Kinetics of chiral phase transition in hot and dense quark matter

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The kinetics of chiral transitions in hot and dense quark matter is studied via a microscopic framework (Nambu-Jona-Lasinio model) and a phenomenological model (Ginzburg-Landau free energy). We focus on the far-from-equilibrium ordering dynamics subsequent to a quench from the massless quark phase to the massive quark phase. The morphology of the ordering system is characterized by the scaling of the order-parameter correlation function. The domain growth process obeys the Allen-Cahn growth law, $L(t) \sim t^{1/2}$. We also study the growth of bubble of the stable massive phase in a background of the metastable massive phase.

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The kinetics of phase transitions, and the ordering process that occurs after a rapid quench in system parameters like temperature and pressure, plays an important and interesting role in different areas of physics \cite{1}. During the transition, the system develops a spatial structure of randomly-distributed domains which evolve with time. This ordering process has been extensively studied in many condensed matter systems like ferromagnets, binary fluids, liquid crystals, etc. In this letter, we study an equally fascinating application in high-energy physics, i.e., the quark hadron phase transition.

Heavy ion collision experiments at high energies produce hot and dense strongly-interacting matter, and provide the opportunity to explore the phase diagram of QCD in the plane of temperature ($T$) and baryon chemical potential ($\mu$). Many model studies \cite{2} as well as recent lattice studies \cite{3} indicate that at sufficiently large baryonic densities, there is a line of first-order transitions in the ($\mu, T$)-plane between a chirally-symmetric phase and a broken-symmetric phase. As one moves along the phase boundary towards higher $T$ and smaller $\mu$, the first-order transition becomes weaker ending in a second-order critical point which had been proposed using on equilibrium thermodynamics \cite{4}, have been recently interpreted within a quasi-stationary framework \cite{5}. In the present letter, we study the far-from-equilibrium kinetics of chiral phase transitions subsequent to a quench from a disordered phase (with vanishing quark condensate) to the ordered phase.

To model chiral symmetry breaking in QCD, we use the two-flavor Nambu-Jona-Lasinio (NJL) model \cite{8} with the Hamiltonian:

$$
\mathcal{H} = \sum_{i,a} \psi^i \left( -i \vec{\sigma} \cdot \vec{\nabla} + \gamma^0 m_i \right) \psi^a - \frac{G}{2} \left( \langle \bar{\psi} \psi \rangle^2 - \langle \bar{\psi} \gamma_5 \psi \rangle^2 \right). \tag{1}
$$

To describe the ground state, we take an ansatz with quark-antiquark condensates \cite{9}:

$$
|\text{vac}\rangle = \exp \left[ \int \frac{d^3 k}{(2\pi)^3} q_i^0(k) \Phi(k) \bar{q}^i_0(-k) - \text{h.c.} \right] |0\rangle. \tag{2}
$$

Here, $q^i, \bar{q}$ are two-component quark and antiquark creation operators, and $|0\rangle$ is the perturbative chiral vacuum. Further, $h_i(k)$ is a variational function related to the quark-antiquark condensate as $\langle \bar{\psi} \psi \rangle = -6/(2\pi)^3 \sum_{i=1}^2 \int \frac{d^3 k}{(2\pi)^3} \sin[2h_i(k)]$. This flavor-dependent function can be determined by minimising the energy at $T = 0$ or the thermodynamic potential at nonzero $T$ and density. Without going into the details which are reported elsewhere \cite{10}, we write down the expression for the thermodynamic potential as

$$
\tilde{\Omega}(M, \beta, \mu) = -\frac{12}{(2\pi)^3} \sum_{i=\pm} \int \frac{d^3 k}{(2\pi)^3} \text{ln} \left[ 1 + \exp(-\beta \omega_i) \right].
$$
Here, we have taken vanishing current quark mass, and introduce \( g = G[1 + 1/(4N_c)] \); \( M = -2g\rho_s \) with \( \rho_s = \langle \bar{\psi}\psi \rangle \) being the scalar density. Further, \( \rho_v = \langle b^4\psi \rangle \) is the vector density, and \( \omega_\pm = (k^2 + M^2)^{1/2} \pm \nu \) where \( \nu = \mu - G\rho_v/N_c \). The resulting phase diagram is shown in Fig. 1.

![FIG. 1: Phase diagram of the NJL model in the (\( \mu, T \)) plane for zero current quark mass. A line of first-order transitions (green online) meets a line of second-order transitions (blue online) at the tricritical point (tcp). We have (\( \mu_{tcp}, T_{tcp} \)) \( \simeq (285.0, 74.9) \) MeV. The dashed lines denote the spinodals for the first-order transitions.](image)

Close to the phase boundary, the thermodynamic potential may be expanded in a power series of the order parameter \( M \):

\[
\hat{\Omega}(M) = \hat{\Omega}(0) + \frac{a}{2} M^2 + \frac{b}{4} M^4 + \frac{d}{6} M^6 + \cdots \equiv F(M).
\]  

One can obtain the coefficients in terms of \( \beta \) and \( \mu \) by Taylor-expanding the thermodynamic potential in Eq. (3). However, we will treat them as phenomenological constants which are adjusted to recover the phase diagram in Fig. 1. After all, we are interested in studying the kinetics of the chiral transition, usually at parameter values far from the critical curve in Fig. 1—where we cannot Taylor-expand \( \hat{\Omega}(M, \beta, \mu) \). The extrema of the potential in Eq. (4) are determined by the gap equation \( F'(M) = -aM + 6M^3 + dM^5 = 0 \). The solutions are \( M_0 = 0 \), and \( M_\pm^2 = (-b \pm \sqrt{b^2 - 4ad})/2d \). For \( b > 0 \), the transition is second-order, analogous to an \( M^4 \)-potential—the stationary points are \( M = 0 \) (for \( a > 0 \)) or \( M = 0, \pm M_+ \) (for \( a < 0 \)). For \( a < 0 \), the preferred equilibrium state is the one with massive quarks. For \( b < 0 \), the solutions of the gap equation are as follows: (i) \( M = 0 \) for \( a > b^2/4d \), (ii) \( M = 0, \pm M_+ \) for \( b^2/4d > a > 0 \), and (iii) \( M = 0, \pm M_+ \) for \( a < 0 \). A first-order transition takes place at \( a_c = 3b^2/16 \) with the order parameter jumping discontinuously from \( M = 0 \) to \( M = \pm M_+ \). The critical point is located at \( b_{tcp} = 0, a_{tcp} = 0 \).

Let us consider the dynamical environment of a heavy-ion collision. As long as the evolution is slow compared to the typical re-equilibration time, the order parameter field will be in local equilibrium. We consider a system which is rendered thermodynamically unstable by a rapid quench from the disordered phase to the ordered phase. The unstable disordered state evolves via the emergence and growth of domains rich in the preferred phase in the coarsening system is inhomogeneous, and we include a surface tension term in the Ginzburg-Landau free energy as follows:

\[
\Omega(M) = \int dx \left[ \frac{a}{2} M^2 + \frac{b}{4} M^4 + \frac{d}{6} M^6 + \frac{K}{2} (\nabla M)^2 \right].
\]  

The evolution of the system is described by the time-dependent Ginzburg-Landau (TDGL) equation

\[
\frac{\partial}{\partial t} M(r, t) = -\Gamma \delta \Omega[M]/\delta M + \theta(r, t),
\]  

modeling the over-damped relaxational dynamics of \( M(r, t) \) to the minimum of \( \Omega[M] \). Here, \( \Gamma \) is the inverse damping coefficient, and \( \theta(r, t) \) is the noise term satisfying the fluctuation-dissipation relations \( \langle \theta(r, t) \rangle = 0 \) and \( \langle \theta(r', t')\theta(r'', t'') \rangle = 2\Gamma T \delta(t' - t'') \delta(t' - t''). \) We use the natural scales of order parameter, space and time to introduce the dimensionless variables, \( M' = \sqrt{|a|/|b|} M', \ r' = \sqrt{K/|a|} r', \ t' = t'/|\Gamma|, \ \theta' = (|a|^{3/2}/|b|^{1/2}) \theta' \). Dropping the primes, we have the dimensionless TDGL equation:

\[
\frac{\partial}{\partial t} M(r, t) = -\Lambda M^3 M + \nabla^2 M + \theta(r, t),
\]  

where \( \lambda = |a|d/b^2 \). For \( T = 15 \) MeV, as \( \mu \) takes values 320 MeV, 330 MeV, 338 MeV and 350 MeV, the corresponding values of \( \lambda \) are 0.065, 0.14, 0.23 and 0.34, which we have obtained by fitting the effective potential Eq. (4) with Eq. (3).

First, we study the ordering dynamics for \( b > 0 \). We solve the Eq. (7) numerically using an Euler discretization scheme. This is implemented on a 3-d lattice of size \( N^3 \) (\( N = 256 \)), with periodic boundary conditions in all directions. The dimensionless mesh sizes
are $\Delta x = 1.0$ and $\Delta t = 0.1$, which satisfy the numerical stability condition. We have further confirmed that the spatial mesh size is sufficiently small to resolve the interface region. The details of the numerics will be reported elsewhere [10].

In Fig. 2 we show the evolution of a disordered initial condition for Eq. (7) with $a < 0$, i.e., a temperature quench through the second-order line in Fig. 1. The initial state consisted of small-amplitude thermal fluctuations about the massless phase $M = 0$. The system rapidly evolves into domains of the massive phase with $M \simeq M_+$ and $M \simeq -M_+$. The interfaces correspond to $M = 0$—their evolution is shown in the snapshots (frames on left) of Fig. 2. The frames on the right show the interface structure in a cross-section of the snapshots. The domains have a characteristic length scale which grows with time. The growth process is analogous to the coarsening dynamics in the TDGL equation with an $M^4$-potential, where growth is driven by kinks with the equilibrium profile $M(z) = \tanh(\pm \sqrt{2}z)$. The order-parameter correlation function $C(r, t)$ shows dynamical scaling or dynamical self-similarity, $C(r, t) = f(r/L)$—the scaling function $f(x)$ has been calculated by Ohta et al. (OJK) [11]. In Fig. 3(a), we demonstrate that $C(r, t)$ for the evolution in Fig. 2 shows the scaling property. Further, the domain scale obeys the Allen-Cahn (AC) growth law, $L(t) \sim t^{1/2}$ [see Fig. 3(b)]. The interface velocity $v \sim dL/dt \sim 1/L$, where $L^{-1}$ measure the local curvature of the interface. This yields the AC growth law.

Next, let us consider the case with $b < 0$. In this case, a first-order chiral transition occurs for $a < a_c = 3|b|^2/(16d)$ (or $\lambda < \lambda_c = 3/16$). For $a < 0$, the potential has a double well structure and the ordering dynamics is analogous to $M^4$-theory i.e., the domain growth scenario is similar to Fig. 5. We focus our attention on a quench from the disordered state ($M = 0$) to $0 < \lambda < \lambda_c$, corresponding to a quench between the first-order line and $S_1$ in Fig. 1. The massless state is now a metastable state of the $M^6$-potential. The chiral transition proceeds via the nucleation and growth of droplets of the preferred phase ($M = \pm M_+$). This nucleation results from large fluctuations in the initial condition or thermal fluctuations during the evolution. In Fig. 4 we show the nucleation and growth process. At early times ($t = 400$) the system is covered with the $M = 0$ phase with small droplets of the preferred phase. These droplets grow in time and coalesce into domains. The subsequent coarsening of these domains is analogous to that in Figs. 2-3. In the late stages, there is no memory of the nucleation which characterized growth during the early stages.

In Fig. 3(a) we show the scaling of correlation function for $\lambda = 0.14$ at four different time steps. OJK function (as for usual $M^4$-free energy) has good agreement with simulation data. (b) The domain size $L(t)$ vs $t$ for $\lambda = 0.14$. Domain growth data is consistent with Allen-Cahn growth law $L(t) \sim t^{1/2}$.
In Fig. 5(b), we plot via the corresponding TDGL equation, and consider both Landau (GL) free energy. We study the chiral kinetics of droplet velocity $v$ phase ($M = 0$) for $\lambda = 0.14$ for different time steps. The inner circle corresponds to a droplet at time, $t = 20$. (b) Shows the plot of droplet velocity $v$ vs. $\lambda$. Circles refer to numerical data while solid line corresponds to predictions from a phase-plane analysis.

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