Graded sum formula for $\tilde{A}_1$-Soergel calculus and the nil-blob algebra.

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Abstract

We study the representation theory of the Soergel calculus algebra $A^C_w := \text{End}_{\text{Blob}}(w)$ over $\mathbb{C}$ in type $\tilde{A}_1$. We generalize the recent isomorphism between the nil-blob algebra $\mathbb{NB}_n$ and a diagrammatically defined subalgebra $A^C_w$ of $A^C_w$ to deal with the two-parameter blob algebra. Under this generalization, the two parameters correspond to the two simple roots for $\tilde{A}_1$. Using this, together with calculations involving the Jones-Wenzl idempotents for the Temperley-Lieb subalgebra of $\mathbb{NB}_n$, we obtain a concrete diagonalization of the matrix of the bilinear form on the cell module $\Delta_w(v)$ for $A^C_w$. The entries of the diagonalized matrices turn out to be products of roots for $\tilde{A}_1$. We use this to study Jantzen type filtrations of $\Delta_w(v)$ for $A^C_w$. We show that, at an enriched Grothendieck group level, the corresponding sum formula has terms $\Delta_w(s_\alpha v)[t(s_\alpha v) - t(v)]$, where $[\cdot]$ denotes grading shift.

1 Introduction

Cellular algebras were introduced by Graham and Lehrer in 1994 in the paper [15] as a framework for studying the non-semisimple representation theory of many finite dimensional algebras.

The motivating examples for cellular algebras were the Iwahori-Hecke algebras of type $A_n$ and Temperley-Lieb algebras, but it has since been realized that many other finite dimensional algebras fit into this framework. For a cellular algebra one has a family of cell modules $\{\Delta(\lambda)\}$, endowed with bilinear forms $\langle \cdot, \cdot \rangle_\lambda$, that together control the representation theory of the algebra in question. Unfortunately, the concrete analysis of these bilinear forms is in general difficult, but in this paper we give a non-trivial cellular algebra over $\mathbb{C}$ for which the bilinear forms $\langle \cdot, \cdot \rangle_\lambda$ can in fact be diagonalized over an integral form of the algebra, thus solving all relevant questions concerning them, and therefore, by cellular algebra theory, concerning the representation theory of the algebra itself.

Our cellular algebra has two origins. Firstly it arises in the diagrammatic Soergel calculus of the Coxeter system $(W, S)$ of type $A_1$ as the endomorphism algebra $A^C_w := \text{End}(w)$ of $w := s_t s_t \cdots$ of length $n$, where $S = \{s, t\}$. An approach to Soergel calculus of universal Coxeter groups, in particular of type $\tilde{A}_1$, has been developed recently by Elias and Libedinsky in [10], see also [8]. For type $A_1$ this approach involves the two-colour Temperley-Lieb algebra but unfortunately the two-colour Temperley-Lieb algebra only captures the degree zero part of $A^C_w$, whereas our interests lie in the full grading on $A^C_w$.

The second origin of our cellular algebra is as a certain idempotent truncation $\mathbb{NB}_{n-1}$ of Martin and Saleur’s blob-algebra from [28]. In [26] the algebras $A^C_w$ and $\mathbb{NB}_{n-1}$ were studied extensively and in particular presentations in terms of generators and relations were found for each of them. Using this it was shown that there is an isomorphism $A^C_w \cong \mathbb{NB}_{n-1}$ where $A^C_w$ is a natural diagrammatically defined subalgebra of $A^C_w$, whose dimension is half the dimension of $A^C_w$. On the other hand, we show in this paper that the representation theory of $A^C_w$ can be completely recovered from the representation theory of $A^C_w$.

Similarly to the original blob-algebra, the diagrams for $\mathbb{NB}_{n-1}$ is given by blobbed (marked) Temperley-Lieb diagrams, as in the following examples

![Examples of blob diagrams](image)

although the rule for multiplying diagrams is different. Following [26], we call $\mathbb{NB}_{n-1}$ the nil-blob algebra, but in fact $\mathbb{NB}_{n-1}$ has also appeared in the literature under the name the dotted Temperley-Lieb algebra, see [35]. In [35] it was shown that the associated dotted Temperley-Lieb category is equivalent to the Bar-Natan category of the thickened annulus and this equivalence was used in the recent work [18] to construct a Kirby color for Khovanov homology.

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An important feature of \( \hat{A}_w \) and \( \mathbb{N}B_{n-1} \), and in fact of all cellular algebras appearing in this paper, is the fact that they are \( \mathbb{Z} \)-graded algebras, with explicitly given degree functions defined in terms of the diagrams. They are \( \mathbb{Z} \)-graded cellular in the sense of Hu and Mathas, see [20].

In this paper our first new result is a construction of integral forms \( A_w \) and \( B_{n-1}^{x,y} \) for \( A_w^C \) and \( \mathbb{N}B_{n-1} \) over the two-parameter polynomial algebra \( R := \mathbb{C}[x,y] \) and a lift of the isomorphism \( A_w^C \cong \mathbb{N}B_{n-1} \) to \( A_w \cong B_{n-1}^{x,y} \). The integral form \( A_w \) is in fact already implicit in the setup for Soergel calculus, using the dual geometric realization of the Coxeter group \( W \) of type \( A_1 \). Under this realization, the parameters \( x \) and \( y \) correspond to the two simple roots for \( W \). The integral form for \( \mathbb{N}B_{n-1} \) is also a well-known object, since it is simply the two-parameter blob-algebra \( B_{n-1}^{x,y} \) with blob-parameter \( x \) and marked loop parameter \( y \). Thus the novelty of our result lies primarily in the isomorphism between these integral forms, which on the other hand has the quite surprising consequence of rendering a Coxeter group theoretical meaning to the two blob algebra parameters for \( B_{n-1}^{x,y} \), since they become nothing but the simple roots for \( W \).

Our main interests lies in the representation theory of \( A_w \) which via the above isomorphism is equivalent to the representation theory of \( B_{n-1}^{x,y} \). For several reasons the representation theory of \( B_{n-1}^{x,y} \) is more convenient to handle. Both algebras are cellular algebras with diagrammatically defined cellular bases, but the straightening rules for expanding the product of two cellular basis elements in terms of the cellular basis are easier in the \( B_{n-1}^{x,y} \) setting. Secondly, there is a natural Temperley-Lieb subalgebra \( T_L \) of \( B_{n-1}^{x,y} \) whose associated restriction functor \( \text{Res} \) is very useful for our purposes. We show in section 4 of our paper that \( \text{Res} \) maps a cell module \( \Delta_{n-1}^B(\lambda) \) for \( B_{n-1}^{x,y} \) to a module with a cell module filtration for \( T_L \) and that the sections of this filtration induce a diagonalization of the bilinear form \( \langle \cdot, \cdot \rangle_{n-1,\lambda}^B \) on \( \Delta_{n-1}^B(\lambda) \).

This leaves us with the task of calculating the values of the bilinear form on these sections. For \( \lambda \geq 0 \) and \( n - 1 = 2k + \lambda \) we show that this task is equivalent to calculating the coefficient of the identity \( 12 \ldots \lambda \) in the expansion of

\[
\begin{array}{ccccccc}
1 & 2 & \cdots & k & 1 & 2 & \cdots \\
\text{JW}_{n-1}
\end{array}
\]

(1.2)

where \( \text{JW}_{n-1} \) is the Jones-Wenzl idempotent for \( T_L \). Similarly, for \( \lambda < 0 \) and \( n - 1 = 2k + |\lambda| \) it is equivalent to calculating the coefficient of \( 12 |\lambda| \ldots \lambda \) in the expansion of

\[
\begin{array}{ccccccc}
1 & 2 & \cdots & k & 1 & 2 & \cdots |\lambda| \\
\text{JW}_{n-1}
\end{array}
\]

(1.3)

The determination of these coefficients constitutes the main calculatory ingredient of our paper and is done in section 5. The result is given in Theorem 5.1 for \( \lambda \geq 0 \) and in Theorem 5.2 for \( \lambda < 0 \). Although one may possibly not have expected Coxeter theory to appear in this calculation, the result turns out to be a nice product of positive roots for \( W \).

This work is partially motivated by the paper [36] by the second author in which a diagonalization of the bilinear form for the cell module for \( A_w \) is obtained, in fact the results of [36] are valid for a general Coxeter system. Unfortunately, as was already mentioned in [36], the diagonalization process in that paper does not work over \( R \) itself, but only over the fraction field \( Q \) of \( R \), since it relies on certain Jucys-Murphy elements for \( A_w \) that are of degree 2, and not 0. As a consequence the \( \mathbb{Z} \)-graded structure on the cell module for \( A_w \) breaks down under the diagonalization process in [36]. The diagonalization process of the present paper, however, which is based on the Jones-Wenzl idempotents that are of degree 0, resolves this problem at least for type \( A_1 \).

In the final section 7 of the paper we use the results of the previous sections to set up a version \( \Delta_{w,d}^{gr,C}(v) \supset \Delta_{w,1}^{gr,1,C}(v) \supset \Delta_{w,2}^{gr,2,C}(v) \supset \ldots \) of the Jantzen filtration formalism for the graded cell module \( \Delta_{w,d}^{gr,C}(v) \), using the
bilinear form $\langle \cdot, \cdot \rangle_w^v$ on $\Delta_{w,v}^{gr,C}(v)$. Since $\langle \cdot, \cdot \rangle_w^v$ is a graded bilinear form, this filtration consists of graded submodules of $\Delta_{w,v}^{gr,C}(v)$, and in our final Corollary 7.12 we show that the following identity holds at enriched Grothendieck group level

$$\sum_{k>0} \langle \Delta_{w,v}^{gr,k,C}(v) \rangle_q = \sum_{\alpha>0} \sum_{v<s,v \leq w} \langle \Delta_{w,v}^{gr,C}(s\alpha v)[l(s\alpha v) - l(v)] \rangle_q$$

(1.4)

where $\alpha > 0$ refers to the positive roots for $W$ and $[\cdot]$ to grading shift. This is our graded Jantzen sum formula. Analogues of (ungraded) Jantzen filtrations with associated sum formulas exist in many module categories of Lie type and give information on the irreducible modules for the category in question, see for example [1], [2], [30]. But although graded representation theories in Lie theory have been known since the beginning of the nineties, see for example [3], [4], [12], [13], [19], [27] and [38], to our knowledge graded sum formulas in the sense of (1.4) have so far not been available, even though they would be very useful for calculating decomposition numbers. We believe (1.4) gives an interesting indication of the possible form of graded sum formulas in representation theory.

The layout of the paper is as follows. In the next section we introduce the notation that shall be used throughout the paper and recall the various algebras that play a role throughout: the Temperley-Lieb algebra $\mathbb{TL}_n$, the blob algebra $\mathbb{NB}_n$, and the Soergel algebra $\mathbb{A}_w$. We also recall how each of them fits into the cellular algebra language. In section 3 we introduce the subalgebra $\mathbb{A}_w$ of $\mathbb{A}_n$ and show the isomorphism $\mathbb{B}_{n-1}^{gr,y} \cong \mathbb{A}_w$ that was mentioned above. We also show how the cellular algebra structure on $\mathbb{A}_w$ induces a cellular structure on $\mathbb{A}_n$ and that there is an isomorphism $\Delta_{n-1}^B(\lambda) \cong \Delta_w(v)$ between the respective cell modules for $\mathbb{B}_{n-1}^{gr,y}$ and $\mathbb{A}_w$. In section 4 we consider a natural filtration of Lie theory have been known since the beginning of the nineties, see for example [14], [27], [29], [31], [32], [33], [34]. In this paper, we shall use the following variation of it.

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2 Blob algebras and Soergel calculus for $\tilde{A}_1$

Throughout we use as ground field the complex numbers $\mathbb{C}$, although several of our results hold in greater generality. We set

$$R := \mathbb{C}[x, y].$$

(2.1)

We consider $R$ to be a (non-negatively) $\mathbb{Z}$-graded $\mathbb{C}$-algebra via

$$\deg(x) = \deg(y) = 2.$$  

(2.2)

In this paper we shall consider several diagram algebras. Possibly the oldest and most studied diagram algebra is the Temperley-Lieb algebra. It arose in statistical mechanics in the seventies. In the present paper we shall use the following variation of it.

**Definition 2.1.** The Temperley-Lieb algebra $\mathbb{TL}_n$ with loop-parameter $-2$ is the $R$-algebra on the generators $U_1, \ldots, U_{n-1}$ subject to the relations

$$U_i^2 = -2U_i \quad \text{if} \quad 1 \leq i < n$$

(2.3)

$$U_iU_jU_i = U_i \quad \text{if} \quad |i - j| = 1$$

(2.4)

$$U_iU_j = U_jU_i \quad \text{if} \quad |i - j| > 1.$$  

(2.5)

The blob algebra was introduced by Martin and Saleur in [28], as a a way of considering boundary conditions in the statistical mechanical model of the Temperley-Lieb algebra. Since its introduction, the blob algebra has been the subject of much research activity in mathematics as well as physics, see for example [14], [27], [29], [31], [32], [33], [34]. In this paper, we shall use the following variation of it.
Definition 2.2. The two-parameter blob algebra $\mathbb{B}_n^{x,y}$, or more precisely the blob algebra with loop-parameter $-2$, marked loop parameter $y$ and blob-parameter $x$, is the $R$-algebra on the generators $U_0, U_1, \ldots, U_{n-1}$ subject to the relations

\begin{align}
U_i^2 &= -2U_i \quad &\text{if } 1 \leq i < n \\
U_iU_jU_i &= U_i \quad &\text{if } |i-j| = 1 \text{ and } i, j > 0 \\
U_iU_j &= U_jU_i \quad &\text{if } |i-j| > 1 \\
U_1U_0U_1 &= yU_1 \\
U_0^2 &= xU_0.
\end{align}

The nil-blob algebra $\mathbb{NB}_n$, that was introduced and studied extensively in [26], may be recovered from $\mathbb{B}_n^{x,y}$ via specialization, that is

$$\mathbb{NB}_n \cong \mathbb{B}_n^{x,y} \otimes_R \mathbb{C}\quad (2.11)$$

where $\mathbb{C}$ is made into an $R$-algebra via $x \mapsto 0$ and $y \mapsto 0$. In other words, $\mathbb{B}_n^{x,y}$ may be considered a deformation of $\mathbb{NB}_n$, and in fact this shall be the point of view of the present paper.

Another interesting specialization of $\mathbb{B}_n^{x,y}$ is $\tilde{T}_n$ defined as $\tilde{T}_n := \mathbb{B}_n^{x,y} \otimes_R \mathbb{C}$ where $\mathbb{C}$ is made into an $R$-algebra via $x \mapsto 1$ and $y \mapsto -2$. Let $I := \langle U_0 - 1 \rangle$ be the two-sided ideal in $\tilde{T}_n$ generated by $U_0 - 1$. Then

$$\tilde{T}_n := \tilde{T}_n/I \quad (2.12)$$

is the Temperley-Lieb algebra from Definition 2.1, but defined over $\mathbb{C}$.

Just as is the case for $\mathbb{NB}_n$, one easily checks that $\mathbb{B}_n^{x,y}$ is a $\mathbb{Z}$-graded algebra.

Lemma 2.3. The rules $\deg(U_i) = 0$ for $i > 0$ and $\deg(U_0) = 2$ define a (non-negative) $\mathbb{Z}$-grading on $\mathbb{B}_n^{x,y}$.

Proof. The relations are easily seen to be homogeneous with respect to $\deg$.

As already indicated, $\mathbb{TL}_n$ and $\mathbb{B}_n^{x,y}$ are diagram algebras. This fact plays an important role in our paper, and let us briefly explain it. The diagram basis for $\mathbb{TL}_n$ consists of $\text{T}\text{emperley-Lieb diagrams}$ on $n$ points, which are planar pairings between $n$ northern points and $n$ southern points of a rectangle. The diagram basis for $\mathbb{B}_n^{x,y}$ consists of blobbed (marked) Temperley-Lieb diagrams on $n$ points, or blob diagrams on $n$ points, which are marked planar pairings between $n$ northern points and $n$ southern points of a rectangle, where only pairings exposed to the left side of the rectangle may be marked, and at most once. There is thus a natural embedding of Temperley-Lieb diagrams into blob diagrams. The multiplication $D_1D_2$ of two diagrams $D_1$ and $D_2$ is given by concatenation of them, with $D_2$ on top of $D_1$. This concatenation process may give rise to internal marked or unmarked loops, as well as diagrams with more than one mark. Internal unmarked loops are removed from a diagram by multiplying it by $-2$, whereas internal marked loops are removed from a diagram by multiplying it by $y$. Finally, any diagram with $r > 1$ marks on a diagram is set equal to the same diagram multiplied by $x^{r-1}$, but with the $(r-1)$ extra marks removed. For example, for

\begin{equation}
D_1 = \includegraphics[scale=0.5]{diagram1}, \quad D_2 = \includegraphics[scale=0.5]{diagram2}
\end{equation}

we have that

\begin{equation}
D_1D_2 = \includegraphics[scale=0.5]{diagram3} = x^2y
\end{equation}

Later on, we shall give many more examples.

For the proof of the isomorphisms between $\mathbb{B}_n^{x,y}$ and its diagrammatic version, one may consult the appendix of [9] or else adapt the more self-contained proof given in [26], and similarly for $\mathbb{TL}_n$. Under the isomorphism we have that

\begin{equation}
1 \mapsto \includegraphics[scale=0.5]{diagram4}
\end{equation}
and that
\[ U_0 \mapsto \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \hline & & & & & & \\ \end{array}, \quad U_i \mapsto \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \end{array}. \quad (2.16) \]

The number of Temperley-Lieb diagrams and blob diagrams on \( n \) points is the Catalan number \( \frac{1}{n+1}\binom{2n}{n} \) and \( \binom{2n}{n} \).

In particular \( \mathbb{T}_n \) and \( \mathbb{B}_n^{x,y} \) are free over \( R \) of rank
\[ \text{rk} \mathbb{T}_n = \frac{1}{n+1}\binom{2n}{n} \quad \text{and} \quad \text{rk} \mathbb{B}_n^{x,y} = \binom{2n}{n}. \quad (2.17) \]

All algebras considered in the present paper fit into the general language of cellular algebras, introduced by Graham and Lehrer.

**Definition 2.4.** Let \( A \) be a finite dimensional algebra over a commutative ring \( k \) with unity. Then a cellular basis structure for \( A \) consists of a triple \((\Lambda, \text{Tab}, C)\) such that \( \Lambda \) is a poset, \( \text{Tab} \) is a function on \( \Lambda \) with values in finite sets and \( C : \prod_{\lambda \in \Lambda} \text{Tab}(\lambda) \times \text{Tab}(\lambda) \to A \) is an injection such that
\[ \{C^\lambda_{st} | s, t \in \text{Tab}(\lambda), \lambda \in \Lambda\} \]
is a \( k \)-basis for \( A \): the cellular basis for \( A \). The rule \((C^\lambda_{st})^* := C^\lambda_{ts}\) defines a \( k \)-linear antihomomorphism of \( A \) and the structure constants for \( A \) with respect to \( \{C^\lambda_{st}\} \) satisfy the following condition with respect to the partial order: for all \( a \in A \) we have
\[ aC^\lambda_{st} = \sum_{u \in \text{Tab}(\lambda)} r_{usa}C^\lambda_{ut} + \text{lower terms} \]
where lower terms means a linear combination of \( C^\mu_{ab} \) where \( \mu < \lambda \) and where \( r_{usa} \in k \).

To make \( \mathbb{T}_n \) fit into this language we choose \( k = R \), \( \Lambda = \Lambda_n := \{n, n-2, \ldots, 1\} \) (or \( \Lambda := \Lambda_n = \{n, n-2, \ldots, 2, 0\} \) if \( n \) is odd (or even), with poset structure inherited from \( \mathbb{Z} \). For \( \lambda \in \Lambda_n \) we choose \( \text{Tab}(\lambda) \) to be Temperley-Lieb half-diagrams with \( \lambda \) propagating lines, that is Temperley-Lieb diagrams on \( \lambda \) northern and \( n \) southern points in which each northern point is paired with a southern point. For \( s,t \in \text{Tab}(\lambda) \) we define \( C^\lambda_{st} \) to be the diagram obtained from gluing \( s \) and the horizontal reflection of \( t \), with \( s \) on the bottom. Here is an example of this gluing process with \( n = 8 \) and \( s,t \in \text{Tab}(2) \).

\[ (s,t) = \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \end{array} \mapsto \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \end{array}. \quad (2.18) \]

**Theorem 2.5.** The above triple \((\Lambda_n, \text{Tab}, C)\) makes \( \mathbb{T}_n \) into a cellular algebra.

**Proof:** This follows directly from the definitions. \( \square \)

To make \( \mathbb{B}_n^{x,y} \) fit into the cellular algebra language we choose \( k = R \), \( \Lambda = \{\Lambda_{\pm n} := \{\pm n, \pm(n-2), \ldots, \pm 1\} \) (or \( \Lambda := \Lambda_{\pm n} = \{\pm n, \pm(n-2), \ldots, \pm 2, 0\} \) if \( n \) is odd (or even), with poset structure given by \( \lambda < \mu \) if \( |\lambda| < |\mu| \) or if \( |\lambda| = |\mu| \) and \( \lambda < \mu \). For example, for \( n = 6 \) we have
\[ \Lambda_{\pm 6} := \{\pm 6, \pm 4, \pm 2, 0\}, \quad 0 < -2 < -4 < 4 < -6 < 6 \quad (2.19) \]

For \( \lambda \in \Lambda_{\pm n} \) we choose \( \text{Tab}(\lambda) \) to be blob half-diagrams with \( |\lambda| \) propagating lines, that is marked Temperley-Lieb diagrams on \( |\lambda| \) northern and \( n \) southern points in which each northern point is paired with a southern point and in which only non-propagating pairings exposed to the left side of the rectangle may be marked. For \( s,t \in \text{Tab}(\lambda) \) we define \( C^\lambda_{st} \) to be the diagram obtained from gluing \( s \) and the horizontal reflection of \( t \), with \( s \) on the bottom, and marking the leftmost propagating line if \( \lambda < 0 \). Here is an example of this gluing process with \( n = 8 \) and \( s,t \in \text{Tab}(-2) \).

\[ (s,t) = \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \end{array} \mapsto \begin{array}{c|c|c|c|c|c|c} \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \hline \hline & & & & & & \\ \end{array}. \quad (2.20) \]

With this notation we have the following Theorem.
\textbf{Theorem 2.6.} The above triple \( (A_{\pm n}, \text{Tab}, C) \) makes \( \mathbb{B}^{x,y}_n \) into a cellular algebra.

\textit{Proof:} This also follows directly from the definitions. \hfill \Box

For \( \lambda \in \Lambda_n \) or for \( \lambda \in \Lambda_{\pm n} \) we define \( t_\lambda \in \text{Tab}(\lambda) \) to be the following half-diagram

\begin{equation}
  t_\lambda = \begin{array}{cccccc}
  1 & 2 & \cdots & k & 1 & 2 \\
  \end{array} \quad |\lambda| \quad (2.21)
\end{equation}

and set \( C_\lambda = C_{t_\lambda} \), that is

\begin{equation}
  C_\lambda = \begin{cases}
    \langle U_1 U_3 \cdots U_{2k-1} \rangle = \begin{array}{cccccc}
      1 & 2 & \cdots & k & 1 & 2 \\
    \end{array} \quad |\lambda| & \text{if } \lambda \geq 0 \\
    (U_1 U_3 \cdots U_{2k-1}) U_{0}(U_2 U_4 \cdots U_{2k}) U_1 U_3 \cdots U_{2k-1} = \begin{array}{cccccc}
      1 & 2 & \cdots & k & 1 & 2 \\
    \end{array} \quad |\lambda| & \text{if } \lambda < 0
  \end{cases} \quad (2.22)
\end{equation}

where \( k = (n - |\lambda|)/2 \). Thus \( C_\lambda \) is an element of \( \mathbb{T}L_n \) or \( \mathbb{B}^{x,y}_n \), depending on the context. With this we can define the two-sided cell ideals \( A^\lambda \) and \( A^{<\lambda} \) of \( \mathbb{T}L_n \) (resp. \( \mathbb{B}^{x,y}_n \)) via

\begin{equation}
  A^\lambda := \{ aC_\mu b | a, b \in \mathbb{T}L_n \) (resp. \( \mathbb{B}^{x,y}_n \)) \text{ and } A^{<\lambda} := \{ aC_\mu b | a, b \in \mathbb{B}^{x,y}_n \text{ (resp. } \mathbb{B}^{x,y}_n \text{) and } \mu < \lambda \}. \quad (2.23)
\end{equation}

It follows from Definition 2.4 that \( A^{<\lambda} \subset A^\lambda \) and so \( A^\lambda / A^{<\lambda} \) is a \( \mathbb{T}L_n \)-bimodule (resp. \( \mathbb{B}^{x,y}_n \)-bimodule). Let \( C_\lambda := C_\lambda + A^{<\lambda} \). We define the cell module \( \Delta_{TL}^\lambda(\lambda) \) for \( \mathbb{T}L_n \) (resp. \( \Delta_{B}^\lambda(\lambda) \) for \( \mathbb{B}^{x,y}_n \)) as

\begin{equation}
  \Delta_{TL}^\lambda(\lambda) := \mathbb{T}L_n C_\lambda \text{ (resp. } \Delta_{B}^\lambda(\lambda) := \mathbb{B}^{x,y}_n C_\lambda \) \subseteq A^\lambda / A^{<\lambda}. \quad (2.24)
\end{equation}

We now have the following Theorem.

\textbf{Theorem 2.7.} Let \( \Delta_{TL}^\lambda(\lambda) \) be the cell module for \( \mathbb{T}L_n \). Then \( \Delta_{TL}^\lambda(\lambda) \) is free over \( R \) with basis \( \{ C_{st} | s \in \text{Tab}(\lambda) \} \) where \( C_{st} := C_{st} + A^{<\lambda} \). A similar statement holds for \( \Delta_{B}^\lambda(\lambda) \).

\textit{Proof:} This follows from the algorithm given in the proof Theorem 2.5 of [26]. \hfill \Box

\textbf{Remark 2.8.} Our definition (2.24) of the cell modules for \( \mathbb{T}L_n \) and \( \mathbb{B}^{x,y}_n \) differs slightly from the definition given in [15], but in view of the Theorem the two definitions coincide.

We now come to Soergel calculus. In [26], an isomorphism \( A_w \cong \mathbb{N} \mathbb{B} \) was established, where \( A_w \) is the endomorphism algebra of a Bott-Samelson object in the Soergel calculus of type \( \tilde{A}_1 \). We aim at generalizing this result to an isomorphism involving \( \mathbb{B}^{x,y}_n \).

To any Coxeter system \( (W, S) \), Elias and Williamson associated in [11], a diagrammatic category \( \mathcal{D}(W, S) \). We fix \( S := \{ s, t \} \) and let \( W \) be the Coxeter group on \( S \) given by

\begin{equation}
  W := \langle s, t | s^2 = t^2 = e \rangle \quad (2.25)
\end{equation}

that is, \( W \) is the affine Weyl group of type \( \tilde{A}_1 \), or the infinite dihedral group with Brudt order \( < \) chosen such that \( 1 \) is the minimal element. The definition of \( \mathcal{D}(W, S) \) depends on the choice of a \textit{realization} \( \mathfrak{h} \) of \( (W, S) \), which is a representation \( \mathfrak{h} \) of \( W \) over \( \mathbb{C} \), arising from a choice of \textit{simple roots} and \textit{simple coroots}, see [11, Section 3.1]. In [26], the realization \( \mathfrak{h} \) was chosen to be the \textit{geometric representation} of \( W \) defined over \( \mathbb{C} \), see [21, 5.3], with coroots \( \alpha^\vee_1, \alpha^\vee_2 \) being a basis for \( \mathfrak{h}^* \) and roots \( \alpha_s, \alpha_t \in \mathfrak{h}^* \) given by

\begin{equation}
  \alpha_s(\alpha_1^\vee) = 2, \quad \alpha_t(\alpha_1^\vee) = -2, \quad \alpha_s(\alpha_1^\vee) = -2, \quad \alpha_t(\alpha_1^\vee) = 2 \quad (2.26)
\end{equation}

that is \( \alpha_s = -\alpha_t \). With this choice of realization of \( (W, S) \), the symmetric algebra of the dual representation is \( R := S(\mathfrak{h}^*) \cong \mathbb{C}[\alpha_s] \), or simply a one-variable polynomial algebra.
In this paper, we choose for realization \( h \) of \((W, S)\) the dual of the geometric representation. To be precise, we choose \( h \) to be the \( \mathbb{C} \)-vector space of dimension two, containing an element \( \alpha^\vee_s = -\alpha_1^\vee \), such that for a basis \( \alpha_s, \alpha_t \) for \( h^* \) the relations (2.26) hold. For this choice of \( h \) we have that

\[
R = S(h^*) = \mathbb{C}[\alpha_s, \alpha_t]
\]

that is \( R \) is a two-variable polynomial algebra. We shall use the identifications

\[
x := \alpha_s, \quad y := \alpha_t
\]

such that the two definitions of \( R \) in (2.27) and (2.1) coincide, but shall in general use the symbols \( x, y \) when referring to \( B_{x,y} \) and \( \alpha_s, \alpha_t \) when referring to Soergel calculus.

We consider \( R \) to be a \( \mathbb{Z} \)-graded algebra where \( \deg(\alpha_s) = \deg(\alpha_t) = 2 \). The action of \( W \) on \( h^* \) is given by the formulas

\[
s\alpha_s = -\alpha_s, \quad s\alpha_t = \alpha_t + 2\alpha_s, \quad t\alpha_t = -\alpha_t, \quad t\alpha_s = \alpha_s + 2\alpha_t.
\]

It extends to an action of \( W \) on \( R \) and so we have the Demazure operators \( \partial_s, \partial_t : R \to R \) defined by:

\[
\partial_s(f) = \frac{f - sf}{\alpha_s}, \quad \partial_t(f) = \frac{f - tf}{\alpha_t}.
\]

We have that

\[
\partial_s(\alpha_s) = \partial_t(\alpha_t) = 2, \quad \partial_s(\alpha_t) = \partial_t(\alpha_s) = -2.
\]

Let us now briefly explain the definition of the diagrammatic Soergel category \( D(W, S) \) for our choices. Let \( \exp \) be the set of expressions over \( S \), that is words \( w = s_i s_i \cdots s_i \) in the alphabet \( S \). We consider the empty expression \( w = \emptyset \) to be an element of \( \exp \).

**Definition 2.9.** Let \((W, S)\) be as above. A Soergel diagram for \((W, S)\) is a finite graph embedded in \( \mathbb{R} \times [0, 1] \). The arcs of a Soergel diagram are coloured by either colour red or colour blue, corresponding to the elements of \( S \). The vertices of a Soergel diagram are of the four possible types indicated below, univalent vertices (dots) and trivalent vertices where all three incident arcs are of the same colour.

\[
\begin{align*}
\text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1);
\end{tikzpicture}} \\
\text{\begin{tikzpicture}
\draw [blue] (0,0) -- (0,1);
\end{tikzpicture}} \\
\text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1) -- (1,0.5);
\end{tikzpicture}} \\
\text{\begin{tikzpicture}
\draw [blue] (0,0) -- (0,1) -- (1,0.5);
\end{tikzpicture}} \\
\end{align*}
\]

A Soergel diagram has its regions, that is the connected components of the complement of the graph in \( \mathbb{R} \times [0, 1] \), decorated by elements of \( R \). For simplicity, we omit the decoration \( 1 \in R \) when drawing Soergel diagrams.

A vertex of an arc of a Soergel diagram that belongs to the boundary of the strip \( \mathbb{R} \times [0, 1] \) is called a boundary point. We say that an arc \( l \) of \( D \) is a boundary dot arc if one of its vertices is a dot and the other one is a boundary point. The left to right reading of the boundary points gives rise to two elements of \( \exp \) called the bottom boundary and top boundary of the diagram, respectively.

**Definition 2.10.** The diagrammatic Soergel category \( D(W, S) \) is the monoidal category whose objects are the elements of \( \exp \) and whose morphisms \( \text{Hom}_D(x, y) \) are the \( R \)-modules generated by all Soergel diagrams with bottom boundary \( x \) and top boundary \( y \), modulo isotopy and modulo the following local relations

\[
\begin{align*}
\text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1);
\end{tikzpicture}} &= \text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1);
\end{tikzpicture}} \\
\text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1) -- (1,0.5);
\end{tikzpicture}} &= \text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1) -- (1,0.5);
\end{tikzpicture}} \\
\text{\begin{tikzpicture}
\draw [red] (0,0) -- (0,1);
\end{tikzpicture}} &= \alpha_s
\end{align*}
\]
where the relations (2.33)–(2.37) also hold if red is replaced by blue.

For $f \in R$ and $D$ a Soergel diagram, the product $fD$ is defined as the diagram obtained from $D$ by replacing the polynomial $g$ of the leftmost region of $D$ by $fg$. The multiplication $D_1D_2$ of diagrams $D_1$ and $D_2$ is given by vertical concatenation with $D_2$ on top of $D_1$, where regions containing two polynomials are replaced by the same regions, but containing the product of these polynomials. The monoidal structure is given by horizontal concatenation. There is a natural $\mathbb{Z}$-grading on $D_{(W,S)}$, extending the grading on $R$, in which dots, that is the first two diagrams in (2.32) have degree 1, and the trivalent, that is the last two diagrams in (2.32), have degree $-1$.

**Remark 2.11.** For more details concerning the definition of $D_{(W,S)}$ one should consult [26] or the original paper [11]. Note that apart from the choice of realization of $(W, S)$, the relations appearing in Definition (2.10) also differ slightly from the ones appearing in the corresponding Definition 3.2 in [26]. To be precise, in Definition 3.2 of [26], there is a final relation

$$fD = 0$$

(2.38)

where $D$ is any Soergel diagram and $f$ is any homogeneous polynomial of strictly positive degree, multiplied on the left on $D$.

**Definition 2.12.** We define $D_{(W,S)}^c$ to be the category obtained from $D_{(W,S)}$ by adding relation (2.38).

**Remark 2.13.** It is shown in [11] that there is an equivalence between $D_{(W,S)}$ and the category of Soergel bimodules for $(W, S)$. It induces an equivalence between $D_{(W,S)}^c$ and the category of Soergel modules.

Let now $n$ be a fixed non-negative integer and define $w \in \text{exp}$ of length $n$ via

$$w := st^ns \cdots$$

(2.39)

such that the first generator of $w$ is $s$ but the last generator depends on the parity of $n$. We then introduce $\hat{A}_w$ as follows

$$\hat{A}_w := \text{End}_{D_{(W,S)}}(w).$$

(2.40)

By the definitions, $\hat{A}_w$ is an $R$-algebra with multiplication $D_1D_2$ given by concatenation of $D_1$ and $D_2$ and scalar product $fD$ by multiplication of $f$ with the polynomial appearing in the leftmost region of $D$. Its one-element is denoted 1, and is as follows

$$1 := \begin{array}{c}
1 \ 2 \ 3 \ \cdots \ \cdots \ n
\end{array}$$

(2.41)

As in [26] we introduce certain elements of $\hat{A}_w$ that shall play a key role throughout. For $i = 1, \ldots, n-2$, let $U_i$ be the following element of $\hat{A}_w$

$$U_i := \begin{array}{c}
1 \ 2 \ 3 \ \cdots \ \cdots \ n
\end{array}$$

(2.42)
and for $i = 0$, set

$$U_0 := \begin{array}{cccc}
1 & 2 & 3 & \cdots & \cdots & n \\
\downarrow & & & & & \\
\downarrow & & & & & \\
\downarrow & & & & & \\
\downarrow & & & & & \\
\downarrow & & & & & \\
\downarrow & & & & & \\
\end{array}. \quad (2.43)$$

## 3 Isomorphism Theorems

The following Theorem is fundamental for this paper.

**Theorem 3.1.** There is a homomorphism of $R$-algebras $\varphi : \mathbb{E}_{n-1}^x \rightarrow \tilde{A}_w$ given by $U_i \mapsto U_i$ for $i = 0, 1, \ldots, n-2$.

**Proof:** The proof is almost identical to the proof of Theorem 3.4 in [26]. We must check that $U_0, U_1, \ldots, U_{n-2}$ satisfy the relations given by the $U_i$'s in Definition (2.2). The verification of the relations (2.6), (2.7) and (2.8) is done exactly as in [26] whereas relation (2.9), for example for $n = 4$, is verified as follows

$$U_1 U_0 U_1 = \begin{array}{cccc}
& & & \cdots & \cdots & \cdots & \cdots & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\end{array} = \alpha_i = yU_1 \quad (3.1)$$

and (2.10), for example for $n = 4$, is verified as follows

$$U_0^2 = \begin{array}{cccc}
& & & \cdots & \cdots & \cdots & \cdots & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\downarrow & & & & & & & \\
\end{array} = \alpha_i = xU_0. \quad (3.2)$$

This proves the Theorem. \qed

For general $(W, S)$ there is a, somewhat unwieldy, recursive procedure for constructing an $R$-basis for the morphisms $\text{Hom}_D(x, y)$, for any $x, y \in \exp$. It is a diagrammatic version of Libedinsky’s *double leaves basis* for Soergel bimodules and the basis elements are also called double leaves. For $W$ the infinite dihedral group, there is however a non-recursive description of the double leaves basis that was used extensively in [26], and that also plays an important role in the present paper.

The double leaves diagram basis elements for $\tilde{A}_w$ are built up from top and bottom ‘half-diagrams’, similarly to Temperley-Lieb and blob diagrams. Let us explain these half-diagrams.

By the isotopy relations for Soergel diagrams, we have the following diagram identities

$$= = = = = = = = \quad (3.3)$$

and we in shall in general represent these diagrams as follows

$$\quad (3.4)$$

The diagrams in (3.4) are called *hanging full birdcages*. We shall also consider *non-hanging full birdcages* that look as follows

$$\quad (3.5)$$
We sometimes omit the word 'full' when referring to 'full birdcages'. We say that the birdcages in (3.3), (3.4) and (3.5) are of colour blue, but shall also allow birdcages of colour red, as follows

We further allow degenerate, non-hanging and hanging, full birdcages as follows

We define the length of a full birdcage to be the number of enclosed dots in it, where the birdcages of length zero are the degenerate ones. A full birdcage which is not degenerate is called non-degenerate. In the following examples, the first birdcage is of length 4 whereas the last two are of length zero.

We also consider top full birdcages, that are obtained from bottom full birdcages by reflecting through a horizontal axis.

We now introduce the operation of replacing a degenerate non-hanging full birdcage, in other words a boundary dot arc, by a non-hanging non-degenerate full birdcage as follows

In the notation of [25], a birdcagecage is any diagram that can be obtained by performing the above operation repeatedly a number of times, for example

A light leaf diagram element $\tilde{D}$ for $\tilde{A}_w$ is by definition a horizontal concatenation of birdcagecages as indicated below in (3.11), with bottom boundary $w$. Zone A consists of a number of non-hanging birdcagecages whereas zone B consists of a number of hanging birdcagecages, but zone C may consist of at most one non-hanging birdcagecage.

The hanging birdcagecages in zone B of a light leaves diagram define an element $\tilde{v} \in W$. In the above example we have $\tilde{v} = \text{tst}$. The double leaves basis of $\tilde{A}_w$ is obtained by running over all $\tilde{v} \leq w$ and over all pairs of light leaves for $\tilde{A}_w$ that are associated with that $\tilde{v}$. For each such pair $(\tilde{D}_1, \tilde{D}_2)$ the second component $\tilde{D}_2$ is reflected through a horizontal axis, and the two components are glued together. The resulting diagram is a double leaf, for example
The fundamental result concerning double leaves is the fact that they form an $R$-basis for $\tilde{A}_w$. In fact, a stronger Theorem holds. Let $\Lambda_w := \{v \in W \mid v \leq w\}$, endowed with poset structure via the Bruhat order $\prec$. For $v \in \Lambda_w$, let $\text{Tab}_w(v)$ be the set of light leaves for $\tilde{A}_w$ defining $v$ in the above sense, and for $D_1, D_2 \in \text{Tab}_w(v)$ let $C_{D_1, D_2}^w \in \tilde{A}_w$ be the double leaf obtained by gluing as above. We now have the following Theorem.

**Theorem 3.2.** The triple $(\tilde{A}_w, \text{Tab}_w(v), C)$ defines a cellular basis structure on $\tilde{A}_w$.

**Proof:** See [25] and [11]. 

Following [26], we now introduce $A_w \subseteq \tilde{A}_w$ as follows.

**Definition 3.3.** Let $A_w$ be the span in $\tilde{A}_w$ of all double leaves with empty zone $C$, or equivalently, $A_w$ is the free $R$-module with basis given by the double leaves of empty zone $C$.

Our next Theorem is an analogue of Theorem 3.8 and Corollary 3.9 of [26], although the proofs of parts b) and c) of the Theorem are different from the proofs of the corresponding statements in [26], since the algebras considered in [26] are $\mathbb{C}$-algebras whereas the algebras in the present paper are $R$-algebras. Therefore, in the present article some extra care is necessary since nonzero coefficients of $R$ need not be invertible. Moreover, the arguments in [26] depend on the linear algebra fact that injective linear transformations $f : V \to W$ between vector spaces $V$ and $W$ of the same finite dimension are isomorphisms. The analogous statement is false for free $R$-modules, which is the main reason why the proofs of the present paper are different from the ones in [26].

**Theorem 3.4.** a) The cardinality of double leaves of empty zone $C$ is $\binom{2n}{n}$ and so $A_w$ is free over $R$ of rank $\text{rk } A_w = \text{rk } B_n^{x,y} = \binom{2n}{n}$.

b) $A_w$ is an $R$-algebra. It is the subalgebra of $\tilde{A}_w$ generated by $U_0, U_1, \ldots, U_{n-2}$.

c) The homomorphism $\phi : B_n^{x,y} \to \tilde{A}_w$ from Theorem 3.1 induces an isomorphism $\phi : B_n^{x,y} \to A_w$.

**Proof:** To show a) we first note that the cardinality of the set of double leaves of empty zone $C$ is given by Definition 3.7 in [26] and Theorem 3.8 c) in [26], and so the statements about $A_w$ are a direct consequence of Definition 3.3 and (2.17).

In order to prove b) we define $A'_w$ to be the subalgebra of $\tilde{A}_w$ generated by $U_0, U_1, \ldots, U_{n-2}$ and must show that $A'_w = A_w$. We first show that $A'_w \supseteq A_w$.

Recall the diagram $C_{\lambda} \in B_n^{x,y}$ from (2.22). For $\lambda \geq 0$ we have that

$$\varphi(C_{\lambda}) = U_1U_3 \cdots U_{2k-1} = \begin{array}{c}
\begin{array}{cccc}
& & & \\
1 & 2 & \cdots & n \\
\end{array}
\end{array} \quad (3.13)$$

and for $\lambda < 0$ we have

$$\varphi(C_{\lambda}) = (U_1U_3 \cdots U_{2k-1})U_0(U_2U_4 \cdots U_{2k})U_1U_3 \cdots U_{2k-1} = \begin{array}{c}
\begin{array}{cccc}
& & & \\
1 & 2 & \cdots & n \\
\end{array}
\end{array} \quad (3.14)$$
We also need the diagrams

\[ \varphi(U_i U_{i+2} \cdots U_{i+2k}) = \begin{array}{c|c|c}
\cdots & \cdots & \cdots \\
i & i+2 & n \\
\end{array} \] (3.15)

Now, multiplying together appropriate diagrams of the form (3.13) and of the form (3.14) we deduce that any diagram of the form

\[ \begin{array}{c|c|c}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array} \] (3.16)

belongs to \( A'_w \) where the number of hanging birdcages on the right is at least one. The number of non-hanging birdcages on top of (3.16) is the same as on the bottom, but we need to break this symmetry, that is, we must show that diagrams \( D \) as in (3.16), but with unequal numbers of top and bottom non-hanging birdcages in zone \( A \), also belong to \( A'_w \). Note that the number of top and the number of bottom non-hanging birdcages in zone \( A \) are always of the same parity and so, in order to break this symmetry in zone \( A \), we first give a procedure for splitting any non-hanging and non-degenerate birdcage in zone \( A \) in three non-hanging birdcages, and still stay in \( A'_w \).

If the non-hanging birdcage is the leftmost one, it can easily be split in three parts via multiplication by \( U_0 \), as illustrated below

\[ \begin{array}{c|c|c}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array} = \begin{array}{c|c|c}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array} \] (3.17)

Note that this belongs to \( A'_w \). In the non-hanging birdcage is not the leftmost one, we first notice that multiplication with appropriate \( U_i \)'s has the effect of `moving' a dot from one birdcage to its neighbouring birdcage, as illustrated below in the following two examples.

\[ \begin{array}{c|c|c}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array} = \begin{array}{c|c|c}
\cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
\end{array} \] (3.18)

Using this we can also `move' any non-hanging birdcage to the leftmost position and then multiply it by \( U_0 \), to split it in three birdcages. Next, we use (3.18) to move the birdcages to the desired positions. The result of this belongs to \( A'_w \) and so the symmetry in zone \( A \) has been broken.

In fact, once we have the right number of birdcages in zone \( A \), we can use successive multiplications of the types given in (3.18), to obtain any combination of desired birdcage lengths, as opposed to (3.17) that always produces degenerate birdcages.
Now in (3.16), we must also break the symmetry with respect to birdcage lengths in zone B. But in order to break this symmetry we proceed just as we did in the last step for zone A, applying (3.18) successively, adjusting the lengths of the relevant birdcages until they are as desired.

All in all we have now shown that any diagram of the form described in (3.19) belongs to $A_w'$, where the number and the lengths of the top and bottom birdcages in zone A may differ, as may the lengths of birdcages in zone B.

$$\begin{align*}
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{align*}$$

(3.19)

Finally, to conclude the proof of $A_w' \supseteq A_w$, we observe that the process, illustrated in (3.9), of replacing a degenerate non-hanging full birdcage by a non-hanging non-degenerate full birdcage can be realized as the multiplication on top or bottom with a diagram of the form (3.15). Below we give an example.

$$\begin{align*}
\begin{array} {c}
\protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{array}
\end{align*}
\end{array} = \begin{array} {c}
\protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{array}
\end{array}.

(3.20)

In order to prove the other inclusion $A_w' \subseteq A_w$, we need to change the argument of the corresponding statement in [26] since it, as already mentioned, depends on dimension arguments in linear algebra (over $\mathbb{C}$).

We first observe that $U_i \in A_w$ for $i = 0, 1, \ldots, n - 2$. Hence, to show $A_w' \subseteq A_w$ it is enough to verify that $A_w$ is invariant under left and right multiplication by the $U_i$'s.

Let therefore $D$ be a diagram for $A_w$. We first choose $i > 0$ and proceed to give a description of the effect of multiplying $U_i$ below on $D$, that is we describe $U_iD$ in terms of $D$. Let $l_1, l_2$ and $l_3$ be the arcs in $D$ that has bottom boundary points $i, i + 1$ and $i + 2$ of $D$, respectively. Without loss of generality we may assume that $l_1$ and $l_3$ are red, and that $l_2$ is blue. Recall that an arc $l$ of $D$ is a boundary dot arc if one of its vertices is a dot and the other one is a boundary point.

**Case 1:** This is the case where $l_1$ and $l_2$ have a common vertex in $D$, or, more generally, that $l_1$ and $l_2$ are connected in $D$. We then have that $l_2$ is a boundary dot arc. In now follows from the isotopy relations (3.3) and the relation $U_i^2 = -2U_i$, see Theorem 3.1, that $U_iD = -2D$, since relation (2.36) implies that the scalar $-2$ moves to the left of $D$. Hence we get that $U_iD \in A_w$. Here is an illustration, where we for simplicity leave out the top part of the diagrams.

$$\begin{align*}
\protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{align*}
\end{array} = \begin{array} {c}
\protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{align*} = -2.

(3.21)

**Case 2:** This is the case where $l_2$ is a boundary dot arc, whereas $l_1$ and $l_3$ are the rightmost and leftmost arcs of birdcagecages, respectively. Note that in this situation the bottom boundary points $i$ and $i + 1$ belong to zone $A$ of $D$. We here get that $U_iD = D_1$ where $D_1$ is the diagram obtained from $D$ by concatenating horizontally the two birdcagecages, with an extra dot in the middle. Since $D_1 \in A_w$, this case is also OK. Here is an illustration, once again without the top parts of the diagrams.

$$\begin{align*}
\begin{array} {c}
\protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{array}
\end{array} = \begin{array} {c}
\protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} \\
\{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}} & \{ & \protect{\begin{array} {c}
\protect{\text{zone A}} \\
\protect{\text{zone B}}
\end{array}}
\end{array}
\end{array} = \alpha_t.

(3.22)
Case 3: This is the case where \( l_2 \) has a trivalent vertex, whereas \( l_1 \) and \( l_3 \) are the rightmost and leftmost arcs of birdcagecages, respectively. In this case we get \( U_i D = D_1 \) where \( D_1 \) is obtained from \( D \) by eliminating \( l_2 \) and joining the birdcagecages involving \( l_1 \) and \( l_3 \), with an extra dot in the middle. Once again, we get that \( D_1 \in A_w \), and so this case is also OK. Here is an illustration.

\[
\begin{array}{ccc}
\includegraphics[width=150px]{case3a} & = & \includegraphics[width=150px]{case3b} \\
\includegraphics[width=150px]{case3c} & = & \includegraphics[width=150px]{case3d}
\end{array}
\]  \quad (3.23)

Case 4: This is the case where \( l_1 \) is the leftmost arc of a birdcagecage, whereas \( l_2 \) has a trivalent vertex \( V \). Let \( B_1 \) be the birdcagecage that lies to the southwest of \( V \) and let \( B_2 \) be the birdcagecage that lies to the southeast of \( V \). Then \( U_i D = D_1 \) where \( D_1 \) is obtained from \( D \) by joining \( B_1 \) and \( l_1 \) and adding a boundary dot arc to the left of \( B_1 \). We have that \( D_1 \in A_w \) and so this case is done. Here is an example, without the top parts of the diagrams.

\[
\begin{array}{ccc}
\includegraphics[width=150px]{case4a} & = & \includegraphics[width=150px]{case4b} \\
\includegraphics[width=150px]{case4c} & = & \includegraphics[width=150px]{case4d}
\end{array}
\]  \quad (3.24)

Case 5: This is the identical to case 4 except that both vertices of \( l_2 \) are supposed to be boundary points. Let \( B \) be the birdcagecage that lies below \( l_2 \). Then \( U_i D = D_1 \) where \( D_1 \) is obtained from \( D \) by splitting \( l_2 \) in two dot boundary arcs, and \( B \) is joined with \( l_1 \). Since \( D_1 \in A_w \) we are done in this case, as well. Here is an example.

\[
\begin{array}{ccc}
\includegraphics[width=150px]{case5a} & = & \includegraphics[width=150px]{case5b} \\
\includegraphics[width=150px]{case5c} & = & \includegraphics[width=150px]{case5d}
\end{array}
\]  \quad (3.25)

Case 6: This is the case where a vertex of \( l_2 \) is a top boundary point of \( D \). Then \( l_1 \) and \( l_2 \) are the rightmost and leftmost arcs of two birdcagecages \( B_1 \) and \( B_2 \). We get that \( U_i D = D_1 \) where \( D_1 \) is obtained from \( D \) by joining \( B_1 \) and \( B_2 \) and splitting \( l_2 \) in two dot boundary arcs. Since \( D_1 \in A_w \) we have that this case is OK as well. Here is an example.

\[
\begin{array}{ccc}
\includegraphics[width=150px]{case6a} & = & \includegraphics[width=150px]{case6b} \\
\includegraphics[width=150px]{case6c} & = & \includegraphics[width=150px]{case6d}
\end{array}
\]  \quad (3.26)

Case 7: This is the case where \( l_2 \) is the leftmost arc of a hanging birdcagecage. This case resembles case 6. We have that \( l_1 \) and \( l_2 \) are the rightmost and leftmost arcs of two birdcagecages \( B_1 \) and \( B_2 \). Then we get that \( U_i D = D_1 \) where \( D_1 \) is obtained from \( D \) by joining \( B_1 \) and \( B_2 \) and replacing \( l_2 \) by a bottom dot boundary arc. We have \( D_1 \in A_w \) and so this case is OK as well. Here is an example.

\[
\begin{array}{ccc}
\includegraphics[width=150px]{case7a} & = & \includegraphics[width=150px]{case7b} \\
\includegraphics[width=150px]{case7c} & = & \includegraphics[width=150px]{case7d}
\end{array}
\]  \quad (3.27)

There are a few remaining cases to consider, but they are all small variations of the cases already studied and so we leave them to the reader.
We next consider \( i = 0 \). Let \( B \) be the leftmost bottom birdcage of \( D \) and let \( l \) be its leftmost bottom arc. If \( D \) is a boundary dot arc, we have \( U_0D = \alpha_0 D \in A_w \). Otherwise we have \( U_0D = \alpha_1 D \) where \( D_1 \) is the birdcage obtained from \( D \) by replacing \( l \) by a boundary dot arc, and here we also have \( U_0D \in A_w \). Here is an illustration of this case.

\[
\begin{array}{c}
\text{Diagram 1} \\
= \quad \text{Diagram 2}
\end{array}
\quad \quad (3.28)
\]

Finally, we observe that the description of the right multiplication \( DU_i \) is completely analogous to the description of \( U_i D \), and so we have concluded the proof of b).

We now give an alternative proof of \( A'_w \subseteq A_w \), adapting the proof in [26] and using one of the main results in [11]. Let \( Q \) be the quotient field of \( R \) and let \( QA_w := Q \otimes_R A_w \), \( QA'_w := Q \otimes_R A'_w \) and \( Q \mathbb{B}^x_{n-1} = Q \otimes_R \mathbb{B}^x_{n-1} \). Since \( A_w \) and \( \mathbb{B}^x_{n-1} \) are torsion free \( R \)-modules, in fact even free, we may view \( A_w \) and \( \mathbb{B}^x_{n-1} \) as \( R \)-submodules of \( QA_w \) and \( QA'_{n-1} \), via the map \( D \mapsto 1 \otimes_R D \). Similarly, we may view \( A'_w \) as an \( R \)-submodule of \( QA'_w \), since \( A'_w \) is torsion free, being a submodule of the free \( R \)-module \( \hat{A}_w \).

Now the inclusion \( A_w \subseteq A'_w \) induces an inclusion \( QA_w \subseteq QA'_w \), since \( Q \otimes_R (\cdot) \) is an exact functor, and the surjection \( \varphi : \mathbb{B}^x_{n-1} \rightarrow A'_w \) induces a surjection \( Q \mathbb{B}^x_{n-1} \rightarrow QA'_w \). Combining this with a), we deduce that \( QA_w = QA'_w \) since both are \( \mathbb{Q} \)-vector spaces of the same dimension \( \binom{2n}{n} \).

Let us now show \( A'_w \subseteq A_w \). As in the first proof of \( A'_w \subseteq A_w \), it is for this enough to check that \( U_i D \in A_w \), whenever \( D \) is a light leaves diagram in \( A_w \). Now using \( QA_w = QA'_w \), we find elements \( q_k \in Q \) such that

\[
U_i D = \sum_{D_k \in A_w} q_k D_k
\quad (3.29)
\]

where \( D_k \) runs over the light leaves basis for \( A_w \). On the other hand, as was shown in Theorem 6.11 of [11], the light leaves diagrams \( \{D_i\} \) for \( \hat{A}_w \) form an \( R \)-basis for \( \hat{A}_w \), and so there also exist \( r_k \in R \) such that

\[
U_i D = \sum_{D_l \in \hat{A}_w} a_l D_l
\quad (3.30)
\]

Comparing (3.29) and (3.29) we deduce that \( q_k \in R \), and so \( U_i D \in A_w \), as claimed.

We remark that, in a suitable sense, the alternative proof of b) is equivalent to the first proof, since the arguments in [11], that hold in the setting of a general Coxeter system \((W,S)\), depend on a case-by-case similar to the one carried out in our first proof.

To prove c) we argue essentially as in [26], although a little extra care has to be exercised since the algebras are defined over a commutative ring rather than a field. But by (2.17), a) and b) we have that \( \varphi \) is a surjective homomorphism between free \( R \)-modules of the same finite rank. On the other hand, \( R \) is a commutative ring with 1 and so indeed \( \varphi \) is an isomorphism, as follows from Vasconcelos’ Theorem, see [39]. The proof of the Theorem is finished. \( \square \)

Suppose that \( w = s_{i_1} s_{i_2} \cdots s_{i_n} \). Then we define \( \Lambda_w \subseteq \hat{\Lambda}_w \) as the subset of all ‘tails’ of \( \hat{\Lambda}_w \), that is

\[
\Lambda_w = \{ v \in \hat{\Lambda}_w | v = s_{i_k} s_{i_{k+1}} \cdots s_{i_n} \text{ for } k = 1, \ldots, n \}. \quad (3.31)
\]

Note that \( w \in \Lambda_w \) but \( 1 \notin \Lambda_w \). For example for \( n = 7 \) we have that \( w = stststs \) and so

\[
\Lambda_w = \{ stststs, tststs, ststs, tsts, sts, ts, s \}. \quad (3.32)
\]

Let \( \Lambda'_w \subseteq \hat{\Lambda}_w \) be the subset obtained from \( \Lambda_w \) by deleting the last generator of each element of \( \hat{\Lambda}_w \). Keeping the example \( n = 7 \) we have \( \Lambda'_w = \{ stst, tstst, stst, tst, t, 1 \} \). With this definition we have

\[
\hat{\Lambda}_w = \Lambda_w \cup \Lambda'_w. \quad (3.33)
\]

The relevance of \( \Lambda_w \) comes from the following Theorem.
**Theorem 3.5.** The triple \((\Lambda_w, \text{Tab}_w(v), C)\) defines a cellular basis structure on \(A_w\).

**Proof:** By part b) of Theorem 3.4 we know that \(A_w\) is a subalgebra of \(\hat{A}_w\) and so the theorem follows immediately from Theorem 3.5 and the fact that for \(D_1, D_2 \in \text{Tab}_w(v)\) and \(v \in \hat{A}_w\) we have that \(C^n_{D_1D_2}\) belongs to \(A_w\) if and only if \(v \in A_w\).

By Theorem 3.4 we know that \(\mathbb{B}^x_{n-1} \cong A_w\) and from Theorem 2.6 and Theorem 3.5 we know that both algebras are cellular. It now seems plausible that the corresponding cell modules are isomorphic as well. This is however not automatic since the cellular structure on a cellular algebra is not unique. Our next aim is to show that the cell modules are indeed isomorphic.

For this we first need to establish a poset isomorphism \(\psi : \Lambda_w \cong \Lambda_{\pm(n-1)}\). The sets \(\Lambda_w\) and \(\Lambda_{\pm(n-1)}\) are both of cardinality \(n\) and the respective order relations are both total, so there is in fact a unique choice of \(\psi\). It is given by the following Lemma, where \(l(\cdot)\) is the usual Coxeter length function on \(W\).

**Lemma 3.6.** Let \(\psi : \Lambda_w \rightarrow \Lambda_{\pm(n-1)}\) be the map defined by

\[
\psi(v) = \begin{cases} 
  l(v) - 1 & \text{if } v \text{ begins with } s \\
  -l(v) & \text{if } v \text{ begins with } t.
\end{cases}
\]  

Then \(\psi\) is an isomorphism of posets. We denote by \(\varphi\) the inverse \(\varphi = \psi^{-1} : \Lambda_{\pm(n-1)} \rightarrow \Lambda_w\).

(The different meanings of the symbol \(\varphi\), for example as the algebra isomorphism \(\mathbb{B}^x_{n-1} \cong A_w\), but also as the poset isomorphism \(\Lambda_{\pm(n-1)} \cong \Lambda_w\), should not give rise to confusion). For example, if \(n = 7\) we have

\[
\varphi(6, -6, 4, -4, 2, -2, 0) = (\text{ststst}, \text{ststst}, \text{ststst}, \text{sts}, \text{sts}, \text{ts}, \text{s}).
\]

For \(v \in \Lambda_w\) we now introduce \(C_v \in A_w\) as the double leaf diagram of the form (3.13) or (3.14) that defines \(v \in W\), that is

\[
C_v = \begin{cases} 
  \begin{array}{c}
  \text{v} \\
  \text{v}
  \end{array} & \begin{array}{c}
  \text{v} \\
  \text{v}
  \end{array},
\end{cases}
\]

where the vertical lines below \(\begin{array}{c}
  \text{v} \\
  \text{v}
  \end{array}\) in each diagram of (3.36) define \(v\).

We have that \(C_v = \varphi(C_\lambda)\) where \(\varphi(\lambda) = v\), that is \(C_v\) corresponds to the blob algebra element \(C_\lambda\) defined in (2.22). Let \(A_v\) and \(A_{<v}\) be the cell ideals in \(A_w\) given by

\[
A_v := \{ ac_v b | a, b \in A_w \} \quad \text{and} \quad A_{<v} := \{ ac_v b | a, b \in A_w \text{ and } u < v \}
\]

and define the cell module

\[
\Delta_w(v) := A_w \overline{C_v} \subset A_v / A_{<v}
\]

where \(\overline{C_v} = C_v + A_{<v}\). Let \(D_v\) be the half-diagram corresponding to \(C_v\). Then we have the following Theorem.

**Theorem 3.7.** \(\Delta_w(v)\) is free over \(R\) with basis \(\{ \overline{C_{D,D_v}} | D \in \text{Tab}_w(v) \}\) where \(\overline{C_{D,D_v}} := C_{D,D_v} + A_{<v}\).

**Proof:** This follows from the algorithm given in the proof of Theorem 3.4. \(\square\)

We then finally obtain the main result of this section.
Theorem 3.8. Suppose that \( \varphi(\lambda) = v \). Then the isomorphism \( \varphi : B_{n-1}^\lambda \to A_w \) induces an isomorphism \( \varphi : \Delta_n^B(\lambda) \to \Delta(\lambda) \) of cell modules.

Proof: Since \( \varphi(C_\lambda) = C_v \) we have that \( \varphi \) induces ideal isomorphisms \( A^\lambda \cong A^v \) and \( A^{\lambda \cdot x} \cong A^{\lambda \cdot y} \) and hence \( A^\lambda / A^{\lambda \cdot x} \cong A^v / A^{\lambda \cdot y} \). But from this we deduce that \( \Delta_w(\mu) \) and \( \Delta_n^B(\lambda) \) are isomorphic under \( \varphi \), as claimed. \( \Box \)

For a general cellular algebra \( \mathcal{A} \) over \( k \) with cell modules \( \{ \Delta(\lambda) \mid \lambda \in \Lambda \} \) there is a canonical bilinear form \( \langle \cdot, \cdot \rangle \) on \( \Delta(\lambda) \) that plays an important role for the representation theory of \( \mathcal{A} \). Let \( \Lambda_0 = \{ \lambda \in \Lambda \mid \langle \cdot, \cdot \rangle \neq 0 \} \) and define for \( \lambda \in \Lambda_0 \) the \( \mathcal{A} \)-module \( L(\lambda) := \Delta(\lambda) / \text{rad}(\langle \cdot, \cdot \rangle) \), where \( \text{rad} \) is the radical of \( \langle \cdot, \cdot \rangle \) in the usual sense of bilinear forms. It is an \( \mathcal{A} \)-submodule of \( \Delta(\lambda) \) because \( \langle \cdot, \cdot \rangle \) is \( \mathcal{A} \)-invariant, that is \( \langle xy, z \rangle = \langle x, yz \rangle \) for all \( x \in \mathcal{A} \) and \( y, z \in \Delta(\lambda) \), where \( * \) is the antihomomorphism of \( \mathcal{A} \) given in Definition 2.4. In the case where \( k \) is a field, the \( \mathcal{A} \)-modules in the set \( \{ L(\lambda) \mid \lambda \in \Lambda_0 \} \) are all irreducible, and each \( \mathcal{A} \)-irreducible module occurs exactly once in the set, see Theorem 3.4 in [15].

Let \( \langle \cdot, \cdot \rangle^w \) be the bilinear form on \( \Delta_w(\nu) \) and let \( \langle \cdot, \cdot \rangle^B \) be the bilinear form on \( \Delta_n^B(\lambda) \). Then we have the following Theorem.

Theorem 3.9. \( \langle \cdot, \cdot \rangle^B_{n-1, \lambda} \) and \( \langle \cdot, \cdot \rangle^w \) are equivalent under \( \varphi \), in other words

\[
\langle \varphi(s), \varphi(t) \rangle^w = \langle s, t \rangle^B_{n-1, \lambda} \quad \text{for } s, t \in \Delta_n^B(\lambda)
\]

(3.39)

where \( \varphi(\nu) = v \).

Proof: A bilinear and invariant form \( \langle \cdot, \cdot \rangle \) on \( \Delta_w(\nu) \) corresponds to an \( \mathcal{A}_w \)-homomorphism \( \Delta_n^B(\lambda) \to \Delta_n^B(\lambda) \) where \( \Delta_n^B(\lambda) \) is the dual of \( \Delta_n^B(\lambda) \). For \( Q \) the fraction field of \( R \) and \( \overline{Q} \) its algebraic closure we set \( \Delta_n^B(\lambda) := \Delta_n^B(\lambda) \otimes_R \overline{Q} \). Then \( \Delta_n^B(\lambda) \) is irreducible and so by Schur’s Lemma \( \Delta_n^B(\lambda) \to \Delta_n^B(\lambda) \) is unique up to a scalar. Hence \( \langle \cdot, \cdot \rangle \) is unique up to a scalar \( \mu \), that is

\[
\langle \psi(a), \psi(b) \rangle^B_{n-1, \lambda} = \mu\langle a, b \rangle^w \quad \text{for } a, b \in \Delta_w(\lambda)
\]

(3.40)

But using \( a = b = C_v \), one checks that \( \mu = 1 \) and so the Theorem follows. \( \Box \)

The purpose of the present paper is to study the form \( \langle \cdot, \cdot \rangle^w \). In view of Theorem 3.9, we can instead study the form \( \langle \cdot, \cdot \rangle^B_{n-1, \lambda} \), which turns out to be easier to handle.

4 Restriction of \( \Delta_n^B(\lambda) \) to \( \mathcal{T}_L \)

As already mentioned, there is an embedding \( \mathcal{T}_L \subset B_{n-1}^\lambda \) which at the diagrammatic level is an embedding of Temperley-Lieb diagrams in blob diagrams. This gives rise to a restriction functor \( \text{Res} = \text{Res}_{\mathcal{T}_L}^B \) from \( B_{n-1}^\lambda \)-modules to \( \mathcal{T}_L \)-modules. In this section we study the application of \( \text{Res} \) on \( \Delta_n^B(\lambda) \).

Recall from Theorem 2.7 that the cell module \( \Delta_n^\lambda \) for \( B_{n-1}^\lambda \) (resp. \( \Delta_n^{\mathcal{T}_L}(\lambda) \)) has basis \( \{ C_{s, t, \lambda} \mid s \in \text{Tab}(\lambda) \} \). From now on, if \( \lambda \geq 0 \) we shall identify \( C_{s, t, \lambda} \) with \( s \) so that we consider the basis for \( \Delta_n^\lambda \) to consist of half-diagrams. For example, the basis of the \( B_{n-1}^\lambda \)-module \( \Delta_n^\lambda(1) \) consists of the following half-diagrams

\[
(4.1)
\]

If \( \lambda < 0 \) we still identify \( C_{s, t, \lambda} \) with \( s \), but with the leftmost propagating line marked. For example, the basis of the \( B_{n-1}^\lambda \)-module \( \Delta_n^\lambda(-3) \) consists of the following half-diagrams

\[
(4.2)
\]

Finally, for the \( \mathcal{T}_L \)-module \( \Delta_n^{\mathcal{T}_L}(\lambda) \) we once again identify \( C_{s, t, \lambda} \) with \( s \). Thus, for example the basis of the \( \mathcal{T}_L \)-module \( \Delta_0^{\mathcal{T}_L}(1) \) consists of the half-diagrams that appear in the first row of (4.1).
In terms of these identifications the action of $D \in \mathbb{B}_n^{y}$ (resp. $D \in T\mathbb{L}_n$) on $D_1 \in \Delta_n^B(\lambda)$ (resp. $D_1 \in \Delta_n^{TL}(\lambda)$) is given by concatenation with $D_1$ on top of $D$, followed by the same reduction process of extra blobs and internal loops, marked or unmarked, that we gave for $\mathbb{B}_n^{y}$ (resp. $T\mathbb{L}_n$) itself. If the result of this does not belong to the span of half-diagrams for $\Delta_n^B(\lambda)$ (resp. $\Delta_n^{TL}(\lambda)$), we have $DD_1 = 0$.

Suppose that $\lambda \in \Lambda_{\pm n}$ and set $k := \frac{n - |\lambda|}{2}$. We then define a filtration $0 = F^{-1}(\lambda) \subset F^0(\lambda) \subset \cdots \subset F^k(\lambda) = \Delta_n^B(\lambda)$ of $\Delta_n^B(\lambda)$ via

$$F^i(\lambda) := \begin{cases} \text{span}_R\{s \in \text{Tab}(\lambda) \text{ has } i \text{ or less blobs}\} & \text{if } \lambda \geq 0 \\ \text{span}_R\{s \in \text{Tab}(\lambda) \text{ has } i + 1 \text{ or less blobs}\} & \text{if } \lambda < 0. \end{cases} \quad (4.3)$$

For example, for $\lambda$ as in (4.1) we have that $F^0(\lambda)$ is the span of the diagrams of the first row, $F^1(\lambda)$ is the span of the diagrams of the first two rows and $F^2(\lambda) = \Delta_n^B(1)$.

The following result is the analogue of Lemma 8.2 from [9].

**Lemma 4.1.**

a) $F^i(\lambda)$ is a $T\mathbb{L}_n$-submodule of $\text{Res}\Delta_n^B(\lambda)$.

b) There is a homomorphism of $T\mathbb{L}_n$-modules $\pi_i : F^i(\lambda) \to \Delta_n^{TL}(|\lambda| + 2i)$ that induces an isomorphism

$$\pi_i : F^i(\lambda)/F^{i-1}(\lambda) \cong \Delta_n^{TL}(|\lambda| + 2i). \quad (4.4)$$

**Proof:** a) follows from the fact that the action of $T\mathbb{L}_n$ does not produce new blobs. To show b), we use the map $\pi_i : F^i(\lambda) \to \Delta_n^{TL}(|\lambda| + 2i)$ that transforms a marked southern arc to two propagating lines, and removes the mark on any propagating line. For example, for $\Delta_n^B(1)$ we have that $\pi_1$ transforms the diagrams of the second row of (4.1) to the following diagrams

$$\bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc \bigcirc. \quad (4.5)$$

As in [9], one readily checks that $\pi_i$ is a homomorphism of $T\mathbb{L}_n$-modules, that induces an isomorphism $\pi_i : F^i(\lambda)/F^{i-1}(\lambda) \cong \Delta_n^{TL}(|\lambda| + 2i)$. □

Our next goal is to construct sections for the $\pi_i$’s from Lemma 4.1. For this we need to recall the Jones-Wenzl idempotent $JW_n$ for $T\mathbb{L}_n$, see [23] and [40]. It is defined as the unique element $JW_n \in T\mathbb{L}_n$ that satisfies the conditions

$$\text{coef}_1(JW_n) = 1 \text{ and } U_iJW_n = JW_nU_i = 0 \text{ for all } i = 1, 2, \ldots, n - 1 \quad (4.6)$$

where $\text{coef}_1(JW_n)$ denotes the coefficient of 1 when $JW_n$ is expanded in the diagram basis for $T\mathbb{L}_n$. We use the standard rectangle notation to indicate $JW_n$, as follows

$$JW_n = \begin{array}{c}
\begin{array}{c}
\ldots
\end{array}
\begin{array}{c}
\ldots
\end{array}
\begin{array}{c}
\ldots
\end{array}
\begin{array}{c}
\text{JW}_n
\end{array}
\begin{array}{c}
\ldots
\end{array}
\begin{array}{c}
\ldots
\end{array}
\begin{array}{c}
\ldots
\end{array}
\end{array} \in T\mathbb{L}_n. \quad (4.7)$$

For example, we have that

$$\text{JW}_2 = \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\end{array} + \frac{1}{2} \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\end{array} \quad (4.8)$$

and

$$\text{JW}_3 = \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\end{array} + \frac{2}{3} \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\begin{array}{c}
\bigcirc
\end{array}
\end{array} + \frac{2}{3} \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\bigcirc
\end{array} + \frac{1}{3} \begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\bigcirc
\end{array} + \frac{1}{3} \begin{array}{c}
\bigcirc
\bigcirc
\end{array} \quad (4.9)$$

Apart from special cases, see for example Corollary 3.7 of [16], there is no closed formula for the coefficient of a diagram in $JW_n$, but there are several, equivalent, recursive formulas that can be used to calculate $JW_n$, for
In the next section we shall explain another recursive formula for $JW_n$ that turns out to be useful for our key calculations.

Important properties of the $JW_n$'s are their idempotency, as already mentioned, and their invariance under vertical reflection as well as horizontal reflection, that is $*$. Furthermore, they satisfy the following absorption property

$$JW_n JW_m = JW_n$$  \hspace{1cm} (4.10)

We now return to the $B_{x,y}$-module $\Delta_{\lambda}^{R}$ and its filtration $\{F_i(\lambda)\}$. For $i = 0, \ldots, k$ we define $e_\lambda^i \in F_i(\lambda)$ as the following element

$$e_\lambda^i = \begin{cases} \phantom{JW} & \text{if } \lambda \geq 0 \\ \phantom{\text{JW}} & \text{if } \lambda < 0 \end{cases} \hspace{1cm} (4.11)$$

that is, the number of blobs on $e_\lambda^i$ is $i$ if $\lambda \geq 0$ and $i + 1$ if $\lambda < 0$. In general, for $D$ any Temperley-Lieb diagram, we define $1^j D$ as the left concatenation of $j$ vertical lines on $D$, and we extend this definition linearly to the Temperley-Lieb algebra itself. For example, for $j = n - |\lambda| - 2i$ we have $1^j JW_{|\lambda|+2i} \in \mathbb{T}_n$ which we depict as follows

$$1^j JW_{|\lambda|+2i} = \begin{bmatrix} \phantom{JW} & \cdots & \phantom{JW} \\ \phantom{\text{JW}} & \cdots & \phantom{\text{JW}} \end{bmatrix} \hspace{1cm} (4.13)$$

With this choice of $j$ we define elements $f_\lambda^i \in F^i(\lambda)$ as follows

$$f_\lambda^i := (1^j JW_{|\lambda|+2i}) e_\lambda^i = \begin{cases} \phantom{JW} & \text{if } \lambda \geq 0 \\ \phantom{\text{JW}} & \text{if } \lambda < 0 \end{cases} \hspace{1cm} (4.14)$$

**Lemma 4.2.** Let $\pi_i : F^i(\lambda) \to \Delta_{\lambda + 2i}^{TL}(\lambda)$ be the homomorphism from Lemma 4.1. We have that

$$\pi_i(e_\lambda^i) = \pi_i(f_\lambda^i). \hspace{1cm} (4.15)$$
Proof: We show that if \( D \neq 1 \) is any diagram appearing in the expansion of \( J W_{\lambda\mid+2i} \) then \((1^j D)e_i^\lambda \in F^{j-1}(\lambda)\), where \( j = n - |\lambda| - 2i \), from which the proof of the Lemma follows since \( \ker \pi_i = F^{j-1}(\lambda) \) and \( \text{coef}_1(J W_{\lambda\mid+2i}) = 1 \).

To prove this claim we first consider \( \lambda \geq 0 \). Any \( D \neq 1 \) in the expansion of \( J W_{\lambda\mid+2i} \) contains at least one arc connecting two northern points. If that arc only involves blobbed arcs in (4.14), illustrated with red below, or only involves vertical lines in (4.14), illustrated with blue below, then \((1^j D)e_i^\lambda \) has strictly less blobs than \( e_i^\lambda \) or has strictly less propagating lines than \( e_i^\lambda \), proving the claim in these cases.

\[
\text{(4.16)}
\]

If the arc involves both the blobbed arcs in (4.14) and the vertical lines in (4.14), then either we will be in the previous case or there will be a, possibly different, arc that involves the last blobbed arc and the first vertical line in (4.14), illustrated with blue below. But then \((1^j D)e_i^\lambda \) has a vertical blobbed line and is zero in \( \Delta_n^B(\lambda) \), finishing the proof of the case \( \lambda \geq 0 \).

\[
\text{(4.17)}
\]

The case \( \lambda < 0 \) is shown in a similar way. \( \Box \)

We now define the \( TL_n \)-module \( S_i(\lambda) \) via

\[
S_i(\lambda) := TL_n f_i^\lambda \subseteq F^i(\lambda).
\]

(4.18)

Recall the bilinear form \( \langle \cdot, \cdot \rangle_{n,\lambda}^B \) on \( \Delta_n^B(\lambda) \). In terms of half-diagrams \( D, D_1 \) for \( \Delta_n^B(\lambda) \), we have that \( \langle D, D_1 \rangle_{n,\lambda}^B \) is given by expanding \( D^*D_1 \) in terms of the diagram basis for \( B_{n,\lambda}^B \) and taking the coefficient of \( 1 \) if \( \lambda > 0 \), the coefficient of \( 0 \) if \( \lambda = 0 \) and the coefficient of \( U_0 \) if \( \lambda < 0 \). For example, for \( \langle \cdot, \cdot \rangle_{5,1}^B \) we have

\[
\left\langle \begin{array}{ccc}
\otimes \\
\otimes \\
\otimes \\
\end{array} \right\rangle_{5,1} = \text{coef}_1 \left( \begin{array}{c}
\bigcirc \\
\bigcirc \\
\end{array} \right) = -2xy.
\]

(4.19)

There is a similar description of the bilinear form \( \langle \cdot, \cdot \rangle_{n,\lambda}^{TL} \) on \( \Delta_n^{TL}(\lambda) \). With this notation we can now prove the following Theorem.

**Theorem 4.3.**

a) **We have** \((e_j^\lambda)^*TL_n f_i^\lambda = 0 \) for \( j < i \).

b) **We have** \( \langle F^{i-1}(\lambda), S_i(\lambda) \rangle_{n,\lambda}^B = 0 \).

c) **We have** \( S_i(\lambda) \cap F^{i-1}(\lambda) = 0 \).

d) **The** \( TL_n \)-modules \( S_i(\lambda) \) and \( \Delta_n^{TL}(\lambda\mid+2i) \) are isomorphic.

**Proof:** To show a) we must check that the following diagram is zero for every diagram \( D \) for \( TL_n \).

\[
\text{(4.20)}
\]

Let \( a := n - |\lambda| - 2i + 1 \) and consider all possible cases for the line \( L \) leaving the \( a \)'th northern point of \( D \), indicated with blue in (4.20).
If $L$ pairs $a$ with a northern point of $D$ which is located to the right of $a$, then $L$ is a northern arc and so we get immediately by (4.6) that (4.20) is zero. If $L$ pairs $a$ with a southern point of $D$ which is located strictly to the right of $a$, then the area to the right of $L$ has strictly more northern than southern points and so there will be a northern arc to the right of $a$. We then conclude once again by (4.6) that (4.20) is zero. This is the situation indicated in the figure below.

If $L$ pairs $a$ with a southern point which is located either directly below or to the left of $a$, then it will be a left endpoint of one of the southern arcs of $D$, since the parities of the numbers of northern and southern points of $D$ to the left of $L$ must be the same. If the right endpoint of that arc is connected to a northern point of $D$, then once again by (4.6) we get that (4.20) is zero, so let us assume that it is connected to a southern point of $D$ via a line $A_1$. That southern point must be a left endpoint of an arc below $D$ since otherwise the number of southern points below $A_1$ would be odd. If the arc is unmarked and its right endpoint is joined to a northern point of $D$, we get once again by (4.6) that (4.20) is zero, so let us suppose that it is joined to another southern point of $D$ via a line $A_2$. Repeating the previous argument, the right endpoint of $A_2$ must be the left endpoint of an arc whose right endpoint, in case the arc is unmarked, is connected by a line $A_3$ to another southern point of $D$. Repeating this argument, we produce a series of southern lines $A_1, A_2, \ldots, A_k$, that finally ends up in either an endpoint of one of the blobbed southern arcs below $D$, or in an endpoint of one of the vertical lines below $D$. But since $j < i$ we then conclude that not all northern points of $D$ to the right of $a$ can be endpoints of propagating lines, and so we conclude by (4.6) that (4.20) is zero. Below we indicate the argument.

Finally, if $L$ pairs $a$ with a northern point of $D$ to the left of $a$, the argument is essentially the same as in the previous case. We indicate it as follows.

To show $b)$, we first observe that, as a $\mathbb{T}_n\mathbb{L}_n$-module, $F^{-1}(\lambda)$ is generated by $\{e_j^\lambda \mid j < i\}$, as follows from Lemma 4.1. In view of this, $b)$ follows from $a)$ and the definition of $\langle \cdot, \cdot \rangle^B_{n,\lambda}$.

We next show $c)$. Suppose that there is $s \in S_i(\lambda) \cap F^{-1}(\lambda) \setminus \{0\}$. To get the desired contradiction, we first claim that the restriction of $\langle \cdot, \cdot \rangle^B_{n,\lambda}$ to $F^{-1}(\lambda)$ is non-degenerate. Indeed, any diagram in $F^{-1}(\lambda)$ can be viewed as a diagram for $\Delta^{\mathbb{T}_n\mathbb{L}_n}(\lambda)$, decorated with certain blobs, see for example (4.1) and (4.2). For $D$ a diagram for $F^{-1}(\lambda)$ we denote by $\mathbb{T}_n\mathbb{L}(D)$ the associated diagram for $\Delta^{\mathbb{T}_n\mathbb{L}_n}(\lambda)$, obtained by removing the blobs. Then via the
specialization $x = 1, y = -2$ given in (2.12), we have that
\[
\left((D, D_1)^{\mathbb{B}}_{n, \lambda}\right)_{x=1, y=-2} = \langle \text{TL}(D), \text{TL}(D_1) \rangle^{\mathbb{TL}}_{n, |\lambda|}
\] (4.24)
as one checks from the definitions. Since $\langle \cdot, \cdot \rangle^{\mathbb{TL}}_{n, |\lambda|}$ is non-degenerate, we now deduce from (4.24) that also the restriction of $\langle \cdot, \cdot \rangle^{\mathbb{B}_{n, \lambda}}$ to $\mathcal{F}^i(\lambda)$ is non-degenerate, as claimed. Hence, for $s \in S_i(\lambda) \cap \mathcal{F}^{i-1}(\lambda) \setminus \{0\}$ there exists an $s_1 \in \mathcal{F}^{i-1}(\lambda)$ such that $(s, s_1)^{\mathbb{B}_{n, \lambda}} \neq 0$, which is in contradiction with b). This proves c).

To show d) we consider the composition
\[
f_i : S_i(\lambda) \subseteq \mathcal{F}^i(\lambda) \longrightarrow \mathcal{F}^i(\lambda)/\mathcal{F}^{i-1}(\lambda) \cong \Delta^{\mathbb{TL}}_n(|\lambda| + 2i)
\] (4.25)
where the last isomorphism is given in Lemma 4.1. In view of Lemma 4.2, we get that $f_i$ is surjective. On the other hand, the kernel of $f_i$ is $S_i(\lambda) \cap \mathcal{F}^{i-1}(\lambda)$ which is zero by c), and so $f_i$ is also injective. The Theorem is proved.

\[\square\]

**Corollary 4.4.** Set as before $k := \frac{n - |\lambda|}{2}$. Then, with respect to $\langle \cdot, \cdot \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda}$, there is an orthogonal direct sum decomposition of $\text{Res} \Delta^{\mathbb{B}_{n, \lambda}}_n(\lambda)$, as follows
\[
\text{Res} \Delta^{\mathbb{B}_{n, \lambda}}_n(\lambda) = S_1(\lambda) \oplus \ldots \oplus S_k(\lambda).
\] (4.26)

*Proof:* Combining Lemma 4.2 with c) of Theorem 4.3, we get that
\[
\text{Res} \mathcal{F}^i(\lambda) = S_i(\lambda) \oplus \mathcal{F}^{i-1}(\lambda)
\] (4.27)
and the Corollary follows by induction on this formula. \[\square\]

The Corollary allows us to diagonalize $\langle \cdot, \cdot \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda}$ as follows. The restriction of $\langle \cdot, \cdot \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda}$ to $S_i(\lambda)$ defines a $\mathbb{TL}_{n}$-invariant bilinear form on $S_i(\lambda)$ which by Schur’s Lemma, arguing as in the proof of Theorem 3.9, must be equivalent to $\langle \cdot, \cdot \rangle^{\mathbb{TL}_{n, |\lambda|}+2i}_{n, |\lambda|+2i}$, that is for $s, t \in S_i(\lambda)$ we have that
\[
\langle s, t \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda} = c_i(x, y) \langle f_i(s), f_i(t) \rangle^{\mathbb{TL}_{n, |\lambda|}+2i}_{n, |\lambda|+2i}
\] (4.28)
for some $c_i(x, y) \in \mathbb{R}$. On the other hand, by our choice of ground field $\mathbb{C}$, we have that $\langle \cdot, \cdot \rangle^{\mathbb{TL}_{n, |\lambda|}+2i}_{n, |\lambda|+2i}$ is equivalent to the standard bilinear form on $\Delta^{\mathbb{TL}}_n(|\lambda| + 2i)$ given by the identity matrix. In other words, there is an $R$-basis $f_1, f_2, \ldots, f_m$ for $\Delta^{\mathbb{TL}}_n(|\lambda| + 2i)$ such that
\[
\langle f_i, f_j \rangle^{\mathbb{TL}_{n, |\lambda|}+2i}_{n, |\lambda|+2i} = \delta_{ki}
\] (4.29)
where $\delta_{ki}$ is the Kronecker delta. Combining (4.28) and (4.29) we then conclude that also $\langle \cdot, \cdot \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda}$ can be diagonalized. To be precise, there is an $R$-basis $B = \{b_1, \ldots, b_m\}$ for $\Delta^{\mathbb{B}_{n, \lambda}}_n(\lambda)$ such that the matrix $M_{n, \lambda}^{\mathbb{B}_{n, \lambda}}(x, y) = M_{n, \lambda}^{\mathbb{B}_{n, \lambda}} := \left(\langle b_i, b_j \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda}\right)_{i, j=1, \ldots, m}$ for $\langle \cdot, \cdot \rangle^{\mathbb{B}_{n, \lambda}}_{n, \lambda}$ has the following form
\[
M_{n, \lambda}^{\mathbb{B}_{n, \lambda}} = \begin{pmatrix}
c_0(x, y)I_{d_0} & 0 & \cdots & 0 \\
0 & c_1(x, y)I_{d_1} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & c_k(x, y)I_{d_k}
\end{pmatrix}
\] (4.30)
where $d_i = \text{dim} \Delta^{\mathbb{TL}_{n, |\lambda|}+2i}_n$ and $I_{d_i}$ is the $d_i \times d_i$-identity matrix, whereas the 0’s are 0-matrices of appropriate dimensions.

5 Determination of $c_i(x, y)$

The purpose of this section is to calculate $c_i(x, y)$’s from (4.28) and (4.30). This is the key calculation of the paper. Quite surprisingly, the result turns out to be given in terms of nice expressions involving the positive roots for $W$.

For $i = 1, 2, \ldots$, we define $\alpha_{x, i}, \alpha_{y, i} \in \mathfrak{h}^*$ via
\[
\alpha_{x, i} := ix + (i - 1)y = i\alpha_s + (i - 1)\alpha_t, \quad \alpha_{y, i} := iy + (i - 1)x = i\alpha_t + (i - 1)\alpha_s.
\] (5.1)
By our choice of realization for $W$, the following formulas for $\alpha_{x,i}$ and $\alpha_{y,i}$ hold, as one proves by induction using (2.29).

\[ \alpha_{x,2i-1} = (st)^i \alpha_s, \quad \alpha_{y,2i} = t(st)^i \alpha_s, \quad \alpha_{y,2i-1} = (ts)^i \alpha_t, \quad \alpha_{x,2i} = s(tst)^i \alpha_t. \] (5.2)

For a general Coxeter system $(W,S)$ with realization $\mathfrak{h}$ we define a root $\beta \in \mathfrak{h}^*$ to be any element of the form $\beta = w\alpha$ where $w \in W$ and $\alpha \in \mathfrak{h}^*$ is a simple root. The formulas in (5.2) show that $\{\pm \alpha_{x,i}, \pm \alpha_{y,i} | i = 1, 2, \ldots\}$ is the set of all roots for $W$, with $\alpha_{x,1} = x = \alpha_s$ and $\alpha_{y,1} = y = \alpha_t$ being the simple roots.

With this notation we can now formulate the following Theorem.

**Theorem 5.1.** a) *Suppose that $\lambda \geq 0$ and that $f_1^\lambda$ is as in (4.14) but with $j = 0$, that is $f_1^\lambda = f_2^\lambda$ where*

\[ f_2^\lambda = \begin{array}{c}
1 \\
2 \\
\vdots \\
k \\
1 \\
2 \\
\vdots \\
n \\
JW_n \\
\end{array} \] (5.3)

Then we have that

\[ \langle f_1^\lambda, f_2^\lambda \rangle_{n,\lambda} = \frac{1}{(k)} \langle \alpha_{x,\lambda+2} \alpha_{x,\lambda+3} \cdots \alpha_{x,\lambda+k+1} \rangle \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,k} \] (5.4)

where $n = 2k + \lambda$. (Note that the right hand side of (5.4) contains $k$ factors $\alpha_{x,l}$ for consecutive $l$’s, and also $k$ factors $\alpha_{y,l}$, for consecutive $l$’s. For $k = 0$, it is set equal to 1. Note also that $\binom{n}{k} = \dim \Delta^B_n(\lambda)$).

b) *Let $c_i(x,y)$ be as in (4.28) and (4.30). Then up to multiplication by a nonzero scalar in $\mathbb{C}$, we have that*

\[ c_i(x,y) = (\alpha_{x,\lambda+2} \alpha_{x,\lambda+3} \cdots \alpha_{x,\lambda+i+1}) \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,i} \] (5.5)

(where, as before, the product is set equal to 1 if $i = 0$).

**Proof:** We first prove a). For simplicity, we set $\beta_{k,\lambda} := \langle f_1^\lambda, f_2^\lambda \rangle_{n,\lambda}$ and must therefore show that $\beta_{k,\lambda}$ satisfies the formula given in (5.4). By definition $\beta_{k,\lambda}$ is the coefficient of 1, or the coefficient of 0, if $\lambda > 0$ or if $\lambda = 0$, of the following diagram

\[ \begin{array}{c}
1 \\
2 \\
\vdots \\
k \\
1 \\
2 \\
\vdots \\
n \\
JW_n \\
\end{array} \] (5.6)

For example, in view of (4.8) we have that

\[ \beta_{1,0} = \begin{array}{c}
JW_2 \\
\end{array} = \begin{array}{c}
\frac{1}{2} \\
\end{array} + \begin{array}{c}
\frac{1}{2} \end{array} = xy + \frac{1}{2} y^2 = \frac{1}{2} \alpha_{x,2} \alpha_{y,1}. \] (5.7)

We claim that $\beta_{k,\lambda}$ satisfies the following ‘deformation’ of the Pascal triangle recursive formula

\[ \beta_{1,0} = \frac{1}{2} \alpha_{x,2} \alpha_{y,1}, \quad \beta_{0,1} = 1, \quad \beta_{k,\lambda} = \beta_{k,\lambda-1} + \frac{k^2 (\alpha_{y,k})^2}{n(n-1)} \beta_{k-1,\lambda}. \] (5.8)

From this the proof of the formula (5.4) in a) follows by induction on $N := k + \lambda$ as follows. The basis of the induction $N = 1$ is immediate from (5.7) and the definitions, so let us assume that (5.4) holds for $N - 1$ and check
it for $N$, using (5.8). We get
\[
\beta_{k,\lambda} = \frac{1}{\binom{n-1}{k}} (\alpha_{x,\lambda+1} \alpha_{x,\lambda+2} \cdots \alpha_{x,\lambda+k}) \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,k}
+ \frac{k^2(\alpha_{y,k})^2}{n(n-1)\binom{n-2}{k-1}} (\alpha_{x,\lambda+2} \alpha_{x,\lambda+3} \cdots \alpha_{x,\lambda+k}) \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,k-1}
\]
\[
= \left( \frac{\alpha_{x,\lambda+1}}{\binom{n-1}{k}} + \frac{k^2\alpha_{y,k}}{n(n-1)\binom{n-2}{k-1}} \right) (\alpha_{x,\lambda+2} \cdots \alpha_{x,\lambda+k}) \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,k}
= \frac{1}{(n-k)(\binom{n}{k})} (n\alpha_{x,\lambda+1} + k\alpha_{y,k}) (\alpha_{x,\lambda+2} \cdots \alpha_{x,\lambda+k}) \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,k}.
\]

On the other hand, using $n = 2k + \lambda$ we get that
\[
n\alpha_{x,\lambda+1} + k\alpha_{y,k} = (2k + \lambda)\alpha_{x,\lambda+1} + k\alpha_{y,k}
= (2k + \lambda) (\lambda + 1) x + \lambda y + k (ky + (k-1)x)
= (\lambda + k) ((\lambda + k + 1)x + (\lambda + k)y) = (\lambda + k)\alpha_{x,\lambda+k+1}
= (n-k)\alpha_{x,\lambda+k+1}.
\]

Inserting this in the last expression of (5.9) we get that
\[
\beta_{k,\lambda} = \frac{1}{\binom{n}{k}} (\alpha_{x,\lambda+2} \alpha_{x,\lambda+3} \cdots \alpha_{x,\lambda+k+1}) \alpha_{y,1} \alpha_{y,2} \cdots \alpha_{y,k}
\]
proving the inductive step.

In order to prove the claim (5.8) we first introduce the following diagrammatic notation for $\beta_{k,\lambda}$
\[
\beta_{k,\lambda} = \begin{array}{c}
\begin{array}{cccccccccc}
1 & \cdots & 1 & k & 1 & \cdots & \lambda & \cdots & \cdots & 1
\end{array}
\end{array}
\]
\[
(5.12)
\]
where the numbers above the diagram indicate the cardinalities of the arcs and the vertical lines. We next recall the following recursive formula for calculating $JW_{n}$, that was already alluded to above
\[
JW_{n} = \begin{array}{c}
\begin{array}{cccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
JW_{n-1} & + & \sum_{j=1}^{n-1} \frac{j}{n}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
\cdots & j & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\end{array}
\]
\[
(5.13)
\]
where the number $j$ indicates the position of the arc. As a matter of fact, (5.13) follows from by repeated applications of (4.10). Let us use it to show the recursive formula (5.8) for $\beta_{k,\lambda}$.

Concatenating on top and on bottom with the blobbed arcs, the first term of (5.13) becomes
\[
\begin{array}{c}
\begin{array}{cccccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{cccccccccc}
JW_{n-1} & = & \beta_{k,\lambda-1}.
\end{array}
\end{array}
\]
\[
(5.14)
\]
We next consider the contribution to $\beta_{k,\lambda}$ from the terms of the sum in (5.13). We first observe that concatenating on top and on bottom with the blobbed arcs, only the terms in the sum in (5.13) where $j \leq 2k$ can contribute to
\( \beta_{k,\lambda} \) since otherwise the concatenation has the form

\[
\begin{array}{c}
\text{JW}_{n-1} \\
\end{array}
\]

\[ (5.15) \]

in which the \( j \)’th northern point is connected to another northern point.

We next consider the contributions for \( j = 1, 3, \ldots, 2k - 1 \). Apart from the coefficients, they are of the form indicated below (in the cases \( j = 1 \) and \( j = 3 \)).

\[
\begin{array}{c}
\text{JW}_{n-1} \\
\end{array}
\]

\[ (5.16) \]

and are in fact all equal to \( y_\beta \beta'_{k,\lambda} \) where \( \beta'_{k,\lambda} \) is the diagram

\[
\begin{array}{c}
\text{JW}_{n-1} \\
\end{array}
\]

\[ (5.17) \]

In these diagrams, the dashed lines once again refer to the summands of the Jones-Wenzl elements where the points are connected as indicated.

We then consider the contributions for \( j = 2, 4, \ldots, 2k - 2 \). They are all of the form indicated below (for \( j = 2 \) and \( j = 4 \))

\[
\begin{array}{c}
\text{JW}_{n-1} \\
\end{array}
\]

\[ (5.19) \]
and are all equal to $x \beta'_{k,\lambda}$ where $\beta'_{k,\lambda}$ is as before in (5.18). Finally, for $j = 2k$ there is no contribution since the corresponding diagram is as follows

Summing up and taking into account the coefficients $\frac{j}{n}$ appearing in (5.13) we get from all this that

$$
\beta_{k,\lambda} = \beta_{k,\lambda-1} + k \frac{(k-1)x + ky}{n} \beta'_{k,\lambda} = \beta_{k,\lambda-1} + \frac{k}{n} \alpha_{y,k} \beta'_{k,\lambda}.
$$

We are therefore faced with the problem of calculating $\beta'_{k,\lambda}$, that is

For this we first observe that by the symmetry properties of the Jones-Wenzl idempotents, we have that $\beta'_{k,\lambda}$ is equal to

We now expand $\mathbf{JW}_{n-1}$ in (5.24) using (5.13). The first term of (5.13) does not contribute to (5.24) so let us consider the contribution of the $j$'th term of the sum of (5.13), where arguing as in (5.15) we see that only $j \leq 2k$ can contribute. Once again, there is a dependency on the parity of $j$. If $j = 1, 3, \ldots, 2k - 1$ the contributions are of the form indicated below (for $j = 1, 3$ and disregarding the coefficients)
and are in fact all equal to \( y\beta_{k-1,\lambda} \). Similarly, for \( j = 2, 4, \ldots, 2k - 2 \) we get contributions of the form

\[
\beta'_{k,\lambda} = \frac{k}{n-1} \alpha_y k \beta_{k-1,\lambda}.
\]  

(5.29)

Combining (5.29) and (5.22) we arrive at the promised recursive formula (5.8) for the \( \beta'_{k,\lambda} \)’s. This proves a). The proof of b) is immediate from a) and the definitions. ■

Using Theorem 5.1 we can now calculate the matrix \( M_{5,1}^\beta \) for \( \langle \cdot, \cdot \rangle^\beta_{n,\lambda} \), see (4.30). We illustrate it on \( M_{5,1}^\beta \). Recall that the diagram basis for \( \Delta^\beta_{5}(1) \) is given in (4.1) and so we get from the Theorem that

\[
M_{5,1}^\beta = \begin{pmatrix}
I_5 & 0 & 0 \\
0 & \alpha_{x,3} \alpha_{y,1} I_4 & 0 \\
0 & \alpha_{x,3} \alpha_{x,4} \alpha_{y,1} \alpha_{y,2} I_1 & 0
\end{pmatrix}
\]  

(5.30)

where the 0’s are 0-matrices of appropriate dimensions.

We now consider the situation where \( \lambda < 0 \). We have the following Theorem.

**Theorem 5.2.** a) Suppose that \( \lambda < 0 \) and that \( f^\lambda_i \) is as in (4.14) but with \( j = 0 \), that is \( f^\lambda_i = f^\lambda_k \) where

\[
f^\lambda_k =
\begin{pmatrix}
1 & 2 \\
\vdots & \ddots
\end{pmatrix}
\]  

(5.31)
Then we have that

$$\left\langle f^k, f^\lambda \right\rangle_{n, \lambda} = \frac{1}{n!} (\alpha_{x,1} \alpha_{x,2} \cdots \alpha_{x,k+1}) \alpha_{y,1+|\lambda|} \alpha_{y,2+|\lambda|} \cdots \alpha_{y,k+|\lambda|}$$  

(5.32)

where \( n = 2k + |\lambda| \). \( \text{(Note that the right hand side of (5.4) contains } k + 1 \text{ factors } \alpha_{x,i} \text{ for consecutive } i \text{'s, but } k \text{ factors } \alpha_{y,i} \text{, for consecutive } i \text{'s. For } k = 0, \text{ it is set equal to } \alpha_{x,1} = x. \text{ Note also that } \binom{n}{k} = \dim \Delta^B_n(\lambda). \)\)

b) Let \( c_i(x, y) \) be as in (4.28) and (4.30). Then up to multiplication by a nonzero scalar in \( \mathbb{C} \), we have that

$$c_i(x, y) = (\alpha_{x,1} \alpha_{x,2} \cdots \alpha_{x,i+1}) \alpha_{y,1+|\lambda|} \alpha_{y,2+|\lambda|} \cdots \alpha_{y,i+|\lambda|}$$  

(5.33)

(where, as before, the product is set equal to \( x \) if \( i = 0 \)).

\[ \text{Proof: } \ \text{The proof is essentially the same as the proof of Theorem 5.1. We leave the details to the reader.} \]

Let us illustrate Theorem 5.2 on \( M^B_{5,-1} \) and \( M^B_{5,-3} \). Recall that the diagram basis for \( \Delta^B_5(-1) \) is obtained from (4.1) by marking the leftmost propagating line of each diagram, whereas the diagram basis for \( \Delta^B_5(-3) \) is given in (4.2). We have

$$M^B_{5,-1} = \begin{pmatrix} \alpha_{x,1} I_5 \\ 0 & \alpha_{x,2} \alpha_{y,2} I_4 \\ 0 & 0 \end{pmatrix}, \quad M^B_{5,-3} = \begin{pmatrix} \alpha_{x,1} I_4 \\ 0 & \alpha_{x,2} \alpha_{y,3} I_1 \end{pmatrix}.$$  

(5.34)

6 Characterization of \( c_i(x, y) \) in Terms of Bruhat Order on \( W \)

We saw in the Theorems 5.1 and 5.2 that \( c_i(x, y) \) has a factorization in terms of roots for \( W \). In this section we describe the reflections in \( W \) that correspond to these roots. It turns out that these reflections can be described nicely in terms of the Bruhat order on \( W \).

Let \( \beta = w\alpha \) be a root for \( W \) where \( \alpha \) is a simple root. Then we define the reflection \( s_\beta \in W \) associated with \( \beta \) via the formula

$$s_\beta := ws_\alpha w^{-1}$$  

(6.1)

where \( s_\alpha \in S \) is the generator associated with \( \alpha \). It is shown in section 5.7 of [21] that \( s_\beta \) only depends on \( \beta \), not on the particular choices of \( w \) and \( \alpha \) such that \( \beta = w\alpha \).

For our \( W \), the reflections \( s_{x,i} \) and \( s_{y,i} \) for the roots \( \alpha_{x,i} \) and \( \alpha_{y,i} \) are given by the formulas of the following Lemma.

Lemma 6.1. Let \( \alpha_{x,i} \) and \( \alpha_{y,i} \) be the positive roots for \( W \) introduced in (5.1) and let \( s_{x,i} \) and \( s_{y,i} \) be the associated reflections, for \( i = 1, 2, \ldots \) Then we have that

$$s_{x,i} = (st)^{i-1} s, \quad s_{y,i} = t(st)^{i-1}.$$  

(6.2)

\[ \text{Proof. } \text{This is immediate from (5.1) and the definitions.} \]

We now have the following Lemma.

Lemma 6.2. Let \( v \in \Lambda_w \) and let \( \lambda := \varphi(v) \in \Lambda_{\pm(n-1)} \) where \( \varphi : \Lambda_w \to \Lambda_{\pm(n-1)} \) is the function defined in Lemma 3.6. Set \( k = \frac{n-1-|\lambda|}{2} \).

a) Suppose that \( \lambda \geq 0 \). Then the set of reflections \( s_\alpha \) in \( W \) satisfying \( v < s_\alpha v \leq w \) is exactly

$$\{s_{x,\lambda+2}, s_{x,\lambda+3}, \ldots, s_{x,\lambda+k+1}\} \cup \{s_{y,1}, s_{y,2}, \ldots, s_{y,k}\}$$  

(6.3)

obtained by transforming the roots of the factors of (5.4) to reflections.

b) Suppose that \( \lambda < 0 \). Then the set of reflections \( s_\alpha \) in \( W \) satisfying \( v < s_\alpha v \leq w \) is exactly

$$\{s_{x,1}, s_{x,2}, \ldots, s_{x,k+1}\} \cup \{s_{y,1+|\lambda|}, s_{y,2+|\lambda|}, \ldots, s_{y,k+|\lambda|}\}$$  

(6.4)

obtained by transforming the roots of the factors of (5.32) to reflections.

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Proof. Let us prove a). Since \( \lambda \geq 0 \) we have that \( v \) begins with \( s \) and that \( \lambda = l(v) - 1 \). Viewing \( v \) as a ‘tail’ of \( w \), see (3.31), there are \( k \) instances in \( w \) of \( s \) to the left of \( v \) and also \( k \) instances of \( t \) to the left of \( v \). Multiplying \( v \) on the right of the reflections \( s_{y,1}, s_{y,2}, \ldots, s_{y,k} \) gives the tails from these \( t \)’s, as illustrated below in (6.5) for \( n = 20 \) and \( \lambda = 9 \)

\[
\begin{array}{c}
v = s t s t s t s t s t s t s t s t s t s t \\
\quad s_{y,1} v \quad s_{y,2} v \quad s_{y,3} v \quad s_{y,4} v \quad s_{y,5} v
\end{array}
\]

(6.5)

and multiplying \( v \) on the right of the reflections \( s_{x,\lambda+2}, s_{x,\lambda+3}, \ldots, s_{x,\lambda+k+1} \), gives the tails from the \( s \)’s, upon deleting the last generator of \( w \), as illustrated below

\[
\begin{array}{c}
v = s t s t s t s t s t s t s t s t s t s t \\
\quad s_{x,11} v \quad s_{x,12} v \quad s_{x,13} v \quad s_{x,14} v \quad s_{x,15} v
\end{array}
\]

(6.6)

These products all satisfy the conditions \( v < s_{a} v \leq w \) of the Lemma, and one also checks that they are the only ones satisfying the conditions. This shows a), and b) is shown the same way. \( \square \)

7 Graded Jantzen filtrations and sum formulas

In this final section we study the representation theory of \( \tilde{A}_w \) and its specialization \( \tilde{A}_w^C := \tilde{A}_w \otimes_R \mathbb{C} \), where the \( R \)-algebra structure on \( \mathbb{C} \) is given by mapping \( \alpha_s^y \) and \( \alpha_t^y \) to 0. Note that \( \tilde{A}_w^C \) has already been studied in the literature, in fact for general \((W,S)\) for example in [36] or in [33].

As already mentioned, \( \tilde{A}_w \) is a cellular algebra and hence also \( \tilde{A}_w^C \) is a cellular algebra, with cell modules \( \Delta^C(y) := \Delta_w(y) \otimes_R \mathbb{C} \) for \( y \in \Lambda_w \). Moreover, as already indicated, \( \tilde{A}_w \) and \( \tilde{A}_w^C \) are \( \mathbb{Z} \)-graded algebras with degree function \( \deg \) given in (2.10). In fact they are graded cellular algebras, that is they fit into the following definition first formulated by Hu and Mathas, see [20].

Definition 7.1. Suppose that \( k \) is a commutative ring with identity and that \( A \) is a \( k \)-algebra which is cellular on the triple \((\Lambda, \text{Tab}, C)\). Suppose moreover that \( A \) is a \( \mathbb{Z} \)-graded algebra via \( A = \bigoplus_{n \in \mathbb{Z}} A_n \). Then we say that \( A \) is a \( \mathbb{Z} \)-graded cellular algebra if for each \( \lambda \in \Lambda \) there is a function \( \deg : \text{Tab}(\lambda) \rightarrow \mathbb{Z} \) such that for \( s, t \in \text{Tab}(\lambda) \) we have that \( C_{st} \in A_{\deg(s) + \deg(t)} \).

We shall in general refer to \( \mathbb{Z} \)-graded cellular algebras simply as graded cellular algebras.

The same degree function \( \deg \) that was used for \( \tilde{A}_w \) also makes \( A_w \) and \( A_w^C \) into \( \mathbb{Z} \)-graded cellular algebras. On the other hand, to make the blob-algebras \( B_{\eta}^C \) and \( \mathbb{NB}_{\eta}^C \) fit into Definition 7.1 we use the function \( \deg : \text{Tab}(\lambda) \rightarrow \mathbb{Z} \) given by

\[
\deg(t) : \text{Tab}(\lambda) \rightarrow \mathbb{Z}, \deg(t) = \begin{cases} 2 \{ \text{number of blobs in } t \} & \text{if } \lambda \geq 0 \\ 2 \{ \text{number of blobs in } t \} + 1 & \text{if } \lambda < 0 \end{cases}
\]

(7.1)

where \( \text{Tab}(\lambda) \) refers to blob half-diagrams as in the paragraph prior to equation (2.20). Thus, when viewing \( \text{Tab}(\lambda) \) as the basis elements for \( \Delta^C(\lambda) \) as in (4.1) and (4.2), the function \( \deg \) assigns degree 1 to a blob on a propagating line, and degree 2 to a blob on a non-propagating line.

For \( A \) a graded cellular algebra over \( k \) we let \( A \)-mod be the category of \( A \)-modules which are free over \( k \) with finite basis consisting of homogeneous elements. For \( M \in A \)-mod we define its graded rank \( \text{rk}_q M \in \mathbb{Z}[q,q^{-1}] \) via

\[
\text{rk}_q M := \sum_{i \in \mathbb{Z}} \text{rk} M_i q^i
\]

(7.2)
where \( \text{rk} M_i \) is the number of basis elements for \( M \) that have degree \( i \).

We shall use the notation \( \Delta^g_w \) and \( \Delta^g_{wC} \) when referring to \( \Delta_w \) and \( \Delta_w^c \) as graded cellular algebras and similarly, for \( v \in \Lambda_w \), we shall use the notation \( \Delta^g_w(v) \) and \( \Delta^g_{wC}(v) \) for the graded cell modules for \( \Delta^g_w \) and \( \Delta^g_{wC} \). On the blob-algebra side, we shall use the notation \( B^g_{n-x-y} \) and \( NB^g_{n} \) for \( B_{x,y} \) and \( N B_n \), when viewed as graded cellular algebras, and shall for \( \lambda \in \Lambda_{\pm} \) use the notation \( \Delta^g_{n-1}(\lambda) \) and \( \Delta^g_{n-1}(\lambda) \) for the graded cell modules for \( B^g_{n-x-y} \) and \( NB^g_{n} \).

The proof of Theorem 3.4 shows that \( B^g_{n-1} \cong A^g_w \) and so also \( NB^g_{n-1} \cong A^g_{wC} \). Recall the map \( \varphi : \Lambda_{\pm}(n-1) \to \Lambda_w \) from Lemma 3.6. Then, similarly, the proof of Theorem 3.8 shows that \( \Delta^g_{n-1}(\lambda) \cong \Delta^g_w(v) \) where \( \varphi(\lambda) = v \) and so also \( \Delta^g_{n-1}(\lambda) \cong \Delta^g_{n-1}(\lambda) \). However, for \( v \) not belonging to the image of \( \varphi \), that is for \( v \in \Lambda^c_w \), see (3.33), we need to work a little bit to get an analogous description of \( \Delta^g_w(v) \) and \( \Delta^g_{n-1}(\lambda) \).

For \( A \) a graded cellular algebra and \( M = \oplus_{i \in \mathbb{Z}} M_i \) a graded \( A \)-module we define the graded shift \( M[k] \) of \( M \) as the graded \( A \)-module given by

\[
M[k] = \oplus_{i \in \mathbb{Z}} M[k]_i \quad \text{where} \quad M[k]_i := M_{i-k}. \tag{7.3}
\]

The first part of the following Lemma has just been mentioned, but we still include it for later reference. The second part of the Lemma essentially says that the \( s \)'s in \( \Lambda^c_w \) do not give rise to 'new' cell modules for \( A^g_w \).

**Lemma 7.2.** Suppose that \( w = w_1 s' \), that is \( s' \) is the last \( S \)-generator of \( w \).

a) For \( v = \varphi(\lambda) \) we have that \( \Delta^g_w(v) \cong \Delta^g_{n-1}(\lambda) \).

b) For \( v \in \Lambda^c_w \) set \( v_1 := vs' \), that is \( l(v_1) = l(v) + 1 \). Then \( \Delta^g_w(v) \cong \Delta^g_{n-1}(\lambda) \) where \( \varphi(\lambda) = v_1 \).

**Proof:** As mentioned, we only need to prove b). Let \( D_1 \) be the following diagram

\[
D_1 := \begin{array}{c|c|c|c|c|c|c}
         & & & & & & \\
\bullet & & & & & & \\
& & & & & & \\
& & v & & & & \\
& & & & & & \\
& v_1 & & & & & \\
& & & & & & \\
\end{array}
\tag{7.4}
\]

(where we suppose that \( s' \) is blue). For any diagram \( D \) for \( \Delta^g_w(v_1) \), we define \( f(D) := DD_1 \), that is \( f(D) \) is obtained from \( D \) by multiplying on top with \( D_1 \). Then \( f \) induces an \( R \)-isomorphism \( \Delta^g_w(v_1) \cong \Delta^g_w(v) \) which is also a module isomorphism since left multiplication commutes with right multiplication. But \( D_1 \) is of degree 1, and so we get that \( f : \Delta^g_w(v_1) \to \Delta^g_w(v)[-1] \) and hence \( \Delta^g_w(v_1) \cong \Delta^g_w(v) \). Combining this with a) we obtain b). \( \square \)

**Lemma 7.3.** a) Suppose that \( \lambda \in \Lambda_{\pm}(n-1) \) with \( \lambda \geq 0 \). Then

\[
\text{rk}_q \Delta^g_{n-1}(\lambda) = \text{rk}_q \Delta^g_{n-1}(\lambda) = \sum_{i=0}^{n-\frac{1}{2}-\lambda} \text{rk} \Delta^g_{n-1}(\lambda + 2i) q^{2i}. \tag{7.5}
\]

b) Suppose that \( \lambda \in \Lambda_{\pm}(n-1) \) with \( \lambda < 0 \). Then

\[
\text{rk}_q \Delta^g_{n-1}(\lambda) = \text{rk}_q \Delta^g_{n-1}(\lambda) = qr \text{rk}_q \Delta^g_{n-1}(-\lambda). \tag{7.6}
\]

**Proof:** This follows immediately from Lemma 4.1. \( \square \)

For \( \lambda \in \Lambda_{\pm}(n-1) \) let \( \{ b_1, b_2, \ldots, b_m \} \) be the \( R \)-basis for \( \Delta^g_{n-1}(\lambda) \) obtained from (4.30) and Theorem 5.1 if \( \lambda \geq 0 \) or from Theorem 5.2 if \( \lambda < 0 \). According to these Theorems \( (b_1, b_i)_{n-1,1}^B \) is a product of positive roots for \( W \) and so \( \{ b_1, b_2, \ldots, b_m \} \) consists of homogeneous elements since \( (\lambda, b_i)_{n-1,1}^B \) is homogeneous. The degree of \( b_i \) is equal to number of roots appearing in \( (b_i, b_i)_{n-1,1}^B \) according to Theorem 5.1 and 5.2.

Let now \( \alpha \) be a positive root for \( W \). We then introduce the following \( B^g_{n-x-y} \)-submodule of \( \Delta^g_{n-1}(\lambda) \)

\[
\Delta^g_{n-1}(\lambda) := \{ a \in \Delta^g_{n-1}(\lambda) \mid \alpha \text{ divides } (a, b)_{n-1,1}^B \text{ for all } b \in \Delta^g_{n-1}(\lambda) \}. \tag{7.7}
\]
From the above remarks we have that $\Delta_{n-1}^{gr,\alpha}(\lambda)$ is a free over $R$ with basis
\[ \{b_i \mid \alpha \text{ is a factor of } \langle b_i, b_i \rangle_{n-1,\lambda} \}. \] (7.8)

The proof of the following Theorem is a compilation of the results from the previous sections.

**Theorem 7.4.** Supposing that $v = \varphi(\lambda)$ and that $v < s_\alpha v \leq w$ we have that
\[ \text{rk}_q \Delta_{n-1}^{gr,\alpha}(\lambda) = \text{rk}_q \Delta_{n-1}^{gr}(s_\alpha v)[l(s_\alpha v) - l(v)]. \] (7.9)

Otherwise, if $v < s_\alpha v \leq w$ is not satisfied, we have $\Delta_{n-1}^{gr,\alpha}(\lambda) = 0$.

**Proof:** Let $k := \frac{n-1}{\lambda}$. Let us first consider the case where $\lambda \geq 0$ and $\alpha = \alpha_{y,i}$ for some $i = 1, 2, \ldots, k$, hence $s_\alpha = t(st)^{-1}$, see Lemma 6.2. By Lemma 7.2 we have that $\Delta_{n-1}^{gr,\beta}(\lambda) \cong \Delta_{n-1}^{gr}(v)$. The distinct roots $\alpha_{x,j}$ and $\alpha_{y,j}$ are irreducible and unassociated elements of $R$ and so it follows from the description in Theorem 5.1 of the matrix $M_{n-1,\lambda}$ in (4.30) that
\[ \text{rk}_q \Delta_{n-1}^{gr,\alpha}(\lambda) = \sum_{j=0}^{k} \text{rk}_q \Delta_{n-1}^{gr}(\lambda + 2j)(2^j) = q^{2^{2-k}} \sum_{j=0}^{k} \text{rk}_q \Delta_{n-1}^{gr}(\lambda + 2j + 2)(2^j). \] (7.10)

Using Lemma 7.3 and Lemma 3.6 we get that (7.10) is equal to
\[ q^{2^k} \text{rk}_q \Delta_{n-1}^{gr}(\lambda + 2i) = q^{2^k-1} \text{rk}_q \Delta_{n-1}^{gr}(\lambda - 2i) = q^{2^k-1} \text{rk}_q \Delta_{n-1}^{gr}(\lambda - 2i) = q^{2^k-1} \text{rk}_q \Delta_{n-1}^{gr}(s_\alpha v). \] (7.11)

But $v$ begins with $s$ since $\lambda \geq 0$ and so the last expression of (7.11) is $\text{rk}_q \Delta_{n-1}^{gr}(s_\alpha v)[l(s_\alpha v) - l(v)]$ which shows the Theorem in this case. For an illustration of $s_\alpha v$, see (6.5).

Let us now consider the case where still $\lambda \geq 0$, but $\alpha = \alpha_{y,\lambda+1+i}$ and hence $s_\alpha = (st)^{\lambda+i} s$, see Lemma 6.2. Then $\alpha$ appears in the same blocks as $\alpha_{y,i}$ did in the previous case, and so we get from (7.10) and (7.11) that
\[ \text{rk}_q \Delta_{n-1}^{gr,\alpha}(\lambda) = q^{2^k-1} \text{rk}_q \Delta_{n-1}^{gr}(t(st)^{i-1}) = q^{2^k} \text{rk}_q \Delta_{n-1}^{gr}((st)^i v) \] (7.12)

where we used Lemma 3.6 and b) of Lemma 7.3 for the second equality. On the other hand, writing $v = wi$ with $l(u) + 1 = l(v)$ we get from Lemma 6.2 and b) of Lemma 7.2 that
\[ \text{rk}_q \Delta_{n-1}^{gr}(s_\alpha v) = \text{rk}_q \Delta_{n-1}^{gr}((st)^i u) = q \text{rk}_q \Delta_{n-1}^{gr}((st)^i v). \] (7.13)

Comparing (7.12) and (7.13) we get
\[ \text{rk}_q \Delta_{n-1}^{gr,\alpha}(\lambda) = q^{2^k-1} \text{rk}_q \Delta_{n-1}^{gr}(s_\alpha v) = q \text{rk}_q \Delta_{n-1}^{gr}(s_\alpha v)[l(s_\alpha v) - l(v)] \] (7.14)

which shows the Theorem in this case as well. The remaining cases of the Theorem are proved with similar techniques.

Our next aim is to generalize Theorem 7.4 to $\Delta_{w}^{gr}(v)$. This is immediate if $v \in \Lambda_w$ since in that case $v \in im \varphi$, whereas for $v \in \Lambda_w$ we have to work a little bit. For $\alpha$ a positive root we first generalize (7.7) in order to get a graded submodule $\Delta_{w}^{gr,\alpha}(v)$ of $\Delta_{w}^{gr}(v)$.

\[ \Delta_{w}^{gr,\alpha}(v) := \{a \in \Delta_{w}^{gr}(v) \mid \alpha \text{ divides } \langle a, b \rangle_{n,v} \text{ for all } b \in \Delta_{w}^{gr}(v) \}. \] (7.15)

We then have the following Theorem.

**Theorem 7.5.** Let $\alpha$ be a positive root for $W$. If $v < s_\alpha v \leq w$ then
\[ \text{rk}_q \Delta_{w}^{gr,\alpha}(v) = \text{rk}_q \Delta_{w}^{gr}(s_\alpha v)[l(s_\alpha v) - l(v)] \] (7.16)

and otherwise $\Delta_{w}^{gr,\alpha}(v) = 0$.

**Proof:** As already mentioned, if $v \in \Lambda_w$ the result follows immediately from Theorem 7.4, so let us assume that $v \in \Lambda_w$. As before we write $w = w_1 s'$, where $s'$ is the last $S$-generator of $w$ and set $v_1 = vs'$. Then $l(v_1) = l(v) + 1$ and $v_1 \in \Lambda_w$. Let $D$ be a diagram basis element for $\Delta_{w}(v)$. Then $D$ has nonempty zone $C$ and we define $D_1$ to be the diagram basis element for $\Delta_{w}(v_1)$ which is obtained from $D$ by making the last non-hanging birdcage,
corresponding to zone C, hanging. Then $D \mapsto D_1$ is a bijection between the diagram basis for $\Delta_w(v)$ and the diagram basis for $\Delta_w(v_1)$. On the other hand, using the definition of the bilinear forms we have that

$$\langle D, C \rangle^w_{n,v} = v(\alpha') \langle D_1, C_1 \rangle^w_{n,v_1} \tag{7.17}$$

where $\alpha'$ is the root corresponding to $s'$ and so the matrix for $\langle \cdot, \cdot \rangle^w_{n,v}$ has the diagonalized form $v(\alpha')M^B_{n-1, \lambda}$ where $\lambda = \varphi(v_1)$. We now assume that $\lambda \geq 0$ such that $M^B_{n-1, \lambda}$ is given by Theorem 5.1. One then checks that $v(\alpha') = \alpha_{x, \lambda+1}$. Moreover, the set of reflections $s_{\alpha}$ in $W$ satisfying $v < s_{\alpha}v \leq w$ is the union of $s_{x, \lambda+1}$ with the set of reflections $s_{\alpha}$ satisfying $v_1 < s_{\alpha}v_1 \leq w$, and hence, in view of Lemma 6.2, it is equal to

$$\{s_{x, \lambda+1}, s_{x, \lambda+2}, s_{x, \lambda+3}, \ldots, s_{x, \lambda+k+1}\} \cup \{s_{y,1}, s_{y,2}, \ldots, s_{y,k}\} \tag{7.18}$$

which is exactly the set of reflections obtained by transforming the roots of the diagonal elements of $v(\alpha')M^B_{n-1, \lambda}$ to reflections.

Let us now show (7.16). If $\alpha$ is a factor of a diagonal element of $M^B_{n-1, \lambda}$ we are reduced to the previous case treated in Theorem 7.4, as follows

$$\text{rk}_q \Delta^{gr,\alpha}(v) = q \text{rk}_q \Delta^{gr,\alpha}_{n-1}(\lambda) = q \text{rk}_q \Delta^{gr}_{n-1}(s_{\alpha}v_1)[l(s_{\alpha}v_1) - l(v_1)] = \text{rk}_q \Delta^{gr}_{n-1}(s_{\alpha}v)[l(s_{\alpha}v) - l(v)] \tag{7.19}$$

where we used Lemma 7.2 for the last step. On the other hand, if $\alpha = \alpha_{x, \lambda+1}$ we have that $s_{\alpha}v_1 = v$ and so we get

$$\text{rk}_q \Delta^{gr,\alpha}(v) = q \text{rk}_q \Delta^{gr}_{n-1}(\lambda) = \text{rk}_q \Delta^{gr}_{n-1}(s_{\alpha}v_1)[l(s_{\alpha}v) - l(v)] \tag{7.20}$$

which shows (7.16) in this case as well. The cases where $\lambda < 0$ are treated with similar techniques. \qed

$$\Delta^{gr,\alpha}(v)$$ is free over $R$ and hence we get immediately a specialized version of Theorem 7.5.

**Corollary 7.6.** Defining $\Delta^{gr,\alpha,C}(v) := \Delta^{gr,\alpha}(v) \otimes_R C$ we have that

$$\text{rk}_q \Delta^{gr,\alpha,C}(v) = \text{rk}_q \Delta^{gr}_{\alpha}(s_{\alpha}v)[l(s_{\alpha}v) - l(v)]. \tag{7.21}$$

The definition of $\Delta^{gr,\alpha}(v)$ in (7.15) is reminiscent of the Jantzen filtration for Verma modules. To make this analogy even stronger we let $R_1 := C[x]$ and define $\Delta^{gr,x}(v) := \Delta^{gr}_{w}(v) \otimes_R R_1$ where $R_1$ is made into an $R$-module via $\alpha_s \mapsto x$ and $\alpha_t \mapsto x$. Then $\{\Delta^{gr,x}(v) | v \in A_w\}$ are graded cell modules for $\Lambda^{gr,x} := A^{gr}_{w} \otimes_R R_1$. In $R_1$ all the roots $\alpha_{x,i}$ and $\alpha_{y,i}$ are non-zero scalar multiples of $x$ and so we define for any $k = 1, 2, \ldots$

$$\Delta^{gr,k,x}(v) := \{a \in \Delta^{gr}_{w}(v) | x^k \text{ divides } (a, b)^w_{n,s} \text{ for all } b \in \Delta^{gr,x}(v)\} \tag{7.22}$$

$$\Delta^{gr,k,C}(v) := \pi(\Delta^{gr,x,k}(v)) \tag{7.23}$$

where $\pi : \Delta^{gr,x}(v) \rightarrow \Delta^{gr,x}(v) \otimes_R C$ is the quotient map: here $C$ is made into an $R_1$-module via $x \mapsto 0$. Then $\Delta^{gr,C}_w(\alpha) \supset \Delta^{gr,1,C}_w(\alpha) \supset \Delta^{gr,2,C}_w(\alpha) \supset \ldots$ is a filtration of graded submodules of $\Delta^{gr,C}_w(\alpha)$ and we have the following Corollary to Theorem 7.5.

**Corollary 7.7.** a) $\Delta^{gr,C}_w(\alpha)/\Delta^{gr,1,C}_w(\alpha)$ is irreducible or zero.

b) The following graded analogue of Jantzen’s sum formula holds

$$\sum_{k>0} \text{rk}_q \Delta^{gr,k,C}(v) = \sum_{\alpha > 0} \text{rk}_q \Delta^{gr,C}(s_{\alpha}v)[l(s_{\alpha}v) - l(v)] \tag{7.24}$$

where $\alpha > 0$ refers to the positive roots of $W$.

As pointed out in the introduction, analogues of ungraded Jantzen filtrations with associated sum formulas exist in many module categories of Lie type and give information on the irreducible modules for the category in question. But although graded representation theories in Lie theory have been known for a long time and would be very useful for calculating decomposition numbers, to our knowledge graded sum formulas have so far not been available. The virtue of Corollary 7.7 is to show the possible form of graded sum formulas in graded representation theory.

It should be noted that in the present $\mathbb{N} \mathbb{E}_n$-situation, the irreducible modules can be read off from Theorem 5.1 and Theorem 5.2 and are in fact Temperley-Lieb cell modules. See also [32] for a different approach to this.
We believe that the equalities in Theorem 7.5 and Corollary 7.6 are valid on module level, but have so far not been able to prove so. But in the remainder of the article we indicate how to generalize them to enriched Grothendieck group level. The methods for this are essentially generalizations to the graded case of the methods in [36], where the corresponding ungraded case is treated.

Let \( A \) be a graded cellular algebra over \( C \). Let \( (A-\text{mod})_q \) be the enriched Grothendieck group for \( A \), that is \((A-\text{mod})_q \) is the Abelian group generated by symbols \( \langle M \rangle_q \), for \( M \) running over the modules in \((A-\text{mod})_q \), subject to the relations \( \langle M \rangle_q = \langle M_1 \rangle_q + \langle M_2 \rangle_q \) whenever there is a short exact sequence \( 0 \to M_1 \to M_2 \to 0 \) in \((A-\text{mod})_q \). The grading shift in \((A-\text{mod})_q \) induces a grading shift in \((A-\text{mod})_q \) via \( \langle M[k] \rangle_q := \langle M \rangle_q[k] \) and so we get a \( \mathbb{Z}[q,q^{-1}] \)-structure on \((A-\text{mod})_q \).

The following is a natural generalization of a cellular category, see [41], to the \( \mathbb{Z} \)-graded case.

**Definition 7.8.** Let \( k \) be a commutative ring with identity and let \( C \) be a \( k \)-linear \( \mathbb{Z} \)-graded category, that is for objects \( m,n \) in \( C \) we have a decomposition

\[
\text{Hom}_C(m,n) = \oplus_{i \in \mathbb{Z}} \text{Hom}_C(m,n)_i.
\]

(7.25)

Suppose that \( C \) is endowed with a duality \( * \). Then \( C \) is called a \( \mathbb{Z} \)-graded cellular category if there exists a poset \( \Lambda \) and for each \( \lambda \in \Lambda \) and each object \( n \) in \( C \) a finite set \( \text{Tab}(n,\lambda) \) which is decomposed as \( \text{Tab}(n,\lambda) = \cup_{i \in \mathbb{Z}} \text{Tab}(n,\lambda)_i \), together with a map \( \text{Tab}(m,\lambda) \times \text{Tab}(n,\lambda) \to \text{Hom}_C(m,n,\langle S, T \rangle) \to C_{ST}^\lambda \), satisfying \( C_{ST}^\lambda \in \text{Hom}_C(m,n)_{i+j} \) if \( S \in \text{Tab}(n,\lambda)_i \) and \( T \in \text{Tab}(n,\lambda)_j \). These data satisfy that \( (C_{ST}^\lambda)^* = C_{TS}^{\lambda^*} \) and that

\[
\{ C_{ST}^\lambda | S \in \text{Tab}(m,\lambda), T \in \text{Tab}(n,\lambda), \lambda \in \Lambda \}
\]

(7.26)

and for all \( a \in \text{Hom}_C(n,p)_i, S \in \text{Tab}(m,\lambda), T \in \text{Tab}(n,\lambda)_k \)

\[
a C_{ST}^\lambda = \sum_{S^i \in \text{Tab}(p,\lambda)_i} r_a(S^i,S) C_{S^i,T}^\lambda \mod A_{1+j+k}^\lambda
\]

(7.27)

where \( A^\lambda \) is the span of \( \{ C_{ST}^\mu | \mu < \lambda, S \in \text{Tab}(m,\mu), T \in \text{Tab}(p,\mu) \} \).

This following simple fact was already mentioned in [36], in the ungraded case. Let \( C \) be a graded cellular category and let \( A \) be a finite subset of the objects of \( C \). Define \( \text{End}_C(A) \) as the direct sum

\[
\text{End}_C(A) := \oplus_{m,n \in A} \text{Hom}_C(m,n).
\]

(7.28)

Then \( \text{End}_C(A) \) has a \( k \)-algebra structure as follows

\[
g \cdot f := \begin{cases} 
g \circ f & \text{if } f \in \text{Hom}_C(m,n), g \in \text{Hom}_C(n,p) \text{ for some } m,n,p \\ 0 & \text{otherwise} \end{cases}
\]

(7.29)

and, in view of (7.25) and (7.27), this is a graded \( k \)-algebra structure. Moreover, we have the following Theorem.

**Theorem 7.9.** Let \( C \) be a graded cellular category and let \( A \) be a finite subset of the objects for \( C \). Define for \( \lambda \in \Lambda \) the set \( \text{Tab}(\lambda) := \cup_{n \in A} \text{Tab}(n,\lambda) \). Let for \( S \in \text{Tab}(\lambda), T \in \text{Tab}(\lambda) \) the element \( C_{ST}^\lambda \in \text{End}_C(A) \) be defined as the inclusion of \( C_{ST}^\lambda \in \text{Hom}_C(m,n) \) in \( \text{End}_C(A) \). Then these data define a graded cellular algebra structure on \( \text{End}_C(A) \).

**Proof:** Just as in the ungraded case considered in [36], this follows immediately from the definitions.

For a general Coxeter system \((W,S)\), it was shown in [11] that the diagrammatic Soergel categories \( D_{(W,S)} \) and \( D_{(W,S)}^C \), see Definition 2.10 and Remark 2.11, are graded cellular categories in the sense of Definition 7.8.

Let us indicate the category which make the categories \( D_{(W,S)} \) and \( D_{(W,S)}^C \), see Definition 2.10 and Remark 2.11, fit into Definition 7.8, for our choice of \((W,S)\). In case of \( D_{(W,S)} \), we use for \( \mathbb{R} \) the ring \( R \), and for the objects and morphisms we use the objects and morphisms given in Definition 2.10. For the poset \( \Lambda \) we use \( W \) itself, endowed with the Bruhat order poset structure. For \( w \in \exp \) starting with \( s \) and in reduced form, that is \( w = w \), we use for \( \text{Tab}(w,v) \) the set of birdcages \( \text{Tab}_w(v) \). For \( w \in \exp \) starting with \( t \) and in reduced form, we use for \( \text{Tab}_w(v) \) the corresponding set of birdcages \( \text{Tab}_w(v) \). For \( w \in \exp \) a general object in \( D_{(W,S)} \) not in reduced form, we use the general light leaves construction from [11]. Since we do not need the details of this construction we skip it at this point. In case of \( D_{(W,S)}^C \) we use the same ingredients as for \( D_{(W,S)} \), except that for \( \mathbb{R} \) we replace \( R \) by \( C \).
Let us now fix a finite subset $W_0 \subseteq W$ of $W$ such that $v \in W_0, u \leq v \implies u \in W_0$, that is $W_0$ is an ideal in $W$. For each $z \in W_0$ we let $\bar{z}$ be its (unique) reduced expression. We then set $W_{\bar{z}} := \{ z \mid z \in W_0 \} \subseteq \text{exp}$ and define
\[
\tilde{A}_{gr, W_0} := \text{End}_{D_0}(W_{\bar{z}}) \quad \text{and} \quad \tilde{A}_{gr, C} := \text{End}_{D_0(\text{ext})}(W_{\bar{z}}).
\] (7.30)
Note that in this setup we recover the graded cellular algebras $\tilde{A}_{gr, W}$ and $\tilde{A}_{gr, C}$ by taking $W_0 := \{ w \}$.

We now have the following Theorem, which is our main reason for changing to the categorical setting.

**Theorem 7.10.** $\tilde{A}_{gr, C}$ is a graded quasi-hereditary algebra over $\mathbb{C}$.

**Proof:** The proof of Theorem 8.5 in [36], corresponding to the ungraded setting, carries over to the present graded setting.

Let $\Delta_{gr, W_0}(v)$ and $\Delta_{gr, C}(v)$ be the graded cell modules for $\tilde{A}_{gr, W}$ and $\tilde{A}_{gr, C}$. There is an $R$-module decomposition
\[
\Delta_{gr, W_0}(v) = \oplus_{z \in W_0} \Delta_{gr}(v)
\] (7.31)
where we use $z$ instead of $w$ to indicate that $\bar{z}$ may begin with $s$ as well as $t$. There is a similar decomposition for $\Delta_{gr, C}(v)$. Let $\langle \cdot, \cdot \rangle_1$ be the bilinear form on $\Delta_{gr, W_0}(v)$. It is orthogonal with respect to the decomposition in (7.31). Mimicking what we did for $\Delta_{gr}(v)$ we choose a root for $W$ and define for $\Delta_{gr, W_0}(v)$ via
\[
\Delta_{gr, W_0}(v) := \{ a \in \Delta_{gr, W_0}(v) \mid a \text{ divides } (a, b)_W \text{ for all } b \in \Delta_{gr, W_0}(v) \}
\] (7.32)
and set
\[
\Delta_{gr, C}(v) := \pi(\Delta_{gr, W_0}(v))
\] (7.33)
where $\pi$ is before. Following [36], we define for $z \in W_0$ projection maps $\varphi_z$ as follows
\[
\varphi_z : \langle \tilde{A}_{gr, C} \mod \rangle_q \to \langle \tilde{A}_{gr, C} \mod \rangle_q, \quad \langle \Delta_{gr, C}(v) \rangle_q \to \langle \Delta_{gr, C}(v) \rangle_q
\] (7.34)
and arguing as in [36], we get the following compatibility at Grothendieck group level
\[
\varphi_z(\langle \Delta_{gr, C}(v) \rangle_q) = \langle \Delta_{gr, C}(v) \rangle_q.
\] (7.35)
We have natural homomorphisms of $\mathbb{Z}[q, q^{-1}]$-modules
\[
\text{rk}_{W_0,q} : \langle \tilde{A}_{gr, C} \mod \rangle_q \to \mathbb{Z}[q, q^{-1}], \quad \langle M \rangle_q \mapsto \text{rk}_q M
\]
\[
\text{rk}_{z,q} : \langle \tilde{A}_{gr, C} \mod \rangle_q \to \mathbb{Z}[q, q^{-1}], \quad \langle M \rangle_q \mapsto \text{rk}_q M.
\] (7.36)
Let $\Phi : \langle \tilde{A}_{gr, C} \mod \rangle_q \to \oplus_{z \in W_0} \mathbb{Z}[q, q^{-1}]$ be the $\mathbb{Z}[q, q^{-1}]$-homomorphism whose $z$'th coordinate is equal to the composite map $\text{rk}_{z,q} \circ \varphi_z$. With this notation we have the following Theorem.

**Theorem 7.11.** $\Phi : \langle \tilde{A}_{gr, C} \mod \rangle_q \to \oplus_{z \in W_0} \mathbb{Z}[q, q^{-1}]$ is an isomorphism of $\mathbb{Z}[q, q^{-1}]$-modules.

**Proof:** Since $\langle \tilde{A}_{gr, C} \mod \rangle_q$ and $\oplus_{z \in W_0} \mathbb{Z}[q, q^{-1}]$ are free $\mathbb{Z}[q, q^{-1}]$-modules of the same rank, it is enough to show that $\Phi$ is surjective, by Vasconcelos’ Theorem once again, see [39]. Let $f = \sum_{z \in W_0} f_z \in \oplus_{z \in W_0} \mathbb{Z}[q, q^{-1}]$ and choose $f_{z_0}$ nonzero and $\bar{z}_0$ minimal with respect to this condition. The $z_0$'th component of $\Phi(\langle \Delta_{gr, C}(z_0) \rangle_q)$ is $\text{rk}_{z_0,q}(\Delta_{z_0}(z_0)) = 1$ and so the $z_0$'th component of $\Phi(\langle \Delta_{gr, C}(z_0) \rangle_q)$ is $f_{\bar{z}_0}$. On the other hand, the $z$'th component of $\Phi(\langle \Delta_{gr, C}(z_0) \rangle_q)$ is $\text{rk}_{z,q}(\Delta_{z}(z_0))_q$ which is nonzero only if $z_0 \leq z$. Hence, we can use induction on $f - \Phi(f_{\bar{z}_0}(\langle \Delta_{gr, C}(z_0) \rangle_q))$ and get that $f \in \text{im}\Phi$, as claimed.

We can now prove the promised Grothendieck group extension of Corollary 7.6.

**Corollary 7.12.** For $v \in W_0$ we have $\Delta_{gr, C}(v) = 0$ unless $w \geq s_\alpha v > v$. If $w \geq s_\alpha v > v$ then
\[
\langle \Delta_{gr, C}(v) \rangle_q = \langle \Delta_{gr, C}(s_\alpha v) \rangle_q \text{ if } (s_\alpha v) - l(v) \rangle_q
\] (7.37)
**Proof.** We apply $\Phi$ to (7.37) and check that both sides are equal. Using (7.35) and Corollary 7.6 we get that the $z$-th component of the left hand side is $rk_q \Delta_{W_0}^{\text{gr}, x}(s_0 v)[l(s_0 v) - l(v)]$ which by definition of $\Phi$ coincides with the right hand side. We then use Theorem 7.11 to conclude the proof. \hfill \Box

Finally, the Grothendieck group extension of Corollary 7.7 is proved with the same techniques, upon changing the ground ring for the category $D_{(W,S)}$ from $R$ to $R_1$. The cell modules for $D_{(W,S)}$ are called $\Delta_{W_0}^{\text{gr}, x}(v)$ and we define for $k = 1, 2, \ldots$

$$\Delta_{W_0}^{\text{gr}, k, x}(v) := \{ a \in \Delta_{W_0}^{\text{gr}, x}(v) | \langle a, b \rangle_{W_0} \in \mathbf{x}^k R_1 \text{ for all } b \in \Delta_{W_0}^{\text{gr}, x}(v) \}$$

and set

$$\Delta_{W_0}^{\text{gr}, k, C}(v) := \pi(\Delta_{W_0}^{\text{gr}, k, x}(v)).$$

Mimicking the proof of Corollary 7.12 we then have the following generalization of Corollary 7.7.

**Corollary 7.13.** The following graded analogue of Jantzen’s sum formula holds:

$$\sum_{k > 0} \langle \Delta_{W_0}^{\text{gr}, k, C}(v) \rangle_q = \sum_{\alpha > 0} \langle \Delta_{W_0}^{\text{gr}, C}(s_\alpha v)[l(s_\alpha v) - l(v)] \rangle_q.$$  \hfill (7.40)

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