Generalized Complex and Dirac Structures on Homogeneous Spaces

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Abstract

Generalized complex geometry [7] has been a subject of recent interest in mathematics and physics and is a general setting for differential geometry. The aim of this paper is to study generalized complex geometry and Dirac geometry [3], [4] on homogeneous spaces. We offer a characterization of equivariant Dirac structures on homogeneous spaces, which is then used to construct new examples of generalized complex structures. We consider Riemannian symmetric spaces, quotients of compact groups by closed connected subgroups of maximal rank, and nilpotent orbits in \( \mathfrak{sl}_n(\mathbb{R}) \). For each of these cases, we completely classify equivariant Dirac structures. Additionally, we consider equivariant Dirac structures on semisimple orbits in a semisimple Lie algebra. Here equivariant Dirac structures can be described in terms of root systems or by certain data involving parabolic subagebras.

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Hitchin’s notion of “generalized complex geometry” \[7\] is a common generalization of complex and symplectic structures which has found several applications in physics and mathematics \[10\], \[7\], \[5\]. For instance, the generalized Kähler structure turns out to be precisely the setting for \(N = 2\) supersymmetric sigma models \[10\]. Generalized complex structures are a special case of complex Dirac structures, a concept defined by Courant and Weinstein \[3\], \[4\] that also includes Poisson structures, integrable distributions, and presymplectic structures.

After a brief exposition on the essential ideas in generalized complex geometry in \(\S\) 2, we partially classify equivariant generalized complex structures on homogeneous spaces and, more generally, equivariant (complex) Dirac structures on homogeneous spaces in \(\S\) 3 and \(\S\) 4. This gives a description of such Dirac structures in terms of linear algebra data. The \(G\)-equivariant generalized complex structures \(L\) on \(G/K\) are in bijection with pairs \((E, \varepsilon)\) of a subalgebra \(E \subset g_C\) and \(\varepsilon \in \wedge^2 E^*\) satisfying certain conditions. This bijection allows us to provide some new examples of generalized complex structures.

Here is a list of results for particular classes of homogeneous spaces.

- \(\clubsuit \) \textit{G compact and K connected of maximal rank} (i.e. \(K\) contains a Cartan subgroup). We completely classify equivariant generalized complex structures. In this setting, there are examples of generalized complex structures which are neither symplectic nor complex.

- \(\spadesuit \) \textit{Semisimple coadjoint orbits in semisimple real Lie algebras}. We describe generalized complex structures in terms of simpler “combinatorial” data that involves only the root system.

- \(\heartsuit \) \textit{Real nilpotent orbits}. Here we restrict ourselves to split semisimple Lie algebras. For the Lie algebra \(\mathfrak{sl}_n(\mathbb{R})\), the only equivariant generalized complex structures are B-transforms of symplectic structures. We hypothesize that this is true for any split semisimple Lie algebra and show that the claim reduces to distinguished orbits in simple Lie algebras.

- \(\diamondsuit \) \textit{Riemannian Symmetric Spaces}. Again we completely classify equivariant generalized complex structures but these turn out to yield little that is new. Every generalized complex structure on a Riemannian symmetric space is essentially a product of complex and (B-transforms of) symplectic structures.
2 Dirac and Generalized Complex Geometry

We introduce the basic definitions and notational conventions used in this paper. For a systematic development of generalized complex structures as well as some of their applications, we refer the reader to [5]. The notation of [5] will most strongly be followed.

For a manifold $M$, generalized geometry is concerned with the bundle $V_M := TM \oplus T^*M$. There is a natural bilinear form on $V_M$, given by the obvious pairing $\langle X + \xi, Y + \eta \rangle = X(\eta) + Y(\xi)$ for sections $X, Y$ of $TM$ and $\xi, \eta$ of $T^*M$. Furthermore, $V_M$ is equipped with the Courant bracket defined by

$$[X + \xi, Y + \eta] = [X, Y] + \iota_X d\eta + \frac{1}{2} d(\iota_X \eta) - \iota_Y d\xi - \frac{1}{2} d(\iota_Y \xi),$$

where $\iota$ denotes contraction in the first variable ($\iota_x \phi = \phi(x, -, \ldots)$). The Courant bracket $[ , ]$ and the bilinear form $\langle , \rangle$ extend $\mathbb{C}$-bilinearly to $(V_M)_\mathbb{C} = V_M \otimes \mathbb{C}$.

**Definition 1.** A generalized almost complex structure on $M$ is a map $\mathcal{J} : V_M \rightarrow V_M$ such that $\mathcal{J}$ is orthogonal with respect to the inner product $\langle , \rangle$ and $\mathcal{J}^2 = -1$. Just as with complex structures, one may consider the $i$-eigenbundle, $D$, of $\mathcal{J}$ in $(V_M)_\mathbb{C}$. The Courant bracket defines an integrability condition $([D, D] \subset D)$ for $\mathcal{J}$ to be called a generalized complex structure. This follows the analogy with almost complex structures; an almost complex structure is a complex structure precisely when its $i$-eigenbundle is integrable with respect to the Lie bracket.

The two canonical examples of generalized complex structures come from complex and symplectic structures. Since $V_M = TM \oplus T^*M$, we can express any map $V_M \rightarrow V_M$ as a block matrix in terms of this decomposition, and we will follow this convention throughout the text. If $J$ is a complex structure,

$$\begin{bmatrix} J & 0 \\ 0 & -J^* \end{bmatrix}$$

is a generalized complex structure. The $i$-eigenbundle of $J$ is $E \oplus Ann(E)$, where $E$ is the $i$-eigenbundle of $J$, and $Ann(E)$ is the annihilator of $E$ in $T^*M$.

For a symplectic structure $\omega$, we get a generalized complex structure

$$\begin{bmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{bmatrix}$$

where $\omega(x) := \omega(x, -)$. The $i$-eigenbundle is the graph of $i\omega$ in $(V_M)_\mathbb{C}$. The fact that the symplectic form $\omega$ is closed implies that this generalized almost complex structure is integrable, hence a generalized complex structure.

The $i$-eigenbundle $D$ of a generalized complex structure $\mathcal{J}$ turns out to be an integrable maximal isotropic subbundle of $(V_M)_\mathbb{C}$, also known as a complex Dirac structure. Thus, the study of generalized geometry now lies in the framework of Dirac structures. With this in mind, we recall the following working definitions for our paper.

**Definition 2.** For any manifold, $M$,

1. A real almost Dirac structure on $M$ is a maximal isotropic subbundle $D$ of $V_M$. A real almost Dirac structure is called a real Dirac structure if it is integrable with respect to the Courant bracket. Similarly, a complex almost Dirac structure is a maximal isotropic subbundle $D \subset (V_M)_\mathbb{C}$, and a complex Dirac structure is an integrable complex almost Dirac structure.

2. A complex Dirac structure $D$ is said to be of constant rank if the projection map $pr : D \rightarrow TM$ is of constant rank.
A generalized (almost) complex structure $\mathcal{J}$ is equivalent to is a complex (almost) Dirac structure $D$ such that $D \cap \overline{D} = 0$. Note that the integrability is a closed condition and that being generalized complex is an open condition. Henceforth we will think of generalized complex structures as complex Dirac structures.

Since the complexification of any Dirac structure is a complex Dirac structure, both generalized complex structures and Dirac structures are complex Dirac structures. Thus, the set of complex Dirac structures contains real Dirac structures and generalized complex structures. Henceforth (almost) Dirac structure will always mean complex (almost) Dirac structure, and we will specify whether it is also real Dirac (i.e. if $\overline{\mathcal{D}} = \mathcal{D}$) if there is any ambiguity.

Most of the complex Dirac structures considered in this paper will be of constant rank. It is checked in [5] that any complex Dirac structure of constant rank is of the form

$$L(E, \varepsilon) := \{X + \xi \in (\mathcal{V}_M)_C \mid X \in E \text{ and } \iota_X \varepsilon = \xi_{\mid E}\},$$

where $E$ is a subbundle of $TM$ and $\varepsilon \in \Gamma(M, \wedge^2 E^*)$. Complex and symplectic structures, for example, are of this form. For a subbundle $E$ of $TM$, we define the differential $d_E : \Gamma(M, \wedge^2 E^*) \rightarrow \Gamma(M, \wedge^3 E^*)$ by the following formula. For sections $X,Y,Z$ of $E$ and $\varepsilon \in \Gamma(M, \wedge^2 E^*)$, we set

$$d_E \varepsilon(X,Y,Z) = \varepsilon(X,[Y,Z]) + \varepsilon(Y,[Z,X]) + \varepsilon(Z,[X,Y]) + X\varepsilon(Y,Z) - Y\varepsilon(X,Z) + Z\varepsilon(X,Y).$$

In other words, $d_E \varepsilon$ is the restriction to $\wedge^3 E$ of the ordinary De Rham differential of any extension $\varepsilon \in \wedge^2 T^* M$ of $\varepsilon$. Gualtieri [5] proves the following useful lemma.

**Lemma 2.1.** A complex almost Dirac structure of constant rank $L(E, \varepsilon)$ is integrable if and only if $E$ is integrable and $d_E \varepsilon = 0$.

**Remark 2.2.** There are several formulations of the integrability condition for an almost Dirac structure $L$ on $M$. For example, the Lie algebroid derivative $d$ defined in [5] extends to $d : \wedge^k \mathcal{V}_M \rightarrow \wedge^{k+1} \mathcal{V}_M^*$. Then it is easily verified that since $L = \text{Ann}(L)$ and $\mathcal{V}_M$ is naturally isomorphic to $\mathcal{V}_M^*$ via the inner product, $L$ is integrable if and only if $dL \subset L \wedge \mathcal{V}_M$. Gualtieri [5] shows that there is a similar condition when one describes $L$ in terms of pure spinors. Since equivariant Dirac structures on a homogeneous space are always of constant rank, lemma [2,1] will be the most useful criterion for determining integrability. The following lemma also allows one to determine integrability in special cases by more familiar criterion—such as having a Poisson bivector.

**Lemma 2.3.** We recall from [2] that the following procedures create Dirac structures from geometric structures on $M$. The first three are real, and (4) and (5) are special cases of generalized complex structures (as we have seen).

1. To an integrable distribution $\mathcal{D} \subset TM$, assign $[\mathcal{D} \oplus \text{Ann}(\mathcal{D})]_C$.
2. To a Poisson structure $\pi \in \Gamma(M, \wedge^2 TM)$, assign $L(\pi, T^* M)$.
3. To a presymplectic structure $\omega \in \Omega^2(M)$, assign $L(TM, \omega)$.
4. To a complex structure $J$, assign $T^{(1,0)} M \oplus T^{*(0,1)} M$, where $T^{(1,0)} M$ and $T^{(0,1)} M$ denote the holomorphic and antiholomorphic tangent bundles respectively with respect to $J$.
5. To a symplectic structure $\omega \in \Omega^2(M)$, assign $L(TM_C, i\omega)$.

**Remark 2.4.** A symplectic form $\omega$ on $M$ determines a complex Dirac structure in one of two ways: $L(TM, \omega)$ and $L(T\mathcal{C} M, -i\omega)$. The former is a real Dirac structure, and the latter is a generalized complex structure.
We recall the notions of pullback and pushforward of linear Dirac structures \cite{5}. For a map \( f : D \rightarrow T^*M \) with constant rank, there is a subbundle \( U \subseteq T^*M \) such that \( D \) is of the form

\[
L(\pi, U) := \{ X + \xi | X|_\pi = \iota_\xi \pi \}.
\]

If \( U = T^*M \), then for \( \pi \in \Gamma(M, \wedge^2 TM) \), \( L(\pi, T^*M) \) is a Dirac structure if and only if \( \pi \) is a Poisson bi-vector \cite{5}, \cite{12}. Now presymplectic structures, complex structures, and Poisson structures can all be considered Dirac structures.

We recall the notions of pullback and pushforward of linear Dirac structures \cite{5}. For a map \( F : V \rightarrow W \) of vector spaces and a subspace \( D \subseteq V \oplus V^* \), define

\[
F_*D = \{ FX + \xi \in W \oplus W^* | X + F^*\xi \in D \}
\]

and for a subspace \( D \subseteq W \oplus W^* \), define

\[
F^*D = \{ X + F^*\xi \in V \oplus V | FX + \xi \in D \} = (F^*), D.
\]

Now let \( f : M \rightarrow N \) be any map of manifolds. For a Dirac structure, \( D \), on \( N \), the pullback \( f^*D \) is defined pointwise by \( (f^*D)_p = (df_p)^* D_{f(p)} \). It is not necessarily itself a Dirac structure.

### 2.1 Twisted Courant Bracket and Automorphisms

In addition to the standard Courant bracket on \( \mathcal{V}_M \), Severa and Weinstein noticed a twisted Courant bracket \([ , ]_H \) for each closed 3-form \( H \) on \( M \), defined as

\[
[X + \xi, Y + \eta]_H = [X + \xi, Y + \eta] + H(X, Y, -)
\]

and \( \mathcal{V}_{M,H} := (TM \oplus T^*M, [ , ]_H, \langle , \rangle) \) so that \( \mathcal{V}_M \) is just \( \mathcal{V}_{M,0} \). For any 2-form \( B \) on \( M \), there is an automorphism of the vector bundle \( TM \oplus T^*M \),

\[
\begin{bmatrix} 1 & 0 \\ B_\eta & 1 \end{bmatrix},
\]

denoted \( e^B \). Indeed, \( e^B \) is an isomorphism \( \mathcal{V}_{M,H+dB} \rightarrow \mathcal{V}_{M,H} \). In other words, \( e^B \) is orthogonal with respect to \( \langle , \rangle \), and \( [e^B u, e^B v]_H = e^B [u, v]_{H+dB} \) for all sections \( u, v \) of \( TM \oplus T^*M \). When \( B \) is closed, then \( e^B \) is an automorphism of \( \mathcal{V}_{M,H} \). This is what we call a B-transform, i.e. an automorphism of \( \mathcal{V}_{M,H} \) of the form \( e^B \) for a closed 2-form \( B \). In fact, the automorphism group of \( \mathcal{V}_{M,H} \) is the semidirect product of the group of diffeomorphisms \( M \rightarrow M \) and closed 2-forms \( Z^2(M) \) \cite{5}. B-transforms are thought of as the symmetries of the Courant bracket. An \( H \)-twisted Dirac structure \( D \subseteq \mathcal{V}_{M,H} \) is simply a maximal isotropic subbundle which is integrable with respect to the \( H \)-twisted Courant bracket.

## 3 GENERALIZED COMPLEX AND DIRAC STRUCTURES ON HOMOGENEOUS SPACES

We partially describe equivariant Dirac and generalized complex structures on a homogeneous space \( G/K \) by giving equivalent data involving only the Lie algebra. The main results are in Theorem \ref{3.11} and Proposition \ref{4.8} where we parameterize equivariant Dirac structures and generalized complex structures by pairs \((E, \varepsilon)\) of a Lie subalgebra \( E \) of \( \mathfrak{g}_C \) and \( \varepsilon \in \wedge^2 E^* \) satisfying some conditions.
3.1 Distributions on Homogeneous Spaces

In this subsection, we recall some well-known background facts for operations on Dirac structures. We would like to classify $G$-invariant distributions on a homogeneous space $G/K$. By pulling back a distribution on $G/K$ to a distribution on $G$ and then considering the subspace of $g$ determined by this distribution on $G$, this will give for each distribution on $G/K$ a subspace of $g$ which uniquely determines the distribution on $G/K$. In particular, this can be used for complex distributions given by complex structures on $G/K$.

Let $f : M \to N$ be a submersion and $D \subset T N$ a distribution on $N$. Define $f^{-1} D$ by $(f^{-1} D)_p = df_p^{-1}(D_{f(p)})$.

**Proposition 3.1.** If $D$ is a distribution on $N$ and $f : M \to N$ is a submersion, then

1. $f^{-1}D$ is a distribution on $M$, and it is integrable if and only if $D$ is integrable.
2. If $M$ and $N$ are $G$-spaces and $f$ is $G$-equivariant, then $D$ is $G$-invariant if and only if $f^{-1}D$ is $G$-invariant.

**Proof.** 1. Both of these statements are local. Since $f$ is a submersion, locally $f$ looks like $p = pr_1 : U \times V \to U$. Now $p^{-1} D = D \oplus TV$, i.e. $p^{-1} D = pr_1^*D + pr_2^*TV$, and both claims are clear.

2. Let $L_g^M$ denote left multiplication by $g \in G$ on $M$, and let $L_g^N$ denote left multiplication by $g$ on $N$. Let $p \in M$. Since $f \circ L_g^M = L_g^N \circ f$, $(df_p)_*(L_g^M)_p(f^{-1}D)_p = ((L_g^N)_p f)_*(D_{f(p)})$. If $f^{-1}D$ is left-invariant, then $D_{g(f(p))} = df_g(p)^*(f^{-1}D)_p = df_g(p)(L_g^M)_p(f^{-1}D)_p = (L_g^N)_pD_{f(p)}$, and $D$ is equivariant. Now suppose that $D$ is equivariant. Then $df_g(p)(L_g^M)_p(f^{-1}D)_p = (L_g^N)_pD_{f(p)} = D_{g(f(p))}$, which shows that $(dL_g)_p(f^{-1}D)_p \subset f^{-1}D_{g(p)}$, so $(dL_g)_p(f^{-1}D)_p = f^{-1}D_{g(p)}$, and $f^{-1}D$ is left-invariant. \(\square\)

**Corollary 3.2.** Let $G$ be any Lie group and $K$ any closed, connected subgroup. There is a bijection between left-invariant integrable distributions on $G$ and Lie subalgebras of $g$, and there is a bijection between $G$-invariant integrable distributions on $G/K$ and Lie subalgebras of $g$ containing $\mathfrak{t}$.

**Proof.** There is a bijection between subspaces $u$ of $g$ and left-invariant distributions $\mathcal{D}$ on $G$ given by $u = D_e$. The Lie algebra structure of $\mathfrak{g}$ is defined by the Lie bracket on left-invariant sections of $TG$. Therefore, the integrable left-invariant distributions correspond to subalgebras. Now by Proposition 3.1 and the first part of this proof, sending a distribution $D$ on $G/K$ to $(\pi^{-1} D)_e$ gives an injective map from $G$-invariant integrable distributions on $G/K$ into subalgebras of $g$ containing $\mathfrak{t}$. It remains to show that any subalgebra $u$ of $\mathfrak{g}$ containing $\mathfrak{t}$ is of the form $(\pi^{-1}D)_e$ for some integrable $G$-invariant distribution $D$ on $G/K$. Given a subalgebra $u \supset \mathfrak{t}$, define $D_{\pi^{-1}(\mathfrak{g})} = (dL_{\pi^{-1}(\mathfrak{g})})_{\mathfrak{t}}u$. Since $u/\mathfrak{t}$ is a subrepresentation of $K$ in $\mathfrak{g}/\mathfrak{t}$, $D$ is a well-defined $G$-invariant distribution on $G/K$. By Proposition 3.1, $\pi^{-1}D$ is a left-invariant distribution on $G$. Since $(\pi^{-1}D)_e = u$ is a subalgebra, $\pi^{-1}D$ is integrable and therefore so is $D$. \(\square\)

**Remark 3.3.** If $K$ were disconnected, Corollary 3.2 would say that there is a bijection between $G$-invariant integrable distributions on $G/K$ and $K$-invariant subalgebras of $g$ containing $\mathfrak{t}$.

3.2 Homogeneous Dirac Structures

**Lemma 3.4.** Let $f : M \to N$ be a submersion and $\mathcal{D}$ be an almost Dirac structure on $N$.

1. If $\mathcal{D} = L(E, \varepsilon)$ is of constant rank, then $f^*\mathcal{D} = L(f^{-1}E, f^*\varepsilon)$, so $f^*\mathcal{D}$ is an almost Dirac structure.
2. $f^*\mathcal{D}$ is integrable if and only if $\mathcal{D}$ is integrable.

**Proof.** (1) We need to check that for all $p \in M$, $L(f^{-1}E, f^*\varepsilon)_p = (df_p)^*L(E, \varepsilon)_{f(p)}$. Hence we must show that for a linear map of vector spaces $\varphi : V \to W$ and $\mathcal{D} = L(E, \varepsilon) \subset W \oplus W^*$, $\varphi^*L(E, \varepsilon) = L(\varphi^{-1}E, \varphi^*\varepsilon)$. First we show that $\varphi^*\mathcal{D}$ is maximal isotropic, as is pointed out in [5]. The fact that $\langle X + \varphi^*\xi, Y + \varphi^*\eta \rangle = \langle \varphi X + \xi, \varphi Y + \eta \rangle$ implies that $\varphi^*\mathcal{D} \subset (\varphi^*\mathcal{D})^\perp$. It is obvious that $\varphi^* Ann(E) \subset Ann(\varphi^{-1}E)$. Since there is an injection $\varphi : V/\varphi^{-1}E \to W/E$, there is a surjection $\varphi^* : Ann(E) \to Ann(\varphi^{-1}E)$.
that By way of counterexample, let a submersion is sufficient but not necessary to ensure that

\[ D \]

\[ \text{fact }[5] \] that a product

\[ E \]

\[ f \]

(2) Note that if \( f \) is not a smooth subbundle of \( V_M \), and the requirement that \( f \) be a submersion is sufficient but not necessary to ensure that \( f^*D \) is a Dirac structure.

By way of counterexample, let \( f : \mathbb{R} \to \mathbb{R}^2 \) be \( x \mapsto (x,0) \), and let \( D = L(E,0) = E \oplus \text{Ann}E \), where \( E = \mathbb{R}(\frac{\partial}{\partial x} + x \frac{\partial}{\partial y}) \). Obviously \( D \) is maximal isotropic. Since \( E \) is a 1-dimensional distribution, it is integrable. Also \( d_E \xi = 0 \). It follows from Lemma \[2.11\] that \( D \) is integrable. Therefore, \( D \) is a Dirac structure. However, the pullback at 0 is \( (f^*D)_0 = df_0^*(D_{(0,0)}) = \mathbb{R} \cdot \frac{\partial}{\partial x} \). whereas if \( x \neq 0 \), \( (f^*D)_x = df^*(D_{(x,0)}) = \mathbb{R} \cdot dx. \)

Therefore \( f^*D \) is not a smooth subbundle of \( V_M \).

Remark 3.5. In general, \( f^*D \) need not even be a smooth subbundle of \( V_M \), and the requirement that \( f \) be a submersion is sufficient but not necessary to ensure that \( f^*D \) is a Dirac structure.

Proposition 3.6. The Courant bracket on \( (V_G)_C \) gives \( g_C \oplus g_C^* \), the structure of a Lie algebra. This is the semidirect product \( g_C \rtimes g_C^* \), where Lie algebra \( g_C \) acts on \( g_C^* \) by the coadjoint representation.

Proof. It must be shown that the Courant bracket is closed on left-invariant sections of \( (V_G)_C \). The Courant bracket is invariant under diffeomorphisms \( G \to G \). Left-invariant sections of \( (V_G)_C \) are invariant under the diffeomorphisms \( L_g, g \in G \). It follows that the Courant bracket preserves left-invariance. For left-invariant sections, the formula for the Courant bracket is easily seen to be the same as \( g_C \rtimes g_C \).

Since \( \{ , \} \) is clearly anti-symmetric, the only thing to be checked is that the restriction of \( \{ , \} \) to left-invariant sections of \( (V_G)_C \) satisfies the Jacobi identity. It is known \[2.12\] that for sections \( A,B,C \) of \( (V_G)_C \),

\[ \text{Jac}(A,B,C) = d(Nij(A,B,C)), \]

where \( Nij(A,B,C) = 1/3(\langle [A,B],C \rangle + \langle [B,C],A \rangle + \langle [C,A],B \rangle) \). Then if \( A, B, C \) are constant sections (meaning left-invariant), \( Nij(A,B,C) \) is constant and therefore \( \text{Jac}(A,B,C) = 0 \).

Definition 3. We say that a maximal isotropic subalgebra of \( g_C^* \rtimes g_C \) is a Dirac Lie subalgebra of \( g_C^* \rtimes g_C \) or a linear Dirac structure on \( g_C \oplus g_C^* \), and a maximal isotropic subspace of \( g_C \oplus g_C^* \) is called a linear almost Dirac structure.

Proposition 3.7. 1. There is a bijection between linear (almost) Dirac structures on \( g_C^* \rtimes g_C \) and left-invariant (almost) Dirac structures on \( G \).

2. For a subspace \( E \subseteq g_C \) and \( \varepsilon \in \wedge^2 E^* \), let \( \tilde{E} \) be the left-invariant distribution on \( E \) determined by \( E \), and let \( \tilde{\varepsilon} \in \wedge^2 E^* \) be the left-invariant 2-form determined by \( \varepsilon \). The linear almost Dirac structure \( L(E,\varepsilon) \) corresponds to the almost Dirac structure \( L(\tilde{E},\tilde{\varepsilon}) \) on \( G \).

Proof. We will show that if \( L \) is a linear almost Dirac subspace of \( g_C \oplus g_C^* \), then it determines a unique left-invariant almost Dirac structure \( \mathcal{L} \) on \( G \) such that \( \mathcal{L}_e = L \) and such that \( \mathcal{L} \) is integrable if and only if \( L \) is a Lie subalgebra of \( g_C^* \rtimes g_C \). We also show that, on the other hand, any left-invariant Dirac structure \( \mathcal{L} \) induces a linear Dirac structure \( \mathcal{L}_e \) of \( g_C^* \rtimes g_C \).
That \( L \) determines a G-invariant almost Dirac structure \( \mathcal{L} \) is obvious. If \( \mathcal{L} \) is integrable, it is apparent from the definition of the Courant bracket on \( \mathfrak{g}_C^* \times \mathfrak{g}_C \) that \( L \) is a subalgebra, for if \( X,Y \in L \), then \([X,Y] = [X,Y]_e \in L \) because \( \mathcal{L} \) is integrable.

Now suppose that \( L \) is a subalgebra. For sections \( X + \xi, Y + \eta \) of \( (\mathcal{V}_G)_C \) and \( f \in C^\infty(G)_C \), it is a general fact of Courant brackets \([5]\) that \([X + \xi, f(Y + \eta)] = f[A,B] + X(f)(Y + \eta) - (X + \xi, Y + \eta)df\). Therefore, if \( \mathcal{L} \) is the left-invariant almost Dirac structure such that \( \mathcal{L}_e = L \), to check integrability of \( \mathcal{L} \), it suffices to check it on a frame of \( \widetilde{\mathcal{L}} \). With this in mind, choose any basis \( X_1,...,X_n \) of \( L \), which provides a left-invariant frame \( \tilde{X}_1,...,\tilde{X}_n \) of \( \mathcal{L} \). By definition of the Lie algebra structure on \( \mathfrak{g}_C^* \times \mathfrak{g}_C \), \([X_i,\tilde{X}_j] = [\tilde{X}_i,X_j]\). But since \( L \) is a subalgebra, \([X_i,X_j] \in L \) and so \([\tilde{X}_i,\tilde{X}_j] \) is a section of \( \mathcal{L} \). Therefore, \( \mathcal{L} \) is integrable.

The final claim is transparent because \( L(\tilde{E},\tilde{\varepsilon})_e = L(E,\varepsilon) \). □

**Definition 4.** A subalgebra \( E \subset \mathfrak{g}_C \) and \( \varepsilon \in \wedge^2E^* \) determine left-invariant \( \tilde{E} \) and \( \tilde{\varepsilon} \) as in Proposition \([3.7]\).

For \( X,Y,Z \in E \), let \( \tilde{X},\tilde{Y},\tilde{Z} \) be the left-invariant vector fields on \( G \) with respective values \( X,Y,Z \) at \( e \in G \). Define the differential \( d_{\tilde{E}} : \wedge^2E^* \to \wedge^3E^* \) by \( d_{\tilde{E}}\varepsilon(X,Y,Z) := d_{\tilde{E}}\tilde{\varepsilon}(\tilde{X},\tilde{Y},\tilde{Z})_e \).

**Remark 3.8.** The differential \( d_{\tilde{E}} \) is given by the following formula:

\[
d_{\tilde{E}}\varepsilon(X,Y,Z) = \varepsilon(X,[Y,Z]) + \varepsilon(Y,[Z,X]) + \varepsilon(Z,[X,Y]).
\]

With this formula, \( d_{\tilde{E}} \) is the Lie algebra differential for \( E \) \([13]\).

**Proposition 3.9.** \( L(E,\varepsilon) \) is a Dirac subalgebra if and only if \( E \) is a subalgebra of \( \mathfrak{g}_C \) and \( d_{\tilde{E}}\varepsilon = 0 \).

**Proof.** This follows from Propositions \([2.1]\) and \([3.7]\).

**Remark 3.10.** If the Lie algebra cohomology of \( E \) in degree 2 vanishes (i.e. \( H^2(E,\mathbb{C}) = 0 \) ), then for \( \varepsilon \in \wedge^2E^* \) we have \( d_{\tilde{E}}\varepsilon = 0 \) if and only if \( \varepsilon = \phi \circ [\cdot,\cdot] \) for some \( \phi \in E^* \). If \( E \) is a semisimple Lie algebra, then \( H^2(E,\mathbb{C}) = 0 \) \([13]\).

### 3.2.1 Classification of Homogeneous Dirac Structures

Throughout the remainder of this section, assume that \( K \) is connected. We establish a bijection between the G-invariant Dirac structures on \( G/K \) and the set of Dirac subalgebras \( L \) of \( \mathfrak{g}_C^* \times \mathfrak{g}_C \) containing \( \xi_C \). This correspondence sends a Dirac structure \( \mathcal{D} \) on \( G/K \) to \((\pi^*\mathcal{D})_e \), and its inverse sends a subalgebra \( L \subset \mathfrak{g}_C^* \times \mathfrak{g}_C \) to the G-invariant Dirac structure determined by \((d\pi)_*,L\).

**Theorem 3.11.** Let \( G \) be a Lie group and \( K \) be a closed, connected subgroup. There is a bijection between the G-invariant (almost) Dirac structures on \( G/K \) and the set of (almost) Dirac subalgebras \( L \) of \( \mathfrak{g}_C^* \times \mathfrak{g}_C \) containing \( \xi_C \). Any Dirac subalgebra of \( \mathfrak{g}_C^* \times \mathfrak{g}_C \) is of the form \( L(E,\varepsilon) \). The G-invariant Dirac structures on \( G/K \) are thus parameterized by pairs \((E,\varepsilon)\), where \( E \) is a Lie subalgebra of \( \mathfrak{g}_C \) containing \( \xi_C \), \( d_{\tilde{E}}\varepsilon = 0 \), and \( \varepsilon \) vanishes on \( \xi \).

Observe that the integrability of an almost Dirac structure corresponds to whether the linear almost Dirac structure (i.e. maximally isotropic subspace) \( L \subset \mathfrak{g}_C^* \times \mathfrak{g}_C \) is a subalgebra or simply a subspace. Equivalently, integrability is the same as asking that \( E \subset \mathfrak{g}_C \) is a subalgebra, rather than simply a subspace, and that \( d_{\tilde{E}}\varepsilon = 0 \).

**Definition 5.** Let \( G \) be a Lie group and \( K \) a closed subgroup. A Dirac pair is a pair \((E,\varepsilon)\) satisfying the conditions of of Theorem \([3.11]\).

Theorem \([3.11]\) will be proven over the course of several lemmas.

**Lemma 3.12.**
1. If $L$ is a Dirac subalgebra of $\mathfrak{g}_C^* \times \mathfrak{g}_C$, then $\mathfrak{t}_C \subset L$ if and only if $pr_C L \subset \text{Ann}(\mathfrak{t}_C)$.

2. If $L = L(E, \varepsilon)$ is a Dirac Lie subalgebra of $\mathfrak{g}_C^* \times \mathfrak{g}_C$ containing $\mathfrak{t}_C$, then $\varepsilon$ is $\text{Ad}(K)$-invariant. In other words, $\varepsilon([k, X], Y) + \varepsilon(X, [k, Y]) = 0$ for all $k \in \mathfrak{t}_C$ and $X, Y \in \mathfrak{g}$. Also $\varepsilon$ vanishes on $\mathfrak{t}_C$.

Proof.

1. Since $L$ is maximal isotropic, $L = L^\perp$, so $L \subset \mathfrak{g}_C \oplus \text{Ann}(\mathfrak{t}_C)$ if and only if $L = L^\perp \supset (\mathfrak{g}_C \oplus \text{Ann}(\mathfrak{t}_C))^\perp = \mathfrak{t}_C$.

2. If $L = L(E, \varepsilon)$ is a Dirac subalgebra of $\mathfrak{g}_C^* \times \mathfrak{g}_C$ containing $\mathfrak{t}_C$, then by part 1 of this lemma, $pr_C L \subset \text{Ann}(\mathfrak{t}_C)$. For any $X \in E$, there exists $X + \eta \in L$. Because $L$ is isotropic, for any $k \in \mathfrak{t}_C \subset L$, $0 = \langle k, X + \eta \rangle = \eta(k) = \varepsilon(X, k)$. Therefore $\varepsilon$ vanishes on $\mathfrak{t}_C$. Now by Proposition 3.13 $\varepsilon([k, X], Y) + \varepsilon(X, [k, Y]) = 0$ for all $k \in \mathfrak{t}_C, X, Y \in \mathfrak{g}$. □

Lemma 3.13. If $D$ is a $G$-invariant almost Dirac structure on $G/K$, then $D$ is $G$-invariant if and only if $\pi^* D$ is $G$-invariant

Proof. Let $X + d\pi_*^*\eta \in \pi^* D$. This means that $d\pi_*^*\eta \in D_{\pi(e)}$. Then $g \cdot (X + d\pi_*^*\eta) = (dL_g)_e X + (dL_g)_e^{-1} \cdot (d\pi_*^*\eta) = (dL_g)_e X + d\pi_*^* \cdot (dL_g)_e^{-1} \cdot \eta$, which is in $\pi^* D$ if and only if $d\pi_*^* \cdot (dL_g)_e X + (dL_g)_e^{-1} \cdot \eta \in D_{\pi(g)}$. Thus, $g \cdot (X + d\pi_*^*\eta) \in \pi^* D$. Therefore $\pi^* D$ is $G$-invariant if and only if $D$ is $G$-invariant. □

Lemma 3.14. Let $L \subset \mathfrak{g}_C^* \times \mathfrak{g}_C$ be any Dirac subalgebra containing $\mathfrak{t}_C$, and let $D \subset (\mathfrak{g}/\mathfrak{t})_C \oplus (\mathfrak{g}/\mathfrak{t})_C^*$ be any $K$-invariant linear Dirac structure. Then

1. $\pi^*(\pi_* L) = L$,
2. $\pi_* (\pi^* D) = D$, and
3. $\pi_* L$ is $K$-invariant. More generally, $L$ is $K$-stable if and only if $\pi_* L$ is $K$-stable.

Proof. Here $\pi$ is used to denote $d\pi_e$.

1. By definition, $\pi^*(\pi_* L) = \{X + \pi^* \eta \mid \pi X + \eta \in \pi_* L\}$. But $\pi X + \eta \in \pi_* L$ if and only if $X + \pi^* \eta \in L$.

So $\pi^*(\pi_* L) = \{X + \pi^* \eta \mid X + \pi^* \eta \in L\} = \{X + \omega \in L \mid \omega = \pi^* \eta \text{ for some } \eta\} = \{X + \omega \in L \mid \omega \in \text{im}(\pi^*)\}$. But $\text{im}(\pi^*) = \text{Ann}(\mathfrak{t}_C)$, so since $\text{proj}_{\mathfrak{g}_C} L \subset \text{Ann}(\mathfrak{t}_C)$, we have $\pi^*(\pi_* L) = L$.

2. By definition, $\pi_* (\pi^* D) = \{\pi X + \eta \mid \pi X + \pi^* \eta \in \pi^* D\}$. Since $X + \pi^* \eta \in \pi^* D$ if and only if $\pi X + \eta \in D$, we find that $\pi_* (\pi^* D) = \{\pi X + \eta \mid \pi X + \eta \in D\} = D$ because $\pi$ is surjective.

3. If we consider the Ad representation (i.e. $\text{Ad} + \text{Ad}^{-1}$ representation) on $\mathfrak{g}_C \oplus \mathfrak{g}_C^*$, then $d\pi_e \circ \text{Ad}(k) = (dl_k)_{\pi(e)} \circ d\pi_e, \text{Ad}(k)^* \circ (d\pi_e)^* = (dl_k)^* \circ (d\pi_e)^*$, and $(d\pi_e)^* \circ (dl_k)^{-1} = \text{Ad}(k)^{-1} \circ (d\pi_e)^*$. We get $k \cdot (X + (d\pi_e)^* \eta) = \text{Ad}(k) X + \text{Ad}(k)^{-1} \circ (d\pi_e)^* \eta = \text{Ad}(k) X + (d\pi_e)^* \circ (dl_k)^{-1} \cdot \eta$, which is in $L$ if and only if $d\pi_e \circ \text{Ad}(k)^{-1} \circ (dl_k)^{-1} \cdot \eta \in \mathfrak{t}_C$. Of course $d\pi_e \circ \text{Ad}(k) + (dl_k)^{-1} \cdot \eta = (dl_k)_{\pi(e)} \circ (d\pi_e) X + (dl_k)^{-1} \cdot \eta = k \cdot (d\pi_e X + \eta)$. So $k \cdot (X + (d\pi_e)^* \eta) \in L$ if and only if $k \cdot (d\pi_e X + \eta) \in \mathfrak{t}_C$. The result is that $L$ is K-stable if and only if $\pi_* L$ is K-stable. But $L$ is necessarily K-stable because $\mathfrak{t}_C \subset L$. □

Now we prove Theorem 3.11.

Proof. First let $L = L(E, \varepsilon) \subset \mathfrak{g}_C^* \times \mathfrak{g}_C$ be a Dirac subalgebra containing $\mathfrak{t}_C$, and let $D = \pi_* L$. By part 3 of Lemma 3.14 $D$ is K-invariant and so defines a G-invariant almost Dirac structure $D$ on $G/K$. Lemma 3.13 implies that since $D$ is equivariant, $\pi^* D$ is a left-invariant almost Dirac structure on $G$. Now by Lemma 3.13 $L = \pi^* \pi_* L = \pi^* D = (\pi^* D)_e$. Hence, $\pi^* D$ is the left-invariant almost Dirac structure determined by $L$. By Proposition 3.14 since $L$ is a subalgebra, $\pi^* D$ is integrable. Then by Proposition 3.14 $D$ is integrable. This shows that from the linear Dirac structure $L$ we obtain an equivariant Dirac structure $D$ on $G/K$, and
In the other direction, an equivariant Dirac structure $\mathcal{D}$ on $G/K$ yields $L = (\pi^* \mathcal{D})_e$. Since $\mathcal{D}$ is an equivariant Dirac structure, so is $\pi^* \mathcal{D}$ by Lemma 3.13. Hence, by Proposition 3.7, $L \subset \mathfrak{g}_C ^* \oplus \mathfrak{g}_C$ is a Dirac subalgebra. It follows from the definition of pullback $\pi^*$ that $\mathfrak{g}_C \subset L$. This shows that from an equivariant Dirac structure $\mathcal{D}$ on $G/K$, we obtain a Linear Dirac structure $L$, which contains $\mathfrak{g}$. To see that the correspondence is a bijection, we need only to observe that $\pi_* L = \pi_*(\pi^* \mathcal{D})_e = \pi_* \pi^*(\mathcal{D}_e) = \mathcal{D}_e$ by Lemma 3.14. This also shows that $L$ is $K$-stable. Naturally, $\mathcal{D}_e$ is the subspace of $\mathfrak{g}_C \oplus \mathfrak{g}_C^*$ which determines the Dirac structure $\mathcal{D}$. This establishes the bijection.

The description in terms of pairs $(E, \varepsilon)$ follows directly from Proposition 3.9 and the fact that any linear Dirac structure, $L \subset \mathfrak{g}_C \times \mathfrak{g}_C$ is of the form $L = L(E, \varepsilon)$. □

### 3.2.2 Real Dirac Structures

The real equivariant Dirac structures on $G/K$ are those $L \subset \mathfrak{g}_C \times \mathfrak{g}_C$ such that $L = \mathcal{T}_G$ or equivalently, those $L$ such that $L = D_C$ for some $D \subset \mathfrak{g} \oplus \mathfrak{g}^*$. But by considering only $\mathcal{V}_G$, $\mathfrak{g}$, and $\mathfrak{t}$ instead of their complexifications, the following theorem follows in exactly the same way as Theorem 3.11.

**Theorem 3.15.** Let $G$ be a Lie group and $K$ be a closed, connected subgroup. There is a bijection between the $G$-invariant (almost) Dirac structures on $G/K$ and the set of (almost) Dirac subalgebras $L$ of $\mathfrak{g}^* \times \mathfrak{g}$ containing $\mathfrak{t}$. Any Dirac subalgebra of $\mathfrak{g}^* \times \mathfrak{g}$ is of the form $L(E, \varepsilon)$. The $G$-invariant Dirac structures on $G/K$ are thus parameterized by pairs $(E, \varepsilon)$, where $E$ is a Lie subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$, $d_E \varepsilon = 0$, and $\varepsilon$ vanishes on $\mathfrak{t}$.

**Corollary 3.16.** By correspondence given in Theorem 3.17 between Linear Dirac structures $L = L(E, \varepsilon) \subset \mathfrak{g} \oplus \mathfrak{g}^*$ and equivariant Dirac structures on $G/K$,

1. The symplectic structures are all $L(\mathfrak{g}, \varepsilon)$ such that $\text{Ker}(\varepsilon^*_L) = \mathfrak{t}$.
2. The presymplectic structures are all $L(\mathfrak{g}, \varepsilon)$.
3. The Poisson structures are all $L(E, \varepsilon)$ such that $\text{Ker}(\varepsilon^*_L) = \mathfrak{t}$.
4. There is a bijection between $G$-invariant real Dirac structures on $G/K$ and $G$-invariant presymplectic structures on the spaces $H/K$ for connected subgroups $H$ of $G$ containing $K$. With this bijection, a linear Dirac structure $L(E, \varepsilon)$ corresponds to a presymplectic structure on $H/K$, where $H$ is the connected Lie subgroup corresponding to Lie subalgebra $E$. Only when $\text{Ker}(\varepsilon^*_L) = \mathfrak{t}$ does $\varepsilon$ gives a symplectic structure. Hence there is a bijection between $G$-invariant real Dirac structures on $G/K$ and $G$-invariant presymplectic structures on the spaces $H/K$ for connected subgroups $H$ containing $K$.

**Proof.** 1. Let $L = L(E, \varepsilon) \subset \mathfrak{g}^* \times \mathfrak{g}$ be a subalgebra containing $\mathfrak{t}$ such that $\text{Ker} \varepsilon^*_L = \mathfrak{t}$, and let $L = L(TG, \tilde{\varepsilon})$ be the left-invariant Dirac structure on $G$ determined by $L$. Then $D = d\pi_* L = L(\mathfrak{t}/\mathfrak{t}, \omega_0)$ gives equivariant Dirac structure $\mathcal{D} = L(T(G/K), \omega)$. By Proposition 3.2, $L(TG, \pi^* \omega) = \pi^* \mathcal{D} = L = L(TG, \tilde{\varepsilon})$, so $\tilde{\varepsilon} = \pi^* \omega$. By Prop. 3.7, $\tilde{\varepsilon}$ is closed and left-invariant. This implies that $0 = d_E \tilde{\varepsilon} = d\tilde{\varepsilon} = d\pi^* \omega = \pi^* d\omega$, whence $d\omega = 0$ since $d\pi^*$ is injective. To show that $\omega$ is non-degenerate, it is enough to show that $\omega_0$ is non-degenerate, by $G$-invariance. But $\varepsilon^*_E = (d\pi^* \omega_0)_{\mathfrak{t}} = (d\pi^* (\omega_0))_L d\pi^*$. Since $d\pi^*$ is surjective and $d\pi^*$ is injective, $\text{Ker}((\omega_0)_{\mathfrak{t}}) = 0$ if and only if $\text{Ker}(\varepsilon^*_E) = \mathfrak{t}$. Conversely, given symplectic $\omega$, the above argument may be run backwards to show that for $\varepsilon = d\pi^* \omega_0$, one has $\text{Ker}(\varepsilon^*_E) = \mathfrak{t}$.

2. This follows in the same manner as part 1. The only difference is, using the notation from part 1 of this proof, $\omega$ may not be non-degenerate.

3. Dirac structures $L(E, \omega) \subset \mathcal{V}_{G/K}$ correspond to $L(d\pi^{-1} E, \pi^* \omega)_e \subset \mathfrak{g} \oplus \mathfrak{g}^*$. Visibly, $\omega$ is non-degenerate if and only if $\omega_{\pi(e)}$ is non-degenerate, which happens exactly when $\text{Ker}((d\pi^* \omega)_E) = \text{Ker}(d\pi^* \omega) = \mathfrak{t}$. 
Now it suffices to show that Dirac structures \( L(E, \omega) \subset \mathcal{V}_{G/K} \) with \( \omega \) non-degenerate are exactly those corresponding to Poisson structures. This should make sense because a Poisson structures gives an integrable distribution \( E \) and a symplectic structure \( \omega \) on the leaves of the foliation determined by \( E \).

To see how this works, let \( L(E, \omega) \subset \mathcal{V}_{G/K} \) be a Dirac structure and \( \omega \) be non-degenerate. Since \( T^*(G/K) \rightarrow E^* \) and \( \omega_\sharp : E^* \rightarrow E^* \), it follows that \( pr_{T^*(G/K)} L(E, \omega) = T^*(G/K) \). Thus, \( L(E, \omega) \) is of the form \( L(\beta, T^*(G/K)) \) for some 2-form \( \beta \) on \( G/K \). It is a result of [1] and [12] that almost Dirac structures of the form \( L(\beta, T^*(G/K)) \) are Dirac structures precisely when \( \beta \) is a Poisson bivector.

4. Follows easily. \( \square \)

4 Generalized Complex Structures on Homogeneous Spaces

We delineate the conditions for \( L \subset g_\mathbb{C} \times g_\mathbb{C} \) or a Dirac pair \( (E, \varepsilon) \) to represent an generalized complex structure in Corollary 4.11 and Proposition 4.8. Again we emphasize that in addition to the closed integrability condition, we now require the genericity condition \( L \cap \mathcal{T} = \mathfrak{t}_\mathbb{C} \).

4.1 Classification

As a corollary to Theorem 3.11 we have:

**Corollary 4.1.** Let \( G \) be a Lie group and \( K \) a closed, connected subgroup. If \( L \subset g_\mathbb{C} \times g_\mathbb{C} \) is a linear (almost) Dirac structure containing \( \mathfrak{t}_\mathbb{C} \), then \( L \) represents a generalized (almost) complex structure on \( G/K \) if and only if \( L \cap \mathcal{T} = \mathfrak{t}_\mathbb{C} \).

**Proof.** If \( \mathcal{D} \) is a equivariant complex Dirac structure on \( G/K \), then \( \mathcal{D} \) is a generalized complex structure if and only if \( \mathcal{D} \cap \mathcal{T} = 0 \), which happens exactly when \( D_{\pi(e)} \cap \mathcal{T}_{\pi(e)} = 0 \). Note that \( \pi^* \) may be applied to arbitrary subspaces and not just maximal isotropic subspaces, so we observe immediately that \( \pi^*(0) = \mathfrak{t}_\mathbb{C} \). On the other hand, if \( \mathcal{V} \) is any nonzero subspace of \( (\mathfrak{g}/\mathfrak{t})_\mathbb{C} \oplus \left((\mathfrak{g}/\mathfrak{t})_\mathbb{C}\right)^* \), it follows from the definition of \( \pi^* \) together with surjectivity of \( d\pi_e \) (and injectivity of its dual) that if \( \pi^* \mathcal{V} = \mathfrak{t}_\mathbb{C} \), \( \mathcal{V} \subset \mathfrak{g}/\mathfrak{t} \), whence \( \mathfrak{t}_\mathbb{C} = \pi^* \mathcal{V} = \pi^{-1} \mathcal{V} \) and so \( \mathcal{V} = 0 \). We conclude that \( \pi^* \mathcal{V} = \mathfrak{t}_\mathbb{C} \) precisely when \( \mathcal{V} = 0 \).

For any two subspaces \( \mathcal{V}, \mathcal{W} \subset (\mathfrak{g}/\mathfrak{t})_\mathbb{C} \oplus \left((\mathfrak{g}/\mathfrak{t})_\mathbb{C}\right)^* \), \( \pi^*(\mathcal{V} \cap \mathcal{W}) \subset \pi^* \mathcal{V} \) implies that \( \pi^*(\mathcal{V} \cap \mathcal{W}) = \pi^* \mathcal{V} \cap \pi^* \mathcal{W} \). Finally, the observation that \( \pi^* \mathcal{V} = \pi^* \mathcal{W} \) completes the proof. \( \square \)

**Proposition 4.2.** In the correspondence of Theorem 3.11 and Corollary 4.1

1. The complex structures are given by all \( L(E, 0) \subset g_\mathbb{C} \oplus g_\mathbb{C}^* \) such that \( E + \mathcal{E} = g_\mathbb{C} \) and \( E \cap \mathcal{E} = \mathfrak{t}_\mathbb{C} \). Thus there is a bijection between \( G \)-invariant complex structures on \( G/K \) and subalgebras \( E \subset g_\mathbb{C} \) such that \( E + \mathcal{E} = g_\mathbb{C} \) and \( E \cap \mathcal{E} = \mathfrak{t}_\mathbb{C} \). This correspondence can be extended to a bijection between \( G \)-invariant almost complex structures on \( G/K \) and subspaces \( E \subset g_\mathbb{C} \) such that \( E + \mathcal{E} = g_\mathbb{C} \) and \( E \cap \mathcal{E} = \mathfrak{t}_\mathbb{C} \).

2. The symplectic structures are all \( L(g_\mathbb{C}, \varepsilon) \) such that \( \text{Ker}(\varepsilon) = \mathfrak{t}_\mathbb{C} \) and \( \varepsilon \) is purely imaginary (i.e. \( \varepsilon = i \omega \) for some real 2-form \( \omega \)).

**Proof.** We know that complex structures are exactly generalized complex structures of the form \( L(E, 0) \). Part (1) now follows from Theorem 3.11 and Corollary 4.1.

(2) This follows in the same way as part (1) of Corollary 4.11. \( \square \)

**Remark 4.3.** The condition that a complex structure be \( G \)-invariant means that the distribution it defines in \( T(G/K) \otimes \mathbb{C} \) is \( G \)-invariant. This is equivalent to requiring that each \( l_g : G/K \rightarrow G/K \) is holomorphic. This equivalence follows directly from the fact that a map \( f : M \rightarrow N \) of complex manifolds is holomorphic if and only if \( df_p(T^1,0)M \subset T^1,0f(p)N \) for all \( p \in M \), where \( T^1,0M \) and \( T^1,0N \) are the holomorphic tangent bundles. Here \( D = T^1,0(G/K) \oplus T^*(0,1)(G/K) \). This is also true for almost complex structures.
Proposition 4.4. In the correspondence of Theorem 3.11 and Corollary 4.1.

1. Any \( L(E, i\omega) \), where \( \omega \) is the restriction to \( E \) of a real 2-form on \( \mathfrak{g} \), gives a symplectic structure on some \( H/K \) for some subgroup \( H \) of \( G \). Specifically, if \( D \) is the subalgebra of \( \mathfrak{g} \) such that \( D_C = E \cap \overline{E} \), then \( H \) is the connected Lie subgroup of \( G \) with Lie algebra \( \mathfrak{k} \).

2. The symplectic structures are all \( L(E, i\omega) \), where \( \omega \in \Lambda^2 \mathfrak{g}^* \).

3. Any generalized complex structure \( L(E, \varepsilon) \) gives a complex structure \( L(E, 0) \) on \( G/H \), where \( H \) is the Lie subgroup from part 1, as long as \( H \) is closed.

Proof. 1. In order for \( \{ = \{ L \) and \( \Lambda^2 \mathfrak{g} \in \mathfrak{g} \), gives a symplectic structure on \( G/K \), it is often

4.2 Analysis of Conditions on \( E \) and \( \varepsilon \)

Although Corollary 4.1 gives a description of equivariant generalized complex structures on \( G/K \), it is often

more useful to describe them in terms of a subalgebra \( \mathfrak{g} \) containing \( \mathfrak{k} \)

because \( E \cap \overline{E} = E \cap \overline{E} \). Since \( D \) is a Lie subalgebra, it corresponds to some Lie subgroup \( H \) containing \( K \). \( \omega \) is clearly non-degenerate on \( D/\mathfrak{d} \) and closed. Therefore it determines a symplectic form on \( H/K \).

2. It is clear that these are the symplectic structures. The condition that \( \text{Ker} \omega_2 = \mathfrak{t} \) is equivalent to

non-degeneracy of the symplectic form, but it is also equivalent to \( L \cap \overline{L} = \mathfrak{t} \).

3. This follows from Proposition 4.2. \( \square \)

4.2 Analysis of Conditions on \( E \) and \( \varepsilon \)

Although Corollary 4.1 gives a description of equivariant generalized complex structures on \( G/K \), it is often

more useful to describe them in terms of a subalgebra \( \mathfrak{g} \) containing \( \mathfrak{e} \)

By \( \mathbb{C} \)-linear extension, \( \mathfrak{g}^* \rightarrow (\mathfrak{g} \mathbb{C})^* \), and this gives an isomorphism \( (\mathfrak{g} \mathbb{C})^* = \mathfrak{g}^* \oplus \mathbb{C} \mathfrak{g}^* \). If \( \alpha \in \mathfrak{g}^*_\mathbb{C} \)

and \( X \in \mathfrak{g} \), it may be easily verified that \( \overline{\alpha}(X) = \overline{\alpha}(X) \), where \( \alpha \mapsto \overline{\alpha} \) denotes conjugation. It follows

immediately that \( \text{Ann}(\overline{\mathfrak{E}}) = \text{Ann}(\mathfrak{E}) \). We define \( \overline{\alpha}(X, Y) = \varepsilon(X, Y) \).

Proposition 4.5. \( \overline{L} = L(\mathfrak{E}, \varepsilon) \).

Proof. The result follows by observing the following equalities.

\( \overline{L} = \{ X + \varepsilon \mid X \in \mathfrak{E}, \varepsilon_{|\mathfrak{E}} = \varepsilon_{\mathfrak{E}} \} \)

\( = \{ X + \varepsilon \mid X \in \mathfrak{E}, \text{ for all } Y \in \mathfrak{E} \varepsilon(Y) = \varepsilon(X, Y) \} \)

\( = \{ X + \varepsilon \mid X \in \mathfrak{E}, \text{ for all } Y \in \mathfrak{E} \varepsilon(Y) = \varepsilon(X, Y) \} \)

\( = \{ X + \varepsilon \mid X \in \mathfrak{E}, \text{ for all } Y \in \mathfrak{E} \varepsilon(Y) = \varepsilon(X, Y) \} \)

\( = L(\mathfrak{E}, \varepsilon). \square \)

Notation : We let the symbol \( \setminus \) denote the difference of two sets.

Proposition 4.6. \( L \cap \overline{L} = \mathfrak{t} \) if and only if

1. \( E + \mathfrak{E} = \mathfrak{g}, \) and \( E + \mathfrak{E} = \mathfrak{g}, \) and

2. \( X \in (\mathfrak{E} \cap E) \setminus \mathfrak{t} \) and \( X + \varepsilon \in L \) implies \( \overline{X} + \overline{\varepsilon} \notin L \) (i.e. \( pr_{\mathfrak{g} \cap \mathbb{C}^*} \rightarrow L \cap \overline{L} = \mathfrak{t} \).

Proof. The first condition is equivalent to \( \text{Ann}(\mathfrak{E}) \cap \text{Ann}(\overline{\mathfrak{E}}) = 0 \), since \( \text{Ann}(\mathfrak{V}) \cap \text{Ann}(\mathfrak{W}) = \text{Ann}(\mathfrak{V} + \mathfrak{W}) \)

for any subspaces \( \mathfrak{V}, \mathfrak{W} \). First suppose that \( L \cap \overline{L} = \mathfrak{t} \). \( \mathfrak{Ann}(\mathfrak{E}) \subset L \) and \( \overline{\mathfrak{Ann}(\mathfrak{E})} = \mathfrak{Ann}(\overline{\mathfrak{E}}) \subset \overline{L} \). This implies that \( \text{Ann}(\mathfrak{E}) \cap \text{Ann}(\overline{\mathfrak{E}}) \subset L \cap \overline{L} = \mathfrak{t} \). But \( \text{Ann}(\mathfrak{E}) \cap \text{Ann}(\overline{\mathfrak{E}}) \subset \mathfrak{g}^*_\mathbb{C} \) and so \( \text{Ann}(\mathfrak{E}) \cap \text{Ann}(\overline{\mathfrak{E}}) = 0 \).

Part (2) is obviously true.

Now suppose that (1) and (2) are true. In order to show that \( L \cap \overline{L} \subset \mathfrak{t} \), let \( A = X + \varepsilon \in L \cap \overline{L} \). We

know that \( X \in E \cap \overline{\mathfrak{E}} \), and (2) implies that \( X \in \mathfrak{t} \). Hence, \( \varepsilon_{\mathfrak{e}}X = 0 \), which implies that \( \varepsilon \in \text{Ann}(\mathfrak{E}) \).

Also \( X + \varepsilon \in L \), so \( \varepsilon \in \text{Ann}(\mathfrak{E}) \) and therefore \( \varepsilon \in \text{Ann}(\overline{\mathfrak{E}}) = \text{Ann}(\overline{\mathfrak{E}}) \). But now \( \varepsilon \notin \text{Ann}(\mathfrak{E}) \cap \text{Ann}(\overline{\mathfrak{E}}) = \text{Ann}(\mathfrak{E} + \overline{\mathfrak{E}}) = \text{Ann}(\mathfrak{g}^*_\mathbb{C}) = 0 \). Therefore \( A = X \). By condition (2), \( X \in \mathfrak{t} \). Thus, \( L \cap \overline{L} = \mathfrak{t} \). \( \square \)
Proposition 4.7. Suppose that $E + \overline{E} = \mathfrak{g}_\mathbb{C}$. Then $L \cap \overline{L} = \mathfrak{t}_\mathbb{C}$ if and only if for all $X \in E \cap \overline{E}$, if $\varepsilon(X, Y) - \varepsilon(X, Y) = 0$ for all $Y \in E \cap \overline{E}$, then $X \in \mathfrak{t}_\mathbb{C}$.

**Proof.** Suppose that $L \cap \overline{L} = \mathfrak{t}_\mathbb{C}$, and let $X \in E \cap \overline{E} \setminus \mathfrak{t}_\mathbb{C}$. If $\psi, \xi \in \mathfrak{g}^*$ such that $\xi|_{E} = \varepsilon_1X$ and $\psi|_{E} = \varepsilon_1\overline{X}$, then the condition on $\varepsilon$ asks that there exists $Y \in E \cap \overline{E}$ such that $\varepsilon(X, Y) \neq \varepsilon(X, Y)$ can be expressed as $\overline{\xi} - \psi \notin \text{Ann}(E \cap \overline{E})$. This is equivalent to $\varepsilon(X, Y) \neq \varepsilon(X, Y)$ or in other words, $\xi(Y) \neq \overline{\psi}(Y)$. This means exactly that $(\overline{\xi} - \psi)(Y) \neq 0$, or $\bar{\xi} - \psi \notin \text{Ann}(E \cap \overline{E})$. Suppose now, for the sake of contradiction, that $\overline{\xi} - \psi \in \text{Ann}(E \cap \overline{E}) = \text{Ann}E + \text{Ann}E$. Then $(\overline{\xi} - \psi) = \alpha + \beta \in \text{Ann}(E + \text{Ann}(E)$, and $\overline{\alpha} \in \text{Ann}(E) = \text{Ann}(E) \subset L$, which implies that $X + \xi - \bar{\beta} \in L$. In this case, $X + \overline{\xi} - \beta \in \overline{L}$. However, $X + \xi - \beta = X + \psi + \alpha + \beta - \beta = X + \psi + \alpha \in L$, whence $X + \overline{\xi} - \beta \in L \cap \overline{L} = \mathfrak{t}_\mathbb{C}$. This is a contradiction, because it was assumed that $X \in E \cap \overline{E} \setminus \mathfrak{t}_\mathbb{C}$. Therefore, the fact that $L \cap \overline{L} = \mathfrak{t}_\mathbb{C}$ implies the desired condition on $\varepsilon$.

Now suppose that for all $X \in E \cap \overline{E}$, if $\varepsilon(X, Y) - \varepsilon(X, Y) = 0$ for all $Y \in E \cap \overline{E}$, then $X \in \mathfrak{t}_\mathbb{C}$. Suppose that $X + \xi \in L \cap \overline{L}$ with $X \in E \cap \overline{E}$. We aim to show that $X \in \mathfrak{t}_\mathbb{C}$. This is sufficient by Proposition 4.6. We know that $\xi|_L = \varepsilon_2X$ and $\overline{\xi}|_L = \varepsilon_1(\overline{X})$ because $X + \xi$ and $X + \overline{\xi}$ both lie in $L$. If $X \notin \mathfrak{t}_\mathbb{C}$, then there exists $Y \in E \cap \overline{E}$ such that $\varepsilon(X, Y) \neq \varepsilon(X, Y)$. However, $\varepsilon(X, Y) = \xi(Y) = \xi((\overline{Y})) = \varepsilon(\overline{X}, Y)$, which is a contradiction. Therefore $X \in \mathfrak{t}_\mathbb{C}$ and $L \cap \overline{L} = \mathfrak{t}_\mathbb{C}$. □

Proposition 4.8. Let $G$ be a Lie group and $K$ a closed, connected subgroup of $G$. There is a bijection between $G$-invariant generalized complex structures on $G/K$ and pairs $(E, \varepsilon)$, $E$ a subalgebra of $\mathfrak{g}_\mathbb{C}$ and $\varepsilon \in \wedge^2E^*$, such that

1. $\mathfrak{t}_\mathbb{C} \subset E$,
2. $E + \overline{E} = \mathfrak{g}_\mathbb{C}$,
3. $d_E\varepsilon = 0$,
4. $\varepsilon(\mathfrak{t}) = 0$, and
5. For $X \in E \cap \overline{E}$, if $\varepsilon(X, Y) - \varepsilon(X, Y) = 0$ for all $Y \in E \cap \overline{E}$, then $X \in \mathfrak{t}_\mathbb{C}$.

**Proof.** The proof is immediate from Proposition 4.7 and 4.6 and Corollary 4.1.

**Definition 6.** For a homogeneous space $G/K$, pairs $(E, \varepsilon)$ satisfying the conditions of Proposition 4.8 are called generalized complex pairs or GC-pairs.

**Remark 4.9.** Conditions (1) and (4) of Proposition 4.8 are conditions for $L(E, \varepsilon)$ to represent an almost Dirac structure on $G/K$. The requirement that this almost Dirac structure is integrable is condition (3) together with the requirement that $E$ is a Lie subalgebra. Finally, conditions (2) and (5) ensure that $L(E, \varepsilon)$ is a generalized complex structure.

**Remark 4.10.** Condition (5) of Proposition 4.8 simply asks that $Ker((\varepsilon - \overline{\varepsilon})_2 : E \cap \overline{E} \rightarrow E \cap \overline{E}) = \mathfrak{t}_\mathbb{C}$. Condition (5) may be stated in yet another way. We may extend $\varepsilon \in \wedge^2E^*$ to some $B \in \wedge^2\mathfrak{g}_\mathbb{C}$. If $B = B_r + iB_i$ is the decomposition of $B$ into real and imaginary parts, then $\varepsilon = \varepsilon_r + \varepsilon_i$, where $\varepsilon_r$ and $\varepsilon_i$ are the real and imaginary parts of $\varepsilon$ respectively (i.e. the restrictions of $B_r$ and $B_i$). Then $\varepsilon(X, Y) - \varepsilon(X, Y) = \varepsilon_i(X, Y)$. Since $\varepsilon_i(\mathfrak{t}_\mathbb{C}) = 0$, $\varepsilon_i$ defines a linear map $\varepsilon_i : E/\mathfrak{t}_\mathbb{C} \times E/\mathfrak{t}_\mathbb{C} \rightarrow E/\mathfrak{t}_\mathbb{C}$. Condition 5 of Prop 4.8 may be restated in the following way: $\varepsilon_i$ is non-degenerate when restricted to $E \cap \overline{E}/\mathfrak{t}_\mathbb{C} \times E \cap \overline{E}/\mathfrak{t}_\mathbb{C}$.

### 4.3 B-transformations

Recall that for a 2-form $B$ on a manifold $X$, the map $TX \xrightarrow{B} \Gamma(T^*X)$ can be extended by 0 on $T^*X$ to give a map $\mathcal{V}_X \rightarrow \mathcal{V}_X$. The exponential of this map is called a $B$-transform and is denoted by $e^B$. For a
Dirac structure $L$ on $X$, applying the Courant algebroid automorphism $e^B$ to $L$ gives another Dirac structure $e^B L$, which we will call the $B$-transform of $L$ under the 2-form $B$. Since we are only considering G-invariant Dirac structures on $G/K$, we only consider those B-transforms which transform equivariant Dirac structures to equivariant Dirac structures.

**Lemma 4.11.** Let $D$ be an equivariant Dirac structure on $G/K$ and $L(E, \varepsilon) = \pi^* D_e \subset \mathfrak{g}_C \oplus \mathfrak{g}_C^*$. 

1. For a 2-form $B \in \Omega^2(G/K)$, $\pi^* (e^B D) = e^{\pi^* B} \pi^* D$.

2. If $\eta \in \wedge^2 \text{Ann}(t) \subset \wedge^2 \mathfrak{g}^*$ and $d\eta = 0$, then $e^0 L(E, \varepsilon) = \pi^* (e^\omega D)$ for some 2-form $\omega$ on $G/K$.

**Proof.** If $V$ is a subspace of $\mathfrak{g}$ or a $K$-invariant subspace of $\mathfrak{g}/\mathfrak{k}$, let $\tilde{V}$ denote the corresponding distribution. Similarly, if $\omega \in \wedge^2 \mathfrak{g}^*$ or $\wedge^2 (\mathfrak{g}/\mathfrak{k})^*$ and $K$-invariant, let $\tilde{\omega}$ denote the corresponding 2-form.

The first claim is that for a 2-form $B \in \Omega^2(G/K)$, $\pi^* (e^B D) = e^{\pi^* B} \pi^* D$. Suppose that $D = L(\tilde{F}, \tilde{\omega})$. Recall that $L(\tilde{E}, \tilde{\varepsilon}) = \pi^* D = L(\pi^{-1} \tilde{F}, \pi^* \tilde{\omega})$. If $i : \tilde{E} \to TG$ and $j : \tilde{F} \to T(G/K)$ are inclusions, then $\pi \circ i = j \circ \pi|_{\tilde{E}}$, $\pi^* (e^B L(\tilde{F}, \tilde{\omega})) = \pi^* L(\tilde{E}, \tilde{\varepsilon} + j^* B) = L(\tilde{E}, \tilde{\varepsilon} + j^* B) = L(\tilde{E}, \tilde{\varepsilon} + i^* \pi^* B) = e^{\pi^* B} L(\tilde{E}, \tilde{\omega})$.

Now let $\eta \in \wedge^2 \text{Ann}(t)$ be closed, so $\eta = d\pi^* B$ for a (unique) $B \in \wedge^2 (\mathfrak{g}/\mathfrak{k})^*$. It has been shown already that $B$ is left $K$-invariant if and only if $\eta$ is Ad-$K$-invariant. However, since $\eta$ is closed and vanishes on $\mathfrak{t}$, $\eta$ is automatically Ad-$K$-invariant, as was stated in Remark 3.12. Therefore, $B$ yields a G-invariant 2-form $\tilde{B}$, and $\pi^* \tilde{B} = \tilde{\eta}$ so that $0 = d\tilde{\eta} = d\pi^* B = \pi^* dB$, which implies $dB = 0$ since $\pi^*$ is injective. From the first part of this proof, $e^0 L(E, \varepsilon) = e^0 L(\tilde{E}, \tilde{\varepsilon})$.

We now provide a sufficient condition so that every B-transformation taking an invariant Dirac structure $D$ on $G/K$ to another invariant Dirac structure is given by a $G$-invariant 2-form.

**Notation:** Recall that for a Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, $H^2(\mathfrak{g}, \mathbb{C})$ denotes the Lie algebra cohomology in degree 2. We will let $Z^2(\mathfrak{g}, \mathbb{C})$ denote degree 2 cocycles in the the complex $\text{Hom}_\mathbb{C}(\wedge^2 \mathfrak{g}, \mathbb{C})$ which defines the Lie algebra cohomology $\text{H}^2(\mathfrak{g}, \mathbb{C})$.

**Proposition 4.12.** Let $D$ be an equivariant Dirac structure on $G/K$ and $L(E, \varepsilon) = \pi^* D_e \subset \mathfrak{g}_C \oplus \mathfrak{g}_C^*$. If $H^2(\mathfrak{g}, \mathbb{C})$ surjects onto $H^2(E, \mathbb{C})$, then every equivariant $B$-transform of $D$ is of the form $e^0 L(E, \varepsilon)$ for some $\eta \in \wedge^2 \mathfrak{g}^*$ such that $d\eta = 0$. For instance, if $E$ is semisimple, then every $B$-transform is of this type.

**Proof.** First we observe that $H^2(\mathfrak{g}, \mathbb{C})$ surjects onto $H^2(E, \mathbb{C})$ if and only if $Z^2(\mathfrak{g}, \mathbb{C})$ surjects onto $Z^2(E, \mathbb{C})$. Let $B \in \Omega^2(G/K)$ be a closed 2-form such that $e^B D$ is G-invariant. By Lemma 4.11, $\pi^* (e^B D) = e^{\pi^* B} \pi^* D$.

In fact, $e^{\pi^* B} \pi^* D$ is determined by its value at $e$. If $\omega = \pi^* B_e$, then $(e^{\pi^* B} \pi^* D)_e = e^B \omega$. Since $e^0 L$ is a Dirac structure, $d_{E\omega}|_{E \times E} = 0$, so by assumption, there exists some $\eta \in Z^2(\mathfrak{g}, \mathbb{C})$ which agrees with $\omega$ on $E \times E$.

If $E$ is semisimple, then by Remark 3.10 any $\varepsilon \in Z^2(E, \mathbb{C})$ is of the form $\phi \circ [\ , \ ]$ for some $\phi \in E^*$. But $\mathfrak{g}^*$ surjects onto $E^*$, so there is some $\phi \in \mathfrak{g}^*$ which restricts to $\phi$. Therefore, letting $\eta = \phi \circ [\ , \ ]$ gives the desired result. □

**4.4 Quotients by Disconnected Subgroups**

So far we have only considered homogeneous spaces $G/K$, where $K$ is a closed, connected subgroup. In this case, equivariant Dirac structures are given by Dirac pairs, described in Theorem 3.11 and generalized complex structures are given by GC-pairs. When $K$ is disconnected, however, in order for a Dirac subalgebra $L \subset \mathfrak{g}_C^* \times \mathfrak{g}_C$ to give a Dirac structure on $G/K$, it must be $K$-invariant. If $K$ is connected, then $K$-invariance follows from $\mathfrak{t}$-invariance, but this is not necessarily the case when $K$ is disconnected.

**Lemma 4.13.** G-invariant generalized complex structures on $G/K$ are given by $K$-stable subalgebras of $\mathfrak{g}_C^* \times \mathfrak{g}_C$ containing $\mathfrak{k}_C$. A Dirac Lie subalgebra $L = L(E, \varepsilon) \subset \mathfrak{g}_C^* \times \mathfrak{g}_C$ is $K$-invariant if and only if $E$ and $\varepsilon$ are $K$-invariant.
Proof. The proof is exactly the same as the proof of Theorem 3.11 except that we must assume that $L$ is $K$-stable, which will no longer automatically follow from $L$ containing $\mathfrak{t}$. A simple calculation shows that $K$ preserves $L$ if and only if $K$ preserves $E$ and $\varepsilon$. $\square$

Thus, the general version of Theorem 3.11 and Proposition 4.8 is:

**Theorem 4.14.** Let $G$ be a Lie group and $K$ a closed subgroup. The $G$-invariant complex Dirac structures on $G/K$ are parameterized by $K$-invariant Dirac pairs $(E, \varepsilon)$, and the $G$-invariant generalized complex structures are parameterized by $K$-invariant GC-pairs.

**Remark 4.15.** If $G$ is connected and $\varepsilon = d_E \phi = \phi \circ [\cdot , ]$ for some $\phi \in E^*$, then $\varepsilon$ is $K$-invariant if and only if $\phi|_{[E,E]}$ is $K$-invariant.

**Remark 4.16.** Let $K^0$ be the identity component of $K$. If $L \supset \mathfrak{k}_C$, then we already noted that $K^0$-invariance is automatic. To check $K$-invariance, one need only check invariance under a discrete subset of $K$, namely invariance under a representative of each coset in $K/K^0$.

5 Quotients of Compact Groups by Connected Subgroups of Maximal Rank

In this section let $G$ be a compact group and $K$ be a connected subgroup containing a Cartan subgroup $C$. We wish to classify the generalized complex structures on $G/K$ by listing all GC-pairs $(E, \varepsilon)$. Subsection 5.1 will focus on the case when $K = C$, and the subsequent subsections will consider any $K \supset C$. We first notice that since $G$ is compact, its Lie algebra $\mathfrak{g}$ is reductive. Since $\mathfrak{t}$ contains a Cartan subalgebra $\mathfrak{c} = \text{Lie}(C)$, $\mathfrak{t}$ contains the center of $\mathfrak{g}$. Thus, GC-pairs for $G$ are the same as those for $G/Z_G$, which is semisimple. Henceforth, we will assume that $G$ is semisimple so that the notation is less cumbersome. We will show that $E$ must be a parabolic subalgebra and that $\varepsilon$ is exact in the sense of Remark 3.10. Proposition 5.11 gives a full explanation. We also determine in Section 5.5 that equivariant generalized complex structures on $G/K$ up to $B$-transform can be thought of as a symplectic structure on a subgroup together with a complex structure on the quotient of $G$ by that subgroup. Finally, we explain the geometric structure of the moduli of equivariant Dirac structures on $G/K$ in the final subsection (Proposition 5.16).

Let $\mathfrak{h} = \mathfrak{t}_C$ and $\mathfrak{l}_G := E \cap \mathfrak{E}$. Conjugation on $\mathfrak{g}_C$ with respect to $\mathfrak{g}$ will be denoted by $\sigma$ or $x \mapsto \overline{x}$. Finally, let $\Delta$ denote the set of roots with respect to the Cartan subalgebra $\mathfrak{h}$.

**Notation:** Let $\mathfrak{g}$ be any semisimple Lie algebra over $\mathbb{R}$ with Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}_C$. If $\alpha$ is a root, we denote a root space $(\mathfrak{g}_C)_\alpha$ by $\mathfrak{g}_{C,\alpha}$ for convenience of notation.

**Remark 5.1.** Any subalgebra $E$ containing $\mathfrak{h}$ is of the form $E = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_{C,\alpha}$. Furthermore, since $[\mathfrak{g}_{C,\alpha},\mathfrak{g}_{C,\beta}] = \mathfrak{g}_{C,\alpha+\beta}$ when $\alpha \neq -\beta$, subalgebras containing $\mathfrak{h}$ are in bijection with closed subsets of $\Delta$. A subset $A \subset \Delta$ is called closed if $A$ has the property that if $\alpha, \beta \in A$ and $\alpha + \beta \in \Delta$, then $\alpha + \beta \in A$.

**Definition 7.** A closed subset $A \subset \Delta$ is called symmetric if $A = -A$, and a subalgebra $\mathfrak{l} \subset \mathfrak{g}_C$ is called symmetric if $\mathfrak{l} = \overline{\mathfrak{l}}$, i.e. if it is defined over the $\mathbb{R}$.

**Remark 5.2.** $\mathfrak{l} \supset \mathfrak{h}$ is symmetric if and only if its corresponding subset $A$ of $\Delta$ is symmetric. Note that $\mathfrak{l}$ may be symmetric but not be a Levi subalgebra.

**Lemma 5.3.** Let $\mathfrak{l}$ be a symmetric subalgebra of $\mathfrak{g}_C$ containing $\mathfrak{h}$, which corresponds to a symmetric subset $A \subset \Delta$.

1. $[\mathfrak{l},\mathfrak{l}] = ([\mathfrak{l},\mathfrak{l}] \cap \mathfrak{h}) \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_{C,\alpha}$, and it is a semisimple Lie algebra.
2. $\mathfrak{l} = Z(\mathfrak{l}) \oplus [\mathfrak{l},\mathfrak{l}]$. 

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Proof. 1. Upon the observation that $A$ is itself a root system, this is a consequence of a theorem of Serre ([3], p.99).

2. The Killing form $\kappa$ is non-degenerate when restricted to $\mathfrak{h}$. Let $\mathfrak{h}_0$ be the orthogonal complement to $\mathfrak{h}' := [\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h}$ with inner product given by the Killing form. If $h \in \mathfrak{h}_0$, then for any $\alpha \in \Delta$, $0 = \kappa(\alpha, h) = \alpha(h)$. Thus, $0 = \alpha(h)X_\alpha = [h, X_\alpha]$, and $h \in Z(\mathfrak{l})$ (since $\mathfrak{h}$ is abelian). On the other hand, $\mathfrak{l} = \mathfrak{h}_0 \oplus [\mathfrak{l}, \mathfrak{l}]$. So if $x = x_0 + x' \in Z(\mathfrak{l})$, then since $x_0 \in Z(\mathfrak{l})$, $x' \in Z(\mathfrak{l})$. However, by part 1, $[\mathfrak{l}, \mathfrak{l}]$ is semisimple, which implies that $x' = 0$. Therefore $Z(\mathfrak{l}) = \mathfrak{h}_0$. □

5.1 Quotients by Cartan Subgroups

In this subsection only we consider the case when $K = C$.

5.1.1 Generalized Complex Structures on G/C

Proposition 5.4. The subalgebras $E$ of $\mathfrak{g}_C$ satisfying $E + \overline{E} = \mathfrak{g}_C$ and $\mathfrak{h} \subset E$ are exactly the parabolic subalgebras.

Proof. Since $E$ is a subalgebra containing $\mathfrak{h}$, $E$ corresponds to some closed subset $A \subset \Delta$. $G$ is compact, so $\sigma$ maps $\mathfrak{g}_{C,\alpha}$ isomorphically into $\mathfrak{g}_{C, -\alpha}$. Therefore $E + \overline{E} = \mathfrak{g}_C$ implies that $A \cup -A = \Delta$. Subalgebras of this form are precisely the parabolic subalgebras ([1] p.174). □

For a parabolic subalgebra $E$ corresponding to a subset $A \subset \Delta$, we will let $\mathfrak{l} = \mathfrak{l}_E$ denote the Levi factor of $E$, i.e. the subalgebra corresponding to roots $A \cap -A$. Let $E$ be a parabolic subalgebra. To the end of showing that $\varepsilon$ is exact for any pair $(E, \varepsilon)$ of Prop 4.8, we put forth the following proposition.

Proposition 5.5. Fix a parabolic subalgebra $E = \mathfrak{h} \oplus \bigoplus_{\alpha \in A} \mathfrak{g}_{C, \alpha}$ of $\mathfrak{g}_C$, and let $V = V_E$ be the vector space $V := \{ \varepsilon \in \Lambda^2 E^* \mid d_E \varepsilon = 0 \text{ and } \varepsilon_{\mathfrak{h} h} = 0 \}$. Choose a base $\Delta_0$ for the roots $\Delta$ such that the corresponding system of positive roots $\Delta_0^+$ lies in $A$ and a basis $\{ X_\alpha \}_{\alpha \in \Delta^+} \cup \{ X_{-\alpha} = \sigma(X_\alpha) \}_{\alpha \in \Delta^+}$ for $\bigoplus_{\alpha \in \Delta} \mathfrak{g}_{C, \alpha}$. Let $\{ X_\alpha^* \in \mathfrak{g}_{C, \alpha} \}_{\alpha \in \Delta}$ be a dual basis.

1. Every $\varepsilon \in V$ is of the form

$$\varepsilon = \sum_{\alpha \in A \cap (-A)} c_\alpha X_\alpha^* \wedge X_{-\alpha}^*$$

for some constants $c_\alpha$, where $c_{-\alpha} = -c_\alpha$.

2. Any $\varepsilon \in V_E$ is exact. In fact, any $\varepsilon \in V_E$ is of the form $\tilde{\phi} \circ [\cdot, \cdot]$, where $\phi \in ([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h})^*$, and $\phi$ is extended by zero to a linear functional $\phi$ on $\mathfrak{g}_C = ([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h}) \oplus Z(\mathfrak{l}) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{C, \alpha}$. Thus $V$ is parameterized by $([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{h})^*$.

Proof. 1. For any $h \in \mathfrak{h}$, $X \in \mathfrak{g}_{C, \alpha}$, $Y \in \mathfrak{g}_{C, \beta}$, and $\varepsilon \in V$,

$$0 = d_E(h, X, Y) = \varepsilon(X,[Y,h]) + \varepsilon(Y,[h,X])$$

$$= \varepsilon(X,-\beta(h)Y) + \varepsilon(Y,\alpha(h)X) = (\beta(h) - \alpha(h))\varepsilon(X,Y).$$

This implies that for $X \in \mathfrak{g}_{C, \alpha}$ and $Y \in \mathfrak{g}_{C, \beta}$, one has $\varepsilon(X,Y) = 0$ unless $\alpha = -\beta$. Therefore, if one chooses a base $\Delta_0$ for the roots $\Delta$ such that $\Delta_0^+ \subset A$ and a basis $\{ X_\alpha \}_{\alpha \in \Delta^+} \cup \{ X_{-\alpha} = \sigma(X_\alpha) \}_{\alpha \in \Delta^+}$ for $\bigoplus_{\alpha \in \Delta} \mathfrak{g}_{C, \alpha}$ along with a dual basis $\{ X_\alpha^* \in \mathfrak{g}_{C, \alpha} \}_{\alpha \in \Delta}$,

$$\varepsilon = \sum_{\alpha \in A \cap (-A)} c_\alpha X_\alpha^* \wedge X_{-\alpha}^*$$

for some constants $c_\alpha$, where $c_{-\alpha} = -c_\alpha$. 16
2. By Lemma 5.3, \([l, l]\) is semisimple. By part (1) of this proposition, \(\varepsilon\) only depends on its restriction to \([l, l]\), so \(\varepsilon|_{[l, l]} = \varepsilon\) by Remark 3.10. Since \(\varepsilon\) is of the form described in part (1), we can assume that \(\phi \in (\mathfrak{l}, \mathfrak{l} \cap \mathfrak{h})^*\) and has been extended to be 0 on all of the root spaces \(\mathfrak{g}_{C, \alpha}, \alpha \in A - A\). We can also extend \(\phi\) by zero to \(\hat{\phi}\) on \(\mathfrak{g}_C = ([l, l] \cap \mathfrak{h}) \oplus \mathbb{Z}(l) \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{C, \alpha}\). Then on \(E\), \(\varepsilon = \hat{\phi} \circ [\ , \]\). Clearly \(\phi \in ([l, l] \cap \mathfrak{h})^*\) can be chosen freely, so that \(V_E = ([l, l] \cap \mathfrak{h})^*\). \(\square\)

For each \(\alpha \in \Delta\), let \(\hat{\alpha}\) denote the associated coroot. That is, if \(\alpha = \kappa(t_{\alpha}, -)\), where \(\kappa\) is the Killing form, then \(\hat{\alpha} = 2t_{\alpha}/(\alpha, \alpha)\). For a symmetric subalgebra \(\mathfrak{l} \subset \mathfrak{g}_C\) that contains \(\mathfrak{h}\), we denote its root system by \(\Delta(l)\).

**Proposition 5.6.** Let \(\varepsilon\) and \(\phi\) be as in Proposition 5.5. Then Condition 5 of Corollary 4.8 is satisfied if and only if either of the following equivalent conditions holds:

1. \(Re(c_{\alpha}) \neq 0\) for all \(\alpha \in \Delta(l)\).
2. \(Re(\phi(\hat{\alpha})) \neq 0\) for every coroot \(\hat{\alpha}\) with \(\alpha \in \Delta(l)\).

**Proof.**

1. Condition 5 of Corollary 4.8 is satisfied exactly when \(\varepsilon(X_{\alpha}, \sigma(X_{\alpha})) - \varepsilon(\sigma(X_{\alpha}), X_{\alpha}) \neq 0\) for all \(\alpha \in \Delta(l)\), which happens if and only if \(0 \neq Re(\varepsilon(X_{\alpha}, \sigma(X_{\alpha})) = Re(c_{\alpha})\) for all \(\alpha \in \Delta(l)\).

2. Let \(u_{\alpha} = [X_{\alpha}, \sigma(X_{\alpha})]\) for \(\alpha \in \Delta^+\). There exists some constant \(c\) such that \(u_{\alpha} = cY_{\alpha}\), where \(Y_{\alpha}\) is the unique vector in \(\mathfrak{g}_{C, -\alpha}\) such that \(\{X_{\alpha}, Y_{\alpha}\} = \hat{\alpha}\).

A quick computation shows that \(c\) is a real number. Since condition 5 of Corollary 4.8 is satisfied precisely when \(Re(\phi(u_{\alpha})) \neq 0\) for all \(\alpha \in \Delta(l)\) (\(\varepsilon = \phi \circ [\ , \]\)), condition 5 of Corollary 4.8 is equivalently satisfied if and only if \(Re(\phi(\hat{\alpha})) \neq 0\) for all \(\alpha \in \Delta(l)\) (\(\varepsilon = \phi \circ [\ , \]\)). \(\square\)

**Remark 5.7.** Since \(t_{\alpha + \beta} = t_\alpha + t_\beta\), we have \(\hat{\alpha} + \hat{\beta} = \frac{1}{(\alpha, \beta)}((\alpha, \alpha)\hat{\alpha} + (\beta, \beta)\hat{\beta})\). The set of \(\phi \in ([l, l] \cap \mathfrak{h})^*\) satisfying the condition of Proposition 5.6 is guaranteed to be nonempty; letting \(c_{\alpha} \in \mathbb{R}^+\) for all \(\alpha \in \Delta_0(l) := \Delta(l) \cap \Delta_0\) provides a \(\phi \in ([l, l] \cap \mathfrak{h})^*\) satisfying the condition of Proposition 5.6.

**Corollary 5.8.** The equivariant generalized complex structures on \(G/C\) are parameterized by pairs \((E, \phi)\) where \(E\) is a parabolic subalgebra of \(\mathfrak{g}_C\) and \(\phi \in ([l, l] \cap \mathfrak{h})^*\) is such that \(Re(\phi(\hat{\alpha})) \neq 0\) for all coroots \(\hat{\alpha}\) with \(\alpha \in \Delta(l)\). Moreover, if we fix a Borel subalgebra \(\mathfrak{b}\) containing \(\mathfrak{h}\), then the equivariant generalized complex structures, up to conjugacy by an automorphism of \(\mathfrak{g}_C\), are parameterized by pairs \((S, \phi)\) where \(S\) varies over subsets of simple roots for \(\mathfrak{b}\) and \(\phi\) is as above.

**Proof.** The result is now direct consequences of the previous results. The only observation that must be made is that any parabolic subalgebra is conjugate by an automorphism of \(\mathfrak{g}_C\) to some parabolic subalgebra containing \(\mathfrak{b}\). This proves the final assertion. \(\square\)

### 5.1.2 Real Dirac Structures

**Lemma 5.9.** There is a bijection between equivariant real Dirac structures on \(G/C\) and pairs \((E, \psi)\) of a subalgebra \(E\) containing \(\mathfrak{c}\) and \(\psi \in ([E, E] \cap \mathfrak{c})^*\) or, equivalently, pairs \((l, \phi)\) of a symmetric subalgebra \(l = E_C\) containing \(\mathfrak{h}\) and \(\phi \in ([l, l] \cap \mathfrak{h})^*\) such that \(\phi([E, E]) \subset \mathbb{R}\).

**Proof.** Subalgebras \(E\) containing \(\mathfrak{c}\) are in bijection with subalgebras \(l\) containing \(\mathfrak{h} = \mathfrak{c}_C\) such that \(\mathfrak{l} = l\) (by sending \(E\) to \(l = E_C\)). These are the symmetric subalgebras containing \(\mathfrak{h}\) and correspond to subsets \(S \subset \Delta\) which are themselves root systems.

Any \(\varepsilon \in \wedge^2 E^*\) may be extended \(\mathbb{C}\)-linearly to \(\varepsilon_C \in \wedge^2 \mathfrak{l}^*\), and \(L(l, \varepsilon_C)\) is a complex Dirac structure. The proof of Proposition 5.5 is still valid, even though \(l\) is symmetric but not necessarily a Levi subalgebra. The result is still that \(\varepsilon_C = \phi \circ [\ , \]\), and \(\phi([E, E]) \subset \mathbb{R}\). We may write \([E, E] \cap \mathfrak{c}_C = [E_C, E_C] \cap \mathfrak{t}_C = [E_C, E_C] \cap l \subset \mathfrak{h}\). Thus, if \(\psi = \phi|_{[E, E] \cap \mathfrak{c}}\), \(\phi = \psi_C\), by which we mean the \(\mathbb{C}\)-linear extension of \(\psi\).
Additionally, \( Z(l) \oplus \oplus_{\alpha \in \Delta \cap \mathcal{C}} \mathfrak{g}_{\alpha} = Z(l) \oplus \oplus_{\alpha \in \Delta \cap \mathcal{C}} \mathfrak{g}_{\alpha} \), whence \( Z(l) \oplus \oplus_{\alpha \in \Delta \cap \mathcal{C}} \mathfrak{g}_{\alpha} = F_{\mathcal{C}} \) for some \( F \subset \mathfrak{g} \). Namely, 
\[
F = Z(l) \oplus \oplus_{\alpha \in \Delta} U_{\alpha},
\]
where \( U_{\alpha} = R\text{-span}(X_{\alpha} + i\sigma(X_{\alpha}), i\sigma(X_{\alpha}) - i\sigma(X_{\alpha})) \). Clearly \([l, F] \subset F\). We may extend \( \psi \) by 0 on \( F \) to \( \tilde{\psi} \in \mathfrak{g}^{\ast} \). Then \( \tilde{\psi} = \psi^{\ast} \). A Dirac structure, therefore, gives a pair \((l, \psi)\) of a symmetric subalgebra \( l \) containing \( \mathfrak{h} \) and \( \psi \in ([E, E] \cap \mathfrak{c})^{\ast} \). Conversely, given such a pair \((l, \psi)\), we may extend \( \psi \) to \( \tilde{\psi} \) as before and let \( E \) be such that \( E_{\mathcal{C}} = l \). This gives \( L(E, \tilde{\psi} \circ [\cdot, \cdot]) \) which obviously corresponds to a Dirac structure.

\[\square\]

Remark 5.10. \( \phi([E, E]) \subset \mathbb{R} \) is equivalent to saying that \( \phi(\tilde{a}) = c_{a} \) is purely imaginary for all \( a \in \Delta(l) \).

5.2 Quotients by Connected Maximal Rank Subgroups

Let \( G \) be a compact, semisimple Lie Group and \( K \) be a subgroup containing a Cartan subgroup \( C \) of \( G \) as before. Let \( \mathfrak{c} = Lie(C) \) and \( \mathfrak{h} = \mathfrak{c}_{C} \). The subalgebra \( \mathfrak{c}_{C} \) corresponds to some root system \( \Delta(\mathfrak{c}_{C}) \subset \Delta \), which will be fixed throughout this section. For a subalgebra \( E \) containing \( \mathfrak{c}_{C} \), denote \( I := E \cap \overline{E} \), and let \( A = A_{1} \subset \Delta \) be the root system of \( I \). The results of Subsection 5.1 easily generalize to any \( K \subset C \).

Proposition 5.11. Let \( G \) be compact and \( K \subset C \) a closed, connected subgroup.

1. Equivariant real Dirac structures on \( G/K \) correspond bijectively to pairs \((E, \psi)\), where \( E \) is a subalgebra containing \( I \) and \( \psi \in Ann_{E/[E, E]^{\ast}}([I, I] \cap \mathfrak{c}) \). Equivalently, these Dirac structures can be described by pairs \((l, \phi)\), where \( I \) is a symmetric subalgebra containing \( \mathfrak{c}_{C} \) and \( \phi \in Ann_{I([I, I] \cap \mathfrak{h})^{\ast}}([\mathfrak{c}_{C}, \mathfrak{c}_{C}] \cap \mathfrak{h}) \) such that \( \phi([l, l] \cap \mathfrak{c}) \subset \mathbb{R} \).

2. The set of invariant generalized complex structures on \( G/K \) correspond bijectively to pairs \((E, \phi)\) where \( E \) is a parabolic subalgebra containing \( \mathfrak{c}_{C} \). \( \phi \in Ann_{I([I, I] \cap \mathfrak{h})^{\ast}}([\mathfrak{c}_{C}, \mathfrak{c}_{C}] \cap \mathfrak{h}) \), and \( Re(\phi(\tilde{a})) \neq 0 \) for all coroots \( \tilde{a} \) with \( a \in \Delta \setminus \Delta(\mathfrak{c}_{C}) \).

Proof. 1. Dirac pairs \((E, \varepsilon)\) are the same as for \( G/C \) except that we require \( I \subset E \) and \( \varepsilon_{I}(l) = 0 \). Any such \( \varepsilon \) is, therefore, of the form \( \varepsilon = \tilde{\phi} \circ [\cdot, \cdot] \) as in Corollary 4.8.

Letting \( Ann_{I([I, I] \cap \mathfrak{h})^{\ast}}([\mathfrak{c}_{C}, \mathfrak{c}_{C}] \cap \mathfrak{h}) := \{ a \in ([I, I] \cap \mathfrak{h})^{\ast} \mid a([\mathfrak{c}_{C}, \mathfrak{c}_{C}] \cap \mathfrak{h}) = 0 \} \), 
\[
\phi \in Ann_{I([I, I] \cap \mathfrak{h})^{\ast}}([\mathfrak{c}_{C}, \mathfrak{c}_{C}] \cap \mathfrak{h}) \iff \phi \in Ann_{I([E, E]^{\ast})^{\ast}}([I, I] \cap \mathfrak{c}) \iff \phi \in Ann_{I([E, E]^{\ast})^{\ast}}([E, E] \cap \mathfrak{c}) \iff \varepsilon_{I}(l) = 0.
\]

2. Any \( L(E, \varepsilon) \subset \mathfrak{g}_{C} \oplus \mathfrak{g}_{C}^{\ast} \) which provides an invariant Dirac structure on \( G/K \) also provides one on \( G/C \), and \( \varepsilon_{I} \) vanishes on \( I \) if and only if \( \phi \in Ann_{I([I, I] \cap \mathfrak{h})^{\ast}}([\mathfrak{c}_{C}, \mathfrak{c}_{C}] \cap \mathfrak{h}) \). Finally, condition 5 of Corollary 4.8 is met if and only if \( Re(\phi(\tilde{a})) \neq 0 \) for all \( a \in \Delta(I) \setminus \Delta(\mathfrak{c}_{C}) \). \( \square \)

5.3 B-Transforms

Lemma 5.12. Every real equivariant Dirac structure on \( G/K \) is the B-transform of some \( L(E, 0) \subset \mathfrak{g} \oplus \mathfrak{g}^{\ast} \). Therefore the equivalence class, under B-transformations, of invariant real Dirac structures on \( G/K \) is parameterized by subalgebras \( E \) of \( \mathfrak{g} \) containing \( I \).

Proof. For a Dirac structure given by \( L(E, \varepsilon) \subset \mathfrak{g} \oplus \mathfrak{g}^{\ast} \) we’ve already seen that \( \varepsilon = i^{\ast}B \) for \( B = \tilde{\psi} \circ [\cdot, \cdot] \), where \( i : E \hookrightarrow \mathfrak{g} \) is inclusion. Thus, \( e^{B}L(E, 0) = L(E, i^{\ast}B) = L(E, \varepsilon) \), and any invariant Dirac structure is equivalent via a B-transformation to some \( L(E, 0) \). \( \square \)

Proposition 5.13. Let \( G \) be compact and \( K \) contain a Cartan subgroup. The following data are equivalent:

1. \( G \)-invariant generalized complex structures on \( G/K \) up to B-transform.

2. Triples consisting of a connected subgroup \( H \) of \( G \) containing \( K \), a \( G \)-invariant complex structure on \( G/H \), and an \( H \)-invariant symplectic structure on \( H/K \).
The goal of this section is to explain the geometric structure of the space \( L \) containing \( G \). Let us first show that for any Lie subalgebra \( \mathfrak{H} \subset \mathfrak{g} \) containing \( \mathfrak{t} \), \( \mathfrak{H} \) is its own normalizer in \( \mathfrak{g}_C \). Because the normalizer \( n_{\mathfrak{g}_C}(\mathfrak{H}) \supset \mathfrak{H}_C \) is a direct sum of \( \mathfrak{h} = \mathfrak{t}_C \) and some root spaces, it is not difficult to see that \( n_{\mathfrak{g}_C}(\mathfrak{H}) = n_{\mathfrak{g}_C}(\mathfrak{H}_C) = \mathfrak{H}_C \) and that therefore \( g(\mathfrak{H}) = \mathfrak{H} \). Now given such a Lie subalgebra \( \mathfrak{H} \), the identity component of \( N_G(\mathfrak{H}) \) is a closed, connected subgroup with Lie algebra \( N_\mathfrak{g}(\mathfrak{H}) = \mathfrak{H} \). Therefore, any Lie subgroup of \( \mathfrak{g} \) containing \( \mathfrak{t} \) is the Lie algebra of a closed connected subgroup \( H \) of \( G \).

Given a parabolic subalgebra \( E \), \( \mathfrak{t}_C \), we find that \( E \cap \mathfrak{t} = \mathfrak{t}_C \) for some Lie subalgebra \( \mathfrak{t} \subset \mathfrak{t} \subset \mathfrak{g} \). We just showed that \( \mathfrak{H}_C = Lie H \) for some closed, connected Lie subgroup \( H \) of \( G \). Then \( E \) defines a \( G \)-invariant complex structure on \( G/H \). In the other direction, given \( H \) and a \( G \)-invariant complex structure on \( G/H \), this is simply a subalgebra \( E \subset \mathfrak{g}_C \) such that \( E \cap \mathfrak{H} = \mathfrak{H}_C \). It is clear that these constructions are inverses of each other. This shows that a choice of a parabolic subalgebra containing \( \mathfrak{t}_C \) is the same as the choice of a complex structure on the quotient of \( G \) by a closed, connected subgroup \( H \). \( \square \)

5.4 Moduli of Complex Dirac structures on \( G/K \)

We fix some notation. Let \( \mathcal{G}_C \) denote the set of generalized complex structures on \( G/K \). Let \( \mathcal{D}^G \) denote the set of complex Dirac structures, and let \( D^G \) denote the set of real Dirac structures. We denote by \( \mathcal{G}_C \), \( \mathcal{G}^G \), and \( D^G \) the ones that are \( G \)-invariant.

The goal of this section is to explain the geometric structure of the space \( \mathcal{D}^G \). We do this by expressing \( \mathcal{D}^G \) as a disjoint union of Euclidean spaces with closure relations. We know that \( D^G \) embeds into \( \mathcal{D}^G \) by complexification and that \( \mathcal{G}_C \subset \mathcal{D}^G \). Additionally, the set \( S := \{ \text{subalgebras of } \mathfrak{g}_C \text{ containing } \mathfrak{t}_C \} \) embeds into \( \mathcal{D}^G \) by sending \( E \mapsto L(E,0) \).

For abstract reasons, we know that \( \mathcal{D}^G \) is a variety. The orthogonal group \( O = O(\mathfrak{g} \oplus \mathfrak{g}^*,(\cdot,\cdot)) \) acts transitively on maximal isotropic subspaces. The set of maximal isotropic subspaces \( \mathcal{L} \) is the quotient of \( O \) by the stabilizer of \( \mathfrak{g}_C \) and so is itself a variety. The maximal isotropic subspaces \( L \) which contain \( \mathfrak{t}_C \) and for which \( [L,L] \subset L \) form a closed algebraic set in this variety. Therefore we may think of \( \mathcal{D}^G \) as a closed subvariety of \( \mathcal{L} \), which is itself a closed subvariety of the Grassmanian Gr(dim(\( g \)), \( \mathfrak{g} \oplus \mathfrak{g}^* \)).

In the cases \( su_3/\mathfrak{t} \) and \( su_4/\mathfrak{t} \) (for Cartan subalgebras \( \mathfrak{t} \)), \( \mathcal{D}^G \) will be described explicitly. To describe \( \mathcal{D}^G \) more generally is more difficult. However, we observe that

\[
\mathcal{D}^G = \bigsqcup_{E \in S} \mathcal{O}_E
\]
where $O_E := \{L(E,\varepsilon) \in \mathbb{C}D^G_{G/K}\}$. Clearly $L(E,\varepsilon) \in \mathbb{C}D^G_{G/K}$ if and only if $\varepsilon \in V_E$ (in the notation of Proposition 5.4). Therefore $O_E \simeq V_E$, which is a complex vector space. This is not very enlightening, however, since it is possible that $O_E \cap O_F \neq \emptyset$ for some $E, F \in \mathfrak{A}$. This description will be enhanced by stating $\text{dim}(O_E)$ for each $E \in \mathfrak{A}$ and all of the closure relations for the $O_E$’s.

A subalgebra $E \in \mathfrak{A}$ corresponds to a subset $A \subseteq \Delta$. $E = E_0 \oplus E'$ (direct sum as vector spaces) where $E_0$ is the subalgebra corresponding to the symmetric subset $A_0 = A \cap -A$ and $E'$ is the direct sum of the root spaces for $A \setminus A_0$. Recall that $\Delta(t_e) \subset \Delta$ is the subset of roots corresponding to $t_e \supset \mathfrak{h}$.

**Proposition 5.14.** With $E = E_0 \oplus E' \in \mathfrak{A}$ as above,

1. $E'$ is an ideal of $E$, and $A_0$ is a root subsystem.
2. $\text{dim}(O_E) = \text{rank}A_0$, and therefore $O_E \simeq \mathbb{C}^{\text{rank}A_0}(\Delta(t_e))$.

**Proof.** Since $A_0$ is the intersection of two closed subsets, it is closed. The fact that reflections leave $A_0$ invariant follows from the fact that for non-proportional roots $\alpha$ and $\beta$, the $\alpha$-string through $\beta$ is unbroken. It is now easily verified that $A_0$ satisfies all of the root system axioms.

To check that $E'$ is an ideal in $E$, let $\alpha \in A_0$ and $\beta \in A \setminus A_0$ such that $\gamma = \alpha + \beta$ is a root. If $\gamma \in A_0$, then $\beta = \gamma - \alpha \in A_0$ (since $A_0$ is symmetric and closed) which is a contradiction. Therefore $[E_0, E'] \subset E'$. Now suppose that $\alpha, \beta \in A \setminus A_0$ and $\gamma = \alpha + \beta \in A$. If $\gamma \in A_0$, then $-\gamma \in A$ and $-\beta = \alpha - \gamma \in A$ which contradicts the fact that $\beta \in A \setminus A_0$. This proves that $[E_0, E'] \subset E'$ and $[E', E'] \subset E'$.

2. This follows from Lemmas 5.3 and 5.5 $\square$

**Proposition 5.15.**

1. There is a continuous action of $W_t := \text{Ann}_H((t_e, t_e) \cap \mathfrak{h})$ on $\mathbb{C}D^G_{G/K}$, and the orbits of $W_t$ are all $O_E$ for $E \in \mathfrak{A}$. The action of $W_t$ is $\phi : L \mapsto e^{\phi[1]}L$. These are complex B-transforms, i.e. the 2-form is allowed to be complex.
2. $\overline{O_E} \cap O_F \neq \emptyset$ if and only if $\overline{O_F} \subset \overline{O_E}$.

**Proof.** Any $\varepsilon \in V_E = \{\omega \in \wedge^2 E^* \mid d_E \omega = 0 \text{ and } \omega(t) = 0\}$ is of the form $\phi \circ [\cdot, \cdot]$ for some $\phi \in \text{Ann}_H((E_0, E_0), (t_e, t_e) \cap \mathfrak{h})$, and $\phi$ may be extended to $\tilde{\phi} \in \text{Ann}_H((t_e, t_e) \cap \mathfrak{h})$ as in Lemmas 5.3, 5.5. Thus, any $L(E, \varepsilon)$ is the B-transform of $E = L(E, 0)$. B-transforms do not change the subalgebra $E$, so orbits are exactly all $O_E$’s.

2. If $x \in \overline{O_E} \cap O_F$. There is a sequence $s_n \in O_E$ converging to $x$. For any $g \in G$, $gs_n$ is a sequence in $O_E$ converging to $gx$. Thus, $gx \in \overline{O_E}$ for all $g \in W_t$, and $O_F \subset \overline{O_E}$ because $O_F$ is an orbit for $W_t$.

**Proposition 5.16.** Let $G$ be compact and $K$ contain a Cartan subgroup. If $E = E_0 \oplus E'$ as above, then $L(F, \omega) \in \mathbb{C}D^G_{G/K}$ lies in $\overline{O_E}$ if and only if $F_0$ is a Levi subalgebra of $E_0$ and $F' = E'$. Moreover, $O_F \cap \overline{O_E} \neq \emptyset$ if and only if $F_0$ is a Levi subalgebra of $E_0$ and $F' = E'$.

**Proof.** First we show that if $x \in \mathbb{C}D^G_{G/K}$ is not an isolated point, then $x \in \overline{O_E}$ for some $E \in \mathcal{S}$, and there is a sequence $L(E, \varepsilon_n)$ converging to $x$. If $x$ is not an isolated point, there is a sequence in $\mathbb{C}D^G_{G/K}$ converging to $x$. But $\mathbb{C}D^G_{G/K}$ is a union of finitely many orbits $\{O_K\}_{K \subseteq \mathcal{S}}$. Since there are finitely many subalgebras E in $\mathcal{S}$, there is a subsequence which lies in exactly one of the orbits.

Suppose that $O_F \cap \overline{O_E} \neq \emptyset$. By part 1 of this lemma and Proposition 5.15, it is enough to assume that there is a sequence $L(E, \varepsilon_n)$ converging to $L(F, 0)$ and then to show that $F_0$ is a Levi subalgebra of $E_0$ and that $F' = E'$. Fix a basis $\{X_\alpha\}_{\alpha \in \Delta}$ of $\mathbb{C}D_{G/K}$ such that $[X_\alpha, X_{-\alpha}] = t_\alpha$ for all $\alpha \in \Delta$. Let $\{X'_\alpha\}$ be a dual
basis.

For a point \( L(E, \varepsilon) \in \mathbb{C}D_G^{G/K} \), write \( \varepsilon \) in coordinates as \( \varepsilon = \sum_{\alpha \in A_0} \alpha \). Let \( \pi = \sum_{\alpha \in T} c_{\alpha} X_\alpha \times X_{-\alpha} \), where \( T = \{ \alpha \in A_0 \mid c_\alpha \neq 0 \} \). We have, in fact, expressed \( \varepsilon \) as \( \varepsilon \in \wedge^2 \mathfrak{g}_C^*, \) which we restrict to \( E \times E \). It therefore makes sense to consider \( \text{Im}(\varepsilon) \subset \mathfrak{g}_C^* \), which is just the span of the dual vectors \( \{ X_\alpha \}_{\alpha \in T} \). We immediately observe that \( L(E, \varepsilon) = L(\pi, \text{Ann}(E) \oplus \text{Im}(\varepsilon)) \) We have a sequence \( L(E, \varepsilon_n) \to L(F, 0) \) with \( \varepsilon_n = \phi_n \circ [\cdot] \) for a sequence \( \phi_n \in \left( \left[ E_0, E_0 \right] \cap \mathfrak{h} \right)^*. \) We may also think of \( \varepsilon_n \) as a system \( \{ c_{\alpha,n} \}_{\alpha \in A_0} \) with \( \alpha \in A_0 \) and \( n \in \mathbb{N} \). For any \( \alpha \in A_0 \), the sequence \( \{ c_{\alpha,n} \} \) may converge to infinity. If it does not, we can choose a bounded subsequence and therefore a convergent subsequence of \( c_{\alpha,n} \). Since \( A_0 \) is finite, there is a subsequence \( \varepsilon_{n_k} \), for which each \( c_{\alpha,n_k} \) is either convergent or goes to infinity. For ease of notation, assume that the original sequence has this property. In order to get \( L(F, 0) \) as the limit, for each \( \alpha \in A_0 \), it must be true that either \( c_{\alpha,n} \to 0 \) or \( |c_{\alpha,n}| \to \infty \). If \( c_{\alpha,n} \to k \neq 0 \), then since \( X_\alpha + c_{\alpha,n} X_\alpha \times X_{-\alpha} \), \( X_{-\alpha} - c_{\alpha,n} X_\alpha \in L(E, \varepsilon_n) \), we would have that \( X_\alpha + k X_{-\alpha}, X_{-\alpha} - k X_\alpha \in L(F, 0) \), which is impossible since \( L(F, 0) \) is isotropic.

We define \( S := \{ \alpha \in A_0 \mid \phi_n(\alpha) \to 0 \} = \{ \alpha \in A_0 \mid \phi_n(t_\alpha) \to 0 \} \). Note that \( \Delta(t_C) \) is contained in \( S \) because \( \phi_n \) vanishes on all coarrows \( \hat{\alpha} \) for \( \alpha \in \Delta(t_C) \) and for all \( n \). Since \( t_{\alpha+\beta} = t_\alpha + t_\beta, S \) is closed, symmetric, and in fact a root subsystem. If \( \alpha_1, \ldots, \alpha_r \) is a base for \( S \), suppose \( \beta = a_1 \alpha_1 + \ldots + a_r \alpha_r \) with all \( a_i \in \mathbb{R} \) then \( \tau_\beta = a_1 t_{\alpha_1} + \ldots + a_r t_{\alpha_r} \) and \( \phi_n(\tau_\beta) \to 0 \). If \( W = \text{span}(S) \subset \text{span}(\Delta) \), then \( W \cap \Delta = S \). It is known that for any subspace \( W \subset \text{span}(\Delta) \), \( W \cap \Delta = S \). Therefore \( S \) defines a Levi subalgebra. Note that \( S = \{ \alpha \in A_0 \mid c_{\alpha,n} \to 0 \} \) because \( \phi_n(t_\alpha) = \varepsilon_n([X_\alpha, X_{-\alpha}]) \). We may replace \( \varepsilon_n \) with a sequence \( \varepsilon_n \) such that \( c_{\alpha,n} = 0 \) for all \( n \) whenever \( \alpha \in S \) and \( c_{\alpha,n} = c_{\alpha,n} \) for all \( n \) whenever \( \alpha \in A_0 \setminus S \).

Then \( L(E, \varepsilon_n) \) and \( L(E, \varepsilon'_n) \) both converge to \( L(F, 0) \). Note that \( L(E, \varepsilon'_n) \) may not be in \( \mathbb{C}D_G^{G/K} \) but it is in the Grassmannian \( Gr(g \oplus g^*, \dim(g)) \) of which \( \mathbb{C}D_G^{G/K} \) is a subvariety.

We readily see that \( \text{Im}(\varepsilon) = \text{span}(X_\alpha)_{\alpha \in A_0} \) for all \( n \), and \( \text{Ann}(E) = \text{span}(X_\alpha)_{\alpha \in \Delta \setminus A} \). If \( \varepsilon_n = \sum_{\alpha \in A_0} c_{\alpha,n} X_\alpha \times X_{-\alpha} \), then \( L(E, \varepsilon'_n) = L(\pi_n, \text{Ann}(E) \oplus \text{Im}(\varepsilon'_n)) = L(\pi_n, \text{span}(X_\alpha \mid \alpha \in (\Delta \setminus A) \cup (A_0 \setminus S))) \), and \( \varepsilon'_n \to 0 \) as \( n \to \infty \). The limit is \( L(0, \text{span}(X_\alpha \mid \alpha \in (\Delta \setminus A) \cup (A_0 \setminus S))) = L(\mathfrak{h} + E' + I_S, 0) \) where \( I_S \) is the Levi subalgebra with root system \( S \). Now \( A' \cup S \) is the root system for a subalgebra \( F \) such that \( S \) corresponds to \( F_0 \) and \( F' = E' \). That \( F = \mathfrak{h} + E' + I_S \) is a subalgebra follows from the fact that \( E' \) is an ideal in \( E \). This shows that every limit point of \( O_E \) is of this form.

It remains to show that any subalgebra \( F \) containing \( t_\mathfrak{g} \) with \( F_0 \) a Levi subalgebra of \( E_0 \) and \( F' = E' \) is a limit point of \( O_E \). Suppose \( F_0 \) corresponds to the root subsystem \( S \subset \Delta \) (Recall \( \Delta(t_{C}) \subset S \)). Since \( S \) represents a Levi subalgebra of \( A_0 \), it is possible to choose a base \( \alpha_1, \ldots, \alpha_r \) for the root subsystem \( S \) which extends to a base \( \alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_n \) of \( A_0 \). By Lemma 5.3, 5.3. Any \( \varepsilon \) for which \( d_E \varepsilon = 0 \) is determined by the constants \( c_1 = c_{\alpha_1}, \ldots, c_n = c_{\alpha_n} \). Choose a sequence \( \varepsilon_n \) such that all \( c_i \in \mathbb{R} \) are negative, \( c_i = 0 \) for \( i \leq r \) and \( c_i \to \infty \) for \( i > r \). Then \( \varepsilon_n = t_{\alpha_i} \) form a basis of \( \mathfrak{h} \cap [E_0, E_0] \) with dual basis given by \( t_i^* \). So \( \varepsilon = (c_1, \ldots, c_n) = (\sum c_i t_i^*) \circ [\cdot] \).

Then \( \text{Im}(\varepsilon) = \text{span}(X_\alpha \mid \alpha \not\in \Delta) \) cannot be expressed purely as \( n_1 \alpha_1 + \ldots + n_r \alpha_r \). Then it converges to \( \text{span}(X_\alpha \mid \alpha \in A_0 \setminus S) \) for all \( n \). Then because \( L(E, \varepsilon_n) = L(\pi_n, \text{Im}(\varepsilon(n)) \oplus \text{Ann}(E)) \) and \( \pi_n \to 0 \), the limit is \( L(0, \text{Im}(\varepsilon(n)) \oplus \text{Ann}(E)) = L(F, 0) \).

5.4.1 The Moduli for SU2 and SU3

We consider the quotients of \( SU_2 \) and \( SU_3 \) by the standard Cartan subgroups and delineate explicitly the moduli of Dirac structures on these spaces.

Lemma 5.17. When \( G = SU_2 \) and \( K \) is the standard Cartan subgroup,
\( \mathbb{C}D_G^{G/K} = CP^1 \cup \{ \text{two points} \} \), and \( G^C_G^{G/K} = \mathbb{C} \setminus i \mathbb{R} \subset \mathbb{C} \subset CP^1 \subset CP^1 \setminus \{ \text{two points} \} \).
Lemma 5.18. Let \( G = SU_3 \) and \( K \) be the standard Cartan subgroup. Then \( \mathcal{CP}^2 \) is the disjoint union of the following connected components:

1. There is an isomorphism \( \overline{\mathcal{O}_{st}} \simeq \{(x, y) \times [u, v] \times [s, t] \in (\mathbb{CP}^1)^3 \mid xv + ytu - yvs = 0\}. \) The closure \( \overline{\mathcal{O}_{st}} \) of \( \mathcal{O}_{st} \) consists of \( \mathcal{O}_{st} \) together with \( \mathcal{O}_E \) for each Levi subalgebra \( E \). The generalized complex structures are \( \mathcal{GC}_{G/K} \cap \overline{\mathcal{O}_{st}} = \mathcal{O}_{st} \cap (\mathbb{C} \setminus i\mathbb{R})^3 \subset \mathbb{C}^3 \subset \mathbb{C}^1 \).

2. Let \( E \) be one of the six proper parabolic subalgebras. Then \( \overline{\mathcal{O}_E} \simeq \mathbb{C}^1 \). The generalized complex structures are \( \mathcal{GC}_{G/K} \cap \overline{\mathcal{O}_E} = \mathcal{O}_E \cap \mathbb{C} \subset \mathbb{C} \subset \mathbb{C} \subset \mathbb{P}^1 \).

3. The twelve remaining subalgebras represent isolated points.

Proof. For \( SU_3 \), \( \mathfrak{su}_3(\mathbb{C}) = \mathfrak{sl}_3(\mathbb{C}) \). Let \( \mathfrak{h} \) be the standard torus consisting of the diagonal matrices, and let \( X_{\alpha,\beta} \) denote the matrix with \( 1 \) in the \((\alpha,\beta)\)-th entry and zeros elsewhere. Let \( \alpha \) denote the root for which the root space contains \( X_{1,2} \), and let \( \beta \) denote the root for which \( X_{2,3} \) is the root space. Write \( \epsilon \in \mathbb{Z}^2(\mathfrak{st}, \mathbb{C}) \) as \( \epsilon = c_\alpha X^\alpha_{\alpha} \wedge X^\alpha_{-\alpha} + c_\beta X^\beta_{\beta} \wedge X^\beta_{-\beta} + c_{\alpha+\beta} X^\alpha_{\alpha+\beta} \wedge X^\beta_{-\alpha-\beta} \) with \( c_{\alpha+\beta} = c_\alpha + c_\beta \). We may think of \( \mathcal{O}_{st} \) as \( \{ (c_\alpha, c_\beta, c_{\alpha+\beta}) \in \mathbb{C}^3 \mid c_\alpha + c_\beta - c_{\alpha+\beta} = 0 \} \). Embedding \( \mathbb{C} \) into \( \mathbb{CP}^1 = \mathbb{C} \cup \infty \) gives \( \mathbb{C}^3 \subset (\mathbb{CP}^1)^3 \).

For a closed subset \( \Phi \subset \Delta \) of roots, we denote by \( E_\Phi \) the corresponding Lie subalgebra which is the sum of \( \mathfrak{h} \) and the root spaces for the roots in \( \Phi \). We will also denote \( \mathcal{O}_{E_\Phi} \) simply by \( \mathcal{O}_\Phi \). We make the following identifications.

\[
\begin{align*}
\mathcal{O}_{\pm \alpha} & = L(E_{\{ \pm \alpha \}}, c_\alpha X^\alpha_{\alpha} \wedge X^\alpha_{-\alpha}) \text{ corresponds to the point } [c_\alpha, 1] \times [0, 1] \subset (\mathbb{CP}^1)^3, \\
\mathcal{O}_{\pm \beta} & = L(E_{\{ \pm \beta \}}, c_\beta X^\beta_{\beta} \wedge X^\beta_{-\beta}) \text{ corresponds to the point } [1, 0] \times [c_\beta, 1] \subset (\mathbb{CP}^1)^3, \\
\mathcal{O}_{\pm (\alpha+\beta)} & = L(E_{\{ \pm (\alpha+\beta) \}}, c_{\alpha+\beta} X^\alpha_{\alpha+\beta} \wedge X^\beta_{-\alpha-\beta}) \text{ corresponds to the point } [0, 1] \times [0, 1] \subset (\mathbb{CP}^1)^3. \\
\mathcal{O}_0 & = L(\mathfrak{h}, 0) \text{ corresponds to the point } [1, 0] \times [1, 0].
\end{align*}
\]

This is a complete list of the Levi subalgebras in \( \mathfrak{sl}_3 \) that contain \( \mathfrak{h} \), all of which lie in \( \overline{\mathcal{O}_{st}} \). A quick computation, using the above identifications for the Levi subalgebras, shows that we can identify \( \mathcal{O}_{st} \) with \( \{ (x, y) \times [u, v] \times [s, t] \in (\mathbb{CP}^1)^3 \mid xv + ytu - yvs = 0\} \). The generalized complex structures are \( \mathcal{GC}_{G/K} \cap \overline{\mathcal{O}_{st}} = \mathcal{O}_{st} \cap (\mathbb{C} \setminus i\mathbb{R})^3 \subset \mathbb{C}^3 \subset \mathbb{C}^1 \) by Proposition 5.6.

There are six proper parabolic subalgebras: \( \pm \{ \pm \alpha, \beta, \alpha + \beta \}, \pm \{ \pm \alpha, \pm \alpha + \beta \}, \) and \( \pm \{ \alpha, -\beta, \pm (\alpha + \beta) \} \). For each of the parabolics \( E \), \( \mathcal{O}_E \simeq \mathbb{C} \), and the limit contains one point. Therefore \( \overline{\mathcal{O}_E} \simeq \mathbb{C}^1 \). For example, if \( E \) is \( \pm \{ \pm \alpha, \beta, \alpha + \beta \} \), then \( \mathcal{O}_E = \{ L(E, cX^\alpha_{\alpha} \wedge X^\beta_{-\alpha}) \} \), which is the same as \( \mathbb{C} \). Letting \( c \to \infty \) gives \( L(\{ \beta, \alpha + \beta \}, 0) \). For each of these parabolic subalgebras, the generalized complex structures are \( \mathcal{GC}_{G/K} \cap \overline{\mathcal{O}_E} = \mathcal{O}_E \subset \mathbb{C} \subset \mathbb{C} \subset \mathbb{C}^1 \) by Proposition 5.6. Since generalized complex pairs only occur for parabolic subalgebras, this provides a complete list of the generalized complex structures. There are six subalgebras which contain root spaces for two roots but are not Levi subalgebras. In this way, each is in the closure of some \( \mathcal{O}_E \) for \( E \) parabolic. Each of these copies of \( \mathbb{CP}^1 \) is a connected component of the moduli space \( \mathcal{CP}^2 \).

There are six subalgebras which contain only one root space, and there are six Borel subalgebras. These are all isolated points. □
6 Semisimple Orbits

We have given a description of Dirac structures on adjoint orbits when $G$ is compact. We now attempt to describe generalized complex structures on more semisimple orbits in more general groups. In the case of a semisimple orbit $O_h$ in a real semisimple Lie algebra, we would like to understand what are GC-pairs $(E, \varepsilon)$ for the homogeneous space $O_h = \text{Int}(g)/Z_G(h)$ (where $\text{Int}(g)$ is the connected Lie subgroup of $\text{Aut}(g)$ with Lie algebra $adg$). GC-pairs turn out to be equivalent to a pair $(A, \phi)$ of a closed subset $A$ of roots and a linear functional on $\Lambda \cap -\Lambda \subset \Delta$ satisfying some conditions (Theorem 6.1 and Corollary 6.2). We go on to describe such closed subsets $A$ to parabolic subalgebras.

First we consider the case when $h$ is a regular semisimple element, i.e. when $h := Z_g(h)$ is a Cartan subalgebra. The notation and formulation of statements is less burdensome for regular elements, but the proofs are essentially the same. As we will see, the results for general semisimple orbits follow immediately once we have done the regular case.

Throughout this section, fix a Cartan involution $\theta$ of $g$. Since any Cartan subalgebra is $\text{Int}(g)$-conjugate to a $\theta$-stable one, we may assume that $h$ is $\theta$-stable so that $h = t \oplus e$ is the Cartan decomposition of $h$. Let $x \mapsto \bar{x}$ or $\sigma$ denote conjugation in $g\mathfrak{c}$ with respect to $g$. Also $\sigma$ will denote conjugation with respect to roots: $(\sigma \alpha)(h) = \sigma(\alpha(h))$ for $\alpha \in \Delta = \Delta(g\mathfrak{c}, h\mathfrak{c})$. The involution $\theta$ extends to a $\mathbb{C}$-linear map on $g\mathfrak{c}$, also denoted by $\theta$. Since $h$ is $\theta$-stable, $\theta$ permutes the roots by $\theta(\alpha) = \alpha(\theta h)$. It is the case that $\theta = -\sigma$ on $h\mathfrak{c}_2$, and $\sigma|_{\mathfrak{g}} = 1, \sigma|_{\mathfrak{h}} = -1$.

We begin with the (simple version) of the main theorem of this section. The full version is Corollary 6.2 which addresses the case when $h$ is an arbitrary semisimple element.

**Theorem 6.1.** Let $h$ be a regular semisimple element in a real semisimple Lie algebra $g$ as above. Then the equivariant generalized complex structures on the adjoint orbit $O_h$ are given by pairs $(A, \phi)$ where $A \subset \Delta$ is a closed subset of roots such that $A \cup \sigma A = \Delta$ and $A \cap \sigma A \subset -A$, and $\phi$ is a linear functional on $\text{span}_\mathbb{C}(\Lambda \cap -\Lambda)$ satisfying $\phi(\alpha) \neq \phi(\sigma(\alpha))$ for all $\alpha \in A \cap \sigma A$.

**Proof.** Equivariant generalized complex structures on $O_h = G/Z_G(h)$, $G = \text{Int}(g)$, are given by pairs GC-pairs $(E, \varepsilon)$. Since $h\mathfrak{c} = E$, $E = h\mathfrak{c} \oplus \bigoplus_{\alpha \in A \cap \sigma A} (g\mathfrak{c})_\alpha$ for some closed subset $A$ of $\Delta$. Since $\sigma$ maps $(g\mathfrak{c})_\alpha$ isomorphically to $(g\mathfrak{c})_{\sigma \alpha}$, Conditions (1) and (2) of Proposition 4.8 are satisfied if and only if if $A \cup \sigma A = \Delta$. Now let $S = A \cap -A$. Then $h\mathfrak{c} = \text{span}_\mathbb{C} S \subset h\mathfrak{c}$. We claim that $d_E \varepsilon = 0$ and $\varepsilon|_{h\mathfrak{c}} = 0$ if and only if $\varepsilon = \tilde{\phi} \circ [\ , \ ]$, where $\tilde{\phi}$ is an extension–extended to be 0 on $(h\mathfrak{c})^\perp$ (determined by the Killing form $\kappa$)–of some $\phi \in (h\mathfrak{c})^*$. Clearly any such $\phi$ gives $\varepsilon = \tilde{\phi} \circ [\ , \ ]$ satisfying $d_E \varepsilon = 0$ and $\varepsilon|_{h\mathfrak{c}} = 0$. It remains to show that any such $\varepsilon$ of this form.

Just as in Proposition 5.6, if $X \in (g\mathfrak{c})_\alpha$ and $Y \in (g\mathfrak{c})_\beta$, then $\varepsilon(X, Y) \neq 0$ only if $\beta = -\alpha$. Therefore if we let $E' = b' \oplus \bigoplus_{\alpha \in S} (g\mathfrak{c})_\alpha$, $\varepsilon$ is determined by $\omega := \varepsilon|_{E' \otimes E'}$. It is still true that $\omega_\sharp(b') = 0$ and $d_{E'} \omega = 0$. However, now there is the advantage that $E'$ is semisimple, which implies that $H^2(E', \mathbb{C}) = 0$. Then since $\omega \in Z^2(E', \mathbb{C})$, $\omega = \phi \circ [\ , \ ]$ for some $\phi \in (E')^*$. However, $\phi$ must vanish on each $(g\mathfrak{c})_\alpha$ because $(g\mathfrak{c})_\alpha = [b', (g\mathfrak{c})_\alpha]$. Thus, $\phi$ is determined by $\phi|_{b'}$.

Finally, we claim that condition (5) of Proposition 4.8 are satisfied if and only if the following two conditions are met:

a.) $A \cap \sigma A \subset -A$

b.) $\phi(\alpha) \neq \phi(\sigma(\alpha))$ for all $\alpha \in A \cap \sigma A$. 

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If a.) and b.) are true, let \( X = h + \sum_{\alpha \in A \cap \sigma A} X_\alpha \notin h_C \). Since \( X \notin h_C, X_\alpha \neq 0 \) for some \( \alpha \in A \cap \sigma A \subset -A \). Choose \( Y \in (g_C)_{-\alpha} \) such that \([X,Y] = \tilde{\alpha}\). This entails that \( \varepsilon(X,Y) = \phi(\tilde{\alpha}) \). On the other hand, \( \overline{\varepsilon(X,Y)} = \phi(\sigma(\tilde{\alpha})) = \phi(\sigma(\alpha)) \neq \phi(\tilde{\alpha}) \) by b.). So condition 5 of Proposition 4.8 is satisfied.

Now suppose that condition 5 of Proposition 4.8 is satisfied and let \( X \in (g_C)_{\alpha} \) for some \( \alpha \in A \cap \sigma A \). Condition 5 implies that there exists \( Y \in E \cap E' \) such that \( \phi([X,Y]) \neq \phi(\sigma[X,Y]) \). Since \( E = h \oplus \bigoplus_{\alpha \in A} (g_C)_{\alpha} \), we may assume that \( Y \in (g_C)_{-\alpha} \) since \( \phi \) (or more precisely \( \tilde{\phi} \)) vanishes on the root spaces. Hence, \( 0 \neq Y \in E \cap E' \cap (g_C)_{-\alpha} \) and so \( (g_C)_{-\alpha} \in E \) and \( -\alpha \in A \). This shows that \( A \cap \sigma A \subset -A \). Also, \( Y \) can be chosen such that \( [X,Y] = \tilde{\alpha} \), so \( \phi(\tilde{\alpha}) = \phi(\tilde{\alpha}) \). \( \Box \)

There is a couple of easy examples:

1.) If \( h \) is a split Cartan subalgebra, then any generalized complex pair \((E,\varepsilon)\) must satisfy \( E = g_C \) since \( \sigma \alpha = \alpha \) for all roots \( \alpha \in \Delta \).

2.) If \( g \) is compact, then this is the example described in Section 5.

If \( h \in g \) is an arbitrary semisimple element, then \( Z := Z_{g_C}(h) \) is a Levi subalgebra containing a Cartan subalgebra \( h \). Then the subalgebra \( Z \) corresponds to some subset \( \Lambda \subset \Delta(g_C,h) \). The situation in this case is almost the same except that we require \( \Lambda \subset A \). This can be stated explicitly in the following theorem.

**Corollary 6.2.** Let \( h \) be a semisimple element in a real semisimple Lie algebra \( g \) as above. As above, let \( \Lambda \subset \Delta(g_C,h) \) be the set of roots corresponding to \( Z(h) \supset h \). The equivariant generalized complex structures on the adjoint orbit \( O_h \) are given by \( Z_G(h)\)-invariant pairs \((A,\phi)\), where \( \Lambda \subset A \subset \Delta \) is a closed subset of roots and \( \phi \in (\text{span}_C(\{\tilde{\alpha} \mid \alpha \in A \cap -A\})^* \), satisfying:

1. \( A \cup \sigma A = \Delta \),
2. \( A \cap \sigma A \subset -A \),
3. \( \phi(\tilde{\alpha}) \neq \phi(\sigma(\tilde{\alpha})) \) for all \( \alpha \in A \cap \sigma A \setminus \Lambda \), and
4. \( \phi(\tilde{\alpha}) = 0 \) for all \( \alpha \in \Lambda \).

**Proof.** GC-pairs \((E,\varepsilon)\) correspond to pairs \((A,\phi)\) as in Theorem 6.1. The only additional requirements are that \( \Lambda \subset A \) and \( \phi(\tilde{\alpha}) = 0 \) for all \( \alpha \in \Lambda \). \( \Box \)

In the remainder of the section we analyze conditions (1) and (2) of Theorem 6.2 and relate them to \( \theta \)-stable parabolic subalgebras.

**Definition 8.** Any closed subset \( A \subset \Delta \) will be called a generalized complex subset of roots if \( A \cup \sigma A = \Delta \) and \( A \cap \sigma A \subset -A \).

Generalized complex subsets correspond to subalgebras \( E \) of \( g_C \) which occur in generalized complex pairs \((E,\varepsilon)\) by Corollary 6.2. It would now be helpful to have some description of generalized complex subsets of roots.

**Lemma 6.3.** If \( A \) is a generalized complex subset, then \( A \cup \theta A \) is the closed subset of roots corresponding to a parabolic subalgebra containing \( h_C \).

**Proof.** A closed subset, \( \Phi \), of roots corresponds to a parabolic subalgebra containing \( h_C \) if and only if \( \Phi \cup -\Phi = \Delta \). \( \Box \). Closed subsets of roots of this type are called parabolic. Let \( \Phi = A \cup \theta A \). First we see that \( \Phi \cup -\Phi = A \cup \theta A \cup -A \cup -\theta A \subset A \cup -\theta A = A \cup \sigma A = \Delta \).

Therefore it only needs to be shown that \( \Phi \) is a closed subset of \( \Delta \). If \( \alpha, \beta \in \Phi \) and \( \alpha + \beta = \gamma \in \Delta \), we must show that \( \gamma \in \Phi \). Since \( A \) and \( \theta A \) are closed, this reduces to the case when \( \alpha \in A \) and \( \beta \in \theta A \). In this case, if \( \gamma \in A \subset \Phi \), then there is nothing to show. Otherwise, \( \gamma \in \sigma A \) because \( A \cup \sigma A = \Delta \). Then \( -\beta \in \sigma A \), and \( \gamma \in \sigma A \) so that \( \alpha = \gamma - \beta \in \sigma A \). Hence, \( \alpha \in A \cap \sigma A \subset -A \). Therefore \( \pm \alpha \in A \). But also \( \alpha \in A \cap \sigma A \subset -A \) implies \( -\sigma \alpha \in A \), whence \( \pm \alpha, \pm \sigma \alpha \in A \). Then \( \alpha \in \theta A \) and so \( \alpha + \beta \in \theta A + \theta A \subset \theta A \). \( \Box \)
If $A$ is a generalized complex subset of $\Delta$, then $\Phi = A \cup \theta A$ is parabolic and $\Psi = \Gamma \cup \Psi$, where $\Gamma = \Phi \cap -\Phi$ and $\Psi = \Phi \setminus \Gamma$. It is clear that if $\Phi$ corresponds to a parabolic subalgebra $p \subset g_C$, then $\Gamma$ corresponds to a Levi factor $I$ of $p$ and $\Psi$ corresponds to the nilpotent radical $u$ of $p$: $p = I \oplus u$.

We easily see that $\Gamma = (A \cup \theta A) \cap (-A \cup -\theta A) = (A \cap -A) \cup (A \cap -\theta A) \cup (-\theta A \cap -A) \cup (-A \cap -\theta A)$. But $A \cap -\theta A = A \cap \sigma A \subset A \cap -A$ and similarly, $\theta A \cap -A \subset A \cap -A$. Therefore $\Gamma = (A \cap -A) \cup (A \cap -A)$.

Since the Levi subalgebra $I$ is reductive, 

$$I = \bigoplus_{i=1}^{n} S_i \oplus Z(I),$$

where each $S_i$ is a simple Lie algebra and $Z(I) \subset h$. Each $S_i$ corresponds to a subset $\Gamma_i \subset \Gamma$. Obviously $\Gamma_i = (A \cap -A) \cap \Gamma_i \cup \theta(A \cap -A) \cap \Gamma_i$. The following lemma will demonstrate that for each $i$, either $\Gamma_i \subset (A \cap -A)$ or $\Gamma_i \subset \theta(A \cap -A)$.

Obviously $\Phi$ is $\theta$-stable, and $\Gamma = (A \cap -A) \cup \theta(A \cap -A)$, so $\Gamma$ is also $\theta$-stable. Hence, $\Psi$ is also $\theta$-stable. Of course, this means that $p$, $u$, and $I$ are also all $\theta$-stable. We know that $\Delta = A \cup \sigma A$ so that $-\Psi \cap \Phi = \emptyset$ implies that $-\Psi \subset \sigma A$. Applying $\sigma$ gives $\Psi \subset A$ because $\Psi$ is $\theta$-stable.

**Lemma 6.4.** Let $g$ be a simple complex Lie algebra containing Cartan subalgebra $h$. If $P$ and $Q$ are two reductive subalgebras containing $h$ such that $P \cap Q = g$, then $P = g$ or $Q = g$. Equivalently, if $\Delta$ is an irreducible root system and $X, Y$ are closed symmetric subsets of $\Delta = \Delta(g, h)$ such that $X \cup Y = \Delta$, then either $X = \Delta$ or $Y = \Delta$.

**Proof.** We prove the latter statement. The first step is to show that $X^c \perp Y^c$, where $X^c = \Delta \setminus X \subset Y \setminus X$. Let $\alpha \in X$, $\beta \in X^c$. Suppose that $\alpha + \beta = \gamma \in \Delta$. If $\gamma \in X$, then $\beta = \gamma - \alpha$. But $X$ is symmetric, so $\alpha \in X \implies -\alpha \in X$. Since $X$ is closed, $\beta = \gamma - \alpha \in (X + X) \cap \Delta \subset X$, which is a contradiction. Therefore, $\Delta \cap (X + X^c) \subset X^c$ and similarly, $\Delta \cap (Y + Y^c) \subset Y^c$. If $\alpha \in X^c$ and $\beta \in Y^c$, then $\alpha + \beta \in X + X^c \subset X^c$, and $\alpha + \beta \neq X + X^c \subset X^c$, which means that $\alpha + \beta \in X^c \cap Y^c = \Delta^c = \emptyset$. Therefore $(X^c + Y^c) \cap \Delta = \emptyset$. If $\alpha \in X^c$, $\beta \in Y^c$, then if $(, ,)$ denotes the usual inner product on roots, $(\alpha, \beta) \geq 0$ because otherwise, $\alpha + \beta$ would be a root. Thus, $(X^c, Y^c) \geq 0$. But $0 \leq (X^c, Y^c) = (X^c, -Y^c) = -(X^c, Y^c) \leq 0$. So $(X^c, Y^c) = 0$.

If $Y^c = 0$, then $Y = \Delta$ and we have proven what we wanted. If, on the other hand, $Y^c \neq 0$, then $Z := (Y^c)^\perp \cap \Delta$ are the roots corresponding to some Levi subalgebra $I$ because if $V$ is the vector space spanned by $\Delta$, $W \cap \Delta$ defines a Levi subalgebra for any subspace $W \subset V$. But since $X^c \perp Y^c$, $X^c \subset Z$, whence $Z^c \subset X$.

There is some parabolic subalgebra $p = I \oplus u$, where $u$ is the nilpotent radical of $p$. Then $u$ is the direct sum of root spaces for some subset of roots $U \subset \Delta$, $p$ corresponds to the roots $Z \cup U$, and $\Delta = Z \cup U \cup -U$. Then since $Z^c \subset X$, $\pm U \subset X$.

In order to show that $X = \Delta$, we must show that $X$ contains all simple roots (with simple roots determined by a Borel subalgebra $b$ containing $u$ and contained in $p$). In this situation, the Levi factor $I$ corresponds to some subset of simple roots. Let $\alpha_0$ be a simple root. Since $g$ is simple, the Dynkin diagram for $\Delta$ is a connected graph, which is a tree. Either $\alpha_0 \in U \subset X$, or it is possible to choose a string $\alpha_0, \alpha_1, \ldots, \alpha_k$ of simple roots such that $\alpha_k \in U$ and $\alpha_0, \alpha_1, \ldots, \alpha_{k-1} \in Z$ and such that $(\alpha_i, \alpha_j) < 0$ if $j = i+1$ and $(\alpha_i, \alpha_j) = 0$ if $j > i+1$. This implies that $\alpha_0 + \alpha_1 + \ldots + \alpha_k \in \Delta$ and also that both $\alpha_1 + \ldots + \alpha_k \in U$ and $\alpha_0 + \alpha_1 + \ldots + \alpha_k \in U$. Hence $\pm (\alpha_1 + \ldots + \alpha_k) \in Z^c \subset X$, from which it follows that $\alpha_0 = (\alpha_0 + \alpha_1 + \ldots + \alpha_k) - (\alpha_1 + \ldots + \alpha_k) \in X + X \subset X$. Therefore every simple root lies in $X$. Because $X$ contains all simple roots and $X = -X = \Delta$ as desired. □

Lemma 6.4 ensures that for each $i$, either $\Gamma_i \subset (A \cap -A)$ or $\Gamma_i \subset \theta(A \cap -A)$. If, in the above notation, $S$ denotes the set of summands $\{S_1, \ldots, S_n\}$ or $\{\Gamma_1, \ldots, \Gamma_n\}$, $\theta$ is a permutation of $S$, and we will set $S' = \{S \in S \mid \theta S = S\}$. More generally, for a parabolic $p$, $S_p$ will denote the set of semisimple summands in $p$. 25
Proposition 6.5. With the above notation, a generalized complex subset $A$ of $\Delta$ is equivalent to the following data:

1. A $\theta$-stable parabolic subalgebra $p \supset h$.
2. Subsets $T, R$ of $S_p$ such that $T \cup \theta T \cup R = S_p$.
3. Reductive subalgebras $q_i$ of $S_i$ with $h \cap S_i \subset q_i \subset S_i$, $S_i \in \theta T$. (The corresponding set of roots in $\Delta(S_i, h \cap S_i)$ will be denoted by $A_i$).

Proof. Given $A$, let $p = A \cup \theta A$, let $T = \{\Gamma_i \mid \Gamma_i \subset A \cap -A \text{ but } \Gamma_i \notin \theta(A \cap -A)\}$, and let $R = \{\Gamma_i \mid \Gamma_i \subset (A \cap -A) \cap \theta(A \cap -A)\}$. Finally, let $q_i$ be the subalgebra containing $S_i \cap h$ and the root spaces $\Gamma_i \cap A \cap -A$ for $S_i \in T$.

Conversely, given the data $\Phi$, $T$, $R$, $q_i$, $\Phi = \Gamma \cup \Psi$, and $\Gamma = \cup \Gamma_i$. Let

$$A = \Psi \cup \{\Gamma_i \mid \Gamma_i \in T \cup R\} \cup \{A_i \mid S_i \in \theta T\}.$$ 

Then $\sigma A = -\Psi \cup -\{\Gamma_i \mid \Gamma_i \in \theta T \cup R\} \cup \{\theta A_i \mid S_i \in \theta T\}$ and $\theta A = \Psi \cup \{\Gamma_i \mid \Gamma_i \in \theta T \cup R\} \cup \{A_i \mid S_i \in \theta T\}$.

Clearly $A \cup \theta A = \Phi$, $A \cup \sigma A = \Delta$, and $A \cap \sigma A = (\bigcup A_i) \cup (\bigcup \theta A_i) \cup R \subset A$. This means that $A$ is a generalized complex subset. □

7 Nilpotent Orbits

We classify generalized complex structures on real nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$ by showing that all are $B$-transforms of symplectic structures. We conjecture that the same is true in any split semisimple Lie algebra over $\mathbb{R}$. We reduce this conjecture to the case of distinguished nilpotent elements in simple Lie algebras.

7.1 Nilpotent orbits in $\mathfrak{sl}_n(\mathbb{R})$

In this subsection $G = SL_n(\mathbb{R})$ and $g = \mathfrak{sl}_n(\mathbb{R})$.

Proposition 7.1. Let $g = \mathfrak{sl}_n(\mathbb{R})$, and let $e \in g$ be any nilpotent element. If $E \subset g_C$ is any Lie subalgebra such that $Z_{g_C}(e) \subset E$ and $E + E^\perp = g_C$, then $E = g_C$.

Proof. First we prove this for the case when $e$ is regular. Since $e$ is regular, there is a standard triple $\{e, h, g\} \subset g$ with the following properties. There is a split Cartan subalgebra $h \subset g$ so that

$$g = h \oplus \bigoplus_{\alpha \in \Delta} g_\alpha$$

There is a Borel $b \subset h$ such that $h \in b$ and $\alpha(h) = 2$ for all simple roots $\alpha$, which are determined by $b$. Also, for every simple root $\alpha$, $pr_{g_\alpha} e \neq 0$.

Now let $E$ be a Lie subalgebra such that $Z_{g_C}(e) \subset E$ and $E + E^\perp = g_C$. If $\dim E = d$, we can view $E \in Gr_d(g_C)$. Define $L = \lim_{s \to -\infty} Ad(exp(s h)) E \in Gr_d(g_C)$. Since $Ad(exp(s h)) Z_{g_C}(e) = Z_{g_C}(e)$ for all $s$, $Z_{g_C}(e) \subset L$. Let $\gamma$ be the unique root of maximum height. Since $E + E^\perp = g_C$, there is a vector $v \in E$ whose projection, $pr_{(g_C)_{-\gamma}} v$, onto the root space $(g_C)_{-\gamma}$ is nonzero. Since $(g_C)_{-\gamma}$ is the eigenspace on which $h$ has the least eigenvalue, $\lim_{s \to -\infty} Ad(exp(s h)) v = x \in (g_C)_{-\gamma}$ in $\mathbb{P}(g_C)$. Hence, $L$ contains $Z_{g_C}(e)$ and $(g_C)_{-\gamma}$ and is stable under $ad_h$.

To prove the proposition, it therefore suffices to show that for any Lie subalgebra $L$ of $g_C$ satisfying $ad_h(L) \subset L$, $Z_{g_C}(e) \subset L$, and $g_C_{-\gamma} \subset L$ must equal all of $g_C$. In fact, we show that we can reduce to the case when $h \in L$. If we prove this for all such $L$ which also contain $h$, then let $L$ be such a Lie algebra.
which is $h$-stable but does not necessarily contain $h$. Then $L' := <h > + L$ satisfies all of the necessary conditions and contains $h$. Therefore, $L' = g_C$. Since $L$ is $h$-stable, $L$ is a non-zero ideal in $g_C$ which was assumed to be simple. In fact, only $g$ was assumed to be simple, but $g$ simple and split implies that $g$ is not complex and therefore $g_C$ is simple. Therefore $L = g_C$. Hence, it suffices to show that if $L \subset g_C$ is a Lie subalgebra such that $h \in L$, $Z_{g_C}(e) \subset L$, and $g_{C- \gamma} \subset L$, then $L = g_C$. To this end, suppose that $L$ is any such subalgebra.

There is a decomposition $L = m \oplus n$, where $n$ is the nilpotent radical of $L$ and $m$ is a reductive subalgebra. Since $h \in L$ is semisimple and $n$ is an ideal, $ad_h$ acts diagonally on $L$ and $n$. Hence $L$ decomposes as a direct sum of its $ad_h$-eigenspaces, $L_i$ ($i \in \mathbb{Z}$). Also $n$ decomposes into a direct sum of its $ad_h$-eigenspaces $n_i$.

First we show that $n_0 = 0$. By definition, $n_0 \subset Z_{g_C}(h) = h_C$. Therefore any elements in $n_0$ act diagonally on $L$ and $n$. Since $n$ is nilpotent, this implies that $n_0 \subset Z(L) \subset Z_L(e, g_{C- \gamma}) \subset Z_{g_C}(e, g_{C- \gamma})$. Thus, to show that $n_0 = 0$, it suffices to show that $Z_{g_C}(e, g_{C- \gamma}) = 0$.

Since $e$ is regular, we may assume that $h$ is the diagonal matrices in $s_l_n$ and

$$e = \sum_{i=1}^{n-1} X_{i,i+1}$$

where $X_{i,j}$ is the matrix with 1 in the $ij$-th entry and 0 elsewhere. Then $X_{n,1}$ is a basis for $g_{C- \gamma}$ if we take the standard Borel (in fact the only Borel containing $e$). It is easily checked now that $Z_{g_C}(e, g_{C- \gamma}) = 0$.

This shows that $Z_{g_C}(e, g_{C- \gamma}) = 0$ and therefore $n_0 = 0$. If $0 \neq x \in n_i$ for $i < 0$, then since $Ker(ad_e) \cap \bigoplus_{i \leq 0} n_i = 0$, $0 \neq ad_e x \in n_0 = 0$, which gives a contradiction. Therefore $n_i = 0$ for $i \leq 0$ and so

$$n = \bigoplus_{i > 0} n_i.$$

However, since $\gamma$ is the highest root,

$$[X_{- \gamma}, g_C] \subset \bigoplus_{i \leq 0} (g_{C})_i.$$

Since $X_{- \gamma} \in L$ and $n$ is an ideal in $L$, this means that $[X_{- \gamma}, n] = 0$.

We have seen that $n \subset Z_{g_C}(X_{- \gamma})$. If $n \neq 0$, then there is some nonzero $x \in n$. $n$ is $ad_e$-stable since it is an ideal of $L$. But since $e$ is nilpotent, $ad_e k x = 0$ for some minimal $k$. Then $0 \neq ad_e k x \in n \cap Z_L(e) \subset Z_L(e, X_{- \gamma}) \subset Z_{g_C}(e, g_{C- \gamma}) = 0$. This is a contradiction. Therefore $n = 0$.

Because $n = 0$, $L = m$, which is reductive. Hence, $L = S \oplus Z$, where $Z$ is the center of $L$ and $S$ is semisimple. Since $[h, e] = 2e, e \in S$. Because $S$ is semisimple, there is a standard triple $\{e, h', f'\} \subset S \subset g_C$. However, when we view $g_C$ as an $sl_2$-module for the triple $\{e, h', f'\}$, $g_C$ is generated by $Z_{g_C}(e)$. Since $Z_{g_C}(e)$ and $\{e, h', f'\}$ are contained in $L$, $L = g_C$.

We have proven the theorem for the case when $e$ is regular. If $e$ is not regular, Subsection [7][2] shows that the theorem still holds. We need only observe that if $l$ is a Levi subalgebra of $s_l_n$, then it is a direct sum of its center and semisimple Lie algebras, each isomorphic to some $sl_m$. □

**Corollary 7.2.** Every equivariant generalized complex structure on $s_l_n(\mathbb{R})$ is a $B$-transform of a symplectic structure.

**Proof.** This now follows from Proposition[7][1] because generalized complex structures of the form $L(TM_C, \varepsilon)$ are precisely the B-transforms of symplectic structures. □

With this in mind, the following proposition describes generalized complex structures on $O_\varepsilon$ for any nilpotent $e \in s_l_n(\mathbb{R})$. 27
Proposition 7.3. The equivariant generalized complex structures on a nilpotent orbit \( O_e \) in \( \mathfrak{sl}_n(\mathbb{R}) \) are given by an open set in \( Z(Z_{\mathfrak{g}_C}(e))^{Z_G(e)} \). Specifically, the equivariant generalized complex structures are parameterized by all \( t = t_r + it_i \in Z(Z_{\mathfrak{g}_C}(e)) \) such that \( Z_{\mathfrak{g}_C}(t_i) = Z_{\mathfrak{sl}_n(\mathbb{C})}(e) \) and such that \( t \) is \( Z_G(e) \)-invariant.

Proof. Let \( \mathfrak{g} = \mathfrak{sl}_n(\mathbb{R}) \). Proposition 7.1 shows that any GC-pair \((E, \varepsilon)\) must in fact satisfy \( E = \mathfrak{g}_C \). Since \( \mathfrak{g}_C \) is semisimple, \( H^2(\mathfrak{g}_C, \mathbb{C}) = 0 \), whence \( \varepsilon = \phi \circ [ , ] \) for some \( \phi \in \mathfrak{g}_C^\ast \). Any such \( \varepsilon \) satisfies \( dE\varepsilon = 0 \). But \( \varepsilon \) must also satisfy \( \varepsilon(Z_{\mathfrak{g}_C}(e), \mathfrak{g}_C) = 0 \). Using the Killing form, \( \mathfrak{g}_C \simeq \mathfrak{g}_C^\ast \), so we may identify \( \phi \) with some \( t \in \mathfrak{g}_C \)

\[
\phi([Z_{\mathfrak{g}_C}(e), \mathfrak{g}_C]) = 0 \\
\iff \kappa(t, [Z_{\mathfrak{g}_C}(e), \mathfrak{g}_C]) = 0 \\
\iff \kappa([t, Z_{\mathfrak{g}_C}(e)], \mathfrak{g}_C) = 0 \\
\iff [t, Z_{\mathfrak{g}_C}(e)] = 0 \\
\iff t \in Z(Z_{\mathfrak{g}_C}(e))
\]

Therefore, \( \varepsilon \) satisfies condition 4 of Proposition 7.3 if and only if \( t \in Z(Z_{\mathfrak{g}_C}(e)) \).

For \((E, \phi \circ [ , ])\) to be a GC-pair, condition 5 of Proposition 7.3 must also be satisfied. Breaking up \( \phi = \phi_r + i\phi_i \) into real and imaginary parts, condition 5 is equivalent to requiring that for all \( x \notin Z_{\mathfrak{g}_C}(e) \), there exists \( y \in \mathfrak{g}_C \) such that \( \phi_i[x, y] \neq 0 \). In other words, we require that \( \phi_i \circ [ , ] ) \) be nondegenerate on \( \wedge^2 \mathfrak{g}_C/Z_{\mathfrak{g}_C}(e) \). Again, when \( \phi_i \) is identified with \( t_i \) via the Killing form, this is equivalent to asking that for each \( x \notin Z_{\mathfrak{g}_C}(e) \), there exists \( y \in \mathfrak{g}_C \) such that \( \kappa([t, x], y) \neq 0 \). But since \( \kappa \) is nondegenerate, this happens exactly when \( [t, x] \neq 0 \) for all \( x \notin Z_{\mathfrak{g}_C}(e) \). This happens if and only if \( Z_{\mathfrak{g}_C}(t_i) \subset Z_{\mathfrak{g}_C}(e) \). However, \( t_i \in Z(Z_{\mathfrak{g}_C}(e)) \); hence \( Z_{\mathfrak{g}_C}(t_i) = Z_{\mathfrak{g}_C}(e) \).

7.2 Reduction to Distinguished Orbits in Simple Lie Algebras

We wish to extend the results in the previous section to nilpotent orbits in arbitrary split semisimple Lie algebras.

For brevity, if \( \mathfrak{g} \) is a split semisimple real Lie algebra and \( e \) is a nilpotent element, let \( P(\mathfrak{g}, e) \) denote the following statement:

If \( E \) is a subalgebra of \( \mathfrak{g}_C \) such that \( Z_{\mathfrak{g}_C}(e) \subset E \) and \( E + \overline{E} = \mathfrak{g}_C \), then \( E = \mathfrak{g}_C \).

Conjecture 1. \( P(\mathfrak{g}, e) \) is true for any split semisimple Lie algebra \( \mathfrak{g} \) and any nilpotent \( e \in \mathfrak{g} \).

The following results show that it suffices to prove the conjecture for distinguished nilpotent orbits in simple, split Lie algebras.

Lemma 7.4. In order to prove \( P(\mathfrak{g}, e) \) for any split semisimple Lie algebra \( \mathfrak{g} \) and any nilpotent \( e \in \mathfrak{g} \), it is enough to prove the result when \( \mathfrak{g} \) is simple.

Proof. First assume that \( P(\mathfrak{g}, e) \) is true whenever \( \mathfrak{g} \) is simple and \( e \in \mathfrak{g} \) is nilpotent. Now let \( \mathfrak{g} \) be any split semisimple Lie algebra so that \( \mathfrak{g} = \oplus \mathfrak{g}_i \) is a direct sum of split simple Lie algebras. One can complete \( e \) to a standard \( \mathfrak{sl}_2 \) triple \( \{e, h, f\} \) such that \( e = \sum e_i \), \( f = \sum f_i \), \( h = \sum h_i \), and each \( \{e_i, h_i, f_i\} \) is a standard triple in \( \mathfrak{g}_i \).

Let \( \pi_i : \mathfrak{g}_C \rightarrow (\mathfrak{g}_i)_C \) denote the projection map. Clearly \( Z(\mathfrak{g}_i)_C(e_i) \subset Z(\mathfrak{g})_C(e) \subset E \), whence \( Z(\mathfrak{g}_i)_C(e_i) \subset \pi_i(E) \). We know that \( (\mathfrak{g}_i)_C = (\mathfrak{g}_i)_C \) and \( E + \overline{E} = \mathfrak{g}_C \), from which it follows that \( \pi_iE + \overline{\pi_iE} = (\mathfrak{g}_i)_C \). But it was assumed that \( P(\mathfrak{g}_i, e_i) \). This implies that \( \pi_iE = (\mathfrak{g}_i)_C \) because \( \mathfrak{g}_i \) is split simple.
Consequently, if \( x_i \in (\mathfrak{g}_i)_\mathfrak{C} \), there exists \( x \in E \) such that \( \pi_i x = x_i \). In particular, there exists \( f' \in E \) such that \( pr_{g(C)} f' = f_i \). Since \( e_i \in Z_{g}(e) \), \( h_i = [e_i, f_i] = [e_i, f'] \in E \). Then it is also the case that \(-2f_i = [h_i, f_i] = [h_i, f'] \in E \). We may conclude that \( f_i \in E \), for each \( f_i \in E \). The centralizer \( Z_{g(C)}(e) \) of \( e \) and \( \{e, h, f \} \) generate \( \mathfrak{g}_C \) as a subalgebra. Therefore \( E = \mathfrak{g}_C \). □

We now aim to show that in order to prove \( P(\mathfrak{g}, e) \) for arbitrary nilpotent elements \( e \), it is enough to show this in the case when \( e \) is a distinguished nilpotent in \( \mathfrak{g}_C \).

Define a split Levi subalgebra of \( \mathfrak{g}_C \) to be any Levi subalgebra of \( \mathfrak{g}_C \) that contains a split Cartan subalgebra of \( \mathfrak{g} \). Note that any split Levi subalgebra \( l \) satisfies \( l = \mathfrak{T} \).

**Lemma 7.5.** Let \( \mathfrak{g} \) be a split semisimple Lie algebra. If \( e \in \mathfrak{g} \) is a nilpotent element that is not distinguished, then there is a proper split Levi subalgebra containing \( e \).

**Proof.** Let \( \{e, h, f\} \subset \mathfrak{g} \) be a standard triple. There is a Cartan involution \( \theta \) of \( \mathfrak{g} \) such that \( \theta e = -f \), \( \theta f = -e \), and \( \theta h = -h \). This is possible when \( \{e, h, f\} \) span \( \mathfrak{g} \) and also in general because any Cartan involution of a semisimple subalgebra may be extended to a Cartan involution of the entire Lie algebra (see 9.4.1 [2]). Let \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) be the corresponding Cartan decomposition and by the theory of \( \mathfrak{sl}_2 \) representations, \( e \) is distinguished in \( \mathfrak{g}_C \) if and only if \( Z_{\mathfrak{g}_C}(e, h) = Z_{\mathfrak{g}_C}(e, h, f) = 0 \). Since \( e \) is not distinguished, there exists a nonzero \( x \in Z_{\mathfrak{g}_C}(e, h) \). Since \( \{e, h, f\} \subset \mathfrak{g} \), this means that there exists a non-zero \( y \in Z_{\mathfrak{g}}(e, h) \). If we let \( u = y - \theta y \), then \( u \in \mathfrak{p} \) and is therefore diagonalizable. Note that \( 0 = \theta([y, h]) = -[\theta y, h] \) so that \( \theta y \in Z_{\mathfrak{g}}(h) \).

Also, \( 0 = \theta([y, f]) = -[\theta y, e] \) so that \( \theta y \in Z_{\mathfrak{g}}(e, h) \) and therefore \( u \in Z_{\mathfrak{g}}(e, h) \). Notice that \( u \) lies in some maximal abelian subspace \( \mathfrak{v} \) of \( \mathfrak{p} \). All maximal abelian subspaces of \( \mathfrak{p} \) are conjugate by the group \( Int(\mathfrak{g}) \) of inner automorphisms. Therefore \( \mathfrak{v} \) is conjugate to a split Cartan subalgebra and must itself be a split Cartan subalgebra. Since \( u \) is semisimple, \( Z_{\mathfrak{g}_C}(u) \) is a Levi subalgebra. It is split since \( \mathfrak{v} \subset Z_{\mathfrak{g}_C}(u) \), and finally \( e \in Z_{\mathfrak{g}_C}(u) \). So \( e \) lies in a a split Levi subalgebra. If \( u \neq 0 \), then \( Z_{\mathfrak{g}_C}(u) \neq \mathfrak{g}_C \), and \( e \) lies in a proper split Levi subalgebra.

It remains to show that there is a \( y \in Z_{\mathfrak{g}}(e, h) \) such that \( u = y - \theta y \neq 0 \). In other words, we need to show that \( Z_{\mathfrak{g}}(e, h) \) is not contained in \( \mathfrak{k} \). To see this, we can embed \( \mathfrak{g} \) into some \( \mathfrak{sl}_n \) by the adjoint map. Henceforth in this proof, we will view \( \mathfrak{g} \) as a subalgebra of \( \mathfrak{sl}_n \). If \( x \in \mathfrak{sl}_n \) is a nilpotent element which is in Jordan form, it is possible to choose a standard triple \( \{x, y, z\} \) whose semisimple element \( z \) is a diagonal matrix.

It can be shown that any semisimple element \( s \in Z_{\mathfrak{sl}_n}(x, z) \) has real eigenvalues. Now, if \( Z_{\mathfrak{g}}(e, h) \subset \mathfrak{k} \), let \( 0 \neq y \in Z_{\mathfrak{g}}(e, h) \). Since \( y \in \mathfrak{k} \), \( y \) has purely imaginary eigenvalues. However, there is an automorphism \( g \in Aut(\mathfrak{sl}_n(\mathbb{R})) \) such that \( g.e = x \) and \( g.h = z \), where \( x \) is in Jordan form and \( z \) is a diagonal matrix. But then \( g.y \in Z_{\mathfrak{sl}_n}(x, z) \) is a semisimple element with purely imaginary eigenvalues. This is a contradiction.

Therefore \( Z_{\mathfrak{g}}(e, h) \) is not contained in \( \mathfrak{k} \), and this completes the proof. □

**Lemma 7.6.** Let \( \mathfrak{g} \) is a split semisimple Lie algebra. Any nilpotent \( e \in \mathfrak{g} \) is contained in a split Levi subalgebra of \( \mathfrak{g}_C \) that is minimal among all maximal subalgebras containing \( e \).

**Proof.** We proceed by induction on \( \dim(\mathfrak{g}) \). The case when \( \dim(\mathfrak{g}) = 3 \) is trivial. Let \( e \in \mathfrak{g} \) as above. If \( e \) is distinguished, then we are done. If not, then there is a split Levi subalgebra \( \mathfrak{l}_C \) containing \( e \) and a split Cartan \( \mathfrak{h} \). Here \( \mathfrak{l}_C \) is the complexification of some \( \mathfrak{l} \subset \mathfrak{g} \). Since \( \mathfrak{l}_C \) is reductive, \( \mathfrak{l} = [l, l] \oplus Z(l) \), and \( e \in \mathfrak{l}' := [l, l] \). If \( e \) is distinguished in \( \mathfrak{l}' \) (which is a split real form of \( \mathfrak{l}_C \)), then \( \mathfrak{l}'_C \) is minimal. Otherwise, there exists a split Levi subalgebra \( \mathfrak{m} \) of \( \mathfrak{l}' \) with \( e \in \mathfrak{m} \subset \mathfrak{l}' \) in which \( e \) is distinguished. This follows by induction hypothesis. The Levi subalgebra \( \mathfrak{m} \) contains a split Cartan \( \mathfrak{s} \), and we wish to show that \( \mathfrak{s} \oplus Z(l) \) is a split Cartan for \( \mathfrak{g} \). Note that \( \mathfrak{h} \cap \mathfrak{l}' \) is a split Cartan subalgebra of \( \mathfrak{l}' \). But \( \mathfrak{s} \) is also a split Cartan subalgebra of \( \mathfrak{l}' \). Therefore there exists \( g \in Int(\mathfrak{l}') \) such that \( g.s = \mathfrak{h} \cap \mathfrak{l}' \). We may view \( g \) as an element of \( Int(\mathfrak{g}) \), so that \( g.(s \oplus Z(l)) = g.s + g.Z(l) = \mathfrak{h} \cap \mathfrak{l}' + Z(l) = \mathfrak{h} \). Now since \( \mathfrak{s} \oplus Z(l) \) and \( \mathfrak{h} \) are conjugate by a member of \( Int(\mathfrak{g}) \), \( \mathfrak{s} \oplus Z(l) \) is a split Cartan subalgebra of \( \mathfrak{g} \). Then \( \mathfrak{m}_C \oplus Z(l) \) is a split Levi subalgebra in which \( e \) is distinguished. That is to say, \( \mathfrak{m}_C \oplus Z(l) \) is a minimal Levi containing \( e \), and it is split. □

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Proposition 7.7. Let $\mathfrak{g}$ be a split semisimple Lie algebra, and let $e \in \mathfrak{g}$ be any nilpotent element. If $\mathfrak{l}$ is a minimal split Levi subalgebra containing $e$, then $P(\mathfrak{l}, \mathfrak{l}, e)$ implies $P(\mathfrak{g}, e)$. Therefore to prove $P(\mathfrak{g}, e)$ for any split semisimple $\mathfrak{g}$ and nilpotent $e \in \mathfrak{g}$, it suffices to prove that for any split semisimple Lie algebra $\mathfrak{s}$ and distinguished nilpotent $x \in \mathfrak{s}$, $P(\mathfrak{s}, x)$.

Proof. Assume that $P(\mathfrak{l}, \mathfrak{l}, e)$, and let $E$ be a subalgebra of $\mathfrak{gC}$ such that $Z_{\mathfrak{gC}}(e) \subset E$ and $E + E^\perp = \mathfrak{gC}$. Let $\mathfrak{h} \subset \mathfrak{l}$ be a split Cartan subalgebra. There exits $h \in \mathfrak{h}$ such that $\mathfrak{l}$ is the 0-eigenspace for $ad_h$. But since $e \in \mathfrak{l}$, $h \in Z_{\mathfrak{gC}}(e) \subset E$. Hence $E$ decomposes as a direct sum of eigenspaces for $ad_h$, and $pr_{\mathfrak{hc}}E = E \cap \mathfrak{lc}$. Here $pr_{\mathfrak{lc}}$ is projection onto the 0-eigenspace, $\mathfrak{lc}$, for $ad_h$. This implies that $pr_{\mathfrak{lc}}E$ is a subalgebra of $\mathfrak{lc}$ containing $Z_{\mathfrak{lc}}(e)$ and such that $pr_{\mathfrak{lc}}E + pr_{\mathfrak{hc}}E = \mathfrak{lc}$.

Note that $\mathfrak{lc} = [\mathfrak{l}, \mathfrak{l}]_C \oplus Z(\mathfrak{l})_C$, and $Z(\mathfrak{l}) \subset Z_{\mathfrak{gC}}(e) \subset E$. Hence, $(E \cap [\mathfrak{l}, \mathfrak{l}]_C) + (E \cap [\mathfrak{l}, \mathfrak{l}]_C) = [\mathfrak{l}, \mathfrak{l}]_C$. In conjunction with the fact that $Z_{[\mathfrak{l}, \mathfrak{l}]_C} \subset E \cap [\mathfrak{l}, \mathfrak{l}]_C$ and $P([\mathfrak{l}, \mathfrak{l}], e)$ implies that $E \cap [\mathfrak{l}, \mathfrak{l}]_C = [\mathfrak{l}, \mathfrak{l}]_C$ and therefore $\mathfrak{lc} \subset E$. This means that $e$ lies in some semisimple Lie subalgebra of $E$, namely $[\mathfrak{l}, \mathfrak{l}]$. It follows that there is a standard triple $\{e, h, f\} \subset E$. But $f$ and $Z_{\mathfrak{gC}}(e)$ together generate all of $\mathfrak{gC}$. □

8 Riemannian Symmetric Spaces

Let $(M, Q)$ be a Riemannian symmetric space $M$ with metric $Q$, and let $G = I(M)_{p_0}$ denote the identity component of the isometry group. We fix a point $p_0 \in M$, and let $K \subset G$ be the subgroup fixing $p_0$. We know that $\mathfrak{k} = Lie K$ is the set of fixed points of an involutive isometry $\theta$ of $\mathfrak{g}$. It is known that $\mathfrak{g}$ decomposes as a direct sum of Lie algebras $\mathfrak{l}_c, \mathfrak{l}_n, \mathfrak{l}_e$, each of which is fixed by $\theta$, where $(\mathfrak{l}_c, \theta), (\mathfrak{l}_n, \theta), (\mathfrak{l}_e, \theta)$ are orthogonal symmetric pairs of the compact, non-compact, and Euclidean type respectively. Each of $\mathfrak{l}_c$ and $\mathfrak{l}_n$ decompose into a direct sum of irreducible orthogonal symmetric pairs (in the sense of Definition 9). Thus, $\mathfrak{g}$ is a direct sum of ideals $\mathfrak{g} = \mathfrak{l}_c \oplus \bigoplus_{i=1}^n \mathfrak{g}_i$, where each $(\mathfrak{g}_i, \theta)$ is an orthogonal symmetric pair of the compact or non-compact type. Thus, the Lie algebra $\mathfrak{g}^* \rtimes \mathfrak{g}$ is a direct sum, which we denote by $\mathfrak{g}^* \rtimes \mathfrak{g} = \mathfrak{u}_c \oplus \bigoplus_{i=1}^n \mathfrak{u}_i$. The main result of this section is the following theorem.

Theorem 8.1. With the above notation, any $G$-invariant generalized complex structure $L$, when viewed, as a subalgebra $L \subset \mathfrak{g}^* \rtimes \mathfrak{g}$ is a direct sum $L = L_c \oplus \bigoplus_{i=1}^n L_i \subset (\mathfrak{u}_c \oplus \bigoplus_{i=1}^n \mathfrak{u}_i)_c$. Each $L_i$ represents either a complex structure or a B-transform of a symplectic structure. When $M$ is simply connected, the generalized complex structure is a product of generalized complex structures on $M = M_c \times \prod_{i=1}^n M_i$.

In Subsections 8.1 and 8.2, we consider an arbitrary Riemannian symmetric pair $(G, K)$ with $G/K \simeq M$. We assume that $G$ is connected and acts on $M$ by isometries, so we have a map $\tau: G \rightarrow I_0(M)$. Just in case $G$ doesn’t act effectively, let $N = Ker(\tau)$, and $G/K \simeq (G/N)/(K/N) \simeq M$. We can show that $K/N$ is compact [3] and replace $(G, K)$ by $(G/N, K/N)$. We therefore assume from the beginning that $G$ acts effectively. Hence, $K$ is compact. Also $\mathfrak{t}$ contains no non-zero ideal of $\mathfrak{g}$ because if it did contain such an ideal $i$ the connected subgroup $I$ corresponding to $i$ would be normal and therefore act trivially on $G/K$, which would contradict that $G$ acts effectively. We now have a closed embedding $G \hookrightarrow I_0(M)$, where $U_0 \subset K \subset U$ (where $U$ is the subgroup fixing $eK \in G/K = M$).

8.1 Irreducible Semisimple Symmetric Spaces

If $(G, K)$ is a Riemannian Symmetric pair with $G$ semisimple, then the Lie algebra $\mathfrak{g}$ has a decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ into $\pm 1$-eigenspaces for an involution $\theta$ corresponding to the pair $(G, K)$. In other words, $(\mathfrak{g}, \theta)$ is an orthogonal symmetric pair with decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$.

Definition 9. An orthogonal symmetric pair $(\mathfrak{g}, \theta)$ is called irreducible if $\mathfrak{g}$ is semisimple, $\mathfrak{t}$ contains no nonzero ideal of $\mathfrak{g}$, and $\mathfrak{p}$ is an irreducible $\mathfrak{t}$-module (e, p as above). A Riemannian symmetric space $(G, K)$ is called irreducible, if $G$ is semisimple and the corresponding orthogonal symmetric pair is irreducible.
Generalized complex structures on $G/K$ are given by GC-pairs $(E, \varepsilon)$. The conditions on $E$ are that $E$ is a subalgebra containing $\mathfrak{t}_C$ and $E + \overline{E} = \mathfrak{g}_C$. It is known that if $\mathfrak{g}$ is noncompact, then $\theta$ is a Cartan involution of $\mathfrak{g}$. If $\mathfrak{g}$ is compact, then the dual orthogonal symmetric pair $(\mathfrak{g}^* = \mathfrak{t} \oplus \mathfrak{p}, \theta^*)$ is non-compact, and $\theta^*$ is a Cartan involution. GC-pairs $(E, \varepsilon)$ for $(\mathfrak{g}, \theta)$ are the same as those for $(\mathfrak{g}^*, \mathfrak{t} \oplus \mathfrak{p}, \theta^*)$ except that condition 5 of Proposition 8.4 is different in the two cases. Moreover, when it is convenient for finding candidates for GC pairs $(E, \varepsilon)$, one may assume that $(\mathfrak{g}, \theta)$ is of noncompact type or of compact type.

**Lemma 8.2.** Let $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}$ be a Cartan decomposition of a real semisimple Lie algebra $\mathfrak{g}$. Suppose that $\mathfrak{p}$ is an irreducible $\mathfrak{t}$-module. Then if $E$ is any subalgebra of $\mathfrak{g}_C$ containing $\mathfrak{t}_C$, then either $E = \mathfrak{g}_C$ or $E = \mathfrak{t}_C \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an irreducible $\mathfrak{t}$-submodule of $\mathfrak{p}_C$ and $\mathfrak{a} \oplus \overline{\mathfrak{a}} = \mathfrak{p}_C$.

**Proof.** Since $\mathfrak{t}_C \subset E$, $E = \mathfrak{t}_C \oplus \mathfrak{a}$ for some subspace $\mathfrak{a}$ of $\mathfrak{p}_C$. The subspace $\mathfrak{a}$ must therefore be a $\mathfrak{t}$-module. But since $\mathfrak{t}_C$ is also a $\mathfrak{t}$-module, so is $\mathfrak{a} \oplus \overline{\mathfrak{a}}$. However, since $\mathfrak{a} \cap \overline{\mathfrak{a}}$ is stable under complex conjugation, $\mathfrak{a} \cap \overline{\mathfrak{a}} = \mathfrak{a}_C$ for some subspace $V \subset \mathfrak{a} \cap \overline{\mathfrak{a}}$. Note that $V$ is also a $\mathfrak{t}$-module, being the intersection of three $\mathfrak{t}$-modules. We assumed that $\mathfrak{p}$ was an irreducible $\mathfrak{t}$-module. Hence, either $V = \mathfrak{p}_C$, in which case $E = \mathfrak{g}_C$, or $V = 0$, in which case $\mathfrak{p}_C = \mathfrak{a} \oplus \overline{\mathfrak{a}}$.

In the second case, we must show that $\mathfrak{a}$ is irreducible. Suppose that there is a proper submodule $W \subset \mathfrak{a}$. This would mean that $W \oplus \overline{W} \subset \mathfrak{p}_C$ is a submodule. Again, though, $W \oplus \overline{W}$ is stable under conjugation, so it is the complexification of some $U \subset \mathfrak{p}$, which must also be a $\mathfrak{t}$-module. Necessarily $U \supsetneq \mathfrak{p}$, which contradicts the fact that $\mathfrak{p}$ is irreducible. □

**Lemma 8.3.** Let $(\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}, \theta)$ be an irreducible semisimple orthogonal symmetric pair corresponding to the Riemannian symmetric pair $(G, K)$. Let $K_0$ denote the connected component of $K$. Either $K_0$ is semisimple, or $K_0 \simeq S^1 \times K_{ss}$, where $K_{ss}$ is semisimple. Equivalently, $\dim(Z(t)) \in \{0, 1\}$.

**Proof.** $\mathfrak{t}$ is compact, hence reductive, so $\mathfrak{t} = Z(\mathfrak{t}) \oplus [\mathfrak{t}, \mathfrak{t}]$, and on the group level, $K_0 = Z_0 K_{ss}$, where $Z_0$ is the connected component of the center of $K_0$. Both $Z_0$ and $K_{ss}$ are closed.

Let $\mathfrak{a} \subset \mathfrak{p}_C$ be an irreducible submodule. Let $Z = Z(\mathfrak{t})$. Because $\mathfrak{t}$ is compact and contains $Z$, $Z$ acts on $\mathfrak{a}$ by simultaneously diagonalizable matrices so that $\mathfrak{a}$ decomposes as a direct sum of eigenspaces for $Z$. Then since $Z$ commutes with $\mathfrak{t}_{ss} = [\mathfrak{t}, \mathfrak{t}]$, $\mathfrak{t}_{ss}$ preserves each of these eigenspaces. Since $\mathfrak{a}$ is irreducible, there can therefore be at most one eigenspace. That is, $Z$ acts by a scalar $\lambda \in Z_{ss}$. on $\mathfrak{a}$: $[x, v] = \lambda(v)x$ for all $x \in Z$, $v \in \mathfrak{a}$. If $\dim(Z) > 1$, there exists a nonzero $x \in \text{Ker}(\lambda)$. This implies that $x \in Z$ and $[x, \mathfrak{a}] = 0$. But also $[x, \overline{x}] = [x, x] = [x, x] = 0$ so that $x \in Z_{ss}$, which gives a contradiction because $\mathfrak{g}_C$ is semisimple. Therefore $\dim(Z) \leq 1$. If $Z = 0$, then $\mathfrak{t}$ is semisimple. If $\dim(Z) = 1$, $Z_0$ (the connected subgroup with Lie algebra $Z$) is a compact connected abelian subgroup of dimension 1 and is therefore isomorphic to $S^1$. □

**Proposition 8.4.** Let $(\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p}, \theta)$ be an irreducible semisimple orthogonal symmetric pair. Then any GC-pair $(E, \varepsilon)$ for $G/K$ must be of one of the following two forms:

1. $E = \mathfrak{g}_C$ and $\varepsilon = \phi \circ [\cdot, \cdot]$, for some $\phi \in \text{Ann}_{\mathfrak{g}_C}(\mathfrak{t}_C, \mathfrak{t}_C) \oplus \mathfrak{p}_C) \simeq Z(\mathfrak{t}_C)^*$. We decompose $\phi$ as $\phi = \phi_r + i\phi_i$ with $\phi_r, \phi_i \in \mathfrak{g}^*$. A sufficient and necessary condition for the existence of a GC-pair is that $\phi_i \circ [\cdot, \cdot]$ is symplectic (non-degenerate).

2. $E = \mathfrak{t}_C \oplus \mathfrak{a}$ with $\mathfrak{a} \oplus \overline{\mathfrak{a}} = \mathfrak{p}_C$ and $\varepsilon = 0$ so that $(E, 0)$ is a complex structure.

Therefore on an irreducible semisimple Riemannian symmetric space $G/K$, the only $G$-invariant generalized complex structures are complex structures and complex symplectic structures (i.e. closed, complex two-forms) for which the imaginary part is symplectic.

**Proof.** 1. Recall that $\mathfrak{g}_C$ semisimple implies $H^2(\mathfrak{g}_C, \mathbb{C}) = 0$. Thus, $d_{\mathfrak{g}_C}\varepsilon = 0$ implies that $\varepsilon = \phi \circ [\cdot, \cdot]$ for some $\phi \in \mathfrak{g}_C^*$. But $\varepsilon(\mathfrak{t}_C) = 0$ implies $\phi(\mathfrak{t}_C, \mathfrak{g}_C) = 0$. Therefore $\phi \in \text{Ann}_{\mathfrak{g}_C}(\mathfrak{t}_C, \mathfrak{g}_C)$. It is easily seen that $[\mathfrak{g}_C, \mathfrak{t}_C] = [\mathfrak{t}_C, \mathfrak{t}_C] \oplus \mathfrak{p}_C$. Note however that $\mathfrak{t}$ is compact, hence reductive, so $\mathfrak{t} = Z(\mathfrak{t}) \oplus [\mathfrak{t}, \mathfrak{t}]$. From this it follows that $\text{Ann}_{\mathfrak{g}_C}(\mathfrak{t}_C, \mathfrak{t}_C) \oplus \mathfrak{p}_C) \simeq Z(\mathfrak{t}_C)^*$. Any such $\varepsilon = \phi \circ [\cdot, \cdot]$ with $\phi \in \text{Ann}_{\mathfrak{g}_C}(\mathfrak{t}_C, \mathfrak{t}_C) \oplus \mathfrak{p}_C)$
Lemma 8.6. In order to satisfy condition 5, let \( x \in p_C \) and suppose that \( \phi([x, y]) - \phi([x, y]) = 0 \) for all \( y \in g_C \). Since, \( 2\hat{\phi}([x, y]) = \phi([x, y]) - \phi([x, y]) \), this means that \( \phi_i([x, y]) = 0 \) for all \( y \in g_C \). Condition 5 is thus satisfied if and only if for each \( x \in p_C \), there is some \( y \in g_C \) such that \( \phi_i([x, y]) \neq 0 \). In other words, this happens when \( \phi_i \cdot [ , ] \) is nondegenerate.

2. We have already seen that if \( E \neq p_C \), then \( E \) is of this form. We have \( g_C = t_C \oplus a \oplus \overline{a} \) and so \( g_C^* \cong t_C^* \oplus a^* \oplus \overline{a} \), and \( E^* \cong t_C^* \oplus a^* \). If \( \varepsilon \in \Lambda^2 g^* \) and \( d_{E^*} = 0 \), we may extend \( \varepsilon \) by 0 on \( g_C \times \overline{a} \) to \( \varepsilon' \in \Lambda^2 g^* \). It can easily be checked that since \( (\varepsilon')_k \) vanishes on \( t_C \oplus \overline{a} \) that \( d_{g_C} \varepsilon' \) vanishes on each of the following sets: \( t_C \times a \times \overline{a}, a \times a \times \overline{a}, t_C \times t_C \times a, a \times a \times a, t_C \times a \times a, \) and \( a \times a \times a \). Furthermore, \( d_{g_C} \varepsilon' = d_{E^*} \varepsilon = 0 \) when restricted to \( (t_C \oplus a)^3 \), and therefore \( d_{g_C} \varepsilon' = 0 \).

Since \( g_C \) is semisimple, \( H^2(g_C, \mathbb{C}) = 0 \), whence \( \varepsilon' = \phi \circ [ , ] \) for some \( \phi \in g_C^* \). Recall that \( \phi \in Ann_{g_C^*}((t_C, \mathbb{C}) \oplus p_C) \cong Z(t_C)^* \). If \( \varepsilon \neq 0 \), then \( \phi \neq 0 \) and so \( \phi(x) \neq 0 \) for some \( x \) spanning \( Z(t_C) \). Also, since \( t_C = Z(t_C) \oplus [t, t]_C \) and \( \varepsilon \neq 0 \), the projection \( pr_{Z(t_C)}([a, a]) = Z(t_C) \). But then \( Z(t)_C = Z(t)_C = pr_{Z(t_C)}([a, a]) = pr_{Z(t_C)}([a, a]) = pr_{Z(t_C)}([a, a]). \) In other words, \( \varepsilon'([a, a]) = \phi([a, a]) = \phi(Z(t_C)) = 0 \). However, \( \varepsilon' \) was constructed to vanish on \( g_C \times \overline{a} \), so this is a contradiction. Therefore \( \varepsilon = 0 \). \( \square \)

Remark 8.5. The proof of Proposition 8.3 shows that any G-invariant Dirac structure on an irreducible Riemannian symmetric space \( G/K \) is a complex structure or a (complex) presymplectic structure.

Lemma 8.6. If there is a G-invariant complex structure on \( G/K, \) then \( g \) is not a complex Lie algebra.

Proof. Suppose that \( g \) is complex and that \( G/K \) has a G-invariant complex structure \( a \subset p_C \) as in part (2) of Lemma 8.3. Then \( g = u_C = u \oplus Ju \) for some compact real form \( u \) of \( g \). By the classification of irreducible semisimple orthogonal symmetric pairs [8], \( g \) is simple, \( t = u \), and \( p = Ju \). It is apparent that \( a \subset p_C = Ju \oplus iJu \) \( J(u \oplus iu) = Ju u \). This implies that \( Ja \subset u_C = u \oplus iu = t_C \). It is also true that \( [u_C, Ja] = J[u_C, a] = J[t_C, a] = Ja \) so that \( Ja \) is an ideal in \( u_C \cong g \). This contradicts the fact that \( g \) is simple. \( \square \)

8.1.1 Irreducible Riemannian Symmetric Spaces of the Non-Compact Type

For this subsection, \( (g = t \oplus p, \theta) \) will be an irreducible semisimple orthogonal symmetric pair of non-compact type associated to a symmetric space \( G/K \). When \( (g, t) \) is of the non-compact type, \( K \) is connected [8], so \( t \)-invariance always implies \( K \)-invariance.

Theorem 8.7. Let \( G/K \) be an irreducible Riemannian symmetric space of the non-compact type. The following are equivalent:

1. \( G/K \) admits a \( G \)-invariant generalized complex structure.

2. \( G/K \) is a Hermitian symmetric space. In particular, \( G/K \) is Kahler with \( G \)-invariant Kahler structure.

3. \( Z_t \neq 0 \).

Theorem 8.7 will be proven in the course of several lemmas.

Lemma 8.8. \( G \)-invariant generalized complex structures on \( G/K \) exist only when \( p_C \) is not an irreducible \( t_C \)-module. In that case, there exists a \( G \)-invariant complex structure on \( G/K \).

Proof. First suppose that \( Z_t \neq 0 \). Let \( 0 \neq z \in Z_t, \) \( ad_z \) acts diagonally on \( p_C \), so \( p_C \) is a direct sum of eigenspaces for \( z \). Since \( z \) commutes with \( t \), \( t \) preserves each eigenspace. Thus, each eigenspace is a \( t \)-module. We have already seen that these are either all of \( p_C \) or complex structures \( a \subset p_C \) as in Lemma 8.3. If there exists such an \( a \), then \( p_C \) is not irreducible. On the other hand, if \( ad_z \) has only one eigenvalue, \( \lambda \) on \( p_C \), then \( p_C, p_C \subset t_C \cap (g_C)_{2\lambda}, \) where \( (g_C)_{2\lambda} \) is the \( 2\lambda \)-eigenspace for \( ad_z \). However,
\( \mathfrak{k}_C \subset (\mathfrak{g}_C)_0 \). This is possible if and only if \( \lambda = 0 \) or \([\mathfrak{p}_C, \mathfrak{p}_C] = 0 \). If \( \lambda = 0 \), then \( z \in Z_g = 0 \), which is a contradiction. Therefore we must have \([\mathfrak{p}_C, \mathfrak{p}_C] = 0 \), whence \([\mathfrak{p}, \mathfrak{p}] = 0 \).

Now supposing that \([\mathfrak{p}_C, \mathfrak{p}_C] = 0 \), the assumption that \( Z_k \neq 0 \) and \( \mathfrak{p}_C \) is irreducible will lead to a contradiction. Since \((g, \theta)\) is irreducible, either \( \mathfrak{g} \) is simple, or \( \mathfrak{g}^* \) (dual symmetric space, not the vector space dual) is simple. If \( \mathfrak{g} \) is simple, then \([\mathfrak{p}, \mathfrak{p}] = 0 \) implies that \([\mathfrak{k}, \mathfrak{k}] \oplus \mathfrak{p} \) is a proper ideal of \( \mathfrak{g} \), which is a contradiction. Therefore, whenever \( Z_k \neq 0 \), \( \mathfrak{p}_C \) is not irreducible, so there exists a complex structure \( \mathfrak{a} \subset \mathfrak{p}_C \). If \( \mathfrak{g} \) is not simple, then \( \mathfrak{g}^* \) is simple, and a similar argument applies.

Now suppose that \( Z_k = 0 \). There can be no \( G \)-invariant symplectic structures because by Lemma 8.4, \( \phi \in \text{Ann}_{\mathfrak{g}_C}(\mathfrak{k}_C, \mathfrak{t}_C, \mathfrak{p}_C) = \text{Ann}_{\mathfrak{g}_C}(\mathfrak{g}_C^*) = 0 \). The only possible generalized complex structures are complex structures. If a complex structure exists, as in Lemma 8.4, then \( \mathfrak{p}_C \) is not irreducible. □

**Lemma 8.9.** If \( Z_k \neq 0 \), then \( \mathfrak{p}_C \) is not irreducible and in fact \( G/K \) has a \( G \)-invariant complex structure.

**Proof.** This is demonstrated in the proof of Lemma 8.8 □

**Lemma 8.10.** If there exists a \( G \)-invariant complex structure on \( G/K \), then \( G/K \) is a Hermitian symmetric space.

**Proof.** Let \( \mathfrak{a} \subset \mathfrak{p}_C \) be a complex structure in the sense of Lemma 8.4. We know that \( \mathfrak{a} \oplus \overline{\mathfrak{a}} = \mathfrak{p}_C \). As usual, \( \kappa \) will denote the Killing form on \( \mathfrak{g} \), which when restricted to \( \mathfrak{p} \times \mathfrak{p} \) is positive definite since \( \mathfrak{g} \) is non-compact (It would be negative definite if \( \mathfrak{g} \) were compact, in which case we could replace \( \kappa \) by \(-\kappa \)). \( \kappa \) gives rise to its complexification \( \kappa_C \) on \( \mathfrak{p}_C \times \mathfrak{p}_C \), which will again be denoted by \( \kappa \), due to the fact that \( \kappa_C \) is none other than the Killing form on \( \mathfrak{g}_C \). Defining \( h(x, y) = \kappa(x, \overline{y}) \) gives a Hermitian form on \( \mathfrak{p}_C \). \( h_{x \times \mathfrak{a}} \) is still a Hermitian form. But since \( \mathfrak{a} \) is a complex structure, the projection \( \pi : \mathfrak{g}_C \rightarrow \mathfrak{a} \) provides an isomorphism \( \pi : \mathfrak{a} \rightarrow \mathfrak{p}_C \), thereby giving a Hermitian metric \( H \) on \( \mathfrak{g}/\mathfrak{t}_C = \mathfrak{p}_C \) (with respect to the complex structure \( J \) defined by \( \mathfrak{a} \)). Then \( H = g + i\omega \), where \( g \) and \( \omega \) are the real and imaginary parts of \( H \), respectively. Because \( H \) is hermitian, \( g \) is positive definite and \( J \) — invariant, and because \( \kappa \) is \( \text{Ad}(K) \)-invariant, so are \( H \) and \( g \).

Finally, we observe that if \( x \in \mathfrak{p} \) and \( k \in \mathfrak{k} \), then since \( \mathfrak{a} \) is stable under \( \mathfrak{t}_C \), \([k, x] - i[k, Jx] = [k, x - iJx] = [k, x] - iJ[k, x] \). It follows that \( J \) commutes with \( \text{ad}(\mathfrak{t}) \). This is all that is needed [8] for \( G/K \) to be a Hermitian symmetric space. □

Theorem 8.7 can now be proved:

**Proof.** (1 \( \Rightarrow \) 2 ) Suppose that \( G/K \) has a non-trivial generalized complex structure. Then by Lemma 8.8, \( G/K \) admits a \( G \)-invariant complex structure. Lemma 8.10 ensures that \( G/K \) is Hermitian. The symplectic form \( \omega \) associated with this Hermitian form is obviously invariant and is known to be symplectic [8] (i.e. \( G/K \) is Kahler). This is also easily checked by verifying that since \( \omega \) is \( G \)-invariant and vanishes on \( \mathfrak{t} \times \mathfrak{g} \), then \( d\omega = 0 \).

(2 \( \Rightarrow \) 3 ) If \( G/K \) is Hermitian, then there is a \( G \)-invariant symplectic structure \( \phi \circ [\cdot, \cdot] \), with \( \phi \in Z(\mathfrak{t})^* \) as in Lemma 8.4. Since this is symplectic, \( \phi \neq 0 \) and therefore \( Z_k \neq 0 \).

(3 \( \Rightarrow \) 1) If \( Z_k \neq 0 \), then Lemma 8.9 implies that \( G/K \) has a complex structure. □

**Proposition 8.11.** Let \((G, K)\) be an irreducible Riemannian symmetric pair of the non-compact type. If \( G/K \) admits any \( G \)-invariant generalized complex structures, then \( CD^G_{G/K} = \mathbb{C}P^1 \cup \{ \text{two points} \} \). The two points are complex structures \( \mathfrak{a}, \overline{\mathfrak{a}} \) as in Lemma 8.4 and \( GC^G_{G/K} = \{ c \in \mathbb{C} \mid Im(c) \neq 0 \} \cup \{ \mathfrak{a}, \overline{\mathfrak{a}} \} \). The correspondence is established in the following way. Fixing a \( G \)-invariant symplectic structure on \( G/K, \mathbb{C}^\times \simeq \{ \mu(T(G/K)_C, \omega) \mid c \in \mathbb{C}^\times \} \), and \( T(G/K)_C \) corresponds to \( 0 \in \mathbb{C} \), while \( T^*(G/K)_C \) corresponds to \( \infty \in \mathbb{C}P^1 \). The complex structures \( \mathfrak{a} \) and \( \overline{\mathfrak{a}} \) are isolated points, whereas the new generalized complex structures are deformations of symplectic ones.
Proof. If \( L = L(E, \varepsilon) \) is a complex Dirac structure, then \( E \) is a subalgebra of \( g_C \) containing \( t_C \). By Lemma 8.2, \( E = t_C \) or \( E = t_C + a \), \( E = t_C \oplus a \), or \( E = g_C \) as in Lemma 8.3.

If \( E = g_C \), then \( \varepsilon = \phi \circ [ , ] \) for some \( \phi \in Z_2^* \), as in Lemma 8.4. By Lemma 8.2, \( Z_2^* \) is 1-dimensional, so if we fix some \( 0 \neq \phi_0 \in Z_2^* \), then all possible pairs \((g_C, \varepsilon)\) giving a complex Dirac structure are given by \( \varepsilon = c \phi_0 \) for some \( c \in \mathbb{C} \). If \( c = 0 \), \( L = T(G/K) \) and if \( c = \infty \), then \( L = T^*(G/K) \) just as in Proposition 5.10. By Lemma 8.4, the generalized complex structures are given by all \( c \) such that \( \text{Im}(c) \neq 0 \).

If \( E = t_C \oplus a \), then \( a \) is a complex structure. We will see that in this case, \( a \) and \( \overline{a} \) are the only complex structures. Let \( b \subset p_C \) be another such complex structure. Again, we have \( b \oplus \overline{b} = p_C \). But also \( b \cap a = 0 \) and \( b \cap \overline{a} = 0 \) since they are intersections of distinct irreducible \( t \)-modules. Consequently, \( b \) is the graph of some \( \mathbb{R} \)-linear isomorphism \( T : a \to \overline{a} \). For all \( k \in t \), \([k, x + Tx] = [k, x] + [k, Tx] = [k, x] + T[k, x] \) because \( b \) is a \( t \)-module. Hence, \( T \) commutes with \( ad_t \) and \( T \) is in fact an isomorphism of \( t \)-modules.

Since \( G/K \) admits a \( G \)-invariant complex structure, \( Z_t \neq 0 \) by Theorem 8.7. Let \( 0 \neq z \in Z_t \). The proof of Lemma 8.5 shows that \( z \) has exactly two eigenvalues on \( p_C \), \( \lambda \) and \( -\lambda \). The \( \lambda \) eigenspace is \( a \) and the \(-\lambda \) eigenspace is \( \overline{a} \). For any \( x \in a \), \(-\lambdaTx = [z, Tx] = T[z, x] = T(\lambda x) = -\lambdaTx \). Since \( T \) is an isomorphism, \( \lambda = 0 \), which is impossible because \( g \) has trivial center (being semisimple). Therefore, there exists no such isomorphism \( T \), and \( a \) and \( \overline{a} \) are the only complex structures. □

8.1.2 Irreducible Riemannian Symmetric Spaces of the Compact Type

Let \((G, K)\) be a Riemannian symmetric pair of the compact type. Theorems 8.7 and 8.11 are still true if \( K \) is connected. This is the case if \( G \) is simply connected [8]. If \( K \) is not connected one must check which \( a \subset p_C \) and \( \omega = \phi \circ [ , ] \) of the previous section are \( K \)-invariant.

8.2 General Riemannian Symmetric Spaces

8.2.1 Semisimple Riemannian Spaces

It is known that any semisimple orthogonal symmetric Lie algebra \((g, \theta)\) is a direct sum of semisimple ideals \( g_i \), preserved by \( \theta \) such that \((g_i, \theta_i) = \theta|_{g_i}\) is itself an orthogonal symmetric pair. Obviously, the subspace \( t \) fixed by \( \theta \) is a direct sum of \( t_i \subset g_i \).

**Lemma 8.12.** Let \((g, \theta)\) be a semisimple orthogonal symmetric pair such that \( g = \bigoplus g_i \) is a direct sum of semisimple ideals \( g_i \), such that \((g_i, \theta_i)\) are all irreducible orthogonal symmetric pairs. Let \( t_C \subset E \subset g_C \) be a subalgebra such that \( E + T = g_C \). Then \( E = \bigoplus E_i \), where \( E_i \subset (g_i)_C \).

**Proof.** Fix some \( i \). Let \( E_i = pr_{(g_i)_C}E \). We wish to show that \( E_i \subset E \). Because each \((g_i)_C\) is an ideal and closed under conjugation, \( E_i \) is a subalgebra of \((g_i)_C\) and \( E_i + E_i' = (g_i)_C \). Due to Lemma 8.3 and the fact that \( g_i \) is irreducible, \( E_i = (t_C)_i + a_i \), where either \( a_i + \overline{a_i} = (p_i)_C \) or \( a_i = (p_i)_C \).

We know that \( t_C = \bigoplus(t_C)_i \subset E \), whence \((t_C)_i \subset E \). So to show that \( E_i \subset E \), it suffices to show that \( a_i \subset E \). But, again since each \( g_j \) is a \( g_i \), \( a_i \subset (t_C)_i \). \( E_i = [(t_C)_i, a_i] \subset [(t_C)_i, E_i] = [(t_C)_i, E] \subset E \). The only thing that needs to be checked is that \( a_i = [(t_C)_i, a_i] \). If \( a_i = (p_i)_C \), then this is obviously true. Otherwise, by Lemma 8.4, \( g_i \) being irreducible means that \( a_i \) is an irreducible \((t_C)_i\)-module. Consequently, \( a_i = [(t_C)_i, a_i] \). □

**Lemma 8.13.** Let \((g, \theta)\) be a semisimple orthogonal symmetric pair such that \( g = \bigoplus g_i \) is a direct sum of semisimple ideals \( g_i \), such that \((g_i, \theta_i)\) are all irreducible orthogonal symmetric pairs. Any GC-pair \((E, \varepsilon)\) is a direct sum of GC-pairs \((E_i, \varepsilon_i)\) for \( g_i \) in the following sense: \( E = \bigoplus E_i \), each \((E_i, \varepsilon_i)\) is a GC-pair for \((g_i, \theta_i)\), and \( \varepsilon = \varepsilon_1 \oplus \varepsilon_2 \ldots \oplus \varepsilon_n \).
Proof. Lemma 8.12 shows that $E$ is a direct sum of the $E_i$’s. It only remains to show that $\varepsilon = \varepsilon_1 \oplus \varepsilon_2 \ldots \oplus \varepsilon_n$, which would follow if we could show that $\varepsilon(a_i, a_j) = 0$ if $i \neq j$. Here $E_i = (\mathfrak{t}_C)_i \oplus \mathfrak{a}_i$. We see that

$$\varepsilon(a_i, a_j) = \varepsilon([(\mathfrak{t}_C)_i, a_i], a_j) = \varepsilon(a_i, [(\mathfrak{t}_C)_i, a_j]) = \varepsilon(a_i, 0) = 0.$$

\[ \square \]

If $G/K$ is a Riemannian symmetric space with $G$ semisimple, then the orthogonal symmetric pair $(\mathfrak{g}, \theta)$ decomposes as $\mathfrak{g} = \oplus \mathfrak{g}_i$, a direct sum of irreducible orthogonal symmetric pairs $(\mathfrak{g}_i, \theta_i)$. But if $G$ is simply connected, this means that $G = \prod G_i$, where $G_i$ is the simply connected Lie group with Lie algebra $\mathfrak{g}_i$. Then $G/K \cong \prod G_i/K_i$ as long as $K$ is also connected. Thus $G/K$ is a product of irreducible semisimple Riemannian symmetric spaces.

Theorem 8.14. Let $G$ be semisimple and $G/K$ a Riemannian symmetric space.

1. If $G$ is simply connected and $K$ is connected, $G/K$ is then a product of irreducible Riemannian symmetric spaces $G_i/K_i$. Any $G$-invariant generalized complex structure on is a product of generalized complex structures on the $G_i/K_i$’s.

2. Even if $G$ is not simply connected, a generalized complex structure on $G/K$ is given by a subalgebra $L = L(E, \varepsilon) \subseteq \mathfrak{g}_C \oplus \mathfrak{g}_C^\perp$, which may still be thought of as a product on the Lie algebra level: $L = L(E_1 \oplus \ldots \oplus E_n, \varepsilon_1 \oplus \ldots \oplus \varepsilon_n) = \oplus L(E_i, \varepsilon_i)$.

Proof. The proof is immediate from Lemmas 8.4, 8.12 and 8.13 $\square$

8.2.2 When $G$ is the Isometry Group

For the remainder of this section, $G$ will refer to the identity component of the isometry group of a Riemannian symmetric space. Let $\mathfrak{g} = \text{Lie}(G)$. The pair $(G, K)$ yields an orthogonal symmetric pair $(\mathfrak{g}, \theta)$. Then $\mathfrak{g}$ decomposes into ideals $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_n \oplus \mathfrak{g}_a$, where $(\mathfrak{g}_c, \theta_c = \theta|_{\mathfrak{g}_c})$, $(\mathfrak{g}_n, \theta_n = \theta|_{\mathfrak{g}_n})$, and $(\mathfrak{g}_a, \theta_a = \theta|_{\mathfrak{g}_a})$ are orthogonal symmetric pairs of the compact, noncompact, and abelian type respectively. We have a decomposition $\mathfrak{g}_c = \mathfrak{t}_c \oplus \mathfrak{p}_c$ and similar decompositions for $\mathfrak{g}_n$ and $\mathfrak{g}_a$.

Lemma 8.15. In notation described above,

1. Let $E$ be a subalgebra of $\mathfrak{g}_C$ containing $\mathfrak{t}_C$ such that $E + \overline{E} = \mathfrak{g}_C$. Then $E = E_c \oplus E_n \oplus E_a$ as Lie algebras, where each summand is contained in $\mathfrak{g}_c$, $\mathfrak{g}_n$, or $\mathfrak{g}_a$.

2. If $(E = E_c \oplus E_n \oplus E_a, \varepsilon)$ is a GC-pair, then $\varepsilon = \varepsilon_c \oplus \varepsilon_n \oplus \varepsilon_a$.

Therefore any GC-pair is of the form $(E_c \oplus E_n \oplus E_a, \varepsilon_c \oplus \varepsilon_n \oplus \varepsilon_a)$.

Proof.

1. Since each summand is an ideal, $E_c = pr(\mathfrak{g}_c)_c E$ is a subalgebra containing $(\mathfrak{t}_c)_C$ and such that $E_c + \overline{E_c} = (\mathfrak{g}_c)_C$. The summand $\mathfrak{g}_c$ is of compact type, so it is semisimple. We have seen in the proof of Lemma 8.12 that $E_c = (\mathfrak{t}_c)_C \oplus \mathfrak{a}_c$ and $\mathfrak{a}_c = [\mathfrak{a}_c, (\mathfrak{t}_c)_C] \subset [(\mathfrak{t}_c)_C, E] \subset [E, E] \subset E$. Therefore $\mathfrak{a}_c \subset E$ and $E_c \subset E$. An identical argument shows that $E_n \subset E$, whence also $E_a \subset E$. Then $E$ is a direct sum of these.

2. This would follow if we could show that $\varepsilon(\mathfrak{a}_c, E_n \oplus E_a) = \varepsilon(\mathfrak{a}_n, E_c \oplus E_a) = \varepsilon(\mathfrak{a}_a, E_c \oplus E_n) = 0$. We first argue that $\varepsilon(\mathfrak{a}_c, E_n \oplus E_a) = 0$. Since $\mathfrak{g}_c$, $\mathfrak{g}_n$, and $\mathfrak{g}_a$ are ideals, $\varepsilon(\mathfrak{a}_c, E_n \oplus E_a) = \varepsilon([\mathfrak{t}_c, \mathfrak{a}_c], E_n \oplus E_a) = \varepsilon(\mathfrak{a}_c, [\mathfrak{t}_c, E_n \oplus E_a]) = \varepsilon(\mathfrak{a}_c, 0) = 0$. An identical argument shows that $\varepsilon(\mathfrak{a}_n, E_c \oplus E_a) = 0$. Now it automatically follows that $\varepsilon(\mathfrak{a}_a, E_c \oplus E_n) = 0$. $\square$

It is well known that any simply connected Riemannian symmetric space $M$ can be expressed as a product $M = M_c \times M_n \times M_a$ of Riemannian symmetric spaces of the compact, noncompact and abelian types in accordance with the decomposition $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_n \oplus \mathfrak{g}_a$. 

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Proposition 8.16. Let \( M = G/K \) be a simply connected Riemannian symmetric space, where \( M = M_c \times M_n \times M_a \) as above and where \( G \) is the identity component of the isometry group. Any equivariant generalized complex structure on \( M \) is a product of generalized complex structures on \( M_c, M_n, \) and \( M_a \). If, however, \( M \) is any Riemannian symmetric space, then a generalized complex structure \( L \subset g_c \oplus g_c^\ast \) may still be thought of as a product since 
\[
L = L(E, \varepsilon) = L(E_c \oplus E_n \oplus E_a, \varepsilon_c \oplus \varepsilon_n \oplus \varepsilon_a) = L(E_c, \varepsilon_c) \oplus L(E_n, \varepsilon_n) \oplus L(E_a, \varepsilon_a).
\]

Proof. This follows from Lemma 8.15. \( \Box \)

Generalized complex structures on Riemannian symmetric spaces of semisimple type (compact and non-compact) have already been described. We now only need to describe generalized complex structures on Riemannian symmetric spaces of abelian type.

\( G/K \) is of the abelian, or Euclidean, type if the associated orthogonal symmetric pair \( (g = \mathfrak{t} \oplus \mathfrak{p}, \theta) \) satisfies the condition that \( \mathfrak{p} \) is an abelian ideal of \( g \).

Lemma 8.17. Let \( G/K \) be of the Euclidean type, and let \( (g = \mathfrak{t} \oplus \mathfrak{p}, \theta) \) be the associated orthogonal symmetric pair. If \( (E, \varepsilon) \) is a GC-pair, then \( E = \mathfrak{t}_C \oplus \mathfrak{a} \), where \( \mathfrak{a} + \mathfrak{a}^\ast = \mathfrak{p}_C \). In fact, any such \( \mathfrak{a} \) is of the form \( \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \), where \( \mathfrak{a}_1 = \mathfrak{a}_I \) and \( \mathfrak{a}_I \cap \mathfrak{a} = 0 \). Furthermore, \( Z^2(E, \mathbb{C}) \cong (\wedge^2 \mathfrak{a}^\ast)^K \), where \( Z^2(E, \mathbb{C}) = \{ \varepsilon \in \wedge^2 \mathfrak{a}^\ast \mid d_E \varepsilon = 0 \} \) and \( (\wedge^2 \mathfrak{a}^\ast)^K \) is the space of \( K \)-fixed points in the \( K \)-representation on \( \wedge^2 \mathfrak{a}^\ast \).

Proof. Obviously, any GC-pair \( (E, \varepsilon) \) must satisfy \( E = \mathfrak{t}_C \oplus \mathfrak{a} \), where \( \mathfrak{a} + \mathfrak{a}^\ast = \mathfrak{p}_C \). That \( \mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}_2 \) follows from the fact that since \( \mathfrak{t} \) is a compactly embedded subalgebra, \( \mathfrak{a} \) decomposes as a direct sum \( \mathfrak{a} = \bigoplus_{i=1}^n \mathfrak{V}_i \) of \( \mathfrak{t} \)-submodules. Letting \( I = \{ i \mid \nabla_i \cap \mathfrak{a} \subset \mathfrak{a} \} \), \( \mathfrak{a}_1 = \bigoplus_{\varepsilon \in I} \mathfrak{V}_i \) and \( \mathfrak{a}_2 = \bigoplus_{\not\in I} \mathfrak{V}_i \).

Now we address the question of which \( \varepsilon \) are admissible. To say that \( \varepsilon \) vanishes on \( \mathfrak{t} \) is simply to say that \( \varepsilon \in \wedge^2 \mathfrak{a}^\ast \). Using the formula for \( d_E \varepsilon \) given in Proposition 3.9 and checking \( d_E \varepsilon \) on each of \( \mathfrak{a} \times \mathfrak{a} \times \mathfrak{a} \), \( \mathfrak{a} \times \mathfrak{a} \times \mathfrak{t} \), \( \mathfrak{a} \times \mathfrak{t} \times \mathfrak{t} \), and \( \mathfrak{t} \times \mathfrak{t} \times \mathfrak{t} \), it is easy to see that \( d_E \varepsilon = 0 \) if and only if \( \varepsilon \) is \( K \)-invariant. \( \Box \)

8.3 Real Dirac Structures

All of the techniques used to describe generalized complex structures carry over to the real case, and the results can be summarized in the following two propositions.

Proposition 8.18. Let \( G/K \) be a semisimple irreducible Riemannian symmetric space. All \( G \)-invariant real Dirac structures on \( G/K \) are presymplectic structures, i.e. are of the form \( L(g, \varepsilon) \). Any such \( \varepsilon \) is of the form \( \varepsilon = \phi \circ [\ , \ ] \), for some \( \phi \in \text{Ann}_{\mathfrak{t}*}(\mathfrak{t}, \mathfrak{t} \oplus \mathfrak{p}) \approx Z(\mathfrak{t})^* \).

Proposition 8.19. Let \( G/K \) be a Riemannian symmetric space. Any real Dirac structure is a product of Dirac structures in the sense described above.

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