ON SOME FUNCTIONAL GENERALIZATIONS OF THE REGULARITY OF TOPOLOGICAL SPACES

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Dedicated to the 60th birthday of M. M. Zarichnyi

Abstract. We introduce and study some generalizations of regular spaces, which were motivated by studying continuity properties of functions between (regular) topological spaces. In particular, we prove that a first-countable Hausdorff topological space is regular if and only if it does not contain a topological copy of the Gutik hedgehog.

In this paper we introduce and study some generalizations of regular spaces, which were motivated by continuity properties of functions between (regular) topological spaces. First we introduce the necessary definitions.

A subset $U$ of a topological space $X$ is called $\theta$-open if each point $x \in U$ has a neighborhood $O_x \subset X$ such that $\overline{O_x} \subset U$. It is clear that each $\theta$-open set is open. Moreover, a topological space is regular if and only if each open subset of $X$ is $\theta$-open.

Lemma 1. Let $U$ be a $\theta$-open subset of a topological space $X$ and $V$ be a $\theta$-open subset of $U$. Then $V$ is $\theta$-open in $X$.

Proof. For each point $x \in V$, the $\theta$-openness of $U$ in $X$ yields an open neighborhood $U_x \subset X$ such that $\overline{U_x} \subset U$. The $\theta$-openness of $V$ in $U$ yields an open neighborhood $V_x \subset U$ such that $\overline{V_x} \subset U$. Now consider the open neighborhood $O_x = V_x \cap U_x$ and observe that $\overline{O_x} \subset \overline{V_x} \cap \overline{U_x} \subset \overline{V_x} \cap U = \overline{U} \subset V$. \hfill \Box

For a function $f : X \to Y$ between topological spaces by $C(f)$ we denote the set of continuity points of $f$.

Definition 2. A function $f : X \to Y$ between topological spaces is called

- scatteredly continuous if for any non-empty subset $A \subset X$ the set $C(f|A)$ is not empty;
- weakly discontinuous if if for any non-empty subset $A \subset X$ the set $C(f|A)$ has non-empty interior in $A$;
- $\theta$-weakly discontinuous if if for any non-empty subset $A \subset X$ the set $C(f|A)$ contains a non-empty $\theta$-open subset of $A$.

So, we have the implications:

$\theta$-weakly discontinuous $\Rightarrow$ weakly discontinuous $\Rightarrow$ scatteredly continuous.

The first and last implications can be reversed for functions with regular domain and range, respectively.

Theorem 3 (trivial). A function $f : X \to Y$ from a regular topological space $X$ to a topological space $Y$ is weakly discontinuous if and only if it is $\theta$-weakly discontinuous.

Theorem 4 (Bokalo). A function $f : X \to Y$ from a topological space $X$ to a regular space $Y$ is scatteredly continuous if and only if it is weakly discontinuous.

A proof the Theorem can be found in [1], [8]. More information on various sorts of generalized continuity can be found in [2]–[12].

Motivated by Theorems 3 and 4, let us introduce the following definition.

Key words and phrases. regular space, quasi-regular space, $sw$-regular space, $w\theta$-regular space, $\theta$-weakly regular space, weakly regular space, locally regular space, the Gutik hedgehog.
**Theorem 6.** A topological space $X$ is $w\theta$-regular if and only if for each subspace $A \subset X$, each non-empty open subset $U \subset A$ contains a non-empty $\theta$-open subset of $A$.

**Proof.** To prove the “if” part, assume that for each subspace $A \subset X$, every non-empty open subset $U \subset A$ contains a non-empty $\theta$-open subset of $A$. To show that the space $X$ is $w\theta$-regular, fix any weakly discontinuous map $f : X \to Y$. To show that $f$ is $\theta$-weakly discontinuous, take any non-empty subset $A \subset X$. Since $f$ is weakly discontinuous, there exists a non-empty open subset $U \subset A$ such that $f|U$ is continuous. By our assumption, $U$ contains a $\theta$-open subspace $V$ of $A$. Since $f|V$ is continuous, the function $f$ is $\theta$-weakly discontinuous.

Now we prove the “only if” part. Assume that the space $X$ is $w\theta$-regular. Given any subset $A \subset X$ and a non-empty open subset $U \subset A$, consider the closures $A$ and $A \setminus U$ of the sets $A$ and $A \setminus U$ in $X$. Observe that $\tilde{U} := A \setminus A \setminus U$ is an open set in $A$ with $\tilde{U} \cap A = U$ and $\tilde{U} \subset U$. Consider the topological sum $Y = \tilde{U} \oplus (X \setminus \tilde{U})$ and observe that the identity map $f : X \to Y$ is weakly discontinuous. The $w\theta$-regularity of the space $X$ ensures that $f$ is $\theta$-weakly discontinuous. Consequently, the closure $\tilde{U}$ of $U$ in $A$ contains a non-empty $\theta$-open subset $V \subset \tilde{U}$ such that $f|V$ is continuous. The continuity of $f|V$ ensures that $V \subset \tilde{U}$. We claim that $V$ is $\theta$-open in $\tilde{U}$. Since $V$ is $\theta$-open in $\tilde{U}$, for any $x \in V$ there exists a neighborhood $O_x$ of $x$ such that $O_x$ is open in $\tilde{U}$ and $O_x \subset \tilde{U} \subset V \subset \tilde{U}$. So, $O_x$ is open in $\tilde{U}$ and hence is open in $A$.

Taking into account that $V$ is a non-empty $\theta$-open subset of $\tilde{U}$, we conclude that $V \cap A \subset \tilde{U} \cap A = U$ is a non-empty $\theta$-open subset of $A$, contained in the set $U$. □

**Problem 7.** Characterize topological spaces, which are $sw$-regular.

We shall prove that $sw$-regular and $w\theta$-regular spaces are preserved by $\theta$-weak homeomorphisms.

**Definition 8.** A bijective function $f : X \to Y$ between topological spaces is called a ($\theta$-)weak homeomorphism if both functions $f$ and $f^{-1}$ are ($\theta$-)weakly discontinuous.

We shall need the following proposition describing the continuity properties of compositions of scatteredly continuous, weakly discontinuous and $\theta$-weakly discontinuous functions.

**Proposition 9.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions between topological spaces.

1. If $f, g$ are weakly discontinuous, then $g \circ f$ is weakly discontinuous.
2. If $f, g$ are $\theta$-weakly discontinuous, then $g \circ f$ is $\theta$-weakly discontinuous.
3. If $f$ is weakly discontinuous and $g$ is scatteredly continuous, then $g \circ f$ is scatteredly continuous.
4. If $f$ is scatteredly continuous and $g$ is $\theta$-weakly discontinuous, then $g \circ f$ is scatteredly continuous.

**Proof.** 1. Assume that $f, g$ are weakly discontinuous. To prove that $g \circ f$ is weakly discontinuous, we need to show that for any non-empty subset $A \subset X$ the set $C(g \circ f|A)$ has non-empty interior in $A$. By the weak discontinuity of $f$, the set $C(f|A)$ contains a non-empty open subset $U \subset A$. By the weak discontinuity of $g$, the set $C(g|f(U))$ contains a non-empty open set $V \subset f(U)$. By the continuity of $f|U$, the set $W = (f|U)^{-1}(V)$ is open in $U$ and hence open in $A$. Since $f(W) \subset V$, the continuity of the restrictions $f|W$ and $g|V$ implies the continuity of the restriction $g \circ f|W$. So, $W \subset C(g \circ f|A)$.

2. Assume that $f, g$ are $\theta$-weakly discontinuous. To prove that $g \circ f$ is $\theta$-weakly discontinuous, we need to show that for any non-empty subset $A \subset X$ the set $C(g \circ f|A)$ contains a non-empty $\theta$-open subset $W \subset A$. By the $\theta$-weak discontinuity of $f$, the set $C(f|A)$ contains a non-empty
A topological space $X$ is called $(\theta)$-regular if it is $(\theta)$-weakly homeomorphic to a regular topological space.

**Definition 13.**
Example 14. Consider the real line $\mathbb{R}$ endowed with the second-countable topology $\tau$ generated by the subbase
$$\{\mathbb{Q}\} \cup \{(-\infty, a), (a, +\infty) : a \in \mathbb{R}\}.$$It can be shown that the topological space $X = (\mathbb{R}, \tau)$ is weakly regular. The identity map $\mathbb{R} \to X$ is scatteredly continuous but not weakly discontinuous, which implies that the space $X$ is not $sw$-regular. On the other hand, the function $\chi : X \to \{0, 1\} \subset \mathbb{R}$ defined by
$$\chi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}; \end{cases}$$is weakly discontinuous but not $\theta$-weakly discontinuous, witnessing that the space $X$ is not $w\theta$-regular. Theorem 15 implies that the space $X$ is not $\theta$-weakly regular.

Theorem 14. A topological space $X$ is $\theta$-weakly regular if and only if each non-empty closed subspace $A \subset X$ contains a non-empty $\theta$-open regular subspace.

Proof. First assume that $X$ is $\theta$-weakly regular and fix any $\theta$-weak homeomorphism $h : X \to Y$ to a regular topological space $Y$.

Given any subspace $A \subset X$, we need to find a non-empty $\theta$-open regular subspace $W \subset A$. Since the map $h$ is $\theta$-weakly discontinuous, there exists a non-empty $\theta$-open subset $U \subset A$ such that $h|U$ is continuous. Since $h^{-1}$ is $\theta$-weakly discontinuous, the non-empty subspace $h(U)$ of $Y$ contains a non-empty $\theta$-open subspace $V$ such that $h^{-1}|V$ is continuous. The continuity of the map $h|U$ implies that the set $W := (h|U)^{-1}(V)$ is $\theta$-open in $U$ and hence $\theta$-open in $A$ (by Lemma 4). The continuity of maps $h|W$ and $h^{-1}|h(W)$ implies that $h|W : W \to h(W)$ is a homeomorphism. The regularity of the topological space $Y$ implies the regularity of its subspace $h(W)$ and the regularity of the topological copy $W$ of $h(W)$. Therefore, $W$ is a required non-empty $\theta$-open regular subspace of $A$.

Now assume that each non-empty closed subspace $A \subset X$ contains a non-empty $\theta$-open regular subspace. Let $A^\theta$ be the union of all $\theta$-open regular subspaces of $A$. It is clear that the subspace $A^\theta$ is $\theta$-open in $A$ and regular. Let $X_0 := X$ and $X_\alpha = \bigcap_{\beta < \alpha} X_\beta \setminus X_\beta^\theta$ for each ordinal $\alpha$. It follows that for any ordinal $\alpha$ with $X_\alpha \neq \emptyset$ the set $X_{\alpha+1} = X_\alpha \setminus X_\alpha^\theta$ is closed in $X_\alpha$ and has non-empty complement $X_{\alpha+1} \setminus X_\alpha = X_\alpha^\theta$. Consequently, $X_\gamma = \emptyset$ for some $\gamma$ and hence $X = \bigcup_{\alpha < \gamma} X_\alpha^\theta$.

Let $Y := \bigoplus_{\alpha < \gamma} X_\alpha^\theta$ be the topological sum of the regular spaces $X_\alpha^\theta$ for $\alpha < \gamma$. It is clear that the space $Y$ is regular and the identity map $i : Y \to X$ is continuous. We claim that the identity map $i^{-1} : X \to Y$ is $\theta$-weakly discontinuous. Given any non-empty subset $A \subset X$ find the smallest ordinal $\beta \leq \gamma$ such that $A \not\subset X_\beta$. Then $A \subset X_\alpha$ for all $\alpha < \beta$, which implies that $\beta$ is a successor ordinal. Write $\beta = \alpha + 1$ for some $\alpha$ and observe that $U = A \cap X_\alpha = A \cap (X_\alpha \setminus X_{\alpha+1})$ is a non-empty $\theta$-open subspace of $A$ such that $i^{-1}|U$ is continuous. This means that $i^{-1}$ is $\theta$-weakly discontinuous and $i : X \to Y$ is a $\theta$-weak homeomorphism of $X$ onto the regular space $Y$.

By analogy we can prove a characterization of weakly regular spaces.

Theorem 17. A topological space $X$ is weakly regular if and only if each (closed) subspace $A \subset X$ contains a non-empty open regular subspace.

A topological space $X$ is called

- quasi-regular if each non-empty open subset of $X$ contains the closure of some non-empty open set in $X$;
- hereditarily quasi-regular if each subspace of $X$ is quasi-regular.

Theorem 15 implies

Corollary 18. Each $w\theta$-regular space is hereditarily quasi-regular.

Theorems 16 and 15 imply:
Corollary 19. Each scattered $T_1$-space is $\theta$-weakly regular and hence is $sw$-regular and $w\theta$-regular.

The $T_1$-requirement in Corollary 19 is essential as shown by the following example.

Example 20. Consider the connected doubleton $D = \{0, 1\}$ endowed with the topology $\{\emptyset, \{0\}, \{0, 1\}\}$. It is clear that $D$ is a scattered space. The function $f : \mathbb{R} \to D$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

is scatteredly continuous but not weakly discontinuous as $C(f) = \mathbb{Q}$ has empty interior in $\mathbb{R}$. Consequently, $D$ is not $sw$-regular and hence not $\theta$-weakly regular.

The identity map $i : D \to \{0, 1\}$ to the discrete doubleton is weakly discontinuous but not $\theta$-weakly discontinuous. This means that $D$ is not $w\theta$-regular.

Definition 21. A topological space $X$ is locally regular if $X$ admits an open cover by regular subspaces.

Theorem 22. Each locally regular topological space $Y$ is $sw$-regular.

Proof. Given a scatteredly continuous map $f : X \to Y$ and a non-empty subset $A \subset X$, we should show that the set $C(f|A)$ has non-empty interior in $A$.

By the scattered continuity of $f$, the map $f|A$ has a continuity point $a \in A$. By our assumption, the point $f(a)$ is contained in an open regular subspace $U \subset Y$. By the continuity of $f$ at $a$, there exists an open neighborhood $O_a \subset A$ of $a$ such that $f(O_a) \subset U$. Since $U$ is regular, the set $C(f|O_a)$ has non-empty interior in $O_a$ and then the set $C(f) \supset C(f|O_a)$ has non-empty interior in $A$. \hfill $\Box$

Example 23. On the real line $\mathbb{R}$ consider the Euclidean topology $\tau_E$ and the topology $\tau$ generated by the subbase

$$\tau_E \cup \{W_n : n \in \omega\} \quad \text{where} \quad W_n = \mathbb{R} \setminus \left\{ \frac{1}{2^m} : m \in \omega, k \geq n \right\}.$$

It can be shown that the space $X = (\mathbb{R}, \tau)$ is $\theta$-weakly regular but not locally regular.

A topological space $X$ is called regular at a point $x \in X$ if any neighborhood of $x$ in $X$ contains a closed neighborhood of $x$ in $X$. A topological space $X$ is called nowhere regular if $X$ is not regular at each point $x \in X$.

Example 24. Let $\tau_E$ be the Euclidean topology of the real line and $\tau$ be the topology generated by the subbase

$$\{(U \cap \mathbb{Q}) \cup \{x\} : x \in U \in \tau_E\}.$$

The space $(\mathbb{R}, \tau)$ is locally regular and hence $sw$-regular. On the other hand, it is nowhere regular, not quasi-regular and not $w\theta$-regular.

Now, we describe the smallest non-regular first-countable Hausdorff space, which is called the Gutik hedgehog. The Gutik hedgehog is the space $\mathbb{N}^{\leq 2} = \mathbb{N}^0 \cup \mathbb{N}^1 \cup \mathbb{N}^2$ endowed with the topology generated by the base

$$\{(x) : x \in \mathbb{N}^2\} \cup \{U_n : n \in \mathbb{N}\} \cup \{U_{n,m} : n, m \in \mathbb{N}\}$$

where

$$U_n = \emptyset \cup \{(i, j) \in \mathbb{N}^2 : i \geq n\} \quad \text{and} \quad U_{n,m} = \{(n)\} \cup \{(n, j) : j \geq m\} \subset \mathbb{N}^1 \cup \mathbb{N}^2$$

for $n, m \in \omega$. Here $\emptyset$ is the unique element of the set $\mathbb{N}^0$. For the first time, the Gutik hedgehog has appeared in the paper [9] of Gutik and Pavlyk.

The following properties of the Gutik hedgehog can be derived from its definition.

Lemma 25. The Gutik hedgehog is first-countable, scattered and locally regular, but not regular.

Moreover, the following theorem shows that the Gutik hedgehog is the smallest space among non-regular first-countable spaces.
Theorem 26. A first-countable Hausdorff space $X$ is not regular if and only if $X$ contains a topological copy of the Gutik hedgehog.

Proof. The “if” part follows from the non-regularity of the Gutik hedgehog.

To prove the “only if” part, assume that a first-countable Hausdorff space $X$ is not regular at some point $x$. Then we can find a neighborhood $U_0 \subset X$ of $x$ that does not contain the closure of any neighborhood $V$ of $x$. Fix a neighborhood base $\{U_n\}_{n \in \mathbb{N}}$ at $x$ such that $U_n \subset U_{n-1}$ for all $n \in \mathbb{N}$. Let $k_1 = 0$, choose any point $x_1 \in \overline{U_{k_1}} \setminus U_0$, and using the Hausdorff property of $X$, find a neighborhood $V_1$ of $x_1$ such that $V_1 \cap U_{k_2} = \emptyset$ for some number $k_2 > k_1$.

Proceeding by induction, we can choose an increasing number sequence $(k_n)_{n \in \omega}$ and a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $X$ such that for every $n \in \mathbb{N}$, the point $x_n$ belongs to $U_{k_n} \setminus U_0$ and has an open neighborhood $V_n$, disjoint with the neighborhood $U_{k_n+1}$ of $x$. Observe that for every $i < n$, we have $x_n \in \overline{U_{k_n}} \subset \overline{U_{k_i}} \subset X \setminus V_i \subset X \setminus \{x_i\}$, which implies that $x_n \notin \{x_i\}_{i<n}$. Replacing $V_n$ by a smaller neighborhood of $x_n$, we can assume that its closure $\overline{V_n}$ does not contain the points $x_1, \ldots, x_{n-1}$.

Since $X$ is first-countable, for every $n \in \mathbb{N}$ we can choose a sequence $\{x_{n,i}\}_{i \in \mathbb{N}}$ of pairwise distinct points in $V_n \cap U_{k_n}$ that converge to $x_n$. Observe that for any $n < m$ the sets $\overline{U_{k_m}} \supset \overline{U_{k_n}} \supset \{x_{n,i}\}_{i \in \mathbb{N}}$ and $V_n \supset \{x_{n,i}\}_{i \in \mathbb{N}}$ are disjoint, which implies that the points $x_{n,i}$, $n, i \in \mathbb{N}$, are pairwise disjoint. Consider the subspace $\tilde{H} := \{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{n,i} : n, i \in \mathbb{N}\}$ and observe that the map $h : H \rightarrow \tilde{H}$, defined by $h(\emptyset) = x$, $h(n) = x_n$ and $h(n, m) = x_{n,m}$ for $n, m \in \mathbb{N}$, is a homeomorphism.

Finally let us draw a diagram of all provable implications between various regularity properties.

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regular

sw-regular ←θ-weakly regular ←wθ-regular

locally regular ←weakly regular ←hereditarily quasi-regular
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Examples [14, 23] and [24] show that none of the implications

weakly regular $\Rightarrow$ sw-regular,

θ-weakly regular $\Rightarrow$ locally regular,

locally regular $\Rightarrow$ wθ-regular

holds in general.

Problem 27. Is each sw-regular space weakly regular? quasi-regular?

Problem 28. Which properties in the diagram are preserved by products?

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References

[1] A. V. Arkhangelskii and B. M. Bokalo, *Tangency of topologies and tangential properties of topological spaces*, Trans. Mosk. Math. Soc. 1993 (1993), 139–163.

[2] R. Baire, *Sur les fonctions de variables reelles*, Annali di Mat. (3) 3 (1899), no. 1, 1–123. DOI: 10.1007/BF02419243

[3] T. Banakh and B. Bokalo, *On scatteredly continuous maps between topological spaces*, Topology Appl. 157 (2010), no. 1, 108–122. DOI: 10.1016/j.topol.2009.04.043

[4] T. Banakh and B. Bokalo, *Weakly discontinuous and resolvable functions between topological spaces*, Hacet. J. Math. Stat. 46 (2017), no. 1, 103–110. DOI: 10.15672/HJMS.2016.399

[5] T. Banakh, B. Bokalo, and N. Kolos, *Topological properties preserved by weakly discontinuous maps and weak homeomorphisms*, Topology Appl. 221 (2017), 91–106. DOI: 10.1016/j.topol.2017.02.036

[6] B. M. Bokalo and N. M. Kolos, *On operations on some classes of discontinuous functions*, Carpathian Math. Publ. 3 (2011), no. 2, 36–48.

[7] B. Bokalo and N. Kolos, *When does SC_p(X) = R^X hold?*, Topology 48 (2009), no. 2–4, 178–181. DOI: 10.1016/j.topol.2009.11.016

[8] B. Bokalo, O. Malanyuk, *On almost continuous mappings* (in Ukrainian), Mat. Stud. 9 (1995), no. 1, 90–93.

[9] O. Gutik, K. Pavlyk, *On pseudocompact topological Brandt Xô-extensions of semitopological monoids*, Topological Algebra Appl. 1 (2013) 60–79.

[10] L. Holá and Z. Piotrowski, *Set of continuity points of functions with values in generalized metric spaces*, Tatra Mt. Math. Publ. 42 (2009), 149–160.

[11] B. Kirchheim, *Baire one star functions*, Real Anal. Exchange 18 (1992/93), no. 2, 385–389.

[12] V. A. Vinokurov, *Strong regularizability of discontinuous functions*, Dokl. Akad. Nauk SSSR 281 (1985), no. 2, 265–269 (Russian).

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