Robust optimal periodic control using guaranteed Euler’s method

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3. Euler’s method and error bounds
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Motivation

- Dynamical systems:
  - in which a function describes the time dependence of a point in a geometrical space.
  - we only know certain observed or calculated states of its past or present state.
  - dynamical systems have a direct impact on human development.

⇒ The importance of studying:
- synchronization
- behavior
- robust control
Motivation

Dynamical systems:
- in which a function describes the time dependence of a point in a geometrical space.
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⇒ The importance of studying:
- synchronization
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- robust control
Robustness

- A control is considered **robust** if the dynamical system still **stable**, which means small perturbations to the solution using this control lead to a new solution that stays **close** to the original solution forever.

- A **stable** system produces a **bounded output** for a given **bounded input**.
An invariant

- The **bounded output** of some periodic **stable** system that is generated by a periodic **robust** control can be considered as an **invariant** from certain $t$.

- An invariant is an **unchanged** object after operations applied to it.
Problematic

Controlled system (by a periodic control)

Invariant

Stability analysis

Initial condition

Generate

Compute
Description of the method

- Given a differential system \( \Sigma : \frac{dy}{dt} = f_u(y) \) of dimension \( n \) controlled by \( u \), an initial point \( y_0 \in \mathbb{R}^n \), a real \( \varepsilon > 0 \), and a ball \( B_0 = B(y_0, \varepsilon) \):

The center of each ball at time \( t \) is the Euler approximate solution \( \tilde{Y}_{y_0}^u(t) \) of the system starting at \( y_0 \), and the radius is a function \( \delta_{\varepsilon}^u(t) \) bounding the distance between \( \tilde{Y}_{y_0}^u(t) \) and an exact solution \( Y_{y_0}^u(t) \) starting at \( B_0 \).

\(^1 B(y_0, \varepsilon) \) is the set \( \{ z \in \mathbb{R}^n \mid \| z - y_0 \| \leq \varepsilon \} \) where \( \| \cdot \| \) denotes the Euclidean distance.
Description of the method

- Given a differential system $\Sigma : \frac{dy}{dt} = f_u(y)$ of dimension $n$ controlled by $u$, an initial point $y_0 \in \mathbb{R}^n$, a real $\varepsilon > 0$, and a ball $B_0 = B(y_0, \varepsilon)^1$

- The tube can be described as $\bigcup_{t \geq 0} B(t)$ where $B(t) \equiv B(\tilde{Y}_u(t), \delta_\varepsilon(t))$.

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Description of the method

- Given a differential system \( \Sigma : \frac{dy}{dt} = f_u(y) \) of dimension \( n \) controlled by \( u \), an initial point \( y_0 \in \mathbb{R}^n \), a real \( \varepsilon > 0 \), and a ball \( B_0 = B(y_0, \varepsilon) \)\(^1\):

\[
\text{To find a } \text{bounded invariant, we look for a positive real } T \text{ such that } B((i+1)T) \subseteq B(IT) \text{ for some } i \in \mathbb{N}. \text{ In case of success, the ball } B(IT) \text{ is guaranteed to contain the "stroboscopic" sequence } \{B(jT)\}_{j=i,i+1,...} \text{ of sets } B(t) \text{ at time } t = iT, (i+1)T, \ldots \text{ and thus constitutes the sought bounded invariant set.}
\]

\(^1\)\(B(y_0, \varepsilon)\) is the set \{\(z \in \mathbb{R}^n \ | \ |z - y_0|| \leq \varepsilon\}\) where \( |\cdot|\) denotes the Euclidean distance.
Euler’s method and error bounds

Let us consider the differential system controlled by $u$:

$$\frac{dy(t)}{dt} = f_u(y(t)).$$

where $f_u(y(t))$ stands for $f(u(t), y(t))$ with $u(t) = u$ for $t \in [0, \tau]$, and $y(t) \in \mathbb{R}^n$ denotes the state of the system at time $t$ where $\tau$ is the integration time-step.

- $Y_{y_0}^u(t)$ denotes the exact continuous solution $y$ of the system at time $t \in [0, \tau]$ under constant control $u$, with initial condition $y_0$.
- $\tilde{Y}_{y_0}^u(t) \equiv y_0 + tf_u(y_0)$ denotes Euler’s approximate value of $Y_{y_0}^u(t)$ for $t \in [0, \tau]$. 
Let us explain the principle of the method to find a control pattern $\pi \in U^k$ based on DP and Euler integration method used in [CF19a; CF19b].

Given $k \in \mathbb{N}$ and $\tau \in \mathbb{R}_{>0}$, we consider the following *finite time horizon optimal control problem*: Find for each $y \in S$

- the *value* $v_k(y)$, i.e.,

$$v_k(y) = \min_{\pi \in U^k} \{ J_k(y, \pi) \} \equiv \min_{\pi \in U^k} \{ \| Y^\pi_y(k\tau) - y_{end} \| \}.$$  

- and an *optimal pattern*:

$$\pi_k(y) := \arg \min_{\pi \in U^k} \{ \| Y^\pi_y(k\tau) - y_{end} \| \}.$$  

The space $S$ is discretized by means of a grid $\mathcal{X}$ such that any point $y_0 \in S$ has an “$\varepsilon$-representative” $z_0 \in \mathcal{X}$. This method is generated by a procedure $PROC^\varepsilon_k$ which, for any $y \in S$, takes its representative $z \in \mathcal{X}$ as input, and returns a pattern $\pi^\varepsilon_k \in U^k$ corresponding to an approximate optimal value of $v_k(y)$ (see [CF19b]).
Proposition

[LCVCF17] Consider the solution $Y_{y_0}^u(t)$ of $\frac{dy}{dt} = f_u(y)$ with initial condition $y_0$ of $\varepsilon$-representative $z_0$ (hence such that $\|y_0 - z_0\| \leq \varepsilon$), and the approximate solution $\tilde{Y}_{z_0}^u(t)$ given by the explicit Euler scheme. For all $t \in [0, \tau]$, we have:

$$\| Y_{y_0}^u(t) - \tilde{Y}_{z_0}^u(t) \| \leq \delta_{\varepsilon}^u(t).$$
Definition

\( \delta_{\varepsilon}^u(t) \) is defined as follows for \( t \in [0, \tau] \):

\[
\text{if } \lambda_u < 0 : \\
\delta_{\varepsilon}^u(t) = \left( \varepsilon^2 e^{\lambda_u t} + \frac{C_u^2}{\lambda_u^2} \left( t^2 + \frac{2t}{\lambda_u} + \frac{2}{\lambda_u^2} \left( 1 - e^{\lambda_u t} \right) \right) \right)^{\frac{1}{2}}
\]

\[
\text{if } \lambda_u = 0 : \\
\delta_{\varepsilon}^u(t) = \left( \varepsilon^2 e^t + C_u^2 (-t^2 - 2t + 2(e^t - 1)) \right)^{\frac{1}{2}}
\]

\[
\text{if } \lambda_u > 0 : \\
\delta_{\varepsilon}^u(t) = \left( \varepsilon^2 e^{3\lambda_u t} + \frac{C_u^2}{3 \lambda_u^2} \left( -t^2 - \frac{2t}{3\lambda_u} + \frac{2}{9 \lambda_u^2} \left( e^{3\lambda_u t} - 1 \right) \right) \right)^{\frac{1}{2}}
\]

where \( C_u \) and \( \lambda_u \) are real constants specific to function \( f_u \), defined as follows:

\[
C_u = \sup_{y \in S} L_u \| f_u(y) \|
\]
Definition

$L_u$ denotes the Lipschitz constant for $f_u$, and $\lambda_u$ is the “one-sided Lipschitz constant” (or “logarithmic Lipschitz constant” [AS14]) associated to $f_u$, i.e., the minimal constant such that, for all $y_1, y_2 \in S$:

$$\langle f_u(y_1) - f_u(y_2), y_1 - y_2 \rangle \leq \lambda_u \|y_1 - y_2\|^2,$$  \hspace{1cm} (H0)

where $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors of $S$.

The constant $\lambda_u$ can be computed using a nonlinear optimization solver (e.g., CPLEX [Cpl09]) or using the Jacobian matrix of $f$.

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[AS14] Z. Aminzare and E. D. Sontag, “Contraction methods for nonlinear systems: A brief introduction and some open problems,” in 53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014, 2014, pp. 3835–3847.

[Cpl09] I. I. Cplex, “V12. 1: User’s manual for cplex,” International Business Machines Corporation, vol. 46, no. 53, p. 157, 2009.
A differential system with bounded uncertainty is of the form

\[
\frac{dy(t)}{dt} = f_u(y(t), w(t)),
\]

with \( t \in \mathbb{R}^n \), states \( y(t) \in \mathbb{R}^n \), and uncertainty \( w(t) \in \mathcal{W} \subset \mathbb{R}^n \) (\( \mathcal{W} \) is compact, i.e., closed and bounded).

- We suppose (see [LCADSC+17]) that there exist constants \( \lambda_u \in \mathbb{R} \) and \( \gamma_u \in \mathbb{R}_{\geq 0} \) such that, for all \( y_1, y_2 \in S \) and \( w_1, w_2 \in \mathcal{W} \):

\[
\langle f_u(y_1, w_1) - f_u(y_2, w_2), y_1 - y_2 \rangle \leq \lambda_u \|y_1 - y_2\|^2 + \gamma_u \|y_1 - y_2\| \|w_1 - w_2\| \quad (H1).
\]

- Instead of computing \( \lambda \) and \( \gamma \) globally for \( S \), it is advantageous to compute them \textit{locally} depending on the subregion of \( S \) occupied by the system state during a considered interval of time.

[LCADSC+17] A. Le Coënt et al., “Distributed control synthesis using Euler’s method,” in Proc. of International Workshop on Reachability Problems (RP’17), ser. Lecture Notes in Computer Science, vol. 247, Springer, 2017, pp. 118–131.
Proposition

\( \delta_{\varepsilon, \mathcal{W}}(t) \) is defined as follows for \( t \in [0, \tau] \):

\[
\begin{align*}
\text{if } \lambda_u < 0 : & \quad \delta_{\varepsilon, \mathcal{W}}(t) = \left( \frac{C^2}{-\lambda_u^4} \left( -\lambda_u^2 t^2 - 2\lambda_u t + 2e^{\lambda_u t} - 2 \right)
+ \frac{1}{\lambda_u^2} \left( \frac{C\gamma_u|\mathcal{W}|}{-\lambda_u} \left( -\lambda_u t + e^{\lambda_u t} - 1 \right) + \lambda_u \left( \frac{\gamma_u^2(|\mathcal{W}|/2)^2}{-\lambda_u} \left( e^{\lambda_u t} - 1 \right) + \lambda_u \varepsilon^2 e^{\lambda_u t} \right) \right) \right)^{1/2} \tag{1} \\
\text{if } \lambda_u > 0 : & \quad \delta_{\varepsilon, \mathcal{W}}(t) = \frac{1}{(3\lambda_u)^{3/2}} \left( \frac{C^2}{\lambda_u} \left( -9\lambda_u^2 t^2 - 2\lambda_u t + 2e^{3\lambda_u t} - 2 \right)
+ 3\lambda_u \left( \frac{C\gamma_u|\mathcal{W}|}{\lambda_u} \left( -3\lambda_u t + e^{3\lambda_u t} - 1 \right) + 3\lambda_u \left( \frac{\gamma_u^2(|\mathcal{W}|/2)^2}{\lambda_u} \left( e^{3\lambda_u t} - 1 \right) + 3\lambda_u \varepsilon^2 e^{3\lambda_u t} \right) \right) \right)^{1/2} \\
\text{if } \lambda = 0 : & \quad \delta_{\varepsilon, \mathcal{W}}(t) = \left( C^2 \left( -t^2 - 2t + 2e^t - 2 \right) + \left( C\gamma|\mathcal{W}| \left( -t + e^t - 1 \right) \right.
+ \left( \gamma^2(|\mathcal{W}|/2)^2(e^t - 1) + \varepsilon^2 e^t \right) \right)^{1/2} \tag{3}
\end{align*}
\]
Biochemical process example

Consider a biochemical process model $Y$ of continuous culture fermentation (see [HLID09]) and initial condition in $B_0 = B(x_0, \varepsilon)$ for some $x_0 \in \mathbb{R}^2$ and $\varepsilon > 0$ (see [BQ20]): Let $Y = (X, S, P) \in \mathbb{R}^3$ satisfies the differential system:

\[
\begin{align*}
\frac{dX}{dt} &= -DX(t) + \mu(t)X(t) \\
\frac{dS}{dt} &= D(S_f(t) - S(t)) - \frac{\mu(t)X(t)}{Y_{x/s}} \\
\frac{dP}{dt} &= -DP + (\alpha\mu(t) + \beta)X(t)
\end{align*}
\]

[HLID09] B. Houska et al., “Approximate robust optimization of time-periodic stationary states with application to biochemical processes,” in CDC, (Dec. 16–18, 2009), Shanghai, China: IEEE, 2009, pp. 6280–6285. DOI: 10.1109/CDC.2009.5400684.

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\end{align*}
\]

where $X$ denotes the biomass concentration, $S$ the substrate concentration, and $P$ the product concentration of a continuous fermentation process. The model is controlled by $S_f \in [S_{f_{\text{min}}} , S_{f_{\text{max}}} ]$. While the dilution rate $D$, the biomass yield $Y_{x/s}$, and the product yield parameters $\alpha$ and $\beta$ are assumed to be constant and thus independent of the actual operating condition, the specific growth rate $\mu : \mathbb{R} \rightarrow \mathbb{R}$ of the biomass is a function of the states:

\[
\mu(t) = \mu_m \frac{\left(1 - \frac{P(t)}{P_m}\right) S(t)}{K_m + S(t) + \frac{S(t)^2}{K_i}}
\]

[HLID09] B. Houska et al., “Approximate robust optimization of time-periodic stationary states with application to biochemical processes,” in CDC, (Dec. 16–18, 2009), Shanghai, China: IEEE, 2009, pp. 6280–6285. DOI: 10.1109/CDC.2009.5400684.

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Consider now the system $\Sigma'$ with uncertainty $w(\cdot) \in \mathcal{W}_0 = [-0.005, 0.005]$ and initial condition $Y_0$:

$$
\begin{align*}
\frac{dX}{dt} &= -DX(t) + \mu(t)X(t) + w \\
\frac{dS}{d\mathcal{S}} &= D(S_f(t) - S(t)) - \frac{\mu(t)X(t)}{Y_{x/s}} + w \\
\frac{dP}{dt} &= -DP + (\alpha\mu(t) + \beta)X(t) + w
\end{align*}
$$

(4)
Biochemical process with uncertainty

Biochemical process with an additive perturbation $\|w\| \leq 0.005$ over 4 periods ($4T = 192$) for $\Delta t = 1/400$ and initial condition $(X(0), S(0), P(0)) = (6.52, 12.5, 22.4)$, with $X(t), S(t), P(t)$ and control $S_f(t)$.

- We have: $B((i_0 + 1)T_0) \subset B(i_0 T_0)$ for $i_0 = 1$. 
Conclusion and Perspectives

Conclusion

- We presented a simple method to generate a bounded invariant for a differential system.
- We have given a simple condition which guarantees that, under a repeated control sequence the system with perturbation is robust.
- The method uses a simple algorithm to compute local rates of contraction in the framework of Euler’s method.
- The method uses a very general criterion of inclusion of one set in another.

Perspectives

- Extend our method in order to take into account the specification of state constraints during the evolution of the system.
Z. Aminzare and E. D. Sontag, “Contraction methods for nonlinear systems: A brief introduction and some open problems,” in *53rd IEEE Conference on Decision and Control, CDC 2014, Los Angeles, CA, USA, December 15-17, 2014*, 2014, pp. 3835–3847.

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[LCDVCF17] A. Le Coënt, F. De Vuyst, L. Chamoin, and L. Fribourg, “Control synthesis of nonlinear sampled switched systems using Euler’s method,” in SNR, (Apr. 22, 2017), ser. EPTCS, vol. 247, Uppsala, Sweden, 2017, pp. 18–33. DOI: 10.4204/EPTCS.247.2.