Three dimensional stationary cyclic symmetric Einstein–Maxwell solutions; black holes∗

Alberto A. García–Díaz†

Departamento de Física,
Centro de Investigación y de Estudios Avanzados del IPN,
Apdo. Postal 14-740,
07000 México DF, México, and
Department of Physics,
University of California,
Davis, CA 95616, USA.

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From a general metric for stationary cyclic symmetric gravitational fields coupled to Maxwell electromagnetic fields within the (2 + 1)-dimensional gravity the uniqueness of wide families of exact solutions is established, among them, all uniform electromagnetic solutions possessing electromagnetic fields with vanishing covariant derivatives, all fields having constant electromagnetic invariants $F_{\mu\nu} F^{\mu\nu}$ and $T_{\mu\nu} T^{\mu\nu}$, the whole classes of hybrid electromagnetic solutions, and also wide classes of stationary solutions are derived for a third order nonlinear key equations. Certain of these families can be thought of as black hole solutions. For the most general set of Einstein–Maxwell equations, reducible to three non–linear equations for the three unknown functions, two new classes of solutions–having anti-de Sitter spinning metric limit–are derived. The relationship of various families with those reported by different authors’ solutions has been established. Among the classes of solutions with cosmological constant a relevant place occupy: the electrostatic and magnetostatic Peldan solutions, the stationary uniform and spinning Clement classes, the constant electromagnetic invariant branches with the particular Kamata–Koikawa solution, the hybrid cyclic symmetric stationary black hole fields, and the non–less important solutions generated via $SL(2, R)$ transformations where the Clement spinning charged solution, the Martinez–Teitelboim–Zanelli black hole solution, and Dias–Lemos metric merit mention.

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†aagarcia@fis.cinvestav.mx
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I. INTRODUCTION

During the last two decades three–dimensional gravity has received some attention, in particular, in topics such as: black hole physics, search of exact solutions, quantization of fields coupled to gravity, cosmology, topological aspects, and others. This interest in part has been motivated by the discovery, in 1992, of the 2 + 1 stationary circularly symmetric black hole solution by Bañados, Teitelboim and Zanelli [1]–the BTZ black hole– see also [2–4], which possesses certain features inherent to 3 + 1 black holes. On the other hand, it is believed that 2 + 1 gravity may provide new insights towards a better understanding of the physics of 3 + 1 gravity. In the framework of exact solutions in 2 + 1 gravity the list of references on the topic is extremely vast; one finds works on point masses, cosmological and perfect fluid solutions, dilaton and string fields, and on electromagnetic fields coupled to gravity, among others.

The purpose of this contribution is to provide a new approach on the search of electromagnetic–gravitational solutions to the Einstein–Maxwell fields of the 2+1 gravity in the presence of a cosmological constant, allowing for stationary and cyclic symmetries. The
search and interpretation of this kind of solutions has been the goal and realm of several authors' investigations starting from quite different perspectives and using a variety of approaches, which sometime have brought about duplication of results and efforts. The main objective of this work is to derive general families of stationary (static) cyclic symmetric solutions to the Einstein–Maxwell field equations, establishing their relationship with known to-date solutions, and to point out the families allowing for black hole interpretation.

The outline of this work is as follows:

Sec. II contains the theorem on the existence of possible classes of electromagnetic fields for stationary cyclic symmetric 2+1 spacetimes; \[ *F = a dt + b d\phi + c g_{rr}/\sqrt{-g} dr \] fully characterizes the families of Maxwell electromagnetic fields. In Sec. III the canonical metrics to be used and the corresponding Einstein–Maxwell equations are explicitly given. Sec. IV is devoted to the determination of the general static solutions. Sec. V deals with the determination of the uniform electromagnetic fields, i.e., those which possess vanishing covariant derivatives \[ F_{\alpha\beta\gamma} = 0. \] The set of stationary solutions for constant invariant \[ F_{\mu\nu} F^{\mu\nu}, \] and consequently, due to the structure of the electromagnetic fields, with constant energy momentum invariants \[ T^\mu_\mu \] and \[ T_{\mu\nu} T^{\mu\nu}, \] are derived in Sec. VI. The so called self (anti)-dual fields are derived in Sec. VII. The determination of the general stationary solution for the electromagnetic field \[ *F = c g_{rr}/\sqrt{-g} dr \] is accomplished in Sec. VIII. A master equation—a single nonlinear fourth order ordinary equation subsequently reducible to a third order one—is established for the determination of the stationary fields having pure electric or magnetic features and particular solutions to it are reported in Sec. IX. In Sec. X we search for general stationary solutions for the electromagnetic field \[ *F = a dt + b d\phi; \] solutions within a wide class of structural functions allowing for logarithms are derived and their uniqueness demonstrated. Sec. XI is devoted to stationary solutions generated via \[ SL(2, R) \]–transformations. Finally, we end with some concluding remarks in Sec. XII.

As it has been stated above, the main objective of this report is to demonstrate, via straightforward integration of the field equations, the completeness of electromagnetic classes of stationary cyclic symmetric solutions. A full characterization of the physical contents of these solutions would require a considerable more extension of this paper, for this reason, some short comments are made in this respect close to those contained in the related references if there are any, and also about the new found families with special emphasis on their black hole feature.

II. ELECTROMAGNETIC FIELD FOR STATIONARY CYCLIC SYMMETRIC 2+1 SPACETIMES

To begin with, we consider a stationary cyclic symmetric spacetime with signature (-,+,+), i.e., a space endowed with stationary symmetry \( k = \partial_t, \ k \cdot k < 0, \) such that \( \mathcal{L}_k g = 0, \) and cyclic symmetry \( m = \partial_\phi, \ m \cdot m > 0, \) such that \( \mathcal{L}_m g = 0, \) with closed integral curves from 0 to \( 2\pi, \) which in turn commute \( [k, m] = 0. \) Hence the Killing vector fields \( k \) and \( m \) generate the group \( SO(2) \times R. \) The electromagnetic field, described by the antisymmetric tensor field \( F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu, \) is assumed to be stationary cyclic symmetric, i.e., \( \mathcal{L}_k F = 0 = \mathcal{L}_m F. \) It should be pointed out that, in contrast to the general 3+1 stationary cyclic symmetric spacetime, any 2+1 stationary cyclic symmetric spacetime is necessarily circular, i.e., the circularity conditions

\[ k \wedge m \wedge dk = 0 = k \wedge m \wedge dm \] (2.1)
are identically fulfilled because of their 4–form character and hence there exists the discrete symmetry when simultaneously $t \rightarrow -t$ and $\phi \rightarrow -\phi$. One may find a coordinate system such that the metric tensor components $g(k \, dr) = 0$ and $g(m \, dr) = 0$, where the coordinate direction $dr$ is orthogonal to the surface spanned by $k \wedge m$. Commonly one introduces the coordinate system $\{t, \phi, r\}$ in (2+1)-dimensional gravity.

The main goal of this section is to demonstrate of the following theorem.

**Theorem:** The general form of stationary cyclic symmetric electromagnetic fields in 2+1 dimensions is given by

$$*F = adt + b d\phi + c \frac{g_{rr}}{\sqrt{-g}} dr,$$  (2.2)

where the constants $a, b$ and $c$ are subjected, by virtue of the Ricci circularity conditions, to the equations

$$ac = 0 = bc,$$  (2.3)

which gives rise to two disjoint branches

$$c \neq 0, *F = c \frac{g_{rr}}{\sqrt{-g}} dr,$$  (2.4)

and

$$c = 0, *F = adt + b d\phi,$$  (2.5)

with its own sub-classes $a = 0$ or $b = 0$.

To establish that the field $*F$ possesses the form given by Eq. (2.2) one uses the source–free Maxwell equations

$$dF = 0 = d*F,$$  (2.6)

where $*$ denotes the Hodge star operation.

Let us evaluate the exterior derivative of the $t$–component $*F(k)$ of $*F$,

$$d*F(k) = d i_k *F = \nabla_k *F - i_k d*F = 0 - 0 \rightarrow *F(k) := a = \text{constant},$$  (2.7)

the first zero arises from the stationary character of the field $F$, while the second one corresponds to the Maxwell equation. Similarly, for the $\phi$–component $*F(m)$ one has

$$d*F(m) = d i_m *F = \nabla_m *F - i_m d*F = 0 - 0 \rightarrow *F(m) := b = \text{constant}.$$  (2.8)

In this manner we have established that the $t$ and $\phi$ components of the dual field $*F$ are constants given correspondingly by $a$ and $b$. The component of $*F$ along the vector direction $\partial_r$ remains to be determined. For this purpose, consider the $t\phi$–component $F(k,m)$ of the field $F$, which can be expressed as $F(k,m) = i_m i_k F = (-i_m i_k *F = i_m * (k \wedge *F) = * (m \wedge k \wedge *F)) = - *F(*(k \wedge m))$, thus its derivative yields

$$dF(k,m) = d(i_m i_k F) = d i_m (i_k F)$$

$$= (\nabla_m - i_m d)(i_k F) = i_k \nabla_m F + i_{[k,m]} F - i_k (\nabla_m - i_k d)F$$

$$= 0 \rightarrow F(k,m) := c = \text{constant}.$$  (2.9)

Since the constant $c$ can be written as $c = - *F(*(k \wedge m))$, to determine it, one evaluates $*(k \wedge m)$. Identify the Killing vectors accordingly with $k = \partial_t$ and $m = \partial_{\phi}$, then

$$*(k \wedge m) = -\sqrt{-g} dr = -\sqrt{-g} g^{rr} \partial_r$$  (2.10)
thus
\[ c = -\ast F(-\sqrt{-g} g^{r\tau} \partial_r) = \sqrt{-g} g^{rr} \ast F(\partial_r). \] (2.11)

Conversely, from the above–mentioned relation one determines the \(r\)–component of the field \(\ast F\), namely \(\ast F(\partial_r) = \sqrt{-g} g^{rr} \ast F(\partial_r)\). In this manner, the structure of \(F\), explicitly given by (2.2), has been established.

The vanishing conditions (2.3) straightforwardly arise from the Ricci circularity conditions \(m \wedge k \wedge R(k) = 0\) and \(k \wedge m \wedge R(m) = 0\). Correspondingly, the vanishing conditions \(ac = 0 = bc\) can be established immediately, as we shall see in the next section, from the Einstein equations \(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = \kappa T_{\mu\nu} - \Lambda g_{\mu\nu}\), where the electromagnetic energy–momentum tensor components are defined through the electromagnetic field \(F_{\mu\nu} = -F_{\nu\mu}\) as \(4\pi T_{\mu\nu} = F_{\mu\sigma}F_{\nu}^{\sigma} - \frac{1}{4}g_{\mu\nu}F_{\alpha\sigma}F^{\alpha\sigma}\). A first formulation of this theorem with an outline of its demonstration has been reported in [5].

### III. GENERAL METRIC AND EINSTEIN EQUATIONS

In general, in \((2 + 1)\)–dimensional gravity any stationary cyclic symmetric metric can be given as
\[
g = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{\phi\phi} d\phi^2 + g_{rr} dr^2.
\] (3.1)

When a Maxwell electromagnetic field is present, the field tensor, as we established previously, possesses the structure
\[
F^{\alpha\beta} = \frac{1}{\sqrt{-g}} \begin{bmatrix} 0 & b & -\frac{c_{0\alpha\beta}}{\sqrt{-g}} \\ -b & 0 & a \\ \frac{c_{0\alpha\beta}}{\sqrt{-g}} & -a & 0 \end{bmatrix},
\] (3.2)

where \(g := \det(g_{\mu\nu})\), which makes apparent the fulfillment of the divergence equation
\[
(\sqrt{-\det(g_{\mu\nu})} F^{\alpha\beta})_{,\beta} = 0
\]

for constants \(a, b,\) and \(c\). The Maxwell electromagnetic energy–momentum tensor is given as usual as
\[
T_{\mu\nu} = \frac{1}{4\pi} (F_{\mu\sigma} F^{\nu\sigma} - \frac{1}{4} g_{\mu\nu} F_{\tau\sigma} F^{\tau\sigma}).
\] (3.3)

**A. Canonical metrics and Einstein–Maxwell equations**

Without loss of generality one can choose the coordinates for a stationary cyclic symmetric \(2 + 1\) metric, developed with respect to the cyclic symmetry \(m = \partial_{\phi}\), in such a way that it becomes
\[
g = -\frac{F(r)}{H(r)} dt^2 + \frac{dr^2}{F(r)} + H(r) [d\phi + W(r) dt]^2.
\] (3.4)
On the other hand, if one chooses the stationary symmetry $k = \partial_t$ as the fundamental Killing field, the stationary cyclic symmetric $2 + 1$ metric can be written as

$$
g = -\frac{F(r)}{h(r)}[dt - \omega(r) d\phi]^2 + h(r) d\phi^2 + \frac{dr^2}{F(r)},
$$

$$
F = F, \quad H = h - \frac{F}{h} \omega^2, \quad W \frac{H}{F} = \omega, \quad h = \frac{HF}{F - W^2 H^2}.
$$

Mostly we will use the metric (3.4) in the forthcoming developments, but occasionally the metric representation (3.5) will be used. When doing so, the derived expressions will be given in terms of the set $\{F(r), h(r), \omega(r)\}$ of structural functions. Omitting the dependence of the structural functions on the variable $r$, the Maxwell electromagnetic field contravariant tensor is given by

$$
(F^{\mu\nu}) = \begin{pmatrix}
0 & b & -\frac{c}{F} \\
-b & 0 & a \\
\frac{c}{F} & -a & 0
\end{pmatrix},
$$

(3.6)

where $a$, $b$, and $c$ are constants related with the character of the field. For instance, if only $b$ is different from zero, while $a$ and $c$ vanish, the field is called (pure) electric field. When $a \neq 0$, $b = 0 = c$, one deals with a pure magnetic field; other possibilities do not receive a particular name. The covariant components $F_{\mu\nu}$ of the field tensor are given by:

$$
F_{tr} = -b/H - W H(a - b W)/F, \quad F_{t\phi} = c, \quad F_{r\phi} = H(a - b W)/F.
$$

(3.7)

The electromagnetic field quadratic invariant $FF := F_{\mu\nu} F^{\mu\nu}$ is given by

$$
FF = -\frac{2c^2}{F} + 2 \frac{H (a - W b)^2}{F} - 2 \frac{b^2}{H}.
$$

(3.8)

Notice that if one uses the vector–potential description of the electromagnetic field

$$
F := \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu = d(A_\mu dx^\mu) =: dA
$$

(3.9)

one would have

$$
F = d \int^r [(\frac{1}{H} - \frac{H}{F}W^2)b + \frac{H}{F} W a] dr \times dt + d \int^r [-\frac{H}{F}W b + \frac{H}{F} a] dr \times d\phi
$$

$$
+ d \frac{1}{2} c(t d\phi - \phi dt) = dA.
$$

(3.10)

The energy–momentum tensor associated with the metric (3.4) occurs to be

$$
(T^\mu_\nu) = \begin{pmatrix}
\frac{b^2 (H^2 W^2 - F) - a^2 H^2 - c^2 H}{8\pi F H} & \frac{ac}{4\pi} & -\frac{a[b(H^2 W^2 - F) - a H^2 W]}{4\pi F H} \\
\frac{cH(W b - a)}{4\pi F^2} & -\frac{b^2 F + H^2 (W b - a)^2 + c^2 H}{8\pi F H} & -\frac{c[b(H^2 W^2 - F) - a H^2 W]}{4\pi F^2 H} \\
\frac{bH(W b - a)}{4\pi F} & \frac{bc}{4\pi} & -\frac{b^2 (H^2 W^2 - F) - a^2 H^2 + c^2 H}{8\pi F H}
\end{pmatrix}
$$

(3.11)
and possesses the trace \( T := T^\mu_\mu \) given by
\[
T = -\frac{1}{8\pi} c^2 + \frac{1}{8\pi} \frac{H(a - Wb)^2}{F} - \frac{1}{8\pi} \frac{b^2}{H} = \frac{1}{16\pi} FF,
\]
and the electromagnetic energy momentum quadratic invariant \( TT := T^\mu_\nu T^\nu_\mu \)
\[
TT = \frac{3}{64\pi^2} \left[ \frac{H^2(a - bW)^2 - b^2F - c^2H}{F^2H^2} \right]^2 = \frac{3}{256\pi^2} FF^2.
\]
The Einstein–Maxwell equations
\[
E^\mu_\nu := R^\mu_\nu - \frac{R}{2} g^\mu_\nu + \Lambda g^\mu_\nu - 8\pi T^\mu_\nu = 0 \quad (3.14)
\]
for a negative cosmological constant \( \Lambda = -\frac{1}{l^2} \) explicitly read:
\[
E^t_t = \frac{1}{2} F H_{r,r} + \frac{1}{4} H F^r_F - \frac{1}{4} F \frac{H F^2}{H^2} H^r_{r,2} + \frac{1}{2} H^2 W W_{r,r} + HW W_{r} H_{,r} + \frac{1}{4} H^2 W_{,r}^2
\]
\[
+ b^2 \frac{F - H^2 W^2}{F H} + \frac{c^2}{F} + \frac{a^2 H}{F} - \frac{1}{l^2}, \quad (3.15a)
\]
\[
E^r_r = -2ca, \quad (3.15b)
\]
\[
E^t_\Phi = \frac{1}{2} W F_{r,r} - FW \frac{H_{,r}}{H} - W \frac{H F^r_F}{H} - 2a^2 \frac{H W}{F} - 2ab \frac{F - H^2 W^2}{F H}
\]
\[
- \frac{1}{2} \left( F + H^2 W^2 \right) \left( W_{,r} + 2W_{r} \frac{H}{H} \right) + FW \frac{H_{,r}^2}{H^2} - H^2 WW_{r}^2, \quad (3.15c)
\]
\[
E^r_t = -2c \frac{H}{F^2} (Wb - a), \quad (3.15d)
\]
\[
E^r_r = \frac{1}{4} H F_{r} - \frac{1}{4} F \frac{H_{,r}^2}{H^2} + \frac{1}{4} H^2 W_{,r}^2 + \frac{b^2}{H} - \frac{c^2}{F} - \frac{H}{F} (bW - a)^2 - \frac{1}{l^2}, \quad (3.15e)
\]
\[
E^r_\Phi = -2ca \frac{H W}{F^2}, \quad (3.15f)
\]
\[
E^t_\Phi = \frac{1}{2} H^2 W_{r,r} + HW W_{r} H_{,r} - 2b \frac{H}{F} (Wb - a), \quad (3.15g)
\]
\[
E^r_\Phi = -2bc, \quad (3.15h)
\]
\[
E^\Phi_\Phi = \frac{1}{2} F_{r,r} - \frac{1}{2} \frac{H_{,r} F}{H} - \frac{3}{4} H F^r_F + \frac{3}{4} F \frac{H_{,r}^2}{H^2} - \frac{1}{2} H^2 W W_{r,r} + HW W_{r} H_{,r}
\]
\[
- \frac{3}{4} H^2 W_{,r}^2 - b^2 \frac{F - H^2 W^2}{F H} + \frac{c^2}{F} - \frac{a^2 H}{F} - \frac{1}{l^2}. \quad (3.15i)
\]
The vanishing of \( E^r_t \) and \( E^\Phi_\Phi \) yields respectively \( ac = 0 = bc \). Therefore one can distinguish the branches:
\( c = 0 \) with \( a \) and \( b \), not vanishing simultaneously; and \( c \neq 0 \) with \( a \) and \( b \) vanishing simultaneously.
In the forthcoming sections we shall deal with the integration and characterization of each branch starting from the simplest static solutions.
B. Complex extension and real cuts

It would be of some interest to add some lines about the complex extension of the metric under consideration. Accomplishing in the metric (3.4) the complex transformations

\[ t \rightarrow i \Phi, \quad \phi \rightarrow -iT, \]  

one arrives at

\[ g_c = \left( \frac{F}{H} - HW^2 \right) d\Phi^2 + \frac{dr^2}{F} - H dT^2 + 2HW dT d\Phi, \]  

which can be brought to the form

\[ g_c = -\frac{F}{\mathcal{H}} dT^2 + \frac{dr^2}{F} + \mathcal{H} (d\Phi + \mathcal{W}dT)^2, \]  

accompanied by the identification

\[ \mathcal{F} = F, \quad \mathcal{H} = \frac{F}{H} - HW^2, \quad \mathcal{W} = \frac{HW}{\mathcal{H}}. \]  

At the level of the field tensor \( F^{\mu\nu} \) one has

\[ F^{\mu\nu} = \begin{bmatrix} 0 & B & -\frac{c}{F} \\ -B & 0 & A \\ \frac{c}{F} & -A & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \ a & \frac{c}{F} \\ i \ a & 0 & ib \\ -\frac{c}{F} & -ib & 0 \end{bmatrix} \]  

thus the following correspondence for the field constants arises

\[ -i \ a \rightarrow B, \quad i \ b \rightarrow A, \quad -c \rightarrow C. \]  

Summarizing, one may say that the role of the Killingian coordinates has been interchanged: the time-like coordinate \( t \) becomes the space-like \( \Phi \)-coordinate, while the cyclic \( \phi \)-coordinate becomes the new time-coordinate \( T \). Correspondingly, one has to think of the tensor components of the participating quantities from this perspective. This procedure can be used to determine new classes of solutions from known ones. For instance, one can generate magnetic solutions from electric ones. The relations arising from this kind of complex transformations have been called “duality mapping” by Cataldo [6], although strictly there is no electric-magnetic duality in 2+1 dimensions.

C. Positive \( \Lambda \) solutions

For completeness and to avoid duplication of works it is worthwhile to notice that solutions for positive cosmological constant are easily obtainable from the anti-de Sitter (\( \Lambda = -1/l^2 \)) ones; First, notice that the Einstein equations for any sign of a cosmological constant \( \Lambda \)-positive or negative-are recovered from Eqs. (3.15) simply by replacing \(-1/l^2 \rightarrow \Lambda \). Next, having at disposal a concrete solution of the Einstein equations mentioned above (3.15), by replacing there \( l^2 \) by \(-l^2 \), one determines the corresponding metric structure for positive cosmological constant \( \Lambda = 1/l^2 \). This replacement is equivalent to accomplishing the complex change \( l \rightarrow il \) in the \( \Lambda < 0 \) solution, nevertheless one ought to take care of possible additional arrangements of constants, if any, and also of possible changes in the signature.

In the publication appears (3.15a) instead of the correct set of equations (3.15).
D. Quasilocal mass, energy and angular momentum

To evaluate quasilocal mass, energy, and angular momentum of spacetimes with asymptotic different from the flat one, in particular the anti-de Sitter, one uses the quasilocal formalism developed in [7, 8].

For a stationary cyclic symmetric metric of the form

\[ g = -N(\rho)^2 \, dt^2 + L(\rho)^{-2} \, d\rho^2 + K(\rho)^2 \left[ d\Phi + W(\rho) \, dT \right]^2, \]

(3.22)

the surface energy density is given by

\[ \epsilon(\rho) = -\frac{L(\rho)}{\pi K(\rho)} \frac{d}{d\rho} K(\rho) - \epsilon_0, \]

(3.23)

where \( \epsilon_0 \) is the reference energy density, which in the case of solutions with negative cosmological constant \( \Lambda = -1/l^2 \) corresponds to the density of the anti–de Sitter spacetime, namely \( \epsilon_0 = -\frac{1}{\pi \rho} \sqrt{1 + \rho^2} \). The momentum density is determined from

\[ j(\rho) = \frac{K(\rho)^2 L(\rho)}{2 \pi N(\rho)} \frac{d}{d\rho} W(\rho). \]

(3.24)

The integral momentum \( J(\rho) \), global energy \( E(\rho) \), and the integral mass \( M(\rho) \) are correspondingly given by

\[ J(\rho) = 2\pi K(\rho) j(\rho), \]
\[ E(\rho) = 2\pi K(\rho) \epsilon(\rho), \]
\[ M(\rho) = E(\rho) N(\rho) - W(\rho) J(\rho). \]

(3.25)

As far as the evaluation of these physical quantities for the studied classes of solutions is concerned, a work on these lines is being developed and will be published elsewhere.

In the publication there is a misprint in \( M(\rho) \); instead of the correct \( N(\rho) \) was typed \( K(\rho) \).

IV. STATIC CYCLIC SYMMETRIC SOLUTIONS FOR MAXWELL FIELDS

In this section, we derive all the static solutions of the Einstein–Maxwell equations (3.15); there are only three families within this class. In the static solution \( W(r) = 0 \), consequently the metric (3.4) becomes

\[ g = -\frac{F(r)}{H(r)} \, dt^2 + \frac{dr^2}{F(r)} + H(r) \, d\phi^2, \]

(4.1)
and the Einstein–Maxwell equations simplify drastically:

\[
\begin{align*}
E_{t}^{t} &= \frac{1}{2} F \frac{H_{r,r}}{H} + \frac{1}{4} H \frac{H_{r} F_{r}}{H} - \frac{1}{4} F \frac{H_{r}^{2}}{H^2} + \frac{b^2}{F} + \frac{c^2}{F} + \frac{a^2 H}{F} - \frac{1}{l^2}, \\
E_{r}^{r} &= \frac{1}{4} H F_{r} - \frac{1}{4} F \frac{H_{r}^{2}}{H^2} + \frac{b^2}{F} - \frac{c^2}{F} - \frac{a^2 H}{F} - \frac{1}{l^2}, \\
E_{\phi}^{\phi} &= \frac{1}{2} F_{r,r} - \frac{1}{2} F \frac{H_{r,r}}{H} - \frac{3}{4} H_{r} F_{r} + \frac{3}{4} F \frac{H_{r}^{2}}{H^2} - \frac{b^2}{F} + \frac{c^2}{F} - \frac{a^2 H}{F} - \frac{1}{l^2}, \\
E_{t}^{r} &= -2 c a, \quad E_{t}^{\phi} = -2 a b \frac{1}{H}, \\
E_{r}^{t} &= 2 a c \frac{H}{F^2}, \quad E_{r}^{\phi} = -2 b c \frac{1}{F H}, \\
E_{\phi}^{t} &= 2 a b \frac{H}{F}, \quad E_{\phi}^{r} = -2 b c,
\end{align*}
\] (4.2)

Each of these $E_{\mu}^{\nu}$-equations has to be equated to zero, therefore one can distinguish the following three families of static solutions:

the electric class: $b \neq 0, a = 0, c = 0$,

the magnetic class: $a \neq 0, b = 0, c = 0$,

the hybrid class: $c \neq 0, a = 0, b = 0$.

In the next subsections we proceed to integrate each class separately.

A. Electrostatic solutions; $b \neq 0, a = 0$

The substraction $E_{t}^{t}(a = 0 = c) - E_{r}^{r}(a = 0 = c)$ yields

\[\frac{d^2}{dr^2} H (r) = 0 \Rightarrow H (r) = C_0 + C_1 r,\] (4.3)

where $C_0$, and $C_1$ are constants of integration; $C_1$ at this stage is assumed to be different from zero, the zero case deserves special attention and will be treated separately. Substituting this structural function $H$ into the equation $E_{t}^{t}(a = 0 = c)$ one arrives at a first–order differential equation for $F$

\[
\frac{H_{r}}{H} F_{r} - \left(\frac{H_{r}}{H}\right)^2 F + 4 \frac{b^2}{H} - \frac{4}{l^2} = 0,
\] (4.4)

which by introducing an auxiliary function $f(r)$ through

\[F(r) = H(r) f(r) = (C_0 + C_1 r) f(r),\]

reduces to the simple equation

\[
\frac{df(r)}{dr} = \frac{4}{C_1 l^2} \frac{C_1 r + C_0 - b^2 l^2}{C_0 + C_1 r}
\] (4.5)

with general integral

\[f = \frac{4}{C_1 l^2} \left(K_0 + C_1 r - b^2 l^2 \ln(C_0 + C_1 r)\right),\] (4.6)
where \( K_0 \) is a new integration constant, into which of course one has incorporated \( C_0 \). Summarizing, one arrives at the metric

\[
g = - \frac{F(r)}{h(r)} dt^2 + \frac{dr^2}{F(r)} + h(r)d\phi^2,
\]

\[
F(r) = \frac{4}{C_1 l^2} \left[ K_0 + h(r) - b^2 l^2 \ln h(r) \right] h(r),
\]

\[
h(r) = C_1 r + C_0.
\]

This solution is characterized by:

the vector field

\[
A = A_t dt = \frac{b}{C_1} \ln h dt,
\]

the electromagnetic field tensors

\[
F^{\mu\nu} = 2b \delta^{[\mu}_t \delta^{\nu]}_r,
F_{\mu\nu} = -2 \frac{b}{h(r)} \delta^{[\mu}_r \delta^{\nu]}_t,
\]

with field invariant

\[
F_{\mu\nu}F^{\mu\nu} = -2 \frac{b^2}{h},
\]

the energy momentum tensor

\[
T^{\mu\nu} = -\frac{b^2}{8\pi h} \left( \delta^{[\mu}_r \delta^{\nu]}_t + \delta^{[\mu}_t \delta^{\nu]}_r - \delta^{[\mu}_\phi \delta^{\nu]}_\phi \right),
\]

with quadratic energy momentum invariant and trace

\[
T^{\mu\nu}T_{\mu\nu} = \frac{3}{64\pi^2 h^2}, T^\mu_\mu = -\frac{1}{8\pi h} b^2.
\]

A familiar representation of the above-mentioned solution is achieved for the choice \( C_0 = 0, C_1 = 2, K_0 = b^2 l^2 \ln 2r_0 \), which yields

\[
g = - \left[ \frac{2r}{l^2} - b^2 \ln \frac{r}{r_0} \right] dt^2 + 2r d\phi^2 + \frac{dr^2}{2r \left[ \frac{2r}{l^2} - b^2 \ln \frac{r}{r_0} \right]},
\]

\[
A = \frac{b}{2} \ln \frac{r}{r_0} dt.
\]

This solution, endowed with mass, electric charge, and radial parameters, allows for a charged black hole interpretation. The mass may assume positive as well as negative values, whereas the charge is not upper bound.
1. Gott–Simon–Alpern, Deser–Mazur, and Melvin electrostatic solution $\Lambda = 0$

According to the existing references Gott, Simon, and Alpern were the first to derive solutions within Maxwell theory in 2+1 gravity [9, 10]; they found, among other things, the electrostatic solution without cosmological constant. Introducing in the above expressions, (4.7) and (4.8), the radial coordinate $\rho$ through $C_0 + C_1 r \rightarrow \rho^2$ together with $t \rightarrow C_1 t/2, K_0 \rightarrow l^2 k_0$, and by letting $1/l^2 \rightarrow 0$ one arrives at the electrostatic solution in the form

$$g = -F dt^2 + \frac{d\rho^2}{F} + \rho^2 d\phi^2,$$

$$F(\rho) = k_0 - 2b^2 \ln \rho = \frac{\kappa}{2\pi} Q^2 \ln \frac{\rho_c}{\rho},$$

$$A = A_t dt = b \ln \rho dt.$$ (4.14)

Some authors refer to the coordinate system $\{t, \rho, \phi\}$, in which the perimeter of the circle equates $2\pi \rho = \int_0^{2\pi} \rho d\phi$, as to the "Schwarzschild" coordinates. Practically at the same time Deser and Mazur [11] published their version of the electrostatic solution for $\Lambda = 0$. Moreover, by then, the work by Melvin [12] was published with the derivation of the electrostatic as well as the magnetostatic solutions for vanishing $\Lambda$. Later, Kogan [13] reported and analyzed the (electro and magneto) static solutions of the (2+1)–dimensional Einstein–Maxwell equations for both positive and negative signs of the gravitational constant $\kappa$: recall that in the three dimensions there is not restriction on its sign. The $r$–coordinate used there was such that $g_{rr} = 1$ for the signature used in the present report.

2. Peldan electrostatic solution with $\Lambda$

The electrostatic solution with cosmological constant in polar coordinates arises from the general expressions above, (4.7) and (4.8), by means of the coordinate and parameter changes $C_0 + C_1 r \rightarrow \rho^2, t \rightarrow C_1 t/2, K_0 \rightarrow -l^2 m$. In this way one obtains

$$g = -F dt^2 + \frac{d\rho^2}{F} + \rho^2 d\phi^2,$$

$$F(\rho) = \frac{\rho^2}{l^2} - m - 2b^2 \ln \rho,$$

$$A = b \ln \rho dt.$$ (4.15)

The corresponding field tensors are given by

$$T^{\mu\nu} = -\frac{b^2}{8\pi \rho^2} \left( \delta_\mu^t \delta_\nu^t + \delta_\mu^\rho \delta_\nu^\rho - \delta_\mu^\phi \delta_\nu^\phi \right),$$

$$F_{\mu\nu} = -\frac{b}{\rho} \delta_\mu^t \delta_\nu^r.$$ (4.16)

To achieve the specific Peldan [14] writing, one has to accomplish the additional identifications $m \rightarrow -C_1, 1/l^2 \rightarrow -\lambda/2, b^2 \rightarrow q^2/4, t \rightarrow C_2 t, \rho \rightarrow r$. 
B. Magnetostatic solutions; \(a \neq 0, b = 0\)

To derive the magnetostatic solution, one starts from the addition \(E_t t + E_r r\) which yields

\[
\frac{d}{dr} \left[ \frac{F}{H} \frac{d}{dr} H - 4 \frac{r}{l^2} \right] = 0, \tag{4.17}
\]

with integral

\[
\frac{F}{H} \frac{d}{dr} H = 4 \frac{r}{l^2} + C_1. \tag{4.18}
\]

The substraction \(E_t t - E_r r\) gives

\[
F^2 \frac{d^2}{dr^2} H + 4a^2 H^2 = 0. \tag{4.19}
\]

Substituting \(F(r)\) from Eq. (4.18) into Eq. (4.19) one arrives at a first–order equation for \(\frac{d}{dr} H\)

\[
(4 + C_1 l^2)^2 \frac{d^2 H}{dr^2} + 4a^2 l^4 \left( \frac{dH}{dr} \right)^2 = 0, \tag{4.20}
\]

which is rewritten as

\[
d \left( \frac{d}{dr} H \right)^{-1} = -a^2 l^4 d(4 + C_1 l^2)^{-1} \tag{4.21}
\]

with first integral

\[
\left( \frac{d}{dr} H \right)^{-1} = \frac{C_2 (4 + C_1 l^2) - a^2 l^4}{4r + C_1 l^2}. \tag{4.22}
\]

A subsequent integration gives

\[
H(r) = \frac{[R(r)l^2 + a^2 l^4 \ln (R(r)l^2) + C_3]}{4C_2^2},
\]

\[
F(r) = R(r) H(r),
\]

\[
R(r) l^2 := C_2 (4 + C_1 l^2) - a^2 l^4, \tag{4.23}
\]

where Eq. (4.18) it has been used to evaluate \(F(r)\). These structural functions completely determine the magnetostatic solution; without any loss of generality, by letting \(C_2 \rightarrow C_1 l^2 / 4\), \(C_1 \rightarrow 4(a^2 l^2 + C_0)/(C_1 l^2)\), \(C_3 \rightarrow K_0 l^2 - a^2 l^4 \ln l^2\), the magnetostatic metric can be given as

\[
g = -h(r) dt^2 + \frac{dr^2}{H(r) h(r)} + H(r) d\phi^2,
\]

\[
H(r) = \frac{4}{C_1^2 l^2} \left[ K_0 + h(r) + a^2 l^2 \ln h(r) \right],
\]

\[
F(r) = H(r) h(r), h(r) := C_1 r + C_0. \tag{4.24}
\]
This solution is characterized by:
the electromagnetic field vector
\[ \mathbf{A} = \frac{a}{C_1} \ln h \, d\phi, \quad (4.25) \]
the electromagnetic field tensors
\[ F^{\mu\nu} = -2a \left[ \phi^\mu \delta^\nu_r + \phi^\mu_r \delta^\nu_\phi + \phi^\mu_\phi \delta^\nu_r \right], \quad (4.26) \]
with field invariant
\[ F_{\mu\nu} F^{\mu\nu} = 2 \frac{a^2}{h(r)}, \quad (4.27) \]
the energy–momentum tensor
\[ T_{\mu\nu} = \frac{a^2}{8\pi h(r)} \left( -\delta^\mu_t \delta^\nu_t + \delta^\mu_r \delta^\nu_r + \delta^\mu_\phi \delta^\nu_\phi \right), \quad (4.28) \]
with energy field invariants
\[ T_{\mu\nu} T^{\mu\nu} = 3 \frac{a^4}{64\pi^2 h(r)^2}, \quad T_{\mu\mu} = \frac{a^2}{8\pi} \frac{1}{h(r)}. \quad (4.29) \]
This class of solutions allows for a hydrodynamics interpretation in terms of a perfect fluid
energy–momentum tensor for a stiff fluid, \( \rho = p \), where \( \rho \) and \( p \) are, respectively, the fluid
energy density and the fluid pressure. In fact, the energy momentum tensor for a perfect
fluid is given by
\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \]
Therefore choosing the fluid 4-velocity along the time direction \( u^\mu = \delta^\mu_t / \sqrt{-g_{tt}} \) one establishes that \( \rho = \frac{a^2 H}{8\pi F^2} = p. \]

1. Peldan magnetostatic solution with \( \Lambda \)

Introducing in the metric (4.24) new coordinates according to \( h = C_1 r + C_0 \rightarrow \rho^2, t \rightarrow t, \phi \rightarrow \phi C_1/2, K_0 \rightarrow k_0 l^2 \) one gets
\[ g = -\rho^2 dt^2 + \frac{d\rho^2}{F(\rho)} + F(\rho) d\phi^2, \]
\[ F(\rho) = k_0 + \frac{\rho^2}{l^2} + 2a^2 \ln \rho, \]
\[ \mathbf{A} = a \ln \rho \, d\phi, \quad (4.30) \]
characterized by the field tensors
\[ T_{\mu\nu} = \frac{a^2}{8\pi \rho^2} \left( -\delta^\mu_t \delta^\nu_t + \delta^\mu_\rho \delta^\nu_\rho + \delta^\mu_\phi \delta^\nu_\phi \right), \quad (4.31) \]
This solution has been derived and analyzed in [14].
2. **Hirschmann–Welch solution with \( \Lambda \)**

Accomplishing in the general magnetic static metric (4.24) the transformations
\[
C_1 r + C_0 \rightarrow (\rho^2 + r_+^2 - ml^2)/l^2 =: h(\rho), \\
2\phi/(C_1 l^2) \rightarrow \phi, \quad a^2 l^4 = \chi^2, K_0 = m, \tag{4.32}
\]
one ends with the Hirmanch–Welch [15] representation of the magnetic solution
\[
g = -\frac{1}{l^2}(\rho^2 + r_+^2 - ml^2)dt^2 + [\rho^2 + r_+^2 + \chi^2 \ln(|h(\rho)|)]d\phi^2 \\
+ \frac{l^2 \rho^2 d\rho^2}{(\rho^2 + r_+^2 - ml^2)[\rho^2 + r_+^2 + \chi^2 \ln(|h(\rho)|)]},
\]
with vector potential
\[
A = \frac{1}{2}\chi \ln(|\rho^2 + r_+^2)/l^2 - m|d\Phi, \tag{4.34}
\]
For \( \rho = 0 \), one determines the constant \( r_+ \) fulfilling
\[
r_+^2 + \chi^2 \ln |r_+^2/l^2 - m| = 0. \tag{4.35}
\]
This solution is endowed with mass, magnetic charge and radial parameters. The coordinate \( \rho \) ranges from zero to infinity. This magnetic solution does not allow the existence of an event horizon since time–like geodesics can reach the origin at finite proper time, while null geodesics approach the origin at finite affine parameter; hence it does not describe a magnetic black hole. Moreover the Ricci tensor, and consequently the curvature tensor, as well as the electromagnetic field, are well behaved in this spacetime.

Cataldo et al. [16] commented on this static circular magnetic solution of the 2+1 Einstein-Maxwell equations, derived previously by other authors, and came to the conclusion that this solution, considered up to that moment as a two-parameter one, is in fact a one-parameter solution, which describes a distribution of a radial magnetic field in a 2+1 anti-de Sitter background spacetime, and that the mass–parameter is just a pure gauge and can be re–scaled to minus one.

3. **Melvin, and Barrow–Burd–Lancaster magnetostatic solution \( \Lambda = 0 \)**

Melvin [12] derived the electric and the magnetic static solutions for vanishing cosmological constant \( \Lambda = 0 \). The corresponding solution can be obtained from (4.30) and written as
\[
g = -\rho^2 dt^2 + \frac{d\rho^2}{F(\rho)} + F(\rho) d\phi^2, \\
F(\rho) = k_0 + 2a^2 \ln \rho, \tag{4.36}
\]
or by introducing \( a^4 r^2 e^{k_0/a^2} = k_0 + a^2 \ln \rho^2 \), and scaling the variables \( t \) and \( \phi \) one brings it to the form
\[
g = e^{r^2} (-dt^2 + dr^2) + r^2 d\phi^2. \tag{4.37}
\]
In the paragraph devoted to stiff perfect fluid, Barrow, Burd, and Lancaster, see \[17\], pointed out that for a fluid aligned along the time-coordinate, ” in (2+1) dimensions the stiff fluid has an energy-momentum tensor identical to that of a static magnetic field”, and they continued with a statement very close to the following: if one sets the electric field components \( F_{0i} = 0 \) and magnetic components \( F_{ij} = \epsilon_{ij} \sqrt{2} \rho \) in the electromagnetic energy-momentum tensor \( T_{\mu\nu} = F_{\mu\lambda} F_{\nu}^{\lambda} - g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} / 4 \) reduces to the perfect fluid energy-momentum tensor \( T_{\mu\nu} = (\rho + p) u_{\mu} u_{\nu} + p g_{\mu\nu} \), with energy density \( \rho \) equalling the pressure \( p \), \( \rho = p \).

**C. Static hybrid \( A = \xi (td\phi - \phi dt) \) solution**

In this case, the integration starts from the combination \( E_{\xi}^\xi(a = 0 = b) + 2E_r^r(a = 0 = b) + E_\phi^\Phi(a = 0 = b) \) which yields

\[
\frac{d^2}{dr^2} F - \frac{8}{l^2} = 0, \tag{4.38}
\]

with integral

\[
F(r) = 4 \frac{(r - r_1)(r - r_2)}{l^2}. \tag{4.39}
\]

As the equation for \( H(r) \) one may consider the first–order equation \( E_r^r(a = 0, b = 0) \), which can be written as

\[
\left( \frac{H_r}{2H} - \frac{F_r}{4F} \right)^2 = \left( \frac{F_r}{4F} \right)^2 - \frac{c^2}{F^2} - \frac{1}{l^2 F^2}. \tag{4.40}
\]

Evaluating the right–hand side of this equation, one arrives at

\[
\left[ \frac{d}{dr} \ln \left( \frac{H}{F^{1/2}} \right) \right]^2 = 4 \frac{(r_2 - r_1)^2 - c^2 l^4}{l^4 F^2}. \]

For definiteness we assume \( r_2 > r_1 \). Accomplishing the integration one obtains

\[
\ln \left( \frac{H}{\sqrt{F}} \right) = \pm \frac{\sqrt{(r_2 - r_1)^2 - c^2 l^4}}{2 (r_2 - r_1)} \times \ln \left( \frac{r - r_1}{r - r_2} \right).
\]

Introducing the constant \( \alpha \) through

\[
\alpha = 1 - \frac{l^4 c^2}{(r_2 - r_1)^2}, \quad c^2 = \frac{(r_2 - r_1)^2 (1 - \alpha)}{l^4}, \tag{4.41}
\]

one obtains \( H(r) \) in the form

\[
H(r) = K_0^2 \sqrt{F(r)} \left( \frac{r - r_1}{r - r_2} \right)^{\pm \frac{\alpha}{2}}. \tag{4.42}
\]
Summarizing, this class of solutions is given by the metric

\[ g = \frac{-F}{H} dt^2 + \frac{1}{F} dr^2 + H d\phi^2, \]

\[ F = \frac{4}{l^2} (r - r_1)(r - r_2), \]

\[ H = \frac{2 K_0^2}{l} (r - r_1)^{(1+\sqrt{\alpha})/2} (r - r_2)^{(1-\sqrt{\alpha})/2}, \]

\[ A = \frac{c}{2} (td\phi - \phi dt). \quad (4.43) \]

The field tensor characterization of this solution is given by

\[ F^{\mu\nu} = -2 c F \delta_{[\mu} t \delta_{\nu]} \phi, \quad F^{\mu\nu} = 2 c \delta_{[\mu} t \delta_{\nu]} \phi, \]

\[ T^{\mu}_{\nu} = \frac{c^2}{8 \pi F} (-\delta^{\mu}_t \delta^{\nu} + \delta^{\mu}_r \delta^{\nu} - \delta^{\mu}_{\phi} \delta^{\nu}_{\phi}), \]

\[ FF = -2 \frac{c^2}{F}, \quad TT = \frac{3}{64 \pi^2 F^2}, \]

\[ T^{\mu}_{\mu} = -\frac{1}{8 \pi F}. \quad (4.44) \]

By scaling transformations of the Killingian coordinates \( \phi \) and \( t \), the arbitrary constant \( K_0 \) can be equated to 1.

1. Cataldo azimuthal electromagnetic solution

Subjecting the metric (4.43) to the coordinate transformations

\[ t = \frac{1}{\sqrt{2}} K_0 t' \sqrt{1+\alpha} / l', \quad \phi = \frac{1}{\sqrt{2} K_0} t' \sqrt{1-\alpha} / \phi', \quad r = \rho^2 + r_1, \quad M := \frac{1}{l^2} (r_2 - r_1), \quad (4.45) \]

dropping primes, one brings the static hybrid metric to the form

\[ g = -\rho^{1+\sqrt{\alpha}} \left( \frac{\rho^2}{l^2} - M \right)^{(1+\sqrt{\alpha})/2} dt^2 + \rho^{1-\sqrt{\alpha}} \left( \frac{\rho^2}{l^2} - M \right)^{(1-\sqrt{\alpha})/2} d\phi^2 + \left( \frac{\rho^2}{l^2} - M \right)^{-1} d\rho^2. \quad (4.46) \]

The electromagnetic field tensor under the above-mentioned transformations becomes

\[ F^{\mu\nu} = M \sqrt{1 - \alpha} \delta_{[\mu} t \delta_{\nu]} \phi, \]

\[ T^{\mu}_{\nu} = \frac{M^2}{32 \pi \rho^2 (\rho^2 / l^2 - M)} \times (-\delta^{\mu}_t \delta^{\nu}_t + \delta^{\mu}_r \delta^{\nu}_r - \delta^{\mu}_{\phi} \delta^{\nu}_{\phi}). \quad (4.47) \]

This solution corresponds to the static charged solution reported in [18], where the name of azimuthal static solution was coined.
V. UNIFORM ELECTROMAGNETIC SOLUTIONS

To determine all uniform electromagnetic solutions, i.e., those possessing vanishing covariant derivatives of \( F_{\mu\nu} \), \( F_{\mu\nu;\sigma} = 0 \), one has to start the integration process from the differential relations arising from these conditions. The hybrid class \( c \neq 0 \) does not allow for such kind of solutions. The other families with \( a \neq 0 \) and (or) \( b \neq 0 \) give rise to non–trivial solutions.

A. General uniform electromagnetic solution for \( a \neq 0 \neq b \)

A class of uniform electromagnetic stationary solutions, for \( a \neq 0 \), \( b \neq 0 \) and \( c = 0 \), can be constructed by demanding the vanishing of the covariant derivatives of the electromagnetic tensor field, \( F_{\mu\nu;\sigma} = 0 \), which yields two independent equations: \( F_{t\phi;t} = 0 \) and \( F_{tr;r} = 0 \). From the last one, one isolates \( dW/dr \)

\[
\frac{dW}{dr} = -b \frac{F}{H^3(a - b W)} \frac{dH}{dr},
\]

which when used in the first equation \( F_{t\phi;t} = 0 \) allows us to write

\[
\frac{dF}{dr} = \frac{F}{H} \frac{dH}{dr} - b^2 \frac{F^2}{H^3(a - b W)^2} \frac{dH}{dr}.
\]

As the next step, one substitutes recursively these first derivatives into the Einstein equations. One gets, among other relations, a simple expression for the equation \( E_{\phi} \)

\[
\frac{d^2 H}{dr^2} = -4 \frac{H^2}{F^2} (a - b W)^2,
\]

which when substituted back into the Einstein equations reduces them to a single relation

\[
F(b^2 l^2 - H) = l^2 H^2 (a - b W)^2,
\]

from which one has

\[
W(r) = \frac{a}{b} \mp \frac{F}{lbH} \sqrt{b^2 l^2 - H}.
\]

Using the relation (5.4) in Eq. (5.3) one obtains

\[
\frac{d^2 H}{dr^2} = -4 \frac{1}{l^2 F} (b^2 l^2 - H).
\]

On the other hand, substituting \( W(r) \) from Eq. (5.5) into Eq. (5.2) one arrives at the relation

\[
\frac{dF}{dr} (b^2 l^2 - H) + \frac{F}{dr} \frac{dH}{dr} = 0,
\]

with integral

\[
F(r) = \frac{b^2 l^2 - H(r)}{l^2 \beta^2}, \quad \beta = \text{constant}.
\]
The substitution of $F(r)$ from (5.8) into Eq. (5.6) yields

$$\frac{d^2 H}{dr^2} = -4\beta^2,$$  \hfill (5.9)

hence

$$H(r) = -2\beta^2 r^2 + c_1 r + c_0.$$  \hfill (5.10)

Consequently the function $W(r)$ becomes

$$W(r) = \frac{a}{b} \pm \frac{1}{l^2 b \beta} \frac{b^2 l^2 - H}{H}.$$  \hfill (5.11)

No restriction arises from the remaining Eq. (5.1).

Thus, we have determined the general uniform electromagnetic stationary cyclic symmetric solution given by the metric and the field vector

$$g = -\frac{b^2 l^2 - H(r)}{l^2 \beta^2 H(r)} dt^2 + \frac{l^2 \beta^2 dr^2}{b^2 l^2 - H(r)} + H(r) \left[ d\phi + \left( \frac{a}{b} \mp \frac{1}{l^2 b \beta} \frac{b^2 l^2 - H}{H} \right) dt \right]^2,$$

$$H(r) = -2\beta^2 r^2 + C_1 r + C_0,$$

$$A = -\beta r \left[ d\phi - \frac{1}{l^2 b \beta} dt \right],$$  \hfill (5.12)

classified by the uniform electromagnetic field tensors

$$F_{\mu\nu} = -2 \frac{1}{l^2 b} \pm \frac{a l^2 \beta}{l^2 b} \delta_{[\mu} \delta_{\nu]} r \pm 2\beta \delta_{[\mu}^r \delta_{\nu]} \phi,$$

$$8\pi l^2 T_{\nu}^\mu = -(1 \pm 2a l^2 \beta) \left( \delta_{[\nu}^r \delta_{\mu]}^\mu - \delta_{[\nu}^\phi \delta_{\mu]}^\phi \right) - \delta_{[\nu}^r \delta_{\mu]}^\mu \mp 2\beta b l^2 \delta_{[\nu}^\phi \delta_{\mu]}^\phi + \frac{2a}{b} (1 \pm a l^2 \beta) \delta_{[\nu}^r \delta_{\mu]}^\phi$$  \hfill (5.13)

with $FF$ invariant $FF = -2/l^2$.

Although the solution above has been derived for $\Lambda = -1/l^2$, the branch corresponding to $\Lambda = 1/l^2$ is achieved from the above expressions by changing $l^2 \rightarrow -l^2$.

**B. Uniform “stationary” electromagnetic $A = r/(bl^2)(dt - \omega_0 d\phi)$ solutions**

Consider now the case $a = 0 = c$ for the metric (3.5)

$$g = -\frac{F}{h} (dt - \omega d\phi)^2 + h d\phi^2 + \frac{dr^2}{F} = -\frac{F}{H} dt^2 + \frac{dr^2}{F} + H (d\phi + W dt)^2,$$

$$F = F, H = h - \frac{F}{h} \omega^2, W = \frac{\omega F}{H h}.$$  \hfill (5.14)

The electromagnetic tensor amounts to $F^{\mu\nu} = 2 b \delta^{[\mu}_{[\nu} \delta^{r]}_{r]}$, and

$$F_{\mu\nu} = -2 \frac{b}{HF} (F - H^2 W^2) \delta^{[\mu}_{[\nu} \delta^{r]}_{r]} + 2 b \frac{HW}{F} \delta^{[\mu}_{[\nu} \delta^\phi_{r]} \delta^{r]}_{r]} = -2 \frac{b}{h(r)} \delta^{[\mu}_{[\nu} \delta^{r]}_{r]} + 2 b \frac{\omega(r)}{h(r)} \delta^{[\mu}_{[\nu} \delta^\phi_{r]} \delta^{r]}_{r]}.$$
The covariant derivatives $F_{\phi r;r}$ and $F_{tr;r}$ of the field $F_{\mu \nu}$ are equal to zero if
\[
\omega(r) = \omega_0, \ h(r) = h_0.
\]

Therefore, the structural functions $\omega(r) = \omega_0$ and $h(r) = h_0$ are constants. The Einstein–Maxwell equations require the fulfillment of the equations
\[
\frac{d^2}{dr^2} F(r) = \frac{4}{l^2} \rightarrow F(r) = \frac{2}{l^2} r^2 + c_1 r + c_0, \ h_0 = b^2 l^2. \tag{5.15}
\]

Consequently the derived solution can be given as
\[
g = -\frac{F}{h_0}(dt - \omega_0 d\phi)^2 + h_0 \, d\phi^2 + \frac{dr^2}{F(r)}, \]
\[
F(r) = \frac{2r^2}{l^2} + c_1 r + c_0, \ h_0 = b^2 l^2, \]
\[
A = \frac{r}{bl^2} (dt - \omega_0 d\phi), \tag{5.16}
\]
and hence by a shifting transformation of the $t$–coordinate the derived metric becomes a static one. The electromagnetic tensors characterizing this uniform “stationary” cyclic symmetric solution are given by
\[
F_{\mu \nu} = -\frac{1}{b l^2} \delta_{[\mu} \delta_{\nu]} r + 2 \frac{\omega_0}{b l^2} \delta_{[\mu}^\phi \delta_{\nu]}^r,
\]
\[
T_{\mu}^\nu = \frac{1}{8 \pi l^2} (-\delta_{[\mu}^{\, t} \delta_{\nu]}^r - \delta_{[\mu}^{\, r} \delta_{\nu]}^r + \delta_{[\mu}^{\, \phi} \delta_{\nu]}^r) + \frac{\omega_0}{4 \pi l^2} \delta_{[\mu}^\phi \delta_{\nu]}^r. \tag{5.17}
\]

This solution can be generated from the static solution, which is given in subsection V C by the metric (5.20), via the transformation $t \rightarrow t - \omega_0 \phi, \ \phi \rightarrow \phi$.

1. Clement uniform “stationary” electromagnetic solution

Clement [19] reported the uniform “stationary” generalization of the electrostatic solution in the form of
\[
g = -\frac{F}{H_0} (dt - \omega_0 d\phi)^2 + \frac{dr^2}{F(r)} + H_0 d\phi^2, \]
\[
F(r) = \frac{2r^2}{l^2} + c_1 r + c_0, \]
\[
A = \frac{1}{\sqrt{H_0} l} r \, (dt - \omega_0 d\phi), \tag{5.18}
\]
which is equivalent to the metric (Cl.25, $\Lambda = -1/l^2$), where we have changed the signature. In Clement’s parametrization $H_0 = \frac{\pi^2 l^2}{4m}$, with $m = 1/(2 \kappa)$.  

On the other hand for \( b \neq 0 \) and positive cosmological constant \( \Lambda = 1/l^2 \) there is no a uniform electromagnetic stationary cyclic symmetric solution; the reason is hidden in the resulting erroneous signature. In the considered case, for the metric (3.5) with structural functions \( F(r), h(r), \omega(r) \) the electromagnetic tensor amounts, for \( a = 0 \), to

\[
F_{\mu\nu} = 2b \delta^\mu_r [\delta^\nu_r], \\
F_{\mu\nu} = -2b/h(r) \delta_\mu^t \delta_\nu^r + 2b \omega(r)/h(r) \delta_\mu^\phi \delta_\nu^r.
\]

Therefore \( F_{\mu\nu};\lambda = 0 \) is achieved for \( h(r) = h_0, \omega(r) = \omega_0 \). The Einstein equations requires

\[
E_{rr} = b^2/h_0 + 1/l^2 = 0 \rightarrow h_0 = -b^2 l^2.
\]

The covariant tensor components \( g_{tt} \) and \( g_{rr} \) explicitly amount to

\[
g_{tt} = \frac{F(r)}{b^2 l^2} > 0, \\
g_{rr} = \frac{1}{F(r)} > 0
\]

which contradicts the adopted signature \( \{-, +, +\} \), therefore this case does not represent a compatible solution.

C. Matyjasek–Zaslavskii uniform electrostatic solution

The sub–branch \( \{a = 0, b \neq 0, W(r) = 0\} \) of uniform electrostatic solutions arises for constant \( H(r), H(r) = H_0 = \text{constant} \). The equation \( E_t^t \) implies that the constant \( H_0 \) has to be

\[
H_0 = \frac{b^2 l^2}{1}. 
\]

The remaining equation \( E_t^\phi \) amounts to

\[
\frac{d^2 F}{d r^2} - \frac{4}{l^2} = 0, \rightarrow F(r) = \frac{2r^2}{l^2} + 4c_1 r + c_0; 
\]

consequently the metric and the field vector become

\[
g = -\frac{F(r)}{b^2 l^2} dt^2 + \frac{dr^2}{F(r)} + b^2 l^2 d\phi^2, \quad A = \frac{r}{b l^2} dt. 
\]

The electromagnetic field tensors of this solution possess constant eigenvalues and also exhibit the uniform character; explicitly they are given by

\[
F_{\mu\nu} = \begin{bmatrix} 0 & -\frac{1}{b l^2} & 0 \\ \frac{1}{b l^2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \dot{T}_{\mu}^\nu = \begin{bmatrix} -\frac{1}{8\pi l^2} & 0 & 0 \\ 0 & -\frac{1}{8\pi l^2} & 0 \\ 0 & 0 & \frac{1}{8\pi l^2} \end{bmatrix},
\]

with constant field invariants given by

\[
FF = \frac{2}{l^2}, \quad TT = \frac{3}{64 \pi^2 l^4}.
\]

Incorporating the constant \( H_0 = b^2 l^2 \) in the new definitions of \( t \) and \( \phi, t/b l \rightarrow t, b l \phi \rightarrow \phi \), one can set \( b l = 1 \) in the metric (5.20).

For the sake of comparison with the previous reports let us introduce hyperbolic functions:

\[
F(r) = 2\frac{r^2}{l^2} + 4c_1 r + c_0 = |c_0 - 2l^2 c_1|^2 \left( \frac{2(r + l^2 c_1)^2}{|l^2|c_0 - 2l^2 c_1|^2} + 1 \right),
\]
\[ +: r + l^2 c_1 = \sqrt{\left| c_0 - 2l^2 c_1^2 \right|} / 2 \sinh(\alpha x), \quad F(r) \to |c_0 - 2l^2 c_1^2| \cosh^2(\alpha x), \]
\[ -: r + l^2 c_1 = \sqrt{\left| c_0 - 2l^2 c_1^2 \right|} / 2 \cosh(\alpha x), \quad F(r) \to |c_0 - 2l^2 c_1^2| \sinh^2(\alpha x), \]
\[ c_0 = 2l^2 c_1^2 : r + l^2 c_1 = \exp \sqrt{2x/l}, \quad F(r) \to \frac{2}{l^2} \exp (2\sqrt{2x/l}). \quad (5.22) \]

The above-mentioned quantities, i.e. metric (5.20) and structural functions (5.22), determine the solution derived in [20]. Expressions (5.20) and (5.22) are equivalent to the Bertotti [21] and Robinson [22] uniform electromagnetic–gravitational field solution, for a constant slice of one of the spatial coordinates; the BR 3 + 1 solution allows for a product of two surfaces of constant curvature as manifold. Moreover, the two–dimensional BR–metric sector \( ds^2 \) reduces to \( r^2 d\phi^2 \) in the 2+1 case, therefore the (2+1)–dimensional uniform electrostatic field can be considered as a dimensional reduction of the 3 + 1 Bertotti–Robinson solution.

With the aim of demonstrating the uniqueness of this class of uniform solutions with \( H(r) = h_0 = \text{constant} \), even in the framework of stationary fields, let us consider the general metric with \( W(r) \):

In the case \( \{a = 0 = c, b \neq 0, H(r) = h_0\} \), the combination of equations \( E_t^t - E_r^r - W(r)E_t^\phi = 2 \frac{W(r)^2 h_0 b^2}{F(r)} \) implies \( W(r) = 0 \) and consequently the gravitational field is static; further integration gives rise to the above–mentioned uniform electrostatic fields.

Case \( \{a \neq 0, b = 0 = c, H(r) = h_0\} \): from \( E_t^t - E_r^r - W(r)E_t^\phi = 2a^2 h_0 / F(r) \) therefore there is no solution here.

Case \( \{a = 0 = b, c \neq 0, H(r) = h_0\} \): the combination \( E_t^t - E_r^r - W(r)E_\phi^t = 2c^2 / F(r) \), hence there is no solution.

**D. Uniform “stationary” electromagnetic A = r/(a l^2)(d\phi + W_0 dt) solutions**

In the case of positive cosmological constant \( \Lambda = 1/l^2 \) there exists a uniform “stationary” magnetic solution with constant \( W(r) = W_0 \).

The electromagnetic tensor possesses the structure

\[ F_{\mu\nu} = -2a \frac{H}{F} W \delta_\mu^{[t} \delta_\nu^{r]} - 2a \frac{H}{F} \delta_\mu^{[\phi} \delta_\nu^{r]}, \]

and its covariant derivatives occurs to be zero if

\[ F_{rt} = -a \frac{H^2 dW}{2F} dr, \]
\[ F_{r\phi} = -\frac{a}{4FH} (-2H^2 \frac{d}{dr} \frac{F}{H} + H^3 \frac{dW}{dr}) \]

vanish. Hence

\[ F(r) = \beta H(r), \quad W(r) = W_0 = \text{constant}, \quad (5.23) \]

and consequently

\[ F_{tr} = -W_0 \frac{a}{\beta}, \quad F_{\phi r} = -\frac{a}{\beta}. \quad (5.24) \]
The Einstein equations reduce to

\[ E_{r}^{r} = -\frac{a^2}{\beta} + \frac{1}{l^2} = 0 \Rightarrow \beta = a^2 l^2, \quad 2 E_{t}^{t} = \frac{d^2}{dr^2} F(r) + \frac{4}{l^2} = 0 \]

\[ \Rightarrow F(r) = -\frac{2}{l^2} r^2 + c_1 r + c_0. \quad (5.25) \]

The metric and fields for the derived solution can be expressed as

\[ g = -a^2 l^2 dt^2 + \frac{dr^2}{F(r)} + \frac{F(r)}{a^2 l^2} (d\phi + W_0 dt)^2, \]

\[ F(r) = -\frac{2r^2}{l^2} + c_1 r + c_0, \]

\[ A = \frac{r}{a l^2} (d\phi + W_0 dt), \quad (5.26) \]

with uniform electromagnetic field tensor is

\[ F_{\mu\nu} = -2 \frac{W_0}{a l^2} \delta^r_{[\mu} \delta^r_{\nu]} - 2 \frac{1}{a l^2} \delta^\phi_{[\mu} \delta^\phi_{\nu]}, \]

and energy–momentum tensor

\[ 8\pi T_{\nu}^{\mu} = \frac{1}{l^2} \left[ -\delta^t_{\nu} \delta^t_{\mu} + \delta^r_{\nu} \delta^r_{\mu} + \delta^\phi_{\nu} \delta^\phi_{\mu} + 2 W_0 \delta^t_{\nu} \delta^t_{\mu} \right]. \]

This solution is equivalent to the Clement’s solution given by Eq. (Cl.26, \( \Lambda = 1/l^2 \)) of [19].

1. No uniform stationary generalization of the magnetostatic solution for \( \Lambda = -1/l^2 \)

On the contrary, as far as to the stationary uniform electromagnetic branch with \( a \neq 0 \) and negative cosmological constant \( \Lambda = -1/l^2 \) is concerned, one establishes that there is no solution at all. Following a similar procedure as the one used in the previous case, where now \( b = 0 \), \( F(r) = \beta H(r) \), \( W(r) = W_0 = \text{constant} \). The Einstein equation \( E_{r}^{r} = -\frac{a^2}{\beta} - \frac{1}{l^2} = 0 \) yields \( \beta = -a^2 l^2 \). The covariant tensor components \( g_{rr} \) and \( g_{\phi\phi} \) explicitly amount to \( g_{rr} = 1/F(r) > 0 \), \( g_{\phi\phi} = -a^2 l^2 F(r) < 0 \), which yields a contradiction with the adopted signature \( \{-,+,+\} \). Hence this case does not represent a solution compatible with the 2+1 metric signature.

VI. CONSTANT ELECTROMAGNETIC INVARIANTS’ SOLUTIONS

This section is devoted to the derivation of the electromagnetic fields coupled to stationary (static) cyclic symmetric 2+1 gravitational fields such that their electromagnetic invariants \( FF, T \) and \( TT \) are constants; because of the proportionality of \( T \) and \( TT \) to \( FF \), it is enough to establish under which conditions

\[ F_{\mu\nu} F^{\mu\nu} = -2 \left( \frac{r^2}{F} \right) + 2 \frac{H}{F} \frac{(a - Wb)^2}{F} - 2 \frac{b^2}{H} \]
vanishes. It should be pointed out that the hybrid $c \neq 0$ class of spacetimes does not allow for constant invariants’ solution when a cosmological constant is present. Therefore it is sufficient to restrict oneself to the case $a \neq 0$ or $b \neq 0$.

In what follows we shall search for sub–classes of solutions with constant electromagnetic invariant, namely those with $\{a \neq 0, b \neq 0, W(r) = W(r)\}$, $\{a = 0, b \neq 0, \omega(r) = \omega_0\}$, and $\{a \neq 0, b = 0, W(r) = W_0\}$ families of solutions.

At this stage it is worthwhile to point out that constant invariant electromagnetic fields contain, as sub–classes, the covariantly constant electromagnetic field solutions, while the inverse statement does not hold.

### A. General constant electromagnetic invariant $F_{\mu\nu}F^{\mu\nu} = 2\gamma$ case for $a \neq 0 \neq b$

Restricting the present section to the study of the cases $a \neq 0$ and $b \neq 0$, the constancy of $F_{\mu\nu}F^{\mu\nu} = 2\gamma$ is guaranteed by

$$\frac{F}{H^2}(b^2 + \gamma H) = (a - bW)^2; \quad (6.1)$$

which yields

$$W(r) = \frac{a}{b} \pm \frac{\sqrt{F(r)}}{bH(r)} \sqrt{b^2 + \gamma H(r)}. \quad (6.2)$$

In general, the following combinations of the Einstein equations give

$$E_t^t + E_\phi^\phi + 2E_r^r = \frac{d^2F}{dr^2} - \frac{8}{l^2} + \frac{b^2F - H^2(a - bW)^2}{FH} = 0,$$

$$2HF(E_t^t - E_r^r - W E_\phi^t) = F^2 \frac{d^2H}{dr^2} + 4H^2(a - bW)^2 = 0. \quad (6.3)$$

Using the relation (6.1) one brings Eq. (6.3) to the form

$$\frac{d^2F}{dr^2} = \frac{8}{l^2} + 4\gamma, \quad \frac{d^2H}{dr^2} + \frac{4\gamma}{F}H = -\frac{4b^2}{F}, \quad (6.4)$$

with solution for $F(r)$

$$F(r) = \left(\frac{4}{l^2} + 2\gamma\right)r^2 + C_1r + C_0 = \left(\frac{4}{l^2} + 2\gamma\right)(r - r_1)(r - r_2). \quad (6.5)$$

and the function $H$, as solution of its second–order equation (6.4), is expressed in terms of hypergeometric functions. Nevertheless there exists a shortcut for the integration of the function $H(r)$ by noticing that $E_r^r$ contains only first derivatives of the structural functions: replacing $W(r)$ from (6.2) in the quoted equation, after extracting square root, one arrives at

$$\gamma F \frac{dH}{dr} - (b^2 + \gamma H) \frac{dF}{dr} = \pm \frac{b}{l} \sqrt{1 + l^2\gamma} \sqrt{F} \sqrt{b^2 + \gamma H},$$
which, when introducing the auxiliary function \( Q(r)^2 := b^2 + \gamma H(r) \), becomes

\[
\frac{d}{dr} \frac{Q}{\sqrt{F}} = 2 \frac{b}{l F} \sqrt{1 + l^2 \gamma} = 0.
\]

Integrating this equation, using \( F(r) \) (6.5), one obtains

\[
Q = \pm bl \sqrt{1 + l^2 \gamma} \ln \frac{r - r_2}{r_2 - r_1} + \beta \sqrt{F(r)}.
\] (6.6)

Fulfilling a single constraint still remains; using the expressions of the function \( W(r) \) and its derivatives from the \( E_{\mu\nu} \)–equations as well as the derivatives of \( F(r) \), together with \( H(r) \) in terms of \( Q(r) \) and the derivative of the latter from (6.6), one gets a single equation

\[
2(1 + l^2 \gamma) (Q(r)^2 + b^2) \sqrt{F} + lb Q(r) \sqrt{1 + l^2 \gamma} \frac{dF}{dr} = 0,
\] (6.7)

which is incompatible with \( Q(r) \) determined in (6.6) except for \( \gamma = -1/l^2 \).

Hence, we conclude that there are no solutions for arbitrary constant electromagnetic invariant \( FF = 2 \gamma \).

1. Constant electromagnetic invariant \( FF = \mp 2/l^2 \) solution

A class of constant electromagnetic invariants’ stationary solutions with \( a \neq 0, b \neq 0 \) and \( \gamma = -1/l^2, FF = -2/l^2 \), arises for

\[
W(r) = \frac{a}{b} \mp \frac{F}{bl H(r)} \sqrt{b^2 l^2 - H}.
\] (6.8)

As in the previous case, \( F(r) \) and \( H(r) \) fulfill the Eq. (6.4) for \( \gamma = -1/l^2 \) and correspondingly their integrals are given by

\[
F(r) = \frac{2}{l^2} r^2 + c_1 r + c_0, \quad H(r) = b^2 l^2 - \beta^2 l^2 F(r).
\]

There are no further constraints from the field equations. Consequently the final result can be written as

\[
g = -\frac{b^2 l^2 - H(r)}{l^2 \beta^2 H(r)} dt^2 + l^2 \beta^2 \frac{dr^2}{b^2 l^2 - H(r)} + H(r) \left[ d\phi + \left( \frac{a}{b} \mp \frac{1}{l^2 b \beta} \frac{b^2 l^2 - H}{H} \right) dt \right]^2,
\]

\[
H(r) = -2\beta^2 r^2 + C_1 r + C_0,
\]

\[
A = -\beta r \left[ d\phi - \frac{1 \pm a l^2 \beta}{l^2 b \beta} dt \right],
\]

which coincides with the uniform solution (5.12). Therefore we have determined a class of uniform constant electromagnetic invariant stationary solutions for both non–vanishing constants \( a \neq 0 \neq b \). Although the solution mentioned above has been derived for \( \Lambda = -1/l^2 \), the branch with positive \( \Lambda = 1/l^2 \) is achieved from the above–mentioned expressions by changing \( l^2 \rightarrow -l^2 \).
2. Vanishing invariant $F_{\mu\nu}F^{\mu\nu} = 0$ solution

The next in simplicity class of solutions corresponds to the family with vanishing invariant $FF = 0, \gamma = 0$. In such a case

$$W(r) = \frac{a}{b} \pm \sqrt{\frac{F(r)}{H(r)}}. \quad (6.9)$$

The first–order equations (6.6) and (6.7) give rise to a single first order equation for $F(r)$, namely

$$\frac{dF}{dr} = \pm \frac{4}{l} \sqrt{F} \rightarrow F(r) = \frac{4}{l^2} (r - C)^2. \quad (6.10)$$

The integration of the equation (6.4) for $H(r)$

$$\frac{d^2H}{dr^2} = -\frac{4b^2}{F}$$

yields

$$H(r) = C_0 + C_1 r + b^2 l^2 \ln (r - C). \quad (6.11)$$

Therefore, the vector field potential occurs to be

$$A = \pm \frac{bl}{2} \ln (r - C) \left( d\phi + \frac{a}{b} dt \right). \quad (6.12)$$

This solution corresponds to a possible representation of the Kamata–Koikawa [23] solution to be treated in detail in Section VII.

It should be pointed out that this solution does not belong to the family of uniform solutions, i.e., the fields possessing vanishing covariant derivatives.

B. Constant electromagnetic invariant $FF = -2/l^2$ solution for $b \neq 0$

The constant electromagnetic invariant solution with $b \neq 0$ can be determined as solution of the Einstein–Maxwell equations by considering the stationary metric in the form

$$g = \frac{F}{h} (dt - \omega d\phi)^2 + h d\phi^2 + dr^2 = -\frac{F}{H} dt^2 + \frac{dr^2}{F} + H (d\phi + W dt)^2,$$

$$F = F, H = h - \frac{F}{h} \omega^2, W = \frac{\omega F}{H h}.$$

Demanding the electromagnetic invariant $FF$ in the case $b \neq 0 = a$ to be constant, one establishes

$$FF = -\frac{2b^2}{h(r)} \rightarrow h(r) = h_0 = \text{constant}. \quad (6.13)$$
Substituting $h(r) = h_0$ into the Einstein equations one obtains from $E^t_\phi$ that

$$\frac{d^2}{dr^2}\omega = -\frac{2}{F}\frac{dF}{dr}\frac{d\omega}{dr}, \quad (6.14)$$

which when used in $E^t_t - E^r_r$ yields

$$\frac{F}{h_0^2}(d\omega)^2 = 0 \rightarrow \omega(r) = \omega_0. \quad (6.15)$$

Replacing $\omega = \omega_0$ and $h = h_0$ in the remaining equations one establishes

$$\frac{d^2}{dr^2}F(r) = \frac{4}{l^2} \rightarrow F(r) = \frac{2}{l^2}r^2 + c_1 r + c_0, \quad h_0 = b^2 l^2 \quad (6.16)$$

Therefore we arrive at a constant electromagnetic invariant solution in the form

$$g = -\frac{F}{h_0}(dt - \omega_0 d\phi)^2 + h_0 d\phi^2 + \frac{dr^2}{F(r)},$$

$$F(r) = \frac{2r^2}{l^2} + c_1 r + c_0, \quad h_0 = b^2 l^2,$$

$$A = \frac{r}{b l^2} (dt - \omega_0 d\phi), \quad (6.17)$$

which in all respects is identical to the uniform electromagnetic solution (5.16) derived in the previous section. Notice that this solution exists only for negative cosmological constant, $\Lambda = -1/l^2$, there is no extension to $\Lambda = 1/l^2$. It is evident that this solution can be generated from the static one, (5.20), via the transformations $t \rightarrow t - \omega_0 \phi, \quad \phi \rightarrow \phi$.

For $\omega_0 = 0$, the above-mentioned metric and field reduce to the Matyjasek–Zaslavskii solutions, see Section (V C), thus this class of constant electromagnetic invariants’ static solutions occurs to be unique with the additional property of being a uniform static solution.

**C. Constant electromagnetic invariant $FF = 2/l^2$ stationary solution for $a \neq 0$**

In the case of positive cosmological constant $\Lambda = 1/l^2$ there exists a constant electromagnetic invariant stationary $a \neq 0$ solution. Requiring the constancy of the electromagnetic invariant $FF = 2a^2H$, one gets

$$FF = 2a^2H \rightarrow H(r) = \beta^2 F(r). \quad (6.18)$$

From $E^t_\phi$ one establishes

$$\frac{d^2}{dr^2}W = -\frac{2}{F}\frac{dF}{dr}\frac{dW}{dr}, \quad (6.19)$$

which when used in $E^t_t - E^r_r$ yields

$$\frac{F}{\beta^4}(\frac{dW}{dr})^2 = 0 \rightarrow W(r) = W_0 \quad (6.20)$$
Using $W = W_0$ and $H(r) = \beta^2 F(r)$ in the remaining Einstein equations, one gets
\[
\frac{d^2}{dr^2} F(r) + \frac{4}{l^2} = 0 \rightarrow F(r) = -\frac{2}{l^2} r^2 + c_1 r + c_0. \tag{6.21}
\]
Therefore we have established that there is a unique constant electromagnetic invariants’ solution given by
\[
g = -a^2 l^2 dt^2 + \frac{dr^2}{F(r)} + \frac{F(r)}{a^2 l^2} (d\phi + W_0 dt)^2,
\]
\[
F(r) = -\frac{2r^2}{l^2} + c_1 r + c_0,
\]
\[
A = \frac{r}{a l^2} (d\phi + W_0 dt), \tag{6.22}
\]
which is identical to the uniform electromagnetic solution (5.26) derived in the previous Section V dealing with uniform electromagnetic solutions. Notice that this solution exists only for positive cosmological constant, $\Lambda = 1/l^2$, there is no $\Lambda = -1/l^2$ solution within this class.

**VII. (ANTI–) SELF–DUAL MAXWELL FIELDS; $FF = 0$**

A particular family of stationary cyclic symmetric solutions arises by demanding the vanishing of the electromagnetic invariant $F_{\mu\nu}F^{\mu\nu}$,
\[
F_{\mu\nu}F^{\mu\nu} = 2 \frac{H(a - b W)^2}{F} - 2 \frac{b^2 H}{F} = 0 \rightarrow W(r) = \frac{a}{b} \pm \sqrt{\frac{F}{H}}. \tag{7.1}
\]
For the above–mentioned $W(r)$, the equation $E_r^r$ gives
\[
l^2 \left( \frac{dF}{dr} \right)^2 - 16 F = 0 \rightarrow F(r) = 4 \frac{(r - C)^2}{l^2}. \tag{7.2}
\]
After the substitution of $W(r)$ and $F(r)$ into the Einstein equations, the remaining equation to be solved amounts to
\[
(r - C)^2 \frac{d^2}{dr^2} H + b^2 l^2 = 0 \rightarrow H(r) = C_0 + C_1 r + b^2 l^2 \ln (r - C).
\]
The gravitational and electromagnetic fields of this solution can be given as
\[
g = -\frac{F}{H} dt^2 + \frac{dr^2}{F} + H(d\phi + W dt)^2,
\]
\[
H(r) = C_0 + C_1 r + b^2 l^2 \ln (r - C), \quad F(r) = 4 \frac{(r - C)^2}{l^2}, \quad W(r) = \frac{a}{b} \pm \sqrt{\frac{F}{H}},
\]
\[
A = \pm \frac{l}{2} \ln (r - C) (a dt + b d\phi). \tag{7.3}
\]
This solution is characterized by the field tensor
\[
F_{\mu\nu} = \pm \frac{l}{(r - C)} (a\delta_{[\mu}^r \delta_{\nu]} - b\delta_{[\mu}^\phi \delta_{\nu]}),
\]
\[
F^{\mu\nu} = 2b\delta_{[\mu}^\phi \delta_{\nu]} - 2a\delta_{[\mu}^r \delta_{\nu]}), \tag{7.4}
\]
with energy–momentum tensor

\[ T_{\mu}{}^{\nu} = \frac{l}{4 \pi (r-C)} \left( -a b \delta_{\mu}{}^{t} \delta_{t}{}^{\nu} + a b \delta_{\mu}{}^{\phi} \delta_{\phi}{}^{\nu} + a^{2} \delta_{\mu}{}^{t} \delta_{\phi}{}^{\nu} - b^{2} \delta_{\mu}{}^{\phi} \delta_{t}{}^{\nu} \right). \]  

(7.5)

Notice that the three invariants \( F_{\mu\nu} F^{\mu\nu} \), \( T_{\mu}{}^{\mu} \) and \( T_{\mu\nu} T^{\mu\nu} \) are equal to zero. Without any loss of generality one can always set \( C = 0 \).

### A. Kamata–Koikawa solution

Kamata and Koikawa \[23\] reported their electrically charged BTZ black hole with negative cosmological constant such that the Maxwell field is self (anti-self) dual, condition which is imposed on the orthonormal basis components of the electric field and the magnetic field. This solution describes an electrically charged extreme black hole with mass \( M \), angular momentum \( J \), and electric charge \( Q \). To achieve their representation one accomplishes in metric (7.3) the substitutions

\[
\begin{align*}
    r &= \rho^2, \quad t \to t \sqrt{Q/2}, \quad \phi \to \phi/\sqrt{Q}, \quad C_1 \to Q, \quad l \to |\Lambda|^{-1/2} C_0 \to -b/\Lambda \ln \rho_0^2, \quad C = \rho_0^2, \\
    H/Q &\to K^2, \quad W Q/2 \to \frac{a Q}{b^2} + N^\phi, \quad F \to 4 \rho^2 L^2,
\end{align*}
\]

(7.6)

arriving at the solution

\[
\begin{align*}
    g &= -\rho^2 L^2 \frac{dt}{K^2} + \frac{d\rho^2}{L^2} + K^2 \left[ d\phi + \left( \frac{a Q}{b^2} + N^\phi \right) dt \right]^2, \\
    L^2 &= |\Lambda|(\rho - \rho_0^2/\rho)^2, \quad K^2 = \rho^2 + \frac{b^2}{Q \Lambda} \ln \left( \frac{\rho^2 - \rho_0^2}{\rho_0^2} \right), \quad N^\phi = \pm \frac{\rho L}{K^2}, \\
    A &= \frac{b}{2 \sqrt{Q |\Lambda|}} \ln \left( \frac{\rho^2 - \rho_0^2}{\rho_0^2} \right) \left( d\phi + \frac{a Q}{b^2} dt \right).
\end{align*}
\]

(7.7)

The electromagnetic field tensors occur to be

\[
\begin{align*}
    F_{\mu\nu} &= \frac{\rho}{\sqrt{|\Lambda|}(\rho^2 - \rho_0^2)} \left( a \sqrt{Q} \delta_{[\mu}^t \delta_{\nu]}^r + 2 \frac{b}{\sqrt{Q}} \delta_{[\mu}^\phi \delta_{\nu]}^r \right), \\
    T_{\mu}{}^{\nu} &= \frac{1}{8 \pi} \frac{1}{\sqrt{|\Lambda|}(\rho^2 - \rho_0^2)} \left[ a b (\delta_{[\mu}^t \delta_{t]}^\nu - \delta_{[\mu}^\phi \delta_{\phi]}^\nu) - \frac{Q a^2}{2} \delta_{[\mu}^\phi \delta_{t]}^\nu - \frac{2 b^2}{Q} \delta_{[\mu}^t \delta_{\phi]}^\nu \right].
\end{align*}
\]

(7.8)

Next, one restores the factor \( \pi G \) in the above–mentioned solution through the identifications of the physical parameters:

\[
\rho_0^2 = 4 \pi G Q^2 / |\Lambda| = \frac{\epsilon}{2 |\Lambda|^{1/2}} J, \quad b = 2 \sqrt{\pi} G Q^{3/2}, \quad a = \pm 4 \sqrt{\pi} G |\Lambda|^{1/2} Q^{1/2},
\]

arriving at the metric (7.7) with structural functions

\[
L^2 = |\Lambda|(\rho - \rho_0^2/\rho)^2, \quad N^\phi = \pm \frac{\rho L}{K^2}, \quad K^2 = \rho^2 + \rho_0^2 \ln \left( \frac{\rho^2 - \rho_0^2}{\rho_0^2} \right)
\]

(7.9)
and electromagnetic field tensors

\[ A = Q \sqrt{\pi |\Lambda|} \ln \left( \frac{\rho^2 - \rho_0^2}{\rho_0^2} \right) \times \left( \frac{1}{\sqrt{|\Lambda|}} d\phi + dt \right), \]

\[ F_{\mu\nu} = -4Q \sqrt{\pi} \frac{G}{\rho^2 - \rho_0^2} \left( \delta_{[\mu}^{\nu} \delta_{\nu]}^r + \frac{1}{\sqrt{|\Lambda|}} \delta_{[\mu}^{\phi} \delta_{\nu]}^r \right), \]

\[ T_{\mu}^{\nu} = \frac{Q^2 G}{\rho^2 - \rho_0^2} \left( -\delta_{\mu}^{\nu} \delta_t^r + \delta_{\mu}^{\phi} \delta_t^r - \frac{1}{\sqrt{|\Lambda|}} \delta_{\mu}^{\phi} \delta_t^r + \sqrt{|\Lambda|} \delta_{\mu}^{\phi} \delta_t^r \right). \] (7.10)

It should be pointed out that Clement [19] also reported a metric expression and electromagnetic vector field describing a solution with vanishing electromagnetic invariants. Comments concerning the mass content of this solution can be found in [24]. This solution is horizonless and consequently does not permit a black hole interpretation.

VIII. GENERAL STATIONARY CYCLIC SYMMETRIC SOLUTION FOR ELECTROMAGNETIC FIELD

The main goal of this section is to derive the stationary cyclic symmetric spacetime corresponding to the case \( c \neq 0 \), i.e., for the vector potential

\[ A = \frac{c}{2} (td\phi - \phi dt). \] (8.1)

It is worthwhile to point out that this case has no analog in stationary axial symmetric spacetimes of the standard 3+1 Einstein–Maxwell theory. The set of field equations is given by: \( \{ E_t^t, E_t^\phi, E_r^r, E_r^\phi, E_\phi^t, E_\phi^\phi \} \). In the forthcoming subsections two main families of solutions exhibiting the hybrid feature of the vector potential are derived.

A. Ayon–Cataldo–Garcia hybrid electromagnetic stationary solution

The starting point in the integration process of the system of field equations is \( E_\phi^t(a = 0 = b) = 0, \) (3.15g), which possesses a first integral of the form

\[ W_r = \frac{J}{H^2}, \] (8.2)

where \( J \) is an integration constant. The combination \( E_t^t + 2E_r^r + E_\phi^\phi \), for \( a = 0 = b \), yields

\[ F_{r,r} - \frac{8}{l^2} = 0, \] (8.3)

which possesses the general solution

\[ F = \frac{4}{l^2} (r - r_1)(r - r_2), \] (8.4)
where \( r_1 \) and \( r_2 \) are constant of integration. Next, using \( W_r \) from Eq. (8.2) in \( E_r' (a = 0 = b) \), (3.15e), one arrives at

\[
\frac{1}{4} \left( \frac{H_r}{H} - \frac{1}{2} \frac{F_r}{F} \right)^2 - \frac{J^2}{4H^2 F} = \frac{F_r^2}{16F^2} - \frac{c^2}{F^2} - \frac{1}{l^2 F}. \tag{8.5}
\]

The evaluation the right-hand side of this equation gives the same result as in the static case, thus one gets

\[
\frac{d}{dr} \ln \left( \frac{H}{F^{1/2}} \right)^2 - \frac{J^2}{H^2 F} = \frac{4(r_2 - r_1)^2 \alpha}{l^4 F^2}, \tag{8.6}
\]

where \( \alpha \) is defined through

\[
c^2 = \frac{(r_2 - r_1)^2}{l^4} (1 - \alpha). \]

From the above equation it becomes apparent that \( H \) can be sought in the form of

\[
H(r) = h(r) \sqrt{F(r)}. \tag{8.7}
\]

Replacing \( H(r) \) from above into (8.6) one obtains an equation for \( h(r) \) which can be given as

\[
\frac{d h}{\sqrt{\alpha_0^2 h^2 + J^2}} = \pm \frac{dr}{F}, \quad \alpha_0 := 2(r_2 - r_1)\sqrt{\alpha}/l^2, \tag{8.8}
\]

with integral

\[
\ln \left( \alpha_0 h + \sqrt{\alpha_0^2 h^2 + J^2} \right) = \ln \left[ k_1 \left( \frac{r - r_1}{r - r_2} \right)^{\pm \sqrt{\alpha}/2} \right], \tag{8.9}
\]

where \( k_1 \) is an integration constant. Therefore \( h(r) \) can be expressed as

\[
h(r) = \frac{l^2 k_1}{4(r_2 - r_1)\sqrt{\alpha}} \left[ \left( \frac{r - r_1}{r - r_2} \right)^{\pm \sqrt{\alpha}/2} - \frac{J^2}{k_1^2} \left( \frac{r - r_1}{r - r_2} \right)^{\pm \sqrt{\alpha}/2} \right]. \tag{8.10}
\]

The integration of the Eq. (8.2) for \( W \) does not present problem.

Summarizing the derived above results, one has that this family of solutions can be given by

\[
g = -\frac{F}{H} dt^2 + \frac{dr^2}{F} + H(d\phi + W dt)^2;
\]

\[
F = \frac{4}{l^2} (r - r_1)(r - r_2),
\]

\[
H(r) = l \frac{\sqrt{(r - r_1)(r - r_2)}}{2K_1(r_2 - r_1)\sqrt{\alpha}} \times \left[ \left( \frac{r - r_1}{r - r_2} \right)^{\pm \sqrt{\alpha}} - K_1^2 J^2 \left( \frac{r - r_1}{r - r_2} \right)^{\pm \sqrt{\alpha}} \right],
\]

\[
W(r) = W_0 \pm \frac{4}{l^2} J K_1^2 \sqrt{\alpha}(r_2 - r_1) \times \left[ \left( \frac{r - r_1}{r - r_2} \right)^{\pm \sqrt{\alpha}} - K_1^2 J^2 \right]^{-1},
\]

\[
A = \frac{c}{2} (td\phi - \phi dt), \tag{8.11}
\]
where the constant $K_1$ stands for $1/k_1$, $K_2 = 1/k_1$, and $W_0$ is an integration constant. Recall that the parameter $\alpha$ is related to $c$, $r_1$, and $r_2$ through $c^2 = \frac{(r_2 - r_1)^2}{l^4}(1 - \alpha)$. Correspondingly, the electromagnetic field tensors are

$$F_{\mu \nu} = 2c\delta[\mu, t_\nu],$$

$$T^\nu_\mu = \frac{c^2}{8\pi F}(-\delta^t_\mu \delta^t_\nu + \delta^r_\mu \delta^r_\nu - \delta^\phi_\mu \delta^\phi_\nu),$$

with invariants

$$F_{\mu \nu}F^{\mu \nu} = -2\frac{c^2}{F}, \quad T^{\mu \nu} = 3\frac{c^4}{64\pi^2 F^2}, \quad T^\mu_\mu = \frac{1}{8\pi} \frac{c^2}{F}.$$

This solution has been reported, for the first time to our knowledge, in [5]. The static hybrid BTZ solution would arise, one has to accomplish the coordinate transformations

$$t = \frac{l}{4K_2} \frac{1}{r_2 - r_1} J^2 T,$$

$$\phi = \Phi - \left(\frac{W_0}{4K_1} \frac{1}{r_2 - r_1} \frac{1}{K_1} \frac{J}{l} \right) T,$$

$$r = \frac{1}{1 - K_2 l K_1} \left( r_1 - r_2 K_2 J^2 - 2 \frac{K_1 l}{J^4} (r_2 - r_1) \rho^2 \right),$$

where with $\{T, \rho, \Phi\}$ are denoted the corresponding BTZ coordinates, which ought to be accompanied with the identification

$$J^2 K_1 = -R(\cdot) := -\left( M l - \sqrt{M^2 l^2 - J^2} \right), \quad M = -\frac{1}{2} \frac{K_1 l^2}{K_2 J^4}.$$

In this way this solution can be given in the standard representation as

$$g = -\frac{\rho^2}{H(\rho)} f(\rho) dT^2 + \frac{d\rho^2}{f(\rho)} + H(\rho) [d\Phi + W(\rho) dT]^2,$$

$$f(\rho) = \frac{\rho^2}{l^2} - M + \frac{J^2}{4\rho^2},$$

$$H(\rho) = \frac{\sqrt{2\rho^2 - l R_- \sqrt{2\rho^2 - l R_+}}}{4\sqrt{\alpha} K_1 \sqrt{M^2 l^2 - J^2}} \left[ J^2 K_2^2 (2\rho^2 - l R_+)^{-\sqrt{\alpha}/2} \left( 2\rho^2 - l R_+ \right)^{-\sqrt{\alpha}/2} \right. \left. - (2\rho^2 - l R_-)^{-\sqrt{\alpha}/2} \left( 2\rho^2 - l R_- \right)^{-\sqrt{\alpha}/2} \right] + \cdots$$

$$W(\rho) = \frac{R_-}{J l} \left[ (2\rho^2 - l R_+)^{\sqrt{\alpha}}(2\sqrt{\alpha} \sqrt{M^2 l^2 - J^2} + R_-)^{\sqrt{\alpha}} R_- - (2\rho^2 - l R_-)^{\sqrt{\alpha}} \right] \times \left[ (2\rho^2 - l R_+)^{\sqrt{\alpha}} R_- - (2\rho^2 - l R_-)^{\sqrt{\alpha}} \right]^{-1}. \quad (8.15)$$

1. The ACG hybrid solution allowing for BTZ limit

To achieve a representation of this hybrid solution in terms of the radial coordinate $\rho$, such that at the limit of vanishing electromagnetic parameter $c = 0 \rightarrow \alpha = 1$, the stationary BTZ solution would arise, one has to accomplish the coordinate transformations

$$t = \frac{l}{4K_1} \frac{1}{r_2 - r_1} J^2 T,$$

$$\phi = \Phi - \left(\frac{W_0}{4K_1} \frac{1}{r_2 - r_1} \frac{1}{K_1} \frac{J}{l} \right) T,$$

$$r = \frac{1}{1 - K_2 l K_1} \left( r_1 - r_2 K_2 J^2 - 2 \frac{K_1 l}{J^4} (r_2 - r_1) \rho^2 \right),$$

where with $\{T, \rho, \Phi\}$ are denoted the corresponding BTZ coordinates, which ought to be accompanied with the identification

$$J^2 K_1 = -R(\cdot) := -\left( M l - \sqrt{M^2 l^2 - J^2} \right), \quad M = -\frac{1}{2} \frac{K_1 l^2}{K_2 J^4}.$$
When the electromagnetic field is turned off, $c = 0 \rightarrow \alpha = 1$, the above metric components reduce to

$$g_{TT} = M - \frac{\rho^2}{r^2}, \quad g_{T\Phi} = \frac{J}{2}, \quad g_{\Phi\Phi} = \rho^2, \quad g_{\rho\rho} = \left(\frac{\rho^2}{r^2} - M + \frac{J^2}{4\rho^2}\right)^{-1},$$

(8.16)

which correspond to the BTZ ones.

This solution possesses mass $M$, angular momentum $J$, electromagnetic parameter $\alpha$, and negative cosmological constant, and describes a black hole.

B. Constant electromagnetic invariants’ hybrid solution for $\Lambda = 0$

This section is devoted to the studied of the hybrid electromagnetic stationary solution with constant electromagnetic invariant $FF$ and by virtue of the field structure, constant $T$ and $TT$. The constant character of $FF = -\frac{2c^2}{F_0}$ is achieved by requiring $F(r) = F_0$, and consequently all electromagnetic invariants equal to constants

$$FF = -\frac{2c^2}{F_0}, \quad TT = \frac{3}{64\pi^2 F_0^2}, \quad T_{\mu}^\nu = -\frac{c^2}{8\pi F_0}.$$ 

(8.17)

Again the integration of the Einstein equations start from $E_\phi\,(a = 0 = b) = 0$, which gives the relation

$$\frac{d}{dr} W(r) = \frac{J}{H(r)^2},$$

(8.18)

for the integration of the function $H(r)$ the substitution of $\frac{d}{dr} W(r)$ and $F(r) = F_0$ into the remaining Einstein equations requires the cosmological constant to vanish, $\Lambda = 0$. Under such condition, the equation for $H(r)$ becomes

$$F_0 \left(\frac{d H(r)}{dr}\right)^2 - J^2 + 4c^2 F_0 H(r)^2 = 0$$

with solution

$$H(r) = \epsilon_H \frac{J}{2c} \sqrt{F_0} \sin \frac{2c}{F_0} (r - C_0), \quad \epsilon_H = \pm 1,$$

(8.19)

which, used in (8.18), after integration yields

$$W(r) = W_0 + \epsilon_W \frac{2c}{J} \cot \frac{2c}{F_0} (r - C_0), \quad \epsilon_W = \pm 1,$$

(8.20)

where $\epsilon_H$ and $\epsilon_W$ assume their signs independently; one has to take care on the ranges of the variable $r$ to guarantee a correct signature. Moreover, notice that the integration constant $C_0$ can be always equated to zero. Therefore the corresponding metric and electromagnetic field vector amount to

$$g = -\frac{F_0}{H} dt^2 + \frac{dr^2}{F_0} + H(d\phi + W dt)^2,$$

$$H = \epsilon_H \frac{J}{2c} \sqrt{F_0} \sin \frac{2c}{F_0} r, \quad W = W_0 + \epsilon_W \frac{2c}{J} \cot \frac{2c}{F_0} r,$$

$$A = \frac{c}{2}(t d\phi - \phi dt).$$

(8.21)
The electromagnetic field tensors are
\[ F_{\mu\nu} = 2\delta_{\mu}^{\parallel} \delta_{\nu}^{\phi}, \]
\[ T_{\mu}^{\nu} = \frac{c^2}{8\pi F_0} \left[-\delta_{\mu}^{\parallel}\delta_{\nu}^{\parallel} + \delta_{\mu}^{\parallel} \phi_{\nu}^{\parallel} - \delta_{\mu}^{\phi} \delta_{\nu}^{\phi}. \right] \tag{8.22} \]

By means of scaling transformations \( F_0 \) can be set always equal to unit, \( F_0 = 1 \), hence this solution is endowed with two effective parameters \( c \) and \( J \).

**IX. STATIONARY CYCLIC SYMMETRIC SOLUTIONS FOR \( a \neq 0 \) OR \( b \neq 0 \)**

This section deals with the search of stationary solutions for the branches where one of the electromagnetic constants is zero, \( a \neq 0 = b \) or \( b \neq 0 = a \). It occurs that for these families the integration problem reduces to find the solution of a master four order (reducible to a third order) nonlinear equation for \( F(r) \), and to fit a differential constraint on the found structural functions \( F(r) \) and \( H(r) (\mathcal{H}(r)) \). The integration of \( W(r) (W(r)) \) is trivial.

**A. Stationary magneto–electric solution for \( a \neq 0 = b \),**

If the structural function \( W(r) \) is different from a constant (the constant case will be treated at the end of this paragraph) then \( E_{\phi}^t \) reads
\[ E_{t}^{\phi} = \frac{d}{dr} \left( H^2 \frac{d}{dr} W \right) = 0, \tag{9.1} \]
which yields
\[ \frac{d}{dr} W = \frac{J}{H^2}. \tag{9.2} \]

The remaining independent Einstein-Maxwell equations arise respectively from combinations \( (4 E_r^r + 2E_t^t + 2E_{\phi}^\phi), (-2H(E_r^r - E_t^t + W E_{\phi}^\phi)/F), \) and \( E_r^r \):
\[ EQ_F = \frac{d^2 F}{dr^2} - 4a^2 \frac{H}{F} - \frac{1}{l^2} = 0, \]
\[ EQ_H = \frac{d^2}{dr^2} H + 4a^2 \frac{H^2}{F^2} = 0, \]
\[ E_r^r = \frac{1}{4H} \frac{d}{dr} \left( \frac{d}{dr} F \right) - \frac{F}{4H} \left( \frac{d}{dr} H \right)^2 + \frac{J^2}{4H^2} - a^2 H \frac{F}{F} - \frac{1}{l^2} = 0. \tag{9.3} \]

The equation \( E_r^r \) can be written in the form
\[ EQ_{H1} = \left( \frac{1}{2H} \frac{d}{dr} H - \frac{1}{4F} \frac{d}{dr} F \right)^2 - \frac{1}{16 F^2} \left( \frac{d}{dr} F \right)^2 + \frac{a^2 H}{F^2} - \frac{1}{4} \frac{J^2}{F H^2} + \frac{1}{l^2 F} = 0. \tag{9.4} \]

On the other hand using \( EQ_F \) one expresses \( H \) in terms of \( F \) and its derivative
\[ H(r) = \frac{1}{4a^2} \left( \frac{d^2 F}{dr^2} - \frac{8}{l^2} \right) F. \tag{9.5} \]
Substituting the above $H(r)$ into $EQ_H$ (9.3) one gets

$$F \frac{d^4 F}{dr^4} + 2 \frac{d^3 F}{dr^3} \frac{dF}{dr} + 2 \left( \frac{d^2 F}{dr^2} \right)^2 - \frac{24}{l^2} \frac{d^2 F}{dr^2} + \frac{64}{l^4} = 0. \quad (9.6)$$

Therefore, integrating, if possible, Eq. (9.6) for $F(r)$, substituting the solution $F(r)$ into Eq. (9.5) one determines $H(r)$. The resulting functions $F(r)$ and $H(r)$ ought to fulfil the Eq. (9.4) or $E_r$ equation from Eq. (9.3). By integrating the linear first order Eq. (9.2) one determines $W(r)$.

The contravariant components of electromagnetic tensor are

$$F^{\mu \nu} = -2 \alpha \delta^\mu_{[\phi} \delta^\nu_{]} r. \quad (9.7)$$

The Eq. (9.6) for $F(r)$ can be reduced to a third–order non-linear equation. In this equation, the problem for deriving solutions in this branch actually resides.

Another possibility arises with the introduction of the auxiliary function $h(r)$ by means of

$$H(r) = F(r)^{1/2} h(r), \quad (9.8)$$

the $EQ_{H1}$ acquires the form

$$EQ_h = - l^2 h^2 \left( \frac{dF}{dr} \right)^2 + 4 l^2 F^2 \left( \frac{dh}{dr} \right)^2 + 16 l^2 a^2 h^3 \sqrt{F} - 4 l^2 J^2 + 16 h^2 F = 0, \quad (9.9)$$

and one could try to determine solutions for this variant.

1. “Stationary” magneto–electric $A = A(r)(d\phi - J_0 dt)$ solution

A particular solution to Eq. (9.6) is given by $F(r)$ from (4.24), namely

$$F(r) = \frac{4 h(r)}{C_1^2 l^2} \left[ K_0 + h(r) + a^2 l^2 \ln h(r) \right], \quad h(r) := C_1 r + C_0. \quad (9.10)$$

which, being substituted into Eq. (9.5), leads to

$$H(r) = \frac{4}{C_1^2 l^2} \left[ K_0 + h(r) + a^2 l^2 \ln h(r) \right]. \quad (9.11)$$

Entering with these particular solutions $F(r)$ and $H(r)$ in the constraint Eq. (9.4) one arrives at

$$\frac{J^2}{F(r)} = 0 \rightarrow W(r) = -J_0 = \text{constant}. \quad (9.12)$$

Summarizing, this solution is given by the same structural functions (4.24) of the magnetostatic solution except that in the present case the function $W(r)$ is a constant. The corresponding metric line element and field vector can be written as

$$g = -h(r)dt^2 + \frac{dr^2}{H(r) h(r)} + H(r)(d\phi - J_0 dt)^2,$$
$$A = \frac{a}{C_1} \ln h(r)(d\phi - J_0 dt). \quad (9.13)$$
The electromagnetic field tensors and their invariants are given by

\[ F_{\mu\nu} = 2 a \delta_{[\nu}^\mu \delta_{\phi]}^r, \quad F_{\mu\nu} = 2 a J_0/\hbar(r) \delta_{[\mu}^t \delta_{\phi]}^r - 2 a/h(r) \delta_{\mu}^\nu \delta_{\phi}^r, \quad FF = \frac{2a^2}{\hbar}, \]  

(9.14)

and

\[ T^\nu_{\mu} = \frac{a^2}{8\pi \hbar} \left[ -\delta^\nu_t \delta^\mu_t + \delta^\nu_r \delta^\mu_r + \delta^\nu_\phi \delta^\mu_\phi - 2 J_0 \delta^\nu_t \delta^\mu_\phi \right], \quad T = \frac{3}{64\pi^2} \frac{a^4}{h^2}. \]  

(9.15)

Because of the structure of the energy–momentum tensor above, this solution can be interpreted as a rigidly rotating perfect fluid

\[ T^\nu_{\mu} = (\rho + p)u^\mu u^\nu + p g_{\mu\nu}, \quad u^\mu = \frac{1}{\sqrt{F/H}}(\delta^\mu_t + J_0 \delta^\mu_\phi), \]

(9.16)

with energy density \( \rho \) and pressure \( p \) given by

\[ \rho = \frac{1}{8\pi} \frac{a^2}{\hbar} = p. \]

This solution can be generated via transformations \( t \to t, \phi \to \phi - J_0 t \) from the magnetostatic solution (4.24).

2. Clement “rotating” electromagnetic \( A = A(r)(d\phi + \omega_0 dt) \) solution

Clement [19] published the dual family of electromagnetic “stationary” cyclic symmetric solutions, Eq. (Cl.24), changing signature and \( V_{\text{Cl}} \to -V \), given by

\[ g = V(d\phi + \omega_0 dt)^2 + \frac{1}{\xi_0^2} \frac{d\rho^2}{2\rho V} - 2\rho dt^2, \]

\[ V = -2\Lambda \rho + \frac{\pi_1}{4m} \ln\left(\frac{\rho}{\rho_0}\right), \]

\[ A = -\frac{\pi_1}{2} \ln\left(\frac{\rho}{\rho_0}\right)(d\phi + \omega_0 dt), \]

(9.17)

where \( m, \pi_1, \xi_0 \) and \( \rho_0 \) are constants, \( \Lambda = \pm 1/l^2 \) stands for the cosmological constant of both signs; for anti-de Sitter \( \Lambda = -1/l^2 \). The parameter \( \omega_0 \) is related to the angular momentum constant.

It is worthwhile to notice that the Clement expressions (9.17) satisfy the 2+1 Einstein–Maxwell equations if \( \xi_0^2 = 1 \) and for \( 2m = 1/\kappa \); for the adopted in the Clement’s convention, \( \kappa \neq 1 \), \( G_{\mu\nu} + \Lambda g_{\mu\nu} = 4\pi \kappa T_{\mu\nu} \), the evaluation of the right hand side of the Einstein equations for the structural functions (9.17), for \( \xi_0^2 = 1 \), yields

\[ G^\nu_{\mu} = \frac{\pi_1}{8m\rho} \left[ -\delta^\nu_t \delta^\mu_t + \delta^\nu_r \delta^\mu_r + \delta^\nu_\phi \delta^\mu_\phi + 2\omega_0 \delta^\nu_t \delta^\mu_\phi \right], \]

while the right hand side amounts to

\[ 4\pi \kappa T^\nu_{\mu} = \frac{\kappa \pi_1^2}{4\rho} \left[ -\delta^\nu_t \delta^\mu_t + \delta^\nu_r \delta^\mu_r + \delta^\nu_\phi \delta^\mu_\phi + 2\omega_0 \delta^\nu_t \delta^\mu_\phi \right]. \]
hence \( 2m = 1/\kappa \).

If one were adopting \( \kappa = 1 \), then modifying the electromagnetic vector \( \mathbf{A} \) to be \( \mathbf{A}_{\text{mod}} = -\frac{\pi}{2\sqrt{2} m} \ln \left( \frac{\rho}{\rho_0} \right) (d\phi + \omega_0 dt) \), one would arrive at the solution in our convention.

It is apparent that this Clement’s solutions correspond to a variant of the solution derived in the previous Section (IX A 1), with the identification \( r \to \rho \) followed by minor scaling transformations of \( t \) and \( \phi \). Notice also that the above generalization with \( W(r) = \omega_0 \neq 0 \) of the magnetostatic solution (4.24) can be determined applying to it \( SL(2, R) \) transformations of the form \( \phi \to \phi + \omega_0 t, t \to t \).

**B. Stationary electro–magnetic solution for \( b \neq 0 = a \)**

A straightforward way to derive the equations and solutions of this class of fields is just by using the complex extension of the stationary magnetic field we derived in the previous subsection taking into account the specific structure of the functions (3.19) of the extended metric (3.18) and the metric components from (9.13) of the magnetic solution together with the formal change \( a^2 \to -b^2 \).

Another close possibility is to accomplish the substitution

\[
F = \mathcal{F}, \quad H = \frac{\mathcal{F}}{\mathcal{H}} - \mathcal{H}W^2, \quad W = \frac{\mathcal{H}W}{H},
\]

(9.18)

in the corresponding Einstein equations for this case \( b \neq 0 = a \), arriving at the following set of independent field equations

\[
EQ_{\mathcal{F}} = \frac{d^2 \mathcal{F}}{dr^2} + 4b^2 \frac{\mathcal{H}}{\mathcal{F}} - 8\frac{1}{l^2} = 0,
\]

\[
EQ_{\mathcal{H}} = \frac{d^2 \mathcal{H}}{dr^2} - 4b^2 \frac{\mathcal{H}^4}{\mathcal{F}^2} = 0,
\]

\[
E'_r = \frac{1}{4\mathcal{H}} \frac{d\mathcal{H}}{dr} \frac{d\mathcal{F}}{dr} - \frac{\mathcal{F}}{4\mathcal{H}^2} \left( \frac{d\mathcal{H}}{dr} \right)^2 + \frac{1}{4\mathcal{H}^2} J^2 + b^2 \frac{\mathcal{H}}{\mathcal{F}} - \frac{1}{l^2} = 0,
\]

\[
\frac{d}{dr} W = \frac{J}{\mathcal{H}^2}.
\]

(9.19)

Continuing with the parallelism, isolating \( \mathcal{H} \) from \( EQ_{\mathcal{F}} \) and replacing it into \( EQ_{\mathcal{H}} \) one obtains

\[
\mathcal{F} \frac{d^4 \mathcal{F}}{dr^4} + 2 \frac{d^3 \mathcal{F}}{dr^3} \frac{d\mathcal{F}}{dr} + 2 \left( \frac{d^2 \mathcal{F}}{dr^2} \right)^2 - 24 \frac{d^2 \mathcal{F}}{dr^2} \frac{d\mathcal{F}}{dr} + \frac{64}{l^4} = 0.
\]

(9.20)

Thus, as before, the first step in the integration of the problem depends upon the Eq. (9.20) for \( \mathcal{F}(r) \), structurally identical to Eq. (9.6). Substituting the solution \( \mathcal{F}(r) \) into \( EQ_{\mathcal{H}} \) from Eq. (9.19) one determines \( \mathcal{H} \). The resulting functions \( \mathcal{F}(r) \) and \( \mathcal{H} \) have to fulfill \( E'_r \) from Eq. (9.19). By integrating the linear first order equation for \( W \) one determines \( W(r) \).
1. “Stationary” electro–magnetic $\mathbf{A} = A(r)(dt + J_0d\phi)$ solution

The only known until now particular solution for $\mathcal{F}(r)$ of Eq. (9.20) and its corresponding solutions for $\mathcal{H}$ and $\mathcal{W}$ are

$$\mathcal{F} = \frac{4}{C_1^2 l^2} \left[ K_0 + h(r) - b^2l^2 \ln h(r) \right] h(r), \quad h(r) := C_1 r + C_0,$$

$$\mathcal{H} = \frac{\mathcal{F}}{h}, \quad \mathcal{W} = -J_0 = \text{constant}. \quad (9.21)$$

Substituting these expressions into Eq. (9.18), one gets

$$F = \mathcal{F} = \mathcal{H} h, \quad H = h - \mathcal{H} J_0^2,$$

$$W = -J_0 \frac{\mathcal{H}}{h - \mathcal{H} J_0^2}, \quad \mathcal{H} := \frac{4}{C_1^2 l^2} \left[ K_0 + h(r) - b^2l^2 \ln h(r) \right], \quad (9.22)$$

therefore, the corresponding metric and field vector can be written as

$$\mathbf{g} = -\mathcal{H}(dt + J_0d\phi)^2 + \frac{dr^2}{H h(r)} + h(r)d\phi^2,$$

$$\mathbf{A} = \frac{b}{C_1} \ln h(r)(dt + J_0d\phi). \quad (9.23)$$

The electromagnetic field tensors and their invariants are given by

$$F^{\mu\nu} = 2b \delta^{[\mu}_t \delta^{\nu]}_r, \quad F_{\mu\nu} = -2 \frac{b}{h(r)} \delta_\mu^{[t} \delta_\nu^r} - 2b \frac{J_0}{h(r)} \delta_\mu^{[\phi} \delta_\nu^r], \quad F F = -2 \frac{b^2}{h},$$

$$T_\nu^\mu = \frac{b^2}{8\pi h} \left[ -\delta_\nu^t \delta_\mu^t - \delta_\nu^r \delta_\mu^r + \delta_\phi^\phi \delta_\mu^\phi - 2 J_0 \delta_\nu^t \delta_\mu^\phi \right], \quad TT = \frac{3}{64\pi^2} \frac{b^4}{h^2}.$$}

As we shall see in the forthcoming section, this stationary electromagnetic solution can be generated via transformations $t \to t + J_0 \phi, \phi \to \phi$ from the electrostatic solution (4.7).

2. Clement “rotating” electromagnetic $\mathbf{A} = A(r)(dt - \omega_0d\phi)$ solution

Also Clement [19] published a class of electromagnetic “stationary” cyclic symmetric metrics, Eq. (Cl.23), changing signature, given by

$$\mathbf{g} = -U(dt - \omega_0d\phi)^2 + \frac{1}{\xi_0^2} \frac{d\rho^2}{2\rho U} + 2\rho d\phi^2,$$

$$U = -2\Lambda\rho - \frac{\pi_0^2}{4m} \ln \left( \frac{\rho}{\rho_0} \right),$$

$$\mathbf{A} = \frac{\pi_0}{2} \ln \left( \frac{\rho}{\rho_0} \right)(dt - \omega_0d\phi), \quad (9.24)$$

where $m, \pi_0, \xi_0$ and $\rho_0$ are constant parameters, $\Lambda = \pm 1/l^2$ stands for the cosmological constant of both signs; for anti–de Sitter $\Lambda = -1/l^2$. The parameter $\omega_0$ is a constant related
to the angular momentum. The evaluation of the right hand side of the Einstein equations for the structural functions (9.24), for \( \xi_0^2 = 1 \), yields

\[
G^\nu_\mu = -\frac{\pi_0^2}{8\pi r} \left[ \delta^\nu_\lambda \delta^\mu_\tau + \delta^\nu_\lambda \delta^\mu_\tau - \delta^\nu_\phi \delta^\mu_{\lambda} - 2 \omega_0 \delta^\phi_\nu \delta^\nu_\tau \right],
\]

while the energy momentum tensor in the left hand side, for the vector \( A \), amounts to

\[
4\pi T^\nu_\mu = -\frac{\pi_0^2}{4\pi r} \left[ \delta^\nu_\lambda \delta^\mu_\tau + \delta^\nu_\lambda \delta^\mu_\tau - \delta^\nu_\phi \delta^\mu_{\lambda} - 2 \omega_0 \delta^\phi_\nu \delta^\nu_\tau \right],
\]

therefore Einstein–Maxwell equations are fulfilled if \( 2m = 1/\kappa \) or, for \( \kappa = 1 \), modifying the electromagnetic vector \( A \) to be \( A_{\text{mod}} = \frac{\pi_0}{2\sqrt{2}m} \ln(\frac{\rho}{\rho_0}) (dt - \omega_0 d\phi) \). Recall that additionally one has to set \( \xi_0^2 = 1 \).

It is clear that this solution is equivalent to the one treated in the previous Section IX B 1 for the identification \( C_1 r + C_0 \rightarrow \rho \) accompanied with minor scaling transformations of \( t \) and \( \phi \).

Notice that this branch of rotating solutions with \( W(r) = \omega_0 \) can be determined from the static electric field solution, i.e., metric (4.7) and vector \( A \) (4.8), via \( SL(2, R) \) transformations: \( t \rightarrow t - \omega_0 \phi, \phi \rightarrow \phi \).

3. Constant \( W \) electric solution

In the case \( W(r) = -J = \text{constant} \) the equation \( E_\phi^t \), from (3.15), reduces to \( b^2 H J/F = 0 \), then \( J = 0 \rightarrow W = 0 \). Hence, the set of equations reduces to the corresponding one of the static case.

X. STATIONARY CYCLIC SYMMETRIC SOLUTIONS FOR \( a \neq 0 \) AND \( b \neq 0 \)

It is clear that the derivation of a general solution to the whole system of Einstein–Maxwell equations (3.15) is far from being an easy task. Nevertheless, it occurs that some simplifications of the system of equations can be achieved by a useful change of the structural functions and combinations of the Einstein equations; the integration problem on the whole for the three structural functions is constrained to three differential equations without any further restrictions. Although we could not find sufficiently general classes of solutions, we were able to determine new families of solutions within particular combinations of elementary functions.

A. Alternative representation of the Einstein equations

Having in mind the derivation of other possible families of Einstein–Maxwell solutions with \( a \) and \( b \) different from zero, it is desirable to have at hand the most simple set of equations. For this purpose, introducing \( W(r) = \Omega(r)/H(r) \), the independent Einstein equations can be written as

\[
EQ_{H2} = F^2 \frac{d^2}{dr^2} H + 4(aH - b\Omega)^2 = 0, \tag{10.1}
\]
\[ EQ_{\Omega} = H F^2 \frac{d^2}{dr^2} \Omega + 4 \Omega (a H - b \Omega)^2 + 4b F (a H - b \Omega) = 0, \quad (10.2) \]

\[ EQ_{F_1} = F \frac{dF}{dr} H \frac{dH}{dr} - F^2 \left( \frac{dH}{dr} \right)^2 - 4 H (a H - b \Omega)^2 + F \left( H \frac{d\Omega}{dr} - \Omega \frac{dH}{dr} \right)^2 + 4F H (b^2 - H/l^2) = 0, \quad \text{correct: equate to zero}, \quad (10.3) \]

\[ EQ_{F_2} = \Omega H^2 \frac{d^2 F}{dr^2} - 2 \Omega H \frac{dH}{dr} \frac{dF}{dr} + 2\Omega \left( \frac{dH}{dr} \right)^2 F - 2 \Omega \left( H \frac{d\Omega}{dr} - \Omega \frac{dH}{dr} \right)^2 = 0. \quad \text{correct: equate to zero}, \quad (10.4) \]

It is worthwhile to point out that the equation \( EQ_{F_2} \) can be considered as an integrability condition of the system of equations; differentiating the \( EQ_{F_1} \) one obtains the second derivative \( \frac{d^2 F}{dr^2} \) together with second derivatives of \( H \) and \( \Omega \), which can be replaced through \( EQ_H \) and \( EQ_{\Omega} \), next substituting \( \frac{d^2 F}{dr^2} \) into \( EQ_{F_2} \) one arrives at an equation of the form \((H \frac{dF}{dr} + F \frac{dH}{dr}) \times EQ_{F_1}\), equal to zero by virtue of the same \( EQ_{F_1} \). Although one can adopt a different point of view; the \( EQ_{F_2} \)–equation arises as the differentiation of \( EQ_{F_1} \) together with the use of \( EQ_H \) and \( EQ_{\Omega} \), and therefore it is not an independent equation.

Using the experience gathered until now, we shall search for particular solutions of the form

\[ F(r) = P(r) + Q(r) \ln(r), \]
\[ H(r) = A(r) + B(r) \ln(r), \]
\[ W(r) = \Omega(r)/H(r), \]
\[ \Omega(r) = V(r) + Z(r) \ln(r), \quad (10.5) \]

where it is assumed the explicit dependence on \( \ln(r) \). Substituting these guessed functions into the quoted system of equations and equating to zero the coefficients of different powers of \( \ln(r) \) one arrives at a very large non–linear system of equations; since the non–trivial Einstein equations are five, then one may expect 40 secondary equations. For instance, from equations arising from the coefficients of \( \ln(r) \) to the seventh power, one has

\[ E^t_t \ln^2 = B^2 Q l^2 (Z \frac{dB}{dr} - B \frac{dZ}{dr})^2 - 2 B^3 Z Q l^2 (Z \frac{d^2 B}{dr^2} - B \frac{d^2 Z}{dr^2}), \]
\[ E^\phi_\phi \ln^2 + 3E^r_r \ln^2 = 2 l^2 B^3 Q (Z \frac{d^2 B}{dr^2} - B \frac{d^2 Z}{dr^2}), \]

hence

\[ Z \frac{d^2 B}{dr^2} - B \frac{d^2 Z}{dr^2} = 0, \quad Z \frac{dB}{dr} - B \frac{dZ}{dr} = 0, \]

therefore

\[ Z(r) = c_1 B(r). \]

After a very lengthy and time–consuming integration process we succeeded in getting two branches of stationary electromagnetic solutions of the Einstein–Maxwell equations. The structural functions \( H \) and \( W \) possess a multiplicative factor \( a/b \) which can be absorbed by re–scaling of the Killingian coordinates according to: \( \sqrt{|a/b|} t \to t \) and \( \sqrt{|b/a|} \phi \to \phi \), \( \sqrt{|a|/|b|} = \pm \alpha \).
B. Stationary electromagnetic solution with BTZ–limit

This class of solutions, depending on three parameters, is given by

\[ g = -\frac{F(r)}{H(r)} dt^2 + \frac{dr^2}{F(r)} + H(r) [d\phi + W(r) dt]^2, \]

\[ F(r) = 4 \frac{r^2}{l^2} + 2 \frac{r}{l} \left( l w_1 + \sqrt{l^2 w_1^2 - 4} \right) [w_0 + W_0 \ln(r)], \]

\[ H(r) = \frac{r}{l} \sqrt{l^2 w_1^2 - 4} - [w_0 + W_0 \ln(r)], \]

\[ W(r) = \Omega(r)/H(r), \quad \Omega(r) := w_0 + w_1 r + W_0 \ln(r), \]

\[ W_0 := -\frac{1}{2} l^2 \alpha^2 \left( l^2 w_1^2 - 2 - l w_1 \sqrt{l^2 w_1^2 - 4} \right), \]

\[ A = \frac{1}{4} \alpha \left( l w_1 - \sqrt{l^2 w_1^2 - 4} \right) (dt - d\phi) \ln(r). \]

\[ F^\mu_\nu = 2 \alpha \left( \delta^\mu_\nu \delta^r_r - \delta^\mu_\nu \phi \delta^r_r \right), \quad F^\mu_\nu = -\frac{\alpha l}{4r} \left( l w_1 - \sqrt{l^2 w_1^2 - 4} \right) \left( \delta^\mu_\nu \phi \delta^r_r - \delta^\mu_\nu \phi \delta^r_r \right), \]

\[ T^\mu_\nu = \frac{\alpha^2}{8 \pi F H} \left[ \begin{array}{ccc} -[F + H^2(1-W^2)] & 0 & 2[F + H^2 W(1-W)] \\ 0 & -[F - H^2(1-W^2)] & 0 \\ -2H^2(1-W) & 0 & [F + H^2(1-W^2)] \end{array} \right], \]

with electromagnetic invariants \( FF = -2 \frac{\alpha^2}{l^2} + 2 \frac{\alpha^2}{l} H (1-W)^2, \) and \( TT = \frac{3}{64 \pi^2} \frac{\alpha^4}{l^4} \left( -F + H^2 (1-W)^2 \right)^2. \)

1. Transformation to BTZ-like coordinates

Since this solution possesses as a limit for \( \alpha = 0 \) the BTZ solution, it is natural to search for new coordinates in which it will become apparent the BTZ standard structure. First one determines the radial transformation \( r = \beta_0 (\rho^2 + \gamma_0); \) since \( g_{rr} \to g_{pp}, \) then

\[ F(r) = 4 \frac{r^2}{l^2} + 2 \frac{r}{l} \left( l w_1 + \sqrt{l^2 w_1^2 - 4} \right) w_0 \to F(\rho) = \frac{\rho^4}{l^2} - M \rho^2 + \frac{J^2}{4}, \]

hence

\[ \gamma_0^2/l^2 + \gamma_0 M + J^2/4 = 0 \to \gamma_0/l = -\frac{Ml}{2} \pm \frac{1}{2} \sqrt{l^2 M^2 - J^2}, \]

\[ w_0 \left( l w_1 + \sqrt{l^2 w_1^2 - 4} \right) = \pm 2 \beta_0 \sqrt{l^2 M^2 - J^2}. \]

Next, the structure of the Killingian transformations is of the form

\[ t = \alpha_t T + \beta_t \Phi, \quad \phi = \delta_t \Phi. \]
Substituting these relations into the metric and comparing the metric components with the corresponding ones of the BTZ–metric one establishes that

\[ w_0 = \frac{1}{J} \sqrt{l^2 M^2 - J^2} \left( -l M + \sqrt{l^2 M^2 - J^2} \right), \quad \beta_0 = -1, \quad w_1 = \frac{2 M}{J}. \quad (10.7) \]

Therefore, the coordinate transformations to be used in the electromagnetic solution in order to get the proper BTZ limit when the electromagnetic \( \alpha \)-parameter is switch off is given by

\[ r = -\rho^2 + \frac{M l^2}{2} + \frac{l}{2} \sqrt{l^2 M^2 - J^2}, \]

\[ t = \frac{1}{\sqrt{2}} (l^2 M^2 - J^2)^{-1/4} \left( \sqrt{\frac{J}{l}} T - l M \sqrt{\frac{I}{J}} \Phi \right), \]

\[ \phi = \frac{1}{\sqrt{2}} \sqrt{\frac{I}{J}} (l^2 M^2 - J^2)^{1/4} \Phi. \quad (10.8) \]

Under these transformations the metric becomes

\[ g = -\frac{\rho^2 F(\rho)}{H(\rho)} dT^2 + \frac{d\rho^2}{F(\rho)^2} + H(\rho) (d\Phi^2 + W(\rho) dT^2), \]

\[ F(\rho) = \frac{\rho^2}{l^2} - M + \frac{J^2}{4\rho^2} - \frac{l \alpha^2}{2 J \rho^2} \left[ J^2 l - 2 \rho^2 R(-) \right] \ln |r|, \]

\[ H(\rho) = \frac{H_n}{H_d}, \]

\[ H_n := -l^6 J^2 \alpha^4 R_{(-)}^2 (\ln |r|)^2 - 2 l^3 J \sqrt{l^2 M^2 - J^2} \left[ -2 \sqrt{l^2 M^2 - J^2} R_{(-)} \rho^2 + M J^2 l^2 \right] \frac{\alpha^2 \ln |r|}{\rho^2} + 4 \rho^2 J^2 \left( \rho^2 - l^2 M \right) (l^2 M^2 - J^2), \]

\[ H_d := 4 J^2 (l^2 M^2 - J^2) \left( \rho^2 - l^2 M \right) - 2 J \sqrt{l^2 M^2 - J^2} R_{(-)}^{3 \alpha^2} \ln |r|, \]

\[ W(\rho) = \frac{\Omega(\rho)}{H_n}, \]

\[ \Omega(\rho) := l^5 J R_{(-)}^{3 \alpha^4} (\ln |r|)^2 + l^2 J^2 \alpha^2 \sqrt{l^2 M^2 - J^2} \left[ J^2 l + 2 l R_{(-)}^2 - 2 R_{(-)} \rho^2 \right] \ln |r| - 2 J^3 \left( \rho^2 - l^2 M \right) (l^2 M^2 - J^2), \]

\[ r := -\rho^2 + \frac{M l^2}{2} + \frac{l}{2} \sqrt{l^2 M^2 - J^2}, \quad R_{(\pm)} := M l \pm \sqrt{l^2 M^2 - J^2}. \quad (10.9) \]

Notice that \( g_{tt} g_{\phi\phi} - g_{\theta\phi}^2 = -\rho^2 F(\rho) \). The correspondence of this function representation of this electromagnetic solution with the BTZ solution in the limit of vanishing electromagnetic parameter \( \alpha \) becomes apparent:

\[ F(\rho) = \frac{\rho^2}{l^2} - M + \frac{J^2}{4\rho^2}, \quad H(\rho) = \rho^2, \quad W(\rho) = -\frac{J}{2 \rho^2}. \]
Thus, this anti-de Sitter solution has three parameters: mass $M$, angular momentum $J$, and electromagnetic parameter $\alpha$. Because of its close similarity to the BTZ solution, it could represent a black hole; a research in this direction is in progress.

C. Stationary electromagnetic solution with BTZ-counterpart limit

The second possible solution in the studied class is given by

$$g = -\frac{\mathcal{F}}{\mathcal{H}}dT^2 + \frac{dr^2}{\mathcal{F}} + \mathcal{H}(d\Phi + \mathcal{W}dT)^2,$$

$$\mathcal{F}(r) = \frac{4r^2}{l^2} + 2\frac{r}{l}(lw_1 + \sqrt{l^2w_1^2 - 4})(w_0 + W_0 \ln(r)),$$

$$\mathcal{H}(r) = \frac{\mathcal{H}_n}{\mathcal{H}_d},$$

$$\mathcal{H}_n = \mathcal{F}(r) - \Omega(r)^2,$$

$$\mathcal{H}_d = \frac{r}{l}\sqrt{l^2w_1^2 - 4} - (w_0 + W_0 \ln(r)),$$

$$\mathcal{W}(r) = \frac{\Omega(r)}{\mathcal{H}(r)}, \quad \Omega(r) = w_0 + W_0 \ln(r) + w_1 r,$$

$$W_0 := \frac{1}{2}l^2\alpha^2\left(l^2w_1^2 - 2 - lw_1\sqrt{l^2w_1^2 - 4}\right).$$

$F^{\mu\nu} = 2\alpha\left(\delta^\mu_{[\nu}\delta^{\nu]_r} + \delta^\mu_{\phi}\delta^{\nu}_r\right)$,

$$T^\mu_{\nu} = \frac{\alpha^2}{8\pi F\mathcal{H}}\begin{bmatrix} -2[F + \mathcal{H}^2(1 - \mathcal{W})] & 0 & -2[F - \mathcal{H}^2\mathcal{W}(1 + \mathcal{W})] \\ 0 & -2[F - \mathcal{H}^2(1 + \mathcal{W})] & 0 \\ 2\mathcal{H}^2(1 + \mathcal{W}) & 0 & [F + \mathcal{H}^2(1 - \mathcal{W})^2] \end{bmatrix}, \tag{10.10}$$

where $w_0$ and $w_1$ are parameters related to mass and angular momentum, while $\alpha$ is an electromagnetic parameter; the electromagnetic invariants are

$$FF = 2\frac{\alpha^2}{F\mathcal{H}}[-F + \mathcal{H}^2(1 + \mathcal{W})^2], \quad TT = \frac{3}{64}\frac{\alpha^4}{\pi^2 F^2 \mathcal{H}^2}[-F + \mathcal{H}^2(1 + \mathcal{W})^2]^2. \tag{10.11}$$

The calligraphic capital letters have been used above to make their relationship evident to those structural functions arising as real cuts of the complex extensions of the studied class of metric, see (3.18). This solution of the Einstein–Maxwell equations can be considered also as a real cut of the complex version of the stationary electromagnetic solution with BTZ–limit given in the previous paragraph; the structural functions $\mathcal{F}$, $\mathcal{H}$, and $\mathcal{W}$ can be constructed according to Eq. (3.19) with $F$, $H$, and $W$ from Eq. (10.5) accompanied by the replacement of the sign in front of $\alpha^2$, $\alpha^2_{el} \rightarrow -\alpha^2_{mg}$. If one searches for the anti-de
Sitter limit of this solution, one would arrive at an alternative real cut of the BTZ–solution, namely to the "BTZ–solution counterpart", for short BTZ–counterpart.

\[
g_c = -\rho^2 \frac{\mathcal{F}}{\mathcal{H}} dT^2 + \frac{d\rho^2}{\mathcal{F}} + \mathcal{H}(d\Phi + \mathcal{W}dT)^2,
\]

\[
\mathcal{F} = \frac{\rho^2}{l^2} - M + \frac{J^2}{4\rho^2}, \quad \mathcal{H} = \frac{\rho^2}{l^2} - M, \quad \mathcal{W} = \frac{J}{2\mathcal{H}}.
\]

(10.12)

Recall that in the above–mentioned metric one can again introduce the radial coordinate by changing

\[
\rho^2 \rightarrow R^2 + M l^2, \quad \mathcal{H} \rightarrow R^2, \quad W \rightarrow \frac{J^2}{4R^2},
\]

\[
\mathcal{F}(\rho) \rightarrow F(R) = \frac{R^2}{l^2} + M + \frac{J^2}{4R^2}.
\]

(10.13)

Since this solution possesses the BTZ–counterpart as a limit for \( \alpha = 0 \), it is pertinent to search for new coordinates in which it will become apparent the BTZ–solution counterpart structure (10.12).

1. Transformation to BTZ-counterpart coordinates

The constants and the coordinate transformations to be used in this case are given by

\[
w_0 = -\frac{1}{J} \sqrt{l^2 M^2 - J^2} \left( l M - \sqrt{l^2 M^2 - J^2} \right), \quad w_1 = 2 \frac{M}{J},
\]

\[
r = -\rho^2 + \frac{M l^2}{2} + \frac{l}{2} \sqrt{l^2 M^2 - J^2},
\]

\[
\phi = \frac{1}{\sqrt{2}} \left( l^2 M^2 - J^2 \right)^{-1/4} \left( M l \sqrt{\frac{l}{J}} T + \sqrt{\frac{J}{l}} \Phi \right),
\]

\[
t = \frac{1}{\sqrt{2}} \sqrt{\frac{l}{J}} \left( l^2 M^2 - J^2 \right)^{1/4} T.
\]

(10.14)

Under these transformations the solution amounts to

\[
g \rightarrow -\rho^2 \frac{F(\rho)}{H(\rho)} dT^2 + \frac{d\rho^2}{F(\rho)} + H(\rho)(d\Phi + W(\rho)dT)^2,
\]

\[
F = \frac{\rho^2}{l^2} - M + \frac{J^2}{4\rho^2} + \frac{l \alpha^2}{2J \rho^2} \left( J^2 l - 2 R(-) \rho^2 \right) \ln |r|,
\]

\[
H(\rho) = \frac{H_n}{H_d},
\]

\[
H_n := -l^6 J^6 \alpha^4 (\ln |r|)^2 + 4 J^3 l^3 \left( l^2 M - \rho^2 \right) \sqrt{l^2 M^2 - J^2 R_{(+)}^2} \alpha^2 \ln |r|
\]

\[
-4 \left( l^2 M - \rho^2 \right)^2 \left( l^2 M^2 - J^2 \right) R_{(+)}^4,
\]

\[
H_d := -2l^6 J^3 \sqrt{l^2 M^2 - J^2 R_{(+)}^2} \alpha^2 \ln |r| + 4 l^2 \left( l^2 M - \rho^2 \right) \left( l^2 M^2 - J^2 \right) R_{(+)}^4,
\]
\[ W(\rho) = \frac{l^2 \Omega(\rho)}{J H_n}, \]
\[ \Omega(\rho) = -l^5 J^6 R_{(+)} \alpha^4 (\ln |r|)^2 \]
\[ + l^2 J^3 \left[ l \left( R_{(+)}^2 + 2J^2 \right) \sqrt{l^2 M^2 - J^2} + (J^2 - R_{(+)}^2) \rho^2 \right] R_{(+)}^2 \alpha^2 \ln |r| \]
\[ + 2 J^2 \left( l^2 M^2 - J^2 \right) (\rho^2 - l^2 M) R_{(+)}^4, \]
\[ r := -\rho^2 + \frac{M l^2}{2} + \frac{l}{2} \sqrt{l^2 M^2 - J^2}, \]
\[ R_{(\pm)} := M l \pm \sqrt{l^2 M^2 - J^2}. \quad (10.15) \]

The correspondence of this representation with the BTZ–solution counterpart in the limit of vanishing electromagnetic parameter \( \alpha \) is evident.

Because of the complexity of the system of equations, we have found very hard to determine other branches, if any, of exact solutions in the general case.

**XI. Generating Stationary Solutions Via \( SL(2, R) \)–Transformations From the Static Solutions**

This section deals with \( SL(2, R) \)–transformations applied on static solutions to construct stationary cyclic symmetric classes of solutions, namely the electric and magnetic stationary families.

**A. \( SL(2, R) \)–transformations**

Let us consider the general metric
\[ g = g_{tt} dt^2 + 2 g_{t \phi} dt d\phi + g_{\phi \phi} d\phi^2 + g_{rr} dr^2, \]
and accomplish here a \( SL(2, R) \) transformations of the Killingian coordinates \( t \) and \( \phi \)
\[ t = \alpha \tilde{t} + \beta \tilde{\phi}, \phi = \gamma \tilde{t} + \delta \tilde{\phi}, \Delta := \alpha \delta - \beta \gamma \neq 0. \quad (11.1) \]

The transformed metric components are given by
\[ g_{\tilde{t} \tilde{t}} = \alpha^2 g_{tt} + 2 \alpha \gamma g_{t \phi} + \gamma^2 g_{\phi \phi}, g_{\tilde{t} \tilde{\phi}} = \alpha \beta g_{tt} + (\alpha \delta + \beta \gamma) g_{t \phi} + \gamma \delta g_{\phi \phi}, \]
\[ g_{\tilde{\phi} \tilde{\phi}} = \beta^2 g_{tt} + 2 \beta \delta g_{t \phi} + \delta^2 g_{\phi \phi}, g_{rr} = g_{rr}, \quad (11.2) \]
while under the considered transformations the electromagnetic field tensor (3.2) becomes
\[ F^{\tilde{\alpha} \tilde{\beta}} = \frac{1}{\sqrt{-\tilde{g}}} \left[ \begin{array}{c} 0 & \tilde{b} & -\tilde{b} \\ \tilde{b} & 0 & \tilde{a} \\ -\tilde{b} & -\tilde{a} & 0 \end{array} \right], \]
\[ \tilde{g} = \det(g_{\tilde{\mu} \tilde{\nu}}), \quad (11.3) \]
where the new constant are given in terms of the original ones through

\[
\begin{align*}
\tilde{\alpha} &= \frac{\alpha a + \gamma b}{\alpha \delta - \beta \gamma}, \quad a = \delta \tilde{\alpha} - \gamma \tilde{b} \\
\tilde{b} &= \frac{\beta a + \delta b}{\alpha \delta - \beta \gamma}, \quad b = -\beta \tilde{a} + \alpha \tilde{b}, \\
\tilde{c} &= \frac{c}{\alpha \delta - \beta \gamma}.
\end{align*}
\] (11.4)

Notice that

\[
\begin{align*}
g_{\tilde{t}\tilde{t}}g_{\tilde{\phi}\tilde{\phi}} - g_{\tilde{t}\tilde{\phi}}^2 &= (g_{tt}g_{\phi\phi} - g_{t\phi}^2)(\alpha \delta - \beta \gamma)^2 = -F \Delta^2,
\end{align*}
\]

therefore, in concrete applications it is more useful to use normalized transformations with \(\Delta = \alpha \delta - \beta \gamma = 1\).

The electromagnetic tensor occurs to be form–invariant under the above–mentioned \(SL(2, R)\)–transformations if the field constants \(a, b\), and \(c\) are identified based on Eq (11.4). This property, on its turn, yields to the form–invariance of the electromagnetic energy–momentum tensor \(T_{\mu \nu} = 1/(4\pi)(F_{\mu \sigma} F^{\nu \sigma} - 1/4 \delta_{\nu}^{\mu} F_{\tau \sigma} F^{\tau \sigma})\), and consequently to the form–invariance of the Einstein-Maxwell equations.

Therefore, starting with an electromagnetic solution in which a single electric \((b \neq 0)\) or magnetic \((a \neq 0)\) field is present, by accomplishing the above–mentioned \(SL(2, R)\)–transformations, one can generate solutions with both electric and magnetic fields \(\tilde{b} \neq 0, \tilde{a} \neq 0\) present. Conversely, if one originally has had a solution endowed with both constant parameters \(a\) and \(b\) then, via transformations, one could achieve a branch of solutions with one single parameter. At this level, one may argue that one deals with one specific solution in its different coordinate representations. But there exists a second point of view in 2+1–gravity: to end with a new solution one has to change the variety, i.e., the topology, requiring the ranges of change of the new variable be, for instance, the same as the ranges of the original variables. This procedure can be considered as a generating solution technique and it has been used to construct stationary solutions starting from static solutions as we shall show in the forthcoming sections.

For the metric (3.4), subjected to the above–mentioned \(SL(2, R)\)–transformations, one gets

\[
\begin{align*}
g_{\tilde{t}\tilde{t}} &= -\alpha^2 \frac{F}{H} + H (\alpha W + \gamma)^2, \\
g_{\tilde{t}\tilde{\phi}} &= -\alpha \beta \frac{F}{H} + H (\delta + \beta W)(\gamma + \alpha W), \\
g_{\tilde{\phi}\tilde{\phi}} &= -\beta^2 \frac{F}{H} + H (\beta W + \delta)^2, \\
g_{rr} &= \frac{1}{F},
\end{align*}
\] (11.5)

hence, the expressions of the new structural functions are given in the form

\[
\begin{align*}
\tilde{H} &= -\beta^2 \frac{F}{H} + H(\delta + \beta W)^2, \quad \tilde{W} \tilde{H} = -\alpha \beta \frac{F}{H} + H(\delta + \beta W)(\gamma + \alpha W), \quad \tilde{F} = F.
\end{align*}
\] (11.6)

The transformed electromagnetic field tensor \(F^{\tilde{\mu}\tilde{\nu}}\), as it should be, exhibits its form–invariant
property
\[
F_\mu^\nu = \begin{bmatrix}
0 & \tilde{b} & -\frac{\tilde{c}}{F} \\
-\tilde{b} & 0 & \tilde{a} \\
\frac{\tilde{c}}{F} & -\tilde{a} & 0
\end{bmatrix},
\]  
(11.7)
where as before the new field constant parameters are related with the old ones according to Eq (11.4).

**B. Transformed electrostatic \( b \neq 0 \) solution**

Starting with the general electrostatic Maxwell solution (4.7) with metric
\[
g = -\frac{F}{H} dt^2 + \frac{1}{F} dr^2 + H d\phi^2,
\]
\[
F(r) = 4 \frac{H(r)}{C_1^2 l^2} \left[ K_0 + H(r) - b^2 l^2 \ln H(r) \right],
\]
\[
H(r) = C_1 r + C_0,
\]  
(11.8)
under normalized \( SL(R, 2) \)–transformations
\[
t = \frac{\alpha}{\sqrt{\Delta}} \tilde{t} + \frac{\beta}{\sqrt{\Delta}} \tilde{\phi},
\]
\[
\phi = \frac{\gamma}{\sqrt{\Delta}} \tilde{t} + \frac{\delta}{\sqrt{\Delta}} \tilde{\phi},
\]
\[
\Delta = \alpha \delta - \beta \gamma \neq 0.
\]  
(11.9)
(in the general (non-normalized) case the same expressions hold except for the absence of \( \Delta \), set simply \( \Delta = 1 \), the new metric, the rotated one, acquires the form
\[
g_{\tilde{\mu} \tilde{\nu}} = \begin{bmatrix}
-\frac{\alpha^2 F}{\Delta H} + \frac{\gamma^2}{\Delta} H & 0 & -\frac{\alpha \beta F}{\Delta H} + \frac{\delta \gamma}{\Delta} H \\
0 & 1 \frac{1}{F} & 0 \\
-\frac{\alpha \beta F}{\Delta H} + \frac{\delta \gamma}{\Delta} H & 0 & -\frac{\beta^2 F}{\Delta H} + \frac{\delta^2}{\Delta} H
\end{bmatrix},
\]  
(11.10)
the electromagnetic field tensor becomes
\[
F_{\tilde{\mu} \tilde{\nu}} = \begin{bmatrix}
0 & \frac{\delta b}{\sqrt{\Delta}} & 0 \\
-\frac{\delta b}{\sqrt{\Delta}} & 0 & \frac{\gamma b}{\sqrt{\Delta}} \\
0 & -\frac{\gamma b}{\sqrt{\Delta}} & 0
\end{bmatrix},
\]  
(11.11)
while the electromagnetic energy–momentum tensor amounts to
\[
T_{\tilde{\mu} \tilde{\nu}} = \begin{bmatrix}
-\frac{1}{8\pi} \frac{(\alpha \delta + \beta \gamma) b^2}{H \Delta} & 0 & \frac{\gamma}{4\pi} \frac{\alpha b^2}{H \Delta} \\
0 & -\frac{1}{8\pi} \frac{b^2}{H} & 0 \\
-\frac{1}{4\pi} \frac{\delta b^2}{H \Delta} & 0 & \frac{1}{8\pi} \frac{(\alpha \delta + \beta \gamma) b^2}{H \Delta}
\end{bmatrix}.
\]  
(11.12)
Explicitly, the new metric is given by the non-zero components
\[
g_{\tilde{\mu}} = -\frac{\alpha^2}{\Delta} \frac{1}{H(r) g_{rr}} + \frac{\gamma^2}{\Delta} H(r),
\]
\[
g_{\tilde{\nu} \tilde{\nu}} = -\frac{\alpha \beta}{\Delta} \frac{1}{H(r) g_{rr}} + \frac{\delta \gamma}{\Delta} H(r),
\]
\[
g_{\tilde{\phi} \tilde{\phi}} = -\frac{\beta^2}{\Delta} \frac{1}{H(r) g_{rr}} + \frac{\delta^2}{\Delta} H(r),
\]
\[
g_{rr} = \frac{1}{4 H(r) \left[ K_0 + H(r) - b^2 l^2 \ln H(r) \right]},
\]
\[
H(r) = C_1 r + C_0.
\]  
(11.13)
For general $SL(2, R)$–transformations, with non-vanishing entries, the electromagnetic field tensor $F^{\mu\nu}$ allows for the presence of both electric and magnetic fields, corresponding to new $b$ and $a$ different from zero.

If one accomplishes the transformation of the dependent variable $r$ to the radial (polar) coordinate $\rho$, arc $= \rho \, d\phi$, one chooses

$$H(\rho) = C_1 \rho + C_0 = \rho^2, \quad C_1 = 2.$$  

(11.14)

1. **Stationary electromagnetic solution**

In particular, for the $SL(2, R)$–transformation

$$t = \tilde{t} - \omega \tilde{\phi}, \quad \phi = \tilde{\phi}, \quad \alpha = 1, \quad \beta = -\omega, \quad \gamma = 0, \quad \delta = 1, \quad \Delta = 1,$$

(11.15)

one obtains a new solution, the rotated one, with metric components

$$g_{\tilde{t}\tilde{t}} = -\frac{4}{C_1^2 l^2} \left[ K_0 + H(\rho) - b^2 l^2 \ln H(\rho) \right], \quad g_{\tilde{t}\tilde{\phi}} = \omega \frac{4}{C_1^2 l^2} \left[ K_0 + H(\rho) - b^2 l^2 \ln H(\rho) \right],$$

$$g_{\tilde{\phi}\tilde{\phi}} = H(\rho) - \omega^2 \frac{1}{H(\rho) g_{rr}}, \quad g_{rr} = \frac{1}{4 H(\rho) \left[ K_0 + H(\rho) - b^2 l^2 \ln H(\rho) \right]},$$

$$H(\rho) = C_1 \rho + C_0.$$  

(11.16)

The electromagnetic field tensor is given by

$$F^{\mu\tilde{\nu}} = \begin{bmatrix} 0 & b & 0 \\ -b & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad T^{\mu\tilde{\nu}} = \begin{bmatrix} -\frac{1}{8\pi \rho^2} b^2 & 0 & 0 \\ 0 & -\frac{1}{8\pi \rho^2} b^2 & 0 \\ \frac{1}{4\pi} b^2 & 0 & \frac{1}{8\pi \rho^2} b^2 \end{bmatrix}.$$  

Therefore, by means of a $SL(2, R)$–transformation applied to the static electric cyclic symmetric $2+1$ Einstein–Maxwell solution one can generate a unique electromagnetic stationary cyclic symmetric solution in the sense of the structure of the field tensor $F^{\mu\nu}$, which is equal in all respects to the electro–magnetic solution determined by the metric (9.23). It is worthwhile to mention that in 1993 Clement reported a solution belonging to this class, see [19], Eq. (Cl.24).

2. **Clement spinning charged BTZ solution**

The so–called Clement’s spinning charged BTZ solution, derived in [25] deserves special attention. It arises as a result of a $SL(2, R)$–transformation of the electrostatic solution given in terms of the radial coordinate $\rho \to r$. Here the main Clement results are reproduced in a way quite close to the cited work.

Setting $C_1 = 2$, which is equivalent to $t \to t C_1/2$, accomplishing the coordinate transformation $h(r) = C_1 r + C_0 \to r^2$, and introducing the definitions $r_0 = \exp(K_0/(2b^2l^2))$, and
\[ b^2 = 4\pi G Q^2, \] the metric (11.8) becomes

\[
g = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2d\phi^2, \quad F(r) = \frac{K_0}{l^2} + \frac{r^2}{l^2} - b^2 \ln r^2 = \frac{r^2}{l^2} - 8\pi GQ^2 \ln \frac{r}{r_0},
\]

\[
A = 2Q \sqrt{\pi G} \ln \frac{r}{r_0} dt. \quad (11.17)
\]

To establish the range of values of \( r_0 \) allowing the existence of a black hole, let us consider \( F(r) \) in the form

\[
F(r) = r^2 l^2 (1 - \frac{k}{r^2} \ln \frac{r^2}{r_0^2}), \quad k = 4\pi GQ^2 l^2, \quad (11.18)
\]

the factor \( 1 - \frac{k}{r^2} \ln \frac{r^2}{r_0^2} \) vanishes in the set of points \( r_h \) determined through the LambertW function, LambertW(x) exp(LambertW(x)) = x, namely

\[
r_h^2 = -k \text{LambertW}(-\frac{r_0^2}{k}), \quad (11.19)
\]

which is positive for \( r_0^2 = k \exp(-1) \epsilon, \quad 0 < \epsilon \leq 1, \) or explicitly

\[
r_0^2 \leq 4\pi GQ^2 l^2/e. \quad (11.20)
\]

Subjecting the metric (11.17) and the vector potential \( A \) to the transformation at uniform angular velocity

\[
t \rightarrow t - \omega \phi, \quad \phi \rightarrow \phi - \frac{\omega}{l^2} t, \quad \alpha = 1, \quad \beta = -\omega, \quad \gamma = -\frac{\omega}{l^2}, \quad \delta = 1, \quad (11.21)
\]

one arrives at the metric

\[
g = -(F(r) - \frac{\omega^2}{l^4} r^2)dt^2 + 2\omega(F(r) - \frac{r^2}{l^2})dtd\phi + (r^2 - \omega^2 F(r))d\phi^2 + \frac{dr^2}{F(r)},
\]

\[
F(r) = \frac{r^2}{l^2} - 4\pi GQ^2 \ln \frac{r^2}{r_0^2},
\]

\[
A = Q \sqrt{\pi G} \ln \frac{r^2}{r_0^2} (dt - \omega d\phi). \quad (11.22)
\]

One could arrive at this result by using the metric components (11.13) with transformation coefficients from (11.21) and setting \( C_1 = 2, \Delta = 1. \)

By choosing the axial symmetry as fundamental, the metric (11.22) can be brought to the form

\[
g = -r^2 (1 - \omega^2/l^2)^2 \frac{F(r)}{\mathcal{H}(r)} dt^2 + \frac{dr^2}{F(r)^2} + \mathcal{H}(r) (d\phi + W(r) dt)^2,
\]

\[
F = F = \frac{r^2}{l^2} - 4\pi GQ^2 \ln \frac{r^2}{r_0^2}, \quad W = \omega \frac{F^2 - r^2/l^2}{\mathcal{H}} = -4\pi GQ^2 \frac{\omega}{\mathcal{H}} \ln \frac{r^2}{r_0^2},
\]

\[
\mathcal{H} = r^2 - \omega^2 F = r^2 (1 - \frac{\omega^2}{l^2}) + \omega^2 4\pi GQ^2 \ln \frac{r^2}{r_0^2}. \quad (11.23)
\]
The Clement spinning charged BTZ solution is endowed with three parameters $Q$, $r_0$, and $\omega$. It allows for a black hole interpretation.

Alternatively, introducing the scaling transformation $r = l/\bar{l} \times \bar{r}$, the definitions $\bar{l}^2 = l^2 - \omega^2$, $|\omega| < l$, and $\bar{r}_0 = l/l \times r_0$, the proper Clement solution, dropping the bar from $r$, is given as

$$g = -r^2 \frac{F(r)}{H(r)} dt^2 + \frac{dr^2}{F(r)} + H(r)[d\phi + W(r) dt]^2,$$

$$F(r) = \frac{r^2}{l^2} - \frac{l^2 - \omega^2}{l^2} 8\pi G Q^2 \ln \frac{r}{r_0}, \quad W(r) = -\frac{\omega}{H(r)} 8\pi G Q^2 \ln \frac{r}{r_0}, \quad H(r) = r^2 + \omega^2 8\pi G Q^2 \ln \frac{r}{r_0},$$

$$A = 2Q \sqrt{\pi G} \ln \frac{r}{r_0} (dt - \omega d\phi).$$

(11.24)

The corresponding electromagnetic fields are given by

$$F_{\mu\nu} = -\frac{4Q}{r} \sqrt{\pi G} (\delta_{[\mu} \delta_{\nu]} - \omega \delta_{[\mu} \delta_{\nu]}^r),$$

$$T_{\mu}^\nu = -\frac{G Q^2 l^2 + \omega^2}{2l^2} \delta_{\mu}^t \delta_{\nu}^t - \frac{G Q^2 l^2 + \omega^2}{2l^2} \delta_{\mu}^\phi \delta_{\nu}^\phi + \frac{G Q^2 l^2 + \omega^2}{2l^2} \delta_{\mu}^\phi \delta_{\nu}^\phi + \frac{G Q^2 l^2 + \omega^2}{2l^2} \delta_{\mu}^\phi \delta_{\nu}^\phi,$$

(11.25)

The length re-scaling was chosen in such a manner that $H \to r^2$, at spatial infinity and $g(11.24) \to g_{\text{BTZ}}(Q = 0)$. According to Clement: one may formally define mass and angular momentum parameters $M(r_1)$ and $J(r_1)$ by identifying, at a given scale $r = r_1$, the values of the structural functions with the corresponding uncharged BTZ values. Nevertheless, the mass and angular momentum defined in this way occur to be $r_1$–dependent and diverge logarithmical as $r_1 \to \infty$. This solution is a black hole if the condition of the form (11.20),

$$\bar{r}_0^2 \leq 4\pi G Q^2 l^2/e,$$

(11.26)

is fulfilled. It possesses two horizons, at which $F(r)$ vanishes, which are roots of the relation

$$r^2 - \bar{l}^2 8\pi G Q^2 \ln \frac{r}{\bar{r}_0} = 0,$$

(11.27)

which are given by the LambertW function, see Eq. (11.19). The largest root determines the event horizon at $r = r_+ = r_h$, while the inner one is a Cauchy horizon at $r = r_-$, with $r_+ > r_- > \bar{r}_0$. Since the metric function $H$ changes sign for a certain value $r = r_c < \bar{r}_0$, similarly as the rotating uncharged BTZ solution, thus there are closed time–like curves in the region inside the radius $r_c$. It is apparent that the metric and the electromagnetic field are singular at $r = 0$.

3. Kamata–Koikawa limit

It should be pointed out that Clement [19] also reported the so called self–dual solution published later in [23]. By accomplishing the limiting transition $\omega \to \pm l \Rightarrow \bar{l} \to 0$, of the
metric structural functions (11.24), while the other parameters $Q$ and $\bar{r}_0$ remain fixed, one arrives then at the metric (11.24) with structural functions and vector field

$$F = \frac{r^2}{l^2}, \quad W = \mp \frac{l}{H} 8\pi GQ^2 \ln \frac{r}{\bar{r}_0}, \quad H = r^2 + l^2 8\pi GQ^2 \ln \frac{r}{\bar{r}_0},$$

$$A = Q \ln \frac{r}{\bar{r}_0} \left( dt \mp l \, d\phi \right),$$

(11.28)

Notice that this solution does not possess horizon; in the limiting transition $\bar{l} \to 0$ the horizon does not survive since it disappears below $\bar{l} = (4\pi GQ^2)^{1/2}$, as quoted by Clement.

The proper KK representation of this one-parameter solution is achieved by accomplishing the radial transformation and scaling of parameters

$$r^2 = r^2_{0K}, \quad r_0 = (4\pi GQ^2)^{1/2},$$

and the subscripts are self-explanatory. It is worthwhile also to notice that a derivation and analysis of the KK solution has been accomplished in [26] too.

4. Martínez–Teitelboim–Zanelli solution

Martínez, Teitelboim and Zanelli, see [27], reported a generalization of the BTZ black hole spacetime equipped with an electric charge $Q$, the mass $M$ and the angular momentum $J$. The main features of this charged black hole, among others, following the quoted paper, are: the total $M$, $J$ and $Q$ which are boundary terms at infinity, the extreme black hole can be thought of as a particle moving with the speed of light, and the inner horizon of the rotating uncharged black hole is unstable under the perturbation of a small electric charge. According to the quoted reference, this charged electrically black hole is pathological in the sense it exists for arbitrary values of the mass and that there is no upper bound on the electric charge.

The starting point is the electrostatic metric (11.8) given in terms of the polar coordinate $r$,

$$H(r) = C_1 r + C_0 \to r^2, \quad C_1 = 2, \quad K_0/l^2 \to -\bar{M}, \quad b^2 \to \frac{1}{4} \bar{Q}^2,$$

(11.30)

therefore $F(r) = \frac{r^2}{l^2} - \bar{M} - \frac{1}{4} \bar{Q}^2 \ln r^2$. Using the metric components (11.13) with transformation coefficients from the “rotation boost” transformation

$$t \to \frac{1}{\sqrt{1 - \omega^2/l^2}} \left( t - \omega \phi \right), \quad \phi \to \frac{1}{\sqrt{1 - \omega^2/l^2}} \left( \phi - \frac{\omega}{l^2} t \right),$$

(11.31)

one arrives at the metric

$$g = - \left[ \frac{r^2}{l^2} - \frac{1}{1 - \omega^2/l^2} (\bar{M} + \frac{\bar{Q}^2}{4} \ln r^2) \right] dt^2 - 2 \frac{\omega}{1 - \omega^2/l^2} (\bar{M} + \frac{\bar{Q}^2}{4} \ln r^2) dt d\phi + \left[ r^2 + \frac{\omega^2}{1 - \omega^2/l^2} (\bar{M} + \frac{\bar{Q}^2}{4} \ln r^2) \right] d\phi^2 + \frac{dr^2}{r^2/l^2 - \bar{M} - \frac{1}{4} \bar{Q}^2 \ln r^2}.$$

(11.32)
This metric can be brought to the form

\[ g = -r^2 \frac{F(r)}{H(r)} dt^2 + \frac{dr^2}{F(r)} + H(r) \left( d\phi + W(r) dt \right)^2, \]

\[ F(r) = \frac{r^2}{l^2} - \bar{M} - \frac{1}{4}\bar{Q}^2 \ln r^2, \quad W(r) = -\frac{\omega}{1 - \omega^2/l^2} \frac{\bar{M} + \frac{1}{4}\bar{Q}^2 \ln r^2}{H(r)}, \]

\[ H(r) = r^2 + \frac{\omega^2}{1 - \omega^2/l^2} (\bar{M} + \frac{1}{4}\bar{Q}^2 \ln r^2). \] (11.33)

The electromagnetic field tensor is given by

\[ F_{\mu\nu} = \frac{\bar{Q}}{r \sqrt{1 - \omega^2/l^2}} \left( \delta_{[\mu} \delta_{\nu]} - \omega l^2 \delta_{[\mu} \delta_{\nu]} \phi \right). \] (11.34)

The angular momentum, charge, and mass can be evaluated via quasilocal definitions, see Section III D; the presence of logarithmic terms in the structural metric functions yields to divergences at infinity of the energy–momentum quantities. As pointed out by the authors, the divergence in the mass can be handled by enclosing the system in a large circle of radius \( r_0 \) in which will be bound \( \bar{M}(r_0) \)–the energy within \( r_0 \)–and the electrostatic energy outside \( r_0 \) given by \(-Q^2 \ln r_0/2\), thus the total mass (independent of \( r_0 \) and finite) is given by \( \bar{M} = M(r_0) - Q^2 \ln r_0/2 \).

C. Transformed magnetostatic solution \( a \neq 0 \) solution

To determine the stationary rotating generalization of the magnetostatic metric (4.24)

\[ g = -\frac{F}{H} dt^2 + \frac{1}{F} dr^2 + H d\phi^2, \]

\[ H(r) = \frac{4}{C_1^2 l^2} \left[ K_0 + h(r) + a^2 l^2 \ln h(r) \right], \quad F(r) = H(r) h(r), \quad h(r) := C_1 r + C_0, \]

one subjects it to \( SL(2, R) \)-transformations

\[ t = \frac{\alpha}{\sqrt{\Delta}} \tilde{t} + \frac{\beta}{\sqrt{\Delta}} \tilde{\phi}, \quad \phi = \frac{\gamma}{\sqrt{\Delta}} \tilde{t} + \frac{\delta}{\sqrt{\Delta}} \tilde{\phi}, \quad \Delta = \alpha \delta - \gamma \beta, \]

giving rise to the rotated new metric in the form

\[ g_{\tilde{\mu}\tilde{\nu}} = \begin{bmatrix} -\frac{\alpha^2 F}{\Delta H} + \frac{\gamma^2}{\Delta} H & 0 & -\frac{\alpha \beta F}{\Delta H} + \frac{\gamma \delta}{\Delta} H \\ 0 & \frac{1}{\tilde{F}} & 0 \\ -\frac{\alpha \beta F}{\Delta H} + \frac{\gamma \delta}{\Delta} H & 0 & -\frac{\beta^2 F}{\Delta H} + \frac{\delta^2}{\Delta} H \end{bmatrix}, \]

which is accompanied with the electromagnetic field tensor

\[ F^{\tau\sigma} = \begin{bmatrix} 0 & a \frac{\beta}{\sqrt{\Delta}} & 0 \\ -a \frac{\beta}{\sqrt{\Delta}} & 0 & a \frac{\alpha}{\sqrt{\Delta}} \\ 0 & -a \frac{\alpha}{\sqrt{\Delta}} & 0 \end{bmatrix}. \] (11.35)
The corresponding Maxwell energy–momentum tensor becomes

\[
T_{\tilde{\mu}\tilde{\nu}} = \begin{bmatrix}
-\frac{(\alpha \beta + \gamma) a^2 H}{8\pi \Delta} & 0 & \frac{\gamma a a^2 H}{4\pi \Delta} \\
0 & \frac{a^2 H}{8\pi F} & 0 \\
-\frac{\delta \beta a^2 H}{4\pi \Delta} & 0 & \frac{(\alpha \beta + \gamma) a^2 H}{8\pi \Delta} \\
\end{bmatrix}. \tag{11.36}
\]

Explicitly, the non-zero metric components are

\[
g_{\tilde{t}\tilde{t}} = -\frac{\alpha^2}{\Delta} h(r) + \frac{\gamma^2}{\Delta} h(r) g_{rr},
g_{\tilde{t}\tilde{\phi}} = -\frac{\alpha \beta}{\Delta} h(r) + \frac{\delta \gamma}{\Delta} h(r) g_{rr},
g_{\tilde{\phi}\tilde{\phi}} = -\frac{\beta^2}{\Delta} h(r) + \frac{\delta^2}{\Delta} h(r) g_{rr},
g_{rr} = \frac{1}{4 h(r) \left[K_0 + h(r) + a^2 l^2 \ln h(r)\right]},
\]

\[
h(r) = C_1 r + C_0. \tag{11.37}
\]

1. Stationary magneto–electric solution

In particular, for the \( SL(2, R) \)–transformation

\[
t = \tilde{t}, \quad \phi = -\frac{\omega}{l^2} \tilde{t} + \tilde{\phi}, \quad \alpha = 1, \quad \beta = 0, \quad \gamma = -\frac{\omega}{l^2}, \quad \delta = 1,
\]

one obtains a new solution with metric components

\[
g_{\tilde{t}\tilde{t}} = -h(r) + \frac{\omega^2}{l^4} h(r) g_{rr},
g_{\tilde{t}\tilde{\phi}} = -\frac{\omega}{l^2} \frac{1}{h(r) g_{rr}},
g_{\tilde{\phi}\tilde{\phi}} = \frac{1}{h(r) g_{rr}},
g_{rr} = \frac{1}{4 h(r) \left[K_0 + h(r) + a^2 l^2 \ln h(r)\right]},
\]

\[
h(r) = C_1 r + C_0, \tag{11.39}
\]

which is equal to the constant \( W = \omega \) stationary magneto–electric solution (9.13). Therefore, by means of a \( SL(2, R) \)–transformation applied to the magnetostatic cyclic symmetric \( (2 + 1) \) Einstein–Maxwell solution one can generate a unique electromagnetic stationary cyclic symmetric solution in the sense of the structure of the field tensor \( F^{\mu\nu} \).

Clement reported a field belonging to this class of solutions, see [19], Eq. (Cl.23).

2. Dias–Lemos magnetic BTZ–solution counterpart

Dias and Lemos [28] published a rotating magnetic solution in 2+1 gravity–the magnetic counterpart of the spinning charged BTZ solution, i.e., a point source generating a magnetic field. Also, it was established that both the static and rotating magnetic solutions possess negative mass and that there is an upper bound for the intensity of the magnetic field source and for the value of the angular momentum.
A simple representation of this solution can be achieved from our transformed magnetic metric (11.37) by setting

\[ t = \tilde{t}, \phi = -\frac{\omega}{l^2} t + \tilde{\phi}, \alpha = 1, \beta = -\omega, \gamma = -\frac{\omega}{l^2}, \delta = 1, \Delta = 1, \]

\[ C_1 = 2, h(r) = C_1 r + C_0 \rightarrow r^2 \]

obtaining

\[ g = -(r^2 - \frac{\omega^2}{l^2} F) dt^2 + (F - \omega^2 r^2) d\phi^2 - 2\omega \frac{F}{l^2} - r^2 dt d\phi + \frac{dr^2}{F(r)}, \]

\[ F = \frac{K_0}{l^2} + \frac{r^2}{l^2} + a^2 \ln r^2. \] (11.40)

The proper Dias–Lemos representation uses a more involved definitions of the transformed \( r \)-coordinate and parameterizations of the \( SL(2, R) \)-transformations:

\[ h(r) = C_1 r + C_0 \rightarrow (\rho^2 + r_+^2 - ml^2)/l^2, C_1 = 2/l^2, \chi^2 = a^2 l^4, \]

\[ t = \sqrt{1 + \omega^2} t - l \omega \tilde{\phi}, \]

\[ \phi = -\frac{\omega}{l} t + \sqrt{1 + \omega^2} \tilde{\phi}, \]

dropping tildes, one has

\[ g = - \left[ h - \frac{\omega^2}{l^2} (m l^2 + \chi^2 \ln |h|) \right] dt^2 - 2 \frac{\omega}{l} \sqrt{\omega^2 + 1} \left[ m l^2 + \chi^2 \ln |h| \right] dt d\phi \]

\[ + [h l^2 + (\omega^2 + 1) (m l^2 + \chi^2 \ln |h|)] d\phi^2 + \frac{l^2 \rho^2 d\rho^2}{(\rho^2 + r_+^2 - ml^2)[(\rho^2 + r_+^2 + \chi^2 \ln |h|]}, \]

\[ h = (\rho^2 + r_+^2 - ml^2)/l^2, \] (11.41)

with vector potential

\[ A = \frac{1}{2} \chi \ln |h(\rho)|[-\frac{\omega}{l} dt + \sqrt{1 + \omega^2} d\Phi]. \] (11.42)

Notice that in the above–mentioned representation, the equation

\[ r_+^2 + \chi^2 \ln |(\rho^2 + r_+^2)/l^2 - m| = 0, \] (11.43)

used in the Eq. (3.2) of [28] was not used here. According to these authors, this rotating magnetic spacetime is null and time–like geodesically complete and as such horizonless.
D. Transformed hybrid static $c \neq 0$ solution

To determine the stationary rotating generalization of the hybrid static solution (4.43)

\[
g = -\frac{F}{H} dt^2 + \frac{1}{F} dr^2 + H d\phi^2,
\]

\[
F = \frac{4}{l^2} (r - r_1)(r - r_2), \quad H = \frac{2}{l} K_0^2 (r - r_1)^{(1 + \sqrt{\alpha_0})/2}(r - r_2)^{(1 + \sqrt{\alpha_0})/2},
\]

\[
A = \frac{c}{2} (td\phi - \phi dt),
\]

one subjects it to the $SL(2, R)$–transformations

\[
t = \frac{\alpha}{\sqrt{\Delta}} \tilde{t} + \frac{\beta}{\sqrt{\Delta}} \tilde{\phi}, \quad \phi = \frac{\gamma}{\sqrt{\Delta}} \tilde{t} + \frac{\delta}{\sqrt{\Delta}} \tilde{\phi}, \quad \Delta = \alpha \delta - \gamma \beta,
\]

arriving at the stationary rotating solution, omitting tildes, given by

\[
g = -\left(\frac{\alpha^2}{\Delta} H - \frac{\gamma^2}{\Delta} H\right) dt^2 + 2\left(-\frac{\alpha \beta}{\Delta} H + \frac{\gamma \delta}{\Delta} H\right) dt d\phi + \left(-\frac{\beta^2}{\Delta} H + \frac{\delta^2}{\Delta} H\right) d\phi^2 + \frac{1}{F} dr^2,
\]

\[
F = \frac{4}{l^2} (r - r_1)(r - r_2), \quad H = \frac{2}{l} K_0^2 (r - r_1)^{(1 + \sqrt{\alpha_0})/2}(r - r_2)^{(1 + \sqrt{\alpha_0})/2},
\]

\[
A = \frac{c}{2} (td\phi - \phi dt).
\]

In the seed hybrid static metric has been used $\alpha_0$ instead of the original $\alpha$ to avoid confusion with the parameters appearing in the $SL(2, R)$–transformations.

Therefore the transformed electromagnetic tensors stay unchanged

\[
F_{\mu \nu} = 2c \delta_{[\mu} \delta_{\nu]},
\]

\[
T_{\mu}^{\nu} = \frac{1}{8\pi} \frac{c^2}{F} (-\delta_{[\mu}^{\tau} \delta_{\nu]}^{\nu} + \delta_{\mu}^{\tau} \delta_{\nu}^{\nu} - \delta_{\mu}^{\phi} \delta_{\phi}^{\nu}).
\]

Hence, by means of a $SL(2, R)$–transformation applied to the hybrid static cyclic symmetric 2+1 Einstein–Maxwell solution, one generates a family of hybrid, stationary cyclic symmetric solutions with structurally unique field tensors $F_{\mu \nu}$ and $T_{\mu}^{\nu}$.

1. Cataldo azimuthal rotating solution

Applying the above–mentioned $SL(2, R)$ transformation to the Cataldo azimuthal electrostatic solution (4.46) one arrives at a new stationary solution having the static BTZ solution as a limit, namely

\[
g = -\left(\frac{\alpha^2}{\Delta} H(-) - \frac{\gamma^2}{\Delta} H(+)\right) dt^2 + 2\left(-\frac{\alpha \beta}{\Delta} H(-) + \frac{\gamma \delta}{\Delta} H(+)\right) dt d\phi + \left(-\frac{\beta^2}{\Delta} H(-) + \frac{\delta^2}{\Delta} H(+)\right) d\phi^2 + \left(\frac{\rho^2}{l^2} - M\right)^{-1} dr^2,
\]

\[
H(+) := \rho^{1+\sqrt{\alpha_0}} \left(\frac{\rho^2}{l^2} - M\right)^{(1-\sqrt{\alpha_0})/2}, \quad H(-) := \rho^{1-\sqrt{\alpha_0}} \left(\frac{\rho^2}{l^2} - M\right)^{(1+\sqrt{\alpha_0})/2}.
\]

(11.45)
The electromagnetic tensors are given by

\[ F_{\mu\nu} = \frac{M\sqrt{1 - \alpha_0}}{2} \delta_{[\mu}^t \delta_{\nu]} \phi, \]

\[ T_{\mu}^\nu = \frac{M^2}{32\pi \rho^2 l^2} \frac{(1 - \alpha_0)}{\rho^2/l^2 - M} (-\delta_{\mu}^t \delta_t^\nu + \delta_{\mu}^\rho \delta_{\rho}^\nu - \delta_{\mu}^\phi \delta_{\phi}^\nu). \]  

(11.46)

In particular, one could choose the rotation boost transformation

\[ t \rightarrow \frac{1}{\sqrt{1 - \omega^2/l^2}} (t - \omega \phi), \quad \phi \rightarrow \frac{1}{\sqrt{1 - \omega^2/l^2}} \left( \phi - \frac{\omega}{l^2} t \right), \]

where the parameter \( \omega \) can be related to the angular momentum constant.

**XII. CONCLUDING REMARKS**

In the framework of the (2+1)-dimensional Einstein–Maxwell theory with cosmological constant different families of exact solutions for cyclic symmetric stationary (static) metrics have been derived. For the static classes and hybrid, static and stationary as well, families their uniqueness is proven by the used integration procedure. The completeness and relationship of all uniform electromagnetic  \( F_{\mu\nu;\alpha} = 0 \), and constant invariant  \( F_{\mu\nu} F^{\mu\nu} = 2\gamma \) solutions is achieved. The uniqueness of the stationary families of solutions has been partially established; various specific branches of solutions in the general case are determined via a straightforward integration. In this systematic approach all known electromagnetic stationary cyclic symmetric solutions are properly identified. It seems to be a rule that electrically charged solutions allow for a black interpretation while for the magnetic classes such black hole feature seems to be absent; a research on this respect is undertaken and will be published elsewhere.

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[1] M. Bañados, C. Teitelboim and J. Zanelli, The black hole in three–dimensional spacetime, *Phys. Rev. Lett.* 69, (1992) 1849 [hep-th/9204099].

[2] M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli, Geometry of the 2+1 black hole, *Phys. Rev.* D 48, (1993) 1506 [gr-qc/9302012].

[3] D. Cangemi, M. Leblanc, R.B. Mann, Gauge formulation of the spinning black hole in (2+1)-dimensional anti–de Sitter space, *Phys. Rev.* D 48, (1993) 3606.

[4] S. Carlip, The (2+1)-dimensional black hole, *Class. Quantum Grav.* 12, (1995) 2853.
[5] A. Ayon–Beato, M. Cataldo, A.A. Garcia, Electromagnetic fields in stationary cyclic symmetric 2+1 gravity, in Proceedings of the 10th. PASCOS04 and Pran Nath Fest, Eds.G. Alverson, E. Barberis, P. Nath, and M.T. Vaughn (World Scientific, 2005), p. 554–558.

[6] M. Cataldo and P. Salgado, Static Einstein–Maxwell solutions in 2+1 dimensions, Phys. Rev. D 54, (1996) 2971.

[7] J.D. Brown and J.W. York, Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D 47, (1993) 1407.

[8] J.D. Brown, J. Creighton and R.B. Mann, Temperature, energy and heat capacity of asymptotically anti–de–Sitter black holes, Phys. Rev. D 50, (1994) 6394.

[9] J.R. Gott, J. Simon, and M. Alpert, General relativity in a (2+1) –dimensional space–time: An electrically charged solution, Gen. Rel. Grav. 18, (1986) 1019.

[10] J.R. Gott, and M. Alpert, Gen. Rel. Grav. 16, (1984) 243.

[11] S. Deser and P.O. Mazur, Static solutions in D=3 Einstein–Maxwell theory, Class. Quantum Grav. 2, (1985) L51.

[12] M.A. Melvin, Exterior solutions for electric and magnetic stars in 2+1 dimensions, Class. Quantum Grav. 3, (1986) 117.

[13] I.I. Kogan, About some exact solutions for 2+1 gravity coupled to gauge fields, Mod. Phys. Lett. A 7,(1992) 2341.

[14] P. Peldan, Unification of gravity and Yang–Mills theory in (2+1) dimensions, Nucl. Phys. B395, (1993) 239.

[15] E.W. Hirschmann and D.L. Welch, Magnetic solutions to 2+1 gravity, Phys. Rev. D 56, (1996) 5596.

[16] M. Cataldo, J. Crisostomo, S. del Campo and P. Salgado, On magnetic solution to 2+1 Einstein–Maxwell gravity, Phys. Lett. B 584, (2004) 123.

[17] J.D. Barrow, A.B. Burd and D. Lancaster, Three–dimensional classical spacetimes, Class. Quantum Grav. 3, (1986) 551.

[18] M. Cataldo, Azimuthal electric field in a static rotationally symmetric (2+1)–dimensional spacetime, Phys. Lett. B 529, (2002) 143.

[19] G. Clement, Classical solutions in three–dimensional Einstein–Maxwell cosmological gravity, Class. Quantum Grav. 10, (1993) L49-L54.

[20] J. Matyjasek and O.B. Zaslavskii, Extremal limit for charged and rotating 2+1-dimensional black holes and Bertotti–Robinson geometry, Class. Quantum Grav. 21, (2004) 4283.

[21] B.Bertotti, Uniform Electromagnetic Field in the Theory of General Relativity, Phys. Rev. 116, (1959) 1331.

[22] I. Robinson, A Solution of the Maxwell–Einstein Equations, Bull. Acad. Pol. Sci. 7, (1959) 351.

[23] M. Kamata and T. Koikawa, The electrically charged BTZ black hole with self (anti-self) dual Maxwell field, Phys. Lett. B 353, 196 (1995); 2+1 dimensional charged black hole with (anti–) self dual Maxwell field, Phys. Lett. B 391, (1997) 196.

[24] K.C.K. Chan, Comment on the calculation of the angular momentum and mass for the (anti–) self–dual charged spinning BTZ black hole, Phys. Lett. B 373, (1996) 296.

[25] G. Clement, Spinning charged BTZ black holes and self–dual particle–like solutions, Phys. Lett. B 367, (1996) 70 [gr–qc/9510025].

[26] M. Cataldo and P. Salgado, Three dimensional extreme black hole with self (anti-self) dual Maxwell field, Phys. Lett. B 448, (1999) 20.

[27] C. Martinez, C. Teitelboim and J. Zanelli, Charged rotating black hole in three spacetime
dimensions, *Phys. Rev. D* **61**, (2000) 104013 [hep-th/9912259].

[28] O.J.C Dias and J.P.S. Lemos, Rotating magnetic solution in three dimensional Einstein gravity, *JHEP* **01(2002), 006** (2002).