Numerical analysis of fractional human liver model in fuzzy environment

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ABSTRACT
Many papers have shown that fractional derivatives are preferable to other operators when the data or information is exact, but this is not the case in practice because we live in an uncertain environment. Fuzzy operators are the best option for modelling in this situation. In this paper, we use the fuzzy fractional Caputo’s derivative to generalize the fractional-order human liver model. We consider both types of H-differentiability (type 1 and type 2). We establish a general procedure of solution under the concept of H-differentiability through fuzzy Laplace transform. We implement the proposed scheme to derive the numerical results of the model. We present the archived theoretical solution via two- and three-dimensional graphs at different values of fractional orders and specific fuzzy triangular initial conditions. We present the evolution of the proposed model for some values of $\varphi_0 \in [0, 1]$ to see the effect of uncertainty on the secretion of Bromsulphthalein in the blood and liver.

1. Introduction
The liver is indeed a vertebrate-only organ, and the primary role of the liver is to separate the blood from the digestive tract until it is distributed to the surrounding tissue. It is found in the right upper quadrant of the abdomen in humans, under the diaphragm.\textsuperscript{[1]} The liver has lobes to the left and right, and the gallbladder is under the liver. In the liver, bile is secreted and winds up in the intestines. Therefore, food is digested, absorbed and processed by liver, pancreas and intestines. For blood clotting and other activities, the liver often makes vital proteins. Regarding the vital role of the liver, there has been increasing interest of many researchers to explain the performance of the liver using suitable mathematical models\textsuperscript{[2,3]}. However, such natural phenomena are well explained by fractional models rather than the classical integer-order models \textsuperscript{[4,5]}. For the first time, Celechovska\textsuperscript{'} modelled human liver using integer-order operators\textsuperscript{[2]}. In\textsuperscript{[2]}, the author explained that relevant information about an individual’s status can be provided from human clinical trials in order to model data. He obtained data by using “Bromsulphthalein (BSP) Excretion Test” for parameter identification. BSP is a dye that is put into the blood, and the liver is the sole organ in the body that absorbs it and secretes it straight into the bile. Let $\tilde{Y}(t)$ and $\tilde{G}(t)$ reflect the sum of BSP at time $t$ in the blood and in the liver, respectively. Then model proposed by Celechovska\textsuperscript{'} using ordinary derivative is as follows:

\begin{align}
\frac{d}{dt} \tilde{Y}(t) &= -\lambda \tilde{Y}(t) + \kappa \tilde{G}(t), \\
\frac{d}{dt} \tilde{G}(t) &= \lambda \tilde{Y}(t) - (\epsilon + \kappa)\tilde{G}(t),
\end{align}

where $(\tilde{Y}(0), \tilde{G}(0)) = (I, 0)$ is the I.C by injecting some quantity $I > 0$ of BSP into the blood at once. $\lambda, \epsilon, \kappa$ are constants which represent the transfer rates, characterized the system and the human status. It is important for the research to discover dynamic equations from scientific findings. Additionally, theoretical physics and mathematics seek to describe and interpret natural phenomena. For over 100 years, applied mathematics has established a comprehensive approximation of the uncertainties concerned with the conduct of a probabilistic method\textsuperscript{[6,7]}. Fractional calculus got tremendous attention of the mathematicians in the current century\textsuperscript{[8–11]}. Fractional operators are extensively used for modelling of different problems which occur in applied sciences. Ahmad et al. investigated Ambartsumian equation under nonsingular FD\textsuperscript{[12]}. Sai-fullah et al. discussed the nature of shock waves produced in the Klein–Gordon Equation under the FD\textsuperscript{[13]}. The FD has also used in the modelling of wastewater\textsuperscript{[14]}. Over the last few decades, new tools have been implemented in fractional calculus, in which unspecified parameters present in models are considered to be
constant over time, but some of them are not constant and focus on various conditions that do not appear in the model. A stochastic approach that is based on the probabilistic knowledge of the system components is used to deal with the uncertainty in these parameters; however, due to the errors and impreciseness this information is complicated. Therefore, researchers used fuzzy fractional differential equations (FFDEs) to handle models with parameters and initial values through various techniques. Such techniques are too connected to the origin of the mathematical model, and can also be extended to other models. Agarwal et al. presented the concept of FFDEs [15]. Allahviranloo and co-workers [16,17] extend this idea to solve FFDE by introducing “Riemann–Liouville H-differentiability” using various techniques. Afterward, Mazandarani and his co-authors [18,19] introduced the “Caputo-type fuzzy fractional derivative” to solve FFDE. Bede and his co-authors work under the generalized differentiability of fuzzy valued function [20,21]. However, in all the above solutions of FFDEs there are three limitations. First, “Hukuhara difference” is not mentioned, which diverges the solution. Secondly, the “Riemann–Liouville derivative” requires initial guess of the fractional derivative of the unknown solution. Thirdly, under the Type-2 fuzzy fractional derivative used in [19] the computational cost is high, however, sufficiently close to the originality of the model. To overcome such limitations new concepts like “Caputo’s fuzzy differentiability”, various numerical methods are investigated by many researchers [21–23]. Determining more reliable solutions for all fuzzy FODEs is really tedious. Mathematicians have made several efforts to solve fuzzy FODEs by using different techniques [22–24]. The Laplace transform (LT) is one of the important transformation in the theory of fuzzy DE. FFDEs, its fuzzy IV and fuzzy BVPs are solved by the method of fuzzy LT. To solve FFDE by using fuzzy LT, problems are reduced to an algebraic problem, the transferring of calculus operations to algebraic operations is very important in applied mathematics known as operational calculus. The fuzzy LT also has the benefit that without pointing out a general solution in the first and achieving non-homogeneous DES in the second, it directly solves problems. In the literature [25], Ahmad et al. investigated a fuzzy COVID-19 model through fuzzy LT. The fuzzy fractional Fisher equation has been solved by fuzzy LT [26]. Ullah et al. computed the solution of the fuzzy population model under fractional operator through fuzzy LT [27]. In the paper [28], Ahmad et al. derived solution of fuzzy dispersive PDEs under fractional operators via fuzzy LT. In our article, we will apply fuzzy LT to get the numerical results of the proposed problem. Inspired by the above literature on FDEs, we investigate model (1) under fuzzy fractional Caputo’s operator. Consider (1) as

\[
\begin{align*}
D^\alpha_0 \tilde{\varphi}(t) &= \lambda \circ \tilde{\varphi}(t) + \kappa \circ \tilde{\varphi}(t), \\
D^\beta_0 \tilde{\varphi}(t) &= \lambda \circ \tilde{\varphi}(t) + (\epsilon + \kappa) \circ \tilde{\varphi}(t),
\end{align*}
\]

associated to fuzzy initial conditions, for \( \varphi_0 \in [0, 1] \),

\[
\begin{align*}
\tilde{\varphi}(0) &= \overline{[\tilde{\varphi}(0, \varphi_0), \tilde{\varphi}(0, \varphi_0)]}, \\
\tilde{\varphi}(0) &= \overline{[\tilde{\varphi}(0, \varphi_0), \tilde{\varphi}(0, \varphi_0)]},
\end{align*}
\]

where \( \tilde{\varphi}(t) \) and \( \tilde{\varphi}(t) \) are fuzzy functions and \( \tilde{\varphi}(0) \) and \( \tilde{\varphi}(0) \) are fuzzy numbers.

2. Preliminaries

Here, we give preliminary concepts of fuzzy fractional calculus. For more details, the reader can find more knowledge in [29,30].

**Definition 2.1 ([31])**: A fuzzy set \( \tilde{\varphi} : \mathbb{R} \rightarrow [0, 1] \) is said to be a fuzzy number if:

(i) Closure of the set \( \{m \in \mathbb{R} : \tilde{\varphi}(m) > 0 \} \) is compact.

(ii) \( \tilde{\varphi} \) is normal (for some \( z_0 \in \mathbb{R} \); \( \tilde{\varphi}(z_0) = 1 \)).

(iii) \( \tilde{\varphi} \) is fuzzy convex \( (\tilde{\varphi}(b \varphi + (1 - b) \varphi) \geq (\tilde{\varphi}(f) \land \tilde{\varphi}(g)) \text{ for all } b \in [0, 1], f, g \in \mathbb{R} \).

(iv) \( \tilde{\varphi} \) is upper semi-continuous on \( \mathbb{R} \).

**Definition 2.2 ([16,31])**: The parametric form of a fuzzy number \( \tilde{\varphi}(\varphi_0) \) is denoted by \([\tilde{\varphi}(\varphi_0), \tilde{\varphi}(\varphi_0)]\), for \( \varphi_0 \in [0, 1] \), having properties:

(i) \( \tilde{\varphi}((\varphi_0)) \leq \tilde{\varphi}((\varphi_0)). \)

(ii) \( \tilde{\varphi}(\varphi_0) \) is non-increasing function, right continuous at 0 and left continuous over \( (0, 1] \).

(iii) \( \tilde{\varphi}(\varphi_0) \) is an non-decreasing function, right continuous at 0 and left continuous over \( (0, 1] \).

If \( \tilde{\varphi} = [\tilde{\varphi}(\varphi_0), \tilde{\varphi}(\varphi_0)] \) and \( \tilde{\rho} = [\tilde{\rho}(\varphi_0), \tilde{\rho}(\varphi_0)] \) belong to \( \mathbb{F} \) (set of fuzzy numbers) and \( k_0 \) be a real number, then fuzzy addition, fuzzy subtraction and scalar multiplication between the two fuzzy numbers are, respectively, defined as

\[
\begin{align*}
\tilde{\varphi} + \tilde{\rho} &= [\tilde{\varphi}(\varphi_0) + \tilde{\rho}(\varphi_0), \tilde{\varphi}(\varphi_0) + \tilde{\rho}(\varphi_0)], \\
\tilde{\varphi} - \tilde{\rho} &= [\tilde{\varphi}(\varphi_0) - \tilde{\rho}(\varphi_0), \tilde{\varphi}(\varphi_0) - \tilde{\rho}(\varphi_0)], \quad k_0 \geq 0, \\
k_0 \circ \tilde{\varphi} &= \begin{cases} 
\{k_0 \tilde{\varphi}(\varphi_0), k_0 \tilde{\varphi}(\varphi_0)\}, & k_0 \geq 0, \\
\{k_0 \overline{\tilde{\varphi}(\varphi_0)}, k_0 \overline{\tilde{\varphi}(\varphi_0)}\}, & k_0 < 0.
\end{cases}
\end{align*}
\]

**Definition 2.3 ([32])**: Let \( t_1, t_2 \in \mathbb{R} \). The Hukuhara difference between \( t_1 \) and \( t_2 \) is denoted by \( t_1 \circ^H t_2 \), if there exists \( t_3 \) such that \( t_1 = t_2 \circ^H t_3 \). Also, \( t_1 \circ^H t_2 \neq t_1 + (-1)t_2 \).
Definition 2.4 ([21]): Assume that \( \tilde{\vartheta} : (a_0, b_0) \to \mathbb{F} \) and \( j_0 \in (\alpha, \beta) \). Then we say that \( \tilde{\vartheta} \) is \( H \)-differentiable at \( j_0 \) if an element \( \tilde{\vartheta}'(j_0) \in \mathbb{F} \) exists, such that:

1. for any \( p > 0 \) sufficiently small, \( \exists \tilde{\vartheta}_p(j_0 + p) \subseteq \tilde{\vartheta}(j_0) \) and
2. \( \tilde{\vartheta}(j_0) \circ \tilde{\vartheta}(j_0 - p) \), such that

\[
\lim_{h \to 0} \frac{\tilde{\vartheta}(j_0 + p) \circ \tilde{\vartheta}(j_0)}{p} = \lim_{h \to 0} \frac{\tilde{\vartheta}(j_0) \circ \tilde{\vartheta}(j_0 - p)}{p} = \tilde{\vartheta}'(j_0),
\]

(2) for any \( p > 0 \) sufficiently small, \( \exists \tilde{\vartheta}(j_0) \subseteq \tilde{\vartheta}(j_0 + p) \), and

\[
\lim_{p \to 0} \frac{\tilde{\vartheta}(j_0) \circ \tilde{\vartheta}(j_0) + p}{p} = \lim_{p \to 0} \frac{\tilde{\vartheta}(j_0) \circ \tilde{\vartheta}(j_0 - p)}{-p} = \tilde{\vartheta}'(j_0).
\]

For easy readability, we can say that \( \tilde{\vartheta} \) is (1)-differentiable on \( a_0, b_0 \) if \( \tilde{\vartheta} \) is differentiable in case of (1) and similarly if \( \tilde{\vartheta} \) is differentiable in case of (2), then \( \tilde{\vartheta} \) is (2)-differentiable on \( a_0, b_0 \).

Theorem 2.5 ([31]): Let \( \tilde{\vartheta} : R \to \mathbb{F} \) be a fuzzy mapping and \( \tilde{\vartheta}(x, x_0) = [\tilde{\vartheta}(x, x_0), \tilde{\vartheta}(x, x_0)] \), for all \( 0 \leq x_0 \leq 1 \).

1. \( [\tilde{\vartheta}'(x)](x_0) = [\tilde{\vartheta}'(x, x_0), \tilde{\vartheta}'(x, x_0)] \), if \( \tilde{\vartheta} \) is a (1)-differentiable function.
2. \( [\tilde{\vartheta}'(x)](x_0) = [\tilde{\vartheta}(x, x_0), \tilde{\vartheta}(x, x_0)] \), if \( \tilde{\vartheta} \) is a (2)-differentiable function.

Definition 2.6 ([31]): The fuzzy Caputo derivative for \( \tilde{\vartheta} \in \mathcal{C}(\mathbb{R}, \mathbb{F}) \) at \( x \) is defined as

\[
\left( CD^\alpha \tilde{\vartheta} \right)(x) = \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - \zeta)^{-\gamma} \frac{d}{d\zeta} \tilde{\vartheta}(\zeta) d\zeta,
\]

where \( \gamma \in (0, 1) \); then \( \tilde{\vartheta} \) is \( C(1 - \gamma) \)-differentiable if (3) holds, while \( \tilde{\vartheta} \) is (1)-differentiable, and \( \tilde{\vartheta} \) is \( C(2 - \gamma) \)-differentiable if (3) holds, while \( \tilde{\vartheta} \) is (2)-differentiable.

Definition 2.7 ([31]): Let \( \tilde{\vartheta} \in \mathcal{C}(\mathbb{R}, \mathbb{F}) \) be such that \( \tilde{\vartheta} = [\tilde{\vartheta}(x_0), \tilde{\vartheta}(x_0)] \), \( 0 \leq x_0 \leq 1 \) and \( t_1 \in (0, b) \) such that the fractional Caputo’s derivative is defined as follows:

\[
D^\alpha \tilde{\vartheta}(t_0) = [D^\alpha \tilde{\vartheta}(t_0), D^\alpha \tilde{\vartheta}(t_0)], \quad 0 \leq \alpha \leq 1,
\]

where

\[
D^\alpha \tilde{\vartheta}(t_0) = \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - \zeta)^{-\gamma} \frac{d}{d\zeta} \tilde{\vartheta}(\zeta) d\zeta.
\]

**Case(1):**

Consider \( \tilde{\vartheta}(t) \) and \( \tilde{\vartheta}(t) \) are \( C[(i - \gamma)] \)-differentiable functions. Then (6) can be written as

\[
\begin{align*}
D^\alpha \tilde{\vartheta}(t_0) & = \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - \zeta)^{-\gamma} \frac{d}{d\zeta} \tilde{\vartheta}(\zeta) d\zeta, \\
D^\alpha \tilde{\vartheta}(t_0) & = \frac{1}{\Gamma(1 - \gamma)} \int_0^t (t - \zeta)^{-\gamma} \frac{d}{d\zeta} \tilde{\vartheta}(\zeta) d\zeta,
\end{align*}
\]
For the infinite series solution, we take

\[
\begin{align*}
\tilde{y}(t, \varphi_0) &= \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0), \\
\tilde{y}(t, \varphi_0) &= \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0).
\end{align*}
\]  

(12)

Using (12) and considering the lower case of (11), we have

\[
\begin{align*}
\sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0) + L^{-1} \left[ \frac{1}{L} \left[ -\lambda \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0) \right] \right. \\
&\quad + \kappa \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0)] \\
&\quad - \left. (\epsilon + \kappa) \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0) \right].
\end{align*}
\]  

(13)

Similarly for upper functions, we get

\[
\begin{align*}
\sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0) + L^{-1} \left[ \frac{1}{L} \left[ -\lambda \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0) \right] \right. \\
&\quad + \kappa \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0)] \\
&\quad - \left. (\epsilon + \kappa) \sum_{n=0}^{\infty} \tilde{y}_n(t, \varphi_0) \right].
\end{align*}
\]  

(14)

Comparing terms of both sides of (13) and (14), we can find the series solution as follows:

\[
\begin{align*}
\tilde{y}_0(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0), \\
\tilde{y}_0(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0).
\end{align*}
\]  

(15)

\[
\begin{align*}
\tilde{y}_1(t, \varphi_0) &= L^{-1} \left[ \frac{1}{L} \left[ -\lambda \tilde{y}_0(t, \varphi_0) + \kappa \tilde{y}_0(t, \varphi_0) \right] \right. \\
&\quad + \left. \lambda \tilde{y}_0(t, \varphi_0) - (\epsilon + \kappa) \tilde{y}_0(t, \varphi_0) \right],
\end{align*}
\]  

(16)
\[\begin{align*}
\tilde{y}_2(t, \varphi_0) &= L^{-1} \left[ \frac{1}{s^2} L \left[ -\lambda \tilde{y}_1(t, \varphi_0) + \kappa \tilde{y}_1(t, \varphi_0) \right] \right], \\
\tilde{y}_2(t, \varphi_0) &= L^{-1} \left[ \frac{1}{s^2} L \left[ -\lambda \tilde{y}_1(t, \varphi_0) + \kappa \tilde{y}_1(t, \varphi_0) \right] \right], \\
\tilde{y}_2(t, \varphi_0) &= L^{-1} \left[ \frac{1}{s^2} L \left[ -\lambda \tilde{y}_1(t, \varphi_0) + (\epsilon + \kappa) \tilde{y}_1(t, \varphi_0) \right] \right], \\
\tilde{y}_2(t, \varphi_0) &= L^{-1} \left[ \frac{1}{s^2} L \left[ -\lambda \tilde{y}_1(t, \varphi_0) + (\epsilon + \kappa) \tilde{y}_1(t, \varphi_0) \right] \right].
\end{align*}\]

Solution for \(C_1(\tilde{y} - \gamma)\)-differentiability is given by

\[\begin{align*}
\frac{d}{dt} \tilde{y}(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0) + \tilde{y}_1(t, \varphi_0) + \cdots \\
\tilde{y}(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0) + \tilde{y}_1(t, \varphi_0) + \cdots \\
\tilde{y}(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0) + \tilde{y}_1(t, \varphi_0) + \cdots \\
\tilde{y}(t, \varphi_0) &= \tilde{y}_0(t, \varphi_0) + \tilde{y}_1(t, \varphi_0) + \cdots
\end{align*}\]

Case(ii):

Let us consider \(\tilde{y}(t)\) and \(\tilde{y}(t)\) are \(C_1(\tilde{y} - \gamma)\)-differentiable, then (6) can be written as

For lower functions:

\[\begin{align*}
L \left[ D_y^t \tilde{y}(t, \varphi_0) \right] &= L \left[ -\lambda \tilde{y}(t, \varphi_0) + \kappa \tilde{y}(t, \varphi_0) \right], \\
L \left[ D_y^t \tilde{y}(t, \varphi_0) \right] &= L \left[ -\lambda \tilde{y}(t, \varphi_0) - (\epsilon + \kappa) \tilde{y}(t, \varphi_0) \right], \\
L \left[ D_y^t \tilde{y}(t, \varphi_0) \right] &= L \left[ -\lambda \tilde{y}(t, \varphi_0) + \kappa \tilde{y}(t, \varphi_0) \right], \\
L \left[ D_y^t \tilde{y}(t, \varphi_0) \right] &= L \left[ -\lambda \tilde{y}(t, \varphi_0) + (\epsilon + \kappa) \tilde{y}(t, \varphi_0) \right].
\end{align*}\]
Using (12) and considering the lower case of (22), we have

\[
\begin{align*}
\sum_{a=0}^{\infty} \tilde{y}_a(t, \psi_0) \\
= \tilde{y}(0, \psi_0) + L^{-1} \left[ \frac{1}{s^\gamma} \int \left[ -\lambda \sum_{a=0}^{\infty} \tilde{y}_a(t, \psi_0) \\
+ \kappa \sum_{a=0}^{\infty} \tilde{y}_a(t, \psi_0) \right] dt \right], \\
\sum_{a=0}^{\infty} \tilde{y}_a(t, \psi_0) \\
= \tilde{y}(0, \psi_0) + L^{-1} \left[ \frac{1}{s^\gamma} \int \left[ \lambda \sum_{a=0}^{\infty} \tilde{y}_a(t, \psi_0) \\
- (\epsilon + \kappa) \sum_{a=0}^{\infty} \tilde{y}_a(t, \psi_0) \right] dt \right].
\end{align*}
\]

Comparing terms on both sides of (23) and (24), we have

\[
\begin{align*}
\tilde{y}_0(t, \psi_0) &= \tilde{y}(0, \psi_0), \\
\dot{\tilde{y}}_0(t, \psi_0) &= \tilde{y}(0, \psi_0), \\
\tilde{y}_0(t, \psi_0) &= \tilde{y}(0, \psi_0), \\
\tilde{y}_0(t, \psi_0) &= \tilde{y}(0, \psi_0),
\end{align*}
\]

and so on. The series solution under $\mathcal{C}([0, 1]-\gamma)$-differentiability is given by

\[
\begin{align*}
\tilde{y}(t, \psi_0) &= \tilde{y}_0(t, \psi_0) + \tilde{y}_1(t, \psi_0) + \cdots \\
\dot{\tilde{y}}(t, \psi_0) &= \dot{\tilde{y}}_0(t, \psi_0) + \dot{\tilde{y}}_1(t, \psi_0) + \cdots \\
\ddot{\tilde{y}}(t, \psi_0) &= \ddot{\tilde{y}}_0(t, \psi_0) + \ddot{\tilde{y}}_1(t, \psi_0) + \cdots
\end{align*}
\]

4. Numerical results and simulations

Let us take the fuzzy initial conditions as $\tilde{y}(0, \psi_0) = [\psi_0 - 1, 1 - \psi_0]$ and $\tilde{y}(0, \psi_0) = [\psi_0 - 1, 1 - \psi_0]$, where $\psi_0 \in [0, 1]$. On using the above procedure we get the following numerical results as

![Figure 1. 2D representation of $\tilde{y}(t, \psi_0)$ under type-1 differentiability at a different fractional order.](image-url)
Figure 2. 2D representation of $\tilde{G}(t, \psi_0)$ under type-1 differentiability at a different fractional order.

Figure 3. 3D representation of $\tilde{Y}(t, \psi_0)$ under type-1 differentiability at a different fractional order.
Systems (27) and (28) represent solutions of the proposed model under $C[(i) - \gamma]$ and $C[(ii) - \gamma]$-differentiability, respectively. We have presented the numerical results of type 1 and 2 differentiability graphically in Figures 1–8. We take few fractional order and uncertainty belongs to $[0, 1]$. The simulations are performed for the parameter values: $\lambda = 0.054736$, $\kappa = 0.0152704$ and $\epsilon = 0.0093906$. We see from the numerical results and graphs that solution of a fuzzy fractional model depends on the type of differentiability. The graphs show that the solution under both types of differentiability is fuzzy functions. To demonstrate the proposed model biologically, we simulate the obtained results (27–28) for a fixed fractional-order $\gamma = 0.8$ and for random values $\psi_0 \in [0, 1]$ (see Figure 9(a)). We set $[\psi_0 - 1, 1 - \psi_0] = \psi_0$ in Equation (27). In Figure 9(a), subfigures (a)–(d) represent the dynamics of the proposed model for various values uncertainty.

Figure 4. 3D representation of $\tilde{G}(t, \psi_0)$ under type-1 differentiability at a different fractional order.

Figure 5. 2D representation of $\tilde{Y}(t, \psi_0)$ under type-2 differentiability at a different fractional order.
\( \varphi_0 = 0.4, 0.6, 0.8, 1 \) respectively. The red colour curves are the BSP in the blood and the blue colour curves are BSP in the liver. The effect of the uncertainty \( \varphi_0 \) on the dynamics of the BSP of blood and liver can be seen in the subfigures (a)–(d). As we see from Figure 9(a), BSP in the blood decreases while BSP in the liver increases. From subfigures (a)–(d), we note that as the uncertainty increases from 0.4 to 1, as a result, rapid increase and decrease occur in the BSP of liver and blood. Also, stability occurs rapidly as uncertainty increases. Hence, we concluded that fuzzy fractional operators are better tools to model a real phenomenon because the fractional order solution and its corresponding curves approach the integers order solution and curves as fractional orders tends to 1. Thus, fuzzy fractional calculus provides a global dynamics of any mathematical model instead of ordinary and fractional order model.
Figure 8. 3D representation of $\tilde{\mathcal{F}}(t, \varphi_0)$ under type-2 differentiability at a different fractional order.

Figure 9. Dynamics of the model at $\gamma = 0.8$ and various particular values of $\varphi_0$. 
5. Conclusion

In this paper, we have analysed the human liver mathematical model by fuzzy fractional Caputo’s derivative. Laplace transforms is one of the important transformation in the theory of fuzzy DE. FFDEs, its fuzzy IV and fuzzy BVPs are solved by the method of the fuzzy Laplace transform. To solve FFDE by using the fuzzy Laplace transform, problems are reduced to an algebraic problem, the transferring of calculus operations to algebraic operations is very important in applied mathematics known as operational calculus. The fuzzy Laplace transformation also has the benefit that without identifying a general solution in the first and achieving non-homogeneous DEs in the second, it directly solves problems. Therefore, we have implemented a fuzzy Laplace transform to established the general procedure for the solution of the proposed model under the concept of H-differentiability. On using the proposed algorithm, we have determined the numerical results of the considered model. The acquired theoretical results have been represented through 2D and 3D graphs for the different values of the fractional order and uncertainty $\psi_0 \in [0, 1]$. We concluded from the figures that fractional DEs, along with fuzziness, provide global dynamics of any physical phenomena. Thus, modelling of the physical problem with fuzzy operators is better in an uncertain environment. However, the proposed method has some limitations. The fuzzy Laplace transform solves only linear equations. It cannot handle the nonlinear problems alone. On the other hand, sometimes H-difference does not hold. Then another approaches like $gH$-differentiability should be considered to study a model in fuzzy setting. In our next project, we will investigate different mathematical models through various fuzzy operators.

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