Three-dimensional Trajectory Tracking Guidance Law Based on Linear Quadratic Regulator

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Abstract. A three-dimensional trajectory tracking guidance law is designed based on the theory of linear quadratic regulator (LQR), which can help to solve the problem of long range trajectory tracking for air-defense missile in high altitude. Firstly, the missile mass model is linearized by using the coordinate variable of X axial. Secondly, the guidance parameters and mathematical model are obtained by designing state variables and control variables. At last, the three-dimensional trajectory tracking guidance law were proposed with theory of LQR. When the trajectory deviation is given, the simulation results show that the guidance law designed in this paper can suppress the disturbance and track ideal trajectory accurately.

Keywords: Three-dimensional trajectory; Trajectory tracking; LQR; Guidance law

1. Introduction
A two-dimensional trajectory tracking guidance law were designed based on the theory of LQR [1]. The literature [2] studied model of hypersonic vehicle and attitude control based on feedback linearization method. A roll angle control method based on LQR control algorithm was put forward for controlling roll angle on the two-dimensional course correction [3]. The trajectory tracking device was designed by LQR, which was used to track the optimal trajectory based on the theory of Gauss pseudospectral [4]. A trajectory tracking guidance law was proposed based on the theory of sliding mode variable structure control, which can help to solve the problem of trajectory tracking for air-defense missile [5]. An aircraft trajectory tracking guidance law were designed by nonlinear spatial inversion [6]. The literatures [7-9] established a nonlinear inverse control law based on distance to land to make the aircraft follow accurately a vertical profile and a desired airspeed. The literature [10] established the control law by using nonlinear dynamic inversion to make the aircraft accurately follow 3D+T desired trajectories. The literature [11] tackled the nonlinear control problem of aircraft trajectory tracking in the framework of multiple linear time-varying constrained control. The literature [12] discussed the aircraft trajectory tracking in three-dimensional space by six-degree-of-freedom nonlinear flight control scheme for unmanned aerial vehicles (UAVs). The literature [13] studied the problem of converting a trajectory tracking controller to a path tracking controller for a nonlinear non-minimum phase longitudinal aircraft model based on LQR. The literature [14] studied a flight path tracking problem for underactuated UAV in 3D space. A decoupling trajectory tracking method for gliding reentry vehicles was presented to improve the reliability of the guidance system in literature [15], and compared with the linear quadratic regulator (LQR) approach in some specific conditions.

The trajectory deviations of the tracking guidance law are basically tens of meters, up to several hundred meters. In this paper, long range tracking of air-defense missile and initial error up to 1 km are considered. The missile mass model is linearized by using the coordinate variable of X axial. A three-dimensional trajectory tracking guidance law is designed based on the theory of linear quadratic
regulator (LQR) by designing normal overload and lateral overload as control variables and selecting \(y, z, \theta, \psi\) as state variables. A simulation example is given to show the effectiveness of the law at last.

2. Mathematical model of missile

Considering a missile as a particle, the mathematical model is as follows:

\[
\begin{align*}
mv &= P \cos \alpha \cos \beta - C_s qS - mg \sin \theta \quad (1-a) \\
\dot{mv} - \dot{v} &= P \sin \alpha + C_s qS - mg \cos \theta \quad (1-b) \\
-mv \cos \theta \psi &= -P \cos \alpha \sin \beta + C_z qS \quad (1-c) \\
\dot{x} &= v \cos \theta \cos \psi \quad (1-d) \\
\dot{y} &= v \sin \theta \quad (1-e) \\
\dot{z} &= -v \cos \theta \sin \psi \quad (1-f)
\end{align*}
\]

Here, \(m\) and \(S\) are quality and reference area of missile, respectively. \(v\) is missile velocity. \(P\) represents engine thrust. \(\theta\) and \(\psi\) are tilt angle and deflection angle of trajectory, respectively. \(q\) represents dynamic pressure calculated by \(q = \frac{1}{2} \rho v^2\). \(\rho\) is the atmospheric density calculated by \(\rho = \rho_0 e^{-\frac{h}{h_0}}\), \(\rho_0 = 1.225 \text{ kg/m}^3\), \(h_0 = 7254.3 \text{ m}\). \(h\), that is, \(y\) represents the altitude of missile. \(C_s, C_y, C_z\) are drag coefficient, lift coefficient and lateral force coefficient. \(g = 9.8 \text{ m/s}^2\).

3. Design of three-dimensional trajectory tracking guidance law

3.1. Description of optimal control for trajectory tracking

Suppose that a linear system can be represented by state vectors:

\[
\dot{x} = Ax + Bu
\]

A state feedback controller is designed to satisfy the desired performance of the system:

\[
\dot{x} = (A - BK)x = A_0x
\]

Performance index:

\[
J = \frac{1}{2} \int_0^T x^T Q x + u^T R u dt
\]

Here, \(Q\) and \(R\) are weight matrices, \(Q\) is positive semidefinite matrix, \(R\) is positive definite matrix. When \(t\) tends to infinity, \(x\) approaches 0, which guarantees the stability of the system. The minimum control variable \(u\) means the optimal control is obtained with the minimum control energy.

The optimal control variable is \(u = -Kx\), equation (4) can be converted into:

\[
J = \frac{1}{2} \int_0^T x^T (Q + K^T R K)x dt
\]

Suppose there is a constant matrix \(P\) so that equation (6) is established:

\[
\frac{d}{dt}(x^T P x) = -x^T (Q + K^T R K)x \Rightarrow \dot{x}^T P x + x^T P \dot{x} + x^T Q x + x^T K^T R K x = 0
\]

Combining equation (3) and equation (6):

\[
x^T A_0^T P x + x^T P A_0 x + x^T Q x + x^T K^T R K x = x^T ((A - BK)^T P + P(A - BK) + Q + K^T R K)x = 0
\]

\(K = R^{-1}B^T P\), equation (7) can be converted into:

\[
PA + A_0^T P x - PBR^{-1}B^T P + Q = 0
\]

Thus, the optimal control variable matrix is \(u = -Kx\). The optimal feedback gain matrix is \(K = R^{-1}B^T P\). \(P\) is the solution of equation (8).

3.2. Design of three-dimensional trajectory tracking guidance law
The three-dimensional trajectory tracking guidance law is designed to make parameters deviations (position, tilt angle and deflection angle of trajectory, etc.) between actual trajectory and ideal trajectory minimum by establishing the linearized state-space equation with the state variable of trajectory parameters deviation. The equation (5) is the performance index. The optimal feedback gain matrix is designed based on the theory of LQR, which can make the least trajectory deviation.

It is necessary to linearize the nonlinear mathematical model of missile to design trajectory tracking guidance law based on the theory of LQR. The speed of missile is uncontrollable. Combining equation (1), the missile mass model is linearized by using the coordinate variable of X axial.

\[
\frac{dy}{dx} = \tan \theta / \cos \psi \quad (9-a)
\]

\[
\frac{dz}{dx} = -\tan \psi \quad (9-b)
\]

\[
\frac{d\theta}{dx} = \frac{P \cdot \sin \alpha + C_r qS - mg \cdot \cos \theta}{mv^2 \cdot \cos \theta \cdot \cos \psi} = \frac{gn_y}{v^2 \cdot \cos \theta \cdot \cos \psi} - \frac{g}{v^2 \cdot \cos \psi} \quad (9-c)
\]

\[
\frac{d\psi}{dx} = -\frac{P \cdot \cos \alpha \cdot \sin \beta + C_r qS}{mv^2 \cdot (\cos \theta)^2 \cdot \cos \psi} = -\frac{gn_y}{v^2 \cdot (\cos \theta)^2 \cdot \cos \psi} \quad (9-d)
\]

State variable matrix:

\[
x = [\Delta y \Delta z \Delta \theta \Delta \psi]^T = \begin{bmatrix} y - y_b \quad z - z_b \quad \theta - \theta_b \quad \psi - \psi_b \end{bmatrix}^T \quad (10)
\]

Subscript b represents the data of ideal trajectory.

The optimal control variable matrix is \( u = [\Delta n_y \Delta n_z]^T \). The linearized state-space equation is:

\[
\frac{d}{dt} \begin{bmatrix} \Delta y \Delta z \Delta \theta \Delta \psi \end{bmatrix}^T = A \begin{bmatrix} \Delta y \Delta z \Delta \theta \Delta \psi \end{bmatrix}^T + B \begin{bmatrix} \Delta n_y \Delta n_z \end{bmatrix}^T \quad (11)
\]

Performance index:

\[
J = \frac{1}{2} \int_0^T \left[ \Delta y \Delta z \Delta \theta \Delta \psi \right]^T Q \left[ \Delta y \Delta z \Delta \theta \Delta \psi \right]^T + \left[ \Delta n_y \Delta n_z \right]^T R \left[ \Delta n_y \Delta n_z \right]^T \, dt \quad (12)
\]

The optimal control variable matrix \( u = -Kx = -K \begin{bmatrix} \Delta y \Delta z \Delta \theta \Delta \psi \end{bmatrix}^T \) can make \( J \) minimum.

The optimal feedback gain matrix:

\[
K = R^{-1} B^T P \quad (13)
\]

\[
P A + A^T P - P B R^{-1} B^T P + Q = 0 \quad (14)
\]

Normal overload and lateral overload of actual trajectory required:

\[
\begin{bmatrix} \Delta n_y \Delta n_z \end{bmatrix}^T = -K \begin{bmatrix} \Delta y \Delta z \Delta \theta \Delta \psi \end{bmatrix}^T \quad (15-a)
\]
\[
\begin{bmatrix}
\Delta n_y \\
\Delta n_z
\end{bmatrix}^T = \begin{bmatrix}
\Delta n_y \\
\Delta n_z
\end{bmatrix}^T + \begin{bmatrix}
\Delta n_y \\
\Delta n_z
\end{bmatrix}^T
\]

(15-b)

3.3. Work Flow for guidance law
When $t$ tends to infinity, state variable matrix approaches zero matrix. Actual trajectory coincides with ideal trajectory and overloads are optimal. The flow chart and working procedure are as follows:

![Flow chart of guidance law](image)

Figure 1. Flow chart of guidance law.

(1) Combing equation (11), (14) and equation (13), the optimal feedback gain matrix is calculated.
(2) Combing equation (10), trajectory data is sampled and optimal control variable matrix is calculated.
(3) According to equation (15) and ideal trajectory, the missile is controlled to fly.

4. Simulation results and analysis
The initial conditions of simulation: $v_0 = 3000 \text{m/s}$, $\theta_0 = 15^\circ$, $\phi_0 = 60^\circ$, $m = 3000 \text{kg}$, $h = 0.001$. The ideal trajectory is 500km long. The initial height of missile is 26km, and the height of hitting target is 20km. Trajectory deviation is $\Delta y = 1000$. Set parameters to $R = \text{diag}[100 \ 100]$, $Q = \text{diag}[0.0007 \ 0.01 \ 1 \ 1]$.

![Trajectory curve](image)

Figure 2. Trajectory curve.

![Deviation curve of position](image)

Figure 3. Deviation curve of position.

![Deviation curve of angles](image)

Figure 4. Deviation curve of angles.

![Deviation curve of overload](image)

Figure 5. Deviation curve of overload.

Table 1. Statistical data of simulation results.
Simulation results analysis: Simulation results is shown in figure 2-5 and table 1. As can be seen, the guidance law can suppress the disturbance and track ideal trajectory in altitude, overloads, tilt angle and deflection angle of trajectory accurately. The actual curves of deviation vary smoothly. The last tilt angle and the ideal angle are no more than 0.001 deg. The last deflection angle and the ideal angle are no more than 0.02 deg. Deviations of normal overload and lateral overload approaches 0. The last position to the target is no more than 6 m, thus performs high precision.

5. Conclusions
Aiming at the problem of long range trajectory tracking for air-defense missile in high altitude, the missile mass model is linearized by using the coordinate variable of X axial. Trajectory deviation can be obtained by comparing trajectory data between actual trajectory and ideal trajectory. According to this, optimal control variable is calculated. Then the optimal control variable is added to the overload of the ideal trajectory to obtain the overload of the actual trajectory required. In the simulation process, the controller parameters of LQR are designed. When the trajectory deviation is given, the simulation results show that the three-dimensional trajectory tracking guidance law designed in this paper based on the theory of LQR can suppress the disturbance and track the reference trajectory accurately.

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