A SUFFICIENT CONDITION FOR NILPOTENCY OF THE COMMUTATOR SUBGROUP

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Abstract: Let $G$ be a finite group with the property that if $a$ and $b$ are commutators of coprime orders, then $|ab| = |a||b|$. We show that $G'$ is nilpotent.

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The following criterion of nilpotency of a finite group was established by Baumslag and Wiegold in [1]:

**Theorem 1.** Let $G$ be a finite group in which $|ab| = |a||b|$ whenever $a$ and $b$ have coprime orders. Then $G$ is nilpotent.

Here $|x|$ stands for the order of an element $x$ in a group $G$. In this article we establish a similar criterion of nilpotency for the commutator subgroup $G'$. Recall that $g \in G$ is a commutator if $g = x^{-1}y^{-1}xy$ for suitable $x, y \in G$. By definition, the commutator subgroup $G'$ is the subgroup of $G$ generated by all commutators. The purpose of this article is to prove

**Theorem 2.** Let $G$ be a finite group in which $|ab| = |a||b|$ whenever $a$ and $b$ are commutators of coprime orders. Then $G'$ is nilpotent.

In view of the above theorem one might suspect that a similar phenomenon holds for other group-words. Recall that a group-word $w = w(x_1, \ldots, x_s)$ is a nontrivial element of the free group $F = F(x_1, \ldots, x_s)$ on free generators $x_1, \ldots, x_s$. A word is a commutator word if it belongs to the commutator subgroup $F'$. Given a group-word $w$, it can be viewed as function defined on every group $G$. The subgroup of $G$ generated by the $w$-values is called the verbal subgroup of $G$ corresponding to the word $w$. It is usually denoted by $w(G)$. The following question might be asked: Let $w$ be a commutator word and let $G$ be a finite group with the property that if $a$ and $b$ are $w$-values of coprime order, then $|ab| = |a||b|$. Is the verbal subgroup $w(G)$ nilpotent?

A similar question for noncommutator words would not be interesting since an easy counterexample is provided just by each nonabelian simple group $G$, say of exponent $e$, and the word $x^n$, where $n$ is a divisor of $e$ such that $e/n$ is prime. Even in the case of commutator words the answer to the question is negative: Kassabov and Nikolov showed in [2] that for every $n \geq 7$ the alternating group $A_n$ admits a commutator word whose every nontrivial value has order 3. We suspect that the answer to the question is positive in case of multilinear commutator words, i.e., the words having a form of a multilinear Lie monomial (for example, $[[x_1, x_2, x_3], [x_4, x_5]]$).

Throughout the sequel $G$ denotes a finite group satisfying the hypothesis of Theorem 2. We denote by $X$ the set of commutators in $G$. As usual, $\pi(K)$ stands for the set of primes dividing the order of a group $K$. The Fitting subgroup of $K$ is denoted by $F(K)$.

**Lemma 3.** Let $x \in X$ and let $N$ be a subgroup normalized by $x$. If $|(x)|, |N| = 1$, then $[x, N] = 1$.

**Proof.** Choose $y \in N$. The order of $[x, y]$ is prime to that of $x$. Therefore we must have $|x[x, y]| = |x|[x, y]$. However $x[x, y] = y^{-1}xy$. This is a conjugate of $x$ and so $|x[x, y]| = |x|$. Hence, $[x, y] = 1$. □

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Lemma 4. If $G$ is soluble, then $G'$ is nilpotent.

Proof. Arguing by induction on $|G|$, we can assume that the second commutator subgroup $G''$ is nilpotent. Suppose that there are two different primes $p \in \pi(G')$ and $q \in \pi(G'')$. Let $P$ be a Sylow $p$-subgroup of $G'$ and $Q$, a Sylow $q$-subgroup of $G''$. It is straightforward from the Focal Subgroup Theorem [3, Theorem 7.3.4] that $P$ is generated by $P \cap X$. By Lemma 3 $[Q, x] = 1$ if $x \in P \cap X$. Therefore $[Q, P] = 1$. It follows that $P$ is normal in $PG''$ and so $P \leq F(PG'')$. Since $PG''$ is normal in $G'$, we conclude that $P \leq F(G)$. This holds for any prime $p$ such that $G''$ is not a $p$-group. Therefore if $G''$ is divisible by at least two different primes, all Sylow subgroups of $G'$ belong to $F(G)$ and so $G'$ is nilpotent. If $G''$ is a $q$-group then the Sylow $q$-subgroup of $G'$ is obviously normal in $G$ and so again we conclude that all Sylow subgroups of $G'$ belong to $F(G)$. The proof is complete. □

Proof of Theorem 2. Suppose now that $G$ is a counterexample of minimal order. So all proper subgroups in $G$ are soluble and we can assume that $G = G'$. Let $R$ be the soluble radical in $G$. It follows that $G/R$ is nonabelian simple. By Lemma 4 $R'$ is nilpotent. Suppose that $G$ is nonsimple and $R \neq 1$ and, for a prime $q$, let $Q$ be the Sylow subgroup of $F(G)$. Let $T$ be the subgroup of $G$ generated by all commutators that are $q'$-elements. By the Focal Subgroup Theorem, all Sylow $p$-subgroups of $G'$ for $p \neq q$ are contained in $T$. Therefore the commutator subgroup of $G/T$ is a $q$-group. Since $G = G'$, we conclude that $G = T$. Combining Lemma 3 with the Focal Subgroup Theorem, we deduce now that $F(G) \leq Z(G)$.

Further, we remark that for every $x \in G$ the subgroup $\langle x, R \rangle$ is soluble and so $\langle x, R' \rangle$ is nilpotent. It follows that $[R, x] \leq F(G) = Z(G)$ and so $R = Z_2(G)$. In particular, $R$ is nilpotent and so $R = Z(G)$. Thus, $G$ is quasisimple.

Since $G$ does not possess a normal 2-complement, it follows from the Frobenius Theorem [3, Theorem 7.4.5] that $G$ contains a 2-subgroup $H$ and an element of odd order $b \in N_G(H)$ such that $[H, b] \neq 1$. By Thompson’s Theorem [3, Theorem 5.3.11] we can assume that $H$ is of nilpotency class at most 2 and $H/Z(H)$ is elementary abelian. We claim that $G$ contains an element $a$ such that $a$ is a 2-element, $a \in X$, and $a$ has order 2 modulo $Z(G)$.

Indeed, since $H$ is of class at most 2, all elements in $[H, g]$ are commutators for every $g \in H$. If for some $g \in H$ the subgroup $[H, g]$ does not lie in $Z(G)$, every element of $[H, g]$, having order 2 modulo $Z(G)$, enjoys the required properties. Therefore we assume that $[H, g]$ lies in $Z(G)$ for every $g \in H$. In particular $H' \leq Z(G)$. It follows that $[H, b] \cap C_G(b) \leq Z(G)$ and all elements in $[H, b]$ are commutators modulo $Z(G)$. If $d \in [H, b]$ such that $d \notin Z(G)$ and $d^2 \in Z(G)$, then $[d, b]$ is as required. This proves the existence of $a$ with the above properties.

Now, we fix such an element $a$. Since $G/Z(G)$ is nonabelian simple, it follows from the Baer–Suzuki Theorem [3, Theorem 3.8.2] that there exists $t \in G$ such that the order of $[a, t]$ is odd. On the one hand, it is clear that $a$ inverts $[a, t]$. On the other hand, by Lemma 3, $a$ must commute with $[a, t]$. This is a contradiction. □

References

1. Baumslag B. and Wiegold J., A Sufficient Condition for Nilpotency in a Finite Group. arXiv:1411.2877v1[math.GR].
2. Kassabov M. and Nikolov N., “Words with few values in finite simple groups,” Q. J. Math., 64, 1161–1166 (2013).
3. Gorenstein D., Finite Groups, Chelsea Publ. Co., New York (1980).

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