Transport Equations for Oscillating Neutrinos

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We derive a suite of generalized Boltzmann equations, based on the density-matrix formalism, that incorporates the physics of neutrino oscillations for two- and three-flavor oscillations, matter refraction, and self-refraction. The resulting equations are straightforward extensions of the classical transport equations that nevertheless contain the full physics of quantum oscillation phenomena. In this way, our broadened formalism provides a bridge between the familiar neutrino transport algorithms employed by supernova modelers and the more quantum-heavy approaches frequently employed to illuminate the various neutrino oscillation effects. We also provide the corresponding angular-moment versions of this generalized equation set. Our goal is to make it easier for astrophysicists to address oscillation phenomena in a language with which they are familiar. The equations we derive are simple and practical, and are intended to facilitate progress concerning oscillation phenomena in the context of core-collapse supernova theory.

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I. INTRODUCTION

Core-collapse supernova explosions, whatever their mechanism, involve neutrino transport and neutrino-matter coupling in a fundamental way [1]. At the extreme densities and temperatures encountered in the unstable stellar core there is prodigious production of neutrinos of all six neutrino species ($\nu_e$, $\bar{\nu}_e$, $\nu_\mu$, $\bar{\nu}_\mu$, $\nu_\tau$, and $\bar{\nu}_\tau$) and the corresponding integrated neutrino flux can be comparable to the dynamical matter flux. The consensus mechanism of explosion centrally involves neutrino heating of the shocked mantle to drive the blast to infinity, leaving behind a cooling and deleptonizing proto-neutron star [2]. Hence, to understand these phenomena in any way requires an understanding of neutrino physics and transport in every particular.

The mathematical description of neutrino transport and transfer frequently starts with a classical Boltzmann equation for the corresponding phase-space density ($f_\nu$) for each species [3]. An equivalent formulation would solve for the specific intensity ($I_\nu$) of the multi-angle, multi-energy-group neutrino beams at every spatial point, at every time, for every species [4, 5]. The solution of this time-dependent, multi-angle, spectral field for all spacetime and for six species is a numerical “grand challenge” of the first order that remains out of computational reach. Nevertheless, Boltzmann equations are the starting points for the various more simplified approaches to neutrino transport that to date have been employed by supernova theorists. These include multi-group, flux-limited diffusion and two-moment closures [6, 7]. Moreover, to lessen the computational burden of the challenging non-spherical hydrodynamic context revealed by current theory to be important, researchers have oftentimes reduced the transport problem even further into multiple one-dimensional radial/spherical solves (the so-called “ray-by-ray” method [8]). Though less computationally demanding, this dimensional compromise may be problematic [2].

However, we now know that neutrinos have mass and oscillate among themselves [9, 10]. In addition, neutrino-matter refraction can lead to resonant conversion between species, even if the vacuum oscillation angles are small [11-13]. Furthermore, it has been shown that neutrino-neutrino refraction effects in the neutrino-rich supernova...
environment can lead to self-oscillation effects \[14\–16\] and in particular “spectral swapping/splitting” \[17\–23\]. These effects are most prominent in supernovae if the mass hierarchy \[24, 25\] is inverted (“IH”, a possibility) or when (rarely) the electron number density is not greater than that of the neutrinos \[26\–28\]. However, the possibility of such a rich set of oscillation behaviors and transformations has introduced new excitement into supernova science \[29\–31\]. At the very least, oscillation phenomena will alter the mix and spectra of supernova neutrinos detected at Earth \[28, 32\–34\].

Neutrino oscillations are purely quantum-mechanical effects that are not captured by the classical Boltzmann equation. However, as was shown by Strack & Burrows \[35\] and Sigl & Raffelt \[36, 37\], the density matrix formalism of quantum mechanics is the most natural generalization of classical transport theory (and its simplifications) with which to incorporate oscillation physics. The angle- and energy-dependent effects of self-refraction are naturally accounted for. Whereas the classical formulation involves only the diagonal components of the density matrix (each component being associated with a given neutrino species), the quantum-mechanical extension adds corresponding equations for the off-diagonal “phase-space densities,” and new sources on the “right-hand-sides” that now couple the various neutrino species. These sources are added to the classical absorption, scattering, and emission sources of classical transport. From the oscillation perspective, the latter classical terms represent the various physical decoherence effects, and comparisons between the magnitudes of these terms and the oscillation coupling terms provide a natural means to determine the potential degrees of decoherence.

In a real sense, the quantum-mechanical equation set merely increases the number of partial differential equations similar to the classical Boltzmann equation that need to be simultaneously solved. This makes it easier for the supernova theorist and/or numericist to generalize their classically developed formalism to include neutrino oscillations in a quantum-mechanically rigorous way. There are no wave functions, and in principle no imaginary quantities need be addressed nor invoked. This provides a much-needed bridge between the oftimes opaque oscillation literature and the practical astrophysicist. “All” that needs to be done is to solve an extended set of coupled Boltzmann-like equations.

It is with this philosophy in mind that we present in this paper the generalized set of neutrino transport equations that contain oscillation physics. We do this for both two-neutrino and three-neutrino variants. We also provide the associated moment equations that might lead to practical and tractable simplifications of this equation set. For the latter, we do not provide the higher-moment closures necessary to employ the moment approach. This art we leave to future work. However, we believe the format of the resultant moment equations is particularly clear and should prove useful. No attempt is made to address the energy and angular resolutions that may be necessary to manifest all the various possible oscillation effects. Nor do we suggest various averaging procedures over neutrino energy, etc. that might obviate the need for high energy or angle resolution. All approaches and studies of oscillation phenomena are burdened with the same concerns and issues. In this spirit, since the various consequences of neutrino oscillations and refraction have been been amply explored in the literature, and Strack & Burrows \[35\] have already demonstrated that the density-matrix approach yields the correct results for a subset of them (see also \[38\]), we won’t in this paper provide any solutions to the equations derived. Rather, we hope that those in the supernova community interested in generalizing their thinking and calculations to incorporate oscillation phenomena with minimal effort will find our results of use.

II. TRANSPORT EQUATION WITH OSCILLATIONS

A. Neutrino Mixing

Neutrinos exhibit mixing because of the discrepancy between flavor and mass states. For two flavors, the transformation from the mass basis to the flavor basis is given by

\[
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix} = U
\begin{pmatrix}
\nu_e \\
\nu_\mu
\end{pmatrix},
\]

where, with mixing angle \(\theta\)

\[
U = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}.
\]

For three flavors, the transformation to the flavor basis is given by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix \[13\] \(U\):

\[
\begin{pmatrix}
\nu_1 \\
\nu_2 \\
\nu_3
\end{pmatrix} = U
\begin{pmatrix}
\nu_e \\
\nu_\mu \\
\nu_\tau
\end{pmatrix},
\]
where in this case, with $c_{ij} = \cos \theta_{ij}$, and $s_{ij} = \sin \theta_{ij}$

$$U = \begin{pmatrix}
1 & 0 & 0 \\
0 & c_{23} & s_{23} \\
0 & -s_{23} & c_{23}
\end{pmatrix} \begin{pmatrix}
c_{13} & 0 & s_{13} e^{-i \delta} \\
0 & 1 & 0 \\
-s_{13} e^{i \delta} & 0 & c_{13}
\end{pmatrix} \begin{pmatrix}
c_{12} & s_{12} & 0 \\
-s_{12} & c_{12} & 0 \\
0 & 0 & 1
\end{pmatrix}.$$ (4)

An example of the three mixing angles and $\Delta m^2$ in the currently accepted range is summarized in Table I for both the normal hierarchy (NH) and the inverted hierarchy (IH) [39, 40]. The CP phase $\delta$ has not been determined.

B. The Heisenberg-Boltzmann Equation

The Heisenberg Boltzmann Equation is given by [35]

$$\frac{\partial F}{\partial t} + \vec{v} \cdot \frac{\partial F}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial F}{\partial \vec{p}} = -i[H, F] + C,$$ (5)

where for three flavors of neutrinos

$$F = \langle \nu | \rho | \nu \rangle = \begin{pmatrix}
\langle \nu_e | \nu_e \rangle & \langle \nu_e | \nu_\mu \rangle & \langle \nu_e | \nu_\tau \rangle \\
\langle \nu_\mu | \nu_e \rangle & \langle \nu_\mu | \nu_\mu \rangle & \langle \nu_\mu | \nu_\tau \rangle \\
\langle \nu_\tau | \nu_e \rangle & \langle \nu_\tau | \nu_\mu \rangle & \langle \nu_\tau | \nu_\tau \rangle
\end{pmatrix},$$ (6)

is the density matrix given by the Wigner phase space density

$$\rho(r, p, t) = \int d^3 R e^{-ip R} \psi^\dagger(r - R, t) \psi(r + R, t),$$ (7)

where $p$ denotes the particle momentum. $C$ denotes the classical scattering, absorption, and emission terms, and is given by:

$$C = \begin{pmatrix}
C_{\nu_e} & 0 & 0 \\
0 & C_{\nu_\mu} & 0 \\
0 & 0 & C_{\nu_\tau}
\end{pmatrix}.$$ (8)

Note the $C$ is diagonal and does not couple different neutrino species. The analogous form of Eq. (6) and (8) for two flavors is straightforward. We note that the last term on the left-hand-side of Eq. (5) represents gravitational redshift, and can be omitted if general relativistic effects on neutrino transport are not considered.

Note that $F$ is Hermitian, and can thus be expanded in unitary groups. We define real quantities $f_{\gamma}$ so that

$$F = \sum_{\gamma=0}^{3} f_{\gamma} \sigma^\gamma$$ (11)

for two flavors and

$$F = \sum_{\gamma=0}^{8} f_{\gamma} \lambda^\gamma$$ (12)

for three flavors, where $\sigma$ and $\lambda$ are the Dirac and Gell-Mann matrices (see Appendix A). Note that the $f_{\gamma}$ for two flavors are not related to those for three flavors. This structure arises from the SU(N) rotation symmetry in neutrino flavor space that the transformations due to neutrino oscillations must satisfy [23, 41, 42]. The Hamiltonian contains the terms

$$H = H_0 + H_e + H_{\nu\nu} - H_{\nu\bar{\nu}},$$ (13)

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$$H = H_0 + H_e + H_{\nu\nu} - H_{\nu\bar{\nu}},$$ (13)
where * denotes complex conjugate. For two flavors, the vacuum and matter Hamiltonians \( H_0 \) and \( H_e \) in the flavor basis take the familiar form

\[
H_0 = \frac{\omega}{2} \sin 2\theta \sigma^1 - \frac{\omega}{2} \cos 2\theta \sigma^3,
\]

and

\[
H_e = \frac{A}{2} \sigma^3,
\]

where the vacuum frequency \( \omega \) \((\langle m_i^2 - m_j^2 \rangle/2p)\) and mixing angle \( \theta \) take usual meanings, and \( A = \frac{\sqrt{2}G_F n_e}{\hbar} \) represents the interaction strength. \( n_e \) is the electron number density.

For three flavors, the traceless vacuum Hamiltonian in the flavor basis is

\[
H_0 = \frac{1}{3} \mathcal{U}^\dagger \left( \begin{array}{ccc} -\Delta_{21} - \Delta_{31} & \Delta_{21} - \Delta_{32} & \Delta_{32} + \Delta_{31} \\ \Delta_{32} + \Delta_{31} & -\Delta_{21} - \Delta_{32} & \Delta_{21} - \Delta_{31} \\ \Delta_{21} - \Delta_{31} & \Delta_{32} + \Delta_{31} & -\Delta_{21} - \Delta_{32} \end{array} \right) \mathcal{U},
\]

where \( \Delta_{ij} = m_i^2 - m_j^2 \) and \( p \) is the particle momentum/energy. From here the expression \( H_0^2 \) in

\[
H_0 = \sum_{\gamma=0}^8 H_0^7 \lambda_\gamma
\]

can be extrapolated\(^2\).

The electron interaction term is

\[
H_e = \frac{\sqrt{2}A}{8} \left( 3\lambda_3 + \sqrt{3}\lambda_8 \right).
\]

One defines the self-interaction strength as

\[
B_\gamma(p, \vec{r}, t) = \frac{\sqrt{2}G_F}{\hbar} \int d^3q \left( 1 - \cos \theta^{pq} \right) f_\gamma(p, \vec{r}, t).
\]

Denoting anti-neutrino quantities with a bar, the self-interaction Hamiltonians are written as

\[
H_{\nu\overline{\nu}} = \sum_{\gamma=0}^3 B_\gamma \sigma^\gamma
\]

and

\[
H_{\overline{\nu}\nu}^* = \sum_{\gamma=0}^8 \overline{B}_\gamma^* \sigma^\gamma,
\]

for two and three flavors, respectively.

\(^2\) For example, ignoring the CP phase, and denoting \( \omega_{31} = \frac{\Delta_{31}}{2p} \), \( \omega_{21} = \frac{\Delta_{21}}{2p} \), \( \omega_{32} = \omega_{31} - \omega_{21} \), \( S_{ij} = \sin 2\theta_{ij} \), and \( K_{ij} = \cos 2\theta_{ij} \), we have

\[
\begin{align*}
2H_0^4 &= (\omega_{31} - \omega_{23} s_{12}^2) (s_{13} + \omega_{23} c_{13} c_{23}) S_{12} + \omega_{21} c_{13} c_{23} S_{12}, \\
2H_0^6 &= (\omega_{31} - \omega_{21} c_{13} c_{23}) (s_{13} + \omega_{23} c_{13} c_{23}) S_{12} - \omega_{21} c_{13} c_{23} S_{12}, \\
2H_0^8 &= -\omega_{21} s_{13} S_{12} K_{23} + \frac{S_{23}}{4} \left[ 4\omega_{31} c_{13}^2 - \omega_{21} (1 + 3K_{12} + 2K_{13} c_{13}^2) \right], \\
2H_0^3 &= \frac{1}{2} \left[ -\omega_{21} - 2\omega_{31} \right] (1 - 3K_{13} + 2K_{23} c_{13}^2) + \omega_{21} (4S_{12} S_{23} s_{13} - K_{12} (6c_{13}^2 + 4K_{13} c_{13}^2)), \\
2H_0^6 &= \frac{1}{24\sqrt{3}} \left[ -\omega_{21} - 2\omega_{31} \right] (1 - 3K_{13} - 6K_{23} c_{13}^2) - 3\omega_{21} (4S_{12} S_{23} s_{13} + K_{12} (6c_{13}^2 + 4K_{13} c_{13}^2)), \\
2H_0^8 &= H_0^6 = H_0^8 = 0.
\end{align*}
\]
C. The Matrix Equations

The transport equation in matrix form thus reads for two flavors, summing from 0 to 3:

\[
\begin{align*}
\frac{\partial F}{\partial t} + \vec{v} \cdot \frac{\partial F}{\partial r} + \dot{\vec{p}} \cdot \frac{\partial F}{\partial \vec{p}} &= 2\epsilon_{\alpha\beta\gamma}(H_0^\alpha + H_e^\alpha + B^\alpha - \bar{B}^{\alpha*})f^\beta\sigma^\gamma + C, \\
\frac{\partial \bar{F}}{\partial t} + \vec{v} \cdot \frac{\partial \bar{F}}{\partial r} + \dot{\vec{p}} \cdot \frac{\partial \bar{F}}{\partial \vec{p}} &= 2\epsilon_{\alpha\beta\gamma}(H_0^\alpha - H_e^\alpha - B^\alpha + \bar{B}^{\alpha*})\bar{f}^\beta\sigma^\gamma + \bar{C},
\end{align*}
\]

(23)

where \(\epsilon_{\alpha\beta\gamma}\) is the anti-symmetric tensor. Again throughout this paper, anti-neutrino quantities are denoted with a bar. Note that the anti-symmetric tensor induces a cross product:

\[
\vec{z} = \vec{x} \times \vec{y} \iff z_i = \epsilon_{ijk}x^jy^k.
\]

Thus, Eq. (23) contains the “flavor pendulum” equation of motion, and this cross-product form motivated the language used to describe collective neutrino oscillations (see, for example, [16, 43, 44] and references therein). (For more on the pendulum analogy, we refer the reader to Appendix B.)

Similarly for three flavors, summing from 0 to 8

\[
\begin{align*}
\frac{\partial F}{\partial t} + \vec{v} \cdot \frac{\partial F}{\partial r} + \dot{\vec{p}} \cdot \frac{\partial F}{\partial \vec{p}} &= 2c_{\alpha\beta\gamma}(H_0^\alpha + H_e^\alpha + B^\alpha - \bar{B}^{\alpha*})f^\beta\lambda^\gamma + C, \\
\frac{\partial \bar{F}}{\partial t} + \vec{v} \cdot \frac{\partial \bar{F}}{\partial r} + \dot{\vec{p}} \cdot \frac{\partial \bar{F}}{\partial \vec{p}} &= 2c_{\alpha\beta\gamma}(H_0^\alpha - H_e^\alpha - B^\alpha + \bar{B}^{\alpha*})\bar{f}^\beta\lambda^\gamma + \bar{C},
\end{align*}
\]

(24)

where \(c_{\alpha\beta\gamma}\) are the SU(3) structure constants given in Appendix A. The SU(3) structure can be similarly understood as a generalized cross product. Eq. (24) can be expanded in the natural basis with Eqs. (6), (8), and (A3). For two flavors, Eq. (23) can be similarly expanded.

D. The Transport Basis

In transport algorithms, it is easiest to keep the diagonal terms in the natural basis, or flavor basis, and put the off-diagonal terms in the SU(N) basis. We call this basis the “transport basis.” \(C\) and \(\bar{C}\) are diagonal and, hence, unchanged. \(\mathcal{F}\) is given in SU(N) expansion in Eqs. (11) and (12). Notice that \(\sigma_{0,3}\) and \(\lambda_{0,3,8}\) are diagonal, and, therefore, we take linear combinations of them to restore the diagonal terms to the flavor basis. For two flavors,

\[
f_0 = \frac{f_{\nu_e} + f_{\nu_\mu}}{2}, \quad f_3 = \frac{f_{\nu_e} - f_{\nu_\mu}}{2},
\]

(25)

or inversely

\[
f_{\nu_e} = f_0 + f_3, \quad f_{\nu_\mu} = f_0 - f_3.
\]

(26)

For three flavors, the relations are

\[
\begin{align*}
f_0 &= \frac{1}{3}(f_{\nu_e} + f_{\nu_\mu} + f_{\nu_\tau}), \\
f_3 &= \frac{1}{2}(f_{\nu_e} - f_{\nu_\mu}), \\
f_8 &= \frac{1}{6\sqrt{3}}(f_{\nu_e} + f_{\nu_\mu} - 2f_{\nu_\tau}),
\end{align*}
\]

(27)

or

\[
\begin{align*}
f_{\nu_e} &= f_0 + f_3 + \sqrt{3}f_8, \\
f_{\nu_\mu} &= f_0 - f_3 + \sqrt{3}f_8, \\
f_{\nu_\tau} &= f_0 - 2\sqrt{3}f_8.
\end{align*}
\]

(28)
The same form also applies to any other diagonal quantities such as self-interaction strength $B$, or $C$. The two-flavor transport equations can, thus, be written as, noting $c_{a,0} = 0$: 

$$
\frac{\partial f_\nu}{\partial t} + \vec{v} \cdot \frac{\partial f_\nu}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f_\nu}{\partial \vec{p}} = 2\epsilon_{a,33}(H^a_0 + H^a_e + B^a - \bar{B}^a) f^{\beta} + C_{\nu\nu},
$$

$$
\frac{\partial f_{\nu\mu}}{\partial t} + \vec{v} \cdot \frac{\partial f_{\nu\mu}}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f_{\nu\mu}}{\partial \vec{p}} = -2\epsilon_{a,33}(H^a_0 + H^a_e + B^a - \bar{B}^a) f^{\beta} + C_{\nu\mu},
$$

$$
\frac{\partial f_\gamma}{\partial t} + \vec{v} \cdot \frac{\partial f_\gamma}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f_\gamma}{\partial \vec{p}} = 2\epsilon_{a,\gamma}(H^a_0 + H^a_e + B^a - \bar{B}^a) f^{\beta}, \quad \text{for} \ \gamma = 1, 2,
$$

and for anti-neutrinos 

$$
\frac{\partial \bar{f}_\nu}{\partial t} + \vec{v} \cdot \frac{\partial \bar{f}_\nu}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial \bar{f}_\nu}{\partial \vec{p}} = 2\epsilon_{a,33}(H^a_0 - H^a_e - B^a + \bar{B}^a) \bar{f}^{\beta} + C_{\nu\nu},
$$

$$
\frac{\partial \bar{f}_{\nu\mu}}{\partial t} + \vec{v} \cdot \frac{\partial \bar{f}_{\nu\mu}}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial \bar{f}_{\nu\mu}}{\partial \vec{p}} = 2\epsilon_{a,33}(H^a_0 - H^a_e - B^a + \bar{B}^a) \bar{f}^{\beta} + C_{\nu\mu},
$$

$$
\frac{\partial \bar{f}_\gamma}{\partial t} + \vec{v} \cdot \frac{\partial \bar{f}_\gamma}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial \bar{f}_\gamma}{\partial \vec{p}} = 2\epsilon_{a,\gamma}(H^a_0 - H^a_e - B^a + \bar{B}^a) \bar{f}^{\beta}, \quad \text{for} \ \gamma = 1, 2.
$$

The two-flavor equations expanded in component form are given in Appendix D. The three-flavor equations in the transport basis are, noting that $c_{a,0} = 0$ or, equivalently, that $f_0 = (f_{\nu\nu} + f_{\nu\mu} + f_{\nu\nu})/3$, is not affected by oscillation: 

$$
\frac{\partial f_\nu}{\partial t} + \vec{v} \cdot \frac{\partial f_\nu}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f_\nu}{\partial \vec{p}} = 2(c_{a,33} + \sqrt{3}c_{a,8\bar{a}})(H^a_0 + H^a_e + B^a - \bar{B}^a) f^{\beta} + C_{\nu\nu},
$$

$$
\frac{\partial f_{\nu\mu}}{\partial t} + \vec{v} \cdot \frac{\partial f_{\nu\mu}}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f_{\nu\mu}}{\partial \vec{p}} = 2(-c_{a,33} + \sqrt{3}c_{a,8\bar{a}})(H^a_0 + H^a_e + B^a - \bar{B}^a) f^{\beta} + C_{\nu\mu},
$$

$$
\frac{\partial f_\gamma}{\partial t} + \vec{v} \cdot \frac{\partial f_\gamma}{\partial \vec{r}} + \dot{\vec{p}} \cdot \frac{\partial f_\gamma}{\partial \vec{p}} = 2c_{a,\gamma}(H^a_0 + H^a_e + B^a - \bar{B}^a) f^{\beta}, \quad \text{for} \ \gamma = 1, 2, 4, 5, 6, 7.
$$

The transformation for $C$ is used in the full SU(N) expansion; see Appendix B.
the direction of momentum and $\Omega_\nu$ denotes solid angle. The scattering, absorption, and emission source terms can be written in simple forms and are discussed in Appendix C. Such a set of equations contains the $n + 1$th moment. Therefore, more information relating the moments is needed, and this is called the closure problem [4, 5]. Typically a closure among the 0th, 1st and 2nd moments is used, and we leave a discussion of possible closures to future work.

For two flavors, setting $\beta = \sqrt{2 G F c}$, we write the 0th and 1st moment equations in the transport basis:

$$\frac{\partial E^{\nu_\alpha}}{\partial t} + \partial^i F^{\nu_\alpha}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\nu_\alpha}_{ij} = 2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) E^{\beta} + \frac{\beta}{c} \int \frac{dq}{q} \left\{ c^2 E^{\alpha}(q) E^{\beta} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right] + C^{(0)}_{\nu_\alpha},$$

$$\frac{\partial E^{\nu_\mu}}{\partial t} + \partial^i F^{\nu_\mu}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\nu_\mu}_{ij} = -2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) E^{\beta} + \frac{\beta}{c} \int \frac{dq}{q} \left\{ c^2 \tilde{E}^{\alpha}(q) E^{\beta} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right] + C^{(0)}_{\nu_\mu},$$

$$\frac{\partial E^{\gamma}}{\partial t} + \partial^i F^{\gamma}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\gamma}_{ij} = 2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) E^{\beta} + \frac{\beta}{c} \int \frac{dq}{q} \left\{ c^2 E^{\alpha}(q) E^{\beta} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right], \quad \text{for } \gamma = 1, 2,$$

and

$$\frac{\partial F^{\nu_\alpha}_{ij}}{\partial t} + \partial^i F^{\nu_\alpha}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\nu_\alpha}_{ij} = 2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) F^{\beta}_{ij} + \beta \int \frac{dq}{q} \left\{ \tilde{E}^{\alpha}(q) F^{\beta}_{ij} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right] + C^{(1)}_{\nu_\alpha},$$

$$\frac{\partial F^{\nu_\mu}_{ij}}{\partial t} + \partial^i F^{\nu_\mu}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\nu_\mu}_{ij} = -2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) F^{\beta}_{ij} + \beta \int \frac{dq}{q} \left\{ \tilde{E}^{\alpha}(q) F^{\beta}_{ij} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right] + C^{(1)}_{\nu_\mu},$$

$$\frac{\partial F^{\gamma}_{ij}}{\partial t} + \partial^i F^{\gamma}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\gamma}_{ij} = 2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) F^{\beta}_{ij} + \beta \int \frac{dq}{q} \left\{ \tilde{E}^{\alpha}(q) F^{\beta}_{ij} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right], \quad \text{for } \gamma = 1, 2,$$

where $C^{(0,1)}_{\nu_\mu}$ are the moments of the scattering and absorption terms, discussed in Appendix C. The $\partial q$ on the left hand side is differentiation with respect to energy and again is related to gravitational redshift. The breve (˘) is a short hand for

$$\tilde{E}_\alpha = E_\alpha(q) - \tilde{E}^{\ast}_\alpha(q),$$

$$\tilde{F}_\alpha = F_\alpha(q) - \tilde{F}^{\ast}_\alpha(q).$$

$E$ and $F$ are the generalized neutrino energy density and flux spectra: the 0th and 1st angular moments of the components of the distribution function/density matrix. For anti-neutrinos, the equations are obtained by the substitutions:

$$H_\nu \rightarrow -H_\nu,$$

$$E_\alpha(q) \rightarrow E^{\ast}_\alpha(q) + \tilde{E}_\alpha(q),$$

$$F_\alpha(q) \rightarrow \tilde{F}^{\ast}_\alpha(q) + \tilde{F}_\alpha(q).$$

For the full expansion in component form, we refer the reader to Appendix D.

For three flavors, the first two transport moment equations in the transport basis are:

$$\frac{\partial E^{\nu_\alpha}}{\partial t} + \partial^i F^{\nu_\alpha}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\nu_\alpha}_{ij} = 2(\epsilon_{\alpha\beta\gamma} + \sqrt{3} \epsilon_{\alpha\beta\delta}) \left[ (H^\alpha_0 + H^\alpha_e) E^{\beta} + \frac{\beta}{c} \int \frac{dq}{q} \left\{ c^2 E^{\alpha}(q) E^{\beta} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right] + C^{(0)}_{\nu_\alpha},$$

$$\frac{\partial E^{\nu_\mu}}{\partial t} + \partial^i F^{\nu_\mu}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\nu_\mu}_{ij} = -2(\epsilon_{\alpha\beta\gamma} + \sqrt{3} \epsilon_{\alpha\beta\delta}) \left[ (H^\alpha_0 + H^\alpha_e) E^{\beta} + \frac{\beta}{c} \int \frac{dq}{q} \left\{ c^2 E^{\alpha}(q) E^{\beta} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right] + C^{(0)}_{\nu_\mu},$$

$$\frac{\partial E^{\gamma}}{\partial t} + \partial^i F^{\gamma}_{ij} + \hat{p}^j \frac{\partial}{\partial c} F^{\gamma}_{ij} = 2 \epsilon_{\alpha\beta\gamma} \left[ (H^\alpha_0 + H^\alpha_e) E^{\beta} + \frac{\beta}{c} \int \frac{dq}{q} \left\{ c^2 E^{\alpha}(q) E^{\beta} - \tilde{F}^{\alpha}_i (q) \cdot F^\beta_i \right\} \right], \quad \text{for } \gamma = 1, 2, 4, 5, 6, 7,$$

(37)

---

4 It is straightforward to show that the moment formalism naturally contains a geometric closure, consistent with the result of [13], when the single-angle approximation is assumed [10].
and
\[ \frac{\partial F_{\nu}^{\nu}}{\partial t} + \partial^i P_{ij}^{\nu} + \dot{\rho}^i \frac{\partial}{\partial\rho} P_{ij}^{\nu} = 2(c_{\alpha\beta\gamma} + \sqrt{3}c_{\alpha\beta\gamma}) \left[ (H_0^\alpha + H_0^\beta) F_i^\beta + \beta \int \frac{dq}{q} \left\{ \tilde{E}^\alpha(q) F_i^\beta - \tilde{F}_j^\beta(q) \cdot P_{ij}^\beta \right\} \right] + C^{(1)}_{\nu}, \]
\[ \frac{\partial F_i^{\nu}}{\partial t} + \partial^j P_{ij}^{\nu} + \dot{\rho}^j \frac{\partial}{\partial\rho} P_{ij}^{\nu} = 2(-c_{\alpha\beta\gamma} + \sqrt{3}c_{\alpha\beta\gamma}) \left[ (H_0^\alpha + H_0^\beta) F_i^\beta + \beta \int \frac{dq}{q} \left\{ \tilde{E}^\alpha(q) F_i^\beta - \tilde{F}_j^\beta(q) \cdot P_{ij}^\beta \right\} \right] + C^{(1)}_{\nu}, \]
\[ \frac{\partial F_{ij}^{\nu}}{\partial t} + \partial^j P_{ij}^{\nu} + \dot{\rho}^j \frac{\partial}{\partial\rho} P_{ij}^{\nu} = 2(-2\sqrt{3}c_{\alpha\beta\gamma}) \left[ (H_0^\alpha + H_0^\beta) F_i^\beta + \beta \int \frac{dq}{q} \left\{ \tilde{E}^\alpha(q) F_i^\beta - \tilde{F}_j^\beta(q) \cdot P_{ij}^\beta \right\} \right] + C^{(1)}_{\nu}, \]
\[ \frac{\partial F_i^{\gamma}}{\partial t} + \partial^j P_{ij}^{\gamma} + \dot{\rho}^j \frac{\partial}{\partial\rho} P_{ij}^{\gamma} = 2c_{\alpha\beta\gamma} \left[ (H_0^\alpha + H_0^\beta) F_i^\beta + \beta \int \frac{dq}{q} \left\{ \tilde{E}^\alpha(q) F_i^\beta - \tilde{F}_j^\beta(q) \cdot P_{ij}^\beta \right\} \right], \text{ for } \gamma = 1, 2, 4, 5, 6, 7. \]

(38)

IV. CONCLUSIONS

In this paper, we have provided a suite of generalized Boltzmann equations, based on the density-matrix formalism, that incorporates the physics of neutrino oscillations for two- and three-flavor oscillations, matter refraction, and self-refraction. The resulting equations are straightforward extensions of the classical transport equations that by their couplings and augmentation to include off-diagonal densities are of a form usefully similar to the classical, diagonal flavor-basis set of partial differential equations. The expanded equation set nevertheless contains the full physics of quantum oscillation phenomena, though it maintains the classical format familiar to supernova astrophysicists who perform traditional neutrino transport simulations. In this way, our formalism provides a bridge between the familiar approaches employed by supernova modelers and the formalisms employed by the pioneers in neutrino oscillation physics. We also provide the corresponding angular-momentum versions of this generalized equation set. Our goal is to make it easier for astrophysicists to address oscillation phenomena in a language with which they are familiar. At the same time, we hope that neutrino oscillation experts interested in incorporating the effects of transport in a natural way may find our formalism of use. While we have not included a discussion of sterile neutrinos or spin flips [47], and have not explored the differences between Majorana and Dirac [48] neutrinos, nor possible neutrino-antineutrino oscillations [47], we believe the equations derived are simple, clear, and practical renditions that will facilitate progress on oscillation phenomena in the context of core-collapse theory.

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Appendix A: SU(2) and SU(3)

The Dirac matrices
\[ \sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]

(A1)
satisfy the SU(2) commutation relations
\[ \left[ \sigma_\alpha, \sigma_\beta \right] = i \epsilon_{\alpha\beta\gamma} \sigma^\gamma, \]

(A2)
where the structure constants are represented by the anti-symmetric tensor \( \epsilon_{\alpha\beta\gamma} \), which in particular vanishes when any of the indices are zero.
The Gell-Mann matrices

\[ \lambda_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \lambda_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \lambda_2 = \begin{bmatrix} 0 & 0 & -i \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \]
\[ \lambda_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \lambda_4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_5 = \begin{bmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, \]
\[ \lambda_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \lambda_7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \]

satisfy the SU(3) commutation

\[ \left[ \frac{\lambda_3}{2}, \frac{\lambda_6}{2} \right] = 3c_{\alpha\beta\gamma} \frac{\lambda_8}{2}. \] (A4)

The structure constants \( c_{\alpha\beta\gamma} \) are anti-symmetric with respect to exchange of pair indices, and in particular \( c_{\alpha\beta0} = 0 \). The non-vanishing components can be specified via \( c_{123} = 2; \ c_{147}, c_{165}, c_{246}, c_{257}, c_{345}, c_{376} = 1; \ c_{678}, c_{458} = \sqrt{3} \). (A5)

**Appendix B: Boltzmann Equations in the Full SU(N) Basis**

The flavor pendulum equation of motion widely discussed in the literature is equivalent to expanding the matrix equation in full SU(N) basis. To write the diagonal terms in the SU(N) basis requires the transformation Eqs. (23)-(28), with \( f \) replaced by the source terms \( C \):

\[ C_0 = \frac{C_{\nu e} + C_{\nu \mu}}{2}, \quad C_3 = \frac{C_{\nu e} - C_{\nu \mu}}{2}, \] (B1)

for two flavors and

\[ C_0 = \frac{1}{3} (C_{\nu e} + C_{\nu \mu} + C_{\nu \nu}), \]
\[ C_3 = \frac{1}{2} (C_{\nu e} - C_{\nu \mu}), \] (B2)
\[ C_8 = \frac{1}{6\sqrt{3}} (C_{\nu e} + C_{\nu \mu} - 2C_{\nu \nu}), \]

for three flavors.

The resulting equations read

\[ \frac{\partial f_\gamma}{\partial t} + \vec{v} \cdot \frac{\partial f_\gamma}{\partial \vec{r}} + \vec{\dot{p}} \cdot \frac{\partial f_\gamma}{\partial \vec{p}} = 2\chi_{\alpha\beta\gamma}(H_\alpha^e + H_\alpha^\mu + B^\alpha - \bar{B}^\alpha)f_\beta + C_\gamma, \]
\[ \frac{\partial \bar{f}_\gamma}{\partial t} + \vec{v} \cdot \frac{\partial \bar{f}_\gamma}{\partial \vec{r}} + \vec{\dot{p}} \cdot \frac{\partial \bar{f}_\gamma}{\partial \vec{p}} = 2\chi_{\alpha\beta\gamma}(H_\alpha^e - H_\alpha^\mu - B^\alpha + \bar{B}^\alpha)\bar{f}_\beta + \bar{C}_\gamma, \] (B3)

where

\[ \begin{align*}
\chi_{\alpha\beta\gamma} &= \epsilon_{\alpha\beta\gamma} \quad \text{for two flavors}, \\
\chi_{\alpha\beta\gamma} &= c_{\alpha\beta\gamma} \quad \text{for three flavors}. \quad \text{(B4)}
\end{align*} \]

The non-zero components of \( C_\gamma \) are given by Eq. (B1) and Eq. (B2), similarly for \( \bar{C}_\gamma \). Writing \( f_\gamma, H_\alpha^e \) and so on as vectors, recognizing \( \chi_{\alpha\beta\gamma} \) as a cross product operator, and denoting \( D_t = \frac{\partial}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{r}} + \vec{\dot{p}} \cdot \frac{\partial}{\partial \vec{p}} \), we write Eq. (B3) as

\[ D_t \bar{f} = 2(\bar{H}_0 + \bar{H}_e + \bar{B} - \bar{B}^*) \times \bar{f} + \bar{C}, \]
\[ D_t \bar{f} = 2(\bar{H}_0 - \bar{H}_e - \bar{B} + \bar{B}^*) \times \bar{f} + \bar{C}. \] (B5)

This is the flavor pendulum equation of motion, taking its name from the analogy with the equation of motion of a classical spinning top in a gravitational field, with an additional term \((\bar{C})\) for classical scattering, absorption, and emission.
Appendix C: Classical Source Terms

The scattering, absorption, and emission terms can be written \[35\], dropping flavor indices, as:

\[
\frac{C}{c\gamma} = \kappa^a \left( \frac{I - I}{1 - f_{eq}} \right) - \kappa^s I + \frac{\kappa^s}{4\pi} \int \Phi(\Omega_p, \Omega_q) I(\Omega_q) d\Omega_q, \tag{C1}
\]

where \(a\) and \(s\) denote absorption and scattering, respectively. \(\gamma\) is given in Table II. \(f_{eq}\) is the equilibrium Fermi-Dirac distribution for the flavor in question. \(I\) is the blackbody specific intensity. The blocking factor \(1 - f_{eq}\) corrects for the filled states in fermion statistics, a mechanism called stimulated absorption, discussed in \[49\]. The coefficients are related to the relevant cross sections \(\sigma_i\) via

\[
\kappa = \sum_i n_i \sigma_i. \tag{C2}
\]

\(\Phi\) represents scattering back into the beam and can be approximated by

\[
\Phi_i(\Omega_p, \Omega_q) = 1 + \delta_i \cos \theta_{pq}, \tag{C3}
\]

where in both formulae \(i\) is an index for the type of scattering or absorption, i.e. the type of particles involved in the scattering or absorption.

With Eq. \(\text{(C1)}\), Eq. \(\text{(C2)}\), and Eq. \(\text{(C3)}\), we have:

\[
\frac{C}{c\gamma} = -\kappa^s I + \kappa^a \left( \frac{I - I}{1 - f_{eq}} \right) + \frac{\kappa^s}{4\pi} \left[ cE(q) + \delta_T \bar{p} \cdot \bar{F}(q) \right], \tag{C4}
\]

where \(\delta_T = \sum_i n_i \sigma_i \delta_i \). But for conservative scattering \(\Phi\) is only a phase function of angles, and thus \(|p| = |q|\). The generalization for inelastic scattering is straightforward. The zeroth moment is

\[
\frac{1}{c\gamma} C^{(0)} = -\kappa^s cE(p) + \kappa^a \left( \frac{4\pi \Gamma - cE(p)}{1 - f_{eq}} \right) + \frac{\kappa^s}{4\pi} \left[ 4\pi cE(p) \right] \tag{C5}
\]

Notice that these are the usual transfer moment equation terms with source \(4\pi \Gamma \kappa^a\) and sink \(-\kappa^s cE\), modulated by the blocking term \(1/(1 - f_{eq})\). The 1st moment is

\[
\frac{1}{c\gamma} C^{(1)i} = -\kappa^s F^i \kappa^a \left( \frac{F^i}{1 - f_{eq}} \right) + \frac{\kappa^s}{4\pi} \left[ \delta_T \frac{4\pi \delta_{ij} F^j}{3} \right] \tag{C6}
\]

\[
= -\kappa^s \frac{1}{1 - f_{eq}} F^i - \kappa^a \left( \frac{F^i}{1 - f_{eq}} \right) \delta_T \frac{4\pi \delta_{ij} F^j}{3} \]

\[
= -\kappa_T F^i. \]

Appendix D: Components of Two-Flavor Transport Equations

For components, we use the following notation for any neutrino variable \(Q\):

\[
\hat{Q}_2 = Q_2 + \bar{Q}_2 \]

\[
\hat{Q}_{1,3} = Q_{1,3} - \bar{Q}_{1,3} \tag{D1}
\]

where again the bar denotes the anti-neutrino.
Thus:

\[ \begin{align*}
\frac{\partial f_{\nu}}{\partial t} + v \cdot \frac{\partial f_{\nu}}{\partial r} + \dot{p} \cdot \frac{\partial f_{\nu}}{\partial p} &= f_2 \omega \sin 2\theta + 2 \left[ f_2 B_1 - f_1 B_2 \right] + C_{\nu}, \\
\frac{\partial f_{\nu}}{\partial t} + v \cdot \frac{\partial f_{\nu}}{\partial r} + \dot{p} \cdot \frac{\partial f_{\nu}}{\partial p} &= -f_2 \omega \sin 2\theta - 2 \left[ f_2 B_1 - f_1 B_2 \right] + C_{\nu}, \\
\frac{\partial f_1}{\partial t} + v \cdot \frac{\partial f_1}{\partial r} + \dot{p} \cdot \frac{\partial f_1}{\partial p} &= -f_3 \omega \sin 2\theta + f_1 \left[ A - \omega \cos 2\theta \right] - 2 \left[ f_3 B_1 - f_2 B_2 \right],
\end{align*} \]  

and

\[ \begin{align*}
\frac{\partial f_{\nu}}{\partial t} + v \cdot \frac{\partial f_{\nu}}{\partial r} + \dot{p} \cdot \frac{\partial f_{\nu}}{\partial p} &= f_2 \omega \sin 2\theta - 2 \left[ f_2 B_1 + f_1 B_2 \right] + C_{\nu}, \\
\frac{\partial f_{\nu}}{\partial t} + v \cdot \frac{\partial f_{\nu}}{\partial r} + \dot{p} \cdot \frac{\partial f_{\nu}}{\partial p} &= -f_2 \omega \sin 2\theta + 2 \left[ f_2 B_1 + f_1 B_2 \right] + C_{\nu}, \\
\frac{\partial f_2}{\partial t} + v \cdot \frac{\partial f_2}{\partial r} + \dot{p} \cdot \frac{\partial f_2}{\partial p} &= -f_3 \omega \sin 2\theta + f_2 \left[ A - \omega \cos 2\theta \right] + 2 \left[ f_3 B_1 - f_1 B_2 \right].
\end{align*} \]  

The corresponding first two-moment equations for neutrinos are

\[ \begin{align*}
\partial_t E_{\nu} + \nabla \cdot \vec{F}_{\nu} + \frac{1}{c} \dot{p} \cdot (\partial_p \vec{F}_{\nu}) &= \kappa (4\pi \Gamma_{\nu} - cE_{\nu}) \\
&= E_2 \omega \sin 2\theta + 2 \frac{\beta}{c^2} \int \frac{dq}{q} \left( c^2 \vec{E}_1(q) E_2 - \vec{F}_1(q) \cdot \vec{F}_2 - c^2 \vec{E}_2(q) E_1 + \vec{F}_1(q) \cdot \vec{F}_2 \right), \\
\partial_t E_{\nu} + \nabla \cdot \vec{F}_{\nu} + \frac{1}{c} \dot{p} \cdot (\partial_p \vec{F}_{\nu}) &= \kappa (4\pi \Gamma_{\nu} - cE_{\nu}) \\
&= -E_2 (A - \omega \cos 2\theta) - 2 \frac{\beta}{c^2} \int \frac{dq}{q} \left( c^2 \vec{E}_3(q) E_2 - \vec{F}_3(q) \cdot \vec{F}_2 - c^2 \vec{E}_2(q) E_3 + \vec{F}_3(q) \cdot \vec{F}_2 \right), \\
\partial_t E_1 + \nabla \cdot \vec{F}_1 + \frac{1}{c} \dot{p} \cdot (\partial_p \vec{F}_1) &= -E_2 (A - \omega \cos 2\theta) - 2 \frac{\beta}{c^2} \int \frac{dq}{q} \left( c^2 \vec{E}_1(q) E_3 - \vec{F}_1(q) \cdot \vec{F}_3 - c^2 \vec{E}_3(q) E_1 + \vec{F}_3(q) \cdot \vec{F}_1 \right), \\
\partial_t E_2 + \nabla \cdot \vec{F}_2 + \frac{1}{c} \dot{p} \cdot (\partial_p \vec{F}_2) &= -E_3 \omega \sin 2\theta + E_1 (A - \omega \cos 2\theta) - 2 \frac{\beta}{c^2} \int \frac{dq}{q} \left( c^2 \vec{E}_1(q) E_3 - \vec{F}_1(q) \cdot \vec{F}_3 - c^2 \vec{E}_3(q) E_1 + \vec{F}_3(q) \cdot \vec{F}_1 \right).
\end{align*} \]  

and

\[ \begin{align*}
\partial_t F_{\nu} + c^2 \partial_j P^{ij\nu} + \dot{p}^j \partial_p P^{ij\nu} + \kappa \Gamma_{\nu} F_{\nu} \\
&= F_3 \omega \sin 2\theta + 2 \beta \int \frac{dq}{q} \left( \vec{E}_1(q) F_2 - \vec{F}_1(q) P^{ij}_{ij} - \vec{E}_2(q) F_1 + \vec{F}_2(q) P^{ij}_{ij} \right), \\
\partial_t F_{\nu} + c^2 \partial_j P^{ij\nu} + \dot{p}^j \partial_p P^{ij\nu} + \kappa \Gamma_{\nu} F_{\nu} \\
&= -F_2 \omega \sin 2\theta - 2 \beta \int \frac{dq}{q} \left( \vec{E}_3(q) F_2 - \vec{F}_3(q) P^{ij}_{ij} - \vec{E}_2(q) F_3 + \vec{F}_2(q) P^{ij}_{ij} \right), \\
\partial_t F_1 + c^2 \partial_j P^{ij1} + \dot{p}^j \partial_p P^{ij1} \\
&= -F_3 \omega \sin 2\theta + c F_2 (A - \omega \cos 2\theta) - 2 \beta \int \frac{dq}{q} \left( \vec{E}_1(q) F_3 - \vec{F}_1(q) P^{ij}_{ij} - \vec{E}_3(q) F_1 + \vec{F}_3(q) P^{ij}_{ij} \right), \\
\partial_t F_2 + c^2 \partial_j P^{ij2} + \dot{p}^j \partial_p P^{ij2} \\
&= -F_3 \omega \sin 2\theta + c F_2 (A - \omega \cos 2\theta) - 2 \beta \int \frac{dq}{q} \left( \vec{E}_1(q) F_3 - \vec{F}_1(q) P^{ij}_{ij} - \vec{E}_3(q) F_1 + \vec{F}_3(q) P^{ij}_{ij} \right).
\end{align*} \]
TABLE I. Neutrino Mixing Parameters

| $\Delta m^2$ | NH | IH |
|--------------|----|----|
| $\Delta m^2_{21}$ | $7.5 \times 10^{-5} \text{eV}^2$ | $7.5 \times 10^{-5} \text{eV}^2$ |
| $\Delta m^2_{32}$ | $2.39 \times 10^{-3} \text{eV}^2$ | $-2.42 \times 10^{-3} \text{eV}^2$ |
| $\Delta m^2_{31}$ | $2.47 \times 10^{-3} \text{eV}^2$ | $-2.34 \times 10^{-3} \text{eV}^2$ |

Mixing angles rad

| $\theta_{12}$ | 0.58 |
| $\theta_{23}$ | 0.67 |
| $\theta_{13}$ | 0.15 |

TABLE II. Definitions of crucial quantities

| Quantity | Definition | Dimension |
|----------|-----------|-----------|
| Wigner Phase Space density | $f(r,p,t) = \int d^3 R \ e^{-ipR} \psi^\dagger(r + \frac{R}{2}, t)\psi(r - \frac{R}{2}, t)$ | $\text{1}$ |
| Specific Intensity | $I_\epsilon = \frac{c^3 |f|^2}{(2\pi \bar{\hbar})^3} \equiv \gamma^{-1} f$ | $L^{-2}T^{-1}$ |
| Energy density | $E(p,\vec{r},t) = \frac{1}{c} \int d\Omega_p I(p,\vec{r},t)$ | $L^{-3}$ |
| Energy flux | $F(p,\vec{r},t) = \int d\Omega_p \ pI(p,\vec{r},t)$ | $L^{-2}T^{-1}$ |
| Momentum tensor | $P^{ij}(p,\vec{r},t) = \frac{1}{c} \int d\Omega_p \ p_i p_j I(p,\vec{r},t)$ | $L^{-3}$ |
| Higher moment tensors | $Q^{(n)} = \int d\Omega_p \ (\hat{p})^n I(p,\vec{r},t)$ | $L^{-2}T^{-1}$ |

$a_\epsilon$ is the particle energy.