Optimal Prediction of Time-to-Failure from Information Revealed by Damage

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We present a general prediction scheme of failure times based on updating continuously with time the probability for failure of the global system, conditioned on the information revealed on the pre-existing idiosyncratic realization of the system by the damage that has occurred until the present time. Its implementation on a simple prototype system of interacting elements with unknown random lifetimes undergoing irreversible damage until a global rupture occurs shows that the most probable predicted failure time (mode) may evolve non-monotonically with time as information is incorporated in the prediction scheme. In addition, both the mode, its standard deviation and, in fact, the full distribution of predicted failure times exhibit sensitive dependence on the realization of the system, similarly to “chaos” in spin glasses, providing a multi-dimensional dynamical explanation for the broad distribution of failure times observed in many empirical situations.

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Systems of connected and interacting elements often fail through a self-organizing cascade process. Predicting the remaining lifetime of a complex structure or the precise time of failure remains an unsolved problem for all applications (engineering structures, materials, earthquakes, grids, networks, groups and so on), notwithstanding its huge importance and overwhelming consequences. Different strategies include deterministic modeling, stochastic one body or many body approaches, computational intelligence methods, and many other classifiers and pattern recognition techniques, all with limitations and lack of sufficient understanding of the underlying physical mechanisms. A major problem is that failure of a given system is highly history- and sample-dependent: in contrast with standard statistical physics, the problem is not to calculate an ensemble averaged thermodynamic property but to obtain a precise statement for each single idiosyncratic realization. This difficulty is bypassed for instance in the strategy which consists in viewing material rupture as a kind of universal critical transition [1] (see however [2]), which is based on the hope, which is partially supported by experiments, that a large system may behave like a typical realization with a kind of self-averaging property. But this misses the real practical challenge which is to detect the possible existence of flaws, either pre-existing or self-organized which are known to create a great variability of the lifetimes from one sample to the next. In addition, existing methods are often mute on the limits of predictability and on the sensitivity to various elements of the system under consideration.

Here, we analyze this prediction problem with a simple prototype of interacting elements with unknown random lifetimes undergoing irreversible damage until a global rupture occurs, the so-called time-dependent hierarchical fiber bundle model [3]. By obtaining the absolute best prediction scheme in a probabilistic sense, we are able to cast new light on the above questions. Consistent with the information usually available in realistic situations, we assume the knowledge of only the statistical properties of the constituting elements but not of their specific realizations. We use the physics of their interaction to develop the prediction scheme. The key idea is to update continuously with time the conditional probability for failure of the global system, conditioned on the information revealed by the damage that has occurred until the present time. Continuously collecting information on the ongoing damage progressively reveals key information on the pre-existing idiosyncratic realization of the system which can be gradually integrated in a better and better probabilistic prediction.

Consider a hierarchical structure of elements with $N$ levels loaded with a stress $\sigma$ per element. The first level is made of the individual elements, the second level is made of pairs of elements, the third level is made of pairs of pairs and so on. This defines a discrete hierarchical tree of local coordination 2 (the results below are easy to extend to any coordination). This topology impacts the dynamics of failure in the following way. When one of the two bundles of a given pair fails, its stress load is transferred instantaneously to the surviving bundle, such that its load is doubled. When this bundle breaks, its load is transferred to the pair of bundles associated to it if this second...
pair is still present. Otherwise, it is transferred to the pair of two pairs linked at the next hierarchical level. Given some stress history $\sigma(t')$, $t' \geq 0$, an element is assumed to break at some fixed random time, where the probability that this random time takes a specific value $t$ is specified by its cumulative distribution function

$$F_0(t) = \int_0^t P_0(t')dt' = 1 - \exp \left\{ -\kappa \int_0^t \sigma(t')^\rho dt' \right\} .$$

(1)

This amounts to considering an element failure as a conditional Poisson process with an intensity which is function of all the past stress history weighted by the stress amplification exponent $\rho > 0$. Applied to material failure, this law captures the physics of failure due to stress corrosion, to stress-assisted thermal activation and to damage. A system of $2^N$ elements is fully specified by attributing to each element $i = 1, ..., 2^N$ the a priori distinct and result from the PDFs of the elementary elements at the first level combined with the specific history a fixed failure time $t_i$ taken from the distribution (1). The failure time $t_i$ is by definition the time at which the element $i$ would have broken if the stress had stayed constant equal to the initial value $\sigma$. But, the elements are coupled through the hierarchical load transfer rule defined above. As a consequence of the hierarchical structure of the load transfers occurring at each rupture, the stress applied to a given element may increase, leading to a shortening of its lifetime. Consider a pair of bundles with lifetimes $t_1 < t_2$. At time $t = t_1$, when the first bundle breaks down, its load is transferred to the second bundle. It is easy to show from (1) that this leads to a reduction of its lifetime to

$$t_{12} = t_1 + \alpha(t_2 - t_1) < t_2 , \quad \alpha = 2^{-\rho} .$$

(2)

This law applies for any realization of lifetimes at all levels within the hierarchy and forms the basis for our derivation below.

In order to mimic a real-life situation, we consider a creep experiment of our hierarchical system, such that at time $0$, a stress $\sigma$ is applied. We have no access to the specific individual lifetimes of the individual constituting elements, only to their probability density function (PDF) $P_0(x)$, as in a real experiment. At time passes, damage occurs, that is, elements break, thus revealing their initial lifetimes or combination thereof. The situation becomes rapidly complicated because of the interactions between the elements through the hierarchical stress-load redistribution, as the damage spreads across the levels of the hierarchy. In a real-life experiment, the damage in a material sample is monitored for instance by acoustic emissions, with both time and space localization. In order to construct our prediction scheme, we just need to construct the prediction scheme for a system of four elements (or bundles) with a priori unknown initial lifetimes $t_1, t_2, t_3$ and $t_4$, whose PDFs are known. In the case where each bundle reduces to an element of level 1, the PDF’s are identical and equal to $P_0(x)$, as we assume that the elements have i.i.d. lifetimes. However, for the case of four bundles of arbitrary level $j > 0$, the PDF’s of their lifetime are a priori distinct and result from the PDFs of the elementary elements at the first level combined with the specific history of the damage until time $t$ undergone by each bundle, as we now explain.

Prediction in absence of revealed damage. Let $P_1(t_i)$ denote the PDF of the lifetimes of element (or bundle) $i$, with $i = 1, ..., 4$. If we knew $t_1$ and $t_2$, we would determine the lifetime of the pair as $t_{1,2} = \min[t_1, t_2] + \alpha (\max[t_1, t_2] - \min[t_1, t_2])$, according to (2) (and similarly for the pair $(3,4)$). But $t_1$ and $t_2$ are unknown, and the best we can do is to calculate the PDF of $t_{1,2}$ at some given time $t$. Conditioned on the fact that no element has failed, we have

$$P_{1,2}(t_{1,2}) = \frac{1}{\alpha} \int_t^{t_{1,2}} dt_1 \tilde{P}_{1,1}(t_1) \tilde{P}_{2,1} \left[ (t_{1,2} - (1 - \alpha)t_1) / \alpha \right] + (1 \leftrightarrow 2) ,$$

(3)

where $\tilde{P}_{i,1}(t_i) = \int_{t_i}^{\sqrt{P_i(x)dx}}$ is the conditional PDF’s of element $i$, given that it has not yet broken at time $t$. The second contribution ($1 \leftrightarrow 2$) in (3) corresponding to $t_1 > t_2$ is obtained from the first contribution corresponding to $t_1 < t_2$ by exchanging the two indices 1 and 2. We check that $P_{1,2}(t_{1,2})$ is normalized to unity over the time interval from $t$ to $\infty$ by using the identity $\int_t^\infty dt_{1,2} \int_t^{t_{1,2}} dt_1 = \int_t^\infty dt_1 \int_t^{\infty} dt_{1,2}$ and the change of variable $t_{1,2} \rightarrow u \equiv \{t_{1,2} - (1 - \alpha)t_1\} / \alpha$. The PDF $P_{1,2,3,4}(t_{1,2,3,4})$ of the lifetimes $t_{1,2,3,4}$ of the group of four elements at time $t$ conditioned on the fact that no element has ruptured until $t$ has the same structure as (3) with the substitutions $1 \rightarrow (1, 2)$ and $2 \rightarrow (3, 4)$.

Using the knowledge that one element failed at time $t^*$. Suppose we record the failure of the element 1 at time $t^*$, i.e., its initially unknown lifetime $t_1$ is suddenly revealed: $t_1 = t^*$. Conditioned on this information revealed at time $t^*$, we know proceed to derive how this impacts the prediction of the lifetime of the four elements, changing $P_{1,2,3,4}(t_{1,2,3,4})$ into a conditional PDF $P_{1,2,3,4}(t_{1,2,3,4} | t^*)$. Indeed, the failure of element 1 at
$t^*$ immediately changes $P_{(1,2),t^*}(t_{(1,2)})$ for the rupture time of the pair (1, 2) (i.e., of element 2 given that element 1 has broken) from expression (4) to

$$P_{(1,2),t^*}(t_{(1,2)}) = \frac{1}{\alpha} \tilde{P}_{2,t^*} \left( \left[ t_{(1,2)}^* - (1 - \alpha)t^* \right] / \alpha \right).$$  

(4)

Expression (4) derives from (3) by replacing $\tilde{P}_{1,t}(t_1)$ by $\delta(t_1 - t^*)$ to express the certain knowledge of the failure time of element 1. It can also be interpreted as the change of the failure time of element 2 from $t_2$ to $t^* + \alpha(t_2 - t^*)$ by the stress transfer from element 1 to element 2 occurring at $t^*$ (with the proper normalization of the distribution). The gain in prediction accuracy described below is due to the fact that the variance of $P_{(1,2),t^*}(t_{(1,2)})$ given by (4) is smaller than that of $P_{(1,2),t}(t_{(1,2)})$ given by (3).

In contrast with the previous case leading to $P_{(1,2,3,4),t^*}(t_{(1,2,3,4)})$ when no failure has occurred yet, the two pairs (1, 2) and (3, 4) do not play a symmetric role and two scenarios can occur for times greater than $t^*$, given that element 1 has broken at $t^*$. Scenario 1 is that element 2 fails first, followed by the rupture of second pair (3, 4). This scenario contains both the case where element 2 breaks first and then (3, 4) and the case when element 3 (or 4) breaks first, then element 2 fails and then element 4. The probability of this scenario is

$$\Pr[t_{(1,2)}^* < t_{(3,4)}^*] = \int_{t^*}^{\infty} dt_{(1,2)}^* \frac{1}{\alpha} \tilde{P}_{2,t^*} \left( \left[ t_{(1,2)} - (1 - \alpha)t^* \right] / \alpha \right) P_{(3,4),t^*} \left( \left[ t_{(1,2,3,4)} - (1 - \alpha)t_{(1,2)}^* \right] / \alpha \right).$$  

(5)

In the final calculation of $\Pr[t_{(1,2)}^* < t_{(3,4)}^*]$, we must use the fact that $P_{(1,2),t^*}(t_{(1,2)})$ is given by (4). Scenario 2 is that the second pair (3, 4) breaks first, followed by the failure of element 2. This occurs with a probability $\Pr[t_{(1,2)}^* > t_{(3,4)}^*] = 1 - \Pr[t_{(1,2)}^* < t_{(3,4)}^*]$.

Conditioned on the fact that the rupture follows the first scenario ($t_{(1,2)}^* < t_{(3,4)}^*$), the PDF for the failure time $t_{(1,2,3,4)}^*$ of the whole four-element system is

$$P_{(1,2,3,4),t^*}(t_{(1,2,3,4)}^*) = \frac{1}{\alpha^2} \int_{t^*}^{\infty} dt_{(1,2)}^* \tilde{P}_{2,t^*} \left( \left[ t_{(1,2)} - (1 - \alpha)t^* \right] / \alpha \right) P_{(3,4),t^*} \left( \left[ t_{(1,2,3,4)} - (1 - \alpha)t_{(1,2)}^* \right] / \alpha \right).$$  

(6)

Here, the PDF for the failure time of the first pair (1, 2) is changed into $P_{(1,2),t^*}(t_{(1,2)})$ given by (4). We can thus rewrite (6) as

$$P_{1,2,3,4,t^*}(t_{(1,2,3,4)}^*) = \frac{1}{\alpha^2} \int_{t^*}^{\infty} dt_{(1,2)}^* \tilde{P}_{2,t^*} \left( \left[ t_{(1,2)} - (1 - \alpha)t^* \right] / \alpha \right) P_{(3,4),t^*} \left( \left[ t_{(1,2,3,4)} - (1 - \alpha)t_{(1,2)}^* \right] / \alpha \right).$$  

(7)

Conditioned on the fact that the rupture follows the second scenario ($t_{(1,2)}^* > t_{(3,4)}^*$), the PDF for the failure time $t_{(1,2,3,4)}^*$ of the whole four-element system is

$$P_{2,3,4,t^*}(t_{(1,2,3,4)}^*) = \frac{1}{\alpha^2} \int_{t^*}^{\infty} dt_{(3,4)}^* P_{(3,4),t^*} \left( t_{(3,4)}^* \right) \tilde{P}_{2,t^*} \left( \frac{t_{(1,2,3,4)}^*}{\alpha^2} - \frac{(1 - \alpha)t^*}{\alpha^2} - \frac{1 - \alpha}{\alpha} t_{(3,4)}^* \right).$$  

(8)

where $P_{(1,2),t^*}(t_{(1,2)}^*)$ is given by (4).

Combining both scenarios yields the PDF for the failure time $t_{(1,2,3,4)}^*$ of the four-element system:

$$P_{(1,2,3,4),t^*}(t_{(1,2,3,4)}^*) = P_{1,2,3,4,t^*}(t_{(1,2,3,4)}^*) + P_{2,3,4,t^*}(t_{(1,2,3,4)}^*).$$  

(9)

where the two terms in the r.h.s. of (9) are given respectively by (7) and (8). We verify that the PDF $P_{(1,2,3,4),t^*}(t_{(1,2,3,4)}^*)$ is normalized to unity as $\int_{t^*}^{\infty} dt_{(1,2,3,4)}^* P_{(1,2,3,4),t^*}(t_{(1,2,3,4)}^*) = \Pr[t_{(1,2)}^* < t_{(3,4)}^*] + \Pr[t_{(1,2)}^* > t_{(3,4)}^*] = 1$, since the integral of (7) gives (4), and the integral of (8) gives the complement to 1, using the identity $\int_{t^*}^{\infty} dy \int_{t^*}^{\infty} dx = \int_{t^*}^{\infty} dy \int_{t^*}^{\infty} dx$ and a change of variable.

Two elements are broken in the same pair (i.e. scenario 1 is fulfilled). Suppose that element 2 breaks at some later time $t^1 > t^*$ before the rupture of the pair (3, 4). This rupture reveals a new information which can be
exploited to improve the prediction of the rupture time of the 4-element bundle. Indeed, expression (9) is changed into

\[ P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) = \frac{P_{(3,4),t^*} \left( \frac{t_{(1,2,3,4)}^{*,\tau} - (1 - \alpha)t^*}{\alpha} \right)}{\int_{t^*}^{\infty} dx \frac{P_{(3,4),t^*} \left( [x - (1 - \alpha)t^*] / \alpha \right)}{\alpha}}, \quad \text{for } t \geq t^!. \]  

(10)

This corresponds to a considerable decrease of uncertainty: first, scenario 2 is now excluded and, second, the distribution \( P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) \) is collapsed similarly to the process leading to (4) at time \( t^* \). The denominator ensures the normalization of \( P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) \) over the interval \([t^*, +\infty]\) and expresses the fact that \( P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) \) is a distribution of failure times conditioned to the failure time being larger than \( t^* \). The PDF \( P_{(3,4),t^*} \) contains the information of whether element 3 or 4 (but not both) have ruptured in the mean time, according to a derivation similar to that leading to (4) after the rupture of element 1.

**Two elements are broken, one in each of the pairs (1,2) and (3,4).** In this case the prediction of the rupture time is given by expression (10) but the knowledge that a element broke in (3,4) means that \( P_{(3,4),t^*} \) has to be replaced by the expression (4), with a change of indices \((1,2) \rightarrow (3,4)\) in (4).

**Three elements are broken with scenario 2.** Suppose that the pair (3,4) breaks at some time \( t^* > t^\rangle \) before the failure of element 2. Then, again the prediction of the rupture time of the 4-element bundle is improved according to

\[ P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) = \frac{\tilde{P}_{2,t^*,t^*} \left( \frac{t_{(1,2,3,4)}^{*,\tau} - (1 - \alpha)t^*}{\alpha} - \frac{1 - \alpha}{\alpha} t^* \right)}{\int_{t^*}^{\infty} dx \tilde{P}_{2,t^*,t^*} \left( \frac{x - (1 - \alpha)t^*}{\alpha} - \frac{1 - \alpha}{\alpha} t^* \right)}, \quad \text{for } t \geq t^!. \]  

(11)

The denominator ensures the normalization of \( P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) \) over the interval \([t^*, +\infty]\) and expresses the fact that \( P_{(1,2,3,4),\tau^*,t^*} \left( t_{(1,2,3,4)}^{*,\tau} \right) \) is a distribution of failure times conditioned to the failure time being larger than \( t^* \).

**Three elements are broken with scenario 1** The prediction of the rupture time is then given by expression (10) but the knowledge that a element broke in (3,4) means that \( P_{(3,4),t^*} \) has to be replaced by the expression (4), with a change of indices \((1,2) \rightarrow (3,4)\) in (4).

It is straightforward to iterate this enumeration for a system of arbitrary size \( 2^N \). Here, we present results obtained for a system of 16 elements, with identical exponential distributions of lifetimes. In order to calculate the PDF of the lifetime \( t_c \) of the whole system, we decompose it into four bundles of 4 elements each, for which we calculate their corresponding PDFs. The four PDFs for each 4-bundle in turn take the role of the \( P_i,t^* \) used in the previous calculations of the PDF for the total bundle of four 4-bundles. It is important to stress that, even though the lifetimes of the individual elements are i.i.d., the PDFs of the four 4-bundles remain the same only as long as no individual element has broken and then diverge as damage grows.

We use these formulas to obtain Figure 4 which shows the PDFs of the lifetime of the total system for a different number \( n \) of broken elements. As damage is revealed, the width of the distribution decreases which means that the uncertainty about when the system will fail decreases. At the same time, the most likely value of the lifetime of the system first increases up to \( n = 6 \) broken elements, after which the damage of the system is so important that a global rupture is imminent and the most likely value of \( t_c \) decreases.

Figure 4 illustrates the concept of the sensitivity of the evolution of the PDF of failure times on the initial randomness (analogous to “chaos” in spinglasses) and documents two different ways by which the “trajectories” of two PDFs can diverge: i) the modes (most probable value) move apart as a function of time; ii) the width also exhibits sensitive dependence on the quenched randomness. Consider e.g. the PDFs represented by the continuous and the dashed line. Their modes were slightly different for \( n = 4 \) broken elements but then moved closer for \( n = 8 \) broken elements. While comparable for \( n = 4 \), their widths have evolved very differently after \( n = 8 \) elements have failed. This illustrates the dependence upon which sub-levels of the hierarchy which have been broken.

This prediction scheme based on incorporating iteratively the information on the unknown pre-existing characteristic of the systems which are revealed by the growing damage does not require a priori a complete knowledge of the dynamics and opens the road to a suite of approximations for real systems involving increasing degrees of model sophistications used in the implementation which should be tested systematically. We expect the concept of multidimensional dependence on initial conditions to remain a robust feature of the prediction of time-to-failure
in many systems, that is, there are several measures of the sensitivity to initial conditions in the divergence of the trajectories of the PDFs of failure times.

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FIG. 1: PDFs of the lifetime $t_c$, shown at different levels of damage for a system that contained initially 16 elements, just after the last element broke. The different curves correspond to increasing numbers $n$ of broken elements: $n = 0$ (fat solid line), $n = 4$ (thin solid line), $n = 6$ (dashed line), $n = 8$ (dash-dotted line), and $n = 12$ (dotted line). Inset: Evolution of the corresponding lifetimes (shown as the bar heights) of the 16 elements with the height representing their lifetimes.
FIG. 2: PDFs of five different systems of 16 elements with different realizations of the initial lifetimes of the individual elements. a) $n = 4$ elements broken, b) $n = 8$ elements broken and c) $n = 12$ elements broken.