ΛCDM jerk parameter in symmetric teleparallel cosmology

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In this paper, we have examined the recently proposed modified symmetric teleparallel gravity, in which gravitational Lagrangian is given by an arbitrary function of non-metricity scalar Q. We have considered the ΛCDM jerk parameter to express the Hubble rate. Moreover, we have used 57 points of Hubble H(z) and 1048 points of Pantheon datasets to constraint our model parameters by means of the Markov Chain Monte Carlo analysis. The mean values and the best fit obtained give consistent Hubble rate and deceleration parameter compared to the observation values. In order to study the current accelerated expansion scenario of the Universe with the presence of the cosmological fluid as a perfect fluid, we have considered two forms of teleparallel gravity. We have studied the obtained field equations with the proposed forms of f(Q) models, specifically, linear f(Q) = αQ + β and non-linear f(Q) = Q + mQ^n models. Next, we have discussed the physical behavior of cosmological parameters such as energy density, pressure, EoS parameter, and deceleration parameter for both model. To ensure the validity of our proposed cosmological models, we have checked all energy conditions. The properties of these parameters confirm that our models describe the current acceleration of the expansion of the Universe. This result is also corroborated by the energy conditions criteria. Finally, the EoS parameter for both models indicates that the cosmological fluid behaves like a quintessence dark energy model.

I. INTRODUCTION

A set of recent observations of Type Ia Supernova (SN Ia) [1, 2], Cosmic Microwave Background (CMB) [3, 4], large scale structure [5, 6], Baryonic Acoustic Oscillations (BAO) [7, 8], and Wilkinson Microwave Anisotropy Probe (WMAP) experiment [9, 10] show unexpected behavior of cosmic expansion. The scientific community believed that cosmic expansion constantly decelerated due to standard Friedmann’s equations in General Relativity (GR), the surprising thing that has been observed is that the Universe is accelerating. The origin of this late-time cosmic acceleration is one of the most important open issues in the scientific community today. The first proposed solution to explain the mystery of the cosmic acceleration within the framework of GR, is the introduction of a new component of energy, of an unknown nature and constitutes a major part of the total content of the Universe called dark energy (DE). DE can lead to negative pressure (p < 0) or equivalently its equation of state (EoS) parameter is negative (ω ≡ p/ρ < 0), where ρ is the energy density of the Universe. According to recent WMAP9 [11] observations, collecting data from H0 measurements, SN Ia, CMB, and BAO, show that the current value of the EoS parameter is ω0 = −1.084 ± 0.063. Also, in the year 2015 Planck collaboration showed that ω0 = −1.006 ± 0.0451 [12] and furthermore, in 2018 it informed that ω0 = −1.028 ± 0.032 [13]. The simplest explanation for DE is the cosmological constant (Λ) with ωΛ = −1 that Einstein introduced into the field equations in another context [14]. Faced with the problems linked to its predicted order of magnitude from quantum gravity compared to the observed value, other alternatives to DE have been proposed such as quintessence DE models with EoS parameter in the range −1 < ω < −1/3 [15] and phantom energy models with EoS parameter ω < −1 [16]. Always within the framework of GR, and motivated by these models other more attractive and interesting dynamical DE models have been proposed such as k-essence [17], Chameleon [18], tachyon [19], Chaplygin gas [20, 21] and little sibling of the big rip [22–26].

Recently, another type of alternative to explain the current speed of the acceleration of the Universe has been emerged whose basic idea is that GR may be valid at certain cosmological limits. This type of alternative
is called modified theories of gravity (MTG), such as \( f(R) \) gravity (where \( R \) is the Ricci scalar), \( f(T) \) gravity (where \( T \) is the torsion), \( f(G) \) gravity (where \( G \) is the Gauss-Bonnet), see these works in this regard [27–29]. According to the standard model of cosmology or the so-called \( \Lambda \)CDM (\( \Lambda \) + Cold Dark Matter), which is the most accepted model today, in the beginning of the Universe photons were the most prevalent and this stage is called the radiation-dominated era, and with the cosmic expansion another stage appeared called the matter-dominated era while DE was slow in the beginning. But later, DE dominated matter as the Universe commences to expand and grow bigger over the time and commence to accelerate from five to six billion years ago [30]. The transition from the early decelerating phase to the current accelerating phase of the Universe is likely due to a cosmic jerk. This transition occurs in the Universe for various models with a positive value of the cosmographic jerk parameter and a negative value of the deceleration parameter [31–35]. The cosmographic jerk parameter is one of the important tools for describing the dynamics of the Universe and is defined as a dimensionless third order derivative of the scale factor of the Universe \( a \) with respect to the cosmic time \( t \). The cosmographic jerk parameter for flat \( \Lambda \)CDM model is given as

\[
\dot{J} = \frac{\ddot{a}}{aH^3} = 1. \tag{1}
\]

where the Hubble parameter, \( H = \frac{\dot{a}}{a} \), represents the expansion rate of the Universe. Several authors have suggested applications of the jerk parameter as a means of reconstructing cosmological models in various cosmological contexts [36–38].

In the space-time manifold, we can outline the gravitational interactions using three types of concepts namely curvature \( R \), torsion \( T \), and non-metricity \( Q \). In the famous theory of GR, the concept responsible for gravitational interactions is the curvature of space-time. Some other two possibilities torsion and non-metricity offer the equivalent description of GR and the corresponding gravity are named teleparallel and symmetric teleparallel equivalent of GR. It can be said that the \( f(R) \) gravity mentioned above is a modification of curvature-founded gravity (GR) with zero torsion and non-metricity. It can also be said that the \( f(T) \) gravity is a modification of torsion-founded gravity i.e. the Teleparallel Equivalent of GR (TEGR), with the non-metricity and curvature disappearing. Finally, we can say that the \( f(Q) \) gravity is a modification of the Symmetric Teleparallel Equivalent of GR (STEGR) with zero torsion and curvature, in other words, \( f(Q) \) gravity is equivalent to GR in flat space [39] (for more details, see Sec. II). Many works on \( f(Q) \) gravity have been published in the literature. Mandal et al. have tested the energy conditions and cosmography in \( f(Q) \) gravity [40, 41], Harko et al. studied the coupling matter in modified \( Q \) gravity by presuming a power-law function [42], Dimakis et al. examined quantum cosmology for a \( f(Q) \) polynomial model [43], see also [44–46].

Our aim in this paper is to exploit the above cosmographic jerk parameter relation for the flat \( \Lambda \)CDM model to construct cosmological models as alternatives to the cosmic acceleration in the framework of \( f(Q) \) gravity (or the so-called symmetric teleparallel gravity) that has been recently proposed [47, 48] in which the non-metricity scalar \( Q \) characterize the gravitational interactions. The acceleration of the expansion of the Universe may also be described by a reconstruction in the context of standard cosmology or in modified gravity. In the current paper, we consider this kind of reconstruction in the case of \( f(Q) \) modified gravity.

The present paper is organized as follows: In Sec. II we discuss some basics of \( f(Q) \) gravity and derive the field equations for the cosmological fluid. Cosmological solutions of the field equations are described with the help of the flat \( \Lambda \)CDM jerk parameter in Sec. III. In the same section, we use Hubble and Pantheon datasets to estimate the model parameters by means of the Markov Chain Monte Carlo and we discussed the energy conditions for \( f(Q) \) gravity in the latter. Further, in Sec. IV we discuss different behaviors of our cosmological models according to the choice of linear and non linear form of \( f(Q) \) function. Finally, Sec. V is used to recapitulate and conclude the results.

II. SOME BASICS OF \( f(Q) \) GRAVITY

In this background, the symmetric metric tensor \( g_{\mu\nu} \) is exploited in the definition of the length of a vector and an asymmetric connection (Weyl-Cartan connection) \( \Sigma^\gamma_{\mu\nu} \) is exploited to defines the covariant derivatives and parallel transport. As mentioned above, the general affine connection in differential geometry can be decomposed into the following three independent components: the Christoffel symbol \( \Gamma^\gamma_{\mu\nu} \), the contortion tensor \( C^\gamma_{\mu\nu} \) and the disformation tensor \( L^\gamma_{\mu\nu} \) and is given by [39]

\[
\Sigma^\gamma_{\mu\nu} = \Gamma^\gamma_{\mu\nu} + C^\gamma_{\mu\nu} + L^\gamma_{\mu\nu}, \tag{2}
\]

where \( \Gamma^\gamma_{\mu\nu} \equiv \frac{1}{2} g^{\gamma\alpha} \left( \partial_\mu g_{\alpha\nu} + \partial_\nu g_{\alpha\mu} - \partial_\alpha g_{\mu\nu} \right) \) is the Levi-Civita connection of the metric \( g_{\mu\nu} \), the contorsion tensor \( C^\gamma_{\mu\nu} \) can be written as \( C^\gamma_{\mu\nu} \equiv \frac{1}{2} T^\gamma_{\mu\nu} + T_{(\mu}(^\gamma_{\nu)} \),
with the torsion tensor defined as $T'_{\mu\nu} = 2\Sigma'^{\gamma}_{\mu\nu}$. Finally, the disformation tensor $L'_{\mu\nu}$ is derived from the non-metricity tensor $Q_{\gamma\mu\nu}$ as

$$L'_{\mu\nu} = \frac{1}{2} \xi'^{\gamma} \left( Q_{\gamma\mu\nu} + Q_{\mu\nu\gamma} - Q_{\gamma\mu\nu} \right).$$  

(3)

In Eq. (3), the non-metricity tensor $Q_{\gamma\mu\nu}$ is a covariant derivative of the metric tensor with regard to the asymmetric connection $\Sigma'^{\gamma}_{\mu\nu}$, i.e. $Q_{\gamma\mu\nu} = \nabla_{\gamma} g_{\mu\nu}$, and it can be obtained

$$Q_{\gamma\mu\nu} = -\partial_{\gamma} g_{\mu\nu} + g_{\nu\sigma} \Sigma'^{\sigma}_{\mu\gamma} + g_{\mu\rho} \Sigma'^{\rho}_{\nu\gamma}. $$  

(4)

The connection is presumed to be torsionless and curvatureless within the current background. It corresponds to the pure coordinate transformation from the trivial connection mentioned in [47]. Thus, for a flat and torsion-free connection, the Weyl-Cartan connection (2) can be parameterized as

$$\Sigma'^{\gamma}_{\mu\nu} = \frac{\partial x^{\gamma}}{\partial \xi^{\rho}} \partial_{\mu} \partial_{\nu} x^{\rho}. $$  

(5)

Here, $\xi^\gamma = \xi^\gamma (x^\mu)$ is an invertible relation. Thus, it is always possible to get a coordinate system so that the connection $\Sigma'^{\gamma}_{\mu\nu}$ vanish. This condition is called coincident gauge and has been used in many studies of STEGR [49] and in this condition the covariant derivative $\nabla_{\gamma}$ reduces to the partial derivative $\partial_{\gamma}$. Thus, in the coincident gauge coordinate, we find

$$Q_{\gamma\mu\nu} = -\partial_{\gamma} g_{\mu\nu}. $$  

(6)

The symmetric teleparallel gravity is a geometric description of gravity equivalent to GR (STEGR) within coincident gauge coordinates in which $\Sigma'^{\gamma}_{\mu\nu} = 0$ and $C'^{\gamma}_{\mu\nu} = 0$, and consequently from Eq. (2) we can conclude that [39]

$$\Gamma'^{\gamma}_{\mu\nu} = -L'_{\mu\nu}. $$  

(7)

The modified Einstein-Hilbert action in symmetric teleparallel gravity can be considered as [47, 48]

$$S = \int \sqrt{-g} d^{4}x \left[ \frac{1}{2} f(Q) + \mathcal{L}_{m} \right], $$  

(8)

where $f(Q)$ can be expressed as the arbitrary function of non-metricity scalar $Q$, $g$ is the determinant of the metric tensor $g_{\mu\nu}$ and $\mathcal{L}_{m}$ is the matter Lagrangian density. The non-metricity tensor $Q_{\gamma\mu\nu}$ and its traces can be written as

$$Q_{\gamma\mu\nu} = \nabla_{\gamma} g_{\mu\nu}, $$  

(9)

$$Q_{\gamma} = Q_{\gamma}^{\mu}_{\mu}, \quad \tilde{Q}_{\gamma} = Q^{\mu}_{\gamma\mu}. $$  

(10)

It is also useful to introduce the superpotential tensor (non-metricity conjugate) as

$$4P'_{\mu\nu} = -Q'_{\mu\nu} + 2Q_{(\mu}^{\gamma}_{\nu)} + Q'_{\gamma\mu\nu} - \tilde{Q}'_{\gamma\mu\nu} - \delta^{\gamma}_{(\mu}Q_{\nu)} , $$  

(11)

where the trace of the non-metricity tensor can be obtained as

$$Q = -Q_{\gamma\mu\nu} P'_{\gamma\mu\nu}. $$  

(12)

The matter energy-momentum tensor is defined as

$$T_{\mu\nu} = -\frac{\delta}{\sqrt{-g}} \frac{\delta \mathcal{L}_{m}}{\delta g^{\mu\nu}}. $$  

(13)

Now, by varying the modified Einstein-Hilbert action (8) with respect to the metric tensor $g_{\mu\nu}$, the gravitational field equations are obtained as

$$\nabla_{\nu} \left( \sqrt{-g} f_{Q} P'_{\gamma\mu\nu} \right) + \frac{1}{2} f g_{\mu\nu} + f_{Q} (P_{\nu\sigma\rho} Q_{\mu}^{\rho\sigma} - 2P_{\nu\rho\sigma} Q_{\mu}^{\rho\sigma}) = -T_{\mu\nu}. $$  

(14)

Here, we consider the standard Friedmann-Lemaitre-Robertson-Walker (FLRW) metric

$$ds^{2} = -dt^{2} + a^{2}(t) \left[ dx^{2} + dy^{2} + dz^{2} \right], $$  

(16)

where $a(t)$ is the scale factor which measures how the distance between two objects varies with time in the expanding Universe. Now, we can find the expression for
a non-metricity scalar $Q$ by considering the trace of the non-metricity tensor for this line element (16) as follow

$$Q = 6H^2.$$  \hfill (17)

In cosmology, the contents of the Universe are often considered to be filled with a perfect fluid, i.e. a fluid without viscosity. In this case, the stress-energy momentum tensor of the cosmological fluid is given by

$$T_{\mu\nu} = (p + \rho)u_{\mu}u_{\nu} + p\delta_{\mu\nu},$$  \hfill (18)

where $p$ represents the isotropic pressure with cosmological fluid and $\rho$ represents the energy density of the Universe. Here, $u^\mu = (1, 0, 0, 0)$ are components of the four velocities of the cosmological fluid.

In order to derive the modified Friedmann equations in $f(Q)$ gravity in the case of a Universe described by the FLRW metric (16), we use Eqs. (14) and (18) which describes a homogeneous and isotropic Universe. We obtain

$$3H^2 = \frac{1}{2f_Q} \left(-\rho + \frac{f}{2}\right),$$  \hfill (19)

$$H + 3H^2 + \frac{f_Q}{f_Q} H = \frac{1}{2f_Q} \left(p + \frac{f}{2}\right),$$  \hfill (20)

where dot (.) represents derivative with respect to cosmic time $t$. The energy conservation equation of the stress-energy momentum tensor writes

$$\dot{\rho} + 3H(\rho + p) = 0.$$  \hfill (21)

Using equations (19) and (20), the energy density of the Universe and its isotropic pressure can be written as

$$\rho = \frac{f}{2} - 6H^2f_Q,$$  \hfill (22)

$$p = \left(H + 3H^2 + \frac{f_Q}{f_Q} H\right) 2f_Q - \frac{f}{2}.$$  \hfill (23)

Now, we can find the cosmological evolution equations in a form similar to the standard Friedman equations of GR, by introducing an effective energy density $\rho_{\text{eff}}$ and an effective isotropic pressure $p_{\text{eff}}$ as

$$\rho_{\text{eff}} = \frac{1}{f_Q} \left(\rho - \frac{f}{2}\right),$$  \hfill (24)

$$p_{\text{eff}} = -2\frac{f_Q}{f_Q} H + \frac{1}{f_Q} \left(p + \frac{f}{2}\right).$$  \hfill (25)

Furthermore, in analogy with the standard GR Friedmann, we can rewrite Eqs. (19) and (20) as

$$3H^2 = \frac{1}{2} \rho_{\text{eff}},$$  \hfill (26)

$$H + 3H^2 = \frac{p_{\text{eff}}}{2}.$$  \hfill (27)

The equations above can be interpreted as an additional component of a modified energy-momentum tensor $T^\text{eff}_{\mu\nu}$ due to the non-metricity terms that behaves as an effective dark energy fluid.

III. COSMOLOGICAL SOLUTIONS AND ENERGY CONDITIONS

The field equations (19)-(20) are a system of two independent equations with four unknowns, namely: $\rho$, $p$, $f$, and $H$. It is therefore difficult to solve it completely without adding other equations or constraints to the model. Here we will build our model using the $\Lambda$CDM jerk parameter. Also in the literature, the jerk parameter provides an alternative method to describe the models close to $\Lambda$CDM. Thus, the general solution of Eq. (1) can be written as [36, 50]

$$a(t) = \left(Ce^{\lambda t} + De^{-\lambda t}\right)^{\frac{1}{3}},$$  \hfill (28)

which is an increasing function of cosmic time describing the accelerated behavior of the Universe. In Eq. (28), $C$, $D$, and $\lambda$ are model parameters that will be constrained by observational data. To obtain cosmological results that give a direct comparison of model predictions with observational data, also, one of the most useful approaches in cosmology is to determine cosmological parameters in terms of cosmological redshift $z$ in lieu of the cosmic time $t$. For this, we use the relation between the scale factor of the Universe and the cosmological redshift as $a(t) = (1 + z)^{-1}$, where the value of the scale factor at present is $a(0) = 1$. However and in order to illustrate our reconstruction $f(Q)$ gravity from the jerk parameter, we assume $C = -D$.

From Eq. (28) we obtain the cosmic time in terms of the cosmological redshift as

$$t(z) = \frac{1}{\lambda} \sinh^{-1}\left(\frac{(1+z)^{-3/2}}{2C}\right).$$  \hfill (29)

Now, using Eq. (28) the Hubble parameter takes the form

$$H(t) = \frac{2\lambda \left(e^{2\lambda t} + 1\right)}{3\left(e^{2\lambda t} - 1\right)}.$$  \hfill (30)
Thus, from Eqs. (29) and (30) we obtain the Hubble parameter in terms of the cosmological redshift as
\[
H(z) = \frac{2\lambda}{3} \left( e^{2\sinh^{-1}\left( \frac{1}{2(1+z)^{1/2}} \right)} + 1 \right).
\] (31)

The deceleration parameter \( q \) that describes the evolution of the Universe can be gained as a function of the cosmological redshift \( z \) as [39]
\[
q(z) = -1 + (1 + z) \frac{dH(z)}{dz}.\] (32)

Using Eqs. (31) and (32), we get
\[
q(z) = -4e^{2\sinh^{-1}\left( \frac{1}{2(1+z)^{1/2}} \right)} + e^{4\sinh^{-1}\left( \frac{1}{2(1+z)^{1/2}} \right)} + 1.
\] (33)

Now, from expressions of the Hubble parameter and the deceleration parameter in terms of the cosmological redshift given by Eqs. (31) and (33), we will concentrate in the next section on the constraint of the model parameters using observational data.

### A. Observational constraints

To constrain the \( \Lambda \)CDM jerk model, we perform the minimization of chi-square function, \( \chi^2 \), using the Markov Chain Monte Carlo (MCMC) algorithm [51] in emcee python library [52], where \( \chi^2 = -2\ln(L_{\text{max}}) \) with \( L \) is the likelihood function. The observational datasets that we use in our analysis are the \( H(z) \) datasets (denoted as OHD) with 57 data points of Hubble parameter in the redshift range \( 0.07 \leq z \leq 2.41 \) [53]. The chi-square of OHD datasets is given by
\[
\chi^2_{\text{OHD}}(C, \lambda) = \sum_{i=1}^{57} \frac{[H_{\text{obs}}(z_i) - H_{\text{th}}(C, \lambda, z_i)]^2}{\sigma^2(z_i)},\] (34)
where \( H_{\text{th}}(C, \lambda, z_i), H_{\text{obs}}(z_i) \) and \( \sigma(z_i) \) are the theoretical values, the observed values predicted by the model of Hubble parameter at redshift \( z_i \) and the standard deviation, respectively. \( C \) and \( \lambda \) are the free parameters of our theoretical model. The 57 data points consist of 31 points of \( H(z) \) measured from the differential age (DA) method [54] and the residual 26 points of \( H(z) \) measured through BAO measurements and other methods [55].

Fig. 1 displays the error bar plot of our \( \Lambda \)CDM jerk model and the well-known \( \Lambda \)CDM model in cosmology, with cosmological constant density parameter \( \Omega_{\Lambda,0} = 0.7 \), matter density parameter \( \Omega_{m,0} = 0.3 \), and \( H_0 = 69 \text{km/s/Mpc} \).

Further, we use the Pantheon datasets with 1048 Supernovae Type Ia (denoted as SNe) in the redshift range \( 0.01 \leq z \leq 2.3 \) [56] with the chi-square,
\[
\chi^2_{\text{SNe}}(C, \lambda) = \sum_{i=1}^{1048} \frac{[H_{\text{obs}}(z_i) - H_{\text{th}}(C, \lambda, z_i)]^2}{\sigma^2(z_i)},\] (35)
where \( H_{\text{obs}}(z_i) \) is the observational distance modulus and \( H_{\text{th}}(C, \lambda, z_i) \) is the theoretical value defined as
\[
H_{\text{th}}(C, \lambda, z_i) = 5\log_{10}D_L(z) + \mu_0,\] (36)
with the luminosity distance \( D_L(z) \) defined by
\[
D_L(z) = (1 + z) \int_0^z \frac{cdz'}{H(z')},\] (37)
and
\[
\mu_0 = 5\log(1/H_0\text{Mpc}) + 25,\] (38)
where \( c \) is the speed of light and \( H_0 \) is the current Hubble rate.

Fig. 2 displays the error bar plot of the \( \Lambda \)CDM jerk model using the 1048 data points of the Pantheon datasets. The figure also shows a comparison between our \( \Lambda \)CDM jerk model and the well-known \( \Lambda \)CDM model in cosmology.

Combining both datasets, the \( \chi^2_{\text{tot}} \) will be the sum of \( \chi^2 \) of the \( H(z) \) datasets, \( \chi^2_{\text{OHD}}, \) and of the Pantheon datasets, \( \chi^2_{\text{SNe}} \)
\[
\chi^2_{\text{tot}} = \chi^2_{\text{OHD}} + \chi^2_{\text{SNe}},\] (39)
Fig. 3 displays the \( 1 - \sigma \) and \( 2 - \sigma \) likelihood contours for the model parameters \( C \) and \( \lambda \) using the \( H(z)\)+Pantheon datasets. The model parameters with the best fit are \( C = 0.316^{+0.018}_{-0.020} \) and \( \lambda = 86.6^{+2.7}_{-2.6} \).

Fig. 4 displays the evolution of the deceleration parameter in relation to constrained values of the model parameters from the combined \( H(z)\)+Pantheon datasets. It seems obvious that our universe recently underwent a transition from a decelerated phase to an accelerated phase. According to the model parameter values constrained by the \( H(z)\)+Pantheon datasets, the transition redshift is \( z_{tr} = 0.7206 \). In addition, the current value of the deceleration parameter is \( q_0 = -0.5746 \) for the \( H(z)\)+Pantheon datasets.
FIG. 1. A good fit to the 57 points of the Hubble datasets is displayed in the plot of $H(z)$ versus the redshift $z$ for our $\Lambda$CDM jerk model, which is shown in red, and $\Lambda$CDM, which is shown in black dashed lines.

FIG. 2. A good fit to the 1048 points of the Pantheon datasets is displayed in the plot of $\mu(z)$ versus the redshift $z$ for our $\Lambda$CDM jerk model, which is shown in red, and $\Lambda$CDM, which is shown in black dashed lines.

B. Energy conditions

The energy conditions (ECs) are a set of alternative conditions that are used to put additional constraints on the validity of the constructed cosmological model and have many applications in theoretical cosmology. For example these conditions play an important role in GR as they help to prove the theorems about the presence of the singularity of space-time and black holes [58]. In the context of this work, ECs are exploited in order to predict the acceleration phase of the universe. Like these conditions can be obtained from the well-known Raychaudhury equations, which forms are [59–61]

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma_{\mu\nu}\sigma^{\mu\nu} + \omega_{\mu\nu}\omega^{\mu\nu} - R_{\mu\nu}u^\mu u^\nu,$$  \hspace{1cm} (40)
FIG. 3. The graphic displays the values of the model parameters $C$ and $\lambda$ that best fit the data from the combined $H(z)$+Pantheon datasets at $1 - \sigma$ and $2 - \sigma$ levels of confidence.

FIG. 4. The graphical behavior of deceleration parameter in terms of $z$ i.e. Eq. (31) with the constraint values from OHD+Pantheon datasets.

\[
\frac{d\theta}{d\tau} = -\frac{1}{2} \theta^2 - \sigma_{\mu\nu} \sigma^{\mu\nu} + \omega_{\mu\nu} \omega^{\mu\nu} - R_{\mu\nu} n^{\mu} n^{\nu}, \tag{41}
\]

where $n^\mu$, $\theta$, $\omega_{\mu\nu}$ and $\sigma^{\mu\nu}$ are the null vector, the expansion factor, the rotation and the shear associated with the vector field $u^\mu$, respectively. In Weyl geometry with the existence of non-metricity scalar $Q$, the Raychaudhury equations take various forms, for more details see [62]. Next, the above equations (40) and (41) fulfill the conditions

\[
R_{\mu\nu} u^\mu u^\nu \geq 0, \tag{42}
\]
\[
R_{\mu\nu} n^\mu n^\nu \geq 0. \tag{43}
\]

Thus, if we examine the perfect fluid distribution of cosmological matter, the ECs for $f(Q)$ gravity are given as follows [40],

- Weak energy conditions (WEC) if $\rho_{\text{eff}} \geq 0, \rho_{\text{eff}} + p_{\text{eff}} \geq 0$.
- Null energy condition (NEC) if $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$.
- Dominant energy conditions (DEC) if $\rho_{\text{eff}} \geq 0, |p_{\text{eff}}| \leq \rho$.
- Strong energy conditions (SEC) if $\rho_{\text{eff}} + 3p_{\text{eff}} \geq 0$.

By taking Eqs. (24) and (25) in the WEC, NEC, and DEC constraints, we can prove that

- Weak energy conditions (WEC) if $\rho \geq 0, \rho + p \geq 0$.
- Null energy condition (NEC) if $\rho + p \geq 0$.
- Dominant energy conditions (DEC) if $\rho \geq 0, |p| \leq \rho$.

These results are in concordance with those of Capozziello et al. [63]. In the case of the SEC condition, we find

\[
\rho + 3 p - 6 \dot{f}_Q H + f \geq 0. \tag{44}
\]

Now, using the above ECs, we can check the validity of our cosmological models in the following sections.

IV. COSMOLOGICAL $f(Q)$ MODELS

In this section, we will discuss the proposed cosmological models and some of their physical properties such as the energy density, pressure and equation of state (EoS) parameter using the general solution of the flat $\Lambda$CDM jerk parameter. In addition, we will verify our cosmological models with the help of the ECs described in the previous section. Here, we will propose two models of $f(Q)$ gravity. In the first model, we will assume a linear form of $f(Q)$ gravity. Then, in the second model we will take a non-linear functional form of $f(Q)$ gravity.

A. Linear model $f(Q) = aQ + \beta$

In this subsection, we presume the following simplest linear form of the $f(Q)$ function i.e.

\[
f(Q) = aQ + \beta \tag{45}
\]
where $\alpha$ and $\beta$ are free parameters. The motivation behind this linear form is the cosmological constant, despite the problems it faces, it is considered to be the most successful model among the alternatives offered in cosmology. The results of this model have been discussed in several contexts [45, 46, 64, 65].

Using Eqs. (22) and (30), we get the energy density of the Universe in the form

$$\rho = \frac{3\beta \left(e^{2\lambda t} - 1\right)^2 - 8\alpha \lambda^2 \left(e^{2\lambda t} + 1\right)^2}{6 \left(e^{2\lambda t} - 1\right)^2}. \quad (46)$$

Again, using Eqs. (23) and (30) we get the isotropic pressure of the Universe as

$$p = \frac{1}{6} \left(8\alpha \lambda^2 - 3\beta\right). \quad (47)$$

Thus, the EoS parameter ($p = \omega \rho$) for our model is

$$\omega = -\frac{\left(e^{2\lambda t} - 1\right)^2 \left(3\beta - 8\alpha \lambda^2\right)}{3\beta \left(e^{2\lambda t} - 1\right)^2 - 8\alpha \lambda^2 \left(e^{2\lambda t} + 1\right)^2}. \quad (48)$$

FIG. 5. The graphical behavior of energy density in terms of $z$ with $\alpha = -0.5$ and $\beta = 2$ for the specific case of $f(Q) = \alpha Q + \beta$.

From Fig. 5, we can see that the energy density of the Universe is a decreasing function of cosmic time (or increasing function of redshift) and remains positive as the Universe expands. Also, they tend to zero in the future. Moreover, the plot for EoS parameter in Fig. 6 shows quintessence-like behavior in the present, converges to the $\Lambda$CDM model in the future and to the dust matter in the past. Also, the present value of the EoS parameter corresponding to the OHD+Pantheon is $\omega_0 = -0.7161$. Now, using Eqs. (46) and (47) in the above ECs, we have plotted the behavior of NEC, DEC, and SEC in terms of cosmological redshift in Fig. 7. From this figure, it can be clearly seen that all the ECs are satisfied while the SEC is violated. This violation of the SEC is the evidence of the validity of the proposed cosmological model, and thus predicts the accelerating phase of the Universe.

FIG. 6. The graphical behavior of EoS parameter in terms of $z$ with $\alpha = -0.5$ and $\beta = 2$ for the specific case of $f(Q) = \alpha Q + \beta$.

FIG. 7. The graphical behavior of ECs in terms of $z$ with $\alpha = -0.5$ and $\beta = 2$ for the specific case of $f(Q) = \alpha Q + \beta$.

B. Non-linear model $f(Q) = Q + mQ^n$

Here, for the second model, we discuss the non-linear functional form of $f(Q)$,

$$f(Q) = Q + mQ^n \quad (49)$$

where $m$ and $n$ are free parameters. Also, this specific form has been considered in many cosmological contexts [41, 44, 65].

Thus, for this specific choice, we get the energy density, the isotropic pressure and the EoS parameter as
\[\rho = \frac{2^{3n-1} - n - 1}{(e^{2\lambda t} - 1)^2} \left( -3m(2n - 1) \left( e^{2\lambda t} - 1 \right)^2 \left( \frac{9}{4} H^2 \right)^n - \lambda^2 3^n 8^{1-n} \left( e^{2\lambda t} + 1 \right)^2 \right), \tag{50}\]

\[p = \frac{3^{-n-1} \left( 3m 8^n (2n - 1) (2 - 4n) e^{2\lambda t} + e^{4\lambda t} + 1 \right) \left( \frac{9}{4} H^2 \right)^n + 8\lambda^2 3^n \left( e^{2\lambda t} + 1 \right)^2}{2 \left( e^{2\lambda t} + 1 \right)^2}, \tag{51}\]

and

\[\omega = -\frac{\left( e^{2\lambda t} - 1 \right)^2 \left( 3m 8^n (2n - 1) (2 - 4n) e^{2\lambda t} + e^{4\lambda t} + 1 \right) \left( \frac{9}{4} H^2 \right)^n + 8\lambda^2 3^n \left( e^{2\lambda t} + 1 \right)^2}{\left( e^{2\lambda t} + 1 \right)^2 \left( 3m 8^n (2n - 1) (e^{2\lambda t} - 1)^2 \left( \frac{9}{4} H^2 \right)^n + 8\lambda^2 3^n \left( e^{2\lambda t} + 1 \right)^2 \right)}, \tag{52}\]

respectively.

From Fig. 8, it is clear that the energy density of the Universe is an increasing function of cosmological redshift and remains positive as the Universe expands. In addition, it tends to zero in the future. Further, the plot for EoS parameter in Fig. 9 shows quintessence-like behavior in the present and converges to the \(\Lambda\)CDM model in the future and to the dust matter in the past. Further, the present value of the EoS parameter corresponding to the OHD+Pantheon is \(\omega_0 = -0.7009\). For this case, using Eqs. (50) and (51) in the above ECs, we have plotted the behavior of NEC, DEC, and SEC in terms of cosmological redshift in Fig. 10. From this figure, it is clear that all the ECs are satisfied but SEC is violated. This violation of the SEC is the evidence of the validity of the proposed cosmological model, and thus predicts the accelerating phase of the Universe.

V. DISCUSSIONS AND CONCLUSIONS

The standard model of cosmology (\(\Lambda\)CDM) is most widely accepted today as it has been able to explain a large number of observed phenomena: the expansion of the Universe, the existence of the cosmic microwave background and the big bang nucleosynthesis. However, as we have pointed out in the introduction, \(\Lambda\)CDM could not explain dark energy (DE) and other issues \([66, 67]\). These puzzles prompted many authors to search for suitable alternatives, and some scientists went so far as to suggest a modification of general relativity, the theoretical basis of the \(\Lambda\)CDM model.

In this paper, we have discussed one of these recently proposed theories which has attracted the attention of many researchers i.e. \(f(Q)\) gravity where the non-metricity \(Q\) is the basis of gravitational interactions with zero curvature and torsion. We have studied a homo-
 homogeneous and isotropic FLRW space-time in the framework of this modified theory and the help of the $\Lambda$CDM jerk parameter. We have briefly described the theory’s mathematical formalism, then we have derived the field equations in the FLRW space-time for the content of the Universe in the form of a perfect fluid. Moreover, we have used 57 data points of Hubble $H(z)$ and 1048 data points to constrain the model parameters. The current Hubble rate and the deceleration parameter derived from the best fit of Markov Chain Monte Carlo are in agreement with those of the Planck data [57]. Moreover, we combined $H(z)$+Pantheon data sets with recently published Pantheon data sets to get the model parameters that fit the data the best. The results of the best fit is $C = 0.316 \pm 0.018$ and $\lambda = 86.6 \pm 2.7$.

Next, we have considered two functional forms of $f(Q)$ gravity, specifically, a linear and a non-linear form. We have analyzed the behavior of different cosmological parameters such as energy density, pressure and EoS parameter for both models. We have also checked all energy conditions in order to ensure the validity of our proposed cosmological models. For both models, Figs. 5 and 8, we have found that the energy density of the Universe is a decreasing function of cosmic time (or increasing function of redshift) and remains positive as the Universe expands. Furthermore, from Figs. 6 and 9, we have observed that the EoS parameter behaves like a quintessence dark energy model in the vicinity of our present time, while in the future and in the past it behaves as the $\Lambda$CDM model and the dust matter, respectively for both models. Finally, from the energy conditions as shown in Figs. 7 and 10 we can conclude that all the energy conditions are satisfied for both models while SEC is violated. The results above demonstrate that our proposed cosmological models are in strong agreement with today’s observations.

**Data availability** There are no new data associated with this article.

**Declaration of competing interest** The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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