The inverse eigenvalue problem for symmetric anti-bidiagonal matrices

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Abstract

The inverse eigenvalue problem for real symmetric matrices of the form

\[
\begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & 0 & * \\
0 & 0 & 0 & \cdots & 0 & * & * \\
0 & 0 & 0 & \cdots & * & * & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & * & \cdots & 0 & 0 & 0 \\
0 & * & * & \cdots & 0 & 0 & 0 \\
* & * & 0 & \cdots & 0 & 0 & 0
\end{bmatrix}
\]

is solved. The solution is shown to be unique. The problem is also shown to be equivalent to the inverse eigenvalue problem for a certain subclass of Jacobi matrices.

1 Introduction

The goal of this paper is to characterize completely the spectra of real symmetric anti-bidiagonal matrices, i.e., matrices of the form

\[
A = \begin{bmatrix}
0 & 0 & \cdots & 0 & a_n \\
0 & 0 & \cdots & a_{n-2} & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_{n-2} & \cdots & 0 & 0 \\
a_n & a_{n-1} & \cdots & 0 & 0
\end{bmatrix}, \quad a_1, \ldots, a_n \in \mathbb{R}. \tag{1}
\]
This work is motivated by the author’s ongoing work on the nonnegative inverse eigenvalue problem and is inspired by well-known results on Jacobi matrices due to Hochstadt [5], [6], Hald [4], Gray and Wilson [3], as well as by the classical connection between the Jacobi matrices and orthogonal polynomials (see, e.g., [1, p. 267]).

The blanket assumption for the rest of the paper is that all \( a_j \) are positive. This restriction is clearly unimportant, since the sign of any \( a_j, j > 1 \), can be changed using a unitary similarity of the form

\[
\text{diag}(\varepsilon_1, \ldots, \varepsilon_n), \quad \varepsilon_j = \pm 1,
\]

and the problem for \( a_1 < 0 \) can be solved by switching from \( A \) to \( -A \). The assumption \( a_j > 0, j = 1, \ldots, n \), is however just right to guarantee uniqueness of a matrix that realizes a given \( n \)-tuple as its spectrum.

2 Definitions and notation

Notation used in the paper is rather standard. The spectrum of a matrix \( A \) is denoted by \( \sigma(A) \). A submatrix of \( A \) with rows indexed by an increasing sequence \( \alpha \) and columns indexed by another sequence \( \beta \) is denoted by \( A(\alpha, \beta) \). For simplicity, a principal submatrix of \( A \) with rows and columns indexed by \( \alpha \) is denoted by \( A(\alpha) \). (A typical choice for such an \( \alpha \) will be \( i:j \), the sequence of consecutive integers \( i \) through \( j \).) The size of a sequence \( \alpha \) is denoted by \( \#\alpha \). If \( \#\alpha = \#\beta \), then \( \det A(\alpha, \beta) \) is denoted by \( A[\alpha, \beta] \); \( \det A(\alpha) \) is denoted by \( A[\alpha] \). The elementary symmetric functions of an \( n \)-tuple \( \Lambda \) are denoted as \( \sigma_j(\Lambda) \). Thus

\[
\sigma_1(\Lambda) := \sum_{j=1}^{n} \lambda_j, \quad \sigma_2(\Lambda) := \sum_{i<j} \lambda_i \lambda_j, \quad \text{etc.}
\]

The term anti-bidiagonal matrix was already introduced. Other requisite definitions are listed next.

A Jacobi matrix is a tridiagonal matrix with positive off-diagonal entries.

A sign-regular matrix of class \( d \leq n \) with signature sequence \( \varepsilon_1, \ldots, \varepsilon_d \), where \( \varepsilon_j = \pm 1 \) for all \( j \), is a matrix satisfying

\[
\varepsilon_j A[\alpha] \geq 0 \quad \text{whenever } \#\alpha = j, \quad j = 1, \ldots, d.
\]

If in addition all minors of order at most \( d \) are nonzero, the matrix is called strictly sign-regular. Finally, if a certain power of a sign-regular matrix of class \( d \) is strictly sign-regular, then the matrix is called sign-regular of class \( d^+ \). A particular case of strict sign regularity is total positivity when all minors of a matrix are positive.

A sequence \( \mu_1 < \cdots < \mu_k \) is said to interlace a sequence \( \lambda_1 < \cdots < \lambda_{k+1} \) if

\[
\lambda_1 < \mu_1 < \lambda_2 < \mu_2 < \cdots < \mu_k < \lambda_k.
\]

3 Results

The following theorem is the main result of this paper.
Theorem 1 A real \( n \)-tuple \( \Lambda \) can be realized as the spectrum of an anti-bidiagonal matrix \( \mathbf{A} \) with all \( a_j \) positive if and only if \( \Lambda = (\lambda_1, \ldots, \lambda_n) \) where
\[
\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1} \lambda_n > 0.
\] (2)

The realizing matrix is necessarily unique.

Proof. Necessity. Let \( J \) denote the antidiagonal unit matrix
\[
J := \begin{bmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{bmatrix}.
\]

Note that \( J \) is sign-regular of class \( n \) with the signature sequence
\[
1, -1, -1, 1, 1, \cdots, (-1)^{\lceil n-1/2 \rceil}.
\] (3)

Next, note that \( B := JA \) is a nonnegative bidiagonal matrix, hence all its minors are nonnegative. Now, by the Cauchy-Binet formula
\[
A[\alpha] = (JB)[\alpha] = \sum \, J[\alpha, \beta]B[\beta, \alpha].
\]

Since the only nonzero minors of \( J \) are principal, we conclude that the matrix \( A \) is sign-regular of class \( n \) with the same signature sequence (3). Since \( A^2 \) is a positive definite Jacobi matrix, a high enough power of \( A^2 \) is totally positive, hence \( A \) is sign-regular of type \( n^+ \).

By a theorem of Gantmacher and Krein \([2, p. 301]\), the eigenvalues of \( A \) therefore can be arranged to form a sequence with alternating signs and strictly decreasing absolute values whose first element is positive, i.e., the spectrum \( \sigma(A) \) satisfies (2).

Sufficiency. First reduce the inverse problem for anti-bidiagonal matrices to the inverse problem for certain Jacobi matrices. Consider a matrix of the form \( \mathbf{A} \). To stress its dependence on \( n \) parameters \( a_1 \) through \( a_n \), let us denote it by \( \mathbf{A}_n \). The argument will involve the collection of all matrices \( \mathbf{A}_n, n \in \mathbb{Z} \), determined by a single sequence \( a_1, a_2, \ldots \). Denote the characteristic polynomial of \( \mathbf{A}_n \) by \( p_n \):
\[
p_n(\lambda) := \det(\lambda I - \mathbf{A}_n).
\]

Expanding it by its first row yields
\[
p_n(\lambda) = \lambda p_{n-1}(\lambda) - a_n^2 p_{n-2}(\lambda), \quad n \geq 2
\]
\[
p_0(\lambda) = 1, \quad p_1(\lambda) = \lambda - a_1,
\] (4) (5)
since the matrix \( \mathbf{A}_{n-1} \) is similar to its reflection about the antidiagonal.

This three-term recurrence relation \( \mathbf{B} \) with initial conditions \( \mathbf{B} \) is also satisfied (see, e.g., \([11, p. 267]\) or check directly) by the characteristic polynomials of the Jacobi matrices
\[
\mathbf{B}_n := \begin{bmatrix}
  a_1 & a_2 & 0 & \cdots & 0 & 0 \\
a_2 & 0 & a_3 & \cdots & 0 & 0 \\
0 & a_3 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & a_n \\
0 & 0 & 0 & \cdots & a_n & 0
\end{bmatrix}
\] (6)
if each of them is expanded by its last row. Thus the inverse eigenvalue problem for anti-bidiagonal matrices $A_n$ is equivalent to the inverse eigenvalue problem for Jacobi matrices $B_n$.

Now comes the crucial step in the proof. Consider expanding the characteristic polynomials of matrices $B_n$ in the opposite order, i.e., starting from the first row. Precisely, let us denote by $q_{n-j+1}$ the characteristic polynomial of the principal submatrix $B_n(j:n)$, with $q_n = p_n$. The corresponding recurrence relation is

$$q_{n}(\lambda) = (\lambda - a_1)q_{n-1}(\lambda) - a_2^2q_{n-2}(\lambda), \quad (7)$$
$$q_{n-j}(\lambda) = \lambda q_{n-j-1}(\lambda) - a_{j+2}^2q_{n-j-2}(\lambda), \quad j = 1, \ldots, n - 2, \quad (8)$$
$$q_0(\lambda) = 1, \quad q_1(\lambda) = \lambda. \quad (9)$$

Let $\Lambda$ be an $n$-tuple satisfying (2). Define the polynomial $q_n$ as

$$q_n(\lambda) := \prod_{j=1}^{n}(\lambda - \lambda_j)$$

and show that one can define polynomials $q_{n-j}$ for all $j = 1, \ldots, n$ so as to meet the requirements (7)–(9). To this end, first define

$$a_1 := \sigma_1(\Lambda), \quad q_{n-1}(\lambda) = \frac{(-1)^n q_0(-\lambda) - q_n(\lambda)}{2a_1}. \quad (10)$$

Note that $a_1 > 0$ due to the properties of $\Lambda$ and that the (monic) polynomial $q_{n-1}$ is even or odd depending on whether $n - 1$ is even or odd. Also note that the coefficient of $\lambda^{n-3}$ in $q_{n-1}$ is equal to

$$\frac{\sigma_3(\Lambda)}{a_1} = \frac{\sigma_3(\Lambda)}{\sigma_1(\Lambda)} < 0.$$

On the other hand, the coefficient of $\lambda^{n-2}$ in $q_n(\lambda)$ is $\sigma_2(\Lambda) < 0$. Therefore, it remains to show that the quantity $\frac{\sigma_3(\Lambda)}{\sigma_1(\lambda)} - \sigma_2(\Lambda)$ is positive, so $a_2$ can be defined as its (positive) square root:

$$a_2 := \sqrt{\frac{\sigma_3(\Lambda)}{\sigma_1(\lambda)} - \sigma_2(\Lambda)}.$$

Indeed, let us prove that

$$\sigma_3 > \sigma_1\sigma_2 \quad (11)$$

by induction. The base case is $n = 3$, where $\lambda_1 > -\lambda_2 > \lambda_3 > 0$. Then (11) reduces to the inequality

$$
\left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}\right)(\lambda_1 + \lambda_2 + \lambda_3) > 1.
$$

(12)

Differentiating the left-hand side of (12), one can check that it is an increasing function of $\lambda_1$ for $\lambda_1 \geq -\lambda_2$. Since the left-hand side is exactly 1 when $\lambda_1 = -\lambda_2$, this proves (12) and therefore proves (11). If $n > 3$, also notice that inequality (11) turns into equality for $\lambda_1 = -\lambda_2$, so it remains to argue that the difference $\sigma_3 - \sigma_1\sigma_2$ is an increasing function of $\lambda_1$ for $\lambda_1 \geq -\lambda_2$. But this is indeed the case, as can be seen by considering symmetric functions of the set $\Lambda' := -\lambda_2, \ldots, -\lambda_n$. Since

$$\sigma_1(\Lambda) = \lambda_1 - \sigma_1(\Lambda'), \quad \sigma_2(\lambda) = -\lambda_1\sigma_1(\Lambda') + \sigma_2(\Lambda'), \quad \sigma_3(\Lambda) = \lambda\sigma_2(\Lambda') - \sigma_3(\Lambda'),$$

we have

the inequality (14) amounts to

$$\lambda_1^2 \sigma_1(\Lambda') - \lambda_1 \sigma_1(\Lambda') + \sigma_1(\Lambda') \sigma_2(\Lambda') - \sigma_3(\Lambda') > 0,$$

and the derivative of the last left-hand side is positive, since $\lambda_1 \geq \sigma_1(\Lambda')$. This completes the proof of (14). Thus, $a_2$ is well-defined.

With these definitions in place, define $q_{n-2}$ from (7), i.e., let

$$q_{n-2}(\lambda) := -\frac{q_n(\lambda) - (\lambda - a_1)q_{n-1}(\lambda)}{a_2^2}.$$

Note that $q_{n-2}$ is a monic polynomial and is odd or even (precisely, it has the same parity as its leading term).

Now show that the roots of $q_{n-1}$ interlace those of $q_n$ and the roots of $q_{n-2}$ interlace those of $q_{n-1}$. Note that the polynomials $p_n(\lambda)$ and $(-1)^{n-1}p_n(-\lambda)$ have the same sign on the intervals

$$(-|\lambda_1|, -|\lambda_2|), (-|\lambda_3|, -|\lambda_4|), \ldots, (|\lambda_2|, |\lambda_1|).$$

Moreover, the sequence of these signs is alternating. The polynomial $q_{n-1}$ defined by (10) therefore has exactly $n - 1$ real zeros, each of them between two consecutive zeros of $q_n$. The implication for the root interlacing of $q_{n-2}$ and $q_{n-1}$ is immediate and is a standard argument on orthogonal polynomials (cf. [1, Section 5.4]). Due to the root interlacing of $q_{n-1}$ and $q_n$ and due to (7), the values of $q_{n-2}$ at the zeros of $q_{n-1}$ form an alternating sequence. Therefore, the roots of $q_{n-2}$ interlace those of $q_{n-1}$.

The rest of the argument is quite straightforward. With $q_{n-j}$ and $q_{n-j-1}$ defined, one defines $q_{n-j-2}$ from (8) making sure that $a_{j+2}^2$ is indeed positive, for each $j = 1, \ldots, n - 2$. The resulting monic polynomials will have alternating parities and interlacing roots. The quantity $a_{j+2}^2$ is to be set equal to the difference between the second elementary symmetric function $\sigma_2$ of the roots of $p_{n-j-1}$ and the second elementary symmetric function of the roots of $p_{n-j}$. With a slight abuse of notation, this may be denoted by

$$a_{j+2}^2 = \sigma_2(p_{n-j-1}) - \sigma_2(p_{n-j}).$$

The roots of either polynomial are symmetric about 0, therefore, the corresponding second elementary symmetric function is simply

$$(-1) \cdot \text{the sum of squares of all positive roots.}$$

By the interlacing property, the sum of squares for $p_{n-j}$ exceeds that for $p_{n-j-1}$, hence $\sigma_2(p_{n-j-1}) - \sigma_2(p_{n-j}) > 0$ and hence $a_{j+2}$ is well defined.

The argument also shows the uniqueness of the realizing matrix (6), therefore the uniqueness of the realizing matrix (14), provided, of course, that $a_j$ are chosen to be positive.

The following corollary was established in the course of the above proof.

**Corollary 2** A real $n$-tuple $\Lambda$ can be realized as the spectrum of a Jacobi matrix (6) if and only if $\Lambda = (\lambda_1, \ldots, \lambda_n)$ where

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1} \lambda_n > 0.$$

The realizing matrix is necessarily unique.
Finally, another simple consequence of Theorem 1 is the following result.

**Corollary 3** Let $\mathcal{M}$ be a real positive $n$-tuple. Then there exists a Jacobi matrix that realizes $\mathcal{M}$ as its spectrum and has an anti-bidiagonal symmetric square root of the form $A$ with all $a_j$ positive.

**Proof.** Let the elements of $\mathcal{M}$ be ordered $\mu_1 > \mu_2 > \cdots > \mu_n (> 0)$. Define

$$\lambda_j := (-1)^{j-1} \sqrt{\mu_j}, \quad j = 1, \ldots, n, \quad \Lambda := (\lambda_j : j = 1, \ldots, n).$$

Then

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1} \lambda_n > 0.$$  

By Theorem 1, there exists a symmetric anti-bidiagonal matrix $A$ with spectrum $\sigma(A) = \Lambda$. But then $B := A^2$ is a Jacobi matrix with spectrum $\mathcal{M}$. \hfill $\Box$

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**References**

[1] G.E. Andrews, R. Askey, R. Roy, *Special Functions*. Cambridge University Press, Cambridge, 1999.

[2] F. R. Gantmacher and M. G. Krein, *Oszillazionsmatrizen, Oszillazionskerne und kleine Schwingungen mechanischer Systeme*. Akademie-Verlag, Berlin, 1960.

[3] L. J. Gray and D. G. Wilson, *Construction of a Jacobi matrix from spectral data*, Linear Algebra Appl., 14 (1976), 131–134.

[4] O. Hald, *Inverse eigenvalue problems for Jacobi matrices*, Linear Algebra Appl., 14 (1976), 63–85.

[5] H. Hochstadt, *On some inverse problems in matrix theory*, Arch. Math. 18 (1967), 201–207.

[6] H. Hochstadt, *On the construction of a Jacobi matrix from spectral data*, Linear Algebra Appl., 8 (1974), 435–446.