Multiple solutions for equations involving bilinear, coercive and compact forms with applications to differential equations

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Abstract

The existence of multiple fixed points for the coercive, bilinear, compact forms defined in the cone in the Banach space. Multiple applications to the integral equations derived from BVPs for differential equations are provided.

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1 Introduction and motivation

It is well known that the quadratic equation

\[ u = au^2 + u_0 \]  

(1)

can have either none, one or two solutions \( u \in \mathbb{R} \), depending on the data \( a > 0 \) and \( u_0 \in \mathbb{R} \). For example, if we assume that

\[ 4au_0 < 1 \]  

(2)
than the existence of two positive solutions of (1) is guaranteed.

In this paper we would like to show that this simple observation can be
generalized if we replace quadratic term $ax^2$ with a bilinear form under suitable
conditions. More specifically, we shall consider the equation in the cone $P$ in
the Banach space $U$ with the norm $|\cdot|$ in the form

$$u = b(u, u) + u_0$$

(3)

for some given element $u_0 \in P$ and bilinear, coercive and compact form $b$ defined
on the product space $P \times P$.

The assumption (2) guaranteeing the existence of two solutions for the
quadratic equation (1) has to be adequately rephrased for (3) as

$$4|b||u_0| < 1$$

(4)

where $|b|$ denotes the norm of the bilinear norm. However, the proof of this
result we shall postpone to the next section.

Let us, however notice that the Banach Fixed Point Theorem for local con-
traction was used extensively to prove the existence of at least one fixed point
(e.g. for the Navier-Stokes equations in [2] or for a Boltzmann equation) for the
bilinear model equation like (3). This fixed point was located in the neighbour-
hood of 0 thus making contraction approach feasible.

To prove the existence of two solutions we shall use another Fixed Point
Theorem due to Krasnosieliski (cf. [3]) which allows to obtain more solutions if
the nonlinear opearator has the required property of “crossing” identity twice,
i.e. by the cone compression and the expansion on some appropriate convex
subsets of the cone.

A quadratic nonlinearity being the simplest possible in the nonlinear world poses many questions about the global existence and uniqueness. There are numerous models which possess such a structure, like the Navier Stokes equation, the Boltzmann equation, the quadratic reaction diffusion equation (cf. [5]), the Smoluchowski coagulation equation or the system modelling chemotaxis and to name but a few. The problem of uniqueness of solutions for these equations attracted a lot of attention and only some partial results are known. In some cases nonuniqueness occurs and the existence of two solutions can be proved. Sometimes one of the solution is a trivial one and then the proof relies on finding a nontrivial one. In these difficult but important models one encounters another problem making our approach not feasible i.e. very common lack of compactness, thus if we would like to make our approach feasible we are forced to consider some truncated baby model compatible with compact bilinear forms.

2 The main result

To prove the result announced in the previous section we shall use the following theorem [3, Theorem 2.3.4] originating from the works of Krasnosielki, cf., e.g., [6].

**Theorem 2.1** Let $E$ be a Banach space, and let $P \subset E$ be a cone in $E$. Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in $E$ such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let operator $A$ satisfy conditions

$$\|Au\| \leq \|u\| \text{ for any } u \in P \cap \partial \Omega_1 \text{ and } \|Au\| \geq \|u\| \text{ for any } u \in P \cap \partial \Omega_2$$
\[ \|Au\| \geq \|u\| \text{ for any } u \in P \cap \partial \Omega_1 \text{ and } \|Au\| \leq \|u\| \text{ for any } u \in P \cap \partial \Omega_2 \]
is satisfied. Then \( A \) has at least one fixed point in \( P \cap (\Omega_2 \setminus \Omega_1) \).

**Theorem 2.2** Assume that, for the given cone \( P \subset E \), the bilinear and compact
form \( b : P \times P \to P \) satisfies the following coercivity condition

\[
\inf_{|u| = 1, u \in P} b(u, u) > 0. \tag{5}
\]

Then for any \( u_0 \in P \) as small as to satisfy (4) the equation (3) admits at least
two solutions in \( P \).

**Proof.** Let us define the operator

\[ Tu = b(u, u) + u_0 \tag{6} \]

then we shall apply Krasnosielksi Theorem once as a cone-compression in the
neighbourhood of zero and secondly as a cone-expansion at infinity.

Notice that we have the following estimates

\[
|Tu| \leq |u_0| + B|u|^2, \\
|Tu| \geq |u_0| - B|u|^2, \tag{7}
\]

where constant \( B = |b| > 0 \) denotes the norm of the bilinear form \( b \), i.e., the
least (smallest??) constant \( B \) satifying for any \( u, v \in \) the inequality

\[ |b(u, v)| \leq B|u||v|. \]

for any \( u, v \in E \).
Hence for sufficiently small $\rho_1 > 0$, i.e., such that $B\rho_1^2 + \rho_1 < |u_0|$ and any $u \in P$ and $|u| = \rho_1$ one has

$$|Tu| \geq |u_0| - B\rho_1^2 > \rho_1 = |u|. \quad (8)$$

Moreover, if we assume that there exists $\rho_2$ such that

$$|u_0| + B\rho_2^2 < \rho_2 \quad (9)$$

then apparently for any $u \in P$ and $|u| = \rho_2$ one has

$$|Tu| \leq |u_0| + B|u|^2 < \rho_2 = |u|. \quad (10)$$

But this can be accomplished if we assume $B\rho_1^2 + \rho_1 < |u_0| < \rho_2 - B\rho_2^2$.

Finally, for sufficiently large values of $\rho_3 > 0$ and any $u \in P$ and $|u| = \rho_3$, due to the coercivity assumption (5), one has

$$b(u, u) \geq C|u|^2 \quad (11)$$

implying

$$|Tu| \geq C\rho_3^2 - |u_0| > \rho_3 = |u|. \quad (12)$$

To be more specific $\rho_3$ has to be so large that $|u_0| < C\rho_3^2 - \rho_3$.

Combining (8) with (10) we get that part of the cone $P$ between the values $\rho_1$ and $\rho_2$ (in the $|\cdot|$ norm) is compressed while between the values $\rho_2$ and $\rho_3$ is expanded yielding the desired two fixed points in each set. Note that it might be necessary to distinguish between $\rho_2$ used in both sets as to prevent both fixed points to glue together.

The last the only assumption which should be made is to guarantee (10) to hold which follows readily from (4).
3 Examples of applications to differential equations

Example 1.

Consider the following BVP for ODE for continuous positive function $f$

\[- u''(t) = (u(t))^2 + f(t), \quad (13)\]

\[u(0) = u(1) = 0. \quad (14)\]

This problem can be formulated in a required form

\[u = b(u, u) + u_0 \quad (15)\]

where the function

\[u_0(t) = \int_0^1 G(t, s)f(s) \, ds \quad (16)\]

for the symmetric, Green function given by $G(t, s) = t(1-s), 0 \leq t \leq s \leq 1$, while the bilinear form $b$ is defined, for any $u, v \in P$, by

\[b(u, v) = \int_0^1 G(t, s)u(s)v(s), \]

and the cone $P$ on which coercivity of $B$ holds can be defined as

\[P = \left\{ u \geq 0 : \inf_{t \in [a, b]} u(t) \geq \min\{a, 1-b\}|u|_\infty \right\}. \]

For more applications to this kind of problems see [8, 7].

Example 2. Consider the following BVP for PDE, where $\Omega$ is an annulus in $\mathbb{R}^n$,

\[\Delta u(x) = (u(x))^2 + f(x), \quad x \in \Omega, \quad (17)\]

\[u|_{\partial \Omega} = 0. \quad (18)\]
Then the problem can be formulated as

\[ u = b(u, u) + u_0 \tag{19} \]

where

\[ u_0(x) = \int_{\Omega} G(x, y)f(y) \, ds \tag{20} \]

for the appropriate, symmetric Green function \( G \). The bilinear form \( B \) is defined by

\[ B(u, v) = \int_0^1 G(x, y)u(y)v(y) \, ds \]

and the cone \( P \) on which coercivity of \( B \) holds, can be expressed as \( P \{ u \geq 0 : \inf_{x \in D} u(x) \geq \gamma |u|_\infty \} \), for \( \overline{D} \subset \Omega \) and some positive \( \gamma \). For more applications to this kind of problems see \[4\].

**Example 3.** Consider for \( u = u(x, t) \) the boundary value problem

\[ u_t = \Delta u + u^2, \quad u(x, 0) = f(x). \tag{21} \]

Then one can formulate this problem as

\[ u = b(u, u) + u_0 \tag{22} \]

where

\[ u_0 = S(t)f, \tag{23} \]

while the bilinear form \( b \) is defined by

\[ b(u, v) = \int_0^t S(t - s)u(s)v(s) \, ds. \tag{24} \]

Note that for any \( t, s \) and the heat semigroup \( S(t) \) one has

\[ C|S(t-s)u(s)^2|_2 \geq |S(t-s)u(s)^2|_1 \geq |u(s)^2|_1 \geq |u(s)|^2_2, \tag{25} \]
hence after integration on \((0, t)\) and taking sup norm with respect to \(t \in [0, T]\), the right hand side can be estimated by
\[
\int_0^T |u(s)|_2^2 \, ds ,
\] (26)
while taking squared integral with respect to \(t \in [0, T]\) the right hand side can be estimated by
\[
T \left( \int_0^T |u(s)|_2^2 \right) \, ds .
\] (27)
Moreover
\[
|S(t)f|_2 \leq C|f|_2 ,
\] (28)
hence the second condition for sup norm in \(t\) follows. However one cannot guarantee the condition for coercivity. Note that due to [5] there are some results guaranteeing non uniqueness results. Our approach cannot yield this second solution, though some nonuniqueness results are known for semilinear parabolic equations [9].

**Example 4. General case.**

Since the crucial assumptions in both examples which guarantee that the cone \(P\) is invariant is the following property of the Green function (or in general the kernel of Hammerstein operator involving the bilinear form \(b(u, v)\)) \(G\)
\[
\inf_{x \in U} G(x, y) \geq \gamma \sup_{x \in V} G(x, y)
\] (29)
where \(\gamma > 0\) is independent of a set \(V\) and its subset \(U\) such that \(\overline{U} \subset V\). Note that this condition holds either if \(V\) is interval or annulus but we were not able to prove it for arbitrary domain, e.g. for a ball in higher dimension.
If (29) holds without any problems one can prove that properly defined cone is invariant under the action of the bilinear form, thus making it coercive. This problem can be illustrated in the second example, since if we replace an annulus with a ball the Green function is not bounded from above and also in the third one for the heat semigroup where the norms cannot be compared. This is the main obstacle in extending this kind of results to physically interesting models like the Navier Stokes equation, the Boltzmann equation or the Smoluchowski coagulation models sharing the property of quadratic nonlinearities.

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