Four approaches to quantization of the relativistic particle

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Abstract

The connection between four different approaches to quantization of the relativistic particle is studied: reduced phase space quantization, Dirac quantization, BRST quantization, and (BRST)-Fock quantization are each carried out. The connection to the BFV path integral in phase space is provided. In particular, it is concluded that the full range of the lapse should be used in such path integrals. The relationship between all these approaches is established.

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1. Introduction

Investigating the problems that arise in the quantization of the parametrized relativistic particle may not seem like a very promising task. After all, it has been known for a long time that to implement relativistic covariance a quantum field theory for this system needs to be developed, so we expect any “first-quantization” approach to ultimately fail. Although this is a reasonable consideration, it is also true that the parametrized relativistic particle, a constrained system, bears a close resemblance to more complex and interesting systems, such as string theory, mini-superspace, and gravity. All of these are constrained systems, and many of the deep issues that arise trying to quantize them play a role in the quantization of the parametrized relativistic particle [1]. In addition, it is not obvious that the reasons that have led us to abandon a “first-quantized” theory for the relativistic particle and turn to quantum field theory, will be valid in the case of gravity. For these reasons, every effort should be made to completely understand what the different methods for quantization of a constrained system yield in the case of the parametrized relativistic particle.

One of the problems that these systems share is that of the range of the “lapse”, which is especially apparent in the covariant path integral approach to quantum gravity [2]. The problem there is that a coordinate space covariant path integral for gravity can be constructed using string theoretic methods, but one of the integration variables, called the lapse, has an undefined range. Relativistic covariance dictates only that the range should be either $(0, \infty)$ or $(-\infty, \infty)$. If it were possible to derive this path integral from a Hilbert space formalism, the range of integration of all the variables would be completely fixed. Understanding the Hilbert space structure of a quantum theory is essential to make sense of an object like a path integral.

The general action for the relativistic particle in a curved spacetime and in a background electro-magnetic field is given by

$$A = \int_{\tau_i}^{\tau_f} d\tau \left( -m\sqrt{g(x^\alpha)_\mu\nu \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} - e \frac{dx^\mu}{d\tau} A_\mu(x^\alpha) \right), \quad (1.1)$$

or by the alternate form

$$A' = \int_{\tau_i}^{\tau_f} d\tau \left( \frac{1}{\lambda(\tau)} g(x^\alpha)_\mu\nu \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + m\lambda(\tau) - e \frac{dx^\mu}{d\tau} A_\mu(x^\alpha) \right). \quad (1.2)$$

The boundary condition needed for an extremum are $x^\mu(\tau_i) = x^\mu_i$ and $x^\mu(\tau_f) = x^\mu_f$. Let us begin studying equation [1.1] in either case the end product is the same, as we will show in a moment. This action is invariant under reparametrizations that do not affect the boundaries: $\tau \rightarrow f(\tau)$ with $f(\tau_i) = \tau_i$ and $f(\tau_f) = \tau_f$, and with $df/d\tau > 0$. If $e = 0$, the action is also invariant under reparametrizations satisfying $\tau \rightarrow f(\tau)$, with $f(\tau_i) = \tau_f$ and $f(\tau_f) = \tau_i$, and with $df/d\tau < 0$. We can think of the full reparametrization group as the direct product of $Z_2$ and the group of reparametrizations connected with the identity, and of this action as carrying an unfaithful representation of the $Z_2$ part of the reparametrization group. The action
of the $Z_2$ part of the reparametrization group can be described by two types of reparametrization functions: $f_+$, which maps $\tau_i$ and $\tau_f$ into themselves, and $f_-$, which maps $\tau_i$ into $\tau_f$ and vice versa. The group multiplication is rules are

$$Z_2 = \begin{cases} f_+ \cdot f_+ = f_+ \\ f_+ \cdot f_- = f_- \\ f_- \cdot f_- = f_+ \end{cases}$$

(1.3)

The full reparametrization group is given by $G = Z_2 \otimes F_+$ where $F_+$ denotes the part connected to the identity. The action in equation 1.2 carries a faithful representation of $G$, with $\lambda(\tau) \rightarrow \hat{f}\lambda(f)$, for an arbitrary $e$.

Defining the momenta in the usual way we find—because of the reparametrization invariance—the first class constraint

$$\Phi = (p_\mu - A_\mu)g^{\mu\nu}(p_\nu - A_\nu) - m^2 \approx 0,$$

(1.4)

and a zero hamiltonian, $H \equiv 0$. The same equations of motion can be obtained from the so-called first order action in the phase space coordinates

$$F = \int_{\tau_i}^{\tau_f} d\tau (p_\mu \dot{x}^\mu + p_\mu \dot{\lambda} - v\Phi),$$

(1.5)

which is invariant under the gauge transformations $\delta x^\mu = \epsilon(\tau)\{x^\mu, \Phi\}$, $\delta p_\mu = \epsilon(\tau)\{p_\mu, \Phi\}$, and $\delta \nu = \dot{\epsilon}(\tau)$, as long as the gauge parameter vanishes at the boundaries, i.e., $\epsilon(\tau_i) = \epsilon(\tau_f) = 0$. It is not hard to see also that this symmetry is the same as the one in the Lagrangian form, with the identification $f(\tau) = \tau + \epsilon(\tau)$. The restriction in the gauge freedom at the boundaries is not usually present in the gauge theories. As explained in reference [17], this twist in the concept of invariance is a consequence of the form of the constraint, which is non-linear in the coordinates conjugate to what is fixed at the boundaries, i.e., the momenta. Under the above gauge transformation, the first order action changes, as a boundary term appears:

$$A \rightarrow A + \epsilon(\tau) (p_\mu \frac{\partial \Phi}{\partial p_\mu} - \Phi)\bigg|_{\tau_i}^{\tau_f}.$$  

(1.6)

This term vanishes if the constraint $\Phi$ is linear in the momenta. We will discuss this point further below.

Using the action in equation 1.2, the situation is just a bit more complicated. First we find the constraint $p_\lambda \approx 0$, since the action is independent of $\dot{\lambda}$. The hamiltonian is not zero, $H' \equiv \lambda \Phi$. The condition $\dot{p}_\lambda = 0$, which is needed to insure that the constraint is preserved by the dynamics [8], implies $\Phi \approx 0$, i.e., the constraint in equation 1.4 appears here as a secondary constraint. The extended hamiltonian is given by $H_E = \lambda \Phi + u \Phi + w p_\lambda$. The dynamics that do not involve the trivial degree of freedom $\lambda$ are thus described as in the earlier case.

The important features of the parametrized relativistic particle system are:

- It is a constrained system with a first class constraint.
• It is a parametrized system. The actions for such systems are not entirely "gauge", since they do not have gauge freedom at the boundaries.

• The constraint surface has a non-trivial topology—it is disconnected and non-compact—and the constraint cannot be made into a momentum by a canonical transformation.

When looking at this list it is useful to keep in mind the parametrized non-relativistic case. It is indeed possible to construct a parametrized theory of the non-relativistic particle, by making $t$ into a dynamical variable \[3, 4\]. The action is given by

$$S = \int_{\tau_i}^{\tau_f} d\tau \, L = \int_{\tau_i}^{\tau_f} d\tau \left( \frac{m}{2} \frac{\dot{x}^2}{\dot{t}} - iV(t, x) \right) \quad (1.7)$$

The constraint in this case is given by $p_t + R(t, x, p_x) \approx 0$, with $R(t, x, p_x) = p_x^2/(2m) + V(t, x)$. This system is not hard to quantize, although it shares the features in the first two items of the list with the relativistic case. The reason is that the constraint has a trivial topology and can be made into a momentum by a canonical transformation. It is well-known that all approaches to quantization of a constrained system are equivalent when it is possible to make the constraints into momenta by a canonical transformation. The last item in the list is the one which really makes things difficult for the relativistic particle, as we will see.

In all these aspects the relativistic particle is similar to string theory, mini-superspace and gravity, and we should understand this simple system before attempting to understand more complicated ones. For example, we will shown that when the electric field is divergence-less, unitary single-particle covariant quantization of the relativistic particle is possible. The result is not very surprising, but relevant may be relevant to quantum gravity. What are the analog conditions to "covariance" and an electric field with no divergence, if any, in quantum gravity?

Aside from this, the parametrized relativistic particle is the simplest non-trivial example in which to study the equivalence of the different quantization methods available today: reduced phase space quantization, Dirac quantization, BRST quantization, and (BRST)-Fock quantization. In fact, these methods are only well-defined for the simplest cases, and studying this system may indicate the proper way to their generalization.

Let us now look more closely at some of the more complicated systems with features in common with the parametrized relativistic particle. As mentioned, mini-superspace models are mathematically very similar to the relativistic particle in a curved background. As an example, consider the Robertson-Walker model described by the metric $ds^2 = -N^2(\tau)dt^2 + q(\tau)d\Omega_3^2$, where $d\Omega_3^2$ is the metric on the unit three-sphere \[4\]. The constraint that appears in this theory is \[4\] $4p^2 - q + 1 \approx 0$. Although similar to the constraint in \[1.4\] this constraint is actually more closely related to the one that appears in the parametrized non-relativistic particle: notice that the constraint surface is simply connected, and from this point of view
quantization should not be very difficult. More general mini-superspace models is

$$S_M = \frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \left( \frac{g_{AB} \dot{Q}^A \dot{Q}^B}{N} + NU(Q) \right), \quad (1.8)$$

where the signature of $g_{AB}$ is Lorentzian. This action is equivalent to $S'_M = -\frac{1}{2} \int_{\tau_i}^{\tau_f} d\tau \sqrt{U(Q)g_{AB} \dot{Q}^A \dot{Q}^B}$. In this homogeneous cosmological models the lapse $N$ is essentially the time-time component of the metric, and the parameter $Q^0$ characterizes the scale of the universe. The other parameters describe spatial anisotropies. The associated constraint in this model is

$$\Phi_M \equiv P_A P_B g^{AB} - U(Q) \approx 0, \quad (1.9)$$

which looks like the one in equation 1.4 with with a coordinate dependent mass.

The action for $(3+1)$-dimensional gravity is given by

$$S_H = \int d^4x \sqrt{\text{det} g} \left( R - 2\Lambda \right). \quad (1.10)$$

In the Arnowitt-Deser-Misner formalism it is assumed that the topology of space-time is of the form $\mathbb{R} \times \Sigma$, and with the metric expressed by $ds^2 = N^2 dt^2 - g_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$ the action becomes, in the canonical formalism ($g$ and $\hat{R}$ stand for the 3-geometry metric and curvature in the time slices, unless specified otherwise by a (4) superscript)

$$S_H = \int dt \int_{\Sigma} d^3x \left( \pi^{ij} \dot{g}_{ij} - N^i \mathcal{H}_i - N \mathcal{H} \right), \quad (1.11)$$

where the constraints are $\mathcal{H}_i, \mathcal{H} \approx 0$, with

$$\mathcal{H}_i = -2\nabla_j \pi^j_i, \quad \mathcal{H} = G_{ijkl} \pi^{ij} \pi^{kl} - \sqrt{g} (R - 2\Lambda), \quad (1.12)$$

and

$$G_{ijkl} = \frac{1}{2\sqrt{g}} (g_{ik} g_{jl} + g_{il} g_{jk} - g_{ij} g_{kl}), \quad (1.13)$$

and where $\pi^{ij}(x)$ is the momentum conjugate to $g_{ij}(x)$, $\{g_{ij}(x), \pi^{kl}(x')\} = \delta_i^k \delta_j^l \delta^3(x - x')$. The linear constraints generate the space diffeomorphisms, while the constraint $\mathcal{H} \approx 0$ generates the dynamics. The variable $N$ is called the lapse function, and is akin to $\lambda$ in the relativistic particle case. The linear constraints can be treated in the reduced phase space context, but the appearance of the quadratic constraint is much more difficult to treat. It is also possible in this case to write a “square-root” action which is very reminiscent of equation 1.14

$$S[g_{ab}, N^a] = \int d\tau \int_{\Sigma} d^3x \sqrt{g} R(U_{ab} U^{ab} - U^2), \quad (1.14)$$

where $U_{ab} \equiv \dot{g}_{ab} - N_{a:b} - N_{b:a}$. A simplified version of this theory is $(2+1)$-dimensional gravity. This is a remarkable system, because it is described by a finite number of degrees of freedom—the constraints eliminate almost all of them.
As discussed in [10, 11], in this case the different quantization schemes available seem to yield substantially different theories. For example [12, 13], it is possible, through a proper choice of “extrinsic time”, to reduce explicitly the theory classically and quantize the finite number of remaining degrees of freedom. The Chern-Simons approach to (2+1)-gravity is also of the “constraint then quantize” kind, and the equivalence of these methods has been established, at least for simple topologies [13]. On the other hand, the Wheeler-DeWitt approach [11], i.e., Dirac quantization, is much less understood.

These examples are meant to highlight the similarities of these systems and the parametrized relativistic particle, and convince the reader that quantization methods should first be tested and completely understood in such a simple theory. Some of the problems that may arise are likely to be already present in this toy model, making their solution more readily apparent.

We will begin by showing that reduced phase space quantization is not possible in general backgrounds, since it leads to the the square-root Schrödinger equation. The behavior of the square-root Schrödinger equation under the Lorentz group has already been studied by Sucher [14], who showed that in general it is not covariant. We will show, however, that this equation transforms properly under Lorentz boosts when the electric field is divergence-less. In the next section we will discuss full Dirac quantization of the relativistic particle, and define an inner product following some simple considerations. Then, we will discuss BRST quantization and define the BRST inner product. Finally, we will generalize and use (BRST)-Fock quantization.

2. Reduced phase space quantization

Reduced phase space quantization, a “constrain, then quantize” approach, provides a clear conceptual framework for quantization of those systems in which it is possible to make the constraints into momenta by a canonical transformation. Notice that if it is possible to perform such a canonical transformation then there exists a well-defined function in phase space that satisfies \( \{ \xi, \Phi \} = 1 \). In the case of the relativistic particle it is impossible to find such a function. The reader can check that in the free case \( \{ \xi, \Phi \} = 0 \) at \( p_\mu = 0 \), for any non-singular \( \xi \). As a consequence, in the reduction process we will encounter problems—we will end up with two classical theories instead of one. This, of course, is due to the disconnectedness of the constraint surface. For simplicity we analyze first the 2-dimensional free case.

To “reduce” the phase space we add another constraint to the system, \( \Upsilon \approx 0 \), the so-called gauge-fixing function, and work with the hamiltonian \( H_{E'} = v(\tau)\Phi + w(\tau)\Upsilon \). We will use here a \( \tau \)-dependent gauge, \( \Upsilon_f = t - f(\tau) \) (this type of gauge-fixing function is sometimes called “canonical”, because it does not involve lagrange multipliers). Next, we need to insure that the constraint and the gauge-fixing term are preserved under the dynamics. The condition \( \Phi \approx 0 \) implies \( w = 0 \), while \( \dot{\Upsilon}_f = \partial \Upsilon_f / \partial \tau + \{ \Upsilon_f, H_E \} \approx 0 \) implies \( v = \dot{f} / 2p_t \), so

\[
H_{E'} = \frac{\dot{f}}{2p_t} \Phi = \frac{\dot{f}}{2p_t} (p_t^2 - p_x^2 - m^2).
\]

(2.1)
The equations of motion in this gauge are \( \dot{t} = \dot{f}, \dot{x} = -\dot{f}p_x/p_t, \dot{p}_t = 0 \) and \( \dot{p}_x = 0 \). We can rewrite the constraint as \( p_t = \pm \sqrt{p_x^2 + m^2} \), and also rewrite

\[
\dot{x} = \pm \{x, f \sqrt{p_x^2 + m^2} \} = \pm f \{x, \sqrt{p_x^2 + m^2} \}. \tag{2.2}
\]

The system is effectively reduced to the coordinates \( x, p_x, p_t \), with a Hamiltonian \( H_f = \pm f \sqrt{p_x^2 + m^2} \). Notice that the Hamiltonian comes with two signs: thus propagation back and forth in time are both described in this system, depending on the initial conditions of the particle. If the initial conditions are such that the particle is in the positive branch of the constraint it will stay there, and evolve with the positive root Hamiltonian. Else, it will move back in time—it will be an anti-particle.

Equivalently, we can use a Dirac bracket formulation to analyze the reduced phase space. The bracket computation with this gauge is easily done, since \( \{\Upsilon_f, \Phi\} = 2p_t \):

\[
\{A, B\}_* = \{A, B\} - (\{A, \Upsilon_f\}, \{A, \Phi\}) \left( \begin{array}{cc} 0 & -\{\Upsilon_f, \Phi\}^{-1} \\ \{\Upsilon_f, \Phi\}^{-1} & 0 \end{array} \right) \left( \begin{array}{c} \{\Upsilon_f, B\} \\ \{\Phi, B\} \end{array} \right), \tag{2.3}
\]

and this yields

\[
\{x, p_x\}_* = 1, \{t, p_t\}_* = 0, \{x, p_t\}_* = \pm \{x, \sqrt{p_x^2 + m^2} \}, \tag{2.4}
\]

with all others zero. The appearance of the two signs in the bracket structure is a clear sign of trouble. The dynamics are given by

\[
\dot{F} = \frac{\partial F}{\partial \tau} + \frac{\partial \Upsilon}{\partial \tau} \{F, \Phi\}, \tag{2.5}
\]

It is important to note that equation 2.2 is gauge-dependent. This is because we have not selected a set of observables, i.e., phase space functions that have zero Poisson bracket with the constraint. Had we done so we would have found that their \( \tau \)-derivative is zero, since as observables they would also commute with the reduced phase space Hamiltonian, \( H_{f'} \). It helps to recall here that the action we start with, in equation 1.1, is not fully gauge-invariant, so it is not surprising that if we now insist on total gauge-invariance, we do not get back to the point that we started with. This is due to the way in which we have defined the problem—with gauge-dependent boundary conditions.

In the free parametrized non-relativistic case this point is especially transparent. Using a canonical transformation we can explicitly separate the set of physical coordinates, call them \( q, p \), from the gauge coordinates, \( Q, P \) (for notational simplicity we discuss the 2D case),

\[
Q = t - t_\lambda, \quad P = p_t + p_x^2/2m \quad (\text{gauge degrees of freedom})
\]

\[
q = p_x(t - t_\lambda) - mx, \quad p = -p_x/m \quad (\text{physical degrees of freedom}) \tag{2.6}
\]

After this canonical transformation, one might expect that the original action may be rewritten as \( S = \int_{t_1}^{t_f} d\tau (\dot{q}p + \dot{Q}P - \omega \Phi) \), but this is incorrect: a gauge-dependent
surface term (the generator of the canonical transformation) is missing. The boundary conditions used in the original action, \(x^\mu(\tau_{i,f}) = x^\mu_{i,f}\), do not involve the physical coordinates of the theory only: \(Q\) and \(-q/m - pQ\) are fixed at the boundaries. The above action does not have an extremum with such boundary conditions; a surface term needs to be added. Indeed, consider the variation of the action:

\[
\delta S = \int_{\tau_i}^{\tau_f} ((\dot{Q} - w)\delta P - \dot{P}\delta Q - P\delta w + q\delta p - \dot{p}\delta q) d\tau + (P\delta Q + p\delta q)\bigg|_{\tau_i}^{\tau_f} \tag{2.7}
\]

With boundary conditions on \(x, t\), the surviving surface term is

\[
(p\delta q)\big|_{\tau_i}^{\tau_f} = - p_x(t - t_\lambda) \frac{\delta p_x}{m} \bigg|_{\tau_i}^{\tau_f} \tag{2.8}
\]

To eliminate it we need to add a surface term to the action, \(B\big|_{\tau_i}^{\tau_f}\), with \((p\delta q + \delta B)\big|_{\tau_i}^{\tau_f} = 0\). A solution is \(B = (t - t_\lambda) \frac{p^2}{2m} = \frac{m}{2} Qp^2\). This term is the origin of the lack of gauge invariance at the boundaries comes from, as it depends on \(Q\). Let us summarize: there is nothing special about the action—or the constraint, as it is simple enough, \(P \approx 0\). However, our insistence on peculiar boundary conditions makes the addition of a gauge dependent boundary term necessary for the existence of an extremum (the generator of the time-dependent canonical transformation needed to go to constant-of-the-motion coordinates, the Principal Function), and gauge invariance at the boundaries is lost. This is not a serious problem, it is just a language problem. If we agree to use boundary condition using constant-of-the-motion coordinates we can leave the surface term out and work with a fully gauge invariant theory (with identical results). This is the action in reference [17].

The proper way to thing about \(t\) is as the coordinate of a reference object we call clock. Of course, this coordinate is pure gauge, an arbitrary function of \(\tau\). Nonetheless, it is not obligatory to make it disappear from the formalism. For instance, we are allowed to say “when the clock is at \(t = t_\lambda\), \(x = x_\lambda\)”, or to rewrite equation 2.2 in the form \(dx/dt = \pm \sqrt{p^2 + m^2}\). The rigorous way to implement this is to work only with observables, such as the ones in equation 2.6. We can recover the \(t\) coordinate in the un-parametrized formalism not through \(\tau\) or \(t\), but through \(t_\lambda\). To summarize: properly speaking, we should only use observables, and recover the concept of time through a shifting between these observables (using the \(p_t\) generator). Shifting between observables can be done by choosing different gauges if we do not restrict the theory to observables, which is what we have done here. Although this is uglier, it is how we started with in the action.

To quantize the reduced phase space we must now choose a sign for the hamiltonian (this choice is equivalent to a choice of a “branch” of the constraint at the beginning of the reduction process). The system is reduced to the spatial coordinates and momenta, and a Schrödinger equation with a square-root hamiltonian (e.g., choosing the negative root, and with the gauge \(t = \tau\)),

\[
i\partial_\theta \Psi(x^\mu) = H(A^\nu(x^\mu), \partial_\mu))\Psi(x^\mu), \tag{2.9}
\]

with \(H = \sqrt{D^\mu D_\mu + m^2} - A_0\), and \(D_\mu = \partial_\mu - iA_\mu\). This equation also appears in Dirac quantization (as the condition for physical states) if a branch of the constraint
is chosen. It is not hard to see that the choice of different gauges is equivalent to performing a \((\tau\text{-dependent})\) canonical transformation in the reduced phase space, or unitary transformations in the quantum theory (leading to different “pictures” of quantum mechanics) [15]. Let us see, however, that in the process of decomposing the constraint we have lost a symmetry that was present in the original action: covariance. How should this equation behave under Lorentz transformations? The wave-function must transform as a relative scalar 3-density of weight 1/2, if we are to construct a probability interpretation using the usual inner product. Thus, under a change of coordinates the wave-function changes by \(\Psi(x^\mu) \rightarrow \gamma^{1/2} \Psi(\Lambda^{-1}_\nu x^\nu)\), and we must check that under this change the wave-function still satisfies the Schrödinger equation. Given equation 2.9, we need

\[
i \Lambda^\mu_0 \frac{\partial}{\partial x^\mu} \Psi(x^\mu) = H(\Lambda^\mu_\nu A^\nu(x^\nu), \Lambda^\mu_\nu \partial_\mu) \Psi(x^\mu).
\] (2.10)

For simplicity, suppose first that \(\Psi(x^\mu)\) satisfies the free case of equation 2.9 \((A_\mu = 0)\). We have to check that

\[
i \Lambda^\mu_0 \frac{\partial}{\partial x^\mu} \Psi(x^\mu) = \sqrt{\Lambda^\mu_i \partial_\mu \Lambda^i_\nu \partial_\nu + m^2} \Psi(x^\mu).
\] (2.11)

Rewriting the right hand side, we obtain

\[
\sqrt{\Lambda^\mu_i \partial_\mu \Lambda^i_\nu \partial_\nu + m^2} \Psi(x^\mu) = \sqrt{\Lambda^\mu_i \partial_\mu \Lambda^i_\nu \partial_\nu - \Lambda^\mu_0 \partial_\mu \Lambda^0_\nu \partial_\nu + m^2} \Psi(x^\mu)
\]

\[
= \sqrt{-\Lambda^\mu_0 \partial_\mu \Lambda^0_\nu \partial_\nu} \Psi(x^\mu),
\] (2.12)

which is consistent. Notice that the crucial element in the derivation was that the square-root equation imply the Klein-Gordon equation,

\[
i \partial_0 \Psi(x^\mu) = \sqrt{\partial^i \partial_i + m^2} \Psi(x^\mu) \Rightarrow [\partial_\mu \partial^\mu + m^2] \Psi(x^\mu) = 0
\] (2.13)

which is true because

\[
\partial_\mu \partial^\mu + m^2 = \left(\partial_0 - \sqrt{\partial^i \partial_i + m^2}\right) \left(\partial_0 + \sqrt{\partial^i \partial_i + m^2}\right),
\] (2.14)

since \([\partial_0, \partial_i] = 0\).

Interaction with a background electro-magnetic field appears through the definition of covariant derivatives (minimal coupling), \(D_\mu = \partial_\mu - i A_\mu\),

\[
i D_0 \Psi(x^\mu) = \sqrt{D^i D_i + m^2} \Psi(x^\mu).
\] (2.15)

One can show [16] that this equation is \(U(1)\) gauge covariant, meaning that if \(\psi(x^\mu)\) is a solution and one makes the changes

\[
\psi(x^\mu) \rightarrow e^{-ie\Lambda(x^\mu)} \psi(x^\mu), \quad A_\mu \rightarrow A_\mu + \partial_\mu \Lambda,
\] (2.16)

the equation is still valid. In reference [14] it was shown that there exist electromagnetic fields for which the square-root equation does not transform properly. Let
us see what requirements the electro-magnetic must satisfy in order to preserve covariance. As before, we have to check that

\[
i \Lambda_0^\mu D_0 \Psi(x^\mu) = \sqrt{\Lambda_0^\mu \Lambda_0^\nu D_\nu} + m^2 \Psi(x^\mu) = \sqrt{D^2 + m^2} \Psi(x^\mu).
\]

A sufficient condition for consistency is that the electric field be divergence-less,

\[
[D_0, D_i D^i] = \nabla \cdot \vec{E} = 0,
\]

as we will now show. Notice that this is a covariant statement: if it holds in one frame, it holds in all of them, because what all it says is that there are no charges (but one can check this explicitly). If this condition is met the Klein-Gordon operator decouples,

\[
[D_0^\Lambda, D^\mu D_\mu] \phi(x) = [D_0^\Lambda, D^\mu D_\mu] \phi(x) = 0.
\]

These two facts now imply that

\[
i \Lambda_0^\mu D_0 \Psi(x^\mu) = \sqrt{\Lambda_0^\mu \Lambda_0^\nu D_\nu} \Psi(x^\mu).
\]

Thus, we conclude that the square-root Schrödinger equation is covariant if there are no charges. In general, however, we see that the process of reducing and then quantizing implies the loss of symmetries that were present in the original action.

### 3. Dirac quantization

As described in [3] we begin by quantizing all the variables in the theory, \( x^\mu \to \hat{x}^\mu, p_\mu \to \hat{p}_\mu \), with the commutator algebra \([\hat{x}^\mu, \hat{p}_\nu] = i g^{\mu \nu}\). The original extended Hilbert space can be described, for example, by

\[
I = \int d^4 x \langle x^\mu | x^\mu \rangle, \quad \langle x^\mu | y^\mu \rangle = \delta^4(x^\mu - y^\mu),
\]

and

\[
I = \int d^4 p \langle p_\mu | p_\mu \rangle, \quad \langle p_\mu | p_\nu \rangle = \delta^4(p_\mu - p_\nu),
\]

with the commutator algebra represented by \(|\psi \rangle \sim \langle x^\mu | \psi \rangle \equiv \psi(x^\mu)\) and \(\hat{x}^\mu \to x, \hat{p}_\mu \to -i \partial / \partial x^\mu\), which implies \(\langle x^\mu | p_\mu \rangle = A \exp(i x^\mu p_\mu)\). The physical “subspace” is defined by the Dirac condition

\[
\hat{\Phi} | \psi^D \rangle = 0,
\]
where some ordering for the constraint has been chosen. The physical states, then, are the solutions to the Klein-Gordon equation (in the context of quantum gravity this equation is known as the Wheeler-DeWitt equation). In the physical space, however, we cannot use the extended state space inner product,

\[
\langle \psi_D | \varphi_D \rangle = \int d^4x \left( \psi_D(x^\mu) \right)^* \varphi_D(x^\mu),
\]  

(3.4)

because it yields divergent norms for the physical states (unless the coordinate space is compact, a case which will not be considered here). Strictly speaking, physical states are not in the extended Hilbert space—the states in the original Hilbert space had finite norms. Thus, we need to redefine a finite inner product in the physical subspace, with the right properties:

A) it satisfies \( (\psi_a | \psi_b) )^* = (\psi_b | \psi_a) \),

B) it is invariant under changes of gauge-fixing,

C) it preserves the original hermiticity structure, and

E) \( (\psi | \psi) \geq 0, \psi = 0 \) only if \( |\psi| = 0 \).

The last requirement is needed to relate amplitudes with probabilities. For the case of a single constraint, this inner product is formally given by

\[
(\psi^D_a, \psi^D_b) = \int dV \left( \psi^D_a(x^\mu) \right)^* \delta(\Upsilon) \{ \Upsilon, \Phi \} \psi^D_b(x^\mu),
\]  

(3.5)

where \( \Upsilon \) is a gauge-fixing function, and the over-brace reminds us that underneath it there is an operator that may need ordering. As it stands, this is just a recipe that is well-defined only for the case of a constraint that can be made into a momentum by a canonical transformation \( \Phi = P \). In such a situation the meaning of this expression is rather simple: classically, since a momentum variable vanishes, say \( P \), its conjugated variable \( Q \) is pure gauge and can be ignored. The quantum version of this is that because of the Dirac condition the physical states do not depend on the coordinate variable in the coordinate representation, and therefore the inner product in coordinate space must not include an integration over \( Q \). The determinant insures that for reasonably chose gauge-fixing functions, the effective delta function is precisely \( \delta(Q - Q_0) \), for some irrelevant constant \( Q_0 \). Notice that, strictly speaking, the absolute value of the determinant should be used, and the resulting inner product would then be positive-definite.

When the constraint cannot be made into a momentum—as it happens in the relativistic particle case—this recipe cannot be interpreted in such simple terms, and is must be taken as a formal recipe that needs to be carefully implemented. We intend to do this now.

An immediate observation is that the operator \( \delta(\Upsilon) \{ \Upsilon, \Phi \} \) should be hermitean in the original inner product. Hermicity of this operator in the original inner product

*This can always be done locally, but in general not globally.
is essential if the “reduced” inner product is to satisfy \((\psi_a, \psi_b)^* = (\psi_b, \psi_a)\), as we wrote above, and this will ensure that states have real norms,
\[
\| \psi \|^2 = (\psi, \psi) = [(\psi, \psi)]^* \in \mathbb{R} \quad (3.6)
\]
This is certainly needed if we are to make contact with the classical world through expectation values of hermitean observables.

Although there are other possibilities, we will use the following prescription for hermicity,
\[
\delta(\hat{Y}) = \frac{1}{2} \left( \delta(\hat{Y}) \{ \hat{Y}, \hat{\Phi} \} + \{ \hat{Y}, \hat{\Phi} \} \delta(\hat{Y}^\dagger) \right). \quad (3.7)
\]
In the case of the free relativistic particle, using \(Y = t - f(\tau)\), this simple inner product prescription translates into
\[
(\psi_a^D, \psi_b^D) = \int dV \ (\psi_a^D)^* \left( \delta(t - f(\tau)) \hat{p}_t + \hat{p}_t^\dagger \delta(t - f(\tau)) \right) \psi_b^D. \quad (3.8)
\]
Now, in the coordinate representation \(\hat{p}_t = -i \frac{d}{dt} = i \frac{d}{dt} = \hat{p}_t^\dagger\), because (the delta function takes care of any boundary condition problems at \(t = \pm \infty\)). After integrating this delta function, we recover the Klein-Gordon inner product,
\[
(\psi_a^D, \psi_b^D) = \int d^3x \ (\psi_a^D(t, x))^* \left( -i \frac{\hat{d}}{dt} + i \frac{\hat{d}}{dt} \right) \psi_b^D(t, x) \bigg|_{t = f(\tau)}, \quad (3.9)
\]
which is real, as promised. In the interacting case, this reasoning leads to the analogous result, with covariant derivatives replacing the regular ones, since the Poisson bracket of the constraint with the gauge-fixing term is
\[
\{ t, \Phi \} = 2 g^{0\nu} (p_\nu - A_\nu). \quad (3.10)
\]
The symmetrized inner product measure is then
\[
\delta(\hat{t} - f(\tau)) \hat{g}^{00}(\hat{p}_\mu - \hat{A}_\mu) + \{ \hat{g}^{00}(\hat{p}_\mu - \hat{A}_\mu) \}^\dagger \delta(\hat{t} - f(\tau)), \quad (3.11)
\]
which leads precisely to the Klein-Gordon inner product in a curved spacetime
\[
(\psi_1, \psi_2) = \int d^3x \sqrt{|g|} g^{0\nu} (\psi_1^* D_\nu \psi_2 - \psi_2^* D_\nu \psi_1). \quad (3.12)
\]
This inner product is related to the \(U(1)\) Nöether current conservation law of the action \(I = \int d^4x \sqrt{|g|} (g^{ab} D_a \psi D_b \psi^* + m^2 \psi \psi^*)\) which is just \(\nabla_a (\psi D^a \psi^* - \psi^* D^a \psi) = 0\).

Another feature of Dirac quantization is that observables in the theory are defined by requiring that they commute with the constraint. For instance, in the case of the parametrized non-relativistic particle, this means that observables satisfy the Heisenberg equation of motion with the “wrong” sign, \(i \frac{\partial \hat{Q}}{\partial t} + [\hat{O}, \hat{H}] = 0\), i.e., they are constants of the motion. The result is that the expectation value of
observables between physical states is $\tau$-independent, and $\tau$ disappears from the formalism. This, on the other hand, was expected: in this formalism “time” is a gauge degree of freedom, and everything in Dirac quantization has been designed to eliminate gauge-dependence. Recall that this feature was also present in the classical reduced phase space discussion. There it was pointed out that to make direct contact with the un-parametrized case the restriction to observables must not be made. This is also true here. However, if the condition for observables is imposed, it is certainly still possible to recover the concept of time. For example, in the parametrized non-relativistic case, the observables associated to position are $\hat{x}_\lambda = \exp(-i\hat{H}(t - t_\lambda))\hat{x}\exp(i\hat{H}(t - t_\lambda))$, and the expectation value between Dirac states of this quantity depends on $t_\lambda$—and not on gauge-fixing. The propagator in this formalism is obtained through the expectation value of eigenstates of $\hat{x}_\lambda$ and $\hat{x}_{\lambda'}$, and not as the coordinate representation of an evolution operator—there really is no time or time-evolution. The same results can be obtained by a trick: to simulate time, make things gauge-dependent, which is why parametrized actions are gauge-dependent.

4. BRST quantization

In this section we will derive the inner product for the relativistic particle from the BRST quantization approach. We will first discuss the correct BRST state space: this results from a derivation of the BRST cohomology using states with well-defined inner products. Once we have this state space, we will select its zero ghost subspace (which consists of two sectors), since zero ghost states are the ones related to expectation values of physical observables. Using these zero ghost states we will then write the expectation value of an operator that acts as a propagator, which leads to the BFV path integral. We then argue that the above expectation value should be used to define an inner product in the zero ghost sectors, and obtain the inner product in each sector.

In BRST quantization the constraints are automatically implemented together with the simplectic structure of the reduced phase space in the theory. The Jacobian factors which must appear in the definitions of brackets or within path integrals, are automatically produced by the formalism through the ghosts.

Recall the following from the classical BRST treatment of a system with a single constraint $\Phi$ [4]. The first objects that are introduced into the extended phase space—if not already present—are the multiplier $\lambda$ and its conjugate momentum $p_\lambda$, i.e., $\{\lambda, p_\lambda\} = 1$. Since $\lambda$ is an arbitrary function of $\tau$, its momentum is constrained, $p_\lambda \approx 0$. We thus have two constraints. To each constraint, the rules say we must associate a canonically conjugate pair of ghosts, $\eta_0, \rho_0$ and $\eta_1, \rho_1$ with $\{\eta_0, \rho_0\} = 1$ and $\{\eta_1, \rho_1\} = 1$—and with the other (super)brackets vanishing (ghosts are odd Grassmann variables [4], e.g., $\eta_0^2 = 0$). The total extended phase space is thus described by $x^\mu, p_\mu, \lambda, p_\lambda, \eta_0, \rho_0, \eta_1, \rho_1$ . The generator of gauge transformations in extended phase space is the BRST generator,

$$\Omega = \eta_0 \Phi + \eta_1 p_\lambda,$$

(4.1)
which has the crucial property \( \{ \Omega, \Omega \} = 0 \). This property encodes the constraint algebra. The dynamics are then generated by the hamiltonian \( \mathcal{H} = h + \{ \mathcal{O}, \Omega \} \) (which reduces to \( \{ \mathcal{O}, \Omega \} \) when \( h = 0 \)), where \( \mathcal{O} \) is a gauge-fixing function. In the case of the relativistic particle, one can check that the reduced phase space equations of motion are correctly produced by the formalism, without having to worry about the dynamics of the constraints as in the reduced phase space formalism.

In the transition to the quantum theory, phase space coordinates become operators, and the (super)Poisson bracket structure is translated into the (super)commutator language, \( \{ A, B \} \rightarrow i\hbar[A, B] \). In this case we have both commutators and anti-commutators. For example, have

\[ [\hat{\Omega}, \hat{\Omega}] = \hat{\Omega}^2 = 0. \] (4.2)

In the parametrized relativistic particle case the state space \( \{|\Psi\rangle\} \) is spanned by the basis \(|x^\mu, \eta_0, \eta_1\rangle\), for example. In the coordinate representation we have

\[ \langle x^\mu, \eta_0, \eta_1 | \Psi \rangle \equiv \Psi(x^\mu, \eta_0, \eta_1) \]
\[ = \psi(x^\mu, \lambda) + \psi^0(x^\mu, \lambda)\eta_0 + \psi^1(x^\mu, \lambda)\eta_1 + \psi^{01}(x^\mu, \lambda)\eta_0\eta_1. \] (4.3)

The inner product in this (extended) space is given by

\[ (\Sigma, \Psi) \equiv \int dtdxd\lambda d\eta_0 d\eta_1 (\Sigma(z^A))^* \Psi(z^A). \] (4.4)

The BRST physical space is defined the BRST generator \( \hat{\Omega} \) properties:

a) \( \hat{\Omega}^\dagger = \hat{\Omega} \), and
b) \( \hat{\Omega}^2 = 0 \),

which it inherits from the classical description: \( \Omega \) is real, and \( \{ \Omega, \Omega \} = 0 \) [4, 18].

The definition consists of two items:

i) the BRST physical condition

\[ \hat{\Omega}|\Psi\rangle_{ph} \equiv 0, \] (4.5)

ii) and the concept of BRST cohomology: we identify

\[ |\Psi\rangle_{ph} \sim |\Psi\rangle_{ph} + \hat{\Omega}|\Delta\rangle, \] (4.6)

since the state \( \hat{\Omega}|\Delta\rangle \) is physical \( (\hat{\Omega}^2 = 0) \), and has zero inner product with any physical state (since \( \hat{\Omega}^\dagger = \hat{\Omega} \), and \( \hat{\Omega}|\Psi\rangle_{ph} = 0 \)). Such states are called null states.

The hamiltonian is given by \( \hat{\mathcal{H}} = \{ \mathcal{O}, \hat{\Omega} \} \), for some gauge-fixing operator \( \mathcal{O} \), and we might guess that it has no effect on physical amplitudes due to the two conditions above. We will see that this is also false.
There are many solutions to the BRST condition, \( \hat{\Omega} |\Psi \rangle \equiv 0 \). Consider for example the zero-ghost number BRST physical states

\[
|\Psi_\Upsilon \rangle = |\psi_{\Upsilon=0}, p_\lambda = \eta_0 = 0 \rangle \sim \psi_0^0, \eta_0, \\
|\Psi_\Phi \rangle = |\psi_{\Phi=0}, \lambda = \rho_0 = \eta_1 = 0 \rangle \sim \psi_1^1, \eta_1.
\]

(4.7)

We use a notation in which \( \psi_{p_\lambda = 0} \), for example, is shorthand for a state that satisfies \( \hat{p}_\lambda \psi = 0 \). These are the BRST-invariant states which are used in the definition of the boundary conditions of the BFV path integral, needed in the Fradkin-Vilkovisky theorem \[4, 18\]. In the coordinate representation, the BRST equation reads

\[
\hat{\Omega} \Psi = (\hat{\eta}_0 \hat{\Phi} + \hat{\eta}_1 \hat{p}_\lambda) \left( \psi + \psi^1 \eta_1 + \psi^0 \eta_0 + \psi^{01} \eta_0 \eta_1 \right) \\
= \hat{\Phi} \psi \eta_0 + \hat{p}_\lambda \psi \eta_1 + \left( \hat{\Phi} \psi^1 - \hat{p}_\lambda \psi^0 \right) \eta_0 \eta_1 \\
= 0.
\]

(4.8)

The general solution to this equation has the components \( \psi = \psi_{p_\lambda = 0}, \psi^1 = \hat{p}_\lambda \varphi + \psi_{p_\lambda = 0}^1, \psi^0 = \hat{\Phi} \varphi + \psi_{p_\lambda = 0}^0 \) and \( \psi^{01} = \text{free} \), with \( \varphi(x^\mu, \lambda) \) arbitrary.

Let us review the logic in the construction of the BRST cohomology. The standard argument is that since \( \hat{\Omega}^2 = 0 \) we must make the identification \( |\Psi \rangle \sim |\Psi \rangle |\Lambda \rangle \). The reasoning is that the states \( \Omega |\Lambda \rangle \) are physical (which is clearly true), and that they decouple from other physical states. This, however, is not true in general, because in the physical space the inner product is not well-behaved. That is, the above argument is based on the assumption that \( \langle \Lambda |\hat{\Omega}^\dagger |\Phi \rangle = 0 \), and this puts restrictions on what \( |\Lambda \rangle \) can be, since in general \( \Omega |\Lambda \rangle \) is not null. The problem can be traced to the following. In BRST quantization one is eventually trying to work with state spaces defined through a condition similar to \( \hat{P} |\Psi \rangle = 0 \), which is selecting something like a zero momentum eigenstate. It is also desired that operators such as \( \hat{P} \) and \( \hat{Q} \) be hermitean in the state space, where the physical states must have finite norms. Finally, the algebra must be preserved, \( \left[ \hat{Q}, \hat{P} \right] = i \). It is not too hard to see that it is not possible to satisfy all these conditions at once. For instance, the quantity \( \langle P = 0 |[\hat{Q}, \hat{P}] |P = 0 \rangle \) is undefined if all the conditions are met. BRST gets around this problem by the use of different sectors in the state space. Instead of having a single state space, there are always pairs of state spaces, and each has its dual, as we will see. By dual of a sector we mean that part of the state space that is not orthogonal to the sector. This is governed by the ghost component of each sector: to obtain a nonzero inner product a \( \eta_0 \eta_1 \) combination is needed—see equation \[4.4\].

For this reason it is incorrect to assume, for example, that one can express a state in the form \( |\Psi_{\Omega=0} \rangle = \psi_{p_\lambda = 0} + \psi_{p_\lambda = 0}^1 \eta_1 + \psi_{p_\lambda = 0}^0 \eta_0 + \psi_{p_\lambda = 0}^{01} \eta_0 \eta_1 \) by a proper choice of \( |\Lambda \rangle \), since these states have an ill-defined norm. However, because all the states in the physical space must have finite norms, it is possible, by the addition
of a null state, to rewrite this solution in the form

$$\Psi_{\Omega = 0} = \rho_{\Phi = 0}^1 + \rho_{\Phi = 0}^0 \eta_1 + \rho_{\Phi = 0}^0 \eta_0 + \rho_{\Phi = 0}^1 \eta_0$$

(4.9)

where $\gamma = \gamma(t)$ is some canonical gauge-fixing, and $\rho = \rho(x^\mu)$. Notice that this state has a well defined norm, as it should, and that in each sector a constraint and a gauge-fixing term are satisfied, while in the dual sectors the roles of the multiplier and original constraint are reversed. Because of this, these states have finite norms. This components include as particular cases the states in equation 4.7—the boundary states in the BFV path integral.

Let us summarize the above discussion with the following statement: let $F$ be the space of states of finite norm; any BRST-invariant state in $F$ can be brought into the form of equation 4.3 by the addition of a null state in $F$.

### 4.1. Inner product in the zero ghost sector

Although we have an inner product in the physical state space, we would like to avoid the use of dual state spaces, and define the theory within a single state sector with a unique Hilbert space. The zero-ghost sectors are associated with the physical observables—which have zero ghost number—so they are the natural candidates for this task. To achieve this goal we need an operator that will map a state in a given sector to one in the dual sector. The operator $\exp(\hat{K}, \hat{\Omega})$ has the right characteristics, as we now explain. Any definition of inner product in BRST theory must take into account the existence of the BFV path integral in phase space [4,18].

The path integral carries information about the states, the inner product, and the hamiltonian. As we discuss below, it is easy to write the physical amplitude in terms of a path integral in phase space. The concept of physical state space enters in the boundary conditions, which arise from the required BRST invariance of the “end” states in the amplitude. This provides the interpretation for the usual BRST boundary conditions: they can be understood in the context of the cohomology of BRST-invariant states. States that implement such boundary conditions belong to the cohomology of the BRST generator. For instance, in the relativistic particle case we can derive the path integral expression for the propagation amplitude from the Hilbert space expression

$$U(t_i, x_i, t_f, x_f) \equiv \langle t_f, x_f, \eta_0 = \rho_1 = p_\lambda = 0 | \hat{U} | t_i, x_i, \eta_0 = \rho_1 = p_\lambda = 0 \rangle$$

(4.10)

by using the states in the zero-ghost cohomology sectors and the propagation operator

$$\hat{U} = e^{-i\Delta\tau \hat{H}}$$

(4.11)

where $\hat{H}$ is the extended super-hamiltonian: $\hat{H} \equiv \{ \hat{O}, \hat{\Omega} \}$. Our notation anticipates that this amplitude will not depend on $\Delta\tau$, which will be the case. To obtain the path integral expression rewrite $(-i\Delta\tau \hat{H} = \Delta\tau [\hat{O}, \hat{\Omega}])$

$$e^{i\Delta\tau [\hat{O}, \hat{\Omega}]} = \lim_{N \to \infty} \left( 1 + \frac{\Delta\tau}{N} [\hat{O}, \hat{\Omega}] \right)^N$$

(4.12)
and, at each $\tau$-division, insert the resolution of the identity in extended phase space

$$
\hat{I} = \int dt dx dp \rho_0 d\rho \ |t, x, p, \eta_0, \rho_0 \rangle \langle t, x, p, \eta_0, \rho_1 |
= \int dp_t dp_x d\lambda \rho_0 d\eta_1 \ |p_t, p_x, \lambda, \rho_0, \eta_1 \rangle \langle p_t, p_x, \lambda, \rho, \eta_1 |
$$

and the projections

$$
\langle t, x, p, \eta_0, \rho_1 | p_t, p_x, \lambda, \rho_0, \eta_1 \rangle = e^{i(t p_t + x p_x + p_\lambda + \eta_0 \rho_0 + \rho_1 \eta_1)},
$$

just as in the unconstrained case.

To be more specific, let us consider the following two types of gauge-fixing terms:

a) **Non-Canonical**: $O_{NC} = \rho_0 \lambda$, which yields

$$
\{ O_{NC}, \Omega \} = \lambda \Phi + \rho_0 \eta_1.
$$

The terminology refers to the fact that the gauge-fixing is carried through the lagrange multiplier.

b) **Canonical**: $O_C = \rho_1 \Upsilon$, where $\Upsilon = t - f(\tau)$. Here

$$
\{ O_C, \Omega \} = \rho_1 \eta_0 \{ \Upsilon, \Phi \} + p_\lambda \Upsilon,
$$

and the resulting gauge-fixing is essentially $\Upsilon = 0$.

Notice that the sum of these two terms is an “anti”-BRST charge, $O_{NC} + O_C = \rho_0 \lambda + \rho_1 \Upsilon$, the result of replacing each coordinate in the BRST generator by its phase space conjugate—with the exception of the quadratic constraint, which has as conjugate the gauge-fixing term. Now, formally

$$
\langle \Psi_a | e^{[\hat{O}_{NC}, \hat{\Omega}]} | \Psi_b \rangle = \langle \Psi_a | \Psi_b \rangle
$$

for physical states—states annihilated by the BRST generator—and this is where the invariance of the amplitude under changes in gauge-fixing is expected to come from. However, this is a formal statement, because the inner product needs regularization. For example, take the BRST invariant states $| \Psi_\phi \rangle$ of equation 4.7. Then

$$
\langle \Psi_\phi | e^{[\hat{O}_C, \hat{\Omega}]} | \Psi_\phi' \rangle =
$$

$$
\langle \psi_{\phi=0}, \lambda = \rho_0 = \eta_1 = 0 | \exp \left( \left( \hat{\rho} \hat{\eta}_0 [\hat{\Upsilon}, \hat{\Phi}] + i \hat{p}_\lambda \hat{\Upsilon} \right) \right) | \psi'_{\phi=0}, \lambda = \rho_0 = \eta_1 = 0 \rangle
$$

which, up to ordering questions, we can guess will turn out to be $\langle \psi_{\phi=0} | \overline{\langle \Upsilon, \Phi \rangle} \delta(\Upsilon) | \psi'_{\phi=0} \rangle$ i.e., the Klein-Gordon inner product. Why? The ghost integrations should just yield the determinant $\overline{\langle \Upsilon, \Phi \rangle}$, and the multiplier degree of freedom the delta function. Ordering questions are important, however, so let us proceed carefully. The amplitude is

$$
\langle \psi_a(t, x), \Phi = 0 = \lambda = \eta_1 = \rho_0 | \exp \left( \hat{\rho} \hat{\eta}_0 [\hat{\Upsilon}, \hat{\Phi}] + i \hat{p}_\lambda \hat{\Upsilon} \right) | \psi_b(t, x), \Phi = 0 = \lambda = \eta_1 = \rho_0 \rangle
$$
\[ = \int d^4x \psi^*_a(t, x) \left[ \int dp_\lambda dp_1 dp_0 \exp \left( \rho_1 \eta_0 [\hat{\Phi}, \hat{\Phi}] + i p_\lambda \hat{\Phi} \right) \right] \psi_b(t, x), \quad (4.19) \]

where in the free case \([\hat{\Phi}, \hat{\Phi}] = [\hat{t}, \hat{p}^2] = i2\hat{p}_t\). To compute this integral we make use of the Campbell-Baker-Hausdorff theorem (see, for example, reference [19]). That is

\[ \ln(e^A e^B) = A + B + \frac{1}{2} [A, B] + \frac{1}{12} [A, [A, B]] - \frac{1}{12} [B, [B, A]] + ... = \quad (4.20) \]

Fortunately the series ends quickly here, and we have

\[ \exp \left( \rho_1 \eta_0 [\hat{\Phi}, \hat{\Phi}] + i p_\lambda \hat{\Phi} \right) = \exp \left( \rho_1 \eta_0 [\hat{\Phi}, \hat{\Phi}] - i \rho_1 \eta_0 \rho_\lambda \right) \exp \left( i p_\lambda \hat{\Phi} \right) = \left( 1 + \rho_1 \eta_0 [\hat{\Phi}, \hat{\Phi}] - i \rho_1 \eta_0 \rho_\lambda \right) \exp \left( i p_\lambda \hat{\Phi} \right). \quad (4.21) \]

The ghost integrations are now easily done, and yield

\[ \int dp_\lambda dp_1 dp_0 \exp \left( \rho_1 \eta_0 [\hat{\Phi}, \hat{\Phi}] + i p_\lambda \hat{\Phi} \right) = i\hat{p}_t \delta(\hat{\Phi}) + i\delta(\hat{\Phi}) \hat{p}_t. \quad (4.22) \]

This is, up to a factor of \( i \), the Klein-Gordon inner product, since

\[ \langle \psi_a | \delta(\hat{t} - f(\tau)) \hat{p}_t | \psi_b \rangle = \int dx dt \langle \psi_a | x, t \rangle \langle x, t | \delta(\hat{t} - f(\tau)) \hat{p}_t | \psi_b \rangle \quad (4.23) \]

which is just

\[ \int dx dt \delta(t - f(\tau)) \langle \psi_a | x, t \rangle \left( -i \frac{\partial}{\partial t} \right) \langle x, t | \psi_b \rangle \quad (4.24) \]

Notice that this inner product involves the hermitized expression of the determinant operator—as discussed in the previous section. It does not yield an absolute value or anything else.

With an electro-magnetic background the result is again immediate:

\[ \int dp_\lambda dp_1 dp_0 \exp \left( \rho_1 \eta_0 [\hat{\Phi}, \hat{\Phi}] + i p_\lambda \hat{\Phi} \right) = i\hat{\Pi}_0 \delta(\hat{\Phi}) + i\delta(\hat{\Phi}) \hat{\Pi}_0 \quad (4.25) \]

because \([\hat{t}, \hat{\Phi}_E] = 2i\hat{\Pi}_0\), and \([\hat{t}, [\hat{t}, \hat{\Phi}_E]] = -2\), which is again a multiple of the identity and commutes, and the series terminates. Is this true for the general interacting case? We can compute

\[ [\hat{t}, \hat{\Phi}] = 2i \hat{g}^{00} (\hat{p}_\mu - \hat{\Lambda}_\mu) + \nabla_\nu \hat{g}^{0\nu} = 2i \hat{g}^{0 \mu} (\hat{p}_\mu - \hat{\Lambda}_\mu) \quad (4.26) \]

and \([\hat{t}, [\hat{t}, \hat{\Phi}]] = -2\hat{g}^{00}\), and again we need \([\hat{g}^{00}, \hat{g}^{0 \mu} (\hat{p}_\mu - \hat{\Lambda}_\mu)] = 0\). This holds provided \(\hat{g}^{0 \nu} [\hat{p}_\nu, \hat{g}^{00}] = 0\). Notice that a natural (covariant) ordering of the constraint operator is assumed.

We should also study the other zero-ghost sector, with the states \(|\Psi_T\rangle\), which satisfy \(\hat{p}_\lambda, \hat{\eta}_0, \hat{\rho}_1 |\Psi_T\rangle_{\rho_\lambda = \rho_0 = 0} = 0\). These are not Dirac states, and appear in the path integrals we evaluate in the particle case [20, 21]. It is easy to see, as we
now show, that using the non-canonical gauge-fixing, the amplitude (from which it is again easy to reach the BFV path integral) becomes
\[
\langle \Psi | e^{[\hat{O}_{NC},\hat{\Omega}]} | \psi' \rangle = \langle \psi \delta(\hat{\Phi}) | \psi' \rangle.
\] (4.27)
i.e., the Hadamard, or on-shell, amplitude:
\[
(\psi,\psi') \equiv \langle \psi \delta(\hat{\Phi}) | \psi' \rangle.
\] (4.28)
Notice that the full-range of the multiplier \(\lambda\) has been used, since in this representation this variable is fully ranged. This is a quite remarkable result: neither the initial nor the final states are physical in the Dirac sense, and this object has no natural interpretation as a physical amplitude within the Dirac approach. It appears naturally in the BRST approach. We will next see how this amplitude appears in (BRST)-Fock quantization as well. For now, notice that this inner product has a lot of nice properties (as long as \(\Phi\) is hermitean), including positivity.

5. Dirac-Fock quantization

We review first the situation for simple constraints [4], i.e., constraints can be canonically transformed to momenta, \(P_1 \approx 0 \approx P_2\). The reason is that this approach is really unknown territory—the prescription is clear only for the simple class of situations in which the constraints can be made into momenta by canonical transformations. To begin with, an even number of constraints is required. Let us state why immediately: we will, in essence, pair the constraints and assign to each combination opposite sign norm states so that their effect in the theory cancels after we select the physical space. Thus, gauge–degrees of freedom effectively disappear from the theory. The key new ingredient here is the appearance of states with negative norms.

Fock quantization begins with the definition of the operators
\[
\hat{a} = \hat{P}_1 + i\hat{P}_2, \quad \hat{a}^\dagger = \hat{P}_1 - i\hat{P}_2,
\] (5.1)
and
\[
\hat{b} = -\frac{i}{2}(\hat{Q}^1 + i\hat{Q}^2), \quad \hat{b}^\dagger = \frac{i}{2}(\hat{Q}^1 - i\hat{Q}^2).
\] (5.2)
The commutation relation that follow from these definitions are
\[
[\hat{a}, \hat{b}^\dagger] = [\hat{b}, \hat{a}] = 1
\] (5.3)
and the rest zero. Notice that it is implied by the notation here that both \(\hat{P}_1\) and \(\hat{P}_2\) are hermitean. For example, \(\hat{a} + \hat{a}^\dagger\) is hermitean, and is equal to \(2\hat{P}_1\). This fact is
crucial for the development of the formalism, and is a subtle assumption—it selects an indefinite inner product when we define the vacuum.

The states on this space are defined by the following construction:

a) It is assumed that there is a “vacuum” state, $|0\rangle$, satisfying the conditions

$$\hat{a}|0\rangle = \hat{b}|0\rangle = 0$$

b) This state is also assumed to have unit norm, $\langle 0|0 \rangle = 1$.

c) The rest of these states are defined by acting on the “vacuum” above with the creation operators.

Before we work out some of the consequences of this prescription, let us look at what it is doing in terms of the $\hat{P}_i$ operators. What is the vacuum? We need a state that satisfies

$$(\hat{P}_1 + i \hat{P}_2)|0\rangle = -\frac{i}{2} (\hat{Q}^1 + i \hat{Q}^2)|0\rangle = 0 \quad (5.4)$$

In a standard Hilbert space there is no such state! The operators $P_1, P_2, Q^1, Q^2$ are assumed to be hermitean, yet if we look at the definition of the vacuum we notice that at some of them need to have imaginary eigenvalues. How can this be? It is not hard to see that an indefinite inner product is needed. For example [4], consider the states $(\hat{a}^\dagger + \hat{b}^\dagger)|0\rangle$, $(\hat{a}^\dagger - \hat{b}^\dagger)|0\rangle$, and $\hat{a}^\dagger|0\rangle$. They have positive, negative, and zero norm respectively. The Fock Hilbert space is not a positive definite inner product space.

Next, we need to impose the condition for physical states. The constraints above are equivalent classically to demanding that $a \approx 0 \approx a^\dagger$. However, we cannot demand this condition from the states: there is no state in our construction satisfying $\hat{a}^\dagger|\psi\rangle = 0$! We can only demand that the physical states satisfy

$$\hat{a}|\psi_F\rangle = 0. \quad (5.5)$$

These states have a well defined norm: the physical space is a true subspace of the original Hilbert space. We do not need to redefine the inner product. Thus, Fock quantization is superior to Dirac quantization in this respect: in Dirac quantization one needs to redefine an inner product in the physical subspace. Also, there is no need for gauge-fixing.

What happened to the other “half” of the constraint? Although the physical states do not satisfy the Dirac condition, the expectation value of the constraints between physical states is always zero. Moreover, we will now see that the physical Fock state space is reduced to a space isomorphic to the Dirac states—the space of states that satisfy the constraints in Dirac quantization, which in this simple example are given by the unique state, $|P_1 = P_2 = 0\rangle$. This is because a physical state in the Fock space is either the vacuum or a linear combination of the vacuum and a physical state that has zero inner product with all the physical states. Indeed, the vacuum is a physical state and the other physical states are given by the so-called null states, $\hat{a}^\dagger n|0\rangle$, since $[\hat{a}, \hat{a}^\dagger] = 0$. These decouple from the physical states,
since $\hat{a}^\dagger \equiv (\hat{a})^\dagger$. Null states are, by definition, physical states with zero norm and zero inner product with any other physical state. Thus, we are left over with a single state, just as in the Dirac approach. What has happened is that the two degrees of freedom have combined and have annihilated each other. This is reflected by the appearance of the null states, which in turn is a consequence of the fact that we have quantized the system with an indefinite inner product (a change of variables will decouple the algebra into a commutator with the “wrong” sign, $[\hat{Q}^\pm, \hat{Q}^\dagger \mp] = -1$, and a regular one $[\hat{Q}^\pm, \hat{Q}^\dagger \pm] = +1$)

How can we generalize the above formalism to more complicated situations? Let us study the case of the relativistic particle. To proceed with Fock quantization we need two constraints: the original constraint, $\Phi \approx 0$, and another one $P_\lambda \approx 0$. Thus, the form of the action which includes the lagrange multiplier is a more natural starting point for Fock quantization. The full state space will be defined using

$$\hat{a} = \hat{p}_t + i\hat{p}_\lambda$$
$$\hat{b} = -\frac{i}{2}(\hat{t} + i\hat{\lambda}) \quad (5.6)$$

and hermitean conjugates, and the usual definition of the vacuum. As long as we use canonical pairs in this definitions, we will obtain the right commutation properties of the creation and annihilation operator. For example, this definition could be changed using a canonical transformation.

The strategy will be as follows:

I) Find an operator $\hat{M}$ such that we can write the constraint as a linear combination of this operator and its hermitean conjugate, for instance

$$\hat{\Phi} = \hat{M} + \hat{M}^\dagger \quad (5.7)$$

Something similar should happen to the other constraint, $P_\lambda \approx 0$. This we will call a splitting.

II) Define the physical states by $\hat{M} |\Psi\rangle = 0$. This will ensure that the expectation value of the constraints in the physical space is zero,

$$p_h \langle \Psi | \hat{\Phi} |\Psi\rangle_{p_h} = p_h \langle \Psi | \hat{P}_2 |\Psi\rangle_{p_h} = 0.$$  

III) The splitting must be such that $\hat{M}$ be a \textit{weakly normal} operator, $[\hat{M}, \hat{M}^\dagger] = \hat{a} \hat{M}$. This insures that the states generated by $\hat{M}^\dagger$ are null (i.e., physical states that have zero inner product with any physical state).

Let us now consider the simple quadratic constraint $\hat{\Phi} = \hat{P}_1^2 - \hat{A}^2$, where we assume that $\hat{A}$ is a hermitean operator that commutes with $\hat{a}, \hat{b}$—in the particle case this corresponds to an electro-magnetic background with zero electric field, for example: $P_1 = p_t$ and $A^2 = \Pi^2 + m^2$. Now rewrite the constraint in terms of the new variables,

$$\hat{\Phi} = \hat{P}_1^2 - \hat{A}^2 = (\hat{a} + \hat{a}^\dagger)^2 - \hat{A}^2 = \frac{1}{4} (\hat{a}^2 + \hat{a}^\dagger)^2 - \hat{A}^2. \quad (5.8)$$

\footnote{It is actually sufficient to be able to express the original constraint as a linear combination of $M$ and $M^\dagger$. See below.}
A natural first “splitting” guess would be \( \hat{M} = \frac{1}{4} \left( \hat{a}^2 + \hat{a}^{\dagger 2} \right) - \frac{1}{2} \hat{A}^2 \). Notice, however, that this definition implies that
\[
P_h \langle \Psi \mid (\hat{M} - \hat{M}^{\dagger}) \mid \Psi \rangle_{P_h} = 0,
\]
which is not quite what we need. So how do we perform the “splitting” of the constraint? Although it is not too hard to see the solution, an elegant solution to the above conditions comes out directly from BRST-Fock quantization, as we discuss next.

6. BRST-Fock quantization

As in regular BRST, in the BRST-Fock approach we define the physical space by \( \hat{\Omega} \mid \Psi \rangle = 0 \), with \( \hat{\Omega} = \hat{\eta}_0 \hat{\Phi} + \hat{\eta}_1 \hat{P}_2 \). Now, in BRST-Fock we define oscillator variables for the ghosts
\[
\hat{c} = \hat{\eta}_0 + i \hat{\eta}_1, \quad \hat{\bar{c}} = \frac{i}{2} (\hat{\rho}_0 + i \hat{\rho}_1), \quad [\hat{c}, \hat{\bar{c}}^{\dagger}] = 1, \quad (6.1)
\]
and hermitean conjugates, as well as those in equations 5.1 and 5.2. There are other ways to define these variables, but this will not alter the results below in a fundamental way. The states are constructed in terms of a unit-norm vacuum \( \mid 0 \rangle \) defined by
\[
\hat{a}, \hat{b}, \hat{c}, \hat{\bar{c}} \mid 0 \rangle = 0. \quad (6.2)
\]
Using these variables the BRST generator becomes
\[
\Omega = \frac{\hat{c} + \hat{\bar{c}}^{\dagger}}{2} \hat{\Phi} + \frac{\hat{c} - \hat{\bar{c}}^{\dagger}}{2i} (\hat{\rho}_0 + i \hat{\rho}_1) = \frac{1}{2} \left( \hat{c} \hat{M}^{\dagger} + \hat{\bar{c}} \hat{M} \right), \quad (6.3)
\]
where \( \hat{M} = (\hat{\Phi} + \frac{\hat{a} - \hat{a}^{\dagger}}{2i})/2 = (\hat{\Phi} + i \hat{P}_\lambda)/2 \). Hence, the physical states satisfy
\[
\hat{c} \mid \Psi \rangle = 0, \quad \hat{M} \mid \Psi \rangle = 0. \quad (6.4)
\]
This ensures that \( \hat{\Omega} \mid \Psi \rangle = 0 \), and that physical states carry no ghost excitations. Notice that this procedure fixes the “splitting” defined in the previous section, and satisfies the properties
\[
\hat{M} + \hat{M}^{\dagger} = \hat{\Phi}, \quad \hat{M} - \hat{M}^{\dagger} = i \hat{P}_2, \quad [\hat{M}, \hat{M}^{\dagger}] = 0. \quad (6.5)
\]
It is possible to define the ghost oscillator variables differently, and find that the states satisfy \( \hat{M}' \psi = 0 \), with \( \hat{M}' = \alpha \hat{\Phi} + \beta \hat{P}_\lambda \). This ambiguity is not serious, and corresponds to canonical transformation of the ghosts.

6.1. The Fock representation in coordinate space

Consider the mixed coordinate-momentum representation in which we write the Fock states in the form \( \varphi = \varphi(p_t, \lambda) \). The vacuum is defined by
\[
(i + i \lambda) \varphi_0(p_t, \lambda) = (i \frac{\partial}{\partial p_t} + i \lambda) \varphi_0(p_t, \lambda) = 0, \quad (6.6)
\]
This is solved by
\[ \varphi_0(\pt, \lambda) = e^{-\lambda \pt}. \] (6.8)

Can we come up with an inner product here that respects the algebra, the hermiticity properties of the operators and that gives unit norm to the vacuum, i.e., can we rewrite the Fock inner product in this representation? The answer is yes, of course. The inner product is given by
\[ \langle \psi_a, \psi_b \rangle \equiv \int_{-i\infty}^{i\infty} d\lambda \int_{-\infty}^{\infty} d\pt \left[ \psi_a(\pt^*, \lambda^*) \right]^* \psi_b(\pt, \lambda). \] (6.9)

First notice that the vacuum is normalizable to unity:
\[ \| \psi_0 \| = \int_{-i\infty}^{i\infty} d\lambda \int_{-\infty}^{\infty} d\pt e^{-2\lambda \pt} = \int_{-\infty}^{\infty} dq \int_{-\infty}^{\infty} d\pe e^{-2iq\pe} = i\pi. \] (6.10)

Next, we can check that the operators are hermitean. This could be troublesome, for example, for the operator \( \hat{t} \sim i\partial_\pt. \) But we can check that the boundary term that appears in the check for hermicity vanishes,
\[ \int_{-i\infty}^{i\infty} d\lambda \int_{-\infty}^{\infty} d\pt i\partial_\pt e^{-2\lambda \pt} = i\pi \int_{-\infty}^{\infty} d\pt \delta^{\prime}(\pt) = 0. \] (6.11)

Similarly, it is also easily checked that \( \hat{\lambda}, \hat{\lambda}, \) and \( \hat{t} \) are also hermitean in this inner product, as they should. This is sufficient to insure that we have a good representation of the Fock algebra, since the whole representation is based on the unit norm of the vacuum, the algebra, and the hermicity properties of the operators.

Now, as we discussed, the condition for physical states in (BRST-) Fock quantization is
\[ (\hat{\phi} + i\hat{t}\hat{\lambda}) \psi = 0. \] (6.12)

In the above representation (and in the free case) this has the general solution
\[ \psi(\pt, \pi) = \varphi(p_\mu) e^{-\lambda(p_\mu \pi^\mu - m^2)}. \] (6.13)

More generally, the solution is given by \( |\Psi_{ph}\rangle = \exp(-\hat{\lambda}\hat{\phi})|\varphi(x^\mu)\rangle. \) Observe that the inner product between two such states is “on-shell”,
\[ \langle \Psi_{a} | \Psi_{b} \rangle = \int d^4 p \left( \Psi_{a}(p_\mu) \right)^* \delta(\hat{\phi}) \Psi_{b}(p_\mu) \]
\[ = \int d^4 x \left( \Psi_{a}(x^\mu) \right)^* \delta(\hat{\phi}) \Psi_{b}(x^\mu) \]
\[ = \langle \Psi_{a} | \delta(\hat{\phi}) | \Psi_{b} \rangle. \] (6.14)

As long as \( \Phi \) is hermitean, this inner product satisfies the property \( \langle \Psi_{a} | \Psi_{b} \rangle^* = (\Psi_{b} | \Psi_{a} \rangle \), and it never yields negative norms. How can we interpret this result? In
the case of the parametrized non-relativistic particle, where Φ = pt + R(t, x, p), the interpretation is simple: because of the delta function in the definition of inner product, the state space is reduced to the states of non-relativistic quantum mechanics \( \Psi_a(x^\mu) = (x^\mu | \Psi) \), with \( |x^\mu_+\rangle = \exp(i\hat{R})|\bar{x}\rangle \). To see why, notice that null states are states of the form \( \varphi_{null} = f(\hat{\Phi})e^{-\lambda \hat{\Phi}}\varphi(\bar{x}) \), with \( f(0) = 0 \). The identification of states that differ by a null state reduces the state space to the zero modes of \( \hat{\Phi} \). One can also see this by noticing that to compute the inner product of any two states it is sufficient to explore the implications of the Fock inner product for the basis \( |x^\mu_+\rangle \), \( \langle x^\mu_f|\delta(\hat{\Phi})|x^\mu_i\rangle = \int d^3p e^{i(\bar{x}_f - \bar{x}_i)\cdot\vec{p} - i(t_f - t_i)}\hat{R} = \langle x^\mu_f|e^{-i\hat{R}(t_f - t_i)}|x^\mu_i\rangle \), (6.15)

which is the non-relativistic propagator. What are the observables here? Observables are defined by asking that they commute with \( \hat{M} \), so as not to make physical states unphysical. For example, the operators \( \hat{X} = \hat{x} + i\hat{\lambda}\partial\hat{\Phi}/\partial\hat{p}_\lambda \) and \( \hat{P}_\lambda = \hat{p}_\lambda - i\hat{\lambda}\partial\hat{\Phi}/\partial\hat{x} \) are observables, and one can check that their action reduces to the usual ones in the reduced state space. For example, notice that

\[
\begin{align*}
\hat{X}|\Psi_{ph}\rangle &= \exp(-\hat{\lambda}\hat{\Phi})\hat{x}|\varphi\rangle \\
\hat{P}_\lambda|\Psi_{ph}\rangle &= \exp(-\hat{\lambda}\hat{\Phi})\hat{p}_\lambda|\varphi\rangle 
\end{align*}
\] (6.16)

In the free parametrized relativistic particle case, the Fock cohomology has two sectors. The “reduced” physical space is described by

\[
\begin{align*}
\psi_+(p_\mu) &= \varphi_+(p_\lambda) \left( p_\mu + \sqrt{p_\lambda^2 + m^2} \right) e^{-\lambda \Phi}, \\
\psi_-(p_\mu) &= \varphi_-(p_\lambda) \left( p_\mu - \sqrt{p_\lambda^2 + m^2} \right) e^{-\lambda \Phi}.
\end{align*}
\] (6.17)

(6.18)

One can check that these two sets of states are physical, not null, and that their difference is not null. The inner product in the reduced physical space is positive definite: both “positive” and “negative” energy states have positive norms.

An important corollary is that the Hadamard Green function, or on-shell amplitude,

\[
\Delta_1(x - y) = \frac{1}{(2\pi)^4} \int d^4k \delta(\Phi) e^{-ik(x-y)},
\] (6.19)

can be written in the explicit BRST-Fock form

\[
i\Delta_1(x - y) = \langle x^\mu_f|\delta(\hat{\Phi})|x^\mu_i\rangle = \langle M = 0, c = 0, x^\mu_f|M = 0, c = 0, x^\mu_i\rangle,
\] (6.20)

with \( M \equiv \Phi + i\rho_\lambda \). As was already show in reference [21], this amplitude can also be rewritten in the explicit BRST form

\[
i\Delta_1(x - y) = \langle \psi_{\varphi=0=\rho=0}\rangle \left[ \mathcal{O}_{NC,\Omega} \right] |\psi_{\varphi=0=\rho=0}\rangle,
\] (6.21)

where these states are in the Fock representation and the gauge-fixing is noncanonical. As explained above, it is simple to rewrite this expression as the BFV...
path integral in phase space and, after integration of momenta, derive the geometric path integral in coordinate space. This tells us that the full-range lapse prescription in the path integral formalism is well grounded in a BRST-Fock Hilbert space quantization.

In the interacting case the analysis of the Fock cohomology is more complicated. Null states are still given by the zero modes of $\hat{\Phi}$, which are not simple to analyze. In the case of an electric field with zero divergence, the situation remains simple, since $\hat{\Phi}$ decouples.

7. Conclusion

Let us summarize our results. Using reduced phase space quantization leads to the loss of symmetries. Reduced phase space quantization is not general enough to be applied to systems with complex phase spaces.

Dirac quantization leads to the Klein-Gordon inner product, but it could also lead to other possibilities, depending on our choice of hermitean ordering in the definition of inner product. This inner product is not positive definite (unless an absolute value is used in the definition of inner product), and has to be redefined, since the physical state space is not really in the original state space. Time disappears from the formalism, but can be recovered with the careful interpretation of the observables.

BRST quantization yields two sectors, one akin to the Dirac state space, with the Klein-Gordon inner product, the other leading to on-shell amplitudes. This formalism is still not totally well-defined and the process seems a bit ad hoc. What happens with the other sectors? Some questions remain. It seems, however, that BRST encodes the Dirac and anti-Dirac approaches to quantization (see below).

In BRST-Fock, gauge-fixing is not need, nor the redefinition of inner product: the physical states reside already in the extended state space—they have finite norms. The arbitrariness in this approach resides in the definition of the vacuum (i.e., the definition of annihilation and creation operators). Two sectors appear in the theory, at least in the free case. The inner product is positive definite inner product, aside form the null states. The lapse has full range. The amplitudes are “on-shell”, although the states do not satisfy the Dirac condition. Time remains in the picture. For instance, in the parametrized non-relativistic case this quantization approach reduces to the usual non-relativistic quantum mechanics. This formalism is the ground and backbone for the path integrals usually discussed in the literature. This representation is identical to the “anti”-Dirac one in the zero-ghost sectors in the BRST section.

The different approaches to quantization then essentially lead to two types of amplitudes: symbolically,

$$\langle \Psi_{\Phi=0}^{a} | \delta(\hat{\Phi}) | \Psi_{\Phi=0}^{b} \rangle, \quad \langle \Psi_{\Phi=0}^{a} | \delta(\hat{\Upsilon}) \{ \hat{\Phi}, \hat{\Upsilon} \} | \Psi_{\Phi=0}^{b} \rangle. \quad (7.1)$$

Both of these yield positive norms for the states (although the needed absolute value does not arise from BRST in a natural way). When the constraint can be made
into a momentum, $P \approx 0$, by a canonical transformation, these amplitudes yield equivalent theories: they essentially reduce to

$$\langle \Psi^a_{Q=0} | \delta(\hat{P}) | \Psi^b_{Q=0} \rangle, \quad \langle \Psi^a_{P=0} | \delta(\hat{Q}) | \Psi^b_{P=0} \rangle$$

(7.2)

and they both eliminate the gauge degrees of freedom from the theory. Thus, if we had started with the constraint $Q \approx 0$, we would have ended up with the same result. If we follow the same logic through with the toy model in which the constrain is $\Phi = P^2 - P_o^2 \approx 0$ ($P_o > 0$), we will also find that the amplitudes in the two approaches,

$$\langle \Psi^a_{Q=Q_o} | \delta(\hat{P}^2 - P_o^2) | \Psi^b_{Q=Q_o} \rangle$$

(7.3)

$$= \langle \Psi^a_{Q=Q_o} | (\delta(\hat{P} - P_o)/P_o + \delta(\hat{P} + P_o)/P_o) | \Psi^b_{Q=Q_o} \rangle,$$

yield, at the end, identical physical spaces—with two separate sectors. Notice that, for example,

$$\langle P = P_o | \delta(\hat{Q} - Q_o) | P = P_o \rangle = \langle P = P_o | Q = Q_o \rangle \langle Q = Q_o | P = P_o \rangle = \langle Q = Q_o | \delta(\hat{P} - P_o) | Q = Q_o \rangle$$

(7.5)

so the equivalence is clear. In BRST theory, as we just mentioned, the two approaches are available, appearing in the different zero-ghost sectors.

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