Well-posedness for SQG sharp fronts with unbounded curvature

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ABSTRACT. Patch solutions for the surface quasigeostrophic (SQG) equation model sharp temperature fronts in atmospheric and oceanic flows. We establish local well-posedness for SQG sharp fronts of low Sobolev regularity, $H^{2+s}$ for arbitrarily small $s > 0$, allowing for fronts with unbounded curvature.

1. Introduction

We study the surface quasigeostrophic (SQG) equation

$$\partial_t \theta(x, t) + u(x, t) \cdot \nabla \theta(x, t) = 0, \quad (x, t) \in \mathbb{R}^2 \times \mathbb{R}_+,$$

(1.1)

$$u = \nabla^\perp \Lambda^{-1} \theta, \quad \Lambda = (-\Delta)^{\frac{1}{2}}.$$

(1.2)

The SQG equation of geophysical significance comes from the 3D quasigeostrophic (QG) equation in the half-space. The QG equation arises in the study of stratified flows in which the Coriolis force is balanced with the pressure and serves as a model in simulations of large-scale atmospheric and oceanic circulation [24]. In the ideal case of zero bulk vorticity, the QG equation reduces to a self-consistent equation on the boundary—the SQG equation (1.1)-(1.2). SQG shares many analogies with the 3D Euler equations [9] and has been studied extensively. In particular, vortex lines for the 3D Euler equations are analogous to levels sets of the scalar $\theta$.

SQG is an active scalar equation, i.e. the transport velocity is given in terms of the transported scalar. A class of active scalar equations, called generalized SQG (gSQG), is given by (1.2) with the velocity$$u = \nabla^\perp \Lambda^{-2+\alpha} \theta, \quad \alpha \in [0, 1].$$
The case $\alpha = 0$ corresponds to the vorticity formulation of the 2D Euler equations. The scalar-to-velocity relation becomes more singular as $\alpha$ increases.

While the SQG equation is locally well-posed in Sobolev spaces $H^s(\mathbb{R}^2)$, $s > 2$, the question of global regularity versus finite-time singularity remains open [9]. This is also the case for gSQG for any $\alpha \in (0, 1)$ [8]. Nevertheless, certain blow-up scenarios have been ruled out; eg. collision of level sets in the development of sharp fronts [12], hyperbolic saddle breakdown [11]. There exist families of non-trivial global solutions [5] but some small initial data provide growth for all time if lack of finite time singularities is assumed [20]. On the other hand, $L^p$ weak solutions are known to exist globally, but their uniqueness is unknown. More precisely, by appealing to the commutator structure of SQG’s nonlinearity in the weak form, global weak solutions were obtained for $L^2$ [25] and lower $L^p$ integrability ($p > 4/3$) [23]. Nonunique weak solutions with negative Sobolev regularity were constructed in [3]. $L^p$ weak solutions include the important class of patch solutions which are given by the characteristic function of an evolving domain $D(t) \subset \mathbb{R}^2$:

$$\theta(x, t) = \begin{cases} \theta^0 & \text{if } x \in D(t), \\ 0 & \text{if } x \notin D(t), \end{cases}$$

(1.3)

where $\theta^0$ is a nonzero constant. In other words, $D(t)$ is the level set $\{\theta(t) = \theta^0\}$. Patch solutions of the 2D Euler equations are also called vortex patches [2]. SQG patches are also called sharp fronts as they model sharp temperature fronts in atmospheric and oceanic flows. Unlike general weak solutions, sharp fronts

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are known to be unique. Precisely, [26] proved local-in-time existence of $C^\infty$ periodic sharp fronts using a Nash-Moser inverse function theorem. Here, the regularity is that of $\partial D(t)$. The first local-in-time existence result for finite regularity [16] was obtained in $H^k$, $k \geq 3$. Uniqueness was subsequently established in [10] for sharp front weak solutions with bounded curvature; namely, $\partial D(t) \in C^{2+\delta}$. Recently, [17] proved the existence of $H^2$ patches for SQG with $\alpha \in (0, 1)$, the proof of which breaks down when $\alpha = 1$ (SQG) and no uniqueness was given for $\alpha \in [\frac{1}{2}, 1)$. Note that for $H^2$ fronts, the curvature is merely $L^2$. In view of the aforementioned results, a natural question is whether SQG sharp fronts with unbounded curvature are well-posed (existence and uniqueness). In the present paper, we affirmatively answer this by establishing local well-posedness of SQG sharp fronts in $H^{2+s}$ with arbitrarily small $s > 0$.

**Theorem 1.1.** Let $s \in (0, \frac{1}{2})$ be arbitrary. For any initial bounded domain $D(0)$ with non self-intersecting $H^{2+s}$ boundary, there exist $T > 0$ and a unique solution $\theta$ of (1.1)-(1.3) on $[0, T]$ such that the boundary of $D(t)$ remains non self-intersecting and belongs to $H^{2+s}$.

The importance of curvature in the study of sharp temperature fronts is supported by numerical analysis and theoretical results. In [13], numerical evidence of curvature blow-up of the front boundary is shown when two distinct particles on the boundary collide at the same point. Later on, [18] proved that uniform control of the curvature prevents these particle collision type singularities. Further numerics [27] indicate curvature blow-up for different singular solutions, specifically a cascade of filament instabilities with width tending to zero in finite time. On the other hand, [17] proved that $H^3$ solutions can be continued provided that the contour is non self-intersecting and its $W^{2,p}$ norm is controlled for some $p > 2$.

Recently there has been a lot of effort to obtain different type of SQG front solutions. Almost sharp fronts were given in [14][15] establishing high order asymptotics of smoothed out sharp fronts to exact sharp fronts. [4] established the first nontrivial global-in-time SQG fronts which are uniformly rotating smooth solutions. Surprisingly there also exist a family of nontrivial analytic stationary fronts [19]. In the case of infinite near planar fronts, recent works [22][21] gave global-in-time solutions of small slope. Existence and construction of co-rotating and traveling pairs for the SQG equations were recently given in [6][7]. The main result in this paper provides the first construction of general fronts with low regularity, allowing unbounded curvature.

### 1.1. Contour dynamics equation and reparametrization

Computing the normal velocity at the patch boundary, an evolution equation for the boundary can be obtained as

$$
\partial_t x(\gamma, t) = \int_T \frac{\partial_{\eta} x(\gamma, t) - \partial_{\eta} x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|} d\eta. \tag{1.4}
$$

See e.g. [16] for a detailed derivation. The singularity in the denominator of (1.4) can be compared to the difference in the parameter variable by $L^\infty$ control of the arc chord quantity,

$$
F(x)(\gamma, \eta, t) = \frac{|\eta|}{|x(\gamma, t) - x(\gamma - \eta, t)|}. \tag{1.5}
$$

This quantity blows up if the patch self-intersects or if the parametrization artificially causes the blow up. To rule out this second possibility, one can reformulate the evolution equation to choose a parametrization such that the modulus of the tangent vector to the boundary only depends on time, i.e.

$$
|\partial_{\gamma} x(\gamma, t)|^2 = A(t). \tag{1.6}
$$

Furthermore, this choice of parameter will yield several useful techniques to deal with the low regularity of the sharp fronts. Adding a scalar multiple of the tangent $\partial_{\gamma} x$ to (1.4) does not change the shape of the patch boundary because it is independent of the normal velocity. Following the derivation in a previous paper [16], the scalar multiple can be computed explicitly. This yields the needed contour evolution equation satisfying the property (1.6):

$$
\partial_t x(\gamma, t) = \int_T \frac{\partial_{\eta} x(\gamma, t) - \partial_{\eta} x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^2} d\eta + \lambda(\gamma, t) \partial_{\gamma} x(\gamma, t), \tag{1.7}
$$
\[ \lambda(\gamma, t) = \frac{\pi + \gamma}{2\pi} \int_T \int_{\pi} \frac{\partial_\gamma (\partial_\gamma x(\gamma) - \partial_\eta x(\gamma - \eta))}{|x(\gamma) - x(\gamma - \eta)|} \cdot \frac{\partial_\gamma x(\gamma)}{|\partial_\gamma x(\gamma)|^2} \, d\eta d\gamma 
\]

\[ - \int_{-\pi}^{\pi} \int_{\pi} \frac{\partial_\gamma (\partial_\gamma x(\xi) - \partial_\eta x(\xi - \eta))}{|x(\xi) - x(\xi - \eta)|} \cdot \frac{\partial_\gamma x(\xi)}{|\partial_\gamma x(\xi)|^2} \, d\eta d\xi. \]  

(1.8)

Using the preceding parametrization, Theorem 1.1 can be restated as

**Theorem 1.2.** Let \( s \in (0, \frac{1}{4}) \) be arbitrary. Let \( x^0(\gamma) \in H^{2+s}(\mathbb{T}) \) such that \( \partial_\gamma x^0 \cdot \partial_\eta^2 x^0 = 0 \) and \( F(x^0) \in L^\infty(\mathbb{T} \times \mathbb{T}) \). There exists \( T > 0 \) depending only on \( \|x^0\|_{H^{2+s}} \) and \( \|F(x^0)\|_{L^\infty} \) such that (1.7)-(1.8) has a unique solution \( x \in C([0, T]; H^{2+s}(\mathbb{T})) \) satisfying \( F(x) \in L^\infty(\mathbb{T} \times \mathbb{T} \times [0, T]) \) and \( x|_{t=0} = x^0 \).

1.2. **On the proof.** The right side of equation (1.7) has a logarithmic singularity \( 1/|\eta| \). This singularity is a reason for the \( H^2 \) existence result in [17] only holding for gSQG with \( \alpha < 1 \), in which case the denominator \( |x(\gamma, t) - x(\gamma - \eta, t)| \) is replaced by \( |x(\gamma, t) - x(\gamma - \eta, t)|^\alpha \). In order to reach \( \alpha = 1 \), our key idea is to work in \( H^{2+s} \) and use the extra \( s \) derivative to compensate the logarithmic singularity. The trade-off is that the control of the fractional Sobolev regularity \( H^{2+s} \) induces various commutators. A priori estimates in \( H^{2+s} \) are established in Sections 2 and 3 using appropriate commutator estimates. These estimates in turn require in particular the control of \( \lambda \) in the Hölder and Sobolev spaces \( C^{1+\varepsilon} \) and \( H^\frac{3}{2} \), whose proof delicately uses the full \( H^{2+s} \) regularity. See subsection 5.1. A priori estimates for the arc chord quantity (1.5) also require new arguments and are obtained in Section 4. All the a priori estimates are gathered in Section 6 to conclude the existence of solutions via regularizing initial data. The regularization procedure requires some care in order to guarantee the property (1.6) carries through the approximation scheme.

The uniqueness part is subtle. Recall that the uniqueness result in [10] for (1.7)-(1.8) was obtained via an \( H^1 \) contraction estimate. For unbounded curvature this approach does not work neither for \( L^2 \) nor \( H^2 \) contraction estimate. To get a glimpse of the difficulty, let us consider one of the more singular terms in the H^2 scheme.

\[ I_{312} = \int \partial_\gamma z(\gamma) \cdot (\frac{1}{|x(\gamma) - x(\gamma - \eta)|} - \frac{1}{|y(\gamma) - y(\gamma - \eta)|}) \, d\eta d\gamma. \]

In the regime of bounded curvature, guaranteed when \( s \in (\frac{1}{2}, 1) \), we have that \( \partial_\gamma^2 x, \partial_\gamma^2 y \in C^s \) with \( \delta = s - \frac{1}{2} \). We can then control

\[ I_{312} \lesssim \|\partial_\gamma z\|_{L^2}^2 \|\partial_\gamma^2 y\| \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \int \frac{1}{|\eta|^{1-\delta}} \, d\eta \]

which is integrable. Note that \( C^2 \) regularity would not suffice for this argument. For \( s \in (0, \frac{1}{2}) \), our idea is to prove contraction estimates not only for the \( H^1 \) norm but also the moduli of the tangential vectors \( |\partial_\gamma x(\gamma, t)| - |\partial_\gamma y(\gamma, t)| \) to the patches. The entire argument has to avoid the use of the \( C^2 \) norm of the solution through detailed decompositions of terms. This is done in Section 5.

**Remark 1.3.** Both a priori and contraction estimates crucially use the full \( H^{2+s} \) regularity. Whether SQG sharp fronts are well-posed in \( H^2 \) remains an open problem. On the other hand, for higher regularity \( H^{2+s} \) with \( s \in (\frac{1}{2}, 1) \), local well-posedness should hold by modifications of our proof for the existence part.

2. A priori estimates for nontangential nonlinearity

For notational simplicity we denote

\[ x_-(\gamma, \eta) = x(\gamma) - x(\gamma - \eta), \]

(2.1)

\[ g = g(\gamma, \eta) = \frac{1}{|x_-(\gamma, \eta)|}. \]

(2.2)
When it is clear that $\gamma$ and $\eta$ are the variables, we use the shorthand
\[ x_\gamma = x_\gamma(\gamma) = x_\gamma(\gamma, \eta). \] (2.3)

We shall also write for fixed $t$,
\[ \|F(x)\|_{L^\infty} = \|F(x)(\cdot, t)\|_{L^\infty(T \times \mathbb{T})}. \] (2.4)

We assume throughout this section that $x \in C([0, T]; H^3)$ is a solution of (1.7)-(1.8). Fixing $s \in (0, \frac{1}{2})$, our goal is to establish a priori estimates for $H^{2+s}$ norm of $x$. For high frequencies, we differentiate $(1.7)$ twice in $\gamma$ to obtain
\[ \partial_\gamma \partial_\gamma^2 x = \int_T \partial_\gamma^3 x - \lambda \partial_\gamma x \partial_\gamma \eta \gamma d\eta + \int_T \partial_\gamma x \partial_\gamma^2 x + 2 \partial_\gamma \lambda \partial_\gamma^2 x + \partial_\gamma^2 \lambda \partial_\gamma x \]
\[ := N_1 + N_2 + N_3 + T_1 + T_2 + T_3. \] (2.5)

We apply $\Lambda^s$ to (2.5), multiply the resulting equation by $\Lambda^s \partial_\gamma^2 x$, then integrate over $\mathbb{T}$, leading to
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda^s \partial_\gamma^2 x\|_{L^2}^2 = \sum_{j=1}^3 N_j + \sum_{j=1}^3 T_j. \] (2.6)

We call $T_j$ tangential terms and $N_j$ nontangential terms. This section is devoted to a priori estimates for the nontangential terms. Tangential terms shall be considered in Section 3.

2.1. Preliminaries. We shall frequently appeal to the identity
\[ x_\gamma \cdot \partial_\gamma x_\gamma = (x_\gamma - \eta \partial_\gamma x(\gamma)) \cdot \partial_\gamma x_\gamma + \frac{1}{2} \eta \|\partial_\gamma x_\gamma\|^2. \] (2.7)

For the proof of (2.7) we use the fact that $|\partial_\gamma x_\gamma(\gamma, t)|^2 = A(t)$ is independent of $\gamma$ to obtain
\[ \partial_\gamma x_\gamma(\gamma, t) \cdot \partial_\gamma x_\gamma(t) = |\partial_\gamma x_\gamma(\gamma, t)|^2 - \partial_\gamma x_\gamma(\gamma, t) \cdot \partial_\gamma x_\gamma(\gamma - \eta, t) \]
\[ = \frac{1}{2} \left( |\partial_\gamma x_\gamma(\gamma, t)|^2 + |\partial_\gamma x_\gamma(\gamma - \eta, t)|^2 - 2 \partial_\gamma x_\gamma(\gamma, t) \cdot \partial_\gamma x_\gamma(\gamma - \eta, t) \right) = \frac{1}{2} \|\partial_\gamma x_\gamma\|^2. \] (2.8)

LEMMA 2.1. For all $\varepsilon \in [0, \frac{1}{2}]$, we have
\[ |\partial_\gamma g| \lesssim \frac{1}{|\eta|^{1-2\varepsilon}} \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|^2_{C^{\frac{1}{2}+\varepsilon}} \] (2.9)

and
\[ \|\partial_\gamma g(\cdot, \gamma, t)\|_{C^{\cdot, \frac{1}{2}+\varepsilon}} \lesssim \frac{1}{|\eta|^{1-s+\varepsilon}} \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|^2_{C^{\frac{1}{2}+s}} + \frac{1}{|\eta|^{1-s}} \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|^2_{C^{\frac{1}{2}+s}} \|\partial_\gamma x\|^2_{C^{\frac{1}{2}+s}}. \] (2.10)

PROOF. In view of (2.7) we have
\[ \partial_\gamma g = \frac{(x_\gamma - \eta \partial_\gamma x(\gamma)) \cdot \partial_\gamma x_\gamma}{|x_\gamma|^3} - \frac{1}{2} \eta \|\partial_\gamma x_\gamma\|^2 = -m_1(\gamma, \eta) - \frac{1}{2} m_2(\gamma, \eta). \] (2.11)

By Taylor expansion, it is readily seen that
\[ |x_\gamma - \eta \partial_\gamma x(\gamma)| \leq |\eta|^{\frac{1}{2}+\varepsilon} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}, \quad |\partial_\gamma x_\gamma| \leq |\eta|^{\frac{1}{2}+\varepsilon} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}. \]

Consequently,
\[ |\partial_\gamma g| \lesssim \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|^2_{C^{\frac{1}{2}+\varepsilon}} \frac{|\eta|^{\frac{3}{2}+\frac{1}{2}+\varepsilon} + |\eta|^{1+2(\frac{1}{2}+\varepsilon)}}{|\eta|^3}, \]
whence (2.9) follows.
We turn to prove (2.10). For $|h| \leq \pi$ we consider the difference $\partial_y g(\gamma, \eta) - \partial_y g(\gamma + h, \eta)$. First,
\[
m_2(\gamma, \eta) - m_2(\gamma + h, \eta) = \eta \frac{|\partial_y x_-(\gamma, \eta)|^2}{|x_-(\gamma, \eta)|^3} - \eta \frac{|\partial_y x_-(\gamma + h, \eta)|^2}{|x_-(\gamma + h, \eta)|^3} \\
= \eta \frac{|\partial_y x_-(\gamma, \eta)|^2 - |\partial_y x_-(\gamma + h, \eta)|^2}{|x_-(\gamma, \eta)|^3} \\
+ \eta |\partial_y x_-(\gamma + h, \eta)|^2 \left( \frac{1}{|x_-(\gamma, \eta)|^3} - \frac{1}{|x_-(\gamma + h, \eta)|^3} \right) \\
= : I + II.
\]

We write
\[
|\partial_y x_-(\gamma, \eta)|^2 - |\partial_y x_-(\gamma + h, \eta)|^2 \\
= \left[ \partial_y x_-(\gamma, \eta) - \partial_y x_-(\gamma + h, \eta) \right] \left[ \partial_y x_-(\gamma, \eta) + \partial_y x_-(\gamma + h, \eta) \right] = A \cdot B.
\]

Clearly,
\[
|B| \lesssim |\eta|^{\frac{1}{2} + \epsilon} \| \partial_y x \|_{C^{\frac{1}{2} + \epsilon}}.
\]

On one hand,
\[
|A| = |\eta| \int_0^1 [\partial_y^2 x(\gamma - r\eta) - \partial_y^2 x(\gamma + h - r\eta)] dr \lesssim |\eta| ||h||^a \| \partial_y^2 x \|_{C^a} \forall a \in (0, 1).
\]

On the other hand, using the mean value theorem gives
\[
|A| = |\partial_y x_-(\gamma, -h) - \partial_y x_-(\gamma - \eta, -h)| \leq |\partial_y x_-(\gamma, -h)| + |\partial_y x_-(\gamma - \eta, -h)| \lesssim |h|^b \| \partial_y x \|_{C^b} \forall b \in (0, 1).
\]

Therefore, regarding $A$ as a linear operator of $x$, we deduce by interpolation that
\[
|A| \lesssim |h|^\theta |\eta|^\theta \| \partial_y x \|_{C^{\theta(1+a)+(1-\theta)b}} \forall \theta \in (0, 1).
\]

Now for any $\epsilon \in [0, \frac{1}{2})$ we choose $\theta = \frac{1}{2} - \epsilon$ and $a, b \in (0, 1)$ satisfying $\theta(1 + a) + (1 - \theta)b = s + \frac{1}{2}$. Then $\theta a + (1 - \theta)b = s + \frac{1}{2} - \epsilon = s + \epsilon$ and
\[
|A| \lesssim |h|^{s+\epsilon} |\eta|^{\frac{1}{2} - \epsilon} \| \partial_y x \|_{C^{s+\frac{1}{2}}}.
\]

Combining this with the above estimate for $B$ we get
\[
|I| \lesssim \|F(x)\|_L^2 \|h|^{s+\epsilon} |\eta|^{\frac{1}{2} - s+\epsilon} \| \partial_y x \|_{C^{s+\frac{1}{2}}}. \tag{2.12}
\]

As for $II$ we first note that
\[
|\partial_y x_-(\gamma + h, \eta)|^2 \lesssim |\eta| \| \partial_y x \|_{C^{\frac{1}{2}}}. \tag{2.13}
\]

Setting $k(\gamma, \eta) = |x_-(\gamma, \eta)|^{-3}$ we have
\[
\partial_y k(\gamma, \eta) = 3 \frac{\partial_y g(\gamma, \eta)}{|x_-(\gamma)|^2}.
\]

Then using the fundamental theorem of calculus and the bound (2.9) for $\partial_y g$ we obtain
\[
|k(\gamma + h, \eta) - k(\gamma, \eta)| \lesssim \frac{|h|}{|\eta|^{2-2s}} \|F(x)\|_{L^\infty}^5 \| \partial_y x \|_{C^{s+\frac{1}{2}}}. \tag{2.14}
\]

We have proved
\[
|II| \lesssim \frac{|h|}{|\eta|^{1-2s}} \|F(x)\|_{L^\infty}^5 \| \partial_y x \|_{C^{s+\frac{1}{2}}}^2 \| \partial_y x \|_{C^{\frac{1}{2}}}. \tag{2.15}
\]
Next, we have
\[ m_1(\gamma, \eta) - m_1(\gamma + h, \eta) \]
\[ = \frac{[x_-(\gamma, \eta) - \eta \partial_\gamma x(\gamma)] \cdot \partial_\gamma x_-(\gamma, \eta)}{|x_-(\gamma, \eta)|^3} - \frac{[x_-(\gamma + h, \eta) - \eta \partial_\gamma x(\gamma + h)] \cdot \partial_\gamma x_-(\gamma + h, \eta)}{|x_-(\gamma + h, \eta)|^3} \]
\[ = \frac{[x_-(\gamma, \eta) - \eta \partial_\gamma x(\gamma)] \cdot \partial_\gamma x_-(\gamma, \eta) - [x_-(\gamma + h, \eta) - \eta \partial_\gamma x(\gamma + h)] \cdot \partial_\gamma x_-(\gamma + h, \eta)}{|x_-(\gamma, \eta)|^3} \]
\[ + [x_-(\gamma + h, \eta) - \eta \partial_\gamma x(\gamma + h)] \cdot \partial_\gamma x_-(\gamma + h, \eta) \left( \frac{1}{|x_-(\gamma, \eta)|^3} - \frac{1}{|x_-(\gamma + h, \eta)|^3} \right) \]
\[ = III + IV. \]

Since
\[ |[x_-(\gamma + h, \eta) - \eta \partial_\gamma x(\gamma + h)] \cdot \partial_\gamma x_-(\gamma + h, \eta)| \leq |\eta|^2 \| \partial_\gamma x \|_{C^{1,1}}^2. \]

Combining this and (2.14) we deduce that IV obeys the same bound as II:
\[ |IV| \lesssim \frac{|h|}{|\eta|^{1 - 2s}} \| F(x) \|_{L^\infty}^5 \| \partial_\gamma x \|_{C^{3/2 + s}}^2. \quad (2.16) \]

We write
\[ III = \frac{1}{|x_-(\gamma, \eta)|^3} \partial_\gamma x_-(\gamma, \eta) \cdot [x_-(\gamma, \eta) - \eta \partial_\gamma x(\gamma) - x_-(\gamma + h, \eta) + \eta \partial_\gamma x(\gamma + h)] \]
\[ + \frac{1}{|x_-(\gamma, \eta)|^3} \left[ \partial_\gamma x_-(\gamma, \eta) - \partial_\gamma x_-(\gamma + h, \eta) \right] \cdot [x_-(\gamma + h, \eta) - \eta \partial_\gamma x(\gamma + h)] \]
\[ = III_1 + III_2. \]

Combining the estimate
\[ |x_-(\gamma + h, \eta) - \eta \partial_\gamma x(\gamma + h)| \leq |\eta|^{3/2} \| \partial_\gamma x \|_{C^{3/2 + s}} \]
and (2.12) we obtain
\[ |III_2| \lesssim \| F(x) \|_{L^\infty}^5 \frac{|h|^s}{|\eta|^{1 - s + \varepsilon}} \| \partial_\gamma x \|_{C^{3/2 + s}}^2 \quad \forall \varepsilon \in (0, \frac{1}{2}). \]

By the mean value theorem,
\[ C := x_-(\gamma, \eta) - \eta \partial_\gamma x(\gamma) - x_-(\gamma + h, \eta) + \eta \partial_\gamma x(\gamma + h) \]
\[ = -\frac{\eta^2}{2} \int_0^1 (1 - r) \| \partial_\gamma^2 x(\gamma - r\eta) - \partial_\gamma^2 x(\gamma + h - r\eta) \| dr \]
and thus
\[ |C| \lesssim \eta^2 |h|^a \| \partial_\gamma^2 x \|_{C^a} \quad \forall a \in (0, 1). \]

On the other hand, since
\[ C = x_-(\gamma, -h) - x_-(\gamma - \eta, -h) - \eta \partial_\gamma x_-(\gamma, -h) = \eta \int_0^1 \partial_\gamma x_-(\gamma - r\eta, -h) dr - \eta \partial_\gamma x_-(\gamma, -h) \]
we get
\[ |C| \lesssim |\eta| |h|^b \| \partial_\gamma x \|_{C^b} \quad \forall b \in (0, 1). \]

By an interpolation argument as for (2.12) we obtain
\[ |C| \lesssim |h|^s \eta^{3/2 - \varepsilon} \| \partial_\gamma x \|_{C^{3/2 + s}} \quad \forall \varepsilon \in (0, \frac{1}{2}). \]
Consequently,
\[
|III| \lesssim \|F(x)\|_{L^\infty}^3 \left| \frac{\eta^{\frac{3}{2}+\varepsilon}}{|\eta|^3} \|\partial_x x\|_{C^{s+\frac{1}{2}}} \|h\|_{L^{\infty}} \right|^{\frac{3}{2}-\varepsilon} \|\partial_x x\|_{C^{s+\frac{1}{2}}} \\
\leq \|F(x)\|_{L^\infty}^3 \left| \frac{h\|\partial_x x\|_{C^{s+\frac{1}{2}}}}{|\eta|^{1-s+\varepsilon}} \|\partial_x x\|_{C^{s+\frac{1}{2}}} \right|^{\varnothing \in [0, \frac{1}{2})}.
\]

We have proved
\[
|III| \lesssim \|F(x)\|_{L^\infty}^3 \left| \frac{h\|\partial_x x\|_{C^{s+\frac{1}{2}}}}{|\eta|^{1-s+\varepsilon}} \|\partial_x x\|_{C^{s+\frac{1}{2}}} \right|^{\varnothing \in [0, \frac{1}{2})},
\]
Putting together (2.13), (2.15), (2.16) and (2.17) we conclude the proof of (2.10).

**Lemma 2.2.** For all \( r \in (0, 1) \) and \( \delta > 0 \), there exists \( C > 0 \) such that
\[
\|\Lambda^r u\|_{L^2} \leq C \|u\|_{B^{1+\delta}_{2,1}} \|v\|_{H^{r-1}}
\]
provided that the right-hand side is finite.

**Proof.** Using the Bony decomposition \( uv = Tu + T_u + R(u, v) \) we write
\[
\Lambda^r u = \Lambda^r T u + \Lambda^r (T_u u) + \Lambda^r R(u, v) - T_{\Lambda^r} u - R(\Lambda^r u, v).
\]
It follows from paraproduct rules that
\[
\|\Lambda^r T u\|_{L^2} \lesssim \|\nabla u\|_{L^\infty} \|v\|_{H^{r-1}} \quad (A.9),
\]
\[
\|\Lambda^r R(u, v)\|_{L^2} \lesssim \|T u\|_{H^r} \lesssim \|u\|_{B^{1+\delta}_{2,1}} \|v\|_{H^{r-1}} \quad (A.8),
\]
\[
\|\Lambda^r (T_u u)\|_{L^2} \lesssim \|R(u, v)\|_{H^r} \lesssim \|u\|_{B^{1+\delta}_{2,1}} \|v\|_{H^{r-1}} \quad (A.3),
\]
\[
\|T_{\Lambda^r} u\|_{L^2} \lesssim \|u\|_{B^{1+\delta}_{2,1}} \|\Lambda^r v\|_{H^{r-1}} \lesssim \|u\|_{B^{1+\delta}_{2,1}} \|v\|_{H^{r-1}} \quad (A.8),
\]
\[
\|R(\Lambda^r u, v)\|_{L^2} \lesssim \|u\|_{B^{1+\delta}_{2,1}} \|\Lambda^r v\|_{H^{r-1}} \lesssim \|u\|_{B^{1+\delta}_{2,1}} \|v\|_{H^{r-1}} \quad (A.3).
\]

Since \( B^{1+\delta}_{2,1} \subset B^{1}_{2,1} \), the proof is complete.

**2.2. Control of \( N_1 \).** By symmetrizing and integration by parts, we have
\[
N_1 = \int_T \Lambda^r \partial^2_x x(\gamma) \tilde{N}_1(\gamma) d\gamma = \frac{1}{2} \int_T \int_T \Lambda^r \partial^2_x x - \partial^3_x x - g d\gamma d\eta
\]
\[
= \frac{1}{2} \int_T \int_T \Lambda^r \partial^2_x x - \Lambda^r \partial^3_x x - g d\gamma d\eta + \frac{1}{2} \int_T \int_T \Lambda^r \partial^2_x x - [\Lambda^r, g] \partial^3_x x d\gamma d\eta
\]
\[
= \frac{1}{2} \int_T \int_T \Lambda^r \partial^2_x x - \partial^3_x x - g d\gamma d\eta + \frac{1}{2} \int_T \int_T \Lambda^r \partial^2_x x - [\Lambda^r, g] \partial^3_x x d\gamma d\eta
\]
\[
= N_{11} + N_{12}.
\]
By virtue of the Lipschitz bound (2.9) for \( g \), we have
\[
|N_{11}| \lesssim \|F(x)\|_{L^\infty}^3 \|\partial_x x\|_{C^{s+\frac{1}{2}}} \int_T \int_T |\Lambda^r \partial^2_x x - |^2 d\gamma d\eta
\]
\[
\lesssim \|F(x)\|_{L^\infty}^3 \|\partial_x x\|_{C^{s+\frac{1}{2}}} \|\Lambda^r \partial^2_x x\|_{L^2}.
\]
Regarding \( N_{12} \) we appeal to the commutator estimate (2.13) and notice that \([\Lambda^r, g] = [\Lambda^r, g - \tilde{g}(0)]\), where \( g = g(\cdot, \eta) \). Then for any \( \delta > 0 \),
\[
\|\Lambda^r \partial^3_x x - \|_{L^2} \lesssim \|g - \tilde{g}(0)\|_{B^{1+\delta}_{2,1}} \|\partial^2_x x\|_{H^r} \lesssim \|\partial_x g\|_{B^{1+\delta}_{2,1}} \|\partial^2_x x\|_{H^r} \lesssim \|\partial_x g\|_{C^{\delta}} \|\partial^2_x x\|_{H^r}.
\]
where we have used Proposition [A.1] in the last inequality. Now invoking the (homogeneous) Hölder estimate (2.10) for $\partial_\gamma g$ with $\varepsilon = 0$ yields

$$\left|N_{12}\right| \lesssim \left(\|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^2 + \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^4 \right) \int_T \left(\int_T \Lambda_\gamma^s \partial_\gamma^2 x \cdot ||\partial_\gamma x||_{H_{2\gamma-1}} d\gamma \right) \frac{d\eta}{|\eta|^{1-s}} \lesssim \left(\|F(\tilde{x})\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^2 + \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^4 \right) \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2}^2.$$  

In view of this and the above estimate for $N_{11}$ we have proved

$$\left|N_1\right| \lesssim \left(\|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^2 + \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^4 \right) \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2}^2. \quad (2.19)$$

### 2.3. Control of $N_2$

$$N_2 = \int_T \Lambda_\gamma^s \partial_\gamma^2 x(\gamma) \cdot \partial_\gamma (\gamma) d\gamma = 2 \int_T \Lambda_\gamma^s \partial_\gamma^2 x(\gamma) \cdot \int_T \Lambda_\gamma^s \partial_\gamma^2 x(\gamma) - \partial_\gamma g) d\eta \leq 2 \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2} \int_T \|\Lambda_\gamma^s (\partial_\gamma^2 x - \partial_\gamma g)\|_{L^2} d\eta.$$  

Combining the paraproduct rules (A.5), (A.6) and (A.8), we get

$$\|\Lambda_\gamma^s (uv)\|_{L^2} \leq \|uv\|_{L^\infty} \lesssim \|v\|_{L^\infty} \|u\|_{H^s} + \|u\|_{L^2} \|v\|_{B^{s+4/2}_{\infty,2}} \lesssim \|v\|_{L^\infty} \|u\|_{H^s} + \|u\|_{L^2} \|v\|_{C^{s+\delta}} \quad (2.20)$$

for any $\delta \in (0, \frac{1}{2})$. Note that we have used Proposition [A.1] to have $\|v\|_{B^{s+4}_{\infty,2}} \simeq \|v\|_{C^{s+\delta}}$. We apply (2.20) with $u = \partial_{\gamma}^2 x(-, \eta)$ and $v = \partial_{\gamma} g(-, \eta)$. Using the $L^\infty$ estimate (2.9) and the Hölder estimate (2.10), we deduce

$$\|\Lambda_\gamma^s (\partial_\gamma^2 x - \partial_\gamma g)\|_{L^2} \lesssim \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2} \|F(x)\|_{L^\infty}^3 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}} \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}} \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}} \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2}^2.$$  

Therefore,

$$\left|N_2\right| \lesssim \left(\|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^2 + \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1+s}_{2\gamma+*}}^4 \right) \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2}^2. \quad (2.21)$$

### 2.4. Control of $N_3$

Clearly,

$$\left|N_3\right| \leq \|\Lambda_\gamma^s \partial_\gamma^2 x\|_{L^2} \int_T \|\Lambda_\gamma^s (\partial_\gamma x - \partial_\gamma^2 g)\|_{L^2} d\eta.$$  

We compute

$$\partial_\gamma x - \partial_\gamma^2 g = 3\partial_\gamma x \frac{(x_\gamma - \partial_\gamma x - \gamma)^2}{|x_\gamma|^3} - \alpha \partial_\gamma x \frac{\partial_\gamma x - \gamma}{|x_\gamma|^3}.$$  

We shall only consider the term $n = \partial_\gamma x - \frac{x_\gamma - \partial_\gamma x}{|x_\gamma|^3}$ because the other terms can be treated similarly. Using $\partial_\gamma x \cdot \partial_\gamma^2 x = 0$ we write

$$n = \frac{\partial_\gamma x - \gamma}{|x_\gamma|^3} x_\gamma - \partial_\gamma^2 x = \frac{\partial_\gamma x - \gamma}{|x_\gamma|^3} (x_\gamma - \eta \partial_\gamma x) \cdot \partial_\gamma^2 x - \frac{\partial_\gamma x - \gamma}{|x_\gamma|^3} \eta \partial_\gamma x_\gamma - \partial_\gamma^2 x (\gamma - \eta) := n_1 + n_2.$$  

By virtue of inequality (2.20) we have

$$\|\Lambda_\gamma^n_1\|_{L^2} \lesssim \|\Lambda_\gamma \partial_\gamma^2 x\|_{L^2} \left(\left\|\partial_\gamma x\right\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \right)$$
for all $\varepsilon \in (0, \frac{1}{2})$. By the mean value theorem,
\[
\left| \frac{\partial_{\gamma} x}{|x_{-}|^{3}}(x_{-} - \eta \partial_\gamma x) \right| \lesssim \frac{|\eta|^{\frac{1}{2}+\varepsilon}\|\partial_{\gamma} x\|_{C^{\frac{1}{2}+\varepsilon}}}{|\eta|^3} \|F(x)\|_{L^\infty} |\eta|^{\frac{1}{2}+\varepsilon}\|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}
\lesssim \frac{1}{|\eta|^{1-2\varepsilon}} \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^2.
\]

On the other hand, as for the term $m_1$ in (2.11) we have (see (2.16) and (2.17))
\[
\left\| \frac{\partial_{\gamma} x}{|x_{-}|^{3}}(x_{-} - \eta \partial_\gamma x) \right\|_{C^{\frac{1}{2}+\varepsilon}} \lesssim \frac{1}{|\eta|^{1-s+\varepsilon}} \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^2
+ \frac{1}{|\eta|^{1-2\varepsilon}} \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^2 \forall \varepsilon \in [0, \frac{1}{2}).
\]

Thus choosing $\varepsilon = \frac{\delta}{2}$ we arrive at
\[
\|A^s n_1\|_{L^\infty} \lesssim \frac{1}{|\eta|^{1-\frac{\delta}{2}}} \|A^s \partial_\gamma^2 x\|_{L^2} \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^2
+ \|\partial_\gamma^2 x\|_{L^2} \left( \frac{1}{|\eta|^{1-\frac{\delta}{2}}} \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^2 + \frac{1}{|\eta|^{1-2\varepsilon}} \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^4 \right).
\]

By a similar argument it can be shown that $n_2$ obeys the same bound, and so does $\partial_{\gamma} x_{-} \partial_\gamma^2 g$. We conclude that
\[
|N_3| \lesssim \left( \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^2 + \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}^4 \right) \|A^s \partial_\gamma^2 x\|_{L^2}^2.
\]

Putting together (2.19), (2.21) and (2.22), we obtain

**Proposition 2.3.** For all $s \in (0, \frac{1}{2})$, there exists a polynomial $P(\cdot, \cdot)$ such that
\[
\sum_{j=1}^{3} |N_j| \leq P(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{\frac{1}{2}+\varepsilon}}) \|\partial_\gamma^2 x\|_{L^2}^2.
\]

**3. A priori estimates for tangential nonlinearity**

In this section, we derive a priori estimates for the tangential terms $T_j$ in (2.6) assuming that $x \in C([0, T]; H^3)$ is a solution of (1.7)-(1.8). This requires good bounds for $\lambda$.

**3.1. Estimates for $\lambda$.** We first recall that
\[
\lambda(\gamma, t) = \frac{\pi + \gamma}{2\pi} \int_{\mathbb{T}} \int_{T} \partial_{\gamma} \left( \frac{\partial_{\gamma} x(\gamma) - \partial_{\gamma} x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} \right) \cdot \frac{\partial_{\gamma} x(\gamma)}{|\partial_{\gamma} x(\gamma)|^2} d\eta d\gamma
- \int_{-\pi}^{\gamma} \int_{T} \partial_{\gamma} \left( \frac{\partial_{\gamma} x(\xi) - \partial_{\gamma} x(\xi - \eta)}{|x(\xi) - x(\xi - \eta)|} \right) \cdot \frac{\partial_{\gamma} x(\xi)}{|\partial_{\gamma} x(\xi)|^2} d\eta d\xi.
\]

Taking derivative with respect to $\gamma$ and recalling that $A(t) = \|\partial_\gamma x(\gamma)\|^2$ depends only on $t$, we obtain
\[
\partial_{\gamma} \lambda(\gamma, t) = \frac{1}{2\pi} \int_{T} \int_{T} \partial_{\gamma} \left( \frac{\partial_{\gamma} x(\gamma) - \partial_{\gamma} x(\gamma - \eta)}{|x(\gamma) - x(\gamma - \eta)|} \right) \cdot \frac{\partial_{\gamma} x(\gamma)}{|\partial_{\gamma} x(\gamma)|^2} d\eta d\gamma
+ \frac{1}{A(t)} \int_{T} \frac{x_{-}(\gamma)}{|x_{-}(\gamma)|^3} \partial_{\gamma} x_{-}(\gamma) \cdot \partial_{\gamma} x(\gamma) d\eta - \frac{1}{A(t)} \int_{T} \partial_{\gamma}^2 x_{-}(\gamma) \cdot \partial_{\gamma} x(\gamma) d\eta,
\]
where $x_{-}(\gamma) = x(\gamma) - x(\gamma - \eta)$. Using the identities $\partial_{\gamma} x(\gamma) \cdot \partial_{\gamma}^2 x_{-}(\gamma) = -\partial_{\gamma} x_{-}(\gamma) \cdot \partial_{\gamma}^2 x(\gamma - \eta)$ and $\partial_{\gamma} x(\gamma) \cdot \partial_{\gamma} x_{-}(\gamma) = \frac{1}{2} |\partial_{\gamma} x_{-}(\gamma)|^2$, we rewrite
\[
\partial_{\gamma} \lambda(\gamma) = \Gamma_1(\gamma) + \Gamma_2(\gamma) + \Gamma_3(\gamma)
\]
where
\[ \Gamma_1(\gamma) = \frac{1}{A(t)} \int_T \frac{\partial_\gamma x(-\gamma) \cdot \partial_\gamma^2 x(\gamma - \eta)}{|x(-\gamma)|} \, d\eta, \] (3.3)

\[ \Gamma_2(\gamma) = \frac{1}{2A(t)} \int_T \frac{|\partial_\gamma x(-\gamma)|^2 x(-\gamma) \cdot \partial_\gamma x(-\gamma)}{|x(-\gamma)|^3} \, d\eta, \] (3.4)

\[ \Gamma_3 = \frac{-1}{2\pi A(t)} \int_T \int_T \frac{\partial_\gamma x(-\gamma)}{|x(-\gamma)|} \cdot \partial_\gamma^2 x(\gamma) \, d\eta d\gamma. \] (3.5)

Note that \( \Gamma_3 \) is a constant in \( \gamma \).

3.1.1. Hölder estimate for \( \partial_\gamma \lambda \). We start with an \( L^\infty \) bound for \( \partial_\gamma \lambda \).

**Lemma 3.1.** For all \( \mu \in (0, \frac{1}{2}) \), there exists \( C > 0 \) such that
\[ \|\lambda\|_{W_1,\infty} \leq C \|F(x)\|_{L^\infty}^3 + \|F(x)\|_{L^\infty}^4 \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \|\partial_\gamma^2 x\|_{L^2}. \] (3.6)

**Proof.** It suffices to prove that \( \|\partial_\gamma \lambda\|_{L^\infty} \) is controlled by the right-hand side of (3.6). Indeed, since \( \lambda(-\pi) = \lambda(\pi) = 0 \), the Poincaré inequality \( \|\lambda\|_{L^\infty} \leq 2\pi \|\partial_\gamma \lambda\|_{L^\infty} \) holds.

From the definition of \( F \) we have
\[ \frac{1}{A(t)} \leq \|F(x(t))\|_{L^\infty}^2. \]

Hence, \( \Gamma_1 \) is bounded as
\[ |\Gamma_1| \leq \|F(x)\|_{L^\infty}^3 \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \int_T \frac{|\partial_\gamma^2 x(\gamma - \eta)|}{|\eta|^\frac{3}{2} - \mu} \, d\eta \lesssim \|F(x)\|_{L^\infty}^3 \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \|\partial_\gamma^2 x\|_{L^2}. \]

On the other hand,
\[ |\Gamma_2| \leq \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \int_T \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} |\eta|\frac{3}{2} + \mu \|F(x)\|_{L^\infty}^2 \|\partial_\gamma^2 x\|_{L^\infty} \int_0^1 \|\partial_\gamma^2 x(\gamma - r\eta)\| \, dr \, d\eta \]
\[ \lesssim \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \int_0^1 \frac{dr}{r^2} \left( \int_T \frac{1}{|\eta|^{1-2\mu}} \, d\eta \right) \frac{1}{2} \|\partial_\gamma^2 x\|_{L^2}, \]

and
\[ |\Gamma_3| \leq \|F(x)\|_{L^\infty}^3 \int_T \int_T \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \frac{d\eta}{|\eta|^{1-\mu}} \|\partial_\gamma^2 x(\gamma)\| \, d\gamma \]
\[ \lesssim \|F(x)\|_{L^\infty}^3 \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \|\partial_\gamma^2 x\|_{L^2}. \]

\[ \square \]

Next, we prove the following Hölder estimate for \( \partial_\gamma \lambda \).

**Proposition 3.2.** For any \( \mu \in (0, \frac{1}{2}) \), if \( a \in (0, 1) \) satisfies
\[ a\left(\frac{1}{2} + \mu\right) < \mu \] (3.7)

then there exists a polynomial \( P(\cdot, \cdot) \) independent of \( x \) such that
\[ \|\partial_\gamma \lambda\|_{C^a(\frac{1}{2}+\mu)} \leq P(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}}) \|\partial_\gamma^2 x\|_{H^\mu}. \] (3.8)
PROOF. We shall prove that the finite difference $\partial_\gamma \lambda(\gamma + h) - \partial_\gamma \lambda(\gamma) = \sum_{i=1}^{2} \Gamma_i(\gamma + h) - \Gamma_i(\gamma)$ can be bounded by a positive power of $h$. The highest order term is

$$\Gamma_1(\gamma + h) - \Gamma_1(\gamma) = \Gamma_{11} + \Gamma_{12} + \Gamma_{13}$$

where

$$\Gamma_{11} = \frac{1}{A(t)} \int_{T} \frac{\partial_\gamma x_-(\gamma)}{|x_-(\gamma)|} \cdot (\partial_\gamma^2 x(\gamma - \eta + h) - \partial_\gamma^2 x(\gamma - \eta))d\eta,$$

$$\Gamma_{12} = \frac{-1}{A(t)} \int_{T} \partial_\gamma x_-(\gamma) \cdot \partial_\gamma^2 x(\gamma - \eta + h) \left( \frac{1}{|x_-(\gamma)|} - \frac{1}{|x_-(\gamma + h)|} \right) d\eta,$$

$$\Gamma_{13} = \frac{-1}{A(t)} \int_{T} (\partial_\gamma x_-(\gamma) - \partial_\gamma x_-(\gamma + h)) \cdot \frac{\partial_\gamma^2 x(\gamma - \eta + h)}{|x_-(\gamma + h)|} d\eta.$$

On one hand, by Cauchy-Schwarz’s inequality

$$|\Gamma_{11}| \leq \|F(x)\|_{L^2_T}^2 \int_{T} \frac{\|\partial_\gamma x\|_{C^{1+\mu}} \|F(x)\|_{L^\infty_T}}{|\eta|^{\frac{3}{2} - \mu}} |\partial_\gamma^2 x(\gamma - \eta + h) - \partial_\gamma^2 x(\gamma - \eta)|d\eta$$

(3.9)

$$\leq C\|F(x)\|_{L^2_T}^2 \|\partial_\gamma x\|_{C^{1+\mu}} \|\partial_\gamma^2 x\|_{L^2_T}.$$ (3.10)

On the other hand, using the mean value theorem before applying Cauchy-Schwarz’s inequality, we have

$$|\Gamma_{11}| \leq \|F(x)\|_{L^2_T}^2 \int_{T} \frac{\|\partial_\gamma x\|_{C^{1+\mu}} \|F(x)\|_{L^\infty_T}}{|\eta|^{\frac{3}{2} - \mu}} \int_{0}^{1} |h||\partial_\gamma^2 x(\gamma - \eta + rh)| dr d\eta$$

(3.9)

$$\leq C\|F(x)\|_{L^2_T}^2 |h| \|\partial_\gamma x\|_{C^{1+\mu}} \|\partial_\gamma^2 x\|_{L^2_T}.$$ (3.10)

Viewing $\Gamma_{11}$ as a linear operator in $\partial_\gamma^2$ and interpolating between (3.9) and (3.10) gives

$$|\Gamma_{11}| \leq C\|F(x)\|_{L^2_T}^2 |h|^\mu \|\partial_\gamma x\|_{C^{1+\mu}} \|\partial_\gamma^2 x\|_{H^\mu}.$$ (3.11)

Next, for $\Gamma_{12}$, we have

$$|\Gamma_{12}| \leq \|F(x)\|_{L^2_T}^2 \int_{T} |\partial_\gamma x_-(\gamma)| |\partial_\gamma^2 x(\gamma - \eta + h)| \frac{|x_-(\gamma + h) - x_-(\gamma)|}{|x_-(\gamma + h)||x_-(\gamma)|} d\eta$$

$$\lesssim \|F(x)\|_{L^2_T}^2 \int_{T} \int_{0}^{1} |\partial_\gamma x_-(\gamma)| \|\partial_\gamma^2 x(\gamma - \eta + h)| \frac{|h||\partial_\gamma x_-(\gamma + rh)|}{|\eta|^{\frac{3}{2} - \mu}} dr d\eta$$

$$\lesssim \|F(x)\|_{L^2_T}^2 \|\partial_\gamma x\|_{C^{1+\mu}} \int_{T} \int_{0}^{1} |\partial_\gamma^2 x(\gamma - \eta + h)| \frac{|h||\partial_\gamma^2 x(\gamma + r'\eta + rh)|}{|\eta|^{\frac{3}{2} - \mu}} dr dr' d\eta,$$

and therefore

$$|\Gamma_{12}| \lesssim \|F(x)\|_{L^2_T}^2 \|\partial_\gamma x\|_{C^{1+\mu}} \|\partial_\gamma^2 x\|_{L^2_T} |h|^\frac{1}{2} \int_{T} |\partial_\gamma^2 x(\gamma - \eta + h)| \frac{d\eta}{|\eta|^{\frac{3}{2} - \mu}}$$

$$\lesssim \|F(x)\|_{L^2_T}^2 |h|^\frac{1}{2} \|\partial_\gamma x\|_{C^{1+\mu}} \|\partial_\gamma^2 x\|_{L^2_T}^2.$$ (3.11)
For $\Gamma_{13}$, choosing $a \in (0, 1)$ such that $a(\frac{1}{2} + \mu) < \mu$, we estimate

$$
\begin{align*}
|\Gamma_{13}| &\leq \|F(x)\|_{L^\infty}^2 \int_T |\partial_\gamma x_-(\gamma) - \partial_\gamma x_-(\gamma + h)| \frac{\partial^2_x x(\gamma - \eta + h)}{\eta^a} d\eta \\
&= \|F(x)\|_{L^\infty}^2 \int_T |\partial_\gamma x_-(\gamma) - \partial_\gamma x_-(\gamma + h)|^a |\partial_\gamma x_-(\gamma) - \partial_\gamma x_-(\gamma + h)|^{1-a} \frac{\partial^2_x x(\gamma - \eta + h)}{\eta^a} d\eta \\
&\lesssim \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \int_T |h|^{a(\frac{1}{2}+\mu)} \frac{\partial^2_x x(\gamma - \eta + h)}{|h|^{(1-a)(\frac{1}{2}+\mu)}} d\eta \\
&\lesssim \|F(x)\|_{L^\infty}^2 |h|^{a(\frac{1}{2}+\mu)} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \|\partial^2_x x\|_{L^2}.
\end{align*}
$$

The finite difference for $\Gamma_2$ is decomposed as

$$
\Gamma_2(\gamma + h) - \Gamma_2(\gamma) = \Gamma_{21} + \Gamma_{22} + \Gamma_{23} + \Gamma_{24}
$$

(3.11)

where

$$
\begin{align*}
\Gamma_{21} &= \frac{1}{2A(t)} \int_T \left( |\partial_\gamma x_-(\gamma + h)|^2 - |\partial_\gamma x_-(\gamma)|^2 \right) x_-(\gamma) \cdot \partial_\gamma x_-(\gamma) d\eta, \\
\Gamma_{22} &= \frac{1}{2A(t)} \int_T \left( |\partial_\gamma x_-(\gamma + h)|^2 |x_-(\gamma + h) - x_-(\gamma)| \cdot \partial_\gamma x_-(\gamma) d\eta, \\
\Gamma_{23} &= \frac{1}{2A(t)} \int_T |\partial_\gamma x_-(\gamma + h)|^2 |x_-(\gamma + h) - \partial_\gamma x_-(\gamma) \partial_\gamma x_-(\gamma) d\eta,
\end{align*}
$$

and

$$
\Gamma_{24} = \frac{1}{2A(t)} \int_T |\partial_\gamma x_-(\gamma + h)|^2 \cdot \partial_\gamma x_-(\gamma + h) \left( \frac{1}{|x_-(\gamma + h)|^3} - \frac{1}{|x_-(\gamma)|^3} \right) d\eta.
$$

We successively bound the $\Gamma_{2j}$ as follows.

$$
\begin{align*}
|\Gamma_{21}| &\leq \|F(x)\|_{L^\infty} \frac{1}{2} \int_T \left( |\partial_\gamma x_-(\gamma + h)| + |\partial_\gamma x_-(\gamma)| \right) \left( |\partial_\gamma x_-(\gamma + h)| - |\partial_\gamma x_-(\gamma)| \right) |\partial_\gamma x_-(\gamma)| d\eta \\
&\lesssim \|F(x)\|_{L^\infty} \int_T \int_0^1 \frac{|\partial_\gamma x|^2_{C^{\frac{1}{2}+\mu}} |h|^{\frac{1}{2}+\mu} |\partial^2_x x(\gamma - r \eta)|}{|\eta|^{\frac{1}{2}-\mu}} dr d\eta \\
&\lesssim \|F(x)\|_{L^\infty} \|h|^{\frac{1}{2}+\mu} \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}} \|x\|_{H^2}.
\end{align*}
$$

$$
\begin{align*}
|\Gamma_{22}| &\leq \|F(x)\|_{L^\infty}^2 \int_T |\partial_\gamma x_-(\gamma + h)|^2 \frac{|x_-(\gamma + h) - x_-(\gamma)| |\partial_\gamma x_-(\gamma)| \partial_\gamma x_-(\gamma) d\eta} \left( |x_-(\gamma)|^3 \right) \\
&\lesssim \|F(x)\|_{L^\infty}^5 \int_T \int_0^1 \frac{|\partial_\gamma x|^3_{C^{\frac{1}{2}+\mu}} |\partial_\gamma x_-(\gamma + rh)|}{|\eta|^{\frac{1}{2}-3\mu}} dr d\eta \\
&\lesssim \|F(x)\|_{L^\infty}^5 \int_T \int_0^1 \frac{|\partial_\gamma x|^4_{C^{\frac{1}{2}+\mu}}}{|\eta|^{1-4\mu}} dr d\eta \lesssim \|F(x)\|_{L^\infty}^5 \|h\| \|\partial_\gamma x|^4_{C^{\frac{1}{2}+\mu}}.
\end{align*}
$$

$$
\begin{align*}
|\Gamma_{23}| &\leq \|F(x)\|_{L^\infty}^2 \int_T |\partial_\gamma x_-(\gamma + h)|^2 \frac{|x_-(\gamma + h) - \partial_\gamma x_-(\gamma + h) - \partial_\gamma x_-(\gamma) d\eta} \left( |x_-(\gamma)|^3 \right) \\
&\lesssim \|F(x)\|_{L^\infty}^5 \|\partial_\gamma x|^3_{C^{\frac{1}{2}+\mu}} \|x\|_{C^1} |h|^{\frac{1}{2}+\mu} \int_T \frac{d\eta}{|\eta|^{1-2\mu}} \\
&\lesssim \|F(x)\|_{L^\infty}^5 \|\partial_\gamma x|^3_{C^{\frac{1}{2}+\mu}} |h|^{\frac{1}{2}+\mu}.
\end{align*}
$$
Finally, for \( a \in (0, 1) \) satisfying \((\frac{1}{2} + \mu)a < 4\mu\), we have

\[
|\Gamma_{24}| \leq \|F(x)\|_{L^\infty}^2 \frac{1}{2} \int_T |\partial_\gamma x_-(\gamma + h)|^2 |x_-(\gamma + h)| |\partial_\gamma x_-(\gamma + h) - x_-(\gamma)|
\times \frac{1}{|x_-(\gamma + h)||x_-(\gamma)|} \left( \frac{1}{|x_-(\gamma + h)|^2} + \frac{1}{|x_-(\gamma + h)||x_-(\gamma)|} + \frac{1}{|x_-(\gamma)|^2} \right) d\eta
\lesssim \|F(x)\|_{L^\infty}^5 \int_T \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}}^3 |x_-(\gamma + h)| - x_-(\gamma) |x_-(\gamma + h) - x_-(\gamma)|^{1-a} \frac{d\eta}{|\eta|^\frac{3}{2}-3\mu},
\]
and therefore

\[
|\Gamma_{24}| \lesssim \|F(x)\|_{L^\infty}^5 |h|^{(\frac{1}{2}+\mu)a} \|x\|_{C^1} |\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}}^4 \int_T \frac{d\eta}{|\eta|^\frac{3}{2}-3\mu-(\frac{1}{2}+\mu)(1-a)}
\lesssim \|F(x)\|_{L^\infty}^5 |h|^{(\frac{1}{2}+\mu)a} \|x\|_{C^1} |\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}}^4.
\]

Gathering the above estimates, we conclude the proof of Proposition 3.2. \( \square \)

3.1.2. Sobolev estimate for \( \partial_\gamma \lambda \). Our goal is to establish the following bound for \( \|\partial_\gamma \lambda\|_{H^{\frac{1}{2}}} \).

**Proposition 3.3.** For any \( \mu \in (0, \frac{1}{2}) \), there exists a polynomial \( P(\cdot, \cdot) \) such that

\[
\|\partial_\gamma \lambda\|_{H^{\frac{1}{2}}} \leq P(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{\frac{1}{2}+\mu}}) \|\partial_\gamma^2 x\|_{L^2}.
\]  

**Proof.** We use the decomposition (3.2) for \( \partial_\gamma \lambda \) together with the double-integral definition of \( \| \cdot \|_{H^{\frac{1}{2}}} \) to have

\[
\|\partial_\gamma \lambda\|_{H^{\frac{1}{2}}}^2 = \int_T \int_T \left| \frac{\partial_\gamma \lambda(\gamma) - \partial_\gamma \lambda(\gamma - \xi)}{|\xi|^2} \right|^2 d\xi d\gamma \leq \sum_{i=1}^3 \bar{\Gamma}_{1i} + \sum_{i=1}^4 \bar{\Gamma}_{2i},
\]

where

\[
\bar{\Gamma}_{11} = \frac{1}{A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{\partial_\gamma x_-(\gamma) - \partial_\gamma x_-(\gamma - \xi)}{|x_-(\gamma)|} \cdot \left( \frac{\partial_\gamma^2 x(\gamma - \eta) - \partial_\gamma^2 x(\gamma - \eta - \xi)}{\partial_\gamma x(\gamma - \eta - \xi)} \right) d\eta \right|^2,
\]

\[
\bar{\Gamma}_{12} = \frac{1}{A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{\partial_\gamma x_-(\gamma) - \partial_\gamma x_-(\gamma - \xi)}{|x_-(\gamma)|} \cdot \partial_\gamma^2 x(\gamma - \eta - \xi) d\eta \right|^2,
\]

\[
\bar{\Gamma}_{13} = \frac{1}{A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{\partial_\gamma x_-(\gamma - \xi)}{|x_-(\gamma)|} \cdot \partial_\gamma^2 x(\gamma - \eta - \xi) \left( \frac{1}{|x_-(\gamma)|} - \frac{1}{|x_-(\gamma - \xi)|} \right) d\eta \right|^2,
\]

and

\[
\bar{\Gamma}_{21} = \frac{1}{2A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{(\partial_\gamma x_-(\gamma))^2 - \partial_\gamma x_-(\gamma - \xi)^2}{|x_-(\gamma)|^3} d\eta \right|^2,
\]

\[
\bar{\Gamma}_{22} = \frac{1}{2A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{\partial_\gamma x_-(\gamma - \xi)^2 - \partial_\gamma x_-(\gamma - \xi)^2}{|x_-(\gamma)|^3} d\eta \right|^2,
\]

\[
\bar{\Gamma}_{23} = \frac{1}{2A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{\partial_\gamma x_-(\gamma - \xi)^2 - \partial_\gamma x_-(\gamma - \xi)^2}{|x_-(\gamma)|^3} d\eta \right|^2,
\]

\[
\bar{\Gamma}_{24} = \frac{1}{2A(t)} \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left| \frac{\partial_\gamma x_-(\gamma - \xi)^2 - \partial_\gamma x_-(\gamma - \xi)^2}{|x_-(\gamma)|^3} d\eta \right|^2.
\]

Using

\[
\partial_\gamma^2 x(\gamma - \eta) - \partial_\gamma^2 x(\gamma - \eta - \xi) = \partial_\eta \{ \partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \xi) + \partial_\gamma x(\gamma - \eta - \xi) \}
\]
we can integrate by parts to obtain
\[ \tilde{\Gamma}_{111} \leq \tilde{\Gamma}_{111} + \tilde{\Gamma}_{112} \]
where
\[ \tilde{\Gamma}_{111} = \frac{1}{A(t)} \int_T d\gamma \int_T \int_T \frac{d\xi}{|\xi|^2} \int_T \frac{d\gamma x(\gamma - \eta)}{|x_-(\gamma)|} \cdot \{ \partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \eta - \xi) \} d\eta \]
\[ \text{(3.13)} \]
and
\[ \tilde{\Gamma}_{112} = \frac{1}{A(t)} \int_T d\gamma \int_T \int_T \int_T \frac{d\xi}{|\xi|^2} \int_T \frac{[x_-(\gamma) \cdot \partial_\gamma x(\gamma - \eta)]\partial_\gamma x_-(\gamma)}{|x_-(\gamma)|^3} \cdot \{ \partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \xi) \} d\eta \]²

Now, for any \( b \in [0, 1] \) we have
\[ |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \xi) + \partial_\gamma x(\gamma - \eta - \xi)| \leq |\xi|^{(\frac{1}{2} + \mu)b} |\eta|^{(\frac{1}{2} + \mu)(1 - b)} \| \partial_\gamma x \|_{C^{\frac{1}{2} + \mu}}. \]
Hence, for \( \frac{1}{1 + 2\mu} < b < 1 \) we obtain
\[ \tilde{\Gamma}_{111} \leq \| F(x) \|_{L^\infty}^2 \int_T d\gamma \int_T \int_T \left| \frac{d\xi}{|\xi|^{2(1 + 2\mu)b}} \right| \int_T \left| \partial_\gamma^2 x(\gamma - \eta) \| \partial_\gamma x \|_{C^{\frac{1}{2} + \mu}} \| F(x) \|_{L^\infty} \frac{d\eta}{|\eta|^{(\frac{1}{2} + \mu)(1 - b)}} \right|^2 \]
\[ \leq \| F(x) \|_{L^\infty}^4 \| \partial_\gamma x \|_{C^{\frac{1}{2} + \mu}}^2 \left| \partial_\gamma^2 x \ast \cdot |^{-1 + (\frac{1}{2} + \mu)(1 - b)} \right|_{L^2}^2 \]
\[ \leq \| F(x) \|_{L^\infty}^4 \| \partial_\gamma x \|_{C^{\frac{1}{2} + \mu}}^2 \| \partial_\gamma^2 x \|_{L^2}^2. \]

On the other hand, writing
\[ \partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \xi) + \partial_\gamma x(\gamma - \eta - \xi) = \eta \int_0^1 \partial_\gamma x(\gamma - r\eta) dr - \eta \int_0^1 \partial_\gamma x(\gamma - \xi - r\eta) dr \]
yields
\[ \tilde{\Gamma}_{112} \lesssim \| F(x) \|_{L^\infty}^4 \int_T d\gamma \int_T \left| \frac{d\xi}{|\xi|^{2(1 + 2\mu)b}} \right| \left( \int_T \int_0^1 dr \left| \partial_\gamma x \right|_{L^\infty} \left| \partial_\gamma x \right|_{C^{\frac{1}{2} + \mu}} \left| \partial_\gamma^2 x(\gamma - r\eta) \right| \frac{d\eta}{|\eta|^{(\frac{1}{2} + \mu)(1 - b)}} \right)^2 \]
\[ \leq \| \partial_\gamma x \|_{L^\infty}^2 \| \partial_\gamma x \|_{C^{\frac{1}{2} + \mu}}^2 \| F(x) \|_{L^\infty}^2 \int_T \int_T \left| \frac{d\xi}{|\xi|^{1 - (\frac{1}{2} + \mu)(1 - b)}} \right| \left| \partial_\gamma^2 x(\gamma - \xi) \right| \frac{d\xi}{r(\frac{1}{2} + \mu)(1 - b)} \left| \partial_\gamma^2 x \ast \cdot |^{-1 + (\frac{1}{2} + \mu)(1 - b)} \right|_{L^2}^2 \]
\[ \leq \| \partial_\gamma x \|_{L^\infty}^2 \| \partial_\gamma x \|_{C^{\frac{1}{2} + \mu}}^2 \| F(x) \|_{L^\infty}^2 \| \partial_\gamma^2 x \|_{L^2}^2. \]

provided that \( \frac{1}{1 + 2\mu} < b < 1. \)

We note that \( \tilde{\Gamma}_{112} \) is similar to \( \tilde{\Gamma}_{111} \) given by (3.13) since
\[ \tilde{\Gamma}_{112} = \int_T d\gamma \int_T \int_T \frac{d\xi}{|\xi|^2} \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma - \eta) - \partial_\gamma x(\gamma - \xi) + \partial_\gamma x(\gamma - \eta - \xi)}{|x_-(\gamma)|} \cdot \partial_\gamma^2 x(\gamma - \eta - \xi) d\eta \]

Regarding \( \tilde{\Gamma}_{113} \), the bound
\[ \left| \frac{1}{|x_-(\gamma)|} - \frac{1}{|x_-(\gamma - \xi)|} \right| \leq \frac{|x_-(\gamma) - x_-(\gamma - \xi)|}{|x_-(\gamma)||x_-(\gamma - \xi)|} \leq \| \partial_\gamma x \|_{L^\infty} \| F(x) \|_{L^\infty}^2 \frac{|\xi|^b}{|\eta|^{1 + b}} \]
implies
\[ \bar{\Gamma}_{13} \lesssim \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \int_T \left( |\partial_\gamma x| + |\partial_\gamma y| \right) F(x) \left| \frac{\partial^2 x(x-\gamma-\eta-\xi)}{|\eta|^{2-\beta}} \right| d\eta \]
\[ \lesssim \|\partial_\gamma x\|_{C^{2+\mu}}^2 \|\partial_\gamma y\|_{L^\infty}^2 \|F\|_{L^\infty}^2 \|\partial^2 x\|_{L^2}^2, \]
provided that \( \frac{1}{2} < b < \frac{1}{2} + \mu. \) \( \bar{\Gamma}_{21} \) can be controlled as
\[ |\bar{\Gamma}_{21}| \lesssim \|F(x)\|_{L^\infty}^2 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \int_T \left( |\partial_\gamma x(\gamma-\eta-\xi)| + |\partial_\gamma x(\gamma-\eta)| \right) \left( |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)| \right) \]
\[ \times \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^{2+\mu}} \frac{d\eta}{|\eta|^{2-\beta}} \]
\[ \lesssim \|F(x)\|_{L^\infty}^6 \|\partial_\gamma x\|_{C^{2+\mu}}^6 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \int_T \frac{|\xi|^{2+\mu}}{|\eta|^{2-\beta}} d\eta \]
\[ \lesssim \|F(x)\|_{L^\infty}^6 \|\partial_\gamma x\|_{C^{2+\mu}}^6 \cdot \]

Next, for \( \frac{1}{2} < b < \frac{1}{2} + 3\mu \) we estimate
\[ |\bar{\Gamma}_{22}| \lesssim \|F(x)\|_{L^\infty}^2 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \int_T \left| x-(\gamma-x-\xi)^b \right| \left| x-(\gamma-x-\xi)^b \right| \]
\[ \times \|\partial_\gamma x\|_{C^{2+\mu}}^3 \|F(x)\|_{L^\infty}^3 \frac{d\eta}{|\eta|^{2-3\mu}} \]
\[ \lesssim \|F(x)\|_{L^\infty}^6 \|\partial_\gamma x\|_{C^{2+\mu}}^6 \]
\[ \lesssim \|F(x)\|_{L^\infty}^6 \|\partial_\gamma x\|_{C^{2+\mu}}^6 \cdot \]

\( \bar{\Gamma}_{23} \) can be bounded as
\[ |\bar{\Gamma}_{23}| \lesssim \|F(x)\|_{L^\infty}^2 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \int_T \left| \partial_\gamma x \right| \left| x \right| \left| F(x) \right| \frac{d\eta}{|\eta|^{2-3\mu}} \]
\[ \lesssim \|F(x)\|_{L^\infty}^8 \|\partial_\gamma x\|_{C^{2+\mu}}^8 \cdot \]

Finally, for \( \frac{1}{2} < b < \frac{1}{2} + 3\mu \) we have
\[ |\bar{\Gamma}_{24}| \lesssim \|F(x)\|_{L^\infty}^2 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left( \int_T \left| \partial_\gamma x(\gamma-x-\xi)^2 \right| \right) \left( \int_T \left| \partial_\gamma x(\gamma-x-\xi)^2 \right| \right) \]
\[ \times \left( \frac{1}{|x-(\gamma-x-\xi)|^2} + \frac{1}{|x-(\gamma-x-\xi)|^2} + \frac{1}{|x-(\gamma-x)|^2} \right) \]
\[ \lesssim \|F(x)\|_{L^\infty}^2 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left( \int_T \left| F(x) \right| \left| \partial_\gamma x \right| \right) \frac{d\eta}{|\eta|^{2-3\mu+a}} \]
\[ \times \left( |x-(\gamma-x-\xi)|^{a} \right) \left( |x-(\gamma-x)|^{1-a} \frac{d\eta}{|\eta|^{2-3\mu+a}} \right)^2, \]
and therefore
\[ |\bar{\Gamma}_{24}| \lesssim \|F(x)\|_{L^\infty}^8 \int_T d\gamma \int_T \frac{d\xi}{|\xi|^2} \left( \int_T \left| \partial_\gamma x \right| \right) \frac{d\eta}{|\eta|^{2-3\mu+a}} \]
\[ \lesssim \|F(x)\|_{L^\infty}^8 \|\partial_\gamma x\|_{C^{2+\mu}}^8 \cdot \]

Gathering the above estimates lead to the desired bound \( (5.1.2). \)

\[ \square \]
3.2. Estimates for tangential terms $T_j$. Recall that

$$T_1 = \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \lambda(\gamma) \cdot \partial_\gamma^2 x(\gamma) d\gamma,$$

$$T_2 = 2 \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \partial_\gamma \lambda(\gamma) \cdot \partial_\gamma^2 x(\gamma) d\gamma,$$

$$T_3 = \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \partial_\gamma^2 \lambda(\gamma) \cdot \partial_\gamma x(\gamma) d\gamma.$$

We first rewrite $T_1$ using commutator and integration by parts as

$$T_1 = \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \lambda(\gamma) \cdot \Lambda^s \partial_\gamma^2 x(\gamma) d\gamma + \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \cdot [\Lambda^s, \lambda(\gamma)] \partial_\gamma^2 x(\gamma) d\gamma$$

$$= -\frac{1}{2} \int_T \partial_\gamma \lambda(\gamma) \Lambda^s \partial_\gamma^2 x(\gamma))^2 d\gamma + \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \cdot [\Lambda^s, \lambda(\gamma)] \partial_\gamma^2 x(\gamma) d\gamma$$

$$:= T_{11} + T_{12}.$$

We fix $\mu \in (0, \frac{1}{2})$ and denote by $P(\cdot, \cdot)$ polynomials that depend only on $\mu$. It follows from the $L^\infty$ bound (3.6) for $\partial_\gamma \lambda$ that

$$|T_{11}| \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{\frac{1}{2} + \mu}})\|\partial_\gamma^2 x\|_{L^2}\|\partial_\gamma^2 x\|_{H^\mu}.$$  

(3.17)

The commutator estimate (3.18) gives

$$|T_{12}| \leq \|\Lambda^s \partial_\gamma^2 x\|_{L^2} \|\Lambda^s, \lambda(\gamma)] \partial_\gamma^2 x(\gamma)\|_{L^2} \lesssim \|\Lambda^s \partial_\gamma^2 x\|_{L^2} \|\partial_\gamma \lambda\|_{C^\varepsilon}$$

for any $\varepsilon > 0$. A combination (3.6) and (3.8) implies that for any $\varepsilon \in (0, \mu)$,

$$\|\partial_\gamma \lambda\|_{C^\varepsilon} \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{\frac{1}{2} + \mu}})\|\partial_\gamma^2 x\|_{H^\mu},$$

whence

$$|T_{12}| \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{\frac{1}{2} + \mu}})\|\partial_\gamma^2 x\|_{H^\mu}\|\Lambda^s \partial_\gamma^2 x\|_{L^2}.$$  

(3.18)

As for $T_2$, we note

$$T_2 = \int_T \Lambda^s \partial_\gamma^2 x(\gamma) \cdot \Lambda^s(\partial_\gamma \lambda(\gamma) \partial_\gamma^2 x(\gamma))) d\gamma \leq \|\Lambda^s \partial_\gamma^2 x\|_{L^2} \|\Lambda^s(\partial_\gamma \lambda \partial_\gamma^2 x)\|_{L^2}.$$ 

Applying (A.5), (A.6) and (A.7) gives the product estimate

$$\|uv\|_{H^s} \lesssim \|u\|_{L^\infty} \|v\|_{H^s} + \|u\|_{H^\frac{s}{2}} \|v\|_{B^{s-\frac{1}{2}}_{2,\infty}} \quad \forall s \in (0, \frac{1}{2}).$$  

(3.19)

Using (3.19) with $u = \partial_\gamma \lambda, v = \partial_\gamma^2 x$ and recalling the embedding $H^s \subset B^s_{2,\infty}$, we deduce

$$|T_2| \lesssim \|\Lambda^s \partial_\gamma^2 x\|_{L^2}^2 (\|\partial_\gamma \lambda\|_{L^\infty} + \|\partial_\gamma \lambda\|_{H^\mu}).$$

Then in view of (3.6) and (3.12), we obtain

$$|T_2| \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{H^{1+\mu}})\|\Lambda^s \partial_\gamma^2 x\|_{L^2}^2.$$  

(3.20)

Now, since $0 = \partial_\gamma |\partial_\gamma x|^2 = 2\partial_\gamma x \partial_\gamma^2 x$ we have

$$T_3 = -\int_T \partial_\gamma^2 \lambda(\gamma) \cdot [\Lambda^{2s}, \partial_\gamma x(\gamma)] \partial_\gamma^2 x(\gamma) d\gamma.$$  

(3.21)

In order to control the commutator, we prove

**Lemma 3.4.** For $s \in (0, \frac{1}{2})$ we have

$$\|[\Lambda^{2s}, \partial_\gamma x] \partial_\gamma^2 x\|_{H^{\frac{1}{2}}} \leq C\|\partial_\gamma x\|_{C^{\frac{1}{2} + \mu}}\|\partial_\gamma^2 x\|_{H^\mu}.$$  

(3.22)
By virtue of Lemma A.4, we have

\[ |\Lambda_{2s} u| v = |\Lambda_{2s} T_u| v + |\Lambda_{2s} (T_v u) + \Lambda_{2s} R(u, v) - T_{\Lambda_{2s} R} u - R(u, \Lambda_{2s} v)|. \]

Combining Propositions 2.3, 3.5 and (3.25), we conclude the a priori estimate for the \( L_2 \) norm.

Next, for the \( L_2 \) bound, we multiply (1.7) by \( x(\gamma) \), integrate, symmetrize then integrate by parts:

\[
\frac{1}{2} \frac{d}{dt} \|x\|_{L_2}^2 = \frac{1}{2} \int_T \int_T \partial_\gamma x_{-}(\gamma, \eta) \cdot x_{-}(\gamma, \eta) \frac{1}{|x_{-}(\gamma, \eta)|} d\eta d\gamma + \int_T \lambda(\gamma) \partial_\gamma x(\gamma) \cdot x(\gamma) d\gamma
\]

\[
= - \frac{1}{4} \int_T \int_T |x_{-}(\gamma, \eta)|^2 \partial_\gamma \frac{1}{|x_{-}(\gamma, \eta)|} d\eta d\gamma - \frac{1}{2} \int \int \partial_\gamma \lambda(\gamma)|x(\gamma)|^2 d\gamma.
\]

Clearly,

\[
\int_T \int_T |x_{-}(\gamma, \eta)|^2 \partial_\gamma \frac{1}{|x_{-}(\gamma, \eta)|} d\eta d\gamma = - \int_T \int \partial_\gamma |x_{-}(\gamma, \eta)| d\gamma d\eta = 0.
\]

An application of (3.6) gives

\[
\frac{d}{dt} \|x\|_{L_2}^2 \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma^2 x\|_{L^\infty}) \|x\|_{L_2}^2.
\]

Combining Propositions 2.3, 3.5 and (3.25), we conclude the a priori estimate for the \( H^{2+s} \) norm.

**Proposition 3.5.** For all \( s \in (0, \frac{1}{2}) \), there exists a polynomial \( \mathcal{P}(\cdot, \cdot) \) such that

\[
\sum_{j=1}^{3} |T_j| \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{L^\infty}) \|\partial_\gamma^2 x\|_{H^{2+s}}^2.
\]

Next, for the \( L^2 \) bound, we multiply (1.7) by \( x(\gamma) \), integrate, symmetrize then integrate by parts:

\[
\frac{d}{dt} \|x\|_{L^2}^2 = \frac{1}{2} \int_T \int T_u + R(u, v) \lambda(\gamma) \partial_\gamma x(\gamma) \cdot x(\gamma) d\gamma
\]

\[
\leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{H^{2+s}}) \|\partial_\gamma^2 x\|_{H^s}^2.
\]
4. Propagation of the arc chord condition

We assume throughout this section that \( x \in C([0, T]; H^3) \) is a solution of \((1.7)-(1.8)\). In order to close the a priori estimates \((2.23), (3.24)\) and \((3.25)\), it remains to bound the arc chord condition in terms of \( \|x\|_{H^{2+s}} \) and itself. To that end, we differentiate \( F(x)(\gamma, \eta, t) = |\eta|_{x_-(\gamma, \eta, t)} \) to have

\[
\partial_t F(\gamma, \eta) = -|\eta| \frac{x_-(\gamma, \eta) \cdot \partial_t x_-(\gamma, \eta)}{|x_-(\gamma, \eta)|^3} = -|\eta| \frac{x_-(\gamma, \eta) \cdot [Q(\gamma) - Q(\gamma - \eta)]}{|x_-(\gamma, \eta)|^3} - |\eta| \frac{x_-(\gamma, \eta) \cdot \partial_\gamma x(\gamma) [\lambda(\gamma) - \lambda(\gamma - \eta)]}{|x_-(\gamma, \eta)|^3} - |\eta| \frac{x_-(\gamma, \eta) \cdot \partial_\gamma x_-(\gamma, \eta) \lambda(\gamma - \eta)}{|x_-(\gamma, \eta)|^3} \tag{4.1}
\]

where

\[
Q(\gamma) = \int_\mathbb{T} \partial_\gamma x_-(\gamma, \xi) g(\gamma, \xi) d\xi, \quad g(\gamma, \xi) = \frac{1}{|x_-(\gamma, \xi)|}.
\]

It is readily seen that

\[
|F_2| \leq |\eta| \frac{\|\partial_\gamma x\|_{L^\infty} |\eta| \|\partial_\gamma \lambda\|_{L^\infty}}{|x_-(\gamma, \eta)|^2} \leq \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma \lambda\|_{L^\infty},
\]

so that invoking \((3.6)\) yields

\[
F_2 \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{1+\mu}}) \|\partial_\gamma^2 x\|_{L^2} \quad \forall \mu \in (0, \frac{1}{2}). \tag{4.2}
\]

Recalling the identity \((2.7)\) for \( x_\gamma \cdot \partial_\gamma x_\gamma \) we obtain

\[
|F_3| \lesssim |\eta|^3 \|F(x)\|_{L^\infty} \frac{|\eta|^2 \|\partial_\gamma x\|_{C^{1/2}}^2 \|\lambda\|_{L^\infty}}{|\eta|^3} \lesssim \|F(x)\|_{L^\infty}^2 \|\partial_\gamma x\|_{C^{1/2}}^2 \|\lambda\|_{L^\infty}. \tag{4.3}
\]

Then, appealing to \((3.6)\) again, \((4.3)\) implies

\[
|F_3| \leq \mathcal{P}(\|F(x)\|_{L^\infty}, \|\partial_\gamma x\|_{C^{1+\mu}}) \|\partial_\gamma^2 x\|_{L^2} \quad \forall \mu \in (0, \frac{1}{2}). \tag{4.4}
\]

For the most difficult term \( F_1 \) we estimate

\[
|F_1| \leq \|F(x)\|_{L^\infty} \left( \int_{|\xi|<|\eta|^2} + \int_{|\xi|>|\eta|^2} \right) \frac{1}{|\xi|} |x_-(\gamma, \eta) \cdot [\partial_\gamma x_-(\gamma, \xi) - \partial_\gamma x_-(\gamma - \eta, \xi)] d\xi := F_{11} + F_{12}.
\]

By the mean value theorem and Cauchy-Schwarz’s inequality, we have

\[
|F_{11}| \leq \|F(x)\|_{L^\infty}^4 \left( \int_{|\xi|<|\eta|^2} + \int_{|\xi|>|\eta|^2} \right) \frac{1}{|\xi|} |\eta| \|\partial_\gamma x\|_{L^\infty} |\xi| \int_0^1 \left( |\partial_\gamma^2 x(\gamma - r\xi)\| + |\partial_\gamma^2 x(\gamma - \eta - r\xi)| \right) dr d\xi
\]

\[
\leq C \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \frac{1}{|\eta|} \int_0^1 \frac{1}{\sqrt{T}} \left( \int_{|\xi|<|\eta|^2} 1 d\xi \right)^{\frac{3}{2}} \left( \int_{|\xi|>|\eta|^2} 1 d\xi \right)^{\frac{3}{2}} dr
\]

\[
= C \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{L^\infty} \|\partial_\gamma^2 x\|_{L^2}.
\]

As for \( F_{12} \) we split

\[
x_-(\gamma, \eta) \cdot [\partial_\gamma x_-(\gamma, \xi) - \partial_\gamma x_-(\gamma - \eta, \xi)]
\]

\[
= [x_-(\gamma, \eta) - \eta \partial_\gamma x(\gamma)] \cdot \partial_\gamma x_-(\gamma, \eta) + \eta \partial_\gamma x(\gamma) \cdot \partial_\gamma x_-(\gamma, \eta) - [x_-(\gamma, \eta) - \eta \partial_\gamma x(\gamma)] \cdot \partial_\gamma x_-(\gamma - \xi, \eta) - \eta \partial_\gamma x(\gamma) \cdot \partial_\gamma x_-(\gamma - \xi, \eta)
\]

\[
= \|\partial_\gamma x\|_{C^{1+\mu}} \|\partial_\gamma^2 x\|_{C^{1/2}} \|\lambda\|_{L^\infty}.
\]
and accordingly $F_{12} = \sum_{j=1}^{4} F_{12j}$. Clearly,

$$|F_{121}| \leq \|F(x)\|_{L^\infty} \frac{1}{|\eta|^2} \int_{|\xi|>|\eta|^2} \frac{1}{|\xi|^2} |\eta|^{1+2s} \|\partial_{\gamma} x\|_{C^{1+s}} |\eta|^{1+2s} \|\partial_{\gamma} x\|_{C^{1+s}} d\xi$$

$$\leq 2 \|F(x)\|_{L^\infty} \|\partial_{\gamma} x\|_{C^{1+s}}^2 |\eta|^{2s} \ln \pi - \ln |\eta|^2$$

$$\leq C \|F(x)\|_{L^\infty} \|\partial_{\gamma} x\|_{C^{1+s}}^2.$$

In view of identity (2.8) and by the same argument, $F_{122}$ and $F_{123}$ obey the same bound as $F_{121}$. We further split $F_{124} = F_{1241} + F_{1242}$ according to

$$\eta \partial_{\gamma} x(\gamma) \cdot \partial_{\gamma} x(\gamma - \xi, \eta) = \eta [\partial_{\gamma} x(\gamma) - \partial_{\gamma} x(\gamma - \xi)] \cdot \partial_{\gamma} x(\gamma - \xi, \eta) + \eta \frac{1}{2} |\partial_{\gamma} x(\gamma - \xi, \eta)|^2,$$

where we have again appealed to (2.8). By the mean value theorem and Cauchy-Schwarz’s inequality,

$$|F_{1241}| \leq C \|F(x)\|_{L^\infty} \int_{T} \|\partial_{\gamma} x\|_{C^{1+s}}^2 \int_{0}^{1} |\partial_{\gamma}^2 x(\gamma - \xi - r\eta)|^2 dr d\xi \leq C \|F(x)\|_{L^\infty} \|\partial_{\gamma} x\|_{C^{1+s}}^2 \|\partial_{\gamma}^2 x\|_{L^2}.$$

Finally, we bound

$$|F_{1242}| \leq C \|F(x)\|_{L^\infty} \frac{1}{|\eta|^2} \int_{|\xi|>|\eta|^2} \frac{1}{|\xi|^2} |\eta|^{1+2s} \|\partial_{\gamma} x\|_{C^{1+s}}^2 d\xi \leq C \|F(x)\|_{L^\infty} \|\partial_{\gamma} x\|_{C^{1+s}}^2 |\eta|^{2s} \ln \pi - \ln |\eta|^2$$

$$\leq C \|F(x)\|_{L^\infty} \|\partial_{\gamma} x\|_{C^{1+s}}^2.$$

We have proved

$$|F_{1}| \leq C \|F(x)\|_{L^\infty} \|\partial_{\gamma} x\|_{H^{1+s}}^2. \quad (4.5)$$

and thus, in view of (4.2) and (4.4), we obtain

PROPOSITION 4.1. For all $s \in (0, \frac{1}{2})$, there exists $\mathcal{P}(\cdot, \cdot)$ such that

$$\left|\frac{d}{dt} F(x)(\gamma, \eta, t)\right| \leq \mathcal{P}(\|F(x)(t)\|_{L^\infty}, \|\partial_{\gamma} x(t)\|_{C^{1+s}}), \|\partial_{\gamma}^2 x(t)\|_{H^s}, \quad t \in [0, T]. \quad (4.6)$$

5. Uniqueness

This section is devoted to the proof of the following stability result which implies the uniqueness of $H^{2+s}$ solutions.

THEOREM 5.1. Suppose that $x$ and $y$ are two solutions of (1.7)-(1.8) in $L^\infty([0, T]; H^{2+s}(\mathbb{T}))$ and satisfy the arc chord condition. Then we have

$$\|x(t) - y(t)\|_{H^1} + \|\partial_{\gamma} x(t) - \partial_{\gamma} y(t)\|_{H^1}^2 \leq e^{Ct} \left( \|x(0) - y(0)\|_{H^1} + \|\partial_{\gamma} x(0) - \partial_{\gamma} y(0)\|_{H^1}^2 \right), \quad (5.1)$$

for any $t \in [0, T]$, where $C$ depends only on $\|x, y\|_{L^\infty([0, T]; H^{2+s}(\mathbb{T}))}$ and $\|F(x), F(y)\|_{L^\infty(\mathbb{T} \times T \times [0, T])}$. Consequently, $x \equiv y$ if $x(0) = y(0)$.

As before we shall write $f = f(\gamma, t)$, $f' = f(\gamma - \eta, t)$, $f_+ = f - f'$ and $f_0 = f_1$ when there is no danger of confusion in writing double integrals with respect to the variables $\gamma$ and $\eta$. Recall from (3.6) that

$$\|\langle \lambda(x), \lambda(y)\rangle\|_{L^\infty([0, T]; W^{1,\infty}(\mathbb{T}))} \leq C. \quad (5.2)$$

Set $z = x - y$. 

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5.1. \( L^2 \) estimates. We have
\[
\frac{1}{2} \frac{d}{dt} \|z\|_{L^2}^2 = \int z \cdot \partial_t z d\gamma = I_1 + I_2,
\]
where
\[
I_1 = \int z \cdot \left( \partial_x \frac{\partial_x x}{|x|} - \frac{\partial_y y}{|y|} \right) d\eta d\gamma, \quad I_2 = \int z \cdot (\lambda(x) \partial_x x - \lambda(y) \partial_y y) d\gamma.
\]
We split \( I_1 = I_{11} + I_{12} \), where
\[
I_{11} = \int z \cdot \int \frac{\partial_x z}{|x|} d\eta d\gamma, \quad I_{12} = \int z \cdot \int \frac{\partial_y y - (1/|x|) \partial_x x}{|y|} d\eta d\gamma.
\]
After symmetrizing, an integration by parts gives
\[
I_{11} = \frac{1}{2} \int \int z \cdot \partial_x \frac{z}{|x|} d\gamma d\eta = \frac{1}{4} \int \int |z| \frac{\partial_x x}{|x|} \frac{\partial_x x}{|x|} d\gamma d\eta.
\]
The identity
\[
f = \eta \int_0^1 \partial_x f(\gamma - r\eta) dr \quad (5.3)
\]
together with Hölder’s inequality (in \( \gamma \)) allows us to get the bound
\[
I_{11} \leq \|F(x)\|_{L^\infty} \|\partial_x x\|_{C^2} \int_0^1 dr \int \frac{d\eta}{|\eta|^2} \int d\gamma |z + |z'|| \partial_x z(\gamma - r\eta) | \leq C \|z\|_{H^1}^2.
\]
For \( I_{12} \) one writes
\[
I_{12} = \int z \cdot \int \frac{\partial_y y - |x|}{|y|} d\eta d\gamma \leq \int \int \frac{|z\partial_y y - |x|}{|x| |y|} d\eta d\gamma,
\]
so that arguing as for \( I_{12} \), we obtain
\[
I_{12} \leq \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \|\partial_x y\|_{C^2} \int_0^1 dr \int \frac{d\eta}{|\eta|^2} \int d\gamma |z\partial_y z(\gamma - r\eta) | \leq C \|z\|_{H^1}^2.
\]
Thus,
\[
I_1 \leq C \|z\|_{H^1}^2. \quad (5.4)
\]
Next, \( I_2 = I_{21} + I_{22} \), where
\[
I_{21} = \int z \cdot \partial_x \lambda(x) d\gamma, \quad I_{22} = \int z \cdot \partial_y \lambda(x) d\gamma.
\]
It follows at once from (5.2) that \( I_{21} \leq C \|z\|_{H^1}^2 \). Similarly, \( I_{21} \leq C \|z\|_{H^1}^2 \), provided that
\[
\|\lambda(x) - \lambda(y)\|_{L^\infty} \leq C \|z\|_{H^1}. \quad (5.5)
\]
Taking this for granted, we obtain \( I_2 \leq C \|z\|_{H^1}^2 \), which in conjunction with (5.4) yields the differential inequality
\[
\frac{d}{dt} \|z\|_{L^2}^2 \leq C \|z\|_{H^1}^2. \quad (5.6)
\]
The remainder of this subsection is devoted to the proof of (5.5). Let us write
\[
\lambda(x) - \lambda(y) = G_1 + G_2,
\]
where
\[
G_1 = \frac{\gamma + \pi}{2\pi} \int \left[ \frac{\partial_x x}{|x|^2} \partial_y \left( \int \frac{\partial_x x}{|x|} d\eta \right) - \frac{\partial_y y}{|y|^2} \partial_x \left( \int \frac{\partial_y y}{|y|} d\eta \right) \right] d\gamma,
\]
and
\[
G_2 = \frac{\gamma + \pi}{2\pi} \int \left[ \frac{\partial_x x}{|x|^2} \partial_y \left( \int \frac{\partial_x x}{|x|} d\eta \right) - \frac{\partial_y y}{|y|^2} \partial_x \left( \int \frac{\partial_y y}{|y|} d\eta \right) \right] d\gamma.
\]
and

\[ G_2 = -\int_{-\pi}^{\pi} \frac{\partial_y x(\eta, t)}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x(\eta, t) - \partial x(\eta - \xi, t)}{|x(\eta, t) - x(\eta - \xi, t)|} \right) d\eta \]  
\[ + \int_{-\pi}^{\pi} \frac{\partial_y y(\eta, t)}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial y(\eta, t) - \partial y(\eta - \xi, t)}{|y(\eta, t) - y(\eta - \xi, t)|} \right) d\eta. \]

We shall prove that both \(|G_1|\) and \(|G_2|\) are bounded by \(C\|z\|_{H^1}\). To that end, we decompose further \(G_1 = G_{11} + G_{12} + G_{13}\), where

\[ G_{11} = \frac{\gamma + \pi}{2\pi} \int \frac{\partial y \gamma z}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} \right) d\gamma, \]

\[ G_{12} = \frac{\gamma + \pi}{2\pi} \int \left( \frac{1}{|\partial y \gamma|} - \frac{1}{|\partial \gamma y|} \right) \partial \gamma y \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} \right) d\gamma, \]

\[ G_{13} = \frac{\gamma + \pi}{2\pi} \int \frac{\partial y \gamma y}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} \right) d\gamma. \]

A further splitting provides \(G_{11} = G_{111} + G_{112}\) where

\[ G_{111} = -\frac{\gamma + \pi}{2\pi} \int \frac{\partial y \gamma z}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} \right) d\gamma, \]

\[ G_{112} = \frac{\gamma + \pi}{2\pi} \int \frac{\partial y \gamma y}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} \right) d\gamma. \]

The less singular term can be estimated analogously to \(I_{11}\):

\[ |G_{111}| \leq \|F(x)\|_{L^\infty} \|\partial \gamma x\|_{C^\frac{1}{2}} \int_0^1 d\gamma \int \frac{d\eta}{|\eta|^\frac{3}{2}} \int d\gamma |\partial \gamma z| |\partial_\gamma^2 x(\gamma - r\eta)| \leq C\|z\|_{H^1}. \]

In \(G_{112}\) we use Cauchy-Schwarz’s inequality as follows:

\[ G_{112} \leq \|F(x)\|_{L^3} \|\partial \gamma x\|_{L^\infty} \int \frac{d\eta}{|\eta|^\frac{3}{2}} \int d\gamma |\partial \gamma z(\gamma)| |\partial_\gamma^2 x(\gamma)| \frac{|\partial_\gamma^2 x(\gamma)|}{|\eta|^{\frac{3}{2} - s}} \frac{1}{|\eta|^{\frac{3}{2} + s}} \]

\[ \lesssim \|F(x)\|_{L^3} \|\partial \gamma x\|_{L^\infty} \|\partial_\gamma^2 x\|_{L^2} \|\partial \gamma z(\gamma)| \|L^{\frac{3}{2}, \eta} \leq C\|\partial \gamma z\|_{L^2}. \]

Noticing

\[ |G_{12}| \leq 2\|F(x)\|_{L^\infty} \int \|\partial \gamma z\| \left| \partial \gamma \left( \int \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} d\gamma \right) \right| d\gamma, \]

we see that \(G_{12}\) can be controlled analogously to \(G_{11}\). Regarding \(G_{13}\) we first integrate by parts using \(\partial_\gamma y \cdot \partial \gamma^2 y = 0\) to obtain

\[ G_{13} = -\frac{\gamma + \pi}{2\pi} \int \frac{\partial_\gamma^2 y}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|x - \gamma, \eta|} - \frac{1}{|y - \gamma, \eta|} \right) d\gamma d\eta = G_{131} + G_{132}, \]

where

\[ G_{131} = -\frac{\gamma + \pi}{2\pi} \int \frac{\partial_\gamma^2 y}{\partial \gamma} \cdot \partial \gamma \left( \frac{1}{|x - \gamma, \eta|} - \frac{1}{|y - \gamma, \eta|} \right) d\gamma d\eta, \]

\[ G_{132} = -\frac{\gamma + \pi}{2\pi} \int \frac{\partial_\gamma^2 y}{\partial \gamma} \cdot \partial \gamma \left( \frac{\partial x - \partial x \gamma, \eta}{|y - \gamma, \eta|} \right) d\gamma d\eta. \]

\(G_{131}\) can be controlled as in \(I_{11}\):

\[ |G_{131}| \leq \|F(y)\|_{L^\infty} \|F(x)\|_{L^\infty} \|\partial \gamma x\|_{C^\frac{1}{2}} \int_0^1 d\gamma \int \frac{d\eta}{|\eta|^\frac{3}{2}} \int d\gamma |\partial_\gamma^2 y| |\partial \gamma z(\gamma - r\eta)| \leq C\|z\|_{H^1}. \]
After symmetrizing,

\[ G_{132} = -\frac{\gamma + \pi}{4\pi} \int \int \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \frac{\partial_\gamma z}{|y|} d\eta d\gamma, \]

so that it can be controlled as in \( G_{112} \).

Next we turn to \( G_2 \). To stay with the integration variables \( \gamma \) and \( \eta \), we change \( G_2(\gamma) \) to \( G_2(\xi) \):

\[ G_2(\xi) = -\int_{-\pi}^{\pi} \frac{\partial_\gamma x}{|\partial_\gamma x|^2} \cdot \partial_\gamma \left( \int_{-\pi}^{\pi} \frac{\partial_\gamma x}{|x|} d\eta \right) d\gamma \]

where

\[ G_{21} = \int_{-\pi}^{\pi} \frac{\partial_\gamma x}{|\partial_\gamma x|^2} \cdot \partial_\gamma x \cdot \frac{\partial_\gamma x}{|x|^3} d\eta d\gamma, \]

\[ G_{22} = -\int_{-\pi}^{\pi} \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \partial_\gamma y \cdot \frac{\partial_\gamma y}{|y|^3} d\eta d\gamma, \]

Recalling the identity (2.7)

\[ x_- \cdot \partial_\gamma x_- = (x_- - \partial_\gamma x \eta) \cdot \partial_\gamma x_- + \frac{\eta}{2} |\partial_\gamma x_-|^2, \]

we split \( G_{21} = \sum_{j=1}^{8} G_{21j} \), where

\[ G_{211} = \int_{-\pi}^{\pi} \frac{\partial_\gamma z}{|\partial_\gamma z|^2} \cdot \partial_\gamma x \cdot \frac{\partial_\gamma x}{|x|^3} d\eta d\gamma, \]

\[ G_{212} = \int_{-\pi}^{\pi} \left( \frac{1}{|\partial_\gamma y|^2} - \frac{1}{|\partial_\gamma y|^2} \right) \partial_\gamma y \cdot \partial_\gamma x \cdot \frac{\partial_\gamma x}{|x|^3} d\eta d\gamma, \]

\[ G_{213} = \int_{-\pi}^{\pi} \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \partial_\gamma x \cdot \frac{\partial_\gamma x}{|x|^3} d\eta d\gamma, \]

\[ G_{214} = \int_{-\pi}^{\pi} \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \partial_\gamma \left( \frac{1}{|x|^3} - \frac{1}{|y|^3} \right) x_- \cdot \partial_\gamma x \cdot d\eta d\gamma, \]

\[ G_{215} = \int_{-\pi}^{\pi} \frac{\partial_\gamma z}{|\partial_\gamma z|^2} \cdot \partial_\gamma y \cdot \frac{\partial_\gamma y}{|y|^3} (z_- - \partial_\gamma z \eta) \cdot \partial_\gamma x \cdot d\eta d\gamma, \]

\[ G_{216} = \int_{-\pi}^{\pi} \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \partial_\gamma y \cdot \frac{\partial_\gamma y}{|y|^3} (y_- - \partial_\gamma y \eta) \cdot \partial_\gamma z \cdot d\eta d\gamma, \]

\[ G_{217} = \int_{-\pi}^{\pi} \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \partial_\gamma \left( \frac{\eta}{2} \right) \partial_\gamma z \cdot \partial_\gamma x \cdot d\eta d\gamma, \]

\[ G_{218} = \int_{-\pi}^{\pi} \frac{\partial_\gamma y}{|\partial_\gamma y|^2} \cdot \partial_\gamma y \cdot \frac{\partial_\gamma y}{|y|^3} \partial_\gamma y \cdot \partial_\gamma z \cdot d\eta d\gamma. \]

Using (5.3) we bound the first two as follows:

\[ |G_{211}| + |G_{212}| \leq C \| \partial_\gamma x \|_{C^\frac{1}{2}} \int_0^1 \frac{d\eta}{|\eta|^\frac{1}{2}} \int d\gamma |\partial_\gamma z| |\partial_\gamma^2 x(\gamma - r\eta)| \leq C \| z \|_{H^1}. \]

Combining (5.3) and (5.9) yields

\[ |G_{2,3}| \leq C \| \partial_\gamma x \|_{C^\frac{1}{2}} \int_0^1 \frac{d\eta}{|\eta|^\frac{1}{2}} \int d\gamma |\partial_\gamma z| + |\partial_\gamma z'| |\partial_\gamma^2 x(\gamma - r\eta)| \leq C \| z \|_{H^1}. \]

Using (5.3) twice allows us to get

\[ |G_{214}| \leq C \| \partial_\gamma y \|_{C^\frac{1}{2}} \int_0^1 \frac{d\eta}{|\eta|^\frac{1}{2}} \int d\gamma |\partial_\gamma z(\gamma - r\eta)| |\partial_\gamma^2 x(\gamma - r\eta)| \leq C \| z \|_{H^1}. \]
Similarly in $G_{215}$, we obtain
\[ |G_{215}| \leq C \|\partial_\gamma y\|_{C^+} \int_0^1 dr \int_0^1 d\tilde{r} \int \frac{dn\tilde{\eta}}{|\gamma - \tilde{\eta}|^2} \int d\gamma (|\partial_\gamma z(\gamma - r\eta)| + |\partial_\gamma z|) \|\partial_\gamma^2 x(\gamma - \tilde{r}\eta)| \leq C \|z\|_{H^1}.
\]
The next term is controlled as
\[ |G_{216}| \leq C \|\partial_\gamma y\|_{C^+} \int_0^1 dr \int \frac{dn\theta}{|\theta|^2} \int d\gamma |\partial_\gamma^2 y(\gamma - r\eta)||(|\partial_\gamma z| + |\partial_\gamma z'|) \leq C \|z\|_{H^1}.
\]
The last two are estimated analogously:
\[
|G_{217}| + |G_{218}| \leq C \|\partial_\gamma x\|_{C^+} + \|\partial_\gamma y\|_{C^+} \int_0^1 dr \int \frac{dn\theta}{|\theta|^2} \int d\gamma |\partial_\gamma^2 y(\gamma - r\eta)||(|\partial_\gamma z| + |\partial_\gamma z'|)
\leq C \|z\|_{H^1}.
\]
We are left with the most singular term $G_{22}$. The identities
\[
\partial_\gamma x \cdot \partial_\gamma^2 x_\gamma = -\partial_\gamma x \cdot \partial_\gamma^2 x' = -\partial_\gamma x_\gamma \cdot \partial_\gamma^2 x'
\]
yields
\[ G_{22} = \int_0^1 \int \frac{\partial_\gamma x_\gamma \cdot \partial_\gamma^2 x'}{|\partial_\gamma x|} d\gamma d\eta,
\]
Hence, we can decompose $G_{22} = \sum_{j=1}^4 G_{22j}$, where
\[
G_{22j} = \int_0^1 \int \frac{\partial_\gamma x_\gamma \cdot \partial_\gamma^2 x'}{|\partial_\gamma^2 x|} d\gamma d\eta,
\]
\[
G_{222} = \int_0^1 \int \frac{\partial_\gamma y_\gamma \cdot \partial_\gamma^2 y'}{|\partial_\gamma^2 y|} d\gamma d\eta,
\]
\[
G_{223} = \int_0^1 \int \frac{\partial_\gamma y_\gamma \cdot \partial_\gamma^2 y'}{|\partial_\gamma^2 y|} (\frac{1}{|\partial_\gamma x|^2} - \frac{1}{|\partial_\gamma y|^2}) d\gamma d\eta,
\]
The more singular $G_{221}$ can be rewritten as
\[
G_{221} = \int_0^1 \int \frac{\partial_\gamma (\int \frac{z_\gamma \cdot \partial_\gamma^2 x'}{|\partial_\gamma x|^2} d\eta)}{d\gamma} - \int z_\gamma \cdot \partial_\gamma (\frac{\partial_\gamma^2 x'}{|\partial_\gamma x|^2} d\eta) d\gamma
\]
\[ = G_{221}^1 + G_{221}^2 + G_{221}^3,
\]
where
\[
G_{221}^1 = \int_0^1 \int \frac{z_\gamma \cdot \partial_\gamma^2 x'}{|\partial_\gamma x|^2} d\eta d\gamma,
\]
\[
G_{221}^2 = \int_0^1 \int \frac{z_\gamma \cdot \partial_\gamma^2 x'}{|\partial_\gamma x|^2} d\eta d\gamma,
\]
\[
G_{221}^3 = \int_0^1 \int \frac{z_\gamma \cdot \partial_\gamma^2 x'}{|\partial_\gamma x|^2} d\eta d\gamma.
\]
Identity \( \mathbb{E} \) together with integration in the variable $\eta$ provides
\[
|G_{221}| \leq 2 \|F(x)\|^3 \|\partial_\gamma x\|_{L^2} \|\partial_\gamma^2 x\|_{L^2} \leq C \|z\|_{H^1}.
\]
Concerning $G_{221}^3$, the fact that $\partial_\gamma^3 x' = \partial_\gamma^2 \partial_\gamma x_\gamma$ and integration by parts in $\eta$ yield
\[
G_{221}^3 = -\int_0^1 \int \frac{z_\gamma \cdot \partial_\gamma^2 x_\gamma}{\partial_\gamma x_\gamma} d\gamma d\eta + \int_0^1 \int \partial_\gamma z_\gamma \cdot \partial_\gamma^2 x_\gamma d\gamma d\eta.
\]
Analogously to $G_{112}$, we get
\[
|G_{221}^2| \leq 2\|F(x)\|_L^2 \int_0^1 dr \int \frac{d\gamma}{|\eta|} \int d\gamma (|\partial_\gamma z(\gamma - r\eta)| + |\partial_\gamma z'| |\partial_\gamma^2 x_\cdot|) \\
\leq C\|\partial_\gamma z\|_{L^2} \|x\|_{H^{2+s}} \leq C\|z\|_{H^1}.
\]
The desired bound for $G_{221}$ is obtained. Regarding $G_{222}$, we similarly write
\[
G_{222} = \int_{-\pi}^\pi \left[ \partial_\gamma \left( \int \frac{\partial_y z}{|\partial_y x|^2 |x_-|} d\eta \right) - \int \partial_\gamma \left( \frac{\partial_y z}{|\partial_y x|^2 |x_-|} \right) \cdot \partial_\gamma z' d\eta \right] d\gamma,
\]
and proceed similarly. The terms $G_{223}$ and $G_{224}$ are easier to control and we omit further details.

5.2. $H^1$ estimates. Our goal is to prove that
\[
\frac{d}{dt} \|\partial_\gamma z\|_{H^1} \leq C(\|\partial_\gamma x\| - |\partial_\gamma y| \|\gamma\|_{H^1} + \|z\|_{H^1}^2).
\] (5.10)

We have
\[
\frac{1}{2} \frac{d}{dt} \|\partial_\gamma z\|_{L^2}^2 = \int \partial_\gamma z \cdot \partial_\gamma z_t d\gamma = I_3 + I_4,
\]
where
\[
I_3 = \int \partial_\gamma z \cdot \int \partial_\gamma \left( \frac{\partial_\gamma^2 x_-}{|x_-|} - \frac{\partial_\gamma y_-}{|y_-|} \right) d\eta d\gamma, \quad I_4 = \int \partial_\gamma z \cdot \partial_\gamma (\lambda(x) \partial_\gamma x - \lambda(y) \partial_\gamma y) d\gamma.
\]

We split further $I_3 = I_{31} + I_{32}$, where
\[
I_{31} = \int \partial_\gamma z \cdot \int \left( \frac{\partial_\gamma^2 x_-}{|x_-|} \right) d\eta d\gamma, \quad I_{32} = \int \partial_\gamma z \cdot \int \left( \frac{\partial_\gamma y_-}{|y_-|} \cdot \frac{\partial_\gamma x_-}{|x_-|} \right) d\eta d\gamma.
\]

Furthermore, $I_3 = I_{311} + I_{312}$, where
\[
I_{311} = \int \partial_\gamma z \cdot \int \left( \frac{\partial_\gamma^2 z_-}{|x_-|} \right) d\eta d\gamma, \quad I_{312} = \int \partial_\gamma z \cdot \int \left( \frac{\partial_\gamma^2 y_-}{|y_-|} \cdot \frac{1}{|x_-|} - \frac{1}{|y_-|} \right) d\eta d\gamma.
\]

We show that all the induced terms are bounded by the right-hand side of (5.10).

Symmetrizing and using (5.9) yield
\[
|I_{311}| = \frac{1}{4} \int \int |\partial_\gamma z_-|^2 \frac{x_- \cdot \partial_\gamma x_-}{|x_-|^3} d\eta d\gamma \leq \|F(x)\|_{L^\infty} \|\partial_\gamma x\|_{C^0} \int \frac{d\eta}{|\eta|^3} 2^s \int d\gamma (|\partial_\gamma z|^2 + |\partial_\gamma z'|^2) \leq C\|\partial_\gamma z\|_{L^2}^2.
\] (5.11)

$I_{312}$ is the most difficult term due to the low regularity. We perform a further splitting,
\[
I_{312} = J_1 + J_2 + J_3,
\] (5.12)

\[
J_1 = \int \partial_\gamma z \cdot \partial_\gamma^2 y \int \left( \frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|} - \frac{1}{|y_-|} - \frac{1}{|\partial_\gamma y||\eta|} \right) d\eta d\gamma,
\]

\[
J_2 = -\int \partial_\gamma z \cdot \int \partial_\gamma^2 y \left( \frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|} - \frac{1}{|y_-|} - \frac{1}{|\partial_\gamma y||\eta|} \right) d\eta d\gamma,
\]

\[
J_3 = \int \partial_\gamma z \cdot \int \partial_\gamma^2 y \left( \frac{1}{|\partial_\gamma x||\eta|} - \frac{1}{|\partial_\gamma y||\eta|} \right) d\eta d\gamma.
\]
Using that the modulus of the tangent vectors only depend on time, we have \(|\partial_\gamma x|=|\partial_\gamma x'|\) and thus

\[
\frac{1}{|x_-|} - \frac{1}{|\partial_\gamma x||\eta|} = (\partial_\gamma x'|\eta| - x_-) - (\partial_\gamma x'|\eta| + x_-) \cdot (\partial_\gamma x'|\eta| + x_-) = \frac{2(\partial_\gamma x'|\eta| - x_-) \cdot (\partial_\gamma x'|\eta| + x_-)}{|x_-| |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|} = \frac{|\partial_\gamma x'|\eta| - x_-|^2}{|x_-| |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|}.
\]

The same decomposition holds for \(y\), implying

\[
J_1 = J_{11} + J_{12}
\]

with

\[
J_{11} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma x'|\eta| - x_-|^2 |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|}{|x_-| |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|} - \frac{|\partial_\gamma y'|\eta| - y_-|^2 |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} \right) d\eta d\gamma,
\]

\[
J_{12} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma y'|\eta| - y_-|^2 |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} - \frac{|\partial_\gamma x'|\eta| - x_-|^2 |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|}{|x_-| |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|} \right) d\eta d\gamma.
\]

Using (53) and that \(\partial_\gamma x \cdot \partial_\gamma^2 x = 0\) gives

\[
2(\partial_\gamma x'|\eta| - x_-) \cdot (\partial_\gamma x'|\eta| + x_-) = 2(\partial_\gamma x' - \partial_\gamma x(\gamma - r\eta)) \cdot (\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)||^2 - |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)||^2),
\]

whence

\[
J_{11} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma x'|\eta| - x_-|^2 |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|}{|x_-| |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|} - \frac{|\partial_\gamma y'|\eta| - y_-|^2 |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} \right).
\]

Now, \(J_{11} = \sum_{j=1}^5 J_{11j}\), where

\[
J_{111} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma z' - \partial_\gamma z(\gamma - r\eta)) \cdot (\partial_\gamma x' - \partial_\gamma x(\gamma - r\eta)) |\eta|}{|x_-| |\partial_\gamma x'|\eta| |(x_- | + |\partial_\gamma x'|\eta)|} \right),
\]

\[
J_{112} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma y' - \partial_\gamma y(\gamma - r\eta)) \cdot (\partial_\gamma z' - \partial_\gamma z(\gamma - r\eta)) |\eta|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} \right),
\]

\[
J_{113} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma y' - \partial_\gamma y(\gamma - r\eta)) |\eta|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} \left( \frac{1}{|x_-|} - \frac{1}{|y_-|} \right) \right),
\]

\[
J_{114} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma y' - \partial_\gamma y(\gamma - r\eta)) |\eta|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} \left( \frac{1}{|\partial_\gamma x'|} - \frac{1}{|\partial_\gamma y'|} \right) \right),
\]

and

\[
J_{115} = \int d\gamma z \cdot \partial_\gamma^2 y \int d\eta \int_0^1 dr \left( \frac{|\partial_\gamma y' - \partial_\gamma y(\gamma - r\eta)) |\eta|}{|y_-| |\partial_\gamma y'|\eta| |(y_- | + |\partial_\gamma y'|\eta)|} \times \left( \frac{1}{|x_-| |\partial_\gamma x'|} - \frac{1}{|y_-| |\partial_\gamma y'|} \right) \right).
\]

In \(J_{111}\) it is possible to use the next mean value identity

\[
\partial_\gamma x' - \partial_\gamma x(\gamma - r\eta)) = \eta(r - 1) \int_0^1 d\rho \partial_\gamma^2 x(\gamma - (\rho + r(1 - \rho))\eta)
\]

to bound as follows

\[
|J_{111}| \leq \|F(x)||L_\infty \int d\gamma |\partial_\gamma z||\partial_\gamma^2 y| \int_0^1 (1 - r) dr \int_0^1 d\rho \times \int d\eta(|\partial_\gamma z'| + |\partial_\gamma z(\gamma - r\eta))||\partial_\gamma^2 x(\gamma - (\rho + r(1 - \rho))\eta)|.
\]
Integrating first in $\eta$ and later in $\gamma$ it is possible to get the desired control:

$$J_{111} \leq C \int_0^1 \int_0^1 \frac{d\rho}{r^2 \rho} \int_0^1 \frac{dp}{(\rho + r(1 - \rho))^\frac{1}{2}} \|\partial_\gamma z\|^2_{L^2} \|\partial_\gamma^2 y\|\
\|\partial_\gamma^2 x\|_{L^2}$$

$$\leq C \int_0^1 \int_0^1 \frac{d\rho}{r^2 \rho} \int_0^1 \frac{dp}{(\rho + r(1 - \rho))^\frac{1}{2}} \|\partial_\eta z\|^2_{L^2} \leq C\|\gamma\|^2_{H^1}.$$ 

The other $J_{11j}$ and $J_{1,2}$ can be treated in the same fashion, giving the desired bound for $J_1$. As for $J_2$, we only consider the following counterpart of $J_{111}$ as the control for other terms follows along the same lines.

$$J_{211} = - \int d\gamma \partial_\gamma z \cdot \int d\eta \partial_\gamma^2 y \int_0^1 \int d\gamma |\partial_\gamma^2 y(\gamma, \eta)| \|\partial_\gamma^2 y(\gamma, \eta)\|.$$ 

The mean value theorem gives

$$\partial_\gamma y - \partial_\gamma y(\gamma, \eta) = r\eta \int_0^1 dp \partial_\gamma^2 y(\gamma + (\rho - 1)r\eta),$$

so that

$$|J_{211}| \leq C \int_0^1 \int_0^1 dp \int d\gamma |\partial_\gamma z| \int d\eta |\partial_\gamma^2 y(\gamma, \eta)| \|\partial_\gamma^2 y(\gamma, \eta)\| = K_1 + K_2,$$

where

$$K_1 = C \int_0^1 \int_0^1 dp \int d\gamma |\partial_\gamma z|^2 \int d\eta |\partial_\gamma^2 y(\gamma, \eta)|,$$

$$K_2 = C \int_0^1 \int_0^1 dp \int d\gamma d\eta |\partial_\gamma z| |\partial_\gamma^2 y(\gamma, \eta)| |\partial_\gamma^2 y(\gamma, \eta)|.$$ 

By Cauchy-Schwarz’s inequality in $\eta$,

$$K_1 \leq C \int_0^1 \int_0^1 \frac{d\rho}{(1 - \rho)^\frac{1}{2}} \|\gamma\|^2_{H^1} \leq C\|\gamma\|^2_{H^1}.$$ 

For $K_2$ we use Cauchy-Schwarz’s inequality in both $\gamma$ and $\eta$ to have

$$K_2 \leq C \int_0^1 \int_0^1 dp \left( \int d\gamma d\eta |\partial_\gamma z|^2 |\partial_\gamma^2 y(\gamma, \eta)|^2 \right)^\frac{1}{2} \left( \int d\gamma d\eta |\partial_\gamma^2 y|^2 |\partial_\gamma z(\gamma, \eta)|^2 \right)^\frac{1}{2}$$

$$\leq C \int_0^1 \int_0^1 \frac{d\rho}{(1 - \rho)^\frac{1}{2}} \int_0^1 \frac{dp}{(1 - \rho)^\frac{1}{2}} \|\partial_\gamma z\|^2_{L^2} \|\partial_\gamma^2 y\|^2_{L^2} \leq C\|\gamma\|^2_{H^1},$$

where we have made the change of variables $\zeta = \gamma - \eta$, $d\gamma = d\zeta$ in the second double integral. Consequently, $|J_{211}| \leq C\|\gamma\|^2_{H^1}$ and thus

$$|J_1| \leq |J_2| \leq C\|\gamma\|^2_{H^1}.$$ 

To bound $J_3$ we recall that $|\partial_\gamma x|$ and $|\partial_\gamma y|$ depend only on $t$, so that as in the control of $G_{112}$,

$$|J_3| \leq \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \|\partial_\gamma z\| - |\partial_\gamma y| \int \frac{d\eta}{|\eta|} \int d\gamma |\partial_\gamma z| |\partial_\gamma^2 y|$$

$$\leq \|F(x)\|_{L^\infty} \|F(y)\|_{L^\infty} \|\partial_\gamma z\| - |\partial_\gamma y| \|\partial_\gamma^2 y\|_{H^s}$$

$$\leq C\|\partial_\gamma x\| - |\partial_\gamma y| \|\partial_\gamma z\|_{L^2}.$$ 

We have proved that $I_{312} = J_1 + J_2 + J_3$ is bounded by

$$|I_{312}| \leq C(\|\partial_\gamma x\| - |\partial_\gamma y| \|\gamma\|_{H^1} + \|\gamma\|^2_{H^1})$$

which in conjunction with (5.11) yields

$$|I_{31}| \leq C(\|\partial_\gamma x\| - |\partial_\gamma y| \|\gamma\|_{H^1} + \|\gamma\|^2_{H^1}).$$ (5.13)
Note that the control quantity \(|\partial_x x| - |\partial_y y||z||_{H^1}\) is only due to \(J_3\).

Regarding \(I_{32}\), we use identity (5.9) to split \(I_{32} = \sum_{j=1}^{8} I_{32j}\), where

\[
I_{321} = -\int \partial_t z \cdot \int \partial \gamma \left( x_+ - \partial_t x\gamma \right) \cdot \partial_x x_+ d\eta d\gamma,
\]

\[
I_{322} = -\int \partial_t z \cdot \int \partial \gamma y_+ \left( x_+ - \partial_t y\gamma \right) \cdot \partial_x x_+ d\eta d\gamma,
\]

\[
I_{323} = -\int \partial_t z \cdot \int \partial \gamma y_- \left( x_- + \partial_t y\gamma \right) \cdot \partial_x x_- d\eta d\gamma,
\]

\[
I_{324} = -\int \partial_t z \cdot \int \partial \gamma y_- \left( x_- - \partial_t y\gamma \right) \cdot \partial_x y_- (|x_-|^{-3} - |y_-|^{-3}) d\eta d\gamma,
\]

\[
I_{325} = -\int \partial_t z \cdot \int \partial \gamma \left( \frac{\eta|\partial_x x_-|^2}{2|x_-|^3} \right) d\eta d\gamma,
\]

\[
I_{326} = -\int \partial_t z \cdot \int \partial \gamma y_- \left( \frac{\eta|\partial_x x_-|^2}{2|x_-|^3} \right) d\eta d\gamma,
\]

\[
I_{327} = -\int \partial_t z \cdot \int \partial \gamma y_- \left( \frac{\eta|\partial_x x_-|^2}{2|x_-|^3} \right) d\eta d\gamma,
\]

and

\[
I_{328} = -\frac{1}{2} \int \partial_t z \cdot \int \partial \gamma \left( \frac{\eta|\partial_x y_-|^2}{2|x_-|^3} \right) d\eta d\gamma.
\]

Analogously to the control of \(G_{112}\) we find \(I_{32} \leq C||z||_{H^1}^2\). Combining this with (5.13) we obtain

\[
|J_3| \leq C\left( ||\partial_t x| - |\partial_t y||||z||_{H^1} + ||z||_{H^1}^2 \right). 
\]

(5.14)

Finally, we write \(I_4 = \sum_{j=1}^{4} I_{4j}\), where

\[
I_{41} = \int \partial_t z \cdot \lambda(x) \partial_t^2 z d\gamma, \quad I_{42} = \int \partial_t z \cdot (\lambda(x) - \lambda(y)) \partial_t^2 y d\gamma,
\]

\[
I_{43} = \int |\partial_t z|^2 \partial_t \lambda(x) d\gamma, \quad I_{44} = \int \partial_t z \cdot \partial_t y (\partial_t \lambda(x) - \partial_t \lambda(y)) d\gamma.
\]

Integration by parts in \(I_{41}\) yields \(I_{41} = -\frac{1}{2} I_{43}\), while

\[
|I_{43}| \leq ||\partial_t z||_{L^2}||\partial_t \lambda(x)||_{L^\infty} \leq C\|z\|_{H^1}^2,
\]

where we have used. By virtue of (5.2), we have

\[
|I_{42}| \leq C||\partial_t^2 y||_{L^2}||\partial_t z||_{L^2}||\lambda(x) - \lambda(y)||_{L^\infty} \leq C\|z\|_{H^1}^2.
\]

Next, we integrate by parts

\[
I_{44} = -\int \partial_t z \cdot \partial_t^2 y (\lambda(x) - \lambda(y)) d\gamma - \int \partial_t^2 z \cdot \partial_t y (\lambda(x) - \lambda(y)) d\gamma
\]

and use the identity \(\partial_t^2 z \cdot \partial_t y = -\partial_t^2 x \cdot \partial_t z\) to have

\[
I_{44} \leq (||\partial_t^2 y||_{L^2} + ||\partial_t^2 x||_{L^2})||\partial_t z||_{L^2}||\lambda(x) - \lambda(y)||_{L^\infty} \leq C\|z\|_{H^1}^2.
\]

It follows that

\[
I_4 \leq C\|z||_{H^1}^2.
\]

(5.15)

A combination of (5.14) and (5.15) leads to (5.10).
5.3. Estimates for $|\partial_x r| - |\partial_y r|$. Identity

$$\frac{d}{dt} |\partial_x r|^2 = \frac{1}{2} \int \partial_x r \cdot \partial_y \left( \int \frac{\partial x}{|x|} \, d\gamma \right) \, d\gamma,$$

and integration by parts provide

$$\frac{d}{dt} |\partial_x r| = -\frac{1}{2\pi} \int \frac{\partial^2 x}{|\partial_x r|} \cdot \int \frac{\partial x}{|x|} \, d\eta d\gamma.$$

This new identity allows to find the following splitting

$$\frac{d}{dt} (|\partial_x r| - |\partial_y r|) = I_5 + I_6 + I_7 + I_8,$$

where

$$I_5 = \frac{-1}{2\pi} \int \left( \frac{1}{|\partial_x r|} - \frac{1}{|\partial_y r|} \right) \partial^2 x \cdot \int \frac{\partial x}{|x|} \, d\eta d\gamma, \quad I_6 = \frac{-1}{2\pi} \int \frac{\partial^2 x}{|\partial_y r|} \cdot \int \frac{\partial x}{|x|} \, d\eta d\gamma,$$

$$I_7 = \frac{-1}{2\pi} \int \frac{\partial^2 y}{|\partial_y r|} \cdot \int \frac{\partial z}{|x|} \, d\eta d\gamma, \quad \text{and} \quad I_8 = \frac{-1}{2\pi} \int \frac{\partial^2 y}{|\partial_y r|} \cdot \int \frac{\partial x}{|x|} (\frac{1}{|x|} - \frac{1}{|y|}) \, d\eta d\gamma.$$

Proceeding as before, we can obtain

$$I_5 \leq \|F(x)\|_{L^\infty} \|\partial_x r\|_{C^{1,2}} \int \frac{d\eta}{|\eta|^{\frac{1}{2}}} \int d\gamma |\partial^2 x| |\partial_x z| \leq C \|z\|_{H^1}.$$

It is possible to integrate by parts in $I_6$ in such a way that

$$I_6 = \frac{1}{2\pi} \int \frac{\partial z}{|\partial_y r|} \cdot \int \frac{\partial^2 x}{|x|} \, d\eta d\gamma - \frac{1}{2\pi} \int \frac{\partial z}{|\partial_y r|} \cdot \int \frac{\partial x}{|x|} \, d\eta d\gamma.$$

Therefore, following the control of $G_{1,1,2}$ (see (5.8)) we bound

$$I_6 \lesssim \|F(y)\|_{L^\infty} \left\{ \|F(x)\|_{L^\infty} \|\partial_z z\|_{L^2} \|\partial^2 x\|_{H^s} + \|F(x)\|_{L^\infty} \|\partial_x r\|_{C^{1,2}} \|\partial_z z\|_{L^2} \|\partial^2 x\|_{L^2} \right\} \leq C \|z\|_{H^1}.$$

Next, we symmetrize $I_7$ as

$$I_7 = \frac{-1}{4\pi} \int \int \frac{\partial^2 y}{|\partial_y r|} \cdot \frac{\partial z}{|x|} \, d\eta d\gamma = \frac{-1}{2\pi} \int \frac{\partial z}{|\partial_y r|} \cdot \int \frac{\partial^2 y}{|x|} \, d\eta d\gamma,$$

so that

$$I_7 \leq \|F(y)\|_{L^\infty} \|F(x)\|_{L^\infty} \|\partial_z z\|_{L^2} \|\partial^2 y\|_{H^s} \leq C \|z\|_{H^1}.$$

Finally,

$$I_8 \lesssim \|F(y)\|_{L^\infty} \|F(x)\|_{L^\infty} \|\partial_y r\|_{C^{1,2}} \int_0^1 \int \|\partial^2 y(\gamma)\| \|\partial_y z(\gamma - r\eta)\| d\gamma \frac{d\eta}{|\eta|^{\frac{1}{2}}} \leq \|F(y)\|_{L^\infty} \|F(x)\|_{L^\infty} \|\partial_y r\|_{C^{1,2}} \|\partial^2 y\|_{L^2} \|\partial_y z\|_{L^2} \leq C \|z\|_{H^1}.$$

We have proved that

$$\left| \frac{d}{dt} (|\partial_x r| - |\partial_y r|) \right| \leq C \|z\|_{H^1}$$

which together with (5.10) implies the closed differential inequality

$$\frac{d}{dt} (\|z\|_{H^1}^2 + |\partial_x r| - |\partial_y r|)^2 \leq C (\|z\|_{H^1}^2 + \|\partial_x r\|^2) (\|z\|^2_{H^1} + \|\partial_x r\|^2). \quad (5.16)$$

The use of Gronwall’s inequality yields the stability estimate (5.11), finishing the proof of Theorem 5.1.
6. Proof of the main result–Theorem 1.2

In this section we construct solutions of system (1.7)–(1.8) with initial data \( x^0(\gamma) \in H^{2+s}, \ s \in (0, \frac{1}{2}) \). The uniqueness has been proved in Theorem [5.1]. The existence is done by regularizing the initial data and applying the existence result for regular \((H^3)\) data established in [16]. Since our a priori estimates crucially use the fact that the tangent vector’s length is independent of the parameter of the curve, the regularization procedure must maintain this property. Assume that the initial closed curve \( x^0(\gamma) \in H^{2+s} \) satisfies the arc chord condition, i.e., \( \|F(x^0)\|_{L^\infty} < \infty \). In addition, upon reparametrizing (see below) we may assume \( \partial_\gamma|\partial_\gamma x^0(\gamma)| = 0 \). First, we mollify \( x^0 \) to get

\[
x^0_\varepsilon(\gamma) = (\Gamma_\varepsilon * x^0)(\gamma),
\]

with \( \Gamma_\varepsilon \) an approximation of the identity. It is clear that \( x^0_\varepsilon \in C^\infty(\mathbb{T}) \) and

\[
\lim_{\varepsilon \to 0} \|x^0_\varepsilon - x^0\|_{H^{2+s}} = 0.
\]

(6.1)

This ensures that for small \( \varepsilon, x^0_\varepsilon \) satisfies the arc chord condition. To obtain the constant length (in parameter) of the tangent vector, we reparametrize using

\[
\phi_\varepsilon : [-\pi, \pi] \to [-\pi, \pi], \quad \phi_\varepsilon(\xi) = -\pi + \frac{2\pi}{L_\varepsilon} \int_{-\pi}^{\xi} |\partial_\gamma x^0_\varepsilon(\gamma)| d\gamma, \quad L_\varepsilon = \int_{-\pi}^{\pi} |\partial_\gamma x^0_\varepsilon(\gamma)| d\gamma.
\]

It is clear that \( \phi_\varepsilon \in C^\infty(\mathbb{T}) \) and its inverse \( \phi_\varepsilon^{-1} \) is well defined. Then the curve

\[
\tilde{x}^0_\varepsilon(\gamma) = x^0_\varepsilon(\phi^{-1}_\varepsilon(\gamma))
\]

satisfies

\[
\tilde{x}^0_\varepsilon \in C^\infty(\mathbb{T}), \quad \partial_\gamma|\partial_\gamma \tilde{x}^0_\varepsilon(\gamma)| = 0.
\]

(In fact, \(|\partial_\gamma \tilde{x}^0_\varepsilon(\gamma)| = \frac{L_\varepsilon}{2\pi}\)). We postpone the proof of the following lemma to the end of this section.

**Lemma 6.1.**

\[
\lim_{\varepsilon \to 0} \|\tilde{x}^0_\varepsilon - x^0\|_{H^{2+s}} = 0.
\]

(6.2)

Now \( \tilde{x}^0_\varepsilon \) is a smooth initial curve satisfying the arc chord condition and \( \partial_\gamma|\partial_\gamma \tilde{x}^0_\varepsilon(\gamma)| = 0 \). Set \( \varepsilon = n^{-1} \) and relabel \( \tilde{x}^0_\varepsilon = \tilde{x}^0_n \). Applying the existence result in [16] we obtain for each \( n \) a solution \( x_n \in C([0, T_n]; H^3) \) satisfying the arc chord condition and \( x_n|t = 0 = \tilde{x}^0_n \). The convergence (6.2) ensures that the sequences \( \|\tilde{x}^0_n\|_{H^{2+s}} \) and \( \|F(\tilde{x}^0_n)\|_{L^\infty([0, T_n])} \) are bounded. Using this and a continuity argument, we deduce from the a priori estimates (3.26) and (4.6) that there exists \( 0 < T < T_n \) for all \( n \) such that the sequences

\[
\|x_n\|_{C([0, T]; H^{2+s})} \quad \text{and} \quad \|F(x_n)\|_{L^\infty([0, T]; L^\infty([0, T] \times \mathbb{T}))}
\]

are bounded. It follows easily from equations (1.7)–(1.8) with the aid of (3.6) that \( \partial_t x_n \) is uniformly bounded in \( L^\infty([0, T]; L^2) \). By virtue of the Aubin-Lions lemma, there exists \( x \in C_w([0, T]; H^{2+s}) \cap C([0, T]; H^2) \) such that (upon extracting a subsequence)

\[
x_n \to x \quad \text{in} \quad L^\infty([0, T]; H^{2+s}) \quad \text{and} \quad x_n \to x \quad \text{in} \quad C([0, T]; H^2).
\]

(6.3)

Using these convergences it can be shown that \( x \) is a solution of (1.7)–(1.8). We note that \( \lambda(x_n) \to \lambda(x) \) in \( L^\infty \) in view of (5.5). Next, we show the continuity in time \( x \in C([0, T]; H^{2+s}) \). Indeed, from (3.26) \( x_n \) satisfies

\[
\|x_n(t)\|_{H^{2+s}}^2 \leq \|\tilde{x}^0_n\|_{H^{2+s}}^2 \exp \left( \int_0^t P(\|F(x_n)(t')\|_{L^\infty}, \|\partial_\gamma^2 x_n(t')\|_{H^s}) dt' \right) \leq \|\tilde{x}^0_n\|_{H^{2+s}}^2 \exp(tC(T))
\]

for all \( t \in [0, T] \). Letting \( n \to \infty \) yields

\[
\|x\|_{L^\infty([0, T]; H^{2+s})}^2 \leq \limsup_{n \to \infty} \|x_n\|_{L^2([0, t]; H^{2+s})}^2 \leq \|x^0\|_{H^{2+s}}^2 \exp(tC(T)), \quad t \in [0, T].
\]
It follows that
\[
\lim_{t \to 0^+} \|x(t)\|_{H^{2+s}} \leq \|x^0\|_{H^{2+s}}
\]
which combined with the weak continuity \(x \in C_w([0, T]; H^{2+s})\) implies that \(x\) is continuous from the right at \(t = 0\) with values in \(H^{2+s}\). Next, for any \(t_0 \in (0, T)\), we consider \(x(t_0)\) as the new initial data. The above argument gives a solution \(\bar{x} \in L^\infty([t_0, t_0 + \delta]; H^{2+s})\) with \(\delta = \delta(t_0) \in (0, T - t_0)\) such that \(\bar{x}\) is continuous from the right at \(t_0\) with values in \(H^{s+2}\). The same property holds for \(x\) since the uniqueness result in Theorem 5.1 implies \(x = \bar{x}\) on \([t_0, t_0 + \delta]\). The left continuity can be obtained similarly using the fact that (1.7), (1.8) are time reversible.

**Proof of Lemma 6.7**

We shall prove that
\[
\lim_{\varepsilon \to 0} \|\phi_\varepsilon(\cdot) - \cdot\|_{H^{2+s}} = 0 \tag{6.4}
\]
which in turn yields same for the inverse
\[
\lim_{\varepsilon \to 0} \|\phi_\varepsilon^{-1}(\cdot) - \cdot\|_{H^{2+s}} = 0
\]
upon using the formula for derivatives of inverse functions. These imply the desired convergence
\[
\|x_\varepsilon - x^0\|_{H^{2+s}} \leq \|x_\varepsilon \circ \phi_\varepsilon^{-1} - x_\varepsilon^0\|_{H^{2+s}} + \|x_\varepsilon^0 - x^0\|_{H^{2+s}} \to 0 \quad \text{as} \quad \varepsilon \to 0.
\]
It can be easily checked that \(\|\phi_\varepsilon(\cdot) - \cdot\|_{L^2} \to 0\). For the highest order derivative, we first compute
\[
\partial_\gamma^2 \phi_\varepsilon(\gamma) = \frac{2\pi}{L_\varepsilon} \frac{\partial_\gamma x_\varepsilon^0(\gamma) \cdot \partial_\gamma^2 x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|}.
\]
Then using the formula
\[
\Lambda^s \partial_\gamma^2 \phi_\varepsilon(\gamma) = c \int_{\mathbb{R}} \frac{\partial_\gamma^2 \phi_\varepsilon(\gamma) - \partial_\gamma^2 \phi_\varepsilon(\gamma - \eta)}{\|\eta\|^{1+s}} d\eta
\]
and the identity
\[
\Lambda^s \left( \frac{2\pi}{L} \frac{\partial_\gamma x_\varepsilon^0(\gamma) \cdot \partial_\gamma^2 x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} \right) = \Lambda^s(0) = 0, \quad L = \int_{-\pi}^{\pi} |\partial_\gamma x_\varepsilon^0(\gamma)| d\gamma = 2\pi |\partial_\gamma x_\varepsilon^0(\gamma)|,
\]
we can decompose
\[
\Lambda^s \partial_\gamma^2 \phi_\varepsilon = \sum_{j=1}^{6} G_j,
\]
where
\[
G_1(\gamma) = 2\pi \left( \frac{1}{L_\varepsilon} - \frac{1}{L} \right) \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} \cdot \Lambda^s \partial_\gamma x_\varepsilon^0(\gamma), \quad G_2(\gamma) = \frac{2\pi}{L} \left( \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} - \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} \right) \cdot \Lambda^s \partial_\gamma x_\varepsilon^0(\gamma),
\]
\[
G_3(\gamma) = \frac{2\pi}{L} \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} \cdot (\Lambda^s \partial_\gamma x_\varepsilon^0 - \Lambda^s \partial_\gamma x_\varepsilon^0),
\]
\[
G_4(\gamma) = \frac{2\pi c}{L} \left( \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} - \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} \right) \frac{\partial_\gamma^2 x_\varepsilon^0(\gamma - \eta)}{\|\eta\|^{1+s}} d\eta,
\]
\[
G_5(\gamma) = \frac{2\pi c}{L} \left( \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} - \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} \right) \frac{\partial_\gamma^2 x_\varepsilon^0(\gamma - \eta)}{\|\eta\|^{1+s}} d\eta,
\]
and
\[
G_6(\gamma) = \frac{2\pi c}{L} \left( \frac{\partial_\gamma x_\varepsilon^0(\gamma)}{\|\partial_\gamma x_\varepsilon^0(\gamma)\|} - \frac{\partial_\gamma x_\varepsilon^0(\gamma - \eta)}{\|\partial_\gamma x_\varepsilon^0(\gamma - \eta)\|} \right) \frac{\partial_\gamma^2 x_\varepsilon^0(\gamma - \eta)}{\|\eta\|^{1+s}} d\eta.
\]
Using the convergence \(5.1\), we can show that \(\|G_j\|_{L^2} \to 0\) as \(\varepsilon \to 0\) for all \(1 \leq j \leq 6\). Indeed, denoting \(B = |\partial_\gamma x^0(\gamma)| > 0\) independent of \(\gamma\), \(5.1\) implies
\[
\exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0), \forall \gamma \in T, \, B_\varepsilon(\gamma) := |\partial_\gamma x^0_\varepsilon(\gamma)| \geq \frac{B}{2}.
\]
Note in addition that \(B^{-1} \leq \|F(x^0)\|_{L^\infty}\) and \(\|B_\varepsilon - B\|_{L^\infty} \to 0\). In particular, \(L_\varepsilon \to L\) and thus \(\|G_1\|_{L^2} \to 0\). As for \(G_2\) we estimate
\[
\|G_2\|_{L^2} \leq \frac{2\pi}{L} \left\| \frac{\partial_\gamma x^0_\varepsilon(\gamma)}{|\partial_\gamma x^0_\varepsilon(\gamma)|} - \frac{\partial_\gamma x^0(\gamma)}{|\partial_\gamma x^0(\gamma)|} \right\|_{L^\infty} \|\Lambda^s \partial_\gamma x^0_\varepsilon\|_{L^2},
\]
where the \(L^\infty\) norm tends to 0 because \(5.1\) implies that \(\partial_\gamma x^0_\varepsilon \to \partial_\gamma x^0\) in \(H^{1+s} \subset C^{1/2+s}\). The term \(G_3\) is obvious. Since \(L_\varepsilon \to L\), in \(G_4\) it suffices to bound
\[
K := \left\| \int \frac{\partial_\gamma x^0_\varepsilon(\gamma)}{|\partial_\gamma x^0_\varepsilon(\gamma)|} \cdot \frac{\partial_\gamma x^0_\varepsilon(\gamma)}{|\partial_\gamma x^0_\varepsilon(\gamma)|} \right\|_{L^2(T)},
\]
\[
\leq \left\| \frac{\partial_\gamma x^0_\varepsilon}{|\partial_\gamma x^0_\varepsilon|} \right\|_{C^{1/2}} \left( \int_0^\pi \left( \int_{1-1}^0 \frac{2|x^0_\varepsilon(\gamma - \eta)|}{|\eta|^{1+s}} \, d\eta \right)^2 d\gamma \right)^{1/2} + 2 \left( \int_0^\pi \left( \int_{|\eta| > 1} \frac{2|x^0_\varepsilon(\gamma - \eta)|}{|\eta|^{1+s}} \, d\eta \right)^2 d\gamma \right)^{1/2},
\]
where
\[
\left\| \frac{\partial_\gamma x^0_\varepsilon}{|\partial_\gamma x^0_\varepsilon|} \right\|_{C^{1/2}} \leq M = M(\|\partial_\gamma x^0_\varepsilon\|_{C^{1/2}}, B).
\]
Minkowski’s inequality allows to bound
\[
K \leq \left( M \int_0^1 \frac{d\eta}{|\eta|^{1+s}} + 2 \int_{|\eta| > 1} \frac{d\eta}{|\eta|^{1+s}} \right) \left( \int_0^\pi \left( \int_{|\eta| > 1} \frac{2|x^0_\varepsilon(\gamma - \eta)|}{|\eta|^{1+s}} \, d\eta \right)^2 d\gamma \right)^{1/2} \leq C(M + 1) \|\partial_\gamma x^0_\varepsilon\|_{L^2},
\]
which is uniformly bounded in \(\varepsilon\). Here and in the remainder of this proof, \(C\) denotes absolute constants. By an analogous argument using that \(B\) is independent of \(\gamma\), we obtain
\[
\|G_6\|_{L^2} \leq \frac{C}{LB} \|\partial_\gamma x^0\|_{C^{1/2}} \|\partial_\gamma^2 x^0_\varepsilon - \partial_\gamma^2 x^0\|_{L^2} \to 0.
\]
As for \(G_5\), using the notation \(x(\gamma, \eta) = x(\gamma) - x(\gamma - \eta)\) we rewrite
\[
G_5(\gamma) = \frac{2\pi}{L} \int \frac{1}{B(\gamma)} \left[ \frac{\partial_\gamma x^0_\varepsilon(\gamma)}{B_\varepsilon(\gamma)} + \partial_\gamma x^0_\varepsilon(\gamma, \eta) \left( \frac{1}{B_\varepsilon(\gamma)} - \frac{1}{B(\gamma - \eta)} \right) \right] \cdot \partial_\gamma x^0_\varepsilon(\gamma - \eta) \left( \frac{1}{B_\varepsilon(\gamma - \eta)} - \frac{1}{B(\gamma - \eta)} \right) \, d\eta \equiv G_{5,1} + G_{5,2} + G_{5,3},
\]
where the splitting is according to the terms in the square brackets. Arguing as in \(G_6\) gives
\[
\|G_{5,1}\|_{L^2} \leq \frac{C}{LB} \|\partial_\gamma x^0_\varepsilon - \partial_\gamma x^0\|_{C^{1/2}} \|\partial_\gamma^2 x^0_\varepsilon\|_{L^2},
\]
\[
\|G_{5,2}\|_{L^2} \leq \frac{C}{L} \|B_\varepsilon^{-1}(\cdot) - B^{-1}\|_{L^\infty} \|\partial_\gamma x^0\|_{C^{1/2}} \|\partial_\gamma^2 x^0_\varepsilon\|_{L^2},
\]
where both right-hand sides tend to 0. For \(G_{5,3}\) we note that
\[
|B_\varepsilon(\gamma) - B_\varepsilon(\gamma - \eta)| \leq C \|\partial_\gamma x^0_\varepsilon\|_{C^{1/2+s}} |\eta|^{1+s},
\]
\[
|B_\varepsilon(\gamma) - B_\varepsilon(\gamma - \eta)| = |B_\varepsilon(\gamma) - B(\gamma) + B(\gamma - \eta) - B_\varepsilon(\gamma - \eta)| \leq 2 \|B_\varepsilon(\cdot) - B\|_{L^\infty},
\]
whence
\[
|B_\varepsilon(\gamma) - B_\varepsilon(\gamma - \eta)| \leq C |\eta|^{1/2} \|\partial_\gamma x^0_\varepsilon\|_{C^{1/2+s}} \|B_\varepsilon(\cdot) - B\|_{L^\infty},
\]
Then using the argument in \(G_6\) yields
\[
\|G_{5,3}\|_{L^2} \leq \frac{C}{LB^2} \|\partial_\gamma x^0\|_{C^{1/2+s}} \|\partial_\gamma^2 x^0_\varepsilon\|_{L^2} \|B_\varepsilon(\cdot) - B\|_{L^\infty} \|\partial_\gamma^2 x^0_\varepsilon\|_{L^2}.
\]
The homogeneous Besov norm is defined by removing $\Delta B_p,r$ for $\varphi W e have
For $k \geq 0$ we define
$$\chi_j(\xi) = \chi\left(\frac{\xi}{2^k}\right), \quad \varphi_j(\xi) = \chi_j(\xi) - \chi_{j-1}(\xi) = \varphi\left(\frac{\xi}{2^k}\right),$$
so that $\varphi_j$ is supported in the annulus $\{2^{k-1} < |\xi| < 2^{k+1}\}$. Clearly
$$\chi(\xi) + \sum_{k=0}^{\infty} \varphi_k(\xi) = 1 \quad \forall \xi \in \mathbb{R}^d.$$
For $f : \mathbb{T}^d \to \mathbb{R}$ and $j \geq 0$ we define the Fourier multipliers
$$\varphi_{-1}(D) f = \chi(D) f, \quad \Delta_j f = \mathcal{F}^{-1}(\varphi_j \mathcal{F}(f)) \quad \forall j \geq 0,$$
$$S_j f = \sum_{-1 \leq k \leq j-1} \Delta_k f = \chi(2^{-j+1}D) f \quad \forall j \geq 0.$$
We have $\varphi_{-1}(D) f = \hat{f}(0)$ and (formally)
$$f = \sum_{j \geq -1} \Delta_j f = \hat{f}(0) + \sum_{j \geq 0} \Delta_j f.$$
For $p, r \in [1, \infty]$ and $s \in \mathbb{R}$ we define the inhomogeneous Besov norm by
$$\|f\|_{B^s_p, r(\mathbb{T}^d)} = \|2^sj \Delta_j f\|_{L^p(\mathbb{T}^d)} \|r\|_{\ell^\infty((-1)^j, \infty)}, \quad \mathbb{N} = \{0, 1, \ldots\}. \quad (A.1)$$
It is well-known that the Besov space $B^s_{2,2}(\mathbb{T}^d)$ coincides with the Sobolev space $H^s(\mathbb{T}^d)$ and
$$\|f\|_{B^s_{2,2}(\mathbb{T}^d)} \approx \|f\|_{H^s(\mathbb{T}^d)}.$$
The homogeneous Besov norm is defined by removing $\Delta_{-1} f = \hat{f}(0)$, i.e.
$$\|f\|_{\dot{B}^s_p, r(\mathbb{T}^d)} = \|2^sj \Delta_j f\|_{L^p(\mathbb{T}^d)} \|r\|_{\ell^\infty(\mathbb{N})}. \quad (A.2)$$
**PROPOSITION A.1.** For $\alpha \in (0, 1)$, the $\alpha$-Hölder (semi)norm is defined by
$$\|f\|_{C^\alpha(\mathbb{T}^d)} = \sup_{x, y \in \mathbb{T}^d, x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}. \quad (A.3)$$
The norms $\|\cdot\|_{\dot{C}^\alpha(\mathbb{T}^d)}$ and $\|\cdot\|_{\dot{B}^\infty_{\infty, \infty}(\mathbb{T}^d)}$ are equivalent. Consequently, the norms $\|\cdot\|_{C^\alpha(\mathbb{T}^d)}$ and $\|\cdot\|_{\dot{B}^\infty_{\infty, \infty}(\mathbb{T}^d)}$ are equivalent, where
$$\|\cdot\|_{C^\alpha(\mathbb{T}^d)} = \|\cdot\|_{\dot{C}^\alpha(\mathbb{T}^d)} + \|\cdot\|_{L^\infty(\mathbb{T}^d)}. \quad (A.4)$$
**PROOF.** Let $x \neq y \in \mathbb{T}^d$ and choose $j_0 \geq -1$ such that $|x - y| \simeq 2^{-j_0}$. Clearly $\Delta_{-1} f(x) - \Delta_{-1} f(y) = 0$. For $0 \leq j \leq j_0$ we use Bernstein’s inequality to obtain
$$|\Delta_j f(x) - \Delta_j f(y)| \leq |x - y| |\nabla \Delta_j f|_{L^\infty} \lesssim 2^j |\Delta_j f|_{L^\infty} |x - y| \lesssim 2^{j(1-\alpha)} \|f\|_{\dot{B}^\infty_{\infty, \infty}(\mathbb{T}^d)} |x - y|.$$
On the other hand, when $j \geq j_0$ we simply estimate
$$|\Delta_j f(x) - \Delta_j f(y)| \leq 2 \|\Delta_j f\|_{L^\infty} \lesssim 2^{-j_0} \|f\|_{\dot{B}^\infty_{\infty, \infty}(\mathbb{T}^d)}.$$
Summing over \( j \geq -1 \) gives
\[
|f(x) - f(y)| \lesssim \sum_{j=0}^{j_0} 2^{j(1-\alpha)} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)} |x - y| + \sum_{j=j_0+1}^\infty 2^{-j\alpha} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)}
\]
\[
\lesssim 2^{j_0(1-\alpha)} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)} |x - y| + 2^{-j_0\alpha} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)}
\]
\[
\lesssim |x - y|^{-1+\alpha} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)} |x - y| + |x - y|^{\alpha} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)}
\]
\[
\lesssim |x - y|^{\alpha} \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)}.
\]
Thus \( \|f\|_{\tilde{C}^\alpha} \lesssim \|f\|_{B^\alpha_{\infty, \infty}(\mathbb{T}^d)} \).

Conversely, for \( j \geq 0 \) we have \( \int_{\mathbb{T}^d} \mathcal{F}^{-1} \varphi_j(x) dx = 0 = \varphi_j(0) = 0 \) and thus
\[
|\Delta_j f(x)| = \left| \int_{\mathbb{T}^d} \mathcal{F}^{-1}(\varphi_j)(x-y)(f(y) - f(x)) dy \right|
\]
\[
= \left| \int_{\mathbb{T}^d} \sum_{k \in \mathbb{Z}^d} \mathcal{F}^{-1}_{\mathbb{R}^d}(\varphi_j)(x-y+2\pi k)(f(y) - f(x)) dy \right|
\]
\[
= \left| \int_{\mathbb{R}^d} \mathcal{F}^{-1}_{\mathbb{R}^d}(\varphi_j)(x-y)(f(y) - f(x)) dy \right|
\]
where we have used the Poisson summation
\[
\mathcal{F}^{-1}(\theta)(x) = \sum_{k \in \mathbb{Z}^d} \mathcal{F}^{-1}_{\mathbb{R}^d}(\theta)(x+2\pi k)
\]
for all Schwartz function \( \theta : \mathbb{R}^d \to \mathbb{C} \). Here \( \mathcal{F}^{-1}_{\mathbb{R}^d}(\theta)(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\xi} \theta(\xi) d\xi \). It follows that
\[
|\Delta_j f(x)| \leq \|f\|_{\tilde{C}^\alpha} \int_{\mathbb{R}^d} |x-y|^{\alpha} \mathcal{F}^{-1}_{\mathbb{R}^d}(\varphi_j)(x-y) dy \lesssim 2^{-j\alpha} \|f\|_{\tilde{C}^\alpha}
\]
and thus \( \|f\|_{\tilde{B}^\alpha_{\infty, \infty}(\mathbb{T}^d)} \lesssim \|f\|_{\tilde{C}^\alpha} \). We have proved the equivalence between \( \| \cdot \|_{\tilde{C}^\alpha(\mathbb{T}^d)} \) and \( \| \cdot \|_{\tilde{B}^\alpha_{\infty, \infty}(\mathbb{T}^d)} \).

Regarding the equivalence between the inhomogeneous norms, it suffices to note that
\[
\|\Delta_{j-1} f\|_{L^\infty} \lesssim \|f\|_{L^\infty}, \quad \|f\|_{L^\infty} \leq \sum_{j \geq 1} \|\Delta_j f\|_{L^\infty} \lesssim \|f\|_{\tilde{B}^\alpha_{\infty, \infty}}.
\]

\[\square\]

**Definition A.2.** For \( a, u : \mathbb{T}^d \to \mathbb{C} \) the paraproduct \( T_a u \) is defined by
\[
T_a u = \sum_{j \geq 1} S_{j-1} a \Delta_j u.
\]

The (formal) Bony decomposition is given by
\[
a u = T_a u + T_u a + R(a, u),
\]
where
\[
R(a, u) = \sum_{j, k \geq -1, |j-k| \leq 1} \Delta_j a \Delta_k u.
\]

Note that \( \text{supp} \mathcal{F}(S_{j-1} a \Delta_j u) \subset \{ 2^{j-1} \leq |\xi| \leq 2^{j+2} \} \) for all \( j \geq 1 \).

**Proposition A.3.** (i) If \( s_1, s_2 \in \mathbb{R} \) such that \( s_1 + s_2 > 0 \) then
\[
\|R(a, u)\|_{H^{s_1 + s_2}} \lesssim \|a\|_{\tilde{B}^\alpha_{\infty, \infty}} \|u\|_{H^{s_2}}.
\]
(ii) For all \( s \in \mathbb{R} \) and \( m > 0 \)
\[
\| T_a u \|_{H^s} \lesssim \| a \|_{L^\infty} \| u \|_{H^s}, \quad (A.6)
\]
\[
\| T_a u \|_{H^s} \lesssim \| a \|_{B_{\infty, \infty}^{-m}} \| u \|_{H^{s+m}}. \quad (A.7)
\]

(ii) For all \( s \in \mathbb{R} \) and \( r > 0 \),
\[
\| T_a u \|_{H^s} \lesssim \| a \|_{H^{-r}} \| u \|_{B_{\infty, 2}^{s+r}}. \quad (A.8)
\]

**Proof.** The inequalities (A.5), (A.6) and (A.7) can be found in Section 2.8, [1]. Let us prove (A.8). For all \( j \geq 1 \),
\[
\| S_{j-1} a \Delta_j u \|_{L^2} \leq \sum_{k=-1}^{j-2} \| \Delta_k a \|_{L^2} \| \Delta_j u \|_{L^\infty}
\]
\[
\leq \left( \sum_{k=-1}^{j-2} 2^{-2rk} \| \Delta_k a \|_{L^2}^2 \right)^{\frac{1}{2}} \left( \sum_{k=-1}^{j-2} 2^{2rk} \right)^{\frac{1}{2}} \| \Delta_j u \|_{L^\infty}
\]
\[
\lesssim \| a \|_{H^{-r}} 2^{rj} \| \Delta_j u \|_{L^\infty}.
\]

Consequently,
\[
\| T_a u \|_{H^s}^2 \lesssim \| a \|_{H^{-r}}^2 \sum_{j \geq 1} 2^{2(s+r)j} \| \Delta_j u \|_{L^\infty}^2 \lesssim \| a \|_{H^{-r}}^2 \| u \|_{B_{\infty, 2}^{s+r}}^2.
\]

**Lemma A.4 ([1] Lemma 2.99).** For \( s, m \in \mathbb{R} \) and \( p, r \in [1, \infty] \), there exists \( C > 0 \) such that
\[
\| [\Lambda^m, T_a] u \|_{L^p_{q,r} (\mathbb{T}^d)} \leq C \| \nabla a \|_{L^\infty (\mathbb{T}^d)} \| u \|_{B_{p,r}^{m-1} (\mathbb{T}^d)}.
\]

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