Topological character of hydrodynamic screening in suspensions of hard spheres: an example of universal phenomenon

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Abstract

Although in the case of polymer solutions the existence of hydrodynamic screening has been theoretically established some time ago, use of the same methods for suspensions of hard spheres thus far have failed to produce similar results. In this work we reconsider this problem. Using superposition of topological and London-style qualitative arguments we prove the existence of screening in hard sphere suspensions. Even though some of these arguments were employed initially for treatments of superconductivity and superfluidity, we find analogs of these phenomena in nontraditional settings such as in colloidal suspensions, turbulence, magnetohydrodynamics, etc. In particular, in suspensions we demonstrate that the hydrodynamic screening is an exact analog of Meissner effect in superconductors. The extent of screening depends on the volume fraction of hard spheres. The zero volume fraction limit corresponds to the normal state. The case of finite volume fractions to the mixed state typical for superconductors of the second kind with such a state becoming fully "superconducting" at the critical volume fraction $\varphi^*$ for which the (zero frequency) relative viscosity $\eta_{\text{relative}}$ diverges. Brady and, independently, Bicerano et al using scaling-type arguments predicted that for $\varphi$ close to $\varphi^*$ the viscosity $\eta_{\text{relative}}$ behaves as $C(1 - \varphi/\varphi^*)^{-2}$ with $C$ being some constant. Their prediction is well supported by experimental data. In this work we explain such a behavior of viscosity in terms of a topological-type transition which mathematically can be made isomorphic to the more familiar Bose-Einstein condensation transition. Because of this, the results and methods of this work are not limited to suspensions. In the concluding section we describe other applications ranging from turbulence and magnetohydrodynamics to high temperature superconductors and QCD, etc.

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1 Introduction

In his 1905-1906 papers on Brownian motion for suspensions of hard spheres, Einstein obtained now famous relation for the self-diffusion coefficient $D_0$ for the noninteracting hard spheres of radius $R$ immersed in a solvent at temperature $T$ [1]:

$$D_0 = \frac{k_B T}{\gamma} = \frac{k_B T}{6\pi R \eta_0}. \quad (1.1)$$

In this formula $\eta_0$ is the viscosity of a pure solvent and $k_B$ is the Boltzmann’s constant. This result is valid only in the infinite dilution limit. In another paper [2] Einstein took into account the effects of finite concentrations and obtained the first nonvanishing correction to $\eta_0$ for small but finite concentrations. It is given by

$$\frac{\eta}{\eta_0} = 1 + 2.5 \varphi + O(\varphi^2) \quad (1.2)$$

with $\varphi$ being the volume fraction $\varphi := \frac{n V}{4 \pi R^3}$. In this formula $n$ is the number of monodisperse hard spheres in the volume $V$. If we formally replace $\eta_0$ by $\eta$ in Eq. (1.1), the obtained result can be cautiously used as a definition for the cooperative diffusion coefficient $D$, i.e.

$$D = \frac{k_B T}{6\pi R \eta}. \quad (1.3)$$

Below, we use symbols $D_0$ for the self-diffusion coefficient and $D$ for the cooperative diffusion coefficient. By combining Eqs. (1.1) - (1.3), we also obtain:

$$\frac{D}{D_0} = 1 - 2.5 \varphi + O(\varphi^2). \quad (1.4)$$

Eq.(1.4) compares well with experimental results, e.g. those discussed in Ref.[3]. Numerous attempts have been made to obtain results like Eq.(1.4) systematically. The above results are restricted by the observation that Stoke’s formula for friction $\gamma$ is applicable only for time scales longer than the characteristic relaxation time $\tau_r$ of the solvent, $\tau_r := \rho R^2/\pi \eta_0$, e.g. see [4]. In this formula $\rho$ is the density of pure solvent. This requirement provides the typical cut-off time scale, while the parameter $R$ serves as a typical space cut-off for the problems we are going to study in this work.

By analogy with the theory of nonideal gases, expansion Eq. (1.2) is referred to as a "virial". Unlike the theory of nonideal gases, where the virial coefficients are known exactly to a very high order [5], values for coefficients in the virial expansion for $\eta$ have been an active area of research to date even in the low concentration regime. A considerable progress was made in obtaining closed form approximations describing the rheological properties of suspensions of hard spheres in a broad range of concentrations [6-8]. Similar results for particles of other geometries are much less complete [7,9]. An extension of these results to solutions of polymers has taken place in parallel with these developments [10]. A noticeable advancements have been made in our understanding of rheology of dilute and semifluid polymer solutions for fully flexible polymers and rigid rods. It should be noted, though, that polymers add further complexities because the connectivity of the polymer chain backbone plays an essential role in calculations of rheological properties of polymer solutions. The effect of chain connectivity on viscoelastic properties of polymer solutions has been an object of extensive discussion, and many of theoretical difficulties encountered in describing these solutions are shared by suspensions of hard spheres. In particular, it is known [11], that particles immersed in a viscous fluid affect the motion of each other both hydrodynamically and by direct interaction (hard core, etc). Since the motion of particles in a fluid is correlated, it contributes to the distribution of local velocities within the fluid. Behavior of many systems (e.g. those listed in Section 6) other than the hard sphere suspensions happens to be closely related or even isomorphic to that noticed in suspensions. This observation makes study of suspensions important in many areas of physics, chemistry and biology. For reasons which will become apparent upon reading, in this work we shall mention only physical applications.

In a polymer solutions when the polymer concentration $\varphi$ increases, it is believed that the hydrodynamic interactions become unimportant due to the effects of hydrodynamic screening [12]. To our

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3The data on page 5 and in Table 2 of this reference support our conjecture.
knowledge, screening has not been established in the theory of hard sphere suspensions. If it would occur in suspensions, the screened particle motion could be affected only by thermal fluctuations (truly Brownian motion!). Such Brownian hard spheres can be described by the short range interacting random walk model [13]. For finite concentrations, we expect the longer range hydrodynamic interactions to be very important. That this is indeed the case, is the central theme of our paper. In what follows, we provide the theoretical arguments in favor of hydrodynamic interparticle interactions and screening which must be present in solutions at non-vanishing concentrations. By exploiting analogies between electrodynamics and fluid mechanics we shall demonstrate that hydrodynamic screening occurs in much the same way as screening of the magnetic field in superconductors. Therefore, mathematically, the description of screening in suspensions is analogous to that for the Meissner effect in superconductors. This observation will allow us to account for a number of interesting properties of suspensions. For instance, the viscosity of hard sphere suspensions is known to diverge beyond some critical concentration \( \varphi^* \). This phenomenon has been observed experimentally and is well documented, e.g. see Refs. [14-18]. All these references are concerned with changes in rheological properties of suspensions occurring with changes in concentration \( \varphi \). Using scaling-type arguments Brady, Ref.[19], and, independently, Bicerano et al, Ref. [20], had found that near \( \varphi = \varphi^* \) the relative viscosity \( \eta/\eta_0 \) diverges as \( \eta/\eta_0 = C(1 - \varphi/\varphi^*)^{-2} \) with \( C \) being some constant. Furthermore, as it is shown by Bicerano et al, such analytical dependence of relative viscosity on concentration \( \varphi \) actually works extremely well for all concentrations. In view of (1.3), it is reasonable to expect vanishing of \( D \) for \( \varphi \to \varphi^* \). This phenomenon was indeed observed in Ref.[21].

Theoretically, the result for relative viscosity was obtained as result of a combined nontrivial use of topological and combinatorial arguments. Such arguments can also be used, for instance, for description of the onset of turbulence in fluids or gases. As described in Ref.[22], such a regime in these substances is characterized by the sharp increase in the viscosity (just like in suspensions). According to Chorin, Ref.[22], Section 6.8, one can think about such an increase as analogous to processes which take place in superfluid \(^4\)He when one goes in temperatures from below to above \( \lambda \)-transition, that is from the superfluid to normal fluid state. Such a transition is believed to be associated with uninhibited proliferation of tangled vortices on any scale. In this work we demonstrate that Chorin’s conjecture is indeed correct. This interpretation is possible only if both topological and combinatorial arguments are rigorously and carefully taken into account. Surprisingly, when this is done, the emerging description becomes isomorphic to that known for the Bose-Einstein condensation transition. Because of this, in addition to turbulence, in concluding section of this work we briefly discuss a number of apparently different physical systems whose behavior under certain conditions resembles that found in colloidal suspensions.

The rest of this paper is organized as follows. In section 2, we introduce notations, discuss experimental data with help of previously found generalized Stokes-Einstein relation [23] and make conjectures about how these results should be interpreted in the case if hydrodynamic screening does exist. Some familiarity with Ginzburg-Landau (G-L) theory of superconductivity is expected for proper understanding of this and the following sections. In Section 3 we study in detail how many particle diffusion processes should be affected if hydrodynamic interactions are taken into account. The major new results of this section are given in Section 3.3. where we rigorously demonstrate that account of hydrodynamic interactions causes modification of Fick’s laws of diffusion in the same way as presence of electromagnetic field causes modification of the Schrodinger’s equation for charged particles. The gauge fields emerging in the modified Fick’s equations are of zero curvature implying involvement of the Chern-Simons topological field theory. The following Section 4 considers in detail the implications of the results obtained in Section 3. The major new result of this section is given in Section 4.4. where we adopted the logic of the ground breaking paper by London and London [24] in order to demonstrate the existence of hydrodynamic screening. Thus, the phenomenon of screening in suspensions is analogous to the Meissner effect in superconductors [25]. In Section 5 we follow the logic of Ginzburg-Landau paper [26] elaborating the work by London brothers and develop similar G-L-type theory for suspensions. The major new result of this section is presented in Section 5.5, in which by using combinatorial and topological methods we reproduce the scaling results by Brady [19] and Bicerano et al, Ref.[20]. In Section 6 we place the obtained results in a much broader context. It is done with help of two key concepts: helicity and force-free fields. They had been in use for some time in areas such as magnetohydrodynamics, fluid, plasma and gas turbulence, classical mechanics written in hydrodynamic formalism but not in superconductivity or colloidal suspensions, etc.
In this section we mention as well other uses of these concepts in disciplines such as high temperature superconductivity, quantum chromodynamics, string theory, non-Abelian fluids, etc. The paper also contains three appendices which are made sufficiently self contained. They are not only very helpful in providing details supporting the results of the main text but also of independent interest.

2 Stokes-Einstein Virial Expansions for a Broad Concentration Range

2.1 General Results

In 1976, Batchelor obtained the following general result for the cooperative diffusion coefficient [27]:

\[ D(\varphi) = \frac{K(\varphi)}{6\pi\eta r} \frac{\varphi}{1 - \varphi} \left(\frac{\partial \mu}{\partial \varphi}\right)_{p,T}, \]  

(2.1)

where \( K(\varphi) \) is the sedimentation coefficient of the particles in suspension and \( \mu \) is the chemical potential. Batchelor obtained for \( K(\varphi) \) the following result:

\[ K(\varphi) = 1 - 6.55\varphi + O(\varphi^2) \]  

(2.2)

so that (2.1) with thus obtained first order result for \( K(\varphi) \) can be used only for low concentrations. In Ref.[3] an attempt was made to extend Batchelor’s results to higher concentrations. This was achieved in view of the fact that

\[ \frac{\varphi}{1 - \varphi} \left(\frac{\partial \mu}{\partial \varphi}\right)_{p,T} = \left(\frac{\partial \Pi}{\partial n}\right)_{p,T}, \]  

(2.3)

where \( \Pi \) is the osmotic pressure. Use of this result in (2.1) produces:

\[ D(\varphi) = \frac{K(\varphi)}{6\pi\eta r} \left(\frac{\partial \Pi}{\partial n}\right)_{p,T}. \]  

(2.4)

The Carnahan-Starling equation of state for hard spheres can be used to obtain the following result for compressibility

\[ \left(\frac{\partial \Pi}{\partial n}\right)_{p,T} = k_B T \frac{[1 + 2\varphi^2 + (\varphi - 4)\varphi^3]}{(1 - \varphi)^4}, \]  

(2.5)

thus converting equation (2.4) into

\[ D(\varphi) = \frac{K(\varphi) k_B T}{6\pi\eta r} \frac{[1 + 2\varphi^2 + (\varphi - 4)\varphi^3]}{(1 - \varphi)^4}. \]  

(2.6)

To be in accord with Batchelor’s result (2.2) at low concentrations, the authors [3] suggested replacing of Eq.(2.2) by

\[ K(\varphi) \approx (1 - \varphi)^{6.55} \]  

(2.7)

which allows us to rewrite (2.6) in the following final form

\[ D/D_0 = (1 - \varphi)^{6.55} \frac{[1 + 2\varphi^2 + (\varphi - 4)\varphi^3]}{(1 - \varphi)^4}. \]  

(2.8)

convenient for comparison with experimental data.

Such a comparison can be found in Fig.12 of Ref [3] where this result is plotted against author’s own experimental data for the cooperative diffusion coefficient. The experimental data within error margins
appears to agree extremely well with the theoretical curve obtainable from Eq.(2.8). However, it should be kept in mind that, in fact, originally Eq.(2.2) was determined only to first order in $\varphi$ (and, therefore, only for the volume fractions less than about 0.05). Therefore, formally, Eq.(2.8) is in accord with Eq.(2.2) only for volume fractions of lesser than about 0.03. Therefore, it is clear from Fig 12 of [3] that to improve the agreement in the whole range of concentrations, a knowledge of a second order in $\varphi$ is desirable in (2.2). This problem can be by passed as follows.

From [3] the viscosity data from the same experiments were obtained so that the data can be fit to the following second order expansion:

$$\frac{\eta}{\eta_0} = 1 + 2.5\varphi + 6.54\varphi^2 + O(\varphi^3).$$

(2.9)

To obtain this result, the authors constrained the first order coefficient to 2.5 to comply with Einstein’s result (1.2) for viscosity. If one considers these data without such a constraint, then one obtains,

$$\frac{\eta}{\eta_0} = 1 + 2.4\varphi + 7.1\varphi^2 + O(\varphi^3).$$

(2.10)

In the paper by Kholodenko and Douglas [23] the following result for the cooperative diffusion coefficient was derived (the generalized Stokes-Einstein relation)

$$\frac{D}{D_0} = \frac{1}{(\eta/\eta_0)} \left[ \frac{S(0,0)}{S_0(0,0)} \right]^{-1/2},$$

(2.11)

where $S(0,0)$ is the $k = 0$, zero angle static scattering form factor. The thermodynamic sum rule for the hard sphere gas produces the following result for this formfactor:

$$\left[ \frac{S(0,0)}{S_0(0,0)} \right]^{-1/2} = 1 + 4\varphi + 7\varphi^2 + O(\varphi^3).$$

(2.12)

By combining Eqs (2.9)-(2.12) the result for cooperative diffusion is obtained:

$$\frac{D}{D_0} = 1 + 1.6\varphi - 3.9\varphi^2 + O(\varphi^3).$$

(2.13)

For the sake of comparison with experiment, we made a numerical fit to the experimental data for higher concentrations obtained in [3] by a polynomial (up to a second order in $\varphi$) with the result\footnote{The correlation coefficient obtained for this fit is 0.97.}:

$$\frac{D}{D_0} = 1 + 1.5505\varphi - 5.3663\varphi^2 + O(\varphi^3).$$

(2.14)

Comparison between Eqs (2.13) and (2.14) shows that the theoretically obtained result, Eq.(2.13), is in good agreement with the experimental data, Eq.(2.14), within error margins. Alternatively, we can use the reciprocal of the empirical expression, Eq.(2.14), in (2.11) to obtain

$$\frac{\eta}{\eta_0} = 1 + 2.45\varphi + 8.5\varphi^2 + O(\varphi^3),$$

(2.15)

which also compares well with the experimental data, Eq.(2.10).

2.2 The generalized Stokes-Einstein relation and the role of hydrodynamic screening

We return now to Eq.(2.11) for further discussion. Based on the results of introductory section, especially on Eqs (1.1) and (1.3), we can formally write:

$$\frac{D}{D_0} = \frac{R}{R^* \eta} \frac{\eta_0}{\eta} = 1 + a_1\varphi + a_2\varphi^2 + O(\varphi^3).$$

(2.17)

The actual values of $a_i$, $i = 1, 2, \ldots$ can be determined using Eq.(2.11) written in the following form
\[ D/D_0 = \left( \frac{\xi}{\xi_0} \right)^{-1} = \frac{1}{(\eta/\eta_0)} \left[ \frac{S(0,0)}{S_0(0,0)} \right]^{-1/2}, \]  

(2.18)

where \( \xi \) is the correlation length, \( \xi_0 \) is the correlation length in the infinite dilution limit \( \xi_0 \sim R \). To justify such a move we need to remind to our readers of some facts from the dynamical theory of linear response. To do so, we borrow some results from our previous work \cite{23}.

Generally, both \( D \) and \( D_0 \) are measured by light scattering experiments. In these experiments the Fourier transform of the density-density correlator

\[ S(\mathbf{R}, \tau) = \langle \delta n(\mathbf{r},t) \delta n(\mathbf{r}',t') \rangle \]  

(2.19)

is being measured. The formfactor, Eq.(2.19), is written with account of translational invariance, requiring the above correlator to be a function of relative distance \( |\mathbf{r} - \mathbf{r}'| = \tau \) only. Time homogeneity, makes it in addition to be a function of \( |t - t'| = \tau \) only. In this expression, \( \langle ... \rangle \) represents an equilibrium thermal average while density fluctuations are given by \( \delta n(\mathbf{r},t) = n(\mathbf{r},t) - \langle n \rangle \). By definition, the Fourier transform of Eq.(2.19) is given by

\[ S(\mathbf{q}, \omega) = \int d\mathbf{r} \int d\tau S(\mathbf{R}, \tau)e^{i\mathbf{q} \cdot \mathbf{r} - i\omega\tau}. \]  

(2.20)

Using this expression, we obtain the initial decay rate \( \Gamma^{(0)}_{\mathbf{q}} \) as follows \cite{10,23}:

\[ \Gamma^{(0)}_{\mathbf{q}} = -\frac{\partial}{\partial \tau} \ln \left[ \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega\tau} S(\mathbf{q}, \omega) \right]_{\tau \to 0^+}. \]  

(2.21)

With help of this result, the cooperative diffusion coefficient is obtained as

\[ D = \frac{\partial}{\partial \mathbf{q}^2} \Gamma^{(0)}_{\mathbf{q}} |_{\mathbf{q} = 0}. \]  

(2.22)

In the limit of vanishingly low concentrations the self-diffusion coefficient is known to be \cite{10}

\[ D_0 = \frac{1}{6} \lim_{t \to \infty} \frac{1}{t} \left\langle (\mathbf{r}(t) - \mathbf{r}(0))^2 \right\rangle, \]  

(2.23)

where \( \langle ... \rangle \) denotes the Gaussian-type average. Following Lovesey \cite{28}, it is convenient to rewrite this result as

\[ D_0 = \frac{1}{3} \int_0^\infty dt \langle \mathbf{v} \cdot \mathbf{v}(t) \rangle \]  

(2.24)

in view of the fact that if

\[ \mathbf{r}(t) - \mathbf{r}(0) = \int_0^t \mathbf{v}(\tau) d\tau \]  

(2.25a)

then,

\[ \left\langle (\mathbf{r}(t) - \mathbf{r}(0))^2 \right\rangle = \left\langle \left( \int_0^t \mathbf{v}(\tau) d\tau \right)^2 \right\rangle = 2 \int_0^t \int_0^t \langle \mathbf{v}(\tau) \cdot \mathbf{v}(\tau') \rangle d\tau d\tau' = 2 \int_0^t d\tau (t - \tau) \langle \mathbf{v}(\tau) \cdot \mathbf{v}(0) \rangle, \]  

(2.25b)

while, by definition, \( D_0 = \frac{1}{6} \lim_{t \to \infty} \frac{1}{t} \left\langle (\mathbf{r}(t) - \mathbf{r}(0))^2 \right\rangle \).

With these definitions in place and taking into account Eq.(2.18), we would like now to discuss in more detail the relationship between \( D \) and \( D_0 \). Using Eq.(2.29) of Ref. \cite{23} we obtain

\[ D = \lim_{\tau \to 0^+, \mathbf{k} \to 0} \frac{1}{3} \int_0^\infty dt' \left\langle \mathbf{j}(0,t) \cdot \mathbf{j}(0,t') \right\rangle \frac{S(0,0)}{S(0,0)}, \]  

(2.26)
where the current \( j \) is given as \( j = \delta n(r, t)\mathbf{v}(r, t) \), provided that the non-slip boundary condition

\[
\mathbf{v}_f(r, t) = \frac{dr}{dt} = \mathbf{v}(t)
\]  

(2.27)
is applied. Here \( \mathbf{v}_f(r, t) \) is the velocity of the fluid and \( r(t) \) is the position of the center of mass of the hard sphere with respect to the chosen frame of reference. Eq.(2.26) is in agreement with Eq.(2.24) in view of the fact that in the limit of zero concentration \( S(0, 0) = 1 \) so that \( < j(0, t) \cdot j(0, t') > \rightarrow (\mathbf{v} \cdot \mathbf{v}(t)) \) as we would like to demonstrate now. For this purpose, in view of Eq.(2.22) it is convenient to rewrite the result Eq.(2.26) is the equivalent form

\[
S(\mathbf{q}, 0) \Gamma_\mathbf{q}^{(0)} = \int_0^\infty dt'' \mathbf{q} \cdot < j(\mathbf{q}, t)j(-\mathbf{q}, t'') > \cdot \mathbf{q} |_{\tau \to 0^+}
\]  

(2.28)
in accord with Eq.(2.15) of Ref.[23]. This relation is very convenient for theoretical analysis. For instance, it is straightforward to obtain \( D \) in the decoupling approximation as suggested by Ferrell [21]. It is given by

\[
\mathbf{q} \cdot < j(\mathbf{q}, t)j(-\mathbf{q}, t') > \cdot \mathbf{q} = \mathbf{q} \cdot \mathbf{v}(\mathbf{q}', t)\mathbf{v}(\mathbf{q} - \mathbf{q}', t') > \cdot \mathbf{q} (\delta n(\mathbf{q} - \mathbf{q}', t)\delta n(\mathbf{q}', t')).
\]

(2.29)

In Section 5.4, we provide proof that the above decoupling is in fact exact. This provides an explanation why it is working so well in real experiments.

In the meantime, we shall consider this decoupling as an approximation. Once such an approximation is made, the problem of calculation of \( D \) is reduced to the evaluation of correlators defined in Eq.(2.29). For the velocity-velocity correlator, the following expression was obtained before (e.g. see Ref.[23], Eq.(2.18)):

\[
< \mathbf{v}_{ik}(t)\mathbf{v}_{jk}(t') > = (2\pi)^3 \delta(\mathbf{k} + \mathbf{k}') \{ \delta_{ij} - \frac{k_ik_j}{k^2} \} \frac{2k_BT}{\eta k^2} \delta(t - t')
\]

\[
= 2k_BT\mathcal{H}_{ij}(\mathbf{k})\delta(t - t')
\]  

(2.30)

with \( i, j = 1 - 3 \). This expression defines the Oseen tensor \( \mathcal{H}_{ij}(\mathbf{k}) \) to be discussed in detail in the next section. The presence of the delta function \( \delta(t - t') \) in Eq.(2.30) makes it possible to look only at the equal time density-density correlator in the decomposition of the \( j \cdot j \) correlator given by Eq.(2.29). Such a correlator also was discussed in Ref.[23] where it is shown to be

\[
(\delta n(\mathbf{k} - \mathbf{k}', t)\delta n(\mathbf{k}', t)) = k_BT \left[ \frac{\partial}{\partial\Pi} < n > \right]_T
\]

(2.31a)

Actually, it is both time and \( k \)-independent since, as is well known, it is the thermodynamic sum rule. That is

\[
S(0, 0) = k_BT \left[ \frac{\partial}{\partial\Pi} < n > \right]_T.
\]

(2.31b)

It is convenient to rewrite this result as follows

\[
k_BT \left[ \frac{\partial}{\partial\Pi} < n > \right]_T = \int d\mathbf{R} S(\mathbf{R}, 0) \equiv S(0, 0),
\]

(2.31c)

implying that

\[
S(\mathbf{R}, 0) = k_BT \left[ \frac{\partial}{\partial\Pi} < n > \right]_T \delta(\mathbf{R}).
\]

(2.32)

In view of Eq.(2.12), we notice that in the limit of vanishing concentrations \( S(0, 0) = 1 \). In such an extreme case the decoupling made in Eq.(2.29) superimposed with the definition, Eq.(2.22), and the fact that

\[
\frac{\partial}{\partial q^2} \ldots = \frac{1}{3} tr \left( \sum_{i,j} \frac{\partial}{\partial q_i} \frac{\partial}{\partial q_j} \ldots \right)
\]
produces the anticipated result, Eq.(2.24), as required. Next, following Ferrell [21], we regularize the delta function in Eq.(2.32). Using an identity
\[\int_0^\infty dx xe^{-x} = e^{-x}\]
the regularized expression for \(S(R,0)\) is obtained as follows
\[S(r,0) = \frac{k_B T}{4\pi \xi^2} \langle n \rangle \left[ \partial \frac{\partial \langle n \rangle}{\partial \Pi} \right]_T \frac{1}{r} e^{-\frac{r}{\xi}}, \tag{2.33}\]
where \(r = |R|\) and the parameter \(\xi\) is proportional to the static correlation length. To use this expression for calculations of \(D\), by employing Eq.(2.26) we have to transform the hydrodynamic correlator, Eq.(2.30), into coordinate form as well. Such a form is given in Eq.(2.33) of Ref.[23] as
\[\langle v(r,t) \cdot v(r',t') \rangle = \frac{k_B T}{\pi \eta} \frac{1}{|r-r'|} \delta(t-t'). \tag{2.34}\]
This expression is written with total disregard of possible effects of the hydrodynamic screening, though. The combined use of Eqs (2.33) and (2.34) in Eq.(2.26) produces the anticipated result
\[D = \frac{k_B T}{3\pi \eta \xi} \tag{2.35}\]
in accord with that obtained by Ferrell, Ref.[21], Eq.(11), provided that we redefine (still arbitrary) the parameter \(\xi\) as \(2\xi\). The result (2.35) also coincides with Eq.(1.3) if we identify \(\xi\) with \(R^*\). Furthermore, by looking at Eq.(2.18) we realize that in the infinite dilution limit we have to replace \(\xi\) by \(\xi_0\) and, accordingly, \(\eta\) by \(\eta_0\). Such an identification leads to the generalized Stokes-Einstein relation in the form given by Eq.(2.18) implying that
\[\frac{\dot{\xi}}{\xi_0} = \left[ \frac{S(0,0)}{S_0(0,0)} \right]^{1/2}. \tag{2.36a}\]
Since we have noticed before that \(S_0(0,0) = 1\) this result can be rewritten as
\[\dot{\xi} = \sqrt{S(0,0)} \xi_0. \tag{2.36b}\]
Suppose now that hydrodynamic interactions are screened to some extent. In such a case the result Eq.(2.34) should be modified accordingly. Thus, we obtain
\[\langle v(r,t) \cdot v(r',t') \rangle = \frac{k_B T}{\pi \eta} \frac{\exp(-\frac{r}{\xi_H})}{r} \frac{1}{|r-r'|} \delta(t-t'), \tag{2.37}\]
where we have introduced the hydrodynamic correlation length \(\xi_H\). If, as we shall demonstrate below, the analogy between hydrodynamic and superconductivity makes sense under some conditions then, using this assumed analogy we introduce the Ginzburg parameter \(\kappa_G\) for this problem via known relation [25]:
\[\xi_H = \kappa_G \xi. \tag{2.38}\]
Using Eqs (2.33), (2.37) and (2.38) in (2.26), the result for \(D\), Eq.(2.35), acquires the following form:
\[D = \frac{k_B T}{6\pi \Sigma \eta} \left(1 + \frac{1}{\kappa_G} \right)^{-1}. \tag{2.39}\]
Since, the adjustable parameter \(\Sigma\) is introduced in Eq.(2.39) quite arbitrarily, we can, following Ferrell, Ref. [21], take full advantage of this fact now. To do so, we notice that from the point of view of the observer, the relation (2.36) holds irrespective of the absence or presence of hydrodynamic screening. Because of this, we write
\[\Sigma(1 + \frac{1}{\kappa_G}) = \dot{\xi} \tag{2.40}\]

\[\text{For more details, see our work, Ref.[23].}\]
so that the Eq.(2.36) used in the generalized Stokes-Einstein relation remains unchanged. By combining Eq.s (2.36b), (2.38) and (2.40) we obtain:

\[ \kappa_G = \frac{\xi_H}{\xi_0} \frac{1}{\sqrt{S(0,0)}} = \frac{\Sigma}{\xi_0} \frac{\kappa_G}{\sqrt{S(0,0)}} (1 + \frac{1}{\kappa_G}) \]  

(2.41a)

or, equivalently,

\[ \frac{\xi_0}{\Sigma} = \frac{1}{\sqrt{S(0,0)}} (1 + \frac{1}{\kappa_G}) \]  

(2.41b)

In this equation the parameter \( \Sigma \) is still undefined. We can define this parameter now based on physical arguments. In particular, let us set \( \Sigma = S(0,0)\xi_0 \). Then, we end up with the equation

\[ 1 + \frac{1}{\kappa_G} = \frac{1}{\sqrt{S(0,0)}} \]  

(2.42)

leading to

\[ \kappa_G = \frac{1}{\sqrt{S(0,0)}} - 1. \]  

(2.43)

To reveal the physical meaning of this equation we use Eq.s (2.36b), (2.38) and (2.43) in order to obtain

\[ \xi_H = \frac{\sqrt{S(0,0)}\xi_0}{\sqrt{S(0,0)}} - 1 \]  

(2.44)

From Eq.(2.12) we notice that by considering the infinite dilution limit \( \varphi \to 0 \), we obtain: \( \xi_H \to \infty \), implying absence of hydrodynamic screening. Consider the opposite case: \( \varphi \to \infty \) (that is, in practice, \( \varphi \) being large). Looking at Eq.(2.12) we notice that in this case \( \xi_H \to 0 \) indicating the complete screening, as expected. Using Eq.(2.42), these results allow us to rewrite the generalized Stokes-Einstein relation, Eq.(2.18), in the equivalent form emphasizing the role of hydrodynamic screening. Thus, we obtain,

\[ \frac{D}{D_0} = \frac{1}{(\eta/\eta_0)} (1 + \frac{1}{\kappa_G}). \]  

(2.45)

### 3 Diffusion processes in the presence of hydrodynamic interactions

#### 3.1 Some facts from the diffusion theory

If \( n \) is the local density, then the flux \( j = n\mathbf{v} \) obeys Fick’s first law:

\[ \mathbf{j} = -D\nabla n, \]  

(3.1)

where \( D \) is the (in general, cooperative) diffusion coefficient, and \( \mathbf{v} \) is the local velocity. Upon substitution of this expression into the continuity equation

\[ \frac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0 \]  

(3.2)

we obtain the diffusion equation commonly known as Fick’s second law

\[ \frac{\partial n}{\partial t} = D\nabla^2 n. \]  

(3.3)
In the presence of some external forces, i.e.
\[ F = -\nabla U, \]
the diffusion laws must be modified. This is achieved by assuming the existence of some kind of friction, i.e. by assuming that there exists a relation
\[ \gamma v = F \]
between the local velocity \( v \) and force \( F \) with the coefficient of proportionality \( \gamma \) being, for instance (in the case of hard spheres), of the type given in Eq.(1.1). With such an assumption, the diffusion current, Eq.(3.1), is modified now as follows
\[ j = -D \nabla n - \frac{n}{\gamma} \nabla U. \]
Such a definition makes sense. Indeed, in the case of equilibrium, when the concentration \( n_{eq} \) obeys the Boltzmann’s law
\[ n_{eq} = n_0 \exp\left(-\frac{U}{k_B T}\right), \]
vanishing of the current in Eq.(3.6) is assured by substitution of Eq.(3.7) into Eq.(3.6) thus leading to the already cited Einstein result, Eq.(1.1), for \( D_0 \). As in the case of Eq.(1.3), we shall assume that for finite concentrations one can still use the Einstein-like result for the diffusion coefficient. With such an assumption, the current \( j \) in Eq.(3.6) acquires the following form 
\[ j = -n_0 \gamma \nabla \left( k_B T \ln n + U \right) \equiv -\frac{n}{\gamma} \nabla \mu, \]
where the last equality defines the nonequilibrium chemical potential \( \mu \), e.g. like that given in Eq.(2.1).
Alternatively, the modified flux velocity \( v_f \) is given now by \(-\frac{\gamma}{\gamma} \nabla \mu\) so that the continuity Eq.(3.2) reads as
\[ \frac{\partial n}{\partial t} + \nabla \cdot (n v_f) = 0. \]
Exactly the same equation can be written for the probability density \( \Psi \) if we formally replace \( n \) by \( \Psi \) in the above equation [10]. Such an interpretation of diffusion is convenient since it allows one to talk about diffusion in terms of the trajectories of Brownian motion of individual particles whose positions \( x_n(t), n = 1, 2, \ldots \) are considered to be as random variables. Then, the probability \( \Psi \) describes such collective Brownian motion process described by the following Schrodinger-like equation
\[ \frac{\partial}{\partial t} \Psi = -\sum_n \frac{\partial}{\partial x_n} (\Psi v_{f n}) \]
in which the velocity \( v_{f n} \) is given by
\[ v_{f n} = -\sum_m L_{nm} \frac{\partial}{\partial x_m} (k_B T \ln \Psi + U). \]
Thus, we obtain our final result
\[ \frac{\partial}{\partial t} \Psi = \sum_{m,n} \frac{\partial}{\partial x_n} L_{nm} (k_B T \frac{\partial}{\partial x_m} \Psi + \frac{\partial U}{\partial x_m} \Psi) \]
adaptable for hydrodynamic extension. For this purpose, we need to remind our readers of some basic facts from hydrodynamics.
3.2 Hydrodynamic fluctuations and Oseen tensor

The analog of Newton’s equation for fluids is the Navier-Stokes equation. It is given by [29]

\[
\frac{\partial}{\partial t} \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla P + \Gamma \nabla^2 \mathbf{v}
\]  

(3.13)

where \( P \) is the hydrodynamic pressure, \( \Gamma = \eta_0/\rho_0 \) is the kinematic viscosity and \( \rho_0 \) is the density of the the pure solvent. At low Reynold’s numbers, the convective term \((\mathbf{v} \cdot \nabla) \mathbf{v}\) can be neglected [29], p.63. We shall also assume that the fluid is incompressible, i.e

\[
\text{div} \, \mathbf{v} = 0.
\]  

(3.14)

Under such conditions the Fourier transformed Navier-Stokes equation can be written as

\[
\frac{\partial}{\partial t} \mathbf{v}_k = -\Gamma k^2 \mathbf{v}_k - \frac{i \mathbf{k}}{\rho} P_k.
\]  

(3.15)

Let us add a fluctuating source term \( f_k \) to the right hand side of Eq.(3.15). Then, using the incompressibility condition, Eq.(3.14), we obtain:

\[
P_k = -i \rho \frac{\mathbf{k} \cdot \mathbf{f}_k(t)}{k^2}.
\]  

(3.16)

Introducing the transverse tensor \( T_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2} \) and decomposing a random force as

\[
f^T_k(t) = \sum_j T_{ij}(k) f_{jk}(t)
\]  

(3.17)

eventually replaces the Navier-Stokes equation by the Langevin-type equation for the transverse velocity fluctuations:

\[
\frac{\partial}{\partial t} \mathbf{v}_k + \Gamma k^2 \mathbf{v}_k = f^T_k(t).
\]  

(3.18)

As is usually done for such type of equations, we shall assume that the random fluctuating forces are Gaussianly distributed. This assumption is equivalent to the statement that

\[
\langle f^T_{ik}(t) f^T_{jk}(t') \rangle = T_{ij}(k) (2\pi)^3 \delta(k + k') \tilde{D} \delta(t - t')
\]  

(3.19)

with parameter \( \tilde{D} \) to be determined. A formal solution of the Langevin-type Eq.(3.18) is given by

\[
\mathbf{v}_k(t) = \mathbf{v}_k(0) e^{-\tilde{\Gamma} t} + \int_0^t dt' \tilde{f}^T_k(t') e^{-\tilde{\Gamma}(t-t')}
\]  

(3.20)

with \( \tilde{\Gamma} = k^2 \Gamma \). Introducing \( \mathbf{v}_k(t) - \mathbf{v}_k(0) e^{-\tilde{\Gamma} t} = \hat{\mathbf{v}}_k(t) \), we obtain

\[
\langle \hat{v}_{ik}(t) \hat{v}_{jk}(t') \rangle = \left\langle \int_0^t dt'' \tilde{f}^T_k(t') e^{-\tilde{\Gamma}(t-t'')} \int_0^{t''} dt''' \tilde{f}^T_{jk}(t''') e^{-\tilde{\Gamma}(t''-t''')} \right\rangle.
\]  

(3.21)

To calculate this correlator, and to determine the parameter \( 2 \tilde{D} \), we consider the equal time correlator first. In such a case the equipartition theorem produces the following result:

\[
\langle \hat{v}_{ik}(t) \hat{v}_{jk}(t) \rangle = T_{ij}(k) (2\pi)^3 \delta(k + k') \frac{k_B T}{\rho}.
\]  

(3.22)
Taking into account Eqs. (3.19) and (3.22) we obtain for the velocity-velocity correlator, Eq.(3.21), the following result

\[
\langle \hat{v}_{ik}(t) \hat{v}_{jk}(t') \rangle = T_{ij}(k)(2\pi)^3 \delta(k+k') \frac{2k_B T \hat{\Gamma}}{\rho \hat{\Gamma}} \exp(-\hat{\Gamma} |t - t'|)
\]

\[
= 2k_B T \mathcal{H}_{ij}(k) \frac{\hat{\Gamma}}{2} \exp(-\hat{\Gamma} |t - t'|).
\] (3.23)

In the limit \( \hat{\Gamma} \to \infty \) the combination \( \frac{2k_B T \hat{\Gamma}}{\rho \hat{\Gamma}} \exp(-\hat{\Gamma} |t - t'|) \) can be replaced by \( \delta(t - t') \). In this limit the obtained expression coincides with already cited Eq.(2.30). Furthermore, the constant \( \tilde{D} \) can be chosen as \( k_B T \rho \). To prove the correctness of these assumptions, we take a Fourier transform (in time variable) in order to obtain

\[
\langle \hat{v}_{ik}(\omega) \hat{v}_{jk}(-\omega) \rangle = 2k_B T \rho \mathcal{H}_{ij}(k) \omega^2 + \hat{\Gamma}^2 T_{ij}(k).
\] (3.24)

This result coincides with Eq.(89.17) of Ref.[25] as required. Here the sign \( \dot{=} \) means “up to a delta function prefactor”. Incidentally, these prefactors were preserved in another volume of Landau and Lifshitz, e.g. see Ref. [30], Eq.(122.12). Since in the limit \( \omega \to 0 \) we reobtain (upon inverse Fourier transform in time) Eq.(2.30), this fact provides the needed justification for replacement of the factor \( \frac{2k_B T \hat{\Gamma}}{\rho \hat{\Gamma}} \exp(-\hat{\Gamma} |t - t'|) \) by \( \delta(t - t') \).

In polymer physics, Ref.[10], typically only this \( \omega \to 0 \) limit is considered, which is equivalent to considering physical processes at time scales much larger than the characteristic time scale \( \tau_r = \rho R^2 / \pi \eta_0 \) mentioned in the Introduction. Although this fact could cause some inconsistencies (e.g. see discussion below), we shall follow the traditional pathway by considering mainly this limit causing us to drop altogether time-dependence in Eq.(3.15) thus bringing it to the form considered in the book by Doi and Edwards, Ref.[10], Eq. (3.III.2). Following these authors, this approximation allows us to specify a random force \( f(r) \) as

\[
f(r) = \sum_n F_n \delta(r - R_n)
\] (3.25)

implying that particle (hard sphere) locations are at the points \( R_n \) so that the fluctuating component of fluid velocity \( v(r) \) at \( r \) is given by

\[
v(r) = \sum_n \mathbf{H}(r - R_n) \cdot F_n
\] (3.26)

with the Oseen tensor \( \mathbf{H}_{ij}(r) \) in the coordinate representation given by

\[
\mathbf{H}_{ij}(r) = \frac{1}{8\pi \eta |r|} (\delta_{ij} + \hat{r}_i \hat{r}_j).
\] (3.27)

In this expression \( \hat{r}_i = \frac{r_i}{|r|} \). In view of Eq.(2.27), we can rewrite Eq.(3.26) in the following suggestive form

\[
v(R_n) = \sum_{m(m \neq n)} \mathbf{H}(R_n - R_m) \cdot F_m,
\] (3.28)

for velocity \( v(R_n) \) of the particle located at \( R_n \).

### 3.3 Fick’s laws in the presence of hydrodynamic interactions. Emergence of gauge fields

By comparing Eqs(3.12) and (3.28) we could write the Fick’s first law explicitly should the Oseen tensor be also defined for \( m = n \). But it is not defined in this case. As in electrostatics, self-interactions must be excluded from consideration. In view of the results of Section 2, the situation can be repaired if we
assume that at some concentrations the hydrodynamic interactions are totally screened. In such a case only the usual Brownian motion of individual particles is expected to survive. With these remarks, Fick’s first law for such hydrodynamically interacting suspensions of spheres can be written now as follows

$$v_{f} = -\sum_{m} \tilde{H}(R_{n} - R_{m}) \cdot \frac{\partial}{\partial R_{m}}(k_{B}T \ln \Psi + U), \quad (3.29)$$

where the redefined Oseen tensor $\tilde{H}_{ij}(R)$ has the diagonal part $\tilde{H}_{ii}(R) = 1/\gamma$ in accord with Eq.(1.1). The potential $U$ comes from short-range non-hydrodynamic interactions between particles, which are always present. Using this result and Eq.(3.10), we finally arrive at the Fick’s second law

$$\frac{\partial \Psi}{\partial t} = \sum_{n,m} \frac{\partial}{\partial R_{n}} \cdot \tilde{H}(R_{n} - R_{m}) \cdot (k_{B}T \frac{\partial \Psi}{\partial R_{m}} + \frac{\partial U}{\partial R_{m}} \Psi) \quad (3.30)$$

in accord with Eq.(3.110) of Ref.[10]. Since this equation contains both diagonal and nondiagonal terms the question arises about its mathematical meaning. That is, we should inquire: under what conditions does the solution to this equation exist? The solution will exist if and only if the above equation can be brought to the diagonal form. To do so, as it is usually done in mathematics, we have to find generalized coordinates in which the above equation will acquire the diagonal form. Although the attempts to do so were made by several authors, most notably, by Kirkwood, e.g. see Ref.[10], chr-3 and references therein, in this work we would like to extend their results to account for effects of gauge invariance.

We begin with the following auxiliary problem: since $\nabla^{2} = \text{div} \cdot \nabla = \nabla \cdot \nabla$, we are interested in finding how this result changes if we transform it from the flat Euclidean space to the space described in terms of generalized coordinates. This task is easy if we take into account that in the Euclidean space

$$\nabla \cdot \nabla = \sum_{i,j} \frac{\partial}{\partial x_{i}} h_{ij} \frac{\partial}{\partial x_{j}}, \quad (3.31)$$

with $h^{ij}$ being a diagonal matrix with unit entries. We notice that the above expression is a scalar and, hence, it is covariant. This means, that we can replace the usual derivatives by covariant derivatives, the metric tensor $h_{ij}$ by the metric tensor $g_{ij}$ in the curved space so that in this, the most general case, we obtain

$$D_{i} g^{ij} D_{j} f(x) = \frac{\partial}{\partial x_{i}} g^{ij} \frac{\partial}{\partial x_{j}} f(x) + g^{kj} \Gamma_{ik}^{j} \frac{\partial}{\partial x_{j}} f(x), \quad (3.32)$$

where summation over repeated indices is assumed, as usual. The covariant derivative $D_{i}$ is defined for a scalar $f$ as $D_{i} f = \frac{\partial}{\partial x_{i}} f$ and for contravariant vector $X^{i}$ as

$$D_{j} X^{i} = \frac{\partial X^{i}}{\partial x_{j}} + \Gamma_{jk}^{i} X^{i} \quad (3.33)$$

with Christoffel symbol $\Gamma_{jk}^{i}$ defined in a usual way of Riemannian geometry. A precise definition of this symbol is going to be given below. Since $\Gamma_{ik}^{i} = \frac{\partial}{\partial x_{k}} \ln \sqrt{g}$, we can rewrite Eq.(3.32) in the following alternative final form

$$\nabla^{2} f = D_{i} g^{ij} D_{j} f(x) = \frac{1}{\sqrt{g} \frac{\partial}{\partial x_{i}}} [g^{ij} \sqrt{g} \frac{\partial}{\partial x_{i}} f] \quad (3.34)$$

so that in Eq.(3.3) the operator $\nabla^{2}$ is replaced now by that given by Eq.(3.34). To make this presentation complete, we have to include the relation

$$g_{ij} = \frac{\partial r^{k}}{\partial q^{i}} \frac{\partial r^{l}}{\partial q^{j}} h_{kl}. \quad (3.35)$$

In the simplest case, when we are dealing with 3 dimensional vectors, so that $r = r(q_{1}, q_{2}, q_{3})$, sometimes it is convenient to introduce vectors

$$e_{i} = \frac{\partial r}{\partial q^{i}}. \quad (3.36)$$
and the metric tensor

\[ g_{ij} = e_i \cdot e_j \]  

(3.37)

with \( \cdot \) being the usual Euclidean scalar product sign. Definitions Eq.(3.35) and (3.37) are obviously equivalent in the present case. Because of this, it is clear that upon transformation to the curvilinear coordinates the Riemann curvature tensor written in terms of \( g_{ij} \) is still zero since it is obviously zero for the \( h_{kl} \). The curvature tensor will be introduced and discussed below. Before doing so, using the example we have just described, we need to rewrite Eq.(3.30) in terms of generalized coordinates. In the present case, we must have \( 3N \) generalized coordinates and the tensor \( h_{kl} \) is not a unit tensor anymore. Our arguments will not change if we replace Eq.(3.30) by that in which the potential \( U = 0 \). Furthermore, we shall adsorb the factor \( k_B T \) into the tensor \( \tilde{H} \) and this redefined tensor we shall use instead of \( h_{kl} \). Evidently, the final result for the Laplacian, Eq.(3.34), will remain unchanged. The question arises: if in the first example the Riemannian curvature tensor remains flat after a coordinate transformation (since the tensor \( h_{kl} \) is the tensor describing the flat Euclidean space), what can be said about the Riemann tensor in the present case? To answer this question consider once again Eq.(3.35), this time with the tensor \( H_{mn} \) instead of \( h_{ij} \). For the sake of argument, let us ignore for a moment the fact that each of the \( q \) is 3-dimensional. Then, we obtain,

\[ g_{\alpha\beta} = \frac{\partial R^k}{\partial q^\alpha} \tilde{H}_{kl} \frac{\partial R^l}{\partial q^\beta}, \]

(3.38a)

where we introduced a set of new generalized coordinates \( \{Q\} = \{q_1, ..., q_N\} \) so that \( R_l = R_l(\{Q\}) \). We shall use Greek indices for new coordinates and Latin for old. In the case of 3 dimensions the above result becomes

\[ g_{\alpha\beta} = \frac{\partial R^k}{\partial q^\alpha} \tilde{H}_{kl} \frac{\partial R^l}{\partial q^\beta}, \]

(3.38b)

with \( \cdot \) being the Euclidean scalar product sign as before. The indices \( k, l, \alpha \) and \( \beta \) now have 3 components each. We are interested in generalized coordinates which make the metric tensor \( g_{\alpha\beta} \) diagonal. By analogy with Eq.(3.36), we introduce now a scalar product

\[ < R \cdot R > \equiv R^k \cdot \tilde{H}_{kl} \cdot R^l \]

(3.39)

so that instead of the vectors \( e_i \) we obtain now

\[ e_\alpha = \frac{\partial R}{\partial q^\alpha} \]

(3.40)

and, accordingly,

\[ g_{\alpha\beta} = < e_\alpha \cdot e_\beta >. \]

(3.41)

The Christoffel symbol can be defined now as

\[ \frac{\partial e_\alpha}{\partial q^\beta} = \Gamma^\gamma_{\alpha\beta} e_\gamma. \]

(3.42)

To find the needed generalized coordinates, we impose an additional constraint

\[ \frac{\partial e_\alpha}{\partial q^\beta} = \frac{\partial e_\beta}{\partial q^\alpha}, \]

(3.43)

compatible with the symmetry of the tensor \( \tilde{H}_{kl} \). By combining Eq.s(3.42) and (3.43) we obtain,

\[ \Gamma^\gamma_{\alpha\beta} e_\gamma = \Gamma^\gamma_{\beta\alpha} e_\gamma \]

(3.44)

implying that \( \Gamma^\gamma_{\alpha\beta} = \Gamma^\gamma_{\beta\alpha} \). That is, the imposition of the constraint, Eq.(3.43), is equivalent to requiring that our new generalized space is Riemannian (that is, without torsion). In such a space we would like to consider the following combination

\[ R_{\alpha\beta} = \frac{\partial^2 e_\alpha}{\partial q^\alpha \partial q^\beta} - \frac{\partial^2 e_\beta}{\partial q^\beta \partial q^\alpha}. \]

(3.45)
Again, using Eq.(3.42) we obtain
\[ \frac{\partial}{\partial q^\alpha} (\Gamma^\gamma_{\alpha\beta} e^\gamma) = \left( \frac{\partial}{\partial q^\alpha} \Gamma^\gamma_{\alpha\beta} \right) e^\gamma + \left( \frac{\partial}{\partial q^\beta} \Gamma^\gamma_{\alpha\beta} \right) e^\gamma. \] (3.46)

Analogously, we obtain
\[ \frac{\partial}{\partial q^\beta} (\Gamma^\gamma_{\beta\alpha} e^\gamma) = \left( \frac{\partial}{\partial q^\alpha} \Gamma^\gamma_{\beta\alpha} \right) e^\gamma + \left( \frac{\partial}{\partial q^\beta} \Gamma^\gamma_{\beta\alpha} \right) e^\gamma. \] (3.47)

Finally, we use Eq.(3.42) in Eq.(3.46) and (3.47) in order to obtain the following result for \( R_{\alpha\beta} \)
\[ R_{\alpha\beta} = \left( \frac{\partial}{\partial q^\alpha} \Gamma^\gamma_{\beta\alpha} \right) e^\gamma - \left( \frac{\partial}{\partial q^\beta} \Gamma^\gamma_{\beta\alpha} \right) e^\gamma + \Gamma^\gamma_{\alpha\beta} \Gamma^\gamma_{\omega\alpha} e^\gamma - \Gamma^\gamma_{\alpha\beta} \Gamma^\gamma_{\omega\alpha} e^\gamma \]
\[ = R^\gamma_{\alpha\beta} e^\gamma. \] (3.48)

The second line defines the Riemann curvature tensor. In the most general case it is given by \( R^\gamma_{\alpha\beta} \). By combining Eqs.(3.40), (3.43), (3.45) and (3.48) we conclude that
\[ \frac{\partial^2 e^\alpha}{\partial q^\alpha \partial q^\beta} = \frac{\partial^2 e^\beta}{\partial q^\beta \partial q^\alpha} \] (3.49)

implying that the Riemann tensor is zero so that the connection \( \Gamma^\gamma_{\alpha\beta} \) is flat. For such a case we can replace the covariant derivative \( D_i \) by \( \nabla_i + A_i \). The vector field \( A_i \) is defined as follows. Introduce a 1-form \( A \) via \( A = A_i dx^i \), \( A_i = A^\alpha_i T^\alpha \), where in the non-Abelian case \( T^\alpha \) is one of infinitesimal generators of some Lie group \( G \) obeying the commutation relations \( [T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma \) of the associated with it Lie algebra. In addition, \( \text{tr}(T^\alpha T^\beta) = \frac{1}{2} \delta^{\alpha\beta} \). The Chern-Simons (C-S) functional \( CS(A) \) producing upon minimization the needed flat connections is given by \( k \) \[ CS(A) = \frac{k}{4\pi} \int_M \text{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \]
\[ = \frac{k}{8\pi} \int_M \varepsilon^{ijk} \text{tr}(A_i (\partial_j A_k - \partial_k A_j) + \frac{2}{3} A_i [A_j, A_k]) \] (3.50)

with \( k \) being some integer. Minimization of this functional produces an equation for the flat connections. Indeed, we have
\[ \frac{8\pi}{k} CS(A + B) = \int_M \text{tr}(B \wedge dA + A \wedge B + 2B \wedge A \wedge A) \]
\[ = 2 \int_M \text{tr}(B \wedge (dA + A \wedge A)), \] (3.51)

where we took into account that
\[ \int_M \text{tr}(A_i dx^i \wedge \frac{\partial B_k}{\partial dx^j} dx^j \wedge dx^k) = \int_M \text{tr}(B_k dx^k \wedge \frac{\partial A_i}{\partial dx^j} dx^j \wedge dx^i). \]

From here, by requiring
\[ \frac{\delta}{\delta B} CS(A + B) = 0 \] (3.52)

we obtain our final result:
\[ dA + A \wedge A = \left( \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} + [A_i, A_j] \right) dx^i \wedge dx^j \equiv F(A) dx^i \wedge dx^j = 0. \] (3.53)

In the last equality we have taken into account that both in the C-S and Yang-Mills theory \( F(A) \) is the curvature associated with connection \( A \). Vanishing of curvature produces Eq.(3.53) for the field \( A \). Irrespective to the explicit form of the field \( A \), we have just demonstrated that, at least in the case when the potential \( U \) in Eq.(3.30) is zero, this equation can be brought into diagonal form provided that the operator \( \nabla_i \) is replaced by \( \nabla_i + A_i \) with the field \( A_i \) to be specified below, in the next section.
4 An interplay between topology and randomness: connections with the vortex model of superfluid $^4$He

4.1 General comments

The C-S functional, Eq.(3.50), whose minimization produces Eq.(3.53) for the field $A$ was introduced into physics by Witten [32] and was discussed in the context of polymer physics in our previous works summarized in Ref.[33]. Since polymer physics of fully flexible polymer chains involves diffusion-type equations [10], the connections between polymer and colloidal physics are apparent. For this reason, we follow Ref.[32] in our exposition and use it as general source of information.

Specifically, as explained by Witten [32], theories based on the C-S functional are known as topological field theories. The averages in these theories produce all kinds of topological invariants (depending upon the generators $T^\alpha$ in the non Abelian case) which are observables for such theories. In the present case the question arises: should we use the non Abelian version of the C-S field theory or is it sufficient to use only its Abelian version, to be defined shortly? Since both versions of C-S theory were discussed in the context of polymer physics in Ref.[33], we would like to argue that, for the purposes of this work, the Abelian version of the C-S theory is sufficient. We shall provide the proof of this fact in this section.

The action functional for the abelian C-S field theory is given by

$$ S_{C-S}^A[A] = \frac{k}{8\pi} \int_M d^3x \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho $$

(4.1)

With such defined functional one calculates the (topological) averages with help of the C-S probability measure

$$ \langle \cdots \rangle_{C-S} = \hat{N} \int D[A] \exp\{iS_{C-S}^A[A]\} \cdots $$

(4.2)

The random objects which are subject to averaging are the Abelian Wilson loops $W(C)$ defined by

$$ W(C) = \exp\{ie \oint_C d\mathbf{r} \cdot \mathbf{A}\}, $$

(4.3)

where $C$ is some closed contour in 3 dimensional space (normally, without self-intersections), and $e$ is some constant ("charge") whose exact value is of no interest to us at this moment. The averages of products of Wilson’s loops (perhaps, forming a link $L$)

$$ W(L) = \prod_{i=1}^n W(C_i) $$

(4.4)

are the main objects of study in such a topological field theory. Substitution of $W(L)$ into Eq.(4.2) produces the following result [32]

$$ \langle W(L) \rangle_{C-S} = \exp\{i \left( \frac{2\pi}{k} \sum_{i,j} e_ie_j lk(i,j) \right) \} $$

(4.5)

with the (Gauss) linking number $lk(i,j)$ defined as

$$ lk(i,j) = \frac{1}{4\pi} \oint_{C_i} \oint_{C_j} [d\mathbf{r}_i \times d\mathbf{r}_j] \cdot \frac{(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} = \frac{k}{4\pi} \int_0^{T_i} \int_0^{T_j} ds_1 ds_2 [\mathbf{v}(s_i) \times \mathbf{v}(s_j)] \cdot \frac{(\mathbf{r}(s_i) - \mathbf{r}(s_j))}{|\mathbf{r}(s_i) - \mathbf{r}(s_j)|^3}, $$

(4.6)

\footnote{E.g. see Eq.(4.12) of Ref.[33].}
Here $T_i$ and $T_j$ are respectively the contour lengths of contours $C_i$ and $C_j$ and $v(s) = \frac{d}{ds}r(s)$. With the Gauss linking number defined in such a way, in view of Eq.(4.5), it should be clear that we must consider as well self-linking numbers $lk(i,i)$. Such a technicality requires us to think about the so called framing operation discussed in some detail in both Ref.s [32] and [33]. We shall ignore this technicality until Section 6 for reasons which will become apparent.

### 4.2 An interplay between the topology and randomness in hydrodynamics

Following Tanaka and Ferrari, Refs [34, 35], we rewrite the Gauss linking number in a more physically suggestive form. For this purpose, we introduce the "magnetic" field $B(r)$ via

$$B(r) = \frac{1}{4\pi} \oint_{C_j} dr_j \times \frac{(r - r(s_j))}{|r - r(s_j)|^3}$$

(4.7)

allowing us to rewrite the linking number $lk(i,j)$ as

$$lk(i,j) = \oint_{C_i} dr_i \cdot B(r_i).$$

(4.8)

Eq.(4.7) for the field $B(r)$ is known in magnetostatics as Biot-Savart law, e.g. see Ref.[36], Eq.(30.14). Because of this, we recognize that

$$\nabla \cdot B = 0$$

(4.9)

and

$$\nabla \times B = j,$$

(4.10)

where $j(r) = \oint_{C} ds v(s) \delta(r - r(s))$. To connect these results with hydrodynamics, we introduce the vector potential $A$ in such a way that

$$\nabla \times A = B.$$  

(4.11)

Using this result in Eq.(4.10) we obtain as well

$$\nabla^2 A = -j$$

(4.12)

in view of the fact that $\nabla \cdot A = 0$. In hydrodynamics we can represent the local fluid velocity following Ref.[37], page 86, as

$$v = \nabla \times A$$

(4.13)

and define the vorticity $\tilde{\omega}$ as

$$\tilde{\omega} = \nabla \times v.$$ 

(4.14)

By analogy with Eq.s (4.10) and (4.12) we now obtain

$$\nabla^2 A = -\tilde{\omega}.$$ 

(4.15)

Hence, to apply previous results to hydrodynamics the following identification should be made:

$$\tilde{\omega} \rightarrow j; v \rightarrow B.$$ 

(4.16)

The kinetic energy $E$ of a fluid in a volume $M$ is given by

$$E = \frac{\rho}{2} \int_M v^2 d^3r.$$ 

(4.17)

We would like now to explain how this energy is related to the above defined linking numbers. For this purpose, we introduce the following auxiliary functional:

$$F[A] = \oint_{C_i} dr_i \cdot A(r_i).$$ 

(4.18)
Use of the theorem by Stokes produces

\[
\mathcal{F}[\mathbf{A}]_{i} = \oint_{C_{i}} d\mathbf{r}_{i} \cdot \mathbf{A}(\mathbf{r}_{i}) = \iint_{S_{i}} d\mathbf{S}_{i} \cdot (\nabla \times \mathbf{A}) = \iint_{S_{i}} d\mathbf{S}_{i} \cdot \mathbf{v} = \iint_{S_{i}} d\mathbf{S}_{i} \oint_{C_{j}} ds_{j} \mathbf{v}(s_{j}) \delta(\mathbf{r}_{i} - \mathbf{r}(s_{j})).
\]  \hspace{1cm} (4.19)

At the same time, for the linking number, Eq.(4.8), an analogous procedure leads to the following chain of equalities

\[
\text{lk}(i, j) = \oint_{C_{i}} d\mathbf{r}_{i} \cdot \mathbf{B}(\mathbf{r}_{j}) = \oint_{C_{i}} d\mathbf{r}_{i} \cdot \mathbf{v}(\mathbf{r}_{j}) = \iint_{S_{i}} d\mathbf{S}_{i} \cdot (\nabla \times \mathbf{v}) = \iint_{S_{i}} d\mathbf{S}_{i} \cdot \tilde{\omega}.
\]  \hspace{1cm} (4.20a)

Since the same vector potential was used in both Eqs.(4.11) and (4.13) we notice that Eqs.(4.12) and (4.15) also imply that

\[
\tilde{\omega} = e\mathbf{e}_{j},
\]  \hspace{1cm} (4.20b)

where \(e\) is some constant to be determined. Because of this, we obtain

\[
e\mathcal{F}[\mathbf{A}]_{i} = \text{lk}(i, j) = e \iint_{S_{i}} d\mathbf{S}_{i} \oint_{C_{j}} ds_{j} \mathbf{v}(s_{j}) \delta(\mathbf{r}_{i} - \mathbf{r}(s_{j})).
\]  \hspace{1cm} (4.21)

Since the obtained equivalence is of central importance for the entire work, we would like to discuss a few additional details of immediate relevance. In particular, from Eq.(4.20b), which we shall call from now on, the London equation (e.g. see the Subsection 4.4 below), it should be clear that the as yet unknown constant \(e\) must have dimensionality of inverse length \(L^{-1}\). This fact should be taken into account when we consider the following dimensionless\(^7\) functional

\[
\mathcal{W}[\mathbf{A}] = \frac{\rho}{2k_{B}T} \int_{M} d^{3}\mathbf{r}(\nabla \times \mathbf{A})^{2} + i\frac{e}{f} \sum_{j} \oint_{C_{j}} d\mathbf{r}_{j} \cdot \mathbf{A}(\mathbf{r}_{j})
\]  \hspace{1cm} (4.22)

and the path integral associated with it, i.e.

\[
\hat{N} \int D[\mathbf{A}] \delta(\nabla \cdot \mathbf{A}) \exp\{-\mathcal{W}[\mathbf{A}]\} = < W(L) >_{T}
\]  \hspace{1cm} (4.23a)

to be compared with Eqs.(4.2) and (4.5). Here the thermal average \(< \cdots >_{T}\) is defined by

\[
< \cdots >_{T} = \hat{N} \int D[\mathbf{A}] \delta(\nabla \cdot \mathbf{A}) \exp\{-\frac{\rho}{2k_{B}T} \int_{M} d^{3}\mathbf{r}(\nabla \times \mathbf{A})^{2}\} \cdots.
\]  \hspace{1cm} (4.23b)

Calculation of this Gaussian path integral is complicated by the presence of a delta constraint (Coulomb gauge) in the path integral measure. Fortunately, this path integral can be found in the paper by Brereton and Shah [38]. Without providing the details, these authors presented the following final result in notations adapted to this work:

\[
< W(L) >_{T} = \exp\{-\frac{1}{2\rho} \left(\frac{e}{f}\right)^{2} \sum_{i,j=1}^{t} \int_{0}^{t} ds_{i} ds_{j} \dot{\mathbf{r}}(s_{i}) \cdot \tilde{\mathbf{H}}[\mathbf{r}(s_{i}) - \mathbf{r}(s_{j})] \cdot \dot{\mathbf{r}}(s_{j})\}.
\]  \hspace{1cm} (4.24)

The Oseen tensor \(\tilde{\mathbf{H}}(\mathbf{R})\) in this expression was previously defined in Eq.(3.27) and the prime on the summation sign means that the diagonal part of this tensor should be excluded. Even though calculations leading to this result are not given in Ref.[38], they can be easily understood field-theoretically. For this purpose, we have to regularize the delta function constraint in the path integral measure in Eq.(4.23) very

\(^{7}\)In view of the fact that the dimensionality of \(e\) is fixed we have introduced a factor \(f\) which makes the functional \(\mathcal{W}[\mathbf{A}]\) dimensionless. This factor will be determined shortly below.
much the same way as Ferrell, Ref.[21], did it in the case of hydrodynamics as we discussed in Section 2. Specifically, we write
\[
\tilde{N} \int D[\sigma(r)] \exp\left(-\frac{1}{2\xi} \int d^3r \sigma^2(r)\right) \int D[A] \delta(\nabla \cdot A - \sigma(r)) \exp\left(-\frac{\rho}{2k_B T} \int_M d^3r (\nabla \times A)^2\right) \cdots
\]
\[
= \tilde{N} \int D[A] \exp\left\{ -\frac{\rho}{2k_B T} \int_M d^3r (\nabla \times A)^2 - \frac{1}{2\xi} \int d^3r \sigma^2(r) \right\} \cdots
\]
\[
= \tilde{N} \int D[A] \exp\left\{ -\frac{\rho}{2k_B T} \int_M d^3r [A_{\mu} (-\delta_{\mu\nu} \nabla^2 - (1 - \frac{1}{\xi}) \partial_\mu \partial_\nu)] A_\nu \right\} \cdots.
\]
(4.25)
with some adjustable regularizing parameter \(\tilde{\xi}\). Also, for the quadratic form (in \(A\)) in the exponent of the last expression we obtain
\[
\int_M d^3r A_\mu [-\delta_{\mu\nu} \nabla^2 - (1 - \frac{1}{\xi}) \partial_\mu \partial_\nu] A_\nu = \int d^3k A_\mu(k) [\delta_{\mu\nu} k^2 - (1 - \frac{1}{\xi}) k_\mu k_\nu] A_\nu
\]
(4.26)
The inverse of the matrix \([\delta_{\mu\nu} k^2 - (1 - \frac{1}{\xi}) k_\mu k_\nu]\) is easy to find following Ramond [39]. Indeed, we write
\[
[\delta_{\mu\nu} k^2 - (1 - \frac{1}{\xi}) k_\mu k_\nu] = [X(k) \delta_{\nu\rho} + Y(k) k_\nu k_\rho] = \delta_{\mu\rho}.
\]
(4.27)
From here the unknown functions \(X(k)\) and \(Y(k)\) can be determined so the inverse matrix is given explicitly by
\[
[X(k) \delta_{\nu\rho} + Y(k) k_\nu k_\rho] = \frac{1}{k^2} \left[ \delta_{\nu\rho} - (1 - \tilde{\xi}) \frac{k_\nu k_\rho}{k^2} \right].
\]
(4.28)
In the limit \(\tilde{\xi} \to 0\) we recover the Oseen tensor (up to a constant \(1/\eta\)) in the \(k\)-space representation in accord with Ref.[10]. These results explain why in the average, Eq.(4.24), there are no diagonal terms. Now we are ready to determine the constant \(c\) introduced in Eq.(4.22).

4.3 Reparametrization invariance and vortex-vortex interactions

The important result for \(< W(L) >_T\) contains random velocities \(\dot{r}(s)\) and thus, seemingly, additional averaging is required. The task now lies in finding the explicit form of this averaging. To do so, several steps are required. To begin, we notice that in the absence of hydrodynamic interactions Eq.(3.30) acquires the following form
\[
\frac{\partial \Psi}{\partial t} = D_0 \sum_n \frac{\partial^2}{\partial R_n^2} \Psi
\]
(4.29)
with diffusion coefficient \(D_0\) defined in Eq.(1.1). If Eq.(4.29) we treat \(\Psi\) as Green’s function (e.g. see Appendix A for details), then it can be formally represented in the path integral form as
\[
\Psi(t; R_1, \ldots, R_N) = \int D[[R(\tau)]] \exp\left(-\frac{1}{4D_0} \int_0^t \sum_{i=1}^N |\dot{r}(\tau_i)|^2 d\tau_i\right).
\]
(4.30)
In this expression we have suppressed the explicit $R$-dependence of the path integral to avoid excessive notation. Hydrodynamic interactions can now be accounted for as follows

$$\mathcal{F} = \frac{1}{\tilde{N}} \int D[A] \exp\left\{-\frac{\rho}{2k_B T} M \right\} d^3r A_\mu \left[-\delta_\mu\nu \nabla^2 - (1 - \frac{i}{\xi}) \partial_\mu \partial_\nu\right] A_\nu$$

$$\times \int D[\{r(\tau)\}] \exp\left\{-\frac{1}{4D_0} \int_0^t \left(\sum_{j=1}^N [\dot{r}(\tau_j)]^2 \right) d\tau_j\right\} \exp\left\{i e \int_0^t \sum_{j=1}^N [\dot{r}(\tau_j)] \cdot A[r(\tau_j)] d\tau_j\right\}$$

$$\equiv <\prod_{j=1}^N \int D[\{r(\tau_j)\}] \exp\left\{-\frac{1}{4D_0} \int_0^t [\dot{r}(\tau_j)]^2 d\tau_j\right\} \exp\left\{i e \int_0^t \dot{r}(\tau_j) \cdot A[r(\tau_j)] d\tau_j\right\} >_T.$$  \hspace{1cm} (4.31)

Perturbative calculation of path integrals of the type

$$I[A; t] = \int D[\{r(\tau_j)\}] \exp\left\{-\frac{1}{4D_0} \int_0^t [\dot{r}(\tau_j)]^2 d\tau_j\right\} \exp\left\{i e \int_0^t \dot{r}(\tau_j) \cdot A[r(\tau_j)] d\tau_j\right\}$$ \hspace{1cm} (4.32)

was considered by Feynman long ago, Ref.[40]. From this paper it follows that the most obvious way to do such a calculation is to write the usual Schrödinger-like equation

$$\left(\frac{\partial}{\partial t} - D_0 (P - ieA)^2\right) G(t, r; t', r') = 0, \hspace{0.5cm} r \neq r'$$ \hspace{1cm} (4.33)

and to take into account that $(P - ieA)^2 = P^2 - ieA \cdot P - ieP \cdot A - e^2A^2 \simeq P^2 - ieA \cdot P + O(A^2)$ (since $P \cdot A = 0$). This result is useful to compare with Eq.(3.32) in order to recognize that the field $A$ is indeed a connection.

To use these results, we would like to rewrite Eq.(3.30) in the alternative form which (for $U = 0$) is given by

$$\frac{\partial \Psi}{\partial t} = D_0 \sum_n \frac{\partial^2}{\partial R_n^2} \Psi + k_B T \sum_{m,n,i,j}' \tilde{H}_{ij}(R_n - R_m) \frac{\partial}{\partial R_{in}} \frac{\partial}{\partial R_{jm}} \Psi.$$ \hspace{1cm} (4.34)

In arriving at this equation we took into account Eq.(3.14). Consider such an equation for $n = 2$. In this case, we rewrite Eq.(4.34) in the style of quantum mechanics, i.e.

$$\left(\frac{\partial}{\partial t} - H_1 - H_2 - V_{12}\right) \Psi = 0$$ \hspace{1cm} (4.35)

in which, as in quantum mechanics, we shall treat $V_{12}$ as a perturbation. The best way of dealing with such problems is to use the method of Green’s functions. For our reader’s convenience we present some facts about this method in Appendix A. Eq.(A.10) of this Appendix provides an equation for the effective potential $\mathcal{V}$. A similar type of equation was obtained in the book by Doi and Edwards, Ref.[10], in Section 5.7.3., who used methods of the effective medium theory. Using this theory they were able to prove the existence of screening for the case of polymer solutions. We shall reach an analogous conclusion about screening in colloidal suspension using different arguments to be discussed in the next subsection.

In the meantime, we would like to provide arguments justifying our previously made approximation: $(P - ieA)^2 \simeq P^2 - ieA \cdot P + O(A^2)$. Using results of Appendix A, we introduce the one-particle Green’s function $G_0$ as a solution to the equation

$$\left(\frac{\partial}{\partial t} - D_0 \frac{\partial^2}{\partial R^2}\right) G_0(R, t; R', t') = \delta(R - R') \delta(t - t').$$ \hspace{1cm} (4.36)

Having in mind the determination of previously introduced factor $f$ (in Eq.(4.22)), it is convenient to rescale the variables in this equation to convert it into dimensionless form. Evidently, the most convenient
choice is \( t = \tau / (D_0 / R_0^2) \) and \( \mathbf{R} = R_0 \mathbf{\hat{R}} \) with \( R_0 \) is the hard sphere radius introduced in Eq.(1.1) and \( \tau \) and \( \mathbf{\hat{R}} \) being dimensionless time and space coordinates. Below, we shall avoid the use of tildas for \( \mathbf{\hat{R}} \) and shall still write \( t \) instead of \( \tau \). The original symbols can be restored whenever they are required. Having this in mind, next we consider the two-particle Green’s function \( G_0 \). In the absence of interactions, it is just the product of two Green’s functions of the type given by Eq.(4.36). As a result, the Dyson-type equation for the full Green’s function for Eq.(4.34) (\( n = 2 \) case) is given by

\[
G(\mathbf{R}_1, \mathbf{R}_2; t; \mathbf{R}_1'', \mathbf{R}_2'', t'') = G_0(\mathbf{R}_1, t; \mathbf{R}_1', t') G_0(\mathbf{R}_2, t; \mathbf{R}_2', t') + \int G_0(\mathbf{R}_1, t; \mathbf{R}_1', t') G(\mathbf{R}_2, t; \mathbf{R}_1', t'; \mathbf{R}_2', t'') V(\mathbf{R}_1', \mathbf{R}_2') G(\mathbf{R}_1', \mathbf{R}_2', t'; \mathbf{R}_2'', t'') d\mathbf{R}_1' d\mathbf{R}_2' dt_1'
\]

(4.37)

in which the potential \( V(\mathbf{R}_1', \mathbf{R}_2') = k_B T \tilde{H}_ij(\mathbf{r}_1 - \mathbf{r}_2) \frac{\partial}{\partial \mathbf{R}_1} \cdot \frac{\partial}{\partial \mathbf{R}_2} \). As before, summation over repeated indices is assumed. Using results of Appendix A and Eq.(4.37) it is possible to write now the equation for the effective potential. In view of the results to be discussed in the next subsection, this is actually unnecessary. Hence, we proceed with other tasks at this point. Specifically, taking into account Eq.(3.27) in which the explicit form of the Oseen tensor is given, we conclude that the nondiagonal part of this tensor can be discarded in the Dyson Eq.(4.37). This is so because of the following obvious identity: \([ (\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{r}_1 ] [(\mathbf{r}_1 - \mathbf{r}_2) \cdot \mathbf{r}_2] + [ (\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{r}_1 ] [(\mathbf{r}_2 - \mathbf{r}_1) \cdot \mathbf{r}_2] = 0\) associated with the scalar products of unit vectors in Eq.(3.27). Evidently, it is always possible to select a coordinate system centered, say, at \( \mathbf{r}_1 \). Alternatively, this result can be easily proven in \( k \)-space taking into account the incompressibility constraint. Furthermore, these observations cause us to write the potential \( V(\mathbf{R}_1, \mathbf{R}_2) \) in the following dimensionful form\(^8\)

\[
V(\mathbf{R}_1, \mathbf{R}_2) = \frac{k_B T}{4\pi \eta} \frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|} \frac{\partial}{\partial \mathbf{R}_1} \cdot \frac{\partial}{\partial \mathbf{R}_2}.
\]

(4.38a)

Using dimensional analysis of Eq.(4.36), this result can be easily rewritten also in dimensionless form. Explicitly, it is given by

\[
V(\mathbf{R}_1', \mathbf{R}_2') = \frac{k_B T}{4\pi \eta R_0 D_0} \frac{1}{|\mathbf{R}_1 - \mathbf{R}_2|} \frac{\partial}{\partial \mathbf{R}_1} \cdot \frac{\partial}{\partial \mathbf{R}_2}
\]

(4.38b)

in which the scalar product can be of any sign. This fact is of importance because of the following.

Using Eq.(4.31) and proceeding with calculations of the path integral following Feynman’s prescriptions \(40\), we obtain exactly the same equation as that given by Eq.(4.37). This observation allows us to determine the constants \( e \) and \( f \) explicitly. In view of the results just obtained, the constant \( e \) can be determined only with accuracy up to a sign. Taking this into account, the value of \( e \) is determined as

\[
 e = \pm \frac{1}{R_0} \sqrt{\frac{D_0 \eta}{4\pi}}
\]

while the constant \( f \) is given by \( D_0 \) in view of the fact that the field \( \mathbf{A} \) in Eq.(4.22) has dimensionality \( L^2 / t \), i.e. that of the diffusion coefficient, while the dimensionality of \( e \) is fixed by the Eq.(4.20b), so that the combination \( ed\mathbf{\tilde{F}}(s) \) is dimensionless.

Using these results and Eq.(4.38), we can rewrite \( < W(L) >_T \) defined by Eq.(4.24) in the following manifestly dimensionless physically suggestive form

\[
< W(L) >_T = \exp\left( -\frac{k_B T}{D_0 8\pi \eta} \sum_{i,j=1}^{N} \sum_{s_i,s_j} \int \int \left| \frac{dr(\tau_i) \cdot dr(\tau_j)}{|r(\tau_i) - r(\tau_j)|} \right| \right)
\]

(4.39)

where we have introduced the dimensionless Ising spin-like variables \( s_i \), playing the role of charges accounting for the sign of the product \( \frac{\partial}{\partial \mathbf{R}_1} \cdot \frac{\partial}{\partial \mathbf{R}_2} \). Since the whole system must be ”electrically neutral”, at this point it is possible to develop the Debye-Hückel-type theory of hydrodynamic screening by analogy with that developed for Coulombic systems, e.g. see Ref.[41]. Nevertheless, below we choose another, more elegant path to arrive at the same conclusions.

\(^8\)Using dimensional analysis performed for Eq.(4.36) the result, Eq.(4.38), can be easily rewritten also in dimensionless form
Before doing so, we notice that there is an important difference between the double integral, Eq.(4.39), and \( \frac{1}{4D_0} \int_t^T [\dot{\mathbf{r}}(\tau_j)]^2 \, d\tau_j \) present in the exponent in Eq.(4.31). While the double integral, Eq.(4.39), is manifestly reparametrization invariant, the diffusion integral is not. This means that we can always reparametrize time in this diffusion integral so that the coefficient \((4D)^{-1}\) can be made equal to any preassigned nonnegative integer. This was effectively done already when we introduced the dimensionless variables in Eq.(4.36). Such inequivalence between these two types of integrals can be eliminated if we replace this diffusion-type integral by that which is manifestly reparametrization-invariant. In such a case the total action is given by

\[
S = m_0 \sum_i \oint d\tau_i \sqrt{\mathbf{r}^2(\tau_i) + \frac{k_B T}{D_0 S \eta} \sum_{i,j=1}^{i', j'} s_i s_j \oint \oint \frac{|d\mathbf{r}(\tau_i) \cdot d\mathbf{r}(\tau_j)|}{|\mathbf{r}(\tau_i) - \mathbf{r}(\tau_j)|} }. \tag{4.40}
\]

It should be noted that use of a symbol \( \oint \) instead of \( \int \) in Eq.(4.40) is a delicate matter. In [33] we demonstrated that in the limit of long times (that is in the limit \( \omega \to 0 \) used in this work) all random walks are asymptotically closed (that is, the Brownian trajectory in this limit becomes very much the same as known for ring polymers)\(^9\). Since the result, Eq.(4.40), is manifestly reparametrization invariant, such a replacement is permissible. Additional explanations are given in Appendix B which we recommend to read only after reading of Section 5.

The constant \( m_0 \) in Eq.(4.40) will be determined in the next section. The form of the action given by Eq.(4.40) is almost identical to that for the action for the superfluid liquid \(^4\)He as discussed in the book by Kleinert [42], page 300. From the same book, it also follows that the Ginzburg-Landau theory of superconductivity also can be recast in the same form. We said "almost identical to" meaning that in these two theories (of superfluidity and superconductivity) the self-interaction of vortices is also allowed so that if the above expression would represent the dual (vortex) description of colloidal suspension dynamics (e.g. see Appendix B), then the prime in the double summation above can be removed since the vortices are allowed to intersect with themselves.

In the direct case, when the focus of attention is on particles, removal of the prime in the double summation in Eq.(4.40) would imply that the Oseen tensor is defined for particles hydrodynamically interacting with themselves. This assumption is not present in the original Doi-Edwards formulation, Ref.[10]. As we noticed already in Eq.(3.29), the diagonal part of the Oseen tensor is associated with self-diffusion. The question therefore arises: can this "almost equivalence" be converted into full equivalence? The main feature of superconductors is the existence of the Meissner (for hard spheres) and dual (for vortices) Meissner effect. In the present case such an effect is equivalent to the existence of hydrodynamic screening. Hence, to prove such an equivalence requires us to prove the existence of hydrodynamic screening for suspensions. Evidently, we cannot immediately use Eq.(4.40) for such a proof. Therefore, in the next subsection we use London-style arguments to arrive at the desired conclusion.

### 4.4 London-style theory of hydrodynamic screening

We begin our proof by taking into account the non-slip boundary condition, Eq.(2.27):

\[
\mathbf{v}(\mathbf{r}, t) = \frac{d\mathbf{r}}{dt} = \mathbf{v}(t). \tag{2.27}
\]

Within the approximations made, we also have to impose the incompressibility requirement

\[
\nabla \cdot \mathbf{v}(\mathbf{r}, t) = 0. \tag{3.14}
\]

Because of this requirement, the current \( \mathbf{j} = \rho \mathbf{v} \) becomes \( \mathbf{j} = n_0 \mathbf{v} \) with the density \( n_0 \) being a constant. Since \( \mathbf{j} \) is a vector, we can always represent it as

\[
\mathbf{j} = \alpha \nabla \psi \tag{4.41}
\]

\(^9\)Additional mathematical results on this property are discussed in Section 6.2.
with suitably chosen scalar $\psi$ and some proportionality constant $\alpha$. To choose such a scalar we take into account that in the present case
\[
\nabla \cdot \mathbf{j} = 0
\]
implying
\[
\nabla^2 \psi = 0.
\]
The vector $\mathbf{j}$ given by Eq.(4.41) is not uniquely defined. It will still obey the Eq.(4.42) if we write
\[
\mathbf{j} = \alpha \nabla \psi \pm g \mathbf{A}
\]
for a vector $\mathbf{A}$ such that $\nabla \cdot \mathbf{A} = 0$. Evidently, a vector obeying Eq.(4.13) by construction possess this property. The choice of the sign "+" or "−" in the above equation can be determined based on the following arguments. Since $\mathbf{j} = n_0 \mathbf{v}$ and since $n_0$ is constant, we can replace Eq.(4.44) by
\[
\mathbf{v} = \alpha \nabla \psi \pm g \mathbf{A}
\]
by suitably redefining constants $\alpha$ and $g$. Next, we assume that $\mathbf{v}$ is a random variable so that on average $\langle \mathbf{v} \rangle = 0$ thus implying
\[
\langle \alpha \nabla \psi \rangle = \pm g \langle \mathbf{A} \rangle = 0.
\]
This equation causes us to choose the sign "−". After this, we can write for the correlator
\[
\langle \mathbf{v} \cdot \mathbf{v} \rangle = \alpha^2 \langle \nabla \psi \cdot \nabla \psi \rangle + g^2 \langle \mathbf{A} \cdot \mathbf{A} \rangle = 2g^2 \langle \mathbf{A} \cdot \mathbf{A} \rangle.
\]
In view of our choice of $\mathbf{A}$, the $\langle \mathbf{A} \cdot \mathbf{A} \rangle$ correlator coincides with that given in the exponent of Eq.(4.24). Now we take into account Eq.(4.20b) where, of course, we replace $\mathbf{j}$ by $\mathbf{v}$ so that using the dictionary, Eq.(4.16), we arrive at
\[
\tilde{\omega} = e \mathbf{v}
\]
(London equation)
\[
\tilde{\omega} = \nabla \times \mathbf{v}
\]
(Maxwell equation).
Such an identification becomes apparent because of the following arguments. Let us use Eq.(4.45) in Eq.(4.48) in order to obtain
\[
\tilde{\omega} = e(\alpha \nabla \psi - e \mathbf{A}).
\]
In this equation we replaced the constant $g$ by $e$. Furthermore, since Eq.(4.49) formally looks like the Fick’s first law, we can as well rewrite this result as
\[
\tilde{\omega} = e(D_0 \frac{2\pi}{2\pi} \nabla \psi - e \mathbf{A}).
\]
By applying to both sides of this equation the curl operator and taking into account Eq.(4.13), we obtain
\[
\nabla \times \tilde{\omega} = -e^2 \mathbf{v}.
\]
Taking into account the Maxwell’s Eq.(4.14) and using it in Eq.(4.51) we obtain as well
\[
\nabla^2 \mathbf{v} = e^2 \mathbf{v}.
\]
Equivalently, we obtain,
\[
\nabla^2 \mathbf{A} = e^2 \mathbf{A}.
\]
Using Eq.s(4.47), (4.52a) and following the same steps as in the Appendix A of our previous work, Ref.[23], we obtain
\[
\langle \mathbf{v}(r) \cdot \mathbf{v}(0) \rangle = \text{const} \left(\frac{r}{\xi_H}\right),
\]
(4.53)
where $\xi_H = e^{-1} = \left(\frac{1}{R_0} \sqrt{\frac{p_0 \rho}{4\pi \eta}}\right)^{-1}$ and the constant in Eq.(4.53) can be obtained from comparison between this equation and Eq.(2.37). The analogous result is also obtained for the $\langle \mathbf{A} \cdot \mathbf{A} \rangle$ correlator.

In accord with Eq.(2.44) we obtain the result of central importance $\xi_H \to \infty$ when $\rho \to 0$, implying absence of screening in the infinite dilution limit. Our derivation explains the rationale behind the identification of Eq.s (4.14) and (4.48) with the Maxwell and London equations in the theory of superconductivity, Ref.[25], pages 174, 175. Evidently, such an identification becomes possible only in view of the topological nature of the London equation, Eq.(4.48), coming from identification of Eq.(4.19) with (4.20a).

## 5 Exotic superconductivity of colloidal suspensions

### 5.1 General Remarks

In the previous section we developed a theory of hydrodynamic screening following ideas of the London brothers, Ref.[24]. As is well known, their seminal work found its most notable application in the theory of ordinary superconductors [25]. At the same time, Eq.(4.40) was originally used in the theory of superfluid $^4$He. In the book by Kleinert [42] it is shown that Eq.(4.40) can be rewritten in such a way that it will acquire the same form as used in the phenomenological Ginzburg-Landau (G-L) theory of superconductivity [25]. We would like to arrive at the same conclusions differently. In doing so we also would like to determine both the physical and mathematical meaning of the parameter $m_0$ which was left undetermined in Eq.(4.40). We shall develop our arguments mainly following the original G-L pathway.

It should be said, though, that in the present case the connections with superconductivity are only in the structure of equations to be derived. The underlying physics is similar but not identical to that for superconductors. Indeed, in the case of superconductors one typically is talking about the superconducting-to-normal transition controlled by temperature. Also, one is talking about the temperature-dependent "critical" magnetic field (the upper and the lower critical magnetic fields in the case of superconductors of the second kind) which destroys the superconductivity. In the present case of colloidal suspensions there is no explicit temperature dependence: the same phenomena can take place at any temperature at which the solvent is not frozen. If we account for short range forces, then, of course, one can study a situation in which such a colloidal suspension is undergoing a temperature-controlled phase transition. Such a case requires a separate treatment and will not be considered in this work. In the present case the phase diagram can be qualitatively described as follows. The infinite dilution limit corresponds to the normal state. The regime of finite concentrations corresponds to a mixed state, typical for superconductors of the second kind, and the dramatic jump in viscosity discussed in the Introduction and in Section 2 corresponds to the transition to the "fully superconducting" state. Such a difference from the usual superconductors brings some new physics into play which may be useful, in other disciplines, e.g. in the high energy physics or turbulence, etc.\(^\text{10}\)

### 5.2 G-L style derivation of equations of superconductivity for colloidal suspensions

We begin with the one of Maxwell’s equations in its conventional form, e.g. as given in Ref.[25], page 181,

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j}. \quad (5.1)$$

In the G-L theory we have for the current $\mathbf{j}$ the following result:

$$\mathbf{j} = -\frac{i\hbar}{2m} (\phi^* \nabla \phi - \phi \nabla \phi^*) - \frac{e^2}{mc} |\phi|^2 \mathbf{A}. \quad (5.2)$$

\(^\text{10}\)E.g. see Section 6.
Both equations can be obtained by minimization of the following (truncated) G-L functional\textsuperscript{11}

\[
\mathcal{F}[\mathbf{A}, \varphi] = \int d^3r \left\{ \frac{1}{8\pi} (\nabla \times \mathbf{A})^2 + \frac{\hbar^2}{4m} \left| (\nabla - \frac{2i\tilde{e}}{\hbar c} \mathbf{A}) \varphi \right|^2 \right\}
\] (5.3)

with respect to \(\mathbf{A}\). Substitution of the ansatz \(\varphi = \sqrt{\frac{n_s}{2}} \exp(i\psi)\) into Eq.(5.2) leads to the current

\[
j = \frac{\epsilon \hbar}{2m} n_s (\nabla \psi - \frac{2\tilde{e}}{\hbar c} \mathbf{A})
\] (5.4a)

to be compared with our Eq.(4.50). Evidently, this result is equivalent to the postulated London equation

\[
\nabla \times j = -\frac{e n_s}{mc} \mathbf{B}.
\] (5.4b)

At the same time, a comparison of Eq.(5.4a) with Eq.(4.50) leads to the following chain of identifications:

\[
\frac{\epsilon \hbar}{2m} n_s \rightarrow eD_0 \quad \text{and} \quad \frac{\epsilon^2}{mc} n_s \rightarrow e^2.
\]

Consequently, we obtain as well:

\[
\frac{\hbar^2}{2m} \rightarrow D_0, \quad \frac{\epsilon^2 n_s}{m} \rightarrow e^2, \quad \frac{4\pi - 1}{2\pi} \frac{2\tilde{e}}{\hbar} \rightarrow \frac{2e}{D_0} n_s
\]

Using these identifications, we can rewrite the functional \(\mathcal{F}[\mathbf{A}, \varphi]\) as follows

\[
\mathcal{F}[\mathbf{A}, \varphi] = \frac{\rho}{2} \int d^3r \left\{ (\nabla \times \mathbf{A})^2 + D_0 \left| (\nabla - i\frac{2\pi e}{D_0} \mathbf{A}) \varphi \right|^2 \right\}.
\] (5.5)

In the traditional setting, the superconducting density \(n_s\) is determined from the full G-L functional

\[
\mathcal{F}[\mathbf{A}, \varphi] = \int d^3r \left\{ \frac{(\nabla \times \mathbf{A})^2}{8\pi} + \frac{\hbar^2}{4m} \left| (\nabla - \frac{2i\tilde{e}}{\hbar c} \mathbf{A}) \varphi \right|^2 \right\} + a |\varphi|^2 + \frac{b}{2} |\varphi|^4,
\] (5.6)

e.g. by minimization with respect to \(\varphi^*\). In fact, to obtain \(n_s\) it is formally sufficient to treat only the case when \(\mathbf{A} = 0\). Indeed, under this condition we obtain

\[
a |\varphi_c|^2 + b |\varphi_c|^4 |\varphi_c| = 0,
\] (5.7)

which has a nontrivial solution only for \(a < 0\). In this case we get \(n_s = \frac{|\varphi_c|}{\tilde{e}}\), provided that \(b > 0\), as usual.

If we use this result back in Eq.(5.6), that is we use \(\varphi = \sqrt{\frac{n_s}{2}} \exp(i\psi)\) in Eq.(5.6) then, the polynomial (in \(\varphi\)) part of the functional becomes a constant. This constant is divergent when the volume of the system goes to infinity. To prevent this from happening another constant term is typically added to the functional \(\mathcal{F}[\mathbf{A}, \varphi]\) so that it acquires the following canonical form

\[
\mathcal{F}[\mathbf{A}, \varphi] = \int d^3r \left\{ \frac{(\nabla \times \mathbf{A})^2}{8\pi} + \frac{\hbar^2}{4m} \left| (\nabla - \frac{2i\tilde{e}}{\hbar c} \mathbf{A}) \varphi \right|^2 \right\} + \frac{b}{2} (|\varphi|^2 - n_s)^2.
\] (5.8)

Then, when \(\varphi_c = \sqrt{\frac{n_s}{2}} \exp(i\psi)\), the polynomial (in \(\varphi\)) part of the functional vanishes and, accordingly, in this limit we require

\[
\int d^3r \left| (\nabla - \frac{2i\tilde{e}}{\hbar c} \mathbf{A}) \varphi_c \right|^2 \rightarrow 0
\] (5.9)

as well. This leads us to the equation

\[
\frac{hc}{i2\tilde{e} \varphi_c} \nabla \varphi_c = \mathbf{A}
\] (5.10a)

or to

\[
\frac{hc}{2\tilde{e}} \nabla \psi = \mathbf{A}.
\] (5.10b)

\textsuperscript{11}This truncation is known in literature as the "London limit".
This equation coincides (on average) with the previously obtained Eq. (4.46) (with redefinitions described above) as required and will be treated further in Section 5.4.

It should be noted though that originally, in London’s theory, Ref. [24], the $n_s$ was left as an adjustable parameter and, hence, microscopically undefined. This is important in our case since the phenomenon of superconductivity can be looked upon (as in thermodynamics) without any reference to spontaneous symmetry breaking, Higgs effect, etc. At the level of G-L theory, the London equations are reproduced with help of the truncated G-L functional. Hence, in principle, in the present case use of the truncated functional, Eq. (5.5), is also sufficient. At the macroscopic mean field level the presence of polynomial terms in the full-G-L functional, Eq. (5.6) and (5.8) seems somewhat artificial. They do not reveal their microscopic origin and are introduced just to fit the data. We would like to use some known facts from the path integral treatments of superconductivity/superfluidity in order to reveal their physical meaning. Such information is also useful for development of the hydrodynamic theory of colloidal suspensions.

5.3 Path integrals associated with the G-L functional

In view of Eq. (4.40), we begin our discussion with the simplest case of the path integral for a single “relativistic” scalar particle.

Following Polyakov, Ref. [43], the Euclidean version of propagator for such a (Klein-Gordon) particle is given by

$$G(x, x') = \int \frac{Dx(\tau)}{Df(\tau)} \exp(-m_0 \int_0^1 d\tau \sqrt{\dot{x}^2(\tau)}), \quad (5.11a)$$

where in the most general case

$$\dot{x}^2(\tau) = g_{\mu\nu}(x) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}. \quad (5.11b)$$

This propagator is of interest in string theory since it represents a reduced form of the propagator for the bosonic string. As in the case of a string, the action of this path integral is manifestly reparametrization-invariant, i.e. invariant under changes of the type $x(\tau) \rightarrow x(f(\tau))$ (with $f(\tau)$ being some nonnegative monotonically increasing function). The path integral measure is designed to absorb this redundancy. The full account of this absorption is cumbersome. Because of this, instead of copying Polyakov’s treatment of such a path integral, we shall adopt a simplified treatment allowing us to recover Polyakov’s final results.

We begin with an obvious well-known identity

$$\left(\frac{1}{4\pi t}\right)^{\frac{d}{2}} \exp\left(-\frac{1}{4} x^2 \right) = \int_{x(0)=0}^{x(t)=x} D[x(\tau)] \exp\left\{-\frac{1}{4} \int_0^t d\tau \left(\frac{dx}{d\tau}\right)^2\right\}. \quad (5.12)$$

This identity is used below as follows. Consider the propagator for the Klein-Gordon (K-G) field given by

$$G(x) = \int \frac{d^d k}{(2\pi)^d} \frac{\exp(i k \cdot x)}{k^2 + m^2}. \quad (5.13)$$

By employing the identity

$$\frac{1}{a} = \int_0^\infty dx \exp(-ax) \quad (5.14)$$
Eq. (5.13) can be rewritten as follows

\[
G(x) = \int_0^\infty dt \exp(-tm^2) \int \frac{d^d k}{(2\pi)^d} \exp(i k \cdot x - tk^2)
\]

\[
= \frac{1}{\mathcal{E}} \int_0^\infty dt \exp(-\mathcal{E} t m^2) \int \frac{d^d k}{(2\pi)^d} \exp(i k \cdot x - \mathcal{E} t k^2)
\]

\[
= \frac{1}{\mathcal{E}} \int_0^\infty dt \exp(-t\mathcal{E} m^2) \left( \frac{1}{4\pi \mathcal{E} t} \right)^{\frac{d}{2}} \exp\left(-\frac{1}{4 \mathcal{E}} \frac{x^2}{t} \right)
\]

\[
= \frac{1}{\mathcal{E}} \int_0^\infty dt \exp(-t\mathcal{E} m^2) \int \mathcal{D} [x(\tau)] \exp\left\{ -\frac{1}{4 \mathcal{E}} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 \right\}
\]

(5.15)

where we used the identity, Eq. (5.12), to obtain the last line and introduced an arbitrary nonnegative parameter \(\mathcal{E}\) for comparison with results by Polyakov. Specifically, using page 163 of the book by Polyakov (and comparing our 3rd line above with the 3rd line of his Eq. (9.63)) we can make the following identifications:

\(\mathcal{E} \leftrightarrow \epsilon, \quad m^2 \leftrightarrow \mu\). Since, according to Polyakov, \(\mu = \epsilon^{-1}(m_0 - \frac{c}{\sqrt{\epsilon}})\) with \(c\) being some constant, we obtain:

\(m_0 = \mathcal{E} m^2 + \frac{c}{\sqrt{\epsilon}}\). That is, the physical mass \(m^2\) entering the K-G equation is obtained as the limit of the expression \((\epsilon \to 0)\)

\[
m = \lim_{m_0 \to m_{cr}} \epsilon^{-\frac{d}{2}}(m_0 - m_{cr})^\frac{\delta}{2}.
\]

(5.16)

Clearly, such an expression is nonnegative by construction. From the last line of Eq. (5.15) it follows that the propagator for the K-G field is just the direct Laplace transform of the nonrelativistic "diffusion" propagator, Eq. (5.12), with the Laplace variable \(m\) playing a role of a mass for such a field. In the Euclidean version of the K-G propagator this mass cannot be negative since in such a case the identity Eq. (5.14) cannot be used so that the connection between the nonrelativistic and the K-G propagators is lost. However, Eq. (5.2) seemingly is for the quantum current while the propagator in Eq. (5.12) is describing Brownian motion, not quantum diffusion. To fix the problem we have to replace time \(t\) in Eq. (5.12) by \(it\) and, accordingly, to make changes in Eq. (5.15). This then converts the Laplace transform into the Fourier transform, provided that the nonrelativistic propagator describes the \(\text{retarded}\) Green’s function. To use the full strength of the apparatus of quantum field theory one needs to use the causal Green’s functions. This is required by the relativistic covariance treating space and time coordinates on the same footing. Once all of these requirements are met, it becomes possible to treat the case of a negative mass.

It should be emphasized at this point that the London-style derivation given in the previous section formally does not require such quantum mechanical analogy. Because of this, the following problem emerges: is it possible to reproduce the functional integral \(\mathcal{F}\) defined by Eq. (4.31) using the truncated G-L functional for superconductivity in the exponent of the associated path integral? We would like to provide an affirmative answer to this question now.

We begin with the partition function \(Z\) for the two-component scalar K-G-type field

\[
\ln Z = -\ln[\det(-\nabla^2 + m^2)]
\]

(5.17)

Since

\[
\ln[\det(-\nabla^2 + m^2)] = tr \left[ \ln(-\nabla^2 + m^2) \right]
\]

(5.18)

and

\[
tr \left[ \ln(-\nabla^2 + m^2) \right] = \int \frac{d^d k}{(2\pi)^d} \ln(k^2 + m^2),
\]

(5.19)
we can use the results of our previous work, Ref.[44], for evaluation of the last expression. Thus, we obtain,

\[ tr \left[ \ln \left( \frac{-\nabla^2 + m^2}{-\nabla^2} \right) \right] = \int_0^\infty \int_0^\infty \right] \frac{d^4 k}{(2\pi)^4} \ln(k^2 + y) \]

\[ = \int_0^\infty \int_0^\infty \frac{d^4 k}{(2\pi)^4} \frac{1}{k^2 + y} = \int dy G(0;y) \]

\[ = \int dt \int dy \exp(-ty) \int_0^{\infty} \mathcal{D}[x(\tau)] \exp\left\{ -\frac{1}{4} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 \right\} \]

\[ = \int dt \int_0^{\infty} \mathcal{D}[x(\tau)] \exp\left\{ -\frac{1}{4} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 \right\}. \quad (5.20) \]

Following the usual practice, we shall write \( \bar{f} \) instead of \( \int \) in the path integral and consider a formal (that is diverging!) expression for the free energy \( F_0 \)

\[ \exp(-F_0) = \ln Z_0 = -\ln[\det(-\nabla^2 + m^2)] = \int dt \int_0^{\infty} \exp(-m^2t) \int \mathcal{D}[x(\tau)] \exp\left\{ -\frac{1}{4} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 \right\}. \quad (5.21) \]

by keeping in mind that this result makes sense mathematically only when the same expression with \( m^2 = 0 \) is subtracted from it as required by Eq.(5.20). Inclusion of the electromagnetic field into this scheme can be readily accomplished now. For this purpose we replace the \( \nabla \) operator by its covariant derivative \( \nabla \rightarrow \nabla \equiv \nabla - ieA \) (we put \( D_0 = 1 \) in view of developments presented in Eq.(5.15)). Using \( \mathbf{D} \) instead of \( \nabla \) in Eq.(5.20) we have to evaluate the following path integral\(^{12}\)

\[ [\det(-\mathbf{D}^2 + m^2)]^{-1} = \int D[\varphi, \varphi'] \exp(-\frac{1}{2} \int d^4r \{ \varphi(-\mathbf{D}^2 + m^2)\varphi' \}). \quad (5.22) \]

For \( A = 0 \) we did this already while for \( A \neq 0 \) we can treat terms containing \( A \) as perturbation. We can do the same for the path integral in Eq.(4.32). This is easy to understand if we realize that

\[ \int_0^\infty \int_0^\infty \exp(-m^2t) I[A; t] \big|_{A=0} = -\frac{d}{dm^2} \int_0^\infty \int_0^\infty \exp(-m^2t) \int \mathcal{D}[x(\tau)] \exp\left\{ -\frac{1}{4} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 \right\}. \quad (5.23) \]

Therefore, the final answer reads as follows

\[ \exp(-F) = \ln Z = -\ln[\det(-\mathbf{D}^2 + m^2)] = \int \frac{dt}{t} \exp(-m^2t) \int \mathcal{D}[x(\tau)] \exp\left\{ -\frac{1}{4} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 + i \frac{e}{f} \mathbf{A} \mathbf{.} \mathbf{x} (\tau) \right\} \]

\[ = \int \frac{dt}{t} \exp(-m^2t) \int \mathcal{D}[x(\tau)] \exp\left\{ -\frac{1}{4} \int_0^t d\tau \left( \frac{dx}{d\tau} \right)^2 \right\} \exp\left\{ -i \frac{e}{f} \int \mathbf{A} \mathbf{.} \mathbf{x} (\tau) \right\}. \quad (5.24) \]

This result demonstrates that applying the operator \( \int \frac{dt}{t} \exp(-m^2t) \) to \( I[A; t] \) defined in Eq.(4.32) makes it equivalent to the "matter" part of truncated G-L functional for superconductivity as needed. This raises

\(^{12}\)For \( m^2 = 0 \) this is just part of the truncated G-L functional.
a question about comparison of the full G-L functional with the "diffusion" path integrals of Section 4. Evidently, this can be done only if in the original diffusion Eq.(3.30) we do not discard the potential $U$. If we do not discard the potential and if, instead, we ignore the hydrodynamic interactions completely, we would end up with the following path integral for interacting Brownian particles in the canonical ensemble

$$
\Xi = \int \prod_{i=1}^{N} D[\mathbf{x}(\tau_i)] \exp\{-\frac{1}{4D_0} \sum_{i=1}^{N} \int_{0}^{t} d\tau_i \left( \frac{d\mathbf{x}}{d\tau_i} \right)^2 - \sum_{i<j}^{N} \int_{0}^{t} d\tau_i \int_{0}^{t} d\tau_j V[\mathbf{x}(\tau_i) - \mathbf{x}(\tau_j)]\}. \tag{5.25}
$$

It is essential that this expression does not contain self-interactions typical for problems involving polymer chains with excluded volume-type interactions. The situation here resembles that encountered when, following Doi and Edwards, Ref.[10], we redefined the Oseen tensor in Eq.s (3.29 ) and (3.30) so that it acquired the diagonal part. In the present case we must require the diagonal part to be zero at the end of calculations. These results, correct for colloidal particles, may become incorrect in the present case for the following reason. From looking either at Eq.(5.24) or Eq.(4.39), we recognize that in these cases we are dealing with assemblies of loops (vortices) which are in one-to one correspondence with diffusing particles. While this topic is studied in detail in the next subsection and Appendix B, here we notice that if Eq.(5.25) is written for such loops, then the excluded volume requirement becomes essential, even for a single loop. Indeed, the existence of such a loop is possible only if the field $\mathbf{A}$ associated with these loops is uniquely defined. This is possible only if the loop contour does not have self-interactions. This is the origin of the excluded volume constraint requirement. With this restriction imposed, we introduce a density

$$
\rho(\mathbf{r}) = \sum_{i=1}^{N} \int_{0}^{t} d\tau_i \delta(\mathbf{x} - \mathbf{x}(\tau_i)) \tag{5.26}
$$

so that the binary potential in Eq.(5.25) can be written as

$$
\frac{1}{2} \sum_{i,j}^{N} \int_{0}^{t} d\tau_i \int_{0}^{t} d\tau_j V[\mathbf{x}(\tau_i) - \mathbf{x}(\tau_j)] = \frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}) V[\mathbf{r} - \mathbf{r}'] \rho(\mathbf{r}'). \tag{5.27}
$$

Then, using the Hubbard-Stratonovich (H-S) identity we obtain,

$$
\exp\left(\frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \rho(\mathbf{r}) V[\mathbf{r} - \mathbf{r}'] \rho(\mathbf{r}')\right) = \mathcal{N} \int D[\psi(\mathbf{r})] \exp\left(-\frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \psi(\mathbf{r}) V^{-1}[\mathbf{r} - \mathbf{r}'] \psi(\mathbf{r}')\right) \exp(i \int d\mathbf{r} \psi(\mathbf{r}) \rho(\mathbf{r})) \tag{5.28}
$$

with $\mathcal{N}$ being a normalization constant (bringing the above identity to the statement $1 = 1$ for $\rho = 0$). Use of this result in Eq.(5.25) in which self exclusion is allowed converts this partition function into the following form (written for the loop ensemble)

$$
\Xi = \mathcal{N} \int D[\psi(\mathbf{r})] \exp\left(-\frac{1}{2} \int d\mathbf{r} \int d\mathbf{r}' \psi(\mathbf{r}) V^{-1}[\mathbf{r} - \mathbf{r}'] \psi(\mathbf{r}')\right) \prod_{i=1}^{N} G_i(0; t | \psi) \tag{5.29}
$$

where

$$
G_i(0; t | \psi) = \mathcal{D}[\mathbf{x}(\tau_i)] \exp\left(-\int_{0}^{t} d\tau_i \left[ \frac{1}{4} \left( \frac{d\mathbf{x}}{d\tau_i} \right)^2 + i e \psi[\mathbf{x}(\tau_i)] \right]\right) \tag{5.30}
$$

In the case of polymers, typically, one uses the delta function-type potential for description of the interactions. This observation is helpful in the present case as well because of the following. Consider the G-L functional, Eq.(5.6), and use the H-S identity for the interaction term

$$
\exp\left(-\frac{b}{2} ||\psi||^2\right) = \mathcal{N} \int D[\psi(\mathbf{r})] \exp\left(-\frac{1}{2b} \int d\mathbf{r} \int d\mathbf{r}' \psi(\mathbf{r}) \psi(\mathbf{r}')\right) \exp(i \int d\mathbf{r} \psi(\mathbf{r}) |\psi|^2). \tag{5.31}
$$

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This allows us to replace the determinant, Eq.(5.22), by the following (more general) determinant

$$\begin{align*}
\left[\det(-D^2 + m^2 + i\psi)\right]^{-1} &= \int D[\varphi, \varphi] \exp\left(-\frac{1}{2} \int d^3r \{\varphi(-D^2 + m^2 + i\psi[r] \varphi)\} \right) \quad (5.32)
\end{align*}$$

which, in view of Eq.(5.24), can be equivalently rewritten as

$$\begin{align*}
\exp(-\mathcal{F}) &= \ln Z = -\ln[\det(-D^2 + m^2 + i\psi)] = \mathcal{H} \int D[\psi(r)] \exp(-\frac{1}{2b} \int dr \int dr' \psi(r)\psi(r')) \\
&\quad \times \int_0^\infty \frac{dt}{t} \exp(-m^2t) \int \mathcal{D}[x(\tau)] \exp\left\{-\frac{1}{4} \int_0^t d\tau \left(\frac{dx}{d\tau}\right)^2\right\} \exp\left\{-ie \oint dx \cdot A[x(\tau)] + i \oint d\tau \psi(x(\tau))\right\} \\
&= \int_0^\infty \frac{dt}{t} \exp(-m^2t) \int \mathcal{D}[x(\tau)] \exp\left\{-\frac{1}{4} \int_0^t d\tau \left(\frac{dx}{d\tau}\right)^2\right\} \\
&\quad \times \exp\left\{-ie \oint dx \cdot A[x(\tau)] - \frac{b}{2} \oint dx' \oint d\tau' \delta(x(\tau) - x(\tau')).\right\} \quad (5.33)
\end{align*}$$

Alternatively, this result can be rewritten as a grand canonical ensemble of selfavoiding loops

$$\begin{align*}
Z[A;b] - 1 &= \sum_{n=1}^{\infty} \frac{1}{n!} \prod_{l=1}^{n} \int \frac{dt_l}{t_l} \exp(-m^2t_l) \int \mathcal{D}[x(\tau_l)] \exp\left\{-\frac{1}{4} \sum_{l=1}^{n} \int_0^t d\tau_l \left(\frac{dx}{d\tau_l}\right)^2\right\} \\
&\quad \times \exp\left\{-ie \oint dx \cdot A[x(\tau_l)]\right\} - \frac{b}{2} \sum_{l,m=1}^{n} \oint d\tau_l \oint d\tau_m \delta(x(\tau_l) - x(\tau_m)). \quad (5.34)
\end{align*}$$

This result is useful to compare with Eq.(4.31). From such a comparison it is evident that Eq.(5.34) is compatible with that obtained previously. It accounts for the effects of non hydrodynamic-type interactions which can be incorporated, in principle, in the diffusion Eq.(3.30) in which the potential $U$ must be specified. Clearly, the use of path integrals makes such a task much simpler. However, even though the above derivation is intuitively appealing, strictly speaking, it cannot be used for a number of reasons. Unlike the G-L functional, Eq.(5.8), which is convenient for studying of topological and nonperturbative effects, Eq.(5.34) makes sense only in perturbative calculations. This means that phenomena such as screening (caused by the Higgs effect) cannot be captured with such a formalism alone. These observations explain why screening effects were found in solutions of polymers but not in colloidal suspensions, Ref. [10,12]. Furthermore, Eq.(5.34) contains a mixture of reparametrization-invariant and non invariant terms. This is questionable mathematically. It would be more logical to have the entire action reparametrization-invariant. We study these issues in some detail in the next subsection.

### 5.4 Reparametrization-invariance and its consequences. London-style analysis

Since path integrals mathematically can seldom be defined rigorously, we would like in this subsection to extend the analysis of Sections 4.2.-4.4 avoiding the use of path integrals. We start with a discussion of the result, Eq.(5.10b), which is the superconducting analog of Eq.(4.46) for suspensions. Since $B = \nabla \times A$, (or $v = \nabla \times A$ in the case of suspensions) we conclude that Eq.(5.10b) causes the first term in the G-L functional Eq.(5.6) (or (5.5)) to vanish in the bulk. Nevertheless, it is perfectly permissible to write

$$\begin{align*}
\frac{hc}{2\varepsilon} \oint d\mathbf{r} \cdot \mathbf{\nabla} \psi = \oint d\mathbf{r} \cdot \mathbf{A}.
\end{align*}$$

and to use Stokes’ theorem

$$\begin{align*}
\oint_C d\mathbf{r} \cdot \mathbf{A} = \iint d\mathbf{S} \cdot (\nabla \times \mathbf{A}) = \iint d\mathbf{S} \cdot \mathbf{B} = \frac{hc}{2\varepsilon}
\end{align*}$$

(5.36)
with \( n = 0, \pm 1, \pm 2, \ldots \). An analogous result for suspensions reads as follows

\[
\oint_C d\mathbf{r} \cdot \mathbf{A} = \int \int dS \cdot (\nabla \times \mathbf{A}) = \int \int dS \cdot \mathbf{v} = n\frac{D_0}{e}.
\]  

(5.37a)

In view of Eqs.(4.19) and (4.20), this result leads also to

\[
\frac{1}{D_0} \int \int dS \cdot \tilde{\omega} = n = \frac{e}{D_0} \oint_C d\mathbf{r} \cdot \mathbf{A}
\]

(5.37b)

which is the same as Eq.(4.21).

These results can be interpreted in a number of ways. For the sake of argument, we would like to explore the more established case of superconductivity first. Following Lund and Regge, Ref.[45], we suppose that the vector potential \( \mathbf{A} \) can be presented as follows

\[
\mathbf{A}(\mathbf{r}) = \frac{k}{4\pi} \oint_{C_1} \frac{1}{|\mathbf{r} - \mathbf{r}(\sigma)|} \left( \frac{\partial}{\partial \sigma} \right) d\sigma = \frac{k}{4\pi} \oint_{C_1} \frac{1}{|\mathbf{r} - \mathbf{r}(\sigma)|} \mathbf{v}(\sigma) d\sigma
\]

(5.38)

with the appropriately chosen constant \( k \). This result easily follows from Eq.(4.15) under the assumption that \( \tilde{\omega}(\mathbf{r}) = k \oint_{C_1} d\sigma \mathbf{v}(\sigma) \delta(|\mathbf{r} - \mathbf{r}(\sigma)|) \) (which is the same as our Eq.(4.20b)). Substitution of this result into Eq.(5.36) produces

\[
\oint_{C_1} d\mathbf{r} \cdot \mathbf{A} = \frac{k}{4\pi} \oint_{C_1} \oint_{C_2} d\sigma d\sigma' \frac{\mathbf{v}(\sigma) \cdot \mathbf{v}(\sigma')}{|\mathbf{r}(\sigma) - \mathbf{r}(\sigma')|^3} = \frac{\hbar c}{2e} n
\]

(5.39)

The obtained result allows us to determine the constant \( k \). To do so we need to demonstrate that the above double integral is a linking number, e.g. see Eq.(4.6). The proof of this result depends upon correctness of the following statement

\[
\oint d\mathbf{r} \cdot \mathbf{A} = \oint d\mathbf{r} \cdot \mathbf{B} = \oint d\mathbf{r} \cdot \nabla \times \mathbf{A} = \frac{\partial}{\partial \sigma} \times \oint_{C_1} \frac{1}{|\mathbf{r} - \mathbf{r}(\sigma)|} \mathbf{v}(\sigma) d\sigma
\]

\[
\oint_{C_1} \oint_{C_2} \frac{\mathbf{v}(\sigma) \cdot \mathbf{v}(\sigma')}{|\mathbf{r}(\sigma) - \mathbf{r}(\sigma')|^3} = \frac{\hbar c}{2e} n
\]

(5.40)

with linking number \( lk(1,2) \) defined in Eq.(4.6). If the above result is correct and the constant \( \kappa \) can be found then, the constant \( k \) can be determined from Eq.(5.39). Hence, the task lies in demonstrating that the nonzero constant \( \kappa \) does exist. To do so we shall use the standard London analysis. Thus, we write

\[
\nabla \times \mathbf{B} = \frac{4\pi}{c} \mathbf{j} \quad \text{(Maxwell equation)}
\]

(5.1)

and

\[
\nabla \times \mathbf{j} = -\frac{en_s}{mc} \mathbf{B} \quad \text{(London equation).}
\]

(5.4b)

Since, \( \mathbf{B} = \nabla \times \mathbf{A} \), and \( \nabla \cdot \mathbf{B} = \nabla \cdot \mathbf{A} = 0 \), we have \( \kappa \mathbf{B} = \mathbf{A} \) so that we obtain

\[
\nabla \times \mathbf{A} = \kappa \frac{4\pi}{c} \mathbf{j}
\]

(5.41)

and, from here

\[
-\nabla^2 \mathbf{A} = \nabla \times \nabla \times \mathbf{A} = \kappa \frac{4\pi}{c} (\nabla \times \mathbf{j}) = -\kappa \frac{4\pi en_s}{mc^2} \mathbf{B} = -\kappa^2 \frac{4\pi en_s}{mc^2} \mathbf{A}
\]

(5.42)
which is the familiar screening-type equation, e.g. see Eq.(4.51b). Since, in the conventional setting the penetration depth $\delta^2$ is known to be $\delta^2 = \left(\frac{4\pi n_e}{mc^2}\right)^{-1}$, we can chose $\kappa^2 = 1$ implying that $\kappa = \frac{hc}{\delta}$. The choice $\kappa = 1$ does not mean of course that the constant $\kappa$ is dimensionless. Because of this, we obtain

$$\frac{1}{4\pi\kappa} \oint_{C_1} \oint_{C_2} d\sigma' d\sigma \frac{v(\sigma) \cdot v(\sigma')}{|r(\sigma) - r(\sigma')|} = \text{lk}(1, 2)$$

(5.43)

in accord with Eq.(4.21). Next, if we take into account screening effects, the conclusions we’ve reached will remain the same due to reparametrization invariance of both sides of Eq.(5.43). Indeed, consider one loop, say $C_1$, going from $-\infty$ to $+\infty$ in the $z$-direction. If we compactify $\mathbb{R}^3$ by adding one point at infinity so that $\mathbb{R}^3$ becomes $S^3$, then such a loop will be closed. Another loop can stay mainly in the x-y plane so that the linking number becomes the winding number, e.g. see Ref.[46], page 134. Under these conditions the screening factor $\exp\left(-\frac{r}{\delta}\right)$ under the integral of the left hand side of Eq.(5.43) is unimportant since we can always arrange our windings in such a way that $r \ll \delta$ for any preassigned nonzero $\delta$ so that the screening factor becomes unimportant.

The above analysis can be extended to the case of colloidal suspensions in view of the results of Sections 4.2 and 4.4. implying that in both superconductivity and colloidal suspensions the phase transition is topological in nature (e.g. in the colloidal case Eq.(4.39) is a topological invariant to be considered in the next subsection). Evidently, such a conclusion cannot be reached by perturbatively calculating the Green’s function in Eq.(4.37).

In Section 5.1 we discussed similarities and differences between superconductors and colloidal suspensions. It is appropriate now to add a few additional details to the emerging picture. In the case of superconductivity correctness of the topological picture depends upon the existence of nontrivial solutions of Eq.(5.42). These are possible only when the parameter $n_s$ is nonzero. When it becomes zero the above picture breaks down. In the case of suspensions the role of the parameter $\kappa^{-1}$ is played by the density-dependent parameter $e$. This can be easily seen if we take into account that dimensional analysis requires us to replace Eq.(5.38) by

$$A(r) = \frac{D_0}{4\pi} \oint_C \frac{1}{|r - r(\sigma)|} v(\sigma) d\sigma$$

(5.44)

so that by employing Eq.(5.37b) we obtain,

$$\frac{e}{4\pi} \oint_{C_1} \oint_{C_2} d\sigma' d\sigma \frac{v(\sigma) \cdot v(\sigma')}{|r(\sigma) - r(\sigma')|} = n = \text{lk}(1, 2)$$

(5.45)

as expected.

5.5 Bose-Einstein-type transition in a system of linked loops

In the Introduction we noted that Chorin, Ref.[22], conjectured that the superfluid-to normal transition in $^4$He is associated with vortices causing a sharp increase in viscosity. In this subsection we would like to demonstrate that, at least for colloidal suspensions, his conjecture is correct: the sharp increase in viscosity is associated with the lambda-type transition. Instead of treating this problem in full generality, i.e. for the nonideal Bose gas, we simplify matters and consider a Bose condensation type transition typical for the ideal Bose gas. It should be noted though that our simplified treatment is motivated only by the fact that it happens to be sufficient for comparison with experimental data. In other cases, such a restriction can be lifted.

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13Emergence of such a screening factor can be easily understood if we replace Eq.(4.15) by Eq.(5.42) with the right hand side given by $\hat{\omega}(r) = k \oint_C v(\sigma) \delta(r - r(\sigma))$ in accord with Eq.(5.38).
To develop such a theory we use the information obtained in the previous subsection augmented by some additional facts needed for completion of our task. In particular, we are interested in the expression for the kinetic energy. Up to a constant it is given by

\[ E = \frac{1}{2} \int d^3r (\nabla \times A)^2 \]  

(5.46)

and is manifestly nonnegative. Using known facts from vector analysis this expression can be rewritten as follows

\[ E = \frac{1}{2} \int d^3r (\nabla \times A) \cdot v = \frac{1}{2} \int d^3r [A \cdot \tilde{\omega} + \text{div}(A \times v)] = \frac{1}{2} \int d^3r A \cdot \tilde{\omega} \]  

(5.47)

In view of Eq.(5.38) we can rewrite this result as

\[ E = \frac{k^2}{2} \oint_{C_1} \oint_{C_2} d\sigma d\sigma' \frac{v(\sigma) \cdot v(\sigma')}{|r(\sigma) - r(\sigma')|} \]  

(5.48)

to be compared with Eq.(5.45). Using such a comparison we arrive at an apparent contradiction: while an expression for \( E \) should be nonnegative, the linking number \( \text{lk}(1,2) \) can be both positive or negative.

If we make the replacement \( r \to -r \) in Eq.(5.48) nothing changes but if we do the same for \( \text{lk}(1,2) \) it changes the sign. Thus, if we want to use \( \text{lk}(1,2) \) in Eq.(5.48) we have to use \( |\text{lk}(1,2)| \). This number was introduced by Arnold and is known in literature as entanglement complexity\(^{14}\). Evidently, in view of this remark, \( n \) in Eq.(5.45) can be only nonnegative. If we require our system to be invariant with respect to rotations of the coordinate frame, Eq.(4.39) should be rewritten according to the procedure developed in our work, Ref.[49]. This means, that we introduce a set of linking numbers: \( n_1, n_2, ..., n_i, ... \) so that for a given \( n \)^{15}, the set of \( \frac{1}{2} n(n - 1) \equiv N \) possible linking numbers can be characterized by the total linking number \( L \), i.e. we have

\[ \sum_{i=1}^{N} n_i = L. \]  

(5.49)

This result can be rewritten alternatively as follows. Let \( C_1 \) be the number of links with linking number 1, \( C_2 \) the number of links with linking number 2 and so on. Then, we obtain

\[ \sum_{i=1}^{L} iC_i = L. \]  

(5.50)

Furthermore, we also must require

\[ \sum_{i=1}^{L} C_i = N \]  

(5.51)

Define the Stirling-type number \( \tilde{S}(L,N) \) via the following generating function\(^{16}\)

\[ \sum_{N=0}^{L} \tilde{S}(L,N) x^N = x(x+1) \cdots (x+L-1). \]  

(5.52)

Set in this definition \( x = 1 \). This then allows us to introduce the probability \( p(L,N) = \tilde{S}(L,N) / L! \) The number \( \tilde{S}(L,N) \) can be easily obtained\(^{17}\) with the result given by

\[ \tilde{S}(L,N) = \prod_{i=1}^{N} \frac{L!}{i!C_i!}. \]  

(5.53)

\(^{14}\)For more details about this number and its many applications can be found in our works, Refs.[47,48].

\(^{15}\)E.g. see Eq.(4.4).

\(^{16}\)The true Stirling number of the first kind \( S(L,N) \) is defined as follows: \( S(L,N) := (-1)^{L-N} \tilde{S}(L,N) \).

\(^{17}\)E.g. see Ref. [49].
With thus obtained results, we are now ready to return to Eq.(4.39) in which we make a rescaling:
\[ r(\tau) \rightarrow R_0 \tilde{r}(\tau), \] with \( \tilde{r}(\tau) \) being dimensionless. After which, using Eq.(5.45) we can rewrite Eq.(4.39) as follows
\[ <W(L)>_T = \exp\left(-\frac{3n_0}{\eta} L\right) \] (5.54)
Evidently, the numerical factor of 3 in the exponent is non-essential and can be safely dropped upon rescaling of \( L \). To use this expression we combine it with Eq.(5.34) in which we have to make some adjustments following Feynman, Ref.[50], pages 62-64. On these pages Feynman discusses a partition function for the ideal Bose gas written in the path integral form. We would like to rewrite his result in the notation of our paper. For this purpose we use Eq.(4.30) in which the path integral is written for a loop and is in discrete form. We obtain,
\[ h(\nu) = \left(\frac{1}{4\pi D_0}\right)^\frac{3\nu}{2} \int \prod_{i=1}^{\nu} d^3 \mathbf{r}_i \exp\left\{-\frac{1}{4D_0} \left[ (\mathbf{r}_1 - \mathbf{r}_2)^2 + (\mathbf{r}_2 - \mathbf{r}_3)^2 + \cdots + (\mathbf{r}_{\nu-1} - \mathbf{r}_\nu)^2 + (\mathbf{r}_\nu - \mathbf{r}_1)^2\right] \right\} \]
\[ = V \left(\frac{1}{4\pi \nu D_0}\right)^\frac{3\nu}{2} \] (5.55)
with \( V = \int d^3 \mathbf{r}_1 \). Under such circumstances the Brownian ring is made out of \( \nu \) links(segments) so that we can identify its length with \( \nu \). In the present case each such ring is linked with another ring thus forming a link with a linking number \( iC_i \), \( i = 0, 1, 2, \ldots \). Since the linking number is independent of the lengths of rings from which it is made, we can take advantage of this fact by identifying the index \( i \) with \( \nu \). By combining Eq.s (5.50)-(5.55) and repeating the same steps as given in Feynman’s lectures we assemble the following dimensionless grand partition function \( \mathcal{F} \)
\[ e^{-\mathcal{F}} = \sum_{C_1, \ldots, C_\nu} \prod_{\nu} \frac{h(\nu)^{C_\nu}}{C_\nu!} \exp\left(-\frac{n_0}{\eta} C_\nu\right) = \sum_{C_1, \ldots, C_\nu} \prod_{\nu} \frac{1}{C_\nu!} (h(\nu)^{\frac{z^\nu}{\nu}})^{C_\nu} = \exp\left(\sum_{\nu} h(\nu)^{\frac{z^\nu}{\nu}}\right). \] (5.56)
Here the "chemical" potential \( z = \exp\left(-\frac{n_0}{\eta}\right) \). Taking the logarithm of both sides of the above equation we obtain the partition function for the ideal Bose gas. Written per unit volume it reads
\[ \mathcal{F} = -\left(\frac{1}{4\pi D_0}\right)^\frac{3}{2} \zeta_{5/2}(z). \] (5.57)
In this expression \( \zeta_\alpha(z) \) is Riemann’s zeta function
\[ \zeta_\alpha(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^\alpha}. \] (5.58)
This function is well defined for \( z < 1 \), i.e. for \( \frac{n_0}{\eta} < \infty \) and is divergent for \( z > 1 \), thus indicating a Bose condensation whose onset is determined by the value \( z = 1 \) (i.e. \( \eta = \infty \)) for which \( \zeta_{5/2}(1) = 1.341 \). If we follow standard treatments, then we obtain for the critical density \( \rho_c \)
\[ \rho_c = \left(\frac{1}{4\pi D_0}\right)^\frac{3}{2} 2.612. \] (5.59)
In view of Eq.(5.55), the obtained result for density has the correct dimensionality. From here the critical volume fraction is: \( \varphi_c = \rho_c \frac{4}{\pi} R_0^3 \). The number 2.612 is just the value of \( \zeta_{3/2}(1) \). This means, that we can
write in the general case
\[ \rho(z) = \left( \frac{1}{4\pi D_0} \right)^{3/2} \zeta_{3/2}(z) \] (5.60)
thus giving us the equation
\[ \rho_c - \rho \rho_c = 1 - \frac{\zeta_{3/2}(z)}{\zeta_{3/2}(1)} \] (5.61)
In the book by London, Ref.[51], we found the following expansion for \( \zeta_{3/2}(z) \) in the vicinity of \( z = 1 \) \((z < 1)\):
\[ \zeta_{3/2}(z) = -3.545\alpha^{3/2} + 2.612 + 1.460\alpha - 0.104\alpha^2 + ... \] (5.62)
where \( \alpha = -\ln z \). Use of this result in Eq.(5.61) produces the following result:
\[ \frac{\eta}{\eta_0} = \left( \frac{1}{3.545} \right)^2 \left( 1 - \frac{\rho}{\rho_c} \right)^{-2} \] (5.63)
in accord with scaling predictions by Brady, Ref. [19], and Bicerano et al.,Ref.[20]. It should be noted though that in view of Eq.(5.54) the actual value of the constant prefactor in Eq.(5.63) is quite arbitrary and can be adjusted with help of experimental data. For instance, by making this prefactor of order unity, Bicerano et al obtained a very good agreement with experimental data in the whole range of concentrations, e.g. see Ref. [20], Fig.4.

6 Discussion and outlook

6.1 General comments

With the exception of the work by De Gennes [52] on phase transition in smectics A, the superconductivity and superfluidity phenomena are typically associated with the domain of low temperature condensed matter physics. This fact remains true even with account of cuprate superconductors, Ref.[54]. The results obtained in this work cause us to look at these phenomena differently. For instance, the previously mentioned relation \( \tilde{\omega}(r) = k \oint C v(\sigma) \delta(r \cdot r(\sigma)) \) used in the work by Lund and Regge, Ref.[45], for fluids, coincides with our Eq.(4.20b) for colloids. The work of Lund and Regge is based on previous work by Rasetti and Regge, Ref.[55], on superfluid He and, therefore, their results are apparently valid only in the domain of low temperatures. This conclusion is incorrect however as shown in the series of papers by Berdichevsky, Ref.s [56,57]. Any ideal (that is Euler-type) incompressible fluid can be treated this way. Furthermore, as results by Chorin, Ref.[22], indicate, the same methods should be applicable for description of the onset of fluid/gas turbulence. In our work the fluid is manifestly nonideal. Nevertheless, in the long time (zero frequencies) limit it can still be treated as if it is ideal.

The most spectacular departure from traditional view on the results by Lund and Regge was recently made in a series of papers by Schief and collaborators, Refs. [58,59]. The latest results elaborating on his work can be found in Ref.[60]. Schief demonstrated that the results of Lund and Regge work well in the case of magnetohydrodynamics, that is, ultimately in the plasma installations designed for controlled thermonuclear synthesis.

The basic underlying physics of all these phenomena can be summarized as follows. In every system which supports knotted structures, the existence of a decoupling of topological properties from the conformational (statistical) properties of flux tubes from which these knots/links are made should be possible. Since this statement is not restricted to a simple Abelian C-S field theory describing knots/links existing in G-L theory, in full generality the theory should include the G-L theory as a special case (as demonstrated above). Accordingly, the minimization of the corresponding truncated G-L functional may or may not lead to London-type equations. We would like to illustrate these general statements by specific examples. This is accomplished below.

\[ ^{18}\text{Lately, however, these ideas have began to be popular in color superconductivity dealing with quark matter [53].} \]
6.2 Helicity and force-free fields imply knotting and linking but not necessarily superconductivity via London mechanism

The concept of helicity has its origin in theory of neutrino, Ref.[61]. An expression $\sigma \cdot p/|p|$ is called helicity. Here $\sigma \cdot p = \sigma_x p_x + \sigma_y p_y + \sigma_z p_z$, and $p_i$ and $\sigma_i$, $i = 1−3$, are being respectively the components of the momentum and Pauli matrices. The eigenvalue equation

$$\left[\sigma \cdot p/|p|\right] \Psi = \lambda \Psi$$

(6.1)

produces eigenvalues $\lambda$ which can be only $\pm 1$. Moffat, Ref.[62], designed a classical analog of the helicity operator. He proposed to use the product $v \cdot \nabla \times v \equiv v \cdot \tilde{\omega}$ for this classical analog. In it, as before, e.g. see Eq.(4.14), the vorticity field $\tilde{\omega}$ is used. Moffat constructed an integral (over the volume $M$)

$$I = \int_M v \cdot \tilde{\omega} \, dV$$

(6.2)

along with two other integrals: the kinematic kinetic energy

$$\frac{2T}{\rho} E = \int_M v^2 \, dV$$

(6.3)

and the rotational kinetic energy

$$\Omega = \int_M \tilde{\omega}^2 \, dV.$$  

(6.4)

Then, he used the Schwarz inequality

$$I^2 \leq E \Omega \quad \text{or} \quad \Omega \geq \frac{I^2}{E}$$

(6.5)

in order to demonstrate that the equality is achieved only if $\tilde{\omega} = \alpha v$ where $\alpha$ is a constant. Since this requirement coincides exactly with our Eq.(4.20b), it is of interest to study this condition further. In particular, under this condition we obtain $\alpha I = E$ which would coincide with our Eq.(5.43) (see also 5.48)) should $I$ be associated with the linking number. Fortunately, this is indeed the case. The proof was given by Arnold and is outlined in Ref.[63], pages 141-146. In view of its physical significance, we would like to discuss it in some detail.

Before doing so, we notice that the condition $\tilde{\omega} = \alpha v$ is known in literature as the force-free condition for the following reason. In electrodynamics, the motion of an electron in a magnetic field is given by (in the system of units in which $m = c = e = 1$)

$$\frac{d\mathbf{v}}{dt} = \mathbf{v} \times \mathbf{B}$$

(6.6a)

while the use of the Maxwell’s equation, our Eq.(4.10), produces as well

$$\mathbf{v} = \nabla \times \mathbf{B} = \alpha \mathbf{B}$$

(6.6b)

Using previously established equivalence $\mathbf{v} \equiv \mathbf{B}$ and substitution of Eq.(6.6b) into Eq.(6.6a) explains why the force-free condition is given by $\tilde{\omega} = \alpha v$. This equation can be looked upon as an eigenvalue equation for the operator $\nabla \times (\cdots)$. From this point of view the force-free equation is totally analogous to its quantum counterpart, Eq.(6.1). Details can be found in Ref.[64].

Going back to Arnold’s proof, we note that according to Moffatt, Ref.[62], page 119,

$$I = \int_M v \cdot \tilde{\omega} \, dV = \frac{1}{4\pi} \int_{V(1)} \int_{V(2)} \frac{\mathbf{R}_{12} \cdot [\tilde{\omega}(1) \times \tilde{\omega}(2)]}{||\mathbf{R}_{12}||^3} \, dV(1) \, dV(2).$$

(6.7)
Clearly, if as is done by Moffatt and others in physics literature (e.g. Lund and Regge, etc.), we assume that the vector potential $A$ can be given in the form of Eq.(5.44), then $I$ indeed becomes the linking number, Eq.(4.6). If, however, we do not make such an assumption, then much more sophisticated methods are required for the proof of this result. Use of these methods is not of academic interest only, as we would like to explain now. According to Kozlov, Ref.[65], the force-free case $\tilde{\omega} = \alpha \mathbf{v}$ belongs to the category of so called vortex motion in the weak sense. There are many other vortex motions for which $\mathbf{v} \times \nabla \times \mathbf{v} \neq 0$. These are vortex motions in the strong sense. Evidently, any relation with superconductivity or superfluidity (which is actually only hinted at this stage in view of results obtained in previous sections) is lost in this (strong) case. But even with the vorticity present in the weak sense this connection is not immediately clear. This is so because of multitude of solutions of the force-free equation as discussed, for example, in Refs.[66, 67]. We would like to discuss only those solutions which are suitable for use in Arnold’s theorem. These solutions can be obtained as follows. Taking the curl of the equation

$$\nabla \times \mathbf{B} = \alpha \mathbf{B}, \quad (6.8)$$

provided that $\nabla \cdot \mathbf{B} = 0$, produces

$$(\nabla^2 + \alpha^2) \mathbf{B} = 0, \quad (6.9)$$

to be compared with our result, Eq.(4.52a). Unlike our case, which is motivated by analogies with superconductivity and superfluidity, in the present case there are many solutions of this equation. We choose only the solution which illustrates the theorem by Arnold. It is given by $\mathbf{v} = (A \sin z + C \cos y, B \sin x + A \cos z, C \sin y + B \cos z)$, where $ABC \neq 0$ and $A, B, C \in \mathbb{R}$. This solution is obtained for $\alpha = 1$.

Following Arnold, we introduce the asymptotic linking number $\Lambda(x_1, x_2)$ via

$$\Lambda(x_1, x_2) = \lim_{T_1, T_2 \to \infty} \frac{1}{4\pi T_1 T_2} \int_0^{T_1} \int_0^{T_2} \mathbf{d}t_1 \mathbf{d}t_2 \frac{(\dot{x}_1(t_1) \times \dot{x}_2(t_2)) \cdot (x_1(t_1) - x_2(t_2))}{\|x_1(t_1) - x_2(t_2)\|^3}. \quad (6.10a)$$

The theorem proven by Arnold states that if the motion described by trajectories $x_1(t_1)$ and $x_2(t_2)$ is ergodic, then

$$\frac{1}{4\pi} \int \int \frac{R_{12} \cdot [\tilde{\omega}(1) \times \tilde{\omega}(2)]}{\|R_{12}\|^3} dV(1)dV(2) = \frac{1}{\sqrt{2}} \int \int \Lambda(x_1, x_2) dV(1)dV(2) = lk(1, 2). \quad (6.10b)$$

That is the function $\Lambda(x_1, x_2)$ on ergodic trajectories is almost everywhere constant. This theorem as such does not imply that this constant is an integer. For us it is important to realize that both Eq.(4.52a) and Eq.(6.9) can produce trajectories minimizing the Schwarz inequality thus leading to the condition $\alpha I = E$ with $I$ being either linking (in the case of suspensions) or self-linking number (depending upon the problem in question) or a combination of both. Because both Eq.(4.52a) and (6.9) cause formation of links, the choice between them should be made on a case-by-case basis. In particular, existence of the Messner effect in superconductors leaves us with no freedom of choice between these two equations. In the case of magnetohydrodynamics/plasma physics the situation is less obvious. In the next subsection we shall argue in favour of superconducting/superfluid choice between these equations. To our knowledge, such a choice was left unused in plasma physics literature.

6.3 Ideal magnetohydrodynamics and superfluidity/superconductivity

In order to discuss the work by Schief, Ref.[58], we would like to remind to our readers of some facts from the work by Lund and Regge (originally meant to describe superfluid $^4\text{He}$) since these fact nicely supplement those presented in previous sections. We already mentioned that Berdichevsky adopted these results for normal fluids, including those which are turbulent. Lund and Regge assumed that the vortex has a finite thickness so that the non-slip boundary condition, Eq.(2.27), should be now amended to account for finite thickness. The amended equation is given by

$$v_i(t) = \frac{\partial x_i}{\partial t} + \frac{\partial x_i}{\partial \sigma} \frac{\partial \sigma}{\partial t}, \quad (6.11)$$
where $\sigma$ parametrizes the coordinate along the vortex line. Eq.(5.38) taken from work by Lund and Regge then implies:

$$
\varepsilon_{ijk} \frac{\partial x_i}{\partial \sigma} \left( \frac{\partial x_k}{\partial t} - v_k \right) = 0.
$$

(6.12)

This equation is treated as an equation of motion by Lund and Regge obtained with help of the following Lagrangian

$$
L = \frac{k \rho}{3} \int C \varepsilon_{ijk} x_i \frac{\partial x_j}{\partial \sigma} \frac{\partial x_k}{\partial t} \, d\sigma - \frac{\rho}{2} \int \sqrt{\delta^2} \, dV.
$$

(6.13)

Since the zero thickness limit of the action for this Lagrangian is given by our Eq.(4.22), which upon integration of the $A$-field leads to the result, Eq.(4.24), the same can be done in the present case and, accordingly, by analogy with the action, Eq.(4.22), which was extended, e.g. see Eq.(4.40), in the present case it can be extended as well so that the final result for the action of the Nambu-Goto bosonic string interacting with electromagnetic-type field reads (using the same signature of space-time as used in Ref.[45])

$$
S = -m \int d\sigma d\tau \sqrt{-g} + f \int A_{\mu\nu} \frac{\partial x_\mu}{\partial \sigma} \frac{\partial x_\nu}{\partial \tau} \, d\sigma d\tau - \frac{1}{4} \int F^2 \, dvol
$$

(6.14)

with

$$
\sqrt{-g} = \left[ - \left( \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\nu}{\partial \sigma} \right) \cdot \left( \frac{\partial x^\mu}{\partial \tau} \frac{\partial x_\alpha}{\partial \tau} \right) + \left( \frac{\partial x^\mu}{\partial \sigma} \frac{\partial x^\nu}{\partial \sigma} \right)^2 \right]^{1/2}.
$$

(6.15)

and $m$ and $f$ being some coupling constants. The metric of the surface enclosing the vortex can be always brought to diagonal form by some conformal transformation\textsuperscript{19}. In such coordinates, variation of the action $S$ produces the following set of equations

$$
m(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \sigma^2}) x_\mu = f \varepsilon^{\mu\nu\lambda\rho} F_{\nu\rho} \frac{\partial x_\mu}{\partial \tau} \frac{\partial x_\lambda}{\partial \sigma} 
$$

(6.16a)

and

$$
\partial^\mu \partial_\mu A^{\alpha\beta} = -2f \int d\sigma d\tau \left( \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} - \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \sigma} \right) \delta^{(4)}(x(\sigma, \tau) - y)
$$

(6.16b)

provided that $\partial_{\mu} A^{\mu\nu} = 0$. Since the last equation is just the wave equation with an external source, the equation of motion for the vortex is Eq.(6.16a). In such a form it was obtained in Ref.[58] describing vortices in ideal magnetohydrodynamics. Under some physically plausible condition it was reduced in the same reference to the equation of motion for the one-dimensional Heisenberg ferromagnet. This result will be discussed further below from a somewhat different perspective.

It should be noted though that Eq.(6.16a) emerges in Ref.[58] under somewhat broader conditions than those allowed by the force-free equation. In view of the content of the next subsection, we would like to reproduce this, more general case, now. For this purpose, we recall that the Euler’s equation for fluids can be written in the form, Ref.[29],

$$
\frac{\partial}{\partial t} \tilde{\omega} = \nabla \times (v \times \tilde{\omega}).
$$

(6.17)

In the case when $\tilde{\omega}$ is time-independent, it is sufficient to require only that

$$
v \times \tilde{\omega} = \nabla \Phi
$$

(6.18)

with $\Phi$ being some (potential) scalar function. In the case of hydrodynamics the equation $\Phi = \text{const}$ is the famous Bernoulli equation. Thus, the force-free condition in this case is equivalent to the Bernoulli condition/equation. In magnetohydrodynamics there is an analog of the Bernoulli equation as explained in Ref.[69]. So, again, the equation $\Phi = \text{const}$ is equivalent to the force-free condition. In the case of magnetohydrodynamics the vortex Eq.(6.16a) is obtained under the condition $\Phi = \text{const}$. Since Eq.(6.16a) describes the vortex filament, the helicity integral, Eq.(6.7), describes either linking, self-linking or both.

\textsuperscript{19}For more details, please see Ref.[68].
In the case of self-linking it is known, e.g. see Ref.s[48,63], that \( lk(1, 1) = Tw + Wr \). Analytically, the writhe \( Wr \) term is expressible as in Eq.(4.6) but with \( C_1 \) and \( C_2 \) now representing the same closed curve. The need for \( Tw \) disappears if the closed curve can be considered to have zero thickness. More accurately, the closed curve should be a ribbon in order to have a nonzero \( Tw \). This is explained in Ref.[63]. With the exception of Appendix C, in this work we have ignored such complications.

6.4 Classical mechanics in the vortex formalism, inertial dynamics of nonrigid bodies and G-L theory of high temperature superconductors

Euler’s Eq.(6.17) can be rewritten in the equivalent form:

\[
\frac{\partial}{\partial t} \mathbf{v} = \mathbf{v} \times \mathbf{\omega} - \nabla \Phi. \tag{6.19}
\]

Following Kozlov, Ref.[65], in the case of Hamiltonian mechanics it is convenient to consider a very similar (Lamb) equation given by

\[
\frac{\partial}{\partial t} \mathbf{u} + (\nabla \times \mathbf{u}) \cdot \mathbf{v} = -\nabla \Phi, \tag{6.20}
\]

in which the vector \( \mathbf{u} \) is such that \( \nabla \cdot \mathbf{u} = 0 \). It can be demonstrated that Hamiltonian dynamics is isomorphic to the dynamics described by the above Lamb equation, provided that we make the following identifications. Let \( \Sigma^n_t \) be a manifold in phase space \( P = T^* M \) admitting a single-valued projection onto a configurational space \( M \). In canonical coordinates \( x \) and \( y \) this manifold is defined by the equation

\[
y = u(x, t). \tag{6.21}
\]

It is not difficult to demonstrate that the manifold \( \Sigma^n_t \) is an invariant manifold for a canonical Hamiltonian \( H(x, y, t) \) if and only if the field \( y \) satisfies the Lamb’s Eq.(6.20) and that \( \Phi(x, t) = H(x, y(x, t), t) \) is a function on \( M \) parametrized by time \( t \) in such a way that

\[
\mathbf{v} = \left. \frac{\partial H}{\partial y} \right|_{y=u} \tag{6.22}
\]

and

\[
\dot{y} = \left. \frac{\partial H}{\partial x} \right|_{y=u} + \frac{\partial u}{\partial t} \tag{6.23}
\]

Relevance of these results to our discussion can be seen when Eq.(6.23) is compared with Eq.(6.11) of Lund and Regge. This comparison shows their near equivalence. In view of this, we would like to exploit this equivalence further by employing it for analysis of the truncated G-L functional analogous to our Eq.(5.3) typically used for phenomenological description of high temperature superconductors [54]. In this case the functional \( \tilde{F}[\mathbf{A}, \varphi] \) should be replaced by

\[
\tilde{F}[\mathbf{A}, \varphi] = \int d^3 r \left\{ \frac{1}{8\pi} (\mathbf{\nabla} \times \mathbf{A})^2 + \frac{\hbar^2}{2m} \left| (\mathbf{\nabla} \varphi - \frac{2\tilde{e}}{\hbar c} \mathbf{A}) \right|^2 + \frac{\hbar^2}{2m} \left| (\mathbf{\nabla} \varphi - \frac{2\tilde{e}}{\hbar c} \mathbf{A}) \right|^2 \right\} \tag{6.24}
\]

with its components lying in the x-y (cuprate) plane and z-plane perpendicular to it. By varying this functional with respect to \( \mathbf{A}_\perp \) and \( \mathbf{A}_\parallel \) separately we obtain respectively the following components for the Maxwell’s equation

\[
\mathbf{\nabla} \times \mathbf{B}_i = \frac{4\pi}{c} \mathbf{j}_i \ (i = \perp \text{ and } \parallel), \tag{6.25}
\]

where

\[
\mathbf{j}_\perp = -\frac{ie\hbar}{2m} (\varphi^* \mathbf{\nabla} \varphi - \varphi^* \mathbf{\nabla} \varphi^*), \quad \text{and} \quad \mathbf{j}_\parallel = -\frac{ie\hbar}{2m} (\varphi^* \frac{d}{dz} \varphi - \varphi \frac{d}{dz} \varphi^*) - \frac{2\tilde{e}^2}{m} |\varphi|^2 \mathbf{A}_\parallel. \tag{6.26}
\]
From here we obtain the phenomenological London-type equations
\[ \nabla \times j_\perp = -\frac{e n_s}{m_\perp c} B_\perp \quad \text{and} \quad \nabla \times j_\parallel = -\frac{e n_s}{m_\parallel c} B_\parallel. \] (6.27)

By combining Eqs (6.25) and (6.27) and using results of our Sections 4.2. and 4.4 we can rewrite these equations in the following suggestive (London-type) form
\[ \tilde{\omega}_\perp = e_\perp v_\perp \quad \text{and} \quad \tilde{\omega}_\parallel = e_\parallel v_\parallel. \] (6.28)

This form allows us to make a connection with the inertial dynamics of a nonrigid body. Following Kozlov, Ref.[65], we consider the motion of a nonrigid body in which particles can move relative to each other due to internal forces. Let the inertia axes of the body be the axes of the moving frame. Let \( K \) be the angular momentum of the body relative to a fixed point and \( \omega \) the angular velocity of the moving trihedron while the inertia matrix \( I \) is \( \text{diag} (I_\perp, I_\perp, I_\parallel) \). The angular momentum and the angular velocity are related by
\[ K = I \omega + \lambda. \] (6.29)
where \( \lambda = (\lambda_\perp, \lambda_\perp, \lambda_\parallel) \) is the gyroscopic torque originating from the motion of particles inside the body.

From here we obtain the Euler equation
\[ \dot{K} = \omega \times K = 0, \] (6.30)
which is a simple consequence of Eq.(6.29). In view of Eqs.(4.45) and (4.48) we can identify Eqs. (6.28) with (6.29) thus formally making Eqs. (6.29), of London type. The hydrodynamic analogy can be in fact extended so that the hydrodynamically looking Lamb-type equation can be easily obtained and analyzed. Details are given in Ref.[65], page 148.

6.5 Dirac monopoles, dual Meissner effect, Abelian projection for QCD and string models

At this point our readers may have already noticed the following. 1. In our derivation of Eq.(5.63) we made screening effects seemingly disappear while the title of our work involves screening. 2. In Eq.(6.14) we introduced the Nambu-Goto string normally used in hadron physics associated with non Abelian Yang-Mills (Y-M) gauge fields. Quantum chromodynamics (QCD) of hadrons and mesons is definitely not the same thing as scalar electrodynamics (that is G-L model) discussed in our work. 3. Variation of the action \( S \) in Eq.(6.14) leading to the string equation of motion, Eq.(6.16a), under some conditions reduces to the equation of motion for the Heinsenberg (anti) ferromagnetic chain, which indeed describes the motion of the vortex filaments [59]. From this reference it follows that such equation of motion, in principle, can be obtained quite independently from the Nambu-Goto string, QCD, etc. In this subsection we demonstrate that the above loose ends are in fact indicative of the very deep underlying mathematics and physics needed for a unified description of all of these phenomena.

The formalism developed thus far in this work suffers from a kind of asymmetry. On one hand, we started with a solution of hard spheres and then we noticed that these hard spheres in solution act as currents (if one is using the magnetic analogy). The famous Biot-Savart law in magnetostatics is causing two currents to be entangled with each other thus creating the Gauss linking number, Eq.(4.8). Thus, it appears that in solution two particles (currents) are always linked (entangled) with each other. That this is indeed the case was noticed long ago as mentioned in the Introduction, e.g. see Ref.[11]. We can treat the vortices causing such linkages as independent objects. This is reflected in the fact that we introduced the vorticity \( \tilde{\omega} (r) \) as \( \tilde{\omega} = k \frac{1}{C} \int C d\sigma v(\sigma) \delta (r - r(\sigma)), \) e.g. read comments after Eq.(5.38). In view of our major equation \( \tilde{\omega} (r) = ev \), we can think either about the velocity (or vorticity) of a particular hard sphere or about the velocity of a particular vortex. Because of this, it is possible to treat both particles and vortices
on the same footing. In such a picture (sketched in Appendix B) one can either eliminate vortices and think about effective interactions between hard spheres or vice versa. In this sense we can talk about the *duality* of descriptions and, hence, about the *dual Meissner effect*—for loops instead of particles\(^{21}\).

Before describing the emerging picture in more detail, we note the following. Consider the expression for vorticity \(\tilde{\omega} = k \oint_C dv(\sigma) \delta(\mathbf{r} - \mathbf{r}(\sigma))\) from the point of view of reparametrization invariance. In particular, since we have a closed contour, we can always choose it as going from infinity to minus infinity (it is easy topologically to wrap it onto a closed contour of any size). For the function \(y = \exp(\sigma)\) we have evidently \(0 \leq y \leq \infty\) when \(\sigma\) varies from \(-\infty\) to \(\infty\). This means that we can replace \(\sigma\) by \(\ln y\) in the expression for the vorticity in order to obtain

\[
\tilde{\omega} = k \int_{-\infty}^{0} dz \mathbf{v}(z) \delta(\mathbf{r} - \mathbf{r}(z)),
\]

which in a nutshell is the same thing as a Dirac monopole, Ref.[70], with charge strength \(k\), so that the vortices can be treated as Dirac monopoles. In Appendix C we provide some facts about Dirac monopoles in relation to vortices. According to Dirac [70] the string attached to such a monopole can either go to infinity (as in the present case) or to another monopole of equal and opposite strength. In our case this means only that when two hard spheres become hydrodynamically entangled, they cannot escape the linkage they formed. This is the (topological) essence of quark confinement in QCD known as *monopole condensation*\(^{22}\). But we are not dealing with QCD in this work! How then we can talk about the QCD? The rationale for this was put forward first by Nambu, Ref.[71]\(^{23}\). In his work he superimposed the G-L and Dirac monopole theories to demonstrate quark confinement for mesons (these are made of just two quarks: quark and antiquark). For this qualitative picture to make sense, there should be some way of reducing QCD to G-L type theory. The feasibility of such an *abelian reduction (projection)* was investigated first by 't Hooft in Ref.[72]. Recent numerical studies have provided unmistakable evidence supporting the idea of quark confinement through monopole condensation, Ref.s [73,74]. Theoretical advancements made since the publication of 't Hooft’s paper took place along two different (opposite) directions. In one direction, recently, Faddeev and Niemi found knot-like topological solitons using a Skyrme-type nonlinear sigma model and conjectured that such a model can correctly represent QCD in the low energy limit [75,76]. That this is indeed the case was established in a series of papers by Cho [77,78] and, more recently, by Kondo, Ref.[79]. In another direction, in view of the fact that, while macroscopically the Meissner effect is triggered by the effective mass of the vector field, microscopically this mass is generated by Cooper pairs [25], it makes sense to look at detection of the excited states of such Cooper-like pairs experimentally. The famous variational BCS treatment of superconductivity contains at its heart the gap equation responsible for the formation of Cooper pairs. The BCS treatment of superconductivity was substantially improved by Richardson, Ref.[80], who solved the microscopic model exactly. His model is known in literature as the Richardson model. Closely related to this model is a model proposed by Gaudin. It is also exactly solvable (by Bethe anstatz methods) [81]. The Gaudin model(s) describes various properties of one dimensional spin chains in the semiclassical limit. Energy spectra of the Gaudin and Richardson models are very similar. In particular, under some conditions they are equidistant, like those for bosonic string models.\(^{24}\) Recently, we were able to find new models associated with Veneziano amplitudes, e.g. see Ref.s [82,83], describing meson-meson scattering processes. In particular, we demonstrated that the Richardson-Gaudin spin chain model producing equidistant spectra can be obtained directly from Veneziano amplitudes. Since the Veneziano amplitudes describe extremely well the meson mass spectrum, and since we demonstrated that the Richardson-Gaudin model (originally used in superconductivity and nuclear physics) can be recovered from combinatorial and analytical properties

\(^{21}\)It should be noted that in the case of usual superconductors one should distinguish between the constant magnetic fields penetrating superconductors and the fields made by vortices. In the case of colloidal suspensions it is also possible to create some steady velocity current and to consider velocity at a given point in the fluid as made of both steady and fluctuating parts.

\(^{22}\)That is the Bose-Einstein-type condensation in view of results of Section 5. This is explained further in Appendix C

\(^{23}\)We discuss his work briefly in Appendix C

\(^{24}\)Also, for monopoles models discussed in Appendix C.
of these amplitudes, this means that the Abelian reduction can be considered as confirmed (at least for mesons) not only numerically but also experimentally.

6.6 Miscellaneous

In Section 3.3. we demonstrated that for colloidal suspensions it is sufficient to use only the Abelian version of the Chern-Simons theory for description of emerging entanglements. There could be other instances where such an Abelian treatment might fail. Examples of more sophisticated non-Abelian fluids were considered in several recent excellent reviews [84,85]. These papers might serve as points of departure for the treatment of more elaborate hydrodynamical problems involving non-Abelian entanglements. Finally, the force-free equation $\vec{\omega} = \alpha \vec{v}$ which is used in our work, is known to possess interesting new physical properties when, instead of treating $\alpha$ as a constant, one treats $\alpha$ as some function of the coordinates. Such treatment can be found in Ref.[86] and involves the use of conformal transformations and invariants recently considered in our work on the Yamabe problem, Ref.[87], and the Poincare’ conjecture, Ref.[68].

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Appendix A. Some facts from the theory of Green’s functions

Consider an equation

$$\left( \frac{\partial}{\partial t} - H \right) \Phi = 0.$$  

(A.1)

Such an equation can be written in the form of an integral equation as follows

$$\Phi(x, t) = \int G_0(x, t; x', t')\Phi_0(x', t')dx'dt'$$  

(A.2)

so that

$$\Phi(x, t \rightarrow t') = \Phi_0(x, t').$$  

(A.3)

Under such conditions, the Green’s function $G_0(x, t; x', t')$ must obey the following equation

$$\left( \frac{\partial}{\partial t} - H \right) G_0(x, t; x', t') = \delta(x - x')\delta(t - t')$$  

(A.4)

provided that $G_0 = 0$ for $t < t'$. In a more complicated situation, when

$$\left( \frac{\partial}{\partial t} - H - V \right) G(x, t; x', t') = \delta(x - x')\delta(t - t')$$  

(A.5)

we can write a formal solution for $G$ in the form of the integral (Dyson’s) equation

$$G(x, t; x', t') = G_0(x, t; x', t') + \int G_0(x, t; x', t')V(x', t')G(x', t'; x'', t'')dx'dt'$$  

(A.6)

or, symbolically, $G = G_0 + G_0VG$. In the case of Eq.(4.35) of the main text, we have to replace Eq.(A.5) by

$$\left( \frac{\partial}{\partial t} - H_1 - H_2 - V_{12} \right) G(x_1, x_2, t; x'_1, x'_2, t') = \delta(x_1 - x'_1)\delta(x_2 - x'_2)\delta(t - t')$$  

(A.7)
and, accordingly, the Dyson type Eq.(A.6) is now replaced by the analogous equation in which now we must have $G_0(x_1, x_2, t; x'_1, x'_2, t') = G_0(x_1, t; x'_1, t')G_0(x_2, t; x'_2, t')$. To check the correctness of such a decomposition we note that for $x \neq x'$ Eq.s (A.1) and (A.4) coincide while for $t \to t'$ integration of Eq.(A.4) over a small domain around zero and taking into account that $G_0 = 0$ for $t < t'$ produces $G_0(x, t \to t'; x', t') = \delta(x - x')$. Repeating these arguments for the two-particle Green’s function and using Eq.(A.7) (with $V_{12} = 0$ ) provides the needed proof of the decomposition of $G_0$ in the two-particle case.

Define now formally the renormalized potential $V$ via

$$G = G_0 + G_0V G_0.$$

(A.8)

Then, by comparing this equation with the original Dyson’s equation for $G$ we obtain

$$G - G_0 = G_0V G_0 = G_0V G = G_0V (G_0 + G_0V G_0).$$

(A.9)

This allows us to write the integral equation for the effective potential $V$ as

$$V = V + VG_0V.$$

(A.10)

Appendix B. Dual treatment of the dynamics of colloidal suspensions and hydrodynamic screening

We begin by first considering screening. The path integral for the functional, Eq.(5.5), can be conveniently rewritten as follows

$$\mathcal{F}[A, \varphi] = \frac{\rho}{2} \int d^3r \left\{ (\nabla \times A)^2 + D_0 \left( \left( \nabla - \frac{2\pi e}{D_0} A \right) \varphi \right)^2 \right\}$$

$$= \frac{\rho}{2} \int d^3r \left\{ (\nabla \times A)^2 + \left( \frac{D_0}{\pi} \right)^2 \left( \nabla \psi - \frac{2\pi e}{D_0} A \right)^2 \right\}$$

(B.1)

upon substitution of the ansatz $\varphi = \frac{\sqrt{2D_0}}{2\pi} \exp(i\psi)$ into first line of Eq.(B.1). Such a substitution is consistent with the current defined in Eq.(4.50).

Since $\nabla \cdot A = 0$, we obtain

$$\left( \nabla \psi - \frac{2\pi e}{D_0} A \right)^2 = \left( \nabla \psi \right)^2 + \left( \frac{2\pi e}{D_0} A \right)^2 - \frac{4\pi e}{D_0} A \cdot \nabla \psi.$$

(B.2)

Consider now the following path integral

$$Z = \int D\{\psi\} \exp\left[ -\frac{1}{2} \left( \frac{D_0}{\pi} \right)^2 \int d^3r \left( \nabla \psi \right)^2 - \frac{4\pi e}{D_0} A \cdot \nabla \psi \right].$$

(B.3a)

Since it is of a Gaussian-type, it can be straightforwardly calculated with the result

$$Z = N \exp\left( \frac{\epsilon^2}{2} A_{\mu} \frac{\partial_{\nu} \partial_{\rho} A_{\nu}}{\nabla^2} \right).$$

(B.3b)

Here $N$ is some (normalization) constant. Using this result and Eq.(B.1) we obtain the following final expression for the partition function for the vector $A$-field with account of constraints

$$\Xi = \int D[A] \exp\left\{ -\frac{\rho}{2k_BT} \int d^3r \left\{ A_{\mu} \left[ -\delta_{\mu\nu} \nabla^2 - (1 - \frac{1}{\xi}) \partial_{\mu} \partial_{\nu} \right] A_{\nu} + \epsilon^2 A_{\mu} (\delta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\nabla^2}) A_{\nu} \right\}. \right.$$ 

(B.4a)

This result is in complete accord with Eq.(4.51b) where for the mass $m$ of the vector field $A$ we obtained: $m = \epsilon$. The above derivation was made without the use of Higgs-type calculations, Ref.[88]. Surely, it is
in accord with these calculations. We would like now to rewrite the obtained result in a somewhat formal (simplified) form as follows:

$$\Xi = \int D[A] \delta(\nabla \cdot A) \exp\left\{ -\frac{\rho}{2k_BT} \int d^3r [ (\nabla \times A)^2 + e^2A^2] \right\}. \quad (B.4b)$$

This will be used below in such simplified form. To avoid extra notation, we also set $\frac{\rho}{k_BT} = 1$. This factor can be restored if needed.

Now we are ready for the dual treatment, which can be done in several ways. For instance, following the logic of Dirac’s paper [70], we replace $\Xi$ by

$$\Xi = \int D[A] \delta(\nabla \cdot A) \exp\left\{ -\frac{1}{2} \int d^3r [ (\nabla \times A + \mathbf{v})^2 + e^2A^2] \right\} \quad (B.5)$$

where $\mathbf{v} = \frac{\tilde{\omega}}{e} = \oint_C d\sigma \mathbf{v}(\sigma) \delta(\mathbf{r} - \mathbf{r}(\sigma))$. Next, we use the Hubbard-Stratonovich-type identity allowing us to make a linearization, e.g.

$$\exp\left\{ -\frac{1}{2} \int d^3r [(\nabla \times A + \mathbf{v})^2] \right\} = \int D[\Psi] \exp\left\{ -\frac{1}{2} \int d^3r \Psi^2 + i \int d^3r ( (\nabla \times A + \mathbf{v}) \cdot \Psi) \right\} \quad (B.6)$$

Then, we take advantage of the fact that $(\nabla \times A) \cdot \Psi = (\nabla \times \Psi) \cdot A + \nabla \cdot (A \times \Psi)$. By ignoring surface terms this allows us to rewrite the above result as follows

$$\int D[\Psi] \exp\left\{ -\frac{1}{2} \int d^3r \Psi^2 + i \int d^3r ( (\nabla \times \Psi) \cdot A + \mathbf{v} \cdot \Psi) \right\} = \int D[\Psi] \exp\left\{ -\frac{1}{2} \int d^3r \Psi^2 + i \int d^3r ( (\nabla \times \Psi) \cdot A + \mathbf{v} \cdot \Psi) \right\} \quad (B.7)$$

Using this result in Eq.(B.5) and using the Hubbard-Stratonovich transformation again we obtain:

$$\Xi = \int D[\Psi] \delta(\nabla \cdot \Psi) \exp\left\{ -\frac{1}{2} \int d^3r [ (\nabla \times \Psi)^2 + e^2\Psi^2] \right\} = \int D[\Psi] \delta(\nabla \cdot \Psi) \exp\left\{ -\frac{1}{2} \int d^3r [\Psi^2 + \frac{1}{e^2} (\nabla \times \Psi)^2] + i \int d^3r \mathbf{v} \cdot \Psi \right\}. \quad (B.8)$$

Since $\exp(i \int d^3r \mathbf{v} \cdot \Psi) = \exp(i \oint_C d\sigma \mathbf{v}(\sigma) \cdot \Psi(\sigma))$ we can use this expression in Eq.(5.34) in order eventually to arrive at the functional of G-L-type (analogous to Eq.(5.6) with obviously redefined constants). The vector field $\Psi$ is now massive. It is convenient to make a replacement: $\Psi \rightarrow e\Psi$ to make the functional for the $\Psi$ field look exactly as in Eq.(B.5) (with $\mathbf{v} = 0$). The above transformations provide a manifestly dual formulation of the colloidal suspension problem. These transformations can be made differently nevertheless. Such an alternative treatment is useful since the end result has relevance to string theory and to the problem of quark confinement in QCD as was first noticed by Nambu, Ref.[71]. This topic is discussed briefly in the next appendix.

Appendix C Nambu string and colloidal suspensions: Some unusual uses of Dirac monopoles.
We begin with Eq.(B.5) but this time we treat it differently. In particular, we have

\[
\Xi = \int D[A] \delta(\nabla \cdot A) \exp\left\{-\frac{1}{2} \int d^3r \left[ (\nabla \times A) + v^2 + e^2 A^2 \right] \right\}
\]

\[
= \int D[A] \delta(\nabla \cdot A) \exp\left\{-\frac{1}{2} \int d^3r v^2 - \int d^3r (\nabla \times A) \cdot v - \frac{1}{2} \int d^3r (\nabla \times A)^2 - \frac{e^2}{2} \int d^3r A^2 \right\}
\]

\[
= \int D[A] \delta(\nabla \cdot A) \exp\left\{-\frac{1}{2} \int d^3r v^2 - \int d^3r (\nabla \times A) \cdot A - \frac{1}{2} \int d^3r (\nabla \times A)^2 - \frac{e^2}{2} \int d^3r A^2 \right\}
\]

\[
= \int D[A] \delta(\nabla \cdot A) \exp\left\{-\frac{1}{2} \int d^3r v^2 + e^2 \sum_{i<j} \oint_{C_i} \oint_{C_j} \frac{dl(\sigma_i) \cdot dl(\sigma_j)}{|r(\sigma_i) - r(\sigma_j)|} \exp\left(-\frac{|r(\sigma_i) - r(\sigma_j)|}{\xi_H}\right) \right\}. \quad (C.1)
\]

The exponent in Eq.(C.1) is useful for comparison with that given in Eq.(4.40). Such a comparison suggests that while the second (linking) term is essentially the same as in Eq.(4.40)\textsuperscript{25}, the first term in the exponent of Eq.(C.1) might be analogous to the "kinetic" string-like term in Eq.(4.40). This line of reasoning can be found in the paper by Nambu \textsuperscript{[71]}. If one ignores quark masses as is usually done in string-theoretic literature, then Eq.(13) of Nambu's paper looks very much like our Eq.(C.1), provided that we identify the first term with the stringy Nambu-Goto term\textsuperscript{26}. To do so, we formally need to use the results of our Sections 5.5 and 6.2. This time, however, we have to allow for self-linking. Also, we have to take into account that for this case the energy and the helicity become the same (up to a constant). Thus, one can consider the helicity instead of energy. A very detailed treatment of helicity was made in the paper by Ricca and Moffatt, Ref.[89], from which it follows that the helicity is ideally suited for the description of self-linking. In such a case we have to deal with closed curves of finite thickness. In fact, it is sufficient to have a closed tube instead of a closed infinitely thin curve. On such a tube one can perform the Dehn surgery by cutting a tube at some section, twisting the free ends through a relative angle $2\pi n_0$, where $n_0$ is some integer, and reconnecting the ends. This operation makes a self-linking proportional to $n_0$. If we agree that the Dehn twists are made only in increments of $\pm 2\pi$, we obtain the "spectrum" which is equidistant and, hence, string-like.

This intuitive picture can be made more quantitative as follows. Taking into account Eq.(5.48), the kinetic term in the exponent of Eq.(C.1) can be tentatively written as follows

\[
\frac{1}{2} \int d^3r v^2 = \frac{e^2}{2} \sum_{C_i, C_j} \oint_{C_i} \oint_{C_j} \frac{v(\sigma) \cdot v(\sigma')}{|r(\sigma) - r(\sigma')|}. \quad (C.2)
\]

This expression suffers from two apparent deficiencies. First, while the second term in the exponent of Eq.(C.1) accounts for screening effects, Eq.(C.2) is written without such an account. Second, since energy and helicity are proportional to each other and since the Dehn surgery can be made only for surfaces, Eq.(C.2) should be modified by replacing infinitely thin contours by tubes. To repair the first problem we follow the book by Pismen, Ref.[90], where on page 186 we find the following information. Consider our Eq.s(4.52a) or (4.52b) and take into account Eq.(4.15). Then, we can write

\[
\nabla^2 A - e^2 A = -e \oint_{C} d\sigma v(\sigma) \delta(\mathbf{r} - \mathbf{r}(\sigma)). \quad (C.3)
\]

The solution of the equation for vector potential $A$, Eq.(5.38), should now be modified to account for screening and boundary effects. The result for energy, Eq.(5.47), now will be changed accordingly so that the screening exponent will emerge in Eq.(C.1). To account for surface effects we recognize that the self-linking expression, Eq.(C.2) is reparametrization invariant. If, instead of infinitely thin contours we consider fluctuating tubes, the reparametrization invariance should survive. The surface analog of the expression $\oint_{C} d\sigma v(\sigma) \delta(\mathbf{r} - \mathbf{r}(\sigma))$ is given in Eq.(6.16b). By introducing the notation

\[
S_{\alpha\beta} = \frac{\partial x^\alpha}{\partial \sigma} \frac{\partial x^\beta}{\partial \tau} - \frac{\partial x^\alpha}{\partial \tau} \frac{\partial x^\beta}{\partial \sigma} \quad (C.4)
\]

\textsuperscript{25}We have mentioned already that the screening is not affecting the topological nature of this term.

\textsuperscript{26}E.g. see Eq.(6.14).
the self-linking term can be brought into the following final form (for just one loop for brevity)

\[
\frac{1}{2} \int d^3 r v^2 = \frac{e^2}{2} \int d\sigma d\tau \int d\sigma' d\tau' S^{\alpha\beta}(\sigma, \tau) \exp\left(-\frac{|r(\sigma, \tau) - r(\sigma', \tau')|}{\xi}\right) S^{\alpha\beta}(\sigma', \tau'), \tag{C.5}
\]

which is just what Nambu obtained. He further demonstrated that such a term can be transformed into \(-m \int d\sigma d\tau \sqrt{-g}\) (e.g. see our Eq.(6.14)) with the constant \(m\) (the string tension) being related to coupling constant(s) of the theory. Since in the limit of infinitely thin tubes results just obtained match those discussed in our Section 5.5, we would like to take advantage of this observation. In Section 5.5 we considered fully flexible (Brownian) loops. From the theory of polymer solutions it is known that such loops can be made of the so called semiflexible polymers whose rigidity is rather weak. Following our work, Ref.[91], the path integrals describing semiflexible polymer chains are given by

\[
I = \int D[u(\tau)] \exp(-S[u(\tau)]) \tag{C.6}
\]

with action \(S[u(\tau)]\) given by

\[
S[u(\tau)] = \frac{\kappa}{2} \int_0^\tau d\tau' \left( \frac{du}{d\tau'} \right)^2 + \int_0^\tau d\tau \lambda(\tau)(u^2(\tau) - 1). \tag{C.7}
\]

The rigidity constant is \(\kappa\). For brevity it will be put equal to one. The Lagrange multiplier \(\lambda\) takes care of the fact that the "motion" is taking place on the surface of a 2-sphere. Minimization of the action \(S\) produces

\[
\frac{d^2}{d\tau^2} u = \lambda u \tag{C.8}
\]

with Lagrange multiplier being determined by the constraint \(\frac{d}{d\tau} u^2 = 0\) thus producing instead of Eq.(C.8) the following result:

\[
\ddot{u} = -(\dot{u} \cdot \dot{u}) u, \tag{C.9}
\]

where \(\dot{u} = \frac{d}{d\tau} u\), etc. In view of the results of this subsection, consider now an immediate extension of the obtained results known as the Neumann model\(^{27}\)

\[
\ddot{u} + Gu = \lambda u, \quad u^2 = 1 \tag{C.10}
\]

for some matrix \(G\) which always can be brought to the diagonal form. By analogy with Eq.(6.8), we can rewrite Eq.(C.10) in the following equivalent form

\[
u \times [\ddot{u} + Gu] = 0 \tag{C.11}
\]

since \(u \times \lambda u = 0\). The above equation is just a special case of the Landau-Lifshitz (L-L) equation describing dynamics of Heisenberg (anti)ferromagnets. In one space and one time dimension the L-L equation reads

\[
\frac{\partial}{\partial t} u = \{u \times [\ddot{u} + Gu]\}, \tag{C.12}
\]

where now \(\ddot{u} = \frac{d^2}{d\tau^2} u\), etc. In Sections 6.3 and 6.5 we mentioned already that L-L equation describes the dynamics of vortex filaments in fluids, plasmas, etc and is also obtainable from the Lund-Regge theory. Following Veselov.[93], consider a special solution of the L-L equation obtained by inserting the ansatz \(u(x, t) = u(x - i\theta t)\) into Eq.(C.12). Such a substitution produces:

\[
-i\theta \ddot{u} = \{u \times [\ddot{u} + Gu]\}, \quad u^2 = 1. \tag{C.13}
\]

\(^{27}\)Some useful details related to Neumann’s model can be found in our work, Ref.[92].
Taking a vector product of both sides of this equation produces

\[ \ddot{\mathbf{u}} + \mathbf{G}\mathbf{u} = \lambda \mathbf{u} + i\theta [\dot{\mathbf{u}} \times \mathbf{u}]. \]  

(C.14)

This equation describes the classical motion of a charged particle in the presence of a Dirac monopole. At the quantum level such a problem was studied in detail by Dunne, Ref.[94], who demonstrated that in the limit \( \theta \to \infty \) the monopole spectrum is equidistant. This result is compatible with the result of Ricca and Moffat [89], and explains the role of monopoles in quark confinement (in view of results of our Section 5.5). Furthermore it corroborates the results of our recent work, Ref.[83], briefly mentioned in Section 6.5., where the spectrum of the 1-d Heisenberg XXX spin chain was recovered directly from the combinatorics of scattering data supplied by uses of Veneziano amplitudes in scattering experiments.

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