Since January 2020 Elsevier has created a COVID-19 resource centre with free information in English and Mandarin on the novel coronavirus COVID-19. The COVID-19 resource centre is hosted on Elsevier Connect, the company's public news and information website.

Elsevier hereby grants permission to make all its COVID-19-related research that is available on the COVID-19 resource centre - including this research content - immediately available in PubMed Central and other publicly funded repositories, such as the WHO COVID database with rights for unrestricted research re-use and analyses in any form or by any means with acknowledgement of the original source. These permissions are granted for free by Elsevier for as long as the COVID-19 resource centre remains active.
A new lifetime family of distributions: Theoretical developments and analysis of COVID 19 data

I. Elbatal
Department of Mathematics and Statistics - College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Saudi Arabia

A R T I C L E   I N F O

Keywords:
Beta G family
Topp–Leone G family
Moments
Entropy
MCMC
Symmetric and asymmetric loss functions
COVID 19 data

A B S T R A C T

In parametric statistical modeling and inference, it is critical to develop generalizations of existing statistical distributions to make them more flexible in modeling real data sets. Thus, this paper contributes to the subject by investigating a new flexible and versatile generalized family of distributions defined from the alliance of the families known as beta-G and Topp–Leone generated (TL-G), inspiring the name of BTL-G family. The characteristics of this new family are studied through analytical, graphical and numerical approaches. Statistical features of the family such as expansion of density function (pdf), cumulative function (cdf), moments (MOs), incomplete moments (IMOs), mean deviation (MDE), and entropy (ENT) are calculated. The model parameters’ maximum likelihood estimates (MaxLEs) and Bayesian estimates (BEs) are provided. Symmetric and Asymmetric Bayesian Loss function have been discussed. A complete simulation study is proposed to illustrate their numerical efficiency. The considered model is also applied to analyze two different kinds of genuine COVID 19 data sets. We show that it outperforms other well-known models defined with the same baseline distribution, proving its high level of adaptability in the context of data analysis.

Introduction

Several Statisticians have recently shown an interest in providing new produced families of continuous distributions by adding one or more extra shape parameters to the baseline model in order to generate models. These extra form factors increase accuracy and flexibility in data fitting. Among the most well-known generators seem to be: the Marshall Olkin G by Marshall et al. [1], beta (B) G (BG) by Eugene et al. [2], Topp Leone odd Lindley G by Reyad et al. [3], Topp–Leone (TL) G (TLG) by Rezaei [4], TL odd log–logistic G by de Brito et al. [5], B WeibullG by Yousof et al. [6], The Fréchet TL G by Reyad et al. [3], the odd Fréchet G by Haq and Elaghrity [7], exponentiated power generalized Weibull power series family of distributions by Alldahlan et al. [8], odd generalized NH -G by Ahmed et al. [9], among others.

One of the continuous distributions that is appealing as a generator is the TL distribution. Topp and Leone [10] suggested this distribution. The characteristics of the TL distribution have been investigated by a number of experts, measures of reliability and stochastic ordering by Ghitany et al. [11]; Kurtosis’ conduct by Kotz and Seier [12]; MOs of order statistics by Genc [13]; stress–strength by Genc [14], among others. Al-Shomrani et al. [15] developed a novel family of distributions dubbed the TLG family of distributions, which is constructed on a one-parameter distribution with bathtub shaped risk rates. The cdf is supplied by

\[ H(x; \theta, \xi) = (1 - G(x; \xi))^\theta, \quad \theta > 0, \; x > 0, \]  

(1)

the corresponding pdf is

\[ h(x; \theta, \xi) = 2\theta g(x; \xi)G(x; \xi)(1 - G(x; \xi))^\theta - 1. \]  

(2)

where \( G(x; \xi) \) is a baseline cdf, \( g(x; \xi) \) is the corresponding pdf, \( \xi \) are the parameters specifying the baseline distribution, \( \xi > 0 \) and \( \theta > 0 \).

The survival of the TL – G family is given by

\[ H(x; \theta, \xi) = 1 - (1 - G(x; \xi))^\theta. \]  

(3)

The BG family has the coming cdf and pdf

\[ F(x) = \frac{1}{B(\lambda, a)} \int_0^{H(x)} w^{(a-1)}(1 - w)^{\lambda-1} dw = \frac{B_H(\lambda, a)}{B(\lambda, a)} = I_{H(x)}(\lambda, a), \]  

(4)

where \( \lambda > 0 \) and \( a > 0 \) are two extra parameters that add skewness to the produced distribution and change the tail weight. The pdf for (4) is expressed as follows:

\[ f(x) = \frac{h(x)}{B(\lambda, a)} H(x)^{\lambda-1} [1 - H(x)]^{a-1}. \]  

(5)

In this article, we create and investigate a novel family of distributions by using two more shape factors in (1) to produce the provided family greater flexibility. We get a new broader family of distributions based

E-mail address: iielbatal@imamu.edu.sa.

https://doi.org/10.1016/j.rinp.2021.104979
Received 13 October 2021; Received in revised form 27 October 2021; Accepted 2 November 2021
Available online 14 November 2021
2211-3797/© 2021 Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).
on the beta TLG (BTLG) family of distributions by inserting (1) in (4). The cpdf and pdf of $BT\text{LG}$ family, respectively, are given by

$$F(x; \theta, \lambda, a, \xi) = I_{(1-G^2)}(x; \xi)\theta(\lambda, a),$$  

(6) and

$$f(x; \theta, \lambda, a, \xi) = \frac{2\theta g(x; \xi)G(x; \xi)}{B(\lambda, a)} \left(1 - \frac{G^2(x; \xi)}{\theta(\lambda, a)}\right)^{a-1}.$$  

(7)

The BTLG family includes various unique members, and it is crucial in a variety of applications (reliability, economics, engineering and other areas of research). If $X$ is a random variable with $BT\text{L} - G$ pdf (7), we use the notation $X \sim BT\text{L}(\theta, \lambda, a, \xi)$. Some special cases of the new family are listed in the following Table 1.

More paper discussed modeling for Covid-19 spread as Atangana and Araz [16,17], Shafiq et al. [18], Atangana [19], Hassan et al. [20], Ibrahim et al. [21], and Sindhu et al. [22,23]. Sindhu et al. [24] discussed analysis of the left censored data from the Pareto type II distribution.

The remainder of the paper is laid out as follows: A useful expansion for the BTLG pdf and some special models are introduced in Section “Useful Expansion”. Various structural properties including quantile function (QF), MOCs, IMOCs, MDEs, residual life (RL) and reversed residual life (RRL) functions are derived in Section “Statistical Properties”. The ENT of proposed family are provided in Section “Entropy”. The following result investigates useful expansions for $F(x)$ and $f(x)$. If $|z| < 1$ and $b > 0$ is a real non-negative, then the following power series holds.

$$ (1 - z)^{b-1} = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(b)}{k!} (1 - z)^k.$$  

(8)

Using the expansion (8) in (7), we get

$$f(x; \theta, \lambda, a, \xi) = \frac{2\theta g(x; \xi)G(x; \xi)}{B(\lambda, a)} \times \left[1 - \left(1 - \frac{G^2(x; \xi)}{\theta(\lambda, a)}\right)^{a-1}\right]^{\theta(b-1)}.$$  

(9)

again using the binomial expansion

$$ (1 - w)^b = \sum_{n=0}^{\infty} (-1)^n \binom{b}{n} w^n,$$  

(10)

insert (10) in (6), the $BT\text{L} - G$ density reduces to

$$f(x; \theta, \lambda, a, \xi) = \sum_{n=0}^{\infty} b_n \delta(x; \xi)G(x; \xi)^n.$$  

(11)

where

$$b_n = \frac{2\theta}{B(\lambda, a)} \sum_{j=0}^{\infty} (-1)^{j+k} \frac{\Gamma(\lambda + j)}{j! \Gamma(\theta(\lambda + j) - k)} \frac{\Gamma(\theta(\lambda + j) - k)}{(\lambda + j)^{2k+1}}.$$  

(12)

Another formula can be extracted from pdf (11) as follows

$$f(x; \theta, \lambda, a, \xi) = \sum_{n=0}^{\infty} d_n \pi_{(n+1)}(x)$$  

(13)

where

$$d_n = \frac{\theta}{m!} \text{ and } \pi_{(n+1)}(x) = \delta g(x; \xi) G(x; \xi)^{b-1}$$

denotes the exponentiated-G (exp-G) density with power parameter $\delta$. Consequently, Eq. (12) represents the $BT\text{L} - G$ density as an infinite linear combinations of exp-G densities. Similarly, the cpdf of the $BT\text{L} - G$ family can

| Sub families | $\theta$ | $\lambda$ | $a$ | Author |
|--------------|---------|----------|----|-------|
| $FL - G$ family | $\theta$ | 1 | 1 | Al-Shomrani et al. [15] |
| $GL - G$ family | $\theta$ | 1 | 1 | New |
| $ET - G$ family | $\theta$ | 1 | 1 | New |

Table 1

| Sub families | $\theta$ | $\lambda$ | $a$ | Author |
|--------------|---------|----------|----|-------|
| $L - G$ family | $\theta$ | 1 | 1 | Al-Shomrani et al. [15] |
| $G - G$ family | $\theta$ | 1 | 1 | New |
| $E - G$ family | $\theta$ | 1 | 1 | New |

Table 2

| Model | Cdf / $G(x; \gamma)$ | pdf / $g(x; \gamma)$ | $\theta(x; \gamma)$ |
|-------|----------------------|----------------------|-------------------|
| Lomax | $1 - (1 + \frac{\gamma}{x})^{-\gamma}$ | $\frac{\gamma}{x^{\gamma+1}} (1 + \frac{\gamma}{x})^{-\gamma-1}$ | $e^{\gamma x}$ |
| Exponential | $1 - e^{-\gamma x}$ | $\gamma e^{-\gamma x}$ | $e^{\gamma x}$ |
| Rayleigh | $1 - e^{-\frac{x}{\gamma}}$ | $\frac{\gamma}{\sqrt{\pi}} e^{-\frac{x^2}{\gamma^2}}$ | $e^{\gamma x}$ |
also be expressed as a mixture of exp-G cdfs where
\[
F(x;\theta,\lambda,\alpha,\xi) = \sum_{m=0}^{\infty} d_m \Pi_{m+1}(x)
\]
where \( \Pi_{m+1}(x) \) is the exp-G cdf with power parameter \((m + 1)\).

Some special models of the BTL-G family

Many special members of the BTL-G family are of potential interest, for tractability of the related functions and flexibility reasons. Some of them are listed in Table 2, defined with their cdf and pdf for the sake of place.

**beta Topp Leone Lomax (BTLLo) distribution**

The pdf of BTLLo distribution are
\[
f(x;\theta,\lambda,\alpha,a) = \frac{2\theta^a(1 + \frac{x}{b})^{-2a}}{B(\lambda, a)} \left(1 - (1 + \frac{x}{b})^{-2a}\right)^{\lambda-1} \times \left[1 - (1 - (1 + \frac{x}{b})^{-2a})^{\alpha-1}\right].
\]  
(13)

**beta Topp Leone Exponential (BTLE) distribution**

The pdf of the BTLE model (for \( x > 0 \)) are
\[
f(x;\theta,\lambda,\alpha,\mu) = \frac{2\theta^\mu e^{-\mu x}}{B(\lambda, a)} \left((1 - e^{-\mu x})\right)^{\lambda-1} \left[1 - (1 - e^{-\mu x})^\alpha\right]^{\theta-1}.
\]  
(14)

**beta Topp Leone Rayleigh (BTLR) distribution**

The pdf of the BTLR model (for \( x > 0 \)) are
\[
f(x;\theta,\lambda,\alpha,\rho) = \frac{2\theta^\rho x e^{-\rho x^2}}{B(\lambda, a)} \left((1 - e^{-\rho x^2})\right)^{\lambda-1} \left[1 - (1 - e^{-\rho x^2})^\alpha\right]^{\theta-1}.
\]  
(15)

Statistical properties

This section shows important distributional and structural properties satisfied by the BTL-G family. The conventional MO and MO generating functions of the BTLG family are developed. The varied orders for the MOs are particularly beneficial for determining device anticipated life time, skewness (SK), and kurtosis (KU) in a given collection of observations occurring in reliability applications.
Moments

The $r_{th}$ ordinary MO of $X$ can be obtained from (12) as follows
\[
\mu_r = E(X^r) = \sum_{m=0}^{\infty} d_m E(W_{(m+1)}^r),
\]
where $W_{(m+1)}^r$ denotes the exp-G random variable with power parameter $(m+1))$. The measures of SK and KU can be derived from the $n_{th}$ central MOs, say $M_n(x)$ of $X$, where
\[
M_n(x) = E(X - \mu_1)^n = \sum_{r=0}^{\infty} \binom{n}{r} (-\mu_1)^{n-r} E(X^r)
\]
where $M_n(x)$ is the MO generating function of $W_{(m+1)}$. Consequently, $M_n(x)$ can be easily determined from the exp-G generating function.

Quantile function and moment generating function

The $BTL-G$ QuF, say $x = Q(u)$ can be obtained by inverting (6) as follows
\[
X = Q_G(u) = G^{-1} \left\{ \left[ 1 - (1 - u^{\frac{1}{\theta}})^{\frac{1}{\gamma}} \right] \right\}.
\]
where $Q_G(u)$ denotes the QuF corresponding to $G(x)$, and $G^{-1}(\cdot)$ is the inverse of the baseline cdf. One of the earliest SK measures to be suggested is the Bowley SK (Kenney and Keeping, [25]) defined by
\[
SK = \frac{Q(\frac{7}{8}) - Q(\frac{5}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.
\]
On the other hand, the Moors KU (Moors, [26]) based on Qus is given by
\[
KU = \frac{Q(\frac{6}{8}) - Q(\frac{2}{8}) + Q(\frac{3}{8}) - Q(\frac{1}{8})}{Q(\frac{6}{8}) - Q(\frac{2}{8})}.
\]
where $Q(\cdot)$ represents the QuF. We used the BTLE distribution as special case of proposed family. Figs. 2 and 3 for BTLE distribution show that measures are meaning of this statement sensitive to outliers and they exist even for distributions without MOs.

The MO generating function of $X$ can be computed from Eq. (12) as follows
\[
M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} d_n M_{(m+1)}(t),
\]
where $M_{(m+1)}(t)$ is the MO generating function of $W_{(m+1)}$. Consequently, $M_X(t)$ can be easily determined from the exp-G generating function.

Further, the incomplete moments play an important role for measuring inequality. For example, the first incomplete moment can be used to obtain the formulas of Lorenz and Bonferroni curves. The $s_{th}$ IMOs...
of $X$ defined by $\phi_{s}(t)$ for any real $s > 0$ can be expressed from (12) as

$$\phi_{s}(t) = \int_{-\infty}^{t} x^{s} f(x) dx = \sum_{m=0}^{\infty} d_{m} \int_{-\infty}^{\infty} x^{s} \pi_{(m+1)}(x) dx$$  \hspace{1cm} (19)$$

Eq. (19) denotes the $s$th IMOs of $\mathcal{W}(m+1)$. The MDEs about the mean $\mu = E(X)$ and the MDEs about the median $M$ are

$$\delta_{1}(x) = E[X - \mu] = 2 \mu_{1} F(\mu_{1}) - 2 \phi_{1}(\mu_{1})$$

and

$$\delta_{2}(x) = E[X - M] = \mu_{1} - 2 \phi_{1}(M)$$

respectively, where $\mu_{1} = E(X)$, $M = \text{median}(X) = Q_{1/2}$, $F(\mu_{1})$ is evaluated from (6) and $\phi_{1}(t)$ is the first complete MO given by (19) with $s = 1$. We can determine $\delta_{1}(x)$ and $\delta_{2}(x)$ by two techniques, the first can be obtained from (12) as $\phi_{1}(t) = \sum_{m=0}^{\infty} d_{m} \mathcal{W}_{(m+1)}(t)$ where $\mathcal{W}_{(m+1)}(t) = \int_{-\infty}^{\infty} x^{1} \pi_{(m+1)}(x) dx$ is the first IMO of the exp-G distribution. The second approach is given by $\phi_{1}(t) = \sum_{m=0}^{\infty} d_{m} \theta_{(m+1)}(t)$ where

$$\theta_{(m+1)}(t) = (m+1) \int_{0}^{G(t)} u^{m+1} \mathcal{Q}(u) du.$$  

**Moments of residual life**

The $r$th order MO of the RL is given by

$$\mu_{r}(t) = E((X - t)^{r} \mid X > t) = \frac{1}{F(t)} \int_{t}^{\infty} (x-t)^{r} f(x) dx, r \geq 1$$

$$= \frac{1}{F(t)} \sum_{m=0}^{\infty} d_{m}^{*} \int_{t}^{\infty} x^{r} \pi_{(m+1)}(x) dx$$

where $d_{m}^{*} = d_{m} \sum_{h=0}^{\infty} \binom{r}{h} (-t)^{r-h}$. The mean RL (MRL) of $BT_{L-G}$ family of distributions can be computed by setting $r = 1$ in the above equation, defined as

$$\mu(t) = E(X_{c}) = E(X \mid X > t).$$

The $r$th order MO of the RRL can be computed by

$$m_{r}(t) = E((t - X)^{r} \mid X \leq t) = \frac{1}{F(t)} \int_{0}^{t} (t-x)^{r} f(x) dx, r \geq 1$$

$$= \frac{1}{F(t)} \sum_{m=0}^{\infty} d_{m}^{*} \int_{0}^{t} x^{r} \pi_{(m+1)}(x) dx$$  \hspace{1cm} (20)$$
I. Elbatal

Results in Physics 31 (2021) 104979

Fig. 8. Boxplot, TTT-plot and estimated hazard of BTLE distribution with empirical hazard for data I.

Fig. 9. Scatter plot Matrices of BTLE distribution for data I.

Table 3

|                | Min. 0.0932 | 1st Qu. 0.3457 | Median 0.5005 | Mean 0.5532 | 3rd Qu. 0.7021 | Max. 1.6286 |
|----------------|-------------|----------------|---------------|-------------|----------------|-------------|
| i \( \theta \) | 3           | 0.0744         | 0.6056        | 1.0523      | 1.2904         | 1.7131      | 5.2288     |
| ii \( \xi \)   | 3           | 0.0019         | 0.0734        | 0.1657      | 0.2321         | 0.3193      | 1.1917     |
| iii \( \alpha \)| 3           | 0.0225         | 0.1946        | 0.3299      | 0.4642         | 0.6385      | 2.3834     |
| iv \( \lambda \)| 3           | 0.0932         | 0.3457        | 0.5005      | 0.5532         | 0.7021      | 1.6286     |

The mean activity time \( \text{MAT} \) of the \( BTLG \) family of distributions can be determined by setting \( r = 1 \) in (20), where

\[
m(t) = E(X(t)) = E(t - X \mid X < t).
\]

Entropy

The Rényi ENT is defined by \( (\rho > 0, \rho \neq 1) \)

\[
I_{\rho}(\rho) = \frac{1}{1 - \rho} \log \left( \int_{-\infty}^{\infty} f^{\rho}(x)dx \right).
\]

Using (7), applying the same procedure of the useful expansion (13) and after some simplifications, we get

\[
f^{\rho}(x) = \sum_{m=0}^{\infty} \Delta_m g(x)^{\rho} G(x)^{\rho m + k - \rho}
\]

where

\[
\Delta_m = \sum_{j,k=0}^{\infty} (-1)^j \left( \frac{x}{\alpha} \right)^j \Gamma(j + 1) \Gamma(j + 1 - \rho) \frac{x^{2k + 1}}{k}
\]

Thus Rényi ENT of \( BTG \) family is

\[
I_{\rho}(\rho) = \frac{\rho}{1 - \rho} \log(\frac{2\theta}{B(\lambda, \alpha)}) + \frac{1}{1 - \rho} \log \left( \sum_{m=0}^{\infty} g(x)^{\rho} G(x)^m dx \right).
\]

Maximum likelihood estimation

The MaxLEs have appealing features and may be used to create confidence intervals and regions, as well as test statistics. Only complete samples are used to calculate the MaxLEs of the parameters of the \( BTG \) family of distributions. Assume \( x_1, \ldots, x_n \) be a random sample of size \( n \) from the \( BTG \) distribution given by (6). Let \( \Phi = (\theta, \lambda, \alpha, \xi)^T \)
be \( q \times 1 \) vector of parameters. The logarithm of likelihood function is

\[
L_n = n \log(2\theta) - n \log B(\lambda, a) + \sum_{i=1}^{n} \log g(x_i; \psi) + \sum_{i=1}^{n} \log G(x_i; \psi)
\]

\[
+ (\lambda - 1) \sum_{i=1}^{n} \log(1 - G(x_i; \psi))
\]

\[
+ (a - 1) \sum_{i=1}^{n} \log \left[ 1 - (1 - G(x_i; \psi))^\psi \right].
\]

The score vector components, say, \( U(\Phi) = \left( \frac{\partial U}{\partial \theta}, \frac{\partial U}{\partial \lambda}, \frac{\partial U}{\partial a}, \frac{\partial U}{\partial \psi} \right)^T \) are given by

\[
U_\theta = \frac{n}{\theta} + \lambda \sum_{i=1}^{n} \log(1 - G(x_i; \psi)) + (a - 1) \sum_{i=1}^{n} \left( \frac{1}{1 - G(x_i; \psi)} \log(1 - G(x_i; \psi)) - 1 \right).
\]

\[
U_\lambda = -n\psi(\lambda) + n\psi(\lambda + a) + \sum_{i=1}^{n} \log \left( 1 - G(x_i; \psi) \right),
\]

\[
U_a = -n\psi(a) + n\psi(\lambda + a) + \sum_{i=1}^{n} \log \left[ 1 - (1 - G(x_i; \psi))^\psi \right],
\]

and

\[
U_\psi \left| \frac{\partial L_n}{\partial \psi} \right| = \sum_{i=1}^{n} \frac{G(x_i; \psi) \log G(x_i; \psi)}{G(x_i; \psi)} + \sum_{i=1}^{n} \frac{G(z_i(x_i; \psi))}{G(x_i; \psi)}
\]

\[
+ (1 - \theta) \sum_{i=1}^{n} \frac{2G(x_i; \psi)G(z_i(x_i; \psi))}{1 - G(x_i; \psi)}
\]

\[
+ \theta(a - 1) \sum_{i=1}^{n} \frac{2G(x_i; \psi)G(z_i(x_i; \psi))(1 - G(x_i; \psi))^\psi - 1}{1 - (1 - G(x_i; \psi))^\psi}.
\]

where \( z_i(x_i; \psi) \) means the derivative of the function \( z \) with respect to \( U \).

Setting these equations to zero, \( U_\theta = U_\lambda = U_a = U_\psi = 0 \), and solving them simultaneously yields the MaxLE (\( \hat{\Phi} \)) of \( \Phi \).

**Bayesian estimation**

As random and parameter uncertainties are represented by a previous joint distribution that is established prior to the data collected on the failure, the Bayesian approach deals with the parameters. The ability to incorporate prior knowledge into research makes the Bayesian method very useful in the analysis of reliability as Sundhu and Atanagana [27], as one of the main problems associated with reliability analysis is the limited availability of data. In the \( \theta, \lambda, a, \) and \( \psi \) parameters, as prior gamma distributions, we have to use the insight before the. The, \( \theta, \lambda, a, \) and \( \psi \) independent joint prior density function can be written as follows:

\[
\Pi(\Phi) \propto \theta^{n-1} \phi^{\alpha-1} a^{\psi-1} e^{-(\psi+\alpha)z + \psi(1 + \frac{\psi-1}{\phi})},
\]

where \( \Phi = (\theta, \lambda, a, \psi)^T \). From the likelihood function and joint prior function, the joint posterior density function of \( \theta, \lambda, a, \) and \( \psi \) is obtained.

The joint posterior of the distribution of BTLE-G family can then be presented as

\[
\Pi(\psi|x) \propto \theta^{n+\alpha-1} \phi^{\alpha-1} a^{\psi-1} e^{-(\psi+\alpha)z + \psi(1 + \frac{\psi-1}{\phi})} \left( \frac{1}{B(\lambda, a)} \right)^n \]

\[
\prod_{i=1}^{n} g(x_i; \psi)G(x_i; \psi) \left( 1 - (1 - G(x_i; \psi))^\psi \right)^{\psi-1}. \]

Using the most common function for symmetric loss, which is a function for squared error loss. Bayes estimators of \( \theta, \lambda, a, \) and \( \psi \) based on

---

**Table 4**

For MaxLE and Bayesian, Bias and MSE for BTLE distribution parameters with changing \( \psi \).

| \( \psi \) | MaxLE Bias | MaxLE MSE | SE Bias | SE MSE | Linex \( \psi = 2 \) Bias | Linex \( \psi = 2 \) MSE | Linex \( \psi = -0.5 \) Bias | Linex \( \psi = -0.5 \) MSE |
|---|---|---|---|---|---|---|---|---|
| 15 | 0.0495 | 0.1138 | -0.0157 | 0.0813 | -0.0606 | 0.0812 | -0.0042 | 0.0821 |
| 30 | 0.1139 | 0.1916 | -0.0110 | 0.0212 | -0.0123 | 0.0211 | -0.0079 | 0.0212 |
| 1.5 | 0.0952 | 0.1814 | 0.0054 | 0.0166 | -0.0051 | 0.0161 | 0.0080 | 0.0168 |
| 60 | 0.3626 | 0.9954 | 0.0233 | 0.0523 | -0.0124 | 0.0458 | 0.0324 | 0.0546 |
| 2.75 | 0.05129 | 0.05904 | -0.0151 | 0.00170 | -0.00240 | 0.00170 | -0.00129 | 0.00170 |
| 100 | 0.05079 | 0.08222 | -0.00467 | 0.00162 | -0.00562 | 0.00165 | -0.00444 | 0.00162 |

The joint posterior of the distribution of BTLE-G family can then be presented as

\[
\Pi(\psi|x) \propto \theta^{n+\alpha-1} \phi^{\alpha-1} a^{\psi-1} e^{-(\psi+\alpha)z + \psi(1 + \frac{\psi-1}{\phi})} \left( \frac{1}{B(\lambda, a)} \right)^n \]

\[
\prod_{i=1}^{n} g(x_i; \psi)G(x_i; \psi) \left( 1 - (1 - G(x_i; \psi))^\psi \right)^{\psi-1}. \]
the squared error loss function are defined by the squared error loss function:

\[ S(\hat{\phi}) = E \left( (\hat{\phi} - \Phi)^2 \right) \]

\[ = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty (\hat{\phi} - \Phi)^2 \Pi(\Phi) d\Phi_1 d\Phi_2 d\Phi_3 d\Phi_4 \]  

(27)

In cases when under-estimate is more dangerous than over-estimation, or when positive and negative estimation errors produce distinct effects, under-estimation should be penalized severely. This fine can be obtained by applying Varian’s LINEX loss function, which was devised as an asymmetric loss function and is defined as

\[ L(\epsilon) = e^{\delta \epsilon} - \Delta e - 1; \quad \epsilon \neq 0, \Delta = \frac{\hat{\phi}}{\Phi} - 1 \quad \text{or} \quad \Delta = \hat{\phi} - \Phi \]  

(28)

where \( \epsilon \) is the shape parameter and \( \hat{\phi} \) is an estimate of the parameter \( \Phi \). When \( \epsilon > 0 \), the loss from under-estimation is greater. When \( \epsilon < 0 \), the loss from under-estimation is greater. The LINEX loss function is approximated as a squared loss function for small values of \( \epsilon \). The risk function associated with \( \hat{\phi} \) under the LINEX loss function is obtained

\[ R(\epsilon) = E(L(\epsilon)). \]  

(29)

It should be noted that the integrals given by (27) cannot be obtained directly. As a consequence, we employ the Markov chain Monte Carlo (MCMC) technique to estimate the value of integrals. Gibbs sampling and more general Metropolis–Hastings within Gibbs samplers are important sub-classes of MCMC techniques. The MCMC method’s most popular applications are the Metropolis–Hastings (MH) algorithm and Gibbs sampling. The MH algorithm, like acceptance–rejection sampling, assumes that a candidate value from a proposal distribution can be generated for each iteration of the algorithm. The MH algorithm, like acceptance–rejection sampling, claims that a candidate value from a proposal distribution can be generated for each iteration of the algorithm. For more examples, see [28], Almetwally et al. [29,30], Sindhu et al. [31–34] and Abd El-Rahim and El-Rahim et al. [35].

Table 5

| \( \zeta \) | MaxLE | SE | Linear \( \epsilon = 2 \) | Linear \( \epsilon = -0.5 \) |
| --- | --- | --- | --- | --- |
| \( \theta \) | \( \theta \) | \( \theta \) | \( \theta \) | \( \theta \) |
| 15 | 0.0178 | 0.1427 | 0.0056 | 0.0532 | 0.0842 | 0.0053 | 0.0866 |
| \( \lambda \) | 0.2914 | 0.4521 | 0.0031 | 0.0356 | 0.0366 | 0.0006 | 0.0125 | 0.0574 |
| \( \alpha \) | 0.3083 | 0.5497 | 0.0167 | 0.0308 | 0.0110 | 0.0272 | 0.0238 | 0.0321 |
| \( \beta \) | 0.1983 | 1.1189 | 0.0028 | 0.0769 | 0.0441 | 0.0763 | 0.0148 | 0.0780 |

In cases when under-estimate is more dangerous than over-estimation, under-estimation should be penalized severely. This fine can be obtained by applying Varian’s LINEX loss function, which was devised as an asymmetric loss function and is defined as

\[ L(\epsilon) = e^{\delta \epsilon} - \Delta e - 1; \quad \epsilon \neq 0, \Delta = \frac{\hat{\phi}}{\Phi} - 1 \quad \text{or} \quad \Delta = \hat{\phi} - \Phi \]  

(28)

where \( \epsilon \) is the shape parameter and \( \hat{\phi} \) is an estimate of the parameter \( \Phi \). When \( \epsilon > 0 \), the loss from under-estimation is greater. When \( \epsilon < 0 \), the loss from under-estimation is greater. The LINEX loss function is approximated as a squared loss function for small values of \( \epsilon \). The risk function associated with \( \hat{\phi} \) under the LINEX loss function is obtained

\[ R(\epsilon) = E(L(\epsilon)). \]  

(29)

It should be noted that the integrals given by (27) cannot be obtained directly. As a consequence, we employ the Markov chain Monte Carlo (MCMC) technique to estimate the value of integrals. Gibbs sampling and more general Metropolis–Hastings within Gibbs samplers are important sub-classes of MCMC techniques. The MCMC method’s most popular applications are the Metropolis–Hastings (MH) algorithm and Gibbs sampling. The MH algorithm, like acceptance–rejection sampling, assumes that a candidate value from a proposal distribution can be generated for each iteration of the algorithm. The MH algorithm, like acceptance–rejection sampling, claims that a candidate value from a proposal distribution can be generated for each iteration of the algorithm. For more examples, see [28], Almetwally et al. [29,30], Sindhu et al. [31–34] and Abd El-Rahim and El-Rahim et al. [35]. To generate random samples of conditional posterior densities from the BTLE-G family, we use the MH within the Gibbs sampling. We need the conditional distribution of the parameters. For more examples, see [28], Almetwally et al. [29,30], Sindhu et al. [31–34] and Abd El-Rahim and El-Rahim et al. [35].

Simulation

We present a brief Monte Carlo simulation research in this section to evaluate the MLE and Bayesian for BTLE distribution parameters. Inverting the cdf formula makes it simple to recreate the BTLE
The inverse cdf is used to generate the random numbers. The simulation results and inverse procedure are acquired using the statistical software R and the (uniroot) library with command stats. Four alternative sets of parameters are used in the simulation:

(i) $\lambda = 0.75$, $\alpha = 0.75$, $\xi = 0.75$ and $\theta = 1.5$ and 3.

(ii) $\theta = 1.5$, $\lambda = 0.75$, $\alpha = 0.75$ and $\xi = 1.5$ and 3.

(iii) $\theta = 2.75$, $\lambda = 0.75$, $\xi = 1.5$ and $\alpha = 1.5$ and 3.

(iv) $\theta = 2.75$, $\alpha = 1.5$, $\xi = 1.5$ and $\lambda = 1.5$ and 3.

The sample size was chosen to be $n = 15, 30, 60, \text{ and } 100$, with 10000 Monte-Carlo replications. The conjugate-gradient algorithm “CG” with “optim” is used to maximize the expression (21). For $l = 1, \ldots, 10000$, the estimates of MaxLE and Bayesian for parameters of BTLE distribution are obtained for each set of Monte-Carlo simulation. The use of assessment techniques like bias and mean square errors (MSE) is taken into account. The following formula is used to calculate
The simulation results of the methods mentioned in this study for Table 3 shows the summary measures of the simulated data. The following conclusions can be drawn from these tables: The Bias and MSE decrease as sample size $n$ increases for actual parameters of the BTLE distribution. In case i, the Minimum (Min.), first quartile (1st Qu.), Median, Mean, third quartile (3rd Qu.), maximum (Max.), Bias and MSE increases as $\theta$ increases. Bayesian estimation is the best estimation method.

### Real data analysis

The COVID-19 data are provided in this section to evaluate the consistency of the BTLE distribution. We compare the proposed model with
Table 7

| λ | MaxLE | SE | Linex c = 2 | Linex c = −0.5 |
|---|-------|----|-------------|----------------|
|   | Bias  | MSE | Bias  | MSE | Bias  | MSE | Bias  | MSE |
| 15 | 0.2011 | 0.4529 | -0.0037 | 0.0297 | -0.0150 | 0.0300 | 0.0308 | 0.0298 |
| 30 | 1.7587 | 0.1648 | 0.1080 | -0.0177 | 0.0211 | -0.0300 | 0.0216 | -0.0145 | 0.0210 |

Table 8

| MLLE | Estimates | SE | CVM | AD | KS | P-value |
|------|-----------|----|-----|----|----|---------|
| BTLE | 0.6916    | 5.367 | 8.433 | 0.0859 | 0.5890 | 0.0905 | 0.6661 |
| EL   | 41.9104   | 35.152 | 109.977 | 0.0932 | 0.6138 | 0.1014 | 0.5246 |
| KW   | 40.8159   | 38.773 | 0.0368 | 0.2316 | 0.3939 | 0.4212 | 0.3536 |

Table 9

| MLE and Bayesian | Estimates | SE | Estimates | SE |
|------------------|-----------|----|-----------|----|
| BTLE             | 6.9216    | 5.367 | 12.6438   | 5.7081 |
| EL               | 9.7557    | 8.433 | 8.7842    | 7.9530 |
| KW               | 0.6293    | 0.4777 | 0.8836    | 0.4131 |
| WITL             | 145.9216  | 77.7439 | 133.4252  | 47.6505 |

Weibull inverse Topp–Leone (OWITL) [Almetwally [38]], New I. Elbatal

We conclude that all the models used are suitable for these data, as the P-value is greater than 0.05, and we can look at the following illustration in two Figs. 6 and 7 that makes this conclusion. In addition, other related models such as exponential Lomax (EL [El-Bassiouny et al. [20]], Kumaraswamy Weibull (KW) Cordeiro et al. [36], Kumaraswamy Inverted Topp–Leone (KITL) [Hassan et al. [37]], Odd Weibull inverse Topp–Leone (OWITL) [Almetwally [38]], New Exponential-X Fréchet (NEXF) [Alzeley et al. [39]] and Weibull-Lomax (WL) [Tahir et al. [40]]. Tables 8 and 10 provide the statistics Cramer von Mises (CVM), Anderson Darling (AD) values, and the Kolmogorov–Smirnov (KS) statistics, along with the P-value for all models fitted on the basis of real data set. For more examples of applications of COVID-19 data, see Almetwally [38], Almetwally et al. [41], Abu El Azm et al. [42], and Almomy et al. [43].

The data I represents 61 days of COVID-19 data of Saudi Arabia, from March 1 to April 30, 2021 [https://covid19.who.int/]. The mortality rate was calculated. The following are this data: 0.0106, 0.0106, 0.0107, 0.0107, 0.0132, 0.0132, 0.0133, 0.0133, 0.0134, 0.0134, 0.0134, 0.0135, 0.0156, 0.0157, 0.0157, 0.0157, 0.0158, 0.0158, 0.0159, 0.0159, 0.0159, 0.0160, 0.0160, 0.0161, 0.0162, 0.0162, 0.0179, 0.0181, 0.0181, 0.0181, 0.0182, 0.0183, 0.0183, 0.0184, 0.0185, 0.0186, 0.0186, 0.0203, 0.0206, 0.0208, 0.0221, 0.0223, 0.0227, 0.0231, 0.0231, 0.0252, 0.0253, 0.0256, 0.0269, 0.0276, 0.0280, 0.0299, 0.0300, 0.0318, 0.0320, 0.0321, 0.0327.

We conclude that all the models used are suitable for these data, as the P-value is greater than 0.05, and we can look at the following illustration in two Figs. 6 and 7 that makes this conclusion. In addition,
we conclude that, when compared to other models, the BTLE is an appropriate model for these data. Also, the estimated PDF with histogram and estimated CDF with empirical CDF plots of BTLE distribution are shown in Fig. 4 and PP-Plot and QQ-Plot of BTLE distribution in Fig. 5.
Additionally, we used the total time on test (TTT) plot (Aarset [44]) to determine the shape of the empirical hazard rare function. The TTT plots of the data I set is given in Fig. 8. Also Fig. 8 show box plot of data I set and conclude these do not have outliers. We also note from the third Fig. 8, that the estimated hazard of BTLE distribution with empirical hazard is increasing, and since the estimated hazard of BTLE distribution with empirical hazard changes almost steadily in the last period, this is due to the inactivity of the virus in this period as it was already.

The data II represents 55 days of COVID-19 data of Saudi Arabia, from July 11 to September 3 of 2021 [https://covid19.who.int/]. The drought mortality rate was calculated. The following are the data: 0.0325, 0.0264, 0.0324, 0.0282, 0.0282, 0.0301, 0.0260, 0.0300, 0.0239, 0.0278, 0.0278, 0.0238, 0.0296, 0.0217, 0.0275, 0.0235, 0.0235, 0.0195, 0.0214, 0.0233, 0.0271, 0.0213, 0.0231, 0.0192, 0.0211, 0.0268, 0.0249, 0.0267, 0.0172, 0.0267, 0.0209, 0.0228, 0.0171, 0.0227, 0.0189, 0.0208, 0.0245, 0.0132, 0.0226, 0.0150,
From Table 10, it is noted that the BTLE model provides the best fit among all competitive distributions because it has the smallest value of \( \chi^2 \). The empirical PDFs, CDFs, QQ plots, PP plots, TTT plots and box plot for data II are displayed in Figs. 11, 12, 13, 14 and 15 respectively. Tables 9 and 10 represent MaxLE and Bayesian estimation based on symmetric loss function. Figs. 9, 10, 16 and 17 show scatter plot matrices and convergence plots of MCMC for parameter estimates of BTLE distribution in the first and second data.

### Summary and conclusions

This article proposes the Beta Topp Leone G family as a generalization of the Beta Topp Leone distribution. Quantile function, moments of residual and reversed residual life, moment generation, incomplete moments, and Entropy are some statistical features of the BTL-G Family. This study proposes a novel generalization of exponential, Lomax, and Rayleigh distributions based on the BTL-G Family. Estimation techniques such as Bayesian and ML are discussed. Under asymmetric and symmetric loss function (LINEX and SE), the Bayesian estimator is derived. The numerical analysis is used for the BTLE distribution, which is a subset of the BTL-G Family. The goal of a Monte Carlo simulation study is to see how well estimations perform. In the vast majority of circumstances, we find that Bayesian estimates outperform identical alternative estimates. Two genuine COVID-19 data sets were collected from Saudi Arabia at separate dates, and they demonstrated that the BTLE distribution is an appropriate model for this data when compared to other competing distributions.

### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

### Data availability

The data used to support the findings of this study are included in this paper.

### Acknowledgment

This research was supported by the Deanship of Scientific Research, Imam Mohammad Ibn Saud Islamic University (IMSIU), Saudi Arabia, Grant No. (21-13-18-066).

### References

[1] Marshall, O. \( a \) new method for adding a parameter to a family of distributions with applications to the exponential and Weibull families. Biometrika 1997;84:61–5.

[2] Eugene N, Lee C, Famoye F. Beta-normal distribution and its applications. Comm Stat Theory Methods 2002;31:497–512.

[3] Rayh H, Alizadeh M, Jamal F, Othman S. The Topp Leone odd Lindley-G family of distributions: Properties and applications. J Stat Manag Syst 2018:21(7):1273–97.

[4] Rezaei S, Sadr B, Alizadeh M, Nadarajah S. Topp-Leone generated family of distributions: Properties and applications. Commun Stat - Theory Methods 2017:46(6):2899–909.

[5] Brito E, Cordeiro GM, Yousof HM, Alizadeh M, Silva GO. Topp-Leone odd log-logistic family of distributions. J Stat Comput Simul 2017:87:3040–58.

[6] Yousof H, Rasekh M, Alizadeh M. The extended odd Fréchet family of distributions: properties, applications and regression modeling. Int J Math Comput 2019:30:1–16.

[7] Ullah AM, Elgarhy M. The odd Fréchet-G family of probability distributions. J Appl Stat Model 2018:7(1):185–201.

[8] Alzahrani MA, Jamal F, Cheema C, Elbatal I, Elgarhy M. Exponentiated power generalized Weibull power series family of distributions: Properties, estimation and applications. PLoS One 2020:15(3):e0230004, 1–25.

[9] Ahmad Z, Elgarhy M, Hamdani GG, Butt NS. Odd generalized NH generated family of distributions with application to exponential model. Pak J Stat Oper Res 2020:1653–71.

[10] Topp CW, Leone FC. A family of J-shaped frequency functions. J Amer Statist Assoc 1955:50(269):209–19.

[11] Ghitany ME, Kotz S, Xie M. On some reliability measures and their stochastic orderings for the Topp-Leone distribution. J Appl Stat 2005:32(7):715–22.

[12] Kotz S, Seier E. Kurtosis of the Topp-Leone distributions. Interstat 2007;1:1–15.

[13] Genc AI. Moments of order statistics of Topp–Leone distribution. Statist Papers 2013:54:117–39.

[14] Genc AI. Estimation of \( P(X > Y) \) with Topp-Leone distribution. J Stat Comput Simul 2013:83(3):326–39.

[15] Al-Shomrani A, Afr A, Shawky A, Hanif S, Shahbaz MQ. Topp-Leone family of distributions: some properties and application. Pak J Stat Oper Res 2016:12(3):443–51.

[16] Atangana A, Arat SI. Modeling third waves of Covid-19 spread with piecewise differential and integral operators: Turkey, Spain and Czechia. 2021, http://dx.doi.org/10.1101/2021.05.20.21257515, medRxiv.

[17] Atangana A, Arat SI. Modeling and forecasting the spread of COVID-19 with stochastic and deterministic approaches: Africa and europe. Adv Difference Eq 2021:2021:1(1):1–10.

[18] Shafq A, Lone SA, Siddhu TN, Khattab Y, Al-Mdallal QM, Muhammad T. A new modified Kies Fréchet distribution: Applications of mortality rate of Covid-19. Results Phys 2021:28:104638.

[19] Atangana A. A novel Covid-19 model with fractional differential operators with singular and non-singular kernels: Analysis and numerical scheme based on Newton polynomial. Alex Eng J 2021:60(4):3781–806.

[20] Hassan AM, Almetwally EM, Ibrahim GM. Kumaramasivam inverted Topp-Leone distribution with applications to COVID-19 data. Comput Mater Contin 2021:58(1):337–58.

[21] Ibrahim GM, Hassan AS, Almetwally EM, Almogny HM. Parameter estimation of alpha power inverted Topp-Leone distribution with application. Intell Autom Soft Comput 2021(2):353–71.

[22] Siddhu TN, Shafq A, Al-Mdallal QM. On the analysis of number of deaths due to Covid-19 outbreak data using a new class of distributions. Results Phys 2021:21:103747.
[23] Sindhu TN, Shafiq A, Al-Mdallal QM. Exponentiated transformation of Gumbel type-II distribution for modeling COVID-19 data. Alex Eng J 2021;60(1):671–89.

[24] Sindhu TN, Aslama M, Shafiq A. Analysis of the left censored data from the Pareto type II distribution. Casp J Appl Sci Res 2013;2(7):53–62.

[25] Kenney JF, Keeping ES. Mathematics of Statistics, Part I. third ed. New Jersey: Princeton; 1962.

[26] Moors JJ. A quantile alternative for kurtosis. J R Stat Soc D 1988;37:25–32.

[27] Sindhu TN, Atangana A. Reliability analysis incorporating exponentiated inverse Weibull distribution and inverse power law. Qual Reliab Eng Int 2021. http://dx.doi.org/10.1002/qre.2864.

[28] Almongy HM, Alshenawy FY, Almetwally EM, Abdo DA. Applying transformer insulation using Weibull extended distribution based on progressive censoring scheme. Axioms 2021;10(2):100.

[29] Almetwally EM, Sabry MA, Alharbi R, Alnagar D, Mubarak SA, Hafez EH. Marshall Olkin alpha power Weibull distribution: Different methods of estimation based on type-I and type-II censoring. Complexity 2021;2021.

[30] Almetwally EM, Sabry MA, Alharbi R, Alnagar D, Mubarak SA, Hafez EH. Marshall–Olkin alpha power Weibull distribution: Different methods of estimation based on type-I and type-II censoring. Complexity 2021;2021. http://dx.doi.org/10.1007/s40745-021-00329-w.

[31] Sindhu TN, Saleem M, Aslam M. Bayesian estimation for Topp Leone distribution under trimmed samples. J Basic Appl Sci Res 2013;3(10):347–60.

[32] Sindhu TN, Khan HM, Hussain Z, Al-Zahrani R. Bayesian inference from the mixture of half-normal distributions under censoring. J Natl Sci Found Sri Lanka 2018;46(4):587–600.

[33] Sindhu TN, Hussain Z, Aslam M. On the Bayesian analysis of censored mixture of two Topp-Leone distribution. Sri Lankan J Appl Stat 2019;19(1).

[34] Sindhu TN, Hussain Z. Mixture of two generalized inverted exponential distributions with censored sample: properties and estimation. Stat Appl–Italian J Appl Stat 2018;30(3):373–91.

[35] Abd El-Raheem AM, Almetwally EM, Mohamed MS, Hafez EH. Accelerated life tests for modified Kies exponential lifetime distribution: binomial removal, transformers turn insulation application and numerical results. AIMS Math 2021;6(5):5222–55.

[36] Cordeiro GM, Ortega EM, Nadarajah S. The Kumaraswamy Weibull distribution with application to failure data. J Franklin Inst B 2010;347(8):1399–429.

[37] Hassan AS, Almetwally EM, Ibrahim GM. Kumaraswamy inverted Topp Leone distribution with applications to COVID-19 data. Comput Mater Continua 2021;68(1):337–58.

[38] Almetwally EM. The odd Weibull inverse Topp Leone distribution with applications to COVID-19 data. Ann Data Sci 2021;1–20. http://dx.doi.org/10.1007/s40745-021-00329-w.

[39] Alzeley O, Almetwally EM, Gemeay AM, Alshaabari HM, Hafez EH, Abu-Mousa MH. Statistical inference under censored data for the new exponential-X Fréchet distribution: Simulation and application to leukemia data. Comput Intell Neurosci 2021;2021.

[40] Tahir MI, Cordeiro GM, Mansoor M, Ziaa M. The Weibull-Lomax distribution: properties and applications. Hacet J Math Stat 2015;44(2):461–80.

[41] Almetwally EM, Alharbi R, Alnagar D, Hafez EH. A new inverted topp-leone distribution: applications to the COVID-19 mortality rate in two different countries. Axioms 2021;10(1):25.

[42] Abu El Azm WS, Almetwally EM, AL-Aziz SN, El-Bagoury AH, Alharbi R, Abo-Kasem OE. A new transmuted generalized lomax distribution: Properties and applications to COVID-19 data. Comput Intell Neurosci 2021;2021. http://dx.doi.org/10.1155/2021/5918511.

[43] Almongy HM, Almetwally EM, Aljohani HM, Alghamdi AS, Hafez EH. A new extended Rayleigh distribution with applications of COVID-19 data. Results Phys 2021;23:104012.

[44] Aarset MV. How to identify a bathtub hazard rate. IEEE Trans Reliabil 1987;36:106–8.