In this paper we study Gödel universe in the framework of $f(R, T)$ modified theories of gravity, where $R$ is the curvature scalar and $T$ the trace of the energy momentum tensor. We demonstrate that Gödel solution occurs in this modified theory and still we suggest a path to understanding the smallness of the cosmological constant.

Keywords: modified gravity; Gödel universe; cosmological constant.

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1. Introduction

The interest to the modified gravity theories is caused by two reasons. The first of them is a quantum one, that is, the problem of constructing the perturbatively consistent gravity theory: it is well known that the usual Einstein gravity is non-renormalizable, whereas, for example, the higher-derivative gravity models such as $R^2$ gravity, suffer from the problem of ghosts, therefore, one needs to develop other models. The second reason is a cosmological one, that is, one must find a consistent gravity theory allowing for explanation of the cosmic acceleration and explain the smallness of the cosmological constant. A number of the theories has been discussed in this context (see for a review [1]). One of the new ideas in this context is the generalization of the well-known concept of the $f(R)$ gravity (for a review, see f.e. [2, 3, 4]) proposing the introduction of a function whose argument is not only the scalar curvature $R$ but also another important scalar, the trace of the energy-momentum tensor of the matter. This concept naturally emerges as a continuation of the idea of $\Lambda(T)$ gravity [5] which claims that the cosmological constant is a function of the trace of the energy-momentum tensor, $T$. The idea that the Lagrangian is a function of $R$ and $T$ has been proposed in [6] where the equations of motion were derived for the several forms of the function $f(R, T)$, the consistency of the Friedmann-Robertson-Walker (FRW) metric for these forms of $f(R, T)$ was verified within this theory.

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and the general relativity limit has been discussed. Some cosmological aspects of the \( f(R, T) \) gravity have been also discussed in \cite{7, 8, 9} where, in particular, it was shown that the matter compatible with the FRW metric displays a quintom-like behaviour for a certain form of \( f(R, T) \). In this context others studies were performed, for example, the laws of the thermodynamics were verified in \cite{10} in \cite{11}, wormhole solutions were studied, finite-time future singularity was discussed in \cite{12} and in \cite{13} was shown that the energy conditions, as known in general relativity, can also be applied in this modified theory. The \( f(R, T) \) modified theory of gravity is an interesting application of the latter \( f(R, L_m) \) gravity proposed in \cite{14}, where the field equations are equivalent to the field equations of the \( f(R) \) gravity for empty space-time, but differ from them, as well as from general relativity, when the matter is present.

As the \( f(R, T) \) gravity has been studied in several cosmological contexts, the problem of consistency of well-known metrics within this theory naturally arises. One of the first problems in this context is whether the Gödel metric \cite{15}, known as a simplest metric allowing for the closed timelike curves (CTCs) is compatible with this theory, or, as is the same, whether the CTCs can be consistent within this theory? This is the problem we address in this paper.

The present paper is organized as follow. In the section 2 we will quickly review the field equations of \( f(R, T) \) gravity. The Gödel universe within this modified theory is verified in the section 3. We discuss and conclude in section 4.

2. Field equations in the \( f(R, T) \) gravity

The action of the \( f(R, T) \) gravity model \cite{5} is

\[
S = \frac{1}{2} \int \sqrt{-g} \left[ f(R, T) + 2L_m \right] d^4x, \tag{1}
\]

where \( f(R, T) \) is an arbitrary function of the scalar curvature \( R \), and of the trace \( T \) of the energy-momentum tensor of the matter. The dependence from \( T \) is investigated as a possible source to introduce exotic fluids or quantum effects (conformal anomaly). The \( L_m \) is the matter Lagrangian. In this work, for the sake of the simplicity, we will use the system of units where \( 8\pi G = 1 \). The energy momentum tensor is defined as

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta(\sqrt{-gL_m}) \frac{\delta g_{\mu\nu}}{\delta g^{\mu\nu}}, \tag{2}
\]

and its trace is \( T = g^{\mu\nu}T_{\mu\nu} \). Assuming that \( L_m \) depends only on the components of \( g_{\mu\nu} \), and not on their derivatives, and using the well-known properties of the variation with respect to the metric components \cite{16}, we get

\[
T_{\mu\nu} = g_{\mu\nu}L_m - 2 \frac{\partial L_m}{\partial g^{\mu\nu}}. \tag{3}
\]
First of all, let us briefly discuss the derivation of the equations of motion in this theory. Varying the action (1) with respect to the metric, we arrive at

$$\delta S = \frac{1}{2} \int \left[ f_R(R, T) \delta R + f_T(R, T) \frac{\delta T}{\delta g} \delta g^{\mu \nu} - \frac{1}{2} g_{\mu \nu} f(R, T) \delta g^{\mu \nu} + \frac{2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu \nu}} \sqrt{-g} d^4x, \right]$$

where

$$f_R \equiv \frac{\partial f}{\partial R}, \quad f_T \equiv \frac{\partial f}{\partial T}.$$  

Here, the variation of the scalar curvature is

$$\delta R = R_{\mu \nu} \delta g^{\mu \nu} + g_{\mu \nu} \Box \delta g^{\mu \nu} - \nabla_\mu \nabla_\nu \delta g^{\mu \nu},$$

and the variation of $T$ can be found as

$$\delta T \delta g^{\mu \nu} = \delta g_{\alpha \beta} T_{\alpha \beta} + g_{\alpha \beta} \frac{\delta T_{\alpha \beta}}{\delta g^{\mu \nu}} = T_{\mu \nu} + \Theta_{\mu \nu},$$

with

$$\Theta_{\mu \nu} \equiv g^{\alpha \beta} \frac{\delta T_{\alpha \beta}}{\delta g^{\mu \nu}}.$$

Assuming that the matter Lagrangian is given by $L_m = -p$, as in (5), we rest with

$$\Theta_{\mu \nu} = -2T_{\mu \nu} - 2g^{\alpha \beta} \frac{\partial^2 L_m}{\partial g^{\alpha \beta} \partial g^{\mu \nu}}.$$  

Now let us study the field equations (8) for the G"odel universe.

3. Testing the G"odel universe in the $f(R, T)$ gravity

Now, let us verify whether the G"odel metric which is written as

$$ds^2 = a^2 \left[ dt^2 - dx^2 + \frac{1}{2} e^{2x} dy^2 - dz^2 + 2e^x dt dy \right],$$

where $a$ is a positive number, is compatible with this gravitational model. The corresponding non-zero components of the Ricci tensor look like

$$R_{00} = 1, \quad R_{02} = R_{20} = e^x, \quad R_{22} = e^{2x},$$

Then,

$$\Theta_{\mu \nu} = -2T_{\mu \nu} - pg_{\mu \nu}.$$  

Assuming that the matter Lagrangian is given by $L_m = -p$, as in (5), we rest with

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Now let us study the field equations (8) for the G"odel universe.
and the scalar curvature is
\[ R = \frac{1}{a^2}. \]  

(14)

We suppose that the energy-momentum tensor has the same structure as in 15, that is
\[ T_{\mu\nu} = 8\pi\rho u_\mu u_\nu + \Lambda g_{\mu\nu}, \]

(15)

where \( u_\mu = (a, 0, ae^x, 0) \) and \( \Lambda \) is the cosmological constant. The non-zero components of the \( T_{\mu\nu} \) are
\[ T_{00} = (8\pi\rho + \Lambda)a^2, \]
\[ T_{02} = (8\pi\rho + \Lambda)a^2 e^x, \]
\[ T_{11} = -\Lambda a^2, \]
\[ T_{22} = \left(8\pi\rho + \frac{\Lambda}{2}\right)a^2 e^{2x}, \]
\[ T_{33} = -\Lambda a^2. \]

(16)

The trace \( T \) of the energy momentum tensor is
\[ T = 8\pi\rho + 4\Lambda. \]

(17)

We will study this compatibility between Gödel metric and \( f(R, T) \) gravity for some special cases:

3.1. \( f(R, T) = f_1(R) + f_2(T) \)

In this case the field equation becomes
\[ f_1'(R)R_{\mu\nu} - [f_1(R) + f_2(T)]\frac{1}{2}g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_1'(R) \]
\[ = T_{\mu\nu} - f_2'(T)T_{\mu\nu} - f_2'(T)\Theta_{\mu\nu}, \]

(18)

where the prime denotes a derivative with respect to the argument.

Assuming that matter is a perfect fluid and, for simplicity, we choose the case where the pressure is zero, \( p = 0 \). We have \( \Theta_{\mu\nu} = -2T_{\mu\nu} \) and we stay with
\[ f_1'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f_1(R) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f_1'(R) \]
\[ = T_{\mu\nu} + f_2'(T)T_{\mu\nu} + \frac{1}{2}g_{\mu\nu}f_2(T). \]

(19)

We can reformulate this equation as an effective Einstein field equation
\[ R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = G_{\text{eff}} T_{\mu\nu} + T_{\mu\nu}^{\text{eff}}, \]

(20)
where
\[ G_{\text{eff}} = \frac{1}{f_1'(R)} \left[ 1 + f_2'(T) \right], \]
\[ T_{\mu\nu}^\text{eff} = \frac{1}{f_1'(R)} \left[ \frac{1}{2} \left( f_1(R) - R f_1'(R) + f_2(T) \right) g_{\mu\nu} - \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_1'(R) \right]. \] (21)

Therefore, the gravitational coupling is dependent on the matter.

Making the following choices \( f_1(R) = R \) and \( f_2(T) = 2\lambda T \), where \( \lambda \) is a constant we obtain
\[ G_{\text{eff}} = 1 + 2\lambda, \]
\[ T_{\mu\nu}^\text{eff} = \lambda T g_{\mu\nu}. \] (22)

Then the equation (20) becomes
\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \left( 1 + 2\lambda \right) T_{\mu\nu} + \lambda T g_{\mu\nu}. \] (23)

The non-zero components of the motion equations, i.e., (00), (11) and (22) are
\[ \frac{1}{2} = 8\pi\rho(1 + 3\lambda)a^2 + \Lambda(1 + 6\lambda)a^2, \]
\[ \frac{1}{2} = -8\pi\lambda a^2 - \Lambda(1 + 6\lambda)a^2, \]
\[ \frac{3}{4} = 8\pi\rho \left( 1 + \frac{5\lambda}{2} \right) a^2 + \frac{\Lambda}{2}(1 + 6\lambda)a^2. \] (24)

The components (00) and (02) are the same, as well as (11) and (33). We find as solution these equations
\[ 8\pi\rho = \frac{1}{a^2(1 + 2\lambda)}, \]
\[ \Lambda = -\frac{1 + 4\lambda}{2a^2(1 + 2\lambda)(1 + 6\lambda)}. \] (25)

If \( \lambda = 0 \) we recovered the result obtained by Gödel in [15], \( 8\pi\rho = \frac{1}{a^2} \) and \( \Lambda = -\frac{1}{2a^2}. \)

3.2. \( f(R, T) = f_1(R) + f_2(R)f_3(T) \)

In this case the field equations \([6]\) for the case of a perfect fluid, with \( p = 0 \) (dust), becomes
\[ \left[ f_1'(R) + f_2'(R)f_3(T) \right] R_{\mu\nu} - \frac{1}{2} f_1'(R) g_{\mu\nu} + \left( g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f_1'(R) = 0, \]
\[ = T_{\mu\nu} + f_2(R)f_3'(T)T_{\mu\nu} + \frac{1}{2} f_2'(R)f_3'(T)g_{\mu\nu}. \] (26)

Again, following the suggestions made in \([6]\), i.e., \( f_1(R) = R \), \( f_2(R) = R \) and \( f_3(T) = \lambda T \), we obtain \( f_1'(R) = f_2'(R) = 1 \) and \( f_3'(T) = \lambda \). Applying to the Gödel metric, and
we assume that the trace of the energy-momentum tensor \( (17) \) is a constant, the field equations are reduced to
\[
(1 + \lambda)R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu} + R\lambda T_{\mu\nu} + \frac{1}{2}R\lambda T g_{\mu\nu}. \tag{27}
\]
The non-zero components, i.e., (00), (11) and (22), form the system
\[
\begin{align*}
\frac{1}{2} &= 8\pi\rho \left( a^2 + \frac{3}{2}\lambda \right) + \Lambda(a^2 + 3\lambda) - \lambda, \\
\frac{1}{2} &= -8\pi\rho \left( \frac{\lambda}{2} \right) - \Lambda(a^2 + 3\lambda), \\
\frac{3}{4} &= 8\pi\rho \left( a^2 + \frac{5\lambda}{4} \right) + \frac{\Lambda}{2}(a^2 + 3\lambda) - \lambda,
\end{align*} \tag{28}
\]
whereas the component (02) is identical to (00), and (33) to (11). Solving this system, we obtain
\[
\begin{align*}
8\pi\rho &= \frac{1 + \lambda}{a^2 + \lambda}, \\
\Lambda &= -\frac{a^2 + \lambda(2 + \lambda)}{2(a^2 + \lambda)(a^2 + 3\lambda)}.
\end{align*} \tag{29}
\]
If \( f_3(T) = 0 \) we recovered the result obtained in \([15]\).

4. Conclusion
To conclude, we found that in certain cases, i.e., equations \([23]\) and \([29]\), the Gödel metric solves the equations of motion in the \( f(R, T) \) gravity. So, the possibility for arising the CTCs taking place in a general relativity can be naturally promoted to the \( f(R, T) \) gravity. We also observe that the cosmological constant depend on the geometry and matter. In principle, if the parameter \( \lambda \) associated with the function \( f(T) \) is small, i.e., \( \lambda << 1 \) and \( \lambda << a \), we recover the usual results of general relativity for the metric Gödel. In the case where \( \lambda >> 1 \) and \( \lambda >> a \), we obtain for first case \( \Lambda \approx -\frac{1}{6a^2\lambda} \) and for the second case we find a value independent of \( \lambda \) for the cosmological constant. Therefore the particular case, \( f(R, T) = f_1(R) + f_2(T) \), for a convenient choice of \( \lambda \) shows itself interesting because it may indicate a path to understanding the smallness of the cosmological constant.

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