Emergent QED$_3$ in an extended weak interacting Kane-Mele-Hubbard model

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We study a Kane-Mele model with weak extended Hubbard interactions. We show that a dynamic massive Maxwell field emerges in the long wavelength limit. In the zero Proca mass limit, the charge confinement in the compact U(1) gauge field is consistent with no gapless excitation in the bulk of the topological insulator. The mass gaps of the photon can be revealed by measuring the dynamic structure function in Bragg scattering. By coupling to an extra fermion, the low energy effective theory turns out to be an emergent “quantum electrodynamics” in 2+1 dimensions with/without a Chern-Simons term. A non-quantized plateau Hall effect and quantum anomalous Hall effect responding to the “electric” field, i.e., the gradient of the spin density, can be observed either individually or combinatorially.

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Introduction. The quantum gauge fields mediate all forces between elementary particles which constitute the world. Those gauge fields which serve the electromagnetic and the weak forces are known as photon and intermediate bosons, respectively, and have been “seen”. The gluons, which carry the strong force, are still hiding, i.e., confinement$^{1,2}$. As a toy model of the confinement, quantum electrodynamics in 2+1 dimensions (QED$_3$) has been extensively studied for near three decades$^{3–5}$. Recently, the quantum simulation of the dynamic gauge theories becomes much interest by means of the cold atoms and other systems$^{6–20}$. Most of these works focused on simulating the lattice gauge theories$^{21–23}$. Non-perturbative effects such as the confinement may be explored in these lattice models. The question is: do these effects survive in the continuum limit?

The continuum gauge fields can emerge from strongly correlated quantum many body systems, e.g., a doped Mott insulator$^{24}$. However, most of these gauge symmetries rely on mean field states as saddle points$^{25}$. The stability of these mean field states was not guaranteed because the quantum fluctuations are often not controllable.

In this Letter, we report an emergent massive Maxwell field (or Proca field) from a two-dimensional fermionic insulating system with a weak interaction. Very recently, Haldane model with quantum anomalous Hall effect$^{26}$ has an experimental realization with cold fermions on optical lattice$^{27}$. We therefore focus on Kane-Mele model$^{28,29}$ and its generalization. Essentially, they are two copies of Haldane model and can be realized by the same technique as that in$^{27}$. The interaction term we consider is rather generic and thus the realization of our model may not be limited to cold atom systems. The key point for this emergent phenomenon is that both the spin-up and spin-down fermions have non-zero Chern numbers$^{30}$. In the long wavelength limit, the Lagrangian for a given spin fermion field reduces to the Thirring model. Integrating over the fermion fields$^{31}$, the final effective theory turns out to be a Proca theory. The photon is massive because the 3-spin current excitations are gapped in the bulk. As the mass tends to nought the theory becomes an emergent Maxwell theory. The nontrivial topology of the lattice fermion model again yields the monopole condensation in this compact gauge theory which results in the charge confinement$^{32}$. The ”photon” is again gapped, in consistent with the insulating behavior of the system. We calculate the density-density correlation functions which can be experimentally measured by Bragg spectroscopy of a cold atom system$^{32}$.

Generalizing the Kane-Mele model by violating the time reversal symmetry (TRS), we find that the low energy effective theory may still has TRS until a Chern-Simons term emerges when two ”fine structure constants” mismatch (see below). Finally, we introduce an extra fermion which interacts with the spinful fermions. In the continuum limit, this extra fermion field plays a role of the matter field in QED$_3$ with/without the Chern-Simons term. When this Chern-Simons term exists, a non-quantized plateau Hall effect (PHE) of the extra fermions can be observed with the the gradient of the spin density as its external ”electric field”. If the extra fermion carries a non-zero Chern number, the induced Chern-Simons term implies a quantum anomalous Hall effect (QAHE). PHE and QAHE can be observed either individually or combinatorially.

The Model. The Kane-Mele model with no Rashba spin-orbital coupling is exactly two-copies of Haldane model on a honeycomb lattice at half filling$^{28,29}$,

$$H_0 = -t \sum_{\langle ij \rangle, \sigma} c_{i \sigma}^\dagger c_{j \sigma} + i \lambda \sum_{\langle ij \rangle, \sigma \sigma'} \nu_{i j} c_{i \sigma}^\dagger \sigma \sigma' c_{j \sigma'}^\dagger,$$

where $c_{i \sigma}^\dagger$ is the spin-$\sigma$ fermion creation operator at site $i$. $\sigma \sigma'$ is the third Pauli matrix. $t$ is the hopping constant
while $\lambda$ is the spin-orbit coupling strength. $\langle ij \rangle_1$ stands for the summation running over the nearest neighbors and $\langle ij \rangle_2$ for the next nearest neighbors. $\nu_{ij} = 1(-1)$ if the fermion makes a left (right) turn to get to the second bond \cite{33}. In order to study the emergent phenomenon, we consider the interaction which is governed by on-site Hubbard and nearest neighbor Hubbard interactions, i.e.,

$$H_{\text{int}} = V \sum_i n_i n_i + U \sum_{\langle ij \rangle_1} n_i n_j,$$

where $V$ and $U$ are small. $n_i = \sum_\sigma c_i^\dagger c_i$ which is the total particle number operator at site $i$. After a series of Hubbard-Stratonovich transformations, one finds that for the weak couplings, only a nearest neighbor Hubbard term

$$\tilde{H}_{\text{int}} = U \sum_{\langle ij \rangle_1, \sigma} c_i^\dagger c_i c_j^\dagger c_j,$$  

contributes to the effective action while other terms shift the fermion mass a little bit because they are irrelevant in the renormalization group sense, which then do not affect the effective theory \cite{For details see Supplementary Materials[33]}. The spin-$\sigma$ fermions interact with a nearest neighbor Hubbard coupling while the different spin fermions are decoupled. The total Hamiltonian $H = H_0 + \tilde{H}_{\text{int}}$ is time reversal symmetric and the third component of spin is conserved.

Long wavelength limit. The Kane-Mele Hamiltonian \cite{11} minimizes at Dirac points $K_\pm = (0, \pm \frac{\pi}{3})$ \cite{26}. Here we take the lattice spacing $a = 1$ and $\hbar = 1$. The mass gap at $K_\pm$ is $\Delta = |m_{\pm,\sigma} c^2| = 3\sqrt{3} |c$ with $m_{\pm,\sigma} c^2 = v \sigma \Delta$ (here $\sigma$ before $\Delta$ equals $\pm$) and $c = \frac{\sqrt{3}}{2} t$ is Fermi velocity; $v = \pm$ is the valley labeling the Dirac points $K_\pm$. At half-filling, the Fermi level lays in the gap. In the long wavelength limit, Eq. (1) can be approximated around $K_\pm$ to the linear order of the momentum $p$

$$H_0 \approx \sum_{p,\sigma,v} \psi_{p,\sigma,v}^\dagger H_0(p, \sigma, v) \psi_{p,\sigma,v},$$

where $\psi_{p,\sigma,v} = (b_{p\sigma v}, u_{p\sigma v})^T$ is the two component fermion operator near the Dirac point $K_v$ and $b, w$ label the two sub-lattice. The Hamiltonian $H_0(p, \sigma, v)$ is a $2 \times 2$ matrix given by

$$H_0(p, \sigma, v) = \begin{pmatrix} c p_y \tau^x - v c p_x \tau^y - m_{\sigma,v} c^2 \tau^z & 0 \\ 0 & -c p_y \tau^x + v c p_x \tau^y + m_{\sigma,v} c^2 \tau^z \end{pmatrix}.$$

The spin Chern number is then defined as the difference of spin dependent Chern numbers \cite{26}, i.e., $\nu = (\nu_- - \nu_+)/2$ and $\nu_\sigma = \frac{1}{2} \text{sgn}(m_{\pm,\sigma} c^2, \sigma) - \text{sgn}(m_{\pm,\sigma} c^2, -\sigma)$. We then find that $\nu_+ = -1$, $\nu_- = 1$ and $\nu = -1$. Now, turn on the interaction. For $U \ll t$, i.e., the weak coupling limit, the interaction is not strong enough to cause a topological phase transition. In the rest of this paper, we restrict in the topological non-trivial phase with $\nu_{\sigma} \neq 0$.

In the continuum limit, the interacting Hamiltonian \cite{33} can be approximated as (see Supplementary Materials \cite{33})

$$\tilde{H}_{\text{int}} \approx \frac{g^2}{2} \sum_\sigma \int d^2 r [j_{\mu\sigma}(r) j_{\mu\sigma}^\dagger(r) + \frac{\sqrt{3}}{2} \nabla n_{\mu\sigma}(r) \cdot \nabla n_{\mu\sigma}(r)],$$

where $g^2 = U$ and the 3-current $j_{\mu\sigma} \approx \sum_v j_{\mu\sigma v}$ with $j_{\mu\sigma v}$ defined nearby $K_v$,

$$j_{0\sigma,v} = j_{\sigma}^0 = \psi_{\sigma v}^\dagger(r) \psi_{\sigma v}(r),$$

$$j_{x\sigma,v} = -j_{\sigma}^x = \psi_{\sigma v}^\dagger(r) r^y \psi_{\sigma v}(r),$$

$$j_{y\sigma,v} = -j_{\sigma}^y = \psi_{\sigma v}^\dagger(r) r^x \psi_{\sigma v}(r).$$

In Eq. (6), we first rewrite Eq. (3) by the summation over plaquettes and then take the continuum limit by setting $r$ in the center of the plaquette. The second term in Eq. (6) after ”$\approx$” is negligible in the low wavelength limit because it is proportional to $p^2$. Thus, in the continuum limit, the Lagrangian is given by

$$\mathcal{L} = \sum_{\sigma} \left( \sum_v \langle \psi_{\sigma,v}(t, r) (c \partial_t + m_{\sigma} c^2)^2 \psi_{\sigma,v}(t, r) - \frac{g^2}{2} j_{\mu\sigma}(t, r) j_{\mu\sigma}^\dagger(t, r) \right),$$

where $m_{\sigma} = \sigma \Delta/c^2$, $\bar{\psi}_{\sigma,v} = \psi_{\sigma,v}^\dagger$, $\tau$ and $\partial_t = i \sigma^z \partial_t - c \tau^x \partial_x - c \tau^y \partial_y$.

Effective field theory. We now introduce the Hubbard-Stratonovich transformation to decouple the current -current interactions,

$$\exp \left\{ \frac{i g^2}{2} \int dt \int d^2 r j_{\mu\sigma} j_{\mu\sigma}^\dagger \right\} = \int \mathcal{D} a_{\mu\sigma} \exp \left\{ -i \int dt d^2 r \left( \frac{1}{2} a_{\mu\sigma} a_{\mu\sigma}^\dagger + g j_{\mu\sigma} a_{\mu\sigma}^\dagger \right) \right\}.$$

Then by using the functional bosonization technique \cite{31}, the Lagrangian reads

$$\mathcal{L}' = \sum_{\sigma} \left[ \frac{1}{2} a_{\mu\sigma} a_{\mu\sigma} + \frac{\alpha}{4\pi} \epsilon_{\mu\nu\rho} a_{\mu\sigma} \partial_{\nu} a_{\rho\sigma} \right],$$

up to the order of $O(\partial / \Delta)$ and here the ”fine structure constant” $\alpha = \frac{\sqrt{3}}{2}$. However, the Chern-Simons terms make the TRS not been explicitly shown. We then define $A_{\mu} = \frac{1}{2 \sqrt{3} \pi} (a_{\mu\uparrow} - a_{\mu\downarrow})$. Integrating over the $a_{\mu\uparrow}$, the effective Lagrangian turns out \cite{33}

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \frac{1}{2} m_A^2 A^\mu A_{\mu},$$

where $F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ and $m_A = \frac{2 \pi}{\alpha}$. Disappearance of the Chern-Simons term is a result of the TRS. This
means that a Proca theory emerges. The efforts to determine the limit of the photon mass are going along a long time [34]. Our results give a playground to see what happens if there is a massive photon. The mass of the gauge field also resembles Meissner effect in superconductivity.

The finite mass of the Proca field is consistent with no gapless excitation in the bulk of a topological insulator. To see this point explicitly, we calculate the current-current correlation functions by adding a source term \( \xi_\mu j^\mu \) to the Lagrangian [3]. One finds that

\[
\langle (j_\mu \uparrow - j_\mu \downarrow)(j_\nu \uparrow - j_\nu \downarrow) \rangle_{\xi \to 0} = \frac{2}{g^2} \left( \frac{q_\mu q_\nu}{q^2 - m_3^2} \right), \tag{12}
\]

where \( q_\mu = (\omega, q) \) and \( q^2 = q_\mu q^\mu = \omega^2 - q^2 \). The non-zero mass ensures this propagator is finite range and so are the current-current correlations. The consistency is checked. The propagator of \( A_\mu \) field is known as

\[
\langle A_\mu(q) A_\nu(-q) \rangle = -\frac{g_{\mu\nu} - q_\mu q_\nu/m_3^2}{q^2 - m_3^2}. \tag{13}
\]

**Instanton effect and confinement.** In this simple Kane-Mele model, we do not consider the monopole effect arising from the periodic condition of the 3-spin current in the lattice model. This is because the Proca mass gap is always finite if \( \nu_\sigma \neq 0 \). The Proca \( m_3 \to 0 \) (or \( \alpha \to \infty \)) means the strong coupling limit, \( U/t \to \infty \). In this limit, the system is already in the topologically trivial Mott phase and no gauge field emerges.

To reach \( m_3 = 0 \), we introduce a third nearest neighbor hopping term to the Kane-Mele model \( H_3 = -t_3 \sum_{(ij)3} \epsilon_{ij}^3 \sigma_i \). We find that when \( t_3/t = 1/3 \), a topological phase transition happens and \( \nu_\sigma = 2 \sigma \) after \( t_3/t > 1/3 \). The Fermi velocity, or "speed of light" \( c \) is proportional to \( t - 2t_3 \). When \( t_3 \to t/2, c \to 0 \). Since \( \nu_\sigma = -\nu_\chi \), the Chern-Simons terms also vanish as required by the TRS. Therefore, Eq. (14) is still correct but now the mass \( m_3 \to 0 \) as \( t_3 \to t/2 \). We then have an emergent Maxwell theory and the system becomes gapless as shown by Eqs. (14) and (13). This is contradictory with the lattice model which is a topological insulator. Thus, we must carefully check the process going to the continuum limit. Because the 3-spin current obeys the periodic boundary condition, the photon field \( A_\mu \) is periodic. In the long wavelength limit, we must take this matter into account, i.e., when we integrate over the fermion fields to arrive at Eq. (10), \( \epsilon_{\mu\nu\rho\sigma\mu\rho/2} = \epsilon_{\mu\nu\rho} \partial^\sigma a_\sigma^\rho \) must be replaced by \( \epsilon_{\mu\nu\rho} (F_\nu^\rho(r, t) - 2\pi n_\nu^\rho(r, t))/2 \), or

\[
H_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho} \to \frac{1}{2} \epsilon_{\mu\nu\rho} (F^{\nu\rho}(r, t) - 2\pi n_\nu^\rho(r, t)), \tag{14}
\]

where \( n_\nu^\rho \) are an anti-symmetric matrix with arbitrary integer elements and the path integrals include the sum over all \( n \) in order to keep the periodicity. This is Polyakov's instanton scenario for the charge confinement

\[ \text{[8]: The pure compact Maxwell theory is in the confinement phase due to the instanton condensation. This is consistent with no gapless bulk excitation in a topological insulator: On one hand, it is well-known that the correlation functions of } H_\mu \text{ field is of a finite correlation length} [8] \]

\[ \langle H_\mu(q) H_\nu(-q) \rangle = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2 - m_3^2}. \tag{15} \]

where the mass \( m \) is determined by the density of the instanton condensate. This gives the charge confinement through the area law of Wilson’s loop argument [8]. On the other hand, the current-current correlation functions when \( m_A \to 0 \) gives rise to

\[ \langle (j_\mu \uparrow - j_\mu \downarrow)(j_\nu \uparrow - j_\nu \downarrow) \rangle = \frac{2}{g^2} \langle (H_\mu(q) H_\nu(-q)) \rangle. \tag{16} \]

Thus, we verify the consistency. This result shows that one can determine the density of instantons by measuring the pick position of the dynamic current-current correlation functions.

**Bragg spectroscopy.** The spin density-density correlation functions in (12) and (13) are measurable. The imaginary Fourier component of the correlation functions, the dynamic structure functions, are given by [33]

\[ S(q_\mu) = \frac{2\pi}{g^2} q^2 \delta(\omega^2 - q^2 - m_3^2), \text{ when } m_3 \neq 0, \tag{17} \]

\[ S(q_\mu) = \frac{2\pi}{g^2} \omega^2 \delta(\omega^2 - q^2 - m_3^2), \text{ when } m_3 = 0. \tag{18} \]

They can be detected by Bragg scattering [32]. In the Bragg scattering process, two laser beams with 3-momentum difference \( q_\mu = k_1 - k_2 \) are sent to the optical lattice trapped cold atom gas. The illuminated cold atom system responds to this perturbation with a spin density fluctuation \( H_\lambda = (V_s/2) (\delta j_\lambda(q) e^{iq\cdot q} + h.c.) \). The Bragg scattered atoms then absorb a momentum. The rate of the momentum transfer can be measured by the time of flight images and relates to the dynamic structure function by \( dP_s/dt = 2\pi q(V_s/2)^2 |S(q_\mu) - S(-q_\mu)| \propto (1 - e^{-\beta q_\mu}) S(q_\mu) \) where \( P_s \) is the total momentum difference between the spin-up and down atoms [32]. Because of the weak coupling \( U < t \), the heating arising from the interaction is negligible and then \( dP_s/dt \) can be evaluated at zero temperature, i.e., \( \propto S(q_\mu) \).

**QED3 with Chern-Simons term.** Emergence of a Chern-Simons term in the gauge theory requires to break the TRS. Distinguishing the hoppings and the couplings by spins, i.e., \( t \to t_{\uparrow \downarrow} \) and \( g \to g_{\uparrow \downarrow} \) will do so. \( \lambda \) can be dependent on spins and the sublattices, i.e., \( \lambda \to \lambda_{\uparrow \downarrow, b_n} \). In addition, an extra fermion \( \chi \) is put in, which may or may not have a non-trivial Chern number and serves as the matter field in the emergent QED3. The interactions between these fermions are designed as follows (here the
tilde again means an effective Hamiltonian in the sense of Eq. (43):

$$\tilde{H}_{\text{int}} = \sum_{(ij)_1} \left\{ U_{\chi} n_{\chi i} n_{\chi j} + \sum_{\sigma} U_{\sigma} n_{\sigma i} n_{\sigma j} + \sum_{\sigma} \sigma U_{\sigma} (n_{\sigma i} n_{\sigma i} + n_{\sigma j} n_{\sigma j} + n_{\sigma i} n_{\sigma j} + n_{\sigma j} n_{\sigma i}) - 2\sigma U_{\sigma} c_\sigma (c_\sigma c_\sigma + c_{\bar{\sigma}} c_{\bar{\sigma}} + h.c.) \right\}$$

$$\approx \sum_{a=1,2} \frac{g_{a}^{2}}{2} \int d^{2}r \langle j_{a}(\mathbf{r}) j_{a}(\mathbf{r}) \rangle + O(q^{2}), \quad (19)$$

where $g_{a}^{2} = U_{\uparrow,\downarrow}; \quad j_{a}(\mathbf{r}) = j_{a}(\mathbf{r}) \pm j_{\mu \chi} \sqrt{U_{\chi}^{(1)}/U_{\chi}^{(1)}}$ with $U_{\chi}^{(1)} = \frac{\alpha_{\uparrow} + \alpha_{\downarrow}}{\alpha_{\uparrow} - \alpha_{\downarrow}} U_{\chi}; \quad \alpha_{\sigma} = g_{a}^{2}/c_{\sigma}$ and $c_{\sigma} = \sqrt{3}t_{\sigma}/2$. $U_{\chi,\sigma}$ is not independent and has been fixed by $U_{\chi,\sigma} = \sqrt{U_{\chi}^{(1)}}/U_{\chi}^{(1)}/3$. The first two terms come from the on-site and nearest neighbor Hubbard interaction as in the weak interacting Kane-Mele model. Most terms in $H_{\text{int}}$ are the on-site and nearest neighbor interaction while the terms in the last line of (19) are correlated hoppings of the spin fermions and the extra fermion. Notice that if we identify $g_{a}$ and $c_{\sigma}$ as the "charges" and "speeds of light", $\alpha_{\sigma}$ are the corresponding "fine structure constants".

In the continuum limit, the generalized model can be reduced to the three component interaction Dirac fermion model. If we integrate over the fermions $\psi_{\sigma}$ by means of the functional bosonization, we obtain the following effective Lagrangian

$$\mathcal{L}_{\text{eff}} = \mathcal{L}_{\chi} + \frac{4\pi^{2} U_{\chi}}{3\alpha_{\uparrow} \alpha_{\downarrow}} j_{\chi}^{\mu} A_{\mu} + \frac{4\pi^{2}}{2\alpha_{\uparrow} \alpha_{\downarrow}} A^{\mu} A_{\mu} + \frac{\pi(\alpha_{\downarrow} - \alpha_{\uparrow})}{\alpha_{\uparrow} \alpha_{\downarrow}} \epsilon_{\mu
u\rho} A_{\mu} \partial_{\nu} A_{\rho} - \frac{1}{4} F^{\mu
u} F_{\mu\nu}, \quad (20)$$

where $\mathcal{L}_{\chi}$ is the free Lagrangian of $\chi$ fermion which carries a "charge" $q_{\chi} = 2\pi \sqrt{U_{\chi}^{(1)}}/3\alpha_{\uparrow} \alpha_{\downarrow}$. Eq. (20) gives rise to a QED$_{3}$ for a Proca field coupling to a matter field $\chi$. We see that when $\alpha_{\uparrow} = \alpha_{\downarrow}$, the TRS emerges in the pure Proca theory even though the lattice model has no TRS, e.g., $t_{\uparrow} \neq t_{\downarrow}$ and $U_{\uparrow} \neq U_{\downarrow}$ but $t_{\uparrow} U_{\uparrow} = t_{\downarrow} U_{\downarrow}$.

**PHE and QAHE.** Now we integrate the $\chi$ fermion field in Eq. (20) to see how the $\chi$ fermion respond to the Proca field. The effective Lagrangian reads

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4} F^{\mu
u} F_{\mu\nu} + \frac{\tilde{\nu} \chi^{2}}{4\pi} \epsilon_{\mu\nu\rho} A_{\mu} \partial_{\nu} A_{\rho} + \frac{1}{2} m_{A}^{2} A^{\mu} A_{\mu}, \quad (21)$$

where $\tilde{\nu} = \nu_{\chi} + \frac{q_{\chi}^{2}}{\chi^{2}}$, $\tilde{\alpha} = \frac{\alpha_{\uparrow} \alpha_{\downarrow}}{\alpha_{\uparrow} - \alpha_{\downarrow}}$, $\nu_{\chi}$ the Chern number of $\chi$, $\alpha_{\chi} = \frac{q_{\chi}^{2}}{\chi^{2}}$ and $c_{\chi} = \sqrt{3}t_{\chi}/2$. Then the Proca equations read

$$\langle j_{\chi}^{\mu} \rangle = \frac{\delta S_{\text{eff}}}{\delta A_{\mu}} = \partial_{\nu} F^{\mu
u} + \frac{\tilde{\nu} \chi}{2\pi} \epsilon_{\mu\nu\rho} \partial_{\nu} A_{\rho} + m_{A}^{2} A^{\mu}, \quad (22)$$

If we assume that there is a static spatial distribution of the densities of the spinful fermions in the bulk, the Proca equations are simplified as

$$\langle j_{\chi} \rangle = \nabla \cdot \mathbf{E} + m_{A}^{2} A_{0}, \quad (23)$$

$$c_{\chi} \langle j_{\chi} \rangle = -\frac{\tilde{\nu} \chi^{2}}{h} F_{\chi}, \quad c_{\chi} \langle j_{\chi} \rangle = -\frac{\tilde{\nu} \chi^{2}}{h} E_{x}. \quad (24)$$

Eq. (23) reflects the divergence of the "electric" field equals to a reduced density, which is the density of the matter field minus $m_{A} A_{0}$ arising from a non-zero $m_{A}$. Eq. (24) shows that there is a Hall plateau with a fraction $\tilde{\nu}$. This consists of the $\nu_{\chi}$ QAHE as $\tilde{\alpha}^{-1} = 0$ and the pure PHE with a fraction $\nu_{0} = \frac{\tilde{\alpha}^{2} \chi^{2}}{\tilde{\alpha}^{2} \chi^{2}}$. The former is associated with the non-trivial topological property of the $\chi$ fermion and implies a gapless chiral edge state. The latter is a novel non-quantized Hall effect with a plateau of the Hall conductivity. Experimental proposal using the cold atom system to observe the edge states of topological Hall effect has been set up [33]. This can be examined experimentally.

The measurement of the Hall conductivity in a cold atom system on optical lattice has provided by driving the optical lattice [24]. In our system, we do not drive the optical lattice because the magnetic and electric field can be "created" by manipulating the spin density distribution.

**Conclusions.** We have studied the Kane-Mele type model with weak on-site and nearest neighbor Hubbard interactions. An emergent QED$_{3}$ theory with Chern-Simons and Proca terms was deduced in the long wavelength limit. We discussed the monopole effect in the zero Proca mass limit. The corresponding mass gaps can be observed by Bragg spectroscopy. We showed that PHE and QAHE can be observed.

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SUPPLEMENTARY MATERIALS

Derivation for the effective Proca theory

Let us start with the following Hamiltonian for our weak interacting Kane-Mele model on a honeycomb lattice at half filling,

$$H = H_{KM} + H_{\text{int}},$$

(1)

with $H_{KM}$ to be the ordinary Kane-Mele Hamiltonian without Rashba interaction.\[1\][2],

$$H_{KM} = H_h + H_{SO} = -t \sum_{\langle ij \rangle_1} \sum_{\sigma} c_{i\sigma}^\dagger c_{j\sigma} + i\lambda \sum_{\langle ij \rangle_2} \sum_{\sigma\sigma'} \nu_{ij} c_{i\sigma}^\dagger \tau_{\sigma\sigma'} c_{j\sigma'},$$

(2)

where the pair $\langle ij \rangle_1$ in the summation stands for the nearest neighbor pair while $\langle ij \rangle_2$ stands for the next nearest neighbor pair. $t$ is the hopping constant while $\lambda > 0$ is the spin-orbit coupling strength. $\nu_{ij} = 1$ for a right turn while $\nu_{ij} = -1$ for a left turn. $c_{\sigma}$ is the fermion creation operator, $\tau^z$ is the third Pauli matrix and $\sigma$ labels the spin index. This model Hamiltonian $H_{KM}$ can be used to describe quantum spin Hall effect which is a time reversal symmetry protected topological phase with a $\mathbb{Z}_2$ index. We consider a generic interaction term $H_{\text{int}}$, i.e., the on site and the nearest neighbor Hubbard interaction,

$$H_{\text{int}} = V \sum_i n_i n_i + U \sum_{\langle ij \rangle_1} n_i n_j,$$

(3)

$V, U > 0$ are the coupling constants and $n_i$ is the fermion occupation operator. We have 3 nearest neighbors on the honeycomb lattice (set the lattice constant to be 1),

$$\vec{b}_1 = \left(\frac{1}{2\sqrt{3}}, \frac{1}{2}\right), \vec{b}_2 = \left(\frac{1}{2\sqrt{3}}, -\frac{1}{2}\right), \vec{b}_3 = (-\frac{1}{\sqrt{3}}, 0),$$

(4)

and 6 next-to-nearest neighbors,

$$\vec{a}_1 = \left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \vec{a}_2 = (0, 1), \vec{a}_3 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right),$$
$$\vec{a}_4 = (-\frac{\sqrt{3}}{2}, \frac{1}{2}), \vec{a}_5 = (0, -1), \vec{a}_6 = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right).$$

(5)

Now we rewrite the Hamiltonian $H_{KM}$ in the momentum space,

$$H_h = -t \sum_{\langle ij \rangle} \sum_{\sigma} (b_{i\sigma}^\dagger b_{j\sigma} + h.c.) =$$
$$\sum_{\vec{k}\sigma} (-t(\sum_i \exp(i\vec{k} \cdot \vec{b}_i))b_{\vec{k}\sigma}^\dagger b_{\vec{k}\sigma} + h.c.),$$

(6)

where we label the positions of two sublattice by $b_i$ and $w_i$, and

$$H_{SO} = i\lambda \sum_{\langle ij \rangle_2} \sum_{\sigma\sigma'} \nu_{ij} \tau_{\sigma\sigma'}(b_{i\sigma}^\dagger b_{j\sigma'} + w_{i\sigma}^\dagger w_{j\sigma'}).$$

$$= 2\lambda \sum_{\vec{k}\sigma} \tau_{\sigma\sigma'}(\sin(\sqrt{3}k_x/2 + k_y/2) - \sin(k_y) - \sin(\sqrt{3}k_x/2 - k_y/2))(b_{\vec{k}\sigma}^\dagger b_{\vec{k}\sigma'} - w_{\vec{k}\sigma}^\dagger w_{\vec{k}\sigma'}).$$

(7)

Now let us consider the low energy behavior around the two independent Dirac points $\vec{K}$ and $-\vec{K}$, with

$$\vec{K} = \left(0, \frac{4\pi}{3}\right).$$

(8)

We have,

$$H_{KM} \approx \sum_{p,\sigma,\nu} \psi_{p,\sigma,\nu}^\dagger \mathcal{H}_0(p, \sigma, \nu) \psi_{p,\sigma,\nu},$$

(9)

where $v = \pm$ is the valley index labeling the Dirac points $K_{\pm}$, $\psi_{p,\sigma,\nu} = (b_{p\sigma\nu}, w_{p\sigma\nu})$ is two component fermion operator near the Dirac point $K_\nu$. The Hamiltonian $\mathcal{H}_0(p, \sigma, \nu)$ is a $2 \times 2$ matrix given by

$$\mathcal{H}_0(p, \sigma, \nu) = \begin{pmatrix} c_p^\dagger \tau^x & -vc_p \tau^y & m_{\nu,\sigma}c^2 \tau^z \end{pmatrix}.$$ (10)

The Chern number of each spin $\sigma$ is,

$$\nu_\sigma = \frac{1}{2}(\text{sgn}(m_{+,\sigma}c^2, \sigma) - \text{sgn}(m_{+,\sigma}c^2, \sigma)).$$ (11)

And in our model, $\nu_+ = -1$ and $\nu_- = 1$. Now let’s focus on the interaction term.

$$H_{\text{int}} = 2V \sum_i n_i n_i + U \sum_{\langle ij \rangle_1} n_i n_j,$$

(12)

Now we take a Hubbard-Stratonovich transformation for $2V \sum_i n_i n_i$ term as an example,

$$\exp(-2V \sum_i n_i n_i) = \int Dd_\uparrow Dd_\downarrow$$
$$
\exp(i \sum_i \frac{1}{2} d_\uparrow d_\downarrow + \sqrt{\nu} n_i d_\uparrow + \sqrt{\nu} n_i d_\downarrow).$$

(13)

Therefore we know that the on site Hubbard interaction term will shift the effective mass, i.e.,

$$m_{+,\uparrow}' = m_{+,\uparrow} + \sqrt{\nu} d_\uparrow,$$
$$m_{-,\uparrow}' = m_{-,\uparrow} - \sqrt{\nu} d_\uparrow,$$
$$m_{+,\downarrow}' = m_{+,\downarrow} + \sqrt{\nu} d_\downarrow,$$
$$m_{-,\downarrow}' = m_{-,\downarrow} - \sqrt{\nu} d_\downarrow.$$ (14)
which will not change the signature of $m_{\epsilon, \sigma}$ as long as the Hubbard interaction constant $V$ is small. Therefore the Chern number $\nu_\sigma$ is unchanged. Similarly after a series of Hubbard-Stratonovich transformations, we find that only

$$\hat{H}_{\text{int}} = U \sum_{(ij)\sigma} e_i^\dagger c_{i\sigma} e_j^\dagger c_{j\sigma},$$

will contribute to the effective action while other terms will shift the fermion mass a little which do not affect the effective theory. In order to take the correct continuum limit, we rewrite $\hat{H}_{\text{int}}$ by summation over the plaquettes (see Fig. 1), i.e.,

$$\hat{H}_{\text{int}} = \frac{U}{2} \sum_{(bw)\in \text{plaq.} \sigma} n_{b\sigma} n_{w\sigma}$$

$$= \frac{U}{2} \sum_{(bw)\in \text{plaq.} \sigma} \left( n_{w_1\sigma} n_{b_2\sigma} + n_{b_3\sigma} n_{w_3\sigma} + n_{w_3\sigma} n_{b_4\sigma} + n_{b_4\sigma} n_{w_2\sigma} + n_{w_2\sigma} n_{b_5\sigma} + n_{b_5\sigma} n_{w_1\sigma} \right).$$

Now we approximate $n_{i\sigma}$ by its value at the center of the plaquette (denote as $r$), for example,

$$n_{w_1\sigma} \approx n_w(r) = \frac{\sqrt{3}}{3} \partial_x n_w(r) + \frac{1}{2} \partial_y n_w(r),$$

$$n_{b_2\sigma} \approx n_b(r) = \frac{\sqrt{3}}{3} \partial_y n_b(r) - \frac{1}{2} \partial_x n_b(r).$$

Therefore after some arithmetics, Eq. (15) becomes,

$$\hat{H}_{\text{int}} = \frac{g^2}{2} \sum_{r, \sigma} (6 n_{b\sigma}(r) n_{w\sigma}(r) + \sqrt{3} \nabla n_{b\sigma}(r) \cdot \nabla n_{w\sigma}(r)),$$

where $g^2 = U$. In the continuum limit, the above term becomes $\frac{g^2}{2} \sum_{\sigma} \int d^3 x j_{b\sigma} j_{w\sigma}$ up to the first order of momentum. The disappearance of the first order terms is guaranteed by the lattice symmetry. Now we use another Hubbard-Stratonovich transformation to decouple this current-current interaction and integrate over the fermion fields $\psi_{\sigma, \mu}$, we have,

$$\mathcal{L}' = \sum_{\sigma} \left[ \frac{1}{2} a_{\sigma}^\dagger a_{\sigma} + \frac{\alpha}{4 \pi} \nu_\sigma \epsilon^{\mu \nu \rho} a_{\mu\sigma} \partial_\nu a_{\rho\sigma} \right],$$

up to the order of $O(\partial/\Delta)$. Here $\alpha = \frac{\pi^2}{g^2}$ is the fine structure constant and $a_{\mu\sigma}$ are the auxiliary fields introduced by the Hubbard-Stratonovich transformation. Define

$$A_\mu = \frac{g^2}{2 \sqrt{2} \pi c} (a_{\mu\uparrow} - a_{\mu\downarrow}),$$

then after some arithmetics, the Lagrangian $\mathcal{L}'$ becomes,

$$L = \frac{1}{2} a_{\mu\uparrow}^\dagger a_{\mu\uparrow} + \frac{1}{2} \frac{8 \pi^2 c^2}{g^4} A^2 - \frac{4 \sqrt{2} \pi c}{g^2} A_\mu a_{\mu\uparrow}^\dagger + d_{\mu\uparrow}^\dagger a_{\mu\uparrow}$$

$$\sqrt{2} \epsilon^{\mu \nu \rho} a_{\mu\uparrow} \partial_\nu A_\rho + \frac{2 \pi c}{g^2} \epsilon^{\mu \nu \rho} A_\mu \partial_\nu A_\rho.$$

Now let us integrate out the $a_{\mu\uparrow}$ field, we get the effective Lagrangian of Proca theory,

$$L = -\frac{F_{\mu \nu} F_{\mu \nu}}{4} + \frac{1}{2} m_A^2 A^\mu A_\mu,$$

with $F_{\mu \nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $m_A = \frac{2 \pi c}{g^2} = \frac{2 \pi}{\lambda}$. 

![FIG. 1: r is the center of the plaquette, the coordinates are set up in the ordinary way, i.e., the horizontal line is the x coordinate while the vertical line is the y coordinate. The distance between nearest neighbors is $\sqrt{3}/3$.](image)

### Calculation for the correlation function

In this subsection, we calculate the correlation function of spin currents $\langle (j_{\mu\uparrow} - j_{\mu\downarrow}) (j_{\nu\uparrow} - j_{\nu\downarrow}) \rangle$ of the weak interacting Kane-Mele model. Define the partition function,

$$Z(\eta_\mu) = \int \mathcal{D} \psi_\uparrow \mathcal{D} \psi_\downarrow \mathcal{D} \bar{\psi}_\downarrow \mathcal{D} \bar{\psi}_\uparrow \mathcal{D} a_{\mu\uparrow} \mathcal{D} a_{\mu\downarrow} \exp \left[ i \int d^3 x \left( \bar{\psi}_\uparrow (i \partial_\nu - M_\nu) \psi_\uparrow + \psi_\downarrow (i \partial_\nu - M_\nu) \bar{\psi}_\downarrow + \frac{1}{2} a_{\mu\uparrow}^\dagger a_{\mu\uparrow}^\dagger a_{\mu\downarrow}^\dagger a_{\mu\downarrow}^\dagger + \frac{1}{2} a_{\mu\uparrow} a_{\mu\uparrow}^\dagger a_{\mu\downarrow} a_{\mu\downarrow}^\dagger + \epsilon^{\mu \nu \rho} a_{\mu\uparrow} \partial_\nu \bar{a}_{\rho\downarrow} \bar{a}_{\mu\downarrow} \right) \right].$$

The partition function $Z(\eta_\mu)$ reduces to the original one when $\eta_\mu = 0$. Redefine $\bar{a}_{\mu\uparrow} = a_{\mu\uparrow} + \eta_\mu$ and $\bar{a}_{\mu\downarrow} = a_{\mu\downarrow} - \eta_\mu$, then integrate the fermion part, we have,

$$Z(\eta_\mu) = \int \mathcal{D} \bar{a}_{\mu\uparrow} \mathcal{D} \bar{a}_{\mu\downarrow} \exp \left[ i \int d^3 x \left( \frac{1}{2} (\bar{a}_{\mu\uparrow} - \eta_\mu)^2 + \frac{1}{2} (\bar{a}_{\mu\downarrow} + \eta_\mu)^2 \right) \right] \left( \frac{g^2}{4 \pi c} \epsilon^{\mu \nu \rho} \bar{a}_{\mu\uparrow} \partial_\nu \bar{a}_{\rho\downarrow} \right).$$
Choose $\nu_1 = -1, \nu_2 = 1$ as in the Letter, then,

\[ \langle (j_{\mu \tau} - j_{\mu \lambda})(j_{\nu \tau} - j_{\nu \lambda}) \rangle = \frac{1}{g^2} \frac{\delta}{\delta \eta^\mu} \frac{\delta}{\delta \eta^\nu} Z(\eta)|_{\eta = 0}/Z(0) = \frac{1}{g^2} (2g_{\mu \nu} + (\tilde{a}_{\mu \nu} - \tilde{a}_{\nu \mu}) Z(\eta)|_{\eta = 0}). \]

(26)

We can calculate $\langle a_{\mu \alpha} a_{\nu \tau} \rangle$ from $Z(0)$ directly,

\[ Z(0) = \int D\eta D\mu D\lambda \exp[i \int d^3x \left( \frac{1}{2} \eta^\mu \eta^\nu - \frac{1}{2} \eta^\mu \eta^\nu + \frac{g^2}{4\pi^2} e^{\mu \nu \rho} \eta^\mu \partial_\rho A_{\nu \lambda} \right) \left( \eta_{\mu \lambda} \right)]. \]

(27)

Therefore,

\[ \langle a_{\mu \alpha} a_{\nu \tau} \rangle = -g_{\mu \tau} + i g_{\mu \sigma} q_{\sigma \tau}, \]

\[ = \frac{m_A^2 g_{\mu \nu} - q_{\mu} q_{\nu} + \imath \epsilon_{\mu \sigma \rho} q_{\sigma} \epsilon_{\nu \rho},}{q^2 - m_A^2}, \]

(28)

where $m_A = 2\pi c/g^2$. Similarly, we have,

\[ \langle a_{\mu \tau} a_{\nu \lambda} \rangle = \langle a_{\mu \lambda} a_{\nu \tau} \rangle = 0, \]

(29)

\[ \langle a_{\mu \lambda} a_{\nu \nu} \rangle = \frac{m_A^2 g_{\mu \nu} - q_{\mu} q_{\nu} + \imath \epsilon_{\mu \sigma \rho} q_{\sigma} \epsilon_{\nu \rho}}{q^2 - m_A^2}, \]

(30)

Then, we get

\[ \langle (j_{\mu \tau} - j_{\mu \lambda})(j_{\nu \tau} - j_{\nu \lambda}) \rangle = \frac{1}{g^2} (2g_{\mu \nu} + \frac{q^2}{2m_A} \langle A_{\mu \lambda} \rangle). \]

(31)

We can also calculate the spin current correlator $\langle (j_{\mu \tau} - j_{\mu \lambda})(j_{\nu \tau} - j_{\nu \lambda}) \rangle$ in an alternative way, i.e., through the emergent effective Proca theory. Define,

\[ \bar{A}_{\mu} = \frac{1}{\sqrt{2m_A}} (a_{\mu \tau} - a_{\mu \lambda}), \]

(32)

and integrate out the $a_{\mu \tau}$ field in Eq. (28), we have,

\[ Z(0) = \int D\bar{A}_{\mu} \exp \left[ i \int d^3x \left( \frac{1}{4} \epsilon^{\mu \nu \rho} \bar{A}_{\mu} \partial_\rho F_{\nu \lambda} + \frac{m_A^2}{2} \bar{A}_{\mu} \bar{A}_{\mu} \right) \right]. \]

(33)

Then,

\[ \langle (j_{\mu \tau} - j_{\mu \lambda})(j_{\nu \tau} - j_{\nu \lambda}) \rangle = \frac{2}{g^2} (g_{\mu \nu} + m_A^2 \langle \bar{A}_{\mu} \bar{A}_{\nu} \rangle) \]

\[ = \frac{2}{g^2} \frac{q^2 g_{\mu \nu} - q_{\mu} q_{\nu}}{q^2 - m_A^2}. \]

(34)

And there is a third way to calculate $\langle (j_{\mu \tau} - j_{\mu \lambda})(j_{\nu \tau} - j_{\nu \lambda}) \rangle$. From the correspondence of Thirring model and Chern-Simons-Maxwell theory [2], we know that

\[ Z(0) = \int D\eta D\mu D\lambda D\bar{A}_{\mu} D\bar{A}_{\lambda} \exp \left[ i \int d^3x \left( \frac{1}{2} \eta^\mu \eta^\nu + \epsilon^{\mu \nu \rho} \eta^\mu \partial_\rho A_{\nu \lambda} + \frac{\pi c}{g^2} \epsilon^{\mu \nu \rho} A_{\mu \lambda} \partial_\rho A_{\nu \lambda} + \frac{1}{2} (\tilde{a}_{\mu \nu} - \tilde{a}_{\nu \mu}) \right) \right]. \]

(35)

If we integrate out $A_{\mu \tau}$ and $A_{\mu \lambda}$ fields, then Eq. (35) will reduce to Eq. (28). Now we integrate out $a_{\mu \tau}$ and $a_{\mu \lambda}$ fields and define $F_{\mu \nu} \equiv \partial_\mu A_{\nu \tau} - \partial_\nu A_{\mu \tau}$, $F_{\mu \nu} \equiv \partial_\mu A_{\nu \lambda} - \partial_\nu A_{\mu \lambda}$, we have,

\[ Z(0) = \int DA_{\mu} DA_{\lambda} \exp \left[ i \int d^3x \left( \frac{1}{4} F^{\mu \nu} F_{\mu \nu} \right) \right]. \]

(36)

Then $j_{\mu \tau} \sim \frac{1}{g} H_{\mu \tau} \equiv \frac{1}{g} \epsilon_{\mu \nu \rho} \partial_\nu A_{\rho \tau}$ and $j_{\mu \lambda} \sim \frac{1}{g} H_{\mu \lambda} \equiv \frac{1}{g} \epsilon_{\mu \nu \rho} \partial_\nu A_{\rho \lambda}$. The propagator for $A_{\mu \sigma}$ field is (with a gauge fixing term $L = -\frac{1}{2} \bar{A}_{\mu} \partial_\mu A_{\mu}$),

\[ \Delta_{\mu \nu} = \langle (g_{\mu \nu} - \imath \epsilon_{\mu \sigma \rho} q_{\sigma}) A_{\sigma \nu} + \imath \epsilon_{\mu \sigma \rho} q_{\sigma} \rangle, \]

(37)

where $m_A = 2\pi c/g^2$. Since when we calculate the correlation function $\langle A_{\mu \sigma}(x) A_{\sigma \nu}(y) \rangle$, we need to add some term, such as $A_{\mu \sigma} J^\rho$, and $J^\rho$ should be conserved, i.e., $\partial_\nu J^\rho = 0$, then the $q_{\mu} q_{\nu}$ term in the propagator does not contribute. Then,

\[ \langle A_{\mu \tau}(g) A_{\nu \tau}(-g) \rangle = \frac{2}{g^2} (g_{\mu \nu} - \imath \epsilon_{\mu \sigma \rho} q_{\sigma}) (A_{\sigma \tau} \hat{A}_{\nu \tau}) \]

(38)

\[ = \frac{1}{g^2} \langle (H_{\mu \tau}(q) - H_{\nu \tau}(q))(H_{\nu \tau}(-q) - H_{\mu \tau}(-q)) \rangle \]

(39)

\[ = \frac{1}{g^2} \epsilon_{\mu \nu \sigma} \epsilon_{\nu \lambda \rho} q^\sigma q^\rho \langle (A_{\sigma \tau} - A_{\lambda \tau}) (A_{\nu \tau} - A_{\lambda \tau}) \rangle \]

(40)

\[ = \frac{2}{g^2} \frac{q^2 g_{\mu \nu} - q_{\mu} q_{\nu}}{q^2 - m_A^2}. \]

(41)

From the above discussion, one can see that these three ways are consistent with each other, and the emergent Proca theory also gives us the right results.

While in $c \to 0$ limit ($m_A \to 0$), the result of $\langle H_{\mu \tau} H_{\nu \lambda} \rangle$ should be replaced by $q_{\mu} q_{\nu} - \frac{q^2 q_{\mu} q_{\nu}}{q^2 - m_A^2 + 4\epsilon}$ due to instanton condensation [4]. Therefore we have,

\[ \langle (j_{\mu \tau} - j_{\mu \lambda})(j_{\nu \tau} - j_{\nu \lambda}) \rangle = \frac{2}{g^2} (g_{\mu \nu} - \frac{q_{\mu} q_{\nu}}{q^2 - m_A^2 + 4\epsilon}), \]

(42)
scattering, which demonstrate the imaginary part of $\langle (j_{0\uparrow} - j_{0\downarrow})(j_{0\uparrow} - j_{0\downarrow}) \rangle$. When $m_A \neq 0$,  
\[ S(q_\mu) = |\text{Im}(j_{0\uparrow} - j_{0\downarrow})(j_{0\uparrow} - j_{0\downarrow})| = \frac{2\pi}{g^2}q^2\delta(\omega^2 - q^2 - m_A^2), \]  
and when $m_A = 0$,  
\[ S(q_\mu) = |\text{Im}(j_{0\uparrow} - j_{0\downarrow})(j_{0\uparrow} - j_{0\downarrow})| = \frac{2\pi}{g^2}\omega^2\delta(\omega^2 - q^2 - \bar{m}^2). \]  
We can see that the peaks of $S(q_\mu)$ differ in these two cases. Experimentally we expect that one can observe a peak at small momentum when the incident frequency is large enough to create an instanton. The peak corresponds to the density of instantons when $c \to 0$.

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