Geometric Perspective of Entropy Function: Embedding, Spectrum and Convexity*

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Abstract

From the perspective of Sen entropy function, we study the geometric and algebraic properties of a class of (extremal) black holes in \( D \geq 4 \) spacetimes. For a given moduli space manifold, we describe the thermodynamic geometry away from attractor fixed point configurations with and without higher derivative corrections. From the notion of embedding theory, the present investigation offers a set of generalized complex structures and associated properties of differentiable manifolds. We have shown that the convexity of arbitrary entropy function can be realized in an extended subfield of the eigenvalues of the Hessian \( B \) of Sen entropy function. Thus, the spectra of \( B \) are analyzed by defining Krull of the corresponding semisymplectic algebras. From the framework of commutative algebra, we find that the convex hull of the eigenvalues defines a generalized spectrum of \( B \). The corresponding complexification is established for finitely many eigenvalues of \( B \). For the minimally extended subfield, we show that the spectrum of \( B \) reduces to the thermodynamic type spectra, at the attractor fixed point(s). In the limit of \( \text{AdS}_2 \times S^{D-2} \) near horizon geometry, the attractor flow analysis offers the stability of arbitrary extremal black hole. From the perspective of string compactifications, our investigation implies a set of deformed S-duality transformations, which contain both the duality invariant charges and monodromy invariant parameters. The role of the algebraic geometry is discussed towards the viewpoints of attractor stability conditions, rational conformal field theory, elliptic curves, deformed quantization(s), moduli manifolds and Calabi-Yau.

Keywords: Entropy Function; Embedding; Spectrum; Convexity; Calabi-Yau; Attractor Stability; S-duality; Deformed Quantization; Generalized Geometry; Higher Derivative Gravity; Black Hole Physics.

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1 Introduction

Definition of the surface gravity follows from the existence of a limiting force, which must be exerted at infinity, in order to hold a unit test mass in place, when approaching the horizon of a static black hole. The surface gravity remains a constant over an imaginary surface inside which there cannot exist an object that can ever escape to the outside world. Such an imaginary surface defines the event horizon of a black hole \[ \text{BH} \].

Mathematically, it has been shown that the surface gravity remains constant over the horizon, for a diffeomorphism covariant Lagrangian \[ \mathcal{L} \]. Further, for any time \( t \) in classical general relativity, the change in the horizon area remains a positive definite quantity \[ A \]. For a given black hole, there exists an equivalence of the energy to the mass, entropy to the horizon area, volume to the angular momentum and the number of moles to the charges. This amounts to the statement that black holes are thermodynamical objects \[ \Omega \]. In particular, we wish to exploit the fact that the black hole entropy \( S_{BH} \) is a thermodynamical quantity. Recently, Sen has obtained a procedure to compute the entropy of a class of extremal black holes, from the entropy function method \[ \mathcal{F} \].

In this direction, our study is extensively applicable to all black branes, as long as the entropy function is known. Moreover, an extension of the entropy function formalism has been made for \[ D_1D_5 \] and \[ D_2D_6N_{S_3} \] non-extremal configurations \[ [15,20] \]. In the throat approximation, these solutions respectively correspond to the Schwarzschild black holes in \[ AdS_3 \times S^3 \times T^4 \] and \[ AdS_3 \times S^2 \times S^1 \times T^4 \].

The entropy function method has emerged as one of the most powerful techniques for calculating the entropy of a large class of extremal black holes in \( D \geq 4 \). Interestingly, such extremal black holes possess \( AdS_2 \times S^{D-2} \) near horizon geometry. It has been known \[ [9,21] \] that the entropy function generalizes the standard attractor mechanism \[ [22,25] \], viz. the horizon data depend only on the charges of the black hole, and remain independent of the corresponding asymptotic values of the scalar fields. Subsequently, the entropy function method provides an interesting front for incorporating various higher derivative corrections. Moreover, such a consideration takes an account of the corrections to the holomorphic prepotential, and thereby it leads to the theory of generalized prepotential \( F \). As examined in section 4, the reason follows from the fact that Legendre transformation of the black hole entropy \( S_{BH} \), when performed with respect to the radial electric fields \( \{ e_i \} \), leads to the definition of the generalized prepotential. It is worth mentioning that \( S_{BH} \) has found further importance from the perspective of the topological string partition function \[ [27,30] \]. On the other hand, the charged black holes in string theory possess the statistical entropy \( S_{BH} = \ln \Omega(\vec{p}, \vec{q}) \), where \( \Omega(\vec{p}, \vec{q}) \) is the degeneracy of the elementary string states \[ [31,34] \]. For the small black holes, it is known \[ [35,36] \] that the leading order entropy vanishes identically and one has \( S_{BH} = 0 \), in the supergravity limit. Most of these formulations involve a heavy use of the supersymmetry algebra in order to analyze the properties of the underlying higher order \( \alpha' \)-corrections.

In contrast to the above, for a given Lagrangian density, Sen entropy function method solely depends on the near horizon field configuration and thus the bosonic content of the theory enables one to evaluate the entropy of an extremal black hole. Interestingly, a class of \( \alpha' \)-corrections to the black hole entropy, arising from an underlying string theory compactification, agrees with the corresponding entropy obtained from the topological string partition function consideration \[ [28,37] \].

For a given black hole configuration, such \( \alpha' \)-corrections are governed by the generalized prepotential \( F \). In this sense, the higher derivative terms play a special role from the perspective of effective string actions. In order to produce the required black hole entropy \( S_{BH} \), the formulation of Sen entropy function not only offers a few terms which appear in the effective Lagrangian density, but it also examines the role of the complete set of higher derivative terms, which could affect the near horizon geometry of an extremal black
hole [43]. Furthermore, it is known that the entropy function method is an analysis of the equations of motion, which does not make a direct use of supersymmetry. Therefore, the evaluation of the entropy only requires to know certain specific structures of the higher derivative terms, which depend on the compactifying internal space. Importantly, the entropy function method remains duality invariant, as the equation of motions are so, thus it provides a duality invariant formula for Wald entropy $S_{BH}$ of the chosen black holes [3][4]. It is worth mentioning that the entropy function method is applicable only to those Lagrangian density $L$, which can be expressed in terms of gauge invariant field strengths and does not explicitly involve the gauge fields of the theory. For example, such a situation arises in the presence of Chern Simons corrections to the Lagrangian density. In such a case, one can try to (i) remove the non-covariant terms, for instance by going to the dual field variables and thereby (ii) bring out the Lagrangian density $L$ into the required form. Interestingly, one of such an example is the BTZ black hole under the Chern Simons and higher derivative $\alpha'$-corrections. In this case, it is known [44] that the gauge field $\{A_i\}$ are expressible in terms of the field strength tensor $F_{\mu\nu}$. Following Wald entropy formula [3][4], if it is not possible to eliminate an explicit gauge field dependence from the Lagrangian density, then the entropy function method needs a further generalization.

From the perspective of the attractor stability of $AdS_2 \times S^{D-2}$ near horizon geometry, it is worth analyzing the physical properties of higher derivative gravity. In this investigation, we consider an arbitrary theory of the higher derivative gravity, which is coupled with the following sets of fields and their respective attractor fixed point horizon values: (i) scalar fields $\{\phi_i\} \rightarrow \{u_i\}$, (ii) electric- magnetic field tensor $F_{\mu\nu}$, with $\{F_{\mu\nu} \rightarrow e_i, F_{\phi_i} \rightarrow \frac{\pi}{2} \sin \theta\}$ and (iii) $AdS_2 \times S^{D-2}$ parameters $\{v_i\} \rightarrow \{v_i^{\infty}\}$. For given horizon data $\{u_i, e_i, p_i, v_i\}$ of an extremal black hole, let $F(\mathbf{u}, \mathbf{v}, \mathbf{p}, \mathbf{\phi})$ be Sen entropy function, then the corresponding attractor entropy $S_{BH}(\mathbf{u}, \mathbf{v})$ can be computed by requiring the existence of asymptotically flat extremal black hole solution. See section 3 for further details. One of the motivational question which we wish to address in this work is the following: Is $AdS_2 \times S^{D-2}$ near horizon geometry a stable attractor? Namely, we examine whether there really exist an interpolating solution between the flat asymptotic geometry and $AdS_2 \times S^{D-2}$ near horizon geometry. The answer is known in certain two derivative theories, see for instance [12][13]. In order to have an interpolating solution between the flat asymptotic geometry and $AdS_2 \times S^{D-2}$ near horizon geometry, the Hessian matrix of the entropy function must be positive definite metric at the critical point(s). Further, it is natural to ask: Does there exist certain generalizations in order to incorporate higher derivative corrected action(s)? As the next step, in order to answer such a question, we offer the algebraic geometric perspective in section 5.

In this short note, we set out to explore the attractor stability of various extremal black holes with a given set of $\alpha'$-corrections. For given entropy function, we define moduli dependent thermodynamic geometry, which as an intrinsic geometry leads to interesting extensions of the thermodynamic configuration, from the viewpoint of the attractor mechanism. In particular, such a geometric extension renders into a moduli dependent (intrinsic) manifold, which are known since the thermodynamic notion of $N = 2$ supergravity theories [22]. Further, the stability of the central charge $Z$ and the corresponding covariant derivatives have been of a constant interest in various supergravity theories [43]. From the perspective of thermodynamic geometry [44][45], there exists an intrinsic Riemannian metric structure, which could be defined as the negative Hessian matrix of the attractor entropy. For a class of extremal and non-extremal black holes in string theory and M-theory [49][50], it has been observed that the thermodynamic geometry offers a well-defined notion of statistical stability and thereby the role of higher derivative corrections at a given attractor. Indeed, the effect of $R^2$ and $R^4$ spacetime corrections are known for certain non-extremal black holes in string theory, see for instance [11][53][54]. From the viewpoint of the intrinsic algebraic geometry, we show that the framework of Sen entropy function method [51][52] offers an efficient characterization of the stability under both the effects of finitely many (i) stringy $\alpha'$-corrections and (ii) quantum loops.

In order to extend the role of the thermodynamic geometry away from the attractor fixed point(s), we need to consider a set of scalar fields $\{\varphi^a\}$. As illustrated in section 2, the Kähler moduli can be explicitly characterized by a set of complex holomorphic coordinates $\{z^i\}$ and thus the Kähler metric can be expressed as $G_{ab} dz^a dz^b = \frac{\partial^2 \mathcal{L}}{\partial z^i \partial \bar{z}^j} dz^i d\bar{z}^j$. Away from the attractor fixed points [53], there exists an embedding $R^k \hookrightarrow R^k \times \mathcal{M}_\varphi$. Geometrically, such a configuration describes a manifold of dimension $k = 1 + 2l$, where $l$ is the number of the electric-magnetic charges. For given thermodynamic variables $\{q_A, p^A\}$, the asymptotic moduli fields $\{\varphi^a_\infty\}$ modify the black holes thermodynamics as $dS = TdS + \psi^A dq_A + \chi_A dp^A - \Sigma_a d\varphi^a_\infty$, where $\varphi^a_\infty \in \mathcal{M}(\varphi^a_\infty)$ [63]. For an arbitrary scalar moduli $\mathcal{M}_\varphi$, one needs to extend the partial derivative $\partial_\varphi$ to the Kähler covariant derivative $\nabla_\varphi$, corresponding to the scalar fields with respect to the Kähler metric $G_{ab}$ on $\mathcal{M}_\varphi$. In the limit when temperature $T \rightarrow 0$, the underlying black hole becomes extremal and thus the components of Ruppenier metric $S_{ab} \rightarrow \infty$. Thus, it turns out that the standard notion of black hole thermodynamics breaks down [63]. In order to deal with black hole thermodynamics near the extreme, we need to consider a renormalized
dependent interacting statistical system. In section 5, we have defined the spectrum and analyzed the convexity of

the basic notion of Riemannian and Symplectic geometries. In section 3, we briefly review the construction of

manifolds.

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S-duality transformations, which contain both the gauge charges and monodromy invariant parameters. In

work is kept to explain the following question: What happens to the thermodynamic intrinsic geometry [46–48],

when one moves away from the attractor fixed point(s)? In the context of AdS$_2 \times S^2$ near horizon geometry,

the effective potential $V_{BH}$ can be further expressed in terms of the central charge $Z_{BH}$ of the extremal black

hole [64]. Subsequently, for given K"ahler potential $K$, the covariant derivatives [64] take the following form

$\nabla_a Z = (\partial_a + \frac{1}{2} \partial_a K) Z$. As the main interest of the present work is to examine the attractor stability of an extremal black hole in arbitrary spacetime dimension $D$, thus it suffices to consider an extreme K"ahler moduli such that the K"ahler potential $K$ remains near the chosen critical point(s), under the fluctuations of the moduli. Subsequently, the examination of the attractor stability of extremal black holes can be considered as the limiting map $\nabla_a \rightarrow \partial_a$. Algebraically, we have shown that such a map involves the replacements $\nabla_a \rightarrow \partial_a$, for a given set of scalars $X_a = \{X_0, X_1, \ldots, X_n\}$. Consequently, there exists a class of shifted S-duality transformations and thus such a prolongation arises as the local transformations on $M := M_0 \otimes M_2 \otimes M_2 \otimes M_2$. See section 4 for an explicit construction of $M$. Indeed, the section 4 shows that the electric and magnetic frames are described by the maps $(0, p_i) \leftrightarrow (q_i, p_i) \rightarrow (q_i, 0)$. This motivates to analyze the question that, under what condition(s), the Hessian matrix $B$ of the entropy function of an extremal black hole remains well-defined and locally convex. In the minimal limit of the field of the eigenvalues of $B$, such a consideration offers both the geometric and algebraic properties, pertaining to the attractor stability of arbitrary $D$ dimensional (extremal) black holes with and without higher curvature spacetime corrections.

For a given entropy function, $B$ defines a semisymplectic geometry, which exhibits the generalized atlas, generalized Legendre transform, generalized symplectomorphism, semi-product and strong stability. Such a geometry describes thermodynamic geometry together with a non-trivial intertwining of the moduli space geometry. Physically, the underlying black hole configuration corresponds to a moduli dependent interacting statistical system. As a set of embeddings and their well-defined compositions, it reduces to the (generalized) symplectic geometry, which possesses a set of generalized complex structures. From the perspective of commutative algebra, we have shown that the convexity of an extremal black hole entropy function can be realized in the minimally extended (sub)field of the eigenvalues of $B$. For a given $M$, it follows that the spectrum of $B$ can be analyzed by defining Krull of the underlying algebra of $B$. Thus, the analysis of such a generalized convexity and spectrum is interesting from the perspective of the commutative algebra. Further, we notice that the convex hull, and thus the underlying complexification of generalized spectrum can be defined in the minimally extended algebraic field, as an extended set of the eigenvalues of $B$. Finally, it is not difficult to show that the above spectra reduce to a thermodynamic like spectrum at the attractor fixed point(s). In this case, the underlying geometry renders to the standard Riemannian geometry at the extremum value of the entropy function. Furthermore, it is possible to demonstrate that both the above mentioned geometries have the same non-zero complexifiable joint spectrum. Herewith, we find an agreement with Sen entropy extremization principle. Physically, such an analysis explains attractor stability of AdS$_2 \times S^{D-2}$ near horizon geometry, pertaining to the chosen extremal black hole. Hereby, we offer an account of the stringy effects, which arise via an introduction of the scalar fields, gauge fields and arbitrary covariant higher derivative terms, in the gravity side. Form the perspective of string theory, the respective structures are exploited in the sections 4 – 6. In the case of the Calabi-Yau (CY) compactification, we have explicitly shown in section 6 that the present consideration implies a set of deformed S-duality transformations, which contain both the gauge charges and monodromy invariant parameters. In principle, this opens up a new avenue towards the compatibility structures and deformed quantizations on CY manifolds.

The rest of the presentation is organized as follows. In section 2, we offer a set of definitions pertaining to the basic notion of Riemannian and Symplectic geometries. In section 3, we briefly review the construction of Sen entropy function method and thereby show its equivalence with the corresponding constrained dynamical system. In section 4, we describe geometric properties of Sen entropy function, which correspond to moduli dependent interacting statistical system. In section 5, we have defined the spectrum and analyzed the convexity
of Sen entropy function of a given black hole, from the perspective of Krull of the underlying algebras. In section 6, we apply the above consideration to the case of Calabi-Yau compactification(s) and provide an algebraic notion to the deformed S-duality transformations. Finally, in section 7, we present perspective remarks and open issues for a future investigation.

2 Real and Complex Geometry

In this section, we provide a brief review of the Riemannian geometry and its topological connection with the symplectic geometry. From the perspective of the subsequent sections, we define a set of needful tools, e.g., existence of closed skewsymmetric forms on an even dimensional Riemannian manifold and review some further developments, see for example the Refs. [65–67]. In particular, we shall offer further refinements of the Kähler geometry in section 6. Importantly, the analysis of the section 6 explores details of the cases pertaining to Calabi-Yau and deformed S-dualities. What follows next that we set up the notations and conventions for a given manifold as follows. Let \( E_n := \{(y_1, y_2, \ldots, y_n) | y_i \in \mathbb{R}\} \) be a set, then \( U \subseteq E_n \) can be defined as an open ball, if there exists a positive radius \( r \) with a given norm on the set \( U \) such that \( U(y, r) := \{x \in E_n | \|x - y\| < r, \forall x \in E_n\} \). Further, for a given \( n \) dimensional Riemannian manifold \( M_n \), we shall consider an open ball, which is centered at the point \( a \) and has a finite radius \( r \), as the set \( B_{M_n}(a, r) = \{x \in M_n | \|x - a\| < r\} \). Thus, an open cover of \( M \subseteq E_n \) can be considered as a finite collection \( \{U_{\alpha}\} \) of the open sets in \( M \), that is endowed in the union \( M := \bigcup_{\alpha} U_{\alpha} \). Subsequently, a subset \( M \) of \( E_n \) is an \( n \)-dimensional smooth manifold, if we are given a finite collection \( \{U_{\alpha}: x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha \} \) such that the following characterizations hold:

(a) The set \( \{U_{\alpha}\} \) forms an open cover of \( M \), where \( U_{\alpha} \) is said to be a co-ordinate neighborhood of \( M \).

(b) We shall assume that each \( x_i^\alpha \) is a \( C^\infty \) real valued function with a finite domain \( U_{\alpha} \), i.e. the map \( x_i^\alpha : U_{\alpha} \rightarrow E_n \) is a \( C^\infty \)-map.

(c) There exists a family of 1-1 maps \( x_{\alpha} : U_{\alpha} \rightarrow E_n \). The 1-1 property of this map shows that the collection \( x_{\alpha}(u) = (x_1^\alpha(u), x_2^\alpha(u), \ldots, x_n^\alpha(u), \ldots) \) offers an existence of the local charts on \( M \), if the range of the mappings are defined over the open sets \( W_{\alpha} \subset E_n \). For a given family of \( C^\infty \)-maps \( \{x_i^\alpha(u)\}, r^{th}-\)local co-ordinate is defined by the local chart \( x_{\alpha} \). Thus, for an arbitrary parametrization \( u \) of \( M \), the coordinate transformations are regular, if (i) Jacobian determinant \( \det(\frac{\partial x^i}{\partial y^j}) \neq 0 \) and (ii) mapping class property \( x_{\alpha} : U_{\alpha} \rightarrow W_{\alpha} \subset E_n \) holds.

(d) Further, if \( (U, x^1) \) and \( (V, x^2) \) are two local charts on the manifold \( M \) such that \( U \cap V = \phi \), then 1-1 nature of the coordinate chart mappings allows us to express the one set of parameters in terms of the other. In particular, we have \( x^1 = x^2(x^2) \) with the following inverse set of the coordinate functions \( x^2 = x^2(x^1) \). Henceforth, we shall assume that the characteristic functions \( x_i^\alpha(u) \) are \( C^\infty \)-maps and thus they form an admissible set of coordinate functions on the manifold \( M \).

For any compact Riemannian manifold \((M, g)\), it follows that there exists \( u := \bigcup_{\alpha=1}^n U_{\alpha} \) such that \( M \subseteq u \). Moreover, the underlying coordinate chart mappings can be expressed as the set \( \{u_{\alpha} : x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha\} \), where \( x_i^\alpha : U_{\alpha} \rightarrow \mathbb{R}^1 \) are \( C^\infty \)-maps \( \forall \alpha \in \{1, 2, \ldots, n\} \). Thus, the collection \( u \) is a proper open cover, whenever \( M = u \). The transition functions \( \{x_i^\alpha \circ x_\beta^{-1}\} \) provide us to go back and forth on \( M \), i.e. we can go from any chosen \( U_{\alpha} := \{x_1^\alpha, x_2^\alpha, \ldots, x_n^\alpha\} \) to other \( U_{\beta} := \{x_1^\beta, x_2^\beta, \ldots, x_n^\beta\} \), as long as the transformations preserve \( \|\frac{\partial x_i^\alpha}{\partial x_j^\beta}\| \neq 0 \). Physically, if \( M \) is contained in some finite cover, then we shall say that \( M \) is a compact manifold. Further, every Riemannian manifold \((M_n, g)\) is endowed with the line element \( ds^2 := g_{ij}dx^idx^j \), which defines distance by the standard inner product structure on the tangent manifold \( T_pM_n \) of the manifold \( M_n \). Moreover, let \( T_1 \) and \( T_2 \) be any two topological vector spaces, then there exists the mapping \( f : T_1 \rightarrow T_2 \), which turns out to be a homeomorphism, if (a) \( f \) is bijective, (b) both \( f \) and \( f^{-1} \) are continuous. In particular, let \( M_1 \) and \( M_2 \) be two such manifolds, then the map \( f : M_1 \rightarrow M_2 \) is a diffeomorphism, if (a) \( f \) is homeomorphism, (b) both the function \( f \) and its inverse \( f^{-1} \) are differentiable. Physically, a square can be viewed as the homeomorphic image of a circle, which is diffeomorphic to the corresponding ellipse.

As mentioned earlier, we wish to examine the nature of the embeddings and spectrum of the associated Hessian matrix of Sen entropy function of the given black hole. Thus, we recall some symplectic geometric properties of an even dimensional Riemannian manifold, in order to examine the geometric and algebraic properties as per the interest of the present work. For the given embeddings, the geometry of closed skewsymmetric forms is essentially topological in nature, and thus we shall often talks about the symplectic topology, in due course of the symplectic geometry. For instance, one of the common feature of the lowest dimensional symplectic geometry is that it is the two dimensional Riemannian geometry, which measures the area of the complex curves, instead of the length of the real curves. In the case of Euclidean spaces, the basic notion of the symplectic geometry can be
described as per the following datum. Consider $R^{2n}$ with the symplectic form $\omega_0 = dx_1 \wedge dy_1 + \ldots + dx_n \wedge dy_n$, then there exists an isomorphism, which could explicitly be given by the formulae: $X = \frac{\partial}{\partial x_j} \rightarrow \iota_X \omega_0 = dy_j$ and $\frac{\partial}{\partial y_j} \rightarrow -dx_j$. This isomorphism turns out to be a rotation through the quarter turn, if both the tangent space $T_x R^{2n}$ and the corresponding cotangent space $T^*_x R^{2n}$ of $R^{2n}$ are identified as: $\frac{\partial}{\partial x_j} \equiv e_{2j-1} = dx_j$ and $\frac{\partial}{\partial y_j} \equiv e_{2j} = dy_j$.

Although, a symplectic form can be regarded as some geometrical structure in nature, however many problems arise, when one posits the existence of a symplectic structure, which reduce to an easier problem in the topology. Thus, the symplectic geometry puts whole of its emphasis on the topological aspects of the geometry. Physically, the origin of the symplectic geometry lies in the fundamentals of classical mechanics and thus the most natural example of a symplectic manifold corresponds to the Euclidean phase space $R^{2n}$, which is the space parameterizing the positions and momenta of a classical system with the $n$ degree of freedoms. The associated symplectic form is defined by $\omega_0 = \sum dp_i \wedge dq_i$, where the position variables of the system are $q_i$ and the corresponding momenta are denoted as $p_i$. In particular, one of the fundamental attention in the symplectic geometry is given to the Darboux theorem, which locally comments that there exists a set of coordinates, in which the symplectic two form can be given by the standard Euclidean symplectic form $\omega_0$. Moreover, the Moser’s method allows, for given any family of the two forms $\{\omega_t; t \in [0,1]\}$ satisfying an appropriate set of hypotheses, viz. an existence of the interpolation between a given symplectic form $\omega_1$ and the standard Darboux symplectic form $\omega_0$, that one can construct a family of diffeomorphisms $\{\varphi_t; t \in [0,1]\}$ such that the following composition holds $\varphi_t^* \omega_1 = \omega_t$. Thus, $\forall t \in [0,1]$, it is possible to exhibit dynamical properties of the given system with the fact that one can diffeomorphically pull back the family $\omega_t$ to standard Darboux form $\omega_0$.

In general, let $M_{2n}$ be a closed even dimensional compact smooth manifold without boundary. Then, there exists a smooth symplectic structure $\omega$ such that $\omega$ is closed, i.e. $d\omega = 0$ and nondegenerate, i.e., $\omega^n = \omega \wedge \ldots \wedge \omega = 0$. For such a 2-form $\omega$ on $M$, the nondegeneracy condition is equivalent to the fact that $\omega$ induces an isomorphism between the vector fields and 1-forms, as the mappings: $T_x M \rightarrow T^*_x M$, for all $X \rightarrow \iota_X \omega = \omega(X, \cdot)$. For any $C^\infty$-function $H : M \rightarrow R$ on a symplectic manifold $(M, \omega)$, the equation $dH(\cdot) = \omega(X_H, \cdot)$ defines the associated Hamiltonian vector field $X_H$ to the corresponding Hamiltonian $H$, and thus the perspective of the quantization. Furthermore, the associated vector field offers a smooth Hamiltonian function on the symplectic manifold $(M, \omega)$. Moreover, there is a family of contractible Riemannian metrics $\{g_J\}$ on the manifold $M$ such that the associated symplectic form $\omega$ is constructed via $\omega$-compatible almost complex structure $J$, i.e., there exists an automorphism $J : TM \rightarrow TM$ such that $J^2 = -Id$, which defines the associated complex vector bundle $TM$. In this case, the compatibility conditions are: (i) symmetry $\omega(x, y) = \omega(Jx, Jy)$ and (ii) non-degeneracy $\omega(x, Jx) > 0, \forall x \neq 0$. These conditions imply that the associated bilinear form $g_J : g_J(x, y) = \omega(x, Jy)$ is the Riemannian metric on the manifold $M$. Thus, we find, for all compatible $\omega$, that the set of possible $J$ is nonempty and contractible. For a detailed treatment of the above notions see [89,99] and the references therein. Interestingly, one of the main aspect of the symplectic structure properties is that there exists its explicit connection with the Riemannian geometry and Kähler geometry. Subsequently, in sections 4 and 5, we examine these notions from the perspective of embedding theory and associated spectra of Sen entropy function of the given black hole.

In the due course of our geometric study, we shall illustrate that there exists the associated notion of generalized Kähler manifolds, which essentially consist of a pair of commuting generalized complex structures. For given complex manifold and the corresponding symplectic manifold, the two generic examples of the generalized complex structures arise as follow. First, the case of the generalized complex structure is simply the standard complex structure, which could be considered as the following bundle maps. Let $M$ be an even dimensional real manifold, which is equipped with the integrable almost complex structure $J : TM \rightarrow TM$, then the automorphism $J_J = \begin{pmatrix} -J & 0 \\ 0 & -J^* \end{pmatrix}$ defines a generalized complex structure on $M$. Similarly, if $\omega$ is the standard symplectic structure on $M$, then $\omega$ can be expressed as the following skew-symmetric map $\omega : TM \rightarrow T^*M$. Thus, for a given even dimensional manifold $M$, the second class of the generalized complex structures can be defined as the automorphism group $J_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix}$.

It is worth mentioning that the above generalized geometric structures remain well-defined on any even dimensional manifold. Further generalized geometric consequences pertaining to Calabi-Yau and symplectic manifolds are discussed in the subsequent sections. From the perspective of the generalized Calabi-Yau manifolds, such structures turn out to be either of the odd type or even type, which can be transformed by the actions of underlying diffeomorphisms and thus an existence of the closed 2-forms on $M$. For the six dimensions
manifolds, it is interesting to note that such generalized geometric notions can be characterized by the critical points of the natural variational problem on certain closed forms \[24,25\]. Subsequently, the moduli space properties may locally be characterized by a well-defined consideration of an ensemble of open sets, which are shown either to be contained in the odd cohomology or in the even cohomology. In this sense, our analysis anticipates the importance of generalized complex structures, which interpolate between the complex and symplectic structures. Thus, the language of the present note naturally fits on the interface of the intrinsic algebraic geometry. In particular, we focus our attention on Calabi-Yau embeddings and the associated generalized S-duality transformations in section 6. Thereby, the perspectives of differential geometry and commutative algebra are jointly examined for Sen entropy function of the extremal black holes.

### 3 Sen Entropy Function

In this section, we present a brief review of Sen entropy function method and thereby offer the attractor entropy of an extremal black hole. Subsequently, we shall invoke the corresponding geometric connection from the perspective of constrained dynamical system. In particular, this section is intended to provide a set of needful tools for the structures, which emerge as certain geometric and algebraic properties of Sen entropy function. In this set up, the extremal black hole shall be viewed as per the following definition. A four dimensional black hole is said to be the extremal solution \[9,21–25\], if (i) it has \(AdS_2 \times S^2\) near horizon geometry, which is in particular known as Robinson-Bertotti vacuum and (ii) the most general near horizon background fields respect \(SO(2,1) \times SO(3)\) symmetry. In this framework, the entropy of an arbitrary extremal black hole can be defined as the limit \(S_{BH}^{ext} := \lim_{r \to 0} (S_{BH})\), where \(h := r_+ - r_-\) and \(S_{BH}\) are respectively the difference between the outer and inner radii of the horizon and the entropy of the corresponding non-extremal black hole. Thus, the extremal limit is reached, when both the outer radius \(r_+\) and inner radius \(r_-\) coincide. Such a limiting procedure is necessary, since an extremal black hole does not possess bifurcate killing horizon \[3–5\]. Henceforth, we shall assume that \(S_{BH}\) is well-defined, for a given regular horizon black hole.

Let us consider an arbitrary theory of the gravity coupled with a set of abelian gauge fields \(\{A_i^{(i)}\}\), scalar fields \(\{\phi_s\}\) and arbitrary combinations of their covariant derivatives. Then, under such a consideration. Refs. \[3–5\] show that the Lagrangian density can be expressed as \(\mathcal{L} := \mathcal{L}[g_{\mu\nu}, Dg_{\mu\nu}, \phi_s, D\phi_s, \ldots, F_{\mu\nu}, DF_{\mu\nu}, \ldots, \gamma]\). Moreover, the Thomos replacement theorem \[3–5\] leads to the fact that the Lagrangian density \(\mathcal{L}\) can be written in a manifestly covariant form and thus it remains independent of the background field \(\gamma\). In particular, let us focus our attention on those higher derivative theories for which the covariant derivatives of all tensor fields vanish. Then, the entropy of the black hole can be computed from Wald formula \[3–5\]. For a given event horizon area \(A_H\), one finds that Wald entropy of the black hole can be expressed as \(S_{BH} = 8\pi \frac{\hat{g}^2}{\phi_s} g_{\mu\nu} g_{s\beta} A_H\).

In fact, the most general solution of the equations of motion, which remain consistent with (i) \(SO(2,1) \times SO(3)\) symmetry and (ii) \(AdS_2 \times S^2\) near horizon geometry, takes the following form: 
\[
ds^2 := g_{\mu\nu} dx^\mu dx^\nu = v_1(-r^2 dt^2 + \frac{1}{r^2}) + v_2(d\theta^2 + \sin^2 \theta d\phi^2),\]
\[\phi_s = u_s\] and \(\{F_{\mu\nu}^{(i)} = e_i, F_{\beta\gamma} = \frac{v_3}{v_1} \sin \theta\}\), where \(v_1, v_2, u_s\) and \(\{e_i\}\) are the constants which label the solution of the equations of motion. In order to do so, let us define the following horizon function \(f(\nabla, \partial, \phi, \varphi) := \int_{S^2} d\theta d\phi \sqrt{-detg}\mathcal{L}, \mathcal{V}(\theta, \phi) \in S^2,\) where \(\{e_i\}\) are \((r,t)\) components of \(F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu\). Further, the electric charges of the theory are measured by \(q_i := \frac{\partial F_{\mu\nu}}{\partial \phi_i}\). The \(\{p_i\}\) denote the magnetic charges of the corresponding black hole. From the perspective of \(AdS_2 \times S^2\) near horizon geometry, we notice further that the function \(f\) is evaluated as an integral over the horizon, which involves only the angular coordinates. Thus, Sen entropy function can be defined as Legendre transform of the above function \(f\) with respect to the electric variables \(\{e_i\}\). In this consideration, Sen entropy function \[9,21\] can in general be expressed as \(F(\nabla, \partial, \phi, \varphi) := 2\pi(\phi_s \frac{\partial F(\nabla, \partial, \phi, \varphi)}{\partial \phi_s} - f(\nabla, \partial, \phi, \varphi)),\) where \(\partial\) denotes inner product between near horizon electric fields \(\varphi\) and the corresponding derivative operators \(\frac{\partial}{\partial \phi}\). Thus, the entropy of the limiting extremal black hole is obtained as the extremum value of the entropy function \(F(\nabla, \partial, \phi, \varphi)|_{\text{extremum}(\varphi, \phi)}\). This shows that Sen entropy function method \[9,21\] is consistent with the equations of the motion. In this sense, it is worth mentioning that Sen entropy function method is a more suggestive manner of the standard attractor mechanism \[22,25\].

Furthermore, the calculation of the entropy (function) of an extremal black hole can be generalized to arbitrary D-dimensional spacetime. To do so, we shall proceed as follows. Let \(\{q_i\}\) be a finite collection of the electric charges associated with the \(D\)-dimensional 1-form gauge fields \(\{A_i\}\) and \(\{p_i\}\) be a finite collection of the constants which label the solution of the equations of motion. In this way, one can choose a local
co-ordinate system such that $AdS_2$ part of the spacetime metric is proportional to $(-r^2 dt^2 + \sum dx_i^2)$, then for all diffeomorphism covariant Lagrangian density $L[g, \phi_s, ...]$, Refs. [3][5] show that the entropy of the black holes satisfies $LT(q_i) = \int_{S^{D-2}} \sqrt{-\det g} L[g, \phi_s, ...]$; where $LT$ denotes Legendre transformation of the entropy $S_{BH}(\vec{q}, \vec{p})$. Here, the above mentioned Legendre transformation is taken with respect to the electric charges $\{q_i\}$. Physically, for a given set of electric charges $\{q_i\}$, the respective conjugate variables $\{e_i\}$ represents a set of radial electric fields, which are associated to $i^{th}$ horizon valued gauge field of the considered black hole configuration.

In particular, let us consider Robinson-Bertotti $AdS_2 \times S^{(D-2)}$ near horizon configuration, such that the parameters $\{v_1, v_2\}$ parameterize the respective sizes of $AdS_2$ and $S^{(D-2)}$. Then, for the given horizon values of electric fields, magnetic fields, scalars fields and vector fields of the extremal black hole, the Legendre transformation of $f = \int_{S^{(D-2)}} \sqrt{-\det g} L$ with respect to $\{e_i\}$ is defined such that we have $q_i := \frac{\partial f}{\partial e_i}$. This implies that $2\pi \times LT_{\{e_i\}}(f) = F$. Thus, Sen entropy function $F(u_s; v_k; q_i, p_0)$ [9][17], as a function of the charges, near horizon moduli and other parameters, if any, can be expressed as $F = 2\pi \times LT_{\{e_i\}}(\int_{S^{(D-2)}} \sqrt{-\det g} L)$. Consequently, it follows, from the equations of motion, that the sizes $\{v_k\}$ of $AdS_2$ and $S^{(D-2)}$ and the near horizon values $\{u_s\}$ of underlying scalar fields $\{\phi_s\}$ can be determined by the extremization of Sen entropy function $F$. For a set of given electric-magnetic charges of the black hole, the above mentioned extremization is performed with respect to both the near horizon moduli $\{u_s\}$ and parameters $\{v_k\}$. In the other words, the equations determining the horizon values of $\{u_s\}$ and $\{v_k\}$ are given by: $\frac{\partial f}{\partial u_s} = 0 \Rightarrow u_s = u^0_s$ and $\frac{\partial f}{\partial v_k} = 0 \Rightarrow v_i = v^0_i$. One of the central result of Sen entropy function method is that the horizon entropy of the corresponding black hole is given by the attractor fixed point formula: $S_{BH} = F(u^0_1, v^0_2; q_i, p_0)$.

From the perspective of the standard attractor mechanism and fixed point behavior of the scalar fields, Sen entropy function method provides an efficient tool to incorporate higher derivative corrections to the attractor equations and thus is an appropriate framework for the calculation of the entropy of an extremal black hole. In particular, one can examine the stability properties of a class of extremal black hole solutions. To do so, let us consider $N = 2$ supergravity theory with a given Lagrangian density $L$, which contains arbitrary finite collection of higher derivative covariant tensor fields. Let $f$ be the reduced Lagrangian density over the horizon of the extremal black hole such that the Legendre transformation of $f$ with respect to $\{e_i\}$ provides the associated Wald entropy, as the extremum of $f$. Further, interesting examples include the followings: (i) $N = 2$ BPS black holes interacting with Weyl tensor multiplet and (ii) non-BPS black holes with $R^2$-corrections. In this case, it is known that the higher derivative $\alpha'$-corrections arise precisely due to the non-holomorphic corrections to the pre-potential [7][11].

From the perspective of $\alpha'$-corrections, an equivalence of Sen entropy function with the corresponding reduced dynamical system follows from the consideration of the function $f = \int_{S^{D-2}} d^D \Omega_L$, as the reduced Lagrangian density over the horizon of the black hole. Indeed, the reduced hamiltonian density can be defined as Sen entropy function $F = \sum e^i q_i - f$, where $q_i := \frac{\partial f}{\partial e_i}$ define constraints on the corresponding dynamical system. As per the present consideration, $\{q_i\}$ and $\{e_i\}$ can be respectively treated as co-ordinates and momenta, for a given entropy function $F(p_i, q_i, e_i, v_i, u_s)$. In the electric and magnetic pictures, these variables are precisely related via the respective Legendre transformations $q_i = \pm \frac{\partial f}{\partial e^i}$ and the inverse Legendra transformations $e^i = \pm \frac{\partial f}{\partial q_i}$, where $\pm$ signify the sign conventions being chosen. Moreover, it is not difficult to see that the magnetic charges $\{p_i\}$ are constrained by the Bianchi identities and thus the corresponding magnetic fields satisfy $B_t \sim p_i, \forall i$. The variables $\{v_i, u_s\}$ define the radii of $AdS_2$ and $S^{(D-2)}$ and the underlying scalar moduli, whose horizon values are fixed by Sen entropy function extremization procedure. In a chosen duality frame of the charges, it thus follows that Sen entropy function method is equivalent to certain constrained dynamical system. For example, in the electric description, such a consideration shows that all the magnetic charges are fixed by Bianchi identities and thus they remain proportional to the corresponding magnetic fields.

What follows next is that Sen entropy function facilitates us to study the fixed point behavior of scalar fields and other parameters. As per the definition of the attractor mechanism, this enables us to compute the macroscopic entropy of arbitrary extremal black hole. Subsequently, we shall examine algebraic and geometric properties of Sen entropy function and show, from the perspective of constrained dynamical systems, that our framework exhibits a set of embeddings and convexity relations. In the next section, we offer such an exposition from the perspective of embedding theory and finite parameter Sen entropy functions.
4 Embeddings and Entropy Function

In this section, we study geometric properties of the embeddings associated with Sen entropy function of an extremal black hole configuration. Subsequently, we define the local charts, atlas, metric tensor, generalized symplectic transformations and the associated Legendre transform on the underlying manifold. From the perspectives of the local and global geometry, one of the central motivation is to investigate the natural structures of $C^\infty$ manifold $\mathcal{M}$, which possesses a Riemannian-like metric leading to the real and complex geometric structure(s), such that the manifold $\mathcal{M}$ of the real dimensions $2n$ can be identified as a symplectic manifold.

Most of the natural structures may be studied by using the tools of the complex geometry, viz. the charts of an atlas are identified as a finite collection of the open subsets in $\mathbb{C}^n$. Thus, along a given fiber, we may assume that the compositions of the charts are described by the holomorphic maps among the finitely many domains of $\mathbb{C}^n$. Further, the very natural structures of $C^\infty$ manifold $\mathcal{M}$ promote us to define the symplectic structures, which we wish to explicate for the case of an even dimensional manifold $\mathcal{M}_{2n}$ and thereby show the existence of a nondegenerate close 2-form $\omega$. For a given $\mathcal{M}_{2n}$, the classical geometry over the complex numbers anticipates Kähler geometry, which, as the geometry of a complex manifold, possesses a compatible Riemannian metric. The concerned algebraic properties have been relegated to the subsequent section. Notice that the conventional Riemannian geometry has an extremely rich set of geometric structures, viz. there exist interesting local structures. However, the corresponding symplectic geometry offers many important elements towards the global structures. Both the above local and global geometric structures intersect in the realm of Kähler geometry, where both of the above two geometric structures remain compatible. Thus, one can determine the one of the structure from the other in a natural way. In particular, the compatibility properties constitute the Kähler structures on the manifold $\mathcal{M}$.

For a black hole, we begin our analysis by recalling the fact that Sen entropy function $F(u_s, v_k, q_i, p_j)$ can essentially be defined as the map $F: \mathcal{M} \to \mathbb{R}$. In particular, for the case of the extremal black holes in $D = 4$, we have the following manifold $\mathcal{M} = M_0 \otimes M_2 \otimes M_{2n}$. In the case of quantized charges $\{q_i, p_j\}$, the base manifold $\mathcal{M}_{2n}$ may be regarded as a symplectic manifold with the symplectic structure $\omega^2 := \Sigma_{i<j}\omega_{ij}dp_i \wedge dq_j$.

As per the theory of differentiable manifolds, $\mathcal{M}_{2n}$ is a symplectic manifold, if there exists a differential 2-form $\omega$ such that the manifold $\mathcal{M}_{2n}$ satisfies the following two properties: (i) the 2-form $\omega$ is nondegenerate, i.e. the matrix $\omega_{ij}(\mathcal{P}, \mathcal{Q})$ is invertible; (ii) the 2-form $\omega$ is closed, which, with the help of the exterior differential operator $d$, may be expressed as $d\omega = 0$. Notice further that the manifold $\mathcal{M}$ is defined as a mixed deformation of $\mathcal{M}_{2n}$, $\mathcal{M}_0$ and $\mathcal{M}_2$ in a given order of the composition, or the converse order. Thus, a consistent interpretation of the manifold $\mathcal{M}$ may be offered as a semisymplectic manifold, whenever the base manifold $\mathcal{M}_{2n}$ is treated as the symplectic manifold. What follows in the sequel that we shall focus our attention on $D = 4$ extremal black hole entropy functions. This is because the case of arbitrary $D$ dimensional black hole spacetimes arises as an obvious extension $\mathcal{M}_0 \to \mathcal{M}_0$, where $\hat{\phi}$ is a finite collection of possible fields and other parameters, if any.

In order to offer an insight of the present consideration, let us first introduce a set of mappings, viz. maps from the space $\mathcal{M}_i$ to the space $\mathcal{M}_j$, such that, for each pair of indices $\{i,j\}$, the resulting composition $\mathcal{D}$ can be defined as the following embedding $\mathcal{M}_{2n} \hookrightarrow^\mathcal{D} \mathcal{M}$. In order to determine a well-defined composition, we may notice that there exist the two sequences of embeddings $\{\mathcal{D}_1\}$ and $\{\mathcal{D}_2\}$, which are respectively associated to the horizon values of the moduli $\{\phi_i\}$ and parameters $\{v_i\}$, for a given set of charges. For this proposition, we may define an embedding $\mathcal{D}$ as the extension map $(\mathcal{P}, \mathcal{Q}, \mathcal{U}, \bar{\mathcal{U}}) \mapsto (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{U}}, \bar{\tilde{\mathcal{U}}})$. Thus, there is a translation map $T: \mathcal{M} \to \mathcal{M}$ such that $(\mathcal{P}, \mathcal{Q}) \mapsto (\tilde{\mathcal{P}}, \tilde{\mathcal{Q}}, \tilde{\mathcal{U}}, \bar{\tilde{\mathcal{U}}})$, for some $\phi_i \in \mathcal{M}_0$, $v_i \in \mathcal{M}_2$. Moreover, it is not difficult to show that $T$ is a diffeomorphism on $\mathcal{M}$. To examine the above mentioned geometric properties, let us consider a finite set of the embeddings $\{\mathcal{D}_1, \mathcal{D}_2\}$, and thereby an explicit definition emerges as the following embedding sequence $\mathcal{M}_{2n} \hookrightarrow_{\mathcal{D}_1} \mathcal{M}_{2n} \otimes \mathcal{M}_{\{\phi_i\}} \hookrightarrow_{\mathcal{D}_2} \mathcal{M}_{2n} \otimes \mathcal{M}_{\{\phi_i\}} \otimes \mathcal{M}_{\{v_i\}}$. For a given pair $\{\mathcal{D}_1, \mathcal{D}_2\}$, there exists a local isomorphism $\sim$, which as the complete embedding of $\mathcal{D}$, furnishes the following well-defined composition $\mathcal{D} := \mathcal{D}_2 \circ \mathcal{D}_1$. As shown in table 1, we notice, from the perspective of a given sequence of embeddings on the manifolds $\mathcal{M}_{\{\phi_i\}}$ and $\mathcal{M}_{\{v_i\}}$, that the well-definiteness property corresponds to the requirement that the embeddings $\{\mathcal{D}_1, \mathcal{D}_2\}$ are defined as per the following compositions: (i) $\mathcal{D}_1 := \mathcal{D}_{1m} \circ \cdots \circ \mathcal{D}_{12} \circ \mathcal{D}_{11}$ and (ii) $\mathcal{D}_2 := \mathcal{D}_{22} \circ \mathcal{D}_{21}$. For a given composition map $\circ$, it is easy to show that the other composition $\mathcal{D}_1 \circ \mathcal{D}_2$ is ill-defined and is not the same as the composition $\mathcal{D} = \mathcal{D}_2 \circ \mathcal{D}_1$. This amounts to the fact that the inverse orientation embedding $\mathcal{D}^{-1} \neq \mathcal{D}$ prefers an order of the compositions and thus the resulting composition $\mathcal{D}$ turns out to be a sequence of finitely many unidirectional embeddings. It is worth mentioning that such an underlying orientation does not make a diffeent physical importance, whenever there exists a local coordinate frame on the manifold $\mathcal{M}$. 

9
near horizon configuration of extremal black holes picks up a duality frame, which could be either in the electric
duality transformations. For a given pair of maps
complex structure, we may note that there always exists a possibilit y of some generalized electric-magnetic
on
or
ρ
z
associated Poisson structure on
the electric-magnetic charges
we find that the deformed metric tensor appears as the direct sum of the associated metric tensors pertaining
to the left hand side of the above bracket is symplectic, whereas the right hand side is symmetric and
that the metric tensor
for a proper symplectic basis, we observe that the above mentione d manifold
M
even dimensional Riemannian manifold
{−→ e
k
,i
→ q
k
} ∈M
may not imply an uniquely defined Euclidean structure. Considering th e manifold
the standard symplectic matrix
n
represents
the standard symplectic operator, which may thus be explicitly given by the matrix
it is natural to associate an interpretation of the semi-Euclidean st ructure on the manifold
M
dilaton field on the manifold
M
fields
ψ
i
,φ
i
. Herewith, from the definition of the Euclidean structure on
M
n
⊗ M
ϕi
↓D
1
v
ϕ
↓M
ϕ
n
p
,q
E
p
SP
spacetime parameters
{v1,v2}. Moreover, we see, for finitely many electrically magnetically charged black holes [22], that the standard symplectic matrix
J = ( 0
0
E
E
- E
0 ) reduces to the Gram matrix Q_{ab} = ( p^2 - p\cdot q q^2 ). Thus, for a proper symplectic basis, we observe that the above mentioned manifold M is locally spanned by a finite sequence
{si}_1^{2n+m+2}, whose components are defined as the union of 2n electric-magnetic charges, m scalar fields
{ϕ_i}_1^{2n}, with near horizon values
{ui}_1^{2n+1} and the two near horizon AdS_2, S^2 spacetime parameters
{v_1,v_2}. As an artifact of the two dimensional manifold M_2, it is easy to see that the metric on M_2 can be reduced to the following diagonal metric:
\nu_{αβ} \sim ϵ_{αβ} e^ϕ. In this case, the resulting function Φ(v_1,v_2) may be considered as the dilaton field on the manifold M_2. In general, for a given entropy function, it follows that there exists a class of semi-ecuclidan structures on M, which are locally spanned by the basis set \{e_ϕ, ϕ\}, as in the usual Euclidean cases. Furthermore, it is follows that they have
J^2 = -E_2n. Herewith, from the definition of the Euclidean structure on M_2n, it is worth mentioning that the associated Poisson structure on M_2n can be expressed as the following bracket [\vec{\phi}, \vec{\psi}]_{RB} = (J_2 \cdot \vec{\phi}, \vec{\psi}). From the fact that the left hand side of the above bracket is symplectic, whereas the right hand side is symmetrical and thus it follows that the operator J must be symplectic. Observe further that the standard complex structure on M_2n is defined by the map z_k \mapsto p_k + i q_k and thus we can define a deformed complex structure by an extension map z_k \mapsto p_k + i q_k or q_k \mapsto q_k + u_k + v_k or an arbitrary (linear) combination of the charges \{p_k,q_k\} such that the map \iota remains well-defined. In particular, the map \iota defines an embedding: M_n \mapsto M_n \otimes M_m \otimes M_2, which, for example, in the magnetic frame could be given by the extension map p_k \mapsto ω p_k + ω u_k + ω v_k. Thus, it is easy to see that the corresponding projection map ρ must be given by the restriction M_n \otimes M_m \otimes M_2 \mapsto \rho \otimes \rho. In turn, the fixed horizon behavior of the black hole may be defined by the composition of translation and restriction maps as the following sequence (p_k, u_k, v_k) \mapsto T^{-1} (p_k, 0, 0) \mapsto ρ (p_k). Under the above proposition of generalized complex structure, we may note that there always exists a possibility of some generalized electric-magnetic
duality transformations. For a given pair of maps \{T, ρ\}, the above consideration leads to the fact that the near horizon configuration of extremal black holes picks up a duality frame, which could be either in the electric

| Table 1: Sequence of embeddings associated with M_{\{ϕ_i\}} and M_{\{v_i\}}.  |
|---------------------------------------------------------------|
| M_{2n} \rightarrow M_{2n} ⊗ R^{1}_{\{ϕ_1\}}                   |
| M_{2n} ⊗ R^{1}_{\{ϕ_2\}} \rightarrow M_{2n} ⊗ M_{\{ϕ\}} ⊗ M_{\{v\}} |
| M_{2n} ⊗ M_{\{ϕ\}} ⊗ R^{1}_{\{v_1\}} \rightarrow M_{2n} ⊗ M_{\{ϕ\}} ⊗ M_{\{v_2\}} |
| M_{2n} ⊗ M_{\{ϕ\}} ⊗ R^{1}_{\{v_2\}} \rightarrow M_{2n} ⊗ M_{\{ϕ\}} ⊗ M_{\{v_1\}} |
| M_{2n} ⊗ M_{\{ϕ\}} ⊗ \cdots ⊗ M_{\{ϕ_m\}} \sim           |

We shall now define an associated Euclidean structure on the manifold M. Let us consider the symplectic system \{ϕ, ψ\}, moduli fields \phi and AdS_2 × S^2 parameters ϕ. As pointed out in Ref. [21], we shall take \{e_ϕ, ϕ\}, on the same footing. Therefore, we can define a vector \vec{ϕ} := \sum_i (p_i e_ϕ + q_i ϕ + u_i e_ϕ + v_i ϕ) such that the semi-Euclidean structure is given by the product (\vec{ϕ}, \vec{ψ}) := \sum_i (p_i ϕ_j q_i + u_i u_j u_i + v_i v_j v_i). At this juncture, it is worth mentioning that the above geometric notions lie in the structures of the M_{ij}. Thus,
sector or in the magnetic sector or in a mixed sector defined as the (linear) combinations of the electric-magnetic charges \( \{ p_\nu, q_\nu \} \).

Herewith, we extend our analysis towards the perspective of dynamics and stability theory. In order to do so, let us restrict our attention the general linear mappings, namely, an algebraic group \( GL(n, K) \), where \( K \) is a finite field, ring or any algebraic object, satisfying the symmetry properties of the metric \( b_{ij} \). To concentrate on the geometric physics of an extremal black hole, we shall offer the analysis when \( K \) is a finite field. In the case when \( K = \mathbb{C} \), such a consideration may be defined by the following sequence of mappings \( \{ f_k \in C \} \) such that \( GL(n, C) := \{ f \in C \} \). In turn, this perspective allows us to define the unitary transformation preserving Hermitian scalar product: \( \mathcal{F}_1, \mathcal{F}_2 := (\mathcal{F}_1, \mathcal{F}_2) + i[\mathcal{F}_1, \mathcal{F}_2] \); where the scalar and skew scalar products are respectively given by the real and imaginary parts of the above scalar product on \( M \). Note that the complex structure, which as defined earlier with the injection \( \iota \), is the same as the one defined with the above scalar and skew scalar products. In order to consider the stability conditions on \( M \), let us first consider the notion that the usual mechanical stability transformations are defined as an automorphism, namely, the dynamics on \( M \) is said to be stable, if \( \forall \epsilon > 0 \), there exist \( \delta > 0 \) and \( (i) \) a map \( g : M \to M \) such that \( \| \mathcal{F} \| < \delta \Rightarrow \| g^N \mathcal{F} \| < \epsilon, \forall N > 0 \). Under the generalized or deformed symplectic transformations, it is worth mentioning that the stability may be defined as an extension of the above stability transformation on \( M_{2n} \). In fact, the required extension can be defined by a deformation of the above symplectic transformation \( g \) and thus a generalized transformation is said to be “strongly stable”, iff every generalized symplectic transformation \( \mathcal{F} \), which is sufficiently close to \( g \), is stable. This shows that there exists a map \( \mathcal{F} \) such that it remains sufficiently close to \( g \), if the matrix elements of the map \( \mathcal{F} \) differs from that of the map \( g \) by less than a sufficiently small number \( \epsilon \), in a fixed local basis \( \{ s_1 \}_{i=1}^{2n+m+2} \) on \( M \). Thus, the stability of an arbitrary \( \mathcal{F} \) on \( M \) can be examined in a preferred local basis \( \{ s_i \}_{i=1}^{2n+m+2} \). Equivalently, the above analysis amounts to the fact that the restriction map \( p \) and translation map \( T \) on the corresponding matrix elements \( b_{ij} \) reveal the geometric nature of the black hole horizon configuration. In particular, we find that the stability of Sen entropy function is given by \( |\mathcal{F}_{ij}| \approx |g_{ij}| + \epsilon \). Thus, the strong stability of a semisymplectic transformation relates it's coefficients to the given local transformations on a stable manifold \( M \).

Let us now construct the generalized symplectic atlas of a deformed symplectic manifold \( M \). Such a construction can essentially be realized by considering the metric \( b_{ij} \). This follows from the fact that every symplectic manifold has some local coordinate basis \( \{ \mathcal{F}, \mathcal{F} \} \). Now, the Darboux theorem offers standard symplectic 2-form \( \omega^2 := d\mathcal{F} \land d\mathcal{F} \), from which various symplectic structures may be obtained accordingly. As per say, the definition of every manifold induces a compatibility condition on the underlying local charts of the atlas of the manifold. This gives a condition on the connecting maps \( \{ s_i^{-1} \circ s_j \} \), in order to go from one chart to the other, in a well-defined way. In order to make coordinate transformations from a given region to the another, the intersection maps need to be well-defined, in certain patches of the coordinate space. In the other words, we can define an atlas of the manifold \( M \) with the norm: \( b^2 = \omega^2 + \mu^2 + \nu^2 \), which may locally be accompanied by the introduction of the corresponding Euclidian coordinate space \( \mathbb{R}^{2n+m+2} = \{ (\mathcal{F}, t) ; \mathcal{F} ; \mathcal{F} \} \) and thereby the norm \( b^2 \) defines nature of the transformation(s) from the one point to the other. Such a generalized symplectic structure may be realized as the generalized canonical transformations, which are defined as \( b^2 \) preserving maps. In particular, one can use the compatibility structures of the composition maps \( \mathcal{U}_{ij} := \{ s_i^{-1} \circ s_j \} \), for some given \( \omega^2 := d\mathcal{F} \land d\mathcal{F} \), \( \mu^2 = d\mathcal{F} \land d\mathcal{F} \), and \( \nu^2 = d\mathcal{F} \land d\mathcal{F} \). Here, the above products respectively correspond to the following matrices \( \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix} \) and \( (\nu_{\alpha\beta} u_{\alpha\beta})_{2 \times 2} \). As defined before, the case of \( M(R) \) follows from the following expansions \( b^2 = dp_1 \land dq_1 + dp_2 \land dq_2 + \ldots + dp_n \land dq_1 + du_1 \land du_1 + du_1 \land du_2 + \ldots + du_1 \land du_m + \ldots + du_m \land du_m + dv_1 \land dv_1 + \ldots + dv_2 \land dv_2 \). In the case when the effective actions are allowed to be complex, it is worth mentioning that the stability analysis needs an extension on the corresponding complex manifold \( M(C) \). From the viewpoint of the complex analysis, the well-definedness of the resulting complex embeddings requires that the composition maps \( \{ s_i^{-1} \circ s_j \} \) should factorize into the analytic and antianalytic sectors.

In order to formulate algebraic properties of the above mentioned geometry, let us focus on the fact that the symplectic structure defines an effective hamiltonian dynamics. As mentioned in section 2, an immediate goal could be to give the precise local and global stability interpretations on \( M \), for a given Sen entropy function \( F(s) \). Subsequently, we shall return to the associated algebraic definition of Legendre transformation in the next section. Before describing the details of the algebraic geometric analysis, it will be prudent to note that \( F \) is said to be strictly convex, if the Hessian matrix \( (d^2 F)_p(s) \gg 0, \forall p \in M \). Let \( F \) be such a strictly convex function on \( M \). Then, for a given \( l \in M^{*} \), there exists a map \( F_l : M \to R \) such that the cotangent structure can be defined by \( F_l(w) := F(w) - l(w) \). It is worth mentioning that the Hessian matrix of the function \( F \) is a quadratic form.
on \( \mathcal{M} \) and thus it can be locally expressed as \((d^2 F)_p(s) := \sum_{i,j} \frac{\partial^2 F}{\partial x^i \partial x^j} s_i s_j \). Such a consideration offers the local definition of the Hessian, which is defined in terms of the basis functions of the manifold \( \mathcal{M} \). Notice further that the corresponding global definition is given by \((d^2 F)_p(s) := \sum_{i,j} F(p + ts) |_{t=0} \). Now, it can be easily verified that both the above frameworks have the same realization of the dynamical stability. In particular, it follows from the fact that \((d^2 F)_p = (d^2 F)_p \Rightarrow F \) is strictly convex \( \Leftrightarrow F \) is strictly convex. It is worth mentioning that the determinant of the Hessian matrix of the considered entropy function \( F(s) \) remains unchanged under the transformations \( b_j \rightarrow b_j - c \), where \( c \) is an arbitrary constant. Herewith, we find that the stability of arbitrary entropy function \( F(s) \) can be defined as the set \( S_F := \{ l \in \mathcal{M}^* \mid F_l \) is stable \}. In this case, we see that the stable set \( S_F \) is an open and convex subset of the underlying cotangent manifold \( \mathcal{M}^* \). For such \( \mathcal{M} \), a suitable definition of the Legendre transform can be offered as follows. For a given entropy function \( F \in C^\infty(\mathcal{M}, K) \), the generalized Legendre transformation is the map \( L_F : \mathcal{M} \rightarrow \mathcal{M}^* \) such that \( p \rightarrow dF_p \in T^*_p \mathcal{M} \approx \mathcal{M}^*; \forall p \in \mathcal{M} \). In this case, it is not difficult to verify that the map \( L_F \) is \((a)\) 1-1, \((b)\) onto and \((c)\) both \( L_F \) and \( L_F^{-1} \) are continuous. Thus, we find that the map \( L_F \) locally results into a homeomorphism. From this perspective, we see that the homeomorphism \( L_F \) becomes a diffeomorphism onto \( S_F \), whenever \( L_F \) is a differentiable map. In particular, it can be easily shown that the following conclusion holds. Let \( F \) be a strictly convex function, then the map \( L_F : \mathcal{M} \rightarrow S_F \) is an isomorphism. In fact, \( L_F \) turns out to be a diffeomorphism onto \( S_F \). Furthermore, the inverse map \( L_F^{-1} : S_F \rightarrow \mathcal{M} \) describes the unique minimum point \( p_l \in \mathcal{M} \) of \( F_l = F - l, \forall l \in S_F \). Thus, it is easy to see that the point \( p \) corresponds to the unique minimum of \( F(w) - dF_p(w) \). We wish to remark that the dual space \( \mathcal{M}^* \), corresponding to the given \( F \), is completely described by the map \( F^* : S_F \rightarrow R \). For some \( l \in \mathcal{M}^* \), the minimum value of \( F(s) \) may be defined as \( F^*(l) = -\min_{p \in S_F} F(p) \). Finally, it turns out that the duality relation between \( L_F^{-1} \) and \( L_F \), are expressible in terms of the orthonormal basis functions of \( \mathcal{M} \) and \( \mathcal{M}^* \). In particular, let \( \{s_j\} \) be the tangent basis and \( \{\tilde{s}^j\} \) be the corresponding cotangent basis of a given \( \mathcal{M} \). Thus, as per the general theory of local vector spaces, we find that the underlying basis functions satisfy the following orthonormality relation \( s_j \cdot \tilde{s}^j = \delta^j_i \). Furthermore, the above consideration implies that we have \( s_j^{-1} = \tilde{s}^j, \forall I \), and hence the isomorphism property follows as the identification \( L_F^{-1} = L_F \). Subsequently, it follows that \( L_F \) and \( L_F^{-1} \) are dual to each other. In the next section, we shall analyze the above properties of Sen entropy function from the perspective of commutative algebra and thereby setup the notions for the Calabi-Yau geometry.

5 Spectrum and Convexity

As mentioned in section 4, we can determine a finite set of embeddings associated with the geometric perspective of \( \mathcal{M} \). The corresponding Hessian matrix of the entropy function can be expressed in terms of the metric \( \omega_{ab} \), for a given set of the electric-magnetic charges, \( u_{ab} \) corresponding to the scalar fields obtained for a chosen compactification and the variables \( v_{ab} \) as the gravity parameters characterizing the spacetime metric tensor of the considered black hole. The geometry thus obtained from the Hessian matrix of the entropy function fits in the general framework of the generalized geometry. As mentioned in section 2, such a geometry may be reduced to the symplectic geometry, with a set of associated complex structures \( J := \{ J^{(i)} \} \) for the case of an even number of the scalar fields. Thus, there exists an even dimensional Riemannian manifold whose metric structure can be defined as the bilinear map: \( g_{IJ} : g_{IJ}(p, q) \rightarrow \omega(p, Jq) \). The case of the odd dimensional \( \mathcal{M} \) corresponds to an odd number of the scalar fields. For every given Hessian matrix \( b_{ij} \), it follows in both the above cases that there exists a Riemannian manifold \( \mathcal{M} \). Further, the associated geometry corresponds to a higher dimensional symplectic manifold \( \mathcal{M} \), iff the dimension of the considered manifold \( \mathcal{M} \) is an even integer.

In this section, the goal is to describe the general procedure for determining the algebraic properties, which arise from the Hessian quadratic form and associated polynomial function of a given Sen entropy function. We begin with the postulates and notions of the basic commutative algebra, see Refs. [79] [81] for more details, in order to properly deal with the concepts of the spectrum and convexity of the underlying geometry of a given entropy function \( F(\vec{x}) \). From this perspective, let Sen entropy function be defined as the map \( F : \mathcal{M} \rightarrow R \) such that \( F(\vec{x}) = a \). Then, the level set, as the inverse of a singleton \( b \), may be defined as a finite set \( F^{-1}(b) = \{ \vec{x} \in \mathcal{M} \mid F(\vec{x}) = a \} \). For the reason which follows subsequently, we shall work with an assignment that there are finitely many co-ordinates \( s_i \in \mathcal{M} \). For a given index \( \Lambda \), let \( s = \{ s_i \} \) be a bounded sequence, then the infinite norm of the vector \( \vec{s} \) may be defined as the following set \( \| s \|_\infty = \sup \{ |s_i| : i \in \Lambda \} \). Moreover, consider \( C(\mathcal{M}_1, \mathcal{M}_2) \ni T : \mathcal{M}_1 \rightarrow \mathcal{M}_2 \), then the supremum norm of such an operator \( T \), for the given norms \( \| \cdot \|_1 \) on \( \mathcal{M}_1 \) and \( \| \cdot \|_2 \) on \( \mathcal{M}_2 \), may be given by \( \| T \|_\infty = \sup \{ |T\|_2 | |u|_1 \leq C \} \). On the other hand, a submanifold \( \mathcal{N} \subseteq \mathcal{M} \) turns out to be compact, if for a given open cover \( u = \bigcup_{s \in \Lambda} U_s \), there exists
some $\Lambda$ such that $\{\alpha_i\}_{i=1}^n \subseteq \Lambda$. In this case, it follows hereby that the submanifold $\mathcal{N}$ is finitely covered, viz. $\mathcal{N} \subseteq \bigcup_{i=1}^n U_i$.

We shall now turn our attention on the analysis of the spectrum and convexity of Sen entropy function $F : \mathcal{M} \to R$ of the associated black hole. This may most conveniently be accomplished by noticing the fact that there exists the quadratic function $(d^2 F)_\mu(s) := \sum_{i=1}^r d^2 F_i(s_i) = \sum_{i=1}^r b_i s_i s_j$, where $r = \dim(\mathcal{M})$.

Let us consider the matrix $B := (b_j)_{r \times r}$, then the stability of the corresponding black hole system may be described by the spectrum of the eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ of $B$. In this interpretation, the convexity of $F$ may be realized in the minimal subfield $K^n$, containing the sequence $\{\lambda_i\}_{i=1}^n$ with a finite completion $\{\bar{\lambda}_i\}_{i=1}^n$. In such cases, one finds that an example of the field $K^n$, associated with black hole entropy function, turns out to be the real field $R^n$ for the configurations whenever the corresponding matrix $B$ ceases to a real symmetric matrix. The case of $C^n$ may be similarly extended with the fact that $B$ belongs to a set of complex hermitian matrices.

From the perspective of the function theory, let us consider a field $K \subset R^n$ as the bounded level set of the (extremal) black hole entropy function $F(u, v, p_i, q_j)$. Then, under the above consideration, Ref. [S2] shows that there exists a convex polynomial hull $\varphi(K)$, which may in turn be defined by the set $\varphi(K) = \{\bar{\lambda} = (\lambda_1, \ldots, \lambda_n) \in K^n | |P(\bar{\lambda})| \leq |P|_K \forall P \in K\}$. Here, we have taken the norm of the polynomial $P$ as the set $\|P\|_K = \sup\{\|P(\bar{\lambda})\| | \bar{\lambda} \in K\}$. In general, this allows that such a polynomial convex hull of $\varphi(K)$ satisfies $\varphi(K) \subseteq K$. In particular, for the case of $\varphi(K) = K$, it may easily be shown that the set $K$ is polynomially convex. Herewith, it is easy to see that $\varphi(K)$ has following properties: (i) $\varphi(K)$ is compact. Considering the fact that $\varphi(K)$ is closed, let $\varepsilon_j$ be a finite set of polynomials satisfying the following relation $\varepsilon_j(\lambda_1, \lambda_2, \ldots, \lambda_n) = \lambda_j$. By taking $P = \varepsilon_j$ into the definition of $\varphi(K)$, it follows, for the case of $\bar{\lambda} \in K\setminus \{\bar{\lambda}: \varepsilon_j(\bar{\lambda}) = \lambda\}$, that we have $|\lambda_j| \leq \sup\{\|P\|_K | \bar{\lambda} \in K\} = C < \infty$. In turn, this implies that $\varphi(K)$ is bounded and thus compact. In particular, this implies that any $p$-convex $K$ defined as the set $\{\lambda_i\}$ is thus compact. (ii) From the definition of $p$-convexity, we may further see that the set $K$ remains $p$-convex, iff there exists a polynomial $P_0$ such that $|P_0(\bar{\lambda}_0)| > |P_0|_K, \bar{\lambda}_0 \in K^n \setminus K$. (iii) Moreover, if $K$ is $p$-convex and $C > 0$, then one finds that there exists a polynomial $Q$ such that $|Q|_K \leq C, \forall \bar{\lambda} \in K^n \setminus K$, where $C < |Q(\bar{\lambda})|$. From the consideration of (ii), it may easily be seen with an appropriate choice of $P_0$ that there exists a $\lambda_0 \in K^n \setminus K$ such that $K$ is $p$-convex. Such a consideration leads to the statement that the polynomial $Q$ can be identified as $Q(\bar{\lambda}) = C P_0(\bar{\lambda})/|P_0|_K$.

On the other hand, from the theory of Banach algebra of several complex variables, Ref. [S2][S3] make known that a compact set $K \subset C$ is $p$-convex, iff $C \subseteq K$ is connected. Thus, in order to consider the convexity of a function of several complex variables, let us define polydisc $P_\xi$ of polyradius $\xi = (\xi_1, \ldots, \xi_n)$ as the set

\[ P_\xi = \{\bar{\lambda} \in K^n | |\lambda_j| \leq \xi_j, \forall j = 1, \ldots, n\}. \]

Then, the polysubset $\Pi \subset P_\xi$ is a $p$-polyhedron in $P_\xi$, if there exist polynomials $\{P_i\}_{i=1}^m$ such that $\Pi = \{\bar{\lambda} \in P_\xi | |P_i(\bar{\lambda})| \leq 1 \forall i = 1, \ldots, m\}$. Now, one may observe that the followings hold: (i) Suppose $m = n$ and $P_j = \xi_j^{-1} \varepsilon_j$, where $\varepsilon_j$ are the polynomials defined by $\varepsilon_j(\bar{\lambda}) = \lambda_j$, then $P_\xi$ is a $p$-polyhedron. (ii) There exists a relation between $p$-convexity and $p$-polyhedron, namely, suppose $\bar{\lambda}_0 \notin K^n \setminus \Pi$ then either we have $|\lambda_0| > \xi_j$ or there exist some polynomials satisfying $|P_0(\bar{\lambda}_0)| > 1$. In the first case, we find that $|\varepsilon_j(\bar{\lambda}_0)| = |\lambda_0| > \xi_j \geq |\varepsilon_j|_{\Pi}$, while in the second case, we have $|P_0(\bar{\lambda}_0)| > 1 \geq |\varepsilon_j|_{\Pi}$. Therefore, in both the above cases, we have $\varepsilon_j \notin \varphi(\Pi) \Rightarrow \varphi(\Pi) = \Pi$ and thus $\Pi$ is $p$-convex. In general, this lead to the fact that every $p$-polyhedron is $p$-convex. From the definition of $p$-convexity and the choice of the polynomial $P_\xi$ $\forall \bar{\lambda}_0 \in K^n \setminus K$, the existence of the above polyhedron implies that we have $|P_\xi(\bar{\lambda}_0)| > 1$ and $|P_\xi|_K \leq 1$. Moreover, from the continuity condition of $P_\xi$, we see that $|P_\xi(\bar{\lambda})| > 1$, as for some $\lambda$, there exists an open neighborhood $\mathcal{N}(\bar{\lambda})$ at $\bar{\lambda} = \bar{\lambda}_0$. Thus, writing $P = P_\xi$, allowing $\bar{\lambda}_0$ into the range $P \setminus G(\subseteq P \setminus K)$ and using the compactness of $P \setminus G$, we have a finite family of $P_\xi(\bar{\lambda})$, which corresponds to the polynomials $P_\lambda$ such that $\bigcup_{i=1}^j N_{\bar{\lambda}_i} \supseteq P \setminus G$. By setting $\Pi = \{\bar{\lambda} \in P | |P_\xi(\bar{\lambda})| \leq |P_\lambda|_K < 1\}$, it may easily be seen that $K \subseteq \Pi$. Next suppose that $\bar{\lambda} \notin G$ with $\bar{\lambda} \notin P$, then of course $\bar{\lambda} \notin K$, as $\Pi \subseteq K$. On the other hand, if $\bar{\lambda} \in P$, then $\bar{\lambda} \notin K \setminus G$ and thus we have $\bar{\lambda} \in N_{\bar{\lambda}_0}$, for some $j$. Hence, it follows that we have $|P_\xi| > 1$, whenever $\bar{\lambda} \in K$, viz. $\Pi \subseteq K$. This amounts to say that the polysubset $\Pi$ can be examined by the
following property. Let $K$ be a p-convex set such that $K \in P_{\mathcal{I}}^2$ and $G$ be an open set with $K \subseteq G \subseteq K^n$, then there exists a polyhedron $\Pi$ such that $K \subseteq \Pi \subseteq G$.

In order to analyze the generalized spectrum of a given entropy function, it requires to recall some basic properties of the abstract algebraic system. A triple $(\mathbb{R}, +, \cdot)$ is said to be a ring if the followings hold: (i) $(\mathbb{R}, +)$ is abelian, (ii) $ab \in \mathbb{R}, \forall a, b \in \mathbb{R}$, (iii) $(a, b) = (a, b), c \in \mathbb{R}$, (iv) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$. Moreover, $(\mathbb{R}, +, \cdot)$ is commutative, if $a \cdot b = b \cdot a$. Subsequently, an associative ring $\mathbb{R}$ becomes an algebra over a field $K$, if $\mathbb{R}$ is a vector space over $K$ such that $\forall a, b \in \mathbb{R}, \alpha \in K$ satisfying the following multiplicative property $\alpha(a \cdot b) = (\alpha a) \cdot b = a \cdot (\alpha b)$. Furthermore, let $A$ be a linear vector space over the field $K$ with $a_1, a_2 \in A$ and $a_1, a_2 \in K$ such that (i) $a_1 a_2 + a_2 a_1 \in A$ (ii) $\alpha(a_1 a_2) = (\alpha a_1) a_2 = \alpha a_2 a_1$. Then, the algebra $A$ becomes a morphism, if there exists a map $A \rightarrow A$. For a given ring $(\mathbb{R}, +, \cdot)$, we shall take an account of the fact that there exists an ideal $I \subseteq (\mathbb{R}, +, \cdot)$, if (i) $a \in I$, then $a \cdot b \in I$; $\forall a, b \in I$ and (ii) $r, a, r \in I; r a, b \in I$. In this case when $J \neq 0$ and $I \subseteq J$, then $J = I$ which will be termed as the maximal ideal of the algebra $A$. Further, along with the other notions of commutative algebra, we wish to focus our attention on the homomorphism, which can be thought as a mapping from one algebraic system to another algebraic system and preserves the given set theoretic structures. For example, in the case of the group theory, the map $\phi$ from a group $G$ to $G$ is said to be homomorphism, if $\forall a, b \in G$ and $\phi(a), \phi(b) \in G$, we have $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$. With this reconsideration, we are in a position to consider the generalized spectrum of arbitrary entropy function, viz. for a finite set of elements, we are in proper position to define the complexification map(s). For an even number of elements, we shall herewith show that the above consideration leads to stability properties of the chosen entropy function, e.g., algebraic fronts of the symplectic structure.

Let us consider unital commutative algebra $A$ with a given unity element $e$. For given set of $a_j \in A$, we may define a vector as $\vec{a} := (a_1, ..., a_n)$. Subsequently, if $\vec{X} = (\lambda_1, ..., \lambda_n)$ is a finite collection of the eigenvalues $\lambda_j \in K$, then $J = I(\vec{X}) = I(\vec{X}, \vec{a})$ is an ideal of $A$ with the following generators $(a_j - \lambda_j e) ; \forall j = 1, ..., n$. For a given algebra $A$, we may thus define the Krull $J = I(\vec{X}) = \sum_{j=1}^{n} a_j \cdot (a_j - \lambda_j e)$. Therefore, the corresponding joint spectrum $\sigma(\vec{a}) = \sigma(a_1, ..., a_n)$ is defined by the set $\sigma(\vec{a}) := \{ \lambda : \lambda \in K^n | I(\vec{X}) \neq A \}$. Notice that the joint spectrum is sometime termed as the simultaneous and generalized spectrum. For $n = 1$, it follows that we have $\vec{a} = a$ and $\sigma(\vec{a}) = \sigma(a)$. Further, it is easy to show that the various properties of the generalized spectrum are summarized as per the followings: (i) The isomorphism $I(\vec{X}) = A$ gives the bijection $\vec{X} \in K^n \setminus \sigma(\vec{a}) \Leftrightarrow \exists \{a_i\}_{i=1}^{n} \in A$ such that $\sum_{j=1}^{n} (a_j - \lambda_j e) b_j = e$. (ii) The necessary and sufficient condition for $I(\vec{X}) \neq A$ implies that $\vec{X} \in \sigma(\vec{a})$, if $\forall \{a_i\}_{i=1}^{n} \in A$, then there exists $\sum_{j=1}^{n} (a_j - \lambda_j e) b_j \notin G_i$; where $G_i$ denote the invertible group elements of the algebra $A$. (iii) If $\vec{X} \in \sigma(\vec{a})$, then $I(\vec{X}) \neq A$. Thus, the Krull $I(\vec{X})$ is contained in the maximal ideal $M$, viz. $(a_j - \lambda_j e) \in I(\vec{X}) \subseteq M$. On the other hand, for some $j$, if $(a_j - \lambda_j e) \in M$, then $I(\vec{X}) \subseteq M$. Hence, $\vec{X} \in \sigma(\vec{a})$, if there exists a maximal ideal $M$ of $A$ such that we have $(a_j - \lambda_j e) \in M; \forall j = 1, ..., n$. Consequently, we have the following physical result, which follows as the corollary of the above properties of the generalized spectrum: $\forall \vec{X} \in \sigma(\vec{a}) \Rightarrow \lambda_j \in \sigma(a_j) ; j = 1, ..., n$ such that $\sigma(\vec{a}) \subseteq \bigotimes_{j=1}^{n} \sigma(a_j)$. For given $\vec{X} \in \sigma(\vec{a})$, the present consideration shows that we have $(a_j - \lambda_j e) \in M \Rightarrow \lambda_j \in \sigma(a_j)$.

We now exploit the algebraic advantages arising from the consideration of the complex number field $C$. Specifically, we wish to explicate the situation for the spectrum of Sen entropy function $F(\vec{a})$. For the complexification of even dimensional strictly real topological algebras, a number of interesting cases arise from the character homeomorphism. From the fact that the complex number field $C$ is algebraically closed, it application turns out to be relatively easier to deal with. For example, there exists a class of matrices and operators, which have complex eigenvalues. Furthermore, there exist many more algebraically closed polynomials over the number field $C$. Although, we loose the algebraic order relations of the corresponding real number field $R$, however, the algebraic closure property, as one of the main benefits of the field $C$, offers us interesting fronts. To illustrate an idea of the power of the complex number field, let us consider the following matrix $\Lambda := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ be $R^2$ basis. It follows that $\Lambda e_1 = e_2$ and $\Lambda e_2 = -e_1$ define the corresponding Cauchy Riemann conditions over the complex number field $C$. Thus, the operator $\Lambda : C^2 \rightarrow C^2$ has a well-defined complex structure. Furthermore, let $\lambda$ be the eigenvalue of $\Lambda$, then it follows that we have $\lambda^2 + 1 = 0$, i.e. $\lambda = \pm i \in C$. Therefore, the associated polynomial $P(\lambda) := \lambda^2 + 1$ is not closed over $R$. However, if possible, let us suppose on the contrary that $\lambda \in R$ with $\lambda^2 = -1$ then $0 \in R$ and $(-\lambda) > 0 \Rightarrow (-\lambda)(-\lambda) > 0 \Rightarrow \lambda^2 > 0$. However, in the case of the real number field, $\lambda^2 < -1 < 0$ is a contradiction. We can prove similar contradictions for the choices $(-\lambda) < 0$ and $\lambda = 0$. Thus, the polynomial $P(\lambda)$ possesses root(s) over the number field.
C, however, it has no roots over the real number field \(R\). Consequently, the polynomial \(P(\lambda)\) is algebraically closed over the complex number field \(C\). This is one of the main motivation of the subsequent consideration that we may complexify the spectrum in an even dimensional number field \(K^{2n}\).

Motivated from the spectrum, let us examine the role of the complexification in the corresponding case of Sen entropy function. In the algebraic setting, an abstract notion of the complexification can be realized as follows. Let \(A\) be an algebra over the number field \(K\), then there exists a homomorphism \(\chi\) of \(A\) onto \(K\) such that its one dimensional representation corresponds to the character of \(A\). Thus, it follows that \(\chi^{-1}(0) = \ker \chi\) is the kernel of \(\chi\). Subsequently, let \(\Delta = \Delta(A)\) denotes the set of all possible characters of \(A\). Then, in general case, we notice further that \(\Delta\) could be empty, in principle. As per the present consideration, a map \(\chi: A \to K\) is the character homomorphism, iff there exists no nontrivial character \(\chi \neq 0\). It follows further that \(\chi = 0\) lead to the fact that there exists an element \(a_0 \in A\) such that \(\chi(a_0) = 0 \neq 0\). Moreover, if \(\beta \in K\), then \(\chi(\beta a_0) = \beta\) implies that \(\chi\) is a surjective map and thus \(\chi\) is the character. Conversely, it is easy to see that, if \(\chi\) is non trivial, then it has a character representation. In general, the perspective of the complexification may be seen as follows. Let \(A\) be a strictly real topological algebra. For a given complexification \(\tilde{A}\), the map \(\Lambda\) : \(\chi \to \tilde{\chi}\) is the homeomorphism such that \(\chi \in \Delta(\Lambda)\) and \(\tilde{\chi} \in \Delta(\tilde{A})\). Similarly, let \(\Lambda_c = \Lambda|_{\Delta_c}\) be a restriction of \(\Lambda\) onto the subspace \(\Delta_c\), then \(\Lambda_c : \Delta_c(\Lambda) \to \Delta_c(\tilde{A})\) is also a homeomorphism. Therefore, for a given real valued function \(f\) on \(A\), an extension function \(\tilde{f}\) can be represented as \(\tilde{f}(s_1 + s_2) = f(s_1) + if(s_2)\), \(\forall s_1, s_2 \in A\), which is the standard canonical extension of a real valued function \(f\) over \(C\). It is not difficult to show that the character \(\tilde{\chi}\) of \(\tilde{A}\) is real valued, if the corresponding restricted character \(\chi = \tilde{\chi}|_A\) is real. Herewith, we observe that \(\Lambda\) is a bijection, since \(\tilde{\chi}(z) = \chi(s_1) + iz(s_2), \forall z = s_1 + is_2\); where \(z \in A; s_1, s_2 \in A\). Moreover, \(\tilde{\chi}\) is continuous, iff \(\chi\) is continuous. Thus, it is not difficult to show that \(\Lambda_c\) is also a bijection. In the above consideration, there exists a map \(\tilde{\chi}\) over \(\Lambda\), whenever \(\chi\) over \(A\), and thus \(\Lambda\) and \(\Lambda_c\) are well-defined homeomorphisms.

Now, let us explicate the above characterization for the joint spectrum of an entropy function. Although we can generalize the spectra for the case of non-extremal black holes, for instance non-extremal \(D_1D_5\) and \(D_2D_6NS_5\) configurations [18][22]. However, for purpose of uniformity, we shall focus our specific attention on Sen entropy function of the extremal black holes. Let \(F(\tilde{\chi})\) be Sen entropy function with finitely many electric-magnetic charges \((Q_1 := (p_1, q_1) \in M_{2n})\), which, from the perspective of S-duality, are considered as symplectic vectors endowed with the standard symplectic 2-form \(\omega\). For a commutative vector \(\tilde{\chi}\) in \(K\) and symplectic vector \(\tilde{\beta}\) in \(\tilde{K}\), the semi-simplectic spectrum \(\sigma(\tilde{\chi}, \tilde{\beta})\) can be examined as per the following consideration. For simplicity, let’s consider a pair \((n_1, n_2)\) such that \(n_1 + n_2 = n\), where \(n_2\) is an even integer with \(\tilde{\chi} = (a_1, ..., a_n)\) over a commutative field \(K\). Whilst, the vector \(\tilde{\beta} = (b_1, ..., b_{n_1+n_2}) \in \tilde{K}\) is defined such that there exists an ensemble of pairs \((p_i, q_i)\), which allows a symplectification over the basis of the algebra \(\tilde{A}\) and satisfies the following splitting \(\tilde{A} := A + \tilde{A}\). In this case, we find that the entropy function \(F\) has a semisympetcticitable algebra \(A\), whose semisympetctic spectrum can be expressed as \(\sigma(\tilde{\chi}) = \sigma(\tilde{\chi}, \tilde{\beta})\). From the perspective of commutative algebra, we notice that the generalized or joint spectrum of \(F\) is defined as per the definition of the convex hull. Thus, it can be easily shown that the complexification of the spectrum of \(F\) can be accomplished in an algebraically extendable field of the eigenvalues of the spectrum of \(\beta\) over the algebra \(A := A + \tilde{A}\).

Before the present section, we will illustrate further consequences of the foregoing spectra. From the perspective of the thermodynamic and attractor configurations, an unified geometric interest is associated to the definition of Sen entropy function and the corresponding attractor entropy of the black hole. For the extremal black hole configurations, it is worth emphasizing that the results following from the definition of the Hessian matrix of \(F(s)\) is shown to be important from the perspective of the intrinsic geometry and commutative algebra. Explicitly, let us focus our attention on Sen entropy extrinzination method, then the above mentioned results concerning the joint spectra of the Hessian matrix of \(F\) offer a guideline principle towards the stability of the black hole. In particular, let us consider the quadratic form of the above entropy function \(F\) and the associated attractor entropy. For the extremal black holes, the spectrum pertaining to the entropy function geometry is the product of the spectra of the attractor entropy and the associated moduli. In the other words, the intrinsic semisympetctic geometry renders to the standard thermodynamic geometry at the attractor horizon configuration, when the entropy function \(F\) is evaluated at the extremum values of the moduli \(\tilde{\chi}\), gauge fields and gravity parameters \(\tilde{\beta}\). The extrinzination of \(F\) determines the attractor values of \(\tilde{\chi}\) and \(\tilde{\beta}\) in terms of the charges of the black hole. For given \(\tilde{\chi}\) and \(\tilde{\beta}\), the corresponding generalized attractor equations are given by \(\frac{\partial F(\tilde{\chi})}{\partial a_{n_i}} = 0\) and \(\frac{\partial F(\tilde{\chi})}{\partial a_{n_i}} = 0\). This yields the critical points \((u_i, v_i) = (u_i^0, v_i^0)\) of the theory. Subsequently, the entropy of the black hole is evaluated as the attractor fixed point value of Sen entropy function \(F(\tilde{\beta})\). Thus, for \(s_i^0 = (u_i^0, v_i^0, q_i, p_i)\), we have the following attractor entropy \(S_{\beta\text{HIL}}(\tilde{\chi}, \tilde{\beta}) = F(\tilde{\chi}^{0}, \tilde{\beta}^0, \tilde{\chi}, \tilde{\beta})\). At the attractor fix points \(s_i^0\), it follows that the nonzero metric elements of \(M\) are alike from the elements of
the standard attractor fixed point geometry.

In order to analyze the spectrum of the attractor flow, let us define \( r_{ij} := \frac{\partial^2 S(Q)}{\partial q_i \partial q_j} \) as the thermodynamic metric tensor associated to the charges \( Q_i := (p_i, q_i) \) and thereby consider the associated matrix \( \mathcal{R} := (r_{ij}) \). As defined in section 4, for given \( b_{ij} := \frac{\partial^2 F(b)}{\partial b_i \partial b_j} \) of the entropy function \( F \), let us consider the matrix \( \mathcal{B} := (b_{ij}) \). Let \( \{ \lambda_i \} \) be the eigenvalues of the matrix \( \mathcal{R} \). As \( n_2 \) is an even integer, thus \( \{ \lambda_i \} \) form the symplectic structure under the complexification. For the matrix \( \mathcal{B} \), let \( \{ \lambda_i \} \) be a finite collection of the eigenvalues of \( \mathcal{B} \), where \( n_2 \) of \( \{ \lambda_i \} \) possesses the same structures, while \( n_1 \) of the eigenvalues belong to the field \( K \). The present consideration offers interesting consequences for both the attractor mechanism and Sen entropy function method. Specifically, such a given theory has a fixed number of charges, thus the joint spectra of both the thermodynamic and extended thermodynamic configurations possesses some nonzero complexifiable spectra \( \sigma(b_1, b_2, \ldots, b_{n_2}) \), which are defined as per the minimal extension of eigenvalues of the matrices \( \mathcal{B} \) and \( \mathcal{R} \). Therefore, the stability of Sen entropy function of an extremal black hole can be geometrically examined by the convexity of eigenvalues of the quadratic function \( \mathcal{B} \). Algebraically, the corresponding spectrum may be analyzed as per the definition of the Krull of the algebra of eigenvalues of \( \mathcal{B} \). Notice further that the transformations to the symplectic and real manifolds and to the associated algebras depend solely on the choice of the pair \( (n_1, n_2) \). Physically, the pair \( (n_1, n_2) \) is fixed by the compactifying manifold. Up to some homomorphism, the spectrum \( \sigma(Q) \) may differ from the attractor valued thermodynamic spectrum \( \sigma(Q) \) of Sen entropy function. It is worth mentioning that there exists a submersion of the intrinsic metric tensor \( g_{\varphi, \psi}(Q) \) such that \( g_{\varphi, \psi}(Q) \). From the perspective of complexity of the eigenvalues, we find that both the above mentioned spectra have the same non-zero joint spectra, up to a homomorphism. In the next section, we shall illustrate that the notion of algebraic geometry offers guidelines pertaining to the stabilization of Sen entropy function.

6 Calabi-Yau Geometry

In this section, we focus our attention on algebraic properties of the two sequences of the embeddings \( D_1, D_2 \). As mentioned in the previous section, we shall explicate here the fact that the stability relations, following from the eigenvalues of \( \mathcal{B} \), yield deformed S-duality transformations. Pertaining to the mentioned issues in sections 4 and 5, we wish to illustrate the case of the Calabi-Yau moduli. For a set of given scalar fields, we present the associated algebraic geometric properties from the perspective of \( A \) and \( B \) modules. As mentioned in section 1, let us begin our analysis by reconsidering the notions from the Refs. [13][14], namely, a \( n \)-dimensional complex manifold \( X \) endowed with local coordinate charts \( \{ z^i, \varphi^j \} \). Let \( \mathcal{B} = B_{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q} dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\varphi^{j_1} \wedge \cdots \wedge d\varphi^{j_q} \) be a \( (p, q) \)-form on \( X \). Then, the exterior derivative operator \( \partial \) is expressed by \( \partial B = \frac{1}{i^n} B_{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q} d\varphi^{j_1} \wedge \cdots \wedge d\varphi^{j_q} \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \). Furthermore, a similar expression follows for the operator \( \overline{\partial} B \). Subsequently, it is easy to see that \( \partial^2 = 0 = \overline{\partial}^2 \). Under the consideration of Dolbeault cohomology, the Hodge theorem amounts to the fact that each cohomology class \( H^p_\varphi(X) \) contains unique harmonic form \( B_k \) such that the Laplacian operator decomposes as per the following \( \Delta \mathcal{B} := \partial \overline{\partial} + \overline{\partial} \partial \Delta \mathcal{B} = 0 \). In the case of \( A \) manifold, the above Laplacian leads to the standard notion that we have \( \Delta = 2\Delta \mathcal{B} = 2\Delta \mathcal{B} \). Thus, the harmonic forms associated with the derivative operators \( \partial \) and \( \overline{\partial} \) imply an isomorphism between the vector spaces \( H^p_\varphi(X) \) and \( H^p_\varphi(X) \). For any \( (p, q) \)-harmonic form \( B_{p,q} \) on \( X \), such an identification implies the existence of de-Rham cohomology. For \( (p, q) \)-form, it turns out that \( B_{p,q} \) satisfy the following Dolbeault sum \( B_p = B_{p,0} + B_{p-1,1} + \cdots + B_{0,p} \). For a given \( p \)-form \( B_p \), an application of the operator \( \Delta \) on \( B_p \) shows that we have \( \Delta B_p = 0 \). Thus, \( \Delta \mathcal{B}_{p_1, p_2} \) becomes a \( (p_1, p_2) \)-form, whenever \( \Delta \mathcal{B} \) preserves the degree of \( B_{p_1, p_2} \). It is easy to see that the principle of mathematical induction on the index \( i \) gives \( \Delta \mathcal{B}_{p_{i-1}, i} = 0, \forall i \). Thus, the vector space pertaining to the harmonic de-Rham \( p \)-form decomposes as the direct sum of the harmonic Dolbeault \((p_1, p_2)\)-form, where \( p_1 + p_2 = p \). Consequently, in the case of the \( A \) manifold, the harmonic forms represent a set of cohomology classes, which uniquely reduces to the following decomposition \( \mathcal{H}_p(X) = \mathcal{H}_p^{0,0}(X) \oplus \mathcal{H}_p^{0,1}(X) \oplus \cdots \oplus \mathcal{H}_p^{p,0}(X) \). In this case, the betti numbers are given by \( b_p = \mathcal{H}_p^{0,0} + \mathcal{H}_p^{0,1} + \cdots + \mathcal{H}_p^{p,0} \), where \( h^{p, q} = \dim(\mathcal{H}_p^{q, p}(X)) \) are the Hodge numbers of \( X \).

The manifold \( X \) becomes a \( A \) manifold, if there exists a complex valued function \( K(z, \overline{\varphi}) \) such that the underlying metric tensor is defined by \( g_{i\overline{j}} = \frac{\partial^2 K}{\partial z^i \partial \overline{\varphi}^j} \). As mentioned in section 2, there exists a non-degenerate closed 2-form \( \omega = 2i\partial \partial K(z, \overline{\varphi}) \) such that \( d\omega = 0 \) for any \( K \). On the other hand, it is well known that the topological invariance of \( A \) manifold imposes a pair of restrictions on the associated Hodge numbers, viz. \( h^{0,2} = h^{2,0} \) and \( h^{p, q} = h^{n-p, n-q} \). For a set of harmonic \((q, p)\) forms on the given \( A \) manifold, the first
restriction corresponds to a mapping of the operator $\partial$ to the operator $\overline{\partial}$. Thus, the correspondence $B \to \overline{B}$ results into an invertible map between the underlying $\partial$ and $\overline{\partial}$ cohomologies. Further, the second restriction could be realized as the map $(A,B) \to F_X A \wedge B$ from the space $H^{p,q} \oplus H^{n-p,n-q}$ into $C$. As the above map is non-degenerate, thus $H^{p,q}$ and $H^{n-p,n-q}$ can be viewed as a pair of dual vector spaces with the same dimension $b^p = b^{n-p}$. In correspondence with the associated de-Rham cohomology, the identification of $p$-form cohomology class with $(n-p)$ cycle homology class leads to the standard Poincaré duality. For a given harmonic class $\Sigma$, the perspective of homology classes follows from the consideration of $(n-p)$ cycle on $\mathcal{H}_{n-p}$. In the limit $p \to 0$, there exists a class of delta function $\delta(\Sigma)$ localizations on $p$-dimensional hypersurfaces. Such a consideration examines the limiting parametric nature of $p$-forms. Considering the fact that $\delta(\Sigma)$ is defined as an integral over $p$-dimensional submanifold, Poincaré duality of $\Sigma$ provides the underlying cohomology classes. The Hodge integers of Kähler manifolds remain symmetric under both the horizontal and vertical reflections.

In order to examine the case of Calabi-Yau, let us consider $n$ complex dimensional kähler manifold endowed with a covariantly constant holomorphic $n$-form $\Omega$ such that there exists a Riemannian manifold with $SU(n)$ holonomy. In particular, let us consider the case of $n$-torus $T^n = \prod_{i=1}^{n}(S^1)^i$. Let $\{X_i, J\}_{i=1}^{n}$ be a set of generating vector fields which preserve complex structure $J$ and $\{JX_i\}_{i=1}^{n}$ as the corresponding trivialization of the tangent bundle $TM$. Given $Z_i := X_i \equiv JX_i, \forall i = 1,2,\ldots,n$, we can define $n$ commuting holomorphic vector fields $\{Z_i\}_{i=1}^{n}$ such that there exists a subset $\mathcal{M}_0 \subset \mathcal{M}$, which is biholomorphically equivalent to an open neighborhood of the set $(S^1)^n \subset (C^*)^n$. Thus, the definition of the holomorphic vector fields follows from the trivialization of the canonical tangent bundle $TM$. Moreover, one can choose $\mathcal{M}_0$ to be fibred over a ball $B_0 \subset B$ such that $\mathcal{M}_0$ is homotopically equivalent to an orbit of $T^n$. It follows from the notion of the Lagrangian manifolds that the Kähler form $\omega$ restricts to the zeroes of the orbits, e.g. $\omega$ has a trivial cohomology class. Hereby, for $\theta \in \Omega^1(\mathcal{M}_0)$, there exists a $(1,1)$ form $\omega = d\theta_0 + \overline{\theta}_0$, which is the higher Dolbeault cohomology groups of the product of open sets vanishes in $C^*$, thus there exists a function $f$ such that $\overline{\theta}_0 = \theta_0$. From the definition of Kähler potential $K = \overline{mf}$, we have $\omega = d\overline{\theta} + \overline{\theta} = 2i\theta K$. In order to obtain $T^n$-invariance of the Kähler potential $K$, we need to take an average over the compact group $T^n$. By defining the map $(z_1, \ldots, z_n) \to (e^{z_1}, \ldots, e^{z_n}) \in (C^*)^n$, one finds the following Kähler potential $K(z_1, \ldots, z_n) = K(z_1 + \overline{z}_1, \ldots, z_n + \overline{z}_n)$. Thus, the Kähler metric can be considered as the function of the real parts of holomorphic coordinates. In this concern, the work of Calabi \cite{73} offers an interesting front for the construction of nonhomogeneous Einstein metrics.

Perspective situations arise further, when the underlying Ricci tensor vanishes identically. In this case, there exists a covariantly constant holomorphic $n$-form $dz_{1} \wedge \ldots \wedge dz_n$, iff the Kähler potential $K$ satisfies the real Ampère-Monge equations, viz. we have $det(\frac{\partial^2 K}{\partial x_i \partial y_j}) = constant$. With the help of the relative coordinates $\{x_i, y_i\}$, the metric can be expressed as $g = \sum_{i,j} \frac{\partial^2 K}{\partial x_i \partial y_j}(dx_i dx_j + dy_i dy_j)$. Considering the fact that $T^n$ acts isometrically on $\mathcal{M}$, one finds that there exists a non-trivial quotient metric on the base space $B$. From the perspective of the moduli space geometry, such a metric can be defined as the quotient metric on special Lagrangian submanifold $\mathcal{M}$. For a set of $T^n$-invariant coordinates $\{x_i\}$, the metric tensor locally reduces into the following form $g = \sum \frac{\partial^2 K}{\partial x_i \partial y_j}(dy_i dy_j)$. Explicitly, the consideration of the local coordinates $z_i = x_i + iy_j$ implies that the orbit of $T^n$ can be viewed as the point in $B$ such that $(x_1, \ldots, x_n) = (e^{y_1}, \ldots, e^{y_n})$. Thus, the set $\{\frac{\partial K}{\partial x_i}, \ldots, \frac{\partial K}{\partial x_n}\}$ characterizes forms whose de-Rham cohomology classes form the integral basis of the first cohomology class. For each fibre $\mathcal{M}_0 \to B_0$, this follows from the fact that $Jdx_i = dy_i$, where $J$ is orthogonal matrix. Moreover, the metric on each of the torus can be reduced to the flat metric $g = \sum (\frac{\partial^2 K}{\partial x_i \partial y_j} dy_i dy_j)$. Herewith, there exists a map from the base space $B_0$ to the moduli space of real flat $n$-tori, which can be identified as the space of $n \times n$ positive definite matrices with $GL(n,Z)$. Up to finitely many modulo actions of $GL(n,Z)$, the subsequent analysis is devoted to examine the fact whether the map $x \to \frac{\partial^2 K}{\partial x_i \partial y_j}$ offers a globally well-defined integral basis $\{\frac{\partial K}{\partial x_i}, \ldots, \frac{\partial K}{\partial x_n}\}$ for $H^1(\mathcal{M}, R)$.

From the perspective of compactification, the above notion demonstrates a lower dimensional realization of string theory. Physically, it is worth mentioning that a local Calabi-Yau manifold $X$ can be defined as the equation $s_1 s_2 = H(s_3, s_4)$ in $C^4$ such that there exists a well-defined holomorphic $(0,3)$ form $\gamma = \frac{ds_3}{s_2} \wedge s_3 \wedge s_4$. Here, the manifold $X$ can be regarded as $C^*$ fibration of the fiber $s_1 s_2 = constant$ over the $s_3 s_4$ plane. Thus, the corresponding 3-cycles on $X$ reduces to the associated 1-cycles on the Riemann Surface $\Sigma : 0 = H(s_3, s_4)$. In fact, the periods of $\gamma$ on $X$ descend down to the periods of the meromorphic 1-form $\Lambda$ on $\Sigma$ such that $\int_{X} \gamma = \int_{\gamma \subset X} \Lambda$, where $\Lambda := s_1 ds_4$. For arbitrary $g$ genus Riemann surface $\Sigma$, there exists $2g$ compact 1-cycles such that the symplectic basis $\{A^i, B_j\}$ satisfies $A^i \cap B_j = \delta_{ij}$, $i,j = 1,2,\ldots,g$. For a general noncompact Riemann surface, we may work with an associated set of general basis, which are defined as the intersections $A^i \cap B_j = n^i_j$, where $n^i_j$ are integers. To proceed further, let us define $q^i = \int_{A^i} \Lambda, p_i = \int_{B_i} \Lambda$ and thus the set $\{q^i\}$ can be
considered as the normalizable moduli of $X$. In the above case, for a given noncompact CY, we observe that the function $H(s_3, s_4)$ depends on a set of non-normalizable complex structure moduli $\{t^a\}$ such that there exists compact 3-cycles $C_α \in H_3(X)$ and 1-cycles on $Σ$, where $t^a = \int_{C_α} A$. Herewith, it follows that the moduli space metric remains non-normalizable in the directions for which the corresponding homology dual cycles $C^α$ are non-compact. Consequently, the parameters $\{t^a\}$ of the model are not themselves the physical moduli. Nevertheless, the present consideration implies a set of deformed S-duality transformations $\tilde{p}_i = A^i j p_j + B_j k q^k + F_{i k l m} t^{i l m}$ and $\tilde{q}^j = C^{j i l} p_l + D_i j k q^k + F_{i j k l m} t^{i j k l m}$, where $\{t^a\}$ can be interpreted as a set of monodromy invariant parameters. In this case, notice further that we have defined the most general extension maps $q^i \to \tilde{q}^i, p_i \to \tilde{p}_i$ such that the monodromy group preserves the symplectic form $d q^i \wedge dp_i$ with $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, Z)$. In the foregoing case, it is easy to show that the basis $\{q^i, p_j\}$ transforms in an unconventional way, while the deformed period matrix $\gamma_{ij} = \frac{∂}{∂q^i} p_j$ transforms in the standard way. Explicitly, this leads to the following period map: $τ \to \tilde{τ} = \frac{A τ + B}{C τ + D}$. Now, it is natural to think of $H^3(X, Z)$ as the classical phase space pertaining to the symplectic form $d q^i \wedge dp_i$. In order to quantize $X$, we may promote $\{q^i, p_j\}$ to their canonical conjugate operators satisfying the following canonical commutation relations $[q^i, p_j] = i \hbar δ^{ij}$. Furthermore, the above method of quantization remains compatible on a general CY, including both the real and complex polarizations. This follows from the fact that $(0,3)$ form $γ$ lives in the complexification $H^3(G, C) = C \otimes H^3(X, R)$. As in the case of the compact CY, the period integrals over $A_1$ and $B_1$-cycles over $CY(3)$ can be expressed in terms of the moduli, for a given $(0,3)$-form $γ$. For example, the background independent theory $[76][77]$ offers further algebraic issues. For the case of the $B$-model topological string theory and almost modular forms, Ref. $[78]$ offers Gromov-Witten invariants of a class of orbifold theories and the corresponding quantum mechanical phase space properties, as the quantum geometry of $H^3(X)$, where $X$ is a Calabi-Yau threefold. In the present consideration, we may hereby define the moduli $s_3$ and $\tilde{s}_3$ as the generalized period integrals over the cycles $A_1$ and $B_1$. For a given $γ$, the quantization may be analyzed by the following integrals: $\int_A γ = s^i$ and $\int_B γ = \tilde{s}_i$. Moreover, for cases $D > 4$, we notice the existence of a larger sequence $\{u_{ij}^{2k} \}_{k=1}^n$, instead of a short sequence: $\{u_{ij}^{2k} \}_{k=1}^m$, where $k > m$. Thus, apart from the above extension of $M_{2m}$ to $M_{2k}$, there is the complete freedom to define the notion of semismplectic geometries and thus an interesting deformation property of the underlying algebraic geometry.

In order to interrelate the properties of the underlying embeddings and the spectra of Sen entropy function, we may proceed as follows. Let $H^{03}(X)$ be the local vector spaces of the Calabi-Yau manifold $X$ and $h^{ij}$ be the dimension of the corresponding representations. Considering the fact that the Hodge integers $\{h^{ij}\}$ form a diamond structure with the symmetries $h^{ij} = h^{ji}$ and $h^{i2} = h^{n-i,n-j}$, it follows that the Hodge integers form standard $Z$-module structures and thus their fractions form the perspective of $Q$-modules. Let the underlying characteristic polynomials be defined as $f^{ij} = a_0 + a_1 x + \ldots + a_n x^n = 0$, then the root of $f^{ij}$ can be expressed by the set $\{r_α\}_{α=1}^n$. In this case, the fact that $h^{ij} \in Z; ∀i, j = 1, 2, \ldots, n$, allows us to define $ζ_{ij} = \{r_1, r_2, \ldots, r_{h^{ij}}\}$, $ζ = \bigcup_{i,j=1}^n ζ_{ij}$ and $T = \bigcap_{i,j=1}^n ζ_{ij}$. Subsequently, let $J = \{x | f(x) = 0, ∀x ∈ X\}$. Then, we observe that $ζ_{ij} = J$, as $f(x) = 0, ∀x = r_α$, where $α = 1, 2, \ldots, h^{ij}$. Moreover, the element $T$ is the maximal ideal generated by the set $ζ_{ij}$. From the theory of $A$ and $B$ modules, let $M$ be $A$-module and $N$ be $B$-module. Then, there exists the following duality map $δ : M → N$. For the case of the free modules, we see that the above duality map gives rise to the standard Poincaré type pairing between the spaces $H^{03}(X)$ and $H_{ij}(X)$. Due to the duality equivalence of $A$ and $B$-modules, we see for given cohomology and homology sequences that there exists a set of mappings $\{t_1, t_2\}$ such that $t_1 : M → H^{03}(X)$ and $t_2 : N → H_{ij}(X)$, and vice-versa. In other words, the maps $t_1$ and $t_2$ are well-defined up to a dual transformation. From the viewpoint of Poincaré duality lemma, it follows that $D : H^{03} → H_{ij}$ and thus we have the existence of the following sequences: $M → t_1 H^{03} → D H_{ij} → t_2 H_{ij-1} → N → t_2 H_{ij-1} → D → M → t_1 H^{03} → H_{ij} → t_2 H_{ij-1} → N → t_2 H_{ij-1} → D → M$. Hence, it is not difficult to show that there exists a well-defined composition of $\{u_1, u_2\}$, viz. the maps $u_1 : M → H_{ij}(X)$ and $u_2 : M → H^{03}(X)$ are well-defined under the composition. Moreover, from the perspective of $A$ and $B$ module theory, we find that $H^{03}$ and $H_{ij}$ are encoded by the maps $\{t_1\}$ and $\{u_1\}$. Herewith, it follows that we have $i) M → t_1 H^{03} → D H_{ij} → t_2 H_{ij} → H_{ij-1} → N → t_2 H_{ij-1} → D → M$ and $ii) N → t_2 H_{ij} → D → H^{03} → H_{ij-1} → t_2 H_{ij} → N → t_2 H_{ij} → D → M$. For the above sequences (i) and (ii), we find that the corresponding block diagram commutes, iff the maps $\{t_1\}$ satisfy $t_1 o D = δ o t_2$.

A relationship between the rational points on $A$-module and the corresponding complex points on the associated Kähler manifold $X$ may be defined as follows. Let $M$ be $A$-module generated by the standard fractional transformations, then the cohomology group of $X$ can be described by the map $t_1 : M → H^{03}(X)$. It is interesting to notice $∀α ∈ C^n$ that the Kähler potential $K(ζ, T)$ defines the characteristic function of a given Kähler manifold $X$. Thus, the minimal polynomial $f(s^1, s^2, \ldots, s^n)$ can be defined as a function in $m$ rational’s $s_α := p_α / q_α$.
such that \( q_\alpha \neq 0 \) and \( p_\alpha, q_\alpha \in \mathbb{Z} \). Let \( \rho_\alpha(s') \) be the corresponding minimal monic polynomial such that the roots of \( f(s') = \prod_\alpha \rho_\alpha(s') \) are defined as the set \( \{ r_1 \} \). Then, for given free \( A \)-module, we find that the rational basis functions satisfy \( \{ s_1, \ldots, s_m \} \rightarrow t_1; \{ e_1, \ldots, e_n, \tau \ldots, \tau_n \} \rightarrow d \{ \xi^1, \ldots, \xi^{d-1} \} \). Subsequently, we observe that the composition of \( t_1 \) and \( d \) is well-defined, whenever \( m = 2n \). For a given \( M \), the dimensional equivalence polynomial \( \text{dim}(M) = h^{1,1} \) implies that \( t_1 \) is an isomorphism. Furthermore, we can map the rational points on \( B \)-module to the homology sequence of \( X \). Notice further that the prolongment \( Q \rightarrow R \) lifts \( s' \in R \) and thus it describes classical properties of the basis set \( \{ s' \} \). Interestingly, such a prolongment plays an important role in revealing the limiting relationships between the classical and quantum moduli spaces. An exact analysis of the above question is left open for a future investigation.

In order to make an equivalence between the algebraic and geometric stability properties of the entropy function of an extremal black hole, we may ask the following questions. From the perspective of algebra, the unit circle \( x^2 + y^2 = 1 \) can be examined by considering the Euclidean ring \( k(x, y)/(x^2 + y^2 - 1) \), where \( (x^2 + y^2 - 1) \) is the prime ideal generated by the unit circle. Such an introduction motivates the study of algebraic geometric properties of black brane entropy functions. What follows next that we wish to consider a set of rings, which are associated with the stability of a given entropy function. Let us consider the Taylor series expansion of the entropy function as a finite polynomial in \( \{ s_i \} \). Then, the attractor stability of the black hole can be examined by the algebraic properties of finite polynomials. Notice further that an entropy function could lead to a non-polynomial expansion, as well. For a single variable complex valued entropy function \( \mathfrak{e} \), such an expansion could lead to the following (in)finite Lorentz series \( f(z) = \ldots + \frac{a_0}{z^2} + \frac{a_1}{z} + \ldots \). Depending on \( a_n \), where \( i \in \bar{A} \subset \mathbb{Z} \) or \( i \in \mathbb{Z} \). Up to an algebraic factor, the above cases can be explored polynomially in \( z \in \mathbb{C} \), whenever one can normalize the Lorentz series expansion of \( f(z) \). To be precise, let us consider an ideal \( (f(z)) \) and define the corresponding generalized Euclidean ring as \( \mathbb{Z}_0 := f(x, y) / (f(x, y)) \). On the other hand, let the underlying geometry be defined by the Poincaré like element \( ds^2 := dzd\zeta/f(z) \). Inductively, we may ask the following question: What are the morphisms that \( \mathbb{Z}_0 \) preserves? In order to extend the case for the generalized real Kähler potential \( K(\alpha, \beta) \), let \( u_1 \) be the near horizon values of the real scalar fields and \( v_1 \) be the near horizon spacetime parameters pertaining to \( AdS_2 \times S^{D-2} \) near horizon geometry of the extremal black hole. Let \( g_{ij} = \partial_i \partial_j K(\alpha, \beta) \) be the metric tensor with the corresponding line element \( ds^2 = g_{ij}d\alpha^i d\alpha^j \), where \( d\alpha^i = (\alpha, \beta) \). Now, define \( \mathfrak{z} = K(\alpha^i) \backslash (K(\alpha^i)) \), where \( (K(\alpha^i)) \) is a polynomial ideal generating the equations of the motion, up to an algebraic factor. Then, the present analysis anticipates an interesting avenue to examine the associated algebraic and geometric structures of the map \( \mathfrak{z} \) and its possible variants. Such an issue pertains to both the attractor stability of the black hole and Sen entropy function.

For a given entropy function \( F(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{t}) \), let the metric tensor of the interest be defined as \( g_{ab} = -\partial_a \partial_b F(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{t}) \), where \( \mathfrak{p} := (\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{t}) \in \mathcal{M}_{2n} \otimes \mathcal{M}_u \otimes \mathcal{M}_v \). As mentioned before, let the manifold \( \mathcal{M}_{2n} \) be symplectic, and \( \mathcal{M}_u \) and \( \mathcal{M}_v \) be the two Riemannian manifolds. Thus, the present consideration opens an avenue to analyze the algebraic properties of the quotient \( \mathfrak{z} = F(\mathfrak{p}) \backslash (F(\mathfrak{p})) \). In this case, we may preserve the symplectic structure with \( [\mathfrak{p}_i, \mathfrak{q}_j] = h \delta_{ij} \), \( \forall (\mathfrak{p}, \mathfrak{q}) \in (\mathcal{M}_{2n}, \omega) \), where \( \omega \) denotes the corresponding non-degenerate closed two form. For a given map \( F : \mathcal{M}_{2n} \otimes \mathcal{M}_u \otimes \mathcal{M}_v \rightarrow \mathcal{R} \), there exists a set of embeddings \( \mathcal{M}_{2n} \rightarrow \mathcal{D}_1, \mathcal{M}_{2n} \otimes \mathcal{M}_u \otimes \mathcal{D}_1, \mathcal{M}_{2n} \otimes \mathcal{M}_u \otimes \mathcal{M}_v \), and \( \mathcal{M}_{2n} \otimes \mathcal{M}_u \otimes \mathcal{M}_v \). Herein, we observe that the map \( \mathcal{D}_1 \) is a symplectic embedding, however \( \mathcal{D}_2 \) is a deformed symplectic embedding. For given manifolds \( \{ \mathcal{M}_{2n}, \mathcal{M}_u, \mathcal{M}_v \} \), such a consideration indicates the existence of an unidirectional composition \( \mathcal{D}_1 \circ \mathcal{D}_2 \) in \( \mathcal{D}_2 \). In general, it follows further that the other composition \( \mathcal{D}_2 \circ \mathcal{D}_1 \) may be as well ill-defined. Thus, an entropy function can geometrically be defined as the set of embeddings \( \{ \mathcal{D}_1, \mathcal{D}_2 \} \) on a given semisymplectic manifold \( \mathcal{M} \). From the perspective of the quotient map \( \mathfrak{z} \), it would be interesting to analyze the algebraic properties of the above embeddings \( \mathcal{D}_1, \mathcal{D}_2 \). In order to study the nature of the underlying moduli space compactification, let us consider the restriction \( (\mathfrak{p}, \mathfrak{q}) \rightarrow (\mathfrak{p}, \mathfrak{q}) \). From the perspective of Sen entropy function and generalized attractor flow properties, we may ask the following question. What are the algebraic properties of the maps \( \mathfrak{z}_1 \) and \( \mathfrak{z} \) under the identification \( K : \mathcal{M} \rightarrow \mathcal{R} \) satisfying the local structure \( \partial_i \partial_j F(\mathfrak{p})|_{(\mathfrak{p}, \mathfrak{q})} = \partial_i \partial_j K(\mathfrak{p}, \mathfrak{q}), \forall (\mathfrak{p}, \mathfrak{q}) \in \mathcal{M} := \mathcal{M}_u \otimes \mathcal{M}_v ? \) On the other hand, let us consider the attractor flow data as the restrictions: \( u_i \rightarrow u_i^0 \) and \( v_i \rightarrow v_i^0 \) whenever \( i = 1, 2, \ldots, n \). Then, such a consideration offer a platform to examine an existence of the submersions of \( \mathcal{M} \) to a symplectic manifold \( \mathcal{M}_{2n} \). As mentioned before, for given \( b_{ij} \), we may consider the associated line element as \( ds^2 := r_{ij}dQ^idQ^j \) with the Riemannian metric \( r_{ij} := \partial_i \partial_j S(\mathfrak{Q}) \) and thereby examine the existence of the natural complex structure \( J \) such that there exists a non degenerate closed two form \( \omega \) and thus a symplectic manifold \( (\mathcal{M}_{2n}, \omega) \). Finally, the properties of \( \omega := S(\mathfrak{Q}) \backslash (S(\mathfrak{Q})) \) could reveal underlying algebraic issues pertaining to the deformed string dualities. Moreover, such a consideration opens up a set of avenues to examine the algebraic properties of \( \mathfrak{z}_1|_{(u_i^0, v_i^0)} \) and \( \omega \). For given attractor values of the moduli fields and electric magnetic charges \( \{ u_i^0, v_i^0, \mathfrak{q}, \mathfrak{p} \} \) of
the black hole, it would be important to explore the morphism properties of the maps $\varpi_1$, $\varpi$ and $\varphi$.

7 Conclusion and Outlook

In this note, we have analyzed geometric and algebraic properties of the Hessian of Sen entropy function, associated with the computation of attractor entropy. For a given extremal black hole in string theory, we have offered explicit construction for the generalized symplectic metric, generalized symplectomorphism, generalized symplectic atlas, generalized symplectic stability, generalized Legendre transformation, semi-product, strong stability conditions and thereby illustrated algebraic underlying properties of the spectrum. In particular, the above analysis of the spectrum and convexity has been realized from Krull of the semisymplectifiable algebra $A$, which arises as the minimally extended (sub)field of the underlying eigenvalues of the Hessian quadratic function $B$, defining the attractor stability of the underlying black hole configuration, as per the notion of a semisymplectic manifold $M$. We find that the semisymplectic geometry, when considered as the minimally extended commutative algebraic system, describes the algebraic geometric properties of the black hole entropy functions with various higher curvature corrections of arbitrary $D$ dimensional black hole spacetime. Geometrically, the above reduction leads to the symplectic geometry with a set of given generalized complex structures, which can be defined as the composition of the two series of the embeddings: $D_1$ and $D_2$. From the perspective of the extremal black hole configurations, our algebraic geometric study explains the attractor stability of $AdS_3 \times S^{D-2}$ near horizon geometry and thus it contains possible effects of the scalar fields, gauge fields and arbitrary higher derivative covariant gravity. It turns out that the Riemannian geometric viewpoints of the underlying entropy function renders to the standard thermodynamic geometry, at the attractor fixed point values of a given entropy function. Both the above two geometries possess the same set of nonzero complexifiable generalized spectra, which we have shown to be in the accordance with standard attractor flow properties. Thus, we find that the embedding perspective of Sen entropy function of a given (extremal) black hole configuration encodes the algebraic geometric characteristic features, as a generalized attractor.

For the shake of mathematical simplicity, we have chosen the entropy function $F(\mathcal{S})$ to be a real map. However, for the case of the complex effective Lagrangian theories, our construction allows a set of geometric transitions, which takes an account of the transition from a given real manifold to the corresponding complex manifold. Thus, our analysis further extends for the complex valued entropy functions. In this sense, the non-holomorphic corrections to the prepotential appear analogously as the corresponding fluctuations of the underlying statistical ensemble. In particular, one can easily construct a complex map $F : M \rightarrow C$ such that the generalized complex or complex semisymplectic structure $b^2$ remains compatible with deformed Cauchy-Riemann conditions on the connection functions $\{s_i^{-1} \circ s_j\}$. Indeed, the corresponding higher derivative corrections can be examined with such a complexified consideration. For a given generalized prepotential, when the above method is applied to the CY geometries, we find a set of deformed S-duality transformations accompanying an inclusion of the monodromy invariant parameters, and thus an internal compatibility with the CY quantization. Refs. [84, 85] offer a quantization of such local space from the perspective of the complex polarization method on the space $H^3(M,C)$ and the corresponding reality equivalences on $H^3(M,R)$. For the case of the semisymplectic transformations, we have illustrated that the concerned extensions can be defined as the composition of finitely many mappings, e.g., an identification map $p_i \mapsto \tilde{p}_i = p_i + u_i + v_i$. Thus, the present analysis offers an explicit realization of shifted S-duality transformations, which describe dynamical behavior of the black hole system away from the attractor fixed point(s).

Moreover, from the perspective of the convexity of the spectrum of an extremal black hole entropy function, we have shown that the corresponding algebraic nature emerges from the minimally extended subfield which contains eigenvalues of the Hessian quadratic function $B$. For a given semisymplectic manifold $M$, the above consideration leads to a deformation of the base space symplectic manifold $M_{2n}$ by the moduli space manifold $M_g$ and the gravity sector manifold $M_k$. Explicitly, our analysis shows that such generalized spectra can be investigated from the perspective of commutative algebra, by defining the associated convex hulls. Herewith, we have established that the complexification can be realized in the minimally extended subfield of the eigenvalues of $B$, which at the attractor fixed point(s), reduces to the standard thermodynamic like configuration and the corresponding algebraic properties follow from the associated reality matrix $R$. We have shown that the above semisymplectic geometry turns out to be a well-defined configuration, which physically corresponds to the moduli dependent interacting statistical system. Mathematically, the above geometry describes a non-trivial deformed manifold and contains the effects of the moduli space $M_g$ and non trivially reduces to the thermodynamic like geometry, at the attractor fixed point(s) of the given black hole.

Finally, the present examination signifies geometric and algebraic structures for the purely nonperturbative,
topological and purely quantum aspects and opens up guidelines towards the microstates counting of a black hole. Thus, such a perspective not only offers the role of the outer product of topological string wave function, but also intends to analyze the full density matrix of underlying quantum configurations. From the perspective of a definite quantum theory, we anticipate that there exist a rich class of rational conformal field theory (rCFT), which map Sen entropy function of a given black hole, as a generalized attractor mechanism, to the corresponding elliptic curves \[ \mathcal{E} \]. Thus, for a given entropy function \[ F(\mathcal{E}) \], it would be interesting to understand how a rCFT configuration generalizes to the elliptic curves, or should the elliptic curves be themselves generalized, in order to yield a consistent rCFT spectrum of the extremal black holes? Secondly, from the viewpoint of flux compactifications \[ \mathcal{F} \], such a proposition is expected to offer simulating platforms in order to understand algebraic geometric properties of \( \mathcal{N} = 2 \) black hole attractors and integer SUSY attractors. In the above exploitations, the Galois groups and Falting height functions are hereby undermined to offer interesting perspectives for nonzero heterotic dilaton-axion fields, apart from some discreet identifications. It would be interesting to push further the algebraic geometric perspectives of the generalized attractors, CFT configurations and Sen entropy function.

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