Embeddings of Schur functions into types $B/C/D$

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Abstract
We consider the problem of embedding the semi-ring of Schur-positive symmetric polynomials into its analogue for the classical types $B/C/D$. If we preserve highest weights and add the additional Lie-theoretic parity assumption that the weights in images of Schur functions lie in a single translate of the root lattice, there are exactly two solutions. These naturally extend the Kirillov–Reshetikhin decompositions of representations of symplectic and orthogonal quantum affine algebras $U_q(\hat{g})$ (some still conjectural, some recently proven).

1 Introduction and Background

Consider the infinite-dimensional vector space $Y$ over $\mathbb{R}$ whose basis elements $\{v_\lambda\}$ are indexed by Young diagrams $\lambda$, i.e. by all partitions of $n$ for all $n \geq 0$. There are two natural ring structures on $Y$, owing to the fact that Young diagrams can be used to index the irreducible representations of the various classical Lie groups. One multiplication arises from the decomposition of tensor products in type $A$; its structure constants are the familiar Littlewood–Richardson coefficients. The other arises in the same way from tensor product decomposition in the other classical types $B$, $C$ and $D$ (remarkably, all three series give the same multiplication).

Our goal is to understand the embeddings of the former ring, $Y_A$, into the latter, $Y_{BCD}$, in which

$$v_\lambda \mapsto v_\lambda + \sum_{\mu < \lambda} m_{\lambda\mu} v_\mu, \quad m_{\lambda\mu} \geq 0.$$

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Here $\mu < \lambda$ denotes the extended dominance order on partitions: given $\lambda = (\lambda_1, \ldots, \lambda_r)$ and $\mu = (\mu_1, \ldots, \mu_s)$, we say $\mu \leq \lambda$ if $\mu_1 + \cdots + \mu_k \leq \lambda_1 + \cdots + \lambda_k$ for $1 \leq k \leq s$, where $\lambda_i = 0$ for $i > r$. (The order is “extended” because we do not require that $|\lambda| = |\mu|$.)

The purpose of this paper is first to present a construction which builds such an embedding out of any polynomial or formal power series $p(x) \in \mathbb{R}[[x]]$ with the property that the symmetric function $\kappa_p := \prod_i p(x_i)/\prod_{1<i<j}(1-x_i x_j)$ has a positive expansion in terms of Schur functions. The resulting embedding acts on $v_\lambda$ by skewing $\lambda$ by $\kappa_p$, in a way we will define precisely in Section 2. Second, we show that this construction yields all of the desired embeddings.

Before we state our final and motivating result, we briefly clarify the connection between $Y$ and representations of classical groups. The Young diagrams $\lambda = (\lambda_1, \ldots, \lambda_r)$ with $\ell(\lambda) := r \leq n$ rows index the dominant integral highest weights, and therefore the irreducible finite-dimensional representations, of $SL(n+1)$, $SO(2n+1)$, and $Sp(2n)$ (types $A$, $B$ and $C$, respectively). The situation in $SO(2n)$ (type $D$) is slightly different; here they parametrize the restrictions to $SO(2n)$ of irreducible representations of $O(2n)$, which split into two conjugate irreducibles if $\lambda_n > 0$.

One remarkable property of this parameterization is that for fixed Young diagrams $\lambda$, $\mu$, the multiset of diagrams giving the summands in the tensor product of the $\lambda$ and $\mu$ representations is independent of $n$, provided $n$ is sufficiently large (in particular, $n > \ell(\lambda) + \ell(\mu)$). Moreover, in this “stable limit” the differences between types $B$, $C$ and $D$ vanish, giving us the ring structure we referred to above as $Y_{BCD}$. For a full discussion of these topics see the fundamental work of Koike and Terada [8], to which we will refer in greater detail later.

Returning to the type of embeddings described in equation (4) above, we can explain the constraints in the language of representation theory. Now our goal is to find a family of representations $\{W(\lambda)\}$ of type $B/C/D$ which are a homomorphic image of the irreducible representations $\{V(\lambda)\}$ of type $A$. The embedding now describes the decomposition of $W(\lambda)$ into irreducibles, and in the representation theoretic setting we naturally demand that the constants $m_{\lambda\mu}$ are nonnegative integers. The Lie theoretic meaning of $\mu < \lambda$ is that $\lambda - \mu$ is a sum of positive roots, guaranteeing that $W(\lambda)$ has $\lambda$ as its maximal weight. This agrees with the above definition of the extended dominance order with the additional constraint that $|\lambda| \equiv |\mu|$ mod 2, to require that the weights $\lambda$ and $\mu$ are in the same translate of the root lattice.

We can now state our final result: given the Lie-theoretic integrality and parity constraints, there are exactly two embeddings. In the above construction, they come from $p(x) = 1$ and $p(x) = \frac{1}{1-x} = 1+x^2+x^4+\cdots$. The Schur function expansions of $\kappa_p$ is as the sum of all $s_\lambda$, where $\lambda$ ranges over all partitions which have only even length columns or rows, respectively. The two embeddings are intertwined by the involution on $Y$ which takes the basis vector $v_\lambda$ to $v_{\lambda'}$, where $\lambda'$ is the transpose Young diagram.

In the special case that $\lambda$ is a rectangle, the resulting representations $W(\lambda)$ coincide with restrictions of certain irreducible representations of the quantum
affine algebra $U_q(\hat{g})$ to $U_q(g)$. These decompositions were originally conjectured by Kirillov and Reshetikhin \cite{Kirillov1990} and recently proved for simply-laced classical $g$ by Chari \cite{Chari1993}; see Section 4 for some details. The two Lie-theoretic embeddings give the decompositions for the orthogonal and symplectic cases. There is evidence to support the natural hope that our two embeddings answer the same quantum groups question when $\lambda$ is not rectangular.

The organization of the rest of this paper is as follows. In Section 2 we present the aforementioned construction, and we state and discuss our main theorem (Theorem 2.3), which asserts that this construction yields all of the embeddings we seek. The proof of the main theorem is presented in Section 3. Finally, in Section 4 we discuss the connections to Lie theory, and in particular the way in which these results generalize the Kirillov–Reshetikhin representations of quantum affine algebras.

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2 Constructing Embeddings

In this section we work in the ring $\Lambda$ of symmetric functions in countably many variables with rational coefficients (or, when necessary, in a completion $\hat{\Lambda}$ where we allow infinite formal sums of basis elements) and follow the standard notation of Macdonald’s book \cite{Macdonald1995}. Recall in particular that the Schur functions $\{s_\lambda\}$, the “universal characters” of the polynomial representations of $GL(n)$, are an orthonormal basis of $\Lambda$ with respect to the standard inner product, and that the map $\phi_A : Y \to \Lambda$ which takes $\lambda$ to $s_\lambda$ is an embedding of the ring structure we called $Y_A$. We use $\emptyset$ to denote the empty partition of zero; $s_\emptyset = 1$ in $\Lambda$.

Extending the Schur function picture to the other classical types, Koike and Terada \cite{Koike1997} defined two additional bases $\{sp_\lambda\}$ and $\{o_\lambda\}$ of the same ring, consisting of analogous universal characters for the symplectic and orthogonal groups. The maps $\phi_C : \lambda \mapsto sp_\lambda$ and $\phi_{BD} : \lambda \mapsto o_\lambda$ are two different embeddings of the ring structure we called $Y_{BCD}$.

We can use these embeddings to rephrase our quest for maps $Y_A \to Y_{BCD}$ from equation (3) in terms of automorphisms of $\Lambda$, with the minor inconvenience that we must break symmetry by picking one of $\phi_C$ or $\phi_{BD}$. We arbitrarily choose $\phi_C$ in this and the next section.

**Theorem 2.1** Let $p(x) = 1 + a_1x + a_2x^2 + \cdots$ be a polynomial or formal power series with $p(0) = 1$, and define

$$\kappa_p = \frac{\prod_{i=1}^\infty p(x_i)}{\prod_{1 \leq i < j < \infty}(1 - x_ix_j)} \in \hat{\Lambda}.$$
Then the map $f_p : \Lambda \to \Lambda$ defined by

$$f_p(s_\lambda) = \phi_C \circ \phi_A^{-1}(\kappa_p^\perp s_\lambda)$$

is a ring homomorphism. Here $\phi_C \circ \phi_A^{-1}$ is the map $s_\mu \mapsto sp_\mu$, while $\kappa_p^\perp$ indicates skewing by $\kappa_p$, the adjoint to multiplication by $\kappa_p$.

**Proof.** We show that the map $f_p = \phi_C \circ \phi_A^{-1} \circ \kappa_p^\perp$ is the automorphism of $\Lambda$ defined on the homogeneous symmetric functions $h_n$ by $h_n \mapsto h_n + a_1 h_{n-1} + a_2 h_{n-2} + \cdots + a_{n-1} h_1 + a_n$. This completely defines the map, as the $h_n$ are an algebraically independent set of generators for $\Lambda$. Although $\kappa_p$ generally has an infinite expansion in terms of Schur functions, recall that $s_\mu^\perp s_\lambda = s_{\lambda/\mu} = 0$ unless $\mu \subseteq \lambda$, so $\kappa_p^\perp s_\lambda \in \Lambda$. Thanks to R. Stanley for showing us a special case of the following argument.

We will rewrite $f_p$ using the inner product $\langle \cdot, \cdot \rangle$ instead of $\phi_C \circ \phi_A^{-1}$ to transform Schur functions to their symplectic analogues. We will need two sets of variables $\{x_i\}$ and $\{y_j\}$; the inner products are all taken with respect to the $x$ variables.

$$f_p(s_\lambda(y)) = \sum_\mu \langle \kappa_p^\perp s_\lambda(x), s_\mu(x) \rangle_x sp_\mu(y)$$

$$= \left\langle s_\lambda(x), \kappa_p \left( \sum_\mu s_\mu(x) sp_\mu(y) \right) \right\rangle_x$$

$$= \left\langle s_\lambda(x), \frac{\prod_i p(x_i)}{\prod_{i<j}(1-x_ix_j)} \frac{\prod_{i<j}(1-x_iy_j)}{\prod_{i<j}(1-x_{i,j})} \right\rangle_x$$

The expansion $\sum_\mu s_\mu(x) sp_\mu(y)$ for $\prod_{i<j}(1-x_ix_j)/\prod_{i,j}(1-x_iy_j)$ is found in [9]. Cancelling, we continue:

$$= \left\langle s_\lambda(x), \prod_i p(x_i) \prod_i \left( \sum_{n \geq 0} h_n(y) x_i^n \right) \right\rangle_x$$

$$= \left\langle s_\lambda(x), \prod_i \sum_{n \geq 0} (h_n(y) + a_1 h_{n-1}(y) + \cdots) x_i^n \right\rangle_x$$

$$= s_\lambda(y) \bigg|_{h_n(y) \mapsto h_n(y) + a_1 h_{n-1}(y) + \cdots}$$

This last equality is the ring homomorphism $h_n(y) \mapsto h_n(y) + a_1 h_{n-1}(y) + \cdots$ applied to the Cauchy identity $\prod_{i,j}(1-x_iy_j)^{-1} = \sum_\mu s_\mu(x) s_\mu(y)$. ■

**Example 2.2** There are three noteworthy cases in which the Schur function expansion of $\kappa_p$ was recognized by Littlewood (11.9 in [8], p. 238):

$$1/ \prod_{i<j} (1-x_ix_j) = \sum_\lambda s_{(2\lambda)}$$

$$1/ \prod_i (1-x_i^2) \prod_{i<j} (1-x_ix_j) = \sum_\lambda s_{2\lambda}$$

$$1/ \prod_i (1-x_i) \prod_{i<j} (1-x_ix_j) = \sum_\lambda s_\lambda$$
That is, for \( p(x) = 1 \), \( p(x) = 1/(1 - x^2) \), and \( p(x) = 1/(1 - x) \), we can expand \( \kappa_p \) as the sum of \( s_\lambda \) for all \( \lambda \) with even column heights, all \( \lambda \) with even row lengths, and all \( \lambda \), respectively.

When \( p(x) = 1 \) and \( f_p \) is the identity, this gives us the Character Interrelation Theorem of Koike and Terada \[8\], which expands \( s_\lambda \) in the \( sp \)-basis. For a concrete example, let us compute \( \kappa_p \) on \( s_\lambda \) for \( \lambda = (322) \). Using the above expansion, we see that \( \kappa_p \) takes \( s_\lambda \) to \( \sum_\mu s_\lambda/\mu \) where \( \mu \) runs over all partitions with even column heights. Since \( s_{\lambda/\mu} = 0 \) unless \( \mu \subseteq \lambda \), the only \( \mu \) which contribute are \( (22) \), \( (11) \), and the empty partition:

\[
\kappa_p s_{(322)} = s_{(322)/(22)} + s_{(322)/(11)} + s_{(322)} = (s_{(3)} + s_{(21)}) + (s_{(311)} + s_{(221)}) + s_{(322)}
\]

So \( s_{(322)} = f_p(s_{(322)}) = sp_{(322)} + sp_{(311)} + sp_{(221)} + sp_{(3)} + sp_{(21)}. \)

We can now state our main theorem.

**Definition.** Say \( p(x) \) is \( \kappa \)-positive if \( \kappa_p \) is \( s \)-positive, \( i.e. \) has all nonnegative coefficients when written in the basis of Schur functions. For instance, the above comments show that \( 1/(1 - x) \), and \( 1/(1 - x^2) \) are all \( \kappa \)-positive.

**Theorem 2.3 (Main Theorem)** Let \( f : \Lambda \to \Lambda \) be a ring homomorphism such that \( f(s_\lambda) = sp_\lambda + \sum_{\mu < \lambda} m_{\lambda\mu} sp_\mu \) for some constants \( m_{\lambda\mu} \).

(a) If all \( m_{\lambda\mu} \geq 0 \) then \( f = f_p \) for some \( \kappa \)-positive \( p \), as in Theorem 2.4.

(b) If we add the Lie-type assumption that all \( m_{\lambda\mu} \) are integers and \( m_{\lambda\mu} = 0 \) unless \( |\lambda| \equiv |\mu| \mod 2 \), then \( p(x) \) is either \( 1 \) or \( 1/(1 - x^2) \).

We will defer the work of proving Theorem 2.3 to the next section. For now we make the following straightforward observations about part (a):

1. The converse is clear: if \( p \) is \( \kappa \)-positive then \( f_p \) certainly takes \( s_\lambda \) to something \( sp \)-positive, since \( s_{\lambda/\mu} \) is always \( s \)-positive.

2. If \( f = f_p \) and all \( m_{\lambda\mu} \geq 0 \) then \( p \) is certainly \( \kappa \)-positive: the only time \( s_\theta \) appears in \( s_\mu \) is when \( \lambda = \mu \), so the coefficient of \( s_\lambda \) in \( \kappa_p \) is \( m_{\lambda\theta} \geq 0 \).

3. As a corollary of the theorem, all \( m_{\lambda\mu} \) being nonnegative implies \( m_{\lambda\mu} \) is zero unless \( \mu \subseteq \lambda \), a much stronger condition than \( \mu \leq \lambda \).

Observations about part (b) and connections to Lie theory will be made in Section 3. In contrast with having only two solutions in the Lie case, the full space of \( \kappa \)-positive polynomials and formal power series is quite large and appears somewhat messy.

**Problem 1** Give a closed-form characterization of the \( \kappa \)-positive \( p(x) \).

Given \( p(x) = 1 + a_1x + a_2x^2 + \cdots \), the coefficient of \( s_\lambda \) in \( \kappa_p \) is some polynomial in the \( a_i \) for each \( \lambda \), and it seems \emph{a priori} possible that no finite set of conditions is equivalent to every one of those polynomials being positive. This problem, in other words, may have no good answer. We make two observations on \( \kappa \)-positivity.
Proposition 2.4 \( p(x) \) is \( \kappa \)-positive whenever \( \prod_i p(x_i) \) is \( s \)-positive.

This is clear because \( \kappa_p \) is the above product multiplied by the \( s \)-positive \( \kappa_1 \).
Stanley \[1\] observed that this \( s \)-positivity condition is equivalent to a question of Schoenberg’s asking which \( p \) are the generating functions for sequences giving rise to totally positive infinite Toeplitz matrices. The answer was proved partially by Aissen, Schoenberg and Whitney \[1\] and completed by Edrei \[4\] and independently by Thoma \[12\]: \( p \) must be of the form

\[
p(x) = e^{\gamma x} \prod_i (1 + \alpha_i x)/\prod_j (1 - \beta_j x)
\]

where \( \alpha_i, \beta_j, \gamma \) are nonnegative real numbers and \( \sum_i \alpha_i, \sum_j \beta_j \) converge. If \( p(x) \) is a polynomial, this is just demanding that all of its roots be real and negative. For an extended discussion see \[12\] (ex. 7.91, pp. 481, 543ff).

The converse does not hold (consider \( 1/(1-x^2) \)), and computational evidence suggests that there are relatively open sets of \( p(x) \) which are \( \kappa \)-positive without meeting the \( s \)-positivity criterion. Consider the generic quadratic \( p(x) = 1 + bx + ax^2 \). For \( a \geq a_0 \) for some \( a_0 \) (perhaps \( a_0 = 1 \)), numerical evidence suggests that the positivity of the coefficients of \( s_{(2^t,1^t)} \) in \( \kappa_p \) for all \( t \) imply that \( b^2 \geq 4a \), so \( \kappa \) and \( s \)-positivity coincide. But for \( a \leq a_1 \approx 0.39816 \ldots \) (the real root of \( 2z^3 + 3z^2 + z - 1 \)), the limiting coefficient seems to be that of \( s_{(32211)} \), which only forces \( b \geq \frac{a^\sqrt{3\pi}}{4a-\pi} \).

Proposition 2.5 The involution \( p(x) \mapsto \frac{1}{1-x^2}p(-x) \) preserves \( \kappa \)-positivity.

Taking the transpose of Young diagrams commutes with both multiplications (\( Y_A \) and \( Y_{BCD} \)), and we will show it induces the above involution on the set of \( \kappa \)-positive \( p \). Let \( \omega \) be the standard involution of \( \Lambda \) (or \( \hat{\Lambda} \)) taking \( s_\lambda \) to \( s_{\lambda'} \). We will say that \( p(x) \) and \( q(x) \) are dual if \( \kappa_p = \omega \kappa_q \). The expansions in terms of Schur functions cited in Example 2.2 show that 1 and \( 1/(1-x^2) \) are dual to one another, while \( 1/(1-x) \) is self-dual.

First note that \( \omega \) acts on the symmetric functions \( \prod_i p(x_i) \) by \( p(x) \mapsto 1/p(-x) \); this can be derived after applying \( f_p \) to the well-known relation \( \sum h_n x^n = (\sum e_n(-x)^n = 1 \). As we observed above, \( \kappa_p = \kappa_1 \prod_i p(x_i) \), and we know \( \omega \kappa_1 = \kappa_1/(1-x^2) \). We conclude that the dual of \( p(x) \) is \( 1/(1-x^2)p(-x) \).

Note that we have shown that \( \kappa_p^\perp \) takes \( e_n \) to \( e_n + b_1 e_{n-1} + b_2 e_{n-2} + \cdots \), and thus that \( f_p \) takes \( s_{p(1^n)} \) to \( s_{p(1^n)} + b_1 s_{p(1^{n-1})} + b_2 s_{p(1^{n-2})} + \cdots \), where \( q(x) = 1 + b_1 x + b_2 x^2 + \cdots \) is dual to \( p \).

Perhaps a classification of \( \kappa \)-positive \( p(x) \) with integer coefficients is possible. These would describe automorphisms of \( \Lambda_Z \), the ring of symmetric functions with integer coefficients (which many authors just call \( \Lambda \)). Given two polynomials \( r(x), t(x) \in \mathbb{Z}[x] \) with \( r(0) = t(0) = 1 \) and all roots real and negative, the \( s \)-positive \( r(x)/t(-x) \) and its dual are \( \kappa \)-positive; we do not know of any other integer examples.
3 Proof of the Main Theorem

In this section we present arguments and calculations to prove Theorem 2.3. The calculations involve finding relations among various $m_{\lambda \mu}$, the coefficients of $sp_\mu$ in $f(s_\lambda)$, generally by applying $f$ to the dual Jacobi–Trudi formula expressing $s_\lambda$ as a determinant of a matrix of elementary symmetric functions $e_n$.

These computations, naturally, will require expanding products $sp_\mu sp_\nu$ into $\sum_\lambda d^\lambda_{\mu \nu} sp_\lambda$, where the symplectic structure constants $d^\lambda_{\mu \nu}$ are not as universally familiar as their type-$A$ analogues, the Littlewood–Richardson coefficients $c^\lambda_{\mu \nu}$.

For what follows it suffices to know that $d^\lambda_{\mu \nu} = c^\lambda_{\mu \nu}$ when $|\lambda| = |\mu| + |\nu|$, and $d^\lambda_{\mu \nu} = 0$ unless $|\lambda| = |\mu| + |\nu| - 2k$ for some $k \in \mathbb{Z}_{\geq 0}$.

The reader disinclined to descend into the depths of occasionally cumbersome computation is invited to skip everything after the statements of Propositions 3.2 and 3.3 until their respective ■s. The logic of the overall argument should not be lost.

Proposition 3.1 Let $f$ be a ring homomorphism such that $f(s_\lambda) = sp_\lambda + \sum_{\mu \neq \lambda} m_{\lambda \mu} sp_\mu$. We will denote $m_{(1^i)(1^j)}$ with $i > j$ by $m_{ij}$.

The $m_{ij}$ completely determine $f$, and each $m_{\lambda \mu}$ can be expressed as a polynomial in the $m_{ij}$. If we say $m_{ij}$ has degree $i - j$, then each monomial of $m_{\lambda \mu}$ is of degree at most $|\lambda| - |\mu|$ in the $m_{ij}$.

When $\lambda = (1^j)$ is a single column, the only $\mu < \lambda$ are $(1^i)$ for $j < i$, so knowing the $m_{ij}$ is equivalent to knowing the image of $s_{(1^i)}$ for all $i$. (We use the dual form of the Jacobi-Trudi identity so often precisely because there are so few $\mu < \lambda$ when $\lambda$ is a single column, but not a single row.) Since $s_{(1^i)}$ is the elementary symmetric function $e_i$ and the $e_i$ generate the ring $\Lambda$, the constants $m_{ij}$ determine the function $f$.

More concretely, the dual Jacobi–Trudi formula says $s_\lambda = det(e_{\lambda' - i + j})$, where $\lambda' = (\lambda'_1, \ldots, \lambda'_r)$ is the transpose of $\lambda$, and the determinant is of an $r \times r$ matrix. We apply $f$ to both sides and find $m_{\lambda \mu}$ is the coefficient of $sp_\mu$ in the determinant of the matrix whose $i, j$th entry is $sp_{(1^i)} + m_{k,k-1} sp_{(1^{k-1})} + \cdots + m_{k,0} sp_0$, where $k = \lambda'_i - i + j$. This is clearly a polynomial in the $m_{ij}$.

If we assign to $m_{ij}$ degree $i - j$ and assign to $sp_\mu$ degree $|\nu|$, then the $i, j$th matrix entry is homogeneous of degree $\lambda'_i - i + j$. This degree can only decrease when we expand products, since $sp_\mu sp_\nu$ is a sum of $sp_\lambda$ with $|\lambda| \leq |\mu| + |\nu|$. So the terms of $f(s_\lambda)$ have degree at most $|\lambda|$, and the coefficient $m_{\lambda \mu}$ is of degree at most $|\lambda| - |\mu|$.

When we construct $f_\mu$ from a formal power series $p(x)$ as in Theorem 2.4, we get $m_{ij} = b_{i-j}$, where $q(x) = 1 + b_1 x + b_2 x^2 + \cdots$ is the dual of $p(x)$ in the sense of Proposition 2.5. Therefore Theorem 2.3(a) is precisely the claim that $m_{ij}$ depends only on $i - j$.

We proceed by induction on $d = i - j$. Fix some integer $d > 0$ for the remainder of this proof. Take some $f$ determined by its $m_{ij}$, and suppose $m_{ij}$ is some constant $b_{i-j}$ whenever $i - j < d$. Our task is to show that $m_{k,k-d}$ is independent of $k$. 


To this end, construct $f_p$ from $p(x)$, the dual of the finite polynomial $q(x) = 1 + b_1 x + \cdots + b_{d-1} x^{d-1}$, as above. This $f_p$ will also remain fixed for the duration: we can think of it as the degree-$d$ polynomial truncation of our arbitrary embedding $f$. Let $m_{\lambda \mu}^{(p)}$ denote the constants describing $f_p$, just as $m_{\lambda \mu}$ describe $f$. By Proposition 3.1, we know that $m_{\lambda \mu}^{(p)} = m_{\lambda \mu}$ whenever $|\lambda| - |\mu| < d$.

**Proposition 3.2** We compute some values of $m_{\lambda \mu}$: for $d \geq 1$ and $k \geq d + 2$,

1. $m_{(k,(1^{k-d})} = (-1)^{k-1}(m_{k,k-d} - 2m_{k-1,k-1-d} + m_{k-2,k-2-d})$.
2. $m_{(k-1,k-1),(k-2,1^{k-d})} = -m_{(k,(1^{k-d})}$.

Since all $m_{\lambda \mu}$ are required to be nonnegative, we conclude that $m_{k,k-d}$ for fixed $d$ is a linear function of $k$.

We begin with part 1, which is most of the work. First note that we want to compute $m_{\lambda \mu}$ where $\mu \not\subseteq \lambda$, so $m_{\lambda \mu}^{(p)} = 0$. Also, since $|\lambda| - |\mu| = d$, we learn from Proposition 3.1 that $m_{\lambda \mu}$ has degree at most $d$, but any term of $m_{\lambda \mu} - m_{\lambda \mu}^{(p)}$ must mention some $m_{ij}$ where $i - j \geq d$. Combining these, we find that $m_{\lambda \mu}$ must be a linear combination of $m_{i,i-d}$ for various values of $i$.

Now consider the $k \times k$ matrix whose $i,j$th entry is $f(s_{(1^{i-j})})$, whose determinant is $f(s_{(1^k)})$ according to dual Jacobi–Trudi. We need only track of determinant contributions linear in the $m_{i,i-d}$, all of which arise from picking a permutation and from each matrix entry choosing either the top-degree summand $sp_{(1^{i-j})}$ $(k-1)$ times) or the lower-degree summand $m_{1-i+j,1-i+j-d}sp_{(1^{i-j-1})}$ (once). In this matrix, we have ones immediately below the main diagonal, and zeros everywhere below those. Thus the only permutations contributing to the determinant correspond to choosing a composition $k_1 + k_2 + \cdots + k_r = k$, as follows: within square diagonal blocks of successive sizes $k_i$, choose all the subdiagonal unit entries and the entry in the upper-right corner of the block. The sign of this permutation is $\prod (-1)^{k_i-1} = (-1)^{k-r}$. From each such term, our desired $m_{\lambda \mu}$ picks up a contribution of $\pm m_{k_i,k_i-d}$ from the block of size $k_i$, for each $1 \leq i \leq r$.

Explaining the proposition’s coefficients of $\pm 1, \mp 2, \pm 1$ for a distinguished block of size $k, k-1, k-2$ respectively is straightforward. The $m_{k,k-d}$ comes from the unique composition with one part of size $k$, and gets sign $(-1)^{k-1}$, while the $2m_{k-1,k-1-d}$ come from compositions $(k-1) + 1$ and $1 + (k-1)$ and have sign $(-1)^{k-2}$. When our distinguished block has size $k-2$ there are five compositions, which we can abbreviate as $*11, 1*1, 11*, *2$ and $2*$, where the $*$ is the block providing the $m_{k-2,k-2-d}$. In three cases this has sign $(-1)^{k-3}$ and in two cases $(-1)^{k-2}$, and the desired coefficient is obtained.

This computation for $k-2$ makes it clear that each composition of $s$ with $t$ parts will contribute $t+1$ terms $m_{k-s,k-s-d}$, corresponding to $t+1$ ways to insert a $*$ designating a block of size $k-s$, with sign depending on the parity of $t$. To show that these all cancel, it suffices to check that $\sum_s (-1)^t(t+1) \text{comp}(s,t) = 0$ for any $s \geq 3$. Here $\text{comp}(s,t)$ denotes the number of compositions of $s$ with
exactly \( t \) parts, equal to \( \binom{s - 1}{r - 1} \), and the resulting identity on binomial coefficients is easily verified by induction.

Part 2 follows from part 1 by observing that \( s_{(k-1,k-1)} \) can be computed via the (normal) Jacobi–Trudi formula as \( s_{(k-1)} s_{(k-1)} - s_{(k)} s_{(k-2)} \). The same argument as above shows that the desired coefficient \( m_{(k-1,k-1),(k-2,1^{k-d})} \) is a linear combination of the \( m_{i,i-d} \). These can arise in \( f(s_{(k-1)}) f(s_{(k-1)}) - f(s_{(k)}) f(s_{(k-2)}) \) only by multiplying a coefficient linear in the \( m_{i,i-d} \) from one factor by a coefficient independent of the \( m_{i,i-d} \) and of maximal degree from the other. On coefficients independent of the \( m_{i,i-d} \), \( f \) agrees with \( f_p \), so in particular \( m_{\lambda,\mu} = 0 \) unless \( \mu \subseteq \lambda \), and the only term of maximal degree is \( s_p \lambda \) itself. Finally, we observe that \( s_{(k-2)} \) is the only \( s_\lambda \) mentioned here which fits inside \( (k-2,1^{k-d}) \), thus we can only find the desired shape in (minus) the product of \( s_p \lambda \) with \( m_{(k),(1^{k-d})} s_p(1^{k-d}) \).

Thus \( m_{(k-1,k-1),(k-2,1^{k-d})} = m_{(k),(1^{k-d})} \) is the negative of the coefficient calculated in part 1. We conclude that \( m_{k,k-d} = 2m_{k-1,k-1-d} + m_{k-2,k-2-d} \) must be zero for all \( k \geq d + 2 \), and therefore that \( m_{k,k-d} \) depends linearly on \( k \), as claimed.

Now let us write the linear function \( m_{i+d,j} \) as \( \alpha_d + j \beta_d \) for some constants \( \alpha_d \) and \( \beta_d \); we will complete our induction by showing \( \beta_d = 0 \). Certainly \( \beta_d \geq 0 \), since \( m_{i+d,j} \) must be nonnegative for arbitrarily large values of \( j \). The following is essentially the same argument with a little extra computation required.

**Proposition 3.3** We compute that \( m_{(k^d),(k-1)^d} = \alpha_d - (k-1)\beta_d \). Since this must be nonnegative for arbitrarily large values of \( k \), we conclude \( \beta_d = 0 \).

Using the same strategy as before, we apply \( f \) to the dual Jacobi–Trudi matrix for \( s_{(k^d)} \). It is again the case that \( m^{(p)}_{(k^d),(k-1)^d} \) is zero: while \( \langle (k-1)^d \rangle \) is indeed contained in \( (k^d) \), it only arises as \( s^\dagger_\lambda s_{(k^d)} \) for \( \mu = \langle 1^d \rangle \), and we chose \( p \) dual to a degree \( d - 1 \) polynomial, ensuring that \( \langle s_{(1^d)}, \kappa_p \rangle = 0 \). Thus the same degree argument as before shows that the desired coefficient is a linear combination of the \( m_{i+d,j} \).

Once again, we know that each contribution comes from taking some term in the determinant of the Jacobi–Trudi matrix and replacing a single factor \( e_r \) with \( m_{r,r-d} e_{r-d} \). Note that \( e_r \) must be replaced in this way if it is above the main diagonal: any superdiagonal \( e_r \) not so replaced has \( r > d \) and \( s_{(k-1)^d} \) cannot possibly appear. Let us consider whether this replacement can take place at the \( i,j \) position in the matrix, where we would replace \( e_{d+j-i} \) with some multiple of \( e_{j-i} \). Certainly we need \( j \geq i \).

Now observe that the cofactor of the \( i,j \) position in the matrix is precisely the dual Jacobi–Trudi matrix for the skew shape \( \langle (k-1)^d, i-1 \rangle / \langle j-1 \rangle \). The only \( s_\lambda \) which appears in the expansion of this skew shape and which is contained in \( \langle (k-1)^d \rangle \) is \( \lambda = \langle (k-1)^{d-1}, k-1-j+i \rangle \) — that is, \( \langle (k-1)^d \rangle \) with a horizontal strip of length \( j-i \) removed from its last row. We must multiply this by \( e_{j-i} \) to get a factor of \( \langle (k-1)^d \rangle \), but the dual Pieri rule says multiplication by \( e_{j-i} \) adds a vertical strip of length \( j-i \). The only way this vertical strip can fill the horizontal hole is if its length \( j-i \) is zero or one.
When \( j - i = 0 \), we are looking at contributions from the identity permutation: there are \( k \) places to replace \( e_d \) with \( m_{d,0}e_0 \). When \( j - i = 1 \) our permutation is one of the \( k - 1 \) adjacent transpositions, and we replace \( e_{d+1} \) with \( m_{d+1,1}e_1 \). Thus \( m_{d(k-1),((k-1)j)} = k m_{d,0} - (k - 1)m_{d+1,1} \). Writing \( m_{j+d,j} \) as \( \alpha_d + j\beta_d \), the coefficient is therefore \( \alpha_d - (k - 1)\beta_d \).

Propositions 3.2 and 3.3 show that \( m_{ij} \) is a function only of \( i - j \). All such embeddings \( f \) are constructed in Theorem 2.1, and the proof of Theorem 2.3(a) is complete.

Theorem 2.3 (b) is now quite simple. The Lie-theoretic parity assumption means \( p(x) \) is an even power series \( 1 + a_2x^2 + a_4x^4 + \cdots \). We quickly compute that \( m_{2(k+1),0} = a_2k - a_{2k+2} \) for all \( k \geq 0 \), which follows at once from \( sp(i)sp(j) \) containing \( sp_0 \) once if \( i = j \), and not at all otherwise. Thus we have \( 1 \geq a_2 \geq a_4 \geq \cdots \geq 0 \). The Lie-theoretic assumption that \( a_2 \) is an integer leaves us with two cases. If \( a_2 = 0 \) then all \( a_i = 0 \) and \( p(x) = 1 \). If \( a_2 = 1 \) then its dual \( q(x) = 1/(1 - x^2)p(-x) \) is \( 1 + O(x^4) \). But \( q(x) \) is also an even function with integral power series, so by the above, \( q(x) = 1 \) and \( p(x) = 1/(1 - x^2) \). There is a unique dual pair of solutions, as claimed.

## 4 Connections to Lie Theory

The motivation for this paper was to generalize some work of Kirillov and Reshetikhin on the characters of certain irreducible finite-dimensional representations of quantum affine algebras. According to Theorem 2.3(b), there is a unique dual pair of maps \( Y_A \rightarrow Y_{BCD} \) with certain properties, which we mentioned have a Lie-theoretic origin. In this section we will explain how those two maps give a generalization of the Kirillov–Reshetikhin characters for the symplectic and orthogonal quantum affine algebras.

The paper in question concerned representation of the quantum affine algebra \( \hat{U}_q(\mathfrak{g}) \), where \( \mathfrak{g} \) is a simple Lie algebra of classical type, and \( \hat{\mathfrak{g}} \) is its corresponding affine Lie algebra. Finite-dimensional representations of \( \hat{U}_q(\mathfrak{g}) \) are not yet well-understood. Due to the embedding \( \hat{U}_q(\mathfrak{g}) \hookrightarrow \hat{U}_q(\hat{\mathfrak{g}}) \), finite-dimensional \( \hat{U}_q(\mathfrak{g}) \) modules can be said to have weights, and any representation of \( \hat{U}_q(\hat{\mathfrak{g}}) \) can be viewed as a module over the semisimple \( U_q(\mathfrak{g}) \), and thus decomposed into a direct sum of \( U_q(\mathfrak{g}) \)-irreducibles. When \( \mathfrak{g} \) is of type \( A_n \), irreducibles of \( U_q(\mathfrak{g}) \) remain irreducible after restriction, but for types \( B/C/D \) this decomposition of irreducibles is nontrivial and not known in general.

An earlier work of Kirillov dealt with \( \mathfrak{g} \) of type \( A \) and investigated the decomposition of tensor products of “rectangular” irreducible representations — that is, representations whose highest weight is a multiple of a fundamental weight, whose Young diagram is a rectangle. The Bethe Ansatz and associated methods from mathematical physics led to a formula for the decomposition; a central step in the proof was the representation-theoretic identity

\[
V(m\omega) \otimes V((m + 1)\omega) \otimes V((m - 1)\omega) \oplus V(m\omega_{\ell-1}) \otimes V(m\omega_{\ell+1}).
\]
Here \( V(m\omega_\ell) \) is the irreducible representation of \( \mathfrak{g} \) with highest weight \( m\omega_\ell \), where \( m \in \mathbb{Z}_{\geq 0} \) and \( \omega_\ell \) for \( 1 \leq \ell \leq \text{rank}(\mathfrak{g}) \) are the fundamental weights of \( \mathfrak{g} \) (so \( m\omega_\ell \) corresponds to a rectangle of \( m \) columns each of height \( \ell \)).

The work of Kirillov and Reshetikhin \( \mathbb{K} \) applied the same approach to the other classical Lie algebras, where the above identity is no longer true for irreducible representations. Instead it holds when we replace the irreducible \( V(m\omega_\ell) \) with certain reducible representation \( W(m\omega_\ell) \), which, according to the mathematical physics origins, ought to be the decomposition of an irreducible \( U_q(\hat{\mathfrak{g}}) \) module, as discussed above. Rephrasing their formulas in the language of Young diagrams, when \( \mathfrak{g} \) is of type \( B_n \) and \( \ell \leq n-1 \) or \( D_n \) and \( \ell \leq n-2 \),

\[
W(m\omega_\ell) = \sum_\lambda V(\lambda)
\]

summing over all shapes \( \lambda \) which can be obtained from the \( \ell \times m \) rectangle by removing vertical \( 2 \times 1 \) dominos; for type \( C_n \) and \( \ell \leq n-1 \) we do the same but remove horizontal dominos instead.

The identity (\( \mathbb{K} \)) holds when \( \omega_{\ell-1} \) and \( \omega_{\ell+1} \) denote the two weights with the same length as and adjacent to \( \omega_\ell \) in the Dynkin diagram of \( \mathfrak{g} \); the paper (\( \mathbb{R} \)) also gave a version for when the \( \omega_\ell \) node is trivalent or is adjacent to a shorter or longer root, and corresponding values of \( W(m\omega_\ell) \) for the \( \ell \) not mentioned above, which we omit here. The connection to \( U_q(\hat{\mathfrak{g}}) \) was conjectural, as it is based on some unproven properties widely believed to hold for the Bethe Ansatz. Vyjayanthi Chari has recently announced a proof \( \mathbb{C} \) using entirely different techniques, in the case that \( \mathfrak{g} \) is simply-laced and the weight \( \omega_\ell \) appears with multiplicity at most 2 in the highest weight of \( \mathfrak{g} \). This proves the Lie theory connection when \( \mathfrak{g} \) is of type \( A \) and \( D \), and for for about half the choices of \( \omega_\ell \) for exceptional \( \mathfrak{g} \) of type \( E \).

The result is quite remarkable: the identity (\( \mathbb{K} \)) implies that the \( W(m\omega_\ell) \) are given in terms of the fundamental \( W(\omega_\ell) \) by the dual Jacobi-Trudi formula, even when \( \mathfrak{g} \) is not of type \( A \). A result of the author (\( \mathbb{V} \)) shows that this requirement is so strong that there is in fact a unique family of representations \( W(m\omega_\ell) \) which obey the identity (including the trivalent and different-length root extensions).

This brings us to the connection with the present work. For \( \mathfrak{g} \) of type \( B/C/D \), the Kirillov–Reshetikhin representations \( W(m\omega_\ell) \) (for \( \ell \ll \text{rank}(\mathfrak{g}) \)) are indeed the restrictions of irreducible \( U_q(\hat{\mathfrak{g}}) \) representations if, and only if, those quantum affine irreducibles obey the type-A Jacobi-Trudi formula. We extend this proposition beyond the case of rectangles.

Theorem (\( \mathbb{C} \)) gave us two distinguished automorphisms, which we will now call \( f_{\mathbb{B}} \) and \( f_{\mathbb{B}} \) on the vector space \( Y \) whose basis vectors are labelled by Young diagrams. These names are mnemonic: \( f_{\mathbb{B}} \) arises from skewing by \( \sum_\lambda s_\lambda \) where \( \lambda \) ranges over all partitions will all even rows, \( i.e. \) partitions built of horizontal dominos, and \( f_{\mathbb{B}} \) similarly for vertical dominos. In the notation of Sections (\( \mathbb{C} \) and \( \mathbb{D} \)), these are pulled back from \( f_p \) for \( p(x) = 1/(1 - x^2) \) and \( p(x) = 1 \), respectively, but this asymmetry is misleading: if we had arbitrarily chosen \( \phi_{BD} : \lambda \mapsto o_\lambda \) instead of \( \phi_C : \lambda \mapsto s_{\lambda} \) to embed our computations in the ring \( \Lambda \), the two \( p(x) \) would be reversed.

**Proposition 4.1** Let \( \mathfrak{g} \) be of type \( B/C/D \) and take some \( \lambda \) with fewer than \( \text{rank}(\mathfrak{g}) - 2 \) parts. Define a representation \( W(\lambda) \) of \( \mathfrak{g} \) as follows:
• For symplectic \( g \) (type \( C \)), let \( W(\lambda) \) have character \( \phi_C \circ f(\lambda) \).
  Equivalently, apply \( o_\mu \mapsto sp_\mu \) to \( s_\lambda \).

• For orthogonal \( g \) (type \( B/D \)), let \( W(\lambda) \) have character \( \phi_{BD} \circ f(\lambda) \).
  Equivalently, apply \( sp_\mu \mapsto o_\mu \) to \( s_\lambda \).

These \( W(\lambda) \) agree with the Kirillov–Reshetikhin values when \( \lambda \) is a multiple of a fundamental weight. If irreducible representations of quantum affine algebras obey type-\( A \) algebraic relations, then \( W(\lambda) \) is isomorphic to an irreducible representation of \( U_q(\hat{g}) \) viewed as a \( U_q(g) \)-module.

Certainly an irreducible representation of \( U_q(\hat{g}) \) viewed as a \( U_q(g) \)-module must have the properties we demanded in Theorem 2.3: the \( m_{\lambda/\mu} \) are multiplicities of \( U_q(g) \)-irreducibles, so must be nonnegative integers, and the weights \( \mu \) that appear must have \( \mu \leq \lambda \) in the Lie-theoretic sense, from considering the lowering operators in \( U_q(\hat{g}) \). The equivalent formulas for the character follow because \( 1/(1-x^2) \) is dual to the identity on \( \Lambda \) and \( \omega(sp_\mu) = o_\mu' \). Note that \( o_\mu \mapsto sp_\mu \) is the ring homomorphism \( h_i \mapsto h_i + h_{i-2} + h_{i-4} + \cdots \), and \( sp_\mu \mapsto o_\mu \) is its conjugate \( e_i \mapsto e_i + e_{i-2} + e_{i-4} + \cdots \).

To compare the Kirillov–Reshetikhin values, recall that if \( \lambda \) is a rectangle, then \( s_{\lambda/\mu} = s_\nu \) where \( \nu \) is the skew shape \( \lambda/\mu \) rotated 180°.

For instance, our calculations in Example 2.2 translated into the language of Lie theory would correspond to a type \( B/D \) decomposition \( W(\omega_1 + 2\omega_3) \simeq V(\omega_1 + 2\omega_3) \oplus V(2\omega_1 + \omega_3) \oplus V(\omega_2 + \omega_3) \oplus V(3\omega_1) \oplus V(\omega_1 + \omega_2) \).

**Problem 2** Prove that these \( W(\lambda) \) are the \( U_q(g) \)-restrictions of irreducible representations of \( U_q(\hat{g}) \).

Note that since we work in the stable limit where \( n \) is assumed to be sufficiently large that it is irrelevant, our results do not propose decompositions of \( W(\lambda) \) when \( \lambda \) is supported on the spin weights.

Until the recent announcement of \([3]\), there were almost no known decompositions to compare with our proposed values of \( W(\lambda) \). The methods used there offer a way to calculate upper bounds on the multiplicities in the decomposition of a certain canonical irreducible representation of \( U_q(\hat{g}) \) (the “minimal affinization” of Chari and Pressley; see \([2]\) for simply-laced \( g \)). Vyjayanthi Chari has kindly verified (private communication) that these upper bounds coincide exactly with our \( W(\lambda) \) for some special cases of type \( D \), e.g. \( \lambda = a\omega_1 + b\omega_3 \) and \( \lambda = \omega_2 + \omega_4 \) (where the decomposition is not multiplicity-free).

**References**

[1] Aissen, M.; Schoenberg, I. J.; Whitney, A. M. On the generating functions of totally positive sequences. *I. J. Analyse Math.* 2 (1952), 93–103.
[2] Chari, V.; Pressley, A. Quantum affine algebras and their representations. *Representations of groups (Banff, AB, 1994)*, 59–78. CMS Conf. Proc. 16, Amer. Math. Soc., Providence, RI, 1995.

[3] Chari, V. On the fermionc formula and the Kirillov–Reshetikhin conjecture. Preprint [math.QA/0006090](http://front.math.ucdavis.edu), 2000.

[4] Edrei, A. Proof of a conjecture of Schoenberg on the generating function of a totally positive sequence. *Canadian J. Math.* 5 (1953), 86–94.

[5] Kirillov, A. N. Completeness of states of the generalized Heisenberg magnet. (Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 134 (1984), transl. in *J. Soviet Math.* 36 (1987), 115–128.

[6] Kirillov, A. N.; Reshetikhin, N. Yu. Representations of Yangians and multiplicities of occurrence of the irreducible components of the tensor product of representations of simple Lie algebras. (Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* 160 (1987), transl. in *J. Soviet Math.* 52 (1990), 3156–3164.

[7] Kleber, M. Polynomial relations among characters coming from quantum affine algebras. *Math. Research Lett.* 5 (1998) no. 6, 731–742.

[8] Koike, K.; Terada, I. Young-diagrammatic methods for the representation theory of the classical groups of type $B_n$, $C_n$, $D_n$. *J. Algebra* 107 (1987), no. 2, 466–511.

[9] Littlewood, Dudley E. *The Theory of Group Characters and Matrix Representations of Groups*, second edition. Oxford University Press, London, 1950.

[10] Macdonald, I. G. *Symmetric Functions and Hall Polynomials*, second edition. Oxford University Press, Oxford, 1995.

[11] Stanley, R. P. *Graph colorings and related symmetric functions: ideas and applications: A description of results, interesting applications, & notable open problems*. *Discrete Math.* 193 (1998), no. 1-3, 267–286.

[12] Stanley, Richard P. *Enumerative Combinatorics, vol. 2*. Cambridge University Press, Cambridge, 1999.

[13] Thoma, E. Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe. (German) *Math. Zeit-schrift* 85 (1964), 40–61.

[14] Veigneau, S. *ACE, an Algebraic Combinatorics Environment for the computer algebra system MAPLE. User’s Reference Manual, Version 3.0*. IGM 98–11, Université de Marne-la-Vallée, 1998. http://weyl.univ-mlv.fr/~ace