Thermal relaxation of magnons and phonons near resonance points in magnetic insulators

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Abstract – We theoretically investigate the energy relaxation rate of magnons and phonons near the resonance points to clarify the underlying mechanism of heat transport in ferromagnetic materials. We find that the simple two-temperature model is valid for the one-phonon/one-magnon process, as the rate of energy exchange between magnons and phonons is proportional to the temperature difference between them, and it is independent of temperature in the high temperature limit. We found that the magnon-phonon relaxation time due to the one-phonon/one-magnon interaction could be reduced to 1.48 µs at the resonance point by applying an external magnetic field. It means that the resonance effect plays a significant role in enhancing the total magnon-phonon energy exchange rate, apart from the higher order interaction processes.

The heat transport in insulators is generally dominated by phonons which are the quanta of lattice vibrations. In magnetic insulators, the spin waves (magnons) could also act as heat carriers\([1,2]\), as observed for yttrium iron garnet (YIG)\([3]\), Nd\(_2\)CuO\(_4\)\([4]\), RbMnF\(_3\)\([5]\), and MnF\(_2\)\([6]\). The notable contribution of magnons to thermal conductivity has also been discovered in the spin ladder compound (Sr,Ca,La)\(_{14}\)Cu\(_{24}\)O\(_{41}\)\([7,8]\). In addition, the characteristics of heat transport due to magnons have been probed by the generation of spin excitations in InGaAs quantum dot system\([9]\). In recent years, the heat current due to spin excitations has also stimulated the field of spin caloritronics\([10]\), resulting in the recent discovery of spin Seebeck effect\([11]\), and spin Peltier effect\([12]\).

Intuitively, the total thermal conductivity (κ\(_T\)) of magnetic insulators could be evaluated by a simple sum of the lattice thermal conductivity (κ\(_p\)) and magnon thermal conductivity (κ\(_m\)) contributions: κ\(_T\) = κ\(_p\) + κ\(_m\). However, magnons will be scattered by phonons and vice versa, thus the interaction between phonons and magnons becomes relevant in determining the total thermal conductivity of magnetic materials. An effective strategy to evaluate the heat transport through both phonons and magnons, together with their interaction, is the two-temperature model\([13,14]\). It was first proposed by Sanders and Walton for the coupled magnon-phonon mode diffusion in the ferrimagnet YIG and the antiferromagnet MnF\(_2\)\([6]\). In their paper, phonons and magnons are assumed to be excited to their equilibrium states with different effective temperatures, with the local energy exchange rate between these two carriers proportional to their temperature difference. They found that thermal conductivity is not determined only by κ\(_m\) and κ\(_p\) but also by the magnon-phonon relaxation time τ\(_{mp}\)\([6]\). Recently, Chen et al. generalized a two-temperature model including the effect of the concurrent magnetization flow by assuming a constant magnon-phonon relaxation time\([15]\). However, there has been no theoretical work to check the validity of the introduction of τ\(_{mp}\). In addition, crucial problems such as the strength of the magnetoelastic coupling, the exact relaxation time\([16,17]\), and their magnetic field dependence have not been fully understood yet.

Taking YIG as a prototype for studying the magnon properties in ferromagnetism\([18,19]\), Sato\([20]\) suggested that the thermal conductivity due to magnon proportional
to $T^2$ could be greater than that of phonons to $T^3$ below 1K by assuming that the mean free paths of phonons and magnons are limited by the boundary condition and comparable in magnitude. Later on, Daugulis \cite{21} experimentally observed a large decrease of thermal conductivity in YIG when applying a magnetic field of 2T at 0.5 K. To explore the field and temperature dependence of thermal transport properties, many theoretical and experimental researches have been carried out. Those have revealed that the magnetoelastic coupling between phonons and magnons plays a significant role for governing the total thermal conductivity \cite{14,22,23}. The strength of the coupling reaches a peak at the resonance condition, where the one-phonon/one-magnon process with the phonon and magnon of same frequency and wave vector becomes relevant \cite{24,25}.

In this Letter, we focus our attention on the resonance behavior of the one-phonon/one-magnon interaction. We derive a simple formula for the rate of energy exchange between magnons and phonons in analogy with the two-temperature model for electron-phonon system \cite{26}. We present the physical conditions to set up the two-temperature model based on the one-phonon/one-magnon interaction process. The field dependence of the temperature relaxation rate and the relaxation time corresponding to the one-phonon/one-magnon interaction is also obtained.

The magnetic Hamiltonian of YIG consists of dipolar, exchange interactions between spins, and the Zeeman splitting due to the external magnetic field $(H)$ along $z$ direction \cite{27}:

$$H_{\text{mag}} = \frac{\mu_0 (g \mu_B)^2}{2} \sum_{i \neq j} |r_{ij}|^2 \hat{S}_i \cdot \hat{S}_j - 3 \left( r_{ij} \cdot \hat{S}_i \right) \left( r_{ij} \cdot \hat{S}_j \right) - J \sum_i S^2 \cdot S^2 - g \mu_B H \sum_i S_i^z,$$

where $\mu_0$ is the vacuum permeability, $\mu_B$ is the Bohr magneton, $g$ is the $g$ factor, $J$ is the exchange integral. The spin $\hat{S}_i = \hat{S}(r_i)$ locates on the site $r_i$, with $S = |\hat{S}_i| = a_0^3 M_s / g \mu_B$, where $a_0$ is the unit cell lattice constant, $M_s$ is the saturation magnetization density, and $r_{ij} = r_i - r_j$. By employing the Holstein-Primakoff transformation, the quantized Hamiltonian for spin excitations could be expressed as \cite{27}:

$$H_{\text{mag}} = \sum_k A_k a_k^\dagger a_k + \frac{1}{2} \left( B_k a_k^\dagger a_k^\dagger + B_k a_k a_k \right),$$

where

$$\frac{A_k}{\hbar} = D F_k + \frac{\gamma \mu_0 H + \gamma \mu_0 M_s \sin^2 \theta_k}{2},$$

$$\frac{B_k}{\hbar} = \frac{\gamma \mu_0 M_s \sin^2 \theta_k}{2} e^{-2i\phi_k},$$

and $a_k^\dagger (a_k)$ is the magnon creation (annihilation) operators with wave vector $k$, $D = 2S J a_0^2$ the exchange stiffness, $\gamma = g \mu_B / h$ the gyromagnetic ratio, $\theta_k = \arccos (k_z / k)$ the polar angle between wave vector $k$ and the magnetic field along $z$ direction, and $\phi$ the azimuthal angle of $k$ in $xy$ plane. In the long-wavelength limit, the form factor $F_k \approx k^2$ \cite{27}. Eq. (2) could be diagonalized by using the Bogoliubov transformation:

$$\begin{bmatrix} a_k \\ a_k^\dagger \end{bmatrix} = \begin{bmatrix} u_k & -v_k \\ v_k & u_k \end{bmatrix} \begin{bmatrix} a_k^\dagger \\ a_k \end{bmatrix},$$

with

$$u_k = \sqrt{\frac{A_k + \hbar \omega_k}{2 \hbar \omega_k}}, v_k = \sqrt{\frac{A_k - \hbar \omega_k}{2 \hbar \omega_k}} e^{2i\phi_k}.$$

The Hamiltonian is then simplified to:

$$H_{\text{mag}} = \sum_k \hbar \omega_k a_k^\dagger a_k,$$

and the dispersion relation for bulk magnons in the long-wavelength limit is:

$$\omega_k = \sqrt{D k^2 + \gamma \mu_0 H \sqrt{D k^2 + \gamma \mu_0 (H + M_s \sin^2 \theta_k)}}$$

The Hamiltonian for one-phonon/one-magnon interaction process has also been derived by Flebus et al. \cite{27}:

$$H_{\text{int}} = \hbar n B_1 \left( \frac{\gamma h^2}{4 M_s \rho} \right)^{\frac{1}{2}} \sum_{k,\lambda} k \omega_{c k}^{\frac{1}{2}} e^{-i\phi} a_k (c_{-k \lambda} + c_{k \lambda}^\dagger) \times (-i \delta_{\lambda 1} \cos 2 \theta_k + i \delta_{\lambda 2} \cos \theta_k - \delta_{\lambda 3} \sin 2 \theta_k) + H.c.,$$

where $n = 1/a_0^3$ is the number density of the magnetic particles in the system, $B_1$ the magnetoelastic constants, $\rho$ the average mass density, $c_{k \lambda}^\dagger (c_{k \lambda})$ the phonon creation (annihilation) operators with wave vector $k$. $\delta$ is the Kronecker delta, and $\lambda = 3$ labels the transverse acoustic (TA) phonon, $\lambda = 3$ labels the longitudinal acoustic (LA) phonon. Under Debye approximation, the phonon dispersion relation is $\omega_{k \lambda} = C_{\lambda} |k|$. Following the procedure of Bogoliubov transformation, Eq. (3) could be expressed in terms of the magnon quasiparticles operators $a_k^\dagger (a_k)$:

$$H_{\text{int}} = \hbar n B_1 \left( \frac{\gamma h^2}{4 M_s \rho} \right)^{\frac{1}{2}} \sum_{k,\lambda} k \omega_{c k}^{\frac{1}{2}} e^{-i\phi} (u_k a_k - u_k a_k^\dagger) \times (c_{-k \lambda} + c_{k \lambda}^\dagger) (-i \delta_{\lambda 1} \cos 2 \theta_k + i \delta_{\lambda 2} \cos \theta_k - \delta_{\lambda 3} \sin 2 \theta_k) + H.c..$$

The decay rate of the magnon and phonon distribution functions $n_m(k)$ and $n_p(k \lambda)$ are:

$$\text{see eq. (10a) and eq. (10b)}$$

where

$$|M_{k, k\lambda}|^2 = \frac{\gamma h^2}{4 M_s \rho} \gamma \mu_0 |k \omega_{k \lambda}|^2 \beta_1, \quad \beta_1 = |u_k + v_k|^2 \cos^2 2 \theta_k, \quad \beta_2 = |u_k + v_k|^2 \cos 2 \theta_k, \quad \beta_3 = |u_k - v_k|^2 \cos^2 2 \theta_k$$

p-2
\[
\frac{\partial n_m(k)}{\partial t} = -\frac{2\pi}{\hbar^2} \sum_{k,\lambda} |M_{k,k\lambda}|^2 \delta(\omega_m - \omega_p) \times \{n_m(k) [1 + n_p(k,\lambda)] - [1 + n_m(k)] n_p(k,\lambda)\}
\] (10a)

\[
\frac{\partial n_p(k,\lambda)}{\partial t} = -\frac{2\pi}{\hbar^2} |M_{k,k\lambda}|^2 \delta(\omega_m - \omega_p) \times \{n_p(k,\lambda) [1 + n_m(k)] - [1 + n_p(k,\lambda)] n_m(k)\}
\] (10b)

\[
v_k^2 \sin^2 2\theta_k.
\]

In this model, we assumed that other collision processes such as phonon-phonon interaction and magnon-magnon interaction are strong enough to keep the local equilibrium, then the distribution functions \(n_m(k)\) and \(n_p(k,\lambda)\) can be replaced by the equilibrium ones \(\{\exp[\hbar\omega(k)/k_BT_m] - 1\}^{-1}\) and \(\{\exp[\hbar\omega(k)/k_BT_p] - 1\}^{-1}\) where magnon and phonon temperatures are noted as \(T_m\) and \(T_p\), respectively.

The energy of magnons and phonons are \(E_m = \sum_k \hbar \omega_k n_m(k)\) and \(E_p = \sum_{k,\lambda} \hbar \omega_{k\lambda} n_p(k,\lambda)\), thus the changing rate of energy becomes:

\[
\text{see eq. (12)}
\]

A Taylor expansion of \(n_m(\omega) - n_p(\omega)\) in terms of \(T_m - T_p\) is

\[
\text{see eq. (13)}
\]

where \(z = \hbar \omega/k_BT\) and \(T = (T_m + T_p)/2\). Since the first term in the right hand side of Eq. (13) is much larger than other higher order terms, only the linear term is enough in the evaluation. Therefore, the energy changing rate is linearly dependent on the temperature difference, and the temperature changing rate becomes:

\[
\frac{\partial T_m}{\partial t} = g_{mp}(T) (T_p - T_m), \quad (16a)
\]

\[
\frac{\partial T_p}{\partial t} = g_{pm}(T) (T_m - T_p), \quad (16b)
\]

with

\[
g_{mp}(T) = \frac{2\pi k_B}{\hbar^2} \sum_{k,\lambda} |M_{k,k\lambda}|^2 \delta(\omega_m - \omega_p) \frac{z^2}{(z^2 - 1)^2}, \quad (17a)
\]

\[
g_{pm}(T) = \frac{2\pi k_B}{\hbar^2} \sum_{k,\lambda} |M_{k,k\lambda}|^2 \delta(\omega_m - \omega_p) \frac{z^2}{(z^2 - 1)^2}, \quad (17b)
\]

At the same time, the temperature difference between \(T_m\) and \(T_p\) will decay exponentially, that

\[
\frac{\partial}{\partial t} \Delta T = -\frac{\Delta T}{\tau_{mp}(T)}, \quad (18)
\]

where \(\Delta T = T_m - T_p\) and \(\tau_{mp}(T) = [g_{mp}(T) + g_{pm}(T)]^{-1}\).

### Table 1: Parameters of magnetoelastic coupling in YIG [27]

| Symbol                      | Value          | Unit  |
|-----------------------------|----------------|-------|
| Lattice constant            | \(a_0\)        | Å     |
| Average mass density        | \(\rho\)       | \(10^3\) Kg m\(^{-3}\) |
| Gyromagnetic constant       | \(\gamma\)     | \(2\pi \times 28\) GHz T\(^{-1}\) |
| Saturation magnetization    | \(\mu_0 M_s\)  | 0.2439 T |
| Exchange stiffness          | \(D\)          | \(7.7 \times 10^{-6}\) m\(^2\) s\(^{-1}\) |
| Magnetoelastic coupling     | \(B_{\lambda}\) | \(2\pi \times 1988\) GHz |
| TA phonon velocity          | \(c_{1,2}\)    | \(3.9 \times 10^3\) m s\(^{-1}\) |
| LA phonon velocity          | \(c_3\)        | \(7.2 \times 10^4\) m s\(^{-1}\) |

We present a numerical calculation of the relaxation rate \(\tau_{mp}^{-1}\) due to magnetoelastic coupling in YIG. The parameters used in our calculation are listed in Table 1. Fig. 1 shows the temperature dependence of relaxation rate in absence of magnetic field. \(\tau_{mp}^{-1}\) decreases rapidly with the increasing of temperature and saturates above 100K, which is mainly attributed to the temperature dependence of specific heat. In the high-temperature limit \((\hbar \omega \ll k_BT)\), \(\omega^2v^2/((\omega^2 - 1)^2) \rightarrow 1\), \(C_m \rightarrow Nk_BT\) and \(C_p \rightarrow 3Nk_BT\), Eq. (17) could be reduced to:

\[
g_{mp}(T) = \frac{2\pi}{N\hbar^2} \sum_{k,\lambda} |M_{k,k\lambda}|^2 \delta(\omega_m - \omega_p), \quad (19a)
\]

\[
g_{pm}(T) = \frac{2\pi}{3N\hbar^2} \sum_{k,\lambda} |M_{k,k\lambda}|^2 \delta(\omega_m - \omega_p), \quad (19b)
\]

and it is obtained that \(\tau_{mp} \rightarrow 0.26 \text{ } \mu\text{s}^{-1}\) at high temperature limit.

In the presence of magnetic field, the relaxation rate at high temperature limit are plotted in Fig. 2 as a function of the strength of magnetic field. Magnetic field could effectively shift the dispersion relation to higher energy. Thus the intersects of magnon and phonon dispersions which satisfy the energy conservation varies with the magnetic field. As a result, for individual phonon modes, the relaxation rate is decreased with the increasing of magnetic field, until reaching the critical magnetic field

\[
\mu_0 H = \frac{(\gamma D\omega_0 M_s \sin^2 \theta - c_z^2)^2}{4\gamma D\omega_0^2 L}.\]

A sharp increasing of relaxation rate is attributed to the tangency of phonon dispersion and magnon dispersion, which maximize the interaction phase space as shown in Fig. 3, where the phase space is defined as

\[
P_{mp} = \frac{1}{3N} \sum_{k,\lambda} \delta(\omega_m(k) - \omega_p(k,\lambda)) \text{.}
\]
\[ \frac{\partial E_m}{\partial t} = -\frac{\partial E_p}{\partial t} = - \frac{2\pi}{\hbar} \sum_{k\lambda} \hbar \omega_{m}(k) |M_{E,k\lambda}|^2 \delta (\omega_m - \omega_p) [n_m (k) - n_p (k\lambda)] \]

\[ n_m(\omega) - n_p(\omega) = \frac{\varepsilon^2 \varepsilon^2}{(\varepsilon^2 - 1)^2} \frac{k_B}{\hbar \omega} (T_m - T_p) + \frac{1}{3!} \left[ \frac{3\varepsilon^4 \varepsilon^4}{2(\varepsilon^2 - 1)^2} - \frac{3\varepsilon^2 \varepsilon^5}{2(\varepsilon^2 - 1)^2} \right] \frac{k_B}{\hbar \omega} (T_m - T_p)^3 + ..., \]

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Thermal relaxation of magnons and phonons near resonance points

Fig. 3: (Color online) The phase space as a function of external magnetic field. The inset shows the dispersion relation of acoustic phonons and magnon as an illustration, where $\theta_k = \pi/2$.

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