Optimal control of anthracnose including a pulse strategy

David Jaures FOTSA MBOGNE
Department of Mathematics and Computer Science
ENSAI, The University of Ngaoundere
Email: mjdaudifots@gmail.com

I. INTRODUCTION

Anthracnose is a phytopathology which attacks several commercial tropical crops such as coffee. The Anthracnose of coffee is known under the name coffee berry disease (CBD) and its pathogen is the Colletotrichum kahawae, an ascomycete fungus. The literature on Anthracnose pathosystem is very large [4], [5], [7], [13], [15], [19], [20]. There are also several attempts of modelling in order to better understand CBD spread and identify efficient control strategies [9], [10], [12], [16], [17], [18], [19], [20], [26]. These approaches to control anthracnose disease. Among them there are genetic methods [4], [5], [7], [15], [17], [22], [23], biological control [11], chemical control [5], [21], [23] and cultivation practices [5], [18], [19], [20]. The chemical method seems more efficient but presents several ecological risks. Moreover, inadequate application of treatments could induce resistance in the pathogen [24]. A dynamical spatial model including chemical control have been proposed and analysed in [15]. The aim of that work was to optimize the used of the chemical control with respect to a given cost functional. The disease dynamics was represented by an inhibition rate $\theta$ satisfying the following problem.

\[
\partial_t \theta = \alpha (t, x) \left( 1 - \frac{1}{(1 - \sigma u (t, x))} \right) + \text{div} \left( A (t, x) \nabla \theta \right),
\]

on $\mathbb{R}^+ \times \Omega$

\[
\left\langle A (t, x) \nabla \theta (t, x), n (x) \right\rangle = 0, \quad \text{on } \mathbb{R}^+ \times \partial \Omega
\]

where $n (x)$ denotes the normal vector on the boundary at $x$ and

\[
\theta (0, x) = \rho (x) \geq 0, \quad x \in \Omega \subseteq \mathbb{R}^3
\]

The main contribution of the present paper is to include into the model above a pulse control strategy representing cultivation practices such as pruning old infected twigs and removing mummified fruits. Indeed, those actions are made at discrete times with a certain rhythm. Although the action of antifungal compounds is continuous their application is not continuous in the time. In the same manner parameters of the model are not necessarily continuous. In addition, this paper assume more weak conditions on the regularity of the parameter of the model than [13]. We then propose the following impulsive model.

\[
\partial_t \theta = \alpha (t, x) \left( 1 - \frac{1}{(1 - \sigma u (t, x))} \right) + \text{div} \left( A (t, x) \nabla \theta \right),
\]

on $\mathbb{R}^+ \setminus \{ (t_i)_{i \in \mathbb{N}} \} \times \Omega$

\[
\theta (t_i^+, x) = (1 - v_i (x)) \theta (t_i, x), \quad i \in \mathbb{N}^+, x \in \Omega
\]

\[
\tau_0 = t_0 = 0 \quad \text{and} \quad \tau_i = \inf \{ t_k > \tau_{i-1}; k > 0 \}
\]

\[
\| \theta (t_k, \cdot) \|_{H^1 (\Omega)} \geq \sigma^*
\]

where $(t_k)_{k \in \mathbb{N}}$ is an increasing sequence in $\mathbb{R}^+$ such that $\lim_{k \to \infty} t_k = \infty$.

\[
\left\langle A (t, x) \nabla \theta (t, x), n (x) \right\rangle = 0, \quad \text{on } \left( \mathbb{R}^+ \setminus \{ (t_i)_{i \in \mathbb{N}} \} \right) \times \partial \Omega
\]

where $n (x)$ designates the normal vector on the boundary at $x$ and

\[
\theta (0, x) = \rho (x) \geq 0, \quad x \in \Omega \subseteq \mathbb{R}^3
\]

Let recall that $\theta (t^+, x)$ and $\theta (t^-, x)$ mean respectively $\lim_{s \to t, s > t} \theta (s, x)$ and $\lim_{s \to t, s < t} \theta (s, x)$. Here, $\theta$ is assumed left continuous with respect to the time $\theta (t, \cdot) = \theta (t^-, \cdot)$.
In the model \(1 \rightarrow 5\) function \(\alpha\) represents the inhibition pressure which could also depend on climatic and environmental conditions \([9], [10], [12]\). It could be an almost periodic function with respect to time, taking into account climatic seasons. The term \(\text{div}(A\nabla \theta)\) refers to the spatial spread of the disease in the open domain \(\Omega \subset \mathbb{R}^2\). The boundary condition \((A\nabla \theta, n) = 0\) where \(A\) is a \(3 \times 3\)-matrix \((a_{ij})\) could be understood as the law steering migration of the disease between \(\Omega\) and its exterior. For instance, if \(A\) is reduce to \(I\) the identity matrix then \((\nabla \theta, n) = 0\) means that the domain \(\Omega\) is isolated or closed. That is \(\Omega\) does not have any exchange with its exterior. In order to give a mathematical sense to the term \((A\nabla \theta, n)\) we assume that \(\partial \Omega \in H^1(\mathbb{R}; \mathbb{R}^2)\).

There is two means of control in the model \(1 \rightarrow 5\). The first one is the chemical control corresponding the effects after application of fungicides. Those effects are modeled by a function \(v\) taking its values in the set \([0, 1]\). \(1 - \sigma\) is a threshold parameter that models the inhibition rate below which it is not possible and not necessary to try any control strategy. It may correspond to the to epidermis penetration by hyphae. The second control measure of anthracnose is cultivational practices which reduce at discrete times the density of the pathogen and therefore inhibition rate \(\theta\).

In the model \(1 \rightarrow 5\) cultivational practices are represented by the sequence \((v_i)_{i \in \mathbb{N}^*} \subset [0, 1]\). \(\sigma^*\) is a parameter that models the critical threshold above which it is necessary to intervene to reduce severity and diffusion of the disease. The sequence \((t_k)_{k \in \mathbb{N}}\) could be viewed as times when usual interventions are possible. It can be in terms of hours, days, months or weeks as it is common in coffee cultivation for instance. Indeed, practically it is not realistic to monitor or intervene the whole time. Thus, based on the hypothesis that any major events could not happen in a "small" interval of time a calender is established. Obviously, something could happen even in a little interval of time. However, we can assume that during that period chemical control still acts. We then assume that the period of activity of chemical treatment is finite and greater than \(\sup_{k \in \mathbb{N}^*} |t_k - t_{k-1}| < \infty\).

The remainder of the paper has the following structure. In section II we study the well-posedness of the model \(1 \rightarrow 5\) and its aggregate version. Section III focuses on the optimal control of anthracnose based only on the pulse strategy. In subsection III-A we look at the optimal control of the aggregate version while in subsection III-B we do the same work for the general model. In subsection III-C we present and discuss some numerical simulations corresponding to results stated in previous subsections. Section IV is devoted to the study of the optimal control of anthracnose using simultaneously pulse and continuous strategies. In subsections IV-A and IV-B we study respectively the optimal control of the aggregate model and the general model. In subsection IV-C we present and discuss some numerical simulations corresponding to results of the two previous subsections. Finally, in section V we discuss the model and some results.

II. WELL-POSEDNESS OF THE MODEL

A. An aggregate version of the model

An aggregate version of the model \(1 \rightarrow 5\) might correspond to the spatially averaged inhibition rate of anthracnose all over the spatial domain of study \(\Omega\). Let \(|\Omega|\) denotes the volume \(\int_\Omega dx\). If we replace \(\alpha, u\) and \(v_i\) respectively by spatial averages \(\frac{1}{|\Omega|} \int_\Omega \alpha (., x) dx\), \(\frac{1}{|\Omega|} \int_\Omega u (., x) dx\) and \(\frac{1}{|\Omega|} \int_\Omega v_i (., x) dx\) but we keep the same notation then the spatial average \(\Theta (t) = \frac{1}{|\Omega|} \int_\Omega \theta (t, x) dx\) satisfies the following impulsive differential equation.

\[d\Theta = (\alpha (1 - \Theta) - \sigma u(t)) \quad \text{on} \quad \mathbb{R}_+^* \setminus \{\tau_i\}_{i \in \mathbb{N}^*} \]

\[\Theta (\tau_i^+) = (1 - v_i) \Theta (\tau_i) \quad i \in \mathbb{N}^* \quad (7)\]

\[\tau_0 = t_0 = 0 \quad \text{and} \quad \tau_i = \inf \{t_k > \tau_{i-1}; k > 0, \Theta (t_k) \geq \sigma^*\} \quad (8)\]

\[\Theta (0) = \Theta_0 \in [0, 1] \quad (9)\]

The aggregate model is simpler to study and exhibit the average behaviour of the general model. On the other hand, tools used for the aggregate model give an idea on the generalizations that are necessary to study the general model.

Let denote by the solution of \((3) \rightarrow (4)\) a piecewise absolutely continuous function \(\Theta\) satisfying in each set \([\tau_k, \tau_{k+1}]\)

\[\Theta (t) = \Theta (\tau_k^+) + \int_{\tau_k}^t \alpha (s) (1 - \Theta (s) / (1 - \sigma u(s))) \, ds \quad (10)\]

We make the following assumptions in order to survey existence and the uniqueness of such a solution.

\[(H1): \alpha \in L^\infty_{\text{loc}}(\mathbb{R}_+^*; \mathbb{R}_+)\]

\[(H2): u \in L^\infty([0, 1])\]

**Proposition 1:** If \(\Theta\) is a solution of \((3) \rightarrow (4)\) then \(\Theta\) is \([0, 1] \rightarrow \mathbb{R}_+^*\)-valued.

**Proof:** Let \(\Theta\) be the solution of \((3) \rightarrow (4)\). Since \(v_i \in [0, 1]\) it suffices to establish that the restriction of \(\Theta\) on \([t_0, t_1]\) is \([0, 1] \rightarrow \mathbb{R}_+^*\)-valued. Let

\[f = \begin{cases} -\Theta, & \Theta < 0 \\ 0, & \Theta \geq 0 \end{cases} \quad \text{and} \quad g = \begin{cases} \Theta - 1, & \Theta > 1 \\ 0, & \Theta \leq 1 \end{cases} \]

We first prove that \(\Theta \geq 0\). Assume that the subset \(U \subset [t_0, t_1]\) is the nonempty interior of the set where \(f = \) is not null. \(f\) is continuous and is null on \(\partial U\) the boundary of \(U\) because the set \([t_0, t_1]\) is convex. For every time \(t \in U\), \(f\) satisfies \(f(\inf U) = 0\) and

\[f(t) = \int_{\inf U}^t (1 - \sigma (s)/(1 - \sigma u(s))) \, ds \leq 0\]

That is a contradiction which means \(U\) is empty. Using again the continuity of \(f\) we get that \(f\) is null on \([t_0, t_1]\). We conclude that \(\Theta\) remains nonnegative.

Now we prove that \(\Theta \leq 1\). Assume that the subset \(V \subset [t_0, t_1]\) is the nonempty interior of the set where \(g = \) is not null.
$g$ is continuous and is null on $\partial V$ because the set $]t_0, t_1]$ is convex. For every time $t \in V$, $g$ satisfies $g(\inf V) = 0$ and

$$
g(t) = \int_{\inf V}^{t} \alpha(s)(1 - (g(s) + 1)/(1 - \sigma u(s))) \, ds
= \int_{\inf V}^{t} \alpha(s)(1 - 1/(1 - \sigma u(s))) \, ds
- \int_{\inf V}^{t} \alpha(s)g(s)/(1 - \sigma u(s)) \, ds
\leq 0
$$

That is a contradiction which means $V$ is empty. Using again the continuity of $g$ we get that $g$ is null on $[t_0, t_1]$. We conclude that $\Theta$ remains less than 1.

**Proposition 2:** The problem (10) has a unique global solution.

**Proof:** It suffices to establish existence of a local solution and use the proposition [1] to conclude the result. It also suffices to restrict ourselves to the set $[t_0^+, t_1]$. The function $(t, x) \in [t_0^+, t_1] \times [0, 1] \rightarrow \alpha(t)(1 - x/(1 - \sigma u(t)))$ is integrable with respect to $t$, lipschitz continuous with respect to $x$ and upper bounded by $\alpha$ which is also integrable with respect to $t$. Then by the Carathéodory theorem (see [3]) there is a local $[0, 1]$ valued solution. Let now prove the uniqueness of solution. Assume that $x$ and $y$ are solutions on $[t_0^+, t_1]$.

$$
|x(t) - y(t)| = |x(t_0^+) - y(t_0^+)|
+ \int_{t_0}^{t} \alpha(s) |y(s) - x(s)|/(1 - \sigma u(s)) \, ds
$$

Using Gronwall lemma we get

$$
|x(t) - y(t)| \leq |x(t_0^+) - y(t_0^+)|
\times \exp \left( \int_{t_0}^{t} \alpha(s)/(1 - \sigma u(s)) \, ds \right)
$$

and more generally if $t \in [t_0^+, t_{k+1}]

$$
|x(t) - y(t)| \leq \left( \prod_{i=0}^{k} |x(t_0) - y(t_0)| \right)
\times \exp \left( \int_{t_0}^{t} \alpha(s)/(1 - \sigma u(s)) \, ds \right)
$$

Thus solution is unique and depends continuously on initial conditions.

It is important to notice that the solution of (10) is continuous with respect to control strategies $u$ and $(v_i)$.

**B. The main model**

Let denote by the solution of (10) a piecewise absolutely continuous $H^2(\Omega; \mathbb{R})$ -valued function $\theta$ satisfying in each set $]\tau_k, \tau_{k+1}]$, $\forall x \in \Omega$

$$
\theta(t, x) = \theta(\tau_k^+, x) + \int_{\tau_k}^{t} \alpha(s,x)(1 - \theta(s,x)/(1 - \sigma u(s,x))) \, ds
+ \int_{\tau_k}^{t} \text{div}(A(s,x)\nabla\theta(s,x)) \, ds
$$

and

$$
\theta(0^-, x) = \rho(x) \in [0, 1], \forall x \in \overline{U} \subseteq \mathbb{R}^3
$$

By the weak solution of (10) we mean a piecewise absolutely continuous $H^1(\Omega; \mathbb{R})$ -valued function $\theta$ satisfying the two last conditions. Additionaly the weak solution shall satisfy in each set $]\tau_k, \tau_{k+1}]$

$$
\int_{\Omega}^{\theta(t, x)} \psi(t, x) \, dx
= \int_{\Omega}^{\theta(\tau_k^+, x)} \psi(\tau_k^+, x) \, dx
- \int_{\Omega \setminus \tau_k} \langle A(s,x) \nabla \theta(s,x), \nabla \psi(s,x) \rangle \, dsdx
+ \int_{\Omega \setminus \tau_k} \alpha(s,x) \psi(s,x)(1 - \theta(s,x)/(1 - \sigma u(s,x))) \, dsdx
$$

where $\psi \in H^1(\Omega; \mathbb{R})$. We make the following assumptions in order to survey existence and the uniqueness of such a solution.

**H3:** $\alpha \in L^\infty_{\text{loc}}(\mathbb{R}^+; L^\infty(\Omega; \mathbb{R}^+))$.

**H4:** $\forall i, j \in \{1, 2, 3\}, a_{ij} \in L^\infty_{\text{loc}}(\mathbb{R}^+; W^{1, \infty}(\Omega; \mathbb{R}))$.

**H5:** $\exists \delta \in \mathbb{R}^+$ such that $\forall t \in \mathbb{R}^+, \forall w \in W^{1, 2}(\Omega; \mathbb{R})$

$$
\int_{\Omega} \langle A(t, x) \nabla w(x), \nabla w(x) \rangle \, dx \geq \delta \int_{\Omega} \langle \nabla w(x), \nabla w(x) \rangle \, dx.
$$

**H6:** $u \in L^\infty(\mathbb{R}^+; L^\infty(\Omega; [0, 1]))$.

**H7:** $\forall i \in \mathbb{N}, v_i \in L^\infty(\Omega; [0, 1])$.

**Proposition 3:** If $\theta$ is a solution of (10) then $\forall t \in \mathbb{R}^+$, $\theta(t, x) \in [0, 1]$ for almost every $x \in \Omega$.

**Proof:** Let $\theta$ be the solution of (10). Since $v_i \in L^\infty(\Omega; [0, 1])$ it suffices to establish the result for the restriction of $\theta$ on $]t_0, t_1]$. Let

$$
f = \begin{cases} 
-\theta, & \theta < 0 \\
0, & \theta \geq 0
\end{cases}
\quad \text{and} \quad
g = \begin{cases} 
\theta - 1, & \theta > 1 \\
0, & \theta \leq 1
\end{cases}
$$

We first prove that $\theta \geq 0$. Assume that a subset $U = U_1 \times U_2 \subset [t_0, t_1] \times \Omega$ is the nonempty interior of a set where $f$ is not null. $f$ is continuous and is null on $\partial U$ the boundary of
V because the set \([t_0, t_1] \times \Omega\) is convex. \(f \left( \inf U_1, . \right) = 0\) and

\[
\frac{1}{2} \partial_t \|f(t, .)\|^2_{L^2(U_2; \mathbb{R})} = \frac{1}{2} \partial_t \int_{U_2} f^2 (t, x) \, dx = \int_{U_2} f (t, x) \partial_t f (t, x) \, dx = -\int_{U_2} \alpha (t, x) f (t, x) \, dx + \int_{U_2} \div \left( A(t, x) \nabla f(t, x) \right) f(t, x) \, dx
\]

\[
\leq 0
\]

Using the Gronwall lemma \(\|f(t, .)\|^2_{L^2(U_2; \mathbb{R})} \leq 0\) and is consequently null. That is a contradiction which means \(U_2\) is empty. We conclude that \(\forall t \in [t_0, t_1], \theta (t, .)\) remains nonnegative almost everywhere in \(\Omega\).

Now we prove that \(\theta \leq 1\). Assume that a subset \(V = V_1 \times V_2 \subset [t_0, t_1] \times \Omega\) is the nonempty interior of a set where \(g\) is not null, \(g\) is continuous and is null on \(\partial V\) the boundary of \(V\) because the set \([t_0, t_1] \times \Omega\) is convex. \(g \left( \inf V_1, . \right) = 0\) and

\[
\frac{1}{2} \partial_t \|g(t, .)\|^2_{L^2(U_2; \mathbb{R})} = \int_{V_2} \alpha (t, x) g (t, x) \left( 1 - 1 / \left( 1 - \sigma u (t, x) \right) \right) \, dx + \int_{V_2} \div \left( A(t, x) \nabla g(t, x) \right) g(t, x) \, dx
\]

\[
\leq 0
\]

Using the Gronwall lemma \(\|g(t, .)\|^2_{L^2(U_2; \mathbb{R})} \leq 0\) and is consequently null. That is a contradiction which means \(U_2\) is empty. We conclude that \(\forall t \in [t_0, t_1], \theta (t, .)\) remains less than 1 almost everywhere in \(\Omega\).

From assumptions (H4)-(H5) the following problem has a unique solution in \(H^1 (\Omega)\) for an arbitrary but fixed time \(t > 0\).

\[
\begin{aligned}
\div \left( A(t, x) \nabla w (x) \right) &= f (x), \forall x \in \Omega \\
\left( A(t, x) \nabla w (x), n (x) \right) &= 0, \forall x \in \partial \Omega
\end{aligned}
\]

where \(f \in L^2 (\Omega)\). Following [2] in theorems 3.6.1 and 3.6.2 we can easily establish that there is an orthonormal complete system \((\varphi_n (t, .))_{n \in \mathbb{N}} \subset L^2 (\Omega)\) of eigenfunctions and eigenvalues \((\lambda_n (t))\) such that \(\forall n \in \mathbb{N},\)

\[
\begin{align*}
\div \left( A(t, x) \nabla \varphi_n (t, x) \right) &= \lambda_n (t) \varphi_n (t, x), \forall x \in \Omega \\
\left( A(t, x) \nabla \varphi_n (t, x), n (x) \right) &= 0, \forall x \in \partial \Omega
\end{align*}
\]

Moreover, \((\varphi_n (t, .))_{n \in \mathbb{N}} \subset H^1 (\Omega)\) and if \(\partial \Omega\) is of class \(C^2\) then \((\varphi_n (t, .))_{n \in \mathbb{N}} \subset H^2 (\Omega)\).

Now we make the following assumption

**(H8):** The sequence \((\varphi_n)\) does not depend on the time (ie \(\varphi_n (t, .) = \varphi_n (., .), \forall t > 0\)).

(H8) could happen if \(A (t, .)\) has the form \(\mu (t) B (.)\) with \(\mu (t) \in \mathbb{R}, \forall t \geq 0\). It will be the case in particular if \(A (t, .) = \mu (t) I\) and therefore \(\div \left( A(t, x) \nabla w \right) = \Delta w\). Here \(I\) denotes the identity matrix of \(\mathbb{R}^2\). Whether (H8) is satisfied a weak solution \(\theta\) of (6) - (9) can be written as the sum \(\sum_{n \geq 0} \theta_n \varphi_n\) where each \(\theta_n\) is an absolutely continuous function of the time and satisfies in each set \([\tau_k, \tau_{k+1}]\),

\[
\begin{align*}
\theta_n (t) &\int_{\Omega} \varphi_n^2 (x) \, dx \\
&= \theta_n (\tau_k^{+}) \int_{\Omega} \varphi_n^2 (x) \, dx \\
&- \int_{\tau_k}^{t} \theta_n (s) \int_{\Omega} \alpha (s, x) \varphi_n^2 (x) / (1 - \sigma u (s, x)) \, dx dx \\
&+ \int_{\tau_k}^{t} \lambda_n (s) \theta_n (s) \int_{\Omega} \varphi_n^2 (x) \, dx dx + \int_{\tau_k}^{t} \int_{\Omega} \alpha (s, x) \varphi_n (x) \, dx dx
\end{align*}
\]

**Theorem 4:** The problem [8] - [11] has a unique global weak solution \(\theta\). Moreover, if \(\Omega\) is of class \(C^2\) then \(\forall t > 0, \theta (t, .) \in H^2 (\Omega)\).

**Proof:** It suffices to establish existence of a local solution and use the proposition[3] to conclude the result. It also suffices to restrict ourselves to the set \([t_0^{+}, t_1]\).

The function

\[
(t, w) \in \left[ t_0^{+}, t_1 \right] \times [0, 1] \rightarrow \lambda_n (t) w + \int_{\Omega} \alpha (s, x) \varphi_n (x) \, dx dx / \left( \int_{\Omega} \varphi_n^2 (x) \, dx \right)
\]

is integrable with respect to \(t\), lipschitz continuous with respect to \(w\). Using proposition[3] and the Carathéodory theorem the problem [11] - [15] has a unique global weak solution \(\theta \in L^2 (\Omega; [0, 1])\). That weak solution depends continuously on initial conditions.

It is important to notice that the solution of [8] - [11] is continuous with respect to control strategies \(u\) and \((v_i)\).
III. Optimal Control Based Only on the Pulse Strategy

In this section we restrict the time of study to the set \([0, T]\) where \(T \in [\tau_k, \tau_{k+1}]\) corresponds to the annual production period with respect to a time unit.

A. The aggregate model

The aim of this subsection is to prove the existence and characterize of strategy \(v^* = (v^*_i)_{i \in \mathbb{N}}\) which minimizes the following cost functional.

\[
J(v) = \int_0^T \Theta(s) \, ds + \sum_{i=0}^k c_i v_i \Theta(\tau_i) + \Theta(T)
\]

where \(c = (c_i)_{i \in \mathbb{N}} \subseteq \mathbb{R}_+^k\) is a sequence of cost ratios related to the use of control. Looking at \(J\) we are just interested by the \(k + 1\) first terms of \(v\) and \(c\).

**Proposition 5:** There is an optimal strategy \(v^* = (v^*_i)_{i \in \mathbb{N}}\) which minimizes \(J\).

**Proof:** The problem can be reduced to find \(v^* = (v^*_i)_{0 \leq i \leq k} \in [0,1]^{k+1}\) with other terms null since they are useless in \(J\). Note that \(0 \leq J \leq 1 + T + \sum_{i=0}^k c_i\). Let \(J^* = \min_{v \in [0,1]^{k+1}} J(v)\). There is a sequence \((v^n)_{n \in \mathbb{N}}\) such that the sequence \((J(v^n))_{n \in \mathbb{N}}\) converges to \(J^*\). Since \([0,1]^{k+1}\) is compact and \(J\) is continuous there is a subsequence \((v^{n_m})\) which converges to \(v^* = (v^*_i)_{0 \leq i \leq k} \in [0,1]^{k+1}\) and \(J(v^*) = J^*\).

In the remaining of the subsection we try to characterize the optimal control strategy in order to compute it. We have mentioned above that the solution of \((6) - (9)\) is continuous with respect to control strategies \(u\) and \((v_i)\). Now we do the additional assumption that the solution of \((6) - (9)\) is gâteaux differentiable with respect to \(v = (v_i)_{0 \leq i \leq k} \in [0,1]^{k+1}\). Let \(\Theta_u\) be the solution of \((6) - (9)\) associated to a chosen control strategy \(u\) \(\Phi = D_u \Theta_u\) and \(J_v = D_v J (v) = \Gamma^v u\). Let \(\Phi\) be the semiflow corresponding to \((6) - (9)\). \(\Phi\) satisfies \(\forall t \in [\tau_i, \tau_{i+1}]\),

\[
\Phi(t, \tau_i, x) = x \exp \left( \int_{\tau_i}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right) + \int_{\tau_i}^t \alpha(s) \exp \left( \int_s^t \frac{\alpha(\tau)}{1 - \sigma u(\tau)} \, d\tau \right) \, ds
\]

and \(\partial_t \Phi(t, \tau_i, x) = \exp \left( -\int_{\tau_i}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)\).

On the other hand

\[
\Theta_u(t) = \Phi(t, \tau_i, \Theta_u(\tau_i^+)) = \Phi(t, \tau_i, v_i \Phi(\tau_i, \tau_i-1, \sigma u_{\theta}(\tau_i-1)))
\]

We have \(\forall t \in [\tau_i, \tau_{i+1}]\),

\[
z_v(t) = \omega^i v_0 \Theta_u(0) \exp \left( -\int_0^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)
\]

and \(\forall t \in [\tau_i, \tau_{i+1}] \), \(i \in \mathbb{N}^*\)

\[
z_v(t) = (\omega^i v_i \Theta_u(\tau_i) + \underline{\omega}^i z_v(\tau_i))
\]

\[
\times \exp \left( -\int_{\tau_i}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)
\]

\[
= \left( \prod_{j=0}^i \omega^j \right) z_v(\tau_0)
\]

\[
\times \exp \left( -\int_0^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)
\]

\[
+ \sum_{j=0}^{i-1} \left( \prod_{\tau_j}^i \omega^j \right) v_j \Theta_u(\tau_j) \times \exp \left( -\int_{\tau_j}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)
\]

\[
= \sum_{j=0}^{i-1} \left( \prod_{\tau_j}^i \omega^j \right) v_j \Theta_u(\tau_j) \times \exp \left( -\int_{\tau_j}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)
\]

and thus \(z_v\) satisfies

\[
dz_v/dt = -\alpha(t) z_v(t) / (1 - \sigma u(t)), \quad \text{on} \quad \mathbb{R}_+^k \setminus (\tau_i)_{i \in \mathbb{N}}
\]

\[
z_v(0) = 0 \in [0,1]
\]

\[
\forall i \in \mathbb{N}^*, \quad z_v(\tau^+_i)
\]

\[
= \sum_{j=0}^{i-1} \left( \prod_{\tau_j}^i \omega^j \right) v_j \Theta_u(\tau_j) \times \exp \left( -\int_{\tau_j}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right)
\]

\[
\forall t \in [0, T], \quad \Psi_{\Lambda}(t) \in \mathbb{R}^{k+1} \text{ is such that} \quad \Psi_{\Lambda}(0) = 0 \quad \text{and} \quad \forall t \in [\tau_i, \tau_{i+1}]\),

\[
\Psi_{\Lambda}(t) = \begin{cases} \prod_{\tau_j}^i \omega^j \Theta_u(\tau_j) \exp \left( -\int_{\tau_j}^t \frac{\alpha(s)}{1 - \sigma u(s)} \, ds \right), & \text{if } 0 \leq j \leq i \\ 0, & \text{if } j > i \end{cases}
\]

**Proposition 6:** If \(v = v^*\) is an optimal strategy then \(\forall v \in V\),

\[
\sum_{i=0}^k (c_i - p_{v^*}(\tau^+_i)) v_i \Theta_u(\tau_i) = J_v \geq 0
\]

where

\[
V = \left\{ v \in \mathbb{R}^{k+1}; \exists \varepsilon > 0; v^* + \varepsilon v \in [0,1]^{k+1} \right\}
\]

and \(p_{v^*}\) is solution of the following adjoint problem

\[
dp_{v^*}/dt = \alpha(t) p_{v^*} / (1 - \sigma u(t)) - 1, \quad \text{if } t \in [0, T] \setminus \{\tau_i\}, \quad i \in [0, k] \cap \mathbb{N}^* \\
p_{v^*}(T) = 1, \quad p_{v^*}(\tau_i) = c_i v^*_i + p_{v^*}(\tau_i^+) (1 - v^*_i)\]
Using integration by part we get
\[ p_{v^*}(t) = (c_i v^*_{i+1} + p_{v^*}(\tau_{i+1}^+)) (1 - v^*_{i+1}) \]
\[ \times \exp\left( -\int_t^{\tau_{i+1}^+} \alpha(s) / (1 - \sigma u(s)) \, ds \right) \]
\[ + \int_t^{\tau_{i+1}^+} \exp\left( -\int_s^\infty \alpha(\tau) / (1 - \sigma u(\tau)) \, d\tau \right) \, ds \]
\[
\text{and } \forall t \in [\tau_k, T] \]
\[ p_{v^*}(t) = \exp\left( -\int_t^T \alpha(s) / (1 - \sigma u(s)) \, ds \right) \]
\[ + \int_t^T \exp\left( -\int_s^\infty \alpha(\tau) / (1 - \sigma u(\tau)) \, d\tau \right) \, ds \]
For an arbitrary but fixed \( v \in V \) and \( \varepsilon > 0 \) sufficiently small we have \( J(v^* + \varepsilon v) \geq J(v^*) \) and consequently
\[ 0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^T (\Theta_{v^* + \varepsilon v}(s) - \Theta_{v^*}(s)) \, ds \]
\[ + \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \sum_{i=0}^k c_i ((v^*_i + \varepsilon v_i) \Theta_{v^* + \varepsilon v}(\tau_i) - v^*_i \Theta_{v^*}(\tau_i)) \]
\[ + \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( (\Theta_{v^* + \varepsilon v}(T) - \Theta_{v^*}(T)) \right) \]
\[ = z_v(T) + \int_0^T z_v(s) \, ds + \sum_{i=0}^k c_i (v^*_i z_v(\tau_i) + v_i \Theta_{v^*}(\tau_i)) \]
\[ = J_v \]
Using integration by part we get
\[ \int_0^T p_{v^*}(s) \, dz_v(s) \]
\[ = \int_0^T p_{v^*}(s) \, dz_v(s) + \sum_{i=0}^{k-1} \int_{\tau_k}^{\tau_{i+1}^+} p_{v^*}(s) \, dz_v(s) \]
\[ = p_{v^*}(T) z_v(T) - p_{v^*}(\tau_{i+1}^+) z_v(\tau_{k+1}^+) \]
\[ + \sum_{i=0}^{k-1} p_{v^*}(\tau_{i+1}^+) z_v(\tau_{i+1}) - p_{v^*}(\tau_i^+) z_v(\tau_i) \]
\[ - \sum_{i=0}^{k-1} \int_{\tau_i}^{\tau_{i+1}^+} z_v(s) \, dp_{v^*}(s) - \int_0^T z_v(s) \, dp_{v^*}(s) \]
\[ = p_{v^*}(T) z_v(T) - p_{v^*}(\tau_{i+1}^+) z_v(\tau_{k+1}^+) \]
\[ + \sum_{i=0}^{k-1} p_{v^*}(\tau_{i+1}^+) z_v(\tau_{i+1}) - p_{v^*}(\tau_i^+) z_v(\tau_i) \]
\[ - \int_0^T (\alpha(s) p_{v^*}(s) / (1 - \sigma u(s)) - 1) \, z_v(s) \, ds \]
and
\[ \int_0^T p_{v^*}(s) \, dz_v(s) \]
\[ = -\int_0^T (\alpha(s) / (1 - \sigma u(s))) p_{v^*}(s) \, z_v(s) \, ds \]
Thus
\[ p_{v^*}(T) z_v(T) - p_{v^*}(\tau_{k+1}^+) z_v(\tau_{k+1}^+) + \int_0^T z_v(s) \, ds \]
\[ + \sum_{i=0}^{k-1} p_{v^*}(\tau_{i+1}^+) z_v(\tau_{i+1}) - p_{v^*}(\tau_i^+) z_v(\tau_i) \]
\[ = 0 \]
and
\[ J_v = z_v(T) + \int_0^T z_v(s) \, ds + \sum_{i=0}^k c_i (v^*_i z_v(\tau_i) + v_i \Theta_{v^*}(\tau_i)) \]
\[ = (1 - p_{v^*}(T)) z_v(T) + \sum_{i=0}^k c_i (v^*_i - p_{v^*}(\tau_i)) z_v(\tau_i) \]
\[ + \sum_{i=0}^k c_i v_i \Theta_{v^*}(\tau_i) \]
\[ = \sum_{i=0}^k (c_i v_i^* - p_{v^*}(\tau_i)) + p_{v^*}(\tau_i^+) \]
\[ + \int_0^T \left( c_i (1 - v_i^*) \right) v_i \Theta_{v^*}(\tau_i) \]
\[ = \sum_{i=0}^k (c_i - p_{v^*}(\tau_i^+)) v_i \Theta_{v^*}(\tau_i) \]

The result above is a version of the maximum principle but it does not give an efficient way to compute the optimal strategy. We propose to use the gradient method to estimate an optimal strategy. That method is described in \([\text{1}]\). From the proof above \( \Gamma_{\overline{J}} \in \mathbb{R}^{k+1} \) is such that \( \forall j \in [0,k] \cap \mathbb{N}_0 \),
\[ \Gamma_{\overline{J}}^j = (c_j - p_{v^*}(\tau_j^+)) \Theta_{v^*}(\tau_j) \]

**B. The main model**

In this subsection we prove the existence and characterize a strategy \( v^* = (v^*_i)_{i \in \mathbb{N}} \) which minimizes the following cost functional.
\[ J(v) = \int_0^T \int_\Omega \Theta(s, x) \, dx \, ds + \int_\Omega \Theta(T, x) \, dx \]
\[ + \sum_{i=0}^k \int_\Omega c_i (v_i(x) \Theta(\tau_i, x) \, dx) \]

where \( \forall i \in \mathbb{N}_0, v_i \in L^{\infty}(\Omega; [0,1]) \), \( c_i \in L^{\infty}(\Omega; \mathbb{R}_+) \), \( \forall x \in \Omega, c(x) = (c_i(x))_{i \in \mathbb{N}_0} \subset \mathbb{R}^k_+ \) is a sequence of cost ratios related to the use of control. Looking at \( J \) we are just interested by the first \( k+1 \) terms of \( v \) and \( c \).

**Theorem 7:** There is an optimal strategy \( v^* = (v^*_i)_{i \in \mathbb{N}} \) which minimizes \( J \).

Before giving a proof to the proposition \([\text{7}]\) we recall the following lemma stated in \([\text{1}], [\text{6}]\).

**Lemma 8:** (Mazur)

Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence in a real Banach space \( X \) that is weakly convergent to \( x \in X \). Then there exists a sequence \( (y_n)_{n \in \mathbb{N}} \subset X \) which converges strongly to \( x \) and such that \( \forall n \in \mathbb{N}, y_n \) is an element of the convex closure of \( (x_n)_{n \in \mathbb{N}} \).

**Proof:** (of theorem \([\text{7}]\)) The problem can be reduced to find \( v^* = (v^*_i)_{0 \leq i \leq k} \in [0,1]^{k+1} \) with other terms null since they are useless in \( J \). Note that \( 0 \leq J \leq \mathbb{R} \left( 1 + T + \sum_{i=0}^k \|c_i\|_{L^{\infty}(\Omega; \mathbb{R})} \right) \). Let \( J^* = \min_v J(v) \).

There is a sequence \( (v^n)_{n \in \mathbb{N}} \subset (L^{\infty}(\Omega; [0,1]))^{k+1} \) such that
the sequence \((J(u^n, v^n))_{n \in \mathbb{N}}\) converges to \(J^*\). The sequence 
\((v^n)_{n \in \mathbb{N}}\) is bounded and there is a subsequence \((v^{n_m})\) which
converges weakly to a strategy \(v^* \in (L^\infty(\Omega; [0, 1]))^{k+1}\).
Using the lemma of Mazur there is a sequence \((\varphi^m)_{n \in \mathbb{N}}\)
in\(\text{conv}((v^n))_{n \in \mathbb{N}}\) \(\subset (L^\infty(\Omega; [0, 1]))^{k+1}\) which
converges strongly to \(v^*\). Since \(J\) is continuous \(J(v^*) = J^*\).

In the remaining of the subsection we characterize the optimal control strategy in order to compute it. The solution of (11) – (13) is continuous with respect to control strategies \((v_i)\).
Now we do the additional assumption that the solution of (11) – (13) is gâteaux differentiable with respect to \(v = (v_i)_{0 \leq i \leq k} \in (L^\infty(\Omega; [0, 1]))^{k+1}\). Let \(\theta_{k}\) be the solution of (11) – (13) associated to a chosen control strategy \(\varphi\),
\(z_v = D_v \varphi, v = \Psi^v\) and \(J_v = D_v J(v)\) \(v = \Gamma^v\). \(z_v\) satisfies
\[
\partial_t z_v = -\alpha(t, x) z_v(t, x) / (1 - \sigma v(t, x)) \\
+ \operatorname{div}(A(t, x) \nabla z_v(t, x))
\]
on \(\mathbb{R}^*_+ \setminus \{\tau_i\} \times \Omega\)
\(z_v(0, x) = 0, x \in \Omega\)
\(\forall i \in \mathbb{N}^*, \)
\(z_v(\tau_i^+, x) = (1 - v_i(x)) z_v(\tau_i^-, x) - v_i(x) \theta_{k}(\tau_i^-, x)\)
\(, x \in \Omega\)
(16)
(17)
(18)

Theorem 9: If \(v = v^*\) is an optimal strategy then \(\forall v \in V\),
\[
\sum_{i=0}^{k} \int_\Omega (c_i(x) - p_{v^*}(\tau_i^+, x)) v_i(x) \theta_{v^*}(\tau_i, x) dx = J_v \geq 0
\]
where \(V = \{v \in (L^\infty(\Omega; \mathbb{R}))^{k+1}; \exists \varepsilon > 0; v^* + \varepsilon v \in (L^\infty(\Omega; [0, 1]))^{k+1}\}\)
and \(p_{v^*}\) is solution of the following adjoint problem
\[
dp_{v^*}/dt = \alpha(t, x) p_{v^*} / (1 - \sigma u(t, x)) \\
- \operatorname{div}(A(t, x) \nabla p_{v^*}(t, x)),
\]
(t, x) \(\in \{0, T\} \setminus \{\tau_i\} \times \Omega\), \(i \in [0, k] \cap \mathbb{N}^*\)
\(\forall x \in \Omega, p_{v^*}(T, x) = 1, \)
\(p_{v^*}(\tau_i, x) = (1 - v_i(x)) p_{v^*}(\tau_i^+, x) + c_i(x) v_i^*(x)\)
\(\langle A(t, x) \nabla p_{v^*}(t, x), n(x) \rangle = 0, \text{ on } \mathbb{R}^*_+ \times \partial \Omega\)
(19)
(20)
(21)

Proof: Using the similar reasoning than the proof of proposition \(\text{[4]}\) there is a unique absolutely solution \(p_{v^*}\) to the problem (19) – (21). For an arbitrary but fixed \(v \in V\) and \(\varepsilon > 0\) sufficiently small we have \(J(v^* + \varepsilon v) \geq J(v^*)\) and consequently
\[
0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^T \int_\Omega (\Theta_{v^*+\varepsilon v}(s, x) - \Theta_{v^*}(s, x)) dx ds
+ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \sum_{i=0}^{k} \int_\Omega c_i(x) ((v_i^* + \varepsilon v_i(x)) - v_i(x)) \times \Theta_{v^*+\varepsilon v}(\tau_i, x) - v_i^*(x) \Theta_{v^*}(\tau_i, x) dx
+ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_\Omega ((\Theta_{v^*+\varepsilon v}(T, x) - \Theta_{v^*}(T, x)) x dx
= \int_\Omega z_v(T, x) dx + \int_0^T \int_\Omega z_v(s, x) dx ds
+ \sum_{i=0}^{k} \int c_i(x) (v_i^*(x) z_v(\tau_i, x)) + v_i(x) \Theta_{v^*}(\tau_i, x) dx
= J_v
\]

Using integration by part we get
\[
\int_0^T p_{v^*}(s, x) z_v(s, x) dt
= p_{v^*}(T, x) z_v(T, x) - \int_0^T z_v(s, x) \partial_t p_{v^*}(s, x) dt
+ \sum_{i=0}^{k-1} p_{v^*}(\tau_{i+1}, x) z_v(\tau_{i+1}, x)
- \sum_{i=0}^{k} p_{v^*}(\tau_i, x) z_v(\tau_i, x)
= p_{v^*}(T, x) z_v(T, x) + \int_0^T z_v(s, x) ds
+ \sum_{i=0}^{k-1} p_{v^*}(\tau_{i+1}, x) z_v(\tau_{i+1}, x)
- \sum_{i=0}^{k} p_{v^*}(\tau_i^+, x) z_v(\tau_i^+, x)
- \int_0^T \alpha(x) p_{v^*}(s, x) z_v(s, x) / (1 - \sigma u(s, x)) ds
\]

and
\[
\int_0^T p_{v^*}(s, x) dz_v(s, x)
= - \int_0^T \alpha(s) p_{v^*}(s, x) z_v(s, x) / (1 - \sigma u(s, x)) ds
+ \int_0^T p_{v^*}(s, x) \partial_t p_{v^*}(s, x) dt
\]

Thus
\[
\int_\Omega p_{v^*}(T, x) z_v(T, x) dx + \int_0^T \int_\Omega z_v(s, x) dx ds
+ \sum_{i=0}^{k-1} \int_\Omega p_{v^*}(\tau_{i+1}, x) z_v(\tau_{i+1}, x) dx
- \sum_{i=0}^{k} \int_\Omega p_{v^*}(\tau_i^+, x) z_v(\tau_i^+, x) dx
= 0
\]
and

$$J_v = \int_{\Omega} z_v (T, x) \, dx + \int_{0}^{T} \int_{\Omega} z_v (s, x) \, dx \, ds$$

\[+ \sum_{i=0}^{k} \int_{\Omega} c_i (x) \left( v_i^+ (x) \right) z_v (\tau_i, x) + v_i (x) \theta_v (\tau_i, x) \, dx\]

\[= \int_{\Omega} \left( 1 - \sum_{i=0}^{k} \int_{\Omega} \left( 1 - c_i (x) \right) v_i (x) \theta_v (\tau_i, x) \, dx \right) z_v (T, x) \, dx\]

\[- \sum_{i=0}^{k} \int_{\Omega} p_v (\tau_i, x) z_v (\tau_i, x) \, dx\]

\[+ \sum_{i=0}^{k} \sum_{i=0}^{\infty} \int_{\Omega} \left( 1 - c_i (x) \right) v_i (x) \theta_v (\tau_i, x) \, dx\]

\[= \sum_{i=0}^{\infty} \int_{\Omega} \left( c_i (x) - p_v (\tau_i, x) \right) v_i (x) \theta_v (\tau_i, x) \, dx\]

For those simulations we used following [13] an inhibition pressure of the form

$$\alpha (t) = a \left( t - b \right)^2 \left( 1 - \cos \left( 2\pi t / c \right) \right),$$

with $b$ and $c$ in $[0, 1]$. This function reflects the seasonality of empirically-based severity index models found in the literature [9, 10, 12]. The particular values used in the simulation were $a = 1$, $b = 0.75$, $c = 0.2$ and $c_i = 1$, $\forall i \in \mathbb{N}$. Indeed, we assume that each pulse intervention has the same cost. That choice could be in general not realistic taking into account special features related to season and other conditions depending on time. However, it is simpler and no reason have been found to choose another special cost since theoretical results remain independently true and we are just illustrating them. We have distinguished two cases of initial conditions. The first one is under the parameter $\sigma^* = 0.2$ and the second is above it.

Let consider in $(L^\infty (\Omega; \mathbb{R}))^{k+1} \supset (L^\infty (\Omega; [0, 1]))^{k+1}$ the inner product

$$\langle v_1, v_2 \rangle = \sum_{i=0}^{k} \int_{\Omega} v_i^1 (x) v_i^2 (x) \, ds$$

with its associated norm

$$\|v\|_2 = \sum_{i=0}^{k} \left( \int_{\Omega} \left( v_i^1 (x) \right)^2 \, ds \right)^{1/2}.$$ 

Note that $\forall v \in (L^\infty (\Omega; \mathbb{R}))^{k+1}$,

$$\|v\|_2 \leq \sqrt{|\Omega|} \times \sum_{i=0}^{k} \sup_{x \in \Omega} \sup_{s \in [0, T]} \left| v_i^j (x) \right|.$$ 

That allows us to find $v^*$ using the topology given by $\|\cdot\|_2$. Since $\Gamma_\theta$ is a linear continuous operator there is a unique $\tau \in L^\infty ([0, T]; \mathbb{R})$ such that

$$\langle \tau, v \rangle = \Gamma_\theta v$$

From the proof above, $\forall j \in [0, k] \cap \mathbb{N}$, $\forall x \in \Omega$,

$$\tau_j (x) = \left( c_j (x) - p_v (\tau_j^+, x) \right) \theta_v (\tau_j, x)$$

$$= \Gamma_\theta (x)$$

After we identify $\Gamma_\theta$ to $\tau$. That identification is useful to implement the gradient method.

C. Computer simulations of the optimal pulse strategy

In this subsection we performed simulations in order to illustrate theoretical results stated above. We only present the aggregate model which is more easier to display graphically and presents the average behaviour of the general model.
**Proposition 10:** There is an optimal strategy \((u^*, v^*)\) which minimizes \(J\).

**Proof:** The problem can be reduced to find \(v^* = (v^*_i)_{0 \leq i \leq k} \in [0,1]^{k+1}\) with other terms null since they are useless in \(J\). Note that 0 ≤ \(J\) ≤ 1 + \(\sup_{s \in [0,T]} C(s)\) \(T + \sum_{i=0}^k c_i\).

Let \(J^* = \min_{(u,v) \in L^\infty([0,T];[0,1]) \times [0,1]^{k+1}} J(u,v)\). There is a sequence \((u^n,v^n)_{n \in \mathbb{N}} \subset L^\infty([0,T];[0,1]) \times [0,1]^{k+1}\) such that the sequence \((J(u^n,v^n))_{n \in \mathbb{N}}\) converges to \(J^*\). The sequence \((u^n,v^n)_{n \in \mathbb{N}}\) is bounded and there is a subsequence \((u^{n_m},v^{n_m})\) which converges weakly to a strategy \((u^*,v^*)\) \(\in L^\infty([0,T];[0,1]) \times [0,1]^{k+1}\). Using the lemma of Mazur there is a sequence \((u^{n_m},v^{n_m})_{n \in \mathbb{N}}\) in \(conv((u^n,v^n)_{n \in \mathbb{N}}) \subset L^\infty([0,T];[0,1]) \times [0,1]^{k+1}\) which converges strongly to \((u^*,v^*)\). Since \(J\) is continuous \(J(u^*,v^*) = J^*\). \(\blacksquare\)

Let \(\hat{\Theta}_{\hat{w},\hat{u}}\) be the solution of (11) – (14) associated to a chosen control strategy \((u,v), z_u = D_u\hat{\Theta}_{\hat{w},\hat{u}} u = \Psi_{\hat{w},\hat{u}} u\) and \(J_u = D_u J(u,v) u = \Gamma_{\hat{w},\hat{u}} u\). Let also \(\Phi\) be the semiflow corresponding to (11) – (14). \(\forall t \in [\tau_i, \tau_{i+1}],\)

\[
\Psi_{\hat{w},\hat{u}} u (t) = z_u (t) = \hat{w} \left( z_u (\tau_i) - \sigma \hat{\Theta}_{\hat{w},\hat{u}} (\tau_i) \int_{\tau_i}^t \alpha(s) u(s) / (1 - \sigma u(s))^2 ds \right) \times \exp \left( - \int_{\tau_i}^t \alpha(s) / (1 - \sigma u(s)) ds \right) \times \int_{\tau_i}^t \alpha(s) \exp \left( - \int_{\tau_i}^s \alpha(\tau) / (1 - \sigma u(\tau)) d\tau \right) \times \int_s^t \alpha(\tau) u(\tau) / (1 - \sigma u(\tau))^2 d\tau ds
\]

In this section we restrict the time of study to the set \([0,T]\) where \(T \in [\tau_k, \tau_{k+1}[,\) corresponds to the annual production period with respect to a time unit.

**IV. Optimal control based both on the continuous and the pulse strategies**

Throughout this subsection we study existence and characterize of a strategy \((u^*, v^*) = (u^*_i, v^*_i)_{i \in \mathbb{N}}\) which minimizes the following cost functional.

\[
J(u,v) = \int_0^T \Theta(s) + C(s) u(s) ds + \sum_{i=0}^k c_i v_i \Theta(\tau_i) + \Theta(T)
\]

where \(C \in L^\infty_{loc}(\mathbb{R}_+;\mathbb{R}_+)\) is almost everywhere positive and \(c = (c_i)_{i \in \mathbb{N}} \subset \mathbb{R}_+.\) \(C\) and \(c\) are time dependent cost ratios related to the use of the control strategy. Looking at \(J\) we are just interested by the \(k + 1\) first terms of \(v\) and \(c\). We can also consider the restriction of \(u\) on \([0,T]\) which belongs to the set \(L^\infty([0,T];[0,1]).\)

Figures above show how efficient is the pulse control strategy. Impulsions are much probable and intensive when the inhibition pressure increase.
\[- v \theta_{\underline{u}, \overline{u}} (\tau_i) \int_{\tau_i}^{t} (\alpha (s) u (s) / (1 - \sigma_u (s))^2 \, ds \\
\times \exp \left( - \int_{\tau_i}^{t} \frac{\alpha (s)}{1 - \sigma_u (s)} \, ds \right) \\
- \int_{\tau_i}^{t} \sigma \alpha (s) \left( \int_{s}^{t} \alpha (\tau) u (\tau) / (1 - \sigma_u (\tau))^2 \, d\tau \right) \\
\times \exp \left( - \int_{s}^{t} \frac{\alpha (\tau)}{1 - \sigma_u (\tau)} \, d\tau \right) \, ds \right) \\
+ z_u (\tau_0) \left( \Pi_{j=0}^{i-1} v_j \right) \exp \left( - \int_{\tau_0}^{t} \alpha (s) / (1 - \sigma_u (s)) \, ds \right) \]

\[= - \sum_{j=0}^{i-1} \left( \Pi_{j=0}^{i-1} v_j \right) \int_{\tau_j}^{\tau_{j+1}} \alpha (s) u (s) / (1 - \sigma_u (s))^2 \, ds \]
\[\times \sigma \theta_{\underline{u}, \overline{u}} (\tau_j) \exp \left( - \int_{\tau_j}^{t} \alpha (s) / (1 - \sigma_u (s)) \, ds \right) \]
\[- \sum_{j=0}^{i-1} \left( \Pi_{j=0}^{i-1} v_j \right) \int_{\tau_j}^{\tau_{j+1}} \int_{s}^{\tau_{j+1}} \alpha (\tau) u (\tau) / (1 - \sigma_u (\tau))^2 \, d\tau \]
\[\times \sigma \alpha (s) \exp \left( - \int_{s}^{\tau_{j+1}} \frac{\alpha (\tau)}{1 - \sigma_u (\tau)} \, d\tau \right) \, ds \]
\[+ \int_{\tau_i}^{t} \alpha (\tau) / (1 - \sigma_u (\tau))^2 \, d\tau \]
\[\times \sigma \alpha (s) \exp \left( - \int_{\tau_i}^{t} \frac{\alpha (s)}{1 - \sigma_u (s)} \, ds \right) \]
\[\times \psi \vartheta_{\underline{u}, \overline{u}} (\tau_i) \exp \left( - \int_{\tau_i}^{t} \alpha (s) / (1 - \sigma_u (s)) \, ds \right) \]

Thus \(z_u\) satisfies

\[dz_u / dt = - \alpha (t) z_u (t) / (1 - \sigma_u (t)) \]
\[- \sigma \alpha (t) u (t) \theta_{\underline{u}, \overline{u}} (t) / (1 - \sigma_u (t)), \text{ on } \mathbb{R}^*_+ \]
\[z_u (0) = 0 \in [0, 1] \]
\[z_u (\tau_i^+) = (1 - \underline{u}) z_u (\tau_i^-) \]

**Proposition 11:** If \((\underline{u}, \overline{u}) = (u^*, v^*)\) is an optimal strategy then \( \forall (u, v) \in V, \)
\[0 \leq J_u + J_v \]
\[= \int_{0}^{T} C (s) u (s) \, ds + \sum_{i=0}^{k} (c_i - p_{u^*} (\tau_i^+)) v_i \theta_{v^*} (\tau_i) \]
\[- \int_{0}^{T} \sigma u (s) \alpha (s) p_{u^*} (s) \theta_{u^* \cdot v^*} (s) / (1 - \sigma_u^* (s))^2 \, ds \]

where
\[V = \{(u, v) \in L^\infty ([0, T]; \mathbb{R}) \times \mathbb{R}^k; \exists \varepsilon > 0; \]
\[(u^* + \varepsilon u, v^* + \varepsilon v) \in L^\infty ([0, T]; [0, 1]) \times [0, 1]^{k+1} \}

and \(p_{u^*} = p_{v^*}\) the solution to the problem \([13] - [15]\).

**Proof:** For an arbitrary but fixed \((u, v) \in V \) and \( \varepsilon > 0 \) sufficiently small we have \(J (u^* + \varepsilon u, v^* + \varepsilon v) \geq J (u^*, v^*) \) and consequently

\[0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{0}^{T} (\theta_{u^* + \varepsilon u, v^* + \varepsilon v} (s) - \theta_{u^* \cdot v^*} (s)) \, ds \]
\[+ \lim_{\varepsilon \to 0^+} \sum_{i=0}^{k} c_i ((v_i^* + \varepsilon v_i) \theta_{u^* + \varepsilon u, v^* + \varepsilon v} (\tau_i) - v_i \theta_{u^* \cdot v^*} (\tau_i)) \]
\[+ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} ((\theta_{u^* + \varepsilon u, v^* + \varepsilon v} (T) - \theta_{u^* \cdot v^*} (T)) \]
\[= z_u (T) + z_v (T) + \int_{0}^{T} (z_u (s) + z_v (s)) \, ds \]
\[+ \sum_{i=0}^{k} c_i (v_i^* (z_v (\tau_i) + z_u (\tau_i)) + v_i \theta_{v^*} (\tau_i)) \]
\[= J_u + J_v \]

Using integration by part we get
\[\int_{0}^{T} p_{u^*} dz_u (s) \]
\[= p_{u^*} (T) z_u (T) - \int_{0}^{T} z_u dp_{u^*} (s) \]
\[- \sum_{i=0}^{k} (p_{u^*} (\tau_i^+) z_u (\tau_i^+) \]
\[- \sum_{i=0}^{k} p_{u^*} (\tau_i^-) z_u (\tau_i^-) \]
\[= p_{u^*} (T) z_u (T) + \int_{0}^{T} z_u (s) \, ds \]
\[- \int_{0}^{T} \alpha (s) p_{u^*} (s) z_u (s) / (1 - \sigma_u^* (s)) \, ds \]
\[+ \sum_{i=1}^{k} (p_{u^*} (\tau_i^+) \alpha (s) (p_{u^*} (s \theta_{u^* \cdot v^*} (s) / (1 - \sigma_u^* (s))^2) \, ds \]

and
\[\Gamma_u = J_u \]
\[= \int_{0}^{T} z_u (s) + C (s) u (s) \, ds + \sum_{i=0}^{k} c_i z_u (\tau_i) + z_u (T) \]
\[= \int_{0}^{T} \Psi_u u (s) + C (s) u (s) \, ds + \sum_{i=1}^{k} c_i (\Psi_u u (\tau_i) + \Psi_{\underline{u}, \overline{u}} (T) \]
\[+ \Psi_{\underline{u}, \overline{u}} (T) \]

and
\[\int_{0}^{T} p_{u^*} dz_u (s) \]
\[= - \int_{0}^{T} \alpha (s) p_{u^*} (s) z_u (s) / (1 - \sigma_u^* (s)) \, ds \]
\[- \int_{0}^{T} \sigma u (s) (s) p_{u^*} (s) \theta_{u^* \cdot v^*} (s) / (1 - \sigma_u^* (s))^2 \, ds \]
Thus
\[ p_{u^*} (T) z_u (T) + \int_0^T z_u (s) \, ds + \sum_{i=1}^k (p_{u^*} (\tau_i) - (1 - u^*_i) p_{u^*} (\tau_i^+)) z_u (\tau_i) \]
\[ + \int_0^T \sigma u (s) \alpha (s) p_{u^*} (s) \Theta_{u^*} (s) / (1 - \sigma u^* (s))^2 \, ds = 0 \]

\[ J_u = \int_0^T z_u (s) + C (s) u (s) \, ds + \sum_{i=1}^k c_i u^*_i z_u (\tau_i) + z_u (T) \]
\[ + \int_0^T z_u (s) + C (s) u (s) \, ds - \int_0^T z_u (s) \, ds + \sum_{i=1}^k \left( c_i v^*_i - p_{u^*} (\tau_i) + (1 - u^*_i) p_{u^*} (\tau_i^+) \right) z_u (\tau_i) \]
\[ - \int_0^T \sigma u (s) \alpha (s) p_{u^*} (s) \Theta_{u^*} (s) / (1 - \sigma u^* (s))^2 \, ds = 0 \]
\[ J_v = \int_0^T C (s) u (s) \, ds \]
\[ - \int_0^T \sigma u (s) \alpha (s) p_{u^*} (s) \Theta_{u^*} (s) / (1 - \sigma u^* (s))^2 \, ds \]
and
\[ J_u + J_v = \int_0^T C (s) u (s) \, ds + \sum_{i=0}^k (c_i - p_{u^*} (\tau_i^+)) v_i \Theta_{u^*} (\tau_i) \]
\[ - \int_0^T \sigma u (s) \alpha (s) p_{u^*} (s) \Theta_{u^*} (s) / (1 - \sigma u^* (s))^2 \, ds \]

From the proof above it follows that
\[ \Gamma_u = \int_0^T C (s) u (s) \, ds \]
\[ - \int_0^T \sigma u (s) \alpha (s) p_{u^*} (s) \Theta_{u^*} (s) / (1 - \sigma u^* (s))^2 \, ds \]

Let consider in \( L^\infty ([0, T]; \mathbb{R}) \) the inner product
\[ \langle u_1, u_2 \rangle = \int_0^T u_1 (s) u_2 (s) \, ds \]
with its associated norm \( \| u \|_3 = \left( \int_0^T u^2 (s) \, ds \right)^{1/2} \). Note that \( \forall u \in L^\infty ([0, T]; \mathbb{R}), \| u \|_3 \leq \sqrt{T} \times \text{ess sup}_{t \in [0, T]} |u (t)|. \)

That allows us to find \( u^* \) using the topology given by \( \| . \|_3. \)

Since \( \Gamma_u \) is a linear continuous operator there is a unique \( \overline{u} \in L^\infty ([0, T]; \mathbb{R}) \) such that
\[ \langle \overline{u}, u \rangle = \Gamma_u u \]
Indeed,
\[ \overline{u} = C - \sigma \alpha p_{u^*} \Theta_{u^*} / (1 - \sigma u^*)^2 \]

We then identify \( \Gamma_{\overline{u}} \) to \( \overline{u} \).

\[ J (u, v) = \int_{\Omega} \int_0^T \theta (s, x) + C (s, x) u (s, x) \, d s d x \]
\[ + \sum_{i=0}^k \int_{\Omega} c_i (x) v_i (x) \theta (\tau_i, x) \, d x + \int \Theta (T, x) \, d x \]

where \( C \in L^\infty_{\text{loc}} (\mathbb{R}_+ \times \Omega; \mathbb{R}_+) \) is almost everywhere positive, \( \forall i \in \mathbb{N}, v_i \in L^\infty (\Omega; [0, 1]), c_i \in L^\infty (\Omega; \mathbb{R}_+), \forall x \in \Omega, c (x) = (c (x_i))_{i \in \mathbb{N}} \subset \mathbb{R}_+ \). C and \( c \) are time dependent cost ratios related to the use of the control strategy. Looking at \( J \)
we are just interested by the \( k + 1 \) first terms of \( v \) and \( c \). We can also consider the restriction of \( u \) on \([0, T]\) which belongs to the set \( L^\infty ([0, T]; [0, 1]) \).

**Theorem 12:** There is an optimal strategy \( (u^*, v^*) = (u^*_i, v^*_i)_{i \in \mathbb{N}} \) which minimizes \( J \).

Proof: The problem can be reduced to find \( v^* = (v^*_i)_{0 \leq i \leq k} \in (L^\infty (\Omega; [0, 1]))^{k+1} \) with other terms null since they are useless in \( J \). Note that
\[ 0 \leq J \leq |\Omega| \left( 1 + \left( 1 + \sup_{s \in [0, T]} C (s) \right) T + \sum_{i=0}^k c_i \right) \]

Let \( J^* = \min_{(u, v) \in \mathcal{L}_{\text{loc}} ([0, T]; [0, 1]) \times (L^\infty (\Omega; [0, 1]))^{k+1}} J (u, v) \).

There is a sequence \( \{u^n, v^n\}_{n \in \mathbb{N}} \subset \mathcal{L}_{\text{loc}} ([0, T]; [0, 1]) \times (L^\infty (\Omega; [0, 1]))^{k+1} \) such that the sequence \( \{J (u^n, v^n)\}_{n \in \mathbb{N}} \) converges to \( J^* \). The sequence \( (u^n, v^n)_{n \in \mathbb{N}} \) is bounded and there is a sequence \( (u^{n_m}, v^{n_m})_{m \in \mathbb{N}} \) in \( \mathcal{L}_{\text{loc}} ([0, T]; [0, 1]) \times (L^\infty (\Omega; [0, 1]))^{k+1} \) which converges strongly to \( (u^*, v^*) \). Since \( J \) is continuous \( J (u^*, v^*) = J^* \).

Let \( \Psi_{\overline{u}} \) be the solution of \( (1) - (5) \) associated to a chosen control strategy \((u, v)\), \( z_u = D_u \theta_{\overline{u}} u = \Psi_{\overline{u}} u \) and \( J_u = D_u J (u, v) u = \Gamma_{\overline{u}} u. z_u \) satisfies
\[ dz_u / dt = -\alpha (t, x) z_u (t, x) / (1 - \sigma u (t, x)) \]
\[ - \sigma u (t, x) u (t, x) \theta_{\overline{u}} (t, x) / (1 - \sigma u (t, x)) \]
\[ + \text{div} \left( A (t, x) \nabla z_u (t, x) \right) , \]
on \( (\mathbb{R}_+ \setminus \{ \tau_i \}_{i \in \mathbb{N}}) \times \Omega \)
\[ z_u (0, x) = 0, x \in \Omega \]
\[ z_u (\tau_i, x) = (1 - \psi (x)) z_u (\tau_i, x), x \in \Omega \]
\[ \Gamma \mathbf{u} = J_u \]
\[ = \int_0^T \int_{\Omega} z_u (s, x) + C (s, x) u (s, x) \, dx \, ds + \sum_{i=0}^k \int_{\Omega} c_i (x) \psi_i (x) \, dx \]
\[ + \int_{\Omega} \Psi u (s, x) + C (s, x) u (s, x) \, dx \, ds + \sum_{i=1}^k \int_{\Omega} c_i (x) \psi_i (x) \Psi u_i (\tau_i, x) \, dx \]
\[ + \int_{\Omega} \Psi u (T, x) \, dx \]

**Theorem 13:** If \((u, v) = (u^*, v^*)\) is an optimal strategy then
\[ \forall (u, v) \in V, \]
\[ 0 \leq J_u + J_v \]
\[ = \int_0^T \int_{\Omega} C (s, x) u (s, x) \, dx \, ds + \sum_{i=0}^k \int_{\Omega} (c_i (x) - p_u^* (\tau_i^+, x)) v_i (x) \]
\[ \times \theta_{u^*, v^*} (\tau_i, x) \, dx \]
\[ - \int_0^T \int_{\Omega} \sigma u (s, x) \alpha (s, x) p_u^* (s, x) \theta_{u^*, v^*} (s, x) \]
\[ / (1 - \sigma u^* (s, x))^2 \, dx \, ds \]

where

\[ V = \{(u, v) \in L^\infty ([0, T]; \mathbb{R}) \times (L^\infty (\Omega; \mathbb{R}))^{k+1} \mid \exists \varepsilon > 0; \]
\[ (u^* + \varepsilon u, v^* + \varepsilon v) \in L^\infty ([0, T]; [0, 1]) \times (L^\infty (\Omega; [0, 1]))^{k+1} \} \]

and \(p_{u^*} = p_{u^*}\) the solution to the problem \([19] - [21]\).

**Proof:** For an arbitrary but fixed \((u, v) \in V\) and \(\varepsilon > 0\) sufficiently small we have \(J (u^* + \varepsilon u, v^* + \varepsilon v) \geq J (u^*, v^*)\)

\[ 0 \leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^T \int_{\Omega} \big( \theta_{u^*, u^*, v^*} (s, x) - \theta_{u^*, v^*} (s, x) \big) \, dx \, ds \]
\[ + \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{\Omega} \bigg( \int_{\Omega} (\varepsilon v_i^* (x) + \varepsilon v_i (x)) \]
\[ \times \theta_{u^*, u^*, v^*} (\tau_i, x) - v_i^* (x) \theta_{u^*, v^*} (\tau_i, x) \bigg) \, dx \]
\[ + \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_0^T \int_{\Omega} \big( \theta_{u^*, u^*, v^*} (T, x) - \theta_{u^*, v^*} (T, x) \big) \, dx \, ds \]
\[ = \int_0^T \int_{\Omega} \big( z_u (T, x) + z_v (T, x) \big) \, dx \, ds \]
\[ + \int_0^T \int_{\Omega} \big( z_u (s, x) + z_v (s, x) \big) \, dx \, ds \]
\[ + \sum_{i=0}^k \bigg( \int_{\Omega} c_i (x) v_i^* (x) z_v (\tau_i, x) \bigg) \]
\[ + \int_{\Omega} c_i (x) z_v (\tau_i, x) \, dx \]
\[ + \int_{\Omega} c_i (x) v_i (x) \theta_{v^*} (\tau_i, x) \, dx \]
\[ = J_u + J_v \]

Using integration by part we get

\[ \int_0^T p_{u^*} (s, x) \partial_t z_u (s, x) \]
\[ = p_{u^*} (T, x) z_u (T, x) - \int_0^T z_u (s, x) \partial_t p_{u^*} (s, x) \]
\[ + \sum_{i=1}^k p_{u^*} (\tau_i, x) z_u (\tau_i, x) \]
\[ - \sum_{i=1}^k (1 - w_i (x)) p_{u^*} (\tau_i^+, x) z_u (\tau_i, x) \]
\[ + \int_0^T z_u (s, x) \, dx \, ds \]
\[ = p_{u^*} (T, x) z_u (T, x) + \int_0^T z_u (s, x) \, dx \, ds \]
\[ - \int_0^T \alpha (s, x) p_{u^*} (s, x) z_u (s, x) / (1 - \sigma u^* (s, x)) \, ds \]
\[ + \sum_{i=1}^k p_{u^*} (\tau_i, x) z_u (\tau_i, x) \]
\[ - \sum_{i=1}^k (1 - w_i (x)) p_{u^*} (\tau_i^+, x) z_u (\tau_i, x) \]
\[ + \int_0^T z_u (s, x) \, dx \, ds \]
\[ = p_{u^*} (T, x) z_u (T, x) + \int_0^T z_u (s, x) \, dx \, ds \]
Thus

\[
\int_\Omega^T p_{u^*} (T, x) z_u (T, x) \, dx \\
+ \int_0^T \int_\Omega^T z_u (s, x) \, dxds \\
+ \sum_{i=1}^k \int_\Omega^T c_i (x, v^*_i (x)) z_u (\tau_i, x) \, dx \\
- \sum_{i=1}^k \int_\Omega^T (1 - \mathcal{U}_i (x)) p_{u^*} (\tau_i^+, x) z_u (\tau_i, x) \, dx \\
+ \int_0^T \int_\Omega^T \sigma u (s, x) \alpha (s, x) p_{u^*} (s, x) \Theta_{u^*, v^*} (s, x) \\
/ (1 - \sigma u^* (s, x))^2 \, dxds
\]

\[= 0\]

\[
J_u = \int_0^T \int_\Omega^T z_u (s, x) + C (s, x) u (s, x) \, ds \\
+ \sum_{i=0}^k \int_\Omega^T c_i (x) v^*_i (x) z_u (\tau_i, x) \, dx \\
+ \int_\Omega^T z_u (T, x) \, dx \\
= \int_0^T \int_\Omega^T z_u (s, x) + C (s, x) u (s, x) \, dxds \\
- \int_0^T \int_\Omega^T z_u (s, x) \, dxds \\
+ \sum_{i=1}^k \int_\Omega^T c_i (x) v^*_i (x) - p_{u^*} (\tau_i, x) z_u (\tau_i, x) \, dx \\
+ \sum_{i=1}^k \int_\Omega^T (1 - \mathcal{U}_i (x)) p_{u^*} (\tau_i^+, x) z_u (\tau_i, x) \, dx \\
- \int_0^T \int_\Omega^T \sigma u (s, x) \alpha (s, x) p_{u^*} (s, x) \Theta_{u^*, v^*} (s, x) \\
/ (1 - \sigma u^* (s, x))^2 \, dxds
\]

\[= \int_0^T \int_\Omega^T C (s, x) u (s, x) \, dxds \\
- \int_0^T \int_\Omega^T \sigma u (s, x) \alpha (s, x) p_{u^*} (s, x) \Theta_{u^*, v^*} (s, x) \\
/ (1 - \sigma u^* (s, x))^2 \, dxds
\]

From the proof above it follows that

\[
\Gamma_{\mathcal{U}} u = \int_0^T \int_\Omega^T C (s, x) u (s, x) \, dxds \\
- \int_0^T \int_\Omega^T \sigma u (s, x) \alpha (s, x) p_{u^*} (s, x) \Theta_{u^*, v^*} (s, x) \\
/ (1 - \sigma u^* (s, x))^2 \, dxds
\]

Let consider in \(L^\infty ([0, T] \times \Omega; \mathbb{R}) \supset L^\infty ([0, T] \times \Omega; [0, 1])\) the inner product

\[\langle u_1, u_2 \rangle = \int_0^T \int_\Omega u_1 (s) u_2 (s) \, dxds\]

with its associated norm \(\| u \|_4 = \left( \int_0^T \int_\Omega u^2 (s) \, dxds \right)^{1/2}\).

Note that \(\forall u \in L^\infty ([0, T] \times \Omega; \mathbb{R})\), \(\| u \|_4 \leq \sqrt{\| \Omega \| T} \times \text{ess sup} |u (t)|\). That allows us to find \(u^*\) using the topology given by \(\| \cdot \|_4\). Since \(\Gamma_{\mathcal{U}}\) is a linear continuous operator there is a unique \(\mathcal{U} \in L^\infty ([0, T] \times \Omega; \mathbb{R})\) such that

\[\langle \mathcal{U}, u \rangle = \Gamma_{\mathcal{U}} u\]

Indeed,

\[\mathcal{U} = C - \sigma \alpha p_{\mathcal{U}} \mathcal{U} / (1 - \sigma \mathcal{U})^2\]

We then identify \(\Gamma_{\mathcal{U}} \) to \(\mathcal{U}\).

C. Computer simulations of the optimal coupled strategy

In this subsection we performed simulations in order to illustrate theoretical results stated in the two subsections above. For the same reasons given in subsection III-C we only present the aggregate model. We keep the same inhibition pressure. The others values used in the simulation were \(C (t) = 1\), \(\forall t \in [0, 1]\) and \(c_i = 1\), \(\forall i \in \mathbb{N}\). Indeed, we assume that each pulse intervention has the same constant cost with continuous chemical control strategy. The hypothesis of constant costs still remains in general not realistic taking into account special features related to season and other conditions depending on time. Moreover, the impact in terms of costs is probably not the same for the chemical approach and the cultivational one. However, assuming that the two strategies have the same cost allows us to compare them in term of effectiveness. We have
also distinguished the same two cases of initial conditions surveyed in [III-C].

Figure 5: Inhibition pressure and continuous control strategy for $\Theta (0) = 0.14$.

Figure 6: Inhibition pressure and pulse control strategy for $\Theta (0) = 0.14$.

Figure 7: Inhibition dynamic rate for $\Theta (0) = 0.14$.

Figure 8: Inhibition pressure and continuous control strategy for $\Theta (0) = 0.40$.

Figure 9: Inhibition pressure and pulse control strategy for $\Theta (0) = 0.40$.

Figure 10: Inhibition dynamic rate for $\Theta (0) = 0.40$.

Figures above presents the impact of control strategies. Moreover, comparing thoses figures with the others displayed in [III-C] we observe that if the costs of chemical an cultiva-
tional strategies are the same then it seems better to use only the cultivational control. In other terms cultivational practices are more effective in controlling the anthracnose than chemical control.

V. DISCUSSION

In this paper a spatial diffusion model of anthracnose control with two different strategies \( (u \text{ and } v) \) has been surveyed. The single chemical strategy \( u \) have been studied in [13]. The added pulse strategy \( v = (v_i)_{i \in \mathbb{N}} \) represents the cultivational practices such as pruning old infected twigs, removing mummified fruits [5], [18], [19], [20], [26]. The model is quite general and conditions of smoothness on parameters are sufficiently weak. Indeed, parameters are just assumed measurable and essentially locally bounded on \( \Omega \). Moreover, we have showed existence and uniqueness of a solution valued in the set \( [0, 1] \) in order for the model to be well formulated mathematically and epidemiologically. We were able to establish the existence and to characterize an optimal control strategy that effectively reduces the inhibition rate compared to the case where no control is used. The proposed method to find an optimal control strategy was the gradient method presented in [11].

Numerical simulations have been performed for aggregate version of the model in order to give relevant interpretations with simple representations. As explained in subsection II-A the aggregate version of the model represents its average behaviour on the bounded domain \( \Omega \). Moreover, we have observed that it is better to use exclusively the pulse cultivational strategy if the related costs are at most equal to those related to continuous chemical strategy including environmental damages. Indeed, the simultaneous use of both strategy gives a lower performance than the exclusively use of the pulse cultivational strategy.

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