REGULARITY CONDITIONS OF 3D NAVIER-STOKES FLOW IN TERMS OF LARGE SPECTRAL COMPONENTS

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Abstract. We develop Ladyzhenskaya-Prodi-Serrin type spectral regularity criteria for 3D incompressible Navier-Stokes equations in a torus. Concretely, for any $N > 0$, let $w_N$ be the sum of all spectral components of the velocity fields whose all three wave numbers are greater than $N$ absolutely. Then, we show that for any $N > 0$, the finiteness of the Serrin type norm of $w_N$ implies the regularity of the flow. It implies that if the flow breaks down in a finite time, the energy of the velocity fields cascades down to the arbitrarily large spectral components of $w_N$ and corresponding energy spectrum, in some sense, roughly decays slower than $\kappa^{-2}$.

1. Introduction

In this paper, we are concerned with the 3D incompressible Navier-Stokes equations in a flat torus:

$$
\begin{aligned}
\frac{\partial u}{\partial t}(x,t) - \nu \Delta u(x,t) + u(x,t) \cdot \nabla u(x,t) + \nabla p(x,t) &= f(x,t), \\
\text{div } u(x,t) &= 0, \\
u(x,0) &= u_0(x)
\end{aligned}
$$

for $x \in \mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Here, $u(\cdot , t) : \mathbb{T}^3 \to \mathbb{R}^3$ is the velocity fields and $p(\cdot , t) : \mathbb{T}^3 \to \mathbb{R}$ is the pressure. If $u_0$ is smooth enough, there exists a unique solution of (1.1) for $0 < t < T$ for some $T > 0$(See [2] and references therein). Then, there arises a natural question whether $T = \infty$ for every smooth $u_0$ or not. This question has not been settled yet. However, if the solution satisfies one of the following conditions up to $t = T$ for some $T > 0$,

1) $\nabla u \in L_r^r([0,T];L^q(\mathbb{T}^3))$, $\frac{2}{r} + \frac{3}{q} = 2$, $q \in [3/2, \infty]$, 

2) $u \in L_r^r([0,T];L^q(\mathbb{T}^3))$, $\frac{2}{r} + \frac{3}{q} = 1$, $q \in [3, \infty]$.

then the solution becomes smooth for $0 < t \leq T$ [5, 8, 9, 10]. The above conditions are called the Serrin conditions. The same is true for $\mathbb{R}^3$ instead of the flat torus $\mathbb{T}^3$.

We are interested in improving the above Serrin conditions in terms of spectral decomposition. Some parts of the Fourier expansion of the velocity fields can be essentially considered as a two dimensional flow and we expect
that a certain genuinely three dimensional part of the velocity fields plays the crucial role for the regularity of the flow. To be specific, let

\[ u = \sum_{k \in \mathbb{Z}^3} c^k e^{2\pi i k \cdot x} \]

be the Fourier expansion of \( u \). For any \( N > 0 \), we call

\[ w_N \equiv Q_N(u) \equiv \sum_{|k_1|,|k_2|,|k_3| > N} c^k e^{2\pi i k \cdot x} \]

a genuine 3D part of \( u \) cut at the mode \( N \). Then, we shall show that if, for any \( q > 1 \),

\[ \liminf_{N \to \infty} \| Q_N(u) \|_{L^{q,r}(0,T;\mathbb{T}^3)} < \infty, \quad 3/q + 2/r = 1 \]

or

\[ \liminf_{N \to \infty} \| \nabla Q_N(u) \|_{L^{q,r}(0,T;\mathbb{T}^3)} < \infty, \quad 3/q + 2/r = 2, \]

then the regularity of \( u \) up to \( t = T \) is guaranteed.

If \( u_0 \) is smooth and \( u \) breaks down firstly at a finite time, \( t = T \), the above condition actually means that \( L^{q,r} \) norm of \( w_N \) blows up as \( t \to T \) for any \( N > 0 \). Taking \( q = 3, r = \infty \), this, in particular, implies the energy density of \( w_N \),

\[ E(\kappa) \equiv \sum_{|k| = \kappa} |c^k|^2 k^2, \]

is not uniformly bounded by \( \kappa^{-\delta} \) for any \( \delta > 2 \) near \( t = T \). In many statistical turbulence theories, the energy spectrum is expected to behave like \( \kappa^{-a} \) with \( a \in [1,2] \). For example, in Markovian Random Coupling Model and Quasi-Normal Markovian Approximation theory, \( E(\kappa) \sim \kappa^{-2} \) is expected (see chapter VII of [7]) and, in Kolomogorov’s homogeneous turbulent theory, \( E(\kappa) \sim \kappa^{-5/3} \) is expected [4, 7]. In this sense, our result actually supports that the breakdown of the solutions is related with the turbulent aspect of the genuine 3D part of the velocity fields.

2. Preliminary

We define \( V \) the set of all smooth functions \( \phi \) which are divergence free and mean zero, i.e., \( V = \{ \phi \in C^\infty(\mathbb{T}^3)^3 : \int \phi dx = 0, \text{div}\phi = 0 \} \). We also define \( H \) and \( V \) by the closure of the set \( V \) with respect to norms \( \| \cdot \|_{L^2(\mathbb{T}^3)} \) and \( \| \cdot \|_{H^1(\mathbb{T}^3)} \) respectively. In the Fourier expansion, the function spaces, \( H \) and \( V \) can be represented as follows:

\[ H = \{ h \in L^2(\mathbb{T}^3) : \hat{h}^0 = 0, k \cdot \hat{h}^k = 0 \text{ for all } k \in \mathbb{Z}^3 \}, \]

\[ V = \{ h \in H^1(\mathbb{T}^3) : \hat{h}^0 = 0, k \cdot \hat{h}^k = 0 \text{ for all } k \in \mathbb{Z}^3 \}. \]

We let \( P \) be the orthogonal projection of \( L^2(\mathbb{T}^3) \) to \( H \). Then the weak solution of Navier-Stoke equation is given in the following way,
Definition 2.0.1. A weak solution of the Navier-Stokes equation (1.1) is a function \( u \in L^2(0,T;V) \cap L^\infty(0,T;H) \) satisfying \( \frac{du}{dt} \in L^1_{loc}([0,T);V') \) and

\[
\begin{aligned}
\left\{ \begin{array}{l}
\left( \frac{du}{dt}, \varphi \right) + (Au, \varphi) + b(u, u, \varphi) = (f, \varphi), \\
u(0) = u_0,
\end{array} \right. \quad \forall \varphi \in V,
\end{aligned}
\]

where \((\cdot, \cdot)\) is the standard inner product of \( L^2(\mathbb{T}^3) \), \( A \) is the Stokes operator given by \( A = P(-\Delta) : \mathcal{D}(A) \to H \), and \( b(\cdot, \cdot, \cdot) \) is a trilinear form given by \( b(\varphi, \psi, \eta) = \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \varphi_i \partial_i \psi_j \eta_j \, dx \). We remark that the test function \( \varphi \) in fact can be chosen in \( L^\infty(0,T;V) \) since all terms in (2.2) remain meaningful. It is well-known that there exists a weak solution of Navier-Stokes equations [6]:

Theorem 2.0.2. There exists at least a weak solution \( u \) of the Navier-Stokes equation (1.1) for every \( u_0 \in H, f \in L^2(0,T;V') \).

This Leray weak solution becomes smooth and unique up to \( t = T \) if \( u \) belongs to the Serrin class, (1) or (2) as we mentioned earlier.

Let \( c_k(t) \) be the Fourier expansion of \( u \) as in (1.2) and \( \lambda_k = 4\pi^2|k|^2 \). Then \( Au \) is

\[
Au = \sum_{k \in \mathbb{Z}^n} c_k \lambda_k e^{2\pi ik \cdot x}.
\]

Also, for any \( 0 \leq s < 1 \), we define \( A^s \) by

\[
A^s u = \sum_{k \in \mathbb{Z}^n} c_k \lambda_k^s e^{2\pi ik \cdot x}.
\]

For any \( N > 0 \), and the solution \( u \) of (1.1), we define \( w_N \) as in (1.3) and \( v_N = u - w_N \) as in (1.4)

\[
v_N = u - w_N = v^1 + v^2 + v^3 = \sum_{|k_1| \leq N} c_k e^{2\pi ik \cdot x} + \sum_{|k_1| > N, |k_2| \leq N} c_k e^{2\pi ik \cdot x} + \sum_{|k_1| > N, |k_2| > N} c_k e^{2\pi ik \cdot x}.
\]

For \( w_N \), we have the following usual Sobolev embedding theorem [1].

Theorem 2.0.3. Let \( n > 1 \) be an integer, \( 2 < q < \infty \), and \( s > 0 \) satisfy

\[
\frac{2s}{n} = \frac{1}{2} - \frac{1}{q}.
\]

Then for any \( h \in L^2(\mathbb{T}^n) \), we have the following,

\[
\|h\|_{L^q} \leq C \left\{ \|(-\Delta)^s h\|_{L^2} + \|h\|_{L^2} \right\}
\]

where \( C \) is a constant depending only on \( n, s, \) and \( q \) and, for \( 0 < s < 1 \),

\[
(-\Delta)^s h(x) = \sum_{k \in \mathbb{Z}^n} \hat{h}_k \lambda_k^s e^{2\pi ik \cdot x}
\]
and \( \lambda_k = 4\pi^2|k|^2 \) as before.

\( \psi_N \) possesses infinite number of spectral components and is of 3-dimensional. However, they satisfy the following 2-dimensional Sobolev embeddings.

**Lemma 2.0.4.** Given any \( s, 0 \leq 2s < 1 \),

\[
\| \psi \|^q_{L^q} \leq C N^{1/2} \| A^s \psi \|^2_{L^2}, \quad i = 1, \cdots, 3
\]

for \( \frac{1}{q} = \frac{1}{2} - s \). Also,

\[
\| \psi_N \|^2_{L^2} \leq C N^{1/2} \| \psi_{N, 1/2} \|^2_{L^2} \| A^1/2 \psi_N \|^{1-2/q}_{L^q}.
\]

**Proof.** We first show (2.9). Since the proof is exactly parallel, we only show it for \( \psi^1 \). Let \( x = (x_1, \overline{x}) \in \mathbb{R} \times \mathbb{R}^2 \), \( k = (k_1, \overline{k}) \in \mathbb{Z} \times \mathbb{Z}^2 \), and

\[
h_{k_1}(\overline{x}) = \sum_{\overline{k} \in \mathbb{Z}^2} c^{(k_1, \overline{k})}(t) e^{2\pi i \overline{k} \cdot \overline{x}}
\]

where \( c^{(k_1, \overline{k})} \) is as in (1.2).

Then \( \psi^1 \) is represented as follows,

\[
\psi^1(x) = \sum_{|k_1| \leq N} e^{2\pi i k_1 x_1} \sum_{\overline{k} \in \mathbb{Z}^2} c^{(k_1, \overline{k})} e^{2\pi i \overline{k} \cdot \overline{x}} = \sum_{|k_1| \leq N} e^{2\pi i k_1 x_1} h_{k_1}(\overline{x}).
\]

From the definition of \( h_{k_1} \), the fractional derivative of \( h_{k_1} \) is given as

\[
(-\Delta)^s h_{k_1}(\overline{x}) = \sum_{\overline{k} \in \mathbb{Z}^2} c^{(k_1, \overline{k})}(t) (\lambda_{\overline{k}})^s e^{2\pi i \overline{k} \cdot \overline{x}}
\]

where \( \lambda_{\overline{k}} = 4\pi^2|\overline{k}|^2 \).

Now we are comparing the \( L^2 \)-norm of \( \| A^s \psi^1(x) \|^2_{L^2} \) and \( \sum_{|k_1| \leq N} \| (-\Delta)^s h_{k_1} \|^2_{L^2} \). Since

\[
\| A^s \psi^1(x) \|^2_{L^2} = \sum_{|k_1| \leq N} \sum_{\overline{k}} \left( |k_1|^2 + |\overline{k}|^2 \right)^s |c^{(k_1, \overline{k})}|^2
\]

\[
= \sum_{|k_1| \leq N} \left\{ \sum_{\overline{k} \neq 0} \frac{(|k_1|^2 + |\overline{k}|^2)^s}{|\overline{k}|^{2s}} |c^{(k_1, \overline{k})}|^2 + |k_1|^{2s} |c^{(k_1, 0)}|^2 \right\},
\]

and \( |k_1| \geq 1 \) in the latter term due to the average zero property of \( u \), we have

\[
\sum_{|k_1| \leq N} \left\{ \| (-\Delta)^s h_{k_1} \|^2_{L^2} + \| h_{k_1} \|^2_{L^2} \right\} \leq C \| A^s \psi^1(x) \|^2_{L^2}.
\]
In particular, by the Minkowski inequality, the Cauchy-Schwarz inequality, and Theorem 2.0.3 with \( n = 2 \), we have

\[
\|v^1\|_{L^q}^2 \leq \left( \sum_{|k_1| \leq N} \|h_{k_1}\|_{L^q} \right)^2 \leq (2N + 1) \sum_{|k_1| \leq N} \|h_{k_1}\|_{L^q}^2 \\
\leq CN \sum_{|k_1| \leq N} \{ \|(-\Delta)^s h_{k_1}\|_{L^2}^2 + \|h_{k_1}\|_{L^2}^2 \}
\]

(2.15)

\[
\leq CN \|A^s v^1(x)\|_{L^2}^2
\]

for \( s = 1/2 - 1/q \). Here, \( C \) is a constant depending only on \( q \).

Next, we show (2.10). By the \( l^{1/s} - l^{1/(1-s)} \) duality, we have

\[
\|A^s v_N\|_{L^2}^2 = \sum_k |c_k|^2 |\lambda_k|^{2s} = \sum_k (|c_k|^2 |\lambda_k|)^{2s} |c_k|^{2(1-2s)}
\]

\[
\leq (\sum_k |c_k|^2 |\lambda_k|)^{2s} (\sum_k |c_k|^{2})^{1-2s}
\]

\[
= \|A^{1/2} v_N\|_{L^2}^{4s} \|v_N\|_{L^2}^{2-4s}
\]

Hence, together with (2.9), we finish the proof.

We finish this section by introducing the theorem.

**Theorem 2.0.5** ([3]). Let \( n \) be any positive integer and \( q, r \) be constants satisfying \( q \in [r, nr/(n - r)] \) if \( r \in [1, n) \), and \( q \in [r, \infty) \), if \( r \geq n \). Then, for all \( \varphi \) such that \( \varphi \in L^r(\mathbb{T}^n) \) and \( \nabla \varphi \in L^r(\mathbb{T}^n) \), we have

\[
\|\varphi\|_{L^q} \leq C(n, r, q) \|\varphi\|_{L^r}^\alpha \|\nabla \varphi\|_{L^r}^{1-\alpha}
\]

where \( \frac{1}{q} = \frac{a}{r} + \left( 1 - \frac{1}{a} \right) \left( \frac{1}{r} - \frac{1}{n} \right) \).

The inequality (2.10) is in fact the inequality (2.16) with \( n = 2 \). This is why \( v \) has 2-dimensional properties.

### 3. Spectral Serrin condition

We now state our main theorem and show it here.

**Theorem 3.0.6.** Suppose that \( u \) is a weak solution of the 3D Navier-Stokes equation (1.1) and satisfies one of the Serrin conditions (1.4) - (1.5). Then \( u \) is a classical solution up to \( t = T \).

3.1. **Proof for (1.5)** with \( q > 3/2 \). By (1.5), there exists \( N > 0 \) such that \( \nabla w_N \in L^q(0, T; L^q(\mathbb{T}^3)) \), \( \frac{2}{r} + \frac{3}{q} = 2 \), \( q \in (3/2, \infty] \). For a weak solution, for almost every \( 0 < s < T \), \( u(s) \in V \). Then, for some \( S \in (s, T] \), \( u \) becomes smooth for \( t \in [s, S) \) by the local unique existence of a strong solution [2]. By this local existence of strong solutions, we only need to show that \( u(S) \in V \) for all such \( s \). Since the proof is exactly the same, we show it replacing \( [s, S) \)
with \([0, T]\) for convenience' sake. This means \(u(t)\) is smooth for \(t < T\). By applying \(\varphi = Au\) to the equation (2.2), we get

\[
\frac{1}{2} \frac{d}{dt} \|A^{1/2}u\|_{L^2}^2 + \nu \|Au\|_{L^2}^2 = -b(u, u, Au) + (f, Au).
\]

Since \((f, Au)\) does not make any technical difficulties, we assume \(f = 0\). For convenience, we denote \(w = w_N\) and \(v = v_N\). By integration by parts, we have

\[
\int (w \nabla u) Au \, dx = \int (w \nabla u) P(-\Delta)u \, dx
\]

\[
= \int w^i \partial_i u^j (\partial_k^2 u^j) \, dx = \int \partial_k w^i \partial_i u^j \partial_k u^j \, dx.
\]

Hence, from the Hölder inequality and Theorem 2.0.5, we have

\[
\int (w \nabla u) Au \, dx \leq \left| \int \partial_k w^i \partial_i u^j \partial_k u^j \, dx \right| \leq \|\nabla w\|_{L^p} \|\nabla u\|_{L^2}^2
\]

\[
\leq \|\nabla w\|_{L^p} \|\nabla u\|_{L^2}^2 \|\nabla^2 u\|_{L^2}^{2-2a} \leq C \|\nabla w\|_{L^p}^1 \|\nabla u\|_{L^2}^2 + \delta \|Au\|_{L^2}^2.
\]

Here \(p\) and \(a\) are chosen to satisfy \(\frac{1}{q} + \frac{2}{p} = 1\) and \(\frac{1}{p} = \frac{a}{2} + (1 - a) \left(\frac{1}{2} - \frac{1}{3}\right)\). It follows that

\[
\frac{3}{q} + \frac{2}{1/a} = \frac{3}{q} + 2(1 - \frac{3}{2q}) = 2
\]

are satisfied for \(q > 3\).

We also have by integration by parts,

\[
\left| \int (v \nabla u) Au \, dx \right| = \left| \int \partial_k v^i \partial_i u^j \partial_k u^j \, dx \right|
\]

\[
\leq \left| \int \partial_k v^i \partial_l v^j \partial_k u^j \, dx \right| + \left| \int \partial_k v^i \partial_i v^j \partial_k w^j \, dx \right| + \left| \int \partial_k v^i \partial_l w^j \partial_k u^j \, dx \right|.
\]

Since

\[
\left| \int \partial_k v^i \partial_l v^j \partial_k u^j \, dx \right| + \left| \int \partial_k v^i \partial_l w^j \partial_k u^j \, dx \right|
\]

\[
\leq \|\nabla v\|_{L^p} \|\nabla v\|_{L^p} \|\nabla w\|_{L^2} + \|\nabla w\|_{L^p} \|\nabla w\|_{L^p} \|\nabla u\|_{L^p}
\]

\[
\leq 2 \|\nabla w\|_{L^p} \|\nabla u\|_{L^2}^2,
\]

we get

\[
\left| \int \partial_k v^i \partial_l v^j \partial_k w^j \, dx \right| + \left| \int \partial_k v^i \partial_l w^j \partial_k u^j \, dx \right| \leq C \|\nabla w\|_{L^p}^1 \|\nabla u\|_{L^2}^2 + 2\delta \|Au\|_{L^2}^2
\]

by repeating arguments in (3.3).
Moreover, because of (2.10), we have

\begin{equation}
\left| \int \partial_k v^i \partial_i v^j \partial_k v^j \, dx \right| = \left| \int (v \nabla v) A v \, dx \right| \leq \|v\|_{L^4} \|\nabla v\|_{L^4} \|Av\|_{L^2}
\end{equation}

(3.8)

\begin{align*}
&\leq C N \|v\|_{L^2}^{1/2} \|\nabla v\|_{L^2} \|Av\|_{L^2}^{3/2} \\
&\leq C N^4 \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^4 + \delta \|Av\|_{L^2}^2.
\end{align*}

By combining (3.1), (3.3), (3.7), and (3.8), we have

\begin{equation}
1 \frac{d}{dt} \|A^{1/2} u\|_{L^2}^2 + \nu \|A u\|_{L^2}^2 \leq C N^4 \left( \|\nabla w\|_{L^q}^{1/\alpha} + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \right) \|\nabla u\|_{L^2}^2 + 4 \delta \|A u\|_{L^2}^2.
\end{equation}

(3.9)

Let \( Y(t) = \|A^{1/2} u\|_{L^2}^2 \) and take \( \delta = \nu / 8 \). Then we have

\begin{equation}
\frac{d}{dt} Y(t) + \nu \|A u\|^2 \leq C N^4 \left( \|\nabla w\|_{L^q}^{1/\alpha} + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \right) Y(t).
\end{equation}

(3.10)

From the above inequality (3.10) and the Gronwall’s inequality, we have

\begin{equation}
Y(t) + \nu \int_0^T \|A u(\tau)\|_{L^2}^2 \, d\tau \leq Y(0) \exp \left\{ C N^4 \int_0^T \|\nabla w\|_{L^q}^{1/\alpha} + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \, d\tau \right\}
\end{equation}

(3.11)

and this inequality tells us that \( u \) is smooth.

3.2. Proof for (1.4) with \( q > 3 \). For given \( q > 3 \), choose \( p \) and \( \alpha \) to satisfy,

\begin{equation}
1 \frac{1}{q} + \frac{1}{p} = \frac{1}{2}, \quad \text{and} \quad \frac{1}{p} = \frac{\alpha}{2} + (1 - \alpha) \left( \frac{1}{2} - \frac{1}{3} \right).
\end{equation}

(3.12)

Then, from the Hölder inequality, we have

\begin{equation}
\int w \nabla u A \, dx \leq \|w\|_{L^q} \|\nabla u\|_{L^p} \|A u\|_{L^2}
\end{equation}

(3.13)

\begin{align*}
&\leq \|w\|_{L^q} \|\nabla u\|_{L^2}^2 \|A u\|_{L^2}^{2-\alpha} \\
&\leq C \|w\|_{L^q}^{2/\alpha} \|\nabla u\|_{L^2}^2 + \delta \|A u\|_{L^2}^2.
\end{align*}

We note that \( q \) and \( 2/\alpha \) satisfy

\begin{equation}
\frac{3}{q} + \frac{2}{2/\alpha} = 1.
\end{equation}

(3.14)

By using (2.10), we have

\begin{equation}
\int v \nabla v A \, dx \leq C \|v\|_{L^4} \|\nabla v\|_{L^4} \|A v\|_{L^2}
\end{equation}

(3.15)

\begin{align*}
&\leq C N \|v\|_{L^2}^{1/2} \|\nabla v\|_{L^2} \|A v\|_{L^2}^{3/2} \\
&\leq C N^4 \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^4 + \delta \|A v\|_{L^2}^2.
\end{align*}
By applying integration by parts twice to \( \int v \nabla w Aw \, dx \), we have
\[
\int v \nabla w Aw \, dx = \sum_{i,j,k=1}^{3} \int v^i \partial_k w^j (-\partial_k^2 w^j) \, dx = \sum_{i,j,k=1}^{3} \int \partial_k v^i \partial_i \partial_k w^j \, dx
\]
\[
= - \sum_{i,j,k=1}^{3} \left\{ \int \partial_k^2 v^i \partial_i w^j \, dx + \int \partial_k v^i \partial_i \partial_k w^j \, dx \right\}.
\]
Hence, by using the similar calculations in (3.13), we can obtain
\[
(3.17) \quad \int v \nabla w Aw \, dx \leq C(\|w\|_{L^6}^{2/\alpha} \|\nabla u\|_{L^2}^2 + \delta \|A u\|_{L^2}^2).
\]
Finally, by integration by parts, we have
\[
(3.18) \quad \int v \nabla w Av \, dx = \sum_{i,j,k=1}^{3} \int v^i \partial_k w^j (-\partial_k^2 v^j) \, dx
\]
\[
= \sum_{i,j,k=1}^{3} \left\{ \int \partial_k v^i \partial_i \partial_k w^j \, dx + \int v^i \partial_i \partial_k w^j \, dx \right\}
\]
\[
= \sum_{i,j,k=1}^{3} \left\{ - \int \partial_k v^i \partial_i w^j \partial_k v^j \, dx + \int v^i \partial_i \partial_k w^j \partial_k v^j \, dx \right\}
\]

The last two terms are bounded by
\[
(3.19) \quad C(\|w\|_{L^6}^{2/\alpha} \|\nabla u\|_{L^2}^2 + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 + 2\delta \|A v\|_{L^2}^2)
\]
by the similar calculation in (3.13) and (3.15).

By combining (3.1), (3.13), (3.15), (3.17), and (3.19), we have
\[
(3.20) \quad \frac{1}{2} \frac{d}{dt} \|A^{1/2} u\|_{L^2}^2 + \nu \|A u\|_{L^2}^2 \leq C N^4 \left( \|w\|_{L^6}^{2/\alpha} + \|v\|_{L^2}^2 \|\nabla v\|_{L^2}^2 \right) \|\nabla u\|_{L^2}^2 + 4\delta \|A u\|_{L^2}^2.
\]

Hence \( \|A^{1/2} u\|_{L^2}^2 \) is bounded and it implies that \( u \) is smooth.

3.3. Proof for (1.4) with \( q = 3 \). We note that if \( w \) satisfies condition (1.4) with \( q = 3 \), then the above calculation does not work because \( a = 0 \). We also remark that \( \nabla w \in L^{3/2, \infty} \) implies \( w \in L^{3, \infty} \). So, we treat only the case (1.4) with \( q = 3 \) separately. We make use of [9] in this case. We denote \( w_N = Q_N(u) \) and \( v_N = P_N(u_N) = (I - Q_N)(u_N) \) as in (1.3). This decomposition is in fact an orthogonal decomposition and the equation satisfied by \( v \) is
\[
\frac{dv}{dt} + \nu Av = P_N f - P_N B(v + w, v + w)
\]
and the equation satisfied by \( w \) is
\[
\frac{dw}{dt} + \nu Aw = Q_N f - Q_N B(v + w, v + w).
\]
We remark that the equation (3.21) tells us that if \( w \) is regular, then the solution \( v \) is also regular. This procedure has not been exposed so far, but we will see how it operates more explicitly in the procedure of the proof. The main idea for \( q = 3 \) is to take \( \varphi = A^{1/3}v \) in the equation (2.22) instead of \( A^{1/2}v \). Then we get the following equation,

\[
\frac{1}{2} \frac{d}{dt} \| A^{1/6} v \|_{L^2}^2 + \nu \| A^{2/3} v \|_{L^2}^2 = -b(u, u, A^{1/3} v).
\]

**Lemma 3.3.3.** If \( \text{Eq. 3.3.1} \) \( \varphi = A^{1/3} v \) in the equation (2.22) instead of \( A^{1/2} v \). Then we get the following equation,

\[
\frac{1}{2} \frac{d}{dt} \| A^{1/6} v \|_{L^2}^2 + \nu \| A^{2/3} v \|_{L^2}^2 = -b(u, u, A^{1/3} v).
\]

**Proposition 3.3.1.** If \( w_N \) satisfies (1.3) with \( q = 3 \), then

\[
A^{1/6} v \in L^\infty(0, T; H), \quad A^{2/3} v \in L^2(0, T; H).
\]

To prove the Proposition 3.3.1, we need to estimate the nonlinear term \( b(u, u, A^{1/3} v) \).

**Lemma 3.3.2.**

\[
\int v \nabla u A^{1/3} v \, dx \leq C \| \nabla u \|_{L^2}^2 \| A^{1/6} v \|_{L^2}^2 + \delta \| A^{2/3} v \|_{L^2}^2.
\]

*Here, \( C \) is a constant depending on \( N \).*

**Proof.** From (2.9) and (2.10), we have

\[
\| v \|_{L^3} \leq C_N \| A^{1/6} v \|_{L^2}, \quad \text{and} \quad \| A^{1/3} v \|_{L^6} \leq C_N \| A^{2/3} v \|_{L^2}.
\]

From above inequalities and the Holder inequality, we have

\[
\int v \nabla u A^{1/3} v \, dx \leq \| v \|_{L^3} \| \nabla u \|_{L^2} \| A^{1/3} v \|_{L^6}
\leq C \| A^{1/6} v \|_{L^2} \| \nabla u \|_{L^2} \| A^{2/3} v \|_{L^2}
\leq C \| \nabla u \|_{L^2}^2 \| A^{1/6} v \|_{L^2}^2 + \delta \| A^{2/3} v \|_{L^2}^2.
\]

**Lemma 3.3.3.**

\[
\int w \nabla u A^{1/3} v \, dx \leq C \| \nabla u \|_{L^2}^2 \| w \|_{L^3}^2 + \delta \| A^{2/3} v \|_{L^2}^2.
\]

*\( C \) is a constant depending on \( N \).*

**Proof.** By the Holder inequality and lemma 2.0.4,

\[
\int w \nabla u A^{1/3} v \, dx \leq \| w \|_{L^3} \| \nabla u \|_{L^2} \| A^{1/3} v \|_{L^6}
\leq C \| w \|_{L^3} \| \nabla u \|_{L^2} \| A^{2/3} v \|_{L^2}
\leq C \| \nabla u \|_{L^2}^2 \| w \|_{L^3}^2 + \delta \| A^{2/3} v \|_{L^2}^2.
\]

□
Proof of Proposition 3.3.1. From (3.23), and from the above two lemmas, we have
(3.30)\
\frac{d}{dt}\|A^{1/6}v\|_{L^2}^2 + \nu\|A^{2/3}v\|_{L^2}^2 \leq C\|\nabla u\|_{L^2}^2 \|A^{1/6}v\|_{L^2}^2 + C\|\nabla u\|_{L^2}^2 \|w\|_{L^3}^2.
by taking \(\delta = \nu/4\).

Hence we get the following by Gronwall’s inequality,
(3.31)\
\|A^{1/6}v\|_{L^2}^2 + \int_0^T \|A^{2/3}v(\tau)\|_{L^2}^2 d\tau \\
\leq \exp \left\{ \int_0^T C\|\nabla u(\tau)\|_{L^2}^2 d\tau \right\} \left[ \|A^{1/6}v(0)\|_{L^2}^2 \right. \\
+ C \sup_{0 \leq \tau \leq T} \|w(\tau)\|_{L^3}^2 \int_0^T \|\nabla u(\tau)\|_{L^2}^2 d\tau \right].
\]

Now, we are ready for the proof for \(q = 3\). By Proposition 3.3.1 and Lemma 2.0.4,
(3.32)\
\sup_{0 \leq t \leq T} \|v(t)\|_{L^3}
is bounded. From this and our assumption, we have \(u \in L^{\infty}(0,T;L^3)\), which is enough by [2].

The following corollary actually implies that the energy spectrum should not be bounded uniformly by \(\kappa^{-2-\epsilon}\).

Corollary 3.3.4. Suppose \(u\) breaks down firstly at \(t = T\). Then, for any \(N > 1\) and any \(\delta > 2\),
\[ \liminf_{t \to T} \sup_{\kappa} \kappa^\delta E(\kappa) = \infty. \]
Proof. Suppose not, then, for some \(N > 1, \delta > 2\) and \(t_0 < T\), we get
\[ \sup_k \left( \kappa^\delta E(\kappa) \right) < C \]
for \(t_0 < t < T\) for some constant \(C\). But then, \(E(\kappa) < C\kappa^{-\delta}\) for \(t_0 < t < T\).

Now, by theorem 3.0.6, 2.0.3 and \(L^2\) boundedness of \(u\), taking \(q = 3\), we have
\[ \infty = \int_0^T \|w_N\|_{L^3} \leq C \int_0^T \|A^{1/4}w_N\|_{L^2}. \]
However,
\[ \|A^{1/4}w_N\|_{L^2} = C \sum_{|k| > N} |k|^2 \|w_N\|_{L^2} \leq C \sum_{\kappa > N} \kappa E(\kappa) < C \]
and we arrive at a contradiction. \(\square\)
Since the solution is smooth at any fixed $t < T$, the spectrum should be bounded by $\kappa^{-2-\epsilon}$ at any fixed $t < T$. Thus, it actually implies that, as $t \to T$, there appears some range of $\kappa$ for which $E(\kappa) > C\kappa^{-2-\epsilon}$ for any $\epsilon$ and any $C > 0$.

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