Liouville theorem for Pseudoharmonic maps from Sasakian manifolds∗†‡

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Abstract

In this paper, we derive a sub-gradient estimate for pseudoharmonic maps from noncompact complete Sasakian manifolds which satisfy CR sub-Laplace comparison property, to simply-connected Riemannian manifolds with nonpositive sectional curvature. As its application, we obtain some Liouville theorems for pseudoharmonic maps. In the Appendix, we modify the method and apply it to harmonic maps from noncompact complete Sasakian manifolds.

1 Introduction

In [10], S. T. Yau derived a well-known gradient estimate for harmonic functions on complete noncompact Riemannian manifolds. By this estimate, he got a Liouville theorem for positive harmonic functions on Riemannian manifolds with nonnegative Ricci curvature. In [2], S. Y. Cheng generalized the method in [10] to harmonic maps. In [3], S. C. Chang, T. J. Kuo and J. Tie modified the method in [10] and applied it to positive pseudoharmonic functions on noncompact Sasakian $(2n+1)$-manifolds. They introduced a new auxiliary function and successfully dealt with the awkward term in Bochner-type formula. As a result, they obtained a sub-gradient estimate and Liouville theorem for positive pseudoharmonic functions.

In this paper, inspired by [2, 3], we derive a sub-gradient estimate for pseudoharmonic maps from noncompact complete Sasakian manifolds which satisfies CR sub-Laplace comparison property (Theorem 5.3). Then we get the Liouville theorem for pseudoharmonic maps (Theorem 5.6). In the Appendix, we apply the method to harmonic maps from noncompact complete Sasakian manifolds and derive a Reeb energy density estimate (Theorem 6.5). From this estimate, we can prove Liouville theorem for harmonic maps on Sasakian manifolds.

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2 Basic Notions

A smooth manifold \( M \) of real dimension \((2n + 1)\) is said to be a CR manifold, if there exists a smooth rank \( n \) complex subbundle \( T_{1,0}M \subset TM \otimes \mathbb{C} \) such that

\[
T_{1,0}M \cap T_{0,1}M = 0
\]

and

\[
[\Gamma(T_{1,0}M), \Gamma(T_{1,0}M)] \subset \Gamma(T_{1,0}M)
\]

where \( T_{0,1}M = \overline{T_{1,0}M} \) is the complex conjugate of \( T_{1,0}M \). If \( M \) is a CR manifold, then its Levi distribution is the real subbundle \( HM = \text{Re}\{T_{1,0}M \oplus T_{0,1}M\} \). It carries a complex structure \( J_b : HM \to HM \), which is given by

\[
J_b(X + iY) = \sqrt{-1}(X - iY)
\]

for any \( X \in T_{1,0}M \). Since \( HM \) is naturally oriented by the complex structure, then \( M \) is orientable if and only if there exists a global non-vanishing 1-form \( \theta \) such that \( \theta(HM) = 0 \). Any such section \( \theta \) is referred to as a pseudo-Hermitian structure on \( M \). The Levi form \( L_\theta \) is given by

\[
L_\theta(Z, W) = -\sqrt{-1}d\theta(Z, W)
\]

for any \( Z, W \in T_{1,0}M \).

**Definition 2.1.** An orientable CR manifold \( M \) with a pseudo-Hermitian structure \( \theta \), denoted by \((M, HM, J_b, \theta)\), is called a pseudo-Hermitian CR manifold if its Levi form \( L_\theta \) is positive definite.

If \( (M, HM, J_b, \theta) \) is strictly pseudoconvex, there exists a unique nonvanishing vector field \( T \), transverse to \( HM \), satisfying \( T_\cdot \theta = 1 \), \( T_\cdot d\theta = 0 \). This vector field is called the characteristic direction of \((M, HM, J_b, \theta)\). Define the bilinear form \( G_\theta \) by

\[
G_\theta(X, Y) = d\theta(X, J_b Y)
\]

for \( X, Y \in HM \). Since \( L_\theta \) and \( G_\theta \) coincide on \( T_{1,0}M \oplus T_{0,1}M \), \( G_\theta \) is also positive definite on \( HM \otimes HM \). This allows us to define a Riemannian metric \( g_\theta \) on \( M \) by

\[
g_\theta(X, Y) = G_\theta(\pi_H X, \pi_H Y) + \theta(X)\theta(Y), \quad X, Y \in TM
\]

where \( \pi_H : TM \to HM \) is the projection associated to the direct sum decomposition \( TM = HM \oplus \mathbb{R}T \). This metric is usually called the Webster metric.

On a strictly pseudoconvex CR manifold, there exists a canonical connection preserving the complex structure and the Webster metric. Actually

**Proposition 2.2** ([5]). Let \( (M, HM, J_b, \theta) \) be a strictly pseudoconvex CR manifold. Let \( T \) be the characteristic direction and \( J_b \) the complex structure in \( HM \) (extending to an endomorphism of \( TM \) by requiring that \( J_bT = 0 \)). Let \( g_\theta \) be the Webster metric. Then there is a unique linear connection \( \nabla \) on \( M \) (called the Tanaka-Webster connection) such that:

(i) The Levi distribution \( HM \) is parallel with respect to \( \nabla \).
\( \nabla J_b = 0, \nabla g_\theta = 0 \).

(iii) The torsion \( T \) of \( \nabla \) satisfies \( T_{\nabla}(X,Y) = 2d\theta(X,Y)T \) and \( T_{\nabla}(T, J_b X) + J_b T_{\nabla}(T, X) = 0 \) for any \( X, Y \in HM \).

The pseudo-Hermitian torsion, denoted \( \tau \), is the \( TM \)-valued 1-form defined by \( \tau(X) = T_{\nabla}(T, X) \). Note that \( \tau(T_{1,0}M) \subset T_{0,1}M \) and \( \tau \) is \( g_\theta \)-symmetric (cf. [5]).

**Proposition 2.3 ([5])**. If \( (M, HM, J_b, \theta) \) is a strictly pseudoconvex CR manifold, the synthetic object \( (J_b, -T, -\theta, g_\theta) \) is a contact metric structure on \( M \). This contact metric structure is a Sasakian structure if and only if the pseudo-Hermitian torsion \( \tau \) is zero.

**Example 2.4 (Heisenberg group)**. The Heisenberg group \( \mathbb{H}^n \) is obtained by \( \mathbb{C}^n \times \mathbb{R} \) with the group law \( (z, t) \cdot (w, s) = (z + w, t + s + 2Im(z \cdot w)) \).

Let us consider the complex vector fields on \( \mathbb{H}^n \),

\[
T_\alpha = \frac{\partial}{\partial z^\alpha} + \sqrt{-1}z^\alpha \frac{\partial}{\partial t}
\]

where \( \frac{\partial}{\partial z^\alpha} = \frac{1}{2}(\frac{\partial}{\partial x^\alpha} - \sqrt{-1}\frac{\partial}{\partial y^\alpha}) \) and \( z^\alpha = x^\alpha + \sqrt{-1}y^\alpha \). The CR structure \( T_{1,0}\mathbb{H}^n \) is spanned by \( \{T_1, \ldots, T_n\} \). There is a pseudo-Hermitian structure \( \theta \) on \( \mathbb{H}^n \) defined by

\[
\theta = dt + 2 \sum_{\alpha=1}^n (x^\alpha dy^\alpha - y^\alpha dx^\alpha).
\]

The Levi form \( L_\theta = 2 \sum_{\alpha=1}^n dz^\alpha \wedge d\bar{z}^\alpha \) is positive definite, so \( (\mathbb{H}^n, H\mathbb{H}^n, J_b, \theta) \) is a strictly pseudo-Hermitian CR manifold. The characteristic direction is \( T = \frac{\partial}{\partial t} \). Moreover, the Tanaka-Webster connection of \( (\mathbb{H}^n, H\mathbb{H}^n, J_b, \theta) \) is flat. Hence the pseudo-Hermitian torsion is zero, and \( (\mathbb{H}^n, H\mathbb{H}^n, J_b, \theta) \) is Sasakian (See [5] for details).

Let \( (M, HM, J_b, \theta) \) be a strictly pseudoconvex CR \((2n+1)\)-manifold. Let \( \{Z_1, \ldots, Z_n\} \) be a local orthonormal frame of \( T_{1,0}M \) defined on the open set \( U \subset M \), and \( \{\theta^1, \ldots, \theta^n\} \) its dual coframe. Then,

\[
d\theta = 2\sqrt{-1} \sum_{\alpha=1}^n \theta^\alpha \wedge \bar{\theta}^\alpha.
\]

Since \( \tau(T_{1,0}M) \subset T_{0,1}M \), one can set \( \tau Z_\alpha = A_\alpha^\beta Z_\beta \) for some local smooth functions \( A_\alpha^\beta : U \to \mathbb{C} \). Denote by \( \{\omega_\alpha^\beta\} \) the Tanaka-Webster connection 1-forms with respect to the frame \( \{T_\alpha\} \), i.e. \( \nabla Z_\alpha = \omega_\alpha^\beta \otimes Z_\beta \). Then the structure equations can be expressed as follows:

\[
d\theta_\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \quad \tau_\alpha \wedge \theta^\alpha = 0, \quad \omega_\alpha^\beta + \omega^\bar{\beta}_\alpha = 0 \quad (2.1)
\]
where \( \tau^\alpha = A^\alpha_\beta \theta^\beta = A_{\alpha\beta} \theta^\beta \) is a local 1-form.

In [5] [11], the authors showed that the curvature form of Tanaka-Webster connection \( \Pi_\beta^\alpha = d\omega_{\beta}^\alpha - \omega_{\beta}^\gamma \wedge \omega_{\gamma}^\alpha \) is given by

\[
\Pi_\beta^\alpha = R^\alpha_{\beta \mu \nu} \theta^\mu \wedge \theta^\nu + W^\alpha_{\beta \mu} \theta^\mu \wedge \theta - W^\alpha_{\beta \mu} \theta^\mu \wedge \theta + 2\sqrt{-1} T_{\beta \mu} \wedge \tau^\alpha - 2\sqrt{-1} T_{\beta \theta} \wedge \tau^\alpha \quad (2.2)
\]

where \( W^\alpha_{\beta \mu} = A_{\beta \mu}^\alpha \) and \( W^\alpha_{\beta \mu} = A_{\mu \beta}^\alpha \). In particular, \( R_{\beta \mu \gamma} = R_{\mu \beta \gamma} \). The pseudo-Hermitian Ric tensor and the \( Tor \) tensor on \( M, 0 M \) are defined by

\[
\text{Ric}(X, Y) = R_{\alpha \beta} X_\alpha Y_\beta = R_{\alpha \beta \gamma} X_\alpha Y_\beta \quad (2.3)
\]

and

\[
\text{Tor}(X, Y) = \sqrt{-1} (X_\alpha Y_\beta A_{\alpha \beta} - X_{\alpha \beta} A_{\alpha \beta}) \quad (2.4)
\]

for \( X = X_\alpha Z_\alpha \in M, Y = Y_\beta Z_\beta \in M \).

Assume that \( (N, h) \) is a Riemannian manifold. Let \( \{ \xi_i \} \) be a local orthonormal frame of \( TN \), and \( \{ \sigma^i \} \) its dual coframe. Denote by \( \{ \eta_j \} \) the connection 1-forms of the Levi-Civita connection \( \nabla \) on \( N \), i.e. \( \nabla \xi_i = \eta_i^j \otimes \xi_j \). Then we have the structure equations

\[
d\sigma^i = \sigma^j \wedge \eta_j^i, \quad d\eta_j^i = \eta_j^l \wedge \eta_l^i + \Omega_j^i, \quad \Omega_j^i = \frac{1}{2} R_j^i k l \sigma^k \wedge \sigma^l, \quad (2.5)
\]

where \( R \) is the curvature of Levi-Civita connection \( \nabla \) in \( (N, h) \).

Suppose that \( (M, HM, J_0, \theta) \) is a strictly pseudoconvex CR \((2n+1)\)-manifold and \( \nabla \) is its Tanaka-Webster connection. Let \( f : M \to N \) be a smooth map and \( f^*TN \) the pullback bundle. Denote

\[
d_0 f = \pi_H df = f^i_\alpha \theta^\alpha \otimes \xi_i + f^i_\alpha \theta^\alpha \otimes \xi_i \in \Gamma(T^*M \otimes f^*TN),
\]

\[
f_0 = df(T) = f_0 \xi_i \in \Gamma(f^*TN) . \quad (2.6)
\]

Let \( \nabla^f \) be the pullback connection in \( f^*TN \) induced by the Levi-Civita connection of \( (N, h) \). Then we can determine a connection \( \nabla^f \) in \( T^*M \otimes f^*TN \) by

\[
\nabla^f_X (\omega \otimes \xi) = \nabla_X \omega \otimes \xi + \omega \otimes \nabla_X^f \xi
\]

for any \( X \in \Gamma(TM), \omega \in \Gamma(T^*M) \) and \( \xi \in \Gamma(f^*TN) \). Under the local frame \( \{ \theta^\alpha, \theta^\alpha \} \) and \( \{ \xi_i \} \), the tensor \( \nabla^f df \) can be expressed by:

\[
\nabla^f df = f^i_\alpha \theta^\alpha \otimes \theta^\beta \otimes \xi_i + f^i_\alpha \theta^\alpha \otimes \theta^\beta \otimes \xi_i + f^i_\alpha \theta^\alpha \otimes \theta^\beta \otimes \xi_i
\]

\[
+ f^i_\alpha \theta^\beta \otimes \theta^\beta \otimes \xi_i + f^i_\alpha \theta \otimes \theta^\alpha \otimes \xi_i + f^i_\alpha \theta \otimes \theta^\alpha \otimes \xi_i
\]

\[
+ f^i_\alpha \theta \otimes \theta \otimes \xi_i + f^i_\alpha \theta \otimes \theta \otimes \xi_i + f^i_\alpha \theta \otimes \theta \otimes \xi_i \quad (2.7)
\]

Denote by \( \nabla^f \) \( df \) the restriction of \( \nabla^f df \) to \( HM \times HM \). Throughout the paper, the Einstein summation convention is used (except in the inequality (2.21)) and the ranges of indices are

\[
\alpha, \beta, \gamma, \mu, \cdots \in \{1, \ldots, n\}, \quad i, j, k, l, \cdots \in \{1, \ldots, \dim N\}
\]

where \( \dim M = 2n + 1 \).
Definition 2.5 ([5]). Let us consider the f-tensor field on $M$ given by

$$\tau(f; \theta, \hat{\nabla}) = \text{trace}_{\theta}(\nabla^d_d f) \in \Gamma(f^*TN).$$

We say that $f$ is pseudoharmonic, if $\tau(f; \theta, \hat{\nabla}) = 0$.

It is known that pseudoharmonic maps are the critical points of the following energy functional (cf. [5]):

$$E_{\Omega}(f) = \frac{1}{2} \int_{\Omega} \text{trace}_{\theta}(\pi_H f^*h) \theta \wedge (d\theta)^n$$

for any compact domain $\Omega \subset\subset M$. With respect to the local frame $\{\theta, \theta^\alpha, \theta^{\bar{\alpha}}\}$ and $\{\xi_i\}$, we have

$$\tau(f; \theta, \hat{\nabla}) = (f^i_{\alpha\bar{\alpha}} + f^i_{\bar{\alpha}\alpha})\xi_i. \quad (2.8)$$

3 Bochner-Type formulas

In [6], A. Greenleaf obtained the commutation relations of smooth functions and established Bochner-type formulas of pseudoharmonic functions. In [7], John M. Lee derived the commutation relations of $(1, 0)$-forms. We shall need the commutation relations of various covariant derivatives of smooth maps and Bochner-type formulas of pseudoharmonic maps.

Lemma 3.1. Let $f : M \to N$ be a smooth map. The covariant derivatives of $df$ satisfy the following commutation relations:

$$f^i_{\alpha\beta} = f^j_{\beta\alpha}, \quad (3.1)$$

$$f^i_{\alpha\bar{\beta}} - f^i_{\bar{\beta}\alpha} = 2\sqrt{-1} f^j_{\alpha\bar{\beta}} \xi_j, \quad (3.2)$$

$$f^i_{0\alpha} - f^i_{\alpha0} = f^j_{\beta\alpha} A^j_{\alpha\beta}, \quad (3.3)$$

and

$$f^i_{\alpha\bar{\gamma} \beta} - f^i_{\alpha\beta \bar{\gamma}} = 2\sqrt{-1} f^j_{\alpha\bar{\gamma}} A^j_{\alpha\beta} - 2\sqrt{-1} f^j_{\bar{\gamma}} A^j_{\alpha\beta} \delta_{\alpha\bar{\gamma}} - f^j_{\alpha\bar{\gamma}} f^k_{\beta j} \hat{R}^i_{kl}, \quad (3.4)$$

$$f^i_{\alpha\bar{\gamma} \bar{\beta}} - f^i_{\alpha\bar{\beta} \bar{\gamma}} = 2\sqrt{-1} f^j_{\alpha\bar{\gamma}} A^j_{\alpha\beta} - 2\sqrt{-1} f^j_{\bar{\gamma}} A^j_{\alpha\beta} \delta_{\alpha\bar{\gamma}} - f^j_{\alpha\bar{\gamma}} f^k_{\beta j} \hat{R}^i_{kl}, \quad (3.5)$$

$$f^i_{\alpha0 \beta} - f^i_{\alpha0 \beta} = f^j_{\gamma A^{\mu}_{\alpha\beta}} R^{\mu}_{\alpha\beta}, \quad (3.6)$$

$$f^i_{\alpha\beta0} - f^i_{\alpha\beta0} = f^j_{\gamma A^{\bar{\nu}}_{\alpha\beta}} \gamma - f^j_{\alpha\gamma} A_{\beta}^{\bar{\mu}} \gamma - f^j_{\bar{\nu}} f^k_{\beta j} \hat{R}^i_{kl}, \quad (3.7)$$

$$f^i_{\alpha0 \bar{\beta}} - f^i_{\alpha0 \bar{\beta}} = f^j_{\gamma A^{\bar{\mu}}_{\bar{\beta}} A_{\alpha}} - f^j_{\alpha\gamma} A_{\bar{\beta}}^{\bar{\mu}} A_{\alpha} - f^j_{\bar{\mu}} f^k_{\beta j} \hat{R}^i_{kl}. \quad (3.8)$$

Proof. The identities (2.6) imply

$$f^*\sigma^i = f^i_{\alpha\theta^\alpha} + f^i_{\alpha\theta^{\bar{\alpha}}} + f^i_{\bar{\alpha}\theta}. \quad (3.9)$$
We take the exterior derivative of (3.9) and use the structure equations (2.1), (2.5) to get
\[
0 = (df^i - f^j_\beta \omega^\beta_\alpha + f^j_\alpha \tilde{\eta}^i_\beta) \wedge \theta^\alpha + (df^i_\alpha - f^j_\beta \omega^\beta_\alpha + f^j_\alpha \tilde{\eta}^i_\beta) \wedge \theta^\beta \\
+ (df^i_0 + f^j_0 \tilde{\eta}^i_\beta) \wedge \theta + f^j_\alpha \theta \wedge \tau^\alpha + f^j_\alpha \theta \wedge \tau^\beta + 2\sqrt{-1} f^j_0 \theta^\beta \wedge \theta^\tilde{\beta},
\]
(3.10)
where \( \tilde{\eta}^i_\beta = f^* \eta^i_\beta \). On the other hand, the second-order covariant derivatives satisfy
\[
df^i_\alpha - f^j_\beta \omega^\beta_\alpha + f^j_\alpha \tilde{\eta}^i_\beta = f^i_{\alpha \beta} \theta^\beta + f^i_{\alpha 0} \theta, \quad (3.11)
df^i_\alpha - f^j_\beta \omega^\beta_\alpha + f^j_\alpha \tilde{\eta}^i_\beta = f^i_{\alpha \beta} \theta^\beta + f^i_{\alpha 0} \theta, \quad (3.12)
df^i_0 + f^j_0 \tilde{\eta}^i_\beta = f^i_{0 \beta} \theta^\beta + f^i_{0 0} \theta. \quad (3.13)
\]
Substituting the above three equations into (3.10) and using \( \tau^\alpha = A^\alpha_\beta \theta^\beta \), we obtain
\[
0 = f^i_{\alpha \beta} \theta^\beta \wedge \theta^\alpha + f^i_{\alpha \beta} \theta^\beta \wedge \theta^\beta + (f^i_{\alpha \beta} - f^j_\beta 0 - 2\sqrt{-1} f^j_\alpha \delta_\alpha_\beta) \theta^\tilde{\beta} \wedge \theta^\alpha \\
+ (f^i_{\alpha 0} - f^j_\beta A^\beta_\alpha) \theta \wedge \theta^\alpha + (f^i_{\alpha 0} - f^j_\beta A^\beta_\alpha) \theta \wedge \theta^\tilde{\beta}.
\]
which (by comparing types) yields (3.1). To prove the next five equations, we differentiate (3.11) and use the structure equations again. Then we obtain
\[
0 = (df^i_{\alpha \beta} - f^j_{\alpha \gamma} \omega^\gamma_\beta - f^j_{\alpha \beta} \omega^\gamma_\alpha + f^j_{\alpha \beta \gamma} \tilde{\eta}^i_\beta) \wedge \theta^\beta \\
+ (df^i_{\alpha \beta} - f^j_{\alpha \gamma} \omega^\gamma_\beta - f^j_{\alpha \beta} \omega^\gamma_\alpha + f^j_{\alpha \beta \gamma} \tilde{\eta}^i_\beta) \wedge \theta^\beta \\
+ (df^i_{\alpha 0} - f^j_{\beta 0} \omega^\beta_\alpha + f^j_{\beta 0} \tilde{\eta}^i_\beta) \wedge \theta + f^j_\beta \Pi^\beta_\alpha - f^j_\beta f^* (\Omega^i_j) \\
+ f^i_{\alpha \beta} A^\beta_\gamma \theta \wedge \theta^\gamma + f^i_{\alpha \beta} A^\beta_\gamma \theta \wedge \theta^\gamma + 2\sqrt{-1} f^i_{\alpha 0} \theta^\beta \wedge \theta^\beta. \quad (3.14)
\]
Since the third-order covariant derivatives of \( f \) is given by
\[
df^i_{\alpha \beta} - f^j_{\alpha \delta} \omega^\delta_\beta - f^j_{\alpha \delta} \tilde{\eta}^i_\beta = f^i_{\alpha \beta} \theta^\beta + f^i_{\alpha 0} \theta, \quad (3.15)
df^i_{\alpha \beta} - f^j_{\alpha \delta} \omega^\delta_\beta - f^j_{\alpha \delta} \tilde{\eta}^i_\beta = f^i_{\alpha \beta} \theta^\beta + f^i_{\alpha 0} \theta, \quad (3.16)
df^i_{\alpha 0} - f^j_{\beta 0} \omega^\beta_\alpha + f^j_{\beta 0} \tilde{\eta}^i_\beta = f^i_{\alpha 0} \theta + f^i_{\alpha 0} \theta^\gamma + f^i_{\alpha 0} \theta^\gamma,
\]
we can substitute them into (3.14) and use (2.2), (2.5) to obtain
\[
0 = \sum_{\gamma<\beta} (f^i_{\alpha \beta \gamma} - f^i_{\alpha \gamma \beta} - 2\sqrt{-1} f^i_{\beta \gamma} A_{\alpha \gamma} + 2\sqrt{-1} f^i_{\gamma \alpha} A_{\beta \gamma} + f^i_{\alpha k} f^k_{\beta} f^j_{\gamma} R^i_{j k l} \theta^\gamma \wedge \theta^\beta \\
+ \sum_{\gamma<\beta} (f^i_{\alpha \beta \gamma} - f^i_{\alpha \gamma \beta} - 2\sqrt{-1} f^i_{\beta \gamma} A_{\alpha \gamma} + 2\sqrt{-1} f^i_{\gamma \alpha} A_{\beta \gamma} + f^i_{\alpha k} f^k_{\beta} f^j_{\gamma} R^i_{j k l} \theta^\gamma \wedge \theta^\beta \\
+ \sum_{\gamma<\beta} (f^i_{\alpha \beta \gamma} - 2\sqrt{-1} f^i_{\mu} A^\mu_\alpha \delta_\beta - 2\sqrt{-1} f^i_{\mu} A^\mu_\beta \delta_\alpha + f^i_{\alpha k} f^k_{\beta} f^j_{\gamma} R^i_{j k l} \theta^\gamma \wedge \theta^\beta \\
+ \sum_{\beta} (f^i_{\alpha \beta 0} - f^i_{\alpha 0 \beta} - f^i_{\gamma} A_{\alpha \beta} + f^i_{\gamma} A_{\beta \alpha} + f^i_{\alpha k} f^k_{\beta} f^j_{\gamma} R^i_{j k l} \theta^\gamma \wedge \theta^\beta \\
+ \sum_{\beta} (f^i_{\alpha \beta 0} - f^i_{\alpha 0 \beta} + f^i_{\gamma} A_{\beta \alpha} + f^i_{\gamma} A_{\alpha \beta} + f^i_{\alpha k} f^k_{\beta} f^j_{\gamma} R^i_{j k l} \theta^\gamma \wedge \theta^\beta) \wedge \theta^\gamma.
\]
which (by comparing types) yields (3.4)-(3.8).

Before introducing the Bochner-type formulas, we recall a property of the sub-Laplace operator $\Delta_b$ (cf. [5]). If $u$ is a $C^2$ function on $M$, then $\Delta_b u = trace_{G_a}(\nabla_b d_b u)$. With respect to the local orthonormal frame $\{T, Z_\alpha, Z_\bar{\alpha}\}$, we have $\Delta_b u = u_{\bar{\alpha}\bar{\alpha}} + u_{\alpha\alpha}$.

**Lemma 3.2.** For any smooth map $f : M \to N$, we have

$$\frac{1}{2} \Delta_b |d_b f|^2 = |\nabla_b^i d_b f|^2 + \langle \nabla_b^i f; \theta, \nabla \rangle, d_b f \rangle - 4(d_b f \circ J_b, \nabla_b^i f_0)$$

$$+ (2Ric - 2(n - 2)Tor)(f^i_\beta Z_\beta, f^i_\bar{\alpha} Z_\bar{\alpha})$$

$$+ 2(f^i_\alpha f^j_\beta f^k_\delta f^l_\gamma \tilde{R}^i_{jkl} + f^i_\alpha f^j_\beta f^k_\delta f^l_\gamma \tilde{R}^i_{jkl}),$$

(3.15)

$$\frac{1}{2} \Delta_b |df(T)|^2 = |\nabla_b^i f_0|^2 + \langle df(T), \nabla_b^i \tau(f; \theta, \nabla) \rangle + 2f^i_\beta f^j_\alpha f^k_\delta f^l_\gamma \tilde{R}^i_{jkl}$$

$$+ 2(f^i_\alpha f^j_\beta A_{\bar{\beta}a} + f^i_\beta f^j_\alpha A_{\bar{\beta}a} + f^i_\alpha f^j_\beta A_{\beta a} + f^i_\beta f^j_\alpha A_{\beta a})$$

(3.16)

where $\nabla_b^i \tau(f; \theta, \nabla)$ and $\nabla_b^i f_0$ are the restriction of $\nabla^i \tau(f; \theta, \nabla)$ and $\nabla^i f_0$ to $HM$.

**Proof.** Since $|d_b f|^2 = 2f^i_\beta f^j_\beta$, we have

$$\frac{1}{2} \Delta_b |d_b f|^2 = (f^i_\beta f^j_\beta)_\beta\beta + (f^i_\bar{\alpha} f^j_\bar{\alpha})_\beta\beta$$

$$= 2(f^i_\alpha f^j_\beta f^i_\bar{\alpha} f^j_\bar{\alpha} + f^i_\beta f^j_\alpha f^i_\bar{\alpha} f^j_\bar{\alpha} + f^i_\bar{\alpha} f^j_\beta f^i_\alpha f^j_\bar{\alpha} + f^i_\bar{\alpha} f^j_\bar{\alpha} f^i_\alpha f^j_\beta + f^i_\bar{\alpha} f^j_\bar{\alpha} f^i_\beta f^j_\beta$$

$$= |\nabla_b^i d_b f|^2 + f^i_\beta f^j_\alpha f^i_\alpha f^j_\beta + f^i_\alpha f^j_\beta f^i_\beta f^j_\alpha + f^i_\beta f^j_\alpha f^i_\bar{\alpha} f^j_\bar{\alpha}$$

Lemma 3.1 implies

$$f^i_\beta f^j_\alpha f^i_\beta f^j_\alpha = f^i_\beta f^j_\alpha f^i_\beta f^j_\alpha = f^i_\beta f^j_\alpha f^i_\beta f^j_\alpha = f^i_\beta f^j_\alpha f^i_\beta f^j_\alpha$$

Substituting them into the previous identity, we obtain

$$\frac{1}{2} \Delta_b |d_b f|^2 = |\nabla_b^i d_b f|^2 + f^i_\beta f^j_\alpha f^i_\alpha f^j_\beta + f^i_\alpha f^j_\beta f^i_\beta f^j_\alpha$$

$$+ 2(\nabla_b^i f^j_\beta f^k_\mu A_{\beta \mu} - f^i_\alpha f^j_\beta f^k_\mu A_{\alpha \mu} - f^i_\beta f^j_\alpha f^k_\mu A_{\beta \mu} + f^i_\beta f^j_\alpha f^k_\mu A_{\alpha \mu})$$

By the identity $(d_b f \circ J_b, \nabla_b^i f_0) = \sqrt{-1}(f^i_\alpha f^j_\beta f^k_\mu A_{\alpha \mu} - f^i_\beta f^j_\alpha f^k_\mu A_{\alpha \mu})$, we get (3.15). The proof of (3.16) is similar. \[\square\]
Lemma 3.3. Let $(M, HM, J_b, \theta)$ be a $(2n + 1)$-Sasakian manifold with

$$\text{Ric}(X, X) \geq -k|X|^2$$  \hspace{1cm} (3.17)

for all $X \in T_{1,0}M$, and some $k \geq 0$. Suppose that $(N, h)$ is a Riemannian manifold with nonpositive sectional curvature. If $f : M \rightarrow N$ is a pseudoharmonic map, then for any $\nu > 0$, we have

$$\Delta_b |d_b f|^2 \geq |\nabla_b^f d_b f|^2 + 2n|f_0|^2 - \left(2k + \frac{32}{\nu}\right) |d_b f|^2 - \frac{1}{2} \nu |\nabla_b^f f_0|^2$$  \hspace{1cm} (3.18)

and

$$\Delta_b |f_0|^2 \geq 2|\nabla_b^f f_0|^2.$$  \hspace{1cm} (3.19)

Proof. Since $f$ is pseudoharmonic, by definition we have $\tau(f; \theta, \hat{\nabla}) = 0$. Because $(M, HM, J_b, \theta)$ is Sasakian, the tensor $\text{Tor}$ is pseudoharmonic, by definition we have $\tau \equiv 0$. Hence, by the assumption on the pseudo-Hermitian Ricci curvature, \eqref{eq:3.17} becomes

$$\begin{align*}
\Delta_b |d_b f|^2 &= 2|\nabla_b^f d_b f|^2 - 8\langle d_b f \circ J_b, \nabla_b^f f_0 \rangle - 2k|d_b f|^2 \\
&\quad + 4\langle f^i_{\alpha \beta} f^j_{\alpha \beta} f^k_{\alpha \beta} \bar{R}^i_{j k l} + f^i_{\alpha \beta} f^j_{\alpha \beta} f^k_{\alpha \beta} \bar{R}^i_{j k l} \rangle.
\end{align*}$$  \hspace{1cm} (3.20)

Using the commutation relation \eqref{eq:3.18}, we can estimate

$$|\nabla_b^f d_b f|^2 = 2 \sum_{\alpha, \beta = 1}^n (f^i_{\alpha \beta} f^i_{\alpha \beta} + f^i_{\alpha \beta} f^i_{\alpha \beta}) \geq 2 \sum_{\alpha = 1}^n f^i_{\alpha \alpha} f^i_{\alpha \alpha}$$

$$\begin{align*}
&= \frac{1}{2} \sum_{\alpha = 1}^n (|f_{\alpha \alpha}^i + f_{\alpha \alpha}^j|^2 + |f_{\alpha \alpha}^i - f_{\alpha \alpha}^j|^2) \\
&\geq \frac{1}{2} \sum_{\alpha = 1}^n |f_{\alpha \alpha}^i|^2 - 2n|f_0|^2.
\end{align*}$$  \hspace{1cm} (3.21)

The second term of the right side of \eqref{eq:3.20} can be controlled by the Schwarz inequality

$$- 8\langle d_b f \circ J_b, \nabla_b^f f_0 \rangle \geq - \frac{32}{\nu} |d_b f|^2 - \frac{1}{2} \nu |\nabla_b^f f_0|^2.$$  \hspace{1cm} (3.22)

To deal with the last term of \eqref{eq:3.20}, we set $e_{\alpha} = \text{Re} df(Z_{\alpha})$ and $\bar{e}_{\alpha} = \text{Im} df(Z_{\alpha})$. Then

$$\begin{align*}
\text{Last term of } \eqref{eq:3.20} &= 4 \langle \bar{R}(df(Z_{\beta}), df(Z_{\alpha})), df(Z_{\beta}), df(Z_{\alpha}) \rangle \\
&\quad + 4 \langle \bar{R}(df(Z_{\beta}), df(Z_{\alpha})), df(Z_{\alpha}), df(Z_{\alpha}) \rangle \\
&= - 4\langle \bar{R}(e_{\alpha}, e_{\beta}), e_{\alpha}, e_{\alpha} \rangle + \langle \bar{R}(e_{\alpha}, e_{\alpha}'), e_{\beta} \rangle + \langle \bar{R}(e_{\alpha}', e_{\beta}), e_{\alpha} \rangle \\
&\quad + \langle \bar{R}(e_{\alpha}', e_{\alpha}'), e_{\beta} \rangle + \langle \bar{R}(e_{\alpha}', e_{\beta}), e_{\alpha} \rangle \\
&\quad \geq 0.
\end{align*}$$  \hspace{1cm} (3.23)

where we have used the assumption that the sectional curvature of $N$ is nonpositive. Substituting \eqref{eq:3.21}, \eqref{eq:3.22} and \eqref{eq:3.23} into \eqref{eq:3.20}, we get \eqref{eq:3.18}.
Observe that
\[
f^i f^j f^k f^l \tilde{R}^i_{\; kl} = \langle \tilde{R}(df(Z_\alpha), df(T)) df(Z_\alpha), df(T) \rangle \\
= - \left( \langle \tilde{R}(e_\alpha, e_0) e_0, e_\alpha \rangle + \langle \tilde{R}(e'_\alpha, e_0) e_0, e'_\alpha \rangle \right) \\
\geq 0.
\] (3.24)

Then (3.19) can be easily proved from (3.16) and (3.24).

From now on, we assume that \((N, h)\) is a simply connected Riemannian manifold with nonpositive sectional curvature. Let \(\rho\) be the distance to a fixed point \(y_0 \in N\). Then \(\rho^2\) is smooth on \(N\). By the Hessian comparison theorem, we have
\[
\text{Hess}(\rho^2) \geq 2h.
\]

For any smooth map \(f : M \rightarrow N\), the chain rule gives that
\[
\Delta_b (\rho^2 \circ f) = d\rho^2(\tau(f; \tilde{\nabla})) + \text{trace}_{G_\theta} \text{Hess}(\rho^2)(db f, db f).
\]

Therefore, we can conclude that if \(f\) is pseudoharmonic, then
\[
\Delta_b (\rho^2 \circ f) \geq 2|db f|^2.
\] (3.25)

4 Cannot-Carathéodory distance

As known, the maximum principle is an important tool to obtain pointwise estimates for solutions of geometric PDEs. In order to use it in Sasakian manifolds, we need some special exhaustion function to construct a cutoff function. A natural choice is the Carnot-Carathéodory distance function.

**Definition 4.1.** Let \((M, HM, J_b, \theta)\) be a strictly pseudoconvex CR manifold. A piecewise \(C^1\)-curve \(\gamma : [0, 1] \rightarrow M\) is said to be horizontal if \(\gamma'(t) \in HM\) whenever \(\gamma'(t)\) exists. The length of \(\gamma\) is given by
\[
l(\gamma) = \int_0^1 |\gamma'|^{1/2}_{C_o} dt.
\]

We define the Cannot-Carathéodory distance between two points \(p, q \in M\) by
\[
d_c(p, q) = \inf \{l(\gamma) | \gamma \in C_{p,q} \}
\]
where \(C_{p,q}\) is the set of all horizontal curves joining \(p\) and \(q\). We say that \((M, HM, J_b, \theta)\) is complete if it is complete as a metric space. A horizontal curve \(\gamma : [0, 1] \rightarrow M\) is called length minimizing geodesic if \(l(\gamma) = d_c(\gamma(0), \gamma(1))\).

Fix \(x_0 \in M\), and set \(r(x) = d_c(x_0, x)\). The Carnot-Carathéodory ball of radius \(R\) centered at \(x_0\) is denoted by \(B_R(x_0) = \{x \in M | r(x) < R\}\).
In [9], R. Strichartz pointed out that if \((M, H, J, b, \theta)\) is complete, then for any \(x_0, x \in M\), there exists at least one length minimizing geodesic \(\gamma : [0, 1] \to M\) joining \(x_0\) and \(x\). Moreover, \(\gamma\) can extend to \((-\infty, \infty)\). We say that \(x\) is a cut point of \(x_0\), if for any \(\epsilon > 0\), \(\gamma|_{[0, 1+\epsilon]}\) is no longer a length minimizing geodesic joining \(x_0\) and \(\gamma(1 + \epsilon)\). The set of all cut points of \(x_0\), denoted by \(\text{cut}(x_0)\), is called the cut locus of \(x_0\). Theorem 1.2 and Proposition 1.2 in [1] assert that the Cannot-Carathéodory distance \(r\) to a reference point \(x_0\) is smooth on \(M \setminus (\text{cut}(x_0) \cup \{x_0\})\).

**Definition 4.2** ([3]). Let \((M, H, J, b, \theta)\) be a noncompact complete Sasakian \((2n + 1)\)-manifold with
\[
\text{Ric}(X, X) \geq -k|X|^2
\]
for all \(X \in T_{1,0}M\) and some \(k \geq 0\). We say that \((M, H, J, b, \theta)\) satisfies CR sub-Laplace comparison property relative to a point \(x_0 \in M\), if there exists a positive constant \(C_1\) such that the Carnot-Carathéodory distance \(r\) to \(x_0\) satisfies
\[
\triangle_b r \leq C_1 \left( \frac{1}{r} + \sqrt{k} \right) \tag{4.1}
\]
on \(M \setminus (\text{cut}(x_0) \cup \{x_0\})\) and where \(r \geq 1\).

**Proposition 4.3** ([4]). There exists a positive constant \(C'_1\) on Heisenberg group \((\mathbb{H}^n, J_b, \theta)\) such that
\[
\triangle_b r \leq \frac{C'_1}{r} \tag{4.2}
\]
on \(M \setminus (\text{cut}(o) \cup \{o\})\). Here \(r\) is the Carnot-Carathéodory distance to the origin \(o\).

Since the pseudohermitian torsion and the pseudohermitian Ricci curvature of Heisenberg group \((\mathbb{H}^n, J_b, \theta)\) are both zero, Proposition 4.3 asserts that the CR sub-Laplace comparison property holds on Heisenberg group.

**5 Sub-Gradient Estimate For Pseudoharmonic Map**

In this section, we will obtain a sub-gradient estimate for pseudoharmonic maps. Let \((M, H, J, b, \theta)\) be a noncompact complete \((2n + 1)\)-Sasakian manifold with CR sub-Laplace comparison property relative to a point \(x_0 \in M\) and
\[
\text{Ric}(X, X) \geq -k|X|^2
\]
for all \(X \in T_{1,0}M\) and some \(k \geq 0\). Suppose that \((N, h)\) is a simply connected Riemannian manifold with nonpositive sectional curvature. We consider a pseudoharmonic map \(f : M \to N\). Let \(\rho\) be the Riemannian distance to \(y_0 = f(x_0)\).
We choose a function $\psi \in C^\infty([0, \infty))$ with the property that
\[ \psi|_{[0,1]} = 1, \quad \psi|_{[2,\infty)} = 0, \quad -C_2 \leq \frac{\psi}{R^{\frac{1}{2}}} \leq 0, \quad |\psi''| \leq C_2. \]

Let $R > 1$ be fixed. By CR sub-Laplacian comparison property, the cutoff function $\eta = \psi(\frac{R}{R})$ satisfies:
\[
\eta^{-1}|db\eta|^2 \leq C_2' \frac{R^2}{R^2}
\]
\[
\Delta_b \eta = \frac{\psi'''}{R^2}|db\eta|^2 + \frac{\psi'}{R} \Delta_b \eta \geq -C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right)
\tag{5.1}
\]
on $M \setminus (cut(x_0) \cup \{x_0\})$. Here $C_2'$ depends only on $C_2$ and $C_1$. Denote $b_R = 2\sup\{\rho \circ f(x) | x \in B_{2R}(x_0)\}$. We construct a smooth function $F(x) : B_{2R}(x_0) \to \mathbb{R}$ by
\[
F(x) = \frac{|db_f|^2 + \mu \eta |f_0|^2}{b_R^2 - \rho^2 \circ f}(x).
\tag{5.2}
\]
The positive coefficient $\mu$ will be determined later.

**Lemma 5.1.** If $r$ is smooth at $x \in B_{2R}(x_0)$ and $(\eta F)(x) \neq 0$, then at $x$, we have
\[
\Delta_b(|dbf|^2 + \mu \eta |f_0|^2) \geq \frac{1}{2} \frac{|dbf|^2 + \mu \eta |f_0|^2|^2}{|dbf|^2 + \mu \eta |f_0|^2}
+ 2n - 6\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) |f_0|^2 - 32 \left( k + \frac{1}{\mu \eta} \right) |df|^2.
\tag{5.3}
\]

**Proof.** First we compute
\[
\Delta_b(\mu \eta |f_0|^2) = \mu \left( \Delta_b \eta |f_0|^2 + 2\langle db \eta, db |f_0|^2 \rangle + \eta \Delta_b |f_0|^2 \right)
= \mu \left( \Delta_b \eta |f_0|^2 + 2\langle db \eta, 2\nabla_b^f f_0, f_0 \rangle + \eta \Delta_b |f_0|^2 \right)
\geq \mu \left( \Delta_b \eta |f_0|^2 - \eta |\nabla_b^f f_0|^2 - 4|f_0|^2 \eta^{-1}|db \eta|^2 + \eta \Delta_b |f_0|^2 \right)
\geq \mu \eta |\nabla_b^f f_0|^2 - 5\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) |f_0|^2.
\]
The last inequality is due to (3.19) and (5.1). Hence by (3.18) with $\nu = \mu \eta$, we have the estimate
\[
\Delta_b(|dbf|^2 + \mu \eta |f_0|^2) \geq \frac{1}{2} \left( |\nabla_b^f df|^2 + \mu \eta |\nabla_b^f f_0|^2 \right) + \frac{1}{2} |\nabla_b^f df|^2
+ \left[ 2n - 5\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right] |f_0|^2 - 32 \left( k + \frac{1}{\mu \eta} \right) |dbf|^2.
\tag{5.4}
\]
Lemma 5.2. If \( x \) is a nonzero maximum point of \( \eta F \), we have the estimate

\[
|d_b|d_b f|^2| \leq 4 |d_b f|^2 |\nabla_b f|^2, 
\]

\[
|d_b|f|^2| \leq 4 |f|^2 |\nabla_b f|^2. 
\]  

(5.5) (5.6)

If \( |d_b f|(x) \neq 0 \) and \( |f_0|(x) \neq 0 \), then at \( x \), we have

\[
\frac{1}{2} \left( \langle \nabla_b f \rangle^2 + \mu \eta |\nabla_b f|^2 \right)
\geq \frac{1}{8} \left( \frac{|d_b|d_b f|^2|}{|d_b f|^2} + \mu \eta |\nabla_b f|^2 \right) + \frac{\mu |d_b \eta|^2}{\eta} |f_0|^2
= \frac{1}{8} \mu |d_b \eta|^2 |f_0|^2.
\]

Substituting this inequality to (5.5), we get (5.3). If \( |d_b f|(x) = 0 \) (or \( |f_0|(x) = 0 \)), we can directly discard the nonnegative term \( \frac{1}{2} \langle \nabla_b f \rangle^2 \) (or \( \frac{\mu}{2} \mu \eta |\nabla_b f|^2 \)) from (5.4) and use the Schwarz inequality (5.5) (or (5.6)) to obtain (5.3)

Let \( x \) be a maximum point of \( \eta F \) on \( B_{2R}(x_0) \). If \( x \) is not in the cut loci of \( x_0 \), then \( \eta \) is smooth near \( x \). If \( x \) is in the cut loci of \( x_0 \), we may remedy \( \eta \) by the following consideration. Since \( (M, HM, J, \theta) \) is complete, there exists a length minimizing geodesic curve \( \gamma : [0, 1] \to M \) which joins \( x_0 \) and \( x \). Let \( \rho \) be a small positive number. Along \( \gamma \), \( x \) is before the cut point of \( \gamma(\rho) \). This guarantees that the modified function \( \tilde{\gamma}(z) = d_{\epsilon}(z, \gamma(\rho)) + \epsilon \) is smooth in the neighborhood of \( x \). Moreover, triangle inequality implies that:

\[
r \leq \tilde{r}, \quad \text{and} \quad r(x) = \tilde{r}(x).
\]

Set \( \tilde{\eta} = \psi(\tilde{r}) \). Then \( \tilde{\eta} \) is smooth near \( x \)

\[
\eta \geq \tilde{\eta}, \quad \text{and} \quad \eta(x) = \tilde{\eta}(x).
\]

This means that \( x \) is still a maximum point of \( \tilde{\eta} F \). Hence, we may assume without loss of generality that \( r \) is already smooth near \( x \).

**Lemma 5.2.** If \( x \) is a nonzero maximum point of \( \eta F \) on \( B_{2R}(x_0) \), then at \( x \), we have the estimate

\[
0 \geq 2n \eta F - 2 34n (k + \frac{1}{\mu}) \int |d_b f|^2 + 2n - 31 \mu \int \frac{|d_b f|^2 + |\nabla_b f|^2}{\frac{R^2}{b^2} - \rho^2 \circ f} F, 
\]

(5.7)

*Proof.* It is obvious that \( x \) is still a maximum point of \( \ln(\eta F) \) on \( B_{2R}(x_0) \). Since \( \Delta_b \) is a degenerate elliptic operator, the maximum principle implies that at \( x \),

\[
0 = d_b \ln(\eta F) = \frac{d_b \eta}{\eta} + \frac{d_b (d_b f^2 + \mu |f_0|^2)}{d_b f^2 + \mu |f_0|^2} + \frac{d_b (\rho^2 \circ f)}{b^2_\ell - \rho^2 \circ f}, 
\]

(5.8)

\[
0 \geq \Delta_b \ln(\eta F) = \frac{\Delta_b \eta}{\eta} - \frac{|d_b \eta|^2}{\eta^2} + \frac{\Delta_b (d_b f^2 + \mu |f_0|^2)}{d_b f^2 + \mu |f_0|^2}
- \frac{|d_b (d_b f^2 + \mu |f_0|^2)|^2}{(d_b f^2 + \mu |f_0|^2)^2} + \frac{\Delta_b (\rho^2 \circ f)}{b^2_\ell - \rho^2 \circ f} + \frac{|d_b (\rho^2 \circ f)|^2}{(b^2_\ell - \rho^2 \circ f)^2}.
\]

(5.9)
By Lemma 5.1, (5.9) becomes

\[
0 \geq \frac{\Delta_b \eta}{\eta} - \frac{|d_b \eta|^2}{\eta^2} - \frac{23}{24} \frac{|d_b (|d_b f|^2 + \mu |f_0|^2)|^2}{(|d_b f|^2 + \mu |f_0|^2)^2} + \frac{|d_b (\rho^2 \circ f)|^2}{(b_R^2 - \rho^2 \circ f)^2} \\
+ \frac{[2n - 6\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right)] |f_0|^2 - 32(\alpha + \frac{1}{\mu}) |d_b f|^2}{|d_b f|^2 + \mu |f_0|^2} + \frac{\Delta_b (\rho^2 \circ f)}{b_R^2 - \rho^2 \circ f}.
\]

Substituting (5.8) in above inequality and using Schwarz inequality: \((\alpha + \beta)^2 \leq 24\alpha^2 + \frac{24}{25}\beta^2\), we obtain

\[
0 \geq \frac{\Delta_b \eta}{\eta} - \frac{|d_b \eta|^2}{\eta^2} - 32 \left( k + \frac{1}{\mu \eta} \right) \frac{|d_b f|^2}{|d_b f|^2 + \mu |f_0|^2} \\
+ \left[ 2n - 6\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right] \frac{|f_0|^2}{|d_b f|^2 + \mu |f_0|^2} + 2 \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f}.
\]

By the estimates (3.25) and (5.1), we have

\[
0 \geq -25C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) F - 32 \left( k + \frac{1}{\mu \eta} \right) \frac{|d_b f|^2}{|d_b f|^2 + \mu |f_0|^2} \\
+ \left[ 2n - 6\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right] \frac{|f_0|^2}{|d_b f|^2 + \mu |f_0|^2} + 2 \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f}.
\]

Hence multiplying both sides by \(\eta F\), we conclude that

\[
0 \geq -25C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) F - 32 \left( \eta k + \frac{1}{\mu} \right) \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f} \\
+ \left[ 2n - 6\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right] \frac{\eta |f_0|^2}{b_R^2 - \rho^2 \circ f} + 2 \eta F \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f}.
\]

Finally, we rewrite (5.2) as

\[
\frac{\eta |f_0|^2}{b_R^2 - \rho^2 \circ f} = \frac{1}{\mu} \frac{F - \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f}}{\mu}
\]

and substitute it into the previous inequality. This procedure yields

\[
0 \geq \left[ 2n - 31\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right] F + \left[ 2 \eta F - \frac{1}{\mu} \left( 2n - 31\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right) - 32 \left( \eta k + \frac{1}{\mu} \right) \right] \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f} \\
\geq \left[ 2n - 31\mu C_2' \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right) \right] \frac{F}{\mu} + \left[ 2 \eta F - \frac{2n}{\mu} - 32(\alpha + \frac{1}{\mu}) \right] \frac{|d_b f|^2}{b_R^2 - \rho^2 \circ f}.
\]

The last inequality is due to \(0 \leq \eta \leq 1\). Since \(n \geq 1\), we get (5.7).

\(\square\)
Now we present our main results.

**Theorem 5.3.** Let \((M, HM, J_b, \theta)\) be a noncompact complete \((2n+1)\)-Sasakian manifold with CR sub-Laplace comparison property relative to a fixed point \(x_0\) and

\[\text{Ric}(X, X) \geq -k|X|^2\]

for all \(X \in T_{1,0}M\), and some \(k \geq 0\). Suppose that \((N, h)\) is a simply connected Riemannian manifold with nonpositive sectional curvature. Assume that \(f : M \to N\) is a pseudoharmonic map. Let \(\rho\) be the Riemannian distance to \(y_0 = f(x_0)\). For any \(R > 1\), set \(b_R = 2 \sup \{ \rho \circ f(x) | x \in B_{2R}(x_0) \}\) and \(a = \frac{R^2}{1 + \sqrt{kR}}\). Then, on \(B_R(x_0)\)

\[|df|^2 + a|f_0|^2 \leq C^3 b_R^2 \left( \frac{1}{a} + k \right) \]

(5.11)

where the constant \(C_3\) only depends on the dimension of \(M\) and \(C_1\).

**Remark 5.4.** Our auxiliary function \(\eta F\) for the maximum principle is slightly different from that one introduced in [3]. In our case, we omit the variable \(t\) in the auxiliary function. This seems to simplify the related estimates even for the pseudoharmonic function case.

**Proof.** Let \(\mu = \frac{n}{31C_2^2} \frac{R^2}{1 + \sqrt{kR}} = \frac{1}{31} a\). We consider the auxiliary function \(F\) given by (5.2). Let \(x\) be a maximum point of \(\eta F\) on \(B_{2R}(x_0)\). We assume \((\eta F)(x) \neq 0\) (Otherwise, the following estimate (5.12) is trivial). Since \(2n - 31 \mu^2 C_2^2 \left( \frac{1}{R^2} + \frac{\rho^2}{R^2} \right) = n > 0\), the last term of the right side in (5.7) is positive. Hence Lemma 5.2 yields

\[\max_{z \in B_{2R}(x_0)} (\eta F)(z) \leq 17n \left( k + \frac{1}{\mu} \right). \]

(5.12)

Since \(\eta(z) = 1\) for \(z \in B_R(x_0)\), this inequality asserts that on \(B_R(x_0)\)

\[|df|^2 + a|f_0|^2 \leq 17n(b_R^2 - \rho^2 \circ f) \left( k + \frac{1}{\mu} \right) \leq 17nb_R^2 \left( k + \frac{1}{\mu} \right).\]

Hence (5.11) can be obtained by choosing a proper constant \(C_3\). \(\square\)

The Reeb energy density is defined by the partial energy density \(\frac{1}{2} |df(T)|^2\). From the sub-gradient estimate (5.11), we can derive an estimate of Reeb energy density for pseudoharmonic maps and get some vanishing results.

**Corollary 5.5.** Let \((M, HM, J_b, \theta)\) be a noncompact complete Sasakian manifold with CR sub-Laplace comparison property relative to a fixed point \(x_0\) and

\[\text{Ric}(X, X) \geq -k|X|^2\]

for all \(X \in T_{1,0}M\) and some \(k \geq 0\). Suppose that \((N, h)\) is a simply connected Riemannian manifold with nonpositive sectional curvature. Assume that \(f :
$M \to N$ is a pseudoharmonic map. Let $\rho$ be the Riemannian distance to $y_0 = f(x_0)$. For any $R > 1$, set $b_R = 2\sup\{\rho \circ f(x) | x \in B_{2R}(x_0)\}$ and $a = \frac{R^2}{1 + \sqrt{kR}}$. Then, on $B_R(x_0)$

$$
|f_0|^2 \leq C_3 b^2_R \left( \frac{2}{R^4} + \frac{3k}{R^2} + \frac{k\sqrt{k}}{R} \right),
$$

(5.13)

In particular,

(i) if $\text{Ric} \geq 0$ (i.e. $k = 0$) and the image of $f$ satisfies:

$$
\lim_{R \to \infty} R^{-2} \sup\{\rho \circ f(x) | x \in B_{2R}(x_0)\} = 0,
$$

then $df(T) = 0$.

(ii) if the pseudohermitian Ricci curvature of $M$ has strictly negative lower bound (i.e. $k > 0$) and the image of $f$ satisfies:

$$
\lim_{R \to \infty} R^{-\frac{3}{2}} \sup\{\rho \circ f(x) | x \in B_{2R}(x_0)\} = 0,
$$

then $df(T) = 0$.

The sub-gradient estimate (5.11) also gives Liouville theorem for pseudoharmonic maps.

**Theorem 5.6.** Let $(M, H M, J_0, \theta)$ be a noncompact complete Sasakian manifold with nonnegative pseudohermitian Ricci curvature, and satisfy CR sub-Laplace comparison property relative to a fixed point $x_0 \in M$. Suppose that $(N, h)$ is a simply connected Riemannian manifold with nonpositive sectional curvature. Assume that $f : M \to N$ is a pseudoharmonic map. Let $\rho$ be the Riemannian distance to $y_0 = f(x_0)$. For any $R > 1$, set $b_R = 2\sup\{\rho \circ f(x) | x \in B_{2R}(x_0)\}$. Then, on $B_R(x_0)$

$$
|d_b f|^2 + R^2 |f_0|^2 \leq C_3 \frac{b^2_R}{R^2}.
$$

In particular, if the image of $f$ satisfies

$$
\lim_{R \to \infty} R^{-1} \sup\{\rho \circ f(x) | x \in B_{2R}(x_0)\} = 0,
$$

then $f$ is a constant map.

Since Heisenberg group $(H^n, H H^n, J_0, \theta)$ satisfies CR sub-Laplace comparison property, Theorem 5.6 can be applied to Heisenberg group.

**Corollary 5.7.** There is no bounded pseudoharmonic map from Heisenberg group $(H^n, H H^n, J_0, \theta)$ to a simply connected Riemannian manifold with nonpositive sectional curvature.
6 Appendix

In this section, we will derive a Reeb energy density estimate for harmonic maps from Sasakian manifolds to Riemannian manifolds. We recall the definition of harmonic maps. Let $(M, H, J, J_0, \theta)$ be a strictly pseudoconvex CR manifold, and let $\nabla^\theta$ be the Levi-Civita connection of $(M, g^\theta)$. Let $(N, h)$ be a Riemannian manifold, and $\nabla$ its Levi-Civita connection. Suppose that $f : M \to N$ is a smooth map. Let $f^*TN$ be the pullback bundle and $\nabla^f$ the pullback connection. We can determine a connection $\nabla^f, \theta$ in $T^*M \otimes f^*TN$ by

$$\nabla^f, \theta_X(\omega \otimes \xi) = \nabla^\theta_X \omega \otimes \xi + \omega \otimes \nabla^f_X \xi$$

for any $X \in \Gamma(TM)$, $\omega \in \Gamma(T^*M)$ and $\xi \in \Gamma(f^*TN)$. So $f$ is harmonic if

$$\tau^\theta(f; \theta, \nabla) = \text{trace}_{g^\theta}(\nabla^f, \theta df) = 0.$$  

With respect to the local orthonormal frame $\{\theta, \theta^\alpha, \theta^\bar{\alpha}\}$ in $T^*M \otimes \mathbb{C}$ and $\{\xi_i\}$ in $TN$, we have

$$\tau^\theta(f; \theta, \nabla)(f) = (f^i_{\alpha\bar{\alpha}} + f^i_{\bar{\alpha}\alpha} + f^i_{00}) \xi_i.$$  

(6.1)

Comparing with the equation (2.8), we obtain

$$\tau^\theta(f; \theta, \nabla)(f) = \tau(f; \theta, \nabla)(f) + \nabla^f_T df(T).$$  

(6.2)

As above, we need a Bochner-type formula for harmonic maps and a special exhaustion function.

**Lemma 6.1.** Let $f : M \to N$ be a smooth map. Then

$$\frac{1}{2} \|df(T)\|^2 = |\nabla^f f_0|^2 + \langle df(T), \nabla^f_T \tau^\theta(f; \theta, \nabla) \rangle + 2f_0^i f_0^j f_0^k \hat{R}_{ijkl}^i + 2(f_0^i f_0^j A_{\beta\bar{\alpha},\alpha} + f_0^i f_0^j A_{\bar{\beta}\alpha,\alpha} + f_0^i f_0^j A_{\beta\bar{\alpha}} + f_0^i f_0^j A_{\bar{\beta}\alpha}).$$  

(6.3)

where $\triangle$ is the Laplacian operator in $(M, g^\theta)$.

**Proof.** On the one hand, we notice that

$$\frac{1}{2} \|df(T)\|^2 = \frac{1}{2} \triangle |df(T)|^2 + \frac{1}{2} (f_0^i f_0^i)_{00} = \frac{1}{2} \triangle |df(T)|^2 + f_0^i f_0^i + f_0^i f_0^i.$$  

(6.4)

On the other hand, by (6.1), we have

$$\langle df(T), \nabla^f_T \tau^\theta(f; \theta, \nabla) \rangle = \langle df(T), \nabla^f_T \tau(f; \theta, \nabla) \rangle + \langle df(T), \nabla^f_T \nabla^f_T df(T) \rangle$$

$$= \langle df(T), \nabla^f_T \tau(f; \theta, \nabla) \rangle + f_0^i f_0^i.$$  

Hence substituting the above equation and (3.16) into (6.4), we get (6.3). \qed
Lemma 6.2. Let $(M, H, J, b, \theta)$ be a Sasakian manifold, and $(N, h)$ a Riemannian manifold with nonpositive sectional curvature. If $f : M \to N$ is a harmonic map, then
\[
\frac{1}{2} \triangle |df(T)|^2 \geq |\nabla f_0|^2.
\] (6.5)

The proof follows from (3.24) and (6.3).

Definition 6.3. Let $(M, H, J, b, \theta)$ be a Sasakian manifold with
\[
\text{Ric}(X, X) \geq -k|X|^2
\]
for any $X \in T_{1,0}M$, and some $k \geq 0$. We say that $(M, H, J, b, \theta)$ satisfies CR Laplace comparison property relative to a fixed point $x_0 \in M$, if there exists a positive constant $C_4$ such that the Carnot-Carathéodory distance $r$ to $x_0$ satisfies
\[
\triangle r \leq C_4 \left( \frac{1}{r} + \sqrt{k} \right)
\] (6.6)
\[
|dr|_{g_0} \leq C_4
\] (6.7)
on $M \setminus (\text{cut}(x_0) \cup \{x_0\})$ and where $r \geq 1$.

On Heisenberg group $(\mathbb{H}^n, H, J, \theta)$, the square of the Carnot-Carathéodory distance function $r$ to the origin has the following expression
\[
[r(z, t)]^2 = \frac{\phi^2}{(\sin \phi)^2} ||z||^2
\] (6.8)
where $||z||^2 = \sum_{\alpha=1}^n |z^\alpha|^2$, $\phi$ is the unique solution of $\chi(\phi)||z||^2 = |t|$ in the interval $[0, \pi)$ and $\chi(\phi) = \frac{\phi}{\sin \phi} - \cot \phi$. See [3, 4] for details.

Proposition 6.4. On Heisenberg group $(\mathbb{H}^n, H, J, \theta)$, there exists a positive constant $C'_4$ such that the Carnot-Carathéodory distance $r$ to the origin $o$ satisfies
\[
\triangle r \leq C'_4 \frac{1}{r}
\] (6.9)
\[
|dr|_{g_o}^2 \leq C'_4
\] (6.10)
on $M \setminus (\text{cut}(o) \cup \{o\})$ and where $r \geq 1$. Therefore, $(\mathbb{H}^n, H, J, \theta)$ satisfies CR Laplace comparison property relative to the origin.

Proof. We first calculate $Tr$ and $TTr$ on $M \setminus (\text{cut}(o) \cup \{o\})$. When $t > 0$, we take the partial derivative along $\frac{\partial}{\partial t}$ of $\chi(\phi)||z||^2 = |t|$ and use the expression of $\chi$. The result is
\[
\frac{\partial \phi}{\partial t} = \frac{1}{2 ||z||^2} \frac{(\sin \phi)^3}{\sin \phi - \phi \cos \phi}.
\]
Therefore,

\[
Tr^2 = \frac{\partial r^2}{\partial t} = \phi,
\]

\[
TTr^2 = \frac{\partial^2 r^2}{\partial t^2} = \frac{1}{r^2} \frac{(\sin \phi)^5}{\phi^2 (\sin \phi - \phi \cos \phi)}.
\]

Since \( TTr^2 = 2r TTr + 2|Tr|^2 \), there exists a constant \( \tilde{C}_4 \) such that

\[
|Tr| \leq \frac{\tilde{C}_4}{r}, \quad |TTr| \leq \frac{\tilde{C}_4}{r^3}.
\] (6.11)

When \( t < 0 \), we can do the similar calculations and obtain the same inequality (6.11). When \( t = 0 \), we can use the continuity property to get the same estimate (6.11), since \( r \) is smooth on \( M \setminus (\text{cut}(o) \cup \{o\}) \). Hence the inequalities (6.11) always hold on \( M \setminus (\text{cut}(o) \cup \{o\}) \). From Proposition (6.11), there exists a constant \( \tilde{C}_4' \) such that

\[
\nabla b r \leq \frac{\tilde{C}_4'}{r}
\] (6.12)

on \( M \setminus (\text{cut}(o) \cup \{o\}) \). Let \( C_4' = 1 + \tilde{C}_4 + \tilde{C}_4^2 + \tilde{C}_4' \). Then

\[
\Delta r = \nabla b r + TTr \leq \frac{C_4'}{r}
\]

\[
|dr|^2 = |dr_r|^2 + (Tr)^2 \leq C_4'
\]

on \( M \setminus (\text{cut}(o) \cup \{o\}) \) and where \( r \geq 1 \).

To derive the Reeb energy density estimate, we need an analogue estimate of (3.25). Assume that \( (N,h) \) is a simply connected Riemannian manifold with nonpositive sectional curvature. Let \( \rho \) be the distance to a fixed point \( y_0 \in N \). If \( f : M \to N \) is harmonic, the Hessian comparison theorem implies

\[
\nabla (\rho^2 \circ f) \geq 2|df|^2.
\] (6.13)

**Theorem 6.5.** Let \( (M,HM,J_0,\theta) \) be a noncompact complete Sasakian manifold with CR Laplace comparison property relative to a fixed point \( x_0 \) and

\[
\text{Ric}(X,X) \geq -k|X|^2
\]

for any \( X \in T_{1,X}M \), and some \( k \geq 0 \). Suppose that \( (N,h) \) is a simply connected Riemannian manifold with nonpositive sectional curvature. Let \( \rho \) be the Riemannian distance to \( y_0 = f(x_0) \). For any \( R > 1 \), set \( b_R = 2 \sup \{ \rho \circ f(x) | x \in B_{2R}(x_0) \} \). Then, on \( B_R(x_0) \)

\[
|df(T)|^2 \leq C_6 b_R^2 \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right)
\] (6.14)

where the constant \( C_6 \) depends only on \( C_4 \). Moreover,
(i) if Ric ≥ 0 (i.e. k = 0) and the image of f satisfies
\[ \lim_{R \to \infty} R^{-1} \sup \{ \rho \circ f(x) | x \in B_{2R}(x_0) \} = 0, \]
then df(T) = 0.

(ii) if the pseudohermitian Ricci curvature of M has strictly negative lower bound (i.e. k > 0) and the image of f satisfies
\[ \lim_{R \to \infty} R^{-\frac{2}{3}} \sup \{ \rho \circ f(x) | x \in B_{2R}(x_0) \} = 0, \]
then df(T) = 0.

Remark 6.6. In [8], R. Petit got a similar vanishing theorem for harmonic maps from compact Sasakian manifolds to Riemannian manifolds with nonpositive sectional curvature.

Proof. The choices of ψ and η are the same as in Section 5. Since \((M, H M, J_b, \theta)\) satisfies CR Laplace comparison property, then η satisfies
\[
\eta^{-1}|d\eta|^2 \leq \frac{C_5}{R^2},
\]
\[
\Delta \eta = \frac{\psi''}{R^2} |dr|^2 + \frac{\psi'}{R} \Delta r \geq -C_5 \left( \frac{1}{R^2} + \frac{\sqrt{k}}{R} \right)
\] (6.15)
on \(M \setminus (cut(x_0) \cup \{x_0\})\). Here \(C_5\) depends only on \(C_4\) and \(C_2\).

Given \(R > 1\), we consider the function \(G : M \to \mathbb{R}\), which is given by
\[ G(x) = \frac{|f_0|^2}{b_R^2 - \rho^2 \circ f}(x). \]
Let \(x\) be a maximum point of \(\eta G\) on \(B_{2R}(x_0)\). If \(x\) is in the cut locus of \(x_0\), then we can modify \(r\) as in Section 5. Without loss of generality, assume that \(r\) is smooth at \(x\) and \((\eta G)(x) \neq 0\). It is obvious that \(x\) is still a maximum point of ln(\(\eta G\)) on \(B_{2R}(x_0)\). Then the maximum principle asserts that at \(x\),
\[
0 = d \ln(\eta G) = \frac{d\eta}{\eta} + \frac{d|f_0|^2}{|f_0|^2} + \frac{d(\rho^2 \circ f)}{b_R^2 - \rho^2 \circ f},
\]
\[
0 \geq \Delta \ln(\eta G) = \frac{\Delta \eta}{\eta} - \frac{|d\eta|^2}{\eta^2} + \frac{\Delta |f_0|^2}{|f_0|^2} - \frac{|d|f_0|^2|^2}{|f_0|^4} + \frac{\Delta (\rho^2 \circ f)}{b_R^2 - \rho^2 \circ f} + \frac{|d(\rho^2 \circ f)|^2}{(b_R^2 - \rho^2 \circ f)^2}.
\] (6.16)
Applying (6.5) and the inequality \(|d|f_0|^2|^2 \leq 4 |f_0|^2 |\nabla^f f_0|^2\) to (6.17), we have
\[
0 \geq \frac{\Delta \eta}{\eta} - \frac{|d\eta|^2}{\eta^2} - \frac{1}{2} \frac{|d|f_0|^2|^2}{|f_0|^4} + \frac{|d(\rho^2 \circ f)|^2}{(b_R^2 - \rho^2 \circ f)^2} + \frac{\Delta (\rho^2 \circ f)}{b_R^2 - \rho^2 \circ f}.
\]
With the aid of Schwarz inequality, we can use (6.16) to estimate the third and fourth terms. The result is

\[ 0 \geq \frac{\triangle \eta}{\eta^2} - 2 \frac{|d\eta|^2}{\eta^2} + \frac{\triangle (\rho^2 \circ f)}{b_R^2 - \rho^2 \circ f}. \]

Therefore combining with (6.13) and (6.15), we conclude that at \( x \),

\[ \frac{|df|^2}{b_R^2 - \rho^2 \circ f} \leq 3C_5 \left( \frac{1}{R^2} + \sqrt{\kappa} \right). \]

Hence by \(|f_0|^2 \leq |df|^2\), we can get an estimate of \( \eta G \):

\[ \max_{z \in B_{2R}(x)} \frac{\eta |f_0|^2}{b_R^2 - \rho^2 \circ f} (z) = (\eta G)(x) = \frac{\eta |f_0|^2}{b_R^2 - \rho^2 \circ f} (x) \leq 3C_5 \left( \frac{1}{R^2} + \sqrt{\kappa} \right). \]

This yields for any \( z \in B_R(x_0) \),

\[ |f_0|^2 (z) \leq 3C_5 \left( b_R^2 - \rho^2 \circ f(z) \right) \left( \frac{1}{R^2} + \sqrt{\kappa} \right) \leq 3C_5 b_R^2 \left( \frac{1}{R^2} + \sqrt{\kappa} \right). \]

Let \( C_6 = \frac{3}{2} C_5 \). The above inequality yields (6.14). The rest of this theorem follows from the estimate (6.14). \(\square\)

The relation (6.2) shows that if \( df(T) = 0 \), then harmonic map is equivalent to pseudoharmonic map. Therefore, Theorem 5.6 asserts the following Liouville theorem.

**Corollary 6.7.** Let \((M, HM, J_b, \theta)\) be a noncompact complete Sasakian manifold with nonnegative pseudohermitian Ricci curvature, and satisfy both CR sub-Laplace comparison property and CR Laplace comparison property relative to a fixed point \( x_0 \in M \). Suppose that \((N, h)\) is a simply connected Riemannian manifold with nonpositive sectional curvature. Assume that \( f : M \to N \) is a harmonic map. Let \( \rho \) be the Riemannian distance to \( y_0 = f(x_0) \). If the image of \( f \) satisfies

\[ \lim_{R \to \infty} R^{-1} \sup \{ \rho \circ f(x) | x \in B_{2R}(x_0) \} = 0, \]

then \( f \) is a constant map.

Proposition 4.3 and Proposition 6.4 state that Heisenberg group satisfies both CR sub-Laplace comparison property and CR Laplace comparison property relative to the origin.

**Corollary 6.8.** There is no bounded harmonic map from Heisenberg group \((H^n, H H^n, J_b, \theta)\) to a simply connected Riemannian manifold with nonpositive sectional curvature.

**Remark 6.9.** If \( n \geq 2 \), then the Levi-Civita connection of Heisenberg group \((H^n, H H^n, J_b, \theta)\) does not have nonnegative Ricci curvature. Thus Corollary 6.8 can not be derived from the results in [2].
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