Nonperturbative renormalization
in a scalar model within Light-Front Dynamics

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Abstract

Within the covariant formulation of Light-Front Dynamics, in a scalar model with the interaction Hamiltonian \(H = -g\psi^2(x)\varphi(x)\), we calculate nonperturbatively the renormalized state vector of a scalar ”nucleon” in a truncated Fock space containing the \(N\), \(N\pi\) and \(N\pi\pi\) sectors. The model gives a simple example of nonperturbative renormalization which is carried out numerically. Though the mass renormalization \(\delta m^2\) diverges logarithmically with the cutoff \(L\), the Fock components of the ”physical” nucleon are stable when \(L \to \infty\).

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1 Introduction

The knowledge of the hadron properties within the framework of Quantum Chromo-Dynamics (QCD) is one of the main issues in strong interaction physics. Several approaches have been pursued in the last twenty years, in particular lattice gauge calculations. Among the alternatives to these calculations, Light-Front Dynamics (LFD) is of particular interest [1]. It has proven successful in many phenomenological applications involving few-body systems in particle and nuclear physics. However, the application of LFD to field theoretical calculations is still in its infancy [2]. The main issue to be solved is the renormalization procedure [3]. In perturbative calculations, the renormalization of the electron self-energy in QED, in standard LFD, is already non-trivial in the sense that it involves non-local counterterms [4]. This unpleasant feature is however a direct consequence of the choice of a preferential direction, the $z$ axis, in the determination of the quantization plane. This can be well understood in the Covariant Formulation of Light-Front Dynamics (CLFD) [5], as shown in ref. [6]. In this formulation, the state vector is defined on the light-front surface given by the equation $\omega \cdot x = 0$, where $\omega$ is the four-vector with $\omega^2 = 0$. The particular case where $\omega = (1, 0, 0, -1)$ corresponds to standard LFD. In CLFD, the counterterm needed to renormalize the electron self energy in first order perturbation expansion is simply dependent on the orientation of the light front, defined by the four-vector $\omega$.

We shall investigate in this article how the question of non-perturbative renormalization can be formulated in CLFD. We shall first derive the general eigenvalue equation, whose solutions are the Fock state components. As a first example, we shall illustrate our strategy with a simple model involving two coupled scalar particles. Namely, the scalar "nucleon" $N$ radiates the scalar "pions" $\pi$. In this simple example, the Fock space is restricted to $N$, $N\pi$ and $N\pi\pi$ states. Represented as a series of graphs in perturbation theory, it contains an infinite number of irreducible contributions to the self energy. They diverge and require renormalization. At large value of the coupling constant this system cannot be solved perturbatively. We show how to determine, in a self-consistent manner, the non-perturbative mass counterterm. This counterterm is then calculated numerically. Models involving two and three constituent bound states were also analyzed in [7].

In a more general field-theoretical framework, in the Light-Front Tamm-Dancoff approximation involving spin 1/2 particles for instance, renormalization is not reduced to the introduction of a mass counter term. In this case, one should introduce sector-dependent counter terms, as shown in refs. [8]. In the two-nucleon sector, for spin 1/2 particles, additional box divergences appear [9]. These divergences and the sector dependent counter terms are absent in the scalar model restricted to the dressed "one-scalar-nucleon" system. Therefore, their analysis is beyond the scope of the present paper. In spite of that, the scalar model is still rather instructive, since the renormalization considered here is not reduced to the perturbative one. We will consider very large coupling constant, excluding the convergence of the perturbative series.
The plan of the article is as follows. In section 2 we establish the general equations of motion for the Fock components. In the truncated Fock space, the corresponding system of equations, which determine the Fock components and the mass renormalization, is detailed in sect. 3. The renormalization of the wave function and of the coupling constant is calculated in sections 4 and 5 respectively. Numerical results are presented and discussed in section 6. We present our conclusions in section 7.

2 Eigenstate equation

We start with the general eigenstate equation for the state vector \( \phi(p) \):

\[
\hat{P}^2 \phi(p) = M^2 \phi(p),
\]

(1)

where

\[
\hat{P}_\mu = \hat{P}_\mu^0 + \hat{P}_\mu^{int}. 
\]

(2)

We have decomposed here the momentum operator \( \hat{P}_\mu \) into two parts: the free one, \( \hat{P}_\mu^0 \), and the interacting one \( \hat{P}_\mu^{int} \), given by:

\[
\hat{P}_\mu^0 = \sum_i \int d_i^\dagger(\vec{k})d_i(\vec{k})k_\mu \ d^3k,
\]

\[
\hat{P}_\mu^{int} = \omega_\mu \int H^{int}(x)\delta(\omega\cdot x) \ d^4x = \omega_\mu \int_{-\infty}^{+\infty} \tilde{H}^{int}(\omega\tau) \frac{d\tau}{2\pi},
\]

(3)

where we have denoted by \( \tilde{H}^{int} \) the Fourier transform of the interaction Hamiltonian:

\[
\tilde{H}^{int}(p) = \int H^{int}(x) \exp(-ip\cdot x) d^4x,
\]

(4)

and \( d_i^\dagger \ (d_i) \) corresponds to the creation (destruction) operator for the various particles under consideration. The explicitly covariant formulation of LFD manifests itself in the fact that \( \hat{P}_\mu^{int} \) in eq. (3) is proportional to \( \omega_\mu \) and is determined by the integral over the light-front plane \( \omega\cdot x = 0 \).

The equations for the Fock components can be obtained from (1), by substituting there the Fock decomposition for the state vector \( \phi(p) \) and calculating the matrix elements of \( \hat{P}^2 \) in the Fock space. With the above expressions for \( \hat{P} \), eq. (1) obtains the form:

\[
\left[ (\hat{P}_\mu^0)^2 + (\omega \cdot \hat{P}_\mu^0) \int \tilde{H}^{int}(\omega\tau) \frac{d\tau}{2\pi} + \int \tilde{H}^{int}(\omega\tau) \frac{d\tau}{2\pi} (\omega \cdot \hat{P}_\mu^0) \right] \phi(p) = M^2 \phi(p).
\]

(5)
In order to simplify this equation, we can use the fact that the operators \((\omega \cdot \hat{P}^0)\) and \(\int \tilde{H}^{\text{int}}(\omega \tau) d\tau\) commute. Indeed, from the commutation relation \([\hat{P}_\mu, \hat{P}_{\nu}] = 0\) we get:

\[
[\omega \cdot \hat{P}, \hat{P}_\nu] = [\omega \cdot \hat{P}^0, \hat{P}_\nu + \hat{P}_{\nu}^{\text{int}}] = [\omega \cdot \hat{P}^0, \hat{P}_{\nu}^{\text{int}}] = 0 .
\]

Moreover, since \(\omega^2 = 0\), we can replace here and below \((\omega \cdot \hat{P}^0)\) by \(\omega \cdot \hat{P}\). We thus obtain:

\[
(\omega \cdot \hat{P}) \int \tilde{H}^{\text{int}}(\omega \tau) \phi(p) d\tau = \int (\omega \cdot \hat{P}) \tilde{H}^{\text{int}}(\omega \tau) \phi(p) d\tau = (\omega \cdot p) \int \tilde{H}^{\text{int}}(\omega \tau) \phi(p) d\tau ,
\]

and the equation (5) is transformed to:

\[
2(\omega \cdot p) \int \tilde{H}^{\text{int}}(\omega \tau) \frac{d\tau}{2\pi} \phi(p) = - \left[ (\hat{P}^0)^2 - M^2 \right] \phi(p) .
\]

The state vector \(\phi(p)\) is now decomposed in Fock components according to:

\[
\phi(p) = (2\pi)^{3/2} \int \phi_1(k_1, p, \omega, \omega \tau) a^\dagger(\vec{k}_1)|0\rangle \delta(4)(k_1 - p - \omega \tau)2(\omega \cdot p)d\tau \frac{d^3k_1}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_1}}}
\]

\[
+ (2\pi)^{3/2} \int \phi_2(k_1, k_2, p, \omega, \omega \tau) a^\dagger(\vec{k}_1)b^\dagger(\vec{k}_2)|0\rangle \delta(4)(k_1 - k_2 - p - \omega \tau)2(\omega \cdot p)d\tau \frac{d^3k_1}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_1}}} \frac{d^3k_2}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_2}}}
\]

\[
\times \delta(4)(k_1 + k_2 + k_3 - p - \omega \tau)2(\omega \cdot p)d\tau \frac{d^3k_1}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_1}}} \frac{d^3k_2}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_2}}} \frac{d^3k_3}{(2\pi)^{3/2} \sqrt{2\varepsilon_{k_3}}} + \cdots ,
\]

where \(\varepsilon_{k_i} = \sqrt{k_i^2 + m_i^2}\) and \(m_i\) is the mass of the particle \(i\) of momentum \(k_i\). We introduce in (6) spinless particles of two types: the ”nucleon” (with creation operator \(a^\dagger\)) and the ”pion” (with creation operator \(b^\dagger\)). The state vector (6) represents the ”dressed nucleon”, consisting of the ”bare nucleon” and the admixture of one, two, \(\ldots\), many ”pions”. Our dressed nucleon is also scalar, therefore the state vector (6) corresponds to zero total angular momentum. This means that the Fock components are scalars and depend only on scalar products of all available four-vectors.

The conservation law for the momenta in each Fock component has the form:

\[
k_1 + k_2 + \cdots + k_n = p + \omega \tau .
\]

Hence, the action of the operator \((\hat{P}^0)^2 - M^2\) in (6) on the state vector \(\phi(p)\) is reduced to the multiplication of each Fock component by the factor \((\sum k_i)^2 - M^2 = 2(\omega \cdot p)\).
If we introduce the notation:

\[ G(p) = 2(\omega \cdot p) \hat{\tau} \phi(p), \]

where \( \hat{\tau} \) is the operator which, acting on a given component \( \phi_i \) of \( \phi \), gives \( \tau \phi_i \). \( G \) has therefore a Fock decomposition which differs from (7) by the replacement of the wave functions \( \phi_i \) by the vertex parts \( \Gamma_i \) given by:

\[ \Gamma_i \equiv 2(\omega \cdot p) \tau \phi_i = (s - M^2)\phi_i, \quad (8) \]

where \( s = (\sum k_i)^2 \). We thus find the eigenvalue equation:

\[ \frac{1}{2\pi} \int \tilde{H}^{int}(\omega \tau) \frac{d\tau}{\tau} G(p) = -G(p) \equiv -\lambda(M^2)G(p). \quad (9) \]

We introduce in (9) the factor \( \lambda(M^2) \) depending on \( M^2 \). The eigenvalue \( M^2 \) is found from the condition \( \lambda(M^2) = 1 \). This equation is quite general and equivalent to the eigenstate equation (3).

3 Equation for the Fock components

3.1 System of coupled integral equation

For the simplified model we consider in this study, we take the following interaction Hamiltonian:

\[ H = -g\psi^2(x)\varphi(x) \quad (10) \]

where the scalar field \( \psi \) with mass \( m \) corresponds to the scalar "nucleon" and the field \( \varphi \) with mass \( \mu \) corresponds to the scalar "pion".

The system of equations for these vertex parts is shown graphically in fig. 4. In order to write down this system of equations, it is enough to apply to the diagrams of fig.
the rules of the graph technique detailed in ref. [5]. We thus find:

\[
\begin{align*}
\Gamma_1(k_1, p, \omega \tau_1) &= \delta m^2 \int \Gamma_1(k'_1, p, \omega \tau') \delta^{(4)}(k'_1 - p - \omega \tau') \theta(\omega \cdot k'_1) \delta(k'^2_1 - m^2) d^4 k'_1 \frac{d\tau'}{\tau' - i0} \\
&+ \frac{g}{(2\pi)^3} \int \Gamma_2(k'_1, k'_2, p, \omega \tau') \delta^{(4)}(k'_1 + k'_2 - p - \omega \tau') \\
&\times \theta(\omega \cdot k'_1) \delta(k'^2_1 - m^2) d^4 k'_1 \theta(\omega \cdot k'_2) \delta(k'^2_2 - \mu^2) d^4 k'_2 \frac{d\tau'}{\tau' - i0} ,
\end{align*}
\]

\begin{align*}
\Gamma_2(k_1, k_2, p, \omega \tau_2) &= g \int \Gamma_1(k'_1, p, \omega \tau') \delta^{(4)}(k'_1 - p - \omega \tau') \theta(\omega \cdot k'_1) \delta(k'^2_1 - m^2) d^4 k'_1 \frac{d\tau'}{\tau' - i0} \\
&+ \delta m^2 \int \Gamma_2(k'_1, k'_2, p, \omega \tau') \delta^{(4)}(k'_1 + k'_2 - p - \omega \tau') \\
&\times \theta(\omega \cdot k'_1) \delta(k'^2_1 - m^2) d^4 k'_1 \frac{d\tau'}{\tau' - i0} \\
&+ \frac{1}{(2\pi)^3} \int \Gamma_3(k'_1, k'_2, k'_3, p, \omega \tau') \delta^{(4)}(k'_1 + k'_2 + k'_3 - p - \omega \tau') \\
&\times \theta(\omega \cdot k'_1) \delta(k'^2_1 - m^2) d^4 k'_1 \theta(\omega \cdot k'_3) \delta(k'^2_3 - \mu^2) d^4 k'_3 \frac{d\tau'}{\tau' - i0} ,
\end{align*}

\begin{align*}
\Gamma_3(k_1, k_2, k_3, p, \omega \tau_3) &= g \int \Gamma_2(k'_1, k'_2, k'_3, p, \omega \tau') \delta^{(4)}(k'_1 + k'_2 - p - \omega \tau') \\
&\times \theta(\omega \cdot k'_1) \delta(k'^2_1 - m^2) d^4 k'_1 \frac{d\tau'}{\tau' - i0} .
\end{align*}

Figure 1: Diagrammatical representation of the eigenvalue equation projected onto the Fock components of the state vector (restricted here to \(N=3\)). For the sake of clarity, we remove all kinematical variables.
Figure 2: Perturbative expansion, in terms of the pion-nucleon coupling constant $g$, of the nucleon self-energy.

Figure 3: First order perturbative expansion of the nucleon self-energy.

The origin of the mass counterterm $\delta m^2$ in these equations is explained below. Since we truncate the Fock space to three particles, we omit in the last equation (11c) the coupling of $\Gamma_3$ to the four-body component $\Gamma_4$. Note that our approximation is not based on a perturbative expansion in terms of the coupling constant. It is based on a decomposition over intermediate states with an increasing number of particles. Once this number is fixed, we solve a non-perturbative problem in terms of $g$. If we iterate the system (11), i.e., express $\Gamma_3$ and $\Gamma_2$ from the third and the second equations and substitute it in the first one, we get for $\Gamma_1$ the sum of the diagrams shown on fig. 2. Of course, this series of graphs corresponds to all the irreducible contributions to the self energy with intermediate states up to $N\pi\pi$. These contributions are iterated again, i.e., they appear repetitively on the nucleon line. In contrast to the case of the intermediate state $N\pi$, which generates only one self energy diagram shown on fig. 3, the number of irreducible contributions generated by the intermediate states up to $N\pi\pi$ is infinite. The system of equations (11) corresponds to the sum of all of them.

In the right-hand side of equations (11a) and (11b) we introduced the counterterm $\delta m^2$ corresponding to the interaction Hamiltonian $H = -\delta m^2 \psi^2(x)$. This term provides mass renormalization. The counterterm is however not introduced in the third equation containing the three-body intermediate state. At this point we have some freedom in the definition of the model Hamiltonian. In principle, the counterterm could be also introduced into the three-body intermediate state, since formally it does not increase the number of particles in the intermediate state. Our motivation in choosing the Hamilto-
nian without the counterterm in the last n-th Fock sector is the following. Consider, for example, the mass operator \(-g^2\Sigma(p^2)\) in second order perturbation theory. It is given by the diagram of fig. 3 with two particles (N and \(\pi\)) in the intermediate state. The counterterm \(\delta m^2\), which is shown by a cross on the nucleon line (one particle N in the intermediate state) just renormalizes this mass operator. Therefore, for the renormalization of the mass operator determined by the sum of (an infinite number of) irreducible diagram with \(n\) particles in the intermediate state, one should consider the graphs, with the counterterm insertions, with \(n-1\) particles in the intermediate states only \([4]\). These diagrams are just generated by the Hamiltonian without the counterterm in the \(n\)-th Fock sector.

We can easily transform eqs. (11) by performing the integrations which do not involve loops, keeping the loop integrals untouched. The result is the following:

\[
\Gamma_1(k_1, p, \omega \tau_1) = \frac{\delta m^2}{2(\omega p)\tau_1} \Gamma_1(k_1, p, \omega \tau_1)
\]
\[+ g \frac{1}{(2\pi)^3} \int \Gamma_2(k'_1, k'_2, p, \omega \tau') \delta^{(4)}(k'_1 + k'_2 - p - \omega \tau') \]
\[\times \theta(\omega k'_1)\delta(k'_1^2 - m^2)d^4k'_1\theta(\omega k'_2)\delta(k'_2^2 - \mu^2)d^4k'_2 \frac{d\tau'}{\tau' - i0}, \tag{12a}\]

\[
\Gamma_2(k_1, k_2, p, \omega \tau_2) = \frac{g}{2(\omega p)\tau_1} \Gamma_1(p_1, p, \omega \tau_1)
\]
\[+ \frac{\delta m^2}{2(\omega p)\tau_2}\Gamma_2(k_1, k_2, p, \omega \tau_2)
\]
\[+ g \frac{1}{(2\pi)^3} \int \Gamma_3(k'_1, k_2, k'_3, p, \omega \tau') \delta^{(4)}(k'_1 + k_2 + k'_3 - p - \omega \tau') \]
\[\times \theta(\omega k'_1)\delta(k'_1^2 - m^2)d^4k'_1\theta(\omega k'_3)\delta(k'_3^2 - \mu^2)d^4k'_3 \frac{d\tau'}{\tau' - i0}, \tag{12b}\]

\[
\Gamma_3(k_1, k_2, k_3, p, \omega \tau_3) = \frac{g}{2(\omega p)\tau_2(1 - x_2)} \Gamma_2(k'_1, k_2, p, \omega \tau'_2)
\]
\[+ \frac{g}{2(\omega p)\tau'_2(1 - x_3)} \Gamma_2(k''_1, k_3, p, \omega \tau''_2), \tag{12c}\]

In eqs. (12a) and (12b), \(2(\omega p)\tau_1 = k_1^2 - p^2 = m^2 - p^2 \to 0\) when \(p^2 \to m^2\). Since \(\tau_1\) is in the denominator, we keep \(p^2 \neq m^2\) and take the limit \(p^2 \to m^2\) in the final equation. In eq. (12b), \(p_1 = p - \omega \tau_1\) with \(2(\omega p)\tau_2 = s_{12} - m^2\) and \(s_{12} = (k_1 + k_2)^2\). In eq. (12c) we use the notations:

\[
2(\omega p)\tau'_2 = s'_{12} - m^2, \quad 2(\omega p)\tau''_2 = s''_{12} - m^2,
\]

where

\[
s'_{12} = (k'_1 + k_2)^2, \quad s''_{12} = (k''_1 + k_3)^2
\]

8
\[
\begin{align*}
\Gamma_1 & = \Gamma_2 + \Gamma_3 \\
\Gamma_2 & = \Gamma_1
\end{align*}
\]

Figure 4: Diagrammatical representation of the eigenvalue equation in first order perturbation theory.

and \( k'_1, k''_1 \) are determined by the conservation laws:

\[
k'_1 = k_1 + k_3 + \omega \tau'_2 - \omega \tau_{123}, \quad k''_1 = k_1 + k_2 + \omega \tau''_2 - \omega \tau_{123},
\]

where \( \omega \tau_{123} \) is the momentum of the spurion line entering the diagram. Everywhere we note \( x_i = \omega \cdot k_i / \omega \cdot p \).

3.2 Reduction to two-particle Fock states

We consider first the approximation in which the state vector (7) contains only the bare nucleon \( N \) and the state \( N\pi \). In this approximation, the system of equations for one-body and two-body Fock components is represented diagrammatically on fig. 4. It is easily obtained by omitting \( \Gamma_3 \) in (11c), together with the counterterm in the equation which determines the last Fock sector \( \Gamma_2 \). This truncation of the Fock space, retaining the minimal number of components, is equivalent to second order perturbation theory.

Rewritten in terms of the variables \( \vec{q} \) and \( \vec{n} \) (see appendix A), this system of equations obtains the simple form:

\[
\begin{align*}
\Gamma_1 & = \frac{\delta m^2}{m^2 - p^2} \Gamma_1 + g \int \sigma(\vec{q}', p^2) \Gamma_2(\vec{q}', \vec{n}) \frac{d^3 q'}{(2\pi)^3}, \\
\Gamma_2(\vec{q}, \vec{n}) & = \frac{g}{m^2 - p^2} \Gamma_1,
\end{align*}
\]

where \( \sigma(\vec{q}, p^2) \) is the integrand which determines the self-energy \( \Sigma(p^2) \):

\[
\begin{align*}
\Sigma(p^2) & = \int \sigma(\vec{q}', p^2) \frac{d^3 q'}{(2\pi)^3}, \\
\sigma(\vec{q}, p^2) & = \frac{1}{2 (s_{12} - p^2 - i0)} \frac{1}{\varepsilon(q, m) \varepsilon(q, \mu)}.
\end{align*}
\]
\[
\Gamma_1 = \Gamma_2 + \Gamma_1 \delta m^2 \\
\Gamma_2 = \Gamma_1 + \Gamma_2 \delta m^2 \\
+ \Gamma_2 + \Gamma_2
\]

Figure 5: Same as Figure 1, but where the last Fock sector \(n=3\) has been expressed in terms of the two-body one.

with

\[
s_{12} = (k_1 + k_2)^2 = [\varepsilon(\vec{q}, m) + \varepsilon(\vec{q}, \mu)]^2, \varepsilon(\vec{q}, m) = \sqrt{m^2 + q^2},
\]

and similarly for \(\varepsilon(\vec{q}, \mu)\). Since \(\Gamma_1\) does not depend on the relative momentum, it follows from eqs. (14) that \(\Gamma_2(\vec{q}, \vec{n})\) does not depend on the relative momentum too, i.e., \(\Gamma_2(\vec{q}, \vec{n}) = const\). Hence, we get:

\[
\left(-1 + \frac{\delta m^2_0}{m^2 - p^2}\right) \Gamma_1 + g \Sigma(p^2) \Gamma_2 = 0,
\]

\[
\frac{g}{m^2 - p^2} \Gamma_1 - \Gamma_2 = 0.
\]

From (17) we find the eigenvalue equation:

\[
p^2 = m^2 - \delta m^2_0 - g^2 \Sigma(p^2).
\]

The counter term \(\delta m^2_0\) is determined from the on-shell condition \(p^2 = m^2\), where \(m\) is the physical mass of the nucleon. This gives:

\[
\delta m^2_0 = -g^2 \Sigma(m^2),
\]

as expected in second order calculation of mass renormalization.

### 3.3 Solution for the three particle system

Consider now the system of equations which incorporates the three-body Fock component \(\Gamma_3\). It was shown graphically on fig. 1. We can easily express \(\Gamma_3\) through \(\Gamma_2\). The system of equations is thus transformed as shown in fig. 5. Its analytical representation has the
form:

\[ \Gamma_1 = \frac{\delta m^2}{m^2 - p^2} \Gamma_1 + g \int \sigma(\vec{q}', p^2) \Gamma_2(\vec{q}', \vec{n}) \frac{d^3 q'}{(2\pi)^3}, \] (20a)

\[ \Gamma_2(\vec{q}, \vec{n}) = \frac{g}{m^2 - p^2} \Gamma_1 + \frac{\delta m^2 + g^2 \Sigma(s_1)}{(s_{12} - p^2) x_k1} \Gamma_2(\vec{q}, \vec{n}) 
+ g^2 \int \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) \sigma(\vec{q}', p^2) \Gamma_2(\vec{q}', \vec{n}) \frac{d^3 q'}{(2\pi)^3}. \] (20b)

Each term in eqs. (20) is represented by a graph on fig. 5. In these equations, \( s \) is given by (16), and:

\[ s_1 = (k_1 - \omega \tau)^2 = m^2 - x_{k1} (s_{12} - p^2), \]
\[ x_{k1} = \frac{\omega \cdot k_1}{\omega \cdot p} = \frac{1}{\sqrt{s_{12}}} [\varepsilon(\vec{q}, m) - \vec{n} \cdot \vec{q}]. \]

In this equation, \( \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) \) is the propagator of the second intermediate state:

\[ \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) = \int \theta[k_1 - k_2'] \delta[(k_1 - k_2 + \omega \tau - \omega \tau')^2 - m^2] \frac{d \tau''}{\tau'' - i0} \]
\[ = \frac{\theta[k_1 - k_2]}{m^2 - (k_1' - k_2 - \omega \tau')^2}. \] (21)

Its expression in terms of the variables \( \vec{q}, \vec{q}', \vec{n} \) can be easily calculated with the kinematics detailed in the appendix.

The system of equation (20) can now be solved by two independent methods.

i) The first one consists in the elimination of \( \Gamma_1 \) in the two equations in (20). The result is an equation for \( \Gamma_2 \) which, at \( p^2 = m^2 \), obtains the form:

\[ \Gamma_2(\vec{q}, \vec{n}) = -\frac{g^2}{\delta m^2} \int \sigma(\vec{q}', p^2) \Gamma_2(\vec{q}', \vec{n}) \frac{d^3 q'}{(2\pi)^3}
+ \frac{g^2 \Sigma(s_1)}{(s_{12} - p^2) x_k1} \Gamma_2(\vec{q}, \vec{n}) 
+ g^2 \int \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) \sigma(\vec{q}', p^2) \Gamma_2(\vec{q}', \vec{n}) \frac{d^3 q'}{(2\pi)^3}
\equiv \lambda(\delta m^2) \Gamma_2(\vec{q}, \vec{n}). \] (22)

Like in eq. (9), we introduce in (22) the factor \( \lambda(\delta m^2) \). The mass renormalization \( \delta m^2 \) will be found from the non-perturbative condition \( \lambda(\delta m^2) = 1 \). ii) The second method consists in the direct elimination of \( \Gamma_2 \) in the two equations (20). We can indeed rewrite eq. (20b) in the following form:

\[ \left[ \Gamma_2(\vec{q}, \vec{n}) - \frac{\delta m^2 + g^2 \Sigma(s_1)}{(s_{12} - p^2) x_k1} \Gamma_2(\vec{q}, \vec{n}) 
- g^2 \int \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) \sigma(\vec{q}', p^2) \Gamma_2(\vec{q}', \vec{n}) \frac{d^3 q'}{(2\pi)^3} \right] = \frac{g}{m^2 - p^2} \Gamma_1. \] (23)
After discretization of the momenta, this equation can be written schematically:

$$A_{ij} \Gamma_2^j = \frac{g}{m^2 - p^2} \Gamma_1 [1]^i,$$  \hspace{1cm} (24)

where [1] is the vector whose every component is 1, and $A_{ij}$ is a two-dimensional matrix obtained after discretization of the momenta (index $i$ for $q, \vec{n} \cdot \vec{q}$ and index $j$ for $q', \vec{n} \cdot \vec{q}'$). The vertex function $\Gamma_2$ can now be expressed in terms of $\Gamma_1$ after a simple matrix inversion:

$$\Gamma_2^i = \frac{g}{m^2 - p^2} \Gamma_1 A^{-1}_{ij} [1]^j.$$  \hspace{1cm} (25)

After insertion in eq.(20a), we end up with an equation involving $\Gamma_1$ only. Since $\Gamma_1$ is a non-zero constant, it can be removed from the equation, leading to an equation for $\delta m^2$ generalizing eq. (19). In the perturbative limit, the matrix $A$ reduces to the unit matrix, and we therefore recover eq. (19) exactly.

## 4 Renormalization of the wave function

### 4.1 Non-perturbative case

We schematically rewrite the state vector $|p\rangle$, given by eq. (7), as:

$$|p\rangle = \phi_1 |N\rangle + \phi_2 |N\pi\rangle + \phi_3 |N\pi\pi\rangle.$$  \hspace{1cm} (7)

It is normalized as follows [5]:

$$\langle p'|p \rangle = 2p_0 \delta^{(3)}(\vec{p} - \vec{p}') .$$  \hspace{1cm} (26)

The Fock components are thus normalized in order to provide the condition (26). Substituting the state vector (7) in the left-hand side of eq. (26), we get, before normalization of the Fock components:

$$\langle p'|p \rangle = Z \ 2p_0 \delta^{(3)}(\vec{p} - \vec{p}') ,$$  \hspace{1cm} (27)

where

$$Z = N_1 + N_2 + N_3 ,$$

with:

$$N_1 = \phi_1 ^2 ,$$  \hspace{1cm} (28a)

$$N_2 = \frac{1}{2(2\pi)^3} \int \phi_2^2(q, \vec{n}) \frac{[\epsilon(q, m) + \epsilon(q, \mu)]}{\epsilon(q, m) \epsilon(q, \mu)} d^3q ,$$  \hspace{1cm} (28b)

$$N_3 = \frac{1}{4(2\pi)^6} \int \phi_3^2(q_1, q_2, q_3, \vec{n}) \delta^{(3)}(\vec{q}_1 + \vec{q}_2 + \vec{q}_3) \times \frac{[\epsilon(q_1, m) + \epsilon(q_2, \mu) + \epsilon(q_3, \mu)]}{\epsilon(q_1, m) \epsilon(q_2, \mu) \epsilon(q_3, \mu)} d^3q_1 d^3q_2 d^3q_3 .$$  \hspace{1cm} (28c)
From $\Gamma_2$ calculated with eq. (22), we can find $\phi_2$ according to:

$$\phi_2(\vec{q}, \vec{n}) = \frac{\Gamma_2(\vec{q}, \vec{n})}{s_{12} - m^2}. \tag{29}$$

The calculation of $N_2$ is then straightforward. To calculate $N_1$ and $N_3$ we should know $\phi_1$ and $\phi_3$. The calculation of $\phi_1$ and $\phi_3$ is explained in appendix B. Note that $\phi_1$ does not depend on any relative momentum. The normalized state vector satisfying the condition (26) obtains the form:

$$\mid p \rangle = \phi_{ren}^1 \mid N \rangle + \phi_{ren}^2 \mid N \pi \rangle + \phi_{ren}^3 \mid N \pi \pi \rangle,$$

where

$$\phi_{ren}^{1,2,3} = \phi_{1,2,3}/\sqrt{Z}. \tag{30}$$

We can introduce the creation operator of the new, composed field, creating directly the state $\mid p \rangle$. It is written schematically as:

$$A^\dagger(\vec{p}) = \phi_{ren}^1 a^\dagger + \phi_{ren}^2 a^\dagger b^\dagger + \phi_{ren}^3 a^\dagger b^\dagger b^\dagger, \tag{31}$$

so that $\mid p \rangle = A^\dagger(\vec{p}) \mid 0 \rangle$. With the renormalized wave functions $\phi_{ren}$, the vacuum expectation value of the commutator: $\langle 0 \mid [A(\vec{p}), A^\dagger(\vec{p}^\prime)] \mid 0 \rangle = 2p_0 \delta^{(3)}(\vec{p} - \vec{p}^\prime)$ is the same as the one-body operators $a, a^\dagger$ (except for the normalization factor $2p_0$). So the state $\mid p \rangle$ which is a dressed state in terms of the bare operators $a^\dagger, b^\dagger$, can be interpreted, in this sense, as an elementary particle in terms of the operator $A^\dagger$.

### 4.2 Perturbative case

The perturbative case is simply obtained from the preceding one by omitting $\phi_3$. Taking into account that from eq. (14) we have $\Gamma_1/(m^2 - p^2) = \phi_1$, we find: $\Gamma_2 = g\phi_1$. Substituting it into (29), we find $\phi_2$ and then, by eq. (28b), we obtain $N_2$:

$$N_2 = \phi_1^2 g^2 I_2, \text{ where } I_2 = \frac{1}{16\pi^3} \int \frac{\sqrt{s} d^3q}{(s_{12} - m^2)^2 \varepsilon(q,m)\varepsilon(q,\mu)}. \tag{32}$$

We thus find the ratio:

$$\frac{N_2}{N_1} = 16\pi m^2 \alpha I_2 \approx 0.38\alpha \tag{33}$$

with $\alpha = \frac{q^2}{10\pi m^2}$. The numerical value of $I_2$ is given for $m = 0.94, \mu = 0.14$. The integral $I_2$ is logarithmically divergent for $\mu \to 0$. The two-body contribution is rapidly decreasing, when the mass of the intermediate particle increases.
5 Renormalization of the coupling constant

As mentioned above, the mass renormalization counterterm $\delta m^2$, for the state vector incorporating $N$ and $N\pi$ states only is given by eq. (19) and coincides with the perturbative result. For the state vector incorporating the states $N$, $N\pi$ and $N\pi\pi$, it is determined by eq. (22). This renormalization constant is infinite when the cutoff tends to infinity.

The coupling constant is also renormalized, though its renormalization is finite for the particular scalar system we are interested in the present study. We show below how this renormalization can be carried out. This renormalization is a “by-product” of the renormalization of the wave function fulfilled in the previous section. Let us consider first the case of the state vector containing $N$ and $N\pi$ states only.

5.1 Truncation to $N$ and $N\pi$ states

In this approximation, the normalization factor $N_2$ is given by eq. (32). We thus get:

$$Z = N_1 + N_2 = \phi_i^2 Z_1 \quad \text{with} \quad Z_1 = 1 + g^2 I_2,$$

where $I_2$ is given by eq. (32). After renormalization the wave function $\phi_2$ turns therefore into:

$$\phi_2 = \frac{g\phi_1}{s - m^2} \rightarrow \phi_2^{\text{ren}} = \frac{g}{\sqrt{Z_1}(s - m^2)} = \frac{g_{\text{ren}}}{s - m^2}$$

where we introduced the renormalized coupling constant

$$g_{\text{ren}} = g/\sqrt{Z_1}.$$  

This value of $g_{\text{ren}}$ can be also represented as the residue of the two-body wave function at $s = m^2$, i.e. the value of $\Gamma_2(q, z)$ at the nonphysical value of $q$ corresponding to $s = m^2$.

One can alternatively define $g_{\text{ren}}$ from the $\pi N$ scattering amplitude determined by $N$ exchange in the $s$-channel:

$$F = \frac{g^2}{m^2 - p^2} + \frac{g^2}{m^2 - p^2} (\delta m^2 + g^2 \Sigma(p^2)) \frac{g^2}{m^2 - p^2} + \ldots = \frac{g^2}{m^2 - p^2 - \delta m^2 - g^2 \Sigma(p^2)}$$

where $p^2 = (k_1 + k_2)^2 - 2(\omega p)\tau$. Near the pole $p^2 = m^2$ we get:

$$F = \frac{g^2}{(m^2 - p^2)(1 + g^2 d\Sigma(p^2)/dp^2|_{p^2=m^2})} = \frac{g_{\text{ren}}^2}{m^2 - p^2}$$

where we introduced $g_{\text{ren}}$ by eq. (36), but with $Z_1$ given by:

$$Z_1 = 1 + g^2 d\Sigma(p^2)\bigg|_{p^2=m^2}.$$
Taking eqs. (15) for $\Sigma$ one can easily check that the equations (34) and (39) determine the same $Z_1$.

Note that the renormalized coupling constant $g_{\text{ren}}$ is finite and it is always smaller than the bare one $g$. When the bare constant $g$ tends to infinity, the constant $g_{\text{ren}}$ remains however finite, but it reaches its maximal value. Expressed in terms of $\alpha = g^2/(16\pi m^2)$ it has the form:

$$\alpha_{\text{ren}}^{\text{max}} = 1/(16\pi m^2 I_2).$$

It does not depend on $g$. According to (33), its numerical value for $m = 0.94$ and $\mu = 0.14$ is: $\alpha_{\text{ren}}^{\text{max}} = 1/0.38 = 2.63$.

5.2 Truncation to $N$, $N\pi$ and $N\pi\pi$ states

In this case the renormalized wave functions are given by eq. (30) with $Z$ determined by eqs. (27,28). Note that the renormalization constant $Z$ can still be represented in the form (39) with $\Sigma(p^2)$ determined by all the irreducible contributions (see appendix C).

Restricting ourselves to the system considered above and having found the renormalized state vector $|p\rangle$, we can get all the physical information (we can for example calculate the electromagnetic form factors, if the particles are charged). So we do not need in practice to define and calculate the renormalized coupling constant $g_{\text{ren}}$. However, it is useful to calculate it for further generalization to the case of particles with spin, when the charge renormalization constant will become infinite.

As we already mentioned, the standard definition of the coupling constant is the residue of the wave function $\phi_2$ at $s = m^2$, i.e., the value of $\Gamma_2(q, z)$ at the non-physical value of $q = i\kappa$ corresponding to $s = m^2$. For a non-relativistic bound state calculation for instance, it is given by $\kappa = \sqrt{m|\epsilon_b|}$, where $\epsilon_b$ is the binding energy of the bound state. In coordinate space it is the coefficient of the asymptotical behavior of the wave function $\psi(r \to \infty) \propto \exp(-\kappa r)$. Since we calculate $\Gamma_2(q, z)$ numerically in the physical region for $q$, it is not easy to find numerically with enough accuracy its interpolation into the non-physical region. However, we do not need to choose the renormalization point $s = m^2$, but we can choose any other renormalization point. We can define for instance the renormalized coupling constant as the value of $\Gamma_2(q, z)$ at $q = 0$:

$$\tilde{g}_{\text{ren}}^2 = \Gamma_2^{\text{ren}}(q = 0, z).$$

Note that $\Gamma_2(q = 0, z)$ does not depend on $z$. In the case of $N + N\pi$ intermediate states $\Gamma_2$ does not depend on $q$ and the renormalized coupling constants $\tilde{g}_{\text{ren}}$ and $g_{\text{ren}}$ coincide with each other.

6 Numerical results

Both methods to solve eqs. (11) are used to cross-check our results. For regularization purposes, we introduce a cutoff $L$, i.e. integrate in (22) over modulae of all the relative
three-momenta $q$ until $q \leq L$. Note that this cut-off procedure preserves rotational invariance. For the masses, we choose the nucleon and pion masses: $m = 0.94$ GeV and $\mu = 0.14$ GeV. The integration over the azimuthal angle is done analytically. The equation is reduced to a matrix form by discretizing the integral. The convergence of the integrals is already obtained for 30 points in the variable $q$ and 15 points in the variable $z$. The points in $q$ where not taken equally spaced, but with a spacing proportional to $h^2$, where $h$ is the equal spacing in the variable $\sqrt{q}$. In the first method, the eigenvalue $\lambda(\delta m^2)$ of the matrix is found numerically and $\delta m^2$ is fixed to get $\lambda \equiv 1$ to less than 0.5%. For the second method, $\delta m^2$ is calculated by a standard iteration procedure, starting from the perturbative result. Since both methods give identical results to less than 1% we only quote the results obtained with the second method.

The results for the dimensionless coupling constant $\alpha = g^2/(16\pi m^2) = 3$ and for different values of the cutoff parameter $L$ are shown in Table 1. We denote by $\delta m^2/\delta m_0^2$ the ratio of the value $\delta m^2$ found from eq. (22), to the perturbative value $\delta m_0^2$, given by eq. (19). The values $N_{1,2,3}$ denote the contributions of the corresponding Fock sectors to the normalization of the state vector. In order to characterize the wave function quantitatively, we calculate also the average value of the (absolute) relative momentum $<q>$, normalized to the two-body Fock component:

$$<q> = \frac{1}{N_2} \frac{1}{(2\pi)^3} \int \phi_2^2(q, \vec{n}) \frac{[\varepsilon(q, m) + \varepsilon(q, \mu)]}{\varepsilon(q, m) \varepsilon(q, \mu)} q d^3q.$$  

![Table 1](image)

Table 1: Numerical results for $\alpha = 3$, as a function of the cut-off $L$.

One clearly sees that though $\delta m^2$ increases logarithmically, the contributions of the Fock components $N_1, N_2, N_3$, as well as the average momentum $<q>$ in the two-body Fock component become stable after $L = 5$. This means that we indeed found numerically the renormalized solution for the wave function.

One can see also that the nonperturbative value of $\delta m^2$ is very close to the perturbative one $\delta m_0^2$. We have checked that the function $\Gamma_2(q, z)$ is almost constant, as expected for the perturbative solution. The solution for $\delta m^2$ remains very close to the perturbative
result also for higher values of the coupling constant. The results for \( L = 200 \) and for different \( \alpha \) between 1 and 1000 are shown in table 2. One can see that \( \delta m^2 \) is very close to the perturbative value even for extremely large coupling constant \( \alpha = 1000 \). As expected, the three-body sector dominates, when the coupling constant increases. However, the ratio \( N_2/N_1 \) is still close to its perturbative value \((N_2/N_1)_0\), given by eq. (33).

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\alpha & 1 & 3 & 10 & 30 & 100 \\
\hline
\delta m^2/\delta m_0^2 & 1.011 & 1.019 & 1.029 & 1.035 & 1.040 \\
\hline
N_1 & 0.661 & 0.325 & 0.081 & 0.016 & 2. E-3 \\
\hline
N_2 & 0.292 & 0.457 & 0.366 & 0.186 & 0.068 \\
\hline
N_3 & 0.047 & 0.218 & 0.553 & 0.798 & 0.930 \\
\hline
N_2/N_1 & 0.441 & 1.41 & 4.51 & 11.91 & 32.9 \\
\hline
(N_2/N_1)_0 & 0.38 & 1.14 & 3.8 & 11.4 & 38 \\
\hline
\end{array}
\]

Table 2: Same as Table 1, but with \( L \) fixed to 200, and \( \alpha \) is varied.

The reason that the solution is close to the perturbative one lies in the super-renormalizability of the scalar theory. The third term in eq. (22) converges and does not require any cutoff, whereas the first two terms are divergent and dominate. Therefore, a very small departure of \( \delta m^2 \) from its perturbative value \( \delta m_0^2 \) can accommodate the finite higher order correction. If the third term can be neglected, the equation is approximately satisfied with the perturbative value of \( \delta m^2 \). In order to show that it is indeed so, we introduce the cutoff in the two divergent terms, but do not introduce it in the third, convergent one. The results for \( \alpha = 3 \) are shown in table 3. For small enough value of \( L \) the two first terms are suppressed, and the solution drastically differs from the perturbative one. When \( L \) increases, the solution becomes closer and closer to the perturbative one.

\[
\begin{array}{|c|c|c|c|}
\hline
L & 0.1 & 1 & 10 \\
\hline
\delta m^2/\delta m_0^2 & 36.3 & 1.55 & 1.07 \\
\hline
\end{array}
\]

Table 3: Test calculation in which the cutoff \( L \) is introduced only in the two first terms in eq. 22 (see text), for \( \alpha = 3 \).

We emphasize that the small deviation of the solution from the perturbative one, even for very large values of the coupling constant, is just a property of the non-perturbative equation (22), which includes the contributions (with one-, two- and three-body intermediate states) to all orders of \( g \). This result cannot be justified in any perturbative expansion in terms of \( g \).
Conclusion

In a first attempt to address the question of non-perturbative renormalization in CLFD, we have investigated in this study a simple, but nevertheless meaningful, model based on two scalar particles. This model is reminiscent of the structure of the physical nucleon in the low energy regime, in terms of bare nucleons coupled to pions.

Using the nice features of CLFD, we have first derived the general eigenstate equation that should be used in order to calculate any physical state vector. We emphasize here that this equation is quite general and is not restricted to the case of scalar particles nor to the restricted Fock space we consider in this study. It should therefore also be used when solving more complex systems as QED or QCD.

The results we obtained for the simple scalar model, with a restricted Fock space expansion up to three particles, are quite encouraging. We obtained, numerically, a renormalized solution using a simple mass counter term. Surprisingly enough, this counter term is not very different from the perturbative, logarithmically divergent, mass counterterm, even for very large values of the coupling constant. We traced back this feature to the nature of our scalar model we start from, which is super-renormalizable. This result however does not imply that higher Fock states are negligible. We find that the first non-trivial Fock component gets larger and larger as the coupling constant increases.

The direct generalization of this study is the investigation of non-perturbative renormalization in scalar QED, following the study of perturbative renormalization in QED already done in ref. [6] in CLFD. This will be the subject of a forthcoming publication.

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A Kinematics

We show in this appendix, how to express the propagator (21) in terms of the variables $\vec{q}, \vec{q}', \vec{n}$.

The two-body wave function depends on the following four-vectors:

$$\phi = \phi(k_1, k_2, p, \omega \tau), \quad k_1 + k_2 = p + \omega \tau.$$  \hspace{1cm} (42)
We introduce the variables:

\[ \vec{q} = L^{-1}(\mathcal{P})\vec{k}_1 = \vec{k}_1 - \frac{\vec{p}}{\sqrt{\mathcal{P}^2}}[k_{10} - \frac{\vec{k}_1 \cdot \vec{p}}{\sqrt{\mathcal{P}^2 + \mathcal{P}_0}}], \quad (43) \]

\[ \vec{n} = L^{-1}(\mathcal{P})\vec{\omega}/|L^{-1}(\mathcal{P})\vec{\omega}| = \sqrt{\mathcal{P}^2}L^{-1}(\mathcal{P})\vec{\omega}/\omega \cdot \vec{p}, \quad (44) \]

where

\[ \mathcal{P} = p + \omega \tau, \quad (45) \]

and \( L^{-1}(\mathcal{P}) \) is the Lorentz boost. The wave function under the integral (see the last diagram in fig. 3) depends on \( k'_1, k'_2, p, \omega \tau' \), and, correspondingly, on the variable:

\[ \vec{q}' = L^{-1}(\mathcal{P}')\vec{k}'_1, \quad (46) \]

where

\[ \mathcal{P}' = p + \omega \tau'. \quad (47) \]

In order to obtain \( \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) \), we should express the four-momenta \( k'_1, k'_2, \omega \tau' \) in \( (k'_1 - k'_2 - \omega \tau')^2 \) in terms of \( \vec{q}, \vec{q}', \vec{n} \). Since under the Lorentz transformations and the rotations of the four-vectors the variables \( \vec{q}, \vec{q}', \vec{n} \) are rotated only, the expression for the scalar \( \Pi(\vec{q}, \vec{q}', \vec{n}, p^2) \) does not depend on the system of reference, and, hence, can be found in the most convenient one. We find it in the system where

\[ \vec{P}' = \vec{k}'_1 + \vec{k}'_2 = \vec{p} + \vec{\omega} \tau' = 0. \]

In this system \( \vec{k}'_1 = \vec{q}' \) and \( k'_{10} = \varepsilon(q', m) \). From the conservation law the vector \( \vec{P} \) is expressed as: \( \vec{P} = \vec{k}_1 + \vec{k}_2 = \vec{\omega}(\tau - \tau') \). We have:

\[ \tau = \frac{s - p^2}{2(\omega p)}, \quad \tau' = \frac{s' - p^2}{2(\omega p)}, \quad (48) \]

where \( \sqrt{s} = \varepsilon(\vec{q}, m) + \varepsilon(\vec{q}, \mu), \sqrt{s'} = \varepsilon(\vec{q}', m) + \varepsilon(\vec{q}', \mu) \). Since \( \vec{\omega} = \omega_0 \vec{n} \) and \( \omega \cdot p = \omega(k'_1 + k'_2) = \omega_0(\varepsilon(\vec{q}', m) + \varepsilon(\vec{q}', \mu)) = \omega_0 \sqrt{s'} \), we find

\[ \vec{P} = \vec{n} \frac{s - s'}{2\sqrt{s'}}, \quad \mathcal{P}_0 = \frac{s + s'}{2\sqrt{s'}}. \quad (49) \]

One can check that \( \mathcal{P}^2 = s \). According to eqs. (43) and (46), the variables \( \vec{q} \) and \( \vec{q}' \) are defined by different Lorentz boosts. From (19) one can see that this difference is the boost in the direction of \( \vec{n} \). This boost does not change the unit vector \( \vec{n} \).
Now let us find \( \vec{k}_2 \). It is obtained by reverting (43):

\[
\vec{k}_2 = L(\mathcal{P})(-\vec{q}) = -\vec{q} + \frac{\vec{p}}{\sqrt{\vec{p}^2}} [\varepsilon(\vec{q}, \mu) - \frac{\vec{q} \cdot \vec{P}}{\sqrt{\vec{P}^2 + P_0}}] .
\] (50)

Eq. (50) is obtained from (43) by replacing in the r.h.s \( \vec{k}_1 \) by \(-\vec{q}\) and by changing the sign at \( \vec{P} \).

Substituting here eqs. (49) for \( P \), we find:

\[
\vec{k}_2 = -\vec{q} + \vec{n}_s - s' \sqrt{\frac{s}{ss'}} \left( \varepsilon(\vec{q}, \mu) - \vec{n} \cdot \vec{q} \sqrt{s - s'} \right) .
\] (51)

and similarly for \( k_{20} \):

\[
k_{20} = \frac{\varepsilon(\vec{q}, \mu) \mathcal{P}_0}{\sqrt{\mathcal{P}^2}} - \frac{\vec{q} \cdot \vec{P}}{\sqrt{\mathcal{P}^2}} = \varepsilon(\vec{q}, \mu) \frac{s + s'}{2 \sqrt{ss'}} - \vec{n} \cdot \vec{q} \frac{s - s'}{2 \sqrt{ss'}} .
\] (52)

From (48) we get:

\[
\vec{ω}_\tau' = \vec{n} \frac{s' - p^2}{2 \sqrt{s'}} , \quad ω_0 \tau' = \frac{s' - p^2}{2 \sqrt{s'}} .
\]

Substituting the above expressions for the four-momenta into \((k'_1 - k_2 - ω τ')^2\), we find that the four-vector squared in the denominator of the propagator (21) is expressed in terms of the variables \( \vec{q}, \vec{q}', \vec{n}, \bar{n} \) as follows:

\[
(k'_1 - k_2 - ω τ')^2 = \left[ \varepsilon(\vec{q}', m) - \varepsilon(\vec{q}, \mu) \frac{s + s'}{2 \sqrt{ss'}} + \vec{n} \cdot \vec{q} \frac{s - s'}{2 \sqrt{ss'}} - \frac{s' - p^2}{2 \sqrt{s'}} \right]^2
- \left[ \vec{q}' + \vec{q} - \vec{n} \frac{s - s'}{2 \sqrt{ss'}} \left( \varepsilon(\vec{q}, \mu) - \vec{n} \cdot \vec{q} \frac{\sqrt{s - s'}}{\sqrt{s + s'}} \right) - \vec{n} \frac{s' - p^2}{2 \sqrt{s'}} \right]^2 .
\] (53)

### B Calculation of \( φ_1 \) and \( φ_3 \)

The Fock components \( φ_1 \) and \( φ_3 \) are calculated as follows. The wave function \( φ_2 \) in (28a) is related to the vertex functions \( Γ_2 \) by eq. (24). Having found numerically the vertex function \( Γ_2(\vec{q}, \bar{n}) = Γ_2(q, z) \) from the equation (22), we then find by (29) the Fock component \( φ_2 \).

With eqs. (12a), reduced to the first equation in (14), we express \( Γ_1 \) through \( Γ_2 \) in the limit \( p^2 \to m^2 \), and then find the component \( φ_1 \):

\[
φ_1 = \frac{g}{δm^2} \int \Sigma_i(\vec{q}', \mu^2) Γ_2(\vec{q}', \bar{n}) \frac{d^3q'}{(2\pi)^3} .
\] (54)
With eqs. (12c), we express \( \Gamma_3 \) through \( \Gamma_2 \) and then find \( \phi_3 \):

\[
\phi_3(\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{n}) = \frac{g \phi_2(q', z')}{(s_{123} - m^2)(1 - x_2)} + \frac{g \phi_2(q'', z'')}{(s_{123} - m^2)(1 - x_3)}.
\]

(55)

Here \( \phi_2(q', z') \) is \( \phi_2(k'_1, k'_2, p, \omega \tau'_2) \) represented through the relative momentum, and \( \phi_2(q'', z'') \) is \( \phi_2(k''_1, k''_2, p, \omega \tau''_2) \).

The wave function \( \phi_2 \) in the r.h.s. of eq. (55) depends on the variables defined in the center of mass of the two-body subsystem. We shall now express these variables in terms of the three-body relative momenta \( \vec{q}_1, \vec{q}_2, \vec{q}_3 \).

The variable \( q' \) is related to \( s'_{12} \) by \( s'_{12} = [\varepsilon(q', m) + \varepsilon(q', \mu)]^2 \). We thus get:

\[
q'^2 = \left[ m^4 + (\mu^2 - s'_{12})^2 - 2m^2(\mu^2 + s'_{12}) \right] / (4s'_{12}) .
\]

where, by taking the first of eqs. (13) squared:

\[
s'_{12} = s_{123} - \frac{s_{13} - m^2}{1 - x_2},
\]

with:

\[
s_{123} = (\varepsilon(q_1, m) + \varepsilon(q_2, \mu) + \varepsilon(q_3, \mu))^2,
\]

\[
s_{13} = (k_1 + k_3)^2 = (\varepsilon(q_1, m) + \varepsilon(q_3, \mu))^2 - q_2^2,
\]

\[
x_1 = \frac{\varepsilon(q_1, m) - \vec{n} \cdot \vec{q}_1}{\sqrt{s_{123}}}, \quad x_{2,3} = \frac{\varepsilon(q_{2,3}, \mu) - \vec{n} \cdot \vec{q}_{2,3}}{\sqrt{s_{123}}}.
\]

(56)

We use the fact that \( \vec{q}_1 + \vec{q}_2 + \vec{q}_3 = \vec{0} \).

The value of \( z' \) is found by comparing expressions (54) for \( x_1 \) with:

\[
x_1 = \frac{\varepsilon(q', m) - z'q'}{\varepsilon(q', m) + \varepsilon(q', \mu)}.
\]

This gives:

\[
z' = [\varepsilon(q', m) - x_1(\varepsilon(q', m) + \varepsilon(q', \mu))] / q',
\]

where \( x_1 \) is given by (56) in terms of \( \vec{q}_1, \vec{q}_2, \vec{q}_3 \).

The values of \( q'', z'' \) are found from the equations:

\[
q''^2 = \left[ m^4 + (\mu^2 - s''_{12})^2 - 2m^2(\mu^2 + s''_{12}) \right] / (4s''_{12}) ,
\]

\[
z'' = [\varepsilon(q'', m) - x_1(\varepsilon(q'', m) + \varepsilon(q'', \mu))] / q'',
\]

where:

\[
s''_{12} = s_{123} - \frac{s_{12} - m^2}{1 - x_3},
\]

with \( s_{12} = (k_1 + k_2)^2 = [\varepsilon(q_1, m) + \varepsilon(q_2, \mu)]^2 - q_3^2 \).
C Proof of eq. (39) in the general case

We show below that the renormalization constant $Z$ can still be represented in the form (39) with $\Sigma(p^2)$ determined by all the irreducible contributions. In this case, the number of irreducible diagrams containing two- and three-body intermediate states is infinite. Some of them are shown in fig. 2. Consider, for example, a contribution containing $n$ successive intermediate states. The corresponding amplitude contains the factor $1/\tau_i$ for any of these states. Like in eq. (15), any factor $1/\tau_i$ turns into $1/(s_i - p^2)$, where $s_i$ is the invariant energy of a given intermediate state and $p$ is the external incoming/outgoing momentum. So, indicating only these factors, we represent this contribution to $\Sigma(p^2)$ as:

$$\Sigma_n(p^2) = \int \prod_{i=1,\ldots,n} \frac{1}{s_i - p^2} \ldots$$

where the dots include all the integrations with the corresponding measures. Derivative over $p^2$ gives the factor $1/(s_i - p^2)^2$.

Now consider the graphs which are the same at the right to a given $i$-th intermediate state and differ from each other by the contributions at the left to this state. The infinite sum of them determines the amplitude of the virtual transition from the initial state $N$ to the states $N\pi$ or $N\pi\pi$. It is the vertex $\Gamma_2$ (if the state $i$ is the two-body state), or $\Gamma_3$ (if the state $i$ is the three-body state). We can then take the sum over all the contributions to the right of this given state $i$ and again obtain $\Gamma_2$ or $\Gamma_3$. So, the result has the form

$$\frac{d\Sigma(p^2)}{dp^2} = \int \frac{\Gamma_2^2}{(s_i - p^2)^2} \ldots + \int \frac{\Gamma_3^2}{(s_i - p^2)^2} \ldots = \int \phi_2^2 \ldots + \int \phi_3^2 \ldots = N_2 + N_3$$

and, after extraction of the common factor $N_1$, we recover eq. (39).

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