THE FREE ABELIAN TOPOLOGICAL GROUP AND THE FREE
LOCALLY CONVEX SPACE ON THE UNIT INTERVAL †

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ABSTRACT. We give a complete description of the topological spaces $X$ such that the
free abelian topological group $A(X)$ embeds into the free abelian topological group $A(I)$ of the closed unit interval. In particular, the free abelian topological group $A(X)$ of any finite-dimensional compact metrizable space $X$ embeds into $A(I)$. The situation turns out to be somewhat different for free locally convex spaces. Some results for the spaces of continuous functions with the pointwise topology are also obtained. Proofs are based on the classical Kolmogorov’s Superposition Theorem.

§1. INTRODUCTION

The following natural question arises as a part of the search for a topologized version of the Nielsen-Schreier subgroup theorem. Let $X$ and $Y$ be completely regular topological spaces; in which cases the free (free abelian) topological group over $X$ can be embedded as a topological subgroup into the free (free abelian) topological group over $Y$? This problem has been treated for a long time [4, 10, 12-17, 21-23, 25, 28], ever since it became clear that in general a topological subgroup of a free (free abelian) topological group need not be topologically free [8, 4, 10]. Recently a complete answer was obtained in the case where $X$ is a subspace of $Y$ and the embedding of free topological groups extends the embedding of spaces [35]. However, we are interested in the existence of an embedding which is not necessarily a “canonical” one. Among the most notable achievements, there are certain sufficient conditions for a subgroup of a free topological group to be topologically free [4, 22] and the following results.

1991 Mathematics Subject Classification. 22A05, 55M10, 46A03.
Key words and phrases. Free abelian topological groups, free locally convex spaces, spaces of continuous functions, dimension, basic functions, Kolmogorov’s Superposition Theorem.
† Dedicated to the memory of Eli Katz.
1 Research Supported by the Israel Ministry of Science.
2 Partially supported by the Internal Grants Committee of the Victoria University of Wellington.
Theorem 1.1 [14]. If $X$ is a closed topological subspace of the free topological group $F(I)$ then the free topological group $F(X)$ is a closed topological subgroup of $F(I)$, where $I$ is the closed unit interval. □

Corollary 1.2 [22]. If $X$ is a finite-dimensional metrizable compact space then $F(X)$ is a closed topological subgroup of $F(I)$. □

The abelian case proved to be more difficult, and the following is the strongest result known to date.

Theorem 1.3 [12]. If $X$ is a countable CW-complex of dimension $n$, then the free abelian topological group on $X$ is a closed subgroup of the free abelian topological group on the closed ball $B^n$. □

Corollary 1.4 [13]. $A(\mathbb{R})$ embeds into $A(I)$ as a closed topological subgroup. □

It is known [29] that the covering dimension of any two free topological bases in a free (abelian) topological group is the same; this result is similar to the well-known property of free bases of a discrete free (abelian) group having the same cardinality, called the rank of the group. Since the rank of a subgroup of a free abelian group cannot exceed the rank of the group itself, it was conjectured [15, 20] that the dimension of a topological basis of a topologically free subgroup of a free abelian topological group $A(X)$ cannot exceed $\dim X$. It remained even unclear whether the group $A(I^2)$ embeds into $A(I)$ [15].

In this paper we prove that if $X$ is a completely regular space then the free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup if and only if $X$ is a submetrizable $k_\omega$-space such that every compact subspace of $X$ is finite-dimensional. Another characterization: $X$ is homeomorphic to a closed topological subspace of the group $A(I)$ itself. In particular, if $X$ is a compact metrizable space of finite dimension, then $A(X)$ embeds into $A(I)$. Thus, the analogy with the non-abelian case is complete. We also study the problem of embedding the free locally convex space $L(X)$ into the free locally convex space $L(I)$ and characterize those $k_\omega$-spaces $X$ admitting such an embedding. Paradoxically, such spaces $X$ are just all compact metrizable finite-dimensional spaces. In particular, the free LCS $L(\mathbb{R})$ does not embed into $L(I)$. Our results provide answers to a number of open problems from [20, 15, 27].

§2. Preliminaries

Definition 2.1 [19, 8, 20]. Let $X$ be a topological space. The (Markov) free abelian topological group over $X$ is a pair consisting of an abelian topological group $A(X)$ and a topological embedding $X \hookrightarrow A(X)$ such that every continuous mapping $f$ from $X$ to an abelian topological group $G$ extends uniquely to a continuous homomorphism $\bar{f}: A(X) \rightarrow G$. □

If $X$ is a completely regular topological space then the free abelian topological group $A(X)$ exists and is algebraically free over the set $X$ [19, 8, 20]. A topological space $X$ is called a $k_\omega$-space [18, 13-17] if there exists a so-called $k_\omega$-decomposition $X = \bigcup_{n \in \mathbb{N}} X_n$, where all $X_n$ are compact, $X_n \subset X_{n+1}$ for $n \in \mathbb{N}$, and a subset $A \subset X$ is closed if and only if all intersections $A \cap X_n$, $n \in \mathbb{N}$, are closed. All locally convex spaces (LCS) in this paper are real.
Definition 2.2 [19, 1, 31, 6, 7, 34]. Let $X$ be a topological space. The free locally convex space over $X$ is a pair consisting of a locally convex space $L(X)$ and a topological embedding $X \hookrightarrow L(X)$ such that every continuous mapping $f$ from $X$ to a locally convex space $E$ extends uniquely to a continuous linear operator $\bar{f} : L(X) \to E$. □

If $X$ is a completely regular topological space then the free locally convex space $L(X)$ exists; the set $X$ forms a Hamel basis for $L(X)$ [31, 6, 7, 34]. The identity mapping $id_X : X \to X$ extends to a canonical continuous homomorphism $i : A(X) \to L(X)$.

Theorem 2.3 [33]. The canonical homomorphism $i : A(X) \hookrightarrow L(X)$ is an embedding of $A(X)$ into the additive topological group of the LCS $L(X)$ as a closed additive topological subgroup. □

In what follows, we will often identify $A(X)$ with a subgroup of $L(X)$ in the above canonical way. Denote by $L_p(X)$ the free locally convex space $L(X)$ endowed with the weak topology.

Theorem 2.4 [6, 7]. Let $X$ be a completely regular space. The canonical mapping $X \hookrightarrow L_p(X)$ is a topological embedding, and every continuous mapping $f$ from $X$ to a locally convex space $E$ with the weak topology extends uniquely to a continuous linear operator $\bar{f} : L_p(X) \to E$. □

The weak dual space to $L(X)$ is canonically isomorphic to the space $C_p(X)$ of all continuous real-valued functions on $X$ with the topology of pointwise (simple) convergence. The spaces $L_p(X)$ and $C_p(X)$ are in duality. Denote by $C_k(X)$ the space of continuous functions endowed with the compact-open topology. A topological space $X$ is called Dieudonné complete [5] if its topology is induced by a complete uniformity. For example, every Lindelöf space is Dieudonné complete. In particular every $k_\omega$-space is Dieudonné complete.

Theorem 2.5 (Arhangel’skii [3]). Let $X$ and $Y$ be Dieudonné complete spaces. If a linear mapping $C_p(X) \to C_p(Y)$ is continuous then it is continuous as a mapping $C_k(X) \to C_k(Y)$. □

The space $L(X)$ admits a canonical continuous monomorphism

$$L(X) \hookrightarrow C_k(C_k(X))$$

Theorem 2.6 (Flood [6, 7], Uspenskii [34]). If $X$ is a $k$-space then the monomorphism $L(X) \hookrightarrow C_k(C_k(X))$ is an embedding of locally convex spaces. □

Let $X$ be a topological space. A collection of continuous functions $h_1, \ldots, h_m$ on $X$ assuming their values in the closed unit interval $I = [0, 1]$ is called basic [26, 32] if every real-valued continuous function $f$ on $X$ can be represented as a sum $\sum_{i=1}^{n} g_i \circ h_i$ of compositions of basic functions with some continuous functions $g_i \in C(I)$.

2.7. Kolmogorov’s Superposition Theorem [11]. The finite-dimensional cube $I^n$ has a finite basic family of continuous real-valued functions. □

Let us recall that for compact metrizable spaces all three main concepts of dimension (the covering, the small inductive and the large inductive ones) coincide.
Lemma 3.2. Let $C$ subspace to the Michael Selection Theorem (Th. 1.4.9 in [36]), the re exists a compact of locally convex spaces. Then $h$ implies that $C$ nally, put $r$ compact subset $< r$

Consider the mapping $E$ mapping of a compact subspace of $C$ for every $t$ mappings $k$ and $p$.

Proof. As a corollary of the Hahn-Banach theorem, the dual linear map $h^* : C_p(Y) → C_p(X)$ to the embedding $h$ is a continuous surjective homomorphism. Theorem 2.5 says that $h^*$ remains continuous with respect to the compact-open topologies on both spaces, and by virtue of the Open Mapping Theorem, $h^* : C_k(Y) → C_k(X)$ is open. Since for every compact subset $C ⊂ X$ the elements of the image $h(C)$ are contained in the linear span of a compact subset of $Y$ [3], one can choose $k_ω$-decompositions $X = \bigcup_{n=1}^{∞} X_n$ and $Y = \bigcup_{n=1}^{∞} Y_n$ in such a way that for every $n ∈ N$ one has $h(sp X_n) ⊂ sp Y_n$. It is easy to see that the restrictions mappings $r_n : C_k(X_n) → C_k(Y)$ and $q_n : C_k(Y_n) → C_k(Y_n)$ are continuous. 

\section*{3. Auxiliary Constructions}

\textbf{Lemma 3.1.} Consider a commutative diagram of Banach spaces and surjective continuous linear mappings:

$$E_1 ←− r_1 E_2 ←− r_2 E_3 ←− r_3 \ldots ←− r_{n-1} E_n ←− r_n \ldots$$

$$π_1 \downarrow \quad \quad \quad π_2 \downarrow \quad \quad \quad π_3 \downarrow \quad \quad \quad π_n \downarrow$$

$$F_1 ←− q_1 F_2 ←− q_2 F_3 ←− q_3 \ldots ←− q_{n-1} F_n ←− q_n \ldots$$

Denote by $E = \varprojlim E_n$ and $F = \varprojlim F_n$ the Fréchet spaces projective limits of corresponding inverse sequences, and by $π : E → F$ the projective limit of the mappings $π_n$, $n ∈ N$. Then every compact subspace $K ⊂ F$ is an image under the mapping $π$ of a compact subspace of $E$.

Proof. Let $K$ be a compact subspace of $F$. Let $K_n = q_n(K)$ for all $n ∈ N$. According to the Michael Selection Theorem (Th. 1.4.9 in [36]), there exists a compact subspace $C_1 ⊂ E_1$ such that $π_1(C_1) = K_1$. Assume now that for all $k ≤ n$ we have chosen compact subspaces $C_k ⊂ E_k$ such that $π_k(C_k) = K_k$ and $r_{k-1}(C_k) = C_{k-1}$.

Consider the mapping $< r_n, π_{n+1} > : x → (r_n(x), π_{n+1}(x))$ from $E_{n+1}$ to $E_n × F_{n+1}$. The subset $Q_n = \{(y, z) : y ∈ C_n, z ∈ K_{n+1}, q_n(z) = π_n(y)\}$ of the space $E_n × F_{n+1}$ is compact, and is contained in the Banach space image of the continuous linear mapping $< r_n, π_{n+1} >$. Therefore, by the Michael Selection Theorem, there exists a compact subset $C_{n+1} ⊂ E_{n+1}$ such that $< r_n, π_{n+1} > (C_{n+1}) = Q_n$. Consequently, $r_n(C_{n+1}) = C_n$, and $q_{n+1}(C_{n+1}) = K_{n+1}$, which completes the recursion step. Finally, put $C = \varprojlim C_n$; this subset of $E$ is compact, and the property $K ⊂ \varprojlim K_n$ implies that $π(C) = K$. \hfill \square

\textbf{Lemma 3.2.} Let $X$ and $Y$ be $k_ω$-spaces. Let $h : L_p(X) → L_p(Y)$ be an embedding of locally convex spaces. Then $h$ is an embedding of locally convex space $L(X)$ into $L(Y)$ as well.

Proof. As a corollary of the Hahn-Banach theorem, the dual linear map $h^* : C_p(Y) → C_p(X)$ to the embedding $h$ is a continuous surjective homomorphism. Theorem 2.5 says that $h^*$ remains continuous with respect to the compact-open topologies on both spaces, and by virtue of the Open Mapping Theorem, $h^* : C_k(Y) → C_k(X)$ is open. Since for every compact subset $C ⊂ X$ the elements of the image $h(C)$ are contained in the linear span of a compact subset of $Y$ [3], one can choose $k_ω$-decompositions $X = \bigcup_{n=1}^{∞} X_n$ and $Y = \bigcup_{n=1}^{∞} Y_n$ in such a way that for every $n ∈ N$ one has $h(sp X_n) ⊂ sp Y_n$. It is easy to see that the restrictions mappings $r_n : C_k(X_n) → C_k(Y)$ and $q_n : C_k(Y_n) → C_k(Y_n)$ are continuous.
surjections, and that \( C(Y) = \lim C_k(Y_n) \) and \( C(X) = \lim C_k(X_n) \). Denote for each \( n \in \mathbb{N} \) by \( \pi_n \) the restriction \( h^*|_{C_k(Y_n)} \). The conditions of Lemma 3.1 are fulfilled, and therefore every compact subset \( K \subset C_k(X) \) is an image under the mapping \( h^* \) of a suitable compact subset of \( C_k(Y) \). Therefore, the continuous linear map \( h^{**} \) dual to \( h^* \) from the space \( C_k(C_k(X)) \) to \( C_k(C_k(Y)) \) is an embedding of \( C_k(C_k(X)) \) into \( C_k(C_k(Y)) \) as a locally convex subspace. Since the restriction of \( h^{**} \) to \( L(X) \) is \( h \), the desired statement follows from Theorem 2.6. \( \square \)

**Lemma 3.3.** Let \( X \) be a compact space and let \( Y \) be a closed subspace of \( X \). Denote by \( \pi \) the quotient mapping from \( X \) to \( X/Y \). Let \( f_k, k = 1, \ldots, n \) be continuous functions on \( X \) such that their restrictions to \( Y \) form a basic family for \( Y \), and let \( g_i, i = 1, \ldots, m \) be a basic family of functions on \( X/Y \). Then the family of functions \( f_1, \ldots, f_n, g_1 \circ \pi, \ldots, g_m \circ \pi \) is basic for \( X \).

**Proof.** Let \( f : X \to \mathbb{R} \) be a continuous function. For a family of continuous functions \( h_1, \ldots, h_n \in C(I) \), the restriction \( f|_Y \) is represented as \( \sum_{k=1}^n h_k(f_k|_Y) = (\sum_{k=1}^n h_k \circ f_k)|_Y \). Denote by \( g : X \to \mathbb{R} \) the continuous function \( f - \sum_{k=1}^n h_k \circ f_k \); since the restriction \( g|_Y \equiv 0 \), the function \( g \) factors through the mapping \( \pi \); that is, there exists a continuous function \( h : X/Y \to I \) with \( g = h \circ \pi \). For some collection \( s_1, \ldots, s_m \) of continuous functions on \( I \) one has \( h = \sum_{i=1}^m s_i \circ g_i \), which means that \( g = \sum_{i=1}^m s_i \circ g_i \circ \pi \). Finally, one has

\[
f = \sum_{k=1}^n h_k \circ f_k + \sum_{i=1}^m s_i \circ g_i \circ \pi,
\]

as desired. \( \square \)

A topological space \( X \) is called submetrizable if it admits a continuous one-to-one mapping into a metrizable space.

**Lemma 3.4.** Let \( X \) be a submetrizable \( k_\omega \)-space with \( k_\omega \)-decomposition \( X = \cup_{n \in \mathbb{N}} X_n \) such that every subspace \( X_n \) is finite-dimensional. Then there exists an embedding of locally convex spaces \( F : L_p(X) \hookrightarrow L_p(Y) \), where \( Y \) is the disjoint sum of countably many copies of the closed unit interval \( I \), such that \( F(A(X)) \subset A(Y) \).

**Proof.** Let \( X = \cup_{n \in \mathbb{N}} X_n \) be a \( k_\omega \) decomposition of \( X \) with \( X_n \subset X_{n+1} \), for all \( n \in \mathbb{N} \). Since every \( X_n \), \( n \in \mathbb{N} \) is a finite-dimensional metrizable compact space, then for any \( n \in \mathbb{N} \) so is the quotient space \( X_{n+1}/X_n \), and one can choose inductively, using Ostrand’s Corollary 2.8 and Lemma 3.3, a countable family of continuous functions \( f_{n,i}, n \in \mathbb{N}, i = 1, \ldots, k_n, k_n \in \mathbb{N} \) from \( X \) to \( I \) such that for each \( n \in \mathbb{N} \) the following are true:

1. the collection \( f_{m,i}, i = 1, \ldots, k_m, m = 1, \ldots, n \), is basic for \( X_n \);
2. \( f_{n+1,i}|_{X_n} \equiv 0 \) for all \( i = 1, \ldots, k_{n+1} \). Denote the above family of functions \( f_{n,i} \) by \( F \), and let \( Y = \oplus_{f \in F} I_f \) be the disjoint sum of countably many copies of the closed unit interval \( I \). For every \( f \in F \) denote by \( 0_f \) the left endpoint of the closed interval \( I_f \) regarded as an element of the free abelian group \( A(Y) \). Define a mapping, \( F \), from \( X \) to the free abelian group \( A(Y) \) by letting

\[
F(x) = \sum_{i=1, \ldots, k_1} f_{1,i}(x) + \sum_{n \geq 2, i=1, \ldots, k_n} (f_{n,i}(x) - 0_{n,i})
\]

for each \( x \in X \). The mapping \( F \) is properly defined, because the first sum is finite and in the second sum all but finitely many terms are vanishing in the quotient space \( X/Y \). Therefore, the family \( F \) is basic for \( X \).
free abelian group $A(Y)$, for every $x \in X$. The restriction of $F$ to every $X_n$ is continuous if being considered as a mapping to the free abelian topological group $A(Y)$, which fact follows from continuity of each mapping $f_{m,i} : X_n \to I_{f_{m,i}} \subset Y$, $m \leq n$, $i = 1, \ldots, k_m$ and the continuity of subtraction and addition in $A(Y)$. Therefore the mapping $F : X \to A(Y)$ is continuous. If being viewed as a continuous mapping from $X$ to the locally convex space $L_p(Y)$, it extends to a continuous linear operator $F : L_p(X) \to L_p(Y)$. Let $h : X \to \mathbb{R}$ be a continuous function. We will show that there exists a continuous linear functional $\bar{h}$ on the linear subspace $\bar{F}(L_p(X))$ such that $\bar{h} \circ F|_X = h$. It would mean that $\bar{F}(L_p(X))$ is isomorphic to $L_p(X)$, as desired. Construct recursively, making use of the properties 1 and 2 above, a countable family of continuous functions $h_{n,i}$, $i = 1, \ldots, k_n$, $n \in \mathbb{N}$ from $I$ to $\mathbb{R}$ such that for every $n \in \mathbb{N}$ and for all $x \in X_n$,

$$h(x) = \sum_{i=1,\ldots,k_m, m \leq n} (h_{m,i} \circ f_{m,i})(x)$$

Let us recall that $f_{n,i}|_{X_i} \equiv 0$ for all $n \geq 2$ and $i = 1, \ldots, k_n$. It is easy to deduce inductively from this fact that for any $n \geq 2$

$$\sum_{i=1,\ldots,k_n} h_{n,i}(0) = 0$$

Define a continuous mapping $H$ from $Y$ to $I$ by letting $H(y) = h_{n,i}(y)$, if $y \in I_{n,i}$, $n \in \mathbb{N}$. Extend $H$ to a continuous linear functional $\bar{H} : L_p(Y) \to \mathbb{R}$ and denote its restriction to $\bar{F}(L_p(X))$ by $\bar{h}$. We claim that $\bar{h} \circ F|_X = h$, or, which is the same, that for every $n \in \mathbb{N}$ one has $\bar{h} \circ F|_{X_n} = h$. Indeed, for an arbitrary $x \in X_n$ one has:

$$(\bar{h} \circ F)(x) = \bar{H}(F(x)) = \bar{H} \left( \sum_{i=1,\ldots,k_1} f_{1,i}(x) + \sum_{2 \leq m \leq n, i=1,\ldots,k_n} (f_{m,i}(x) - 0_{m,i}) \right)$$

$$= \sum_{i=1,\ldots,k_1} H(f_{1,i}(x)) + \sum_{2 \leq m \leq n, i=1,\ldots,k_n} H(f_{m,i}(x) - 0_{m,i})$$

$$= \sum_{i=1,\ldots,k_m, m \leq n} (h_{m,i} \circ f_{m,i})(x) - \sum_{2 \leq m \leq n, i=1,\ldots,k_n} h_{m,i}(0) = h(x) - 0 = h(x).$$

\[\square\]

§4. Main results

**Theorem 4.1.** For a completely regular space $X$ the following are equivalent.

(i) The free abelian topological group $A(X)$ embeds into $A(I)$ as a topological subgroup.

(ii) The free topological group $F(X)$ embeds into $F(I)$ as a topological subgroup.

(iii) $X$ is homeomorphic to a closed topological subspace of $A(I)$.

(iv) $X$ is homeomorphic to a closed topological subspace of $F(I)$.

(v) $X$ is homeomorphic to a closed topological subspace of $\mathbb{R}^\infty$.

(vi) $X$ is a $k_\omega$-space such that every compact subspace of $X$ is metrizable and finite-dimensional.
(vii) $X$ is a submetrizable $k_\omega$-space such that every compact subspace of $X$ is finite-dimensional.

**Proof.** ($i$) $\Rightarrow$ ($iii$): since the space $X$ is Lindelöf (as a subspace of $A(I)$, see [2]) and hence Dieudonné complete, the group $A(X)$ is complete in its two-sided uniformity [33] and therefore closed in $A(I)$; but $X$ is closed in $A(X)$. ($ii$) $\Leftrightarrow$ ($iv$): see [14]. ($iii$) $\Leftrightarrow$ ($v$) $\Leftrightarrow$ ($iv$): follows from the result of Zarichnyõ [37]: the free topological group $F(I)$ and the free abelian topological group $A(I)$ are homeomorphic to open subsets of $\mathbb{R}^\infty$. ($v$) $\Rightarrow$ ($vi$): the space $\mathbb{R}^\infty = \lim_{\rightarrow} \mathbb{R}^n$ is a $k_\omega$-space such that every compact subspace of it is metrizable and finite-dimensional, and this property is inherited by closed subsets. ($vi$) $\Leftrightarrow$ ($vii$): see [16]. ($vii$) $\Rightarrow$ ($i$): Let $X$ be a submetrizable $k_\omega$-space such that every compact subspace of $X$ is finite-dimensional. According to Lemma 3.4, there exists an embedding of locally convex spaces $F : L_p(X) \hookrightarrow L_p(Y)$, where $Y$ is the disjoint sum of countably many copies of the closed unit interval $I$, such that $F(A(X)) \subset A(Y)$. By virtue of Lemma 3.2, $F$ is also an embedding of locally convex spaces $L(X) \hookrightarrow L(Y)$. Its restriction to $A(X)$ is an embedding of topological groups (Theorem 2.3). Now apply Corollary 1.4. □

**Theorem 4.2.** For a $k_\omega$-space $X$ the following conditions are equivalent.

(i) The free locally convex space $L(X)$ embeds into $L(I)$ as a locally convex subspace.

(ii) The free locally convex space with the weak topology, $L_p(X)$, embeds into $L_p(I)$ as a locally convex subspace.

(iii) The space $C_p(X)$ is a quotient linear topological space of $C_p(I)$.

(iv) $X$ is a finite-dimensional metrizable compact space.

**Proof.** ($ii$) $\Leftrightarrow$ ($iii$): just dual forms of the same statement about two locally convex spaces having their weak topology. ($i$) $\Rightarrow$ ($iv$): Suppose $X$ is a noncompact $k_\omega$-space. Since ($ii$) and by the same token ($iii$) hold, then by virtue of Theorem 2.5 the Fréchet non-normable space $C_k(X)$ is an image of the Banach space $C_k(I)$ under a surjective continuous linear mapping, which is open by virtue of the Open Mapping Theorem – a contradiction. Now the space $X$, being compact, is contained in the subspace $sp_n(Y)$ of $L(Y)$ formed by all words of the reduced length $\leq n$ over $Y$ for some $n \in \mathbb{N}$ [34]. But the space $sp_n(Y)$ is a union of countably many closed subspaces each of which is homeomorphic to a subspace of the $n$-th Tychonoff power of the space $\mathbb{R} \times [X \oplus (-X) \oplus \{0\}]$ [2]. Therefore, $sp_n(Y)$ is finite-dimensional. Finally, submetrizability of $X$ follows from the same property of $L(I)$ (the latter space admits a continuous one-to-one isomorphism into the free Banach space over $I$, [1, 6, 7]). ($iv$) $\Rightarrow$ ($ii$): it follows from Lemmas 3.4 and 3.3 that $L_p(X)$ embeds as a locally convex subspace into the free locally convex space in the weak topology over a disjoint sum of finitely many homeomorphic copies of the closed interval. The latter LCS naturally embeds into $L_p(I)$. ($ii$) $\Rightarrow$ ($i$): apply Lemma 3.3. □

**Remark 4.3.** Surprising as it may seem, the free locally convex space $L(\mathbb{R})$ does not embed into $L(I)$, in spite of the existence of canonical embeddings $A(\mathbb{R}) \hookrightarrow L(\mathbb{R})$ and $A(I) \hookrightarrow L(I)$ and a (non-canonical one) $A(\mathbb{R}) \hookrightarrow A(I)$. It is another illustration to the well-known fact that not every continuous homomorphism to the additive group of reals from a closed additive subgroup of an (even normable) LCS extends to a continuous linear functional on the whole space. Such a misbehaviour is also to blame — at least partly — for apparent lack of progress in attempts to make the Pontryagin-van Kampen duality work for free abelian topological groups [24, 30].
Remark 4.4. The problem of characterization of covering dimension of a completely regular space $X$ in terms of the linear topological structure of the space $C_p(X)$ still remains open (cf. [9]). However, now we can describe those metrizable compact spaces having finite dimension.

Corollary 4.5. A metrizable compact space $X$ is finite-dimensional if and only if the space $C_p(X)$ is a quotient linear topological space of $C_p(I)$. □

Perhaps, the dimension of $X$ can be described in terms referring to the linear topological structure of the space $C_p(X)$ with the help of a characterization of dimension in the language of basic functions due to Sternfeld [32].

Remark 4.6. Our results also provide answers to three problems from the book Open Problems in Topology [27].

Problem 511. Is $A(I^2)$ topologically isomorphic with a subgroup of $A(I)$? 

Yes (cf. Theorem 4.1).

Problem 1046. Assume that $C_p(X)$ can be mapped by a linear continuous mapping onto $C_p(Y)$. Is it true that $\dim Y \leq \dim X$? What if $X$ and $Y$ are compact?

Problem 1047. Assume that $C_p(X)$ can be mapped by an open linear continuous mapping onto $C_p(Y)$. Is it true that $\dim Y \leq \dim X$? What if $X$ and $Y$ are compact?

No, in all four cases (cf. Corollary 4.5).

Acknowledgment

The third author (V.G.P.) thanks the University of Wollongong for hospitality extended in August-September 1992.

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