SO and Sp Chern-Simons at Large N

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We study the large $N$ limit of $SO(N)$ and $Sp(N)$ Chern-Simons gauge theory on $S^3$ and identify its closed string dual as topological strings on an orientifold of the small resolution of the conifold. Applications to large $N$ dualities for $\mathcal{N} = 1$ supersymmetric gauge systems in 4 dimensions are also discussed.
1. Introduction

The aim of this note is to extend the large $N$ duality conjecture of [1] which relates $SU(N)$ Chern-Simons gauge theory on $S^3$ to topological strings on the small resolution of the conifold, to the case of the $SO(N)$ and $Sp(N)$ gauge theories. The basic idea is to act by orientifolding on the duality of [1] and obtain the new duality for $SO(N)$ and $Sp(N)$ cases. The main subtlety arises in the exact identification of the parameters on both sides. This we fix by various consistency checks.

The conjecture of [1] has been embedded in type II superstring theory in [2]. Furthermore this duality has been restated in purely geometric setup by embedding type IIA superstring in M-theory [3]. Similarly we can raise the same questions for the case of $SO$ and $Sp$ gauge group. In particular we show how to embed this duality in type IIA superstrings and interpret it in purely geometric terms by further embedding it in M-theory.

The organization of this paper is as follows: In section 2 we review the Large $N$ conjecture for $SU(N)$ Chern-Simons theory. In section 3 we propose a large $N$ conjecture for $SO(N)$ and $Sp(N)$ Chern-Simons theory on $S^3$. In section 4 we consider the partition function of the Chern-Simons theory for these classes of gauge groups and perform a large $N$ expansion. In section 5 we compare the results with expectations based on the conjectured large $N$ dual. In section 6 we consider connections with $\mathcal{N} = 1$ systems in 4 dimensions.

2. The Large $N$ conjecture for $SU(N)$ Chern-Simons Theory

In this section we briefly review the conjecture of [1] which relates large $N$ limit of $SU(N)$ Chern-Simons gauge theory on $S^3$ to a particular topological string. The conjecture in [1] states that the Chern-Simons gauge theory on $S^3$ with gauge group $SU(N)$ and level $k$ is equivalent, at large $N$, to the closed topological string theory of $A$-type on the $S^2$ blown up conifold geometry with

$$g_s = \frac{2\pi i}{k + N}, \quad t = \frac{2\pi i N}{k + N},$$

(2.1)

where $g_s$ is the string coupling constant and $t$ is the Kähler modulus of the blown-up $S^2$. The coupling constant $g_{CS}$ of the Chern-Simons theory, after taking into account the finite renormalization, is related to $g_s$ as $g_s = g_{CS}^2$. Therefore the Kähler moduli $t$ given by (2.1) is the ’t Hooft coupling $g_{CS}^2 N$ of the Chern-Simons theory.
The geometric motivation of the conjecture is based on starting with the topological strings on conifold geometry $T^* S^3$ and putting many branes on $S^3$, for which we get a large $N$ limit of Chern-Simons on $S^3$ supported on the brane $[4]$. The conjecture states that in the large $N$ limit the branes disappear but the geometry gets deformed and an $S^2$ gets blown up.

The conjecture has been checked for the free energy to all orders in the $1/N$ expansion (since both sides are computable) as well as a large class of Wilson loop expectation values $[3] [4] [5] [6]$. In particular the partition function $Z_{SU}(S^3)$ of the $SU(N)$ Chern-Simons gauge theory on $S^3$ is given by

$$Z_{SU}(S^3) = e^{i \frac{\pi}{2} N} \frac{(N-1)^N}{(k+N)^{N/2}} \sqrt{\frac{k+N}{n}} \prod_{s=1}^{N-1} \left[ 2 \sin \left( \frac{N}{k+N} \right) \right]^{N-s}.$$  \hspace{1cm} (2.2)

The large-$N$ expansion of $\log Z_{SU}(S^3)$ is given by

$$Z_{SU}(S^3) = \exp \left[ \sum_{g=0}^{\infty} g_s^{2g-2} F_g(t) \right],$$ \hspace{1cm} (2.3)

where $g_s$ and $t$ as in (2.1),

$$F_0 = -\zeta(3) + \frac{i \pi^2}{6} t - i \left( m + \frac{1}{4} \right) \pi t^2 + \frac{i}{12} t^3 + \sum_{n=1}^{\infty} n^{-3} e^{-nt}$$

$$F_1 = \frac{1}{24} t + \frac{1}{12} \log (1 - e^{-t}),$$ \hspace{1cm} (2.4)

with $m$ being some integer, and for $g \geq 2$,

$$F_g = \frac{(-1)^{g-1}}{2g(2g-2)} B_g \left[ \frac{(-1)^{g-1}}{(2\pi)^{2g-2}} 2\zeta(2g-2) - \frac{1}{(2g-3)!!} \sum_{n=1}^{\infty} n^{2g-3} e^{-nt} \right].$$ \hspace{1cm} (2.5)

Here $B_g$ is the Bernoulli number. It turns out that the expressions (2.4) and (2.5) for $F_g$ are exactly those of the $g$-loop topological string amplitude on the resolved conifold. These expressions can be derived, as was done in [9], from the target space viewpoint by identifying what the topological strings compute in Type IIA compactification on the corresponding Calabi-Yau space. They can also be derived using the mathematical definition of topological string amplitudes [10].

The geometric transition underlying this large $N$ duality is the conifold transition which is reviewed below.
2.1. Conifold transition

Conifold is described by $z_1 z_4 - z_2 z_3 = \mu$. It is a non-compact Calabi Yau manifold. This manifold is also described by $T^* S^3$. The topology of the manifold is that of a deformed cone with base $= S^2 \times S^3$. To see this by substituting

\begin{align*}
  z_1 &= y_1 + iy_2 \\
  z_2 &= -y_3 + iy_4 \\
  z_3 &= y_3 + iy_4 \\
  z_4 &= y_1 - iy_2
\end{align*}

the conifold equation becomes $y_1^2 + y_2^2 + y_3^2 + y_4^2 = \mu$. Taking the real section of this we notice that there is an $S^3$ (and the imaginary parts give the cotangent directions). As $\mu = 0$, $S^3$ shrinks to zero size and the manifold becomes singular. The singularity can be removed by what is called a “small resolution” which is to replace the origin $z_i = 0$ by an $S^2$. In particular the $S^2$ is parameterized by a complex coordinate $z$ which is defined by $z_1 = zz_2$ or $z_3 = zz_4$ (note that the two are consistent because $z_1 z_4 - z_2 z_3 = 0$). The resulting manifold (in some patch) is now described by three complex coordinates $(z, z_1, z_4)$. This is what is called the conifold transition. We go from having an $S^3$ submanifold at the tip of the cone to having an $S^2$. The topological string for $SU(N)$ Chern-Simons gauge theory is obtained by having $N$ D3 branes wrapping the $S^3$. At large $N$ the better description is in terms of the blowup geometry where the $S^3$ is replaced by $S^2$ of finite size and the branes have disappeared.

3. The $SO(N)$ and $Sp(N)$ Chern-Simons Duals

In this section we extend the duality of [1] to the case of $SO$ and $Sp$ gauge groups. The basic idea is to start with the duality of [1] for the $SU(N)$ gauge group and orientifold both sides. On the side of the conifold with finite size $S^3$ the orientifolding should fix the $S^3$. This leads, depending on the choice of the sign for worldsheets with crosscaps, to an $SO(N)$ or $Sp(N)$ Chern-Simons gauge theory, as is familiar in the context of D-branes.

We consider the following involution in the $S^3$ conifold geometry:

\begin{align*}
  z_1 &\rightarrow \bar{z}_4; & z_2 &\rightarrow -\bar{z}_3
\end{align*}
Note that this $Z_2$ operation leaves the $S^3$ invariant. Orientifolding the conifold geometry by this involution and placing D-branes on $S^3$ leads to $SO(N)$ or $Sp(N)$ Chern-Simons gauge theories on $S^3$.

To find out what the dual of this theory is, all we have to do is to orientifold the dual for the $SU(N)$. In other words we have to orientifold the conifold after the transition where we have a blown up $S^2$. To see how this involution acts on the blownup geometry consider the coordinate chart given by $(z_1, z_4, z)$. From the identification of $z = z_1/z_2$, we see that the involution (3.1) maps to

$$(z_1, z_4, z) \rightarrow (\bar{z}_4, \bar{z}_1, -\frac{1}{\bar{z}}). \tag{3.2}$$

Thus we obtain the dual topological strings by orientifolding the $S^2$ blown up geometry by this involution. Note that the orientifolding action takes $z$ which is the coordinate describing the blown up $S^2$ to $-\frac{1}{\bar{z}}$. This makes the $S^2$ into $RP^2$. So now the target space geometry on the closed string side has an $RP^2$ instead of $S^2$. Note also that this orientifolding has no fixed points (and thus no orientifold planes). The next thing to do is to find the precise map between parameters of the two sides which we will now turn to.

3.1. Large $N$ expansion parameters

We need to identify the parameters of the gauge theory with parameters for strings propagating in the blow up of the conifold geometry. This involves identification of the string coupling constant as well as the size of the $S^2$ with gauge theory parameters. At the tree level the gauge theory coupling constant, which should be identified with string coupling is $1/k$. But just as in the $SU(N)$ case one expects a shift in $k$. In particular from the gauge theory side $\frac{2\pi i}{(k+c_g)}$ is the renormalised coupling. Thus it is natural to identify the string coupling constant, also on the blow up side with that, i.e.

$$g_s = \frac{2\pi i}{k + c_g}$$

for $SO(N)$, we have $c_g = N - 2$ and for $Sp(N)$ it is $c_g = \frac{N}{2} + 1$ (where in the $Sp$ case $N$ is even, and the rank of it is $N/2$). One also has to identify the volume of $S^2$ with some function of ‘t Hooft parameter. In the case of $SU(N)$ the natural identification was $t = Ng_s = Ng_M^2$. It turns out, however, that the natural match in the case of $SO$ and $Sp$ groups is slightly different and we find

$$t = (N + a)g_s.$$
Moreover for $SO$ case $a = -1$ and for $Sp$ case $a = +1$.

To motivate the replacement of $N$ by $N + a$ we proceed as follows: The dual string theory doesn’t see the number of D branes but it sees the amount of D-brane “flux”. This notion is not very precise in the case of the topological string because there is no gauge field coupled to D-brane flux. Nevertheless we will use the intuition based on D-branes in ordinary superstrings and bosonic strings to find the net D-brane “flux”, which is to replace $N$ on the dual gravity side. This comes from the fact that if we have orientifold planes, they do carry D-brane flux. In particular if we have an orientifold plane of dimension $r$, in a string theory which has critical dimensions $d$, the D-brane charge carried by the orientifold plane is

$$a = \mp 2^{d-r}$$

where $\mp$ depends on the choice of the sign for diagrams with crosscap. In particular if we have $SO$ groups the $-$ sign applies and if we have $Sp$ group the $+$ sign applies. If we apply this to topological strings with $d = 6$ and noting that the orientifold plane for us is $S^3$ which has dimension $r = 3$, we learn that

$$a = \mp 1.$$  

This motivates our choice of the identification of the size of $S^2$ with gauge theory parameters. In the next section we show why these identifications are also natural from the point of view of gravitational anomalies for Chern-Simons theory.

3.2. Anomaly Analysis

Since the Chern-Simons theory is a topological theory one expects that it is independent of the background metric. The classical lagrangian is independent of the background metric. But it was argued [11] that at quantum level there is a term which does depend on the background metric. It is of the form:

$$\frac{i\pi c}{12} \int_{S^3} (w \wedge dw) + 2/3(w \wedge w \wedge w) \quad (3.3)$$

Where $w$ is the spin connection and $c$ refers to the central charge of the WZW current algebra at level $k$ for the group that appears in the Chern-Simons theory. $c$ is given by $c = \frac{k(dim(G))}{k+c_g}$. Here $dim(G)$ refers to the number of generators that the gauge group has. So for $SO(N)$ case:

$$c_g = N - 2; \quad dim(G) = \frac{(N^2 - N)}{2} \quad (3.4)$$
As discussed in [1] the existence of gravitational anomaly should be accompanied by terms from topological string amplitudes which contain the characteristic class $R \wedge R$. These terms can only enter if constant maps contribute. The relevant topological string in the present context is the one on the resolved conifold geometry modded out by the orientifold action. Let us write the anomaly term above in terms of the parameters $t$ and $g_s$ using $(N-1) = t/g_s$ and $g_s = 2\pi i/(k + N - 2)$. Then the coefficient of the anomaly term becomes

$$\frac{i\pi c}{12} = \frac{t^2(2\pi i - t)}{48g_s^2} + \frac{2\pi it}{48g_s} + \frac{t}{48}$$

(3.5)

From the above expression we notice that the sphere level term (i.e. $O(\frac{1}{g_s^2}$ term) and torus term (i.e. $O(1)$) is exactly half of the SU(N) answer, as should be the case because there should be no difference between their contribution before or after orientifolding (except for an overall factor of 1/2 as will be discussed in my detail later). The term of $O(\frac{1}{g_s})$ should be interpreted as the contribution from the worldsheet geometry being $RP^2$. Since the target is also $RP^2$ there are no relevant “constant maps” (what we mean is that there is no constant map from $P^1 \to P^1$ which is invariant under the $Z_2$ action which acts on both sides by $z \to -1/\overline{z}$). Such a term would have shown up at order $t^2/g_s$ which is indeed absent, as it should (note that the order $t$ terms is somewhat ambiguous as we need to take at least two derivatives of the $RP^2$ amplitude to fix the symmetries). Thus the above identification of parameters is consistent with the expectations based on anomalies on the dual closed string theory side. A similar story repeats for $Sp(N)$ with $a = +1$.

4. SO and Sp Chern-Simons Gauge Theories at Large $N$

In [11] the partition function of a Chern Simons theory with gauge group $G$ on base manifold $S^3$ was computed as $S_{00}$, i.e., a particular element of the modular transformation matrix associated to the corresponding WZW model. On the other hand $S_{00}$ is known for arbitrary groups and is given by

$$Z = S_{00} = |P/Q|^{-1/2(k + c_g)}^{-r/2} \prod_{\alpha \in \Delta^+} 2Sin\left(\frac{\pi(\alpha, \rho)}{(k + c_g)}\right).$$

(4.1)

Here $\alpha$ runs over the positive roots of the Lie algebra, $c_g$ is the dual Coxeter number and

$$\rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha$$

(4.2)
is the Weyl vector of the Lie Algebra. Here $P$ is the weight lattice and $Q$ is the root lattice. $|P/Q|$ refers to the cardinality of the quotient space and $r$ is the rank of the group. Note that $(\alpha, \rho)$ is an integer or half integer for every $\alpha$. Let $(\alpha, \rho)$ take the value $j$, $f(j)$ times as $\alpha$ runs over all positive roots. Then

$$Z = |P/Q|^{-1/2} (k + c_g)^{-r/2} 2^{\Delta^+} \prod_j \sin \left( \frac{\pi j}{k + c_g} \right)^{f(j)}. \quad (4.3)$$

Consequently, the free Energy is

$$F = -\log(Z) = 1/2 \log |P/Q| + \frac{r}{2} \log(k + c_g) - |\Delta^+| \log 2 - \sum_j f(j) \log \left( \sin \left( \frac{\pi j}{k + c_g} \right) \right). \quad (4.4)$$

We will compute the sum in (4.4) to find an explicit formula for $F$ for the case of $SO(N)$ and $Sp(N)$ gauge groups. Define

$$\lambda = -it = \frac{2\pi(N + a)}{(k + c_g)} \quad (4.5)$$

where $a = -1$ for $SO(N)$ gauge group and $a = +1$ for $Sp(N)$. The crucial piece in Free energy is the last term in (4.4); (the other terms will introduce a minor modification of the final result which we will note below). Let us continue calling this term by $F$. In other words we write

$$F = -\sum_j f(j) \log \left( \sin \left( \frac{\pi j \lambda}{2\pi(N + a)} \right) \right) \quad (4.6)$$

Using the product formula for $\sin(\pi x)$

$$\sin(\pi x) = \pi x \prod_{n=1}^{\infty} \left(1 - \left(\frac{x}{n}\right)^2\right) \quad (4.7)$$

(4.6) becomes

$$F = -\sum_j f(j) \sum_{p=1}^{\infty} \ln(1 - \frac{j^2 \lambda^2}{4(N + a)^2 p^2 \pi^2}) - \sum_j f(j) \log \left( \frac{j \lambda}{2(N + a)} \right) \quad (4.8)$$

The last term in (4.8) again is simple and we will incorporate it in the computation at the end. Let us concentrate on the first term, and still call it by $F$. On using the expansion

$$\log(1 - a) = -\sum_{m=1}^{\infty} \frac{a^m}{m} \quad (4.9)$$

first term in (4.8) becomes

$$\sum_{m=1}^{\infty} \sum_j f(j) \left( \frac{j \lambda}{2\pi(N + a)} \right)^{2m} \frac{\zeta(2m)}{m}. \quad (4.10)$$

In order to evaluate $F$ we must thus evaluate the sum

$$\sum_j f(j) j^{2m}, \quad (4.11)$$

which we will now turn to.
4.1. SO(N) with N Even

The computation depends on which group we are dealing with and it is simplest for the simply laced group SO(N) with N even, which we will first compute. In order to evaluate \( F \) we must thus evaluate the sum

\[
\sum_{j=1}^{N-2} f(j)j^{2m}. \tag{4.12}
\]

\( f(j) \) may be computed by considerations of the root lattice of SO(N) with the result

\[
f(j) = \begin{cases} 
\frac{(N+1-j)}{2} & \text{j odd < N/2} \\
\frac{(N-j)}{2} & \text{j odd \geq N/2} \\
\frac{(N-j-2)}{2} & \text{j even < N/2} \\
\frac{(N-1-j)}{2} & \text{j even \geq N/2.}
\end{cases}
\tag{4.13}
\]

Thus the summation (4.12) reduces to

\[
\sum_{j=1}^{N-2} \frac{(N-1-j)}{2} j^{2m} = \sum_{j=1}^{(N/2-1)} j^{2m} - 2^{2m-1} \sum_{j=1}^{(N/2-1)} j^{2m}. \tag{4.14}
\]

Consequently, (4.10) takes the form

\[
F = \sum_{m=1}^{\infty} \sum_{j=1}^{N-2} \frac{(N-1-j)}{2} j^{2m} \frac{\lambda}{2\pi(N-1)} \frac{2m}{j^{2m}} \frac{\zeta(2m)}{m} \\
+ \sum_{m=1}^{\infty} (1-2^{2m-1}) \sum_{j=1}^{(N/2-1)} j^{2m} \frac{\lambda}{2\pi(N-1)} \frac{2m}{j^{2m}} \frac{\zeta(2m)}{m}. \tag{4.15}
\]

Performing the summation over \( j \) using the formulae

\[
\sum_{j=1}^{k} j^l = \frac{(k+\frac{1}{2})^{l+1}}{l+1} + \sum_{g=1}^{\left\lfloor \frac{l}{2} \right\rfloor} \frac{2^{1-2g}}{(l+1)(2g)} \left( l+1 \right)^{l+1} \left( -1 \right)^{g-1} B_g \left( 1 - 2^{2g-1} \right) \left( k + \frac{1}{2} \right)^{l+1-2g}
\]

\[
\sum_{j=1}^{k} j^l = \frac{(k+1)^{l+1}}{l+1} - \frac{1}{2} (k+1)^l + \frac{1}{l+1} \sum_{g=1}^{\left\lfloor \frac{l}{2} \right\rfloor} \frac{(l+1)}{2g} \left( l+1 \right)^{l+1} \left( -1 \right)^{g-1} B_g \left( k+1 \right)^{l+1-2g}
\]
we find that the first term in the (4.15) is exactly half of the SU(N) answer and is given by the first three terms below and the second term above leads to the last two terms below:

\[
F = \sum_{m=1}^{\infty} (N-1)^2 \frac{\zeta(2m)}{2m(2m+1)(2m+2)} \left( \frac{\lambda}{2\pi} \right)^{2m} - \\
\sum_{m=1}^{\infty} \frac{\zeta(2m)}{4m} \left( \frac{\lambda}{2\pi} \right)^{2m} + \\
\sum_{g=2}^{\infty} (N-1)^{2-2g} \frac{(-1)^g \beta_g}{4g(2g-2)} \sum_{m=1}^{\infty} \zeta(2g-2+2m) \left( \frac{2g-3+2m}{2m} \right) \left( \frac{\lambda}{2\pi} \right)^{2g-2+2m} + \\
\sum_{m=1}^{\infty} (N-1) \left( \frac{1-2^{2m-1}}{2^{2m}} \right) \frac{\zeta(2m)}{2m(2m+1)} \left( \frac{\lambda}{2\pi} \right)^{2m} + \\
\sum_{m=1}^{\infty} 2(N-1)^{1-2g} (1-2^{2m-1}) \left( \frac{\lambda}{2\pi} \right)^{2m} \left( \frac{1}{2} \right)^{2m+1-2g} \frac{\zeta(2m)}{2m(2m+1)} \left( \frac{2m+1}{2g} \right) (-1)^{g-1} \beta_g (2^{1-2g} - 1). 
\]

The first three terms in the above expression correspond to half of the SU(N) answer and is given by (2.4) and (2.5). We rewrite those results here with the correct factor of half. So the first term in (4.16) gives the tree level answer which is:

\[
F_0 = \frac{-\zeta(3)}{2} + \frac{i\pi^2}{12} t - i \left( \frac{m}{2} + \frac{1}{8} \right) \pi t^2 + \frac{i}{24} t^3 + \sum_{n=1}^{\infty} \frac{n^{-3}}{2} e^{-nt} \quad (4.17) 
\]

The second term in (4.16) gives the one loop answer and is given by:

\[
F_1 = \frac{1}{48} t + \frac{1}{24} \log \left( 1 - e^{-t} \right) \quad (4.18) 
\]

The third term in (4.16) gives the higher loop answer and is given by:

\[
F_g = \frac{(-1)^{g-1}}{4g(2g-2)} \beta_g \left[ \frac{(-1)^{g-1}}{(2\pi)^{2g-2}} 2\zeta(2g-2) - \frac{1}{(2g-3)!} \sum_{n=1}^{\infty} n^{2g-3} e^{-nt} \right]. \quad (4.19) 
\]

The fourth term corresponding to \( O(1/g_s) = O(N) \) term (which will correspond to the worldsheet \( RP^2 \) contribution is:

\[
(N-1) \sum_{m=1}^{\infty} \left( \frac{1-2^{2m-1}}{2^{2m}} \right) \frac{\zeta(2m)}{2m(2m+1)} \left( \frac{\lambda}{2\pi} \right)^{2m}. \quad (4.20) 
\]

\[
\frac{(N-1)2\pi}{\lambda} \sum_{m=1}^{\infty} \left( \frac{2^{2m+1} - 1}{2^{2m+1}} \right) \frac{\zeta(2m)}{2m(2m+1)} \left( \frac{\lambda}{2\pi} \right)^{2m+1} \quad (4.21) 
\]

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Writing the above expression in terms of \( g_s \) and \( \lambda \)

\[
\frac{2\pi i}{g_s} \sum_{m=1}^{\infty} \left( \frac{2\zeta(2m)}{2m(2m+1)} \left( \frac{\lambda}{4\pi} \right)^{2m+1} - \frac{\zeta(2m)}{4m(2m+1)} \left( \frac{\lambda}{2\pi} \right)^{2m+1} \right)
\]

The above expression can be rewritten as:

\[
\frac{1}{g_s} \left\{ \sum_{n=\text{odd}..}^{\infty} \frac{1}{n^2} \exp(-nt/2) + at + b \right\}
\]

The above term also includes the contribution of the terms we dropped before which only enter into the terms \( at + b \).

Finally, the last term in (4.16), upon substituting \( m = g + p - 1 \), takes the form

\[
\sum_{g=1}^{\infty} 2(N - 1)^{1-2g} \frac{(-1)^{g-1}B_g}{2g(2g-1)} (1 - 2^{2g-1})
\]

\[
\sum_{p=1}^{\infty} \left( \frac{\lambda}{4\pi} \right)^{2g+p-1} \zeta(2g+2p-2) \left( \frac{2g+2p-3}{2g-2} \right) - \frac{1}{2} \left( \frac{\lambda}{2\pi} \right)^{2g+p-1} \zeta(2g+2p-2) \left( \frac{2g+2p-3}{2g-2} \right)
\]

After doing the summation over \( p \), the \( \lambda \) dependent part in (4.24) becomes

\[
\sum_{g=1}^{\infty} \left( \frac{\lambda}{N-1} \right)^{2g-1} \frac{(-1)^{g-1}B_g}{4g(2g-1)} (1 - 2^{2g-1}) \left\{ \sum_{n=\text{odd}Z} \frac{1}{(2\pi n + \lambda)^{2g-1}} - \sum_{n=\text{even}Z} \frac{1}{(2\pi n + \lambda)^{2g-1}} \right\}
\]

This, when written in terms of string variables i.e. \( g_s \) and Kahler parameter \( t \) the above expression becomes

\[
\sum_{g=1}^{\infty} (g_s)^{2g-1} \frac{(-1)^{g-1}B_g}{4g(2g-1)} (1 - 2^{2g-1}) \left\{ \sum_{n=\text{odd}Z} \frac{1}{(2\pi in + t)^{2g-1}} - \sum_{n=\text{even}Z} \frac{1}{(2\pi in + t)^{2g-1}} \right\}
\]

In terms of the worldsheet instantons (i.e.\( \exp(-t) \)) this expression becomes

\[
\sum_{g=1}^{\infty} (g_s)^{2g-1} \frac{(-1)^{g-1}B_g}{(2g)!} \left( 1 - \frac{1}{2^{2g-1}} \right) \sum_{k=\text{odd}1,3..}^{\infty} k^{2g-2} \exp(-kt/2)
\]

In terms of the worldsheet instanton i.e. Kahler parameter \( t \) and \( \exp(-t) \) this becomes, including the \( O(1/g_s) \) term,

\[
\frac{1}{2} \left\{ \sum_{n=1}^{\infty} \left( \frac{\exp(-nt/2)}{2n\sin(n(g_s/2))} \right) - \sum_{n=1}^{\infty} (-1)^n \left( \frac{\exp(-nt/2)}{2n\sin(n(g_s/2))} \right) \right\} + \frac{at + b}{g_s} + c
\]

where \( a, b, c \) include the contribution of the terms we neglected above.
4.2. SO(N) theory when N is odd

We do the calculation for SO(N) where N is odd; Here also we define \( c_g \) and \( \lambda \) in the same way as in the previous case i.e.

\[
c_g = N - 2; \quad \lambda = \frac{2\pi(N - 1)}{(k + N - 2)}
\]

\[
g_s = \frac{2\pi i}{(k + N - 2)}; \quad t = i\lambda
\]

\( f(j) \) in this case differs from the previous case and is given by

\[
f(j) = \begin{cases} 
1 & j = \frac{2k-1}{2} \ (k = 1, \ldots, \frac{(N-1)}{2}) \\
\frac{(N-1-j)}{2} & j \text{ even} \\
\frac{(N-2-j)}{2} & j \text{ odd}
\end{cases}
\]

Thus the summation \( (4.12) \) reduces to

\[
\sum_{j=1}^{N-2} \frac{(N - 1 - j)}{2} j^{2m} + \frac{(1 - 2^{2m-1})}{2^{2m}} \sum_{j=\text{odd},3..}^{N-2} j^{2m}
\]

So carrying out the summation in the expression

\[
\sum_j f(j) j^{2m} \left( \frac{\lambda}{2\pi(N - 1)} \right)^{2m} \frac{\zeta(2m)}{m}
\]

We first sum over \( j \) and then substitute \( m = g + p - 1 \) and then sum over \( p \) to get exactly the same expression as was obtained in \( SO(N) \) where \( N \) is even, as is expected for a consistent large \( N \) analysis of \( SO(N) \) gauge theory, which should not be sensitive to the parity of \( N \).

4.3. Sp(N) theory

We use the notations in which \( N \) is even.

\[
c_g = \frac{N}{2} + 1; \quad \lambda = \frac{2\pi(N + 1)}{(k + \frac{N}{2} + 1)}
\]

and

\[
g_s = \frac{\pi i}{(k + \frac{N}{2} + 1)}; \quad t = i\frac{\lambda}{2}
\]
\( f(j) \) in this case is given by the following expression

\[
f(j) = \begin{cases} 
\frac{(N-1-j)}{2} & j \text{ odd} \leq \frac{N}{2} \\
\frac{(N-j)}{2} & j \text{ even} < \frac{N}{2} \\
\frac{(N+1-j)}{2} & j \text{ odd} > \frac{N}{2} \\
\frac{(N+2-j)}{2} & j \text{ even} > \frac{N}{2}
\end{cases} \tag{4.36}
\]

Thus the summation \( (4.12) \) reduces to

\[
\sum_{j=1}^{N} \left( \frac{(N+1-j)}{2} \right) j^{2m} + (2^{2m-1} - 1) \sum_{j=1}^{N/2} j^{2m} \tag{4.37}
\]

Comparing this to \( (4.14) \) we see that except for the sign change on the second term, and replacing \( N - 1 \) by \( N + 1 \), it is exactly the same result, and we can thus readily write the result, which is summarized below.

4.4. Summary

Let us now summarize what we have found. For the case of \( S O(N) \) we have found that the partition function of the Chern-Simons theory on \( S^3 \) can be written in terms of the natural variables of the closed topological string \( (g_s, t) = (\frac{2\pi i}{k+N-2}, i \frac{N-1}{k+N-2}) \) as

\[
F_{SO}(g_s, t) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-nt}}{n[2 \sin(n g_s/2)]^2} + \frac{1}{2} \left\{ \sum_{n=1}^{\infty} \left( \frac{e^{-nt/2}}{2n \sin(n g_s/2)} \right) - \sum_{n=1}^{\infty} (-1)^n \left( \frac{e^{-nt/2}}{2n \sin(n g_s/2)} \right) \right\} \tag{4.38}
\]

where there is in addition a finite polynomial of order three in \( t \). The \( Sp \) answer is similar to the \( SO \) case and is given in terms of \( (g_s, t) = (\frac{2\pi i}{k+N+1}, i \frac{N+1}{k+N+1}) \) as

\[
F_{Sp}(g_s, t) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{e^{-nt}}{n[2 \sin(n g_s/2)]^2} - \frac{1}{2} \left\{ \sum_{n=1}^{\infty} \left( \frac{e^{-nt/2}}{2n \sin(n g_s/2)} \right) - \sum_{n=1}^{\infty} (-1)^n \left( \frac{e^{-nt/2}}{2n \sin(n g_s/2)} \right) \right\} \tag{4.39}
\]

again up to a finite polynomial in \( t \). Note that

\[
F_{Sp}(g_s, t) = F_{SO}(g_s, t + 2\pi i) \tag{4.40}
\]

5. Dual Topological string Interpretation

Now we try to interpret these results in the context of topological strings on the resolved conifold, modded out by a \( Z_2 \) orientifold. Namely we consider the \( O(-1) \) +
$O(-1)$ geometry over $\mathbf{P}^1$ modded out by an antiholomorphic involution, which in a local coordinate chart looks as $(z_1, z_4, z) \to (z_4, z_1, \frac{-z}{\bar{z}})$, where $z_1, z_4$ are coordinates along the fiber and $z$ is the coordinate along $\mathbf{P}^1$. On the worldsheet theory, we consider all closed orientable and non-orientable Riemann surfaces. It is well known that the non-orientable ones can be obtained from orientable Riemann surfaces by including one or two “crosscaps”, where a crosscap corresponds to a disc removed from the oriented Riemann surface and where the boundary points of the disc are identified by a reflection. The notion of orientifolding means that we consider maps from the Riemann surfaces to the target for which the worldsheet $Z_2$ involution at crosscaps are compatible with the anti-holomorphic $Z_2$ involution in the target. The Euler characteristic of a closed Riemann surface of genus $g$ is $\chi_0 = 2 - 2g$. The non-orientable ones obtained by adding one crosscap to a genus $g$ surface has Euler characteristic $\chi_1 = 1 - 2g$, and the non-orientable ones obtained by adding two crosscaps have $\chi_2 = -2g$. The difference between the $SO$ theories and the $Sp$ theories is that the Riemann surfaces with an odd number of crosscaps have a different relative minus sign. Note that the string partition function is weighted with $g_s^{\chi}$ and so we see that for a non-orientable Riemann surface with a single crosscap we have only odd powers of $g_s$, whereas for even number of crosscaps the power of $g_s$ is even.

Now we are ready to analyze the predictions (4.38) and (4.39) for closed topological string amplitudes. Note that the first term in both of them is given by half of the $SU(N)$ answer. Since the $SU(N)$ answer is given by orientable Riemann surfaces, and that is also part of what we should sum over here, that is as expected. We can also explain the overall factor of 1/2: When we mod out by a symmetry of order $|G|$ the genus $g$ amplitudes of closed Riemann surfaces will get an extra weight of $1/|G|^g$ (the Hilbert space interpretation is the projection operator acting for each handle). Here $|G| = 2$. Since the genus $g$ amplitude for orientable Riemann surface as genus $g$ is weighted by $(g_s^{SU})^{2g-2}$, by redefining the string coupling $g^{SO} = g^{SU}/\sqrt{2}$ we see that we get an overall factor of 1/2 in front of the answer we got for the closed orientable string theory. This explains the first terms in (4.38) and (4.39).

In addition we have to consider non-orientable Riemann surfaces with one or two crosscaps. However, the extra term in (4.38) and (4.39) have only odd powers of $g_s$ which implies that they correspond to non-orientable Riemann surfaces with a single crosscap.

1 Three crosscaps can be traded for a single crosscap and a handle.
Thus the partition function in this background corresponding to an even number of crosscaps must be zero. This is remarkably consistent with the fact that the terms with odd power of $g_s$ differ just by an overall sign between the $SO$ (4.38) and the $Sp$ (4.39) cases, as is expected for Riemann surfaces with an odd number of crosscaps.

Let us consider the amplitude corresponding to $RP^2$ (i.e. $g = 0$ with one crosscap). This corresponds to keeping the term of order $1/g_s$ in (4.38) (which up to an overall sign is the same as that for (4.39)):

$$
\frac{1}{g_s} \left\{ \sum_{n=odd,3}^{\infty} \frac{1}{n^2} \exp(-nt/2) + at + b \right\}.
$$

The exponential terms should correspond to holomorphic maps from $\mathbb{P}^1$ to $\mathbb{P}^1$ which are invariant under the simultaneous operation of $z \rightarrow -1/z$ on both $\mathbb{P}^1$’s. It is easy to see that this corresponds to the identity map and its odd covers (for example the $2n$ order cover $z(z') = (z')^{2n}$ or any other even order cover is not compatible with the $Z_2$ actions). This is a strong check for our proposed conjecture.

5.1. Schwinger Interpretation

We can try to check the predictions for the topological string amplitudes by connecting the predictions of topological strings to superpotential terms in superstring propagation on the corresponding CY, as was done in [9] for the orientable case. The case at hand is similar to [5] and we will consider a similar embedding for this purpose. Consider type IIA superstring on the resolved conifold geometry $O(-1) + O(-1)$ over $\mathbb{P}^1$ times $R^4$. Mod out by the orientifold action we have discussed for the internal Calabi-Yau, which also acts as $(-1,-1,1,1)$ on $R^4$. This gives a theory in $1+1$ dimensions (the invariant directions for the orientifold action in $R^4$), with 4 supercharges. This is similar to [5] except that there, instead of orientifolding one put some $D4$ branes in the resolved conifold geometry whose worldvolume consists of a Lagrangian submanifold of the Calabi-Yau times an $R^2$ subspace of Minkowski space. Thus the same arguments as [5] applies to this case. In particular turning on graviphoton field strength, relates the topological string amplitudes to Schwinger-like one loop computations of particles coupled to a background of constant field strength. The particles being related to wrapped D2 branes (with some number of D0 branes bound to them). The steps are exactly as in [5] and so we will not repeat them here.
The three different terms in (4.38) can now be interpreted accordingly. The first term corresponds to a D2 brane wrapped over $\mathbb{P}^1$, of BPS mass $t$, and propagating in $\mathbb{R}^4$ (which explains in particular the fact that there are two powers of $\sin$ in the denominator). The next two terms should correspond to particles moving in 2-dimensions, as there is only a single power of $\sin$ in the denominator. Moreover its BPS mass is $t/2$. This can be easily understood: Due to orientifolding the geometry if we consider a single $D2$ brane, whose worldline passes through the fixed point of $Z_2$ action in $\mathbb{R}^4$ which wraps only half of the $\mathbb{P}^1$ whose boundaries are identified due to the $Z_2$ action, it gives rise to a particle in 2 dimensions, with BPS mass $t/2$. The fact that there are two terms of this type in (4.38) is also easily understood. We can consider putting a single $D0$ brane dissolved in the $D2$ brane. Due to the $Z_2$ action this counts as a fractional brane with $1/2$ units of $D0$ brane charge. In other words the existence of two terms reflects the fact that the unit of $D0$ brane charge has changed due to the $Z_2$ orientifold action. Notice that one differs from the other (up to an overall sign) by shifting of $t \rightarrow t + 2\pi i$, which is what is expected for the effect of an extra $D0$ brane. The explanation of the relative sign between the two terms as well as between (4.38) and (4.38) must be due to changing the fermion number assignment for these particles.

6. Connection With $\mathcal{N} = 1$ Systems in $D = 4$

We can embed this duality into type IIa superstring and deduce some duality involving an $\mathcal{N} = 1$ gauge system in four dimensions. For the case of $SU(N)$ this was done in [2]. This was also recently reinterpreted as a geometric duality by embedding of type IIa strings in M-theory [3]. The situation at hand involves modding out by an extra $Z_2$ operation on both sides, and so it should go through.

Let us discuss briefly how this works: Consider type IIa strings in the deformed conifold background $T^*S^3$ with $N$ units of $D6$ branes wrapped on $S^3$. We mod this out by an orientifold $Z_2$ which in the internal Calabi-Yau preserves the $S^3$ and acts trivially on $\mathbb{R}^4$. This gives rise to $SO(N)$ or $Sp(N)$ gauge group living on the brane depending on the choice of the sign for the crosscap. In particular this leads to a sector of the theory with $\mathcal{N} = 1$ supersymmetric $SO(N)$ or $Sp(N)$ Yang-Mills theory.

In [2], which corresponds to the $SU(N)$ Yang-Mills theory, the moduli field associated to the (complexified) blow up mode $\mathbb{P}^1$ was identified as the vev of gaugino bilinear $t = S = g_s Tr W^2$. The lowest component of the field gets a vev which means that we have
$\langle Tr \lambda^2 \rangle \neq 0$. The value of the modulus was determined by extremizing a superpotential $W$ which depends on $N$, and the bare coupling of the gauge theory. In the limit that the bare coupling is small we have a decoupled gauge theory system and the superpotential becomes essentially

$$W = NS\log S + aS$$

and $dW/dS = 0$ gives $S^N = \text{const}$ which is in agreement with the $N$ vacua expected for the $\mathcal{N} = 1$ Yang-Mills theory. The term proportional to $N$ above, comes from the fact that there are $N$ units of $RR$ flux piercing through the $S^2$.

In the case at hand we are acting by orientifolding on both the gauge theory side and its dual. On the dual side we have no branes left and we have the type IIa background on the orientifold of the blownup resolution of the conifold times Minkowski space, where the orientifold acts in the internal part as described before and acts trivially on the Minkowski spacetime. The story is similar to that in [2] with two minor differences: First of all the flux is halved by the orientifolding operation and also shifted by the fact that on the gauge theory side we also have an orientifold which does carry RR-charge of $\mp 4$ units, for $SO, Sp$ cases respectively which leads to $\frac{N}{2} \rightarrow \frac{N}{2} \mp 2$. In addition we will now also have a superpotential coming from the $RP^2$ worldsheet (the term proportional to $1/g_s$). In the small $t = S$ limit this gives $\pm S\log S$ and so altogether in the decoupled limit we have

$$W = (\frac{N}{2} \mp 1)S\log S + aS$$

which gives $\frac{N}{2} \mp 1$ vacua. This agrees with the expected answer for the $O(N)$ and $Sp(N)$ case [12]. Also whether there are $N \mp 2$ vacua or half as many depends on the normalization assigned to $W$ and may reflect the ambiguities in the global choices for groups [12] which would be interesting to better understand in connection with the global issues in realization of gauge group in string theory.

6.1. Embedding in M-theory

One can also follow [3] and embed this construction in M-theory in the context of a $G_2$ holonomy manifold which is topologically $R^4 \times S^3$ modded out by a discrete group. This is easiest to do for the case of $SO$(even) which is the only case we will consider here.\footnote{The non-simply laced case can be obtained by introducing a suitable $Z_2$ involution.}
The only modification compared to $[3]$ in deriving the large $N$ duality for type IIa strings is that there is an extra $Z_2$ action before and after the $S^3$ flop. We write the 7-fold as

$$|z_1|^2 + |z_2|^2 - |z_3|^2 - |z_4|^2 = V$$

where the $S^3$ before the flop ($V > 0$) is identified with the locus $z_3 = z_4 = 0$. Mod out by the Dihedral group generated by

$$(z_3, z_4) \rightarrow (\omega z_3, \omega^{-1} z_4)$$

$$(z_3, z_4) \rightarrow (z_4, -z_3)$$

where $\omega^{N-4} = 1$. We are assuming $N$ is even. Note that by introducing a complex conjugate variable for $z_4 \rightarrow \overline{z}_4$, this action can also be viewed as

$$(z_3, z_4) \rightarrow (\omega z_3, \omega z_4)$$

$$(z_3, z_4) \rightarrow (\overline{z}_4, -\overline{z}_3)$$

This way of writing it is easier to use in inferring its dual description.

The fixed locus of this action is $S^3$ and it has a D-singularity corresponding to $SO(N)$ gauge symmetry. The flopped geometry gives the dual theory. This is where $V < 0$. In this case the group action has no fixed points.

To interpret this in terms of type IIa string we have to choose the ‘11-th’ direction. We choose it to be identified with the circle $(z_3, z_4) \rightarrow (e^{i\theta} z_3, e^{i\theta} z_4)$. In this identification it is easy to see that with $V > 0$ we have $N$ D6 branes, which have been orientifolded with $S^3$ times the Minkowski space, being identified as the orientifold plane. For $V < 0$ we have the same fibration giving rise to a Hopf fibration $S^3 \rightarrow S^2$ where the complex coordinate on $S^2$ is identified with $z = z_3/z_4$. Moreover the fact that the group we are modding out has a cyclic element of order $N - 4$ in the direction of the eleventh circle, implies that in type IIa perspective we have $N - 4$ units of RR flux through the $S^2$ (note that it does not act on $z$). The extra $Z_2$ generator acts by taking $z \rightarrow -1/\overline{z}$ (and acting also in some way over the fiber), which we identify with the orientifold action we have discussed above in the context of topological strings. The identification of parameters in the M-theory, ignoring Euclidean M2 brane instantons leads, as in $[3]$ to the formula $t = -V/(N - 4)g_s$ where $V$ is the volume of $S^3$ before transition and $t$ is the size of the $P^1$, all in type IIa string units. This would naively suggest $N - 4$ vacua. This in fact would agree with the
naive formula one would get from Type IIa string perspective if one ignores worldsheet instantons. However as discussed above for small $t$ the extra superpotential term, coming from the $RP^2$ worldsheet geometry, which lifts up to Euclidean M2 brane instantons in M-theory, will shift this to $N - 2$ vacua.

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