ON ASYMPTOTICALLY LINEAR ELLIPTIC PROBLEMS

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Abstract. In this paper we prove the existence of a signed ground state solution in the mountain pass level for a class of asymptotically linear elliptic problems, even when the nonlinearity is just continuous in the second variable. The (strongly) resonant and non-resonant cases are discussed. A multiplicity result is also proved when \( f \) is odd with respect to the second variable.

1. INTRODUCTION

We are interested in the existence of ground state and other nontrivial solutions to the following class of semilinear problems

\[
\begin{cases}
-\Delta u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where \( \Omega \subset \mathbb{R}^N \) is a bounded smooth domain, \( N \geq 1 \), \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function which is asymptotically linear at the origin and at infinity, that is,

\[
\alpha(x) = \lim_{t \to 0} 2F(x, t)/t^2 \quad \text{and} \quad \eta(x) = \lim_{|t| \to \infty} 2F(x, t)/t^2
\]

uniformly in \( x \in \Omega \).

It is well known in the literature that asymptotically linear problems can be classified as resonant at infinity (if \( \lambda_m(\eta) = 1 \), for some \( m \in \mathbb{N} \)) or non-resonant at infinity (if \( \lambda_m(\eta) \neq 1 \), for all \( m \in \mathbb{N} \)), where, throughout this paper, \( \lambda_m(\theta) \) denotes the \( m \)-th eigenvalue of the problem

\[
\begin{cases}
-\Delta u = \lambda \theta(x) u & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega.
\end{cases}
\]

In particular, observe that if \( \theta(x) = \theta \) is a nontrivial constant, then \( \lambda_m(\theta) = \lambda_m/\theta \), where \( \lambda_m \) denotes the \( m \)-th eigenvalue of Laplacian operator with Dirichlet boundary condition. In fact, the resonant case is subdivided depending on how small at infinity is the function

\[
g(x, t) = \eta(x)t - f(x, t).
\]

As observed in [5], the smaller \( g \) is at infinity, the stronger resonance is. The worst situation is when, for a.e. \( x \in \Omega \),

\[
\lim_{|t| \to \infty} g(x, t) = 0 \quad \text{and} \quad \lim_{|t| \to \infty} \int_0^t g(x, s) ds = \beta(x) \neq \infty.
\]

In this case, we say that problem (P) is strongly resonant. One of the very first works dealing with this situation is [5] where, Bartolo, Benci and Fortunato show the existence of multiple
solutions for strongly resonant problems in the presence of some symmetry in the autonomous nonlinearity. Their proofs are based on a deformation theorem and pseudo-index theory.

Besides [5], there exist so many other works dealing with problem (P). Without any intention to be complete, we refer the reader to some papers and references therein. The existence of solution for problem (P) was investigated under different conditions, for instance, by Ahmad [1], Amann and Zehnder [4], Ambrosetti [2], Dancer [6], De Figueiredo and Miyagaki [8], Li and Willem [12], Liu and Zou [14] and Struwe [16]. Multiplicity results for problem (P) were also investigated by Li and Willem [13], Liu and Zhou [11], Su [18] and Su and Zhao [19]. It is essential to point out that in the great majority of previous references, the nonlinearity is assumed to be differentiable (or even \( C^1 \)) in the second variable, being this assumption crucial in their arguments.

P. Bartolo, V. Benci and D. Fortunato [5] studied (P) when \( f(x,t) = f(t) \) is a smooth function, satisfying:

\[(BBF1) \lim_{|t| \to \infty} (\lambda_m t - f(t))t = 0;\]
\[(BBF2) \lim_{t \to \infty} \int_{-\infty}^{t} (\lambda_m s - f(s))ds = 0;\]
\[(BBF3) \int_{-\infty}^{t} (\lambda_m s - f(s))ds \geq 0, \text{ for all } t \in \mathbb{R}.\]

Under these assumptions, the authors were able to prove the existence of solution to (P). Observe that \((BBF1)\) implies that \( f(t)/t \to \lambda_m \) “faster” than \( t^2 \to \infty \) as \( |t| \to \infty \). More recently, Gongbao Li and Huan-Song Zhou [11] relaxed the differentiability of \( f \) by considering problem (P) under the following assumptions:

\[(LZ1) f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R});\]
\[(LZ2) \lim_{t \to 0} f(x,t)/t = 0 \text{ uniformly in } x \in \Omega;\]
\[(LZ3) \lim_{|t| \to \infty} f(x,t)/t = l \text{ uniformly in } x \in \Omega, \text{ where } l \in (0, \infty) \text{ is a constant, or } l = \infty \text{ and } |f(x,t)| \leq c_1 + c_2|t|^{q-1}, \text{ for some positive constants } c_1, c_2 \text{ and } q \in (2, 2^*);\]
\[(LZ4) f(x,t)/t \text{ is nondecreasing in } t \geq 0 \text{ for almost every } x \in \Omega;\]
\[(LZ5) f(x,-t) = -f(x,t) \text{ for all } (x,t) \in \Omega \times \mathbb{R}.\]

By using a symmetric version of the mountain pass theorem for \( C^1 \)-functionals, the authors were able to prove the following result:

**Theorem 1.1.** Assume that \( f(x,t) \) satisfies conditions \((LZ1)-(LZ3)\) and \((LZ5)\), and \( l > \lambda_m \), then the following hold:

(i) If \( l \in (\lambda_m, \infty) \) is not an eigenvalue of \(-\Delta\) with zero Dirichlet data, then problem (P) has at least \( k \) pairs of nontrivial solutions in \( H_0^1(\Omega) \).

(ii) Suppose that condition

\[(IF) \lim_{|t| \to \infty} [(1/2)f(x,t)t - F(x,t)] = \infty\]

is satisfied, where \( F \) is the primitive of \( f \). Then the conclusion of (i) holds even if \( l = \lambda_m \) is an eigenvalue of \(-\Delta\) with zero Dirichlet data.

(iii) If \( l = \infty \) in condition \((LZ3)\) and \((LZ4)\) holds, then problem (P) has infinitely many nontrivial solutions.

Let \( g \) be given in \((1.3)\) and \( G \) its primitive, since

\[(1/2)f(x,t)t - F(x,t) = G(x,t) - (1/2)tg(x,t),\]
it follows that if \(tg(x, t) = t^2[\eta(x) - f(x, t)/t] \to 0\) as \(|t| \to \infty\), then

\[
\lim_{|t| \to \infty} [(1/2)f(x, t)t - F(x, t)] = \lim_{|t| \to \infty} G(x, t) = \beta(x).
\]

Consequently, in the resonant case (that is \(\lambda_m(\eta) = 1\) for some \(m\)), the limit in (1.5) determines the degree of resonance of problem (P). For instance, in [5] we have \(\beta(x) = \beta \in \mathbb{R}\), thus the problem is strongly resonant. On the other hand, the result for the resonant situation in [11] does not cover the strongly resonant case when \(f(x, t)/t \to \eta(x) = l = \lambda_m\) “faster” than \(t^2 \to \infty\) as \(|t| \to \infty\). In fact, since \((fF)\) is assumed, it follows from (1.5) that \(\beta(x) \equiv \infty\).

In this paper we complement previous results. In fact, in the sequel, we suppose that \(f\) satisfies the following hypotheses:

\begin{itemize}
  \item[(f_1)] \(t \to f(x, t)/|t|\) is increasing (a. e. in \(\Omega\)) and \(\alpha^+, \eta^+ \neq 0\);
  \item[(f_2)] \(\lambda_m(\eta) < 1 < \lambda_1(\alpha)\), for some \(m \geq 1\),
\end{itemize}

where \(\alpha^+\) and \(\eta^+\) are the positive parts of functions \(\alpha\) and \(\beta\) defined in (1.1). It is important to point out that, by \(\alpha^+, \eta^+ \neq 0\) and [7], there exist the positive eigenvalues \(\lambda_m(\eta)\) and \(\lambda_1(\alpha)\).

Assuming \((f_1) - (f_2)\) we have provided existence of ground state solution and multiplicity for (P) in both cases: the non-resonant case \((NRC)\) and resonant case \((RC)\) (see Theorem 4.5), that is:

\begin{itemize}
  \item[(NRC)] \(\lambda_{m+k}(\eta) \neq 1\) for all \(k \in \mathbb{N}\),
  \item[(RC)] \(\lambda_{m+k}(\eta) = 1\) for some \(k \in \mathbb{N}\).
\end{itemize}

In particular, since in the case \((RC)\) we have not assumed hypothesis \((fF)\), our result cover strongly resonant nonlinearities, even when \(f(x, t)/t \to \lambda_m\) “faster” than \(t^2 \to \infty\) as \(|t| \to \infty\), see Section 5.

Finally, replacing assumptions \((f_1) - (f_2)\) by

\begin{itemize}
  \item[(f'_1)] \(\sup_{t \in A}[\mathcal{F}(. , t)/|t|] < \infty\), for each bounded set \(A \subset \mathbb{R}\), and \(\alpha^+, \eta^+ \neq 0\);
  \item[(f'_2)] \(\lambda_m(\alpha) < 1 < \lambda_1(\eta)\), for some \(m \geq 1\),
\end{itemize}

we still have been able to prove the existence of multiple solution for (P), see Theorem 4.6.

In Theorems 4.5 and 4.6, some progresses are obtained regarding the previous works. In what follows, we enumerate the main contributions: (1) Condition \((fF)\) is not required. Instead, we are imposing condition \((\beta)\), which is certainly weaker than \((fF)\). In our best knowledge, condition \((\beta)\) has not appeared in previous papers and allows us to study the existence of ground state solutions for some classes of strongly resonant problems which had not been treated. In a first moment, due to the presence of the number \(\tau_m\), whose dependence on \(f\) is not so explicit, assumption \((\beta)\) may seem difficult to be checked. In order to ensure its viability, we provide in the section 5 a concrete problem (P) for which \((\beta)\) holds and \((fF)\) is not verified;

(2) Since \(\alpha, \eta\) and \(\beta\) depend on \(x\) and \(f\) is not differentiable, our assumptions are more general than a large part of previous papers (which usually require these functions be constants); (3) We provide an unified approach to deal concurrently the non-resonant case, the strong and the non-strong resonant cases.

Our approach is based on the Nehari method, which consists in minimizing the energy functional \(I\) over the so called Nehari manifold \(\mathcal{N}\), a set which contains all the nontrivial solutions of the problem. Although this method has been carefully treated by A. Szulkin and T. Weth [20] for the case of nonlinearities which satisfies superquadraticity conditions at infinity, it is not a trivial task to apply it for problems involving asymptotically linear nonlinearities. In order to cite the main difficulties, we point out that the method consists
in proving the existence of a homeomorphism $\gamma$ between $\mathcal{N}$ and a submanifold $\mathcal{M}$ of $H_0^1(\Omega)$. Despite the absence of a differentiable structure in $\mathcal{N}$, such a homeomorphism allows us to define a $C^1$-functional $\Psi$ on $\mathcal{M}$ with very useful properties. However, differently of the problem for superquadratic nonlinearities, in which $\mathcal{M}$ is the unit sphere $S$ of $H_0^1(\Omega)$, for asymptotically linear nonlinearities, it is not exactly clear who is the suitable manifold $\mathcal{M}$. In fact, after a careful study (see Lemma 2.1, Propositions 3.1 and 3.2) we are able to prove that $\mathcal{M} = S_A := S \cap A$ is a noncomplete submanifold of $H_0^1(\Omega)$, where $A := \{ u \in H_0^1(\Omega) : \|u\|^2 < \int_\Omega \eta(x)u^2dx \}$. This fact brings additional problems. Indeed, it is important to assure that minimizing sequences $\{u_n\}$ for $\Psi$ are not near the boundary of $S_A$. In [20], this step is strongly based in the fact that $f$ has a superquadratic growth at infinity, what implies that $\{\Psi(u_n)\}$ tends to infinity as the distance from $\{u_n\}$ to the boundary tends to zero. In our case, the behaviour of $\{\Psi(u_n)\}$ at infinity, as $dist(u_n, \partial S_A) \to 0$, is indefinite. This fact makes difficult, for example, to know how to extend $\Psi$ to $\overline{S_A}$ in order to apply the Ekeland variational principle, which is crucial to prove that $\{u_n\}$ can be seen as Palais-Smale sequence.

The paper is organized as follows. In Section 2 we present the variational background. In Section 3 we deeply study the Nehari manifold and some of its topological features. In Section 4 we state and prove our main results. Finally, in Section 5, hypothesis $(\beta)$ is discussed in a concrete problem.

2. Preliminaries

Our main goal in this section is to introduce some variational background for $(P)$. We start denoting by $I : H_0^1(\Omega) \to \mathbb{R}$ the energy functional associated to problem $(P)$, given by

$$I(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u)dx,$$

where $\|u\|^2 = \int_\Omega |\nabla u|^2dx$ and $F(x, t) = \int_0^t f(x, s)ds$. It is well known that $I \in C^1(H_0^1(\Omega), \mathbb{R})$ and

$$I'(u)\varphi = \int_\Omega \nabla u \nabla \varphi dx - \int_\Omega f(x, u)\varphi dx.$$

Thus, critical points of $I$ are weak solutions of $(P)$.

The Nehari manifold associated to the functional $I$ is the set

$$\mathcal{N} = \{ u \in H_0^1(\Omega) \setminus \{0\} : \|u\|^2 = \int_\Omega f(x, u)dx \}.$$

Since $f$ is just a Carathéodory function, we cannot ensure that $\mathcal{N}$ is a smooth manifold. Moreover, $S$ denotes the unit sphere in $H_0^1(\Omega)$ and

$$A := \{ u \in H_0^1(\Omega) : \|u\|^2 < \int_\Omega \eta(x)u^2dx \}.$$

Now, it is important to fix some notation. Throughout this paper we denote by $e_j$ a normalized (in $H_0^1(\Omega)$ norm) eigenfunction associated to $\lambda_j(\eta)$ and use the symbol $[u \neq 0]$ to denote the set $\{x \in \Omega : u(x) \neq 0\}$. Moreover, $|A|$ will always denote the Lebesgue measure of a measurable set $A \subset \mathbb{R}^N$, $S(\Omega)$ and $|\theta|_{\infty}$ denote, respectively, the best constant of the continuous embedding from $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$ and the $L^\infty$-norm of a function $\theta$, $\chi(\theta) = \sum_{j=1}^m dim V_{\lambda_j(\theta)}$ is the sum of the dimensions of the first $m$ eigenspaces $V_{\lambda_j(\theta)}$ associated to $(1.2)$.

Finally, $S_\chi(\theta)$ denotes the unit sphere of $\bigoplus_{k=1}^m V_{\lambda_k(\theta)}$ and $\theta^+$ denotes the positive part of a function $\theta$. 

Lemma 2.1. Suppose that $f$ satisfies $(f_1) - (f_2)$. Then, the following claims hold:

(i) The set $\mathcal{A}$ is open and nonempty;
(ii) $\partial \mathcal{A} = \{ u \in H^1_0(\Omega) : \|u\|^2 = \int_{\Omega} \eta(x) u^2 dx \}$;
(iii) $\mathcal{A}^c = \{ u \in H^1_0(\Omega) : \|u\|^2 \geq \int_{\Omega} \eta(x) u^2 dx \}$;
(iv) $\mathcal{N} \subset \mathcal{A}$;
(v) $\mathcal{S} \cap \mathcal{A} \neq \emptyset$.

Proof. (i) By $(f_2)$, if $u$ is an eigenfunction associated to $\lambda_j(\eta)$, for some $j \in \{1, \ldots, m\}$, then $u$ belongs to $\mathcal{A}$. Moreover, $\mathcal{A} = \varphi^{-1}(-\infty, 0)$, where $\varphi : H^1_0(\Omega) \to \mathbb{R}$ is the continuous function $\varphi(u) = \|u\|^2 - \int_{\Omega} \eta(x) u^2 dx$. Items (ii)-(iii) are immediate consequences of the definition of $\mathcal{A}$.

(iv) If $u \in \mathcal{N}$ then,
\[
\|u\|^2 = \int_{\{u \neq 0\}} \left[ \frac{f(x,u)}{u} \right] u^2 dx.
\]
By $(f_1)$, we conclude
\[
\|u\|^2 < \int_{\Omega} \eta(x) u^2 dx.
\]
Showing that $u \in \mathcal{A}$.

(v) It is enough to choose an eigenfunction $e_j$ associated to $\lambda_j(\eta)$ and normalized in $H^1_0(\Omega)$, whatever $j \in \{1, \ldots, m\}$. For sure, we have $e_j \in \mathcal{S} \cap \mathcal{A}$. \qed

In what follows, we will denote $\mathcal{S}_A := \mathcal{S} \cap \mathcal{A}$. Since $\mathcal{S}$ is a $C^1$-manifold of $H^1_0(\Omega)$ and, by Lemma 2.1, $\mathcal{A}$ is open set of $H^1_0(\Omega)$ whose boundary has intersection with $\mathcal{S}$; it follows that $\mathcal{S}_A$ is a noncompact $C^1$-manifold of $H^1_0(\Omega)$. Moreover, from (ii) and (iii), it is clear that $\partial \mathcal{S}_A = \{ u \in \mathcal{S} : 1 = \int_{\Omega} \eta(x) u^2 dx \}$ and $\mathcal{S}_A^c = \{ u \in \mathcal{S} : 1 \geq \int_{\Omega} \eta(x) u^2 dx \}$. The following property of functions in $\partial \mathcal{S}_A$ plays an important role in the existence of solution for $(P)$, see Lemmas 3.6 and 4.3.

Lemma 2.2. The following inequality holds true:
\[
\inf_{u \in \partial \mathcal{S}_A} |[u \neq 0]| \geq (\mathcal{S}(\Omega)/|\eta|_\infty)^{N/2}.
\]

Proof. By using Hölder inequality, it follows that, for each $u \in \partial \mathcal{S}_A$
\[
1 \leq |\eta|_\infty \int_{\{u \neq 0\}} u^2 dx \leq |\eta|_\infty |u|_{L^2}^2 |[u \neq 0]|^{2/N}.
\]
By continuous Sobolev embedding from $H^1_0(\Omega)$ into $L^{2^*}(\Omega)$,
\[
1 \leq |\eta|_\infty (1/\mathcal{S}(\Omega)) |[u \neq 0]|^{2/N}.
\]
Therefore,
\[
|[u \neq 0]| \geq (\mathcal{S}(\Omega)/|\eta|_\infty)^{N/2}, \forall \ u \in \partial \mathcal{S}_A.
\]
The result is proved. \qed

Next lemma provides some consequences of hypothesis $(f_1)$ which will be useful later on.

Lemma 2.3. Suppose that $(f_1)$ holds. Then, a.e. in $\Omega$,

(i) $t \mapsto (1/2)f(x,t) t - F(x,t)$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$;
(ii) $t \mapsto F(x,t)/t^2$ is increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$;
(iii) $f(x,t)/t > 2F(x,t)/t^2$ for all $t \in \mathbb{R}\backslash\{0\}$. 

Proof. (i) Without loss of generality, we can suppose \( t_1 > t_2 > 0 \). Then, a.e. in \( \Omega \),
\[
\frac{1}{2} f(x, t_1) t_1 - F(x, t_1) = \frac{1}{2} f(x, t_1) t_1 - F(x, t_2) - \int_{t_2}^{t_1} \left[ \frac{f(x, s)}{s} \right] s \, ds
\]
\[
> \frac{1}{2} f(x, t_1) t_1 - F(x, t_2) - \frac{f(x, t_1)}{t_1} \int_{t_2}^{t_1} s \, ds
\]
\[
= \frac{1}{2} f(x, t_1) t_1 - F(x, t_2) - \frac{f(x, t_1)}{t_1} (t_1^2 - t_2^2)
\]
\[
= \frac{f(x, t_1) t_2^2}{2} - F(x, t_2)
\]
\[
> \frac{1}{2} f(x, t_2) t_2 - F(x, t_2),
\]
where it was used \((f_1)\) in the last two inequalities. The other case is analogous. Items (ii) and (iii) follows from (i).

From Lemma 2.3\((i)\), the maps \( \beta : \Omega \to (0, \infty] \) given by
\[
(2.1) \quad \beta(x) = \lim_{|t| \to \infty} \left( (1/2) f(x, t) t - F(x, t) \right)
\]
is well defined, and, it might not be integrable.

3. Topological aspects of the Nehari manifold

The main goal of this section is to study some topological features of the Nehari manifold under hypotheses \((f_1) - (f_2)\) and the behaviour of the energy functional \( I \) on \( N \).

Proposition 3.1. Suppose that \( f \) satisfies \((f_1) - (f_2)\) and let \( h_u : [0, \infty) \to \mathbb{R} \) be defined by \( h_u(t) = I(tu) \).

(i) For each \( u \in A \), there exists a unique \( t_u > 0 \) such that \( h'_u(t) > 0 \) in \((0, t_u)\), \( h'_u(t_u) = 0 \) and \( h'_u(t) < 0 \) in \((t_u, \infty)\). Moreover, \( tu \in N \) if, and only if, \( t = t_u \).

(ii) for each \( u \in A^c \), \( h'_u(t) > 0 \) for all \( t \in (0, \infty) \).

Proof. (i) First observe that \( h_u(0) = 0 \). Moreover, for each \( u \in A \), we have
\[
(3.1) \quad \frac{h_u(t)}{t^2} = \frac{1}{2} \|u\|^2 - \int_{|u| \neq 0} \left[ \frac{F(x, tu)}{(tu)^2} \right] u^2 \, dx.
\]
Thus, from \((f_1) - (f_2)\), L’Hospital rule and Lebesgue Dominated Convergence Theorem, it follows that
\[
\lim_{t \to 0} \frac{h_u(t)}{t^2} = \frac{1}{2} \left( \|u\|^2 - \int_{\Omega} \alpha(x) u^2 \, dx \right) > 0
\]
and
\[
\lim_{t \to \infty} \frac{h_u(t)}{t^2} = \frac{1}{2} \left( \|u\|^2 - \int_{\Omega} \eta(x) u^2 \, dx \right) < 0.
\]
Showing that
\[
\lim_{t \to 0} \frac{h_u(t)}{t^2} = \frac{h_u(t)}{t^2} = 0
\]
is positive for \( t \) small and
\[
\lim_{t \to \infty} \frac{h_u(t)}{t^2} = \lim_{t \to \infty} \frac{h_u(t)}{t^2} = -\infty.
\]
Since $h_u$ is a continuous function, previous arguments imply that there exists a global maximum point $t_u > 0$ of $h_u$. Now, we are going to show that $t_u$ is the unique critical point of $h_u$. In fact, supposing that there exist $t_1 > t_2 > 0$ such that $h_u'(t_1) = h_u'(t_2) = 0$, we obtain

$$0 = \int_{[u \neq 0]} \left[ \frac{f(x, t_1 u)}{t_1 u} - \frac{f(x, t_2 u)}{t_2 u} \right] u^2 dx,$$

and, by $(f_1)$, $t_1 = t_2$. The result follows.

(ii) If $u \in A^c$, then $\|u\|^2 \geq \int_{\Omega} \eta(x)u^2 dx$. Thus, it follows from $(f_1)$ that

$$h_u'(t) = \|u\|^2 - \int_{[u \neq 0]} \frac{f(x, tu)}{tu} u^2 dx \geq \int_{[u \neq 0]} \left[ \eta(x) - \frac{f(x, tu)}{tu} \right] u^2 dx > 0, \ \forall \ t > 0.$$

Consequently, $h_u'(t) = t(h_u'(t)/t) > 0$ for all $t \in (0, \infty)$.

\[\Box\]

Remark 1. It is an immediate consequence of previous proposition that, for each $u \in A$ and $s \in (0, \infty)$, $t_{su} = t_u/s$. Moreover, it is clear that $u \in \mathcal{N}$ if, and only if, $t_u = 1$.

Proposition 3.2. Suppose that $f$ satisfies $(f_1) - (f_2)$. Then, the following claims hold:

(A1) $\tau_m := \inf_{u \in \mathcal{S}(\eta)} t_u > 0$;

(A2) $\zeta_W := \max_{u \in W} t_u < \infty$, for all compact set $W \subset \mathcal{S}_A$;

(A3) Map $\hat{m} : A \to \mathcal{N}$ given by $\hat{m}(u) = t_u u$ is continuous and $m := \hat{m}|_{\mathcal{S}_A}$ is a homeomorphism between $\mathcal{S}_A$ and $\mathcal{N}$. Moreover, $m^{-1}(u) = u/\|u\|$.

Proof. (A1) It is a consequence of the proof of Lemma 2.1 that $\mathcal{S}(\eta) \subset \mathcal{S}_A$. Thus, suppose that there exists $\{u_n\} \subset \mathcal{S}_A$ such that $t_n := t_{u_n} \to 0$. In this case, we get $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$. It follows from $(f_1)$ and Lebesgue Dominated Convergence Theorem, that

$$(3.2) \quad \int_{\Omega} \left[ \frac{f(x, t_n u_n)}{t_n u_n} \right] \chi_{[u_n \neq 0]} u_n^2 dx \to \int_{\Omega} \alpha(x)u^2 dx.$$

From Proposition 3.1, we have

$$(3.3) \quad 1 = \int_{\Omega} \left[ \frac{f(x, t_n u_n)}{t_n u_n} \right] \chi_{[u_n \neq 0]} u_n^2 dx, \ \forall \ n \in \mathbb{N}.$$

Now, by

$$\chi_{[u_n \neq 0]}(x) \to 1 \text{ a.e. in } [u \neq 0] \text{ and } \chi_{[u_n \neq 0]}(x) \to 0 \text{ a.e. in } [u = 0],$$

and (3.2), passing to the limit as $n$ tends to infinity in (3.3), we get

$$1 = \int_{\Omega} \alpha(x)u^2 dx.$$

If $\alpha = 0$, we have a clear contradiction. Otherwise, the inequality

$$1 \leq (1/\lambda_1(\alpha)) \|u\|^2 \leq 1/\lambda_1(\alpha),$$

contradicts $(f_2)$.

(A2) Suppose that there exists $\{u_n\} \subset W$ such that $t_n := t_{u_n} \to \infty$. Since $W$ is compact, passing to a subsequence, we obtain $u \in W$ such that $u_n \to u$ in $H_0^1(\Omega)$. Since

$$1 = \|u_n\|^2 = \int_{\Omega} \left[ \frac{f(x, t_n u_n)}{t_n u_n} \right] \chi_{[u_n \neq 0]} u_n^2 dx, \ \forall \ n \in \mathbb{N},$$

...
Last inequality implies that $u$ passing to the lower limit as $n \to \infty$, showing that $u \in S_A$, leading us to a contradiction, since $u \in W \subset S_A$.

(A3) We first show that $\hat{m}$ is continuous. Let $\{u_n\} \subset A$ and $u \in A$, be such that $u_n \to u$ in $H^1_0(\Omega)$. From Remark 1 ($\hat{m}(tu) = \hat{m}(w)$ for all $w \in A$ and $t > 0$), we can assume, without loss of generality, that $\{u_n\} \subset S_A$. Thus,

$$t_n = t_n\|u_n\|^2 = \int_{\Omega} f(x,t_n u_n)u_n dx,$$

where $t_n := t u_n$. From (A1) and (A2), it follows that, up to a subsequence, $t_n \to t > 0$. Thence, passing to the limit as $n \to \infty$ in (3.4), we have

$$t = t\|u\|^2 = \int_{\Omega} f(x,tu)udx,$$

showing that $\hat{m}(u_n) = t_n u_n \to tu = \hat{m}(u)$. The second part of (A3) is immediate.

\[\square\]

**Lemma 3.3.** Suppose that $(f_1) - (f_2)$ holds. Then, $I(u) > 0$, for all $u \in N$.

**Proof.** For any $u \in S_A$, we get

$$I(t_u u) = \int_{\Omega} \left[ \frac{1}{2} \frac{f(x,t_u u)}{t_u u} - \frac{F(x,t_u u)}{(t_u u)^2} \right] (t_u u)^2 dx.$$

The result follows from Lemma 2.3(iii).

\[\square\]

In what follows, let us consider the maps $\hat{\Psi} : A \to \mathbb{R}$ and $\Psi : S_A \to \mathbb{R}$, given by

$$\hat{\Psi}(u) = I(\hat{m}(u))$$

and $\Psi := \hat{\Psi}|_{S_A}$. These maps will be very important in our arguments mainly because of their properties, which will be presented in the next result. The proof of such a result is a consequence of Proposition 3.2 and the details can be found in [20].

**Proposition 3.4.** Suppose that $f$ satisfies $(f_1) - (f_2)$. Then,

(i) $\hat{\Psi} \in C^1(A, \mathbb{R})$ and

$$\hat{\Psi}'(u)v = \hat{m}(u)\|v\|, \forall u \in A \text{ and } \forall v \in H^1_0(\Omega).$$

(ii) $\Psi \in C^1(S_A, \mathbb{R})$ and

$$\Psi'(u)v = m(u)\|v\|, \forall u \in T_A S_A.$$

(iii) If $\{u_n\}$ is a $(PS)_c$ sequence for $\Psi$ then $\{m(u_n)\}$ is a $(PS)_c$ sequence for $I$. If $\{u_n\} \subset N$ is a bounded $(PS)_c$ sequence for $I$ then $\{m^{-1}(u_n)\}$ is a $(PS)_c$ sequence for $\Psi$. 
(iv) \( u \) is a critical point of \( \Psi \) if, and only if, \( m(u) \) is a nontrivial critical point of \( I \). Moreover,

\[
c_N := \inf_{u \in \mathcal{N}} I(u) = \inf_{u \in \mathcal{A}} \max_{t > 0} I(tu) = \inf_{u \in \mathcal{A}} \max_{t > 0} I(tu) = \inf_{u \in \mathcal{A}} \Psi(u).
\]

**Remark 2.** It is a consequence of Lemma 3.3 that \( c_N \geq 0 \). Moreover, if \( c_N \) is achieved then it is positive.

Since \( \mathcal{S}_A \) can be non-complete, we need to be careful about the behaviour of minimizing sequences for \( \Psi \) near the boundary. Next result helps us in this direction.

**Proposition 3.5.** Suppose that \((f_1) - (f_2)\) hold. If \( \{u_n\} \subset \mathcal{S}_A \) is such that \( \text{dist}(u_n, \partial \mathcal{S}_A) \to 0 \), then there exists \( u \in H^1_0(\Omega) \setminus \{0\} \) such that \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \), \( tu_n \to \infty \) and

\[
\liminf_{n \to \infty} \Psi(u_n) \geq \int_{\{u \neq 0\}} \beta(x)dx,
\]

where the maps \( \beta \) was defined in (2.1).

**Proof.** Since \( \{u_n\} \subset \mathcal{S}_A \) is bounded, up to a subsequence, there exists \( u \in H^1_0(\Omega) \) with \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \). On the other hand, from \( \text{dist}(u_n, \partial \mathcal{S}_A) \to 0 \), there exists \( \{z_n\} \subset \partial \mathcal{S}_A \) such that \( \|u_n - z_n\| \to 0 \) as \( n \to \infty \). Thus,

\[
\left| \int_{\Omega} \eta(x)u_n^2dx - 1 \right| = \left| \int_{\Omega} \eta(x)(u_n^2 - z_n^2)dx \right| \\
\leq |\eta|_\infty |u_n + z_n|_2 |u_n - z_n|_2 \\
\leq (2|\eta|_\infty / \lambda_1)\|u_n - z_n\|.
\]

Therefore,

\[
\int_{\Omega} \eta(x)u_n^2dx \to 1.
\]

By using compact embedding from \( H^1_0(\Omega) \) in \( L^2(\Omega) \), it follows that

\[(3.5) \quad 1 = \int_{\Omega} \eta(x)u^2dx.\]

Thus

\[(3.6) \quad u \neq 0.\]

Suppose by contradiction that, for some subsequence, \( \{t_{u_n}\} \) is bounded. In this case, passing again to a subsequence, there exists \( t_0 > 0 \) (see Proposition 3.2(A1)) such that

\[(3.7) \quad t_{u_n} \to t_0.\]

It follows from (3.6), (3.7) and

\[
t_{u_n} = \int_{\Omega} f(x, t_{u_n}u_n)u_ndx, \quad \forall \ n \in \mathbb{N},
\]

that

\[
1 = \int_{\{u \neq 0\}} \frac{f(x, t_0u)}{t_0u} u^2dx.
\]
Combining last equality and \((f_1)\), we have
\[
1 < \int_{\Omega} \eta(x)u^2dx. 
\]
Clearly, \((3.5)\) and \((3.8)\) contradicts each other, showing that \(t_{u_n} \to \infty\). Since the behaviour at infinity of \(\{t_{u_n}u_n\}\) is indetermined on \([u = 0]\), in next inequality, we restrict our arguments to the set \([u \neq 0]\). In fact, it follows from Lemma 2.3, that
\[
\lim_{n \to \infty} \inf \Psi(u_n) = \lim_{n \to \infty} \inf \int_{\Omega} \left[ \frac{1}{2}f(x,t_{u_n}u)tu_n - F(x,t_{u_n}u) \right] dx 
\geq \lim_{n \to \infty} \inf_{[u \neq 0]} \int_{[u \neq 0]} \left[ \frac{1}{2}f(x,t_{u_n}u)tu_n - F(x,t_{u_n}u) \right] dx 
= \int_{[u \neq 0]} \beta(x)dx. 
\]

The following hypothesis is certainly weaker than \((fF)\) and will be considered in our next result
\[
\beta \quad \text{inf ess}_{x \in \Omega} \beta(x) > \frac{|\eta|^{N/2} \tau_m}{2\lambda_1(\eta - \alpha)S(\Omega)^{N/2}}, 
\]
where \(\tau_m\) is defined in Proposition 3.2(i).

Lemma 3.6. Suppose that \(f\) satisfies \((f_1)\) – \((f_2)\) and \((\beta)\). Then
\[
c_N < \inf_{u \in \partial S_A} \int_{[u \neq 0]} \beta(x)dx. 
\]

Proof. It follows from \((f_1)\) that, for each \(u \in S_A\),
\[
c_N \leq \Psi(u) = \int_{\Omega} \left[ \frac{1}{2}f(x,t_uu)tu - F(x,t_uu) \right] dx 
\leq (1/2) \int_{\Omega} [\eta(x) - \alpha(x)]m(u)^2dx 
\leq [1/2\lambda_1(\eta - \alpha)]^2u. 
\]

On the other hand, by \((2.2)\), for each \(u \in \partial S_A\)
\[
\int_{[u \neq 0]} \beta(x)dx \geq |[u \neq 0]| \inf \text{ess}_{x \in \Omega} \beta(x) \geq (S(\Omega)/|\eta|_\infty)^{N/2} \inf \text{ess}_{x \in \Omega} \beta(x). 
\]

The result follows now from \((\beta)\), \((3.9)\) and \((3.10)\). \(\square\)

Based in previous results, in the next illustration, we try to give an idea about some possible topological configuration of the sets \(\mathcal{N}, A\) and \(S_A\). The set \(A\) is represented as the nonempty interior of a “cone” in \(H^1_0(\Omega)\) which intersects the unit sphere in the set \(S_A\) and contains the Nehari set \(\mathcal{N}\). As stated in Propositions 3.2 and 3.5, the Nehari set is homeomorphic to \(S_A\), unbounded and asymptote the boundary of \(A\) at infinity.
Before stating next result, we remember hypotheses (NRC) and (RC) stated in the introduction.

(NRC) \( \lambda_{m+k}(\eta) \neq 1 \) for all \( k \in \mathbb{N} \);

(RC) \( \lambda_{m+k}(\eta) = 1 \) for some \( k \in \mathbb{N} \).

**Proposition 3.7.** Suppose that \( f \) satisfies \((f_1) - (f_2)\).

(i) If (NRC) holds, then \( \Psi \) satisfies the \((PS)_c\) condition in \( S_A \), for all \( c \geq c_N \);

(ii) If (RC) and \( (\beta) \) hold, then \( \Psi \) satisfies the \((PS)_c\) condition in \( S_A \), for all \( c \in [c_N, \inf_{u \in \partial S_A} \int_{u \neq 0} \beta(x)dx] \).

**Proof.** By Proposition 3.2(A3) and Proposition 3.4(iii), it is enough to show that \( I \) satisfies the \((PS)_c\) condition on \( N \) for \( c \in [c_N, \inf_{u \in \partial S_A} \int_{u \neq 0} \beta(x)dx] \). For this, let \( \{u_n\} \subset N \) be a \((PS)_c\) sequence for the functional \( I \). We are going to prove that \( \{u_n\} \) is bounded in \( H^1_0(\Omega) \). In fact, suppose by contradiction that, up to a subsequence, \( \|u_n\| \to \infty \). Define \( v_n := u_n/\|u_n\| = m^{-1}(u_n) \in S_A \). Thus \( \{v_n\} \) is bounded in \( H^1_0(\Omega) \) and

\[
\Psi(v_n) \to c.
\]

Consequently, there exists \( v \in H^1_0(\Omega) \) such that

\[
v_n \rightharpoonup v \quad \text{in} \quad H^1_0(\Omega).
\]
Suppose \( v = 0 \). Since \( \{ \Psi(v_n) \} \) is bounded, it follows that there exists \( C > 0 \) such that
\[
C > \Psi(v_n) = I(tv_n) \geq I(tv_n) = (1/2)t^2 - \int_{\Omega} F(x, tv_n) \, dx, \quad \forall \ t > 0.
\]
From \((f_1) - (f_2)\) and compact embedding, passing to the limit as \( n \to \infty \) in \((3.13)\), we get
\[
C \geq (1/2)t^2, \quad \forall \ t > 0,
\]
a clear contradiction. Thereby, we conclude that
\[
(3.14) \quad v \neq 0.
\]
Now, since \( \{ u_n \} \subset N \) is a \((PS)\) sequence for \( I \), we get
\[
o_n(1) + \int_{\Omega} \nabla u_n \nabla w \, dx = \int_{\Omega} f(x, u_n) w \, dx, \quad \forall \ w \in H^1_0(\Omega).
\]
Dividing last equality by \( \| u_n \| \), we have
\[
(3.15) \quad o_n(1) + \int_{\Omega} \nabla v_n \nabla w \, dx = \int_{\Omega} \left[ f(x, \| u_n \| v_n) \right] \chi_{[v_n \neq 0]}(x) v_n \, w \, dx + \int_{[v=0]} \left[ f(x, \| u_n \| v_n) \right] \chi_{[v_n \neq 0]}(x) v_n \, w \, dx.
\]
By \((f_1)\), \((3.12)\) and Lebesgue Dominated Convergence Theorem, it follows that
\[
(3.16) \quad \int_{[v=0]} \left[ f(x, \| u_n \| v_n) \right] \chi_{[v_n \neq 0]}(x) v_n \, w \, dx \to 0.
\]
On the other hand, by \((3.12)\) and Lebesgue Dominated Convergence Theorem, we get
\[
(3.17) \quad \int_{[v \neq 0]} \left[ f(x, \| u_n \| v_n) \right] \chi_{[v_n \neq 0]}(x) v_n \, w \, dx \to \int_{\Omega} \eta(x) v \, w \, dx.
\]
It follows from \((3.16)\) and \((3.17)\) that, passing to the limit as \( n \to \infty \) in \((3.15)\), we obtain
\[
(3.18) \quad \int_{\Omega} \nabla v \nabla w \, dx = \int_{\Omega} \eta(x) v \, w \, dx, \quad \forall \ w \in H^1_0(\Omega).
\]
Now we have to consider two cases:

(i) If \( \lambda_{m+k}(\eta) \neq 1 \), for all \( k \in \mathbb{N} \), it follows from \((3.18)\) that \( v = 0 \). But this is a contradiction with \((3.14)\). Therefore \( \{ u_n \} \) is bounded in \( H^1_0(\Omega) \).

(ii) If \( \lambda_{m+k}(\eta) = 1 \), for some \( k \in \mathbb{N} \), then \((3.18)\) implies that \( v = e_{m+k} \), where \( e_{m+k} \) is some eigenfunction associated to \( \lambda_{m+k}(\eta) \). From \((3.18)\), it follows also that
\[
\int_{\Omega} \eta(x) v^2 \, dx = \| v \|^2 \leq \liminf_{n \to \infty} \| u_n \|^2 = 1,
\]
that is, \( v \in S^c_A \). Suppose that
\[
(3.19) \quad \int_{\Omega} \eta(x) v^2 \, dx < 1.
\]
Since
\[
(3.20) \quad tv_n = \| tv_n \| = \| u_n \| \to \infty,
\]
by arguing as in (3.16) and (3.17), we obtain
\[
\int_{\Omega} \left[ \frac{F\left(\|u_n\|v_n\right)}{\left(\|u_n\|v_n\right)^2} \right] v_n^2 dx \to \frac{1}{2} \int_{\Omega} \eta(x)v^2 dx.
\]
Thus, passing to the limit as \( n \to \infty \) in the identity
\[
\Psi(v_n) = \|u_n\|^2 \left\{ \frac{1}{2} - \int_{\Omega} \left[ \frac{F\left(\|u_n\|v_n\right)}{\left(\|u_n\|v_n\right)^2} \right] v_n^2 dx \right\}
\]
and using (3.21), we conclude that \( \Psi(v_n) \to \infty \), what is a contradiction with (3.11). Consequently,
\[
1 = \|v\|^2 = \int_{\Omega} \eta(x)v^2 dx,
\]
showing that \( v \in \partial S_A \) and
\[
\|v_n\| \to \|v\|.
\]
By using (3.12) and (3.23), we conclude that \( v_n \to v \) in \( H^1_0(\Omega) \). By invoking Proposition 3.5, we have
\[
c \geq \int_{\{v \neq 0\}} \beta(x) dx,
\]
with \( v = e_{m+k} \in \partial S_A \). Since \( (f_1) - (f_2) \) and \( (\beta) \) hold, last equality contradicts Lemma 3.6, showing that \( \{u_n\} \) is bounded.

Since \( \{u_n\} \) is a bounded sequence, there exists \( u \in H^1_0(\Omega) \) such that \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \), up to a subsequence. Thus, to finish the proof, it remains us to prove that \( \|u_n\| \to \|u\| \). For this, it is sufficient to note that since \( \{u_n\} \) is a \( (PS)_c \) sequence, we have
\[
o_n(1) + \int_{\Omega} \nabla u_n \nabla u dx = \int_{\Omega} f(x, u_n) u dx.
\]
Passing to the limit as \( n \to \infty \) in the previous equality, we get
\[
\|u\|^2 = \int_{\Omega} f(x, u) u dx.
\]
Then (3.25) and Lebesgue’s convergence theorem imply that
\[
\|u_n\|^2 = \int_{\Omega} f(x, u_n) u_n dx = \int_{\Omega} f(x, u) u dx + o_n(1) = \|u\|^2 + o_n(1).
\]
\( \square \)

4. Multiplicity of solutions

In this section, among other things, we are going to prove that the problem \( (P) \) has as many pairs of solutions as we want, provided that function \( f \) is odd with respect to the second variable and the size of \( \lambda_m(\eta) \) or \( \lambda_m(\alpha) \), for \( m \) large enough, is controlled from above.

Our main results in this section will be proved through the Krasnosel’skii’s genus theory. Thus, we start by defining some preliminaries notations:
\[
\gamma_j := \{ B \in \mathcal{E} : B \subset S_A \text{ and } \gamma(B) \geq j \},
\]
where
\[
\mathcal{E} = \{ B \subset H^1_0(\Omega) \setminus \{0\} : B \text{ is closed and } B = -B \}
\]
and \( \gamma : \mathcal{E} \to \mathbb{Z} \cup \{\infty\} \) is the Krasnosel’skii’s genus function, which is defined by
\begin{equation}
\gamma(B) = \begin{cases} 
n := \min\{m \in \mathbb{N} : \text{there exists an odd map } \varphi \in C(B, \mathbb{R}^m \setminus \{0\})\}, \\
\infty, & \text{if there exists no map } \varphi \in C(B, \mathbb{R}^m \setminus \{0\}), \\
0, & \text{if } B = \emptyset.
\end{cases}
\end{equation}

It is important to point that \( \gamma_j \) is well defined, since \( \mathcal{S}_A = -\mathcal{S}_A \).

In the sequel we state some standard properties of the genus which will play a role in this work. More information about this subject can be found, for instance, in [3] or [10].

**Lemma 4.1.** Let \( B \) and \( C \) be sets in \( \mathcal{E} \).

(i) If \( x \neq 0 \), then \( \gamma(\{x\} \cup \{-x\}) = 1 \);
(ii) If there exists an odd map \( \varphi \in C(B, C) \), then \( \gamma(B) \leq \gamma(C) \). In particular, if \( B \subset C \) then \( \gamma(B) \leq \gamma(C) \).
(iii) If there exists an odd homeomorphism \( \varphi : B \rightarrow C \), then \( \gamma(B) = \gamma(C) \). In particular, if \( B \) is homeomorphic to the unit sphere in \( \mathbb{R}^n \), then \( \gamma(B) = n \).
(iv) If \( B \) is a compact set, then there exists a neighborhood \( K \in \mathcal{E} \) of \( B \) such that \( \gamma(B) = \gamma(K) \).
(v) If \( \gamma(C) < \infty \), then \( \gamma(B \setminus C) \geq \gamma(B) - \gamma(C) \).
(vi) If \( \gamma(A) \geq 2 \), then \( A \) has infinitely many points.

From now on, we denote by \( s_m \) the sum of the dimensions of all eigenspaces \( V_j \) associated to eigenvalues \( \lambda_j(\eta) \), where \( 1 \leq j \leq m \).

**Lemma 4.2.** Suppose that \( f(x, \cdot) \) is odd a.e. in \( \Omega \) and satisfies \( (f_2) \). The following statements hold true:

(i) \( \gamma_{s_m} \neq \emptyset \);
(ii) \( \gamma_1 \supset \gamma_2 \supset \ldots \supset \gamma_{s_m} \);
(iii) If \( \varphi \in C(\mathcal{S}_A, \mathcal{S}_A) \) and is odd, then \( \varphi(\gamma_j) \subset \gamma_j \), for all \( 1 \leq j \leq s_m \);
(iv) If \( B \in \gamma_j \) and \( C \in \mathcal{E} \) with \( \gamma(C) \leq s < j \leq s_m \), then \( B \setminus C \in \gamma_{j-s} \).

**Proof.** (i) Let \( \mathcal{S}_{s_m} \) be the \( (s_m \)-dimensional) unit sphere of \( V_1 \oplus V_2 \oplus \ldots \oplus V_m \). From \( (f_2) \), it is clear that \( \mathcal{S}_{s_m} \subset \mathcal{S}_A \). Moreover, from Lemma 4.1(iii), we have \( \gamma(\mathcal{S}_{s_m}) = s_m \), showing that \( \mathcal{S}_{s_m} \subset \gamma_{s_m} \). (ii) It is immediate. (iii) It follows directly from Lemma 4.1(ii). (iv) It is a consequence of Lemma 4.1(v). \( \square \)

Now, for each \( 1 \leq j \leq s_m \), we define the following minimax levels
\begin{equation}
c_j = \inf_{B \in \gamma_j} \sup_{u \in B} \Psi(u).
\end{equation}

**Lemma 4.3.** Suppose that \( f(x, \cdot) \) is odd a.e. in \( \Omega \), satisfies \( (f_1) - (f_2) \) and \( (\beta) \). Then,
\begin{equation}
0 < c_N = c_1 \leq c_2 \leq \ldots \leq c_{s_m} < \inf_{u \in \partial \mathcal{S}_A} \int_{[u \neq 0]} \beta(x) dx.
\end{equation}
Proposition 4.4. Suppose that $\gamma$ that is a contradiction. Then (4.3).

Thus, 
\begin{equation}
\frac{2}{c_m} \leq \max_{u \in S} \Psi(u) \leq \frac{2}{c_m},
\end{equation}

where $c_m$ was defined in Proposition 3.2. The result follows now from Lemma 2.2, (\beta) and (4.3).

Next proposition is crucial to ensure the multiplicity of solutions.

**Proposition 4.4.** Suppose that $f(x, \cdot)$ is odd a.e. in $\Omega$ and satisfies $(f_1) - (f_2)$. If $c_j = \ldots = c_{j+p} = c$, $j + p \leq s_m$ and (NRC), or (RC) and (\beta), occurs then $\gamma(K_c) \geq p + 1$, where $K_c := \{v \in S_A : \Psi(v) = c$ and $\Psi'(v) = 0\}$.

**Proof.** Suppose that $\gamma(K_c) \leq p$. It follows from Proposition 3.7 and Lemma 4.3(i) that $K_c$ is a compact set. Thus, by Lemma 4.1(iv), there exists a compact neighborhood $K \in E$ of $K_c$ such that $\gamma(K) \leq p$. Defining $M := K \cap S_A$, we derive from Lemma 4.1(ii) that $\gamma(M) \leq p$. Despite the noncompleteness of $S_A$ we still can use Theorem 3.11 in [17] (see also Remark 3.12 in [17]) to ensure the existence of an odd homeomorphisms family $\eta(\cdot, t)$ of $S_A$ such that, for each $u \in S_A$,
\begin{equation}
\eta(u, 0) = u
\end{equation}

and
\begin{equation}
t \mapsto \Psi(\eta(u, t))
\end{equation}

is non-increasing.

Observe that, although $S_A$ is non-complete, from Proposition 3.5, Lemma 4.3 and (4.5), for $\varepsilon > 0$ small enough, map $\eta(u, \cdot)$ is well defined in $[0, \infty)$, for each $u \in S_A : \Psi(u) \leq c_m + \varepsilon$. Indeed, suppose by contradiction that $\eta(u, t_0) \in \partial S_A$ for some $u \in S_A : \Psi(u) = c_m + \varepsilon$ and $t_0 > 0$, where $\varepsilon \in (0, \inf_{u \in \partial S_A} \int_{[u \neq 0]} \beta(x)dx - c_m)$. Then, by (4.4), Proposition 3.5 and Lemma 4.3
\begin{equation}
\Psi(\eta(u, 0)) = \Psi(u) \leq c_m + \varepsilon < \int_{[\eta(u, t_0) \neq 0]} \beta(x)dx \leq \lim inf_{t \to t_0} \Psi(\eta(u, t)).
\end{equation}

Thus, there exists $0 < t_s < t_0$, such that $\eta(u, t_s) \in S_A$ and
\begin{equation}
\Psi(\eta(u, 0)) < \Psi(\eta(u, t_s)).
\end{equation}

However, last inequality contradicts (4.5). Thus, third claim of Theorem 3.11 in [17] holds, namely,
\begin{equation}
\eta(\Psi_{c+\varepsilon} \setminus M, 1) \subset \Psi_{c-\varepsilon}.
\end{equation}

Let us choose $B \in \gamma_{j+p}$ such that $\sup_{B} \Psi \leq c + \varepsilon$. From Lemma 4.2(iv), $\overline{B \setminus M} \in \gamma_j$. It follows again from Lemma 4.2(iii) that $\eta(\overline{B \setminus M}, 1) \in \gamma_j$. Therefore, by (4.6) and definition of $c$, we have
\begin{equation}
c \leq \sup_{\eta(\overline{B \setminus M}, 1)} \Psi \leq c - \varepsilon,
\end{equation}

that is a contradiction. Then $\gamma(K_c) \geq p + 1$. 

\qed
We are now ready to prove the following two multiplicity results.

**Theorem 4.5.** Suppose that \( f \) satisfies (\( f_1 \)) – (\( f_2 \)) and (NRC), or (RC) and (\( \beta \)). The following statements hold:
(i) If \( m = 1 \), then there exists a signed mountain pass ground-state solution for problem (P).
(ii) If \( f(x,\cdot) \) is odd a.e. in \( \Omega \), then problem (P) has at least \( \chi(\eta) \) pairs of nontrivial solutions with positive energy.

**Proof.** (i) Let \( \{u_n\} \subset N \) be such that \( I(u_n) \to c_N \). Remember that \( v_n := u_n/\|u_n\| \in S_A \) (see Proposition 3.2(\( A_3 \))) and

\[
\Psi(v_n) \to c_N.
\]

We are going to prove the existence of a sequence \( \{\hat{v}_n\} \) which is a (PS)\( c_N \) sequence for the functional \( \Psi \). For this, let \( \bar{S}_A \) be the closure of \( S_A \) in \( H^1_0(\Omega) \) and consider the map \( \Upsilon : \bar{S}_A \to \mathbb{R} \cup \{\infty\} \) defined by

\[
\Upsilon(u) = \begin{cases} 
\Psi(u) & \text{if } u \in S_A, \\
\int_{[u \neq \emptyset]} \beta(x)dx & \text{if } u \in \partial S_A.
\end{cases}
\]

It follows from Lemma 3.6 that \( c_N = \inf_{u \in \bar{S}_A} \Upsilon(u) \).

Let us show that \( \Upsilon \) is lower semicontinuous. In fact, let \( \{u_n\} \subset \bar{S}_A \) and \( u \in \bar{S}_A \) be such that \( u_n \to u \) in \( H^1_0(\Omega) \). If \( u \in S_A \) then, for \( n \) large enough, \( \Upsilon(u_n) = \Psi(u_n) \) and

\[
\Upsilon(u_n) = \Psi(u_n) \to \Psi(u) = \Upsilon(u),
\]

because \( \Psi \) is continuous. On the other hand, if \( u \in \partial S_A \), we have two cases to consider. If there exists a subsequence \( \{u_n\} \subset S_A \) then, by Proposition 3.5,

\[
\Upsilon(u) = \int_{[u \neq 0]} \beta(x)dx \leq \liminf_{n \to \infty} \Psi(u_n) = \liminf_{n \to \infty} \Upsilon(u_n).
\]

If there exists a subsequence \( \{u_n\} \subset \partial S_A \) then, from \( \beta(x) \geq 0 \) and

\[
\chi_{[u_n \neq 0]}(x) \to 1 \text{ a.e. in } [u \neq 0],
\]

we have

\[
\Upsilon(u) = \int_{[u \neq 0]} \beta(x)dx \leq \liminf_{n \to \infty} \int_{[u_n \neq 0]} \beta(x)dx = \liminf_{n \to \infty} \Upsilon(u_n).
\]

This, in turn, shows that \( \Upsilon \) is a lower semicontinuous map.

Since \( \bar{S}_A \) is a complete metric space (with metric provided by the norm of \( H^1_0(\Omega) \)) and \( \Upsilon \) is bounded from below (see Lemma 3.3), it follows from Theorem 1.1 in [9] that for each \( \varepsilon, \lambda > 0 \) small enough and \( u \in \Upsilon^{-1}[c_N, c_N + \varepsilon] \) there exists \( v \in \bar{S}_A \) such that

\[
c_N \leq \Upsilon(v) \leq \Upsilon(u), \quad \|u - v\| \leq \lambda \text{ and } \Upsilon(w) > \Upsilon(v) - (\varepsilon/\lambda)\|v - w\|, \forall w \neq v.
\]

On the other hand, it follows from Lemma 3.6 that, for \( \varepsilon \) small enough,

\[
\Upsilon^{-1}[c_N, c_N + \varepsilon] = \Psi^{-1}[c_N, c_N + \varepsilon], \forall v \in S_A \text{ and } \Upsilon(v) = \Psi(v).
\]

Passing to a subsequence, it follows from (4.7) that we can choose \( u = v_n, \varepsilon = 1/n^2 \) and \( \lambda = 1/n \) to get \( \hat{v}_n \in S_A \), satisfying

\[
\Psi(\hat{v}_n) \to c_N, \quad \|v_n - \hat{v}_n\| \to 0
\]
(4.11) \[ \Upsilon(w) > \Psi(\hat{v}_n) - (1/n)\|\hat{v}_n - w\|, \forall w \neq \hat{v}_n. \]

Let \( \gamma_n : (-\delta_n, \delta_n) \rightarrow \mathcal{S}_A \) be a differentiable curve, with \( \delta_n > 0 \) small enough, such that \( \gamma_n(0) = \hat{v}_n \) and \( \gamma_n'(0) = z \in T_{\hat{v}_n}(\mathcal{S}_A) \). Choosing \( w = \gamma_n(t) \), it follows from (4.11) that
\[
-\left[ \Psi(\gamma_n(t)) - \Psi(\gamma_n(0)) \right] < (1/n)\|\gamma_n(t) - \gamma_n(0)\|.
\]
By Mean Value Theorem, there exists \( c \in (0, t) \), such that
\[
\|\gamma_n(t) - \gamma_n(0)\| \leq \|\gamma_n'(c)\| |t|.
\]
It follows from (4.12) and (4.13) that, multiplying both sides by \( 1/t \) and passing to the limit of \( t \to 0^+ \), we get
\[
(4.14) \quad -\Psi'(\hat{v}_n)z \leq \frac{1}{n}\|z\|.
\]
Since \( z \in T_{\hat{v}_n}(\mathcal{S}_A) \) is arbitrary, by linearity, we have
\[
|\Psi'(\hat{v}_n)z| \leq \frac{1}{n}\|z\|.
\]
Therefore,
\[
(4.15) \quad \|\Psi'(\hat{v}_n)\|_* \to 0,
\]
as \( n \to \infty \), and, by (4.10), we conclude that \( \{v_n\} \) is a \((PS)_{c_N}\) sequence for \( \Psi \). It follows from Lemma 3.6 and Proposition 3.7 that there exists \( v \in \mathcal{S}_A \) such that, passing to a subsequence, \( v_n \to v \) in \( H^1_0(\Omega) \). Thus \( \Psi'(v) = 0 \) and \( \Psi(v) = c_N \). Defining \( u := m(v) \in N \) and using Proposition 3.4(iv), we conclude that \( I'(u) = 0 \) and \( I(u) = c_N \).

To show that \( u \) does not change sign, observe that if \( u^+ \neq 0 \), then \( u^\pm \in N \). Thus,
\[
(4.16) \quad c_N = I(u) = I(u^+) + I(u^-) \geq 2c_N,
\]
which is a clear contradiction. Therefore, it follows that either \( u^+ = 0 \) or \( u^- = 0 \) and, consequently, \( u \) is a signed solution.

(ii) First of all, note that the levels \( 0 < c_j < \infty \) are critical levels of \( \Psi \). In fact, suppose by contradiction that \( c_j \) is regular for some \( j \). Invoking Theorem 3.11 in [17], with \( \beta = c_j, \varepsilon = 1, N = 0 \), there exist \( \varepsilon > 0 \) and a family of odd homeomorphisms \( \eta(\cdot, t) \) satisfying the properties of the referred theorem. Choosing \( B \in \gamma_j \) such that \( \sup_B \Psi < c_j + \varepsilon \) and arguing as in the proof of Proposition 4.4 we get a contradiction.

Finally, if levels \( c_j, 1 \leq j \leq s_m \), are different from each other, it follows from Proposition 3.4(iv) that the result is proved. On the other hand, if \( c_j = c_{j+1} \equiv c \) for some \( 1 \leq j \leq s_m \), it follows from Proposition 4.4 that \( \gamma(K_c) \geq 2 \). Combining last inequality with Lemma 4.1(vi) and Proposition 3.4(iv), we conclude that \((P)\) has infinitely many pairs of nontrivial solutions. The result now is proved.

\[ \square \]

**Theorem 4.6.** Suppose that \( f(x, \cdot) \) is odd a.e. in \( \Omega \) and satisfies \((f'_1) - (f'_2)\). The following statements hold:

(i) If \( m = 1 \), then problem \((P)\) has a nontrivial solution;
(ii) If \( f(x, \cdot) \) is odd a.e. in \( \Omega \), then problem \((P)\) has at least \( \chi(\alpha) \) pairs of nontrivial solutions with negative energy.
**Proof.** Since \((f'_1)\) and \((f'_2)\) are satisfied, we are going to prove that, in any case, \(I\) is coercive and bounded from below. For that, let \(\{u_n\} \subset H^1_0(\Omega)\) be a sequence with \(\|u_n\| \to \infty\). If \(v_n := u_n/\|u_n\|\), then
\[
\frac{I(u_n)}{\|u_n\|^2} = \frac{1}{2} - \int_\Omega \left[ \frac{F(x, u_n)}{\|u_n\|^2} \right] dx.
\]
Observe that, up to a subsequence,
\[
v_n \rightharpoonup v \text{ in } H^1_0(\Omega),
\]
\[
\chi_{[v_n \neq 0]}(x) \to 1 \text{ a.e. in } [v \neq 0],
\]
\[
\chi_{[v_n = 0]}(x) \to 0 \text{ a.e. in } [v = 0]
\]
and
\[
\int_\Omega \left[ \frac{F(x, u_n)}{\|u_n\|^2} \right] dx = \int_{[v \neq 0]} \left[ \frac{F(x, \|u_n\|v_n)}{\|u_n\|^2 v_n^2} \right] v_n^2 \chi_{[v_n \neq 0]}(x) dx + \int_{[v = 0]} \left[ \frac{F(x, \|u_n\|v_n)}{\|u_n\|^2 v_n^2} \right] v_n^2 \chi_{[v_n = 0]}(x) dx.
\]
Thus, from \((f'_1)\) and Lebesgue Dominated Convergence Theorem
\[
\int_\Omega \left[ \frac{F(x, u_n)}{\|u_n\|^2} \right] dx \to \frac{1}{2} \int_\Omega \eta(x)v^2 dx.
\]
Therefore, by weighted Poincaré inequality (see Proposition 1.10 in \([7]\))
\[
\frac{I(u_n)}{\|u_n\|^2} \to \frac{1}{2} \left( 1 - \int_\Omega \eta(x)v^2 dx \right) \geq \frac{1}{2} \left( 1 - \frac{1}{\lambda_1(\eta)} \|v\|^2 \right).
\]
Finally, convergence \((??)\) and the weak lower semicontinuity of the norm imply that \(\|v\| \leq 1\). Consequently,
\[
\frac{I(u_n)}{\|u_n\|^2} \to \frac{1}{2} \left( 1 - \int_\Omega \eta(x)v^2 dx \right) \geq \frac{1}{2} \left( 1 - \frac{1}{\lambda_1(\eta)} \right) > 0,
\]
where last inequality comes from \((f'_2)\). This proves that \(I\) is coercive. Since \(f\) is Carathéodory, it follows that \(I\) is weakly lower semicontinuous. Consequently, \(I\) is bounded from below and it has a minimum point \(u_* \in H^1_0(\Omega)\) which is a nontrivial solution of \((P)\). In fact, if \(e_1\) is the positive eigenfunction (normalized in \(H^1_0(\Omega)\)) associated to the first eigenvalue \(\lambda_1(\alpha)\), then
\[
I(te_1) = \frac{1}{2} - \int_\Omega \left[ \frac{F(x, te_1)}{(te_1)^2} \right] e_1^2 dx, \forall t > 0.
\]
It follows from \((f'_1) - (f'_2)\) and Lebesgue Dominated Convergence Theorem that
\[
(4.17) \quad \lim_{t \to 0^+} \frac{I(te_1)}{t^2} = \frac{1}{2} \left[ 1 - \int_\Omega \alpha(x)e_1^2 dx \right] = \frac{1}{2} \left[ 1 - \frac{1}{\lambda_1(\alpha)} \right] < 0.
\]
This shows that there exist \(\varepsilon, t_*> 0\) small enough such that
\[
I(u_*) \leq I(te_1) \leq -\varepsilon t_*^2.
\]
This proves item \((i)\).

\((ii)\) Now, observe that for each \(u \in S_{\chi(\alpha)}\) and \(t > 0\),
\[
\frac{I(tu)}{t^2} = \frac{1}{2} - \int_{[u \neq 0]} \left[ \frac{F(x, tu)}{(tu)^2} \right] u^2 dx.
\]
It follows from \((f'_1)\) and Lebesgue Dominated Convergence Theorem that
\[
(4.18) \quad \lim_{t \to 0^+} \frac{I(tu)}{t^2} = \frac{1}{2} \left[ 1 - \int_\Omega \alpha(x)u^2 dx \right].
\]
Let $\dim V_{\lambda_i(\alpha)}$ be the dimension of the eigenspace $V_{\lambda_i(\alpha)}$ and $\{e_{ij}\} \subset \oplus_{k=1}^m V_{\lambda_k(\alpha)}$ the associated orthonormal basis of eigenfunctions. Since $u$ can be written as
\[ u = \sum_{i=1}^m \sum_{j=1}^{\dim V_{\lambda_i(\alpha)}} u_{ij}e_{ij}, \]
with
\[ \sum_{i=1}^m \sum_{j=1}^{\dim V_{\lambda_i(\alpha)}} u_{ij}^2 = 1, \]
we conclude that
\[ \lim_{t \to 0^+} \frac{I(tu)}{t^2} = \frac{1}{2} \left[ 1 - \sum_{i=1}^m \sum_{j=1}^{\dim V_{\lambda_i(\alpha)}} u_{ij}^2 \int_\Omega \alpha(x)e_{ij}^2 \, dx \right] = \frac{1}{2} \left[ 1 - \sum_{i=1}^m \sum_{j=1}^{\dim V_{\lambda_i(\alpha)}} \frac{u_{ij}^2}{\lambda_i(\alpha)} \right]. \]

Since $\lambda_i(\alpha) \leq \lambda_m(\alpha)$ for all $i \in \{1, \ldots, m\}$ and $u \in S_{\chi(\alpha)}$,
\[ \lim_{t \to 0^+} \frac{I(tu)}{t^2} \leq \frac{1}{2} \left[ 1 - \frac{1}{\lambda_m(\alpha)} \sum_{i=1}^m \sum_{j=1}^{\dim V_{\lambda_i(\alpha)}} u_{ij}^2 \right] = \frac{1}{2} \left[ 1 - \frac{1}{\lambda_m(\alpha)} \right] < 0, \]
where the last inequality comes from $(f'_2)$. Therefore, there exist $\varepsilon, \delta > 0$ such that
\[ I(tu) = \left( I(tu)/t^2 \right) t^2 \leq -\varepsilon t^2, \]
for all $0 < t < \delta$ and $u \in S_{\chi(\alpha)}$. Fixing $0 < t_* < \delta$, we have
\[ \sup_{w \in t_* S_{\chi}} I(w) < 0. \]
Since $I$ is coercive, it is standard to prove that it satisfies the $(PS)_c$ condition. Finally, as $I$ is an even $C^1$-functional and $I(0) = 0$, it follows from Theorem 9.1 in [15], that $I$ has at least $\chi(\alpha)$ pairs of critical points.

\[ \square \]

5. ON THE ASSUMPTION $(\beta)$

In this section we are interested in providing a concrete example of function $f$ which satisfies hypothesis $(\beta)$ considered in Theorem 4.5, but does not satisfy assumption $(fF)$. In order to fix some ideas, let us consider $\eta > \lambda_m$ arbitrarily fixed, where $\lambda_m$ denotes the first eigenvalue of the laplacian operator with Dirichlet boundary condition. Let also $A$, the open set
\[ A = A_{\eta} = \{ u \in H_0^1(\Omega) : \|u\|^2 < \eta \int_\Omega u^2 \, dx \}. \]
Let $u_* \in S_{\chi(\eta)} \cap L^\infty(\Omega) \subset S_A$ and $\theta := |u_*|_\infty$. We are going to consider the problem
\[ \text{(CP)} \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \]
with $\Omega \subset \mathbb{R}^N$, $N \geq 1$, a bounded smooth domain,
\[ f(t) = \begin{cases} t|t|^\alpha & \text{if } |t| \leq \theta, \\ \eta \frac{t^5}{|t|^\alpha} & \text{if } |t| > \theta, \end{cases} \]
and
\[ a = \theta^3 (\eta - \theta) \]
is such that \( f \) is a continuous function (and non-differentiable in \(-\theta\) and \(\theta\)).

Observe that for \( \eta \) large enough, we still have \( u_* \in S_A \) because \( \eta_1 \geq \eta_2 > \lambda_1 \) implies \( A_{\eta_2} \subset A_{\eta_1} \). Moreover, for \( \eta \) large enough, \( a \) is a positive constant.

Some simple calculations show us that \( f \) is an odd function satisfying \((f_1) - (f_2)\), with \( \alpha(x) = 0 \), \( \eta(x) = \eta \), \( \lambda_1(\eta - \alpha) = \lambda_1/\eta \) and \( \beta(x) = \beta \), where

\[
\beta = \lim_{|t| \to \infty} \left[ \frac{1}{2} f(t)t - F(t) \right]
\]

\[
= \frac{\eta}{2} \left( \theta^2 - \sqrt{a} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) \right) - \frac{\theta^3}{3} + \lim_{|t| \to \infty} \left\{ \frac{1}{2} \left[ \frac{\eta t^6}{a + t^4} \right] - \left[ \frac{\eta}{2} \left( t^2 - \sqrt{a} \arctan \left( \frac{t^2}{\sqrt{a}} \right) \right) \right] \right\}
\]

\[
= \frac{\eta \theta^2}{2} - \frac{\eta \sqrt{a}}{2} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) - \frac{\theta^3}{3} + \frac{\pi \eta \sqrt{a}}{4}.
\]

Thus, hypothesis \((fF)\) in \([11]\) is not verified. Note that \( f(t)/t \to 0 \) faster than \( t^2 \to \infty \) as \( |t| \to \infty \), that is,

\[
\lim_{|t| \to \infty} \left[ |\eta - f(t)/t|^2 \right] = \eta \lim_{|t| \to \infty} \left[ 1 - \frac{t^4}{a + t^4} \right] t^2 = 0.
\]

Let \( I : H^1_0(\Omega) \to \mathbb{R} \) be the energy functional of \((CP)\). Since \( f \) satisfies \((f_1) - (f_2)\), Proposition 3.1 holds for \( I \). That is, for each \( u \in A \), there exists a unique \( t_u > 0 \) such that \( t_u u \in N \), where \( N \) is the Nehari manifold associated to \( I \). On the other hand, from definition of \( f \), it is clear that, for \( \eta \) large and \( 0 < t < 1 \),

\[
I(tu_*) = \frac{t^2}{2} ||u_*||^2 - \frac{t^3}{3} \int_{\Omega} |u_*|^3 dx
\]

and

\[
(5.2) \quad \alpha_u'(t) = J'(tu_*)(u_*) = t - \int_{\Omega} f(x,tu_*)u_* dx = t - t^2 \int_{\Omega} |u_*|^3 dx.
\]

Since \( u_* \in S_A \), by Hölder’s inequality, we get

\[
1 < \eta \int_{\Omega} u_*^2 dx \leq \eta |\Omega|^{1/3} \left( \int_{\Omega} |u_*|^3 dx \right)^{2/3}.
\]

Showing that

\[
(5.3) \quad \int_{\Omega} |u_*|^3 dx > \frac{1}{\eta^{3/2}|\Omega|^{1/2}}.
\]

It follows from (5.3) that

\[
(5.4) \quad 0 < t_* = \frac{1}{\int_{\Omega} |u_*|^3 dx} < \frac{1}{\eta^{3/2}|\Omega|^{1/2}} < 1,
\]

for \( \eta \) large. Therefore, from (5.2) and (5.4),

\[
\alpha_u'(t_*) = 0,
\]

and by uniqueness

\[
(5.5) \quad t_u = t_*.
\]
In order to apply Theorem 4.5 to (CP), it is enough to prove that assumption (β) holds. In the case of problem (CP), it is equivalent to
\[
\frac{\eta}{2} \theta^2 - \frac{\eta \sqrt{a}}{2} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) - \frac{\theta^3}{3} + \frac{\pi \eta \sqrt{a}}{4} > \frac{\eta^2 \tau_m^{N/2}}{2 \pi (\Omega)^{N/2}},
\]
or yet,
\[
S(\Omega)^{N/2} \left( \frac{\theta^2}{\eta} - \frac{\sqrt{a}}{\eta} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) - \frac{2 \theta^3}{3 \eta^2} + \frac{\pi \sqrt{a}}{2 \eta} \right) > \tau_m^{N/2},
\]
where \(\tau_m\) is the number defined in Proposition 3.2. In order to prove that (5.6) holds, it is enough to show that
\[
S(\Omega)^{N/2} \left( \frac{\theta^2}{\eta} - \frac{\sqrt{a}}{\eta} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) - \frac{2 \theta^3}{3 \eta^2} + \frac{\pi \sqrt{a}}{2 \eta} \right) > \eta^{N/2},
\]
However, it is a straightforward consequence of (5.1) that, for \(\eta\) large enough, we have
\[
\frac{\pi \sqrt{a}}{2 \eta} - \frac{\sqrt{a}}{\eta} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) - \frac{2 \theta^3}{3 \eta^2} \geq C_1 \frac{1}{\eta^{1/2}} - C_2 \frac{\arctan \left( \frac{C_3}{\eta^{1/2}} \right)}{\eta} \geq C_4 \frac{1}{\eta^{1/2}} \geq C_5 \frac{1}{\eta},
\]
for some positive constants \(C_1, C_2, C_3, C_4\) and \(C_5\). It follows from (5.7) and (5.8) that, for \(\eta\) large enough
\[
S(\Omega)^{N/2} \left( \frac{\theta^2}{\eta} - \frac{\sqrt{a}}{\eta} \arctan \left( \frac{\theta^2}{\sqrt{a}} \right) - \frac{2 \theta^3}{3 \eta^2} + \frac{\pi \sqrt{a}}{2 \eta} \right) \geq \frac{C_5}{\eta^{1/2}}.
\]
Thus, from (5.4), (5.5) and (5.9), to prove (5.6), it is enough to show that
\[
\frac{C_5}{\eta^{1/2}} > \frac{1}{\eta^{3N/4} |\Omega|^{N/4}},
\]
for \(\eta\) large. Since \(N \geq 1\), last equality ever occurs. Showing that \(f\) satisfies (β). Consequently, Theorem 4.5 can be applied to conclude the multiplicity of solutions for problem (CP).

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