Braided Rivers and Superconducting Vortex Avalanches

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Abstract

Magnetic vortices intermittently flow through preferred channels when they are forced in or out of a superconductor. We study this behavior using a cellular model, and find that the vortex flow can make braided rivers strikingly similar to aerial photographs of braided fluvial rivers, such as the Brahmaputra. By developing an analysis technique suitable for characterizing a self-affine (multi)fractal, the scaling properties of the braided vortex rivers in the model are compared with those of braided fluvial rivers. We suggest that avalanche dynamics leads to braiding in both cases.

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Magnetic flux penetrates type II superconductors in thin filaments, or vortices, which can move when an electrical current is applied. Since vortex motion creates electrical resistance and destroys superconductivity, these materials are often produced with defects that tend to pin vortices. Experiments [1] and molecular dynamics (MD) simulations [2–4] have indicated that vortices can move through the pinning landscape in preferred channels. Vortices can also move intermittently in time, or “avalanche”, as they are forced in or out of a superconductor [5,6]. Similar behavior has also been seen in simulations [2–4,7]. Here we use a simple cellular model [7] to mimic the experiments in Ref. [5,6], and find that the intermittent vortex flow can make an intricate braided pattern (Fig. 1), similar to aerial photographs of braided fluvial rivers, such as the Brahmaputra, Aichilik, or Hulahula [8,9].

Braided rivers [8] form a separate class of hydrological systems, distinct from single-channel rivers and dendritic river networks. They are made of alluvial channels meeting and dividing, separated by bars and islands. Their deposits are important reservoirs of oil, gas, heavy minerals, etc. Sapozhinikov and Foufoula-Georgiou (SFG) [9] have shown that braided rivers of different scales with different hydrological characteristics exhibit self-affine scaling, which may be universal.

Here we extend the SFG method and show that braided rivers in the vortex model exhibit self-affine multifractal behavior over two decades in length scale. We compare our results with the correlation integral method of SFG, which measures the scale dependence of one particular moment of the flow, and propose that our multifractal analysis technique, which probes all moments, be used to further characterize braided fluvial rivers as well as other braided systems. We suggest that the stick-slip dynamics of vortices leading to avalanches of magnetic flux in superconductors may be analogous to pulse load transport [10] of avalanching sediment in fluvial systems, and lead to braiding in both cases, despite their vastly different length scales and microscopic descriptions. Comparing our measurements with those of SFG (where possible) leads us to speculate that these braided systems could represent a single universality class of dynamic critical phenomena.

The cellular model [7] describes the motion of driven vortices at large physical length
scales, and includes basic features of vortex dynamics: over-damped motion of vortices, repulsive interactions between vortices, and attractive pinning interactions at defects in the material. As in experiments, vortices are slowly pushed into the system at one boundary and allowed to leave at the other boundary. Within a range of parameters, the model evolves into a self-organized critical state \cite{11} with avalanches of all sizes, as explained in detail in \cite{7}. Self-organized criticality has also been observed experimentally \cite{5,6} for a limited range of temperatures and magnetic fields, and in MD simulations of the microscopic equations of motion \cite{12}.

We briefly summarize the model. Consider a two-dimensional honeycomb lattice where each cell \(x\) has three nearest neighbors, and is occupied by an integer number of vortices \(m(x)\). The force to move a vortex from \(x\) to \(y\) is

\[
F_{x\rightarrow y} = V_{\text{pin}}(y) - V_{\text{pin}}(x) + [m(x) - m(y) - 1] + r[m(x1) + m(x2) - m(y1) - m(y2)] .
\]

The nearest neighbor cells of \(x\) are \(y, x1,\) and \(x2\), and the nearest neighbors cells of \(y\) are \(x, y1,\) and \(y2\), and \(r < 1\). The pinning potential \(V_{\text{pin}}(x)\) is equal to \(V_{\text{max}}\) with probability \(p\) and 0 with probability \(1-p\). The numerical results presented here are for \(r = 0.1, V_{\text{max}} = 2.0,\) and \(p = 0.1\). However, the results are robust over a limited range of the parameters, as discussed below. In one iteration, at each cell a single vortex moves one lattice unit when the force in that direction is positive. If more than one unstable direction exists, one of them is chosen randomly \cite{13}. A vortex reaching the right edge of the system is removed. Periodic boundary conditions are applied at the top and bottom of the lattice. An avalanche is initiated by adding a vortex to a stable configuration on the left edge of the system. It proceeds by repeatedly updating until the configuration is again stable. Then a new vortex is added to the system.

The spatial variation of the flow is measured in terms of the number of vortices moving in each cell, averaged over a very long time interval representing many vortices flowing through the system. Fig. 1. represents a “time-lapsed” photograph of vortex motion. Rather
than exhibiting uniform flow, the vortices clearly have preferred channels to move in. In sufficiently large systems, the channels form an intricate braided pattern. In fact braiding was observed, at smaller length scales, from MD simulations \([3,4]\), for a range of parameter values. Outside of this range, branched, “Hortonian” structures resembling dendritic river networks were observed. Nevertheless, a direct comparison, using techniques developed to characterize braided systems \([3]\), with braided fluvial rivers \([8]\) has not previously been made.

For a given configuration of pinning centers in our model, the observed braided pattern is absolutely stable, independent of initial conditions, including a system empty of vortices, and a system that initially had a uniform average slope greater than the critical slope. However, the pattern changes if the pinning centers are moved. Thus, in some superconductors maintained at sufficiently low temperatures, we predict that a braided river pattern would exist that would be stable and reproducible for a given sample, if one averaged the flow over a long time interval. In fact, Battacharya and Higgins \([14]\) have reported sample dependent “fingerprints” in IV measurements which is likely an experimental verification of this behavior.

Qualitatively, the braided vortex river pattern we observe resembles patterns of interconnected channels formed by water flowing over non-cohesive sediment. In fact, braiding has been proposed to be the fundamental instability of laterally unconstrained free surface flow over cohesionless beds, and has been found to be a robust feature in simulations of river flow with sediment transport that includes both erosion and redeposition \([15]\). Interestingly, where no redeposition occurs, as in mountainous regions with high slope, branched, dendritic networks are obtained.

In order to examine the possible relationship between these two types of braided systems, we make a quantitative analysis of braided vortex rivers, and compare the results with those previously measured from aerial photographs and radar imagery \([3]\) of different fluvial rivers. Although the pinning sites are randomly distributed in our model, with no extended spatial correlations, the braided structure itself shows long-range correlations. Since each individual flow event, or avalanche, has no characteristic scale, the sum of those events, as revealed by
the pattern of flow in Fig. 1, also is scale-free up to the size of the system.

A braided river can be characterized by its behavior under a change of scale. The braided vortex river is a self-affine multifractal over the range of length scales we are able to study. The self-affinity is due to the fact that the average flow is anisotropic since it follows the average slope of the vortex pile. This makes the direction of flow and perpendicular to flow distinct. In order to reproduce the same structure under a change of scale one has to scale the two directions in a different manner. This effect is also seen in braided fluvial rivers, as sediment flows down an average slope due to gravity. The multifractality reflects spatial intermittency. Intermittency here means that almost all cells in the model contain at least a small portion of the vortex flow, but the flow is highly concentrated into a subset of the cells, with the most highly concentrated flow, dominating the highest moments, occurring in filamentary structures indicated in yellow in Fig. 1. If each moment of the flow pattern exhibits scaling with a different dimension then the flow is multifractal.

A standard mathematical characterization of the scaling properties of braided rivers is made by partitioning the pattern, such as Fig. 1, into rectangular cells, each one of size $X \times Y$ elementary cells. If $P_i$ is defined to be the fraction of the overall flow that falls into $i$th rectangle of the partitioning, then the qth moment of the probability partition is

$$M_q(X,Y) = \sum_i P_i^q \text{ for } q \geq 0,$$

where the sum is over all rectangles of size $X \times Y$ needed to cover the entire pattern. The spectrum of fractal dimensions $\{D_q\}$ can be calculated by determining the scaling behavior of $M_q(X,Y)$ when $X \sim Y \sim L$ by

$$M_q(L) \sim L^{-(1-q)D_q}.$$

Because of the singular nature of this definition, special care must be used to define $M_q$ for $q = 0$ and $q = 1$. The exponents $D_0$, $D_1$, and $D_2$ correspond to the capacity, information, and correlation dimensions, respectively. The results of our analysis of the braided vortex rivers are shown in Fig. 2. For the largest system size we could study,
each moment, $M_q$, exhibits scaling with a well-defined dimension, $D_q$, over two decades in length scale. The measured dimensions vary continuously from $D_0 = 2.0$ to $D_8 \approx 1.4$; such a variation indicates that the pattern is “multi-fractal”. The value of $D_0$ is to be expected since virtually all of the elementary cells contain a non-zero portion of flow. The lower values of $D_q$ for the higher moments, $q$, arise from the filamentary structure of highly concentrated activity, as mentioned above. The dimension $D_2$ has been measured for braided fluvial rivers with a variety of different types of sediment beds over a wide range of length scales, obtaining $D_2 = 1.5 - 1.7$. This range is relatively close to the values ($D_2 \sim 1.8$) we have measured for different realizations of our model. As discussed later, our accuracy is not sufficient to ascertain whether the small apparent difference is significant or not. A multifractal analyses of braided fluvial rivers using the method described here could determine if they are multifractal or not.

In order to establish and characterize the anisotropic scaling properties of the braided pattern, we have adapted and extended the method of SFG, which was developed to characterize the self-affine structure of braided fluvial rivers. We extend their method to take into account the fact that the self-affine structure can also be multifractal, as demonstrated above. The most general scaling of a pattern is found by determining the scaling of $M_q(X, Y)$ when the rectangle dimension $X \times Y$ scales such that $X \sim Y^{\nu_x/\nu_y}$. In this case, the scaling hypothesis is

$$\frac{M_q(X_1, Y_1)}{M_q(X_2, Y_2)} = \left(\frac{X_1}{X_2}\right)^{(q-1)\nu_x(q)} = \left(\frac{Y_1}{Y_2}\right)^{(q-1)\nu_y(q)}.$$  \hspace{1cm} (2)

Defining $z_q = \log M_q$, $x = \log X$, and $y = \log Y$, and following the arguments of Ref. [17], for each $q$, we find

$$(q - 1) \nu_x(q) \frac{dz_q}{dx} + (q - 1) \nu_y(q) \frac{dz_q}{dy} = 1$$  \hspace{1cm} (3)

Taking the logarithmic derivatives of the moment of the probability partition $dz_q/dx$ and $dz_q/dy$ numerically at a number of different points $(x, y)$, the results can be the fit into (3) to obtain $\nu_x(q)$ and $\nu_y(q)$. The results of this analysis on the braided vortex river patterns
are shown in Fig. 3. Note that the exponents $\nu_x$ and $\nu_y$ vary with $q$. This variation is consistent with the variation of $D_q$ described above. In simple terms, it means that the different fractals which are formed by different moments of the flow also exhibit different self-affine properties, with the highest moments, which are dominated by long, filamentary objects, being more anisotropic than the lower moments.

Previous analyses on different braided fluvial rivers have determined $\nu_x(2) = 0.72 - 0.77$ and $\nu_y(2) = 0.47 - 0.52$. From the results in Fig. 3, we obtain $\nu_x(2) = 0.67 \pm 0.03$ and $\nu_y(2) = 0.44 \pm 0.03$ for braided vortex rivers, where the error bars only represent statistical errors based on the data shown. However, there are also potentially significant sources of systematic error due to finite size limitations. Noticing boundary effects apparent in Fig. 1, we analyze the scaling only in a strip in the center, typically beginning and ending 80 lattice cells inside the left and right edges. We have varied both the strip width and the numerical differentiation scheme and observed the variation in the apparent scaling dimensions. We estimate the systematic error to be approximately 10% of the quoted value for all exponents. Within error, including systematic error, our results for $\nu_x(2)$ and $\nu_y(2)$ are consistent with those measured for different braided fluvial rivers, although we cannot rule out that the exponents are different. The anisotropic exponents $\nu_x$ and $\nu_y$ for other values of $q$, have not been measured for braided fluvial rivers, which would be necessary to make firmer statements about the quantitative similarities or differences between these two systems. Nevertheless, since the three exponents where we can compare do not exhibit differences outside the error limits, we are lead to speculate that the braided structures in the two systems may represent a single universality class.

In this regard, it is important to note that we have also varied the parameters of our model and studied the resulting river structure. Over a range of parameters, $r \in [0.1, 0.2]$, $V_{\text{max}} \in [1.0, 5.0]$, and $p \in [0.1, 0.4]$, we observe a similar braided pattern with fractal and multifractal exponents that lie within the error bounds of the results reported here. However, sufficiently outside this range of parameters it is likely that the morphology of the river structure is different. For example, at weaker pinning strengths the rivers become broad and
the fractal structure disappears. At stronger pinning strengths, the braided structure also seems to disappear and the pattern formed by the rivers more closely resembles dendritic river networks. These findings concerning the different river morphologies are similar to those reported for MD simulations [4]. However, further study is required to confirm these results.

It has previously been postulated that braiding of fluvial rivers is due to a self-organized critical process [19], but a mechanism for this to occur has not been identified. Are there avalanches in fluvial rivers that could lead to self-organization and produce the observed braiding? In fact there are. “Pulses” in bedload transport have been observed to occur on all spatial and temporal scales up to those limited by the size of the river studied [10]. We have observed analogous pulses in our model by measuring the vortex flow through individual lattice cells as a function of time. Perhaps vortices of magnetic flux are analogous to sediment in fluvial rivers. The elementary stick-slip process is that of sediment slipping and then resticking at some other point [15], like our intermittently moving vortices. The elementary slip event can dislodge nearby sediment leading to a chain reaction of slip events, or avalanches. Sediment transport can be triggered when the local sediment slope is too high; the same is true for vortices in a superconductor. Thus, in both magnetic flux and fluvial rivers it appears that the braiding could emerge through a stick-slip process consisting of avalanches of all sizes.

In should also be noted that although there is disorder present in fluvial rivers, due to inhomogeneities in the bedload, it is different than the disorder due to pinning in superconductors. The location of the pinning centers is quenched, whereas the location of bedload inhomogeneities in fluvial rivers are annealed at some time scale. Thus the braided patterns produced in our simulations are fixed, while the braided patterns observed in fluvial rivers are not [19]. Nevertheless, despite this difference, the resulting braided patterns are similar.

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FIGURES

FIG. 1. **Braided Rivers of Superconducting Vortices.** A “time-lapsed” photograph of vortex motion with average flow from left to right. The lattice size is $600 \times 500$. Sites containing an average amount of flow are shown in red. Yellow sites have a flow level greater than 20 times the average. Dark blue sites have almost no vortex flow, although virtually every site has some minimal amount. The intricate braiding pattern is remarkably similar to the pattern formed by braided fluvial rivers.

FIG. 2. **Multi-fractal scaling exponents.** The scaling of moments as a function of $L$, calculated from Fig. 1. The slopes of the lines are the exponents $D_q$. Shown here, from bottom to top, are results for $q = 0, 1, 2, 3, 4, 6$. Inset shows the dimensions $D_q$ as a function of $q$ for five different realizations of braided rivers resulting from the same parameters, with different locations for the pins.

FIG. 3. **Self-affine multi-fractal exponents.** Results for the anisotropic multi-scaling exponents $\nu_x$ and $\nu_y$ as a function of $q$ for five different braided river patterns. Results were calculated from the same data sets used in the inset in Fig. 2. The dark filled symbols correspond to the results for $\nu_x$, while open symbols correspond to the results for $\nu_y$. The results for each of the five different patterns are indicated with unique symbols. The average over the five runs is indicated by the circles connected by the thick line.
This figure "fig1.gif" is available in "gif" format from:

http://arxiv.org/ps/cond-mat/9901228v2
