A SHARP ERROR ESTIMATE OF PIECEWISE POLYNOMIAL COLLOCATION FOR NONLOCAL PROBLEMS WITH WEAKLY SINGULAR KERNELS

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Abstract. As is well known, using piecewise linear polynomial collocation (PLC) and piecewise quadratic polynomial collocation (PQC), respectively, to approximate the weakly singular integral

\[ I(a,b,x) = \int_a^b \frac{u(y)}{|x-y|^\gamma} \, dy, \quad x \in (a,b), \quad 0 < \gamma < 1, \]

have the local truncation error \( O(h^2) \) and \( O(h^4 - \gamma) \). Moreover, for Fredholm weakly singular integral equations of the second kind, i.e., \( \lambda u(x) - I(a,b,x) = f(x) \) with \( \lambda \neq 0 \), also have global convergence rate \( O(h^2) \) and \( O(h^{4-\gamma}) \) in [Atkinson and Han, Theoretical Numerical Analysis, Springer, 2009].

Formally, following nonlocal models can be viewed as Fredholm weakly singular integral equations

\[ \int_a^b u(x) - u(y) \frac{dy}{|x-y|^\gamma} = f(x), \quad x \in (a,b), \quad 0 < \gamma < 1. \]

However, there are still some significant differences for the models in these two fields. In the first part of this paper we prove that the weakly singular integral by PQC have an optimal local truncation error \( O(h^{4-\eta^i}) \), where \( \eta^i = \min\{x_i - a, b - x_i\} \) and \( x_i \) coincides with an element junction point. Then a sharp global convergence estimate with \( O(h) \) and \( O(h^3) \) by PLC and PQC, respectively, are established for nonlocal problems. Finally, the numerical experiments including two-dimensional case are given to illustrate the effectiveness of the presented method.

Key words. Nonlocal problems, weakly singular kernels, piecewise polynomial collocation, convergence analysis

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1. Introduction. In this paper we study an error estimate of the piecewise linear polynomial collocation (PLC) and piecewise quadratic polynomial collocation (PQC) for the nonlocal problems with a weakly singular kernels, whose prototype equation is [1, 4, 14, 16]

\[ \int_a^b \frac{u(x) - u(y)}{|x-y|^\gamma} \, dy = f(x), \quad x \in (a,b), \quad 0 < \gamma < 1 \]

with Dirichlet boundary conditions \( u(a) = u_a \) and \( u(b) = u_b \). Such as nonlocal problems [1,11] have been used to model very different scientific phenomena occurring in various applied fields, for example in materials science, biology, particle systems, image processing, coagulation models, mathematical finance, etc. [1,4].

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Formally, the nonlocal models (1.1) can be viewed as Fredholm weakly singular integral equations of the second kind [2, 3, 21], i.e.,

\[(\ast) \quad \lambda u(x) - \int_a^b \frac{u(y)}{|x-y|^\gamma} dy = f(x), \quad x \in (a, b), \quad 0 < \gamma < 1\]

with a nonzero complex number \( \lambda \in \mathbb{C} \). However, there are still some significant differences for the models in these two fields. For example, the inverse operators of Fredholm integral equations (\( \ast \)) are uniformly bounded, see Theorem 12.5.1 of [3] or [2]; but nonlocal model (1.1) is unbounded. From perspective of error analysis, it is shown that the Fredholm integral equations (\( \ast \)) have \( O(h^2) \) convergence [3, p. 522] by PLC and \( O(h^{4-\gamma}) \) convergence [3, p. 525] by PQC. Such a situation does not take place for model (1.1), even for the case \( \gamma = 0 \). Later in the section 4, we prove an optimal global convergence estimate with \( O(h^2) \) by PLC and \( O(h^3) \) by PQC, respectively, for model (1.1). In fact, the convergence rate for model (1.1) with PLC remains to be proved in [16].

The first key step of error analysis for models (1.1) is to study the following integral with the weakly singular kernels, being defined as

\[(1.2) \quad I(a, b, x) = \int_a^b \frac{u(y)}{|x-y|^\gamma} dy, \quad x \in (a, b), \quad 0 < \gamma < 1.\]

It should be noted that the integral (1.2) can be decomposed into Abel-Liouville integrals (often also called Riemann-Liouville fractional integrals) [12] and Weyl fractional integral [13] if they depart from the constant coefficient \( 1/\Gamma(1-\gamma) \).

Among various techniques for solving integral equations, collocation methods are among the simplest [21], which is only needed one-fold of integration and is much simpler to implement on a computer. Piecewise polynomial collocation methods for the integral (1.2) have been extensively examined by many authors. As is well known, for weakly singular \( (0 < \gamma < 1) \) integral (1.2), an optimal error estimate with \( O(h^2) \) was proved by PLC and only \( O(h^3) \) convergence was established by PQC in [2]. Up to now, the quasi-optimal error estimate with \( O(h^{4-\gamma}) \) convergence was provided by PQC, see [2] or [3, p. 525]. A few years later, the error estimate of the Newton-Cotes rules (piecewise polynomial collocation) for hypersingular \( (\gamma \geq 1) \) integrals (1.2) was first studied in [11]. Later, the superconvergence estimate of the Hadamard finite-part (hypersingular) integral is discussed in [19, 20] and a class of collocation-type methods are developed in [10]. Recently, fractional hypersingular integral equations and nonlocal diffusion equations with PLC is studied in [21] and a general Newton-Cotes rules for fractional hypersingular integrals have been developed in [8]. It should be noted that there are still some differences for the hypersingular integral and weakly singular integral equations. For example, the stiffness matrix of hypersingular integral is a strictly diagonally dominant M-matrix [21], however, it is not possessed for the weakly singular integral equations by PLC.

Numerical methods for the nonlocal problems (1.1) have been proposed by various authors. There are already the second-order convergence results for model (1.1) by linear FEM [5, 18] and for peridynamic or nonlocal problems with the horizon parameter by PLC [6, 15, 21]. As with our previous reviews, it seems to be second-order convergence for nonlocal model (1.1) as well as Fredholm weakly singular integral equations (\( \ast \)) by PLC. Unfortunately, the numerical result of (1.1) with \( \gamma = 1 \) shows that the convergence rate seems to be close to 1.5 by PLC [16] although it remains to
be proved. In this work, inspired by these observations, we will provide the rigorous convergence error estimate with $O(h)$ by PLC for the nonlocal model \eqref{1.1}, even for the case $\gamma = 0$. How about PQC? We have known that there exists the quasi-optimal error estimate with $O(h^{4-\gamma})$ convergence for \eqref{1.2} by PQC in \cite{3} p. 525 or \cite{7}. However, it is still not an optimal error estimate when the singular point coincides with an element junction point. Developed the techniques of hypersingular integral \cite{8,10,13}, we will provide an optimal error estimate with $O(h^{4-\gamma})$, $\eta_i = \min \{ x_i - a, b - x_i \}$ for the integral \eqref{1.2} with weakly singular kernels by PQC. Then the main purpose of the paper is the derivation of an optimal global convergence estimate with $O(h^3)$ for nonlocal problems \eqref{1.1} by PQC.

The paper is organized as follows. In the next section, we provide the discretization schemes for the integral \eqref{1.2} and nonlocal model \eqref{1.1}, respectively. In Section 3, we study the local truncation error for integral \eqref{1.2} by PLC and PQC. The global convergence rate for nonlocal model \eqref{1.1} by PLC and PQC, respectively, are detailed proved in Section 4. To show the effectiveness of the presented schemes, results of numerical experiments are reported in Section 5. In particularity, some simulations for two-dimensional nonlocal problems with nonsmooth kernels in nonconvex polygonal domain are performed. Finally, we conclude the paper with some remarks on the presented results.

2. Collocation method and numerical schemes. To elucidate the superconvergence phenomenon, we use the piecewise linear and quadratic polynomial collocation method to approach the nonlocal model \eqref{1.1}. Let us first consider the weakly singular integral \eqref{1.2}.

2.1. Collocation method for integral \eqref{1.2}. In \cite{2} the author already provided integral formulas to compute the weakly singular integral \eqref{1.2} by the piecewise polynomial collocation. Here, for the sake of theorems, we should explicitly express the coefficients of the quadrature schemes by integral formulas.

Case I: PLC for integral \eqref{1.2}. Let $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$ be a partition with the uniform mesh step $h = (b - a)/N$. Let the piecewise linear basis function $\phi_j(x)$ be defined by \cite{3} p. 484. Then the piecewise linear interpolation $I_1(a, b, x)$ of \eqref{1.2} is

$$I_1(a, b, x_i) = \int_a^b \sum_{j=0}^{N} u(x_j) \phi_j(y) \frac{dy}{|x_i - y|^\gamma} = \sum_{j=0}^{N-1} \left[ \frac{u(x_{j+1}) - u(x_j)}{|x_{j+1} - x_j|^\gamma} \right] \int_{x_j}^{x_{j+1}} \phi_j(y) \frac{dy}{|x_i - y|^\gamma}$$

i.e.,

$$I_1(a, b, x_i) = \sigma_h \gamma \left[ \sum_{j=1}^{N-1} g_{i-j} u(x_j) + \alpha_i u(x_0) + \alpha_{N-i} u(x_N) \right]$$

with $\sigma_h, \gamma = \frac{h^{1-\gamma}}{(2-\gamma)|1-\gamma|}$. Using integral formulas of \cite{2}, we can explicitly derive the internal values coefficients $g_0 = 2$, $g_k = (k+1)^2 - 2k^2 - \gamma + (k-1)^2 - \gamma, k \geq 1$; and the boundary values coefficients $\alpha_i = (i-1)^2 - i^2 - \gamma + (2-\gamma)i^{1-\gamma}, i = 1, 2, \ldots, N - 1$.

Case II: PQC for integral \eqref{1.2}. Let $a = x_0 < x_{\frac{1}{2}} < x_1 < \cdots < x_{\frac{N-1}{2}} < x_N = b$ be a partition with the uniform mesh step $h = (b - a)/N$. Let the piecewise
quadratic basis function $\varphi_j(y)$ or $\varphi_{j+\frac{1}{2}}(y)$ be given in \textsuperscript{[3]} p. 499. Let $u_Q(y)$ be the piecewise Lagrange quadratic interpolant of $u(y)$, i.e.,

$$u_Q(y) = \sum_{j=0}^{N} u(x_j) \varphi_j(y) + \sum_{j=0}^{N-1} u(x_{j+\frac{1}{2}}) \varphi_{j+\frac{1}{2}}(y).$$

Then we have the following piecewise quadratic interpolation $I_2(a, b, x)$ of (1.2)

$$I_2(a, b, x) = \int_{a}^{b} \frac{u_Q(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy$$

$$= \sum_{j=0}^{N-1} u(x_j) \int_{x_{j-1}}^{x_{j+1}} \frac{\varphi_j(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy + u(x_0) \int_{x_{0}}^{x_{1}} \frac{\varphi_0(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy + u(x_N) \int_{x_{N-1}}^{x_N} \frac{\varphi_N(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy + \sum_{j=0}^{N-1} u(x_{j+\frac{1}{2}}) \int_{x_{j}}^{x_{j+1}} \frac{\varphi_{j+\frac{1}{2}}(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy$$

with $1 \leq i \leq 2N - 1$. We divide (2.3) into two parts as follows

$$I_2(a, b, x_i) = \eta_{h, \gamma} \left[ \sum_{j=1}^{N-1} m_{i-j} u(x_j) + \sum_{j=0}^{N-1} q_{i-j-\frac{1}{2}|i|} u(x_{j+\frac{1}{2}}) + \beta_i u(x_0) + \beta_{N-i} u(x_N) \right], \ i = 1, 2, \cdots, N - 1;$$

and

$$I_2(a, b, x_{i+\frac{1}{2}}) = \eta_{h, \gamma} \left[ \sum_{j=1}^{N-1} p_{i+j-\frac{1}{2}|j|} u(x_j) + \sum_{j=0}^{N-1} n_{i-j} u(x_{j+\frac{1}{2}}) + \gamma_i u(x_0) + \gamma_{N-i} u(x_N) \right], \ i = 0, 1, \cdots, N - 1$$

with $\eta_{h, \gamma} = \frac{h^{1-\gamma}}{(3-\gamma)(2-\gamma)(1-\gamma)}$.

Here, from integral formulas of \textsuperscript{[2]}, we can explicitly compute $m_0 = 2(1+\gamma)$ and $m_k = 4 \left[ (k+1)^3(3-\gamma)-(k-1)^3(3-\gamma) \right] - (3-\gamma) \left[ (k+1)2^{-\gamma} + 6k^{2-\gamma} + (k-1)2^{-\gamma} \right], \ k \geq 1$; $p_0 = 4 \left[ \left( \frac{3}{2} \right)^{3-\gamma} - \left( \frac{1}{2} \right)^{3-\gamma} \right] - (3-\gamma) \left[ \left( \frac{3}{2} \right)^{2-\gamma} + 3 \left( \frac{1}{2} \right)^{2-\gamma} \right], \ p_k = m_{k+\frac{1}{2}}, \ k \geq 1$. Moreover, $q_k = -8 \left[ (k+1)^3(3-\gamma)-k^{3-\gamma} \right] + 4(3-\gamma) \left[ (k+1)2^{-\gamma} + k^{2-\gamma} \right], \ k \geq 1$; and $n_0 = (2-\gamma)2^{\gamma+1}, \ n_k = q_{k-\frac{1}{2}}, \ k \geq 1$. The boundary values coefficients $\beta_i = 4 \left[ i^{3-\gamma} - (i-1)^3(3-\gamma) \right] - (3-\gamma) \left[ 3i^{2-\gamma} + (i-1)^2(3-\gamma) \right] + (3-\gamma)(2-\gamma)i^{1-\gamma}, 1 \leq i \leq N-1$ and $\gamma_0 = (2-\gamma)(1-\gamma)2^{\gamma-1}, \ \gamma_i = \beta_{i+\frac{1}{2}}, \ i \geq 1$.

\textbf{2.2. Collocation method for nonlocal model (1.1).} Based on the discussion of the integral (1.2), we now provide the numerical schemes for nonlocal model (1.1).

\textbf{Case I: PLC for nonlocal model (1.1).} From (2.1), Eq. (1.1) reduces to

$$\int_{a}^{b} \frac{u(x_i)}{|x_i - y|^\gamma} dy - I_2(a, b, x_i) = f(x_i) + R_i, \ i = 1, 2, \cdots, N - 1,$$
where the local truncation error $R_i = O(h^2)$ will be proved in Lemma 3.1. Let $u_i$ be the approximated value of $u(x_i)$ and $f_i = f(x_i)$. Then the discretization scheme is

$$
(2.7) \quad \sigma_{h,\gamma} \left[ d_i u_i - \sum_{j=1}^{N-1} g_{i-j}|u_j| \right] = f_i + \sigma_h^1 (\alpha_i u_0 + \alpha_{N-i} u_N), \quad 1 \leq i \leq N - 1.
$$

Here the coefficients $\sigma_{h,\gamma}, \alpha_i, g_{i-j}$ are given in (2.1), and

$$
d_i = (2 - \gamma) \left[ i^{1-\gamma} + (N - i)^{1-\gamma} \right].
$$

For the convenience of implementation, we use the matrix form of the grid functions

$$
U = (u_1, u_2, \cdots, u_{N-1})^T, \quad F = (f_1, f_2, \cdots, f_{N-1})^T,
$$

therefore, E.q. (2.7) can be rewritten as

$$
(2.8) \quad \sigma_{h,\gamma} (D - G) U = F + \sigma_{h,\gamma} H,
$$

where $D = \text{diag} (d_1, d_2, \cdots, d_{N-1})$, $G = \text{toeplitz} (g_0, g_1, \cdots, g_{N-2})$ and

$$
H = (\alpha_1, \alpha_2, \cdots, \alpha_{N-1})^T u_0 + (\alpha_{N-1}, \alpha_{N-2}, \cdots, \alpha_1)^T u_N.
$$

**Case II: PQC for nonlocal model** (1.1). From (2.3), we can rewrite (1.1) as

$$
(2.9) \quad \int_a^b \frac{u(x_2)}{x_2 - y} dy - I_2 (a, b, x_2) = f(x_2) + R_2, \quad 1 \leq i \leq 2N - 1.
$$

Here we will prove that the local truncation error is $R_2 = O \left( h^4 \left( \eta_2 \right)^{-\gamma} \right)$ in Theorem 3.1. Let $u_{\frac{i}{2}}$ be the approximated value of $u(x_{\frac{i}{2}})$ and $f_{\frac{i}{2}} = f(x_{\frac{i}{2}})$. According to (2.3)-(2.9), then the discretization scheme is the following systems

$$
\eta_{h,\gamma} \left[ d_i u_i - \sum_{j=1}^{N-1} m_{i-j}|u_j| - \sum_{j=0}^{N-1} q_{i-j}|u_j| \right] = f_i + \eta_{h,\gamma} (\beta_i u_0 + \beta_{N-i} u_N)
$$

for $1 \leq i \leq N - 1,$

$$
(2.10) \quad \eta_{h,\gamma} \left[ d_{i+\frac{1}{2}} u_{i+\frac{1}{2}} - \sum_{j=1}^{N-1} p_{i+\frac{1}{2}-j}|u_j| - \sum_{j=0}^{N-1} n_{i-j}|u_j| \right]
$$

$$
= f_{i+\frac{1}{2}} + \eta_{h,\gamma} (\gamma_i u_0 + \gamma_{N-i-1} u_N)
$$

for $0 \leq i \leq N - 1,$

where

$$
d_{\frac{i}{2}} = (3 - \gamma)(2 - \gamma) \left( \left( \frac{i}{2} \right)^{1-\gamma} + \left( N - \frac{i}{2} \right)^{1-\gamma} \right), \quad i = 1, 2, \cdots, 2N - 1,
$$

and the coefficients $\eta_{h,\gamma}, \beta_i, \gamma_i, m_{i-j}, n_{i-j}, p_{i+\frac{1}{2}-j}, q_{i-j}|u_j|$ are given in (2.4) and (2.5). For the convenience of implementation, we use the matrix form of
the grid functions $U = \left( u_1, u_2, \cdots, u_{N-1}, u_\frac{1}{2}, u_\frac{3}{2}, \cdots, u_\frac{N-1}{2} \right)^T$ and similarly for $F$. Therefore, we can be rewrite (2.10) as the following systems

\begin{equation}
\eta_{h, \gamma} AU = F + \eta_{h, \gamma} K
\end{equation}

with

$$A = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} - \begin{bmatrix} M & Q \\ P & N \end{bmatrix}.$$ 

Here $D_1 = \text{diag} \left( d_1, d_2, \ldots, d_{N-1} \right)$, $D_2 = \text{diag} \left( d_\frac{1}{2}, d_\frac{3}{2}, \ldots, d_{\frac{N-1}{2}} \right)$,

$M = \text{toeplitz} \left( m_0, m_1, \ldots, m_{N-2} \right)$, $N = \text{toeplitz} \left( n_0, n_1, \ldots, n_{N-1} \right)$,

and

$$K = (\beta_1, \beta_2, \ldots, \beta_{N-1}, \gamma_0, \gamma_1, \ldots, \gamma_{N-1})^T u_0$$

$$+ (\beta_{N-1}, \beta_{N-2}, \ldots, \beta_1, \gamma_{N-1}, \gamma_{N-2}, \ldots, \gamma_0)^T u_N.$$ 

The rectangular matrices $P, Q$ are defined by

$$P = \begin{bmatrix} p_0 & p_1 & p_2 & \cdots & p_{N-3} & p_{N-2} \\ p_0 & p_0 & p_1 & \cdots & p_{N-4} & p_{N-3} \\ p_1 & p_0 & p_0 & \cdots & p_{N-5} & p_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ p_{N-4} & p_{N-5} & p_{N-6} & \cdots & p_0 & p_1 \\ p_{N-3} & p_{N-4} & p_{N-5} & \cdots & p_0 & p_0 \\ p_{N-2} & p_{N-3} & p_{N-4} & \cdots & p_1 & p_0 \end{bmatrix}_{N \times (N-1)}$$

and

$$Q = \begin{bmatrix} q_0 & q_0 & q_1 & \cdots & q_{N-4} & q_{N-3} & q_{N-2} \\ q_1 & q_0 & q_0 & \cdots & q_{N-5} & q_{N-4} & q_{N-3} \\ q_2 & q_1 & q_0 & \cdots & q_{N-6} & q_{N-5} & q_{N-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ q_{N-3} & q_{N-4} & q_{N-5} & \cdots & q_0 & q_0 & q_1 \\ q_{N-2} & q_{N-3} & q_{N-4} & \cdots & q_1 & q_0 & q_0 \end{bmatrix}_{(N-1) \times N}$$

3. Local truncation error for integral (1.2). As is well known, an optimal error estimate with $O(h^2)$ was proved by PLC and only $O(h^3)$ convergence was established by PQC in [3]. To the best of our knowledge, the quasi-optimal error estimate with $O(h^{4+\gamma})$ convergence was provided by PQC, see [7] or [3, p. 525]. However, it is still not an optimal error estimate when the singular point coincides with an element junction point. Based on the idea of [8] [10] [19], we next provide an optimal error estimate $O(h^{4+\gamma})$, $\gamma_i = \min \left\{ x_i - a, b - x_i \right\}$ for the integral (1.2) by PQC.

Using Lagrange interpolation and the property of weakly singular of integral (1.2), we obtain the following local truncation error for integral (1.2) by PLC.

\textbf{Lemma 3.1.} \cite{2} Let $I(a, b, x_i)$ and $I_1(a, b, x_i)$ be defined by (1.2) and (2.1), respectively. If $u(x) \in C^2[a, b]$, then

$$|I(a, b, x_i) - I_1(a, b, x_i)| = O(h^2).$$
3.1. A few technical Lemmas. Let us first introduce some lemmas, which will be used to estimate the local truncation error for integral \([1.2]\) by PQC.

**Lemma 3.2.** Let \(0 < \gamma < 1\), \(u(y) \in C^4[a,b]\) and \(u_Q(y)\) be defined by \([2.2]\). Then

\[
Q_{\frac{i}{4}} := \int_{x_{\frac{i}{4}+1}}^{x_{\frac{i}{4}+1}} \frac{u(y) - u_Q(y)}{x_{\frac{i}{4}+1} - y} dy = O(h^{5-\gamma}),
\]

where \(i\) is a positive integer number, \(\left\lfloor \frac{i}{2} \right\rfloor\) and \(\left\lceil \frac{i}{2} \right\rceil\) denotes the greatest integer that is less than or equal to \(\frac{i}{2}\) and the least integer that is greater than or equal to \(\frac{i}{2}\), respectively.

**Proof.** If \(i\) is even, we have

\[
\int_{x_{\frac{i}{4}}}^{x_{\frac{i}{4}+1}} \frac{(y - x_{\frac{i}{4}})(y - x_{\frac{i}{4}+1})(y - x_{\frac{i}{4}+1})}{y - x_{\frac{i}{4}}} dy = h^{4-\gamma} \int_0^1 t(t - \frac{1}{2})(t - 1) dt,
\]

\[
\int_{x_{\frac{i}{4}-1}}^{x_{\frac{i}{4}}} \frac{(y - x_{\frac{i}{4}-1})(y - x_{\frac{i}{4}})(y - x_{\frac{i}{4}})}{(x_{\frac{i}{4}} - y)^\gamma} dy = -h^{4-\gamma} \int_0^1 t(t - \frac{1}{2})(t - 1) dt.
\]

From Taylor expansion, there exist \(\xi_{\frac{i}{4}} \in [x_{\frac{i}{4}}, x_{\frac{i}{4}+1}]\) and \(\xi_{\frac{i}{4}-1} \in [x_{\frac{i}{4}-1}, x_{\frac{i}{4}}]\) such that

\[
u(y) - u_Q(y) = \frac{u^{(3)}(\xi_{\frac{i}{4}})}{3!} (y - x_{\frac{i}{4}})(y - x_{\frac{i}{4}+1})(y - x_{\frac{i}{4}+1}) \quad \forall y \in [x_{\frac{i}{4}}, x_{\frac{i}{4}+1}];
\]

and

\[
u(y) - u_Q(y) = \frac{u^{(3)}(\xi_{\frac{i}{4}-1})}{3!} (y - x_{\frac{i}{4}-1})(y - x_{\frac{i}{4}})(y - x_{\frac{i}{4}}) \quad \forall y \in [x_{\frac{i}{4}-1}, x_{\frac{i}{4}}].
\]

Then

\[
Q_{\frac{i}{4}} = \int_{x_{\frac{i}{4}-1}}^{x_{\frac{i}{4}+1}} \frac{\nu(y) - u_Q(y)}{x_{\frac{i}{4}+1} - y} dy = \int_{x_{\frac{i}{4}}}^{x_{\frac{i}{4}+1}} \frac{\nu(y) - u_Q(y)}{y - x_{\frac{i}{4}}} dy + \int_{x_{\frac{i}{4}}}^{x_{\frac{i}{4}+1}} \frac{u(y) - u_Q(y)}{y - x_{\frac{i}{4}}} dy
\]

\[
= \left( \frac{u^{(3)}(\xi_{\frac{i}{4}})}{3!} - \frac{u^{(3)}(\xi_{\frac{i}{4}-1})}{3!} \right) h^{4-\gamma} \int_0^1 t(t - \frac{1}{2})(t - 1) dt
\]

\[
= \frac{\gamma}{12(\gamma - 1)(3 - \gamma)(2 - \gamma)} h^{4-\gamma} = O(h^{5-\gamma}).
\]

If \(i\) is odd, it yields

\[
\int_{x_{\frac{i}{4}}}^{x_{\frac{i}{4}+1}} \frac{(y - x_{\frac{i}{4}})(y - x_{\frac{i}{4}+1})(y - x_{\frac{i}{4}+1})}{y - x_{\frac{i}{4}}} dy = h^{4-\gamma} \int_0^{\frac{1}{2}} t(t - \frac{1}{2}) dt,
\]

\[
\int_{x_{\frac{i}{4}-1}}^{x_{\frac{i}{4}}} \frac{(y - x_{\frac{i}{4}-1})(y - x_{\frac{i}{4}})(y - x_{\frac{i}{4}})}{(x_{\frac{i}{4}} - y)^\gamma} dy = -h^{4-\gamma} \int_0^{\frac{1}{2}} t(t - \frac{1}{2}) dt.
\]
Using Taylor expansion, there exist \( \xi \in \left[ x_{i+1}, x_{i+1} \right] \)

\[
\begin{align*}
    u(y) - u_Q(y) &= \frac{u^{(3)}(\xi)}{3!} (y - x_{i+\frac{1}{2}})(y - x_{i}) (y - x_{i+\frac{1}{2}}) \quad \forall \ y \in \left[ x_{i+1}, x_{i+1} \right].
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
    Q_2 &= \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{u(y) - u_Q(y)}{x_{i+\frac{1}{2}} - y} dy = \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{u(y) - u_Q(y)}{(x_{i+\frac{1}{2}} - y)} dy + \int_{x_{i+\frac{1}{2}}}^{x_{i+1}} \frac{u(y) - u_Q(y)}{y - x_{i+\frac{1}{2}}} dy\nonumber\nonumber\nonumber\nonumber\nonumber
    &= \frac{u^{(3)}(\xi)}{3!} h^{4-\gamma} \left( \int_0^{\frac{1}{2}} \left( t + \frac{1}{2} \right) \left( t - \frac{1}{2} \right) dt - \int_0^{\frac{1}{2}} \left( t + \frac{1}{2} \right) \left( t - \frac{1}{2} \right) dt \right) = 0.
\end{align*}
\]

The proof is completed. \( \square \)

**Lemma 3.3.** Let \( 0 < \gamma < 1 \), \( u(y) \in C^4[a, b] \) and \( u_Q(y) \) be defined by (2.2). Then

\[
Q_1 := \int_{x_0}^{x_{\frac{1}{2}}} \frac{u(y) - u_Q(y)}{x_{\frac{1}{2}} - y} dy
\]

\[
= -h^{4-\gamma} \cdot u^{(3)}(x_{\frac{1}{2}}) \sum_{m=1}^{[\frac{1}{2}] - 1} \int_0^1 \frac{t(1 - \frac{1}{2})(t - 1)}{(\frac{1}{2} - m + t)\gamma} dt + O(h^4) \left( x_{\frac{1}{2}} - a \right)^{1-\gamma} + O(h^{5-\gamma}),
\]

where \( i \) is a positive integer number and \([\frac{1}{2}]\) denotes the least integer that is greater than or equal to \( \frac{1}{2} \).

**Proof.** Since \( x_{\left[ \frac{1}{2} \right]} - x_0 = 0 \) with \( i = 1, 2 \), it yields \( Q_1 = 0 \). Then we just need to estimate \( Q_1 \) with \( i \geq 3 \). For any \( y \in [x_{m-1}, x_m] \), using Taylor expansion, there exist \( \xi_m \in [x_{m-1}, x_m] \) such that

\[
u(y) - u_Q(y) = \frac{u^{(3)}(\xi_m)}{3!} (y - x_{m-1}) (y - x_{m}) (y - x_m).
\]

For the sake of simplicity, we take \( w(\xi) = \frac{u^{(3)}(\xi)}{3!} \) and

\[
w(\xi_m) = [w(\xi_m) - w(x_m)] + w(x_m) \\
= [w(\xi_m) - w(x_m)] + w(x_m) (x_{\frac{1}{2}}), \quad \eta_m \in [x_m, x_{\frac{1}{2}}].
\]

Then

\[
\begin{align*}
    Q_1 &= \sum_{m=1}^{[\frac{1}{2}] - 1} \int_{x_{m-1}}^{x_m} \frac{w(\xi_m)}{x_{\frac{1}{2}} - y} \left( \frac{y - x_{m-1}}{y - x_{m}} \right) (y - x_m) dy := J_1 + J_2 + J_3
\end{align*}
\]

with

\[
\begin{align*}
    J_1 &= \sum_{m=1}^{[\frac{1}{2}] - 1} [w(\xi_m) - w(x_m)] \int_{x_{m-1}}^{x_m} \frac{(y - x_{m-1}) (y - x_{m}) (y - x_m)}{(x_{\frac{1}{2}} - y)^\gamma} dy; \\
    J_2 &= w(x_{\frac{1}{2}}) \sum_{m=1}^{[\frac{1}{2}] - 1} \int_{x_{m-1}}^{x_m} \frac{(y - x_{m-1}) (y - x_{m}) (y - x_m)}{(x_{\frac{1}{2}} - y)^\gamma} dy; \\
    J_3 &= \sum_{m=1}^{[\frac{1}{2}] - 1} w(\eta_m) (x_m - x_{\frac{1}{2}}) \int_{x_{m-1}}^{x_m} \frac{(y - x_{m-1}) (y - x_{m}) (y - x_m)}{(x_{\frac{1}{2}} - y)^\gamma} dy.
\end{align*}
\]
Using integration by parts and \( \int_0^1 \tau (\tau - \frac{1}{2}) (\tau - 1) d\tau = 0 \), it yields

\[
\int_{x_{m-1}}^{x_m} \frac{(y - x_{m-1}) (y - x_m)}{(x_{\frac{1}{2}} - y)\gamma} \, dy = h^{4-\gamma} \int_0^1 t \left( t - \frac{1}{2} \right) (t-1) \, dt
\]

(3.1)

\[
= -h^{4-\gamma} \int_0^1 \frac{t (t - \frac{1}{2}) (t-1)}{(\frac{1}{2} - m + t)^{1+\gamma}} \, dt = -\gamma h^{4-\gamma} \int_0^1 \frac{t (\tau - \frac{1}{2}) (\tau - 1) \, d\tau}{(\frac{1}{2} - m + t)^{1+\gamma}}. 
\]

Moreover, we have

\[
\sum_{m=1}^{[\frac{i}{2}]-1} \int_0^1 \frac{h^i \tau (\tau - \frac{1}{2}) (\tau - 1) \, d\tau}{(\frac{1}{2} - m + t)^{1+\gamma}} \, dt \leq \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{(\frac{1}{2} - m + t)^{1+\gamma}} \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{(i - 2m)^{1+\gamma}} \leq 2^{1+\gamma} \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{m^{1+\gamma}} \leq 2^{1+\gamma} \left( 1 + \frac{1}{\gamma} \right). 
\]

(3.2)

Here, for the last inequality, we use

\[
\sum_{m=1}^{i-2} \frac{1}{m^{1+\gamma}} = 1 + \sum_{m=1}^{i-2} \frac{1}{m^{1+\gamma}} = 1 + \sum_{m=2}^{m} \frac{1}{m^{1+\gamma}} \leq 1 + \sum_{m=2}^{m} \frac{1}{x^{1+\gamma}} \leq 1 + \frac{1}{\gamma}. 
\]

From (3.1) and (3.2), it leads to

\[
|J_1| \leq h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \gamma 2^{1+\gamma} \left( 1 + \frac{1}{\gamma} \right) = O(h^{5-\gamma});
\]

\[
J_2 = -h^{4-\gamma} \cdot w \left( x_{\frac{1}{2}} \right) \sum_{m=1}^{[\frac{i}{2}]-1} \int_0^1 \frac{t (t-\frac{1}{2}) (t-1)}{(\frac{1}{2} - m + t)^{1+\gamma}} \, dt.
\]

Next we estimate the error term \( J_3 \). Using (3.1) and (3.2), we have

\[
|J_3| \leq \gamma h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{(\frac{1}{2} - m)^{1+\gamma}} \int_0^1 \frac{1}{(\frac{1}{2} - m + t)^{1+\gamma}} \, dt 
\]

\[
\leq \gamma h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \sum_{m=1}^{[\frac{i}{2}]-1} \left( \frac{i}{2} - m \right)^{-\gamma}. 
\]

We can check

\[
\sum_{m=1}^{[\frac{i}{2}]-1} \left( \frac{i}{2} - m \right)^{-\gamma} = \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{m^{\gamma}} = \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{m^{\gamma}} \leq \sum_{m=1}^{[\frac{i}{2}]-1} \frac{1}{x^{1+\gamma}} \leq 1 + \frac{1}{\gamma}. 
\]

\[
= \int_0^{[\frac{i}{2}]-1} \frac{1}{x^{1+\gamma}} \, dx \leq 1 + \frac{1}{1-\gamma} \left( \frac{i}{2} \right)^{1-\gamma} = h^{\gamma-1} \left( \frac{i}{2} \right)^{1-\gamma}, \text{ } i \text{ is even.} 
\]
On the other hand, if \( i \) is an odd, we have

\[
\sum_{m=1}^{\left\lfloor \frac{i}{2} \right\rfloor - 1} \left( \frac{i}{2} - m \right)^{-\gamma} = \sum_{m=1}^{\left\lfloor \frac{i}{2} \right\rfloor} \frac{1}{(m - \frac{1}{2})^\gamma} = \left( \frac{1}{2} \right)^{-\gamma} + \sum_{m=2}^{\left\lfloor \frac{i}{2} \right\rfloor} \int_{m - \frac{1}{2}}^{m - \frac{1}{2}} \frac{1}{(m - \frac{1}{2})^\gamma} \, dx \\
\leq \left( \frac{1}{2} \right)^{-\gamma} + \int_{\frac{1}{2}}^{\frac{i}{2} - 1} \frac{1}{x^\gamma} \, dx \leq 2 \left( \frac{1}{2} \right)^{1-\gamma} + \frac{1}{1-\gamma} \left( \frac{i}{2} \right)^{1-\gamma} \leq 3 - 2\gamma \left( \frac{i}{2} \right)^{1-\gamma}
\]

\[
= \frac{h^{\gamma-1}(3-2\gamma)}{1-\gamma} \left( \frac{x}{2} - a \right)^{1-\gamma}.
\]

It implies that

\[
|J_3| \leq \gamma h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \frac{h^{\gamma-1}(3-2\gamma)}{1-\gamma} \left( \frac{x}{2} - a \right)^{1-\gamma} = \mathcal{O} \left( h^4 \right) \left( \frac{x}{2} - a \right)^{1-\gamma}.
\]

The proof is completed. \( \Box \)

**Lemma 3.4.** Let \( 0 < \gamma < 1 \), \( u(y) \in C^4[a,b] \) and \( u_Q(y) \) be defined by (2.2). Then

\[
Q_r := \int_{\frac{x}{2} + 1}^{x_i} \frac{u(y) - u_Q(y)}{x - y} \, dy
\]

\[
= h^{4-\gamma} \cdot w \left( \frac{x}{2} \right) \sum_{m=1}^{\left\lfloor \frac{i}{2} \right\rfloor - 1} \int_0^1 \frac{t(t-\frac{1}{2})(t-1)}{(\frac{1}{2} - m + t)^\gamma} \, dt + \mathcal{O} \left( h^4 \right) \left( \frac{x}{2} - a \right)^{1-\gamma} + \mathcal{O} \left( h^{5-\gamma} \right),
\]

where \( i \) is a positive integer number and \( \left\lfloor \frac{i}{2} \right\rfloor \) denotes the greatest integer that is less than or equal to \( \frac{i}{2} \).

**Proof.** Since \( x_{\left\lfloor \frac{i}{2} \right\rfloor + 1} = x_i \) with \( i = 1, 2 \), it yields \( Q_r = 0 \). Then we just need to estimate \( Q_r \) with \( i \geq 3 \). For any \( y \in [x_m, x_{m+1}] \), using Taylor expansion, there exist \( \xi_m \in [x_m, x_{m+1}] \) such that

\[
u(y) - u_Q(y) = \frac{u^{(3)}(\xi_m)}{3!} (y - x_m) \left( y - x_{m+\frac{1}{2}} \right) (y - x_{m+1}).
\]

For the sake of simplicity, we take \( w(\xi_m) = \frac{u^{(3)}(\xi_m)}{3!} \) and

\[
w(\xi_m) = \left[ w(\xi_m) - w(x_m) \right] + w(x_m)
\]

\[
\left[ w(\xi_m) - w(x_m) \right] + w \left( \frac{x}{2} \right) + w'(\eta_m) \left( x_m - \frac{x}{2} \right), \eta_m \in [x_m, x_{\frac{1}{2}}].
\]

Then

\[
Q_r = \sum_{m=\left\lfloor \frac{i}{2} \right\rfloor + 1}^{i-1} w(\xi_m) \int_{x_m}^{x_{m+1}} \frac{(y - x_m) \left( y - x_{m+\frac{1}{2}} \right) (y - x_{m+1})}{(y - x_{\frac{1}{2}})^\gamma} \, dy := J_1 + J_2 + J_3
\]
Moreover, from (3.2), we have

\[ \tilde{J}_1 = \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} [w(\xi_m) - w(x_m)] \int_{x_m}^{x_{m+1}} \frac{(y - x_m)(y - x_{m+\frac{1}{2}})}{(y - x_{\frac{i}{2}})^\gamma} dy; \]

\[ \tilde{J}_2 = w(\frac{x}{2}) \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \int_{x_m}^{x_{m+1}} \frac{(y - x_m)(y - x_m + \frac{1}{2})}{(y - x_{\frac{i}{2}})^\gamma} dy; \]

\[ \tilde{J}_3 = \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} w'(\eta_m) (x_m - \frac{x}{2}) \int_{x_m}^{x_{m+1}} \frac{(y - x_m)(y - x_{m+\frac{1}{2}})}{(y - x_{\frac{i}{2}})^\gamma} dy. \]

Using integration by parts and \( \int_0^1 \tau (\tau - \frac{1}{2}) (\tau - 1) d\tau = 0 \), it yields

\[ \int_{x_m}^{x_{m+1}} \frac{(y - x_m)(y - x_{m+\frac{1}{2}})}{(y - x_{\frac{i}{2}})^\gamma} dy = h^{4-\gamma} \int_0^1 \frac{t (t - \frac{1}{2}) (t - 1)}{(m - \frac{i}{2} + t)^{1+\gamma}} d\tau = \gamma h^{4-\gamma} \int_0^1 \frac{1}{(m - \frac{i}{2} + t)^{1+\gamma}} dt. \]

Moreover, from (3.2), we have

\[ \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \int_0^1 \left| \int_0^1 \tau (\tau - \frac{1}{2}) (\tau - 1) d\tau \right| \frac{1}{(m - \frac{i}{2} + t)^{1+\gamma}} dt \leq \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \int_0^1 \frac{1}{(m - \frac{i}{2} + t)^{1+\gamma}} dt = \left[ \frac{i}{2} \right]^{-1} \int_0^1 \frac{1}{(\frac{i}{2} - m + t)^{1+\gamma}} dt \leq 2^{1+\gamma} \left( 1 + \frac{1}{\gamma} \right). \]

According to the above equations, there exists

\[ |\tilde{J}_1| \leq h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| 2^{1+\gamma} \left( 1 + \frac{1}{\gamma} \right) = O(h^{5-\gamma}); \]

\[ \tilde{J}_2 = h^{4-\gamma} w(\frac{x}{2}) \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \int_0^1 \frac{t (t - \frac{1}{2}) (t - 1)}{(m - \frac{i}{2} + t)^{1+\gamma}} dt \]

\[ = h^{4-\gamma} \cdot w(\frac{x}{2}) \sum_{m=1}^{\lfloor \frac{i}{2} \rfloor - 1} \int_0^1 \frac{t (t - \frac{1}{2}) (t - 1)}{(\frac{i}{2} - m + t)^{1+\gamma}} dt. \]

Next we estimate the error term \( \tilde{J}_3 \). From (3.1) and (3.2), we have

\[ |\tilde{J}_3| \leq \gamma h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \sum_{m=\lfloor \frac{i}{2} \rfloor + 1}^{i-1} \left( m - \frac{i}{2} \right) \int_0^1 \frac{1}{(m - \frac{i}{2} + t)^{1+\gamma}} dt \]

\[ = \gamma h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \sum_{m=1}^{\lfloor \frac{i}{2} \rfloor - 1} \left( \frac{i}{2} - m \right) \int_0^1 \frac{1}{(\frac{i}{2} - m + t)^{1+\gamma}} dt \]

\[ \leq \gamma h^{5-\gamma} \max_{\eta \in [a,b]} |w'(\eta)| \sum_{m=1}^{\lfloor \frac{i}{2} \rfloor - 1} \left( \frac{i}{2} - m \right)^{-\gamma}. \]
The similar arguments can be performed as (3.3), we get

\[ |\tilde{J}_3| \leq \gamma h^{5-\gamma} \max_{\eta \in [a, b]} |w'(\eta)| \frac{h^{4-\gamma}}{1-\gamma} \left( x_\frac{1}{2} - a \right)^{1-\gamma} = \mathcal{O}(h^4) \left( x_\frac{1}{2} - a \right)^{1-\gamma}. \]

The proof is completed. \( \square \)

**Lemma 3.5.** Let \( 0 < \gamma < 1, u(y) \in C^4[a, b] \) and \( u_Q(y) \) be defined by (2.2). Then

\[ Q'_c := \int^{x_N}_{x_i} \frac{u(y) - u_Q(y)}{x_{\frac{1}{2}} - y} dy = \mathcal{O} \left( h^4 \left( x_\frac{1}{2} - a \right)^{-\gamma} \right), \quad 1 \leq i \leq N - 1. \]

**Proof.** For any \( y \in [x_m, x_{m+1}] \), using Taylor expansion, there exist \( \xi_m \in [x_m, x_{m+1}] \) such that

\[ u(y) - u_Q(y) = \frac{u^{(3)}(\xi_m)}{3!} (y - x_m) \left( y - x_{m+\frac{1}{2}} \right) (y - x_{m+1}). \]

For the sake of simplicity, we taking \( w(\xi_m) = \frac{u^{(3)}(\xi_m)}{3!} \). Using integration by parts and \( \int^1_0 \left( \tau - \frac{1}{2} \right) (\tau - 1) d\tau = 0 \), we have

\[
\left| \int_{x_m}^{x_{m+1}} \frac{(y - x_m) (y - x_{m+\frac{1}{2}}) (y - x_{m+1})}{(y - x_{\frac{1}{2}})^{1+\gamma}} dy \right| = h^4 \gamma \left| \int_0^1 \frac{t (t - \frac{1}{2}) (t - 1)}{(t + m - \frac{1}{2})^{1+\gamma}} dt \right|
\]

Moreover,

\[
\sum_{m=1}^{N-1} \int_0^1 \frac{1}{(t + m - \frac{1}{2})^{1+\gamma}} dt = \frac{1}{\gamma} \sum_{m=1}^{N-1} \left[ \left( m - \frac{i}{2} \right)^{-\gamma} - \left( 1 + m - \frac{i}{2} \right)^{-\gamma} \right]
\]

\[
= \frac{1}{\gamma} \left[ \left( \frac{i}{2} \right)^{-\gamma} - \left( N - \frac{i}{2} \right)^{-\gamma} \right] \leq \frac{1}{\gamma} \left( \frac{i}{2} \right)^{-\gamma} = \frac{1}{\gamma} h^\gamma \left( x_{\frac{1}{2}} - a \right)^{-\gamma}.
\]

According to the above equations, we have

\[
|Q'_c| \leq h^{4-\gamma} \max_{\eta \in [a, b]} |w(\xi)| \sum_{m=1}^{N-1} \int_0^1 \frac{1}{(t + m - \frac{i}{2})^{1+\gamma}} dt
\]

\[
\leq h^{4-\gamma} \max_{\eta \in [a, b]} |w(\xi)| h^\gamma \left( x_{\frac{1}{2}} - a \right)^{-\gamma} = \mathcal{O} \left( h^4 \left( x_{\frac{1}{2}} - a \right)^{-\gamma} \right).
\]

The proof is completed. \( \square \)

**Lemma 3.6.** Let \( 0 < \gamma < 1, u(y) \in C^4[a, b] \) and \( u_Q(y) \) be defined by (2.2). Then

\[ Q''_c := \int_{x_0}^{x_{N-1}} \frac{u(y) - u_Q(y)}{x_{\frac{1}{2}} - y} dy = \mathcal{O} \left( h^4 \left( x_{\frac{1}{2}} - a \right)^{-\gamma} \right), \quad N + 1 \leq i \leq 2N - 1. \]
Proof. For any $y \in [x_m, x_{m+1}]$, using Taylor expansion, there exist $\xi_m \in [x_m, x_{m+1}]$ such that
\[
 u(y) - u_Q(y) = \frac{u^{(3)}(\xi_m)}{3!} (y - x_m) \left( y - x_{m+\frac{1}{2}} \right) (y - x_{m+1}).
\]
For the sake of simplicity, we taking $w(\xi_m) = \frac{u^{(3)}(\xi_m)}{3!}$. Using integration by parts and $\int_0^1 \tau (\tau - \frac{1}{2}) (\tau - 1) d\tau = 0$, we have
\[
 \left| \int_{x_m}^{x_{m+1}} \frac{(y - x_m) (y - x_{m+\frac{1}{2}}) (y - x_{m+1})}{(x_{\frac{1}{2}} - y)^\gamma} dy \right| = h^{4-\gamma} \left| \int_0^1 \frac{t (t - \frac{1}{2}) (t - 1)}{(\frac{1}{2} - m - t)^{1+\gamma}} dt \right|
\]
\[
 = \gamma h^{4-\gamma} \left| \int_0^1 \frac{1}{(\frac{1}{2} - m - t)^{1+\gamma}} dt \right| \leq \gamma h^{4-\gamma} \int_0^1 \frac{1}{(\frac{1}{2} - m - t)^{1+\gamma}} dt.
\]
Moreover,
\[
 \sum_{m=0}^{i-N-1} \int_0^1 \frac{1}{(\frac{1}{2} - m - t)^{1+\gamma}} dt = \frac{1}{\gamma} \sum_{m=0}^{i-N-1} \left[ (\frac{1}{2} - m - 1)^{-\gamma} - (\frac{1}{2} - m)^{-\gamma} \right]
\]
\[
 = \frac{1}{\gamma} \left[ (N - \frac{i}{2})^{-\gamma} - (\frac{i}{2})^{-\gamma} \right] \leq \frac{1}{\gamma} \left( N - \frac{i}{2} \right)^{-\gamma} = \frac{1}{\gamma} h^\gamma \left( b - x_{\frac{1}{2}} \right)^{-\gamma}.
\]
According to the above equations, we have
\[
 |Q_c| = \left| \sum_{m=0}^{i-N-1} w(\xi_m) \int_{x_m}^{x_{m+1}} \frac{(y - x_m) (y - x_{m+\frac{1}{2}}) (y - x_{m+1})}{(x_{\frac{1}{2}} - y)^\gamma} dy \right|
\]
\[
 \leq \gamma h^{4-\gamma} \max_{\eta \in [a,b]} |w(\xi)| \sum_{m=0}^{i-N-1} \int_0^1 \frac{1}{(\frac{1}{2} - m - t)^{1+\gamma}} dt
\]
\[
 \leq h^{4-\gamma} \max_{\eta \in [a,b]} |w(\xi)| h^\gamma \left( b - x_{\frac{1}{2}} \right)^{-\gamma} = \mathcal{O} \left( h^4 \left( b - x_{\frac{1}{2}} \right)^{-\gamma} \right).
\]
The proof is completed. $\square$

3.2. Local truncation error for integral $[1,2]$ with PQC. According to the above results, we obtain the following.

Theorem 3.7. Let $I(a, b, x_{\frac{1}{2}})$ and $I_2(a, b, x_{\frac{1}{2}})$ be defined by (1.2) and (2.4), respectively. Let $0 < \gamma < 1$, $u(y) \in C^4[a,b]$ and $u_Q(y)$ be defined by (2.2). Then
\[
 |I(a, b, x_{\frac{1}{2}}) - I_2(a, b, x_{\frac{1}{2}})| = \int_a^b \frac{u(y) - u_Q(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy = \mathcal{O} \left( h^4 \left( \eta_{\frac{i}{2}} \right)^{-\gamma} \right) + \mathcal{O}(h^5-\gamma)
\]
with $\eta_{\frac{i}{2}} = \min \left\{ x_{\frac{1}{2}} - a, b - x_{\frac{1}{2}} \right\}$, $i = 1, 2, \cdots, 2N - 1$.

Proof. If $x_{\frac{1}{2}} \leq \frac{b-a}{2}$, then
\[
 \int_a^b \frac{u(y) - u_Q(y)}{|x_{\frac{1}{2}} - y|^\gamma} dy = Q_l + Q_{\frac{i}{2}} + Q_r + Q_c
\]
with
\[
\begin{align*}
Q_l :&= \int_{x_0}^{\frac{x_1}{2} - 1} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy, \\
Q_r :&= \int_{\frac{x_1}{2} + 1}^{x_1} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy, \\
Q_\frac{1}{2} :&= \int_{\frac{x_1}{2} - 1}^{\frac{x_1}{2} + 1} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy,
\end{align*}
\]

According to Lemmas 3.2-3.5, we obtain
\[
|R_i| = \int_a^b \frac{u(y) - u_Q(y)}{|x_\frac{1}{2} - y|^\gamma} \, dy = O \left( h^4 \left( x_\frac{1}{2} - a \right)^{-\gamma} \right) + O(h^{5-\gamma}).
\]

If \( x_\frac{1}{2} \geq \frac{b - a}{2} \), then
\[
\int_a^b \frac{u(y) - u_Q(y)}{|x_\frac{1}{2} - y|^\gamma} \, dy = \tilde{Q}_c + \tilde{Q}_l + \tilde{Q}_r
\]

with
\[
\begin{align*}
\tilde{Q}_c :&= \int_{x_0}^{x_{i-N}} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy, \\
\tilde{Q}_l :&= \int_{x_{i-N}}^{\frac{x_{i+1}}{2} - 1} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy, \\
\tilde{Q}_r :&= \int_{\frac{x_{i+1}}{2} + 1}^{x_{i+1}} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy, \\
\tilde{Q}_{\frac{1}{2}} :&= \int_{\frac{x_{i+1}}{2} - 1}^{\frac{x_{i+1}}{2} + 1} \frac{u(y) - u_Q(y)}{x_\frac{1}{2} - y} \, dy.
\end{align*}
\]

According to the Lemma 3.6 and the similar arguments can be performed as Lemmas 3.2-3.4, we have
\[
|R_i| = \int_a^b \frac{u(y) - u_Q(y)}{|x_\frac{1}{2} - y|^\gamma} \, dy = O \left( h^4 \left( b - x_\frac{1}{2} \right)^{-\gamma} \right) + O(h^{5-\gamma}).
\]

The proof is completed. \( \square \)

**Remark 3.1.** If \( s \) is not an element junction point, e.g., \( s \in (x_{\frac{1}{2}}, x_{\frac{i+1}{2}}) \), the similar arguments can be performed as Theorem 3.7 by PQC, we have
\[
|I(a, b, s) - I_2(a, b, s)| = \int_a^b \frac{u(y) - u_Q(y)}{|s - y|^\gamma} \, dy = O(h^{4-\gamma}),
\]

which coincides with [7] or [8 p. 525].

**4. Global convergence rate for nonlocal problems (1.1).** In [10] remains to be proved the convergence error estimate by PLC. Inspired by this observations, we derive an optimal global convergence estimate for such nonlocal problems with \( O(h) \) and \( O(h^2) \) by PLC and PQC, respectively.

**4.1. Global convergence rate for model (1.1) with PLC.** A symmetric positive definite matrix with positive entries on the diagonal and nonpositive off-diagonal entries is called an \( M \)-matrix. Then we have the following.

**Lemma 4.1.** Let matrix \( A = D - G \) be defined by (2.8). Then \( A \) is an \( M \)-matrix.

Proof. Let \( A = \{a_{i,j}\}_{i,j=1}^{N-1} \) with \( N \geq 2 \). From (2.8) and (2.1) and Taylor expansion, we have

\[
a_{i,j} = \begin{cases} d_i - g_0 > 0, & i = j, \\ -g_{|i-j|} < 0, & i \neq j. \end{cases}
\]

We next prove the matrix \( A \) is strictly diagonally dominant by rows. Using \( 1 \equiv \sum_{j=0}^{N} \phi_j(x) \), it yields

\[
\int_a^b \frac{1}{|x_i - y|} dy - \int_a^b \sum_{j=1}^{N-1} \phi_j(y) \frac{1}{|x_i - y|} dy = \int_a^b \phi_0(x) \frac{1}{|x_i - y|} dy + \int_a^b \phi_N(x) \frac{1}{|x_i - y|} dy
\]

\[
= \sigma_{h,\gamma} \rho_i \geq \frac{(2 - \gamma)(1 - \gamma)}{2} \sigma_{h,\gamma} \left[ \frac{1}{i^7} + \frac{1}{(N-i)^7} \right] = \frac{h^{1-\gamma}}{2} \left[ \frac{1}{i^7} + \frac{1}{(N-i)^7} \right]
\]

with

\[
\rho_i = [(i - 1)^2 - i^2 - (2 - \gamma)i^{1-\gamma} + (N - i - 1)^2 - (N - i)^2 - (2 - \gamma)(N - i)^{1-\gamma}].
\]

From

\[
(i - 1)^2 - i^2 - (2 - \gamma)i^{1-\gamma} = i^{2-\gamma} \left[ (1 - \frac{1}{i})^{2-\gamma} - 1 + (2 - \gamma) \frac{1}{i} \right]
\]

\[
= i^{2-\gamma} \left[ \frac{(2 - \gamma)(1 - \gamma)}{2} \frac{1}{i^2} + (2 - \gamma)(1 - \gamma) \sum_{n=1}^{\infty} \prod_{k=1}^{n} \frac{k + \gamma - 1}{(n + 2)!} \frac{1}{i^{n+2}} \right]
\]

\[
\geq i^{2-\gamma} \left[ \frac{(2 - \gamma)(1 - \gamma)}{2} \frac{1}{i^2} \right] = \frac{(2 - \gamma)(1 - \gamma)}{2i^7} > 0;
\]

and

\[
(N - i - 1)^2 - (N - i)^2 - (2 - \gamma)(N - i)^{1-\gamma} \geq \frac{(2 - \gamma)(1 - \gamma)}{2(N - i)^7} > 0,
\]

thus we have

\[
\sum_{j=1}^{N-1} a_{i,j} = \rho_i > 0, \quad i = 1, 2 \ldots N - 1.
\]

From the Gerschgorin circle theorem [9, p. 388], the eigenvalues of \( A \) are in the disks centered at \( a_{i,i} \) with radius \( r_i \), i.e., the eigenvalues \( \lambda \) of the matrix \( A \) satisfy

\[
|\lambda - a_{i,i}| \leq r_i = \sum_{j=1,j \neq i}^{N-1} |a_{i,j}|,
\]

which yields

\[
\lambda_{\min}(A) \geq \min_{i=1}^{N} \{a_{i,i} - r_i\} = \min_{i=1}^{N} \rho_i
\]

\[
= \min \left\{ \frac{(2 - \gamma)(1 - \gamma)}{2i^7} + \frac{(2 - \gamma)(1 - \gamma)}{2(N - i)^7} \right\}, \quad i = 1, 2 \ldots N - 1.
\]
The proof is completed. ☐

**Theorem 4.2.** Let $u_i$ be the approximate solution of $u(x_i)$ computed by the discretization scheme (2.8). Let $\varepsilon_i = u(x_i) - u_i$. Then

$$||u(x_i) - u_i||_\infty = O(h).$$

**Proof.** Let $\varepsilon_i = u(x_i) - u_i$ with $\varepsilon_0 = \varepsilon_N = 0$. Subtracting (2.8) from (2.6), we get

$$h^{1-\gamma} \frac{1}{(2-\gamma)(1-\gamma)} \cdot A \varepsilon = R,$$

where $\varepsilon = [\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_{N-1}]^T$ and similarly for $R$ with $R_i = O(h^2)$ in Lemma 3.1.

Let $|\varepsilon_i| := ||\varepsilon||_\infty = \max_{1 \leq i \leq N-1} |\varepsilon_i|$ and $A = \{a_{i,j}\}_{i,j=1}^{N-1}$. From Lemma 4.1 it yields $a_{i,i} > 0$ and $a_{i,j} < 0$, $i \neq j$ and

$$|R_{i0}| \geq \sigma_{h,\gamma} \left| a_{i0,i0} |\varepsilon_i| + \sum_{j=1,j \neq i}^{N-1} a_{i0,j} |\varepsilon_j| \right| = \sigma_{h,\gamma} \left[ a_{i0,i0} |\varepsilon_i| - \sum_{j=1,j \neq i}^{N-1} |a_{i0,j}| |\varepsilon_j| \right].$$

From (4.1) and Lemma 3.1, we have

$$S_{i0} \geq \frac{h^{1-\gamma}}{2} \left[ \frac{1}{\varepsilon_0} + \frac{1}{(N-i-1)} \right] \geq h^{1-\gamma} N^{-\gamma} = \frac{h}{(b-a)} \quad \text{and} \quad R_{i0} = O(h^2).$$

Then

$$||\varepsilon||_\infty = |\varepsilon_i| \leq \frac{|R_{i0}|}{S_{i0}} = O(h).$$

The proof is completed. ☐

**Remark 4.1.** Fredholm integral equations of the second kind model (\(\star\)) holds $||u(x_i) - u_i||_\infty = O(h^2)$ by PLC, see [2] and [3, p. 522]. If $\lambda = 1$ and $\gamma = 0$, the model (\(\star\)) is equivalent to the nonlocal model (1.1) with $\gamma = 0$, i.e., $\int_a^b u(x) - u(y) \, dy = f(x)$.

From Theorem 2.2 it leads to the interesting results $||u(x_i) - u_i||_\infty = O(h)$.

**4.2. Global convergence rate for model (1.1) with PLC.** We next consider the properties of the stiffness matrix $A$ in (2.11).

**Lemma 4.3.** Let the matrices $M$, $N$, $P$, $Q$ be defined by (2.11). Then $M$, $N$, $P$, $Q$ are positive matrices.

**Proof.** Using Taylor expansion, we have

$$(1 + z)^{\alpha} = 1 + \alpha z + \frac{\alpha(\alpha - 1)}{2!} z^2 + \frac{\alpha(\alpha - 1)(\alpha - 2)}{3!} z^3 + \cdots$$

(4.3)

$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha + 1 - k}{n!} z^n, \quad |z| \leq 1, \quad \alpha > 0.$$
We first estimate the elements of $M$. From (2.4) and (4.3), it yields $m_0 > 0$ and
\[
m_i = 4i^{3-\gamma} \left[ \left( 1 + \frac{1}{z} \right)^{3-\gamma} - \left( 1 - \frac{1}{z} \right)^{3-\gamma} \right] \\
- (3-\gamma) i^{2-\gamma} \left[ \left( 1 + \frac{1}{z} \right)^{2-\gamma} + 6 + \left( 1 - \frac{1}{z} \right)^{2-\gamma} \right] \\
= 2 \sum_{n=1}^{\infty} \prod_{k=1}^{2n+1} \frac{3 - 2n}{(2n+1)!} i^{2-2n-\gamma} \\
= 2(3 - \gamma)(2 - \gamma)(1 - \gamma) i^{-\gamma} \left[ \frac{1}{6} - \sum_{n=1}^{\infty} \prod_{k=1}^{2n}(1 - k - \gamma) \frac{2n - 1}{(2n+3)!} i^{-2n} \right] \\
\geq 2(3 - \gamma)(2 - \gamma)(1 - \gamma) i^{-\gamma} \left( \frac{1}{6} - \frac{7}{60} \right) > 0, \quad i \geq 1,
\]
since
\[
\sum_{n=1}^{\infty} \prod_{k=1}^{2n}(1 - k - \gamma) \frac{2n - 1}{(2n+3)!} i^{-2n} \\
\leq \sum_{n=1}^{\infty} \frac{(2n)!}{(2n+3)!} (2n - 1) \frac{1}{60} + \sum_{n=2}^{\infty} \frac{(2n-1)}{(2n+3)(2n+2)(2n+1)} \\
\leq \frac{1}{60} + \sum_{n=2}^{\infty} \frac{1}{(2n+1)(2n+3)} \leq \frac{1}{60} + \frac{1}{10} = \frac{7}{60}.
\]
Now we estimate the elements of $P$. From (2.5) and (4.3) and the above estimate of $m_i$, we have
\[
p_i = 4z^{3-\gamma} \left[ \left( 1 + \frac{1}{z} \right)^{3-\gamma} - \left( 1 - \frac{1}{z} \right)^{3-\gamma} \right] \\
- (3-\gamma) z^{2-\gamma} \left[ \left( 1 + \frac{1}{z} \right)^{2-\gamma} + 6 + \left( 1 - \frac{1}{z} \right)^{2-\gamma} \right] \\
\geq \frac{1}{10} (3 - \gamma)(2 - \gamma)(1 - \gamma) z^{-\gamma} > 0 \quad \text{with} \quad z = i + \frac{1}{2}, \quad i \geq 1.
\]
On the other hand, using (2.6) and (4.3), we obtain
\[
p_0 \geq 4 \left[ \left( \frac{3}{2} \right)^{3-\gamma} - \left( \frac{1}{2} \right)^{3-\gamma} \right] \\
- (3-\gamma) \left[ \left( \frac{3}{2} \right)^{2-\gamma} + 3 \left( \frac{1}{2} \right)^{2-\gamma} \right] - (3-\gamma)(2 - \gamma) \left( \frac{1}{2} \right)^{1-\gamma} \\
= (3 - \gamma)(2 - \gamma)(1 - \gamma) z^{-\gamma} \left[ \frac{1}{6} + \sum_{n=1}^{\infty} \prod_{k=1}^{n}(k - 1 + \gamma) \frac{n^2 + 2n + 1}{(n+3)!} z^{-n} \right] \\
\geq \frac{1}{6} (3 - \gamma)(2 - \gamma)(1 - \gamma) z^{-\gamma} > 0 \quad \text{with} \quad z = \frac{3}{2}.
\]
We next estimate the elements of $Q$. From (2.4) and (4.3), we obtain
\[
q_i = -8(i + 1)^{3-\gamma} \left[ 1 - \left( 1 - \frac{1}{i+1} \right)^{3-\gamma} \right] \\
+ 4(3 - \gamma)(i + 1)^{2-\gamma} \left[ 1 + \left( 1 - \frac{1}{i+1} \right)^{2-\gamma} \right] \\
= 4(3 - \gamma)(2 - \gamma)(1 - \gamma)(i + 1)^{2-\gamma} \left[ \frac{1}{6} + \sum_{n=1}^{\infty} \prod_{k=1}^{n} (k - 1 + \gamma) \frac{n + 1}{(n + 3)!} (i + 1)^{-n} \right] \\
\geq \frac{2}{3}(3 - \gamma)(2 - \gamma)(1 - \gamma)(i + 1)^{-\gamma} > 0, \quad i \geq 0.
\]

We last estimate the elements of $N$. From (2.5) and (4.3), it yields $n_0 > 0$ and
\[
n_i = -8(z + 1)^{3-\gamma} \left[ 1 - \left( 1 - \frac{1}{z+1} \right)^{3-\gamma} \right] \\
+ 4(3 - \gamma)(z + 1)^{2-\gamma} \left[ 1 + \left( 1 - \frac{1}{z+1} \right)^{2-\gamma} \right] \\
\geq \frac{2}{3}(3 - \gamma)(2 - \gamma)(1 - \gamma)(z + 1)^{-\gamma} > 0 \quad \text{with} \quad z = i - \frac{1}{2}, \quad i \geq 1.
\]

The proof is completed. \(\blacksquare\)

**Lemma 4.4.** Let $0 < \gamma < 1$ and $1 \leq i \leq 2N - 1$. Then
\[
\int_{a}^{b} \frac{\varphi_0(x)}{|x_{\frac{i}{2}} - y|^\gamma} \, dy \geq \frac{1}{6}(1 - \gamma)h \left( x_{\frac{i}{2}} - a \right)^{-\gamma};
\]
and
\[
\int_{a}^{b} \frac{\varphi_N(x)}{|x_{\frac{i}{2}} - y|^\gamma} \, dy \geq \frac{1}{6}(1 - \gamma)h \left( b - x_{\frac{i}{2}} \right)^{-\gamma}.
\]

**Proof.** If $i = 1$, then
\[
\int_{a}^{b} \frac{\varphi_0(x)}{|x_{\frac{i}{2}} - y|^\gamma} \, dy = \int_{x_{\frac{i}{2}}}^{\frac{x_{\frac{i}{2}} - y - x_{\frac{i}{2}} - y}{(x_{\frac{i}{2}} - y)^\gamma}} \, dy + \int_{\frac{x_{\frac{i}{2}}}{2}}^{\frac{x_{\frac{i}{2}} - y - x_{\frac{i}{2}} - y}{(y - x_{\frac{i}{2}})^\gamma}} \, dy = h^{1-\gamma} \frac{4}{3 - \gamma} \frac{1}{2^{3-\gamma}}.
\]
Let $\eta_{h,\gamma}$ be given in (2.4) and $i \geq 2$. Using Taylor expansion (4.3), it yields
\[
\int_{a}^{b} \frac{\varphi_0(x)}{|x_{\frac{i}{2}} - y|^\gamma} \, dy = \eta_{h,\gamma} \left[ 4 \left( \left( \frac{i}{2} \right)^{3-\gamma} - \left( \frac{i}{2} - 1 \right)^{3-\gamma} \right) \\
-(3 - \gamma) \left( \left( \frac{i}{2} \right)^{2-\gamma} + \left( \frac{i}{2} - 1 \right)^{2-\gamma} \right) + (3 - \gamma)(2 - \gamma) \left( \frac{i}{2} \right)^{1-\gamma} \right] \\
= \eta_{h,\gamma}(3 - \gamma)(2 - \gamma)(1 - \gamma) \left( \frac{i}{2} \right)^{-\gamma} \frac{1}{6} \left[ 1 - \gamma \sum_{n=5}^{\infty} \prod_{k=3}^{n-3} (k - 1 + \gamma)(n - 4) \frac{(i/2)^{3-n}}{n!} \right] \\
\geq \frac{1}{6} \eta_{h,\gamma}(3 - \gamma)(2 - \gamma)(1 - \gamma)^2 \left( \frac{i}{2} \right)^{-\gamma} = \frac{1}{6}(1 - \gamma)h \left( x_{\frac{i}{2}} - a \right)^{-\gamma}.
\]
Here, for the last inequality, we use
\[
\sum_{n=5}^{\infty} \prod_{k=2}^{n-3} (k - 1 + \gamma)(n - 4) \left( \frac{i}{2} \right)^{3-n} \frac{1}{n!} = \sum_{n=5}^{\infty} \frac{(n-3)!}{n!} < \sum_{n=5}^{\infty} \frac{(n-3)!}{n!} = \frac{1}{6}.
\]

On the other hand, there exists
\[
\int_{a}^{b} \frac{\varphi_N(x)}{\sqrt{x^2 - y}} dy = \int_{a}^{b} \frac{\varphi_0(x)}{\sqrt{y - x^2}} dy \geq \frac{1}{6}(1 - \gamma)h \left( b - x^2 \right)^{-\gamma}.
\]

The proof is completed. \(\square\)

**Lemma 4.5.** Let
\[
A = \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} - \begin{bmatrix} M & Q \\ P & N \end{bmatrix},
\]
then \(A\) is strictly diagonally dominant by rows.

**Proof.** From Lemma 4.4 we know that \(M, N, P, Q\) are positive matrices. From Lemma 4.4 and the property of the interpolation operator, i.e.,
\[
1 = \sum_{j=0}^{N} \varphi_j(x) + \sum_{j=0}^{N-1} \varphi_{j+\frac{1}{2}}(x),
\]
it yields
\[
S_2 := \int_{a}^{b} \frac{1}{\sqrt{x^2 - y}} dy - \int_{a}^{b} \frac{\varphi_0(x)}{\sqrt{y - x^2}} dy \geq \frac{1}{6}(1 - \gamma)h \left( b - x^2 \right)^{-\gamma}.
\]

Using (2.3), (2.9) with \(u(x) \equiv 1\), we can rewrite the above equation as the following
\[
(4.4) \quad \eta_{h,\gamma} \left\{ \begin{bmatrix} D_1 & 0 \\ 0 & D_2 \end{bmatrix} - \begin{bmatrix} M & Q \\ P & N \end{bmatrix} \right\} U = \eta_{h,\gamma} K = S,
\]
where \(U = (1, 1, \cdots, 1)^T\) and \(S = \left( S_1, S_2, \cdots, S_{N-1}, S_{\frac{1}{2}}, S_{\frac{3}{2}}, \cdots, S_{\frac{N-\frac{1}{2}}{2}} \right)^T\). The proof is completed. \(\square\)

**Remark 4.2.** From Lemma 4.5 we know that the matrix \(A\) is nonsingular [17, p. 23] and the linear system (2.11) has a unique solution.

**Theorem 4.6.** Let \(u_i\) be the approximate solution of \(u(x_i)\) computed by the discretization scheme (2.10). Then
\[
||u(x_i) - u_i||_{\infty} = O(h^3).
\]
Proof. Let $\epsilon_i = u(x_i) - y_{i}$ with $\epsilon_0 = \epsilon_N = 0$. Subtracting (2.10) from (2.9), we get

$$\eta_{h,\gamma} \cdot \mathbf{A} \epsilon = R$$

with $\epsilon = \begin{pmatrix} \epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}, \epsilon_{\frac{1}{2}}, \epsilon_{\frac{3}{2}}, \ldots, \epsilon_{N-\frac{1}{2}} \end{pmatrix}^T$ and similarly for $R$.

Upon relabeling and reorienting the vectors $\epsilon$ and $R$ as

$$\bar{\epsilon} = \begin{pmatrix} \epsilon_{\frac{1}{2}}, \epsilon_1, \epsilon_2, \ldots, \epsilon_{N-1}, \epsilon_{N-\frac{1}{2}} \end{pmatrix}^T,$$

$$\bar{R} = \begin{pmatrix} R_{\frac{1}{2}}, R_1, R_2, \ldots, R_{N-1}, R_{N-\frac{1}{2}} \end{pmatrix}^T,$$

then the above equation can be recast as

$$\eta_{h,\gamma} \cdot \bar{\mathbf{A}} \bar{\epsilon} = \bar{R}.$$ 

Let $|\epsilon|_\infty := ||\epsilon||_\infty = \max_{1 \leq i \leq 2N-1} |\epsilon_i|$ and $\bar{\mathbf{A}} = \{a_{i,j}\}_{i,j=1}^{2N-1}$. From Lemma and 4.4, it yields $a_{i,i} > 0$ and $a_{i,j} < 0$, $i \neq j$ and

$$|R_{\frac{1}{2}}| = \eta_{h,\gamma} \left| a_{i_0,i_0} \epsilon_{\frac{1}{2}} + \sum_{j=1,j\neq i_0}^{2N-1} a_{i_0,j} \epsilon_j \right| \geq \eta_{h,\gamma} \left[ a_{i_0,i_0} |\epsilon_{\frac{1}{2}}| - \sum_{j=1,j\neq i_0}^{2N-1} |a_{i_0,j}| |\epsilon_j| \right]$$

$$\geq \eta_{h,\gamma} \left[ a_{i_0,i_0} |\epsilon_{\frac{1}{2}}| - \sum_{j=1,j\neq i_0}^{2N-1} |a_{i_0,j}| |\epsilon_{\frac{1}{2}}| \right] = \eta_{h,\gamma} \left[ a_{i_0,i_0} - \sum_{j=1,j\neq i_0}^{2N-1} |a_{i_0,j}| \right] |\epsilon_{\frac{1}{2}}| = S_{\frac{1}{2}} |\epsilon_{\frac{1}{2}}|.$$ 

According to Lemma 4.4 and Theorem 3.7, we have

$$S_{\frac{1}{2}} \geq \frac{1}{6} (1 - \gamma) h \left[ \left( x_{\frac{1}{2}} - a \right)^{-\gamma} + \left( b - x_{\frac{1}{2}} \right)^{-\gamma} \right],$$

and

$$R_{\frac{1}{2}} = O \left( h^4 \eta_{\frac{1}{2}}^{-\gamma} \right) = \max \left\{ \left( x_{\frac{1}{2}} - a \right)^{-\gamma}, \left( b - x_{\frac{1}{2}} \right)^{-\gamma} \right\} O \left( h^4 \right).$$

Then

$$||\epsilon||_\infty = \left| \epsilon_{\frac{1}{2}} \right| \leq \frac{|R_{\frac{1}{2}}|}{S_{\frac{1}{2}}} = O \left( h^3 \right).$$

The proof is completed. \( \square \)

5. Numerical results. In this section, we numerical verify the above theoretical results including convergence rates. In particularly, some simulations for two-dimensional nonlocal problems with nonsmooth kernels in nonconvex polygonal domain are performed.
SHARP ERROR ESTIMATE FOR NONLOCAL PROBLEMS

5.1. Numerical example for 1D. In this subsection, the $l_\infty$ norm is used to measure the numerical errors.

EXAMPLE 5.1. To numerically confirm the result of Lemma 3.1 and Theorem 3.7, we consider the integral (1.2) with $a = 0$, $b = 1$. Here the test function is $u(x) = e^x$ and define $f(x)$ accordingly.

Table 5.1: Example 5.1. The errors of numerical scheme (2.1) with PLC.

| $\gamma$ | $h$  | $x = h$ error | order | $x = 1/3$ error | order | $x = 1/2$ error | order |
|---------|------|---------------|-------|----------------|-------|----------------|-------|
| 0.3     | 1/64 | 4.7106e-05    |       | 5.9699e-05     |       | 6.0480e-05     |       |
| 0.3     | 1/128| 1.1667e-05    |       | 2.0135         |       | 3.1563e-05     | 1.9959|
| 0.3     | 1/256| 2.8996e-06    |       | 2.0085         |       | 3.7975e-06     | 1.9974|
| 0.3     | 1/512| 7.2218e-07    |       | 2.0054         |       | 9.5039e-07     | 1.9985|
| 0.7     | 1/64 | 9.3738e-05    |       | 3.2028e-04     |       | 1.5912e-04     |       |
| 0.7     | 1/128| 2.3178e-05    |       | 2.1416         |       | 4.0977e-05     | 1.9572|
| 0.7     | 1/256| 5.7477e-06    |       | 2.1232         |       | 1.0487e-05     | 1.9662|
| 0.7     | 1/512| 1.4280e-06    |       | 1.9730         |       | 2.6712e-06     | 1.9730|

Table 5.2: Example 5.1. The errors of numerical scheme (2.3) with PQC.

| $\gamma$ | $h$  | $x = h$ error | order | $x = 1/3$ error | order | $x = 1/2$ error | order |
|---------|------|---------------|-------|----------------|-------|----------------|-------|
| 0.3     | 1/64 | 2.0549e-10    |       | 2.4848e-09     |       | 1.2613e-11     |       |
| 0.3     | 1/128| 1.6878e-11    |       | 3.7410         |       | 4.0820         |       |
| 0.3     | 1/256| 1.3696e-12    |       | 3.6672         |       | 4.0112         |       |
| 0.3     | 1/512| 1.1147e-13    |       | 3.7202         |       | 4.1154         |       |
| 0.7     | 1/64 | 1.5922e-09    |       | 1.6352e-07     |       | 3.3888e-10     |       |
| 0.7     | 1/128| 1.5608e-10    |       | 3.3084         |       | 4.0011         |       |
| 0.7     | 1/256| 1.5652e-11    |       | 3.2951         |       | 3.9993         |       |
| 0.7     | 1/512| 1.5730e-12    |       | 3.3028         |       | 3.9649         |       |

5.2. Numerical example for 2D. In this subsection, the $l_\infty$ norm and the discrete $L^2$-norm, respectively, are used to measure the numerical errors.
Example 5.3. Let us consider the following two-dimensional nonlocal problems

\[
\int_{\Omega} \frac{u(x, y) - u(\bar{x}, \bar{y})}{\sqrt{(x - \bar{x})^2 + (y - \bar{y})^2}} \, d\bar{x}d\bar{y} = f(x, y),
\]

where the nonconvex polygonal domain is a five-point star domain \( \Omega \) in \((0, 2) \times (0, 2)\), and the exact solution is \( u(x, y) = e^{x^2} \cos(\pi y) \). Then the nonhomogeneous boundaries condition and source function \( f(x, y) \) are defined accordingly.

In Fig. 5.1, the triangulations when \( h = 1/4 \) and \( h = 1/8 \) are depicted.

![Fig. 5.1: The space meshes of Example 5.3: (a) h=1/4, (b) h=1/8](image)

Table 5.5 and Table 5.6 show that the orders of accuracy are \( O(h) \) and \( O(h^3) \) by PLC and PQC, respectively, in a nonconvex polygonal domain. Here \( \| \cdot \|_{\infty} \) denotes the \( \ell_{\infty} \) norm and \( \| \cdot \|_{L^2} \) denotes the discrete \( L^2 \)-norm.

### 6. Conclusion

In this work, we first derive an optimal error estimate for weakly singular integral (1.2) by PQC when the singular point coincides with an element junction point. Then the sharp error estimate of piecewise linear and quadratic polynomial
collocation for nonlocal problems \((1.1)\) are provided. Hopefully, an optimal error estimate of the \(k\)th-order Newton-Cotes rule \(O(h^k)\) for odd \(k\) and \(O(h^{k+1})\) for even \(k\) can be obtained of nonlocal model \((1.1)\) by following the idea given in this paper. Moreover, it is also provided a few technical analysis for two-dimensional nonlocal problems with singular kernels or other nonsmooth kernels.

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