Asymptotic properties of the hyperbolic metric on the sphere with three conical singularities *

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Abstract

The explicit formula for the hyperbolic metric \( \lambda_{\alpha, \beta, \gamma}(z) |dz| \) on the thrice-punctured sphere \( \mathbb{P}\{z_1, z_2, z_3\} \) with singularities of order \( \alpha, \beta, \gamma \leq 1 \) with \( \alpha + \beta + \gamma > 2 \) at \( z_1, z_2, z_3 \) was given by Kraus, Roth and Sugawa in [10]. In this paper we investigate the asymptotic properties of the higher order derivatives of \( \lambda_{\alpha, \beta, \gamma}(z) \) near the singularity and give some more precise description for the asymptotic behavior.

1 Introduction

From the viewpoint of the theory of partial differential equations and functions, the hyperbolic metric, also called the Poincaré metric, plays an important role in metric spaces. It facilitates describing the hyperbolic geometry on some domains in different ways. Since the Gaussian curvature of the hyperbolic metric is a negative constant, it can be regarded as the extremal metric of a class of regular conformal metrics with strictly negative Gaussian curvature functions. This kind of conformal metric is more general, and it was discussed by Heins in [4], Kraus, Roth and Ruscheweyh in [9].

Equipped with the hyperbolic metric, a punctured domain is more complicated than a simply connected domain. An elementary case for the punctured domain is the hyperbolic metric \( \lambda_{D^*}(z)|dz| \) on the (once-)punctured unit disk \( D\{0\} \), which is defined by

\[
\lambda_{D^*}(z)|dz| = \frac{|dz|}{2|z| \log(1/|z|)}
\]

with the Gaussian curvature \( -4 \). It is induced by the hyperbolic metric

\[
\lambda_{D}(z)|dz| = \frac{|dz|}{1 - |z|^2}
\]

on the unit disk \( D \). The expression for the hyperbolic metric on the twice-punctured unit disk is not achieved yet, and there are only a few estimates for the density function, pre-Schwarzian and Schwarzian, see [5] [6] [13] for details.

In the thrice-punctured sphere \( \mathbb{P}\{z_1, z_2, z_3\} \) with singularities of order \( \alpha, \beta, \gamma \leq 1 \) at \( z_1, z_2, z_3 \), if \( \alpha + \beta + \gamma > 2 \), the hyperbolic metric \( \lambda_{\alpha, \beta, \gamma}(z)|dz| \) can be expressed in terms

*Keywords. Conical singularities, hyperbolic metrics.
of special functions in $\mathbb{C}\\{0, 1\}$, see [10]. Kraus, Roth and Sugawa used the Liouville equation

$$\Delta u = 4e^{2u} \quad (1.1)$$

to obtain the explicit formula of $\lambda_{\alpha, \beta, \gamma}(z)$. For equation (1.1), Liouville proved in [11] that, in any disk $D$ contained in the punctured unit disk $\mathbb{D}\{0\}$ every solution $u$ to (1.1) can be written as

$$u(z) = \log \left( \frac{|f'(z)|}{1 - |f(z)|^2} \right), \quad (1.2)$$

where $f$ is a holomorphic function in $D$. Kraus, Roth and Sugawa at first obtained the expression of function $f$ in equation (1.2) using the hypergeometric differential equation

$$zw''(z) + [\alpha - (\alpha + \beta - 1)z]w'(z) - \frac{(\alpha + \beta - \gamma)(\alpha + \beta + \gamma - 2)}{4}w(z) = 0,$$

and then gave the explicit formula for $\lambda_{\alpha, \beta, \gamma}(z)$. Note that only the special equation (1.1) is involved here since $\lambda_{\alpha, \beta, \gamma}(z)$ has the Gaussian curvature $-4$. We concern the estimate for the derivatives of $\lambda_{\alpha, \beta, \gamma}(z)$ near the origin and give a stronger result than the estimates in [15].

Then we discuss the so-called Minda-type theorems. In 1997, Minda [12] studied the behavior of the hyperbolic metric in a neighborhood of a puncture on the plane domain using the uniformization theorem for up to second order derivatives. His results can be extended to higher order derivatives of a conformal metric with negative curvatures on an arbitrary hyperbolic region, see [15]. However, near the origin, if the order $\alpha < 1$, this kind of limits may not exist. We prove that for the hyperbolic metric $\lambda_{\alpha, \beta, \gamma}(z)$, the limits in Minda-type always exist, and give the recurrence formula for them.

2 Preliminaries

For complex numbers $a, b, c$ with $c \neq 0, -1, -2, \ldots$, the Gaussian hypergeometric function is defined as

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}, \quad |z| < 1,$$

where $(a)_n$ is the Pochhammer symbol, namely, $(a)_0 = 1$ and

$$(a)_n = a(a + 1) \cdots (a + n - 1)$$

for $n = 1, 2, 3, \ldots$. It is continued analytically to the slit plane $\mathbb{C}\{1, +\infty\}$. Its derivative is given by

$$\frac{d}{dz}F(a, b, c; z) = \frac{ab}{c}F(a + 1, b + 1, c + 1; z). \quad (2.1)$$

We can immediately obtain

$$\frac{d^n}{dz^n}F(a, b, c; z) = \frac{(a)_n(b)_n}{(c)_n}F(a + n, b + n, c + n; z). \quad (2.2)$$
We have
\[ F(a, b, c; z) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F(a, b, a + b - c + 1; 1 - z) \]
\[ + (1 - z)^c - a - b \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)} F(c - a, c - b, c - a - b + 1; 1 - z) \quad (2.3) \]
for \(|\arg(1 - z)| < \pi\), where \(\Gamma(z)\) is the gamma function, see 15.3.6 in [1]. Each term of (2.3) has a pole when \(c = a + b \pm n\), \(n = 0, 1, 2, \ldots\), and this case is covered by
\[ F(a, b, a + b + n; z) = \frac{\Gamma(n)\Gamma(a + b + n)}{\Gamma(a + n)\Gamma(b + n)} \sum_{j=0}^{n-1} \frac{(a)_j(b)_j}{j!(1 - n)_j} (1 - z)^j \]
\[ - \frac{\Gamma(a + b + n)}{\Gamma(a)\Gamma(b)} (z - 1)^n \sum_{j=0}^{\infty} \frac{(a + n)_j(b + n)_j}{j!(j + n)!} (1 - z)^j \log(1 - z) \]
\[ - \Psi(j + 1) - \Psi(j + n + 1) + \Psi(a + j + n) + \Psi(b + j + n), \quad (2.4) \]
for \(|\arg(1 - z)| < \pi\), \(|1 - z| < 1\), where \(\Psi(z) = \Gamma'(z)/\Gamma(z)\) is the digamma function, see 15.3.11 in [1], and we take the convention that \(\sum_{j=a}^{b} = 0\) if \(b < a\) here and after. The behavior of the hypergeometric function near \(z = 1\) satisfies
\[
\begin{align*}
F(a, b, c; 1) &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \quad \text{if } a + b < c, \\
F(a, b, a + b; z) &= \frac{1}{B(a, b)} \left( \log \frac{1}{1 - z} + R(a, b) \right) (1 + O(1 - z)), \quad (2.5) \\
F(a, b, c; z) &= (1 - z)^c - a - b F(c - a, c - b, c; z), \quad \text{if } a + b > c.
\end{align*}
\]
Here
\[ B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a + b)} \quad (2.6) \]
is the beta function and
\[ R(a, b) = 2\Psi(1) - \Psi(a) - \Psi(b) \quad (2.7) \]
with \(\Psi(x) = \Gamma'(x)/\Gamma(x)\) being the digamma function. The asymptotic formula in (2.5) for the case \(a + b = c\) is due to Ramanujan, see [1, 3].

In the domain \(G \subseteq \mathbb{C}\), every positive, upper semi-continuous function \(\lambda : G \to (0, +\infty)\) induces a conformal metric on \(G\). We denote the metric by \(\lambda(z)|dz|\), see [1, 3]. We \(\lambda(z)|dz|\) the metric and \(\lambda(z)\) the density to avoid any ambiguity. A conformal metric \(\lambda(z)|dz|\) on a domain \(G \subseteq \mathbb{C}\) is said to be regular, if its density \(\lambda(z)\) is positive and twice continuously differentiable on \(G\), i.e. \(\lambda(z) > 0\) and \(\lambda(z) \in C^2(G)\). For a domain \(G \subseteq \mathbb{C}\) equipped with a conformal metric \(\lambda(z)|dz|\), the distance function associating two points \(z, \zeta \in G\) is defined by
\[ \rho_\lambda(z, \zeta) := \inf_i \int_i \lambda(z)|dz|, \quad (2.8) \]
where the infimum is taken over all rectifiable paths $\iota$ in $G$ joining $z$ and $\zeta$. We call $(G, \rho_\lambda)$ a metric space. The metric $\lambda(z) |dz|$ is said to be complete on $G$ if $(G, \rho_\lambda)$ is a complete metric space. The Gaussian curvature $\kappa_\lambda(z)$ of the regular conformal metric $\lambda(z) |dz|$ is defined by

$$\kappa_\lambda(z) = -\frac{\Delta \log \lambda(z)}{\lambda(z)^2},$$

where $\Delta$ denotes the Laplace operator. For the definition of the Gaussian curvature in more general case, see [14].

The basic property of Gaussian curvature is its conformal invariance. That means, given a (regular) conformal metric $\lambda(z) |dz|$ on a domain $G$ and a holomorphic mapping $f : \Omega \to G$ on a Riemann surface $\Omega$, the pullback

$$f^* \lambda(w) |dw| := \lambda(f(w)) |f'(w)| |dw|$$

is still a (regular) conformal metric on $\Omega \setminus \{\text{critical points of } f\}$ with Gaussian curvature

$$\kappa_{f^* \lambda}(w) = \kappa_\lambda(f(w)).$$

Here $\Omega \setminus \{\text{critical points of } f\}$ is a punctured domain, the critical points of $f$ are the source of punctures. If the neighborhood of a puncture carries some special structure as given below, we say this puncture is a singularity.

Let $\mathbb{P}$ denote the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and let $\Omega \subseteq \mathbb{P}$ be a subdomain. For a point $p \in \Omega$, let $z$ be local coordinates such that $z(p) = 0$. We say a conformal metric $\lambda(z) |dz|$ on the punctured domain $\Omega^* := \Omega \setminus \{p\}$ has a conical singularity of order $\alpha \leq 1$ at the point $p$, if, in local coordinates $z$,

$$\log \lambda(z) = \begin{cases} 
  -\alpha \log |z| + v(z) & \text{if } \alpha < 1 \\
  -\log |z| - \log \log(1/|z|) + w(z) & \text{if } \alpha = 1,
\end{cases}$$

(2.9)

where $v(z), w(z) = O(1)$ as $z(p) \to 0$ with $O$ and $o$ being the Landau symbols throughout our study. For $u(z) := \log \lambda(z)$, the order $\alpha$ of $\lambda(z) |dz|$ at the conical singularity $p$ is again the order of $u(z)$ at the conical singularity $\log p$. We call the point $p$ a corner of order $\alpha$ if $\alpha < 1$ and a cusp if $\alpha = 1$. It is evident that the cusp is the limit case of the corner.

The hyperbolic metric $\lambda_\Omega(z) |dz|$ on a domain $\Omega$ is a complete metric with some negative constant Gaussian curvature, here we take the constant to be $-4$. By (2.8), the hyperbolic distance between $z, \zeta \in \Omega$ is

$$d_\Omega(z, \zeta) := \inf \int_\iota \lambda_\Omega(z) |dz|$$

and the infimum is always attained; the hyperbolic line passing through $z$ and $\zeta$ is the path for which the infimum is attained. From the conformal invariance of Gaussian curvature we know that, the hyperbolic metric $\lambda_\Omega(z) |dz|$ on any domain $\Omega$ induces a hyperbolic metric on some domain which is conformally equivalent to $\Omega$. The hyperbolic metric is a kind of metric of special interest because it is the unique maximal conformal metric in the sense of conformal invariance, see [2, 4]. The following result gives the explicit formula of the hyperbolic metric on twice-punctured plane $\mathbb{C} \setminus \{0, 1\}$. The terminology *generalized hyperbolic metric* is motivated by the fact that if all singularities are cusps, then we can get back the standard hyperbolic metric on the punctured sphere $\mathbb{P} \setminus \{z_1, \ldots, z_n\}$, see [10].
Theorem A ([10]) Let $0 < \alpha$, $\beta < 1$ and $0 < \gamma \leq 1$ such that $\alpha + \beta + \gamma > 2$. Then the generalized hyperbolic density on the thrice-punctured sphere $\mathbb{P}\{0, 1, \infty\}$ of orders $\alpha, \beta, \gamma$ at $0, 1, \infty$, respectively, can be expressed by

\[
\lambda_{\alpha, \beta, \gamma}(z) = \frac{1}{|z|^\alpha|1-z|^\beta} \cdot \frac{K_3}{K_1 |\varphi_1(z)|^2 + K_2 |\varphi_2(z)|^2 + 2\text{Re}(\varphi_1(z)\varphi_2(\bar{z}))} \tag{2.10}
\]

in the twice-punctured plane $\mathbb{C}\{0, 1\}$, where

\[
K_1 := \frac{\Gamma(c-a)\Gamma(c-b)}{\Gamma(c)\Gamma(c-a-b)}, \quad K_2 := \frac{\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(1-c)\Gamma(a+b+1-c)},
\]

\[
K_3 := \sqrt{\frac{\sin(\pi a)\sin(\pi b)}{\sin(\pi(c-a))\sin(\pi(c-b))}} \cdot \frac{\Gamma(a+b+1-c)\Gamma(c)}{\Gamma(a)\Gamma(b)}
\]

and

\[
\varphi_1(z) = F(a,b,c;z), \quad \varphi_2(z) = F(a,b,a+b-c+1;1-z),
\]

\[
\varphi_3(z) = F(a-c+1,b-c+1,2-c;z),
\]

with

\[
a = \frac{\alpha + \beta - \gamma}{2}, \quad b = \frac{\alpha + \beta + \gamma - 2}{2}, \quad c = \alpha; \tag{2.13}
\]

\[
\delta = \frac{\Gamma(c)}{\Gamma(2-c)} \left( \frac{\Gamma(1-a)\Gamma(1-b)\Gamma(a+1-c)\Gamma(b+1-c)}{\Gamma(a)\Gamma(b)\Gamma(c-a)\Gamma(c-b)} \right) \frac{1}{2}.
\]

The Gaussian curvature of $\lambda(z)$ defined by (2.10) and (2.11) is $-4$. Note that $\varphi_1$ and $\varphi_3$ are analytic in $\mathbb{C}\{1, +\infty\}$, $\varphi_2$ is analytic in $\mathbb{C}\{\infty\}$.

Expressions (2.10) and (2.11) are equal to each other. Denote $\log \lambda(z) := \log \lambda_{\alpha, \beta, \gamma}(z)$ for short, and

\[
\partial^n := \frac{\partial^n}{\partial z^n}, \quad \bar{\partial}^m := \frac{\partial^m}{\partial \bar{z}^m}
\]

for $n \geq 1$. The following theorem is a general estimate for $\log \lambda(z)$ near the singularities.

Theorem B ([15]) For $\lambda(z)$ as in (2.10) with order $\alpha \in (0, 1]$, let $u(z) = \log \lambda(z)$. Then for $m, n \geq 1$,

(i) $\lim_{z \to 0} z^n \partial^n u(z) = \frac{\alpha}{2}(-1)^n(n-1)! = \lim_{z \to 0} z^n \bar{\partial}^m u(z)$,

(ii) $\lim_{z \to 0} z^m z^n \bar{\partial}^m \partial^n u(z) = 0$.

We can estimate the higher order derivatives of a conformal density function $\lambda(z)$ directly. The following result is of Minda-type.
**Theorem C** ([15]) Let $\lambda(z)|dz|$ be a regular conformal metric on a domain $\Omega \subseteq \mathbb{C}$ with an isolated singularity at $z = p$. Suppose that the curvature $\kappa : \Omega \to \mathbb{R}$ has a Hölder continuous extension to $\Omega \cup \{p\}$ such that $\kappa(p) < 0$ and the order of $\log \lambda$ is $\alpha = 1$ at $z = p$. Then

(i) $\lim_{z \to p} (z - p)|z - p|\log(1/|z - p|)\lambda_z(z) = -\frac{1}{2\sqrt{-\kappa(p)}}$,

(ii) $\lim_{z \to p} (z - p)^2|z - p|\log(1/|z - p|)\lambda_{zz}(z) = \frac{3}{4\sqrt{-\kappa(p)}}$,

(iii) $\lim_{z \to p} |z - p|^3\log(1/|z - p|)\lambda_{z\bar{z}}(z) = \frac{1}{4\sqrt{-\kappa(p)}}$.

Theorem C was given only for the order $\alpha = 1$. When the order $\alpha < 1$, the analogous limit

$$\lim_{z \to 0} |z|^\alpha \lambda(z)$$

(2.15)

does not necessarily exist. But for the hyperbolic density $\lambda_{\alpha, \beta, \gamma}(z)$, if $0 < \alpha < 1$, expression (2.11) shows that the limit (2.15) exists. The following theorem is due to Kraus, Roth and Sugawa in [10]. They did not give the explicit formula of (2.15), but it is easy to deduce that from Corollary 4.4 in their paper.

**Theorem D** For the hyperbolic density $\lambda_{\alpha, \beta, \gamma}$ given in (2.11), if $0 < \alpha < 1$, then we have

$$\lim_{z \to 0} |z|^\alpha \lambda(z) = \frac{\delta}{1 - \delta^2}(1 - \alpha)$$

(2.16)

where $\delta$ is as in (2.14), $a$, $b$ and $c$ are as in (2.13).

### 3 Case $0 < \alpha < 1$

In this section we consider the hyperbolic metric when the order $0 < \alpha < 1$. We again let $\lambda(z) := \lambda_{\alpha, \beta, \gamma}(z)$. For the hyperbolic density function $\lambda(z)$, we can only consider the asymptotic behavior near the origin. By the expression of $\lambda(z)$, we know that the singularity $z = 1$ is the same as the origin. As for the infinity, we can change the coordinates by a conformal function, say, $z \mapsto 1/z$, to map $\infty$ onto 0. But some calculation is involved, so it is convenient to consider the case near the origin. In expression (2.10), for orders $0 < \alpha$, $\beta < 1$ and $0 < \gamma \leq 1$, the real parameters $\alpha, \beta, \gamma$ given by condition (2.13) satisfy

$$-\frac{1}{2} < a < 1, \quad -1 < b < \frac{1}{2}, \quad 0 < c < 1.$$  

At first we give a lemma for future use.

**Lemma 3.1** In the expression for $\lambda(z)$ as in (2.10) with order $\alpha \in (0, 1)$, let

$$M(z) := K_1|\varphi_1(z)|^2 + K_2|\varphi_2(z)|^2 + 2\text{Re}(\varphi_1(z)\varphi_2(\bar{z}))$$

$$= (K_1\varphi_1(\bar{z}) + \varphi_2(\bar{z}))\varphi_1(z) + (K_2\varphi_2(\bar{z}) + \varphi_1(\bar{z}))\varphi_2(z).$$

(3.1)
Then for $a, b$ and $c$ are defined in (2.13), $K_1$ and $K_2$ are defined in (212),

(1) $\lim_{z \to 0} \partial M(z) = \frac{ab}{c} \left( K_1 - \frac{1}{K_2} \right)$ for $0 < \alpha < 1/2$,

(2) $\partial M(z) = 2ab \left( K_1 - \frac{1}{K_2} \right) + 2K_2 \left( \frac{\Gamma(c)\Gamma(a + b - c + 1)}{\Gamma(a)\Gamma(b)} \right) \frac{z}{|z|} + O \left( \frac{1}{|z|^2} \right)$ for $\alpha = 1/2$,

(3) $\lim_{z \to 0} z^n |z|^{2\alpha - 2} \partial^n M(z) = \frac{(-1)^{n-1}(c)_{n-1}K_2}{1 - c} \left( \frac{\Gamma(c)\Gamma(a + b - c + 1)}{\Gamma(a)\Gamma(b)} \right) \frac{z}{|z|}$ for $n \geq 2$ if $0 < \alpha \leq 1/2$ and $n \geq 1$ if $1/2 < \alpha < 1$,

(4) $\lim_{z \to 0} z^m z^n |z|^{2\alpha - 2} \partial^m \partial^n M(z) = (-1)^{n+m}(c)_{n-1}(c)_{m-1}K_2 \left( \frac{\Gamma(c)\Gamma(a + b - c + 1)}{\Gamma(a)\Gamma(b)} \right) \frac{z}{|z|}$ for $m, n \geq 1$.

**Remark.** Case (2) can be expressed by $\partial M(z) = O(1)$. It is easy to see that there is no non-vanishing limit such as in (3) holds for $n = 1$ and $\alpha = 1/2$, even if it is multiplied by a power of $z/\bar{z}$.

**Proof of Theorem 3.1.** Since $\varphi_1(z), \varphi_2(z)$ are analytic in $\mathbb{C} \setminus [1, +\infty), \mathbb{C} \setminus (-\infty, 0]$ respectively, then we have $\partial^n \varphi_1(z) = \partial^n \varphi_1(\bar{z})$ for $z \in \mathbb{C} \setminus [1, +\infty), \partial^n \varphi_2(z) = \partial^n \varphi_2(\bar{z})$ for $z \in \mathbb{C} \setminus (-\infty, 0]$. For limit (1), we have

$$\partial M(z) = (K_1 \varphi_1(\bar{z}) + \varphi_2(z))\partial \varphi_1(z) + (K_2 \varphi_2(\bar{z}) + \varphi_1(z))\partial \varphi_2(z).$$

From properties (2.2),

$$\partial \varphi_1(0) = \frac{ab}{c}, \quad (3.2)$$

and from (2.3),

$$\varphi_2(0) = F(a, b, a + b - c + 1; 1) = -\frac{1}{K_2}, \quad \varphi_1(0) = 1, \quad (3.3)$$

provided that $a + b < a + b - c + 1$, so

$$K_1 \varphi_1(0) + \varphi_2(0) = K_1 - K_2^{-1}. \quad (3.4)$$

Now we consider the term $(K_2 \varphi_2(\bar{z}) + \varphi_1(z))\partial \varphi_2(z)$, which satisfies

$$\lim_{z \to 0} (K_2 \varphi_2(z) + \varphi_1(z)) = 0.$$

Note that

$$\varphi_2(z) = F(a, b, a + b - c + 1; 1 - z) \frac{\Gamma(a + b - c + 1)\Gamma(1 - c)}{\Gamma(b - c + 1)\Gamma(a - c + 1)} F(a, b, c, z) + z^{1-c} \frac{\Gamma(a + b - c + 1)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} F(b - c + 1, a - c + 1, 2 - c; z)$$

$$= -\frac{1}{K_2} \varphi_1(z) + z^{1-c} \frac{\Gamma(a + b - c + 1)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} F(b - c + 1, a - c + 1, 2 - c; z)$$

$$= -\frac{1}{K_2} \varphi_1(z) + z^{1-c} \frac{\Gamma(a + b - c + 1)\Gamma(c - 1)}{\Gamma(a)\Gamma(b)} F(b - c + 1, a - c + 1, 2 - c; z)$$

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for $|\arg(z)| < \pi$, which means $\varphi_1(z)$ and $\varphi_2(z)$ are related, so

$$K_2\varphi_2(z) + \varphi_1(z) = -K_2z^{1-c} \frac{\Gamma(a + b - c + 1)\Gamma(c)}{\Gamma(a)\Gamma(b)} F(b - c + 1, a - c + 1, 2 - c; z) \tag{3.5}$$

and

$$\lim_{z \to 0} \frac{K_2\varphi_2(z) + \varphi_1(z)}{z^{1-c}} = -K_2 \frac{\Gamma(a + b - c + 1)\Gamma(c)}{\Gamma(a)\Gamma(b)}. \tag{3.6}$$

Near the origin, by (2.2), for $n \geq 1$,

$$\frac{\partial^n \varphi_2(z)}{(a + b - c + 1)_n} = \frac{(a)_n(b)_n}{(a + b - c + 1)_n} (-1)^n F(a + n, b + n, a + b - c + 1 + n; 1 - z). \tag{3.7}$$

By property (2.3),

$$F(a + n, b + n, a + b - c + 1 + n; 1 - z) = \frac{\Gamma(a + b - c + 1 + n)\Gamma(1 - c - n)}{\Gamma(b - c + 1)\Gamma(a - c + 1)} F(a + n, b + n, c + n; z)$$

$$+ z^{1-c-n} \frac{\Gamma(a + b - c + 1 + n)\Gamma(c + n - 1)}{\Gamma(a + n)\Gamma(b + n)} F(b - c + 1, a - c + 1, 2 - c - n; z)$$

for $|\arg(z)| < \pi$, then near the origin, substituting the above into (3.7), we have

$$\frac{\partial^n \varphi_2(z)}{(a + b - c + 1)_n} = \frac{(a)_n(b)_n}{(a + b - c + 1)_n} (-1)^n \frac{\Gamma(a + b - c + 1 + n)\Gamma(1 - c - n)}{\Gamma(b - c + 1)\Gamma(a - c + 1)} F(a + n, b + n, c + n; z)$$

$$+ \frac{(-1)^n \Gamma(a + b - c + 1)\Gamma(c + n - 1)}{\Gamma(a)\Gamma(b)} F(b - c + 1, a - c + 1, 2 - c - n; z), \tag{3.8}$$

which leads to the limit

$$\lim_{z \to 0} z^{n+c-1}{\partial^n \varphi_2(z)} = (-1)^n(c)_{n-1} \frac{\Gamma(a + b - c + 1)\Gamma(c)}{\Gamma(a)\Gamma(b)}. \tag{3.9}$$

Letting $n = 1$ in (3.8) and combining with (3.5), we have

$$\lim_{z \to 0} (K_2\varphi_2(z) + \varphi_1(z))\partial \varphi_2(z) = 0$$

if $0 < c = \alpha < \frac{1}{2}$. Thus

$$\lim_{z \to 0} \partial M(z) = \lim_{z \to 0} (K_1\varphi_1(z) + \varphi_2(z))\partial \varphi_1(z) = \frac{ab}{c} \left( K_1 - \frac{1}{K_2} \right)$$

provided (3.2) and (3.4).

For (2), we note that (3.5) and (3.8) is still valid for $n = 1$, $\alpha = 1/2$, combining with (3.2) and (3.3) we have (2) hold.

For case (3),

$$\partial^n M(z) = (K_1\varphi_1(z) + \varphi_2(z))\partial^n \varphi_1(z) + (K_2\varphi_2(z) + \varphi_1(z))\partial^n \varphi_2(z). \tag{3.10}$$
From properties (2.2),

\[
\partial^n \varphi_1(0) = \frac{(a)_{n(b)n}}{(c)n}
\]  

(3.11)

for \( n \geq 1 \). Since \( n > 2\alpha - 2 \) for all \( n \geq 2 \) and \( 0 < \alpha < 1 \), from (3.11) and (3.4), we know that the limit (3) is only decided by the term \((K_2 \varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial^n \varphi_2(z)\). Combining with (3.10), (3.9) and (3.6), we have

\[
\lim_{z \to 0} z^m |z|^{2\alpha - 2} \partial^m M(z) = \lim_{z \to 0} z^m |z|^{2\alpha - 2} (K_2 \varphi_2(\bar{z}) + \varphi_1(\bar{z}))\partial^n \varphi_2(z)
\]

\[
= \lim_{z \to 0} \frac{K_2 \varphi_2(\bar{z}) + \varphi_1(\bar{z})}{z^{1-\alpha}} \partial^n \varphi_2(z)
\]

\[
= \lim_{z \to 0} \frac{K_2 \varphi_2(\bar{z}) + \varphi_1(\bar{z})}{z^{1-\alpha}} \cdot \lim_{z \to 0} z^{n+c-1} \partial^n \varphi_2(z)
\]

\[
= (-1)^{n-1}(c)_{n-1} K_2 \left( \Gamma(c) \Gamma(a + b - c + 1) \right) \left( \frac{\Gamma(a + b - c + 1) \Gamma(c)}{\Gamma(a) \Gamma(b)} \right)^2
\]

as in (3).

For (4), if \( m \geq 1 \), \( n \geq 1 \), we have

\[
\partial^m \varphi_1(z) = (K_1 \partial^m \varphi_1(\bar{z}) + \partial^m \varphi_2(\bar{z}))\partial^n \varphi_1(z) + (K_2 \partial^m \varphi_2(\bar{z}) + \partial^n \varphi_1(\bar{z}))\partial^n \varphi_2(z).
\]

Since

\[
\lim_{z \to 0} z^{n+c-1} \partial^n \varphi_1(z) = 0,
\]

then

\[
\lim_{z \to 0} z^m z^n |z|^{2\alpha - 2} \partial^m \varphi_1(z) = \lim_{z \to 0} \frac{z^m z^n}{|z|^{2-2c}} K_2 \partial^m \varphi_2(\bar{z})\partial^n \varphi_2(z)
\]

\[
= \lim_{z \to 0} K_2 \frac{z^m}{z^{1-c}} \partial^m \varphi_2(\bar{z}) \cdot \frac{z^n}{z^{1-c}} \partial^n \varphi_2(z)
\]

\[
= (-1)^{m+n}(c)_{m-1}(c)_{n-1} K_2 \left( \Gamma(c) \Gamma(a + b - c + 1) \right) \left( \frac{\Gamma(a + b - c + 1) \Gamma(c)}{\Gamma(a) \Gamma(b)} \right)^2
\]

as in (4). \qed

The following result is a specific version of Theorem B.

**Theorem 3.2** For \( \lambda(z) := \lambda_{\alpha, \beta, \gamma}(z) \) as in (2.10) with order \( \alpha \in (0, 1) \), let \( u(z) := \log \lambda(z) \). Then for \( m, n \geq 1 \),

(i) \( \lim_{z \to 0} z^n \partial^n u(z) = \frac{\alpha}{2} (-1)^n (n-1)! = \lim_{z \to 0} z^n \partial^n u(z) \),

(ii) \( \lim_{z \to 0} z^m z^n |z|^{2\alpha - 2} \partial^m \partial^n u(z) = \frac{(-1)^{n+m}(c)_{n-1}(c)_{m-1} K_2^2}{K_1 K_2 - 1} \left( \frac{\Gamma(c) \Gamma(a + b - c + 1) \Gamma(c)}{\Gamma(a) \Gamma(b)} \right)^2 \).

**Remark.** Theorem B was proved for the order \( 0 < \alpha \leq 1 \) in [15] with a different limit for the mixed differential, while Theorem 3.2 is given for the order \( 0 < \alpha < 1 \) and we
prove it in a different way for the completeness of this paper. The proof of Theorem 3.2 also can be taken to be an application of Lemma 3.1. For the hyperbolic density $\lambda(z)$ with order $\alpha = 1$, we can also prove Theorem B directly by discussing the properties of hypergeometric functions.

**Proof of Theorem 3.2.** We note that

$$u(z) = -\alpha \log |z| - \beta \log |1 - z| + \log K_3 - \log M(z)$$

with $M(z)$ as in (3.1). At first we consider $\partial^n \log M(z)$. From (3.3),

$$M(0) = K_1 \varphi_1(0) + \varphi_2(0) = K_1 - \frac{1}{K_2}. \quad (3.12)$$

We can calculate that $M(0) > 0$ for any $a, b, c$ as in (2.13). From Lemma 3.1 we have

$$\lim_{z \to 0} z^k \partial^k M(z) = 0$$

for all $k \geq 1$ and $0 < \alpha < 1$. It is easy to observe that $\partial^n \log M(z)$ is a linear combination of products of $\partial^k M$ with $k \leq n$, so when $n \geq 1,$

$$\lim_{z \to 0} z^n \partial^n \log M(z) = 0.$$ 

Since

$$\partial^n \log |1 - z| = -\frac{(n - 1)!}{2(1 - z)^n}, \quad \partial^n \log |z| = \frac{(-1)^{n-1} (n - 1)!}{2z^n},$$

then the first equality in (i) holds.

For the second equality, note that $u(z)$ is real-valued,

$$\lim_{z \to 0} z^n \bar{\partial}^n u(z) = \lim_{z \to 0} z^n \bar{\partial}^n m(z) = \frac{\alpha}{2} (-1)^n (n - 1)!,$$

Therefore (i) is valid.

Now we discuss the term $\bar{\partial}^m \partial^n \log M(z)$ to complete the proof. Since $\bar{\partial}^m \partial^n \log M(z)$ is a linear combination of products of $\bar{\partial}^t \partial^k M/M$ with $0 \leq t \leq m, 0 \leq k \leq n$, so Lemma 3.1 implies that

$$\lim_{z \to 0} z^m z^n |z|^{2\alpha - 2} \prod_{j=2}^{N} \frac{\bar{\partial}^t_j \partial^k_j M(z)}{M(z)} = 0,$$

where $2 \leq N \leq m + n, 1 \leq t_j \leq m$ and $1 \leq k_j \leq n$ for every index $j, 2 \leq j \leq N$. Thus

$$\lim_{z \to 0} z^m z^n |z|^{2\alpha - 2} \bar{\partial}^m \partial^n \log M(z) = \lim_{z \to 0} z^m z^n |z|^{2\alpha - 2} \frac{\bar{\partial}^m \partial^n M(z)}{M(z)} = \left(\frac{-1)^{n-m} (c)_{n-1} (c)_{m-1} K_2^2}{K_1 K_2 - 1} \frac{\Gamma(c) \Gamma(a + b - c + 1)}{\Gamma(a) \Gamma(b)} \right)^2.$$

Note that $\bar{\partial}^m \partial^n \log |1 - z| = 0, \bar{\partial}^m \partial^n \log |z| = 0$, thus (ii) holds. 

For the hyperbolic metric, the following result corresponding to Theorem C holds.
Theorem 3.3 For $m, n \geq 0$, $0 < \alpha < 1$ and $\lambda(z)$ as in (2.10), the limit

$$l_{m, n} := \frac{1}{m!n!} \lim_{z \to 0} |z|^\alpha z^m z^n \bar{\partial}^m \partial^n \lambda(z)$$

exists. Let

$$l_{0, 0} = l := \lim_{z \to 0} |z|^\alpha \lambda(z) = \frac{\delta}{1 - \delta^2} (1 - \alpha) \quad (3.13)$$

by Theorem [13] then the numbers $l_{m, n}$ satisfy the following

(i) $l_{m, n} = \left( -\frac{\alpha}{n} \right) \left( -\frac{\alpha}{m} \right) l$,

(ii) $l_{m, n} = l_{n, m}$,

where

$$\left( \tau \atop j \right) = \frac{\tau(\tau - 1) \cdots (\tau - j + 1)}{j!}$$

is the binomial coefficient.

Proof. Since

$$\partial \lambda(z) = \lambda(z) \partial u(z) \quad (3.14)$$

we have

$$\partial^n \lambda(z) = \sum_{j=0}^{n-1} \left( \begin{array}{c} n - 1 \\ j \end{array} \right) \partial^{n-j} u(z) \partial^j \lambda(z)$$

by induction, where $\partial^0 \lambda(z) = \bar{\partial}^0 \lambda(z) = \lambda(z)$. Then

$$l_{0, n} = \frac{1}{n!} \lim_{z \to 0} \sum_{j=0}^{n-1} \left( \begin{array}{c} n - 1 \\ j \end{array} \right) z^{n-j} \bar{\partial}^{n-j} u(z) \cdot |z|^\alpha z^j \bar{\partial}^j \lambda(z).$$

From the existence of $\lim_{z \to 0} z^{n-j} \bar{\partial}^{n-j} u(z)$ and $l$, it is known that $l_{0, n}$ exists. By (ii) in Theorem 3.2, we have

$$\lim_{z \to 0} z^m z^n \bar{\partial}^m \partial^n u(z) = 0.$$

So we can write $l_{m, n}$ as a sum of the terms not containing any mixed derivatives of $u(z)$,

$$l_{m, n} = \frac{1}{m!n!} \lim_{z \to 0} \sum_{j=0}^{n-1} \left( \begin{array}{c} n - 1 \\ j \end{array} \right) z^{n-j} \bar{\partial}^{n-j} u(z) |z|^\alpha z^m z^j \bar{\partial}^n \partial^j \lambda(z), \quad (3.15)$$

thus the existence of $l_{0, n}$ guarantees $l_{m, n}$ exists.

If $m = 0$, $n = 1$, then (3.13) and (3.14) give

$$l_{0, 1} = \lim_{z \to 0} |z|^\alpha z \partial \lambda(z) = \lim_{z \to 0} |z|^\alpha \lambda(z) \cdot z \partial u(z) = -\frac{\alpha}{2} l,$$

which is a real number, so $l_{1, 0} = l_{0, 1} = l_{0, 1}$. Note that

$$\bar{\partial}^n \lambda(z) = \sum_{j=0}^{n-1} \left( \begin{array}{c} n - 1 \\ j \end{array} \right) \bar{\partial}^{n-j} u(z) \partial^j \lambda(z), \quad (3.16)$$
then \( l_{n,0} = l_{0,n} \) by induction. From (3.15), (3.16), and (i) of Theorem 3.2, we have

\[
l_{m,n} = \sum_{j=0}^{n-1} \frac{1}{m!} \frac{1}{j!(n-1-j)!} \lim_{z \to 0} z^{n-j} \partial^{n-j} u(z) \cdot |z|^\alpha z^m z^j \partial^m \partial^j \lambda(z)
\]

\[
= \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{m!} \frac{1}{j!(n-1-j)!} \lim_{z \to 0} z^{n-j} \partial^{n-j} u(z) \cdot \lim_{z \to 0} |z|^\alpha z^m z^j \partial^m \partial^j \lambda(z)
\]

\[
= \frac{1}{n} \sum_{j=0}^{n-1} \alpha(-1)^{n-j} \frac{1}{m!} \frac{1}{j!(n-1-j)!} \lim_{z \to 0} |z|^\alpha z^m z^j \partial^m \partial^j \lambda(z) = \frac{\alpha}{2n} \sum_{j=1}^{n-1} (-1)^{n-j} l_{m,j}.
\]

Then

\[
n \cdot l_{m,n} = \frac{\alpha}{2} \sum_{j=0}^{n-2} (-1)^{n-j} l_{m,j} - \frac{\alpha}{2} l_{m,n-1} = -(n-1)l_{m,n-1} - \frac{\alpha}{2} l_{m,n-1}.
\]

Since \( l_{0,n} = l_{n,0} \),

\[
l_{m,n} = \frac{-\frac{\alpha}{2} - n + 1}{n} l_{m,n-1} = \left( -\frac{\alpha}{2} \right) l_{0,m} = \left( -\frac{\alpha}{2} \right) \left( -\frac{\alpha}{2} \right) l_{0,0}.
\]

Thus (i) holds and (ii) follows immediately from (i). \( \square \)

The following estimate is for the general case.

**Theorem E** ([15]). Let \( \kappa : \mathbb{D} \to \mathbb{R} \) be a locally Hölder continuous function with \( \kappa(0) < 0 \). If \( u : \mathbb{D}^+ \to \mathbb{R} \) is a \( C^2 \)-solution to \( \Delta u = -\kappa(z) e^{2u} \) in \( \mathbb{D}^+ \), then \( u \) has an order \( \alpha \in (-\infty, 1] \).

If, in addition, \( \kappa(z) \in C^{n-2, \nu}(\mathbb{D}^+) \) for an integer \( n \geq 3 \) and \( 0 < \nu \leq 1 \), then \( u(z) \in C^{n, \nu}(\mathbb{D}^+) \) by the regularity theorem. If the order \( 0 < \alpha < 1 \), then for the remainder function \( v(z) \) and for \( n_1, n_2 \geq 1 \) such that \( n_1 + n_2 = n \), near the origin, we have

\[
\partial^n v(z), \ \partial^n \bar{v}(z), \ \bar{v}^{n_1} \partial^{n_2} v(z) = O(|z|^{2-2\alpha-n}).
\]

From the proof of Theorem 3.2, we can provide a way to verify the sharpness of Theorem E and also Theorem 1.1 in [8]. We state the result as the following theorem.

**Theorem 3.4** For \( m, n \geq 1 \) and \( \lambda \) as in (2.10) with the order \( 0 < \alpha < 1 \), then near the origin, the remainder function \( v(z) \) satisfies

(i) \( \lim_{z \to 0} \partial v(z) = \frac{ab}{c} \) for \( 0 < \alpha < 1/2 \),

(ii) \( \partial v(z) = 2ab + \frac{2K^2}{K_1 K_2 - 1} \left( \frac{\Gamma(c) \Gamma(a + b - c + 1)}{\Gamma(a) \Gamma(b)} \right)^2 \frac{\bar{z}}{|z|} + O \left( |z|^\frac{1}{2} \right) \) near the origin for \( \alpha = 1/2 \),

(iii) \( \lim_{z \to 0} z^{n}|z|^{2\alpha-2} \partial^n v(z) = \frac{(-1)^{n-1} c \Gamma(c) (1 - c) (K_1 K_2 - 1)}{(1 - c) (K_1 K_2 - 1)} \left( \frac{\Gamma(c) \Gamma(a + b - c + 1)}{\Gamma(a) \Gamma(b)} \right)^2 \) for \( n \geq 2 \) if \( 0 < \alpha \leq 1/2 \) and \( n \geq 1 \) if \( 1/2 < \alpha < 1 \),

(iv) \( \lim_{z \to 0} z^{m} z^{|n}|z|^{2\alpha-2} \bar{v} \partial^m v(z) = \frac{(-1)^{n+m} c \Gamma(c) (m-1) K_2^2}{K_1 K_2 - 1} \left( \frac{\Gamma(c) \Gamma(a + b - c + 1)}{\Gamma(a) \Gamma(b)} \right)^2 \) for \( m, n \geq 1 \) and \( 0 < \alpha < 1 \).
Proof. Since for \( \lambda(z) \) in (2.10), \( v(z) = -\beta \log |1 - z| + \log K_3 - \log M(z) \) and
\[
\partial^n \log |1 - z| = \frac{-(n - 1)!}{2(1 - z)^n},
\]
we consider \( \partial^n \log M(z) \) only. From the proof of Theorem 3.2, the limits \( \lim_{z \to 0} \partial v(z) \) and \( \lim_{z \to 0} z^n |z|^{2\alpha - 2} \partial^n v(z) \) both depend solely on the term \( \partial^n M(z) \). Thus by Lemma 3.1 and (3.12),
\[
\lim_{z \to 0} \partial v(z) = \lim_{z \to 0} \partial M(z), \quad \lim_{z \to 0} z^n |z|^{2\alpha - 2} \partial^n v(z) = \lim_{z \to 0} z^n |z|^{2\alpha - 2} \frac{\partial^n M(z)}{M(z)}.
\]
So we obtain the four cases above corresponding to ones in Lemma 3.1. □

4 Case \( \alpha = 1 \)

If \( \alpha = 1 \), the formula for \( \lambda_{1, \beta, \gamma}(z) \) is to be understood in the limit sense \( \lim_{\alpha \to 1} \). So when \( \alpha = c = 1 \), we have
\[
K_3 = \frac{1}{B(a, b)} := \frac{1}{B}, \quad K_2 = 0, \quad S := \frac{\pi \sin(\pi(a + b))}{\sin \pi a \sin \pi b}, \quad K_1 = -\frac{S}{B},
\]
\[
\varphi_1(z) = F(a, b, 1; z), \quad \varphi_2(z) = F(a, b, a + b; 1 - z).
\]

Then
\[
\lambda_{1, \beta, \gamma}(z) = \frac{1}{|z| |1 - z|^\beta} \frac{K_3}{K_1|\varphi_1(z)|^2 + \varphi_1(z)\varphi_2(\bar{z}) + \varphi_1(\bar{z})\varphi_2(z)},
\]
and the remainder function of \( u(z) \) near the origin is
\[
w(z) = -\beta \log |1 - z| + \log K_3 - \log M(z) + \log \log(1/|z|).
\]
The assumption of Theorem A and (2.13) show that \( a \) and \( b \) satisfy
\[
0 < a < 1, \quad 0 < b < 1/2, \quad 0 < a + b < 1.
\]
The function
\[
2R - S = 4\Psi(1) - 2\Psi(a) - 2\Psi(b) - \pi \cot \pi a - \pi \cot \pi b
\]
is of special interest where \( R := R(a, b) \) is as in (2.7) and \( S \) is given by (4.1). Let \( G(x) := 2(\Psi(1) - \Psi(x)) - \pi \cot \pi x \). For the Gamma function \( \Gamma \) and \( 0 < x < 1 \), we have \( \Gamma(x)\Gamma(1 - x) = \pi / \sin \pi x \). Taking the logarithmic derivatives of both sides leads to
\[
\frac{\Gamma'(x)}{\Gamma(x)} - \frac{\Gamma'(1 - x)}{\Gamma(1 - x)} = -\pi \cot \pi x.
\]
which means $G(x) = G(1 - x)$. The fact that the digamma function is negative and decreasing on $(0, 1)$ implies that $G(x) > 0$ when $0 < x < 1$. Since $2R - S = G(a) + G(b)$, then $2R - S > 0$ for all $a$, $b$ given by (2.13).

For any conformal metric $\lambda(z)|dz|$ with the negative Gaussian curvature and the remainder function $w(z)$ defined by (2.9), we have the following result to describe the asymptotic behavior of $w(z)$ near the origin.

**Theorem F** ([15]) Let $\kappa : \mathbb{D} \to \mathbb{R}$ be a locally Hölder continuous function with $\kappa(0) < 0$. If $u : \mathbb{D}^* \to \mathbb{R}$ is a $C^2$-solution to $\Delta u = -\kappa(z)e^{2u}$ in $\mathbb{D}^*$, then $u$ has an order $\alpha \in (-\infty, 1]$. If, in addition, $\kappa(z) \in C^{n-2,\nu}(\mathbb{D}^*)$ for an integer $n \geq 3$, $0 < \nu \leq 1$, then $u(z) \in C^{n,\nu}(\mathbb{D}^*)$ by the regularity theorem. If order $\alpha = 1$, then for the remainder function $w(z)$ and for $n_1, n_2 \geq 1$, $n_1 + n_2 = n$, near the origin, we have

\[
\begin{align*}
\frac{\partial^n w(z)}{\partial z^n}, \quad \frac{\partial^n w(z)}{\partial \bar{z}^n} & = O(|z|^{-n} \log^{-2}(1/|z|)), \\
\frac{\partial^{n_1} \partial^{n_2} w(z)}{\partial z^{n_1} \partial \bar{z}^{n_2}} & = O(|z|^{-n} \log^{-3}(1/|z|)).
\end{align*}
\]

We can verify the sharpness of Theorem F by use of $\lambda_{1,\beta,\gamma}(z)$ as in (4.2). Furthermore, for $\lambda(z)$, we can obtain its precise estimate for higher order derivatives of $w(z)$ near the origin. In fact, we have the following result stronger than Theorem F.

**Theorem 4.1** Let $\lambda(z) := \lambda_{1,\beta,\gamma}(z)$ as in (4.2) with $\beta$ and $\gamma$ satisfying the condition in Theorem A and $w(z)$ be the remainder function as in (4.3). Then for $m, n \geq 1$, we have

(i) $\lim_{z \to 0} z^n \log^2(1/|z|) \frac{\partial^n w(z)}{\partial z^n} = \frac{(-1)^n(n-1)!}{4}(G(a) + G(b))$,

(ii) $\lim_{z \to 0} z^n \log^3(1/|z|) \frac{\partial^n w(z)}{\partial \bar{z}^n} = \frac{(-1)^n(n-1)!}{4}(G(a) + G(b))$,

where the function $G$ is defined by (4.5) and $a$, $b$ are given by (2.13).

**Proof.** For the remainder function given by (4.3), we discuss $\log \log(1/|z|)$ and $\log M(z)$ separately. At first, consider the higher order derivatives of $\log \log(1/|z|)$. By induction we know that

\[
\frac{\partial^n \log \log(1/|z|)}{\partial z^n} = \sum_{j=1}^{n} \frac{C_j^{(n)}}{z^n \log^{j}(1/|z|)}
\]

with constant $C_j^{(n)}$ for $1 \leq j \leq n$. Here we only need the first two terms of $\partial^n \log \log(1/|z|)$ for future use. As for the pure derivative $\partial^n \log(1/|z|)$ with $n \geq 1$, set $A_n := C_1^{(n)}$ and $B_n := C_2^{(n)}$, so

\[
\frac{\partial^n \log \log(1/|z|)}{\partial z^n} = \frac{A_n}{z^n \log(1/|z|)} + \frac{B_n}{z^n \log^2(1/|z|)} + \sum_{j=3}^{n} \frac{C_j^{(n)}}{z^n \log^{j}(1/|z|)}.
\]

then the following recurrent relations hold,

$A_1 = -\frac{1}{2}$, $B_1 = 0$, $B_2 = 1$.
\(A_n = -(n-1)A_{n-1}, \quad B_n = -(n-1)B_{n-1} + \frac{1}{2}A_{n-1}.\)

Thus
\[
A_n = \frac{(-1)^n}{2}(n-1)!, \quad B_n = \frac{(-1)^{n-1}}{4}(n-1)! \sum_{j=1}^{n-1} \frac{1}{j}.
\]

For the mixed derivative case with \(n \geq 1, m \geq 1,\) we fix \(n,\) so by induction,

\[
\partial^m \partial^n \log \log(1/|z|) = \sum_{j=1}^{m} \frac{C_j^{(m,n)}}{z^m z^n \log^{j+1}(1/|z|)}
\]

with constant \(C_j^{(m,n)}\) for \(1 \leq j \leq m.\) Set \(C_m := C_1^{(m,n)}\) and \(D_m := C_2^{(m,n)},\) we have

\[
\partial^m \partial^n \log(1/|z|) = \frac{C_m}{z^m z^n \log^2(1/|z|)} + \frac{D_m}{z^m z^n \log^3(1/|z|)} + \sum_{j=3}^{m} \frac{C_j^{(m,n)}}{z^m z^n \log^{j+1}(1/|z|)}.
\]

Then

\[
C_1 = \frac{1}{2}A_n, \quad D_1 = B_n, \quad C_m = -(m-1)C_{m-1}, \quad D_m = -(m-1)D_{m-1} + C_{m-1}.
\]

Therefore

\[
C_m = \frac{(-1)^{m+n-1}}{4}(m-1)!(n-1)!, \quad D_m = \frac{(-1)^{m+n}}{4}(m-1)!(n-1)! \left( \sum_{j=1}^{n-1} \frac{1}{j} + \sum_{j=1}^{m-1} \frac{1}{j} \right).
\]

To estimate the derivatives of \(\log M(z),\) we first calculate \(\frac{\partial^n M(z)}{M(z)}\) for \(n \geq 1.\) Since

\[
\partial^n \varphi_1(z) = \frac{(a)_n(b)_n}{n!} F(a+n, b+n, n+1; z),
\]

\[
\partial^n \varphi_2(z) = (-1)^n \frac{(a)_n(b)_n}{(a+b)_n} F(a+n, b+n, a+b+n; 1-z),
\]
s
so

\[
\partial^n \varphi_1(0) = \frac{(a)_n(b)_n}{n!},
\]

and near the origin, by (2.5) we have

\[
\partial^n \varphi_2(z) = \frac{(a)_n(b)_n}{(a+b)_n} (-1)^n z^n F(b, a, a+b+n; 1-z).
\]

Considering (2.4) shows that

\[
F(b, a, a+b+n; 1-z) = \frac{\Gamma(a+b+n)\Gamma(n)}{\Gamma(a+n)\Gamma(b+n)} + O(|z| \log |z|)
\]
near the origin. Thus
\[
\partial^n \varphi_2(z) = \frac{(-1)^n (a_n b_n)}{z^n} \left( \frac{\Gamma(a+b+n)\Gamma(n)}{\Gamma(a+n)\Gamma(b+n)} + O(|z| \log |z|) \right)
\]
\[
= \frac{(-1)^n(n-1)!}{Bz^n} + O\left( \frac{\log |z|}{|z|^{n-1}} \right).
\]

Property (2.5) gives
\[
\varphi_2(z) = \frac{1}{B} \left( \log \frac{1}{z} + R \right) (1 + O(z)),
\]

summarizing the estimates above, so we can obtain
\[
\partial^n M(z) = (K_1 \varphi_1(\bar{z}) + \varphi_2(\bar{z})) \partial^n \varphi_1(z) + \varphi_1(\bar{z}) \partial^n \varphi_2(z)
\]
\[
= \frac{(a_n b_n)}{B} \log \frac{1}{z} + \frac{(-1)^n(n-1)!}{Bz^n} + O\left( \frac{\log |z|}{|z|^{n-1}} \right)
\]
and
\[
M(z) = K_1 \varphi_1(\bar{z}) \varphi_1(z) + \varphi_2(\bar{z}) \varphi_1(z) + \varphi_1(\bar{z}) \varphi_2(z)
\]
\[
= \frac{2 \log(1/|z|)}{B} \left( 1 + \frac{2R-S}{2\log(1/|z|)} + O(|z|) \right)
\]

near the origin. Then
\[
\frac{\partial^n M(z)}{M(z)} = \partial^n M(z) \left. \frac{B}{2\log(1/|z|)} \left( 1 - \frac{2R-S}{2\log(1/|z|)} + O(|z|) \right) \right|
\]
\[
= \frac{(-1)^n(n-1)!}{2z^n \log(1/|z|)} \left( \frac{(-1)^n(n-1)!}{4z^n \log^2(1/|z|)} \right) + O\left( \frac{1}{|z|^{n-1}} \right).
\]  \quad (4.10)

We note that
\[
\tilde{\partial}^m \partial^n M(z)
\]
\[
= (K_1 \tilde{\partial}^m \varphi_1(\bar{z}) + \tilde{\partial}^n \varphi_2(\bar{z})) \partial^n \varphi_1(z) + \tilde{\partial}^m \varphi_1(\bar{z}) \partial^n \varphi_2(z)
\]
\[
= \frac{(-1)^m(m-1)! (a_m b_m)}{Bz^m} \frac{(-1)^n(n-1)! (a_n b_n)}{m!} + O\left( \frac{\log |z|}{|z|^{m-1}} \right) + O\left( \frac{\log |z|}{|z|^{n-1}} \right),
\]

the same technique leads to
\[
\frac{\tilde{\partial}^m \partial^n M(z)}{M(z)} = O\left( \frac{1}{|z|^m \log(1/|z|)} \right),
\]  \quad (4.11)

where \( \tau = \max\{m, n\} < m + n \).

Now we can consider derivatives of \( w(z) \). In the pure derivative case,
\[
\partial^n w(z) = \frac{\beta(n-1)!}{2(1-z)^n} - \partial^n \log M(z) + \partial^n \log(1/|z|).
\]  \quad (4.12)
Since the coefficients of the first two differential terms in \( \partial^n \log \log (1/|z|) \) are already known as \( A_n \) and \( B_n \), now we discuss \( \partial^n \log M(z) \). Note that \( \partial^n \log M(z) \) is a linear combination of finitely many terms of the form

\[
\prod_{j=1}^{k} \frac{\partial^{n_j} M(z)}{M(z)}
\]

for \( 1 \leq k \leq n \). When \( k = 1 \), term (4.13) is corresponding to the first term in the third line of (4.10) with \( n = 1 \), and it will be canceled by \( A_n \) given in (4.6). So we should look at the second term which contains \( z^{-n} \log^{-2} (1/|z|) \) for the higher order derivatives, while the higher power terms in (4.10) are ignored for a moment. For (4.13), estimate (4.10) shows that

\[
\prod_{j=1}^{k} \frac{\partial^{n_j} M(z)}{M(z)} = O \left( \frac{1}{z^n \log^k (1/|z|)} \right)
\]

for \( n = \sum_{j=1}^{k} n_j \), therefore, to generate the \( z^{-n} \log^{-2} (1/|z|) \) term, \( k \) is at most 2, thus the \( z^{-n} \log^{-2} |z| \) term of \( \partial^n \log M(z) \) only appears in

\[
\frac{\partial^n M}{M} = \frac{1}{2} \sum_{j=1}^{n-1} \binom{n}{j} \frac{\partial^j M \partial^{n-j} M}{M^2}.
\]

For every \( 1 \leq j \leq n-1 \), we have

\[
\frac{\partial^j M \partial^{n-j} M}{M^2} = (-1)^n (j-1)!(n-j-1)! \left( \frac{1}{2} \right)^n 2^{n-1} \log^2 |z| + O \left( \frac{1}{|z|^n \log^3 (1/|z|)} \right).
\]

Denote the coefficient of \( z^{-n} \log^{-2} |z| \) in \( \sum_{j=1}^{n-1} \binom{n}{j} (\partial^j M \partial^{n-j} M/M^2) \) by \( k_n \), then (4.10) leads to

\[
k_n = \sum_{j=1}^{n-1} \binom{n}{j} (-1)^n (j-1)!(n-j-1)! = \frac{(-1)^n (n-1)!}{4} \sum_{j=1}^{n-1} \frac{n}{j(n-j)}
\]

\[
= \frac{(-1)^n (n-1)!}{4} \sum_{j=1}^{n-1} \left( \frac{1}{j} + \frac{1}{n-j} \right)
\]

\[
= \frac{(-1)^n (n-1)!}{2} \sum_{j=1}^{n-1} \frac{1}{j} = -2B_n.
\]

Note that the term \( z^{-n} \log^{-1} (1/|z|) \) only appears in \( \partial^n M(z)/M(z) \), and (4.6), (4.10), (4.12) show that \( z^{-n} \log^{-1} (1/|z|) \) actually is canceled in \( \partial^n w(z) \). In combination with (4.7) and (4.14), we obtain

\[
\lim_{z \to 0} z^n \log^2 (1/|z|) \partial^n w(z) = \frac{(-1)^n (n-1)!}{4} \left( G(a) + G(b) \right) + \frac{1}{2} k_n + B_n
\]

\[
= \frac{(-1)^n (n-1)!}{4} \left( G(a) + G(b) \right),
\]

thus (i) holds.

For the mixed derivatives case,

\[
\partial^m \partial^n w(z) = -\partial^m \partial^n \log M(z) + \partial^m \partial^n \log \log (1/|z|).
\]
Since the coefficients of the first two terms in \( \tilde{\partial}^m \partial^n \log \log(1/|z|) \) are given as \( C_m \) and \( D_m \), now we consider \( \tilde{\partial}^m \partial^n \log M(z) \). It is known that \( M(z) = M(\bar{z}) \) and \( \tilde{\partial}^m \partial^n \log M(z) = \partial^m \partial^n \log M(\bar{z}) \). Thus without loss of generality we may assume \( m \leq n \). Similarly as in the pure derivative case, there will be some cancelation for the term containing \( z^{-n} \bar{z}^{-m} \log^{-2}(1/|z|) \), so the coefficient of \( z^{-n} \bar{z}^{-m} \log^{-3}(1/|z|) \) is desired. The term containing \( z^{-n} \bar{z}^{-m} \log^{-3}(1/|z|) \) must the product of at most three terms in the forms of \( \partial^m M/M \) or \( \tilde{\partial}^m M/M \). Estimate (4.11) and (4.10) imply that the term \( z^{-n} \bar{z}^{-m} \log^{-3}(1/|z|) \) of \( \tilde{\partial}^m \partial^n \log M(z) \) only appears in

\[
- \frac{\partial^m M \partial^n M}{M^2} + \frac{\partial^m M}{M} \sum_{j=1}^{n-1} \binom{n}{j} \frac{\partial^j M \partial^{n-j} M}{M^2} + \frac{\partial^n M}{M} \sum_{j=1}^{m-1} \binom{m}{j} \frac{\partial^j M \partial^{m-j} M}{M^2}
\]

for \( m \geq 1, n \geq 1 \). Then

\[
\tilde{\partial}^m \partial^n \log M(z) = \frac{t_{m,n}}{z^n \bar{z}^m \log^3(1/|z|)} + O \left( \frac{1}{|z|^{m+n+1} \log^3(1/|z|)} \right)
\]

with

\[
t_{m,n} = \left( G(a) + G(b) + \sum_{j=1}^{n-1} \frac{1}{j} + \sum_{j=1}^{m-1} \frac{1}{j} \right) \frac{(-1)^{m+n}(m-1)!(n-1)!}{4}, \tag{4.15}
\]

which can be obtained by the technique similar to the one applied to \( k_n \) in (4.14). Note that the term \( \bar{z}^{-m} z^{-n} \log^{-2}(1/|z|) \) only occurs in \( \tilde{\partial}^m \partial^n M/M^2 \) with the coefficient \( (-1)^{m+n}(n-1)!(m-1)!/4 \), comparing with (4.8) shows that there is no term of \( \bar{z}^{-m} z^{-n} \log^{-2}(1/|z|) \) left in the expression for \( \tilde{\partial}^m \partial^n w(z) \). Thus for \( m \geq 1, n \geq 1 \), by (4.9) and (4.15), we obtain

\[
\lim_{z \to 0} z^n \bar{z}^m \log^3(1/|z|) \tilde{\partial}^m \partial^n w(z) = -t_{m,n} + D_m = \frac{(-1)^{m+n-1}(n-1)!m-1)!}{4} (G(a) + G(b))
\]

This completes the proof and verifies the sharpness of Theorem F. \( \Box \)

When the order \( \alpha = 1 \), there is an analogue of Theorem 3.2 and Theorem 3.3, see [15], also [8]. Here we list them as following ones without proof.

**Theorem 4.2** For \( \lambda(z) := \lambda_{1, \beta, \gamma}(z) \) as in (4.2), let \( u(z) := \log \lambda(z) \). Then for \( m, n \geq 1 \),

(i) \( \lim_{z \to 0} z^n \partial^n u(z) = \frac{1}{2} (-1)^n (n-1)! = \lim_{z \to 0} z^n \tilde{\partial}^n u(z) \),

(ii) \( \lim_{z \to 0} z^n \bar{z}^m \log^2(1/|z|) \tilde{\partial}^m \partial^n u(z) = \frac{(-1)^{n+m}(n-1)!m-1)!}{4} \).

**Theorem G** ([15]) For \( m, n \geq 0, \alpha = 1 \) and \( \lambda(z) \) as in (4.2), the limit

\[
l'_{m,n} := \frac{1}{n!m!} \lim_{z \to 0} |z|^n \log(1/|z|) \bar{z}^m z^n \tilde{\partial}^m \partial^n \lambda(z)
\]

exists. Moreover, the numbers \( l'_{m,n} \) satisfy the following

(i) \( l'_{0,0} := \lim_{z \to 0} |z| \log(1/|z|) \lambda(z) = \frac{1}{2} \),

(ii) \( l'_{m,n} = \frac{1}{2} \left( \begin{array}{c} -\frac{1}{2} \\ n \end{array} \right) \left( \begin{array}{c} -\frac{1}{2} \\ m \end{array} \right) \),

(iii) \( l'_{m,n} = l'_{n,m} \).
Acknowledgement. I would like to thank Prof. Toshiyoki Sugawa for his helpful comments, suggestion and encouragement. I also want to thank Rintaro Ohno for his considerable time reading through each draft.

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