Ulam Type Stability of $A$-Quadratic Mappings in Fuzzy Modular $*$-Algebras

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Abstract: In this paper, we find the solution of the following quadratic functional equation
\[ \sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^{n} Q\left(\sum_{j \neq i} x_j - (n - 1)x_i\right), \]
which is derived from the gravity of the $n$ distinct vectors $x_1, \ldots, x_n$ in an inner product space, and prove that the stability results of the $A$-quadratic mappings in $\mu$-complete convex fuzzy modular $*$-algebras without using lower semicontinuity and $\beta$-homogeneous property.

Keywords: fuzzy modular $*$-algebras; modular $*$-algebras; $A$-quadratic derivation; $\Delta_2$-condition; $\beta$-homogeneous property

1. Introduction

A concept of stability in the case of homomorphisms between groups was formulated by S.M. Ulam [1] in 1940 in a talk at the University of Wisconsin. Let $(G_1, \ast)$ be a group and let $(G_2, \odot, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality
\[ d(h(x \ast y), h(x) \odot h(y)) < \delta \]
for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with
\[ d(h(x), H(x)) < \epsilon \]
for all $x \in G_1$?

The first affirmative answer to the question of Ulam was given by Hyers [2,3] for the Cauchy functional equation in Banach spaces as follows: Let $X$ and $Y$ be Banach spaces. Assume that $f : X \rightarrow Y$ satisfies
\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon \]
for all $x, y \in X$ and for some $\epsilon \geq 0$. Then, there exists a unique additive mapping $T : X \rightarrow Y$ such that
\[ \|f(x) - T(x)\| \leq \epsilon \]
for all $x \in X$. A number of mathematicians were attracted to this result and stimulated to investigate the stability problems of various (functional, differential, difference, integral) equations in some spaces [4–11].

In 2007, Nourouzi [12] presented probabilistic modular spaces related to the theory of modular spaces. Fallahi and Nourouzi [13,14] investigated the continuity and boundedness of linear operators
defined between probabilistic modular spaces in the probabilistic sense. After then, Shen and Chen [15]
following the idea of probabilistic modular spaces and the definition of fuzzy metric spaces based on
George and Veeramani’s sense [16], applied fuzzy concept to the classical notions of modular spaces.
Using Khamsi’s fixed point theorem in modular spaces [17], Wongkum and Kumam [18] proved
the stability of sextic functional equations in fuzzy modular spaces equipped necessarily with lower
semicontinuity and $\beta$-homogeneous property.

In a recent paper [11], Ulam stability of the following additive functional equation
\[
\sum_{1 \leq i < j \leq n} f \left( \frac{x_i + x_j}{m} + \sum_{l=1}^{n-m} x_l \right) = \frac{n - m + 1}{n} \left( \frac{n}{m} \right) \sum_{i=1}^{n} f(x_i).
\]
was investigated in modular algebras without using the lower semicontinuity and Fatou property.

In the present paper, concerning the stability problem for the following functional equation
\[
\sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^{n} Q \left( \sum_{j \neq i} x_j - (n - 1)x_i \right)
\]
which is derived from the gravity of the $n$-distinct vectors in an inner product space, we investigate
the stability problem for $A$-quadratic mappings in $\mu$-complete convex fuzzy modular $*$-algebras of the
following functional equation without using lower semicontinuity and $\beta$-homogeneous property.

2. Preliminaries

Proposition 1. Let $X_1, X_2, \cdots, X_n$ ($n \geq 3$) be distinct vectors in a finite $n$-dimensional Euclidean space $E$.
Putting $G := \frac{1}{n} \sum_{i=1}^{n} X_i$, the gravity of the $n$ distinct vectors, then we get the following identity
\[
\sum_{1 \leq i < j \leq n} \|X_i X_j\|^2 = n \sum_{i=1}^{n} \|X_i G\|^2,
\]
which is equivalent to the equation
\[
\sum_{1 \leq i < j \leq n} \|X_i - X_j\|^2 = \sum_{i=1}^{n} \left\| \sum_{j \neq i} X_j - (n - 1)X_i \right\|^2
\] (1)
for any distinct vectors $X_1, X_2, \cdots, X_n$.

Employing the above equality (1), we introduce the new functional equation:
\[
\sum_{1 \leq i < j \leq n} Q(x_i - x_j) = \sum_{i=1}^{n} Q \left( \sum_{j \neq i} x_j - (n - 1)x_i \right)
\] (2)
for a mapping $Q : U \to V$ and for all vectors $x_1, \cdots, x_n \in U$, where $U$ and $V$ are linear spaces and
$n \geq 3$ is a positive integer.

From now on, we introduce some basic definitions of fuzzy modular $*$-algebras.

Definition 1. [18] A triangular norm (briefly, t-norm) is a function $\odot : [0, 1] \times [0, 1] \to [0, 1]$ satisfies the
following conditions:

1. $\odot$ is commutative, associative;
2. $a \odot 1 = a$;
3. $a \odot b \leq c \odot d$, whenever $a, b, c, d \in [0, 1]$ with $a \leq b, c \leq d$. 
Three common examples of the \( t \)-norm are (1) \( a \circ_M b = \min\{a, b\} \); (2) \( a \circ_P b = a \cdot b \); (3) \( a \circ_L b = \max\{a + b - 1, 0\} \). For more example, we refer to [19]. Throughout this paper, we denote that
\[
\prod_{i=1}^{n} x_i := x_1 \circ \cdots \circ x_n
\]
for all \( x_1, \cdots, x_n \in [0, 1] \).

**Definition 2.** [18] Let \( X \) be a complex vector space and \( \circ \) a \( t \)-norm, and \( \mu : X \times (0, \infty) \to [0, 1] \) be a function.

(a) The triple \((X, \mu, \circ)\) is said to be a fuzzy modular space if, for each \( x, y \in X \) and \( s, t > 0 \) and \( a, \beta \in [0, \infty) \) with \( a + \beta = 1 \),

\begin{align*}
(FM1) \ & \mu(x, t) > 0; \\
(FM2) \ & \mu(x, t) = 1 \text{ for all } t > 0 \text{ if and only if } x = \theta; \\
(FM3) \ & \mu(x, t) = \mu(-x, t); \\
(FM4) \ & \mu(ax + \beta y, s + t) \geq \mu(x,s) \circ \mu(y,t); \\
(FM5) \ & \text{the mapping } t \to \mu(x, t) \text{ is continuous at each fixed } x \in X;
\end{align*}

(b) alternatively, if (FM-4) is replaced by

\[
(FM4-1) \mu(ax + \beta y, s + t) \geq \mu(x, \frac{s}{\beta}) \circ \mu(y, \frac{t}{a}), \quad \text{(where } a, \beta \neq 0); \]

then we say that \((X, \mu, \circ)\) is a convex fuzzy modular.

Now, we extend the properties (FM4) and (FM4-1) in real fields to complex scalar field acting on the space \( X \), as follows:

\begin{align*}
(FM4)' \ & \mu(ax + \beta y, s + t) \geq \mu(x, s) \circ \mu(y, t); \text{ for } a, \beta \in \mathbb{C} \text{ with } |a| + |\beta| = 1, \\
(FM4-1)' \ & \mu(ax + \beta y, s + t) \geq \mu(x, \frac{s}{\beta}) \circ \mu(y, \frac{t}{a}) \text{ for } a, \beta \in \mathbb{C} \text{ with } |a| + |\beta| = 1.
\end{align*}

Next, we introduce the concept of fuzzy modular algebras based on the definition of fuzzy normed algebras [20,21]. If \( X \) is algebra with fuzzy modular \( \mu \) subject to \( \mu(xy, st) \geq \mu(x, s) \circ \mu(y, t) \) for all \( x, y \in X \) and \( s, t \in (0, \infty) \), then we say \((X, \mu, \circ)\) is a fuzzy modular algebra. In addition, a fuzzy modular algebra \( X \) is a fuzzy modular *-algebra if the fuzzy modular \( \mu \) satisfies \( \mu(z^*, t) = \mu(z, t) \) for all \( z \in X, t > 0 \).

**Example 1.** Let \((X, \rho)\) be a modular *-algebra ([22]) and \( \circ \) defined by \( a \circ b := a \circ_M b \). For every \( t \in (0, \infty) \), define \( \mu(x, t) = \frac{1}{1 + \rho(x)} \) for all \( x \in X \). Then, \((X, \mu, \circ)\) is a (convex) fuzzy modular *-algebra.

**Definition 3.** (1) We say that \((X, \mu, \circ)\) is \( \beta \)-homogeneous if, for every \( x \in X, t > 0 \) and \( \lambda \in \mathbb{R} \setminus \{0\} \),

\[
\mu(\lambda x, \ell) = \mu\left(x, \frac{\ell}{|\lambda|^\beta}\right), \quad \text{where } \beta \in (0, 1].
\]

(2) Let \( n \in \mathbb{N} \). We say that \((X, \mu, \circ)\) satisfies \( \Delta_n \)-condition if there exist \( \kappa_n \geq n \) such that

\[
\mu(nx, \ell) \geq \mu\left(x, \frac{\ell}{\kappa_n}\right), \quad \forall x \in X.
\]

**Remark 1.** Let \((X, \mu, \circ)\) be \( \beta \)-homogeneous for some fixed \( \beta \in (0, 1] \). Then, we observe that

\[
\mu(2x, \ell) = \mu\left(x, \frac{\ell}{2^\beta}\right) \geq \mu\left(x, \frac{\ell}{\kappa_2}\right)
\]

for all \( x \in X \) and all \( \kappa_2 \geq 2 \geq |2|^\beta \). Thus, \( \beta \)-homogeneous property implies \( \Delta_2 \)-condition.
Example 2. Let \( \rho : \mathbb{R} \to \mathbb{R}, \mu : \mathbb{R} \times (0, \infty) \to (0, 1] \) be defined by \( \rho(x) = x^2 \) and \( \mu(x, t) = \frac{t}{1 + t^2} \).

Then, we can check that \( (\mu, \circ_M) \) is a convex fuzzy modular on \( \mathbb{R} \) but \( (\mathbb{R}, \mu, \circ_M) \) does not satisfy \( \beta \)-homogeneous property. Let \( \kappa_2 \geq 4 \). Then,

\[
\mu(2x, t) = \frac{t}{1 + \rho(2x)} = \mu\left(x, \frac{t}{4}\right) \geq \mu\left(x, \frac{t}{\kappa_2}\right)
\]

for all \( x \in \mathbb{R} \). Thus, \( (\mathbb{R}, \mu, \circ) \) satisfies \( \Delta_2 \)-condition with \( \kappa_2 \geq 4 \) but is not \( \beta \)-homogeneous.

Definition 4. Let \( (X, \mu, \circ) \) be a fuzzy modular space and \( \{x_n\} \) be a sequence in \( X_\rho \).

(1). \( \{x_n\} \) is said to be \( \mu \)-convergent to a point \( x \in X \) if for any \( t > 0 \),

\[
\mu(x - x_n, t) \to 1
\]

as \( n \to \infty \).

(2). \( \{x_n\} \) is called \( \mu \)-Cauchy if for each \( \varepsilon > 0 \) and each \( t > 0 \), there exists \( n_1 \) such that, for all \( n \geq n_1 \) and all \( p > 0 \), we have \( \mu(x_{n+p} - x_n) > 1 - \varepsilon \).

(3). If each Cauchy sequence is convergent, then the fuzzy modular space is said to be complete.

3. Fuzzy Modular Stability for \( \mathcal{A} \)-Quadratic Mappings

First of all, we find out the general solution of (1.3) in the class of mappings between vector spaces.

Theorem 1. Let \( U \) and \( V \) be vector spaces. A mapping \( Q : U \to V \) satisfies the functional Equation (2) for each positive integer \( n > 2 \) if and only if there exists a symmetric biadditive mapping \( B : U \times U \to V \) such that

\[ Q(x) = B(x, x) \]

for all \( x \in U \).

Proof. Let \( Q \) satisfy Equation (2). One finds that \( Q(0) = 0 \) and \( Q(ax) = a^2 Q(x) \) by changing \((x, y)\) to \((0, 0)\) and \((x, 0)\) in (3), respectively, where \( a := n - 1 \) is a positive integer with \( a \geq 2 \). Putting \( x_1 := x, x_2 := y \) and \( x_i := 0 \) for all \( i = 3, \cdots, n \) in (2), we get

\[
Q(x - ay) + Q(ax - y) + (a - 1)Q(x + y)
= (a + 1)Q(x - y) + (a^2 - 1)[Q(x) + Q(y)]
\]

for all \( x, y \in U \). Using [23] [Theorem 1], we obtain that \( Q \) is a generalized polynomial map of degree at most 4. Therefore,

\[
Q(x) = A_0 + A_1(x) + A_2(x, x) + A_3(x, x, x) + A_4(x, x, x, x)
\]

for all \( x \in U \), where \( A_k : U^k \to V \) is a \( k \)-additive symmetric map (\( k = 1, \cdots, 4 \)) and \( A_0 \in V \). Since \( a \) is an integer, we get

\[
(a^2 - 1)A_0 + (a^2 - a)A_1(x) + (a^2 - a^3)A_2(x, x) + (a^2 - a^4)A_4(x, x, x, x) = 0
\]

for all \( x \in U \) by \( Q(ax) = a^2 Q(x) \). This yields that \( Q(x) = A_2(x, x) \) for all \( x \in U \). \( \square \)

Let \( \mathcal{A} \) be a complex \( \ast \)-algebra with unit and let \( M \) be a left \( \mathcal{A} \)-module. We call a mapping \( Q : M \to \mathcal{A} \) an \( \mathcal{A} \)-quadratic mapping if both relations \( Q(ax) = aQ(x)a^* \) and \( Q(x + y) + Q(x - y) = 2Q(x) + 2Q(y) \) are fulfilled for all \( a \in \mathcal{A}, x, y \in M \) [24]. For the sake of convenience, we define the following:
Let $\mathcal{D}_uf(x_1, \cdots, x_n) := \sum_{1 \leq i < j \leq n} f(u x_i - u x_j) - \sum_{i=1}^n u f \left( \sum_{j \neq i} x_j - (n-1)x_i \right) u^*$, $\epsilon_i(x) := \epsilon(0, \cdots, 0, x_{i-th}, 0, \cdots, 0)$, and $\mathcal{J} := \begin{cases} \{1, \cdots, n-1\} \times \{1, \cdots, n\}, & \text{if } n > 3, \\ \{2\} \times \{1, 2, 3\}, & \text{if } n = 3. \end{cases}$

In addition, let $\circ$ be defined by minimum $t$-norm and $A^M$ be the set of all mapping from $M$ to $A$, $Q_A(M, A)$ be the set of all $A$-quadratic mappings from $M$ to $A$.

Now, we present a stability of the $A$-quadratic mapping concerning Equation (2) in $\mu$-complete convex fuzzy modular $*$-algebras without using $\beta$-homogeneous properties.

**Theorem 2.** Let $(A, \mu, \circ)$ be $\mu$-complete convex fuzzy modular $*$-algebra with norm $\| \cdot \|$ and $M$ be a left $A$-module, $(X, \mu', \circ)$ fuzzy modular space, $U(A)$ the unitary group of $A$. Assume that there exist two mappings $f \in A^M$ and $\epsilon \in X^{M*}$ such that
\[
\mu(\mathcal{D}_uf(x_1, \cdots, x_n), t) \geq \mu'(\epsilon(x_1, \cdots, x_n), t),
\]
for all $(x_1, \cdots, x_n) \in X^n$, $u \in U(A)$, where $2 \leq 2\beta < (n-1)^2$, and either $f$ is measurable or $f(tx)$ is continuous in $t \in R$ for each fixed $x \in M$. Then, there exists a unique mapping $Q \in Q_A(M, A)$ that satisfies Equation (2) and the inequality
\[
\mu \left( f(x) + \frac{(n-1)f(0)}{2} - Q(x), t \right) \geq \Phi \left( x, \frac{(n-1)^2t}{2\beta} \right)
\]
for all $x \in M$ and $t > 0$, where
\[
\Phi(x, t) := \max_{(i,j) \in \mathcal{J}} \left\{ \mu' \left( \epsilon_j(-x), \frac{(n-1)^2t}{6} \right) \circ \mu' \left( \epsilon_i(x), \frac{(n-1)^2nt}{6(n^2 - (i+1)n + 1)} \right) \circ \mu' \left( \epsilon_{i+1}(x), \frac{(n-1)^2nt}{6(n^2 - (i+1)n + 1)} \right) \right\}.
\]

**Proof.** Define a mapping $g : M \to A$ by $g(x) := f(x) + \frac{(n-1)f(0)}{2}$ for all $x \in M$. Then, for each $x \in M$, the following equation is obtained:
\[
g((n-1)x) - (n-1)^2g(x) = \mathcal{D}_uf(-x) + \left( \frac{n^2 - (i+1)n + 1}{n} \right) [\mathcal{D}_1f_i(x) - \mathcal{D}_1f_{i+1}(x)]
\]
for all $i = 1, \cdots, n-1$ and for all $j = 1, \cdots, n$, where 
\[
\mathcal{D}_1f_i(x) = \mathcal{D}_1f(0, \cdots, 0, x_{i-th}, 0, \cdots, 0).
\]
For each fixed \((i, j) \in J\), one obtains from \(\sum_{k=1}^{m} \frac{1+2\cdot 2^{\frac{k}{3}\beta t}}{\beta t} \leq 1\) that

\[
\mu \left( g(x) - \frac{g((n-1)^{mx})}{(n-1)^{2m}}, t \right) \geq \mu \left( \sum_{k=1}^{m} \frac{(n-1)^{k}g((n-1)^{k-1}x) - g((n-1)^{k}x)}{(n-1)^{2k}}, \sum_{k=1}^{m} \frac{t}{2^k} \right)
\]

\[
\geq \prod_{k=1}^{m} \left( \mu' \left( \epsilon_{k}(-x), \frac{(n-1)^{2k}t}{3 \cdot 2^k \beta^{k-1}} \right) \right)
\]

\[
\circ \mu' \left( \epsilon_{i}(x), \frac{(n-1)^{2k}t}{3 \cdot 2^k \beta^{k-1}(m^2 - (i+1)n + 1)} \right)
\]

\[
\circ \mu' \left( \epsilon_{i+1}(x), \frac{(n-1)^{2k}t}{3 \cdot 2^k \beta^{k-1}(n^2 - (i+1)n + 1)} \right)
\]

\[
= \mu' \left( \epsilon_{i}(-x), \frac{(n-1)^{2k}t}{6(n^2 - (i+1)n + 1)} \right) \circ \mu' \left( \epsilon_{i}(x), \frac{(n-1)^{2k}t}{6(n^2 - (i+1)n + 1)} \right)
\]

for all \(t > 0\) and \(x \in M, m \in \mathbb{N}\). Then, it follows from the above inequality that

\[
\mu \left( g(x) - \frac{g((n-1)^{mx})}{(n-1)^{2m}}, t \right) \geq \Phi(x, t)
\]

for all \(x \in M\) and \(t > 0\). Therefore, we prove from this relation that, for any integers \(m, p\),

\[
\mu \left( \frac{g((n-1)^{mx})}{(n-1)^{2m}} - \frac{g((n-1)^{m+p}x)}{(n-1)^{2(m+p)}}, t \right) \geq \mu \left( \frac{g((n-1)^{mx})}{(n-1)^{2m}} - \frac{g((n-1)^{p} \cdot (n-1)^{mx})}{(n-1)^{2m}}, \frac{(n-1)^{2m}t}{(n-1)^{2m}} \right)
\]

\[
\geq \Phi((n-1)^{mx}, (n-1)^{2m}, t) \geq \Phi \left( x, \left( \frac{(n-1)^{2}}{eta} \right)^{m} \right)
\]

for all \(t > 0, x \in M\). Since the right-hand side of the above inequality tends to 1 as \(m \to \infty\), the sequence \(\left\{ \frac{g((n-1)^{mx})}{(n-1)^{2m}} \right\}\) is \(\mu\)-Cauchy and thus converges in \(A\). Hence, we may define a mapping \(Q : M \to A\) as

\[
Q(x) := \mu - \lim_{m \to \infty} \frac{g((n-1)^{mx})}{(n-1)^{2m}} \left( \epsilon \lim_{m \to \infty} \mu \left( \frac{g((n-1)^{mx})}{(n-1)^{2m}}, t \right) = 1 \right)
\]

for all \(x \in M\) and \(t > 0\). In addition, we claim that the mapping \(Q\) satisfies (2). For this purpose, we calculate the following inequality:

\[
\mu \left( \frac{D_{n}Q(x_{1}, \cdots, x_{n})}{L}, t \right) \geq \prod_{1 \leq i < j \leq n} \left( \mu \left( \frac{Q(ux_{i} - ux_{j}) - g((n-1)^{m}(ux_{i} - ux_{j}))}{(n-1)^{2m}}, \frac{Lt}{2^{i+j}} \right) \right)
\]

\[
\circ \mu \left( \frac{uQ(\sum_{j=1}^{n} x_{j} - nx_{j})u^{*} - ug((n-1)^{m}(\sum_{j=1}^{n} x_{j} - nx_{j}))u^{*}}{(n-1)^{2m}}, \frac{Lt}{2^{i+j}} \right)
\]

\[
\circ \mu' \left( \epsilon(x_{1}, \cdots, x_{n}), \left( \frac{(n-1)^{2}}{\beta} \right)^{m} \cdot \frac{Lt}{2^{i+j}} \right)
\]
for all \( x \in M, u \in \mathcal{U}(A), m \in \mathbb{N}, t > 0 \), where \( L := \frac{n^3 - n^2 + 2n + 2}{2} \). This means that 
\[ D_u Q(x_1, \cdots, x_n) = 0 \]
for all \( x_1, \cdots, x_n \in M, u \in \mathcal{U}(A) \). Hence, the mapping \( Q \) satisfies (2) and so \( Q((n-1)x) = (n-1)^2 Q(x) \) for all \( x \in M \). It follows that
\[
\mu \left( Q(x) - g(x), t \right) \geq \mu \left( \frac{Q((n-1)x) - g((n-1)^{m+1}x)}{(n-1)^{2m+2}} \right. \\
+ \sum_{k=1}^{m} \left( \frac{(n-1)^2 g((n-1)^{k-1}x) - g((n-1)^k x), t)}{(n-1)^{2k}} \right. \\
\geq \Phi \left( (n-1)x, \frac{(n-1)^2 t}{2} \right) \bigg) \prod_{k=1}^{m} \Phi \left( x, \frac{(n-1)^2 t}{2^{k-1}} \right) \\
\geq \Phi \left( x, \frac{(n-1)^2}{2\beta} t \right)
\]
for all \( x \in M, t > 0 \).

To prove the uniqueness, let \( Q' \) be another mapping satisfying (2) and
\[
\mu \left( g(x) - Q'(x), t \right) \geq \Phi \left( x, \frac{(n-1)^2}{2\beta} t \right)
\]
for all \( x \in M \). Thus, we have
\[
\mu \left( \frac{1}{2} (Q(x) - Q'(x)), t \right) \geq \mu \left( \frac{Q((n-1)m x) - g((n-1)^m x), t)}{(n-1)^{2m}} \right. \\
\bigg) \prod_{k=1}^{m} \Phi \left( x, \frac{(n-1)^2 t}{2^{k-1}} \right) \\
\geq \Phi \left( x, \frac{(n-1)^{2m}}{2\beta} t \right)
\]
for all \( x \in M, t > 0 \). Taking the limit as \( m \to \infty \), then we conclude that \( Q(x) = Q'(x) \) for all \( x \in M \).

Under the assumption that either \( f \) is measurable or \( f(tx) \) is continuous in \( t \in \mathbb{R} \) for each fixed \( x \in M \), the quadratic mapping \( Q \) satisfies \( Q(tx) = t^2 Q(x) \) for all \( x \in M \) and for all \( t \in \mathbb{R} \) by the same reasoning as the proof of \([25]\). That is, \( Q \) is \( \mathbb{R} \)-quadratic. Let \( P := \frac{u^4 - 2u^3 + 3u^2 - 3u + 14}{4} \).
Putting \( x_1 := -(n-1)^k x \) and \( x_i := 0 \) for all \( i = 2, \cdots, n \) in (4) and dividing the resulting inequality by \( (n-1)^{2k} \), we have
\[
\mu \left( \frac{1}{P} \left( n(n-1)Q(-ux) - uQ((n-1)x)u^* - (n-1)uQ(-x)u^* \right), 4t \right) \\
\geq \mu \left( Q(-ux) - \frac{g(-u(n-1)^k x)}{(n-1)^{2k}}, \frac{Pt}{n(n-1)} \right) \\
\bigg) \prod_{k=1}^{m} \Phi \left( uQ((n-1)^k x)u^* - u^* g((n-1)^k x)u^*, \frac{Pt}{n(n-1)} \right) \\
\bigg) \prod_{k=1}^{m} \Phi \left( D_u f(-(n-1)^k x, 0, \cdots, 0), (n-1)^{2k} Pt \right) \\
\bigg) \prod_{k=1}^{m} \Phi \left( f(0), \frac{4(n-1)^{2k} Pt}{(n-2)(n-1)n(n+1)} \right)
\]
for all \( x \in M, u \in \mathcal{U}(A), t > 0 \). Taking \( k \to \infty \) and using the evenness of \( Q \), we obtain that \( Q(ux) = uQ(x)u^* \) for all \( x \in M \) and for each \( u \in \mathcal{U}(A) \). The last relation is also true for \( u = 0 \).
Now, let $a$ be a nonzero element in $A$ and $K$ a positive integer greater than $4\|a\|$. Then, we have $\frac{\|a\|}{K} < \frac{1}{4} < 1 - \frac{3}{2}$. By [26] [Theorem 1], there exist three elements $u_1, u_2, u_3 \in \mathcal{U}(A)$ such that $3K = u_1 + u_2 + u_3$. Thus, we calculate in conjunction with [27] [Lemma 2.1] that

$$Q(ax) = Q\left(\frac{K}{3} \frac{a}{K} x\right) = \left(\frac{K}{3}\right)^2 Q(u_1 x + u_2 x + u_3 x)$$

$$= \left(\frac{K}{3}\right)^2 B(u_1 x + u_2 x + u_3 x, u_1 x + u_2 x + u_3 x)$$

$$= \left(\frac{K}{3}\right)^2 (u_1 + u_2 + u_3)B(x, x)(u_1' + u_2' + u_3')$$

$$= \left(\frac{K}{3}\right)^2 \frac{a}{K} Q(x) \frac{a^*}{K} = aQ(x)a^*$$

for all $a \in A(a \neq 0)$ and for all $x \in M$. Thus, the unique $\mathbb{R}$-quadratic mapping $Q$ is also $A$-quadratic, as desired. This completes the proof. $\square$

**Corollary 1.** Let $(\mathcal{A}, \rho)$ be a $\rho$-complete convex modular $*$-algebra with norm $\| \cdot \|$ and $M$ be a left $A$-module, $\mathcal{U}(A)$ the unitary group of $A$. Assume that there exist two mappings $f \in \mathcal{A}^M$ and $\varepsilon \in \mathbb{R}^M$ such that

$$\rho(D_u f(x_1, \ldots, x_n)) \leq \varepsilon(x_1, \ldots, x_n),$$

(7)

$$\varepsilon((n - 1)x_1, \ldots, (n - 1)x_n) \leq \beta \varepsilon(x_1, \ldots, x_n)$$

for all $(x_1, \ldots, x_n) \in X^n, u \in \mathcal{U}(A)$, where $2 \leq 2\beta < (n - 1)^2$, and either $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M$. Then, there exists a unique mapping $Q \in Q_{\mathcal{A}}(M, \mathcal{A})$ which satisfies Equation (2) and the inequality

$$\rho\left(f(x) + \frac{(n - 1)f(0)}{2} - Q(x)\right) \leq \frac{12\beta}{(n - 1)^4} \min_{(i,j) \in J} \left\{ \max\left\{ \frac{(n^2 - (i + 1)n + 1)}{n \varepsilon_i(x)}, \frac{(n^2 - (i + 1)n + 1)}{n \varepsilon_{i+1}(x)} \right\} \right\}$$

(8)

for all $x \in M$.

**Proof.** Let $X = \mathbb{R}$ with the fuzzy modular $\mu' : X \times (0, \infty) \to \mathbb{R}$ as

$$\mu'(z,t) = \frac{t}{t + |z|}$$

for all $z \in \mathbb{R}, t > 0$. In addition, define the following convex fuzzy modular $\mu$ as

$$\mu(y,t) = \frac{t}{t + \rho(y)}$$

for all $y \in M, t > 0$. As noted in Example 1, $(\mathcal{A}, \mu, \circ_M)$ is a $\mu$-complete convex fuzzy modular $*$-algebra and $(\mathbb{R}, \mu', \circ_M)$ is a fuzzy modular space. The result follows from the fact that (4) and (5) are equivalent to (7) and (8), respectively. $\square$

**Corollary 2.** Let $(\mathcal{A}, \| \cdot \|)$ be a Banach $*$-algebra and $M$ be a left $A$-module and $\theta > 0$, $p \in (0, 2 - \log_3 2)$. Assume that there exists a mapping $f \in \mathcal{A}^M$ such that

$$\|D_u f(x_1, \ldots, x_n)\| \leq \theta(\|x_1\|^p + \cdots + \|x_n\|^p)$$
for all \((x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(A)\), and either \(f\) is measurable or \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in M\). Then, there exists a unique quadratic mapping \(Q \in Q_A(M, A)\) which satisfies Equation (2) and the inequality

\[
\|f(x) + \frac{(n-1)f(0)}{2} - Q(x)\| \leq \frac{12}{(n-1)^4} \varepsilon \|x\|^p
\]

for all \(x \in M\), where \(\varepsilon\) is a real number defined by

\[
\varepsilon := \begin{cases} 
\min \left\{ \frac{n^2 - (i+1)n + 1}{n} \right\} \geq 1, & \text{if } n > 3, \\
1, & \text{if } n = 3.
\end{cases}
\]

**Proof.** Letting \(\varepsilon(x_1, \cdots, x_n) := \varepsilon(\|x_1\|^p + \cdots + \|x_n\|^p)\), \(\beta := (n-1)^p\) and applying Corollary 1, we obtain the desired result, as claimed. \(\Box\)

Next, we provide an alternative stability theorem of Theorem 2 equipped with \(\Delta_{n-1}\)-condition in \(\mu\)-complete convex fuzzy modular \(*\)-algebras.

**Theorem 3.** Let \((A, \mu, \circ)\) be a \(\mu\)-complete convex fuzzy modular \(*\)-algebra with \(\Delta_{n-1}\)-condition and norm \(\|\cdot\|\) and \(M\) be a \(A\)-left module, \((X, \mu', \circ)\) fuzzy modular space. Assume that there exist two mappings \(f \in A^M\) and \(\varepsilon \in X^M\) such that

\[
\mu(D_n f(x_1, \cdots, x_n), t) \geq \mu'(\varepsilon(x_1, \cdots, x_n), t),
\]

\[
\mu'(\varepsilon_{\frac{x_1}{n-1}}, \cdots, \frac{x_n}{n-1}), t) \geq \mu'(\varepsilon_{\frac{x_1}{n-1}}, \cdots, x_n, \gamma t)
\]

for all \((x_1, \cdots, x_n) \in X^n, u \in \mathcal{U}(A)\), where \((n-1)^2 \gamma > 2x_{n-1}^2\), and either \(f\) is measurable or \(f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in M\). Then, there exists a unique mapping \(Q \in Q_A(M, A)\) which satisfies Equation (2) and the inequality

\[
\mu(f(x) - Q(x), t) \geq \Psi\left(x, \frac{(n-1)t}{2\kappa_{n-1}}\right)
\]

for all \(x \in M, t > 0\), where

\[
\Psi(x, t) = \max_{(i,j) \in J} \left\{ \mu'(\varepsilon_i(-x), \frac{\gamma(n-1)^2 t}{6\kappa_{n-1}^2}, \frac{\gamma(n-1)^2 nt}{6\kappa_{n-1}(n^2 - (i+1)n + 1)})
\right\}
\]

**Proof.** Letting \((x_1, \cdots, x_n) := (0, \cdots, 0)\) in (9) and using it, we get

\[
\mu'(\varepsilon_{0, \cdots, 0}, t) \geq \mu'(\varepsilon(0, \cdots, 0), \gamma^m t)
\]

for all \(t > 0, m \in \mathbb{N}\). Thus, \(\varepsilon(0, \cdots, 0) = 0\) and

\[
\mu\left(\frac{(n-1)^2}{2} f(0), t\right) = \mu(D_n \delta(0, \cdots, 0), t) \geq \mu'(\varepsilon(0, \cdots, 0), t) = 1
\]
for all $t > 0$, which implies $f(0) = 0$. From Equation (6), we get the following equality

$$f(x) - (n - 1)^2 f\left(\frac{x}{n - 1}\right) = D_1 f_i \left(- \frac{x}{n - 1} + \left(\frac{n^2 - (i + 1)n + 1}{n}\right)\right) \left[D_1 f_i \left(\frac{x}{n - 1}\right) - D_1 f_{i+1} \left(\frac{x}{n - 1}\right)\right]$$

(11)

for all $(i, j) \in \mathcal{J}$. Using (11) and $\Delta_{n-1}$-condition of $\mu$, one gets

$$\mu\left(f(x) - (n - 1)^2 f\left(\frac{x}{(n - 1)^m}\right), t\right) \geq \mu\left(\sum_{k=1}^{m} \left(\frac{(n - 1)^{4k-2}}{(n - 1)^{2k}} \left(f\left(\frac{x}{(n - 1)^k}\right) - (n - 1)^2 f\left(\frac{x}{(n - 1)^k}\right)\right) \sum_{k=1}^{m} \frac{t}{2^k}\right) \right.$$

$$\geq \prod_{k=1}^{m} \left(\mu^\prime \left(\epsilon_i(-x), \left(\frac{\gamma(n - 1)^2}{2\kappa_{n-1}}\right)^k \cdot \frac{\kappa_{n-1}^2 t}{3} \right) \right.\left.\mu^\prime \left(\epsilon_j(x), \left(\frac{\gamma(n - 1)^2}{2\kappa_{n-1}}\right)^k \cdot \frac{\kappa_{n-1}^2 t}{3(n^2 - (i + 1)n + 1)} \right)\right)$$

$$\bigg.$$

$$= \mu^\prime \left(\epsilon_i(-x), \frac{\gamma(n - 1)^2 t}{6\kappa_{n-1}^2(n^2 - (i + 1)n + 1)} \right) \bigg.$$

\(\bigg.$$

$$= \mu^\prime \left(\epsilon_j(x), \frac{\gamma(n - 1)^2 t}{6\kappa_{n-1}^2(n^2 - (i + 1)n + 1)} \right) \bigg.$$

(12)

for all $x \in M, t > 0, (i, j) \in \mathcal{J}$. This relation leads to

$$\mu\left(f(x) - (n - 1)^2 f\left(\frac{x}{(n - 1)^m}\right), t\right) \geq \Psi(x, t)$$

(12)

for all $x \in M$ and $t > 0$. Now, replacing $x$ by $\frac{x}{(n - 1)^m}$ in (12), we have

$$\mu\left(\frac{(n - 1)^2 f\left(\frac{x}{(n - 1)^m}\right)}{(n - 1)^m} - (n - 1)^{2m+2p} f\left(\frac{x}{(n - 1)^{m+p}}\right), t\right) \geq \mu\left(f\left(\frac{x}{(n - 1)^m}\right) - (n - 1)^{2p} f\left(\frac{x}{(n - 1)^{m+p}}\right), \frac{t}{\kappa_{n-1}^m}\right) \geq \Psi\left(\frac{x}{(n - 1)^m}, \frac{t}{\kappa_{n-1}^m}\right) \geq \Psi\left(x, \left(\frac{\gamma^2}{\kappa_{n-1}^m}\right)^{m} t\right)$$

which converges to zero as $m \to \infty$. Thus, $\{(n - 1)^2 f(x/(n - 1)^m)\}$ is $\mu$-Cauchy for all $x \in M$, and so it is $\mu$-convergent in $\mathcal{A}$ since the space $\mathcal{A}$ is $\mu$-complete. Thus, we may define a mapping $Q : M \to \mathcal{A}$ as

$$Q(x) := \mu - \lim_{m \to \infty} (n - 1)^2 f\left(\frac{x}{(n - 1)^m}\right)$$

$$\iff \lim_{m \to \infty} (n - 1)^2 \mu \left(Q(x) - f\left(\frac{x}{(n - 1)^m}\right), t\right) = 1$$
for all $x \in M$ and all $t > 0$. Using $\Delta_{n-1}$-condition and convexity of $\mu$, we find the following inequality

$$
\mu \left( f(x) - Q(x), t \right) \geq \mu \left( f(x) - (n-1)^{2m}f\left( \frac{x}{(n-1)^{2m}} \right), (n-1)t \right) \circ \mu \left( \frac{x}{(n-1)^{2m}} - Q(x), \frac{(n-1)t}{2k_{n-1}} \right)
$$

$$
\geq \Psi \left( x, \frac{(n-1)t}{2k_{n-1}} \right)
$$

for all $x \in M, t > 0$ and for enough large $m \in \mathbb{N}$. By the similar way of the proof of Theorem 2, we get $Q$ is $A$-quadratic functional equation.

To prove the uniqueness, let $T$ be another $A$-quadratic mapping satisfying (10). Then, we get $T((n-1)^m x) = (n-1)^{2m}T(x)$ for all $x \in M$ and all $m \in \mathbb{N}$. Thus, we have

$$
\mu \left( \frac{T(x) - Q(x)}{2}, t \right) \geq \mu \left( \frac{T(x)}{(n-1)^m} - f\left( \frac{x}{(n-1)^m} \right), \frac{t}{2k_{n-1}} \right) \circ \mu \left( f\left( \frac{x}{(n-1)^m} \right) - Q\left( \frac{x}{(n-1)^m} \right), \frac{t}{2k_{n-1}} \right)
$$

$$
\geq \Psi \left( \frac{x}{(n-1)^m}, \frac{(n-1)t}{2k_{n-1}} \right) \geq \Psi \left( x, \frac{(n-1)\gamma t}{2k_{n-1}} \right)
$$

Taking the limit as $m \to \infty$, then we conclude that $T(x) = Q(x)$ for all $x \in M$. This completes the proof. \[\square\]

**Corollary 3.** Let $(A, \rho)$ be a $\rho$-complete convex modular $*$-algebra with $\Delta_{n-1}$-condition and norm $\| \cdot \|$. Assume that there exist two mappings $f \in A^M$ and $\varepsilon \in \mathbb{R}^M$ such that

$$
\rho(D_n f(x_1, \ldots, x_n)) \leq \varepsilon(x_1, \ldots, x_n),
$$

$$
\varepsilon \left( \frac{x_1}{n-1}, \ldots, \frac{x_n}{n-1} \right) \leq \frac{1}{\gamma} \varepsilon(x_1, \ldots, x_n)
$$

for all $(x_1, \ldots, x_n) \in X^n, n \in U(A)$, where $\gamma(n-1)^2 > 2k_{n-1}$ and either $f$ is measurable or $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in M$. Then, there exists a unique mapping $Q \in Q_A(M, A)$ which satisfies Equation (2) and the inequality

$$
\rho(f(x) - Q(x)) \leq \frac{12\kappa_{n-1}^3}{\gamma(n-1)^3} \min_{(i,j) \in J} \left\{ \max \left\{ \varepsilon_i(-x), \frac{(n^2 - (i+1)n + 1)\varepsilon_i(x)}{n} \right\}, \frac{(n^2 - (i+1)n + 1)\varepsilon_{i+1}(x)}{n} \right\}
$$

for all $x \in M$.

**4. Conclusions**

We have studied a quadratic functional equation from the gravity of the $n$-distinct vectors and obtained the solution of the quadratic functional equation and investigated the stability results of a $A$-quadratic mapping on $\mu$-complete convex fuzzy modular $*$-algebras without using $\beta$-homogeneous property and lower semicontinuity. Furthermore, as corollaries, we have presented the stability results of the $A$-quadratic mapping in $\rho$-complete convex modular $*$-algebras and Banach $*$-algebras, respectively.
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