GLOBAL ASYMPTOTIC STABILITY IN A CLASS OF NONLINEAR DIFFERENTIAL DELAY EQUATIONS

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ABSTRACT. An essentially nonlinear differential equation with delay serving as a mathematical model of several applied problems is considered. Sufficient conditions for the global asymptotic stability of a unique equilibrium are derived. An application to a physiological model by M. C. Mackey is treated in detail.

1. Introduction. This paper concerns the qualitative analysis of a class of scalar essentially nonlinear differential delay equations. The equations are given by

\[ x'(t) = F(x(t - \tau)) - G(x(t)) \]  

where \( F \) and \( G \) are continuous real-valued functions.

Equations of this type have attracted a significant interest in recent years due to their frequent appearance in a wide range of applications. They serve as mathematical models describing various real life phenomena in mathematical biology, population dynamics and physiology, electrical circuits and laser optics, economics and life sciences, others. See papers [3, 7, 8, 10, 11, 13, 15] and references therein for a partial list of applications and further details.

There is a significant body of theoretical mathematical research on differential delay equation (1) done in the past 20-30 years. They address various aspects of the dynamics in such equations including global asymptotic stability of equilibria, existence of periodic solutions, complicated behavior and chaos, among others. However, most of it deals with the case of linear function \( G \), i.e. \( G(x) = bx, b > 0 \).

Papers [1, 7, 11, 18] represent a partial list of related references.

The principal problem of our interest in this paper is the global asymptotic stability of equilibria in equation (1). We treat the case when both functions \( F \) and \( G \) are nonlinear, which we brand as "essentially nonlinear". For the simpler case of linear \( G \) there are several papers that deal with the global asymptotic stability, see for example [7, 8] and further references therein. Several methods and approaches have been used to tackle this problem and a good variety of results has been derived.

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Very little research was done for the case of nonlinear $G$. However, namely equations with the non-linear $G$ have appeared recently in several important applications. We refer the reader to paper [12] where a model of hematopoietic cell replication and control is considered, and to paper [11] for additional models where both $F$ and $G$ are nonlinear, as well as to monograph [5] for additional applications and references. There are only a few publications on the global asymptotic stability in such equations; the ones we are aware of are [5, 8]. The new mathematical results of this paper mainly concern the case of nonlinear $G$. We address problems of global asymptotic stability of unique trivial (zero) or positive equilibrium. We further extend and improve some of the above mentioned results in [5, 8], including the case of non-monotone $G$.

The paper consists of three principal parts. The first one concerns necessary preliminaries and general properties of solutions of equation (1) and the corresponding difference equations and one-dimensional maps. In the second part, the main aspect under discussion is the global asymptotic stability of a unique positive equilibrium. We establish a global asymptotic stability result for such equations which makes use of the global attractivity in a limiting one-dimensional map. The limiting map can in general be a multi-valued one. This way we extend the known global stability results earlier proved in [5, 7] to this new class of differential delay equations. The third part concerns applications of the mathematical theory of these equations to several real life models. One of the applications is a class of physiological population models proposed by M.C. Mackey in a series of papers. We prove in particular global stability results for the model. Other applications are in economics and related fields.

2. Assumptions and Preliminaries. In this subsection we present some basic properties and mathematical results on differential delay equation (1). The following hypotheses on the nonlinearities $F$ and $G$ will be assumed in different combinations throughout the paper.

A. $F$ and $G$ are defined and continuous on the positive semiaxis $\mathbb{R}^+ = \{ x | x \geq 0 \}$ with their images in $\mathbb{R}^+$, i.e. $F, G \in C(\mathbb{R}^+, \mathbb{R}^+); G(0) = 0$ and $F(0) \geq 0$.

B. Hypothesis A is satisfied. In addition, there exist positive numbers $m$ and $M$ such that $G(x)$ is increasing in $(0, m) \cup (M, +\infty)$, $F(x) > G(x)$ for $x \in (0, m)$ and $F(x) < G(x)$ for $x \in (M, +\infty)$. Moreover $\lim_{x \to +\infty} G(x) = +\infty$.

C. There exists a unique value $x = x_* > 0$ such that $F(x) = G(x)$. All assumptions of the hypothesis B are satisfied.

Equation (1) can be transformed, via the change of the independent variable $t = \tau s$, to the form $\mu y'(s) = F(y(s-1)) - G(y(s))$, where $\mu = 1/\tau$ and $y(s) = x(\tau s)$. It is a standard form of singularly perturbed differential delay equations with the normalized delay $\tau = 1$ [7]. Therefore, we will be considering the differential delay equation

$$\mu x'(t) = F(x(t-1)) - G(x(t)), \quad \mu = \frac{1}{\tau} \tag{2}$$

as an equivalent form of equation (1).

We shall assume that for every initial function $\phi \in C := C([-1, 0], \mathbb{R}^+)$ there exists a unique solution $x = x(t, \phi)$ of equations (2) defined for all $t \geq 0$. We do not address in detail this question of global existence of solutions of equation (2). We only note that the results are multiple and readily available in the literature (see e.g. [2, 1] and further references therein).
Under hypothesis C the constant solution \( x(t) = x_* \) is the only positive equilibrium of equation (72). When \( F(0) = 0 \) equation (72) also admits the trivial equilibrium \( x(t) \equiv 0 \). The latter will be the case in some actual models from applications considered in this paper.

Note that there is a trivial possibility for the nonlinearities \( F \) and \( G \) satisfying the assumption A that \( F(0) = G(0) = 0 \) and the \( x \equiv 0 \) is the only equilibrium of equation (72). The dynamical behavior in equation (72) is rather trivial then, as it will be shown later (in particular for some applied models).

The significance of the general assumptions contained in hypothesis A is seen from the following simple statement.

**Proposition 1.** (Positive invariance). Suppose that hypothesis A is satisfied and let \( \phi \in C, \mu > 0 \) be arbitrary. Then the corresponding solution \( x = x(t, \phi) \) of equation (72) satisfies

\[
x(t) > 0 \quad \forall t \geq 0.
\]

**Proof.** This is a well known property of equation (1) which is contained in various forms in several other papers, in particular in paper [7] for the case of linear \( G(x) = bx, b > 0 \) (see also [11]). It is easily established by simple reasoning, which we provide here for the sake of completeness and subsequent references in the paper. Indeed, if on the contrary \( x(t_0) = 0, x(t) > 0 \quad \forall t < t_0, t_0 > 0 \) then there exists a sequence \( \{t_n\} \uparrow t_0 \) such that \( x'(t_n) < 0 \). Since \( G(0) = 0, F(x(t_0 - 1)) > 0 \) and \( \mu x'(t_n) = F(x(t_n - 1)) - G(x(t_n)) \) one arrives to a contradiction by taking the limit in the latter as \( n \to \infty \). As it is seen from the proof any solution \( x(t) \) with arbitrary initial function \( \phi \in C = C(\mathbb{R}^+, \mathbb{R}^+) \) not only cannot become negative but also cannot reach the zero level in a finite time. \( \square \)

The limiting case \( \mu \to 0^+ (\tau \to \infty) \) in equation (72) corresponds to the implicit difference equation

\[
F(x(t - 1)) - G(x(t)) = 0,
\]

which can also be written in the form

\[
F(x_n) - G(x_{n+1}) = 0. \tag{3}
\]

Note that in the case of monotone \( G \), when the inverse function \( G^{-1} \) exists, the latter can be explicitly resolved for \( x_{n+1} \)

\[
x_{n+1} = G^{-1}(F(x_n)). \tag{4}
\]

In the case of non-monotone \( G \) equation (3) implicitly defines a multi-valued difference equation. We shall denote it by

\[
x_{n+1} \in \Phi(x_n), \tag{5}
\]

where scalar function \( \Phi \) is generally multi-valued. In this paper we shall restrict our considerations to the case when \( \Phi \) can assume only a finite number of values. This restriction results from the case of \( G \) being piecewise monotone in \( \mathbb{R}^+ \) with a finite number of the monotonicity branches.

As usual, a sequence \( \{x_n\} \) will be called a solution of difference equation (72) if \( G(x_{n+1}) = F(x_n) \forall n \in \mathbb{Z}^+ \). Therefore, the solution \( \{x_n\} \) satisfies all three equations (3), (5), and (1) (if \( G^{-1} \) exists for the latter). Given \( x_0 \), due to the non-monotonicity of \( G \), there can be several values of \( x_{n+1} \) which satisfy equation (3). They all are included in (5) as images of \( x_n \) under the multi-valued map \( \Phi \).
Introduce next several definitions associated with the multi-valued map $\Phi$. For related basics of general theory of one-dimensional dynamical systems see e.g. [17]. As usual, an interval $I$ is called invariant under $\Phi$ if for arbitrary $x \in I$ one also has $\Phi(x) \subseteq I$. This means, in particular, that for every $x \in I$ any solution $y$ of the equation $G(y) = F(x)$ belongs to $I$. Given equation (3) and the corresponding map $\Phi$, an invariant interval $I := [a, b]$ will be called $p$-invariant if function $G$ is increasing at the endpoints $a$ and $b$. More precisely,

**Definition 2.1.** A bounded invariant interval $I := [a, b]$ is called $p$-invariant if there exists $\delta > 0$ such that $G(x)$ is increasing on the intervals $[a, a + \delta]$ and $[b - \delta, b]$.

The case when $F$ and $G$ are defined outside the interval $I$, and function $G$ is monotone increasing on an entire neighborhood of either $a$ or $b$ ($[a - \delta, a + \delta]$ or $[b - \delta, b + \delta]$ for some $\delta > 0$), is obviously sufficient for the $p$-invariance of the interval $I$. A sufficient condition for the $p$-invariance is the case when $G'(a) > 0$ and $G'(b) > 0$. Note that in the case $b = +\infty$ the $p$-invariance can also be defined by assuming that there exists $M_0 > a$ such that $G(x)$ is increasing in $[M_0, \infty)$.

**Definition 2.2.** A $p$-invariant interval $I := [a, b]$ will be called attracting if there exists its open neighborhood $U_0(I) = (a - \delta, b + \delta)$ such that $\Phi(U_0(I)) \subseteq U_0(I)$ and $\cap_{n\geq0} \Phi^n(U_0(I)) = \{x, \}$.

In the case of monotone increasing $G$ this definition coincides with the corresponding notion of the attracting interval for the map $\Phi = G^{-1} \circ F$ (see related details in [17, 17] for the case $G(x) = x$).

**Definition 2.3.** An infinite sequence of intervals $\{I_n, n \in \mathbb{N}\}$ imbedded into each other, $I_1 \supset I_2 \supset I_3 \supset \ldots$, will be called squizzing if the following holds:

(a) each interval $I_n, n \in \mathbb{N}$ is $p$-invariant;
(b) they are mapped into each other under $\Phi$

$$\Phi(I_n) \subseteq I_{n+1}, \forall n \in \mathbb{N};$$

(c) they have a single point in common

$$\cap_{n\geq1} I_n = \{x, \}.$$  

Note that one can easily deduce that the point $x = x, \$ in the above definition is a fixed point of the map $\Phi$.

3. Principal Results. This section contains main mathematical results on the dynamical behavior of solutions of differential delay equation (1)/(2). They primarily concern the global asymptotic stability of equilibria.

Assume that map $\Phi$ has an invariant interval $I \subseteq \mathbb{R}^+$, and introduce a subset $\mathcal{C}_I := C([-1, 0], I)$ of initial functions in $\mathcal{C}$ which range is within the interval $I$. The following invariance principle holds for solutions of differential delay equation (2) with the initial values in $\mathcal{C}_I$.

**Proposition 2.** (Invariance Property) Let $I := [a, b]$ be a closed bounded $p$-invariant interval of the multi-valued map $\Phi$. For every initial function $\phi \in \mathcal{C}_I$ and arbitrary $\mu > 0$ the corresponding solution $x = x(t, \phi)$ of equation (2) satisfies $x(t) \in I \forall t \geq 0$.

**Proof.** The proof is similar to that of Proposition 1. Indeed, assume that $\phi \in \mathcal{C}_I$ and let $t_0$ be the first exit point of the solution $x(t)$ from the interval $I$. To be definite,
It show now that there exists time such that for every initial some first exit time of the solution from interval other papers (see e.g. corresponding solution Lemma 3.1. Proposition 3.

way as the invariance property above (Proposition 2.4. The following statement can be found in an equivalent form or deduced from We shall provide a proof that is essentially different from those found elsewhere. Proposition 3. (Permanence) Assume hypothesis (i) to hold. Then equation (2) is permanent.

We shall provide a proof that is essentially different from those found elsewhere. It is based on the squeezing property of a p-invariant interval based on Definition 2.3. The proof uses the following Lemma, which is also a crucial statement in the proof of the main global stability result (see Theorem 3.2 below). The Lemma represents a significant independent interest on its own.

Lemma 3.1. Suppose K and L are p-invariant closed bounded intervals such that K ⊇ L and φ(K) ⊆ L. If none of the endpoints of the interval L is a fixed point then for every initial function φ ∈ Cκ where there exists time t1 = t1(φ) such that the corresponding solution x = x(t, φ) satisfies x(t) ≥ m(∀t ≥ t0). The boundedness (uniform) is meant in the natural way that x(t) ≤ M(∀φ ∈ C and all t ≥ t0 for some t0(φ)). Equation (2) (or a respective biological model behind it) is said to be permanent if it is persistent and uniformly bounded.

The following statement can be found in an equivalent form or deduced from other papers (see e.g. [9]), in particular for the case of linear G(x) = bx, b > 0 ([5, 7]).

Proposition 3. (Permanence) Assume hypothesis (B) to hold. Then equation (2) is permanent.

We shall provide a proof that is essentially different from those found elsewhere. It is based on the squeezing property of a p-invariant interval based on Definition 2.3. The proof uses the following Lemma, which is also a crucial statement in the proof of the main global stability result (see Theorem 3.2 below). The Lemma represents a significant independent interest on its own.

Lemma 3.1. Suppose K and L are p-invariant closed bounded intervals such that K ⊇ L and φ(K) ⊆ L. If none of the endpoints of the interval L is a fixed point then for every initial function φ ∈ Cκ there exists time t1 = t1(φ) such that the corresponding solution x = x(t, φ) of equation (2) satisfies x(t) ∈ L(∀t ≥ t1. Two cases are possible:

(a) φ(0) ∈ L. Then x(t, φ) ∈ L for all t ≥ 0. This is proved exactly the same way as the invariance property above (Proposition 3). By assuming there exists a first exit time of the solution from interval L one arrives at a contradiction. We leave the related details to the reader.

(b) φ(0) ∈ K \ L. To be definite assume x(0) = φ(0) ∈ [inf K, inf L]. We shall show now that there exists time t1 > 0 such that x(t1 , φ) = inf L. Assume this is not true. Then x(t) ∈ [inf K, inf L] ∀t ≥ 0. There are two possibilities.

(b1). x(t) is monotone and there is a limit x0 := limt→∞ x(t). Clearly, in this case limt→∞ x'(t) = 0 and one arrives at G(x0) = F(x0) which means that x0 is
a fixed point. This contradicts to the assumption that \( \Psi(K) \subseteq L \) and the points \( \inf K \) and \( \inf L \) are not fixed points.

(b2). \( x'(t_0) = 0 \) for some \( t_0 > 0 \). Then, the point \( x_0 := x(t_0) \in [\inf K, \inf L] \) is a fixed point. This as in the case of (b1) leads to a contradiction.

Thus, there exists a finite \( t_1 > 0 \) such that \( x(t_1, \phi) = \inf L \). Since \( \phi(t) \in K \), and \( x(t, \phi) \in [\phi(0), \inf L] \) for \( t \geq 0 \) one has that \( \psi(t) := x(t + t_1, \phi) \in K \forall t \in [-\tau, 0] \). Since \( L \) is \( p \)-invariant, \( \psi(0) = \inf L \), and \( \Psi(K) \subseteq L \), one then has that the corresponding solution \( x(t, \psi) \) satisfies \( x(t) \in L \forall t \geq 0 \). Again, this is shown exactly the same way as the proof of the invariance property, Proposition 2. We leave the details to the reader. The case \( x(0) \in [\sup L, \sup K] \) is completely analogous to the one considered above. This completes the proof.

Remark 1. Note that Lemma 3.1 extends in the obvious way to the case when one of the endpoints of both intervals \( K \) and \( L \) is a fixed point. This will be a subcase of considerations in Section 4.

To prove the permanence (see Proposition 3 above) we now observe the following. Since the assumption \( B \) is satisfied, and \( G \) is monotone increasing in some neighborhood of 0 and infinity, there exist a sufficiently small \( m > 0 \) and a sufficiently large \( M > 0 \) such that the interval \( I = [m, M] \) is \( p \)-invariant. Therefore, the set \( C_t \) is invariant under equation (2). Given an arbitrary initial function \( \phi \in C \) one sets \( J_0 := [\min \phi, \max \phi] \). There exists a \( p \)-invariant interval \( J \supseteq J_0 \) such that \( \Phi^m(J) \subseteq I \) for some \( m \in \mathbb{N} \). Then, by repeated application of Lemma 3.1, the corresponding solution \( x(t) = x(t, \phi) \) satisfies \( x(t) \in I \) for all sufficiently large \( t \), and the permanence follows.

Note that the assumption \( G(x) < F(x) \) for \( x \in (0, m) \) implies the persistence only; while the assumption \( G(x) > F(x) \) and \( G \) is monotone increasing for \( x \in [M, \infty) \) imply the boundedness.

Theorem 3.2. (Global Asymptotic Stability) Suppose that hypothesis \( C \) is satisfied. Assume in addition that there exists a squeezing sequence of \( p \)-invariant intervals \( \{I_n, n \geq 1\} \) containing the unique non-trivial fixed point \( x_\ast \) of map \( \Phi \). Then the constant solution \( x(t) = x_\ast \) of differential delay equation (2) is globally asymptotically stable for all values of \( \mu > 0 \). That is, for every initial function \( \psi \in C_{t_0} \) and arbitrary \( \mu > 0 \) the corresponding solution \( x(t) = x(t, \psi, \mu) \) of equation (2) satisfies

\[
\lim_{t \to \infty} x(t) = x_\ast.
\]

Proof. It follows by induction from Lemma 3.1 with \( I_n := K, I_{n+1} := L, n \in \mathbb{N} \).

4. Applications: Blood Cell Production Model of M.C. Mackey. This section deals with the population model of M.C. Mackey [12]. An essentially nonlinear differential equation with delay of the form (1) was proposed in [11, 12] as a mathematical model of blood cell production for the case of chronic myelogenous leukemia. The equation reads

\[
\frac{dx}{dt} = k \beta(x(t - \tau))x(t - \tau) - [\beta(x(t)) + \delta]x(t),
\]

where the nonlinear function \( \beta \) is a monotone Hill function

\[
\beta(x) = \beta_0 \left( \frac{1}{1 + x^n} \right)
\]
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and \( \beta_0, k = 2e^{-\gamma}, n, \delta \) are all positive constants defined by the physiological process behind. In this subsection we provide a detailed analysis of model (i) based on the given nonlinearities \( F \) and \( G \)

\[
F(x) = k\beta_0 \frac{x}{1 + x^n}, \quad G(x) = x \left[ \beta_0 \frac{1}{1 + x^n} + \delta \right]
\]

and values of the parameters \( \beta_0, k, n, \delta \). We establish conditions for the global asymptotic stability of the equilibria. Our results are complementary to those recently obtained in [1,2].

We first make several simple observations about the involved nonlinearities \( F \) and \( G \).

For \( 0 < n \leq 1 \) function \( F \) is increasing with \( \lim_{x \to \infty} F(x) = \infty \) when \( n < 1 \) and \( \lim_{x \to \infty} F(x) = k\beta_0 \) when \( n = 1 \). For \( n > 1 \) function \( F \) is unimodal with the only critical point \( x_{cr} = 1/(\sqrt[n]{n-1}) \) and the absolute maximum value \( F_{cr} := F(x_{cr}) = k\beta_0/n/(n-1) \). Also, \( \lim_{x \to -\infty} F(x) = 0 \) when \( n > 1 \).

An easy calculation shows that \( G(x) \) is either monotone increasing for all \( x \in \mathbb{R}^+ \) or it has two local extreme values \( x_1 \) and \( x_2 \) such that \( G(x) \) is increasing in \([0, x_1] \cup [x_2, \infty)\) and decreasing in \([x_1, x_2]\). Function \( G \) is monotone increasing in \( \mathbb{R}^+ \) if and only if \( \beta_0(n-1)^2 < 4\delta \). When \( \beta_0(n-1)^2 > 4\delta \) it has the two local extreme points \( x_1 \) and \( x_2 \). The values of \( x_1 \) and \( x_2 \) are given by

\[
1 - \left[ (n-1)\beta_0 \pm \sqrt{(n-1)^2\beta_0^2 - 4n\delta\beta_0} \right] \frac{1}{2}\delta.
\]

We shall also need to refer to the respective values of function \( G : G_1 = G(x_1), G_2 = G(x_2) \) (these expressions are easily found but are somewhat lengthy to write down explicitly in terms of the parameters).

Later in this subsection we shall be referring to the respective branches of \( y = G(x) \) (its graph). The first branch is defined on the interval \([0, x_1]\) where \( G(x) \) is monotone increasing with the range \([0, G_1]\). \( G(x) \) is decreasing on its second branch with the domain \( x \in [x_1, x_2] \) and the range \([G_2, G_1]\). The third branch is defined for \( x \in [x_2, \infty) \) where it is increasing with the range \([G_2, \infty)\). \( x_1 \) is the only local maximum and \( x_2 \) is the only local minimum of \( G(x) \) for \( x \in \mathbb{R}^+ \).

Depending on parameter values the proposed model admits one or two steady states, \( x(t) \equiv 0 \) and \( x(t) \equiv x_* \), where

\[
x_* = \left( \beta_0 \frac{k-1}{\delta} - 1 \right)^{1/n}.
\]

Proposition 4. The nontrivial equilibrium \( x_* \) exists if and only if \( k > 1 + \delta/\beta_0 \).

When \( k \leq 1 + \delta/\beta_0 \) equation (i) has the trivial equilibrium \( x(t) \equiv 0 \) only.

The equilibrium \( x_* \) is found from solving the equation \( F(x) = G(x) \), and it is given by formula (10). It is easy to check that the condition \( k \leq 1 + \delta/\beta_0 \) is equivalent to \( F'(0) \leq G'(0) \), and therefore, \( F(x) < G(x) \) for all \( x \in \mathbb{R}^+ \).

Next statement describes cases when the unique trivial equilibrium \( x(t) \equiv 0 \) is globally asymptotically stable.

Proposition 5. Suppose that \( k \leq 1 + \delta/\beta_0 \) holds. The trivial equilibrium \( x(t) \equiv 0 \) is globally asymptotically stable if either one of the following two assumptions is satisfied

1. \( x_{cr} < x_2 \) and \( F(x_{cr}) < G(x_2) \).

Proposition 6. If the unique trivial equilibrium of equation (i) is not globally asymptotically stable, then it is unstable. More precisely, there exists a positive number \( \epsilon \) such that for any \( x_0 \in (0, \epsilon) \), the solution \( x(t) \) of equation (i) with \( x(0) = x_0 \) satisfies

\[
x(t) > \epsilon \quad \text{for some } t > 0.
\]

Proposition 7. Consider the equation

\[
\frac{dx}{dt} = F(x) - G(x),
\]

where \( F \) and \( G \) are defined as above. If \( k > 1 + \delta/\beta_0 \), then there is no exponential attraction. More precisely, there exist positive constants \( M \) and \( c \) such that

\[
x(t) \leq M e^{ct} \quad \text{for all } t > 0.
\]
Proof. The proof is based on application of Theorem 3.2. To be definite, consider a subcase of the second case, \( x_{cr} \geq x_2 \) and \( F(x_{cr}) > G(x_2) \), which represents a more involved situation. Case one and the other subcases of the second case are treated similar and are left to the reader.

One can define \( x_1 := x_{cr} \) and \( !_1 := [0, Xc]^{-1} \) as an initial choice. Since \( F(x_{cr}) > G(x_2) \), there exists exactly one value \( x^2 \in [x_2, x_{cr}] \) such that \( G(x^2) = F(x_{cr}) \). Let \( I_2 := [0, x^2] \). Then, by construction, \( \Phi(I_1) = I_2 \) and both intervals \( I_1 \) and \( I_2 \) are \( p \)-invariant. Likewise, there exists \( x^3 \in [x_2, x_{cr}] \) such that \( x^3 < x^2 \), \( F(x^3) = G(x^2) \), and \( F(x^3) > G(x_2) \). One sets then \( I_3 := [0, x^3] \), with \( \Phi(I_2) = I_3 \) and both intervals \( I_2 \) and \( I_3 \) being \( p \)-invariant. One continues this procedure and finds the last point \( x^m \in [x_2, x_{cr}] \) such that \( F(x_{m-1}) = G(x_m) \) and \( F(x_m) < G(x_2) \). One sets then \( !_m := [0, x^m] \), with \( \Phi(I_{m-1}) = I_m \) and both \( I_{m-1} \) and \( I_m \) being \( p \)-invariant.

For the next interval \( I_{m+1} \), one can set \( I_{m+1} := [0, x_2] \). Note that this is not the only choice for the squizzing sequence of \( p \)-invariant intervals, however, a convenient one. Since \( \max\{F(x), x \in [0, x_m]\} < G(x_2) \) the pre-image of the interval \( I_{m+1} \) under \( \Phi \) is entirely defined by the first monotone branch of \( G \). In fact, the map \( \Phi \) itself is given by \( \Psi = G^{-1} \circ F \) there. Therefore, one can proceed then by choosing \( I_{i+1} = \Phi^{-1}(I_i), i \geq m + 1 \). Since \( G(x) > F(x) \) and both \( F \) and \( G \) are monotone in \( [0, x_i] \) one has that \( \cap_{i \geq m+1} I_i = \{0\} \).

Therefore, the entire sequence of constructed above intervals \( I_{i, i} \geq 1 \) is a squizzing sequence of imbedded intervals. We note now that for any interval \( I_0 \) of the form \( [0, a] \), \( a \geq x_{cr} \), one has \( \Phi(I_0) \subseteq I_1 \). Therefore, for arbitrary initial function \( \phi \in C_{I_0} \) one shows that \( \lim_{t \to \infty} x(t, \phi) = 0 \) by using Theorem 3.2. This completes the proof of Proposition 5.

Remark 2. Note that the remaining subcase not covered by the above Proposition 5 is when \( x_{cr} < x_2 \) and \( F(x_{cr}) > G(x_2) \). One can still show in this case that the corresponding map \( \Phi \) is globally attracting on \( \mathbb{R}^+ \). However, the construction of a squizzing sequence of embedded \( p \)-invariant intervals does not look possible due to the fact that both functions \( F \) and \( G \) are decreasing on the interval \( [\max\{x_1, x_{cr}\}, x_2] \).

Next statement describes cases when the unique positive equilibrium \( x(t) = x_* \) is globally asymptotically stable. Those cases are based on the existence of a squizzing sequence of \( p \)-invariant intervals for the corresponding multi-valued map \( \Phi \).

Proposition 6. Suppose that \( k > 1+\delta/\beta_0 \) holds. The positive equilibrium \( x(t) = x_* \) is globally asymptotically stable if either one of the following three conditions is satisfied:

1. \( F \) and \( G \) are increasing for all \( x \in \mathbb{R}^+ \);
2. \( x_* \leq x_{cr} \);
3. \( G \) is increasing for all \( x \in \mathbb{R}^+ \) and \( x_* \) is globally attracting fixed point of the map \( \Phi := G^{-1} \circ F \).

Proof. We briefly describe the main initial steps of the construction of the squizzing sequence of embedded \( p \)-invariant intervals in each of the cases, leaving remaining details to the reader.

1. The map \( \Phi \) is one-to-one and defined by \( \Phi = G^{-1} \circ F \). With \( x_* > 0 \) given by (16) the initial interval \( I_1 \) can be chosen as \( I_1 := [a, b] \) where \( a \) and \( b \)
are arbitrary positive numbers such that $0 < a < x_1 < b < \infty$. One then defines $I_{i+1} := \Phi(I_i), i \in \mathbb{N}$.

(2). One can choose $x^2 := \min\{x_{or}, x_1\}$ and set $I_1 := [m, x^2]$ where $m > 0$ is arbitrary and such that $m < x_1$. Since both $F$ and $G$ are monotone increasing on $I_1$, the map $\Phi$ is defined by $\Phi := G^{-1} \circ F$. The squizzing sequence of imbedded intervals is constructed then by $I_{i+1} := G^{-1} \circ F(I_i), i \in \mathbb{N}$. By Theorem 3.2 equation (i) is globally asymptotically stable on the set $C_{I_1}$.

One can further show that equation (i) can be globally asymptotically stable on a larger set than $C_{I_1}$, depending on the mutual shape of nonlinearities $F$ and $G$. In particular, when either $F(x_{or}) < G(x_2)$ or $x_{or} > x_2$ holds, there always exists a p-invariant interval $I_0 := [0, b]$ such that $\Phi(I_0) \subseteq I_1$, where $b$ can be chosen such that $b \geq \max\{x_{or}, x_2\}$. This leads to the global asymptotic stability of equation (i) on the entire set $C$.

(3). Map $\Phi$ is well-defined in this case by $\Phi(x) = G^{-1}(F(x))$. One can choose $I_1 := [m, M]$ and such that $\Phi(I_1) \subset I_1$. Here $m > 0$ is sufficiently small so that both $F$ and $G$ are increasing in $[0, m]$ and $\max\{F(m), G(m)\} < \min\{F(x_{or}), G(x_2)\}$; $M > \max\{x_{or}, x_2\}$ is sufficiently large so that $G(M) > \max\{G(x_1), F(x_{or})\}$. One defines the sequence of squizzing intervals as before by $I_{i+1} := G^{-1} \circ F(I_i), i \in \mathbb{N}$.

Note that in case (3) the global asymptotic stability of the positive equilibrium $x(t) \equiv x_*$ also follows from results in papers [5, 7].

**Concluding Remarks and an Open Problem.** This work initiates a study of the global asymptotic stability in the essentially nonlinear differential delay equation (1)/(2) in terms of its limiting difference equation (i), or equivalently, in terms of the multi-dimensional map $\Phi$. The new results extend those previously obtained in [7] for the case of linear $G(x) = x$. The price of dealing with the case of general nonlinear $G(x)$ is a more strong assumption on the attractive nature of the fixed points 0 and $x_*$ of the limiting map $\Phi$, which is the existence of imbedded sequence of p-invariant squizzing intervals. In the prior studied linear case $G(x) \equiv x$ the mere global attractivity of the fixed point $x_*$ was sufficient for the global asymptotic stability of the steady state $x(t) \equiv x_*$ of equation (1)/(2) (for all $\mu = 1/\tau > 0$). These considerations motivate the following open problem.

**Open Problem.** Investigate whether it is true or not that the global attractivity in the limiting multi-valued one-dimensional map $\Phi$ is sufficient for the global asymptotic stability of the unique equilibrium in equation (1)/(2). Derive sharper conditions for the global asymptotic stability than those given by Theorem 3.2 in terms of the existence of the squizzing sequence of embedded p-invariant intervals.

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