On an integrable system of $q$-difference equations satisfied by the universal characters: its Lax formalism and an application to $q$-Painlevé equations

Teruhisa TSUDA
Faculty of Mathematics, Kyushu University, Hakozaki, Fukuoka 812-8581, Japan.
tudateru@math.kyushu-u.ac.jp

Abstract

The universal character is a generalization of the Schur function attached to a pair of partitions. We study an integrable system of $q$-difference equations satisfied by the universal characters, which is an extension of the $q$-KP hierarchy and is called the lattice $q$-UC hierarchy. We describe the lattice $q$-UC hierarchy as a compatibility condition of its associated linear system (Lax formalism) and explore an application to the $q$-Painlevé equations via similarity reduction. In particular a higher-order analogue of the $q$-Painlevé VI equation is presented.

1 Introduction

The universal character, defined by Koike [7], is a polynomial attached to a pair of partitions (Young diagrams) and is a generalization of the Schur function. The universal character describes the irreducible rational character of the general linear group, while the Schur function, as is well known, does the irreducible polynomial character of the group. The Schur function is significant also for the study of integrable systems; in fact, Sato [9] showed that the KP hierarchy, an important class of nonlinear partial differential equations of soliton type, is exactly an infinite-dimensional integrable system characterized by the Schur function. Needless to say, the KP hierarchy provides a basic prototype in the field of integrable systems involving the Painlevé equations. Furthermore, variants of the KP hierarchy including difference and $q$-difference versions have been extensively studied as well as the original one.

On the other hand, in [10] the author proposed an extension of the KP hierarchy, called the UC hierarchy, as an infinite-dimensional integrable system characterized by the universal character. For instance, the whole set of homogeneous polynomial solutions of the UC hierarchy is in one-to-one correspondence with the set of the universal characters. Also in [12] a $q$-difference analogue of the hierarchy was studied in connection with the $q$-Painlevé equations; it is an integrable system of $q$-difference equations originated from certain quadratic relations among the universal characters and was named the lattice $q$-UC hierarchy ($q$-LUC), since whose dependent variables ($\tau$-functions) are arranged on a two-dimensional lattice $\mathbb{Z}^2$; cf. [11] [14]. However, some basic properties, e.g., Lax formalism, of $q$-LUC as an integrable system have been unclear so far.
The aim of this paper is to provide an alternative formulation of $q$-LUC and explore its application to the $q$-Painlevé equations. In Sect. [2] we first reformulate $q$-LUC as a compatibility condition of its auxiliary system of linear equations (Lax formalism). Based on the Lax formalism, we then describe $q$-LUC as a system of $q$-difference evolution equations for appropriate dependent variables, which naturally includes the $q$-KP hierarchy (see, e.g., [4]) as a special case. Next, in Sect. [3] we consider a reduction of $q$-LUC by requiring its solutions to satisfy certain homogeneity and periodicity (a similarity reduction). As a result, we obtain a class of invertible, or rather, birational discrete dynamical systems of $q$-Painlevé type. For example, we present a higher-order extension of the $q$-Painlevé VI equation in Sect. [3]. Observing the universal characters be consistent with the similarity reduction, we can immediately construct particular solutions of the dynamics in terms of the universal characters; cf. [12].

## 2 Universal characters and $q$-difference integrable systems

We begin with recalling the definition of the universal characters. The lattice $q$-UC hierarchy ($q$-LUC) takes the form of bilinear $q$-difference equations that arises from quadratic relations among the universal characters. In this section we present its associated Lax formalism; that is, we reformulate $q$-LUC as a compatibility condition of an auxiliary system of linear $q$-difference equations.

### 2.1 Universal character and lattice $q$-UC hierarchy

A partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a sequence of non-negative integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ and that $\lambda_i = 0$ for $i \gg 0$. The number of $\lambda_i = 0$ is called the length of $\lambda$. For a pair of partitions $\lambda, \mu$, the universal character $S_{[\lambda, \mu]}$ is a polynomial in (infinitely many) variables $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ and is defined by the twisted Jacobi-Trudi formula:

$$ S_{[\lambda, \mu]}(x, y) = \det \begin{pmatrix} p_{\mu l i + j} & p_{\mu l i - j} & \cdots & p_{\mu l i + j - l'} \\ p_{\lambda l i + j} & p_{\lambda l i - j} & \cdots & p_{\lambda l i + j - l'} \\ \vdots & \vdots & \ddots & \vdots \\ p_{\lambda l i + j - l + 1} & p_{\lambda l i - j - 1} & \cdots & p_{\lambda l i + j - l' + 1} \end{pmatrix}_{l \leq i \leq j \leq l'}, $$

(2.1)

with $l = l(\lambda)$ and $l' = l(\mu)$. Here $p_n(x)$ ($n \in \mathbb{Z}$) is determined by the generating function $\sum_{n=0}^{\infty} p_n(x) z^n = \exp \left( \sum_{n=1}^{\infty} x_n z^n \right)$ and $p_n = 0$ for $n < 0$. In the case where $\mu = 0$, we have

$$ S_{[\lambda, 0]}(x, y) = \det(p_{\lambda l - i - j}(x)) =: S_\lambda(x), $$

which is exactly the Schur function. If we count the degree of each variable as $\deg x_n = n$ and $\deg y_n = -n$, then $S_{[\lambda, \mu]}(x, y)$ is a homogeneous polynomial of degree $|\lambda| - |\mu|$, where $|\lambda| = \lambda_1 + \cdots + \lambda_i$. The universal character $S_{[\lambda, \mu]}$ describes the irreducible character of a rational representation of the general linear group corresponding to a pair of partitions $[\lambda, \mu]$, while the Schur function $S_\lambda$ does that of a polynomial representation corresponding to a partition $\lambda$; see [7] for details.

**Example 2.1.** We have $p_0 = 1$, $p_1(x) = x_1$, $p_2(x) = x_2 + x_1^2/2$, $p_3(x) = x_3 + x_2 x_1 + x_1^3/6$, and so on. When $\lambda = (2, 1)$ and $\mu = (1)$, the universal character reads

$$ S_{[2,1,1,1]}(x, y) = \begin{vmatrix} p_{1}(y) & p_{0}(y) & p_{-1}(y) \\ p_{1}(x) & p_{2}(x) & p_{3}(x) \\ p_{-1}(x) & p_{0}(x) & p_{1}(x) \end{vmatrix} = \left( \frac{x_1^3}{3} - x_3 \right) y_1 - x_1^2, $$

which is a homogeneous polynomial of degree $|\lambda| - |\mu| = 3 - 1 = 2$.  

Let $I \subset \mathbb{Z}_{>0}$ and $J \subset \mathbb{Z}_{<0}$ be finite indexing sets and $t_i (i \in I \cup J)$ the independent variables. Let $T_i = T_{i,q}$ be the $q$-shift operator defined by

$$T_{i,q}(t_i) = \begin{cases} \frac{q}{q^n-1} (i \in I) \\ q^{-1} t_i (i \in J) \end{cases} \quad \text{and} \quad T_{i,q}(t_j) = t_j (i \neq j).$$

For unknown functions $\tau_{m,n} = \tau_{m,n}(t)$ in $t_i$ arranged on the lattice $(m,n) \in \mathbb{Z}^2$, the following system of $q$-difference equations is called the lattice $q$-UC hierarchy ($q$-LUC):

$$t_i T_i(\tau_{m,n+1}) T_j(\tau_{m+1,n}) - t_j T_j(\tau_{m,n+1}) T_i(\tau_{m+1,n}) = (t_i - t_j) T_i T_j(\tau_{m,n}) \tau_{m+1,n+1} \quad (2.2)$$

where $i, j \in I \cup J$.

The $q$-LUC was introduced in [12] and it is, as seen below, originated from the quadratic relations satisfied by the universal characters. For convenience, we shall extend the universal procedure like (2.1), to be defined for a pair of arbitrary sequences of integers $[\lambda, \mu]$. Note that one can associate a unique partition $\hat{\lambda}$ with any given sequence of integers $\lambda$ by applying successively a procedure like $(\ldots, k, \ldots) \mapsto (\ldots, l-1, k+1, \ldots)$; we have $S_{[\lambda,\mu]} = \pm S_{[\lambda,\mu]} = \pm S_{[\lambda,\mu]}$. Define the function $s_{[\lambda,\mu]} = s_{[\lambda,\mu]}(t)$ by

$$s_{[\lambda,\mu]}(t) = S_{[\lambda,\mu]}(x,y)$$

via the change of variables

$$x_n = \frac{\sum_{i \in J} t_i^n - q^n \sum_{j \in J} t_j^n}{n(1-q^n)} \quad \text{and} \quad y_n = \frac{\sum_{i \in J} t_i^n - q^n \sum_{j \in J} t_j^n}{n(1-q^n)}. \quad (2.3)$$

**Proposition 2.2 ([12]).** We have

$$t_i T_i(s_{[\lambda,(k',\mu)]}) T_j(s_{[\lambda,(k,\mu)]}) - t_j T_j(s_{[\lambda,(k',\mu)]}) T_i(s_{[\lambda,(k,\mu)]}) = (t_i - t_j) T_i T_j(s_{[\lambda,\mu]}) s_{[\lambda,(k',\mu)]}(t)$$

for any integers $k, k'$ and partitions $\lambda, \mu$.

**Remark 2.3 (Symmetry of $q$-LUC).** If a set of functions $\{\tau_{m,n}(t)\}_{m,n}$ is a solution of $q$-LUC, $(2.2)$, so is $\{c^{-d_{m,n}} \tau_{m,n}(ct)\}_{m,n}$ for arbitrary constants $c \in \mathbb{C}$ and $d_{m,n} \in \mathbb{C}$ with $d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$. Accordingly it seems reasonable to take a particular interest in the fixed solutions with respect to such a scaling symmetry. For example, the universal character $s_{[\lambda,\mu]}$ gives a homogeneous solution of $q$-LUC as seen from Prop. 2.2, and by definition it satisfies $s_{[\lambda,\mu]}(ct) = c^{\lambda - \mu} s_{[\lambda,\mu]}(t)$. Such self-similar solutions will be investigated in Sect. 3 below.

**Remark 2.4.** Suppose $|I| + |J| \geq 3$. Then it can be verified from (2.2) as a necessary condition that

$$\begin{align*}
(t_i - t_j) T_i T_j(T_{k,m}T_{k+1,n}) + (t_j - t_k) T_j T_k(T_{k,m}T_{k+1,n}) \\
+ (t_k - t_i) T_k T_i(T_{k,m}T_{k+1,n}) = 0
\end{align*} \quad (2.4)$$

for any $i, j, k \in I \cup J$. This, $(2.4)$, is nothing but the bilinear equation introduced previously in [11] and named the $q$-UC hierarchy (though this terminology is a little confusing).
2.2 Lax formalism

In order to derive the associated linear system from the bilinear equation (2.2) of q-LUC, we need to prepare an extra variable $z$ called the spectral parameter. Through the change of variables (2.3), we regard $\tau_{m,n}(t)$ as a function in $(x, y)$ and write $\tau_{m,n}(t) = \bar{\tau}_{m,n}(x, y)$. Let us now introduce the wave function $\psi_{m,n} = \psi_{m,n}(t, z)$, a function in $z$ and $t_i (i \in I \cup J)$, defined by

$$\psi_{m,n} = \frac{1}{(-z)^m} \prod_{i \in I} (z^{-1} t_i; q^2) \prod_{j \in J} (z^{-1} t_j; q^{-1}) \bar{\tau}_{m,n}(x, y),$$

where we adopt the following convention of $q$-shifted factorial:

$$(a; q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i),$$

and $[z] = (z, z^2/2, z^3/3, \ldots)$. In view of (2.2), we see that the wave function solves the linear system of $q$-difference equations:

$$T_i(\psi_{m,n}) = u_{i,m,n} \psi_{m,n+1} + t_i \psi_{m+1,n} \quad (i \in I \cup J)$$

(2.6)

where

$$u_{i,m,n} = \frac{T_i(\tau_{m+1,n}) \tau_{m,n+2}}{T_i(\tau_{m+1,n}) \tau_{m+1,n+1}}.$$

(2.7)

**Proposition 2.5.** The compatibility condition $T_i T_j = T_j T_i$ of (2.6) yields

$$T_j(u_{i,m,n}) = u_{i,m,n+1} \frac{t_i u_{j,m+1,n} - t_j u_{i,m+1,n}}{t_i u_{j,m,n+1} - t_j u_{i,m,n+1}} \quad (i \neq j),$$

(2.8)

the nonlinear evolution equations for variables $u_{i,m,n} = u_{i,m,n}(t)$.

**Proof.** We will write the $q$-shift of a function $F = F(t)$ as $T_i T_j \cdots T_j(F) =: F^{(i_1, i_2, \ldots, i_r)}$ for brevity. Applying $T_j$ to (2.6), we have

$$T_j T_i(\psi_{m,n}) = u_{i,m,n}^{(j)} \psi_{m,n+1}^{(j)} + t_i \psi_{m+1,n}^{(j)}$$

$$= u_{i,m,n}^{(j)} (u_{j,m,n+1} \psi_{m,n+2} + t_j \psi_{m+1,n+1}) + t_i (u_{j,m+1,n} \psi_{m+1,n+1} + t_j \psi_{m+2,n})$$

$$= u_{i,m,n}^{(j)} u_{j,m,n+1} \psi_{m,n+2} + (t_j u_{i,m,n}^{(j)} + t_i u_{j,m,n+1}) \psi_{m+1,n+1} + t_j \psi_{m+2,n}.$$

Hence the compatibility condition $T_j T_i = T_i T_j$ is equivalent to

$$u_{i,m,n}^{(j)} u_{j,m,n+1} = u_{i,m,n}^{(i)} u_{i,m,n+1},$$

(2.9a)

$$t_j u_{i,m,n}^{(j)} + t_i u_{j,m,n+1} = t_j u_{i,m,n}^{(i)} + t_i u_{j,m,n+1}.$$  

(2.9b)

Accordingly (2.8) holds.

Conversely, we shall deduce the bilinear form (2.2) of q-LUC from the compatibility condition (2.9) of the linear system (2.6). Taking an auxiliary variable

$$w_{m,n} = \frac{\tau_{m+1,n}}{\tau_{m,n+1}},$$

(2.10)
we can then write (see (2.7))
\[ u_{i,m,n} = \frac{T_i(w_{m,n})}{w_{m,n+1}}. \]

At the level of variables \( w_{m,n} \) the first condition (2.9a) reduces to trivial and the second (2.9b) becomes
\[ w_{m,n}^{(i,j)} = \frac{W_{m,n+1}^{(i,j)} - t_j W_{m+1,n}^{(i,j)}}{W_{m+1,n+1} - t_j W_{m+1,n+1}}. \tag{2.11} \]

Substituting (2.10) into (2.11), we obtain
\[ \frac{t_i \tau_{m+1,n+1}^{(i,j)} - t_j \tau_{m+1,n+1}^{(i,j)}}{\tau_{m+1,n+1}^{(i,j)}} = (\text{LHS})_{(m,n)\rightarrow(m-1,n+1)} \cdot \]

Consequently the above formula can be decomposed as
\[ t_i \tau_{m+1,n+1}^{(i,j)} - t_j \tau_{m+1,n+1}^{(i,j)} = \alpha(m,n) \tau_{m+1,n+1}^{(i,j)}, \]
where \( \alpha(m,n) \) being an arbitrary function such that \( \alpha(m - 1, n + 1) = \alpha(m, n) \). Assume \( \tau_{m,n} \equiv 1 \) (for \( \forall m, n \)) to be a solution. Then we get \( \alpha(m, n) = t_i - t_j \) and arrive at (2.2) as desired.

**Remark 2.6 (Comparison with \( q \)-KP hierarchy).** As predicted from the fact that the universal character is a generalization of the Schur function, the lattice \( q \)-UC hierarchy gives a natural extension of the \( q \)-KP hierarchy. Consider the case where \( \tau_{m,n}(t) \) does not depend on \( n \); thus, \( \psi_{m,n} \) and \( u_{i,m,n} \) do not also. Rename the dependent variables as
\[ \rho_m := \tau_{m,n}, \quad \phi_m := \psi_{m,n} \quad \text{and} \quad v_{i,m} := \frac{T_i(\rho_{m+1})\rho_m}{T_i(\rho_{m+1})\rho_{m+1}} = u_{i,m,n} \tag{2.12} \]
for avoiding confusion. Then (2.2), (2.6) and (2.8) reduce respectively to
\[ t_i T_i(\rho_m) T_j(\rho_{m+1}) - t_j T_j(\rho_m) T_i(\rho_{m+1}) = (t_i - t_j) T_i T_j(\rho_m) \rho_{m+1}, \tag{2.13} \]
\[ T_i(\phi_m) = v_{i,m} \phi_m + t_i \phi_{m+1}, \tag{2.14} \]
and
\[ T_j(v_{i,m}) = v_{i,m} t_j v_{j,m+1} - t_j v_{i,m+1}, \tag{2.15} \]
the bilinear form, the associated linear system, and the nonlinear expression of the \( q \)-KP hierarchy; cf. [4].

**Remark 2.7 (Lax matrices of \( q \)-LUC with \((M,N)\)-periodicity).** Suppose that the \((M,N)\)-periodic condition: \( \tau_{m,n} = \tau_{m+M,n} = \tau_{m,n+N} \) holds. Take an \( MN \)-vector
\[ \Psi = \left( \psi_{1,1}, \psi_{2,1}, \ldots, \psi_{M,1}, \psi_{1,2}, \psi_{2,2}, \ldots, \psi_{M,2}, \ldots, \psi_{1,N}, \psi_{2,N}, \ldots, \psi_{M,N} \right). \]

The linear equation (2.6) can therefore rewritten into the matrix equation:
\[ T_i(\Psi) = B_i \Psi, \tag{2.16} \]
where

\[
B_i = \begin{pmatrix}
O_{M(N-1)\times M} & u_{i,1,1} & u_{i,2,1} & \cdots & u_{i,M-1,1} \\
u_{i,1,N} & u_{i,2,N} & \cdots & \cdots & u_{i,M,N-1} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
u_{i,M,N} & \cdots & \cdots & \cdots & O_{M\times M(N-1)}
\end{pmatrix} + t_i \begin{pmatrix}
\Lambda \\
\cdots \\
\cdots \\
\cdots \\
\Lambda
\end{pmatrix}
\]

and

\[
\Lambda = \begin{pmatrix}
0 & 1 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
(-z)^{-M} & 1 & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots 
\end{pmatrix} M.
\]

The compatibility condition of (2.16) is expressed as

\[
T_i(B_j)B_i = T_j(B_i)B_j. 	ag{2.17}
\]

If \(N = 1\), the situation above reduces to that of the \(q\)-KP hierarchy again.

### 3 Associated birational dynamics and \(q\)-Painlevé equations

While the equation (2.8) given in Prop. 2.5

\[
T_j(u_{i,m,n}) = u_{i,m,n+1} \frac{t_j u_{i,m+1,n} - t_j u_{i,m+1,n+1}}{t_j u_{i,m,n+1} - t_j u_{i,m,n+1}} \quad (i \neq j),
\]

describes a time evolution (\(q\)-shift) of the variable \(u_{i,m,n}\) with respect to \(t_j\), there are some problems:

(i) the time evolution for \(i = j\) is undefined;

(ii) the inverse transformation \(T_i^{-1}\) is also undefined.

In this section we demonstrate how to settle these problems by means of the view point of similarity reduction. To be accurate, if we impose some homogeneity and periodicity on the dependent variables (appropriately chosen), we can describe the time evolution in terms of invertible or rather birational mappings. Interestingly enough, the resulting discrete dynamical systems give rise to \(q\)-difference Painlevé equations.

#### 3.1 The case of \(q\)-KP hierarchy with \(J = \emptyset\)

As a prototypical example, we first review the result [4] in the case of the \(q\)-KP hierarchy (\(q\)-KP). We will continue on the convention used in Remark 2.6.

Let \(I = \{1, 2, \ldots, L\}\) and \(J = \emptyset\). Impose on the variables \(\rho_m = \rho_m(t)\) the \(M\)-periodic condition: \(\rho_{m+M} = \rho_m\) and the homogeneity condition: \(\rho_m(qt) = q^{dm} \rho_m(t) \quad (d_m \in \mathbb{C})\). Concerning the variables \(v_{i,m}\), the constraint above implies that

\[
v_{i,m+M} = v_{i,m} \quad \text{and} \quad T_1 T_2 \cdots T_L(v_{i,m}) = v_{i,m}.
\]
In parallel, the wave function $\phi_m = \phi_m(t, z)$ comes to satisfy $(-z)^M \phi_{m+M} = \phi_m$ and $q^n \phi_m(qt, qz) = \phi_m(t, z)$. Now take the dependent variables

$$x_{i,m} = x_{i,m}(t) := t_i^{-1} T_{i+1} T_{i+2} \cdots T_L(v_{i,m}) \quad (i \in I, \ m \in \mathbb{Z}/M \mathbb{Z})$$

(3.1)

and let $\mathbb{C}(x)$ be the field of rational functions in variables $x_{i,m}$.

**Theorem 3.1** (Kajiwara–Noumi–Yamada [4]). The action of $T_j$ on variables $x_{i,m}$ is given in terms of birational transformations, that is, $T_j(x_{i,m}), T_j^{-1}(x_{i,m}) \in \mathbb{C}(x)$ for any $i, j, m$.

We shall illustrate this theorem with $I = \{1, 2\}$ case. Write

$$x_m := x_{1,m} = t_1^{-1} T_2(v_{1,m}), \quad y_m := x_{2,m} = t_2^{-1} v_{2,m}, \quad x'_m := t_2^{-1} T_1(v_{2,m}), \quad y'_m := t_1^{-1} v_{1,m}.$$ 

The compatibility condition of (2.14) yields (cf. (2.9))

$$x_m y_m = x'_m y'_m \quad \text{and} \quad x_m + y_{m+1} = x'_m + y'_{m+1} \quad (m \in \mathbb{Z}/M \mathbb{Z})$$

which provide a birational mapping $r : (x, y) \to (x', y')$ of the form:

$$x'_m = y_m \frac{P_{m-1}}{P_m}, \quad y'_m = x_m \frac{P_m}{P_{m-1}},$$

where $P_m$ is a polynomial in $(x, y)$ given as

$$P_m(x, y) = \sum_{a=1}^M \prod_{i=1}^{a-1} x_{i+m} \prod_{i=a+1}^M y_{i+m}.$$ 

Note that in the above formula the suffixes $i$ of $x_i$ and $y_i$ are regarded as elements of $\mathbb{Z}/M \mathbb{Z}$, namely, $x_{i+M} = x_i$ etc. In addition, we introduce a permutation $\pi : x_m \leftrightarrow y_m$ of variables. Apply $r \circ \pi$ to variables $x_m$ and $y_m$. Therefore we can verify that $r \circ \pi(x_m) = (y_m) = y'_m = T_2^{-1}(x_m) = qT_1(x_m)$ and $r \circ \pi(y_m) = r(x_m) = x'_m = T_1(y_m)$. Here notice that we have taken into account the similarity condition $T_1 T_2(v_{i,m}) = v_{i,m} (i = 1, 2)$. Summarizing above, the birational action of $T_1 : \mathbb{C}(x, y) \to \mathbb{C}(x, y)$ is given as follows:

$$T_1(x_m) = q^{-1} x_m \frac{P_m}{P_{m-1}}, \quad T_1(y_m) = y_m \frac{P_{m-1}}{P_m} \quad (m \in \mathbb{Z}/M \mathbb{Z})$$

(3.2)

This discrete dynamical system looks $2M$-dimensional. However, (i) if $M$ is odd, it possesses $M + 1$ conserved quantities $x_m y_m (1 \leq m \leq M)$ and $\prod_{i=1}^M x_i$; (ii) if $M$ is even, it does $M + 2$ ones $x_m y_m, \prod_{i=1}^M x_i$ and $\prod_{i=1}^{M/2} x_{2j} y_{2j-1}$. Hence the dimension of the dynamics (3.2) is essentially $M + 1$ (resp. $M + 2$) if $M$ is odd (resp. even). As is known, (3.2) provides a $q$-analogue of the higher order Painlevé equation of type $A_{M-1}^{(1)}$ [8] or the $M$-periodic closing of the Darboux chain [1] [15]. In this paper we call (3.2) the $q$-Painlevé equation of type $A_{M-1}^{(1)}$, denoted by $qP(A_{M-1}^{(1)})$. Note that $qP(A_{2}^{(1)})$ and $qP(A_{3}^{(1)})$, which are of two dimensions, correspond to the fourth and fifth Painlevé equations, respectively.

**Remark 3.2.** About Theorem 3.1 If $I = \{1, 2, \ldots, L\}$, $J = \emptyset$ and the $M$-periodic condition is imposed, the birational action of $T_i$ is generally governed by an affine Weyl group of type $A_{L-1}^{(1)} \times A_{M-1}^{(1)}$; see [3] and also [6]. Note that the resulting discrete time flows $T_i (1 \leq i \leq L)$ define a multi-variable version of a discrete Painlevé equation which is called the $q$-Painlevé system of type $(L, M)$; see [4] Sect. 3 for its explicit description. It is still an interesting open question how to control, by the use of some Weyl groups, the time evolutions $T_i$ of $q$-KP (not to mention that of $q$-LUC) in the case where $J \neq \emptyset$. 

7
3.2 From $q$-KP/UC hierarchy to $q$-Painlevé equations: an overview

As already mentioned, a certain homogeneity and periodic constraint of $q$-KP gives rise to a class of birational dynamical systems of $q$-Painlevé type. Analogously, it is known that $q$-LUC also admits a similar type of reductions to some $q$-Painlevé equations. Let us summarize the known results about how $q$-KP or $q$-LUC corresponds to the $q$-Painlevé equations.

Recall that $q$-KP possesses the following data: $M$ and $(|I|, |J|)$, where $M$ ($1 \leq M \leq \infty$) is the order of periodicity on the dependent variables, i.e., $\rho_{m+M} = \rho_m$ and $(|I|, |J|)$ specifies the set $\{t_i \ (i \in I \cup J)\}$ of time variables. The homogeneity constraint condition reads

$$\prod_{i \in I} T_i \prod_{j \in J} T_j^{-1}(\rho_m) = d_m\rho_m \quad (d_m \in \mathbb{C}).$$

As explained in Sect. 3.1, if we choose $I = \{1, 2\}$ and $J = \emptyset$ and $M \geq 3$ (general), then the time evolution of $T_1$ (or $T_2$) of $q$-KP yields the $q$-Painlevé equation of type $A_{M-1}^{(1)}$, denoted by $q$-$P(A_{M-1}^{(1)})$, via the homogeneity constraint (3.3). Also, it is known [13] that a $q$-analogue of the third Painlevé equation, $q$-$P_{III}$, can be derived from $q$-KP with $(|I|, |J|) = (3, 0)$ and two-periodicity. We sum up in Table 1 below the corresponding data of $q$-KP to each $q$-Painlevé equation:

| $M$ : periodicity | $(|I|, |J|)$ | $T$ : time evolution of $q$-Painlevé equation | Ref. |
|------------------|------------|---------------------------------------------|-----|
| $M \geq 3$      | (2, 0)     | $T = T_1 : q$-$P(A_{M-1}^{(1)})$; $M = 3 \Rightarrow q$-$P_{IV}$; $M = 4 \Rightarrow q$-$P_{V}$ | [4] |
| 2                | (3, 0)     | $T = T_1 : q$-$P_{III}$                     | [13]|

Likewise, $q$-LUC has the data: $(M, N)$ and $(|I|, |J|)$, where $M$ and $N$ ($1 \leq M, N \leq \infty$) represent the period, i.e., $\tau_{m+M,n} = \tau_{m,n+N} = \tau_{m,n}$ and $(|I|, |J|)$ specifies the set of time variables. Note that $q$-LUC with $N$ (or $M$) = 1 is equivalent to $q$-KP. The homogeneity constraint reads

$$\prod_{i \in I} T_i \prod_{j \in J} T_j^{-1}(\tau_{m,n}) = d_{m,n}\tau_{m,n} \quad (d_{m,n} \in \mathbb{C})$$

with the balancing condition $d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$; see Remark 2.3. The results for $q$-LUC are summarized as follows:

| $(M, N)$ : periodicity | $(|I|, |J|)$ | $T$ : time evolution of $q$-Painlevé equation | Ref. |
|-----------------------|------------|---------------------------------------------|-----|
| (2, 2)                | (2, 2)     | $T = T_1T_2 : q$-$P_{VI}$                  | [12, 14]|
| (3, 3)                | (3, 0)     | $T = T_1T_2^{-1} : q$-$P(E_6^{(1)})$       | [12]|
| $(M, M)$ ($M \geq 2$) | (2, 2)     | $T = T_1T_2^{-1}$ (with $t_{-2}/t_{-1} = q^{1/2}$) : $q$-$P(A_{2M-1}^{(1)})$ | [11]|

It is worth mentioning that $q$-$P(A_{2M-1}^{(1)})$ ($M \geq 2$) can be derived from $q$-KP or, alternatively, $q$-LUC. We refer $q$-$P_{VI}$ to be the $q$-analogue of the sixth Painlevé equation due to Jimbo–Sakai [2]. Note that $q$-$P_{V}$ can also be derived from the $q$-analogue of the three-wave resonant system, as shown by Kakei–Kikuchi [5].

Remark 3.3. We remember that the universal characters $S_{[\lambda, \mu]}$ (resp. the Schur functions $S_J$) are homogeneous solutions of $q$-LUC (resp. $q$-KP) and readily compatible with the reduction constraints (3.4) (resp. (3.3)) under consideration; see Prop. 2.2 and Remark 2.3. Hence it is immediate to construct special solutions of $q$-Painlevé equations in terms of $S_{[\lambda, \mu]}$ or $S_J$; see [4, 11, 12, 13, 14] for details.
In the rest of this paper, we will be devoted to the reduction procedure from $q$-LUC to certain birational dynamical systems of $q$-Painlevé type. Firstly, in Sect. 3.3 we deal with the case where $J = \emptyset$ and $(M, N)$ is general, which in fact can be done in a similar manner as the $q$-KP case; cf. Sect. 3.1. Secondly, in Sect. 3.4 we consider in particular the case where $(|I|, |J|) = (2, 2)$ and $(M, N)$ is general and then present a higher-order analogue of $q$-$P_{V_1}$ as a result.

### 3.3 The case of lattice $q$-UC hierarchy with $J = \emptyset$

Assume $J = \emptyset$. In this case the argument can be proceeded along quite a parallel way with the case of $q$-KP (cf. Sect. 3.1); though we will here demonstrate only the case $|I| = 2$ for simplicity. Let $I = \{1, 2\}$ and $J = \emptyset$. Impose on the variables $\tau_{m,n} = \tau_{m,n}(t_1, t_2)$ the $(M, N)$-periodic condition: $\tau_{m+M,n} = \tau_{m,n+N} = \tau_{m,n}$ and the homogeneity condition: $T_1 T_2 (\tau_{m,n}) = q^{d_{m,n}} \tau_{m,n}$, where $d_{m,n} \in \mathbb{C}$ are constant parameters satisfying $d_{m,n} + d_{m+1,n+1} = d_{m+1,n} + d_{m,n+1}$. Concerning the variables $u_{i,m,n}$, the constraint above implies that

$$u_{i,m,n} = u_{i,m+M,n} = u_{i,m,n+N} \quad \text{and} \quad T_1 T_2 (u_{i,m,n}) = q^{d_{m,n} + d_{m,n+1} - d_{m+1,n+1} - d_{m+1,n+1}} u_{i,m,n}.$$  \hspace{1cm} (3.5)

Introduce the dependent variables

$$x_{m,n} := t_2^{-1} T_2 (u_{1,m,n}), \quad y_{m,n} := t_1^{-1} u_{2,m,n+1},$$

and also auxiliary variables

$$x'_{m,n} := t_2^{-1} T_1 (u_{2,m,n}), \quad y'_{m,n} := t_1^{-1} u_{1,m,n+1}.$$  \hspace{1cm}

The compatibility condition of (2.6) yields the formulae (recall (2.9)):

$$x_{m,n} y'_{m,n} = x_{m,n}' y_{m,n}' \quad \text{and} \quad x_{m,n} + y_{m+1,n-1} = x_{m,n}' + y_{m+1,n-1}' \quad (m \in \mathbb{Z}/M\mathbb{Z}, n \in \mathbb{Z}/N\mathbb{Z})$$

which provide a birational mapping $r : (x, y) \mapsto (x', y')$ given by

$$x_{m,n}' = y_{m,n} \frac{P_{m-1,n+1}}{P_{m,n}}, \quad y_{m,n}' = x_{m,n} \frac{P_{m,n}}{P_{m-1,n+1}}.$$  \hspace{1cm}

Here

$$P_{m,n}(x, y) = \sum_{a=1}^{L} \prod_{i=1}^{a-1} x_{m+i,n-i} \prod_{i=a+1}^{L} y_{m+i,n-i}$$

and $L$ is the least common multiple of $(M, N)$. Prepare the mappings $\pi : x_{m,n} \leftrightarrow y_{m,n}$ and $\sigma : (x_{m,n}, y_{m,n}) \mapsto (x_{m,n-1}, y_{m,n+1})$. Apply $r \circ \pi \circ \sigma$ to variables $x_{m,n}$ and $y_{m,n}$. Therefore we see that $r \circ \pi \circ \sigma (x_{m,n}) = r \circ \pi (x_{m,n-1}) = r(y_{m,n-1}) = y_{m,n-1}' = T_2^{-1}(x_{m,n})$ and $r \circ \pi \circ \sigma (y_{m,n}) = r \circ \pi (y_{m,n+1}) = r(x_{m,n+1}) = x_{m,n+1}' = T_1(y_{m,n})$. In view of the similarity condition (3.5), we observe that $T_1(x_{m,n}) = q^{d_{m+1,n} - d_{m,n+2} - d_{m+1,n+1} + d_{m+1,n+1} - 1} T_2^{-1}(x_{m,n})$. Finally, the birational action of $T_1 : \mathbb{C}(x, y)$ turns out to be given as follows:

$$T_1(x_{m,n}) = q^{d_{m+1,n} + d_{m,n+2} - d_{m+1,n+1} - d_{m+1,n+1} - 1} x_{m,n-1} \frac{P_{m,n-1}}{P_{m-1,n}}, \quad (3.6a)$$

$$T_1(y_{m,n}) = y_{m,n+1} \frac{P_{m-1,n+2}}{P_{m,n+1}} \quad (3.6b)$$

for $m \in \mathbb{Z}/M\mathbb{Z}$ and $n \in \mathbb{Z}/N\mathbb{Z}$. 
3.4 The case of lattice $q$-UC hierarchy with $J \neq \emptyset$

Let $I = \{1, 2\}$ and $J = \{-1, -2\}$. We impose on $\tau_{m,n} = \tau_{m,n}(t)$ the homogeneity condition: $\tau_{m,n}(qt) = q^{d_{m,n}}\tau_{m,n}(t)$, where $d_{m,n} \in \mathbb{C}$ fulfills the balance $d_{m,n} + d_{m+1,n+1} = d_{m,n+1} + d_{m+1,n}$. Accordingly, the variables $w_{m,n} = \tau_{m+1,n}/\tau_{m,n+1}$ satisfy

$$T_1T_2(w_{m,n}) = c_{m,n}T_1T_2(w_{m,n}) \quad \text{where} \quad c_{m,n} = q^{d_{m+1,n} - d_{m,n+1}}. \quad (3.7)$$

We will often abbreviate the $q$-shift $T_i \cdots T_jT_1^{-1} \cdots T_j^{-1}(F)$ of a function $F = F(t)$ to $F(t_{1, \ldots, i}l_{j, \ldots, j})$. Let us now choose the dependent variables as

$$f_{m,n} = \frac{w_{m,n}^{(1)}}{w_{m,n}^{(-1)}}, \quad g_{m,n} = \frac{w_{m,n}^{(1,-1)}}{w_{m,n}^{(-1)}} \quad \text{with} \quad d_{m,n_1} = \frac{w_{m,n_1}^{(1,-1)}}{w_{m,n_1}^{(-1)}}. \quad (3.8)$$

Consider the field $\mathbb{K}(f, g)$ of rational functions in $f_{m,n}$ and $g_{m,n}$ with $\mathbb{K}$ being a certain coefficient field.

Theorem 3.4. The action of $T := T_1T_2$ on variables $f_{m,n}$ and $g_{m,n}$ is given in terms of birational transformations, that is, $T^{\pm 1}(f_{m,n}), T^{\pm 1}(g_{m,n}) \in \mathbb{K}(f, g)$ for any $m, n$.

Proof. Recall $(2.11)$ the functional equation satisfied by $w_{m,n}$:

$$w_{m,n}^{(i,j)} = \frac{w_{m,n+1}^{(i)}w_{m,n+1}^{(j)}t_1w_{m,n+1}^{(j)} - t_jw_{m,n+1}^{(i)}}{t_1w_{m,n+1}^{(j)} - t_jw_{m,n+1}^{(i)}} \quad (3.9)$$

where $i \neq j$ and $i, j \in I \cup J = \{1, 2, -1, -2\}$.

First we shall calculate $T_1T_2(f_{m,n})$. Apply $T_{-1}$ to $(3.9)$ with $(i, j) = (1, -2)$. We then obtain

$$\frac{1}{c_{m,n}}T_1T_2(w_{m,n}^{(1)}) = w_{m,n}^{(1,-2)} = \frac{w_{m,n}^{(1,-1)}w_{m,n+1}^{(-1)} - t_1w_{m,n}^{(1)}}{w_{m,n+1}^{(-1)}} \quad (3.10)$$

Applying $T_1$ to $(3.9)$ with $(i, j) = (2, -1)$, we have

$$T_1T_2(w_{m,n}^{(-1)}) = w_{m,n}^{(2,-1)} = \frac{c_{m,n+1}w_{m,n+1}^{(-1)}w_{m,n+1}^{(1)} - t_{-1}w_{m,n+1}^{(-1)}}{w_{m,n+1}^{(-1)}} \quad (3.11)$$

Note that we have used $w_{m,n}^{(1,2)} = c_{m,n}w_{m,n}^{(-1,-2)}$ between the first and second lines. Combining $(3.10)$ with $(3.11)$ leads to

$$\frac{T_1T_2(f_{m,n})}{f_{m+1,n+1}} = \frac{c_{m,n}}{c_{m,n+1}} \left( g_{m,n+1} - \frac{t_{-1}}{t_{-2}} \right) \left( g_{m,n+1} + \frac{t_{1}}{t_{2}} \right). \quad (3.12)$$
Next we shall concern \((T_1 T_2)^{-1}(g_{m,n})\). Notice that \(w_{m+1,n+1} = c_{m+1,n+1}(T_1 T_2)^{-1}(w_{m+1,n+1}^{(-1,-2)})\). It therefore follows from (3.9) with \((i, j) = (1, -1)\) that

\[
(T_1 T_2)^{-1}(w_{m+1,n+1}^{(-1,-2)}) = \frac{1}{c_{m+1,n+1}} w_{m+1,n+1}^{(-1)} t_1 - t_1 f_{m+1,n}.
\] (3.13)

By applying \(T_2^{-1}T_1^{-1}\) to (3.9) with \((i, j) = (2, -2)\), we have

\[
w_{m,n}^{(-1,-2)} = \frac{w_{m+1,n+1}^{(-1)} w_{m+1,n}^{(-1,2)}}{w_{m+1,n+1}^{(-1)[2]}} q^{-1} t_2 w_{m+1,n}^{(-1,-2)[2]} - t_2 w_{m+1,n}^{(-1)}.
\] (3.14)

Observe that \(w_{m,n}^{(-1,-2)[2]} = w_{m,n}^{(1)}/c_{m,n}\) and \(w_{m+1,n+1}^{(-1)[2]} = (T_1 T_2)^{-1} w_{m+1,n+1}^{(1,-1)}\). We then verify from (3.14) that

\[
(T_1 T_2)^{-1} w_{m+1,n+1}^{(1,-1)} = \frac{1}{c_{m+1,n}} w_{m+1,n}^{(-1)} t_2 f_{m+1,n} - q c_{m+1,n} t_2.
\] (3.15)

If we put (3.13) and (3.15) together, we arrive at

\[
\frac{g_{m,n}}{(T_1 T_2)^{-1}(g_{m+1,n+1})} = \frac{c_{m+1,n}}{c_{m+1,n+1}} \left( \frac{f_{m+1,n} - \frac{\alpha}{\gamma}}{f_{m+1,n}^{(1)}/c_{m+1,n+1}} \right) \left( f_{m+1,n+1} - q c_{m+1,n+1} \frac{\alpha}{\gamma} \right).
\] (3.16)

Finally, by virtue of (3.12) and (3.16), it is clear that \(T = T_1 T_2\) acts on \(f_{m,n}\) and \(g_{m,n}\) as a birational mapping.

Let us slightly refine the birational dynamics, (3.12) and (3.16), constructed above in Theorem (3.4). Put

\[
\alpha = \frac{t_1}{t_2}, \quad \beta = \frac{t_1}{t_2}, \quad \gamma = \frac{t_1}{t_1}, \quad \delta = \frac{t_2}{t_2}.
\]

We have then a birational dynamical system \(T: (\alpha, \beta, \gamma, \delta; f_{m,n}, g_{m,n}) \mapsto (q \alpha, \beta/q, \gamma, \delta/q; f_{m,n}^{(1)}, g_{m,n})\).

\[
\overline{f}_{m,n} = \frac{c_{m,n}}{c_{m+1,n+1}} \left( g_{m,n+1} - \alpha \right) \left( g_{m,n+1} - c_{m,n+1} \beta \right) f_{m+1,n+1},
\]

(3.17a)

\[
\overline{g}_{m,n} = \frac{c_{m,n}}{c_{m+1,n+1}} \left( f_{m,n+1} - \gamma \right) \left( f_{m,n+1} - c_{m,n+1} \delta \right) g_{m,n+1+1},
\]

(3.17b)

with \(\alpha \delta / \beta \gamma = 1\) and \(c_{m,n} = q^{d_{m+1,n} - d_{m,n+1}}\).

From now on, we shall impose the \((M, N)\)-periodicity on the suffixes \((m, n)\) of the variables. Let \(L\) denote the least common multiple of \((M, N)\), and recall (3.8). Hence the dynamics (3.17) turns out to possess the \(2MN/L\) conserved quantities:

\[
\prod_{i=1}^{L} f_{m+i,n-i} = \prod_{i=1}^{L} g_{m+i,n-i} = 1.
\] (3.18)
Moreover, if $M = N = 2$ then the dynamics (3.17) is actually closed in two variables, e.g., $f = f_{1,1}$ and $g = g_{1,2}$:

\[
\begin{align*}
\bar{f} &= \frac{c_{1,1} (\alpha g - 1)(g - c_{1,2}\beta)}{f (g - \alpha)(\beta g - c_{1,2})}, \\
\bar{g} &= \frac{c_{1,2} (q\gamma \bar{f} - 1)(\bar{f} - c_{1,1}\delta)}{g (\bar{f} - q\gamma)(\delta \bar{f} - c_{1,1})},
\end{align*}
\]

where $(\alpha, \beta, \gamma, \delta) = (q\alpha/\beta, q\gamma, \delta/\beta)$ and $\alpha\delta/\beta\gamma = 1$. This coincides with the $q$-analogue of the sixth Painlevé equation ($q$-$P_{VI}$); cf. [2]. For this reason we regard (3.17) as a higher-order extension of $q$-$P_{VI}$.

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