Solutions of the spatially-dependent mass Dirac equation with the spin and pseudo-spin symmetry for the Coulomb-like potential

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Abstract

We study the effect of spatially dependent mass function over the solution of the Dirac equation with the Coulomb potential in the (3 + 1)-dimensions for any arbitrary spin-orbit $\kappa$ state. In the framework of the spin and pseudospin symmetry concept, the analytic bound state energy eigenvalues and the corresponding upper and lower two-component spinors of the two Dirac particles are obtained by means of the Nikiforov-Uvarov method, in closed form. This physical choice of the mass function leads to an exact analytical solution for the pseudospin part of the Dirac equation. The special cases $\kappa = \pm 1$ ($l = \bar{l} = 0$, i.e., s-wave), the constant mass and the non-relativistic limits are briefly investigated.

Keywords: Dirac equation, spin symmetry, pseudospin symmetry, bound states, Coulomb potential, spatially-dependent mass, Nikiforov-Uvarov method.

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I. INTRODUCTION

Within the framework of the Dirac equation the spin symmetry arises if the magnitude of the attractive scalar potential $\mathcal{S}(r)$ and repulsive vector potential are nearly equal, $\mathcal{S}(r) \sim V(r)$ in nuclei (i.e., when the difference potential $\Delta(r) = V(r) - \mathcal{S}(r) = A = \text{constant}$). However, the pseudospin symmetry occurs when $\mathcal{S}(r) \sim -V(r)$ are nearly equal (i.e., when the sum potential $\Sigma(r) = V(r) + \mathcal{S}(r) = A = \text{constant}$) [1-3]. The spin symmetry is relevant for mesons [4]. The pseudospin symmetry concept has been applied to many systems in nuclear physics and related areas [2-7]. It has also been used to explain features of deformed nuclei [8], the super-deformation [9] and to establish an effective nuclear shell-model scheme [5,6,10]. The pseudospin symmetry introduced in nuclear theory refers to a quasi-degeneracy of the single-nucleon doublets and can be characterized with the non-relativistic quantum numbers $(n, l, j = l + 1/2)$ and $(n - 1, l + 2, j = l + 3/2)$, where $n$, $l$ and $j$ are the single-nucleon radial, orbital and total angular momentum quantum numbers for a single particle, respectively [5,6]. The total angular momentum is given as $j = \tilde{l} + \tilde{s}$, where $\tilde{l} = l + 1$ is a pseudo-angular momentum and $\tilde{s} = 1/2$ is a pseudospin angular momentum. In real nuclei, the pseudospin symmetry is only an approximation and the quality of approximation depends on the pseudo-centrifugal potential and pseudospin orbital potential [11]. In ref. [12], Alhaidari et al. investigated in detail the physical interpretation on the three-dimensional Dirac equation in the presence of exact spin symmetry limitation $\Delta(r) = 0$ and pseudospin symmetry limitation $\Sigma(r) = 0$.

Some authors have applied the pseudospin symmetry on several physical potentials, such as the harmonic oscillator [12-16], the Woods-Saxon potential [17], the Morse potential [18-20], the Hulthén potential [21], the Eckart potential [22-24], the molecular diatomic three-parameter potential [25], the Pöschl-Teller potential [26] and the Rosen-Morse potential [27].

On the other hand, the problem of the spatially-dependent effective mass is presenting a growing interest along the last few years [28-31]. Many authors have used different methods to study the partially exactly solvable and exactly solvable Schrödinger, Klein-Gordon (KG) and Dirac equations in the presence of variable mass having a suitable mass distribution functions in $1D, 3D$ and/or any arbitrary $D$-dimensional cases for various type potentials [32-45]. In the context of spatially-dependent mass, we have used and applied a recently
proposed approximation scheme [45] for the centrifugal term to find a quasi-exact analytic bound-state solution of the radial KG equation with spatially-dependent effective mass for scalar and vector Hulthén potentials in any arbitrary dimensional space $D$ and orbital angular momentum quantum number $l$ within the framework of the Nikiforov-Uvarov (NU) method [46,47]. In addition, the $s$-wave bound state solution of the (1+1)-dimensional KG equation with mass inversely proportional to the distance from the force center for the inversely linear potential was obtained [48]. Two particular cases were studied, the exact spin symmetry when the vector potential and the scalar potential are equal in magnitude $S(r) = V(r)$ and the pseudospin symmetry when the vector potential is equal to the scalar potential in magnitude but not in sign $S(r) = -V(r)$. Very recently, the study of ref. [48] has also been extended by us [49] to study the bound-state solutions of the (3+1)-dimensional KG equation with position-dependent bosonic mass function $m(r) = m_0 (1 + \lambda br^{-1})$, where $r \neq 0$ for the attractive scalar Coulomb-like potential $S(r) = -\kappa_s r^{-1}$ with $\kappa_s = \hbar c q_s$ being the coupling constant, taking into consideration the general mixings of scalar and vector Lorentz structure potential.

In the present paper, our aim is to study the exact solution of the spatially dependent Dirac equation with the Coulomb-like field potential for any arbitrary spin-orbit quantum number $\kappa$. Under the conditions of the spin symmetry $S(r) \sim V(r)$ and pseudospin symmetry $S(r) \sim -V(r)$, we investigate the bound state energy eigenvalues and corresponding upper and lower spinor wave functions in the framework of the NU method. We also show that the spin and pseudo-spin symmetry Dirac solutions can be reduced to the $S(r) = V(r)$ and $S(r) = -V(r)$ Klein-Gordon solutions [49] in the cases of exact spin symmetry limitation $\Delta(r) = 0$ and pseudospin symmetry limitation $\Sigma(r) = 0$, respectively. Further, the solutions of the Dirac equation with constant mass can be also generated from the general solution when the constant $b$ in the mass function is set to zero.

The paper is organized as follows. In sect. 2, we outline the NU method. Section 3 is devoted for the analytic bound state solutions of the (3+1)-dimensional Dirac equation with spatially dependent mass function for the quantum system obtained by means of the NU method. The spin symmetry and pseudo-spin symmetry solutions are investigated. In sect. 3, we study the cases $\kappa = \pm 1$ ($l = \tilde{l} = 0$, $s$-wave), the constant mass and the non-relativistic limit and compare with other wave equations and models. Finally, the relevant conclusions are given in sect. 4.
II. NU METHOD

The NU method is briefly outlined here and details can be found in ref. [46]. This method was proposed to solve the second-order differential equation of hypergeometric-type:

\[ \psi''_n(r) + \frac{\bar{\tau}(r)}{\sigma(r)} \psi'_n(r) + \frac{\bar{\sigma}(r)}{\sigma^2(r)} \psi_n(r) = 0, \]  

(1)

where \( \sigma(r) \) and \( \bar{\sigma}(r) \) are polynomials, at most, of second-degree, and \( \bar{\tau}(r) \) is a first-degree polynomial. In order to find a particular solution for eq. (1), let us decompose the wave function \( \psi_n(r) \) as follows:

\[ \psi_n(r) = \phi(r)y_n(r), \]  

(2)

and use

\[ [\sigma(r)\rho(r)]' = \tau(r)\rho(r), \]  

(3)

to reduce eq. (1) to the form

\[ \sigma(r)y''_n(r) + \tau(r)y'_n(r) + \lambda y_n(r) = 0, \]  

(4)

with

\[ \tau(r) = \bar{\tau}(r) + 2\pi(r), \quad \tau'(r) < 0, \]  

(5)

where the prime denotes the differentiation with respect to \( r \). One is looking for a family of solutions corresponding to

\[ \lambda = \lambda_n = -n\tau'(r) - \frac{1}{2}n(n-1)\sigma''(r), \quad n = 0, 1, 2, \ldots, \]  

(6)

The \( y_n(r) \) can be expressed in terms of the Rodrigues relation:

\[ y_n(r) = \frac{B_n}{\rho(r)} \frac{d^n}{dr^n} [\sigma^n(r)\rho(r)], \]  

(7)

where \( B_n \) is the normalization constant and the weight function \( \rho(r) \) is the solution of the differential equation (3). The other part of the wave function (2) must satisfy the following logarithmic equation

\[ \frac{\phi'(r)}{\phi(r)} = \frac{\pi(r)}{\sigma(r)}. \]  

(8)

By defining

\[ k = \lambda - \pi'(r), \]  

(9)

one obtains the polynomial
\[
\pi(r) = \frac{1}{2} [\sigma'(r) - \tilde{\tau}(r)] \pm \sqrt{\frac{1}{4} [\sigma'(r) - \tilde{\tau}(r)]^2 - \tilde{\sigma}(r) + k\sigma(r)},
\]  

(10)

where \( \pi(r) \) is a parameter at most of order 1. The expression under the square root sign in the above equation can be arranged as a polynomial of second order where its discriminant is zero. Hence, an equation for \( k \) is being obtained. After solving such an equation, the \( k \) values are determined through the NU method.

In this context, we may also derive a parametric generalization from the NU method valid for most potentials under consideration. To do this, we begin by writing the hypergeometric equation in general parametric form as

\[
[r (c_3 - c_4 r)]^2 \psi''_n(r) + [r (c_3 - c_4 r) (c_1 - c_2 r)] \psi'_n(r) + \left(-\xi_1 r^2 + \xi_2 r - \xi_3\right) \psi_n(r) = 0,
\]

(11)

with

\[
\tilde{\tau}(r) = c_1 - c_2 r,
\]

(12)

\[
\sigma(r) = r (c_3 - c_4 r),
\]

(13)

\[
\tilde{\sigma}(r) = -\xi_1 r^2 + \xi_2 r - \xi_3,
\]

(14)

where the coefficients \( c_i \ (i = 1, 2, 3, 4) \) and the analytic expressions \( \xi_j \ (j = 1, 2, 3) \) are calculated for the potential model under consideration. Overmore, comparing eq. (11) with it’s counterpart eq. (1), we obtain the analytic polynomials, energy equation and wave functions together with the associated coefficients expressed in general parameteric form in Appendix A.

III. ANALYTIC SOLUTION OF THE DIRAC-COULOMB-LIKE PROBLEM

In spherical coordinates, the spatially-dependent mass Dirac equation for fermionic massive spin-1/2 particles interacting with arbitrary scalar potential \( S(r) \) and the time-component \( V(r) \) of a four-vector potential can be expressed as [27,50-53]

\[
[c\alpha \cdot \mathbf{p} + \beta (m(r)c^2 + S(r)) + V(r) - E] \psi_{nn}(\mathbf{r}) = 0, \quad \psi_{nn}(\mathbf{r}) = \psi_{nn}(r, \theta, \phi),
\]

(15)

where \( E \) is the relativistic energy of the system, \( m(r) \) is the spatially-dependent mass of the fermionic particle, \( \mathbf{p} = -i\nabla \) is the momentum operator, and \( \alpha \) and \( \beta \) are \( 4 \times 4 \) Dirac
matrices, which have the following forms, respectively [50-53]

\[ \alpha = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad P = -i\hbar \nabla, \]

(16)

where \( I \) denotes the \( 2 \times 2 \) identity matrix and \( \sigma \) are three-vector Pauli spin matrices

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

(17)

For a particle in a spherical (central) field, the total angular momentum operator \( J \) and the spin-orbit matrix operator \( \hat{K} = -\beta (\sigma \cdot L + I) \) commute with the Dirac Hamiltonian, where \( L \) is the orbital angular momentum operator. For a given total angular momentum \( j \), the eigenvalues of \( \hat{K} \) are \( \kappa = -(j + 1/2) \) for aligned spin \((s_{1/2}, p_{3/2}, \text{ etc.})\) and \( \kappa = j + 1/2 \) for unaligned spin \((p_{1/2}, d_{3/2}, \text{ etc.})\). The spinor wave functions can be classified according to the radial quantum number \( n \) and the spin-orbit quantum number \( \kappa \) and can be written using the Pauli-Dirac representation:

\[ \psi_{n\kappa}(r) = \frac{1}{r} \begin{pmatrix} F_{n\kappa}(r)Y_{jm}(\theta, \phi) \\ iG_{n\kappa}(r)Y_{\tilde{j}m}(\theta, \phi) \end{pmatrix}, \]

(18)

where \( F_{n\kappa}(r) \) and \( G_{n\kappa}(r) \) are the radial wave functions of the upper- and lower-spinor components, respectively, \( Y_{jm}(\theta, \phi) \) and \( Y_{\tilde{j}m}(\theta, \phi) \) are the spherical harmonic functions coupled to the total angular momentum \( j \) and it’s projection \( m \) on the z axis. The orbital and pseudo-orbital angular momentum quantum numbers for spin symmetry \( l \) and pseudospin symmetry \( \tilde{l} \) refer to the upper- and lower-components, respectively. For a given spin-orbit quantum number \( \kappa = \pm 1, \pm 2, \cdots \), the orbital angular momentum and pseudo-orbital angular momentum are given by \( l = |\kappa + 1/2| - 1/2 \) and \( \tilde{l} = |\kappa - 1/2| - 1/2 \), respectively. The quasi-degenerate doublet structure can be expressed in terms of a pseudo-spin angular momentum \( \tilde{s} = 1/2 \) and pseudo-orbital angular momentum \( \tilde{l} \) which is defined as \( \tilde{l} = l + 1 \) for aligned spin \( j = \tilde{l} - 1/2 \) and \( \tilde{l} = l - 1 \) for unaligned spin \( j = \tilde{l} + 1/2 \). For example, \((3s_{1/2}, 2d_{3/2})\) and \((3\tilde{p}_{1/2}, 2\tilde{p}_{3/2})\) can be considered as pseudospin doublets.

Substituting eq. (18) into eq. (15), we obtain two radial coupled Dirac equations for the spinor components

\[ \left( \frac{d}{dr} + \kappa \right) F_{n\kappa}(r) = (m(r)c^2 + E_{n\kappa} - \Delta(r)) G_{n\kappa}(r), \]

(19a)
\[
\left( \frac{d}{dr} - \frac{\kappa}{r} \right) G_{n\kappa}(r) = (m(r)c^2 - E_{n\kappa} + \Sigma(r)) F_{n\kappa}(r),
\]  

(19b)

where \( \Delta(r) = V(r) - S(r) \) and \( \Sigma(r) = V(r) + S(r) \) are the difference and sum potentials, respectively. Eliminating \( G_{n\kappa}(r) \) in eq. (19a) and \( F_{n\kappa}(r) \) into eq. (19b), we get two second-order differential equations for the upper and lower spinor components as

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{1}{\hbar^2 c^2} \left[ U_- (r) U_+(r) + \frac{g_-(r)}{U_-(r)} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \right] \right\} F_{n\kappa}(r) = 0,
\]

(20a)

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa - 1)}{r^2} - \frac{1}{\hbar^2 c^2} \left[ U_- (r) U_+(r) + \frac{g_+(r)}{U_+(r)} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \right] \right\} G_{n\kappa}(r) = 0,
\]

(20b)

where \( U_- (r) = m(r)c^2 + E_{n\kappa} - \Delta(r) \) and \( U_+(r) = m(r)c^2 - E_{n\kappa} + \Sigma(r) \), are the difference and the sum functions, respectively. Also, \( g_-(r) = e^2 \frac{dm(r)}{dr} - \frac{d\Delta(r)}{dr} \) and \( g_+(r) = e^2 \frac{dm(r)}{dr} + \frac{d\Sigma(r)}{dr} \) being the derivative of the mass function minus the difference potential and the derivative of the mass function plus the sum potential, respectively. From the above equations, the energy eigenvalues depend on the quantum numbers \( n \) and \( \kappa \), and also the pseudo-orbital angular quantum number \( \tilde{l} \) according to \( \kappa(\kappa - 1) = \tilde{l}(\tilde{l} + 1) \), which implies that \( j = \tilde{l} \pm 1/2 \) are degenerate for \( \tilde{l} \neq 0 \).

At this stage, we take the vector potential in the form of an attractive Coulomb-like field as

\[
\Sigma(r) = V(r) = -\frac{\hbar cq_v}{r}, \quad q_v = q, \quad r \neq 0,
\]

(21)

where \( q_v \) is being a vector dimensionless real parameter coupling constant and \( \hbar c \) is being a constant with \( J.fm \) dimension. Equations (20a) and (20b) can not be solved analytically because of the last term in the equations, we find it convenient to solve the mathematical relation \( e^2 \frac{dm(r)}{dr} = -\frac{d\Sigma(r)}{dr} = -\frac{dV(r)}{dr} \) for the sake of eliminating this term. We find out that mass function should be taken as

\[
m(r) = m_0 + m_1/r,
\]

(22a)

\[
m_1 = m_0 \lambda_0 b, \quad \lambda_0 = \hbar/m_0 c,
\]

(22b)

with \( m_0 \) and \( m_1 \) are the integration constant (rest mass of the fermionic particle) and the perturbed mass, respectively. Furthermore, \( b \) is the dimensionless real constant to be set to zero for the constant mass case \( (i.e., m_1 = 0) \) and \( \lambda_0 \) is the compton-like wavelength in \( fm \) units. It is worth mentioning that the above choice of the mass function of Coulombic form \([48,49]\) is mostly suitable for modeling the well-known pseudo-Coulomb (Kratzer-type)
potential [54-56]. The interaction field has much impact on the choice of the mass function which, in the present case, is inversely proportional to the distance between the two nuclei at short distances \( m(r) \sim \frac{1}{r} \) and constant at long distances \( m(r \to \infty) \approx m_0 \).

### A. Spin symmetric solution of the Coulomb potential

In the case of exact spin symmetry \( S(r) \sim V(r) \) \((d\Delta/dr = 0, \text{i.e., } \Delta(r) = A = \text{constant})\), eq. (20a) can be approximately written as

\[
\left\{ \frac{d^2}{dr^2} - \frac{\kappa(\kappa + 1)}{r^2} - \frac{1}{\hbar^2c^2} \left[ m(r)c^2 + E_{n\kappa} - A \right] \left[ m(r)c^2 - E_{n\kappa} + \Sigma(r) \right] \right\} F_{n\kappa}(r) = 0, \quad (23)
\]

where \( \kappa = l \) for \( \kappa < 0 \) and \( \kappa = -(l + 1) \) for \( \kappa > 0 \). The spin symmetric energy eigenvalues depend on \( n \) and \( \kappa \), i.e., \( E_{n\kappa} = E(n, \kappa (\kappa + 1)) \). In the last equation, the choice of \( \Sigma(r) = 2V(r) \to V(r) \) as mentioned in ref. [12] enables one to reduce the resulting relativistic solutions into their non-relativistic limit under appropriate transformations. However, if we set \( \Sigma(r) = 2V(r) \), this allows us to compare the resulting Dirac’s solutions with those ones derived before in refs. [47-49] regarding the solution of the KG equation for equal mixings \( S(r) = V(r) \) (i.e., \( \Delta = 0 \)).

Substituting eqs. (21) and (22) into eq. (23), allows us to decompose the Dirac equation and then leading to obtain the Schrödinger-like equation in the spherical coordinates for the upper-spinor component \( F_{n\kappa}(r) \),

\[
\left[ \frac{d^2}{dr^2} + \left( -\varepsilon_{n\kappa}^2 \frac{r^2}{r^2} + \frac{\beta r - \gamma}{r^2} \right) \right] F_{n\kappa}(r) = 0, \quad (24)
\]

where we have defined

\[
\varepsilon_{n\kappa} = \frac{1}{\hbar c} \sqrt{m_0^2c^4 - E_{n\kappa}^2 - A (m_0c^2 - E_{n\kappa})} > 0, \quad (25a)
\]

\[
\beta = \frac{1}{\hbar c} \left[ 2q (m_0c^2 + E_{n\kappa} - A) + b \left( A - 2m_0c^2 \right) \right], \quad (25b)
\]

\[
\gamma = b(b - 2q) + \kappa (\kappa + 1). \quad (25c)
\]

For instance, when \( A = 0 \), the following constraint \( E_{nl} < m_0c^2 \) must be fulfilled for bound state solutions. The quantum condition is obtained from the finiteness of the solution at infinity and at the origin point (i.e., \( F_{n\kappa}(0) = F_{n\kappa}(\infty) = 0 \)). In order to solve eq. (24) by
means of the NU method, we should compare it with eq. (1). The following values for the parameters are found as

\[ \tilde{\tau}(r) = 0, \quad \sigma(r) = r, \quad \tilde{\sigma}(r) = -\varepsilon_{n\kappa}^2 r^2 + \beta r - \gamma. \]  

(26)

Further, inserting these values into eq. (10), we obtain

\[ \pi(r) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4\gamma + 4(k - \beta)r + 4\varepsilon_{n\kappa}^2 r^2}. \]  

(27)

The discriminant of the square root must be set equal to zero, i.e., \( 1 + 4\gamma + 4(k - \beta)r + 4\varepsilon_{n\kappa}^2 r^2 = 0 \). Hence, we find the following two constants:

\[ k_{1,2} = \beta \pm \varepsilon_{n\kappa} \sqrt{1 + 4\gamma}. \]  

(28)

In this regard, we can find the possible functions for \( \pi(r) \) as

\[ \pi(r) = \begin{cases} \frac{1}{2} \pm \left[ \varepsilon_{n\kappa}r + \frac{1}{2} \sqrt{1 + 4\gamma} \right] & \text{for } k_1 = \beta + \varepsilon_{n\kappa} \sqrt{1 + 4\gamma}, \\ \frac{1}{2} \pm \left[ \varepsilon_{n\kappa}r - \frac{1}{2} \sqrt{1 + 4\gamma} \right] & \text{for } k_2 = \beta - \varepsilon_{n\kappa} \sqrt{1 + 4\gamma}. \end{cases} \]  

(29)

According to the NU method, one of the four values of the polynomial \( \pi(r) \) is just proper to obtain the bound states because \( \tau(r) \) has a negative derivative. Therefore, the selected forms of \( \pi(r) \) and \( k \) take the following particular values

\[ \pi(r) = \frac{1}{2} \left( 1 + \sqrt{1 + 4\gamma} \right) - \varepsilon_{n\kappa}r, \quad k = \beta - \varepsilon_{n\kappa} \sqrt{1 + 4\gamma}, \]  

(30)

to obtain

\[ \tau(r) = 1 + \sqrt{1 + 4\gamma} - 2\varepsilon_{n\kappa}r, \quad \tau'(r) = -2\varepsilon_{n\kappa} < 0, \]  

(31)

where prime denotes the derivative with respect to \( r \). In addition, after using eqs. (6) and (9) together with the assignments given in eqs. (26), (30) and (31), the following expressions for \( \lambda \) are obtained as

\[ \lambda_n = \lambda = 2n\varepsilon_{n\kappa}, \quad n = 0, 1, 2, \cdots, \]  

(32)

\[ \lambda = \beta - \varepsilon_{n\kappa} \left( 1 + \sqrt{1 + 4\gamma} \right). \]  

(33)

Now, taking \( \lambda_n = \lambda \), we can solve the above equations to obtain the energy equation for the Coulomb-like potential with spin-symmetry in the Dirac theory,

\[ m_0^2 c^4 - E_{nn}^2 - A (m_0 c^2 - E_{n\kappa}) = \left[ \frac{q (m_0 c^2 + E_{n\kappa} - A) + b \left( \frac{\sqrt{2}}{2} - m_0 c^2 \right)}{n + \delta + 1} \right]^2, \]  

(34)
where
\[ \delta = \sqrt{\left(\frac{1}{2} + \kappa\right)^2 + b(b - 2q)} - \frac{1}{2}. \] (35)

For spatially-dependent mass case, i.e., \( b \neq 0 \), the bound state solutions of the system are determined by means of the parameters \( q \) and \( b \). It is not difficult to conclude that all bound-states appear in pairs, two energy solutions are valid for the particle \( E^p = E^+_{nk} \) and the second one corresponds to the anti-particle energy \( E^a = E^-_{nk} \) in the Coulomb-like field.

Let us now find the corresponding eigenfunctions for this system. Using eqs. (3) and (8), we find
\[ \rho(r) = r^{2\delta+1}e^{-2\varepsilon_{nk}r}, \] (36)
\[ \phi(r) = r^{\delta+1}e^{-\varepsilon_{nk}r}, \] (37)
and further substituting eqs. (26) and (36) into eq. (7), we get
\[ y_n(r) = A_n r^{-(2\delta+1)} e^{2\varepsilon_{nk}r} \frac{d^n}{dr^n} \left[ r^{2\delta+n+1} e^{-2\varepsilon_{nk}r} \right] \sim L_n^{2\delta+1}(2\varepsilon_{nk}r), \] (38)
where \( L_n^{\alpha}(x) \) is the generalized Laguerre polynomials. The relation \( F_{nk}(r) = \phi(r)y_n(r) \) gives the radial upper spinor wave function:
\[ F_{nk}(r) = \mathcal{N} r^{\delta+1} e^{-\varepsilon_{nk}r} L_n^{2\delta+1}(2\varepsilon_{nk}r). \] (39)

The above upper-spinor component satisfies the restriction condition for the bound states, i.e., \( \delta > 0 \) and \( \varepsilon_{nk} > 0 \). Using the normalization condition \( \int_0^\infty u(r)^2 dr = 1 \) and the orthogonality relation of the generalized Laguerre polynomials \( \int_0^\infty x^\alpha e^{-x} \left[ L_n^{(\alpha)}(x) \right]^2 dx = (2n + \alpha + 1) \frac{\Gamma(n+\alpha+1)}{n!} \), the normalizing factor \( \mathcal{N} \) can be found as \([57-59]\)
\[ \mathcal{N} = \sqrt{\frac{n! (2\varepsilon_{nk})^{2\delta+3}}{2(n + \delta + 1)\Gamma(n + 2\delta + 2)}}, \] (40)
where \( \varepsilon_{nk} \) and \( \delta \) are given in eqs. (25a) and (35), respectively.

On the other hand, it is worth noting that if we take the particular values of the coefficients displayed in Table 1, then the expressions given in Appendix A can be calculated for eq. (24) and tested with the ones calculated before in a very simple way.

Let us now study some special cases of much concern to the reader. The following cases are found to be identical with the solution of the spatially-dependent mass and constant mass KG equation for the case spin symmetry case \( S(r) = V(r) \) (\( \Delta(r) = \Sigma(r) = 0 \)) \([29,51]\).
(1) If we consider the case when \( S(r) = V(r) \), i.e., \( q_s = q_v = q \) or \( A = 0 \), then eq. (34) can be reduced to the following forms:

\[
E_{p\kappa} = \frac{[q(b - q) + B_{n\kappa}\sqrt{B_{n\kappa}^2 - b(b - 2q)}]}{q^2 + B_{n\kappa}^2} m_0c^2,
\]

and

\[
E_{a\kappa} = \frac{[q(b - q) - B_{n\kappa}\sqrt{B_{n\kappa}^2 - b(b - 2q)}]}{q^2 + B_{n\kappa}^2} m_0c^2,
\]

where \( B_{n\kappa} \) is defined by

\[
B_{n\kappa} = \left( n + \frac{1}{2} + \sqrt{\left( \frac{1}{2} + \kappa \right)^2 + b(b - 2q)} \right).
\]

Obviously, the bound-state solutions of the particle and anti-particle do exist.

(2) When the mass is constant, i.e., \( b = 0 (m_1 = 0) \), we have

\[
E_{p\kappa} = \frac{(n + \kappa + 1)^2 - q^2}{(n + \kappa + 1)^2 + q^2} m_0c^2 \quad \text{and} \quad E_{a\kappa} = -m_0c^2, \quad n = 0, 1, 2, \ldots, \kappa = \pm 1, \pm 2, \ldots \tag{43}
\]

which gives

\[
E_{p\kappa} = \begin{cases} 
\frac{(n-l)^2-q^2}{(n-l)^2+q^2} m_0 c^2, & \kappa > 0 \\
\frac{(n+l+1)^2-q^2}{(n+l+1)^2+q^2} m_0 c^2, & \kappa < 0 
\end{cases} \quad \text{and} \quad E_{a\kappa} = -m_0 c^2, \quad l = 0, 1, 2, \ldots \tag{44}
\]

where \( n \) and \( \kappa \) signify the usual radial and spin-orbit quantum numbers. The particle has bound state solution whereas anti-particle has continuum solution for all states. Further, the wave functions for particle turn to be

\[
F_{n\kappa}^p(r) = \sqrt{\frac{n!}{2(n + \kappa + 1)\Gamma(n + 2\kappa + 2)}} r^\kappa \exp \left( -\frac{2}{\hbar c} \sqrt{m_0^2 c^4 - E_{n\kappa}^2} r \right) \times L_n^{2\kappa+1} \left( \frac{2}{\hbar c} \sqrt{m_0^2 c^4 - E_{n\kappa}^2} r \right), \quad \kappa > 0, \ E_{n\kappa}^p < m_0 c^2, \tag{45}
\]

whereas the anti-particle has no wave functions.

(3) When the potential coupling constant is set to \( q = b/2 \), the spectrum of the spatially-dependent mass Dirac particle in potential field \( q_s = q_v = q \) is similar to the spectrum of constant mass Dirac particle in the potential fields \( q_s = -q_v \), that is,

\[
E_{n\kappa}^p = m_0 c^2, \tag{46a}
\]
\[ E_{n\kappa}^a = -\frac{(n + \kappa + 1)^2 - q^2 m_0 c^2}{(n + \kappa + 1)^2 + q^2 m_0 c^2}. \quad (46b) \]

(4) When the potential coupling constant is set to \( q = b/2 \), the spectrum of the spatially-dependent mass Dirac particle in potential field \( q_s = -q_v \) is similar to the spectrum of constant mass Dirac particle in the potential fields \( q_s = q_v = q \), that is,

\[ E_{n\kappa}^p = \frac{(n + \kappa + 1)^2 - q^2 m_0 c^2}{(n + \kappa + 1)^2 + q^2 m_0 c^2}, \quad (47a) \]

\[ E_{n\kappa}^q = -m_0 c^2. \quad (47b) \]

B. Pseudospin symmetric solution of the Coulomb potential

In the presence of the pseudospin symmetry \( S(r) \sim -V(r) \) (i.e., \( d\Sigma(r)/dr = 0 \), or \( \Sigma(r) = A = \text{constant} \), eq. (20b) can be exactly written as

\[ \left\{ \frac{d^2}{dr^2} - \frac{\kappa (\kappa - 1)}{r^2} - \frac{1}{h^2 c^2} \left[ m(r)c^2 + E_{n\kappa} - \Delta(r) \right] \left[ m(r)c^2 - E_{n\kappa} + A \right] \right\} G_{n\kappa}(r) = 0, \quad (48) \]

where the energy eigenvalues \( E_{n\kappa} \) depend only on \( n \) and \( \kappa \), i.e., \( E_{n\kappa} = E(n, \kappa(\kappa - 1)) \). We take the difference of the potential as \( 2V(r) \) which allows us to compare our final results with the ones calculated for the KG equation derived in ref. [49]. Substituting

\[ \Delta(r) = V(r) = -\frac{\hbar c q}{r}, q_v = -q, r \neq 0, \]

together with eq. (22) into the last equation, we obtain a Schrödinger equation for the lower component \( G_{n\kappa}(r) \),

\[ \left[ \frac{d^2}{dr^2} + \left( -\tilde{\varepsilon}_{n\kappa}^2 r^2 + \tilde{\beta} \tilde{r} - \tilde{\gamma} \right) \right] G_{n\kappa}(r) = 0, \quad G_{n\kappa}(0) = G_{n\kappa}(\infty) = 0, \quad (49) \]

where we have defined

\[ \tilde{\varepsilon}_{n\kappa} = \frac{1}{\hbar c} \sqrt{m_0^2 c^4 - E_{n\kappa}^2 + A (m_0 c^2 + E_{n\kappa})} > 0, \quad (50a) \]

\[ \tilde{\beta} = -\frac{1}{\hbar c} \left[ 2q \left( m_0 c^2 - E_{n\kappa} + A \right) + b \left( 2m_0 c^2 + A \right) \right], \quad (50b) \]

\[ \tilde{\gamma} = b (b + 2q) + \kappa (\kappa + 1). \quad (50c) \]

To avoid repetition in the solution of eq. (49), a first inspection for the relationship between the present set of parameters \( (\tilde{\varepsilon}_{n\kappa}, \tilde{\beta}, \tilde{\gamma}) \) and the previous set \( (\varepsilon_{n\kappa}, \beta, \gamma) \) tells us that
the negative energy solution for pseudospin symmetry, where \( S(r) = -V(r) \), can be obtained directly from those of the positive energy solution above for spin symmetry using the parameter map [50-52]:

\[
F_{n\kappa}(r) \leftrightarrow G_{n\kappa}(r), V(r) \rightarrow -V(r) \text{ (or } q \rightarrow -q), \quad E_{n\kappa} \rightarrow -E_{n\kappa} \text{ and } A \rightarrow -A. \tag{51}
\]

Following the previous results with the above transformations, we finally arrive at the energy equation,

\[
m_0^2 c^4 - E_{n\kappa}^2 + A (m_0 c^2 + E_{n\kappa}) = \left[ \frac{q (m_0 c^2 - E_{n\kappa} + A) + b (m_0 c^2 + \frac{4}{2})}{n + \frac{1}{2} + \sqrt{(\kappa - \frac{1}{2})^2 + b(b + 2q)}} \right]^2, \tag{52}
\]

and lower-spinor component wave function,

\[
G_{n\kappa}(r) = \sqrt{\frac{n! (2\tilde{\varepsilon}_{n\kappa})^{2n+3}}{2(n + \eta + 1)\Gamma(n + 2\eta + 2)} r^{\eta+1} e^{-\tilde{\varepsilon}_{n\kappa} r} L_{\eta}^{2\eta+1}(2\tilde{\varepsilon}_{n\kappa} r),} \tag{53}
\]

where \( \eta = \sqrt{(\kappa - \frac{1}{2})^2 + b(b + 2q) - \frac{1}{2}} \). The finiteness of the above wave functions for the bound states is achieved once \( \eta > 0 \) and \( \tilde{\varepsilon}_{n\kappa} > 0 \). Let us investigate the following specific cases of interest.

(1) Considering the case when \( S(r) = -V(r) \), i.e., \( q_s = -q_v = q \) or \( A = 0 \), then eq. (52) can be reduced to the following forms:

\[
E_{n\kappa}^{p} = \frac{-q (b + q) + B_{n\kappa} \sqrt{B_{n\kappa}^2 - b(b + 2q)}}{q^2 + B_{n\kappa}^2} m_0 c^2, \tag{54a}
\]

\[
E_{n\kappa}^{a} = \frac{-q (b + q) - B_{n\kappa} \sqrt{B_{n\kappa}^2 - b(b + 2q)}}{q^2 + B_{n\kappa}^2} m_0 c^2, \tag{54b}
\]

where \( B_{n\kappa} \) is defined by

\[
B_{n\kappa} = \left( n + \frac{1}{2} + \sqrt{(\kappa - \frac{1}{2})^2 + b(b + 2q)} \right). \tag{55}
\]

We note that the bound state solutions of the particle and anti-particle are available.

(2) When the mass is constant, i.e., \( b = 0 \), then we obtain

\[
E_{n\kappa}^{p} = \frac{(n + \kappa)^2 - q^2}{(n + \kappa)^2 + q^2} m_0 c^2, \quad E_{n\kappa}^{a} = -m_0 c^2, \quad n = 0, 1, 2, \ldots, \quad \kappa = \pm 1, \pm 2, \ldots, \tag{56}
\]
and the lower-spinor wave functions:

\[ G_{n\kappa}(r) = \sqrt{\frac{n!}{2(n + \kappa)\Gamma(n + 2\kappa)}} r^{\kappa} e^{-\frac{1}{2c} \sqrt{m_0^2 c^4 - E_{n\kappa}^2}} \times L_n^{2\kappa-1} \left( \frac{2}{\hbar c} \sqrt{m_0^2 c^4 - E_{n\kappa}^2} r \right), \]

where \( \kappa = 1, 2, 3, \cdots \), and \( E_{n\kappa} < m_0 c^2 \).

\section*{IV. DISCUSSIONS}

In this section, we are going to study two special cases of the energy eigenvalues given by eqs. (34) and (52). First, let us study the s-wave case \( l = 0 \) (\( \kappa = -1 \)) and \( \tilde{l} = 0 \) (\( \kappa = 1 \)) case

\[ m_0^2 c^4 - E_{n,-1}^2 - A (m_0 c^2 - E_{n,-1}) = \left[ \frac{2q(m_0 c^2 + E_{n,-1} - A) + b (A - 2m_0 c^2)}{2n + 1 + \sqrt{1 + 4b(b - 2q)}} \right]^2, \]

\[ m_0^2 c^4 - E_{n,1}^2 + A (m_0 c^2 + E_{n,1}) = \left[ \frac{2q(m_0 c^2 - E_{n,1} + A) + b (A + 2m_0 c^2)}{2n + 1 + \sqrt{1 + 4b(b + 2q)}} \right]^2. \]

For constant mass case, if one set \( A = 0 \) and \( b = 0 \) into eqs. (58) and (59), we obtain for spin and pseudo-spin symmetric Dirac theory,

\[ E_{n,-1} = \frac{(n + 1)^2 - q^2}{(n + 1)^2 + q^2} m_0 c^2, \]

\[ E_{n,1} = -\frac{(n + 1)^2 - q^2}{(n + 1)^2 + q^2} m_0 c^2. \]

respectively, which are identical with the s-wave results in ref. [48] for \( S(r) = V(r) \) and \( S(r) = -V(r) \), respectively. Let us now discuss the non-relativistic limit of the energy eigenvalues and wave functions of our solution for the spatially dependent mass and for the constant mass Schrödinger equation. If we take \( A = 0 \) and put \( 2q \to q \) [12], i.e., \( S(r) = V(r) = \Sigma(r) \), the non-relativistic limits of energy equation (34) and wave functions (39) under the following transformations \( E_{n\kappa} + m_0 c^2 \approx 2\mu/\hbar^2 \) and \( E_{n\kappa} - m_0 c^2 \approx E_{nl} \) [27,50] become

\[ E_{nl} = -\frac{\mu}{2\hbar^2} \left( n + \frac{1}{2} + \sqrt{\frac{1}{2} + l^2 + b(b - 2q)} \right), \]

respectively.
\[ F_{nl}(r) = N r^{l+1} e^{-\sqrt{2\mu E_{nl} r} L_n^{(2l+1)}} \left( 2 \sqrt{-\frac{2\mu}{\hbar^2} E_{nl} r} \right), \]

respectively, where

\[ N = \sqrt{\frac{n! \left( 2 \sqrt{-\frac{2\mu}{\hbar^2} E_{nl}} \right)^{2l+3}}{2 (n + l + 1) \Gamma (n + 2l + 2)}}. \]

Furthermore, when mass becomes constant \((b = 0)\), we find the well-known non-relativistic energy spectrum and wave functions of the Coulomb problem,

\[ E_{nl} = -\frac{\mu}{2\hbar^2} \frac{q^2}{(n + l + 1)^2}, \quad q = Z\alpha \]

which are identical to those given in refs. [60-62].

V. CONCLUSIONS

To summarize, we have presented the bound state solutions of the spatially-dependent mass Dirac equation with the Coulomb-like field potential under the conditions of the spin symmetry and pseudospin symmetry. We have obtained an explicit expressions for energy eigenvalues and associated wave functions for arbitrary spin-orbit \(\kappa\) state with a specific choice for the mass function that provides an exact solution to the pseudospin symmetric Dirac equation and an approximate solution for the spin symmetric Dirac equation. This suitable choice of mass function which is in the form of Coulomb-like potential enables us to solve the spatially-dependent Dirac equation analytically and reduces it to the constant mass solution as well as when \(b = 0\) \((m_1 = 0)\). The most stringent interesting result is that the present spin and pseudo-spin symmetries can be easily reduced to the previously found Klein-Gordon solution once \(S(r) = V(r)\) and \(S(r) = -V(r)\) \((i.e., \text{when } A = 0)\), respectively. The resulting solutions of the wave functions are being expressed in terms of the generalized Laguerre polynomials. Obviously, when the coupling potential parameters are adjusted to some specific values, particularly when \(q = b/2\), the spectra of the mass
varying Dirac particle for the case $q_s = q_v$ ($q_s = -q_v$) turn to become similar to the spectra of the constant mass Dirac particle for the case $q_s = -q_v$ ($q_s = q_v$), respectively. In the limit of constant mass ($b = 0$), the solution for the energy eigenvalues and wave functions are reduced to those ones given in literature. Also, when spin-orbit quantum number $\kappa = 0$, the problem reduces to $s$-waves solution as in [48].

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Appendix A: Parametric Generalization of the NU Method

Our systematical derivations hold for any potential form.

(i) The analytic results for the key polynomials [47]:

\[ \pi(r) = c_5 + \sqrt{c_9} - \frac{1}{c_3}(c_4\sqrt{c_9} + \sqrt{c_{10}} - c_3c_6)r, \]  
\[ k = -\frac{1}{c_3^2}(c_3c_8 + 2c_4c_9 + 2\sqrt{c_9c_{10}}), \]  
\[ \tau(r) = c_3 + 2\sqrt{c_9} - \frac{2}{c_3}(c_3c_4 + c_4\sqrt{c_9} + \sqrt{c_{10}})r, \]  
\[ \tau'(r) = -\frac{2}{c_3}(c_3c_4 + c_4\sqrt{c_9} + \sqrt{c_{10}}) < 0. \]  

(ii) The energy equation:

\[ c_2n - (2n + 1)c_6 + \frac{1}{c_3}(2n + 1)(\sqrt{c_{10}} + c_4\sqrt{c_9}) \]
\[ + n(n-1)c_4 + \frac{1}{c_3^2}(c_3c_8 + 2c_4c_9 + 2\sqrt{c_9c_{10}}) = 0. \]  

(iii) The wave functions:

\[ \rho(r) = r^{c_{11}}(c_3 - c_4r)^{c_{12}}, \]  
\[ \phi(r) = r^{c_{13}}(c_3 - c_4r)^{c_{14}}, \]  
\[ y_n(r) = P_n^{(c_{11},c_{12})}(c_3 - c_4r), \]  
\[ \psi_n(r) = \phi(r)y_n(r) = A_n r^{c_{13}}(c_3 - c_4r)^{c_{14}}P_n^{(c_{11},c_{12})}(c_3 - 2c_4r), \]  

where \( P_n^{(a,b)}(c_3 - c_4r) \) are the Jacobi polynomials and \( A_n \) is a normalizing factor.

When \( c_4 = 0 \), the Jacobi polynomial turn to be the generalized Laguerre polynomial and the constants relevant to this polynomial change are

\[ \lim_{c_4 \to 0} P_n^{(c_{11},c_{12})}(c_3 - 2c_4r) = L_n^{c_{11}}(c_{15}r), \]  
\[ \lim_{c_4 \to 0} (c_3 - c_4r)^{c_{14}} = e^{-c_{16}r}, \]  

where \( L_n^{c_{11}}(c_{15}r) \) are the generalized Laguerre polynomials and \( B_n \) is a normalizing factor.

(iv) The relevant coefficients \( c_i \) \((i = 5, 6, \cdots, 16)\) are given as follows:

\[ c_5 = \frac{1}{2}(c_3 - c_1), \quad c_6 = \frac{1}{2}(c_2 - 2c_4), \quad c_7 = c_6^2 + \xi_1, \]
\[ c_8 = 2c_5c_6 - \xi_2, \ c_9 = c_5^2 + \xi_3, \]
\[ c_{10} = c_4 (c_3c_8 + c_4c_9) + c_3^2c_7, \]
\[ c_{11} = \frac{2}{c_3} \sqrt{c_9}, \ c_{12} = \frac{2}{c_3c_4} \sqrt{c_{10}}, \]
\[ c_{13} = \frac{1}{c_3} (c_5 + \sqrt{c_9}), \ c_{14} = \frac{1}{c_3c_4} (\sqrt{c_{10}} - c_4c_5 - c_3c_6), \]
\[ c_{15} = \frac{2}{c_3} \sqrt{c_{10}}, \ c_{16} = \frac{c_{15}}{2}. \]
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TABLE I: The specific values for the parametric constants necessary for calculating the energy eigenvalues and eigenfunctions of the spin symmetry Dirac wave equation (23).

| Constant | Analytic value | Constant | Analytic value |
|----------|----------------|----------|----------------|
| $c_1$    | 0              | $c_2$    | 0              |
| $c_3$    | 1              | $c_4$    | 0              |
| $c_5$    | $\frac{1}{2}$ | $c_6$    | 0              |
| $c_7$    | $\varepsilon_{nk}^2$ | $c_8$    | $-\beta$      |
| $c_9$    | $\frac{1}{4} + \gamma$ | $c_{10}$ | $\varepsilon_{nk}^2$ |
| $c_{11}$ | $1 + 2\delta$ | $c_{12} = c_{15}$ | $2\varepsilon_{nk}$ |
| $c_{13}$ | $1 + \delta$  | $c_{14} = c_{16}$ | $\varepsilon_{nk}$ |
| $\xi_1$ | $\varepsilon_{nk}^2$ | $\xi_2$ | $\beta$      |
| $\xi_3$ | $\gamma$      |          |                |