The generator of spatial diffeomorphisms in the Koslowski–Sahlmann representation

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Abstract

A generalization of the representation, underlying the discrete spatial geometry of loop quantum gravity, to accommodate states labelled by smooth spatial geometries, was discovered by Koslowski and further studied by Sahlmann. We show how to construct the diffeomorphism constraint operator in this Koslowski–Sahlmann (KS) representation from suitable connection and triad dependent operators. We show that the KS representation supports the action of hitherto unnoticed ‘background exponential’ operators which, in contrast to the holonomy-flux operators, change the smooth spatial geometry label of the states. These operators are shown to be quantizations of certain connection dependent functions and play a key role in the construction of the diffeomorphism constraint operator.

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1. Introduction

Loop quantum gravity (LQG) is based on the canonical quantization of a classical Hamiltonian description of gravity in terms of an $SU(2)$ connection and its conjugate electric (triad) field. The basic operators of the theory are holonomies of the connection around loops in the Cauchy slice $\Sigma$ and electric fluxes through surfaces therein. The LQG representation yields a discrete spatial geometry at the quantum level as exemplified by the discrete spectra of triad dependent operators such as the volume and the area [1].

In order to capture the effective smoothness of classical spatial geometries, Koslowski [2] constructed a representation of the holonomy-flux algebra in which the flux operator action is augmented by a ‘classical flux’ contribution and the holonomy operator action is unchanged. This representation was further studied with a view towards imposing the $SU(2)$ Gauss law and the diffeomorphism constraints of gravity by Sahlmann [3]. In this Koslowski–Sahlmann (KS) representation, states acquire an additional label corresponding to a smooth ‘background’ triad field, the additional contribution to the flux operator coming precisely from the flux of this background electric field. As seen in equations (2.3)–(2.4) below, neither the holonomy...
not the flux operators change this background field. This feature gives rise to the following conundrum.

Finite spatial diffeomorphisms are unitarily represented in the KS representation [3] and their action on a state labelled by some background triad field yields a state labelled with a new background field, the new label being the image of the old one under the spatial diffeomorphism. On the other hand, the recent construction of the diffeomorphism constraint in LQG [4] from the basic holonomy-flux operators of LQG in such a way that it generates diffeomorphisms, suggests that a similar construction should exist in the KS representation. Since the diffeomorphism constraint depends on the connection and electric field, and since the holonomy-flux variables separate points on the phase space of connections and electric fields, one may expect that, similar to [4], the diffeomorphism constraint operator should be expressed in terms of (a limit of) holonomy-flux operators. However, this is at odds with the feature mentioned in the previous paragraph because diffeomorphisms do change the background triad field whereas the holonomy and flux operators do not.

In this work we resolve this conundrum by identifying a set of connection dependent quantities which, upon quantization, yield operators which do change the background triad. Using these operators together with the holonomy-flux operators yields a satisfactory construction of the diffeomorphism constraint. In section 2, we review the KS representation, identify the new ‘background exponential’ functions and construct their quantization. In section 3, we indicate how to construct the diffeomorphism constraint from the enlarged holonomy-flux-background exponential algebra. Section 4 contains our concluding remarks and a description of results recently obtained such as the crucial role of background exponentials in an improved treatment of the imposition of the Gauss law and diffeomorphism constraints [5] and the construction of the KS representation for two dimensional parameterized field theory (PFT) in such a way as to support its dynamics [6].

In what follows we assume familiarity with standard LQG, and, in section 3, with the results of [4].

### 2. Background exponentials

The classical phase space variables for LQG are the connection $A_i^a$, $i$ being an internal $SU(2)$ Lie algebra valued index, and the conjugate densitized electric (triad) field $E_i^a$ with Poisson brackets $\{A_i^a(x), E_i^b(y)\} = 8\pi \gamma G \delta(x,y) \delta_i^b$ where $\gamma$ is the Barbero–Immirzi parameter. We shall choose units such that $8\pi \gamma G = \hbar = c = 1$.

The kinematical Hilbert space of standard LQG is spanned by the orthonormal basis of spin network states $|s\rangle$. Let the dense domain of the finite linear span of spinnets be $D$. Let $\hat{O}$ be an operator from $D$ to $D$ so that $\hat{O} |s\rangle$ is a finite linear combination of spinnets i.e. $\hat{O} |s\rangle = \sum_i O_i |s_i\rangle$ where $O_i$ are the complex coefficients in the sum over the spinnets $|s_i\rangle$. It is useful to introduce the notation $|\hat{O} s\rangle$ to denote this linear combination of spinnets so that we have that

$$|\hat{O} s\rangle := \hat{O} |s\rangle = \sum_i O_i |s_i\rangle.$$  \hspace{1cm} (2.1)

The KS kinematic Hilbert space is then spanned by states which have, in addition to their LQG spinet label, an additional label $E_i^a$ where $E_i^a$ is a smooth ‘background’ electric field. We denote such a state by $|s, E\rangle$. These states for all $s$, $E$ provide an orthonormal basis for the KS kinematic Hilbert space so that the inner product between two such KS spinnets in this Hilbert space is

$$\langle s', E' | s, E \rangle = \langle s | s' \rangle \delta_{E^a_i, E'^a_i}.$$  \hspace{1cm} (2.2)
where $\langle s|s'\rangle$ is just the standard LQG inner product and the second factor is the Kronecker delta which vanishes unless the two background fields agree in which case it equals unity.

The holonomy-flux operators act on the KS spinnets as:

$$\hat{h}_{a}^{A}|s, E\rangle = [\hat{h}_{a}^{A}, s, E\rangle, \quad (2.3)$$

$$\hat{E}_{s}(f)|s, E\rangle = [\hat{E}_{s}(f)s, E\rangle + E_{s}(f)|s, E\rangle. \quad (2.4)$$

Here, our notation is as follows. $\hat{h}_{a}^{A}$ is the $A, B$ component of the holonomy operator around the loop $\alpha$ in the $j = \frac{1}{2}$ representation. $\hat{E}_{s}(f)$ is the electric flux operator obtained by integrating the electric field with respect to the Lie algebra valued function $f'$ over the surface $S$ i.e.

$$\hat{E}_{s}(f) = \int_{S} f' \hat{E}_{s}^{a} \eta_{abc}. \quad (2.5)$$

where $\eta_{abc}$ is the Levi-Civita tensor of weight $-1$. We have used an obvious generalization of the notation of equation (2.1) wherein given an operator $\hat{O}$ with action $\hat{O}|s\rangle = \sum_{i} O_{i}^{(*)}|s_{i}\rangle$ in standard LQG, we have defined the state $|\hat{O}s, E\rangle$ in the KS representation through

$$|\hat{O}s, E\rangle := \sum_{i} O_{i}^{(*)}|s_{i}, E\rangle. \quad (2.6)$$

Similar to equation (2.5), in equation (2.4), $E_{s}(f)$ denotes the flux of $E'$ smeared with $f'$ through the surface $S$ i.e.

$$E_{s}(f) = \int_{S} f' E_{s}^{a} \eta_{abc}. \quad (2.7)$$

Thus, the action of the electric flux operator in the KS representation obtains, in addition to the standard LQG like first term, an extra background electric flux contribution. We note here that neither the holonomy nor the flux operators change the background electric field label of the state.

Next, we define an operator $\hat{H}_{E}$ which changes the background electric field by translating it through the amount $E_{s}^{a}$ so that:

$$\hat{H}_{E}|s, E_{0}\rangle = |s, E + E_{0}\rangle. \quad (2.8)$$

It is easily verified that this operator commutes with the holonomy operators, that its commutator with the flux operator is

$$[\hat{H}_{E}, \hat{E}_{s}(f)] = -E_{s}(f)\hat{H}_{E}, \quad (2.9)$$

and that two such operators $\hat{H}_{E_{a}}, \hat{H}_{E_{b}}$ commute with each other.

Next, we note that these operators can be constructed as quantizations of certain classical functions which we call `background exponentials’ for reasons which are obvious from the considerations below. Each such function depends only on the connection, is labelled by some background electric field $E_{s}^{a}$ and is denoted by $H_{E_{a}}$. We define $H_{E}$ as:

$$H_{E}(A) = \exp \left( i \int_{\Sigma} A_{a}^{e} E_{s}^{a} \right). \quad (2.10)$$

Clearly, these functions Poisson commute with each other as well as with the holonomies. The only additional non-trivial Poisson bracket is

$$\{H_{E}(A), E_{s}(f)\} = iE_{s}(f)H_{E}(A). \quad (2.11)$$

It follows that the operator action (2.8) provides a representation of the Poisson bracket (2.11) through equation (2.9).

In summary: the operator $\hat{H}_{E}$ defined in equation (2.8) is the quantum correspondent of the classical phase space ‘background exponential’ function $H_{E}(A)$ defined in equation (2.10).
3. The diffeomorphism constraint

As shown in [3], in order for the holonomy-flux algebra to transform correctly under the unitary action of finite spatial diffeomorphisms, it is essential that the background electric field label of any KS state transforms to its diffeomorphic image under this action. More precisely, as shown in [3], the unitary operator \( \hat{U}(\phi) \) corresponding to the finite diffeomorphism \( \phi \) acts on KS spinnets through:

\[
\hat{U}(\phi) |s, E\rangle = |\hat{U}(\phi)s, \phi^*E\rangle = |\phi \circ s, \phi^*E\rangle,
\]

where \( \phi \circ s \) is just the standard LQG diffeomorphic image of the spinnet \( s \) and \( \phi^*E \) is the push forward of the background electric field under \( \phi \). In the classical theory, diffeomorphisms along the vector field \( N_a \) are generated by the diffeomorphism constraint \( D(\vec{N}) \) where:

\[
D(\vec{N}) = \int_{\Sigma} \langle \mathcal{L}_{\vec{N}} A_{ib} \rangle E_i^b.
\]

In [4], this operator was constructed in standard LQG out of the standard holonomy-flux operators by first constructing finite triangulation approximants which are well defined on the kinematic Hilbert space and then defining their continuum limit action on the Lewandowski–Marolf (LM) habitat [8]. In the KS representation, a construction involving only the holonomy-flux operators cannot generate diffeomorphisms of the background electric field label because, from equations (2.3)–(2.4), these operators do not change the background electric field. We now show that finite triangulation approximants to the diffeomorphism constraint which involve the background exponential operators can be constructed in such a way as to correctly generate the action of diffeomorphisms in the KS representation.

We start by rewriting \( D(\vec{N}) \) as

\[
D(\vec{N}) = \int_{\Sigma} \langle \mathcal{L}_{\vec{N}} A_{ib} \rangle E_i^b = \int_{\Sigma} \langle \mathcal{L}_{\vec{N}} A_{ib} \rangle (E_i^b - E_i^b) + \int_{\Sigma} \langle \mathcal{L}_{\vec{N}} A_{ib} \rangle E_i^b.
\]

where have defined \( \mathcal{D}^E(\vec{N}) \) as

\[
\mathcal{D}^E(\vec{N}) := \int_{\Sigma} \langle \mathcal{L}_{\vec{N}} A_{ib} \rangle (E_i^b - E_i^b).
\]

Next, let \( \delta \) be some small parameter and let \( \mathcal{D}^E(\vec{N}) \) be an approximant to \( \mathcal{D}^E(\vec{N}) \) so that

\[
\lim_{\delta \to 0} \mathcal{D}^E(\vec{N}) = \mathcal{D}^E(\vec{N}).
\]

We define the quantity \( D_\delta(\vec{N}) \) as:

\[
D_\delta(\vec{N}) = \frac{\langle e^{\delta \int_{\Sigma} \langle \mathcal{L}_{\vec{N}} A_{ib} \rangle (E_i^b - E_i^b)} \mathcal{D}^E(\vec{N}) \rangle - 1}{i \delta},
\]

where \( \phi(\vec{N}, \delta) \) is the diffeomorphism generated by the shift vector field \( \vec{N} \) so that \( \phi(\vec{N}, \delta) \) translates points in \( \Sigma \) along the orbits of \( \vec{N} \) by an affine amount \( \delta \). It is then straightforward to verify that

\[
\lim_{\delta \to 0} D_\delta(\vec{N}) = D(\vec{N})
\]

so that the expression (3.7) defines an approximant to \( D(\vec{N}) \).

1 We use the notation and conventions of Wald [7].
Similar to [4], our strategy is to identify the parameter $\delta$ as characterizing the fineness of a triangulation $T_i$ of $\Sigma$ and to construct the operator corresponding to $D_\delta(\bar{N})$ in such a way that it takes the form

$$\hat{D}_\delta(\bar{N}) = \frac{\hat{U}(\phi(\bar{N}, \delta)) - 1}{i\delta}$$

on the dense domain $\mathcal{D}_{KS}$ of the finite linear span of KS spinnets. Accordingly, given any KS spinnet $|s, E\rangle$ we construct the quantization of each of the factors $(1 + i\delta\hat{D}_\delta^E(\bar{N}))$ and $e^{\delta\int_{\mathcal{D}}A_0(\phi(\bar{N}, \delta))}e^{(E')^2 - E^2}$ in equation (3.7) in turn, as follows.

We note that:

(i) Equation (2.4) may be rewritten as:

$$\langle \hat{E}_S(f) - E_S(f) | s, E \rangle = |\hat{E}_S(f) s, E \rangle.$$  

Equation (3.10) shows that the action of the operator $\hat{E}_S(f) - E_S(f)$ on the state $|s, E\rangle$ in the KS representation is isomorphic to the action of the electric flux operator $\hat{E}_S(f)$ on the spinnet $|s\rangle$ in the standard LQG representation.

(ii) From equations (2.5) and (2.7), it follows that

$$\hat{E}_S(f) - E_S(f) = \int_s f(\hat{E}_i^u - E_i^u)_{nabc}.$$  

Next, recall that in [4] a finite triangulation approximant $D_\delta^{LOG}(\bar{N})$ to the diffeomorphism constraint was obtained in terms of holonomies along small edges and fluxes through small surfaces, the size of these small objects going to zero in the continuum limit. It is easy to see that if, in the expression for $D_\delta^{LOG}(\bar{N})$ in [4], we substitute each occurrence of an electric flux through a small surface by the difference of the electric flux and the background electric flux through the same surface, we obtain a finite triangulation approximant to $D_\delta^E(\bar{N})$. Indeed this can be readily inferred from equations (3.5), (3.11) together with a quick perusal of [4].

Let us denote the phase space dependence of any function $O$ by $O[A, E]$. Using this notation, the discussion of the previous paragraph implies that we may choose the finite triangulation approximant $D_\delta^F(\bar{N}) = D_\delta^{LOG}(\bar{N})[A, E]$ as:

$$D_\delta^F(\bar{N})[A, E] = D_\delta^{LOG}(\bar{N})[A, E - E].$$

Next, recall that in standard LQG we have, from [4] that:

$$\hat{D}_\delta^{LOG}(\bar{N})|s\rangle = \frac{\hat{U}(\phi(\bar{N}, \delta)) - 1}{i\delta}|s\rangle,$$

where the finite triangulation operator $\hat{D}_\delta^{LOG}(\bar{N})$ is defined by replacing the holonomies and fluxes in the classical approximant, $D_\delta^{LOG}(\bar{N})[A, E]$, by the corresponding operators in standard LQG.

It then follows from (i), (ii) above (see equations (3.10), (3.11)) and equation (3.13) that we have that

$$\hat{D}_\delta^F(\bar{N})|s, E\rangle = \left|\frac{\hat{U}(\phi(\bar{N}, \delta)) - 1}{i\delta} s, E\right\rangle.$$

\[\text{In [4], the diffeomorphism constraint is split into a vector constraint and Gauss law contribution. The finite triangulation vector constraint approximant is dealt with in rigorous detail and the finite triangulation Gauss law approximant semi-heuristically, the assumption being that the treatment of the latter can be improved so as to construct it in terms of holonomy-flux variables. Under this assumption we expect that our treatment here should follow by replacing, in both approximants, every occurrence of the electric flux by the flux difference.}\]
which implies that the action of the second factor in equation (3.7) on the KS spinnet \(|s, E]\) is:
\[
(1 + i\hat{D}_s^E(\vec{N})) |s, E\rangle = i\hat{U}(\phi(\vec{N}, \delta)) |s, E\rangle.
\]
(3.15)

Next, we turn our attention to the first factor, \(e^{i\int_{\Sigma} (\phi(\vec{N}, \delta) - E)A_s}\), in equation (3.7). From equation (2.10) this factor is just a background exponential with background field \((\phi(\vec{N}, \delta) - E)B_s\). We then obtain equation (3.16) on the KS spinnet.

It follows from equations (3.16), (3.7), (2.8) and (3.15) that
\[
\hat{D}_s(\vec{N}) |s, E\rangle = \frac{\hat{H}_{(\phi(\vec{N}, \delta)_E)}(1 + i\hat{D}_s^E(\vec{N})) - 1}{i\delta} |s, E\rangle
\]
(3.17)
\[
= \frac{\hat{H}_{(\phi(\vec{N}, \delta)_E)}\hat{U}(\phi(\vec{N}, \delta))s, E\rangle - |s, E\rangle}{i\delta}
\]
(3.18)
\[
= \frac{\frac{\hat{U}(\phi(\vec{N}, \delta))s, \phi(\vec{N}, \delta)^*_E(E)\rangle - |s, E\rangle}{i\delta}
\]
(3.19)
\[
= \frac{\hat{U}(\phi(\vec{N}, \delta)) - 1}{i\delta} |s, E\rangle
\]
(3.20)
where the final line follows from the unitary action of finite diffeomorphisms in the KS representation [3]. Thus, we have obtained equation (3.9) in terms of the holonomy, flux and, crucially the background exponential operators of the KS representation. Indeed, as explicitly seen above, it is the background exponential contribution which moves the background field by the diffeomorphism \(\phi(\vec{N}, \delta)\).

The next step would be to take the continuum limit of the above equation on an appropriate generalization of the LM habitat. We expect that, in analogy to the standard LM habitat, elements of this ‘LMKS’ habitat would be complex linear mappings on the finite linear span of KS spinnets \(D_{KS}\). Each habitat state would be a finite linear combination of certain elementary basis states \(\Psi_{f, s, E}\). Here the label \([s, E]\) is the diffeomorphism equivalence class of the KS spinnet labels \((s, E)\) so that every element of \([s, E]\) is the image of \((s, E)\) by some diffeomorphism. The label \(f\) is the ‘vertex smooth’ function associated with \(s\) (see [8]) and \(g\) is a complex functional (with suitable functional differentiability properties) on the space of density weight one Lie algebra valued vector fields. Since the main purpose of this work was to demonstrate the key role of the background exponential operators in the KS representation, we leave the working out of these ideas for the future.

4. Concluding remarks

In this paper we showed how to construct finite triangulation approximants to the diffeomorphism constraint operator in the KS representation. Since the background electric fields labelling the KS spinnets are mapped to their diffeomorphic images by the unitary action of finite diffeomorphisms and since the holonomy-flux operators do not change this background field label, it is imperative to introduce phase space functions whose operator correspondents do change these labels, and to use such operators in the construction of the desired finite triangulation approximants. Here we showed that these operators are exactly the background exponential operators constructed in section 2.

It turns out that the background exponentials play a crucial role in the imposition of the Gauss law and diffeomorphism constraints in the KS representation. These results [5]
constitute an improvement over the pioneering works of Koslowski and Sahlmann [2, 3, 9] and we shall report them elsewhere. Clearly the inclusion of background exponentials constitutes an enlargement of the holonomy-flux algebra of LQG. Preliminary work [10] suggests that, just as in LQG, the commutative part of this algebra generated by the holonomies and background exponentials can be completed to a commutative \( \mathbb{C}^* \) algebra, that the Gel’fand spectrum of this \( \mathbb{C}^* \) algebra is the topological completion of the space of smooth connections and that the KS Hilbert space can be obtained as the space of square integrable functions over the spectrum with respect to an appropriately defined ‘KS’ measure. We will report on these results, subject to confirmation, elsewhere.

Our aim is to generalize the KS representation to the asymptotically flat case using many of the results discussed above so as to incorporate the asymptotically flat boundary conditions on the phase space variables in quantum theory. In particular, since the triad field asymptotes to a smooth, flat triad at infinity, we expect that the smooth background triad fields of the KS representation facilitate the imposition of such boundary conditions in quantum theory. While we do expect progress towards the construction of the KS representation at the kinematic, \( SU(2) \) gauge invariant and spatial diffeomorphism invariant levels in the spatially compact and asymptotically flat cases, the construction and imposition of the Hamiltonian constraint, just as in LQG, remains an open issue. In order to build intuition for dynamical issues in the case of a compact Cauchy slice, Sengupta [6] has analysed the KS representation for the toy model of PFT on the Minkowskian cylinder, successfully generalizing the considerations of [12] to the case of the KS representation. On the other hand, PFT on the Minkowskian plane provides a simplified setting for a generalization of the KS representation to asymptotically flat gravity. It turns out that the boundary conditions on the (spatial derivatives of the) embedding variables in planar PFT bear a resemblance to those for the triad field in asymptotically flat gravity. Work on this generalization of the KS representation to planar PFT is also in progress by Sengupta [11].

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