Tripartite Neutrino Mass Matrix

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Abstract

The $3 \times 3$ Majorana neutrino mass matrix is written as a sum of 3 terms, i.e. $M_\nu = M_A + M_B + M_C$, where $M_A$ is proportional to the identity matrix and $M_{B,C}$ are invariant under different $Z_3$ transformations. This $M_\nu$ is very suitable for understanding atmospheric and solar neutrino oscillations, with $\sin^2 2\theta_{atm}$ and $\tan^2 \theta_{sol}$ fixed at 1 and 0.5 respectively, in excellent agreement with present data. It has in fact been proposed before, but only as an ansatz. This paper uncovers its underlying symmetry, thus allowing a complete theory of leptons and quarks to be constructed.
With the recent experimental progress in measuring atmospheric [1] and solar [2] neutrino oscillations, the mass-squared differences of the 3 active neutrinos and their mixing angles are now known with some precision. Typical values at 90% confidence level are [3]

\[(\Delta m^2)_{\text{atm}} \sim (1.3 - 3.0) \times 10^{-3} \text{ eV}^2, \quad \sin^2 2\theta_{\text{atm}} \sim 0.88 - 1, \quad (1)\]

\[(\Delta m^2)_{\text{sol}} \sim (6 - 9) \times 10^{-5} \text{ eV}^2, \quad \tan^2 2\theta_{\text{sol}} \sim 0.33 - 0.76. \quad (2)\]

These few numbers have inspired the writing of hundreds of papers on the structure of the resulting $3 \times 3$ Majorana neutrino mass matrix $\mathcal{M}_\nu$. Is the problem that complicated? Perhaps not, if it is looked at with the proper perspective.

Motivated by the idea that $\mathcal{M}_\nu$ should satisfy [4, 5]

\[UM_\nu U^T = \mathcal{M}_\nu, \quad (3)\]

where $U$ is a specific unitary matrix, a very simple form of $\mathcal{M}_\nu$ is here proposed:

\[\mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B + \mathcal{M}_C, \quad (4)\]

where

\[\mathcal{M}_A = A \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{M}_B = B \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad \mathcal{M}_C = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (5)\]

Since the invariance of $\mathcal{M}_A$ requires only $U_A U_A^T = 1$, $U_A$ can be any orthogonal matrix. As for $\mathcal{M}_B$ and $\mathcal{M}_C$, they are both invariant under the $Z_2$ transformation [6, 7]

\[U_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad U_2^2 = 1, \quad (6)\]

and each is invariant under a $Z_3$ transformation, i.e. $U_B^3 = 1$ and $U_C^3 = 1$, but $U_B \neq U_C$.

Specifically,

\[U_B = \begin{pmatrix} -1/2 & -\sqrt{3}/8 & -\sqrt{3}/8 \\ \sqrt{3}/8 & 1/4 & -3/4 \\ \sqrt{3}/8 & -3/4 & 1/4 \end{pmatrix}, \quad U_C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \quad (7)\]
Note that $U_B$ commutes with $U_2$, but $U_C$ does not. If $U_C$ is combined with $U_2$, then the non-Abelian discrete symmetry $S_3$ is generated.

First consider $C = 0$. Then $\mathcal{M}_\nu = \mathcal{M}_A + \mathcal{M}_B$ is the most general solution of

$$U_B \mathcal{M}_\nu U_B^T = \mathcal{M}_\nu,$$  

and the eigenvectors of $\mathcal{M}_\nu$ are $\nu_e$, $(\nu_\mu + \nu_\tau)/\sqrt{2}$, and $(\nu_\mu - \nu_\tau)/\sqrt{2}$ with eigenvalues $A - B$, $A - B$, and $A + B$ respectively. This explains atmospheric neutrino oscillations with $\sin^2 2\theta_{atm} = 1$ and

$$\left(\Delta m^2\right)_{atm} = (A + B)^2 - (A - B)^2 = 4BA.$$  

Now consider $C \neq 0$. Then in the basis spanned by $\nu_e$, $(\nu_\mu + \nu_\tau)/\sqrt{2}$, and $(\nu_\mu - \nu_\tau)/\sqrt{2}$,

$$\mathcal{M}_\nu = \begin{pmatrix}A - B + C & \sqrt{2}C & 0 \\ \sqrt{2}C & A - B + 2C & 0 \\ 0 & 0 & A + B\end{pmatrix}.$$  

The eigenvectors and eigenvalues become

$$\nu_1 = \frac{1}{\sqrt{6}}(2\nu_e - \nu_\mu - \nu_\tau), \quad m_1 = A - B,$$  

$$\nu_2 = \frac{1}{\sqrt{3}}(\nu_e + \nu_\mu + \nu_\tau), \quad m_2 = A - B + 3C,$$  

$$\nu_3 = \frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), \quad m_3 = A + B.$$  

This explains solar neutrino oscillations as well with $\tan^2 \theta_{sol} = 1/2$ and

$$\left(\Delta m^2\right)_{sol} = (A - B + 3C)^2 - (A - B)^2 = 3C(2A - 2B + 3C).$$

Whereas the mixing angles are fixed, the proposed $\mathcal{M}_\nu$ has the flexibility to accommodate the three patterns of neutrino masses often mentioned, i.e.

(I) the hierarchical solution, e.g. $B = A$ and $C << A$;

(II) the inverted hierarchical solution, e.g. $B = -A$ and $C << A$;
the degenerate solution, e.g. $C \ll B \ll A$.

In all cases, $C$ must be small. Therefore $\mathcal{M}_\nu$ of Eq. (4) satisfies Eq. (8) to a very good approximation, and $Z_2 \times Z_3$ as generated by $U_2$ and $U_B$ should be taken as the underlying symmetry of this model.

Since $\mathcal{M}_C$ is small and breaks the symmetry of $\mathcal{M}_A + \mathcal{M}_B$, it is natural to think of its origin in terms of the well-known dimension-five operator

$$L_{eff} = \frac{f_{ij}}{2\Lambda} (\nu_i \phi^0 - l_i \phi^+)(\nu_j \phi^0 - l_j \phi^+) + H.c.,$$

(15)

where $(\phi^+, \phi^0)$ is the usual Higgs doublet of the Standard Model and $\Lambda$ is a very high scale. As $\phi^0$ picks up a nonzero vacuum expectation value $v$, neutrino masses are generated, and if $f_{ij}v^2/\Lambda = C$ for all $i, j$, $\mathcal{M}_C$ is obtained. Since $\Lambda$ is presumably of order $10^{16}$ to $10^{18}$ GeV, $C$ is of order $10^{-3}$ to $10^{-5}$ eV. Using Eq. (14) and Eq. (2), $A - B + 3C/2$ is then of order $10^{-2}$ to 1 eV. This range of values is just right to encompass all three solutions mentioned above.

As for the form of $\mathcal{M}_C$, it may be understood as coming from effective universal interactions among the leptons at the scale $\Lambda$. For example, if Eq. (15) has $S_3$ symmetry as generated by $U_2$ and $U_C$, the most general form of $\mathcal{M}_C$ would be

$$\mathcal{M}_C = C \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + C' \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (16)$$

However, the $C'$ term can be absorbed into $\mathcal{M}_A$, so again $\mathcal{M}_\nu$ of Eq. (4) is obtained. This form of the neutrino mass matrix has in fact been discussed as an ansatz in a number of recent papers [9] [10] [11] [12] [13].

Consider now $\mathcal{M}_\nu$ of Eq. (4) rewritten as

$$\mathcal{M}_\nu = (A + C) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - B \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + C \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \quad (17)$$
Note that each of the above four matrices is a group element of $S_3$. This is the recent proposal of Harrison and Scott [12]. The difference here is that the underlying symmetry of $\mathcal{M}_\nu$ has been identified, thus allowing a complete theory of leptons and quarks to be constructed.

Going back to $U_B$ of Eq. (7), its eigenvectors and eigenvalues are

\begin{align}
\frac{1}{\sqrt{2}}(\nu_\mu - \nu_\tau), & \quad \lambda_1 = 1, \\
\frac{i}{\sqrt{2}}\nu_e + \frac{1}{2}(\nu_\mu + \nu_\tau), & \quad \lambda = \omega, \\
\frac{i}{\sqrt{2}}\nu_e - \frac{1}{2}(\nu_\mu + \nu_\tau), & \quad \lambda = \omega^2,
\end{align}

where $\omega = e^{2\pi i/3}$. To accommodate this $Z_3$ symmetry in a complete theory, the Standard Model of particle interactions is now extended [5] to include three scalar doublets $(\phi_0^i, \phi_i^-)$ and one very heavy triplet $(\xi^{++}, \xi^+, \xi^0)$. The leptonic Yukawa Lagrangian is given by

\[ \mathcal{L}_Y = h_{ij}[\xi^0 l_i l_j - \xi^+(\nu_i l_j + l_i \nu_j)/\sqrt{2} + \xi^{++}l_i l_j] + f_{ik}(\nu_i \phi_0^j - \nu_j \phi_0^i)l^c_k + H.c., \]

where, under the $Z_3$ transformation,

\begin{align}
(\nu, l)_i & \rightarrow (U_B)_{ij}(\nu, l)_j, \quad l^c_k \rightarrow l^c_k, \\
(\phi^0, \phi^-)_i & \rightarrow (U_B)_{ij}(\phi^0, \phi^-)_j, \quad (\xi^{++}, \xi^+, \xi^0) \rightarrow (\xi^{++}, \xi^+, \xi^0).
\end{align}

This means

\[ U_B^T h U_B = h, \quad U_B^T f^k U_B = f^k, \]

resulting in

\[ h = \begin{pmatrix} a - b & 0 & 0 \\ 0 & a & -b \\ 0 & -b & a \end{pmatrix}, \quad f^k = \begin{pmatrix} a_k - b_k & d_k & d_k \\ -d_k & a_k & -b_k \\ -d_k & -b_k & a_k \end{pmatrix}. \]

Note that $h$ has no $d$ terms because it has to be symmetric. Note also that both $h$ and $f$ are invariant under $U_2$ of Eq. (6). Whereas the neutrino mass matrix $\mathcal{M}_A + \mathcal{M}_B$ is obtained with $A = 2a\langle \xi^0 \rangle$ and $B = 2b\langle \xi^0 \rangle$, the charged-lepton mass matrix $\mathcal{M}_l$ linking $l_i$ to $l^c_k$ has
each of its 3 columns given by

$$(\mathcal{M}_l)_{ik} = \begin{pmatrix} (a_k - b_k)v_1 + d_k(v_2 + v_3) \\ -d_kv_1 + a_kv_2 - b_kv_3 \\ -d_kv_1 - b_kv_2 + a_kv_3 \end{pmatrix}, \quad (26)$$

where $v_i \equiv \langle \phi_i^0 \rangle$. Assume $d_k, b_k << a_k$ and $v_{1,2} << v_3$, then all elements in the first, second, and third rows are of order $a_kv_1 + d_kv_3$, $a_kv_2 - b_kv_3$, and $a_kv_3$ respectively. It is clear that they may be chosen to be of order $a_kv_1 + d_kv_3$, $a_kv_2 - b_kv_3$, and $a_kv_3$ respectively. It is clear that they may be chosen to be of order $m_e$, $m_\mu$, and $m_\tau$, in which case $\mathcal{M}_l$ will become nearly diagonal by simply redefining the $l_k^c$ basis. The mixing matrix $V_L$ in the $l_i$ basis (such that $V_L^\dagger \mathcal{M}_L^\dagger V_L^\dagger$ is diagonal) will be very close to the identity matrix with off-diagonal terms of order $m_e/m_\mu$, $m_e/m_\tau$, and $m_\mu/m_\tau$. This construction allows $\mathcal{M}_\nu$ of Eq. (4) to be in the $(\nu_e, \nu_\mu, \nu_\tau)$ basis as a very good approximation. The small deviation is also desirable for obtaining a nonzero but small value of $U_{e3}$, which is restricted by reactor data \cite{14} to be less than about 0.16 in magnitude. The consequences of having three Higgs doublets in this model are very similar to those discussed in Ref. [5] and are repeated here below.

The Yukawa couplings of the three Higgs doublets are given by Eq. (25). Taking the limit that only $v_3$ is nonzero, the charged-lepton mass matrix is simply given by

$$\mathcal{M}_l = v_3 \begin{pmatrix} d_1 & d_2 & d_3 \\ -b_1 & -b_2 & -b_3 \\ a_1 & a_2 & a_3 \end{pmatrix}, \quad (27)$$

whereas $\phi_1^0$ and $\phi_2^0$ couple to $l_i l_j^c$ according to

$$\begin{pmatrix} a_1 - b_1 & a_2 - b_2 & a_3 - b_3 \\ -d_1 & -d_2 & -d_3 \\ -d_1 & -d_2 & -d_3 \end{pmatrix}, \quad \begin{pmatrix} d_1 & d_2 & d_3 \\ a_1 & a_2 & a_3 \\ -b_1 & -b_2 & -b_3 \end{pmatrix}, \quad (28)$$

respectively. Assuming the hierarchy $d_k << b_k << a_k$ and rotating $\mathcal{M}_l$ of Eq. (27) in the $l_j^c$ basis to define the state corresponding to $\tau$, it is clear from Eq. (28) that the dominant coupling of $\phi_1^0$ is $(m_\tau/v_3)e\tau^c$ and that of $\phi_2^0$ is $(m_\tau/v_3)\mu\tau^c$. Other couplings are at most of order $m_\mu/v_3$ in this model, and some are only of order $m_e/v_3$. The smallness of flavor
changing decays in the leptonic sector is thus guaranteed, even though they should be present and may be observable in the future.

Using Eq. (28), we see that the decays $\tau^- \to e^- e^+ e^-$ and $\tau^- \to e^- e^+ \mu^-$ may proceed through $\phi^0_1$ exchange with coupling strengths of order $m_\mu m_\tau / v^2_3 \simeq (g^2 / 2) (m_\mu m_\tau / M_W^2)$, whereas the decays $\tau^- \to \mu^- \mu^+ \mu^-$ and $\tau^- \to \mu^- \mu^+ e^-$ may proceed through $\phi^0_2$ exchange also with coupling strengths of the same order. We estimate the order of magnitude of these branching fractions to be

$$B \sim \left( \frac{m_\mu^2 m_\tau^2}{m_{1,2}^4} \right) B(\tau \to \mu \nu \nu) \simeq 6.1 \times 10^{-11} \left( \frac{100 \text{ GeV}}{m_{1,2}} \right)^4,$$

(29)

which is well below the present experimental upper bound of the order $10^{-6}$ for all such rare decays [15].

The decay $\mu^- \to e^- e^+ e^-$ may also proceed through $\phi^0_1$ with a coupling strength of order $m_\mu^2 / v^2_3$. Thus

$$B(\mu \to eee) \sim \frac{m_\mu^4}{m_1^4} \simeq 1.2 \times 10^{-12} \left( \frac{100 \text{ GeV}}{m_1} \right)^4,$$

(30)

which is at the level of the present experimental upper bound of $1.0 \times 10^{-12}$. The decay $\mu \to e \gamma$ may also proceed through $\phi^0_2$ exchange (provided that $Re \phi^0_2$ and $Im \phi^0_2$ have different masses) with a coupling of order $m_\mu m_\tau / v^2_3$. Its branching fraction is given by [16]

$$B(\mu \to e \gamma) \sim \frac{3 \alpha}{8 \pi} \frac{m_\tau^4}{m_{eff}^4},$$

(31)

where

$$\frac{1}{m_{eff}^2} = \frac{1}{m_{2R}^2} \left( \ln \frac{m_{2R}^2}{m_\tau^2} - \frac{3}{2} \right) - \frac{1}{m_{2I}^2} \left( \ln \frac{m_{2I}^2}{m_\tau^2} - \frac{3}{2} \right).$$

(32)

Using the experimental upper bound [17] of $1.2 \times 10^{-11}$, we find $m_{eff} > 164 \text{ GeV}$.

In the quark sector, if we use the same 3 Higgs doublets for the corresponding Yukawa couplings, the resulting $up$ and $down$ mass matrices will be of the same form as Eq. (26). Because the quark masses are hierarchical in each sector, we will also have nearly diagonal
mixing matrices as in the case of the charged leptons. This provides a qualitative understanding in our model of why the charged-current mixing matrix linking up quarks to down quarks has small off-diagonal entries.

Once $\phi_0^1$ or $\phi_0^2$ is produced, its dominant decay will be to $\tau^\pm e^\mp$ or $\tau^\pm \mu^\mp$ if each couples only to leptons. If they also couple to quarks (and are sufficiently heavy), then the dominant decay products will be $t\bar{u}$ or $t\bar{c}$ together with their conjugates. As for $\phi_0^3$, it will behave very much as the single Higgs doublet of the Standard Model, with mostly diagonal couplings to fermions. It should also be identified with the $\phi$ of Eq. (15).

In conclusion, a simple tripartite neutrino mass matrix has been proposed, where the first two parts, $\mathcal{M}_A + \mathcal{M}_B$, are invariant under $U_B$ of Eq. (7) with $U_B^3 = 1$. The third part $\mathcal{M}_C$ is considered as a small perturbation which is democratic in the $(\nu_e, \nu_\mu, \nu_\tau)$ basis. The resulting sum of the three parts is invariant under $U_2$ of Eq. (6) with $U_2^2 = 1$ and fixes $\sin^2 2\theta_{atm} = 1$ and $\tan^2 \theta_{sol} = 0.5$; but it also has 3 free parameters $A, B, C$ which determine the 3 neutrino mass eigenvalues as given in Eqs. (11) to (13). This structure with the underlying symmetry $Z_3 \times Z_2$ is supported in the context of a complete theory of leptons (that may be extended to quarks) which includes one very heavy Higgs triplet and three Higgs doublets at the electroweak scale, with experimentally verifiable properties.

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