Published in Chaos, Solitons and Fractals, 12(2001)1431-1437

Incomplete statistics and nonextensive generalizations of statistical mechanics

Qiuping A. Wang

Institut Supérieur des Matériaux du Mans,
44, Avenue F.A. Bartholdi, 72000 Le Mans, France
(March 21, 2022)

Statistical mechanics is generalized on the basis of an information theory for inexact or incomplete probability distribution. A parameterized normalization is proposed and leads to a nonextensive entropy. The resulting incomplete statistical mechanics is proved to have the same theoretical characteristics as Tsallis one which is based on the conventional normalization.

02.50.-r, 05.20.-y, 05.30.-d, 05.70.-a

I. INTRODUCTION

We need a generalized statistical mechanics to replace Boltzmann-Gibbs-Shannon (BGS) one because BGS theory is not capable of interpreting some observed results of physical systems with complicated strong interaction, long range correlation, longtime memory or with noneuclidean and nonsmooth space-time structure such as in the theory of fractal space-time and Cantorian $E^\infty$. Examples of such systems exist everywhere, from cosmic systems, ordinary optical, magnetic or electronic materials around us, fractal and chaotic systems, till microscopic systems such as nuclei. For detailed comments on the breakdowns of BGS theory and on the possible solutions, the reader is advised to read references [5,6] and the references there-in. Complicated physical systems are often nonextensive. For this kind of systems, we already have Tsallis nonextensive statistical mechanics [7,8] which has been successfully used to interpret some peculiar physical phenomena and fractal structures. Tsallis entropy is given as follows:

$$S_q = -k \frac{1 - \sum_{i=1}^{v} p_i^q (q \in R)}{1 - q}$$ (1)

where $p_i$ is the probability of the state $i$ among $v$ possible ones

$$\sum_{i=1}^{v} p_i = 1.$$ (2)

The independence of two probability distributions $[p_1(A)...p_i(A)...p_v(A)]$ et $[p_1(B)...p_j(B)...p_u(B)]$ is defined as usual by

$$p_i(A)p_j(B) = p_{ij}(A \cup B)$$ (3)

where $p_{ij}(A \cup B)$ is the probability for the state $i$ and $j$ to be occupied at the same time. We obviously have $\sum_{i=1}^{v} \sum_{j=1}^{u} p_{ij}(A \cup B) = 1$. We see that Tsallis statistics, as BGS one, is constructed within Kolmogorov algebra of complete probability distribution [10]. On the other hand, the expectation value $<O>$ of an observable $\hat{O}$ is supposed to be calculated in the following way,

$$<O> = \frac{\sum_{i=1}^{v} p_i^q O_i}{\sum_{i=1}^{v} p_i^q}$$ (4)

where $O_i$ is the value of $\hat{O}$ at the state $i$.

I want to emphasize here that, as mentioned above, up to now, all statistical theories (BGS, Tsallis, etc) are based on the information theory for complete probability distribution according to 'complete random variables' (CRV)

\[1\] This kind of “independence” for correlated subsystems can be interpreted by the existence of thermodynamic equilibrium in the composite system [8].
A CRV, ξ, takes distinct values \( X = \{x_1, x_2, ..., x_v\} \) (also referred to as \( v \) events or states) with probabilities \( P = \{p_1, p_2, ..., p_v\} \) and \( X \) constitutes a complete ensemble \( \Omega_x \) defined by all possible values of \( \xi \). A simple example of CRV is the position of a particle in a closed box. All the possible values of the position constitute a complete ensemble of positions defined in the box. We call \( \xi \) an independent CRV if all its values are independent and incompatible (exclusive). In this case, \( P \) is called a complete distribution and satisfies the requirement of Eq. (2).

We can say that all statistical theories constructed within Kolmogorov algebra of complete probability distribution should be logically applied to physical systems of which all the possible physical states are well-known and for which we can in practice count all of the states to carry out the calculation of probability or of whatever physical quantities. This often requires that we can find the exact hamiltonian and also the exact solutions of the equation of motion of the systems to know all possible states and to obtain the exact values of physical quantities.

The generalization of BGS theory in the present paper is based on the information theory for incomplete probability distribution with incomplete random variables (IRV), that is the number \( w \) of the possible values \( \{x_1, x_2, ..., x_w\} \) of IRV is greater or smaller than \( v \). Another case where the present generalization can be applied is that, though \( w = v \), one can not calculate the exact probability distribution \( P = \{p_1, p_2, ..., p_v\} \). In physics, the above two cases are possible if we do not know or can not write analytically all the interactions in a system. In these cases, the solution of the equation of motion is no longer exact, and we can no more in practice count all the possible states and calculate in a precise way any physical quantity of the system even for the countable (well-known) states. So the calculation of the probability \( p_i \) is not exact either. Within this hypothesis, Eq. (4) should be written as

\[
\sum_{i=1}^{w} p_i = Q \neq 1. \tag{5}
\]

where \( w \) is only the number of the countable states given by the solution of the equation of motion and can be greater or smaller than \( v \), the real number of states of the physical system under consideration. \( Q \) should depend on the way to find the countable states and probabilities, or in other words, on the neglected interactions.

The idea of the present generalization of BGS statistics is that we recognize the inadequacy of our knowledge of physical systems, i.e. Eq. (4), and try to tackle the problems in a approximate way allowing us to work within Kolmogorov algebra of probability as BGS statistical mechanics does. We also require that the BGS statistics be a special case of the generalized formalisms. As mentioned above, the present nonextensive generalization is carried out with the hypothesis of an incomplete ensemble of states and of an inexact probability distribution. The reader will find that this incomplete statistics has the same theoretical features as Tsallis one for complete ensemble of states.

II. INCOMPLETE NORMALIZATION

Now let us postulate

\[
\sum_{i=1}^{w} \frac{p_i}{Q} = \sum_{i=1}^{w} p_i^q \tag{6}
\]

so that

\[
\sum_{i=1}^{w} p_i^q = 1, \quad (q \in [0, \infty]) \tag{7}
\]

since \( p_i < 1 \), we have to set \( q \in [0, \infty] \). Eq. (6) now becomes:

\[
< O > = \sum_{i=1}^{w} p_i^q O_i. \tag{8}
\]

Don’t forget that the \( w \) states are only the well-known (countable) ones and therefore do not constitute a complete ensemble of states of the system under consideration. \( w \) can be greater or smaller than the real number of all the possible states, depending on the approximations we use to find the analytic expression of hamiltonian and the solution of the equation of motion of the system.

Eq. (6) is a kind of redistribution of the effect of neglected interactions (or of the unknown events) on the known events. This is quite normal because the known events and their probability distribution are closely related to the
neglected interactions. This *q-deformation* of probability $p_i^q$ is not a new invention. It was the choice of almost all the authors who intended to generalize Shannon information theory [10,11]. It is used by Rényi to calculate the *q-order measure of information* and related to an *average gain* of lost information due to the incomplete distribution. $p_i^q$ is also used in the fractal theory to favor contributions from events with relatively high values (when $q > 1$) or low values (when $q < 1$) in the calculation of multifractal measure [11]. In any case, Eq. (5) is a (reasonable) simplification of Eq. (3) which allows, we will find later in this paper, to work within the mathematical framework of BGS statistical mechanics.

In the simplified incomplete normalization Eq. (3), $p_i^q$ can be called the *effective probability* of the event $i$ and $p_i$ the *real probability* which is physically useful only when $q = 1$ for the cases where no interaction is neglected and $w$ is the total number of events in a complete ensemble. When $q$ is different from unity, the difference ($q - 1$) is related to the neglected interactions. We can see later in this paper that $q$ is related to the extra information (entropy) compared to that we obtain in BGS framework. $q$ can also be related in explicit ways to other basic quantities such as internal energy or free energy of the system.

### III. INCOMPLETE INFORMATION THEORY

It is well known that the conventional (extensive) information theory is based on the following postulates concerning the missing information $I(N)$ to determine the state of a system Ω of $N$ elements [10]:

1. $I(1) = 0$ (no missing information if there is only one event)
2. $I(\kappa) = 1$ (information unity)
3. $I(N) < I(N + 1)$ (more information with more elements)
4. $I(\prod_{i=1}^{N} N_i) = \sum_{i=1}^{N} I(N_i)$ (additivity)
5. $I(N) = I_v + \sum_{i=1}^{N} p_i I(N_i)$ (additivity of information measure in two steps)

where $v$ is the number of all the subsystems $\Omega_i$ with $N_i$ elements and $p_i = \frac{N}{N_i}$ is the probability to find an element in $\Omega_i$ (equiprobability). $I_v$ is the missing information to determine in what subsystem an element will be found. The postulates n°1 to n°4 lead to Hartley formula of information measure [10]:

$$I(N) = \ln N.$$  \hspace{1cm} (9)

Then the postulate n°5 can yield Shannon formula for $I_v$ and entropy.

For nonextensive systems, the additivity postulate n°4 should become, for two systems with $N_1$ and $N_2$ elements, respectively:

$$I(N_1 \times N_2) = I(N_1) + I(N_2) + f(I(N_1), I(N_2))$$  \hspace{1cm} (10)

where the form of the function $f(I(N_1), I(N_2))$ is of central importance for all the rest of this nonextensive information theory. Considering the necessity to come back to Hartley formula in a special case and in order to find the easiest way out, we naturally think of the *q-deformed logarithmic function* $\frac{N^{1-q} - 1}{1-q}$ which $\to \ln N$ when $q \to 1$. So I postulate:

$$I(N) = \frac{N^{1-q} - 1}{1-q}.$$  \hspace{1cm} (11)

This generalized Hartley formula corresponds to the following postulates:

1. $I(N_1 \times N_2) = I\{1 + (1 - q)\frac{1}{1-q}\} = 1$
2. $I(N) < I(N + 1)$
3. $I(N_1 \times N_2) = I(N_1) + I(N_2) + (1 - q)I(N_1) \times I(N_2)$ (non-additivity or nonextensivity).

As for postulate n°5, I put

$$I(N) = I_v + \sum_{i=1}^{w} p_i^q I(N_i).$$

Notice that, $p_i$ in the postulate n°5 is replaced in the postulate n°5 by $p_i^q$ due to the incomplete normalization.

---

2This product form of the nonextensive term is also a consequence of the existence of thermodynamic equilibrium in the composite system [11]
From Eq. (11) and the postulate \( n^5 n_{ex} \), we straightforwardly obtain \( I_w \), the information measure given by an incomplete probability distribution \( \{ p_1, p_2, ..., p_w \} \) defined by \( p_i = N_i / N \) :

\[
\frac{N^{1-q} - 1}{1-q} = I_w + \sum_{i=1}^{w} p_i^q \frac{N^{1-q} - 1}{1-q}
\]

(12)

which yields

\[
I_w \propto -\sum_{i=1}^{w} p_i^q p_i^{1-q} - 1.
\]

(13)

IV. NONEXTENSIVE GENERALIZATION OF BGS STATISTICS

Now we postulate for the nonextensive entropy :

\[
S_q = -k \sum_{i=1}^{w} p_i^q \frac{N^{1-q} - 1}{1-q}
\]

(14)

or

\[
S_q = -k \left[ \sum_{i=1}^{w} p_i - \sum_{i=1}^{w} p_i^q \right] / 1-q
\]

(15)

or even more simply

\[
S_q = k \frac{1 - \sum_{i=1}^{w} p_i}{1-q}
\]

(16)

where \( q > 0 \) is required by the incomplete normalization Eq. (7). It can be easily verified that all the properties of Tsallis entropy (nonnegativity, concavity, pseudo-additivity, etc) [7] are preserved by this entropy [Eq. (14) to (16)] because it is nothing but the Tsallis one with a new normalization condition.

For microcanonical ensemble, we extremize \( S_q \) with the condition in equation (7) and obtain \( p_i^q = 1/w \) and

\[
S_q = k \frac{w}{q - 1} - 1
\]

(17)

which tends to \( S_1 = klnw \) in the \( q \to 1 \) limit.

For canonical ensemble, maximum entropy Eq.(16) with Eqs. (7) and (8) (for energy) as constraints, i.e.

\[
\delta \left[ \frac{S_q}{k} + \frac{\alpha}{1-q} \sum_{i=1}^{w} p_i^q - \alpha \beta \sum_{i=1}^{w} p_i^q E_i \right] = 0
\]

(18)

yields

\[
p_i = \frac{[1 - (1 - q)\beta(E_i)]^{1-q}}{Z_q}
\]

(19)

with

\[
Z_q = \left[ \sum_{i}^{w} [1 - (1 - q)\beta E_i]^{1-q} \right]^{1/q}
\]

(20)

To obtain Legendre transformations, we take Eq. (14) and replace \( p_i^{1-q} \) by equation (13), remembering equation (7) and (8), we obtain
\[ S_q = k \frac{Z_q^{q-1} - 1}{q-1} + k\beta Z_q^{q-1}U_q \]  

(21)

which, with the help of the thermodynamic relation \( \frac{1}{T} = \frac{\partial S_q}{\partial U_q} \), leads to

\[ \beta = \frac{Z_q^{1-q}}{kT} \]  

(22)

and

\[ F_q = U_q - TS_q = -kT \frac{Z_q^{q-1} - 1}{q-1} \]  

(23)

The \( U_q - Z_q \) relation is a little complicated. From equation (8) and (19), it can be recast as follows

\[ U_q = \frac{1}{Z_q} \frac{\partial}{\partial \beta} Z_q' \]  

(24)

where \( Z_q' \) is given by

\[ Z_q' = \sum_i w_i [1 - (1 - q)\beta E_i]^{\frac{1}{1-q}}. \]  

(25)

As in Tsallis’ case, it is straightforward to verify that all above relations reduce to those of BGS case in the \( q \to 1 \) limit.

Now we will discuss some points concerning the nonextensivity of the system. The generalized Hartley formula Eq. (11) or the nonadditivity postulate \( n^{nex} \) suggests that, for two subsystems \( A \) and \( B \) of a system \( C = A + B \):

\[ N_{ij}(C) = N_i(A) \times N_j(B). \]  

(26)

and \( N(C) = N(A) \times N(B) \). These relations assume the factorization of the joint probability \( p_{ij} \) or \( p_{ij}^q \):

\[ p_{ij}^q(C) = p_i^q(A)p_j^q(B) \]  

(27)

which in turn leads to the nonextensivity of entropy,

\[ S_q(A + B) = S_q(A) + S_q(B) + \frac{q-1}{k}S_q(A)S_q(B). \]  

(28)

Eq. (27) is in fact the definition of the independence of the effective probability \( p_i^q(A) \) or \( p_j^q(B) \). Considering the distribution Eq. (19), we easily get

\[ E_{ij}(A + B) = E_i(A) + E_j(B) + (q-1)\beta E_i(A)E_j(B) \]  

(29)

and

\[ U_q(A + B) = U_q(A) + U_q(B) + (q-1)\beta U_q(A)U_q(B). \]  

(30)

Eqs. (27), (29) and (30) tell us that if \( p_i(A) \) and \( p_j(B) \) are independent, the two systems \( A \) and \( B \) are dependent on each other and correlated by Eqs. (29) or (30). But for two independent systems \( A \) and \( B \) with \( E_{ij}(C) = E_i(A) + E_j(B) \), we lose Eq. (27) and, strictly speaking, can no more find the relation between \( U_q(C), U_q(A) \) and \( U_q(B) \), unless we put \( q = 1 \) and come back to BGS case. As for this problem of correlation, Abe [12] has studied \( N \)-body problem with ideal gas model. He concluded that, in thermodynamic limits or in the limit of big \( N \) (particle number), the correlation term in Eq. (29) and (30) would vanish. The suppression of the correlation, in addition, allows to establish the zeroth law of thermodynamics within the framework of nonextensive statistical mechanics with escort probability [12]. We mention here that the reader can find an exact establishment of the zeroth law based on Eq. (28) and (30) without neglecting the energy correlation [13].
V. CONCLUSION

In conclusion, the conventional BGS statistical mechanics is generalized on the basis of the idea that we sometimes can not know all the possible physical states or the exact probability distribution of a complicated physical system and so that we have to use the suitable information theory for incomplete probability distribution. The most important step of this generalization is the incomplete normalization \( \sum_i p_i^q = 1 \) with a free parameter \( q \) (positive) which is dependent on the neglected interactions. We have seen that, apart from some minor differences, the present incomplete scenario of nonextensive statistics has the same characteristics as Tsallis scenario for complete probability distribution. On the other hand, we would like to indicate here that, with the hypothesis of incomplete distribution, the parameter \( q \) is logically related to the quantity \( Q \) in Eq.(5) and so to the interactions neglected in the Hamiltonian of the system. For example, for a microcanonical ensemble, we can write : \( q = 1 - \frac{\ln Q}{\ln p} \) or \( Q = w^{\frac{1}{q-1}} \). In addition, \( q \) can be related to internal energy or to other basic quantities of physical systems \([14]\). So the understanding of nonextensive thermostatistics under the angle of incomplete information may be of interest for interpreting particular values of \( q \) for different complicated physical systems \([1-4]\).

VI. ACKNOWLEDGMENTS

I acknowledge with great pleasure the very useful discussions with Professors Constantino Tsallis and Alain Le Méhauté on some points of this work. Thanks are also due to Professor M.S. El Naschie, Dr. Laurent Nivanen, Dr. François Tsobnang and Dr. Michel Pezeril for valuable comments.

[1] Kröger H, Fractal geometry in quantum mechanics field theory and spin systems, Physics Report 2000;323(2):82-181
[2] El Naschie MS, On the electroweak missing angle in \( E^\infty \) Chaos, Solitons & Fractals, 2000;11(11):1803-1807
[3] Nottale L, Scale relativity and fractal space-time Chaos, Solitons & Fractals, 1996;7(6):877-938
[4] Ahmed E, Hegazi AS, On infinitesimally deformed algebra and fractal space-time theory Chaos, Solitons & Fractals, 2000;11(11):1759
[5] C. Tsallis, Chaos, Solitons & Fractals, 6(1995)539.
[6] Silvio R.A. Salinas and C. Tsallis, Brazilian Journal of Physics(special issue: Nonextensive Statistical Mechanics and Thermodynamics), 29(1999).
[7] C. Tsallis, J. Statis. Phys., 52(1988)479; EMF. Curado and C. Tsallis, J. Phys.A:Math.Gen., 24 (1991)L69.
[8] C. Tsallis, R.S. Mendes and A.R. Plastino, Physica A, 261(1999)534
[9] Qiuping A. Wang, Laurent Nivanen, Alain Le Méhauté and Michel Pezeril, Note on Abe’s general pseudoadditivity for nonextensive systems, e-print : cond-mat/0111541 (submitted); S. Abe, Phys. Rev.E, 63(2001)061105
[10] A. Rényi, Calcul de probabilité,(Paris, Dunod, 1966)P522.
[11] Jens Feder, Fractal,(New York, Plenum, 1988)P80.
[12] S. Abe, Physica A, 269(1999)403-409.
[13] Q.A. Wang, M. Pezeril, L. Nivanen, A. Le Méhauté, Nonextensive distribution and factorization of the joint probability, Chaos, Solitons & Fractals, 13(2001)131; e-print : cond-mat/0010294
[14] Q.A. Wang, L. Nivanen and A. Le Méhauté, 25th Middle European conference on statistical physics,(Nancy, March 2000)(preprint)
Figure captions:
Figure 1) Variation of nonextensive microcanonical entropy $S_q$ with countable state number $w$ for different $q$ value. We see that $S_q$ increases with increasing $q$. 