AFFINE DEFORMATIONS OF A THREE-HOLED SPHERE

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Abstract. Associated to every complete affine 3-manifold $M$ with nonsolvable fundamental group is a noncompact hyperbolic surface $\Sigma$. We classify such complete affine structures when $\Sigma$ is homeomorphic to a three-holed sphere. In particular, for every such complete hyperbolic surface $\Sigma$, the deformation space identifies with two opposite octants in $\mathbb{R}^3$. Furthermore every $M$ admits a fundamental polyhedron bounded by crooked planes. Therefore $M$ is homeomorphic to an open solid handlebody of genus two. As an explicit application of this theory, we construct proper affine deformations of an arithmetic Fuchsian group inside $\text{Sp}(4,\mathbb{Z})$.

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A complete affine manifold is a quotient
\[ M = E / \Gamma \]
where \( E \) is an affine space and \( \Gamma \subset \text{Aff}(E) \) is a discrete group of affine transformations of \( E \) acting properly and freely on \( E \). When \( \dim E = 3 \), Fried-Goldman [18] and Mess [28] imply that either:
- \( \Gamma \) is solvable, or
- \( \Gamma \) is virtually free.

When \( \Gamma \) is solvable, \( M \) admits a finite covering homeomorphic to the total space of a fibration composed of points, circles, annuli and tori. The classification of such structures in this case is straightforward [18]. When \( \Gamma \) is virtually free, the classification is considerably more interesting. In the early 1980s Margulis discovered [26, 27] the existence of such structures, answering a question posed by Milnor [29].

**Conjecture.** Suppose \( M^3 \) is a 3-dimensional complete affine manifold with free fundamental group. Then \( M \) is homeomorphic to an open solid handlebody.

The purpose of this paper is to prove this conjecture in the first non-trivial case.

By Fried-Goldman [18], the linear holonomy homomorphism
\[ \text{Aff}(E^3) \twoheadrightarrow \text{GL}(3, \mathbb{R}) \]
embeds \( \Gamma \) as a discrete subgroup of a subgroup of \( \text{GL}(3, \mathbb{R}) \) conjugate to the orthogonal group \( \text{O}(2,1) \). Thus \( M \) admits a complete flat Lorentz metric and is a (geodesically) complete flat Lorentz 3-manifold. Thus we henceforth restrict our attention to the case \( E \) is a 3-dimensional Lorentzian affine space \( E^3_1 \). A Lorentzian affine space is a simply connected geodesically complete flat Lorentz 3-manifold, and is unique up to isometry.

Furthermore \( L(\Gamma) \) is a Fuchsian group acting properly and freely on the hyperbolic plane \( H^2 \). We model \( H^2 \) on a component of the two-sheeted hyperboloid
\[ \{ v \in \mathbb{R}^3_1 \mid v \cdot v = -1 \}, \]
or equivalently its projectivization in \( \mathbb{P}(\mathbb{R}^3_1) \). (Compare [20].) The quotient
\[ \Sigma := H^2 / L(\Gamma) \]
is a complete hyperbolic surface homotopy-equivalent to \( M \), naturally associated to the Lorentz manifold \( M \).
We prove the above conjecture in the case that the surface Σ is homeomorphic to a three-holed sphere.

Margulis [26, 27] discovered proper actions by bounding (from below) the Euclidean distance that elements of Γ displace points. Our more geometric approach constructs fundamental polyhedra for affine deformations in the spirit of Poincaré’s theorem on fundamental polyhedra for hyperbolic manifolds.

This approach began with Drumm [11], who constructed fundamental polyhedra from crooked planes to show that certain affine deformations Γ acts properly on all of \( E^3_1 \). A crooked plane is a polyhedron in \( E^3_1 \) with four infinite faces, adapted to the invariant Lorentzian geometry of \( E^3_1 \). Specifically, representing the hyperbolic surface Σ as an identification space of a fundamental polygon for the generalized Schottky group \( L(Γ) \subset O(2,1) \), we construct a fundamental polyhedron for certain affine deformations Γ bounded by crooked planes [11]. We call such a fundamental polyhedron a crooked fundamental polyhedron.

Conjecture. Suppose \( \dim(E^3_1) = 3 \) and \( Γ \subset \text{Aff}(E^3_1) \) is a discrete group acting properly on \( E^3_1 \). Suppose that Γ is not solvable. Then some finite-index subgroup of Γ admits a crooked fundamental domain.

We prove this conjecture when Σ is homeomorphic to a three-holed sphere.

Let \( Γ_0 \subset O(2,1) \) be a Fuchsian group. Denote the corresponding embedding

\[
ρ_0 : Γ_0 \hookrightarrow O(2,1) ⊂ GL(3, \mathbb{R})
\]

An affine deformation of \( Γ_0 \) is a representation

\[
Γ_0 \xrightarrow{ρ} \text{Aff}(E^3_1)
\]

satisfying \( L \circ ρ = ρ_0 \). We refer to the image Γ of ρ as an affine deformation as well.

An affine deformation is proper if the affine action of \( Γ_0 \) on \( E^3_1 \) defined by ρ is a proper action. Clearly an affine deformation Γ which admits a crooked fundamental polyhedron is proper.

Theorem (Drumm). Every free discrete Fuchsian group \( Γ_0 \subset O(2,1) \) admits a proper affine deformation.

Actions of free groups by Lorentz isometries are the only cases to consider. Fried-Goldman [18] reduces the problem to when \( Γ_0 \) is a Fuchsian group, and Mess [28] implies \( Γ_0 \) cannot be cocompact. Thus, after passing to a finite-index subgroup, we may assume that \( Γ_0 \) is free.
The linear representation $\rho_0$ is itself an affine deformation, by composing it with the embedding

$$\text{GL}(3, \mathbb{R}) \hookrightarrow \text{Aff}(E_1^3).$$

Slightly abusing notation, denote this composition by $\rho_0$ as well. Two affine deformations are translationally equivalent if they are conjugate by a translation in $E_1^3$. An affine deformation is trivial (or radiant) if and only if it is translationally conjugate to the affine deformation $\rho_0$ constructed above. Equivalently, an affine deformation is trivial if it fixes a point in the affine space $E_1^3$.

Let $\mathbb{R}_1^3$ denote the vector space underlying the affine space $E_1^3$, considered as a $\Gamma_0$-module via the linear representation $\rho_0$. The space of translational equivalence classes of affine deformations of $\rho_0$ identifies with the cohomology group $H^1(\Gamma_0, \mathbb{R}_1^3)$. For each $g \in \Gamma_0$, define the translational part $u(g)$ of $\rho(g)$, as the unique translation taking the origin to its image under $\rho(g)$. That is, $u(g) = \rho(g)(0)$, and

$$x \overset{\rho(g)}{\mapsto} \rho_0(g)(x) + u(g).$$

The map $\Gamma_0 \overset{u}{\rightarrow} \mathbb{R}_1^3$ is a cocycle in $Z^1(\Gamma_0, \mathbb{R}_1^3)$, and conjugating $\rho$ by a translation changes $u$ by a coboundary.

The classification of complete affine structures in dimension 3 therefore reduces to determining, for a given free Fuchsian group $\Gamma_0$, the subset of $H^1(\Gamma_0, \mathbb{R}_1^3)$ corresponding to translational equivalence classes of proper affine deformations.

Margulis [26, 27] introduced an invariant of the affine deformation $\Gamma$, defined for elements $\gamma \in \Gamma$ whose linear part $L(\gamma)$ is hyperbolic. Namely, $\gamma$ preserves a unique affine line $C_\gamma$ upon which it acts by translation. Furthermore $C_\gamma$ inherits a canonical orientation. As $C_\gamma$ is spacelike, the Lorentz metric and the canonical orientation determines a unique orientation-preserving isometry

$$\mathbb{R} \overset{j_\gamma}{\rightarrow} C_\gamma.$$

The Margulis invariant $\alpha(\gamma) \in \mathbb{R}$ is the displacement of the translation $\gamma|_{C_\gamma}$ as measured by $j_\gamma$: $j_\gamma(t) \overset{\gamma}{\rightarrow} j_\gamma(t + \alpha(\gamma))$ for $t \in \mathbb{R}$.

Margulis’s invariant $\alpha$ is a class function on $\Gamma_0$ which completely determines the translational equivalence class of the affine deformation [16, 17]. Charette and Drumm [6] extended Margulis’s invariant to parabolic transformations. However, only its sign is well defined for parabolic transformations. To obtain a precise numerical value one
requires a decoration of $\Gamma_0$, that is, a choice of horocycle at each cusp of $\Sigma$.

If $\Gamma$ is an affine deformation of $\Gamma_0$ with translational part $u \in \mathbb{Z}^1(\Gamma_0, \mathbb{R}^3)$, then we indicate the dependence of $\alpha$ on the cohomology class $[u] \in H^1(\Gamma_0, \mathbb{R}^3)$ by writing $\alpha = \alpha_{[u]}$.

Let $\Gamma_0$ be a Fuchsian group whose corresponding hyperbolic surface $\Sigma$ is homeomorphic to a three-holed sphere. Denote the generators of $\Gamma_0$ corresponding to the three ends of $\partial \Sigma$ by $g_1, g_2, g_3$. Choose a decoration so that the generalized Margulis invariant defines an isomorphism

$$H^1(\Gamma_0, \mathbb{R}^3) \longrightarrow \mathbb{R}^3$$

$$[u] \longmapsto \begin{bmatrix} \alpha_{[u]}(g_1) \\
\alpha_{[u]}(g_2) \\
\alpha_{[u]}(g_3) \end{bmatrix} : = \begin{bmatrix} \alpha_1([u]) \\
\alpha_2([u]) \\
\alpha_3([u]) \end{bmatrix}.$$

**Theorem A.** Let $\Gamma_0, \Sigma_0, \mu_1, \mu_2, \mu_3$ be as above. Then $[u] \in H^1(\Gamma_0, \mathbb{R}^3)$ corresponds to a proper affine deformation if and only if

$$\mu_1([u]), \mu_2([u]), \mu_3([u])$$

all have the same sign. Furthermore in this case $\Gamma$ admits a crooked fundamental domain and $M$ is homeomorphic to an open solid handlebody of genus two.

For purely hyperbolic $\Gamma_0$, Theorem A was proved by Cathy Jones in her doctoral thesis [24], using a different method.

In the case that $\Sigma$ is a three-holed sphere, Theorem A gives a complete description of the deformation space and the topological type. As three-holed spheres are the building blocks of all compact hyperbolic surfaces, the present paper plays a fundamental role in our investigation of affine deformations of hyperbolic surfaces of arbitrary topological type. We conjecture that when $\Sigma$ is homeomorphic to a two-holed projective plane or one-holed Klein bottle, the deformation space will again be defined by finitely many inequalities. However, in all other cases, the deformation space will be defined by infinitely many inequalities. For example, when $\Sigma$ is homeomorphic to a one-holed torus, the deformation space is a convex domain with fractal boundary [23].

Margulis’s opposite sign lemma [26, 27] (see Abels [1] for a beautiful exposition) states that uniform positivity (or negativity) of $\alpha(\gamma)$ is necessary for properness of an affine deformation. In [22, 19] uniform positivity was conjectured to be equivalent to properness. Theorem A implies this conjecture when $\Sigma$ is a three-holed sphere with geodesic boundary. In that case only the three $\gamma$ corresponding to $\partial \Sigma$ need to be checked. However, when $\Sigma$ has at least one cusp, Theorem A
provides counterexamples to the original conjecture. If the generalized Margulis invariant of that cusp is zero, and those of the other ends are positive, then \( \alpha(\gamma) > 0 \) for all hyperbolic elements \( \gamma \in \Gamma \). Other counterexamples are given in [23].

We apply Theorem A to construct a proper affine deformation of an arithmetic group in \( \text{SL}(2, \mathbb{Z}) \) inside \( \text{Sp}(4, \mathbb{Z}) \). Here \( \text{Aff}(E^3_1) \) is represented as the subgroup of \( \text{Sp}(4, \mathbb{R}) \) stabilizing a Lagrangian plane \( L_\infty \) in a symplectic vector space \( \mathbb{R}^4 \) defined over \( \mathbb{Z} \). Its unipotent radical \( U \) is the subgroup of \( \text{Sp}(4, \mathbb{R}) \) which preserves \( L_\infty \), acts identically on \( L_\infty \), and acts identically on the quotient \( \mathbb{R}^4/L_\infty \). The parabolic subgroup \( \text{Aff}(E^3_1) \) is the normalizer of \( U \) in \( \text{Sp}(4, \mathbb{R}) \). Furthermore \( \text{Aff}(E^3_1) \) acts conformally on a left-invariant flat Lorentz metric on \( U \). This model of Minkowski space embeds in the conformal compactification of \( E^3_1 \), the Einstein universe (see [2]) upon which \( \text{Sp}(4, \mathbb{R}) \) acts transitively.

**Theorem B.** Choose three positive integers \( \mu_1, \mu_2, \mu_3 \). Let \( \Gamma \) be the subgroup of \( \text{Sp}(4, \mathbb{Z}) \) generated by

\[
\begin{pmatrix}
-1 & -2 & \mu_1 + \mu_2 - \mu_3 & 0 \\
0 & -1 & 2\mu_1 & -\mu_1 \\
0 & 0 & -1 & 0 \\
0 & 0 & 2 & -1
\end{pmatrix},
\begin{pmatrix}
-1 & 0 & -\mu_2 & -2\mu_2 \\
0 & 2 & -1 & 0 \\
0 & 0 & -1 & -2 \\
0 & 0 & 0 & -1
\end{pmatrix},
\begin{pmatrix}
1 & x & y \\
0 & 1 & y & z \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Let \( U := \exp(\Phi) \subset \text{Sp}(4, \mathbb{R}) \) be the connected unipotent subgroup consisting of matrices

\[x, y, z \in \mathbb{R}\]. Then:

- \( \Gamma \) normalizes \( U \);
- The resulting action of \( \Gamma \) on \( U \) is proper and free;
- \( \Gamma \) acts isometrically with respect to a left-invariant flat Lorentz metric on \( U \);
- The quotient orbifold \( U/\Gamma \) is homeomorphic to an open solid handlebody of genus two.

Our result complements Goldman-Labourie-Margulis [21] when the hyperbolic surface \( \Sigma \) is convex cocompact. In that case the space of proper affine deformations identifies with an open convex cone in \( H^1(\Gamma_0, \mathbb{R}^3_1) \) defined by the nonvanishing of an extension of Margulis’s invariant to geodesic currents on \( \Sigma \).
This cone is the interior of the intersection of half-spaces defined by the functionals

\[ H^1(\Gamma_0, \mathbb{R}^3_1) \to \mathbb{R} \]

\[ [u] \mapsto \alpha[u](g) \]

for \( g \in \Gamma_0 \). In general we expect this cone to be the union of open regions corresponding to combinatorial configurations realized by crooked planes, thereby giving a crooked fundamental domain for each proper affine deformation. Jones [24] used standard Schottky fundamental domains to fill the open cone with such regions. Here we decompose \( \Sigma \) into two ideal triangles, obtaining a single combinatorial configuration which applies to all proper affine deformations.

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1. Lorentzian geometry

This section summarizes needed technical background on the geometry of Minkowski (2+1)-spacetime, its isometries and Margulis’s invariant of hyperbolic and parabolic isometries. For details, variants and proofs, see [1, 6, 7, 9, 12, 14, 16, 19].

Let \( \mathbb{E}^3_1 \) denote Minkowski (2+1)-spacetime, that is, a simply connected complete three-dimensional flat Lorentzian manifold. Alternatively \( \mathbb{E}^3_1 \) is an affine space whose underlying vector space \( \mathbb{R}^3_1 \) of translations is a Lorentzian inner vector space, a vector space with an inner product

\[ \mathbb{R}^3_1 \times \mathbb{R}^3_1 \to \mathbb{R} \]

\[ (v, w) \mapsto v \cdot w \]

of signature \((2, 1)\).

A vector \( x \in \mathbb{R}^3_1 \) is:

- null if \( x \cdot x = 0 \);
- timelike if \( x \cdot x < 0 \);
- spacelike if \( x \cdot x < 0 \).

A spacelike vector \( x \) is unit spacelike if \( x \cdot x = 1 \). A null vector is future-pointing if its third coordinate is positive – this corresponds to choosing a connected component of the set of timelike vectors, or a time-orientation.
Define the *Lorentzian cross-product* as follows. Choose an orientation on $\mathbb{R}^3$. Let
\[
\mathbb{R}_1^3 \times \mathbb{R}_1^3 \times \mathbb{R}_1^3 \xrightarrow{\text{Det}} \mathbb{R}
\]
denote the alternating trilinear form compatible with the Lorentzian inner product and the orientation: if $(v_1, v_2, v_3)$ is a positively-oriented basis, with
\[
v_i \cdot v_j = 0 \text{ if } i \neq j, \quad v_1 \cdot v_1 = v_2 \cdot v_2 = -v_3 \cdot v_3 = 1
\]
then
\[
\text{Det}(v_1, v_2, v_3) = 1.
\]
The Lorentzian cross-product is the unique bilinear map
\[
\mathbb{R}_1^3 \times \mathbb{R}_1^3 \xrightarrow{\boxtimes} \mathbb{R}_1^3
\]
satisfying
\[
u \cdot (v \boxtimes w) = \text{Det}([u \, v \, w]).
\]
The following facts are well known (see for example Ratcliffe [30]):

**Lemma 1.1.** Let $u, v, x, y \in \mathbb{R}_1^3$. Then:
\[
u \cdot (x \boxtimes y) = x \cdot (y \boxtimes u)
\]
\[
(u \boxtimes v) \cdot (x \boxtimes y) = (u \cdot y)(v \cdot x) - (u \cdot x)(v \cdot y).
\]

For a spacelike vector $v$, define its *Lorentz-orthogonal plane* to be:
\[
v^\perp = \{x \mid x \cdot v = 0\}.
\]
It is an *indefinite plane*, since the Lorentzian inner product restricts to an inner product of signature $(1, 1)$. In particular, $v^\perp$ contains two null lines. The two future-pointing linear independent vectors of Euclidean length 1 in this set are denoted $v^-$ and $v^+$ and are chosen so that $(v^-, v^+, v)$ is a positively oriented basis for $\mathbb{R}_1^3$.

A basis $(a, b, c)$ of $\mathbb{R}_1^3$ is positively oriented if and only if
\[
(a \boxtimes b) \cdot c > 0.
\]

**Lemma 1.2.** Let $v \in \mathbb{R}_1^3$ be a unit spacelike vector. Then:
\[
v \boxtimes v^+ = v^+
\]
\[
v^- \boxtimes v = v^-.
\]
For the proof, see Charette-Drumm [7].

Let $G$ denote the group of all affine transformations that preserve the Lorentzian scalar product on the space of directions; $G$ is isomorphic to $O(2,1) \ltimes \mathbb{R}_1^3$. We shall restrict our attention to those transformations whose linear parts are in $\text{SO}(2,1)^0$, thus preserving orientation and
The three-holed sphere. As above, \( L \) denotes the projection onto the linear part of an affine transformation.

Suppose \( g \in \SO(2, 1)^0 \) and \( g \neq I \).

- \( g \) is hyperbolic if it has three distinct real eigenvalues;
- \( g \) is parabolic if its only eigenvalue is 1;
- \( g \) is elliptic if it has no real eigenvalues.

Denote the set of hyperbolic elements in \( \SO(2, 1)^0 \) by \( \Hyp_0 \) and the set of parabolic elements by \( \Par_0 \).

We also call \( \gamma \in G \) hyperbolic (respectively parabolic, elliptic) if its linear part \( L(\gamma) \) is hyperbolic (respectively parabolic, elliptic). Denote the set of hyperbolic elements in \( G \) by \( \Hyp \) and the set of parabolic transformations by \( \Par \).

Let \( \gamma \in \Hyp \cup \Par \). The eigenspace \( \text{Fix}(L(\gamma)) \) is one-dimensional. Let \( v \in \text{Fix}(L(\gamma)) \) be a non-zero vector and \( x \in E^3_1 \). Define:

\[
\tilde{\alpha}_v(\gamma) := (\gamma(x) - x) \cdot v.
\]

The following facts are proved in [1, 6, 7, 16, 19, 22]:

- \( \tilde{\alpha}_v(\gamma) \) is independent of \( x \);
- \( \tilde{\alpha}_v(\gamma) \) is identically zero if and only if \( \gamma \) fixes a point;
- For any \( \eta \in G \) with \( h = L(\eta) \),
  \[
  \tilde{\alpha}_{h(v)}(\eta \gamma \eta^{-1}) = \tilde{\alpha}_v(\gamma)
  \]
  where \( v \in \text{Fix}(g) \) and \( h = L(\eta) \);
- For any \( n \in \mathbb{Z} \),
  \[
  \tilde{\alpha}_v(\gamma^n) = |n|\tilde{\alpha}_v(\gamma).
  \]

A linear transformation \( g \) induces a natural orientation on \( \text{Fix}(g) \) as follows.

**Definition 1.3.** Let \( g \in \Hyp_0 \cup \Par_0 \). A vector \( v \in \text{Fix}(g) \) is positive relative to \( g \) if and only if

\[
(v, x, gx)
\]

is a positively oriented basis, where \( x \) is any null or timelike vector which is not an eigenvector of \( g \).

The sign of \( \gamma \) is the sign of \( \tilde{\alpha}_v(\gamma) \), where \( v \) is any positive vector in \( \text{Fix}(g) \). For \( n < 0 \) the orientation of \( \text{Fix}(g^n) \) reverses, so \( \gamma \) and \( \gamma^{-1} \) have equal sign.

**Lemma 1.4** [26, 27, 6]. Let \( \gamma_1, \gamma_2 \in \Hyp \cup \Par \) and suppose \( \gamma_1 \) and \( \gamma_2 \) have opposite signs. Then \( \langle \gamma_1, \gamma_2 \rangle \) does not act properly on \( E^3_1 \).
Let $\Gamma_0 \subset O(2, 1)$ be a free group and $\rho$ an affine deformation of $\Gamma_0$:  

\[ \rho(g)(x) = g(x) + u(g) \]

where $x \in \mathbb{R}_1^3$. Then $\Gamma_0 \xrightarrow{u} \mathbb{R}_1^3$ is a cocycle of $\Gamma_0$ with coefficients in the $\Gamma_0$-module $\mathbb{R}_1^3$, corresponding to the linear action of $\Gamma_0$. As affine deformations of $\Gamma_0$ correspond to cocycles in $Z^1(\Gamma_0, \mathbb{R}_1^3)$, translational conjugacy classes of affine deformations comprise the cohomology group $H^1(\Gamma_0, \mathbb{R}_1^3)$.

If $g \in \text{Hyp}_0$, set $x_0^g$ to be the unique positive vector in $\text{Fix}(g)$ such that $x_0^g \cdot x_0^g = 1$. If $g \in \text{Par}_0$, choose a positive vector in $\text{Fix}(g)$ and call it $x_0^g$.

Let $u \in Z^1(\Gamma_0, \mathbb{R}_1^3)$. Reinterpreting the Margulis invariant as a linear functional on the space of cocycles $Z^1(\Gamma_0, \mathbb{R}_1^3)$, set:

\[ \Gamma_0 \xrightarrow{\alpha[u]} \mathbb{R} \quad g \mapsto \tilde{\alpha}_{x_0^g}(\gamma), \]

where $\gamma = \rho(g)$ is the affine deformation corresponding to $u(g)$. As the notation indicates, $\alpha[u]$ only depends on the cohomology class of $u$, since $\tilde{\alpha}_{x_0^g}$ is a class function.

2. Hyperbolic geometry and the three-holed sphere

Let $\Sigma$ denote a complete hyperbolic surface homeomorphic to a three-holed sphere. Each of the three ends can either flare out (that is, have infinite area) or end in a cusp. In the former case, a loop going around the end will have hyperbolic holonomy, and parabolic holonomy in the latter case. We consider certain geodesic laminations on the surface from which we will construct crooked fundamental domains.

Fixing some arbitrary basepoint in $\Sigma$, let $\Gamma_0$ denote the image under the holonomy representation of the fundamental group of $\Sigma$. We may thus identify $\Sigma$ with $\mathbb{H}^2/\Gamma_0$.

The fundamental group of $\Sigma$ is free of rank two and admits a presentation

\[ \Gamma_0 = \langle g_1, g_2, g_3 \mid g_3g_2g_1 = \mathbb{I} \rangle, \]

where the $g_i$ correspond to the components of $\partial \Sigma$ and may be hyperbolic or parabolic.

For the rest of the paper, unless otherwise noted, the $g_i$ and their affine deformations $\gamma_i$ are indexed by $i = 1, 2, 3$ with addition in $\mathbb{Z}/3\mathbb{Z}$.

If $g_i$ is hyperbolic, it admits a unique invariant axis $l_i \subset \mathbb{H}^2$ which projects to an end of the three-holed sphere. For $g_i$ parabolic, we think of this invariant line as shrunk to a point on the ideal boundary. For
Figure 1. The invariant lines for \( g_1, g_2, g_3 \), with direction indicated by the arrows.

hyperbolic \( g_i \), set \( x_i^+, x_i^- \) to be its attracting and repelling fixed points, respectively; if \( g_i \) is parabolic, set \( x_i^+ = x_i^- \) to be its unique fixed point.

Since \( \Gamma_0 \) is discrete, the \( l_i \)'s are pairwise disjoint. Furthermore, substituting inverses if necessary, we assume for convenience that the direction of translation along the axes is as in Figure 1. (In this case, all three \( g_i \)'s are hyperbolic.)

The three arcs in \( \mathbb{H}^2 \) respectively joining \( x_i^+ \) to \( x_{i+1}^+ \) project to a geodesic lamination of \( \Sigma \) as drawn in Figures 2 and 3.

We shall adopt the following model for \( \mathbb{H}^2 \) in terms of Lorentzian affine space \( \mathbb{E}^3_1 \). A future-pointing timelike ray is a ray \( q + \mathbb{R}_+ w \), where \( q \in \mathbb{E}^3_1 \) and \( w \in \mathbb{R}^3_1 \) is a future-pointing timelike vector. Parallelism defines an equivalence relation on future-pointing timelike rays, and points of \( \mathbb{H}^2 \) identify with equivalence classes of future-pointing timelike rays.

Denote by \([q + \mathbb{R}_+ w]\) the point in \( \mathbb{H}^2 \) corresponding to the equivalence class of the ray \( q + \mathbb{R}_+ w \).

Geodesics in \( \mathbb{H}^2 \) identify with parallelism classes of indefinite affine planes; a point in \( \mathbb{H}^2 \) is incident to a geodesic if and only if the corresponding future-pointing timelike ray and indefinite affine plane are parallel. A half-space \( H \) in \( \mathbb{E}^3_1 \) bounded by an indefinite affine plane determines a half-plane \( \mathcal{H} \subset \mathbb{H}^2 \). A point \([q + \mathbb{R}_+ w]\) in \( \mathbb{H}^2 \) lies in \( \mathcal{H} \) if and only if \( q + \mathbb{R}_+ w \) intersects \( H \) in a ray, that is, \( q + tw \in H \) for \( t \gg 0 \).
Figure 2. Three lines in $\mathbb{H}^2$ joining endpoints of the invariant axes $l_i$. On the right, the induced lamination of $\Sigma$.

Figure 3. Three lines in $\mathbb{H}^2$ joining endpoints of $l_i$, with $g_2$ parabolic and $l_2$ an ideal point.
Dually, geodesics in \( H^2 \) correspond to spacelike lines, since the Lorentz-orthogonal plane of a spacelike vector is indefinite. In fact, if \( l = \mathbb{R}x^0_y \), then the null vectors \( (x^0_y)^\pm \) respectively project to the ideal points \( x^\pm \).

Furthermore spacelike vectors correspond to oriented geodesics, or, equivalently, to half-planes in \( H^2 \). A spacelike vector spans a unique spacelike ray, which contains a unique unit spacelike vector \( v \). The corresponding half-plane is

\[
\mathcal{H}(v) := \{ [q + \mathbb{R}_+w] \in H^2 \mid w \cdot v \geq 0 \}.
\]

Extending terminology from \( H^2 \) to \( \mathbb{R}^3 \), say that two spacelike vectors \( u, v \in \mathbb{R}^3 \) are:

- ultraparallel if \( u \parallel v \) is spacelike;
- asymptotic if \( u \parallel v \) is null;
- crossing if \( u \parallel v \) is lightlike.

3. Crooked planes and half-spaces

Crooked planes are Lorentzian analogs of equidistant surfaces. We will think of a triple of crooked planes as the natural extension of a lamination. We will see how to get pairwise disjoint crooked plane triples, yielding proper affine deformations of the linear holonomy. In this section, we define crooked planes and discuss criteria for disjointness.

Here is a somewhat technical, yet important, point. What we call crooked planes and half-spaces should really be called positively extended crooked planes and half-spaces. We require crooked planes to be positively extended when the signs of the Margulis invariants are positive. But for the case of negative Margulis invariants, we must use negatively extended crooked planes. As the arguments are essentially the same up to a change in sign change, for the rest of the paper we will restrict to the case of positive signs. The curious reader should consult [15]. (In that paper the crooked planes are called positively or negatively oriented).

Given a null vector \( x \in \mathbb{R}^3 \), set \( \mathcal{P}(x) \) to be the set of (spacelike) vectors \( w \) such that \( w^+ \) is parallel to \( x \). This half-plane in the Lorentz-orthogonal plane \( x^\perp \) is a connected component of \( x^\perp \setminus \langle x \rangle \). If \( v \) is a spacelike vector, then

\[
v \in \mathcal{P}(v^+)
\]

\[
-v \in \mathcal{P}(v^-).
\]

Let \( p \in E^3_1 \) be a point and \( v \in \mathbb{R}^{2,1} \) a spacelike vector. Define the crooked plane \( \mathcal{C}(v, p) \subset E^3_1 \) with vertex \( p \) and direction vector \( v \) to be
the union of two wings
\[ p + \mathcal{P}(v^+) \]
\[ p + \mathcal{P}(v^-) \]
and a stem
\[ p + \{ x \in \mathbb{R}^3_1 \mid v \cdot x = 0, x \cdot x \leq 0 \} \].

Each wing is a half-plane, and the stem is the union of two quadrants in a spacelike plane. The crooked plane itself is a piecewise linear submanifold, which stratifies into four connected open subsets of planes (two wings and the two components of the interior of the stem), four null rays, and a vertex.

**Definition 3.1.** Let \( v \) be a spacelike vector and \( p \in \mathbb{E}^3_1 \). The crooked half-space determined by \( v \) and \( p \), denoted \( H(v, p) \), consists of all \( q \in \mathbb{E}^3_1 \) such that:

- \((q - p) \cdot v^+ \leq 0\) if \((q - p) \cdot v \geq 0\);
- \((q - p) \cdot v^- \geq 0\) if \((q - p) \cdot v \leq 0\);
- Both conditions must hold for \( q - p \in v^\perp \).

Observe that \( C(v, p) = C(-v, p) \). In contrast, the crooked half-spaces \( H(v, p) \) and \( H(-v, p) \) are distinct spaces. Their union and intersection are respectively:

\[ H(v, p) \cup H(-v, p) = \mathbb{E}^3_1 \]
\[ H(v, p) \cap H(-v, p) = C(v, p) = C(-v, p) \].

Crooked half-spaces in \( \mathbb{E}^3_1 \) determine half-planes in \( \mathbb{H}^2 \) as follows. As in the preceding section, a point in \( \mathbb{H}^2 \) corresponds to the equivalence class of a future-pointing timelike ray.

**Lemma 3.2.** Let \( p, q \in \mathbb{E}^3_1 \) and \( v, w \in \mathbb{R}^3_1 \) spacelike. Suppose that \( H(v, p) \) is a crooked half-space and that \( w \cdot v \neq 0 \). Then \( q + tw \in \text{int}(H(v, p)) \) for \( t >> 0 \) if and only if \( \lfloor q + tw \rfloor \in \text{int}(\mathcal{F}(v)) \).

**Proof.** It suffices to consider the case that \( p = 0 \) and

\[ v = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \],

that is,

\[ H(v, p) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid y + z \geq 0 \text{ if } x \geq 0 \text{ and } y - z \geq 0 \text{ if } x \leq 0 \right\} \].
By applying an automorphism preserving $H(v, p)$, we may assume

$$q = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}, \quad w = \begin{bmatrix} d \\ 0 \\ 1 \end{bmatrix}.$$ 

where $|d| < 1$.

Set $q(t) := q + tw$. For any value of $d$, $q(t)$ eventually satisfies

$$y + z = y_0 + z_0 + t > 0$$

for $t \gg 0$ and $y - z < 0$. The point $[q + tw]$ lies in the interior $\text{int}(f_J(v))$ when $d > 0$. In this case, $q(t)$ eventually satisfies

$$x = x_0 + td > 0.$$ 

Thus $q(t) \in \text{int}(H(v, p))$.

Conversely, if $[q + tw] \in f_J(-v)$, then $d < 0$. If $t \gg 0$, then $x < 0$. Therefore $q(t) \notin \text{int}(H(v, p))$ as desired. \hfill \Box

4. DISJOINTNESS OF CROOKED HALF-SPACES

By [11, 13] (see [10] for another exposition), the complement of a disjoint union of crooked half-spaces with pairwise identifications of its boundary defines a fundamental polyhedron for the group generated by the identifications. This section develops criteria for when two crooked half-spaces are disjoint. Lemma 4.2 reduces disjointness of crooked half-spaces to disjointness of crooked planes. We need only consider pairs of crooked half-spaces in the case of ultraparallel or asymptotic vectors: when $u$ and $v$ are crossing $C(u, p)$ and $C(v, p)$ always intersect [15]. Theorem 4.3 and Theorem 4.5 provide criteria for disjointness for crooked planes, and were established in [15]. Their respective corollaries, Corollary 4.4 and Corollary 4.6, provide more useful criteria in terms of the direction vectors.

**Definition 4.1.** Spacelike vectors $v_1, \ldots, v_n \in \mathbb{R}^3_1$ are consistently oriented if and only if, whenever $i \neq j$,

- $v_i \cdot v_j < 0$;
- $v_i \cdot v_j^\perp \leq 0$.

The second requirement implies that the $v_i$ are pairwise ultraparallel or asymptotic. Equivalently, $v_i, v_j, i \neq j$ are consistently oriented if and only if the interiors of the half-planes $f_J(v_i)$ and $f_J(v_j)$ are pairwise disjoint. (See [20], §4.2.1 for details.)

**Lemma 4.2.** Suppose $u, v$ are consistently oriented, $p \in \mathbb{E}^3_1$ and $C(u, p)$ and $C(v, p + w)$ are disjoint. Then $C(v, p + w) \subset H(-u, p)$. 
Theorem 4.3. Let \( \mathbb{H} \) be the set of future-pointing timelike rays on \( \mathcal{C}(v, p + w) \) lie on the stem of \( \mathcal{C}(v, p + w) \) and correspond to the geodesic \( \partial \mathcal{H}(v) \).

Suppose that \( \mathcal{C}(v, p + w) \subset \mathbb{H}(u, p) \). The future-pointing timelike rays on \( \mathcal{C}(v, p + w) \) lie on the stem of \( \mathcal{C}(v, p + w) \) and correspond to the geodesic \( \partial \mathcal{H}(v) \).

Since a future-pointing timelike ray on \( \mathcal{C}(v, p + w) \) lies entirely in \( \mathbb{H}(u, p) \), Lemma 3.2 implies that

\[
\partial \mathcal{H}(v) \subset \mathcal{H}(u).
\]

Since \( u, v \) are consistently oriented, the half-spaces \( \mathcal{H}(u) \) and \( \mathcal{H}(v) \) are disjoint, and \( \mathcal{H}(v) \subset \mathcal{H}(u) \), a contradiction. Thus \( \mathcal{C}(v, p + w) \subset \mathbb{H}(u, p) \) as desired.

\[\Box\]

Corollary 4.4. Let \( \mathcal{C}(v_1, \mathcal{C}(v_2, p_1) \) and \( \mathcal{C}(v_2, p_2) \) are disjoint if and only if

\[
(p_2 - p_1) \cdot (v_1 \boxtimes v_2) > |(p_2 - p_1) \cdot v_2| + |(p_2 - p_1) \cdot v_1|.
\]

Proof. Rescaling if necessary, assume \( v_1, v_2 \) are unit spacelike. By Lemmas 1.1 and 1.2,

\[
v_i^+ \cdot (v_i \boxtimes v_j) = v_i^+ \cdot v_j
\]

\[
v_i^- \cdot (v_i \boxtimes v_j) = -v_i^- \cdot v_j.
\]

for \( i \neq j \). Consequently:

\[
(p_2 - p_1) \cdot (v_1 \boxtimes v_2) = -(a_2 v_2^- + b_2 v_2^+) \cdot v_1 - (a_1 v_1^- + b_1 v_1^+) \cdot v_2
\]

\[
= -a_2 v_2^- \cdot v_1 - b_2 v_2^+ \cdot v_1 - a_1 v_1^- \cdot v_2 - b_1 v_1^+ \cdot v_2
\]

\[
> |(a_2 v_2^- - b_2 v_2^+) \cdot v_1| + |(a_1 v_1^- - b_1 v_1^+) \cdot v_2|.
\]

The above inequality follows because each term in the previous expression is positive (since \( v_1, v_2 \) are consistently oriented). Finally:

\[
|(p_2 - p_1) \cdot v_2| = |(a_1 v_1^- - b_1 v_1^+) \cdot v_2|
\]

\[
|(p_2 - p_1) \cdot v_1| = |(a_2 v_2^- - b_2 v_2^+) \cdot v_1|.
\]

\[\Box\]
Alternatively, \( C(v_1, p_1) \) and \( C(v_2, p_2) \) are disjoint if and only if \( p_2 - p_1 \) lies in the cone spanned by the four vectors
\[
v_2^-, -v_2^+, -v_1^-, v_1^+.
\]
In fact, we allow \( a_1 = b_1 = 0 \) or \( a_2 = b_2 = 0 \) since \( p_2 - p_1 \) would still lie in the open cone. If three of the four coefficients \( a_i, b_i \) are zero, then the crooked planes intersect in a single point, on the edges of the stems.

Assume now that \( v_1, v_2 \in \mathbb{R}^3_1 \) are consistently oriented, asymptotic vectors. Assume, without loss of generality:
\[
v_1^- = v_2^+.
\]

**Theorem 4.5.** Let \( v_1 \) and \( v_2 \) be consistently oriented, asymptotic vectors such that \( v_1^- = v_2^+ \), and \( p_1, p_2 \in E_1^3 \). The crooked planes \( C(v_1, p_1) \) and \( C(v_2, p_2) \) are disjoint if and only if:
\[
\begin{align*}
(p_2 - p_1) \cdot v_1 &< 0, \\
(p_2 - p_1) \cdot v_2 &< 0, \\
(p_2 - p_1) \cdot (v_1^+ \boxtimes v_2^-) &> 0.
\end{align*}
\]

(4)

As in the ultraparallel case, Theorem 4.5 provides criteria for when \( C(v_1, p_1) \) and \( C(v_2, p_2) \) are disjoint.

**Corollary 4.6.** Let \( v_1, v_2 \in \mathbb{R}^3_1 \) be consistently oriented, asymptotic vectors such that \( v_1^- = v_2^+ \). Suppose
\[
p_i = a_i v_i^- - b_i v_i^+,
\]
where \( a_i, b_i > 0 \) for \( i = 1, 2 \). Then \( C(v_1, p_1) \) and \( C(v_2, p_2) \) are disjoint.

**Proof.** Set
\[
v_i^- \boxtimes v_i^+ = \kappa_i^2 v_i,
\]
for \( i = 1, 2 \). Then:
\[
\begin{align*}
(p_2 - p_1) \cdot v_1 &= a_2 v_2^- \cdot v_1 < 0 \\
(p_2 - p_1) \cdot v_2 &= b_1 v_1^+ \cdot v_2 < 0
\end{align*}
\]
and:
\[
(p_2 - p_1) \cdot (v_1^+ \boxtimes v_2^-) = -b_2 v_2^+ \cdot (v_1^+ \boxtimes v_2^-) - a_1 v_1^- \cdot (v_1^+ \boxtimes v_2^-) \\
&= -b_2 \kappa_2^2 (v_1^- \cdot v_2) - a_1 \kappa_1^2 (v_2^+ \cdot v_1) \\
&> 0.
\]

(5)
As in the ultraparallel case, we obtain disjoint crooked planes if and only if \( p_2 - p_1 \) lies in a cone spanned by three rays. In Equation (5), we allow \( b_2 = 0 \) or \( a_1 = 0 \) simply because \( v_1^- = v_2^+ \). If \( a_2 = 0, b_1 = 0 \) or \( a_1 = b_2 = 0 \), then the crooked planes intersect in a null ray.

5. Crooked fundamental domains

Now look at how collections of pairwise disjoint crooked planes correspond to groups acting properly on \( \mathbb{H}^3_{1} \). Let \( v, v' \in \mathbb{R}^3_{1} \) be two spacelike vectors. Suppose \( \gamma \in G \) and \( p, p' \in \mathbb{H}^3_{1} \) satisfy:

\[
\gamma(C(v, p)) = C(v', p').
\]

Then \( \gamma(p) = p' \) and \( L(\gamma)(v) \) is a scalar multiple of \( v' \). In particular, \( \gamma(H(v, p)) \) is one of the two crooked half-spaces bounded by \( C(v', p') \).

**Theorem 5.1.** Suppose that \( H(v_i, p_i) \) are \( 2n \) pairwise disjoint crooked half-spaces and \( \gamma_1, \ldots, \gamma_n \in \Gamma \) such that for all \( i \),

\[
\gamma_i(H(v_{-i}, p_{-i})) = \mathbb{E}^3_{1} \setminus \text{int}(H(v_i, p_i)).
\]

Then \( \langle \gamma_1, \ldots, \gamma_n \rangle \) acts freely and properly on \( \mathbb{E}^3_{1} \) with fundamental domain

\[
\mathbb{E}^3_{1} \setminus \bigcup_{-n \leq i \leq n} \text{int}(H(v_i, p_i)).
\]

**Proof.** By the assumption

\[
\gamma_i(H(v_{-i}, p_{-i})) = \mathbb{E}^3_{1} \setminus \text{int}(H(v_i, p_i)),
\]

the vectors \( v_{\pm i} \) either cross or are parallel to \( x_0^0 \gamma_i \). The theorem is shown in [11, 13], assuming, in the case of hyperbolic \( \gamma_i \), that the vector \( v_i \) crosses the fixed vector \( x_0^0 \gamma_i \). (The vectors \( v_i \) are parallel to \( x_0^0 \gamma_i \) for parabolic \( \gamma_i \).

However, the methods used in [11, 13] extend to the case of hyperbolic generators with \( v_{\pm i} \) parallel to \( x_0^0 \gamma_i \). In particular, the compression of a tubular neighborhood around lines which touch a boundary crooked plane at a point in particular transverse directions is bounded from below. \( \square \)

These fundamental domains notably differ from the standard construction (as in [13]). A crooked fundamental domain \( \Delta \) in \( \mathbb{E}^3_{1} \) for \( \Gamma \) determines a polygon \( \delta \) in \( \mathbb{H}^2 \) for \( L(\Gamma) \); the stems of \( \partial \Delta \) define lines in \( \mathbb{H}^2 \) bounding \( \delta \). However, while \( \Gamma \cdot \Delta = \mathbb{E}^3_{1} \), the union \( L(\Gamma) \cdot \delta \) may only be a proper open subset of \( \mathbb{H}^2 \). In the present case, this is the universal covering of the interior of the convex core of \( \Sigma \). The convex core is an incomplete hyperbolic surface bounded by three closed geodesics. In
contrast, the flat Lorentz manifold $E^3_1/\Gamma$ is complete. While the hyperbolic fundamental domains $L(\gamma)(\delta)$ only fill a proper subset of $H^2$, the crooked fundamental domains $\gamma(\Delta)$ fill all of $E^3_1$.

Theorem 5.1 extends to the case when two of the crooked planes intersect in a single point.

**Lemma 5.2.** Let $v_{-2}, v_{-1}, v_1, v_2 \in \mathbb{R}^3_1$ be consistently oriented vectors and suppose $p_{-1}, p_1, p_2 \in E^3_1$ satisfy:

\[
\mathcal{C}(v_{-2}, p_{-1}) \cap \mathcal{C}(v_2, p_2) = \emptyset
\]

\[
\mathcal{C}(v_{-1}, p_{-1}) \cap \mathcal{C}(v_1, p_1) = \emptyset
\]

\[
\mathcal{C}(v_1, p_1) \cap \mathcal{C}(v_2, p_2) = \emptyset.
\]

Then there exists $p_{-2} \in E^3_1$ such that the crooked planes $\mathcal{C}(v_{-2}, p_{-2})$ and $\mathcal{C}(v_2, \gamma_2(p_{-2}))$ are each disjoint from $\mathcal{C}(v_1, p_1)$.

**Proof.** Let $H(v_0, p_{-1})$ be the smallest crooked half-space containing both $H(v_{-2}, p_{-1})$ and $H(v_{-1}, p_{-1})$. Then

\[
H(v_0, p_{-1}), H(v_1, p_1), H(v_2, p_2)
\]

are pairwise disjoint. Disjointness of crooked planes is an open condition. Therefore there exists $\epsilon > 0$ such that for any $u \in \mathbb{R}^3_1$ of Euclidean norm less than $\epsilon$, the crooked plane $\mathcal{C}(v_2, p_2 + u)$ remains disjoint from $\mathcal{C}(v_0, p_{-1})$ and $\mathcal{C}(v_1, p_1)$. Corollaries 4.4 and 4.6 imply the existence of a $p_{-2}$ such that $\mathcal{C}(v_{-2}, p_{-2})$ is disjoint from both $\mathcal{C}(v_0, p_{-1})$ and $\mathcal{C}(v_{-1}, p_{-1})$. The set of choices being closed under positive rescaling, one can choose $p_{-2}$ close enough to $p_{-1}$ so that $\gamma_2(p_{-2})$ is within an $\epsilon$-neighborhood of $p_2$.

Lemma 4.2 implies:

\[
\mathcal{C}(v_{-2}, p_{-2}) \subset H(v_0, p_{-1}).
\]

In particular, $\mathcal{C}(v_{-2}, p_{-2})$ is disjoint from each $\mathcal{C}(v_1, p_1)$ and $\mathcal{C}(v_2, \gamma_2(p_{-2}))$ as claimed. \qed

6. **The space of proper affine deformations**

Recall the presentation of the fundamental group of $\Sigma$ in Equation (2). We parametrize the space of translational conjugacy classes $H^1(\Gamma_0, \mathbb{R}^3_1)$ of affine deformations of $\Gamma_0$ by Margulis invariants corresponding to $g_1, g_2, g_3$. Positivity of the three signs will guarantee a triple of crooked planes arising from the lamination described in §2. (Alternatively, if the signs are all negative, use negatively extended crooked planes [15] as mentioned in [3]). The existence of such a crooked polyhedron thereby completes the proof of Theorem A.

We begin with the parametrization of $H^1(\Gamma_0, \mathbb{R}^3_1)$. 
Lemma 6.1. Let \( \pi \) denote a free group of rank two with presentation
\[
\langle A_1, A_2, A_3 \mid A_1A_2A_3 = I \rangle.
\]
Let \( \pi \twoheadrightarrow \text{SO}(2,1)^0 \) be a homomorphism such that \( \rho_0(A_i) \in \text{Hyp}_0 \cup \text{Par}_0 \) for \( i = 1, 2, 3 \). Suppose that \( \rho(\pi) \) is not solvable. For each \( i \) choose a vector \( v_i \in \text{Fix}(\rho_0(A_i)) \) positive with respect to \( \rho_0(A_i) \) and define
\[
H^1(\Gamma_0, \mathbb{R}^3_1) \xrightarrow{\mu} \mathbb{R}^3
\]
\[
[u] \mapsto \tilde{\alpha}_{v_i}(\rho(A_i)) = u(A_i) \cdot v_i
\]
where \( \rho \) is the affine deformation corresponding to \( u \). Then
\[
H^1(\Gamma_0, \mathbb{R}^3_1) \xrightarrow{\mu} \mathbb{R}^3
\]
\[
\mu : [u] \mapsto \begin{bmatrix} \mu_1([u]) \\ \mu_2([u]) \\ \mu_3([u]) \end{bmatrix}
\]
is a linear isomorphism of vector spaces.

Of course this lemma is much more general than our specific application. In our application \( \rho_0 \) is an isomorphism of \( \pi = \pi_1(\Sigma) \) onto the discrete subgroup \( \Gamma_0 \subset \text{SO}(2,1)^0 \), and corresponds to a complete hyperbolic three-holed sphere \( \text{int}(\Sigma) \). The generators \( A_1, A_2, A_3 \) correspond to the three components of \( \partial \Sigma \).

The proof of Lemma 6.1 is postponed to the Appendix.

As in §1 choose a positive vector \( x^0_i := x^0_{g_i} \in \text{Fix}(g_i) \), further requiring that \( x^0_i \) be unit spacelike when \( g_i \) is hyperbolic. With this fixed choice of positive vectors:
\[
\mu_i([u]) = \alpha_{[u]}(g_i).
\]

We will now show that every positive cocycle \( (\mu_1, \mu_2, \mu_3) \in Z^1(\Gamma_0, \mathbb{R}^3_1) \) corresponds to a triple of mutually disjoint crooked planes arising from the geodesic lamination described in §2.

By a slight abuse of notation, set \( x^\pm_i = (x^0_i)^\pm \) and \( x^*_i = x^-_i = x^0_i \) when \( g_i \) is parabolic. The three consistently oriented unit spacelike vectors
\[
v_i = \frac{-1}{x^+_i \cdot x^+_i} x^+_i \boxtimes x^+_i
\]
correspond to the arcs joining \( x^+_i \) to \( x^+_i \) in \( \mathbb{H}^2 \).

Lemma 6.2. For \( i = 1, 2, 3 \), choose \( a_i, b_i > 0 \). For
\[
p_i := a_i x^+_i - b_i x^+_i
\]
the crooked planes \( \mathcal{C}(v_i, p_i) \) are pairwise disjoint.
Theorem A. Consider the following four crooked planes:

\[ \mathcal{C}(v_3, p_3), \mathcal{C}(g_1(v_3), p_1) \subset H(v_1, p_1) \]
\[ \mathcal{C}(v_2, p_2), \mathcal{C}(g_2^{-1}(v_2), p_1) \subset H(v_1, p_1) \]

Then apply Lemma 5.2 to obtain a crooked fundamental domain for the cocycle \( u \) such that \( u(g_i) = p_i - p_{i-1}, i = 1, 2, 3. \)

Every positive cocycle arises in this way. Indeed, compute the Margulis invariant for the above cocycle \( u \):

\[
\begin{align*}
\mu_1 &= (p_1 - p_3) \cdot x_1^0 \\
&= (a_1 x_1^+ - b_1 x_2^+ - a_3 x_3^+ + b_3 x_4^+) \cdot (\frac{-(x_1^+ \otimes x_1^+)}{x_1^+ \cdot x_1^+}) \\
&= (-b_1 x_2^+ - a_3 x_3^+) \cdot x_1^0
\end{align*}
\]

(Omit the second line when \( g_1 \) is parabolic.)

Recall that every product \( \beta_{i,j} = -x_i^+ \cdot x_j^0 > 0 \). In matrix form:

\[
\begin{bmatrix}
\mu_1 \\
\mu_2 \\
\mu_3
\end{bmatrix} = \begin{bmatrix}
0 & \beta_{2,1} & 0 & 0 & \beta_{3,1} & 0 \\
\beta_{1,2} & 0 & 0 & \beta_{3,2} & 0 & 0 \\
0 & 0 & \beta_{2,3} & 0 & 0 & \beta_{1,3}
\end{bmatrix} \begin{bmatrix}
a_1 \\
b_1 \\
a_2 \\
b_2 \\
a_3 \\
b_3
\end{bmatrix}
\]
and every positive triple of values \((\mu_1, \mu_2, \mu_3)\) may be realized by choosing appropriate positive values of \(a_i, b_i\). Explicitly, for \(i = 1, 2, 3\), choose \(p_i, q_i > 0\) with \(p_i + q_i = 1\), and define

\[
\begin{bmatrix}
    a_1 \\
    b_1 \\
    a_2 \\
    b_2 \\
    a_3 \\
    b_3
\end{bmatrix} =
\begin{bmatrix}
    p_2 \mu_2 / \beta_{12} \\
    q_1 \mu_1 / \beta_{21} \\
    p_3 \mu_3 / \beta_{23} \\
    q_2 \mu_2 / \beta_{32} \\
    p_1 \mu_1 / \beta_{31} \\
    q_3 \mu_3 / \beta_{13}
\end{bmatrix}.
\]

The proof of Theorem A is complete. \(\square\)

7. Embedding in an arithmetic group

As an application, we construct examples of proper affine deformations of a Fuchsian group as subgroups of the symplectic group \(\text{Sp}(4, \mathbb{R})\).

Consider a 4-dimensional real symplectic vector space \(S\) with a lattice \(S\mathbb{Z}\) such that the symplectic form takes values \(\mathbb{Z}\) on \(S\mathbb{Z}\). Fix an integral Lagrangian 2-plane \(L_\infty \subset S\), that is, a Lagrangian 2-plane generated by \(L_\infty \cap S\mathbb{Z}\). Our model for Minkowski space will be the space \(L_\infty\) of all Lagrangian 2-planes \(L \subset S\) transverse to \(L_\infty\). The underlying Lorentzian vector space is the space of linear maps \(S/L_\infty \to L_\infty\) which are self-adjoint in the sense described below. We denote by \(\text{Aut}(S) \cong \text{Sp}(4, \mathbb{R})\) the group of linear symplectomorphisms of \(S\). We denote by \(\text{Aut}(L_\infty) \cong \text{GL}(2, \mathbb{R})\) the group of linear automorphisms of the vector space \(L_\infty\).

The set of two-dimensional subspaces \(L \subset S\) transverse to \(L_\infty\) admits a simply transitive action of the vector space \(\text{Hom}(S/L_\infty, L_\infty)\), as follows. Denote the inclusion and quotient mappings by

\[
L_\infty \overset{\iota}{\hookrightarrow} S \overset{\Pi}{\twoheadrightarrow} S/L_\infty
\]

respectively. Let \(L\) be a 2-plane transverse to \(L_\infty\) and \(\phi \in \text{Hom}(S/L_\infty, L_\infty)\). Define the action \(\phi \cdot L\) of \(\phi\) on \(L\) as the graph of the composition

\[
L \overset{\Pi}{\twoheadrightarrow} S/L_\infty \overset{\phi}{\to} L_\infty \overset{\iota}{\hookrightarrow} S,
\]

that is,

\[
\phi \cdot L := \{v + \iota \circ \phi \circ \Pi(v) \mid v \in L\}.
\]

The vector group \(\text{Hom}(S/L_\infty, L_\infty) \cong \mathbb{R}^4\) acts simply transitively on the set of 2-planes \(L\) transverse to \(L_\infty\) as claimed.

Such a 2-plane \(L\) is Lagrangian if and only if the corresponding linear map \(\phi\) is self-adjoint as follows. Since \(S\) is 4-dimensional and \(L_\infty \subset S\) is Lagrangian, the symplectic structure on \(S\) defines an isomorphism of \(S/L_\infty\) with the dual vector space \(L_\infty^*\). Let \(\phi \in \text{Hom}(S/L_\infty, L_\infty)\)
be a linear map. Its transpose $\phi^T \in \text{Hom}(L_\infty^*, (S/L_\infty)^*)$ is the map induced by $\phi$ on the dual spaces. Its adjoint $\phi^* \in \text{Hom}(S/L_\infty, L_\infty)$ is defined as the composition

$$S/L_\infty \xrightarrow{\cong} L_\infty^* \xrightarrow{\phi^T} (S/L_\infty)^* \xrightarrow{\cong} L_\infty$$

and the isomorphisms above arise from duality between $S/L_\infty$ and $L_\infty$. If $L \in L_\infty$, and $\phi \in \text{Hom}(S/L_\infty, L_\infty)$, then $\phi \cdot L$ is Lagrangian if and only if $\phi = \phi^*$, that is, $\phi$ is self-adjoint. In this case $\phi$ corresponds to a symmetric bilinear form on $S/L_\infty$.

Let $\Phi \cong \mathbb{R}^3$ denote the vector space of such self-adjoint elements $\phi$ of $\text{Hom}(S/L_\infty, L_\infty)$. Then $L_\infty$ is an affine space with underlying vector space of translations $\Phi$.

Choose a fixed $L_0 \in L_\infty$. The symplectic form defines a nondegenerate bilinear form

$$L_0 \times L_\infty \longrightarrow \mathbb{R}$$

under which $L_0$ and $L_\infty$ are dual vector spaces and $S = L_\infty \oplus L_0$. The restriction $\Pi|_{L_0}$ induces an isomorphism

$$L_0 \xrightarrow{\cong} S/L_\infty.$$ 

Given $\phi \in \Phi$, a self-adjoint endomorphism of $\text{Hom}(S/L_\infty, L_\infty)$, the linear transformation of $S = L_0 \oplus L_\infty$ defined by the exponential map

$$U_\phi := \exp \left( 0 \oplus (\phi \circ \Pi|_{L_0}) \right)$$

is a unipotent linear symplectomorphism of $S$ which:

- acts identically on $L_\infty$;
- induces the identity on the quotient $S/L_\infty$.

Indeed, the exponential map is an isomorphism of the vector group $\Phi$ onto the subgroup of the linear symplectomorphism group of $S$ satisfying the above two properties.

Every linear automorphism $A$ of $L_\infty$ extends to the linear symplectomorphism of $S = L_\infty \oplus L_0$:

$$\sigma(A) := A \oplus (A^T)^{-1}.$$ 

Such linear symplectomorphisms stabilize the Lagrangian subspaces $L_\infty$ and $L_0$, and the image of $\text{Aut}(L_\infty)$ is characterized by these properties. In particular $\text{Aut}(L_\infty)$ normalizes the group $\exp(\Phi)$ corresponding to translations. These two subgroups generate the subgroup of linear symplectomorphisms of $S$ which stabilize $L_\infty$.

The vector space $\Phi$ has a natural Lorentzian structure as follows. Identify $\Phi$ with the vector space $S_2$ of $2 \times 2$ symmetric matrices. The
bilinear form

\[ S_2 \times S_2 \longrightarrow \mathbb{R} \]

\[ X \cdot Y \longmapsto \frac{\text{tr}(XY) - \text{tr}(X)\text{tr}(Y)}{2} \]

is a Lorentzian inner product of signature \((2,1)\). If \(A \in \text{Aut}(L_\infty)\), then

\[ AX \cdot AY = (\det A)^2 X \cdot Y \]

so the subgroup \(\text{SAut}(L_\infty)\) of unimodular automorphisms acts isometrically with respect to this inner product. In this way \(L_\infty\) is a model for Minkowski space and \(\text{SAut}(L_\infty)\) acts by linear isometries. In particular, \(\exp(\Phi)\) corresponds to the group of translations.

We describe this explicitly by matrices. Consider \(\mathbb{R}^4\) with standard basis vectors \(e_k\) for \(1 \leq k \leq 4\). Endow \(\mathbb{R}^4\) with the symplectic form such that:

\[ \omega(e_1, e_3) = -\omega(e_3, e_1) = 1 \]
\[ \omega(e_2, e_4) = -\omega(e_4, e_2) = 1 \]

and all other \(\omega(e_i, e_j) = 0\). That is,

\[ \omega(u, v) := u^T J v \]

where

\[ J := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}. \]

Define the complementary pair of Lagrangian planes:

\[ L_\infty := \langle e_1, e_2 \rangle \]
\[ L_0 := \langle e_3, e_4 \rangle. \]

Thus \((e_3, e_4)\) is the basis of \(L_0\) dual to the basis \((e_1, e_2)\).

Vectors in Minkowski space correspond to self-adjoint linear transformations \(L_\infty \rightarrow L_0 \cong L_\infty^*\), that is, \(2 \times 2\) symmetric matrices as follows. A symmetric matrix

\[ \psi(x, y, z) := \begin{bmatrix} x & y \\ y & z \end{bmatrix} \]

corresponds to a vector in Minkowski space with quadratic form

\[ -\det(\psi) = xz - y^2. \]
The unipotent symplectomorphism corresponding to a symmetric matrix \( \psi(x, y, z) \in S_2 \) is:

\[
U_{\psi(x, y, z)} := \exp \begin{pmatrix}
0 & 0 & x & y \\
0 & 0 & y & z \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & x & y \\
0 & 1 & y & z \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]
where \( x, y, z \in \mathbb{R} \). These correspond to the translations of Minkowski space, and comprise the subgroup \( U \subset Sp(4, \mathbb{R}) \).

The reductive subgroup \( Aut(L_{\infty}) \cong GL(2, \mathbb{R}) \) embeds in \( Aut(S) \cong Sp(4, \mathbb{R}) \) as follows: let

\[
A := \begin{bmatrix}
a & b \\
c & d
\end{bmatrix} \in GL(2, \mathbb{R}) \cong Aut(L_{\infty})
\]
with determinant \( \Delta := \det(A) \). The corresponding linear symplectomorphism preserving the decomposition \( S = L_{\infty} \oplus L_0 \) is:

\[
\sigma(A) := \begin{bmatrix}
a & b & 0 & 0 \\
c & d & 0 & 0 \\
0 & 0 & d/\Delta & -c/\Delta \\
0 & 0 & -b/\Delta & a/\Delta
\end{bmatrix}.
\]

These correspond to linear conformal transformations of Minkowski space. The subgroup \( SAut(L_{\infty}) \) of unimodular automorphisms of \( L_{\infty} \) corresponds to the group of linear isometries of Minkowski space.

The subgroup of \( Aut(S) \) generated by \( U \) and \( SAut(L_{\infty}) \) is a semidirect product \( U \rtimes SAut(L_{\infty}) \) and acts by conjugation on the normal subgroup \( U \). This action corresponds to the action of the group of affine isometries of Minkowski space.

We construct subgroups of \( Sp(4, \mathbb{Z}) \) which act properly on the \( S_2 \) model of \( E^3_1 \). The linear parts and translational parts of Lorentzian transformations of \( S_2 \) are associated with elements of \( Sp(4, \mathbb{R}) \). The level two congruence subgroup \( \Gamma_0 \) of \( SL(2, \mathbb{Z}) \) is generated by

\[
g_1 := -\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad g_2 := -\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}, \quad g_3 := -\begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix},
\]
subject to the relation \( g_1 g_2 g_3 = \mathbb{I} \). It is freely generated by \( g_1 \) and \( g_2 \). All three \( g_i \) are parabolic and the quotient hyperbolic surface \( \Sigma := H^2/\Gamma_0 \) is a three-punctured sphere. The symmetric matrices

\[
v_1 := \begin{bmatrix}
-2 & 0 \\
0 & 0
\end{bmatrix}, \quad v_2 := \begin{bmatrix}
0 & 0 \\
0 & -2
\end{bmatrix}, \quad v_3 := \begin{bmatrix}
-2 & -2 \\
-2 & -2
\end{bmatrix}
\]
define positive fixed vectors with respect to \( g_1, g_2, g_3 \) respectively. The triple \((v_1, v_2, v_3)\) defines a decoration of \( \Sigma \).
An affine deformation of $\Gamma_0$ is defined by two arbitrary vectors $u_1, u_2 \in S_2$ as translational parts:

$$ u_1 := \begin{bmatrix} a_1 & b_1 \\ b_1 & c_1 \end{bmatrix}, \quad u_2 := \begin{bmatrix} a_2 & b_2 \\ b_2 & c_2 \end{bmatrix}. $$

Thus the affine transformations with linear part $g_i$ and translational part $u_i$ are:

$$ \gamma_1 := \begin{bmatrix} 1 & 0 & a_1 & b_1 \\ 0 & 1 & b_1 & c_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix} $$

$$ \gamma_2 := \begin{bmatrix} 1 & 0 & a_2 & b_2 \\ 0 & 1 & b_2 & c_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix} $$

and

$$ \gamma_3 := \begin{bmatrix} 1 & 0 & a_3 & b_3 \\ 0 & 1 & b_3 & c_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 2 & 1 \end{bmatrix} $$

where $\gamma_3 = (\gamma_1 \gamma_2)^{-1}$,

- $a_3 = -a_1 - a_2 + 4b_1 - 4c_1$,
- $b_3 = -2a_1 - 2a_2 + 7b_1 - b_2 - 6c_1$, and
- $c_3 = -4a_1 - 4a_2 + 12b_1 - 4b_2 - 9c_1 - c_2$.

The corresponding Margulis invariants taken with respect to $v_1, v_2, v_3$ are:

$$ \mu_1 = c_1 $$

$$ \mu_2 = a_2 $$

$$ \mu_3 = c_1 + c_2 - 2b_1 + 2b_2 + a_1 + a_2. $$

By Theorem A, the affine deformation $\Gamma := \langle \gamma_1, \gamma_2 \rangle$ acts properly with crooked fundamental domain whenever

$$ \mu_1 > 0 $$

$$ \mu_2 > 0 $$

$$ \mu_3 > 0. $$

Furthermore, taking $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{Z}$ implies $\Gamma \subset \text{Sp}(4, \mathbb{Z})$. 
Here are some explicit examples. Consider, for example, the slice for translational conjugacy defined by $b_1 = b_2 = c_2 = 0$. Choose three positive integers $\mu_1, \mu_2, \mu_3$. Take

\[
\begin{align*}
a_1 &= \mu_3 - \mu_1 - \mu_2 \\
c_1 &= \mu_1 \\
a_2 &= \mu_2,
\end{align*}
\]

that is, let

\[
\gamma_1 := \begin{bmatrix}
1 & 0 & \mu_3 - \mu_1 - \mu_2 & 0 \\
0 & 1 & 0 & \mu_1 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & -2 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 2 & -1
\end{bmatrix}
\]

and

\[
\gamma_2 := \begin{bmatrix}
1 & 0 & \mu_2 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 0 & 0 & 0 \\
0 & 2 & -1 & 0 \\
0 & 0 & -1 & -2 \\
0 & 0 & 0 & -1
\end{bmatrix}
\]

The proof of Theorem B is complete. \qed

**APPENDIX. PROOF OF LEMMA 6.1**

We return to the parametrization of the cohomology $H^1(\Gamma_0, \mathbb{R}^3_1)$ by the three generalized Margulis invariants $\mu_1, \mu_2, \mu_3$ associated to the respective generators $g_1, g_2, g_3$ associated to components of $\partial \Sigma$. When $g_i$ is parabolic, choose a positive vector $v_i$ to define $\mu_i$. We must show that the triple $\mu = (\mu_1, \mu_2, \mu_3)$ defines an isomorphism

\[
H^1(\Gamma_0, \mathbb{R}^3_1) \longrightarrow \mathbb{R}^3.
\]

Under the double covering $\text{SL}(2, \mathbb{R}) \longrightarrow \text{SO}(2, 1)^0$, lift $\rho_0$ to a representation $\pi \longrightarrow \text{SL}(2, \mathbb{R})$. The condition that $\rho_0(\pi)$ is not solvable implies that the representation $\tilde{\rho}_0$ on $\mathbb{R}^3$ is irreducible. By a well-known classic theorem (see, for example, Goldman [20]), such a representation is determined up to conjugacy by the three traces

\[
a_i := \text{tr}(\tilde{\rho}_0(A_i)).
\]
and, choosing $b_3$ such that $b_3 + 1/b_3 = a_3$, we may conjugate $\tilde{\rho}_0$ to the representation defined by:

\[
\tilde{\rho}_0(A_1) = \begin{bmatrix} a_1 & -1 \\ 1 & 0 \end{bmatrix}, \quad \tilde{\rho}_0(A_2) = \begin{bmatrix} 0 & -b_3 \\ 1/b_3 & a_2 \end{bmatrix}, \quad \tilde{\rho}_0(A_3) = \begin{bmatrix} b_3 & -a_1 c_3 + a_2 \\ 0 & 1/b_3 \end{bmatrix}.
\]

(7)

Since $\pi$ is freely generated by $A_1, A_2$, a cocycle $\pi \xrightarrow{u} \mathbb{R}_1^3$ is completely determined by two values $u(A_1), u(A_2) \in \mathbb{R}_1^3$. Furthermore, since $\rho_0(\pi)$ is nonsolvable, the coboundary map

\[
\mathbb{R}_1^3 \xrightarrow{\partial} Z^1(\Gamma_0, \mathbb{R}_1^3)
\]

is injective. Therefore the vector space $H^1(\Gamma_0, \mathbb{R}_1^3)$ has dimension three.

To show that the linear map $\mu$ is an isomorphism, it suffices to show that $\mu$ is onto. To this end, it suffices to show that for each $i = 1, 2, 3$ there is a cocycle $u \in Z^1(\Gamma_0, \mathbb{R}_1^3)$ such that $u(A_i) \neq 0$ and $u(A_j) = 0$ for $j \neq i$. By cyclic symmetry it is only necessary to do this for $i = 1$.

Under the local isomorphism $\text{SL}(2, \mathbb{R}) \xrightarrow{\mu} \text{SO}(2, 1)^0$, the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ maps to the Lie algebra $\mathfrak{so}(2, 1)$ which in turn maps isomorphically to the Lorentzian vector space $\mathbb{R}_1^3$. (Compare [22, 19, 8].) If $g \in \text{SL}(2, \mathbb{R})$ is hyperbolic or parabolic, then a neutral eigenvector $x_0(g)$ is a nonzero multiple of the traceless projection

\[
\hat{g} := g - \frac{\text{tr}(g)}{2} I.
\]

Define a cocycle for the representation $\tilde{\rho}_0$ defined in (7) by:

\[
\mu_1(u) := \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mu_2(u) := \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \mu_3(u) := \begin{bmatrix} 0 & 0 \\ -1/c & 0 \end{bmatrix}.
\]

Then $\mu_1(u) \neq 0$ but $\mu_2(u) = \mu_3(u) = 0$ as claimed. The proof of Lemma 6.1 is complete.

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