An elementary proof of convergence to the mean-field equations for an epidemic model

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Abstract

It is common to use a compartmental, fluid model described by a system of ordinary differential equations (ODEs) to model disease spread. In addition to their simplicity, these models are also the mean-field approximations of more accurate stochastic models of disease spread on contact networks. For the simplest case of a stochastic susceptible-infected-susceptible (SIS) process (infection with recovery) on a complete network, it has been shown that the fraction of infected nodes converges to the mean-field ODE as the number of nodes increases. However the proofs are not simple, requiring sophisticated probability, partial differential equations (PDE), or infinite systems of ODEs. We provide a short proof in this case for convergence in mean-square on finite time intervals using a system of two ODEs and a moment inequality.

1 Introduction

Interactions among individuals impact the transmission of infectious diseases and the structure of the contact network imposes an important constraint on the transmission dynamics (Keeling and Eames 2005). Models used in epidemiology to study disease transmission include mean-field models using ordinary differential equation (ODE) (Anderson and May 1991; Eames and Keeling 2002), Markov or state-based models (Bailey 1975), and individual-based simulation models (Goodreau et al. 2012; Keeling and Eames 2005). These capture the contact network structure at different complexity levels and thus differ in analytical tractability, computational performance, and accuracy (Keeling and Grenfell 2000; Keeling and Eames 2005; Bansal et al. 2007). When choosing a

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particular type of model, whether mean-field models or individual-based simulations, the researcher should not only take into account the disease being transmitted, the size of the target population, the availability of data about the underlying network structure, the desired level of analytical tractability, and computational performance, but also the accuracy of the chosen model (Bansal et al. 2007; Keeling 2005; Keeling and Eames 2005).

The vast majority of studies, which assessed the accuracy of mean-field models, only relied on numerical experiments (Simon and Kiss 2012). However, the rigorous analysis of the accuracy of mean-field models requires deriving a mathematical link between mean-field approximations and the exact Markov models. Kurtz (1970, 1971) are the first to show this (see also Ethier and Kurtz (2005)). They use operator semigroup and martingale techniques to prove that the exact models converge in probability over finite time-intervals to the solution of the mean-field ODE, using as a specific example the stochastic SIS-process on a complete network with increasing number of nodes. Simon and Kiss (2012) summarize the proof in Ethier and Kurtz (2005) without going into the technical details.

Simon et al. (2011) showed for the same setting as ours (i.e., a stochastic SIS-process on a complete network) based on the idea outlined in Diekmann and Heesterbeek (2000), that with increasing number of nodes the expected infected fraction converges to the solution of the mean-field ODE model using a partial differential equation (PDE) approach. This approach exploits the fact, that for a large number of nodes $n$, the discrete probability distribution $x_k(t)$ of the Markov model (i.e., $x_k(t)$ denotes the probability of having $k$ infected nodes) can be approximated using a continuous density function $\rho(t, k/n)$. The convergence is then proved by showing that the expected infected fraction of the resulting first-order PDE converges both to the solution of the mean-field model and to the expected fraction of the Markov model for large $n$. A more detailed description of this proof is given in Chapter 2 in Taylor (2012).

Motivated by the first-order PDE approach and the stochastic approach, which both draw from multiple different mathematical areas, Simon and Kiss (2012) introduced an ODE-based approach to show that the expected infected fraction of the Markov model converges uniformly on finite time-intervals to the solution of the mean-field ODE model. Establishing an infinite homogeneous linear system of ODEs to approximate all moments of the probability distribution $x_k(t)$, Simon and Kiss (2012) show using a perturbation theorem for the infinite system of ODEs that the mean-field
solution is an upper bound of the expected infected fraction in the Markov model. Further, they provide a bound dependent on the network size $n$ for the absolute difference in the expected infected fraction between the mean-field model and the Markov model. Note that neither the PDE nor the infinite ODE approach show that the distribution converges to the mean-field model.

In this paper, we provide a short proof showing that as the population size increases, the dynamics of the infected fraction of the stochastic SIS-process on a complete graph converges uniformly in mean square on finite time-intervals to the mean-field ODE model. Note that convergence in mean-square implies convergence in probability and hence in distribution, thus providing a stronger result than previous proofs of Kurtz (1970, 1971), Ethier and Kurtz (2005), Simon et al. (2011), and Simon and Kiss (2012). More importantly, we only use a system of two ODEs, basic ODE techniques, and Jensen’s inequality in our proof. We hope our short proof using only elementary tools opens up the field to more applied researchers that might not be comfortable using martingale or PDE theory, provides the basis for further mean-field convergence results, and increases understanding about the performance of mean-field models.

2 Theorem

Consider a complete graph of $n$ nodes and let $X_n(t)$ be a Markov process describing which nodes in the network are infected at time $t$. An infected node infects each neighbor at a rate $\tau/n$ and recovers at a rate $\gamma$. We let $I_n(t) := \text{num}_t(X_n(t))$ be the number of infected nodes and $i_n(t) := I_n(t)/n$ the infected fraction. We may sometimes drop the dependence on $t$ and $n$.

Our goal is to prove the following theorem. It shows that the dynamics of the expectation of the infected fraction converges in mean-square on finite time-intervals to the mean-field approximation as the population increases.

**Theorem 1.** If $i_n(0) = u$ for all $n$ where $u \in [0, 1]$, then as $n \to \infty$, $i_n(t)$ converges uniformly in mean square, i.e.,

$$E[|i_n(t) - y(t)|^2] \to 0$$

on any time finite interval $[0, T]$ to the solution of the mean-field equations:

$$y' = \tau y(1 - y) - \gamma y, \quad y(0) = u.$$  \hspace{1cm} (1)
3 Proof of Theorem 1

Since the theorem holds trivially for $i_n(0) = u = 0$, we now assume $u > 0$. To deal with another technicality, the mean-field equations (1) have a well-known closed-form solution, which is unique because the right hand side of (1) is smooth (i.e., continuously differentiable). Since $y$ is deterministic, we will show mean square convergence by proving that $E[i_n(t)] \to y(t)$ and $E[i_n(t)^2] \to y(t)^2$ (or equivalently $\text{Var}[i_n(t)] \to 0$), uniformly on $[0, T]$.

Let $SI(t) := \text{num}_{SI}(X(t))$ be the number of edges in the network with both a susceptible and infected endpoint. Our approach is shorter and more elementary than the original proof by Rand (1999), but see also Simon et al. (2011) for a more formulaic proof.

**Proposition 2.** $E[I]' = (\tau/n)E[SI] - \gamma E[I]$.

Proof. In network configuration $x$, the system can either transition to states $x^+$ with $\text{num}_I(x^+) = \text{num}_I(x) + 1$ or to states $x^-$ with $\text{num}_I(x^-) = \text{num}_I(x) - 1$ (i.e., only to states with $\pm 1$ infected nodes). The aggregate rate for the first is $\gamma \text{num}_I(x)$ and $(\tau/n) \text{num}_{SI}(x)$ for the second, proving that

$$\lim_{h \to 0} \frac{E[I(h)|I(0) = x] - \text{num}_I(x)}{h} = (\tau/n) \text{num}_{SI}(x) - \gamma \text{num}_I(x).$$

Multiplying both sides by $P[X = x]$ and summing over $x$ (unproblematic since it is a finite sum) proves the claim. 

In the case of a complete graph, $SI = I(n - I)$. Substituting into Proposition 2 and using $i = I/n$, we have

$$E[i]' = \tau(E[i] - E[i^2]) - \gamma E[i].$$

(2)

Applying Jensen’s inequality, $E[i^2] \leq E[i^2]$,

$$E[i]' \leq \tau E[i](1 - E[i]) - \gamma E[i].$$

(3)

Then a standard ODE comparison theorem, Lemma 3, justifies our intuition that the solution to the mean-field equations Theorem 1 is an upper bound: $E[i(t)] \leq y(t)$ for $t \geq 0$. Though the approach is new, this is not a new result and can also be found for example in the proof of Theorem 3.1 of Ganesh et al. (2005) or Proposition 5.3 of Simon and Kiss (2012).
Lemma 3. Suppose that \( a(t) \) and \( b(t) \) are scalar; \( f \) is smooth; \( a(0) \leq b(0) \); and \( a'(t) \leq f(a(t)) \) and \( b'(t) = f(b(t)) \) for \( 0 \leq t \leq T \). Then \( a(t) \leq b(t) \) for \( 0 \leq t \leq T \).

Proof. See any graduate ODE text such as Theorem 6.1 in Hale (2009).

We obtain a lower bound by examining \( E[i^2]' \). The transition rates and times of \( I(t)^2 \) are the same as those of \( I(t) \) (see the proof of Proposition 2), just now the jump sizes are \((I \pm 1)^2 - I^2 \) instead of \( \pm 1 \). Thus,

\[
E[I^2]' = \frac{\tau}{n}E[SI((I + 1)^2 - I^2)] + \gamma E[I((I - 1)^2 - I^2)].
\]

(4)

So, in the case of a complete graph, \( SI = I(n - I) \), and using \( i = I/n \),

\[
E[i^2]' = 2\tau(E[i^2] - E[i^3]) - 2\gamma E[i^2] + (1/n)(\tau(E[i] - E[i^2]) + \gamma E[i]).
\]

(5)

Applying Jensen’s inequality, \( E[i^2]^{1.5} \leq E[i^3] \),

\[
E[i^2]' \leq 2\tau(E[i^2] - E[i^3]) - 2\gamma E[i^2] + (1/n)(\tau(E[i] - E[i^2]) + \gamma E[i]).
\]

(6)

Combining (2) and the right hand side of (6) we have the following system:

\[
z_1' = g_1(z_1, z_2) := \tau(z_1 - z_2) - \gamma z_1,
\]

(7a)

\[
z_2' = g_2(z_1, z_2) := 2\tau(z_2 - z_2^{1.5}) - 2\gamma z_2 + (1/n)(\tau(z_1 - z_2) + \gamma z_1),
\]

(7b)

\[
z_1(0) = u, \quad z_2(0) = u^2.
\]

(7c)

The right hand sides are smooth and thus unique solutions exists for every \( n \). Our intuition is that \( z_1, z_2, z_3, \ldots \) are scalar; \( f \) is smooth; \( a(0) \leq b(0) \); and \( a'(t) \leq f(a(t)) \) and \( b'(t) = f(b(t)) \) for \( 0 \leq t \leq T \). Then \( a(t) \leq b(t) \) for \( 0 \leq t \leq T \).

See Figure 1. In the Appendix we give an alternate proof that does not require a phase space diagram.

Since we already have the bound \( E[i_n(t)] \leq y(t) \) for \( t \geq 0 \), the next step is to show that
Figure 1: Phase space diagram. The blue curve \((z_{1,n}(t), z_{2,n}(t))\) is a trajectory of the vector field \((z'_{1,n} = g_1(z_{1,n}, z_{2,n}), z'_{2,n} = g_2(z_{1,n}, z_{2,n}))\) and is always to the left of the black curve \((E[i_{n}(t)], E[i_{n}^2(t)])\) since
\[
E[i_{n}'] = g_1(E[i_{n}], E[i_{n}^2]) \quad \text{and} \quad E[i_{n}^2] = g_2(E[i_{n}], E[i_{n}^2]).
\]

\(z_{1,n}(t) \to y(t)\) and \(z_{2,n}(t) \to y(t)^2\) uniformly on \([0, T]\). Substituting \(\epsilon = 1/n\), the right hand side of (7a)-(7b) is also smooth in \(\epsilon\) and thus by the results about the continuous dependence on parameters of ODE solutions, the right hand side of (7) converges uniformly for any \([0, T]\) as \(n \to \infty\) to the solution of the system:

\[
\begin{align*}
\bar{z}'_1 &= \tau(\bar{z}_1 - \bar{z}_2) - \gamma \bar{z}_1, \\
\bar{z}'_2 &= 2\tau(\bar{z}_2 - \bar{z}_2^{1.5}) - 2\gamma \bar{z}_2, \\
\bar{z}'_1 &= u, \quad \bar{z}_2 = u^2.
\end{align*}
\]

(8a)

(8b)

(8c)

The reasons for existence and uniqueness here are the same as for (7). Note that \(\bar{z}_1 = y\) and \(\bar{z}_2 = y^2\) is a solution to (8) because (8a) and (8c) become (1) and the right hand side of (8b) ends up being \(2yy'\), which equals \((y^2)'\) as we require.
4 Conclusion

We gave a short proof showing that as the population size increases, the dynamics of the infected fraction of the stochastic SIS-process on a complete graph converges uniformly in mean-square on finite time-intervals to the mean-field ODE model. Using the ODE from Proposition 2 to describe the dynamics of the first moment (the expected number of nodes infected) and Jensen’s inequality to approximate the second moment in this equation, we showed that the solution of the mean-field ODE model provides an upper bound to the expected infected fraction. Further, we established a lower bound on the expected fraction infected with the blue curve of the phase space diagram in Figure 1 or by (12) in the appendix. The lower bound is from a system of two ODEs: the ODE describing the dynamics of the first moment combined with an ODE describing the dynamics of the second moment where the third moment in this equation was approximated using Jensen’s inequality. Finally, uniform convergence in mean-square was shown by proving that the lower bound (i.e., the solution of the system of two ODEs approximating the dynamics of the first two moments) converges to the infected fraction and its square in the mean-field ODE model. Thus, the approach presented provides both a stronger convergence result compared to previous approaches showing convergence in probability or convergence of the expected value as well as a shorter, more elementary proof.

Mean-field models appear in many fields, not just epidemic modeling, and if such an elementary approach to proving convergence is generalizable, it would have many users. However, one limitation of our approach is that Lemma 3 about ODE inequalities only holds for scalar ODEs and the argument we used about the path being to the left only works in two dimensions. In cases where a mean-field model leads to a bound involving a system of three or more ODEs, some other argument would be needed. Developing such arguments to generalize this elementary approach beyond the SIS-model to more complicated epidemic models is an area for future work.

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Appendix

Here is a sketch of an alternate proof that avoids need for a phase space diagram. However, it also cannot be extended to higher dimensions. Starting around (6):

Proof. Applying Jensen’s inequality, 

\[ E[i^2]' \leq 2\tau(E[i^2] - E[i^2]^{1.5}) - 2\gamma E[i^2] + (2/n)(\tau E[i](1 - E[i]) + \gamma E[i]), \]  

(9)

and then applying the bound \( 0 \leq E[i] \leq 1 \), we obtain

\[ E[i^2]' \leq 2\tau(E[i^2] - E[i^2]^{1.5}) - 2\gamma E[i^2] + (2/n)(\tau + \gamma). \]  

(10)

Applying Lemma 3, we have \( E[i_n^2] \leq z_{2,n} \), where \( z_{2,n} \) solves the initial value problem:

\[ z_{2,n}' = 2\tau(z_{2,n} - z_{2,n}^{1.5}) - 2\gamma z_{2,n} + (2/n)(\tau + \gamma), \quad z_{2,n}(0) = u. \]  

(11)

Now going back to (2), we have a lower bound for \( E[i]' \),

\[ E[i]' \geq \tau(E[i] - z_{2,n}) - \gamma E[i] \]  

(12)

Applying Lemma 3 here by treating \( z_{2,n}(t) \) as an exogenous function, we have \( E[i_n] \geq z_{1,n} \), where \( z_{1,n} \) solves the initial value problem:

\[ z_{1,n}' = \tau(z_{1,n} - z_{2,n}) - \gamma z_{1,n}, \quad z_{1,n}(0) = u. \]  

(13)

Now we need to show that \( z_{1,n} \to y \). Using an argument similar to that for (8), \( z_{2,n} \) converges uniformly for any \([0,T]\) as \( n \to \infty \) to \( y^2 \), since that is unique solution to

\[ z_{2}' = 2\tau(z_{2} - z_{2}^{1.5}) - 2\gamma z_{2}, \quad z_{2}(0) = u. \]  

(14)
Now the right hand side of (13) converges to the right hand side of

\[ \ddot{z}_1' = \tau(\dot{z}_1 - y^2) - \gamma \dot{z}_1, \quad \ddot{z}_1(0) = u, \]

for which \( y \) is the unique solution. Thus, using the same argument as above, \( z_{1,n} \to y \). \( \square \)