On the Laplacian of strong power graphs of finite groups

A. K. Bhuniya * and Sudip Bera

Department of Mathematics, Visva-Bharati, Santiniketan-731235, India.
anjankbhuniya@gmail.com, sudipbera517@gmail.com

Abstract

Let \( G \) be a finite group of order \( n \). The strong power graph \( P_s(G) \) of \( G \) is the undirected graph whose vertices are the elements of \( G \) such that two distinct vertices \( a \) and \( b \) are adjacent if \( a^{m_1} = b^{m_2} \) for some positive integers \( m_1, m_2 < n \). In this article we give a complete characterization of Laplacian spectrum, and find the permanent of the Laplacian matrix of the strong power graph \( P_s(G) \) for any finite group \( G \).

Keywords: finite groups; strong power graphs; Laplacian spectrum; algebraic connectivity; permanent

AMS Subject Classifications: 05C50; 05C25

1 Introduction

Let \( G \) be a group of \( n \) elements. The strong power graph \( P_s(G) \) of \( G \) is a simple undirected graph whose vertices consists of the elements of \( G \) and two distinct vertices \( a \) and \( b \) are adjacent in \( P_s(G) \) if \( a^{m_1} = b^{m_2} \) for some positive integers \( m_1, m_2 < n \). Thus a finite group \( G \) is noncyclic if and only if \( P_s(G) \) of is complete. The idea of a strong power graph was introduced by Singh and Manilal [16] as a generalization of power graphs of a finite group. Now research on power graphs associated with a finite group has already gained a momentum [1], [3], [12].

Here we study Laplacian spectrum of the strong power graph \( P_s(G) \) of any finite group \( G \). For any graph \( \Gamma \) let \( A(\Gamma) \) be the adjacency matrix and \( D(\Gamma) \) be the diagonal matrix of vertex degrees. Then the Laplacian matrix of \( \Gamma \) is defined as \( L(\Gamma) = D(\Gamma) - A(\Gamma) \). Clearly \( L(\Gamma) \) is a real symmetric matrix and it is well known that \( L(\Gamma) \) is a positive semidefinite matrix with 0 as the smallest eigen
value. Thus we can assume that the Laplacian eigen values are $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n = 0$. Among all Laplacian eigen values of a graph, one of the most popular is the second smallest, called by Fiedler [6], the algebraic connectivity of a graph. It is a good parameter to measure, how well a graph is connected. For example, a graph is connected if and only if its algebraic connectivity is non-zero. The Laplacian matrix of a graph and its eigen values known as Laplacian spectrum are used in different area in mathematics, viz. discrete mathematics, combinatorial optimizations, etc; and to interpret several physical and chemical problems. According to Mohar [13] the Laplacian eigen values are more intuitive and much more important than the eigen values of the adjacency matrix. Here we find the Laplacian characteristic polynomial of $\mathbb{Z}_n$, the group of all integers modulo $n$. If $G$ is a cyclic group of order $n$ then $G \simeq \mathbb{Z}_n$ which implies that $P_s(G) \simeq P_s(\mathbb{Z}_n)$. Thus we get Laplacian spectrum of every finite cyclic group. Also we have characterized the same for any finite non-cyclic group. All these details on the Laplacian spectrum are given in Section 2. Moreover we have derived an explicit formula for the permanent of the Laplacian matrix of strong power graphs of any finite group in Section 3. Let $A = (a_{ij})$ be a square matrix of order $n$, then the permanent of $A$ is denoted by $\text{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where $S_n$ is the set of all permutations of $1, 2, \cdots, n$. The permanent function was introduced by Binet and independently by Cauchy in 1812. The development of the permanent theory was inspired by a famous conjecture posed by van der Waerden in 1926. Permanent theory has many role in probability and statistics, and from the theory of permanent of nonnegative matrices several applications of Alexandroff’s inequality are illustrated. We refer to [14] and [15] for more on permanents. Also we refer to [11] for group theoretic background.

2 Laplacian spectrum of the strong power graphs of finite groups

First we find the characteristic polynomial of the Laplacian matrix associated with the strong power graph $P_s(\mathbb{Z}_n)$ of the cyclic group $\mathbb{Z}_n$.

**Theorem 2.1.** For each positive integer $n \geq 2$,

$$\Theta(P_s(\mathbb{Z}_n)), x) = x(x - n)^{n-\phi(n)-1}(x - n + \phi(n) + 1)(x - n + 1)^{\phi(n)-1}.$$  

**Proof.** Let $\bar{s}_i, i = 1, 2, \cdots, m = n - \phi(n) - 1$ be the non generators of $\mathbb{Z}_n$. We index the rows and columns of the Laplacian matrix $L(P_s(\mathbb{Z}_n))$ in order by the non generators $\bar{s}_i (i = 1, 2, \cdots m)$ and the
of $\Theta(L_s(Z_n))$ expanding through the first row we get

$$
\begin{pmatrix}
n - 1 & -1 & \cdots & -1 & -1 & \cdots & -1 \\
-1 & n - 1 & \cdots & -1 & -1 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & n - 1 & -1 & \cdots & -1 \\
-1 & -1 & \cdots & -1 & n - 2 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & -1 & -1 & \cdots & n - 2 \\
-1 & -1 & \cdots & -1 & 0 & \cdots & 0
\end{pmatrix}
$$

Each row and column sum of the above matrix is zero. Then the characteristic polynomial of

$$L(P_s(Z_n))) = \Theta(L(P_s(Z_n))), x) =$$

$$
\begin{vmatrix}
x - (n - 1) & \cdots & 1 & 1 & \cdots & 1 & 1 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cdots & x - (n - 1) & 1 & \cdots & 1 & 1 \\
1 & \cdots & 1 & x - (n - 2) & \cdots & 1 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & \cdots & 1 & 1 & \cdots & x - (n - 2) & 0 \\
1 & \cdots & 1 & 0 & \cdots & 0 & x - (n - \phi(n) - 1)
\end{vmatrix}
$$

Multiply the first row of $\Theta(L(P_s(Z_n))), x)$ by $(x - 1)$ and apply the row operation $R_1' = R_1 - R_2 - R_3 - \cdots - R_{(n-1)} - R_n$. Then expanding the resulting determinant through the first row we get,

$$\Theta(L(P_s(Z_n))), x) = \frac{x(x - n)}{x - 1} \Theta(L_{s_1}(P_s(Z_n))), x),$$

where $\Theta(L_{s_1,s_2,\ldots,s_i}(P_s(Z_n))), x)$ is the determinant obtained form $\Theta(L(P_s(Z_n))), x)$ deleting the rows and columns corresponding to the non generators $s_1, s_2, \ldots, s_i$. Similarly multiplying the first row of $\Theta(Ls_1(P_s(Z_n))), x)$ by $(x - 2)$, applying the row operation $R_1' = R_1 - R_2 - R_3 - \cdots - R_{(n-1)}$ and expanding through the first row we get

$$\Theta(Ls_1(P_s(Z_n))), x) = \frac{(x - 1)(x - n)}{x - 2} \Theta(Ls_1, s_2(P_s(Z_n))), x),$$

and so

$$\Theta(L(P_s(Z_n))), x) = \frac{x(x - n)^2}{x - 2} \Theta(Ls_1, s_2(P_s(Z_n))), x).$$

Continuing in this way we get

$$\Theta(L(P_s(Z_n))), x) = \frac{x(x - n)^{(n-\phi(n)-1)}}{(x - n + \phi(n) + 1)} \Theta(Ls_1, s_2, \ldots, s_m(P_s(Z_n))), x),$$
where

\[ \Theta(L\bar{s}_1, \bar{s}_2 \cdots \bar{s}_m(P_\mathbb{Z}(\mathbb{Z}_n)), x) = \begin{vmatrix} x - \lambda & 1 & 1 & \cdots & 0 \\ 1 & x - \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & x - m \end{vmatrix} , \]

\[ \lambda = n - 2, \ m = n - \phi(n) - 1 \] and order of the determinant is \( \phi(n) + 1 \). Now expanding \( \Theta(L\bar{s}_1, \bar{s}_2 \cdots \bar{s}_m(P_\mathbb{Z}(\mathbb{Z}_n)), x) \) with respect to last row we get

\[ \Theta(L\bar{s}_1, \bar{s}_2 \cdots \bar{s}_m(P_\mathbb{Z}(\mathbb{Z}_n)), x) = (-1)^{2\phi(n)}(x - m) \begin{vmatrix} x - \lambda & 1 & \cdots & 1 \\ 1 & x - \lambda & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & x - \lambda \end{vmatrix} 
= (x - n + \phi(n) + 1)(x - \lambda + \phi(n) + 1 - 2)(x - \lambda - 1)^{\phi(n) - 1} 
= (x - n + \phi(n) + 1)^2(x - n + 1)^{\phi(n) - 1} . \]

Thus \( \Theta(L(P_\mathbb{Z}(\mathbb{Z}_n)), x) = x(x - 1)^{n - \phi(n) - 1}(x - n + \phi(n) + 1)(x - n + 1)^{\phi(n) - 1} . \)

Let \( G \) be a cyclic group of order \( n \), then \( G \) is isomorphic to \( \mathbb{Z}_n \); and the strong power graphs \( P_\mathbb{Z}(G) \) and \( P_\mathbb{Z}(\mathbb{Z}_n) \) of \( G \) and \( \mathbb{Z}_n \) respectively are isomorphic. Hence the graphs \( P_\mathbb{Z}(G) \) and \( P_\mathbb{Z}(\mathbb{Z}_n) \) have the same Laplacian spectrum. So by Theorem 2.1 we have:

**Proposition 2.2.** If \( G \) is a cyclic group of order \( n \), then the laplacian spectrum of \( P_\mathbb{Z}(G) \) is

\[
\begin{pmatrix}
0 & n & n - \phi(n) - 1 & n - 1 \\
1 & n - \phi(n) - 1 & 1 & \phi(n) - 1
\end{pmatrix}
\]

For any non cyclic group \( G \), the strong power graph \( P_\mathbb{Z}(G) \) is complete \([5]\). So their Laplacian spectrum is given by :

**Proposition 2.3.** Let \( G \) be a noncyclic group of order \( n \), then the Laplacian spectrum of \( P_\mathbb{Z}(G) \) is

\[
\begin{pmatrix}
0 & n \\
1 & n - 1
\end{pmatrix}
\]

The algebraic connectivity of a graph \( \Gamma \), denoted by \( a(\Gamma) \), is the second smallest Laplacian eigenvalue of \( \Gamma \) \([6]\). Now the algebraic connectivity has received special attention due to its huge applications on connectivity problems, isoperimetric numbers, genus, combinatorial optimizations and many other problems. Thus by Proposition 2.2 and Proposition 2.3 we have:
Corollary 2.4. Let $G$ be a group of order $n > 3$.

1. If $G$ is a cyclic group then $a(\mathcal{P}_s(G)) = n - \phi(n) - 1$.

2. If $G$ is a noncyclic group then $a(\mathcal{P}_s(G)) = n$.

Another important application of Laplacian spectrum is on the number of spanning trees of a graph. A spanning tree $T$ of a graph $\Gamma$ is a subgraph which is a tree having same vertex set is same as $\Gamma$. If $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n = 0$ are the Laplacian eigenvalues of a graph $\Gamma$ of $n$-vertices, then the number of spanning trees of $\Gamma$ is denoted by $\tau(\Gamma)$ is given by $\frac{\lambda_1 \lambda_2 \cdots \lambda_{n-1}}{n}$ [Theorem 4.11; [2]]. Thus from the Proposition 2.2 and Proposition 2.3 we have:

Corollary 2.5. Let $G$ be a group of order $n$.

1. If $G$ is a cyclic group then $\tau(\mathcal{P}_s(G)) = n^n - \phi(n)^2 (n - \phi(n) - 1)(n - 1)^{\phi(n) - 1}$.

2. If $G$ is a noncyclic group then $\tau(\mathcal{P}_s(G)) = n^{n-2}$.

The graph energy is defined in terms of the spectrum of the adjacency matrix. Depending on the well-developed spectral theory of the Laplacian matrix, recently Gutman et. al [7] have defined the Laplacian energy of a graph $\Gamma$ with $n$ vertices and $m$ edges as: $LE(\Gamma) = \sum_{i=1}^{n} |\lambda_i - \frac{2m}{n}|$, where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n = 0$ are the Laplacian eigen values of the graph $\Gamma$. This definition has been adjusted so that the Laplcian energy becomes equal to the energy for any regular graph. For various properties of Laplacian energy we refer [8], [10], [9]. From Proposition 2.2 and Proposition 2.3 we have

Corollary 2.6. Let $G$ be a finite group of order $n$.

1. If $G$ is cyclic then $LE(\mathcal{P}_s(G)) = 2(n - 1) - \frac{4\phi(n)}{n}$.

2. If $G$ is noncyclic then $LE(\mathcal{P}_s(G)) = 2(n - 1)$.

3 Permanent of the Laplacian of strong power graph

Let us recall the definition of permanent of a square matrix. For any square matrix $A = (a_{ij})$ of order $n$, the permanent of A is denoted by $\text{per}(A)$ and defined by $\text{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$. It is quite difficult to determine the permanent of a square matrix. In this section we have determined the permanent of the Laplacian matrix of strong power graph of any finite group explicitly. Our method is based on the following observation. Let $A = (a_{ij})$ be a matrix of order $n$, then $\text{per}(A)$ is equal to the coefficient of $x_1x_2\cdots x_n$ in the expression $(a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_1n)(a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_2n)(a_{31}x_1 + a_{32}x_2 + \cdots a_{3n}x_3n)\cdots (a_{nn}x_1 + a_{n2}x_2 + \cdots a_{nn}x_nn)$. In this section, we have assumed that $A$ is a symmetric matrix.
\( \sum_{r=1}^{m+n} (-1)^{r-1}(m+n-r)\left[ \binom{m+n-1}{r-1} + (n-1) \binom{m+n-2}{r-1} \right]. \)

\[ F(x_1, x_2, \ldots, x_n) = a_0 + \sum_{r=1}^{m+n} (-1)^{r-1}(m+n-r)\left[ \binom{m+n-1}{r-1} + (n-1) \binom{m+n-2}{r-1} \right]. \]

Proof. The required permanent is the coefficient of \( x_1 x_2 \cdots x_{m+n+1} \) in \( F(x_1, x_2, \ldots, x_{m+n+1}) = (X - x_1)(X - x_2) \cdots (X - x_n)(X - x_{m+1} + x_{m+n+1})(X - x_{m+2} + x_{m+n+1}) \cdots (X - x_{m+n} + x_{m+n+1})(X_{m+1} + x_{m+2} + \cdots x_{m+n}), \) where \( X = x_1 + x_2 + \cdots x_{m+n}. \)

Now we have \( F(x_1, x_2, \ldots, x_{m+n+1}) = \prod_{i=1}^{m} (X - x_i) \sum_{i=1}^{n} (X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n}), \)

\[ C_{x_1 x_2 \cdots x_{m+n+1}}(F(x_1, x_2, \cdots x_{m+n+1})) \]

\[ = C_{x_1 x_2 \cdots x_{m+n} \prod_{i=1}^{m} (X - x_i) \sum_{i=1}^{n} (X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n})}. \]

\[ = nC_{x_1 x_2 \cdots x_{m+n}}(X - x_1) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n}) + \cdots 
\]

\[ C_{x_1 x_2 \cdots x_{m+n}}((X - x_1) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n})). \]

Now, \( (X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n}) = X^{m+n-2} - X^{m+n} - \sum_{i \neq (m+1)} x_i + \cdots + (-1)^{m+n-1} x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}, \) shows that \( C_{x_1 x_2 \cdots x_{m+n}}((X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n})) = \sum_{r=1}^{m+n} (-1)^{r-1}(m+n-r)! \binom{m+n-1}{r-1}, \) and \( (X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n}) = X^{m+n-1} - X^{m+n} - \sum_{i \neq (m+2)} x_i + \cdots + (-1)^{m+n-1} x_1 x_2 \cdots x_m x_{m+1} x_{m+3} \cdots x_{m+n} \) shows that \( C_{x_1 x_2 \cdots x_{m+n}}((X - x_1)(X - x_2) \cdots (X - x_m)(X - x_{m+1})(X - x_{m+2}) \cdots (X - x_{m+n})) = \sum_{r=1}^{m+n} (-1)^{r-1}(m+n-r)! \binom{m+n-2}{r-1}. \)
Also \( C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}}((X - x_1)(X - x_2) \cdots (X - x_{m+2})(X - x_{m+4}) \cdots (X - x_{m+n})) \)
\( = C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}}((X-x_1)(X-x_2) \cdots (X-x_{m+3})(X-x_{m+5}) \cdots (X-x_{m+n})) = \cdots = C_{x_1 x_2 \cdots x_m x_{m+2} \cdots x_{m+n}}((X-x_1)(X-x_2) \cdots (X-x_m) \cdots (X-x_{m+n-1})) \)
\( = \sum_{r=1}^{m+n} (-1)^{r-1}(m+n-r)! \left( \begin{array}{c} m+n-1 \\ r-1 \end{array} \right). \)

Hence the permanent of the adjacency matrix of the stated graph is
\[
n\left[ \sum_{r=1}^{m+n} (-1)^{r-1}(m+n-r)! \left\{ \left( \begin{array}{c} m+n-1 \\ r-1 \end{array} \right) + (n-1) \left( \begin{array}{c} m+n-2 \\ r-1 \end{array} \right) \right\} \right].
\]

\[
\square
\]

Let \( G \) be a cyclic group of order \( n \). Then \( G \) has \( m = \phi(n) \) generators none of which is adjacent to the identity element \( e \) of \( G \). Thus the set \( G \setminus e \) of all \( n-1 \) non-identity vertices forms a clique and identity \( e \) is adjacent to each of the \( n-\phi(n)-1 \) non-identity non-generators. Hence from Lemma 3.1 it follows immediately that:

**Theorem 3.2.** Let \( G \) be a cyclic group of order \( n \), then the permanent of adjacency matrix of strong power graph of \( G \) is
\[
(n-\phi(n)-1)\left[ \sum_{r=1}^{n-1} (-1)^{r-1}(n-1-r)! \left\{ \left( \begin{array}{c} n-2 \\ r-1 \end{array} \right) + (n-2-\phi(n)) \left( \begin{array}{c} n-3 \\ r-1 \end{array} \right) \right\} \right].
\]

Now we compute the permanent of the Laplacian matrix of a graph and hence compute the permanent of Laplacian matrix of strong power graph of any finite group.

**Lemma 3.3.** The permanent of the Laplacian matrix of any graph \( \Gamma \) of type \( m + n + 1 \) is
\[
\sum_{r=1}^{m+n} \{-1\}^{m+n-r}(m+n-r)!F_r(d) + (d-m+1) \sum_{i+j=m+n} \left( \begin{array}{c} m \\ i \end{array} \right) \left( \begin{array}{c} n \\ j \end{array} \right) (d+2)^i (d+1)^j,
\]
where \( F_r(d) = \sum_{i+j=r-1} \left( \begin{array}{c} m \\ i \end{array} \right) (d+2)^i (d+1)^j \{ n \left( \begin{array}{c} n-1 \\ j \end{array} \right) + n(n-1) \left( \begin{array}{c} n-2 \\ j \end{array} \right) \} - (d-m+1)(m+n-r+1) \left( \begin{array}{c} n \\ j \end{array} \right) \) and \( d = m+n-1 \).

**Proof.** Consider \( F(x_1, x_2, \cdots, x_{m+n+1}) = \prod_{i=1}^{m}(X + (d+1)x_i) \prod_{i=1}^{n}(X + (d+2)x_{m+i} - x_{m+n+1})((d-m+1)x_{m+n+1} + \sum_{i=1}^{n} x_{m+i}), \) where \( X = -(x_1 + x_2 + \cdots + x_{m+n}) \). Then the permanent of the Laplacian matrix of \( \Gamma \) is \( C_{x_1 x_2 \cdots x_{m+n+1}}(F(x_1, x_2, \cdots, x_{m+n+1})) \) which is equal to \( C_{x_1 x_2 \cdots x_{m+n+1}}(\prod_{i=1}^{m}(X + (d+1)x_i) \prod_{i=1}^{n}(X + (d+2)x_{m+i} - x_{m+n+1}) \prod_{i=1}^{n} x_{m+i}) \)
\[(d+1)x_i \prod_{i=1}^{n} (X + (d+2)x_{m+i} - x_{m+n+1} + (\sum_{i=1}^{n} - x_{m+i})) + C_{z_{1}x_{2} \cdots x_{m+n}((d-m+1) \prod_{i=1}^{m} X + (d+1)x_i) \prod_{i=1}^{m} (X + (d+2)x_{m+i})) = (d-m+1)[X^{m+n} + X^{m+n-1} \sum f_1(d)x_i + X^{m+n-2} \sum f_{12}(d)x_1x_2 + \cdots + f_{123 \cdots m}(d)x_1x_2 \cdots x_{m+n}],\]

where \(f_{123 \cdots j}(d)\) is a product of some \((d+1)\) and \((d+2)\) which is clear from the context. So \(C_{z_{1}x_{2} \cdots x_{m+n}((d-m+1) \prod_{i=1}^{m} (X + (d+1)x_i) \prod_{i=1}^{m} (X + (d+2)x_{m+i})) = (d-m+1) \sum_{r=1}^{m+n+1} (-1)^{m+n-r+1}(m+n-r+1)!f_r(d),\)

where \(f_r(d) = \sum_{i+j=r-1} \binom{m}{i} \binom{n}{j} (d+2)^i(d+1)^i.\) Now proceeding similarly as in the proof of Lemma 3.1 we get \(C_{z_{1}x_{2} \cdots x_{m+n+1}}(\prod_{i=1}^{m} (X + (d+1)x_i) \prod_{i=1}^{m} (X + (d+2)x_{m+i} - x_{m+n+1} + (\sum_{i=1}^{n} - x_{m+i})) = n \sum_{r=1}^{m+n+1} (-1)^{m+n-r}(m+n-r)! \sum_{i+j=r-1} \binom{m}{i} \binom{n-1}{j} + (n-1) \binom{n-2}{j} (d+2)^i(d+1)^i].\)

Hence the permanent of the Laplacian matrix of \(\Gamma\) is

\[
\sum_{r=1}^{m+n+1} \binom{m}{i} \binom{n}{j} (d+2)^i(d+1)^i \{n \binom{n-1}{j} + n(n-1) \binom{n-2}{j} - (d-m+1)(m+n-r+1) \binom{n}{j}\}.
\]

\[\square\]

The strong power graph of any cyclic group of order \(n\) is a graph of type \(m+n+1\). So we have:

**Theorem 3.4.** Let \(G\) be a cyclic group of order \(n\), then the permanent of Laplacian matrix of strong power graph of \(G\) is

\[
\sum_{r=1}^{n-1} \{(-1)^{n-r-1}(n-r-1)!F_r(d)\} + (n-\phi(n)-1) \sum_{i+j=n-1} \binom{\phi(n)}{i} \binom{n-\phi(n)-1}{j} n^i(n-1)^j,
\]

where \(F_r(d) = \sum_{i+j=r-1} \binom{\phi(n)}{i} n^i(n-1)^j \{(n-\phi(n)-1) \binom{n-\phi(n)-1}{j} + (n-\phi(n)-1)(n-\phi(n)-3) \binom{n-\phi(n)-1}{j} - (n-\phi(n)-1)(n-r) \binom{n-\phi(n)-1}{j}\}.

For any noncyclic group \(G\) of order \(n\) the strong power graph \(P_s(G)\) is complete. Hence we have:

**Theorem 3.5.** The permanent of the Laplacian matrix of the strong power graph \(P_s(G)\) of a noncyclic group \(G\) of order \(n\) is \((-1)^n n!\left(1 - \frac{n}{1!} + \frac{n^2}{2!} - \frac{n^3}{3!} + \cdots + (-1)^n \frac{n^n}{n!}\right)\).
References

[1] J. Abawajya, A. Kelareva, M. Chowdhury, Power graphs: A survey, Electronic Journal of Graph Theory and Applications, 1 (2) (2013), 1-22.

[2] R. B. Bapat, Graphs and matrices, Second edition, Hindustan Book Agency, 2014.

[3] A. Richard Brualdi, Dragos Cvetkovic, A Combinatorial approch to matrix theory and its application, CRC Press, 2009.

[4] I. Chakrabarty, S. Ghosh, M. K. Sen, Undirected power graphs of semigroups, Semigroup Forum 78(2009)410-426.

[5] S. Chattopadhyay, P. Panigrahi, On Laplacian spectrum of power graphs of finite cyclic and dihedral groups, Linear and multilinear Algebra (2014).

[6] M. Fielder, Algebraic connectivity of graphs, Czechoslovak Math. J. 23(1973) 298-305.

[7] I. Gutman, B. Zhou, Laplacian energy of a graph, Lin. Algebra Appl 414(2006) 29-37.

[8] I. Gutman, B. Zhou, B. Furtula, The Laplacian energy like invariant is an energy like invarient, MATCH Commun. Math. Comput. Chem 64(2010)85-96.

[9] Y. Hou, Unicyclic graphs with minimal energy, J. Math. Chem 29(2001)163-168.

[10] Y. Hou, Z. Teng, C. W. Woo On the spectral redius, k degree and the upper bound of energy in a graph, MATCH Commun.Math. Comput. Chem 57(2007)341-350.

[11] T. W. Hungerford, Algebra, Gratuets Text in Mathematics, New York(NY), Springer-Verlag, 73(1974).

[12] A. R. Moghaddamfar, S. Rahbariyian, W. J. Shi, Certain properties of power graph associated with a finite group, Journal of Algebra and its Applications, 13(7) (2014) 18 pages.

[13] B. Mohar, The Laplacian spectrum of graphs, In: Y. Alavi, G. Chartrand, Oellermann OR, A. J. Schwenk, editors. Graph Theory, combinatorics, and applications, New york(NK):Wiley, 2(1991) 871-898.

[14] H. Minc, Permanents, Addition-Wesley, Reading, Mass, 1978.

[15] H. Minc, Theory of permanents, Linear and Multilinear Algebra 12(1983)227-263.

[16] G. Suresh Singh, K. Manilal, Some Generalities on Power Graphs and Strong Power Graphs, Int. J. Contemp. Math Sciences 5(55)(2010)2723-2730.

[17] D. B. West, Introduction to Graph theory, 2nd ed.pearson education, 2001.