomes of the order of unity. That determines the length scales, where the dynamics becomes diffusive. The mean-free path is
\[ l \sim \frac{v_F}{T} \ln(\eta_0 V^2), \quad (43) \]

where \( \eta_0 \) is the original value of the coefficient \( \eta \), proportional to the spin-phonon coupling constant (Eq. (31)).

The estimate of the mean-free path allows us to estimate the diffusion coefficient as
\[ D = l v_F, \quad \text{so that} \]
\[ D \sim \frac{v_F^2}{T} \ln\left(\frac{J/V}{c}\right). \quad (44) \]

VI. CONCLUSIONS

We now briefly review our results. We started from 1D Heisenberg Hamiltonian Eq. (4). Our goal was to calculate the spin-spin correlation function at long times and non-zero temperature, and check whether or not it had a diffusive pole Eq. (1) in some region in phase space. To perform such a calculation we mapped our original problem onto 1D fermion model Eq. (4) using the Jordan-Wigner transformation Eq. (3). We used the usual diagrammatic technique to calculate a few first orders of the perturbation theory (shown on Fig. 2 and Fig. 3). In order to go beyond that simple approximation we bosonised the fermion model. To get the long-time asymptotic of the spin-spin correlator, we had to combine bosonisation with Keldysh technique, which allows one to calculate real-time correlation functions at finite temperatures without any need in analytic continuation.

The bosonisation procedure allowed us to find the general form of higher order corrections and to sum up the perturbation series. It turned out that in each order of perturbation the
correction to the spin-spin correlator had the same form Eq. (26), therefore we concluded, that the exact spin-spin correlation function do not acquire the diffusive pole from the summation of the perturbation series. This result is also known in the theory of the Sine-Gordon model (which coincides with our boson model), where it has been found that due to the infinite number of conserved charges excitations are propagating and not diffusive.

We also check whether these results (the absence of spin diffusion in the perturbation theory) are robust with respect to small dissipation effects present in real physical systems. Specifically, we considered the effect of a weak spin-phonon interaction Eq. (29). We mapped the full problem (including the spin-phonon interaction) to the bosonic model. We found that the interaction with phonons leads to the boson self-energy Eq. (31), the imaginary part of which is proportional to the constant, $\eta$, at small momenta. Further, we applied the renormalisation group analysis; we integrated out “fast” degrees of freedom and showed that this constant $\eta$ grows moderately while the cut-off is larger than temperature (Eq. (41)), but grows exponentially after the cut-off becomes smaller than the temperature (Eq. (42)). At scales where the imaginary part of the spin-spin correlator becomes of the order of the real part the spin dynamics becomes purely diffusive. By associating the mean-free path with the scale on which the renormalisation procedure breaks down (namely, the renormalisation Eq. (42) becomes of the order of unity) we estimate the diffusion coefficient Eq. (44). Restoring the original units and estimating the spin-phonon coupling constant $J'$ from the expansion of the exchange integral Eq. (28), we have

$$D \sim \frac{\pi^2 (J\alpha)^2}{\hbar k_B T} \ln \left(\frac{J\alpha}{\hbar c}\right).$$

(45)

where $\alpha$ is the lattice spacing and $c$ is the speed of sound.

Thus we found that the presence of the spin-phonon interaction changes the long-time behaviour of the spin-spin correlator, which becomes diffusive.
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APPENDIX A

We present here the boson and fermion Green’s functions, which we are using in our calculations. It is the certain similarity between them, that had inspired the bosonisation. As in the usual zero-temperature bosonisation we need the Green’s functions in the space-time representations. We perform the bosonisation in the original Keldysh basis, but for simplicity we here calculate Green’s functions in the rotated basis and then transform them back.

The retarded fermion Green’s function in the momentum space is

\[ G_R(\epsilon, k) = \frac{1}{\epsilon + k + i0} \]  \hspace{1cm} (46)

where ”-” is for the ”left” and ”+” for the ”right” movers. For the simplification of the formulas the Fermi velocity is set equal to unity. The Fourier integral, which one has to calculate in order to transform the function Eq. (46) to the real space is formally divergent at large momenta. As usual in 1D calculations we introduce a cut-off \( \alpha \) by adding the exponent \( e^{-\alpha |k|} \) to the integral. Thus we have
\[ G_R(x,t) = \int \frac{d\epsilon dk}{(2\pi)^2} e^{ikx - i\epsilon t} \frac{1}{\epsilon \mp k + i0} e^{-\alpha|k|}. \] (47)

Now we have the perfectly converging integral and get

\[ G_R(x,t) = \frac{-\theta(t)}{2\pi} \frac{2i\alpha}{(x \mp t)^2 + \alpha^2}, \] (48)

where

\[ \theta(t) = \begin{cases} 1, & \text{if } t > 0 \\ 0, & \text{if } t < 0 \end{cases} \]

Similarly, the advanced function is

\[ G_A(x,t) = \frac{\theta(-t)}{2\pi} \frac{2i\alpha}{(x \mp t)^2 + \alpha^2} \] (49)

The third Keldysh function in this basis for the right movers in momentum space is

\[ F(\epsilon, k) = -2\pi i \tanh \frac{k}{2T} \delta(\epsilon - k). \] (50)

The Fourier integral in converging and we get

\[ F(x,t) = \frac{T}{\sinh \pi T(x - t)}. \] (51)

The function for the left movers has the opposite sign of the time variable. To be more careful with the pole one has to add \( i\alpha \) to the space coordinate in Eq. (51).

That completes the calculation of the fermions Green’s functions in the rotated Keldysh basis

\[ G_{rot} = \begin{pmatrix} 0 & G_R \\ G_A & F \end{pmatrix}. \] (52)
To return to the original basis, which is needed for bosonisation, one has to perform the rotation

\[
G = \begin{pmatrix} G^{++} & G^{+-} \\ G^{-+} & G^{--} \end{pmatrix} = RG_{\text{rot}}R^{-1},
\]

where the rotation matrix is given by

\[
R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.
\]

We now calculate the boson functions. We start again with the rotated basis. The retarded Green’s function in the momentum space is

\[
D_R(\omega, q) = \frac{1}{\omega^2 - q^2 + 2i\delta\omega}.
\]

Again we have to introduce the cut-off \( \alpha \). It is essential for the purposes of bosonisation to do it in exactly the same way as for the case of fermions, Eq. (47). That way we get

\[
D_R(x,t) = -\frac{i}{4\pi} \theta(t) \ln\frac{(x-t+i\alpha)(x+t-i\alpha)}{(x-t-i\alpha)(x+t+i\alpha)}.
\]

For the advanced function we get the same logarithm

\[
D_A(x,t) = \frac{i}{4\pi} \theta(-t) \ln\frac{(x-t+i\alpha)(x+t-i\alpha)}{(x-t-i\alpha)(x+t+i\alpha)}.
\]
The third function in this basis, $D_F$ contains delta function as its fermion counterpart Eq. (50)
\[ D_F(\omega, q) = -\frac{i\pi}{\omega} \coth \frac{\omega}{2T} \left( \delta(\omega - q) + \delta(\omega + q) \right), \] (58)

but we have to introduce the cut-off here. In real space we get
\[ D_F(x, t) = \frac{i}{2\pi} \ln \frac{(\pi T \alpha)^2}{\sinh \pi T(x - t) \sinh \pi T(x + t)}, \] (59)

where the zeros of the denominator should be treated in exactly the same way as in the fermion case (see text after Eq. (51)).

One can clearly see, that the fermion functions and the arguments of the logarithm in the boson functions are constructed from the same elements. That is why the bosonisation works. To get exactly the same results in the fermion and boson perturbation series we have to turn to the original Keldysh basis Eq. (53) and calculate physical quantities, like the density-density correlation function, which is independent of Keldysh indices and therefore of the choice of the calculational technique. Then the boson and the fermion versions are the same in the limit $\alpha = 0$ (the physical quantities should not depend on that cut-off).

**APPENDIX B**

We start from the functional integral for the density-density correlation function Eq. (11), where the kinetic term contains now non-zero imaginary part Eq. (selfen). We are interested in the infrared asymptotic of that correlator. Therefore we divide the boson field $\phi_\mu$ in two part - "fast" and "slow"
\[ \phi_\mu = \phi^s_\mu + \phi^f_\mu. \] (60)

The integral Eq. (11) then becomes
\[ \Pi(x_1 - x_2) = \frac{1}{Z} \int \mathcal{D}[\phi^s_1, \phi^s_2] \mathcal{D}[\phi^f_1, \phi^f_2] \partial_1 \phi^s_{\alpha}(x_1) \tau_{\alpha\beta} \partial_1 \phi^s_{\beta}(x_2) \exp(-A), \quad (61) \]

where the prefactor contains only slow degrees of freedom since we are looking for the infrared asymptotic. We can separate the integral over fast variables as

\[ I = \frac{1}{Z''} \int \mathcal{D}[\phi^f_1, \phi^f_2] \exp \left( \int \frac{d^2k'}{(2\pi)^2} \left[ \phi^{f*}_\mu B_{\mu\nu} \phi^f_\nu \right. \right. \]

\[ -V' \left( \cos 2\beta \phi^s_1 \cos 2\beta \phi^f_1 - \sin 2\beta \phi^s_1 \sin 2\beta \phi^f_1 - \cos 2\beta \phi^s_2 \cos 2\beta \phi^f_2 
+ \sin 2\beta \phi^s_2 \sin 2\beta \phi^f_2 \right) \bigg] \bigg). \quad (62) \]

The density integral becomes now

\[ \Pi(x_1 - x_2) = \frac{1}{Z'} \int \mathcal{D}[\phi^s_1, \phi^s_2] \partial_1 \phi^s_{\alpha}(x_1) \tau_{\alpha\beta} \partial_1 \phi^s_{\beta}(x_2) \]

\[ \exp \left( \int \frac{d^2k'}{(2\pi)^2} \left[ \phi^{s*}_\mu B_{\mu\nu} \phi^s_\nu \right. \right. \]

\[ \exp \left( \int \frac{d^2k'}{(2\pi)^2} \left[ \phi^{s*}_\mu B_{\mu\nu} \phi^s_\nu \right) \exp \left( \ln M \right), \quad (63) \right. \]

where \( \ln M \) plays now a role of renormalisation of the imaginary part of the kinetic operator.

We now calculate the integral over the fast degrees of freedom Eq. (62). We expand the exponent in Eq. (62) up to the first non-trivial order in perturbation (which is actually all we need, noting the results of Section [IV]) and get the Gaussian integral without any prefactor

\[ M = \frac{V'^2}{2} \int \mathcal{D}[\phi^f_1, \phi^f_2] \exp \left( \int \frac{d^2k'}{(2\pi)^2} \left[ \phi^{f*}_\mu B_{\mu\nu} \phi^f_\nu \right) \right) \int d^2x_1 d^2x_2 \]

\[ \left[ \cos 2\beta \phi^s_1(x_1) \cos 2\beta \phi^f_1(x_1) - \sin 2\beta \phi^s_1(x_1) \sin 2\beta \phi^f_1(x_1) - \cos 2\beta \phi^s_2(x_1) \cos 2\beta \phi^f_2(x_1) 
+ \sin 2\beta \phi^s_2(x_1) \sin 2\beta \phi^f_2(x_1) \right] \]

\[ \left[ \cos 2\beta \phi^s_1(x_2) \cos 2\beta \phi^f_1(x_2) - \sin 2\beta \phi^s_1(x_2) \sin 2\beta \phi^f_1(x_2) - \cos 2\beta \phi^s_2(x_2) \cos 2\beta \phi^f_2(x_2) 
+ \sin 2\beta \phi^s_2(x_2) \sin 2\beta \phi^f_2(x_2) \right]. \quad (64) \]
We can now calculate the functional integral. The result is

\[
M = \frac{V^2}{4} \int d^2 x_1 d^2 x_2 \left\{ \cos 2\beta \phi_1^s(x_1) \cos 2\beta \phi_1^s(x_2) + \sin 2\beta \phi_1^s(x_1) \sin 2\beta \phi_1^s(x_2) \right. \\
\exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2 k'}{(2\pi)^2} D_k^- \left( 1 - e^{ikx} \right) \left( 1 - e^{-ikx} \right) \right) \\
+ \left. \left[ \cos 2\beta \phi_2^s(x_1) \cos 2\beta \phi_2^s(x_2) + \sin 2\beta \phi_2^s(x_1) \sin 2\beta \phi_2^s(x_2) \right] \right. \\
\exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2 k'}{(2\pi)^2} D_k^+ \left( 1 - e^{ikx} \right) \left( 1 - e^{-ikx} \right) \right) \\
- \left[ \cos 2\beta \phi_1^s(x_1) \cos 2\beta \phi_2^s(x_2) + \sin 2\beta \phi_1^s(x_1) \sin 2\beta \phi_2^s(x_2) \right] \right. \\
\exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2 k'}{(2\pi)^2} \left( D_k^{++} + D_k^{--} - D_k^{+-} e^{ikx} - D_k^{-+} e^{-ikx} \right) \right) \\
- \left[ \cos 2\beta \phi_2^s(x_1) \cos 2\beta \phi_1^s(x_2) + \sin 2\beta \phi_2^s(x_1) \sin 2\beta \phi_1^s(x_2) \right] \right. \\
\exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2 k'}{(2\pi)^2} \left( D_k^{++} + D_k^{--} - D_k^{+-} e^{ikx} - D_k^{-+} e^{-ikx} \right) \right) \right\}, \quad (65)
\]

where \( x = x_1 - x_2 \) and \( k_0 \) is the momentum cut-off, delimiting fast variables from slow.

The remaining integrals are similar to those, calculated in the regular perturbation series. We have the Fourier transform of the boson Green’s functions to the real space in the exponent, then take the exponential and Fourier transform back to the momentum space. The difference is that we now have another set of Green’s functions - with non-zero imaginary part - and the momentum integration in limited by the cut-off \( k_0 \). The result of this integration will now depend on the cut-off. For the retarded and advanced functions we get

\[
D_R(x, t; k_0) = \frac{i}{4\pi} \theta(t) e^{-\eta t} 2i \text{Im} \left\{ E_1 \left[ k_0 (\eta t - i(x + t)) \right] - E_1 \left[ k_0 (\eta t + i(x - t)) \right] \right\}, \quad (66)
\]
\[ D_A(x, t; k_0) = -\frac{i}{4\pi} \theta(-t) e^{-\eta|t|} 2i \text{Im}\left\{E_1\left[k_0 (\eta t - i(x + t))\right] - E_1\left[k_0 (\eta t + i(x - t))\right]\right\}, \]

(67)

where \( E_1 \) is the exponential integral. The result for the third Keldysh function \( D_F \) depends on whether the cut-off is larger or smaller than the temperature. For \( k_0 > T \) we get

\[ D_F(x, t; k_0 > T) = -\frac{i}{2\pi} \frac{1}{2} \text{Re}\left[E_1\left[k_0 (\alpha + i(x + t))\right] + E_1\left[k_0 (\alpha + i(x - t))\right]\right] \]

(68)

so that at the origin

\[ D_F(0; k_0 > T) = \frac{i}{2\pi} \ln(k_0\alpha) \]

(69)

which is a small number. For the other case, \( k_0 < T \), in the limit of large \( x \) and \( t \) and assuming that original \( \eta \) is much less than the temperature we have

\[ D_F(x, t; k_0 < T) = -\frac{iT}{\pi k_0} \left(\frac{\sin k_0(x + t)}{k_0(x + t)} + \frac{\sin k_0(x - t)}{k_0(x - t)}\right), \]

(70)

and at the origin is

\[ D_F(0; k_0 < T) = -\frac{i}{\pi k_0} \frac{T}{k_0}, \]

(71)

which is extremely big.

We can now proceed with the renormalisation of the imaginary part coefficient \( \eta \). To do that we expand Eq. (65) in boson fields and get their bilinear combination
\[ M = \frac{(V'\beta)^2}{4} \int d^2x_1 d^2x_2 \]

\[
\left\{ \phi^s_1(x_1)\phi^s_2(x_2) \exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2k'}{(2\pi)^2} D^-_k - \left(1 - e^{ikx}\right) \left(1 - e^{-ikx}\right) \right) \right.

\[ + \phi^s_2(x_1)\phi^s_1(x_2) \exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2k'}{(2\pi)^2} D^+_k + \left(1 - e^{ikx}\right) \left(1 - e^{-ikx}\right) \right) \]

\[ - \phi^s_1(x_1)\phi^s_1(x_2) \exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2k'}{(2\pi)^2} \left(D^+ + D^- - D^+e^{ikx} - D^-e^{-ikx}\right) \right) \]

\[ - \phi^s_2(x_1)\phi^s_2(x_2) \exp \left( i\beta^2 \int_{k' > k_0} \frac{d^2k'}{(2\pi)^2} \left(D^+ + D^- - D^+e^{ikx} - D^-e^{-ikx}\right) \right) \}

\text{(72)}

which can be abbreviated as

\[ M = \frac{(V'\beta)^2}{4} \int d^2x_1 d^2x_2 \left[ \phi^s_\alpha(x_1)R_{\alpha\beta}(x)\phi^s_\beta(x_2) \right]. \text{(73)} \]

where as usual \( x = x_1 - x_2 \).

The imaginary part coefficient \( \eta \) is most transparent in the rotated Keldysh basis Eq. (52). The element of the matrix \( R_{\alpha\beta}(x) \) in the rotated basis which corresponds to the renormalisation of \( \eta \) in the retarded function in terms of original elements is

\[ R_R = \frac{1}{2} \left( R_{11}(x) - R_{22}(x) - R_{12}(x) + R_{21}(x) \right). \text{(74)} \]

For the case of the large cut-off this yields after the Fourier transform \( i\omega \eta \frac{k_0}{k_0^0} \exp(-\frac{\beta^2}{2\alpha} \ln k_0\alpha) \), so that the change in \( \eta \) is
\[ \Delta \eta = \eta \frac{V^2}{4(k_0 \alpha)^4}, \]  

(75)

since \( \beta = \sqrt{4\pi} \). Here as always, \( \alpha \) is the lattice spacing which cuts-off the large momentum integration. So here we have a power-law rise in \( \eta \). When the cut-off is less that the temperature we gain different exponential, so that now

\[ \Delta \eta = \eta \exp \left( \frac{4T}{k_0} \right) \frac{V^2}{4(k_0 \alpha)^4}. \]  

(76)

That means, that on the scales of momenta less that the temperature, the imaginary part coefficient \( \eta \) experiences a tremendous growth, and we immediately get the diffusive dynamics.
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