Factorization of the R-matrix. I.

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Abstract. We study the general rational solution of the Yang-Baxter equation with the symmetry algebra $s'(3)$. The R-operator acting in the tensor product of two arbitrary representations of the symmetry algebra can be represented as the product of the simpler "building blocks" (R-operators). The R-operators are constructed explicitly and have simple structure. We construct in such a way the general rational solution of the Yang-Baxter equation with the symmetry algebra $s'(3)$. To illustrate the factorization in the simplest situation we treat also the $s'(2)$ case.
1 Introduction

The Yang-Baxter equation and its solutions play a key role in the theory of the completely integrable quantum models [1, 2, 3, 6]. The general solution of the Yang-Baxter equation (R-matrix) is the operator $R(u)$ acting in a tensor product $V_1 \otimes V_2$ of two linear spaces. The explicit expression for the R-matrix can be obtained by the following method [7, 8]. The Yang-Baxter equation is reduced to the simpler defining equation for the R-matrix [7]

$$R_{12}(u \otimes v)L_1(u)L_2(v) = L_2(v)L_1(u)R_{12}(u \otimes v)$$

where $L(u)$ is the Lax operator and by some conditions the defining equation is reduced to the recurrence relation for the function of one variable. Let us consider the $s'(2)$-invariant R-matrix [7] for example. The tensor product of two $s'(2)$-modules has the simple direct sum decomposition

$$V_1 \otimes V_2 = \bigoplus_{n=0}^{\infty} V_{\gamma_1+\gamma_2+n}$$

so that one obtains the spectral decomposition of R-matrix in the following general form

$$R_{\gamma_1 \gamma_2}(u) = \bigoplus_{n=0}^{\infty} R_n(u) P_n$$

where operator $P_n$ is the projector on the space $V_{\gamma_1+\gamma_2+n}$ in the tensor product $V_{\gamma_1} \otimes V_{\gamma_2}$. The defining equation results in the recurrence relation for the function $R_n(u)$

$$(u + \gamma_1 + \gamma_2 + n) R_{n+1}(u) = (u + \gamma_1 + \gamma_2 + n) R_n(u)$$

which has the solution [6]

$$R_n(u) = (1)^n \frac{(u + \gamma_1 + \gamma_2 + n)}{(u + \gamma_1 + \gamma_2 + n)}$$

(1.0.1)
This method is generally applicable provided the direct sum decomposition of the tensor product $V_1 \otimes V_2$ has no multiplicities [3]. In generic situation for the algebras of higher rank ($s'(3)$ is the first nontrivial example) one obtains the system of complicated recurrence relations for the function of several variables [7,17,13] which is difficult to solve.

We suggest the natural factorized expression for the general $R$-matrix. It can be represented as the product of the simple "building blocks" (R-operators). The main idea is very simple and can be illustrated on the $s'(2)$-example. The Lax operator depends on two parameters: spin of representation $\vee$ and spectral parameter $u$. It is useful to switch to another two parameters $u_1 = u + \vee$ and $u_2 = u - \vee$ and extract the operator of permutation $P_{12}$ from the $R$-matrix $R_{12} = P_{12}R_{21}$. The defining equation for the operator $R_{12}$ has the form

$$R_{12} = L(u_1;u_2)L_2(v_1;v_2) = L_1(v_1;v_2)L_2(u_1;u_2) = R_{12}.$$

The operator $R_{12}$ interchanges $u_1$ with $v_1$ and $u_2$ with $v_2$ in the product of two Lax-operators. Let us perform this operation in two stages. In the first stage we interchange the parameter $u_1$ with $v_1$ only. The parameters $u_2$ and $v_2$ remain the same. In this way one obtains the natural defining equation for the $R_1$-operator

$$R_1 = L(u_1;u_2)L_2(v_1;v_2) = L_1(v_1;u_2)L_2(u_1;v_2) = R_1.$$

In the second stage we interchange $u_2$ with $v_2$ but the parameters $u_1$ and $v_1$ remain the same. The defining equation for the $R_2$-operator is

$$R_2 = L(u_1;u_2)L_2(v_1;v_2) = L_1(u_1;v_2)L_2(u_1;v_2) = R_2.$$

These equations appear much simpler than initial defining equation for the $R$-operator and the solution can be obtained in a closed form. Finally we construct the "composite object" $R$-matrix from the simplest "building blocks" (R-operators $R_{12} = P_{12}R_1R_2$). Note this factorization is different from the ones used by V.Drinfeld [4] and the $R$-operator is different from the Drinfeld twist $F$. In the present paper we shall consider two examples of such factorization. As an illustrative example we derive the factorized expression for the $R$-matrix with symmetry algebra $s'(2)$. Next we work out in details the first nontrivial example of the $R$-matrix with symmetry algebra $s'(3)$. It seems that the whole construction can be generalized to the case of the $R$-matrix with symmetry algebra $s'(n)$. There exist operators $R_1; R_2; \ldots; R_n$ and the general $R$-matrix can be expressed in the factorized form $R_{12} = P_{12}R_1R_2$.

The presentation is organized as follows. In Section 2 we collect the standard facts about the algebra $s'(2)$ and its representations. We represent the lowest weight modules by polynomials in one variable ($z$) and the $s'(2)$-generators as $R$-order differential operators. We derive the defining relation for the general $R$-matrix, i.e. the solution of the Yang-Baxter equation acting on tensor products of two arbitrary representations, the elements of which are polynomials in $z_1$ and $z_2$. Next we introduce the natural defining equations for the $R$-operators and show that the general $R$-matrix can be represented as the product of such more simple operators. In the Section 3 we follow the same strategy for the algebra $s'(3)$. We represent the $s'(3)$ lowest weight modules by polynomials in three variables ($x; y; z$) and the $s'(3)$-generators as $R$-order differential operators. Next we derive the defining relation for the general $R$-matrix and show how it can be solved using the $R$-operators. Finally, in Section 4 we summarize.
2 The general $\mathfrak{sl}(2)$-invariant R-matrix

In this section we consider the simplest situation when the symmetry algebra of the Yang-Baxter equation is $s'(2)$. We serve it as the illustrative example and present all calculations in the great details.

2.1 $\mathfrak{sl}(2)$ lowest weight modules

In this, preparatory, section we collect some facts about $s'(2)$ lowest weight modules and the notations. The Lie algebra $s'(2)$ has three generators $S$:

$$[S;S] = S \ ; \ [S;S] = 2S$$

the central element (Casimir operator $C_2$) being

$$C_2 = S^2 \ ; \ S + S;S ; \ [C_2;S] = [C_2;S] = 0$$

The Verma module $V$ is the generic lowest weight $s'(2)$-module with the lowest weight $^2C$ and Casimir $C_2 = \left(\begin{array}{cc}1 & 0 \\ 0 & 1 \end{array}\right)$. As a linear space $V$ is spanned by the basis $fv_k g_{k-0}^n$

$$v_k = S^k v_0 \ ; \ S v_k = (1+k)v_k \ ; \ S v_k = k(2'+k-1)v_k$$

where the vector $v_0$ is the lowest weight vector: $S v_0 = 0$; $S v_0 = v_0$. The module $V$ is irreducible, except for $'= n/2$ where $n \in 0;1;2;3$ and there exists an $(n+1)$-dimensional invariant subspace $V_n$. $V$ is spanned by $fv_k g_{k-0}^n$. We shall extensively use the representation $V$ of $s'(2)$ in the nine-dimensional space $C[z]$ of polynomials in variable $z$ with the standard monomial basis $z^k$ and lowest weight vector $v_0 = 1$. The action of $s'(2)$ in $V$ is given by the first-order differential operators:

$$S = z\partial + ' \ ; \ S = \emptyset \ ; \ S_* = z^2\emptyset + 2'z$$

or the group-like elements (global transformations)

$$S(z) = (z) \ ; \ e^{S} (z) = (z) \ ; \ e^{S+} (z) = (1 - z) \ ; \ e^{S'} (z) = (1 \ - z)^{2'} = \frac{1}{z} - \frac{z}{1} \quad (2.1.2)$$

The generating function for the basis vectors of Verma module $V$,

$$e^{S'} \ ; \ V = \sum_{k=0}^{\infty} \frac{X^k}{k!} v_0 = \sum_{k=0}^{\infty} \frac{X^k}{k!} v$$

can be calculated in closed form in the functional representation $V$ using (2.1.2)

$$e^{S+} (1 \ - z)^{2'} = \sum_{k=0}^{\infty} \frac{X^k}{k!} (2') z^k \ ; \ (2')_k = \frac{(2'+k)}{(2')}$$
2.2 Yang-Baxter equation and Lax operator

Let $V_1$, $V_2$, and $V_3$ be lowest weight $s'(2)$-modules and consider the three operators $R_{1,2}(u)$ which are acting in $V_1 \otimes V_2$. The Yang-Baxter equation is the following three term relation

\[ R_{1,2}(u)R_{2,3}(v)R_{1,3}(u) = R_{2,3}(v)R_{1,3}(u)R_{1,2}(v) \]  

We seek the general $s'(2)$-invariant solution $R_{1,2}(u)$ of this equation. The natural way to start from the simplest solutions of Yang-Baxter equation and derive the defining equation for the general R-operator [7]. First we put $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{2}$ in (2.2.1), consider the restriction to the invariant subspace $C^2 \otimes C^2 \otimes C^2$ and obtain the equation

\[ R_{12}(u)R_{13}(v)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u) \]

where the operator $R_{12}(u)$ acts on the first and second copy of $C^2$ in the tensor product $C^2 \otimes C^2 \otimes C^2$ and similarly for the other R-operators. The solution is the Yang's $s'(2)$-invariant R-matrix [23]

\[ R_{12}(u) = u + P_{12} \]

where $P_{12}$ is the permutation operator in $C^2 \otimes C^2$. Next we choose $\gamma_1 = \gamma_2 = \gamma_3 = \frac{1}{2}$ and consider the restriction to the invariant subspace $V \otimes C^2 \otimes C^2$. The restriction of the operator $R_{1,2}(u)$ to the space $V$, $C^2$ coincides up to normalization and the shift of the spectral parameter with the fundamental Lax-operator

\[ L(u) : V \otimes C^2 ! V \otimes C^2 \]

and the Yang-Baxter equation defines the commutation relations for the Lax-operators [6]

\[ L^{(1)}(u) \otimes L^{(2)}(u)R_{12}(v) = R_{12}(v)L^{(2)}(u)L^{(1)}(u) \otimes v \]
where \( L^{(1)}(u) \) is the operator which acts nontrivially on the space \( V \), and the first copy of \( C^2 \) in the tensor product \( V \otimes C^2 \) and \( L^{(2)}(u) \) is the operator which acts nontrivially on the \( V \) and the second copy of \( C^2 \). The solution is given (up to additive constant) by the Casimir operator \( C_2 \) for the tensor product of representations \( V \otimes C^2 \). \[7\]

\[
L(u) \quad u + 2S \quad s + S \quad s_+ + S_\quad s = u + S \quad S = u + \frac{s}{2} + z\bar{z} \quad \frac{\theta}{2} + 2z u \quad \frac{\theta}{2}
\]

where \( s/s_+ \) are the generators in the fundamental representation \[2.1,3\] and \( s_S \) are generators \[2.1.1\] in the generic representation \( V \). The Lax operator acts in the space \( C[z] \otimes C^2 \) and despite the compact notation \( L(u) \) depends really on two parameters: spin \( \frac{s}{2} \) and spectral parameter \( u \). We shall use extensively the parametrization \( u_1 = u + \frac{s}{2} u_2 \) and show all parameters explicitly. There exists very useful factorized representation for the L-operator \[3\]

\[
L(u_1; u_2) = \begin{pmatrix} u_1 + z\bar{z} & 0 & \theta & 0 \\ z\bar{z} + (u_1 \ u_2)z & u_2 & z\bar{z} \\ 1 & 0 & u_2 & z \end{pmatrix} (2.2.2)
\]

The L-operator is \( s(2) \) invariant by construction and as a consequence we obtain the useful equality

\[
1 \quad 0 \\ 0 \quad 1 \\ L(u) \quad 1 \quad 0 \\ 1 = e^s \quad L(u) \quad e^s = e^\theta \quad L(u) \quad e^\theta (2.2.3)
\]

Indeed, we have

\[
(s + S) \quad L(u) = L(u) \quad (s + S) \quad e^s \quad L(u) \quad e^s = L(u) \quad e^s \quad L(u) \quad e^s \quad e^s = 1 \quad 0 \\ 1 \quad 1
\]

Finally we put \( \frac{s}{2} = \frac{1}{2} \) in \[2.2.1\] and consider the restriction on the invariant subspace \( V \). In this way one obtains the defining equation for the operator \( R_{\frac{1}{2} \frac{1}{2}}(u) \) \[7,18\]

\[
R_{\frac{1}{2} \frac{1}{2}}(u \ v) L_1(u) L_2(v) = L_2(v)L_1(u)R_{\frac{1}{2} \frac{1}{2}}(u \ v)
\]

The operator \( L_k \) acts nontrivially on the tensor product \( V \otimes C^2 \) which is isomorphic to \( C[z] \otimes C^2 \) and the operator \( R_{\frac{1}{2} \frac{1}{2}}(u) \) acts nontrivially on the tensor product \( V \otimes V \) which is isomorphic to \( C[z_1] \otimes C[z_2] = C[z_1; z_2] \).

### 2.3 The general R-matrix

Now we are going to solve the defining equation for the general \( R \)-matrix. It is useful to extract the operator of permutation \( P_{12} \)

\[
P_{12} \quad (z_1; z_2) = (z_2; z_1) \quad (z_1; z_2) \quad 2 C[z_1; z_2]
\]

from the \( R \)-operator \( R_{\frac{1}{2} \frac{1}{2}}(u) = P_{12} R_{\frac{1}{2} \frac{1}{2}}(u) \) and solve the defining equation for the \( R \)-operator

\[
R(u_1; u_2 \, v_1; v_2) L_1(u_1; u_2) L_2(v_1; v_2) = L_1(v_1; v_2) L_2(u_1; u_2) R(u_1; u_2 \, v_1; v_2)
\]

To avoid misunderstanding we present this equation in explicit form

\[
R(u_1; u_2 \, v_1; v_2) = \begin{pmatrix} u_1 + z_1 \theta_1 & \theta_1 & v_1 + z_2 \theta_2 & \theta_2 \\ z_1^2 \theta_1 + (u_1 \ u_2)z_1 & u_2 & z_1 \theta_1 & z_2^2 \theta_2 + (v_1 \ v_2)z_2 & v_2 & z_2 \theta_2
\end{pmatrix}
\]
Proposition 1 There exists operator $R_1$ which is the solution of defining equations

$$R_1 L_1 (u_1; u_2) L_2 (v_1; v_2) = L_1 (v_1; u_2) L_2 (u_1; v_2) R_1 \quad (2.3.1)$$

$$R_1 = R_1 (u_1 \dot{y}_1; v_2); \quad R_1 (u_1 \dot{y}_1; v_2) = R_1 (u_1 + \dot{y}_1 + ; v_2 + )$$

and these requirements $x$ the operator $R_1$ up to overall normalization constant $z_{21} = z_2 z_1$

Proposition 2 There exists operator $R_2$ which is the solution of defining equations

$$R_2 L_1 (u_1; u_2) L_2 (v_1; v_2) = L_1 (v_1; u_2) L_2 (u_1; v_2) R_2 \quad (2.3.2)$$

$$R_2 = R_2 (u_1; u_2 \dot{y}_2); \quad R_2 (u_1; u_2 \dot{y}_2) = R_2 (u_1 + ; u_2 + \dot{y}_2 + )$$

and these requirements $x$ the operator $R_2$ up to overall normalization constant $z_{12} = z_1 z_2$

Proposition 3 The operator $R$ can be factorized in a following way

$$R (u_1; u_2 \dot{y}_1; v_2) = R_1 (u_1 \dot{y}_1; u_2) R_2 (u_1; u_2 \dot{y}_2) = R_2 (v_1; u_2 \dot{y}_2) R_1 (u_1 \dot{y}_1; v_2) \quad (2.3.3)$$

Note that the $R$-operators change the spins of $s^*(2)$-representations

$$R_1 : V_1 V_2 \ldots V_{1+1}; \quad \theta_1 = \frac{u_1}{v_1} \quad (2.3.4)$$

$$R_2 : V_1 V_2 \ldots V_{2+1}; \quad \theta_2 = \frac{u_2}{v_2} \quad (2.3.5)$$

but the general R-matrix $R_{1,2} (u) = P_{12} R_1 (u_1 \dot{y}_1; u_2) R_2 (u_1; u_2 \dot{y}_2)$ appears automatically $s^*(2)$-invariant.

The factorization of the $R$-operator can be proven using the simple pictures. The operator $R$ interchanges all parameters in the product of two $L$-operators. The operator $R_2$ interchanges the parameters $u_2$ and $v_2$ only and the operator $R_1$ interchanges the parameters $u_1$ and $v_1$. Using the operator $R_1 R_2$ it is possible to interchange parameters $u_1 v_1$ and $u_2 v_2$ in two steps so that we obtain the first equality in (2.3.3) as the condition of commutativity for the diagram
It is possible to exchange the parameters in the opposite order so that the second equality in (2.3.3) is the condition of commutativity for the diagram

\[
\begin{array}{ccc}
L_1(v_1;u_2) & L_2(v_1;v_2) & R_2(v_1;u_2)
\end{array}
\]

The defining system of equations for the R-operator is the system of differential equations of the second order. It can be reduced to the simpler system of equations of the first order which clearly shows the s'(2)-covariance of the R-operator.

Lemma 1 The system of equations (2.3.1) for the operator R_1 is equivalent to the system

\[
R_1 [L_1(u_1;u_2) + L_2(v_1;v_2)] = [L_1(v_1;u_2) + L_2(u_1;v_2)] \quad R_1 \quad z = z_1 \quad R_1 \quad (2.3.6)
\]

Lemma 2 The system of equations (2.3.2) for the operator R_2 is equivalent to the system

\[
R_2 [L_1(u_1;u_2) + L_2(v_1;v_2)] = [L_1(u_1;v_2) + L_2(v_1;u_2)] \quad R_2 \quad z = z_2 \quad R_2 \quad (2.3.7)
\]

Note that the relations in the first place are simply the rules of commutation of R-operators with s'(2)-generators written in a compact form. In explicit notations it is exactly the relations (2.3.4), (2.3.5). The s'(2)-invariance of R-matrix follows directly from the properties of R-operators.

Proof of the Lemmas 2 and the Proposition 2. As an example we shall consider the operator R_2 and all calculations for the operator R_1 are very similar. We are going to prove that the defining equation (2.3.2) is equivalent to the system (2.3.1) and derive the explicit formula for the operator R_2. First we show that the system (2.3.1) is the direct consequence of the eq. (2.3.2). Let us make the shift u_1^1 \rightarrow u_1 + \ , u_2^1 \rightarrow u_2 + \ , v_1^1 \rightarrow v_1 + \ , v_2^1 \rightarrow v_2 + \ in the defining equation (2.3.2). The R-operator is invariant under this shift, L-operators transform simply

\[
L_1 \quad L_1 + \quad L_2 \quad L_2 + \quad L + (\quad) \quad 1 \quad 0
\]

so that we derive as consequence of eq. (2.3.2)

\[
R_2 [L_1(u_1;u_2) + L_2(v_1;v_2)] = [L_1(u_1;v_2) + L_2(v_1;u_2)] \quad R_2 \quad z = z_2 \quad R_2
\]
\[ R_2 \mathbf{L}(u_1; u_2) \begin{pmatrix} 1 \\ z_2 \end{pmatrix} = L_1(u_1; v_2) \begin{pmatrix} 1 \\ z_2 \end{pmatrix} R_k \]

It is easy to prove that the last equation is the direct consequence of the first and the second ones. Next we show that from the system of equations \((2.3.7)\) follows eq. \((2.3.2)\). This will be almost trivial if we rewrite the equations \((2.3.7)\) and \((2.3.2)\) in equivalent form using the \(s^r(2)\)–invariance of the \(L\)-operator and the commutativity of \(R_2\) and \(z_2\). We start from the equation \((2.3.2)\) for the operator \(R_2\) and use the factorized expression \((2.2.2)\) for the \(L_2\)-operator.

Next we perform the similarity transformation \(M^{-1} M\) of this matrix equation

\[ R_2 \begin{pmatrix} v_1 & 1 \\ 0 & v_2 \end{pmatrix} = L_1(u_1; v_2) \begin{pmatrix} v_1 & 1 \\ 0 & v_2 \end{pmatrix} R_k \]

The \(s^r(2)\)-invariance of \(L\)-operator allows to rewrite the matrix \(M^{-1} L_1 M\) in the form \((2.2.3)\)

\[ M^{-1} L_1(u_1; u_2) M = e^{z_2 \theta_1} L_1(u_1; u_2) e^{z_2 \theta_1} \]

so that we have the simple equation for the operator \(r e^{z_2 \theta_1} R_k e^{z_2 \theta_1}\) (note that \(rz_2 = z_2 r\))

\[ r \mathbf{L}(u_1; u_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = L_1(u_1; v_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} R_k \]

This system of equations is equivalent to the system \((2.3.2)\) written in terms of \(r\). To derive the system of equations which is equivalent to the system \((2.3.7)\) written in terms of \(r\) we repeat the same trick with the shift of parameters and obtain the equations

\[ r L_1(u_1; u_2) + \begin{pmatrix} v_1 & 1 \\ 0 & v_2 \end{pmatrix} = L_1(u_1; v_2) + \begin{pmatrix} v_1 & 1 \\ 0 & v_2 \end{pmatrix} r \]

\[ r \mathbf{L}(u_1; u_2) = L_1(u_1; v_2) \]

The second equation is the evident consequence of the first one. We must prove that the system \((2.3.8)\) is equivalent to the system \((2.3.9)\). First step we factorize in \((2.3.8)\) the matrix \(\text{diag}(v_1, 1; 1)\) from the right so that the parameter \(v_1\) disappears from the equation

\[ r \mathbf{L}(u_1; u_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = L_1(u_1; v_2) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \]

It is easy to see that there are two new equations in comparison with \((2.3.5)\)

\[ r \mathbf{L}(u_1; u_2) \begin{pmatrix} \theta_1 \\ v_2 \end{pmatrix} = L_1(u_1; v_2) \begin{pmatrix} \theta_1 \\ u_2 \end{pmatrix} r \]

In explicit form the defining system \((2.3.9)\) contains three equations

\[ rz_1 \theta_1 = z_1 \theta_1 r; rz_2 = \theta_2 r; rz_1 (z_1 \theta_1 + u_1 u_2) = z_1 (z_1 \theta_1 + u_1 v_2) r \]
The "down"-equation in (2.3.10) is the simple consequence of equation \( r \theta_1 = z_1 \theta_1 \) \( r \) from the system (2.3.11) and the "up"-equation

\[
r(z_1 \theta_1 + u_1 \ v_2) \theta_1 = (z_1 \theta_1 + u_1 \ u_2) \theta_1 \ r
\]
is equivalent to the last equation from the system (2.3.11). Indeed we have

\[
r z_1 (z_1 \theta_1 + u_1 \ u_2) = z_1 (z_1 \theta_1 + u_1 \ v_2) r ! \quad r z_1 (z_1 \theta_1 + u_1 \ u_2) \theta_1 = z_1 (z_1 \theta_1 + u_1 \ v_2) r \theta_1 !
\]

! \( z_1 (z_1 \theta_1 + u_1 \ u_2) \theta_1 r = z_1 (z_1 \theta_1 + u_1 \ v_2) \theta_1 ! \quad (z_1 \theta_1 + u_1 \ u_2) \theta_1 r = r(z_1 \theta_1 + u_1 \ v_2) \theta_1
\]

We have proved the equivalence of the system (2.3.8) and (2.3.9) and therefore the equivalence of the systems (2.3.11) and (2.3.12). It remains to solve the solution. First we solve the equations (2.3.11) for operator \( r \). The solution of equation \( rz_2 = z_2 r \) and the first two equations from the system (2.3.11) is \( r = r[z_1 \theta_1] \). Then the last equation leads to the recurrence relation which has the simple solution

\[
r[z_1 \theta_1 + 1] (z_1 \theta_1 + u_1 \ u_2) = r[z_1 \theta_1] (z_1 \theta_1 + u_1 \ v_2) =) \quad r[z_1 \theta_1] \frac{(z_1 \theta_1 + u_1 \ v_2)}{(z_1 \theta_1 + u_1 \ u_2)}
\]

Finally we derive the expression for the operator \( R_2 \) from the Proposition

\[
R_2 = e^{z_2 \theta_1} r e^{z_2 \theta_1} \frac{(z_{12} \theta_1 + u_1 \ v_2)}{(z_{12} \theta_1 + u_1 \ u_2)};
\]

3 The general \( sl(3) \)-invariant \( R \) - matrix

In the previous section we have constructed the general solution of the Yang-Baxter equation with symmetry algebra \( s'(2) \) and proved that it has simple factorized structure. Next we shall consider the first nontrivial example of the symmetry algebra of the rank two. In fact, we repeat step by step all calculations from the previous section and show that the general solution of the Yang-Baxter equation with the symmetry algebra \( s'(3) \) has the very similar structure.

3.1 \( sl(3) \) lowest weight modules

The algebra \( s'(3) \) has eight generators \( T_{ab} \); \( a; b = 1; 2; 3 \) with condition \( T_{11} + T_{22} + T_{33} = 0 \). The commutation relations have the standard form \( [T_{ab}; T_{cd}] = \delta_{ab} T_{ad} - \delta_{ad} T_{ab} \)

We shall use the following generators of the Cartan subalgebra

\[
H_1 = T_{11} \quad T_{22} \quad H_2 = T_{22} \quad T_{33} \quad T_{11} = \frac{2}{3} H_1 + \frac{1}{3} H_2 \quad T_{22} = \frac{1}{3} H_2 + \frac{1}{3} H_1 \quad T_{33} = \frac{1}{3} H_1 + \frac{2}{3} H_2
\]

There are two central elements \( C_2 = \frac{P}{abcd} T_{ab} T_{bc} \) and \( C_3 = \frac{P}{abc} T_{ab} T_{bc} T_{ca} \). The Verma module is the generic lowest weight \( s'(3)-m \) module \( V = (m; n) \). As a linear space \( V \) is obtained by application of operators \( T_{12}; T_{13}; T_{23} \) to the lowest weight vector \( a_0 \)

\[
T_{21} a_0 = T_{31} a_0 = T_{32} a_0 = 0 \quad H_1 a_0 = m \quad a \quad H_2 a_0 = n \quad a
\]
We shall use the representation $V$ of $s'(3)$ in the finite-dimensional space $C[x; y; z]$ of polynomials in variables $x; y; z$ and lowest weight vector $a_0 = 1$ \cite{10, 12}. The action of $s'(3)$ in $V$ is given by the first-order differential operators. Lowering (decreasing the polynomial degree) operators have the form

$$T_{21} = \theta_x; \ T_{31} = \theta_y; \ T_{32} = \theta_z \ x \theta_y$$

and generate the following global transformations

$$e^{T_{21}} (x; y; z) = (x + y; z); \ e^{T_{31}} (x; y; z) = (x; y + z)$$

$$e^{T_{32}} (x; y; z) = (x; y; x + z)$$

Rising (increasing the polynomial degree) operators

$$T_{12} = x^2 \theta_x, \ y x \theta_y, \ z \theta_z + y \theta z + n x; \ T_{23} = z^2 \theta_z, \ y \theta_x + m z$$

$$T_{13} = y^2 \theta_y, \ x y \theta_x, \ z(y + x z) \theta_z + (n + m) y + m x z$$

generate the global transformations

$$e^{T_{12}} (x; y; z) = [1 + x]^n \ x \frac{x}{1 + x}; \ \frac{y}{1 + x}; \ z + (y + x z)$$

$$e^{T_{23}} (x; y; z) = [1 + z]^m \ x \ \frac{y}{1 + z}; \ \frac{z}{1 + z}$$

$$e^{T_{13}} (x; y; z) = [1 + y]^n \ [1 + (y + x z)]^m \ \frac{x}{1 + y}; \ \frac{y}{1 + y}; \ \frac{z}{1 + y}; \ (y + x z)$$

Two remaining elements of the Cartan subalgebra:

$$H_1 = 2x \theta_x + y \theta_y, \ z \theta_z \ n; \ H_2 = 2z \theta_z + y \theta_y, \ x \theta_x \ m$$

generate the transformations:

$$H_1 (x; y; z) = n \ x; \ y; \ z; \ H_2 (x; y; z) = m \ z; \ y; \ z$$

Using these formulae it is possible to derive the closed expression for the generating functions of the basis vectors

$$e^{T_{12}} e^{T_{23}} e^{T_{13}} 1 = [1 + x + y]^n \ [1 + z + ( + ) (y + x z)]^m$$

The power expansion of these generating functions in $;$ gives the elements of the basis. It is evident that for the generic $m \neq M; \ n \neq N$ the module $\mathcal{V}$ is an irreducible lowest weight $s'(3)$-module isomorphic to $V$ but for the special values of the spin $m = M; \ n = N$ there exists the finite-dimensional invariant subspace $\mathcal{V}_{M, N}$ with dimension

$$\dim \mathcal{V}_{M, N} = \frac{1}{2} (M + 1) (N + 1) (M + N + 2)$$

For generic $M; N \neq 0$ the space $\mathcal{V}_{M, N}$ is invariant subspace in the space of polynomials in three variables $C[x; y; z].$ In the case $N = 0$ we have

$$e^{T_{12}} e^{T_{23}} e^{T_{13}} 1 = [1 + z + ( + ) (y + x z)]^M$$
so that one obtains the invariant subspace $V_{M,0}$ with dimension $\dim V_{M,0} = \frac{1}{2} (M + 1)(M + 2)$ in the space of polynomials in two variables $C \{y + xz; z\}$. In the case $M = 0$ we have

$$e^{T_{12}}e^{T_{23}}e^{T_{13}}l = [1 + x + y]$$

so that one obtains the invariant subspace $V_{0\|0}$ with dimension $\dim V_{0\|0} = \frac{1}{2} (N + 1)(N + 2)$ in the space of polynomials in two variables $C \{x; y\}$. We shall use the three-dimensional representation $V_{(1,0)} \cong C^3$. In the basis

$$e_1 = T_{13} \quad l = y + xz; \quad e_2 = T_{23} \quad l = z; \quad e_3 = 1$$

the $s'(3)$-generators take the form

$$
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
$$

There exists the second three-dimensional representation $V_{(0,1)} \cong C^3$. In the basis

$$e_1 = 1; \quad e_2 = T_{12} \quad l = x; \quad e_3 = T_{13} \quad l = y$$

the $s'(3)$-generators take the similar form but $t_{ik} ! = t_{ki} ! = h_1 ! = h_2 ! = h_2 ! = h_2$.

### 3.2 Yang-Baxter equation and Lax operator

The Yang-Baxter equation is the following three-term relation

$$R_{12}(u \quad v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u \quad v)$$

for the operators $R_{12}(u) : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$. As in the $s'(2)$-case we start from the simplest solutions of Yang-Baxter equation and derive the defining equation for the general $R$-operator. First we put $k = 1 = 2 = 3 = (1;0)$ in Yang-Baxter equation and consider the restriction to the invariant subspace $C^3 \otimes C^3 \otimes C^3$. We obtain the equation

$$R_{12}(u \quad v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u \quad v) \quad (3.2.1)$$

where the operator $R_{12}(u)$ acts on the first and second copy of $C^3$ in the tensor product $C^3 \otimes C^3 \otimes C^3$ and similarly for the other $R$-operators. The solution is well known [1,13]

$$R_{12}(u) = u + P_{12}$$

where $P_{12}$ is the permutation operator in $C^3 \otimes C^3$. Secondly we choose $1 = 2 = (1;0); \quad 3 = (m; n)$ and consider the restriction to the invariant subspace $C^3 \otimes C^3 \otimes V$. The restriction
of the operator $R_{12} (u)$ to the space $C^3\ V_0$ coincides up to normalization and shift of spectral parameter with the Lax-operator

$$L (u) : C^3 \ V_0 \rightarrow C^3 \ V_0$$

and the Yang-Baxter equation coincides with the fundamental commutation relations for the Lax-operator $L_{12} [13,14,15]$

$$R_{12} (u \ v) L^{(1)} (u) L^{(2)} (v) = L^{(2)} (v) L^{(1)} (u) R_{12} (u \ v)$$

where $L^{(1)} (u)$ is the operator which acts nontrivially on the first copy of $C^3$ and $V_0$ in the tensor product $C^3 \otimes C^3 \ V_0$ and $L^{(2)} (u)$ is the operator which acts nontrivially on the second copy of $C^3$ and $V_0$. The solution coincides up to additive constant with the Casimir operator $C_2$ for the representation $C^3 \otimes C^3 \ V_0$. We shall use the Lax-operator $L (u)$

$$L (u) = \begin{pmatrix} 0 & \frac{2}{3} H_1 + \frac{1}{3} H_2 + u & T_{21} & T_{31} & 1 \\ \frac{1}{3} H_2 & \frac{2}{3} H_1 + u & T_{32} & A \\ \frac{1}{3} H_1 & \frac{2}{3} H_2 + u & 0 \\ \end{pmatrix}$$

in defining equation for the general $R$-operator. The Lax-operator $L (u)$ depends on three parameters $u,m,n$. We shall use the parametrization

$$u_1 = u + \frac{m + 2n}{3} \quad ; \quad u_2 = u + \frac{n + 2m}{3} \quad ; \quad u_3 = u + \frac{m + 2n}{3} \quad ; \quad m = u_3 \quad u_2 \quad 1 \quad ; \quad n = u_2 \quad u_1 \quad 1$$

and show this parameters explicitly. The Lax operator $L (u_1;u_2;u_3)$ in the functional representation $V_0$ has the form

$$L (u_1;u_2;u_3) = \begin{pmatrix} 0 & x \theta_x + y \theta_y + u_1 + 2 & 0 & 0 & 1 \\ L_{21} & 0 & x \theta_x + z \theta_z + u_2 + 1 & 0 & \theta_y \\ L_{31} & \frac{1}{L_{32}} & 0 & y \theta_y & z \theta_z + u_3 \\ \end{pmatrix}$$

$$L_{21} = x^2 \theta_x + xy \theta_y + (xz+y) \theta_z + (u_2 \ u_1 \ 1)x \quad ; \quad L_{32} = y \theta_x + z^2 \theta_z + (u_3 \ u_2 \ 1)z$$

$$L_{31} = x y \theta_x + y^2 \theta_y + z (xz+y) \theta_z + (u_3 \ u_2 \ 1)x + (u_3 \ u_1 \ 2)y$$

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and similar to s'(2)-case there exists the factorized representation for the L-operator

\[ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ u_1 & z & 0 \end{pmatrix} \begin{pmatrix} x & 1 & 0 \\ 0 & A & 0 \\ u_2 & 0 & A \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ y & z & 1 \\ u_3 + y + z & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \] (3.2.3)

The L-operator is s'(3)-invariant by construction and as consequence one obtains the useful equality

\[ M^1 L(u) M = \hat{S} L(u) S \quad S = e^{xy} e^{yz} e^{zx} ; \quad M = a \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ b & c & 1 \end{pmatrix} \] (3.2.4)

Finally we put \( i = 1 \) in Yang-Baxter equation

\[ R_{12} (u \quad v) R_{13} (u) R_{23} (v) = R_{23} (v) R_{13} (u) R_{12} (u \quad v) \]

change the notation of the representation spaces \( 2 \quad 1 = (m_1 ; n_1) \); \( 3 \quad 1 = (m_2 ; n_2) \)

and consider the restriction to the invariant subspace \( V_1 \quad V_2 \). In this way one obtains the defining equation for the R-operator

\[ L_1 (u \quad v) L_2 (u) R_{12} (v) = R_{12} (v) L_2 (u) L_1 (u \quad v) \]

The operator \( L_k \) acts nontrivially on the tensor product \( C^3_{k_1 ; k_2 ; k_3} \) and the operator \( R_{12} (u) \) acts nontrivially on the tensor product \( V_1 \quad V_2 \) which is isomorphic to \( C \[ k_1 ; k_2 ; k_3 \] \). Note that obtained defining equation is slightly different from the ones we have used in s'(2)-case. The defining equation which is similar to s'(2) case is

\[ R_{12} (v \quad u) L_1 (u) L_2 (v) = L_2 (v) L_1 (u) R_{12} (v \quad u) \] (3.2.5)

There exists the well known automorphism \([14,15]\) of the Yang-Baxter equation \( R_{12} (u) \) \( R_{12} (u) \) for the most complicated algebras the action of this automorphism is nontrivial. To proceed in close analogy with s'(2) case we shall use the defining equation (3.2.5) so that we derive the expression for the operator \( R_{12} (v \quad u) \).

### 3.3 The General R-matrix

Now we are going to the solution of the defining equation for the general R-matrix. It is useful to extract the operator of permutation

\[ P_{12} : C \[ k_1 ; k_2 \] \quad C \[ k_1 ; k_2 \] \quad C \[ k_1 ; k_2 \] \quad C \[ k_1 ; k_2 \] \quad C \[ k_1 ; k_2 \] \]

\[ P_{12} (k_1 ; y_1 ; z_1 \quad k_2 ; y_2 ; z_2) = (k_2 ; y_2 ; z_2 \quad k_1 ; y_1 ; z_1) \]

from the R-operator \( R_{12} (v \quad u) = P_{12} R (u ; v) \) and solve the defining equation for the R-operator. The main defining equation for the R-operator is

\[ R (u ; v) L_1 (u_1 ; u_2 ; u_3) L_2 (v_1 ; v_2 ; v_3) = L_1 (v_1 ; v_2 ; v_3) L_2 (u_1 ; u_2 ; u_3) R (u ; v) \] (3.3.1)
\[ u_1 = u \quad 2 \quad \frac{m_1 + 2n_1}{3} ; \quad u_2 = u \quad 1 + \frac{n_1 m_1}{3} ; \quad u_3 = u + \frac{n_1 + 2m_1}{3} \]
\[ v_1 = v \quad 2 \quad \frac{m_2 + 2n_2}{3} ; \quad v_2 = v \quad 1 + \frac{n_2 m_2}{3} ; \quad v_3 = v + \frac{n_2 + 2m_2}{3} \]

The operator \( R \) can be represented as the product of the simpler "elementary building blocks" - \( R \)-operators.

**Proposition 4** There exists operator \( R_1 \) which is the solution of the defining equations

\[ R_1 L_1 (u_1; u_2; u_3) L_2 (v_1; v_2; v_3) = L_1 (v_1; u_2; u_3) L_2 (u_1; v_2; v_3) R_1 \quad (3.3.2) \]

\[ R_1 = R_1 (u_1 \dot{y}_1; v_2; v_3) ; \quad R_1 (u_1 \dot{y}_1; v_2; v_3) = R_1 (u_1 + \dot{y}_1 + \dot{v}_2 + \dot{v}_3 + \dot{v}_3) \]

and these requirements \( \times \) the operator \( R_1 \) up to overall normalization constant

\[ R_1 S_1 = \frac{(x \Theta_x + u_1 \dot{v}_2 + 1)}{(x \Theta_x + 1)} e^{x \Theta_x} \quad \frac{(y \Theta_y + u_1 \dot{v}_3 + 1)}{(y \Theta_y + 1)} e^{y \Theta_x} \quad \frac{(x \Theta_x + 1)}{(x \Theta_x + 1)} \]

\[ S_1 = e^{v_1 \dot{z}_1 \Theta_y} e^{v_2 \dot{z}_2 \Theta_y} e^{v_3 \dot{z}_3 \Theta_y} \]

\( x = x_2 ; \quad y = y_2 + x_2 z_2 ; \quad z = z_2 ; \quad \Theta_x = \dot{z}_x ; \quad \dot{z}_y ; \quad \Theta_y = \dot{z}_y ; \quad \Theta_z = \dot{z}_z ; \quad x_2 \dot{z}_y \]

**Proposition 5** There exists operator \( R_2 \) which is the solution of the defining equations

\[ R_2 L_1 (u_1; u_2; u_3) L_2 (v_1; v_2; v_3) = L_1 (u_1; v_2; u_3) L_2 (v_1; u_2; v_3) R_2 \quad (3.3.3) \]

\[ R_2 = R_2 (u_1 u_2 \dot{y}_2; v_3) ; \quad R_2 (u_1 u_2 \dot{y}_2; v_3) = R_2 (u_1 + u_2 + \dot{y}_2 + \dot{v}_3 + \dot{v}_3) \]

and these requirements \( \times \) the operator \( R_2 \) up to overall normalization constant

\[ R_2 S_2 = \frac{(z_2 \Theta_z + u_2 \dot{v}_3 + 1)}{(z_2 \Theta_z + 1)} e^{z_2 \Theta_x} \quad \frac{(x_1 \Theta_x + u_1 \dot{v}_2 + 1)}{(x_1 \Theta_x + 1)} e^{x_1 \Theta_x} \quad \frac{(z_2 \Theta_z + 1)}{(z_2 \Theta_z + 1)} \]

\[ S_2 = e^{v_2 \dot{z}_1 \Theta_y} e^{v_2 \dot{z}_2 \Theta_y} e^{v_2 \dot{z}_3 \Theta_y} \]

**Proposition 6** There exists operator \( R_3 \) which is the solution of the defining equations

\[ R_3 L_1 (u_1; u_2; u_3) L_2 (v_1; v_2; v_3) = L_1 (u_1; v_2; u_3) L_2 (u_1; v_2; v_3) R_3 \quad (3.3.4) \]

\[ R_3 = R_3 (u_1 u_2 u_3 \dot{y}_3) ; \quad R_3 (u_1 u_2 u_3 \dot{y}_3) = R_3 (u_1 + u_2 + u_3 + \dot{y}_3 + \dot{y}_3) \]

and these requirements \( \times \) the operator \( R_3 \) up to overall normalization constant

\[ R_3 S_3 = \frac{(x_1 \Theta_x + u_2 \dot{v}_3 + 1)}{(x_1 \Theta_x + 1)} e^{x_1 \Theta_x} \quad \frac{(y_1 \Theta_y + u_1 \dot{v}_3 + 1)}{(y_1 \Theta_y + 1)} e^{y_1 \Theta_x} \quad \frac{(z_1 \Theta_z + 1)}{(z_1 \Theta_z + 1)} \]

\[ S_3 = e^{v_2 \dot{z}_1 \Theta_y} e^{v_2 \dot{z}_3 \Theta_y} e^{v_3 \dot{z}_3 \Theta_y} \]

**Proposition 7** The \( R \)-operator can be factorized as follows

\[ R (u; v) = R_1 (u_1; v_1; u_2; u_3) R_2 (u_1 u_2; v_2; u_3) R_3 (u_1 u_2 u_3; v_3) \quad (3.3.5) \]
There exist six equivalent ways to represent \( R \) in an factorized form which differ by the order of \( R \)-operators and their parameters. All three expressions and the proof of the factorization of the \( R \)-operator can be obtained using the pictures similar to \( s^r(2) \)-case.

The defining system of equations for the \( R \)-operator can be reduced to the simpler system which clearly shows the property of \( s^r(3) \)-covariance of the \( R \)-operator.

**Lemma 3** The defining equation (3.3.2) for the operator \( R_1 \) is equivalent to the system of equations

\[
R_1 \left[ L_1 (u_1; u_2; u_3) + L_2 (v_1; v_2; v_3) \right] = \left[ L_1 (v_1; u_2; u_3) + L_2 (v_1; u_2; v_3) \right] R_1 \tag{3.3.6}
\]

\[
R_1 x_1 = x_1 R_1 ; \quad R_1 y_1 = y_1 R_1 ; \quad R_1 z_1 = z_1 R_1
\]

\[
R_1 \begin{pmatrix} @ & x_2 & @ y_1 \end{pmatrix} = \begin{pmatrix} @ z_2 & x_2 & @ y_2 \end{pmatrix} \tag{3.3.7}
\]

**Lemma 4** The defining equation (3.3.3) for the operator \( R_2 \) is equivalent to the system of equations

\[
R_2 \left[ L_1 (u_1; u_2; u_3) + L_2 (v_1; v_2; v_3) \right] = \left[ L_1 (u_1; v_2; u_3) + L_2 (v_1; u_2; v_3) \right] R_2 \tag{3.3.8}
\]

\[
R_2 y_1 + x_1 z_1 = (y_1 + x_1 z_1) R_2 ; \quad R_2 z_1 = z_1 R_2 ; \quad R_2 x_2 = x_2 R_2 ; \quad R_2 y_2 = y_2 R_2
\]

**Lemma 5** The defining equation (3.3.4) for the operator \( R_3 \) is equivalent to the system of equations

\[
R_3 \left[ L_1 (u_1; u_2; u_3) + L_2 (v_1; v_2; v_3) \right] = \left[ L_1 (u_1; u_2; v_3) + L_2 (v_1; v_2; u_3) \right] R_3 \tag{3.3.9}
\]

\[
R_3 x_2 = x_2 R_3 ; \quad R_3 y_2 = y_2 R_3 ; \quad R_3 z_2 = z_2 R_3
\]

\[
R_3 \begin{pmatrix} @ x_1 & z_2 & @ y_1 \end{pmatrix} = \begin{pmatrix} @ x_1 & z_2 & @ y_1 \end{pmatrix} \tag{3.3.10}
\]

The relations in the first line are simply the rules of commutation of \( R \)-operators with \( s^r(3) \)-generators written in a compact form. In explicit notations we have for \( \gamma_1 = \left( m_1 ; n_1 \right) \) and \( \gamma_2 = \left( m_2 ; n_2 \right) \)

\[
R : V_1 V_2 \mapsto V_2^0 V_1^0
\]

\[
R_1 : \begin{pmatrix} 0 & 1 \end{pmatrix} = (m_1 ; n_1 + 1) ; \quad \begin{pmatrix} 0 & 2 \end{pmatrix} = (m_2 ; n_2 + 1) ; \quad 1 = u_1 ; v_1
\]

\[
R_2 : \begin{pmatrix} 0 & 1 \end{pmatrix} = (m_1 + 2 ; n_1 + 2) ; \quad \begin{pmatrix} 0 & 2 \end{pmatrix} = (m_2 + 2 ; n_2 + 2) ; \quad 2 = u_2 ; v_2
\]

\[
R_3 : \begin{pmatrix} 0 & 1 \end{pmatrix} = (m_1 + 3 ; n_1 + 3) ; \quad \begin{pmatrix} 0 & 2 \end{pmatrix} = (m_2 + 3 ; n_2 + 3) ; \quad 3 = u_3 ; v_3
\]

The \( s^r(3) \)-invariance of \( R \)-matrix follows directly from the properties of \( R \)-operators so that the general \( R \)-matrix \( R^{12}_{12} (v ; u) = P_{12} R (u ; v) \) is automatically \( s^r(3) \)-invariant.

**Proof.** We shall consider the operators \( R_3 \) and \( R_2 \). All calculations for the operator \( R_1 \) are very similar to the \( R_3 \)-case. Now we are going to the proof of equivalence of defining equation (3.3.4) to the system (3.3.5) and derivation of explicit formula for the operator \( R_3 \). First we show that the system (3.3.3) is the direct consequence of the eq. (3.3.4). Let us make the shift \( u_k \mapsto u_k + ; v_1 \mapsto v_1 + ; v_2 \mapsto v_2 + ; v_3 \mapsto v_3 + \) in the defining equation (3.3.4). The \( R \)-operator is invariant under this shift and \( L \)-operators transform as follows

\[
L_1 \mapsto L_1 + 1 ; \quad L_2 \mapsto L_2 + 1 + ( \theta x_2 0 0 A + ( \theta x_2 1 0 A ) y_2 = 0 0 x_2 z_2 z_2 = 0
\]

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The obtained after all equation is valid for arbitrary parameters, and as consequence we derive the system (3.3.3) and equation (3.3.10). Next we show that from the system of equations (3.3.3), (3.3.10) follows eq. (3.3.4). This will be almost evident if we rewrite these equations in equivalent form, using the $s^3$-invariance of the $L$-operator and the commutativity of $R_3$ and $x_2; y_2; z_2$. We substitute the factorized representation (3.2.4) for the operator $L_2$ in the equation (3.3.4) for the operator $R_3$ ($D_{x_2} = \Theta_{x_2} \ z_2 \Theta_{y_2}$)

\[
\begin{pmatrix}
0 & v_1 & D_{x_2} \Theta_{y_2} \\
0 & v_2 & \Theta_{y_2} A \ M \\
0 & v_3 & 0
\end{pmatrix} = L_1 (u_1; u_2; v_3) M @ 0 v_2 \Theta_{z_2} A \ M \ 1 \ R_3
\]

and perform the similarity transformation $M^{-1} = 1 \ 0 \ 0 \ 0 \ S^{-1} \ L \ S$; $S = \Theta_{x_1} x_1; \Theta_{y_1} y_1$; $\Theta_{x_1} x_1$; $\Theta_{y_1} y_1$; $M = \Theta_{x_2} \ x_2 \ 1 \ 0 \ A$

\[
y_2 \ z_2 \ 1
\]

we derive the equation for the transformed operator $r = S \ R_3 \ S^{-1}$

\[
r \ L (u_1; u_2; u_3) = L_1 (u_1; u_2; v_3) L (v_1; v_2; u_3) \ r \quad (3.3.11)
\]

where

\[
L (v_1; v_2; v_3) \ S @ 0 v_2 \Theta_{x_2} A \ S^{-1} = @ 0 v_2 \Theta_{x_2} \ D_{z_1} A \ ; D_{z_1} = \Theta_{z_1} x_1 \Theta_{y_1}
\]

\[
\begin{pmatrix}
0 & v_3 & D_{x_2} \Theta_{y_2} \\
0 & v_2 & \Theta_{y_2} A \\
0 & v_3 & 0
\end{pmatrix} = L_1 (u_1; u_2; v_3) @ 0 v_2 \Theta_{x_2} \ D_{z_1} A \ ; D_{z_1} = \Theta_{z_1} x_1 \Theta_{y_1}
\]

To derive the system of equations which is equivalent to the system (3.3.3), (3.3.10) written in terms of $r$ we repeat the same trick with the shift of parameters and obtain the system of equations

\[
r \ L (u_1; u_2; u_3) = L_1 (u_1; u_2; v_3) + L (v_1; v_2; u_3) \ r \quad (3.3.12)
\]

\[
r \ L (u_1; u_2; u_3) @ 0 1 A = L_1 (u_1; u_2; v_3) @ 0 1 A \ r \quad (3.3.13)
\]

It is evident that all equations of the system (3.3.3) contained in the equation (3.3.12) except only one (12)-equation $r \Theta_{x_1} = \Theta_{x_1} \ r$. We use the system of equation

\[
r \ L (u_1; u_2; u_3) = L_1 (u_1; u_2; v_3) \ r \ ; r \Theta_{x_1} = \Theta_{x_1} \ r \quad (3.3.14)
\]

as defining system for operator $r$. It is the system of equations (3.3.3), (3.3.10) written in terms of operator $r$. Returning to the system (3.3.11) (which is (3.3.4) written in terms of $r$) we note that it is possible to factorize the matrix $\text{diag} (v_1; v_2; 1)$ from the right

\[
r \ L (u_1; u_2; u_3) @ 0 1 D_{x_2} \Theta_{y_2} \ A = L_1 (u_1; u_2; v_3) @ 0 1 \Theta_{x_2} \ D_{z_1} A \ r
\]

\[
\begin{pmatrix}
0 & v_3 & D_{x_2} \Theta_{y_2} \\
0 & v_2 & \Theta_{y_2} A \\
0 & v_3 & 0
\end{pmatrix} = L_1 (u_1; u_2; v_3) @ 0 v_2 \Theta_{x_2} \ D_{z_1} A \ ; D_{z_1} = \Theta_{z_1} x_1 \Theta_{y_1}
\]
In comparison with (3.3.14) there are three new equations only
\[
\begin{align*}
0 & \quad 1 & \quad 0 & \quad 1 \\
\theta_y & & \theta_z & & \theta_y \\
L_r(u_1;u_2;u_3) & + x_1 \theta_y & = & L_1(u_1;u_2;v_3) & + x_1 \theta_y \\
& & & & r \\
& & & & v_3 \\
& & & & u_3
\end{align*}
\]
Indeed the system (3.3.14) contains the equations \([r;\theta_x] = [r;\theta_y] = [r;\theta_z] = [r;\theta_x] = 0\) and by conditions (3.3.13) we obtain only three new equations. It is possible to show that these equations follow from the system (3.3.14) so that it remains to find the solution of the system of equations (3.3.14). First of all \([r;x_1] = [r;y_1] = [r;z_1] = [r;\theta_x] = [r;\theta_y] = [r;\theta_z] = 0\) and therefore the operator \(r\) depends on the variables \(x_1;y_1;z_1\) only. There are six equations and for simplicity we change \(z_1;y_1;z_1 \rightarrow x;y;z\)
\[
\begin{align*}
\theta_x + y \theta_y & = (x \theta_x + y \theta_y) r \quad \theta_x = \theta_x r \\
\theta_x + z^2 \theta_z & = (u_2 - u_3 + 1) z \\
x^2 \theta_x + x y \theta_y & = y \theta_x + z^2 \theta_z + (u_2 - u_3 + 1) x \\
x y \theta_x + z^2 \theta_z & = (u_2 - u_3 + 1) z + y (y \theta_y + z \theta_z + u_1 - u_3 + 2) \\
\end{align*}
\]
We look the general solution in the form
\[
r = a [z \theta_z] \quad b [y \theta_y] \quad c [z \theta_z]
\]
where
\[
c [z \theta_z] = \frac{(z \theta_z + 1)}{(z \theta_z + u_2 - u_3 + 1)}
\]
It is evident that the solutions of the equations (3.3.15). The equations (3.3.16 - 3.3.18) lead to the recurrence relations for the functions \(a [z \theta_z]\) and \(b [y \theta_y]\)
\[
\begin{align*}
a [z \theta_z + 1] & = (z \theta_z + u_2 - u_3 + 1) a [z \theta_z] \\
b [y \theta_y + 1] & = (y \theta_y + u_1 - u_3 + 1) b [y \theta_y]
\end{align*}
\]
The solution of these equations is
\[
\begin{align*}
a [z \theta_z] & = \frac{(z \theta_z + u_2 - u_3 + 1)}{(z \theta_z + 1)} \\
b [y \theta_y] & = \frac{(y \theta_y + u_1 - u_3 + 1)}{(y \theta_y + 1)}
\end{align*}
\]
Collecting everything together we obtain the expression for the operator \(R_3\) from the Proposition. All calculations for the operator \(R_1\) are very similar and finally one arrives to the system which coincides with the system (3.3.15), (3.3.16) after the change of variables and parameters. In a such way one obtains the expression for the operator \(R_1\) from the Proposition. It remains to prove the equivalence of defining equation (3.3.3) to the system (3.3.3) and derive the explicit formula for the operator \(R_2\). First we show that the system (3.3.3) is the direct consequence of the eq. (3.3.3). Let us make the shift \(u_1 \rightarrow u_1 + v_1\) ; \(u_2 \rightarrow u_2 + v_2\) ; \(u_3 \rightarrow u_3 + v_3\) ; \(v_1 \rightarrow v_1 + v_2\) ;
\[ v_1 + ; v_2 + ; v_3 + \text{ in the defining equation (3.3.3) for the operator } R_2. \text{ The } R \text{-operator is invariant under this shift and } L \text{-operators transform as follows:} \]

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
y_1 + x_1z_1 & z_1 & 1 \\
\end{bmatrix} \\
\begin{bmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
y_2 & 0 & 0 \\
\end{bmatrix}
\]

The obtained after all equation has to be satisfied for arbitrary \( \theta \), and as consequence we derive the system (3.3.8). Next we show that from the systems of equations (3.3.8) follow eq. (3.3.3). This will be almost evident if we rewrite these equations in equivalent form using the \( s'3 \)-invariance of the \( L \)-operator and the commutativity of \( R_2 \) and \( z_1; y_1 + x_1z_1; x_2; y_2 \). We start from the equation (3.3.3) and make the two similarity transformations of the defining equation (3.3.3) using simple matrices which commute with operator \( R_2 \). After all these transformations the defining equation (3.3.3) for the \( R_2 \)-operator in factorized form looks as follows:

\[
\begin{bmatrix}
0 & 1 & 0 & 0 \\
y_1 + z_1x_1 & z_1 & 1 \\
\end{bmatrix}
\]

Next step we rewrite this equation in terms of the transformed operators \( r \) and \( s \):

\[
R_2 = S^{-1} r S; S = [e^{x_1z_1}; e^{y_1}; e^{x_2}; e^{y_2}] \\
0 & 1 & 0 & 0 \\
y_1 & 0 & 1 \\
\end{bmatrix} (3.3.19)
\]

where

\[
L_1 (u_1; u_2; u_3) = \begin{bmatrix}
0 & u_1 + 1 + x_1e^{x_1} & e^{x_1} & e^{y_1} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & z_1e^{y_1} & z_1 & e^{y_2} & e^{y_3} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} (3.3.20)
\]

To derive the system of equations which is equivalent to the system (3.3.8) written in terms of \( r \) we repeat the same trick with the shift of parameters and obtain

\[
r \otimes (u_1; u_2; u_3)m + m L_2 (v_1; v_2; v_3) = [L_1 (u_1; v_2; u_3)m + m L_2 (v_1; u_2; v_3)] (3.3.20)
\]
This system results in a simple equations. First of all we have \( [r; y_1] = [r; z_1] = [r; x_2] = [r; y_2] = 0 \) from the very beginning and the system contain the equations

\[
[r; y_2] = [r; z_1] = [r; x_2] z_2[0; y_2] = 0 \quad (3.3.21)
\]

\[
r(x_1[0; x_1] + y_1[0; y_1]) = (x_1[0; x_1] + y_1[0; y_1]) r \quad r(x_1[0; x_1] z_2[0; z_2]) = (x_1[0; x_1] z_2[0; z_2]) r \quad (3.3.22)
\]

\[
r x_1^2[0; x_1] + (u_2 u_1) x_1 y_1[0; x_2] = x_1^2[0; x_1] + (v_2 u_1) x_1 y_1[0; x_2] r \quad (3.3.23)
\]

\[
r z_2^2[0; z_2] + (v_3 v_2) z_2 y_1[0; x_1] = z_2^2[0; z_2] + (v_3 v_2) z_2 y_1[0; x_1] r \quad (3.3.24)
\]

Returning to the system 3.3.19 which is equivalent to the system 3.3.3 written in terms of \( r \) we note that it is possible to factorize the matrix \( \text{diag}(v_1; 1; 1) \) from the right and the matrix \( \text{diag}(1; 1; u_3) \) from the left so that the parameters \( v_1 \) and \( u_3 \) disappear from equation. The equivalence obtained system of equations to the system 3.3.21 - 3.3.24 can be proven by straightforward calculations. Now we are going to the solution of the defining system of equations. We look the general solution in the form

\[
r = a(z_2[0; z_2]) \text{ e}^x \quad b(k[0; x_1]) \text{ e}^x \quad c(z_2[0; z_2])
\]

where

\[
c(z_2[0; z_2]) = \frac{(z_2[0; z_2] + 1)}{(z_2[0; z_2] + v_3 + 1)} ;
\]

It is the evident solution of the equations 3.3.21, 3.3.22. The equations 3.3.17 and 3.3.18 lead to the recurrence relations for the functions \( a(z_2[0; z_2]) \) and \( b(k[0; x_1]) \)

\[
a(z_2[0; z_2] + 1) (z_2[0; z_2] + 1) = (z_2[0; z_2] + u_2 v_3 + 1) a(z_2[0; z_2])
\]

\[
b(k[0; x_1] + 1) (k[0; x_1] + u_2 v_3 + 1) = (k[0; x_1] + u_2 v_3 + 1) b(k[0; x_1])
\]

The solution of these equations is

\[
a(z_2[0; z_2]) = \frac{(z_2[0; z_2] + u_2 v_3 + 1)}{(z_2[0; z_2] + 1)} ;
\]

\[
b(k[0; x_1]) = \frac{(k[0; x_1] + u_2 v_3 + 1)}{(k[0; x_1] + 1)}
\]

and collecting everything together we obtain the expression for the operator \( R_{1,2} \) from the proposition.

### 4 Conclusions

We have shown that the general \( R \)-matrix can be represented as the product of the simple "building blocks" ( \( R \)-operators. In the present paper we have demonstrated how this factorization arises in the simplest situations of the symmetry algebra \( s'(2) \) and \( s'(3) \). As a byproduct we derived useful representation for the \( s'(2) \)-invariant \( R \)-matrix

\[
R_{1,2}(u) = P_{12} R_1 (u_1; u_2) R_2 (u_1; u_2) \quad P_{12} = \frac{(z_{21}[0; z_2] + 2y_1)}{(z_{21}[0; z_2] + y_1 + y_2)} \quad \frac{(z_{12}[0; z_2] + 2y_1)}{(z_{12}[0; z_2] + y_1 + y_2 + u)}
\]
It is possible to represent the general $s'(2)$-invariant R-matrix in many equivalent forms. The spectral decomposition of the R-matrix (1.0.1) was obtained in the paper [7]. In functional representation R-matrix is some integral operator acting on the space of polynomials. The symbol and kernel of this integral operator was calculated in [13] and in [19] correspondingly. Here we obtain the representation for the R-matrix in the operator form. The similar operator expression for the Hamiltonian of the XXX spin chain was used in [20,21,22].

In the $s'(3)$ case we have derived the explicit expression for the general R-matrix in the factorized form

$$R_{12}^{12}(v;u) = P_{12}R_{1}(u_1;v_1;u_2;u_3)R_{2}(u_1;u_2;v_2;u_3)R_{3}(u_1;u_2;u_3;v_3):$$

The R-operators in many respects are very similar to the general R-matrix but are much more simpler. In some sense the R-matrix is composite object and R-operators are elementary building blocks. Using the simple pictures it is possible to derive the general system of defining relations for the R-operators. First, there exist nontrivial commutation relations for the R-operators acting in different spaces. Second, there are three term relations for the R-operators which play the same role as Yang-Baxter relation for the R-matrix. Using defining relations for R-operators it is possible to derive all relations for the general R-matrix including the Yang-Baxter equation. All this will be discussed elsewhere.

It seems that all these results can be generalized to the symmetry algebra $s'(n)$ where should exists the factorization of the general R-matrix on the product of simple R-operators. In the second part of this work we shall show that the same factorization take place for the general rational solution of the Yang-Baxter equation with the supersymmetry algebra $s'(2j)$.

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References.

[1] P.P. Kulish and E.K. Sklyanin, "On the solutions of the Yang-Baxter equation" Zap Nauchn Sem. LOM I 95 (1980) 129

[2] M. Jimbo, "Introduction to the Yang-Baxter equation", Int. J. Mod. Phys. A 4, (1983) 3759

"Yang-Baxter equation in integrable systems", M. Jimbo ed., Adv. Ser. Math. Phys., 10, World Scientific (Singapore) 1990

[3] V.G. Drinfeld, "Hopf algebras and Yang-Baxter equation", Soviet Math. Dokl. 32 (1985), 254

V.G. Drinfeld, "Quantum Groups" in "Proc. Int. Congress Math., Berkeley, 1986", A.M.S., Providence RI (1987), p 798

[4] V.G. Drinfeld, "Quasi-Hopf algebras", Leningrad Math. J. 1 (1990) 1419
[5] V. Terras, "Drinfeld twists and Functional Bethe Ansatz", Lett. Math. Phys. 48 (1999) 263

H. Pfeifer, "Factorizing twists and the universal R-matrix of the Yangian \( Y(s^2) \)", J. Phys. A Math. Gen. 33 (2000), 8929

[6] P. P. Kulish and E. K. Sklyanin, "Quantum spectral transform method. Recent developments", Lect. Notes in Physics, v 151, (1982), 61,

L.D. Faddeev, "How Algebraic Bethe Ansatz works for integrable model", Les-Houches lectures 1995, [hep-th/9605187]

E.K. Sklyanin, "Quantum Inverse Scattering Method. Selected Topics", in "Quantum Group and Quantum Integrable Systems" (Nankai Lectures in Mathematical Physics), ed. M o-Lin Ge, Singapore World Scientific, 1992, pp. 63-97; [hep-th/9211111]

[7] P. P. Kulish, N. Yu. Reshetikhin and E. K. Sklyanin, "Yang-Baxter equation and representation theory", Lett. Math. Phys. 5 (1981) 393-403

[8] N. Mackay, "Rational R-matrices in irreducible representations", J. Phys. A 24 (1991) 4017

R.-B. Zhang, M. Gould and A. Bracken, "From the representation of the braid group to solutions of the Yang-Baxter equation", Nucl. Phys. B 354, (1991) 625

M. Gould and Y.-Z. Zhang, "R-matrices and the tensor product graph method" [hep-th/0205071]

[9] E. K. Sklyanin, private communication

[10] D. P. Zhelobenko, "Compact Lie Groups and their Representations", AMS, Providence, Rhode Island (1973)

[11] J. J. de Swart, "The Octet Model and its Clebsch-Gordan Coefficients", Rev. Mod. Phys. 35, (1963) 916

[12] M. Shifman, "ITEP Lectures on Particle Physics and Field Theory", (1999) v2 pp 775-875 World Scientific Lect. Notes Phys. 62

[13] P. P. Kulish and N. Yu. Reshetikhin, "On GL(3)-invariant solutions to the Yang-Baxter equation and the associated quantum system s", Zap. Nauchn. Sem. LOMI 120 (1982) 92

[14] A. Molv "Yangians and their applications" (2002) [math.QA/0211288]

[15] A. Molv, M. Nazarov and G. Olshanski "Yangian and classical Lie algebras", Russian Math. Surveys 51:2 (1996), 205-282

[16] P. P. Kulish, "Yang-Baxter equation and reflection equations in integrable models", [hep-th/9507070]

[17] S. J. Ishlaukas and P. P. Kulish, "Spectral resolution of the su(3)-invariant solutions to the Yang-Baxter equation", Zap. Nauchn. Sem. LOMI 1145 (1985) 3

[18] E. K. Sklyanin, "Classical limits of the Yang-Baxter equation", J. Soviet Math. 40, (1988) 93

[19] S. Derkachov, D. Karakhanyan, R. Kirsten "Universal R-matrix as integral operator" Nucl. Phys. B 618, (2001) 589
[20] L.N. Lipatov, "High-energy asymptotics of multicolor QCD and exactly solvable lattice models", JETP Lett. 59 (1994) 596
L.N. Lipatov, "Duality symmetry of reggeon interactions in multicolor QCD", Nucl Phys. B 548, (1999) 328.

[21] L.D. Faddeev and G. P. Korchemsky, "High-energy QCD as a completely integrable model", Phys Lett B 342 (1995) 311.

[22] D. Karakhanian and R. Kirschner, "Conserved currents of the three-reggeon interaction", hep-th/9902147; "High-energy scattering in gauge theories and integrable spin chains", hep-th/9902031, Fortschr. Phys. 48, (2000) 139.

[23] C.-N. Yang "Some exact results for the many-body problem in one dimension with repulsive delta-function interaction", Phys Rev Lett. 19 (1967), 1312.