On some realizable metabelian 5-groups

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Abstract

Let $G$ be a 5-group of maximal class and $\gamma_2(G) = [G, G]$ its derived group. Assume that the abelianization $G/\gamma_2(G)$ is of type $(5, 5)$ and the transfers $V_{H_1 \rightarrow \gamma_2(G)}$ and $V_{H_2 \rightarrow \gamma_2(G)}$ are trivial, where $H_1$ and $H_2$ are two maximal normal subgroups of $G$. Then $G$ is completely determined with the isomorphism class groups of maximal class. Moreover the group $G$ is realizable with some fields $k$, which is the normal closure of a pure quintic field.

Key words: Groups of maximal class, Metabelian 5-groups, Transfer, 5-class groups.

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1 Introduction

The coclass of a $p$-group $G$ of order $p^n$ and nilpotency class $c$ is defined as $cc(G) = n - c$, and a $p$-group $G$ is called of maximal class, if it has $cc(G) = 1$. These groups have been studied by various authors, by determining there classification, the position in coclass graph and the realization of these groups. Blackburn’s paper [1], is considered as reference of the basic materials about these groups of maximal class. Eick and Leedhan-Green in [5] gave a classification of 2-groups. Blackburn’s classification in [1], of the 3-groups of coclass 1 implies that these groups exhibit behaviour similar to that proved for
2-groups. The 5-groups of maximal class have been investigated in detail in \[2\] \[3\] \[4\] \[5\] \[13\].

With an arbitrary prime \( p \geq 2 \), let \( G \) be a metabelian \( p \)-group of order \( |G| = p^n \) and \( cc(G) = 1 \), where \( n \geq 3 \). Then \( G \) is of maximal class and the commutator factor group \( G/\gamma_2(G) \) of \( G \) is of type \((p,p)\) \[11\], \[11\]. By \( G^{(n)}(z,w) \) we denote the representative of an isomorphism class of the metabelian \( p \)-groups \( G \), which satisfies the relations of theorem 2.1 with a fixed system of exponents \( a, w \) and \( z \).

In this paper we shall prove that some metabelian 5-groups are completely determined with the isomorphism class groups of maximal class, furthermore they can be realized.

For that we consider \( k = \mathbb{Q}(\sqrt[p]{p}, \zeta_5) \), the normal closure of the pure quintic field \( \Gamma = \mathbb{Q}(\sqrt[p]{p}) \), and also a cyclic Kummer extension of degree 5 of the 5th cyclotomic field \( k_0 = \mathbb{Q}(\zeta_5) \), where \( p \) is a prime number, such that \( p \equiv -1 (\text{mod } 25) \). According to \[6\], if the 5-class group of \( k \), denoted \( C_{k,5} \), is of type \((5,5)\), we have that the rank of the subgroup of ambiguous ideal classes under the action of \( \text{Gal}(k/k_0) = \langle \sigma \rangle \), denoted \( C^{(\sigma)}_{k,5} \), is rank \( C^{(\sigma)}_{k,5} = 1 \). Whence by class field theory the relative genus field of the extension \( k/k_0 \), denoted \( k^* = (k/k_0)^* \), is one of the six cyclic quintic extensions of \( k \).

By \( F_5^{(1)} \) we denote the Hilbert 5-class field of a number field \( F \). Let \( G = \text{Gal}\left((k^*)^{(1)}_5/k_0\right) \), we show that \( G \) is a metabelian 5-group of maximal class, and has two maximal normal subgroups \( H_1 \) and \( H_2 \), such that the transfers \( V_{H_1-\gamma_2(G)} \) and \( V_{H_2-\gamma_2(G)} \) are trivial. Moreover \( G \) is completely determined with the isomorphism class groups of maximal class. The theoretical results are underpinned by numerical examples obtained with the computational number theory system PARI/GP \[15\].

2 PRELIMINARY

Let \( G \) be a metabelian \( p \)-group of order \( p^n \), \( n \geq 3 \), with abelianization \( G/\gamma_2(G) \) is of type \((p,p)\), where \( \gamma_2(G) = [G,G] \) is the commutator group of \( G \). The subgroup \( G^p \) of \( G \), generated by the \( p \)-th powers is contained in \( \gamma_2(G) \), which therefore coincides with the Frattini subgroups \( \phi(G) = G^p/\gamma_2(G) = \gamma_2(G) \).

According to the basis theorem of Burnside, the group \( G \) can thus be generated by two elements \( x \) and \( y \), \( G = \langle x, y \rangle \). If we declare the lower central series of \( G \) recursively by

\[
\begin{align*}
\gamma_1(G) &= G \\
\gamma_j(G) &= [\gamma_{j-1}(G), G] \quad \text{for } j \geq 2,
\end{align*}
\]

Then we have Kaloujnine’s commutator relation \( [\gamma_j(G), \gamma_l(G)] \subseteq \gamma_{j+l}(G) \), for \( j, l \geq 1 \), and for an index of nilpotence \( c \geq 2 \) the series

\[
G = \gamma_1(G) \supset \gamma_2(G) \supset \ldots \supset \gamma_{c-1}(G) \supset \gamma_c(G) = 1
\]

becomes stationary.

The two-step centralizer

\[
\chi_2(G) = \{ g \in G \mid [g, u] \in \gamma_4(G) \text{ for all } u \in \gamma_2(G) \}
\]

of the two-step factor group \( \gamma_2(G)/\gamma_4(G) \), that is the largest subgroup of \( G \) such that \( [\chi_2(G), \gamma_2(G)] \subset \gamma_4(G) \). It is characteristic, contains the commutator subgroup \( \gamma_2(G) \). Moreover \( \chi_2(G) \) coincides with
$G$ if and only if $n = 3$. For $n \geq 4$, $\chi_2(G)$ is one of the $p + 1$ normal subgroups of $G$.

Let the isomorphism invariant $k = k(G)$ of $G$, be defined by $[\chi_2(G), \gamma_2(G)] = \gamma_{n-k}(G)$, where $k = 0$ for $n = 3$ and $0 \leq k \leq n - 4$ if $n \geq 4$, also for $n \geq p + 1$ we have $k = \min\{n - 4, p - 2\}$.

$k(G)$ provides a measure for the deviation from the maximal degree of commutativity $[\chi_2(G), \gamma_2(G)] = 1$ and is called defect of commutativity of $G$.

With a further invariant $e$, it will be expressed, which factor $\gamma_j(G)/\gamma_{j+1}(G)$ of the lower central series is cyclic for the first time \cite{12}, and we have $e + 1 = \min\{3 \leq j \leq m | 1 \leq |\gamma_j(G)/\gamma_{j+1}(G)| \leq p\}$.

In this definition of $e$, we exclude the factor $\gamma_2(G)/\gamma_3(G)$, which is always cyclic. The value $e = 2$ is characteristic for a group $G$ of maximal class.

### 2.1 On the 5-class group of maximal class

Let $G$ be a metabelian 5-group of order $5^n$, $n \geq 4$, such that $G/\gamma_2(G)$ is of type $(5, 5)$, then $G$ admits six maximal normal subgroups $H_1, ..., H_6$, which contain the commutator group $\gamma_2(G)$ as a normal subgroup of index 5. We have that $\chi_2(G)$ is one of the groups $H_i$. We fix $\chi_2(G) = H_1$. We have the following theorem

**Theorem 2.1.** Let $G$ be a metabelian 5-group of order $5^n$, $n \geq 4$, with the abelianization $G/\gamma_2(G)$ is of type $(5, 5)$ and $k = k(G)$ its invariant defined before. Assume that $G$ is of maximal class, then $G$ can be generated by two elements, $G = \langle x, y \rangle$, be selected such that $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$.

Let $s_2 = [y, x] \in \gamma_2(G)$ and $s_j = [s_{j-1}, x] \in \gamma_j(G)$ for $j \geq 3$. Then we have:

1. $s^5_j s^1_{j+1} s^1_{j+2} s^5_{j+3} s^3_{j+4} = 1$ for $j \geq 2$.
2. $x^5 = s^w_{n-1}$ with $w \in \{0, 1, 2, 3, 4\}$.
3. $y s^2_{2} s^3_{3} s^5_{4} s^5_{5} = s^z_{n-1}$ with $z \in \{0, 1, 2, 3, 4\}$.
4. $[y, s_2] = \prod_{i=1}^{k} s^a_{n-i}$ with $a = (a_{n-1}, ..., a_{n-k})$ exponents such that $0 \leq a_{n-i} \leq 4$.

**Proof.** See \cite{11}, Theorem 2 for $p = 5$. \hfill $\Box$

The six maximal normal subgroups $H_1, ..., H_6$ are arranged as follow:

$H_1 = \langle y, \gamma_2(G) \rangle = \chi_2(G)$, $H_i = \langle xy^{i-2}, \gamma_2(G) \rangle$ for $2 \leq i \leq 6$. The order of the abelianization of each $H_i$, for $1 \leq i \leq 6$, is given by the following theorem.

**Theorem 2.2.** Let $G$, $H_i$ and the invariant $k$ as before. Then for $1 \leq i \leq 6$, the order of the commutator factor groups of $H_i$ is given by:

1. If $n = 2$ we have : $|H_i/\gamma_2(H_i)| = 5$ for $1 \leq i \leq 6$.
2. If $n \geq 3$ we have : $|H_i/\gamma_2(H_i)| = 5^2$ for $2 \leq i \leq 6$, and $|H_1/\gamma_2(H_1)| = 5^{n-k-1}$

**Proof.** See \cite{111}, Theorem 3.1 for $p = 5$. \hfill $\Box$
Lemma 2.1. Let \( G \) be a 5-group of order \(|G| = 5^n, n \geq 4\). Assume that the commutator group \( G/\gamma_2(G) \) is of type \((5,5)\). Then \( G \) is of maximal class if and only if \( G \) admits a maximal normal subgroup with factor commutator of order \( 5^2 \). Furthermore \( G \) admits at least five maximal normal subgroups with factor commutator of order \( 5^2 \).

Proof. Assume that \( G \) is of maximal class, then by theorem 2.2 we conclude that \( G \) has five maximal normal subgroups with the order of commutator factor is \( 5^2 \) if \( n \geq 4 \), and has six when \( n = 3 \). Conversely, Assume that \( cc(G) \geq 2 \), the invariant \( e \) defined before is greater than 3, and since each maximal normal subgroup \( H \) of \( G \) verify \(|H/\gamma_2(H)| \geq 5^e \) we get that \(|H/\gamma_2(H)| > 5^2\). \( \square \)

2.2 On the transfer concept

Let \( G \) be a group and let \( H \) be a subgroup of \( G \). The transfer from \( G \) to \( H \) can be decomposed as follows: Also we note \( \bar{V} \) instead of \( V_{G \rightarrow H} \).

\[
\begin{array}{ccc}
G & \rightarrow & H/\gamma_2(H) \\
\downarrow & & \downarrow \bar{V} \\
G/\gamma_2(G) & \rightarrow & H/\gamma_2(H)
\end{array}
\]

Figure 1: Transfer diagram

Definition 2.1. Let \( G \) be a group, \( H \) be a normal subgroup of \( G \), and let \( g \in G \) such that, \( f \) is the order of \( gH \) in \( G/H \), \( r = \frac{|G:H|}{f} \) and \( g_1, ..., g_r \) be a representative system of \( G/H \), then the transfer from \( G \) to \( H \), noted \( V_{G \rightarrow H} \), is defined by:

\[
V_{G \rightarrow H} : G/\gamma_2(G) \rightarrow H/\gamma_2(H) \\
g/\gamma_2(G) \rightarrow \prod_{i=1}^{r} g_i^{-1} g^l g_i \gamma_2(H)
\]

In the special case that \( G/H \) is cyclic group of order 5 and \( G = \langle h, H \rangle \), then the transfer \( V_{G \rightarrow H} \) is given as:

1. If \( g \in H \); then \( V_{G \rightarrow H}(g\gamma_2(G)) = g^{1+h+h^2+h^3+h^4} \gamma_2(H) \)

2. \( V_{G \rightarrow H}(h\gamma_2(G)) = h^5\gamma_2(H) \)

3 MAIN RESULTS

In this section we investigate the purely group theoretic results to determine the invariants of metabelian 5-group of maximal class developed in theorem 2.1. Furthermore we show that a such metabelian 5-group is realized by the Galois group of some fields tower.
3.1 Invariants of metabelian 5-group of maximal class

In this paragraph, we keep the same hypothesis on the group $G$ and the generators $G = \langle x, y \rangle$, such that $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$. The six maximal normal subgroups of $G$ are as follows: $H_1 = \chi_2(G) = \langle y, \gamma_2(G) \rangle$ and $H_i = \langle xy^{i-2}, \gamma_2(G) \rangle$ for $2 \leq i \leq 6$.

In the case that the transfers from two subgroups $H_i$ and $H_j$ to $\gamma_2(G)$ are trivial, we can determine completely the 5-group $G$.

**Proposition 3.1.** Let $G$ be a metabelian 5-group of maximal class of order $5^n$, $n \geq 4$. If the transfers $V_{\chi_2(G)\rightarrow \gamma_2(G)}$ and $V_{H_2\rightarrow \gamma_2(G)}$ are trivial, then $n \leq 6$ and $\gamma_2(G)$ is of exponent 5. Furthermore:

- If $n = 6$ then $G \sim G_6^{(1,0)}$ where $a = 0$ or 1.
- If $n = 5$ then $G \sim G_5^{(5)}$ where $a = 0$ or 1.
- If $n = 4$ then $G \sim G_0^{(4)}(0,0)$.

**Proof.** Assume that $n \geq 7$, then $\gamma_5(G) = \langle s_5, \gamma_6(G) \rangle$, because $G$ is of maximal class and $|\gamma_5(G)/\gamma_6(G)| = 5$. By [1], lemma 3.3] we have $y^5 s_5 \in \gamma_6(G)$, thus $\gamma_5(G) = \langle s_5, \gamma_6(G) \rangle = \langle y^5 s_5 s_5^5, \gamma_6(G) \rangle = \langle y^5, \gamma_6(G) \rangle$, and since $V_{\chi_2(G)\rightarrow \gamma_2(G)}(y) = y^5 = 1$, because the transfers are trivial by hypothesis, we get that $\gamma_5(G) = \gamma_6(G)$, which is impossible, whence $n \leq 6$ and According to [1], lemma 3.2] $\gamma_2(G)$ is of exponent 5.

If $n = 6$, we have $V_{\chi_2(G)\rightarrow \gamma_2(G)}$ and $V_{H_2\rightarrow \gamma_2(G)}$ are trivial, so by theorem 2.1] we obtain $x^5 = s_5^w = 1$ which imply $w = 0$, because $0 \leq w \leq 4$. Since $\gamma_2(G)$ is of exponent 5, we have $s_5^2 = 1$ and by theorem 2.1] the relation $s_4^5 5_6^5 5_7^5 5_8^5 s_5 = 1$ gives $s_5^2 = 1$, also $s_3^5 5_6^5 5_7^5 5_8^5 s_7 = 1$ gives $s_3^5 = 1$. We replace in $y^5 s_2^1 5_3^1 5_4^1 5_5 s_5 = s_5^2$ and we get $s_5 = s_5^z$, whence $z = 1$. We have $[\chi_2(G), \gamma_2(G)] \subset \gamma_6-k(G) \subset \gamma_4(G)$ then $6 - k \geq 4$, and $0 \leq k \leq 2$, thus $[y, s_2] = s_4^{a_0}$, $a = (\alpha, \beta)$. If $k = 0$, then $a = 0$ and $G \sim G_0^{(6)}(1,0)$, if $k = 1$ then $a = 1$ and $G \sim G_0^{(6)}(1,0)$ and if $k = 2$ then $G \sim G_0^{(6)}(1,0)$.

If $n = 5$, we have $[\chi_2(G), \gamma_2(G)] \subset \gamma_5-k(G) \subset \gamma_4(G)$ then $5 - k \geq 4$, and $0 \leq k \leq 1$. We have $s_5^4 = 1$, $s_2^5 = s_3^5 = 1$ and $[y, s_2] = s_4^w$. the relation $y^5 s_2^1 5_3^1 5_4^1 5_5 s_5 = s_4^z$ imply $s_4^z = 1$ so $z = 0$. As $n = 6$ we obtain $w = 0$. If $k = 0$ then $G \sim G_0^{(5)}(0,0)$ and if $k = 1$ $G \sim G_0^{(5)}(0,0)$.

If $n = 4$, Since $[\chi_2(G), \gamma_2(G)] \subset \gamma_5-k(G) \subset \gamma_4(G)$ we have $4 - k \geq 4$, and $k = 0$, thus $[y, s_2] = 1$, i.e $a = 0$. By the same way in this case we have $w = z = 0$, therefore $G \sim G_0^{(4)}(0,0)$.

**Proposition 3.2.** Let $G$ be a metabelian 5-group of maximal class of order $5^n$. If the transfers $V_{H_2\rightarrow \gamma_2(G)}$ and $V_{H_i\rightarrow \gamma_2(G)}$, $3 \leq i \leq 6$, are trivial, then we have:

- If $n = 5$ or $6$ then $G \sim G_0^{(5,0)}(0,0)$.
- If $n \geq 7$ then $G \sim G_0^{(5)}(0,0)$.

**Proof.** If $n = 5$ or 6, by [1], theorem 1.6] we have $[\chi_2(G), \gamma_2(G)] = 1$ and $[\chi_2(G), \gamma_2(G)] \subset \gamma_4(G)$ elementary, and $(\gamma_2(\chi_2(G)))^5 = 1$ and $\prod_{j = 2}^{3} [\gamma_j(G), \gamma_4(G)] = 1$, we conclude that $(xy)^5 = x^5 y^5 s_2^1 5_3^1 5_4^1 5_5 s_5$ and we have $y^5 s_2^1 5_3^1 5_4^1 5_5 s_5 = s_4^w - 1$ then $(xy)^5 = x^5 s_4^w - 1$ and since $V_{H_2\rightarrow \gamma_2(G)}$ and $V_{H_i\rightarrow \gamma_2(G)}$ are trivial then $(xy)^5 = x^5 = s_4^w - 1 = s_4^w - 1 = 1$, thus $z = w = 0$. Since $[\chi_2(G), \gamma_2(G)] = \gamma_n-k \subset \gamma_4(G)$ we have $n - k \geq 4$, whence $0 \leq k \leq 2$ because $n = 5$ or 6 then $G \sim G_0^{(5)}(0,0)$.
If \( n \geq 7 \), according to corollary page 69 of [1] we have, \((\gamma_j(\chi_2(G)))^5 = \gamma_{j+4}(G)\) for \( j \geq 2 \), and since \( y^5 s_2^{10} s_3^{10} s_4^5 s_5 = s_{n-1}^z \) we obtain:

\[
y^5 = s_{n-1}^z s_5^{-1} s_4^{-1} s_3^{-10} s_2^{-10} \equiv s_{n-1}^z s_5^{-1} \mod \gamma_6(G)
\]

because \( s_5^2 \in \gamma_6(G) \), \( s_3^5 \in \gamma_6(G) \) and \( s_4^3 \in \gamma_6(G) \), and since \( n \geq 7 \) we have \( s_{n-1} \in \gamma_6(G) \), therefor \( V = V_{H_3 \to \gamma_4(G)}(y) \equiv s_5^{-1} \mod \gamma_6(G) \). Thus \( \text{Im}(V) \subset \gamma_5(G) \), In fact \( \text{Im}(V) = \gamma_5(G) \), and also we have \( y \notin \ker(V) \) and \( \forall f \geq 2 \ y^k s_f \notin \ker(V) \). The kernel of \( V \) is formed by elements of \( \gamma_2(G) \) of exponent 5, its exactly \( \gamma_{n-4}(G) \), and since \( G \) is of maximal class then the rank of \( \gamma_2(G) \) is 2 and \( \gamma_2(G) \) admits exactly 25 elements of exponent 5, these elements form \( \gamma_{n-4}(G) \). We conclude that \( |\chi_2(G)/\gamma_2(\chi_2(G))| = |\gamma_{n-4}(G)| \times |\gamma_5(G)| = 5^4 \times 5^{n-5} = 5^{n-1} = |\chi_2(G)| \), whence \( \chi_2(G) \) is abelian because \( \gamma_2(\chi_2(G)) = 1 \), consequently \( [y, s_2] = 1 \), thus \( a = 0 \). As the cases \( n = 5 \) or \( 6 \) we obtain \((xy)^5 = x^5 s_{n-1}^z \), therefor \( z = w = 0 \), hence \( G \sim G^{(n)}_0(0,0) \).

In the case when \( V_{H_i \to \gamma_2(G)} \) and \( V_{H_j \to \gamma_2(G)} \), \( 4 \leq i \leq 6 \) are trivial, according to [1], theorem 1.6] we have \((xy^\mu)^5 = x^5 (y^5 s_2^{10} s_3^{10} s_4^5 s_5)^\mu = s_{n-1}^{w+\mu z} \) with \( \mu = 2, 3, 4 \), then we can admit the same reasoning to prove the result.

\[\square\]

**Proposition 3.3.** Let \( G \) be a metabelian 5-group of maximal class of order \( 5^n \). If the transfers \( V_{H_i \to \gamma_2(G)} \) and \( V_{H_j \to \gamma_2(G)} \), where \( i, j \in \{3, 4, 5, 6\} \) and \( i \neq j \), are trivial, then we have: \( G \sim G^{(n)}_0(0,0) \).

**Proof.** Assume that \( H_i = \langle xy^\mu_1, \gamma_2(G) \rangle \) and \( H_j = \langle xy^\mu_2, \gamma_2(G) \rangle \) where \( \mu_1, \mu_2 \in \{1, 2, 3, 4\} \) and \( \mu_1 \neq \mu_2 \). According to [1], theorem 1.6] we have already prove that \((xy^\mu_1)^5 = s_{n-1}^{w+\mu_1 z} \) and \((xy^\mu_2)^5 = s_{n-1}^{w+\mu_2 z} \). Since \( V_{H_i \to \gamma_2(G)} \) and \( V_{H_j \to \gamma_2(G)} \) are trivial, we obtain \( s_{n-1}^{w+\mu_1 z} = s_{n-1}^{w+\mu_2 z} = \) therefor \( w + \mu_1 z = w + \mu_2 z \equiv 0(\mod 5) \) and since 5 does not divide \( \mu_1 - \mu_2 \) we get \( z = 0 \) and at the same time \( w = 0 \). To prove \( a = 0 \) we admit the same reasoning as proposition 3.2.

\[\square\]

### 3.2 APPLICATION

Through this section we denote by:

- \( p \) a prime number such that \( p \equiv -1(\mod 25) \).
- \( k_0 = \mathbb{Q}(\xi_5) \) the 5\( ^{th} \) cyclotomic field, \( (\xi_5 = e^{2\pi i/5}) \).
- \( k = k_0(\sqrt[5]{p}) \) a cyclic Kummer extension of \( k_0 \) of degree 5.
- \( C_{k,5} \) the 5-ideal class group of \( k \).
- \( k^* = (k/k_0)^* \) the relative genus field of \( k/k_0 \).
- \( F_5^{(1)} \) the absolute Hilbert 5-class field of a number field \( F \).
- \( G = \text{Gal}\left((k^*)_5^{(1)}/k_0\right) \).

We begin by the following theorem.
Theorem 3.1. Let $k = \mathbb{Q}(\sqrt[5]{\gamma}, \zeta_5)$ be the normal closure of a pure quintic field $\mathbb{Q}(\sqrt[5]{\gamma})$, where $p$ a prime congruent to $-1$ modulo 25. Let $k_0$ be the the 5th cyclotomic field. Assume that the 5-class group $C_{k,5}$ of $k$, is of type $(5,5)$, then $Gal(k^*/k_0)$ is of type $(5,5)$, and two sub-extensions of $k^*/k_0$ admit a trivial 5-class number.

Proof. By $C_{k,5}^{(\sigma)}$ we denote the subgroup of ambiguous ideal classes under the action of $Gal(k/k_0) = \langle \sigma \rangle$.

According to [3], theorem 1.1, in this case of the prime $p$ we have rank $C_{k,5}^{(\sigma)} = 1$, and by class field theory, since $[k^* : k] = |C_{k,5}^{(\sigma)}|$, we have that $k^*/k$ is a cyclic quintic extension, whence $Gal(k^*/k_0)$ is of type $(5,5)$.

Since $p \equiv -1(\text{mod } 25)$, then $p$ splits in $k_0$ as $p = \pi_1\pi_2$, where $\pi_1$, $\pi_2$ are primes of $k_0$. By [7], theorem 5.15 we have explicitly the relative genus field $k^*$ as $k^* = k(\sqrt[5]{\pi_1^{a_1}\pi_2^{a_2}}) = k(\sqrt[5]{\pi_1\pi_2}, \sqrt[5]{\pi_1^{a_1}\pi_2^{a_2}})$ with $a_1, a_2 \in \{1, 2, 3, 4\}$ such that $a_1 \neq a_2$. Its clear that the extension $k^*/k_0$ admits six sub-extensions, where $k$ is one of them, and the others are $k_0(\sqrt[5]{\pi_1^{a_1}\pi_2^{a_2}})$, $k_0(\sqrt[5]{\pi_1^{a_1+1}\pi_2^{a_2+1}})$, $k_0(\sqrt[5]{\pi_1^{a_1+2}\pi_2^{a_2+2}})$, $k_0(\sqrt[5]{\pi_1^{a_1+3}\pi_2^{a_2+3}})$ and $k_0(\sqrt[5]{\pi_1^{a_1+4}\pi_2^{a_2+4}})$. Since $a_1, a_2 \in \{1, 2, 3, 4\}$, we can see that the extensions $L_1 = k_0(\sqrt[5]{\pi_1})$ and $L_2 = k_0(\sqrt[5]{\pi_2})$ are sub-extensions of $k^*/k_0$.

In [7, section 5.1], we have an investigation of the rank of ambiguous classes of $k_0(\sqrt[5]{\gamma})/k_0$, denoted $t$. We have $t = d + 4^* - 3$, where $d$ is the number of prime divisors of $x$ in $k_0$, and $4^*$ an index defined as [7, section 5.1]. For the extensions $L_i/k_0$, $(i = 1, 2)$, we have $d = 1$ and by [7, theorem 5.15] we have $4^* = 2$, hence $t = 0$.

By $h_5(L_i)$, $(i = 1, 2)$, we denote the class number of $L_i$, then we have $h_5(L_1) = h_5(L_2) = 1$. Otherwise $h_5(L_i) \neq 1$, then there exists an unramified cyclic extension of $L_i$, denoted $F$. This extension is abelian over $k_0$, because $[F : k_0] = 5^2$, then $F$ is contained in $(L_i/k_0)^*$ the relative genus field of $L_i/k_0$. Since $[(L_i/k_0)^* : L_i] = 5^t = 1$, we get that $(L_i/k_0)^* = L_i$, which contradicts the existence of $F$. Hence the 5-class number of $L_i$, $(i = 1, 2)$, is trivial.

In what follows, we denote by $L_1$ and $L_2$ the two sub-extensions of $k^*/k_0$, which verify theorem 3.1. and by $\tilde{L}$ the three remaining sub-extensions different to $k$. Let $G = Gal((k^*)^{(1)}_5/k_0)$, we have $\gamma_2(G) = Gal((k^*)^{(1)}_5/k^*)$, then $G/\gamma_2(G) = Gal(k^*/k_0)$ is of type $(5,5)$, therefore $G$ is metabelian 5-group with factor commutator of type $(5,5)$, thus $G$ admits exactly six maximal normal subgroups as follows:

$$H = Gal((k^*)^{(1)}_5/k), H_{L_i} = Gal((k^*)^{(1)}_5/L_i), (i = 1, 2), \tilde{H} = Gal((k^*)^{(1)}_5/\tilde{L})$$

With $\gamma_2(G)$ is one of them.

Now we can state our principal result.

Theorem 3.2. Let $G = Gal((k^*)^{(1)}_5/k_0)$ be a 5-group of order $5^n$, $n \geq 4$, then $G$ is a metabelian of maximal class. Furthermore we have:

- If $\chi_2(G) = H_{L_i}(i = 1, 2)$ then: $G \sim G_0^{(n)}(z, 0)$ with $n \in \{4, 5, 6\}$ and $a, z \in \{0, 1\}$.
- If $\chi_2(G) = \tilde{H}$ then: $G \sim G_0^{(n)}(0, 0)$ with $n = 5$ or 6.

$G \sim G_0^{(n)}(0, 0)$ with $n \geq 7$ such that $n = s + 1$ where $h_5(\tilde{L}) = 5^s$.

Proof. Let $G = Gal((k^*)^{(1)}_5/k_0)$ and $H = Gal((k^*)^{(1)}_5/k)$ its maximal normal subgroup, then $\gamma_2(H) = Gal((k^*)^{(1)}_5/k^{(1)}_5)$, therefor $H/\gamma_2(H) = Gal(k^{(1)}_5/k) \simeq C_{k,5}$, and as $C_{k,5}$ is of type $(5,5)$ by hypothesis...
we get that $|H/\gamma_2(H)| = 5^2$. Lemma 3.1 imply that $G$ is a metabelian 5-group of maximal class, generated by two elements $G = \langle x, y \rangle$, such that, $x \in G \setminus \chi_2(G)$ and $y \in \chi_2(G) \setminus \gamma_2(G)$. Since $\chi_2(G) = \langle y, \gamma_2(G) \rangle$, we have $\chi_2(G) \neq H$. Otherwise we get that $|H/\gamma_2(H)| = 5^2$ which contradict theorem 2.1.

According to theorem 3.1 we have $h_5(L_1) = h_5(L_2) = 1$ then the transfers $V_{H_{L_i} \to \gamma_2(G)}$ are trivial. If $\chi_2(G) = H_{L_i}$ the results are nothing else than proposition 3.1. If $\chi_2(G) = \tilde{H}$ and $n = 4$ then $\gamma_4(G) = 1$ and $[\chi_2(G), \gamma_2(G)] = \gamma_2(\tilde{H})$, also $[\chi_2(G), \gamma_2(G)] = \gamma_4(G) = 1$ then $\chi_2(\tilde{H}) = 1$, whence $\tilde{H}$ is abelian. Consequently $\tilde{H}/\gamma_2(\tilde{H}) = C_{L,5}$, so $h_5(\tilde{L}) = |\tilde{H}| = 5^3$ because its a maximal subgroup of $G$. Since $\tilde{L}$ and $k$ have always the same conductor, we deduce that $h_5(k)$ and $h_5(\tilde{L})$ verify the relations $5^5h_L = uh_1^4$ and $5^5h_k = uh_1^4$, given by C. Parry in [14], where $u$ is a unit index and a divisor of $5^6$. Using the 5-valuation on these relations we get that $h_5(\tilde{L}) = 5^8$ where $s$ is even, which contradict the fact that $h_5(\tilde{L}) = 5^2$, hence $n \geq 5$.

The results of the theorem are exactly application of propositions 3.2, 3.3. According to proposition 3.2 if $n \geq 7$ we have $|\chi_2(G)| = 5^{n-1}$ and since $h_5(\tilde{L}) = |\tilde{H}/\gamma_2(\tilde{H})| = |\tilde{H}| = 5^{n-4} = 5^s$ we deduce that $n = s + 1$. □

4 Numerical examples

For these numerical examples of the prime $p$, we have that $C_{k,5}$ is of type $(5, 5)$ and rank $C_{k,5}^{(\sigma)} = 1$, which mean that $k^*$ is cyclic quintic extension of $k$, then by theorem 3.2 we have a completely determination of $G$. We note that the absolute degree of $(k^*)_{5}^{(1)}$ surpass 100, then the task to determine the order of $G$ is definitely far beyond the reach of computational algebra systems like MAGMA and PARI/GP.

Table 1: $k = \overline{\mathbb{Q}}(\sqrt[5]{p}, \zeta_5)$ with $C_{k,5}$ is of type $(5, 5)$ and rank $C_{k,5}^{(\sigma)} = 1$.

| $p$ | $p \pmod{25}$ | $h_{k,5}$ | $C_{k,5}$ | rank ($C_{k,5}^{(\sigma)}$) |
|-----|----------------|----------|-----------|-----------------|
| 149 | -1             | 25       | (5, 5)    | 1               |
| 199 | -1             | 25       | (5, 5)    | 1               |
| 349 | -1             | 25       | (5, 5)    | 1               |
| 449 | -1             | 25       | (5, 5)    | 1               |
| 559 | -1             | 25       | (5, 5)    | 1               |
| 1249| -1             | 25       | (5, 5)    | 1               |
| 1499| -1             | 25       | (5, 5)    | 1               |
| 1949| -1             | 25       | (5, 5)    | 1               |
| 1999| -1             | 25       | (5, 5)    | 1               |
| 2099| -1             | 25       | (5, 5)    | 1               |
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