UNIFORM REGULARITY FOR FREE-BOUNDARY NAVIER-STOKES EQUATIONS WITH SURFACE TENSION

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Abstract. We study the zero-viscosity limit of free boundary Navier-Stokes equations with surface tension in $\mathbb{R}^3$ thus extending the work of Masmoudi and Rousset [1] to take surface tension into account. Due to the presence of boundary layers, we are unable to pass to the zero-viscosity limit in the usual Sobolev spaces. Indeed, as viscosity tends to zero, normal derivatives at the boundary should blow-up. To deal with this problem, we solve the free boundary problem in the so-called Sobolev conormal spaces (after fixing the boundary via a coordinate transformation). We prove estimates which are uniform in the viscosity. And after inviscid limit process, we get the local existence of free-boundary Euler equation with surface tension. In a forthcoming work, we discuss how we can take the simultaneous limit of zero viscosity and surface tension $\varepsilon$.

1. Introduction

The water-wave problem has been studied for several decades from several different points of view. First the local existence for the water wave problem without surface tension was shown by Beale [6]. And for the same problem with surface tension, local existence was studied by Allain [6] and Tani [7] in case of two dimensions and three dimensions respectively. Moreover, with surface tension, global regularity was also studied by Beale [8].

In the case where the fluid is assumed to be inviscid and irrotational, the problem can be thought as a problem on the boundary. Recently, global regularity was achieved by S.Wu [11] and by Germain, Masmoudi and Shatah [12]. In the general case (where the vorticity may be non-zero) there are several papers by Christodoulou and Lindblad [13], and Lindblad [14], Coutand and Shkoller, Shatah and Zheng, and Masmoudi and Rousset where local well-posedness of an inviscid rotational fluid is proven.

In this paper we consider the vanishing viscosity limit for the water wave problem when surface tension is taken into account. The inviscid limit problem of free boundary was studied by Masmoudi and Rousset in [1] without taking surface tension into account. When surface tension is not taken into account, the boundary, $h$, has same regularity as the velocity, $u$, (say $H^m$). In the process of doing high order energy estimates, one loses half a derivative due to some commutators. That commutator comes from $D^m \nabla \varphi$, where $\varphi$ is harmonic extension of $h$ to the interior of the domain, which is $\frac{1}{2}$ more regular than $h$. The main idea of the paper [1] is to use Alinhac’s good unknown which reduce the order via a critical cancellation. And, because of boundary layer, we expect, near the boundary, $u^\varepsilon$ behaves like $u^\varepsilon \sim u(t, x) + \sqrt{\varepsilon} U(t, y, z/\sqrt{\varepsilon})$, where $u$ is solution of Euler equation, $U$ is a some profile, and $y$ is 2-d horizontal variable. So, for high order sobolev space, we cannot hope to get interval of time independent of $\varepsilon$, which is crucial to get strong compactness of solution sequences. Hence we consider a Sobolev conormal space, in which we expect to maintain boundedness of the Lipschitz norm as well as boundedness of higher order co-normal derivatives on an interval of time independent of $\varepsilon$.

Now Let’s consider the similar case with surface tension. We should still use Sobolev conormal spaces like in [1], but we don’t need Alinhac’s good unknown, since our $h$ is expected to have $m + 1$ regularity due to surface tension. Our main problem comes from the fact that the pressure term in the Euler equations becomes less regular when surface tension is introduced. We thus encounter commutators with order $m + \frac{3}{2}$, which we cannot control. For this reason, we decided to do energy estimates using space and time derivatives. This helps because time derivatives actually count for $3/2$ space derivatives on the boundary (this is deduced by studying the properties of the Dirichlet to Neumann mapping). Using this fact, we can derive local existence for a time interval independent to $\varepsilon$. And last, we deduce the solution of free-boundary Euler equation (subject to surface tension) as $\varepsilon \to 0$, using strong compactness argument.
### 1.1. Free-boundary Navier-Stokes equations

We solve incompressible free-boundary Navier-Stokes equations under the gravity with unbounded domain. Also assume that the above of fluid is vacuum.

\[
\partial_t u + u \cdot \nabla u + \nabla p = \varepsilon \Delta u, \quad x \in \Omega_t, \quad t > 0
\]

\[
\nabla \cdot u = 0, \quad x \in \Omega_t
\]

where \( \Omega_t \) is domain which occupied by fluid. We write fluid boundary as \( h \), so

\[
\Omega_t = \{ x \in \mathbb{R}^3, \quad x_3 < h(t, x_1, x_2) \}
\]

First boundary condition is moving boundary condition (or called kinematic boundary condition), which describe fluid particles do not cross the boundary.

\[
\partial_t h = u(t, x_1, x_2, h(t, x_1, x_2)) \cdot N, \quad (x_1, x_2) \in \mathbb{R}^2
\]

where \( N = (\nabla h, 1) \). Second boundary condition is the continuity of stress tensor on the boundary.

\[
p^h \hat{n} - 2\varepsilon (Su)^h \hat{n} = gh - \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}, \quad \text{on} \quad \partial \Omega_t
\]

where \( \hat{n} = N/|N| \) and \( Su \) is symmetric part of the gradient of \( u \)

\[
Su = \frac{(\nabla u) + (\nabla u)^T}{2}
\]

and \( \eta \) is surface tension constant. In this paper, we solve system (1.1)-(1.4).

### 1.2. Parametrization to a Fixed domain

We rewrite the problem from the frame \( \Omega(t) \) to \( S = \{(x, y, z) | z < 0 \} \), the fixed domain. This can be done by diffeomorphism \( \Phi(t, \cdot) \),

\[
\Phi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) \to \Omega_t
\]

\[
x = (y, z) \mapsto (y, \varphi(t, y, z))
\]

We use function \( v \) and \( q \) for velocity and pressure on fixed domain \( S \).

\[
v(t, x) = u(t, \Phi(t, x)), \quad q(t, x) = p(t, \Phi(t, x))
\]

We have to decide \( \varphi(t, \cdot) \) so that \( \Phi(t, \cdot) \) be a diffeomorphism. (Surely, \( \partial_z \varphi \geq 0 \), because it is diffeomorphism)

There are many ways to take \( \varphi \). One easy option is to set \( \varphi(t, y, z) = z + \eta(t, y) \). But this is not proper to our case. (see [1]) Instead of this one, we take a smoothing diffeomorphism as like [1].

\[
\varphi(t, y, z) = A z + \eta(t, y, z)
\]

To ensure that \( \Phi(0, \cdot) \) is a diffeomorphism, \( A \) should be picked so that

\[
\partial_z \varphi(0, y, z) \geq 1, \quad \forall x \in S
\]

and \( \eta \) is given by extension of \( h \) to domain \( S \), defined by

\[
\tilde{\eta}(\xi, z) = \chi(z\xi) \hat{h}(\xi)
\]

where \( \chi \) is a smooth, compactly support function which is 1 on \( B(0,1) \). This smoothing diffeomorphism was used in [3],[4], and also in [1]. For this extension, \( \varphi \) (of course \( \eta \) also.) is \( \frac{1}{2} \) better than \( h \). This will be explained in the section 2.

We also should define derivative of \( v \) in \( S \), so that measure \( \partial_t u \) in \( \Phi_t \). Then we rewrite our equations (1.1)-(1.4) as equations in a fixed domain \( S \). Using change of variable, we get

\[
(\partial_t u)(t, y, \phi) = (\partial_t v - \frac{\partial_t \varphi}{\partial_x \varphi} \partial_z v)(t, y, z) \quad i = t, 1, 2
\]

\[
(\partial_3 u)(t, y, \phi) = \frac{1}{\partial_z \varphi} \partial_z v)(t, y, z)
\]

So we define the following operator as like in [1].

\[
\partial^x_t = \partial_t - \frac{\partial \varphi}{\partial_x \varphi} \partial_z, \quad i = t, 1, 2, \quad \partial^x_z = \frac{1}{\partial_z \varphi} \partial_z
\]
This definition implies that \( \partial_i u \circ \Phi = \partial_i^\varepsilon v, \quad i = t, 1, 2, 3 \)

Hence our equations in \( S \) are,

(1.12) \[ \partial_i^\varepsilon v + v \cdot \nabla^\varepsilon v + \nabla^\varepsilon q = \varepsilon \triangle^\varepsilon v, \quad x \in S, \; t > 0 \]

(1.13) \[ \nabla^\varepsilon \cdot v = 0, \quad x \in S \]

(1.14) \[ \partial_t h = v(t, x_1, x_2, h(t, x_1, x_2)) \cdot N, \quad (x_1, x_2) \in \mathbb{R}^2 \]

(1.15) \[ \partial^\varepsilon h = \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}, \quad \text{on} \; S \]

### 1.3. Functional Framework and Notations.

We introduce conormal space and some function space that is proper to our problem. First we define sobolev conormal derivative as

(1.16) \[ Z_1 = \partial_x, \quad Z_2 = \partial_y, \quad Z_3 = \frac{z}{1 - 2} \partial_z, \quad Z^\alpha = Z^{(\alpha_1, \alpha_2, \alpha_3)} \]

From now on, we use the following symbol.

\[ Z^m = \partial^k Z^\alpha \quad \text{for some} \; (k + |\alpha| = m) \]

There are many combinations that satisfy \( k + |\alpha| = m \), but we will sum all cases later, so we don’t have to distinguish each cases. About norms in conormal space, as usual,

(1.17) \[ |f|_{\mathcal{H}_{s,m}}^2 \equiv \sum_{|\alpha| \leq s} |Z^\alpha f|_{L^2}^2, \quad \|f\|_{\mathcal{H}_{s,m}} \equiv \sum_{|\alpha| \leq s} |Z^\alpha f|_{L^\infty} \]

In this paper we abbreviate the notation as \( | \cdot |_s = | \cdot |_{H^s}, \quad \| \cdot \|_{s} = \| \cdot \|_{H^s} \) and \( \| \cdot \| = | \cdot |_{L^2} \). And similarly \( | \cdot |_{s, \infty} = | \cdot |_{W^{s, \infty}} \) and \( \| \cdot \|_{s, \infty} = \| \cdot \|_{W^{s, \infty}} \). Now we define proper function space \( X^{m,s} \) and \( Y^{m,s} \) as follows.

**Definition 1.1.** We define the space \( X^{m,s} \) and \( Y^{m,s} \) by the norm

(1.18) \[ |h|_{X^{m,s}}^2 \equiv \sum_{(k,\alpha), k + |\alpha| \leq m} |\partial^k D^\alpha h|_s^2 \]

(1.19) \[ \|u\|_{X^{m,s}}^2 \equiv \sum_{(k,\alpha), k + |\alpha| \leq m} \|\partial^k Z^\alpha u\|_s^2 \]

(1.20) \[ |h|_{Y^{m,s}} \equiv \sum_{(k,\alpha), k + |\alpha| \leq m} |\partial^k D^\alpha h|_{s,\infty} \]

(1.21) \[ \|u\|_{Y^{m,s}} \equiv \sum_{(k,\alpha), k + |\alpha| \leq m} \|\partial^k Z^\alpha u\|_{s,\infty} \]

### 1.4. Main Result.

We will use \( L^\infty \) type space which is order of \( \frac{m}{2} \). But, in conormal space, there is no proper sense of half order. So, for convenience, \( \frac{m}{2} \) means, \( \frac{m}{2} \) when \( m \) is even, and \( \frac{m}{2} - \frac{1}{2} \) when \( m \) is odd.

**Theorem 1.2.** For fixed sufficiently large \( m, (m \geq 6) \), let initial data are given so that

(1.22) \[ \sup_{\varepsilon \in (0,1)} \left( \sum_{k=0}^{m} \| \varepsilon^k v_0^\varepsilon \|_{H^{m+k} + |h_0^\varepsilon|_{H^{\frac{m}{2} + 1}}} \right) \leq R \]

and satisfy compatibility conditions

(1.23) \[ \Pi \left( (S^\varepsilon \partial^j_1 v^\varepsilon(0)) \right) \hat{n} = 0, \quad 0 \leq j \leq m, \quad \Pi \equiv I - \hat{n} \otimes \hat{n} \]

Then for \( \forall \varepsilon \in (0,1) \), there exist \( T > 0 \) (independent to \( \varepsilon \)), and some \( C > 0 \), such that there exist a unique solution \( (v^\varepsilon, h^\varepsilon) \) on \( [0, T] \), and the following energy estimate hold.

(1.24) \[ \sup_{t \in [0,T]} \left( \| v^\varepsilon(t) \|_{X^{m,0}} + |h^\varepsilon(t)|_{X^{m,1}} + \| \partial_2 v^\varepsilon(t) \|_{X^{m-1,1}} + \| \partial_2 v^\varepsilon \|_{Y^{\frac{m}{2},0}} \right) < C \]

(1.25) \[ \varepsilon \int_0^T \left( \| \nabla v^\varepsilon(t) \|_{X^{m,0}}^2 + \| \nabla \varepsilon v^\varepsilon(t) \|_{X^{m-1,0}}^2 \right) dt < C \]
Next we get a unique solution of free boundary Euler equation via zero-viscosity limit.

**Theorem 1.3.** Let’s assume that we
\[
\lim_{\varepsilon \to 0} \left( \|v_0^\varepsilon - v_0\|_{L^2(S)} + \|h_0^\varepsilon - h_0\|_{H^1(\partial S)} \right) = 0
\]
where \((v_0^\varepsilon, h_0^\varepsilon)\) and \((v_0, h_0)\) satisfy assumption of Theorem 1.2. Then there exist \((v, h)\) such that those are in
\[
v \in L^\infty([0, T], H^m_c(S)), \quad \partial_z v \in L^\infty([0, T], H^{m-1}_c(S)), \quad h \in L^\infty([0, T], H^{m+1}_c(\mathbb{R}^2))
\]
and
\[
\limsup_{\varepsilon \to 0} \left( \|v_\varepsilon - v\|_{L^2(S)} + \|v_\varepsilon - v\|_{L^\infty(S)} + \|h_\varepsilon - h\|_{H^1(\mathbb{R}^2)} + \|h_\varepsilon - h\|_{W^{1,\infty}(\mathbb{R}^2)} \right) = 0
\]
This is the unique solution of free boundary Euler equation.

\[
\partial_t^e v + (v \cdot \nabla) v + \nabla p = 0, \quad \nabla \cdot v = 0
\]
with free boundary
\[
\partial h = v \cdot N
\]
on the boundary.

1.5. **Scheme of Proof.** We briefly explain main idea of this paper in several steps.

**Remark 1.4.** In this paper \(\Lambda(\cdot, \cdot)\) means an increasing continuous function in all its arguments and may vary every line to line. And we also choose \(\Lambda\) so that \(\Lambda(0) = 0\).

1.5.1. **Energy estimate of \(v\) and \(h\).** First let’s apply \(Z^m_\varepsilon = \partial_x^3 \partial_y^2 \left( \frac{z}{\varepsilon z} \partial_z \right)^{\alpha_z}\), then our estimate looks like,
\[
E_0 \doteq \|v\|^2_{H^m} + |h|^2_{H^{m+1}} + \varepsilon \int_0^t \|\nabla v\|^2_{H^m} \leq C_0 + \Lambda(R) \int_0^t \left( E_0(s) + \|\nabla v\|_{H^{m-1}} + |h|_{H^{m+1}} \right) ds
\]
where, \(C_0\) is some terms of initial conditions, \(R\) contains \(E_0\) and some low order \(L^\infty\)-type terms. But \(|h|_{H^{m+1}}\), on the RHS cannot controlled by \(E_0\). To estimate \(|h|_{H^{m+1}}\), we use Dirichlet-Neumann estimate. We decompose pressure so that \(q^S\) solves \(v_t + (\nabla q^S) = 0\). Then, using
\[
v_t^b + (\nabla q^S)^b = 0
\]
and, by kinematic boundary condition,
\[
h_{tt} = v_t^b \cdot N + v^b \cdot N_t
\]
So,
\[
h_{tt} = - (\nabla q^S)^b \cdot N + v^b \cdot N_t
\]
Since \((q^S)^b \sim \Delta h\), we can get \(h_{tt} \sim \nabla \Delta h\), so it seems like \(\partial_t h \sim \partial_x^2 h\). So, for high order, then \(|h|_{H^{m+\frac{1}{2}}} \sim |\partial_t h|_{H^m}\). Hence our next step energy estimate is gained by applying \(\partial_t Z^{-1}_\varepsilon\)
\[
E_1 \doteq \|\partial_t v\|^2_{H^{m-1}} + |\partial_t h|^2_{H^m} + \varepsilon \int_0^t \|\nabla \partial_t v\|^2_{H^{m-1}} \leq C_0 + \Lambda(R) \int_0^t \left( E_1(s) + \|\nabla \partial_t v\|_{H^{m-2}} + |\partial_t h|_{H^{m+\frac{1}{2}}} \right) ds
\]
where, in this case, \(R\) contains \(E_0\) and \(E_1\) and some low order \(L^\infty\)-type terms. Since \(E^1\) contains \(|\partial_t h|_{H^m}\), it controls \(|h|_{H^{m+\frac{1}{2}}}\), the bad commutator in previous step energy estimate (1.30). We perform this process repeatedly, until \(E_m\) step, where we get
\[
E_2 \doteq \|\partial_t^2 v\|^2_{H^{m-2}} + |\partial_t^2 h|^2_{H^{m-1}} + \varepsilon \int_0^t \|\nabla \partial_t^2 v\|^2_{H^{m-2}}
\]
\[\leq C_0 + \Lambda(R) \int_0^t \left( E_2(s) + \| \nabla \partial_t^2 v \|_{H_{m-2}^0} + | \partial_t^2 h |_{H_{m-\frac{3}{2}}} \right) ds \]

\[\vdots \]

(1.35)

\[E_\tau \doteq ||\partial_t^m v||_{H_{m-\tau}}^2 + | \partial_t^m h |_{H_{m-\tau+1}}^2 + \varepsilon \int_0^t \| \nabla \partial_t^m v \|_{H_{m-\tau}}^2 \]

\[\leq C_0 + \Lambda(R) \int_0^t \left( E_\tau(s) + \| \nabla \partial_t^m v \|_{H_{m-\tau}}^2 + | \partial_t^m h |_{H_{m-\tau+\frac{1}{2}}} \right) ds \]

\[\vdots \]

(1.36)

\[E_{m-1} \doteq ||\partial_t^{m-1} v||_{H_{m-1}}^2 + | \partial_t^{m-1} h |_{H_{m-2}}^2 + \varepsilon \int_0^t \| \nabla \partial_t^{m-1} v \|_{H_{m-1}}^2 \]

\[\leq C_0 + \Lambda(R) \int_0^t \left( E_{m-1}(s) + \| \nabla \partial_t^{m-1} v \|_{H_{m-1}}^2 + | \partial_t^{m-1} h |_{H_{m-2}^2} \right) ds \]

If we sum all above estimates, then \(E_0 + E_1 + \cdots + E_{m-2} + E_{m-1}\) controls every high order terms of \(h\) except \(| \partial_t^{m-1} h |_{H_{m-\frac{3}{2}}}\). And \(\| \nabla v \|\)-type terms will be treated in normal derivative section later.

1.5.2. Energy estimate for all-time derivatives. In the last step, we take only \(\partial_t^m\) to the equation. If we use same estimate as above, we encounter \(| \partial_t^{m+1} h |_{H_{m+2}}\). Then we need \(| \partial_t^{m+1} h |_{L^2}\) but there is no hope to get closed energy estimate for \(| \partial_t^{m+1} h |_{L^2}\). So we show that in this step, estimate is closed. Last step energy is

\[E^m \doteq ||\partial_t^m v||_{L^2}^2 + | \partial_t^m h |_{H^1}^2 + \varepsilon \int_0^t \| \nabla \partial_t^m v \|_{L^2}^2 \]

When \(Z^m\) has spatial derivatives, the worst commutator come from two parts.

1) First part is \(\partial_t^m q\). High-order commutators come from commutator between \(\partial_t^m\) and \(\partial_t^m v\) and their product with \(\partial_t^m q\). i.e.

\[(1.37) \quad \int_S (\nabla \phi \cdot \partial_t^m v) \partial_t^m q \]

So, from \(\partial_t^m (\nabla \phi \partial_t v)\),

\[(1.38) \quad \text{High order commutators } \sim (\partial_t^m \nabla \phi \partial_t v + \partial_t^m \nabla \phi \partial_t v + \partial_t \nabla \phi \partial_t^{-1} \partial_t v) \partial_t^m q \]

Since \(|q| \sim |h|^\frac{3}{2}\), (because of surface tension term) each terms in parenthesis should gain \(\frac{1}{2}\) derivative so that \(|\partial_t^m q|_{-\frac{3}{2}}\) is good to control. First two terms in parenthesis can get \(\frac{1}{2}\) derivative by the property of \(\phi\) that it is one-half more regular than \(h\). For the third term, we integrate by parts for both space and time, to interchange \(\partial_t\) of \(v\) and \(\partial_t\) of \(q\). Then \(\partial_t \partial_t^m q\) is not bad, since \(\partial_t\) is once-half order less than \(\partial_t\) in the aspect of \(h\). This is impossible when \(Z^m\) contains at least one spatial derivatives. For example, let \(Z^m = \partial_t^k \partial_t^l\). If one spatial \(\partial_t\) hit \(\nabla \phi\) and all \(\partial_t^{-1} \partial_t^\ell\) hit \(\partial_t v\), so the highest order commutator terms are like,

\[(1.39) \quad (\partial_t^k \partial_t^l \nabla \phi \partial_t v + \partial_t^k \partial_t^l \nabla \phi \partial_t v + \partial_t^k \partial_t^l \nabla \phi \partial_t v + \partial_t^k \partial_t^l \nabla \phi \partial_t v) \partial_t^m q \]

First three terms are okay to control. We can divide as \(\sim |v|_{k} |v|_{1} \). In the 4th term, \(v\) has optimal regularity as its own, so there is no way to absorb \(\frac{1}{2}\) order from \(\partial_t^k \partial_t^l v\), since \(\partial_t^k \partial_t^l v\) itself has optimal regularity with respect to its energy function.

2) Second part is about commutators from surface tension term. (mean curvature). For general case,

\[(1.40) \quad \int_{\partial S} \left( \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot (\nabla h, \nabla Z^m h)}{\sqrt{1 + |\nabla h|^2}} \right) \cdot \nabla (v^b \cdot Z^m N) dA \sim |\nabla Z^m h|_{\frac{3}{2}}^2 \sim |Z^m h|_{\frac{3}{2}}^2 \]
When $Z^m = \partial_t^m$, we should control terms like,

$$ \int_{\partial S} \frac{\nu \cdot \hat{N}}{\sqrt{1 + |\nabla h|^2}} \partial_i (\partial_{tt}^m h) \partial_j (\partial_{tt}^m h) $$

where $i, j, k = 1, 2$ and at least one is different to other two. So they vanished by divergence condition at result.

$$ \sim \int_{\partial S} \partial_i (\partial_{tt}^m h) \partial_j (\partial_{tt}^m h) \sim \int_{\partial S} \partial_i (\partial_{tt}^m h)^2 \sim 0 $$

Consequently, in the last step,

$$ E_m \leq \| \partial_{tt}^m v \|_{L^2}^2 + |\partial_{tt}^m h|_{H^1}^2 + \varepsilon \int_0^t \| \nabla \partial_{tt}^m v \|_{L^2}^2 $$

$$ \leq C_0 + \Lambda(R) \int_0^t (E_m(s) + \| \nabla \partial_{tt}^m v \|_{L^2} + |\partial_{tt}^m h|_{H^1}) \, ds $$

So, except $\nabla v$ term, in the aspect of $h$, it does not require anymore step. Considering all steps, we now sum all $m + 1$ energy estimate.

### 1.5.3. $L^2$-type normal estimate

Our commutators contains $\| \partial_z v \|_{X^{-1,0}}$, which cannot be controlled, since until above, our $v$ has only conormal regularity. As like in [1], we make energy estimate of $S_n$, tangential part of $S^2 v n$, because $\partial_z v \sim S_n$. (Instead of $\partial v$, it is suffice to estimate $S_n$.) We get the estimate of $S_n$

$$ \| S_n \|_{X^{-1,0}}^2 + \varepsilon \int_0^t \| \nabla S_n \|_{X^{-1,0}}^2 \leq C_0 + \Lambda(R) \int_0^t (E(s) + \| S_n(s) \|_{X^{-1,0}}) \, ds $$

### 1.5.4. $L^\infty$-type normal estimate

Next, we estimate $L^\infty$-type estimate, which is included in $R$ above. As similar, we estimate $S_n$ instead of $\partial_z v$. Main difficulty is commutator between $Z^2$ and Laplacian. We consider thin layer near the boundary and reparameterize so that $\partial_z^2 \partial_z^2$ look like $\partial_{zz}$. And then, we change advection term as

$$ \partial_t \rho + (w_2(t, y, 0), zw_3(t, y, 0)) \cdot \nabla \rho - \varepsilon \partial_{zz} = l.o.t $$

We do not apply simple maximal principle for convection-diffusion equation. We apply Duhamel’s principle using Green’s evolution kernel. Then we can conclude

$$ \| \partial_z v \|_{Y^{\frac{3}{2}, 0}}^2 \leq \| \partial_z (0) \|_{Y^{\frac{3}{2}, 0}}^2 + \Lambda(R) \int_0^t \varepsilon \| \partial_{zz} v \|_{X^{-1,0}}^2 $$

### 1.5.5. Uniform Existence and Uniqueness

As far, we made every necessary energy estimate, especially RHS of energy estimate is independent of $\varepsilon$, provided energy remains bounded. So, using preliminary existence result of A.Tani [7] and strong compactness argument, we get local existence. For uniqueness, it is suffice to do $L^2$-estimate for $\delta v \hat{=} v_1 - v_2$. Then by Gronwall’s inequality, we get the result.

### 1.6. Comparing the problem with and without surface tension

Surface tension is, overall, a regularizing force in the water wave problem; however, it introduces several (perhaps unexpected) difficulties. Here we want to elaborate upon the differences between the paper of Masmoudi-Rousset [1] (the case where no there is no surface tension) and our result (where surface tension is taken into account). In terms of the of the basic functional framework, both works use Sobolev conormal spaces due to the presence of boundary layers. However, there are big differences between these two works. First, let’s look at a scheme of [1] (no surface tension case). When we have no surface tension, $m$-order energy estimate contains $|h|^m$. The main problem which the authors faced in [1] is the presence of certain high order commutators. To get around this problem, the authors made use of Alinhac’s good unknown which allowed them to close the energy estimates. They use the good unknowns: $V^a = Z^a v - \partial_t^a v Z^a q$ and $Q^a = Z^a q - \partial_t^a q Z^a q$, because, with this new variable, bad commutator $Z^a N$ is disappears. The second major problem is in estimating one normal derivative near the boundary its optimal regularity is $m - 2$, not $m - 1$, because of the regularity of $h$. $(|S_n|_{m-1}$ estimate require $|h|_{m+\frac{1}{2}}$ which show lack of $\frac{1}{2}$ regularity.) This is why, [1] requires quiet complicate analysis (need to estimate $|v|_{L^2,h^{m-1}}$) to control $\partial_z v$ of commutator. Both of these problems would disappear if the surface were more regular.

Meanwhile, in the surface tension case (this paper), in the $m$th-order energy estimate has $|h|_{m+1}$ in its
energy. So now one doesn’t need Alinhac’s good unknown. Nevertheless, we also lack of $\frac{1}{2}$ order ($|b|_{m+\frac{1}{2}}$ appears on the commutators) because of the pressure. Because we use conormal spaces,

$$\int_S Z^m v \cdot \nabla^\varphi Z^m q$$

make high order commutator about pressure $q$ in $S$. (which is vanished in case of standard sobolev space derivatives $D^m$, by divergence free condition) Since $q^b \sim \Delta h$, $q \sim \partial h^2 h$. As mentioned above scheme, it is bounded by taking time derivatives, and this step is continued until we take only time derivatives, which does not have any harmful commutators. In this case, normal derivatives are easier to deal with, since $\|S_n\|$ has optimal $m-1$ order regularity, by which we can close energy estimate.

About $L^\infty$ estimate for $S_n$, [1] require $\varepsilon \|\partial s v\|_{L^\infty}$. But we do not need $\varepsilon \|\partial s v\|_{Y^{1,0}}$. This is because, $\varepsilon \|\partial s v\|_{L^\infty}$ appears by Alinhac’s unknown which include $\partial v Z^a$.  

2. Basic Propositions

2.1. Basic propositions. We construct some proposition to estimate commutators.

Proposition 2.1. For $m \in 2\mathbb{N}$, we get the following estimates.

(2.1) $\|Z^m(\omega v)\| \lesssim \|u\|_{X^s_1,0} \|v\|_{Y^s_1,0} + \|v\|_{X^s_1,0} \|u\|_{Y^s_1,0}.$

(2.2) $\|Z^m, u|v\| \lesssim \|u\|_{X^s_1,0} \|v\|_{Y^s_1,0} + \|v\|_{X^s_1,0} \|u\|_{Y^s_1,0}.$

(2.3) $\|Z^m, u, v\| \lesssim \|u\|_{X^s_1,0} \|v\|_{Y^s_1,0} + \|v\|_{X^s_1,0} \|u\|_{Y^s_1,0}.$

Proof. We cannot use general Leibnitz Rule since $Z_3 = \frac{1}{\sqrt{2}} \partial_z$, but every order of derivatives of $\frac{1}{\sqrt{2}}$ is uniformly bounded, so we can use similar estimate if we use $\lesssim$ instead of $\leq$.

(2.4) $\|Z^m(\omega v)\| \leq \sum_{|\beta, \gamma|, |\beta| + |\gamma| \leq m} \|Z^\beta u Z^\gamma v\| + \sum_{|\beta| \leq |\gamma|} \|Z^\beta u Z^\gamma v\| \\
\leq \sum_{|\beta| \geq |\gamma|} \|Z^\beta u \| Z^\gamma v \|_{L^\infty} + \sum_{|\beta| \leq |\gamma|} \|Z^\beta u \| Z^\gamma v \| \lesssim \|u\|_{X^s_1,0} \|v\|_{Y^s_1,0} + \|v\|_{X^s_1,0} \|u\|_{Y^s_1,0}.$

(2.5) $\|Z^m, u|v\| \lesssim \sum_{|\beta| + |\gamma| = m, \beta \neq 0} \|Z^\beta u Z^\gamma v\| \lesssim \|u\|_{X^s_1,0} \|v\|_{Y^s_1,0} + \|v\|_{X^s_1,0} \|u\|_{Y^s_1,0}.$

(2.6) $\|Z^m, u, v\| \lesssim \sum_{|\beta| + |\gamma| = m, \beta \neq 0, \gamma \neq 0} \|Z^\beta u Z^\gamma v\| \lesssim \|u\|_{X^s_1,0} \|v\|_{Y^s_1,0} + \|v\|_{X^s_1,0} \|u\|_{Y^s_1,0}.$

Remark 2.2. The idea is that for each term, put $L^2$ norm to higher derivative term, and give $L^\infty$ norm to low order term. In conormal derivatives, there is no proper notion of rational derivative, so $Z_3^{1/2}$ does not make sense. We deal when $m$ is even, so that $\frac{m}{2}$ is also a integer, but our result also work for odd $m$, because we are suffice to give $\frac{m-1}{2}$ orders to $L^\infty$ and $\frac{m+1}{2}$ orders to $L^2$. So in this paper, $\frac{m}{2}$ means, integer $\frac{m-1}{2}$, when $m$ is odd. But for convenience, we abuse notation. It does not make any problem because, if we pick $m$ as sufficiently large, these $L^\infty$ type terms will be controlled by energy which has order $m$.

The followings are anisotropic embedding and trace property for conormal space.

Proposition 2.3. 1) $s_1 \geq 0, s_2 \geq 0$ such that $s_1 + s_2 > 2$ and $u$ such that $u \in H^{s_1}_{tan}, \partial_z u \in H^{s_2}_{tan}$, we have the anisotropic sobolev embedding:

(2.7) $\|u\|_{H^s_{tan}} \lesssim \|\partial_z u\|_{H^{s_2}_{tan}} \|u\|_{H^{s_1}_{tan}}.$

2) For $u \in H^1(S)$, we have the trace estimate:

(2.8) $\|u(\cdot, 0)\|_{H^s(\beta^2)} \lesssim \|\partial_z u\|_{H^{s_2}_{tan}} \|u\|_{H^{s_1}_{tan}}.$

with $s_1 + s_2 = 2s \geq 0$

Proof. see [1].
2.2. **Estimate of \( \eta \).** As explained before, the reason we choose smoothing diffeomorphism is that \( \eta \) is \( \frac{1}{2} \) better than \( h \). And this fact is crucial in later, because this term can accommodate \( \frac{1}{2} \) instead of pressure, i.e.

\[
\int_S (\nabla \varphi) q \leq \| \nabla \varphi \|_\frac{1}{2} \| q \|_{-\frac{1}{2}} \sim |\nabla h|_{L^2} \| q \|_{-\frac{1}{2}}
\]

We defined diffeomorphism so that at initial time, \( \partial_2 \varphi(0, y, z) \geq 1 \). \( \partial_2 \varphi \) should be positive during our estimates, so our estimate is valid during on \([0, T]\) such that

\[
\partial_2 \varphi(t, y, z) \geq c_0, \quad \forall t \in [0, T]
\]

for some \( c_0 \).

**Proposition 2.4.** For \( \eta \), we obtain the following estimates.

\[
|\nabla \eta|_{H^s(S)} \leq C_s \| h \|_{s+\frac{1}{2}}
\]

(2.10)

\[
|\nabla \eta|_{X_{m,0}} \leq C_s \| h \|_{X_{m,\frac{1}{2}}}
\]

(2.11)

Moreover for \( L^\infty \), we get

\[
\forall s \in \mathbb{N}, \quad |\eta|_{W^{s,\infty}} \leq C_s \| h \|_{s,\infty}
\]

(2.12)

\[
\forall s \in \mathbb{N}, \quad |\eta|_{Y_{m,0}} \leq C_s \| h \|_{Y_{m,0}}
\]

(2.13)

**Proof.** The first thing is from [1], and \( |\nabla \partial_2 \eta|_{H^s(S)} \leq C_s |\partial_2 h|_{s+\frac{1}{2}} \) is also trivial by definition of \( \eta \). So, by summing all cases, we get the second inequality. For \( L^\infty \) type estimate, the third inequality is from [1], and the last thing is also trivial by definition of \( \eta \).

The following lemma is useful to estimate, since we will see many terms like \( \frac{u}{\partial_2 \varphi} \).

**Lemma 2.5.** We have the following estimate.

\[
\left\| Z^m \frac{u}{\partial_2 \varphi} \right\| \lesssim \Lambda \left( \frac{1}{c_0}, |\nabla \varphi|_{Y_{\frac{1}{2},0}}, |h|_{Y_{\frac{1}{2},0}} \right) \left( \| u \|_{X_{m,0}} + |h|_{X_{m,\frac{1}{2}}} \right)
\]

(2.14)

**Proof.** \( F(x) = x/(A+x) \) is a smooth function of which all order derivatives are bounded when \( A+x \geq c_0 > 0 \). So,

\[
\left\| Z^m \frac{u}{\partial_2 \varphi} \right\| = \left\| Z^m \left( \frac{u}{A} \frac{u}{A} F(\partial_2 \eta) \right) \right\| \lesssim \| u \|_{X_{m,0}} + \| Z^m(uF(\partial_2 \eta)) \| \lesssim \| u \|_{X_{m,0}} + \| F(\partial_2 \eta) \|_{Y_{\frac{1}{2},0}} + \| u \|_{Y_{\frac{1}{2},0}} \| F(\partial_2 \eta) \|_{X_{m,0}}
\]

Meanwhile,

\[
\left\| F(\partial_2 \eta) \right\|_{X_{m,0}} \leq \Lambda \left( \frac{1}{c_0}, |\nabla \varphi|_{Y_{\frac{1}{2},0}} \right) |\partial_2 \eta|_{X_{m,0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y_{\frac{1}{2},0}} \right) |h|_{X_{m,\frac{1}{2}}}
\]

(2.15)

\[
\left\| F(\partial_2 \eta) \right\|_{Y_{\frac{1}{2},0}} \leq \Lambda \left( \frac{1}{c_0}, |\nabla \varphi|_{Y_{\frac{1}{2},0}} \right) \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y_{\frac{1}{2},0}} \right)
\]

Hence, we get the result.

\[
\left\| F(\partial_2 \eta) \right\|_{Y_{\frac{1}{2},0}} \leq \Lambda \left( \frac{1}{c_0}, |\nabla \varphi|_{Y_{\frac{1}{2},0}} \right) \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y_{\frac{1}{2},0}} \right)
\]

2.3. **Dissipation term control.** We comment a kind of version of Korn’s inequality for \( S^\varphi \). From [1], we have

**Proposition 2.6.** If \( \partial_2 \varphi \geq c_0 \), \( \| \nabla \varphi \|_{L^\infty} + \| \nabla^2 \varphi \|_{L^\infty} \leq \frac{1}{c_0} \) for some \( c_0 > 0 \), then there exists \( \Lambda_0 = \Lambda \left( \frac{1}{c_0} \right) > 0 \) such that for every \( v \in H^1(S) \), we have

\[
\| \nabla v \|_{L^2(S)}^2 \leq \Lambda_0 \left( \int_S |S^\varphi v|^2 dV_t + \| v \|^2 \right)
\]

(2.16)

where

\[
S^\varphi v = \frac{1}{2} (\nabla^\varphi v + (\nabla^\varphi v)^T)
\]

**Proof.** See proposition 2.9 in [1].
Applying this inequality to general $Z^m v$, we can induce
\[ \| \nabla v \|^2_{X^{m,0}} \leq \int_S |S^\gamma v|^2_{X^{m,0}} + \|v\|^2_{X^{m,0}} \]
So, we our energies (form in previous section) can be bounded by $\|S^\gamma v\|$-type terms, which appear when we do energy estimates.

3. Equations of $(Z^m v, Z^m h, Z^m q)$

3.1. Commutator estimate.

Proposition 3.1. Let, $i = t, 1, 2, 3$,
\[
Z^m (\partial^i \varphi) = \partial^i (Z^m f) + C^m_i (f) \]
Then we have,
\[
\|C^m_i (f)\| \lesssim \Lambda \left( \frac{1}{c_0} \| \nabla f \|_{Y^{\frac{3}{2},0}} + |h|_{Y^{\frac{3}{2},1}} \right) \left( \| \nabla f \|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} \right)
\]
Proof. For $i = t, 1, 2$
\[
Z^m \left( \partial_t f - \frac{\partial \varphi}{\partial z \varphi} \partial_z f \right) = \partial_t (Z^m f) - Z^m \left( \frac{\partial \varphi}{\partial z \varphi} \partial_t f \right)
\]

Now we estimate three terms using above propositions and lemmas.
\[
\left\| \left[ Z^m, \frac{\partial \varphi}{\partial z \varphi}, \partial_z f \right] \right\| \lesssim \left\| \frac{\partial \varphi}{\partial z \varphi} \right\|_{X^{m,-1,0}} \left\| \partial_z f \right\|_{Y^{\frac{3}{2},0}} + \left\| \frac{\partial \varphi}{\partial z \varphi} \right\|_{Y^{\frac{3}{2},0}} \left\| \partial_z f \right\|_{X^{m,-1,0}}
\]
\[
\lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{3}{2},1}} + \left\| \partial_z f \right\|_{Y^{\frac{3}{2},0}} \right) \left( |h|_{X^{m-1,\frac{1}{2}}} + \left\| \partial_z f \right\|_{X^{m-1,0}} \right)
\]
\[
\left\| \left( Z^m \frac{\partial \varphi}{\partial z \varphi} \right) \partial_z f \right\| \lesssim \left\| \partial_z f \right\|_{L^\infty} \left\| Z^m \frac{\partial \varphi}{\partial z \varphi} \right\|
\]
\[
\lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{3}{2},1}} + \left\| \partial_z f \right\|_{L^\infty} \right) |h|_{X^{m,\frac{1}{2}}}
\]
\[
\left\| \frac{\partial \varphi}{\partial z \varphi} \left[ Z^m, \partial_z f \right] \right\| \lesssim \left\| \frac{\partial \varphi}{\partial z \varphi} \right\|_{L^\infty} \left\| \sum_{|\beta| \leq m-1} \partial_\beta \partial_z (Z^m f) \right\| \lesssim \Lambda \left( \frac{1}{c_0}, \left\| \partial_z f \right\|_{Y^{\frac{3}{2},0}} + |h|_{1,\infty} \right) \left\| \partial_z f \right\|_{X^{m-1,0}}
\]
By summing these three terms, we get the result. For $i = 3$,
\[
Z^m \left( \frac{\partial \varphi}{\partial z \varphi} \right) = \left[ Z^m, \frac{1}{\partial z \varphi}, \partial_z f \right] + \left( Z^m \frac{1}{\partial z \varphi} \right) \partial_z f + \frac{1}{\partial z \varphi} \left( Z^m \partial_z f \right)
\]
\[
\lesssim \left\| Z^m (\partial^i \varphi) \right\| + \left\| Z^m \left( \frac{1}{\partial z \varphi} \right) \partial_z f \right\| + \left\| Z^m \left( \frac{1}{\partial z \varphi} \right) \right\| \left\| \partial_z f \right\| \left\| \partial_z f \right\|
\]
We just replace $\partial \varphi$ as 1, so the control is same.

3.2. Interior Equations. Applying $Z^m$ to our system, and using commutator estimates, we get following result.
3.2.1. Pressure.

\begin{equation}
Z^m(\nabla^q q) = \nabla^q(Z^m q) - (C_1^m(q), C_2^m(q), C_3^m(q)) = \nabla^q(Z^m q) - C^m(q)
\end{equation}

Then, by above proposition, we get

\begin{equation}
\|C^m(q)\| \lesssim A\left(\frac{1}{c_0}, \|\nabla q\|_{Y^{\frac{3}{2}, 0}} + |h|_{Y^{\frac{3}{2}, 1}}\right)\left(\|\nabla q\|_{X^{m-1, 0}} + |h|_{X^{m, \frac{3}{2}}}\right)
\end{equation}

3.2.2. Divergence-free.

\begin{equation}
Z^m(\nabla^\varphi \cdot v) = \nabla^\varphi \cdot (Z^m v) - \sum_{i=1}^3 C_i^m(v) = \nabla^\varphi \cdot (Z^m v) - C^m(d)
\end{equation}

and easily,

\begin{equation}
\|C^m(d)\| \lesssim A\left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{3}{2}, 0}} + |h|_{Y^{\frac{3}{2}, 1}}\right)\left(\|\nabla v\|_{X^{m-1, 0}} + |h|_{X^{m, \frac{3}{2}}}\right)
\end{equation}

3.2.3. Transportation. Using divergence free condition, we have

\begin{equation}
\partial_t^e + (v \cdot \nabla^\varphi) = \partial_t + (v_y \cdot \nabla_y) + \frac{1}{\partial_z \varphi} (v \cdot N^\varphi - \partial_t \varphi) \partial_z
\end{equation}

Applying $Z^m$,

\begin{equation}
Z^m(\partial_t^e + v \cdot \nabla^\varphi) v = (\partial_t^e + v \cdot \nabla^\varphi)(Z^m v) + T^m(v)
\end{equation}

where $V_z = \frac{1}{\partial_z \varphi} (v \cdot N^\varphi - \partial_t \varphi)$ and $N^\varphi = (-\nabla \varphi, 1)$

\begin{equation}
T^m(v) = \sum_{i=1}^2 \{\partial_t v \cdot Z^m v_i + [Z^m, v_i, \partial_t v]\} + [Z^m, V_z, \partial_t v] + (Z^m V_z) \cdot \partial_z v + V_z [Z^m, \partial_z] v
\end{equation}

Using propositions and lemmas, we have

\begin{equation}
\|T^m(v)\| \lesssim A\left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{3}{2}, 0}} + |h|_{Y^{\frac{3}{2}, 1}}\right)\left(\|\nabla v\|_{X^{m-1, 0}} + \|v\|_{X^{m, 0}} + |h|_{X^{m-1, \frac{3}{2}}}\right)
\end{equation}

3.2.4. Diffusion.

\begin{equation}
2Z^m \nabla^\varphi \cdot (S^\varphi v) = 2 \nabla^\varphi \cdot Z^m(S^\varphi v) - D^m(S^\varphi v)
\end{equation}

where $D^m(S^\varphi v)_{ij} = 2C_j^m(S^\varphi v)_{ij}$

and

\begin{equation}
2Z^m(S^\varphi v) = Z^m(\partial_t^e v_j + \partial_t^e v_i) = 2S^\varphi(Z^m v) + (C_1^m(v_j) + C_2^m(v_i)) = 2S^\varphi(Z^m v) + \Theta^m(v)
\end{equation}

where $\Theta^m(v)_{ij} = C_i^m(v_j) + C_j^m(v_i)$

So, estimate of $\Theta^m(v)$ is same of $C^m(v)$,

\begin{equation}
\|\Theta^{k, \alpha}(v)\| \lesssim A\left(\frac{1}{c_0}, \|\nabla v\|_{Y^{\frac{3}{2}, 0}} + |h|_{Y^{\frac{3}{2}, 1}}\right)\left(\|\nabla v\|_{X^{m-1, 0}} + |h|_{X^{m, \frac{3}{2}}}\right)
\end{equation}

3.2.5. Conclusion. By far, we get the follow result.

\begin{equation}
(\partial_t^e + v \cdot \nabla^\varphi)(Z^m v) + \nabla^\varphi(Z^m q) - 2\varepsilon \nabla^\varphi \cdot S^\varphi(Z^m v) = \varepsilon \nabla^\varphi \cdot \Theta^m(v) + C^m(q) - T^m(v) - \varepsilon D^m(S^\varphi v)
\end{equation}

\begin{equation}
\nabla^\varphi \cdot (Z^m v) = C^m(d)
\end{equation}

3.3. Boundary Equations. Especially, $\alpha_3 = 0$ because we are in the boundary, conormal norms. And all norms are on $\partial S = \mathbb{R}^2$
3.3.1. Kinematic boundary.

\[ Z^m(\partial_t h - v^b \cdot N) = \partial_t (Z^m h) - \{ [Z^m, v^b, N] + (Z^m v^b) \cdot N + v^b \cdot (Z^m N) \} = 0 \]
\[ \partial_t (Z^m h) - (Z^m v^b) \cdot N - v^b \cdot (Z^m N) - C^m (KB) = 0 \quad \text{where} \quad C^{k,\alpha} (KB) = [Z^{k,\alpha}, v^b, N] \]

\[ \|C^m (KB)\| = \| [Z^m, v^b, N] \| \lesssim \Lambda \left( \|v^b\|_{Y^{\frac{p}{2}, 0}} + \|N\|_{Y^{\frac{p}{2}, 0}} \right) \left( \|v^b\|_{X^{m-1, 0}} + \|N\|_{X^{m-1, 0}} \right) \]
By trace inequality,
\[ \lesssim \Lambda \left( \|\nabla v\|_{Y^{\frac{p}{2}, 0}} + |h|_{Y^{\frac{p}{2}, 1}} \right) \left( \|v\|_{X^{m, 0}} + \|\nabla v\|_{X^{m-1, 0}} + |h|_{X^{m-1, 1}} \right) \]

3.3.2. Continuity of Stress tensor.

**Lemma 3.2.** control of $\nabla v(\cdot , 0)$ by $v^b$

We have the following estimate.

\[ |\nabla v(\cdot , 0)|_{X^{s, 0}} \lesssim \Lambda \left( \frac{1}{c_0} |h|_{Y^{\frac{p}{2}, 1}} + \|v\|_{Y^{\frac{p}{2}, 1}} \right) \left( |\partial_t v_1 + \partial_2 v_2|_{X^{s, 1}} + |v(\cdot , 0)|_{X^{s, 1}} \right) \]

**Proof.** We divide $\nabla v$ as normal part and tangential part. But for $\partial_1 v$ and $\partial_2 v$, result is obvious. So, we only focus on $\partial_2 v$. Firstly, from the divergence free condition $\nabla \cdot v = 0$,

\[ \partial_2 v \cdot n = \frac{1}{|N|} (A + \partial_2 \eta) (\partial_1 v_1 + \partial_2 v_2) \quad \text{where} \quad |N| = \sqrt{1 + |\nabla \eta|^2} \]

On the boundary,

\[ |\partial_2 v \cdot n|_{X^{s, 0}} \lesssim \Lambda \left( \frac{1}{c_0} |h|_{Y^{\frac{p}{2}, 1}} + \|v\|_{Y^{\frac{p}{2}, 1}} \right) \left( |\partial_t \eta(\cdot , 0)|_{X^{s, 0}} + |v(\cdot , 0)|_{X^{s, 1}} \right) \]

\[ \lesssim \Lambda \left( \frac{1}{c_0} |h|_{Y^{\frac{p}{2}, 1}} + \|v\|_{Y^{\frac{p}{2}, 1}} \right) \left( |h|_{X^{m, 1}} + |v(\cdot , 0)|_{X^{s, 1}} \right) \]

using estimate of $\eta$ and trace inequality.

Now, let define $\Pi = I - n \otimes n$ (tangential part of vector). We have the following compatibility condition,

\[ \Pi (S^b \nu) = 0 \]
on the boundary. So,

\[ 2S^b v \tilde{N} = \frac{1}{\partial_2 \varphi} (1 + |\nabla h|^2) (\partial_2 v) - (\partial_1 h \partial_1 v + \partial_2 h \partial_2 v) + \frac{\partial_1 v \cdot N}{0} N + \frac{1}{\partial_2 \varphi} (\partial_2 v \cdot N) N = 0 \]

\[ \partial_2 v(\cdot , 0) = \frac{\partial_2 \varphi}{1 + |\nabla h|^2} \left\{ (\partial_1 h \partial_1 v + \partial_2 h \partial_2 v) - \frac{\partial_1 v \cdot N}{0} N - \frac{1}{\partial_2 \varphi} (\partial_2 v \cdot N) N \right\} \]

We take $\Pi | \cdot |_{X^{s, 0}}$ and use above $|\partial_2 v \cdot n|_{X^{s, 0}}$ estimate again. So we get the same estimate. By adding normal part and tangential part, we finish lemma. 

Now we return to the Stress-continuity condition

\[ Z^m \left\{ \left( q^b - g h + \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) N \right\} - 2\varepsilon Z^m ((S^b v)^b N) = 0 \]

So,

\[ Z^m q^b - g Z^m h - 2\varepsilon (S^b (Z^m v))^b - (\Theta^m (v))^b \right\} N + \{ q^b - g h - 2\varepsilon (S^b v)^b \} Z^m N \]

\[ + \eta \left( \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) Z^m N + \eta \left( \nabla \cdot Z^m \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) N + C^m (B) = 0 \]

where,

\[ C^m (B) = -C^m (B)_1 + C^m (B)_2 \]
\[= -2\varepsilon[Z^m, (S^e v)^b] + \left[Z^m, q^b - gh + \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} N\right] \]

With estimates of
\[
\|C^m(B)\|_1 = 2\varepsilon \left\| \left[Z^m, (S^e v)^b, N\right] \right\|
\]
\[
\lesssim 2\varepsilon \Lambda \left( \frac{1}{c_0}, |h|_{\tilde{Y}_{q^0}} + \|\nabla v\|_{\tilde{Y}_{q^0}} \right) \left( |h|_{X^{m-1,1}} + |v^b|_{X^{m-1,0}} \right)
\]

Then by above lemma,
\[
\lesssim 2\varepsilon \Lambda \left( \frac{1}{c_0}, |h|_{\tilde{Y}_{q^0}} + \|\nabla v\|_{\tilde{Y}_{q^0}} \right) \left( |h|_{X^{m-1,1}} + v^b \right)
\]
\[
\|C^m(B)\|_2 = 2\varepsilon \left\| \left[Z^m, (S^e v)^b \hat{n} \cdot \hat{n}, N\right] \right\|
\]

Similarly as \(C^m(B)_1\), we get the same estimate, so be \(C^m(B)\).

We also estimate \(C^m(S)\), where
\[
Z^m \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} = \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \times \nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} + C^m(S)
\]

which is consist of low order polynomials in terms of \(h\). Take a term in this \(C^m(S)\), then we take \(L^2\) norm for the highest order, and \(L^\infty\) to others. For \(m\) large \((m > 2)\), \(L^\infty\) can be controlled by highest order term by sobolev embedding. So,
\[
\|C^m(S)\| \lesssim \Lambda \left( |h|_{X^{m,1}} \right)
\]

### 3.3.3. Conclusion.

Kinematic boundary condition becomes,
\[
\partial_t (Z^m h) - (Z^m v^b) \cdot N - v^b \cdot (Z^m N) = C^m(KB)
\]

where
\[
\|C^m(KB)\| \lesssim \Lambda \left( \|\nabla v\|_{\tilde{Y}_{q^0}} + |h|_{\tilde{Y}_{q^0}} \right) \left( |v|_{X^{m,0}} + \|\nabla v\|_{X^{m-1,0}} + \|h\|_{X^{m-1,1}} \right)
\]

Continuity of stress tensor condition becomes,
\[
\left\{ Z^m q^b - gZ^m h - \varepsilon S^e (Z^m v) - \varepsilon (\Theta^m(v))^b \right\} N + \{q^b - gh - 2\varepsilon (S^e v)\} Z^m N + \eta \left( \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) Z^m N
\]
\[
+ \eta \left( \nabla \cdot \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \times \nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} + C^m(S) \right) N + C^m(B) = 0
\]

where,
\[
\|C^m(S)\| \lesssim \Lambda \left( |h|_{X^{m-1,1}} \right)
\]
\[
\|C^m(B)\| \lesssim 2\varepsilon \Lambda \left( \frac{1}{c_0}, |h|_{\tilde{Y}_{q^0}} + \|\nabla v\|_{\tilde{Y}_{q^0}} \right) \left( |h|_{X^{m-1,1}} + |v^b|_{X^{m-1,1}} \right)
\]
\[
\left\| (\Theta^m(v))^b \right\| \lesssim \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{\tilde{Y}_{q^0}} + |h|_{\tilde{Y}_{q^0}} \right) \left( |v^b|_{X^{m-1,1}} + |h|_{X^{m-1,1}} \right)
\]
4. Pressure Estimates

We linearly divide $q = q^E + q^{NS} + q^S$, where $q^E$ solves

$$ (v \cdot \nabla) v + \nabla q^E = 0 $$

$q^E |_{z=0} = gh$

$q^{NS}$ solves

$$ \nabla q^{NS} = 2 \varepsilon \nabla \cdot (S^2 v) $$

$q^{NS} |_{z=0} = 2 \varepsilon (S^2 v) \hat{n} \cdot \hat{n}$

$q^S$ solves

$$ \partial_t^2 v + \nabla q^S = 0 $$

$q^S |_{z=0} = -\eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}$

These equations can be transformed into elliptic equation.

Gradient becomes,

$$ \nabla f = \begin{pmatrix} \partial_1 f \partial_2 f \partial_3 f \\ \partial_1 \phi \partial_2 \phi \partial_3 \phi \\ \partial_1 \phi \partial_2 \phi \partial_3 \phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & -\partial_1 \phi \\ 0 & 1 & -\partial_2 \phi \\ 0 & 0 & -\partial_3 \phi \end{pmatrix} \begin{pmatrix} \partial_1 f \\ \partial_2 f \\ \partial_3 f \end{pmatrix} = \frac{1}{\partial_z \phi} P^* \nabla f $$

where

$$ P = \begin{pmatrix} \partial_1 \phi & 0 & 0 \\ 0 & \partial_2 \phi & 0 \\ -\partial_3 \phi & -\partial_3 \phi & 1 \end{pmatrix} $$

Similarly, divergence becomes,(easy to check.)

$$ \nabla \cdot v = \frac{1}{\partial_z \phi} \nabla \cdot (P v) $$

So, we get easily,

$$ \Delta f = \nabla \cdot (\nabla f) = \frac{1}{\partial_z \phi} \nabla \cdot (P \nabla f) = \frac{1}{\partial_z \phi} \nabla \cdot (E \nabla f) $$

where

$$ E = \frac{1}{\partial_z \phi} P P^* = \begin{pmatrix} \partial_1 \phi & 0 & -\partial_1 \phi \\ 0 & \partial_2 \phi & -\partial_2 \phi \\ -\partial_3 \phi & -\partial_3 \phi & \frac{1}{\partial_z \phi} \left( \partial_1 \phi \right)^2 + \left( \partial_2 \phi \right)^2 \end{pmatrix} $$

We start with two lemmas about elliptic-Dirichlet boundary problem. These are very similar to those of [1], with some slight modification for our functionspace. First is nonhomogeneous problem with homogeneous boundary data.

**Lemma 4.1.** For the system in $S$,

$$ -\nabla \cdot (E \nabla \rho) = \nabla \cdot F. \quad \rho(t, y, 0) = 0 $$

Then we have the estimates.

$$ \|\nabla \rho\|_{X^{k,0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{2,1}} + |h|_{2,\infty} + |h|_3 + \|\nabla \cdot F\|_{H^1_{tan}} + \|F\|_{H^2_{tan}} \right) (\|F\|_{X^{k,0}} + |h|_{X^{k-1,1}}) $$

$$ \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{2,1}} + |h|_{2,\infty} + |h|_3 + \|F\|_{H^2} \right) (\|F\|_{X^{k,0}} + |h|_{X^{k-1,1}}) $$
Proof. First, we know the basic result,

\[
\| \nabla \rho \| \leq \Lambda \left( \frac{1}{c_0} \right) \| F \|
\]

We apply \( Z^\alpha \) to the equation, but divergence structure is broken, since \( Z_3 \) and \( \partial_z \) does not commute. So we apply \( \tilde{Z}_3 \), so that

\[
\tilde{Z}_3 f = Z_3 f + \frac{1}{(1-z)^2} f
\]

then, we have,

\[
\tilde{Z}_3 \partial_z = \partial_z Z_3
\]

Now we apply \( \tilde{Z}^\alpha = Z_i^\alpha Z_1^\alpha Z_2^\alpha \tilde{Z}_3^\alpha \) to the equation, then

\[
\nabla \cdot (Z^\alpha (E \nabla \rho)) = \nabla \cdot \left( Z^\alpha F + (\tilde{Z}^\alpha - Z^\alpha) F_h - (\tilde{Z}^\alpha - Z^\alpha) (E \nabla \rho)_{\partial z} \right)
\]

where \( F_h = (F_1, F_2, 0) \). And again,

\[
\nabla \cdot (E \nabla (Z^\alpha \rho)) = \nabla \cdot \left( Z^\alpha F + (\tilde{Z}^\alpha - Z^\alpha) F_h - (\tilde{Z}^\alpha - Z^\alpha) (E \nabla \rho)_{\partial z} \right) + \nabla \cdot C
\]

where

\[
C = -(E[Z^\alpha, \nabla] \rho) - \left( \sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta, \gamma} Z^\beta E \cdot Z^\gamma \rho \right)
\]

Since \( Z^\alpha \rho \) is also zero on the boundary,

\[
\| \nabla \rho \|_{X^{k,0}} \leq \Lambda \left( \frac{1}{c_0} \right) \left( \| F \|_{X^{k,0}} + \| E \nabla \rho \|_{X^{k-1,0}} + \| E[Z^\alpha, \nabla] \rho \| + \left\| \sum_{\beta+\gamma=\alpha, \beta \neq 0} c_{\beta, \gamma} Z^\beta E \cdot Z^\gamma \rho \right\| \right)
\]

3 terms on the RHS can be estimated as follow.

\[
\| E \nabla \rho \|_{X^{k-1,0}} \lesssim \Lambda \left( \frac{1}{c_0} \right) \left( \| h \|_{Y^{k,1}} + \| \nabla \rho \|_{Y^{k,1}} \right) \left( \| \nabla \rho \|_{X^{k-1,0}} + \| h \|_{X^{k-1,1}} \right)
\]

and, since \( [Z^\alpha, \partial_z] = \sum_{|\beta| \leq |\alpha|-1} c_{\alpha, \beta} \partial_z (Z^\beta) \),

\[
\left\| \sum_{|\beta|=|\alpha|-1} c_{\alpha, \beta} \partial_z (Z^\beta \rho) \right\| \lesssim \Lambda \left( \frac{1}{c_0} \right) \left( \| h \|_{Y^{k,1}} + \| \nabla \rho \|_{Y^{k,1}} \right) \left( \| \nabla \rho \|_{X^{k-1,0}} + \| h \|_{X^{k-1,1}} \right)
\]

Using these 3 estimates we get,

\[
\| \nabla \rho \|_{X^{k,0}} \lesssim \Lambda \left( \frac{1}{c_0} \right) \left( \| h \|_{Y^{k,1}} + \| \nabla \rho \|_{Y^{k,1}} \right) \left( \| F \|_{X^{k,0}} + \| \nabla \rho \|_{X^{k-1,0}} + \| h \|_{X^{k-1,1}} \right)
\]

Now, we can use induction for \( \| \nabla \rho \|_{X^{k-1,0}} \) until \( \| \nabla \rho \|_{L^\infty} \), and \( \| \nabla \rho \|_{L^\infty} \) can be estimated as in the [1](6.21)

\[
\| \nabla \rho \|_{L^\infty} \leq \Lambda \left( \frac{1}{c_0} \right) \left( \| h \|_{L^\infty} + \| h \|_{H^{2,\infty}} \right) \left( \| \nabla \cdot F \|_{H^{2,\infty}} + \| F \|_{H^{2,\infty}} \right)
\]

Consequently, we get our result. \( \square \)

Indeed, estimate for standard sobolev space is also available, but since \( F \) contains \( v \), \( F \) can be estimated in conormal space. This is why we made estimate in conormal spaces.

Second is homogeneous problem with nonhomogeneous boundary data.
Lemma 4.2. For the system in $S$,

$$- \nabla \cdot (E \nabla \rho) = 0. \quad \rho(t, y, 0) = f^b$$

Then we have the estimates.

$$|\nabla \rho|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + |f^b|_{Y^{\frac{1}{2}, 0}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + |f^b|_{X^{k, \frac{1}{4}}} \right)$$

**Proof.** We divide $\rho$ under the form $\rho = \rho^H + \rho^r$, where $\rho^H$ absorb the boundary data, and $\rho^r$ solves

$$- \nabla \cdot (E \nabla \rho^r) = \nabla \cdot (E \nabla \rho^H), \quad \rho^r (t, y, 0) = 0$$

We choose $\rho^H$ as

$$\tilde{\rho^H} (\xi, z) = \chi(z \xi) f^b$$

Then using proposition in section 2 (harmonic extension), we get

$$|\nabla \rho^H|_{X^{k, 0}} \lesssim C_s |f^b|_{X^{k, \frac{1}{4}}}$$

$$|\rho^H|_{Y^{k, 0}} \lesssim C_s |f^b|_{Y^{k, 0}}$$

Because we can deduce estimate in standard sobolev space for above lemma, we have

$$|\nabla \rho^r|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + \|\nabla \cdot (E \nabla \rho^H)\|_{H^1_{tan}} + \|E \nabla \rho^H\|_{H^2_{tan}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + |f^b|_{X^{k, \frac{1}{4}}} \right)$$

$$\lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + \|E \nabla \rho^H\|_{H^2} \right) \left( |h|_{X^{k, \frac{1}{4}}} + |f^b|_{X^{k, \frac{1}{4}}} \right)$$

where on the right term,

$$|E \nabla \rho^H|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 0}} + |f^b|_{Y^{\frac{1}{2}, 0}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + |f^b|_{X^{k, \frac{1}{4}}} \right)$$

Consequently, these implies

$$|\nabla \rho^r|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + |f^b|_{Y^{\frac{1}{2}, 0}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + |f^b|_{X^{k, \frac{1}{4}}} \right)$$

We obtained estimates of $|\nabla \rho^r|_{X^{k, 0}}$ and $|\nabla \rho^H|_{X^{k, 0}}$, so get the result. 

These two lemmas give estimate of $q^{NS}$ and $q^S$.

**Proposition 4.3.** Estimate of $q^{NS}$.

$$|\nabla q^{NS}|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + |\nabla v|_{Y^{\frac{1}{2}, 0}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + \varepsilon |h|_{X^{k, \frac{1}{4}}} + \varepsilon \|\nabla v\|_{X^{k, 1}} + \varepsilon \|v\|_{X^{k, 2}} \right)$$

**Proof.** Applying above lemma 4.2, $f^b = 2\varepsilon (S^r v)^b \tilde{n} \cdot \tilde{n}$,

$$|\nabla q^{NS}|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + ||S^r v|^b \tilde{n} \cdot \tilde{n}|_{Y^{\frac{1}{2}, 0}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + \varepsilon |h|_{X^{k, \frac{1}{4}}} + \varepsilon \|S^r v\|_{X^{k, \frac{1}{4}}} \right)$$

$$\lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + \varepsilon \|S^r v\|_{X^{k, \frac{1}{4}}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + \varepsilon |h|_{X^{k, \frac{1}{4}}} + \varepsilon \|\nabla v(\cdot, 0)\|_{X^{k, \frac{1}{4}}} \right)$$

Using lemma in section 3 and trace inequality,

$$\lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + \varepsilon \|S^r v\|_{X^{k, \frac{1}{4}}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + \varepsilon |h|_{X^{k, \frac{1}{4}}} + \varepsilon \|\nabla v\|_{X^{k, 1}} + \varepsilon \|v\|_{X^{k, 2}} \right)$$

**Proposition 4.4.** Estimate of $q^S$.

$$|\nabla q^S|_{X^{k, 0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{\frac{1}{2}, 1}} + |h|_{2, \infty} + |h|_{3} + \eta |h|_{Y^{\frac{1}{2}, 2}} \right) \left( |h|_{X^{k, \frac{1}{4}}} + \eta |h|_{X^{k, \frac{1}{4}}} \right)$$
Proof. Applying above lemma 4.2, \( f^b = -\eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \),

\begin{equation}
|\nabla q^S|^{X_{k,0}} \lesssim \Lambda \left( \frac{1}{c_0} |h|_{Y,1} \frac{h}{h} + |h|_{2,\infty} + |h|_{3} + \left| \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{Y,1} \frac{h}{h} + \left| \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{X^{k,\frac{1}{2}}} \right) \lesssim \Lambda \left( \frac{1}{c_0} |h|_{Y,1} + |h|_{2,\infty} + |h|_{3} + \eta |h|_{Y,2} \right) \left( |h|_{X^{k,\frac{1}{2}}} + \eta |h|_{X^{k,\frac{1}{2}}} \right)
\end{equation}

To estimate \( q^E \), we cannot use above lemma 4.1 directly. This is because, by divergence free condition, \( \nabla \cdot (v \cdot \nabla v) = \nabla v : (\nabla v)^T \) that is, one derivative for \( v \) is canceled. This gives 1 more regularity to \( q^E \) and this fact make it possible to estimate \( \|S_n\|_{X^{m-1,0}} \) in later section. Now we divide \( q^E = q^E_1 + q^E_2 \) as, where \( q^E_1 \) solves

\begin{equation}
-\Delta \varphi q^E_1 = 0, \quad q^E_1(t, y, 0) = gh
\end{equation}

and \( q^E_2 \) solves

\begin{equation}
-\Delta \varphi q^E_2 = (\nabla \varphi v) : (\nabla \varphi v)^T, \quad q^E_2(t, y, 0) = 0
\end{equation}

First, estimate of \( q^E_1 \) comes from Lemma 4.2 easily.

\begin{equation}
\left\| \nabla q^E_1 \right\|_{X^{k,0}} \lesssim \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_{3} + \left| \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{Y,1} \right) |h|_{X^{k,\frac{1}{2}}}
\end{equation}

Second, for \( q^E_2 \), by applying \( Z^k \), we have

\begin{equation}
-\nabla \cdot (E \nabla Z^k q^E_2) = \nabla \cdot (E[Z^k, \nabla]q^E_2 + [Z^k, E] \nabla q^E_2) + [Z^k, \nabla] \cdot (E \nabla q^E_2) + \sum_{i,j} Z^k (\partial_i \varphi v_j \partial_j \varphi v_i)
\end{equation}

Again, we write \( q^E_2 = q^E_{2,1} + q^E_{2,2} \) where

\begin{equation}
-\nabla \cdot (E \nabla Z^k q^E_{2,1}) = \nabla \cdot (E[Z^k, \nabla]q^E_{2,1} + [Z^k, E] \nabla q^E_{2,1})
\end{equation}

\begin{equation}
-\nabla \cdot (E \nabla Z^k q^E_{2,2}) = [Z^k, \nabla] \cdot (E \nabla q^E_{2,2}) + \sum_{i,j} Z^k (\partial_i \varphi v_j \partial_j \varphi v_i)
\end{equation}

\( q^E_{2,1} \) can be estimated by Lemma 4.1, and \( q^E_{2,2} \) can be estimated by

\begin{equation}
\left\| Z^k q^E_{2,2} \right\|_{H^2} \lesssim \left\| Z^k, \nabla \cdot (E \nabla q^E_{2,2}) + \sum_{i,j} Z^k (\partial_i \varphi v_j \partial_j \varphi v_i) \right\| + \left\| Z^k q^E_{2,2} \right\|_{L^2}
\end{equation}

Then by using induction for \( k \) and using basic \( L^2 \) estimate for \( q^E_{2,2} \), we just use (4.11), we can finish the estimate. Now we put together all estimates of \( q^E_1, q^E_{2,1}, q^E_{2,2} \) to get,

**Proposition 4.5.** Estimate of \( q^E \)

\begin{equation}
\left\| \nabla q^E \right\|_{X^{k,0}} \lesssim \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_{3} + \left| \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{Y,1} + \left\| \nabla v \right\|_{Y,1} \right) \left( |h|_{X^{k+1,1}} + \left\| \nabla v \right\|_{X^{k+1,0}} \right)
\end{equation}

We should also estimate \( L^\infty \)-type terms of \( Z^k \). In fact, for \( q^N \) and \( q^S \), we can use sobolev embedding. For \( q^E \), since we can estimate \( \left\| Z^k q^E \right\|_{H^2}, \left\| \partial_{zz} q^E \right\| \) type term can be estimated.

**Proposition 4.6.** \( L^\infty \) type estimate for pressure.

\begin{equation}
\left\| \nabla q \right\|_{Y^{k,0}} \lesssim \Lambda \left( \frac{1}{c_0} |h|_{2,\infty} + |h|_{3} + \left| \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|_{Y,1} + \left\| \nabla v \right\|_{Y,1} \right) \left( |h|_{X^{k+1,1}} + \varepsilon \left\| \nabla v \right\|_{X^{k+2,1}} + \left\| \nabla v \right\|_{X^{k+1,0}} \right)
\end{equation}
Proof. Using anisotropic sobolev embedding,
\[
\| \nabla q^E \|_{Y_{k,0}} \lesssim \| \partial_z \nabla q^E \|_{X_{k+1,0}} \| \nabla q^E \|_{X_{k+2,0}}
\]
Hence, we should estimate \( \| \partial_z \nabla q^E \|_{X_{k+1,0}} \). Meanwhile, \( \| Z^{k+1} q^E \|_{H^2} \) can be estimated by standard theory. For \( q^{NS} \) and \( q^S \), standard sobolev embedding can be used.
\[
\| \nabla q \|_{Y_{k,0}} \lesssim \| \nabla q^E \|_{Y_{k,0}} + \| \nabla q^{NS} \|_{Y_{k,0}} + \| \nabla q^S \|_{Y_{k,0}}
\]
\[
\lesssim \| \nabla q^E \|_{X_{k+2,0}} \| \partial_z \nabla q^E \|_{X_{k+1,0}} + \| \nabla q^{NS} \|_{X_{k+2,0}} + \| \nabla q^S \|_{X_{k+2,0}}
\]
and for \( \| \partial_z \nabla q^E \|_{X_{k+1,0}} \), we see that \( \| Z^{k+1} q^E \|_{H^2} \) can be estimated by standard theory. Using above lemmas, we can get our result. \( \square \)

5. Energy Estimates

We perform energy estimate on \( S \). Our terms have forms of
\[
(5.1) \quad \int_S f g dV_i \quad \text{whered} V_i = \partial_z \phi(t, y, z) dy dz
\]
So we need the following lemma.

**Lemma 5.1.**
\[
(5.2) \quad \int_S \partial_t^i f g dV_i = - \int_S f \partial_t^i g dV_i + \int_{z=0} f g N_i dy, \quad i = 1, 2, 3
\]
\[
(5.3) \quad \int_S \partial_t^i g dV_i = \partial_i \int_S f g dV_i - \int_S f \partial_t^i g dV_i - \int_{z=0} f g \partial_i h
\]
Proof. see [1]. \( \square \)

**Corollary 5.2.** Let \( v(t, \cdot) \) is a vector field on \( S \), such that \( \nabla^\varphi \cdot v = 0 \), then for every smooth \( f, g \) and smooth vector field \( u, w \), we have the following estimates.
\[
(5.4) \quad \int_S (\partial_t^i f + v \cdot \nabla^\varphi f) f dV_i = \frac{1}{2} \partial_i \int_S |f|^2 dV_i - \frac{1}{2} \int_{z=0} \partial_t (h - v \cdot N) dy
\]
\[
(5.5) \quad \int_S (\Delta^\varphi f) g dV_i = - \int_S \nabla^\varphi f \cdot \nabla^\varphi g dV_i + \int_{z=0} \nabla^\varphi f \cdot Ng dy
\]
\[
(5.6) \quad \int_S \nabla^\varphi \cdot (S^\varphi u) \cdot wdV_i = - \int_S S^\varphi u \cdot S^\varphi wdV_i + \int_{z=0} (S^\varphi u N) \cdot wdV
\]
Proof. see [1]. \( \square \)

**Lemma 5.3.** For any smooth solution \( v, h \), we have the basic energy identity.
\[
(5.7) \quad \frac{d}{dt} \left( \int_S |v|^2 dV_i + g \int_{z=0} |h|^2 dy + 2\eta \int_{\partial S} \sqrt{1 + |\nabla^\varphi h|^2} dy \right) + 4\varepsilon \int_S |S^\varphi v|^2 dV_i = 0
\]
Proof. Using above corollaries,
\[
(5.8) \quad \frac{d}{dt} \int_S |v|^2 dV_i = 2 \int_S \nabla^\varphi \cdot (2\varepsilon S^\varphi v - q) \cdot vdV_i
\]
\[
= 2 \int_{\partial S} \left( -gh N \cdot v + \eta \nabla \cdot \frac{\nabla^\varphi N \cdot v}{\sqrt{1 + |\nabla^\varphi h|^2}} \right) dy
\]
\[
= -g \frac{d}{dt} \int_{\partial S} |h|^2 dy - 2\eta \frac{d}{dt} \int_{\partial S} \sqrt{1 + |\nabla^\varphi h|^2} \nabla h dy
\]
\[
= -g \frac{d}{dt} \int_{\partial S} |h|^2 dy - 2\eta \frac{d}{dt} \int_{\partial S} \sqrt{1 + |\nabla^\varphi h|^2} \nabla h dy
\]
As we commented, we work on time interval $\partial_t \varphi$ should be positive and $|h|_{2, \infty}$ should be bounded. So we do calculate energy estimate on an interval of time $[0, T^\varepsilon]$ for which we assume.

\begin{equation}
\partial_t \varphi \geq c_0, \quad |h|_{2, \infty} \leq \frac{1}{c_0}, \quad \forall t \in [0, T^\varepsilon]
\end{equation}

**Proposition 5.4.** Let $t \in [0, T^\varepsilon]$, and define for this $t$,

\begin{equation}
\Lambda_{m, \infty} = |h|_V + \|\nabla v\|_V \tag{5.10}
\end{equation}

then, every smooth solutions satisfy the following for every $m \in \mathbb{Z}$.

\begin{equation}
\|v(t)\|_{X_{m+1}}^2 + |h(t)|_{X_{m-1}}^2 + \varepsilon \int_0^t \|\nabla(v(s))\|_{X_{m-1}}^2 ds \lesssim \left(\|v(0)\|_{X_{m-1}}^2 + |h(0)|_{X_{m-1}}^2\right)
+ \int_0^t \Lambda \left(\|v\|_{X_{m+1}}^2 + \|\nabla v\|_{X_{m-1}}^2 + |\partial_t h|_{X_{m-1}}^2 + |h|_{X_{m-1}}^2\right) ds
\end{equation}

**Proof.** Use above lemma and our new equations. Then basic estimate becomes,

\begin{equation}
\frac{d}{dt} \int_S |Z^m v|^2 dV_t + 4\varepsilon \int_S |S^\varepsilon(Z^m v)|^2 dV_t = 2 \int_{z=0} \left(2\varepsilon S^\varepsilon(Z^m v) - (Z^m q)\right) N \cdot (Z^m v) dV + R_S + R_C
\end{equation}

where

\begin{equation}
R_S = 2\varepsilon \int_S \{\nabla^2 \cdot \Theta^m(v) - D^m(S^\varepsilon v)\} \cdot Z^m v dV_t
\end{equation}

\begin{equation}
R_C = 2 \int_S \{C^m(q) - C^m(T)\}(Z^m v) + C^m(d)(Z^m q) dV_t
\end{equation}

And we use stress-conti boundary condition to RHS integral term. From boundary equation, we have

\begin{equation}
2 \int_{z=0} \left(2\varepsilon S^\varepsilon(Z^m v) - (Z^m q)\right) N \cdot (Z^m v) dy
= 2 \int_{z=0} \left\{-gZ^m h + \eta \left(\nabla \cdot \left(\frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot \nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} + C^m(S)\right)\right)\right\} N \cdot (Z^m v) dy
+ 2 \int_{z=0} \left\{(q - gh)I - 2\varepsilon(S^\varepsilon v) + \eta \nabla \cdot \left(\frac{\nabla h}{\sqrt{1 + |\nabla h|^2}}\right)\right\} (Z^m N) \cdot (Z^m v) dy + R_B
\end{equation}

where

\begin{equation}
R_B = 2 \int_{z=0} \left(C^m(B) - \varepsilon (\Theta^m(v))^b N \cdot (Z^m v) dy
\end{equation}

The highest order of h part is,

\begin{equation}
2\varepsilon \int \nabla \left(\frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h \cdot \nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} + C^m(S)\right) (N \cdot Z^m v) dy
\end{equation}

We use Kinematic Boundary condition on the $(N \cdot Z^m v)$ then focus on

\begin{equation}
-2\eta \int_{\partial S} \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} \cdot \nabla (\partial_t(Z^m h))
\end{equation}

because, this gives

\begin{equation}
2\eta \int_{\partial S} \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} \cdot \nabla (\partial_t(Z^m h)) = \eta \frac{d}{dt} \int_{\partial S} \frac{|\nabla Z^m h|^2}{1 + |\nabla h|^2} + \eta \int_{\partial S} \frac{|\nabla Z^m h|^2}{1 + |\nabla h|^2} < \nabla h, \nabla \partial_t h >
\end{equation}

So, whole second part becomes,

\begin{equation}
- \left\{\eta \frac{d}{dt} \int_{\partial S} \frac{|\nabla Z^m h|^2}{1 + |\nabla h|^2} + \eta \int_{\partial S} \frac{|\nabla Z^m h|^2}{1 + |\nabla h|^2} < \nabla h, \nabla \partial_t h >\right\} + P_1
\end{equation}
We integrate in time, under assuming $|h|_{1,\infty}$ is bounded, we get,

\begin{align}
(5.20) \quad & \|Z^m v(t)\|_{L^2(\mathcal{S})}^2 + \eta \|Z^m \nabla h(t)\|_{L^2(\partial \mathcal{S})}^2 + g \|Z^m h(t)\|_{L^2(\partial \mathcal{S})}^2 + 4\epsilon \int_0^t \|S^\varepsilon(Z^m v(s))\|_{L^2(\mathcal{S})}^2 ds \\
& \lesssim \Lambda \left( \frac{1}{c_0}, \|Z^m v(0)\|_{L^2(\mathcal{S})}^2 + \eta \|Z^m \nabla h(0)\|_{L^2(\partial \mathcal{S})}^2 + 4\epsilon \right) \int_0^t \|S^\varepsilon(Z^m v(s))\|_{L^2(\mathcal{S})}^2 ds \\
& \quad + \eta \Lambda \left( \frac{1}{c_0} \right) \int_0^t \int_{\partial \mathcal{S}} |\nabla Z^m h|^2 |\nabla h, \nabla \partial h| \, dAdS
\end{align}

where $R_S, R_C, R_B$ were defined above, and

\begin{align}
(5.21) \quad & P_1 \doteq 2\eta \int_{\partial \mathcal{S}} \left( \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h < \nabla h, \nabla Z^m h >}{\sqrt{1 + |\nabla h|^2}} + C^m(S) \right) \cdot \nabla (v^b \cdot (Z^m N) + C^m(KB)) dy \\
& \quad + 2\eta \int_{\partial \mathcal{S}} \left( \frac{\nabla h < \nabla h, \nabla Z^m h >}{\sqrt{1 + |\nabla h|^2}} - C^m(S) \right) \cdot \nabla (\partial_t(Z^m h)) dy
\end{align}

\begin{align}
(5.22) \quad & P_2 \doteq 2 \int_{z=0} \left\{ q - gh - 2\varepsilon(S^\varepsilon) + \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right\} (Z^m N) \cdot (Z^m v) dy
\end{align}

\begin{align}
(5.23) \quad & P_3 \doteq 2\eta \int_{z=0} Z^m h \left( v^b \cdot Z^m N + C^m(KB) \right) dy
\end{align}

Now we should estimate above six terms.

1) $R_B$

\begin{align}
(5.24) \quad & |R_B| = 2 \left| \int_{z=0} \left( C^m(B) - \varepsilon (\Theta^m(v))^b (v)N \right) \cdot (Z^m v) dy \right| \\
& \lesssim \left\| C^m(B) - \varepsilon (\Theta^m(v))^b N \right\|_{L^2(\partial \mathcal{S})} \|Z^m v\|_{L^2(\partial \mathcal{S})} \\
& \lesssim \varepsilon \Lambda \left( \frac{1}{c_0}, |h|_{Y^m, -\frac{1}{2}} + \|\nabla v\|_{Y^m, 0} \right) \left( |h|_{X^m, -1, \frac{1}{2}} + |v^b|_{X^m, \frac{1}{2}} \right) \|v\|_{X^m}
\end{align}

2) $P_2$

\begin{align}
(5.25) \quad & P_2 \doteq 2 \int_{z=0} \left\{ q - gh - 2\varepsilon(S^\varepsilon) + \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right\} (Z^m N) \cdot (Z^m v) dy \\
& = 2 \left\{ q^{NS}_{|z=0} I - 2\varepsilon(S^\varepsilon) \right\} (Z^m N) \cdot (Z^m v) dy = 4\varepsilon \int_{z=0} \left\{ (S^\varepsilon) \nhat \cdot \nhat I - (S^\varepsilon) \right\} (Z^m N) \cdot (Z^m v) dy
\end{align}

So,

\begin{align}
(5.26) \quad & |P_2| = 2 \left| \int_{z=0} \left\{ q - gh - 2\varepsilon(S^\varepsilon) + \eta \nabla \cdot \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right\} (Z^m N) \cdot (Z^m v) dy \right| \\
& \lesssim 2\varepsilon \left| Z^m N \right|_{-\frac{1}{2}} \left( \left| (S^\varepsilon) \nhat \cdot \nhat I - (S^\varepsilon) \right| \right)_{1, \infty} \\
& \lesssim \varepsilon \left| h \right|_{X^m, -\frac{1}{2}} \left| v^b \right|_{X^m, \frac{1}{2}} \left| (S^\varepsilon) \nhat \cdot \nhat I - (S^\varepsilon) \right|_{1, \infty} \\
& \lesssim \varepsilon \Lambda \left( \frac{1}{c_0}, |h|_{Y^m, -\frac{1}{2}} + \|\nabla v\|_{Y^m, 0} \right) \left| h \right|_{X^m, -\frac{1}{2}} \left| v^b \right|_{X^m, \frac{1}{2}}
\end{align}

3) $P_3$

\begin{align}
(5.27) \quad & P_3 \doteq 2\eta \int_{z=0} Z^m h \left( v^b \cdot Z^m N + C^m(KB) \right) dy
\end{align}
Let's deal 2nd part of $P_3$.

So, we have only two types of integral. ($m = m_1 + m_2$ and both are non-zero indices.)

For $I_1$, we give $L^2$ estimate to $\partial_z (Z^m v_i)$, $L^2$ to bigger $m_i$, and $L^\infty$ to smaller $m_i$. So we get

Hence, we have

6) $P_1$

Let's deal 2nd part of $P_1$

(5.28) $|P_3| \leq \left| \int_{s=0}^{Z^m h \cdot (v^b \cdot Z^m N + C^m (KB))} \right| \leq \left| Z^m h \cdot (v^b \cdot Z^m N + C^m (KB)) \right|_{L^1(\partial S)}$

(5.29) $R_C = 2 \int_S \{ C^m(q) - T^m(v) \} (Z^m v) + C^m(d)(Z^m q) dV_t$

(5.30) $|R_C| \lesssim A \left( \frac{1}{e_0} \right) \left| \left( C^m(d) \right|_{L^2} \left| Z^m q(t) \right|_{L^2} + \left| T^m(v) \right|_{L^2} \left| Z^m v(t) \right|_{L^2} + \left| C^m(q) \right|_{L^2} \right| \left| Z^m v(t) \right|_{L^2}$

(5.31) $R_S = 2 e \int_S \{ \nabla v^y \cdot \Theta^m(v) - D^m(S^v) \} \cdot Z^m v dV_t$

(5.32) $I_1 = \int_{\partial S} \partial_z (Z^m v_i) Z_{12} (S^v)_{i j} Z_{22} \left( \frac{\partial \phi}{\partial z} \right)$

(5.33) $I_2 = \int_{\partial S} (Z^m v_i) Z_{12} (S^v)_{i j} Z_{22} \left( \frac{\partial \phi}{\partial z} \right)$

(5.34) $|I_1| \lesssim A \left( \frac{1}{e_0} \right) \left| \nabla v \right|_{L^2} \left( \left| \nabla v \right|_{L^2} + |S^v|_{L^2} + |h|_{L^2} \right)$

(5.35) $|I_2| \lesssim A \left( \frac{1}{e_0} \right) \left| \nabla v \right|_{L^2} \left( \left| \nabla v \right|_{L^2} + |S^v|_{L^2} + |h|_{L^2} \right)$

(5.36) $|R_S| \lesssim e A \left( \frac{1}{e_0} \right) \left| \nabla v \right|_{L^2} \left( \left| \nabla v \right|_{L^2} + |S^v|_{L^2} + |h|_{L^2} \right)$

(5.37) $P_1 = 2 \eta \int_{\partial S} \left( \frac{\nabla Z^m h}{1 + |\nabla h|^2} - \frac{\nabla h \cdot \nabla Z^m h}{1 + |\nabla h|^2} + C^m(S) \right) \cdot \nabla (v^b \cdot (Z^m N + C^m (KB))) dy$

(5.38) $\int_{\partial S} \left( \frac{\nabla h \cdot \nabla Z^m h}{1 + |\nabla h|^2} - C^m(S) \right) \cdot \nabla (\partial_i (Z^m h)) dy$
\[
\frac{d}{dt} \int_{\partial S} < \nabla h, \nabla Z^m h >^2 \sqrt{1 + |\nabla h|^2} \, \text{d}y - \int_{\partial S} < \nabla h, \nabla Z^m h > \sqrt{1 + |\nabla h|^2} \, \text{d}y
\]

\[
+ \frac{3}{2} \int_{\partial S} < \nabla h, \nabla \partial_t h > \sqrt{1 + |\nabla h|^2} \, \text{d}y < < \nabla h, \nabla Z^m h >^2 \, \text{d}y
\]

1st term of right hand side can be absorbed to energy(left hand side) when \(|h|_{1,\infty}\) is bounded. So we estimate 2nd and 3rd terms. Both 2nd and 3rd terms are controlled by

\[
(5.39) \quad \Lambda (|h|_{Y^{2,1}}) \, |h|^2_{X^{m,1}}
\]

And,

\[
(5.40) \quad \left| \int_{\partial S} \nabla C^m(S) \cdot (\partial_t Z^m h) \, \text{d}y \right| = \left| \int_{\partial S} \nabla \cdot C^m(S) \partial_t Z^m h \, \text{d}y \right| \\
\leq \| \nabla \cdot C^m(S) \| \, |\partial_t h|_{X^{m,0}}
\]

For 1st part of \(P_1\), it can be controlled by

\[
(5.41) \quad \left\| \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{\nabla h < \nabla h, \nabla Z^m h >}{\sqrt{1 + |\nabla h|^2}} + C^m(S) \right\| \left\| v^b \cdot (Z^m N) + C^m(KB) \right\| \frac{1}{2}
\]

\[
\lesssim \Lambda \left( \frac{1}{c_0} |h|_{Y^{2,1}} + \|\nabla v\|_{Y^{2,0}} \right) \left( \|v\|_{X^{m,0}} + \|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} \right)
\]

Hence, by putting together 1st and 2nd part, we get the estimate of \(P_1\).

\[
(5.42) \quad |P_1| \lesssim \eta \Lambda \left( \frac{1}{c_0} |h|_{Y^{2,1}} + \|\nabla v\|_{Y^{2,0}} \right) \left( \|v\|_{X^{m,0}} + \|\nabla v\|_{X^{m-1,0}} + |h|_{X^{m,\frac{1}{2}}} + |\partial_t h|_{X^{m,0}} \right)
\]

7) last integral of the energy estimate.

\[
(5.43) \quad \eta \int_{\partial S} \left| \nabla Z^m h \right| \|\nabla h, \nabla \partial_t h \| \, \text{d}A
\]

\[
\lesssim \eta \|\nabla h, \nabla (v^b \cdot N) \|_{L^\infty} \|h|_{X^{m,1}} \lesssim \eta \Lambda \left( \frac{1}{c_0} |h|_{Y^{2,1}} + \|\nabla v\|_{Y^{2,0}} \right) \|h|_{X^{m,\frac{1}{2}}}
\]

Now, we can gather all estimates. We write \(|h|_{Y^{2,1}} + \|\nabla v\|_{Y^{2,0}} = \Lambda_{m,\infty}\), and use trace estimate to

\[
|v^b|_{X^{m,\frac{1}{2}}} \sim \|\nabla v\|_{X^{m,0}}
\]

\[
(5.44) \quad \left\| Z^m v(t) \right\|_{L^2(S)}^2 + \eta \left\| Z^m \nabla h(t) \right\|_{L^2(\partial S)}^2 + g \left\| Z^m h(t) \right\|_{L^2(\partial S)}^2 + 4 \varepsilon \int_0^t \left\| S^c(Z^m v(s)) \right\|_{L^2(S)}^2 \, \text{d}s
\]

\[
\lesssim \left( \left\| (Z^m v(0)) \right\|_{L^2(S)}^2 + \eta \left\| (Z^m \nabla h(0)) \right\|_{L^2(\partial S)}^2 + g \left\| Z^m h(0) \right\|_{L^2(\partial S)}^2 \right)
\]

\[
+ \int_0^t \Lambda \left( \frac{1}{c_0} \Lambda_{m,\infty} + \|\nabla q\|_{Y^{2,0}} \right) \left( \|v\|_{X^{m,0}} + \|\nabla v\|_{X^{m-1,0}} + |\partial_t h|^2_{X^{m,0}} + |h|^2_{X^{m,\frac{1}{2}}} + g \|q\|_{X^{m,0}} + \|\nabla q\|_{X^{m-1,0}} \right) \, \text{d}s
\]

We will claim that \(\partial_t^{3/2} \partial_t \) later. So, energy estimate of \(Z^{m+1,\alpha_\xi}\) can be controlled by \(Z^{m+1,\alpha_\xi}\), so our case must not include \(Z^m = \partial_t^m\) case. Hence, we consider only the case that at least one of \(Z^m\) is spatial derivatives. So our function space is \(X^{m-1,1}\). We sum all of these cases to get

\[
(5.45) \quad \left\| v(t) \right\|_{X^{m-1,1}} + \|h(t)\|_{X^{m-1,1}}^2 + 4 \varepsilon \int_0^t \left\| S^c v(s) \right\|_{X^{m-1,1}}^2 \, \text{d}s \lesssim \left( \left\| v(0) \right\|_{X^{m-1,1}} + \|h(0)\|_{X^{m-1,1}}^2 \right)
\]

\[
+ \int_0^t \Lambda \left( \frac{1}{c_0} \Lambda_{m,\infty} + \|\nabla q\|_{Y^{2,0}} \right) \left( \|v\|_{X^{m-1,1}}^2 + \|\nabla v\|_{X^{m-2,1}}^2 + |\partial_t h|^2_{X^{m-1,1}} + |h|^2_{X^{m-1,\frac{1}{2}}} + g \|q\|_{X^{m-1,1}} + \|\nabla q\|_{X^{m-2,1}} \right) \, \text{d}s
\]

On the LHS, we can use proposition 2.6, to replace \(S^c v\) into \(\nabla v\) under the assumption of \(|h|_{2,\infty}\) is bounded and \(\partial_t \varphi\) is positive. And on the RHS using pressure estimates of previous section, we can deduce,

\[
(5.46) \quad \left\| v(t) \right\|_{X^{m-1,1}}^2 + \|h(t)\|_{X^{m-1,2}}^2 + 4 \varepsilon \int_0^t \left\| \nabla v(s) \right\|_{X^{m-1,1}}^2 \, \text{d}s \lesssim \left( \left\| v(0) \right\|_{X^{m-1,1}}^2 + \|h(0)\|_{X^{m-1,1}}^2 \right)
\]
Proposition 6.1. We use pressure estimate on this estimate. Moreover we also use Young’s inequality to separate dissipation type term \( \varepsilon \int_0^t \| \nabla v(t) \|_{X^{m-1,1}} \), and then make it absorbed into LHS. At result, we get

\[
\| v(t) \|_{X^{m-1,1}}^2 + \| h(t) \|_{X^{m-1,2}}^2 + \varepsilon \int_0^t \| \nabla v(s) \|_{X^{m-1,1}}^2 \, ds \lesssim \left( \| v(0) \|_{X^{m-1,1}}^2 + |h(0)|_{X^{m-1,1}}^2 \right) + \int_0^t \Lambda \left( \frac{1}{c_0}, \Lambda, \infty \right) \left\{ \| v \|_{X^{m-1,1}}^2 + \| \nabla v \|_{X^{m-2,1}}^2 + |\partial_t h|_{X^{m-1,1}}^2 + |h|_{X^{m-1,2}}^2 \right\} \, ds
\]

So, ends proposition.

In the next section, we estimate for only time-differentiated space. By summing with above estimate, we get the estimate for norm \( \| \cdot \|_{X^{m,0}} \).

6. Energy Estimates of All Time-Derivatives

Basically, we lose \( \frac{1}{2} \) derivatives in commutator. So we would need Dirichlet-Neumann operator estimate to use the fact, \( \partial_t \sim \partial_t^2 \) on the boundary. In fact, the worst commutator appears when all time derivatives hit the commutator. If at least one time derivative does not hit the term, then it is \( \frac{1}{2} \) better, because \( \partial_t \sim \partial_t^2 \). So, when we take only time derivatives, \( \partial_t^m \), commutator can not absorb all the time derivatives. At result, the last step energy estimate would not produce bad commutator. In fact, bad commutator occur only in \( R^c \) and \( P_1 \), so we are suffice to estimate this two terms, when \( Z^m = \partial_t^m \).

**Proposition 6.1.** When \( Z^m = \partial_t^m \), \( C^m(f) \) can be estimated as follow.

\[
\| C^m(f) \| \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{m+1}} + \| \partial_z f \|_{Y^{m+1}} \right) \left( \| \partial_t^m h \|_{L^\infty} + \| \partial_t^{m-1} \partial_z f \|_{L^\infty} \right)
\]

**Proof.** Since, \( \partial_t \) commutes with \( \partial_z \), we get the following.

\[
\partial_t^m (\partial_t^2 f) = \partial_t^2 (\partial_t^m f) + C^m(f)
\]

\[
C^m(f) = - \left[ \partial_t^m, \frac{\partial \varphi}{\partial z}, \partial_z f \right] - \left( \partial_t^m \frac{\partial \varphi}{\partial z} \right) \partial_z f
\]

And we get easily,

\[
\left\| \left[ \partial_t^m, \frac{\partial \varphi}{\partial z}, \partial_z f \right] \right\| \lesssim \left\| \partial_t^{m-1} \frac{\partial \varphi}{\partial z}, \partial_z f \right\| + \left\| \partial_t^m \frac{\partial \varphi}{\partial z} \right\| \| \partial_t^{m-1} \partial_z f \| \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{m+1}} + \| \partial_z f \|_{Y^{m+1}} \right) \left( \| \partial_t^{m-1} h \|_{L^\infty} + \| \partial_t^{m-1} \partial_z f \|_{L^\infty} \right)
\]

Putting together, we get the result.

Now our aim is to get energy estimate when \( Z^m = \partial_t^m \).

**Proposition 6.2.** Let’s define \( \| (\partial_t h_0) f \| = \sum_{i=0}^m \| \partial_t^i f \| \), and

\[
e_0^{m-1}(t) = \sum_{i=0}^{m-1} e_i(t) = \| v(t) \|_{X^{m-1,1}}^2 + |h(t)|_{X^{m-1,2}}^2
\]

(Sum of previous step’s energy except dissipation term). Now let \( \partial_t^{k,\alpha} = \partial_t^m \), then we have the following energy estimate.

\[
\| \partial_t^m v(t) \|_{L^2(S)}^2 + |\partial_t^m h(t)|_{H^1(\partial S)}^2 + \varepsilon \int_0^t \| S^\alpha (\partial_t^m v)(s) \|_{L^2(S)}^2 \, ds \lesssim \left( e_0^{m-1}(0) + \| (\partial_t^m v)(0) \|_{L^2(S)}^2 + |(\partial_t^m h)(0)|_{H^1(\partial S)}^2 \right) + C(e_0^{m-1}(t)) t
\]
Firstly, we look at $P_1$. Since, we can control $\frac{1}{c_0}$. We lose $\frac{1}{c_0}$ derivative in two terms, $R_C$ and $P_1$. Among $R_C$, the highest order commutators come from $\int_S C^m(q^S)(\partial_t^m v) dV_1$ and $\int_S C^m(d)(\partial_t^m q^S) dV_1$. Firstly, we look at $\int_S C^m(q^S)(\partial_t^m v) dV_1$. From the proposition above,

$$
\int_S C^m(q^S)(\partial_t^m v) dV_1 \leq \|C^m(q^S)\| \|\partial_t^m v\| \lesssim \Lambda \left( \frac{1}{c_0} h + \|\partial_t^m q^S\| + \|\partial_t^m v\| \right)
$$

Secondly, about $\int_S C^m(d)(\partial_t^m q^S) dV_1$, we divide $C_i^m(v)$ as (when $i = 1, 2$, and $i = 3$ case is also same.)

$$
C_i^m(v) = -\partial_t^m \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t v \partial_i v \partial_t^m q^S \lesssim |\partial_t v|_{H^4} \left| \frac{\partial_i \varphi}{\partial z \varphi} \right| \left\| \partial_t^m q^S \right\|_{H^{-\frac{1}{2}}}
$$

1) $\int_S \partial_t^m \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t v \partial_i v \partial_t^m q^S \lesssim |\partial_t v|_{H^4} \left| \frac{\partial_i \varphi}{\partial z \varphi} \right| \left\| \partial_t^m q^S \right\|_{H^{-\frac{1}{2}}} \lesssim \Lambda \left( \frac{1}{c_0} h_{H^1} + \left\| \partial_t^m q^S \right\|_{H^{-\frac{1}{2}}} \right)

2) $\int_S \partial_t^m \partial_i v \cdot \partial_t^m q^S$

This is not high order term, so it is trivial.

3) Since $\partial_t \sim \partial_z$ inside the domain, we cannot give $-\frac{1}{c_0}$ derivative to $\partial_t^m \partial_z v$. So we interchange $\partial_t$ of $v$ and $\partial_z$ of $q^S$ by integration by part both in time and space. By taking intergral for time, we get,

$$
\int_0^t \int_S \partial_t \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t^m \partial_z v \partial_t^m q^S dAdS = \int_0^t \int_S \partial_t \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t^m v \partial_t^m \partial_z q^S dAdS + \int_0^t \int_S \partial_t \partial_t \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t^m v \partial_t^m q^S dAdS
$$

$$
= - \int_0^t \int_S \partial_t \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t^m \partial_z \partial_t^m v \partial_t^m q^S dAdS - \int_0^t \int_S \partial_t \partial_t \left( \frac{\partial_i \varphi}{\partial z \varphi} \right) \partial_t^m v \partial_t^m q^S dAdS
$$

$$
\lesssim \int_0^t \Lambda \left( \frac{1}{c_0} h_{H^1} \right) \left( \left\| \partial_t^m v \right\|^2 + \left\| \partial_t^m \partial_z q^S \right\|^2 \right) dS
$$

4) Remaining term is easy to deal.

$$
\int_S \tilde{C}_{\partial^m d}(d)\partial_t^m q^S dA \lesssim \left\| \tilde{C}_{\partial^m d}(d) \right\|_{H^4} \left\| \partial_t^m q^S \right\|_{H^{-\frac{1}{2}}}
$$

Now we deal $P_1$ part. For $P_1$, in previous section, $\int_{\partial S} \nabla h, \nabla h^\partial > \sqrt{1 + |\nabla h|^2} < \nabla h, \nabla \partial_t^m h >$ was treated. Highest order term will be absorbed in energy. So, we investigate other terms.

1) Since, we can control $\|\nabla C^m(KB)\|_

$$
\int_{\partial S} \left( \frac{\nabla \partial_t^m h}{\sqrt{1 + |\nabla h|^2}} \right) \left( \frac{\nabla h < \nabla h, \nabla \partial_t^m h >}{\sqrt{1 + |\nabla h|^2}} + C^m(S) \right) \nabla C^m(KB) dy
$$
is also good to control.

2)  
\begin{equation}
(6.14) \quad - \int_0^t \int_{\partial S} C^m(S) \cdot \nabla (\partial_t^{m+1} h) dA ds = - \left[ \int_{\partial S} C^m(S) \cdot \nabla (\partial_t^m h) dA \right]_0^t + \int_0^t \int_{\partial S} \partial_t C^m(S) \cdot \nabla \partial_t^m h dA ds \\
\lesssim - \left[ \int_{\partial S} C^m(S) \cdot \nabla (\partial_t^m h) dA \right]_0^t + \int_0^t \| \partial_t C^m(S) \| \| \partial_t^m \nabla h \| ds
\end{equation}

3)  
\begin{equation}
(6.15) \quad \int_{\partial S} \frac{\partial_t^m \nabla h}{\sqrt{1 + \| \nabla h \|^2}} \cdot \nabla (v^b \cdot \partial_t^m N) dA \\
\lesssim \int_{\partial S} \frac{1}{2 \sqrt{1 + \| \nabla h \|^2}} \nabla \cdot \left\{ (v^b_1, v^b_2) | \nabla \partial_t^m h |^2 \right\} dA + \Lambda (\| \|_{L^\infty} + |h|_{2, \infty}) \| \partial_t^m \nabla h \| \\
ung \lesssim \Lambda (\| \|_{L^\infty} + |h|_{2, \infty}) \| \partial_t^m \nabla h \|
\end{equation}

since, first term is zero, which means highest order vanishes by divergence theorem.

4)  
\begin{equation}
(6.16) \quad \int_{\partial S} C^m(S) \cdot \nabla (v^b \cdot \partial_t^m N) dA = \int_{\partial S} \nabla \cdot \left( (v^b \cdot \partial_t^m N) C^m(S) \right) dA - \int_{\partial S} \nabla \cdot C^m(S) (v^b \cdot \partial_t^m N) dA \\
\lesssim \Lambda (\| \|_{L^\infty} + \| \partial_t^m \nabla h \|)
\end{equation}

5)  
\begin{equation}
(6.17) \quad \int_{\partial S} < \nabla h, \nabla \partial_t^m h > \lesssim \Lambda (\| \|_{L^\infty} + \| \partial_t^m \nabla h \|)
\end{equation}

We should controls like

\begin{equation}
(6.18) \quad \int_{\partial S} \frac{\mu}{\sqrt{1 + \| \nabla h \|^2}} \partial_t (\partial_t^m h) \partial jk (\partial_t^m h)
\end{equation}

where \( i, j, k = 1, 2 \) and at least one is different to other two. So WLOG, by divergence theorem, we can change into these form,

\begin{equation}
(6.19) \quad \sim \int_{\partial S} \partial_t (\partial_t^m h) \partial ij (\partial_t^m h) \sim \int_{\partial S} \partial_j | \partial_t (\partial_t^m h) |^2 \sim 0
\end{equation}

So, similar with 3), we get only low order terms, (up to \( \| \partial_t^m \nabla h \| \)) with some trivial finite order terms. Lastly, we deal terms which come from integration by parts. First we define,

\begin{equation}
(6.20) \quad e_0^{m-1}(t) \doteq \sum_{i=0}^{m-1} e_i(t) = \| v(t) \|^2_{X_{m-1,1}} + \| h(t) \|^2_{X_{m-1,2}}
\end{equation}

(which is non-dissipation energy terms of previous steps), then, all terms like, \( \left[ \int_{\partial S} \partial_t \left( \frac{\partial \varphi}{\partial x^i} \right) \partial_t^{m-1} v \partial_t^{m-1} \partial_i q^S \right]_0^t \), can be estimated by

\begin{equation}
(6.21) \quad \left[ \int_{\partial S} \partial_t \left( \frac{\partial \varphi}{\partial x^i} \right) \partial_t^{m-1} v \partial_t^{m-1} \partial_i q^S \right]_0^t \lesssim \Lambda (e_0^{m-1}(0)) + \Lambda (e_0^{m-1}(t)) \lesssim \Lambda (e_0^{m-1}(0)) + C(e_0^{m-1}(t)) t
\end{equation}

where \( C(e_0^{m-1}(t)) \) is some constant depending on \( E^m(t) \). This is possible, since for integration by parts, integrands are one derivative less. And since we can choose function \( \Lambda \) so that \( \Lambda(0) = 0 \). Finally by considering all together, we can get the result. \( \square \)

At result, from this estimate, when we apply \( \partial_t^m \) to the equation, commutator does not require \( \partial_t^{m+1} \).
7. **Dirichlet-Neumann Operator estimate on the boundary**

In this section, we claim that, on the boundary \( \partial_x^{3/2} h \) can be controlled by \( \partial_t h \) with help of some low order terms, so that we can close the energy estimate. We start with section with a lemma which is needed to prove the next proposition.

**Lemma 7.1.** There exists \( c > 0 \) such that for every \( h \in W^{1, \infty}(\mathbb{R}^2) \) with \( 1 - \| h \|_{L^\infty} \geq \delta \) for some \( \delta > 0 \) we have

\[
(G[h], v) \geq c(1 + \| h \|_{W^{1, \infty}(\mathbb{R}^2)})^{-2} \left\| \frac{| \nabla |}{(1 + | \nabla |)^{1/2}} \right\|_{L^2(\mathbb{R}^2)}, \forall v \in H^\frac{1}{2}(\mathbb{R}^2)
\]

Here, \( G[h]v \) means Dirichlet-Neumann operator.

**Proof.** See proposition 3.4 of [2]. \( \square \)

**Proposition 7.2.** When \( | h |_{1, \infty} \) is bounded, \( h \) enjoys the following estimate.

\[
\int_0^t |Z^m \nabla h|^2_{H^\frac{1}{2}(\partial S)} \, ds \lesssim \int_0^t \Lambda \left( \frac{1}{c_0}, |Z^m \nabla \partial_t h(s)| + \|Z^m \nabla v(s)\| \right) \, ds + \Lambda \left( \frac{1}{c_0} \right) C (|Z^m \nabla \partial_t h(t)|) \, t
\]

**Proof.** From kinematic boundary condition \( h_t = v^b \cdot N \), we get \( \partial_t h = v^b \cdot N + v^b \cdot N_t \) We apply \( Z^m \) to this equation, where (\( \alpha_3 = 0 \), because we are on the boundary.

\[
\partial_t (Z^m h) = (Z^m v^b) \cdot N + v^b \cdot (Z^m N) + [Z^m, v^b, N]
\]

Meanwhile, from definition of \( q^S \),

\[
v^b + (\nabla q^S)^b = \left( \frac{\partial \phi}{\partial \varphi} \partial_z v + \frac{\nabla \varphi}{\partial \varphi} \partial_z q^S \right)^b = (Re)^b
\]

We apply \( Z^m \), so get

\[
(Z^m v^b) + (\nabla Z^m q^S)^b = Z^m \left( \frac{\partial \phi}{\partial \varphi} \partial_z v + \frac{\nabla \varphi}{\partial \varphi} \partial_z q^S \right)^b = Z^m (Re)^b
\]

Then we replace \( Z^m v^b \) so we get

\[
(Z^m h)_{tt} = \left\{ -(\nabla Z^m q^S)^b + Z^m (Re)^b \right\} \cdot N + v^b \cdot (Z^m N) + [Z^m, v^b, N] + (Z^m v^b) \cdot N_t + v^b \cdot (Z^m N_t) + [Z^m, v^b, N_t]
\]

From boundary value,

\[
-(Z^m q^S)^b = \eta \nabla \cdot Z^m \left( \frac{\nabla h}{\sqrt{1 + | \nabla h|^2}} \right)
\]

\[
= \eta \left\{ \nabla \cdot \frac{\nabla Z^m h}{\sqrt{1 + | \nabla h|^2}} - \frac{\left< \nabla h, \nabla Z^m h \right>}{\sqrt{1 + | \nabla h|^2}} \nabla h + \nabla \cdot C^m(S) \right\} \approx \eta V^b
\]

Using Dirichlet-Neumann operator symbol, we have

\[
(Z^m h)_{tt} = \eta G[h] V^b + Z^m (Re)^b \cdot N + v^b \cdot (Z^m N) + (Z^m v^b) \cdot N_t + v^b \cdot (Z^m N_t) + [Z^m, v^b, N] + [Z^m, v^b, N_t] = \eta G[h] V^b + l.o.t
\]

where \( l.o.t \) means low order terms.

\[
l.o.t = Z^m (Re)^b \cdot N + v^b \cdot (Z^m N) + (Z^m v^b) \cdot N_t + v^b \cdot (Z^m N_t) + [Z^m, v^b, N] + [Z^m, v^b, N_t]
\]

Now we do dot product with \( V^b \) and integrate for time \( t \).

1) LHS becomes,

\[
\int_0^t (Z^m h_{tt} \cdot V^b) \, ds = \int_0^t \left( (Z^m h)_{tt} \cdot \nabla \cdot Z^m \left( \frac{\nabla h}{\sqrt{1 + | \nabla h|^2}} \right) \right) \, ds
\]
\[
= \int_0^t \int_{\partial S} (\nabla Z^m h_t) \cdot \left( Z^m \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) d\text{Ads} - \left[ \int_{\partial S} \nabla Z^m h_t \partial h \left( Z^m \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \right]_0^t
\]

Let’s define the last term as \( R \).

\[
(7.10) \quad = \int_0^t \int_{\partial S} (\nabla Z^m h_t) \cdot \partial h \left( \frac{\nabla Z^m h}{\sqrt{1 + |\nabla h|^2}} - \frac{<\nabla h, \nabla Z^m h>}{\sqrt{1 + |\nabla h|^2}} \nabla h + C^m(S) \right) d\text{Ads} - R
\]

After some integration by parts (for both space and time),

\[
(7.11) \quad \int_0^t ((Z^m h)_{tt}, V^b) ds \lesssim \int_0^t \Lambda \left( \frac{1}{c_0} \|Z^m \nabla \partial_t h(s)\| \right) ds + \Lambda \left( \frac{1}{c_0} \|Z^m \nabla \partial_t h(0)\| \right) + \Lambda \left( \frac{1}{c_0} \right) C \left( \|Z^m \nabla \partial_t h(t)\| \right) t
\]

2) on the RHS, using the lemma above,

\[
(1 + |h|_{1,\infty})^{-2} \left( \frac{|\nabla|}{|1 + |\nabla||} V^b \right)^2 \lesssim \left( G[h]V^b, V^b \right)
\]

Note that,

\[
(7.12) \quad \left( \frac{|\nabla|}{|1 + |\nabla||} V^b \right)^2 = \left( \frac{|\xi|}{|1 + |\xi||} \right)^2 \xi, \xi \left( Z^m \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}} \right)^2 \]

(since, \( \left| Z^m \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}} \right| \gtrsim \left| Z^m \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}} \right| \))

We now integrate w.r.t time and move low order term to opposite side, we get,

\[
(7.13) \quad \int_0^t (1 + |h|_{1,\infty})^{-2} \left| Z^m \frac{-\nabla h}{\sqrt{1 + |\nabla h|^2}} \right|^2 \lesssim \int_0^t (G[h]V^b, V^b) ds + \int_0^t \Lambda \left( \frac{1}{c_0}, |Z^m h|_{1/2} \right) ds
\]

Hence by putting LHS and RHS, (and extract highest order in the LHS and give all other low order terms to RHS) then we have,

\[
(7.14) \quad \int_0^t \left| Z^m \nabla h \right|_{H^2(\partial S)}^2 ds \lesssim \int_0^t \Lambda \left( \frac{1}{c_0}, \|Z^m \nabla \partial_t h(s)\| \right) ds + \Lambda \left( \frac{1}{c_0}, \|Z^m \nabla \partial_t h(0)\| \right)
\]

\[
+ \Lambda \left( \frac{1}{c_0} \right) C \left( \|Z^m \nabla \partial_t h(t)\| \right) t + \int_0^t \int_{\partial S} (l.o.t)V^b d\text{Ads}
\]

Last term, \( \int_0^t \int_{\partial S} (l.o.t)V^b d\text{Ads} \) does not produce any harmful terms, since if we give \( L^2 \) norm to \( l.o.t \) then the highest parts gives,

\[
Z^m N_t \sim |Z^m \nabla \partial_t h|
\]

\[
Z^m v^b \sim \|Z^m \nabla v\|
\]

\[
Z^m (Re)^b \sim |h|_{X^m} \frac{1}{4} \frac{1}{2} = |h|_{X^m.2}
\]

Hence the result follows.

\[
(7.15) \quad \int_0^t \left| Z^m \nabla h \right|_{H^2(\partial S)}^2 ds \lesssim \int_0^t \Lambda \left( \frac{1}{c_0}, |Z^m \nabla \partial_t h(s)| + \|Z^m \nabla v(s)\| \right) ds
\]

\[
+ \Lambda \left( \frac{1}{c_0}, |Z^m \nabla \partial_t h(0)| \right) + \Lambda \left( \frac{1}{c_0} \right) C \left( \|Z^m \nabla \partial_t h(t)\| \right) t
\]
8. Normal derivative estimate

From above energy estimate, we should control $\|\partial_z v\|_{X^{m-1,0}}$. But it is hard to estimate $\partial_z v$ directly. Instead we estimate $S_n$, which is tangential part of $S^\varphi v\hat{n}$.

\begin{equation}
S_n = \Pi (S^\varphi v\hat{n}) \quad \text{where} \quad \Pi = I - \hat{n} \otimes \hat{n}
\end{equation}

First, we show that instead of $\partial_z v$, we are suffice to estimate $S_n$.

**Lemma 8.1.** We have the following normal part estimate of $\partial_z v$.

\begin{equation}
\|\partial_z v \cdot \hat{n}\|_{X^{m-1,0}} \lesssim \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{Y^{m,0}} \right) \left( \|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}} \right)
\end{equation}

**Proof.** From divergence free condition, we have,

\begin{equation}
\partial_z v \cdot \hat{n} = \frac{1}{\sqrt{1 + |\nabla \varphi|^2}} \partial_z \varphi (\partial_1 v_1 + \partial_2 v_2)
\end{equation}

Applying $Z^{m-1}$ and using basic propositions, we easily get

\begin{equation}
\|Z^{m-1}(\partial_z v \cdot \hat{n})\| \lesssim \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{Y^{m,0}} \right) \left( \|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|v\|_{X^{m-1,1}} \right)
\end{equation}

Using this lemma, we can estimate $\partial_z v$.

**Lemma 8.2.**

\begin{equation}
\|\partial_z v\|_{X^{m-1,0}} \lesssim \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{Y^{m,0}} \right) \left( \|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}} \right)
\end{equation}

**Proof.**

\begin{equation}
2S^\varphi \varphi \hat{n} = (\nabla u) \hat{n} + (\nabla u)^T \hat{n} = (\nabla u) \hat{n} + g^{ij}(\partial_j v \cdot \hat{n}) \partial_y^i
\end{equation}

And from divergence free condition,

\begin{equation}
\partial_N u = \frac{1 + |\nabla \varphi|^2}{\partial_z \varphi} \partial_z v - \partial_1 \varphi \partial_1 v - \partial_2 \varphi \partial_2 v
\end{equation}

to obtain,

\begin{equation}
\|\partial_z v\|_{X^{m-1,0}} \lesssim \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{Y^{m,0}} \right) \left( \|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|\partial_z v \cdot \hat{n}\|_{X^{m-1,0}} + \|S^\varphi v\hat{n}\|_{X^{m,0}} \right)
\end{equation}

and

\begin{equation}
S^\varphi \hat{n} = S_n - (\hat{n} \otimes \hat{n}) (S^\varphi v\hat{n})
\end{equation}

Now we use previous lemma to get

\begin{equation}
\|\partial_z v\|_{X^{m-1,0}} \lesssim \Lambda \left( \frac{1}{c_0}, \|\nabla v\|_{Y^{m,0}} \right) \left( \|v\|_{X^{m,0}} + |h|_{X^{m-1,1}} + \|S_n\|_{X^{m-1,0}} \right)
\end{equation}

Now we estimate $S_n$. As like in [1], we take $\nabla \varphi$ to the navier-stokes equation.

\begin{equation}
\partial^\varphi \nabla \varphi \cdot v + (v \cdot \nabla \varphi) \nabla \varphi \cdot v + (\nabla \varphi)^2 + (D^\varphi)^2 q - \varepsilon \Delta \varphi \nabla \varphi \cdot v = 0
\end{equation}

where $(D^\varphi)^2$ is Hessian matrix. We also take symmetric part of the equation, then using both equations,

\begin{equation}
\partial^\varphi S^\varphi v + (v \cdot \nabla \varphi) S^\varphi v + \frac{1}{2} \left( (\nabla v)^2 + ((\nabla v)^T)^2 \right) + (D^\varphi)^2 q - \varepsilon \Delta \varphi (S^\varphi v) = 0
\end{equation}

By taking tangential operator, $\Pi$,

\begin{equation}
\partial^\varphi S_n + (v \cdot \nabla \varphi) S_n - \varepsilon \Delta \varphi (S_n) = F_S
\end{equation}

where $F_S$ is commutator,

\begin{equation}
F_S = F_S^1 + F_S^2 + F_S^3
\end{equation}
\begin{equation}
F_S^1 = -\frac{1}{2} \Pi \left( (\nabla^2 v)^2 + \left( (\nabla^2 v)^T \right)^2 \right) \hat{n} + \left( \partial_t \Pi + v \cdot \nabla^2 \Pi \right) S^r v \hat{n} + \Pi S^r v (\partial_t \hat{n} + v \cdot \nabla \hat{n})
\end{equation}

\begin{equation}
F_S^2 = -2 \varepsilon \partial_t \Pi \partial_t^3 (S^r v \hat{n}) - 2 \varepsilon \Pi (\partial_t^3 (S^r v) \partial_t^3 \hat{n}) - \varepsilon (\Delta^2 \Pi) S^r v \hat{n} - \varepsilon \Pi S^r \Delta^2 \hat{n}
\end{equation}

\begin{equation}
F_S^3 = -\Pi \left( (D^2)^2 q \right) \hat{n}
\end{equation}

We will apply $Z^{m-1}$ to the equation, so we need to estimate $\| F_S^1 \|_{X^{m-1,0}}$, $\| F_S^2 \|_{X^{m-1,0}}$, $\| F_S^3 \|_{X^{m-1,0}}$. In [1], optimal estimate order was $m-2$, because of regularity of $h$. In our case $h$ is 1 better so we guess that $m-1$ order estimate is possible, but then $Z^{m-1} F_S^3$ has $m+1$ order of $p$, which we cannot control. So we use divergence free condition to show that highest order in $\int_S Z^{m-1} S_n \cdot Z^{m-1} F_S^3 dV_t$ is vanished.

For $Z^{m-1} F_S^1$, using our basic propositions and lemmas,

\begin{equation}
\| Z^{m-1} F_S^1 \|_{L^2(S)} \leq \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) \left( \| \nabla v \|_{X^{m-1,0}} + |h|_{X^{m-1,1}} + \| v \|_{X^{m,0}} \right)
\end{equation}

Similary, for $Z^{m-1} F_S^2$,

\begin{equation}
\| Z^{m-1} F_S^2 \|_{L^2(S)} \leq \varepsilon \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) \left( \| \partial_z v \|_{X^{m-1,0}} + \| \partial_z v \|_{X^{m,0}} + |h|_{X^{m-1,1}} \right)
\end{equation}

By using Young’s inequality,

\begin{equation}
ab \leq \frac{a^2}{2\delta} + \frac{b^2}{2}, \quad \forall \delta > 0
\end{equation}

$\varepsilon \int_S \| \nabla S_n \|_{X^{m-1,0}}$ can be absorbed by energy. For $|h|_{X^{m-1,1}}$, it can be controlled by $|h|_{X^{m,1}}$, by Dirichlet-Neumann operator estimate.

Lastly, for $Z^{m-1} F_S^3$, we just investigate the highest order part of this term, and show it vanished. Remaining terms are not difficult to estimate, like in $Z^{m-1} F_S^1$ and $Z^{m-1} F_S^2$.

\begin{equation}
\int_S Z^{m-1} S_n \cdot Z^{m-1} F_S^3 dV_t = \int_S Z^{m-1} (I - \hat{n} \otimes \hat{n}) \left( (\nabla^2 v) + (\nabla^2 v)^T \right) \hat{n} \cdot Z^{m-1} (I - \hat{n} \otimes \hat{n}) \left( (D^2)^2 q \right) \hat{n}
\end{equation}

The highest order terms in the following type is vanishes, since

\begin{equation}
\int_S \Pi Z^{m-1} (\nabla^2 v)^T \hat{n} \cdot \nabla^2 (\nabla^2 q) \hat{n}
\end{equation}

then by integral by part,

\begin{equation}
\sim \int_S \Pi Z^{m-1} \nabla^2 \cdot (\nabla^2 v)^T \hat{n} \cdot (\nabla^2 q) \hat{n} = 0
\end{equation}

since, $\nabla^2 \cdot (\nabla^2 v)^T$ is zero by divergence free condition veolcity. One another highest term is transpose part of above.

\begin{equation}
\int_S \Pi Z^{m-1} (\nabla^2 v) \hat{n} \cdot \nabla^2 (\nabla^2 q) \hat{n}
\end{equation}

We take transpose to integrand then we get, ($\text{Hessian } (D^2)^2$ is symmetric)

\begin{equation}
\sim \int_S \hat{n}^T Z^{m-1} (\nabla^2 v)^T \Pi \cdot \nabla^2 (\nabla^2 Z^{m-1} q)
\end{equation}

then again, by integration by parts,

\begin{equation}
\sim \int_S \hat{n}^T Z^{m-1} \nabla^2 \cdot (\nabla^2 v)^T \Pi (\hat{n}^T \nabla^2 Z^{m-1} q) = 0
\end{equation}

And Surely, low order terms will be controlled by same as $Z^{m-1} F_S^3$.

Now we make high order estimate. By taking $Z^n$, with $|a| = m - 1$, we have

\begin{equation}
\partial_t Z^n S_n + (v \cdot \nabla^2) Z^n S_n - \varepsilon \Delta^2 Z^n S_n = Z^n (F_S) + C_S
\end{equation}
where $C_S$ is commutator. As like in [1], we divide $C_S$ into,

\begin{equation}
C_S^1 = [Z^α v_y] \cdot \nabla_y S_n + [Z^α, V_z] \partial_z S_n = C_{S_{z}} + C_{S_{z}}, \quad C_S^2 = -\varepsilon[Z^α, \Delta^γ]S_n
\end{equation}

Since $(Z^α S_n)_{z=0} = 0$, we get the following,

\begin{equation}
\frac{1}{2}\int_S |Z^α S_n|^2 dV_t + \varepsilon \int_S |\nabla^α Z^α S_n|^2 dV_t = \int_S (Z^α F_S + C_S) \cdot Z^α S_n dV_t
\end{equation}

Estimate of $C_{S_{z}}$ is easy, we get,

\begin{equation}
\|C_{S_{z}}\| \leq \Lambda \left(\frac{1}{c_{β}} A_{m,∞}\right) (\|S_n\|_{X^{m−1,0}} + \|v\|_{X^{m−1,0}} + \|\partial_z v\|_{X^{m−2,0}})
\end{equation}

To estimate $C_{S_{z}}$, it is not easy, because it contains $C_{S_{z}}$, which is not controlled yet. We give $\partial_z$ to $V_z$ by integration by part. From the commutator, we have to control the terms like,

\begin{equation}
\|Z^β V_z \partial_z Z^γ S_n\|
\end{equation}

where $|β| + |γ| \leq m−1$, $|γ| \leq m−2$ or equivalently $|β| \neq 0$. We interchange $\partial_z$ and $Z_3$ by

\begin{equation}
Z^β V_z \partial_z Z^γ S_n = \frac{1−z}{z} Z^β V_z Z_3 Z^γ S_n
\end{equation}

then by commutation between $\frac{1−z}{z}$ and $Z^β$, we encounter the terms like this, where $c_β$ is some nice, bounded function and $|β| \leq |β|$. 

\begin{equation}
c_β Z^β \left(\frac{1−z}{z} V_z\right) Z_3 Z^γ S_n
\end{equation}

If $\beta = 0$,

\begin{equation}
\|c_β Z^β \left(\frac{1−z}{z} V_z\right) Z_3 Z^γ S_n\| \lesssim \|S_n\|_{X^{m−1,0}}
\end{equation}

If $\beta \neq 0$,

\begin{equation}
\|c_β Z^β \left(\frac{1−z}{z} V_z\right) Z_3 Z^γ S_n\| \lesssim \|Z \left(\frac{1−z}{z} V_z\right)\|_{Y^{m+1,0}} + \|S_n\|_{X^{m−1,0}} + \|S_n\|_{Y^{m+1,0}} \|Z \left(\frac{1−z}{z} V_z\right)\|_{X^{m−2,0}}
\end{equation}

First, we see that,

\begin{equation}
\|Z \left(\frac{1−z}{z} V_z\right)\|_{Y^{m+1,0}} \lesssim \|V_z\|_{Y^{m+1,0}} + \|\partial_z V_z\|_{Y^{m+1,0}}
\end{equation}

and,

\begin{equation}
\|Z \left(\frac{1−z}{z} V_z\right)\|_{X^{m−2,0}} \lesssim \|\nabla V_z\|_{X^{m−2,0}} + \left\|\frac{1}{z(1−z)} V_z\right\|_{X^{m−2,0}} = \left\|\frac{1−z}{z} Z V_z\right\|_{X^{m−2,0}} + \left\|\frac{1}{z(1−z)} V_z\right\|_{X^{m−2,0}}
\end{equation}

So we should estimate the terms that look like,

\begin{equation}
\left\|\frac{1−z}{z} Z^ξ Z V_z\right\|, \quad \left\|\frac{1}{z(1−z)} Z^ξ V_z\right\|
\end{equation}

where $|ξ| \leq m−2$. To estimate these two types of terms, we use the following lemma.

**Lemma 8.3.** If $f(0) = 0$, we have the inequalities,

\begin{equation}
\int_{−∞}^{0} \frac{1}{z^2(1−z)^2} |f(z)|^2 dz \lesssim \int_{−∞}^{0} |\partial_z f(z)|^2 dz
\end{equation}

\begin{equation}
\int_{−∞}^{0} \left(\frac{1−z}{z}\right)^2 |f(z)|^2 dz \lesssim \int_{−∞}^{0} \left(|f(z)|^2 + |\partial_z f(z)|^2\right) dz
\end{equation}

**Proof.** See [1], Lemma 8.4
Using above lemma, we have
\begin{equation}
\left\| \frac{1}{z} Z^\xi Z V_z \right\|^2 \lesssim \| Z^\xi Z V_z \|^2 + \| \partial_z (Z^\xi Z V_z) \|^2
\end{equation}
\begin{equation}
\left\| \frac{1}{z(1-z)} Z^\xi V_z \right\| \lesssim \| \partial_z Z^\xi V_z \|
\end{equation}
So,
\begin{equation}
\| C_{S_1} \| \lesssim \| Z V_z \|_{X^{m-2,0}} + \| \partial_z Z V_z \|_{X^{m-2,0}} + \| \partial_z V_z \|_{X^{m-2,0}}
\end{equation}
Combining with \( C_{S_3} \), we have
\begin{equation}
\| C_{S_1}^1 \| \leq \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) (\| S_n \|_{X^{m-1,0}} + \| \gamma \|_{X^{m-1,0}} + \| \partial_z V_z \|_{X^{m-1,0}})
\end{equation}
\begin{equation}
\leq \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) (\| S_n \|_{X^{m-1,0}} + \| \gamma \|_{X^{m-1,0}} + \| h \|_{X^{m-1,1}})
\end{equation}
For \( C_{S_1}^2 \), we have
\begin{equation}
\varepsilon Z^\alpha (\Delta^\gamma S_n) = \varepsilon Z^\alpha \left( \frac{1}{\partial^2 \varphi} \nabla \cdot (E \nabla S_n) \right) = \varepsilon \frac{1}{\partial^2 \varphi} Z^\alpha (\nabla \cdot (E \nabla S_n)) + C_{S_1}^2
\end{equation}
\begin{equation}
= \varepsilon \frac{1}{\partial^2 \varphi} \nabla \cdot Z^\alpha (E \nabla S_n) + C_{S_2}^2 + C_{S_1}^2
\end{equation}
\begin{equation}
= \varepsilon \frac{1}{\partial^2 \varphi} \nabla \cdot (E \nabla Z^\alpha S_n) + C_{S_3}^2 + C_{S_4}^2 + C_{S_1}^2
\end{equation}
\begin{equation}
= \varepsilon \Delta^\varphi (Z^\alpha S_n) + C_{S_1}^2
\end{equation}
So, we define
\begin{equation}
C_{S_1}^2 = C_{S_1}^2 + C_{S_2}^2 + C_{S_3}^2
\end{equation}
where
\begin{equation}
C_{S_1}^2 \doteq \varepsilon \left[ Z^\beta, \frac{1}{\partial^2 \varphi} \right] \nabla \cdot (E \nabla S_n), \quad C_{S_2}^2 \doteq \varepsilon \frac{1}{\partial^2 \varphi} [Z^\alpha, \nabla] \cdot (E \nabla S_n), \quad C_{S_3}^2 \doteq \varepsilon \frac{1}{\partial^2 \varphi} \nabla \cdot ([Z^\alpha, E \nabla] S_n)
\end{equation}
1) \( C_{S_1}^2 \)
We need to estimate like,
\begin{equation}
\varepsilon \int_S Z^\beta \left( \frac{1}{\partial^2 \varphi} \right) Z^\gamma (\nabla \cdot (E \nabla S_n)) \cdot Z^\alpha S_n dV_i
\end{equation}
where \(|\beta| + |\gamma| = \alpha, \beta \neq 0\). Then again by commutator between \( Z^\gamma \) and \( \nabla \), the forms becomes like the following forms.
\begin{equation}
\varepsilon \int_S Z^\beta \left( \frac{1}{\partial^2 \varphi} \right) \partial_j Z^\gamma (E \nabla S_n) \cdot Z^\alpha S_n dV_i
\end{equation}
where \(|\gamma| \leq |\gamma|\). Now we do integrate by part, so get
\begin{equation}
\left| \varepsilon \int_S Z^\beta \left( \frac{1}{\partial^2 \varphi} \right) \partial_j Z^\gamma (E \nabla S_n) \cdot Z^\alpha S_n dV_i \right|
\end{equation}
\begin{equation}
\leq \left| \varepsilon \int_S \partial_j Z^\beta \left( \frac{1}{\partial^2 \varphi} \right) Z^\gamma (E \nabla S_n) \cdot Z^\alpha S_n dV_i \right| + \left| \varepsilon \int_S Z^\beta \left( \frac{1}{\partial^2 \varphi} \right) Z^\gamma (E \nabla S_n) \cdot \partial_j Z^\alpha S_n dV_i \right|
\end{equation}
Using basic propositions and dividing each terms into \( L^\infty, L^2, L^2 \), then we get,
\begin{equation}
\left| \int_S C_{S_1}^2 \cdot Z^\alpha S_n dV_i \right| \lesssim \varepsilon \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) (\| \nabla Z^\alpha S_n \| + \| S_n \|_{X^{m-1,0}} + \| h \|_{X^{m-1,1}})
\end{equation}
2) \( C_{S_2}^2 \)
We need to estimate like,
\begin{equation}
\varepsilon \int_S \partial_j Z^\beta (E \nabla S_n) \cdot Z^\alpha S_n dV_i
\end{equation}
where $\beta \leq m - 2$. Then again by integration by parts, we can get the same estimate like $C^2_{S_1}$

\[(8.48) \quad \left| \int_S C^2_{S_2} \cdot Z^\alpha S_n dV_t \right| \lesssim \varepsilon \Lambda \left( \frac{1}{c_0}, \Lambda_\infty \right) (\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-1,0}} + |h|_{X^{m,1}}) \]

3) $C^2_{S_3}$

We give $\nabla \cdot Z^\alpha S_n$ by integration by parts, then easily,

\[(8.49) \quad \left| \int_S C^2_{S_3} \cdot Z^\alpha S_n dV_t \right| \lesssim \varepsilon \|Z^\alpha, E\nabla|S_n\| \|\nabla Z^\alpha S_n\| \lesssim \varepsilon \Lambda \left( \frac{1}{c_0}, \Lambda_\infty \right) (\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-1,0}} + |h|_{X^{m,1}}) \]

Combining this and estimate for $L$

\[(8.50) \quad \left| \int_S C^2_{S} \cdot Z^\alpha S_n dV_t \right| \leq \varepsilon \Lambda \left( \frac{1}{c_0}, \Lambda_\infty \right) (\|\nabla Z^\alpha S_n\| + \|S_n\|_{X^{m-1,0}} + |h|_{X^{m,1}}) \]

We give

\[(8.51) \quad \frac{1}{2} \frac{d}{dt} \int_S |Z^\alpha S_n|^2 dV_t + \frac{\varepsilon}{2} \int_S |\nabla^2 Z^\alpha S_n|^2 dV_t \]

\[\leq \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) \left\{ \|S_n\|_{X^{m-1,0}} + \|s\|_{X^{m-1,1}} + \|v\|_{X^{m,0}} + \varepsilon (\|\nabla S_n\|_{X^{m-1,0}} + |h|_{X^{m-1,0}} + |h|_{X^{m-1,0}}) \right\} \]

Now we integrate for time and sum for all indices of $\alpha$, and then use Young’s inequality (take $\delta$ sufficiently small if needed.) to make dissipation on RHS is absorbed by LHS. Then we can get the following result.

**Proposition 8.4. Energy estimate for $S_n$.**

\[(8.52) \quad \left\| S_n(t) \right\|^2_{X^{m-1,0}} + \varepsilon \int_0^t \|\nabla S_n(s)\|^2_{X^{m-1,0}} ds \]

\[\leq \left\| S_n(0) \right\|^2_{X^{m-1,0}} + \int_0^t \Lambda \left( \frac{1}{c_0}, \Lambda_{m,\infty} \right) \left( \|S_n\|_{X^{m-1,0}} + \|v\|_{X^{m,0}} + |h|_{X^{m,1}} + |h|_{X^{m-1/2}} \right) ds \]

Note that $|h|_{X^{m-1,0}}$ was treated by Dirichlet-Neumann operator already.

Now, we start with the basic $L^2$ energy estimate for $S_n$.

\[(8.53) \quad \frac{1}{2} \frac{d}{dt} \int_S |S_n|^2 dV_t + \varepsilon \int_S |\nabla^2 S_n|^2 dV_t = \int_S F_S \cdot S_n dV_t \]

where the boundary condition is

\[(8.54) \quad (S_n)_{z=0} = 0 \]

9. $L^\infty$ Type Estimate

In the previous section, we estimated $L^2$-type norm of $\partial_z v$. We also should estimate $L^\infty$-type norm of $\partial_z v, \|\nabla v\|_{Y^{k,0}}$. Again, instead of $\partial_z v$, we estimate $S_n$.

**Lemma 9.1.** We have the following estimate for normal part of $\partial_z v$.

\[(9.1) \quad \left\| \partial_z v \cdot n \right\|_{Y^{k,0}} \lesssim \Lambda \left( \frac{1}{c_0}, |h|_{Y^{k,0}} \right) \|v\|_{X^{k+1,0}} \]

**Proof.** From divergence free condition,

\[(9.2) \quad \partial_z v \cdot N = \partial_z \varphi (\partial_1 v_1 + \partial_2 v_2) \]

We take $\|\cdot\|_{Y^{k,0}}$ so get the result. \Box

Similar to the previous section, we use

\[2S\varphi v\hat{n} = (\nabla u) \hat{n} + (\nabla u)^T \hat{n} = (\nabla u) \hat{n} + g^{ij} (\partial_j v \cdot \hat{n}) \partial_i \varphi, \]

and divergence free condition,

\[\partial_N u = \frac{1 + |\nabla \varphi|^2}{\partial_z \varphi} \partial_z v - \partial_1 \varphi \partial_1 v - \partial_2 \varphi \partial_2 v \]
So we obtain,
\begin{equation}
\|\partial_z v\|_{Y^{k,0}} \lesssim \Lambda \left( \frac{1}{\epsilon_0}, |\epsilon| Y^{k,0} \right) \left( \|v\|_{Y^{k+1,0}} + \|S^\epsilon v \tilde{n}\|_{Y^{k,0}} \right)
\end{equation}
and since,
\begin{equation}
S^\epsilon v \tilde{n} = S_n + (\tilde{n} \otimes \hat{n}) (S^\epsilon v \tilde{n})
\end{equation}
by using above lemma for normal part of \(\partial_z v\),
\begin{equation}
\|S^\epsilon v \tilde{n}\| \lesssim \|S_n\| + \Lambda \left( \frac{1}{\epsilon_0}, |\epsilon| Y^{k,0} \right) \|\partial_z v \cdot \tilde{n}\|_{Y^{k,0}} + \|v\|_{Y^{k,0}}
\end{equation}
Proposition 9.2. We have the following.
\begin{equation}
\|\partial_z v\|_{Y^{k,0}} \lesssim \Lambda \left( \frac{1}{\epsilon_0}, |\epsilon| Y^{k,0} \right) \left( \|S_n\|_{Y^{k,0}} + \|v\|_{X^{k+3,0}} + \|S_n\|_{X^{k+2,0}} \right)
\end{equation}
Note that for sufficiently small \(k\) (than \(m\)), then \(\partial_z v\) and \(S_n\) are equivalent in \(L^\infty\)-type norm. Above proposition implies we are suffice to estimate \(\|S_n\|_{Y^{k+3,0}}\), instead of \(\|\partial_z v\|_{Y^{k+3,0}}\). So as we did in previous section, we use equation for \(S_n\) with Dirichlet boundary condition. Main difficulty in this section is commutator between \(Z_3 = \frac{i}{\epsilon} \partial_z\) and \(\Delta^\epsilon\). This commutator was not a problem in basic \(L^2\)-type energy estimate of \(v\) and \(S_n\), because the highest order commutator, which looks like \(\sim \epsilon Z^\alpha \partial_z S_n\), can be absorbed into dissipation term in the energy. But, in \(L^\infty\) estimate, we use the following maximal principle for convection-diffusion equation. That is for equation for \(S_n\),
\begin{equation}
\partial_t^\epsilon S_n + (v \cdot \nabla^\epsilon) S_n - \epsilon \Delta^\epsilon S_n = F_S
\end{equation}
we have the following \(L^\infty\)-type estimate which does not have dissipation in energy term.
\begin{equation}
\|S_n(t)\|_{L^\infty} \leq \|S_n(0)\|_{L^\infty} + \int_0^t \|F_S(s)\|_{L^\infty} ds
\end{equation}
So, if we have the commutator which have 1-more derivative than \(Z^\alpha S_n\), we cannot control them with energy, although it has \(\epsilon\) as its coefficient. Note that, \(L^\infty\) terms cannot be controlled by sobolev conormal space. That means, standard sobolev embedding does not hold for conormal space in general. This is because of behavior of near the boundary. But, away from the boundary \(\frac{i}{\epsilon} \partial_z\) is not zero, and its all order derivative for \(z\) is always uniformly bounded. Now, we divided conormal function into two parts, one is supported near the boundary and another is supported away from boundary. Then 2nd stuff are easy to be controlled by sobolev embedding. For the first stuff, we deform the coordinate so that locally \(\partial_z\) look like \(\partial_z\). Then \(\partial_z\) commute with \(\partial_{zz}\), so it does not generate any harmful (which has 1-more order than \(L^\infty\)-type energy) commutator. This clever idea is introduced in [15] (and also in [1]). We introduce this system briefly and use similar arguments to get the result. First, we start with very simple lemma, which means away from the boundary sobolev conormal is just like standard sobolev.

Lemma 9.3. For any smooth cut-off function \(\tilde{\chi}\) such that \(\tilde{\chi} = 0\) in a vicinity of \(z = 0\), we have for \(m > k + 3/2\):
\begin{equation}
\|\tilde{\chi} f\|_{W^{k,m}} \lesssim \|f\|_{H^{m}}
\end{equation}
Now we decompose \(Z^k S_n\) as
\begin{equation}
\|Z^k S_n\|_{L^\infty} \leq \|\chi Z^k S_n\|_{L^\infty} + \|v\|_{Y^{k+1,0}}
\end{equation}
To estimate \(\|v\|_{Y^{k+1,0}}\), we use the following proposition.
Proposition 9.4. We have the following estimate.

\[(9.9) \quad \|v\|_{Y^{k, 0}} \lesssim A \left( \frac{1}{c_0}, \|\nabla v\|_{Y^{1, 0}} + \|v\|_{X^{k+2, 0}} + |h|_{X^{k+1, 1}} + \|S_n\|_{X^{k+1, 0}} \right)\]

Proof. Using anisotropic sobolev embedding,

\[\|Z^i v\|^2_{L^\infty} \lesssim \|\partial_2 v\|_{X^{k+1, 0}} \|v\|_{X^{k+2, 0}}\]

and using lemma of previous section,

\[\|\partial v\|_{X^{k+1, 0}} \lesssim A \left( \frac{1}{c_0}, \|\nabla v\|_{\frac{k+2}{2}} \right) \left( \|v\|_{X^{k+2, 0}} + |h|_{X^{k+1, 1}} + \|S_n\|_{X^{k+1, 0}} \right)\]

Then using induction for \(\|\nabla v\|_{\frac{k+2}{2}}\), until it become 1. And notice that \(\|v\|_{X^{k+2, 0}}\) is absorbed by estimate of \(\|\partial v\|_{X^{k+1, 0}}\). \(\square\)

See that above proposition means that we are suffice to estimate \(Z^i S_n\) only near the boundary, so now we introduce modified coordinate which was introduced in [15] and [1]. Let, define transformation \(\Psi\),

\[(9.10) \quad \Psi(t, \cdot) : S = \mathbb{R}^2 \times (-\infty, 0) \to \Omega_t \]

\[x = (y, z) \mapsto \left( \begin{array}{c} y \\ h(t, y) \end{array} \right) + z\hat{n}^b(t, y)\]

where \(\hat{n}^b\) is unit normal at the boundary, \((-\nabla h, 1)/|N|\). To show that this is diffeomorphism near the boundary, we check

\[D\Psi(t, \cdot) = \left( \begin{array}{ccc} 1 & 0 & -\partial_1 h \\ 0 & 1 & -\partial_2 h \\ \partial_1 h & \partial_2 h & 1 \end{array} \right) + \left( \begin{array}{ccc} -z\partial_{11} h & -z\partial_{12} h & 0 \\ -z\partial_{21} h & -z\partial_{22} h & 0 \\ 0 & 0 & 1 \end{array} \right)\]

This is diffeomorphism near the boundary since norm of second matrix is controlled by \(|h|_{2, \infty}\). So, we restrict \(\Psi(t, \cdot)\) on \(\mathbb{R}^2 \times (-\delta, 0)\) so that it is diffeomorphism. (\(\delta\) is depend on \(c_0\). Of course, think that above support separation was done by \(\chi(z) = \kappa(\frac{z}{\lambda_0})\). Now we write laplacian \(\nabla^\phi\) with respect to Riemannian metric of above parametrization. Riemannian metric becomes,

\[(9.11) \quad g(y, z) = \begin{pmatrix} \hat{g}(y, z) & 0 \\ 0 & 1 \end{pmatrix}\]

where \(\hat{g}\) is \(2 \times 2\) block matrix. And with this metric, laplacian becomes,

\[(9.12) \quad \triangle g = \partial_{zz} f + \frac{1}{2} \partial_{z}(\ln|g|)\partial_z f + \triangle \hat{g} f\]

where

\[(9.13) \quad \triangle \hat{g} f = \frac{1}{|\hat{g}|^2} \sum_{1 \leq i, j \leq 2} \partial_{y^i}(\hat{g}^{ij})|\hat{g}|^2 \partial_{y^j} f\]

where \(\hat{g}^{ij}\) is inverse matrix element of \(\hat{g}\). We now solve problem in domain of \(\Psi\). We restrict \(S^\psi v\) near the boundary and parametrize them via \(\Psi\). Let

\[(9.14) \quad S^\chi = \chi(z) S^\psi v\]

where \(\chi(z) = \kappa(\frac{z}{\lambda_0}) \in [0, 1]\) where \(\kappa\) is smooth and compactly supported near the boundary, taking value 1 there. Equation for \(S^\chi\) is

\[(9.15) \quad \partial^\chi t S^\chi + (v \cdot \nabla^\phi) S^\chi - \varepsilon \triangle^\phi S^\chi = F_{S^\chi}\]

where

\[F_{S^\chi} = F^\chi + F_v\]

\[F^\chi = (V_z \partial z) S^\psi v - \varepsilon \nabla^\phi \chi \cdot \nabla^\phi S^\psi v - \varepsilon \triangle^\phi \chi S^\psi v\]

\[F_v = -\chi(D^\phi)^2 q - \frac{\lambda}{2}((\nabla^\phi v)^2 + (\nabla^\phi v)^t)^2\]

Note that \(F^\chi\) is supported away from the boundary. Rewrite this function on our new frame by taking \(\Phi^{-1} \circ \Psi\). We define,

\[(9.16) \quad S^\Psi(t, y, z) = S^\chi(t, \Phi^{-1} \circ \Psi)\]
and $S^\Psi$ solves
\begin{equation}
\partial_t S^\Psi + w \cdot \nabla S^\Psi - \varepsilon(\partial_{zz} S^\Psi + \frac{1}{2}\partial_z (\ln |g|) \partial_z S^\Psi + \triangle g S^\Psi) = F_{S^\Psi}(t, \Phi^{-1} \circ \Psi)
\end{equation}
where
\[ w = \tilde{\chi}(D\Psi)^{-1}(v(t, \Phi^{-1} \circ \Psi) - \partial_t \Psi) \]
where $\tilde{\chi}$ is slightly larger support so that $\tilde{\chi} S^\Psi = S^\Psi$ and as like $S^X$, $S^\Psi$ is also only supported near the boundary. In this frame $S_n$ correspond to $S^\Psi$, which is defined as following,
\begin{equation}
S_n(t, y, z) = \Pi^b(t, y) S^\Psi_n(t, y) = \Pi^b(t, y) S^X(t, \Phi^{-1} \circ \Psi) \hat{n}^b(t, y)
\end{equation}
where $\Pi^b = I - \hat{n}^b \otimes \hat{n}^b$, (tangential operator at the boundary, so they are independent to $z$.) Then equation for $S_n^\Psi$ becomes,
\begin{equation}
\partial_t S_n^\Psi + w \cdot \nabla S_n^\Psi - \varepsilon(\partial_{zz} S_n^\Psi + \frac{1}{2}\partial_z (\ln |g|) \partial_z S_n^\Psi) = F_n^\Psi
\end{equation}
where
\[ F_n^\Psi = \Pi^b F_{S^\Psi} \hat{n}^b + F_n^{\Psi,1} + F_n^{\Psi,2} \]
where
\[ F_n^{\Psi,1} = ((\partial_t + w_y \cdot \nabla y)\Pi^b) S^\Psi \hat{n}^b + \Pi^b S^\Psi (\partial_t + w_y \cdot \nabla y) \hat{n}^b \]
\[ F_n^{\Psi,2} = -\varepsilon \Pi^b (\triangle g S^\Psi) \hat{n}^b \]
with zero-boundary condition at $z = 0$. Note that $S_n = S_n^\Psi$ on the boundary. We will estimate $S_n^\Psi$ instead of $S_n$, to validate this, we should show that equivalence of these two terms. Firstly, by definition of $S_n^\Psi$,
\begin{equation}
||S_n^\Psi||_{Y^{k,0}} \leq \Lambda(||h||_{Y^{k+1,0}})||\Pi^b S^\Psi v \hat{n}^b||_{Y^{k,0}}
\end{equation}
and since $|\Pi - \Pi^b| + |\hat{n} - \hat{n}^b| = O(z)$ near the boundary $z = 0$,
\begin{equation}
||S_n^\Psi||_{Y^{k,0}} \leq \Lambda(\frac{1}{c_0}||S_n||_{Y^{k,0}} + ||v||_{Y^{k+1,0}})
\end{equation}
Now, we apply anisotropic sobolev embedding to the last term,
\[ ||S_n^\Psi||_{Y^{k,0}} \leq \Lambda(\frac{1}{c_0}||S_n||_{Y^{k,0}} + ||\partial_z v||_{X^{k+2,0}} + ||v||_{X^{k+3,0}}) \]
For $||\partial_z v||_{X^{k+2,0}},$ we use Lemma 8.2 inductively, (to reduce the order of $||\nabla v||_{Y^{\frac{k}{2},0}}$, to get
\begin{equation}
||S_n^\Psi||_{Y^{k,0}} \leq \Lambda(\frac{1}{c_0}||v||_{X^{k+3,0}} + ||S_n||_{X^{k+2,0}} + ||h||_{X^{k+2,0}} + ||S_n||_{Y^{k,0}})
\end{equation}
Since we choose sufficiently smaller $k$ than $m$, this estimate is okay. For opposite direction, we can do similarly to get
\begin{equation}
||S_n||_{Y^{k,0}} \leq \Lambda(\frac{1}{c_0}||v||_{X^{k+3,0}} + ||S_n||_{X^{k+2,0}} + ||h||_{X^{k+2,0}} + ||S_n^\Psi||_{Y^{k,0}})
\end{equation}
So, we finish equivalence argument.
Now we should apply $Z^k$ to the system (9.17). As in [1], applying tangential derivative (Z1, Z2) is not that harmful, but commutator between Z3 and Laplacian is still a problem. Critical observation in [1] is the following Lemma. (Lemma 9.6 in [1]).

**Lemma 9.5. (Lemma 9.6 in [1])** Consider $\rho$ a smooth solution of
\begin{equation}
\partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{H}, \quad z < 0, \quad \rho(t, y, 0) = 0, \quad \rho(t = 0) = \rho_0
\end{equation}
for some smooth vector field $w$ such that $w_3$ vanishes on the boundary. Assume that $\rho$ and $\mathcal{H}$ are compactly supported in $z$. Then we have the estimate:
\[ ||Z_i \rho(t)||_\infty \lesssim ||Z_i \rho_0||_\infty + ||\rho_0||_\infty + \int_0^t (||w||_{L^2} + ||\partial_{zz} w_3||_{L^\infty})(||\rho||_{1,\infty} + ||\rho||_4 + ||\mathcal{H}||_{1,\infty}), \quad i = 1, 2, 3 \]
Lemma 9.7. For evolution operator 
\[ j \]
lower 
\[ \parallel \]
\[ (9.33) \]
\[ z \]
need to consider 
\[ Note \ that \ in \ fact \ there \ exist \ terms \ like \ (9.32) \]
\[ For \ the \ full \ non-homogeneous \ system, \ by \ Duhamel's \ formula, \ (9.31) \]
\[ \partial \]
\[ Now, \ suppose \ that \ \rho \]
\[ \parallel \]
\[ \parallel \]
\[ \partial_z \ln |g| \parallel_z \parallel z \]
is compactly supported in \( \mathbb{R}^3 \), \( Z_3 \Psi \) and \( \rho \) are equivalent, i.e
\[ \rho \parallel \Psi \parallel_{Y^{k,0}} \leq \Lambda \parallel h \parallel_{Y^{k+1,0}} \parallel \Psi \parallel_{Y^{k,0}} \]
\[ \parallel \alpha \parallel_{Y^{k,0}} \leq \Lambda \parallel h \parallel_{Y^{k+1,0}} \parallel \alpha \parallel_{Y^{k,0}} \]
Hence, instead of \( S_{\Psi} \), we estimate \( \rho \). Also note that equation of \( \rho \) is applicable above lemma. Now we extend above lemma to high order.

Lemma 9.6. (High order version) Consider \( \rho \) a smooth solution of
\[ \partial_t \rho + w \cdot \nabla \rho - \varepsilon \partial_{zz} \rho = |g|^{\frac{1}{2}} (F_{\Psi} \parallel F_{\Psi} + F_{\Psi}) \parallel \mathcal{H} \]
where
\[ F_{\Psi} = \frac{\rho}{|g|^{\frac{1}{2}}} (\partial_t + w \cdot \nabla - \varepsilon \partial_{zz}) |g|^{\frac{1}{2}} \]
which shows that \( \varepsilon \partial_z \ln |g| \parallel_z \parallel z \]
is removed. And trivially, \( Z_3 \Psi \) and \( \rho \) are equivalent, i.e
\[ \parallel \rho \parallel_{Y^{k,0}} \leq \Lambda \parallel h \parallel_{Y^{k+1,0}} \parallel \Psi \parallel_{Y^{k,0}}, \quad \parallel \Psi \parallel_{Y^{k,0}} \leq \Lambda \parallel h \parallel_{Y^{k+1,0}} \parallel \Psi \parallel_{Y^{k,0}} \]
Hence, instead of \( S_{\Psi} \), we estimate \( \rho \). Also note that equation of \( \rho \) is applicable above lemma. Now we extend above lemma to high order.

Lemma 9.6. (High order version) Consider \( \rho \) a smooth solution of
\[ \partial_t \rho + w \cdot \nabla \rho = \varepsilon \partial_{zz} \rho + \mathcal{H}, \quad \rho(t, y, 0) = 0, \quad \rho(t = 0) = \rho_0 \]
for some smooth vector field \( w \) such that \( w_3 \) vanishes on the boundary. Assume that \( \rho \) and \( \mathcal{H} \) are compactly supported in \( z \). Then we have the estimate:
\[ \parallel Z^k \rho \parallel_{L^\infty} \lesssim \parallel \rho_0 \parallel_{Y^{k,0}} + \int_0^t \parallel (\partial_z \partial_y w_3) + \parallel \partial_{zz} w_3 \parallel_{Y^{k,0}} \parallel \rho \parallel_{X^{k+3,0}} + \parallel \mathcal{H} \parallel_{Y^{k,0}} d\tau \]
Proof. Applying conormal derivatives to equation generate bad commutator which come from between \( Z_3 \) and Laplacian. So, we rewrite equation as
\[ \partial_t \rho + z \partial_z w_3(t, y, 0) \partial_z \rho + w_y(t, y, 0) \cdot \nabla_y \rho - \varepsilon \partial_{zz} \rho = \mathcal{H} - \mathcal{R} \parallel G \]
where \( w_3 = 0 \) on the boundary
\[ \mathcal{R} = (w_y(t, y, z) - w_y(t, y, 0)) \cdot \nabla_y \rho + (w_3(t, y, z) - z \partial_z w_3(t, y, 0)) \partial_z \rho \]
We use evolution operator \( S(t, \tau) \) for homogeneous solution of above system. Let,
\[ \rho(t, y, z) = S(t, \tau) f_0(y, z), \quad f(\tau, y, z) = f_0(y, z) \quad (initial \ condition) \]
solves,
\[ \partial_t \rho + z \partial_z w_3(t, y, 0) \partial_z \rho + w_y(t, y, 0) \cdot \nabla_y \rho - \varepsilon \partial_{zz} \rho = 0, \quad z > 0, \quad t > \tau, \quad \rho(t, y, 0) = 0 \]
For the full non-homogeneous system, by Duhamel's formula,
\[ \rho(t) = S(t, 0) \rho_0 + \int_0^t S(t, \tau) G(\tau) d\tau \]
Now, suppose that \( \rho \) is compactly supported in \( z \), near the boundary) and \( z < 0 \), then
\[ \parallel Z^k_3 \rho \parallel_{L^\infty} \lesssim \sum_{j=1}^{k} \sum_{i=1}^{j} \parallel z^j \partial_z^i \rho \parallel_{L^\infty} \]
Note that in fact there exist terms like \( \parallel z^j \partial_z^i \rho \parallel_{L^\infty} \). But since \( \rho \) has compact support near the \( z = 0 \), we only need to consider \( z \) near the boundary, so higher \( z \) terms are not harmful. At result, we are suffice to estimate lower \( j \) terms. To estimate each terms on RHS, we should control each \( \parallel z^j \partial_z^i \rho \parallel_{L^\infty} \).

Lemma 9.7. For evolution operator \( S \) as above, we have following estimate.
\[ \parallel z^j \partial_z S(t, \tau) \rho_0 \parallel_{L^\infty} \lesssim \parallel \rho_0 \parallel_{L^\infty} + \sum_{i_1 + i_2 = i} \parallel z^{j-i_1} \partial_z^{i_2} \rho_0 \parallel_{L^\infty} \]
Proof. Basically we follow the method of Lemma 9.6 in [1]. Let \( \rho(t, y, z) = S(t, \tau)\rho_0(y, z) \) solves homogeneous system of (9.31). We extend this variables to whole space by

\[
\partial_t \tilde{\rho} + z \partial_z w_3(t, y, 0) \partial_z \tilde{\rho} + w_y(t, y, 0) \cdot \nabla_y \tilde{\rho} - \varepsilon \partial_{zz} \tilde{\rho} = 0, \quad z \in \mathbb{R}
\]

with initial condition \( \tilde{\rho}(\tau, y, z) = \rho_0(y, z) \).

By introducing \( \mathcal{E} \), which solves,

\[
\partial_t \mathcal{E} = w_y(t, \mathcal{E}, 0), \quad \mathcal{E}(\tau, \tau, y) = y
\]

and define,

\[
g(t, y, z) = \rho(t, \mathcal{E}(t, y, z), z)
\]

Then \( g \) solves,

\[
\partial_t g + z \gamma(t, y) \partial_z g - \varepsilon \partial_{zz} g = 0, \quad z \in \mathbb{R}, \quad g(\tau, y, z) = \tilde{\rho}_0(y, z)
\]

where

\[
\gamma(t, y) = \partial_z w_3(t, \mathcal{E}(t, y, z), 0)
\]

By using Fourier transform, we get explicit form of the solution,

\[
g(t, y, z) = \int_\mathbb{R} k(t, \tau, y, z - z') \tilde{\rho}_0(y, e^{-\Gamma(t)} z') dz'
\]

where

\[
k(t, \tau, y, z - z') = \frac{1}{\sqrt{4\pi \varepsilon}} \int_\tau^{t+\varepsilon} e^{2z(\Gamma(s) - \Gamma(t))} ds \cdot \int_\mathbb{R} k(t, \tau, y, z) dz = 1
\]

\[
\Gamma(t) = \int_\tau^t \gamma(s, y) ds
\]

We note that,

\[
z^j \partial_z g = \int_\mathbb{R} \left( z^j \partial_z k(t, \tau, y, z - z') \right) \tilde{\rho}_0(y, e^{-\Gamma(t)} z') dz'
\]

\[
= \int_\mathbb{R} \left( (z^j - z^{j'}) \partial_z k(t, \tau, y, z - z') + (-1)^i z^{i'} \partial_z k(t, \tau, y, z - z') \right) \tilde{\rho}_0(y, e^{-\Gamma(t)} z') dz'
\]

Now, since \( k \) is Gaussian,

\[
\int_\mathbb{R} \left| (z^j - z^{j'}) \partial_z k \right| dz' \lesssim 1
\]

So, using integration by parts on the 2nd term, we can deduce

\[
\| z^j \partial_z \tilde{\rho} \|_{L^\infty} \lesssim \| \tilde{\rho}_0 \|_{L^\infty} + \left\| \int_\mathbb{R} k(t, \tau, y, z - z') \left\{ j z_{j-1} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') + z^{j'} \partial_z \tilde{\rho}_0(y, e^{-\Gamma(t)} z') e^{-\Gamma(t)} \right\} dz' \right\|_{L^\infty}
\]

\[
\lesssim \cdots \lesssim \| \tilde{\rho}_0 \|_{L^\infty} + \left\| \int_\mathbb{R} k(t, \tau, y, z - z') \left\{ \sum_{i_1 + i_2 = i} (z')^{i_1} \partial_z^{i_1} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') e^{-iz\Gamma(t)} \right\} dz' \right\|_{L^\infty}
\]

\[
\lesssim \| \tilde{\rho}_0 \|_{L^\infty} + \sum_{i_1 + i_2 = i} \left\| \int_\mathbb{R} k(t, \tau, y, z - z') \left\{ (z')^{i_1} \partial_z^{i_1} \tilde{\rho}_0(y, e^{-\Gamma(t)} z') \right\} dz' \right\|_{L^\infty}
\]

By relation of \( \rho \) and \( g \), we get

\[(9.42) \quad \| z^j \partial_z \tilde{\rho} \|_{L^\infty} \lesssim \| z^j \partial_z \tilde{\rho} \|_{L^\infty} \lesssim \| \tilde{\rho}_0 \|_{L^\infty} + \sum_{i_1 + i_2 = i} \| z^{j-i} \partial_z^{i_1} \tilde{\rho}_0 \|_{L^\infty}
\]

\[\square\]
Now we apply $Z^k_3$ to Duhamel's formula to get

\begin{equation}
Z^k_3 \rho(t) = Z^k_3(S(t, \tau) \rho_0) + \int_0^t Z^k_3(S(t, \tau)G(\tau))d\tau
\end{equation}

Using above Lemma 9.7 twice on the RHS,

\begin{equation}
\|Z^k_3 \rho(t)\|_{L^\infty} \lesssim \sum_{i=1}^k \sum_{j=1}^i \left\{ \|\rho_0\|_{L^\infty} + \sum_{j_1+j_2=j} \|z^{j_1-j_2} \partial_z \rho_0\|_{L^\infty} \right\}

+ \sum_{i=1}^k \sum_{j=1}^i \int_0^t \left\{ \|G\|_{L^\infty} + \sum_{j_1+j_2=j} \|z^{j_1-j_2} G\|_{L^\infty} \right\} d\tau
\end{equation}

Using the fact that $\rho$ (also $\rho_0$) and $G$ are compactly supported in $z$, we have

\[ \|Z^k_3 \rho(t)\|_{L^\infty} \lesssim \|\rho_0\|_{W^{k,\infty}} + \int_0^t \|G\|_{W^{k,\infty}} d\tau \]

We also note that other tangential derivatives cases also holds. (This is easier than $Z_3$ case.) Hence

\begin{equation}
\|Z^k_3 \rho(t)\|_{L^\infty} \lesssim \|\rho_0\|_{Y^{k,0}} + \int_0^t \|G\|_{Y^{k,0}} d\tau
\end{equation}

Let’s estimate $\|R\|_{Y^{k,0}}$. Since $\rho$ is compactly supported in $z$ (near the boundary), using Taylor’s series and inserting function $\zeta(z) \equiv \frac{1}{1 - z}$ (inserting this function is very useful, because existence of $\zeta(z)$ enables us to control using conormal derivatives of $\rho$), we have

\begin{equation}
\|R\|_{Y^{k,0}} \leq \|(w_y(t, y, z) - w_y(t, y, 0)) \cdot \nabla_y \rho\|_{Y^{k,0}} + \||(w_3(t, y, z) - z \partial_z w_3(t, y, 0)) \partial_z \rho\|_{Y^{k,0}}

\leq \|\partial_z w_y\|_{Y^{k,0}} \|\zeta(z)\|_{Y^{k+1,0}} + \|\partial_z w_3\|_{Y^{k,0}} \|\zeta^2(z)\|_{Y^{k,0}} \|\partial_z \rho\|_{Y^{k,0}}

\leq (\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_z w_3\|_{Y^{k,0}}) (\|\zeta(z)\|_{Y^{k+1,0}} + \|\zeta^2(z)\|_{Y^{k,0}}) \|\partial_z \rho\|_{Y^{k,0}}
\end{equation}

Using anisotropic sobolev embedding for conormal derivatives,

\begin{equation}
\|\zeta(z)\|_{Y^{k+1,0}} \lesssim \|\zeta(z)\|_{X^{k+1,2}} + \|\partial_z (\zeta(z)\rho)\|_{X^{k+1,1}} \lesssim \|\partial_z (\zeta(z)Z^{k+2}\rho)\|_{L^2}

\lesssim \|\rho\|_{X^{k+3,0}} + \|\zeta'(z)\|_{X^{k+2,0}} + \|\rho\|_{X^{k+3,0}} \lesssim \|\rho\|_{X^{k+3,0}}
\end{equation}

and $\zeta(z)$ is nice bounded function for all order of derivatives, so at result,

\begin{equation}
\|R\|_{Y^{k,0}} \lesssim (\|\partial_z w_y\|_{Y^{k,0}} + \|\partial_z w_3\|_{Y^{k,0}}) \|\rho\|_{X^{k+3,0}}
\end{equation}

Combining with (9.45), we finish the proof.

Now, we are ready to get energy estimate for $\|S_n\|_{Y^{k,0}}$.

\textbf{Proposition 9.8.} Let’s define non-dissipation type energy $E_m$ as

\begin{equation}
E_m \doteq \Lambda \left( \frac{1}{c^0_\square}, \|v\|_{X^{m,0}} + |h|_{X^{m+1,0}} + \|\partial_z v\|_{X^{m-1,0}} + \|\partial_z v\|_{Y^{m,0}} \right)
\end{equation}

(Note that this is equivalent with

\[ Q_m \doteq (\|v\|_{X^{m,0}} + |h|_{X^{m+1,0}} + \|S_n\|_{X^{m-1,0}} + \|S_n\|_{Y^{m,0}}) \]

We have the following estimate for $\|S_n\|_{Y^{m,0}}$.

\begin{equation}
\|S_n(t)\|_{Y^{m,0}} \leq \|S_n(0)\|_{Y^{m,0}} + \varepsilon \int_0^t E_m d\tau \leq \|S_n\|_{X^{m-1,0}}
\end{equation}
Proof. We already transformed $S_n$ equation into equivalent-$\rho$ equation system (9.26). From the result of Lemma 9.6, we should estimate the following four terms. Here, $k = \frac{\Psi}{\rho}$, and $m$ is sufficiently large.

$$
\|\rho\|_{Y^{k+3,0}}, \quad \|\partial_x w_y\|_{Y^{k,0}}, \quad \|\partial_x w_z\|_{Y^{k,0}}, \quad \|\mathcal{H}\|_{Y^{k,0}}
$$

1) $\|\rho\|_{Y^{k+3,0}}$ and $\|\partial_x w_y\|_{Y^{k,0}}$ are trivially controlled by $\mathcal{E}_m$ by definition of $\rho$.

3) $\|\partial_x w_3\|_{Y^{k,0}}$ : By definition of $w$,

$$
w = \chi(D\Psi)^{-1}(v(t, \Phi^{-1}\circ \Psi) - \partial_t \Psi)
$$

For first term,

$$
\|\partial_x (\chi(D\Psi)^{-1}\partial_t \Psi)\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, \|h\|_{Y^{k,0}} + |\partial_t h|_{Y^{k,0}}\right) \lesssim \mathcal{E}_m
$$

For second, by using commutator,

$$
\|\partial_x (\chi(D\Psi)^{-1}v_3(t, \Phi^{-1}\circ \Psi))\|_{Y^{k,0}} \lesssim \|\chi \partial_x z (v(t, \Phi^{-1}\circ \Psi) \cdot \hat{n}^b_t)\|_{Y^{k,0}} + \mathcal{E}_m
$$

where we used

$$
(\chi(D\Psi(t, y, 0))^{-1}f)_3 = f \cdot \hat{n}^b_t
$$

Main difficulty is two normal derivatives. Meanwhile, by definition, $v(t, \Phi^{-1}\circ \Psi) = u(t, \Psi) = u^\Psi$ and $u$ is divergence free is zero. In the local coordinate, divergence free condition gives,

$$
\partial_x u^\Psi \cdot \hat{n}^b_t = -\frac{1}{2}\partial_z (\ln |g|)u^\Psi \cdot \hat{n}^b_t - \nabla g \cdot u^\Psi_t
$$

which means one normal derivative is replaced by tangential derivative. So,

$$
\|\chi \partial_x z (v(t, \Phi^{-1}\circ \Psi) \cdot \hat{n}^b_t)\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, \|\partial_x u^\Psi\|_{Y^{k,0}} + |\partial_t h|_{Y^{k+3,0}}\right) \lesssim \mathcal{E}_m
$$

In conclusion, $\|\partial_x w_3\|_{Y^{k,0}} \lesssim \mathcal{E}_m$.

4) $\|\mathcal{H}\|_{Y^{k,0}}$ : We have

$$
\|F_n^\Psi\|_{Y^{k,0}} \leq \|F_n^\Psi,1\|_{Y^{k,0}} + \|F_n^\Psi,2\|_{Y^{k,0}} + \|\Pi^b F_n S^\Psi \hat{n}^b_t\|_{Y^{k,0}}
$$

For first term,

$$
\|F_n^\Psi,1\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}, \|\partial_x h|_{Y^{k+1,0}} + |h|_{Y^{k+2,0}} + \|w|_{Y^{k,0}}\right) \|\partial_x v\|_{Y^{k,0}} \lesssim \mathcal{E}_m
$$

Second,

$$
\|F_n^\Psi,2\|_{Y^{k,0}} \lesssim \varepsilon \mathcal{E}_m \|S_n\|_{Y^{k+2,0}} \lesssim \mathcal{E}_m (\varepsilon\|S_n\|_{X^{k+2,0}} + \varepsilon \|\partial_x S_n\|_{X^{k+1,0}})
$$

Note that RHS can be bounded by dissipation type energy.

Third,

$$
\|\Pi^b F_n S^\Psi \hat{n}^b_t\|_{Y^{k,0}} \lesssim \mathcal{E}_m \Lambda\left(\frac{1}{c_0}, 1 + \varepsilon\|S_n\|_{Y^{k+2,0}} + \|\Pi^b ((D^c)^2 q) \hat{n}^b_t\|_{Y^{k,0}}\right)
$$

$\varepsilon\|S_n\|_{Y^{k+2,0}}$ was treated as second term, and $\|\Pi^b ((D^c)^2 q) \hat{n}^b_t\|_{Y^{k,0}}$ can be estimated since $\Pi^b \partial_z \sim Z_3$.

$$
\|\Pi^b ((D^c)^2 q) \hat{n}^b_t\|_{Y^{k,0}} \lesssim \Lambda\left(\frac{1}{c_0}\right) \|\nabla q\|_{Y^{k+1,0}}
$$

$$
\lesssim \mathcal{E}_m (1 + \varepsilon\|S_n\|_{X^{k+4,0}})
$$

where last term can be treated similarly as above. (by anisotropic sobolev embedding) We did all estimate to apply Lemma 9.6 and we get finally

$$
\|S_n(t)\|_{Y^{\Psi,0}}^2 \leq \|S_n(0)\|_{Y^{\Psi,0}}^2 + \varepsilon \int_0^t \mathcal{E}_m \|\nabla S_n\|_{X^{m-1,0}}
$$

\[\square\]
10. Uniform Regularity and Local Existence

To get local existence of free-boundary Navier-Stokes equations with viscosity $\varepsilon$, we use existence theory by A. Tani [7], and combine our propositions to get uniform regularity. This procedure will be very similar to that of [1]. First we fix $m \geq 6$ and viscosity $\varepsilon$, and pick an initial data $(v_0^\varepsilon, h_0^\varepsilon)$ so that

$$I_m(0) = \sum_{k=0}^{m} \| \varepsilon^k v_0^\varepsilon \|_{H^{m+k}} + \| h_0^\varepsilon \|_{H^{\frac{m}{2}+1}} < \infty$$

where, $v_0^\varepsilon = \varepsilon^0(0, y, z)$ and same for $h$. First we regularize $v_0^\varepsilon$ by parameter $\delta$, so that $v_0^{\varepsilon, \delta} \in H^{l+1}(S)$, where $l \in (\frac{2}{3}, 1)$. Then by [7], for this initial condition, there is a time interval $T^{\varepsilon, \delta}$ such that on $[0, T^{\varepsilon, \delta}]$, we have a unique solution $v \in W^{l+2, \frac{1}{3}+1}_2([0, T^{\varepsilon, \delta}] \times S) = L^2([0, T^{\varepsilon, \delta}], H^{l+2}(S)) \cap H^{\frac{1}{2}+1}(S)$. Then by parabolic regularity, for $T \in [0, T^{\varepsilon, \delta}]$,

$$\Psi_m(T) = \sup_{[0, T]} \left( \| v \|_{X_{m, 0}}^2 + \| h \|_{X_{m, 1}}^2 + \| \partial_x v \|_{X_{m-1, 0}}^2 + \| \partial_x v \|_{Y_{1, 0}}^2 \right) + \varepsilon \int_0^T \left( \| \nabla v \|_{X_{m, 0}}^2 + \| \nabla \partial_x v \|_{X_{m-1, 0}}^2 \right) < \infty$$

and on the same interval, (by taking $T^{\varepsilon, \delta}$ smaller, if needed.)

$$\partial_x \varphi(T) \geq c_0, \quad |h(T)|_{2, \infty} \leq \frac{1}{c_0}$$

Now let’s suppose $\Psi_m(T_0) < \infty$ for some $T_0$, then using Stoke’s regularity on $[T_0^\varepsilon, T_0]$, we know that $v(T_0) \in H^{l+1}(S)$, so by considering this as initial condition again, we know that it can be extended to some $T_1 > T_0$. We have to show that this extension is uniform in $\varepsilon$ and $\delta$ using our propositions.

Instead of $\Psi_m(t)$, we use $\hat{\Psi}_m(t)$, where

$$\hat{\Psi}_m(T) = \sup_{[0, T]} \left( \| v \|_{X_{m, 0}}^2 + \| h \|_{X_{m, 1}}^2 + \| S_n \|_{X_{m-1, 0}}^2 + \| S_n \|_{Y_{1, 0}}^2 \right) + \varepsilon \int_0^T \left( \| S_n \|_{X_{m, 0}}^2 + \| \nabla S_n \|_{X_{m-1, 0}}^2 \right)$$

(In fact, all our propositions were written in terms of $\hat{\Psi}$.) These two $\Psi_m$ and $\hat{\Psi}_m$ are equivalent as explained in Lemma 8.2. (The opposite direction is trivial.)

To derive uniform time interval, we choose $R$ and $c_0$ so that, $\frac{1}{c_0} << R$, and define,

$$T^{\varepsilon, \delta}_* = \sup \left\{ T \in [0, 1] \ s.t \ \hat{\Psi}_m(t) \leq R, \ |h(t)|_{2, \infty} \leq \frac{1}{c_0}, \ \partial_x \varphi(t) \geq c_0, \ \forall t \in [0, T] \right\}$$

Suppose $\hat{\Psi}_m(T) \leq R$ then, by proposition 5.4, 6.2, 7.2, 8.4, 9.8, and we denote $\Lambda \left( \frac{1}{c_0}, \cdot \right) = \Lambda_0(\cdot)$

$$\Psi_m(T) \leq \Lambda_0(I_m(0)) + \Lambda_0(R)T$$

We note that

$$\| v^\varepsilon(0) \|_{X_{m, 0}} + \| h^\varepsilon(0) \|_{X_{m, 1}} + \| \partial_x v^\varepsilon(0) \|_{X_{m-1, 0}} + \| \partial_x v^\varepsilon(0) \|_{Y_{1, 0}}$$

should be controlled by the size of initial data. So we use the relation $\partial_t h^\varepsilon \sim \partial_x^{3/2} h^\varepsilon$ and the equation estimated at time zero,$$

\partial_t Z^\alpha u(0) = \varepsilon Z^\alpha \cdot \nabla u_0 - Z^\alpha (u_0 \cdot \nabla) u_0 - Z^\alpha \cdot \nabla p_0

Taking several time derivatives and estimating in $H^s$ spaces will gives the control by $I_m(0)$. We note that since initial data have standard sobolev regularity, normal derivative term and $L^\infty$ terms are also bounded by definition and embedding respectively. We have

$$\| v^\varepsilon(0) \|_{X_{m, 0}} + \| h^\varepsilon(0) \|_{X_{m, 1}} + \| \partial_x v^\varepsilon(0) \|_{X_{m-1, 0}} + \| \partial_x v^\varepsilon(0) \|_{Y_{1, 0}} \lesssim \Lambda_0(I_m(0))$$

We note that in proposition 9.8, we can see that $\| S_n \|_{Y_{1, 0}}$ can be controlled by energy. Of course, all of these are valid during,

$$|h(t)|_{2, \infty} \leq |h(0)|_{2, \infty} + \Lambda_0(R)T, \quad \forall t \in [0, T]$$

and since we’ve chosen $A$ in diffeomorphism to be 1, at initial time,

$$\partial_x \varphi(t) \geq 1 - \int_0^t \| \partial_t \nabla \eta \|_{L^\infty} \geq 1 - \Lambda_0(R)T, \quad \forall t \in [0, T]$$
From above three inequality, we see that RHS is independent to \(\varepsilon\) and \(\delta\), so are possible to choose \(R = \Lambda (|h_0|_{L^2}, I_m(0))\) which satisfies that there exist \(T_*\) (independent to \(\varepsilon\), \(\delta\)) such that \(\forall t \in [0, T_*]\),

\[
\hat{\Psi}_m(t) \leq \frac{R}{2}, \quad |h(t)|_{L^\infty} \leq \frac{1}{2c_0}, \quad \partial_z \varphi(t) \geq c_0 + \frac{1 - c_0}{2} > c_0
\]

(10.10)

This implies \(T_* < T^{\varepsilon, \delta}\), hence \(T_*\) implies there exist uniform time, independent to \(\varepsilon, \delta\). Since \(\hat{\Psi}_m(T_*)\) is uniformly bounded in \(\delta\), we can pass the limit, \(\delta \to 0\), by strong compactness argument. Before finishing local existence section, we note about compatibility condition. Since our solution space include \(partial^j_i, j \leq m\), our initial condition should have information about \(\partial^j_i \varphi^\varepsilon(0)\) and \(\partial^j_i h^\varepsilon(0)\). Theses satisfy compatibility condition for Stress-continuity condition, so \((S^\varepsilon \partial^j_i \varphi^\varepsilon(0)) \hat{n}\) must not have tangential part.

(10.11)

\[
\Pi (S^\varepsilon \partial^j_i \varphi^\varepsilon(0)) \hat{n} = 0
\]

11. Uniqueness

11.1. Uniqueness for Navier-Stokes. We prove uniqueness of Theorem 1.2. As usual, we consider two solution sets \((v^\varepsilon_1, \varphi^\varepsilon_1, q^\varepsilon_1), (v^\varepsilon_2, \varphi^\varepsilon_2, q^\varepsilon_2)\) with same initial condition and proper compatibility conditions. Then on the interval \([0, T^\varepsilon]\), we have uniform bounds of energy,

\[
\hat{\Psi}_m(T^\varepsilon) \leq R, \quad i = 1, 2
\]

Let,

\[
\bar{v}^\varepsilon = v^\varepsilon_1 - v^\varepsilon_2, \quad \bar{\varphi}^\varepsilon = \varepsilon, \quad \bar{\eta}_i = h^\varepsilon_i - h^\varepsilon_2, \quad \bar{q}^\varepsilon = q^\varepsilon_1 - q^\varepsilon_2,
\]

We will make system of equations for \((\bar{v}^\varepsilon, \bar{\varphi}^\varepsilon, \bar{q}^\varepsilon)\) and do energy estimate. By divergence free condition, \(\begin{bmatrix} \varphi^\varepsilon_i \cdot v^\varepsilon_i = 0 \end{bmatrix}\)

\[
(\partial_t + v^\varepsilon_i \cdot \nabla_y + V^\varepsilon_z \partial_z) v^\varepsilon_1 + \nabla \delta^\varepsilon \varphi^\varepsilon_1 - \varepsilon \Delta \delta^\varepsilon v^\varepsilon_1 = 0
\]

Then we have equation about \((\bar{v}^\varepsilon, \bar{\varphi}^\varepsilon, \bar{q}^\varepsilon)\). First for Navier-Stokes,

\[
(\partial_t + v^\varepsilon_{1,1} \cdot \nabla_y + V^\varepsilon_z \partial_z) \bar{v}^\varepsilon + \nabla \delta^\varepsilon \bar{q}^\varepsilon - \varepsilon \Delta \delta^\varepsilon \bar{v}^\varepsilon = F
\]

where

\[
F = (v^\varepsilon_{1,2} - v^\varepsilon_{2,1}) \cdot \nabla_y v^\varepsilon_2 + (V^\varepsilon_z \partial_z - v^\varepsilon_1) \partial_z v^\varepsilon_2 - \left( \frac{1}{\partial_z \varphi^\varepsilon_2} - \frac{1}{\partial_z \varphi^\varepsilon_1} \right) (P^\varepsilon_1 \nabla q^\varepsilon_2 + \frac{1}{\partial_z \varphi^\varepsilon_2} ((P_2 - P_1) h^\varepsilon_2)
\]

\[
+ \varepsilon \left( \frac{1}{\partial_z \varphi^\varepsilon_2} - \frac{1}{\partial_z \varphi^\varepsilon_1} \right) \nabla \cdot (E_1 \nabla v^\varepsilon_2) + \varepsilon \frac{1}{\partial_z \varphi^\varepsilon_2} \nabla \cdot ((E_2 - E_1) \nabla v^\varepsilon_2)
\]

For divergence-free condition,

\[
\nabla \delta^\varepsilon \cdot \bar{v}^\varepsilon = \left( \frac{1}{\partial_z \varphi^\varepsilon_2} - \frac{1}{\partial_z \varphi^\varepsilon_1} \right) \nabla \cdot (P_1 v^\varepsilon_2) - \frac{1}{\partial_z \varphi^\varepsilon_2} \nabla \cdot ((P_2 - P_1) v^\varepsilon_2)
\]

For Kinematic boundary condition,

\[
\partial_t \bar{\varphi}^\varepsilon - (v^\varepsilon)^h_{y,1} \cdot \nabla h + ((v^\varepsilon)^h_{y,1})^b - (v^\varepsilon)^h_{y,2}) = - ((v^\varepsilon)^b_{y,2} - (v^\varepsilon)^h_{y,1}) \cdot \nabla h^\varepsilon_2
\]

Continuity of stress tensor condition becomes,

\[
\bar{q} \hat{n}_1 - 2\varepsilon (S^\varepsilon \bar{v}^\varepsilon) \hat{n}_1 = g \bar{\varphi}^\varepsilon - \eta \nabla \cdot \left( \frac{\nabla \bar{\varphi}^\varepsilon}{\sqrt{1 + |\nabla \bar{\varphi}^\varepsilon|^2}} \right)
\]

\[
+ 2\varepsilon (S^\varepsilon \bar{v}^\varepsilon) \hat{n}_1 + 2\varepsilon (S^\varepsilon \bar{v}^\varepsilon) (\hat{n}_1 - \hat{n}_2) - \eta \nabla \cdot \left( \frac{1}{\sqrt{1 + |\nabla h^\varepsilon_1|^2}} - \frac{1}{\sqrt{1 + |\nabla h^\varepsilon_2|^2}} \right) \nabla h^\varepsilon_2
\]

Using above 4 equations, we get \(L^2\)- energy estimate,(since initial condition is zero, no initial term appear on RHS)

\[
\|\bar{v}^\varepsilon(t)\|_{L^2}^2 + \|\bar{\varphi}^\varepsilon(t)\|_{L^2}^2 + \varepsilon \int_0^t \|\nabla \bar{v}^\varepsilon\|_{L^2}^2 \leq \Lambda(R) \int_0^t \left( \|\bar{v}^\varepsilon(s)\|_{L^2}^2 + \|\bar{\varphi}^\varepsilon(s)\|_{H^2}^2 \right) ds
\]
We skip detail calculation, since we can use our previous energy estimates basically. But, in above equations for \((\vec{v}, \vec{h}, \vec{q})\), RHS does not have low order than \(L^2\) energy. However we have uniform bound of \(m\)-order energy, so we can extract bad high order terms into \(\Lambda(R)\). To estimate \(\|\vec{h}(s)\|_{H^\frac{3}{2}}\), we don’t have to take time derivatives as like in previous section, since we already have bounded high order energy. And moreover, we don’t need uniform estimate in \(\varepsilon\), since we are dealing about for fixed \(\varepsilon\). So, we estimate \(\varepsilon \|\vec{h}(s)\|_{H^\frac{3}{2}}^{2}\)

**Lemma 11.1.** For every \(m \in \mathbb{N}, \varepsilon \in (0, 1)\), we have the estimate

\[
(11.7) \quad \varepsilon \|h(t)\|_{L^2}^2 \leq \varepsilon \|h(0)\|_{L^2}^2 + \varepsilon \int_0^t \|\vec{v}\|_{L^2}^2 + \int_0^t \Lambda_{1, \infty} \left(\|v\|_{L^2}^2 + \varepsilon \|h\|_{L^2}^2\right) ds
\]

where

\[
\Lambda_{1, \infty} = \Lambda \left(\|\nabla h\|_{L^\infty} + \|v\|_{L^\infty}\right)
\]

**Proof.** See proposition 3.4 in [1].

This is true for our case, since it comes from Kinematic boundary condition. We can also apply this lemma, to \(\vec{h}\) case,(surely, \(\vec{h}(0) = 0\)) and then combine with above \(L^2\) estimate, then we get the following estimate,(non-uniform in \(\varepsilon\))

\[
(11.8) \quad \|\vec{v}(t)\|_{L^2}^2 + \|\vec{h}(t)\|_{H^\frac{3}{2}}^2 + \varepsilon \|\vec{h}(t)\|_{H^\frac{3}{2}}^2 + \varepsilon \int_0^t \|\nabla \vec{v}(\tau)\|_{L^2}^2 ds \leq \frac{\Lambda(R)}{\varepsilon} \int_0^t \left(\|\vec{v}(\tau)\|_{L^2}^2 + \varepsilon \|\vec{h}(\tau)\|_{H^\frac{3}{2}}^2\right) ds
\]

Then we can use Gronwall’s inequality to get uniqueness. So finish uniqueness part of theorem 1.2.

11.2. **Uniqueness for Euler.** Since our estimate in above subsection(uniqueness for Navier-Stokes) is not uniform in \(\varepsilon\), result cannot be applied to Euler equation. As like in Navier-Stokes case, let we have two solutions \((v_1, h_1, q_1), (v_2, h_2, q_2)\) with same initial condition. Suppose,

\[
(11.9) \quad \sup_{[0,T]} \left(\|v_i\|_{L^\infty} + \|\partial_z v_i\|_{L^\infty} + \|\partial_z v_i\|_{W^{1, \infty}} + \|h_i\|_{L^\infty}\right) \leq R, \quad i = 1, 2
\]

(This is true from result in Theorem 1.2) Define \(\vec{v} = v_1 - v_2, \quad \vec{h} = h_1 - h_2, \quad \vec{q} = q_1 - q_2\) and we write equation of \((\vec{v}, \vec{h}, \vec{q})\), as before. Euler equation becomes,

\[
(11.10) \quad (\partial_t + v_{y,1} \cdot \nabla y + v_{z,1} \partial_z) \vec{v} + \nabla \vec{v} \cdot \vec{q} = F'
\]

where

\[
F' = (v_{y,2} - v_{y,1}) \cdot \nabla v_2 + (V_{z,2} - V_{z,1}) \partial_z v_2 - \left(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1}\right) \left(P_1 v_2 - 1 \partial_z \varphi_1 \nabla \cdot (P_2 - P_1) v_2\right)
\]

For divergence-free condition,

\[
\nabla \vec{v} \cdot \vec{v} = -\left(\frac{1}{\partial_z \varphi_2} - \frac{1}{\partial_z \varphi_1}\right) \nabla \cdot (P_1 v_2) - \frac{1}{\partial_z \varphi_2} \nabla \cdot (\nabla \cdot (P_2 - P_1) v_2)
\]

For Kinematic boundary condition,

\[
\partial_t \vec{h} - v_{y,1} \cdot \nabla h + (v_{z,1} - v_{z,2}) = - (v_{y,2} - v_{y,1}) \cdot \nabla h_2
\]

Continuity of stress tensor condition becomes,

\[
\vec{q} \hat{h}_1 = g \vec{h} - \eta \nabla \cdot \left(\frac{\nabla \vec{h}}{\sqrt{1 + |\nabla \vec{h}|^2}}\right) - \eta \nabla \cdot \left(\frac{1}{\sqrt{1 + |\nabla \vec{h}|^2}} - \frac{1}{\sqrt{1 + |\nabla h_2|^2}} \right) \nabla h_2
\]

By performing basic \(L^2\)-estimate, as similarly above, (we skip detail here)

\[
(11.11) \quad \|\vec{v}(t)\|_{L^2}^2 + \|\vec{h}(t)\|_{H^\frac{3}{2}}^2 \leq \Lambda(R) \int_0^t \left(\|\vec{v}(s)\|_{L^2}^2 + \|\vec{h}(s)\|_{H^\frac{3}{2}}^2\right) ds
\]

We should control \(\|v\|_{L^2}\) on RHS. But, since there are no dissipation on LHS, we cannot make it absorbed. Instead, we use vorticity. Let’s define vorticity \(\omega = \nabla \times v\) (which is equivalent to \(\omega = (\nabla \times u)(t, \Phi)\)). We have

\[
\omega \times \hat{n} = \frac{1}{2} (D^2 v \hat{n} - (D^2 v)^T \hat{n})
\]
\[ S^2 \ddot{v} - (D^2 v)^T \dot{n} = \frac{1}{2} \partial_t u - g^{ij} (\partial_j v \cdot \dot{n}) \partial_i, \]

Hence, it is suffice to estimate \( \omega \) instead of \( \partial_2 v \), i.e.

\[ \| \partial_2 v \|_{L^2} \leq \Lambda(R) \left( \| \omega \|_{L^2} + \| v \|_1 + |h|^2 \right) \]

To estimate \( \omega \), we use vorticity equation.

\[ (\partial_t \omega + v \cdot \nabla \omega) \omega_i = (\omega \cdot \nabla v) v_i \]

\( L^2 \) energy estimate of \( \omega \) is

\[ \| \omega(t) \|_{L^2}^2 \leq \Lambda(R) \int_0^t (|\dot{h}(s)|^2_1 + |\nabla \dot{v}(s)|^2_1 + \| \partial_2 \omega(s) \|_{L^2}^2 + |\nabla \omega(s)|_{L^2}^2) \, ds \]

We also should control \( |h|^2_{L^2} \). As similar to Dirichlet-Neumann estimate we can control this by \( |\partial_2 h|^2_{L^2} \).

And, from kinematic boundary condition of \( \ddot{h} \), we easily get

\[ |\partial_2 \ddot{h}(t)|_{L^2}^2 \leq \Lambda(R) \left( \| \nabla \dot{v} \|_{L^2}^2 + |h|^2_{H^1} \right) \]

Then we can use Gronwall’s inequality to get uniqueness. So finish uniqueness part of theorem 1.2 and theorem 1.3.

12. INVISCID LIMIT

In this section we send \( \varepsilon \) to zero, and get a unique solution of free boundary Euler equation. For \( \varepsilon \in (0, 1] \) and \( T \leq T_* \), we have uniform energy boundness

\[ \Psi_m(T) = \sup_{[0,T]} \left( \| v \|^2_{X^{m,0}} + |h|^2_{X^{m,1}} + \| \partial_2 v \|^2_{X^{m-1,1}} + \| \partial_2 v \|^2_{Y^{m,0}} \right) \leq R \]

So we have uniform boundness for \( v^\varepsilon \) in \( L^\infty([0,T], H^{m-1}_{co,loc}) \) and \( h^\varepsilon \) in \( L^\infty([0,T], H^{m+1}) \). And, by Rellich-Kondrachov theorem, we have, for each \( t \), compactness of \( v^\varepsilon(t) \) in \( H^{m-1}_{co,loc} \) and \( h^\varepsilon(t) \) in \( H^{m}_{loc} \). And from our energy function, we have a uniform boundness of \( \partial_2 u(t) \) in \( H^{m-1}_{co,loc} \) and of \( \partial_2 h^\varepsilon(t) \) in \( H^{m}_{loc} \) for \( v \in [0,T] \). Now by Arzela-Ascoli theorem, we have subsequence \( v^{\varepsilon_n}, h^{\varepsilon_n} \), such that

\[ v^{\varepsilon_n} \rightarrow v, \text{ strongly in } C([0,T], H^{m-1}_{co,loc}) \]

\[ h^{\varepsilon_n} \rightarrow h, \text{ strongly in } C([0,T], H^{m}_{loc}) \]

About pressure, from pressure section, we have boundness of \( \nabla q^\varepsilon \) in \( L^2([0,T] \times S) \), so get some \( q \) such that,

\[ \nabla q^{\varepsilon_n} \rightarrow \nabla q, \text{ weakly in } L^2([0,T] \times S) \]

and limit functions (\( v, h, \nabla q \)) satisfy

\[ \sup_{[0,T]} \left( \| v \|^2_{H^{m-1}_{co}} + |h|^2_{H^{m+1}} + \| \partial_2 v \|^2_{H^{m+1}_{co}} + \| \partial_2 v \|^2_{W^{2,\infty}} \right) \leq R \]

Now, we can pass to the limit and get the fact that \( (v, h, \nabla q) \) is a weak solution of Euler equation.(interior).

For boundary condition, first we can assume that the trace(boundary function),

\[ v^{\varepsilon_n}(t, y, 0) \rightarrow v^b, \text{ weakly in } L^2([0,T] \times S) \]

for some \( v^b \). In kinematic boundary condition, \( v^b \) is linear and we have strong convergence of \( h \), so kinematic boundary condition is satisfied weakly surely. Next, for continuity of stress tensor condition, by bounded lipschitz norm of \( v^{\varepsilon_n} \), 2\( \varepsilon(Su)\dot{n} \rightarrow 0 \) in weak limit process. And, limit of surface tension part is trivial by strong convergence of \( h \). Hence, in the weak sense,

\[ q^b = gh - \eta \nabla \left( \frac{\nabla h}{\sqrt{1 + |\nabla h|^2}} \right) \]

Hence we have a weak solution \( (v, h) \) which is local strong \( H^{m-1}_{co} \times H^{m}_{co} \) - convergence of \( (v^{\varepsilon_n}, h^{\varepsilon_n}) \).(and global for weak convergence in \( L^2 \times H^1 \)) Moreover this limit is unique by previous section. Meanwhile, we
can get strong convergence (non-local) in $L^2 \times H^1$. To get this, we just investigate norm convergence. For $t \in [0,T]$, using basic $L^2$-energy estimate and uniform boundness of high order energy,

$$
(12.6) \quad \left( \|v^\varepsilon_n J^\varepsilon_n(t)\|_{L^2}^2 + g |h^\varepsilon_n(t)|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h^\varepsilon_n(t)|^2} \right)
- \left( \|v_0^\varepsilon J_0^\varepsilon\|_{L^2}^2 + g |h_0^\varepsilon|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h_0^\varepsilon|^2} \right) \leq \varepsilon_n \Lambda(R) \to 0, \text{ as } \varepsilon_n \to 0
$$

where $J^\varepsilon = (\partial_v \varphi^\varepsilon)^{1/2}$ and $\varepsilon_n$ on the RHS come from dissipation of energy estimate. We assume that $\|v_0 - v_0\|_{L^2} \to 0$, $\|h_0^\varepsilon - h_0\|_{H^1} \to 0$ in statement of theorem 1.3 and

$$
\|\partial_v \varphi\|_{t=0} \to 0 \quad \text{uniform energy boundness, and anisotropic embedding.}
$$

This implies

$$
(12.7) \quad \lim_{\varepsilon \to 0} \left( \|v_0^\varepsilon J_0^\varepsilon\|_{L^2}^2 + g |h_0^\varepsilon|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h_0^\varepsilon|^2} \right) = \|v_0 J_0\|_{L^2}^2 + g |h_0|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h_0|^2}
$$

Lastly, using energy conservation in Euler equation ($\varepsilon = 0$, in basic $L^2$-estimate), we get

$$
(12.8) \quad \|v_0 J_0\|_{L^2}^2 + g |h_0|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h_0|^2} = \|v J\|_{L^2}^2 + g |h|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h|^2}
$$

Finally we get norm convergence.

$$
(12.9) \quad \lim_{\varepsilon \to 0} \left( \|v^\varepsilon J^\varepsilon(t)\|_{L^2}^2 + g |h^\varepsilon(t)|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h^\varepsilon(t)|^2} \right) = \|v J\|_{L^2}^2 + g |h|_{L^2}^2 + 2\eta \sqrt{1 + |\nabla h|^2}
$$

With weak convergence, this implies strong convergence to $(v,J,h)$ in $L^2 \times H^1$. And strong convergence of $(u^\varepsilon, h^\varepsilon)$ means $(e^\varepsilon, h^\varepsilon)$ strongly in $L^2 \times H^1$ without $J$ $L^\infty$-type convergence can be done by $L^2$ convergence, uniform energy boundness, and anisotropic embedding.

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