Abstract

We analyze the extension of the well known relation between Brownian motion and Schrödinger equation to the family of Lévy processes. We propose a Lévy–Schrödinger equation where the usual kinetic energy operator – the Laplacian – is generalized by means of a pseudodifferential operator whose symbol is the logarithmic characteristic of an infinitely divisible law. The Lévy–Khintchin formula shows then how to write down this operator in an integro–differential form. When the underlying Lévy process is stable we recover as a particular case the recently proposed fractional Schrödinger equation. A few examples are finally given and we find that there are physically relevant models (such as a form of the relativistic Schrödinger equation) that are in the domain of the possible Lévy–Schrödinger equations.

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1 Introduction

That the Schrödinger equation can be linked to some underlying stochastic process is well known since longtime. This idea has received along the years a number of different formulations: from the Feynman path integral [1], through the Bohm–Vigier model [2], to the Nelson stochastic mechanics [3, 4]. In all these models the underlying stochastic process powering the random fluctuations is a Gaussian Brownian motion, and the focus of the interest is the (non relativistic) Schrödinger equation of quantum mechanics. This particular choice is understandable because on the one hand the Gaussian Brownian motion is the the most natural and widely explored example of Markov process available, and on the other hand its connection with the Schrödinger equation has always lent the hope of a finer understanding of quantum mysteries.
In the framework of stochastic mechanics, however, this standpoint can be considerably broadened since in fact this theory is a model for systems more general than quantum mechanics: a general dynamical theory of Brownian motion that can be applied to several physical problems [5, 6, 7]. On the other hand in recent years we have witnessed a considerable growth of interest in non Gaussian stochastic processes, and in particular in the Lévy processes [8, 9, 10, 11, 12]. This is a field that was initially explored in the 30's and 40's of last century [13, 14, 15], but that achieved a full blossoming of research only in the last decades, also as a consequence of the tumultuous development of computing facilities. This interest is witnessed by the large field of the possible applications of these more general processes from statistical mechanics [4] to mathematical finance [10, 16, 17]. In the physical field, however, the research scope is presently rather confined to a particular kind of Lévy processes: the stable processes and the corresponding fractional calculus [18, 19], while in the financial domain a vastly more general type of processes is at present in use. For instance it must be recalled here that the possibility of widening the perspective of the Schrödinger–Brownian pair has already been considered in recent years [20], but the Schrödinger equation has been generalized only to a fractional Schrödinger equation. No one at our knowledge, instead, has tried to use the more general Lévy infinitely divisible process to widen the horizon of the applications of a generalized Schrödinger equation to other physical systems.

The appeal of the stable distributions is justified by the properties of scaling and self-similarity displayed by the corresponding processes, but it must also be remarked that these distributions show a few features that partly impair their usefulness as empirical models. First of all the non gaussian stable laws always have infinite variance. This makes them rather suspect as a realistic tool and prompts the introduction of truncated stable distributions which, however, are no longer stable. Then the range of the $x$ decay rates of the probability density functions can not exceed $x^{-3}$, and this too introduces a particular rigidity in these models. For example there is no way to approximate a gaussian Wiener process by means of some suitable sequence of non-gaussian stable processes. On the other hand the more general Lévy processes are generated by infinitely divisible laws and do not necessarily show these disturbing features, but they can be more difficult to analyze and to simulate [21, 22, 23]. Beside the fact that they do not have natural scaling properties, the probability density function of their increments could be explicitly known only at one time instant. In fact while their time evolution can always be explicitly given in terms of characteristic functions, their marginal densities may not be calculable. This is a feature, however, that they share with most stable processes, since the probability density functions of the non gaussian stable laws are explicitly known only in precious few cases. On the other hand some new applications in the physical domain for Lévy, infinitely divisible but not stable processes begin to emerge: in particular the statistical characteristics of some recent model of the collective motion in the charged particle accelerator beams seem to point exactly in the direction of some kind of Student infinitely divisible process [6, 24].

This paper is devoted to a discussion of a generalization of the Schrödinger equation which takes into account the entire family of the Lévy processes: we will propose an equation where the infinitesimal generator of the Brownian semigroup (the Laplacian) is substituted by the more general generator of a Lévy semigroup. As it happens
this will be a pseudodifferential operator (as in the fractional case), and the Lévy–Khintchin formula will give us the opportunity to write it down in the form of an explicit integro–differential operator by putting in evidence its continuous (Gaussian) and its jumping (non Gaussian) parts. The advantages of this formulation are many: first of all the widening of the increment laws from the stable to the infinitely divisible case will offer the possibility of having realistic, finite variances. Moreover both the possible presence of a Gaussian component in the Lévy–Khintchin formula, and the wide spectrum of decay velocities of the increment probability densities will afford the possibility of having models with differences from the usual Brownian (and usual quantum mechanical, Schrödinger) case as small as we want. In this sense we could speak of small corrections to the quantum mechanical, Schrödinger equation. Last but not least we will see that there are examples of non stable Lévy processes which are connected to a particular form of the quantum, relativistic Schrödinger equation: an important link that was missing in the original Nelson model. It seems in fact that – as was already pointed out a few years ago [25] – we can only recover some kind of relativistic quantum mechanics if we widen the field of the underlying stochastic processes at least to that of the selfdecomposable Lévy processes. To avoid formal complications we will confine our discussion to the case of processes in just one spatial dimension: generalizations will be straightforward.

2 A heuristic discussion

Let us start from the non relativistic, free Schrödinger equation associated to its propagator or Green function $G(x,t|y,s)$ (see for example [1])

$$ih\partial_t\psi(x,t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x,t)$$

$$G(x,t|y,s) = \frac{1}{\sqrt{2\pi i(t-s)\hbar/m}} e^{-\frac{(x-y)^2}{2i(t-s)\hbar/m}}$$

$$\psi(x,t) = \int_{-\infty}^{+\infty} G(x,t|y,s) \psi(y,s) \, dy$$

and compare it with the Fokker–Planck equation of a Wiener process (Brownian motion) with diffusion coefficient $D$, $pdf$ (probability density function) $q(x,t)$ and transition $pdf$ $p(x,t|y,s)$ (see for example [24])

$$\partial_t q(x,t) = D \partial_x^2 q(x,t)$$

$$p(x,t|y,s) = \frac{1}{\sqrt{4\pi (t-s)D}} e^{-\frac{(x-y)^2}{4(t-s)D}}$$

$$q(x,t) = \int_{-\infty}^{+\infty} p(x,t|y,s) q(y,s) \, dy$$

It is apparent that there is a simple, formal procedure transforming the two structures one into the other:

$$D = \frac{\hbar}{2m}, \quad t \rightarrow it$$
It is well known that this is just the result of an analytic continuation in the complex plane. There are of course important differences between $G$ and $p$. For example while $p$ and $q$ are well behaved pdf’s, $G$ is not a wave function, as can be seen also from a simple dimensional argument. This simple symmetry can then be deceptive, but a better understanding of its true meaning can be achieved either by means of the Feynman path integration with a free Lagrangian of the usual quadratic form, or through the Madelung decomposition [27] of (1) and its subsequent stochastic mechanical model [3, 4]. Our aim here is to generalize to distributions other than Gaussian this simple shortcut from Wiener process to Schrödinger equation, and to analyze its most immediate consequences. We postpone to a subsequent paper a more detailed analysis in the framework of stochastic mechanics.

Let us see first of all what kind of role the gaussian distribution plays in our Wiener–Schrödinger scheme. The pdf and the chf (characteristic function) of a Gaussian law $\mathcal{N}(0,a^2)$

$$q(x) = \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}}, \quad \varphi(u) = e^{-a^2u^2/2}$$

satisfy the relations

$$\varphi(u) = \int_{-\infty}^{+\infty} q(x) e^{iux} dx, \quad q(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(u) e^{-iux} du.$$  \hspace{1cm} (7)$$

Then formally the propagator (2) and the transition pdf (5) respectively have as chf’s

$$e^{-t$}$D(t-s)u^2 = [\varphi(u)]^{i(t-s)/\tau}, \quad e^{-D(t-s)u^2} = [\varphi(u)]^{(t-s)/\tau}$$

where now $\varphi(u) = e^{-D\tau u^2} = e^{-\tau h u^2/2m}$ is the chf of a Gaussian law $\mathcal{N}(0,2D\tau)$, and $\tau$ is a time constant introduced in order to have dimensionless exponents. From (7) we then have

$$G(x,t|y,s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\varphi(u)]^{i(t-s)/\tau} e^{-iBu(\Delta x)} du,$$  \hspace{1cm} (8)$$

$$p(x,t|y,s) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} [\varphi(u)]^{(t-s)/\tau} e^{-iu(\Delta x)} du.$$  \hspace{1cm} (9)$$

In both cases the starting point is the same chf $\varphi$ of the centered, normal law $\mathcal{N}(0,2D\tau)$. Then we consider the chf $[\varphi(u)]^{(t-s)/\tau}$ of the $t-s$ stationary increments of the Wiener process, we pass to the imaginary time variables ($t \leftrightarrow it$), and finally we get from (8) the Schrödinger propagator (2). The time scale $\tau$ incorporated in the initial normal law disappears in the subsequent steps when we generate the chf of the increments. This feature is common to all the stable laws and is the embodiment of the stable processes selfsimilarity.

The equation (1) and (4) can now be easily deduced respectively from (8) and (9). For instance from (2) and (8) we have

$$\psi(x,t) = \int_{-\infty}^{+\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{+\infty} e^{-iDu^2(t-s)} e^{-iu(\Delta x)} du$$
and then we can write

\[
\begin{align*}
  i\partial_t \psi(x,t) &= \int_{-\infty}^{+\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{+\infty} Du^2 e^{-iDu^2(t-s)} e^{-iu(x-y)} du \\
  &= D \int_{-\infty}^{+\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{+\infty} (i\partial_x)^2 e^{-iDu^2(t-s)} e^{-iu(x-y)} du \\
  &= -D \partial_x^2 \int_{-\infty}^{+\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{+\infty} e^{-iDu^2(t-s)} e^{-iu(x-y)} du = -D \partial_x^2 \psi(x,t)
\end{align*}
\]

which – but for a factor \( h \) – is the free, non relativistic Schrödinger equation (11).

We are interested now in reproducing these well known steps starting with the \( \text{chf} \) of a non Gaussian law. Take now an infinitely divisible – in general non Gaussian – law with \( \text{chf} \ \varphi(u) \), and let \( \eta(u) = \ln \varphi(u) \) be its \( \text{lch} \) (logarithmic characteristic). In the following we will restrict us to centered laws, and we will justify this choice in the subsequent sections. \textit{Infinite divisibility} essentially is the property of a \( \text{chf} \) \( \varphi \) which guarantees that also \( \varphi^{t/\tau} \) is a legitimate \( \text{chf} \) for every real \( t \). About the infinitely divisible laws and their intimate relation with the Lévy processes there is a vast literature (see for example [8,15], and for a short introduction [28]). The law of the increment of the corresponding Lévy process then is \( [\varphi(u)]^{(t-s)/\tau} \) and its transition pdf is (9) with our – possibly non Gaussian – infinitely divisible \( \text{chf} \) \( \varphi \). Then, following the procedure previously outlined for the Wiener–Schrödinger equation, the wave function propagator is (3) with our new \( \varphi \), and hence from (3) the time evolution is ruled by

\[
\psi(x,t) = \int_{-\infty}^{+\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{+\infty} [\varphi(u)]^{i(t-s)/\tau} e^{-iu(x-y)} du.
\]

The differential equation can then be deduced as in the Gaussian case and is

\[
\begin{align*}
  i\partial_t \psi(x,t) &= \int_{-\infty}^{+\infty} dy \frac{\psi(y,s)}{2\pi} \int_{-\infty}^{+\infty} \frac{\ln[\varphi(u)]}{\tau} [\varphi(u)]^{i(t-s)/\tau} e^{-iu(x-y)} du \\
  &= -\frac{1}{\tau} \eta(\partial_x) \psi(x,t) \tag{10}
\end{align*}
\]

where now \( \ln[\varphi(\partial_x)] = \eta(\partial_x) \) is a pseudodifferential operator with symbol \( \eta(u) \) that is defined through the use of the Fourier transforms [10,11,29,30]. A pseudodifferential operator \( L \) on a suitable set of functions \( h(x) \), is associated to a function \( \ell(u) \) called the symbol of \( L \), and operates in the following way: if the Fourier transform of \( h \) is

\[
\hat{h}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} h(x) e^{-iux} dx \tag{11}
\]

then

\[
(Lh)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \ell(u) \hat{h}(u) e^{iux} du. \tag{12}
\]

Remark that the definition (11) of Fourier transform of a function \( h(x) \) is traditionally slightly different from the definition (11) of the \( \text{chf} \) of a law. When the symbol is a polynomial of degree \( n \)

\[
\ell(u) = \sum_{k=1}^{n} a_k (iu)^k
\]
then \( L \) is a simple differential operator of order \( n \)

\[
L = \sum_{k=1}^{n} a_k \partial_x^k
\]

as can be easily seen from the properties of the Fourier transforms. However, even if \( \ell(u) \) is not a polynomial, equation (12) defines an operator which is called pseudodifferential. We will now analyze the properties and the role of our pseudodifferential operator \( \eta(\partial_x) \) to see if (10) can reasonably be considered as a generalized Schrödinger equation.

### 3 Semigroups and generators

Let \( X(t) \) be a one dimensional Lévy process, namely a process with stationary and independent increments, and \( X(0) = 0 \) almost surely. The chf of its increments on a time interval \( \Delta t \) then is \( |\varphi(u)|^{\Delta t/\tau} \) where \( \varphi(u) \) is an infinitely divisible law, and \( \tau \) a time scale parameter (see for example [8] and [11] for details about Lévy processes). It is well known that \( \eta(u) = \ln \varphi(u) \) is the lch of an infinitely divisible law if and only if it satisfies the Lévy–Khintchin formula [11]

\[
\eta(u) = i\gamma u - \frac{\beta^2}{2} u^2 + \int_{\mathbb{R}} \left[ e^{iux} - 1 - iuxI_{[-1,1]}(x) \right] \nu(dx)
\]

where \( \gamma, \beta \in \mathbb{R} \), \( I_A \) is the indicator 0–1 function of the set \( A \), and \( \nu(\cdot) \) is the Lévy measure of our infinitely divisible law, namely a measure on \( \mathbb{R} \) such that \( \nu(\{0\}) = 0 \) and

\[
\int_{\mathbb{R}} (x^2 \wedge 1) \nu(dx) < +\infty.
\]

The integrals involving the Lévy measure \( \nu \) should then in general be calculated on \( \mathbb{R} - \{0\} \) to take into account its behavior near \( y = 0 \). The triplet \((\gamma, \beta, \nu)\) completely determines the Lévy process and is also called its characteristic triplet. There are a few equivalent formulations of this important result [8]. In particular the truncation function \( I_{[-1,1]}(x) \) can be chosen in several different ways; this choice, however, will affect only the value of \( \gamma \), while \( \beta \) and \( \nu \) would be left unchanged.

To every Lévy process is associated a semigroup \((T_t)_{t \geq 0}\) acting on the space \( D \) of the measurable, bounded functions [11]: if \( f \in D \), we have \( (T_t f)(x) = \mathbb{E}[f(X(t) + x)] \), where \( \mathbb{E} \) is the expectation. The infinitesimal generator \( A \) of the semigroup (see [11] p. 131) is now defined on the domain \( D_A \) of the functions \( f \in D \) such that the limit (in norm on \( D \))

\[
Af = \lim_{t \to 0^+} \frac{T_t f - f}{t}
\]

exists. It can be proved (see [11] p. 139) that the generators of a Lévy process are pseudodifferential operators that can be extended to the Schwartz space \( S \) of the rapidly decreasing functions. In particular we find that the symbol of \( A \) is nothing else than
\( \eta(u) \), namely \( A = \eta(\partial_x) \), and that from the Lévy–Khintchin formula \(^{[13]}\) we have

\[
\eta(\partial_x)f(x) = \gamma(\partial_x f)(x) + \frac{\beta^2}{2}(\partial_x^2 f)(x) + \int\limits_{\mathbb{R}} [f(x+y) - f(x) - y(\partial_x f)(x)I_{[-1,1]}(y)] \nu(dy).
\]

In other words our pseudodifferential operator \( \eta(\partial_x) \) of equation \(^{[10]}\) is the generator of the underlying Lévy process, and thanks to the Lévy–Khintchin formula it also has an explicit expression in terms of integro–differential operators.

The generator \( A = \eta(\partial_x) \) can finally be extended to \( L^2(\mathbb{R}) \) which is a Hilbert space, so that we can also discuss its self-adjointness. In particular if \( X(t) \) is a Lévy process, then its infinitesimal generator \( A = \eta(\partial_x) \) will be self-adjoint in \( L^2(\mathbb{R}) \) if and only if \( X(t) \) is centered and symmetric, namely if the symbol \( \eta(u) \) is real with

\[
\eta(u) = -\frac{\beta^2}{2}u^2 + \int\limits_{\mathbb{R}} (\cos ux - 1) \nu(dx)
\]

where \( \nu(\cdot) \) is a symmetric Lévy measure (see \(^{[11]}\) p. 154). A Lévy measure is symmetric when \( \nu(B) = \nu(-B) \) for every Borel measurable set, where \( -B = \{x; -x \in B\} \). As a consequence the self-adjoint generators of the centered and symmetric Lévy processes enjoy the following simplified, integro–differential form

\[
(Af)(x) = [\eta(\partial_x)f](x) = \frac{\beta^2}{2}(\partial_x^2 f)(x) + \int\limits_{\mathbb{R}} [f(x+y) - f(x)] \nu(dy).
\]

It is also possible to show that \( -\eta(\partial_x) \) is not only self-adjoint, but also positive on \( L^2(\mathbb{R}) \) in the sense that for every \( f \in L^2(\mathbb{R}) \) we have \( -(f, \eta(\partial_x)f) \geq 0 \), and this is equivalent to say that the spectrum of \( -\eta(\partial_x) \) lies entirely in \([0, +\infty)\).

We come back now to our problem of proposing a generalized, Lévy–Schrödinger equation. Let \( X(t) \) be a centered, symmetric Lévy process with pdf and transition pdf: we know that there is a centered, infinitely divisible law with chf \( \varphi(u) = e^{\eta(u)} \) such that the chf of the stationary increments \( \Delta X(t) = X(t + \Delta t) - X(t) \) is

\[
[\varphi(u)]^{\Delta t/\tau} = e^{\eta(u)\Delta t/\tau}
\]

for a suitable time scale parameter \( \tau \). The transition pdf then is \(^{[9]}\), so that by means of the substitution \( t \leftrightarrow \tau \) we get the propagator \( G \) of equation \(^{[8]}\). A wave function \( \psi \) then evolves following \(^{[10]}\), and since our process is centered and symmetric the generator \( \eta(\partial_x) \) is self-adjoint and has the integro–differential expression \(^{[15]}\). That means that our proposed equation takes the form

\[
i\partial_t \psi(x,t) = -\frac{\eta(\partial_x)}{\tau} \psi(x,t) = \frac{\beta^2}{2\tau} \partial_x^2 \psi(x) - \frac{1}{\tau} \int\limits_{\mathbb{R}} [\psi(x+y) - \psi(x)] \nu(dy).
\]

When \( X(t) \) is a Gaussian Wiener process we know that \( \eta(u) = -\beta^2u^2/2 \) and \( A = \eta(\partial_x) = \beta^2/2 \partial_x^2 \). Then the process evolution equation is reduced to the Fokker–Planck equation \(^{[11]}\) with \( D = \beta^2/2\tau \), and we have argued that in this case the usual non relativistic, free Schrödinger equation can be obtained by means of the substitution \(^{[7]}\). In
fact this amounts to take the \( -\hbar \eta (\partial_x)/\tau = -\hbar D \partial_x^2 \) as the kinetic energy operator. We propose here to extend this shortcut also to the case of non Gaussian Lévy processes. Apparently this is just a formal analogy, but there are at least two ways to make it more compelling: the Feynman path integral, and the Nelson stochastic mechanics. Here we only take for granted this association between generators and kinetic energy operators in order to establish the form of the free Lévy–Schrödinger equation and discuss its first consequences; we instead postpone to a subsequent paper a more rigorous derivation based on the application of Nelson stochastic mechanics to Lévy processes. We will remember however that the method of Feynman path integrals has already been used in the particular case of the fractional Schrödinger equations \[20\] to obtain the same association, albeit in a more restricted case: that of the stable laws, a particular class of infinitely divisible laws. Note that we will adopt here only the formal substitution \( t \leftrightarrow i t \), but we will not impose \( D = \hbar/2m \) because our generalizations are not necessarily supposed to be some kind of quantum mechanics, but will rather describe a dynamical theory of Lévy processes in the spirit of Nelson stochastic mechanics. We can also introduce a constant \( \alpha \) with the dimensions of an action, so that our proposed free Lévy–Schrödinger equation \[16\] becomes

\[
\begin{align*}
   i\alpha \partial_t \psi(x,t) & = H_0 \psi(x,t) = \frac{\alpha}{\tau} \eta(\partial_x) \psi(x,t) \\
   & = -\frac{\beta^2}{2\tau} \partial_x^2 \psi(x,t) - \frac{\alpha}{\tau} \int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)] \nu(dy)
\end{align*}
\]

(17)

where now the free hamiltonian operator \( H_0 \) has the dimensions of an energy. This integro-differential hamiltonian \( H_0 \) is self-adjoint and positive on \( L^2(\mathbb{R}) \) so that it is a good kinetic energy operator. Everything that can be deduced about the usual Schrödinger equation from the positivity and self-adjointness of \( H_0 \) can also be of course extended to our Lévy–Schrödinger equation \(17\). In particular the conservation of the probability in the sense that, if \( |\psi|^2 \) plays the role of the position pdf, then the norm \( ||\psi||^2 \) will be constant. We could finally add a potential \( V(x) \) to \(17\) and get a complete Lévy–Schrödinger equation

\[
\begin{align*}
   i\alpha \partial_t \psi(x,t) & = H \psi(x,t) = -\frac{\alpha}{\tau} \eta(\partial_x) \psi(x,t) + V(x) \psi(x,t).
\end{align*}
\]

(18)

where the hamiltonian is now \( H = H_0 + V \).

4 Discussion and examples

The free Lévy–Schrödinger hamiltonian of equation \(17\) contains two parts: the usual kinetic energy \( -\frac{\beta^2}{2\tau} \partial_x^2 \) related to the Gaussian part of the process; and the jump part which is given by means of an integral with a symmetric Lévy measure \( \nu \). Of course, depending on the nature of the underlying process, the equation \(17\) can contain these components in different mixtures. If the underlying process is purely Gaussian then the Lévy measure \( \nu \) vanishes and \(17\) is reduced to the usual Schrödinger equation. On the other hand when the initial process is totally non Gaussian, then \( \beta = 0 \) and we get a pure jump Lévy–Schrödinger equation. In general both terms are present and if
for instance we introduce $\omega = 1/\tau$ and choose $\alpha = \hbar$ and $\beta^2 = \alpha \tau / m$, then (17) takes the form

$$i\hbar \partial_t \psi(x,t) = -\frac{\hbar^2}{2m} \partial_x^2 \psi(x,t) - \hbar \omega \int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)] \nu(dy).$$

Here the jump term can be considered as a correction to the usual Schrödinger equation and its weight, the energy $\hbar \omega$, is at present a free parameter. In particular for $\beta = 0$ we get pure jump Schrödinger equations of the form

$$i\partial_t \psi(x,t) = -\omega \int_{\mathbb{R}} [\psi(x+y,t) - \psi(x,t)] \nu(dy). \quad (19)$$

Of course these remarks emphasize the fact that the explicit form of the Lévy–Schrödinger equation will depend on the choice of the particular Lévy measure $\nu$.

### 4.1 Stationary free solutions

Let us consider first the stationary solutions of (17): taking

$$\psi(x,t) = e^{-iEt/\alpha} \phi(x), \quad i\alpha \partial_t \psi(x,t) = E \psi(x,t)$$

we have that the spatial part $\phi(x)$ will be solution of

$$H_0 \phi(x) = -\frac{\alpha \beta^2}{2\tau} \phi''(x) - \frac{\alpha}{\tau} \int_{\mathbb{R}} [\phi(x+y) - \phi(x)] \nu(dy) = E \phi(x) \quad (20)$$

For the plane wave solutions

$$\phi(x) = e^{\pm iux}$$

and because of the symmetry of the Lévy measure $\nu$, equation (20) becomes

$$E \phi(x) = \left[ -\frac{\alpha \beta^2}{2\tau} u^2 - \frac{\alpha}{\tau} \int_{\mathbb{R}} (e^{\pm iuy} - 1) \nu(dy) \right] \phi(x) = \frac{\alpha}{\tau} \int_{\mathbb{R}} (\cos uy - 1) \nu(dy) \phi(x) = -\frac{\alpha}{\tau} \eta(u) \phi(x)$$

and hence it is satisfied when between $E$ and $u$ the following relation holds

$$E = -\frac{\alpha}{\tau} \eta(u)$$

Here $u$ is a wave number, while we are used to look for a relation between energy $E$ and momentum $p$. If then we posit $p = \alpha u$ the energy–momentum relation for our free Lévy–Schrödinger equation is

$$E = -\frac{\alpha}{\tau} \eta \left( \frac{p}{\alpha} \right) \quad (21)$$
4.2 Some classes of infinitely divisible laws

We will explore now a few examples of Lévy–Schrödinger equations associated to Lévy processes. For a short summary of the concepts used here see for example [28]. We will consider the chf’s, Lévy measures and infinitesimal generators of centered, symmetric, infinitely divisible laws so that (14) and (15) hold. The form of the Lévy measure $\nu$ is then instrumental to explicitly show how the pseudo-differential generator $\eta(\partial_x)$ works. It would then be useful to list several classes of infinitely divisible laws in a growing order of generality:

1. **Stable laws**: here we have [8 10]

$$\eta(u) = -\frac{(a|u|)^\lambda}{\lambda}; \quad 0 < \lambda \leq 2,$$

with the important particular cases

$$\eta(u) = \begin{cases} 
-a^2 u^2/2 & \text{Gauss law (}\lambda = 2); \\
-a|u| & \text{Cauchy law (}\lambda = 1). 
\end{cases}$$

Stable laws are selfdecomposable and hence their Lévy measures are absolutely continuous [8 10] so that

$$\nu(dx) = W(x) dx$$

with

$$W(x) = \frac{B}{|x|^{\lambda+1}}, \quad B = \begin{cases} 
0 & \lambda = 2; \\
\frac{a}{\pi} & \lambda = 1; \\
-\frac{a^\lambda}{2\lambda \cos \frac{\pi}{2} \Gamma(-\lambda)} & \lambda \neq 1, 2. 
\end{cases}$$

The infinitesimal generator, which in the Gauss case ($\lambda = 2$) simply is

$$\frac{a^2}{2} (\partial_x^2 f)(x),$$

for $0 < \lambda < 2$ becomes the pseudo-differential operator

$$[\eta(\partial_x)f](x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (a|u|)^\lambda e^{iu x} \hat{f}(u) du = B \int_{-\infty}^{+\infty} \frac{f(x+y) - f(x)}{|y|^{\lambda+1}} dy$$

which can also be expressed symbolically in terms of the fractional derivatives [19]

$$(\partial_x^\lambda f)(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} |u|^\lambda e^{iu x} \hat{f}(u) du$$

$$= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (-\partial_x^2)^{\lambda/2} e^{iu x} \hat{f}(u) du = \left[-(-\partial_x^2)^{\lambda/2} f\right](x).$$

2. **Selfdecomposable (non stable) laws**: they are an important sub-family of infinitely divisible laws. Two examples are (for details and other examples see [22, 28])

$$\eta(u) = \begin{cases} 
-\lambda \ln(1 + a^2 u^2), & \text{Variance-Gamma (}\lambda > 0); \\
1 - \sqrt{1 + a^2 u^2}, & \text{Relativistic quantum mechanics.} 
\end{cases}$$
which have no Gaussian part ($\beta = 0$ in the Lévy–Khintchin formula) and produce pure jump processes. Their Lévy measures have densities \cite{22, 28}

$$W(x) = \begin{cases} \lambda|x|^{-1}e^{-|x|/a} & \text{VG;} \\ \frac{1}{(\pi|x|)^{-1}}K_{\frac{1}{2}}(|x|/a) & \text{Relativistic q.m.} \end{cases}$$

where $K_\lambda(z)$ is a modified Bessel function. Remark that, while for the VG we can explicitly write the pdf

$$q(x) = \frac{2}{a^2 \Gamma(\lambda)\sqrt{2\pi}} \left(\frac{|x|}{a}\right)^{\lambda-\frac{1}{2}} K_{\lambda-\frac{1}{2}}\left(\frac{|x|}{a}\right)$$

we have no elementary expressions for the Relativistic q.m. pdf

$$q(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{1-\sqrt{1+a^2u^2}} e^{iax} du$$

which can then be calculated only numerically.

3. \emph{Infinitely divisible (non selfdecomposable) laws}: The classical example of an infinitely divisible, non selfdecomposable law is the Poisson law of intensity $\lambda$, but the corresponding Lévy process would not be symmetric. If however we take the chf $\chi(u)$ of a centered, symmetric law, then the corresponding \emph{compound} Poisson process will be centered and symmetric with check

$$\eta(u) = \lambda (\chi(u) - 1) .$$

In the analysis of the corresponding Lévy measure we must remember that now we can no longer take for granted that $\nu$ is absolutely continuous. For example if the jump size can take only two values $\pm a$ ($a > 0$) with equal probabilities $1/2$, then the chf $\chi(u) = \cos au$ has no pdf, $\eta(u) = \lambda(\cos au - 1)$, and $\nu(dx) = \lambda F(dx)$ where the cumulative distribution

$$F(x) = \frac{\Theta(x-a) + \Theta(x+a)}{2}$$

is a symmetric, two–steps function, and $\Theta(x)$ is the 0–1 Heaviside function. If on the other hand $\chi$ is a law with a pdf $g(x)$, it is possible to show that also the Lévy measure $\nu$ is absolutely continuous with a density

$$W(x) = \lambda g(x) .$$

This completely specify the associated Lévy process on the basis of the Poisson intensity $\lambda$, and of the law of the jump sizes.

4.3 \textbf{Examples of Lévy–Schrödinger equations}

We can now analyze a few examples of Lévy–Schrödinger equations based on the laws listed above. The first two cases have an apparent physical meaning, while at present the other three (Variance–gamma, stable and compound Poisson laws) are short of an immediate interpretation.
1. **Non relativistic, quantum, free particle**: this is the well known case of the Gaussian Wiener process with \( \eta(u) = -\beta^2 u^2 / 2 \) giving rise to the usual Schrödinger equation (1) for a suitable identification of the parameters. In this case the energy–momentum relation (21) is

\[
E = -\frac{\alpha}{\tau} \left( -\frac{\beta^2}{2} \frac{p^2}{\alpha^2} \right) = \frac{\beta^2}{2\alpha \tau} p^2
\]

and with \( \alpha = \hbar \) and \( \beta^2 = \alpha \tau / m \) we get as usual

\[
E = \frac{p^2}{2m}, \quad p = hu.
\]

2. **Relativistic, quantum, free particle**: it is interesting to remark at this point that there is a Lévy process which is connected to the relativistic Schrödinger equation, in the same way as the Wiener process is connected to the non relativistic Schrödinger equation. Take the non stable, selfdecomposable law \( \eta(u) = 1 - \sqrt{1 + a^2 u^2} \), and use the following identifications

\[
\frac{\alpha}{\tau} = mc^2, \quad a = \frac{\hbar}{mc}, \quad p = hu
\]

to find from (21)

\[
E = -mc^2 \eta(u) = \sqrt{m^2 c^4 + p^2 c^2} - mc^2
\]

which is the relativistic total energy less the rest energy \( mc^2 \): namely the kinetic energy. The corresponding Lévy–Schrödinger equation is now

\[
i\hbar \partial_t \psi(x, t) = \left[ \sqrt{m^2 c^4 - c^2 \hbar^2 \partial_x^2} - mc^2 \right] \psi(x, t).
\]

Since the constant \(-mc^2\) can be reabsorbed by means of a phase factor \( e^{imc^2 t/\hbar} \), the wave equation finally is

\[
i\hbar \partial_t \psi(x, t) = \sqrt{m^2 c^4 - c^2 \hbar^2 \partial_x^2} \psi(x, t)
\]

which is the simplest form of a relativistic, free Schrödinger equation [31]. It is interesting to note that the Lévy process behind the relativistic equation (26) is a pure jump process with an absolutely continuous Lévy measure with pdf

\[
W(x) = \frac{1}{\pi |x|} K_1 \left( \frac{|x|}{a} \right) = \frac{1}{\pi |x|} K_1 \left( \frac{mc|x|}{h} \right)
\]

so that equation (26) can also be written as

\[
i\hbar \partial_t \psi(x, t) = -mc^2 \int_{\mathbb{R}} \frac{\psi(x + y, t) - \psi(x, t)}{\pi |y|} K_1 \left( \frac{mc|y|}{h} \right) dy
\]

From the form of the relativistic energy (25) also the usual relativistic corrections to the classical energy–momentum relation for small values of \( p/c \) follow:

\[
E = mc^2 \left( \sqrt{1 + \frac{p^2}{m^2 c^2}} - 1 \right) = \frac{p^2}{2m} - \frac{p^4}{8m^3 c^2} + o(p^5).
\]
Finally if we consider $E = H(p)$ as the Hamiltonian function of a relativistic free particle from the Hamilton equations we get

$$\dot{q} = \partial_p H = \frac{p}{m} \frac{1}{\sqrt{1 + p^2/m^2c^2}}$$

and here too we can see the relativistic correction to the classical kinematic relation $p = m\dot{q}$.

3. Variance–Gamma laws: for the particular case $\lambda = \frac{1}{2}$ we have

$$\eta(u) = -\frac{1}{2} \ln(1 + a^2u^2), \quad q(x) = \frac{1}{a\pi} K_0 \left( \frac{|x|}{a} \right), \quad W(x) = \frac{e^{-|x|/a}}{2|x|}.$$  

The Variance–Gamma processes are pure jump processes with no Gaussian part in the Lévy–Khintchin formula (13) ($\beta = 0$), so that the Lévy–Schrödinger equation becomes

$$i\alpha \partial_t \psi(x, t) = -\frac{\lambda\alpha}{\tau} \int_{\mathbb{R}} \frac{\psi(x + y, t) - \psi(x, t)}{|y|} e^{-|y|/a} dy$$

By choosing

$$\alpha = \frac{ma^2}{\tau}, \quad p = \frac{ma^2}{\tau} u$$

we have the following energy–momentum relation (for $p\tau/ma \to 0$)

$$E = \frac{ma^2}{2\tau^2} \ln \left( 1 + \frac{\tau^2p^2}{m^2a^2} \right) = \frac{p^2}{2m} - \frac{\tau^2}{2ma^2} \frac{p^4}{m^2} + o(p^5)$$

while with the identification $E = H(p)$ we can also recover the kinematic relations between $p$ and $\dot{q}$:

$$\dot{q} = \frac{p/m}{1 + \frac{\tau^2p^2}{a^2m^2}} = \frac{p}{m} - \frac{\tau^2p^3}{a^2m^4} + o(p^4)$$

It is apparent then that again these equations are corrections to the classical relations.

4. Stable laws: for the non Gaussian stable laws we have the lch (22) and the Lévy measure pdf (23) where $0 < \lambda < 2$. With a couple of exception (Cauchy and Lévy laws), however, there are no elementary formulas for their pdf’s $q(x)$. Their Lévy–Schrödinger equation is

$$i\alpha \partial_t \psi(x, t) = \frac{\alpha}{\tau} \frac{a^\lambda}{2\lambda \cos \frac{\lambda\pi}{2} \Gamma(-\lambda)} \int_{\mathbb{R}} \frac{\psi(x + y, t) - \psi(x, t)}{|y|^\lambda+1} dy$$

and with the identifications (27) the energy–momentum relations become

$$E = \frac{\alpha}{\tau} \frac{(a|u|)^\lambda}{\lambda} = \frac{2^{\lambda/2}}{\lambda} \left( \frac{ma^2}{\tau^2} \right)^{1-\lambda/2} \left( \frac{p^2}{2m} \right)^{\lambda/2}$$  \hspace{1cm} (28)
This however looks not as a correction of the classical formula as in the other cases considered, but rather as a completely different formula. The same can be said of the kinematic relations between $\dot{q}$ and $p$ which are now

$$\dot{q} = \frac{p}{m} \left( \frac{p^2 \tau^2}{m^2 a^2} \right)^{\lambda/2 - 1}$$  \hspace{1cm} (29)

The relations (28) and (29) are still another reason to consider not advisable to restrict an inquiry on Lévy processes and Schrödinger equation only to the family of stable processes.

5. Compound Poisson process: for the symmetric, compound Poisson process with the Lévy measure (24) the pure jump Lévy–Schrödinger equation (19) greatly simplifies as

$$i \partial_t \psi(x, t) = -\frac{\lambda \omega a^2}{2} \left[ \psi(x + a, t) - 2 \psi(x, t) + \psi(x - a, t) \right].$$

We can now show that the usual Schrödinger equation (1) can always be recovered as a limit case of this Poisson–Schrödinger equation. If $\psi$ is twice differentiable in $x$, we know that for $a \to 0^+$ we have

$$\psi(x \pm a, t) = \psi(x, t) \pm a \partial_x \psi(x, t) + \frac{a^2}{2} \partial_x^2 \psi(x, t) + o(a^2)$$

and hence

$$i \partial_t \psi(x, t) = -\frac{\lambda \omega a^2}{2} \partial_x^2 \psi(x, t) + \lambda o(a^2).$$

Now, if, as $a \to 0^+$, also $\lambda \to +\infty$ in such a way that $\lambda a^2 \to b^2$, then we have

$$i \partial_t \psi(x, t) = -\frac{\omega b^2}{2} \partial_x^2 \psi(x, t)$$

where $\omega b^2/2$ has the dimensions of a diffusion coefficient: namely, in the limit, we get a Wiener–Schrödinger equation of the type (1). This procedure, for instance, could also be used to introduce small corrections in the coefficient $\hbar^2/2m$ of a Schrödinger equation.

4.4 Perfectly rigid walls

An example of solution of the complete Lévy–Schrödinger equation (18) can easily be obtained in the case of a system confined between two perfectly rigid walls symmetrically located at $x = \pm L/2$. The discussion is similar to that of Section 4.1, but for the boundary conditions which now require that the solutions vanish at $x = \pm L/2$. As a consequence the solutions are the usual trigonometric functions with discrete eigenvalues

$$E_n = -\frac{\alpha}{\tau} \eta(u_n), \quad u_n = \frac{n \pi}{L}, \quad n = 1, 2, \ldots$$

The form of the eigenvalue sequence will depend on the $\eta$. If the underlying process is a Wiener process we have $\eta(u) = -\beta^2 u^2/2$, and hence

$$E_n = \frac{\alpha \beta^2 \pi^2}{2 \tau L^2} n^2$$
which, with the identifications $\alpha = \hbar$ and $\beta^2 = \alpha \sigma / m$, coincides with the usual quantum mechanical result. On the other hand for an underlying Variance–Gamma noise we find

$$E_n = \frac{\lambda \alpha}{\tau} \ln \left(1 + \frac{a^2 \pi^2}{L^2 n^2}\right).$$

Finally for a symmetric, compound Poisson noise with intensity $\lambda$ and equiprobable jump sizes $\pm a$ we get

$$E_n = \frac{\lambda \alpha}{\tau} \left(1 - \cos \frac{a \pi n}{L}\right).$$

Apparently this no longer is a monotone increasing sequence: $E_n$ goes up and down between $0$ and $2 \alpha \lambda / \tau$. If $a / L$ is rational the sequence is periodic; on the other hand when $a / L$ is irrational there are no coincident eigenvalues in the sequence.

5 Conclusions

We have discussed the possibility of generalizing the relation between Brownian motion and Schrödinger equation by formally associating the kinetic energy of a more general system to the generator of a symmetric Lévy process, namely to a pseudodifferential operator whose symbol is the $\text{lch}$ of an infinitely divisible law. This amounts to suppose, then, that this new Lévy–Schrödinger equation is based on an underlying Lévy process that can have both Gaussian and jumping components.

In recent years other extensions of the Schrödinger equation have been put forward in the same spirit of our Lévy–Schrödinger equation. In particular we refer to several papers about fractional Schrödinger equations [20] that explored the use of fractional calculus in a kind of generalized quantum mechanics. As it is clear from the previous sections, however, this is the particular case when our underlying process is stable. The extension to the infinitely divisible, non stable processes, on the other hand, is meaningful because there are significant cases that are now in the domain of our Lévy–Schrödinger picture. In particular we have shown that the simplest form of a relativistic, free Schrödinger equation can be deduced from a particular type of selfdecomposable, non stable process. Moreover in many instances the new energy–momentum relations can be seen as corrections to the classical relations for small values of certain parameters. It must also be remembered that our model is not tied to the use of processes with infinite variance: the variances can be chosen to be finite even in a purely non Gaussian model, and can then be used as a measure of the dispersion.

It is important now to explicitly give in full detail a derivation of the Lévy–Schrödinger equation from either the Feynman integrals or a generalized stochastic mechanics. This seems to be possible because the techniques of the stochastic calculus applied to Lévy processes are today in full development [8, 9, 10, 11, 12], and at our knowledge there is no apparent, fundamental impediment along this road. At present our approach lacks this rigorous discussion of how the Lévy–Schrödinger equation comes out from the evolution equations of the Lévy processes. We have confined ourselves to give only a few heuristic arguments based on both the identification of the process generators as the kinetic energy operators, and the analytic continuation of the time variable $t$ to its imaginary counterpart $it$. We hinted, however, to two possible ways
of giving a more rigorous derivation: we can first of all follow the Feynman integral road. In this case we should bear in mind that the relations among kinetic energy and momentum are no longer the usual relations: this is important to correctly write the Lagrangian in the Feynman integral. Alternatively we can try to generalize Nelson stochastic mechanics by adding a suitable dynamics to our Lévy processes, and this will be the subject of a future paper.

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