On the codimension of subalgebras of the algebra of matrices over a field

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Abstract

In this paper we provide an elementary and easy proof that a proper subalgebra of the matrix algebra $K^{n,n}$, with $n \geq 3$ and $K$ an arbitrary field, has dimension strictly less than $n^2 - 1$.

Introduction

The aim of this paper is to give an elementary proof of the following result:

**Theorem 1.** If $A$ is a proper subalgebra of $K^{n,n}$, the algebra of $n \times n$ matrices with entries in an arbitrary field $K$, and if $n \geq 3$, then $\dim(A) \leq n^2 - 2$, where $\dim(A)$ is the dimension of $A$ as a vector space over $K$.

When $K$ is an algebraically closed field, this result can be deduced and generalized by the following well known Theorem of Burnside (see [4], [5], [6] and [7] for elementary proofs):

**Burnside’s Theorem.** If $A$ is an irreducible subalgebra of $K^{n,n}$, with $n \geq 1$ and $K$ is an algebraically closed field, then $A = K^{n,n}$.

Recall that a subalgebra $A \subseteq K^{n,n}$ is said to be irreducible if the only linear subspaces $U \subseteq K^n$ invariant under all the elements of $A$, i.e. such that $AU \subseteq U$ for all $A \in A$, are $\{0\}$ and $K^n$. 
Then, if $\mathcal{A}$ is a proper subalgebra of $\mathbb{K}^{n,n}$, with $\mathbb{K}$ an algebraically closed field, there exists a non trivial subspace $U$ of $\mathbb{K}^n$ that is invariant under all the elements of $\mathcal{A}$ with $0 < \dim(U) = r < n$. This immediately implies $\dim(\mathcal{A}) \leq n^2 - r(n - r) \leq n^2 - n + 1$, refining significantly the bound in Theorem 1.

So Theorem 1 gives less information on the dimension of a proper subalgebra of $\mathbb{K}^{n,n}$ than the consequences of Burnside’s Theorem, but it holds over an arbitrary field.

From another perspective the reason for which there do not exist codimension 1 subalgebras of $\mathbb{K}^{n,n}$ resides on Wedderburn’s Theorem providing the decomposition of $\mathbb{K}^{n,n}$ into the direct sum of its semi-simple and radical parts. Over an algebraically closed field the semi simple part is a direct sum of matrix algebras while the dimension of the radical is controlled by Gerstenhaber’s Theorem on the dimension of linear subspaces of nilpotent matrices, see [3]. Putting together these not trivial facts one obtains a different proof of the above inequality $\dim(\mathcal{A}) \leq n^2 - n + 1$. This approach is present in [1], where this inequality is also proved to be true over a field of characteristic zero, thus requiring less restrictive hypothesis about $\mathbb{K}$. A, not trivial, proof of Theorem 1 with $\mathcal{A}$ over an arbitrary field, can be also found as a consequence of the results contained in [2].

In conclusion we post our novelty only on the simple and elementary proof of Theorem 1 but not on its contents which are surely well known to any expert in the field.

**Proof of Theorem 1**

Suppose that $\dim(\mathcal{A}) = n^2 - 1$. Since $\dim(\mathbb{K}^{n,n}) = n^2$, given a basis for the vector space $\mathbb{K}^{n,n}$, the subspace $\mathcal{A}$ is represented by only one homogeneous equation in the associated coordinates. We consider the standard basis of $\mathbb{K}^{n,n}$ which consists of the matrices $E_{ij}$ such that $(E_{ij})_{k,l} = \delta_{ik} \cdot \delta_{jl}$, where $\delta$ is the Kronecker delta.
We have:

\[(E_{ij} \cdot E_{kl})_{m,q} = \sum_{p=1}^{n} (E_{ij})_{m,p} (E_{kl})_{p,q} = \sum_{p=1}^{n} \delta_{im}\delta_{jp}\delta_{kp}\delta_{lq} = \delta_{im}\delta_{kj}\delta_{lq}.\]

If \(k \neq j\) then \(E_{ij} \cdot E_{kl} = 0\), while, if \(k = j\), from

\[(E_{ij} \cdot E_{jl})_{m,q} = \delta_{im}\delta_{lq} = (E_{il})_{m,q},\]

it follows that \(E_{ij} \cdot E_{jl} = E_{il}\).

We now suppose the subspace \(\mathcal{A}\) is given by the following equation:

\[a_{11}x_{11} + \ldots + a_{1n}x_{1n} + a_{21}x_{21} + \ldots + a_{nn}x_{nn} = 0\]

(1)

where \(a_{ij} \in \mathbb{K}\) for all \(i, j = 1, \ldots, n\).

We want to show that \(a_{ij} = 0\) for all \(i\) and \(j\). To this aim, for all \(i, j, k\) and \(l \in \{1, \ldots, n\}\), we define the matrices

\[D_{ij}^{kl} = a_{kl}E_{ij} - a_{ij}E_{kl}.\]

By construction \(D_{ij}^{kl}\) satisfies the equation of \(\mathcal{A}\), and therefore we have \(D_{ij}^{kl} \in \mathcal{A}\) for all \(i, j, k\) and \(l \in \{1, \ldots, n\}\).

We now consider a triple of pairwise distinct indices \(i, j\) and \(k\) (we can find them because \(n \geq 3\)). \(D_{ik}^{ij}\) and \(D_{kj}^{ij}\) belong to \(\mathcal{A}\), which is closed under matrix multiplication. Therefore we deduce:

\[\mathcal{A} \ni D_{ik}^{ij} \cdot D_{kj}^{ij} = (a_{ij}E_{ik} - a_{ik}E_{ij}) \cdot (a_{ij}E_{kj} - a_{kj}E_{ij}) =
\]

\[= a_{ij}^2 E_{ik}E_{kj} - a_{ij}a_{kj}E_{ik}E_{ij} - a_{ik}a_{ij}E_{ij}E_{kj} + a_{ik}a_{kj}E_{ij}^2 = a_{ij}^2 E_{ij},\]

where the last equality follows from the properties of the matrices \(E_{ij}\) and from the choice of \(i, j, k\). The matrix \(a_{ij}^2 E_{ij}\) belongs to \(\mathcal{A}\), so it must satisfy the equation (1). From this it follows that \(a_{ij}^2 = 0 \Rightarrow a_{ij} = 0\).
We have just proved that \( a_{ij} = 0 \) for all \( i \neq j \), so the equation of \( \mathcal{A} \) can be written in the form:

\[
a_{11}x_{11} + a_{22}x_{22} + \ldots + a_{nn}x_{nn} = 0.
\]  

(2)

From this equation we can deduce that, for all \( i = 1, \ldots, n \) and \( k \neq i \), the matrices \( P_i = a_{ii}E_{ik} \) and \( Q_i = a_{ii}E_{ki} \) belong to \( \mathcal{A} \).

Then we have:

\[
P_i \cdot Q_i = a_{ii}^2E_{ik}E_{ki} = a_{ii}^2E_{ii} \in \mathcal{A}.
\]

The matrix \( a_{ii}^2E_{ii} \) belongs to \( \mathcal{A} \), so it must satisfy the equation (2). From this it follows that \( a_{ii}^3 = 0 \Rightarrow a_{ii} = 0 \) for all \( i = 1, \ldots n \).

We proved that \( a_{ij} = 0 \) for all \( i \) and \( j \), in contradiction with the assumption that \( \mathcal{A} \) can be represented by only one equation.
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References and Notes

[1] A. L. Agore, The maximal dimension of unital subalgebras of the matrix algebra, arXiv:14030773.

[2] A. L. Agore, G. Militaru, Extending structures, Galois groups and supersolvable associative algebras, accepted for publication in Monatshefte fur Mathematik.

[3] M. Gerstenhaber, On nilalgebras and linear varieties of nilpotent matrices, in I. Amer. J. Math, 80 (1958), 614-622.

[4] I. Halperin, P. Rosenthal, Burnside’s theorem on algebras of matrices, in Amer. Math. Monthly, 87 (1980), 810.

[5] T. Y. Lam, A theorem of Burnside on matrix rings, in Amer. Math. Monthly, 105 (1998), 651-653.

[6] V. Lomonosov, P. Rosenthal, The simplest proof of Burnside’s theorem on matrix algebras, in Linear Algebra Appl., 383 (2004), 45-47.

[7] E. Rosenthal, A remark on Burnside’s theorem on matrix algebras, in Linear Algebra Appl., 63 (1984), 175-177.

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