Boundary rod elements for modeling the railway at the bridge crossing under the action of the speed railway load

I I Ivanchenko
Russian University of Transport, 9, Obraztsov str., Moscow, 127994, Russian Federation
E-mail: ivaii011@mtu-net.ru

Abstract. New rod boundary elements on an elastic foundation (finite elements of large length, when approximating the displacements by a set of linear and trigonometric functions) are proposed for calculating the path oscillations outside the bridge and on the bridge, combined with a double-track beam bridge connected through a layer with a high-speed rolling stock movement. The selection of interlayer material is related to the dynamics of the bridge. To build the methodology, the articles published earlier by the author are used: a step-by-step procedure for solving the problems of unsteady dynamics of structures and the method of "nodal accelerations" to take into account the action on the structures of a moving inertial load.

Introduction
Issues of the bridges’ dynamics on high-speed railways (HSR) remain relevant [1-10]. When calculating the elements of the bridge passage, including the bridge structure and the upper track structure on the bridge and outside the bridge (Figure 1 (a-b)), the path is modeled by a beam on an elastic-viscous Winkler base, consisting of a set of consecutively located rod boundary elements, when they are marked by a symbol $e_h^*$ (Figure 2a), including, for bridges, specialized elements, their marking $\tilde{e}_h$ (Figure 2b), consisting of two rods connected by an elastic-viscous gasket, i.e. rail track, denoting it through $\tilde{e}_h$, and a double-track bridge beam. Only the vertical dynamics of the rolling stock is considered.

System development of differential equations and step procedure construction for its solution
To solve the problem in this statement, a technique is used to study the dynamics of spatial core systems presented in [6, 7]. A flat rail model is considered. A system of wagons is introduced as a moving load, at a speed of movement $V$, denoting them through $h^0=1,..,m_0$, where $m_0$ is the number of wagons in the train, and through - wagons that are currently on the rail track. Oscillations of the system of bar elements $\{e^*_h, \tilde{e}_h\}$ are considered from the perspective of the complex motion theory. The number of elements is denoted by $m$, and the number of nodes uniting the elements is shown by $n$. The operation of transposing matrices is denoted by the symbols $\left[ \right]'$, the points over the symbols are time differentiation. After numbering the elements and nodes, the node with a lower serial number is denoted by $i$, by $k$ - with the highest number for the elements $e^*_h$ and $\tilde{e}_h$ ($h=1,..,m$). The right ones are
introduced: basic -0,X,Y,Z, local \( \bar{O}_{ih}, \bar{x}_{ih}, \bar{y}_{ih}, \bar{z}_{ih} \) and moving \( \bar{O}_{ih}, \bar{x}_{ih}, \bar{y}_{ih}, \bar{z}_{ih} \) coordinate systems. Local coordinate systems for \( \{e^h\}, \{\tilde{e}^h\} \) are oriented so that for each undeformed element, the axis \( \bar{O}_{ih}, \bar{y}_{ih} \) is directed from node \( \tilde{i} \) to node \( k \). The origin of the moving coordinate system \( \bar{O}_{ih} \) is connected with the node \( \tilde{i} \) and, in addition, the plane \( \bar{O}_{ih}, \bar{y}_{ih}, \bar{z}_{ih} \) is torsionally connected with the cross section of the rod in the node \( \tilde{i} \). An axis \( \bar{O}_{ih}, \bar{y}_{ih} \) at each moment of time will be considered passing through a node \( k \) for a given element. We will denote the system \( \{e^h\}, \{\tilde{e}^h\} \) further by \( \{e^h\} \).

Introduced into consideration are column vectors of displacements and forces with components in local coordinate systems assigned to undeformed elements \( e_h \), \( h=1,...,m \):

\[
\bar{q} = \begin{bmatrix} q_1, ..., q^h, ..., q_m \end{bmatrix} \quad \text{is a vector} (N_m \text{ order}) \quad \text{of absolute nodal displacements for} \quad \{e_h\};
\]

\[
\bar{f} = \begin{bmatrix} f_1, ..., f^h, ..., f_m \end{bmatrix} \quad \text{is the vector of nodal forces in all} \quad e_h \quad \text{after the partition of the system} \quad \{e_h\};
\]

\[
\bar{p} = \begin{bmatrix} p^1, ..., p^h, ..., p^m \end{bmatrix} \quad \text{is the vector of external forces and moments acting on} \quad e_h; \quad q^h = \begin{bmatrix} q^h_1, q^h_k \end{bmatrix} \quad \text{is the displacement vector of the nodes} \quad \tilde{i}, k \quad \text{of the element} \quad e_h \quad \text{in absolute motion}; \quad f^h = \begin{bmatrix} f^h_1, f^h_k \end{bmatrix} \quad \text{is the vector of displacements of the nodes} \quad \tilde{i}, k \quad \text{of the element} \quad e_h \quad \text{in relative motion}; \quad f^h = \begin{bmatrix} f^h_1, f^h_k \end{bmatrix} \quad \text{is the vector of displacements of the nodes} \quad \tilde{i}, k \quad \text{of the element} \quad e_h \quad \text{in the figurative motion}; \quad q^h = \begin{bmatrix} q^h_1, q^h_k \end{bmatrix} \quad \text{is the vector of the forces acting at the nodes} \quad \tilde{i}, k \quad \text{of the element} \quad e_h \quad \text{after the dismemberment of the system}; \quad q^h = \begin{bmatrix} q^h_1, q^h_k \end{bmatrix} \quad \text{is the vector of generalized portable displacements} \quad e_h; \quad p^h = \begin{bmatrix} p^h_1, p^h_k \end{bmatrix} \quad \text{is the vector of distributed external forces and moments acting in the general case on the spatial element} \quad e_h; \quad q^{rh}_{\ell} = \begin{bmatrix} q^{rh}_{\ell 1}, ..., q^{rh}_{\ell N} \end{bmatrix} \quad \text{is the vector of relative displacements of any point} \quad e_h. \quad \text{Here, the number of sub vectors} \quad N - \text{is determined by the number of differential operators used to describe the dynamic model} \quad e_h \quad \text{in relative motion.}
\]

\[
S = Vt
\]

**Figure 1. (a, b, c) Model of the system “bridge crossing - high-speed train”**
The column vectors of displacements with components in the main coordinate system are introduced for node numbering: \( \mathbf{f} = 1, \ldots, n \); \( \mathbf{q} = [q_1, \ldots, q_p, \ldots, q_n] \) is the vector (\( N_n \) order) of nodal displacement \( \{e_i\} \); \( \mathbf{q}_f \) is the vector of nodal linear and angular displacements in \( \mathbf{j} \); \( \mathbf{q}_S = [q_{S1}, \ldots, q_{S_p}, \ldots, q_{S_n}]' \) is the vector of given nodal displacements \( \{e_i\} \). It is believed that \( q^h_f, q^h_k, q^h_f, f^h_k, q^h_{S+j} \) are column vectors with six components, for example, \( q^h_f = [q^h_{f1}, \ldots, q^h_{f1}, \ldots, q^h_{f6}]' \); \( f^h_k = [f^h_{k1}, \ldots, f^h_{k1}, \ldots, f^h_{k6}]' \). The first three positive components of them in ascending order \( h \) mean the values of displacements or, respectively, nodal forces directed along \( \tilde{0}_{ih} \tilde{x}_{ih}, \tilde{0}_{ih} \tilde{y}_{ih}, \tilde{0}_{ih} \tilde{z}_{ih} \) (similarly in the main coordinate system), the following three positive components are the angles of rotation of the nodes or, respectively, the nodal moments relative to these or parallel axes when choosing traditional positive direction. When considering special cases, for example, a flat finite element, the excess components in the listed vectors are deleted, while the remaining components retain the same indexing.

The first group of equations providing equilibrium in the nodes of the core system can be written in the form [7, 13]

\[
E_1 (\mathbf{f})' \mathbf{f} = 0 \quad \Rightarrow \quad N_n - \beta_1 \quad \text{(1)}
\]

where \( \mathbf{f} \) is the matrix of the connection of vectors; \( \mathbf{q} = \Gamma \mathbf{q} \); \( E_1 \) is the matrix which allows to extract equations from relation (1) for those nodes and in those directions where displacements are not specified [13]; \( \beta_1 \) is the number of specified nodal displacements.

For a flat element \( \{e_i, e_h\} \) that works only on bending, the nodal displacements and forces are denoted as follows, preserving the numbering of the elements \( \mathbf{m} \) and \( \mathbf{n} \) in accordance with the conditions for the notation introduced above for the spatial element [6,7]

\[
q^h = [q^h_{1m}, q^h_{1h}] = u(0, t), q^h_{1m} = \partial u(0, t)/\partial y, q^h_{1n} = u(\ell_h, t), q^h_{1n} = \partial u(\ell_h, t)/\partial y]' \quad \text{(2)}
\]

\[
f^h = [f^h_{1m}, f^h_{1h}, f^h_{2m}, f^h_{2h}, f^h_{3m}, f^h_{3n}]' \quad \text{for} \quad \mathbf{m} = 3, \quad \mathbf{n} = 4
\]

To compose the remaining groups of equations describing the movement of a system of elements \( e^h \) and \( \tilde{e}^h \) at first, the derivation of the equations of motion is considered for individual rail elements and adding to them the forces and moments from their interaction with neighboring elements after dismembering the system and the distributed inertia forces at the complex movement of the elastic elements.

**Figure 2.** (a, b) Models for boundary bar elements \( e^h \) (a) and \( \tilde{e}^h \) (b)

The decomposition method for \( \tilde{e}^h \) is used, the rail - \( \tilde{e}^h \) and the span modeled by a thin-walled rod are considered separately (Figure 1 (a, b)). The dynamic equilibrium equations are used for \( e^h \) and \( \tilde{e}^h \) subject to the action of a moving system of forces corresponding to the complete dynamic vertical reactions of the wheel sets, taking into account the Voight attenuation.
For the element $\tilde{h}$ and $e^{h}$ vibration of the rail track (relative) and span is described by the equations:

$$
EJ_{1s}\frac{\partial^{2}u^{h}}{\partial t^{2}} + \bar{p}_{i}EJ_{1s}\frac{\partial^{2}u^{h}}{\partial t^{2}} + m_{i}\frac{\partial^{2}u^{h}}{\partial t^{2}} + k_{i}\tilde{u} + \mu_{i}\frac{\partial u^{h}}{\partial t} = e\times M^{h}_{zi} \times (\partial \delta(y-c) / \partial \tilde{c}) + e\times M^{h}_{zi} \times (\partial \delta(y-c) / \partial \tilde{c}) - m_{i}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) - k_{i}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) - \\
- \mu_{i}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) + \sum_{k=1}^{n} \delta(y - C_{k}) \times (P_{h} + R^{h}_{\tilde{h}} )
$$

(3a)

$$
\bar{p}_{i}EJ_{1s}\frac{\partial^{2}\theta^{h}}{\partial y^{2}} - GJ_{d}\frac{\partial^{2}\theta^{h}}{\partial y^{2}} - \bar{\mu}_{i}GJ_{d}\frac{\partial^{2}\theta^{h}}{\partial y^{2}} + \tilde{k}_{i}u \times d^{*} - \bar{\mu}_{i} \frac{\partial u^{h}}{\partial t} \times d^{*} = \tilde{k}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \\
+ \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \tilde{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*}
$$

(3b)

$$
EJ_{s}\frac{\partial^{2}v^{h}}{\partial y^{2}} + \bar{\mu}_{i}EJ_{s}\frac{\partial^{2}v^{h}}{\partial y^{2}} + \tilde{p}_{i}EJ_{s}\frac{\partial^{2}v^{h}}{\partial y^{2}} - \bar{\mu}_{i} \frac{\partial u^{h}}{\partial t} \times d^{*} = \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \tilde{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*} + \bar{\mu}(\tilde{q}^{h}_{i,3} + y(\tilde{q}^{h}_{i,3} - \tilde{q}^{h}_{i,3})/\ell_{h} ) \times d^{*}
$$

(3c)

where $\tilde{u} = u^{h} - \chi \times (v + \theta \times d^{*})$; $EJ_{1s}$, $EJ_{s}$ bending stiffness of rail and beam; $GJ_{d}$ torsional rigidity of the beam; $\tilde{p}_{i}$ moment of inertia of a unit of beam length; $F_{i}$ cross-sectional area; $\bar{\mu}_{i}$, $\tilde{\mu}_{i}$ viscosity coefficients; $\delta(z - C_{k})$ delta function; $v^{h}(z, t)$ vertical displacement of the beam; $u^{h}(z, t)$ vertical relative displacement for $\tilde{h}$; $\tilde{\theta}(z, t)$ angle of the cross-sectional gate relative to the center of the beam bend; $C_{k} = Vt - \bar{C}_{k}$ the abscissa of the wheel set; $\bar{C}_{h}$, the distance from the first pair of wheels to the wheel set with the number $k_{1}$; $V$ the speed of the train; $P_{k_{1}}$ - axial load from a pair of wheels with a number $k_{1}$; $N^{*}_{h}$ - the number of wheel sets currently on the element $\tilde{h}$; $d_{z}$ the horizontal distance from the axis of the bridge to the axis of the upper structure of the path on the bridge (Figure 1b); $\ell_{h}$ the length of the span or element; $e^{h}$ - nodal vertical accelerations in nodes $v$ and $k$ for the rail of the elements $\tilde{h}$ and $\tilde{h}$; $\tilde{k}$, $\mu_{i}$, $\tilde{\mu}_{i}$, $\tilde{\mu}$, $\bar{\mu}$ - the bed coefficients for the rail and the elastic layer between the rail and the beam; $\tilde{R}^{h}_{i,2}, M^{h}_{i,2,3}$ nodal forces and bending moments in the end sections of the elements $e^{h}$ and $\tilde{h}$; $R^{h}_{i,3}$ - dynamic additives to static reactions of wheel pairs with a number $k_{1}$ located on the element $\tilde{h}$; $\varepsilon = 1$ orientation coefficient, for rotation angles and bending moments [7]; $\chi = 1$ auxiliary coefficient; $k_{1} = \bar{k}$, $\mu_{i} = \bar{\mu}$ for $e^{h}$; $k_{1} = \bar{k}$, $\mu_{i} = \bar{\mu}$ for $\tilde{h}$.

To approximate the displacements for the rail and for the span, the sets of functions will be used:

$$
u^{h}(y, t) = u^{h}(y, t) + (u^{h}(0, t) + y(u^{h}(\ell_{h}, t) - u^{h}(0, t))/\ell_{h} )
$$

(4)

$$
u^{h}(y, t) = \sum_{i=1}^{N} W_{i}(\eta)q_{i}^{h}(t), \quad \theta^{h}(y, t) = \sum_{i=1}^{N} W_{i}(\eta)q_{i}^{h}(t), \quad v^{h}(y, t) = \sum_{i=1}^{N} W_{i}(\eta)q_{i}^{h}(t)
$$

(5)

where $W_{i}(\eta)$ - generalized coordinates defining the fields of vertical relative, absolute angular and vertical displacements of the span.
Note that for an element $e_h^*$ it is sufficient to select only equation (3a) with $\tilde{u}(y,t) = u_h(y,t)$, setting $k_e = \tilde{k}$ and $\mu_e = \tilde{\mu}$.

The second group of equations describes the portable movement of elements $e_h^*$ and $\tilde{e}_h^*$, the group is formed using the d’Alembert principle, i.e. by compiling two equations of dynamic equilibrium (equations of projections and moments) with respect to the axes $oz$ and $ox$ passing through the center of mass of the undeformed rod parallel to the coordinate system basic for the rod and taking into account the nodal reactions $\vec{R}_{12}^h$, $M_{12}^h$ - external influences on the beam, the portable and relative fields of inertia forces distributed elastic base reactions, i.e. we form a boundary element for $e_h^*$ and $\tilde{e}_h^*$ in the vertical plane of bending. As a result, the equations of dynamic equilibrium for $e_h^*$ and $\tilde{e}_h^*$ in matrix form in a figurative motion have the form [7]

$$
\begin{align}
\left[H^h\right]\ddot{q}^h - M^h \dot{q}^h + \sum_{i=1}^{N^h} s^i \left(p^i + R^i\right) = 0, \\
\left[M^h\right] \ddot{q}^h - \left(G^h\right) \dot{q}^h + \left(K^h\right) q^h + \left(\Gamma^h\right) q^h = 0,
\end{align}
$$

(6a)

$$
\begin{align}
\left(H^h\right)\dot{y} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\ell_h/2 & 1 & \ell_h/2 & 1 \end{bmatrix}, \\
\left(M^h\right) = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ -1/\ell_h & 0 & 1/\ell_h & 0 \end{bmatrix}, \\
\left(M^h\right) = m^h \begin{bmatrix} \ell_h \\ y/\ell_h \end{bmatrix}
\end{align}
$$

(6b)

where $M^h = m_1 l_h/12, J_{cx} = M l_h^2/12, \eta = y/\ell_h, s^h, M^h$ - matrices for the formation of projections in $\vec{O}_h, \vec{X}_h, \vec{Y}_h, \vec{Z}_h$ the main vectors and main moments relative to the center of mass $e_h^*$ and $\tilde{e}_h^*$, accordingly, from $p^h, R^h, s^h, M^h$ and from the distributed inertia forces in relative motion; $M^h$ and $M^h \to K^h$ and $K^h$ when replacing them $m_1$ with $\tilde{k}, M^h$ and $M^h \to \Gamma_{0}^{h}$ and $\Gamma_{0}^{h}$ when replacing them $m_1$ with $\tilde{\gamma}$.

Let’s present the system (3a) - (3c) in matrix form, using the discretization of the problem by a spatial variable, using the Galerkin method. After substituting (5) in (3a) - (3c) and integrating the terms with respect to the variable $y$ within $[0,l]$, taking into account the orthogonality of the functions $W_i(y)$, for the system (3a) - (3c) we have the equation in the form

$$
\begin{align}
\mathbf{M}_i \ddot{q}_{ij} + \mathbf{C}_i \dot{q}_{ij} + \mathbf{K}_i q_{ij} = \mathbf{D}_i, \\
\mathbf{K}_i = \begin{bmatrix} K^i_{\overline{m}, \overline{m}} \\ \mathbf{C}_i = \begin{bmatrix} C^i_{\overline{m}, \overline{m}} \end{bmatrix}, \mathbf{q}_i = \begin{bmatrix} q_i^h \tilde{q}_i^h \eta_i^h \end{bmatrix}, \mathbf{D}_i = \begin{bmatrix} D_i^1 & D_i^2 & D_i^3 \end{bmatrix}
\end{align}
$$

(7)

To solve the problem, the time step procedure from [6] is used. The problem is discretized by time, introducing integer $t_j$ and half nodes $t_{j+1/2}$. The values $u(y,t), v(y,t), \theta(y,t)$ at the moment $t_{j+1/2}$ can be expressed at the step $[t_{j}, t_{j+1}]$ through the initial conditions at the moment $t_j$ and through the action of the moving force systems $\tilde{R}_{12}^{h,j+1/2} + p_{h}$, the form

$$
\begin{align}
\ddot{q}_{i,j+1} = \mathbf{U}_i \ddot{q}_{i,j} + \mathbf{U}_{2i} \ddot{q}_{i,j} + \mathbf{G}_i \mathbf{D}_{i,j+1/2}, \\
\dot{q}_{i,j+1} = \ddot{q}_{i,j} + \tilde{q}_{i,j+1/2}, \\
q_{i,j+1} = \sum_{i=1}^{N^h} W_i(y) \left[ \mathbf{q}_{i,j} + \ddot{q}_{i,j} + \mathbf{U}_i \ddot{q}_{i,j} + \mathbf{U}_{2i} \ddot{q}_{i,j} + \mathbf{G}_i \mathbf{D}_{i,j+1/2} \right] \Delta t_j/2
\end{align}
$$

(8)

where

$$
\mathbf{G}_i = \mathbf{M}_i + \begin{bmatrix} \mathbf{K}_i \Delta t_j/2 + \mathbf{C}_i \Delta t_j/4 \end{bmatrix}, \\
\mathbf{U}_i = \begin{bmatrix} u_0^{(i)} \bar{u}_{0,i} \\ \bar{u}_{0,i+1,2.3,3} \end{bmatrix}, \\
\mathbf{U}_{2i} = \begin{bmatrix} u_2^{(2)} \bar{u}_{2,i} \bar{u}_{2,i+1,2.3} \end{bmatrix}, \\
\tilde{q}_{i,j+1} = \mathbf{U}_i \tilde{q}_{i,j} + \mathbf{U}_{2i} \tilde{q}_{i,j} + \mathbf{G}_i \mathbf{D}_{i,j+1/2}, \\
\ddot{q}_{i,j+1} = \mathbf{U}_i \ddot{q}_{i,j} + \mathbf{U}_{2i} \ddot{q}_{i,j} + \mathbf{G}_i \mathbf{D}_{i,j+1/2,2} \Delta t_j
$$
The following group of equations ensures the continuity of nodal displacements in local coordinate systems. Composing the equality \( q^h = q^h_e + q^h_r \) and substituting into it (6b), we have the fourth group of equations in the form

\[
(E - H^\dagger H^h)q^h = \prod q^h_r,
\]

where \( q^h_r = [u^h_r(0, t), \partial u^h_r(0, t)/\partial y, u^h_r(\ell^h_r, t), \partial u^h_r(\ell^h_r, t)/\partial y]' \); \( \Pi \) - matrix of transformation of vectors \( q^h_r \) in \( q^h_r \). Similar expressions can be written for the moving nodes of the subsystem \( \{\epsilon^h_e, \tilde{e}^h_h\} \) in the form:

\[
q^h_m = q^h_m e + q^h_m r
\]

where \( q^h_m e, q^h_m r \) - are column vectors of figurative and relative vertical displacements of moving nodes.

To take into account the action of the rolling load when studying the vibrations of the "bridge-train" system, we introduce the expressions for the complete acceleration of the moving contact points of the rolling stock and the span:

\[
\ddot{q} = \frac{\partial^2 u(V(t))}{\partial t^2} + 2V \frac{\partial^2 u(V(t))}{\partial t \partial y} + V^2 \frac{\partial^2 u(V(t))}{\partial y^2}
\]

where \( u(y, t) \) is the vertical deflection of the rail (elements \( \epsilon^h_e, \tilde{e}^h_h \); \( y \) is the coordinate along the length of the beam. As a function \( u(V(t), t) \) in solving the problem in (11) the function \( u^h(y, t) \) from (4) is involved when using the time step procedure from [6] and repeating the algorithm for determining \( \ddot{q} \) from [7] for \( \epsilon^h_e \) and \( \tilde{e}^h_h \) at the step \([t, t+1]\). As a result, at step \([t, t+1] \), the expressions for the full vertical accelerations \( \ddot{q}^h_\ell (\ell^h_\ell = 1, \ldots, N^h_\ell) \) of the contact points (with numbers \( \ell^h_\ell \)) of the moving load and the rail track have the form:

\[
\ddot{q}^h_{\ell, j+1/2} = \ddot{q}^h_{\ell, i} (q^h_{1,3,i+1/2,j}, M^h_{1,x,i+1/2}, \ddot{q}^h_{3,j+1/2}, M^h_{2,x,j+1/2}, \ddot{R}^h_{i,j+1/2}, \ddot{R}^h_{j+1/2}, \ddot{R}^h_{j+1/2}, R_{i,j+1/2}, R_{i,j+1/2})
\]

(12)

Let’s discretized the system of equations (6a) and (9) in time, combining system (10) with it. As a result, at the step \([t, t+1]\) for \( \{\epsilon^h_e, \tilde{e}^h_h\} \) we have:

\[
(H^\dagger)^{\ell, i} = M^h_{\ell, j} q^h_{\ell, j} + \sum_{k=1}^{M} \left[ \frac{1}{2} M^h_{\ell, j} q^h_{\ell, j} + \Gamma^h_{\ell, j} q^h_{\ell, j} + \frac{1}{2} (K^h_{\ell, j} q^h_{\ell, j} + \Gamma^h_{\ell, j} q^h_{\ell, j}) \right] d\eta
\]

\[
(E - H^\dagger H^h)q^h_{\ell, j+1} = \prod q^h_r
\]

(13)

(14)

As a result, after combining systems (13) - (15) and (12) with the initial conditions \( q^h_{\ell, j}, q^h_{\ell, j}, \ddot{q}^h_{\ell, j}, q^h_{\ell, j}, q^h_{\ell, j}, q^h_{\ell, j}, q^h_{\ell, j}, q^h_{\ell, j}, q^h_{\ell, j}, \ell = 1,2 \), known at the step \([t, t+1]\), we have:

\[
A_{\ell} \ddot{q}^h_{\ell, j+1} + B_{\ell} \ddot{q}^h_{\ell, j+1} = C_{\ell, h, 0} + C_{\ell, h, p}
\]

(16)
where \( \vec{f}_{j+1/2} = [\vec{f}^h_j, \vec{R}^h_{z,j+1/2}] \) and \( \vec{R}_{z,j+1/2} = [\vec{R}^{h}_{2,z} M_{2,z}^h \vec{R}^{h}_{2,z} M_{2,z}^h \vec{R}^{h}_{2,z} \cdots, \vec{R}^{h}_{z,N_z}] \).

\( \vec{f}^h_{j+1/2} \) are vectors respectively of nodal reactions during the bending of the element \( \vec{e}_h \) and dynamic additives to the static reactions of the wheel sets of train cars located on \( \vec{e}_h \) at the step \([t_j, t_{j+1}]\).

\( \vec{q}^h_{j+1/2} = [\vec{q}^h_{j, z}, \vec{q}^h_{j, t}] \) and \( \vec{q}^h_{j, z} \) are vectors respectively of the nodal accelerations \( \vec{e}_h \) and vertical accelerations of the contact points of the moving load and the span at the step \([t_j, t_{j+1}]\); \( A_h, B_h \) are matrices of system (16) with dimensions \([4 + N_h^*] \times (4 + N_h^*) \).

From formulas (16) for an element \( \vec{e}_h \) it is possible to obtain formulas for an element on an elastic basis \( \vec{e}^*_h \); for this put in (3a), (7) and (12), respectively \( \chi = 0, \delta = 0, k = \bar{k}, \mu = \bar{\mu} \) combining respectively the system of equations (12) - (15) and (10).

The action of a moving load on the system \( \{e^*_h, \vec{e}^*_h \} \) is considered. The system is modeled by a mechanical system with 10 degrees of freedom (Figure 1a) [3]. Let’s assume that the initial conditions for the “composition-path-bridge” system are zero, and the parameters that determine the position in a 0, \( X, Y, Z \) system moving at a constant speed \( v \) are counted from their values in static equilibrium.

For the carriage system, we have:

\[
\begin{align*}
M_x \ddot{\vec{q}}_x + C_x \dot{\vec{q}}_x + K_x \vec{q}_x &= \vec{R}_x, \\
\vec{R}_x &= \Pi_x \vec{R}_x
\end{align*}
\]

(17)

where \( \vec{q}_x = [\vec{q}_x^E] \) is a block vector with the size \([10 \times m_h]\) of independent generalized coordinates defining in the system 0, \( X, Y, Z \); \( M_x = [M_x^r] \) is block-diagonal matrices of mass, \( C_x = [C_x^r] \) is block-diagonal matrices of damping, \( K_x = [K_x^r] \) is block-diagonal matrices of stiffness for \( 0, X, Y, Z \); \( M^r_x, C^r_x, K^r_x \) – blocks of matrices \( M_x, C_x, K_x \), corresponding to the crew with the number \( r \), forming the system (17); \( \Pi_x \) – matrix of the connection of vectors \( \vec{R}_x \) and \( \vec{R}_x \); \( \vec{R}_x \) is a vector of dynamic additives to static reactions at points of contact \( \{e^*_h\} \) with the carriageway (system \( \{e^*_h, \vec{e}^*_h\} \)) and a web that is rigid outside the path. We take into account the elastic-viscous bonds between the wheels of the contact and the roadway. These bonds model the contact stiffness of the wheels.

\[
\begin{align*}
\vec{R}_x = C_x (\vec{q}_{k_o} - \vec{q}_{c_k}) + \gamma_1 (\vec{q}_{k_o} - \vec{q}_{c_k})
\end{align*}
\]

(18)

where \( \vec{q}_{k_o} \) – the vector of displacements at the contact points of elastic-viscous elements simulating contact stiffness with the roadway; the vector of displacements of the wheels of the carriage (vector trickle \( \vec{q}_c \); \( C_1, \gamma_1 = \nu \times C_1 \)) – stiffness and viscosity of the bonds modeling the deformation of the wheel disk [3]. After discretization of (17) and (18) and the use of the step procedure used above and in [6], the relationship between the dynamic additives to the static reactions of the wheels in the form is:

\[
\begin{align*}
\vec{R}_{t,j+1/2} &= W^0 \vec{q}_{t,j+1/2} + L^0, \\
\vec{q}_{t,j+1/2}^0 &= \vec{q}_{k_o}
\end{align*}
\]

(19)

where \( W^0 \) and \( L^0 \) are presented in [7].

For moving nodes of the system \( \{e^*_h, \vec{e}^*_h, e^0_{R^*}\} \) at the step \([t_j, t_{j+1}]\)

\[
\begin{align*}
\vec{R}_{t,j+1/2}^* + \vec{R}_{t,j+1/2}^* = 0, \\
\vec{R}_{k_o} = [\vec{R}_x^*]
\end{align*}
\]

(20)

We take into account the boundary conditions for the rod system at the step \([t_j, t_{j+1}]\) in the form
Let us consider a test example of the action in the middle of a beam of a sudden force $P$ applied to the construction $\{e^*_h, \tilde{e}_h, e^0_h\}$ depicted in Figure 3.
Figure 3. Bridge crossing model \( \{e_h \} \) for testing a step procedure.

Figure 4. Rail deflections under the force (line 1) and the beam - q (line 2), under the action \( \theta(t)P \) to \( \{e_h \} \).

Figure 4 shows the changes, depending on the time: \( Y(m) \) – deflection at the rail - beams in cent under force (line 1, at \( J_{1x} = 0.7 \times 10^{-4} \text{m}^4; E_1 = 2.1 \times 10^8 \text{kN} / \text{m}^2 \) and \( Y = q \) - deflection in the middle of the supporting beam with parameters from [12] (line 2 at \( J_x = 5.78 \text{m}^4; E_2 = 0.33 \times 10^8 \text{kN} / \text{m}^2 \) under the action of the force \( P = 170 \text{kN} \) (at \( d_0 = 0 \text{ Fig.1} \) and (at \( d_x = 2.35 \text{m Fig.1} \), \( k = 30000 \text{kN} / \text{m}^2 \), \( m_1 = 0.6 \text{t} / \text{m} \), \( \Delta t_j = 0.000125 \text{s} \), \( m = 3 \), \( n = 4 \), \( N = 141 \), \( \mu = 3.6 \text{kNs} / \text{m}^2 \), \( \tilde{\mu}_1 = 0 \), \( \ell_{h0} = 24 \text{m} \). It can be noted that the period of oscillations of the carrier beam in the numerical solution using (24), (line 2, Figure 4) coincides with the period of oscillations at a frequency of \( \nu_1 = 5.72 \text{Hz} \), calculated by the well-known formula for the articulated beam. An estimate of the maximum displacement \( q = Y \) (line 2, in Figure 4) confirms the equality, at \( \tilde{\mu}_1 = 0 \), \( q = 2\tilde{q}_c \) where \( q_c = P\ell^3/(48EJ_{1x}) = 0.000256 \text{m} \) is the static deflection of the articulated support beam. The change \( Y((\text{line 1 at Figure 4}) \), confirms the oscillations of the rail near the static deflection \( \tilde{q}_c = P/(8\beta^2EJ_{1x}) = 0.00239 \text{m} \) under the force of an infinite beam on an elastic foundation, at \( \beta = (k_1/4EJ_{1x})^{1/4} \). The movement of the composition is considered along a rail track, including sections of rails lying on spans, i.e. the dynamics of the system \( \{e_{h0}^*, \tilde{e}_{h0}, e_{h0}^0 \} \) , with \( m = 8 \), \( n = 9 \), \( m_0 = 7 \) at \( \ell_{h0} = 24 \text{m} \) is considered. The composition parameters are presented in [11], for spans with a span of \( \ell_{h} = 24 \text{m} \) in [12], for a rail track, in a test example. At Figure 5 (a, b) (movement along one path, at \( V = 360 \text{ km} / \text{h} \) respectively, the dependences of the dynamic addition \( R(\text{kN}) \) to the static reaction of the first wheel pair of the first carriage and the vertical displacement \( Y(m) \) of the indicated wheel pair on the position of the composition \( (S = Vt, \text{ Figure1a}) \) during its movement on the considered section of the path. Note that the first wheel set enters the spans and then moves in the interval of change \( S \{96, 144\}(m) \) (Figure 5a, b).
Figure 5 (a, b) The change in the dynamic additive R (Figure 5a) and the vertical displacement Y (Figure 5b) of the wheel pair from the composition S position (Figure 1a) when moving along the bridge.

The presented technique allows determining the fields of displacements, accelerations, and dynamic forces in the “bridge – path – high-speed train” system, which remains an urgent task for the development of high-speed railways (HSR).

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