Uniqueness of the static Einstein–Maxwell spacetimes with a photon sphere

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Received 7 May 2015, revised 25 June 2015
Accepted for publication 3 July 2015
Published 30 July 2015

Abstract
We consider the problem of uniqueness of static and asymptotically flat Einstein–Maxwell spacetimes with a photon sphere $P^3$. We are using a naturally modified definition of a photon sphere for electrically charged spacetimes with the additional property that the one-form $\mathbf{F}_\xi$ is normal to the photon sphere. For simplicity we are restricting ourselves to the case of zero magnetic charge and assume that the lapse function regularly foliates the spacetime outside the photon sphere. With this information we prove that $P^3$ has constant mean curvature and constant scalar curvature. We also derive a few equations which we later use to prove the main uniqueness theorem, i.e. the static asymptotically flat Einstein–Maxwell spacetimes with a non-extremal photon sphere are isometric to the Reissner–Nordström one with mass $M$ and electric charge $Q$ subject to $Q^2 \leq M^9$. Keywords: photon sphere, Einstein, Maxwell, uniqueness

1. Introduction

Photon spheres are a well-known prediction of General Relativity and the generalized theories of gravitation. They are regions of spacetime where light can be confined to closed orbits. Photon spheres are expected to be a characteristic of ultracompact objects such as black holes, neutron stars, wormholes and naked singularities [1–16]. They are closely related to gravitational lensing and thus play an important role in astronomy and astrophysics. From an astrophysical perspective a photon sphere is a time-like hypersurface on which the light bending angle is unboundedly large [4, 6]. As an illustration we can consider the Schwarzschild black hole spacetime with mass $M$. It is well known that the photons with an impact parameter $u$ close to the critical value $u_{cr} = 3\sqrt{3}M$, corresponding to a closest approach $r = 3M$, experience a large deflection which can exceed $2\pi$. In other words, for $u$
very close to \( u_c \) (but with \( u > u_c \)) the photons can complete many loops around the hole before reaching the observer. The light rays with \( u < u_c \) will be captured by the black hole. The photons with \( u = u_c \) will make an infinite number of loops on the circular orbit \( r = 3M \). The photon circular orbits with \( r = 3M \) are unstable null geodesics and they define the photon sphere of the Schwarzschild black hole. For an arbitrary static and spherically symmetric spacetime with a metric

\[
dx^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}\sin^2\theta d\phi^2, \tag{1}\]

the photon sphere is given by the equations [4, 5]

\[
g_{tt} \partial_t g_{\theta\theta} = g_{\theta\theta} \partial_t g_{tt}. \tag{2}\]

As in the Schwarzschild case, any null geodesic initially tangent to the photon sphere remains tangent to it. For the famous Reissner–Nordström black hole spacetime with mass \( M \) and charge \( Q (M^2 \geq Q^2) \), equation (2) shows that the photon sphere lying outside the event horizon is given by the time-like hypersurface \( r = r_{ps} \) with [5]

\[
r_{ps} = 3 + \frac{\sqrt{9 - 8\frac{Q^2}{M^2}}}{2}M. \tag{3}\]

For more examples, we refer the reader to [5].

It was shown in [7] that the presence of a photon sphere around an ultracompact object is a sufficient condition for the appearance of the so-called relativistic images (images due to light deflection by angles \( > 3\pi/2 \), as defined by Virbhadra and Ellis [4]), which are vitally important for observational astrophysics [4, 6]. The relativistic images appear as a sequence of a (theoretically) infinite number of images on both sides of the optic axis due to large deflections of light near the photon sphere in addition to the pair of primary and secondary images observed due to light deflection in a weak gravitational field.

Photon spheres are objects with very specific characteristics. One of them is that the lapse function in static spacetimes is constant on the photon sphere. Another is that the photon sphere is a totally umbilic hypersurface with constant mean and scalar curvatures as well as constant surface gravity [17, 18]. These properties make them similar to event horizons. This naturally leads to the question of whether photon spheres can be used in the classification of solutions to a given gravitational theory\(^3\). This has indeed been done for the Schwarzschild spacetime. Namely it has been proven recently that static asymptotically flat solutions to the Einstein equations in vacuum with mass \( M \) possessing a photon sphere are isometric to the Schwarzschild solution [17]. A similar uniqueness theorem has also been proven for the static and asymptotically flat solutions to the Einstein-scalar field equations [18]. In general this uniqueness question is harder than the one for black hole horizons [19], because the class of spacetimes with a photon sphere is much larger than the one with an event horizon.

In our paper, we consider static and asymptotically flat vacuum solutions to the Einstein–Maxwell system of equations containing a photon sphere. We restrict ourselves to the case of zero magnetic charge for simplicity and we define the photon sphere to have one more property compared to the vacuum case, namely the electric field to be orthogonal to the photon sphere. We prove that these spacetimes with \( M^2 \neq Q^2 \) are isometric to the Reissner–Nordström solution with mass \( M \) and electric charge \( Q \) subject to the condition \( \frac{Q^2}{M^2} \leq \frac{3}{8} \).

\(^3\) It is worth mentioning that photon spheres were used in defining the notion of weakly, marginally strongly and strongly naked singularities [4, 7]. Based on this notion, Virbhadra [11] proposed a new cosmic censorship hypothesis which does not generically allow existence of marginally strongly as well as strongly naked singularities. However, it allows weakly naked singularities.
2. Preliminary definitions and equations

In the present paper we consider Einstein–Maxwell gravity described by the following action:

\[ S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left( \mathcal{R} - F_{\mu\nu} F^{\mu\nu} \right), \]  

(4)

where the spacetime manifold is denoted by \((\mathcal{L}^4, g)\), \(\mathcal{R}\) is the Ricci scalar curvature and \(F\) is the Maxwell tensor. From this action we get the field equations:

\[ \mathcal{R}_{\mu\nu} = 2 \left( F_{\mu\rho} F^{\rho}_{\phantom{\rho}\nu} - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right), \]  

(5)

\[ d \star F = 0, \]  

(6)

\[ dF = 0, \]  

(7)

where \(\star\) is the Hodge star operator.

We are considering static spacetimes, which means that there exists a smooth Riemannian manifold \((M^3, g)\) and a smooth lapse function \(N : M^3 \rightarrow \mathbb{R}^+\) such that

\[ \mathcal{L}^4 = \mathbb{R} \times M^3, \ g = -N^2 dt^2 + g. \]  

(8)

Let \(\xi = \frac{\partial}{\partial t}\) be the time-like Killing vector. Then we define staticity of the Maxwell field,

\[ \mathcal{L}_\xi F = 0. \]  

(9)

We shall focus on the purely electric case with \(t_\xi \star F = 0\).

The spacetimes we are considering are also asymptotically flat. Asymptotic flatness is defined in the usual way. The spacetime is asymptotically flat if there exists a compact set \(K \subset M^3\) such that \(M^3 \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus \bar{B}\) where \(\bar{B}\) is the closed unit ball centered at the origin in \(\mathbb{R}^3\) and such that

\[ g = \delta + O(r^{-1}), \quad N = 1 - \frac{M}{r} + O(r^{-1}), \]  

(10)

with respect to the standard radial coordinate on \(\mathbb{R}^3\). The asymptotic expansion of the electromagnetic field can be derived easily and we have

\[ F = -\frac{Q}{r^2} dr \wedge dr + O(r^{-3}). \]  

(11)

As usual \(M\) and \(Q\) are the mass and the electric charge, respectively. We will focus our study on the physically interesting case with \(M > 0\) and \(Q \neq 0\).

We also need to define the photon sphere. We start with the definition of a photon surface [5].

**Definition 2.1.** An embedded time-like hypersurface \((P^3, p) \hookrightarrow (\mathcal{L}^4, g)\) is called a photon surface if any null geodesic initially tangent to \(P^3\) remains tangent to \(P^3\) as long as it exists.

Next we define a photon sphere.

**Definition 2.2.** Let \((P^3, p) \hookrightarrow (\mathcal{L}^4, g)\) be a photon surface. Then \(P^3\) is called a photon sphere if the lapse function \(N\) is constant on \(P^3\) and the one-form \(t_\xi F\) is normal to \(P^3\).
We will make an additional technical assumption. We will assume that the lapse function $N$ regularly foliates the spacetime outside the photon sphere, i.e.

$$\rho^{-2} = g\left(\nabla N, \nabla N\right) \neq 0$$

outside the photon sphere. The spatial part of this exterior region will be denoted by $M_3^{\text{ext}}$ and by definition it has as inner boundary the intersection $\Sigma$ of the outermost photon sphere with the time slice $M^3$. By definition $\Sigma$ is given by $N = N_0$ for some $N_0 \in \mathbb{R}^+$. As a consequence of our assumption all level sets $N = \text{const}$, including $\Sigma$, are topological spheres and $M_3^{\text{ext}}$ is topologically $S^2 \times \mathbb{R}$.

We define the electric field one-form $E$ by

$$E = -i_\xi F,$$

and it satisfies $\delta E = 0$ as a consequence of the field equations and the electromagnetic staticity. Since $M_3^{\text{ext}}$ is simply connected this implies the existence of an electric potential $\Phi$ such that $E = d\Phi$. Using the electric field one-form we can write an explicit expression for $F$,

$$F = -N^{-2} \xi \wedge d\Phi.$$  

By definition, the electric field $E$ is normal to the photon sphere and therefore the electrostatic potential $\Phi$ is constant on the photon sphere. Since the electrostatic potential is defined up to a constant, without loss of generality we shall set $\Phi_0 = 0$.

Using the form of the metric (8) and the form of the Maxwell tensor we can obtain the dimensionally reduced static Einstein–Maxwell field equations:

$$\delta \Delta N = N^{-1} \nabla^i \Phi \nabla_i \Phi,$$

$$\delta R_{ij} = N^{-1} \nabla_i \nabla_j N + N^{-2} \left( g_{ij} \nabla^k \Phi \nabla_k \Phi - 2 \nabla_i \Phi \nabla_j \Phi \right),$$

$$\delta \nabla^i \left( N^{-1} \nabla_i \Phi \right) = 0.$$  

In [20] Israel derived a divergence identity for the lapse function and the electrostatic potential which in our notation is given by

$$\nabla^i \left[ 2 \Phi \nabla_i N - \left( N + N^{-1} \Phi^2 \right) \nabla_i \Phi \right] = 0.$$  

The Gauss theorem applied to this identity leads to the following functional dependence between the lapse function $N_0$ and the electrostatic potential $\Phi_0$ on $\Sigma$

$$N_0^2 = \Phi_0^2 - 2 \frac{M}{Q} \Phi_0 + 1,$$

where we have taken into account that $\Phi$ and $N$ are constant on $\Sigma$ and that $\Phi_0 = 0$. Moreover, this functional dependence holds not only on $\Sigma$ but also on the whole $M_3^{\text{ext}}$, namely

$$N^2 = \Phi^2 - 2 \frac{M}{Q} \Phi + 1.$$  

In order to prove this we can use the following divergence identity:

$$N W_i W^i = \frac{1}{2} \nabla^i \left[ -N^2 + \Phi^2 - 2 \frac{M}{Q} \Phi + 1 \right] W_i.$$  

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where \( w_i \) is given by

\[
  w_i = -\nabla_i N + N^{-1} \left( \Phi - \frac{M}{Q} \right) \nabla_i \Phi. \tag{22}
\]

This identity is a consequence of the dimensionally reduced field equations. Integrating this identity on \( M_{\text{ext}}^3 \) with the help of the Gauss theorem and taking into account (19) and the asymptotic behavior of \( N \) and \( \Phi \) we obtain that \( w_i = 0 \) on \( M_{\text{ext}}^3 \). Hence we find \( N^2 = \Phi^2 - 2M\Phi + C \) with \( C \) being a constant. From the asymptotic behavior of \( N \) and \( \Phi \) we conclude that \( C = 1 \) which proves (20). By the maximum principle for elliptic partial differential equations and by the asymptotic behavior of \( N \) for \( r \to \infty \) we obtain for the values of \( N \) on \( M_{\text{ext}}^3 \) the following inequality:

\[
  N_0 \leq N < 1. \tag{23}
\]

3. Auxiliary equations and theorems

For an isometric embedding \((A^n, a) \hookrightarrow (B^{n+1}, b)\) with unit normal \( \eta \) and second fundamental form \( I_{XY} b^Y b^X \), there are a few other equations we will use.

- The Codazzi equation, which reads
  \[
  b \left( \partial^a R(X, Y, \eta, Z) \right) = \left( \partial^a \nabla X I \right)(Y, Z) - \left( \partial^a \nabla Y I \right)(X, Z) \tag{24}
  \]
  for all \( X, Y, Z \in \Gamma(TA^n) \).

- The contracted Gauss equation, which reads
  \[
  \partial^a R = 2\tau \partial^a R(\eta, \eta) = R - \tau (a^a R^a)^2 + \tau |HI|^2, \tag{25}
  \]
  where \( \tau = b(\eta, \eta) \).

- An equation satisfied for every smooth function \( f: B^{n+1} \to \mathbb{R} \), if \( \tau = 1 \), which reads
  \[
  \partial^a f = \Delta f + \nabla^2 f(\eta, \eta) + (a^a I^a)\eta(f). \tag{26}
  \]

The results in this section are not only vital for the proof of our main theorem but are also interesting for their own. First we recall a result by Claudel et al \[5\].

**Theorem 3.1.** Let \((P^3, p) \hookrightarrow (\Sigma^4, g)\) be an embedded time-like hypersurface. Then \(P^3\) is a photon surface if and only if it is totally umbilic (i.e. its second fundamental form is pure trace).

The second fundamental form of \(P^3\) can then be written as \( h = \frac{\partial}{\partial p} \), where \( \partial \) is the mean curvature of \(P^3\). Using this we can prove a theorem concerning the mean and scalar curvatures of the photon sphere \(P^3\) in our spacetime.

**Theorem 3.2.** Let \((\Sigma^4, g, F)\) be a static, asymptotically flat Einstein–Maxwell spacetime possessing a photon sphere \((P^3, p) \hookrightarrow (\Sigma^4, g)\). Then \(P^3\) has constant mean curvature (CMC) and constant scalar curvature (CSC).
Proof. To prove the theorem we will first use the Codazzi equation (24) for $(P^3, p) \hookrightarrow (\mathbb{L}^4, g)$ with unit normal $\nu$:

$$g(\mathfrak{R}(X, Y, \nu), Z) = (\mathcal{V}_{\xi} h)(Y, Z) - (\mathcal{V}_{\eta} h)(X, Z),$$

where $X, Y, Z \in \Gamma(TP^3)$. Now we contract the slots $X$ and $Z$ and obtain

$$\mathfrak{R}(Y, \nu) = (1 - 3)Y\left(\frac{\delta}{3}\right).$$

Using the Einstein–Maxwell equations to calculate the left-hand side of (28) we get

$$\mathfrak{R}_{\alpha\beta}Y^\alpha\nu^\beta = \frac{1}{t_E}E_1^2t_EY_1Y_\nu + 2\frac{1}{t_E}E_2Y_1E_\nu.$$  

Hence, taking into account that $E^\mu$ is normal to $P^3$ and that $\xi \in \Gamma(TP^3)$, it follows that $t_EY_1 = 0$ and $t_1E_\nu = 0$. Finally, we obtain

$$0 = (1 - 3)Y\left(\frac{\delta}{3}\right),$$

which means that $P^3$ has CMC since $Y$ is an arbitrary tangent vector to $P^3$.

Next we need to prove that $P^3$ has CSC. To do this we will use the contracted Gauss equation (25).

$$\mathfrak{R} = -2\tau \mathfrak{R}(\nu, \nu) = \mathfrak{R} - \tau \left(\rho tr h\right)^2 + \tau |h|^2,$$

$$= \mathfrak{R} - \delta^2 + \frac{\delta^2}{3},$$

$$= \mathfrak{R} - \frac{2}{3}\delta^2.$$  

From the field equations we know that the Ricci scalar $\mathfrak{R}$ vanishes, leading to

$$\mathfrak{R} = \frac{2}{3}\delta^2 - 2\mathfrak{R}(\nu, \nu).$$

Here we can calculate $\mathfrak{R}(\nu, \nu)$, again using the field equations.

$$\mathfrak{R}(\nu, \nu) = 2\left(\frac{1}{t_E}E_1^2t_1E_\nu - \frac{1}{2}\frac{1}{t_E}E_2^2t_1E\right).$$

Since $E$ is normal to $P^3$ we have $t_EE_\nu = E_\nu$. Then (33) takes the form

$$\mathfrak{R}(\nu, \nu) = -\frac{E_\nu^2}{N^2}.$$  

Finally, we obtain that $P^3$ has CSC, given by the expression

$$\mathfrak{R} = \frac{2}{3}\delta^2 + \frac{E_\nu^2}{N^2}.$$  

We will show below that $E_\nu$ is constant on $P^3$ which shows that $\mathfrak{R}$ is constant. This completes the proof. \qed
Next we will derive a few more useful relations. We start off by computing the second fundamental form \( h \) of \((\Sigma, \sigma) \leftrightarrow (M^3, g)\) with unit normal \(\nu\). Let \(X, Y, Z \in \Gamma(T\Sigma)\) be arbitrary tangent vectors to \(\Sigma\). Then

\[
h(X, Y) = g\left(\nabla_X \nu, Y\right) = g\left(\nabla_Y \nu, X\right) = \frac{\delta}{3} p(X, Y) = \frac{\delta}{3} \sigma(X, Y).
\]

(36)

Thus we see that \(\Sigma\) is totally umbilic and has CMC \(H = \frac{2}{3} \delta\). Now we make use of the Codazzi equation for \((\Sigma, \sigma) \leftrightarrow (M^3, g)\).

\[
g \left(\mathcal{R}(X, Y, \nu), Z\right) = \left(\nabla^X h\right)(Y, Z) - \left(\nabla^Y h\right)(X, Z),
\]

\[
= \nabla_X \left(\frac{\delta}{3}\right) \sigma(Y, Z) - \nabla_Y \left(\frac{\delta}{3}\right) \sigma(X, Z).
\]

(37)

After contracting the \(X\) and \(Z\) slots we obtain

\[
\mathcal{R}(Y, \nu) = 0,
\]

(38)
due to \(\delta\) being constant on \(\Sigma\). We will use this result to prove that \(\nu(N)\) is constant on \(\Sigma\). For this purpose we calculate the Lie derivative, using the field equations,

\[
\mathcal{L}_N \nu = \frac{\partial}{\partial t} \nu = \rho \frac{\partial}{\partial \tau} \nu(N) = \rho \frac{\partial}{\partial \tau} \nu(N),
\]

\[
= 2N \mathcal{R}(\nu, X) - 2N^{-1} \nu(X) \nabla^X \phi \nabla_N \phi + 4N^{-1} \nabla_N \phi \nabla^X \phi,
\]

\[
= 2N \mathcal{R}(\nu, X) = 0.
\]

(39)

As a direct consequence of (20) we obtain that \(E_\nu\) is also constant on \(\Sigma\) and thus on \(P^3\).

Next we use equation (26), again for \((\Sigma, \sigma) \leftrightarrow (M^3, g)\),

\[
\Delta N = \Delta N + \nabla^X (\nu) + \left(\sigma h\right)(\nu) \nu(N).
\]

(40)

By the Einstein–Maxwell equations, (40) transforms into

\[
2N^{-1} E E = N \mathcal{R}(\nu, \nu) + 2N^{-1} (i_E \nu)_H + \nu(N) H.
\]

(41)

The contracted Gauss equation gives

\[
\mathcal{R} = 2 \mathcal{R}(\nu, \nu) = \mathcal{R} = \frac{H^2}{2}.
\]

(42)

Finally, contracting the field equations we get

\[
\mathcal{R} = 2N^{-1} i_E E.
\]

(43)

We can combine (41-43) to get

\[
N \mathcal{R} = 2N^{-1} E^2 + 2 \nu(N) H + \frac{1}{2} NH^2.
\]

(44)

We next integrate (44) over \(\Sigma\).

\[
\int_\Sigma N \mathcal{R} d\mu = \int_\Sigma 2N^{-1} E^2 d\mu + \int_\Sigma 2\nu(N) H d\mu + \int_\Sigma \frac{1}{2} NH^2 d\mu.
\]

(45)

Let us denote the area of \(\Sigma\) by \(A_\Sigma\). With this, using the Gauss–Bonnet theorem and noting that for two-dimensional manifolds \(R = 2K\), where \(K\) is the Gaussian curvature, we can
transform (45) so that it becomes
\[ N_0 = \frac{E_0^2 A_0}{N_0} + \frac{1}{4\pi} \left[ \nu(N) \right] H A_\Sigma + \frac{1}{16\pi} H^2 A_\Sigma. \] (46)

Next we will use the Komar definition for the spacetime mass,
\[ M = -\frac{1}{8\pi} \int_{\Sigma^M} g^{\alpha\beta} dS_{\alpha\beta}. \]
\[ = -\frac{1}{8\pi} \int_{\Sigma} g^{\alpha\beta} dS_{\alpha\beta} - \frac{1}{4\pi} \int_{M_{ext}} g_{\rho\sigma} \eta_\rho \sqrt{g} \, d^3y. \] (47)

The first term here is the mass of the photon sphere \( M_P \). Using the field equations we transform the second term and arrive at the expression
\[ M = M_P + Q \Phi_0. \] (49)

Similarly we can calculate explicitly the expression for the mass of the photon sphere, arriving at \( M_P = \frac{1}{4\pi} \left[ \nu(N) \right] A_\Sigma \). With this we rewrite (46) to become
\[ N_0 = \frac{1}{4\pi} \frac{E_0^2 A_0}{N_0} + \frac{M_P H + \frac{1}{16\pi} N_0 H^2 A_\Sigma.} {16\pi} \] (50)

Finally, we will use the contracted Gauss equation again, this time for \((\Sigma^2, \sigma) \leftrightarrow (P^3, \nu)\) with a unit normal \( \eta \),
\[ \mathcal{R} + 2 \mathcal{R}(\eta, \eta) = \mathcal{R}. \] (51)

Remembering (35) and using \( \mathcal{R}(\eta, \eta) = 0 \), we get
\[ \mathcal{R} = \mathcal{R} = \frac{2}{3} \eta_\Sigma + 2 \frac{E_0^2}{N^2} = \frac{3}{2} H^2 + 2 \frac{E_0^2}{N^2}. \] (52)

We integrate (52) over \( \Sigma \), again taking into account the Gauss–Bonnet theorem,
\[ 1 = \frac{3}{16\pi} H^2 A_\Sigma + \frac{1}{4\pi} \frac{E_0^2 A_\Sigma}{N^2}. \] (53)

We can make use of the definition of the electric charge to write it in a form containing the function \( E_0 \), i.e. \( Q = \frac{A_\Sigma E_0}{4\pi N^2} \). Then after a few calculations involving (49), (50) and (53) we arrive at
\[ 1 = \frac{4\pi Q^2}{A_\Sigma} + \frac{3}{2} (M - Q \Phi_0) H. \] (54)

Another very useful relation that can be easily obtained from (50) and (53) is the following:
\[ 2 \nu(N) = N_0 H. \] (55)

4. Uniqueness theorem

We first define the notion of non-extremal photon sphere.
**Defintion 4.1.** A photon sphere is called non-extremal if
\[
\frac{1}{4\pi} H^2 A_{\Sigma} \neq 1. \tag{56}
\]

In the case of Einstein–Maxwell equations, using the relations derived in the previous section, it is not difficult to show that the photon sphere is non-extremal only if \( M^2 \neq Q^2 \).

The main result of the present paper is the following.

**Theorem 4.1.** Let \((\mathcal{L}_{\text{ext}}^4, g, F)\) be a static and asymptotically flat spacetime with given mass \( M \) and charge \( Q \), satisfying the Einstein–Maxwell equations and possessing a non-extremal photon sphere as an inner boundary of \( \mathcal{L}_{\text{ext}}^4 \). Assume that the lapse function regularly foliates \( \mathcal{L}_{\text{ext}}^4 \). Then \((\mathcal{L}_{\text{ext}}^4, g, F)\) is isometric to the Reissner–Nordström spacetime with mass \( M \) and charge \( Q \) subject to the inequality \( \frac{Q^2}{M^2} \leq \frac{9}{8} \).

**Proof.** In proving the theorem we shall follow [18] with some technical modifications due to the fact that the target space metric for the Einstein–Maxwell equations is Lorentzian in contrast to [18], where the target space metric is Riemannian.

Let us consider the 3-metric \( \gamma_{ij} \) on \( M^3_{\text{ext}} \) defined by
\[
\gamma_{ij} = N^2 g_{ij}. \tag{57}
\]

Rewriting the dimensionally reduced static Einstein–Maxwell equations in terms of the new metric \( \gamma_{ij} \) we have
\[
\begin{align*}
R(\gamma)_{ij} &= 2D_i \ln(N)D_j \ln(N) - 2N^{-2}D_i \Phi D_j \Phi, \\
D_iD^i \ln(N) &= N^{-2}D_i \Phi D^i \Phi, \\
D_i \left( N^{-2} D^i \Phi \right) &= 0. \tag{58}
\end{align*}
\]

Further reduction can be achieved by taking in account that \( N \) and \( \Phi \) are functionally dependent via (20). However instead of using \( N \) or \( \Phi \) it is convenient to use another potential \( \tilde{\lambda} \) defined by
\[
\tilde{\lambda} = -N^{-2} \ln(\Phi), \quad \tilde{\lambda}_{\infty} = 0. \tag{59}
\]

In terms of this new potential the dimensionally reduced static Einstein–Maxwell equations become
\[
\begin{align*}
R(\gamma)_{ij} &= 2 \left( \frac{M^2}{Q^2} - 1 \right) D_i \tilde{\lambda} D_j \tilde{\lambda}, \\
D_i D^i \tilde{\lambda} &= 0. \tag{60}
\end{align*}
\]

In fact, the potentials \( N \) and \( \Phi \) parameterize the coset \( SL(2, \mathbb{R})/SO(1, 1) \) with a metric
\[
G_{AB} d\Phi^A d\Phi^B = N^{-2} dN^2 - N^{-2} d\Phi^2 \tag{62}
\]
and it is not difficult to see that \((N(\tilde{\lambda}), \Phi(\tilde{\lambda})\)) is a geodesic on \( SL(2, \mathbb{R})/SO(1, 1) \) with
\[
G_{AB} \frac{d\Phi^A}{d\tilde{\lambda}} \frac{d\Phi^B}{d\tilde{\lambda}} = \frac{M^2}{Q^2} - 1. \tag{63}
\]
Depending on the ratio $\frac{Q^2}{M^2}$ we have three types of geodesics which we will formally call ‘spacelike’ for $\frac{Q^2}{M^2} < 1$, ‘time-like’ for $\frac{Q^2}{M^2} > 1$ and ‘null’ for $\frac{Q^2}{M^2} = 1$. The null geodesics correspond to extremal photon spheres and that is why they will not be considered here. The other types of geodesics have to be considered separately.

**Case $\frac{Q^2}{M^2} < 1$.**

In studying the case of spacelike geodesics we shall use the ‘affine parameter’ $\lambda$ given by

$$\lambda = \sqrt{\frac{M^2}{Q^2} - 1} \tilde{\lambda}.$$  \hspace{1cm} (64)

We proceed further by considering the inequalities [18]

$$\int_{M_{ext}^i} D^i \left[ \Omega^{-1} \left( \Gamma D_i \chi - \chi D_i \Gamma \right) \right] \sqrt{\gamma} d^3 x \geq 0$$  \hspace{1cm} (65)

and

$$\int_{M_{ext}^i} D^i \left( \Omega^{-1} D_i \chi \right) \sqrt{\gamma} d^3 x \geq \int_{M_{ext}^i} D^i \left[ \Omega^{-1} \left( \Gamma D_i \chi - \chi D_i \Gamma \right) \right] \sqrt{\gamma} d^3 x,$$  \hspace{1cm} (66)

where $\chi$, $\Gamma$ and $\Omega$ are defined by

$$\chi = \left( \chi^{\mu} D_{\mu} \chi \right)^{1/2}, \quad \Gamma = \frac{1 - e^{2i}}{1 + e^{2i}}, \quad \Omega = \frac{4e^{2i}}{(1 + e^{2i})^2}.$$  \hspace{1cm} (67)

The equalities in (65) and (66) hold if and only if the Bach tensor $R(\gamma)_{ijkl}$ vanishes [18].

After rather long and unpleasant calculations with the help of Gauss theorem, and taking into account (55), one can show that the first inequality (65) is equivalent to

$$\Phi_0^2 = \frac{3}{2} \frac{M}{Q} \Phi_0 + \frac{1}{2} \leq 0,$$  \hspace{1cm} (68)

while the second inequality (66) gives

$$\Phi_0^2 = \frac{3}{2} \frac{M}{Q} \Phi_0 + \frac{1}{2} \geq 0.$$  \hspace{1cm} (69)

Hence we conclude that

$$\Phi_0^2 = \frac{3}{2} \frac{M}{Q} \Phi_0 + \frac{1}{2} = 0$$  \hspace{1cm} (70)

and therefore $R(\gamma)_{ijkl} = 0$. Since the Bach tensor vanishes we conclude that the metric $\gamma_{ij}$ is conformally flat. Obviously the same holds for the metric $g_{ij}$.

If we denote the surfaces of constant $N$ embedded in $M^3$ by $\Sigma_N$, $(\Sigma_N, \sigma) \leftrightarrow (M^3, g)$, we can write our spacetime metric in the form:

$$g = -N^2 dt^2 + \rho^2 dN^2 + \sigma_{AB} dx^A dx^B.$$  \hspace{1cm} (71)
Using the dimensionally reduced field equations one can show that
\[
R(g)_{ijk}R(g)^{ijk} = \frac{8}{N^4\rho^4} \left[ \frac{M^2 - Q^2}{M^2 - Q^2 + Q^2N^2} \right]^2 \\
\times \left[ h^{\Sigma AB} - \frac{1}{2} H^{\Sigma \Sigma AB} \right] \left[ h^{\Sigma AB} - \frac{1}{2} H^{\Sigma \Sigma AB} \right] + \frac{1}{2\rho^2} \sigma^{AB} \partial_A \rho \partial_B \rho \right], \tag{72}
\]
where \( h^{\Sigma AB} \) is the second fundamental form of \( \Sigma_N \) and \( H^{\Sigma \Sigma AB} \) is its trace. Then, for \( \frac{Q}{M^2} < 1 \), we conclude that
\[
h^{\Sigma AB} = \frac{1}{2} H^{\Sigma \Sigma \Sigma AB}, \quad \partial_A \rho = 0. \tag{73}
\]
The space geometry is therefore spherically symmetric. Then one can easily show that the spacetime is isometric to the Reissner–Nordström spacetime with \( \frac{Q}{M^2} < 1 \). This can be done by direct computation or by using Birkhoff’s theorem for the Einstein–Maxwell equations.

Equation (70) determines the value of the electrostatic potential and the lapse function (via (20)) on the photon sphere. In fact equation (70) has two solutions:
\[
\Phi_0^\pm = \frac{3}{4} \left( \frac{M}{Q} \pm \sqrt{\frac{M^2}{Q^2} - \frac{8}{9}} \right), \tag{74}
\]
corresponding to two photon spheres. In the present paper we consider only the outermost photon sphere given by \( \Phi_0^+ \).

**Case** \( \frac{Q}{M^2} > 1 \).

Here we shall use the affine parameter \( \lambda = \sqrt{1 - \frac{M^2}{Q^2} \lambda} \) with \(-\frac{\pi}{2} < \lambda < \frac{\pi}{2}\). As in the previous case we consider the inequalities (65) and (66), but with different functions \( \Gamma \) and \( \Omega \), namely
\[
\Gamma = \tan(\lambda), \quad \Omega = 1 + \Gamma^2 = \cos^{-2}(\lambda). \tag{75}
\]
In the case under consideration the first inequality (65) is equivalent to (68) while the second one to (69). Hence we have that (70) is satisfied and therefore \( R(g)_{ijk} = 0 \), i.e. the metric \( g_{ij} \) is conformally flat. The same arguments as in the previous case show that the spatial geometry is spherically symmetric and that the spacetime is isometric to the Reissner–Nordström spacetime with \( 1 < \frac{Q}{M^2} \leq \frac{9}{8} \). Note that according to equation (74) a photon sphere exists only for \( \frac{Q}{M^2} \leq \frac{9}{8} \). This is just the inequality first obtained by other means by Claudel et al [5] in their study of the photon spheres in Reissner–Nordström metric.

**5. Discussion**

In the current paper we have proven that the static asymptotically flat solutions to the Einstein–Maxwell equations with mass \( M \) and electric charge \( Q \) possessing a non-extremal photon sphere are isometric to the Reissner–Nordström spacetime with the same mass and charge. We have used simple and physically motivated assumptions, namely that the lapse function foliates the spacetime outside of the photon sphere and that the photon sphere is defined with one additional property (compared to the vacuum Einstein case). This property states that the electric field is normal to the photon sphere. It should be noted that our theorem
does not cover the extremal case which is rather subtle and needs more sophisticated techniques.

In seeking mathematical generality one can ask whether the condition that the lapse function foliates the spacetime can be relaxed and one can consider a priory non-connected photon spheres along the lines of [21]. Our preliminary studies show that this could be done in the case $\frac{Q^2}{M^2} < 1$ (the ‘black hole’ case). In the general case however, the price we have to pay in order to relax the mentioned condition is the increase in technical details and complexity. What’s more, if we relax the condition that the lapse function foliates the spacetime then we have to assume in addition that the spacetime is simply connected. Nevertheless, we intend to study this problem and the problem of extremal photon spheres in future publications.

Acknowledgments

S Y would like to thank the Research Group Linkage Programme of the Alexander von Humboldt Foundation for the support. The partial support by the COST Action MP1304, by Bulgarian NSF grant DFNI T02/6 and by Sofia University Research Grant N70/2015 is also gratefully acknowledged.

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