Sharp spherically averaged Strichartz estimates for the Schrödinger equation

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Abstract
We prove generalized Strichartz estimates with weaker angular integrability
for the Schrödinger equation. Our estimates are sharp except some endpoints.
Then we apply these new estimates to prove scattering for the 3D Zakharov
system with small data in the energy space with low angular regularity. Our
results improve the results obtained recently in Guo Z et al (2014 Generalized
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1. Introduction

In this paper, we continue the previous work [12] to study the generalized Strichartz estimates
for the following Schrödinger-type dispersive equations

\[ i \partial_t u + D^a u = 0, \quad u(0, x) = f(x) \]  

(1.1)

where \( u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}, \quad D = \sqrt{-\Delta}, \quad a > 0. \) We are mainly concerned with the following estimates

\[ \| e^{itD} P_{\xi} f \|_{L_t^2 L_x^\infty} \lesssim \| f \|_{L_x^\infty} \]  

(1.2)

where \( P_{\xi} f \approx \frac{1}{\xi} \hat{f} \) (see the end of this section for the precise definition). Here the norm
\( L_t^\infty L_x^\infty \) for function \( u(t, x) \) on \( \mathbb{R} \times \mathbb{R}^d \) is defined as follows

...
The purpose of this paper is to study the sharp range for \((q, p)\) such that the estimate (1.2) holds. Then we apply these estimates to the 3D Zakharov system.

Two typical examples are of particular interest: one is the wave equation \((a = 1)\), the other is the Schrödinger equation \((a = 2)\). The space time estimates which are called Strichartz estimates address the estimates

\[
\|e^{it\Delta} P_0 f\|_{L^q_t L^p_x} \lesssim \|f\|_{L^2_t x}.
\]  

Strichartz [39] first proved (1.3) for the case \(q = p\) and then the estimates were substantially extended by various authors, e.g. [9, 23] for \(a = 1\), and [8, 43] for \(a = 2\). It is now well-known (see [19]) that the optimal range for (1.3) is the admissible condition: if \(a = 1\),

\[
2 \leq q, p \leq \infty, \quad \frac{1}{q} \leq \frac{d - 1}{2} \left(\frac{1}{2} - \frac{1}{p}\right), \quad (q, p, d) = (2, \infty, 3);
\]  

(WA)

and if \(a \neq 1\),

\[
2 \leq q, p \leq \infty, \quad \frac{1}{q} \leq \frac{d - 1}{2} \left(\frac{1}{2} - \frac{1}{p}\right), \quad (q, p, d) = (2, \infty, 2).
\]

(SA)

However, if \(d \geq 2\) and \(f\) is radial, then (1.3) holds for a wider range of \((q, p)\) (for example, see [21, 32, 35]). For the wave equation \((a = 1)\), the optimal range for (1.3) under the radial symmetry assumption is (see [21, 35, 38], see also [15])

\[
2 \leq q, p \leq \infty, \quad \frac{1}{q} < (d - 1)(\frac{1}{2} \left(\frac{1}{2} - \frac{1}{p}\right)) \quad \text{or} \quad (q, p) = (\infty, 2).
\]

(RWA)

For the Schrödinger-type equation \((a \neq 1)\), it was known that (1.3) holds under the radial symmetry assumption (RSA) if the following condition holds

\[
2 \leq q, p \leq \infty, \quad \frac{1}{q} \leq (d - 1)(\frac{1}{2} \left(\frac{1}{2} - \frac{1}{p}\right)), \quad (q, p, d) = (2, \frac{4d - 2}{2d - 3}).
\]

(RSA)

The range (RSA) is optimal except the radial endpoint \((q, p) = (2, \frac{4d - 2}{2d - 3})\) which still remains open. The RSA was first obtained in [15] except some endpoints improving the results in [32] and the remaining endpoint estimates were later obtained in [5, 18] independently.

There are two kinds of analogous results in the non-radial case. The first is to consider the estimate with additional angular regularity (see the end of this section for the definition of \(H^0_{\omega, A}\))

\[
\|e^{it\Delta} P_0 f\|_{L^q_t L^p_x} \lesssim \|f\|_{H^0_{\omega, A}}.
\]  

in which some angular regularity is traded off by the extension of the admissible range, see [5, 17, 38]. The second one is to consider the estimate (1.2) that we study in this paper. From the viewpoint of applications, the estimate (1.2) works better than (1.4), because there is no loss of angular regularity. The spherically averaged Strichartz norm was used in [41] to obtain
the endpoint case of Strichartz estimate for the 2D Schrödinger equation (see [24] for the 3D wave equation). For the wave equation ($a = 1$), it was known that (1.2) also holds for $(q, p)$ satisfying the (RWA) (see [5, 17, 34]). If $a = 1$, [12] showed that (1.2) holds for $(q, p)$ satisfying the (RWA), improving the results in [17]. Moreover, when $a > 1$, [12] also showed (1.2) holds for $(q, p)$ belonging to a wider range than the (RWA), but the sharp range is unknown.

The main result of this paper is

**Theorem 1.1.** Let $a > 1$. Then (1.2) holds if $(q, p)$ satisfies

(i) $d = 2$:

$$2 \leq q, p \leq \infty, \quad \frac{1}{q} < (d - \frac{1}{2})(\frac{1}{2} - \frac{1}{p}) \text{ or } (q, p) = (\infty, 2).$$

(ii) $d \geq 3$: $(q, p)$ satisfies (RSA).

By the theorem above, we see that the optimal range for (1.2) is obtained except the endpoint line $\frac{1}{q} = (d - \frac{1}{2})(\frac{1}{2} - \frac{1}{p})$ for $d = 2$, and the endpoint $(q, p) = (2, \frac{4a - 7d - 2}{2d - 4})$ for $d \geq 3$. The basic ideas of proving theorem 1.1 are the same as in [12], namely to do the space dyadically localized estimates by exploiting the decay and oscillatory effect of a family of Bessel functions uniformly. The key differences in this paper are: (1) we prove better uniform estimates for Bessel functions (indeed, we give a uniform expansion) in the transitive region; (2) we treat the oscillatory integral operator related to the Bessel function at a finer scale so that we can catch more subtle oscillatory effects; (3) to get the endpoint for $d \geq 3$, we exploit some ‘almost orthogonality’ to overcome some logarithmic summation difficulty.

Besides its own interest, (1.2) plays important roles in the nonlinear problems, e.g. in [12] where the authors proved scattering for the 3D Zakharov system for small data in the energy space with one additional angular regularity. In this paper, we use the new estimates in theorem 1.1 to improve the angular regularity. Consider the 3D Zakharov system:

\[
\begin{align*}
\dot{u} - \Delta u &= nu, \\
\dot{n}/\alpha^2 - \Delta n &= -\Delta |u|^2
\end{align*}
\]

with the initial data

$$u(0, x) = u_0, \quad n(0, x) = n_0, \quad \dot{n}(0, x) = n_1$$

where $(u, n)(t, x) : \mathbb{R}^{1+3} \to \mathbb{C} \times \mathbb{R}$, and $\alpha > 0$ denotes the ion sound speed. The system was introduced by Zakharov [44] as a mathematical model for the Langmuir turbulence in unmagnetized ionized plasma. It preserves $\|u(t)\|_{L^2}$ and the energy

$$E = \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{|D^{-1}n|^2/\alpha^2 + |n|^2}{2} - n|u|^2 \, dx.$$  

The natural energy space for initial data is

$$(u_0, n_0, n_1) \in H^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times H^{-1}(\mathbb{R}^3).$$

The Zakharov system has been extensively studied, see [2–4, 10, 11, 13, 14, 20, 22, 25–27, 29, 30, 31, 33, 40], and the introduction of [12].
Since the global well-posedness for (1.6) with small data in the energy space was proved by Bourgain–Colliander [4], the long time behavior of the solutions has been a very interesting problem. For this problem, the first result was obtained in [13] that scattering holds for small energy data under the RSA. Later, global dynamics below the ground state in the radial case was obtained in [14]. In the non-radial case, for data with sufficient regularity and decay, and with suitable small norm, scattering was obtained in [16]. In [12], the authors proved scattering for small data in the energy space with one order angular regularity. In this paper we prove

**Theorem 1.2.** Let \( s > 3/4 \). Assume \( \| (u_0, n_0, n_1) \|_{\tilde{H}^s_p \times \tilde{H}^s_p \times \tilde{H}^{-1/4}_p} = \varepsilon \) for \( \varepsilon > 0 \) sufficiently small. Then the global solution \( (u, n) \) to (1.6) belongs to \( C^0_t \tilde{H}^{s+3/2}_p \times \tilde{H}^{s+3/2}_p \cap C^0_t \tilde{H}^{-1/4}_p \), and scatters in this space.

Theorem 1.2 was proved in [12] for \( s = 1 \). We improve the angular regularity to \( s > 3/4 \) by using the new estimates of theorem 1.1. To deal with the fractional derivative on the sphere, we transfer it to the fractional derivative on \( SO(3) \) (see the appendix). We remark that \( s > 3/4 \) reaches a limitation of our method. To remove the angular regularity, new ideas should be developed.

### 1.1. Notation

Finally we close this section by listing our notation used, which closely follows that in [12].

- We denote \( \mathbb{N} = \mathbb{Z} \cap [0, \infty) \), \( \mathbb{N}^+ = \mathbb{Z} \cap (0, \infty) \).
- \( \mathcal{F}(f) \) and \( \hat{f} \) denote the Fourier transform of \( f \), \( \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \xi \cdot x} f(x) dx \). For \( a \geq 1 \), \( S_a(f) = e^{i a \xi^2} = F^{-1} e^{i a \xi^2} F \).
- \( \eta : \mathbb{R} \to [0, 1] \) is an even, non-negative smooth function which is supported in \( \{ \xi : |\xi| \leq 8/5 \} \) and \( \eta \equiv 1 \) for \( |\xi| \leq 5/4 \). For \( k \in \mathbb{Z} \), \( \chi_k(\xi) = \eta(\xi/2^k) - \eta(\xi/2^{k-1}) \) and \( \chi_{k+1}(\xi) = \eta(\xi/2^k) \).
- \( P_s \) denote the Besov-type space given by the norm \( \| f \|_{P_s} = \| \Lambda f \|_{L^p} \) for \( s \) is the usual Lebesgue space, and \( \| f \|_{P_s} = \| f \|_{L^p} \).
- \( L^p(\mathbb{R}^d) \) denotes the usual Lebesgue space, and \( L^p(\mathbb{R}^+ : r^{-1/4} dr) \).
- \( L^p_0 \) are Banach spaces defined by the following norms \( \| f \|_{L^p_0} = \| f \|_{L^p} \).
- \( \tilde{H}^{s}_p \) are the fractional Sobolev spaces on \( \mathbb{R}^d \).
- \( B^{s}_{p, q} \) denotes the Besov-type space given by the norm \( \| f \|_{B^{s}_{p, q}} = \left( \sum_{k \in \mathbb{Z}} 2^{ks} \| P_k f \|_{L^p_0}^q \right)^{1/q} \).

0 \leq \alpha \leq 1, \( H^{s, \alpha}_p \) is the space with the norm \( \| f \|_{H^{s, \alpha}_p} = \| \Lambda f \|_{L^p} \), and the spaces \( \tilde{H}^{s, \alpha}_p \) and \( \tilde{B}^{s, \alpha}_{p, q} \) are defined similarly.

For simplicity, we denote \( H^{s, \alpha} = H^{s}_2 \) and \( \tilde{H}^{s, \alpha} = \tilde{H}^{s}_2 \) and \( \tilde{B}^{s, \alpha} = \tilde{B}^{s}_2 \).

Let \( X \) be a Banach space on \( \mathbb{R}^d \). \( L_t^p X \) denotes the space-time space on \( \mathbb{R} \times \mathbb{R}^d \) with the norm \( \| u \|_{L_t^p X} = \| u(t, \cdot) \|_{L_x^p} \).
2. Spherically averaged Strichartz estimates

2.1. Uniform estimates for Bessel functions

To prove theorem 1.1, we need to deal with a family of Bessel functions which are defined by

\[ J_\nu(r) = \frac{(r/2)^\nu}{\Gamma(\nu + 1/2)\pi^{1/2}} \int_0^1 e^{ir(1 - t^2)^{\nu - 1/2}} dt, \quad \nu > -1/2. \]

As one of the most important special functions, Bessel functions were studied extensively, for example, see [28, 42]. For example, we require uniform decay (see section 10.20.4 in [28])

**Lemma 2.1.** Assume \( \nu > r \). Then we have

\[ |J_\nu(r)| + |J'_\nu(r)| \leq C(r^{-1/3} + r^{-1/2})^{-1/4}. \]  

(2.1)

**Lemma 2.2.** Assume \( \nu \in \mathbb{N}, \nu > r \), and \( \lambda > r^{1+\varepsilon} \) for some \( \varepsilon > 0 \). Then for any \( K \in \mathbb{N} \)

\[ |J_\nu(r)| + |J'_\nu(r)| \leq C_K r^{-K\varepsilon}. \]

(2.2)

**Proof.** We only need to estimate \( J_\nu(r) \), since \( J_\nu(r) = J_{-\nu}(r) = J_{-\nu}(r) \) (see p 45, equation (2) in [42]). If \( \nu > 2r \), then integrating by parts \( K \) times we can get \( |J_\nu(r)| \leq |r^{-K}| \leq r^{-K} \). Now we assume \( \nu \leq 2r \). Let \( \phi(x) = r \sin x - \nu \). Then \( \phi'(x) = r \cos x - \nu \). By the assumption \( |\phi'(x)| > r^{1+\varepsilon} \). Define the operator \( D_\nu \) by

\[ D_\nu f = -\partial_x (\phi(x))^{-1} f. \]

Then

\[ J_\nu(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r\sin x - \nu x)} \partial_x (e^{i(r\sin x - \nu x)}) dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(r\sin x - \nu x)} D_\nu^K(1) dx. \]

It suffices to show

\[ |D_\nu^K(1)| \leq C_K r^{-K\varepsilon}. \]

(2.3)

We prove (2.3) by induction on \( K \). For \( K = 0 \), the bound is trivial. Now we consider \( K = 1 \). We have

\[ \partial_x (\phi(x))^{-1} = \phi'(x)^{-1} \frac{r \sin x}{\phi'(x)}. \]

Let \( g(x) = \frac{r \sin x}{\phi'(x)} \). By calculus we see

\[ |g(x)| \leq g(\arccos \frac{r}{\nu}) = \frac{r}{\sqrt{\nu^2 - r^2}} \leq r^{1/2}. \]

(2.4)

Thus by assumption the case \( K = 1 \) is proved. Now we assume (2.3) holds for \( K \) by using the bound (2.4). If \( g \) is not a factor of \( D_\nu^K(1) \), then

\[ |D_\nu^{K+1}(1)| \sim |D_\nu^K(1)\phi'(x)^{-1} g(x)| \leq r^{-(K+1)\varepsilon}. \]
If \( g \) is a factor of \( D^K_0(1) \), then
\[
D^{K+1}_0(1) = CD^K_0(1)\phi'(x)^{-1}g(x) + \tilde{G}
\]
where \( G \) is given by \( D^K_0(1) \) but with one factor \( g \) replaced by \( \phi' \). The latter has better bound \( r^{4+\varepsilon} \) than \( g \). So by induction \[ |G| \lesssim r^{-(K+1)} \]. We complete the proof of the lemma. \( \square \)

Uniform decay of Bessel functions is not sufficient for our purpose. We also need to exploit the oscillations. We will need asymptotic expansions for large \( r \) uniformly with respect to \( \nu \).

Such expansions were given in section 10.20.4 in [28]. Similar techniques were also used in [1].

Lemma 2.3 (Asymptotical property, [1, 28]). Let \( \nu > 10 \) and
\[
|\theta(r) - \nu \pi / 4| \lesssim \frac{\nu}{r^{3/2}} + \frac{1}{r} L_{\nu,1/2}(r) + r^{-1/2} l_{\nu,\infty}(r).
\]

(2) Let \( x_0 = \arccos \frac{\nu}{r} \). For any \( K \in \mathbb{N} \) we have
\[
h(\nu, r) \lesssim \frac{r^{3/2} + \frac{1}{r} L_{\nu,1/2}}{r^{3/2} + \frac{1}{r} L_{\nu,1/2}} L_{\nu,\infty}(r) + r^{-1/2} l_{\nu,\infty}(r).
\]

Moreover, if \( \nu \in \mathbb{Z} \), we have the better estimate
\[
|h(\nu, r)| \lesssim \frac{r^{3/2} + \frac{1}{r} L_{\nu,1/2}}{r^{3/2} + \frac{1}{r} L_{\nu,1/2}} L_{\nu,\infty}(r) + r^{-3/2} l_{\nu,\infty}(r).
\]

2.2. Reduction to the one dimensional estimates

In this section, we prove theorem 1.1 by improving the proof in [12]. First, we recall the reductions and some results in [12] for the readers’ convenience. To prove (1.2), it is equivalent to show
\[
\|T_{\alpha}f\|_{L^2} \lesssim \|f\|_{L^2}
\]
where
\[ T_n f(t, x) = \int_{\mathbb{R}^d} e^{i(t \cdot \xi + t \cdot \xi')} \chi_d(|\xi|) f(\xi) d\xi. \]

Now we expand \( f \) by the orthonormal basis \( \{ Y^k_j \} \), \( k \geq 0, 1 \leq l \leq d(k) \) of spherical harmonics with \( d(k) = C_{n+k-1}^k = C_{n+k-3}^{k-2} \), such that
\[ f(\xi) = f(\rho \sigma) = \sum_{k \geq 0} \sum_{1 \leq l \leq d(k)} a^k_l(\rho) Y^k_l(\sigma). \]

Using the identities (see [36])
\[ \tilde{Y}^k_l(\rho \sigma) = c_{d,k} \rho^{-d-2}\int J_\nu(\rho \sigma) Y^k_l(\sigma) \]
where \( c_{d,k} = (2\pi)^{d/2-1}, \nu = \nu(k) = \frac{d-2+2k}{2} \), then we get
\[ T_n f(t, x) = \sum_{k,l} c_{d,k} T^n_k(a^k_l(t, |x|)) Y^k_l(x/|x|) \]
where
\[ T^n_k(h)(t, r) = r^{-d-2} \int e^{-ir^d} J_\nu(\rho \sigma) h(\rho) d\rho. \]

Here \( J_\nu(\rho) \) is the Bessel function. Thus (2.5) becomes
\[ \| T^n_k(a^k_l) \|_{L^q_t L^p_x} \lesssim \| (a^k_l(\rho)) \|_{L^\nu_t L^\nu_x}. \quad (2.6) \]

To prove (2.6), it is equivalent to show
\[ \| T^n_k(h) \|_{L^q_t L^p_x} \lesssim \| h \|_{L^2} \quad (2.7) \]
with a bound independent of \( \nu \), since \( q, p \geq 2 \).

By the classical Strichartz estimates (see the endpoint estimates in \([19, 24]\)), we can get
\[ \| 1_{r \geq 100} T^n_k(h) \|_{L^q_t L^p_x} \lesssim \| h \|_{L^2}. \]
Thus it remains to show
\[ \| 1_{r \geq 1} T^n_k(h) \|_{L^q_t L^p_x} \lesssim \| h \|_{L^2} \quad (2.8) \]
with a bound independent of \( \nu \). For any \( R \gg 1 \), define
\[ S^n_R(h)(t, r) = \chi_R \left( \frac{r}{R} \right) \int e^{-ir^d} J_\nu(\rho \sigma) h(\rho) d\rho. \]

Then
\[ \| 1_{r \geq 1} T^n_k(h) \|_{L^q_t L^p_x} \lesssim \sum_{j \geq 5} 2^{j} \| S^n_R(h) \|_{L^q_t L^p_x}. \]

Then to prove (2.8), it suffices to show for some \( \delta > 0 \)
\[ R^{-d-1} \frac{d-2}{2} \| S^n_R(h) \|_{L^q_t L^p_x} \lesssim CR^{-\delta} \| h \|_{L^2}. \quad (2.9) \]
where \( C \) is independent of \( \nu \). By interpolation, we only need to show (2.9) for \( (q, p) = (2, p) \).

The difficulty in (2.9) is to obtain a uniform bound as \( \nu \to \infty \). We need to exploit the uniform properties of the Bessel function with respect to \( \nu \). By the uniform decay of the Bessel function presented in lemma 2.1, one can show
Lemma 2.4 (Lemma 2.4, [12]). Assume $a > 0$. For $\nu > 10$, $R \gtrsim 1$, $2 \leq p \leq \infty$
\[
\|S_{R}^{a\nu}(h)\|_{L^{2p}_{t}L^{p}_{x}} \lesssim \|h\|_{L^{2p}_{t}}, \quad \|S_{R}^{a\nu}(h)\|_{L_{x}^\infty L_{t}^{p}} \lesssim \|h\|_{L^{2p}_{t}} . \tag{2.10}
\]

This lemma gives the sharp estimates for the wave equation $a = 1$. For the Schrödinger case $a > 1$, there is some more oscillatory effect to exploit. To do so, in [12], $S_{R}^{a\nu}$ is decomposed into three operators $S_{R}^{a\nu}(h) = \sum_{j=1}^{3}S_{R,j}(h)$ where
\[
S_{R,j}(h) = \chi_{0}\left(\frac{r}{R}\right)\int e^{-ip\nu}j_{0}(rp)\gamma_{j}(\frac{rp - \nu}{\lambda})\chi_{0}(\rho)h(\rho)\,d\rho
\]
with $\gamma_{1}(x) = \eta(x)$, $\gamma_{2}(x) = (1 - \eta(x))1_{x < 0}$, and $\gamma_{3}(x) = (1 - \eta(x))1_{x > 0}$. By some uniform stationary phase analysis, the following lemma was proved in [12]

Lemma 2.5 (Lemma 2.5, [12]). Assume $R \gtrsim 1$, $\lambda \gtrsim 100R^{1/3}$, $2 \leq p \leq \infty$. Then
\[
\|S_{R,j}^{a\nu}(h)\|_{L^{2p}_{t}L^{p}_{x}} \lesssim \lambda^{1/4}R^{-1/4}\|h\|_{L^{2p}_{t}}, \\
\|S_{R,j}^{a\nu}(h)\|_{L_{x}^\infty L_{t}^{p}} \lesssim \left(\lambda^{-1/4}R^{1/4}2^{\beta} + R^{-1/2}\right)\|h\|_{L^{2p}_{t}}, \\
\|S_{R,j}^{a\nu}(h)\|_{L_{x}^\infty L_{t}^{p}} \lesssim \left(\lambda^{-1/4}R^{1/4}2^{\beta} + R^{-1/2}\right)\|h\|_{L^{2p}_{t}}.
\]

2.3. The improvement: non-endpoint

To prove theorem 1.1, we will refine the estimates for $S_{R,3}$ and $S_{R,2}$. We prove

Lemma 2.6. Assume $R \gtrsim 1$, $\lambda \gtrsim 100R^{1/3}$, $2 \leq p \leq \infty$. Then for any $K \in \mathbb{N}$
\[
\|S_{R,K}^{a\nu}(h)\|_{L_{x}^\infty L_{t}^{p}} \lesssim \left(R^{-1/4}2^{\beta} + (R^{-1/2}\lambda^{1/2}2^{K})\left(\lambda^{-1/4}R^{1/4}2^{\beta} + R^{-1/2}\right)\right)\|h\|_{L^{2p}_{t}}. \tag{2.11}
\]

Moreover, if $\nu \in \mathbb{Z}$ we have
\[
\|S_{R,K}^{a\nu}(h)\|_{L_{x}^\infty L_{t}^{p}} \lesssim \left(R^{-1/4}2^{\beta} + (R^{-1/2}\lambda^{1/2}2^{K})\left(\lambda^{-1/4}R^{1/4}2^{\beta} + R^{-1/2}\right)\right)\|h\|_{L^{2p}_{t}}. \tag{2.12}
\]

Proof. By interpolation, we only need to show the estimates for $p = 2, \infty$. By the support of $\gamma_{3}$ we have $rp > \nu + \lambda > \nu + \nu^{1/4}$ in the support of $\gamma_{3}(\frac{2\nu - \nu^{1/4}}{\lambda})$. Thus we use the lemma 2.3, and decompose
\[
S_{R,j}^{a\nu}(h) := M_{R,j}^{a\nu}(h) + E_{R,j}^{a\nu}(h)
\]
where
\[
M_{R,j}^{a\nu}(h) = \chi_{0}\left(\frac{r}{R}\right)\int e^{-ip\nu}e^{i\theta(rp)} + e^{-ip\nu}e^{i\theta(rp)}2^{-1/4}\gamma_{j}(\frac{rp - \nu}{\lambda})\chi_{0}(\rho)h(\rho)\,d\rho, \\
E_{R,j}^{a\nu}(h) = \chi_{0}\left(\frac{r}{R}\right)\int e^{-ip\nu}h_{0}(\nu, rp)\gamma_{j}(\frac{rp - \nu}{\lambda})\chi_{0}(\rho)h(\rho)\,d\rho
\]
with $\theta(r), h_{0}(\nu, r)$ given in lemma 2.3.
Step 1. The estimate for $M_{R,3}$. We only estimate
\[ M_{R,3}(h) = \chi(\frac{r}{R}) \int e^{-i(\rho \varphi + \rho \varphi \gamma \rho \varphi)} \left( \frac{r \rho - \nu}{\lambda} \right) \chi_0(\rho) h(\rho) d\rho, \]
since the other term is similar. It is easy to see
\[ \int \chi \rho_\nu \gamma \rho_\nu \lambda \chi \rho \rho \rho = -\nu \rho \theta \rho \rho \theta \rho \rho = -\frac{1}{2} \]
by Plancherel’s equality in $t$. It suffices to show
\[ ||M_{R,3}(h)||_{L^1_t} \lesssim ||h||_2. \] Let $\gamma(x) = \chi(x) \cdot 1_{x>0}$. We decompose further $M_{R,3}(h) = \sum_{k: R^{2/3} \leq 2^k} M_{R,3,k}(h)$ where
\[ M_{R,3,k}(h) = \chi(\frac{r}{R}) \int e^{-i(\rho \varphi + \rho \varphi \gamma \rho \varphi)} \left( \frac{r \rho - \nu}{2^k} \right) \chi_0(\rho) h(\rho) d\rho. \] It suffices to prove
\[ ||M_{R,3,k}(h)||_{L^1_t} \lesssim 2^{k/4} R^{-3/8} ||h||_2. \] By $TT^*$ argument, it suffices to prove
\[ ||M_{R,3,k}(M_{R,3,k})^*(f)||_{L^1_t} \lesssim 2^{k/4} R^{-3/4} ||f||_{L^1_t}. \] The kernel for $M_{R,3,k}(M_{R,3,k})^*$ is
\[ K(t-t',r,r') = \int e^{-i(t-t')\rho \varphi - \rho \varphi + \rho \varphi \gamma \rho \varphi} \left( \frac{r \rho - \nu}{2^k} \right) \chi_0(\rho) \gamma \left( \frac{r' \rho - \nu}{2^k} \right) \chi_0(\rho) h(\rho) d\rho. \] Obviously, we have a trivial bound
\[ |K| \lesssim 2^{-k/2} R^{-1/2}. \] Recall $\theta(r) = (r^2 - \nu^2)^{1/2} - \nu \arccos \frac{r}{\nu}$, then direct computation shows
\begin{align*}
\theta'(r) &= (r^2 - \nu^2)^{1/2} r^{-1}, \\
\theta''(r) &= (r^2 - \nu^2)^{-1/2} - (r^2 - \nu^2)^{1/2} r^{-2} = (r^2 - \nu^2)^{-1/2} \nu^2 r^{-2}, \\
\theta'''(r) &= (r^2 - \nu^2)^{-3/2} \nu^2 r^{-3} - 3 \frac{2 \nu^2}{r^2}. 
\end{align*} Denoting $G = \frac{\chi_0(\rho) \gamma(\frac{r \rho - \nu}{2^k}) \gamma(\frac{r' \rho - \nu}{2^k}) \chi_0(\rho) \gamma(\frac{r' \rho - \nu}{2^k}) \chi_0(\rho)}{(r \rho - \nu^2)^2 (r' \rho - \nu^2)^2} \chi_0(\rho)$, $\phi_2 = t \rho^\alpha - \theta(r \rho) + \theta(\rho')$. Then
\[ \partial_t \rho_1 = \rho(t) - \theta (r(t)) + \theta(r(t)) r' = 2t - \frac{\rho (r^2 - \rho^2)}{\sqrt{r^2 - \rho^2}} \]

\[ \partial_t^2 \rho_1 = a(a - 1) t \rho^2 - \theta (r(t)) - \theta(r(t)) r' + \frac{r^2 \rho^2 - \nu^2 + \sqrt{r^2 - \rho^2}^2}{\nu^2} \]

\[ \partial_t^3 \rho_2 = -\theta (r(t)) r^3 + \theta(r(t)) r' - \frac{r^2 \rho^2 - \nu^2 + \sqrt{r^2 - \rho^2}^2}{\nu^2} \]

We divide the proof into two cases.

**Case 1.** \( R \lesssim \nu \)

The key observation here is that if \( |\partial_t \phi_2| \ll |t| \), then \( |\partial_t^2 \phi_2| \gtrsim |t| R^{-k} \) on the support of \( G \).

Indeed, if \( |\partial_t \phi_2| \ll |t| \), then \( t \) and \( \sqrt{r^2 - \rho^2 + \sqrt{r^2 - \rho^2}^2} \) have the same size and comparable size. This observation is not true if \( a < 1 \), in which case \( \partial_t \phi_2 \) and \( \partial_t^2 \phi_2 \) can both be small.

Note that on the support of \( G \), one has

\[ \frac{|r^2 - \rho^2|}{\sqrt{r^2 - \rho^2}^2 - \nu^2 + \sqrt{r^2 - \rho^2}^2} \leq 2^{1/2} R^{1/2}. \]

If \( |t| \lesssim 2^{1/2} R^{1/2} \), we divide \( K \)

\[ K = \int e^{-i \phi G} \eta_0 \left( \frac{100 \partial \phi_2}{t} \right) \rho \| \right) + \int e^{-i \phi G} \left( 1 - \eta_0 \left( \frac{100 \partial \phi_2}{t} \right) \right) \rho \| \right) \]

By the van der Corput lemma (proposition 2, section 1 of chapter VIII in [37]), we obtain

\[ |I| \lesssim |t|^{-1/2} \left( \int |\partial_t G| \rho \left( \frac{100 \partial \phi_2}{t} \right) \right) \rho \| \right) + \int |G| \rho \left( \frac{100 \partial \phi_2}{t} \right) \left( \frac{100 \partial \phi_2}{t} \right) \rho \| \right) \]

where for the first term, we estimate it as \( K_3 \), while for the second term, we only need to observe that \( \eta_0 \left( \frac{100 \partial \phi_2}{t} \right) \) has fixed sign depending only on \( t \).

For \( L_2 \), without loss of generality, we assume \( r^2 - r^2 > 0 \). Then integrating by parts, we get

\[ |L_2| \lesssim \int \left| \partial_t \frac{100 \partial \phi_2}{t} \right| G \left( 1 - \eta_0 \left( \frac{100 \partial \phi_2}{t} \right) \right) \rho \| \right) \]

\[ \lesssim \int \left| \partial_t^2 \frac{100 \partial \phi_2}{t} \right| G \left( 1 - \eta_0 \left( \frac{100 \partial \phi_2}{t} \right) \right) \rho \| \right) \]

\[ + |t|^{-1} \left( \int |\partial_t G| \rho \left( \frac{100 \partial \phi_2}{t} \right) \left( \frac{100 \partial \phi_2}{t} \right) \rho \| \right) \]

\[ \lesssim \int \left| \frac{100 \partial \phi_2}{t} \right| G \left( 1 - \eta_0 \left( \frac{100 \partial \phi_2}{t} \right) \right) \rho \| \right) \]

\[ \lesssim |t|^{-1} 2^{1/2} R^{-1/2} \]

1677
where we used the fact that $\partial^2_{\rho^2}\phi_2$ changes the sign at most once.

If $|t| \gg 2^{4/3} R^{1/2}$, we have $|\partial_\rho(\phi_2)| \sim |t|$. Thus integrating by parts, we get

$$|K| \lesssim \int |\partial_\rho(\phi_2)^{-1}\partial_\rho((\partial_\rho\phi_2)^{-1}G)|\,d\rho$$
$$\lesssim \int |(\partial_\rho\phi_2)^{-3}\partial_\rho_3\phi_2 G|\,d\rho + \int |(\partial_\rho\phi_2)^{-2}\partial_\rho^2 G|\,d\rho$$
$$+ \int |(\partial_\rho\phi_2)^{-3}\partial_\rho^2\phi_2\partial_\rho G|\,d\rho + \int |(\partial_\rho\phi_2)^{-4}(\partial_\rho^2\phi_2)^2 G|\,d\rho$$
$$:= I_1 + I_2 + I_3 + I_4.$$  

As for $I_2$, we can obtain

$$I_2 + I_3 + I_4 \lesssim |t|^{-2} 2^{-4/3} R^{-1/2} R^2 2^{-2k} \lesssim |t|^{-2} 2^{-8k/3} R^{3/2}.$$  

For $I_1$, we have

$$I_1 \lesssim |t|^{-3} \lambda^{-1/2} R^{-1/2} \int (-\theta^m(r\rho)^2 - \theta^m(r'\rho)^2)\gamma^2(\frac{r\rho}{\lambda} - \nu)\gamma^2(\frac{r'\rho}{\lambda} - \nu)\,d\rho$$
$$\lesssim |t|^{-3} \lambda^{-1/2} R^{-1/2} \sup_{\rho, r > \rho, r' > \rho} \theta^m(r\rho)^2 \lesssim |t|^{-3} 2^{-1/3} R.$$  

Thus, eventually we get

$$|K| \lesssim |t|^{-1/2} R^{-1/2} |t|^{1/2} + (|t|^{-2} 2^{-8k/3} R^{3/2} + |t|^{-3} 2^{-1/3} R)\,d\rho \lesssim |t|^{-3} 2^{-1/3} R^{1/3}$$

which implies $\|K\|_{L^\infty_t L^\infty_x} \lesssim 2^{1/4} R^{-3/4}$ as desired, if $2k \gtrsim R^{1/3}$.

**Case 2.** $R \gg \nu$.

In this case, we may assume $2^k \sim R$. We also observe that if $|\partial_\rho(\phi_2)| \ll |t|$, then $|\partial^2_{\rho^2}\phi_2| \gtrsim |t|$ on the support of $G$. The rest of the proof is the same as case 1.

**Step 2.** The estimate for $E'_{R,3}$.  

First, we have for any $f \in L^2$

$$\|E'_{R,3}(f)\|_{L^2_t L^\infty_x} \lesssim \|S'_{R,3}(f)\|_{L^2_t L^\infty_x} + \|M'_{R,3}(f)\|_{L^2_t L^\infty_x} \lesssim \|f\|_{L^2}.$$  

On the other hand, using the decay estimate of $h(\nu, r)$, we get

$$\|E'_{R,3}(f)\|_{L^2_t L^\infty_x} \lesssim (\lambda^{-5/4} R^{1/4} + R^{-1/2})\|f\|_{L^2}.$$  

(2.16)

In the case $d = 2, \nu \in \mathbb{Z}$, thus by the better decay of $h(\nu, r)$ given by Lemma 2.3, part (2), we can get

$$\|E'_{R,3}(f)\|_{L^2_t L^\infty_x} \lesssim (\lambda^{-5/4} R^{1/4} + R^{-1})\|f\|_{L^2}.$$  

(2.17)

Thus, the lemma with $K = 0$ is proved by interpolation.

To show the case $K \gg 1$, we need to analyze $E'_{R,3}$ more carefully. Using the expansion in lemma 2.3, we can divide

$$E'_{R,3}(f) = \sum_{k=1}^K E'_{R,3,k}(f) + E'_{R,3,k}$$
where $\tilde{E}_{R,3,k}$ is the term $E_{R,3,k}$ with $h$ replaced by $\tilde{h}$. Arguing as before, we see $E_{R,3,k}^\nu$ has the same bound as $M_{R,3}^\nu$. For $\tilde{E}_{R,3,k}$, we can obtain the bound similarly as $E_{R,3}^\nu$. This completes the proof.

For the case $d = 2$, we also need to refine the estimate for $S_{R,2}^\nu$. By lemma 2.2 and Sobolev embedding we get

**Lemma 2.7.** Let $\nu \in \mathbb{Z}$, $R \geq 1$, $2 \leq p \leq \infty$. If $\lambda \geq 100R^{2+\epsilon}$ for some $\epsilon > 0$, then for any $N > 0$ there exists $C_{N,\epsilon}$ such that

$$\|S_{R,2}^\nu(h)\|_{L^p_t L^q_x} \leq C_{N,\epsilon} R^{-N\epsilon} \|h\|_{L^2_x}.$$  

Now we are ready to prove theorem 1.1 for the non-endpoint range, namely assuming

$$2 \leq q, p \leq \infty, \quad \frac{1}{q} < (d - \frac{1}{2}) \left(1 - \frac{1}{p}\right) \text{ or } (q, p) = (\infty, 2).$$

(2.19)

It suffices to show (2.9). First, we consider $d \geq 3$. From lemmas 2.5 and 2.6 by taking $\lambda = R^{2+\epsilon}$ and $K = 0$, we get

$$R^{\frac{d-1}{p}} \frac{d-2}{2} \|S_{R,2}^{\nu_0}(h)\|_{L^p_t L^q_x} \lesssim R^{\frac{d-1}{p}} \frac{d-2}{2} \|h\|_{L^2_x} \lesssim R^{1-d} \|h\|_{L^2_x}$$

for some $\delta > 0$ if $\frac{4d-2}{2d-3} < p < \frac{2d}{d-1}$. For $d = 2$, we can prove theorem 1.1 similarly by using lemmas 2.5–2.7 by taking $\lambda = R^{13+\epsilon}$ for $\epsilon > 0$ sufficiently small and $K$ sufficiently large.

**2.4. The improvement: endpoint**

It remains to prove theorem 1.1 for $(q, p)$ lying on the endpoint line for $d \geq 3$, namely $\frac{1}{q} = (d - \frac{1}{2}) \left(1 - \frac{1}{p}\right)$. From the proof in the previous section, assuming $\lambda > R^{2+\epsilon}$ we know the logarithmic difficulty only appears in the summation $\sum_{R} \tilde{M}_{R,3}^\nu(h)$ (this is not true for $d = 2$). We will exploit some orthogonality to overcome this logarithmic difficulty. A similar technique was also used in [5] and [18]. It suffices to show

$$\left\| \sum_{R} r^{\frac{d-1}{p}} \frac{d-2}{2} \tilde{M}_{R,3}^\nu(h) \right\|_{L^q_t L^2_x} \lesssim \|h\|_{L^2_x}$$

(2.20)

with a uniform bound with respect to $\nu$. By TT’ argument, we see that (2.20) is equivalent to

$$\left\| \sum_{R, r} r^{\frac{d-1}{p}} \frac{d-2}{2} \tilde{M}_{R,3}^\nu(M_{R,3}^\nu)(r \frac{d-1}{p} \frac{d-2}{2} g) \right\|_{L^q_t L^2_x} \lesssim \|g\|_{L^q_t L^2_x}.$$  

(2.21)

The key ingredient to prove (2.21) is the following observation:

**Lemma 2.8.** Assume $d \geq 2$, $R \gg R' > 1$ and $(q, p)$ satisfies $2 \leq q, p \leq \infty$, $\frac{1}{q} = (d - \frac{1}{2}) \left(1 - \frac{1}{p}\right)$, $(q, p) = (2, \frac{4d-2}{2d-1})$. Then $\exists \epsilon > 0$ such that

$$\|r^{\frac{d-1}{p}} \frac{d-2}{2} \tilde{M}_{R,3}^\nu(M_{R,3}^\nu)(r \frac{d-1}{p} \frac{d-2}{2} g)\|_{L^q_t L^2_x} \lesssim (R'/R)^\epsilon \|g\|_{L^q_t L^2_x}.$$  

(2.22)

**Proof.** By interpolation, it suffices to show lemma 2.8 for $q = p = q_0 = \frac{4d+2}{2d-1}$. We may assume $R \gg R' \geq \nu$. Then we decompose
where \( \tilde{M}_{R,k}^\nu \) is given by (2.13). We can write
\[
M_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* g = \sum_{k: R^{k+1} < 2^k \leq R} M_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* g
\]
where
\[
\tilde{M}_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* g = \int \tilde{K}(t-t',r,r') g(t',r') dr' dr'.
\]
By the stationary phase method as for \( \tilde{K} \) in the proof of lemma 2.6, we obtain
\[
|\tilde{K}| \lesssim (\frac{R}{R^2/2^k})^{-1/2} R^{-1/2} R^{-1/4} 2^{-k/4} \lesssim R^{-1} R^{-1/2}.
\]
Thus we get
\[
\| r^{1/2} \tilde{M}_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* (r^{1/2} \frac{d}{dr} g) \|_{L^2_t L^2_x} \lesssim (RR')^{\frac{d-2}{2}} R^{-1/2} \|g\|_{L^2_t L^2_x}.
\]
Interpolating with the following bound
\[
\| r^{1/2} \tilde{M}_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* (r^{1/2} \frac{d}{dr} g) \|_{L^2_t L^2_x} \lesssim \frac{1}{4} \| (\tilde{M}_{R,k}^\nu)^* (r^{1/2} \frac{d}{dr} g) \|_{L^2_t L^2_x} \lesssim \frac{1}{4} R^{-1/2} R^{1/4} R^{-1/4} \|g\|_{L^2_t L^2_x},
\]
we obtain
\[
\| r^{\frac{d-1}{2}} \tilde{M}_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* (r^{\frac{d-1}{2}} \frac{d}{dr} g) \|_{L^2_t L^2_x} \lesssim \frac{1}{4} (RR')^{\frac{d-2}{2}} R^{-1/2} R^{1/4} R^{-1/4} \|g\|_{L^2_t L^2_x}.
\]
Therefore, summing over \( k \) we complete the proof of lemma 2.8.}

Now we prove (2.21). We have
\[
\sum_{R, R'} r^{\frac{d-1}{2}} \tilde{M}_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* (r^{\frac{d-1}{2}} \frac{d}{dr} \frac{d}{dr} g) \|_{L^2_t L^2_x} \lesssim \left( \sum_{R} \left( \sum_{R'} r^{\frac{d-1}{2}} \tilde{M}_{R,k}^\nu (\tilde{M}_{R,k}^\nu)^* (r^{\frac{d-1}{2}} \frac{d}{dr} \frac{d}{dr} \chi(r/R) g) \right)^2 \right)^{1/2} \lesssim \| \chi(r/R) g \|_{L^2_t L^2_x}^{1/2}.
\]
where we used lemma (2.8) in the second inequality.

Interpolating theorem 1.1 with the classical Strichartz estimates we get
Corollary 2.9. Assume $a > 1, d \geq 2$, $\frac{2d}{d - 2} > p > \frac{4d - 2}{2d - 3}$. Let $\beta(p) = \frac{2(p - 1)}{(4 - p)d + 2p - 2}$. Then
\[ \|e^{i\xi D} P_0 f \|_{L^2_t L^q_x} \lesssim \|f\|_{L^2_x} \]
where $a - \varepsilon$ denotes $a - \varepsilon$ for any $\varepsilon > 0$.

2.5. Counter-example

Finally, we use the Knapp example to obtain some necessary conditions for the Strichartz estimates with mixed angular-radius integrability, namely
\[ \rho \omega_{\beta} \lesssim L^p f(\rho) + tD LL L^0 \]
where $a$ denotes $a - \varepsilon$ for any $\varepsilon > 0$.

Corollary 2.10. Assume $a > 10$ and (2.23) holds. Then $2 \leq \frac{d - 1}{p} + \frac{d - 1}{s}$. As a consequence, $\beta(p)$ in corollary 2.9 is sharp.

Proof. Take
\[ D = \{ \xi = (\xi, \xi') \in \mathbb{R}^d : |\xi - 1| \leq \delta, |\xi'| \leq \delta \}. \]
Let $\hat{f} = 1_D(\xi)$. Then $\|f\|_{L^2_x} \sim \delta^{d/2}$, and
\[ \int_{\mathbb{R}^d} e^{i\xi \xi} f(\xi + \eta_0(\xi)) f(\xi) d\xi = e^{i(t + x) \xi} \int_{\mathbb{R}^d} e^{i\xi(\xi' - \xi') - \xi'} e^{i\xi(\xi' - 1)} e^{i\xi(\xi' + x)} e^{i\xi(\xi' - 1)} e^{i\xi(\xi' - x)} d\xi'. \]
Since in $D$ we have
\[ ||\xi - \xi'|| \leq ||\xi'|| \lesssim \delta^2, \quad |\xi| - 1 - a(\xi - 1) - 1 \lesssim |\xi - 1|^2 \lesssim \delta^2, \]
then for $(t, x) \in E = \{ |t| \lesssim \delta^{-2}, |ta + x| \lesssim \delta^{-1}, |x| \lesssim \delta^{-1} \}$, we have
\[ \left| \int_{\mathbb{R}^d} e^{i\xi(\xi' - \xi') \eta_0(\xi)} f(\xi) d\xi \right| \sim |D| \sim \delta^d. \]
By simple geometric observation we see that
\[ E \supseteq E' = \{ |t| \sim \delta^{-2}, \rho \in (\delta^{-2}, \delta^{-1} + \delta^{-1}), |\theta| < \delta \} \]
where $\theta$ is the central angle. Therefore, (2.23) implies
\[ \delta^d \delta^{d - 1/2} \delta^{-2} \frac{d - 1}{p} \delta^{-1/2} \delta^{-2} \lesssim \delta^{d/2}, \]
which implies immediately that $2 \leq \frac{d}{2} - \frac{2d - 1}{p} + \frac{d - 1}{s}$ by taking $\delta \ll 1$. 

Finally, as in [12] we use the Christ–Kiselev lemma ([6]) to get the inhomogeneous linear estimates.

Corollary 2.11. Assume $a > 1, (q, p), (\bar{q}, \bar{p})$ both satisfy
\[ 2 \leq q, p \leq \infty, \quad \frac{1}{q} < \left( d - \frac{1}{2} \right) \left( \frac{1}{2} - \frac{1}{p} \right). \]
and \( q > q' \). Then
\[
\left\| \int_0^t e^{i(t-s)\mathbf{D}^\alpha}\mathbf{P}\mathbf{f}(s) \right\|_{L^q_x L^{\tilde{q}}_t L^{\tilde{q}}_z} \lesssim \|\mathbf{f}\|_{L^q_x L^{\tilde{q}}_t L^{\tilde{q}}_z}.
\]

3. Applications to 3D Zakharov system

This section is devoted to proving theorem 1.2. We follow the ideas in [13] and [12]. The new difficulty is to handle the fractional derivatives on the sphere in accordance with the Fourier multiplier; we will transfer it to \( SO(3) \). After normal form reduction (see [13]), the Zakharov system (1.6) is equivalent to
\[
(i\partial_t + D^2)(u - \Omega(N, u)) = (Nu)_{H^1+H^\alpha, L^2} + \Omega(D|u|^2, u) + \Omega(N, Nu),
\]
\[
(i\partial_t + \alpha D)(N - D\tilde{\Omega}(u, u)) = D|u|^2_{H^1+H^\alpha, L^2} + D\tilde{\Omega}(Nu, u) + D\tilde{\Omega}(N, u) \tag{3.1}
\]
where \( N = n - iD^{-1}\dot{n}/\alpha \), \( \Omega, \tilde{\Omega} \) are bilinear Fourier multiplier operators with symbols \( \omega, \tilde{\omega} \) respectively, where
\[
\omega(\xi, \eta) = \sum_{k\in\mathbb{Z}} \sum_{\log \alpha > 1} |\chi_k(\xi)\chi_{\xi k - q}(\eta)|^2, \\
\tilde{\omega}(\xi, \eta) = \sum_{k\in\mathbb{Z}} \sum_{\log \alpha > 1} |\chi_k(\xi)\chi_{\xi k - q}(\eta) + \chi_0(\eta)\chi_{\xi k - q}(\xi)|^2.
\]
Here the bilinear Fourier multiplier operator with symbol \( m \) on \( \mathbb{R}^6 \) is the bilinear operator \( T_m \) defined by
\[
T_m(f, g)(\alpha) = \int_{\mathbb{R}^6} m(\xi, \eta)\tilde{F}(\xi, \eta)\tilde{G}(\xi, \eta)e^{i\alpha\cdot\xi}d\eta.
\]
Following [12, 13], for \( u \) and \( N \), we use the Strichartz norms with angular regularity:
\[
u \in X = (D)^{-1}(L^q_x H^{0,3}_\omega \cap L^{q}_{\eta} B^{1/4+\varepsilon, \omega}_{p, q, \eta} \cap L^{q}_{\eta} B^{0,3, \omega}_{p, q, \eta}), \tag{3.2}
\]
\[
N \in Y = L^q_x H^{0,3}_\omega \cap L^{q}_{\eta} B^{-1/4-\varepsilon, \omega}_{p, q, \eta}, \tag{3.3}
\]
for fixed \( 0 < \varepsilon \ll 1 \), where
\[
\frac{1}{q(\varepsilon)} = \frac{1}{4} + \frac{\varepsilon}{3}, \quad \frac{1}{\gamma(\varepsilon)} = \frac{3}{8} + \frac{5\varepsilon}{6}. \tag{3.4}
\]
The condition \( 0 < \varepsilon \ll 1 \) ensures that
\[
\frac{10}{3} < q(\varepsilon) < 4 < q(-\varepsilon) < \infty, \tag{3.5}
\]
such that the norms in (3.2) and (3.3) satisfy the condition in theorem 1.1.

Lemma 3.1. Let \( 1 \leq p_1, p_2 \leq \infty \) and \( 1/p = 1/p_1 + 1/p_2 \). Assume \( m(\xi, \eta) \) is a symbol on \( \mathbb{R}^6 \), \( m(A\xi, A\eta) = m(\xi, \eta) \) for any \( A \in \text{SO}(3) \), \( m \) is bounded and satisfies for all \( \alpha, \beta, \theta_1, \theta_2 \in \mathbb{R} \)
\[
|\partial_x^\alpha \partial_\eta^\beta m(\xi, \eta)| \leq C_{\alpha, \beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|}, \quad \xi, \eta \neq 0.
\]
Then for \( q > 1 \), \( q_1, q_2 \in (1, \infty) \), \( q_1, q_2 \in (1, \infty) \), and \( 1/q = 1/q_1 + 1/q_2 = 1/q_1 + 1/q_2 \).

1682
\[ \|T_m(P_k f, P_k g)\|_{C^3} \lesssim \|f\|_{C^2} \|g\|_{C^2} + \|f\|_{C^2} \|g\|_{C^2} \]

holds for any \( k_1, k_2 \in \mathbb{Z} \), with a uniform constant \( C \).

**Proof.** We can write

\[ T_m(P_k f, P_k g)(x) = \int K(x - y, x - y') f(y)g(y')dy' \]

where the kernel is given by \( K(x, y) = \int m(\xi, \eta) \chi(k\xi) \chi(k\eta) e^{i\xi z + i\eta y} d\xi d\eta \). From the assumption on \( m \), and integration by parts, we get a pointwise bound of the kernel:

\[ |K(x, y)| \lesssim 2^{k(k + 1 + |2^{k(x + y')}| + 2^{k(k + 1 + |2^{k(y)|})} \cdot \quad (3.6) \]

Since \( m(A, \eta) = m(\xi, \eta) \), then \( K(A, Ay) = K(x, y) \) for any \( A \in SO(3) \). Then we have

\[ \|T_m(P_k f, P_k g)\|_{C^3} \lesssim \|T_m(P_k f, P_k g)\|_{C^3} + \|D^0_m T_m(P_k f, P_k g)\|_{C^3} \]

\[ = I + II. \]

We only consider the term \( II \) since term \( I \) can be handled in an easier way. By the fractional derivative on \( SO(3) \) (see the appendix) we get

\[ II = \|A(D^0_m T_m(P_k f, P_k g))\|_{L^3} \leq \|D^0_m A(T_m(P_k f, P_k g))\|_{L^3} \]

\[ = \left\| \int K(x - y, x - y') D^0_m[f(Ay)g(Ay')]dy' \right\|_{L^3} \]

\[ \lesssim \|f\|_{C^2} \|g\|_{C^2} + \|f\|_{C^2} \|g\|_{C^2} \]

where we used Lemma A.1 in the last step. \( \square \)

Now we follow the proof with slight modifications in [13] to prove theorem 1.2. It suffices to prove the nonlinear estimates. Fix \( s > 3/4 \). The following two lemmas can be proved similarly as lemma 3.2–3.3 in [13]. The main difference is that we use lemma 3.1 for every bilinear dyadic piece.

**Lemma 3.2 (Bilinear terms I).**

1. For any \( N \) and \( u \), we have

\[ \|\langle N u \rangle u\|_{L^3} \lesssim \|\langle N \rangle u\|_{L^3} \|\langle D \rangle u\|_{L^3} \]

2. If \( 0 < \theta < 1, \frac{1}{q} = \frac{1}{2} - \frac{\theta}{2} + \frac{\epsilon}{3} \), then for any \( N \) and \( u \)

\[ \|\langle N u \rangle u\|_{L^3} \lesssim \|\langle N \rangle u\|_{L^3} \|\langle D \rangle u\|_{L^3} \]

**Lemma 3.3 (Bilinear terms II).**

1. For any \( u \), we have

\[ \|\langle u \rangle u\|_{L^3} \lesssim \|u\|_{L^3} \|\langle D \rangle u\|_{L^3} \]

2. If \( 0 < \theta < 1, \frac{1}{q} = \frac{1}{2} - \frac{\theta}{2} + \frac{\epsilon}{3} \), then
\[ \|D(u\tilde{u})\|_{L^2([a,b])} \lesssim \|D(u)\|_{L^2_t[H^{s+\frac{3}{2}}_{\omega,\omega}]} \]  

**Remark 1.** In application, we will use lemma 3.2 (1) and lemma 3.3 (2) by fixing \(0 < b_0 \ll 1\) such that by this choice \((\tilde{q}, \tilde{r})\) is admissible to apply corollary 2.9.

For the boundary terms and cubic terms, we can use lemma 3.1 to prove the estimates similarly as lemmas 3.5 and 3.7 obtained in [13].

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**Appendix. Sobolev spaces on \(SO(3)\)**

In the appendix, we collect some facts about the Sobolev spaces on \(SO(3)\). It is well-known that \(G = SO(3)\) is a compact Lie group, with a Haar measure \(\mu\). Let \(X_1(t), X_2(t), X_3(t)\) denote the subgroups with \(x, y, z\) as the axes of rotations, namely

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos t & -\sin t \\
0 & \sin t & \cos t
\end{pmatrix} \begin{pmatrix}
\cos t & 0 & -\sin t \\
0 & 1 & 0 \\
\sin t & 0 & \cos t
\end{pmatrix} \begin{pmatrix}
\cos t & -\sin t & 0 \\
-\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad t \in \mathbb{R}.
\]

Let \(X_j\) be the vector fields induced by \(X_j(t), j = 1, 2, 3\), namely for \(f \in C^\infty(G)\)

\[ X_j(f) = \frac{d}{dt} \bigg|_{t=0} f(AX_j(t)). \]

Define the Laplacian operator \(\Delta\) on \(G\) by \(\Delta = \sum_j X_j^2\) and \(\Delta_h = (-\Delta)^{1/2}\). Then we define the Sobolev space \(H^s(G)\) by the norm \(\|f\|_{H^s(G)} = \|f\|_{L^2(G)} + \|\Delta_h f\|_{L^2(G)}\). \(\Delta_\omega\) denotes the Laplace–Beltrami operator on the unit sphere \(S^2\) endowed with the standard metric \(g\) and measure \(\omega\) and \(D_\omega = (-\Delta_\omega)^{1/2}\). Let \(\tilde{X}_1 = x_2 \partial_{x_3} - x_3 \partial_{x_2}, \tilde{X}_2 = x_3 \partial_{x_1} - x_1 \partial_{x_3}, \tilde{X}_3 = x_1 \partial_{x_2} - x_2 \partial_{x_1}\). It is well-known that for \(f \in C^2(\mathbb{R}^3)\)

\[ \Delta_\omega(f)(x) = \sum_{1 \leq j \leq 3} \tilde{X}_j^2(f)(x). \]

For \(f \in C^2(\mathbb{R}^3), A \in SO(3)\), define the action \(A(f) = f(Ax)\). It is easy to see that \(X_j[f(Ax)] = A\tilde{X}_j(f)(A^{-1}x)\), and hence \((-\Delta)^{y}[f(Ax)] = A((-\Delta_\omega)^{y}f)\). Moreover, we have \(\int_{S^2} f(x) \, d\omega = \int_{G} f(Ax) \, d\mu\). Therefore, we can transfer freely the fractional derivatives between \(S^2\) and \(SO(3)\). We will use the fractional Leibniz rule on \(SO(3)\) which was proved in [7].

**Lemma A.1 (Theorem 4, [7]).** Let \(\alpha \geq 0, p_1, p_2 \in (1, \infty)\) and \(r, p_2, q_1 \in (1, \infty)\) such that \(1/r = 1/p_1 + 1/q_1, i = 1, 2\). Then

\[ \|D^s_\omega(fg)\|_{L^r(G)} \lesssim \|f\|_{p_1} \|D^s_\omega g\|_{q_1} + \|g\|_{p_2} \|D^s_\omega f\|_{p_2}. \]
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