The Segment Number: 
Algorithms and Universal Lower Bounds for Some Classes of Planar Graphs

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Abstract. The segment number of a planar graph $G$ is the smallest number of line segments needed for a planar straight-line drawing of $G$. Dujmović, Eppstein, Suderman, and Wood [CGTA'07] introduced this measure for the visual complexity of graphs. There are optimal algorithms for trees and worst-case optimal algorithms for outerplanar graphs, 2-trees, and planar 3-trees. It is known that every cubic triconnected planar $n$-vertex graph (except $K_4$) has segment number $n/2 + 3$, which is the only known universal lower bound for a meaningful class of planar graphs.

We show that every triconnected planar 4-regular graph can be drawn using at most $n + 3$ segments. This bound is tight up to an additive constant, improves a previous upper bound of $7n/4 + 2$ implied by a more general result of Dujmović et al., and supplements the result for cubic graphs. We also give a simple optimal algorithm for cactus graphs, generalizing the above-mentioned result for trees. We prove the first linear universal lower bounds for outerpaths, maximal outerplanar graphs, 2-trees, and planar 3-trees. This shows that the existing algorithms for these graph classes are constant-factor approximations. For maximal outerpaths, our bound is best possible and can be generalized to circular arcs.

1 Introduction

A drawing of a given graph can be evaluated by various quality measures depending on the concrete purpose of the drawing. Classic examples of such measures include drawing area, number of edge crossings, neighborhood preservation, and stress of the embedding. More recently, Schulz [14] proposed the visual complexity of a drawing, determined by the number of geometric objects (such as line segments or circular arcs) that the drawing consists of. It has been experimentally verified that people without mathematical background tend to prefer drawings with low visual complexity [8]. The visual complexity of a graph drawing depends on the drawing style, as well as on the underlying graph properties. A well-studied measure of the visual complexity of a graph is its segment number, introduced by Dujmović, Eppstein, Suderman, and Wood [1]. It is defined as follows. Recall that a straight-line drawing of a graph maps (i) the vertices of the graph injectively to points in the plane and (ii) the edges of the graph to straight-line segments that
connect the corresponding points. A segment in such a drawing is a maximal set of edges that together form a line segment. For a given straight-line drawing \( \Gamma \) of a graph, the set of segments it induces is unique. The cardinality of that set is the segment number of \( \Gamma \). The segment number, \( \text{seg}(G) \), of a planar graph \( G \) is the smallest segment number over all crossing-free straight-line drawings of \( G \).

**Previous work.** Dujmović et al. [1] pointed out two natural lower bounds for the segment number: (i) \( \eta(G)/2 \), where \( \eta(G) \) is the number of odd-degree vertices of \( G \), and (ii) the slope number, \( \text{slope}(G) \), of \( G \), which is defined as follows. The slope number \( \text{slope}(\Gamma) \) of a straight-line drawing \( \Gamma \) of \( G \) is the number of different slopes used by any of the straight-line edges in \( \Gamma \). Then \( \text{slope}(G) \) is the minimum of \( \text{slope}(\Gamma) \) over all straight-line drawings \( \Gamma \) of \( G \). Dujmović et al. also showed that any tree \( T \) admits a drawing with \( \text{seg}(T) = \eta(T)/2 \) segments and \( \text{slope}(T) = \Delta(T)/2 \) slopes, where \( \Delta(T) \) is the maximum degree of a vertex in \( T \). These drawings, however, use exponential area. Recall that an outerplanar graph is a plane graph that can be drawn such that all vertices lie on the outer face. The weak dual graph of an outerplane graph is its dual graph without the vertex corresponding to the outer face; it is known to be a tree. An outerplane graph whose weak dual is a path is called an outerpath. A maximal outerplanar graph is an outerplanar graph with the maximum number of edges. Dujmović et al. showed that every maximal outerplanar graph \( G \) with \( n \) vertices admits an outerplanar straight-line drawing with at most \( n \) segments. They showed that this is worst-case optimal. They also gave (asymptotically) worst-case optimal algorithms for 2-trees and plane (where the combinatorial embedding and outer face is fixed) 3-trees. Finally, they showed that every triconnected planar graph with \( n \) vertices can be drawn using at most \( 5n/2 - 3 \) segments. For the special cases of triangulations and 4-connected triangulations, Durocher and Mondal [2] improved the upper bound of Dujmović et al. to \((7n - 10)/3\) and \((9n - 9)/4\), respectively. The former bound implies a bound of \((16n - 3m - 28)/3\) for arbitrary planar graphs with \( n \) vertices and \( m \) edges. Kindermann et al. [7] observed that this implies that \( \text{seg}(G) \leq (8n - 14)/3 \) for any planar graph \( G \) if \( m > (8n - 14)/3 \) this follows from the bound, otherwise any drawing of \( G \) is good enough. Samee et al. [13] gave a constructive linear-time algorithm that computes the segment-number of a series-parallel graph of maximum degree 3. Mondal et al. [11] and Igamberdiev et al. [6] showed that every cubic triconnected planar graph (except \( K_4 \)) has segment number \( n/2 + 3 \). Hüllnitschmidt et al. [5] showed that trees, maximal outerplanar graphs and planar 3-trees admit drawings on a grid of polynomial size, using slightly more segments. Later, Kindermann et al. [7] improved some of these bounds.

**Other related work.** Okamoto et al. [12] investigated variants of the segment number. For planar graphs in 2D, they allowed bends. For arbitrary graphs, they considered crossing-free straight-line drawings in 3D and straight-line drawings with crossings in 2D. They showed that all segment number variants are \( \exists \mathbb{R} \)-complete to compute, and they gave upper and existential lower bounds of the segment number variants for cubic graphs. The arc number, \( \text{arc}(G) \), of a graph \( G \)
Table 1: Bounds on the segment number for subclasses of planar graphs. By existential upper bound we mean an upper bound for the universal lower bound. Here, \( \eta \) is the number of odd-degree vertices and \( \gamma = 3c_0 + 2c_1 + c_2 \), where \( c_i \) is the number of simple cycles with exactly \( i \) cut vertices.

| Graph class          | Universal lower bound | Existential upper bound | Existential lower bound | Universal upper bound |
|----------------------|----------------------|-------------------------|------------------------|-----------------------|
| planar conn.         | 1                    | 1                       | 2n - 2                 | (8n - 14)/3           |
| planar 3-conn.       | \( \sqrt{2n} \)     | O(\( \sqrt{n} \))       | 2n - 6                 | 5n/2 - 3              |
| planar 3-conn. 4-reg. | \( O(\sqrt{n}) \)   | \( O(\sqrt{n}) \)       | n                      | n + 3                 |
| planar 3-conn. 3-reg. | \( n/2 + 3 \)       | —                       | —                      | \( n/2 + 3 \)          |
| triangulation        | \( O(\sqrt{n}) \)   | 1                       | 2n - 2                 | (7n - 10)/3           |
| 4-conn. triangulation | \( O(\sqrt{n}) \)   | \( O(\sqrt{n}) \)       | 2n - 6                 | (9n - 9)/4            |
| planar 3-trees       | \( n + 4 \)         | \( n + 7 \)             | \( 2n - 2 \)           | \( 3n/2 \)             |
| 2-trees              | \( (n + 7)/5 \)     | \( (5n + 24)/13 \)      | \( 3n/2 - 2 \)         | \( 3n/2 \)             |
| maximal outerplanar  | \( n + 7/5 \)       | \( (5n + 24)/13 \)      | n                      | n                     |
| maximal outerpath    | \( [n/2] + 2 \)     | \( [n/2] + 2 \)         | n                      | n                     |
| ser.-par. max-deg. 3 | \( \eta/2 + \gamma \) | —                       | —                      | \( \eta/2 + \gamma \) |
| cactus graphs        | \( \eta/2 + \gamma \) | —                       | —                      | OPT                   |

Contribution and outline. First, we show that every triconnected planar 4-regular graph with \( n \) vertices can be drawn using at most \( n + 3 \) segments (note that there are \( 2n \) edges); see Sect. 2. This bound is tight up to an additive constant, improves a previous upper bound of \( 7n/4 + 2 \) implied by a more general result [1, Thm. 15] of Dujmović et al., and supplements the result for cubic graphs due to Mondal et al. [11] and Igamberdiev et al. [6]. Our algorithm works even for plane graphs and produces drawings that are convex, that is, the boundary of each face corresponds to a convex polygon. Our algorithm is recursive and relies on a decomposition of the graph along carefully chosen paths, which might be of independent interest. We also give a simple optimal (cf. Table 1) algorithm for cactus graphs (App. A), generalizing the result of Dujmović et al. for trees.

We prove the first linear universal lower bounds for maximal outerpaths (\( [n/2] + 2 \); see Sect. 3), maximal outerplanar graphs as well as 2-trees (\( (n + 7)/5 \); see App. D), and planar 3-trees (\( n + 4 \); see App. E). This makes the corresponding algorithms of Dujmović et al. constant-factor approximation algorithms. For maximal outerpaths, our bound is best possible and can be generalized to circular arcs. For planar 3-trees, the bound is best possible up to the additive constant. We close with some open problems in Sect. 3.

Known and new results are summarized in Table 1. Many statements are marked with a (clickable) “⋆”. We had to defer their full proofs to the appendix.
Notation and terminology. All graphs in this paper are simple (i.e., we do not allow parallel edges or self-loops). For any graph \( G \), let \( V(G) \) denote \( G \)'s vertex set, and let \( E(G) \) denote \( G \)'s edge set. Let \( G \) be a planar and connected graph, and let \( \Gamma \) be a planar drawing of \( G \). The boundary \( \partial f \) of each face \( f \) of \( \Gamma \) can be uniquely described by a counterclockwise sequence of edges. If \( G \) is biconnected, then \( \partial f \) is a simple cycle (otherwise, \( \partial f \) can visit vertices and edges multiple times). The collection of the boundaries of all faces of \( \Gamma \) is called the combinatorial embedding of \( \Gamma \). The unique unbounded face of \( \Gamma \) is called its outer face; the remaining faces are called internal. Vertices (edges) belonging to the boundary of the outer face are called outer (edges); the remaining vertices (edges) are called internal. A plane graph is a planar graph equipped with a combinatorial embedding and a distinguished outer face. A path in a plane graph is internal if its edges and interior vertices do not belong to its outer face.

2 Triconnected 4-Regular Planar Graphs

This section is concerned with the segment number of 3-connected 4-regular planar graphs. We establish a universal upper bound of \( n + 3 \) segments, which we complement with an existential lower bound of \( n \) segments, where \( n \) denotes the number of vertices.

Overview. Towards the upper bound, we will show that each graph of the considered class admits a drawing where all but three of its vertices are placed in the interior of some segment. In such a drawing, each of these vertices is the endpoint of at most two segments. The claimed bound then follows from the fact that each segment has exactly two endpoints.

To construct the desired drawings, we follow a strategy that has already been used in an algorithm by Hong and Nagamochi [4], which was sped up by Klemz [10]. Both algorithms generate convex drawings of so-called hierarchical plane st-graphs, but they can also be applied to “ordinary” plane graphs. In this context, the algorithmic framework is as follows: the input is an internally (defined below, see Definition 1) 3-connected plane graph \( G \) and a convex drawing \( \Gamma^o \) of the boundary of its outer face. The task is to extend \( \Gamma^o \) to a convex drawing of \( G \). The main idea of both algorithms is to choose a suitable internal vertex \( y \) of the given graph \( G \) and compute three disjoint (except for \( y \)) paths \( P_1, P_2, P_3 \) from \( y \) to the outer face. Each of these paths is then embedded as a straight-line segment so that \( \Gamma^o \) is dissected into three convex polygons, for an illustration see Fig. 1a. The graphs corresponding to the interior of these polygons can now be handled recursively. To ensure that a solution exists, the computed paths (as well as the paths corresponding to the segments of \( \Gamma^o \)) need to be archfree, meaning that they are not arched by an internal face: a path \( P \) is arched by a face \( a \) between \( u, v \in V(\partial a) \cap V(P) \) if the subpath \( P_{uv} \) of \( P \) between \( u \) and \( v \) is interior-disjoint from \( \partial a \), see Fig. 1a. Indeed, if \( a \) is internal, then such a path \( P \) cannot be realized as a straight-line segment in a convex drawing since the interior of the segment \( uv \) has to be disjoint from the realization of \( a \). We follow the idea of
dissecting our graphs along archfree paths. However, to ensure that each internal vertex is placed in the interior of some segment, the way in which we construct our paths is necessarily quite different. Specifically, we will show that a large subfamily of the considered graph class can be dissected along three archfree paths that are arranged in a windmill pattern as depicted in Fig. 2a.

Existence of convex drawings. It is well-known that a plane graph admits a convex drawing if and only if it is a subdivision of an internally 3-connected graph \[3, 4, 16, 17\]. There are multiple ways to define this property and it will be convenient to refer to all of them. Therefore, we use the following characterization; for proofs see, e.g., Kleist et al. \[9\].

**Definition 1.** Let \(G\) be a plane 2-connected graph. Let \(f_o\) denote its outer face. Then \(G\) is called internally 3-connected if and only if the following equivalent statements are satisfied:

1. Inserting a new vertex \(v\) in \(f_o\) and adding edges between \(v\) and all vertices of \(\partial f_o\) results in a 3-connected graph.
2. From each internal vertex \(w\) of \(G\) there exist three paths to \(f_o\) that are pairwise disjoint except for the common vertex \(w\).
3. Every separation pair \(u, v\) of \(G\) is external, i.e., \(u\) and \(v\) lie on \(\partial f_o\) and every connected component of the subgraph of \(G\) induced by \(V(G) \setminus \{u, v\}\) contains a vertex of \(\partial f_o\).

**Observation 1 (\[3\], folklore)** Let \(G\) be an internally 3-connected plane graph and let \(C\) be a simple cycle in \(G\). The closed interior \(C^-\) of \(C\) is an internally 3-connected plane graph.

In the context of our recursive strategy, we face a special case of the following problem: given an internally 3-connected plane graph \(G\) and a convex drawing \(\Gamma^o\) of the boundary of its outer face, extend \(\Gamma^o\) to a convex drawing of \(G\). It is known that such an extension exists if and only if each segment of \(\Gamma^o\) corresponds to an archfree path of \(G\) \[3, 4, 16, 17\]. Hence, we say that \(\Gamma^o\) is compatible with \(G\) if and only if it satisfies this property.

**Construction of archfree paths.** The following lemma gives rise to a strategy for transforming a given internal path into an archfree path:

**Lemma 1 (\[4\], Lem. 1).** Let \(G\) be an internally 3-connected plane graph, \(f\) an internal face of \(G\). Any subpath \(P\) of \(\partial f\) with \(|E(P)| \leq |E(\partial f)| - 2\) is archfree.
Fig. 1: (a) Splitting $\Gamma$ along three straight-line paths. The subpolygon containing arch $a$ cannot be extended to a convex drawing of its subgraph. (b) Left-aligned path $L_G(P)$ of $P = (s, \ldots, t)$.

**left-aligned** path $L_G(P)$ of $P$ is obtained be exhaustively applying the following modification (for an illustration see Fig. 1b): suppose that an internal face $a$ arches $P$ from the left between two vertices $u, v$ such $u$ precedes $v$ along $P$. Transform $P$ by replacing its $uv$-subpath with the $uv$-path obtained by walking along $\partial a$ in counterclockwise direction from $u$ to $v$. The **right-aligned** path $R_G(P)$ is defined symmetrically.

**Lemma 2** ([3, Lemma 5, Corollary 6]). Let $G$ be an internally 3-connected plane graph. Let $P = (s, \ldots, t)$ be a subpath of a simple internal directed path $P'$ between two distinct outer vertices of $G$. Then:
- $L_G(P)$ ($R_G(P)$) is a simple internal st-path not arched from the left (right).
- If $P$ is not arched from the right (left) by an internal face, then $L_G(P)$ ($R_G(P)$) is archfree.
- $R_G(L_G(P))$ ($L_G(R_G(P))$) is archfree.

**Existence of archfree windmills.** Recall that our plan is to dissect our given (internally) 3-connected graph along three archfree paths that form a windmill pattern; see Fig. 2a.

**Definition 2.** Let $G$ be an internally 3-connected graph and let $f_o$ denote its outer face. For $i = 1, 2, 3$, let $P_i = (o_i, \ldots, q_i)$ be a simple path in $G$. We call $(P_1, P_2, P_3)$ a windmill of $G$ if and only if all of the following properties hold (all indices are considered modulo 3):
- (W1) The vertices $o_1, o_2, o_3$ are pairwise distinct and belong to $\partial f_o$.
- (W2) For $i = 1, 2, 3$, no vertex of $V(P_i) \setminus \{o_i\}$ belongs to $\partial f_o$.
- (W3) For $i = 1, 2, 3$, no interior vertex of $P_i$ belongs to $P_{i+1}$.
- (W4) For $i = 1, 2, 3$, the endpoint $q_i$ is an interior vertex of $P_{i+1}$.

If $(P_1, P_2, P_3)$ is a windmill of $G$, we call it archfree if $P_1, P_2, P_3$ are archfree.

A necessary condition for the existence of an archfree windmill is the existence of a strictly internal face (a face without outer vertices). For the considered graph class we show that the condition is sufficient. The following lemma is the main technical contribution of this section:

**Lemma 3** ([3]). Let $G$ be an internally 3-connected plane graph of maximum degree 4 with a strictly internal face $f$. Then $G$ contains an archfree windmill.
Proof (sketch). Let \( f_o \) be the outer face of \( G \). By means of the internal 3-connectivity of \( G \) and Lem. 2, it can be shown that there are three pairwise disjoint archfree paths \( P_i = (o_1, \ldots, f_i), i \in \{1, 2, 3\} \) between \( \partial f_o \) and \( \partial f \) as depicted in Fig. 3. We now walk along \( \partial f \) in a clockwise fashion and append appropriate parts of \( \partial f \) to the paths \( P_1, P_2, P_3 \) to obtain an initial windmill \( (P_1^{cw}, P_2^{cw}, P_3^{cw}) \) as illustrated in Fig. 3. Specifically, we extend each \( P_i \) by the \( f_i f_{i+1} \) subpath of \( \partial f \) that does not contain \( f_{i+2} \) (indices are considered modulo 3). This windmill is not necessarily archfree, but its paths can only be arched in a controlled way: suppose that \( P_i^{cw} \) is arched by an internal face \( a_i^{cw} \). The subpath of \( P_i^{cw} \) that belongs to \( \partial f \) is archfree by Lem.. Combined with the fact that \( P_i \) is archfree, it follows that \( a_i^{cw} \) arches \( P_i^{cw} \) between some vertex \( s_i^{cw} \in V(P_i) \setminus \{f_i\} \) and a vertex \( t_i^{cw} \in V(P_i^{cw}) \setminus V(P_i) \). Moreover, by planarity, \( a_i^{cw} \) has to arch \( P_i^{cw} \) from the left, as illustrated in Fig. 3. We remark that there might be multiple “nested” faces that arch \( P_i^{cw} \). W.l.o.g., we use \( a_i^{cw} \) to denote the “outermost” one, that is, the unique arch whose boundary replaces a part of \( P_i^{cw} \) in the left-aligned path \( Q_i^{cw} = L_G(P_i^{cw}) \), see Fig. 3. The paths of \( (Q_i^{cw}, Q_2^{cw}, Q_3^{cw}) \) are now archfree by Lem. 2 though, \( (W4) \) from Definition 2 is satisfied only for exactly those \( i \in \{1, 2, 3\} \) where the arch \( P_i^{cw} \) is archfree. For each \( Q_i^{cw} \) where \( (W4) \) is violated, we append the \( f_{i+1} t_{i+1}^{cw} \)-path of \( \partial f \) that does not contain \( f_i \), see Fig. 3. This modification maintains the archfreeness by planarity and Lem. 1. However, the resulting path triple \( (R_1^{cw}, R_2^{cw}, R_3^{cw}) \) might still not be a windmill: suppose that a path \( R_i^{cw} \) is not archfree and its arching face \( a_i^{cw} \) is big, that is, \( t_i^{cw} = f_{i+1} \), while additionally the path \( P_i^{cw} \) is archfree (this is the case for \( i = 1 \) in Fig. 3). Then \( (W3) \) from Definition 2 is violated for \( R_i^{cw} \) and \( (W4) \) is violated for \( R^{cw}_i \). Suppose that \( (R_1^{cw}, R_2^{cw}, R_3^{cw}) \) is indeed not a windmill. We construct a path triples \( (P_1^{cw}, P_2^{cw}, P_3^{cw}), (Q_1^{cw}, Q_2^{cw}, Q_3^{cw}), \) and \( (R_1^{cw}, R_2^{cw}, R_3^{cw}) \) in a symmetric fashion by walking around \( \partial f \) in counterclockwise direction. If \( (R_1^{cw}, R_2^{cw}, R_3^{cw}) \) is also not a windmill, it follows that both \( (R_1^{cw}, R_2^{cw}, R_3^{cw}) \) and \( (R_1^{cw}, R_2^{cw}, R_3^{cw}) \) contain a path that is arched by a big face. By planarity and the degree bounds, we can now argue that there is exactly one \( i \in \{1, 2, 3\} \) such that both \( (P_i^{cw}) \) and \( (P_i^{cw}) \) are arched by big faces while both \( (P_1^{cw}) \) and \( (P_i^{cw}) \) are archfree, which is illustrated in Fig. 3: for \( i = 1 \). Assume w.l.o.g. that \( i = 1 \) and that \( s_i^{cw} \) is not closer to \( o_1 \) on \( P_1 \) than \( s_i^{cw} \). In view of the previous observations, it is now easy to argue that the paths of \( (Q_1^{cw}, P_2^{cw}, P_3^{cw}) \) are archfree and satisfy all windmill properties with the exception of \( (W4) \) for

Fig. 2: (a) A windmill \((P_1, P_2, P_3)\). (b) The 3-connected case in the proof of Thm. 1.
Fig. 3: Evolution of the three paths in the proof of Lem. 3.
i = 3. We restore (W4) by appending the $f_1 s_1^w$-subpath of $P_1$ to $P_{cw}^3$, see Fig. 3.

By means of the degree bounds, it can be argued that (W2) and (W4) are maintained for $i = 3$. The resulting path $S_{cw}^3$ might now be arched (from the left, by planarity), which can be remedied by applying Lem. 2, see Figures 3g and h. By means of the degree bounds and planarity arguments, it can be shown that this modification maintains all windmill properties.

A plane graph $G$ is **internally 4-regular** if all of its internal vertices have degree 4 and its outer vertices have degree at most 4. In Lem. 3 we established that the existence of an internal face suffices for the existence of an archfree windmill. By means of simple counting arguments, it can be shown that this condition is satisfied if $G$ has a triangular outer face.

**Lemma 4 (⋆).** Let $G$ be an internally 3-connected plane graph that is internally 4-regular. Let $f_o$ denote the outer face of $G$ and assume $|\partial f_o| = 3$. Then $G$ has a strictly internal face.

**Theorem 1 (⋆).** Let $G$ be an internally 3-connected internally 4-regular plane graph and let $\Gamma^o$ be a compatible convex drawing of its outer face. There exists a convex drawing $\Gamma$ of $G$ that uses $\Gamma^o$ as the realization of the outer face where each internal vertex of $G$ is contained in the interior of some segment of $\Gamma$.

**Proof (sketch).** Our goal is to (recursively) compute coordinates for the internal vertices to obtain the desired drawing of $G$. The base case of the recursion is that $G$ contains no internal edges, in which case there is nothing to show. Assume that $G$ is 3-connected – we deal with the case where $G$ is not 3-connected in the appendix. If $|V(\Gamma^o)| \geq 4$, then there exist two distinct outer vertices $u, v$ that do not belong to a common segment of $\Gamma^o$, see Fig. 2. By 3-connectivity and Lem. 2 they are joined by an archfree internal path $P$. We split $\Gamma^o$ into two simple convex polygons along $P$ and handle the two corresponding subgraphs recursively. If $|V(\Gamma^o)| = 3$, then $G$ contains an archfree windmill $(P, S, Q)$ by Lems. 3 and 4. Since the three outer endpoints of $P, S, Q$ do not belong to a common segment of $\Gamma^o$, we can embed them in a straight-line fashion such that $\Gamma^o$ is dissected into four simple convex polygons, see Fig. 2. We handle the corresponding four subgraphs recursively.

**Universal upper bound.** Recall that to establish the claimed upper bound, it suffices to create a drawing where all but three of the vertices of the graph are drawn in the interior of some segment. To achieve this goal, we can now draw the outer face of the graph as a triangle and then apply Thm. 1.

**Theorem 2 (⋆).** Every 3-connected internally 4-regular plane graph $G$ admits a convex drawing on at most $n + 3$ segments where $n$ is the number of vertices.

**Existential lower bound.** For a graph $G$, let $G^2$ denote the square of $G$, that is, $V(G) = V(G^2)$, and two vertices in $G^2$ are adjacent if their distance in $G$ is at most 2. For $n \geq 6$, the square of the $n$-cycle, $C_n^2$, is 4-regular and triconnected.

**Proposition 1 (⋆).** For even $n \geq 6$, $C_n^2$ is planar and $\text{seg}(C_n^2) \geq n$.  

Fig. 4: An outerpath represented by a pseudo 2-arc arrangement. The internal edges $e_1, \ldots, e_6$ of arc $\alpha$ subdivide the outerpath into bays $H_0, \ldots, H_6$. For our charging, we count crossings of $\alpha$ with other arcs (indicated by red crosses).

3 Maximal Outerpaths

Next, we generalize segments and arcs to pseudo $k$-arcs and give a universal lower bound for the number of pseudo $k$-arcs in drawings of maximal outerpaths.

We call the sequence of vertices $v_1, v_2, \ldots, v_n$ of a maximal outerpath $G$ its stacking order if for each $i$, the graph $G_i$ induced by the vertices $v_1, v_2, \ldots, v_i$ is a maximal outerpath. An arrangement of pseudo $k$-arcs is a set of curves in the plane such that any two of the curves intersect at most $k$ times. (If two curves share a tangent, this counts as two intersections.) We forbid self-intersections, but for $k \geq 2$ we allow a pseudo $k$-arc to be closed.

To show the bound, we present a charging scheme that assigns internal edges to pseudo $k$-arcs. Any drawing of a maximal outerpath has exactly $n - 3$ internal edges. A pseudo $k$-arc is long if it contains at least $k + 1$ internal edges; otherwise it is short. Let $\text{arc}_k$ denote the number of pseudo $k$-arcs, and let $\text{arc}_k^i$ denote the number of pseudo $k$-arcs with $i$ internal edges. The internal edges of a long arc $\alpha$ subdivide the outerpath into subgraphs $H_0, H_1, \ldots, H_\ell$ called bays; see Fig. 4. Given a drawing $\Gamma$ of a maximal outerpath, we denote the sub-drawings of $G_3, G_4, \ldots, G_n$ within $\Gamma$ by $\Gamma_3, \Gamma_4, \ldots, \Gamma_n$, respectively. A pseudo $k$-arc $\alpha$ is incident to a face $f$ if $\alpha$ contains an edge incident to a vertex of $f$. We say that $\alpha$ is active in $\Gamma_i$ if $\alpha$ is incident to the last face that has been added.

Lemma 5 (⋆). For any $i \in \{3, \ldots, n\}$, a partial outerpath drawing $\Gamma_i$ contains at most one active long pseudo $k$-arc.

We do a 2-round assignment to assign each internal edge to a pseudo $k$-arc. We start with the round-1 assignment. Let $I$ denote the set of internal edges of long pseudo $k$-arcs starting at the $(k + 1)$-th internal edge (as for the first $k$ internal edges an arc is still short). We assign all $n - 3$ internal edges except for the edges in $I$ to their own pseudo $k$-arcs:

$$(n-3)-|I| = k \text{arc}_k^{\geq k} + (k-1)\text{arc}_k^{k-1} + \cdots + \text{arc}_k^1 = k \text{arc}_k - \sum_{i=0}^{k-1}(k-i)\text{arc}_k^i$$ (1)

Now we describe the round-2 assignment. There, we charge internal edges to crossings, which we can charge in turn to pseudo $k$-arcs. A crossing is a triplet $(\alpha, \beta, p)$ that consists of two pseudo $k$-arcs $\alpha$ and $\beta$ and a point $p$ at which
\(\alpha\) and \(\beta\) intersect. Consider the number of crossings in an outerpath drawing that involve a long arc \(\alpha\). Suppose \(\alpha\) has \(\ell\) internal edges \((\ell > k)\). For each bay \(H \in \{H_1, \ldots, H_{|I|}\}\), there are \(\geq 2\) crossings of \(\alpha\) with other arcs – one at the first and one at the last vertex of \(H\); see the red crosses in Fig. 4. We charge the surplus internal edges of the long arcs to the other pseudo \(k\)-arcs that are involved in the crossings we counted.

These vertices are individual for each pair of consecutive bays. Hence, the crossings must all be different as well. Note that a tangential point may be shared by some \(H_j\) and \(H_{j+2}\) (for \(j \in \{1, \ldots, \ell-3\}\)); see \(H_2\) and \(H_4\) in Fig. 4 for an example. However, we can still charge a crossing to each of \(H_j\) and \(H_{j+2}\) since a tangential point counts for two crossings. An exception are \(H_0\) and \(H_{\ell}\). They may share a crossing with \(H_1\) and \(H_{\ell-1}\), respectively, as \(H_0\) and \(H_5\) do in Fig. 4. Hence, we count only one crossing for \(H_0\) and \(H_{\ell}\). Therefore, for each internal edge \(e\) of \(I\) we have two individual crossings of the preceding bay, e.g., in Fig. 4

\[H_2\] provides two crossings for \(e_3\). We denote the set of these crossings by \(C\). Note that the crossings of \(H_0, H_1, \ldots, H_{k-1},\) and \(H_{\ell}\) are not included in \(C\) since the succeeding internal edges are not contained in \(I\). Clearly, we know that \(2|I| = |C|\).

Next, we give an upper bound for \(|C|\) in terms of \(\text{arc}_k\). The main argument we exploit is that, by definition, each pseudo \(k\)-arc can participate in at most \(k\) crossings with the (current) long arc. However, we need to be a bit careful for the case when one long pseudo \(k\)-arc becomes inactive and a new pseudo \(k\)-arc becomes long, i.e., we consider the transition between one long arc to a new long arc. For each long arc, we count the crossings in \(H_k, H_{k+1}, \ldots, H_{\ell}\) as described before. We may get additional crossings because potentially any pseudo \(k\)-arc could intersect each long arc \(k\) times. To compensate for the double counting at transitions, we introduce the transition loss \(t_k\). Moreover, we cannot count crossings of the first long arc with (other) long arcs, and we do not count the crossings of the very first bay and the very last bay. This yields the following.

\[
2|I| = |C| \leq k \cdot (\text{arc}_k - 1) + \frac{-(2k - 1)}{2} + 1 + t_k
\]  

Equation (2)

Plugging Eq. (2) into Eq. (1) we obtain the following general formula, which gives a lower bound on the number of pseudo \(k\)-arcs for any outerpath.

\[
\text{arc}_k \geq (2n - 6 + 2 \cdot \sum_{i=0}^{k} (k - i) \cdot \text{arc}_k^i - t_k) / (3k) + 1
\]  

Equation (3)

Since this formula still contains unresolved variables, we now resolve \(t_k\).

Lemma 6. There is a loss of at most one crossing per transition from one long pseudo \(k\)-arc to another long pseudo \(k\)-arc. Hence, \(t_k \leq \max(0, \text{arc}_k^{\geq k} - 1) \leq \text{arc}_k^{\geq k} = \text{arc}_k - \sum_{i=0}^{k} \text{arc}_k^i\), where \(\text{arc}_k^{\geq k}\) is the number of long pseudo \(k\)-arcs.

By Lem. 6 and Eq. (3)

\[
\text{arc}_k \geq (2n + 3k - 6 + \sum_{i=0}^{k} (2k - 2i + 1) \cdot \text{arc}_k^i) / (3k + 1).
\]  

Equation (4)
Fig. 5: Families of maximal outerpaths with (a) \(n/2 + 2\) segments (matching the lower bound in Thm. 3), (b) \(n/3 + 1\) circular arcs, and (c) \((5n + 18)/16 < n/3\) pseudo 2-arcs.

Since this general formula still contains the unresolved variables \(\text{arc}_k^i\), we plug in specific values of \(k\) and prove lower bounds on \(\text{arc}_k^i\). We start with \(k = 1\), i.e., outerpath drawings on pseudo segments.

**Lemma 7** (⋆). For \(k = 1\) and \(n \geq 3\), in any outerpath drawing either \(\text{arc}_1^0 \geq 3\) or \(\text{arc}_1^1 \geq 2\) and \(\text{arc}_1^2 \geq 3\).

Using Lem. 7, we fill the gaps in Eq. (4) for \(k = 1\) and obtain Thm. 3.

**Theorem 3** (⋆). For any \(n\)-vertex maximal outerpath \(G\), \(\text{seg}(G) \geq \left\lfloor \frac{n}{2} \right\rfloor + 2\).

For \(k = 2\), i.e, for (pseudo) circular arcs, Eq. (4) leads to the following bound.

**Theorem 4** (⋆). For any \(n\)-vertex maximal outerpath \(G\), \(\text{arc}(G) \geq \left\lceil \frac{2n}{7} \right\rceil\).

For \(k > 2\), it is not obvious how to generalize circular arcs. Still, we can make a similar statement for curve arrangements, which follows directly from Eq. (4).

**Proposition 2.** Let \(G\) be an \(n\)-vertex maximal outerpath drawn on a curve arrangement in the plane s.t. curves intersect pairwise \(\leq k\) times, can be closed, but do not self-intersect. Then, the number \(\text{arc}_k^i(G)\) of curves required is \(\left\lceil \frac{2n + 3k - 6}{3k + 1} \right\rceil\).

The infinite families of examples in Prop. 3 and Fig. 5 show that our bounds for segments and arcs are tight. This implies, somewhat surprisingly, that, at least for worst-case instances, using pseudo segments requires as many elements as using straight line segments. Whether this also holds for pseudo circular arcs and circular arcs is an open question. With circular arcs, we could not beat a bound of \(n/3\), which we could do for pseudo circular arcs.

**Proposition 3** (⋆). For every \(r \in \mathbb{N}\), maximal outerpaths \(P_r, Q_r, U_r\) exist s.t.

(i) \(P_r\) has \(2r + 6\) vertices and \(\text{seg}(P_r) \leq r + 5 = \frac{2n}{2} + 2\),

(ii) \(Q_r\) has \(3r\) vertices and \(\text{arc}(Q_r) \leq r + 1 = \frac{2n}{3} + 1\),

(iii) \(U_r\) has \(16r + 6\) vertices and \(\text{arc}_2(U_r) \leq 5r + 3 = \frac{5n + 18}{16} \approx 0.3125n\).

**Open Problems.** The most prominent open problem is to close the gaps in Table 1. Since circular-arc drawings are a natural generalization of straight-line drawings, it is natural to also ask about the maximum ratio between the segment number and the arc number of a graph. We make some initial observations regarding this question in App. F. Finally, what is the complexity of deciding whether the arc number of a given graph is strictly smaller than its segment number?
The Segment Number: Algorithms and Universal Lower Bounds

References

1. V. Dujmović, D. Eppstein, M. Suderman, and D. R. Wood. Drawings of planar graphs with few slopes and segments. *Comput. Geom. Theory Appl.*, 38(3):194–212, 2007. doi:10.1016/j.comgeo.2006.09.002

2. S. Durocher and D. Mondal. Drawing plane triangulations with few segments. *Comput. Geom. Theory Appl.*, 77:27–39, 2019. doi:10.1016/j.comgeo.2018.02.000

3. S. Hong and H. Nagamochi. Convex drawings of graphs with non-convex boundary constraints. *Discret. Appl. Math.*, 156(12):2368–2380, 2008. doi:10.1016/j.dam.2007.10.012

4. S. Hong and H. Nagamochi. Convex drawings of hierarchical planar graphs and clustered planar graphs. *J. Discrete Algorithms*, 8(3):282–295, 2010. doi:10.1016/j.jda.2009.05.003

5. G. Hültenschmidt, P. Kindermann, W. Meulemans, and A. Schulz. Drawing planar graphs with few geometric primitives. *J. Graph Alg. Appl.*, 22(2):357–387, 2018. doi:10.7155/jgaa.00473

6. A. Igamberdiev, W. Meulemans, and A. Schulz. Drawing planar cubic 3-connected graphs with few segments: Algorithms & experiments. *J. Graph Algorithms Appl.*, 21(4):561–588, 2017. doi:10.7155/jgaa.00430

7. P. Kindermann, T. Mechedlidze, T. Schneck, and A. Symvonis. Drawing planar graphs with few segments on a polynomial grid. In D. Archambault and C. D. Tóth, editors, *Proc. 27th Int. Symp. Graph Drawing & Netw. Vis.*, volume 11904 of LNCS, pages 416–429. Springer, 2019. doi:10.1007/978-3-030-35802-0_32

8. P. Kindermann, W. Meulemans, and A. Schulz. Experimental analysis of the accessibility of drawings with few segments. *J. Graph Alg. Appl.*, 22(3):501–518, 2018. doi:10.7155/jgaa.00474

9. L. Kleist, B. Klemz, A. Lubiw, L. Schlipf, F. Staals, and D. Strash. Convexity-increasing morphs of planar graphs. *Comput. Geom.*, 84:69–88, 2019. doi:10.1016/j.comgeo.2019.07.007

10. B. Klemz. Convex drawings of hierarchical graphs in linear time, with applications to planar graph morphing. In P. Mutzel, R. Pagh, and G. Herman, editors, *29th Annual European Symposium on Algorithms, ESA 2021, September 6-8, 2021, Lisbon, Portugal (Virtual Conference)*, volume 204 of LIPIcs, pages 57:1–57:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2021. doi:10.4230/LIPIcs.ESA.2021.57

11. D. Mondal, R. I. Nishat, S. Biswas, and M. S. Rahman. Minimum-segment convex drawings of 3-connected cubic plane graphs. *J. Comb. Optim.*, 25(3):460–480, 2013. doi:10.1007/s10878-011-9390-6

12. Y. Okamoto, A. Ravsky, and A. Wolff. Variants of the segment number of a graph. In D. Archambault and C. D. Tóth, editors, *Proc. 27th Int. Symp. Graph Drawing & Network Vis. (GD’19)*, volume 11904 of LNCS, pages 430–443. Springer, 2019. doi:10.1007/978-3-030-35802-0_33

13. M. A. H. Samee, M. J. Alam, M. A. Adnan, and M. S. Rahman. Minimum segment drawings of series-parallel graphs with the maximum degree three. In I. G. Tollis and M. Patrignani, editors, *Proc. 16th Int. Symp. Graph Drawing*, volume 5417 of LNCS, pages 408–419. Springer, 2008. doi:10.1007/978-3-642-00219-9_40

14. A. Schulz. Drawing graphs with few arcs. *J. Graph Alg. Appl.*, 19(1):393–412, 2015. doi:10.7155/jgaa.00360
15. R. Tarjan. Depth-first search and linear graph algorithms. *SIAM Journal on Computing*, 1(2):146–160, 1972. doi:10.1137/0201010

16. C. Thomassen. Plane representations of graphs. In J. A. Bondy and U. S. R. Murty, editors, *Progress in Graph Theory*, pages 43–69. Academic Press, 1984.

17. W. T. Tutte. Convex representations of graphs. *Proceedings of the London Mathematical Society*, s3-10(1):304–320, 1960. doi:10.1112/plms/s3-10.1.304
Appendix

In the following, we provide full proofs and omitted content. First, we introduce some notation that we use throughout the appendix.

Recall that a cactus is a connected graph where any two simple cycles share at most one vertex. A graph $G$ is a $k$-tree if it admits a stacking order $v_1, v_2, \ldots, v_n$ of the vertices together with a sequence of graphs $G_{k+1}, G_{k+2}, \ldots, G_n = G$ such that (i) $G_{k+1}$ is a clique on $\{v_1, \ldots, v_{k+1}\}$; and (ii) for $k + 2 \leq i$, the graph $G_i$ is obtained from $G_{i-1}$ by making $v_i$ adjacent to all vertices of a $k$-clique in $G_{i-1}$. A vertex placement in step (ii) is called a stacking operation. Similarly, we call the sequence of vertices $v_1, v_2, \ldots, v_n$ of a maximal outerplanar graph $G$ its stacking order if for each $i$ the graph $G_i$ induced by the vertices $v_1, v_2, \ldots, v_i$ is a maximal outerplanar graph. If $G$ is an outerpath, each $G_i$ is an outerpath.

In a straight-line drawing $\Gamma$ of a graph $G$, each segment terminates at two vertices. Let $s$ be a segment in $\Gamma$, and let $v$ be an endpoint of $s$. Geometrically speaking, we could extend $s$ at $v$ into a face $f$. We say that $s$ has a port at $v$ in $f$. We call $v$ open if $v$ has at least one port and closed otherwise. Let port($\Gamma$) be the number of ports in $\Gamma$, and let port($G$) be the minimum number of ports over all straight-line drawings of $G$. Observe that, for any planar graph $G$, it holds that $\text{seg}(G) = \text{port}(G)/2$. Hence, in a drawing of $G$, counting segments is equivalent to counting ports.

A An Algorithm for Cactus Graphs

We first state a lower bound for the segment number of cactus graphs. Then, we give a recursive algorithm that produces drawings meeting the bound precisely.

Lemma 8. Let $G$ be a cactus graph, let $\eta$ be the number of odd-degree vertices of $G$, and let $\gamma = 3c_0 + 2c_1 + c_2$, where $c_i$ is the number of simple cycles with exactly $i$ cut vertices in $G$. Then $\text{seg}(G) \geq \eta/2 + \gamma$.

Proof. If $G$ is a tree, then $\gamma = 0$ and $\text{seg}(G) = \eta/2$, as shown by Dujmović et al. [1]. If $G$ is a cycle, then all vertices have degree 2 (that is, $\eta = 0$). Moreover, $c_0 = 1$ and $c_1 = c_2 = 0$. A cycle can be drawn as a triangle (but not with less than three segments), that is, $\text{seg}(G) = 3$.

So assume that $G$ is neither a tree nor a cycle. Then $G$ contains at least one cycle and each cycle has at least one cut vertex, that is, $c_0 = 0$. Let $\Gamma$ be any straight-line drawing of $G$. Every odd-degree vertex of $G$ has a port in $\Gamma$. Hence, $\Gamma$ has at least $\eta$ ports.

Additionally, each cycle $f$ of $G$ is a simple polygon in $\Gamma$. In other words, $f$ is incident to at least three segments in $\Gamma$. If $f$ contains exactly two cut vertices, the drawing of $f$ must contain a bend at some vertex of $f$ that is not a cut vertex, that is, a degree-2 vertex. This increases the number of ports by 2. Similarly, if $f$ contains exactly one cut vertex, the drawing of $f$ must contain two bends at degree-2 vertices, which increases the number of ports by 4. In total, $\Gamma$ has at least $\eta + 4c_1 + 2c_2$ ports or $\eta/2 + 2c_1 + c_2$ segments. Since $c_0 = 0$, we have $\text{seg}(G) \geq \eta/2 + 3c_0 + 2c_1 + c_2$ as claimed. □
It is not difficult, but somewhat technical to draw a given cactus such that the lower bound in the above lemma is met exactly. For an idea of how we proceed, refer to Fig. 6.

**Theorem 5.** Given a cactus graph $G$, we can compute $\text{seg}(G)$ in linear time. Within the same timebound, we can draw $G$ using $\text{seg}(G)$ many segments. If $G$ is given with an outerplanar embedding, the drawing will respect the given embedding.

**Proof.** If $G$ is a tree, we can use the linear-time algorithm of Dujmović et al. [1], which yields a drawing with $\eta/2$ segments, which is optimal. If $G$ is a simple cycle, we can draw $G$ as a triangle, which again is optimal. Otherwise, $c_0 = 0$. In this case, which we treat below, we draw $G$ with $\eta/2 + 2c_1 + c_2$ segments, which is optimal according to Lem. 8.

We draw $G$ recursively, treating its biconnected components as units. Note that, in a cactus graph, the biconnected components (called blocks) are exactly its simple cycles and the edges that do not lie on any simple cycle. The block-cut tree of a connected graph $H$ has a node for each cut vertex and a node for each block. A block node and a cut-vertex node are connected by an edge in the tree if, in $H$, the block contains the cut vertex.

We compute the block-cut tree of $G$, which can be done in linear time [15], and root it at a block node that corresponds to a simple cycle $f$. We start by drawing this block as a regular $p$-gon $P$, where $p$ is the maximum of 3 and the number of cut vertices of $f$. Let $2r$ be the edge length of $P$, and let $\alpha$ be the interior angle at each corner of $P$. Then $\alpha = 180^\circ \cdot (p - 2)/p$.

![Fig. 6: Recursive approach for drawing cactus graphs. Vertex $v$ is a cut vertex of $f$ (or a degree-2 vertex if $f$ has less than three cut vertices). After $f$ has been drawn, the algorithm recursively draws the subgraph $G(v)$ into $C_{v,r}$ such that $v$ has a port if and only if $\text{deg}(v)$ is odd.](image)
For each cut vertex \( v \) of \( f \), we recursively draw the subgraph \( G(v) \) of \( G \) corresponding to the subtree that hangs off \( v \) in the block-cut tree; see Fig. 6. We draw \( G(v) \) into the interior of the circle \( C_{v,r} \) of radius \( r \) centered at \( v \). (Within this circle, we use only the complement of \( P \).) For each pair of cut vertices, the interiors of the corresponding circles are disjoint; hence, the drawing of \( G \) has no edge crossings if the drawings of the subgraphs are crossing-free. Our drawing of \( G \) will have the following property. Each odd-degree vertex has exactly one port and, in every simple cycle of \( G \) with \( j < 3 \) cut vertices, there are exactly \( 3 - j \) degree-2 vertices with two ports. This implies that the total number of segments in our drawing meets the bound in Lem. 8 precisely.

Let \( d = \deg(v) - 2 \), and let \( N(v) = \{v_1, \ldots, v_{d+2}\} \) be the neighborhood of \( v \).
Let \( v_{d+1} \) and \( v_{d+2} \) be the two neighbors of \( v \) that lie on \( f \) (in clockwise order before and after \( v \) on \( f \)) and have already been placed. Let \( v_1, \ldots, v_{d+2} \) be ordered clockwise around \( v \). We assume that neighbors that belong to the same simple cycle are consecutive in this ordering. (Note that this is the case if \( G \) is given with a fixed outerplane embedding.) We now define a set \( W \) of vertices in \( G(v) \) for which we may call our algorithm recursively. Initially, \( W \) is empty. For \( i \in \{1, \ldots, d\} \), if \( v_i \) and \( v \) do not lie on the same simple cycle, then set \( w_i = v_i \) and add \( w_i \) to \( W \). Now let \( f' \) be the simple cycle that contains \( v, v_i \) and another neighbor of \( v \), say, \( v_{i+1} \). If \( f' \) does not contain a cut vertex other than \( v_i \), set \( w_i = v_i \) and add \( w_i \) to \( W \). Otherwise, let \( w_i \) be the cut vertex of \( G \) closest to \( v_i \) in \( G(v) - v \). If \( v_{i+1} \) has the same closest cut vertex \( w_j \), then, if \( w_j \neq v_i \), set \( w_i = v_i \), otherwise set \( w_{i+1} = v_{i+1} \) and \( w_i = w_{i+1} \) and all other cut vertices of \( f' \) (if any) to \( W \) (except \( v \)).

We now place the vertices in \( W \) on the circle \( C_{v,r/2} \). If \( d \) is odd, then we place \( w_{(d+1)/2} \) on the line that bisects the angle \( \angle v_{d+1}v_{d+2} \); namely such that \( w_{(d+1)/2} \) lies opposite of this angle (as \( w_3 \) in Fig. 6). For the remainder of this proof, we assume for simplicity that \( d \) is even. Then \( d \geq 2 \) and we place \( w_d/2 \) on the line \( v_{d+1}v_{d+2} \) and \( w_{d/2+1} \) on the line \( v_{d+1}v_{d+2} \). We place the remaining neighbors in pairs on opposite sides of lines through \( v \) such that these lines equally partition the angle space in the double wedge \( W_{v,\alpha} \) (light yellow in Fig. 6) that is bounded by the lines \( v_{d+1}v_1 \) and \( v_{d+2}v_2 \) and does not contain the angle \( \angle v_{d+1}v_1v_{d+2} \). The angular distance between two consecutive edges incident to \( v \) is then \( \beta = (360\degree - 2\alpha)/d \).

We draw each simple cycle \( f' \) that contains \( v \) and two neighbors \( v_i \) and \( v_{i+1} \) of \( v \) as a simple polygon that connects \( v \) to \( w_i \) to potential further cut vertices of \( f' \) (in their order along \( f' \)) to \( w_{i+1} \) to \( v \).

Now we define, for each newly placed vertex \( w \in W \) with \( \deg(w) > 2 \), values \( \alpha' \) and \( r' \) so that we can draw the graph \( G(w) \) recursively. To this end, if \( v \) and \( w \) lie on the same simple cycle \( f' \), let \( \alpha' \) be the interior angle of \( w \) in \( f' \), and let \( r' \) be the distance of \( w \) to the closest vertex in \( V(f) \cap W \) divided by 2. Otherwise, let \( \alpha' = 0 \) and set \( r' \) such that \( C_{w,r'} \) fits into a wedge centered at \( w \) that has an angle of \( \beta \) at its apex \( v \); see, for example, \( w_3 \) in Fig. 6.

Our invariant is that, in each recursive call for \( G(w') \), we have \( 0 \leq \alpha' < 180\degree \) and \( r' > 0 \). This ensures that our drawing has no crossings. To finish the proof, note that the segments that we draw end only in odd-degree vertices (one port
each) or in degree-2 vertices (two ports each) of simple cycles that have less than three cut vertices.

Concerning the running time, it is easy to see that each recursive call of the algorithm runs in time linear in the size of the subgraph of $G$ that the current call draws without further recursion. Hence, the overall running time is linear in the size of $G$ (including the computation of the block-cut tree).

Note that the algorithm in the proof of Thm. 5 can draw a cactus with a fixed outerplane embedding such that its embedding is maintained. Unfortunately, the drawing area can be at least exponential, even if the embedding is not fixed.

B Proofs Omitted in Section 2 (4-Regular Planar Graphs)

Observation 1 (⋆, folklore) Let $G$ be an internally 3-connected plane graph and let $C$ be a simple cycle in $G$. The closed interior $C^-$ of $C$ is an internally 3-connected plane graph.

Proof. Clearly, Property (I2) of Definition 1 carries over from $G$ to $C^-$. ☐

Lemma 3 (⋆). Let $G$ be an internally 3-connected plane graph of maximum degree 4 with a strictly internal face $f$. Then $G$ contains an archfree windmill.

Proof. Let $f_o$ be the outer face of $G$. We begin by constructing three disjoint archfree paths between $\partial f_o$ and $\partial f$, as illustrated in Fig. 7a. We plan to use these paths, as well as parts of $\partial f$ to construct a windmill (see Fig. 7b). We then apply Lem. 2 to make its paths archfree (illustrated in Fig. 7c and Fig. 8a). This may destroy the windmill properties, but it does so in a controlled way, which allows us to successively modify our paths to restore the windmill properties while maintaining the archfreeness.

Claim 1 $G$ contains three simple paths $P_i = (o_i, \ldots, f_i), i \in \{1, 2, 3\}$ such that

(P1) $P_1, P_2, P_3$ are pairwise vertex-disjoint,
(P2) for $i = 1, 2, 3$, the endpoint $o_i$ belongs to $\partial f_o$,
(P3) for $i = 1, 2, 3$, the endpoint $f_i$ belongs to $\partial f$,
(P4) for $i = 1, 2, 3$, the interior vertices of $P_i$ belong to neither $\partial f$ nor $\partial f_o$, and
(P5) $P_1, P_2, P_3$ are archfree in $G$.

Proof (of Claim 1). For illustrations refer to Fig. 7a. To show that $P_1, P_2, P_3$ exist, we add a new vertex $v_f$ into $f$ and add edges between $v_f$ every vertex of $\partial f$. It is easy to see that this modification retains Property (I1) of Definition 1 and, hence, the resulting graph $G'$ is internally 3-connected. By Property (I2) of Definition 1 applied to $v_f$ in $G'$, it follows that $G$ contains three simple paths $P'_i = (o_i, \ldots, f_i), i \in \{1, 2, 3\}$, that satisfy Properties (I1) - (I4). For $i = 1, 2, 3$, we consider $P'_i$ to be directed from $o_i$ to $f_i$ and define $P_i = R_G(L_G(P'_i))$. By Lem. 2 the paths $P_1, P_2, P_3$ satisfy Property (P5). To see that the remaining
properties are also satisfied, we argue as follows: let $C$ be the simple cycle formed by the paths $P_2, P_3$, the $o_2o_3$-path on $\partial f_\circ$ that passes through $o_1$, and the $f_2f_3$-path on $\partial f$ that passes through $f_1$. The closed interior $C^-$ of $C$ is internally 3-connected by Obs. 1. Note that $R_{C^-}(L_{C^-}(P'_i)) = P'_1$ since the only internal face of $G$ that is not an internal face of $C^-$ but shares a vertex with $P'_i$ is $f$, which has only a single vertex in common with $P'_i$ by Properties (P2)–(P4). By Lem. 2 applied to $C^-$ and $P'_1$, $P_1$ is an internal $o_1f_1$-path of $C^-$. Hence, the paths $P_1, P'_2, P'_3$ satisfy Properties (P1)–(P4) in $G$. With analogous arguments, we see that $P_1, P_2, P_3$ and, finally, $P_1, P_2, P_3$ also satisfy Properties (P1)–(P4). □

Without loss of generality, assume that $a_1, o_2, o_3$ appear on $\partial f$ in clockwise order, as depicted in Fig. 7b. For $i = 1, 2, 3$, we append the $f_if_{i+1}$-path on $\partial f$ that does not pass through $f_{i-1}$ to $P_i$ and call the resulting simple path $P_i^{cw}$ (all indices are considered modulo 3), see Fig. 7b. Symmetrically, for $i = 1, 2, 3$, we append the $f_if_{i+1}$-path on $\partial f$ that does not pass through $f_{i+1}$ to $P_i$ and call the resulting simple path $P_i^{cw}$. By construction, both $(P_1^{cw}, P_2^{cw}, P_3^{cw})$ and $(P_1^{cw}, P_2^{cw}, P_3^{cw})$ are windmills in $G$. If one of them is archfree, we are done. So assume otherwise.

We will now deform parts of $(P_1^{cw}, P_2^{cw}, P_3^{cw})$ (or $(P_1^{cw}, P_2^{cw}, P_3^{cw})$) to obtain the desired archfree windmill. To this end, we introduce some notation, for illustrations refer to Fig. 7b: suppose that a path $P_i^{cw}, i \in \{1, 2, 3\}$ is arched by a face $a_i^{cw}$. The subpath $P_i$ of $P_i^{cw}$ is archfree by Claim 1. Moreover, the $f_if_{i+1}$-subpath of $P_i$ is also archfree by Lem. 1. Consequently, the face $a_i^{cw}$ arches $P_i^{cw}$ between some vertex $s_i^{cw} \in V(P_i) \setminus \{f_i\}$ and a vertex $t_i^{cw} \in V(P_i^{cw}) \setminus V(P_i)$. We consider $P_i^{cw}$ to be directed such that $a_i$ is its source. By planarity, the face $a_i^{cw}$ has to arch from the left. We remark that there might be multiple faces that arch $P_i^{cw}$ (from the left), in which case these faces have to be “nested” (as depicted in Fig. 7b). Without loss of generality, we may assume that $a_i^{cw}$ is the “outermost” of these arches. More precisely, we assume that $a_i^{cw}$ is the unique arch such that a $s_i^{cw}t_i^{cw}$-path on $\partial a_i^{cw}$ replaces the $s_i^{cw}t_i^{cw}$-path on $P_i^{cw}$ in $L_G(P_i^{cw})$. We say that $a_i^{cw}$ is big if $t_i^{cw} = f_{i+1}$ (in Fig. 7b) the arch $a_i^{cw}$ is big, while $a_3^{cw}$ is not). Symmetrically, we define the expressions $a_i^{cw}, s_i^{cw}, t_i^{cw}$ for each path $P_i^{cw}, i \in \{1, 2, 3\}$ that is arched (from the right), and we say that $a_i^{cw}$ is big if $t_i^{cw} = f_i$. For $i = 1, 2, 3$, let $D_i$ denote the simple cycle that is formed by $P_i, P_{i+1}$, the $f_if_{i+1}$-path on $\partial f$ that does not pass through $f_{i+2}$ and the $o_if_{i+1}$-path on $\partial f_o$ that does not pass through $o_{i+2}$ (the closed interior $D_i^-$ of $D_i$ is indicated in Fig. 7a).

For $i = 1, 2, 3$, we define $Q_i^{cw} = L_G(P_i^{cw})$, for illustrations refer to Fig. 7c.

**Claim 2** The paths $Q_1^{cw}, Q_2^{cw}, Q_3^{cw}$ are archfree. Properties (W1)–(W3) from Definition 2 hold for $(Q_1^{cw}, Q_2^{cw}, Q_3^{cw})$. Property (W4) from Definition 2 is satisfied for exactly those $i \in \{1, 2, 3\}$ where the arch $a_{i+1}^{cw}$ is undefined.

**Proof** (of Claim 2). Lem. 2 implies that the paths are archfree (recall that $P_i^{cw}$ is not arched from the right) and that the Properties (W1) and (W2) from Definition 2 carry over from $(P_1^{cw}, P_2^{cw}, P_3^{cw})$ to $(Q_1^{cw}, Q_2^{cw}, Q_3^{cw})$. □
To see that Property \((W3)\) also carries over, we can argue as follows: let \(C_i\) be the simple cycle formed by the paths \(P_{i+1}, P_{i+2}\), the \(\alpha_{i+1}\alpha_{i+2}\)-path on \(\partial f_{o}\) that passes through \(o_i\), and the \(f_{i+1}f_{i+2}\)-path on \(\partial f\) that does not pass through \(f_1\). The closed interior \(C_i^-\) of \(C_i\) is internally 3-connected by Obs. 1. Note that \(L_{C_i^-}(P_{cw}^i) = Q_{cw}^i\) since the internal faces of \(G\) that do not belong to \(C_i^-\) intersect \(P_{cw}^i\) only in \(f_{i+1}\). Hence, by Lem. 2 applied to \(C_i^-\) and \(P_{cw}^i\) and by construction, the part of \(Q_{cw}^i\) that does not belong to \(P_{cw}^i\) is located in the interior of the cycle \(D_i\). By construction, the interior of \(D_i\) is disjoint from \(P_{cw}^1, P_{cw}^2, \) and \(P_{cw}^3\). Moreover, \(D_1, D_2, D_3\) are pairwise interior disjoint. Consequently, Property \((W3)\) from Definition 2 carries over from \((P_{cw}^1, P_{cw}^2, P_{cw}^3)\) to \((Q_{cw}^1, Q_{cw}^2, Q_{cw}^3)\), as claimed.

Finally, Property \((W4)\) is violated for those \(i \in \{1, 2, 3\}\) where the arch \(\alpha_{i+1}^{cw}\) exists since in this case the endpoint \(f_{i+1}\) of \(Q_{cw}^i\) is not on \(Q_{cw}^{i+1}\). In contrast,
Property [W4] is satisfied for those \( i \in \{1, 2, 3\} \) where the arch \( a_i^{cw} \) undefined since in this case the endpoint \( f_{i+1} \) of \( Q_i^{cw} \) is on \( Q_i^{cw} (= P_i^{cw}) \). □

If \((Q_1^{cw}, Q_2^{cw}, Q_3^{cw})\) or \((Q_1^{cw}, Q_2^{cw}, Q_3^{cw})\) (which is defined symmetrically and for which a symmetric version of Claim 2 holds) is an archfree windmill, we are done. So assume otherwise. By Claim 2, the only violated property is [W4]. To remedy the situation, we will now construct three paths \( R_i^{cw}, R_i^{cw}, R_i^{cw} \) by appending appropriate parts of \( \partial f \) to the corresponding paths in \((Q_1^{cw}, Q_2^{cw}, Q_3^{cw})\).

For each \( i \in \{1, 2, 3\} \) where the arch \( a_i^{cw} \) is undefined, we set \( R_i^{cw} = Q_i^{cw} \). For each \( i \in \{1, 2, 3\} \) where the arch \( a_i^{cw} \) exists, we append the \( f_{i+1}f_{i+1}^{cw} \)-path on \( \partial f \) that does not contain \( f_i \) to \( Q_i^{cw} \) and denote the resulting path by \( R_i^{cw} \), for illustrations refer to Fig. 7d.

**Claim 3** The paths \( R_1^{cw}, R_2^{cw}, R_3^{cw} \) are archfree. Properties [W1] and [W2] from Definition 2 hold for \((R_1^{cw}, R_2^{cw}, R_3^{cw})\). Property [W3] from Definition 2 is violated for exactly those \( i \in \{1, 2, 3\} \) where \( P_i^{cw} \) is arched by a big face and \( a_i^{cw} \) is undefined. Property [W4] from Definition 2 violated exactly those \( i \in \{1, 2, 3\} \) where \( P_i^{cw} \) is arched by a big face and \( a_i^{cw} \) is undefined.

**Proof** (of Claim 3). Let \( i \in \{1, 2, 3\} \). To see that \( R_i^{cw} \) is archfree, recall that \( Q_i^{cw} \) is archfree by Claim 2. Hence, if \( R_i^{cw} = Q_i^{cw} \), there is nothing to show, so assume otherwise. The \( f_{i+1}f_{i+1}^{cw} \) subpath of \( R_i^{cw} \) is archfree by Lem. 1. Therefore, if \( R_i^{cw} \) is arched by some internal face \( a \neq f \), then \( a \) has to arch \( R_i^{cw} \) between some vertex in \( V(R_i^{cw}) \setminus V(Q_i^{cw}) \) and some vertex in \( V(Q_i^{cw}) \) \( \{f_i\} \), which is impossible by planarity (specifically, the boundary of \( a \) would have to cross the cycle \( D_{i+1} \)). Moreover, by construction, \( R_i^{cw} \) is not arched by \( f \). Hence, \( R_i^{cw} \) is archfree, as claimed.

Clearly, Properties [W1] and [W2] of Definition 2 carry over from \((Q_1^{cw}, Q_2^{cw}, Q_3^{cw})\).

Regarding Property [W3], let \( q_i \) denote the (internal) endpoint of \( R_i^{cw} \) that is not \( a_i \) for \( j = 1, 2, 3 \). By construction, \( q_i \) belongs to \( P_i^{cw} \). Hence, Property [W4] is violated if and only if \( q_i \) is an interior vertex of \( R_i^{cw} \), which, by construction, is the case if and only if \( P_i^{cw} \) is arched by a big face and \( q_i^{cw} \) is undefined (in which case \( q_i = t_i^{cw} = f_{i+2} = q_{i+1} \), for an illustration refer to Fig. 7d with \( i = 3 \).

Regarding Property [W3], we argue in two steps: let \( I_i^{old} \) be the set of interior vertices of \( R_i^{cw} \) that are also interior vertices of \( Q_i^{cw} \), i.e., \( I_i^{old} = V(Q_i^{cw}) \setminus \{a_i, f_i\} \). Further, let \( I_i^{new} \) be the set of interior vertices of \( R_i^{cw} \) that are not interior vertices of \( Q_i^{cw} \). Property [W3] holds if and only if none of these sets intersects \( V(Q_i^{cw}) \).

We first consider the set \( I_i^{new} \). If \( R_i^{cw} = Q_i^{cw} \), then \( I_i^{new} = \emptyset \) and, hence, \( I_i^{new} \cap V(R_i^{cw}) = \emptyset \). Otherwise, \( I_i^{new} = \{f_{i+1} \} \cup (V(R_i^{cw}) \setminus \{V(Q_i^{cw}) \setminus t_{i+1} \}) \). By construction, this set is disjoint from both \( V(Q_i^{cw}) \) and \( V(Q_{i+1}^{cw}) \). Hence, \( I_i^{new} \cap V(R_i^{cw}) = \emptyset \).

It remains to consider the set \( I_i^{old} \). By Property [W3] for \( Q_i^{cw} \), we have \( I_i^{old} \cap V(Q_i^{cw}) = \emptyset \). So if \( I_i^{old} \cap V(R_i^{cw}) \neq \emptyset \), then \( I_i^{old} \cap (V(R_i^{cw}) \setminus V(Q_i^{cw}) \setminus t_{i+1} \) \) \neq \emptyset. All vertices in \( I_i^{old} \) belong to the closed interior \( D_i^- \) of the cycle \( D_i \). The \( f_{i+2}t_i^{cw} \)-path on \( \partial f \) intersects \( D_i^- \) if and only if \( P_i^{cw} \) is arched by a big face (i.e., \( t_i^{cw} = f_i \), for an illustration refer to Fig. 7d with \( i = 3 \).
namely in \( f_1 \). Hence, Property \([W3]\) is violated for \( i \) if and only if \( P_{i+2}^{cw} \) is arched by a big face and \( a_1^{cw} \) is undefined (in which case \( Q_i^{cw} = P_{i+1}^{cw} \) and, hence, \( f_i \in V(R_i^{cw}) \)).

If \((R_1^{cw}, R_2^{cw}, R_2^{cw})\) or \((P_1^{cw}, P_2^{cw}, R_2^{cw})\) (which is defined symmetrically and for which a symmetric version of Claim 3 holds) is an archfree windmill, we are done. So assume otherwise. By Claim 3, it follows that both \({P_1^{cw}, P_2^{cw}, P_3^{cw}}\) and \({P_1^{cw}, P_2^{cw}, P_3^{cw}}\) contain a path that is arched by a big face. Specifically, we may assume without loss of generality that the arch \( a_1^{cw} \) exists and is big (i.e., \( \ell_1^{cw} = f_2 \) and \( a_2^{cw} \) is undefined; for an illustration refer to Fig. 8). By planarity, the path \( P_2^{cw} \) cannot be arched (the boundary of \( a_1^{cw} \) would have to cross the boundary of \( a_1^{cw} \)). Moreover, the path \( P_3^{cw} \) cannot be arched by a big face since this would imply \( \deg(f_2) \geq 5 \), contradicting the degree bounds. Hence, the arch \( a_1^{cw} \) of \( P_1^{cw} \) exists and is big (by assumption and Claim 3). Note that, by planarity, the path \( P_3^{cw} \) cannot be arched (the boundary of \( a_1^{cw} \) would have to cross the boundary of \( a_1^{cw} \)). Without loss of generality, we may assume that \( s_1^{cw} \) is not closer to \( o_1 \) on \( P_1 \) than \( s_1^{cw} \) (otherwise, we can argue symmetrically), for illustrations refer to Figures 8a and d.

Claim 4 The paths \( Q_i^{cw}, P_2^{cw}, P_3^{cw} \) are archfree. Properties \([W1]\)–\([W3]\) from Definition 2 hold for \( (Q_i^{cw}, P_2^{cw}, P_3^{cw}) \). Property \([W4]\) from Definition 2 is satisfied for \( i = 1, 2, 3 \), but violated for \( i = 3, 3 \).

Proof (of Claim 4). Since neither \( P_2^{cw} \) nor \( P_3^{cw} \) are arched, Claim 3 implies the claimed properties hold for \( (Q_i^{cw}, P_2^{cw}, P_3^{cw}) \). Moreover, \( P_2^{cw} = Q_2^{cw} \) and \( P_3^{cw} = Q_3^{cw} \), which proves the claim. \( \square \)

We now append the \( f_1 s_1^{cw} \)-subpath of \( P_1 \) to \( P_3^{cw} \) and denote the resulting path by \( S_1^{cw} \), for an illustration see Fig. 8.

Claim 5 The paths \( Q_i^{cw} \) and \( P_2^{cw} \) are archfree. Properties \([W1]\)–\([W4]\) from Definition 2 hold for \( (Q_i^{cw}, P_2^{cw}, S_1^{cw}) \).

Proof (of Claim 5). To see that Properties \([W2]\) and \([W4]\) hold for \( i = 3 \), we argue as follows: towards a contradiction, assume that \( s_1^{cw} \in \partial f_1 \), i.e., \( s_1^{cw} = o_1 \). By our assumption about the positions of \( s_1^{cw} \) and \( s_1^{cw} \) on \( P_1 \), it follows that \( s_1^{cw} = s_1^{cw} = o_1 \). However, this implies that \( \deg(o_1) \geq 5 \), contradicting the degree bounds.

Clearly, by construction, the remaining properties of \( (Q_i^{cw}, P_2^{cw}, S_1^{cw}) \) guaranteed by Claim 4 carry over to \( (Q_i^{cw}, P_2^{cw}, S_1^{cw}) \).

If \( (Q_i^{cw}, P_2^{cw}, S_1^{cw}) \) is an archfree windmill, we are done, so assume otherwise. By Claim 5, it follows that \( S_1^{cw} \) is arched by an internal face. The paths \( P_1^{cw} \) and \( P_1 \) are archfree, so an internal face \( a \) that is arching \( S_1^{cw} \) has to do so between some vertex in \( t \in V(S_1^{cw}) \setminus V(P_1^{cw}) \) and some vertex in \( s \in V(P_1^{cw}) \setminus \{f_1\} \). By planarity, it is not possible that \( a \) arches \( S_1^{cw} \) from the right, so \( a \) arches \( S_2^{cw} \) from the left. As in the definition of the arches \( a_1^{cw} \), there may be multiple nested arches that arch \( S_1^{cw} \) from the left, and we assume \( a \) to be the “outermost” one.
More precisely, we assume that \( a \) is this unique arch such that a \( st \)-path on \( \partial a \) replaces the \( st \)-path on \( S^w_3 \) in \( L_G(S^w_3) \).

As the final modification, we replace \( S^w_3 \) by \( T^w_3 = L_G(S^w_3) \), for an illustration see Figures 8c and d.

**Claim 6** \( (Q^w_1, P^w_2, T^w_3) \) is an archfree windmill.

**Proof (of Claim 6).** By Claim 5, the paths \( Q^w_1 \) and \( P^w_2 \) are archfree and Properties \( (W1)–(W4) \) from Definition 2 hold for \( (Q^w_1, P^w_2, S^w_3) \). By Lem. 2, \( T^w_3 \) is archfree. To conclude the proof, it remains to show that Properties \( (W1)–(W4) \) hold for the set \( (Q^w_1, P^w_2, T^w_3) \).

Properties \( (W1) \) and \( (W2) \) are preserved from \( (Q^w_1, P^w_2, S^w_3) \) (by Lem. 2 for \( i = 3 \)). To show that the remaining two properties hold, we first show that \( s \notin V(F_3) \setminus \{ f_3 \} \). If \( s^w \neq s^w_1 \) (see Fig. 8), this is clear by planarity (the boundary of \( a \) would have to cross the boundary of \( a^w_1 \)). Towards a
contradiction, assume that \( s_{cw}^1 = s_{ccw}^1 \) (see Fig. 8c) and \( s \in V(P_3) \setminus \{f_3\} \). By planarity, it follows that \( t = s_{cw}^1 = s_{ccw}^1 \) (otherwise, the boundary of \( a \) would have to cross the boundary of \( a_{cw}^1 \)). However, this implies that \( \deg(t) \geq 5 \), contradicting the degree bounds. So indeed, \( s \notin V(P_3) \setminus \{f_3\} \) as claimed.

Clearly, Property (W4) for \( i = 1, 3 \) is preserved from \((Q_{cw}^1, P_{cw}^2, S_{cw}^3)\) (by Lem. 2 for \( i = 3 \)). Since \( s \notin V(P_3) \setminus \{f_3\} \), it follows that the endpoint \( f_3 \) of \( P_{cw}^2 \) is not an interior vertex of the st-path on \( S_{cw}^3 \). Consequently, Property (W4) is also preserved for \( i = 2 \).

It remains to establish Property (W3). It is clearly preserved for \( i = 1 \). Consider the simple cycle \( C' \) formed by \( Q_{cw}^1 \), the \( f_2f_3 \)-path on \( \partial f \) that does not pass through \( f_1 \), \( P_3 \), and the \( o_1o_2 \)-path on \( f_o \) that does not pass through \( o_2 \). By Obs. 1, the closed interior \( C'− \) of \( C' \) is internally 3-connected. By construction, all vertices of \( Q_{cw}^1 \) and \( P_{cw}^2 \) belong to the closed exterior of \( C' \). By planarity, there is no internal face of \( G \) that is not an internal face of \( C'− \) and incident to more than one vertex of the st-subpath of \( S_{cw}^3 \). Hence, by Lem. 2 applied to \( C'− \) and the st-subpath of \( S_{cw}^3 \), it follows that Property (W3) is preserved for \( i = 2 \) and \( i = 3 \). \( \square \)

By Claim 6, there is an archfree windmill, which concludes the proof. \( \square \)

**Lemma 4** (⋆). Let \( G \) be an internally 3-connected plane graph that is internally 4-regular. Let \( f_o \) denote the outer face of \( G \) and assume \( |\partial f_o| = 3 \). Then \( G \) has a strictly internal face.

Proof. Let \( n, m, f \) denote the number of vertices, edges, and faces of \( G \), respectively. The existence of a separation pair would clearly violate Property (I3) of Definition 1. Hence, the graph is 3-connected and it follows that each vertex of \( \partial f_o \) has degree 3 or 4. The handshaking lemma implies that the number \( \eta \) of odd degree vertices is even. Towards a contradiction, assume that \( G \) has no strictly internal face, i.e., each face is incident to one of the vertices of \( \partial f_o \). By 3-connectivity, this implies that \( f = 7 − \eta \). By Euler’s polyhedron formula, the handshaking lemma, and internal 4-regularity, we obtain:

\[
\begin{align*}
n - m + f &= 2 \\
\iff n - (2n - \frac{\eta}{2}) + (7 - \eta) &= 2 \\
\iff n &= 5 - \frac{\eta}{2}
\end{align*}
\]

If \( \eta = 2 \), it follows that \( n = 4 \), contradicting the internal 4-regularity. If \( \eta = 0 \), then \( G \) is a 4-regular graph on 5 vertices, i.e., it is isomorphic to \( K_5 \); a contradiction to the fact that \( G \) is planar. \( \square \)

**Theorem 1** (⋆). Let \( G \) be an internally 3-connected internally 4-regular plane graph and let \( \Gamma_o \) be a compatible convex drawing of its outer face. There exists a convex drawing \( \Gamma \) of \( G \) that uses \( \Gamma_o \) as the realization of the outer face where each internal vertex of \( G \) is contained in the interior of some segment of \( \Gamma \).
Proof. The coordinates of the outer vertices of $G$ are already fixed. Our goal is to (recursively) compute coordinates for the internal vertices to obtain the desired drawing of $G$. The base case of the recursion is that $G$ contains no internal edges, in which case there is nothing to show. So assume that $G$ has at least one internal edge. Without loss of generality, we may assume that $G$ contains no (outer) degree-2 vertices whose outer angle in $I^o$ is $\pi$ (we can just iteratively merge the two incident edges of such vertices, compute the drawing, and then reinsert the removed vertices at their prescribed coordinates). We distinguish two main cases:

Case 1: $G$ is not 3-connected. We distinguish two subcases:

Case 1.1: $G$ contains a degree-2 vertex $v$. For illustrations refer to Fig. 9a. By internal 4-regularity, $v$ belongs to $V(I^o)$. By our assumption about degree-2 vertices, its outer angle in $I^o$ is reflex. Let $u$ and $w$ denote the two neighbors of $v$. Note that if $uw \in E$, then it belongs to the boundary of the (triangular) internal face incident to $v$ since otherwise $u, w$ would form a separation pair that separates the interior of the cycle $uvw$ from the outer face; contradicting Property (I3) of Definition 1. If $uw \in E$ we set $G'_1 = G$. Otherwise, we add the edge $uw$ in the internal face incident to $v$ and call the resulting graph $G'_1$. Adding an internal edge to an internally 3-connected graph clearly preserves Property (I2) of Definition 1, so, in both cases, $G'$ is internally 3-connected. We delete $v$ from $G'_1$ and call the resulting graph $G_1$. This modification preserves Property (I2) of Definition 1 so $G_1$ is internally 3-connected.

Since $I^o$ is compatible, the vertices $u$ and $w$ cannot belong to a common segment $s$ of $I^o$ (otherwise the internal face of $G$ incident to $v$ would arch $s$). Consequently, we can replace the edges $uw$ and $vw$ of $I^o$ with the edge $uw$ to obtain a simple convex polygon $I^o_1$, which is a convex drawing of the outer face of $G_1$. By Lem. 1 the edge $uw$ is archfree in $G_1$. Combined with the fact that $I^o$ is compatible with $G$, it follows that $I^o_1$ is compatible with $G_1$.

We recursively compute the coordinates of the internal vertices in a convex drawing of $G_1$ with $I^o_1$ as the realization of the outer face such that all internal vertices of $G_1$ are placed in the interior of some segment. Since each internal vertex of $G$ is also an internal vertex of $G_1$, these coordinates combined with the coordinates of $v$ correspond to the desired drawing of $G$.

Case 1.2: $G$ contains no degree-2 vertex. Let $u$ be a vertex that belongs to a separation pair in $G$. By Property (I3) of Definition 1 all vertices that belong to separation pairs are on the boundary of the outer face $f_o$ of $G$. Let $v$ be the first vertex encountered when walking from $u$ along $\partial f_o$ in clockwise direction such that $u, v$ is a separation pair, for an illustration see Fig. 9b. By 2-connectivity, there is an internal face $f$ such that $u, v \in V(\partial f)$. The boundary $\partial f$ contains two simple interior disjoint $uv$-paths. By the case assumption and the definition of $v$, at least one of these paths, say $P'$, is internal. In fact, if $|E(P')| = |E(\partial f)| - 1$, then the other path just consists of a single edge and is therefore also internal. So in any case, the boundary $\partial f$ contains an internal $uv$-path $P$ with $|E(P)| \leq |E(\partial f)| - 2$, which, by Lem. 1 is archfree.

The boundary $\partial f_o$ contains two interior disjoint paths $P_1, P_2$ between $u$ and $v$. Each of these two paths forms a simple cycle together with $P$. The closed interior
of each of these two cycles describes an internally 3-connected plane graph by Obs. 1. We denote these two graphs by $G_1$ and $G_2$ such that $G_i$, $i \in \{1, 2\}$ has $P_i$ on its outer face. We define $\Gamma_i^o$ to be the polygon resulting from replacing the part of $\Gamma^o$ that corresponds to $P_2$ with $P$ drawn as a straight-line segment, see Fig. 9. The drawing $\Gamma^o_2$ is defined analogously. Since $\Gamma^o$ is compatible with $G$, the vertices $u, v$ cannot belong to a common segment $s$ of $\Gamma^o$ (otherwise, since there are no degree-2 vertices (with outer angle $\pi$), $s$ would be arched by the internal face $f$). Hence, $\Gamma^o_1$ and $\Gamma^o_2$ correspond to simple (convex) polygons. Moreover, since $\Gamma^o$ is compatible and $P$ is archfree, $\Gamma^o_1$ and $\Gamma^o_2$ are compatible with $G_1$ and $G_2$, respectively. We recursively compute the coordinates of the internal vertices in convex drawings of $G_1$ and $G_2$ with outer face $\Gamma^o_1$ and $\Gamma^o_2$, respectively, where each internal vertex is placed in the interior of some segment. Since, additionally, the interior vertices of $P$ are contained in the interior of the segment corresponding to $P$, the combination of these drawings corresponds to the desired drawing of $G$.

Case 2: $G$ is 3-connected. We distinguish two subcases:

Case 2.1: $|V(\Gamma^o)| \geq 4$. For illustrations refer to Fig. 10. Then there exist two distinct outer vertices $u, v$ that do not belong to a common segment of $\Gamma^o$. By 3-connectivity, $G$ contains an internal $uv$-path $P'$. Consequently, by Lem. 2 $G$ contains an archfree internal $uv$-path $P$. The boundary of the outer face of $G$ contains two interior disjoint paths $P_1, P_2$ between $u$ and $v$. Each of these two paths forms a simple cycle together with $P$. The closed interior of each of these two cycles describes an internally 3-connected plane graph by Obs. 1. We denote these two graphs by $G_1$ and $G_2$ such that $G_i$, $i \in \{1, 2\}$ has $P_i$ on its outer face. We define $\Gamma_i^o$ to be the polygon resulting from replacing the part of $\Gamma^o$ that corresponds to $P_2$ with $P$ drawn as a straight-line segment. The drawing $\Gamma^o_2$ is defined analogously. By definition of $u$ and $v$, $\Gamma^o_1$ and $\Gamma^o_2$ correspond to simple (convex) polygons. More, since $\Gamma^o$ is compatible and $P$ is archfree, $\Gamma^o_1$ and $\Gamma^o_2$ are compatible with $G_1$ and $G_2$, respectively. We recursively compute the coordinates of the internal vertices in convex drawings of $G_1$ and $G_2$ with outer face $\Gamma^o_1$ and $\Gamma^o_2$, respectively, where each internal vertex is placed in the interior of
some segment. Since, additionally, the interior vertices of \( P \) are contained in the interior of the segment corresponding to \( P \), the combination of these drawings corresponds to the desired drawing of \( G \).

![Fig. 10: a Case 2.1 and (b) Case 2.2 in the proof of Thm. 1](image)

**Case 2.2:** \( |V(\Gamma_o)| = 3 \). By Lem. 4, \( G \) contains a strictly internal face. Thus, by Lem. 3, it contains an archfree windmill \((P, Q, S)\). The paths \( P, Q, S \) dissect \( G \) into four plane graphs \( G_1, G_2, G_3, G_4 \), which are internally 3-connected by Obs. 1. The outer endpoints of \( P, Q, S \) correspond to the exactly three vertices of \( \Gamma_o \). Consequently, they do not belong to a common segment of \( \Gamma_o \). Hence, it is possible to draw each of \( P, Q, S \) as a straight-line segment such that the polygon \( \Gamma_o \) is dissected into four simple convex polygons, as depicted in Fig. 10b. Each of these polygons corresponds to a convex drawing of the outer face of one of \( G_1, G_2, G_3, G_4 \). Moreover, these drawings are compatible with their respective subgraphs since \( \Gamma_o \) is compatible and \( P, S, Q \) are archfree. We recursively draw \( G_1, G_2, G_3, G_4 \) into their respective compatible convex polygons in a convex fashion such that each of their internal vertices is placed in interior of some segment. Since, additionally, all internal vertices of \( G \) that belong to \( P, Q, S \) have been drawn in the interior of one of the segments corresponding to \( P, Q, S \), the combination of the four drawings corresponds to the desired drawing of \( G \).

It is easy to see that the proof of Thm. 1 corresponds to a polynomial-time algorithm. In fact, it seems very plausible that it can be implemented in quadratic time, though, we have not worked out the details yet.

**Theorem 2 (\( \star \)).** Every 3-connected internally 4-regular plane graph \( G \) admits a convex drawing on at most \( n + 3 \) segments where \( n \) is the number of vertices.

**Proof.** We create a convex drawing \( \Gamma_o \) of the outer face of \( G \) on exactly 3 segments. Let \( u, v, w \) be the three vertices of \( \Gamma_o \) whose outer angles are reflex. By 3-connectivity, none of the segments of \( \Gamma_o \) can correspond to a path that is arched by an internal face. Consequently, \( \Gamma_o \) is compatible with \( G \) and, by
Thm. 1, we can create a convex drawing $\Gamma$ of $G$ that uses $\Gamma^o$ as the outer face such that each vertex in $V(G) \setminus \{u, v, w\}$ is drawn in the interior of some segment of $\Gamma$. Hence, for each vertex $x \in V(G) \setminus \{u, v, w\}$ at most two segments of $\Gamma$ have $x$ as an endpoint. For each vertex $y \in \{u, v, w\}$ at most four segments of $\Gamma$ have $y$ as an endpoint. Since each segment has exactly two endpoints, it follows that the number of segments is at most $\frac{2(n-3)+4}{2} = n + 3$, which concludes the proof. 

Proposition 1. For even $n \geq 6$, $C_n^2$ is planar and $\text{seg}(C_n^2) \geq n$.

Proof. Suppose that, for $n \geq 6$, the graph $C_n^2$ has a drawing $\Gamma$ with at most $n-1$ segments. For $i \in \{0, 2, 4\}$, let $n_i$ be the number of vertices in $C_n^2$ with $i$ ports. Clearly, $n = n_0 + n_2 + n_4$. The drawing $\Gamma$ has $2n_2 + 4n_4$ ports and hence $n_2 + 2n_4$ segments. If $n_4 \leq n_0$, then $\Gamma$ has $n_2 + 2n_4 \geq n_2 + (n_0 - n_4) = n$ segments, which would contradict our assumption. Hence, $n_0 > n_4$. Each vertex on the convex hull of $C_n^2$ has zero ports, which implies that $n_4 \geq 2$. This in turn yields that $n_2 = n - n_0 - n_4 \leq n - 7$.

We label the vertices of $C_n^2$ such that $(v_1, v_2, \ldots, v_n)$ forms the simple cycle $C_n$. Since $n_0 > n_4$, there must be two indices $1 \leq j < \ell \leq n$ such that vertices $v_j$ and $v_\ell$ have zero ports and every vertex $v_k$ with $j < k < \ell$ has two ports. Let $V' = N(v_j) \cup \{v_j, \ldots, v_\ell\} \cup N(v_\ell)$ and $n' := |V'|$. We have $n' \leq n - 1$ since $V'$ contains at most $n - 7$ vertices (with two ports) strictly between $v_j$ and $v_\ell$, plus six further vertices. Hence, w.l.o.g., we can choose our labeling of $C_n^2$ such that $V' = \{v_{j-2}, \ldots, v_{\ell+2}\}$. Let $G' = G[V']$, but drop any edge that connects one of the first two with one of the last two vertices. Then $G'$ is isomorphic to the outerpath $R_{n'}$ where every vertex has degree at most 4. Dujmović et al. have shown that $\text{seg}(R_{n'}) = n'$. The graph $G'$, however, has only $2n' - 2$ ports: all vertices have two ports, except for $v_j$ and $v_\ell$ with zero ports and $v_{j-1}$ and $v_{\ell+1}$ (both of degree 3) with at most three ports. This contradicts the fact that $\text{seg}(R_{n'}) = n'$.

Remark 1. As a universal lower bound for the class of 3-connected 4-regular planar graphs, note that in each vertex either at least two segments end or two segments cross. In order to generate $n$ vertices, we need at least $\Omega(\sqrt{n})$ segments as Dujmović et al. observed. It is not hard to see that (grid-like) 3-connected 4-regular planar graphs with segment number $O(\sqrt{n})$ exist.

C Proofs Omitted in Section 3 (Maximal Outerpaths)

Lemma 5. For any $i \in \{3, \ldots, n\}$, a partial outerpath drawing $\Gamma_i$ contains at most one active long pseudo $k$-arc.

Proof. Suppose that $\Gamma_i$ contains two pseudo $k$-arcs $\alpha$ and $\beta$ that are both active and long. Let $\alpha$ have its first internal edge before $\beta$. For $\beta$ to become long, $\beta$ must have $k+1$ internal edges, while $\alpha$ remains active. Let $H_0, H_1, \ldots, H_\ell$ be the
subgraphs into which the internal edges of $\beta$ subdivide the complete outerpath drawing $\Gamma$; see Fig. 4. Now for $\alpha$ to leave $H_0$, $\alpha$ needs either to enter $H_1$ (which requires an intersection between $\alpha$ and $\beta$) or to enter $H_2$ (which requires a tangential point of $\alpha$ at $\beta$ and is counted as two intersections). For $\alpha$ to be active when $\beta$ is long, $\alpha$ needs to reach $H_{k+1}$ (or some $H_j$ with $j > k+1$). This however, requires at least $k+1$ intersection points between $\alpha$ and $\beta$, a contradiction to the definition of pseudo $k$-arcs.

Lemma 6 (⋆). There is a loss of at most one crossing per transition from one long pseudo $k$-arc to another long pseudo $k$-arc. Hence, $t_k \leq \max\{0, \arck - 1\} \leq \arck = \arc_k - \sum_{i=0}^{k} \arck_i$, where $\arck$ is the number of long pseudo $k$-arcs.

Proof. Of course, the loss cannot be negative and the number of transitions from one long arc to the other is $\arck - 1$.

Summing up the losses over all pseudo $k$-arcs of the drawing, we obtain $t_k$. Being counted in a crossing with a long arc more than $k$ times is no contradiction to the definition of pseudo $k$-arcs because the long arc may change. We distinguish two cases for the transition of a long arc $\alpha$ (with internal edge $e_1, e_2, \ldots$ and subgraphs $H_1, \ldots, H_\ell$) to a long arc $\beta$ (with internal edge $e'_1, e'_2, \ldots$ and subgraphs $H'_1, H'_2, \ldots$).

Fig. 11: Cases for the transition of one long pseudo segment $\alpha$ to another long pseudo segment $\beta$.

In the first case, $e_\ell$ precedes $e'_k$; see Fig. 11a. Say an arc $\gamma$ has been counted in $q$ crossings with a long arc before reaching $e'_k$. (If there has been a transition of a long arc before, we have already subtracted its loss and hence we assume $q \leq k$.) Next we show that $\gamma$ is part of at most $k - q + 1$ counted crossings with $\beta$. When $\gamma$ reaches $e'_k$, it must have intersected $\beta$ already at least $q - 1$ times. This is due to the fact that $k - 1 \geq q - 1$ internal edges of $\beta$ precede $e'_k$ and while
an arc is active, it intersects all arcs of the internal edges. We know that \( \gamma \) has intersected \( \alpha \) (or a previous long arc) \( q \) times, so it has been in at least the last \( (q - 1) \) bays of \( \alpha \) (or a previous long arc). By then, \( \beta \) has also already been active and also has been in at least the last \( (q - 1) \) bays of \( \alpha \) (or a previous long arc).

In any bay \( H \), all arcs that leave \( H \) intersect all other arcs that leave \( H \) at least once. Hence, \( \beta \) and \( \gamma \) have intersected pairwise at least \( q - 1 \) times. This means that \( \gamma \) can be part of at most \( q + (k - q + 1) = k + 1 \) counted crossings with a long arc – regarding all long arcs up to and including \( \beta \). It remains to argue that there is at most one arc \( \gamma \) with \( k + 1 \) counted crossings with a long arc per transition. Suppose there was another arc \( \gamma' \) with the same property, which has been in \( q' \) crossings with long arcs before \( \gamma \). Then, without loss of generality, \( \gamma \) has been in the crossing with \( \alpha \) at \( e_\ell \). However, \( \gamma' \) has also intersected \( \alpha \) at \( e_\ell \) but without being counted in a crossing. So, \( \gamma' \) has been in the last \( q' \) \( H \)s of \( \alpha \) together with \( \beta \) and contributes at most \( k - q' \) crossings with \( \beta \).

In the second case, \( e_\ell \) succeeds \( e'_k \); see Fig. 11. If we started counting crossings with \( \beta \) in bay \( H'_{k+1} \) instead of \( H_k \), we would have the same situation as in the first case. Now consider the counted crossing of \( H'_{k+1} \) at \( e'_{k+1} \). Similar to the first case, if an arc \( \delta \) reaches this crossing and was part of counted crossings before, it has intersected \( \alpha \) at \( e_\ell \). Again, only one of \( \gamma \) and some other pseudo \( k \)-arc \( \gamma' \) can contribute the crossing with \( \alpha \) at \( e_\ell \) and then be part of \( k + 1 \) crossings with long edges. For the counted crossing of bay \( H'_{k} \) at \( e'_k \), we cannot rule out the possibility that the involved arc \( \delta \) is part of more than \( k \) crossings. So, we consider this crossing as being lost, but then there exists a crossing of \( \alpha \) and \( \beta \) that has not been counted – namely at the common vertex of \( e_\ell \) and \( e'_k \).

Therefore, also in the second case we have a loss of at most one counted crossing. □

**Lemma 7**. For \( k = 1 \) and \( n \geq 3 \), in any outerpath drawing either \( \text{arc}_1 \geq 3 \) or \( (\text{arc}_0 \geq 2 \text{ and arc}_1 \geq 3) \).

**Proof.** Consider \( v_1 \) and \( v_n \), i.e., the first and the last vertex in the stacking order of \( G \). Each of them is incident to two pseudo segments. If they would lie on only one pseudo segment \( S \), \( S \) would intersect the pseudo segment connecting the two neighbors of \( v_1 \) (or \( v_n \)) twice.

First, we show that \( v_1 \) and \( v_n \) have at least one incident pseudo segment with zero internal edges each (Case 0). Without loss of generality, assume that \( v_1 \) is incident to the pseudo segments \( S_l \) and \( S_r \), both have at least one internal edge, and in the stacking order of the outerpath, the first internal edge \( e \) of \( S_r \) precedes the first internal edge of \( S_l \); see Fig. 12a. The path of faces reaches the face \( f \) when passing over \( e \). However, \( S_l \) is not incident to \( f \) and becomes inactive. (\( S_l \) cannot be incident to \( f \) because then \( S_l \) and \( S_r \) would intersect twice or \( v_1 \) would have a degree \( > 2 \).) Therefore, \( S_l \) has zero internal edges. The same holds when traversing the outerpath backwards starting at \( v_n \).

Using this property, we now can make the following case distinction.

**Case 1:** \( v_1 \) and \( v_n \) are incident to the same pseudo segment \( S \) having zero internal edges. Let the other pseudo segments being incident to \( v_1 \) and \( v_n \) be \( S_1 \)
and $S_n$, respectively (clearly, they are distinct); see Fig. 12(b). This means that $S$ is incident to all faces in the outerpath. So, if $S_1$ or $S_n$ had an internal edge, they would intersect $S$ a second time. Hence, $S_1$ and $S_n$ have also zero internal edges and we have at least three pseudo segments with zero internal edges in total.

**Case 2:** $v_1$ and $v_n$ are incident to the same pseudo segment $S$ having one internal edge. Let the other pseudo segments being incident to $v_1$ and $v_n$ be $S_1$ and $S_n$, respectively (clearly, they are distinct); see Fig. 12(c). Since $S$ has an internal edge, $S_1$ and $S_n$ have zero internal edges. Consider the face $f$ following the internal edge $e$ of $S$. Beside $S$, the two other distinct bounding pseudo segments of $f$ are $S_2$ and $S_3$. Let $S_3$ have an internal edge $e_2$ following $e$ along the sequence of internal faces. All faces of the outerpath are incident to $S$, hence
\(S_3\) cannot have a second internal edge as it intersects \(S\) incident to \(f\). Similarly, \(S_2\) can have at most one internal edge \(e_1\) when it intersects \(S\) incident to \(f\). Thus, \(S_1\) and \(S_n\) have zero internal edges, while \(S\), \(S_2\), and \(S_3\) have at most one internal edge each.

**Case 3:** \(v_1\) and \(v_n\) are incident to the same pseudo segment \(S\) having at least two internal edges. As in Case 2, when the sequence of faces of the outerpath passes over \(S\), there are two pseudo segments \(S_2\) and \(S_3\) each having at most one internal edge. We have this situation at least twice – we denote the next corresponding pair of segments that has at most one internal edge per pseudo segment by \(S_4\) and \(S_5\). Observe that maybe \(S_3 = S_4\); see Fig. 12f. Then, however, \(S_2 \neq S_3\) as otherwise \(S_2\) and \(S_3\) would intersect twice. Therefore, we have two pseudo segments with zero internal edges (\(S_1\) and \(S_n\)) and we have at least three pseudo segments with at most one internal edge (\(S_2\), \(S_3\), and \(S_5\)).

**Case 4:** \(v_1\) and \(v_n\) are incident to four distinct pseudo segment and exactly one of these pseudo segments has at least two internal edges. This case is similar to Case 2 and Case 3. Without loss of generality, let the segments of \(v_1\) and \(v_n\) be \(S_{1,l}\), \(S_{1,r}\) and \(S_{n,l}\), \(S_{n,r}\), respectively, and let \(S_{1,t}\) have at least two internal edges; see Fig. 12d. Consider the first internal edge \(e\) of \(S_{1,l}\) and the face \(f\) preceding \(e\) along the sequence of faces. Let the other pseudo segments bounding \(f\) be \(S_1\) and \(S_2\) and let the internal edge \(e_1\) for entering \(f\) be contained in \(S_1\). Because until the sequence of faces passes over \(S_{1,l}\) a second time, all faces are neighboring \(S_{1,l}\). Hence, \(S_1\) and \(S_2\) have at most one internal edge each. Moreover, observe that neither \(S_{n,l}\) nor \(S_{n,r}\) can be equal to \(S_1\) or \(S_2\) as otherwise they would intersect \(S_{1,l}\) twice. This gives us our bound – the three pseudo segments with at most one internal edge are \(S_1\), \(S_2\), and one of \(S_{n,l}\) and \(S_{n,r}\).

**Case 5:** \(v_1\) and \(v_n\) are incident to four distinct pseudo segment and two of these pseudo segments have at least two internal edges. We have a very similar situation as in Case 4, but now we have \(S_1\) and \(S_2\) in the forward direction and \(S'_1\) and \(S'_2\) symmetrically in the backward direction; see Fig. 12g. Let \(S_{1,l}\) and \(S_{n,l}\) be the segments originating at \(v_1\) and \(v_n\), respectively, that have at least two internal edges each. We have to be a bit more careful about the case that \(S_{1,l}\) and \(S_{n,l}\) intersect. However, even in this case \(S_1\), \(S_2\), \(S'_1\), and \(S'_2\) are four distinct pseudo segments since the first internal edge \(e\) of \(S_{1,l}\) precedes all internal edges of \(S_{n,l}\) and the last internal edge \(e'\) of \(S_{n,l}\) succeeds all internal edges of \(S_{1,l}\) (otherwise the drawing would not be an outerpath).

**Case 6:** \(v_1\) and \(v_n\) are incident to four distinct pseudo segment and each of them has at most one internal edge. If three of them have zero internal edges, we are done. So assume that the pseudo segment \(S_t\) of \(v_1\) (and one pseudo segment of \(v_n\)) has one internal edge \(e\); see Fig. 12h. Consider the face \(f\) preceding \(e\) in the sequence of faces in the outerpath. Beside \(S_t\), let \(f\) be bounded by \(S_1\) and \(S_2\). The key insight is that \(S_1\) and \(S_2\) pass over \(S_t\) at \(e\), but on the other side of \(S_t\), they cannot intersect a second time and so the path of faces in the outerpath can yield another internal edge at most for one of \(S_1\) and \(S_2\). Hence, either \(S_1\) has at most one internal edge (when entering \(f\)) or \(S_2\) has zero internal edges, which provides our bound. We have to be careful about the case that \(S_1\) or \(S_2\)
are pseudo segments of \( v_n \). Note that not both of them can reach \( v_n \) because then they would intersect a second time. If \( S_2 \) reaches \( v_n \), then \( S_1 \) is our third pseudo segment with at most one internal edge. If \( S_1 \) reaches \( v_n \), then \( S_2 \) is our third pseudo segment without any internal edges.

\[ \square \]

Theorem 3 (⋆). For any \( n \)-vertex maximal outerpath \( G \), \( \text{seg}(G) \geq \lfloor \frac{n}{2} \rfloor + 2 \).

Proof. Clearly, \( \text{seg}(G) \geq \text{arc}_1(G) \). Hence, it suffices to show \( \text{arc}_1(G) \geq \lfloor \frac{n}{2} \rfloor + 2 \).

We plug in the result from Lem. 6 into Eq. (4) for \( k = 1 \) and use Lem. 7 to observe

\[
\text{arc}_1 \geq \frac{2n - 3 + 3 \text{arc}_0 + \text{arc}_1}{4} = \frac{n}{2} + \frac{3 \text{arc}_0 + \text{arc}_1 - 3}{4} \geq \frac{n + 3}{2}
\]

As we cannot have partial (pseudo) segments, we can round up to \( \lceil \frac{n}{2} + 3 \rceil = \lfloor \frac{n}{2} \rfloor + 2 \).

\[ \square \]

Theorem 4 (⋆). For any \( n \)-vertex maximal outerpath \( G \), \( \text{arc}(G) \geq \lceil \frac{2n}{7} \rceil \).

Proof. Clearly, \( \text{arc}(G) \geq \text{arc}_2(G) \). Hence, it suffices to show \( \text{arc}_2(G) \geq \lceil \frac{2n}{7} \rceil \).

For \( k = 2 \) and Eq. (4), we plug in the result from Lem. 6 and we get

\[
\text{arc}_2 \geq \frac{2n + 5 \text{arc}_0^2 + 3 \text{arc}_1^2 + 1 \text{arc}_2^2}{7} \geq \frac{2n}{7}.
\]

Since we can only have an integral number of arcs, we can round up this value.

\[ \square \]

Proposition 3 (⋆). For every \( r \in \mathbb{N} \), maximal outerpaths \( P_r, Q_r, U_r \) exist s.t.

(i) \( P_r \) has \( 2r + 6 \) vertices and \( \text{seg}(P_r) \leq r + 5 = \frac{r}{2} + 2 \),

(ii) \( Q_r \) has \( 3r \) vertices and \( \text{arc}(Q_r) \leq r + 1 = \frac{r}{2} + 1 \),

(iii) \( U_r \) has \( 16r + 6 \) vertices and \( \text{arc}_2(U_r) \leq 5r + 3 = \frac{5n + 18}{16} \approx 0.3125 n \).

Proof. (i) Consider Fig. 5a. In the base case (\( m = 0 \)), there obviously is a drawing with six vertices on five line segments. When we increase \( m \) by one, we add a line segment going through the central vertex and increasing the number of vertices by two.

(ii) Consider Fig. 5b, where \( m = 6 \). The main structure is a long horizontal line segment (this is a circular arc with radius \( \infty \)). In the base case (\( m = 2 \)), we have two more circular arcs that look like the first and the last arc in Fig. 5b – two of their vertices are shared with each other, which gives us 6 vertices in total. When we increase \( m \) by one, we add a circular arc as in Fig. 5b. It has six vertices, where three of them are new.

(iii) Consider Fig. 5c. In the base case (\( k = 0 \)), we have only the first three and the last three vertices (in purple) using three pseudo 2-arcs. When we increase \( k \) by one, we add the colored part \( k \) times (to show the repeating pattern, there is another copy in gray). This colored part has 16 vertices, it extends three pseudo 2-arcs and introduces five new pseudo 2-arcs. Observe that each pair of pseudo 2-arcs intersects at most twice.

\[ \square \]
D Maximal Outerplanar Graphs and 2-Trees

Consider a straight-line drawing $\Gamma_G$ of a 2-tree $G$. The main idea for a universal lower bound for 2-trees (and for its subclass of maximal outerplanar graphs) is that $G$ either has many degree-2 vertices and thus requires many segments (recall that, in a 2-tree, all faces are triangles, hence degree-2 vertices cannot be closed) or $G$ can be obtained by gluing few outerpaths for which we know (tight) universal lower bounds on the segment number. By gluing we mean the following. Let $G$ be a 2-tree and $P$ a maximal outerpath. Let $f_G$ be a triangle of $G$ that is not incident to a degree-2 vertex and let $f_P$ be a triangle of $P$ that is incident to a degree-2 vertex (i.e., $f_P$ is the first or last triangle of $P$). Let $\Gamma_P$ be a straight-line drawing of $P$. Then we define the gluing of $\Gamma_P$ to $\Gamma_G$ as the straight-line drawing $\Gamma_{G\oplus P}$ of the 2-tree $G \oplus P$ obtained by identifying $f_P$ and $f_G$; see Fig. 13. Note that $|V(G \oplus P)| = |V(G)| + |V(P)| - 3$. In $\Gamma_{G\oplus P}$, we call $f_G$ and $f_P$ the gluing faces of $G$ and $P$, respectively.

Unfortunately, for gluing outerpaths, we cannot directly employ Thm. 3 because it does not tell us how many ports we lose when gluing. Therefore, we first investigate the distribution of ports within a straight-line drawing of a maximal outerpath. We will see that, by some careful counting arguments, we lose only few (counted) ports when gluing outerpaths. We start by formally proving some auxiliary properties; see Fig. 14.

**Lemma 9.** Let $P$ be a maximal outerpath given with a stacking order, and let $v$ be a vertex of $P$. Then, in any outerplanar straight-line drawing of $P$, all of the following holds.

(P1) If $\deg(v) = 2$ or $\deg(v)$ is odd, then $v$ is open.

(P2) If $\deg(v) \geq 5$, then $v$ is succeeded by $\deg(v) - 4$ many neighbors of degree 3, which we call companions.

(P3) If $\deg(v) \geq 6$ and $v$ is closed, then $v$ has a companion with three ports, which we call bend companion.

(P4) If subsequent vertices $u$ and $v$ both have degree 4, then at least one of $u$ and $v$ is open.
For property (P3) in Lem. 9, observe that vertex \( v_i \) (which has degree 6) is (a) either open or (b) has a bend companion (here \( v_{i+1} \)) with three ports; (c) for property (P4) note that two subsequent degree-4 vertices \( u \) and \( v \) cannot both be closed because of the two triangles they form with their common neighbors \( x \) and \( y \).

(P5) Let \( v \) be stacked upon the edge \( uw \) and \( u,v \) be subsequent vertices. If \( v \) is closed, \( \deg(v) = 4, \deg(u) = 3, \) and \( \deg(w) = 5, \) then either \( u \) or \( w \) has at least three ports.

Proof. We consider each of the statements individually.

(P1) If \( \deg(v) \) is odd, the claim is trivial. Otherwise, \( v \) and its two neighbors form a triangle in any 2-tree and cannot be collinear.

(P2) Since \( v \) has degree at least five, constructing \( P \) with a sequence of stacking operations involves \( \deg(v) - 2 \) consecutive stacking operation on edges incident to \( v \). Consequently, all succeeding neighbors of \( v \), except for the last two, must have degree three.

(P3) Let \( v = v_i \). Consider the companions \( v_{i+1}, \ldots, v_{i+\deg(v_i)-4} \) of \( v \). Suppose neither of them has three ports (two is not possible since they have degree 3), then (at least) \( \deg(v_i) - 2 \) neighbors of \( v_i \) are collinear and thus result in a triangle \( T \) with \( v \) at one corner and these neighbors on the opposing side of \( T \); see Fig. 15a. Then, however, \( v_i \) cannot be closed since \( \deg(v_i) - 2 > \deg(v_i)/2 \) and at most two segments can pass through \( v_i \), which is a contradiction. Hence, one of the companion vertices of \( v \) has three ports; see Figure 15b.

(P4) Let \( x, u, v, y \) be a stacking subsequence in \( P \) where both \( u \) and \( v \) have degree four. Assume, for the sake of contradiction, that there exists a planar straight-line drawing of \( P \) where both \( u \) and \( v \) are closed. Let \( s \) be the segment that contains the edge \( uv \). Then \( s \) intersects at \( u \) the segment \( s' \) that contains \( xu \), and \( s \) intersects at \( v \) the segment \( s'' \) that contains \( xv \); see Fig. 16a. Observe that \( s' \) and \( s'' \) need to intersect again in \( y \) since \( P \) is a maximal outerpath and both \( u \) and \( v \) are closed. However, this would only be possible if \( x, u, v, \) and \( y \) are collinear; which is a contradiction to the drawing being a planar straight-line drawing.

(P5) If \( u \) has three ports, we are done. Otherwise, \( u \) has only one port; see Fig. 16b. Then note that \( u \) and \( v \) need to be collinear with a successor \( v' \) of \( v \) and predecessor \( u' \) of \( u \). Observe that these four vertices are adjacent to \( w \). However, since \( w \) has degree 5, only one of the edges \( \{u',w\}, \{u,w\}, \{v,w\}, \{v',w\} \)
can be extended at \( w \). (In Fig. 16b, the edge \( \{v', w\} \) lies on a segment passing through \( w \).) Therefore, \( w \) has at least 3 ports.

\[ \square \]

Fig. 15: Configurations in the proof of Lem. 9 where \( v \) has degree 6 (or a higher even degree).

Fig. 16: Configurations in the proof of Lem. 9 where \( v \) is closed and has degree 4.

**Proposition 4.** Let \( P \) be a maximal outerpath with \( n \geq 4 \) vertices. Then \( \text{port}(P) \geq n + 1 \). Moreover, for any planar straight-line drawing of \( P \), we can find an injective assignment of ports to vertices such that every port is assigned to its own vertex or to a neighboring vertex.

**Proof.** Given any straight-line drawing \( \Gamma_P \) and any stacking order \( \langle v_1, \ldots, v_n \rangle \) of \( P \), we describe an assignment of ports to vertices in their vicinities such that no two ports are assigned to the same vertex. This immediately proves that \( \text{port}(P) \geq n \). For the one remaining port, observe that \( v_1 \) has an additional unassigned port.
Let $i \in \{1, \ldots, n\}$. We consider different situations for vertex $v_i$. Each situation is illustrated by a vertex in the example shown in Fig. 17. If $v_i$ is open (such as $v_1$, $v_2$, or $v_4$ in Fig. 17), we assign one of the ports to itself. If $v_i$ is closed, then $\deg(v_i)$ is even and at least 4 by (P1). First assume $\deg(v_i) \geq 6$ (such as $v_9$, $v_{13}$, $v_{16}$ in Fig. 17). Then, by (P3), we know that $v_i$ has a bend companion $v_j$ with three ports. Only one of the three ports of $v_j$ is assigned to $v_j$ itself, so we assign one of the remaining ports of $v_j$ to $v_i$. (In Fig. 17 such a port would be supplied by $v_{11}$, $v_{14}$ and $v_{18}$, respectively.)

If $\deg(v_i) = 4$, then either $\deg(v_{i-1}) = 4$ (such as $v_4$ preceding $v_5$) or $\deg(v_{i-1}) = 3$ (such as $v_{14}$ preceding $v_{12}$) since, by (P2), $v_{i-1}$ has degree at most 4. In the former case, $v_{i-1}$ has at least two ports by (P4) and we can assign one of the ports to $v_i$ (such as $v_4$ to $v_5$). In the latter case, we distinguish three subcases. If $v_i = v_3$ in the stacking order of $P$, then $v_2$ has degree 3 and cannot be closed (as $v_2$ and $v_3$ in Fig. 17). If $v_i = v_4$ in the stacking order of $P$, then $v_2$ or $v_3$ has three ports. Otherwise, observe that the common neighboring predecessor of $v_{i-1}$ and $v_i$ has degree at least 5; hence one of (P3) or (P5) applies (see $v_9$, $v_{11}$, and $v_{12}$ in Fig. 17; this is the only case where a vertex provides ports for itself and two other vertices). \(\square\)

Proposition 4 implies a universal lower bound of \((n+1)/2\) for the segment number of an $n$-vertex outerpath. In Thm. 3 we improve this by a constant.

**Theorem 6.** Let $G$ be a 2-tree (or a maximal outerplanar graph) with $n$ vertices. Then $\text{seg}(G) \geq (n+7)/5$.

**Proof.** For now, assume that $G$ is a maximal outerplanar graph. We consider the case that $G$ is a 2-tree at the end of this proof. If the weak dual $T$ of $G$ has at least $(n+7)/5$ leaves, we are done since $G$ has at least as many segments as $T$ has leaves.

Otherwise, let $P = \{P_1, \ldots, P_p\}$ be a minimum-size set of maximal outerpaths such that when we define $G_1 = P_1$ and $G_i = G_{i-1} \oplus P_i$, for $i \in \{2, \ldots, p\}$, we get that $G = G_p$. In other words, we can obtain $G$ by $p-1$ consecutive gluing
operations of the paths in $P$. Note that $p$ is at most $(n + 7)/5 - 1 = (n + 2)/5$ because $P_1$ contains two leaves and, for $i \in \{2, \ldots, p\}$, $P_i$ contains one leaf of $T$.

Next, we show a lower bound on the number of ports on any straight-line drawing of $G$. To this end, we use the assignment of ports to vertices that we established in Prop. 4 and apply it to each outerpath $P_i$ in $P$. Further, we use the stacking order of $P_i$ that starts at the degree-2 vertex of $P_i$ that is not incident to the gluing face of $P_i$. For $i \in \{1, \ldots, p\}$, let $n_i = |V(P_i)|$. Note that $|V(G)| = \sum_{i=1}^{p} n_i - 3(p - 1)$.

First, we compute $\text{port}(P)$, the sum of ports counted for $P_1, \ldots, P_p$:

$$\text{port}(P) = \sum_{i=1}^{p} \text{port}(P_i) \geq \sum_{i=1}^{p} (n_i + 1) = n + 3(p - 1) + p = n + 4p - 3$$

Second, we analyze the number of counted ports that we lose by the $p - 1$ gluing operations. Consider the gluing operation $G_i = G_{i-1} \oplus P_i$ and let $f_{G_{i-1}}$ and $f_P$, respectively, be the gluing faces identified to face $f$ of $G_i$.

Observe that we counted three ports at $f_P$ since neither $v_n$ nor one of its neighbors needs to assign a port to another vertex (we assign only ports to vertices coming later in the stacking order except for bend companions, but the last three vertices cannot be bend companions). We assume to lose all of these three ports when gluing. This means that every vertex has at most as many counted ports in $G_i$ as it had in $G_{i-1}$. For the ports lost at $f_{G_{i-1}}$, observe that the vertex that is identified with $v_n$ at $P_i$ cannot lose any ports. The other two vertices are neighbors in $G_{i-1}$. In the assignment that we established in Prop. 4, any two such vertices provide ports for at most four vertices in total. We assume also to lose all of these ports, which results in a total loss of at most seven ports per gluing operation. Hence, with $p \leq (n + 2)/5$, we get

$$\text{seg}(G) = \frac{\text{port}(G)}{2} = \frac{\text{port}(P) - \text{loss}}{2} \geq \frac{n + 4p - 3 - (7p - 7)}{2} \geq \frac{n + 7}{5}.$$ 

It remains to consider the case that $G$ is a 2-tree. As for maximal outerplanar graphs, we can also construct a 2-tree by gluing multiple outerpaths. Similar to leaves in the dual drawing, each attached outerpath provides at its ending a vertex of degree two with two ports. The only exception is that we are not restricted on gluing to the outside – we may also draw an outerpath within an inner face of the current 2-tree drawing.

A difficulty is how to identify the faces $f_{G_{i-1}}$ and $f_P$ if we want to draw the rest of $P$ within this unified face. However, consider an outerplanar straight-line drawing of $P$ where we “flip” the last vertex $v_n$ over the rest of the drawing such that the drawing remains planar; see Fig. 18. Clearly, the number of ports in the drawing of the maximal outerpath $P$ did not change and the assignment scheme from Prop. 4 is still applicable. We may use such flips also along inner edges of an outerpath drawing to obtain a “folded” outerpath drawing with the same properties. Hence, we can apply gluing operations to inner faces with at most the same loss as analyzed before. \[ \Box \]
Fig. 18: Straight-line drawing of a maximal outerpath where we “flip” \( v_i \) over the rest of the drawing such that the resulting drawing remains planar. This way, we can append maximal outerpaths to inner faces of 2-trees.

We remark that, though we get the same lower bound for maximal outerplanar graphs and 2-trees, the actual (tight) numbers might be different. In other words, maybe there are 2-trees requiring less segments than any maximal outerplanar graph with the same number of vertices. This is because our current analysis is most likely not tight as we see by comparison with our existential upper bound.

For an existential upper bound of maximal outerplanar graphs, consider the construction in Fig. 19. It defines a family of graphs \( G_1, G_2, \ldots \) where the base graph \( G_1 \) has 16 vertices and admits a drawing \( \Gamma_{G_1} \) with eight segments. From \( G_{i-1} \) to \( G_i \), we glue a scaled and rotated copy of \( \Gamma_{G_1} \) to the drawing of \( G_{i-1} \) (gluing faces are shaded). With each step, we get 13 more vertices with only 5 more segments and hence the following result.

**Proposition 5.** For every \( k \in \mathbb{N} \), \( G_k \) has \( n_k = 13k + 3 \) vertices and \( \text{seg}(G_k) \leq 5k + 3 = (5n_k + 24) / 13 \).

### E Planar 3-Trees

In this section we study the segment number of planar 3-trees. For a 3-tree \( G \) with \( n \geq 6 \) and an arbitrary planar straight-line drawing \( \Gamma \) of \( G \), we observe that we can assign at least (i) one port to each internal face of \( \Gamma \) and (ii) twelve ports to the outer face of \( \Gamma \); see Fig. 20. By Euler, any \( n \)-vertex triangulation has \( 2n - 5 \) internal faces. Hence, \( \Gamma \) has \( 2n + 7 \) ports. This yields the following bound, which is tight up to a constant.

**Theorem 7.** Let \( G \) be a planar 3-tree with \( n \geq 6 \) vertices, then \( \text{seg}(G) \geq n + 4 \).

**Proof.** For claim (i), consider a sequence of stacking operations that starts with a drawing of \( K_4 \) and yields \( \Gamma \). Let \( v \) be the current vertex in this process, and let \( f \) be the face into which \( v \) is stacked. Let \( V(f) = \{x, y, z\} \) be the set of vertices incident to \( f \), and let \( f_x, f_y, \) and \( f_z \) be the three newly created faces such that \( V(f_x) = \{v, y, z\} \) etc.; see Fig. 20. Since \( f \) is a triangle, no two of the edges \( xv, \)
Fig. 19: The max. outerplanar graph $G_3$ with 42 vertices drawn on 18 segments. ($G_1$ in black)

$xy$, $yz$ can share a segment. Thus, $v$ has three ports. In particular, the segment $xv$ points into $f_x$, $yv$ points into $f_y$, and $zv$ points into $f_z$. We assign the ports of $v$ accordingly to $f_x$, $f_y$, and $f_z$. When the stacking process ends with $\Gamma$, each internal face of $\Gamma$ has a port assigned to it.

For claim (ii), note that the number of ports on the outer face equals the degree sum of the three vertices on the outer face. Thus, $K_4$ has nine ports. The next (fifth) vertex in the stacking sequence is incident to two vertices on the outer face and hence contributes two more ports. Similarly, the sixth vertex contributes at least one more port; see Fig. 20. Hence, in total, the outer face has at least twelve ports. (Note that this bound is tight since any further vertex can be stacked into an internal face that is not adjacent to the outer face.)

To finish the proof, we treat the remaining small graphs. For $n = 5$, we have one port less on the outer face, and there exists a drawing of this unique graph using eight segments (see Fig. 20) without vertex $v$). It is easy to verify the claim for $n = 4$. $\square$

In Fig. 21, we sketch an infinite family of $n$-vertex planar 3-trees that clearly has segment number $n + 7$. This yields an existential upper bound as formalized in Prop. 6. Hence, the universal lower bound in Thm. 7 is tight up to an additive constant of 3.
Fig. 20: (a) Stacking a vertex $v$ into an internal face $f = xyz$ creates a port in each new face ($f_x$, $f_y$, and $f_z$); (b) a planar 3-tree with $n \geq 6$ vertices has at least twelve ports on the outer face $abc$.

**Proposition 6.** For every $k \geq 1$ there exists a 3-tree $T_k$, whose construction is illustrated in Fig. 21, with $n = 4k + 8$ vertices and $\text{seg}(T_k) \leq 4k + 15 = n + 7$. □

**Proof.** Consider Fig. 21. We start by drawing the outer triangle $v_1, v_2, v_3$ on three segments. As fourth vertex, we add the central vertex $x$ introducing three more segments. Now the fifth and sixth vertex, $u$ and $w$, can re-use the line segments of $xv_1$ and $xv_2$ and, consequently, add only four more segments. The seventh and eighth vertex, $y$ and $z$, can re-use a segment of $uv_3$ and $wv_3$, respectively. Moreover, they share a segment for the edges $yx$ and $zx$, which results in three more segments. This gives us 13 vertices for the base construction.

Now in $k$ rounds, we iteratively stack four vertices into the faces $uyx$, $wxz$, $v_1xy$, and $v_2zx$. We stack along four new (black) line segments (see e.g. $v_1v_9$ in Fig. 21) such that the final drawing uses four more segments going through $y$, $z$, and two through $x$ per iteration (colored line segments in Fig. 21). So, for the former we get four more segments and for the latter $4k$ segments. We can re-use the segments of $uy$, $wz$, $v_3y$, and $v_2z$ for one edge each, which saves us two more segments. Together with the 13 segments of the base construction, this means

$$\text{seg}(T_k) \leq 13 + 4 + 4k - 2 = 4k + 15 = n + 7.$$ □

**F The Ratio of Segment Number and Arc Number**

Since circular-arc drawings are a natural generalization of straight-line drawings, it is natural to also ask about the maximum ratio between the segment number and the arc number of a graph. In this section, we make some initial observations regarding this question. Clearly, $\text{seg}(G)/\text{arc}(G) \geq 1$ for any graph $G$. Note that $\text{seg}(K_3)/\text{arc}(K_3) = 3$, but it remains open whether there is a graph family, or even any other connected planar graph, that reaches or exceeds this ratio. We investigate the ratio for two classes of planar graphs. We construct families
Proposition 7. For \( r \in \mathbb{N} \), let \( P_r \) be the maximal outerpath from Prop. 3; see Fig. 5a. Then, \( \lim_{r \to \infty} \frac{\text{seg}(P_r)}{\text{arc}(P_r)} = 1 \).

Proof. Consider Fig. 5a for a drawing of \( P_r \) on \( n/2 + 2 \) segments where \( n \) is the number of vertices of \( P_r \). Observe that the central vertex has degree \( (n-1) \) and, thus, is contained in at least \( n/2 \) different arcs in any arc-drawing. Hence the segment number and the arc number of \( P_n \) differ by at most a constant of 2. \( \square \)
Proposition 8. For every positive integer $k$, let $Q_k$ be the maximal outerpath with $n_k = 3k + 3$ vertices shown in Fig. 5. Then $\lim_{k \to \infty} \frac{\text{seg}(Q_k)}{\text{arc}(Q_k)} \geq 2$.

Proof. The outerpath $Q_k$ contains $n_k/3 - 2$ degree-6 vertices and for each of them two degree-3 neighbors, with at least one port each. The degree-6 vertices either have 2 ports themselves or their bend companions have three ports. In either case, we find 4 ports for each degree-6 vertex. The remaining six vertices around the first and the last face have at least 10 ports. Therefore, $\text{seg}(Q_k) \geq 2n_k/3 + 1$. Fig. 5 yields that $\text{arc}(Q_k) \leq n_k/3 + 1$ and hence $\frac{\text{seg}(Q_k)}{\text{arc}(Q_k)} \geq 2 - 2/(k+2)$. \qed

Proposition 9. For $k \geq 2$, let $T_k$ be the planar 3-tree shown in Fig. 21. Then $\lim_{k \to \infty} \frac{\text{seg}(T_k)}{\text{arc}(T_k)} \leq 4/3$.

Proof. See Fig. 21 for a drawing of $T_k$ on $4k + 11$ segments. Let $v$ be the unique vertex of degree $n - 1$ and $u, w$ be the two degree-$(n/2 - 1)$ vertices.

There is a set of $4k + 2$ unique paths, one half from $u$ to $v$ and the other from $v$ to $w$. Each of these paths needs to be covered by at least one arc. Obviously no arc may cover more than 2 paths. Now observe that any arc covering one path on each side connects all three vertices, such that only one such arc may exist. Hence, of the remaining 4k paths we may only cover two with the same arc, if both lie on the same side of $v$. However every such arc must have the same tangent in $v$ in order not to cross with the other paths, such that we may only do this on one side of $v$. Hence $k$ arcs may suffice for one side, but the other needs 2k arcs, such that a total of $3k + 1$ arcs are necessary. Hence $\frac{\text{seg}(T_k)}{\text{arc}(T_k)} \leq 4/3 + 29/(9k + 3)$ \qed

Proposition 10. For every odd $n \geq 5$, let $S_n$ be the planar 3-tree shown in Fig. 22. Then $\lim_{n \to \infty} \frac{\text{seg}(S_n)}{\text{arc}(S_n)} \geq 2$.

Proof. Fig. 22 shows a drawing of $S_n$ on $(n + 1)/2$ arcs. Now observe that there are two vertices of degree $(n - 1)$ which lie on $n - 3$ different segments since they may only share one segment. Hence $\frac{\text{seg}(S_n)}{\text{arc}(S_n)} \geq 2 - 8/(n + 1)$. \qed