DEGENERATIONS OF COMPLEX DYNAMICAL SYSTEMS II:
ANALYTIC AND ALGEBRAIC STABILITY

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Abstract. The first article in this series exhibited uniqueness of the weak limit of the
equilibrium measures for a degenerating 1-parameter family of rational functions on the
Riemann sphere. Here we construct a convergent countable-state Markov chain that com-
putes the limit measure. Our technique is combinatorial in nature and may be applied to
compute the location of mass for the equilibrium measure of a non-Archimedean rational
function, under a certain stability hypothesis. As a byproduct, we deduce that meromorphic
maps preserving the fibers of a rationally-fibered complex surface are algebraically stable
after a proper modification.

1. Introduction

In the preceding article [10], we provided a new link between the dynamics of a degenerat-
ing 1-parameter family \( f_t \) of complex rational functions and the dynamics of the associated
rational function \( f \) on the Berkovich projective line \( \mathbf{P}^1 = \mathbf{P}^1_{\mathbb{L}} \). Our main tool was a quantiza-
tion technique: the family \( f_t \) gave rise to a sequence of partitions of \( \mathbf{P}^1 \), and the dynamics of
\( f \) relative to these partitions provided information on the limit of the equilibrium measures
\( \mu_t \) of \( f_t \) as \( t \to 0 \).

In the present article, we refine this quantization technique by constructing a discrete
dynamical system that, under a certain stability hypothesis, allows us to compute the limiting
measures. Specifically, the quantized equilibrium measure for a rational function \( f : \mathbf{P}^1 \to \mathbf{P}^1 \)
is the stationary distribution for an explicit countable-state Markov chain. (See, e.g., [18]
for definitions and relevant background for this type of random process.) The stability
hypothesis parallels the notion of algebraic stability for dynamics on complex surfaces; we
obtain a corollary on the existence of algebraically stable modifications for meromorphic
maps preserving the fibers of a rational fibration.

More precisely, let \( k \) be an algebraically closed field that is complete with respect to a
nontrivial non-Archimedean valuation. For the next result, we do not need any hypothesis
on the characteristic or residue characteristic of \( k \). Let \( f \) be a rational function of degree
\( d \geq 2 \) with coefficients in \( k \). A vertex set \( \Gamma \) — a finite set of type II points in \( \mathbf{P}^1_k \) — gives
rise to a partition of \( \mathbf{P}^1_k \) consisting of the elements of \( \Gamma \) and the connected components of the
complement \( \mathbf{P}^1_k \setminus \Gamma \). If the pair \( (f, \Gamma) \) is “analytically stable” (see Definition 2.7), then there
is an explicit countable subset \( J \) of the partition that contains the Julia set for \( f \) and satisfies
the following property: For any pair of subsets \( U, V \in J \), the number of pre-images \( \# (f^{-1}(y) \cap V) \) is
independent of \( y \in U \), when counted with multiplicities. To each pair of subsets \( U, V \in J \),

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we define a quantity $P_{U,V} \in [0,1]$ by
\[
P_{U,V} = \frac{\# (f^{-1}(y) \cap V)}{d} \quad (y \in U).
\]
Let $P$ be the $|\mathcal{J}| \times |\mathcal{J}|$ matrix with $(U,V)$-entry $P_{U,V}$. Recall that a stationary probability vector for $P$ is a row vector $\nu : \mathcal{J} \to [0,1]$ whose entries sum to 1 and which satisfies $\nu P = \nu$.

**Theorem C.** Let $k$, $f$, $\Gamma$ be as above. Suppose that the pair $(f, \Gamma)$ is analytically stable and that the Julia set for $f$ is not contained in the finite set $\Gamma$. Then $P$ is the transition matrix for a countable state Markov chain with a unique stationary probability vector $\nu : \mathcal{J} \to [0,1]$. The rows of $P^n$ converge pointwise to $\nu$, and the $U$-entry of $\nu$ satisfies $\nu(U) = \mu_f(U)$ for each $U \in \mathcal{J}$, where $\mu_f$ is the equilibrium measure for $f$.

We provide example computations of $P$ and the stationary measure $\nu$ in Section 5, and we explain how Theorem C may be viewed as a generalization of the methods in [9] for degenerating families of complex rational maps.

**Remark 1.1.** The hypothesis that the Julia set is not contained in $\Gamma$ is necessary; see Example 5.2. However, it fails if and only if there is a totally $f$-invariant point $\zeta \in \Gamma$, in which case $\mu_f$ is supported at $\zeta$. This condition is easily verified in practice.

**Remark 1.2.** The transition matrix $P$ is a discretized version of the pullback operator $\frac{1}{d} f^* \mu$ acting on probability measures on $\mathbb{P}^1$.

In Section 2, we will define and develop the notion of analytic stability for a rational function $f \in k(z)$ and a vertex set $\Gamma \subset \mathbb{P}^1_k$. This facilitates the construction of the Markov chain $P$. Its $(U,V)$-entry may be interpreted as the probability of randomly passing from state $U$ to state $V$ under $f^{-1}$. Convergence of the Markov chain $P$ is then essentially a combinatorial reformulation of the equidistribution result of Favre and Rivera-Letelier on iterated pre-images [16].

The most remarkable fact is that such a combinatorial description of the dynamics of $f$ exists at all, and the analytic stability hypothesis is absolutely critical in this respect. However, it is not a major restriction for many applications. We show that any vertex set $\Gamma$ may be augmented to be analytically stable, under a suitable hypothesis on the field of definition of the pair $(f, \Gamma)$ and the ramification of $f$. Following Trucco [22], we say that a rational function $f$ is **tame** if its ramification locus is contained in the connected hull of the (type I) critical points. For example, if the residue characteristic of $k$ is zero, or if the residue characteristic is $p > \deg(f)$, then $f$ is tame.

**Theorem D.** Let $\ell$ be a discretely valued field, and let $f \in \ell(z)$ be a tame rational function of degree $d \geq 2$. Let $k$ be a minimal complete and algebraically closed non-Archimedean extension of $\ell$. For any vertex set $\Gamma$ in $\mathbb{P}^1_k$, there exists a vertex set $\Gamma'$ containing $\Gamma$ such that $(f, \Gamma')$ is analytically stable. Moreover, if every element of $\Gamma$ is $\ell$-split (Definition 3.2), then one may take $\Gamma'$ to have the same property.

A key ingredient in the proof of Theorem D is a classification of Fatou components for Berkovich dynamical systems due to Rivera-Letelier [21]. The proof also uses a “No Wandering Domains” result of Benedetto [4], from which we deduce that all Type II points in the Julia set are preperiodic (under the hypotheses of Theorem D); see Proposition 3.9. The
general strategy of the proof should carry over to $p$-adic fields, but the corresponding result on wandering domains is not known in full generality in residue characteristic $p$ [3].

As an additional application of the previous theorems, we look at the question of when a meromorphic map $F : X \dashrightarrow X$ of a complex surface can be resolved to be algebraically stable by a proper modification of $X$. Recall that $F$ is algebraically stable if there does not exist a curve $C$ such that $F^n(C)$ is collapsed to an indeterminacy point of $F$ for some $n \geq 1$. The maps we consider preserve a rational fibration, and so locally take the form

$$(t, x) \mapsto (t, f_t(x)),$$

where $f_t$ is a meromorphic family of (complex) rational functions, with $t$ in the unit disk $\mathbb{D}$. Our Theorem D implies the following statement, in which $\hat{\mathbb{C}}$ denotes the Riemann sphere.

**Theorem E.** Let $f_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic family of rational functions for $t \in \mathbb{D}$ with $\deg(f_t) = d \geq 2$ for $t \neq 0$. Let $\pi : X \to \mathbb{D}$ be a normal connected surface with projective fibers such that $\pi^{-1}(\mathbb{D}^*) \cong \mathbb{D}^* \times \hat{\mathbb{C}}$ and $X_0 = \pi^{-1}\{0\}$ is reduced. Consider the rational map $F : X \dashrightarrow X$ defined by $(t, z) \mapsto (t, f_t(z))$. There exists a proper modification $Y \to X$ that restricts to an isomorphism over $\mathbb{D}^*$ such that the induced rational map $F_Y : Y \dashrightarrow Y$ is algebraically stable.

In the context of complex projective surfaces, algebraic stability of $F$ is equivalent to the condition that $(F^*)^n = (F^n)^*$ for all $n \geq 1$ as operators on the Picard group. If $F$ is bimeromorphic, then Diller and Favre have shown that an algebraically stable modification always exists [11]. Favre provided examples of monomial maps that show that this is not necessarily the case when $F$ is not birational [14]. However, Favre and Jonsson have shown that for polynomial maps $F : \mathbb{C}^2 \to \mathbb{C}^2$, a projective compactification $X$ of $\mathbb{C}^2$ always exists for which some iterate of $F : X \dashrightarrow X$ is algebraically stable [15]. We note that the latter article uses dynamics on a valuation space similar to the Berkovich line in order to deduce the existence of an algebraically stable resolution; the dynamics, and consequently their arguments, are of a very different flavor than those used here. The new article of Diller and Lin also addresses the existence of algebraically stable modifications for rational maps on surfaces [12].

Finally, we observe that there is a connection between our notion of analytic stability and the arithmetic-dynamical notion of weak Néron model [17, 8]. Let $\ell$ be a discretely valued field with valuation ring $\mathcal{O}$, and let $f \in \ell(z)$ be a rational function. A weak Néron model for $(\mathbb{P}^1_\ell, f)$ is a pair $(\mathcal{X}, F)$ consisting of a regular semistable $\mathcal{O}$-scheme $\mathcal{X}$ and a rational map $F : \mathcal{X} \to \mathcal{X}$ such that:

- The generic fiber of $\mathcal{X}$ is $\mathbb{P}^1_\ell$;
- $F$ restricts to $f$ on the generic fiber; and
- $F$ restricts to a morphism on the smooth locus $\mathcal{X}^{\text{sm}}$.

(This definition is slightly stronger than the one in *loc. cit.*, but the proofs implicitly assume they are equivalent.) Using the arguments in the present paper, one can show the following: Given a weak Néron model for $(\mathbb{P}^1_\ell, f)$, one may associate a canonical vertex set $\Gamma$ such that $(f, \Gamma)$ is analytically stable, and the associated probability vector of Theorem C has finite support. The converse is not true, though, as illustrated by Examples 5.3 and 5.4. Both examples deal with a quadratic polynomial $f$ that admits repelling fixed points defined over the field $\ell = \mathbb{C}(\sqrt{t})$, which preclude the existence of a weak Néron model for $(\mathbb{P}^1_\ell, f)$ [17].
Benedetto and Hsia have independently found the connection between weak Néron models and certain special vertex sets [5].

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2. Markov chains and equilibrium measures

Our goal in this section is to prepare and prove Theorem C. In §2.1 we provide a count of preimages in simple domains in terms of locally defined multiplicities. We define a discrete counterpart to the Fatou and Julia sets in §2.2, and in §2.3 we define the notion of analytic stability and explore some of its properties. The final subsection contains the proof of Theorem C.

Convention. Throughout this section, we let \( k \) be an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value. Note that we do not assume anything about the characteristic or residue characteristic of \( k \). The Berkovich projective line over \( k \) will be denoted \( \mathbb{P}^1_k \) for brevity. An open disk \( D \) in \( \mathbb{P}^1_\mathbb{C} \) is an open set with a unique boundary point, say \( x \). If \( \vec{v} \in T\mathbb{P}^1_\mathbb{C} \) is the inward tangent vector defining \( D \), we write \( D = D(\vec{v}) \). For a nonconstant rational function \( f \in k(z) \), we write \( m_f \) and \( s_f \) for the local degree and surplus multiplicity, respectively. (See, e.g., [13, §3].) For an open disk \( D = D(\vec{v}) \) in \( \mathbb{P}^1 \) with boundary point \( x \), we set \( \bar{f}(D) := D(Tf(\vec{v})) \), where \( Tf : T\mathbb{P}^1_\mathbb{C} \to T\mathbb{P}^1_\mathbb{C} \) is the action on the tangent space. Note that \( \bar{f}(D) = f(D) \) if and only if the surplus multiplicity \( s_f(D) \) is zero.

2.1. Counting preimages. A simple domain \( V \subset \mathbb{P}^1 \) is an open set with finitely many boundary points, all of which must be of type II or type III. Equivalently, a simple domain is the intersection of finitely many open disks \( V_i \), where \( \partial V_i = \{ x_i \} \) is a type II or III point for each index \( i \). If each \( V_i \) has inward tangent vector \( \vec{v}_i \) at \( x_i \), then we can also write \( V = \cap D(\vec{v}_i) \).

Proposition 2.1. Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational function of degree \( d \geq 1 \), and let \( V \) be a simple domain written as the intersection of \( n \) open disks \( V_1, \ldots, V_n \). Then for any \( y \in \mathbb{P}^1 \),

\[
\# (f^{-1}(y) \cap V) = \sum_{i : y \in f(V_i)} m_f(V_i) + \sum_{i=1}^{n} s_f(V_i) - d(n-1),
\]

where all preimages are counted with multiplicity.

Remark 2.2. The image of \( V \) under \( f \) determines a partition of \( \mathbb{P}^1 \) into sets

\[
V_{f,I} := \bigcap_{i \in I} \bar{f}(V_i) \setminus \bigcup_{i \notin I} \bar{f}(V_i)
\]

as \( I \) ranges over all subsets of \( \{1, \ldots, n\} \). The proposition implies that the function \( y \mapsto \# (f^{-1}(y) \cap V) \) is constant on each of these subsets.
Proof. For each \( y \in \mathbb{P}^1 \), let \( I(y) \) denote the (possibly empty) set of indices in \( \{1, \ldots, n\} \) such that \( i \in I(y) \) if and only if \( y \in f(V_i) \). Properties of the local degree and surplus multiplicity of an open disk [13, Prop. 3.10] imply that

\[
\# (f^{-1}(y) \cap V_i) = \varepsilon(i, I(y)) \cdot m_f(V_i) + s_f(V_i)
\]

for each \( i = 1, \ldots, n \). Here \( \varepsilon(\cdot, I) \) denotes the indicator function for a set \( I \) of indices.

Choose an index \( j \in \{1, \ldots, n\} \). The set of pre-images \( f^{-1}(y) \cap V \) is precisely the set of pre-images in \( V_j \) less the pre-images in the complement of \( V_i \) for each index \( i \neq j \). Therefore,

\[
\# (f^{-1}(y) \cap V) = \# (f^{-1}(y) \cap V_j) - \sum_{i \neq j} \# (f^{-1}(y) \cap V_i^c)
\]

\[
= \varepsilon(j, I(y)) m_f(V_j) + s_f(V_j) - \sum_{i \neq j} \left( d - \varepsilon(i, I(y)) m_f(V_i) + s_f(V_i) \right)
\]

\[
= \sum_{i \in I(y)} m_f(V_i) + \sum_{i=1}^n s_f(V_i) - d(n - 1). \quad \Box
\]

2.2. **Vertex sets.** We now define a discretized counterpart to the Fatou and Julia sets. It is not an exact analogue, though the canonical measure for \( f \) is always supported inside the discrete counterpart of the Julia set.

**Definition 2.3.** A **vertex set** for \( \mathbb{P}^1 \) is a finite nonempty set of type II points, which we typically denote by \( \Gamma \). The connected components of \( \mathbb{P}^1 \setminus \Gamma \) will be referred to as **\( \Gamma \)-domains**. As a special case, when a \( \Gamma \)-domain has one boundary point, we call it a **\( \Gamma \)-disk**. Write \( S(\Gamma) \) for the partition of \( \mathbb{P}^1 \) consisting of the elements of \( \Gamma \) and all of its \( \Gamma \)-domains.

**Definition 2.4.** Let \( \Gamma \) be a vertex set for \( \mathbb{P}^1 \), and let \( f: \mathbb{P}^1 \to \mathbb{P}^1 \) be a rational function with \( \deg(f) \geq 2 \). A \( \Gamma \)-domain \( U \) will be called an **\( F \)-domain** if \( f^n(U) \cap \Gamma \) is empty for all \( n \geq 1 \), and otherwise \( U \) will be called a **\( J \)-domain**. If \( U \) is a \( \Gamma \)-disk, it will be called an **\( F \)-disk** or a **\( J \)-disk**, respectively. Write \( J(\Gamma) \subset S(\Gamma) \) for the subset consisting of all \( J \)-domains and the elements of \( \Gamma \).

While the partition \( S(\Gamma) \) is typically uncountable, the set \( J(\Gamma) \) is much smaller.

**Lemma 2.5.** For a given vertex set \( \Gamma \), the set \( J(\Gamma) \) is countable.

**Proof.** Since \( \Gamma \) is finite, it suffices to show that the set of \( J \)-domains is countable. For each integer \( n \geq 1 \), write

\[
J(\Gamma)_n = \{ U \text{ a } J\text{-domain} : f^n(U) \cap \Gamma \neq \emptyset, f^i(U) \cap \Gamma = \emptyset \text{ for } i = 1, \ldots, n - 1 \}. 
\]

By definition, each \( J \)-domain lies in some \( J(\Gamma)_n \). Since \( \Gamma \) is finite, and since each vertex has at most \( d^n \) distinct pre-images under \( f^n \), it follows that \( J(\Gamma)_n \) is finite for each \( n \geq 1 \). \( \Box \)

**Proposition 2.6.** The Julia set for \( f \) is contained in the union of the sets in \( J(\Gamma) \).

**Proof.** If \( U \) is an \( F \)-domain, then \( \Gamma \) does not intersect the union of the forward iterates \( \bigcup_{n \geq 0} f^n(U) \), so that \( U \) must lie in the Fatou set for \( f \). Now the union of the \( F \)-domains is contained in the Fatou set for \( f \), and taking complements shows that the Julia set is contained in the union of \( \Gamma \) with all of the \( J \)-domains. \( \Box \)
2.3. **Analytic stability.** The following definition parallels the one used in the theory of rational self-maps of complex surfaces. (We will discuss the correspondence in Section 4.)

**Definition 2.7.** Let $\Gamma$ be a vertex set for $\mathbb{P}^1$, and let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a rational function with $\deg(f) \geq 2$. One says that the pair $(f, \Gamma)$ is **analytically stable** if for each $\zeta \in \Gamma$, either $f(\zeta) \in \Gamma$ or $f(\zeta) \in U$ for some $F$-domain $U$.

**Convention.** For the remainder of Section 2, we will fix a vertex set $\Gamma$ and a rational function $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ such that $(f, \Gamma)$ is analytically stable.

**Lemma 2.8.** For each $F$-domain $U$ there is another $F$-domain $V$ such that $f(U) \subset V$.

**Proof.** Suppose that $f(U)$ is not contained in an $F$-domain. Then it is either contained in a $J$-domain $V$, or it is not fully contained in any $\Gamma$-domain. In the former case, no boundary point of $U$ can map into $V$ since $(f, \Gamma)$ is analytically stable. Hence $f(U) = V$, which forces $U$ to be a $J$-domain. In the latter case $f(U)$ must contain an element of $\Gamma$, which again implies that $U$ is a $J$-domain. In either case we have reached a contradiction. $\square$

**Lemma 2.9.** Let $U, V \in \mathcal{J}(\Gamma)$ be $J$-domains or vertices. Then the function

$$y \mapsto \#(f^{-1}(y) \cap V)$$

is constant on $U$, where pre-images are counted with multiplicities.

**Proof.** If $U$ is an element of $\Gamma$, then $U$ is a singleton and the result is evident. If $U$ is a $J$-domain, then $f(\Gamma) \cap U = \emptyset$ by analytic stability, and $\#(f^{-1}(y) \cap V) = 0$ whenever $V \in \Gamma$.

Now suppose that $U$ and $V$ are $J$-domains. We may write $V$ as the intersection of $n$ open disks $V_1, \ldots, V_n$. By Proposition 2.1 (and the following remark), it suffices to show that $U$ lies in one of the subsets $V_{i,I}$ for an index set $I \subset \{1, \ldots, n\}$. If no such index set exists, then $U$ must contain a boundary point of some $V_{i,J}$, say $y$, since $U$ is connected. But $y$ is the boundary point of one of the open disks $\tilde{f}(V_j)$, so that the vertex $\zeta \in \partial V_j$ satisfies $y = f(\zeta) \in U$. That is, $f(\Gamma) \cap U \neq \emptyset$, a contradiction to analytic stability. $\square$

**Definition 2.10.** For each pair of elements $U, V \in \mathcal{J}(\Gamma)$, we define an integer $m_{U,V} \in \{0, \ldots, d\}$ as follows. For $y \in U$, set

$$m_{U,V} = \#(f^{-1}(y) \cap V).$$

By Lemma 2.9, $m_{U,V}$ is independent of the choice of $y$.

The multiplicities $m_{U,V}$ are only well defined in the presence of analytic stability. Moreover, they are combinatorial in the sense that they can be computed via local mapping degrees and surplus multiplicities at points of $\Gamma$. More precisely, Proposition 2.1 and the proof of Lemma 2.9 show that:

$$m_{U,V} = \begin{cases} m_f(V) & \text{for } U, V \in \Gamma \text{ with } f(V) = U \\ \sum_{\zeta \in V : f(\zeta) = U} m_f(\zeta) & \text{for } U \in \Gamma \text{ and } J\text{-domain } V \\ \sum_{i: U \subset f(V_i)} m_f(V_i) + \sum_{i=1}^n s_f(V_i) - d(n-1) & \text{for } J\text{-domains } U \text{ and } V = \bigcap_{i=1}^n V_i \\ 0 & \text{otherwise} \end{cases}$$
The (non-negative integer valued) quantities \( m_f(V), m_f(V_i), \) and \( s_f(V_i) \) may be determined algorithmically via reductions of \( f \) in various coordinates; we stress that this is a finite computation.

**Lemma 2.11.** For each \( U \in \mathcal{J}(\Gamma) \), we have
\[
\sum_{V \in \mathcal{J}(\Gamma)} m_{U,V} = \deg(f).
\]

**Proof.** Let \( y \in U \) be any point. Observe that, by analytic stability and Lemma 2.8, each pre-image of \( y \) must be a vertex or else lie in a \( J \)-domain. The result now follows:
\[
\sum_{V \in \mathcal{J}(\Gamma)} m_{U,V} = \sum_{V \in \mathcal{J}(\Gamma)} \#(f^{-1}(y) \cap V) = \#f^{-1}(y) = \deg(f).
\]

2.4. The proof of Theorem C. We restate Theorem C now that we have all of the necessary definitions in hand.

**Theorem C.** Let \( k \) be an algebraically closed field that is complete with respect to a nontrivial non-Archimedean absolute value. Let \( f : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) be a rational function defined over \( k \) of degree \( d \geq 2 \), and let \( \Gamma \) be a vertex set in \( \mathbb{P}^1 \). Suppose that \((f, \Gamma)\) is analytically stable and that the Julia set for \( f \) is not contained in \( \Gamma \). Writing \( \mathcal{J} = \mathcal{J}(\Gamma) \) for the set of \( J \)-domains and vertices of \( \Gamma \), we define a \(|\mathcal{J}| \times |\mathcal{J}|\) matrix \( P \) with \((U,V)\)-entry
\[
P_{U,V} = \frac{m_{U,V}}{d}.
\]
Then \( P \) is the transition matrix for a countable state Markov chain with a unique stationary probability vector \( \nu : \mathcal{J} \rightarrow [0,1] \). The rows of \( P^n \) converge pointwise to \( \nu \), and the \( U \)-entry of \( \nu \) satisfies \( \nu(U) = \mu_f(U) \) for each \( U \in \mathcal{J} \), where \( \mu_f \) is the equilibrium measure for \( f \).

The following lemma makes the necessary connection between iterated pullback via \( f \) and matrix multiplication.

**Lemma 2.12.** Fix a \( J \)-domain or vertex \( U_0 \in \mathcal{J}(\Gamma) \). For each \( n \geq 1 \), each \( V \in \mathcal{J}(\Gamma) \), and each \( y \in U_0 \), we have
\[
\sum_{x \in V} m_{f^n}(x) = \sum_{U_1,\ldots,U_{n-1} \in \mathcal{J}(\Gamma)} m_{U_0,U_1} \cdot m_{U_1,U_2} \cdot m_{U_2,U_3} \cdots m_{U_{n-1},V}.
\]

**Proof.** For ease of notation, let us write \( \mathcal{J} = \mathcal{J}(\Gamma) \). The proof proceeds by induction. The case \( n = 1 \) follows immediately from the definitions:
\[
\sum_{x \in V} m_f(x) = \#(f^{-1}(y) \cap V) = m_{U_0,V}.
\]

Now suppose that the result holds for some \( n \geq 1 \), and let us deduce it for \( n+1 \). We decompose the morphism \( f^{n+1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1 \) as \( f^n \circ f \). Since the local degree \( m_f(\cdot) \) is multiplicative, we see that
\[
\sum_{x \in V} m_{f^{n+1}}(x) = \sum_{z : f^n(z) = y} m_{f^n}(z) \left( \sum_{x : f(x) = z} m_f(x) \right).
\]
Let $z_n$ be a solution to the equation $f^n(z) = y$. As $y \in U_0$ and $U_0 \in \mathcal{J}(\Gamma)$, it follows that $z_n$ is either or a vertex or else lies in some $J$-domain, say $U_n$. Grouping the terms of the first sum above according to the distinct $J$-domains and vertices $U_n \subset f(V)$, we find that

$$\sum_{x \in V} m_{f^{n+1}}(x) = \sum_{U_n \in \mathcal{J}} \sum_{z : f^n(z) = y \in U_n} m_{f^n}(z) \cdot \# (f^{-1}(z) \cap V)$$

$$= \sum_{U_n \in \mathcal{J}} m_{U_n,V} \left( \sum_{z : f^n(z) = y \in U_n} m_{f^n}(z) \right)$$

$$= \sum_{U_n \in \mathcal{J}} m_{U_n,V} \sum_{U_1, \ldots, U_{n-1} \in \mathcal{J}} m_{U_0,U_1} \cdot m_{U_1,U_2} \cdot m_{U_2,U_3} \cdots m_{U_{n-1},U_n}$$

$$= \sum_{U_1, \ldots, U_n \in \mathcal{J}} m_{U_0,U_1} \cdot m_{U_1,U_2} \cdot m_{U_2,U_3} \cdots m_{U_{n},V}.$$ The second equality follows from the independence of $\# (f^{-1}(z) \cap V)$ in $z \in U_n$, while the second to last equality uses the induction hypothesis. Note that $m_{U_n,V} = 0$ if $U_n$ is not contained in the image $f(V)$.

**Proof of Theorem C.** For ease of exposition, we present the proof in several steps.

**Step 1:** The quantities $P_{U,V}$ define a countable state Markov chain on the state space $\mathcal{J}$. It is clear the $0 \leq P_{U,V} \leq 1$ for all $U,V \in \mathcal{J}$, and we have already seen that $\mathcal{J}$ is countable (Lemma 2.5). It remains to show that for each $U \in \mathcal{J}$, we have

$$\sum_{V \in \mathcal{J}} P_{U,V} = 1.$$ This is precisely the content of Lemma 2.11.

**Step 2:** The vector $\omega_f : \mathcal{J} \to [0,1]$ given by $\omega_f(U) = \mu_f(U)$ is a stationary probability vector for $P$. We have already seen that each $F$-domain lies in the Fatou set of $f$ (Proposition 2.6), so that $\mu_f$ does not charge $F$-domains. Hence $\sum_{U \in \mathcal{J}} \omega_f(U) = 1$. The measure $\mu_f$ satisfies the pullback relation $f^* \mu_f = d \cdot \mu_f$ as Borel measures on $\mathbb{P}^1$ [16]. Integrating this formula against the characteristic function of $V$ yields

$$\mu_f(V) = \frac{1}{d} \int_{\mathbb{P}^1} \# (f^{-1}(y) \cap V) \mu_f(y) = \sum_{U \in \mathcal{J}} \frac{m_{U,V}}{d} \cdot \mu_f(U) = \sum_{U \in \mathcal{J}} P_{U,V} \cdot \mu_f(U).$$ Evidently this is equivalent to $\omega = \omega P$.

**Step 3:** For each $n \geq 1$, each pair of subsets $U,V \in \mathcal{J}$, and each $y \in U$, we have

$$\left( P^n \right)_{U,V} = \int_V d^{-n} (f^*)^n \delta_y.$$ This statement is equivalent to the one given in Lemma 2.12.

**Step 4:** The invariant measure $\mu_f$ does not charge $\Gamma$. In general, $\mu_f$ can only charge a point of $\mathbb{P}^1$ if $f$ has simple reduction [2, Cor. 10.47], in which case $\mu_f = \delta_\zeta$ for some type II point.
\( \zeta \in \mathbb{P}^1 \). Here the Julia set is \( \mathbf{J}(f) = \{ \zeta \} \). Our hypotheses ensure that \( \zeta \not\in \Gamma \) if \( f \) has simple reduction. So \( \mu_f(\Gamma) = 0 \).

**Step 5:** The matrix powers \( P^n \) converge entry-by-entry to \( 1 \omega_f \), where \( 1 \) is the column vector of 1’s and \( \omega_f(\U) = \mu_f(\U) \) as in Step 4. Equivalently, \( (P^n)_{\U,V} \to \mu_f(V) \) as \( n \to \infty \), where \( (P^n)_{\U,V} \) is the \((\U, V)\)-entry of \( P^n \).

Fix a \( J \)-domain or vertex \( \U \) and choose any point \( y \in \U \) that is not a \( k \)-rational exceptional point for \( f \). Favre and Rivera-Letelier’s equidistribution of iterated pre-images \([2, \text{ Thm. 10.36}]\) shows that \( d^{-n} (f^n)^* \delta_y \to \mu_f \) weakly as \( n \to \infty \). (This is weak converge of Borel measures on \( \mathbb{P}^1 \)). Since \( \mu_f \) does not charge the boundary of \( V \) (Step 4), the result in Step 3 shows that

\[
(P^n)_{\U,V} = \int_{\V} d^{-n} (f^n)^* \delta_y \to \mu_f(V).
\]

**Step 6:** If \( \nu \) is a probability vector such that \( \nu P = \nu \), then \( \nu = \omega_f \) (as in Step 2). By induction, we have \( \nu = \nu P^n \) for each \( n \geq 1 \). But then letting \( n \to \infty \) gives

\[
\nu = \nu P^n \to \nu(1 \omega_f) = (\nu 1) \omega_f = \omega_f.
\]

Note that passage to the limit involves interchanging the (possibly infinite) sum defining \( \nu P^n \) and the limit of the sequence \( P^n \). This is justified by dominated convergence and the fact that each entry of \( P^n \) is bounded above by 1. Associativity is a similar consideration.

\[
\square
\]

3. **Analytically stable augmentations of vertex sets**

The goal of this section is to prove that (under suitable hypotheses) a vertex set in the Berkovich projective line can be enlarged to yield one that is analytically stable with respect to a given rational function. Our main application will be in the setting of non-Archimedean fields of residue characteristic zero, so we will concentrate our efforts on tame rational functions.

**Convention.** Throughout this section, we let \( \ell \) be a discretely valued field with completion \( \hat{\ell} \), and we let \( k \) be the completion of an algebraic closure of \( \hat{\ell} \). We write \( \mathbb{P}^1 = \mathbb{P}^1_k \). For \( a \in k \) and \( r \in \mathbb{R}_{>0} \), we write \( \mathcal{D}(a, r)^c \) and \( \mathcal{D}(a, r) \) for the open and closed Berkovich disks centered at \( a \) of radius \( r \), respectively.

**Remark 3.1.** For our application to rational maps on fibered complex surfaces, we will take \( \ell \) to be the field of complex functions that are meromorphic at the origin in \( \mathbb{C} \). Then \( \hat{\ell} = \mathbb{C}((t)) \) is the field of formal Laurent series in a local coordinate \( t \), and \( k = \mathbb{L} \) is the completion of the field of formal Puiseux series.

**Definition 3.2.** We say that a type II point \( \zeta \in \mathbb{P}^1 \) is \( \ell \)-split if it is of the form \( \zeta = \zeta_{a,r} \) for some \( a \in \ell \) and \( r \in [\ell^\times] \) — i.e., if it is the supremum norm associated to an \( \ell \)-rational closed disk \( \{ x \in k : |x - a| \leq r \} \). A vertex set \( \Gamma \) is \( \ell \)-split if each of its elements is \( \ell \)-split.

**Remark 3.3.** By a density argument, a type II point is \( \ell \)-split if and only if it is \( \hat{\ell} \)-split.

**Remark 3.4.** A type II point \( \zeta \) canonically induces a norm on the function field \( k(T) \) of \( \mathbb{P}^1_k \). Following Berkovich, we write \( |f|_\zeta \) for the norm of \( f \in k(T) \); it is valued in \( |k^\times| \). One can show that \( \zeta \) is \( \ell \)-split if and only if this induced norm satisfies \( |f|_\zeta \in |\ell^\times| \) for all \( f \in \ell(T) \).
We restate Theorem D using the conventions and terminology of this section.

**Theorem D.** Let \( f \in \ell(z) \) be a tame rational function of degree \( d \geq 2 \), and let \( \Gamma \) be a vertex set in \( \mathbb{P}^1 \). Suppose that \( \Gamma \) is \( \ell \)-split. Then there exists an \( \ell \)-split vertex set \( \Gamma' \) containing \( \Gamma \) such that \((f, \Gamma')\) is analytically stable.

**Remark 3.5.** For each vertex set \( \Gamma \subset \mathbb{P}^1 \), there is a finite (discretely valued) extension \( \ell' / \ell \) such that \( \Gamma \) is \( \ell' \)-split. The equivalence of this statement of Theorem D with the one in the introduction follows immediately from this observation.

The proof is given in §3.2 after recalling several definitions and preliminary results. The overall strategy is to use Rivera-Letelier’s classification of Fatou components to organize the augmentation of \( \Gamma \). If a vertex \( \zeta \) lies in the Julia set for \( f \), then \( \zeta \) must be preperiodic, and we may append its forward orbit to \( \Gamma \). If a vertex \( \zeta \) lies in the Fatou set for \( f \), then we augment \( \Gamma \) with elements of the forward orbit of \( \zeta \) along with several other carefully chosen points, depending on the type of Fatou component to which \( \zeta \) belongs. We refer the reader to [2, §10] for background on non-Archimedean Fatou/Julia theory.

### 3.1. Periodic Fatou components and Julia points.

Versions of the next two results were proved by Rivera-Letelier for rational functions over \( \mathbb{C}_p \) [21, §4,5]. Some parts of the proofs carry over verbatim to the case of residue characteristic zero; additional work was done by Kiwi for the parts that do not [20].

A rational function \( f \in k(z) \) acts on \( \mathbb{P}^1 \) by an open map. Therefore, the image of a Fatou component is again a Fatou component. Moreover, the Fatou set of \( f \) agrees with the Fatou set of any iterate \( f^n \). So to classify periodic Fatou components for \( f \), it will suffice to study the fixed ones.

As in the complex setting, if \( U \) is a Fatou component such that \( f^n(U) = U \), and if \( U \) contains an attracting periodic point \( p \), then we say that \( U \) is the **immediate basin of attraction** for \( p \). Following Kiwi, we say that a periodic Fatou component of period \( n \) is a **Rivera domain** if \( f^n \) induces an automorphism of \( U \). (If \( f \) is tame, it is equivalent to ask that \( f^n \) induce a bijection of \( U \) onto itself.)

**Proposition 3.6** (Rivera-Letelier/Kiwi). Let \( f \in k(z) \) be a tame rational function of degree \( d \geq 2 \), and let \( U \) be a fixed Fatou component. Then \( U \) is either a Rivera domain or an immediate basin of attraction for a type I fixed point.

**Proposition 3.7** (Rivera-Letelier/Kiwi). Let \( f \in k(z) \) be a tame rational function of degree \( d \geq 2 \), and let \( U \) be a fixed Rivera domain. Then \( U \) is an open affinoid, and its boundary consists of finitely many repelling periodic orbits of type II points.

The next proposition is the “No Wandering Domains” result that was alluded to in the Introduction. Note that \( f \) must be defined over the discretely valued subfield \( \ell \subset k \). Note also that, while Benedetto’s results in [4] assume that \( k \) has residue characteristic 0, the proofs only use that the rational functions in question are tame. (See [4, Thm. 5.1] and its proof.)

**Proposition 3.8** (Benedetto). Let \( f \in \ell(z) \) be a tame rational function of degree \( d \geq 2 \). If \( U \) is a wandering Fatou component, then \( f^n(U) \) is a disk with periodic type II boundary point for all \( n \gg 0 \). Moreover, if \( U \) contains an element of \( \mathbb{P}^1(\ell) \), then the boundary point of \( f^n(U) \) is \( \ell \)-split.
The following result is a very powerful consequence of the hypothesis that $f$ is defined over a discretely valued subfield. (Compare Example 5.5.)

**Proposition 3.9.** Let $f \in \ell(z)$ be a tame rational function of degree $d \geq 2$. Then every type II point of the Julia set of $f$ is preperiodic.

**Proof.** Without loss, we may assume that $\ell$ is complete. Consider a complete extension $\ell'/\ell$ with transcendental residue extension and trivial value group extension. (For example, let $\| \cdot \|$ be the Gauss norm on the Tate algebra $\ell(\tau)$ in the variable $\tau$; write $\ell'$ for its fraction field, which is a complete non-Archimedean field with the same value group as $\ell$ and residue field $\tilde{\ell} = \tilde{\ell}(\tau)$, a rational function field.)

Let $k'$ be a minimal complete and algebraically closed non-Archimedean field containing both $k$ and $\ell'$. It is isomorphic to the completion of an algebraic closure of $\ell'$. Let $i : \mathbf{P}^1_k \hookrightarrow \mathbf{P}^1_{k'}$ be the natural continuous embedding. (See [13, §4].) Writing $f_{k'}$ for the induced map on $\mathbf{P}^1_{k'}$, we have that $i \circ f = f_{k'} \circ i$. In particular, $i$ induces an identification of Julia sets $J(f_{k'}) = i(J(f))$. For ease of notation, we drop the use of the embedding map $i$ and view $\mathbf{P}^1_k$ as a closed subset of $\mathbf{P}^1_{k'}$.

Let $\zeta \in J(f)$ be a type II point. Since residue fields $\tilde{k} \subset \tilde{k}'$, there is a tangent direction $\tilde{v} \in T_{\tilde{k}',\zeta} \setminus T_{\tilde{k},\zeta}$ such that the associated open disk $D(\tilde{v})$ does not meet the Julia set. That is, $D(\tilde{v})$ is a Fatou component. If it is a preperiodic component, evidently its boundary point $\zeta$ is also preperiodic. Otherwise, $D(\tilde{v})$ is a wandering component, and Proposition 3.8 applies since $\ell'$ is discretely valued. \hfill $\square$

The next result will allow us to control the orbits of vertices in Rivera domains.

**Lemma 3.10.** Let $f \in k(z)$ be a tame rational function of degree $d \geq 2$, and let $U$ be a Rivera domain for $f$. Then the set of periodic points in $U$ is closed and connected (in $U$).

**Proof.** Replacing $f$ with an iterate if necessary, it suffices to assume that $U$ is fixed. Write $\Sigma(U)$ for the skeleton of $\overline{U}$ — i.e., the connected hull of the boundary of the closure of $U$. As $\partial U$ is a finite nonempty set and $f$ acts on $U$ by an automorphism, the skeleton $\Sigma(U)$ is fixed pointwise by some iterate of $f$. Without loss, we may assume that $\Sigma(U)$ is itself fixed pointwise. Note that the connected components of $U \setminus \Sigma(U)$ are open disks. To prove the lemma, it suffices to show that for each such disk $D$, the periodic locus in $D$ is closed and connected. For then the complement of the periodic locus in $U$ is a collection of disjoint open disks.

Suppose that $D$ is a connected component of $U \setminus \Sigma(U)$. Note that the boundary point of $D$ is fixed. If $D$ is not periodic, then the periodic locus in $D$ is simply its boundary point. If $D$ is periodic, then we may assume without loss that it is fixed by $f$. Let $\eta$ be the boundary point of $D$. Observe that if $x \in D$ is any periodic point, say with period $n$, then the entire segment $[x, \eta]$ must be fixed by $f^n$. For $f^n$ is unramified along $[x, \eta]$; hence, $f^n([x, \eta])$ and $[x, \eta]$ have the same length and the same endpoints, and so they must agree pointwise. It follows that the set of periodic points in $D$ is connected.

To show that the periodic locus in $\overline{D}$ is closed, it suffices to show that there are only finitely many periods that can occur for a periodic point in $D$. Indeed, the set of points in $\overline{D}$ with period dividing a given integer $n$ is closed, being the solutions to the equation $f^n(z) = z$. In fact, we will show something stronger: the set of periods that can occur for a periodic point in $\overline{D}$ contains at most two elements. Make a change of coordinate so that
\[ D = \mathcal{D}(0, 1)^{-}, \] in which case \( \eta \) is the Gauss point. Since \( f \) is an automorphism of \( \mathcal{D}(0, 1)^{-} \), we may expand \( f \) as
\[
f(z) = a_0 + a_1 z + a_2 z^2 + \cdots,
\] where \( |a_i| \leq 1 \) for all \( i \), \( |a_0| < 1 \), and \( |a_1| = 1 \). Let \( \lambda \) be the image of \( a_1 \) in the residue field \( \bar{k} \). If \( \lambda \) is not a root of unity, we will show that 1 is the only possible period for a periodic point in \( D \). If \( \lambda \) is an \( e^{th} \) root of unity, then \( \{1, e\} \) are possible periods.

Let \( y \in D \) be a periodic point of period \( n \geq 1 \). Let \( x_1 \in [y, \eta] \) be the closest fixed point to \( y \). Note that \( x_1 \) is of type II and \( x_1 \neq \eta \) because the tangent vector \( \bar{v} \in TP^{1}_\eta \) pointing toward 0 is fixed by \( f \). If \( y = x_1 \), then \( n = 1 \), and we are finished. Otherwise, the direction \( \bar{v} \in TP^{1}_{x_1} \) containing \( y \) is periodic and non-fixed.

Writing \( x_1 = \zeta_{b, \rho} \) for some \( b, \rho \in k \cap \mathcal{D}(0, 1)^{-} \), we make a change of coordinate in (3.1) to obtain the action of \( f \) on \( TP^{1}_{x_1} \):
\[
\rho^{-1} f(b + \rho z) - b \rho^{-1} = \frac{f(b) - b}{\rho} + a_1 z + \epsilon(z),
\]
where \( \epsilon(z) \) is a series whose coefficients all have absolute value strictly smaller than 1. Since \( x_1 \) is fixed, we find that \( |f(b) - b| \leq \rho \). Let \( \beta \) be the image of \( \rho^{-1}(f(b) - b) \) in the residue field of \( k \). Reducing the above expression modulo the maximal ideal of \( k^0 \) shows that, in appropriate coordinates, the action of \( f \) on \( TP^{1}_{x_1} \) is given by
\[
z \mapsto \beta + \lambda z.
\]

Since there exists a periodic non-fixed direction at \( x_1 \), \( \lambda \) must be a nontrivial root of unity. Let \( e > 1 \) be the multiplicative order of \( \lambda \).

We claim that \( y \) has period \( e \). If not, then let \( x_e \) be the point closest to \( y \) that is fixed by \( f^e \). Then \( x_e \) is of type II, and \( x_e \neq x_1 \) since \( f^e \) fixes the direction \( \bar{v} \in TP^{1}_{x_1} \) pointing toward \( y \). We know \( y \) lies in a periodic non-fixed direction at \( x_e \). But \( f^e \) acts on \( TP^{1}_{x_e} \) by \( z \mapsto z + \beta' \) for some \( \beta' \in \bar{k} \) by a computation analogous to the one in the previous paragraph. This tangent map has no periodic non-fixed direction. Hence \( y = x_e \) and \( y \) has period \( e \). \( \square \)

### 3.2. Existence of analytically stable augmentations.

The goal of this section is to prove the following more precise version of Theorem D. Recall that any \( F \)-domain \( U \) has the property that \( f(U) \) is contained in some \( F \)-domain (Lemma 2.8). The **forward orbit** of \( U \) is the set of \( F \)-domains \( V \) such that \( f^n(U) \subset V \) for some \( n \geq 0 \). We say that an \( F \)-domain is **wandering** if it has infinite forward orbit.

**Theorem 3.11.** Let \( f \in \ell(z) \) be a tame rational function of degree \( d \geq 2 \), and let \( \Gamma \) be an \( \ell \)-split vertex set in \( P^1 \). There exists an \( \ell \)-split vertex set \( \Gamma' \supset \Gamma \) such that for each \( \zeta \in \Gamma' \), exactly one of the following is true:

- \( f(\zeta) \in \Gamma' \);
- \( f(\zeta) \) lies in a wandering \( F \)-disk (relative to \( \Gamma' \)) with periodic boundary point in \( \Gamma' \); or
- \( f(\zeta) \) lies in an \( F \)-disk (relative to \( \Gamma' \)) that contains an attracting type I periodic point.

In particular, \( (f, \Gamma') \) is analytically stable.

Setting \( \Gamma_0 = \Gamma \), we will successively construct subsets \( \Gamma_n \supset \Gamma_{n-1} \) for which fewer of the vertices in \( \Gamma_n \) fail to meet one of the properties in the statement of the theorem. Or rather,
for ease of notation, we will speak of “enlarging the vertex set \( \Gamma \)” at every step in this inductive procedure and dispense with the subscripts entirely.

Before beginning the proof in earnest, we note that a vertex \( \zeta \in \Gamma \) is either in the Julia set \( J(f) \), or else it lies in a Fatou component \( U \). Then either \( U \) is a wandering component, or else there exists \( m \geq 0 \) such that \( f^m(U) \) is a Rivera domain or the basin of attraction of a periodic point (Proposition 3.6). We will treat each of these cases separately, and then conclude by proving that what we have accomplished is sufficient for the theorem.

**Step 1:** Julia vertices. Each element of \( \Gamma \cap J(f) \) is preperiodic (Proposition 3.9). Enlarge \( \Gamma \) by adjoining the forward orbit of all such elements; now \( \zeta \in \Gamma \cap J(f) \) implies \( f(\zeta) \in \Gamma \).

**Step 2:** Wandering Fatou components. Suppose that \( \Gamma \) has nonempty intersection with a wandering Fatou component, say \( U \). Let \( \zeta_1, \ldots, \zeta_s \) be the vertices in the grand orbit of \( U \). Then there exist integers \( n_1, \ldots, n_s \geq 0 \) such that \( f^{n_1}(\zeta_1), \ldots, f^{n_s}(\zeta_s) \) lie in the same Fatou component, say \( V \), and \( f^m(V) \cap \Gamma = \emptyset \) for all \( m \geq 1 \). We may further assume that \( V \) is a disk whose boundary point is periodic and \( \ell \)-split (Proposition 3.8); let \( O_V \) be the orbit of the boundary of \( V \). Adjoin

\[
O_V \quad \text{and} \quad \bigcup_{j=1}^s \left\{ f(\zeta_j), f^2(\zeta_j), \ldots, f^{n_j}(\zeta_j) \right\}
\]

to \( \Gamma \). Then \( \Gamma \) is still \( \ell \)-split and finite. By construction, \( f^m(V) \) is an \( F \)-disk for all \( m \geq 1 \). If \( \zeta \) is a vertex in the grand orbit of \( U \) such that \( f(\zeta) \not\in \Gamma \), then our construction shows \( f(\zeta) \in f(V) \), a wandering \( F \)-disk.

**Step 3:** Attracting basins. Suppose that \( \Gamma \) has nonempty intersection with a preperiodic component \( U \) such that \( f^m(U) \) is the immediate basin of attraction of a \( (\text{type I}) \) periodic point \( x \). For ease of exposition, we will explain the case where \( x \) is a fixed point; the more general setting requires added notational effort only.

We claim that \( x \in \mathbb{P}^1(\ell) \), where \( \ell \) is the closure of \( \ell \) in \( k \). Indeed, since \( \Gamma \) has nontrivial intersection with the basin of attraction of \( x \), any small disk \( D \) about \( x \) will contain \( f^n(\zeta) \) for some vertex \( \zeta \) and all \( n \gg 0 \). In particular, there exists a sequence of elements \( a_n \in \ell \) and radii \( r_n \in |\ell^\times| \) such that \( f^n(\zeta) = \zeta_{a_n,r_n} \), and such that the associated disk \( D(a_n,r_n) \) is contained in \( D \) for all \( n \gg 0 \). It follows that \( a_n \to x \).

Write \( V = f^m(U) \) for the immediate basin of attraction of \( x \), and let \( D \) be a small closed disk about \( x \) in \( V \). We assume that \( D \) is chosen small enough that \( \Gamma \cap \Gamma \) is empty and that \( f(D) \not\subset D \). By the previous paragraph, we may further shrink \( D \) if necessary in order to assume that its boundary point \( \zeta_D \) is \( \ell \)-split. Let us adjoin \( \zeta_D \) to \( \Gamma \), so that \( \Gamma \cap \Gamma = \{ \zeta_D \} \).

By enlarging \( \Gamma \) with sufficiently many iterates of its elements lying in the grand orbit of \( V \), we may assume that (1) \( \Gamma \cap \Gamma = \{ \zeta_D \} \), and (2) if \( \zeta \in \Gamma \) lies in the grand orbit of \( V \), but \( f(\zeta) \not\in \Gamma \), then \( f(\zeta) \in D \). By (1) and the fact that \( f(D) \not\subset D \), each connected component of \( D \setminus \{ \zeta_D \} \) is an \( F \)-disk.

**Step 4:** Rivera domains. Now suppose that \( \Gamma \) has nonempty intersection with the grand orbit of a Rivera domain \( U \). For ease of exposition, we will explain the case where \( U \) is a fixed Rivera domain. By enlarging \( \Gamma \) with sufficiently many iterates of its elements, we may assume that if \( \zeta \in \Gamma \) lies in the grand orbit of \( U \), but \( f(\zeta) \not\in \Gamma \), then \( \zeta \) lies in \( U \setminus P \), where \( P \) is the periodic locus of \( f \) in \( U \). By Lemma 3.10, \( P \) is closed and connected in \( U \). Thus \( U \setminus P \) is a disjoint union of open disks.
Let $D$ be a connected component of $U \setminus P$. Observe that $f^m(D) \cap f^n(D) = \emptyset$ for all $m > n \geq 0$. Indeed, the boundary point of $D$ is periodic, but $D$ cannot itself be periodic. For otherwise it would contain periodic points close to its boundary, in contradiction to the fact that it is disjoint from the periodic locus. Moreover, if $D \cap \Gamma$ is nonempty, then $D$ contains an element $a \in \mathbb{P}^1(\ell)$, and hence so do each of its iterates $f^n(D)$. Without loss, we may change coordinates in order to assume that the Rivera domain $U$ is contained in the unit disk $D(0,1)^-$. Now the diameter of $D$ is the difference $|a - f^n(a)| \in |\ell^\tau|$, where $n$ is the period of the boundary of $D$. That is, the boundary point of $D$ is periodic and $\ell$-split. The argument given in Step 2 may now be applied to $D$ (in place of $V$) in order to show that if $\zeta \in \Gamma$ is a vertex in the grand orbit of $D$ such that $f(\zeta) \notin \Gamma$, then $f(\zeta)$ lies in a wandering $F$-disk with periodic boundary point.

**Step 5:** Conclusion. After applying Step 1, we may assume that if $\zeta \in \Gamma$ and $f(\zeta) \notin \Gamma$, then $\zeta$ lies in the Fatou set of $f$. There are finitely many grand orbits of Fatou components that meet $\Gamma$. After applying Step 2, we see that if $\zeta \in \Gamma$ and $f(\zeta) \notin \Gamma$, then either $f(\zeta)$ lies in a wandering $F$-disk, or else $\zeta$ lies in the grand orbit of a periodic Fatou component. After applying Step 3, we conclude that if $\zeta \in \Gamma$ and $f(\zeta) \notin \Gamma$, then $f(\zeta)$ lies in an $F$-disk containing an attracting periodic point, or it lies in a wandering $F$-disk with periodic boundary point, or else $\zeta$ lies in the grand orbit of a Rivera domain. Note that in Step 3, we have only added vertices in grand orbits of attracting basins, so this has no impact on the $F$-domains we created in Step 2. Finally, after applying Step 4 we have the conclusion of the theorem. Again, note that we have only added vertices in grand orbits of Rivera domains, which does not impact the work from Steps 2 or 3. This concludes the proof of Theorem 3.11 (and therefore also of Theorem D).

4. Algebraically stable resolutions of complex surfaces

In this section, we prove Theorem E. Throughout, we will write $\hat{\mathbb{C}}$ for the Riemann sphere (in order to distinguish it from the complex scheme $\mathbb{P}^1_\mathbb{C}$). Recall the statement:

**Theorem E.** Let $f_t : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a meromorphic family of rational functions for $t \in \mathbb{D}$ such that $\deg(f_t) = d \geq 2$ when $t \neq 0$. Let $\pi : X \to \mathbb{D}$ be a normal connected proper fibered surface such that $\pi^{-1}(\mathbb{D}^*) \cong \mathbb{D}^* \times \hat{\mathbb{C}}$ and $X_0 = \pi^{-1}\{0\}$ is reduced. Consider the rational map $F : X \dashrightarrow X$ defined by $(t,z) \mapsto (t,f_t(z))$. There exists a proper modification $Y \to X$ that restricts to an isomorphism over $\mathbb{D}^*$ such that the induced rational map $F_Y : Y \dashrightarrow Y$ is algebraically stable.

**Remark 4.1.** The hypotheses of the theorem are trivially satisfied if one takes $X = \mathbb{D} \times \hat{\mathbb{C}}$.

To apply our result on analytically stable augmentations of vertex sets in the Berkovich projective line, we must connect the dynamics of rational functions on $\mathbb{P}^1_\mathbb{L}$ with the dynamics of rational maps on fibered surfaces. The connection between vertex sets and formal semistable fibrations of curves over complete valuation rings of height 1 is well understood [1, 7], and we will describe a modification of that theory that applies in our setting. An alternate approach involving moving frames, as in Kiwi’s article [19], may also be feasible.

4.1. Models and vertex sets. We now recall and extend the discussion from [10, §5.1] on the relationship between (degenerating) families of rational curves over a complex disk and vertex sets in $\mathbb{P}^1_\mathbb{L}$. As we will need to identify families over disks of varying sizes, it will be
Remark 4.3. When discussing models, we will typically suppress mention of the map $\eta$ and simply write $X$.

Remark 4.4. In the language of complex geometry, a model $(X, \eta)$ may be interpreted as a normal connected projectively fibered surface $\pi : X \to \mathbb{D}_\varepsilon$ for some small $\varepsilon > 0$, where $\mathbb{D}_\varepsilon$ is the complex disk of radius $\varepsilon$. The isomorphism $\eta$ corresponds to a trivialization $\pi^{-1}(\mathbb{D}_\varepsilon^*) \cong \mathbb{D}_\varepsilon^* \times \hat{\mathbb{C}}$. Note that shrinking $\varepsilon$ does not change the (isomorphism class of the) model, as we are only concerned with its germ structure.

Let $X$ be a model of $\mathbb{P}^1_\ell$. We claim that $X$ gives rise, canonically, to a vertex set $\Gamma_X \subset \mathbb{P}^1 = \mathbb{P}^1_{\mathbb{L}^0}$. The local ring of $\mathbb{D}$ at the origin is contained inside $\mathbb{L}^0$, and hence so is its completion. By completing along the central fiber $X_0$ and base extending to $\mathbb{L}^0$, we obtain an admissible formal scheme $\mathfrak{X}$ over $\mathbb{L}^0$ with generic fiber $\mathbb{P}^1$. Note that since $X_0$ is reduced, it may be identified with the special fiber $\mathfrak{X}_s$ as $\mathbb{C}$-schemes. Let

$$\text{red}_X : \mathbb{P}^1 \to X_0$$

be the canonical surjective reduction map [6, 2.4.4]. Let $\eta_1, \ldots, \eta_r$ be the generic points of the irreducible components of the special fiber $X_0$. There exist unique type II points $\zeta_1, \ldots, \zeta_r \in \mathbb{P}^1$ such that $\text{red}_X(\zeta_i) = \eta_i$ for $i = 1, \ldots, r$. The desired vertex set is $\Gamma_X = \{\zeta_1, \ldots, \zeta_r\}$. Since the model $X$ is defined over $\text{Spec} \ell$, each of the vertices $\zeta_i$ is $\ell$-split.

Proposition 4.5. The association $X \mapsto \Gamma_X$ induces a bijection between the collection of isomorphism classes of models of $\mathbb{P}^1_\ell$ and the collection of $\ell$-split vertex sets in $\mathbb{P}^1$. Moreover, the following are true:

1. Fix a model $X$. For each closed point $x \in X_0$, the formal fiber $\text{red}_X^{-1}(x)$ is a $\Gamma_X$-domain. The association $x \mapsto \text{red}_X^{-1}(x)$ induces a bijection between points of the $\mathbb{C}$-scheme $X_0$ and elements of $\mathcal{S}(\Gamma_X)$.

2. If $X$ and $X'$ are models of $\mathbb{P}^1$, then $X$ dominates $X'$ if and only if $\Gamma_X \supset \Gamma_{X'}$.

Sketch of proof. We have already shown that $X \mapsto \Gamma_X$ gives a well-defined $\ell$-split vertex set in $\mathbb{P}^1$. Functoriality of formal completion, the generic fiber construction, and the reduction map construction shows that if $X'$ and $X$ are isomorphic as models, then $\Gamma_{X'} = \Gamma_X$. Thus we have a well defined map between the desired collections of objects. The map is injective because models are determined by their formal fibers [7, Lem. 3.10].
We now sketch the proof that \( X \mapsto \Gamma_X \) is surjective. Fix a vertex set \( \Gamma \). The argument in [1, Thm. 4.11] carries over to our setting mutatis mutandis and produces a formal model \( \mathcal{X} \) over \( \hat{\ell} = \mathbb{C}((t)) \) with associated vertex set \( \Gamma \). The main idea is that the vertex set \( \Gamma \) allows us to define gluing data for \( \mathbb{P}^1 \) consisting of a finite union of closed affinoids with Shilov boundary in \( \Gamma \). In order to pass to algebraic models, we need only observe that the canonical models of the closed affinoids in the proof of [1, Thm. 4.11] are formal completions of algebraic models over \( \ell \). This is where the \( \ell \)-split hypothesis comes in. More precisely, if \( T = \mathbb{L}(z) \) is the standard single variable Tate algebra, then the affinoid algebras in question are of the form

\[
T(Y_1, \ldots, Y_m)/((z - a_i)Y_i - c_i : i = 1, \ldots, m),
\]

for some \( a_i, c_i \in \mathcal{O} \). The associated \( \mathcal{O} \)-algebra is given by

\[
\mathcal{O}[z, Y_1, \ldots, Y_m]/((z - a_i)Y_i - c_i : i = 1, \ldots, m).
\]

As in the formal case, the associated local models over \( \mathcal{O} \) glue to give a global model \( X \). A direct computation shows that \( X \) is normal with reduced central fiber, and by construction we have \( \Gamma_X = \Gamma \).

Now fix a model \( X \). The formal fiber \( \text{red}_X^{-1}(x) \) is open for each closed point \( x \in X_0 \) by anticontinuity of the reduction map. Since each one is disjoint from the vertex set \( \Gamma_X \), it follows that \( \text{red}_X^{-1}(x) \) is a \( \Gamma_X \)-domain, and that \( x \mapsto \text{red}_X^{-1}(x) \) defines a bijection between the points of the \( \mathbb{C} \)-scheme \( X_0 \) and the collection of \( \Gamma_X \)-domains and vertices \( \mathcal{S}(\Gamma_X) \).

The proof of the final claim about domination and vertex set containment follows exactly as in the formal case; see [1, Thm. 4.11]. \( \square \)

4.2. Proof of Theorem E. Suppose that \( f_t \) is a meromorphic family of rational functions with \( \deg(f_t) = d \geq 2 \) for \( t \neq 0 \), and that \( X \) is a model of \( \mathbb{P}^1_{\hat{\ell}} \). We identify \( X \) with a normal fibered complex surface over a small disk as in Remark 4.4. Define a rational map \( F : X \dashrightarrow X \) by \( F(t, z) = (t, f_t(z)) \) for all \( t \neq 0 \). Let \( f : \mathbb{P}^1 \to \mathbb{P}^1 \) be the rational function determined by viewing the parameter \( t \) as an element of the field \( \mathbb{L} \).

**Lemma 4.6.** The action of \( F \) on \( X_0 \) and the action of \( f \) on \( \mathcal{S}(\Gamma_X) \) are compatible in the following sense: If \( x, x' \in X_0 \) are points (closed or generic) and \( U_x = \text{red}_X^{-1}(x) \) and \( U_{x'} = \text{red}_X^{-1}(x') \) are the corresponding \( \Gamma_X \)-domains or vertices, then

\[
F(x) = x' \quad \text{if and only if} \quad f(U_x) \subset U_{x'}.
\]

In particular, \( x \in X_0 \) is an indeterminacy point for \( F \) if and only if \( f(U_x) \) contains an element of the vertex set \( \Gamma_X \).

**Proof.** If \( F \) were a morphism, this would follow immediately from functoriality of reduction. To apply this argument, we begin by resolving the indeterminacy of \( F \). Let \( \rho : Y \to X \) be a model dominating \( X \) such that the rational map \( F \) extends to a morphism \( \bar{F} : Y \to X \) satisfying \( \bar{F} = F \circ \rho \) when the right side is defined. Functoriality of the reduction map gives a commutative diagram

\[
\begin{array}{ccc}
\mathbb{P}^1 & \xrightarrow{f} & \mathbb{P}^1 \\
\text{red}_Y & \downarrow & \text{red}_Y \\
Y_0 & \xrightarrow{\bar{F}} & X_0 \\
\end{array}
\]
Here we write \( F_0 \) for the induced morphism on central fibers. For \( y \in Y_0 \), let us write \( U_y = \text{red}_{\mathcal{F}}^{-1}(y) \). It follows that
\[
(4.1) \quad \bar{F}_0(y) = x' \quad \text{if and only if} \quad f(U_y) \subset U_{x'}.
\]
Since domination of models corresponds to vertex set containment, we may partition the \( \Gamma_X\)-domain \( U_X \) as
\[
U_x = \bigcup_{y \in \rho^{-1}(x)} U_y.
\]
Applying (4.1) simultaneously to all \( y \in \rho^{-1}(x) \) gives the desired result. \( \square \)

We are now ready to prove Theorem E. By a gluing construction, it suffices to produce a proper modification \( Y \rightarrow X \) over a smaller disk; in particular, we may work with models over \( \text{Spec} \mathcal{O} \) throughout. Identify \( X \) with a model of \( \mathbb{P}^1_\ell \), and let \( \Gamma_X \subset \mathbb{P}^1 \) be the associated vertex set. There exists an \( \ell \)-split vertex set \( \Gamma' \) containing \( \Gamma_X \) such that the pair \((f, \Gamma')\) is analytically stable (Theorem D). By Proposition 4.5, there is a model \( Y \) of \( \mathbb{P}^1_\ell \) that dominates \( X \) and has vertex set \( \Gamma_Y = \Gamma' \). The rational map \( F : X \dashrightarrow X \) extends to a rational map \( F_Y : Y \dashrightarrow Y \), which we claim is algebraically stable. This is clear for horizontal curves (i.e., those that project onto \( \text{Spec} \mathcal{O} \)). So suppose that \( C \subset Y_0 \) is an irreducible curve such that \( F_Y^n(C) \) is collapsed to an indeterminacy point for \( F_Y \), say \( y \). Let \( \zeta \in \Gamma_Y \) be the vertex such that \( \text{red}_Y(\zeta) \) is the generic point of \( C \), and let \( U = \text{red}_{\mathcal{F}}^{-1}(y) \in \mathcal{S}(\Gamma_Y) \) be the \( \mathcal{S}(\Gamma_Y) \)-domain corresponding to \( y \). By the preceding lemma, we see \( f^n(\zeta) \in U \) and that \( f(U) \) meets the vertex set \( \Gamma_Y \). This means \( U \) is a \( J \)-domain for \( \Gamma_Y \). But the pair \((f, \Gamma_Y)\) is analytically stable, so we have a contradiction.

5. Examples

In this final section, we provide a collection of examples to illustrate Theorems C and D. We also show that the results fail without hypotheses on the Julia set (in Theorem C) and the field of definition (in Theorem D). We conclude with a comparison of analytic stability and the indeterminacy condition in the first author’s earlier work [9].

To relate our discussion to complex dynamics and the previous article [10], our examples are defined over \( \mathbb{L} \), the completion of the field of Puiseux series in the parameter \( t \). Note that \( |t| = \exp(-1) < 1 \). The Gauss point of \( \mathbb{P}^1 \) (corresponding to the sup-norm on the unit disk) will be denoted by \( \zeta_g \). For \( a \in \mathbb{L} \) and \( r \in \mathbb{R}_{>0} \), we write \( \mathcal{D}(a, r)^- \) and \( \mathcal{D}(a, r) \) for the open and closed Berkovich disks centered at \( a \) of radius \( r \), respectively.

5.1. A straightforward computation with Theorem C. Consider the pair \((f, \Gamma)\) given by
\[ f(z) = z - 1 + t/z, \quad \Gamma = \{ \zeta_g \}, \]
with degree \( d = 2 \). Then \( f(\zeta_g) = \zeta_g \), and the action on the tangent space \( T\mathbb{P}^1_{\zeta_g} = \mathbb{P}^1(\mathbb{C}) \) is given by the translation \( Tf(z) = z - 1 \). The pair \((f, \Gamma)\) is analytically stable, and the Julia set is not contained in \( \Gamma \). The second preimage of \( \zeta_g \) lies in the disk \( D = \mathcal{D}(0, 1)^- \); in fact, the disk \( D \) has surplus multiplicity \( s_f(D) = 1 \). The \( J \)-domains are disks of the form \( U_a = \mathcal{D}(a, 1)^- \) for \( a = 0, 1, 2, \ldots \). The transition matrix \( P \) for the associated Markov chain is
given by

\[
\begin{pmatrix}
\zeta_g & U_0 & U_1 & U_2 & U_3 & U_4 & \cdots \\
\zeta_g & 1/2 & 1/2 & 0 & 0 & 0 & 0 & \cdots \\
U_0 & 0 & 1/2 & 1/2 & 0 & 0 & 0 & \cdots \\
U_1 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & \cdots \\
U_2 & 0 & 1/2 & 0 & 0 & 1/2 & 0 & \cdots \\
U_3 & 0 & 1/2 & 0 & 0 & 0 & 1/2 & \cdots \\
U_4 & 0 & 1/2 & 0 & 0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Upon computing a few powers \( P^n \), it is not difficult to see that the unique stationary probability vector in this case is

\[
\pi = (0 \ 1/2 \ 1/4 \ 1/8 \ 1/16 \ \cdots ).
\]

5.2. **The failure of Theorem C without a hypothesis on \( J(f) \).** Theorem C requires a hypothesis that the Julia set of \( f \) is not contained in the vertex set \( \Gamma \). To see it is necessary to make some assumption of this kind, consider the pair \((f, \Gamma)\) given by

\[
f(z) = \frac{1}{z^2}, \quad \Gamma = \{ \zeta_g, \zeta_{0,|t|}, \zeta_{0,|t|^{-1}} \}.
\]

Then \( f \) has good reduction, so that \( J(f) = \{ \zeta_g \} \subset \Gamma \). Moreover, if \( D_+ \) (resp. \( D_- \)) is the open disk with boundary point \( \zeta_{0,|t|^{-1}} \) (resp. \( \zeta_{0,|t|} \)) and containing \( 0 \) (resp. \( \infty \)), then \( f(D_+) \subset D_- \) and \( f(D_-) \subset D_+ \). Hence \((f, \Gamma)\) is analytically stable.

Let \( A_\pm \) be the open annuli with boundary points \( \zeta_g \) and \( \zeta_{0,|t|^{-1}} \), respectively. Then \( A_\pm \) are the only \( J \)-domains, so that

\[
J(\Gamma) = \{ A_+, A_-, \zeta_g, \zeta_{0,|t|}, \zeta_{0,|t|^{-1}} \}.
\]

As \( f \) has local degree 2 along the segment \([0, \infty]\), we find that the transition matrix \( P \) for our Markov chain is given by the matrix

\[
\begin{pmatrix}
A_+ & A_- & \zeta_g & \zeta_{0,|t|} & \zeta_{0,|t|^{-1}} \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Now \( P^{2n} = P^2 \) and \( P^{2n+1} = P \) for \( n \geq 1 \). As \( P^2 \neq P \), the powers of \( P \) do not converge. Moreover, there are two independent stationary vectors for \( P \), namely

\[
(1/2 \ 1/2 \ 0 \ 0 \ 0) \quad \text{and} \quad (0 \ 0 \ 1 \ 0 \ 0).
\]

5.3. **An analytically unstable example modified by the procedure of Theorem D.** Consider the pair \((f, \Gamma)\) with

\[
f(z) = z^2 + \frac{1}{t}, \quad \Gamma = \{ \zeta_g \}.
\]

Set \( U = \mathbb{P}^1 \setminus \mathcal{D}(0, 1) \). One computes that \( f(\zeta_g) = \zeta_{1/t, 1} \in U \), and that \( f(U) = \mathbb{P}^1 \setminus \mathcal{D}(1/t, 1) \). Hence \( \zeta_g \in f(U) \), and we conclude that \((f, \Gamma)\) is not analytically stable.
We observe that $\infty$ is an attracting fixed point for $f$, and $\zeta_{0,1/|t|}$ lies in its basin of attraction. Following the proof of Theorem D, we define

$$\Gamma' = \Gamma \cup \{\zeta_{0,1/|t|}\} = \{\zeta_g, \zeta_{0,1/|t|}\}.$$ 

Set $D = \mathbb{P}^1 \setminus \mathcal{D}(0, 1/|t|)^{-}$. Then $f(D) \subset D$, so that the connected components of $D \setminus \{\zeta_{0,1/|t|}\}$ are $F$-disks. Since $f(\zeta_{0,1/|t|})$ lies in one of these disks, the pair $(f, \Gamma')$ is analytically stable.

Let $A$ be the open annulus with boundary points $\zeta_g$ and $\zeta_{0,1/|t|}$. It is the unique $J$-domain for $\Gamma'$, so that

$$J' = \{A, \zeta_g, \zeta_{0,1/|t|}\}.$$ 

The transition matrix $P'$ is given by

$$
\begin{pmatrix}
A & \zeta_g & \zeta_{0,1/|t|} \\
\zeta_g & 1 & 0 & 0 \\
\zeta_{0,1/|t|} & 1 & 0 & 0 \\
\end{pmatrix}
$$

Evidently $(P')^n = 1\nu$ for all $n \geq 1$, where $\nu = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, so that all of the mass of $\mu_f$ is contained in $A$.

---

**Figure 1.** A schematic representation of the state spaces $J'$ and $J''$ in Examples 5.3 and 5.4.

---

5.4. The quadratic polynomial family, again. Continuing with the previous example, $f(z) = z^2 + 1/t$, we can enlarge the vertex set $\Gamma'$ in order to obtain further information about the location of the mass of $\mu_f$. Looking at the Newton polygon of $z^2 - z + 1/t$, we see that $f$ has fixed points $p_{\pm}$ with $|p_{\pm}| = 1/|\sqrt{t}|$, and they are easily seen to be repelling. Let us define

$$\Gamma'' = \{\zeta_g, \zeta_{0,1/|t|}, \zeta_{0,1/|\sqrt{t}|}\}.$$
As \( f(\zeta_{0,1/|t|}) = \zeta_{0,1/|t|} \), the pair \((f, \Gamma')\) is also analytically stable. Let \( V_\pm \) be the open disks with boundary point \( \zeta_{0,1/|t|} \) and containing \( p_\pm \), respectively. The set of states in this case is

\[ \mathcal{J}'' = \{ V_+, V_-, \zeta_g, \zeta_{0,1/|t|}, \zeta_{0,1/|\sqrt{t}|} \}, \]

and the transition matrix \( P'' \) is given by

\[
\begin{pmatrix}
V_+ & V_- & \zeta_g & \zeta_{0,1/|t|} & \zeta_{0,1/|\sqrt{t}|} \\
V_+ & \begin{array}{cc}
1/2 & 1/2 \\
1/2 & 1/2
\end{array} & 0 & 0 & 0 \\
V_- & \begin{array}{cc}
1/2 & 1/2 \\
1/2 & 1/2
\end{array} & 0 & 0 & 0 \\
\zeta_g & 0 & 0 & 0 & 0 \\
\zeta_{0,1/|t|} & 1/2 & 1/2 & 0 & 0 \\
\zeta_{0,1/|\sqrt{t}|} & 1/2 & 1/2 & 0 & 0
\end{pmatrix}
\]

Now we have \((P'')^n = 1 \nu\) for \( n \geq 2 \), where \( \nu = (1/2 \ 1/2 \ 0 \ 0 \ 0) \), so that

\[ \mu_f(V_+) = \mu_f(V_-) = 1/2. \]

5.5. **The failure of Theorem D without suitable hypotheses.** The existence of analytically stable augmentations does not hold for arbitrary pairs \((f, \Gamma)\) over arbitrary non-Archimedean fields, even in residue characteristic zero. For example, suppose that \( f \) is a Lattès map with non-simple reduction. Then the Julia set of \( f \) is an interval in \( \mathbb{P}^1 \) and \( f \) acts on the Julia set by a tent map. (See [16, §5.1].) Let \( \zeta \) be a Julia point with infinite orbit in the Julia set. By passing to an algebraically closed and complete extension of \( \mathbb{L} \), we may take \( \zeta \) to be a type II point.

Let \( \Gamma \) be any vertex set containing \( \zeta \). Every \( \Gamma \)-domain \( U \) that meets the Julia set will be a \( J \)-domain (Proposition 2.6). As the orbit of \( \zeta \) is infinite, it must intersect a \( J \)-domain. Hence \((f, \Gamma)\) cannot be analytically stable.

5.6. **The computations of [9].** The results in the present paper may be viewed as a natural generalization of [9]. The space of rational maps of degree \( d \geq 2 \) on the Riemann sphere \( \hat{\mathbb{C}} \) sits inside \( \overline{\text{Rat}_d} = \mathbb{P}^{2d+1} \), the space of all pairs \((F, G)\) where \( F, G \in \mathbb{C}[X, Y] \) are homogeneous of degree \( d \). We set \( H = \gcd(F, G) \), so that \((F, G) = H \cdot \phi \), where \( \phi \) describes an endomorphism of \( \hat{\mathbb{C}} \). The iteration map \( f \mapsto f^n \), from \( \overline{\text{Rat}_d} \) to \( \overline{\text{Rat}_{d^n}} \), is indeterminate along a subvariety \( I(d) \subset \partial \text{Rat}_d \), independent of \( n \geq 2 \); the set \( I(d) \) consists of all elements \( H \cdot \phi \) for which \( \phi \) is constant and its value is a root of the polynomial \( H \).

Let \( \mu_f \) denote the measure of maximal entropy for the rational function \( f : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \). The main result of [9] states that the map of measures, \( f \mapsto \mu_f \), extends continuously from \( \text{Rat}_d \) to a boundary point \( H \cdot \phi \in \partial \text{Rat}_d \) if and only if \( H \cdot \phi \) lies outside of \( I(d) \).

**Proposition 5.1.** Let \( f_t \) be a degenerating 1-parameter family of rational functions of degree \( d \geq 2 \), and write \( f \) for the associated Berkovich dynamical system. Then the limit \( f_0 \) lies outside \( I(d) \) if and only if the pair \((f, \{\zeta_g\})\) is analytically stable.

The formulas in [9] for the mass of the limit measure \( \mu_0 = \lim_{t \to 0} \mu_{f_t} \) may be deduced from our Markov chain description in Theorem C.

**Proof.** Write \( f_0 = H \cdot \phi \in \partial \text{Rat}_d \). Then \( f(\zeta_g) = \zeta_g \) on \( \mathbb{P}^1 \) if and only if the reduction map (given by the limit function \( \phi \)) is nonconstant. If \( \phi \) is constant, then the \( d \) roots of \( H \) coincide with the \( d \) disks in \( \mathbb{P}^1 \) containing the preimages of \( \zeta_g \), counted with multiplicities. (Compare
The constant value of $\phi$ coincides with the Berkovich disk $U$ containing the image $f(\zeta_g)$. Consequently, $f_0 \in I(d)$ if and only if $U$ contains a preimage of $\zeta_g$, and $(f, \{\zeta_g\})$ fails to be analytically stable. On the other hand, if $\phi$ is constant but $f_0 \notin I(d)$, then $f(U) \subset U$, so that $U$ is an $F$-disk and $(f, \{\zeta_g\})$ is analytically stable.

\[\square\]

References

[1] Matthew Baker, Sam Payne, and Joseph Rabinoff. On the structure of nonarchimedean analytic curves. To appear in the Proceedings of the 2011 Bellairs Workshop in Number Theory, 2103.
[2] Matthew Baker and Robert Rumely. Potential theory and dynamics on the Berkovich projective line, volume 159 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2010.
[3] Robert L. Benedetto. $p$-adic dynamics and Sullivan’s no wandering domains theorem. Compositio Math., 122(3):281–298, 2000.
[4] Robert L. Benedetto. Wandering domains and nontrivial reduction in non-Archipedean dynamics. Illinois J. Math., 49(1):167–193, 2005.
[5] Robert L. Benedetto and Liang-Chung Hsia. Weak Néron models for Lattès maps. Privately communicated preprint, 2013.
[6] Vladimir G. Berkovich. Spectral theory and analytic geometry over non-Archimedean fields, volume 33 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1990.
[7] Siegfried Bosch and Werner Lütkebohmert. Stable reduction and uniformization of abelian varieties. I. Math. Ann., 270(3):349–379, 1985.
[8] Jean-Yves Brinod and Liang-Chung Hsia. Weak Néron models for cubic polynomial maps over a non-Archimedean field. Acta Arith., 153(4):415–428, 2012.
[9] Laura DeMarco. Iteration at the boundary of the space of rational maps. Duke Math. J., 130(1):169–197, 2005.
[10] Laura DeMarco and Xander Faber. Degenerations of complex dynamical systems. Preprint, arXiv:1302.4769v2 [math.DS], 2013.
[11] J. Diller and C. Favre. Dynamics of bimeromorphic maps of surfaces. Amer. J. Math., 123(6):1135–1169, 2001.
[12] Jeffrey Diller and Jan-Li Lin. Rational surface maps with invariant meromorphic two forms. Preprint, arXiv:1308.2567 [math.AG], 2013.
[13] Xander Faber. Topology and geometry of the Berkovich ramification locus for rational functions. Manuscripta Math., doi:10.1007/s00229-013-0611-4, 2013.
[14] Charles Favre. Les applications monomiales en deux dimensions. Michigan Math. J., 51(3):467–475, 2003.
[15] Charles Favre and Mattias Jonsson. Dynamical compactifications of $\mathbb{C}^2$. Ann. of Math. (2), 173(1):211–248, 2011.
[16] Charles Favre and Juan Rivera-Letelier. Théorie ergodique des fractions rationnelles sur un corps ultramétrique. Proc. Lond. Math. Soc. (3), 100(1):116–154, 2010.
[17] Liang-Chung Hsia. A weak Néron model with applications to $p$-adic dynamical systems. Compositio Math., 100(3):277–304, 1996.
[18] John G. Kemeny, J. Laurie Snell, and Anthony W. Knapp. Denumerable Markov chains. Springer-Verlag, New York, second edition, 1976. With a chapter on Markov random fields, by David Griffeath, Graduate Texts in Mathematics, No. 40.
[19] Jan Kiwi. Rescaling limits of complex rational maps. arXiv:1211.3397 [math.DS], preprint, 2012.
[20] Jan Kiwi. Rivera domains. Personal communication, 2013.
[21] Juan Rivera-Letelier. Dynamique des fonctions rationnelles sur des corps locaux. Astérisque, (287):xv, 147–230, 2003. Geometric methods in dynamics. II.
[22] Eugenio Trucco. Wandering Fatou components and algebraic Julia sets. To appear in the Bull. Soc. Math. Fr., 2012.
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