Analytic expression for the pull-or-jerk experiment

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Abstract
This work focuses on a theoretical analysis of a well-known inertia demonstration, which uses a weight suspended by a string with an extra string that hangs below the weight. The point in question is which string, the upper or lower, will break when pulling down on the bottom edge of the lower string at a constant pulling speed. An analytic solution for the equation of motion allows us to identify the critical value of the pulling speed, beyond which the string breaking varies from one to the other. The analysis also provides us with a phase diagram that illustrates the interplay between the pulling speed and the string’s elasticity in the pull-or-jerk experiment.

Keywords: inertia ball demonstration, harmonic oscillator, breaking string

1. Introduction

Suppose that a large and heavy object (weight) is suspended from a ceiling by a long light string, and a similar string is suspended from the bottom of the weight. If you jerk the lower string downward, the lower string breaks, but the upper string survives, and the weight remains suspended from the ceiling. In contrast, if you slowly pull the lower string downward, only the upper string breaks, and the weight falls off. These two contrasting observations, in other words a ‘pull-or-jerk’ experiment, originate from the inertia of the suspended weight; the inertia of the weight prevents the upper string from breaking when the lower string is jerked. In fact, this experiment has been repeatedly used in courses of elementary physics as a concise demonstration of inertia.

The underlying mechanism of the pull-or-jerk experiment is described below in a qualitative manner. A quick jerk causes a rapid increase in the tension of the lower string. Nevertheless, the inertia of the weight prevents an abrupt acceleration of the weight, which results in a further elongation in the lower string. Eventually, the tension in the lower string reaches a maximum value, which results in the lower string breaking when jerked quickly. In
contrast, a slow pull will break only the upper string because of the non-equivalence in the tensions between the two strings. The tension in the upper string is a sum of the restoring force plus the gravitational force on the weight, while the tension in the lower string only equals the restoring force. Therefore, the upper string will reach its maximum tension first when the lower string is pulled slowly.

Despite the qualitative, stereotypical explanation given above, the story does not end there. Using stretchable strings make what we observe richer. As such, no matter how slowly or quickly we pull the lower string, the elastic properties of the two strings will cause an oscillatory motion of the weight along the vertical direction. The oscillation then induces a time modulation in the tensions of the two strings. As a consequence, the problem is beyond a two-alternative choice, i.e., either a jerk or slow pull. Depending on the pulling speed and elastic properties of the strings, how each string breaks may vary. Thus, we are required to analytically solve the equation of motion and derive the time variation in the tensions, which will provide a good gedanken experiment and facilitate a student’s understanding of mechanics. Nevertheless, there have only been a few attempts at solving the problem in a quantitative manner [1–4].

This contribution provides an analytic description of the pull-or-jerk experiment by solving the equation of motion of the system. The analysis is based on the assumption that the bottom edge of the lower string moves linearly downward with time; this constant-pulling-speed condition is different from the existing studies, in which the pulling force acting on the bottom of the lower string varies linearly [1, 3, 4] or sinusoidally [2] with time. Special attention is paid to a closed-form solution for the critical pulling speed that causes the string to break from upper to lower (or vice versa). Physical interpretation of key parameters that determine the critical pulling speed is also clarified.

2. Model

Figure 1 shows a schematic diagram of the theoretical model considered herein. A weight with mass \( m \) is suspended by a stretchable string with a spring constant \( k_1 \) and original length \( L_1 \); another string with \( k_2 \) and \( L_2 \) is attached to the weight and dangled below it. The two strings can be assumed to be made of the same material without any loss of generality. Although they have the same Young’s modulus value, their spring constants should be in proportion to their respective lengths. We further assume that each string is massless and obeys Hooke’s law up to the failure at the maximum tension \( f_{\text{max}} \). The tension \( f_i \) (\( i = 1, 2 \)) in the \( i \)th string is expressed by

\[
f_i = k_i \Delta L_i \text{ (at } f_i < f_{\text{max}} \text{)},
\]

where \( \Delta L_i \) is an increment in the \( i \)th string’s length. Because the strings are massless, the tension in each string is the same throughout.

A pull-or-jerk experiment is initiated by pulling down on the bottom of the lower string with a constant speed \( V \). Because of the elastic properties of the two strings, the position of the weight \( x \) oscillates with time superposed on a linear ramp. During the downward pulling (i.e., at \( t > 0 \)), the length of the lower string is expressed by \( L_2 + Vt - x \), as seen from the right panel in figure 1. Hence, the tension that arises in the lower string is given by

\[
f_2 = k_2 \left[ (L_2 + Vt - x) - L_2 \right] = k_2(Vt - x).
\]
The tension of the upper string is also dependent on $x$ and is expressed as

\[ f_1 = k_1 x + mg. \]  \hfill (3)

Ignoring damping effects for simplicity, we obtain the equation of motion with respect to the weight as

\[ m \frac{d^2 x}{dt^2} = -(k_1 x + mg) + mg + k_2(Vt - x), \]  \hfill (4)

which can be rewritten as

\[ \frac{d^2 x}{dt^2} + \frac{k_1 + k_2}{m} x = \frac{k_2 V}{m} t. \]  \hfill (5)

Imposing the initial condition of $x = 0$ and $dx/dt = 0$ at $t = 0$, we obtain the solution of equation (5) as

\[ x = - \frac{k_2 V}{m \omega^2} \sin \omega t + \frac{k_2 V}{m \omega^2} t, \]  \hfill (6)

with the definition of

\[ \omega = \sqrt{\frac{k_1 + k_2}{m}}. \]  \hfill (7)

The solution given by equation (6) implies that the tensions $f_1$ and $f_2$ are in the form

\[ f_1 = - \frac{k_1 k_2 V}{m \omega^5} \sin \omega t + \frac{k_1 k_2 V}{k_1 + k_2} t + mg, \]

\[ f_2 = \frac{k_2^2 V}{m \omega^5} \sin \omega t + \frac{k_1 k_2 V}{k_1 + k_2} t. \]  \hfill (8)

For convenience, we introduce a normalized (i.e., dimensionless) pulling speed $v^*$, which was defined by

\[ \]
\[ \omega = k_2 V / m g \]  

(9)

in addition to the two parameters

\[ P_1 = k_2 g / \omega^2, \quad P_2 = k_2 g / \omega^2, \quad \left( \frac{P_1}{mg} = \frac{k_1}{k_1 + k_2}, \quad \frac{P_1 + P_2}{mg} = 1 \right). \]  

(10)

The physical meanings of \( v^* \), \( P_1 \), and \( P_2 \) will be discussed later. Using this nomenclature, equation (8) can be rewritten in a simple form as

\[ f_1 = v^* P_1 (ot - \sin ot) + mg, \]
\[ f_2 = v^* P_1 \cdot ot + v^* P_2 \cdot \sin ot, \]  

(11)

or normalized as

\[ \frac{f_1}{mg} = v^* \frac{P_1}{mg} (ot - \sin ot) + 1, \]  

(12)
\[ \frac{f_2}{mg} = v^* \frac{P_1}{mg} \left( ot + \frac{P_2}{P_1} \sin ot \right). \]  

(13)

These analytic formulations of \( f_1 \) and \( f_2 \) provide an important result on the pulling speed \( v^* \). Suppose that \( f_1 = f_2 \) at a specific time \( t = t_c \) that satisfies the relation

\[ -v^* P_1 \sin ot_c + mg = v^* P_2 \sin ot_c. \]  

(14)

This implies that

\[ \sin ot_c = \frac{1}{v^* \cdot \frac{mg}{P_1 + P_2}} = \frac{1}{v^*}. \]  

(15)

Therefore, \( v^* \geq 1 \) is necessary for the existence of \( t = t_c \), across which the magnitude relationship between \( f_1 \) and \( f_2 \) is reversed. In other words, \( v^* \geq 1 \) is a necessary condition for the occurrence of the transition from a ‘slow-pull’ to ‘sharp-jerk’ situation (and vice versa).

Figure 2 shows the time evolution of the normalized tensions \( f_1 / mg \) and \( f_2 / mg \) with respect to \( ot \). The parameter \( P_1 / mg \) is fixed as 0.3 in figure 2(a) and 0.8 in figure 2(b); the relation of \( P_2 / mg = 1 - P_1 / mg \) follows from the definition. When \( v^* < 1 \), two oscillating curves of \( f_1 / mg \) and \( f_2 / mg \) have no crossing point (see the inset of figure 2), which is expected from equation (15). Because \( f_1 > f_2 \) at any moment for the slow-pull case, the upper string must break regardless of the \( f_{max} \) value. This scenario fails if we speed up the pulling such that \( v^* > 1 \). For the jerk case, the two curves of \( f_1 / mg \) and \( f_2 / mg \) intersect periodically with time, and \( f_2 \) can exceed \( f_1 \) at a certain moment. As a consequence, the breaking string depends on the value of \( f_{max} \) as well as \( v^* \). These dependences are the main issue that we address in the subsequent discussion.

3. Physical interpretation of parameters introduced

Before proceeding with the argument, we discuss the physical meanings of \( v^* \), \( P_1 \), and \( P_2 \) that were introduced in equation (11). From their definitions, we can say that
\( v^* = \frac{k_2 V \cdot (1/\omega)}{mg} \), \( P_i = \frac{k_i}{k_1 + k_2} mg \quad [i = 1, 2]. \)

Namely, the normalized velocity \( v^* \) is the ratio between the gravitational force (i.e., \( mg \)) and the restoring force (i.e., \( k_2 V \cdot [1/\omega] \)) of the lower string elongated by a constant rate \( V \).
During a time $t / \omega$. $P_1$ and $P_2$ are the restoring forces of the upper and lower strings, respectively, under the condition that the system is clamped vertically at both ends (see figure 3); the downward displacement $\xi$ of the weight causes the upward forces $k_1 \xi$ and $k_2 \xi$ to exert a force on the weight. At equilibrium, the balanced forces are expressed as

$$mg = (k_1 + k_2) \xi.$$  

(18)

Thus, it follows that

$$k_i \xi = \frac{k_i}{k_1 + k_2} mg = P_i, \quad [i = 1, 2]$$  

(19)

which indicates that $P_i$ is the upward force $k_i \xi$ mentioned above.

4. Critical pulling speed

We are ready to derive a closed-form expression for the critical pulling speed. Equations (11) and (15) tell us that $f_2$ at $t = t_c$ can be written as

$$f_2 (t = t_c) = P_2 + v^* P_1 \sin^{-1} \left( \frac{1}{v^*} \right).$$  

(20)

The two strings will break simultaneously at $t = t_c$ if $f_2 (t = t_c)$ is equal to $f_{\text{max}}$. Therefore, the pull-to-jerk phase boundary is expressed by a series of points $\{v^*, f_{\text{max}}\}$ in the $v^* - f_{\text{max}}$ space, at which the following relation is satisfied

$$P_2 + v^* P_1 \sin^{-1} \left( \frac{1}{v^*} \right) = f_{\text{max}},$$  

(21)
or equivalently

\[
\frac{1}{v^*} = \sin \left( \frac{1}{v^*} \cdot \frac{f_{\text{max}} - P_2}{R_1} \right). \tag{22}
\]

To numerically solve equation (22) with respect to \(v^*\), we define

\[
G(v^*) = \sin \left( \frac{1}{v^*} \cdot \frac{f_{\text{max}} - P_2}{R_1} \right) - \frac{1}{v^*}. \tag{23}
\]

which has dimensionless parameters \(f_{\text{max}}/mg\) and \(R_1/mg\); note again that \(P_2/mg = 1 - (R_1/mg)\). Roots of the equation \(G(v^*) = 0\) vary in response to a change in \(f_{\text{max}}\). Therefore, collecting all the roots with increasing \(f_{\text{max}}\) allows us to draw the phase diagram of the pull-or-jerk experiment in the \(v^*-f_{\text{max}}\) space. In the actual calculation, the roots of \(G(v^*) = 0\) are detected using a conventional Newton–Raphson method.

Figure 4 shows the profile of \(G(v^*)\) for various conditions of \(f_{\text{max}}/(mg)\). The same value of \(R_1/(mg)\) as in figure 2(a) is chosen for clarity. For \(f_{\text{max}}/(mg) \leq 1\), \(G(v^*) < 0\) for every \(v^*\); therefore, no root of \(G(v^*) = 0\) can be found. This absence of a root is obvious from both the mathematics and physics viewpoints. From the mathematics viewpoint, \(G(v^*)\) for
\( f_{\text{max}} / (mg) \leq 1 \) is reduced to \( G(v^*) = \sin(c/v^*) - (1/v^*) \) with a constant \( 0 < c \leq 1 \); thus, it is always negative\(^1\). From the physics viewpoint, when \( f_{\text{max}} \leq mg \), the upper string does not sustain the weight from the outset, and the experiment itself fails; hence, there is no room for the lower string to break. Consequently, the pull-to-jerk transition arises when \( f_{\text{max}} / (mg) > 1 \), which is signified by the presence of crossing points of the curve \( G(v^*) \) with the \( v^* \) axis in figures 4(b)–(d). At \( f_{\text{max}} / (mg) = 2 \) in figure 4(b), the crossing point at \( v^* \sim 1.7 \), which is marked by a solid circle, determines the critical pulling speed; a quicker pull causes the lower string to break, and a slower pull causes the upper string to break. Interestingly, the number of the crossing points monotonically increases with the increase in \( f_{\text{max}} \). In addition, they entirely shift to a larger \( v^* \) with increasing \( f_{\text{max}} \), as traced by solid circles that indicate the rightmost crossing point for each \( f_{\text{max}} / (mg) \). This multiplicity in the roots of \( G(v^*) = 0 \) and their monotonic shift with a increase in \( f_{\text{max}} \) are what we have earlier called in the introduction the ‘richness’ of the pull-or-jerk experiment that is often overlooked in elementary physics courses.

### 5. Phase diagram

Figure 5 shows a phase diagram describing the domains of the upper-string-break phase (designated by ‘U’) and lower counterpart (‘L’) in the \( v^*-f_{\text{max}} \) space. The undermost domain below the \( f_{\text{max}} / (mg) = 1 \) line, which is marked by ‘Irr,’ corresponds to an irrelevant situation in which the string is too fragile to sustain the weight, and the experiment fails. We observe

\(^1\) This fact is readily proved by drawing the graphs of \( y = x \) and \( y = \sin(cx) \) with \( 0 < c \leq 1 \). You will find \( \sin(cx) < x \) at any \( x > 0 \).
that multiple branches of the U-domain spread out from bottom left to top right in a radial manner, and gaps between adjacent branches are occupied by L-domains. All the plots along the boundary curves correspond to the roots of \( G(v^*, f_{\text{max}}) = 0 \); the two particular points highlighted in figure 5 by solid circles (coloured in red) are those depicted in figures 4(b) and (c). Each boundary curve asymptotically approaches a straight line for \( v^* \gg 1 \), whereas in the vicinity of \( v^* = 1 \) they are curvilinear.

An interesting observation in figure 5 is the proportional relation in the slope of the asymptotic straight lines at \( v^* \gg 1 \). To describe the relation, we label the lines as \( j = 0, 1, 2, \ldots \) from the bottom, such that the 0th (horizontal) and 1st (slanted) lines sandwich the lowermost L-domain immediately above the Irr region. Next, we denote the slope of the \( j \)th asymptotic straight line by \( \gamma_j \). It is then found that

\[
\gamma_0; \gamma_1; \gamma_2; \ldots = 0; 1; 2; 3; \ldots. \tag{24}
\]

This proportionality rule holds even when other values of \( R_j/(mg) \) are considered.

Further, the position of the intercept that we obtain if extrapolating the asymptotic lines into the \( f_{\text{max}} \)-axis is noteworthy. Denoted by \( \eta_j \), i.e., the intercept of the \( j \)th line, it appears that \( \eta_j = 1 \) for \( j = 0, 2, 4, \ldots \) and \( \eta_j \) is equal to a constant less than unity for \( j = 1, 3, 5, \ldots \). Namely, there are only two possibilities for the intercepts on the \( f_{\text{max}} \)-axis into which the asymptotic lines extrapolate. This phenomenon is visualized in figure 5 by superposing thin lines coloured in blue (for \( j = 0, 2, 4, \ldots \)) and yellow (for \( j = 1, 3, 5, \ldots \)).

The two interesting features as to the slope \( \gamma_j \) and intercept \( \eta_j \) can be accounted for by considering the properties of \( G(v^*) \) given by equation (23). Suppose that \( v^* \) and \( f_{\text{max}} \) satisfy either of the two relations below

\[
\frac{f_{\text{max}} - P_1}{P_1} = 2j\pi v^* + 1, \tag{25}
\]

or

\[
\frac{f_{\text{max}} - P_1}{P_1} = (2j + 1)\pi v^* - 1. \tag{26}
\]

In the former case, we have

\[
G(v^*) = \sin \left[ \frac{1}{v^*} \cdot (2j\pi v^* + 1) \right] - \frac{1}{v^*} = \sin \left( \frac{1}{v^*} + 2j\pi \right) - \frac{1}{v^*} = \sin \left( \frac{1}{v^*} \right) - \frac{1}{v^*}, \tag{27}
\]

which converges to zero for \( v^* \gg 1 \). Similarly, in the latter case, we have

\[
G(v^*) = \sin \left( \frac{1}{v^*} \cdot [(2j + 1)\pi v^* - 1] \right) - \frac{1}{v^*} = \sin \left( (2j + 1)\pi - \frac{1}{v^*} \right) - \frac{1}{v^*} = \sin \left( \frac{1}{v^*} \right) - \frac{1}{v^*}, \tag{28}
\]

which again converges to zero for \( v^* \gg 1 \). Namely, equations (25) and (26) represent the asymptotic lines depicted in figure 5, the points of which satisfy \( G(v^*, f_{\text{max}}) \approx 0 \) at \( v^* \gg 1 \).

For pedagogical merit, we comment here that a function \( \sin(x)/x \), often called a sinc function, is associated with spherical Bessel functions. A spherical Bessel function of the first order is commonly denoted by \( j_0(x) = \sqrt{\pi/2x} J_0(x) \) with the first-order Bessel function \( J_0(x) \), and we can prove that \( j_0(x) = (\sin x)/x \).

\[\]
We next note that equations (25) and (26) can be rewritten as

\[
\frac{f_{\text{max}}}{mg} = 2j \frac{R}{mg} \pi v^* + 1
\]  

(29)

and

\[
\frac{f_{\text{max}}}{mg} = (2j + 1) \frac{R}{mg} \pi v^* + 1 - 2 \frac{R}{mg}
\]

(30)

respectively. Therefore, the slope \(\gamma_j\) and intercept \(\eta_j\) for the \(j\)th asymptotic lines can be written as

\[
\gamma_j = j \frac{R}{mg} \pi \quad \text{for } j = 0, 1, 2, \ldots
\]

(31)

and

\[
\eta_j = \begin{cases} 
1 & \text{for } j = 0, 2, 4, \ldots \\
1 - 2 \frac{R}{mg} & \text{for } j = 1, 3, 5, \ldots
\end{cases}
\]

(32)

These results explain the proportionality relation in \(\{\gamma_j\}\), and the alternative choices in \(\eta_j\) that were demonstrated.

6. Experimental feasibility

The following discussion helps us to check the feasibility of a classroom experiment for verifying our theoretical results.

The first concern will be realistic values of \(f_{\text{max}}/mg\) obtained in experiments. To estimate it, suppose that we use commercially available strings: a sewing fibre made of polyester and a fishing line made of polyvinylidene fluoride, both of which we are familiar with in our daily life, are cases in point. These strings are typically endowed with the maximum strength, \(f_{\text{max}}\), of a few tens of Newtons (N). For instance, the author has found a sewing thread with \(f_{\text{max}} = 12\) N and a fishing line of \(f_{\text{max}} = 30\) N in a DIY store nearby. This indicates that, if we use a weight of mass \(m = 500\) g in the experiment, we have \(f_{\text{max}}/mg \approx 2.45\) for the sewing thread and \(f_{\text{max}}/mg \approx 6.12\) for the fishing line. These values of \(f_{\text{max}}/mg\) are comparable to those we have considered in theoretical discussion (see figures 4 and 5).

Another concern should be the realistic value of \(v^*\), which can be explored by considering the spring constant, \(k\), of the strings used in the experiment. Although the value of \(k\) depends on the natural lengths of the strings, we can estimate it as a few hundreds Newtons per metre (N/m) if the length of one metre or less is assumed. In fact, the author has confirmed using a simple balance that the sewing thread of 50 cm in length has a spring constant of \(k = 170\) N m\(^{-1}\) and the fishing line of 1 m has \(k = 120\) N m\(^{-1}\). Therefore, if we set up the pulling speed of \(V = 1\) m s\(^{-1}\) and the weight of mass \(m = 500\) g, we obtain \(v^* = kV/(mg) = 170 \times 1/(0.5 \times 26.1 \times 9.8) = 1.33\) and \(v^* = 120 \times 1/(0.5 \times 21.9 \times 9.8) = 1.12\) for the sewing-thread-based apparatus and the fishing-line-based apparatus, respectively. What is more, since the value of \(k\) can be tuned by changing the string’s length, we can cover a wide range of \(v^*\) that has been considered in our theoretical discussion.

The final point to be noted is how to realize a constant-pulling-speed condition. One idea is to use a rotating wheel driven by a gear motor with a few rotations per minute. Suppose that
the bottom end of the lower string is attached to the wheel and its rotational axis is fixed horizontally. Then, constant rate rotation of the wheel results in a desired downward pulling with constant rate. Another possible way is to use a freely rotating wheel made of steel, for example, that possesses a large moment of inertia around the rotation axis. The latter way requires us to install a velocity sensor and a force sensor with the apparatus to measure the wheel’s rotation speed and the string’s tension, respectively. In either case, inexpensive and small size devices will suffice to build the experimental equipment.

7. Conclusion

We have theoretically described the well-known inertia demonstration based on a weight and two strings. Under a constant-pulling-speed condition, we have derived a closed-form expression for the critical pulling speed at a given maximum tension. A phase diagram illustrating which string will break has also been established. The author thinks that it is pedagogical to follow the present argument and compare it with those obtained under other pulling conditions [1–4].

8. Suggested problems

It is reasonable to propose the following questions for further work related to this thesis.

• Consider an upside-down situation where you pick up the top of a string on which hangs a weight, and a similar string connects the bottom of the weight and floor. Can you derive a closed-form expression for the critical speed of upward pulling? What phase diagram do you obtain?
• What occurs if you lay down the apparatus on a friction-free floor? In the system, the left-hand edge of the string left to the weight is fixed to a sidewall, and the right-hand edge of the string right to the weight is what you are pulling horizontally. Gravitational force on the weight no longer contributes in this case.
• Carry out a study similar to this article for a one-dimensional chain of many weights, in which the weights and strings are arranged alternately in the vertical direction from top to bottom. When pulling downward the bottom of the lowest string with a constant speed, is it possible for all the strings to break simultaneously? You can start a double-weight system as the simplest example.

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