Abstract. It is shown that the Bellman function method can be applied to study the $L^p$-norms of general operators on martingales, i.e., of operators that are not necessarily martingale transforms. Informally, we provide a single Bellman-type function that "encodes" the $L^p$-boundedness of "almost all" operators from Gundy’s extrapolation theorem. As examples of such operators, we consider the Haar transforms and the operator whose $L^p$-boundedness underlies Rubio de Francia’s inequality for the Walsh system.

1. Introduction

Gundy’s theorem [4] of 1968 can be seen as a martingale analog of the principle that general Calderón–Zygmund operators are bounded. Consider, for example, Rubio de Francia’s inequality [12]. As discussed in the introduction of [8], it can be thought of as a one-sided analog of Parseval’s identity in $L^p$. This inequality can be reduced to the $L^p$-boundedness of a certain Calderón–Zygmund operator. On the other hand, its analog [11] for the Walsh basis, the martingale counterpart of the Fourier basis, follows from the version [7] of Gundy’s theorem for operators on vector-valued martingales. This illustrates the generality of Gundy’s theorem because the mentioned Calderón–Zygmund operator arising in Rubio de Francia’s original considerations [12] has a vector-valued kernel that satisfies only a very weak and subtle smoothness condition.

On the other hand, in the paper [2] of 1984, Burkholder proves the $L^p$-boundedness of the martingale transforms using another approach borrowed from stochastic optimal control. The martingale transforms are a special case of operators from Gundy’s theorem. They can be considered as a martingale analog of the Hilbert transform, the most basic example of a Calderón–Zygmund operator. Nevertheless, Burkholder’s work [2] is now regarded as a real breakthrough in harmonic analysis because of his method: it gives a deep insight into the structure of the estimated norms.
$L^p$-norms and, in particular, allows him to calculate sharp constants in the corresponding $L^p$-inequalities. His approach gives rise to a new theory \[5, 10, 13\], which establishes a deep connection between harmonic analysis, stochastic processes, differential geometry, and partial differential equations. The methodology suggested by Burkholder is now commonly referred to as the Bellman function method in harmonic analysis.

In this paper, we implement the program that was partly announced (without any proofs) in the short report \[1\]: we show that Burkholder’s method can be extended to “almost all” operators from the vector Gundy’s theorem \[7\] and provide a single Bellman-type function that “encodes” their $L^p$-boundedness. In particular, this applies to the martingale Rubio de Francia operator from \[11\].

2. Notation and preliminaries

Let $\mathcal{A}$ be at most a countable set of indices. By $l^2_\mathcal{A}$ we denote the corresponding $l^2$-space where elements of sequences are indexed by $\alpha \in \mathcal{A}$. Further, $L^p$ and $L^p(l^2_\mathcal{A})$ denote the Lebesgue spaces, respectively, of scalar-valued and vector-valued functions on the interval $[0,1)$.

Suppose $n$ runs over $\mathbb{Z}_{\geq 0}$ and $\{\mathcal{F}_n\}$ is a filtration of the Borel algebra over $[0,1)$. We put $\mathcal{F}_\infty \overset{\text{def}}{=} \sigma(\bigcup_n \mathcal{F}_n)$. For $f \in L^1(l^2_\mathcal{A})$, we denote $E_m f \overset{\text{def}}{=} E[f | \mathcal{F}_m]$, $m \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

Concerning the operators $E_m$ for vector-valued functions and the properties of vector-valued martingales discussed below, see, e.g., \[3, \text{Chapter V}\]. A sequence $\{f_n\}$ of $l^2_\mathcal{A}$-valued functions $f_n = \{f_n^\alpha\}_{\alpha \in \mathcal{A}} \in L^1(l^2_\mathcal{A})$ is called a martingale if $E_m f_n = f_m$ for $m \leq n$. The $L^p$-norm of a martingale is defined as

$$\|\{f_n\}\|_{L^p} \overset{\text{def}}{=} \sup_n \|f_n\|_{L^p},$$

and any function $f \in L^p(l^2_\mathcal{A})$, $1 \leq p < \infty$, generates a martingale $\{E_n f\}$ such that $E_m f \overset{L^p}{\rightarrow} E_\infty f$ and $\|\{E_n f\}\|_{L^p} = \|E_\infty f\|_{L^p}$.

We call a martingale simple if $f_{n+1} \equiv f_n$ for all sufficiently large $n$. We also impose on $\{\mathcal{F}_n\}$ the regularity condition \[7, \text{condition (R)}\]: $\mathcal{F}_n$ are finite and the measures of their atoms decrease, as $n$ increases, no faster than a geometric progression. We present a version of Gundy’s theorem for vector-valued martingales that is formulated and proved in \[7, \text{Theorem 1}\] in somewhat greater generality. The original scalar theorem can be found in \[4\].

**Theorem (R. F. Gundy).** Let $T$ be an operator that transforms simple martingales $f = \{f_n\}$ into scalar-valued measurable functions and has the following properties.

- (G1) $|T(f + g)| \leq C_1(|Tf| + |Tg|)$.
- (G2) $\|Tf\|_{L^2} \leq C_2 \|f\|_{L^2}$.
- (G3) If $f$ satisfies the relations $\Delta_0 f \overset{\text{def}}{=} f_0 \equiv 0$ and

$$\Delta_n f \overset{\text{def}}{=} f_n - f_{n-1} = 1_{e_n} \Delta_n f$$

for $n > 0$ and some $e_n \in \mathcal{F}_{n-1}$, then

$$\{|Tf| > 0\} \subset \bigcup_{n>0} e_n.$$
For such an operator, we have
\[ |\{(Tf) > \lambda\}| \leq C(C_1, C_2, \{\mathcal{F}_n\}) \lambda^{-1} \|f\|_{L^1} \quad \text{for} \quad \lambda > 0. \]

By “⊂” we denote the relation “is a dyadic subinterval of”. Further, we consider, along with [0,1), an arbitrary bounded interval \(I \subseteq \mathbb{R}\) and deal only with filtrations constructed from its dyadic subintervals. For any subinterval \(J \subseteq I\), we denote its left and right halves by \(J^\pm\) and introduce the \(l_2^2\)-valued functions \(\mathbb{I}_J\), \(\alpha \in \mathcal{A}\): the component of \(\mathbb{I}_J^\alpha\) with index \(\alpha\) is the indicator function \(\mathbb{1}_J\) and the other components are zero. Then the \(l_2^2\)-valued Haar functions
\[ h_0^\alpha \equiv |I|^{-1/2} \mathbb{1}_I^\alpha \quad \text{and} \quad h_j^\alpha \equiv |J|^{-1/2}(\mathbb{1}_j^\alpha - \mathbb{1}_{j-}^\alpha), \quad J \subseteq I, \quad \alpha \in \mathcal{A}, \quad (1) \]
form an orthonormal basis in \(L^2(I, l_2^2)\). We drop the index \(\alpha\) in (1) in situations where we are in the scalar setting.

By \(\mathcal{L}(I, l_2^2)\) we denote the set of all linear operators that transform finite linear combinations of the vector-valued Haar functions \(\mathbb{1}_I\) into measurable scalar functions. Next, we introduce a subset of \(\mathcal{L}(I, l_2^2)\) that is a somewhat more regular analog of the class of operators from Gundy’s theorem.

**Definition 1.** We say that an operator \(T \in \mathcal{L}(I, l_2^2)\) belongs to the class \(\mathcal{G}(I, l_2^2)\) if it has the following properties.

\(\text{R1}\) \(\|Tf\|_{L^2} \leq \|f\|_{L^2}\).

\(\text{R2}\) The operator \(T\) does not enlarge the supports of the basis functions:
\[ \text{supp}Th_j^\alpha \subseteq J \quad \text{for} \quad \alpha \in \mathcal{A} \text{ and } J \subseteq I. \]

Let \(\{\mathcal{F}_n\}\) be the Haar filtration on [0,1) where the atomic intervals are bisected one by one, from left to right:
\[ \mathcal{F}_0 = \sigma\{0, 1\}, \quad \mathcal{F}_1 = \sigma\{[0, \frac{1}{2}), [\frac{1}{2}, 1]\}, \quad \mathcal{F}_2 = \sigma\{[0, \frac{1}{4}), [\frac{1}{4}, \frac{1}{2}), [\frac{1}{2}, \frac{3}{4}), [\frac{3}{4}, 1]\}, \]
\[ \mathcal{F}_3 = \sigma\{[0, \frac{1}{8}), [\frac{1}{8}, \frac{1}{4}), [\frac{1}{4}, \frac{3}{8}), [\frac{3}{8}, \frac{1}{2}), [\frac{1}{2}, \frac{5}{8}), [\frac{5}{8}, \frac{3}{4}), [\frac{3}{4}, \frac{7}{8}), [\frac{7}{8}, 1]\}, \quad \mathcal{F}_4 = \sigma\{[0, \frac{1}{16}), ...\}. \]

For this specific filtration, we prove that the only additional regularity, assumed by Definition 1 as compared to (G1)–(G3) is that \(T\) is linear and contractive.

**Proposition 1.** Consider functions \(f \in \text{span}\{h_0^\alpha, h_j^\alpha\}_{\alpha \in \mathcal{A}, J \subseteq [0,1)}\). For an operator \(T \in \mathcal{L}([0,1), l_2^2)\), condition (G2) is equivalent to condition (G3) for martingales \(\{f_n\} \equiv \{E_n f\}\), where \(E_n f\) are calculated with respect to the Haar filtration introduced above.

**Proof.** Suppose \(T\) satisfies (G2) Consider \(f\) such that
\[ \Delta_0 f \equiv E_0 f = \sum_{\alpha \in \mathcal{A}} (f, h_0^\alpha) h_0^\alpha \equiv 0. \]
We have
\[ \Delta_n f \equiv E_n f - E_{n-1} f = \sum_{\alpha \in \mathcal{A}} (f, h_j^\alpha) h_j^\alpha, \]
where \(J_n\) is the interval that is bisected when switching from \(\mathcal{F}_{n-1}\) to \(\mathcal{F}_n\). Thus, the set \(e_n = \{\Delta_n f \neq 0\}\) is empty if \((f, h_j^\alpha) = 0\) for all \(\alpha \in \mathcal{A}\), or equals \(J_n\) otherwise. Due to (G2) the following relation holds, implying (G3)
\[ Tf = \sum_{n} \mathbb{1}_{e_n} \sum_{\alpha \in \mathcal{A}} (f, h_j^\alpha) Th_j^\alpha. \]

The reverse implication is obvious: we only need to apply (G3) to \(h_j^\alpha\). \(\square\)
We can treat $T \in \mathcal{G}(I, l_n^2)$ as bounded linear operators from $L^2(I, l_n^1)$ to $L^2(I)$. We can also apply Gundy’s theorem to obtain the weak-type $(1, 1)$ estimate for them. Thus, relying on the Marcinkiewicz interpolation theorem, we can prove that these operators are uniformly bounded in any $L^p$, $1 < p \leq 2$. On the other hand, the generality of the class $\mathcal{G}(I, l_n^2)$ is, to a large extent, close to the generality of conditions [G1] [G3].

For sequences $\varepsilon = \{\varepsilon_J\}_{J \subseteq [0, 1)}$ of numbers $\varepsilon_J \in \{-1, 1\}$, consider the operators $T_\varepsilon \in \mathcal{G}([0, 1), \mathbb{R})$ defined by the formula

$$T_\varepsilon f \overset{\text{def}}{=} (f, h_0) h_0 + \sum_{J \subseteq [0, 1)} \varepsilon_J (f, h_J) h_J.$$ 

Such operators are called martingale transforms. Relying on the Bellman function method borrowed from stochastic optimal control, Burkholder suggested an alternative proof [2] of their $L^p$-boundedness that gives a deep insight into the structure of the norms $\|T_\varepsilon\|_{L^p \rightarrow L^p}$. In particular, it allowed him to calculate $\sup_\varepsilon \|T_\varepsilon\|_{L^p \rightarrow L^p}$.

Our goal is to show that the Bellman function method can be extended to the whole class $\mathcal{G}(I, l_n^2)$. Our considerations are somewhat similar to ones in [9] where a Bellman-type function is built for Burkholder’s problem. But since we consider much more general problem, everything becomes substantially more complicated. We provide only one of the variety of appropriate Bellman-type functions, but this is enough to demonstrate the applicability of Burkholder’s theory.

3. Motivating examples

First, we provide two examples of operators in $\mathcal{G}$ that are not martingale transforms: exact Bellman function from [2] or Bellman-type function from [9] cannot be used to prove their $L^p$-boundedness. Throughout this section, we assume that $I = [0, 1)$ and that $\mathcal{F}_n$ is the standard dyadic filtration:

$$\mathcal{F}_n = \sigma\left\{ \left[ \frac{k}{2^n}, \frac{k+1}{2^n} \right) \mid 0 \leq k < 2^n \right\}, \quad n \in \mathbb{Z}_{\geq 0}.$$ 

Here, in contrast to the Haar filtration, all the atomic intervals in $\mathcal{F}_{n-1}$ are bisected at once (not one by one) when switching to $\mathcal{F}_n$.

**Haar transforms.** For each $n \in \mathbb{Z}_{\geq 0}$, we can build an operator $H_n \in \mathcal{L}([0, 1), \mathbb{R})$ that establishes a one-to-one correspondence between the finite Haar basis and the indicator basis in the subspace of $\mathcal{F}_n$-measurable functions, i.e. between

$$\{h_0, h_J\}_{J \subseteq [0, 1), |J| \geq 2^{-n+1}} \quad \text{and} \quad \{|e|^{-1/2} \mathbb{1}_e\}_{e \subseteq [0, 1), |e| = 2^{-n}},$$

while mapping other Haar functions to zero or to themselves and satisfying [R2]. Indeed, put $H_n h_0 \overset{\text{def}}{=} h_0$. Next, suppose we have built $H_n$. For $|J| \geq 2^{-n+1}$ and for 0 substituted for $J$, we set

$$H_{n+1} h_J \overset{\text{def}}{=} |e^+|^{-1/2} \mathbb{1}_{e^+},$$

where $e$ is such that $H_n h_J = |e|^{-1/2} \mathbb{1}_e$. For $|J| = 2^{-n}$, we set

$$H_{n+1} h_J \overset{\text{def}}{=} |J^-|^{-1/2} \mathbb{1}_{J^-}.$$ 

Each operator $H_n$ acts on an $\mathcal{F}_n$-measurable function as the corresponding Haar transform and gives a step function constructed from its Haar coefficients.
Furthermore, $H_n$ is either a contraction or a unitary operator, depending on how we handle the Haar functions $h_J$ with $|J| < 2^{-n+1}$, and thus it satisfies \((R1)\). Hence, we have the following fact.

**Proposition 2.** The operators $H_n$ belong to $G([0, 1), \mathbb{R})$.

The Rubio de Francia operator. Consider the Walsh basis \{w_n\} consisting of all possible products of Rademacher functions:

- set $w_0 \equiv 1_{[0, 1)}$;
- for any index $n > 0$, consider its dyadic decomposition $n = 2^{k_1} + \cdots + 2^{k_s}$, $k_1 > k_2 > \cdots > k_s \geq 0$, and set

$$w_n(x) \equiv \prod_{i=1}^s r_{k_i+1}(x),$$

where $r_k(x) = \text{sign} \sin 2^k \pi x$.

The system \{w_n\} resembles in its properties the Fourier basis (see, e.g., \[6, \S 4.5\]) and can be considered as its discrete analog.

1. In particular, the following direct analog of Rubio de Francia’s inequality \[12\] for the Walsh system is proved in \[11\].

**Theorem.** Let \{f^n\} be at most a countable collection of functions with Walsh spectra supported in pairwise disjoint intervals $I_n \subset \mathbb{Z}_{\geq 0}$:

$$f^n = \sum_{m \in I_n} (f^n, w_m) w_m.$$

For $1 < p \leq 2$, we have

$$\left\| \sum_n f^n \right\|_{L^p} \leq C_p \left\| \{f^n\} \right\|_{L^p(\ell^2)},$$

where $C_p$ does not depend on \{f^n\} or \{I_n\}.

Our second example is the operator whose $L^p$-boundedness underlies inequality \[2\]. In order to introduce it, we need certain simple and well-known properties of the Walsh functions.

(W1) For a function $f \in L^1$, its martingale differences $\Delta_n f$ with respect to \{F_n\} coincide with the Walsh multipliers associated with the indicator functions of the intervals $D_0 \equiv \{0\}$ and $D_n = \{2^{n-1}, 2^{n-1} + 1, \ldots, 2^n - 1\}$, $n > 0$:

$$\Delta_0 f = (f, h_0) h_0 = (f, w_0) w_0,$$

$$\Delta_n f = \sum_{J \subseteq [0, 1)} (f, h_J) h_J = \sum_{m \in D_n} (f, w_m) w_m. \quad (3)$$

(W2) For $a, b \in \mathbb{Z}_{\geq 0}$, we have the “exponential” property $w_a w_b \equiv w_{a+b}$, where $a+b$ means the bitwise XOR operation: the corresponding bits in the binary decompositions of $a$ and $b$ are summed modulo 2.

Suppose multi-indices $(j, k)$ run over a subset $A \subseteq \mathbb{Z}_{\geq 0}^2$ and numbers $a_{j,k} \in \mathbb{Z}_{\geq 0}$ are such that the sets $a_{j,k} + D_k$ are pairwise disjoint. Consider functions

$$f = \{f^{j,k}\}_{(j,k) \in A} \in L^2(\ell^2_A).$$

\[1\]It actually is the Fourier basis in the sense of abstract harmonic analysis if we identify $[0, 1)$ with the Cantor dyadic group in the right way.
We introduce an operator \( G \) that transplants parts of the Walsh spectra of \( f^{j,k} \) into \( a_{j,k} + D_k \) and combines the results into a single function:

\[
Gf \equiv \sum_{(j,k) \in A} w_{a_{j,k}} \Delta_k f^{j,k}.
\]

The paper [11] mainly consists of combinatorial arguments that reduce estimate (2) to the estimate

\[
\|Gf\|_{L^p} \leq C_p \|f\|_{L^p(I^j_2)}, \quad 1 < p \leq 2,
\]

where \( C_p \) depends only on \( p \). The operator \( G \) satisfies the conditions of Gundy’s theorem for the standard dyadic filtration. But we can easily switch to the Haar filtration. Namely, we have the following proposition.

**Proposition 3.** The operator \( G \) belongs to \( \mathcal{G}(I^j_2) \).

*Proof.** Since \( a_{j,k} + D_k \) are pairwise disjoint, Parseval’s identity for the Walsh basis and properties [W1] and [W2] imply [R1]

\[
\|Gf\|_{L^2}^2 = \sum_{(j,k) \in A} \|\Delta_k f^{j,k}\|_{L^2}^2 \leq \sum_{(j,k) \in A} \|f^{j,k}\|_{L^2}^2 = \|f\|_{L^2(I^j_2)}^2.
\]

Since we can express \( \Delta_k f^{j,k} \) in terms of \( h_0 \) and \( h_J \) as in (3), we have

- if \( k > 0 \) and \( |J| = 2^{-k+1} \), then \( G h_J^{j,k} = w_{a_{j,k}} h_J \);
- \( G \) vanishes on all other \( h_J^{j,k} \).

This implies [R2].

\[\square\]

### 4. Main results

Henceforth, we suppose \( I \subseteq \mathbb{R} \), \( 1 < p \leq 2 \), and \( \frac{1}{p} + \frac{1}{q} = 1 \). For \( \varphi \in L^1(I) \), we denote \( \langle \varphi \rangle_I \equiv \frac{1}{|I|} \int_I \varphi \). We agree that for vectors \( x, y \in I^2_2 \), the notation \( xy \) means their inner product, and \( |x| \) means the \( l^2 \)-norm of \( x \). For

\[
f = \{f^a\}_{a \in A} \in L^2(I, I^2_2),
\]

we set

\[
\langle f \rangle_I \equiv \{\langle f^a \rangle_I\}_{a \in A} \quad \text{and} \quad \text{osc}_I^2(f) \equiv \left| \langle f - \langle f \rangle_I \rangle^2 \right|_I = \langle |f|^2 \rangle_I - \langle |f| \rangle_I^2.
\]

Suppose \( T \in \mathcal{G}(I, I^2_2) \). We have \( \|T^* g\|_{L^2(I, l^2_2)} = \|T g\|_{L^2(I, l^2_2)} \leq 1 \), and thus the inequality

\[
\langle g^2 \rangle_I - \text{osc}_I^2(T^* g) \geq \left| \langle T^* g \rangle_I \right|^2
\]

holds for any \( g \in L^2(I) \). It becomes an equality if \( T^* \) is an isometry.

We introduce the Bellman function

\[
B(x) \equiv \sup \left\{ \left| \langle g T [f - \langle f \rangle_I] \rangle \right|_I \left| \langle f \rangle_I = x_1, \langle g^2 \rangle_I - \text{osc}_I^2(T^* g) = x_2, \langle |f|^p \rangle_I = x_3, \langle |g|^q \rangle_I = x_4 \right. \right\},
\]

where \( x = (x_1, x_2, x_3, x_4) \in I^2_2 \times \mathbb{R}^3_{\geq 0} \), and the supremum is taken over \( f \in L^2(I, I^2_2), \ g \in L^2(I), \) and \( T \in \mathcal{G}(I, I^2_2) \) satisfying the identities after the vertical bar. It is easy to check that \( B \) does not depend on the choice of \( I \).

Let \( \Omega_B \) consist of all the points \( x \) for which the supremum in (5) is taken over a non-empty set. Applying Jensen’s inequality in vector and scalar forms (or Hölder’s inequality together with Minkowski’s integral inequality for the \( l^2_2 \)-norm), we obtain

\[
\Omega_B \subseteq \Omega_p \equiv \left\{ x \in I^2_2 \times \mathbb{R}^3_{\geq 0} \left| |x_1|^p \leq x_3, \ x_2 \leq x_4^{2/q} \right. \right\}.
\]
We introduce the class $K^p(l^2_a)$ of Bellman-type functions.

**Definition 2.** We say that a function $B \in C(\Omega_p)$ belongs to the class $K^p(l^2_a)$ if it satisfies the following boundary condition and geometric concave-type condition.

(B1) If $|x_1|^p = x_3$ then $B(x) \geq 0$.

(B2) If for $x, x^\pm \in \Omega_p$ and $\Delta \in \mathbb{R}$ we have

$$\frac{x^+ + x^-}{2} - x = (0, \Delta^2, 0, 0),$$

then

$$B(x) \geq \frac{|x^+ - x^-|}{2} \Delta + \frac{B(x^+) + B(x^-)}{2}. \quad (7)$$

Our first theorem states that any Bellman-type function majorizes the true Bellman function.

**Theorem 1.** If $B \in K^p(l^2_a)$, then $B(x) \leq B(x)$ for all $x \in \Omega_B$.

Next, we provide a specific representative of the class $K^p(l^2_a)$. For $y \in \mathbb{R}^4_{\geq 0}$, we define the function

$$B_0(y) \overset{\text{def}}{=} 2(y_3 + y_4) - y_2^p \frac{q/2}{q} - \delta_p \begin{cases} y_1^{2-p} y_2 + y_1^{2-p-2t_p(p-1)} y_2^{t_p+1}, & y_1^p \geq y_2^{q/2}; \\ \frac{2}{p}(2 - p - t_p(p - 1)) y_2^{q/2}, & y_1^p \leq y_2^{q/2}. \end{cases} \quad (8)$$

**Theorem 2.** There exist parameters $t_p \geq 0$, $\delta_p > 0$, and a constant $C_p > 0$ such that the restriction of the function

$$B(x) \overset{\text{def}}{=} C_p B_0(|x_1|, x_2, x_3, x_4) \quad (9)$$

to $\Omega_p$ belongs to $K^p(l^2_a)$.

Theorems 1 and 2 have the following consequence.

**Corollary 1.** If $T \in G([0, 1], l^2_a)$, then for $f \in L^2(l^2_a)$, we have

$$\|Tf\|_{L^p} \leq (2p^{1/p} q^{1/q} C_p + 1) \|f\|_{L^p}. \quad (10)$$

Finally, our third theorem concerns properties (B1) and (B2) for the Bellman function itself.

**Theorem 3.** Let $B_0$ be the function that is defined by (5) in the situation where $l^2_a = \mathbb{R}$ and $T$ runs only over unitary operators in $G(I, \mathbb{R})$. Then $\Omega_{B_0} = \Omega_p$ and $B_0$ satisfies (B1) and (B2).

Theorems 1 and 3 lead to the following conjectures (which may help to calculate the true Bellman function $B$):

- the supremum in (5) is attained on unitary operators;
- the function $B$ is the pointwise infimum of the functions from $K^p(l^2_a)$. 


5. Guessing the candidate.

In this section, we briefly describe how we guess $B_0$. First, applying Taylor expansion to (7) and differentiating with respect to $\Delta$, we infer that property (B2) is associated with the differential inequality

$$d^2 B[y](dy) \leq \frac{(dy_1)^2}{2 \partial y_2} B(y) \leq 0.$$  \hfill (10)

On the left, we calculate the Hessian at $y \in \mathbb{R}_+^4$ and apply it, as a quadratic form, to an arbitrary vector $dy \in \mathbb{R}^4$. Using the ideas of [9], we come to our first guess:

$$B(y) = 2(y_3 + y_4) - y_1^p - y_2^q/2.$$  \hfill (11)

For this function, condition (10) takes the form

$$-p(p-1)y_1^{p-2}(dy_1)^2 - \frac{q}{2}(\frac{q}{2} - 1)y_2^{q/2-2}(dy_2)^2 \leq - \frac{(dy_1)^2}{y_2^{q/2-1}}.$$  \hfill (12)

Therefore, we see that $B$ satisfies (11) only for $y$ such that $y_1^{p-2}y_2^{q/2-1} \geq 1$ or, what is the same, where $y_2^{q/2} \geq y_1^p$. In order to fix this, we can try to add a correction term that makes $B$ “more concave”, similarly to how it is done in [9]. We refer to \{\{y_2^{q/2} = y_1^p\}\} as the critical curve. We try to add $-\delta_p y_1^{q-2p} y_2$ below the critical curve and to add

$$-\delta_p \left( \frac{2}{q} y_2^{q/2} + \frac{2-p}{p} y_1^p \right)$$

above the critical curve (the latter expression comes from Young’s inequality). However, a direct computation (which is quite long) shows that the resulting function satisfies (11) only for $q \leq 4 \Leftrightarrow p \geq 4/3$! In order to overcome this restriction, we add the term $-\delta_p y_1^{q-2p} y_2 |y_2|^{p-1}$ below the critical curve (and the corresponding term above). And this is how we come up with the Bellman candidate.

6. Proof of Theorem 1

Fix $x \in \Omega_B$ and consider $f \in L^2(I, \ell_4^1)$, $g \in L^2(I)$, and $T \in \mathcal{G}(I, \ell_4^1)$ such that $x = x^t$, where $x^t = (x^t_1, x^t_2, x^t_3, x^t_4) \triangleq \langle (f)_j, (g)_j \rangle$, \(\langle |f|^p \rangle_j, \langle |g|^q \rangle_j\), \(J \subseteq I\).

We also introduce

$$\delta_J \triangleq \{ |J|^{-1/2}(g, Th_0^q) \}_{\alpha \in A} \quad \text{and} \quad \Delta_J \triangleq |\delta_J|.$$  \hfill (13)

We have

$$\text{osc}_J^2(T^*g) = \frac{1}{|J|} \sum_{Q \subseteq J, \alpha \in A} (g, Th_0^q) ^2$$  \hfill (14)

and

$$\Delta_J^2 = \frac{x^t_{J^+} + x^t_{J^-}}{2} - x^t_J.$$  \hfill (15)

We note that property (R2) and equation (14) imply that the inequality $x^t_J \geq 0$ (see [4]) is inherited by all $x^t_{J^\pm}$, $J \subseteq I$, and this is the very place where we need (R2).

We also obtain

$$\frac{|x^t_{J^+} - x^t_{J^-}|}{2} \Delta_J \geq \frac{x^t_{J^+} - x^t_{J^-}}{2} \cdot \delta_J = \frac{1}{|J|} \sum_{\alpha \in A} (f, h_0^j(g, Th_0^q)).$$
Applying inequality (7) \( k \) times, we obtain
\[
B(x) \geq \frac{1}{|I|} \sum_{J \subseteq I, \alpha \in A} (f, h_\alpha^g)(g, Th_\alpha^g) + \sum_{J \subseteq I, |J| = 2^{-k} |I|} \frac{B(x^J)}{2^k}. \tag{12}
\]

We denote the first and second terms in (12) by \( U_k \) and \( V_k \), respectively. Since the operator \( T \) and the inner product are continuous in \( L^2 \), we have
\[
U_k \to \langle g TP f \rangle_I,
\]
where \( P \) is the orthogonal projection onto \( \text{span} \left( \{ h_\alpha^g \}_{\alpha \in A, J \subseteq I} \right) \).

By \( L^\infty_{00}(I, l_2^2) \) we denote the subspace of \( L^\infty(I, l_2^2) \) consisting of bounded vector functions with a finite number of non-zero components. Assume for a while that \( g \in L^\infty(I) \) and \( f \in L^\infty_{00}(I, l_2^2) \). We introduce the step function
\[
x^k(t) = (x_1^k(t), x_2^k(t), x_3^k(t), x_4^k(t))
\]
that takes values \( x^J \) on the intervals \( J \subseteq I, |J| = 2^{-k} \). We note that the functions \( x_4^k(t) \) form a bounded submartingale. Thus, by the Lebesgue differentiation theorem and by Doob’s martingale convergence theorem, we have
\[
x^k \xrightarrow{a.e.} (f, \eta, |f|^p, |g|^q),
\]
where \( \eta \) is a function in \( L^1(I) \). All \( x^k \) are uniformly bounded vector functions whose non-zero components take values in a finite-dimensional subspace of \( l_2^2 \). These values form a compact set in this subspace and, therefore, the continuous function \( B \) is bounded on the values of \( x^k \). Lebesgue’s dominated convergence theorem and the boundary condition (B1) imply that
\[
V_k = \int_I B(x^k(t)) \, dt \to \int_I B(f(t), \eta(t), |f(t)|^p, |g(t)|^q) \, dt \geq 0.
\]

We arrive at the inequality
\[
B(x) \geq \langle g TP f \rangle_I.
\]

Now we drop the assumptions \( f \in L^\infty_{00}(I, l_2^2) \) and \( g \in L^\infty(I) \) and, instead, consider sequences \( f_n \) and \( g_m \) in these spaces that tend to \( f \) and \( g \) in the \( L^2 \)- and \( L^q \)-norms, respectively. We have
\[
B((f_n)_I, (g_m)_I) - \text{osc}_I^2(T^* g_m), \langle |f_n|^p \rangle_I, \langle |g_m|^q \rangle_I \geq \langle g_m TP f_n \rangle_I.
\]
Relying on the continuity of \( T, T^*, P \), and the inner product in \( L^2 \), we can pass to the limit as \( n, m \to \infty \). \( \square \)

7. Proof of Theorem \( \Box \)

**Lemma 1.** Suppose \( y_1, y_2 > 0 \) and \( y_1^p \geq y_2^{q/2} \). For any \( a_1, a_2, b_1, b_2 \in \mathbb{R} \) such that
\[
g(a_2 - a_1) = 2p(b_1 - b_2) \quad \text{and} \quad a_2 - a_1 \geq 0,
\]
we have \( y_1^{a_1} y_2^{b_1} \leq y_1^{a_2} y_2^{b_2} \).

**Proof.** Raising both parts of \( y_1^p \geq y_2^{q/2} \) to the power \( \frac{a_2 - a_1}{p} \), we obtain the desired inequality. \( \square \)
Lemma 2. There exist parameters $t_p \geq 0$, $\delta_p > 0$, and a constant $c_p > 0$ such that the function $B_0$, defined by (8), belongs to $C^1(\mathbb{R}^4_{\geq 0})$ and we have

$$d^2B_0(y)(dy) \leq c_p \frac{(dy_1)^2}{2 \partial_{y_2}B_0(y)} \leq 0$$

for any vector $dy \in \mathbb{R}^4$ and any point $y \in \mathbb{R}^4_{\geq 0}$ where $y_1, y_2 \neq 0$ and $y_1^p \neq y_2^{q/2}$.

Proof. The direct calculation of $\partial_{y_1}B_0$ and $\partial_{y_2}B_0$ implies that $B_0 \in C^1(\mathbb{R}^4_{\geq 0})$ and that $\partial_{y_2}B_0 \leq 0$ for any $t_p, \delta_p \geq 0$. It remains to prove that

$$\left( \begin{array}{cc} \partial_{y_1}^2B_0 - \frac{c_p}{2 \partial_{y_2}B_0} & \partial_{y_1}y_{1}B_0 \\ \partial_{y_1}y_{2}B_0 & \partial_{y_2}^2B_0 \end{array} \right) \leq 0,$$

provided $y_1, y_2 \neq 0$ and $y_1^p \neq y_2^{q/2}$. By direct calculations, we have $\partial_{y_2}B_0 \leq 0$ for any $t_p, \delta_p \geq 0$. Thus, it suffices to choose $t_p, \delta_p$, and $c_p$ such that

$$2 \partial_{y_2}B_0 (\partial_{y_1}y_{1}B_0)^2 - c_p \partial_{y_2}^2B_0 \geq 0,$$

and then to prove that

$$2 \partial_{y_2}B_0 (\partial_{y_1}y_{1}B_0)^2 - c_p \partial_{y_2}^2B_0 \geq 0. \quad (13)$$

First, suppose $y_1^p \leq y_2^{q/2}$. Then inequality (13) takes the form

$$y_1^{p-2}y_2^{q/2-1}[p(p-1) + 2\delta_p(2 - t_p)p(p-1)] \geq c_p.$$

In the case being considered, we have $y_1^{p-2}y_2^{q/2-1} = (y_1^{-p}y_2^{q/2})^{1-2/q} \geq 1$ and $\partial_{y_1}y_{1}B_0 = 0$. Therefore, inequalities (13) and (14) hold, for example, when

$$t_p \leq \frac{2 - p}{2(p-1)} = \frac{q}{2} - 1,$$

and $0 \leq t_p \leq 1$. Direct calculations give

$$\partial_{y_2}B_0(y) = -\left( \frac{q}{2} y_2^{q/2-1} + \delta_p y_1^{-p} + (1 + t_p) \delta_p y_1^{-p}y_2 \right),$$

$$\partial_{y_1}^2B_0(y) = -\left( p(p-1) y_1^{-p} + (2 - p)(1 - \delta_p) y_1^{-p}y_2 + s_p(s_p - 1) \delta_p y_1^{-p}y_2 \right).$$

Using direct computation and estimates (15), we obtain

$$\partial_{y_2}B_0(y) \partial_{y_1}^2B_0(y) \geq \delta_p p(p-1) + \frac{q}{2} \beta_1 y_1^{-p} y_2^{q/2-1} + \delta_p \beta_2 y_1^{-p}y_2^{q/2-1},$$

where

$$\beta_1 = p(p-1) - \delta_p(2 - p)(p-1) - \delta_p s_p(1 - s_p),$$

$$\beta_2 = p(p-1)(1 + t_p) - \delta_p(2 - p)(p-1)(2 + t_p) - \delta_p s_p(1 - s_p)(2 + t_p).$$

By putting $c_p = \delta_p p(p-1)$ and by taking a sufficiently small $\delta_p$, we come to (13).
It remains to prove (14). Fact 1 implies \( y_1^{s_1} y_2^{t_1} \leq y_1^{2-p} \) and \( y_1^{s_1-1} y_2^{t_1} \leq y_1^{1-p} \). Relying on these inequalities, we obtain
\[
\begin{align*}
\partial_{y_2} B_0(y) &\geq -\left( \frac{q}{2} y_2^{q/2-1} + \delta_p (2 + t_p) y_1^{2-p} \right), \\
\partial_{y_1} \partial_{y_2} B_0 & = -\delta_p \left( (2 - p) y_1^{1-p} + s_p (1 + t_p) y_1^{s_1-1} y_2^{t_1} \right) \\
& \geq -\delta_p \left( (2 - p + s_p (1 + t_p)) y_1^{1-p} \right).
\end{align*}
\]
Therefore, the expression in (14) is greater than
\[
c_p \frac{q}{2} (\frac{q}{2} - 1) y_2^{q/2-2} + c_p \delta_p (1 + t_p) t_p y_1^{s_1} y_2^{t_1} - \delta_p^2 \beta_3 y_1^{2-2p} y_2^{q/2-1} - \delta_p^3 \beta_4 y_1^{4-3p}, \tag{16}
\]
where
\[
\beta_3 = q (2 - p + s_p (1 + t_p))^2 \quad \text{and} \quad \beta_4 = 2 (2 + t_p) (2 - p + s_p (1 + t_p))^2.
\]
Again, Fact 1 implies \( y_1^{2-2p} y_2^{q/2-1} \leq y_2^{q/2-2} \) and \( y_1^{4-3p} \leq y_1^{s_1} y_2^{t_1} \). Therefore, expression (16) is greater than
\[
(c_p \frac{q}{2} (\frac{q}{2} - 1) - \delta_p^2 \beta_3) y_2^{q/2-2} + (c_p \delta_p (1 + t_p) t_p - \delta_p^3 \beta_4) y_1^{s_1} y_2^{t_1}.
\]
Taking \( c_p = \delta_p (p - 1) \), \( t_p = \min \{ 1, \frac{q}{2} - 1 \} \), and a sufficiently small \( \delta_p > 0 \), we finish the proof.

\[\textbf{Lemma 3.} \text{ Suppose } A \text{ is finite. There exist parameters } t_p \geq 0, \delta_p > 0, \text{ and a constant } C_p > 0 \text{ such that the function } B, \text{ defined by (9), belongs to } C^1(I_A^2 \times \mathbb{R}_+^3) \text{ and we have}
\]
\[
d^2 B[x](dx) \leq \frac{|dx|^2}{2 \partial_{x_2} B(x)} \leq 0
\tag{17}
\]
for any vector \( dx \in I_A^2 \times \mathbb{R}_+^3 \) and any point \( x \in I_A^2 \times \mathbb{R}_+^3 \) where \( |x_1|, x_2 \neq 0 \) and \( |x|^p \neq x_2^{q/2} \).

\[\text{Proof.} \text{ Let } x_1 = \{ x_1^\alpha \}_{\alpha \in A} \text{ and } r \overset{\text{def}}{=} (|x_1|, x_2, x_3, x_4) \in \mathbb{R}_+^4. \text{ First, we note that for any } \alpha \in A, \text{ the function}
\]
\[
\partial_{x_1} B(x) = C_p \partial_{y_1} B_0(r) \frac{x_1^\alpha}{|x|}
\]
is continuous where \( x_1 \neq 0 \). We also have
\[
\left| \partial_{x_1} B(x) \right| \leq C_p \left| \partial_{y_1} B_0(r) \right| \rightarrow 0 \quad \text{as} \quad x_1 \rightarrow 0.
\]
Thus, \( B \in C^1(I_A^2 \times \mathbb{R}_+^3) \). Next, we immediately obtain
\[
\partial_{x_2} B(x) = C_p \partial_{y_2} B_0(r) \leq 0.
\]
It remains to prove the first inequality in (17). As in (9), we obtain
\[
d^2 B[x](dx) = C_p d^2 B_0[r](d|x_1|, dx_2, dx_3, dx_4) + C_p \partial_{y_1} B_0(r) d^2|x_1|.
\]
Here we mean that the differentials \( d|x_1| \) and \( d^2|x_1| \) are calculated at \( x_1 \) and are applied to \( dx_1 \):
\[
d|x_1| = dx_1 \cdot e \quad \text{and} \quad d^2|x_1| = \frac{|Q dx_1|^2}{|x_1|},
\]
where \( e \overset{\text{def}}{=} \frac{x_1}{|x_1|} \) and \( Qz \overset{\text{def}}{=} z - (z \cdot e) e \). Applying Lemma 2 and putting \( C_p = \frac{1}{\sqrt{c_p}} \), we get
\[
d^2 B[x](dx) \leq \frac{(d|x_1|)^2}{2 \partial_{x_2} B(x)} + \frac{2 \partial_{y_1} B_0(r) \partial_{y_2} B_0(r)}{c_p|x_1|} \frac{|Q dx_1|^2}{2 \partial_{x_2} B(x)}.
\]
We have \((dx_1 \cdot e)^2 + |Qdx_1|^2 = |dx_1|^2\). By direct calculations, we get
\[
\partial_{u_i} B_0(r) \partial_{u_j} B_0(r) \geq \delta_p |x_1|.
\]
Thus, any \(\delta_p > 0\) and \(c_p \leq 2\delta_p p\) give the desired result. 

Now we are ready to prove that \(\mathcal{B}_{\{0\}} \in \mathcal{K}^p(\Omega^2)\). In order to satisfy the boundary condition \(\text{(B1)}\) for \(\mathcal{B}\) on \(\Omega_p\), we only need to take a sufficiently small \(\delta_p\) and to apply Young’s inequality.

Next, we prove that \(\mathcal{B}\) satisfies the concave-type condition \(\text{(B2)}\) for all \(x\) and \(x^\pm\) in \(\Omega^2 \times \mathbb{R}^3_{\geq 0}\). We have
\[
\partial_{x_2} B(x_1, x_2) \triangleq \partial_{x_2} B(x) = -\begin{cases}
\gamma_1 \frac{x_2^{q/2-1}}{2} + \gamma_2 |x_1|^{2-p} + \gamma_3 |x_1|^{2-p-2(p-1)} x_2^{(p-1)}, & |x_1|^p \geq x_2^{q/2}; \\
\gamma_4 \frac{x_2^{q/2-1}}{2}, & |x_1|^p \leq x_2^{q/2},
\end{cases}
\]
where the constants \(\gamma_i > 0\) depend only on \(p\). Whenever we deal with the function \(\partial_{x_2} B\) below, we drop the variables \(x_1\) and \(x_4\) from the notation, because \(\partial_{x_2} B\) does not depend on them. We denote
\[
x^\tau = (x_1^+, x_2^+, x_3^+, x_4^+) \triangleq \frac{1 + \tau}{2} x^+ + \frac{1 - \tau}{2} x^-,
\]
\[
\phi(\rho) \triangleq \mathcal{B}(x^0 - (0, \rho \Delta^2, 0, 0)), \quad \text{and} \quad \Phi(\tau) \triangleq \mathcal{B}(x^\tau),
\]
where \(\tau \in [-1, 1]\) and \(\rho \in [0, 1]\). We have
\[
\mathcal{B}(x) - \frac{\mathcal{B}(x^+) + \mathcal{B}(x^-)}{2} = Q + R,
\]
where
\[
Q \triangleq \phi(1) - \phi(0) \quad \text{and} \quad R \triangleq \Phi(0) - \frac{\Phi(-1) + \Phi(1)}{2}.
\]
Calculating and reducing the interval of integration, we obtain
\[
Q = \int_0^1 \phi'(\rho) \, d\rho \geq -\Delta^2 \int_0^{1/2} \partial_{x_2} B(x_1^0, x_2^0) - \rho \Delta^2 \, d\rho.
\]
Since \(x_2^0 \geq \Delta^2\), we have \(x_2^0 - \rho \Delta^2 \geq \frac{3}{2} x_2^0\) for \(\rho \leq \frac{1}{2}\). On the other hand, if there exists \(\rho' \leq \frac{1}{2}\) such that \(|x_2^0|^p = (x_2^0 - \rho' \Delta^2)^{q/2}\), then \(|x_2^0|^p \geq (x_2^0 - \rho \Delta^2)^{q/2}\) for all \(\rho \leq \frac{1}{2}\). In any case, we obtain \(-\partial_{x_2} B(x_1^0, x_2^0 - \rho \Delta^2) \sim -\partial_{x_2} B(x_1^0)\) for \(\rho \leq \frac{1}{2}\). Thus, we have
\[
Q \geq -c_p' \Delta^2 \partial_{x_2} B(x_1^0). \quad (18)
\]

Now we consider the term \(R\). Without loss of generality, we may assume that \(\mathcal{A}\) is finite. Indeed, we can approximate \(x_1\) and \(x_1^\pm\) by sequences that contain only a finite number of non-zero elements. After that we can, due to the continuity of \(\mathcal{B}\), pass to the limit in \((7)\). We can process situations where \(x_1^\pm \equiv 0\) or where \(x_3^\pm \equiv 0\) similarly: we can separate one of the points \(x_1^\pm\) (or of the points \(x_2^\pm\), respectively) from zero and again pass to the limit in \((7)\). In particular, these remarks allow us to regard the function \(\Phi'\) as absolutely continuous. Indeed, the continuous function \(\Phi'\) is differentiable on a cofinite set, and, as it can be seen below, \(\Phi''(\tau) \leq 0\) on this set. The former implies the Luzin N property for \(\Phi'\), and the latter implies that
Φ′ is decreasing and, therefore, is of bounded variation. All this suffices for the absolute continuity of Φ′.

Next, we note that for any vectors h and b in $\ell_2^q \times \mathbb{R}^3$, we have the following general relation. If we define, where possible, the function $\Psi(\tau) \equiv \mathcal{B}(\tau h + b)$, then we have

$$
\Psi''(\tau) = d^2 \mathcal{B}[\tau h + b](h),
$$

(19)

where the right-hand side exists. Further, we set $h \equiv \frac{x^+ - x^-}{2}$. Employing Taylor’s formula in the integral form, relation (19), and Lemma 3, we obtain

$$
\int_{-1}^{1} \Phi''(\tau) (1 - |\tau|) d\tau = -\frac{1}{2} \int_{-1}^{1} d^2 \mathcal{B}[x^\tau](h) (1 - |\tau|) d\tau
$$

$$
= -\frac{|h_1|^2}{4} \int_{-1}^{1} \frac{1 - |\tau|}{\partial_{x^2} \mathcal{B}(x^\tau)} d\tau.
$$

We have $|h_2| \leq x_0^0$. First, we consider the case $|h_1| \leq |x_0^0|$. For $\tau \in [-1/2, 1/2]$, we have $x_2^\tau \simeq x_2^0$ and $|x_1^\tau| \simeq |x_1^0|$. If there exists $\tau' \in [-1/2, 1/2]$ such that $|x_1^{\tau'}|^p = (x_2^{\tau'})^{q/2}$, then $|x_1^\tau|^p \simeq (x_2^\tau)^{q/2}$ for all $\tau \in [-1/2, 1/2]$. This implies

$$
R \geq -c''_p |h_1|^2 / 4 \partial_{x^2} \mathcal{B}(x^0).
$$

Combining this inequality with (18), we obtain the estimate

$$
Q + R \geq \sqrt{c'_p c''_p} |h_1| \Delta.
$$

(20)

Next, suppose $|h_1| \geq |x_0^0|$. We have

$$
|x_1^\tau| \leq 2|h_1| \quad \text{for} \quad \tau \in [-1, 1].
$$

(21)

Let $S \subset [-1, 1]$ be the set of all $\tau$ such that $x_2^\tau \simeq x_2^0$ and $|S| = 1$ and $x_2^\tau \leq 2x_2^0$ for $\tau \in S$. If for all $\tau \in S$ we have $|x_1^\tau|^p \leq (x_2^\tau)^{q/2}$, then we can estimate the integral over $S$ in the same way as for the case $|h_1| \leq |x_0^0|$. Suppose there exists $\tau' \in S$ such that $|x_1^{\tau'}|^p > (x_2^{\tau'})^{q/2}$. Then we have $|x_1^\tau|^p > (\frac{1}{2} x_2^\tau)^{q/2}$ for all $\tau \in S$. This implies

$$
(2|h_1|)^p \geq 2^{-q/2}(x_2^0)^{q/2} \geq 2^{-q/2} \Delta^2 \quad \text{for} \quad \tau \in S.
$$

(22)

Estimates (21) and (22) implies

$$
R \geq c'''_p |h_1|^p |h_1|^{p-1} \geq c'''_p |h_1| \Delta.
$$

(23)

Since it is always true that (20) or (23) holds, we can adjust the constant $C_p$ and obtain inequality (7) for $\mathcal{B}$. □

8. Proof of Corollary 1

Suppose $g \in L^2$ and

$$
x \equiv \left( \langle f \rangle_{[0, 1]}, \|g\|^2_{L^2} - \text{osc}_{[0, 1]}^2(T^*g), \|f\|^2_{L^p}, \|g\|^2_{L^q} \right).
$$
Let $\lambda > 0$. By the homogeneity of $B$ and by Theorems 1 and 2 we obtain
\[
\langle gTf \rangle_{[0,1]} - \langle f \rangle_{[0,1]} \langle T^*g \rangle_{[0,1]} \leq B(x_1, x_2, x_3, x_4) = B(\lambda x_1, \lambda^{-2} x_2, \lambda^p x_3, \lambda^{-q} x_4)
\leq B(\lambda x_1, \lambda^{-2} x_2, \lambda^p x_3, \lambda^{-q} x_4) \leq 2C_p(\lambda^p x_3 + \lambda^{-q} x_4).
\]
\{(24)\}

In order to guess optimal $\lambda$, we need to solve the equation $\partial_\lambda [\lambda^p x_3 + \lambda^{-q} x_4] = 0$. We obtain
\[
\lambda = \left( \frac{q x_4}{p x_3} \right)^{1/p+q}. \tag{25}
\]

By Jensen’s (or Hölder’s) inequality, we have
\[
\left| \langle f \rangle_{[0,1]} \langle T^*g \rangle_{[0,1]} \right| \leq \|f\|_{L^p} \|T^*g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L^q} \leq \|f\|_{L^p} \|g\|_{L_q^*}. \tag{26}
\]

Combining (24), (25), and (26) for $g$ and $-g$, we obtain
\[
|\langle g, Tf \rangle| \leq (2p^{1/p}q^{1/q}C_p + 1) \|f\|_{L^p} \|g\|_{L^q^*}.
\]

This finishes the proof. \hfill \square

9. PROOF OF THEOREM 3

Jensen’s (or Hölder’s) inequality immediately implies that $\Omega_{B_0} \subseteq \Omega_p$. In order to prove that $\Omega_p \subseteq \Omega_{B_0}$, it suffices to set $T = \text{id}_{L^2}$ and to choose, for $x \in \Omega_p$, functions $f$ and $g$ such that $\langle f \rangle_I = x_1$, $\langle g \rangle_I = \sqrt{2}$, $\|f\|_I = x_3$, and $\|g\|_I = x_4$. The desired functions can be easily found in the form
\[
f(t) = \begin{cases} 
x_1 + a, & t \in I^+; \\
x_1 - a, & t \in I^-,
\end{cases}
\quad g(t) = \begin{cases} 
\sqrt{x_2} + b, & t \in I^+; \\
\sqrt{x_2} - b, & t \in I^-,
\end{cases}
\]
where $a, b \in \mathbb{R}$ are some chosen parameters.

Since the function $|\cdot|^p$ is strictly convex, the case $\|f\|_I^p \leq \|f\|_I^p$ of Jensen’s inequality becomes the equality if and only if $f = \text{const}$. Thus, we have \{B1\} for $B_0$.

It remains to prove \{B2\}. Consider $x$, $x^\pm \in \Omega_p$ and $\Delta \in \mathbb{R}$ that are related with each other by \{6\}. For any $\varepsilon > 0$, there exist functions $f^\pm, g^\pm \in L^2(I^\pm)$ and unitary operators $T^\pm \in \mathcal{S}(I^\pm, \mathbb{R})$ that generate $x^\pm$ and realize the supremum in $B_0$ to an accuracy of $\varepsilon$:
\[
\langle g^\pm T^\pm [f^\pm - \langle f^\pm \rangle_{I^\pm}] \rangle_{I^\pm} \geq B_0(x^\pm) - \varepsilon. \tag{27}
\]
Due to the unitarity of $T^\pm$, we have
\[
x_2^\pm = |I^\pm|^{-1} \langle g^\pm, T^\pm h_0^\pm \rangle^2, \tag{28}
\]
where $h_0^\pm \equiv |I^\pm|^{-1/2} \mathbb{1}_{I^\pm}$. In addition, we can always choose $T^\pm$ such that
\[
\langle g^\pm, T^\pm h_0^\pm \rangle \geq 0. \tag{29}
\]

We immediately see that the functions
\[
f(t) \equiv \begin{cases} 
f^+(t), & t \in I^+; \\
f^-(t), & t \in I^-,
\end{cases}
\quad g(t) \equiv \begin{cases} 
g^+(t), & t \in I^+; \\
g^-(t), & t \in I^-,
\end{cases}
\]

Therefore, $\Omega_p \subseteq \Omega_{B_0}$, as required.
generates $x_1$, $x_3$ and $x_4$. Now we construct an appropriate unitary operator $T ∈ G(I, R)$. We set $Th_J = T^± h_J$, $J ⊆ I^±$, and build $Th_0$ and $Th_I$ in the form

\[
Th_0 = ξ_1 T^+ h_0^+ + ξ_2 T^− h_0^−;
\]
\[
Th_I = ξ_3 T^+ h_I^+ − ξ_1 T^− h_I^−.
\]

(30)

It is easy to see that $\{Th_0, Th_I\}_{J ⊆ I}$ is an orthonormal basis in $L^2(I)$ (and, therefore, $T$ is unitary), provided

\[
ξ_1^2 + ξ_2^2 = 1.
\]

(31)

If, in addition, we manage to choose $Th_I$ so that

\[
\Delta = |I|^{−1}(g, Th_I),
\]

(32)

then, first, relations (6) and (11) will imply

\[
x_2 = ⟨g^2⟩_I − \frac{osc_2^2((T^+)^*[g^+]) + osc_2^2((T^−)^*[g^−])}{2} − Δ^2 = ⟨g^2⟩_I − osc_2^2(T^*[g])
\]

and, second, inequalities (27) will imply

\[
⟨g T[f − ⟨f⟩_I]⟩_I = \frac{1}{|I|}((f, h_I)(g, Th_I) + \sum_{J ⊆ I^+, J ⊆ I^−} (f, h_J)(g, Th_J)) ≥ \frac{x_1^+ − x_1^−}{2} Δ + \frac{B_0(x^+) + B_0(x^-)}{2} − ε.
\]

In such a case, $g$ and $T$ generate $x_2$ and, since $Δ$ and $ε$ are arbitrary, we see that $[B2]$ holds for $B_0$.

It remains to prove that (32) is attainable. Substituting (30) into (32) and using (28) and (29), we come to the equation

\[
ξ_2 \sqrt{x_2^+} − ξ_1 \sqrt{x_2^−} = \sqrt{2} Δ.
\]

(33)

It is easy to calculate that the system of equations (31) and (33) is solvable exactly when

\[
\frac{x_2^+ + x_2^−}{2} − Δ^2 ≥ 0.
\]

This is true due to (6), and we are done. □

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