Degenerations of Surface Scrolls and the Gromov-Witten Invariants of Grassmannians

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Abstract: We describe an algorithm for computing certain characteristic numbers of rational normal surface scrolls using degenerations. As a corollary we obtain an efficient method for computing the corresponding Gromov-Witten invariants of the Grassmannians of lines.

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1 Introduction

This paper investigates the enumerative geometry of rational normal surface scrolls in $\mathbb{P}^N$ using degenerations. We obtain an effective algorithm for computing certain characteristic numbers of balanced scrolls. Surface scrolls can be interpreted as curves in the Grassmannian of lines $G(1,N)$. Using our algorithm we calculate the corresponding Gromov-Witten invariants of $G(1,N)$. We work over the field of complex numbers $\mathbb{C}$.

Motivation. By the characteristic number problem we mean the problem of computing the number of varieties in $\mathbb{P}^n$ of a given ‘type’ [e.g. curves of degree $d$ and genus $g$] that meet the ‘appropriate’ number of general linear spaces so that the expected dimension is zero. This problem has attracted a lot of interest since the 19th century (see [Sc], [Kl2]). In the last decade, motivated by the work of string theorists and Kontsevich, there has been significant progress on the problem for curves (see [CH], [V1], [V2] for references).

In comparison the characteristic numbers of higher dimensional varieties are harder to compute, hence have received less attention. In this paper we start a more systematic study of the characteristic numbers of higher dimensional varieties using degenerations. Here we restrict our attention to rational surface scrolls, although most of the techniques apply with little change to higher dimensional scrolls [C2] and can be modified to apply to Del Pezzo surfaces [C1].

The enumerative geometry of scrolls is also attractive for its connection to the Gromov-Witten theory of $G(1,N)$. Localization techniques and the associativity relations in the quantum cohomology ring lead to recursive algorithms that compute the invariants, but these algorithms are usually inefficient. For example, using FARSTA [K3], a computer program that computes Gromov-Witten invariants from associativity relations, it takes several months to determine cubic and quartic invariants of $G(1,5)$. The algorithm we prove here allows us to compute many of these invariants by hand (§8, 9).

Notation. Let $\overline{M}_{0,n}(G(1,N),d)$ denote the Kontsevich moduli space of $n$-pointed genus 0 stable maps to $G(1,N)$ of Plücker degree $d$. Let $\text{Hilb}(\mathbb{P}^N, S^d)$ denote the component of the Hilbert scheme whose general point corresponds to a smooth rational normal surface scroll $S$ of degree $d$ in $\mathbb{P}^N$.

Results. The main results of this paper are the following:

- To calculate the characteristic numbers of scrolls, we specialize the linear spaces meeting the scrolls to a general hyperplane $H$. We prove that a general, non-degenerate, reducible limit of balanced scrolls incident to the linear spaces consists of the union of two balanced scrolls meeting along a line—provided that the limit of the hyperplane sections in $H$ remains non-degenerate. (§6) The precise statements are given in Theorems 6.8 and 6.9.

- By successively breaking the scrolls to smaller degree scrolls, we obtain a recursive algorithm for computing the characteristic numbers of balanced scrolls in $\mathbb{P}^N$ incident to linear spaces of small dimension (§8). Theorem 8.1 summarizes the result.

Example: For instance, the algorithm easily shows that the number of scrolls of degree $n$ in $\mathbb{P}^{n+1}$ containing $n + 5$ general points and meeting a general $n - 3$ plane is $(n - 1)(n - 2)$ (§5).

- As a corollary, we obtain an efficient method for computing the corresponding Gromov-Witten invariants of $G(1,N)$. The proof also yields a method for computing
some Gromov-Witten invariants of $\mathcal{F}(0, 1; N)$, the partial flag variety of pointed lines in $\mathbb{P}^N$ (§9).

**The method.** Our method is a degeneration method inspired by [CH] and especially [V2]. The prototypical example answers the question how many lines meet 4 general lines $l_1, \cdots, l_4$ in $\mathbb{P}^3$. If $l_1$ and $l_2$ lie in a plane $P$, then the answer is easy to see. Let $q = l_1 \cap l_2$. The two solutions are the line in $P$ through $q_3 \cap P$ and $q_4 \cap P$ and the intersection of the two planes $q_3$ and $q_4$. To answer the original question we can specialize two of the lines to a plane. If we know how many of the original solutions approach each of the two special solutions we can answer the problem. Our algorithm carries out this classical idea for rational normal scrolls.

The solution of an enumerative problem by degenerations has two steps. We specialize linear spaces meeting the scrolls to a general hyperplane one at a time. First, we identify the limiting positions of the scrolls.

We prove that non-degenerate limits of scrolls are unions of scrolls where any two adjacent components share a common fiber. These limit surfaces occur as images of trees of Hirzebruch surfaces (Proposition 4.1). We describe the trees that occur as limits of scrolls $S_{k,l}$.

Not every tree of scrolls smooths to $S_{k,l}$. The specializations of $S_{k,l}$ contain a connected degree $k$ curve whose components are rational curves in section classes on the scrolls. The existence of a degree $k$ curve with these properties turns out to be sufficient for a union of two scrolls of total degree $k + l$ to smooth to $S_{k,l}$ (Proposition 4.4).

In §6 we carry out a detailed dimension count to identify which unions of scrolls occur as limits under some non-degeneracy assumptions. The dimension calculations are considerably harder for surfaces than for curves because the need to trace both the hyperplane section and the directrix of the surface forces us to work with a non-convex space. Nonetheless, we prove that in a general one-parameter family if the surfaces and their hyperplane sections remain non-degenerate, balanced scrolls break into unions of balanced scrolls.

Once we determine the limits, we need to determine their multiplicities. We reduce the calculations to the case of curves by constructing a smooth morphism from the space of scrolls to $\overline{M}_{0,n}(\mathbb{P}^N, d)$ and pulling-back the relations between cycles in these spaces [V2] to the space of scrolls (§7). We give many examples to illustrate how the algorithm works in §5.

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2 Preliminaries on Scrolls

This section provides a summary of basic facts about rational scrolls and systems of divisors on them; for more details consult [Bv] Ch. 4, [Fr] Ch. 5 or [GH] §3 Ch. 4.

Rational normal scrolls. Let \( k \leq l \) be two non-negative integers with \( l > 0 \). We will denote a rational normal surface scroll of bidegree \( k, l \) by \( S_{k,l} \). \( S_{k,l} \) is a rational surface of degree \( k + l \) in \( \mathbb{P}^{k+l+1} \). We now recall its construction.

Fix two rational normal curves of degrees \( k \) and \( l \) in \( \mathbb{P}^{k+l+1} \) with disjoint linear spans. Fix an isomorphism between the curves. \( S_{k,l} \) is the surface swept by the lines joining the points corresponding under the isomorphism. The degree \( k \) curve is called the directrix. If the directrix reduces to a point, we obtain \( S_{0,l} \), the cone over a rational normal \( l \) curve. We will call a scroll balanced if \( l - k \leq 1 \), and perfectly balanced if \( k = l \). A perfectly balanced scroll has a one-parameter family of directrices.

Rational normal scrolls are non-degenerate surfaces of minimal degree in projective space. Conversely,

**Proposition 2.1** ([GH] p.525) Every non-degenerate irreducible surface of degree \( m - 1 \) in \( \mathbb{P}^m \) is a rational normal scroll or the Veronese surface in \( \mathbb{P}^5 \).

Hirzebruch surfaces. The Hirzebruch surface \( F_r, r \geq 0 \), is the projectivization of the vector bundle \( \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(r) \) over \( \mathbb{P}^1 \). In this paper the projectivization of a vector bundle \( \mathbb{P} V \) will mean the one-dimensional subspaces of \( V \).

The Picard group of \( F_r, r > 0 \), is generated by two classes: the class \( f \) of a fiber \( F \) of the projective bundle and the class \( e \) of the unique section \( E \) with negative self-intersection. The intersection pairing is given by

\[
f^2 = 0, \quad f \cdot e = 1, \quad e^2 = -r.
\]

The surface \( F_0 \) does not have a section with negative self-intersection; however, the same description holds for its Picard group. The canonical class of \( F_r \) is

\[
K_{F_r} = -2e - (r + 2)f.
\]

**The automorphism group of \( F_r \).** The automorphism group of \( F_r \), for \( r > 0 \), surjects onto \( \mathbb{P} \mathbb{GL}_2(\mathbb{C}) \). The kernel is the semidirect product of \( \mathbb{C}^* \) with \( H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(r)) \) where the former acts on the latter by multiplication. Consequently, the dimension of the automorphism group of \( F_r \) is \( r + 5 \). \( F_0 \) is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \). The automorphism group of \( \mathbb{P}^1 \times \mathbb{P}^1 \) is the semidirect product of \( \mathbb{P} \mathbb{GL}_2(\mathbb{C}) \) with \( \mathbb{Z}/2\mathbb{Z} \), hence has dimension 6.

The relation between the scrolls and the Hirzebruch surfaces is provided by

**Lemma 2.2** The scroll \( S_{k,l} \) is the image of the Hirzebruch surface \( F_{l-k} \) under the complete linear series \( |\mathcal{O}_{F_{l-k}}(e + f)| \). For \( k \neq 0, l \), the image of the curve \( E \) is the unique rational normal \( k \) curve on the scroll. The fibers \( F \) are mapped to lines. Irreducible curves in the class \( e + (l - k)f \) map to rational normal \( l \) curves with linear span disjoint from the linear span of the image of \( E \).

**Section classes.** During degenerations of scrolls it will be essential to determine the limits of their hyperplane sections. When scrolls become reducible, their hyperplane sections remain in section classes.
**Definition 2.3** On a Hirzebruch surface $F_r$, a cohomology class of the form $e + mf$ is called a section class.

Irreducible curves in a section class are sections of the projective bundle. More generally, any curve in a section class consists of a section union some fibers. On $S_{k,l}$ the sections of degree at most $k + l$ are rational normal curves. In particular, the irreducible hyperplane sections are rational normal $k + l$ curves. Our description of the cohomology ring of $F_r$ and Lemma 2.2 imply that

**Lemma 2.4** A curve of degree $d$ on a scroll $S_{k,l}$ that has intersection multiplicity one with fibers is an element of the linear series $|e + (d - k)f|$ on $F_{l-k}$.

**Cohomology calculations.** Since $O_E(e + mf) \cong O_{P^1}(m - r)$ the long exact sequence associated to the sequence

$$0 \to O_{F_r}(mf) \to O_{F_r}(e + mf) \to O_E(e + mf) \to 0$$

implies that if $m \geq r - 1$, then

$$H^1(F_r, O_{F_r}(mf)) \to H^1(F_r, O_{F_r}(e + mf))$$

is surjective; and if $0 \leq m \leq r - 1$, then

$$h^1(O_{F_r}(e + mf)) = h^1(O_{F_r}(mf)) + h^1(O_{P^1}(m - r)).$$

Since $F_r$ is a rational surface $H^1(F_r, O_{F_r}) = 0$. Using the exact sequence

$$0 \to O_{F_r}(mf) \to O_{F_r}((m + 1)f) \to O_{P^1} \to 0$$

we conclude that $H^1(O_{F_r}(mf)) = 0$ for $m \geq 0$ by induction on $m$. We, thus, compute the dimensions of all the cohomology groups for the line bundles $O_{F_r}(e + mf)$, $m \geq 0$.

**Lemma 2.5** The projective dimension of the linear series $|e + mf|$ on $S_{k,l}$, $m \geq 0$ is given by

$$r(e + mf) = \max(k - l + 2m + 1, m).$$

(1)

**Remark:** The preceding discussion proves that when $m < l - k$ the only curves in the section classes $e + mf$ consist of the directrix $E$ union $m - k$ fibers. However, when $m \geq l - k$, the same dimension count implies that there must be irreducible curves in the class $e + mf$.

**Lemma 2.6** The dimension of the locus in the Hilbert scheme whose general point represents a smooth scroll $S_{k,l}$ in $P^N$ is

$$(k + l + 2)N + 2k - 4 - \delta_{k,l}.$$
Since $\mathbb{P}GL(N+1)$ acts transitively on the non-degenerate scrolls $S_{k,l}$, Kleiman’s theorem assures us that if we pick general linear subspaces $\Lambda_i \subset \mathbb{P}^N$ of codimension $c_i$ such that
\[
\sum_i (c_i - 2) = (k + l + 2)N + 2k - 4 - \delta_{k,l}
\]
then there will be finitely many scrolls $S_{k,l}$ meeting all $\Lambda_i$. In the rest of the paper, we address the question of determining this number. Since we will appeal to Kleiman’s theorem \cite{Kl1} frequently, we recall it for the reader’s convenience.

**Theorem 2.7 (Kleiman)** Let $G$ be an integral algebraic group scheme, $X$ an integral algebraic scheme with a transitive $G$ action. Let $f : Y \to X$ and $g : Z \to X$ be two maps of algebraic schemes. For each rational element $s$ of $G$, denote by $sY$ the $X$-scheme given by $y \mapsto sf(y)$.

There exists a dense open subset $U$ of $G$ such that for every rational element in $U$, the fibered product $(sY) \times_X Z$ is either empty or equidimensional and its dimension is the expected dimension
\[
dim(Y) + \dim(Z) - \dim(X).
\]
Furthermore, for a dense open set this fibered product is regular.

**Remark:** Although we stated the theorem in the language of schemes, its proof holds without change for algebraic stacks.

### 3 A Compactification of the Space of Scrolls

In this section we describe a compactification of the space of rational scrolls given by the Kontsevich moduli space of genus 0 stable maps to the Grassmannian.

**Scrolls as curves in the Grassmannian.** To study the geometry of scrolls it is useful to think of them as rational curves in the Grassmannian $G(1,N)$ of lines in $\mathbb{P}^N$.

$S_{k,l}$ is a projective bundle over $\mathbb{P}^1$. The fibers of the projection map $\pi : S_{k,l} \to \mathbb{P}^1$ are lines in $\mathbb{P}^N$. Hence, $\pi$ induces a rational curve of Plücker degree $k + l$ in $G(1,N)$. More explicitly, consider the incidence correspondence

\[
\Phi = \{(p, [L]) : p \in \mathbb{P}^1, [L] \in G(1,N), L \subset S_{k,l}, \pi(L) = p \} \subset \mathbb{P}^1 \times G(1,N).
\]

The image of $\Phi$ under the projection of $\mathbb{P}^1 \times G(1,N)$ to the second factor gives us the required rational curve $C \subset G(1,N)$.

Conversely, given an irreducible, reduced rational curve $C$ of degree $k+l$ in $G(1,N)$ we can construct a rational ruled surface of degree $k + l$ in $\mathbb{P}^N$. Consider the projectivization of the tautological bundle $T$ of $G(1,N)$ over the curve $C$

\[
\Psi = \{([L_c], p) : p \in L_c, c \in C \} \subset C \times \mathbb{P}^N \subset G(1,N) \times \mathbb{P}^N.
\]

Projection to the second factor gives a surface $S$ of degree $k + l$ in $\mathbb{P}^N$. If the span of $S$ is $\mathbb{P}^{k+l+1}$, then by Proposition \ref{prop:scrolls} the surface is a rational normal scroll. If the span of $S$ is smaller, then $S$ is the projection of a rational normal scroll from a linear subspace of $\mathbb{P}^N$.

**Non-degenerate curves.** The span of the surface has dimension smaller than $k+l+1$ if and only if the curve is contained in a $G(1,r)$ for some $r < k + l + 1$. We will refer
to rational curves \( C \subset \mathbb{G}(1, N) \) which do not lie in any \( \mathbb{G}(1, r) \) for \( r < k + l + 1 \) as non-degenerate rational curves in the Grassmannian.

**Non-isomorphic scrolls of the same degree.** The correspondence between rational curves in \( \mathbb{G}(1, N) \) and scrolls in \( \mathbb{P}^N \) does not yet differentiate between non-isomorphic scrolls that have the same degree. The automorphism group of \( \mathbb{G}(1, N) \) does not act transitively on non-degenerate rational curves. The restriction of the tautological bundle \( T \) of \( \mathbb{G}(1, N) \) to different curves can have different splitting types.

Let \( \phi : \mathbb{P}^1 \rightarrow C \) be the normalization of \( C \). Consider the vector bundle \( V = \phi^* T \) on \( \mathbb{P}^1 \).

**Definition 3.1** We define the degree \( k \) of the summand of minimal degree in the decomposition of \( V \rightarrow \mathbb{P}^1 \) to be the directrix degree of \( C \).

The directrix degree distinguishes curves associated to non-isomorphic scrolls. Suppose \( C \subset \mathbb{G}(1, N) \) is an irreducible, non-degenerate curve of directrix degree \( k \), then the projectivization of \( V \rightarrow \mathbb{P}^1 \) is isomorphic to \( \mathbb{F}_{l-k} \). The reverse construction shows that there is a natural bijection between the set of scrolls \( S\kappa,l \) in \( \mathbb{P}^N \) and the set of non-degenerate rational curves of degree \( k + l \) and directrix degree \( k \) in \( \mathbb{G}(1, N) \).

\( S_{0,1} \) and \( S_{1,1} \). Unlike other scrolls, \( \mathbb{P}^2 \) and a smooth quadric \( Q \subset \mathbb{P}^3 \) have more than one scroll structure. \( \mathbb{P}^2 \) can be given the structure of \( S_{0,1} \) in a two parameter family of ways depending on the choice of the vertex point. The quadric surface has two distinct \( S_{1,1} \) structures depending on the choice of ruling on the quadric surface. The correspondence between scrolls and rational curves in the Grassmannian differentiates between these scroll structures.

**A compactification of the space of scrolls.** Using the preceding discussion we can compactify the space of \( S\kappa,l \) using the Kontsevich space of stable maps.

Let \( S \subset \text{Hilb}(\mathbb{P}^N, k+l) \times \mathbb{P}^1 \) denote the component (with its reduced induced structure) of the Hilbert scheme which parametrizes rational normal scrolls. Let \( S \subset S \) denote the open subscheme whose points represent reduced, irreducible, non-degenerate scrolls. Let \( C \subset \mathbb{M}_{0,0}(\mathbb{G}(1, N), k+l) \) be the locus in the Kontsevich moduli scheme of stable maps whose points represent injective maps from an irreducible \( \mathbb{P}^1 \) to a non-degenerate curve in \( \mathbb{G}(1, N) \) of Plücker degree \( k + l \). This locus is contained in the automorphism free locus.

**Theorem 3.2** When \( k + l > 2 \), there is a natural isomorphism between \( S \) and \( C \) taking the locus of \( S\kappa,l \) to maps to curves of directrix degree \( k \).

**Proof:** Projection to the second factor from the incidence correspondence \( \Phi \) induces a morphism from \( S \) to \( C \). We already observed that this morphism is a bijection on points. Since \( C \) is a smooth, quasi-projective variety \( \mathbb{P}^1 \), Zariski’s Main Theorem implies that this morphism is an isomorphism. \( \square \)

**Remark 1.** When \( k + l \leq 2 \), Theorem 3.2 is still valid if instead of the Hilbert scheme we use the space of pointed planes for \( k + l = 1 \) and the space of quadric surfaces with a choice of ruling when \( k + l = 2 \).

**Remark 2.** Theorem 3.2 implies that \( S \) is smooth. Note that the Hilbert scheme can be singular along subloci of \( S \). For example, the Hilbert scheme of quartic scrolls is singular along the locus of rational quartic cones—the component corresponding to Veronese surfaces meets the component of scrolls along that locus.
Theorem 3.2 provides us with a compactification of the space of scrolls $S_{k,l}$. For balanced scrolls we can take the Kontsevich moduli space of stable maps. A Zariski-open set of $\overline{M}_{0,0}(\mathbb{G}(1,N),k+l)$ corresponds to maps from an irreducible $\mathbb{P}^1$ to a curve of directrix degree $\left\lfloor \frac{k+l}{2} \right\rfloor$. For other scrolls we take the scheme-theoretic closure of the locus of maps from $\mathbb{P}^1$ to $\mathbb{G}(1,N)$ whose image has directrix degree $k$.

4 Limits of Scrolls in One Parameter Families

In this section we describe the limits of scrolls and their section classes in one parameter families.

One parameter families of scrolls. Let $X \to B$ denote a flat family of surfaces over a smooth, connected base curve $B$. We assume that every member of the family except for the central fiber $X_0 \to b_0 \in B$ is a scroll $S_{k,l}$. To simplify the statements we assume that the surface underlying $X_0$ is still non-degenerate. This assumption can be weakened by considering projections.

Proposition 4.1 The special fiber $X_0$ is a connected surface whose irreducible components are scrolls $S_{k_i,l_i}$. $X_0$ is the image of a union of Hirzebruch surfaces whose dual graph is a connected tree. The indices $k_i,l_i$ satisfy the constraints:

1. $\sum_i (k_i + l_i) = k + l$
2. $\sum_i k_i \leq k$

Proof: Since the family $X \to B$ is flat the central fiber $X_0$ has to be a connected surface of degree $k + l$. The family $X \to B$ gives rise to a family of curves $Y \to B \setminus b_0$ in $\mathbb{G}(1,N)$, hence to a curve in $\overline{M}_{0,0}(\mathbb{G}(1,N),k+l)$. Since the latter is complete we can extend the family over $b_0$ by a stable map to $\mathbb{G}(1,N)$. The projectivization of the pull back of the tautological bundle maps to $\mathbb{P}^N$ giving a family that agrees with $X$ except possibly at $X_0$. There is a unique scheme structure on the image that makes the family flat. Since over a smooth base curve there is a unique way to complete a family to a flat family ([Ha] III.9.8), this family must coincide with our original family. Hence the underlying surface of $X_0$ is still non-degenerate. This assumption can be weakened by considering projections.

Proposition 4.1 raises the question of which unions of scrolls can be the limits of $S_{k,l}$. We now describe two standard constructions of families of $S_{k,l}$ breaking into a collection of $S_{k_i,l_i}$. Using these inductively we can degenerate $S_{k,l}$ to a tree of surfaces with any $k_i,l_i$ satisfying the numerical conditions of Proposition 4.1. However, we cannot smoothly all such trees to an $S_{k,l}$.
Example: Cones provide the simplest counterexample. The limit of a family of cones is a union of cones whose vertices coincide. We can take two quadric cones meeting along a line, but whose vertices are distinct. This surface cannot be deformed to an $S_{0,4}$.

Construction 1. For any $k \geq r \geq 0$ there exists a flat family of scrolls $S_{k,l}$ specializing to $S_{k-r,l+r}$. To construct such a family it suffices to exhibit a flat family of vector bundles $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l)$ degenerating to $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(k+l-r)$. Since $r < k$ there exists a non-trivial injective bundle map from $\mathcal{O}_{\mathbb{P}^1}(r)$ to $\mathcal{O}_{\mathbb{P}^1}(k)$ giving rise to the extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(r) \to \mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l) \to \mathcal{O}_{\mathbb{P}^1}(k+l-r) \to 0.$$ 

This extension gives rise to a family $E_t$ of vector bundles whose general member is $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l)$, but $E_0 \cong \mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(k+l-r)$. Pick the one-dimensional subspace of $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-(k+l-r)) \otimes \mathcal{O}_{\mathbb{P}^1}(r))$ which contains the extension in question. This provides us with a family which gives $\mathcal{O}_{\mathbb{P}^1}(k) \oplus \mathcal{O}_{\mathbb{P}^1}(l)$ when $t \neq 0$ and $\mathcal{O}_{\mathbb{P}^1}(r) \oplus \mathcal{O}_{\mathbb{P}^1}(l+k-r)$ when $t = 0$.

For a more geometric description of a family of $S_{k,l}$ degenerating to $S_{k-1,l+1}$ consider a surface $S_{k,l+1}$. When we project the surface from a point away from the directrix, we obtain $S_{k,l}$. However, when we project the surface from a point on the directrix we obtain $S_{k-1,l+1}$. Now projecting $S_{k,l+1}$ from the points along a curve that meets the directrix in isolated points, we obtain the desired family. This construction easily generalizes to $r > 1$.

Construction 2. There exists a family of scrolls $S_{k,l}$ degenerating to the union of $S_{k_1,l_1}$ and $S_{k_2,l_2}$ with $k_1 + k_2 = k$. We think of scrolls as projectivizations of vector bundles over $\mathbb{P}^1$. We choose a flat family of $\mathbb{P}^1$s with smooth total space over the unit disk whose general member is smooth, but whose central fiber has two components meeting transversely at one point. Given a line bundle $\mathcal{O}_{\mathbb{P}^1}(k)$ on the general fiber there is always a limit line bundle on the special curve. However, the limit does not have to be unique. Limits differ by twists of one of the components of the reducible fiber. We can get any splitting of $k$ on the two components. A similar consideration applies for $\mathcal{O}_{\mathbb{P}^1}(l)$. Taking the desired splitting and projectivizing gives us the desired family of Hirzebruch surfaces.

Remark: Since $\overline{M}_{0,0}(G(1,N),k+l)$ is an irreducible, smooth Deligne-Mumford stack, every union of scrolls whose dual graph is a connected tree can be smoothed to an $S_{k,l}$ for some $k$ and $l$. However, the minimal $k$ depends on the alignment of directrices on the reducible surface (see Example preceding Construction 1). This is the phenomenon we would like to analyze next.

The limits of section classes. Let $\mathcal{X} \to B$ be a flat family of scrolls subject the hypotheses in the first paragraph of §4. Let $\mathcal{C} \to B$ be a flat family of curves such that $\mathcal{C}_b \subset \mathcal{X}_b$ is a smooth curve in a section class for $b \neq b_0$. We say a curve on $S_{0,l}$ is in a section class if it is the image of a curve in a section class on $F_l$.

Proposition 4.2 The limit $\mathcal{C}_{b_0}$ restricts to a section class on each component of $\mathcal{X}_b$.

Proof: By Proposition 4.1 the central fiber $\mathcal{X}_0$ is the union of scrolls, so it is meaningful to require the restriction of a curve to a component to be in a section class. Since meeting the fibers is a closed condition, $\mathcal{C}_{b_0}$ meets each fiber. To see that it does not meet the general fiber more than once (away from the cone point of any $S_{0,l}$), consider the one parameter family $\mathcal{Y} \to B$ of rational curves in $G(1,N)$.
corresponding to our family of surfaces. Every component of the central fiber in this family is reduced. Hence, the total space of the family cannot be singular along an entire component. Each curve $C_b$, for $b \neq b_0$, maps isomorphically to $V_b$. By Zariski’s Connectedness Theorem the fibers of the map are connected over the smooth locus. The proposition follows. □

Fact from Intersection Theory. Finally, we recall a fact from intersection theory (see [Ful] chapter 12) that will be helpful in determining limits of section classes. Let $C_1$ and $C_2$ be two flat families of curves contained in a flat family of surfaces $X \to B$ over a smooth base curve. Assume that the general fiber of the family is smooth and that on the general fiber $C_{1,b}$ and $C_{2,b}$ are smooth curves that meet transversely at $\gamma$ points. Let $I_0 \subset X_0$ denote the set of isolated points of intersection of $C_{1,0}$ and $C_{2,0}$ contained in the smooth locus of $X_0$. Let $i_p$ denote the intersection multiplicity at the point $p$.

Lemma 4.3 The intersection multiplicities satisfy the inequality

$$\gamma \geq \sum_{p \in I_0} i_p(C_{1,0}, C_{2,0}).$$

Limits of directrices. Proposition 4.2 allows us to determine the limits of directrices as scrolls degenerate. To ease the exposition first assume that the central fiber of Proposition 4.2 allows us to determine the limits of directrices. Proposition 4.4

$\text{Proposition 4.4}$

allows us to determine the limits of directrices of a perfectly balanced scroll, we mean ‘a’ directrix. The intersection multiplicities satisfy the inequality

Lemma 4.3

The case $k_1 + k_2 < k$. Since we are assuming that $k_1 + l_2 \leq k_2 + l_1$, $D$ must consist of the directrix in $S_{k_1,l_1}$ and a section of degree $k - k_1$ in $S_{k_2,l_2}$. Set $j = \cdots$
Consider a family of \( S_{k,l+k-k_1-k_2} \) over a small disk in the complex plane degenerating to the union of \( S_{k_1,l_1} \) and \( S_{k-k_1,m} \) as described in Construction 2. By our discussion the directrices must specialize to the union of the directrices. Now choose \( k - k_1 - k_2 \) general sections of the family that all pass through general points of \( S_{k-k_1,j} \). (Observe that the total space of the family we exhibited in Construction 2 is generically smooth on every component of the special fiber, so we can select such sections.) Projecting the family along these sections gives a family of \( S_{k,l} \) having the desired numerical properties.

We need to verify that we can get all sections \( C \) of degree \( k - k_1 \) on \( S_{k_2,l_2} \) as the projection of the directrix of \( S_{k-k_1,j} \) from suitable points on the surface. On \( S_{k_2,l_2} \) blow up \( k - k_1 - k_2 \) general points \( p_i \) on \( C \). Let \( \Xi \) be the sum of the exceptional divisors. The linear series \( e + (k - k_1 - k_2)f - \Xi \) maps the blow up to projective space as \( S_{k-k_1,j} \). This map contracts the fibers passing through \( p_i \) and maps \( C \) to the directrix. Projecting \( S_{k-k_1,j} \) from the points corresponding to the image of the contracted fibers projects \( S_{k-k_1,j} \) to \( S_{k_2,l_2} \) and the directrix onto \( C \). This concludes the construction.

Now we treat the case when \( D \) contains a multiple of the common line \( L \). As in the previous case we can assume that \( D \) consists of two sections and \( L \) with multiplicity \( m \). \( D \) must contain the directrix in \( S_{k_1,l_1} \) and a section of degree \( k-k_1-m \) in \( S_{k_2,l_2} \). Using the previous case inductively, we can find a family of scrolls \( S_{k+m,l+m} \) degenerating to a chain of \( S_{k_1,l_1} \) \( m \) quadric surfaces and \( S_{k_2,l_2} \) such that their directrices specialize to the directrix of \( S_{k_1,l_1} \), a section of degree \( k-k_1-m \) in \( S_{k_2,l_2} \) and a chain of conics on the quadric surfaces connecting these two curves. Choose \( 2m \) general sections which specialize to a pair of points on a fiber on each quadric surface. Projecting the family from those sections gives the desired family. \( \square \)

**Remark:** We can inductively extend Proposition 4.4 to the case when the central fiber contains more than two components. The following theorem summarizes the conclusion:

**Theorem 4.5** Suppose a one parameter family of scrolls \( S_{k,l} \) specializes to the union \( \bigcup_{i=1}^r S_{k_i,l_i} \). Then the limit of the directrices is a connected curve of degree \( k \) whose restriction to each surface is in a section class. Conversely, given any connected curve \( C \) of degree \( k \leq \sum_i (k_i + l_i)/2 \) whose restriction to each component is in a section class, there exists a one parameter family of \( S_{k,l} \) specializing to the reducible surface such that the limit of the directrices is \( C \).

**Limits of other section classes.** When a family \( \mathcal{X} \rightarrow B \) of scrolls \( S_{k,l} \) specializes to a union of two scrolls \( S_{k_1,l_1} \cup S_{k_2,l_2} \), then the flat limit of curves in a section class \( e + mf, m \geq l - k \), are connected curves of total degree \( m + k \) that restrict to section classes. Suppose that the total space of the family is smooth. The curves give a line bundle \( L \) over \( \mathcal{X} \setminus X_0 \). We can always extend this line bundle to the entire family. However, when the central fiber is reducible, this extension is not unique. Twisting by the components of the central fiber give different extensions.

For concreteness, suppose the line bundle \( L \) is the pull-back of \( \mathcal{O}_{\mathbb{P}^N}(1) \) to \( \mathcal{X} \setminus X_0 \). Let \( L_0 \) be the line bundle over \( X_0 \) arising from any extension of \( L \). If the restriction of \( L_0 \) to each component is effective, then the degree of \( L_0|_{S_{k_1,l_1}} \) ranges between \( k_1 \) and \( k + l - k_2 \). One component of the limit curves corresponds to hyperplanes not containing either of the components of the limit scroll. The other limits correspond to hyperplane sections by hyperplanes containing one of the scrolls and tangent to a certain order to the other one along their common line.
Example: When a family of scrolls $S_{2,4}$ specialize to $S_{1,1} \cup S_{1,3}$, a possible limit of the hyperplane sections restricts to a degree 5 curve on $S_{1,1}$ and the directrix on $S_{1,3}$. The dimension of such curves is 8. However, the dimension of the hyperplane sections was only 7. Consequently, not all quintics can arise as the limit of hyperplane sections of our family of scrolls. Lemma 4.3 provides the answer. On the smooth surfaces the directrices have intersection number 2 with the hyperplane sections. The limit of the directrices is the union of the directrix $D$ of $S_{1,3}$ and the directrix $L$ on $S_{1,1}$ meeting $D$. A general quintic meeting $D$ meets $L$ transversely in 3 other points. We conclude that a quintic can be part of a limit of the hyperplane section only if it has contact of order 2 with $L$ at their point of intersection on the common fiber.

The following lemma, which is an immediate consequence of Lemma 4.3 and Proposition 4.2, summarizes the general situation.

Lemma 4.6 Suppose a one parameter family of scrolls $S_{k,l}$ specializes to a union $\bigcup_{i=1}^r S_{k_i,l_i}$. Then the limit of curves in a section class $e + mf$ specialize to connected curves of degree $m + k$. Their restrictions to each surface lie in a section class. Furthermore, the sum of the intersection multiplicities of a limit curve with the limit of the directrices at isolated points of their intersection on the smooth locus of the surface cannot exceed $m + k - l$.

5 Examples

Example A. Counting cubic scrolls in $\mathbb{P}^4$. Since by Lemma 2.0 there is an 18 dimensional family of cubic scrolls $S_{1,2}$ in $\mathbb{P}^4$, there are finitely many containing $9 - n$ general points and meeting $2n$ general lines.

A1. Cubic scrolls containing 9 general points. We specialize the nine points one by one to a fixed hyperplane $H$ in $\mathbb{P}^4$ (see Figure 2). We can take $H$ to be the span of four of the points.
**Step I.** Specialize a fifth point \( p_5 \) to a general point of \( H \). There are no reducible scrolls at this stage containing all the points. Any reducible scroll would be the union of a quadric and a plane meeting along a line. Since no six of the points lie in a \( \mathbb{P}^3 \), the quadric could contain at most 5 of the points. However, the remaining 4 points do not lie on a plane.

**Step II.** Specialize a sixth point \( p_6 \) to \( H \). Now there is a reducible solution: the plane \( P \) spanned by the 3 points outside \( H \) and the unique quadric in \( H \) containing \( H \cap P \) and the six points in \( H \). However, at this stage there might still be irreducible solutions. Their hyperplane section in \( H \) must be the unique twisted cubic \( C \) that contains the 6 points in \( H \).

**Step III.** Specialize a seventh point \( p_7 \) to a general point of \( H \). Bezout’s theorem forces the scrolls to break into a union of a quadric surface and a plane. The quadric \( Q \) must contain the twisted cubic \( C \) and \( p_7 \). Since the plane and the quadric meet in a line, \( Q \) must also contain the point of intersection \( q \) of \( H \) with the line spanned by the points outside \( H \). This determines the quadric uniquely. The plane is also determined because it must contain the line in the quadric through \( q \) which meets \( C \) only once: recall by Lemma 4.2 the curve \( C \) is in a section class.

Later we will check that both of the solutions occur with multiplicity 1. This will prove that there are 2 cubic scrolls containing 9 points in \( \mathbb{P}^4 \).

**A2. Cubic scrolls in \( \mathbb{P}^4 \) containing 6 points and meeting 6 lines.** Here we sketch the degenerations required to see that there are 1140 cubic scrolls in \( \mathbb{P}^4 \) containing 6 points and meeting 6 lines until we reduce the problem to a straightforward problem about quadrics and planes (see Figure 3).

**Step I.** We specialize 5 points and a line \( l_1 \) to a fixed hyperplane \( H \). This is the first stage where reducible solutions exist. There can be a quadric \( Q \) contained in \( H \) and a plane \( P \) outside \( H \) meeting \( Q \) in a line. There are 4 possibilities: of the 5 lines outside \( H \), 4, 3, 2, or 1 of them can meet \( Q \) and 1, 2, 3, or 4 remaining lines, respectively, can meet \( P \).

If 4 of the lines meet \( Q \), then \( Q \) is determined uniquely. \( P \) can be any of the 4 planes meeting the remaining line, containing the remaining point and meeting \( Q \) in a line. Since \( l_1 \) meets \( Q \) in 2 points, each of the solutions count twice for the choice of point. Finally, we have a factor of 5 for the choice of which 4 lines among the 5 meet the quadric \( Q \). We express this as \( 5 \times 2 \times 4 = 40 \) where the first multiple is the combinatorial multiplicity for the choice of lines, the second multiple is for the choice of point and the last number is the number of surfaces satisfying the incidence conditions. The analysis of the other three cases is similar.

At this stage some scrolls can remain irreducible. Their hyperplane section \( C \) in \( H \) is then a twisted cubic containing the 5 points and meeting \( l_1 \).

**Step II.** We specialize another line \( l_2 \) to \( H \). There are new reducible solutions.

**Case i.** There can be a plane \( P \) in \( H \) and a quadric \( Q \) meeting it along a line. \( P \) must be one of the 10 planes spanned by 3 of the points in \( H \). Finally, \( Q \) must contain the other two points in \( H \), meet \( P \) in a line, and meet the lines and the point \( p \) lying outside \( H \). Further specialization shows that there are 6 such quadrics. Hence we get 60 solutions.

**Case ii.** There can be a quadric \( Q \) in \( H \) and a plane \( P \) outside meeting \( Q \) in a line. There are four possibilities: 1, 2, 3 or 4 of the lines can meet \( P \). Let us analyze the case when 3 lines meet \( Q \). The limit hyperplane section \( C \) contained in \( Q \) must meet 5 points and \( l_1 \). Among the the quadrics containing 8 points only a finite number contains such a twisted cubic. To determine the number we specialize the conditions on the cubic curve.
Take a general 2 plane $\Pi$ and specialize 3 of the points and $l_1$ to $\Pi$. By Bezout’s theorem $C$ has to break into a conic and a line. The line can be any of the three lines containing two of the points in $\Pi$ or it can be the line joining the two points outside of $\Pi$. Once we require $Q$ to contain any of the lines, it is uniquely determined. When there is a line $l$ in $\Pi$, the limit of $C$ meets $l_1$ only in $l \cap l_1$. When the line $l$ is outside $\Pi$, the limit of $C$ has a conic in $\Pi$ which must meet $l_1$ in 2 points, hence we get a multiplicity of 2. The other cases are analogous.

Figure 3: Example A2. Cubic scrolls in $\mathbb{P}^4$ containing 6 points and meeting 6 lines.

By Lemma 4.2 the curve $C$ must be in a section class in $Q$, hence should meet the fibers only once. When counting quadric surfaces, one has to be careful to distinguish between the rulings.

**Step III.** Finally, there can still be irreducible scrolls. In that case their hyperplane section must be one of the 5 twisted cubics in $H$ containing 5 points and meeting $l_1$ and $l_2$ (see §2.3 of [V2]). The analysis is similar to the previous cases.

**Example B. Counting quadric surfaces in $\mathbb{P}^4$.** The degeneration method allows us to count different types of scrolls. We illustrate this by counting quadric surfaces and quadric cones in $\mathbb{P}^4$ (see Figure B).

**B1. Quadric surfaces in $\mathbb{P}^4$ containing 3 points and meeting 7 lines. Step I.** Specialize the three points $p_1, p_2, p_3$ and a line $l_1$ to the hyperplane $H$. At this
stage there is a unique quadric contained in $H$ satisfying all the incidence conditions. It counts with multiplicity 2 for the choice of intersection point with $l_1$.

If a quadric is not contained in $H$, its hyperplane section must lie in the plane $\Pi$ spanned by the points $p_1, p_2, p_3$ and must meet $l_1$ at $l_1 \cap \Pi$.

**Step II.** We specialize $l_2$. The quadric can lie in $H$. If not, the hyperplane section must be the unique conic containing $p_i$ and $l_i \cap \Pi$. Specializing a third line $l_3$ forces the quadrics to either become reducible or to lie in $H$. We obtain a total of 9 quadric surfaces containing 3 points and meeting 7 lines.

**Figure 4:** Example B. Counting quadric surfaces containing 3 points and meeting 7 lines and quadric cones containing 3 points and meeting 6 lines.

**B2. Quadric cones in $\mathbb{P}^4$ containing 3 points and meeting 6 lines.** We compare the case of quadric cones to the case of quadric surfaces.

**Step I.** Specialize the points $p_1, p_2, p_3$ and the line $l_1$ to $H$. The cone can lie in $H$. There are 4 quadric cones in $\mathbb{P}^3$ containing 8 general points. Each solution counts with multiplicity 2 for the choice of intersection point with $l_1$.

**Step II.** If a cone does not lie in $H$, its hyperplane section in $H$ must lie in the plane $\Pi$ spanned by $p_i$, so it must meet $p_i$ and $q = l_1 \cap \Pi$. We specialize another line $l_2$ to $H$. The cone can lie in $H$. Again there are 4 solutions each counted with multiplicity 2.

There can also be reducible solutions: the union of $\Pi$ and one of the three planes that meet $\Pi$ in a line and meet the three lines not contained in $H$. This case is delicate. The limit of the hyperplane sections is a conic containing $p_i$ and $q$. The two planes are images of Hirzebruch surfaces $F_1$ whose directrices are contracted. The image conic is in the class $e + 2f$ on the $F_1$ it lies in. It also meets the directrix of the other $F_1$. Hence, the limit of the hyperplane section has to contain $p_i$ and $q$ and
be tangent to the line common to the planes at the limit of the vertices of the nearby cones. There are two conics containing 4 points and tangent to a line in \( \mathbb{P}^2 \). The two points of tangency give us the possible limiting positions of the vertices of our original family of cones. So each of the pairs of planes can be the limit of cones in two ways depending on the choice of the vertex point. We thus get a count of 6.

**Step III.** If the cone is neither reducible nor contained in \( H \), then the hyperplane section in \( H \) is determined. Specializing a third line \( l_3 \) forces the cones to break or to lie in \( H \). The calculations are analogous to the previous case. We obtain a total of 30 quadric cones containing 3 points and meeting 6 lines.

**Example C. Counting scrolls of degree \( n \) in \( \mathbb{P}^{n+1} \) containing \( n + 5 \) points and meeting an \( n - 3 \) plane.** We give a final example to illustrate the types of recursive formulae one might hope to obtain from our method of counting. Observe that by Lemma 2.6 there will be finitely many scrolls \( S_{\lfloor \frac{n+1}{2} \rfloor} \) in \( \mathbb{P}^{n+1} \) containing \( n + 5 \) points and meeting an \( n - 3 \) plane \( P \). Let us denote this number by \( S(n) \).

Figure 5: Counting degree \( n \) scrolls in \( \mathbb{P}^{n+1} \) containing \( n + 5 \) points and meeting an \( n - 3 \) plane.

**Step I.** We specialize the points to a hyperplane \( H \). The first irreducible solution occurs when \( n + 3 \) of the points lie in \( H \): a plane outside \( H \) and a scroll of degree \( n - 1 \) in \( H \) meeting the plane in a line. The degree \( n - 1 \) scroll has to meet \( q \), the point of intersection of \( H \) with the span of the two points outside \( H \). If the scroll meets \( P \), then we reduce to the same problem in degree one less, so the number is \( S(n-1) \).

If the plane meets \( P \), then the scroll must contain a line \( l \) in \( P' = H \cap P \) containing \( q \). We have to count scrolls of degree \( n - 1 \) in \( \mathbb{P}^n \) containing a line through a point in a \( \mathbb{P}^{n-2} \) and containing an additional \( n + 3 \) points in general position. An easy specialization shows that there are \( n - 2 \) such scrolls.

If the scrolls remain irreducible, then their hyperplane section \( C \) in \( H \) is determined. When we specialize the \( n - 3 \) plane \( P \) to \( H \) the scroll breaks into a degree \( n - 1 \) scroll union a plane. We are reduced to counting degree \( n - 1 \) scrolls containing \( C \), a point and meeting \( P \). It is easy to see that there are \( n - 2 \) such scrolls. Solving the recursion we conclude that there are \( (n-1)(n-2) \) degree \( n \) scrolls in \( \mathbb{P}^{n+1} \) containing \( n + 5 \) points and meeting an \( n - 3 \) plane.

We will now justify the calculations made above by making the necessary dimension counts and multiplicity calculations. In a table at the end of the paper we will provide some other characteristic numbers of surfaces.
In this section we describe the set theoretical limits of surface degenerations under the assumption that the surfaces and their successive hyperplane sections remain non-degenerate. We will compute the dimension of images of maps from trees of Hirzebruch surfaces to $\mathbb{P}^N$ and determine the codimension one loci in the Hilbert scheme of scrolls when we require one of their points to lie in a fixed hyperplane. The calculation in this section will be purely set theoretic. We collect our notation here for the reader’s convenience.

**Notation:** Let $H$ and $\Pi$ be general hyperplanes in $\mathbb{P}^N$. In our algorithm we will specialize a linear space meeting either a surface or a curve on a surface to a hyperplane.

The dimension of a linear space will be indicated by a subscript. We will often omit the dimension from the notation.

Let $\{\Delta_i\}_{i=1}^l$, $\{\Sigma_i\}_{i=1}^l$ and $\{\Lambda_i\}_{i=1}^l$ be three collections of general linear subspaces of $\mathbb{P}^N$. These will be the linear spaces that we have not yet specialized to the hyperplane and meet the surface, meet the marked curve and meet and contain a fiber, respectively.

Let $\{\Gamma_j\}_{j=1}^I$ and $\{\Omega_j\}_{j=1}^I$ be collections of general linear subspaces of $H$ and $\Pi$, respectively. These will be the linear spaces that we have already specialized to the hyperplane.

$\Delta(0), \ldots, \Delta(M)$ will denote a partition of the $I$ linear spaces $\Delta_i$ into $M+1$ parts. We will use analogous notation for the other linear spaces.

**Spaces of maps.** We now define a sequence of spaces of maps from Hirzebruch surfaces to $\mathbb{P}^N$. Intuitively they will correspond to scrolls with two marked curves on them. The scrolls will meet certain linear spaces. The curves will meet some others. In addition some fibers of the scroll will be required to lie in a linear space and meet another linear space.

**Definition of $MS_H$.** Let

\[
MS_H(\mathbb{P}^N; k, l; C(k+1), D; \{\lambda_i\}_{i=1}^l, \{q_i\}_{i=1}^l, \{p_j \in C\}_{j=1}^l; \pi : \pi^{-1}(H) = C; \\
(\pi(\lambda_i) \subset \Lambda_i, \pi(\lambda_i) \cap \Lambda_i \neq \emptyset)_{i=1}^l, (\pi(q_i) \in \Delta_i)_{i=1}^l, (\pi(p_j) \in \Gamma_j)_{j=1}^l)
\]

be the set of maps, up to isomorphism, from a Hirzebruch surface $\mathbb{F}_{l-k}$, with two marked sections $(C, D)$, $Y$ marked fibers and $I$ marked points, to $\mathbb{P}^N$ whose image is an $S_{k,l}$ not contained in $H$ such that

1. $\pi(C) = \pi(S) \cap H$.
2. $D$ is the directrix (or if $k = l$, a choice of directrix).
3. The images of the marked fibers $\lambda_i$ are contained in the linear spaces $\Lambda_i$ and meet the linear spaces $\Lambda_i$.
4. $p_i$ are marked points on the surface and $q_j$ are marked points on the curve $C$.

We assume that they are distinct points whose images lie in the linear spaces $\Delta_i$ and $\Gamma_j$, respectively.

**Definition of $MS_\Pi$.** Similarly let

\[
MS_\Pi(\mathbb{P}^N; k, l; C(d), D; \{\lambda_i\}_{i=1}^l, \{q_i\}_{i=1}^l, \{q'_i \in C\}_{i=1}^l, \{p_j \in C\}_{j=1}^l; \pi : (\pi(\lambda_i) \\
\subset \Lambda_i, \pi(\lambda_i) \cap \Lambda_i \neq \emptyset)_{i=1}^l, (\pi(q_i) \in \Delta_i)_{i=1}^l, (\pi(q'_i) \in \Sigma_i)_{i=1}^l, (\pi(p_j) \in \Omega_j)_{j=1}^l)
\]
be the set of maps, up to isomorphism, from a Hirzebruch surface \( F_{l-k} \), with two marked sections \((C, D)\), \( Y \) marked fibers and \( I \) marked points, to \( \mathbb{P}^N \) whose image is an \( S_{k,t} \) such that

1. \( \pi(C) \) has degree \( d \).
2. \( D \) is the directrix (or if \( k = l \), a choice of directrix).
3. The images of the marked fibers \( \lambda_i \) are contained in the linear spaces \( \Lambda_i \) and meet the linear spaces \( \Lambda^i \).
4. \( q_i \) are marked points on the surface, \( q_j \) are marked points on the curve \( C \) whose images are not contained in \( \Pi \) and \( p_j \) are marked points on the curve \( C \) that map to \( \Pi \). We assume that they are distinct points whose images lie in the linear spaces \( \Delta^i \) and \( \Omega^j \), respectively.

We can compactify both \( MS_H \) and \( MS_{\Pi} \) in a manner analogous to §3. For concreteness we explain the construction for \( MS_H \). The case of \( MS_{\Pi} \) is identical. Let \( \mathbb{F}(0, 0, 1; N) \) denote the variety of two-pointed lines in \( \mathbb{P}^N \). A curve in \( \mathbb{F}(0, 0, 1; N) \) is determined by three degrees, the degrees \( (d_0, d_1) \) of the two projections \( \gamma_0, \gamma_1 \) to \( \mathbb{P}^N \) and the degree \( d_2 \) of the projection \( \gamma_2 \) to \( \mathbb{G}(1, N) \). Given a map \( \pi \) in \( MS \) from a surface \( S \) with sections \( C \) and \( D \), we get a stable map to \( \mathbb{F}(0, 0, 1; N) \) by sending \((S, C, D, \pi)\) to the map \( \sigma \) from \( C \) given by \( \sigma(p) = (\pi(p), \pi(D \cap F_p), [\pi(F_p)]) \) where \( F_p \) is the fiber through \( p \). We have to enhance this correspondence to mark the points on the surface. To do that we simply take the \( I \)-th fold fiber product of the universal family \( \mathcal{H} \) over the stack \( \mathcal{M}_{0,j_H+Y}(\mathbb{F}(0, 0, 1; N), (d_0, d_1, d_2)) \). We will denote the closure of \( MS_H \) and \( MS_{\Pi} \) in these stacks as \( \overline{MS}_H \) and \( \overline{MS}_{\Pi} \), respectively. When we do not want to distinguish between them we will use the notation \( \overline{M} \).

**Definition of the Divisors** \( D_H, D_{\Pi} \). In \( \overline{MS}_H \) requiring the stable map to map \( p_I \) into \( H \) defines a Cartier divisor. We will denote this divisor by \( D_H \). Similarly, \( \overline{MS}_{\Pi} \) has a Cartier divisor \( D_{\Pi} \) defined by requiring the image of \( q'_J \) to lie in \( \Pi \). Colloquially, the surfaces in the divisors are the surfaces we see after we specialize one of the points to a hyperplane.

**Definitions of** \( X_H \) and \( X_{\Pi} \). Let

\[
X_H(\mathbb{P}^N; (k_i, l_i; C(d_i), D(e_i), \lambda(i), q(i), p(i) \in C(i))_{i=0,1}; \pi)
\]

be the set of maps, up to isomorphism, from a pair of Hirzebruch surfaces \( F_{l-k} \), meeting transversely along a fiber with the usual markings such that

1. \( S_0 := \pi(F_{l-k_0}) \subset H \) is a scroll \( S_{k_0,l_0} \) contained in \( H \).
2. \( S_1 := \pi(F_{l-k_1}) \) is a scroll \( S_{k_1,l_1} \) not contained in \( H \) and which meets \( H \) transversally along the line that joins it to \( S_0 \).
3. \( C(d_i) \) is a section of degree \( d_i \) and \( D(e_i) \) is a section of degree \( e_i \) on \( S_i \) such that \( C(0) \cup C(1) \) and \( D(0) \cup D(1) \) form a connected curve.
4. \( \lambda(i), q(i) \) and \( p(i) \) is a partition of the marked fibers and points to the two components and they satisfy the same incidence and containment relations as in the definition of \( MS_H \).

The definition of \( X_{\Pi} \) is similar, but with \( H \) replaced by \( \Pi \) and the names of the linear spaces and points modified as in the definition of \( M_{\Pi} \). Denote the corresponding stacks in \( \overline{MS}_H \) and \( \overline{MS}_{\Pi} \) by \( X_H \) and \( X_{\Pi} \), respectively.

**Definitions of** \( W_H \) and \( W_{\Pi} \). Let

\[
W_H(\mathbb{P}^N; (k_i, l_i; C(d_i), D(e_i), \lambda(i), q(i), p(i) \in C(d_i))_{i=0,1}; \pi)
\]

be the set of maps, up to isomorphism, from the union of \( M + 1 \) Hirzebruch surfaces with the usual markings to \( \mathbb{P}^N \) such that
1. All the components \( S_i \) for \( i > 0 \) are attached along distinct fibers to a central component \( S_0 \).
2. \( \pi(S_0) \) is a cone not contained in \( H \).
3. \( C(0) \) and \( D(0) \) both contain the directrix of \( S_0 \).
4. \( \pi(S_i) \subset H \) for all other \( i \).
5. The fibers and marked points are distributed according to the partition and meet and lie in their designated linear spaces as in the definition of \( MS_H \).

The definition of \( W_{II} \) is obtained by making the usual modifications. We will denote the corresponding stacks by \( \mathcal{W}_H \) and \( \mathcal{W}_{II} \).

**Subschemes of \( \text{Hilb}(\mathbb{P}^N, S^{k+l}) \).** Let \( \text{Hilb}(\mathbb{P}^N, S^{k+l}) \) denote the component of rational normal scrolls of degree \( k + l \) in the Hilbert scheme. Since the image surfaces in the spaces of maps we defined have generically the same Hilbert polynomial, they all map to \( \text{Hilb}(\mathbb{P}^N, S^{k+l}) \).

**Dimension counts.** With this preparation we can begin the dimension counts.

**Proposition 6.1.** Let \( A \) be a reduced, irreducible substack of \( \overline{M} \) and let \( p \) be any of the labeled points. Then there exists a Zariski-open subset \( U \) of the dual projective space \( \mathbb{P}^{N^*} \) such that for all hyperplanes \( [H] \in U \) the intersection \( A \cap \{ \pi(p) \in H \} \) is either empty or reduced of dimension \( \dim A - 1 \).

**Proof:** \( \mathbb{P}^N \) is a homogeneous space under the action of the group \( \mathbb{P}GL(N + 1) \). The proposition follows from Theorem 2.3. In the notation of the theorem take \( f : H \to \mathbb{P}^N \) be the immersion of a hyperplane. Let \( g : A \to \mathbb{P}^N \) be evaluation morphism at \( p \). \( \square \)

**Proposition 6.2.** Let \( A \) be a reduced, irreducible substack of \( \overline{M} \) and let \( \lambda \) be any of the marked fibers. Then there exists a Zariski-open subset \( U \) of the dual projective space \( \mathbb{P}^{N^*} \) such that for all hyperplanes \( [H] \in U \) the intersection \( A \cap \{ \pi(\lambda) \in H \} \) is either empty or reduced of dimension \( \dim A - 2 \).

**Proof:** Consider the Grassmannian \( \mathbb{G}(1, N) \) under the group action \( \mathbb{P}GL(N + 1) \). Let \( f : \Sigma_{1,1}(H) \to \mathbb{G}(1, N) \) be the immersion of the Schubert cycle of lines contained in \( H \). Let \( g : A \to \mathbb{G}(1, N) \) be the evaluation morphism. The proposition follows from Theorem 2.3 and the fact that the cycle \( \Sigma_{1,1}(H) \) has codimension 2 in the Grassmannian. \( \square \)

**Remark.** In the notation of the Proposition 6.1 or 6.2, we can further deduce that if \( B \) is a proper, closed substack of \( A \) then every component of \( B \cap \{ \pi(p) \in H \} \) is a proper closed substack of a component of \( A \cap \{ \pi(p) \in H \} \) by using the dimension statement in Proposition 6.1 or 6.2 for every component of \( B \).

**Caution and Convention:** For the rest of this section, a directrix of a perfectly balanced scroll will be considered an ordinary section class. The directrix class of a perfectly balanced scroll behaves differently than the directrix class of other scrolls dimension theoretically. Instead of noting this exception in every theorem we declare that a perfectly balanced scroll does not have any directrices.

**The dimension of the building blocks.** We now compute the dimension of the locus of maps, up to isomorphism, from an irreducible Hirzebruch surface \( F_{l-k} \) with two marked irreducible sections \( C \) and \( D \) to \( \mathbb{P}^N \) such that

(i) the image of \( F_{l-k} \) is a scroll \( S_{k,l} \) that has contact of order \( m_i \) along fibers \( \lambda_i \), \( 1 \leq i \leq m \), with a fixed hyperplane \( H \),
(ii) the marked curves \( C, D \) have degrees \( d \) and \( e \), respectively. In case these curves are distinct we assume that they have contact of order \( n \) with each other along distinct fibers \( \lambda_1, \cdots, \lambda_k, \lambda_{m+1}, \cdots, \lambda_{m+n} \) \((k \leq m)\).

**Proposition 6.3** If \( \sum_{i=1}^{m} m_i > k \), then this dimension is

\[
N(k + l + 2) + k - l - 5 - \delta_{k,l} + m + \alpha, \text{ where }
\]

(i) \( \alpha = 0 \) if \( C \) and \( D \) lie in \( H \);

(ii) \( \alpha = 2d - k - l + 1 \) if \( C \) and \( D \) coincide, but do not lie in \( H \);

(iii) \( \alpha = 2e - k - l + 1 + n - \sum_{i=1}^{n+m} n_i \) if \( C \) lies in \( H \), but \( D \) does not;

(iv) \( \alpha = 2d + 2e - 2k - 2l + 2 + n - \sum_{i=1}^{n+m} n_i \) if \( C \) and \( D \) are distinct and do not lie in \( H \).

If \( \sum_{i=1}^{m} m_i \leq k \), then the dimension is

\[
N(k + l + 2) + 2k - 4 - \delta_{k,l} - 2 \sum_{i=1}^{m} m_i + m + \alpha, \text{ where }
\]

(i) \( \alpha = 0 \) if \( C \) and \( D \) both coincide with the directrix or are contained in \( H \);

(ii) \( \alpha = n - \sum_{i=1}^{n+m} n_i \) if one of \( C \) or \( D \) coincides with the directrix and the other

(iii) \( \alpha = 2d - k - l + 1 \) if \( C \) and \( D \) coincide but are distinct from the directrix and
do not lie in \( H \);

(iv) \( \alpha = 2d - k - l + 1 + n - \sum_{i=1}^{n+m} n_i \) if \( D \) coincides with the directrix or lies in
\( H \) and \( C \) is a section distinct from them;

(v) \( \alpha = 2d + 2e - 2k - 2l + 2 + n - \sum_{i=1}^{n+m} n_i \) if \( C \) and \( D \) do not coincide; are distinct
from the directrix and do not lie in \( H \).

**Proof:** The map from \( F_{l-k} \) is given by \( N+1 \) sections \( s_0, \cdots, s_N \) in the class \( O_{F_{l-k}}(e + lf) \). We assume \( s_0 \) corresponds to the hyperplane \( H \). In case the sections \( C \) and \( D \) do not coincide, we choose the \( n \) lines along which \( C \) and \( D \) meet. Since there is a one parameter family of fibers this gives us a choice of \( n \) dimensions. Finally, we mark two sections \( C \) and \( D \) on the surface which have the required incidence with each other along the chosen fibers. We need to know the dimension of pairs of sections with specified incidence along specified fibers.

**Lemma 6.4** On \( F_{l-k} \) the projective dimension of pairs of distinct irreducible sections

\((s_1, s_2)\) in the classes \( e + m_1f, e + m_2f, m_1 \leq m_2 \), respectively, having contact of order

\( n \), \( \sum_i n_i \leq m_1 + m_2 + k - l \), with each other along specified fibers is

\[
2m_1 + 2m_2 + 2k - 2l + 2 - \sum_{i} n_i.
\]

unless \( m_1 = 0 \) and \( k \neq l \). In the latter case the dimension is

\[
2m_2 + k - l + 1 - \sum_{i} n_i.
\]

**Proof:** Set \( u = m_1 + m_2 + k - l \) and \( t = 2m_2 + k - l + 1 \). We can pick the section in the class \( e + m_1f \) arbitrarily. By Lemma 2.3 the dimension of these sections is

\( 2m_1 + k - l + 1 \), unless \( m_1 = 0 \) and \( k \neq 0 \). In the latter case the dimension is 0. We can embed the surface \( F_{l-k} \) with the section class \( e + m_2f \). In this embedding the first chosen section class is a rational normal curve of degree \( u \). To choose a curve in the
second section class with certain tangency conditions at specified points is equivalent to choose a hyperplane in $\mathbb{P}^d$ with the required tangency conditions to the rational normal curve along the points of its intersection with the specified fibers. Since the curves are distinct and irreducible the hyperplane will not contain the first curve. Hence, its intersection with the span of the first curve will also be a hyperplane.

A rational normal curve is embedded by a complete linear system on $\mathbb{P}^3$. Any two positive divisors of degree $d$ on $\mathbb{P}^3$ are linearly equivalent. A hyperplane having the specified order of contact is given by $n_1p_1 + \cdots + n_rp_r + D$ where $D$ is any sum of points $m_1 + k_1 - \sum_i n_i$. Such hyperplanes form a $u - \sum_i n_i$ dimensional linear subspace of $(\mathbb{P}^r)^*$. For each such linear space the hyperplanes of $\mathbb{P}^d$ that contain it is an irreducible linear space of dimension $m_2 - m_1 + 1$ in $(\mathbb{P}^r)^*$.

Using Lemma 6.4 we can complete the proof of Proposition 6.3. Note that if $\sum_{i=1}^m m_i > k$, then the directrix of the surface has to be contained in $H$ and we must have $\sum_{i=1}^m m_i = l$. In this case the dimension for the choice of $s_0$ is $m + 1$. By Lemma 2.5 the dimension for the choice of each of the other $N$ section classes is $k + l + 2$.

Finally, we have to choose the sections $C$ and $D$.

- If they both coincide with the directrix, there is nothing to choose.
- If they coincide, but are in a different section class, by Lemma 2.5 we should add $2d - k - l + 1$.
- If $C$ coincides with the directrix, then we have nothing further to choose. If $D$ does not coincide with the directrix, then by Lemma 6.4 the choice for its dimension is $2e - k - l + 1 - \sum_{i=1}^m n_i$.
- If $C$ and $D$ are distinct section classes different from the directrix, Lemma 6.4 provides us the dimension.

If $\sum_{i=1}^m m_i \leq k$, then the hyperplane in $H$ contains a section class different from the directrix. The class in $H$ residual to the lines must be $e + (l - \sum_{i=1}^m m_i)f$. Hence, by Lemma 2.5 the dimension of $s_0$ is $k + l + 2 - \sum_{i=1}^m m_i + m + 1$. The rest of the calculation is analogous to the previous case. We must also projectivize and subtract the dimension of the automorphism group of $S_{k,l}$. The proposition follows.

Gluing scrolls. We now prove that gluing scrolls along fibers to form a tree imposes the expected number of conditions. Since the fibers are $\mathbb{P}^1$ there is a three dimensional family of isomorphisms which glue the two fibers. We will often have to match section classes in different surfaces along the fibers. This will impose additional conditions on the choice of isomorphism.

Let $X \subset \mathbb{P}^N$ be a projective variety. Let $F_X(s)$ be an irreducible, reduced sub-scheme of its Fano scheme of $s$ dimensional linear spaces. Let $(p_i)_{i=1}^q$ denote $q \leq s + 1$ points in general linear position. Let $H(X)$ denote the $\mathbb{P}GL(N + 1)$ orbit of $[X]$ in the Hilbert scheme. For any $[Z] \in H(X)$, let $F_Z(s)$ be the translation of $F_X(s)$ in the Grassmannian by the element that takes $X$ to $Z$.

**Lemma 6.5** Let $X_1, X_2 \subset \mathbb{P}^N$ be two projective varieties. In the incidence correspondence

$$I := \{(Z_1, \lambda_1, (p_j^1)_{j=1}^q, Z_2, \lambda_2, (p_j^2)_{j=1}^q) : Z_i \in H(X_i), \lambda_i \in F_{Z_i}(s), p_j^1 \in \lambda_1, p_j^2 \in \lambda_2 \subset H(X_1) \times \mathbb{G}(s, N) \times (\mathbb{P}^N)^q \times H(X_2) \times \mathbb{G}(s, N) \times (\mathbb{P}^N)^q \}
$$

the locus $J$ defined by $\lambda_1 = \lambda_2$ and $p_j^1 = p_j^2$ for all $1 \leq j \leq q$ has codimension $(s + 1)(N - s) + sq$.
Proof: $J$ is contained in the locus where $\lambda_1 = \lambda_2$. Restricted to this locus $J$ can be seen as the projection that forgets the points on $\lambda_2$. Since the choice of $q$ points in $\mathbb{P}^s$ has dimension $sq$, the fibers of this projection are all irreducible of dimension $sq$. Hence, to conclude the lemma it suffices to settle the case $q = 0$. Consider the projection from $I$ to the product of the Grassmannians $\mathbb{G}(s,N) \times \mathbb{G}(s,N)$. By assumption all fibers are projectively equivalent under the diagonal action of $\mathbb{PGL}(N+1)$, so they all have the same dimension. We conclude that the codimension of $J$, which is the pull back of the diagonal, is equal to the codimension of the diagonal in $\mathbb{G}(s,N) \times \mathbb{G}(s,N)$. □

Forming trees. Let $T$ be a connected tree with $v$ vertices numbered from 1 to $v$. Suppose $e_1$ of its edges are labeled 1, $e_2$ of its edges are labeled 2 and the rest are labeled 0. Let $\nu_i$ be the valence of the $i$-th vertex.

Let $V$ be a $v$-tuple of scrolls where on the $i$-th one there are $\nu_i$ marked lines with two distinct points on each. Let $W$ be the subscheme of $V$ corresponding to the $v$-tuples that form the tree $T$. That is if vertex $i$ is adjacent to vertex $j$, then a marked line of the scroll in the $i$-th position coincides with a marked line of the scroll in the $j$-th position and if the edge between the vertices is labeled by 1 or 2, one or both of the points, respectively, coincide.

Proposition 6.6 The codimension of $W$ in $V$ is

$$(2N - 2)(v - 1) + e_1 + 2e_2.$$ 

Proof: When there are only two surfaces, the proposition is a special case of Lemma 6.5. To prove the proposition in general induct on the number of vertices by removing a root and using Lemma 6.5 again. Note that the same argument applies when the lines joining a surface to two adjacent surfaces coincide. □

Main Enumerative Theorems: We now state and prove the main enumerative theorems under the assumption that the surfaces, their limit hyperplane sections and the marked points on the curves remain non-degenerate.

For an open subset of $\mathcal{M}$ the image of the maps have the same Hilbert polynomial. This induces a rational morphism to the Hilbert scheme. The subspaces $\mathcal{X}$ and $W$ whose general points correspond to maps with non-degenerate image also admit rational maps to the same Hilbert scheme.

Definition 6.7 A divisor of $\mathcal{M}$ whose general point corresponds to a map with non-degenerate image is called enumeratively relevant if its image in the Hilbert scheme has codimension one in the image of $\mathcal{M}$.

To determine the characteristic numbers of scrolls we count the number of reduced points in the locus of the Hilbert scheme corresponding to the type of scroll we are interested in. General one parameter families of conditions lead to one parameter families of solutions. We analyze the boundary divisors that these one parameter families meet. Viewed from this perspective only enumeratively relevant divisors of $\mathcal{M}$ contribute to the enumerative calculations.

Contracted components can add moduli to maps from trees of Hirzebruch surfaces without changing the number of moduli of the image surfaces. In order to eliminate these extra moduli we define the modified tree.

The modified tree. Let $\pi$ be a map from a tree $\tilde{T}$ of Hirzebruch surfaces. We will refer to vertices corresponding to surfaces that are contracted by $\pi$ as contracted vertices and to subtrees consisting entirely of contracted vertices as contracted subtrees.
The *modified tree* $T$ is defined to be the same as $\tilde{T}$ if the map $\pi$ does not contract any surfaces. If $\pi$ contracts some surfaces, we modify $\tilde{T}$ as follows:

1. If a maximal, connected, contracted subtree abuts only one non-contracted vertex, we simply remove it.
2. If a maximal, connected, contracted subtree abuts exactly two non-contracted vertices, we remove the subtree and join the two vertices by an edge.
3. If a maximal, connected contracted subtree abuts three or more non-contracted vertices, then we remove the contracted tree and join the adjacent vertices to a node. We mark the node by $c$, the number of non-contracted vertices adjacent to the contracted tree. The reader can think of the node as a fiber common to more than two surfaces in the image of $\pi$. We will call such lines junctions (Figure 6).

![Figure 6: The modified tree.](image)

We call a subtree of a modified tree connected, if every vertex has an edge to some other vertex in the subtree or connects to a node that another vertex in the subtree is connected to. The image of a connected subtree is connected in codimension 1.

Although Theorems 6.8 and 6.9 initially look long and complicated, in fact they state that only very few types of behavior occur when we carry out the degenerations. Making all the behavior precise, unfortunately, requires many cases. Colloquially, the theorems assert that when we specialize a linear space $\Delta^I$ to a hyperplane $H$, some of the balanced scrolls satisfying the incidences can remain outside $H$ (then the hyperplane section of the scroll in $H$ meets $\Delta^I$), some can lie in $H$ (the scroll meets the intersection of the linear spaces with $H$) and some can break into a union of two balanced scrolls one of which is contained in $H$—provided the surface and the limit of the hyperplane sections in $H$ remain non-degenerate. In addition, the theorems specify that the scrolls can become reducible only after there are enough linear conditions to determine the limit of the hyperplane sections in $H$ and that $\Delta^I$ needs to meet the component contained in $H$.

**Theorem 6.8** Every enumeratively relevant component of the divisor

$$D_H \subset \overline{\mathcal{MS}}_H(N; k, l, C(k + l), D, Y, I, J_H)$$

whose general member corresponds to a map where the set theoretic images of the surface and $C$ have maximal span is one of the following
1. $\overline{MS}_H(N; k, l, C(k + l), D, Y, I - 1, J_H + 1 : \Gamma^{J_H + 1} = \Delta^I)$
2. $\overline{MS}_H(N - 1; k, l, C(k + l), D, \tilde{Y}, \tilde{I}, J_H : \tilde{\Delta}^I = \Delta^I \cap H, (\tilde{\Delta}^I = \Delta^I \cap H)_{i=1}^{l-1}$
3. $\Delta^I = \Delta^I, J_H \geq k + l + 1$
4. $\mathcal{X}(\mathbb{P}^n; (k_0, l_0), C(k_0 + l_0 + 1), D(k_0) : I \in \Delta(0), J_H(0) \geq k_0 + l_0 + 2)$,
   $(k_1 = k - k_0, l_1 = l - l_0, C(k_1 + l_1 - 1), D(k_1))$
5. $\mathcal{W}(\mathbb{P}^n; (k_i, l_i), C(d_i), D(k_i), i = 0, d_0 = k_0 + l_0 - M, \text{ for } i > 0$
   $d_i = k_i + l_i + 1, I \in \Delta(i), J_H(i) \geq k_i + l_i + 2)$.

In addition, if $k = l$ there can be
6. $\mathcal{X}(\mathbb{P}^n; (k_0, k_0 + 1; C(2k_0 + 2), D(k_0 + 1) : I \in \Delta(0), J_H(0) \geq 2k_0 + 3)$,
   $(k - k_0 - 1, k - k_0, C(2k - 2k_0 - 1), D(k - k_0))$
7. $\mathcal{X}(\mathbb{P}^n; (k_0, k_0 + 1; C(2k_0 + 2), D(k_0) : I \in \Delta(0), J_H(0) \geq 2k_0 + 3)$,
   $(k - k_0 - 1, k - k_0, C(2k - 2k_0 - 1), D(k - k_0 + 1))$

**Theorem 6.9** Let $k + l \leq d \leq k + l + 1$. Every enumeratively relevant component of
$D_H \subset \overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, I', J_H)$
where the curve and the limit divisor cut out on $C$ by $\Pi$ remains non-degenerate is of the form
1. $\overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, I' - 1, J_H + 1 : \Omega^{J_H + 1} = \Sigma^I')$
2. $\overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, I' - 1, J_H : q^I' = p_j, \dim \Omega^j = \dim \Sigma^I' = N - 2)$
3. $\overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, \tilde{I}' - 1, J_H + 1 : \pi(C) \subset \Pi,
   J_H = d \leq k + l, \Sigma^I' = \Sigma^I \cap \Pi$)
4. $\overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, I' - 1, J_H + 1 : \pi(C) \subset \Pi, J_H = d,
   \Delta^I = \Delta^I \cap \Pi, \Sigma^I = \Sigma^I \cap \Pi$)
5. $\overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, I' - 1, J_H + 1 : C = \tilde{C} \cup F, \pi(\tilde{C}) \subset \Pi,
   J_H = d, q^I' \subset \tilde{C})$
6. $\overline{MS}_H(\mathbb{P}^n; k, l, C(d), D, Y, I, I' - 1, J_H + 1 : C = \tilde{C} \cup F, \pi(F) \subset \Pi,
   J_H \geq 2, p_j, p_j', q^I' \subset \tilde{F}(F))$
7. $\overline{MS}_H(\mathbb{P}^n; k, l, C(l), D, Y, I, I' - 1, J_H + 1 : C = D \cup F_1 \cdots \cup F_{l-k},
   q^I' \in \pi(D \cup F_1 \cdots \cup F_{l-k}) \subset \Pi, J_H \geq k + r + 1)$
8. $\overline{MS}_H(\mathbb{P}^n; k, l, C(l), D, Y, I, I' - 1, J_H + 1 : k > 0, C = D \cup F_1 \cdots \cup F_{l-k},$
   $q^I' \in \pi(F_1 \cup \cdots \cup F_{l-k}) \subset \Pi, J_H \geq 2r)$
9. $\mathcal{X}(\mathbb{P}^n; (k_0, l_0), C(k_0 + l_0 + 1), D(k_0) : I \in \Delta(0), J_H(0) \geq k_0 + l_0 + 2)$,
   $(k_1 = k - k_0, l_1 = l - l_0, C(d - k_0 - l_0 - 1), D(k_1))$.

In addition if $k = l$ we can have
10. $\mathcal{X}(\mathbb{P}^n; (k_0, k_0 + 1; C(2k_0 + 2), D(k_0 + 1) : I' \in \Delta(0), J_H(0) \geq 2k_0 + 3)$,
    $(k - k_0 - 1, k - k_0, C(d - 2k_0 - 2), D(k - k_0))$.
11. $\mathcal{X}(\mathbb{P}^n; (k_0, k_0 + 1; C(2k_0 + 2), D(k_0) : I' \in \Delta(0), J_H(0) \geq 2k_0 + 3)$,
    $(k - k_0 - 1, k - k_0, C(d - 2k_0 - 2), D(k - k_0 + 1))$. 

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We will prove both theorems by dimension counts. The arguments become fairly intricate because we need to account for competing phenomena. In a tree of scrolls requiring three surfaces to meet along the same line costs dimension. On a scroll requiring a curve to contain a line also costs dimension. However, if a curve contains the common line of three surfaces, then that voids the matching conditions between pieces of a curve on different components of a surface. The task at hand is to prove that the gain is always less than the cost.

**Proof:** By repeatedly using Propositions 6.1 and 6.2 and the remark following them, it suffices to prove Theorem 6.8 when $I = 1, Y = 0$ and Theorem 6.9 when $I = Y = 0, I' = 1$. The general case then follows by adding marked points and marked fibers and requiring them to lie in the intersection of general hyperplanes.

**Proof of Theorem 6.8** By Proposition 6.3 the dimension of $M_H$ is

$$N(k + l + 2) + 2k - 4 + (J_H + 2).$$

The last term $J_H + 2$ corresponds to the choice of $J_H$ points on $C$ and a point on the surface. We can ignore these terms until the end of the calculation, where they will help select the enumeratively relevant divisors.

Let $U$ be a component of $D_H$. Since we are interested only in the enumeratively relevant components, we can work with the modified tree $T$.

I. **No component lies in $H$.** In this case, the hyperplane section meets $\Delta_I$, i.e. $q_I \in C$. Hence the last term decreases by 1 to $J_H + 1$. Since a reducible surface or a more unbalanced surface that is the limit of scrolls $S_{k,l}$ has dimension at most $N(k + l + 2) + 2k - 5$, the only divisor where the map $\pi$ does not map a component of the surface to $H$ is given by 1.

II. **The entire surface lies in $H$.** Since any limit of scrolls that lies in $H$ is the limit of scrolls that lie in $H$. In this case the dimension of the surface with a choice of hyperplane section and directrix is at most $N(k + l + 2) + 2k - 5$ with equality if and only if the surface is an $S_{k,l}$. Since the additional term we add is still $J_H + 2$ we conclude that the only divisors has the form 2. Furthermore, suppose $J_H < k + l + 1$. Then there is a positive dimensional space of hyperplane sections that pass through the chosen $J_H$ points. Such a divisor is not enumeratively relevant.

III. From now on we can assume that $\pi$ maps at least one component of the surface into $H$ and keeps at least one component outside $H$.

**Using the non-degeneracy assumption to simplify $T$.** Since we are assuming that the image surface spans $\mathbb{P}^{k+l+1}$, each of the non-contracted components map to rational normal scrolls. Moreover, the span of any subsurface connected in codimension 1 of degree $d$ must be $\mathbb{P}^{d+1}$. Since the limit curve $C$ is non-degenerate each of its non-contracted components maps to a rational normal curve. We conclude that the restriction of $C$ to any subsurface of degree $d$ connected in codimension 1 has degree at most $d + 1$ since such a surface can span at most $\mathbb{P}^{d+1}$.

The surfaces outside form a connected tree and have multiplicity 1 with $H$ along their lines contained in $H$. Let $T_i$ be a maximal, connected subtree of the modified tree $T$ where all the vertices correspond to surfaces mapped into $H$. At most one surface outside $H$ can be connected to $T_i$. Otherwise, the restriction of $C$ would have degree at least 2 more than the total degree of the surfaces in $T_i$ contradicting the previous observation. As a corollary we conclude that the surfaces outside $H$ form a single connected tree. Again by the non-degeneracy of $C$ the surfaces outside have multiplicity 1 along any line they meet $H$ including the lines where they are connected.
to surfaces inside. Moreover, a line joining two surfaces outside $H$ is not contained in $H$.

**Reminder:** We remind the reader that a line of the image surface common to more than two components is called a *junction*. A line where exactly two components meet is an *ordinary common line*.

**Bounding the dimension of maps.** We now calculate the dimensions of various loci of maps.

**Notation:** We assume that $\pi$ is a map whose image forms a contracted tree $T$ of scrolls with $v$ vertices and $\sigma$ nodes marked $c_1, \ldots, c_\sigma$. We assume that $T$ has

- $t$ maximal connected subtrees $T_1, \ldots, T_t$ with $v_1, \ldots, v_t$ vertices whose image under $\pi$ lies in $H$ and
- one connected subtree $T_0$ with $v_0$ vertices all of which lie outside $H$. Let the total number of vertices be $v$.

We assume that the $i$-th vertex is marked by two integers $(k_i, l_i)$ to denote that the surface corresponding to that vertex is $S_{k_i, l_i}$. We must, of course, have $\sum k_i + l_i = k + l$.

In addition, each vertex has the information of two limit curves $C, D$ associated to it.

We assume that

- Both $C$ and $D$ contain $\sigma_1$ junctions $F_1, \ldots, F_{\sigma_1}$ and $\mu_1$ ordinary common lines $G_1, \ldots, G_{\mu_1}$ where $D$ contains them with multiplicity $f_i$ and $g_i$, respectively,
- $D$ contains $\mu_2$ additional ordinary common lines and $\sigma_2$ junctions with multiplicities $f_i$ and $g_i$, respectively, and
- $C$ contains $\mu_3$ additional ordinary common lines and $\sigma_3$ junctions.

- On the $j$-th surface apart from the ordinary common lines and nodes, $C$ and $D$ contain $\zeta_j$ common fibers, $C$ contains $\xi_j$ additional fibers and $D$ contains $\psi_j$ additional fibers. We allow $D$ to contain the common fibers with multiplicity $\rho_{ji}$ and the additional fibers with multiplicity $\phi_{ji}$.

- Finally, on surfaces where the section parts of $C$ and $D$ are distinct, the sections have contact of order $m_{i,j}$ with each other along fibers common to at least two surfaces and contact of order $n_{i,j}$ along other fibers common to both $C$ and $D$.

**Caution:** Recall our convention that a *perfectly balanced scroll* does not have any directrices.

**Summation convention.** $\sum_{EE}, \sum_{CC}, \sum_{CE}, \sum_{ED}, \sum_{CD}$ will denote summation over indices for which both $C$ and $D$ contain the directrix, both $C$ and $D$ contain the same section class, $D$ contains the directrix, $C$ contains the directrix, or they both contain different section classes, respectively. We will also decorate the summation notation with $I$ or $O$ to denote summing over only the surfaces inside or the surfaces outside.

**The dimension of the building blocks.** We have to arrange our surfaces outside so that their intersection with $H$ is governed by the data of $C$. In addition, we have to choose $D$ in all the surfaces and $C$ in the surfaces inside. Using Proposition 6.3 we see that the dimension of the data of the surfaces $S_{k_i, l_i}$ together with the curves $C$ and $D$ on them prior to the gluing conditions is
Gluing the pieces. So far we have the dimension of \( C \) in meet the common fibers of two surfaces, except when the common fiber is contained join, except when the sections already meet the fiber at the same point, i.e. \( m > 0 \). We now introduce some more notation to record all these different possibilities.

More Notation. Note that \( T \) has \( v - 1 - \sum_{i=1}^{\sigma}(c_i - 1) \) ordinary common lines. Suppose

- in \( \alpha_1 \) of them surfaces with distinct sections meet and \( C \) and \( D \) both contain the common fiber; in \( \alpha_2 \) of them distinct surfaces with distinct sections meet and only \( D \) contains the fiber; and in \( \alpha_3 \) of them surfaces with distinct sections meet and \( C \) contains the fiber;

- in \( \beta_1 \) of them surfaces with common sections meet and both \( C \) and \( D \) contain the fiber; in \( \beta_2 \) of them only \( D \) and \( \beta_3 \) of them only \( C \) contains the fiber.

Note that there must be \( \mu_1 - \alpha_1 - \beta_1 \) ordinary common lines where a surface with distinct sections meet a surface with common sections and both \( C \) and \( D \) contain the fiber. Similarly, for the other two cases.

Other than these fibers suppose that there are \( \kappa_1 \) ordinary fibers where surfaces with distinct sections meet and have \( m_{ij} > 0 \), \( \kappa_2 \) ordinary fibers where surfaces with distinct sections meet and \( m_{ij} = 0 \). \( \kappa_3 \) fibers where surfaces with distinct sections meet those with common sections and \( \kappa_4 \) fibers where surfaces with common sections meet. We remark that

\[
v - 1 - \sum_{i=1}^{\sigma}(c_i - 1) = \sum_{i=1}^{3}\mu_i + \sum_{i=1}^{4}\kappa_i
\]

Suppose at each of the first \( \sigma_1 + \sigma_2 + \sigma_3 \) junctions where either \( C \) or \( D \) or both contain the fiber \( \tilde{c}_i \) surfaces with common sections meet \( c'_i \) surfaces with distinct sections. Suppose in the next \( \sigma_4 \) junctions \( \tilde{c}_i \) surfaces with common sections meet \( c'_i \) surfaces with distinct sections. Of course, we must have \( c_i = \tilde{c}_i + c'_i \). In the next \( \sigma_5 \) junctions suppose that surfaces with distinct sections, but with \( m > 0 \) meet. Finally, in the last \( \sigma_6 \) junctions surfaces with distinct sections and \( m = 0 \) meet. Note that \( \sum_{i=1}^{6}\sigma_i = \sigma \).

Choosing the lines to glue along. On each surface we have to choose the lines along which we will glue it to other surfaces. For gluing two surfaces outside or two

\[
\sum_{j=1}^{n}(N(k_j + l_j + 2) + k_j - l_j - 6 - \delta_{k_j,l_j}) + \sum_{E \in O}(l_j + 1) + \sum_{C \in O}(d_j + 2) \\
+ \sum_{E \in D \mid O}(2e_j - k_j + 2) + \sum_{C \in E \mid O}(d_j + 2) + \sum_{C \in D \mid O}(d_j + 2e_j - k_j - l_j + 3) + \sum_{O}\psi_j \\
+ \sum_{C \in C \mid D}(2d_j - k_j - l_j + 1) + \sum_{E \in D \mid I}(2e_j - k_j - l_j + 1) + \sum_{C \in E \mid I}(2d_j - k_j - l_j + 1) \\
+ \sum_{C \in D \mid I}2(d_j + e_j - k_j - l_j + 1) + \sum_{I}(\zeta_j + \xi_j + \psi_j) - \sum_{E \in D \mid C \in E \mid D \mid I}i(m_{ji} + n_{ji})
\]
surfaces inside this gives us a choice of 2 for each connection except at the junctions where this over counts by $c_i - 2$. For gluing a surface outside and a surface inside we have only a choice of 1 dimension because the surface outside has only finitely many lines in $H$.

**Combining the gluing conditions.** When we glue the sections if in one surface the sections coincide, then on the other surface we must have $m > 0$ or the two sections must coincide, hence we only have 1 point to match. In case the sections do not match on either of the two surfaces, we have two points to match, unless $m > 0$ on one, hence both of them. By Lemma 6.5 and Proposition 6.6 we conclude that we have to add the following gluing conditions to our previous expression

$$
\sum_{i=1}^{t} (6 - 2N)(v_i - 1) + (4 - 2N)(v_0 - 1) + (5 - 2N)t + \sum_{i=1}^{\sigma} (2 - c_i)
- \mu_2 - \mu_3 - \kappa_1 - 2\kappa_2 - \kappa_3 - \kappa_4 - \sum_{i=\sigma_1+1}^{\sigma} (c_i - 1) - \sum_{i=\sigma_6+1}^{\sigma} 2(c_i - 1)
$$

**Simplifying the formulae.** Now we simplify these expressions using the facts that $D$ has degree $k$, $C$ has degree $k + l$ and that the total surface has degree $k + l$. In the computations I found it helpful to express these facts in terms of the following equations

$$
k = \sum_{EE,CE} k_j + \sum_{CE,ED} e_j + \sum_{i,j} (\rho_{ji} + \phi_{ji}) + \sum_{i=1}^{\sigma} f_i + \sum_{i=1}^{\mu_1 + \mu_2} g_i
$$

$$
k + l = \sum_{ED,EE} k_j + \sum_{CE,CC} d_j + \sum_{j} (\zeta_j + \xi_j) + \mu_1 + \mu_3 + \sigma_1 + \sigma_3
$$

$$
k + l = t + \sum_{I} (k_j + l_j) + \sum_{O} (\zeta_j + \xi_j) + \sum_{EE,ED,O} k_j + \sum_{CE,CC,O} d_j
$$

Finally, we have to use the fact that the curves cannot have total intersection multiplicity larger than $k$ at isolated points on the smooth locus of the surface. If we assume that the total intersection multiplicity is $k - w$ for some nonnegative $w$, we obtain the relation:

$$
k - w = \sum_{ED} (e_j + \xi_j + \phi_j - l_j) + \sum_{CE} (d_j + \xi_j + \phi_j - l_j)
+ \sum_{CD} (e_j + \xi_j + \phi_j + d_j - k_j - l_j) - \sum_{j} \left( \sum_{i} m_{ji} + \sum_{i} n_{ji} \right)
$$

**Final formula.** It is straightforward, but messy algebra to conclude that the locus of maps with the tree we described has dimension at most 27.
\[ N(k + l + 2) + 2k - 3 - v_0 - w + \sum_{CC,CD} (k_j - l_j) - \sum_j \delta_{k_j,l_j} - \sum_{EE} 1 + \sum_{CD} 1 \]
\[ - \sum_{ED,EE} (\xi_j + \sum_i \phi_{ji}) + \sum_j \psi_j - \sum_{j} \sum_i (\rho_{ji} + \phi_{ji}) - \sum_{i=1}^{\sigma_1+\sigma_2} f_i - \sum_{i=1}^{\mu_1+\mu_2} g_i \]
\[ - \mu_3 + \sigma - \sigma_1 - \sigma_3 - \sum_{i=\sigma_1+1}^{\sigma - \sigma_6} (c_i - 1) - \sum_{i=\sigma - \sigma_6+1}^\sigma 2(c_i - 1) - \kappa_2 \]

**Interpreting the formula.** We compare \(N(k + l + 2) + 2k - 4\) with the formula to see when the formula is exactly one less.

- Since \(k_j - l_j - \delta_{k_j,l_j} \leq -1\), the surfaces of the form \(CD\) contribute less than or equal to 0 to the sum with equality if and only if the surface is balanced.
- Note that \(\psi_j - \sum_i \phi_{ji} \leq 0\).
- For each junction between 1 and \(\sigma - \sigma_6\) the contribution is \(-1\) or less and for every other junction the contribution is less than \(-2\).
- All the other terms after \(2k\), if they exist, are strictly negative.

Since \(v_0 \geq 1\),

\[ N(k + l + 2) + 2k - 3 - v_0 \leq N(k + l + 2) + 2k - 4 \]

we conclude that

- \(\sigma_6 = 0\).
- There can be at most one surface of type \(EE\) or \(CC\) since these surfaces contribute \(-1\) or less to the sum.
- If the surface contains a subsurface of type \(EE\) or \(CC\), \(C\) and \(D\) cannot contain any junctions, ordinary common lines, or any fibers on surfaces of type \(CE, ED, CD\).

Using the equations above we can reexpress \(w\) as

\[ w = \sum_{CC} (2d_j - k_j - l_j) + \sum_{EE} (k_j - l_j) + \mu_1 + \mu_3 + \sigma_1 + \sigma_3 + \sum_j \xi_j \]
\[ + \sum_{i=1}^{\sigma_1+\sigma_2} f_i + \sum_{i=1}^{\mu_1+\mu_2} g_i + \sum_{CC,EE} (\xi_j + \phi_{ji}) + \sum (m_i + n_i) \]

- A codimension 1 locus satisfying our assumptions does not contain any components of type \(CC\). If there is a surface of type \(CC\), then \(w\) is at least 1 since \(m \geq 1\) for the components abutting the component of type \(CC\) and \(2d_j - k_j - l_j \geq 0\).

**Suppose there are no surfaces of type \(EE\).** Then the surfaces are of types \(CE, ED\) or \(CD\). Any ordinary common line or junction contained in the curves contributes less than or equal to \(-2\). Hence, \(C\) and \(D\) cannot contain any junctions or ordinary common lines. Each time we glue two of these surfaces either \(\kappa_2 = 1\) or \(m \geq 1\) for both of the surfaces. We conclude that at a general point in a codimension one locus there must be exactly two surfaces, one inside \(S_{k_1,l_1}\) and one outside \(S_{k_0,l_0}\), and the curves \(C\) and \(D\) meet the common fiber at distinct points. Since \(C\) and \(D\) cannot contain fibers (this would contribute less than or equal to \(-1\)), their restriction to each component must be a section. The degree of the component of \(C\) in the surface outside is \(k_0 + l_0 - 1\) and the degree of the component in the surface inside is \(k_1 + l_1 + 1\).

The surfaces have the types \((CD, CE), (CE, ED)\) or \((CE, CE)\).
• If the type is \((CE, CE)\), then \(k = k_1 + k_2\).
• If a surface is of type \(ED\), then that surface must be outside \(H\). Since the directrix of the surface has degree one less than its degree the surface must be a plane. (Remember the quadric surface does not have any directrices.) The degree of \(D\) is at least 1 and it is required to pass through a point. Hence the choice of \(D\) contributes a one-parameter family to our dimension. Such a component cannot be enumeratively relevant unless \(k = l\). In the latter case a perfectly balanced scroll breaks into a plane and another balanced scroll.
• If there is a surface of type \(CD\), it must be balanced. Either the scroll is perfectly balanced and \(D\) is a directrix, in which case \(k = k_1 + k_2\) or the choice of \(D\) contributes at least 1 to the dimension. It contributes exactly one when the surface is balanced and not perfectly balanced and the degree of \(D\) is one larger than the degree of the directrix. Such a case can be enumeratively relevant only if \(k = l\). In that case we must have \(k = k_1 + k_2 + 1\) and a perfectly balanced scroll breaks into a union of two balanced scrolls.

**Suppose that there is a component \(S\) of type \(EE\).** As we already observed in this case there are no junctions and the curves \(C, D\) do not contain any ordinary common lines or lines on the surfaces of other types. Furthermore, we must have \(w = 0\). Hence, the equation for \(w\) simplifies to

\[0 = k_S - l_S + \xi_S + \sum \phi_{ji} + \sum_i (m_{ji} + n_{ji}).\]

The degree of the hyperplane section restricted to \(S\) is \(k_S + \xi_S\). Since any surface emanating from \(S\) can account for at most one degree by our non-degeneracy assumption, we conclude that there must be at least \(l_S - \xi_S\) surfaces adjacent to \(S\). They are surfaces of type \(CE, ED\) or \(CD\), so \(m \geq 1\) for each of them. Since \(w = 0\), we conclude that there are exactly \(l_S - \xi_S\) surfaces, all adjacent to \(S\). The curves \(C\) and \(D\) meet each other simply along the lines joining the other surfaces to \(S\). \(S\) lies outside \(H\) and all the other surfaces are in \(H\). Moreover, \(k_S = 0\), so \(S\) is a cone.

The surfaces inside have to be of type \(CE\) or \(CD\). If they are all of type \(CE\), then \(k = \sum k_i\). If some surfaces are of type \(CD\), either all of them are perfectly balanced and \(D\) is a directrix. Hence, \(k = \sum k_i\). Or \(k = l\) the surface is balanced, the degree of \(D\) is one more than the degree of the directrix. There can be at most one such surface. In this case \(S\) must be a plane and there are only two components.

Finally, we have to add the choices for the \(J_H\) points and the point \(q_I\). We must have \(q_I\) in a surface inside, otherwise we lose 1 dimension. Finally, to be enumeratively relevant \(C\) should not move in a linear system. We conclude that \(J_H(I) \geq k_I + l_I + 2\) for each of the surfaces inside. (In fact, \(J_H(I) \geq k_I + l_I + 3\) in case the surface outside is a plane.) This completes the proof. \(\square\)

**Proof of theorem 6.9.** It suffices to prove the case \(I = Y = 0, I' = 1\). The dimension of \(M_{3\Pi}\) is

\[(N + 1)(k + l + 2) + 2d - 2l - 5.\]

Let \(U\) be a component of \(D_{\Pi}\). We have \(J_{\Pi} \leq d\).

1. Suppose the surface is irreducible. If the curve is also irreducible, then \(q_I \in C \cap \Pi\). Here \(q_I\) can also coincide with one of the \(p_j\). Of course, we mean that there is a map to \(\mathbb{F}(0,0;1;N)\) with a contracted component and the marked points move to the contracted component. We will continue using the more suggestive notation \(q_I = p_j\) since this is more convenient for counting purposes. If the curve does not lie in \(\Pi\),
these are divisors provided that the surface remains an $S_{k,l}$. If the curve lies in $\Pi$, then either $d \leq k + l$ or the surface also lies in $\Pi$. In either case this locus is a divisor precisely when $J_{\Pi} = d$.

If the curve becomes reducible, then it must be the union of a section of degree $d'$ together with $d - d'$ fibers. We denote this curve as $\tilde{C} \cup F_1 \cup \cdots \cup F_{d-d'}$. If $\tilde{C}$ is contained in $\Pi$, then the locus is a divisor when $d' = d - 1$ and $J_{\Pi} = d$ and the remaining fiber is not contained in $\Pi$ or $d = l$ and the directrix union $r$ fibers are contained in $\Pi$ and $J_{\Pi} = k + r + 1$.

If only fibers are contained in $\Pi$, then either there must be a unique fiber and two points among the $J_{\Pi}$ must specialize to it or $d = l$ and the curve must break to the union of the directrix with lines $r$ of which are contained in $\Pi$. In the latter case $2r$ of the points in $J_{\Pi}$ must specialize pairwise to the lines..

2. We can assume that both the curve and the surface break. (We include the case when the surface breaks into planes unions scrolls and the image of the curve does not ‘really’ break.) We will perform a calculation analogous to the previous case to find the divisors. We work with the modified tree $T$.

**Using the non-degeneracy assumption to simplify $T$.** Since the hyperplane section of $C$ in $\Pi$ is non-degenerate, the components of $C$ outside $\Pi$ meet $\Pi$ transversely. Each tree of curves inside can meet at most one curve outside. In particular, the curves outside form a connected tree. The surfaces outside have contact of order one with $\Pi$ along their lines contained in $\Pi$ except possibly when the surface is a cone and the section part of $C$ reduces to the vertex. However, in the latter case forcing the cone to have higher order of contact with $\Pi$ strictly lowers the dimension, so we can ignore this case.

**Three or more curves.** Unfortunately, in addition to the cases we considered in the previous proof, now the surfaces outside $\Pi$ can have sections that lie in $\Pi$ or remain outside $\Pi$. Fortunately, due to our non-degeneracy assumptions we do not need to record the incidence data of $C$ with the hyperplane. If we remove the non-degeneracy assumption, it is easy to construct situations where the directrix has high order of contact with the hyperplane and the curve has high order of contact with both the hyperplane and the directrix at different points. It is a very hard problem to determine the limits in this generality. This is the main obstruction for carrying out our algorithm in general.

**Notation:** We preserve the notation and conventions of the previous proof. We decorate our sums for the surfaces outside $H$ with $\ast \subset H$ to signify summation over the incidies where $\ast$ is in $H$. We decorate the summation by $'$ if none of $E_j, C_j, D_j$ is contained in $H$. In that case, we let $x_j$ denote the number of lines that the surface has in $H$. Finally, we decorate our lines with $H$ and $\not\subset H$ in case the line belongs to the part of $C$ or $D$ that lies in a surface outside $H$, and the line is in $H$ or not, respectively.

**Building blocks:** By Proposition 6.3 the dimension prior to gluing is

$$
\sum_{j=1}^v (N(k_j + l_j + 2) - 6 + k_j - l_j - \delta_{k_j,l_j}) + \sum_{EE|O} (k_j + l_j - x_j + 2) \\
+ \sum_{EE\subset H} (l_j + 1) + \sum_{CC\ast\subset H} (d_j + 2) + \sum_{CC,CE|O} (2d_j - k_j + 2) + \sum_{EE\not\subset H} (e_j + 2)
$$
The degree of smooth points of the surface has to be chosen one more point than the total degree of the curves in the tree. Once we choose that the points remain non-degenerate for each connected tree of curves in $\Pi$, we choose the interpretation.

The terms $(2d_j - x_j + 3) + \sum_{CC,CE} (2e_j - k_j + 2) + \sum_{CD} (d_j + 2e_j - k_j - l_j + 3)$

$+ \sum_{CC,CE} (2e_j - k_j - l_j + 3) + \sum_{CD} (2e_j + 2d_j - 2k_j - l_j + 3)$

$+ \sum_{CD} (2d_j + 2e_j - k_j - l_j - x_j + 4) + \sum_{ED} (2e_j - x_j + 3)$

$+ \sum_{ED} (2d_j - k_j - l_j + 1) + \sum_{CE} (2d_j - k_j - l_j + 1)$

$+ \sum_{CC} (2d_j - k_j - l_j + 1) + \sum_{CD} (2d_j + e_j - k_j - l_j + 1) + \sum_{I} (\zeta_j + \xi_j + \psi_j)$

$+ \sum_{O} (\xi_j,\zeta H + \zeta_j,\xi H + \psi_j,\zeta H) - \sum_{ED,CD} \sum_{i} (m_{ji} + n_{ji})$

**Final Formula.** The gluing conditions are identical to the previous case. We retain the notation we used there. Finally we use the facts that the degree of $C$ is $d$, the degree of $D$ is $k$ and that their intersection multiplicities at isolated points along the smooth points of the surface has to be $d - l - w$ for some positive $w$. Combining everything we obtain

$\sum_{j} (k_j + l_j) - t - \sum_{j} \delta_{k_j,l_j} + \sum_{CC,CD} (k_j - l_j) + \sum_{O} \psi_j + \sum_{O} \phi_j + \sum_{i} (\xi_j + \zeta_j + \psi_j) + \sigma - \sigma_1 - \sigma_3$

$- \sum_{i} (\zeta_j,\zeta H + \zeta_j,\xi H + \zeta_j,\zeta H) - \sum_{O} (x_j - 1)$

**Interpretation.** This expression gives us the dimension of the tree of scrolls before we choose the $J_H$ points that are the marked points of $C \cap \Pi$. Since we are assuming that the points remain non-degenerate for each connected tree of curves in $H$ we can choose one more point than the total degree of the curves in the tree. Once we choose the points we compare the above expression with $(N + 1)(k + l + 2) + 2d - 2l - 5$. The curves in a tree of surfaces in $\Pi$ form a connected tree. Consequently, there can be two types of connected trees of curves in $\Pi$: Trees that contain a curve of a surface contained in $\Pi$ and trees that do not. We refer to these trees as trees of the first and second kind, respectively.

- The terms $\sum_{j} (k_j + l_j) + t$ and $d_j + \zeta_j,\zeta H + \zeta_j,\zeta H$ taken over the curves and surfaces contained in a tree of the first kind add up to at least the degree of the curve contained in it. Hence with the choice of points the contribution of each such tree is at most 1 with equality if and only if the degree of the curve in each tree of surfaces inside is the maximum allowed. We can also assume that the trees on the surfaces in $\Pi$ do
not get connected because otherwise $t$ contributes negatively. Note that junctions or ordinary common lines do not change the conclusion and in fact continue to contribute negatively. We can further assume that trees of the first kind do not have any curves or fibers (except for junctions or ordinary common lines) contained in surfaces outside $\Pi$. If such a curve exists, there must be an additional surface outside contributing to $v_0$, hence bringing the contribution of the tree to at most 0.

Surfaces outside have to connect these $t$ trees of curves. Let us call such surfaces connecting surfaces. A connecting surface cannot have a section part of $C$ in $\Pi$ since otherwise it would connect the trees inside. If it has $D$ in $\Pi$, then $-e_j$ would contribute an amount more than the number of trees it connects. If it has $E$ in $\Pi$, the number of trees the surface connects would be at most one more than $k_j$. Together with the contribution from $v_0$, this would annul the contribution of all those trees. Finally, if the surface has no special sections, then it must contain a line for every surface it connects, hence together with $v_0$ the term $1 - x_j$ annuls the contribution of the trees it connects.

The argument for trees of the second kind is analogous but easier. We conclude that the choice of points at most exactly cancel the contribution of the terms discussed.

- If there is a surface outside which contains no special sections and no lines in $\Pi$, the contributions of $v_0$ and $(1 - x_j)$ exactly cancel each other out.
- Since $k_j - l_j - \delta_k l_j \leq -1$ a component of type $CC$ contributes $-1$ or less and a component of type $CD$ contributes 0 or less to the sum with equality if and only if they are balanced. A component of type $EE$ contributes $-1$. We conclude that there can be at most one surface of type $EE$ or $CC$.

To continue our analysis we express $w$ as in the previous proof. By arguments analogous to the ones given there we have that

- a codimension 1 locus does not contain any components of type $CC$.
- if the surface does not contain a subsurface of type $EE$, there can be at most 2 components. The curves $C$ and $D$ do not contain any fibers and the degree of the curve in the surface inside is $k_I + l_I + 1$. The enumeratively relevant codimension one loci have $k_1 + k_2 = k$, unless $k = l$ and $k_1 + k_2 = k - 1$.
- a subsurface of type $EE$ is not contained in $\Pi$. Suppose a surface contains a subsurface $S$ of type $EE$. There cannot be any junctions or ordinary common lines in this case. All the other surfaces are of type $CE$, $CD$, or $ED$. By an argument similar to the previous case, we see that there cannot be a component of type $EE$ in case the curve has degree $k + l + 1$. In case the degree of the curve is $k + l$ there can be components of type $EE$ only in the case described in the previous theorem. However, then all the curves inside are connected and we conclude that there can be at most one other component. One can continue the analysis to see what components would appear if the degree of the curve were smaller. When one would like to apply the algorithm in cases we will not explain in this paper, this extension becomes useful.

\section{Multiplicity Calculations}

In this section we carry out the multiplicity calculations needed for enumerative computations involving balanced scrolls. The philosophy, motivated by Vakil’s work on curves, is that any multiplicity should be reflected in the local structure of the limit surface.

Under the hypotheses of Theorems 6.8 and 6.9 when we specialize a linear space to $H$ or $\Pi$, balanced scrolls incident to the linear space break into at most two balanced scrolls. By Proposition 4.4 the limit of the directrices is uniquely determined by

\section*{7 Multiplicity Calculations}

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the surfaces. For multiplicity calculations it is more convenient to reformulate the degeneration problem in the space of scrolls where we only mark the hyperplane section.

Let $\overline{\mathcal{M}}_H(P^N; k, l; C, \{\lambda_j\}_{j=1}^r, \{q_i\}_{i=1}^r, \{p_j\}_{j=1}^r)$, or $\overline{\mathcal{M}}_H$ for short, be defined like the corresponding space $\overline{\mathcal{M}}_{\mathcal{H}}$ in §6 except that now do not mark the directrix. More explicitly, an open set in $\overline{\mathcal{M}}_H$ corresponds to maps from a Hirzebruch surface into $P^N$ as a scroll $S_{k,l}$, where the marked curve $C$ maps to the hyperplane section in $H$ and the marked fibers and points are required to lie in various linear spaces. We compactify the space as in §3. In a similar fashion define the space $\overline{\mathcal{M}}_{\Pi}$ and the loci $\lambda'_H$ and $\lambda''_H$ corresponding to $\overline{\mathcal{M}}_{\Pi}$ and $\lambda_H$ and $\lambda'_H$, where again the only difference is that we do not mark the directrix. The spaces $\overline{\mathcal{M}}_H$ and $\overline{\mathcal{M}}_{\Pi}$ have natural Cartier divisors $D'_H$ and $D''_H$ defined by requiring the point $p_I$ and $q'_I$ to lie in $H$ and $\Pi$, respectively.

Since the limits of directrices are determined uniquely by the surfaces, Theorems 6.8 and 6.9 describe the enumeratively relevant components of $D'_H$ and $D''_H$ subject to the non-degeneracy assumptions. We would like to compute the multiplicity of the Cartier divisor along each of the Weil divisors appearing in the list.

Lemma 7.1 Let $S_{k,l}$ be a non-singular scroll in $P^N$. Let $\nu := \nu_{S_{k,l}/P^N}$ denote its normal bundle in $P^N$. Suppose $D$ is a divisor in a section class $e + mf$ for $m \leq l + 1$, then

1. $H^i(S_{k,l}, \nu) = 0$, for $i \geq 1$.
2. $H^i(S_{k,l}, \nu \otimes O_{S_{k,l}}(-D)) = 0$ for $i \geq 1$.

Proof: The line bundles $O_{S_{k,l}}$, $O_{S_{k,l}}(1)$, and $O_{S_{k,l}}(1) \otimes O_{S_{k,l}}(-D)$ have no higher cohomology and by Serre duality $h^2(S_{k,l}, O_{S_{k,l}}(-D)) = 0$. Consequently, the Euler sequence for $P^N$

$$0 \to O_{P^N} \to \oplus_{i=1}^{n+1} O_{P^N}(1) \to T_{P^N} \to 0$$

implies that $H^i(S_{k,l}, T_{P^N} \otimes O_{S_{k,l}}) = 0$ and $H^i(S_{k,l}, T_{P^N} \otimes O_{S_{k,l}}(-D)) = 0$ for $i \geq 1$.

The standard exact sequence

$$0 \to T_{S_{k,l}} \to T_{P^N} \otimes O_{S_{k,l}} \to \nu \to 0$$

implies that

$h^i(S_{k,l}, \nu) = h^{i+1}(S_{k,l}, T_{S_{k,l}})$ and $h^i(S_{k,l}, \nu(-D)) = h^{i+1}(S_{k,l}, T_{S_{k,l}}(-D))$.

When $i = 2$ the right hand sides immediately vanish. When $i = 1$ they also vanish by Serre duality. □

Theorem 7.2 When $l - k \leq 1$, the components

1. $\overline{\mathcal{M}}_H(N; k, l, C, Y, I - 1, J_H + 1)$
2. $\overline{\mathcal{M}}_H(N - 1; k, l, C, Y, I, J_H)$
3. $\mathcal{A}'(P^N; (k_0, l_0, C(k_0 + l_0 + 1)), (k_1, l_1, C(k + l - k_0 - l_0 - 1)))$, satisfying the constraints listed in 1, 2, 3, 5 and 6 of Theorem 6.8 occur with multiplicity one in

$$D'_H \subset \overline{\mathcal{M}}(N; k, l, C, Y, I, J_H).$$
Proof: By Propositions 6.1 and 6.2 it suffices to restrict to the case $Y = 0$, $I = 1$, $\Gamma^j = H$ for all $j$. We can then recover the general case by slicing with general hyperplanes.

To determine the multiplicity for the first locus we can assume that $\Delta^j$ has dimension $N - 3$. Consider the family obtained by rotating $\Delta^j$ into $H$. Let $\Delta^j(t)$ denote this family. We assume that $\Delta^j(0) \subset H$. Consider the family

$$\{(t, S_{k,l}, p_1, \cdots, p_{J_H}) : S \cap \Delta(t) \neq \emptyset\}$$

of balanced scrolls $S_{k,l}$ which meet $\Delta(t)$ and have $J_H$ marked points in $H$. The multiplicity under consideration is the same as the multiplicity of the divisor $t = 0$ in this family. The family admits a rational map to the space of rational normal curves with $J_H$ marked points by sending each marked surface to the hyperplane section in $H$. This map is defined over the locus under discussion and is smooth over that locus by Lemma 7.1.

Here we are using the fact that to show that a morphism of stacks $A \to B$ where $B$ is smooth and $A$ is equidimensional and smooth it suffices to show that the Zariski tangent space to the fiber is of dimension $\dim A - \dim B$.

The divisor whose multiplicity we are trying to determine occurs as a component of the pull-back of the divisor of rational curves in $H = \mathbb{P}^{N-1}$ that meet an $N - 3$ dimensional linear space. Since the latter divisor is reduced and the morphism is smooth we conclude that the multiplicity is one in this case.

To determine the multiplicity of the other loci it is more convenient to look at the case when $\Delta^j = \mathbb{P}^N$. In that case the marking of the point $q_I$ gives the universal surface over the loci in question. To compute the multiplicity we can forget the marking of $q_I$. Note that in all the loci described the hyperplane section $C$ is uniquely determined by the surface and the marked points. We get a map from the space of surfaces to the space of rational curves by sending the map from the surface to the map from $C$ to $H$. This is smooth as in the previous case by Lemma 7.1 and the standard normal bundle sequence relating the normal bundle of the reducible surface to that of the union. When $C$ is reducible, the divisor in question is a component of the pull-back of a boundary divisor of $\overline{\mathcal{M}}_{0, J_H}(\mathbb{P}^{N-1}, k+l)$ under a smooth morphism, hence reduced.

Finally in case 2 and when the surface breaks into a plane union a balanced scroll it is easy to see that the multiplicity is one by direct computation using the determinantal representation of scrolls. □

**Theorem 7.3** When $l - k \leq 1$, the components $\overline{\mathcal{M}}_{11}$ and $\mathcal{X}'_{11}$ of

$$D'_1 \subset \overline{\mathcal{M}}_{11}(N; k, l, C, Y, I, I', J_H)$$

listed in 1–11 of Theorem 6.9 occur with multiplicity one in $D'_1$.

**Proof:** It suffices to consider the case $Y = I = 0$, $I' = 1$. We reduce this case to Theorem 6.2 in [2]. $\overline{\mathcal{M}}_{11}(\mathbb{P}^N, k, l, C, 0, 0, 1, J_H)$ admits a morphism $\iota$ to $\overline{\mathcal{M}}_{0, J_H+1}(\mathbb{P}^N, d)$ which sends $(S, C, \pi)$ to $\pi : C \to \mathbb{P}^N$ and stabilizes. In light of §3, we can interpret this morphism—at least in an open set containing the loci we are interested in—as the morphism induced between the corresponding Kontsevich spaces by the projection from $\mathbb{F}(0, 1; N)$ to $\mathbb{P}^N$.

We claim that $\iota$ is smooth at all the loci covered by the theorem. Both the image and the domain of $\iota$ are equidimensional and smooth along the loci we are interested in. It suffices to check that the fiber is smooth. We need to compute the dimension.
of the Zariski tangent space to the fiber. The Zariski tangent space to the fiber at a point \((S,C)\), where \(S\) is a Hirzebruch surface and \(C\) is a curve in a section class that lie in one of the loci covered by the theorem, is given by \(H^0(S, \nu_S(-C))\). If the surface is smooth, then we can conclude that the morphism is smooth by Lemma 7.1.

If \(S = S_1 \cup S_2\) has two components meeting transversely along their common line \(L\), the claim follows from Lemma 7.1 when we use the standard exact sequence for normal bundles.

\[ D' \circ \Pi \] is the pull-back of the divisor \(D_H\) in \([V^2]\) by \(\iota\). Since \(\iota\) is smooth our theorem follows from Theorem 6.2 in \([V^2]\). □

Finally, to conclude the multiplicity calculations we recall Proposition 2.18 in \([V^3]\). This proposition asserts that if to the data of a zero dimensional locus \(M\), we add a linear space of dimension \(N-2\) that meets the surface or a linear space of dimension \(N-1\) that meets a marked curve, then the degree of the stack multiplies by the degree of the surface or the degree of the marked curve, respectively.

8 A Simple Enumerative Consequence

In this section we describe an application of Theorems 6.8 and 6.9 to the characteristic numbers of balanced scrolls. We impose enough point conditions to ensure that we can satisfy the non-degeneracy assumptions of the theorems. When counting scrolls of degree \(n\) in \(\mathbb{P}^{n+1}\), the hypotheses of Theorem 6.1 require that at least \(n+4\) of the linear spaces are points. It is possible to strengthen the theorem at the expense of complicating the algorithm.

**Theorem 8.1** Suppose \(0 \leq l - k \leq 1\). Let \(\{\Delta^i_{a_i}\}_{i=1}^I\) be a set of linear spaces of dimension \(a_i < N-2\) in \(\mathbb{P}^N\) in general linear position such that

1. \(\sum_{i=1}^I N - 2 - a_i = (k + l + 2)N + 2(k - 4 - \delta_{k,l})\)
2. \(a_I \leq N - k - l - 1\)
3. \(a_i = 0 \text{ for } 1 \leq i \leq k + l + 1\)
4. \(a_{k+l+2+j} \leq N - k - l - 1, \text{ for } 0 \leq j \leq a_I + 1\)

Then there exists an algorithm which computes the number of scrolls \(S_{k,l}\) meeting \(\{\Delta^i_{a_i}\}_{i=1}^I\).

**Proof:** We now describe the algorithm and prove that it terminates without stepping outside the bounds of our non-degeneracy assumptions. We begin with the case \(N = k + l + 1\) and reduce the more general case to it later.

**Step 1.** Specialize the \(\Delta^i\), except for \(\Delta^I\), one by one to general linear spaces of a general hyperplane \(H\) in order of increasing dimension until a reducible solution appears.

In our case the first \(k + l + 3\) linear spaces and \(\Delta^I\) are points. We can take \(H\) to be the span of the first \(k + l + 1\) points. We claim that after we specialize \(\Delta^{k+l+2}\) to \(H\) the scrolls incident to all \(\Delta^i\) are still irreducible balanced scrolls.

The hyperplane section in \(H\) has to meet the first \(k + l + 1\) points, so it is non-degenerate. Similarly, since the scroll needs to meet \(\Delta^I\), it spans \(\mathbb{P}^{k+l+1}\). By Theorem 6.8 if there is a reducible scroll, then the component of the scroll in \(H\) meets \(\Delta^{k+l+2}\)
and contains $d + 2$ of the first $k + l + 1$ points, where $d$ is the degree of the scroll. Since this is impossible the claim follows. We repeat step 1 by specializing $\Delta^{k+l+3}$ to $H$.

Theorem 6.8 still applies. In this case there are 2 possibilities.

**Case i.** Some scrolls can remain irreducible. Then their hyperplane section in $H$ is the unique rational normal curve containing the $k + l + 3$ points in $H$. Repeat Step 1 by specializing $\Delta^{k+l+4}$. Theorem 6.8 still applies and this case can no longer occur. Proceed to the next possibility.

**Case ii.** Some scrolls can become reducible. By Theorem 6.8 the only reducible scrolls can be a balanced scroll of degree $k + l - 1$ in $H$ and a plane outside.

- If we arrived at Case ii after passing through Case i first, this is clear since otherwise the limit hyperplane section would be reducible. In this case the degree $k + l - 1$ scroll contains the rational normal curve and meets $\Delta^{k+l+4}$. The plane contains $\Delta'$. The rest of the conditions are distributed among the two. We need to consider each way of partitioning the other conditions in such a way that they do not impose more conditions on either of the components than they can satisfy. (If we do not satisfy the last clause, the algorithm will give 0.)

- If we are in Case ii right after having specialized $\Delta^{k+l+3}$, then the component in $H$ needs to meet $\Delta^{k+l+3}$ and contain at least $d + 2$ of the first $k + l + 2$ points if its degree is $d$. The only possibility is $d = k + l - 1$. The scroll of degree $k + l - 1$ contains the $k + l + 3$ points in $H$. The plane contains $\Delta'$. We consider each partition of the rest of the conditions. In either case proceed to step 2.

**Crucial Point:** Proposition 4.4 implies that if a balanced scroll breaks into a union of two balanced scrolls the gluing conditions on the directrices are automatically satisfied. Therefore, we reduced the problem of counting scrolls of degree $k + l$ to counting pairs (Plane, Scroll of degree $k + l - 1$) meeting along a line and in addition satisfying the conditions described in Case ii. Using Step 2, which we now describe, we further reduce the problem to counting degree $k + l - 1$ scrolls.

Note that the scroll of degree $k + l - 1$ needs to contain a line of the plane outside $H$. The plane in turn contains a point and meets some linear spaces.

**Step 2.** Use Schubert calculus to reexpress the conditions on the plane (or if one were to apply the algorithm more generally, the conditions on the linear space containing the scroll outside $H$) as multiples of Schubert cycles.

After Step 2, the plane is required to contain a point ($\Delta'$), meet a linear space $\Lambda_1$ in a line, and lie in a linear space $\Lambda_2$. In turn the common line between the two scrolls is required to meet the linear space $\Lambda_1 \cap H$ and lie in $\Lambda_2 \cap H$. We have reduced the problem to counting degree $k + l - 1$ balanced scrolls in $\mathbb{P}^{k+l}$ satisfying the conditions in Case ii) and containing a fiber lying in a linear space and meeting another linear space.

- If we arrived at this stage without going through Case i, we are done by induction. The steps so far only used Theorem 6.8 which allows for our new condition without changing the conclusions. In addition the scroll contains $k + l + 3$ points. We can go back to Step 1 and run the process from the beginning.

- If we arrived at this stage after passing through Case i, we have to count scrolls of degree $k + l - 1$ in $\mathbb{P}^{k+l}$ containing a rational normal curve $C$ of degree $k + l$, meeting some linear spaces and containing a fiber which lies in a linear space and meets some other linear space. Proceed to step 3.

**Step 3.** Specialize a linear space meeting the rational normal curve of degree $k + l$ to a general hyperplane $\Pi$ in order of increasing dimension, but always keeping a point
outside \( \Pi \), until the curve or the surface becomes reducible.

In our case Step 3 amount to breaking \( C \) into a rational normal curve of degree \( k + l - 1 \) union a general line. Theorem 6.9 applies since the surface still spans \( \mathbb{P}^{k+l} \) and the limit of the hyperplane section of the curve in \( \Pi \) is non-degenerate. We conclude that the surface cannot break after this degeneration. If \( \Delta^{k+l+4} \) had codimension more than 2, specialize it to \( \Pi \) after which the surface has to necessarily break into a plane containing \( l \) union a degree \( k + l - 1 \) surface. Go back and repeat Steps 2 and 3. If \( \Delta^{k+l+4} \) has codimension 2, specialize a different linear space again in order of increasing dimension. Go back to Step 2. We have reduced the problem to a problem of one degree lower in \( \mathbb{P}^{k+l-1} \).

Inductively, we reduce the problem to a problem of counting planes with conditions of meeting a linear space, containing lines in a linear space or containing a conic. Finally, Theorems 7.2 and 7.3 dictate the multiplicities with which each case occurs.

This concludes the description of the algorithm when \( N = k + l + 1 \).

When \( N > k + l + 1 \), the algorithm is almost identical and quickly reduces to the case \( N = k + l + 1 \). Start by specializing \( \Delta^i \) for \( i \leq k + l + 2 \) to a general hyperplane. By the assumption 3, the first \( k + l + 1 \) span \( P \), a \( \mathbb{P}^{k+l} \). The hyperplane section of the scrolls in \( H \) have to lie in \( P \). After we specialize \( \Delta^{k+l+2} \), there are two possibilities.

1. If the scroll can lies in \( H \), we are done by induction.
2. If the scroll does not lie in \( H \), \( \Delta^{k+l+2} \) meets \( P \) in a point. At this stage there cannot be any reducible scrolls by the argument given above. Specialize \( \Delta^{k+l+3} \) to \( H \).
   - If the scroll lies in \( H \), we are again done by induction.
   - If the scroll becomes reducible, proceed to Step 2 in the above process since then must be a scroll of degree \( k + l - 1 \) in \( P \).
   - If the scroll remains irreducible and outside \( H \), its hyperplane section in \( P \) is determined. After the next degeneration proceed with Step 2 in the above process. This concludes the proof. \( \square \)

Remark 1. Although to satisfy the non-degeneracy assumptions the algorithm dictates an order of specialization, the enumerative numbers are independent of the order. By specializing the conditions in different orders one can solve more problems. For example, to find the number \( n \) of cubic scrolls in \( \mathbb{P}^3 \) that contain a fixed twisted cubic and five general points, we can count cubic scrolls in \( \mathbb{P}^4 \) that contain a twisted cubic, a point and meet 4 lines. If we specialize the point to the hyperplane of the twisted cubic, some of the limits become degenerate. Comparing the number to the one obtained from our algorithm, we conclude that \( n = 21 \).

Remark 2. If we remove the non-degeneracy assumption in Theorem 6.8 there are components of \( D_H \) whose general point corresponds to a map with image a scroll \( S_0 \) of degree \( d_0 \) in \( H \) with many scrolls \( S_i \) outside \( H \) attached to it. The scrolls outside \( H \) can have contact of order \( m \) with \( H \) along their common lines with \( S_0 \). Moreover, the components do not have to remain balanced. New multiplicities appear: the divisors where the components of the scrolls have higher tangency with \( H \) appear with higher multiplicity. The limit of the directrices usually have tangency conditions with the limits of the hyperplane sections.

Even if we enlarge the class of problems to include these, at the next stage worse degenerations appear. Once the surface breaks again, we need to record the new hyperplane section which in turn can have various tangencies with both the directrix and the old hyperplane section. The analogue of Lemma 6.4 is not true for more than two curves. When the number of curves exceeds two, I do not know a complete list of the limits.
Remark 3. We can ask for the characteristic numbers of \( S_{k,l} \) when \( l - k > 1 \). Theorems 6.8 and 6.9 do not require the scrolls to be balanced. They determine the set-theoretic limits of unbalanced scrolls. In fact, Step 1 in the algorithm of Theorem 8.1 can be carried out for unbalanced scrolls the same way. However, the crucial observation that there are no matching conditions on the directrices of balanced scrolls no longer holds for unbalanced scrolls. After Step 1 of the algorithm we cannot reduce ourselves to the problem of counting smaller degree scrolls. In addition the limit of the hyperplane section has to meet the directrix along the special fiber.

One can reprove Theorems 6.8 and 6.9 by including this condition. The proof is identical, only the statement and the interpretation change. New divisors appear where the hyperplane section contains the special fiber or the directrix thus voiding the incidence condition. However, it becomes harder to trace this condition during a long degeneration.

Finally, the multiplicity statements become harder for unbalanced scrolls. Cones, especially, exhibit unexpected multiplicities. However, in small degree one can compute the characteristic numbers of unbalanced scrolls (see Example B2). We note that it is easy to see that each of the degenerations in Example B2 occur with multiplicity one by writing explicit first-order deformations, hence we omit a detailed argument.

9 The Gromov-Witten Invariants of \( G(1, N) \)

In this section we explain the relation between the characteristic numbers of balanced scrolls and the Gromov-Witten invariants of \( G(1, N) \).

Gromov-Witten Invariants. Recall that \( \overline{M}_{0,n}(X, \beta) \), the Kontsevich spaces of stable maps, come equipped with \( n \) evaluation morphisms \( \rho_1, \ldots, \rho_n \) to \( X \), where the \( i \)-th evaluation morphism takes the point \([C, p_1, \ldots, p_n, \mu]\) to the point \( \mu(p_i) \) of \( X \). Given classes \( \gamma_1, \ldots, \gamma_n \) in the Chow ring \( A^*X \) of \( X \), the Gromov-Witten invariant associated to these classes is defined by

\[
I_\beta(\gamma_1, \ldots, \gamma_n) = \int_{\overline{M}_{0,n}(X, \beta)} \rho_1^*(\gamma_1) \cup \cdots \cup \rho_n^*(\gamma_n).
\]

If \( X \) is a homogeneous space \( X = G/P \) and \( \gamma_i \) are fundamental classes of pure dimensional subvarieties \( \Gamma_i \) of \( X \), then there is a close connection between the enumerative geometry of \( X \) and the Gromov-Witten invariants given by Lemma 14 in \cite{FP}. We reproduce this lemma for the reader’s convenience. Assume

\[
\sum_{i=1}^n \text{codim}(\Gamma_i) = \dim(X) + \int_\beta c_1(T_X) + n - 3.
\]

Let \( g\Gamma_i \) denote the \( g \) translate of \( \Gamma_i \) for some \( g \in G \).

Lemma 9.1 Let \( g_1 \cdot \cdot \cdot, g_n \in G \) be general elements, then the scheme theoretic intersection

\[
\rho_1^{-1}(g_1\Gamma_1) \cap \cdots \cap \rho_n^{-1}(g_n\Gamma_n)
\]

is a finite number of reduced points supported in \( M_{0,n}(X, \beta) \) and the Gromov-Witten invariant equals the cardinality of this set

\[
I_\beta(\gamma_1, \ldots, \gamma_n) = \# \rho_1^{-1}(g_1\Gamma_1) \cap \cdots \rho_n^{-1}(g_n\Gamma_n).
\]
In the case of $M_{0,n}(\mathcal{G}(1,N), k+l)$ using Kleiman’s theorem we can, in fact, conclude that the intersection in $\mathfrak{D}$ is supported in the locus of maps to non-degenerate curves of directrix degree $\lfloor (k+l)/2 \rfloor$.

Assume the $\Gamma_i$ are Schubert cycles of the form $\Sigma_{a_i}$, the cycle of lines meeting an $a_i$ dimensional linear space. By Theorem 3.2 the cardinality of the intersection in $\mathfrak{D}$ is equal to the number of balanced scrolls meeting general linear spaces of dimension $a_i$, $1 \leq i \leq n$. We thus obtain the following corollary to Theorem 8.1:

**Corollary 9.2** Let $\Gamma_i = \Sigma_{a_i}$. Assume $a_i$ satisfy the conditions of Theorem 8.1. Then the algorithm described in Theorem 8.1 provides an algorithm for computing $I_{k+l}(\gamma_1, \cdots, \gamma_n)$

**Remark.** One has to exercise caution when translating the number of quadric surfaces to degree 2 Gromov-Witten invariants of Grassmannians. Our algorithm counts actual quadric surfaces. Since quadric surfaces can be seen as scrolls in two distinct ways depending on the choice of ruling, the Gromov-Witten invariant is twice the number of quadric surfaces.

A closer analysis of the algorithm in Theorem 8.1 shows that it computes the number of balanced scrolls of degree $k+l$ containing a section class of degree $k+l$ or $k+l+1$ subject to the non-degeneracy assumptions. By an argument similar to the one just given, our algorithm computes certain Gromov-Witten invariants of $F(0,1;N)$. For a sample of different approaches to the Gromov-Witten invariants of Grassmannians and Flag manifolds see [Ci], [BKT] or [Tam].

We conclude with a table of characteristic numbers of surfaces. We use the notation $n(N; k,l; a_0, a_1, \cdots, a_k)$ to denote the number of $S_{k,l}$ in $\mathbb{P}^N$ that meet $a_0$ points, $a_1$ lines, $\cdots$, $a_k$ $k$-planes.

| $n(4;1,1;4,5)$ | $n(4;1,2;9,0)$ |
|----------------|----------------|
| $= 1$          | $= 2$          |
| $n(4;1,1;3,7)$ | $n(4;1,2;8,2)$ |
| $= 9$          | $= 17$         |
| $n(4;1,1;2,9)$ | $n(4;1,2;7,4)$ |
| $= 64$         | $= 138$        |
| $n(4;1,1;1,11)$| $n(4;1,2;6,6)$ |
| $= 430$        | $= 1140$       |
| $n(4;0,2;4,4)$ | $n(4;1,2;5,8)$ |
| $= 4$          | $= 9770$       |
| $n(4;0,2;3,6)$ | $n(5;1,2;4,5,1)$|
| $= 30$         | $= 58$         |
| $n(4;0,2;2,8)$ | $n(5;1,2;4,4,3)$|
| $= 190$        | $= 423$        |
| $n(5;1,1;3,0,8)$| $n(5;2,2;9,1)$ |
| $= 48$         | $= 6$          |

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