ANALYTIC TORSION FOR BORCEA-VOISIN THREEFOLDS

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Abstract. In their study of genus-one string amplitude $F_1$, Bershadsky-Cecotti-Ooguri-Vafa discovered a remarkable identification between holomorphic Ray-Singer torsion and instanton numbers for Calabi-Yau threefolds. The holomorphic torsion invariant for Calabi-Yau threefolds corresponding to $F_1$ is called BCOV invariant. In this paper, we establish an identification between the BCOV invariants of Borcea-Voisin threefolds and another holomorphic torsion invariants for $K3$ surfaces with involution. We also introduce BCOV invariants for certain class of Calabi-Yau orbifolds including Borcea-Voisin orbifolds. We make a comparison of BCOV invariants between Borcea-Voisin orbifolds and their crepant resolutions.

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Introduction

In [3], [4], Bershadsky-Cecotti-Ooguri-Vafa discovered a remarkable identification between holomorphic Ray-Singer torsion and instanton numbers for Calabi-Yau threefolds. For a Calabi-Yau threefold $X$, they introduced the following combination of Ray-Singer analytic torsions [30], [6]

$$T_{BCOV}(X, \gamma) = \exp\{ - \sum_{p,q} (-1)^{p+q} pq \zeta_{p,q}'(0) \},$$

where $\zeta_{p,q}(s)$ is the spectral zeta function of the Laplacian acting on the $(p,q)$-forms on $X$ with respect to a Ricci-flat Kähler metric $\gamma$. Although $T_{BCOV}(X, \gamma)$ itself is not an invariant of $X$, its correction [12]

$$\tau_{BCOV}(X) = \text{Vol}(X, \gamma)^{-3+\frac{\chi_X}{12}} \text{Vol}_{L^2}(H^2(X, \mathbb{Z}), [\gamma])^{-1} T_{BCOV}(X, \gamma)$$

is an invariant of $X$, where $\text{Vol}_{L^2}(H^2(X, \mathbb{Z}), [\gamma])$ is the covolume of the lattice $H^2(X, \mathbb{Z}) \subset H^2(X, \mathbb{R})$ with respect to the $L^2$-metric on the cohomology group induced by $\gamma$ and $\chi(X)$ is the topological Euler number of $X$. We call $\tau_{BCOV}(X)$ the BCOV invariant of $X$. Because of its invariance property, $\tau_{BCOV}$ gives rise to
a function on the moduli space of Calabi-Yau threefolds. The function $-\log \tau_{\text{BCOV}}$ is identified with the physical quantity $F_1$. Then the remarkable conjecture of Bershadsky-Cecotti-Ooguri-Vafa [3] can be formulated as follows.

Let $\Delta^* \subset \mathbb{C}$ be the unit punctured disc. Let $\pi : X \to (\Delta^*)^n$ be a large complex structure limit of Calabi-Yau threefolds [27] with fiber $X_s = \pi^{-1}(s)$, $s \in (\Delta^*)^n$ and let $Y$ be the mirror Calabi-Yau threefold corresponding to the large complex structure limit point $0 \in \Delta^n$. Define the function $F_1^{\text{top}}(t)$ on the complexified Kähler cone $H^2(Y, \mathbb{R}) + i \mathcal{K}_Y$ as the infinite product [3]

$$F_1^{\text{top}}(t) = \epsilon^{2\pi i \int_c t \wedge c_2(Y)} \prod_{d \in H_2(Y, \mathbb{Z}) \setminus \{0\}} \left(1 - \epsilon^{2\pi i t(d)}\right)^{n_2(d)} \prod_{\nu \geq 1} \left(1 - \epsilon^{2\pi i \nu t(d)}\right)^{n_1(d)}$$

where $\mathcal{K}_Y$ is the Kähler cone of $Y$ and the numbers $\{n_2(d)\}$ are certain curve counting invariants of $Y$. Then the conjecture of Bershadsky-Cecotti-Ooguri-Vafa claims that $F_1^{\text{top}}(t)$ converges when $\exists t \gg 0$ and that the following equality holds near the large complex structure limit point $0 \in \Delta^n$

\[(0.1) \quad \tau_{\text{BCOV}}(X_s) = C \left\| F_1^{\text{top}}(t(s))^2 \left(\frac{\Xi_s}{(A_0^s, \Xi_s)}\right)^{3+n+\frac{s(X_s)}{2}} \otimes \left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_n}\right)_{t(s)}\right\|^2,\]

where $C$ is a constant, $n = h^{2,1}(X_s) = h^{1,1}(Y)$, $A_0^s \in \Gamma((\Delta^*)^n, R^2\pi_s^*\mathcal{Z})$ is $\pi_1((\Delta^*)^n)$-invariant, $\{\Xi_s\}_{s \in (\Delta^*)^n} \in \Gamma((\Delta^*)^n, \pi_*\mathcal{K}_{X/(\Delta^*)^n})$ is a nowhere vanishing relative canonical form of the family $\pi : X \to (\Delta^*)^n$, $t(s) = (t_1(s), \ldots, t_n(s))$ is the system of canonical coordinates [27] on $\Delta^n$ and $\| \cdot \|$ is the Hermitian metric induced from the $L^2$-metric on $\pi_*\mathcal{K}_{X/(\Delta^*)^n}$ and the Weil-Petersson metric on the holomorphic tangent bundle on $(\Delta^*)^n$. After the works of Klemm-Marino [18], Maulik-Pandharipande [24], Zinger [40], the curve counting invariants $\{n_2(d)\}$ appearing in $F_1^{\text{top}}(t)$ are expected to be the Gopakumar-Vafa invariants of $Y$.

To our knowledge, this conjecture of Bershadsky-Cecotti-Ooguri-Vafa is still widely open. Although the curvature equation characterizing $\tau_{\text{BCOV}}$ on the moduli space of Calabi-Yau threefolds is known [6], [3], [11], [12], because of the global nature of the differential equation and because the global structure of the moduli space of Calabi-Yau threefolds is not well understood in general, it is difficult to derive an explicit expression of the function $\tau_{\text{BCOV}}$ in the canonical coordinates near the large complex structure limit point. (See [12], [42] for some Calabi-Yau threefolds whose BCOV invariants are explicitly expressed on the moduli space.)

In the present paper, generalizing [12], [42], we shall give more examples of Calabi-Yau threefolds whose BCOV invariant can be explicitly expressed on a certain locus of their moduli space. The Calabi-Yau threefolds which we mainly treat in the present paper are those studied by Borcea [8] and Voisin [37]. In their study of mirror symmetry, Borcea and Voisin introduced a class of Calabi-Yau threefolds which are defined as the natural resolution

$$\tilde{X}_{(S, \theta, T)} \to X_{(S, \theta, T)} = (S \times T)/\theta \times (-1)_T.$$ 

Here, $(S, \theta)$ is a 2-elementary $K3$ surface, i.e., a $K3$ surface equipped with an anti-symplectic holomorphic involution [11], [40], and $T$ is an elliptic curve. We call $\tilde{X}_{(S, \theta, T)}$ (resp $X_{(S, \theta, T)}$) the Borcea-Voisin threefold (resp. orbifold) associated with $(S, \theta, T)$. By the computation of the Hodge numbers of $\tilde{X}_{(S, \theta, T)}$ and hence the
dimension of its Kuranishi space \[37\], a generic deformation of \(\tilde{X}_{(S, \theta, T)}\) is no longer of Borcea-Voisin type unless \(S^3\), the fixed-point-set of the \(\theta\)-action on \(S\), is either empty or rational. Hence, in most cases, the Borcea-Voisin locus forms a proper subvariety of the deformation space of \(\tilde{X}_{(S, \theta, T)}\).

By construction, once their deformation type is fixed, Borcea-Voisin threefolds are parametrized by a certain Zariski open subset of the locally symmetric variety associated with the product of the period domain for 2-elementary K3 surfaces and that of elliptic curves. We regard this locally symmetric variety as the moduli space of Borcea-Voisin threefolds of fixed deformation type. By Nikulin \[28\], \[29\], the deformation type of \((S, \theta)\) and hence that of \(\tilde{X}_{(S, \theta, T)}\) is determined by the isometry class of the invariant sublattice of \(H^2(S, \mathbb{Z})\) with respect to the \(\theta\)-action
\[
H^2(S, \mathbb{Z})^\theta = \{ l \in H^2(S, \mathbb{Z}); \ \theta^* l = l \}.
\]

For this reason, the isometry class of \(H^2(S, \mathbb{Z})^\theta\) is called the type of \((S, \theta)\) and also the type of \(\tilde{X}_{(S, \theta, T)}\). There exist 75 distinct types \[28\], \[29\]. We often identify a type with its representative, hence a primitive 2-elementary Lorentzian sublattice of the K3-lattice.

In their study of duality in string theory \[10\], Harvey-Moore studied \(F_1\) for a special class of Borcea-Voisin threefolds, called Enriques-Calabi-Yau threefolds, and gave an identification between \(F_1\) for Enriques-Calabi-Yau threefolds and the equivariant determinant of Laplacian for K3 surfaces with fixed-point-free involution. (See also \[12\], \[41\].) After Harvey-Moore, in \[40\], another holomorphic torsion invariant was constructed for 2-elementary K3 surfaces of type \(M\) (cf. Section \[2.2\]):
\[
\tau_M(S, \theta) = \text{Vol}(S, \kappa)^{\frac{14 - r(M)}{4}} \tau_{\mathbb{Z}_l}(S, \kappa)(\theta) \text{Vol}(S^\theta, \kappa|_{S^\theta}) \tau(S^\theta, \kappa|_{S^\theta}),
\]
where \(\kappa\) is a \(\theta\)-invariant Ricci-flat Kähler form on \(S\), \(\tau_{\mathbb{Z}_l}(S, \kappa)(\theta)\) is the equivariant analytic torsion of \((S, \theta)\) with respect to \(\kappa\), \(r(M)\) is the rank of \(M\), and \(\tau(S^\theta, \kappa|_{S^\theta})\) is the analytic torsion of \((S^\theta, \kappa|_{S^\theta})\). By its invariance property, \(\tau_M\) is regarded as a function on the moduli space of 2-elementary K3 surfaces of type \(M\). In \[40\], \[44\], the automorphic property of the function \(\tau_M\) was established:
\[
\tau_M = ||\Phi_M||^{-1/2\nu},
\]
where \(\nu > 0\) is a certain integer and \(||\Phi_M||\) is the Petersson norm of an automorphic form \(\Phi_M\) on the moduli space vanishing exactly on the discriminant locus with multiplicity \(\nu\). Recently, in \[43\], \[21\], the structure of \(\Phi_M\) as an automorphic form on the moduli space of 2-elementary K3 surfaces of type \(M\) is determined. Namely, except for possible two types, \(\Phi_M\) is always the tensor product of an explicit Borcherds product and an explicit Siegel modular form. (See Section \[2.4\])

In this way, we have two invariants \(\tau_{\text{BCOV}}(\tilde{X}_{(S, \theta, T)})\) and \(\tau_M(S, \theta)\), which can be identified by Harvey-Moore \[16\] when \(S/\theta\) is an Enriques surface. Our first result is an extension of this identification of Harvey-Moore to arbitrary types of Borcea-Voisin threefolds and the corresponding 2-elementary K3 surfaces.

**Theorem 0.1.** There exists a constant \(C_M\) depending only on the isometry class \(M\) such that for every Borcea-Voisin threefold \(\tilde{X}_{(S, \theta, T)}\) of type \(M\),
\[
\tau_{\text{BCOV}}(\tilde{X}_{(S, \theta, T)}) = C_M \tau_M(S, \theta)^{-4} ||\eta(T)||^{24}^2,
\]
where \(||\eta(T)||\) is the value of the Petersson norm of the Dedekind \(\eta\)-function evaluated at the period of \(T\).
We remark that Theorem 0.1 was conjectured in [42, Conj. 5.17]. After the conjecture of Bershadsky-Cecotti-Ooguri-Vafa (0.1), Theorem 0.1 clarifies the significance of the invariant $\tau_M$ in mirror symmetry. Since $\Phi_M$ is expressed as the product of an explicit Borcherds lift and an explicit Siegel modular form, $\tau_{BCOV}$ is also explicitly determined on the moduli space of Borcea-Voisin threefolds.

Since the BCOV torsion makes sense for Calabi-Yau orbifolds, it is natural to ask the possibility of extending the construction of BCOV invariants to Calabi-Yau orbifolds. In this paper, we give a partial answer to this question by constructing BCOV invariants for a class of Calabi-Yau orbifolds including Borcea-Voisin orbifolds. This extended BCOV invariant of the orbifold $X(S,\theta,T)$ is denoted by $\tau_{orb}^{BCOV}(X(S,\theta,T))$. Then a natural question is a comparison of the two BCOV invariants $\tau_{BCOV}(\tilde{X}(S,\theta,T))$ and $\tau_{orb}^{BCOV}(X(S,\theta,T))$. In Section 6, following Harvey-Moore [16], we shall prove the following.

**Theorem 0.2.** There exists a constant $C'_M$ depending only on $M$ such that for every Borcea-Voisin orbifold $X(S,\theta,T)$ of type $M$,

$$\tau_{BCOV}(X(S,\theta,T)) = C'_M \tau_M (S,\theta)^{-4} ||\eta(T)||^{24}.$$ 

**Corollary 0.3.** There exists a constant $C''_M$ depending only on $M$ such that for every Borcea-Voisin orbifold $X(S,\theta,T)$ of type $M$,

$$\tau_{BCOV}(\tilde{X}(S,\theta,T)) = C''_M \tau_{orb}^{BCOV}(X(S,\theta,T)).$$

Our proof of Corollary 0.3 is indirect in the sense that it is a consequence of Theorems 0.1 and 0.2. A direct proof of Corollary 0.3 along the line of [42, Question 5.18] is strongly desired.

To prove Theorem 0.1, we compare the complex Hessians of $\log \tau_{BCOV}$ and $\log(\tau_M^{-4} ||\eta^{24}||^2)$ to conclude that $\log[\tau_{BCOV}/(\tau_M^{-4} ||\eta^{24}||^2)]$ is a pluriharmonic function on the moduli space of Borcea-Voisin threefolds of type $M$. Then we must prove that $\log \tau_{BCOV}$ and $\log(\tau_M^{-4} ||\eta^{24}||^2)$ have the same singularity on the discriminant locus of the moduli space. By the locality of the singularity of $\log \tau_{BCOV}$ established in [45] and the corresponding result for $\log \tau_M$ in [40], we can verify that $\tau_{BCOV}$ and $\tau_M^{-4} ||\eta^{24}||^2$ have the same singularity on the discriminant locus by computing their singularities for some particular examples of Borcea-Voisin threefolds.

As in the case of ordinary BCOV invariants, the construction of BCOV invariants for certain Calabi-Yau orbifolds is an application of the curvature theorem of Bismut-Gillet-Soulé [6] for Quillen metrics and its equivariant extension by Ma [22]. Proof of Theorem 0.2 is parallel to the one given by Harvey-Moore [16].

This paper is organized as follows. In Section 1 we recall BCOV invariants and their basic properties. In Section 2 we recall the holomorphic torsion invariant $\tau_M$ for 2-elementary $K3$ surfaces and its explicit formula. In Section 3 we prove Theorem 0.1. In Section 4 we study the variation of equivariant BCOV torsion. In Section 5 we extend the BCOV invariant to a class of Calabi-Yau orbifolds of dimension 3. In Section 6 we prove Theorem 0.2.

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1. **BCOV invariants for Calabi-Yau threefolds**

A compact connected Kähler manifold $X$ of dimension 3 is called a *Calabi-Yau threefold* if the following conditions are satisfied

1. $K_X \cong \mathcal{O}_X$,
2. $H^1(X, \mathcal{O}_X) = H^2(X, \mathcal{O}_X) = 0$.

If a compact Kähler orbifold of dimension 3 satisfies (1), (2), then $X$ is called a *Calabi-Yau orbifold*. It is classical that every Calabi-Yau threefold is projective algebraic and that every Kähler class of a Calabi-Yau threefold contains a unique Ricci-flat Kähler form \[^{39}\].

**1.1. Analytic torsion.** Let $V$ be a compact Kähler manifold of dimension $n$ with Kähler metric $g = \sum_{i,j} g_{ij} \, dz_i \otimes d\bar{z}_j$. Then the Kähler form of $g$ is defined as $\gamma := \sqrt{-1} \sum_{i,j} g_{ij} \, dz_i \wedge d\bar{z}_j$. Let $(\xi, h)$ be a holomorphic Hermitian vector bundle on $V$. Let $\Box_q = (\bar{\partial} + \partial^*)^2$ be the Laplacian acting on the $(0, q)$-forms on $V$ with values in $\xi$. Let $\sigma(\Box_q)$ be the spectrum of $\Box_q$ and let $E(\Box_q; \lambda)$ be the eigenspace of $\Box_q$ corresponding to the eigenvalue $\lambda \in \sigma(\Box_q)$. Then the spectral zeta function of $\Box_q$ is defined as

$$
\zeta_q(s) := \sum_{\lambda \in \sigma(\Box_q) \setminus \{0\}} \lambda^{-s} \dim E(\Box_q; \lambda).
$$

It is classical that $\zeta_q(s)$ converges absolutely on the half-plane $\Re(s) > \dim V$, that $\zeta_q(s)$ extends to a meromorphic function on $\mathbb{C}$ and that $\zeta_q(s)$ is holomorphic at $s = 0$. Ray-Singer \[^{30}\] introduced the notion of *analytic torsion*.

**Definition 1.1.** The analytic torsion of $(\xi, h)$ is defined as the real number

$$
\tau(V, \xi) := \exp\{-\sum_{q \geq 0} (-1)^q q \zeta_q'(0)\}.
$$

In mirror symmetry, the following combination of analytic torsions introduced by Bershadsky-Cecotti-Ooguri-Vafa \[^{4}\] plays a crucial role. Write $\Box_{p,q} = (\bar{\partial} + \partial^*)^2$ for the Laplacian acting on the $(p, q)$-forms on $V$ and $\zeta_{p,q}(s)$ for its spectral zeta function.

**Definition 1.2.** The *BCOV torsion* of $(V, \gamma)$ is the real number defined as

$$
T_{\text{BCOV}}(V, \gamma) := \exp\{-\sum_{p,q \geq 0} (-1)^{p+q} pq \zeta_{p,q}'(0)\} = \prod_{p \geq 0} \tau(X, \Omega^p_X)(-1)^{p^2}.
$$

The BCOV torsion itself is not an invariant of a Calabi-Yau threefold, even if the Kähler form $\gamma$ is Ricci-flat. By adding a small correction term, it becomes an invariant of a Calabi-Yau threefold \[^{12}\]. Let us recall its construction.

**1.2. The BCOV invariant.** Let $X$ be a Calabi-Yau threefold and let $\gamma$ be a Kähler form on $X$, which is not necessarily Ricci-flat. Let $\eta$ be a nowhere vanishing canonical form on $X$. Let $c_3(X, \gamma)$ be top Chern form of $(TX, \gamma)$. In this paper, we follow the convention in Arakelev geometry. Hence

$$
\text{Vol}(X, \gamma) := \parallel \gamma \parallel_{L^2, \gamma}^3 = (2\pi)^{-3} \int_X \frac{\gamma^3}{3!}, \quad \parallel \eta \parallel_{L^2}^2 = (2\pi)^{-3} \int_X (\sqrt{-1})^3 \eta \wedge \bar{\eta}.
$$
We define the covolume \( \text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma]) \) of the lattice \( H^2(X, \mathbf{Z}) \subset H^2(X, \mathbf{R}) \) as the volume of the compact real torus \( H^2(X, \mathbf{R})/H^2(X, \mathbf{Z}) \) with respect to the \( L^2 \)-metric on \( H^2(X, \mathbf{R}) \) induced from \( \gamma \)

\[
\text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma]) := \text{Vol}(H^2(X, \mathbf{R})/H^2(X, \mathbf{Z}), [\gamma]) = \det \langle (e_i, e_j)_{L^2, \gamma} \rangle,
\]

where \( \{e_i\}_{1 \leq i \leq h_2(X)} \) is a basis of \( H^2(X, \mathbf{Z}) \). Here \( (e_i, e_j)_{L^2, \gamma} \) is defined as

\[
(e_i, e_j)_{L^2, \gamma} := (2\pi)^{-3} \int_X \mathcal{H}e_i \wedge \tau(\mathcal{H}e_j),
\]

where \( \mathcal{H}e_i \) is the harmonic representative of \( e_i \) with respect to \( \gamma \) and \( \tau \) denotes the Hodge star-operator with respect to \( \gamma \). Notice that the \( L^2 \)-metric on \( H^2(X, \mathbf{R}) \) induced from \( \gamma \) depends only on the polarization such that \( \gamma \) depends only on the cohomology class \( [\gamma] \in H^2(X, \mathbf{R}) \).

**Definition 1.3.** The BCOV invariant of \( X \) is the real number defined by

\[
\tau_{BCOV}(X) := \text{Vol}(X, \gamma)^{-3 + \frac{\chi(X)}{2}} \text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])^{-1} T_{BCOV}(X, \gamma) \times \exp \left[ -\frac{1}{12} \int_X \log \left( \sqrt{-1} \frac{\eta \wedge \bar{\eta}}{\tau^3/3!} \frac{\text{Vol}(X, \gamma)}{||\eta||_{L^2}^3} \right) c_3(X, \gamma) \right],
\]

where \( \chi(X) = \int_X c_3(X) \) is the topological Euler number of \( X \).

**Remark 1.4.** The Kähler form \( \gamma \) on \( X \) is Ricci-flat if and only if

\[
\sqrt{-1} \frac{\eta \wedge \bar{\eta}}{\gamma^3/3!} = \frac{||\eta||_{L^2}^3}{\text{Vol}(X, \gamma)}.
\]

When \( \gamma \) is Ricci-flat, we get a simpler expression

\[
\tau_{BCOV}(X) = \text{Vol}(X, \gamma)^{-3 + \frac{\chi(X)}{2}} \text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma])^{-1} T_{BCOV}(X, \gamma).
\]

We also remark that \( \text{Vol}(X, \gamma) \) and \( \text{Vol}_{L^2}(H^2(X, \mathbf{Z}), [\gamma]) \) are constant under polarized deformations of Calabi-Yau threefolds \([12\, \text{Lemma 4.12}].\) Hence, for each moduli space of polarized Ricci-flat Calabi-Yau threefolds \( \mathcal{M} \) (cf. \([13, 32, 36]\)), there is a constant \( C_M > 0 \) depending only on the polarization such that

\[
\tau_{BCOV}(X) = C_M T_{BCOV}(X, \gamma)
\]

for all \( (X, \gamma, c_1(L)) \in \mathcal{M} \). Here \( L \) is an ample line bundle on \( X \) giving the polarization and \( \gamma \) is a Ricci-flat Kähler form on \( X \) with \( [\gamma] = c_1(L) \). In this sense, the BCOV invariant can be identified with the BCOV torsion for polarized Ricci-flat Calabi-Yau threefolds.

**Theorem 1.5.** For a Calabi-Yau threefold \( X \), \( \tau_{BCOV}(X) \) is independent of the choice of a Kähler form on \( X \). Namely, \( \tau_{BCOV}(X) \) is an invariant of \( X \).

**Proof.** See \([12\, \text{Th. 4.16}]\). \( \square \)

After Bershadsky-Cecotti-Ooguri-Vafa and Theorem \([13]\), \( \tau_{BCOV} \) is regarded as a function on the moduli space of Calabi-Yau threefolds.
1.3. Singularity of BCOV invariants. In this subsection, we recall some results in \cite{12, 45} about the boundary behavior of BCOV invariants, which are applications of the Bismut-Lebeau embedding theorem for Quillen metrics \cite{7}.

Let \( \pi: \mathcal{X} \to B \) be a surjective morphism from an irreducible projective fourfold \( \mathcal{X} \) to a compact Riemann surface \( B \). Assume that there exists a finite subset \( \Delta_\pi \subset B \) such that \( \pi|_{\Delta_\pi}: \mathcal{X}|_{\Delta_\pi} \to B \setminus \Delta_\pi \) is a smooth morphism and such that \( X_t := \pi^{-1}(t) \) is a Calabi-Yau threefold for all \( t \in B \setminus \Delta_\pi \).

**Theorem 1.6.** For every \( 0 \in \Delta_\pi \), there exists \( \alpha = \alpha_0 \in \mathbb{Q} \) such that

\[
\log \tau_{\text{BCOV}}(X_t) = \alpha \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0),
\]

where \( t \) is a local parameter of \( B \) centered at \( 0 \in \Delta_\pi \).

**Proof.** See \cite{45} Ths. 0.1 and 0.2. \( \square \)

Next we recall the *locality* of the logarithmic singularity of BCOV invariants. Let \( \mathcal{X} \) and \( \mathcal{X}' \) be normal irreducible projective fourfolds. Let \( B = \mathcal{X} \) and \( B' = \mathcal{X}' \) be compact Riemann surfaces. Let \( \pi: \mathcal{X} \to B \) and \( \pi': \mathcal{X}' \to B' \) be surjective holomorphic maps. Let \( \Sigma_{\pi}|_{\mathcal{X} \setminus \text{Sing} \mathcal{X}} \) be the closure of the critical locus of \( \pi|_{\mathcal{X} \setminus \text{Sing} \mathcal{X}} \) and \( \Sigma_{\pi'}|_{\mathcal{X}' \setminus \text{Sing} \mathcal{X}'} \) be the closure of the critical locus of \( \pi'|_{\mathcal{X}' \setminus \text{Sing} \mathcal{X}'} \).

Define the critical loci of \( \pi \) and \( \pi' \) as

\[
\Sigma_\pi := \text{Sing} \mathcal{X} \cup \overline{\Sigma_{\pi}|_{\mathcal{X} \setminus \text{Sing} \mathcal{X}}} \quad \Sigma_{\pi'} := \text{Sing} \mathcal{X}' \cup \overline{\Sigma_{\pi'}|_{\mathcal{X}' \setminus \text{Sing} \mathcal{X}'}}
\]

and the discriminant loci of \( \pi \) and \( \pi' \) as \( \Delta_\pi := \pi(\Sigma_\pi) \) and \( \Delta_{\pi'} := \pi'(\Sigma_{\pi'}) \), respectively. Let \( 0 \in \Delta_\pi \) and \( 0' \in \Delta_{\pi'} \). Let \( V \) (resp. \( V' \)) be a neighborhood of 0 (resp. 0') in \( B \) (resp. \( B' \)) such that \( \mathcal{X} \cong \Delta \) and \( \mathcal{X} \cap \Delta_\pi = \{0\} \) (resp. \( \mathcal{X}' \cong \Delta \) and \( \mathcal{X}' \cap \Delta_{\pi'} = \{0\} \)). We assume the following:

(A1) \( \Delta_\pi \neq B \), \( \Delta_{\pi'} \neq B' \), \( \dim \Sigma_\pi \leq 2 \), \( \dim \Sigma_{\pi'} \leq 2 \), and \( X_0 \) and \( X'_0 \) are irreducible.

(A2) \( X_t \) and \( X'_t \) are Calabi-Yau threefolds for all \( t \in B \setminus \Delta_\pi \) and \( t' \in B' \setminus \Delta_{\pi'} \).

(A3) \( \pi^{-1}(V) \setminus \Sigma_\pi \) carries a nowhere vanishing canonical form \( \Xi \). Similarly, \( (\pi')^{-1}(V') \setminus \Sigma_{\pi'} \) carries a nowhere vanishing canonical form \( \Xi' \).

(A4) The function germ of \( \pi \) near \( \Sigma_\pi \cap \pi^{-1}(V) \) and the function germ of \( \pi' \) near \( \Sigma_{\pi'} \cap (\pi')^{-1}(V') \) are isomorphic. Namely, there exist a neighborhood \( O \) of \( \Sigma_\pi \cap \pi^{-1}(V) \), a neighborhood \( O' \) of \( \Sigma_{\pi'} \cap (\pi')^{-1}(V') \), and an isomorphism \( \varphi: O \to O' \) such that \( \pi|O = \pi' \circ \varphi|O' \).

For \( b \in B \) and \( b' \in B' \), we set \( X_b := \pi^{-1}(b) \) and \( X'_b := (\pi')^{-1}(b') \). For \( b \in B \setminus \Delta_\pi \) and \( b' \in B' \setminus \Delta_{\pi'} \), the BCOV invariants \( \tau_{\text{BCOV}}(X_b) \) and \( \tau_{\text{BCOV}}(X'_b) \) are well defined. Let \( 0 \in \Delta_\pi \) and \( 0' \in \Delta_{\pi'} \). A local parameter of \( B \) (resp. \( B' \)) centered at 0 (resp. \( 0' \)) is denoted by \( t \). Hence \( t \) is a generator of the maximal ideal of \( \mathcal{O}_{B,0} \) and that of \( \mathcal{O}_{B',0} \).

**Theorem 1.7.** If the assumptions (A1)–(A4) are satisfied, then \( \log \tau_{\text{BCOV}}(X_t) \) and \( \log \tau_{\text{BCOV}}(X'_t) \) have the same logarithmic singularities at \( t = 0 \):

\[
\lim_{t \to 0} \frac{\log \tau_{\text{BCOV}}(X_t)}{\log |t|} = \lim_{t \to 0} \frac{\log \tau_{\text{BCOV}}(X'_t)}{\log |t|}.
\]

In particular,

\[
\log \tau_{\text{BCOV}}(X_t) - \log \tau_{\text{BCOV}}(X'_t) = O(\log(-\log |t|)) \quad (t \to 0).
\]

**Proof.** See \cite{45} Th. 4.1. \( \square \)
1.4. Algebraic section on the moduli space corresponding to $\tau_{BCOV}$. In this subsection, we prove that $\tau_{BCOV}$ is expressed as the norm of a certain meromorphic section of a holomorphic line bundle on the moduli space of Calabi-Yau threefolds.

Let $\mathfrak{M}$ be a coarse moduli space of polarized Calabi-Yau threefolds [13, 32, 36]:

(i) Every point of $\mathfrak{M}$ corresponds to an isomorphism class of a polarized Calabi-Yau threefold.

(ii) For any polarized family of Calabi-Yau threefolds $(\pi: \mathcal{X} \to B, \mathcal{L})$, where $\mathcal{L}$ is a relatively ample line bundle on $\mathcal{X}$, the classifying map $B \ni b \mapsto [(X_b, L_b)] \in \mathfrak{M}$ is holomorphic. Here $X_b := \pi^{-1}(b)$, $L_b := \mathcal{L}|_{X_b}$ and $[(X_b, L_b)]$ is the isomorphism class of $(X_b, L_b)$.

Since every holomorphic line bundle $L$ on a Calabi-Yau threefold $X$ extends to a holomorphic line bundle on the Kuranishi family of $X$, any $[(X, L)] \in \mathfrak{M}$ has a neighborhood isomorphic to $\text{Def}(X)/\text{Aut}(X, L)$, where $\text{Def}(X)$ is the Kuranishi space of $X$.

Let $\mathcal{M}$ be a component of $\mathfrak{M}$. Since $\text{Def}(X)$ is smooth [32, 35] and since $\text{Aut}(X, L)$ is finite, $\mathcal{M}$ is an orbifold. Set $\mathcal{M}_{\text{reg}} := \mathcal{M} \setminus \text{Sing} \mathcal{M}$. Then $\mathcal{M}_{\text{reg}}$ is a complex manifold. Since $\mathcal{M}$ is quasi-projective by Viehweg [36, Th. 1.13], there exists a projective manifold $\overline{\mathcal{M}}$ containing $\mathcal{M}_{\text{reg}}$ as a dense Zariski open subset such that $\mathcal{D} := \overline{\mathcal{M}} \setminus \mathcal{M}_{\text{reg}}$ is a normal crossing divisor of $\overline{\mathcal{M}}$. For $[(X, L)] \in \mathcal{M}$, we set $h^{1,2} := h^{1,2}(X) = \dim \mathcal{M}$ and $\chi := \chi(X)$, where $\chi(X)$ is the topological Euler number of $X$. Then $h^{1,2}$ and $\chi$ are constant on $\mathcal{M}$.

1.4.1. Hodge bundles and the Kodaira-Spencer maps. For $[(X, L)] \in \mathcal{M}$, let $f: (\mathfrak{X}, X) \to (\text{Def}(X), [X])$ be the Kuranishi family of $X$. Let $\mathcal{H}^3$ be the holomorphic vector bundle on $\mathcal{M}_{\text{reg}}$ such that $\mathcal{H}^3_{[(X, L)]} = H^3(X, \mathbb{C})$ and let $0 = \mathcal{F}^4 \subset \mathcal{F}^3 \subset \mathcal{F}^2 \subset \mathcal{F}^1 \subset \mathcal{F}^0 = \mathcal{H}^3$ be the Hodge filtration. We have

$$\mathcal{H}^3|_{\text{Def}(X)} = R^3f_*\mathcal{O}_{\text{Def}(X)}; \quad \mathcal{F}^p/\mathcal{F}^{p+1}|_{\text{Def}(X)} \cong R^{3-p}f_*\Omega^p_{\mathfrak{X}/\text{Def}(X)},$$

where $\Omega^p_{\mathfrak{X}/\text{Def}(X)} := \Lambda^p\Omega^1_{\mathfrak{X}/\text{Def}(X)}$ and $\Omega^1_{\mathfrak{X}/\text{Def}(X)} := \Omega^1_{\mathfrak{X}/\text{Def}(X)}$. The line bundle $\lambda := \mathcal{F}^3$ on $\mathcal{M}_{\text{reg}}$ is called the Hodge bundle. Then $\lambda|_{\text{Def}(X)} = f_*K_{\mathfrak{X}/\text{Def}(X)}$.

By Kawamata [17, Th. 17], there exists a finite covering $\phi: \mathcal{M} \to \overline{\mathcal{M}}$ with branch locus $\mathcal{D}$ such that the monodromy operator on $\phi^*\mathcal{H}^3$ along $\phi^{-1}(\mathcal{D})$ is unipotent. Under this assumption, by the nilpotent orbit theorem of Schmid [31, Th. 4.12], the vector bundles $\phi^*\mathcal{H}^3$ and $\phi^*\mathcal{F}^p$ extend to holomorphic vector bundles $\widetilde{\mathcal{H}}^3$ and $\widetilde{\mathcal{F}}^p$ on $\overline{\mathcal{M}}$, respectively. We define line bundles $\widetilde{\lambda}$ and $\widetilde{\mu}$ on $\overline{\mathcal{M}}$ by $\widetilde{\lambda} := \widetilde{\mathcal{F}}^3$, $\widetilde{\mu} := \det(\widetilde{\mathcal{F}}^2/\widetilde{\mathcal{F}}^3)$.

Then $\widetilde{\lambda}|_{\phi^{-1}(\mathcal{M}_{\text{reg}})} = \phi^*\lambda$.

Define the holomorphic vector bundle $N$ on $\mathfrak{X}$ by the exact sequence:

$$0 \longrightarrow \Theta_{\mathfrak{X}/\text{Def}(X)} := \ker f_* \longrightarrow \Theta_{\mathfrak{X}} \longrightarrow N \longrightarrow 0.$$

The projection $f_*: N \to f^*\Theta_{\text{Def}(X)}$ induces an isomorphism $\iota: f_*N \cong \Theta_{\text{Def}(X)}$. Let $\delta: f_*N \to R^1f_*\Theta_{\mathfrak{X}/\text{Def}(X)}$ be the connecting homomorphism. The Kodaira-Spencer
map is defined as the composite
\(\rho_{\text{Def}}(X) := \delta \circ \iota^{-1}: \Theta_{\text{Def}(X)} \rightarrow R^1\delta \tau X/\text{Def}(X) = R^1\delta \tau X^{\wedge}/\text{Def}(X) \otimes (\delta \tau X/\text{Def}(X))^\vee \cong (\mathcal{F}^2/\mathcal{F}^3) \otimes (\mathcal{F}^3)^\vee|_{\text{Def}(X)}.\)

Then the Kodaira-Spencer map induces an isomorphism of holomorphic vector bundles on \(\mathcal{M}_{\text{reg}}:\)
\(\rho: \Theta_{\mathcal{M}_{\text{reg}}} \rightarrow (\mathcal{F}^2/\mathcal{F}^3) \otimes (\mathcal{F}^3)^\vee.
\)

The isomorphism \(\rho\) and its lift
\(\phi^* \rho: \Theta_{\mathcal{M}}|_{\phi^{-1}(\mathcal{M}_{\text{reg}})} \rightarrow (\mathcal{F}^2/\mathcal{F}^3) \otimes (\mathcal{F}^3)^\vee|_{\phi^{-1}(\mathcal{M}_{\text{reg}})}\)
are again called the Kodaira-Spencer map. Since
\[\det(\phi^* \rho) \in H^0 \left( \phi^{-1}(\mathcal{M}_{\text{reg}}), \det((\mathcal{F}^2/\mathcal{F}^3) \otimes (\mathcal{F}^3)^\vee) \otimes (\det \Theta_{\mathcal{M}})^\vee \right)\]
has at most algebraic singularity along \(\phi^{-1}(\mathcal{D})\), \(\det(\phi^* \rho)\) is a meromorphic section of the line bundle \(\tilde{\mu} \otimes \tilde{\lambda}^{-h^{1,2}} \otimes K_{\mathcal{M}}\). Hence we have the canonical isomorphism
\[(1.2) \quad K_{\mathcal{M}}^{-1} \otimes \mathcal{O}_{\mathcal{M}}(\text{div} \det(\phi^* \rho)) \cong \tilde{\mu} \otimes \tilde{\lambda}^{-h^{1,2}}.\]

1.4.2. Weil-Petersson metric and its boundary behavior. Since the third cohomology group of Calabi-Yau threefolds consists of primitive cohomology classes, \(\mathcal{F}^p/\mathcal{F}^{p+1}\) is equipped with the \(L^2\)-metric, which is independent of the choice of a Kähler metric on each fiber. In particular, so is the Hodge bundle \(\tilde{\lambda}\). This metric is denoted by \(h_{L^2}\) or \((\cdot, \cdot)_{L^2}\). The \(L^2\)-metric \((\cdot, \cdot)_{L^2}\) on \((\mathcal{F}^p/\mathcal{F}^{p+1})|_{\{(X,L)\}} = H^q(X, \Omega_X^p), p+q = 3,\) is expressed by
\[\langle u, v \rangle_{L^2} := (\sqrt{-1})^{p-q}(-1)^3 \int_X u \wedge \overline{v}.
\]
The Hermitian metric on the line bundle \(\det(\mathcal{F}^p/\mathcal{F}^{p+1})\) induced from \(h_{L^2}\) is denoted by \(h_{L^2}\) or \(\|\cdot\|_{L^2}\). The Weil-Petersson forms on \(\mathcal{M}_{\text{reg}}\) and \(\mathcal{M}\) are defined as
\[\omega_{\text{WP}} = c_1(\lambda, h_{L^2}), \quad \tilde{\omega}_{\text{WP}} = c_1(\tilde{\lambda}, h_{L^2}).\]
Then \(\phi^* \omega_{\text{WP}} = \tilde{\omega}_{\text{WP}}|_{\phi^{-1}(\mathcal{M}_{\text{reg}})}\). Let \(\eta_{X/\text{Def}(X)} \in H^0(\text{Def}(X), \delta \tau X/\text{Def}(X))\) be a nowhere vanishing holomorphic section and define the function \(\|\eta_{X/\text{Def}(X)}\|^2_{L^2} \otimes \text{Def}(X)\) by
\[\|\eta_{X/\text{Def}(X)}\|^2_{L^2}(\{X_l\}) := \|\eta_{X/\text{Def}(X)}\|_{X_l}^2\]
Since \(\omega_{\text{WP}} = c_1(\lambda, h_{L^2})\), we have
\[\omega_{\text{WP}} = -dd^c \log\|\eta_{X/\text{Def}(X)}\|^2_{L^2} = c_1(\delta \tau X/\text{Def}(X), h_{L^2}).\]
By e.g. [31, Th. 2] and [12, Def. 4.3], we get
\[(1.5) \quad \omega_{\text{WP}}(u, v) = \frac{\langle \rho_{\text{Def}(X)}(u) \otimes \eta, \rho_{\text{Def}(X)}(v) \otimes \eta \rangle_{L^2}}{\|\eta\|^2_{L^2}}\]
for all \(u, v \in \Theta_{\text{Def}(X)}|_{\{X\}} = H^1(X, \Theta_X)\), where \(\eta \in H^0(X, K_X) \setminus \{0\}\) and the numerator is the cup-product pairing between \(H^1(X, \Omega_X^2)\) and \(\overline{H^1(X, \Omega_X^2)} = H^2(X, \Omega_X^1)\).

By the expression [L5], \(\omega_{\text{WP}}\) is a real analytic \((1, 1)\)-form on \(\Theta_{\text{Def}(X)}\).

Let
\[\text{Ric}_{\omega_{\text{WP}}} := c_1(\Theta_{\text{Def}(X)}, \omega_{\text{WP}})\]
be the Ricci-form of the Weil-Petersson form on \(\Theta_{\text{Def}(X)}\).
The differential equation satisfied by $\varphi_{\text{d}}$ the closed positive $(1,1)$-normal crossing divisor $\mathcal{M}_{12}$. From this isometry, we have the following equality of $(1,1)$-forms on $\mathcal{M}_{12}$:

$\omega_{\mathcal{M}} = \sqrt{-1} \sum_{i=1}^{k} \frac{dt_i \wedge \overline{dt_i}}{|t_i|^2 (\log |t_i|)^2} + \sqrt{-1} \sum_{j=k+1}^{h^{1,2}} dt_j \wedge \overline{df_j}$

Let $\text{Ric}_{\mathcal{M}}$ be the Ricci form of $\mathcal{M}$, namely the first Chern form of the holomorphic tangent bundle equipped with the Weil-Petersson metric $(\Theta_{\varphi^{-1}(\mathcal{M}_{12})}, \mathcal{M}_{12})$. Then $\text{Ric}_{\mathcal{M}} = \varphi^{*} \text{Ric}_{\mathcal{M}}$. By (1.3), the Kodaira-Spencer map induces an isometry of holomorphic Hermitian vector bundles on $\varphi^{-1}(\mathcal{M}_{12})$:

$\varphi^{*} p: (\mathcal{M}_{12}, \mathcal{M}_{12}) \cong ((\mathcal{F}^{2}/\mathcal{F}^{3}) \otimes (\mathcal{F}^{3})^c, (\cdot, \cdot)_{L^2} \otimes \| \cdot \|_{L^2}^2)$.

From this isometry, we have the following equality of $(1,1)$-forms on $\varphi^{-1}(\mathcal{M}_{12})$:

$\text{Ric}_{\mathcal{M}} + h^{1,2} \mathcal{M}_{12} = c_1((\mathcal{F}^{2}/\mathcal{F}^{3}, (\cdot, \cdot)_{L^2}) = c_1(\mathcal{M}_{12}, \| \cdot \|_{L^2})$.

By Lu [20] Th. 1.1, $\text{Ric}_{\mathcal{M}} + (h^{1,2} + 3) \mathcal{M}_{12}$ is a positive $(1,1)$-form on $\varphi^{-1}(\mathcal{M}_{12})$. By Cattani-Kaplan-Schmid [10] Prop. 5.22, Proof of Cor. 5.23 again and the above expression, the $(1,1)$-form $\text{Ric}_{\mathcal{M}} + (h^{1,2} + 3) \mathcal{M}_{12}$ has Poincaré growth near the normal divisor $\varphi^{-1}(\mathcal{D})$. Hence, on $\mathcal{U}$ as above, we have the following estimate:

$0 \leq \{\text{Ric}_{\mathcal{M}} + (h^{1,2} + 3) \mathcal{M}_{12}\}|_{\mathcal{U}} \leq C \omega_{\mathcal{M}}$.

We identify the $(1,1)$-forms $\mathcal{M}_{12}$, $\text{Ric}_{\mathcal{M}} + (h^{1,2} + 3) \mathcal{M}_{12}$ on $\varphi^{-1}(\mathcal{M}_{12})$ with the closed positive $(1,1)$-currents on $\mathcal{M}$ defined as their trivial extensions from $\varphi^{-1}(\mathcal{M}_{12})$ to $\mathcal{M}$.

1.4.3. The differential equation satisfied by $\log \tau_{\mathcal{M}}$ on $\mathcal{M}$. As usual, we define $d^c = \frac{1}{4\pi i} (\partial - \overline{\partial})$ for complex manifolds. Hence $dd^c = \frac{1}{2\pi \sqrt{-1}} \overline{\partial} \partial$. By the curvature formula for Quillen metrics [10], the complex Hessian of $\tau_{\mathcal{M}}$ can be computed as follows on the Kuranishi space of $X$, where $[\langle X, L \rangle] \in \mathcal{M}$:

**Theorem 1.8.** The following equality of $(1,1)$-forms on $(\text{Def}(X), [X])$ holds:

\[dd^c \log \tau_{\mathcal{M}} = \text{Ric}_{\mathcal{M}} + \left( h^{1,2} + 3 \right) \omega_{\mathcal{M}}. \]

In particular, $\tau_{\mathcal{M}} \in C^\omega(\text{Def}(X))$.

**Proof.** See [12] Th. 4.14. \hfill \Box

This differential equation can be globalized to the compactified moduli space $\mathcal{M}$.

For this, firstly, we prove the following:

**Theorem 1.9.** Let $\varphi^{-1}(\mathcal{D}) = \bigcup_{i \in I} \mathcal{D}_i$ be the irreducible decomposition. Then there exists $\alpha_i \in \mathbb{Q}$ such that the following equality of $(1,1)$-currents on $\mathcal{M}$ holds:

\[12dd^c (\varphi^{*} \log \tau_{\mathcal{M}}) = (36 + 12h^{1,2} + \chi) \omega_{\mathcal{M}} + 12 \text{Ric}_{\mathcal{M}} - \sum_{i \in I} \alpha_i \delta_{\mathcal{D}_i}.\]
Proof. For \( i \in I \), set \( \bar{D}^0_i := \bar{D}_i \setminus \bigcup_{j \neq i} \bar{D}_j \). For every \( p \in \bar{D}^0_i \), there is a neighborhood \( U \) of \( p \) in \( \bar{M} \) and an isomorphism \( U \cong \Delta^{h_{1,2}} \) such that \( U \cap \bar{D}^0_i \cong \{z_1 = 0\} \cap \Delta^{h_{1,2}} \) and \( U \cap \bar{D}^0_j = \emptyset \) for \( j \neq i \). We write \( z' := (z_2, \ldots, z_{k,2}) \). We define \( Z := \bigcup_{i \neq j} \bar{D}_i \cap \bar{D}_j \).

Since \( \omega_{WP} \) and \( \text{Ric} \omega_{WP} + (h_{1,2} + 3) \omega_{WP} \) are closed positive \((1,1)\)-currents on \( \Delta^{h_{1,2}} \), there exist by [42, Lemma 5.9] plurisubharmonic functions \( \tilde{\psi}_1, \tilde{\psi}_2 \) on \( \Delta^{h_{1,2}} \) such that
\[
(1.8) \quad dd^c \tilde{\psi}_1 = \omega_{WP}, \quad dd^c \tilde{\psi}_2 = \text{Ric} \omega_{WP} + (h_{1,2} + 3) \omega_{WP}
\]
as currents on \( \Delta^{h_{1,2}} \). Since \( \omega_{WP} \) and \( \text{Ric} \omega_{WP} + (h_{1,2} + 3) \omega_{WP} \) have Poincaré growth, there exist constants \( C_5, C_4 > 0 \) such that
\[
(1.9) \quad 0 \leq dd^c \tilde{\psi}_1 \leq C_3 \omega_p, \quad 0 \leq dd^c \tilde{\psi}_2 \leq C_4 \omega_p.
\]
Set
\[
Q(z_1, z') := -\log(-\log|z_1|^2) + \|z'\|^2.
\]
Since \( \omega_p = dd^c Q \), we deduce from (1.8), (1.9) that the following functions are plurisubharmonic on \( \Delta^{h_{1,2}} \):
\[
\tilde{\psi}_1, \quad C_3 Q - \tilde{\psi}_1, \quad \tilde{\psi}_2, \quad C_4 Q - \tilde{\psi}_2.
\]
In particular, these functions are bounded from above on a neighborhood of \( 0 \in \Delta^{h_{1,2}} \). Hence there exist constants \( C_5, C_6 > 0 \) such that
\[
C_5 \{ -\log(-\log|z_1|) - 1 \} \leq \tilde{\psi}_i(z_1, z') \leq C_6 \quad (i = 1, 2).
\]

By Theorems [43, 1.6] and (1.10), there exists \( \alpha(z') \in \mathbb{Q} \) such that on \( \Delta^* \times \Delta^{h_{1,2} - 1} \)
\[
- dd^c \left( \phi^* \log \tau_{BCOV} + \frac{\chi}{12} \tilde{\psi}_1 + \tilde{\psi}_2 \right) = 0,
\]
\[
\left| \left( \phi^* \log \tau_{BCOV} + \frac{\chi}{12} \tilde{\psi}_1 + \tilde{\psi}_2 \right) (z_1, z') - \alpha(z') \log|z_1|^2 \right| \leq C(z') \log(-\log|z_1|),
\]
where \( C(z') \) is a (possibly discontinuous, unbounded) positive function on \( \Delta^{h_{1,2} - 1} \). By [12, Lemma 5.9], there exists \( \alpha_i \in \mathbb{Q} \) and a plurisubharmonic function \( h \in C_C(\Delta^{h_{1,2}}) \) such that
\[
(1.11) \quad \phi^* \log \tau_{BCOV}(z_1, z') + \frac{\chi}{12} \tilde{\psi}_1(z_1, z') + \tilde{\psi}_2(z_1, z') = \alpha_i \log|z_1|^2 + h(z_1, z').
\]
By (1.11), we get the following equation of currents on \( U \)
\[
-12 dd^c (\phi^* \log \tau_{BCOV}) = (36 + 12h_{1,2} + \chi) \omega_{WP} + 12 \text{Ric} \omega_{WP} - \alpha_i \delta_{\bar{P}_i}.
\]
Since \( p \) is an arbitrary point of \( \bar{D}^0_i \), we get the equation of currents on \( \hat{M} \setminus Z \):
\[
(1.12) \quad -12 dd^c (\phi^* \log \tau_{BCOV}) = (36 + 12h_{1,2} + \chi) \omega_{WP} + 12 \text{Ric} \omega_{WP} - \sum_{i \in I} \alpha_i \delta_{\bar{P}_i}.
\]

Let \( p \in Z \). Since the right hand side is a linear combination of closed positive \((1,1)\)-currents on \( \hat{M} \), there is a neighborhood \( V \) of \( p \) in \( \hat{M} \) such that the right hand side of (1.12) has a potential function \( R \) on \( V \). Namely \( R \) is a plurisubharmonic function on \( V \) satisfying the following equation of currents on \( U \)
\[
(1.13) \quad dd^c R = (36 + 12h_{1,2} + \chi) \omega_{WP} + 12 \text{Ric} \omega_{WP} - \sum_{i \in I} \alpha_i \delta_{\bar{P}_i}.
\]
By (1.12), (1.13), $12\phi^*\log \tau_{BCOV} + R$ is a pluriharmonic function on $V \setminus Z$. Since $Z$ has codimension $\geq 2$ in $V$, $12\phi^*\log \tau_{BCOV} + R$ extends to a pluriharmonic function on $V$. Thus (1.12) holds on $V$. Since $p \in Z$ is arbitrary, (1.12) holds on $\mathcal{M}$.

By the definition of trivial extension of currents, $\frac{1}{\deg \phi} \chi_{\omega_{\mathcal{M}}} \psi_{\mathcal{M}}$ coincides with the trivial extension of $\omega_{\mathcal{M}}$ from $\mathcal{M}_{\text{reg}}$ to $\mathcal{M}$. This trivial extension is again denoted by $\omega_{\mathcal{M}}$. Similarly, $\frac{1}{\deg \phi} \phi_*=\{\text{Ric }\omega_{\mathcal{M}}+(h^{1,2}+3)\omega_{\mathcal{M}}\}$ coincides with the trivial extension of $\text{Ric }\omega_{\mathcal{M}}+(h^{1,2}+3)\omega_{\mathcal{M}}$ from $\mathcal{M}_{\text{reg}}$ to $\mathcal{M}$. Set $\mathcal{D}_i := \text{Supp }\phi(\mathcal{D}_i)$. Then $\phi_i := \phi|_{\mathcal{D}_i}$ is a surjective map from $\mathcal{D}_i$ to $\mathcal{D}_i$ and we have the equation of currents $\phi_*\delta_{\mathcal{D}_i} = \deg \phi_i \cdot \delta_{\mathcal{D}_i}$. Applying $\phi_*$ to the both sides of (1.7), we get the following:

**Corollary 1.10.** The following equation of currents on $\mathcal{M}$ holds:

$$-12d\Phi \log \tau_{BCOV} = (36+12h^{1,2}+\chi)\omega_{\mathcal{M}}+12\text{Ric }\omega_{\mathcal{M}}-\sum_{i \in I} \frac{\deg \phi_i}{\deg \phi} \delta_{\mathcal{D}_i}.$$

**1.4.4. The section on $\mathcal{M}$ corresponding to $\tau_{BCOV}$.** Let $\mathcal{M}$ be a complex manifold. For a unitary character $\chi \in \text{Hom}(\pi_1(M), U(1))$ of the fundamental group of $M$, the corresponding holomorphic line bundle on $M$ is denoted by $[\chi]$. Since $\chi$ is unitary, $[\chi]$ is equipped with the canonical Hermitian metric induced from the standard norm $| \cdot |$ on $C$. This Hermitian metric on $[\chi]$ is denoted by $| \cdot |$.

**Lemma 1.11.** Let $D$ be a divisor on $M$. Let $F$ be a positive function on $M \setminus D$ such that $\log F \in L^1_{\text{loc}}(M)$ satisfies the following equation of currents on $M$:

$$d\Phi \log F = \delta_D.$$  

Then there exists a unitary character $\chi \in \text{Hom}(\pi_1(M), U(1))$ and a meromorphic section $s$ of $[\chi]$ such that

$$F = |s|^2, \quad \text{Supp }\text{div}(s) \subset D.$$  

**Proof.** Set $\omega := \partial \log F$. Then $\omega$ is a logarithmic $1$-form on $M$, whose residue with respect to any irreducible component of $D$ is an integer. Let $\mathcal{U} := \{U_\alpha\}_{\alpha \in A}$ be an open covering of $M$ such that all $U_\alpha$ and $U_\alpha \cap U_\beta$ are contractible. Fix a reference point $p_\alpha \in U_\alpha \setminus D$ and define

$$f_\alpha(z) := \exp \left( \int_{p_\alpha}^{z} \omega \right).$$

Since the residues of $\omega$ are integral, $f_\alpha$ is a well-defined meromorphic function on $U_\alpha$ such that $d \log f_\alpha = \omega$ on $U_\alpha \setminus D$. Since $d \log f_\alpha = \partial \log F$ on $U_\alpha \setminus D$ and since $F$ is real-valued, we get $d \log |f_\alpha|^2 = d \log F$ on $U_\alpha \setminus D$. By multiplying an appropriate non-zero constant to $f_\alpha$, we may assume that $F = |f_\alpha|^2$ on $U_\alpha$.

Since $d \log f_\alpha = d \log f_\beta$ on $(U_\alpha \cap U_\beta) \setminus D$, there exists $c_{\alpha\beta} \in C$ such that $\log(f_\alpha/f_\beta) = c_{\alpha\beta}$ on $(U_\alpha \cap U_\beta) \setminus D$. Set $\epsilon_{\alpha\beta} := \exp(c_{\alpha\beta}) \in C^\times$. Since $|f_\alpha| = |f_\beta| = F^{1/2}$ on $U_\alpha \cap U_\beta$, we get $|\epsilon_{\alpha\beta}| = 1$ and $\{\epsilon_{\alpha\beta}\} \in H^1(M, U(1))$. We define $\mathcal{L}$ as the holomorphic line bundle on $M$ associated to the cocycle $\{\epsilon_{\alpha\beta}\} \in H^1(M, U(1))$. Namely, there is a nowhere vanishing canonical holomorphic section $s_\alpha \in H^0(U_\alpha, \mathcal{L})$ such that $s_\alpha = c_{\alpha\beta}^{-1} \cdot s_\beta$ on $U_\alpha \cap U_\beta$. There is a character $\chi \in \text{Hom}(\pi_1(M), U(1))$ such that $\mathcal{L} \cong [\chi]$. Since $\chi$ is unitary, $\mathcal{L}$ is equipped with the canonical Hermitian metric $| \cdot |$ such that $|s_\alpha| = 1$ for all $\alpha$. Since $f_\alpha = \epsilon_{\alpha\beta} f_\beta$ on $U_\alpha \cap U_\beta$, we get a
meromorphic section \( s \) of \( \mathcal{L} \) with \( \text{div}(s) = D \) by setting \( s|_{U_\alpha} = s_\alpha f_\alpha \). On every \( U_\alpha \), we have \( |s|^2 = |f_\alpha|^2 = F \). This completes the proof. \( \square \)

In what follows, for \( a, b, c \in \mathbb{Z} \) and a unitary character \( \chi \in \text{Hom}(\pi_1(\widetilde{M}), U(1)) \), the line bundle \( \lambda^a \otimes \overline{\mu}^b \otimes [\chi]^c \) is equipped with the Hermitian metric induced from the \( L^2 \)-metrics on \( \lambda \) and \( \overline{\mu} \) and the metric \( |\cdot| \) on \([\chi]\). This Hermitian metric is denoted by \( \| \cdot \| \).

**Theorem 1.12.** There exist \( \ell \in \mathbb{Z}_{>0} \), a unitary character \( \chi \in \text{Hom}(\pi_1(\widetilde{M}), U(1)) \) and a meromorphic section \( \sigma \) of the line bundle \( \widetilde{\lambda}^{(36+\chi)\ell} \otimes \overline{\mu}^{-12\ell} \otimes [\chi] \) such that the following equality of functions on \( \phi^{-1}(\mathcal{M}_{\text{reg}}) \) holds:

\[
\phi^* \tau^{12\ell}_{\text{BCOV}} = \| \sigma \|^2.
\]

**Proof.** Set \( \xi := \widetilde{\lambda}^{(36+\chi)} \otimes \overline{\mu}^{-12} \). Let \( \Psi \) be a non-zero meromorphic section of \( \xi \) and let \( E := \text{div}(\Psi) \). By Theorem [1.9] and the Poincaré-Lelong formula, \( \phi^* \log \tau_{\text{BCOV}} \) and \( \| \Psi \|^2 \) satisfy the following equations of currents on \( \widetilde{M} \):

\[
-12dd^c(\phi^* \tau_{\text{BCOV}}) = (36 + 12h^{1,2} + \chi) \overline{\omega}_{\text{WP}} + 12 \text{Ric}_{\text{WP}} - \sum_{i \in I} \alpha_i \delta_{D_i},
\]

\[
-dd^c \log \| \Psi \|^2 = (36 + 12h^{1,2} + \chi) \overline{\omega}_{\text{WP}} + 12 \text{Ric}_{\text{WP}} - \delta_E.
\]

Set \( \Delta := \sum_{i \in I} \alpha_i D_i - E \). On \( \widetilde{M} \), we get

\[
-dd^c \log \left[ \tau^{12}_{\text{BCOV}}/\| \Psi \|^2 \right] = -\delta_\Delta.
\]

Let \( \ell \in \mathbb{Z}_{>0} \) be an integer such that \( \ell \Delta \) is an integral divisor. By Lemma [1.11] applied to \( F := (\tau^{12}_{\text{BCOV}}/\| \Psi \|^2)^\ell \), there is a character \( \chi \in \text{Hom}(\pi_1(\widetilde{M}), U(1)) \) and a meromorphic section \( s \) of \([\chi]\) with \( \text{div}(s) = \ell \Delta \) such that

\[
|s|^2 = F = (\tau^{12}_{\text{BCOV}}/\| \Psi \|^2)^\ell.
\]

Hence \( \sigma := \Psi^\ell \otimes s \) is a meromorphic section of \( \xi^\ell \otimes [\chi] \) with divisor

\[
\text{div}(\sigma) = \text{div}(\Psi^\ell \otimes s) = \ell E + \ell \Delta = \ell \sum_{i \in I} \alpha_i D_i,
\]

such that \( \tau^{12\ell}_{\text{BCOV}} = \| \Psi^\ell \otimes s \|^2 = \| \sigma \|^2 \). This completes the proof. \( \square \)

Since \( \mathcal{O}_{\widetilde{M}}(\text{div}(\det \phi^* \rho)))|_{\phi^{-1}(\mathcal{M}_{\text{reg}})} \cong \mathcal{O}_{\widetilde{M}} \) and since

\[
\lambda^{(36+\chi)\ell} \otimes \overline{\mu}^{-12\ell} \otimes \mathcal{O}_{\widetilde{M}}(\chi) \cong \lambda^{(36-12h^{1,2}+\chi)\ell} \otimes K^{12\ell}_{\widetilde{M}} \otimes \mathcal{O}_{\widetilde{M}}(-12\text{div}(\det \phi^* \rho)) \otimes [\chi]
\]

by [1.2], \( \sigma|_{\phi^{-1}(\mathcal{M}_{\text{reg}})} \) is a nowhere vanishing holomorphic section of the line bundle

\[
\lambda^{(36-12h^{1,2}+\chi)\ell} \otimes K^{12\ell}_{\widetilde{M}} \otimes [\chi] = \phi^* (\lambda^{(36-12h^{1,2}+\chi)\ell} \otimes K^{12\ell}_{\widetilde{M}} \otimes [\chi])
\]
on \( \phi^{-1}(\mathcal{M}_{\text{reg}}) \). A corresponding statement for \( A \)-model can be found in [3 Eq. (14)].
2. Holomorphic torsion invariant for 2-elementary K3 surfaces

2.1. Lattices, domains of type IV and orthogonal modular varieties. A free \(\mathbb{Z}\)-module of finite rank equipped with a non-degenerate integral symmetric bilinear form is called a lattice. For a lattice \(L\), its rank is denoted by \(r(L)\) and its automorphism group is denoted by \(O(L)\). The set of roots of \(L\) is defined as 
\[ \Delta_L := \{ d \in L; \langle d, d \rangle = -2 \}. \]
For a non-zero integer \(k \in \mathbb{Z}\), \(L(k)\) denotes the \(\mathbb{Z}\)-module \(L\) equipped with the rescaled bilinear form \(k\langle \cdot, \cdot \rangle\). The dual lattice of \(L\) is defined as \(L^\vee := \text{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \subset L \otimes \mathbb{Q}\). The finite Abelian group \(A_L := L^\vee / L\) is called the discriminant group of \(L\), which is equipped with the discriminant form, i.e., the \(\mathbb{Q}/2\mathbb{Z}\)-valued quadratic form \(q_\ell := \langle \ell, \ell \rangle \mod 2\mathbb{Z}\) for \(\ell = x + L \in A_L\).

If \(A_L \cong (\mathbb{Z}/2\mathbb{Z})^l\) for some \(l \in \mathbb{Z}_{\geq 0}\), then \(L\) is said to be 2-elementary. For an even 2-elementary lattice \(L\), there exists a unique vector \(1_L\), called the characteristic vector, such that \(\langle \varpi, \varpi \rangle_L = (1_L, \varpi)_L \mod \mathbb{Z}\) for all \(\varpi \in A_L\).

Let \(L\) be an even 2-elementary lattice. We define \(l(L) := \text{rank}_{\mathbb{Z}} A_L\). The parity \(\delta(L)\) of \(q_\ell\) is defined as follows: \(\delta(L) = 0\) (resp. \(\delta(L) = 1\)) if \(q(A_L) \subset \mathbb{Z}\) (resp. \(q(A_L) \not\subset \mathbb{Z}\)). If \(L\) is indefinite, the isometry class of \(L\) is determined by the triplet \((\text{sign}(L), l(L), \delta(L))\) (cf. \([28]\)). We define \(U := (\mathbb{Z}^2, (01^T, 10^T))\). For complex simple Lie algebras \(A_\ell, B_\ell, C_\ell, E_\ell\), their root lattices are denoted by \(\Lambda_\ell, \mathcal{D}_\ell, \mathfrak{E}_\ell\) and are assumed to be negative-definite. The K3-lattice is defined as the even unimodular lattice
\[ L_{K3} := U \oplus U \oplus U \oplus \mathbb{E}_8 \oplus \mathbb{E}_8. \]
For a lattice \(\Lambda\) of \(\text{sign}(\Lambda) = (2, r(\Lambda) - 2)\), we define
\[ \Omega_\Lambda := \{ [\eta] \in \mathbb{P}(\Lambda \otimes \mathbb{C}); \langle \eta, \eta \rangle_\Lambda = 0, \langle \eta, \bar{\eta} \rangle_\Lambda > 0 \}. \]
Then \(\Omega_\Lambda\) consists of two connected components \(\Omega_\Lambda = \Omega^+_\Lambda \sqcup \Omega^-_\Lambda\), each of which is isomorphic to a bounded symmetric domain of type IV of dimension \(r(\Lambda) - 2\). The projective \(O(\Lambda)\)-action on \(\Omega_\Lambda\) is proper and discontinuous. The quotient
\[ M_\Lambda := \Omega_\Lambda / O(\Lambda) = \Omega^+_\Lambda / O^+(\Lambda) \]
is an analytic space of dimension \(r(\Lambda) - 2\), where \(O^+(\Lambda)\) is the subgroup of index 2 of \(O(\Lambda)\) preserving \(\Omega^+_\Lambda\). The discriminant locus of \(M_\Lambda\) is the reduced \(O(\Lambda)\)-invariant divisor of \(\Omega_\Lambda\) defined as
\[ D_\Lambda := \sum_{d \in D_\Lambda / \pm 1} H_d, \quad H_d := \{ [\eta] \in \Omega_\Lambda; \langle d, \eta \rangle = 0 \}. \]

We define
\[ \Omega^0_\Lambda := \Omega_\Lambda \setminus D_\Lambda, \quad M^0_\Lambda := \Omega^0_\Lambda / O(\Lambda). \]
By Baily-Borel, \(M_\Lambda\) and \(M^0_\Lambda\) are irreducible, normal quasi-projective varieties. Let \(M^*_\Lambda\) be the Baily-Borel compactification of \(M_\Lambda\) and define the boundary locus by
\[ B_\Lambda := M^*_\Lambda \setminus M_\Lambda. \]
Then \(\dim B_\Lambda = 1\) if \(r(\Lambda) \geq 4\) and \(\dim B_\Lambda = 0\) if \(r(\Lambda) = 3\). When \(M\) is a primitive 2-elementary Lorentzian sublattice of \(\mathbb{L}_{K3}\) with \(r(M) \geq 18\), then \(B_M\) is irreducible by [11] Prop. 11.7.

For simplicity, assume the splitting of lattices
\[ \Lambda = \mathbb{U}(N) \oplus L, \]
where \(N \in \mathbb{Z}_{>0}\) and \(L\) is an even Lorentzian lattice. Let
\[ C_L := \{ x \in L \otimes \mathbb{R}; \langle x, x \rangle_L > 0 \} \]
be the positive cone of $L$, which consists of connected components $C_L = C_L^+ \sqcup C_L^-$. We identify the tube domain $L \otimes \mathbb{R} + \sqrt{-1}C_L$ with $\Omega_\Lambda$ via the map

$$L \otimes \mathbb{R} + \sqrt{-1}C_L \ni z \rightarrow \left[ \left( -\frac{\langle z, z \rangle_L}{2}, \frac{1}{N} \right), z \right] \in \Omega_\Lambda.$$  

The Kähler form of the Bergman metric on $L \otimes \mathbb{R} + \sqrt{-1}C_L$ is the positive $(1, 1)$-form defined as

$$\omega_\Lambda(z) := -dd^c \log \langle \Im z, \Im z \rangle_L.$$  

Via the identification (2.1), we regard $\omega_\Lambda$ as the $(1, 1)$-form on $\Omega_\Lambda$.

2.2. 2-elementary $K3$ surfaces and their holomorphic torsion invariants.

2.2.1. 2-elementary $K3$ surfaces and their moduli space. Let $S$ be a $K3$ surface. Then $H^2(S, \mathbb{Z})$ endowed with the cup-product pairing is isometric to the $K3$-lattice. Namely, there exists an isometry of lattices $\alpha : H^2(S, \mathbb{Z}) \cong L_{K3}$.

Let $\theta : S \rightarrow S$ be a holomorphic involution. The pair $(S, \theta)$ is called a 2-elementary $K3$ surface if $\theta$ is anti-symplectic in the following sense:

$$\theta^*|_{H^0(S, K_S)} = -1.$$  

The type of an anti-symplectic involution $\theta$ is defined as the isometry class of the invariant sublattice

$$H^2(S, \mathbb{Z})^+, \quad H^2(S, \mathbb{Z})^\pm := \{ v \in H^2(S, \mathbb{Z}); \theta^*v = \pm v \}.$$  

By Nikulin [28], $\alpha(H^2(S, \mathbb{Z})^+) \subset L_{K3}$ is a primitive 2-elementary Lorentzian sublattice. Since the embedding of a primitive 2-elementary Lorentzian lattice into $L_{K3}$ is unique up to an action of $O(L_{K3})$ by [28], the type of an anti-symplectic holomorphic involution is independent of the choice of an isometry $\alpha : H^2(S, \mathbb{Z}) \cong L_{K3}$. By Nikulin [28, 29], the topological type of an anti-symplectic holomorphic involution on a $K3$ surface is determined by its type in the sense that if $(S, \theta)$ and $(S', \theta')$ are two 2-elementary $K3$ surfaces of the same type, then $(S', \theta')$ is deformation equivalent to $(S, \theta)$. By [28, 29], there exists 75 distinct types of 2-elementary $K3$ surfaces.

As a consequence of the global Torelli theorem for $K3$ surfaces, the moduli space of 2-elementary $K3$ surfaces of type $M$ was constructed as follows for any primitive 2-elementary Lorentzian sublattice $M \subset L_{K3}$.

Let $(S, \theta)$ be a 2-elementary $K3$ surface of type $M$ and let $\alpha : H^2(S, \mathbb{Z}) \cong L_{K3}$ be an isometry with $\alpha(H^2(S, \mathbb{Z})^-) = M^-$. Then the period of $(S, \theta)$ is defined by

$$\pi_M(S, \theta) := \left[ \alpha(H^0(S, K_S)) \right] \in \mathcal{M}_{M^+} = \Omega_{M^+}/O(M^+) .$$  

In fact, any point of the discriminant locus $\overline{\mathcal{D}}_{M^+}$ is never realized as the period of a 2-elementary $K3$ surface of type $M$. By [30, Th. 1.8], $\mathcal{M}_{M^+}$ is a coarse moduli space of 2-elementary $K3$ surfaces of type $M$ via the period map.
2.2.2. The fixed point set of a 2-elementary $K3$ surface. Let $(S, \theta)$ be a 2-elementary $K3$ surface of type $M$. Set

$$S^\theta := \{ x \in S; \theta(x) = x \}.$$ 

The topology of $S^\theta$ was determined by Nikulin \cite{29} as follows:

**Proposition 2.1.**  
(1) If $M \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, then $S^\theta = \emptyset$ and the quotient $S/\theta$ is an Enriques surface.

(2) If $M \cong \mathbb{U} \oplus \mathbb{E}_8(2)$, then $S^\theta = C^{(1)}_1 \sqcup C^{(1)}_2$, where $C^{(1)}_1$ and $C^{(1)}_2$ are elliptic curves.

(3) If $M \not\cong \mathbb{U}(2) \oplus \mathbb{E}_8(2), \mathbb{U} \oplus \mathbb{E}_8(2)$, then

$$S^\theta = C^{g(M)} \sqcup E_1 \sqcup \ldots \sqcup E_{k(M)},$$

where $C^{(g)}$ is a compact Riemann surface of genus $g$ and $E_i$ is a smooth rational curve with

$$g(M) := 11 - \frac{r(M) + l(M)}{2}, \quad k(M) = \frac{r(M) - l(M)}{2}.$$

When $M \cong \mathbb{U}(2) \oplus \mathbb{E}_8(2)$, $M$ and $M^\perp$ are called exceptional in this paper. Notice that, when $M \cong \mathbb{U} \oplus \mathbb{E}_8(2)$, the total genus of $S^\theta$ is still given by $g(M) = 2$, so that the first equality of (2.3) remains valid.

**Warning:** In \cite{43}, the lattices $\mathbb{U} \oplus \mathbb{E}_8(2)$ and $\mathbb{U} \oplus \mathbb{U} \oplus \mathbb{E}_8(2)$ are also called exceptional. In the present paper, these lattices are not exceptional.

Let $\mathcal{G}_g$ be the Siegel upper half-space of degree $g$ and let

$$A_g := \mathcal{G}_g / \text{Sp}_g(\mathbb{Z})$$

be the Siegel modular variety of degree $g$, which is the coarse moduli space of principally polarized Abelian varieties of dimension $g$. Let $\omega_{A_g}$ be the Kähler form on $A_g$ in the sense of orbifolds induced from the $\text{Sp}_g(\mathbb{Z})$-invariant Kähler form on $\mathcal{G}_g$.

$$\omega_{\mathcal{G}_g} := -dd^c \log \det \Im \tau$$

After (2.2), we define the Torelli map $\mathcal{T}_M^0 : \mathcal{M}_{M^\perp}^0 \to A_{g(M)}$ by

$$\mathcal{T}_M^0 (\pi_M(S, \theta)) := [\Omega(S^\theta)],$$

where $\Omega(S^\theta) \in \mathcal{G}_g$ is the period of $S^\theta$ with respect to some symplectic basis of $H_1(S^\theta, \mathbb{Z})$ and $[\Omega(S^\theta)] \in A_{g(M)}$ is the corresponding point of the Siegel modular variety. Let $\Pi_{M^\perp} : \Omega_{M^\perp} \to \mathcal{M}_{M^\perp}$ be the projection. We define the holomorphic map $J_M^0 : \Omega_{M^\perp}^0 \to A_{g(M)}$, again called the Torelli map, by

$$J_M^0 := \mathcal{T}_M \circ \Pi_{M^\perp}.$$

By Borel’s extension theorem, $J_M^0$ extends to a holomorphic map from $\Omega_{M^\perp}^0 \cup D_{M^\perp}^0$ to $A_{g(M)}$, the Satake compactification of $A_g$, where $D_{M^\perp}^0$ is the dense Zariski open subset of $D_{M^\perp}$ defined as

$$D_{M^\perp}^0 := \bigcup_{d \in \Delta_{M^\perp}} H_d^0, \quad H_d^0 := H_d \setminus \bigcup_{d \in \Delta_{M^\perp} \setminus \{ \pm d \}} H_d.$$

This extension of $J_M^0$ is denoted by $J_M^*$. Then the semi-positive $(1, 1)$-form $(J_M^*)^* \omega_{A_{g(M)}}$ on $\Omega_{M^\perp}^0$ extends trivially to a closed positive $(1, 1)$-current on $\Omega_{M^\perp}$. The trivial extension of $(J_M^*)^* \omega_{A_{g(M)}}$ from $\Omega_{M^\perp}^0$ to $\Omega_{M^\perp}$ is denoted by $J_M^* \omega_{A_{g(M)}}$. 
2.2.3. A holomorphic torsion invariant for 2-elementary $K3$ surfaces.

**Definition 2.2.** Let $\eta$ be a nowhere vanishing holomorphic 2-form on $S$. Let $\gamma$ be an $\theta$-invariant Kähler form on $S$. Define

$$
\tau_M(S, \theta) := \text{Vol}(S, \gamma) \frac{r(M) - 6}{4} \tau_{\mathbb{Z}}(S, \gamma)(\theta) \text{Vol}(S^0, \gamma|_{S^0}) \tau(S^0, \gamma|_{S^0}) \times \exp \left[ \frac{1}{8} \int_{S^0} \log \left( \frac{\eta \wedge \bar{\eta}}{\gamma^2/2!} \cdot \frac{\text{Vol}(S, \gamma)}{\|\eta\|^2_{L^2}} \right) \bigg|_{S^0} c_1(S^0, \gamma|_{S^0}) \right].
$$

Here $\text{Vol}(S^0, \gamma|_{S^0})$ and $\tau(S^0, \gamma|_{S^0})$ are multiplicative with respect to the decomposition into connected components.

By [41], $\tau_M(S, \theta)$ is independent of the choices of $\eta$ and $\gamma$ and is determined by the period of $(S, \theta)$. In particular, $\tau_M(S, \theta)$ is an invariant of $(S, \theta)$. If $\gamma$ is an $\iota$-invariant Ricci-flat Kähler form on $S$, then

$$
\frac{\eta \wedge \bar{\eta}}{\gamma^2/2!} = \frac{\|\eta\|^2_{L^2}}{\text{Vol}(S, \gamma)}
$$

and hence

$$
\tau_M(S, \theta) = \text{Vol}(S, \gamma) \frac{r(M) - 6}{4} \tau_{\mathbb{Z}}(S, \gamma)(\theta) \cdot \text{Vol}(S^0, \gamma|_{S^0}) \tau(S^0, \gamma|_{S^0}).
$$

Define the $O(M^\perp)$-invariant smooth function $\overline{\tau}_M$ on $\Omega^0_{M^\perp}$ by

$$
\overline{\tau}_M := \Pi^*_{M^\perp} \tau_M.
$$

By [41] Eq. (7.1), [41] Th. 5.2], the following equation of $(1, 1)$-currents on $\Omega_{M^\perp}$ holds

$$
dd^c \log \overline{\tau}_M = \frac{r(M) - 6}{4} \omega_{M^\perp} + J^*_{M^\perp} \omega_{\Omega(M)} - \frac{1}{4} \delta_{D_{M^\perp}}.
$$

2.3. Borcherds products for 2-elementary lattices. Following [43], [21], we attach a Borcherds product for even 2-elementary lattice $\Lambda$ of signature $(2, r(\Lambda) - 2)$.

Let $\mathcal{H} \subset \mathbb{C}$ be the complex upper half-plane. Recall that the Dedekind $\eta$-function and the Jacobi theta series are holomorphic functions on $\mathcal{H}$ defined as

$$
\eta(\tau) = e^{2\pi i \tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad \vartheta_{\Lambda^+_1 + \frac{1}{2}}(\tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i (n + \frac{1}{2})^2 \tau} \quad (k = 0, 1).
$$

We define modular forms $\phi_k^{(0)}(\tau), \phi_k^{(1)}(\tau) \in \mathcal{O}(\mathcal{H})$ for $\Gamma_0(4)$ and the series $\{c_k^{(0)}(l)\}_{l \in \mathbb{Z}}$, $\{c_k^{(1)}(l)\}_{l \in \mathbb{Z} + k/4}$ by

$$
\phi_k^{(0)}(\tau) := \eta(\tau)^{-8} \eta(2\tau)^8 \eta(4\tau)^{-8} \vartheta_{\Lambda^+_1}(\tau)^k = \sum_{l \in \mathbb{Z}} c_k^{(0)}(l) q^l,
$$

$$
\phi_k^{(1)}(\tau) := -16\eta(2\tau)^{-10} \eta(4\tau)^8 \vartheta_{\Lambda^+_1 + \frac{1}{2}}(\tau)^k = \sum_{l \in \mathbb{Z} + k/4} 2c_k^{(1)}(l) q^l.
$$

For $i \in \mathbb{Z}/4\mathbb{Z}$, we set

$$
\psi_k^{(i)}(\tau) := \sum_{l \equiv i \mod 4} c_k^{(0)}(l) q^{l/4}.
$$

Let $\Lambda$ be a primitive 2-elementary sublattice of $\mathbb{L}_{K3}$ with $\text{sign}(\Lambda) = (2, r(\Lambda) - 2)$. Let $\{e_i\}_{i \in \mathbb{A}}$ be the standard basis of the group ring $\mathbb{C}[\mathbb{A}]$. Let $\text{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $\text{SL}_2(\mathbb{Z})$, which is generated by $S := (\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \sqrt{7})$.
and \( T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) (cf. [9 Sect. 2]). By [9 Sect. 4], the Weil representation \( \rho_A : \text{Mp}_2(\mathbf{Z}) \to \text{GL}(\mathbf{C}[A]) \) is defined as

\[
\rho_A(T) e_\gamma := e^{\pi \sqrt{-T q_A(\gamma)}} e_\gamma, \quad \rho_A(S) e_\gamma := \left( \frac{\sqrt{-1} - \frac{\lambda + 1}{\lambda + 1}}{|A|^1/2} \right) \sum_{\delta \in A} e^{-2\pi \sqrt{-T h_\delta(\gamma, \delta)}} e_\delta,
\]

where \( b_A : A_A \times A_A \to \mathbf{Q}/\mathbf{Z} \) is the discriminant bilinear form associated to \( q_A \). By [33 Th. 7.7], the \( \mathbf{C}[A] \)-valued holomorphic function on \( \mathfrak{f} \)

\[
F_{\lambda}(\tau) := \phi_{12-\tau(\lambda)}^{(0)}(\tau) e_0 + 2 \frac{\tau(\lambda) - \tau(\lambda)}{2} \sum_{\gamma \in A} \psi_{12-\tau(\lambda)}^{(2\tau(\gamma))}(\tau) e_\gamma + \phi_{12-\tau(\lambda)}^{(1)}(\tau) e_{1_{\lambda}}
\]

is a modular form for \( \text{Mp}_2(\mathbf{Z}) \) of weight \((4 - \tau(\lambda))/2\) of type \( \rho_A \). Namely, for every \((\frac{a}{b}, \sqrt{ct} + d) \in \text{Mp}_2(\mathbf{Z})\), we have the equality of \( \mathbf{C} [A] \)-valued holomorphic functions on \( \mathfrak{f} \)

\[
F_{\lambda}(\tau) = \sqrt{ct + d}^{4 - \tau(\lambda)} \rho_A \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \sqrt{ct + d} F_{\lambda}(\tau).
\]

To get the integrality of the Fourier coefficients of \( F_{\lambda} \), we set

\[
\tilde{F}_{\lambda}(\tau) := \begin{cases} 
F_{\lambda}(\tau) & (\tau(\lambda) \leq 16), \\
2^{\tau(\lambda) - 16} F_{\lambda}(\tau) & (\tau(\lambda) \geq 16).
\end{cases}
\]

By [21 Sect. 7], \( \tilde{F}_{\lambda}(\tau) \) has integral Fourier expansion. By Borcherds [9 Th. 13.3], the Borcherds lift of \( \tilde{F}_{\lambda}(\tau) \) with respect to \( \Lambda \)

\[
\Psi_{\lambda}(\cdot, \tilde{F}_{\lambda})
\]

is an automorphic form on \( \Omega_{\Lambda} \) for \( O^+(\Lambda) \). For the weight and the singularities of \( \Psi_{\lambda}(\cdot, \tilde{F}_{\lambda}) \), see [21 Th. 7.1].

The infinite product expansion of \( \Psi_{\lambda}(\cdot, \tilde{F}_{\lambda}) \) is given as follows. By the classification of primitive 2-elementary sublattices of \( \mathbb{L}_{K3} \), there exist \( N \in \{1, 2\} \) and an even 2-elementary Lorentzian lattice \( L \) such that

\[
\Lambda = U(N) \oplus L.
\]

By [9 Th. 13.3], \( \Psi_{\lambda} \) is expressed as the following infinite product on \( L \otimes \mathbf{R} + \sqrt{-1} \mathfrak{w} \) under the identification (2.1)

\[
\Psi_{\lambda}(z, \tilde{F}_{\lambda}) = e^{2\pi \sqrt{-T(q(z))}} \prod_{\lambda \in L, \lambda \cdot \mathfrak{w} > 0, \lambda^2 \geq -2} \left( 1 - e^{2\pi \sqrt{-T(\lambda, z)}} \right)^{\tau(0)}_{12 - \tau(\lambda)}(\lambda^2/2)
\]

\[
\times \prod_{\lambda \in 2L^\vee, \lambda \cdot \mathfrak{w} > 0, \lambda^2 \geq -2} \left( 1 - e^{2\pi \sqrt{-T(\lambda, z)}} \right)^{\tau(0)}_{12 - \tau(\lambda)}(\lambda^2/2)
\]

\[
\times \prod_{\lambda \in (1_L + L), \lambda \cdot \mathfrak{w} > 0, \lambda^2 \geq -4} \left( 1 - e^{2\pi \sqrt{-T(\lambda, z)}} \right)^{\tau(1)}_{12 - \tau(\lambda)}(\lambda^2/2),
\]

where the cone \( \mathfrak{w} \subset L \otimes \mathbf{R} \) is called a Weyl chamber of \( F_L \) and the vector \( \rho \in L \otimes \mathbf{Q} \) is called the Weyl vector of \( F_L \) (cf. [9 Sect. 10]), and \( F_L \) is the \( \mathbf{C}[\Lambda] \)-valued modular form of type \( \rho_L \) defined by [9 p.512].
Write \( w(\Lambda) \) for the weight of \( \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \). Then the Petersson norm of \( \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \) is the \( C^\infty \) function on \( L \otimes \mathbb{R} + \sqrt{-1} \mathbb{C} \), defined by

\[
\| \Psi_\Lambda(z, \bar{F}_\Lambda) \|^2 := \langle \Im z, \Im z \rangle^{w(\Lambda)} |\Psi_\Lambda(z, \bar{F}_\Lambda)|^2
\]

and we set

\[
\| \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \| = \begin{cases} \| \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \| & (r(\Lambda) \leq 16) \\ \| \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \|^2 - (r(\Lambda) - 16) & (r(\Lambda) \geq 16). \end{cases}
\]

Regarded as a function on \( \Omega_\Lambda \) via \( \text{(2.1)} \), \( \| \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \|^2 \) is an \( O(\Lambda) \)-invariant \( C^\infty \) function on \( \Omega_\Lambda \). Hence \( \| \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \|^2 \) descends to a function on \( \mathcal{M}_\Lambda \) with possible singularities on the polar locus of \( \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \). In what follows, we regard \( \| \Psi_\Lambda(\cdot, \bar{F}_\Lambda) \|^2 \) as a function on \( \mathcal{M}_\Lambda \).

2.4. An explicit formula for \( \tau_M \). We recall the main result of \( [21] \). Let \( a, b \in \{ 0, 1/2 \}^g \). The pair \((a, b)\) is said to be even if \( 4^g \cdot a \cdot b \in 2\mathbb{Z} \). For even \((a, b)\), the corresponding Riemann theta constant \( \theta_{a, b}(\Omega) \) is defined as the theta series

\[
\theta_{a, b}(\Omega) := \sum_{n \in \mathbb{Z}^g} \exp \left( \pi \sqrt{-1} \left( n + a \right) \Omega (n + a) + 2\pi \sqrt{-1} \left( n + a \right) b \right).
\]

For \( \Omega \in \mathfrak{S}_g \), we define \( \chi_g(\Omega) \) and \( \Upsilon_g(\Omega) \) by

\[
\chi_g(\Omega) := \prod_{(a, b) \text{ even}} \theta_{a, b}(\Omega), \quad \Upsilon_g(\Omega) = \chi_g(\Omega)^8 \sum_{(a, b) \text{ even}} \theta_{a, b}(\Omega)^{-8}.
\]

It is classical that \( \chi_g^8 \) and \( \Upsilon_g \) are Siegel modular forms of weight \( 2g + 1 \) and \( 2g + 1 - 4 \), respectively. Hence their Petersson norms

\[
\| \chi_g(\Omega)^8 \|^2 := (\det \Omega)^{2g + 1} \left| \chi_g(\Omega)^8 \right|^2
\]

\[
\| \Upsilon_g(\Omega) \|^2 := (\det \Omega)^{2g + 1 - 4} \left| \Upsilon_g(\Omega)^8 \right|^2
\]

are \( Sp_{2g}(\mathbb{Z}) \)-invariant functions on \( \mathfrak{S}_g \). We regard \( \| \chi_g \|^2, \| \Upsilon_g \|^2 \in C^\infty(A_g) \) in what follows.

**Theorem 2.3.** Let \( M \) be a primitive 2-elementary Lorentzian sublattice of \( L_{K3} \) with \( (r(M), \delta(M)) \neq (2, 0) \). Set \( r := r(M) \) and \( g := g(M) \). Then there exists a constant \( C_M \) depending only on \( M \) such that the following equality holds for every 2-elementary \( K3 \) surface \((S, \theta)\) of type \( M \):

1. If \((r, \delta) \neq (2, 0), (10, 0)\), then

\[
\tau_M(S, \theta)^{-2r + 1} = C_M \left\| \Psi_M(\pi_M(S, \theta), F_M)(\cdot, \bar{F}_M) \right\|^{2g} \cdot \left| \chi_g(\Omega(S^\theta))^8 \right|^2.
\]

2. If \((r, \delta) = (10, 0)\), then

\[
\tau_M(S, \theta)^{-2r - 1} \| \Psi_M(\pi_M(S, \theta), F_M)(\cdot, \bar{F}_M) \|^{2g - 1} \cdot \left| \Upsilon_g(\Omega(S^\theta))^8 \right|^2 = C_M \left\| \Psi_M(\pi_M(S, \theta), F_M)(\cdot, \bar{F}_M) \right\|^{2g - 1} \cdot \left| \chi_g(\Omega(S^\theta))^8 \right|^2.
\]

**Proof.** See \( [21] \) Th. 0.1.}

**Remark 2.4.** If \((r(M), \delta(M)) = (2, 0)\), then \( M^\perp \) is isometric to either \( U \) or \( U(2) \). For one of these two lattices, an explicit formula for \( \tau_M \) is given in \( [21] \) Sect. 9.
3. BCOV invariants for Borcea-Voisin threefolds

In Section 3, $M \subset \mathbb{L}_{K3}$ denotes a primitive 2-elementary Lorentzian sublattice.

**Definition 3.1.** Let $(S, \theta)$ be a 2-elementary $K3$ surface and let $T$ be an elliptic curve. The orbifold

$$X_{(S, \theta, T)} := \frac{S \times T}{\theta \times (-1)}.$$ 

is called a *Borcea-Voisin orbifold* associated with $(S, \theta, T)$. The type of a Borcea-Voisin orbifold $X_{(S, \theta, T)}$ is defined as that of $(S, \theta)$. Hence the type of $X_{(S, \theta, T)}$ is an isometry class of a primitive 2-elementary Lorentzian sublattice of $\mathbb{L}_{K3}$.

Let $T[2] = T(-1)^T$ be the set of points of order 2 of $T$. Then

$$\text{Sing} X_{(S, \theta, T)} = S^\theta \times T[2]$$

is the 4 copies of $S^\theta$.

Let $X_{(S, \theta, T)}$ be a Borcea-Voisin orbifold of type $M$ and let

$$p : \tilde{X}_{(S, \theta, T)} \to X_{(S, \theta, T)}$$

be the blowing-up of $\text{Sing}(X_{(S, \theta, T)}) = S^\theta \times T[2]$. Then $\tilde{X}_{(S, \theta, T)}$ is a Calabi-Yau threefold, called a *Borcea-Voisin threefold* of type $M$, whose mirror symmetry was studied by Borcea [8], Voisin [37] and Gross-Wilson [15]. The main result of this paper is the following:

**Theorem 3.2.** There exists a constant $C_M$ depending only on the lattice $M$ such that for every Borcea-Voisin orbifold $X_{(S, \theta, T)}$ of type $M$,

$$\tau_{\text{BCOV}}(\tilde{X}_{(S, \theta, T)}) = C_M \tau_M (S, \theta)^{-4} \left\| \eta(T) \right\|_{24}^2,$$

where $\left\| \eta(T) \right\|$ is the Petersson norm of the Dedekind $\eta$-function evaluated at the period of $T$.

Comparing Theorem 3.2 and [12, Th. 13.3], [42, Th. 5.7], Theorem 3.2 was already verified when $M^\perp$ is one of the following lattices:

$$\mathbb{I}_{2,m}(2) \quad (m = 0, 4 \leq m \leq 9), \quad U \oplus U(2) \oplus \mathbb{E}_8(2).$$

Here $\mathbb{I}_{2,m}$ is an odd unimodular lattice of signature $(2, m)$.

**Corollary 3.3.** If $(r, \delta) \neq (2, 0)$, i.e., $M \neq U, U(2)$, then there exists a constant $C_M$ depending only on $M$ such that the following equality holds for every Borcea-Voisin threefold $\tilde{X}_{(S, \theta, T)}$ of type $M$:

1. If $(r, \delta) \neq (2, 0), (10, 0)$, then

(3.1)

$$\tau_{\text{BCOV}}(\tilde{X}_{(S, \theta, T)})^{2^{g-1}(2^g+1)} = C_M \left\| \Psi_M (\pi_M (S, \theta), F_{M^\perp}) \right\|_{2^g}^2 \cdot \left\| \chi_g \left( \Omega(S^\theta) \right) \right\|_{2^g}^2 \times \left\| \eta(\Omega(T)) \right\|_{24}^{2^{g}(2^g+1)}.$$
(2) If \((r, \delta) = (10, 0)\), then
\[
\tau_{BCOV} \left( X_{(S, \theta, T)} \right) = C_M \| \Psi_{M^+} (\pi_M (S, \theta), F_{M^+}) \|^{2g(2g+1)-2} \| \gamma_g (\Omega (S^\theta)) \|^{2(2g-1)}
\]
\[
\times \| \eta (\Omega (T))^{24} \|^{2(g^2-1)(2g-1)-1}.
\]

\[\text{(3.2)}\]

**Proof.** The result follows from Theorems 2.3 and 3.2.

3.1. **A variational formula for** \(\tau_{BCOV}\). We keep the notation in Section 2.

The modular curve \(X(1)\) is the quotient of \(\mathfrak{H}\) defined as
\[
X(1) := \mathfrak{H}/\text{SL}_2(\mathbb{Z}).
\]

Let \(\omega_{hyp}\) be the Kähler form of the Poincaré metric on \(\mathfrak{H}\)
\[
\omega_{hyp} = -dd^c \log \Im \tau.
\]

Let \(\varpi : \mathfrak{H} \to X(1) = \mathfrak{H}/\text{SL}_2(\mathbb{Z})\) be the projection. Define \(\tau_{\text{ell}} \in C^\infty (\mathfrak{H})\) by
\[
\bar{\tau}_{\text{ell}} := \varpi^* \tau_{\text{ell}}.
\]

Then \(\bar{\tau}_{\text{ell}}\) is \(\text{SL}_2(\mathbb{Z})\)-invariant and satisfies the following equation of \((1, 1)\)-forms
\[
- dd^c \log \bar{\tau}_{\text{ell}} = -\omega_{hyp}.
\]

By definition, Borcea-Voisin threefolds of type \(M\) are parametrized by the product \(\mathcal{M}_M \times X(1)\). Hence \(\tau_{BCOV} \in C^\infty (\mathcal{M}_M \times X(1))\).

**Theorem 3.4.** Regard \(\tau_{BCOV}\) as an \(O(M^+) \times \text{SL}_2(\mathbb{Z})\)-invariant \(C^\infty\) function on \(\Omega^0 \times \mathfrak{H}\). Then the following equations of \((1, 1)\)-forms on \(\Omega^0 \times \mathfrak{H}\) hold:
\[
- dd^c \log \tau_{BCOV} = (r(M) - 6)pr_1^* \omega_{M^+} + 4pr_1^* J_M \omega_{A(M)} + 12pr_2^* \omega_{hyp}.
\]

In particular, \(\log \frac{\tau_{BCOV}}{(\tau_{BCOV})^4(\| \eta^{24} \|^2)}\) is a pluriharmonic function on \(\mathcal{M}_M \times X(1)\).

**Proof.** Take a 2-elementary \(K3\) surface \((S, \theta)\) of type \(M\) and an elliptic curve \(T\). Let \(f : (S, \theta) \to \text{Def}(S, \theta)\) be the Kuranishi family of \((S, \theta)\) and let \(g : T \to \text{Def}(T)\) be the Kuranishi family of \(T\). Set \(\bar{X} := X_{(S, \theta, T)}\) and let \(\bar{f} : (X, \bar{X}) \to (\text{Def}(\bar{X}), [\bar{X}])\) be the Kuranishi family of \(\bar{X}\). We have the embedding of germs
\[
\mu : \text{Def}(S, \theta) \times \text{Def}(T) \to \text{Def}(\bar{X}), \quad \mu(s, t) := [X_{(S, \theta, T)}].
\]

for \(s \in \text{Def}(S, \theta)\), \(t \in \text{Def}(T)\), where \((S_s, \theta_s) = f^{-1}(s)\) and \(T_t = g^{-1}(t)\).

(Step 1) Let
\[
E := p^{-1}(\text{Sing}(X_{(S, \theta, T)})) = p^{-1}(S^\theta \times T[2])
\]
be the exceptional divisor of \(p : X_{(S, \theta, T)} \to X_{(S, \theta, T)}\). Then \(E\) is a \(\mathbb{P}^1\)-bundle over \(S^\theta \times T[2]\), whose structure is given as follows. Let \(N := N_{(S^\theta \times T[2])/(S \times T)}\) be the normal bundle of \(S^\theta \times T[2]\) in \(S \times T\). Then \(\text{O}_{\mathbb{P}(N)}(-1) \subset \mathbb{P}(N) \times N\). The projection \(\text{O}_{\mathbb{P}(N)}(-1) \to N\) is the blowing-up of the zero section of \(N\) and
\[
\text{O}_{\mathbb{P}(N)}(-2) = \text{O}_{\mathbb{P}(N)}(-1)/\pm 1 \to N/\pm 1
\]
is a crepant resolution. Let \(N_{E/\bar{X}}\) be the normal bundle of \(E\) in \(\bar{X}\). Then
\[
E = \mathbb{P}(N_{(S^\theta \times T[2])/(S \times T)}), \quad N_{E/\bar{X}}|_E = \text{O}_E(-2).
\]
We set $p_E := p|_E$. Then $p_E : E = \mathcal{P}(N) \to S^8 \times T[2]$ is the projection of $\mathbb{P}^1$-bundle.

Let $i : E \hookrightarrow \tilde{X}$ be the inclusion. By Voisin [37, Lemme 1.7], we have the decomposition

$$H^1(\tilde{X}, \Omega^2_{\tilde{X}}) = \left[H^1(S, \Omega^2_S)^\perp \otimes H^0(T, K_T)\right] \oplus \left[H^0(S, K_S) \otimes H^1(T, \mathcal{O}_T)\right]$$

(3.6)

$$\oplus i_* p_E^* H^0(S^8 \times T[2], \Omega^1_{S^8 \times T[2]}),$$

which is orthogonal with respect to the $L^2$-metric on $H^1(\tilde{X}, \Omega^2_{\tilde{X}})$. The Kodaira-Spencer map induces the following isomorphisms

$$\rho_1 : \Theta_{\text{Def}(S, \Theta)} \cong H^1(S, \Theta_S)^\perp \otimes H^0(S, K_S)^\vee$$

(3.7)

$$\cong \left[H^1(S, \Omega^2_S)^\perp \otimes H^0(T, K_T)\right] \otimes H^0(\tilde{X}, K_{\tilde{X}})^\vee,$$

(3.8)

$$\rho_2 : \Theta_{\text{Def}(T), [T]} \cong H^1(T, \Theta_T) = H^1(T, \mathcal{O}_T) \otimes H^0(T, K_T)^\vee$$

$$\cong \left[H^0(S, K_S) \otimes H^1(T, \mathcal{O}_T)\right] \otimes H^0(\tilde{X}, K_{\tilde{X}})^\vee.$$ 

By (3.6), (3.7), (3.8), we get the following canonical identification

$$\rho_3 : \Theta_{\text{Def}(F), [\tilde{X}]} \cong H^1(\tilde{X}, \Theta_{\tilde{X}}) \cong H^1(S, \Omega^2_S)^\perp \otimes H^0(\tilde{X}, K_{\tilde{X}})^\vee$$

(3.9)

$$\Theta_{\text{Def}(S, \Theta)} \oplus \Theta_{\text{Def}(T), [T]} \oplus \left[i_* p_E^* H^0(S^8 \times T[2], \Omega^1_{S^8 \times T[2]}) \otimes H^0(\tilde{X}, K_{\tilde{X}})^\vee\right],$$

where the first isomorphism is given by the Kodaira-Spencer map and the last decomposition is orthogonal with respect to the Weil-Petersson form $\omega_{\text{WP}}$. By (1.3), we have

$$\left(\mu^* \omega_{\text{WP}}\right)|_{\Theta_{\text{Def}(S, \Theta)} \oplus \Theta_{\text{Def}(T)}} = \text{pr}_1^* \omega_{\text{M}} + \text{pr}_2^* \omega_{\text{hyp}}.$$ 

(Step 2) Let $(\mathcal{G}, h_\mathcal{G})$ be the automorphic vector bundle of rank $g$ on $\mathfrak{S}_g$ equipped with the Hermitian structure induced from the polarization such that

$$f_\mathcal{G} \Omega^1_{\text{Def}(S^g)} \otimes h_{L^2} = J^*_M(\mathcal{G}, h_\mathcal{G}).$$

(\mathfrak{G} is the relative cotangent bundle of the universal family of principally polarized Abelian varieties over $\mathfrak{S}_g$.) Let us see the isometry of Hermitian vector spaces

$$\left(i_* p_E^* H^0(S^8 \times T[2], \Omega^1_{S^8 \times T[2]}) \otimes H^0(\tilde{X}, K_{\tilde{X}})^\vee, \omega_{\text{WP}}\right)$$

(3.10)

$$\cong \left(H^0(S^8 \times T[2], \Omega^1_{S^8 \times T[2]}), 2h_{L^2}\right) \otimes \left(H^0(\tilde{X}, K_{\tilde{X}}), h_{L^2}\right)^\vee.$$

Recall that $i_* : H^*(E) \to H^{*-2}(\tilde{X})$ is defined as the dual of $i^* : H^*(\tilde{X}) \to H^*(E)$ with respect to the Poincaré duality pairing: For all $\varphi \in H^*(\tilde{X})$, $\psi \in H^{4-*}(E)$,

$$\int_{\tilde{X}} i_* \psi \wedge \varphi = 2\pi \int_E \psi \wedge i^* \varphi.$$

Let $\Phi \in H^2_c(N_{E/\tilde{X}})$ be the Thom form of Mathai-Quillen (cf. [2 (1.37)]). Identify $N_{E/\tilde{X}}$ with a tubular neighborhood of $E$ in $\tilde{X}$ and regard $\Phi$ as a $C^\infty$ closed 2-form on $\tilde{X}$ supported on the closure of $N_{E/\tilde{X}}$ by this identification. Since $i_* \psi = [\Phi] \wedge \psi$
and $i^*\Phi = 2\pi c_1(N_{E/X})$ by $\cite{2}$ (1.8)], we deduce from $\eqref{3.5}$ that for all $\omega, \omega' \in H^0(S^\theta \times T[2], \Omega^{1}_{S^\theta \times T[2]})$

$$
\langle i_\ast p^*_E \omega, i_\ast p^*_E \omega' \rangle_{L^2} = -\frac{\sqrt{-1}}{(2\pi)^3} \int_{\mathcal{X}} (i_\ast p^*_E \omega) \wedge [\Phi] \wedge p^*_E \omega',
$$

$$
= -\frac{\sqrt{-1}}{(2\pi)^2} \int_{E} p^*_E (\omega \wedge \overline{\omega'}) \wedge i_\ast [\Phi]
$$

$$
= -\frac{\sqrt{-1}}{2\pi} \int_{E} p^*_E (\omega \wedge \overline{\omega'}) \wedge c_1(O_E(-2))
$$

$$
= \frac{\sqrt{-1}}{\pi} \int_{S^\theta \times T[2]} \omega \wedge \overline{\omega'} = 2\langle \omega, \omega' \rangle_{L^2},
$$

where we used the projection formula to get the 4-th equality. This verifies $\eqref{3.12}$. By $\eqref{3.10}$, $\eqref{3.12}$, we have an isometry of holomorphic Hermitian vector bundles on $\text{Def}(S, \theta) \times \text{Def}(T)$

$$
\mu^* \left( \text{Def}(\mathcal{X}), \omega_{WP} \right) \cong \left( \text{Def}(S, \theta), \omega_{M^\perp} \right) \oplus \left( \text{Def}(T), \omega_{hyp} \right)
$$

$$
\oplus \left[ J^*_M (\mathcal{H}^{\oplus 4}, 2h_{\mathcal{H}^{\oplus 4}} \otimes (f_\ast K_{/\text{Def}(\mathcal{X})}, h_{L^2}^\vee) \right].
$$

(Step 3) Since

$$\text{Ric} \omega_{WP} = c_1(\text{Def}(\mathcal{X}), \omega_{WP}), \quad c_1(\mathcal{H}, 2h_{\mathcal{H}}) = \omega_{A^\theta}, \quad \omega_{WP} = c_1(f_\ast K_{/\text{Def}(\mathcal{X})}, h_{L^2}),$$

we get by $\eqref{3.13}$

$$\mu^* \text{Ric} \omega_{WP} = c_1 \left( \text{Def}(S, \theta), \omega_{M^\perp} \right) + c_1 \left( \text{Def}(T), \omega_{hyp} \right)
$$

$$
+ J^*_M c_1(\mathcal{H}^{\oplus 4}, 2h_{\mathcal{H}^{\oplus 4}}) - \text{rk}(\mathcal{H}^{\oplus 4})c_1(f_\ast K_{/\text{Def}(\mathcal{X})}, h_{L^2})
$$

$$
= c_1(\Omega_{M^\perp}, \omega_{M^\perp}) + c_1(\mathcal{H}, \omega_{\mathcal{H}}) + 4J^*_M \omega_{\mathcal{H}} - 4g \mu^* \omega_{WP}
$$

$$
= - (\dim \Omega_{M^\perp}) : \omega_{M^\perp} - 2\omega_{\mathcal{H}} + 4J^*_M \omega_{\mathcal{H}} - 4g(\omega_{M^\perp} + \omega_{\mathcal{H}})
$$

$$
= - (\dim \Omega_{M^\perp} + 4g) \omega_{M^\perp} - (4g + 2)\omega_{\mathcal{H}} + 4J^*_M \omega_{\mathcal{H}}.
$$

To get the third equality, we used the Einstein property of the bounded symmetric domains equipped with the Bergman metric

$$c_1(\Omega_{M^\perp}, \omega_{M^\perp}) = - (\dim \Omega_{M^\perp}) : \omega_{M^\perp}, \quad c_1(\mathcal{H}, \omega_{\mathcal{H}}) = - 2\omega_{\mathcal{H}}.
$$

Set $r := r(M)$. Then $\dim \Omega_{M^\perp} = r(M^\perp) - 2 = 20 - r$. Since

$$h^{2,1}(\mathcal{X}) - 4g = 21 - r, \quad \frac{\chi(\mathcal{X})}{12} = r - 10$$
by [37, Cor. 1.8], we deduce from [3, 14] and the curvature formula for the BCOV invariant [12, Th. 4.14] that

$$-dd^c \log \bar{\tau}_{\text{BCOV}} = \left( h^{2,1}(X) + \frac{\chi(X)}{12} + 3 \right) J^*\omega_{WP} + \mu^*\text{Ric}\omega_{WP}$$

(3.15)

$$= \left( h^{2,1}(X) + \frac{\chi(X)}{12} + 3 - \dim \Omega_{M^\bot} - 4g \right) \omega_{M^\bot}$$

$$+ \left( h^{2,1}(X) + \frac{\chi(X)}{12} + 3 - 4g - 2 \right) \omega_S + 4J^*\omega_{\mathcal{S}_S}$$

This completes the proof of (3.4). □

Set

$$F^{M^\bot} := \log \left[ \tau_{\text{BCOV}}/(\tau_{\text{M}}^{-2}||\eta^{24}||^2) \right] \in C^\infty (\mathcal{M}^0_{M^\bot} \times X(1)).$$

Then Theorem [3, 2] is equivalent to the assertion that $F^{M^\bot}$ is a constant function on $\mathcal{M}^0_{M^\bot} \times X(1)$. In the rest of Section 3.4 we study the possible singularities of $F^{M^\bot}$.

Let $\varpi: \mathcal{H} \to X(1)$ be the projection and let $\mathcal{P}_{M^\bot} := \Pi_{M^\bot} \times \varpi$ be the projection from $\Omega_{M^\bot} \times \mathcal{H}$ to $\mathcal{M}_{M^\bot} \times X(1)$.

**Proposition 3.5.** For any $d \in \Delta_{M^\bot}$, there exists $\alpha(d) \in Q$ such that

$$-dd^c \left[ \mathcal{P}_{M^\bot}^* F^{M^\bot} \right] = \sum_{d \in \Delta_{M^\bot} / \pm 1} \alpha(d) \delta_{H_d \times \mathcal{H}}.$$  

Here $\alpha(g \cdot d) = \alpha(d)$ for all $g \in O(M^\bot)$. In particular, $\partial F^{M^\bot}$ is a logarithmic 1-form on $\mathcal{M}_{M^\bot} \times X(1)$ with possible pole along $\overline{\mathcal{M}}_{M^\bot} \times X(1)$.

**Proof.** Let $d \in \Delta_{M^\bot}$ and let $z \in \mathcal{H}$. Let $\gamma: \Delta \to \Omega_{M^\bot}$ be a holomorphic curve intersecting $H_d$ transversally at $\gamma(0) \in H_d^0$. By Theorem [1, 6] and [12, Prop. 5.5], there exists $a_{\gamma,d,z} \in Q$ such that

$$\log \tau_{\text{BCOV}} (\mathcal{P}_{M^\bot}^* (\gamma(t), z)) = a_{\gamma,d,z} \log |t|^2 + O (\log (\log |t|)) \quad (t \to 0).$$ 

By [40, Th. 6.5], we have

$$\log \left[ \tau_{\mathcal{M}} (\gamma(t))^{-4}||\eta(z)||^2 \right] = \frac{1}{2} \log |t|^2 + O(1) \quad (t \to 0).$$ 

By (3.17), (3.18), we get

$$F^{M^\bot} (\mathcal{P}_{M^\bot}^* (\gamma(t), z)) = (a_{\gamma,d,z} - \frac{1}{2}) \log |t|^2 + O (\log (\log |t|)) \quad (t \to 0).$$ 

Since $\mathcal{P}_{M^\bot}^* F^{M^\bot}$ is pluriharmonic on $\Omega_{M^\bot} \times \mathcal{H}$, the constant $a_{\gamma,d,z} - \frac{1}{2}$ depends only on $d \in \Delta_{M^\bot}$ by [12, Lemma 5.9]. Hence we can define $\alpha(d) := -(a_{\gamma,d,z} - \frac{1}{2})$. Then (3.19) follows from (3.17) and [12, Lemma 5.9]. The property $\alpha(g \cdot d) = \alpha(d)$ for all $g \in O(M^\bot)$ is a consequence of the $O(M^\bot)$-invariance of $\mathcal{P}_{M^\bot}^* F^{M^\bot}$. □

Recall that $\mathcal{M}^*_{M^\bot}$ is the Baily-Borel compactification of $\mathcal{M}_{M^\bot}$ and $\mathcal{B}_{M^\bot} = \mathcal{M}^*_{M^\bot} \setminus \mathcal{M}_{M^\bot}$ is its boundary locus.
Lemma 3.6. Let $C \subset M_{M^+}$ be an irreducible curve such that $C \neq B_{M^+}$. For any $z \in X(1)$, $F^{M^+}|_{C \times \{z\}}$ has at most logarithmic singularities at $(C \cap B_{M^+}) \times \{z\}$. Namely, for any $x \in (C \cap B_{M^+}) \times \{z\}$, there exists $\alpha \in \mathbb{Q}$ such that
\[ F^{M^+}(t, z) = \alpha \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0), \]
where $t$ is a local parameter of $C$ centered at $x$.

Proof. By [44] Th. 3.1, there is an irreducible curve $B$, a finite surjective map $\varphi : B \to C$, a smooth projective threefold $S$, a surjective holomorphic map $f : S \to B$, and a holomorphic involution $\theta : S \to S$ preserving the fibers of $f$ with the following properties:

(i) There is a non-empty Zariski open subset $B^0 \subset B$ such that $(S_b, \theta_b)$ is a 2-elementary K3 surface of type $M$ for all $b \in B^0$. Here, $S_b := f^{-1}(b)$ and $\theta_b := \theta|_{S_b}$.

(ii) $\varphi : B^0 \to C \cap M_{M^+}$ is the period map for the family of 2-elementary K3 surfaces $f : (S, \theta)|_{B^0} \to B^0$ of type $M$.

Let $T$ be an elliptic curve. Let $X \to (S \times T)/(\theta \times -1_T)$ be a resolution and let $g : X \to B$ be the map induced from the map $f \circ \text{pr}_1 : S \times T \to B$. We may assume that $X_b := g^{-1}(b)$ is the Borcea-Voisin threefold $X_{(S_b, \theta_b, T)}$ for all $b \in B^0$.

Let $p \in \varphi^{-1}(C \cap (M_{M^+} \setminus M_{M^+}))$ and let $(U, s)$ be a coordinate neighborhood of $B$ centered at $p$. By Theorem 1.6 applied to the family of Calabi-Yau threefolds $g : X \to B$, there exists $\beta \in \mathbb{Q}$ such that at $s \to 0$,
\[ \varphi^*(\log \tau_{BCOV})(s) = \log \tau_{BCOV}(X_s) = \beta \log |s|^2 + O(\log(-\log |s|)). \]
By (2.6) and (3.20), $F^{M^+}|_{C \times \{z\}}$ has at most logarithmic singularities at $(C \cap B_{M^+}) \times \{z\}$. This completes the proof. \qed

Set $\pm \sqrt{-1}\infty := X^*(1) \setminus X(1)$.

Lemma 3.7. For any $[\eta] \in M_{M^+}$, $F^{M^+}|_{[\eta] \times X(1)}$ has at most a logarithmic singularity at $(\eta, \pm \sqrt{-1}\infty)$.

Proof. The result follows from [42] Prop. 5.6 and Theorems 6.1 and 6.2. \qed

3.2. Examples verifying Theorem 3.2. In this subsection, we give some examples of Borcea-Voisin threefolds which verifies Theorem 3.2.

Lemma 3.8. Let $F$ be a real-valued pluriharmonic function on $M_{M^+} \times X(1)$. Assume that for any $[\eta] \in M_{M^+}$, $F|_{[\eta] \times X(1)}$ has at most logarithmic singularity at $(\eta, \pm \sqrt{-1}\infty)$.

(1) If $r(M) \leq 17$, then $F$ is a constant function.

(2) If $r(M) \geq 18$, assume moreover that for every complete irreducible curve $C \subset M_{M^+}$ with $C \neq B_{M^+}$ and for every $z \in X(1)$, $F|_{C \times \{z\}}$ has at most logarithmic singularity at any point of $(C \cap B_{M^+}) \times \{z\}$. Namely, there exists $\alpha \in \mathbb{Q}$ such that
\[ F([\eta], j) = \alpha \log |j|^2 + O(\log(-\log |j|)) \quad (j \to \pm \sqrt{-1}\infty), \]
where $j : X^*(1) \cong \mathbb{P}^1$ is the isomorphism given by $j$-invariant. Then $F$ is a constant function.
Proof. (1) Since \( r(M) \leq 17 \), \( B_{M^\perp} \) has codimension \( \geq 2 \) in \( M_{M^\perp}^* \). By Grauert-Remmert [14 Satz 4] and the normality of \( M_{M^\perp}^* \), \( F \) extends to a pluriharmonic function on \( M_{M^\perp}^* \times X(1) \). By the compactness of \( M_{M^\perp}^* \), there exists a function \( \psi \) on \( X(1) \) such that \( F = \text{pr}_2^* \psi \). Since \( F \) is pluriharmonic, \( \psi \) must be a harmonic function on \( X(1) \) because \( \psi = F|_{[\eta] \times X(1)} \), where \( [\eta] \in M_{M^\perp}^0 \). Since \( F|_{[\eta] \times X(1)} \) has at most logarithmic singularity at \(( [\eta], +\infty ) \), \( \psi \) has at most logarithmic singularity at \( +\infty \). Namely, there exists \( \alpha \in \mathbb{R} \) such that
\[
\psi(j) = \alpha \log |j|^2 + O(\log \log |j|) \quad (j \to \infty).
\]
Since \( \psi \) is harmonic on \( X(1) \), it follows from (3.21) that \( \partial \psi \) is a logarithmic 1-form on \( X^*(1) \cong \mathbb{P}^1 \) with possible pole at \( j = \infty \) and with residue \( A \). By the residue theorem applied to \( \partial \psi \), we get \( A = 0 \). Hence \( \partial \psi \) is a holomorphic 1-form on \( X^*(1) \).

As a result, \( \partial \psi = 0 \) on \( X^*(1) \), so that \( \psi \) is a constant function on \( X(1) \). This proves that \( F = \text{pr}_2^* \psi \) is a constant function on \( M_{M^\perp}^* \times X(1) \).

(2) Let \( z \in X(1) \). By assumption and [12 Lemma 5.9], \( F|M_{M^\perp}^* \times \{ z \} \) has at most logarithmic singularity along \( B_{M^\perp} \). Hence \( \partial F|M_{M^\perp}^* \times \{ z \} \) is a logarithmic 1-form on \( M_{M^\perp}^* \times \{ z \} \) with possible pole along the irreducible divisor \( B_{M^\perp} \). We set
\[
\alpha := \text{Res}_{B_{M^\perp}} \partial F|_{M_{M^\perp}^* \times \{ z \}}(\partial F|_{M_{M^\perp}^* \times \{ z \}}).
\]
Let \( C \subset M_{M^\perp}^* \) be a complete irreducible curve with \( C \not\subset B_{M^\perp} \) and \( C \cap B_{M^\perp} \neq \emptyset \). By the residue theorem applied to the logarithmic 1-form \( \partial F|_{C \times \{ z \}} \) on \( C \times \{ z \} \), we get \( \alpha \cdot \#(C \cap B_{M^\perp}) = 0 \). Hence \( \alpha = 0 \), so that \( F|M_{M^\perp}^* \setminus \text{Sing} M_{M^\perp}^* \) is a pluriharmonic function on \( M_{M^\perp}^* \setminus \text{Sing} M_{M^\perp}^* \). By [14 Satz 4] and the normality of \( M_{M^\perp}^* \), \( F|M_{M^\perp}^* \times \{ z \} \) extends to a pluriharmonic function on \( M_{M^\perp}^* \). By the compactness of \( M_{M^\perp}^* \), \( F|M_{M^\perp}^* \times \{ z \} \) is a constant function on \( M_{M^\perp}^* \). This implies the existence of a harmonic function \( \psi \) on \( X(1) \) such that \( F = (\text{pr}_2^*)^* \psi \). By the same argument as in (1), \( \psi \) is a constant function on \( X(1) \). This proves that \( F = \text{pr}_2^* \psi \) is a constant function on \( M_{M^\perp}^* \times X(1) \). This completes the proof. \( \square \)

Theorem 3.9. If \( r(M) \leq 17 \) and \( D_{M^\perp} \) is irreducible, then Theorem 3.2 holds.

Proof. Let \( \alpha \in \mathbb{Q} \) be the residue of the logarithmic 1-form \( \partial F_{M^\perp} \) along \( D_{M^\perp} \times X(1) \). Since \( r(M) \leq 17 \) and hence \( B_{M^\perp} \) has codimension \( \geq 2 \) in \( M_{M^\perp}^* \), there is an irreducible complete curve \( C \subset M_{M^\perp}^* \) such that \( C \cap B_{M^\perp} = \emptyset \) and \( C \cap D_{M^\perp} \neq \emptyset \). Let \( z \in X(1) \) be an arbitrary point. Since \( \partial F_{M^\perp}|_{C \times \{ z \}} \) is a logarithmic 1-form on \( C \times \{ z \} \) with residue \( \alpha \) at any pole of \( \partial F_{M^\perp}|_{C \times \{ z \}} \), the total residue of \( \partial F_{M^\perp}|_{C \times \{ z \}} \) is a non-zero multiple of \( \alpha \). By the residue theorem, we get \( \alpha = 0 \). Thus \( \partial F_{M^\perp} \) is a holomorphic 1-form on \( M_{M^\perp} \times X(1) \), so that \( F_{M^\perp} \) is a pluriharmonic function on \( M_{M^\perp} \times X(1) \). Now, the result follows from Lemma 3.8 (1). \( \square \)

Theorem 3.2 for special \( M \) mentioned at the beginning of Section 5 is a particular example of Theorem 3.9. 3.3. The behavior of BCOV invariants near the discriminant locus.

3.3.1. Ordinary singular families of 2-elementary K3 surfaces. Let \( S \) be a smooth complex threefold and let \( p: S \to \Delta \) be a proper surjective holomorphic function without critical points on \( S \setminus p^{-1}(0) \). Let \( \theta: S \to S \) be a holomorphic involution preserving the fibers of \( p \). Set \( S_t = p^{-1}(t) \) and \( \theta_t = \theta|_{S_t} \) for \( t \in \Delta \). Then \( p: (S, \theta) \to \)
Δ is called an ordinary singular family of 2-elementary K3 surfaces if p has a unique, non-degenerate critical point on S0 and if (Sτ, θτ) is a 2-elementary K3 surface for all τ ∈ Δ*. Let o ∈ S be the unique critical point of p. By [40], Sect. 2.2, there exists a system of coordinates (U, (w1, w2, w3)) centered at o such that
\[ \iota(w) = (-w_1, -w_2, -w_3) \quad \text{or} \quad (w_1, w_2, -w_3), \quad z \in U. \]
If \( \iota(w) = (-w_1, -w_2, -w_3) \) on \( U \), \( \iota \) is said to be of type \((0, 3)\). If \( \iota(w) = (w_1, w_2, -w_3) \) on \( U \), \( \iota \) is said to be of type \((2, 1)\).

**Lemma 3.10.** There exists a system of local coordinates \((z_1, z_2, z_3)\) of \( Z \) centered at \( o \in Z \) and a coordinate \( t \) of \( \Delta \) centered at \( 0 \in \Delta \) with the following properties.

1. If \( \iota \) is of type \((0, 3)\), then
   \[ \iota(z_1, z_2, z_3) = (-z_1, -z_2, -z_3), \quad p(z_1, z_2, z_3) = (z_1)^2 + (z_2)^2 + (z_3)^2. \]
2. If \( \iota \) is of type \((2, 1)\), then
   \[ \iota(z_1, z_2, z_3) = (z_1, z_2, -z_3), \quad p(z_1, z_2, z_3) = (z_1)^2 + (z_2)^2 + (z_3)^2. \]

**Proof.** The proof is standard and is omitted. \( \square \)

### 3.3.2. Two local models of critical points. We introduce two local models of critical points appearing in certain degenerations of Borcea-Voisin threefolds.

Let \( B \subset C^3 \) be the unit ball of radius 1. Let \( T \) be an elliptic curve. Define involutions \( \iota^{(2,2)}, \iota^{(0,4)} \) on \( B \times T \) by
\[ \iota^{(2,2)}(z, w) = (z_1, z_2, -z_3, -w), \quad \iota^{(0,4)}(z, w) = (-z_1, -z_2, -z_3, -w), \]
where \( z = (z_1, z_2, z_3) \in B \) and \( w \in T \). Set
\[ \mathcal{V}^{(2,2)} := (B \times T)/\iota^{(2,2)}, \quad \mathcal{V}^{(0,4)} := (B \times T)/\iota^{(0,4)}. \]

Then \( \mathcal{V}^{(2,2)} \) and \( \mathcal{V}^{(0,4)} \) are orbifolds. Since the nowhere vanishing canonical form \( dz_1 \wedge d z_2 \wedge d z_3 \wedge dw \) on \( B \times T \) is invariant under the \( \iota^{(2,2)} \) and \( \iota^{(0,4)} \)-actions, it descends to a nowhere vanishing canonical form in the sense of orbifolds on \( \mathcal{V}^{(2,2)} \) and \( \mathcal{V}^{(0,4)} \), respectively. We write \( \Xi^{(2,2)} \) (resp. \( \Xi^{(0,4)} \)) for the nowhere vanishing canonical form on \( \mathcal{V}^{(2,2)} \) (resp. \( \mathcal{V}^{(0,4)} \)) induced by \( dz_1 \wedge d z_2 \wedge d z_3 \wedge dw \).

For \( (z, w) \in B \times T \), write \( [(z, w)]^{(2,2)} \in \mathcal{V}^{(2,2)} \) and \( [(z, w)]^{(0,4)} \in \mathcal{V}^{(0,4)} \) for the images of \( (z, w) \) by the projections \( B \times T \to \mathcal{V}^{(2,2)} \) and \( B \times T \to \mathcal{V}^{(0,4)} \), respectively.

**Case 1.** Set \( \Sigma := \{ [(z, w)]^{(2,2)} \in \mathcal{V}^{(2,2)} ; z_1 = z_2 = 0 \} \). Then \( \text{Sing} \mathcal{V}^{(2,2)} = \Sigma \). Let \( \sigma^{(2,2)} : \mathcal{V}^{(2,2)} \to \mathcal{V}^{(2,2)} \) be the blowing-up along \( \Sigma \), which is a resolution of the singularities of \( \mathcal{V}^{(2,2)} \). Define \( F^{(2,2)} \in \mathcal{O}(\mathcal{V}^{(2,2)}) \) by
\[ F^{(2,2)}([(z, w)]^{(2,2)}) := (z_1)^2 + (z_2)^2 + (z_3)^2 \]
and set
\[ F^{(2,2)} := F^{(2,2)} \circ \sigma^{(2,2)} \in \mathcal{O}(\mathcal{V}^{(2,2)}). \]

Set \( W := \{(u, v, r) \in C^3 ; uv - r^2 = 0 \} \). Since \( C^2/\pm 1 \cong W \) via the map \( \pm(z_3, w) \mapsto ((z_3)^2, w^2, z_3 w) \), we have an isomorphism of germs \( (\mathcal{V}^{(2,2)}, x) \cong (C^2 \times W, (0, 0)) \) for any \( x \in \text{Sing} \mathcal{V}^{(2,2)} \). Under this identification of germs, the function germ \( F^{(2,2)} : (C^4/\iota^{(2,2)}, 0) = (C^2 \times W, (0, 0)) \to (C, 0) \) is expressed as
\[ F^{(2,2)}(z_1, z_2, u, v, r) = (z_1)^2 + (z_2)^2 + u, \]
where \( (z_1, z_2) \in C^2, (u, v, r) \in W \).
Let $\sigma: (\tilde{W}, E) \to (W, 0)$ be the blowing-up at the origin, where $E = \sigma^{-1}(0) \cong \mathbb{P}^1$. The isomorphism $(V^{(2,2)}, x) \cong (\mathbb{C}^2 \times W, (0, 0))$ induces an isomorphism of germs $(\tilde{V}^{(2,2)}, \tilde{x}) \cong (\mathbb{C}^2 \times \tilde{W}, (0, \zeta))$ for any $\tilde{x} \in (\sigma^{(2,2)})^{-1}(x)$, where $\zeta \in \sigma^{-1}(0)$ is the point corresponding to $\tilde{x}$. Let $\{U_0, U_1, U_2\}$ be the open covering of $\mathbb{C}^2 \times \tilde{W}$ defined as

$$U_0 := \mathbb{C}^2 \times \sigma^{-1}\{u \neq 0\}, \quad U_1 := \mathbb{C}^2 \times \sigma^{-1}\{v \neq 0\}, \quad U_2 := \mathbb{C}^2 \times \sigma^{-1}\{r \neq 0\}.$$ 

By (3.24), $\tilde{F}^{(2,2)}$ has no critical points on $U_0 \cup U_2$. On $U_1$, we have the system of coordinates $(z_1, z_2, v, s := r/v)$. Since $(z_1, z_2, v, s) = (z_1)^2 + (z_2)^2 + u = (z_1)^2 + (z_2)^2 + \frac{r^2}{v} = (z_1)^2 + (z_2)^2 + vs^2$, we get

$$\Sigma_{\tilde{F}^{(2,2)}} = \{z_1 = z_2 = s = 0\} \cap U_1.$$ 

Hence $\dim \Sigma_{\tilde{F}^{(2,2)}} = 1$. In particular, the divisor $(\tilde{F}^{(2,2)})^{-1}(0)$ is irreducible.

Since the dualizing sheaf of $\tilde{W}$ is trivial, so is the dualizing sheaf of $\mathbb{C}^2 \times \tilde{W}$. Since $\Sigma^{(2,2)}$ is a nowhere vanishing section of the dualizing sheaf of $\mathbb{C}^2 \times \tilde{W}$, $(\sigma^{(2,2)})^* \Sigma^{(2,2)}$ extends to a nowhere vanishing canonical form on $\tilde{V}^{(2,2)}$. If $\Upsilon$ is a nowhere vanishing canonical form on $V^{(2,2)} \setminus \Sigma^{(2,2)}$, then $\Upsilon/\Sigma^{(2,2)}$ is a nowhere vanishing holomorphic function on $V^{(2,2)} \setminus \Sigma^{(2,2)}$, which extends to a nowhere vanishing holomorphic function on $V^{(2,2)}$. Hence $(\sigma^{(2,2)})^* \Upsilon$ is also a nowhere vanishing canonical form on $\tilde{V}^{(2,2)}$.

(Case 2) Let $\omega_1, \omega_2, \omega_3 \in T$ be non-zero points of order 2. Then

$$\text{Sing } V^{(0,4)} = \{[(0, 0)]^{(0,4)}, [(0, \omega_1)]^{(0,4)}, [(0, \omega_2)]^{(0,4)}, [(0, \omega_3)]^{(0,4)}\}$$

consists of 4 isolated quotient singularities isomorphic to $(\mathbb{C}^4/\pm 1, 0)$.

Define $F^{(0,4)} \in \mathcal{O}(V^{(0,4)})$ by

$$F^{(0,4)}([(z, w)]^{(0,4)}) := (z_1)^2 + (z_2)^2 + (z_3)^2$$

By this expression, we have

$$\Sigma_{F^{(0,4)}} = \{[(0, w)]^{(0,4)} \in V^{(0,4)}; w \in T\}.$$ 

In particular, $\dim \Sigma_{F^{(0,4)}} = 1$. Since the inverse image of $(F^{(0,4)})^{-1}(0)$ in $\mathbb{B} \times T$ is irreducible, $(F^{(0,4)})^{-1}(0)$ is an irreducible divisor of $V^{(0,4)}$.

3.3.3. A degenerating family of Borcea-Voisin threefolds and BCOV invariants, I.

Let $S$ and $S'$ be smooth irreducible projective threefolds. Let $\theta: S \to S$ and $\theta': S' \to S'$ be holomorphic involutions on $S$ and $S'$, respectively. Let $B$ and $B'$ be compact Riemann surfaces and let $p: S \to B$ and $p': S' \to B'$ be surjective holomorphic maps. Let $\Delta \subset B$ and $\Delta' \subset B'$ be the discriminant loci of $p: S \to B$ and $p': S' \to B'$, respectively. Let $p \in \Delta$ and $p' \in \Delta'$. For $b \in B$ and $b' \in B'$, set $S_b := p^{-1}(b)$ and $S'_{b'} := (p')^{-1}(b')$. We assume the following:

1. $\theta$ and $\theta'$ preserve the fibers of $p$ and $p'$, respectively. Set $\theta_b := \theta|_{S_b}$ and $\theta'_{b'} := \theta'|_{S'_{b'}}$ for $b \in B$ and $b' \in B'$.

2. There exist primitive 2-elementary Lorentzian sublattices $M, M' \subset L_{K3}$ such that $(S_b, \theta_b)$ is a 2-elementary $K3$ surface of type $M$ for all $B \setminus \Delta$ and $(S'_{b'}, \theta'_{b'})$ is a 2-elementary $K3$ surface of type $M'$ for all $B' \setminus \Delta'$. 
(3) There is a neighborhood $U$ of $p$ in $B$ such that

$$p: (p^{-1}(U), \theta|_{p^{-1}(U)}) \to U$$

is an ordinary singular family of 2-elementary $K3$ surfaces of type $M$. Similarly, there is a neighborhood $U'$ of $p'$ in $B'$ such that

$$p': ((p')^{-1}(U'), \theta'|_{(p')^{-1}(U')}) \to U'$$

is an ordinary singular family of 2-elementary $K3$ surfaces of type $M'$. Let $T$ be an elliptic curve. We set

$$t := \theta \times (-1)_T, \quad t' := \theta' \times (-1)_T.$$

and

$$\mathcal{X} := (S \times T)/t, \quad \mathcal{X}' := (S' \times T)/t'.$$

Let $\pi: \mathcal{X} \to B$ and $\pi': \mathcal{X}' \to B'$ be the projections induced from the projections $p: S \to B$ and $p': S' \to B'$, respectively. Since $S \times T$ (resp. $S' \times T$) is a complex manifold, the set of fixed points of the $t$-action (resp. $t'$-action) on $S \times T$ (resp. $S' \times T$), i.e., $S^0 \times T$ (resp. $(S')^0 \times T$) is the disjoint union of complex submanifolds.

Let $Z := (S^0 \times T[2])^{\text{hol}}$ and $Z' := ((S')^0 \times T[2])^{\text{hol}}$ be the horizontal components. Namely, $Z$ is the union of those connected components of $S^0 \times T[2]$ which are flat over $B$. Similarly, $Z'$ is the union of those connected components of $(S')^0 \times T[2]$ which are flat over $B'$. Then $Z$ and $Z'$ are complex submanifolds of $S \times T$ and $S' \times T$ of codimension 2, respectively.

Let $\sigma: \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blowing-up of $\mathcal{X}$ along $Z$ and let $\sigma': \widetilde{\mathcal{X}}' \to \mathcal{X}'$ be the blowing-up of $\mathcal{X}'$ along $Z'$. We set

$$\widetilde{\pi} := \pi \circ \sigma: \widetilde{\mathcal{X}} \to B, \quad \widetilde{\pi}' := \pi' \circ \sigma': \widetilde{\mathcal{X}}' \to B'.$$

By construction, $\widetilde{\pi}^{-1}(b) = \widetilde{\xi}_{(S_b, \theta_b, T)}$ for $b \in U \setminus \{p\}$ and $((\widetilde{\pi}')^{-1}(b')) = \widetilde{\xi}_{(S'_{b'}, \theta'_{b'}, T)}$ for $b' \in U' \setminus \{p'\}$. Fix isomorphisms of germs $(U, p) \cong (C, 0)$ and $(U', p') \cong (C, 0)$. Then, for $t \in C$ with $0 < |t| \ll 1$, $\widetilde{\xi}_{(S, \theta, T)}$ is a Borcea-Voisin threefold of type $M$ and $\widetilde{\xi}_{(S', \theta, T)}$ is a Borcea-Voisin threefold of type $M'$.

**Theorem 3.11.** If $\theta|_{p^{-1}(U)}$ and $\theta'|_{(p')^{-1}(U')}$ have the same type, then

$$\lim_{t \to 0} \frac{\log \tau_{\text{BCOV}}(\widetilde{\xi}_{(S, \theta, T)})}{\log |t|^2} = \lim_{t' \to 0} \frac{\log \tau_{\text{BCOV}}(\widetilde{\xi}_{(S', \theta', T)})}{\log |t'|^2}.$$

In particular, as $t \to 0$,

$$\log \tau_{\text{BCOV}}(\widetilde{\xi}_{(S, \theta, T)}) - \log \tau_{\text{BCOV}}(\widetilde{\xi}_{(S', \theta', T)}) = O (\log(-\log |t|)).$$

**Proof.** By Theorem 4.7 it suffices to verify conditions (A1), (A2), (A3), (A4) in Section 4.3 for the families $\pi: \widetilde{\mathcal{X}} \to B$ and $\pi': \widetilde{\mathcal{X}}' \to B'$. Since $\pi^{-1}(b) = \widetilde{\xi}_{(S_b, \theta_b, T)}$ for $b \in U \setminus \{p\}$ and $(\pi')^{-1}(b') = \widetilde{\xi}_{(S'_{b'}, \theta'_{b'}, T)}$ for $b' \in U' \setminus \{p'\}$, condition (A2) holds.

Set $\theta_b := \theta|_{p^{-1}(U)}$ and $\theta_{b'} := \theta'|_{(p')^{-1}(U')}$. Let $o$ be the unique critical point of $p|_{p^{-1}(U)}$ and let $o'$ be the unique critical point of $(p')|_{(p')^{-1}(U')}$. (Step 1) By Lemma 3.10 there exist a neighborhood $U$ of $o$ in $S$ and a system of coordinates $(z_1, z_2, z_3)$ on $U$ centered at $o$ such that $(U, (z_1, z_2, z_3)) = (B, (z_1, z_2, z_3))$ and such that either $(3.22)$ or $(3.23)$ holds on $U$. Similarly, there exist a neighborhood $U'$ of $o'$ in $S'$ and a system of coordinates $(z_1, z_2, z_3)$ on $U'$
centered at $\sigma'$ such that $(U', (z_1, z_2, z_3)) = (B, (z_1, z_2, z_3))$ and such that either (3.22) or (3.23) holds on $U'$. Since $\theta_U$ and $\theta_U'$ have the same type, we get

$$\langle U \times T \rangle / \iota \cong \langle U' \times T \rangle / \iota' \cong \begin{cases} \mathcal{V}^{(2,2)} & \text{if } \theta_U, \theta_U' \text{ are of type (2,1)}, \\ \mathcal{V}^{(0,4)} & \text{if } \theta_U, \theta_U' \text{ are of type (0,3)}. \end{cases}$$

(3.25)

Define open subsets $O \subset \tilde{X}$ and $O' \subset \tilde{X}'$ by

$$O := \sigma^{-1}((U \times T) / \iota), \quad O' := (\sigma')^{-1}((U' \times T) / \iota').$$

By (3.26), we get the following isomorphism

$$O \cong O' \cong \begin{cases} \tilde{\mathcal{V}}^{(2,2)} & \text{if } \theta_U, \theta_U' \text{ are of type (2,1)}, \\ \mathcal{V}^{(0,4)} & \text{if } \theta_U, \theta_U' \text{ are of type (0,3)}. \end{cases}$$

By (3.22), (3.23), we get under the isomorphism (3.25)

$$\langle \tilde{U} \times T \rangle / \tilde{\iota} \cong \langle \tilde{U}' \times T \rangle / \tilde{\iota}' \cong \begin{cases} \tilde{\mathcal{V}}^{(2,2)} & \text{if } \theta_U, \theta_U' \text{ are of type (2,1)}, \\ \mathcal{V}^{(0,4)} & \text{if } \theta_U, \theta_U' \text{ are of type (0,3)}. \end{cases}$$

(3.26)

Under the isomorphism (3.26), we get by (3.27) the following isomorphism of pairs:

$$\varpi | (U \times T) / \iota = \varpi' | (U' \times T) / \iota' = \begin{cases} \tilde{\mathcal{V}}^{(2,2)} & \text{if } \theta_U, \theta_U' \text{ are of type (2,1)}, \\ \mathcal{V}^{(0,4)} & \text{if } \theta_U, \theta_U' \text{ are of type (0,3)}. \end{cases}$$

(3.27)

This verififies condition (A4) in Section 1.3.

(Step 2) Since $S_0$ and $S_0'$ are singular $K3$ surfaces with a unique ordinary double point as its singular set, the divisors $\varpi^{-1}(0)$ and $(\varpi')^{-1}(0)$ are irreducible by the descriptions of $(\tilde{\mathcal{V}}^{(2,2)}, \tilde{\mathcal{F}}^{(2,2)})$ and $(\mathcal{V}^{(0,4)}, \mathcal{F}^{(0,4)})$ in Section 3.3.2. Similarly, by the descriptions of $(\tilde{\mathcal{V}}^{(2,2)}, \tilde{\mathcal{F}}^{(2,2)})$ and $(\mathcal{V}^{(0,4)}, \mathcal{F}^{(0,4)})$ in Section 3.3.2, we get

$$\dim \Sigma_{\varpi} = \dim \Sigma_{\varpi'} = 1.$$ (3.29)

This verifies condition (A1) in Section 1.3.

(Step 3) By choosing $U$ and $U'$ sufficiently small, we may assume by [40] Lemma 2.3 that $p^{-1}(U)$ and $(p')^{-1}(U')$ carry nowhere vanishing canonical forms $\xi$ and $\xi'$, respectively, such that $\theta^* \xi = -\xi$ and $(\theta')^* \xi' = -\xi'$. Then $\xi \wedge dw$ and $\xi' \wedge dw$ are nowhere vanishing canonical forms on $p^{-1}(U) \times T$ and $(p')^{-1}(U') \times T$, respectively, such that $\iota^* (\xi \wedge dw) = \xi \wedge dw$ and $(\iota')^* (\xi' \wedge dw) = \xi' \wedge dw$. Hence $\xi \wedge dw$ (resp. $\xi' \wedge dw$) induces a nowhere vanishing canonical form $\Xi$ (resp. $\Xi'$) in the sense of orbifolds on $(U \times T) / \iota$ (resp. $(U' \times T) / \iota'$).

Let $\theta_U$ and $\theta_U'$ be of type (2,1). Since every nowhere vanishing canonical form on $(U \times T) / \iota$ (resp. $(U' \times T) / \iota'$) lifts to a nowhere vanishing canonical form on $O$ (resp. $O'$) via $\sigma$ (resp. $\sigma'$) by Section 3.3.2 Case 1, $\sigma^* \Xi$ (resp. $(\sigma')^* \Xi'$) is a nowhere vanishing canonical form on $O$ (resp. $O'$).

Let $\theta_U$ and $\theta_U'$ be of type (0,3). Then $O \cong O' \cong \mathcal{V}^{(0,4)}$ by (3.26). Hence $\Xi$ (resp. $\Xi'$) is a nowhere vanishing canonical form on $O \setminus \Sigma_{\varpi}$ (resp. $O' \setminus \Sigma_{\varpi'}$). This verifies condition (A3) in Section 1.3. Since conditions (A1), (A2), (A3), (A4) are verified for $\varpi : \tilde{X} \to B$ and $\varpi' : \tilde{X}' \to B'$, the result follows from Theorem 1.7. \(\square\)
3.3.4. A degenerating family of Borcea-Voisin threefolds and BCOV invariants, II.
Let $\mathcal{S}$ be a smooth projective threefold equipped with a holomorphic involution $\theta: \mathcal{S} \to \mathcal{S}$. Let $B$ be a compact Riemann surface and let $p: \mathcal{S} \to B$ be a surjective holomorphic map. Let $\Delta \subset B$ be the discriminant locus of $p: \mathcal{S} \to B$ and let $p \in \Delta$.
We assume the following:

1. $\theta$ preserves the fibers of $p$ and the pair $(S_b, \theta_b)$ is a 2-elementary $K3$ surface of type $M$ for all $B \setminus \Delta$, where $S_b := p^{-1}(b)$ and $\theta_b := \theta|_{S_b}$.
2. There is a neighborhood $U$ of $q$ in $B$ such that $p: (p^{-1}(U), \theta|_{p^{-1}(U)}) \to U$ is an ordinary singular family of 2-elementary $K3$ surfaces of type $M$.

**Theorem 3.12.** Let $T$ be an elliptic curve. If $\theta|_{p^{-1}(U)}$ is of type $(2, 1)$, then

$$
\log \tau_{BCOV}(\tilde{X}(S, \theta, T)) = \frac{1}{2} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0).
$$

**Proof.** Set $M := U \oplus E_8(2)$. Then $\mathcal{T}_{M_{\mathcal{S}}} \mathcal{S}$ is irreducible by [43, Prop. 11.6].

(Step 1) By [40, Th. 2.8], there exist a smooth projective threefold $\mathcal{S}'$ equipped with a holomorphic involution $\theta': \mathcal{S}' \to \mathcal{S}'$, a pointed compact Riemann surface $(B', \mathcal{p}')$ equipped with a neighborhood $U'$ of $\mathcal{p}'$, and a surjective holomorphic map $p': \mathcal{S}' \to B'$ with the following properties:

1. $\theta'$ preserves the fibers of $p'$.
2. $p': (S'_{U}, \theta'|_{S'_{U}}) \to U'$ is an ordinary singular family of 2-elementary $K3$ surfaces of type $M$.

Since $M = U \oplus E_8(2)$, the fixed-point-set $(S'_{U})^{\theta'}_{p'}$ consists of two disjoint elliptic curves for all $b' \in U' \setminus \{\mathcal{p}'\}$. Assume that $\theta'_{U'}$ is of type $(0, 3)$. Then the set of fixed points of the $\theta'_{U'}$-action on $S'_{U} = (p')^{-1}(\mathcal{p}')$ consists of two disjoint elliptic curves and the isolated point $\text{Sing } S'_{\mathcal{p}'}$. Let $\mu: S'_{\mathcal{p}'} \to S'_{\mathcal{p}'}$ be the minimal resolution. By [43, Th. 2.3 (1)], the involution $\theta'_{U'}$ on $S'_{U}$ lifts to an involution $\bar{\theta}'_{\mathcal{p}'}$ on $\tilde{S}'_{\mathcal{p}'}$ and the pair $(\tilde{S}'_{\mathcal{p}'}, \theta'_{U'})$ is a 2-elementary $K3$ surface. Since $\theta'_{U'}$ is of type $(0, 3), \mu^{-1}(\text{Sing } S'_{\mathcal{p}'}) \cong \mathbb{P}^1$ is a component of the fixed-point-set $(\tilde{S}'_{\mathcal{p}'})^{\theta'}_{\mathcal{p}'}$. Thus $(\tilde{S}'_{\mathcal{p}'})^{\theta'}_{\mathcal{p}'}$ consists of two elliptic curves and a $(-2)$-curve. By Proposition 2.1, this is impossible. Hence $\theta'_{U'}$ is of type $(2, 1)$. (Even though $U \oplus E_8(2)$ is one of the exceptional lattices in the sense of [43, the proof of [43, Th. 2.3 (2)] remains valid. The fact that $\theta'_{U'}$ is of type $(2, 1)$ also follows from [43, Th. 2.3 (2)].)

(Step 2) Since the elliptic curve $T$ is fixed, there exists by Theorem 3.9 a constant $C$ such that

$$
\log \tau_{BCOV}(\tilde{X}(S', \theta', T)) = -4 \log \tau_M(S', \theta') + C \quad (\forall t \in U' \setminus \{\mathcal{p}'\}).
$$

Since $p'|_{U'}: (S'|_{U'}, \theta') \to U'$ is an ordinary singular family of 2-elementary $K3$ surfaces, we get by [40, Th. 6.5]

$$
\log \tau_M(S', \theta') = -\frac{1}{8} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0).
$$

By (3.30), (3.31), we get

$$
\log \tau_{BCOV}(\tilde{X}(S', \theta', T)) = \frac{1}{2} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0).
$$
Since \( \theta_{1,2} \) is of type (2,1), it follows from (3.32) and Theorem 3.11 that
\[
\log \tau_{\text{BCOV}}(\tilde{X}(S_i,\theta_i),T_t) = \log \tau_{\text{BCOV}}(\tilde{X}(S_i',\theta_i'),T_t) + O(\log(-\log |t|))
\]
\[
= \frac{1}{2} \log |t|^2 + O(\log(-\log |t|))
\]
as \( t \to 0 \). This completes the proof. \( \Box \)

**Theorem 3.13.** Let \( T \) be an elliptic curve. If \( \theta|_{T'} \) is of type \((0,3)\), then
\[
\log \tau_{\text{BCOV}}(\tilde{X}(S_i,\theta_i),T_t) = \frac{1}{2} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0).
\]

**Proof.** Set \( M := U(2) \oplus E_8(2) \). Then \( \overline{D}_{M+} \) is irreducible by [43 Prop. 11.6].

(Step 1) By [40 Th. 2.8], there exist a smooth projective threefold \( S' \) equipped with a holomorphic involution \( \theta': S' \to S' \), a pointed compact Riemann surface \( (B',p') \) equipped with a neighborhood \( U' \) of \( p' \), and a surjective holomorphic map \( p': S' \to B' \) with the following properties:

1. \( \theta' \) preserves the fibers of \( p' \).
2. \( p': (S'_U,\theta'|_{S'_U}) \to U' \) is an ordinary singular family of 2-elementary \( K3 \) surfaces of type \( M \).

Let \( \Delta' \) be the discriminant locus of \( p': S' \to B' \). Since \( M = U(2) \oplus E_8(2) \), \( \theta_{1,2}' \) has no fixed points on \( S'_{b'} \) for all \( b' \in B' \setminus \Delta' \) by Proposition 2.1. Hence \( (S')^{\theta'} \) has no horizontal components, which implies that \( \theta_{1,2}' := \theta'|_{S'_{b'}} \) is of type \((0,3)\). (Since the lattice \( U(2) \oplus E_8(2) \) is exceptional, [43 Th. 2.3 (2)] does not apply in this case.)

(Step 2) Let \( t \) be a local parameter of \( B' \) centered at \( p' \) and set \( S'_t := (p')^{-1}(t) \) and \( \theta'_t := \theta'|_{S'_t} \). Since the elliptic curve \( T \) is fixed, there exist by Theorem 3.9 a constant \( C \) such that

(3.33) \[
\log \tau_{\text{BCOV}}(\tilde{X}(S'_t,\theta'_t),T_t) = -4 \log \tau_M(S'_b,\theta'_b) + C \quad (\forall t \in U' \setminus \{p'\}).
\]

Since \( p'|_{U'}: (S'|_{U'},\theta') \to U' \) is an ordinary singular family of 2-elementary \( K3 \) surfaces, we get by [40 Th. 6.5]

(3.34) \[
\log \tau_M(S'_b,\theta'_b) = -\frac{1}{8} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0).
\]

By (3.33), (3.34), we get

(3.35) \[
\log \tau_{\text{BCOV}}(\tilde{X}(S_i,\theta_i'),T_t) = \frac{1}{2} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0).
\]

Since \( \theta_{1,2}' \) is of type \((0,3)\), it follows from (3.33) and Theorem 3.11 that

\[
\log \tau_{\text{BCOV}}(\tilde{X}(S_i,\theta_i),T_t) = \log \tau_{\text{BCOV}}(\tilde{X}(S_i',\theta_i'),T_t) + O(\log(-\log |t|))
\]
\[
= \frac{1}{2} \log |t|^2 + O(\log(-\log |t|))
\]
as \( t \to 0 \). This completes the proof. \( \Box \)

### 3.3.5. The singularity of BCOV invariants near the discriminant locus.

**Theorem 3.14.** Let \( M \) be a primitive 2-elementary Lorentzian sublattice of \( L_{K3} \). Let \( C \subset M_{M+} \) be a compact Riemann surface intersecting \( \overline{D}_{M+} \) transversally at sufficiently general point \( p \in C \cap \overline{D}_{M+} \). Let \( \gamma: (C,0) \to (C,p) \) be an isomorphism of germs. Let \( c \in X(1) \). Then the following holds

(3.36) \[
\log \tau_{\text{BCOV}}(\gamma(t),c) = \frac{1}{2} \log |t|^2 + O(\log(-\log |t|)) \quad (t \to 0),
\]
where $\tau_{BCOV}(\gamma(t), c)$ is the BCOV invariant of the Borcea-Voisin threefold associated with the 2-elementary $K3$ surface with period $\gamma(z)$ and the elliptic curve with period $c$.

**Proof.** If (3.36) holds for one isomorphism $\varpi: (C, 0) \to (C, p)$, then it holds for every isomorphism of germs $\gamma: (C, 0) \to (C, p)$. Hence it suffices to prove (3.36) for one particular isomorphism $\varpi: (C, 0) \to (C, p)$.

By [10] Th. 2.8, there is a family of 2-elementary $K3$ surfaces $\rho: (S, \theta) \to B$ of type $M$ (with degenerate fibers) over a pointed compact Riemann surface $(B, q)$ with the following properties:

1. Let $\Delta \subset B$ be the discriminant locus of $\rho: S \to B$ and let $\varpi: B \setminus \Delta \ni b \to \pi_M(S_b, \theta_b) \in \mathcal{M}_M^0$ be the period map for $\rho: (S, \theta) \to B$, where $S_b := \rho^{-1}(b)$ and $\theta_b := \theta|_{S_b}$. Then $\varpi$ extends to a surjective holomorphic map from $B$ to $C$ with $p = \varpi(q)$ such that $\varpi$ is non-degenerate at $q$. In particular, $\varpi: B \to \mathcal{M}_M$ intersects $T_{\mathcal{M}_M}$ transversally at $p = \varpi(q)$.

2. There is a neighborhood $U$ of $q$ in $B$ such that $\rho: (p^{-1}(U), \theta|_{p^{-1}(U)}) \to U$ is an ordinary singular family of 2-elementary $K3$ surfaces of type $M$.

Let $T$ be an elliptic curve with period $c$. Let $(U, t)$ be a coordinate neighborhood of $q$ in $B$. By choosing $U$ sufficiently small, it follows from (1) that $\varpi$ induces an isomorphism of germs $(U, q)$ and $(C, p)$. By construction,

$$\tau_{BCOV}(\varpi(t), c) = \tau_{BCOV}(\tilde{X}(S_t, \theta_t, T)).$$

Since $\rho: (p^{-1}(U), \theta|_{p^{-1}(U)}) \to U$ is an ordinary singular family of 2-elementary $K3$ surfaces, we get the following equation of currents on $\mathcal{M}_M \times X(1)$

$$-dd^c \log \tau_{BCOV} = (r(M) - 6)p^t \omega_M + 4pr^t \omega_A(M) + 12pr^t \omega_{hyp} - \frac{1}{2} \tau_{BCOV} \times X(1) = 0.$$ 

Since $\log(\tau_M^t \|\eta^4\|_2^2)$ satisfies the same equation of currents on $\mathcal{M}_M \times X(1)$ by [10] Th. 6.5, we get the following equation of currents on $\mathcal{M}_M \times X(1)$

$$-dd^c \log(\tau_{BCOV} / (\tau_M^t \|\eta^4\|_2^2)) = 0.$$ 

Hence $F^{M+t} = \log(\tau_{BCOV} / (\tau_M^t \|\eta^4\|_2^2))$ is a pluriharmonic function on $\mathcal{M}_M \times X(1)$. By Lemma 3.7, $F^{M+t}$ verifies the assumption of Lemma 3.8.

If $r(M) \leq 17$, then $F^{M+t}$ is a constant function on $\mathcal{M}_M \times X(1)$ by Lemma 3.8 (1). The assertion is proved when $r(M) \leq 17$.

Let $r(M) \geq 18$. Then $\mathcal{B}_M = \mathcal{M}_M^* \setminus \mathcal{M}_M^0$ is an irreducible divisor of $\mathcal{M}_M^*$ by [43] Prop. 11.7. By Lemma 3.6, $F^{M+t}$ is a pluriharmonic function on $\mathcal{M}_M \times X(1)$ with at most logarithmic singularity along $\mathcal{B}_M \times X(1)$. By Lemma 3.8 (2), $F^{M+t}$ is a constant function on $\mathcal{M}_M \times X(1)$. This proves the assertion when $r(M) \geq 18$. This completes the proof. □

4. **Equivariant BCOV torsion and its curvature**

4.1. **Equivariant analytic torsion.** Following Bismut [9], we recall equivariant analytic torsion. Let $(X, g_X)$ be a compact Kähler manifold. Let $G$ be a finite group of holomorphic automorphisms of $X$ preserving $g_X$. We write $\hat{G}$ for the set
of irreducible representations of $G$. The character of $W \in \hat{G}$ is denoted by $\chi_W$. Since $G$ preserves $g_X$, $G$ acts on $A^{p,q}(X)$, the space of $C^\infty (p,q)$-forms on $X$.

Let $\square_{p,q} = (\bar{\partial} + \partial)^2$ be the Laplacian of $(X, g_X)$ acting on $A^{p,q}(X)$. Let $\sigma(\square_{p,q})$ be the spectrum of $\square_{p,q}$ and let $E(\square_{p,q}; \lambda)$ be the eigenspace of $\square_{p,q}$ corresponding to the eigenvalue $\lambda \in \sigma(\square_{p,q})$. Since $G$ preserves $g_X$ and hence $G$ acts on $E(\square_{p,q}; \lambda)$, we get the orthogonal splitting

$$E(\square_{p,q}; \lambda) = \bigoplus_{W \in \hat{G}} \text{Hom}_G(W, E(\square_{p,q}; \lambda)) \otimes W.$$ 

Define the equivariant $\zeta$-function of $\square_{p,q}$ as the class function on $G$

$$\zeta_{p,q,G}(s)(g) := \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \text{Tr} [g|_{E(\square_{p,q}; \lambda)}]$$

$$= \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \sum_{W \in \hat{G}} \chi_W(g) \dim \text{Hom}_G(W, E(\square_{p,q}; \lambda)).$$

It is classical that $\zeta_{p,q,G}(s)(g)$ converges absolutely on the half-plane $\Re s > \dim X$, extends to a meromorphic function on $C$, and is holomorphic at $s = 0$. The 

\textit{equivariant analytic torsion} $T_{BCOV}$ of $(X, g_X, G)$ is the class function on $G$ defined as

$$\tau_G(X, g_X)(g) := \exp\{- \sum_{q \geq 0} (-1)^q \zeta_{0,q,G}(0)(g)\}$$

and the \textit{equivariant BCOV torsion} $(X, g_X)$ is the class function on $G$ defined as

$$T_{BCOV,G}(X, g_X)(g) := \exp\{- \sum_{p,q \geq 0} (-1)^{p+q} pq \zeta_{p,q,G}(0)(g)\}.$$

Set

$$Y := X/G$$

and let $p: X \to Y$ be the projection. Let $g_Y$ be the Kähler metric on $Y$ in the sense of orbifolds induced from $g_X$. Let $A^{p,q}(Y)$ be the space of $C^\infty (p,q)$-forms on $Y$ in the sense of orbifolds and let $\square_{p,q}^{orb} = (\bar{\partial} + \partial)^2$ be the Laplacian of $(Y, g_Y)$ acting on $A^{p,q}(Y)$. We can define the spectral zeta function $\zeta_{p,q}^{orb}(s)$ of $\square_{p,q}^{orb}$ in the same way as before. Then $\zeta_{p,q}^{orb}(s)$ extends to a meromorphic function on $C$ and is holomorphic at $s = 0$, so that the BCOV torsion $T_{BCOV}(Y, g_Y)$ of $(Y, g_Y)$ can be defined by the same formula as in Definition $1.2$.

We set

$$A^{p,q}(X)^G := \{ \varphi \in A^{p,q}(X); g^* \varphi = \varphi (\forall g \in G) \}$$

and $\square_{p,q}^G := \square_{p,q}|_{A^{p,q}(X)^G}$. Since $p^*: A^{p,q}(Y) \to A^{p,q}(X)^G$ is an isomorphism with $\square_{p,q}^G \circ p^* = p^* \circ \square_{p,q}^{orb}$, the two operators $\square_{p,q}^{orb}$ and $\square_{p,q}^G$ are isospectral. Hence

$$\zeta_{p,q}^{orb}(s) = \sum_{\lambda \in \sigma(\square_{p,q}) \setminus \{0\}} \lambda^{-s} \dim E(\square_{p,q}; \lambda)^G = \frac{1}{|G|} \sum_{g \in G} \zeta_{p,q,G}(s)(g).$$

By this expression, we have

$$T_{BCOV}(Y, g_Y)^{|G|} = \prod_{g \in G} T_{BCOV,G}(X, g_X)(g).$$
4.2. **Characteristic forms.** For a holomorphic Hermitian vector bundle \((E, h)\) over a complex manifold \(M\), we denote by \(c_p(E, h) \in A^{1,1}(M, \text{End}(E))\) the curvature form of \((E, h)\) with respect to the holomorphic Hermitian connection. The Chern form of \((E, h)\) is defined by

\[
c(E, h) = 1 + \sum_{p \geq 1} (-1)^p c_p(E, h) := \det \left( I_E - \frac{\sqrt{-1}}{2\pi} R(E, h) \right) \in \bigoplus_{p \geq 0} A^{p,p}(M),
\]

where \(c_p(E, h) \in A^{p,p}(M)\). The Todd and Chern character forms of \((E, h)\) are the differential forms on \(M\) defined as

\[
\text{Td}(E, h) := \det \left( \frac{\sqrt{-1}}{2\pi} R(E, h) \right) - \exp(-\frac{\sqrt{-1}}{2\pi} R(E, h)) \right) \in \bigoplus_{p \geq 0} A^{p,p}(M),
\]

\[
\text{ch}(E, h) := \text{Tr} \left[ \exp \left( \frac{\sqrt{-1}}{2\pi} R(E, h) \right) \right] \in \bigoplus_{p \geq 0} A^{p,p}(M).
\]

Let \(g: M \to M\) be a holomorphic isomorphism of order \(n\) and let \(M^g := \{ x \in M; g(x) = x \}\) be the set of fixed points of \(g\). Assume that \(g\) preserves \((E, h)\). We have the splitting of holomorphic vector bundles on \(M^g\)

\[(4.2)\]

\[
E|_{M^g} = \bigoplus_{\theta} E(\theta), \quad \theta \in \{ 0, \frac{2\pi}{n}, \ldots, \frac{2\pi(n-1)}{n} \}
\]

where \(E(\theta) := \{ v \in E; g(v) = e^{\sqrt{-1}\theta} v \}\) is the eigenbundle of \(E|_{M^g}\) with respect to the \(g\)-action. Moreover, the splitting \((4.2)\) is orthogonal with respect to the Hermitian metric \(h\). Let \(h_{E(\theta)} := h|_{E(\theta)}\).

Define the equivariant Todd and Chern character forms of \((E, h)\) with respect to the \(g\)-action as

\[
\text{Td}_g(E, h) := \text{Td}(E(0), h_{E(0)}) \prod_{\theta \neq 0} \text{Td} \left( \frac{\sqrt{-1}}{2\pi} R(E(\theta), h_{E(\theta)}) + \sqrt{-1}\theta \right),
\]

\[
\text{ch}_g(E, h) := \text{Tr} \left[ g \cdot \exp \left( \frac{\sqrt{-1}}{2\pi} R(E, h) \right) \right] = \sum_{\alpha} g \cdot \exp \left( \frac{\sqrt{-1}}{2\pi} R(E(\theta), h_{E(\theta)}) \right),
\]

where, for an \((n,n)\)-matrix \(A\),

\[
\frac{\text{Td}}{e} (A + \sqrt{-1}\theta) := \det \left[ \frac{I_n}{I_n - e^{-\sqrt{-1}\theta} \exp(-A)} \right].
\]

By definition, \(\text{Td}_g(E, h)\) and \(\text{ch}_g(E, h)\) are forms in \(\bigoplus_{p \geq 0} A^{p,p}(M^g)\).

4.3. **A variational formula for equivariant BCOV torsion.** In the rest of this section, as an application of the curvature theorem of Bismut-Gillet-Soulé [6] for Quillen metrics and its equivariant extension by Ma [22], we shall derive the curvature formula for the equivariant BCOV torsion.
4.3.1. Set up. Let 
\[ f : \mathcal{X} \to B \]
be a locally-projective smooth morphism from a complex manifold \( \mathcal{X} \) to a complex manifold \( B \) such that \( X_b := f^{-1}(b) \) is a connected *threefold*.

Let \( G \) be a finite *Abelian* group of automorphisms of \( \mathcal{X} \) such that \( f : \mathcal{X} \to B \) is \( G \)-equivariant with respect to the trivial \( G \)-action on \( B \). Hence \( G \) is an Abelian group of fiber-preserving automorphisms of the family \( f : \mathcal{X} \to B \). The order of \( g \in G \) is denoted by \( n_g = \text{ord}(g) \).

Set \( \mathcal{Y} := \mathcal{X}/G \), which is equipped with the projection 
\[ f : \mathcal{Y} \to B \]
induced from the one on \( \mathcal{X} \). Then the fibers of \( f : \mathcal{Y} \to B \) are the orbifolds \( Y_b := X_b/G, b \in B \).

For \( g \in G \), let \( \mathcal{X}^g \) (resp. \( X_b^g \)) be the set of fixed points of \( g \) (resp. \( X_b \)). Assume that for all \( g \in G \) and \( x \in X_b^g \),
\[ g \in \text{SL}(T_x X_b) \cong \text{SL}(\mathbb{C}^3). \]
If 1 is an eigenvalue of \( g \in \text{SL}(T_x X_b) \) and \( g \neq 1 \), then the possible multiplicity is 1 by [4.3], so that \( X_b^g \) consists of at most finitely many compact Riemann surfaces and isolated points. Since \( f : \mathcal{X} \to B \) is smooth, it follows from the \( G \)-equivariance of \( f \) that \( \mathcal{X}^g \) is a complex submanifold of \( \mathcal{X} \) flat over \( B \). If \( X_3^g \) is a component of \( \mathcal{X}^g \), then \( f : X_3^{g} \to B \) is a proper holomorphic submersion with connected fiber. We write \( \mathcal{X}^{g,\langle i \rangle} \) for the union of all components of \( \mathcal{X}^g \) of relative dimension \( i \) and set
\[ \mathcal{X}^{g,\langle i \rangle} := \coprod_{g \in G \setminus \{1\}} \mathcal{X}^{g,\langle i \rangle}. \]

Let \( \{ C_\lambda \}_{\lambda \in \Lambda} \) be the set of connected components of \( \mathcal{X}^{g,\langle 1 \rangle} \) and let \( \{ p_\alpha \}_{\alpha \in \Lambda} \) be the set of connected components of \( \mathcal{X}^{g,\langle 0 \rangle} \). For connected components \( C = C_\lambda \subset \mathcal{X}^{g,\langle 0 \rangle}, \lambda \in \Lambda \) and \( p = p_\alpha \subset \mathcal{X}^{g,\langle 0 \rangle} \), we set
\[ G_C = \{ g \in G; C \subset \mathcal{X}^g \} = \{ g \in G; g|_C = \text{id}_C \}, \]
\[ G_p = \{ g \in G; p \subset \mathcal{X}^g \} = \{ g \in G; g|_p = \text{id}_p \}. \]
Then \( G_C \) and \( G_p \) are subgroups of \( G \). Since \( G_C \subset \text{SL}(N_{C/X}) \) and since \( G_C \) is Abelian, it follows from the classification of finite subgroups of \( \text{SL}(\mathbb{C}^2) \) that \( G_C \) is a cyclic group. We set
\[ n_C := |G_C| \quad \text{and} \quad n_\lambda := |G_{C_\lambda}|. \]

4.3.2. Admissible action. For an Abelian subgroup \( \Gamma \) of \( \text{SL}(\mathbb{C}^3) \), define
\[ \Gamma^0 = \{ g \in \Gamma \setminus \{1\}; \det(g - 1_{\mathbb{C}^3}) \neq 0 \}. \]
Since \( \Gamma \) is Abelian, there exist \( \chi_1, \chi_2, \chi_3 \in \text{Hom}(\Gamma, \mathbb{C}^*) \) such that
\[ \Gamma \cong \{ \text{diag}(\chi_1(g), \chi_2(g), \chi_3(g)); g \in \Gamma \}, \quad \chi_1 \chi_2 \chi_3 = 1. \]

**Definition 4.1.** Let \( \Gamma \) be an Abelian subgroup of \( \text{SL}(\mathbb{C}^3) \). If \( \Gamma^0 = \emptyset \) or if
\[ \sum_{g \in \Gamma^0} \frac{\chi_1(g)}{(1 - \chi_1(g))^2} = \sum_{g \in \Gamma^0} \frac{\chi_2(g)}{(1 - \chi_2(g))^2} = \sum_{g \in \Gamma^0} \frac{\chi_3(g)}{(1 - \chi_3(g))^2}, \]
Lemma 4.3. The eigenvalues of the $X$ with respect to the $N^K$ differentials of the family $p$ (resp. $p_k$) where $g$ we get

Assume $g$ we get

Following standard relations of Chern forms:

Proof. For simplicity, write $h^c$ the Hermitian metrics induced from $g$ we get

Then the vector bundles $\Theta_X$ be an admissible Abelian subgroup of $SL(m)$, the $G$-action on $X$ is said to be admissible.

4.3.3. Some vector bundles and their characteristic forms. Let $g \in G$. Let $\Theta_{X/B}$ (resp. $\Theta_{X/B}$) be the relative holomorphic tangent bundle of the family $p: X \to B$ (resp. $p: X^g \to B$). Let $\Omega^1_{X/B}$ (resp. $\Omega^1_{X/B}$) be the vector bundle of relative Kähler differentials of the family $p: X \to B$ (resp. $p: X^g \to B$). Let $N_{X^g/X}$ (resp. $N^*_{X^g/X}$) be the normal (resp. conormal) bundle of $X^g$ in $X$. We have

Since $\Theta_{X^g/B}$ (resp. $N_{X^g/X}$) is the (resp. union of) eigenbundle(s) of $\Theta_{X/B}|_{X^g}$ with respect to the $g$-action corresponding to the eigenvalue 1 (resp. $\neq 1$), we get

Similarly, we have the splitting

Lemma 4.3. The eigenvalues of the $g$-action on $N_{X^g/X}|_{X^g(1)}$ are of the form

where $k \in \{1, \ldots, n_g - 1\}$ may depend on the component of $X^g(1)$.

Proof. For simplicity, write $n$ for $n_g$. Set $\zeta_n := \exp(2\pi \sqrt{-1}/n)$. Since $g$ has order $n$ and $g \in SL(N_{X^g/X}|_{X^g(1)})$, its eigenvalues are of the form $\{e^{\lambda_k} \zeta_n^{-k}, 0 \leq k < n\}$. Assume $g^m = 1$ on $N_{X^g/X}|_{X^g(1)}$ for some $1 \leq m < n$. Since $g^m = 1$ on $\Theta_{X|X^g(1)}$, we get $g^m = 1$ on $X$, so that $m = nl$ for some $l \in \mathbb{Z}$. This contradicts the choice of $m$. Hence $g^m \neq 1$ on $N_{X^g/X}|_{X^g(1)}$ for any $1 \leq m < n$, which implies $(n, k) = 1$.

Let $h_{X/B}$ be an $\iota$-invariant Hermitian metric on $\Theta_{X/B}$, which is fiberwise Kähler. Then the vector bundles $\Theta_{X^g/B}$, $\Omega^1_{X^g/B}$, $\Omega^p_{X^g/B}$, $N_{X^g/X}$, $N^*_{X^g/X}$ are equipped with the Hermitian metrics induced from $h_{X/B}$, which are denoted by $h_{X^g/B}$, $h_{\Omega^1_{X^g/B}}$, $h_{\Omega^p_{X^g/B}}$, $h_{N_{X^g/X}}$, $h_{N^*_{X^g/X}}$, respectively. Then the splittings (4.3), (4.4) are orthogonal with respect to the metrics $h_{X/B}$ and $h_{\Omega^1_{X^g/B}}$, respectively.

In what follows, we write $c_i(X/B)$, $c_i(X^g/B)$, $c_i(\Omega^p_{X/B})$, $c_i(\Omega^p_{X^g/B})$, $c_i(N_{X^g/X})$, $c_i(N^*_{X^g/X})$ for the $i$-th Chern forms of $(\Theta_{X/B}, h_{X/B})$, $(\Theta_{X^g/B}, h_{X^g/B})$, $(\Omega^p_{X/B}, h_{\Omega^p_{X/B}})$, $(\Omega^p_{X^g/B}, h_{\Omega^p_{X^g/B}})$, $(N_{X^g/X}, h_{N_{X^g/X}})$, $(N^*_{X^g/X}, h_{N^*_{X^g/X}})$, respectively. We have the following standard relations of Chern forms:

$$c_1(X/B)|_{X^g} = c_1(X^g/B) + c_1(N_{X^g/X}),$$
By (4.10), we get

\[ c_1(\Omega^1_{X/B}) = -c_1(X/B), \quad c_1(\Omega^1_{X^s/B}) = -c_1(X^s/B), \]

\[ c_1(N_{X^s/X}^*) = -c_1(N_{X^s/X}), \quad c_2(N_{X^s/X}^*) = c_2(N_{X^s/X}). \]

4.3.5. **Lemma 4.4.** The following identity of differential forms on \( X \) holds:

\[
[Td(\Theta_{X/B}, h_{X/B})] \sum_{q \geq 0} (-1)^q q \, ch(\Omega^q_{X/B}, h_{\Omega^q_{X/B}})]^{(\leq 8)}
\]

\[
= -c_2(X/B) + \frac{3}{2} c_3(X/B) - \frac{1}{12} c_1(X/B) c_3(X/B).
\]

**Proof.** See [4, p.374]. \( \square \)

4.3.5. **The equivariant characteristic forms: the case of relative dimension 1.** Let \( C \) be a component of \( X^{(1)}_C \). For \( b \in B \), we set

\[ C_b := C \cap X_b. \]

Let \( g \in G_C \setminus \{1\} \). Then \( n_g | n_C \). By Lemma 4.3 there exists \( k_g \) with \( (k_g, n_g) = 1 \) such that the \( g \)-action on \( N_{C/X} \) induces the splitting

\[ N_{C/X} = N_{C/X}(\theta) \oplus N_{C/X}(\theta), \quad \theta = \frac{2k_g \pi}{n_g}. \]

We have the corresponding splitting

\[ N_{C/X}^* = N_{C/X}^*(\theta) \oplus N_{C/X}^*(-\theta) \]

such that

\[ N_{C/X}^*(\theta) = (N_{C/X}(\theta))^*, \quad N_{C/X}^*(-\theta) = (N_{C/X}(\theta))^*. \]

Set

\[ \lambda_1 := c_1(N_{C/X}(\theta), h_{N_{C/X}(\theta)}), \quad \lambda_2 := c_1(N_{C/X}(\theta), h_{N_{C/X}(\theta)}). \]

Then

\[ \lambda_1 + \lambda_2 = c_1(N_{C/X}, h_{N_{C/X}}) = c_1(X/B)_{|C} - c_1(C/B). \]

By (4.10), we get

\[ c_1(N_{C/X}^*(\theta), h_{N_{C/X}(\theta)}) = -\lambda_2, \quad c_1(N_{C/X}^*(-\theta), h_{N_{C/X}(\theta)}) = -\lambda_1. \]

Following Bershadsky-Cecotti-Ooguri-Vafa [4, p.374], we compute the characteristic form

\[
Td_g(\Theta_{X/B}, h_{X/B})] \sum_{q \geq 0} (-1)^q q \, ch_q(\Omega^q_{X/B}, h_{\Omega^q_{X/B}})]_{|C}
\]

for \( g \in G_C \setminus \{1\} \). Set

\[ \zeta_g := e^{\sqrt{-1} \theta}. \]

\[ Td_g(\Theta_{X/B}, h_{X/B})_{|C} = Td(\Theta_{X^s/B}, h_{X^s/B})_{|C} \cdot \frac{1}{1 - \zeta_g e^{-\lambda_1}} \cdot \frac{1}{1 - \zeta_g e^{-\lambda_2}}. \]

**Proof.** The result follows from the definition of the equivariant Todd form and the relation (4.12). \( \square \)
Lemma 4.6. The following equality of differential forms on $C$ holds:

$$
\sum_{p \geq 0} (-1)^p p \text{ch}_g(\Omega_{X/B}^p, h_{\Omega_{X/B}^p})|_c = \frac{d}{dt} \bigg|_{t=1} (1-t e^{-c_1(X/B)})(1-t c_g^{-1} e^{-\lambda_1})(1-t c_g e^{-\lambda_2}).
$$

Proof. Since

$$
\Omega_{X/B}^1|_c = \Omega_{C/B}^1 \oplus N_{C/X}^* = \Omega_{C/B}^1 \oplus N_{C/X}^*(\theta) \oplus N_{C/X}^*(-\theta),
$$

it follows from the definition of the equivariant Chern character and (4.12) that

$$
\text{ch}_g(\Omega_{X/B}^1, h_{X/B})|_c = \text{ch}(\Omega_{c/B}^1, h_{\Omega_{c/B}^1}) + e^{-\sqrt{-1} \theta} \text{ch}(N_{C/X}^*(\theta), h_{N_{C/X}^*(\theta)})
\quad + e^{-\sqrt{-1} \theta} \text{ch}(N_{C/X}^*(-\theta), h_{N_{C/X}^*(-\theta)})
\quad = e^{-c_1(C/B)} + \zeta_g e^{-\lambda_2} + \zeta_g^{-1} e^{-\lambda_1}.
$$

Since

$$
\Omega_{X/B}^2|_c = \left(\Omega_{C/B}^1 \otimes N_{C/X}^*(\theta)\right) \oplus \left(\Omega_{C/B}^1 \otimes N_{C/X}^*(-\theta)\right) \oplus \det N_{C/X}^*,
$$

we get

$$
\text{ch}_g(\Omega_{X/B}^2, h_{X/B})|_c = e^{-\sqrt{-1} \theta} \text{ch}(\Omega_{C/B}^1 \otimes N_{C/X}^*(\theta), h_{\Omega_{C/B}^1} \otimes h_{N_{C/X}^*(\theta)})
\quad + e^{-\sqrt{-1} \theta} \text{ch}(\Omega_{C/B}^1 \otimes N_{C/X}^*(-\theta), h_{\Omega_{C/B}^1} \otimes h_{N_{C/X}^*(-\theta)})
\quad + \text{ch}(\det N_{C/X}^*, h_{\det N_{C/X}^*})
\quad = \zeta_g e^{-c_1(C/B)} - \lambda_2 + \zeta_g^{-1} e^{-c_1(C/B)} - \lambda_1 + e^{-\lambda_1 - \lambda_2}.
$$

Since

$$
\Omega_{X/B}^3|_c = \Omega_{X/B}|_c \otimes \det N_{C/X}^*,
$$

we get

$$
\text{ch}_g(\Omega_{X/B}^3, h_{X/B})|_c = \text{ch}(\Omega_{C/B}^1 \otimes \det N_{C/X}^*, h_{\Omega_{C/B}^1} \otimes h_{\det N_{C/X}^*})
\quad = \text{ch}(\Omega_{C/B}^1, h_{\Omega_{C/B}^1}) \text{ch}(\det N_{C/X}^*, h_{N_{C/X}^*})
\quad = e^{-c_1(C/B)} - \lambda_1 - \lambda_2.
$$

By (4.13), (4.14), (4.15), we get

$$
\sum_{p \geq 0} (-1)^p p \text{ch}_g(\Omega_{X/B}^p, h_{\Omega_{X/B}^p})|_c
\quad = -(e^{-c_1(C/B)} + \zeta_g e^{-\lambda_2} + \zeta_g^{-1} e^{-\lambda_1})
\quad + 2(\zeta_g e^{-c_1(C/B)} - \lambda_2 + \zeta_g^{-1} e^{-c_1(C/B)} - \lambda_1 + e^{-\lambda_1 - \lambda_2}) - 3 e^{-c_1(C/B)} - \lambda_1 - \lambda_2
\quad = \frac{d}{dt} \bigg|_{t=1} (1-t e^{-c_1(X/B)})(1-t c_g^{-1} e^{-\lambda_1})(1-t c_g e^{-\lambda_2}).
$$

This completes the proof. □
Lemma 4.7. The following equality of differential forms on $\mathcal{C}_\lambda$ holds:

$$
\left[ \text{Td}_g(\theta_X/B, h_X/B) \sum_{p \geq 0} (-1)^p p \text{ch}_g(\Omega^p_X/B, h_{\Omega^p_X/B}) \right]^{(\leq 4)} =
- 1 + \frac{3}{2} c_1(C/B) + \frac{\zeta_g}{(1 - \zeta_g)^2} c_1(X/B)c_1(C/B) - \left\{ \frac{1}{12} + \frac{\zeta_g}{(1 - \zeta_g)^2} \right\} c_1(C/B)^2.
$$

Proof. For simplicity, write $\zeta$ for $\zeta_g$. By Lemmas 4.5 and 4.6, we get

$$
\left[ \text{Td}_g(\theta_X/B, h_X/B) \sum_{p \geq 0} (-1)^p p \text{ch}_g(\Omega^p_X/B, h_{\Omega^p_X/B}) \right]^{(\leq 4)} =
\frac{d}{dt}\bigg|_{t=1} \left[ c_1(X/B)(1 - t e^{-c_1(X/B)}) (1 - t \zeta e^{-\lambda_1})(1 - t \zeta e^{-\lambda_2}) \right]^{(\leq 4)}
= \left[ \frac{c_1(X/B)}{1 - e^{-c_1(X/B)}} \right]^{(\leq 4)} - \left[ c_1(X/B) \left\{ \frac{\zeta e^{-\lambda_1}}{1 - \zeta e^{-\lambda_1}} + \frac{\zeta e^{-\lambda_2}}{1 - \zeta e^{-\lambda_2}} \right\} \right]^{(\leq 4)}
= - \left( 1 - \frac{1}{2} c_1(X/B) + \frac{1}{12} c_1(X/B)^2 \right)
- c_1(X/B) \left\{ \frac{1}{1 - \zeta} - \frac{\zeta}{(1 - \zeta)^2} \lambda_1 \right\} - c_1(X/B) \left\{ \frac{\zeta}{1 - \zeta} - \frac{\zeta}{(1 - \zeta)^2} \lambda_2 \right\}
= -1 + \frac{3}{2} c_1(C/B) - \frac{1}{12} c_1(C/B)^2 + \frac{\zeta}{(1 - \zeta)^2} c_1(X/B)c_1(N^*_\chi, h_{N^*_\chi})
= -1 + \frac{3}{2} c_1(C/B) + \frac{\zeta}{(1 - \zeta)^2} c_1(X/B)c_1(C/B) - \left\{ \frac{1}{12} + \frac{\zeta}{(1 - \zeta)^2} \right\} c_1(C/B)^2,
$$

where we used (4.11) to get the 4-th and 5-th equalities. \hfill \Box

4.3.6. The equivariant characteristic forms: the case of relative dimension 0. Let $p$ be a component of $\mathcal{X}^{(1)}$. Then

$$
p: B \ni b \to p \cap X_b \in X_b
$$
is regarded as a section of the map $f: \mathcal{X} \to B$.

Let $g \in G_p \setminus \{1\}$. On $p$, $N_{p/\chi}$ splits into the eigenbundles of the $g$-action

$$
N_{p/\chi} = N_{p/\chi}(\theta_1) \oplus N_{p/\chi}(\theta_2) \oplus N_{p/\chi}(\theta_3), \quad \theta_i = \frac{2\pi k_i}{n_g}.
$$

We get the corresponding splitting of $N^*_{p/\chi}$

$$
N^*_{p/\chi} = N^*_{p/\chi}(\theta_1) \oplus N^*_{p/\chi}(\theta_2) \oplus N^*_{p/\chi}(\theta_3)
$$
such that $N^*_{p/\chi}(\theta_i) = (N_{p/\chi}(\theta_i))^*$. For $i = 1, 2, 3$, set

$$
\lambda_i := c_1(N_{p/\chi}(\theta_i), h_{N_{p/\chi}(\theta_i)}).
$$

Then we have the following relations

$$
c_1(N^*_{p/\chi}(\theta_1), h_{N^*_{p/\chi}(\theta_1)}) = -\lambda_1,
$$

(4.16) \hspace{1cm} \lambda_1 + \lambda_2 + \lambda_3 = c_1(N_{p/\chi}, h_{N_{p/\chi}}) = c_1(X/B)|_p.

Set

$$
\zeta_i := e^{\sqrt{-1}\theta_i}.
$$
Lemma 4.8. The following equality of differential forms on \( p \) holds:

\[
\text{td}_p(\Theta_{X/B}, h_{X/B})|_p = \prod_{i=1}^{3} \frac{1}{1 - \zeta_i^{-1} e^{-\lambda_i}}.
\]

Proof. The result follows from the definition of the equivariant Todd form. \( \square \)

Lemma 4.9. The following equality of differential forms on \( p \) holds:

\[
\sum_{p \geq 0} (-1)^p p \text{ch}_p(\Omega^p_{X/B}, h_{\Omega^p_{X/B}})|_p = \frac{d}{dt} \left| \prod_{i=1}^{3} (1 - t \zeta_i^{-1} e^{-\lambda_i}) \right|.
\]

Proof. Since \( \Omega^1_{X/B}|_p = N_p^+(-\theta_1) \oplus N_p^+(\theta_2) \oplus N_p^+(\theta_3) \), it follows from the definition of the equivariant Chern character form that

\[
(4.17) \quad \text{ch}_p(\Omega^1_{X/B}, h_{X/B})|_p = \sum_{i=1}^{3} \zeta_i^{-1} e^{-\lambda_i}.
\]

Since \( \Omega^2_{X/B}|_p = \bigoplus_{i \neq j} N_p^+(\theta_i) \otimes N_p^+(\theta_j) \), we get

\[
(4.18) \quad \text{ch}_p(\Omega^2_{X/B}, h_{X/B})|_p = \sum_{i \neq j} e^{-\sqrt{-1}(\theta_i + \theta_j)} e^{-\lambda_i - \lambda_j} = \sum_{i \neq j} \zeta_i^{-1} \zeta_j^{-1} e^{-\lambda_i - \lambda_j},
\]

Since \( \Omega^3_{X/B}|_p = \det N_p^+ \), we get

\[
(4.19) \quad \text{ch}_p(\Omega^3_{X/B}, h_{X/B})|_p = e^{-\lambda_1 - \lambda_2 - \lambda_3}.
\]

By \( (\ref{4.17}) \), \( (\ref{4.18}) \), \( (\ref{4.19}) \), we get the result. \( \square \)

Lemma 4.10. The following equality of differential forms on \( p \) holds:

\[
\left[ \text{td}_p(\Theta_{X/B}, h_{X/B}) \sum_{p \geq 0} (-1)^p p \text{ch}_p(\Omega^p_{X/B}, h_{\Omega^p_{X/B}})|_p \right]^{(\leq 2)} = \sum_{i=1}^{3} \left( \frac{1}{1 - \zeta_i} + \frac{\zeta_i}{(1 - \zeta_i)^2} \lambda_i \right).
\]

Proof. By Lemmas 4.8 and 4.9 we get

\[
\left[ \text{td}_p(\Theta_{X/B}, h_{X/B}) \sum_{p \geq 0} (-1)^p p \text{ch}_p(\Omega^p_{X/B}, h_{\Omega^p_{X/B}})|_p \right]^{(\leq 2)} = \left[ \prod_{i=1}^{3} \frac{1 - t \zeta_i^{-1} e^{-\lambda_i}}{1 - \zeta_i^{-1} e^{-\lambda_i}} \right]^{(\leq 2)} = \left[ - \sum_{i=1}^{3} \frac{\zeta_i^{-1} e^{-\lambda_i}}{1 - \zeta_i^{-1} e^{-\lambda_i}} \right]^{(\leq 2)}
\]

\[
= \sum_{i=1}^{3} \left( \frac{1}{1 - \zeta_i} + \frac{\zeta_i}{(1 - \zeta_i)^2} \lambda_i \right).
\]

This proves the lemma. \( \square \)

Lemma 4.11. If the \( G \)-action on \( X \) is admissible, then for any component \( p \subset X^{(0)} \)

\[
\left[ \sum_{g \in G_p} \text{td}_g(\Theta_{X/B}, h_{X/B}) \sum_{p \geq 0} (-1)^p p \text{ch}_p(\Omega^p_{X/B}, h_{\Omega^p_{X/B}})|_p \right]^{(1,1)} = \delta(G_p) c_1(X/B)|_p.
\]
Proposition 4.12.
If the \( R \)-\( p \)-equivariant holomorphic vector bundle on \( B \) is denoted by \( \lambda \), then for \( g \) such that \( \lambda \) is admissible, the following equality holds

\[
\sum_{g \in G_p} \chi(g)^{(i)} \log T_{\text{BCOV}}(\mathcal{X}/B)(g) = \delta(G_p) \sum_{i=1}^{3} \frac{\chi^{(i)}(g)}{(1 - \chi^{(i)}(g))^2} \lambda_i = \delta(G_p) (\lambda_1 + \lambda_2 + \lambda_3),
\]

we get the result. \( \square \)

4.3.7. The curvature of the orbifold BCOV torsion. Let \( T_{\text{BCOV},G}(\mathcal{X}/B)(g) \) and \( T_{\text{BCOV}}(\mathcal{Y}/B) \) be the \( C^\infty \) functions on \( B \) defined as

\[
T_{\text{BCOV},G}(\mathcal{X}/B)(g) := T_{\text{BCOV}}(\mathcal{X}/B|_{G}))(g),
\]

\[
\log T_{\text{orb}}(\mathcal{Y}/B)(b) := \log T_{\text{BCOV}}(Y_b/G, h_{\mathcal{X}/B}(Y_b)) = \frac{1}{|G|} \sum_{g \in G} \log T_{\text{BCOV},G}(X_b, h_{\mathcal{X}/B}(X_b))(g)
\]

for \( b \in B \). Hence

\[
T_{\text{BCOV},G}(\mathcal{Y}/B)^{|G|} = \prod_{g \in G} T_{\text{BCOV},G}(\mathcal{X}/B)(g).
\]

By the \( G \)-equivariance of \( f: \mathcal{X} \to B \), the direct image sheaf \( R^q f_* \Omega^p_{\mathcal{X}/B} \) is a \( G \)-equivariant holomorphic vector bundle on \( B \). For \( W \in \hat{G} \), set

\[
(R^q f_* \Omega^p_{\mathcal{X}/B})_W := \text{Hom}_G(W, R^q f_* \Omega^p_{\mathcal{X}/B}) \otimes W
\]

and let

\[
(4.20) \quad R^q f_* \Omega^p_{\mathcal{X}/B} = \bigoplus_{W \in \hat{G}} (R^q f_* \Omega^p_{\mathcal{X}/B})_W
\]

be the isotypical decomposition of the vector bundle \( R^q f_* \Omega^p_{\mathcal{X}/B} \). We set

\[
(R^q f_* \Omega^p_{\mathcal{X}/B})^G := \{ \varphi \in R^q f_* \Omega^p_{\mathcal{X}/B}; g \cdot \varphi = \varphi (\forall g \in G) \}.
\]

The \( L^2 \)-metrics on \( R^q f_* \Omega^p_{\mathcal{X}/B} \), \( (R^q f_* \Omega^p_{\mathcal{X}/B})_W \), \( (R^q f_* \Omega^p_{\mathcal{X}/B})^G \) with respect to the metric \( h_{\mathcal{X}/B} \) are denoted by \( h_{L^2} \).

**Proposition 4.12.** If the \( G \)-action on \( \mathcal{X} \) is admissible, then the following equality of \( (1, 1) \)-forms on \( B \) holds

\[
- \ddc \log T_{\text{BCOV},G}(\mathcal{Y}/B) = - \frac{1}{12|G|} f_* \{ c_{1}(\mathcal{X}/B) c_{3}(\mathcal{X}/B) \} - \frac{1}{12|G|} \sum_{x \in A} (n^2_x - 1) f_* \{ c_{1}(\mathcal{X}/B) c_{1}(\mathcal{X}/B) \}
\]

\[
+ \frac{1}{12|G|} \sum_{x \in A} (n^2_x - n_x) f_* \{ c_{1}(\mathcal{X}/B) c_{3}(\mathcal{X}/B) \}^2 + \frac{1}{|G|} \sum_{x \in A} \delta(G_p, x) f_* \{ c_{1}(\mathcal{X}/B) \}
\]

\[- \sum_{p,q} (-1)^{p+q} p c_{1} \left( (\det R^q f_* \Omega^p_{\mathcal{X}/B})^G, h_{L^2} \right).\]
Proof. By the curvature formulae for Quillen metrics \([6]\) and equivariant Quillen metrics \([22]\) applied to the \(G\)-equivariant morphism \(f: \mathcal{X} \to B\) equipped with the \(G\)-equivariant vector bundles \(\Omega_{\mathcal{X}/B}^p\) \((p \geq 0)\), we get

\[
- dd^c \log T^\text{orb}_{BCOV}(\mathcal{Y}/B) = \frac{1}{|G|} \sum_{g \in G} - dd^c \log T_{BCOV,C}(\mathcal{X}/B)(g)
\]

(4.21)

\[
\sum_{p \geq 0} (-1)^{p+q} \frac{1}{|G|} \sum_{W \in G} \sum_{g \in G} \frac{\chi_W(g)}{\text{rk} W} c_1(\det(R^q f_* \Omega_{\mathcal{X}/B}^p W, h_{L^2}).
\]

For \(g \in G \setminus \{1\}\), we have the decomposition into connected components

\[
\chi^g = \left( \Pi_{\lambda \in \Lambda; \ g \in G_{c_\lambda} \setminus \{1\}} \chi^\lambda \left( \Pi_{\alpha \in A; \ g \in G_{c_a}^\alpha} \right) p_{\alpha} \right).
\]

Hence we get by (4.21)

(4.22)

\[
- dd^c \log T^\text{orb}_{BCOV}(\mathcal{Y}/B)
\]

\[
= \frac{1}{|G|} \left\{ f_{\lambda}(\Theta_{\mathcal{X}/B}, h_{\mathcal{X}/B}) \sum_{p \geq 0} (-1)^{p+q} \frac{1}{|G|} \sum_{W \in G} \sum_{g \in G} \frac{\chi_W(g)}{\text{rk} W} c_1(\det(R^q f_* \Omega_{\mathcal{X}/B}^p W, h_{L^2}) \right\}^{(1,1)}
\]

\[
+ \frac{1}{|G|} \sum_{\lambda \in \Lambda} \sum_{q \in G_{c_\lambda} \setminus \{1\}} \left\{ f_{\lambda}(\Theta_{\mathcal{X}/B}, h_{\mathcal{X}/B}) \sum_{p \geq 0} (-1)^{p+q} \frac{1}{|G|} \sum_{W \in G} \sum_{g \in G} \frac{\chi_W(g)}{\text{rk} W} c_1(\det(R^q f_* \Omega_{\mathcal{X}/B}^p W, h_{L^2}) \right\}^{(1,1)}
\]

\[
+ \frac{1}{|G|} \sum_{a \in A} \sum_{g \in G_{c_a}} \left\{ \frac{1}{12} \sum_{q \in G_{c_a}} \left\{ f_{\lambda}(\Theta_{\mathcal{X}/B}, h_{\mathcal{X}/B}) \sum_{p \geq 0} (-1)^{p+q} \frac{1}{|G|} \sum_{W \in G} \sum_{g \in G} \frac{\chi_W(g)}{\text{rk} W} c_1(\det(R^q f_* \Omega_{\mathcal{X}/B}^p W, h_{L^2}) \right\}^{(1,1)}
\]

(4.11)

Substituting the equalities in Lemmas \([4.4, 4.7, 4.11]\) into (4.22), we get

(4.23)

\[
- dd^c \log T^\text{orb}_{BCOV}(\mathcal{Y}/B) = - \frac{1}{12 |G|} f_{\lambda}(\chi_{\mathcal{X}/B} c_3(\mathcal{X}/B))
\]

\[
+ \frac{1}{|G|} \sum_{\lambda \in \Lambda} \sum_{g \in G_{c_\lambda} \setminus \{1\}} \frac{\zeta_g}{(1 - \zeta_g)^2} f_{\lambda}(\chi_{\mathcal{X}/B} c_1(\mathcal{C}_{\lambda}/B))
\]

\[
- \frac{1}{|G|} \sum_{\lambda \in \Lambda} \sum_{g \in G_{c_\lambda} \setminus \{1\}} \left\{ \frac{1}{12} + \frac{\zeta_g}{(1 - \zeta_g)^2} \right\} f_{\lambda}(\chi_{\mathcal{C}_{\lambda}/B)^2)}
\]

\[
+ \frac{1}{|G|} \sum_{a \in A} \sum_{g \in G_{c_a}} \delta(G_{c_a}) \ f_{\lambda}(\chi_{\mathcal{X}/B} c_1(\mathcal{X}/B))
\]

\[
- \sum_{p \neq q} (-1)^{p+q} p c_1(\det(R^q f_* \Omega_{\mathcal{X}/B}^p W, h_{L^2}).
\]

Since \(G_{c_\lambda}\) is a cyclic group of order \(n_{\lambda}\), we get the result from (4.23) by substituting the following classical formula for the Dedekind sum into (4.23)

(4.24)

\[
\sum_{k=1}^{n_{\lambda}-1} \frac{\zeta_k}{(1 - \zeta_k)^2} = - \frac{n_{\lambda}^2 - 1}{12},
\]
where \( \zeta = \exp(2\pi \sqrt{-1}m/n) \), \((m,n) = 1\).

4.4. The Torelli map. For a component \( C_{\lambda} \subset X^{(1)}_K \), we set \( C_{b,\lambda} := C_{\lambda} \cap X_b \) and \( h_\lambda := \dim H^0(C_{b,\lambda}, K_{C_{b,\lambda}}) \). The Torelli map for the family \( f: C_{\lambda} \to B \) is defined by

\[
J_{C_{\lambda}/B} : B \ni b \to [\text{Jac}(C_{b,\lambda})] \in \mathcal{A}_{h_\lambda},
\]

where \([\text{Jac}(C_{b,\lambda})]\) denotes the isomorphism class of the Jacobian variety \( \text{Jac}(C_{b,\lambda}) \) of \( C_{b,\lambda} \).

Let \( h_{L^2} = h_{f_*K_{C_{\lambda}/B}} \) be the \( L^2 \)-metric on \( f_*K_{C_{\lambda}/B} \), which is independent of the choice of \( h_{X/B} \). Let \( \varphi_1^{(\lambda)}, \ldots, \varphi_{b_1}^{(\lambda)} \in H^0(B, f_*K_{C_{\lambda}/B}) \) be a basis of \( f_*K_{C_{\lambda}/B} \) as a free \( O_B \)-module. For \( b \in B \), set \( \varphi_i^{(\lambda)}(b) := \varphi_i^{(\lambda)}|_{C_{b,\lambda}} \). Then \( \{ \varphi_1^{(\lambda)}(b), \ldots, \varphi_{b_1}^{(\lambda)}(b) \} \) is a basis of \( H^0(C_{b,\lambda}, K_{C_{b,\lambda}}) \) such that

\[
\left\| \varphi_1^{(\lambda)}(b) \wedge \cdots \wedge \varphi_{b_1}^{(\lambda)}(b) \right\|_2^2 = \det \left( \frac{-1}{2\pi} \int_{C_{b,\lambda}} \varphi_i^{(\lambda)}(b) \wedge \varphi_j^{(\lambda)}(b) \right)_{1 \leq i, j \leq b_1}.
\]

By this expression and the definition of \( \omega_{A_{b_1}} \), we get the following equality of \((1,1)\)-forms on \( B \)

\[
(4.25) \quad c_1(f_*K_{C_{\lambda}/B}, h_{L^2}) = J^*_{C_{\lambda}/B} \omega_{A_{b_1}}.
\]

Let \( \tau(C_{\lambda}/B) \) and \( \text{Vol}(C_{\lambda}/B) \) be the \( C^\infty \) function on \( B \) defined as

\[
\tau(C_{\lambda}/B)(b) := \tau(C_{b,\lambda}, h_{X/B}|_{C_{b,\lambda}}),
\]

\[
\text{Vol}(C_{\lambda}/B)(b) := \text{Vol}(C_{b,\lambda}, h_{X/B}|_{C_{b,\lambda}}) = \frac{1}{2\pi} \int_{C_{b,\lambda}} \gamma_{X/B}|_{C_{b,\lambda}}
\]

for \( b \in B \), where \( \gamma_{X/B} \) is the Kähler form associated with \( h_{X/B} \).

Proposition 4.13. The following equality of \((1,1)\)-forms on \( B \) holds

\[
-dd^c \log \tau(C_{\lambda}/B) = [f_*\text{Td}(\Theta_{C_{\lambda}/B}, h_{X/B}|_{\Theta_{C_{\lambda}/B}})]^{(1,1)} + dd^c \log \text{Vol}(C_{\lambda}/B) - J^*_{C_{\lambda}/B} \omega_{A_{b_1}}.
\]

Proof. By the curvature formula for Quillen metrics \([\text{6}]\) applied to the family of curves \( f: C_{\lambda} \to B \), we get

\[
(4.26) \quad -dd^c \log \tau(C_{\lambda}/B) = [f_*\text{Td}(\Theta_{C_{\lambda}/B}, h_{X/B}|_{\Theta_{C_{\lambda}/B}})]^{(1,1)} - \sum_q (-1)^q c_1 \left( R^q f_*\mathcal{O}_{C_{\lambda}}, h_{L^2} \right)
\]

\[
= [f_*\text{Td}(\Theta_{C_{\lambda}/B}, h_{X/B}|_{\Theta_{C_{\lambda}/B}})]^{(1,1)} + dd^c \log \text{Vol}(C_{\lambda}/B)
\]

\[
- J^*_{C_{\lambda}/B} \omega_{A_{b_1}},
\]

where the last equality follows from the Serre duality and \((4.25)\). \(\square\)
Theorem 4.14. If the $G$-action on $X$ is admissible, then the following equality of $(1,1)$-forms on $B$ holds

\[-dd^c \log T^\text{orb}_{\text{BCOV}}(X/B) + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n_\lambda^2 - n_\lambda) \, dd^c \log \{ \tau(C_\lambda/B) \text{Vol}(C_\lambda/B) \} \]

\[= - \frac{1}{12 |G|} \sum_{\lambda \in \Lambda} (n_\lambda^2 - 1)(f_{C_\lambda})_* \{ c_1(X/B) \} + \frac{1}{|G|} \sum_{\alpha \in A} \delta(G_{p_\alpha}) \{ f_{|p_\alpha} \} \{ c_1(X/B) \} \]

\[+ \frac{1}{|G|} \sum_{\alpha \in A} \delta(G_{p_\alpha}) \{ f_{|p_\alpha} \} \{ c_1(X/B) \} \]

\[+ \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n_\lambda^2 - n_\lambda) J^G_{C_\lambda/B} \omega_{A_{\lambda_\alpha}}.\]

Proof. Since

\[\left[ f_* \text{Td}(\theta_{C_\lambda/B}, h_{X/B}|_{\theta_{C_\lambda/B}}) \right]^{(1,1)} = \frac{1}{12} c_1(C_\lambda/B, h_{X/B}|_{C_\lambda/B})^2,\]

the result follows from Propositions 4.12 and 4.13. \qed

5. BCOV invariants for certain Calabi-Yau orbifolds

5.1. BCOV invariants for certain Calabi-Yau orbifolds. Let $X$ be a smooth projective threefold with trivial canonical line bundle. Let $G$ be a finite Abelian group of automorphisms of $X$ such that

\[(5.1) \quad H^0(X, \Omega^p_X)^G = H^0(X, \Omega^p_X) \quad (p = 0, 3), \quad H^3(X, \Omega^p_X)^G = 0 \quad (p = 1, 2).\]

We define $X_\Sigma^{(i)}$ ($i = 0, 1$) in the same way as before in Sect 4.3.1. Namely, $X_\Sigma^{(i)}$ is the disjoint union of connected components of $X^g$ of dimension $i$ for some $g \in G \setminus \{1\}$. Let $\{C_\lambda\}_{\lambda \in \Lambda}$ and $\{p_\alpha\}_{\alpha \in A}$ be the connected components of $X_\Sigma^{(1)}$ and $X_\Sigma^{(0)}$, respectively. Then $\{C_\lambda\}_{\lambda \in \Lambda}$ is the collection of compact Riemann surfaces of $X$ such that $C_\lambda \subset X^g$ for some $g \in G \setminus \{1\}$. Similarly, $\{p_\alpha\}_{\alpha \in A}$ is the collection of points of $X$ such that $p_\alpha$ is a connected component of $X^g$ for some $g \in G \setminus \{1\}$.

For any $x \in X$, write $G_x$ for the stabilizer of $x$ in $G$. The $G$-action on $X$ is said to be admissible if $G_x$ is an admissible Abelian subgroup of $\text{SL}(T_x, X)$ in the sense of Definition 4.11 for all $x \in X^{(0)}$.

Let $G$ be a admissible finite Abelian subgroup of automorphisms of $X$. By (5.1), $X^g$ is the disjoint union of finitely many compact Riemann surfaces and finite points for any $g \in G \setminus \{1\}$.

For $C_\lambda$, we can associate the finite subgroup of $G$

\[G_{C_\lambda} := \{ g \in G; g|_{C_\lambda} = \text{id}_{C_\lambda} \}.\]

Since $G_{C_\lambda}$ is a finite Abelian subgroup of $\text{SL}(\mathbb{C}^2)$, $G_{C_\lambda}$ is a cyclic group of order

\[n_\lambda := |G_{C_\lambda}|.\]

Since the $G$-action on $X$ is admissible, $G_{p_{\alpha}}$ is an Abelian subgroup of $\text{SL}(T_{p_{\alpha}}, X)$ with

\[\delta(G_{p_{\alpha}}) = \sum_{g \in G_{p_{\alpha}}} \frac{\chi_i(g)}{1 - \chi_i(g)^2} \quad (i = 1, 2, 3),\]
where $\chi_1, \chi_2, \chi_3 \in \text{Hom}(G_{p_\alpha}, \mathbb{C}^*)$ are such that $g = \text{diag}(\chi_1(g), \chi_2(g), \chi_3(g))$ for all $g \in G_{p_\alpha}$ in an appropriate coordinates of $X$ near $p_\alpha$. For any $g \in G \setminus \{1\}$, we get
\[ X^g = (\Pi_{\lambda \in \Lambda; g \in G_{C_{\lambda}} \setminus \{1\}} C_{\lambda}) \Pi_{\alpha \in \Lambda; g \in G_{p_\alpha}} p_\alpha. \]

We define
\[ (5.2) \quad \tilde{\chi}(X, G) := \chi(X) + \sum_{\lambda \in \Lambda} \left( n_\lambda^2 - 1 \right) \chi(C_{\lambda}) - 12 \sum_{\alpha \in \Lambda} \delta(G_{p_\alpha}), \]
where $\chi(\cdot)$ denotes the topological Euler number.

Let $\gamma$ be a $G$-invariant Kähler form on $X$ and let $\eta \in H^0(X, \Omega^1_X) \setminus \{0\}$ be a nowhere vanishing canonical form on $X$. We introduce
\[ A(X, \gamma) := \exp \left[ \frac{1}{12} \int_X \log \left( \sqrt{-1} \frac{\eta \wedge \overline{\eta}}{\|\eta\|^2_{L^2}} \cdot \frac{\text{Vol}(X, \gamma)}{\|\eta\|^3_{L^2}} \right) c_3(X, \gamma) \right], \]
\[ A(C_{\lambda}, \gamma|_{C_{\lambda}}) := \exp \left[ \frac{1}{12} \int_{C_{\lambda}} \log \left( \sqrt{-1} \frac{\eta \wedge \overline{\eta}}{\|\eta\|^2_{L^2}} \cdot \frac{\text{Vol}(X, \gamma)}{\|\eta\|^3_{L^2}} \right) c_1(C_{\lambda}, \gamma|_{C_{\lambda}}) \right], \]
\[ A(p_\alpha, \gamma|_{p_\alpha}) := \exp \left[ \frac{1}{12} \log \left( \sqrt{-1} \frac{\eta \wedge \overline{\eta}}{\|\eta\|^3_{L^2}} \cdot \frac{\text{Vol}(X, \gamma)}{\|\eta\|^3_{L^2}} \right) \right]. \]

**Definition 5.1.** For a $G$-invariant Kähler form $\gamma$ on $X$, define the orbifold BCOV invariant of $(X/G, \gamma)$ by
\[ \tau_{\text{BCOV}}^{\text{orb}}(X/G, \gamma) := T_{\text{BCOV}}^{\text{orb}}(X/G, \gamma) \text{Vol}(X, \gamma)^{-3 + \frac{2(X, \gamma)}{3!}} \text{Vol}_{L^2}(H^2(X, \mathbb{Z})^G, [\gamma])^{-1} A(X, \gamma)^{\frac{1}{12}} \]
\[ \times \prod_{\lambda \in \Lambda} \{ \tau(C_{\lambda}, \gamma|_{C_{\lambda}}) \text{Vol}(C_{\lambda}, \gamma|_{C_{\lambda}}) \}^{-\frac{(n_\lambda^2 - 1)}{12}} A(C_{\lambda}, \gamma|_{C_{\lambda}})^{\frac{(n_\lambda^2 - 1)}{12}} \]
\[ \times \prod_{\alpha \in \Lambda} A(p_\alpha, \gamma|_{p_\alpha})^{-\frac{12}{12}} \delta(G_{p_\alpha}). \]

When $G = \{1\}$, $\tau_{\text{BCOV}}^{\text{orb}}(X/G, \gamma)$ is exactly the BCOV invariant of $X$. In this sense, the orbifold BCOV invariant is an extension of the ordinary BCOV invariant.

If the Kähler form $\gamma$ on $X$ is Ricci-flat, then for any $\eta \in H^0(X, K_X) \setminus \{0\}$,
\[ \sqrt{-1} \frac{\eta \wedge \overline{\eta}}{\|\eta\|^3_{L^2}} = \frac{\gamma^3/3!}{\text{Vol}(X, \gamma)}. \]

Hence $A(X, \gamma) = A(C_{\lambda}, \gamma|_{C_{\lambda}}) = A(p_\alpha, \gamma|_{p_\alpha}) = 1$ and we get a simpler expression of $\tau_{\text{BCOV}}^{\text{orb}}(X/G, \gamma)$ in this case:
\[ \tau_{\text{BCOV}}^{\text{orb}}(X/G, \gamma) = T_{\text{BCOV}}^{\text{orb}}(X/G, \gamma) \text{Vol}(X, \gamma)^{-3 + \frac{2(X, \gamma)}{3!}} \text{Vol}_{L^2}(H^2(X, \mathbb{Z})^G, [\gamma])^{-1} \]
\[ \times \prod_{\lambda \in \Lambda} \{ \tau(C_{\lambda}, \gamma|_{C_{\lambda}}) \text{Vol}(C_{\lambda}, \gamma|_{C_{\lambda}}) \}^{-\frac{(n_\lambda^2 - 1)}{12}}. \]

In this section, we shall prove that $\tau_{\text{BCOV}}^{\text{orb}}(X/G, \gamma)$ is independent of the choice of a $G$-invariant Kähler form $\gamma$ on $X$. Hence $\tau_{\text{BCOV}}^{\text{orb}}(X/G, \gamma)$ is an invariant of $X/G$. In the rest of this section, we study the variational formula for $\tau_{\text{BCOV}}^{\text{orb}}$. 
Set up Let \( f : \mathcal{X} \to B \) be a locally-projective smooth morphism from a complex manifold \( \mathcal{X} \) to a complex manifold \( B \) such that \( X_b := f^{-1}(b) \) is a connected threefold with trivial canonical bundle

\[ K_{X_b} \cong \mathcal{O}_{X_b}, \quad \forall b \in B. \]

Let \( G \) be a finite Abelian group acting holomorphically on \( \mathcal{X} \) such that the projection \( f : \mathcal{X} \to B \) is \( G \)-equivariant with respect to the trivial \( G \)-action on \( B \). Hence \( G \) preserves the fibers of \( f \). Moreover, we assume that for all \( g \in G \) and \( b \in B \),

\[ g^*|_{H^0(X_b, K_{X_b})} = 1, \quad H^0(X_b, \Omega^1_{X_b})^G = H^0(X_b, \Omega^2_{X_b})^G = \{0\} \]

and that the \( G \)-action on \( \mathcal{X} \) is admissible. Then (5.3) is satisfied by the first condition of (5.3).

By the \( G \)-equivariance of the family \( f : \mathcal{X} \to B \), the locally free sheaf \( R^q f_* \Omega^p_{\mathcal{X}/B} \) is equipped with the natural \( G \)-action. By the second condition of (5.3) and the Hodge symmetry, we have the following vanishing for \( p \neq q \) and \( p + q \neq 3 \)

\[ (R^q f_* \Omega^p_{\mathcal{X}/B})^G = 0. \]

In what follows, we keep the notation in Sections 4.3 and 4.4.

Let \( b \in B \) be an arbitrary point and set \( X := X_b \). Let \( \mu : (\mathcal{X}, X) \to (\text{Def}(X), [X]) \)

be the Kuranishi family of \( X \). By (34), (Def(\( X \)), [\( X \)]) is smooth of dimension \( h^1(\Theta_X) = h^{1,2}(X) \) and is identified with the set germ at 0 \( \in H^1(X, \Theta_X) \). Since \( G \) is finite, there is a \( G \)-invariant Hermitian metric on \( X \). By making use of this \( G \)-invariant Hermitian metric in the construction of Kuranishi family [19, Chap. X], we see that the \( G \)-action on \( X \) lifts to holomorphic \( G \)-actions on \( \mathcal{X} \) and \( \text{Def}(X) \) so that \( \mu : \mathcal{X} \to \text{Def}(X) \) is \( G \)-equivariant.

Let

\[ B := \text{Def}(X)^G = \{ x \in \text{Def}(X) ; g(x) = x (\forall g \in G) \} \]

be the set of fixed points of the \( G \)-action. We define the map of germs

\[ \mu : (B, b) \to (B, [X]), \quad \mu(b) := [X_b], \]

where \( [X_b] \in \text{Def}(X) \) is the point corresponding to the complex structure on \( X_b \). Then the family of threefolds with \( G \)-action \( f : (\mathcal{X}, G) \to B \) is induced from the Kuranishi family by the map \( \mu \).

5.2. The first Chern form of the relative tangent bundle. Let \( \omega^{GW}_{\text{WP}} \) be the positive (1,1)-form on \( B \) induced from the Weil-Petersson form \( \omega_{\text{WP}} \) (cf. Section 4.2), i.e.,

\[ \omega^{GW}_{\text{WP}} := \omega_{\text{WP}}|_B. \]

Let \( \eta_{X/\text{Def}(X)} \in H^0(\text{Def}(X), f_* K_{X/\text{Def}(X)}) \) be a nowhere vanishing holomorphic section and set \( \eta_{X/B} := \mu^*(\eta_{X/\text{Def}(X)}) \). Since

\[ \eta_{X/B} = \eta_{X/\text{Def}(X)}|_{\mu(B)}, \]

we deduce from (1.4) that

\[ \mu^* \omega^{GW}_{\text{WP}} = -dd^c \log \| \eta_{X/B} \|^2_{L^2} = c_1(f_* K_{X/B}, h_{L^2}). \]
By [12] Eq. (4.2) and [5.3], the following equality of (1,1)-forms on \( \mathcal{X} \) holds:

\[
c_1(\mathcal{X}/B) = -f^* \left\{ \mu^* \omega^G_{\mathcal{W}P} + dd^c \log \text{Vol}(\mathcal{X}/B) \right\}
\]

(5.6)

\[+ dd^c \log \left\{ \frac{1}{1} \frac{\eta_{\mathcal{X}/B} \wedge \eta_{\mathcal{X}/B}}{\eta_{\mathcal{X}/B}^{1/3}} f^* \left( \frac{\text{Vol}(\mathcal{X}/B)}{\|\eta_{\mathcal{X}/B}\|_{L^2}^3} \right) \right\}.\]

5.3. The curvature of \((R^q f_* \Omega^p_{\mathcal{X}/B})^G\): the case \( p + q = 3 \). Recall that the Kodaira-Spencer map

\[
\rho: \Theta_{\text{Def}(X)} \to R^1 f_* \Omega^2_{\mathcal{X}/\text{Def}(X)} \otimes (f_* K_{\mathcal{X}/\text{Def}(X)})^G
\]

was defined in Section 1.4.1. By (1.1), \( \rho \) is \( G \)-equivariance. Hence we get an isomorphism of holomorphic vector bundles on \( \mathcal{B} \)

\[
\rho: (\Theta_{\mathcal{B}}, \omega^G_{\mathcal{W}P}) \cong ((R^1 f_* \Theta^G_{\mathcal{X}/\text{Def}(X)}), h_{L^2})
\]

(5.7)

\[\cong ((R^1 f_* \Omega^2_{\mathcal{X}/\text{Def}(X)})^G, h_{L^2}) \otimes ((f_* K_{\mathcal{X}/\text{Def}(X)})^G, h_{L^2}^{-1}).\]

Recall that the Ricci from of \((\mathcal{B}, \omega^G_{\mathcal{W}P})\) is defined as

\[\text{Ric} \omega^G_{\mathcal{W}P} := c_1(\Theta_{\mathcal{B}}, \omega^G_{\mathcal{W}P}).\]

Lemma 5.2. Set \( h^{p,q}(X)^G := \dim H^q(X, \Omega^p_X)^G \). When \( p + q = 3 \), the following equality of (1,1)-forms on \( \mathcal{B} \) holds:

\[
c_1((R^q f_* \Omega^p_{\mathcal{X}/\text{Def}(X)})^G, h_{L^2}) = \begin{cases} 
\omega^G_{\mathcal{W}P} & (p, q) = (3, 0), \\
\text{Ric} \omega^G_{\mathcal{W}P} + h^{1,2}(X)^G \omega^G_{\mathcal{W}P} & (p, q) = (2, 1), \\
-\text{Ric} \omega^G_{\mathcal{W}P} - h^{1,2}(X)^G \omega^G_{\mathcal{W}P} & (p, q) = (1, 2), \\
-\omega^G_{\mathcal{W}P} & (p, q) = (0, 3).
\end{cases}
\]

Proof. The result for \((p, q) = (3, 0)\) follows from [5.5]. By [5.7], we get the result for \((p, q) = (2, 1)\):

\[
c_1((R^1 f_* \Omega^2_{\mathcal{X}/\text{Def}(X)})^G, h_{L^2}) = c_1(((\Theta_{\text{Def}(X)})^G, \omega^G_{\mathcal{W}P})
\]

\[+ \text{rk} (\Theta_{\text{Def}(X)})^G \cdot c_1((f_* K_{\mathcal{X}/\text{Def}(X)}), h_{L^2}) \mid_{\mathcal{B}} \]

\[= \text{Ric} \omega^G_{\mathcal{W}P} + h^{1,2}(X)^G \omega^G_{\mathcal{W}P}.\]

Since the Serre duality \((R^q f_* \Omega^p_{\mathcal{X}/\text{Def}(X)})^G \cong R^{3-q} f_* \Omega^{3-p}_{\mathcal{X}/\text{Def}(X)}\) is a \( G \)-equivariant isometry with respect to the \( L^2 \)-metrics on the direct image sheaves, the results for \((p, q) = (1, 2), (0, 3)\) follow from those for \((p, q) = (2, 1), (3, 0)\).

Lemma 5.3. When \( p + q = 3 \), the following equality of (1,1)-forms on \( B \) holds:

\[
c_1((R^q f_* \Omega^p_{\mathcal{X}/B})^G, h_{L^2}) = \begin{cases} 
\mu^* \omega^G_{\mathcal{W}P} & (p, q) = (3, 0), \\
\mu^* \text{Ric} \omega^G_{\mathcal{W}P} + h^{1,2}(X)^G \mu^* \omega^G_{\mathcal{W}P} & (p, q) = (2, 1), \\
-\mu^* \text{Ric} \omega^G_{\mathcal{W}P} - h^{1,2}(X)^G \mu^* \omega^G_{\mathcal{W}P} & (p, q) = (1, 2), \\
-\mu^* \omega^G_{\mathcal{W}P} & (p, q) = (0, 3).
\end{cases}
\]
Proof. Since \( H^*(X_b, \Omega^p_{X/B})^G \), \( b \in B \), is primitive when \( p + q = 3 \) and since the \( L^2 \)-metric on the primitive cohomology coincides with the cup-product, the map of germs \( \mu: (B, b) \to (\text{Def}(X), [X]) \) induces an isometry of holomorphic Hermitian vector bundles on \((B, b)\)

\[
((R^2 f_* \Omega^p_{X/B})^G, h_{L^2}) = \mu^* ((R^2 f_* \Omega^p_{X/\text{Def}(X)})^G, h_{L^2}).
\]

Hence the result follows from Lemma \[5.2\] \( \square \)

5.4. The curvature of \((R^2 f_* \Omega^p_{X/B})^G\): the case \( p = q \). For \( b \in B \), the covolume of the lattice \( H^2(X_b, \mathbb{Z})^G \subset H^2(X_b, \mathbb{R})^G \) with respect to the \( L^2 \)-inner product is defined in the same way as in Section \[1.2\] If \( \{e_1(b), \ldots, e_r(b)\} \) is a basis of \( H^2(X_b, \mathbb{Z})^G / \text{Torsion} \), then

\[
\text{Vol}_{L^2}(H^2(X_b, \mathbb{Z})^G, \gamma_b) := \text{det} \left( \langle e_i(b), e_j(b) \rangle_{L^2, \gamma_b} \right) = \|e_1(b) \wedge \cdots \wedge e_r(b)\|^2_{L^2, \gamma_b},
\]

where \( r := \text{rk}_\mathbb{R} H^2(X_b, \mathbb{Z}) \) and \( \langle \cdot, \cdot \rangle_{L^2, \gamma_b} \) denotes the \( L^2 \)-inner product on \( H^2(X_b, \mathbb{R})^G \) with respect to \( \gamma_b \). Then \( \langle \cdot, \cdot \rangle_{L^2, \gamma_b} \) depends only on the Kähler class \( [\gamma_b] \). We define \( \text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G \in C^\infty(B) \) by

\[
\text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G(b) := \text{Vol}_{L^2}(H^2(X_b, \mathbb{Z})^G, \gamma_b), \quad b \in B.
\]

By \[12\] Lemma 4.12, \( \text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G \) is constant on \( B \) if the relative Kähler form \( \gamma_{X/B} \) is induced from a Kähler form on \( X \), i.e., if there is a Kähler form \( \gamma_X \) on \( X \) satisfying \( \gamma_{X/B} = \gamma_X|_{\gamma_{X/B}} \).

**Lemma 5.4.** The following equality of \((1,1)\)-forms on \( B \) holds:

\[
c_1((R^2 f_* \Omega^p_{X/B})^G, h_{L^2}) = \begin{cases} 
-\text{dd}^c \log \text{Vol}(X/B) & (p = 0), \\
-\text{dd}^c \log \text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G & (p = 1), \\
\text{dd} f \log \text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G & (p = 2), \\
\text{dd}^c \log \text{Vol}(X/B) & (p = 3).
\end{cases}
\]

**Proof.** Since \((f_* \mathcal{O}_X)^G = f_* \mathcal{O}_X = \mathcal{O}_B \cdot 1 \) and since \( \|1\|^2_{L^2} = \text{Vol}(X/B) \), we get the result for \( p = 0 \). Similarly, since \((f_* \Omega^2_{X/B})^G = (R^2 f_* \mathcal{O}_X)^G = 0 \) by \[13\], we get by the Hodge decomposition \((R^1 f_* \Omega^1_{X/B})^G = (R^2 f_* \mathbb{Z})^G \otimes \mathbb{Z} \mathcal{O}_B \). Set \( r := \text{rk}_\mathbb{R} H^2(X_b, \mathbb{Z}) \) and let \( L_1, \ldots, L_r \) be \( G \)-invariant holomorphic line bundles on \( X_b \) such that \( \{c_1(L_1), \ldots, c_1(L_r)\} \) is a basis of \( H^2(X_b, \mathbb{Z})^G / \text{Torsion}(H^2(X_b, \mathbb{Z})^G) \). Since \((R^2 f_* \mathcal{O}_X)^G = 0 \) by \[13\], it follows from the exactness of the sequence \((R^2 f_* \mathcal{O}_{X/B})^G \to (R^2 f_* \mathbb{Z})^G \to (R^2 f_* \mathcal{O}_X)^G \) that every \( L_i \) extends to a \( G \)-invariant holomorphic line bundle \( \mathcal{L}_i \) on \( X \). Then \((R^1 f_* \Omega^1_{X/B})^G = \mathcal{O}_B c_1(\mathcal{L}_1) + \cdots + \mathcal{O}_B c_1(\mathcal{L}_r) \), so that \( \det((R^1 f_* \Omega^1_{X/B})^G) = \mathcal{O}_B \cdot c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_r) \). Hence

\[
c_1((R^1 f_* \Omega^1_{X/B})^G, h_{L^2}) = -\text{dd}^c \log \|c_1(\mathcal{L}_1) \wedge \cdots \wedge c_1(\mathcal{L}_r)\|^2_{L^2} = -\text{dd}^c \log \text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G.
\]

This proves the assertion for \( p = 1 \). The assertions for \( p = 2, 3 \) follow from those for \( p = 1, 0 \) by the Serre duality. See \[12\] p.200 for the details. \( \square \)

**Corollary 5.5.** The following equality of \((1,1)\)-forms on \( B \) holds:

\[
\sum_{p,q} (-1)^p q c_1 \left( (R^1 f_* \Omega^p_{X/B})^G, h_{L^2} \right) = -\mu^* \text{Ric} \omega^G_{WP} - (h^{1,2}(X))^G + 3 \mu^* \omega^G_{WP} + \text{dd}^c \log \left\{ \text{Vol}(X/B)^3 \text{Vol}_{L^2}(R^2 f_* \mathbb{Z})^G \right\}.
\]
Proof. The result follows from Lemmas 5.3 and 5.4.

5.5. The direct images of some Chern forms.

**Definition 5.6.** Define $C^\infty$ functions $A(\chi/B)$, $A(\mathcal{C}_\lambda/B)$, $A(p_\alpha/B)$ on $B$ by

\[
A(\chi/B) := \exp \left[ -\frac{1}{12} f_* \left\{ \log \left( \frac{\eta_{\chi/B} \wedge \eta_{\chi/B}}{\gamma_{\chi/B}/3!} \right) \cdot f^* \frac{\text{Vol}(\chi/B)}{\|\eta_{\chi/B}\|_{L^2}} \right\} \right],
\]

\[
A(\mathcal{C}_\lambda/B) := \exp \left[ -\frac{1}{12} f_* \left\{ \log \left( \frac{\eta_{\chi/B} \wedge \eta_{\chi/B}}{\gamma_{\chi/B}/3!} \right) \cdot f^* \frac{\text{Vol}(\chi/B)}{\|\eta_{\chi/B}\|_{L^2}} \right\} \right],
\]

\[
A(p_\alpha/B) := \exp \left[ -\frac{1}{12} f_* \left\{ \log \left( \frac{\eta_{\chi/B} \wedge \eta_{\chi/B}}{\gamma_{\chi/B}/3!} \right) \cdot f^* \frac{\text{Vol}(\chi/B)}{\|\eta_{\chi/B}\|_{L^2}} \right\} \right].
\]

Notice that $A(\chi/B)$ differs from the corresponding function in [12, Def. 4.1] by the factor $\text{Vol}(\chi/B)^{\chi(\chi)/12}$.

**Proposition 5.7.** The following equality of $(1,1)$-forms on $\chi$ holds:

\[
f_* \{ c_1(\chi/B)c_3(\chi/B) \} = -\chi(\chi) \left\{ \mu^* \omega_{WP}^G + dd^c \log \text{Vol}(\chi/B) \right\} - 12 dd^c \log A(\chi/B).
\]

**Proof.** See [12, p.197 Eq (4.3)]. Notice again that our $A(\chi/B)\text{Vol}(\chi/B)^{-\chi(\chi)/12}$ is equal to $A(\chi/B)$ in [12, Def. 4.1].

**Proposition 5.8.** The following equality of $(1,1)$-forms on $B$ holds:

\[
(f|_{\mathcal{C}_\lambda})_* \{ c_1(\chi/B)c_1(\mathcal{C}_\lambda/B) \} = -\chi(\mathcal{C}_\lambda) \left\{ \mu^* \omega_{WP}^G + dd^c \log \text{Vol}(\chi/B) \right\} - 12 dd^c \log A(\mathcal{C}_\lambda/B).
\]

**Proof.** By (5.6), we get

\[
(f|_{\mathcal{C}_\lambda})_* \{ c_1(\chi/B)c_1(\mathcal{C}_\lambda/B) \}
= -\left( f|_{\mathcal{C}_\lambda} \right)^* \{ \mu^* \omega_{WP}^G + dd^c \log \text{Vol}(\chi/B) \} \wedge c_1(\mathcal{C}_\lambda/B)
+ (f|_{\mathcal{C}_\lambda})_* \left[ dd^c \log \left( \frac{\sqrt{-1} \eta_{\chi/B} \wedge \eta_{\chi/B}}{\gamma_{\chi/B}/3!} \right) \cdot f^* \left( \frac{\text{Vol}(\chi/B)}{\|\eta_{\chi/B}\|_{L^2}} \right) \right] \wedge c_1(\mathcal{C}_\lambda/B)
\]

\[
= -\chi(\mathcal{C}_b) \left\{ \mu^* \omega_{WP}^G + dd^c \log \text{Vol}(\chi/B) \right\} - 12 dd^c \log A(\mathcal{C}_\lambda/B).
\]

To get the second equality, we used the projection formula and the commutativity of $dd^c$ and $(f|_{\mathcal{C}_\lambda})_*$. 

**Proposition 5.9.** The following equality of $(1,1)$-forms on $B$ holds:

\[
(f|_{p_\alpha})_* \{ c_1(\chi/B) \} = -\mu^* \omega_{WP}^G - dd^c \log \text{Vol}(\chi/B) - 12 dd^c \log A(p_\alpha/B).
\]

**Proof.** The result follows from (5.6) and the projection formula.
5.6. The curvature formula. Set $\mathcal{Y} := \mathcal{X}/G$. As a relative version of $\tau_{\text{BCOV}}^{\text{orb}}$, we introduce a function $\tau_{\text{BCOV}}^{\text{orb}}(\mathcal{Y}/B) \in C^\infty(B)$ by

$$
\tau_{\text{BCOV}}^{\text{orb}}(\mathcal{Y}/B) :=
T_{\text{BCOV}}^{\text{orb}}(\mathcal{Y}/B) \text{Vol}(\mathcal{X}/B)^{-3+\frac{5}{12}|\mathcal{X}/G|} \left(\text{Vol}_{L^2}(R^2 f_s Z)^G\right)^{-1} A(\mathcal{X}/B)^{3/2} \times \prod_{\lambda \in \Lambda} \left\{ \tau(C_\lambda/B) \text{Vol}(C_\lambda/B) - \frac{n^2 - n_\lambda}{12} \right\}^{-1/2} A(C_\lambda/B)^{1/2} \prod_{\alpha \in A} A(p_\alpha/B) \left(\text{Vol}(\mathcal{Y}/B)^{2} + 3\right)^{1/2} \mu^* \omega_{\text{WP}}^G + \mu^* \text{Ric}_{\text{WP}}^G + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - n_\lambda) J_{C_\lambda/B}^* \omega_{A_{h\lambda}}.
$$

**Theorem 5.10.** The following equality of $(1,1)$-forms on $B$ holds

$$
-dd^c \log \tau_{\text{BCOV}}^{\text{orb}}(\mathcal{Y}/B) = \left(\frac{\chi(X,G)}{12|G|} + h^{1,2}(X)^G + 3\right) \mu^* \omega_{\text{WP}}^G + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - n_\lambda) J_{C_\lambda/B}^* \omega_{A_{h\lambda}}.
$$

**Proof.** By Theorem 4.14 we get

$$
-dd^c \log T_{\text{BCOV}}^{\text{orb}}(\mathcal{Y}/B) + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - n_\lambda) dd^c \log \{\tau(C_\lambda/B)\text{Vol}(C_\lambda/B)\}
$$

$$
= -\frac{1}{12|G|} f_s \{c_1(\mathcal{X}/B) c_3(\mathcal{X}/B)\} - \frac{1}{12|G|} \sum_{\lambda \in \Lambda} (n^2 - 1)(f|_{C_\lambda})_* \{c_1(\mathcal{X}/B) c_1(C_\lambda/B)\}
$$

$$
+ \frac{1}{|G|} \sum_{\alpha \in A} \delta(G_{p_\alpha}) (f|_{p_\alpha})_* \{c_1(\mathcal{X}/B)\} - \sum_{p,q} (-1)^{p+q} p c_1((\det R^2 f_s \Omega_{X/B}^p G^G h_{L^2})
$$

$$
+ \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - n_\lambda) J_{C_\lambda/B}^* \omega_{A_{h\lambda}}.
$$

By substituting the formulae in Propositions 5.7 5.8 5.9 and Corollary 5.5 into (5.9), we get

$$
-dd^c \log T_{\text{BCOV}}^{\text{orb}}(\mathcal{Y}/B) + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - n_\lambda) dd^c \log \{\tau(C_\lambda/B)\text{Vol}(C_\lambda/B)\}
$$

$$
= \left(\frac{\chi(X)}{12|G|} + \frac{1}{12|G|} \sum_{\lambda \in \Lambda} (n^2 - 1) \chi(C_\lambda) - \frac{1}{|G|} \sum_{\alpha \in A} \delta(G_{p_\alpha}) + h^{1,2}(X)^G + 3\right) \mu^* \omega_{\text{WP}}^G
$$

$$
+ \mu^* \text{Ric}_{\text{WP}}^G + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - n_\lambda) J_{C_\lambda/B}^* \omega_{A_{h\lambda}}
$$

$$
+ \left(\frac{\chi(X)}{12|G|} + \frac{1}{12|G|} \sum_{\lambda \in \Lambda} (n^2 - 1) \chi(C_\lambda) - \frac{1}{|G|} \sum_{\alpha \in A} \delta(G_{p_\alpha}) - 3\right) dd^c \log \text{Vol}(\mathcal{X}/B)
$$

$$
- dd^c \log \text{Vol}_{L^2}(R^2 f_s Z)^G + \frac{1}{|G|} dd^c \log A(\mathcal{X}/B) + \frac{1}{|G|} \sum_{\lambda \in \Lambda} (n^2 - 1) dd^c \log A(C_\lambda/B)
$$

$$
- \frac{12}{|G|} \sum_{\alpha \in A} \delta(G_{p_\alpha}) dd^c \log A(p_\alpha/B).
$$

Substituting (5.2) into (5.10), we get the result. □
Theorem 5.11. If the $G$-action on $X$ is admissible, then $\tau_{BCOV}^{\text{orb}}(X/G, \gamma)$ is independent of the choice of a $G$-invariant Kähler form $\gamma$ on $X$.

Proof. Let $\gamma_0$ and $\gamma_\infty$ be two $G$-invariant Kähler forms on $X$, which are identified with the corresponding Hermitian metrics. Set $\mathcal{X} := X \times \mathbb{P}^1$ and let $f := pr_2: \mathcal{X} \to \mathbb{P}^1$ be the projection. Then $f: \mathcal{X} \to \mathbb{P}^1$ is a trivial family with fiber $X$ and the $G$-action on $X$ extends to the $G$-action on $\mathcal{X}$. Define the Hermitian metric $\gamma_{\mathcal{X}/\mathbb{P}^1} := \{\gamma_t\}_{t \in \mathbb{P}^1}$ on $\Theta_{\mathcal{X}/\mathbb{P}^1}$ by $\gamma_t := (\gamma_0 + |t|^2\gamma_\infty)/(1 + |t|^2)$. Then $\gamma_{\mathcal{X}/\mathbb{P}^1}$ is fiberwise Kähler. We define the function $\tau_{BCOV}^{\text{orb}}(\mathcal{Y}/\mathbb{P}^1)$ by

$$\tau_{BCOV}^{\text{orb}}(\mathcal{Y}/\mathbb{P}^1)(t) := \tau_{BCOV}^{\text{orb}}(\mathcal{Y}, \gamma_t).$$

By Theorem 5.11 applied to the trivial family $f: (\mathcal{X}, \iota \times \text{id}_{\mathbb{P}^1}) \to \mathbb{P}^1$, we get on $\mathbb{P}^1$ $-dd^c \log \tau_{BCOV}^{\text{orb}}(\mathcal{Y}/\mathbb{P}^1) = 0$.

Hence $\tau_{BCOV}^{\text{orb}}(\mathcal{Y}/\mathbb{P}^1)$ is a harmonic function on $\mathbb{P}^1$, it is a constant. Hence $\tau_{BCOV}^{\text{orb}}(\mathcal{Y}, \gamma_0) = \tau_{BCOV}^{\text{orb}}(\mathcal{Y}, \gamma_\infty). \quad \Box$

Definition 5.12. Let $X$ be a smooth projective threefold with trivial canonical line bundle. Let $G$ be a finite Abelian group of automorphisms of $X$ whose action on $X$ is admissible and which satisfies

$$H^0(X, \Omega^p)^G = 0 \quad (p = 1, 2), \quad H^0(X, \Omega^p)^G = H^0(X, \Omega^p) \quad (p = 0, 3).$$

Then the (orbifold) BCOV invariant of $X/G$ is defined as

$$\tau_{BCOV}^{\text{orb}}(X/G) := \tau_{BCOV}^{\text{orb}}(X/G, \gamma),$$

where $\gamma$ is an arbitrary $G$-invariant Kähler form on $X$.

It is very likely that one can extend BCOV invariants for certain non-global Calabi-Yau orbifolds by using the anomaly formula for orbifolds [23, Th. 0.1] instead of curvature theorem. The details are left to the reader.

In Section 6 we shall focus on the case where $G$ is generated by an involution $\iota: X \to X$ such that

$$\iota^*|_{H^0(X, K_X)} = 1, \quad \iota^*|_{H^0(X, \Omega_X)} = \iota^*|_{H^0(X, \Omega_X)} = -1.$$

Set $Y := X/\iota$. Let $\gamma$ be a Kähler form on $Y$ in the sense of orbifolds and let $\eta$ be a nowhere vanishing canonical form on $Y$. In this case, the orbifold BCOV invariant of $Y$ is given by

$$\tau_{BCOV}^{\text{orb}}(Y) = \text{Vol}(Y, \gamma)^{-3 + \frac{\chi(Y) + \chi(Sing Y)}{12}} \text{Vol}_{L^2}(H^2(Y, \mathbb{Z}), [\gamma])^{-1} T_{BCOV}(Y, \gamma)$$

$$\times \exp \left\{ -\frac{1}{12} \int_Y \log \left( \frac{\sqrt{-1} \eta \wedge \bar{\eta}}{\gamma^{3/2}} \right)(p, \gamma) \right\}$$

$$\times \text{Vol}(\text{Sing} Y, \gamma|_{\text{Sing} Y})^{-1} \tau(\text{Sing} Y, \gamma|_{\text{Sing} Y})^{-1}$$

$$\times \exp \left\{ -\frac{1}{8} \int_{\text{Sing} Y} \log \left( \frac{\sqrt{-1} \eta \wedge \bar{\eta}}{\gamma^{3/2}} \right)(p, \gamma) \right\} c_1(\text{Sing} Y, \gamma|_{\text{Sing} Y}),$$

because $2\chi(Y) = \chi(X) + \chi(Sing Y)$. If the Kähler form $\gamma$ on $Y$ is Ricci-flat, then

$$(5.11) \quad \tau_{BCOV}^{\text{orb}}(Y) = \text{Vol}(Y, \gamma)^{-3 + \frac{\chi(Y) + \chi(Sing Y)}{12}} \text{Vol}_{L^2}(H^2(Y, \mathbb{Z}), [\gamma])^{-1} T_{BCOV}(Y, \gamma)$$

$$\times \text{Vol}(\text{Sing} Y, \gamma|_{\text{Sing} Y})^{-1} \tau(\text{Sing} Y, \gamma|_{\text{Sing} Y})^{-1}.$$
6. BCOV invariants for Borcea-Voisin orbifolds

In Section 6, we reduce the BCOV invariants of Borcea-Voisin orbifolds of type $M$ to the holomorphic torsion invariants of 2-elementary $K3$ surfaces of the same type.

6.1. Analytic torsion for elliptic curves. Let $T$ be an elliptic curve and let $\gamma$ be a Kähler form on $T$. Let $\xi$ be a non-zero holomorphic 1-form on $T$. We set

$$\tau_{\text{ell}}(T) := \text{Vol}(T, \gamma) \tau(T, \gamma) \exp \left[ \frac{1}{12} \int_T \log \left( \sqrt{-1} \frac{\xi \wedge \bar{\xi}}{\gamma} \right) c_1(T, \gamma) \right].$$

Since $\chi(T) = \int_T c_1(T, \gamma) = 0$, $\tau(T, \gamma)$ is independent of the choice of $\xi$.

Let $H$ be the complex upper half-plane. Let $T = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}$, $\tau \in H$. The normalized flat Kähler metric on $T$ is given by $dz \otimes d\bar{z}/\Im \tau$ and its Kähler form is given by $\gamma_T = \sqrt{-1} dz \wedge d\bar{z}/\Im \tau$. Since $\text{Vol}(T, \gamma_T) = (2\pi)^{-1} \int_T \gamma_T = \pi^{-1}$, we have

$$\tau_{\text{ell}}(T) = \pi^{-1} \tau(T, \gamma_T).$$

The Laplacian $\square_{0,0}^T$ of $(T, \gamma_T)$ acting on $C^\infty(T)$ is given by

$$\square_{0,0}^T = -\Im \tau \frac{\partial^2}{\partial z \partial \bar{z}} = -\frac{\Im \tau}{4} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

For $(m, n) \in \mathbb{Z}^2$, we set

$$\nu_{m,n}(T) := \frac{\pi^2}{\Im \tau} |m\tau + n|^2.$$

Then

$$\sigma(\square_{0,0}^T) = \{ \nu_{m,n}(T); (m, n) \in \mathbb{Z}^2 \}.$$

**Theorem 6.1.** For every elliptic curve $T$,

$$\tau_{\text{ell}}(T) = \left( 4\pi \| \eta(\Omega(T)) \|_4 \right)^{-1}.$$

**Proof.** Let $\zeta_{0,0}^T(s)$ be the spectral zeta function of $\square_{0,0}^T$. By the Kronecker limit formula [38, p.75 Eq. (17)], we have

$$\frac{d}{ds} \zeta_{0,0}^T(s) = \frac{d}{ds} \sum_{(m, n) \in \mathbb{Z}^2 \backslash\{0\}} \frac{(\Im \tau)^s}{|m\tau + n|^{2s}} = -\log \left( 4\pi \| \eta(\tau) \|_4 \right).$$

This, together with (6.1) and the anomaly formula [38 Th. 0.2], yields the result. \qed

6.2. An observation of Harvey-Moore for Borcea-Voisin orbifolds.

**Theorem 6.2.** Let $(S, \theta)$ be a 2-elementary $K3$ surface of type $M$ and let $T$ be an elliptic curve. Then the following equality holds:

$$\tau_{\text{BCOV}}(X_{(S, \theta, T)}) = 2^{12-l(M)} (2\pi)^{2r(M)+2} \tau_M(S, \theta)^{-4} \tau_{\text{ell}}(T)^{-12}.$$

Theorem 6.2 was proved by Harvey-Moore [16 Sect. V] when $\theta: S \to S$ is free from fixed points, equivalently when the type of $X_{(S, \theta, T)}$ is $U(2) \oplus E_6(2)$. Since some of the details of the proof were omitted in [16 Sect. V], we given them here.

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ANALYTIC TORSION FOR BORCEA-VOISIN THREEFOLDS 53
6.2.1. Symmetries of the spectrum of Laplacians. Let $\gamma_T$ and $\gamma_S$ be Ricci-flat Kähler forms on $T = C / \mathbb{Z} + \tau \mathbb{Z}$ and $S$, respectively, such that

$$\gamma_T := \sqrt{-1} \frac{dz \wedge d\bar{z}}{3_T}, \quad \theta^* \gamma_S = \gamma_S.$$  

We set

$$\gamma := \text{pr}_1^* \gamma_S + \text{pr}_2^* \gamma_T.$$  

Then $\gamma$ is a $\theta \times (-1)_T$-invariant Ricci-flat Kähler form on $S \times T$ and is regarded as a Ricci-flat Kähler form on $X_{(S, \theta, T)} = (S \times T) / \theta \times (-1)_T$ in the sense of orbifolds. Let $\eta$ and $\xi$ be nowhere vanishing canonical forms on $S$ and $T$, respectively. Then $\eta \wedge \xi$ is a nowhere vanishing canonical form on $S \times T$, which we identify the corresponding nowhere vanishing canonical form on $X_{(S, \theta, T)}$. By the Ricci-flatness of $\gamma_S$ and $\gamma_T$, the canonical forms $\eta$, $\xi$, $\eta \wedge \xi$ are parallel with respect to the Levi-Civita connections associated to the Kähler forms $\gamma_S$, $\gamma_T$, $\gamma$, respectively. For simplicity, write $X$ for $X_{(S, \theta, T)}$ when there is no possibility of confusion.

Let $\zeta_{p,q}(s)$ be the spectral zeta function of $\Box_{p,q}$ acting on $A^{p,q}(X)$, the space of smooth $(p, q)$-forms on $X$ in the sense of orbifolds.

**Lemma 6.3.** The following equality of meromorphic functions on $\mathbb{C}$ holds

$$\sum_{p,q \geq 0} (-1)^{p+q} pq \zeta_{p,q}(s) = 9 \zeta_{0,0}(s) - 6 \zeta_{1,0}(s) + \zeta_{1,1}(s).$$  

**Proof.** By the ellipticity of the Dolbeault complex $(A^{p,*}(X), \bar{\partial})$, we have the following equality of spectral zeta functions for all $p \geq 0$

$$\zeta_{p,0}(s) - \zeta_{p,1}(s) + \zeta_{p,2}(s) - \zeta_{p,3}(s) = 0. \tag{6.2}$$

Since $\eta \wedge \xi$ is parallel and nowhere vanishing, the Laplacians $\Box_{p,0}$ and $\Box_{p,3}$ are isospectral by the map $\otimes (\eta \wedge \xi) : A^{p,0}(X) \ni \varphi \mapsto \varphi \cdot (\eta \wedge \xi) \in A^{p,3}(X)$. In particular, $\zeta_{p,0}(s) = \zeta_{p,3}(s)$, which, together with (6.2), implies $\zeta_{p,1}(s) = \zeta_{p,2}(s)$. Hence

$$\zeta_{p,q}(s) = \zeta_{p,3-q}(s) \tag{6.3}$$

for all $0 \leq p, q \leq 3$. Since $\Box_{p,0} \varphi = \Box_{q,3-q} \varphi$ for all $\varphi \in A^{p,q}(X)$, we get for all $p, q \geq 0$

$$\zeta_{p,q}(s) = \zeta_{q,p}(s). \tag{6.4}$$

By (6.3), (6.4), every $\zeta_{p,q}(s)$ is equal to one of $\zeta_{0,0}(s)$, $\zeta_{1,0}(s)$, $\zeta_{1,1}(s)$. Replacing $\zeta_{p,q}(s)$ by the corresponding $\zeta_{0,0}(s)$ or $\zeta_{1,0}(s)$ or $\zeta_{1,1}(s)$, we get the result. \(\square\)

6.2.2. The spectrum of various Laplacians. Let $\Box_{p,q}^S$ (resp. $\Box_{p,q}^T$) be the Laplacian acting on $A^{p,q}(S)$ (resp. $A^{p,q}(T)$) with respect to $\gamma_S$ (resp. $\gamma_T$). Set

$$A^{p,q}(S)^\pm := \{ \varphi \in A^{p,q}(S) : \theta^* \varphi = \pm \varphi \}, \quad \Box_{p,q}^S := \Box_{p,q}^S | A^{p,q}(S)^\pm,$$

$$A^{p,q}(T)^\pm := \{ \varphi \in A^{p,q}(T) : (-1_T)^* \varphi = \pm \varphi \}, \quad \Box_{p,q}^T := \Box_{p,q}^T | A^{p,q}(T)^\pm.$$  

For simplicity, write $\Box_{p,q}^{S,\pm} = \Box_{0,0,0}^{S,\pm}$ and $\Box_{p,q}^{T,\pm} = \Box_{0,0,0}^{T,\pm}$. Set $h^{1,1}(S)^\pm := \dim \ker \Box_{p,q}^{S,\pm}$. Let $\sigma(\Box_{p,q}^{S})$ (resp. $\sigma(\Box_{p,q}^{T})$) be the set of eigenvalues of $\Box_{p,q}^{S,\pm}$ (resp. $\Box_{p,q}^{T,\pm}$) counted with multiplicities. We can write

$$\sigma(\Box_{p,q}^{S,\pm}) = \sigma(\Box_{0,0,0}^{S,\pm}) = \{ 0 \} \coprod \{ \lambda_i^+(S) \}_{i \in I}, \quad \sigma(\Box_{p,q}^{T,\pm}) = \sigma(\Box_{0,0,0}^{T,\pm}) = \{ \lambda_j^-(S) \}_{j \in J},$$

where $\lambda_i^+(S) > 0$ and $\lambda_j^-(S) > 0$ for all $i \in I$ and $j \in J$. In what follows, for a subset $S \subset \mathbb{R}$, the notation $\nu \cdot S$ implies that every element of $S$ has multiplicity $\nu$.  

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**Ken-Ichi Yoshikawa**
Lemma 6.4. The set \( \sigma(\square_{p,q}^{S,\pm}) \) is given as follows:

1. \( \sigma(\square_{1,0}^{S,\pm}) = \sigma(\square_{0,1}^{S,\pm}) = \sigma(\square_{0,0}^{S,\pm}) = \{ \lambda_j^+(S) \}_{i \in I} \bigoplus \{ \lambda_j^-(S) \}_{j \in J} \)
2. \( \sigma(\square_{1,1}^{S,\pm}) = h^{1,1}(S) \cdot \{ 0 \} \bigoplus 2 \cdot \{ \lambda_j^+(S) \}_{i \in I} \bigoplus 2 \cdot \{ \lambda_j^-(S) \}_{j \in J} \)

Proof. Since the Dolbeault complex \((A^\bullet, \bar{\partial})\) is elliptic and since

\[
(6.5) \quad \sigma(\square_{p,q}^{S,\pm}) = \sigma(\square_{p,q}^{S,\mp})
\]

via the map \( \otimes \eta : A^{p,0}(S)^{\pm} \ni \varphi \to \varphi \cdot \eta \in A^{p,2}(S) \), we get

\[
\sigma(\square_{0,1}^{S,\pm}) = (\sigma(\square_{0,1}^{S,\pm}) \setminus \{ 0 \}) \bigoplus (\sigma(\square_{0,2}^{S,\pm}) \setminus \{ 0 \}) = (\sigma(\square_{0,0}^{S,\pm}) \setminus \{ 0 \}) \bigoplus (\sigma(\square_{1,0}^{S,\mp}) \setminus \{ 0 \}) = \{ \lambda_j^+(S) \}_{i \in I} \bigoplus \{ \lambda_j^-(S) \}_{j \in J}.
\]

By taking the complex conjugation, we get \( \sigma(\square_{1,0}^{S,\pm}) = \{ \lambda_j^+(S) \}_{i \in I} \bigoplus \{ \lambda_j^-(S) \}_{j \in J} \). This proves (1). By the ellipticity of the Dolbeault complex \((A^{1,\bullet}, \bar{\partial})\) and (6.5), we get

\[
\sigma(\square_{1,1}^{S,\pm}) = h^{1,1}(S) \cdot \{ 0 \} \bigoplus \sigma(\square_{1,2}^{S,\pm}) = h^{1,1}(S)^{\pm} \cdot \{ 0 \} \bigoplus \sigma(\square_{1,0}^{S,\mp}) \bigoplus \sigma(\square_{0,1}^{S,\pm}) = h^{1,1}(S)^{\pm} \cdot \{ 0 \} \bigoplus \{ \lambda_j^+(S) \}_{i \in I} \bigoplus 2 \cdot \{ \lambda_j^-(S) \}_{j \in J},
\]

where the last equality follows from (1). This proves (2).

For \( T = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z} \) and \((m, n) \in \mathbb{Z}^2\), recall that \( \nu_{m,n}(T) = (\Im \tau / \pi)^{-2} |m\tau + n|^2 \). Then \( \sigma(\square_{0,0}^{S,\pm}) = \{ \nu_{m,n}(T) : (m, n) \in \mathbb{Z}^2 \} \). The following is classical (cf. [30, p. 166]).

Lemma 6.5. Set \((\mathbb{Z}^2)^* := \mathbb{Z}^2 \setminus \{(0, 0)\} \). Then

\[
\sigma(\square_{1,0}^{T,\pm}) = \sigma(\square_{1,1}^{T,\pm}) = \{ 0 \} \bigoplus \{ \nu_{m,n}(T) : (m, n) \in (\mathbb{Z}^2)^* / \{ \pm 1 \} \}
\]

\[
\sigma(\square_{1,0}^{T,\pm}) = \sigma(\square_{1,1}^{T,\pm}) = \{ 0 \} \bigoplus \{ \nu_{m,n}(T) : (m, n) \in (\mathbb{Z}^2)^* / \{ \pm 1 \} \}
\]

6.2.3. Various spectral \( \zeta \)-functions. Let \( \zeta^{S,\pm}_{p,q}(s) \) and \( \zeta^{T,\pm}_{p,q}(s) \) be the spectral zeta functions of \( \square_{p,q}^{S,\pm} \) and \( \square_{p,q}^{T,\pm} \), respectively. Then

\[
\zeta^{S,\pm}_{p,q}(s) = \sum_{i \in I} \lambda_i^\pm(S)^{-s}, \quad \zeta^{S,\pm}_{0,0}(s) = \sum_{j \in J} \lambda_j^\pm(S)^{-s}
\]

and

\[
\zeta^{T,\pm}_{p,q}(s) = \zeta^{T,\pm}_{0,0}(s) = \sum_{(m, n) \in (\mathbb{Z}^2)^* / \{ \pm 1 \}} (\Im \tau / \pi)^{2s} |m\tau + n|^2 |s| = \sum_{(m, n) \in (\mathbb{Z}^2)^* / \{ \pm 1 \}} \nu_{m,n}(T)^{-s}.
\]

Following Harvey-Moore [10], we set

\[
\mu^+(s) := \sum_{i \in I, (m, n) \in (\mathbb{Z}^2)^* / \{ \pm 1 \}} (\lambda_i^+(S) + \nu_{m,n}(T))^{-s},
\]

\[
\mu^-(s) := \sum_{j \in J, (m, n) \in (\mathbb{Z}^2)^* / \{ \pm 1 \}} (\lambda_j^-(S) + \nu_{m,n}(T))^{-s}.
\]

Lemma 6.6. The following equality of meromorphic functions on \( \mathbb{C} \) holds:

\[
\zeta_{0,0}(s) = \zeta^{T,\pm}_{p,q}(s) + \zeta^{S,\pm}_{p,q}(s) + \mu^+(s) + \mu^-(s).
\]
Proof. Since the decomposition
\[ A^{0,0}(X)^+ = [A^{0,0}(T)^+ \otimes A^{0,0}(S)^+] \oplus [A^{0,0}(T)^- \otimes A^{0,0}(S)^-] \]
is orthogonal and is preserved by \( \square_{0,0} \), we get
\begin{equation}
(6.6) \quad \sigma(\square_{0,0}) = \sigma(\square_{0,0}|_{A^{0,0}(T)^+ \otimes A^{0,0}(S)^+}) \Pi \sigma(\square_{0,0}|_{A^{0,0}(T)^- \otimes A^{0,0}(S)^-}).
\end{equation}
Since
\begin{equation}
(6.7) \quad \square_{p,q}(\varphi \otimes \psi) = (\square^S_{p',q'} \varphi) \otimes \psi + \varphi \otimes (\square^T_{p',q''} \psi)
\end{equation}
for all \( \varphi \in A^{p',q'}(S) \) and \( \psi \in A^{p'',q''}(T) \) with \( p = p' + p'' \), \( q = q' + q'' \), it follows from (6.7) and Lemmas 6.4 and 6.5 that
\begin{equation}
(6.8) \quad \sigma(\square_{0,0}|_{A^{0,0}(T)^+ \otimes A^{0,0}(S)^+}) = \{0\} \Pi (\sigma(\square^T_{+}) \setminus \{0\}) \Pi (\sigma(\square^S_{+}) \setminus \{0\}) \Pi \{\lambda_i^+(S) + \nu_{m,n}(T) ; i \in I, (m,n) \in (Z^2)^*/ \pm 1\},
\end{equation}
\begin{equation}
(6.9) \quad \sigma(\square_{0,0}|_{A^{0,0}(T)^- \otimes A^{0,0}(S)^-}) = \{\lambda_j^-(S) + \nu_{m,n}(T) ; j \in J, (m,n) \in (Z^2)^*/ \pm 1\}.
\end{equation}
The result follows from (6.6), (6.8), (6.9).

Lemma 6.7. The following equality of meromorphic functions on \( C \) holds:
\[ \zeta_{1,0}(s) = \zeta^{T,+}(s) + \zeta^{S,+}(s) + 2\zeta^{S,-}(s) + 3\mu^+(s) + 3\mu^-(s). \]
Proof. Since the decomposition
\[ A^{1,0}(X)^+ = [A^{1,0}(T)^+ \otimes A^{0,0}(S)^+] \oplus [A^{1,0}(T)^- \otimes A^{0,0}(S)^-] \]
is orthogonal and is preserved by \( \square_{1,0} \), we get
\begin{equation}
(6.10) \quad \sigma(\square_{1,0}) = \sigma(\square_{1,0}|_{A^{1,0}(T)^+ \otimes A^{0,0}(S)^+}) \Pi \sigma(\square_{1,0}|_{A^{1,0}(T)^- \otimes A^{0,0}(S)^-})
\begin{align*}
\quad & \Pi \sigma(\square_{1,0}|_{A^{0,0}(T)^+ \otimes A^{1,0}(S)^+}) \Pi \sigma(\square_{1,0}|_{A^{0,0}(T)^- \otimes A^{1,0}(S)^-}).
\end{align*}
\end{equation}
By (6.7) and Lemmas 6.4 and 6.5 we get
\begin{align*}
(6.11) & \quad \sigma(\square_{1,0}|_{A^{1,0}(T)^+ \otimes A^{0,0}(S)^+}) = \sigma(\square^T_{+}) \Pi \{\lambda_i^+(S) + \nu_{m,n}(T) ; i \in I, (m,n) \in (Z^2)^*/ \pm 1\}, \\
(6.12) & \quad \sigma(\square_{1,0}|_{A^{1,0}(T)^- \otimes A^{0,0}(S)^-}) = \sigma(\square^S_{-}) \Pi \{\lambda_j^-(S) + \nu_{m,n}(T) ; j \in J, (m,n) \in (Z^2)^*/ \pm 1\}, \\
& \quad \sigma(\square_{1,0}|_{A^{0,0}(T)^+ \otimes A^{1,0}(S)^+}) = \sigma(\square^S_{+}) \Pi \{\lambda_j^+(S) + \nu_{m,n}(T) ; j \in J, (m,n) \in (Z^2)^*/ \pm 1\}
\begin{align*}
\quad & \Pi \{\lambda_j^-(S) + \nu_{m,n}(T) ; j \in J, (m,n) \in (Z^2)^*/ \pm 1\}, \\
(6.13) & \quad \sigma(\square_{1,0}|_{A^{0,0}(T)^- \otimes A^{1,0}(S)^-}) = \{\lambda_j^-(S) + \nu_{m,n}(T) ; j \in J, (m,n) \in (Z^2)^*/ \pm 1\}. \\
& \quad \Pi \{\lambda_j^-(S) + \nu_{m,n}(T) ; j \in J, (m,n) \in (Z^2)^*/ \pm 1\}.
\end{align*}
The result follows from (6.10), (6.11), (6.12), (6.13), (6.14).
Lemma 6.8. The following equality of meromorphic functions on $\mathbb{C}$ holds:
\[ \zeta_{1,1}(s) = 21\zeta^{T,+}(s) + 5\zeta^{S,+}(s) + 4\zeta^{S,-}(s) + 9\mu^{+}(s) + 9\mu^{-}(s). \]

Proof. Since the decomposition
\[ A^{1,1}(X)^+ = [A^{1,1}(T)^+ \otimes A^{0,0}(S)^+] \oplus [A^{1,1}(T)^- \otimes A^{0,0}(S)^-] \]
\[ \oplus [A^{1,0}(T)^+ \otimes A^{0,1}(S)^+] \oplus [A^{1,0}(T)^- \otimes A^{0,1}(S)^-] \]
\[ \oplus [A^{0,1}(T)^+ \otimes A^{1,0}(S)^+] \oplus [A^{0,1}(T)^- \otimes A^{1,0}(S)^-] \]
is orthogonal and is preserved by $\square_{1,1}$, we get
\[ \sigma(\square_{1,1}) = \sigma(\square^+_{1,1}) \Pi \sigma(\square^-_{1,1}) \]
(6.15)

By (6.7) and Lemmas 6.4 and 6.5 we get
\[ \sigma(\square_{1,1}|_{A^{1,1}(T)^+ \otimes A^{0,0}(S)^+}) = \{0\} \Pi (\sigma(\square^+_{1,1}) \backslash \{0\}) \Pi (\sigma(\square^-_{1,1}) \backslash \{0\}) \Pi \left\{ \lambda^+_i(S) + \nu_{m,n}(T); i \in I, (m,n) \in (\mathbb{Z}^2)^*/\pm 1 \right\} , \]
(6.16)

\[ \sigma(\square_{1,1}|_{A^{1,1}(T)^- \otimes A^{0,0}(S)^-}) = \{\lambda^-_j(S) + \nu_{m,n}(T); j \in J, (m,n) \in (\mathbb{Z}^2)^*/\pm 1 \} , \]
(6.17)

\[ \sigma(\square_{1,1}|_{A^{0,1}(T)^+ \otimes A^{1,0}(S)^+}) = \{\lambda^+_i(S) + \nu_{m,n}(T); i \in I, (m,n) \in (\mathbb{Z}^2)^*/\pm 1 \}
\Pi \left\{ \lambda^-_j(S) + \nu_{m,n}(T); j \in J, (m,n) \in (\mathbb{Z}^2)^*/\pm 1 \right\} , \]
(6.18)

\[ \sigma(\square_{1,1}|_{A^{0,1}(T)^- \otimes A^{1,0}(S)^-}) = \{\lambda^-_j(S) + \nu_{m,n}(T); j \in J, (m,n) \in (\mathbb{Z}^2)^*/\pm 1 \} , \]
(6.19)
Notice that of the factor 2 in (6.25), (6.26) and the corresponding equalities in [12, p. 253]. By (6.25), (6.26), we get

\[ \text{Vol}(X) \]

\[ \square \]

(6.3). Since \( l = l(M) \) and \( l l(M) \) for simplicity. By the definition of \( l = l(M) \), we have \( |A_M| = |M' / M| = 2 \). Since \( \text{sign}(M) = (1, r - 1) \), there exists an integral basis \( \{e_1, \ldots, e_r\} \) of \( H^2(S, \mathbb{Z}) \) such that

\[ \det ((e_i, e_j))_{1 \leq i, j \leq r} = (-1)^{r-1}2^l, \]

where \( \langle \cdot, \cdot \rangle \) is the intersection pairing on \( H^2(S, \mathbb{Z}) \). Then \( \{e_1, \ldots, e_r, \gamma_T \} \) is an integral basis of \( H^2(S \times T, \mathbb{Z}) \) with respect to \( \gamma \). By the same computations as in [12, p. 253], we get

\[ \langle e_i, e_j \rangle_{L^2, \gamma} = (2\pi)^{-3} \left( \frac{2(e_i[\gamma_S]e_j[\gamma_S])}{\langle \gamma_S, \gamma_S \rangle} - \langle e_i, e_j \rangle \right), \]

\[ \langle e_i, [\gamma_T] \rangle_{L^2, \gamma} = 0, \quad \langle [\gamma_T], [\gamma_T] \rangle_{L^2, \gamma} = \frac{\langle \gamma_S, [\gamma_S] \rangle}{(2\pi)^3 \cdot 2^r} = \frac{\text{Vol}(S, \gamma_S)}{2\pi}. \]

Notice that \( \int_T \gamma_T = 2 \), whereas \( \int_T \gamma_T = 1 \) in [12, §13.3]. This explain the difference of the factor 2 in (6.25), (6.26) and the corresponding equalities in [12, p. 253]. By (6.25), (6.26), we get

\[ \text{Vol}_{L^2}(H^2(S \times T, \mathbb{Z})^+, \gamma) = \det \left( \langle e_i, e_j \rangle_{L^2, \gamma}, \langle e_i, [\gamma_T] \rangle_{L^2, \gamma}, \langle [\gamma_T], [\gamma_T] \rangle_{L^2, \gamma} \right) \]

\[ = (-2\pi)^{-3r} \frac{\text{Vol}(S, \gamma_S)}{2\pi} \det \left( \langle e_i, e_j \rangle - \frac{2(e_i[\gamma_S]e_j[\gamma_S])}{\langle \gamma_S, \gamma_S \rangle} \right) \]

\[ = (1)^{-3r}(2\pi)^{-3r-1}\text{Vol}(S, \gamma_S) \cdot (-1) \cdot \det (e_i, e_j) \]

\[ = (2\pi)^{-3r-1}2^l \cdot \text{Vol}(S, \gamma_S). \]
Since \( \text{Sing} X_{(S,\theta,T)} = (S \times T)^{\theta \times (-1)\tau} = S^\theta \times T[2] \) is the 4 copies of \( S^\theta \) and since \( \gamma|_{\text{Sing} X_{(S,\theta,T)}} = \gamma|_{S^\theta} \) for every component of \( (S \times T)^{\theta \times (-1)\tau} \), we get

\[
\text{Vol} \left( \text{Sing} X_{(S,\theta,T)}, \gamma|_{\text{Sing} X_{(S,\theta,T)}} \right) \tau \left( \text{Sing} X_{(S,\theta,T)}, \gamma|_{\text{Sing} X_{(S,\theta,T)}} \right)
= \left\{ \text{Vol}(S^\theta, \gamma|_{S^\theta}) \tau(S^\theta, \gamma|_{S^\theta}) \right\}^4.
\]

By Lemmas 6.8 and 6.8, we get

\[
\sum_{p,q \geq 0} (-1)^{p+q} \zeta_{p,q}(s) = 24\zeta^{T,+}(s) + 8 \left\{ \zeta^{S,+}(s) - \zeta^{S,-}(s) \right\}.
\]

Let \( \zeta^T(s) \) be the spectral zeta function of \( \square^T_{0,0} \). Since \( \zeta^{T,+}(s) = \zeta^{T,-}(s) = \zeta^T(s)/2 \), we get by (6.29)

\[
T_{BCOV}(X_{(S,\theta,T)}, \gamma) = \exp \left( -12 \frac{d}{ds} \bigg|_{s=0} \zeta^T(s) - 8 \frac{d}{ds} \bigg|_{s=0} \left\{ \zeta^{S,+}(s) - \zeta^{S,-}(s) \right\} \right)
= \tau_{Z_2(S,\gamma|S)}(\theta)^{-4} \tau(T,\gamma_T)^{-12},
\]

where the second equality follows from [11] Lemma 4.3 and the definition of \( \tau(T,\gamma_T) \). Since \( \gamma, \gamma_S, \gamma_T \) are Ricci-flat, it follows from (5.11), (2.4), (6.1), (6.24), (6.27), (6.28), (6.30) that

\[
\tau_{\text{orb}}_{BCOV}(X_{(S,\theta,T)})
= \frac{1}{T_{BCOV}(X_{(S,\theta,T)}, \gamma) \tau \left( \text{Sing} X_{(S,\theta,T)}, \gamma|_{\text{Sing} X_{(S,\theta,T)}} \right)^{-1}}
\times \text{Vol}(X_{(S,\theta,T)}, \gamma|_{\text{Sing} X_{(S,\theta,T)}})^{-1}
\times \text{Vol} \left( X_{(S,\theta,T)}, \gamma|_{\text{Sing} X_{(S,\theta,T)}} \right)^{-1}
= \tau_{Z_2(S,\gamma|S)}(\theta)^{-4} \tau(T,\gamma_T)^{-12} \left\{ \text{Vol}(S^\theta, \gamma|_{S^\theta}) \tau(S^\theta, \gamma|_{S^\theta}) \right\}^{-4}
\times \frac{1}{\text{Vol}(S, \gamma|S)/2\pi} \cdot (2\pi)^{3+1/2} \cdot \tau_{\text{ell}}(T)^{-12}.
\]

Here we used the equalities \( \chi(S \times T) = 0 \), \( \chi(S^\theta \times T[2]) = 8(r - 10) \) and \( \tau_{\text{ell}}(T) = \pi^{-1} \tau(T,\gamma_T) \) to get the last equality.

6.4. A counter example to an analogue of Theorem 6.2 It seems that an analogue of Theorem 6.2 for global Calabi-Yau orbifolds \( X/G \) does not hold in general. The pair of quintic threefolds and its automorphism group \( G \cong (\mathbb{Z}/5\mathbb{Z})^3 \) in [20] seems to give a counter example.

For \( \psi \in C \), let \( X_\psi \subset P^4 \) be the quintic threefold defined by the equation

\[
(z_1)^5 + (z_2)^5 + (z_3)^5 + (z_4)^5 + (z_5)^5 - 5\psi z_1 z_2 z_3 z_4 z_5 = 0.
\]

On \( X_\psi \), we have the projective action of the following group \( G \subset \text{PSL}(C^5) \)

\[
G = \{ \text{diag}(a_1, a_2, a_3, a_4, a_5); a_i \in \mu_5, a_1 a_2 a_3 a_4 a_5 = 1 \}/\mu_5 I_5 \cong (\mathbb{Z}/5\mathbb{Z})^3,
\]

where \( \mu_5 \subset C^* \) is the group of 5-th roots of 1 and \( I_5 \) is the 5 × 5-identity matrix. The \( G \)-action on \( X_\psi \) is admissible and \( X_\psi/G \) is a Calabi-Yau orbifold for \( \psi \not\in \mu_5 \).

The set of fixed curves for the \( G \)-action on \( X_\psi \) is given by \( \{ C_{ijk} \}_{1 \leq i < j < k \leq 5} \)

\[
C_{ijk} = \{ \{ z \in X_{\psi}; z_l = 0 (\forall l \neq i, j, k) \} \}.
\]

Then \( G_{C_{ijk}} \cong \{ \text{diag}(\zeta, \zeta^{-1}, 1); \zeta \in \mu_5 \} \cong \mathbb{Z}/5\mathbb{Z} \) and \( C_{ijk} \) is a curve of genus 6.
The set of isolated fixed points of the $G$-action on $X_{\psi}$ is given by \{$S_{ij}\}_{1 \leq i < j \leq 5}:
\[ S_{ij} = \{ [z] \in X_{\psi}; z_l = 0 (\forall l \neq i,j) \}. \]

Here $S_{ij}$ consists of 5 distinct points $\{p^{(1)}_{ij}, \ldots, p^{(5)}_{ij}\}$ and $S_{ij} \subset C_{ijk}$. One can verify that $G_{p^{(k)}_{ij}} \cong \{ \text{diag}(\zeta, \zeta', (\zeta \zeta')^{-1}); \zeta, \zeta' \in \mu_5 \} \cong (\mathbb{Z}/5\mathbb{Z})^{\oplus 2}$. By this expression, $G_{p^{(k)}_{ij}} \subset \text{SL}(\mathbb{C}^3)$ is admissible for all $p^{(k)}_{ij}$ and hence the $G$-action on $X_{\psi}$ is admissible. Since $G^0_{p^{(k)}_{ij}} = \{ \text{diag}(\zeta, \zeta', (\zeta \zeta')^{-1}); \zeta, \zeta' \in \mu_5 \setminus \{1\}, \zeta' \neq \zeta^{-1} \}$, we get by (4.24)
\[ \delta(G_{p^{(k)}_{ij}}) = 3 \times \sum_{\zeta \in \mu_5 \setminus \{1\}} \frac{\zeta}{(1 - \zeta)^2} = -6. \]

Since $h^{2,1}(X_{\psi})^G = 1$, $\chi(X_{\psi}) = -200$, we get
\[ \bar{\chi}(X_{\psi}, G) = \chi(X_{\psi}) + \sum_{1 \leq i < j < k \leq 5} (5^2 - 1) \chi(C_{ijk}) - 12 \sum_{1 \leq i < j \leq 5} \sum_{k=1}^5 \delta(G_{p^{(k)}_{ij}}) = 1000. \]

Since $h^{1,2}(X_{\psi})^G = 1$ and since $C_{ijk}$ and $S_{ij}$ do not deform as $\psi$ varies, we deduce from Theorem 6.10, the following differential equation on $C \setminus \mu_5$:
\[ -dd^c \log \tau_{BCOV}^G(X_{\psi}/G) = \frac{14}{3} \omega_{WP}^G + \text{Ric} \omega_{WP}^G. \]

On the other hand, let $\tilde{X}_{\psi}$ be any crepant resolution of $X_{\psi}/G$, which is called a quintic mirror threefold. Since $h^{2,1}(\tilde{X}_{\psi})^G = 1$, $\chi(\tilde{X}_{\psi}) = 200$, it follows from [12, Th. 4.14] that $\tau_{BCOV}(\tilde{X}_{\psi})$ satisfies the following differential equation on $C \setminus \mu_5$:
\[ -dd^c \log \tau_{BCOV}(\tilde{X}_{\psi}) = \frac{62}{3} \omega_{WP}^G + \text{Ric} \omega_{WP}^G. \]

Comparing (6.32) and (6.33), we conclude that there exists no constant $C$ such that for all $\psi \in C \setminus \mu_5$
\[ \tau_{BCOV}^G(X_{\psi}/G) = C \tau_{BCOV}(\tilde{X}_{\psi}). \]

In particular, an analogue of Theorem 6.2 does not hold for the Calabi-Yau orbifolds $X_{\psi}/G$.

Possibly, our definition of BCOV invariants for Calabi-Yau orbifolds given in Definition 5.1 may be too naive for Theorem 6.2 to be true in general.

**Question 6.9.** Can one extend BCOV invariants for general Calabi-Yau orbifolds so that the following analogue of Theorem 6.2 holds?
\[ \tau_{BCOV}^G(X) = C(X, \tilde{X}) \tau_{BCOV}(\tilde{X}). \]

Here $X$ is a Calabi-Yau orbifold of dimension three, $\tilde{X}$ is its (possibly some specific) crepant resolution, and $C(X, \tilde{X})$ is a constant depending only on the topologies of $X$ and $\tilde{X}$.

Question 6.9 is closely related to another conjecture [12, Conjecture 4.17] claiming the birational invariance of BCOV invariants. We refer the reader to [24] for a current progress on this latter conjecture.
References

[1] Alexeev, V., Nikulin, V.V. Del Pezzo and K3 surfaces, MSJ Memoires 15, Math. Soc. Japan (2006)
[2] Berline, N., Getzler, E., Vergne, M. Heat Kernels and Dirac Operators, Springer, Berlin (1992).
[3] Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C. Holomorphic anomalies in topological field theories, Nuclear Phys. B 405 (1993), 279–304.
[4] Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C. Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (1994), 311–427.
[5] Bismut, J.-M. Equivariant immersions and Quillen metrics, J. Differential Geom. 41 (1995), 53–157.
[6] Bismut, J.-M., Gillet, H., Soulé, C. Analytic torsion and holomorphic determinant bundles I, II, III, Commun. Math. Phys. 115 (1988), 49–78, 79–126, 301–351.
[7] Bismut, J.-M., Lebeau, G. Complex immersions and Quillen metrics, Publ. Math. IHES 74 (1991), 1–297.
[8] Borcea, C. K3 surfaces with involution and mirror pairs of Calabi–Yau manifolds, Mirror Symmetry II (B. Green and S.-T. Yau eds) AMS/International Press (1997), 717–743.
[9] Borcea, C. Equivariant analytic torsion, a new definition, Commun. Math. Phys. 130 (1982), 39–63.
[10] Broustet, A., Cartier, P. Coarse moduli space for compact Kähler manifolds, Publ. RIMS, Kyoto Univ. 10 (1975), 977–1005.
[11] Cattani, E., Kaplan, A., Schmid, W. Degeneration of Hodge structures, Ann. of Math. 123 (1986), 457–535.
[12] Fang, H., Lu, Z. Generalized Hodge metrics and BCOV torsion on Calabi-Yau moduli, J. reine angew. Math. 588 (2005), 49–69.
[13] Fang, H., Lu, Z., Yoshikawa, K.-I. Analytic torsion for Calabi–Yau threefolds, J. Differential Geom. 80 (2008), 175–250.
[14] Fujiki, A. Coarse moduli space for compact Kähler manifolds, Publ. RIMS, Kyoto Univ. 20 (1984), 977–1005.
[15] Grauert, H., Remmert, R. Plurisubharmonische funktionen in komplexen Räumen, Math. Zeit. 65 (1956), 175–194.
[16] Gross, M., Wilson, P.M.H. Mirror symmetry via 3-tori for a class of Calabi-Yau threefolds, Math. Ann. 309 (1997), 505–531.
[17] Harvey, J., Moore, G. Exact gravitational threshold correction in the Ferrara-Harvey-Strominger-Vafa model, Phys. Rev. D 57 (1998), 2329–2336.
[18] Kawamata, Y. Characterization of abelian varieties, Compositio Math. 43 (1981), 253–276.
[19] Klemm, A., Marino, M. Counting BPS states on the Enriques Calabi-Yau, Commun. Math. Phys. 280 (2008), 27–76.
[20] Kuranishi, M. Deformations of compact complex manifolds, Les Presses de l’Université de Montréal, (1971).
[21] Lu, Z. On the Hodge metric of the universal deformation space of Calabi-Yau threefolds, J. Geom. Anal. 11 (2001), 103–118.
[22] Ma, S., Yoshikawa, K.-I. K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space IV, in preparation
[23] Ma, X. Submersions and equivariant Quillen metrics, Ann. Inst. Fourier 50 (2000), 1539–1588.
[24] Ma, X. Orbifolds and analytic torsion, Trans. Amer. Math. Soc. 357 (2005), 2205–2233.
[25] Maulik, D., Pandharipande, R. New calculations in Gromov-Witten theory, Pure Appl. Math. Q. 4 (2008) 469–500.
[26] Morrison, D. Mirror symmetry and rational curves on quintic threefolds: a guide for mathematicians, J. Amer. Math. Soc. 6 (1993), 223–247.
[27] Morrison, D. Compactifications of moduli spaces inspired by mirror symmetry, Astérisque 218 (1993), 243–271.
[28] Nikulin, V.V. Finite automorphism groups of Kähler K3 surfaces, Trans. Moscow Math. Soc. 38 (1980), 71–135.
[29] Nikulin, V.V. Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections, J. Soviet Math. 22 (1983), 1401–1476.
[30] Ray, D.B., Singer, I.M. Analytic torsion for complex manifolds, Ann. of Math. 98 (1973), 154–177.
[31] Schmid, W. Variation of Hodge structure: The singularities of the period mapping, Invent. Math. 22 (1973), 211–319.
[32] Schumacher, G. Moduli of polarized Kähler manifolds, Math. Ann. 269 (1984), 137–144.
[33] Siu, Y.-T. Analyticity of sets associated to Lelong numbers and the extension of closed positive currents, Invent. Math. 27 (1974), 53–156.
[34] Tian, G. Smoothness of the universal deformation space of Compact Calabi-Yau manifolds and its Peterson-Weil metric, Mathematical Aspects of String Theory (ed. S.-T. Yau), World Scientific (1987), 629–646.
[35] Todorov, A. The Weil-Petersson geometry of the moduli space of SU(n ≥ 3) (Calabi-Yau) manifolds I, Commun. Math. Phys. 126 (1989), 325–346.
[36] Viehweg, E. Quasi-Projective Moduli for Polarized Manifolds, Springer, Berlin (1995).
[37] Voisin, C. Miroirs et involutions sur les surfaces K3, Journée de Géométrie Algébrique d’Orsay (Orsay, 1992), Astérisque 218 (1993), 273–323.
[38] Weil, A. Elliptic Functions According to Eisenstein and Kronecker, Classics in Mathematics, Springer, Berlin (1999).
[39] Yau, S.-T. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère Equation, I, Commun. Pure Appl. Math. 31 (1978), 339–411.
[40] Yoshikawa, K.-I. K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space, Invent. Math. 156 (2004), 53–117.
[41] Yoshikawa, K.-I. Real K3 surfaces without real points, equivariant determinant of the Laplacian, and the Borcherds \( \Phi \)-function, Math. Zeit. 258 (2008), 213–225.
[42] Yoshikawa, K.-I. Calabi-Yau threefolds of Borcea–Voisin, analytic torsion, and Borcherds products, From Probability to Geometry (II), Volume in honor of Jean-Michel Bismut, ed. by X. Dai, R. Leandre, X. Ma, W. Zhang, Astérisque 328 (2009), 351–389.
[43] Yoshikawa, K.-I. K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space II, J. reine angew. Math. 677 (2013), 15–70.
[44] Yoshikawa, K.-I. K3 surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space III, Math. Zeit. 272 (2012), 175–190.
[45] Yoshikawa, K.-I. Degenerations of Calabi-Yau threefolds and BCOV invariants, preprint arXiv:1409.0127 (2014).
[46] Zinger, A. The reduced genus 1 Gromov-Witten invariants of Calabi-Yau hypersurfaces, J. Amer. Math. Soc. 22 (2009), 691–737.

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