THEORY OF SERIES IN THE $\mathcal{A}$-CALCULUS AND THE N-Pythagorean Theorem

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August 15, 2017

Abstract

In this paper we study sequences, series, power series and uniform convergence in the $\mathcal{A}$-Calculus. Here $\mathcal{A}$ denotes an associative unital real algebra. We say a function is $\mathcal{A}$-differentiable if it is real differentiable and its differential is in the regular representation of the algebra. We show the theory of sequences and numerical series resembles the usual theory, but, the proof to establish this claim requires modification of the standard arguments due to the submultiplicativity of the norm on $\mathcal{A}$. In contrast, the theorems concerning divergence of power series over $\mathcal{A}$ are modified notably from the standard theory. We study how the ratio, root and geometric series results are modified due to both the submultiplicativity of the norm and the calculational novelty of zero-divisors. Despite these difficulties, we find natural generalizations of the usual theorems of real analysis for uniformly convergent series of functions. We establish the usual calculations with power series transfer nicely to the $\mathcal{A}$-calculus. Then we use power series to define sine, cosine, hyperbolic sine, hyperbolic cosine and the exponential. Many standard identities are derived for entire functions over arbitrary commutative $\mathcal{A}$. Finally, special functions are introduced and we derive the $N$-Pythagorean Theorem which produces $\cos^2 \theta + \sin^2 \theta = 1$ and $\cosh^2 \theta - \sinh^2 \theta = 1$ as well as a host of other less known identities.

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1 Introduction and overview

We use $A$ to denote a real unital associative algebra of finite dimension. Elements of $A$ are known as $A$-numbers. We study calculus where real numbers have been replaced by $A$-numbers. The resulting calculus we refer to as $A$-calculus. Our typical goal is to find theorems which apply to as large a class of real associative algebras as possible.

Setting aside complex analysis, calculus over more general number systems have been studied from about 1890 to the present time. There are too many papers to list. To see how our current framework relates to the existing literature please see [4].

The goal of this paper is to study sequences, series and power series over an algebra. This serves as a foundation for an ongoing project to develop $A$-calculus generalizations of the standard calculational techniques. We should mention N. BeDell has written three supplementary algebra papers [1], [2] and [3] which provide further algebraic discussion of zero-divisors, logarithms and the $N$-Pythagorean Theorem proved in Section 7 of this paper. Then in the sequel to this paper [6] one of the authors and N. BeDell present the theory of $A$-ordinary differential equations.

In Section 2 we review the major developments from [5]. In particular, we discuss representations, submultiplicativity of the norm, the definition and theory of differentiation over an algebra and select theorems from integral calculus over $A$. Many further examples, proofs and motivations can be found in [5] and we encourage the reader to consult that paper before digesting this current work.

Section 3 develops the theory of sequences over $A$ following [10]. We show the usual arithmetic of limits transfers nicely to sequences in $A$. The results are quite natural, however, the submultiplicativity complicated the usual proofs from real or complex analysis.

Numerical series over $A$ are covered in Section 4. We found that the usual elementary convergence and divergence tests are meaningful over an algebra. The $n$-th term test, comparison test, absolute convergence, root and ratio test all naturally generalize over $A$. The Cauchy Criterion is meaningful and the Cauchy product exists to multiply series where at least one is absolutely convergent. Once more, the proofs required significant modification due to the submultiplicativity of the norm.

In Section 5 we study power series in $A$. We attempt to generalize the elementary convergence and divergence theorems of real calculus. We find the Root and Ratio Test need significant modification. One aspect of the modification is that the submultiplicative constant $m_A$ appears in convergence results. For example, where $R = 1/\alpha$ in the usual calculus we found $R = 1/m_A \alpha$. Also, the geometric series $1 + z + z^2 + \cdots$ converges for $\|z\| < 1/m_A$. A second, and initially perplexing, modification is seen in the absence of divergence cases for the Root and Ratio Tests for power series over $A$. The multiplicative property of the norm in real or complex analysis is important to obtain the boundary between divergence and convergence of a power series. Submultiplicativity of the norm on $A$ spoils the usual
argument for divergence for both the Root and Ratio Tests. However, we also understand
the reason for this modification in view of the phenomenon seen in Example 5.5. Zero divi-
sors allow new domains of convergence which are not seen in single-variave analysis over \( \mathbb{R} \) or \( \mathbb{C} \). Uniform convergence and Weierstrauss \( M \)-Test are studied. The standard theorems
concerning sequences of uniformly convergent functions hold for \( \mathcal{A} \)-differentiable sequences of functions. The integral of the limiting function is the limit of the integrals, however, the
result for derivatives is not as simple. See Theorem 5.19 which mirrors the usual theorem
of real analysis. We show the term-by-term derivative of a power series in \( \mathcal{A} \) is indeed the
derivative of the given power series. Given an entire function on \( \mathbb{R} \) we show there exists
a unique entire extension to \( \mathcal{A} \). We show entire functions are absolutely convergent on \( \mathcal{A} \)
and uniformly convergent on any finite ball in \( \mathcal{A} \). Finally, we show the product of entire
functions is again entire. Indeed, the entire functions on \( \mathcal{A} \) form an algebra.

Transcendental functions such as the exponential, sine, cosine and hyperbolic sine and cosine
are covered in Section 6. Theorem 5.23 indicates the definitions we offer are inescapable. We
find the usual identities for the elementary functions on \( \mathbb{R} \) extend naturally to any commu-
tative unital algebra \( \mathcal{A} \).

Section 7 reverses the direction of study from that of Section 6. We ask, given a specific
choice of \( \mathcal{A} \), which functions appear naturally? In particular, we study functions which
appear as component functions of the exponential. We call these the special functions of \( \mathcal{A} \).
We find a theorem we call the \( N \)-Pythagorean Theorem which provides an identity which
holds for the special functions. This theorem makes the identities \( \cos^2 \theta + \sin^2 \theta = 1 \) and
\( \cosh^2 \phi - \sinh^2 \phi = 1 \) part of an chain of such identities.

2 Review of \( \mathcal{A} \) calculus

In this Section we have two main goals. First, to provide necessary background to understand
the new theory developed in the later sections. Second, to alert the reader to some of the
major results which are already established in \([5]\). Please consult \([5]\) for references and
discussion of how our work connects to the existing literature.

2.1 Algebra and the regular representations

We say \( \mathcal{A} \) is an algebra if \( \mathcal{A} \) is a finite-dimensional real vector space paired with a function
\( \star : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) which is called multiplication. In particular, the multiplication map
satisfies the properties below:

(i.) bilinear: \( (cx + y) \star z = c(x \star z) + y \star z \) and \( x \star (cy + z) = c(x \star y) + x \star z \) for all
\( x, y, z \in \mathcal{A} \) and \( c \in \mathbb{R} \),

(ii.) associative: for which \( x \star (y \star z) = (x \star y) \star z \) for all \( x, y, z \in \mathcal{A} \) and,

(iii.) unital: there exists \( 1 \in \mathcal{A} \) for which \( 1 \star x = x \) and \( x \star 1 = x \).

\(^{1}\text{we write (}\mathcal{A}, \star\text{) to emphasize the pairing where helpful}\)
We say $x \in \mathcal{A}$ is an $\mathcal{A}$-number. If $x \ast y = y \ast x$ for all $x, y \in \mathcal{A}$ then $\mathcal{A}$ is commutative.

The left-multiplication by $x$ is the map $L_x : \mathcal{A} \to \mathcal{A}$ defined by $L_x(y) = x \ast y$ for all $y \in \mathcal{A}$. Observe, by associativity of $\mathcal{A}$,

$$L_x(y) = x \ast 1 \ast y = L_x(1) \ast y. \quad (1)$$

A linear transformation $T : \mathcal{A} \to \mathcal{A}$ is right $\mathcal{A}$ linear if $T(x \ast y) = T(x) \ast y$ for all $x, y \in \mathcal{A}$. We say the set $\mathcal{R}_\mathcal{A}$ of all right $\mathcal{A}$ linear transformations forms the regular representation of $\mathcal{A}$. Since $\mathcal{A}$ is unital the regular representation is isomorphic to $\mathcal{A}$. The isomorphism from $\mathcal{A}$ to $\mathcal{R}_\mathcal{A}$ is denoted map and we find it convenient to use $\#_\mathcal{A}$ for $\text{map}^{-1}$. In particular,

$$\text{map}(x) = L_x \quad \& \quad \#_\mathcal{A}(T) = T(1). \quad (2)$$

The idea here is that $\#_\mathcal{A}(T)$ provides the $\mathcal{A}$ number which corresponds to $T$. If $\beta$ is a basis for $\mathcal{A}$ then the matrix regular representation of $\mathcal{A}$ with respect to $\beta$ is

$$M_\mathcal{A}(\beta) = \{[T]_{\beta,\beta} \mid T \in \mathcal{R}_\mathcal{A}\} \quad (3)$$

where $[T]_{\beta,\beta}$ denotes the matrix of $T$ with respect to the basis $\beta$. In the case $\mathcal{A} = \mathbb{R}^n$ we may forego the $\beta$ notation and write

$$M_\mathcal{A} = \{[T] \mid T \in \mathcal{R}_\mathcal{A}\} \quad (4)$$

for the regular representation of $\mathcal{A}$. There is a natural isomorphism of $\mathcal{A}$ and $M_\mathcal{A}$: If $\beta = \{v_1, \ldots, v_n\}$ is a basis for $\mathcal{A}$ where $v_1 = 1$ then

$$M(x) = [[x]_\beta[[x \ast v_2]_\beta] \cdots [[x \ast v_n]_\beta] \quad (5)$$

where $[x]_\beta$ is the coordinate vector of $x$ with respect to $\beta$. In many applications we consider the case $\mathcal{A} = \mathbb{R}^n$ with $\beta = \{e_1, \ldots, e_n\}$ the usual standard basis such that $e_1 = 1$. Given these special choices we obtain much improved formula

$$M(x) = [x \ast e_2 \cdots \ast e_n]. \quad (6)$$

**Example 2.1.** The complex numbers are defined by $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$ where $i^2 = -1$. We denote $a + ib = [a, b]^T$ corresponding to our identifications $e_1 = 1$ and $e_2 = i$. Notice, $(a + ib)e_2 = (a + ib)i = ia - b = [-b, a]^T$. Thus, $M(a + ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a typical element of the regular matrix representation of $\mathbb{C}$ which we denote $M_\mathbb{C}$.

**Example 2.2.** The hyperbolic numbers are given by $\mathcal{H} = \mathbb{R} \oplus j\mathbb{R}$ where $j^2 = 1$. Identifying $e_1 = 1$ and $e_2 = j$ we have $a + bj = [a, b]^T$. Moreover,

$$(a + bj)e_2 = (a + bj)j = aj + b = [b, a]^T. \quad (7)$$

Therefore, $M(a + bj) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ is a typical matrix in $M_\mathcal{H}$.

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$^2$isomorphic as associative real algebras, we say $(\mathcal{A}, \ast)$ and $(\mathcal{B}, \cdot)$ are isomorphic if there is a linear bijection $\Psi : \mathcal{A} \to \mathcal{B}$ for which $\Psi(x \ast y) = \Psi(x) \cdot \Psi(y)$ for all $x, y \in \mathcal{A}$.
We say \( x \in \mathcal{A} \) is a **unit** if there exists \( y \in \mathcal{A} \) for which \( x \ast y = y \ast x = 1 \). The set of all units is known as the **group of units** and we denote this by \( \mathcal{A}^\times \). We say \( a \in \mathcal{A} \) is a **zero-divisor** if \( a \neq 0 \) and there exists \( b \neq 0 \) for which \( a \ast b = 0 \) or \( b \ast a = 0 \). Let \( \text{zd}(\mathcal{A}) = \{ x \in \mathcal{A} \mid x = 0 \text{ or } x \text{ is a zero-divisor} \} \).

**Example 2.3.** and \( \text{zd}(\mathcal{H}) = \{ a + bj \mid a^2 = b^2 \} \) whereas \( \mathcal{H}^\times = \{ a + bj \mid a^2 \neq b^2 \} \). The reciprocal of an element in \( \mathcal{H}^\times \) is simply

\[
\frac{1}{a + bj} = \frac{a - bj}{a^2 - b^2}
\]

this follows from the identity \((a + bj)(a - bj) = a^2 - b^2 \) given \( a^2 - b^2 \neq 0 \). Let \( \mathcal{B} = \mathbb{R} \times \mathbb{R} \) with \((a, b)(c, d) = (ac, bd)\) for all \((a, b), (c, d) \in \mathcal{B} \). We can show that

\[
\Psi(a, b) = a \left( \frac{1 + j}{2} \right) + b \left( \frac{1 - j}{2} \right) \quad \text{and} \quad \Psi^{-1}(x + jy) = (x + y, x - y)
\]

provide an isomorphism of \( \mathcal{H} \) and \( \mathbb{R} \times \mathbb{R} \). In [3] examples are given which show how this isomorphism can be used to solve the quadratic equation in \( \mathcal{H} \) and to derive d’Alembert’s solution to the wave equation.

### 2.2 Submultiplicative norms

The division algebras \( \mathbb{R}, \mathbb{C} \) and \( \mathbb{H} \) can be given a **multiplicative norm** where \( \|x \ast y\| = \|x\| \|y\| \). Generally we can only find **submultiplicative norm**.

**Example 2.4.** If \( \mathcal{H} \) is given norm \( \|x + jy\| = \sqrt{x^2 + y^2} \) then \( \|zw\| \leq 2\|z\|\|w\| \).

If \( \mathcal{A} \) is an algebra over \( \mathbb{R} \) with basis \( \{ v_1, \ldots, v_n \} \) then define **structure constants** \( C_{ijk} \) by

\[
v_i \ast v_j = \sum_{k=1}^{n} C_{ijk} v_k \quad \text{for all } 1 \leq i, j, k \leq n.
\]

For proof of what follows see [3].

**Theorem 2.5.** (submultiplicative norm) If \( \mathcal{A} \) is an associative \( n \)-dimensional algebra over \( \mathbb{R} \) then there exists a norm \( \| \cdot \| \) for \( \mathcal{A} \) and \( m_\mathcal{A} > 0 \) for which \( \|x \ast y\| \leq m_\mathcal{A} \|x\| \|y\| \) for all \( x, y \in \mathcal{A} \). Moreover, for this norm we find \( m_\mathcal{A} = C(n^2 - n + 1)\sqrt{n} \) where \( C = \max\{ C_{ijk} \mid 1 \leq i, j, k \leq n \} \).

**Corollary 2.6.** If \( \|x \ast y\| \leq m_\mathcal{A}\|x\|\|y\| \) for \( x, y \in \mathcal{A} \) then \( \|z^n\| \leq m_\mathcal{A}^n\|z\|^n \) for each \( z \in \mathcal{A} \) and \( n \in \mathbb{N} \).

**Corollary 2.7.** Suppose \( m_\mathcal{A} > 0 \) is a real constant such that \( \|x \ast y\| \leq m_\mathcal{A}\|x\|\|y\| \) for all \( x, y \in \mathcal{A} \). If \( b \in \mathcal{A}^\times \) and \( a \in \mathcal{A} \) then \( \| a \| \| b \| \leq m_\mathcal{A} \| a \| \| b \| \| .

### 2.3 Differential calculus on \( \mathcal{A} \)

The definition of differentiability with respect to an algebra variable is open to some debate. There seem to be two main approaches:

**D1:** define differentiability in terms of an algebraic condition on the differential,
D2: define differentiability in terms of a deleted-difference quotient

In [3] it is shown that these definitions are interchangeable on an open set in the context of a commutative semisimple algebra. However, it is also shown that in there exist D1 differentiable functions which are nowhere D2. Hence, we prefer to use D1 as it is more general. Following [3] we define differentiability with respect to an algebra variable as follows:

**Definition 2.8.** Let $U \subseteq \mathcal{A}$ be an open set containing $p$. If $f : U \to \mathcal{A}$ is a function then we say $f$ is $\mathcal{A}$-differentiable at $p$ if there exists a linear function $d_pf \in \mathcal{R}_\mathcal{A}$ such that

$$
\lim_{h \to 0} \frac{f(p + h) - f(p) - d_pf(h)}{||h||} = 0. \tag{10}
$$

In other words, $f$ is $\mathcal{A}$-differentiable at a point if its differential at the point is a right-$\mathcal{A}$-linear map. Equivalently, given a choice of basis, $f$ is $\mathcal{A}$-differentiable if its Jacobian matrix is found in the matrix regular representation of $\mathcal{A}$. If $\mathcal{A}$ has basis $\beta = \{v_1, \ldots, v_n\}$ has coordinates $x_1, \ldots, x_n$ then $d_pf(e_j) = \frac{\partial f}{\partial x_j}(p)$. Suppose $v_1 = 1$ then $d_pf(1) = \frac{\partial f}{\partial x_1}(p)$. Observe right linearity of the differential indicates $d_pf(v_j) = d_pf(1 \ast v_j) = d_pf(1) \ast v_j$ hence for each $p$ at which $f$ is $\mathcal{A}$ differentiable we find:

$$
\frac{\partial f}{\partial x_j}(p) = \frac{\partial f}{\partial x_1}(p) \ast v_j. \tag{11}
$$

These are the $\mathcal{A}$-Cauchy Riemann Equations. There are $n - 1$ equations in $\mathcal{A}$ which amount to $n^2 - n$ scalar equations. If the $\mathcal{A}$-CR equations hold for a continuously differentiable $f$ at $p$ then we have that $d_pf \in \mathcal{R}_\mathcal{A}$.

Next we wish to explain how to construct the derivative function $f'$ on $\mathcal{A}$. We are free to use the isomorphism between the right $\mathcal{A}$ linear maps and $\mathcal{A}$ as to define the derivative at a point for via $f'(p) = \#(d_pf)$. This is special to our context. In the larger study of real differentiable functions on an $n$-dimensional space no such isomorphism exists and it is not possible to identify arbitrary linear maps with points.

**Definition 2.9.** Let $U \subseteq \mathcal{A}$ be an open set and $f : U \to \mathcal{A}$ an $\mathcal{A}$-differentiable function on $U$ then we define $f' : U \to \mathcal{A}$ by $f'(p) = \#(d_pf)$ for each $p \in U$.

Equivalently, we could write $f'(p) = d_pf(1)$ since $\#(T) = T(1)$ for each $T \in \mathcal{R}_\mathcal{A}$. Many properties of the usual calculus hold for $\mathcal{A}$-differentiable functions.

**Proposition 2.10.** For $f$ and $g$ both $\mathcal{A}$-differentiable at $p$,

(i.) $(f + g)'(p) = f'(p) + g'(p),

(ii.) for constant $c \in \mathcal{A}$, $(c \ast f)'(p) = c \ast f'(p),

(iii.) Given $\mathcal{A}$ is commutative, $(f \ast g)'(p) = f'(p) \ast g(p) + f(p) \ast g'(p).

(iv.) $(f \ast g)'(p) = f'(g(p)) \ast g'(p).
Theorem below gives us license to convert equations in $A$.

Let $f$ be differentiable at $x$ and $p$ then $\frac{\partial f}{\partial \zeta_j} = 0$ for $j = 2, \ldots, n$. In other words, another way we can look at $A$-differentiable functions is that they are functions of $\zeta$ alone.

We are also able to find an $A$-generalization of Wirtinger's calculus. In [5] we introduce conjugate variables $\bar{\zeta}_2, \ldots, \bar{\zeta}_n$ for $A$ and find for commutative algebras if $f : A \to A$ is $A$-differentiable at $p$ then $\frac{\partial f}{\partial \zeta_j} = 0$ for $j = 2, \ldots, n$. In other words, another way we can look at $A$-differentiable functions is that they are functions of $\zeta$ alone.

The theory of higher derivatives is also developed in [5].

**Definition 2.11.** Suppose $f$ is a function on $A$ for which the derivative function $f'$ is $A$-differentiable at $p$ then we define $f''(p) = (f')'(p)$. Furthermore, supposing the derivatives exist, we define $f^{(k)}(p) = (f^{(k-1)})'(p)$ for $k = 2, 3, \ldots$.

Naturally we define functions $f'', f''', \ldots, f^{(k)}$ in the natural pointwise fashion for as many points as the derivatives exist. Furthermore, with respect to $\beta = \{v_1, \ldots, v_n\}$ where $v_1 = 1$, we have $f'(p) = d_p f(1) = \frac{\partial f'}{\partial x_1}(p)$. Thus, $f' = \frac{\partial f}{\partial x_1}$. Suppose $f''(p)$ exists. Note,

$$f''(p) = (f')'(p) = \#(d_p f''(1)) = \frac{\partial f'}{\partial x_1} = \frac{\partial^2 f}{\partial x_1^2}(p). \quad (12)$$

Thus, $f'' = \frac{\partial^2 f}{\partial x_1^2}$. By induction, we find if $f^{(k)}$ exists then $f^{(k)} = \frac{\partial^k f}{\partial x_1^k}$. Furthermore, if $f : A \to A$ is $k$-times $A$-differentiable then

$$\frac{\partial^k f}{\partial x_1 \partial x_2 \cdots \partial x_k} = \frac{\partial^k f}{\partial x_1^k} v_{i_1} v_{i_2} \cdots v_{i_k}. \quad (13)$$

Theorem below gives us license to convert equations in $A$ to partial differential equations which every component of an $A$-differentiable function must solve!

**Theorem 2.12.** Let $U$ be open in $A$ and suppose $f : U \to A$ is $k$-times $A$-differentiable. If there exist $B_{i_1 i_2 \ldots i_k} \in \mathbb{R}$ for which $\sum_{i_1 i_2 \ldots i_k} B_{i_1 i_2 \ldots i_k} v_{i_1} v_{i_2} \cdots v_{i_k} = 0$ then

$$\sum_{i_1 i_2 \ldots i_k} B_{i_1 i_2 \ldots i_k} \frac{\partial^k f}{\partial x_1 \partial x_2 \cdots \partial x_k} = 0. \quad (14)$$

**Example 2.13.** Since $i^2 = -1$ in $\mathbb{C}$ it follows for $z = x + iy$ that complex differentiable $f$ have $f_{yy} = -f_{xx}$. We usually see this in notation $f = u + iv$ and the observation $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$. The real and imaginary parts of a complex differentiable function are harmonic because $1 + i^2 = 0$.

**Example 2.14.** Since $j^2 = 1$ in $\mathbb{H}$ it follows for $z = x + iy$ that $\mathbb{H}$-differentiable $f$ have $f_{yy} = f_{xx}$. If $f = u + iv$ then $u_{xx} - u_{yy} = 0$ and $v_{xx} - v_{yy} = 0$. Thinking of $y$ as time and $x$ as position, the partial differential equation $u_{xx} = u_{yy}$ is a unit-speed wave equation. Component functions of hyperbolic differentiable functions are solutions to the wave equation in one dimension!
For complex algebras the Cauchy Integral Formula links differentiation and integration in such a way that one complex derivative’s existence requires all higher complex derivative’s likewise exist. We consider many examples where \( \mathcal{A} \) does not permit such a simplification. However, if we know \( f \) is smooth in the real sense and once \( \mathcal{A} \)-differentiable then it is shown in [5] that \( f \) allows infinitely many \( \mathcal{A} \)-derivatives. Taylor’s Formula for \( \mathcal{A} \) is also given:

**Theorem 2.15. (Taylor’s Formula for \( \mathcal{A} \)-Calculus:)** Let \( \mathcal{A} \) be a commutative, unital, associative algebra over \( \mathbb{R} \). If \( f \) is real analytic at \( p \in \mathcal{A} \) then

\[
f(p + h) = f(p) + f'(p) \star h + \frac{1}{2} f''(p) \star h^2 + \cdots + \frac{1}{k!} f^{(k)}(p) \star h^k + \cdots
\]

where \( h^2 = h \star h \) and \( h^{k+1} = h^k \star h \) for \( k \in \mathbb{N} \).

### 2.4 Integral calculus on \( \mathcal{A} \)

Integration along curves in \( \mathcal{A} \) is defined in [5] in much the same fashion as \( \mathbb{C} \). If \( \zeta : [t_0, t_1] \to \mathcal{A} \) is differentiable parametrization of a curve \( C \) and \( f \) is continuous near \( C \) then

\[
\int_C f(\zeta) \star d\zeta = \int_{t_0}^{t_1} f(\zeta(t)) \star \frac{d\zeta}{dt} dt. \tag{15}
\]

**Theorem 2.16.** Let \( C \) be a rectifiable curve with arclength \( L \). Suppose \( ||f(\zeta)|| \leq M \) for each \( \zeta \in C \) and suppose \( f \) is continuous near \( C \). Then

\[
|| \int_C f(\zeta) \star d\zeta || \leq m_\mathcal{A} M L
\]

where \( m_\mathcal{A} \) is a constant such that \( ||z \star w|| \leq m_\mathcal{A} \ ||z|| \ ||w|| \) for all \( z, w \in \mathcal{A} \).

Let us conclude with a list of notable results given in [5]. The order in which these results are derived is perhaps surprising. In fact, the next result is last:

**Theorem 2.17. (Fundamental Theorem of Calculus Part I:)** Let \( C \) be a differentiable curve from \( \zeta_0 \) to \( \zeta \) in \( U \subseteq \mathcal{A} \) where \( U \) is an open simply connected subset of \( \mathcal{A} \). Assume \( f \) is \( \mathcal{A} \) differentiable on \( U \) then

\[
\frac{d}{d\zeta} \int_C f(\eta) \star d\eta = f(\zeta).
\]

**Theorem 2.18. (Fundamental Theorem of Calculus Part II:)** Suppose \( f = \frac{dF}{d\zeta} \) near a curve \( C \) which begins at \( P \) and ends at \( Q \) then

\[
\int_C f(\zeta) \star d\zeta = F(Q) - F(P).
\]

**Theorem 2.19.** Let \( f : U \to \mathcal{A} \) be a function where \( U \) is a connected subset of \( \mathcal{A} \) then the following are equivalent:

(i) \( \int_{C_1} f \star d\zeta = \int_{C_2} f \star d\zeta \) for all curves \( C_1, C_2 \) in \( U \) beginning and ending at the same points,
(ii.) $\int_C f \ast d\zeta = 0$ for all loops in $U$.

(iii.) $f$ has an antiderivative $F$ for which $\frac{dF}{d\zeta} = f$ on $U$.

**Theorem 2.20. (Cauchy’s Integral Theorem for $A$:)** If $U \subseteq A$ is simply connected then $\int_C f \ast d\zeta = 0$ for all loops $C$ in $U$ if and only if $f$ is $A$-differentiable on $U$.

### 3 Sequences

In this section we will discuss sequences and the concept of limits and convergence in for a real associative algebra of finite dimension. To do this, we will generalize many of the theorems from real analysis to our context. Much of our generalization parallels Rudin’s *Principles of Mathematical Analysis* [10].

**Definition 3.1.** A function $f : \mathbb{N} \to A$ is called a sequence in $A$. If $f(n) = x_n$, for $n \in \mathbb{N}$, then it is customary to denote the sequence $f$ by the symbol $\{x_n\}$. The values of $f$, that is, the elements $x_n$, are called the terms of the sequence.

Convergence of sequences is measured in terms of the norm $\| \cdot \|$ on $A$.

**Definition 3.2.** Let $A$ be a real associative algebra with a norm $\| \cdot \|$. A sequence $\{p_n\}$ in $A$ is said to converge if there is a point $p \in A$ with the following property: For every $\varepsilon > 0$ there is an integer $M$ such that $n \geq M$ implies that $\|p_n - p\| < \varepsilon$. In this case we also say that $\{p_n\}$ converges to $p$, or that $p$ is the limit of $\{p_n\}$, and we write $p_n \to p$, or $\lim_{n \to \infty} p_n = p$. If $\{p_n\}$ does not converge, then it is said to diverge.

**Example 3.3.** Consider the sequence $\{p_n\}$ in $\mathbb{C}$ defined by $p_n = \frac{in+1+i}{n}$ for all $n \in \mathbb{N}$. Recall $| \cdot |$ defined by $|x + iy| = \sqrt{x^2 + y^2}$ provides a norm on $\mathbb{C}$. Consider,

$$|p_n - i| = \left| \frac{in + 1 + i}{n} - i \right| = \left| \frac{in + 1 + i - in}{n} \right| = \frac{|1 + i|}{n} = \frac{\sqrt{2}}{n}. \quad (16)$$

But, given $\varepsilon > 0$, we know from the Archimedean property of real numbers that there exists $M \in \mathbb{N}$ such that $\frac{1}{M} < \frac{\varepsilon}{\sqrt{2}}$. Thus, for all $n \geq M$, we have:

$$|p_n - i| = \frac{\sqrt{2}}{n} \leq \frac{\sqrt{2}}{M} < \sqrt{2} \left( \frac{\varepsilon}{\sqrt{2}} \right) = \varepsilon. \quad (17)$$

Thus $\{p_n\}$ converges to $i$.

Since $A$ is a vector space, a sequence in $A$ is a sequence of vectors over $\mathbb{R}$. Part (iii.) of the Theorem below explains that the convergence of a vector sequence is tied to the convergence of its component sequences relative to a basis. In contrast, Parts (i.), (ii.) and (iv.) of the Theorem below directly resemble the usual results for real sequences, the linearity and multiplicativity of limits, following Theorem 3.3 of [10].
Theorem 3.4. Suppose $\mathcal{A}$ is an associative algebra paired with a submultiplicative norm $\| \cdot \|$ and a basis $\{ v_1, \ldots, v_N \}$ such that $\| v_i \| = 1$ and for all $x = \sum_{i=1}^{N} x^i v_i, |x^i| \leq \| x \|$ for all $i$. Suppose also that $\{ s_n \}, \{ t_n \}$ are sequences in $\mathcal{A}$ with $s_n = \sum_{i=1}^{N} s^i_n v_i, t_n = \sum_{j=1}^{N} t^j_n v_j$, and $\lim_{n \to \infty} s_n = s, \lim_{n \to \infty} t_n = t$ where $s = \sum_{i=1}^{N} s^i, t = \sum_{j=1}^{N} t^j$. Then:

(i.) $\lim_{n \to \infty} (s_n + t_n) = s + t$,

(ii.) $\lim_{n \to \infty} (\alpha \star s_n) = \alpha \star s$, for any number $\alpha \in \mathcal{A}$,

(iii.) $\lim_{n \to \infty} s_n = s$ if and only if $\lim_{n \to \infty} s^i_n = s^i$ for all $i = 1, 2, \ldots, N$,

(iv.) $\lim_{n \to \infty} (s_n \star t_n) = s \star t$.

Proof: begin with (i.) and (ii.) suppose $s_n \to s$ and $t_n \to t$ with respect to $\| \cdot \|$ on $\mathcal{A}$ as described in the Theorem. For $\alpha \neq 0$, given $\varepsilon > 0$, there exist integers $N_1, N_2$ such that

\[ n \geq N_1 \implies \| s_n - s \| < \frac{\varepsilon}{2m_A \| \alpha \|}; \] \hspace{1cm} (18)

\[ n \geq N_2 \implies \| t_n - t \| < \frac{\varepsilon}{2}. \]

If $N_3 = \max\{ N_1, N_2 \}$ then $n \geq N_3$ implies

\[ \| (\alpha \star s_n + t_n) - (\alpha \star s + t) \| \leq \| \alpha \star (s_n - s) \| + \| t_n - t \| \]
\[ \leq m_A \| \alpha \| \| (s_n - s) \| + \| t_n - t \| \]
\[ < m_A \| \alpha \| \frac{\varepsilon}{2m_A \| \alpha \|} + \frac{\varepsilon}{2} \]
\[ = \varepsilon. \]

Let $\alpha = 1$ to obtain (i.) and let $t_n = 0$ for all $n \in \mathbb{N}$ to obtain (ii.) for $\alpha \neq 0$. If $\alpha = 0$, then for any $\varepsilon > 0$ we note $\| \alpha \star s_n - \alpha \star s \| = \| 0 - 0 \| = 0$ hence $\lim_{n \to \infty} (\alpha \star s_n) = \alpha \star s$.

Next we give the proof of (iii.):

($\Rightarrow$) Assume $s_n \to s$. For $\varepsilon > 0$, suppose there exists $M \in \mathbb{N}$ such that $n \geq M$ implies $\| s_n - s \| < \varepsilon$. Thus $n \geq M$ implies $|s^i_n - s^i| \leq \| s_n - s \| < \varepsilon$ and we find $s^i_n \to s^i$ for all $i = 1, \ldots, N$.

($\Leftarrow$) Assume $s^i_n \to s^i$ for all $i = 1, 2, \ldots, N$. Given $\varepsilon > 0$, choose $M_1, M_2, \ldots M_N \in \mathbb{N}$ such
that \( n \geq M_i \) implies \( |s_n^i - s^i| < \frac{\epsilon}{M} \). If \( M = \max\{M_1, \ldots, M_N\} \) and \( n \geq M \) then we find

\[
\|s_n - s\| = \left\| \sum_{i=1}^{N} (s_n^i - s^i)v_i \right\| \\
\leq \sum_{i=1}^{N} \|(s_n^i - s^i)v_i\| \\
= \sum_{i=1}^{N} |s_n^i - s^i||v_i| \\
< \sum_{i=1}^{N} \frac{\epsilon}{\sqrt{N}}(1) \\
= \epsilon.
\]

Therefore, \( s_n \rightarrow s \) and this completes the proof of (iii.).

The proof (iv.) requires some calculation. Let \( C_{ijk} \) be constants such that \( v_i \star v_j = \sum_k C_{ijk}v_k \) and \( C = \max\{|C_{ijk}| \mid 1 \leq i, j, k \leq n\} \). Applying the triangle inequality we find:

\[
\|v_i \star v_j\| \leq \sum_{k=1}^{N} |C_{ijk}| \|v_k\| = CN
\]

as \( \|v_k\| = 1 \) and \( \sum_{k=1}^{N} 1 = N \). Calculate:

\[
\|s_n \star t_n - s \star t\| = \left\| \sum_{i,j} (s_n^i v_i) \star (t_n^j v_j) - \sum_{i,j} (s^i v_i) \star (t^j v_j) \right\| \\
= \left\| \sum_{i,j} (s_n^i t_n^j - s^i t^j)(v_i \star v_j) \right\| \\
\leq \sum_{i,j} |s_n^i t_n^j - s^i t^j| \|v_i \star v_j\| \\
\leq \sum_{i,j} |s_n^i t_n^j - s^i t^j| CN
\]

where we have used Equation (21) in the last step. Observe that

\[
s_n^i t_n^j - s^i t^j = (s_n^i - s^i)(t_n^j - t^j) + s^i(t_n^j - t^j) + t^j(s_n^i - s^i)
\]
and apply it to Equation 22 as to obtain:

\[
\|s_n * t_n - s * t\| = C N \sum_{i,j} \left| (s^i_n - s^j)(t^i_n - t^j) + s^i(t^i_n - t^j) + t^j(s^i_n - s^j) \right|
\]

(24)

\[
\leq C N \left( \sum_{i,j} \left| s^i_n - s^j \right| |t^i_n - t^j| + \sum_{i,j} \left| s^i \right| |t^i_n - t^j| + \sum_{i,j} \left| t^j \right| \left| s^i_n - s^j \right| \right)
\]

\[
\leq C N \left( \sum_{i,j} \left| s^i_n - s^j \right| |t^i_n - t^j| + \sum_{i,j} M_s |t^i_n - t^j| + \sum_{i,j} M_t |s^i_n - s^j| \right)
\]

\[
\leq C N \left( \sum_{i,j} \left| s^i_n - s^j \right| |t^i_n - t^j| + \sum_{i,j} N M_s |t^i_n - t^j| + \sum_{i,j} N M_t |s^i_n - s^j| \right)
\]

where \(M_s = \max\{s^1, \ldots, s^N\}\) and \(M_t = \max\{t^1, \ldots, t^N\}\). Let \(\varepsilon > 0\). Assume \(s_n \to s\) and \(t_n \to t\) thus by (iii.) we know the component sequences \(s^i_n \to s^i\) and \(t^i_n \to t^i\). It follows we may choose \(N_1^i, N_2^i, N_3^i, N_4^i \in \mathbb{N}\) for \(i = 1, \ldots, N\) such that

\[
n \geq N_1^i \implies |s^i_n - s^i| < \frac{\sqrt{\varepsilon N}}{N^2 \sqrt{2C}},
\]

(25)

\[
n \geq N_2^i \implies |t^i_n - t^i| < \frac{\sqrt{\varepsilon N}}{N^2 \sqrt{2C}},
\]

\[
n \geq N_3^i \implies |s^i_n - s^i| < \frac{\varepsilon}{4N^2 M_s C},
\]

\[
n \geq N_4^i \implies |t^i_n - t^i| < \frac{\varepsilon}{4N^2 M_t C}.
\]

If \(N_5 = \max\{N_j^i \mid j = 1, 2, 3, 4, i = 1, \ldots, N\}\) then following Equation 24 and 25 we find

\[
\|s_n * t_n - s * t\| \leq C N \left( \sum_{i,j=1}^N \frac{\sqrt{\varepsilon N}}{N^2 \sqrt{2C}} + \frac{\sqrt{\varepsilon N}}{N^2 \sqrt{2C}} + \sum_{j=1}^N \frac{N M_s \varepsilon}{4N^2 M_s C} + \sum_{i=1}^N \frac{N M_t \varepsilon}{4N^2 M_t C} \right)
\]

(26)

\[
= \frac{\varepsilon}{2N^2} \sum_{i,j=1}^N 1 + \frac{\varepsilon}{4N} \sum_{j=1}^N 1 + \frac{\varepsilon}{4N} \sum_{i=1}^N 1
\]

\[
= \varepsilon.
\]

Therefore, \(s_n * t_n \to s * t\). \(\Box\)

Continuity of functions on \(A\) can be described sequentially.

**Theorem 3.5.** The following are equivalent:

1. For all \(\varepsilon > 0\), there exists \(\delta > 0\) such that \(0 < \|z - z_0\| < \delta\) implies \(\|f(z) - f(z_0)\| < \varepsilon\).

2. For all \(\{z_n\}\) such that \(z_n \to z_0\) as \(n \to \infty\), \(\lim_{n \to \infty} f(z_n) = f(z_0)\).

**Proof:** (i.) \(\Rightarrow\) (ii.): Suppose (i.) and let \(\{z_n\}\) be a sequence in \(A\) such that \(\lim_{n \to \infty} z_n = z_0\). Let \(\varepsilon > 0\). Choose \(\delta > 0\) such that \(\|z - z_0\| < \delta\) implies \(\|f(z) - f(z_0)\| < \varepsilon\). As \(\{z_n\}\) converges to \(z_0\), we know there exists \(M \in \mathbb{N}\) such that \(n > M \Rightarrow \|z_n - z_0\| < \delta\). Observe, if
On any algebra \(\mathcal{A}\) is true.\(^{(i.)}\) is a Cauchy sequence and there exists \(\varepsilon_0 > 0\) such that for any \(\delta > 0\), there exists \(z^*\) such that \(\|z^* - z_0\| < \delta\) and \(\|f(z^*) - f(z_0)\| \geq \varepsilon_0\). In particular, this holds for \(\delta = 1/n\). For each \(n \in \mathbb{N}\) we choose an element \(z_n\) such that \(\|z_n - z_0\| < 1/n\) and \(\|f(z_n) - f(z_0)\| \geq \varepsilon_0\) and thus construct the sequence \(\{z_n\}\). Therefore, \(z_n \to z_0\) and \(\lim_{n \to \infty} f(z_n) \neq f(z_0)\). But this is a contradiction to \((ii.)\) and conclude that \((i.)\) is true. □

**Example 3.6.** On any algebra \(\mathcal{A}\) with a submultiplicative norm described above, the function \(f: \mathcal{A} \to \mathcal{A}\) defined by \(f(z) = z^2 = z \ast z\) for all \(z \in \mathcal{A}\) is continuous. Indeed, for any \(z_0 \in \mathcal{A}\) and any sequence \(\{z_n\}\) which converges to \(z_0\), we have: using \((iv.)\) of Theorem 3.4,

\[
\lim_{n \to \infty} f(z_n) = \lim_{n \to \infty} z_n \ast z_n = (\lim_{n \to \infty} z_n) \ast (\lim_{n \to \infty} z_n) = z_0 \ast z_0 = z_0^2 = f(z_0)
\]

which, by Theorem 3.5, shows continuity at every point of \(\mathcal{A}\) and thus continuity on \(\mathcal{A}\).

We close this section with a valuable concept, based on Definition 3.8 of [10].

**Definition 3.7.** A sequence \(\{p_n\}\) in an algebra \(\mathcal{A}\) with a norm \(\|\cdot\|\) is said to be a Cauchy sequence if for every \(\varepsilon > 0\) there is an integer \(M\) such that if \(n, m \geq M\) then \(\|p_n - p_m\| < \varepsilon\).

**Theorem 3.8.** (Cauchy Criterion) If \(\mathcal{A}\) is an associative algebra paired with a submultiplicative norm \(\|\cdot\|\) and a basis \(\{v_1, \ldots, v_N\}\) such that for all \(x = \sum_{i=1}^{N} x^i v_i, |x^i| < \|x\|\) for each \(i = 1, \ldots, N\), then a sequence \(\{p_n\}\) in \(\mathcal{A}\) is convergent if and only if it is Cauchy.

**Proof:** \((\Rightarrow)\) Suppose the sequence \(\{p_n\}\) converges to \(p\). Given \(\varepsilon > 0\), there exists an integer \(M\) such that \(n \geq M\) implies \(\|p_n - p_m\| < \varepsilon/2\). Suppose \(m, n \geq N\)

\[
\|p_n - p_m\| = \|p_n - p + (p - p_m)\| \leq \|p_n - p\| + \|p_m - p\| < \frac{\varepsilon}{2} + \varepsilon = \varepsilon.
\]

Thus \(\{p_n\}\) is a Cauchy sequence.

\((\Leftarrow)\) Suppose the sequence \(\{p_n\}\) is Cauchy. Thus, given \(\varepsilon > 0\), there exists and integer \(M > 0\) such that \(n, m > N\) implies \(\|p_n - p_m\| < \varepsilon\). Consider, \(p_n - p_m = \sum_{i=1}^{N} (p^i_n - p^i_m)v_i\). Thus, given the properties of the basis of \(\mathcal{A}\), \(p^i_n - p^i_m < \|p_n - p_m\| < \varepsilon\) for \(i = 1, \ldots, N\). We have shown all the component sequences \(\{p^i_n\}\) are Cauchy. We know from real analysis that a real sequence converges if and only if it is Cauchy. Thus, for each \(i = 1, \ldots, N\), \(\{p^i_n\}\) is a real Cauchy sequence and there exists \(p^i \in \mathbb{R}\) for which \(p^i_n \to p^i\). Thus, defining \(p = \sum_{i=1}^{N} p^i v_i\) we find \(p_n \to p\) by part \((iii.)\) of Theorem 3.3 □

If \(\mathcal{A}\) meets the Cauchy Criterion, we say that it is a complete algebra.

**Example 3.9.** \(\mathbb{R}^{n \times n}\) with \(\|A\| = \sqrt{\text{trace}(A^t A)}\) is a complete algebra since its norm has the necessary property with respect to the basis of unit-matrices \(\{E_{ij}\}\) defined by \((E_{ij})_{kl} = \delta_{ik}\delta_{jl}\). It is not hard to show \(\|A_{ij}\| \leq \|A\|\) and \(\|E_{ij}\| = 1\) for all \(i, j\).

All the examples we study are complete since it is known that any finite dimensional vector space over \(\mathbb{R}\) is complete. For example, see Theorem 2.4-2 on page 73 of [9].
4 Series

We now consider sequences of sums and the limits of these sequences. The definition and the two theorems that follow mirror Rudin’s 3.21-3.23 in [10].

Definition 4.1. Given a sequence \( \{a_n\} \) in an algebra \( \mathcal{A} \), we call the sequence \( \{s_n\} \) where
\[
s_n = \sum_{k=1}^{n} a_k
\]
the sequence of partial sums of \( \{a_n\} \). We denote \( \lim_{n \to \infty} s_n = \sum_{k=1}^{\infty} a_k \) or \( \sum a_k \)
and call this limit an infinite series. If \( \{s_n\} \) converges to \( s \), we say that the series converges and write \( \sum_{k=1}^{\infty} a_k = s \). If \( \{s_n\} \) diverges, we say that the series diverges.

Notice if \( m \geq n \) and \( s_n = \sum_{k=1}^{n} a_k \) then
\[
s_m - s_n = \sum_{k=n+1}^{m} a_k = \sum_{k=m}^{n} a_k.
\]
(29)

Hence the Cauchy criterion (Theorem 3.8) applied to the sequence of partial sums provides the following useful test for convergence of a series:

Theorem 4.2. (Cauchy criterion for series) The series \( \sum a_k \) in \( \mathcal{A} \) converges if and only if for every \( \varepsilon > 0 \) there exists \( M \in \mathbb{N} \) such that \( m \geq n \geq M \) implies \( \left\| \sum_{k=n}^{m} a_k \right\| \leq \varepsilon \).

In particular, if \( \sum a_k \) converges then by taking \( m = n \), we find \( \|a_n\| \leq \varepsilon \) for all \( n \geq M \). Therefore, we find:

Theorem 4.3. (n-th Term Test) If \( \sum a_n \) converges then \( \lim_{n \to \infty} a_n = 0 \).

We now develop some theorems to test for convergence of series. The next theorem follows 3.25 of [10], with a modified second part.

Theorem 4.4. (Comparison Test) If \( \|a_n\| \leq c_n \) for \( n \geq N_0 \), where \( N_0 \) is some fixed integer, and if \( \sum c_n \) converges, then \( \sum a_n \) converges. Likewise, if there exists \( b_n \in [0, \infty) \) for which \( \sum b_n \) diverges and \( b_n \leq \|a_n\| \) then \( \sum \|a_n\| \) diverges.

Proof: Suppose \( \varepsilon > 0 \). There exists \( M \geq N_0 \) such that \( m \geq n \geq M \) implies \( \sum_{k=n}^{m} c_k \leq \varepsilon \), by the Cauchy criterion. Hence
\[
\left\| \sum_{k=n}^{m} a_k \right\| \leq \sum_{k=n}^{m} \|a_k\| \leq \sum_{k=n}^{m} c_k \leq \varepsilon.
\]
(30)

Therefore \( \sum a_k \) converges as we have shown it satisfied Cauchy criterion for series. For the divergent case, since \( \sum b_n \) diverges it follows its sequence of partial sums are unbounded and hence \( \|a_n\| \) also has an unbounded sequence of partial sums hence \( \sum_{n} \|a_n\| \) diverges. □

Example 4.5. In \( \mathbb{R}^{2 \times 2} \) where
\[
\left\| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\| = \sqrt{a^2 + b^2 + c^2 + d^2},
\]
the series \( \sum_{n=1}^{\infty} \left[ \frac{1/n^2}{3/n^2} \frac{2/n^2}{4/n^2} \right] \) converges. For every \( n \), we have
\[
\left\| \begin{pmatrix} 1/n^2 & 2/n^2 \\ 3/n^2 & 4/n^2 \end{pmatrix} \right\| = \frac{1+4+9+16}{n^2} = \frac{30}{n^2}.
\]
But, we know from real analysis that \( \sum_{n=1}^{\infty} \frac{30}{n^2} \) converges, so the result follows by the Comparison Test.
Definition 4.6. If the series $\sum \|a_n\|$ converges then $\sum a_n$ is said to converge absolutely.

Naturally, we recover the standard meaning of absolute convergence for real series given the choice $A = \mathbb{R}$ with $\|x\| = \sqrt{x^2}$.

Example 4.7. In $\mathbb{C}$, the series $\sum \frac{i^n}{n}$ converges, as can be shown in complex analysis, but it does not converge absolutely. Indeed, $\sum |\frac{i^n}{n}| = \sum \frac{1}{n}$ does not converge.

Theorem 4.8. If $\sum \|a_n\|$ converges then $\sum a_n$ converges. In other words, absolute convergence implies ordinary convergence for series in $A$.

Proof: Suppose $\sum \|a_n\|$ converges. Let $n > m$

$$0 \leq \left\| \sum_{k=m}^{n} a_k - \sum_{k=m}^{m} a_k \right\| = \left\| \sum_{k=m}^{n} a_k \right\| \leq \sum_{k=m}^{n} \|a_k\|$$

(31)

Since $\sum \|a_n\|$ converges we know by Cauchy Criterion $\sum_{k=m}^{n} \|a_k\| \to 0$. Hence,

$$\left\| \sum_{k=m}^{n} a_k - \sum_{k=m}^{m} a_k \right\| \to 0$$

(32)

and by Theorem 4.2 we find $\sum a_k$ converges. □

The next two theorems mirror 3.33 and 3.34 of [10].

Theorem 4.9. (Root Test) Given $\sum a_n$, put $\alpha = \limsup_{n \to \infty} \sqrt[n]{\|a_n\|}$. Then:

(i.) if $\alpha < 1$, $\sum a_n$ converges absolutely;

(ii.) if $\alpha > 1$, $\sum a_n$ diverges;

(iii.) if $\alpha = 1$, the test gives no information.

Proof: If $\alpha < 1$, we can choose $\beta$ such that $\alpha < \beta < 1$, and an integer $M$ such that

$$\sqrt[n]{\|a_n\|} < \beta$$

(33)

for $n \geq M$. That is, $n \geq M$ implies

$$\|a_n\| < \beta^n$$

(34)

Since $0 < \beta < 1$ we recognize the geometric series $\sum \beta^n$ converges. Convergence of $\sum \|a_n\|$ follows from the comparison test (Theorem 4.4). To prove (ii.) suppose $\alpha > 1$ then by definition of limsup there exists a subsequence $\sqrt[n]{\|a_{n_k}\|} \to \alpha > 1$ thus $a_{n_k} \to 0$ hence $\sum a_n$ diverges. (iii.) the usual examples from real calculus suffice. □

Example 4.10. In $\mathbb{C}$, the series $\sum (\frac{i \cos(n)}{2})^n$ converges by the Root Test. Indeed, we have:

$$\alpha = \limsup_{n \to \infty} \sqrt[n]{\left|\frac{i \cos(n)}{2}\right|^n} = \limsup_{n \to \infty} \left|\frac{i \cos(n)}{2}\right| = \limsup_{n \to \infty} \frac{\cos(n)}{2} = \frac{1}{2} < 1.$$
Theorem 4.11. *(Ratio Test)* The series \( \sum a_n \)

(i.) converges absolutely if \( \limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} < 1 \),

(ii.) diverges if \( \frac{\|a_{n+1}\|}{\|a_n\|} \geq 1 \) for all \( n \geq n_0 \), where \( n_0 \) is some fixed integer.

**Proof:** If condition (i.) holds, we can find \( \beta < 1 \), and an integer \( M \), such that
\[
\frac{\|a_{n+1}\|}{\|a_n\|} < \beta \quad (36)
\]
for \( n \geq M \). In particular,
\[
\|a_{M+1}\| < \beta \|a_M\| \Rightarrow \|a_{M+2}\| < \beta \|a_{M+1}\| < \beta^2 \|a_M\| \Rightarrow \cdots \Rightarrow \|a_{M+p}\| < \beta^p \|a_M\|. \quad (37)
\]
Thus, \( \|a_n\| < \|a_M\| \beta^{-M} \beta^n \) for \( n \geq M \), and it follows from the comparison test that \( \sum \|a_n\| \) converges, since \( \sum \beta^n \) converges, and thus we obtain (i.).

To understand (ii.) suppose \( \|a_{n+1}\| \geq \|a_n\| \neq 0 \) for \( n \geq n_0 \), it is easily seen that the condition \( a_n \to 0 \) fails and thus (ii.) follows by the \( n \)-th term test. \( \square \)

Care should be taken with part (ii.), the absence of a limiting process is significant. The knowledge that \( \lim a_{n+1}/a_n = 1 \) implies nothing about the convergence of \( \sum a_n \). The series \( \sum 1/n \) and \( \sum 1/n^2 \) demonstrate this.

**Example 4.12.** In the quaternions, the series \( \sum \frac{(1+i+j+k)^n}{n!} \) converges absolutely by the Ratio Test. Note,
\[
\alpha = \limsup_{n \to \infty} \frac{\|\frac{(1+i+j+k)^{n+1}}{(n+1)!}\|}{\|\frac{(1+i+j+k)^{n}}{n!}\|} = \limsup_{n \to \infty} \frac{\alpha^{n+1}}{(n+1)!} = \limsup_{n \to \infty} \frac{2}{n+1} = 0 < 1. \quad (38)
\]

Finally, we examine the convergence of sums and products of series.

**Theorem 4.13.** If \( \sum a_n \) and \( \sum b_n \) converge and \( c \in A \) then
\[
\sum (a_n + b_n) = \sum a_n + \sum b_n \quad \& \quad \sum c \star a_n = c \star \sum a_n.
\]

**Proof:** Suppose there exist \( A, B \in A \) for which \( \sum a_n = A \) and \( \sum b_n = B \). Partial sums
\[
A_n = \sum_{k=0}^{n} a_k \quad \& \quad B_n = \sum_{k=0}^{n} b_k \quad (39)
\]
have \( A_n \to A \) and \( B_n \to B \). By Theorem 3.4 we calculate for \( c \in A \),
\[
c \star A_n + B_n \to c \star A + B. \quad (40)
\]
This proves the theorem. \( \square \)

The multiplication of two series is understood in terms of the Cauchy product.
Definition 4.14. Given $\sum a_n = A$, and $\sum b_n = B$, we set
$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$
for $n \in \mathbb{N}$ and call $\sum c_n$ the product of the two given series.

Absolute convergence is useful to study the existence of the product of two series. In particular, we show that the product of a convergent series with an absolutely convergent series converges to the Cauchy product:

**Theorem 4.15.** Suppose

(i.) $\sum_{n=0}^{\infty} a_n$ converges absolutely,

(ii.) $\sum_{n=0}^{\infty} a_n = A \in \mathcal{A}$ and $\sum_{n=0}^{\infty} b_n = B \in \mathcal{A},$

(iii.) $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ for all $n \in \mathbb{N}.$

Then $\sum_{n=0}^{\infty} c_n = A \ast B.$

**Proof:** assume (i.), (ii.) and (iii.) from the statement of the Theorem are given. Let
$$A_n = \sum_{k=0}^{n} a_k, \quad B_n = \sum_{k=0}^{n} b_k, \quad C_n = \sum_{k=0}^{n} c_k, \quad \beta_n = B_n - B. \quad (41)$$

Then notice we can rearrange the terms in the finite sum $C_n$ as follows:

$$C_n = a_0 b_0 + (a_0 b_1 + a_1 b_0) + \cdots + (a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0) \quad (42)$$
$$= a_0 B_n + a_1 B_{n-1} + \cdots + a_n B_0$$
$$= a_0 (B + \beta_n) + a_1 (B + \beta_{n-1}) + \cdots + a_n \beta_0$$
$$= A_n \ast B + a_0 \beta_n + a_1 \beta_{n-1} + \cdots + a_n \beta_0.$$  \(\gamma_n\)

We wish to show that $C_n \rightarrow A \ast B.$ Since $A_n \ast B \rightarrow A \ast B$, it suffices to show that $\gamma_n \rightarrow 0.$ Since $\sum a_n$ converges absolutely, we know there exists $\alpha \in [0, \infty)$ for which
$$\alpha = \sum_{n=0}^{\infty} ||a_n||. \quad (43)$$

Let $\varepsilon > 0$ be given. By (ii.), $\beta_n \rightarrow 0.$ Hence we can choose $M \in \mathbb{N}$ such that $||\beta_n|| \leq \frac{\varepsilon}{\alpha m_A}$ for $n \geq M$, in which case
$$||\gamma_n|| \leq ||a_n \ast \beta_0 + \cdots + a_{n-M} \ast \beta_M|| + ||a_{n-M-1} \ast \beta_{M+1} + \cdots + a_0 \ast \beta_n|| \quad (44)$$
$$\leq ||a_n \ast \beta_0 + \cdots + a_{n-M} \ast \beta_M|| + m_A ||a_{n-M-1}|| ||\beta_{M+1}|| + \cdots + m_A ||a_0|| ||\beta_n||$$
$$\leq ||a_n \ast \beta_0 + \cdots + a_{n-M} \ast \beta_M|| + m_A ||a_{n-M-1}|| \frac{\varepsilon}{\alpha m_A} + \cdots + m_A ||a_0|| \frac{\varepsilon}{\alpha m_A}$$
$$\leq ||a_n \ast \beta_0 + \cdots + a_{n-M} \ast \beta_M|| + (||a_{n-M-1}|| + \cdots + ||a_0||) \frac{\varepsilon}{\alpha}$$
$$\leq ||a_n \ast \beta_0 + \cdots + a_{n-M} \ast \beta_M|| + \varepsilon.$$
Fix $M$ and let $n \to \infty$, we find
\[
\limsup_{n \to \infty} \|\gamma_n\| \leq \varepsilon
\]
since $a_k \to 0$ as $k \to \infty$. Since $\varepsilon$ is arbitrary, we find $\gamma_n \to 0$ and the Theorem follows. \(\square\)

If $\mathcal{A}$ is not commutative then it is possible that $A \star B \neq B \star A$. However, it is clear that a similar argument could be given if we were instead given the absolute convergence of $\sum b_n$. Consequently, the product of two convergent series converges to the product of their sums if at least one of the two series converges absolutely.

## 5 Power series

**Definition 5.1.** Suppose there exists $z_0 \in \mathcal{A}$ and $c_0, c_1, c_2, \cdots \in \mathcal{A}$ and
\[
f(z) = \sum_{n=0}^{\infty} c_n \star (z - z_0)^n
\]
for each $z \in \mathcal{A}$ for which the series converges. Then we say $f(z)$ is a **power series centered at** $z_0$ **in** $\mathcal{A}$ **with coefficients** $c_n$.

The domain of a power series is controlled by both its coefficients and its center. If we change the center while holding the coefficients fixed then the domain is modified by translation.

**Lemma 5.2.** If $f(z) = \sum_{n=0}^{\infty} c_n \star (z - z_1)^n$ converges on $U \subseteq \mathcal{A}$ then $g(z) = \sum_{n=0}^{\infty} c_n \star (z - z_2)^n$ converges on $z_2 - z_1 + U = \{z_2 - z_1 + u \mid u \in U\}$. If $\sum_{n=0}^{\infty} c_n \star z^n$ converges on $U$ then $\sum_{n=0}^{\infty} c_n \star (z - z_0)^n$ converges on $z_0 + U$.

**Proof:** Suppose $x = z_2 - z_1 + z$ for $z \in U$ then $x - z_2 = z - z_1$ hence
\[
g(x) = \sum_{n=0}^{\infty} c_n \star (z - z_1)^n = f(z).
\]
Since and $f(z)$ exists for $z \in U$ we find $g(x)$ exists for each $x \in z_2 - z_1 + U$. Finally, set $z_1 = 0$ and $z_2 = z_0$ to obtain the last claim of the Lemma. \(\square\)

The result below differs from the usual Root Test of real or complex series in that the test does not guarantee divergence for $\|z\| > R$.

**Theorem 5.3. (Root Test for Power Series)** Given an algebra $\mathcal{A}$ with $\|x \star y\| \leq m_\mathcal{A} \|x\| \|y\|$ for all $x, y \in \mathcal{A}$ and power series $\sum c_n \star (z - z_0)^n$ in $\mathcal{A}$, let
\[
\alpha = \limsup_{n \to \infty} \sqrt[n]{\|c_n\|} \quad \& \quad R = \frac{1}{m_\mathcal{A}^\alpha}.
\]
Then $\sum c_n \star (z - z_0)^n$ is absolutely convergent for $\|z - z_0\| < R$. Moreover, if $\alpha = 0$ then $\sum c_n \star (z - z_0)^n$ converges absolutely on $\mathcal{A}$.
Proof: Let us study power series $\sum c_n z^n$ in $A$ centered at 0. Let $a_n = c_n z^n$ and seek to apply Theorem 4.9. Consider:

$$
\limsup_{n \to \infty} \sqrt[n]{\|a_n\|} = \limsup_{n \to \infty} \sqrt[n]{\|c_n z^n\|} \\
\leq \limsup_{n \to \infty} \sqrt[n]{m_A \|z^n\| \|c_n\|} \\
\leq \limsup_{n \to \infty} \sqrt[n]{m_A \|z^n\| |c_n|} \quad \text{(applied Proposition 2.6)} \\
= m_A \|z\| \limsup_{n \to \infty} \sqrt[n]{|c_n|} \\
= \frac{\|z\|}{R}.
$$

Hence, the series is absolutely convergent if $\|z\| < R$. If $\alpha = 0$ then Theorem 4.9 provides absolute convergence at each $z \in A$. Finally, Lemma 5.2 completes the proof for $z_o \neq 0$. □

In the theory of power series over the real or complex numbers the root test provides a boundary between points of convergence and divergence. However, we will see in Example 5.5 the appearance of zero divisors makes it is possible to find additional points of convergence beyond those indicated by the root test. Fortunately, we are able to offer a result which improves Theorem 5.3:

**Theorem 5.4.** Suppose that $\sum c_n (z - z_o)^n$ converges for all $\|z - z_o\| < R$ for some $R > 0$. Then if $\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|}$ then $\alpha \leq \frac{1}{m_A R}$.

**Proof:** Suppose $\varepsilon > 0$, then if $\|z - z_o\| = R - \varepsilon$ we have $\sum c_n (z - z_o)^n$ converges. Thus, by the numerical root test (Theorem 4.9),

$$
\limsup_{n \to \infty} \sqrt[n]{\|c_n z^n\|} \leq 1. \quad (48)
$$

However,

$$
\limsup_{n \to \infty} \sqrt[n]{\|c_n z^n\|} \leq \limsup_{n \to \infty} \sqrt[n]{m_A \|c_n\| |z^n|} = m_A \|z\| \limsup_{n \to \infty} \sqrt[n]{|c_n|} = m_A (R - \varepsilon) \alpha. \quad (49)
$$

Thus $\alpha \leq \frac{1}{m_A (R - \varepsilon)}$. Since this is true for all $\varepsilon > 0$, it guarantees that $\alpha \leq \frac{1}{m_A R}$. □

The result above rules out the possibility that we could have convergence in a ball for some series and yet fail to meet the root test criterion. In contrast, the convergence region extends past a ball in what follows:

**Example 5.5.** Consider $f(z) = \sum_{n=0}^{\infty} (1 + j)z^n$ over the hyperbolic numbers $H = \mathbb{R} \oplus j\mathbb{R}$ where $j^2 = 1$. Observe $c_n = 1 + j$ for all $n$ hence $\|c_n\| = \sqrt{2}$ and the root test applies:

$$
\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|} = \lim_{n \to \infty} 2^{1/2n} = 1. \quad (50)
$$
Therefore, the Root Test provides for absolute convergence of this series where \( \|z\| < 1/m_A \alpha = 1/\sqrt{2} \) as \( \|zw\| \leq \sqrt{2}\|z\|\|w\| \) for hyperbolic numbers. Yet, consider \( z = a(1-j) \) where \( a \in \mathbb{R} \),

\[
f(a(1-j)) = (1+j)(1)+a(1+j)(1-j)+a^2(1+j)(1-j)^2 + \cdots = 1+j
\]
as \( (1+j)(1-j) = 0 \). Therefore, \( f(z) \) converges along the line \( y = -x \) in the hyperbolic numbers where \( z = x+iy \). Moreover, we can show this result extends to a whole band of nearby hyperbolic numbers. Consider points near \( y = -x \) of the form \( w = a + \varepsilon + j(\varepsilon - a) \).

Such a point is distance \( \sqrt{2}|\varepsilon| \) from \( y = -x \). It is helpful to note the following identity:

\[
(1+j)(x+yyj) = (1+j)(x+y) \quad \& \quad (1+j)(x+yyj)^n = (1+j)(x+y)^n
\]

for all \( n \in \mathbb{N} \). Thus, for \( \varepsilon, a \in \mathbb{R} \),

\[
\|(1+j)w^n\| = \|(1+j)(a+\varepsilon+\varepsilon-a)^n\| = \sqrt{2}|2\varepsilon|^n.
\]

Using the root test for the numerical series \( \sum (1+j)w^n \), we obtain

\[
\alpha = \limsup_{n \to \infty} \sqrt[2]{{\|c_n\|}} = \lim_{n \to \infty} |2\varepsilon|2^{1/2n} = 2|\varepsilon|.
\]

This series converges for \( |\varepsilon| < \frac{1}{2} \) and thus \( f(z) \) converges for all \( z \) at a distance of \( \sqrt{2}|\varepsilon| < \frac{1}{\sqrt{2}} \) from the line \( y = -x \). Observe, this domain of convergence includes the disk \( \|z\| < \frac{1}{\sqrt{2}} \) which we obtained via the root test, and, an infinite band of width \( \sqrt{2} \) about the \( y = -x \) line.

Note, \( m_A = 1 \) for \( A = \mathbb{R} \) with \( |x| = \sqrt{x^2} \) or \( A = \mathbb{C} \) with \( \|z\| = \sqrt{z\bar{z}} \) hence the result below is a natural extension of the usual geometric series from real or complex analysis; the radius of the domain for a geometric series in \( A \)-calculus depends inversely on \( m_A \).

Theorem 5.6. For any commutative\(^3\) algebra \( A \), the geometric series \( \sum z^n \) converges to \( (1-z)^{-1} \) for all \( z \in A \) such that \( 1-z \in A^\times \) and \( \|z\| < \frac{1}{m_A} \).

Proof: The fact that the geometric series converges follows from the root test. To show that it converges to \( (1-z)^{-1} \), let \( S_n = \sum_{k=0}^{n} z^k \) and \( z \) meeting the conditions in the theorem. Observe,

\[
S_n(1-z) = S_n - zS_n
\]

\[
= 1+z+z^2+ \cdots +z^n - z - z^2 - \cdots - z^n - z^{n+1}
\]

\[
= 1 - z^{n+1}
\]

and hence \( S_n = (1-z^{n+1})(1-z)^{-1} \) for all \( n \) for which \( (1-z)^{-1} \in A^\times \). Thus,

\[
\|S_n - (1-z)^{-1}\| = \|(1-z^{n+1})(1-z)^{-1} - (1-z)^{-1}\|
\]

\[
= \|z^{n+1}(1-z)^{-1}\|
\]

\[
\leq m_A\|z^{n+1}\|\|(1-z)^{-1}\|
\]

\[
\leq m_A^{n+1}\|z\|^{n+1}\|(1-z)^{-1}\| \quad (\text{applied Proposition 2.6})
\]

which can be made sufficiently small for \( \|z\| < \frac{1}{m_A} \). \( \square \)

However, the domain defined in the theorem above i may fail to be maximal. Consider:

\(^3\)this theorem is likely true in the noncommutative context as well.
Example 5.7. Consider $\mathcal{A} = \mathbb{R} \times \mathbb{R}$ where $(a, b) \star (x, y) = (ax, by)$ hence $(1, 0) \star (0, 1) = (0, 0)$ are zero-divisors. Observe $\mathcal{A}^\times = \{(a, b) \mid ab \neq 0\}$. Consider $z = (x, y)$ and

$$
\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} (x^n, y^n) = \left( \sum_{n=0}^{\infty} x^n, \sum_{n=0}^{\infty} y^n \right).
$$

(57)

The component series are geometric series with radii $x$ and $y$ respective. The series $\sum_{n=0}^{\infty} z^n$ converges when both its component series converge. In particular, we need $|x| < 1$ and $|y| < 1$. We find the geometric series in $\mathbb{R} \times \mathbb{R}$ converges on the square $(-1, 1)^2$. If we set $\|(x, y)\| = \sqrt{x^2 + y^2}$ then it can be shown that $\|z \star w\| \leq \|z\| \|w\|$ hence $m_A = 1$ for $\mathcal{A} = \mathbb{R} \times \mathbb{R}$ thus Theorem 5.9 only provides convergence on the disk $\|z\| < 1$.

In what follows we use the result of Example 5.7 to derive a result which is directly related to Example 5.7 via the isomorphism of Proposition 2.3.

Example 5.8. In Example 5.3 we saw $f(z) = \sum_{n=0}^{\infty} (1 + j)z^n$ converged on the infinite band

$$
B_+ = \{x + jy \mid -1 - x \leq y \leq 1 - x\}.
$$

(58)

By entirely similar arguments, $g(z) = \sum_{n=0}^{\infty} (1 - j)z^n$ will converge on

$$
B_- = \{x + jy \mid -1 + x \leq y \leq 1 + x\}.
$$

(59)

Note, $h(z) = f(z) + g(z)$ is defined for each $z \in B_+ \cap B_-$.

In particular,

$$
h(z) = 2 \sum_{n=0}^{\infty} z^n
$$

(60)

hence we find the geometric series in the hyperbolic numbers converges on a diamond with vertices $\pm 1, \pm j$. Notice, Theorem 5.9 merely provides convergence on the inscribed disk; $\|z\| < \frac{1}{m_\mathcal{A}} = \frac{1}{\sqrt{2}}$. Furthermore, the diamond with vertices $\pm 1$ and $\pm j$ is the image of the square $[-1, 1]^2$ under the linear isomorphism $\Psi$ of Proposition 2.3.

The interplay between $\mathbb{R} \times \mathbb{R}$ with the direct product and the hyperbolic numbers $\mathbb{R} \oplus j\mathbb{R}$ is illustrative of an important calculational technique. It is often wise to exchange a problem in analysis in one algebra for a more lucid problem in an isomorphic algebra.

Up to this point many of our theorems are likely true for noncommutative algebras. However, to discuss fractions in noncommutative algebras we would need to consider left and right divisors. We leave the noncommuting case to a future work.

Theorem 5.9. (Ratio Test for Series with Unit-Coefficients) Suppose $\mathcal{A}$ is an algebra with $\|x \star y\| \leq m_\mathcal{A}\|x\|\|y\|$ for all $x, y \in \mathcal{A}$ and power series $\sum c_n \star (z - z_0)^n$ where $c_n \in \mathcal{A}^\times$ for all $n$. Let

$$
\alpha = \limsup_{n \to \infty} \left\| \frac{c_{n+1}}{c_n} \right\| \quad \text{and} \quad R = \frac{1}{m_\mathcal{A}^2 \alpha}.
$$

Then $\sum c_n \star (z - z_0)^n$ is absolutely convergent for $z - z_0 \in \mathcal{A}^\times$ with $\|z - z_0\| < R$. Moreover, if $\alpha = 0$ then $\sum c_n \star (z - z_0)^n$ converges absolutely on $\mathcal{A}$.
Proof: Suppose \( z_0 = 0 \) and set \( a_n = c_n \ast z^n \) where \( c_n, z \in \mathcal{A} \times \) and note \( a_n = c_n \ast z^n \in \mathcal{A} \times \) hence we are free to apply the ratio test for numerical series in \( \mathcal{A} \):

\[
\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} = \limsup_{n \to \infty} \frac{\|c_{n+1} \ast z^{n+1}\|}{\|c_n \ast z^n\|} = \limsup_{n \to \infty} m_A \frac{\|c_{n+1} \ast z^{n+1}\|}{\|c_n \ast z^n\|} = m_A \limsup_{n \to \infty} \frac{\|c_{n+1}\|}{\|c_n\|} \leq m_A \limsup_{n \to \infty} \frac{\|c_{n+1}\|}{\|c_n\|} \|z\| \leq m_A^2 \|z\| \limsup_{n \to \infty} \frac{\|c_{n+1}\|}{\|c_n\|} = m_A^2 \|z\| \alpha.
\]

Since \( R = \frac{1}{m_A^2} \) the condition \( m_A^2 \|z\| \alpha < 1 \) is equivalent to \( \|z\| < R \) and Theorem 4.11 allows us to conclude that the series converges absolutely for all \( z \in \mathcal{A} \times \) such that \( \|z\| < R \). Finally, apply Lemma 5.2 to extend the proof to \( z_0 \neq 0 \).

Example 5.10. Consider \( \mathcal{H} = \mathbb{R} \oplus j\mathbb{R} \) where \( j^2 = 1 \). Form the hyperbolic power series

\[ f(z) = \sum_{n=0}^{\infty} (1 + j)n!z^n. \]  

Observe,

\[ f(c(1 - j)) = \sum_{n=0}^{\infty} (1 + j)n!c^n(1 - j)^n = 1 + j \]  

as \((1 - j)(1 + j) = 0\) implies the terms with \( n \geq 1 \) all vanish. Thus the power series converges on \( S = \{c(1 + j) \mid c \in \mathbb{R}\} \). However, if \( z \notin S \) then \( (1 + j)n!z^n \in \mathcal{H}^\times \) and it can be shown \( \lim_{n \to \infty} (1 + j)n!z^n \neq 0 \) thus \( f(z) \) diverges outside \( S \).

The Example above generalizes to other algebras. We can construct power series which converge on the zero-divisors or some subset of the zero-divisors and yet diverge everywhere else. Zero-divisors are simply beyond the scope of the ratio test for general power series in \( \mathcal{A} \). Furthermore, Example 5.5 illustrates that the domain of convergence is not governed by root test alone. Interesting things can happen along zero divisors in the algebra. For example, we suspect \( f(\zeta) = \sum_n (1 + j + j^2)\zeta^n \) where \( \zeta = x + yj + zj^2 \) and \( j^3 = 1 \) converges on the infinite slab of thickness 2 centered about the plane \( x + y + z = 0 \).

In contrast to Theorem 5.9 the Theorem below can give us information about the convergence of the series at zero-divisors of the algebra.

Theorem 5.11. (Ratio Test for Series with Real Coefficients) Let \( \mathcal{A} \) be an algebra with \( \|x \ast y\| \leq m_A \|x\|\|y\| \) for all \( x, y \in \mathcal{A} \). power series \( \sum c_n \ast (z - z_0)^n \) where \( 0 \neq c_k \in \mathbb{R} \) for all \( k \in \mathbb{N} \), put

\[
\alpha = \limsup_{n \to \infty} \frac{|c_{n+1}|}{|c_n|}, \quad R = \frac{1}{m_A \alpha}.
\]
Then \( \sum c_n \ast (z - z_0)^n \) converges absolutely for \( \|z - z_0\| < R \). Moreover, if \( \alpha = 0 \) then
\( \sum c_n \ast (z - z_0)^n \) converges absolutely on \( A \).

**Proof:** Set \( z_0 = 0 \) to begin. If \( c_n \in \mathbb{R} \) then
\( \|c_n \ast z^n\| = |c_n| \|z^n\| \). Put \( a_n = c_n \ast z^n \), and work towards applying the ratio test (Theorem 4.11):

\[
\limsup_{n \to \infty} \frac{\|a_{n+1}\|}{\|a_n\|} = \limsup_{n \to \infty} \frac{\|c_{n+1}z^{n+1}\|}{\|c_nz^n\|} = \limsup_{n \to \infty} \frac{|c_{n+1}| \|z^{n+1}\|}{|c_n| \|z^n\|} \leq \limsup_{n \to \infty} \frac{|c_{n+1}| |m_A| \|z\| \|z^n\|}{|c_n| \|z^n\|} \\
\leq m_A \|z\| \limsup_{n \to \infty} \frac{|c_{n+1}|}{|c_n|} = m_A \|z\| \alpha.
\]

Since \( R = \frac{1}{m_A \alpha} \) we find \( m_A \|z\| \alpha < 1 \) provides \( \|z\| < R \). Thus, applying Theorem 4.11, the series converges for all \( z \in A \) such that \( \|z\| < R \). To conclude we apply Lemma 5.2 to extend the proof to \( z_o \neq 0 \). \( \square \)

Next we compare and contrast the convergence indicated by Theorem 5.3, 5.9 and 5.11.

**Example 5.12.** Consider \( \sum_{n=1}^{\infty} \frac{3^n}{n} z^n \) for \( z \in \mathcal{H} = \mathbb{R} \oplus j\mathbb{R} \) with \( j^2 = 1 \). Notice, \( c_n = \frac{3^n}{n} \) are real and recall \( m_A = \sqrt{2} \) for the hyperbolic numbers we consider here. Use \( R_x \) to denote the radius of convergence suggested by Theorem \( x \). We find:

\[
\left\| \frac{c_{n+1}}{c_n} \right\| = \frac{3n}{n+1} \rightarrow 3 \Rightarrow R_{5.11} = \frac{1}{6} \text{ and } R_{5.9} = \frac{1}{3\sqrt{2}}
\]

and,

\[
\sqrt[n]{|c_n|} = \sqrt[n]{\frac{3^n}{n}} = \frac{3}{n^{1/n}} \rightarrow 3 \Rightarrow R_{5.11} = \frac{1}{3\sqrt{2}}.
\]

Naturally, the root test Theorem 5.3 and the real-coefficient ratio test Theorem 5.11 both provide convergence of the series for \( \|z\| < \frac{1}{3\sqrt{2}} \). However, Theorem 5.9 only indicates convergence for \( \|z\| < \frac{1}{6} \). There is nothing illogical about this as Theorem 5.9 is silent concerning the divergence of the series.

Next we study sequences and series of functions. In particular, we apply these results to power functions to gain insight into the theory of power series in \( A \).

**Definition 5.13.** Suppose \( \{f_n\} \) is a sequence of functions \( f_n : E \to A \) where \( E \subseteq A \), and suppose the sequence of numbers \( \{f_n(z)\} \) converges for every \( z \in E \). Then \( f(z) = \lim_{n \to \infty} f_n(z) \) defines \( f : E \to A \). We say that \( \{f_n\} \) converges to \( f \) **pointwise**.

Pointwise convergence does not always guarantee properties of the sequence transfer to the limit. For example, it is possible to have the limit function of a sequence of continuous functions which is discontinuous. To remedy this shortcoming of pointwise convergence we the stronger criteria of uniform convergence:
Definition 5.14. We say that a sequence of functions \( \{f_n\} \) on \( E \subseteq \mathcal{A} \) converges uniformly to \( f : E \to \mathcal{A} \) if for every \( \varepsilon > 0 \) there is an integer \( M \) such that \( n \geq M \) implies \( \|f_n(z) - f(z)\| < \varepsilon \) for all \( z \in E \).

Similar terminology is given for series of functions on \( A \). We say that the series \( \sum f_n(z) \) converges uniformly on \( E \) if the sequence \( \{s_n\} \) of partial sums defined by

\[
\sum_{i=1}^{n} f_i(z) = s_n(z)
\]

converges uniformly on \( E \). There is also a Cauchy criterion for uniform convergence:

**Theorem 5.15.** A sequence of functions \( \{f_n\} \) defined on \( E \subseteq \mathcal{A} \) converges uniformly on \( E \) if and only if for every \( \varepsilon > 0 \) there exists \( M \in \mathbb{N} \) such that \( m, n \geq M \) and \( z \in E \) implies \( \|f_n(z) - f_m(z)\| < \varepsilon \).

**Proof:** If \( f_n \to f \) uniformly on \( E \) then there exists an integer \( M \) such that \( n \geq M \) and \( z \in E \) imply

\[
\|f_n(z) - f(z)\| < \frac{\varepsilon}{2}.
\]

Suppose \( n, m \geq M \) and \( z \in E \) and consider that

\[
\|f_n(z) - f_m(z)\| \leq \|f_n(z) - f(z)\| + \|f_m(z) - f(z)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Conversely, suppose the Cauchy condition holds. As \( \mathcal{A} \) is complete, the sequence \( \{f_n(z)\} \) converges for every \( z \), to a limit we may call \( f(z) \). Thus \( f_n \to f \) on \( E \). It remains to show this convergence is uniform. Let \( \varepsilon > 0 \) be given, and choose \( M \in \mathbb{N} \) such that \( \|f_n(z) - f_m(z)\| < \varepsilon \). Fix \( n \) and let \( m \to \infty \). Since \( f_m(z) \to f(z) \) as \( m \to \infty \), this gives

\[
\|f_n(z) - f(z)\| < \varepsilon
\]

for every \( n \geq M \) and \( z \in E \) hence \( f_n \to f \) uniformly on \( E \). \( \square \)

Weierstrauss' taught us that uniform convergence of series of functions on \( E \subseteq \mathbb{C} \) can be derived from the existence of a majorizing series. In particular, if a convergent numerical series bounds the values of the function on \( E \) then the series of functions converges uniformly on \( E \). This is often known as Weierstrauss M-test.

**Theorem 5.16.** (Weierstrauss M-Test for \( \mathcal{A} \)) Suppose \( \{f_n(z)\} \) is a sequence of functions defined on \( E \). If \( \sum M_n \) is a convergent series in \( \mathbb{R} \) and \( \|f_n(z)\| \leq M_n \) for all \( z \in E \) and \( n \in \mathbb{N} \) then \( \sum f_n \) converges uniformly on \( E \).

**Proof:** Assume \( \|f_k(z)\| \leq M_k \) for each \( k \in \mathbb{N} \) and note for \( m \geq n \):

\[
\left\| \sum_{i=1}^{m} f_i(z) - \sum_{i=1}^{n} f_i(z) \right\| = \left\| \sum_{i=n+1}^{m} f_i(z) \right\| \leq \sum_{i=n}^{m} \|f_i(z)\| \leq \sum_{i=n}^{m} M_i
\]

for each \( z \in E \). Furthermore, by convergence of \( \sum M_n \), for each \( \varepsilon > 0 \) we may select \( M \in \mathbb{N} \) for which \( m, n \geq M \) imply \( \sum_{i=n}^{m} M_i < \varepsilon \). Consequently, the conditions of Theorem 5.15 are met and we conclude \( \sum f_n \) converges uniformly on \( E \). \( \square \)
Theorem 5.17. If \( \sum c_n (z - z_o)^n \) is a power series on \( \mathcal{A} \) and

\[
\alpha = \limsup_{n \to \infty} \sqrt[n]{|c_n|} \quad \& \quad R = \frac{1}{m_A \alpha}.
\]

then \( \sum_n c_n (z - z_o)^n \) is uniformly absolutely convergent for \( \|z - z_o\| \leq R - \varepsilon \) for any \( \varepsilon \in (0, R) \). If \( R \) is infinite, then the series is uniformly absolutely convergent for \( \|z\| \leq L \) for any \( L > 0 \).

Proof: absolute convergence is given by Theorem 5.3. Let \( \varepsilon \in (0, R) \) and choose \( z_1 \) such that \( z_1 - z_o = m_{\mathcal{A}}(R - \varepsilon) > 0 \). Note \( z_1 - z_o \) is by construction real and \( z_1 - z_o < m_{\mathcal{A}}R = 1/\alpha \) hence:

\[
\limsup_{n \to \infty} \sqrt[n]{|c_n| (z_1 - z_o)^n} = (z_1 - z_o) \limsup_{n \to \infty} \sqrt[n]{|c_n|} < \alpha/\alpha = 1.
\]

Therefore, \( \sum_n |c_n (z_1 - z_o)^n| \) converges by Theorem 4.9. For, \( z \) with \( \|z - z_o\| \leq R - \varepsilon \),

\[
\|c_n (z - z_o)^n\| \leq m_{\mathcal{A}}|c_n||z - z_o|^n
\]

\[
\leq \|c_n\| m_{\mathcal{A}}(R - \varepsilon)^n
\]

\[
= \|c_n\|(z_1 - z_o)^n
\]

\[
= \|c_n \ast (z_1 - z_o)^n\|
\]

for each \( n \) thus by the Weierstrass M-Test (5.16) the theorem follows \( \Box \).

Integration in \( \mathcal{A} \) and uniform limits can be interchanged:

Theorem 5.18. Suppose \( C \) be a piecewise smooth curve of length \( L < \infty \) in \( \mathcal{A} \) and suppose \( U \) is an open set containing \( C \). If \( \{f_j\} \) is a sequence of continuous \( \mathcal{A} \)-valued functions on \( U \), and if \( \{f_j\} \) converges uniformly to \( f \) on \( U \) then \( \int_C f_j(z) \ast dz \) converges to \( \int_C f(z) \ast dz \).

Proof: suppose \( \epsilon > 0 \). Note, uniform convergence of \( \{f_j\} \) to \( f \) implies there exists \( N \in \mathbb{N} \) for which \( n \geq N \) implies \( \|f_n(z) - f(z)\| < \frac{\epsilon}{m_A L} \) for all \( z \in U \). Since \( C \subset U \) we have the same estimate for points on \( C \). Furthermore, by the treatment of integration in [3],

\[
\left\| \int_C f_n(z) \ast dz - \int_C f(z) \ast dz \right\| = \left\| \int_C (f_n(z) - f(z)) \ast dz \right\| \leq m_A \frac{\epsilon}{m_A L} L = \epsilon. \quad \Box
\]

Observe: if \( f_n \to f \) uniformly near \( C \) then \( \lim_{n \to \infty} \int_C f_n \ast dz = \int_C (\lim_{n \to \infty} f_n) \ast dz \).

In complex analysis we learn one consequence of Cauchy’s Integral Formula is that a uniformly convergent sequence of complex-differentiable functions has a limit function which is likewise complex-differentiable. However, in the absence of Cauchy’s Integral Formula, no such luxury is available in the study of differentiability of the limit function. We face the usual difficulty of real analysis which is nicely addressed by Dieudonné in result 8.6.3 of [7]. We show Dieudonné’s result extends naturally to \( \mathcal{A} \)-calculus: we provide sufficient conditions for a sequence of \( \mathcal{A} \)-differentiable functions to have an \( \mathcal{A} \)-differentiable limit function:
Theorem 5.19. Let $U$ be an open connected subset of $\mathcal{A}$, $f_n : U \to \mathcal{A}$ an $\mathcal{A}$-differentiable mapping of $U$ for each $n \in \mathbb{N}$. Suppose that:

(i.) there exists one point $z_0 \in U$ such that the sequence $\{f_n(z_0)\}$ converges in $\mathcal{A}$,

(ii.) for every point $a \in U$, there is a ball $B(a)$ of center $a$ contained in $U$ and such that in $B(a)$ the sequence $\{f_n\}$ converges uniformly.

Then for each $a \in U$, the sequence $\{f_n\}$ converges uniformly in $B(a)$; moreover, if, for each $z \in U$, $f(z) = \lim_{n \to \infty} f_n(z)$ and $g(z) = \lim_{n \to \infty} f'_n(z)$, then $g(z) = f'(z)$, for each $z \in U$.

To be clear, when we write $f'(z)$ this indicates the $\mathcal{A}$-derivative of $f$. Hence, in part, the Theorem asserts $f_n \to f$ where $f$ is $\mathcal{A}$ differentiable. Furthermore, for each $z \in U$:

$$\frac{d}{dz} \left( \lim_{n \to \infty} f_n(z) \right) = \lim_{n \to \infty} \left( \frac{df_n}{dz}(z) \right).$$

Proof: suppose $f_n : U \to \mathcal{A}$ is an $\mathcal{A}$-differentiable mapping on an open connected $U \subseteq \mathcal{A}$. In addition, suppose conditions (i.) and (ii.) hold. Notice that $\mathcal{A}$-differentiable implies Frechet differentiable. Hence, by result 8.6.3 in [7] we find uniform convergence of $\{f_n\}$ as described in the Theorem. Dieudonné uses the notation $f'(x)$ for the Frechet derivative of $f$ at $x$. We change notation and write $Df(x)$ for the Frechet derivative of $f$ at $x$. Hence, by (8.6.3) in [7], if for each $z \in U$, $f(z) = \lim_{n \to \infty} f_n(z)$ and $g(z) = \lim_{n \to \infty} Df_n(z)$, then $g(z) = Df(z)$, for each $z \in U$. Let $\{v_1, v_2, \ldots, v_N\}$ serve as a basis for $\mathcal{A}$ with coordinates $x_1, x_2, \ldots, x_n$. By the definition of partial derivative, for each $i = 1, 2, \ldots, N$,

$$Df(z)(v_i) = \frac{\partial f}{\partial x_i}(z) \quad \& \quad Df_n(z)(v_i) = \frac{\partial f_n}{\partial x_i}(z) \tag{75}$$

thus [7] provides the existence of the Frechet derivative as well as the following identity for the partial derivatives:

$$\frac{\partial}{\partial x_i} \left( \lim_{n \to \infty} f_n(z) \right) = \lim_{n \to \infty} \left( \frac{\partial f_n}{\partial x_i}(z) \right). \tag{76}$$

It remains to show $f = \lim_{n \to \infty} f_n$ is $\mathcal{A}$-differentiable on $U$. From (ii.) we know $f_n$ is $\mathcal{A}$-differentiable hence $f_n$ satisfy the symmetric CR-equations.

$$\frac{\partial f_n}{\partial x_i} * v_j = \frac{\partial f_n}{\partial x_j} * v_i. \tag{77}$$

Hence, using the symmetric $\mathcal{A}$-CR equations and Equation [76] we derive:

$$\frac{\partial f}{\partial x_i} * v_j = \frac{\partial}{\partial x_i} \left[ \lim_{n \to \infty} f_n \right] * v_j \tag{78}$$

$$= \lim_{n \to \infty} \left[ \frac{\partial f_n}{\partial x_i} \right] * v_j \tag{79}$$

$$= \lim_{n \to \infty} \left[ \frac{\partial f_n}{\partial x_i} * v_j \right] \tag{80}$$

$$= \lim_{n \to \infty} \left[ \frac{\partial f_n}{\partial x_j} * v_i \right].$$

---

\(^4\)If $v_1 = 1$ then symmetric CR-equations reduce to the usual CR-equations $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial x_i} * v_j$. 

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Consequently, \( \frac{\partial f}{\partial x_t} \star v_j = \frac{\partial f}{\partial x_j} \star v_i \) and thus \( f \) is \( \mathcal{A} \)-differentiable with \( g(z) = f'(z) \). \( \square \)

Theorem 5.19 allows us to establish the \( \mathcal{A} \)-differentiability of power series in \( \mathcal{A} \). In particular, if an \( \mathcal{A} \)-series converges on an open ball about its center then we find the derivative of the series exists on the same open ball and it is obtained via termwise differentiation:

**Corollary 5.20.** If \( \sum c_n \star (z - z_0)^n \) is a power series on \( \mathcal{A} \) which converges for \( \|z - z_0\| < R \) then \( \frac{d}{dz} \sum c_n \star (z - z_0)^n = \sum nc_n \star (z - z_0)^{n-1} \) for each \( z \in \mathcal{A} \) with \( \|z - z_0\| < R \).

**Proof:** We begin with \( z_0 = 0 \). Assume the series \( \sum c_n \star z^n \) converges for \( \|z\| < R \). Let \( U = \{ z \in \mathcal{A} \mid \|z\| < R \} \) and note \( U \) is an open set. Define \( f_n(z) = \sum_{k=0}^{n} c_k \star z^k \) for \( n = 0, 1, \ldots \) and \( z \in U \). Observe \( \frac{df_n}{dz} = \sum_{k=1}^{n} kc_k \star z^{k-1} \) for \( z \in U \) and \( 0 \in U \) with \( \{ f_n(0) \} = \{ c_0 \} \) is convergent. If \( a \in U \) then note \( B_r(a) = \{ z \mid \|z - a\| < r \} \subseteq U \) where \( r = \min\{\|a\|/2, |R - \|a\||/2\} \). We need to show \( \{ f'_n \} \) converges uniformly on \( B_r(a) \). Since \( \sqrt{n} \to 1 \) as \( n \to \infty \), we have

\[
\limsup_{n \to \infty} \sqrt[n]{\|c_n\|} = \limsup_{n \to \infty} \sqrt{\|c_n\|} = \alpha
\]

hence Theorem 5.3 provides that \( \sum nc_n \star z^{n-1} \) converges for \( \|z\| < \frac{1}{m_{\mathcal{A}}} \). But, by the given convergence of \( \sum c_n \star z^n \) on \( \|z\| < R \) we also have \( \alpha \leq \frac{1}{m_{\mathcal{A}}} \) by Theorem 5.4. Thus \( R \leq \frac{1}{m_{\mathcal{A}}} \) and we find that \( \sum nc_n \star z^{n-1} \) converges for \( \|z\| < R \). Therefore, by Theorem 5.17 we find \( \{ f'_n \} \) is uniformly convergent for \( \|z\| < R - \varepsilon \) for any \( \varepsilon \in (0, R) \). Thus \( \{ f'_n \} \) converges uniformly on \( B_r(a) \) as we are free to adjust \( \varepsilon \) such that \( B_r(a) \subseteq B_{R - \varepsilon}(0) \). In summary, we have satisfied conditions (i.) and (ii.) of Theorem 5.19 and the Corollary follows from the identifications \( f(z) = \sum c_n \star z^n \) and \( g(z) = \sum nc_n \star z^{n-1} \) for which \( f'(z) = g(z) \) on \( U \).

Finally, we apply Lemma 5.2 to shift the \( z_0 = 0 \) result to \( z_0 \neq 0 \). \( \square \)

Higher derivatives follow by iteration.

**Corollary 5.21.** If \( f(z) = \sum c_n \star (z - z_0)^n \) is a power series on \( \mathcal{A} \) which converges for \( \|z - z_0\| < R \) then then \( f \) has derivatives of all orders in \( \|z - z_0\| < R \), which are given by

\[
f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1)c_n \star (z - z_0)^{n-k}.
\]

Moreover, \( f^{(k)}(z_0) = k! c_k \) for \( k = 0, 1, 2, \ldots \).

**Proof:** The first equation follows from successively applying Corollary 5.20 to \( f, f', f'', \ldots \). Then, evaluate at \( z = z_0 \) to obtain \( f^{(k)}(z_0) = k! c_k \). \( \square \)

We close this section with a discussion of entire functions on an \( \mathcal{A} \).

**Definition 5.22.** A function \( f : \mathcal{A} \to \mathcal{A} \) is called entire if it can be written as a power series \( \sum a_n \star z^n \) which converges on all of \( \mathcal{A} \).

**Theorem 5.23.** If \( f(x) = \sum a_n x^n \) is an entire function on the reals, then there exists a unique entire extension to \( \mathcal{A} \). The extension has the form \( \hat{f}(z) = \sum a_n z^n \) for each \( z \in \mathcal{A} \).

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Proof: Let \( f \) be an entire function on the reals. Thus, its radius of convergence is infinite, and by the real root test we have:

\[
\limsup_{n \to \infty} \sqrt[n]{|a_n|} = 0 \quad (80)
\]

Let \( \tilde{f}(z) = \sum a_n z^n \) for each \( z \in \mathcal{A} \). Since the coefficients of the extended function \( \tilde{f}(z) \) are the same as those of \( f(x) \) we find \( \tilde{f}(z) \) is entire by Theorem 5.3. If \( g(z) = \sum b_n * z^n \) is entire function on \( \mathcal{A} \) for which \( g|_{\mathbb{R}} = f = \tilde{f}|_{\mathbb{R}} \) then \( g^{(n)}(0) = \tilde{f}^{(n)}(0) \) for \( n = 0, 1, 2, \ldots \). Hence, \( b_n = a_n \) for \( n = 0, 1, 2, \ldots \) by Corollary 5.21. Thus the extension \( \tilde{f}(z) = \sum a_n z^n \) is the unique extension to \( \mathcal{A} \). \( \square \)

Theorem 5.24. If \( f \) is an entire function on \( \mathcal{A} \), then we have

\[
\limsup_{n \to \infty} \sqrt[n]{\|c_n\|} = 0
\]

and thus \( f \) is uniformly absolutely convergent for \( \|z\| < L \) for all \( L > 0 \).

Proof: Since \( f \) is entire, it follows that \( \sum c_n z^n \) converges on \( \mathcal{A} \) and hence, by Theorem 4.3 we have

\[
\lim_{n \to \infty} c_n z^n = 0 \Rightarrow \lim_{n \to \infty} \|c_n z^n\| = 0 \quad \forall z \in \mathcal{A}. \quad (81)
\]

Given \( \varepsilon > 0 \), there exist \( N \in \mathbb{N} \) such that for each \( n \geq N \),

\[
\left\|c_n \left(\frac{1}{\varepsilon}\right)^n\right\| < 1 \Rightarrow \|c_n\| < \varepsilon^n \Rightarrow \sqrt[n]{\|c_n\|} < \varepsilon. \quad (82)
\]

Thus, \( \limsup_{n \to \infty} \sqrt[n]{\|c_n\|} = 0 \) and by Theorem 5.17 we reach the desired result. \( \square \)

Corollary 5.25. The set of entire functions on \( \mathcal{A} \) is an algebra, and the product of two entire functions \( \sum a_n * z^n \) and \( \sum b_n * z^n \) is equal to \( \sum c_n * z^n \) where \( c_n = \sum_{k=0}^{n} a_k * b_{n-k} \).

Proof: It is clear that this set is a real vector space, so all we must show is that the product of two entire functions \( \sum a_n * z_n \) and \( \sum b_n * z_n \) is also entire. By Theorem 5.17 these functions are absolutely convergent on all of \( \mathcal{A} \), and thus by Theorem 4.15 their product will converge on all of \( \mathcal{A} \) and will be

\[
\left( \sum a_n * z^n \right) * \left( \sum b_n * z^n \right) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k * z^k * b_{n-k} * z^{n-k} \right) \quad (83)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k * b_{n-k} * z^n \right)
\]

\[
= \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} a_k * b_{n-k} \right) * z^n
\]

\[
= \sum_{n=0}^{\infty} c_n * z^n \text{ where } c_n = \sum_{k=0}^{n} a_k * b_{n-k}.
\]

Thus, we have the desired result. \( \square \)
6 Transcendental functions

In this section we let $\mathcal{A}$ denote a real associative finite dimensional commutative algebra $\mathcal{A}$. Theorem 5.23 encourages us to take the power series formulation of elementary functions as fundamental. If we want to recover standard elementary functions by restriction to $\mathbb{R} \subseteq \mathcal{A}$ then our definitions must be given. We provide concrete definitions for the exponential, sine, cosine, hyperbolic sine and hyperbolic cosine over any $\mathcal{A}$. We also provide proofs for identities of these functions which equally well apply to a myriad of choices for $\mathcal{A}$.

6.1 Exponential

Definition 6.1. For each $z \in \mathcal{A}$, we define

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$  

Theorem 6.2. Let $\mathcal{A}$ be a commutative, unital, associative algebra over $\mathbb{R}$.

(i.) $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ is entire on $\mathcal{A}$,

(ii.) $\exp(z) \ast \exp(w) = \exp(z + w)$ for all $z, w \in \mathcal{A}$,

(iii.) $\exp(0) = 1$ and $\exp(-z) = \exp(z)^{-1}$ hence $\exp(z) \in \mathcal{A}^\times$.

Proof: (i.) Identify the coefficients of the exponential are $c_n = \frac{1}{n!}$, which gives us

$$\limsup_{n \to \infty} \sqrt[n]{\|c_n\|} = \limsup_{n \to \infty} \frac{1}{\sqrt[n]{n!}} = 0.$$  

(84)

Thus, by Theorem 5.17 this series uniformly converges absolutely for all $z \in \mathcal{A}$.

(ii.) By Theorem 4.13, we have for each $z, w \in \mathcal{A}$:

$$\exp(z) \ast \exp(w) = \left(\sum_{n=0}^{\infty} \frac{z^n}{n!}\right) \ast \left(\sum_{n=0}^{\infty} \frac{w^n}{n!}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{z^k}{k!} \ast \frac{w^{n-k}}{(n-k)!}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{z^k}{n!} \ast \frac{w^{n-k}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{z^k}{n!} \ast \frac{w^{n-k}}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}$$

$$= \exp(z + w).$$
Clearly $exp(0) = 1$ and since for each $z \in A$ we have

$$exp(z) \ast exp(-z) = exp(z - z) = exp(0) = 1,$$

thus $exp(z) \in A^\times$ for each $z \in A$. Moreover, $exp(-z) = exp(z)^{-1}$. □

The following Theorem should not be surprising.

**Theorem 6.3.** For each $z \in A$, $\frac{d}{dz} exp(z) = exp(z)$.

**Proof:** From Corollary 5.20, we know that $\frac{d}{dz} exp(z)$ exists and

$$\frac{d}{dz} exp(z) = \sum_{n=1}^{\infty} \frac{n z^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = exp(z)$$

for all $z \in A$. □

To appreciate the content of this section we expand the exponential into component functions of several interesting algebras.

**Example 6.4.** Let $A = \mathbb{R}$ then $exp(x) = e^x$ is the usual exponential of real calculus.

**Example 6.5.** Let $A = \mathbb{C}$ then $exp(x + iy) = exp(x) exp(iy) = e^x \cos y + ie^x \sin y$. Setting $u + iv = exp(x + iy)$ we find $u = e^x \cos y$ and $v = e^x \sin y$. The component functions of the exponential are solutions to both the Cauchy Riemann equations $u_x = v_y$ and $v_x = -u_y$ and Laplace’s Equation $\phi_{xx} + \phi_{yy} = 0$.

**Example 6.6.** Let $A = \mathbb{H}$ then $exp(x + jy) = exp(x) exp(jy) = e^x (\cosh y + j \sinh y)$. Setting $u + jv = exp(x + jy)$ provides $u = e^x \cosh y$ and $v = e^x \sinh y$. These functions satisfy the hyperbolic Cauchy Riemann equations $u_x = v_y$ and $u_y = v_x$ and both are solutions to the wave equation $\phi_{xx} - \phi_{yy} = 0$.

**Example 6.7.** Let $A = \mathbb{R} \oplus \epsilon \mathbb{R}$ with $\epsilon^2 = 0$. Then $exp(x + \epsilon y) = exp(x) exp(\epsilon y)$. Calculate

$$exp(\epsilon y) = 1 + \epsilon y + \frac{1}{2} \epsilon^2 y^2 + \cdots = 1 + \epsilon y$$

hence $exp(x + \epsilon y) = e^x + \epsilon ye^x$. The component functions of the exponential are $e^x$ and $\epsilon ye^x$ for this nilpotent algebra.

### 6.2 Hyperbolic sine and cosine

As in the usual calculus, the hyperbolic sine and cosine appear as the odd and even pieces of the exponential function.

**Definition 6.8.** For each $z \in A$, we define

$$cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}, \quad \text{and} \quad sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.$$
Theorem 6.9. The series defining $\cosh(z)$ and $\sinh(z)$ uniformly converge absolutely on $\mathcal{A}$.

Proof: For $\cosh(z)$ we have coefficients $c_n = 1/(2n)!$ hence calculate
\begin{equation}
\limsup_{n \to \infty} n^{\sqrt{\|c_n\|}} = \limsup_{n \to \infty} \frac{1}{\sqrt{(2n)!}} = 0
\end{equation}
and for $\sinh(z)$ we have coefficients $c_n = 1/(2n + 1)!$ hence calculate:
\begin{equation}
\limsup_{n \to \infty} n^{\sqrt{\|c_n\|}} = \limsup_{n \to \infty} \frac{1}{\sqrt{(2n + 1)!}} = 0
\end{equation}

Thus, by the Theorem 5.17 we find both series converge absolutely for each $z \in \mathcal{A}$. □

Theorem 6.10. Hyperbolic sine and cosine have the following properties:

(i.) $\cosh(0) = 1$ and $\sinh(0) = 0$,

(ii.) $\cosh(-z) = \cosh(z)$ and $\sinh(-z) = -\sinh(z)$ for each $z \in \mathcal{A}$,

(iii.) $\frac{d}{dz}\cosh(z) = \sinh(z)$ and $\frac{d}{dz}\sinh(z) = \cosh(z)$ for each $z \in \mathcal{A}$.

Proof: Observe (i.) and (ii.) follow immediately from Definition 6.8. Using Corollary 5.20 we derive (iii.) as follows:
\begin{align*}
\frac{d}{dz}\cosh(z) &= \sum_{n=1}^{\infty} \frac{2n z^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty} \frac{z^{2n-1}}{(2n-1)!} = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!} = \sinh(z) \\
\frac{d}{dz}\sinh(z) &= \sum_{n=0}^{\infty} \frac{(2n+1)z^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = \cosh(z). \quad \Box
\end{align*}

Sums and products of these series will also be entire, giving us the following:

Theorem 6.11. $\cosh^2(z) - \sinh^2(z) = 1$ for all $z \in \mathcal{A}$.

Proof: Let $g(z) = \cosh^2(z) - \sinh^2(z)$ for all $z \in \mathcal{A}$. By Theorems 6.10 6.9 and the chain-rule,
\begin{equation}
g'(z) = 2\cosh(z) \ast \sinh(z) - 2\sinh(z) \ast \cosh(z) = 0
\end{equation}
for all $z \in \mathcal{A}$. Since $\mathcal{A}$ is a connected it follows $g(z)$ is constant. Moreover,
\begin{equation}
g(0) = \cosh^2(0) - \sinh^2(0) = 1
\end{equation}
hence the Theorem follows. □

Theorem 6.12. For all $z \in \mathcal{A}$,

(i.) $e^z = \cosh(z) + \sinh(z),$

(ii.) $\cosh(z) = \frac{1}{2}(e^z + e^{-z}),$

(iii.) $\sinh(z) = \frac{1}{2}(e^z - e^{-z}).$
Proof: item (i.) is verified directly from the definitions:

\[
\cosh(z) + \sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n + 1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = e^z. \tag{94}
\]

Hence (ii.) follows from (i.) by simple calculation:

\[
\frac{1}{2}(e^z + e^{-z}) = \frac{1}{2}(\cosh(z) + \sinh(z) + \cosh(-z) + \sinh(-z)) \tag{95}
\]
\[
= \frac{1}{2}(\cosh(z) + \sinh(z) + \cosh(z) - \sinh(z))
\]
\[
= \cosh(z).
\]

Likewise (iii.) follows from (i.)

\[
\frac{1}{2}(e^z - e^{-z}) = \frac{1}{2}(\cosh(z) + \sinh(z) - \cosh(-z) - \sinh(-z)) \tag{96}
\]
\[
= \frac{1}{2}(\cosh(z) + \sinh(z) - \cosh(z) + \sinh(z))
\]
\[
= \sinh(z).
\]

Alternatively, we could have differentiated (ii.) to obtain (iii.). □

Theorem 6.13. For all \(z, w \in \mathcal{A}\)

(i.) \(\cosh(z + w) = \cosh(z) \star \cosh(w) + \sinh(a) \star \sinh(w)\),

(ii.) \(\sinh(z + w) = \sinh(z) \star \cosh(w) + \cosh(z) \star \sinh(w)\).

Proof: For all \(z, w \in \mathcal{A}\) apply Theorems 6.12 and 6.2 part (ii.)

\[
cosh(z + w) = \frac{1}{2}(e^{z+w} + e^{-(z+w)}) \tag{97}
\]
\[
= \frac{1}{2}(e^z \star e^w + e^{-z} \star e^{-w})
\]
\[
= \frac{1}{2}(e^z \star e^w + e^z \star e^{-w} - e^z \star e^{-w} + e^z \star e^{-w})
\]
\[
= \frac{1}{2}(e^z \star (e^w + e^{-w}) - (e^z - e^{-z}) \star e^{-w})
\]
\[
= \frac{1}{2}(2e^z \star \cosh(w) - 2\sinh(z) \star e^{-w})
\]
\[
= c(z) \star c(w) + s(z) \star c(w) - s(z) \star c(-w) - s(z) \star s(-w)
\]
\[
= c(z) \star \cosh(w) + \sinh(z) \star \sinh(w)
\]

where we used the abbreviated notation \(c(z) = \cosh(z)\) and \(s(z) = \sinh(z)\) in the next to last line. Fix \(w\) and differentiate with respect to \(z\) to find

\[
\sinh(z + w) = \sinh(z) \star \cosh(w) + \cosh(z) \star \sinh(w).
\]

Thus the adding angles formulas for hyperbolic functions exist for \(\mathcal{A}\). □
6.3 Sine and cosine

In certain contexts we could use the imaginary unit $i$ for which $i^2 = -1$ to aid in the definition of sine and cosine. However, there are many algebras without such an imaginary unit hence we provide a treatment which only requires the theory of power series to establish the structure of trigonometric functions. Once more, we find identities for trigonometric function which transcend the choice of $\mathcal{A}$. Our arguments here are parallel those found in Section 68 of [8].

**Definition 6.14.** For each $z \in \mathcal{A}$, we define

$$
\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \& \quad \sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
$$

**Theorem 6.15.** The series defining $\cos(z)$ and $\sin(z)$ converge absolutely on $\mathcal{A}$.

**Proof:** For $\cos(z)$ note $c_n = (-1)^n/(2n)!$ and for $\sin(z)$ the coefficient $c_n = (-1)^n/(2n+1)!$ consequently the proof for Theorem 6.9 equally well applies here. □

**Theorem 6.16.** Sine and cosine over $\mathcal{A}$ have the following properties:

(i.) $\cos(0) = 1$ and $\sin(0) = 0$, 

(ii.) $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$ for each $z \in \mathcal{A}$,

(iii.) $\frac{d}{dz} \cos(z) = -\sin(z)$ and $\frac{d}{dz} \sin(z) = \cos(z)$ for each $z \in \mathcal{A}$.

**Proof:** follows from argument nearly identical to those given for Theorem 6.10. □

The following identity is well-known for $\mathbb{R}$ or $\mathbb{C}$, but it just as well applies to sine and cosine over any $\mathcal{A}$:

**Theorem 6.17.** $\cos^2(z) + \sin^2(z) = 1$ for all $z \in \mathcal{A}$.

**Proof:** Let $g(z) = \cos^2(z) + \sin^2(z)$ for all $z \in \mathcal{A}$. By Theorems 6.15 and 6.16 and the chain-rule, we have for all $z \in \mathcal{A}$

$$
g'(z) = -2\cos(z) \ast \sin(z) + 2\sin(z) \ast \cos(z) = 0. \quad (98)
$$

Thus $g(z)$ is constant on all of $\mathcal{A}$. Since $g(0) = \cos^2(0) + \sin^2(0) = 1$, the result follows. □

**Theorem 6.18.** If $f$ is a function on a connected subset $E$ of $\mathcal{A}$, satisfying $f''(z) = -f(z)$ for all $z \in E$ and $f(0) = 0$, $f'(0) = b \in \mathcal{A}$, then $f(z) = b \ast \sin(z)$ for all $z \in E$.

**Proof:** Let

$$
U(z) = f(z) \ast \sin(z) + f'(z) \ast \cos(z) \quad \& \quad V(z) = f(z) \ast \cos(z) - f'(z) \ast \sin(z). \quad (99)
$$

Apply the product rule to obtain:

$$
U'(z) = f'(z) \ast \sin(z) + f(z) \ast \cos(z) + f''(z) \ast \cos(z) - f'(z) \ast \sin(z) \quad (100)
$$

$$
= f(z) \ast \cos(z) - f(z) \ast \cos(z) = 0.
$$
Thus, as $E$ is connected, $U$ and $V$ are constant. Thus $U(z) = U(0) = b$ and $V(z) = V(0) = 0$ for all $z$ in $E$. Hence,

$$\forall z \in E.$$

Therefore,

$$f(z) = f(0) = b \neq 0.$$

Combining Equations 103 and 102 with Theorem 6.17 we derive:

$$g(z) = f(0) = b.$$

**Theorem 6.19.** If $f$ is a function on a connected subset $E$ of $\mathcal{A}$, satisfying $f''(z) = -f(z)$ for all $z \in E$ and $f(0) = a$, $f'(0) = b \in \mathcal{A}$, then $f(z) = a \cdot \cos(z) + b \cdot \sin(z)$ for all $z \in E$.

**Proof:** Let $g(z) = f(z) - a \cdot \cos(z)$. Then $g''(z) = -g(z)$ for all $z \in E$ and $g(0) = 0$, $g'(0) = b$, so by Theorem 6.18 we find $g(z) = b \cdot \sin(z)$ thus $f(z) = a \cdot \cos(z) + b \cdot \sin(z)$. □

We now arrive at the angle addition formulas for trigonometric functions on $\mathcal{A}$:

**Theorem 6.20.** For $z, w \in \mathcal{A}$,

(i.) $\sin(z + w) = \sin(z) \cdot \cos(w) + \sin(w) \cdot \cos(z)$

(ii.) $\cos(z + w) = \cos(z) \cdot \cos(w) - \sin(z) \cdot \sin(w)$

**Proof:** Fix $w \in \mathcal{A}$ and let $f(z) = \sin(z + w)$ for all $z \in \mathcal{A}$. Then $f''(z) = -f(z)$ for all $z \in \mathcal{A}$ and $f(0) = \sin(w)$ and $f'(0) = \cos(w)$, so by Theorem 6.19 we have

$$f(z) = \sin(w) \cdot \cos(z) + \cos(w) \cdot \sin(z)$$

for all $z, w \in \mathcal{A}$ which proves (i.). Continue to hold $w$ fixed and differentiate (i.) with respect to $z$ to find:

$$\cos(z + w) = \cos(z) \cdot \cos(w) - \sin(z) \cdot \sin(w)$$

thus (ii.) holds true. □

These are helpful to uncover the component function content of sine or cosine over $\mathcal{A}$.
Example 6.21. Consider $\mathcal{H} = \mathbb{R} \oplus j\mathbb{R}$ where $j^2 = 1$. For $x + jy \in \mathcal{H}$ we calculate:

$$
\cos(x + jy) = \cos(x)\cos(jy) - \sin(x)\sin(jy).
$$

But, as $(jy)^{2n} = j^{2n}y^{2n} = y^2$ and $(jy)^{2n+1} = (j)^{2n+1}y^{2n+1} = jy^{2n+1}$ hence $\cos(jy) = \cos(y)$ and $\sin(jy) = j\sin(y)$. Consequently,

$$
\cos(x + jy) = \cos(x)\cos(y) - j\sin(x)\sin(y).
$$

Differentiate with respect to $x$ holding $y$ fixed and find

$$
\sin(x + jy) = \sin(x)\cos(y) + j\cos(x)\sin(y).
$$

You might recognize the products of sine and cosine as well-known solutions to the unit-speed wave-equation $\phi_{xx} = \phi_{yy}$. This is no accident, the unit-speed wave equation is the generalized Laplace equation for $\mathcal{H}$ and we know the component functions of an $\mathcal{H}$-differentiable function solve the generalized Laplace equation of $\mathcal{H}$.

7 The N-Pythagorean theorem

In Section 6 we studied functions whose properties were not tied to a particular choice of algebra. We saw how exponentials, cosine, sine, cosh and sinh all enjoy properties which hold in a multitude of algebras. The direction of the current section is quite the opposite. We now consider a method of obtaining new functions which are particular to our choice of algebra. We call such functions the special functions of $\mathcal{A}$.

7.1 Special functions of an algebra

We shall begin by examining how trigonometric and hyperbolic functions appear as the special functions of the complex and hyperbolic numbers respective.

The basis for the complex numbers is $\{1, i\}$ where $i^2 = -1$. We obtain the trigonometric functions by taking the exponential as follows. For all $x$ in $\mathbb{R}$,

$$
e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} i^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} i^{2n} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} i^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \cos(x) + isin(x).
$$
The real and imaginary parts of this exponential are the sine and cosine functions. Following Theorem 5.23 we find unique extensions of the real trigonometric functions to the complex trigonometric functions. The extended functions are the special functions of $\mathbb{C}$.

Likewise, as the basis for the hyperbolic numbers is $\{1, j\}$ where $j^2 = 1$, we have for $x \in \mathbb{R}$,

$$e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} = \sum_{n=0}^{\infty} j^n \frac{x^n}{n!} = \sum_{n=0}^{\infty} j^{2n} \frac{x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} j^{2n+1} \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + j \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} = \cosh(x) + jsinh(x).$$

Extending hyperbolic cosine and sine to $\mathcal{H}$ we obtain the special functions of $\mathcal{H}$. Notice, the series defining hyperbolic cosine and sine appear naturally as component series of $\exp(jx)$.

Let us generalize the observations above for a unital algebra with generator $\varepsilon$:

**Theorem 7.1.** Let $\varepsilon \in \mathcal{A}$ be a generator of a basis $\{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{N-1}\}$ of $\mathcal{A}$. There exist functions $f_1, f_2, \ldots, f_N : \mathbb{R} \to \mathbb{R}$ for which

$$e^{\varepsilon x} = f_1(x) + \varepsilon f_2(x) + \cdots + \varepsilon^{N-1} f_N(x)$$

for all $x \in \mathbb{R}$. Moreover, for each $i = 1, \ldots, n$, there exist real constants $c_{ij}$ such that $f_i(x) = \sum_{j=0}^{\infty} c_{ij} x^j$. That is, $f_1, \ldots, f_n$ are entire on $\mathbb{R}$. Furthermore, the functions $f_1, \ldots, f_n$ uniquely extend to $\mathcal{A}$ via the rules $f_i(z) = \sum_{j=0}^{\infty} c_{ij} z^j$ for each $z \in \mathcal{A}$.

**Proof:** Theorem 6.2 provides the map $z \mapsto \exp(z)$ is an entire function on $\mathcal{A}$. Thus $\exp(x\varepsilon) = \sum_{k=0}^{\infty} \frac{(x\varepsilon)^k}{k!}$ converges for each $x \in \mathbb{R}$. Let $f_i^n : \mathbb{R} \to \mathbb{R}$ be the component functions with respect to the basis $\{1, \varepsilon, \ldots, \varepsilon^{N-1}\}$ for $\mathcal{A}$. That is, define partial sum functions $f_i^n$ by:

$$\sum_{k=0}^{n} \frac{(x\varepsilon)^k}{k!} = f_1^n(x) + \varepsilon f_2^n(x) + \cdots + \varepsilon^{N-1} f_N^n(x).$$

On the other hand, define components of the function $x \mapsto \exp(x\varepsilon)$ by:

$$\exp(x\varepsilon) = f_1(x) + \varepsilon f_2(x) + \cdots + \varepsilon^{N-1} f_N(x)$$
for each $x \in \mathbb{R}$. Since $\sum_{k=0}^{n} \frac{(xe)^k}{k!} \to \exp(x\varepsilon)$ as $n \to \infty$ we find $f_i^n(x) \to f_i(x)$ as $n \to \infty$ for each $x \in \mathbb{R}$. It follows $f_1, f_2, \ldots, f_N$ are entire on $\mathbb{R}$; there exist real constants $c_{ij}$ for which $f_i(x) = \sum_{j=0}^{\infty} c_{ij} x^j$. Theorem 5.23 provides $f_i$ extends uniquely to an entire function on $A$ for $i = 1, 2, \ldots, N$. $\square$

**Definition 7.2.** Given $A$ and $f_1, \ldots, f_N : A \to A$ as discussed in Theorem 7.1 we say $f_1, f_2, \ldots, f_N$ are the special functions of $A$.

**Example 7.3.** The 3-hyperbolic numbers have for basis $\{1, j, j^2\}$, where $j^3 = 1$. Let $x \in \mathbb{R}$,

$$e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!},$$

$$= j^n \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= j^3 \sum_{n=0}^{\infty} \frac{x^3^n}{(3n)!} + j^3 \sum_{n=0}^{\infty} \frac{x^3^{n+1}}{(3n+1)!} + j^3 \sum_{n=0}^{\infty} \frac{x^3^{n+2}}{(3n+2)!}$$

$$= 1^n \sum_{n=0}^{\infty} \frac{x^n}{(3n)!} + j \sum_{n=0}^{\infty} \frac{x^{n+1}}{(3n+1)!} + j^2 \sum_{n=0}^{\infty} \frac{x^{n+2}}{(3n+2)!}$$

Therefore, the special functions of the 3-hyperbolic numbers are defined by:

$$f_1(z) = \cosh_3(z) = \sum_{n=0}^{\infty} \frac{z^{3n}}{(3n)!},$$

$$f_2(z) = \sinh_{31}(z) = \sum_{n=0}^{\infty} \frac{z^{3n+1}}{(3n+1)!},$$

$$f_3(z) = \sinh_{32}(z) = \sum_{n=0}^{\infty} \frac{z^{3n+2}}{(3n+2)!}.$$

We will elaborate on the notation presently, but the following properties can be easily deduced:

$$\cosh_3(0) = 1, \quad \sinh_{31}(0) = \sinh_{32}(0) = 0,$$

$$\frac{d}{dz} \cosh_3(z) = \sinh_{32}(z), \quad \frac{d}{dz} \sinh_{32}(z) = \sinh_{31}(z), \quad \frac{d}{dz} \sinh_{31}(z) = \cosh_3(z).$$

**Example 7.4.** A somewhat simpler example is that of the algebra with basis $\{1, \varepsilon, \varepsilon^2\}$, where $\varepsilon^3 = 0$. The series for the exponential truncates nicely:

$$e^{\varepsilon x} = \sum_{n=0}^{\infty} \frac{(\varepsilon x)^n}{n!},$$

$$= 1 + \varepsilon x + \frac{1}{2} \varepsilon^2 x^2 + \sum_{n=3}^{\infty} \frac{\varepsilon^n x^n}{n!}$$

$$= 1 + \varepsilon x + \frac{1}{2} \varepsilon^2 x^2.$$
Thus we find special functions for the 3-null numbers are

\[ f_1(z) = 1, \quad f_2(z) = z, \quad f_3(z) = \frac{1}{2}z^2 \]  

(118)

for all \( z \in \mathbb{R} \oplus \varepsilon \mathbb{R} \oplus \varepsilon^2 \mathbb{R} \).

### 7.2 The N-trigonometric and N-hyperbolic functions

We now consider specific algebras whose special functions have a remarkable property.

**Definition 7.5.** Let \( N \in \mathbb{N} \).

(i.) \( \{1, j, j^2, \ldots, j^{N-1}\} \), where \( j^N = -1 \) as the \( N \)-complex numbers.

(ii.) Likewise, the algebra spanned by the basis \( \{1, j, j^2, \ldots, j^{N-1}\} \), where \( j^N = 1 \) are defined as the \( N \)-hyperbolic numbers.

Observe, 2-complex numbers are the ordinary complex numbers and 2-hyperbolic numbers form the usual hyperbolic numbers.

**Definition 7.6.** For a given \( N \in \mathbb{N} \), the \( N \)-trigonometric functions are defined on an algebra \( \mathcal{A} \) as follows:

\[
\cos_N(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{Nn}}{(Nn)!} \quad \text{and} \quad \sin_{Nj}(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{Nn+j}}{(Nn+j)!}
\]

for \( j = 1, 2, \ldots, N - 1 \). The \( N \)-hyperbolic functions are defined on an algebra \( \mathcal{A} \) as follows:

\[
\cosh_N(z) = \sum_{n=0}^{\infty} \frac{z^{Nn}}{(Nn)!} \quad \text{and} \quad \sinh_{Nj}(z) = \sum_{n=0}^{\infty} \frac{z^{Nn+j}}{(Nn+j)!}
\]

for \( j = 1, 2, \ldots, N - 1 \).

We used the above notation in Equation [115].

**Theorem 7.7.** We observe that:

1. the \( N \)-trigonometric functions are special functions of the \( N \)-complex numbers,
2. the \( N \)-hyperbolic functions are special functions of the \( N \)-hyperbolic numbers.
Proof: Begin with (1.). Suppose \( j^N = -1 \). We have, for all \( x \in \mathbb{R} \): 

\[
e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} \]  

\[
= \sum_{n=0}^{\infty} \frac{(jx)^{Nn}}{(Nn)!} + \sum_{n=0}^{\infty} \frac{(jx)^{Nn+1}}{(Nn+1)!} + \cdots + \sum_{n=0}^{\infty} \frac{(jx)^{Nn+N-1}}{(Nn+N-1)!} 
\]  

\[
= \sum_{n=0}^{\infty} \frac{(j^N)^n x^{Nn}}{(Nn)!} + \sum_{n=0}^{\infty} \frac{j(j^N)^n x^{Nn+1}}{(Nn+1)!} + \cdots + \sum_{n=0}^{\infty} \frac{j^{N-1}(j^N)^n x^{Nn+N-1}}{(Nn+N-1)!} 
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{Nn}}{(Nn)!} + j \sum_{n=0}^{\infty} \frac{(-1)^n x^{Nn+1}}{(Nn+1)!} + \cdots + j^{N-1} \sum_{n=0}^{\infty} \frac{(-1)^n x^{Nn+N-1}}{(Nn+N-1)!} 
\]

\[
= \cos_N(x) + jsin_N(x) + \cdots + j^{N-1}sin_N(x). 
\]

Thus \( f_1 = \cos_N, f_2 = \sin_N, \ldots, f_N = \sin_N \).

To prove (2.), let \( j^N = 1 \). We have, for all \( x \in \mathbb{R} \):

\[
e^{jx} = \sum_{n=0}^{\infty} \frac{(jx)^n}{n!} \]  

\[
= \sum_{n=0}^{\infty} \frac{(jx)^{Nn}}{(Nn)!} + \sum_{n=0}^{\infty} \frac{(jx)^{Nn+1}}{(Nn+1)!} + \cdots + \sum_{n=0}^{\infty} \frac{(jx)^{Nn+N-1}}{(Nn+N-1)!} 
\]

\[
= \sum_{n=0}^{\infty} \frac{(j^N)^n x^{Nn}}{(Nn)!} + \sum_{n=0}^{\infty} \frac{j(j^N)^n x^{Nn+1}}{(Nn+1)!} + \cdots + \sum_{n=0}^{\infty} \frac{j^{N-1}(j^N)^n x^{Nn+N-1}}{(Nn+N-1)!} 
\]

\[
= \sum_{n=0}^{\infty} \frac{x^{Nn}}{(Nn)!} + j \sum_{n=0}^{\infty} \frac{x^{Nn+1}}{(Nn+1)!} + \cdots + j^{N-1} \sum_{n=0}^{\infty} \frac{x^{Nn+N-1}}{(Nn+N-1)!} 
\]

\[
= \cosh_N(x) + jsinh_N(x) + \cdots + j^{N-1}sinh_N(x). 
\]

So \( f_1 = \cosh_N, f_2 = \sinh_N, \ldots, f_N = \sinh_N \). \( \square \)

### 7.3 Pythagorean functions

We suppose \( \mathcal{A} = \text{span}\{1, j, \ldots, j^{N-1}\} \) is concretely \( \mathbb{R}^N \) in this section. In particular, this means we identify \( 1 = (1,0,\ldots,0), j = (0,1,0,\ldots,0) \) and so forth. This choice of notation makes the regular representation particularly simple. If \( z \in \mathcal{A} \) then

\[
\mathbf{M}(z) = [z | jz | j^2z | \cdots | j^{N-1}z]. \]  

If \( x \in \mathbb{R} \) then \( x \mapsto \det(\mathbf{M}(e^{jx})) = \det( [e^{jx} | j e^{jx} | j^2 e^{jx} | \cdots | j^{N-1} e^{jx}] ) \) defines a function on \( \mathbb{R} \). In fact, this is an entire function on \( \mathbb{R} \) hence it permits a unique extension to \( \mathcal{A} \). We define:

**Definition 7.8.** For \( \mathcal{A} = \mathbb{R}^N \) with basis \( 1, j, j^2, \ldots, j^{N-1} \) we define the Pythagorean Function of \( \mathcal{A} \) by \( \mathcal{P}_\mathcal{A}(z) = \det( [e^{jx} | j e^{jx} | j^2 e^{jx} | \cdots | j^{N-1} e^{jx}] ) \). We also call the restriction of \( \mathcal{P}_\mathcal{A} \) to \( \mathbb{R} \) the real Pythagorean function of \( \mathcal{A} \).
Notice Pythagorean function is manifestly a formula which involves the special functions of $\mathcal{A}$. Once more it is instructive to examine how this construction unfolds for complex and hyperbolic numbers.

**Example 7.9.** In $\mathbb{C}$, as $e^{ix} = \cos(x) + i \sin(x)$ and $ie^{ix} = i\cos(x) - \sin(x)$ we find the Pythagorean function:

$$
P_\mathbb{C}(z) = \det \begin{bmatrix} \cos(z) & -\sin(z) \\ \sin(z) & \cos(z) \end{bmatrix} = \cos^2(z) + \sin^2(z). \tag{122}
$$

Observe $P_\mathbb{C}(z) = 1$ for all $z \in \mathbb{C}$.

**Example 7.10.** In $\mathcal{H}$, as $e^{jx} = \cosh(x) + j \sinh(x)$ and $je^{jx} = j \cosh(x) + \sinh(x)$ we obtain the Pythagorean function:

$$
P_\mathcal{H}(z) = \det \begin{bmatrix} \cosh(z) & \sinh(z) \\ \sinh(z) & \cosh(z) \end{bmatrix} = \cosh^2(z) - \sinh^2(z). \tag{123}
$$

Thus, we again find $P_\mathcal{H}(z) = 1$ for all $z \in \mathcal{H}$.

Indeed, these examples are part of a larger pattern; the Pythagorean function provides an identity for the special functions of $\mathcal{A}$. This remains true in less known algebras.

**Theorem 7.11. (N-Pythagorean Theorem)** If $\mathcal{A} = \text{span}\{1, j, \ldots, j^{N-1}\}$ where $j^N = \pm 1$ or 0 has Pythagorean function $P_\mathcal{A}(z)$ defined by extending $P(x) = \det[M(e^{jx})]$ to $\mathcal{A}$ then $P_\mathcal{A}(z) = 1$ for all $z \in \mathcal{A}$.

We prove this result in §7A. Next we consider how the $N$-Pythagorean Theorem applies to the 3-hyperbolic numbers.

**Example 7.12.** Consider the 3-hyperbolic numbers: $\mathcal{A} = \mathbb{R} \oplus j\mathbb{R} \oplus j^2\mathbb{R}$ where $j^3 = 1$. By definition, $e^{jx} = \cosh_3(z) + j \sinh_{31}(x) + j^2 \sinh_{32}(x)$ hence $je^{jx} = j \cosh_3(z) + j^2 \sinh_{31}(x) + j \sinh_{32}(x)$ and $j^2 e^{jx} = j^2 \cosh_3(z) + \sinh_{31}(x) + j \sinh_{32}(x)$. Therefore,

$$
P_\mathcal{A}(z) = \det \begin{bmatrix} \cosh_3(z) & \sinh_{31}(z) & \sinh_{32}(z) \\ \sinh_{31}(z) & \cosh_3(z) & \sinh_{32}(z) \\ \sinh_{32}(z) & \sinh_{31}(z) & \cosh_3(z) \end{bmatrix} \tag{124}
$$

$$
= \cosh_3^3(z) + \sinh_{31}^3(z) + \sinh_{32}^3(z) - 3\cosh_3(z)\sinh_{31}(z)\sinh_{32}(z).
$$

It is not too difficult to show $\frac{dP_\mathcal{A}}{dz} = 0$ hence the fact that $P_\mathcal{A}(0) = 1$ implies $P_\mathcal{A}(z) = 1$ on the entire set of 3-hyperbolic numbers.

**Example 7.13.** Consider the dual numbers $\mathcal{A} = \mathbb{R} \oplus \eta\mathbb{R}$ with $\eta^2 = 0$. Observe, $e^{\eta x} = 1 + \eta x$ hence $\eta e^{\eta x} = \eta$ and so

$$
P_\mathcal{A}(z) = \det \begin{bmatrix} 1 & 0 \\ z & 1 \end{bmatrix} = 1. \tag{125}
$$

The $N$-Pythagorean Theorem is not especially exciting in the case of $j^N = 0$. 

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7.4 Proof of N-Pythagorean theorem

To prove the $N$-Pythagorean Theorem we prove the identity holds for $\mathbb{R}$ then we argue Theorem 5.23 extends the result to $A$.

We begin by writing an explicit formula for the real Pythagorean function:

$$\mathcal{P}_A(x) = \det(M(e^{jx})) = \det\left[ e^{jx_1}e^{jx_2} \cdots e^{jN-1}e^{jx} \right]. \tag{126}$$

If $f: \text{dom}(f) \subseteq A \rightarrow A$ is $A$-differentiable and $g: \mathbb{R} \rightarrow A$ is real differentiable then there is a chain rule for the composite $f \circ g: \mathbb{R} \rightarrow A$. In particular,

$$\frac{d}{dx}(f(g(x))) = \frac{df}{dg}(g(x)) \frac{dg}{dx}. \tag{127}$$

Here $g = g_1 + jg_2 + \cdots + j^{N-1}g_N$ has $\frac{dg}{dx} = \frac{dg_1}{dx} + j\frac{dg_2}{dx} + \cdots + j^{N-1}\frac{dg_N}{dx}$. Consider $f(z) = e^z$ and $g(x) = jx$ then we know $\frac{dg}{dx} = e^x$ and it is simple to calculate $\frac{dg}{dx} = j$. Hence,

$$\frac{d}{dx} (e^{jx}) = je^{jx}. \tag{128}$$

Multiply by $j^{k-1}$ and note:

$$j^{k-1} \frac{d}{dx} (e^{jx}) = j^{k-1}je^{jx} \Rightarrow \frac{d}{dx} (j^{k-1}e^{jx}) = j^k e^{jx}. \tag{129}$$

In particular, $\frac{d}{dx} \text{Col}_k(M(e^{jx})) = \text{Col}_{k+1}(M(e^{jx}))$ for $k = 1, \ldots, N - 1$. However,

$$\frac{d}{dx} (j^{N-1}e^{jx}) = j^N e^{jx} \tag{130}$$

provides $\frac{d}{dx} \text{Col}_N(M(e^{jx})) = \kappa \text{Col}_1(M(e^{jx}))$ where $\kappa = \pm 1$ for $j^N = \pm 1$ and $\kappa = 0$ for $j^N = 0$.

The main point here is that every column in the regular representation of $e^{jx}$ has a derivative which is proportional to another column in the representation.

In order to compress the notation a bit let us set $M(e^{jx}) = B = [B_1| \cdots |B_N]$ which gives $\mathcal{P}_A(x) = \det(B)$. The formula for the determinant below makes manifest the fact the determinant is a multilinear function of its columns:

$$\det(B) = \sum_{i_1 \cdots i_N}^{N} \epsilon_{i_1i_2 \cdots i_N} B_{i_1}B_{i_2} \cdots B_{i_N}. \tag{131}$$

Here $\epsilon_{i_1 \cdots i_N}$ is the completely antisymmetric symbol where $\epsilon_{12 \cdots N} = 1$. By the $N$-fold product rule we find:

$$\frac{d}{dx} (\det(B)) = \sum_{i_1 \cdots i_N = 1}^{N} \epsilon_{i_1i_2 \cdots i_N} \frac{dB_{i_1}}{dx} B_{i_2} \cdots B_{i_N} + \sum_{i_1 \cdots i_N = 1}^{N} \epsilon_{i_1i_2 \cdots i_N} B_{i_1} \frac{dB_{i_2}}{dx} \cdots B_{i_N} + \cdots + \sum_{i_1 \cdots i_N = 1}^{N} \epsilon_{i_1i_2 \cdots i_N} B_{i_1}B_{i_2} \cdots \frac{dB_{i_N}}{dx} \tag{132}$$

$$= \det[B_2|B_3| \cdots |B_N] + \det[B_1|B_3|B_4| \cdots |B_N] + \cdots + \det[B_1|B_2| \cdots |\kappa B_1]$$

$$= 0.$$
Thus $x \mapsto \det(M(e^{jx}))$ is a constant function on $\mathbb{R}$. Notice $x = 0$ maps to $\det(I) = 1$ hence $\mathcal{P}_A|_\mathbb{R} = 1$. Finally, we use Theorem 5.23 to find $\mathcal{P}_A(z) = 1$ for all $z \in A$. □

8 Acknowledgements

The authors are thankful to N. BeDell for helpful comments on a rough draft of this article. We should mention that W.S. Leslie provided an alternate, purely algebraic, proof of the N-Pythagorean Theorem in private communication.

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