THE BOCHNER-TYPE FORMULA AND THE FIRST EIGENVALUE OF THE SUB-LAPLACIAN ON A CONTACT RIEMANNIAN MANIFOLD

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Abstract. Contact Riemannian manifolds, with not necessarily integrable complex structures, are the generalization of pseudohermitian manifolds in CR geometry. The Tanaka-Webster-Tanno connection on such a manifold plays the role of Tanaka-Webster connection in the pseudohermitian case. We prove the contact Riemannian version of the pseudohermitian Bochner-type formula, and generalize the CR Lichnerowicz theorem about the sharp lower bound for the first nonzero eigenvalue of the sub-Laplacian to the contact Riemannian case.

1. Introduction

Lichnerowicz [20] obtained a sharp lower bound for the first eigenvalue of the Laplacian-Beltrami operator on a compact Riemannian manifold with a lower Ricci bound, and Obata [22] characterized the case of equality. On a pseudohermitian manifold, the sub-Laplacian is the counterpart of the Laplacian-Beltrami operator. The CR analogue of the Lichnerowicz theorem states that for a $(2n+1)$-dimensional pseudohermitian manifold, $n \geq 3$, satisfying

\[ \text{Ric}(X, X) + \frac{n+1}{2} \text{Tor}(X, X) \geq \kappa h(X, X), \tag{1.1} \]

the first nonzero eigenvalue of the sub-Laplacian is greater than or equal to $n\kappa/(n+1)$. This result was first proved by Greenleaf [13]. But due to a mistake in calculation pointed out in [6] and [12], the coefficient $\frac{n+1}{2}$ in (1.1) was mistaken to be $\frac{2}{3}$. The corresponding results for $n = 2$ and $n = 1$ were obtained later in [18] and [8], respectively. The CR Obata-type theorem was conjectured in [6], which states that if $n\kappa/(n+1)$ is an eigenvalue of the sub-Laplacian on a pseudohermitian manifold, then it is the standard CR structure on the unit sphere in $\mathbb{C}^{n+1}$. This is proved under some additional conditions (cf. [6], [7], [16] and references therein) and without conditions in [19]. There is also a quaternionic contact version of Lichnerowicz theorem [14] (see e.g. [3], [15] and [29] for the quaternionic contact manifolds). In this paper, we generalize the CR Lichnerowicz theorem to the contact Riemannian case.

A $(2n+1)$-dimensional manifold $M$ is called a contact manifold if it has a real 1-form $\theta$, called a contact form, such that $\theta \wedge d\theta^n \neq 0$ everywhere on $M$. There exists a unique vector field $T$, the Reeb vector field, such that $\theta(T) = 1$ and $T \cdot d\theta = 0$. It is well known that given a contact manifold $(M, \theta)$, there are a Riemannian metric $h$ and a $(1,1)$-tensor field $J$ on $M$ such that

\[ h(X, T) = \theta(X), \]
\[ J^2 = -Id + \theta \otimes T, \]
\[ d\theta(X, Y) = h(X, JY), \tag{1.2} \]

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for any vector fields $X$ and $Y$ (cf. p. 278 in [1]). We call $J$ an \textit{almost complex structure}. Once $h$ is fixed, $J$ is uniquely determined. $(M, \theta, h, J)$ is called a contact Riemannian manifold.

Let $TM$ be the tangent bundle and $CTM$ be its complexification. Denote $HM := Ker(\theta)$, the horizontal subbundle. $\mathbb{C}HM$ has a unique subbundle $T^{(1,0)}M$ such that $JX = iX$ for any $X \in \Gamma(T^{(1,0)}M)$. Here and in the following, $\Gamma(S)$ denotes the space of all sections of a vector bundle $S$. Set $T^{(0,1)}M = T^{(1,0)}M$. For any $X \in \Gamma(T^{(0,1)}M)$, we have $JX = -iX$. $J$ is called \textit{integrable} if $[\Gamma(T^{(1,0)}M), \Gamma(T^{(1,0)}M)] \subset \Gamma(T^{(1,0)}M)$. In particular if $J$ is integrable, $J$ is called a CR structure and $(M, \theta, h, J)$ is called a pseudohermitian manifold. On a contact Riemannian manifold there exists a distinguished connection introduced in [23], called the Tanaka-Webster-Tanno connection (or TWT connection briefly). In the pseudohermitian case, this connection is exactly the Tanaka-Webster connection. The Tanno tensor is defined as $Q = \nabla J$. $J$ is integrable if and only if the Tanno tensor $Q = 0$ (cf. Proposition 2.1 in [23]). We can also define the sub-Laplacian operator $\Delta_b$. Since there is no obstruction to the existence of the almost complex structure $J$, contact Riemannian structures exist naturally on any contact manifold and analysis on it has potential applications to the geometry of contact manifolds (cf. [2], [24] and [25] and references therein).

Since the Tanno tensor $Q$ is a (1,2)-tensor, $Q_X := Q(X, \cdot)$ and $\nabla Q(X, X)$ are (1,1)-tensors. Define invariants of contact Riemannian structures:

$$Q_1(X, X) = -2Re(i\text{-trace}\{Y \rightarrow \nabla_Y Q(X, X)\}),$$

$$Q_2(X, X) = \langle Q_X, Q_X \rangle, \quad Q_3(X, X) = \text{trace}\{Y \rightarrow Q_X \circ Q_X(Y)\}, \quad (1.3)$$

where $\langle \cdot, \cdot \rangle$ is the inner product on (1,1)-tensor induced by $h$, and

$$Tor(X, X) = 2Re(ih(\tau_X, X)), \quad (1.4)$$

for any $X \in T^{(1,0)}M$, where $\tau_X$ is the Webster torsion defined as $\tau_X(X) = \tau(T, X)$, $X \in TM$.

Let $\nabla u$ be the gradient of $u$ with respect to the metric $h$, i.e., $h(\nabla u, X) = Xu$ for any $X \in \Gamma(TM)$. Set $\nabla_H u = \pi_H \nabla u$, where $\pi_H$ is the orthogonal projection to $HM$. And $\partial_b u$ is the orthogonal projection of $\nabla_H u$ to $T^{(1,0)}M$. Our main result is as follows.

\textbf{Theorem 1.1.} On a $(2n+1)$-dimensional contact Riemannian manifold, $n \geq 2$, we have the contact Riemannian Bochner-type formula

$$\Delta_b(\|\partial_b u\|^2) = 2\|\nabla^2 u\|^2 - 4\nabla^2 u(T, J\nabla_H u) - 2n Tor(\partial_b u, \partial_b u) + 2Ric(\partial_b u, \partial_b u)$$

$$+ h\left(\nabla_H u, \nabla_H(\Delta_b u)\right) + Q_1(\partial_b u, \partial_b u) - \frac{1}{2}Q_2(\partial_b u, \partial_b u), \quad (1.5)$$

for any $u \in C^\infty_0(M)$.

\textbf{Theorem 1.2.} Suppose that on a compact $(2n+1)$-dimensional contact Riemannian manifold, $n \geq 2$, there exists some positive constant $\kappa$ such that

$$Ric(X, X) - (n+1)Tor(X, X) + \frac{1}{2}Q_1(X, X) - \frac{2n+7}{8(n-1)}Q_2(X, X) + \frac{3}{2(n-1)}Q_3(X, X) \geq \kappa h(X, X), \quad (1.6)$$
for any $X \in T^{(1,0)}M$, where Ric is the Ricci tensor. Then the first nonzero eigenvalue $\lambda_1$ of $\Delta_b$ satisfies

$$\lambda_1 \geq \frac{\kappa n}{n + 1}.$$ 

Note that the coefficients of $Tor$ in (1.5) and (1.6) are different from that (1.1) by a factor $-2$ (cf. [10] and [19]). This is because that in our definition (1.2), $d\theta(X,Y) = h(X,JY)$, while in pseudohermitian case, people usually use $d\theta(X,Y) = 2h(JX,Y) = -2h(X,JY)$. When $Q \equiv 0$, Theorem 1.1 and 1.2 coincide with the CR Bochner-type formula and CR Lichnerowicz theorem, respectively (see e.g. [6], [10], [12], [13], [19]). It is quite interesting to characterize the equality case of (1.6).

In Section 2, we introduce some basic preliminaries, including the TWT connection, the torsion tensor, the curvature tensor and the Tanno tensor. If we choose an orthonormal $T^{(1,0)}M$ frame, there are some simpler relations for the connection coefficients, the Tanno tensor and the structure equations, which will make our calculation easier.

When given an orthonormal $T^{(1,0)}M$ frame, we have $\Gamma_{\alpha\beta} = -\frac{i}{2}Q_{\beta\alpha}$, which vanish in the pseudohermitian case. But in the general case, it may not always vanish. Therefore there exists extra terms involving such connection coefficients in our formulae, e.g. the Bochner-type formula and various integral identities, which will make our calculation more complicated than the pseudohermitian case. The main difficulties of generalizing results to the contact Riemannian case come from handling such extra terms.

In section 3, we introduce the second- and third-order covariant derivatives and their commutation formulae with respect to an orthonormal $T^{(1,0)}M$ frame. In Section 4, we prove the Bochner-type formula on a contact Riemannian manifold. This formula differs from the pseudohermitian case by terms involving the Tanno tensor. And it coincides with the CR Bochner-type formula (cf. e.g. Proposition 9.5 in [10] or Theorem 6 in [19]) when the almost complex structure $J$ is integrable. Similarly to pseudohermitian case, the term $\nabla^2 u(T,J\nabla_H u)$ in the Bochner-type formula can be controlled by using two integral identities. But here, in one identity, we have to use another identity to handle extra terms depending on the Tanno tensor $Q$. It is done in Section 5. In Section 6, with the preparation above, we prove the main Theorem 1.2.

2. Connection coefficients, torsions and curvatures on contact Riemannian manifolds

2.1. TWT connection, the Tanno tensor and the orthonormal $T^{(1,0)}M$ frame.

**Proposition 2.1.** (cf. (7)-(10) in [4]) On a contact Riemannian manifold $(M,\theta,h,J)$, there exists a unique linear connection such that

$$\nabla\theta = 0, \quad \nabla T = 0,$$

$$\nabla h = 0,$$

$$\tau(X,Y) = 2d\theta(X,Y)T, \quad X,Y \in \Gamma(HM),$$

$$\tau(T,JZ) = -J\tau(T,Z), \quad Z \in \Gamma(TM),$$

where $\tau$ is the torsion of $\nabla$.

$\nabla$ is called the TWT connection. The Tanno tensor $Q$ (cf. (10) in [10]) is defined as

$$Q(X,Y) := (\nabla_Y J)X, \quad \text{for } X,Y \in \Gamma(TM).$$
We extend $h$, $J$ and $\nabla$ to the complexified tangent bundle by $\mathbb{C}$-linear extension:
\[
h(X_1 + iY_1, X_2 + iY_2) := h(X_1, X_2) - h(Y_1, Y_2) + i(h(X_1, Y_2) + h(X_2, Y_1)),
\]
\[
J(X_1 + iY_1) := JX_1 + iJY_1,
\]
\[
\nabla_{(X_1+iY_1)}(X_2+iY_2) := \nabla_X X_2 - \nabla_Y Y_2 + i(\nabla_X Y_2 + \nabla_Y X_2).
\]
for any $Z_j = X_j + iY_j \in \mathcal{C}TM$, $j = 1, 2$.

**Proposition 2.2.** Let $W_0 := T$, the Reeb vector. We can choose a local $T^{(1,0)}M$-frame $\{W_j\} = \{W_a, W_\alpha\} \subseteq \{W_a, W_\alpha, T\}$ with $W_\alpha \in T^{(1,0)}M$, $W_\bar{\alpha} = \overline{W_a} \in T^{(0,1)}M$ on a neighborhood $U$ such that
\[
h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}; \quad h_{\bar{\alpha}\beta} = \bar{\delta}_{\alpha\beta}; \quad h_{\alpha\bar{\beta}} = 0.
\]
We call this frame an orthonormal $T^{(1,0)}M$-frame.

**Proof.** Note that (1.2) leads to
\[
JT = 0, \quad \theta(JX) = 0,
\]
\[
h(X, Y) = h(JX, JY) + \theta(X)\theta(Y), \quad d\theta(X, JY) = -d\theta(JX, Y),
\]
for any $X, Y \in TM$ (cf. p. 351 in \[28\]). Choose a vector field $X_1$ in $\Gamma(TM)$ such that $h(X_1, X_1) = \frac{1}{2}$ and let $X_{n+1} := JX_1$. Then $h(X_1, X_{n+1}) = h(X_1, JX_1) = d\theta(X_1, X_1) = 0$, i.e. $X_{n+1}$ is automatically orthogonal to $X_1$, and by third identity in (2.3), we get $h(X_{n+1}, X_{n+1}) = h(JX_1, JX_1) = h(X_1, X_1) = \frac{1}{2}$. We choose $X_2$ orthogonal to $\text{span}\{X_1, JX_1\}$, and define $X_{n+2} := JX_2$. Repeating the procedure, we find a local orthogonal basis $X_1, \cdots, X_{2n}$ with $h(X_a, X_b) = \frac{1}{2}\delta_{ab}$ and $JX_\alpha = X_{\alpha+n}$. Now define
\[
W_a := X_a - iX_{a+n}, \quad W_\bar{\alpha} := \overline{W_a}.
\]
It is direct to see that $JW_\alpha = iW_\alpha$ and $JW_\bar{\alpha} = -iW_\bar{\alpha}$. Namely, $W_\alpha \in T^{(1,0)}M$ and $W_\bar{\alpha} \in T^{(0,1)}M$. Then by Remark 2.1 for the complex extension we get $h(W_\alpha, W_\beta) = h(X_\alpha - iX_{\alpha+n}, X_\beta - iX_{\beta+n}) = 0$, $h_{\alpha\bar{\beta}} = 0$ and $h(W_\alpha, W_\bar{\beta}) = \delta_{\alpha\bar{\beta}}$, $h_{\bar{\alpha}\beta} = \bar{\delta}_{\alpha\beta} = \delta_{\alpha\bar{\beta}}$. \hfill $\square$

**Remark 2.1.** (1) For the multi-index, we adopt the following index conventions in this paper.
\[
\alpha, \beta, \gamma, \rho, \lambda, \mu, \cdots \in \{1, \cdots, n\}, \quad a, b, c, d, e, \cdots \in \{1, 2, \cdots, 2n\},
\]
\[
j, k, l, r, s, \cdots \in \{0, 1, \cdots, 2n\}, \quad \bar{\alpha} = \alpha + n.
\]
(2) In this paper, the Einstein summation convention will be used. Moreover, if indices $\alpha$ and $\bar{\alpha}$ both appear in low (or upper) indices, then the index $\alpha$ will be taken summation, e.g. $h_{\alpha\bar{\alpha}} = \sum_\alpha h_{\alpha\alpha}$.

From now on, we choose a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$. In particular, by (1.2), $h(T, W_a) = h(W_a, T) = \theta(W_a) = 0$ and $h(T, T) = \theta(T) = 1$. We denote $h_{ab} = h(W_a, W_b)$ and use $h_{ab}$ and its inverse matrix to lower and raise indices. Let $\{\theta^a, \theta^\beta, \theta\}$ denote the dual coframe to $\{W_a, W_\alpha, T\}$, i.e., $\theta^\beta(W_a) = \delta_{\alpha\beta}, \theta^\beta(W_\bar{\alpha}) = \theta^\beta(T) = 0, \theta^\beta := \overline{\theta^\beta}$, and $\theta(W_\alpha) = \theta(W_{\bar{\alpha}}) = 0$, $\theta(T) = 1$. Set $\theta^0 := \theta$. The connection 1-form with respect to $\{W_j\}$ is given by $\nabla W_j = \omega^k_j \otimes W_k$, and set $\omega^k_j := \Gamma^k_{ij}\theta^i$, i.e. $\nabla W_j = \Gamma^k_{ij} W_k$. By (2.1), we get $\theta(\nabla X) = h(\nabla X, T) = d(h(X, T)) - h(X, \nabla T) = 0$ for any $X \in TM$, namely $\Gamma^0_{ij} = 0$. And $\nabla T = 0$ implies $\Gamma^0_{0j} = 0$.

By the dual argument, we have
\[
\nabla W_j \theta^k = -\Gamma^k_{ij} \theta^j.
\]
And for any \((r,s)\)-tensor \(\varphi\) with components \(\varphi^{j_{1}\cdots j_{r}}_{k_{1}\cdots k_{s}} = \varphi(\theta^{k_{1}}, \cdots, \theta^{k_{r}}, W_{j_{1}}, \cdots, W_{j_{s}})\), covariant derivatives of \(\varphi\) are given by

\[
\varphi^{j_{1}\cdots j_{r}}_{k_{1}\cdots k_{s}, i} = W_{i}(\varphi^{j_{1}\cdots j_{r}}_{k_{1}\cdots k_{s}}) + \sum_{t=1}^{r} \Gamma^{k_{t}}_{ij_{t}} \varphi^{j_{1}\cdots j_{t-1}j_{t+1}\cdots j_{r}}_{k_{1}\cdots k_{s-1}k_{s+1}\cdots k_{s}} - \sum_{t=1}^{s} \Gamma^{j_{t}}_{ik_{t}} \varphi^{j_{1}\cdots j_{r}}_{k_{1}\cdots k_{t-1}k_{t+1}\cdots k_{s}}.
\]

(2.5)

Denote the components of the almost complex structure \(J\) by \(J_{j}^{l}\). We write \(Q(W_{j}, W_{k}) = (\nabla_{W_{j}} J)W_{j} = Q_{j,k}W_{j}\). Equivalently, \(Q_{j,k} : J_{j}^{l}\). Applying (2.5), we have

\[
Q_{j,k}^{l} = W_{k}J_{j}^{l} + \Gamma_{ks}^{l}J_{s}^{s} - \Gamma_{kj}^{s}J_{s}^{l},
\]

(2.6)

Proposition 2.3. (cf. (16)-(18) in [3]) With respect to a local \(T^{(1,0)}\)-frame \(\{W_{j}\}\), the components of tensor \(Q\) has the following property:

\[
Q_{j\alpha}^{\gamma} = 0, \quad \Gamma_{\alpha\beta}^{\gamma} = 0, \quad Q_{j\beta}^{\gamma} = 0, \quad Q_{\gamma\beta}^{\gamma} = 0, \quad \Gamma_{\alpha\beta}^{\gamma} = -\frac{i}{2}Q_{j\beta}^{\gamma},
\]

(2.7)

In particular, only the components \(\Gamma_{j\beta}^{\gamma}\) of tensor \(Q\) are non-vanishing. In (2.7), \(Q_{ij}^{0} = 0\) follows from \(Q_{j}^{0}W_{k} = Q(T, W_{j}) = (\nabla_{W_{j}} J)T = \nabla_{W_{j}} (JT) - J\nabla_{W_{j}} T \equiv 0\) by (2.1) and (2.23) for any \(W_{j}\). And \(Q_{i\alpha}^{0} = 0\) follows from setting \(X = W_{i}, Y = T, Z = W_{j}\) in identity (cf. (15) in [1]):

\[
2h(Q(X, Y), Z) = h(N^{(1)}(X, Z) - \theta(X)N^{(1)}(T, Z) + \theta(Z)N^{(1)}(X, T), JY),
\]

(2.8)

for any \(X, Y, Z \in T M\), where

\[
N^{(1)} = [J, J] + 2(d\theta) \otimes T, \quad [J, J](X, Y) = J^{2}[X, Y] + [JX, JY] - J[JX, Y] - J[X, JY],
\]

to get \(Q_{i\alpha}^{0}h_{kl} = 0\) by \(JT \equiv 0\) in (2.5). For \(Q_{ij}^{0} = 0\), since we already have \(Q_{i\alpha}^{0} = 0\), it remains to prove \(Q_{\alpha\beta}^{0} = 0\). This follows from \(Q_{\alpha\beta}^{0} = \theta\left(Q(W_{\alpha}, W_{\beta})\right) = \theta\left((\nabla_{W_{\beta}} J)W_{\alpha}\right) = h(T, (\nabla_{W_{\alpha}} J)W_{\alpha}) - h(T, J\nabla_{W_{\alpha}} W_{\beta}) = h\left(W_{\alpha}, J\nabla_{W_{\beta}} W_{\alpha}\right) = 0\) by \(W_{\alpha}, W_{\beta}\) horizontal.

Remark 2.2. In pseudohermitian case, \(\Gamma_{\alpha\beta}^{\gamma} = 0\) by the Tanaka-Webster connection preserving \(T^{(1,0)}M\). But in general case, \(\Gamma_{\alpha\beta}^{\gamma} = -\frac{i}{2}Q_{\beta\alpha}^{\gamma}\) may not vanish.

Recall that only the components \(Q_{\alpha\beta}^{\gamma}\) of tensor \(Q\) are non-vanishing. So by definition (1.3), with respect to a local orthonormal \(T^{(1,0)}\)-frame \(\{W_{j}\}\), we have

\[
Q_{1}(X, X) = -2Re\left(i\text{-trace}\{W_{j} \rightarrow \nabla W_{j}Q(X, X)\}\right) = -2Re\left(iQ_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta}\right) = -2Re\left(iQ_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta}\right) = i\left(Q_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta} - Q_{\beta\alpha\gamma}^{j}X^{\beta}X^{\alpha}\right),
\]

(2.9)

\[
Q_{2}(X, X) = h_{ij}h_{kl}Q_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta} = h_{ij}h_{kl}Q_{\gamma\alpha\beta}^{j}X^{\gamma}X^{\alpha}X^{\beta} = h_{ij}h_{kl}Q_{\gamma\alpha\beta}^{j}X^{\gamma}X^{\alpha}X^{\beta} = Q_{\alpha\beta\gamma}^{j}Q_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta},
\]

\[
Q_{3}(X, X) = \text{trace}\{W_{j} \rightarrow Q_{W_{\alpha}}W_{\beta}W_{\gamma}(W_{j})X^{\alpha}X^{\beta}\} = \text{trace}\{W_{j} \rightarrow Q_{\beta\gamma}^{j}Q_{\alpha\beta\gamma}^{j}(W_{j})X^{\alpha}X^{\beta}\} = Q_{\alpha\beta\gamma}^{j}Q_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta} = Q_{\alpha\beta\gamma}^{j}Q_{\alpha\beta\gamma}^{j}X^{\alpha}X^{\beta},
\]
for any \( X = X^\alpha W_\alpha \in T^{(1,0)} M \). According to (2.6) and \( \Gamma^\gamma_{\alpha\beta} = 0 \) in (2.7), we get the components
\[
Q^\gamma_{\alpha\beta,\rho} = W_\rho Q^\gamma_{\alpha\beta} - \Gamma^\gamma_{\rho\alpha} Q^\gamma_{\beta\epsilon} - \Gamma^\gamma_{\rho\beta} Q^\gamma_{\alpha\epsilon} + \Gamma^\gamma_{\rho\epsilon} Q^\gamma_{\alpha\beta} = W_\rho Q^\gamma_{\alpha\beta} - \Gamma^\mu_{\rho\alpha} Q^\gamma_{\beta\mu} - \Gamma^\mu_{\rho\beta} Q^\gamma_{\mu\alpha} + \Gamma^\gamma_{\rho\mu} Q^\gamma_{\alpha\beta},
\] (2.10)
of tensor \( \nabla Q \).

**Proposition 2.4.** With respect to a local orthonormal \( T^{(1,0)} M \)-frame \( \{ W_j \} \), we have
\[
\Gamma^\gamma_{\alpha\beta} = -\Gamma^\beta_{\alpha\gamma}, \quad \Gamma^\gamma_{\alpha\beta} = -\Gamma^\beta_{\alpha\gamma},
\]
and their conjugation.

**Proof.** By (2.7) and Proposition 2.2, we have
\[
\Gamma^\gamma_{\alpha\beta} = \Gamma^\alpha_{\beta\gamma} = h(\nabla_{W_\alpha} W_\beta, W_\gamma) = W_\alpha(h_{\beta\gamma}) - h(W_\beta, \nabla_{W_\alpha} W_\gamma) = -h_{\beta\gamma} \Gamma^\alpha_{\alpha\gamma} = -\Gamma^\gamma_{\alpha\beta}.
\]
\( \Gamma^\gamma_{\alpha\beta} = -\Gamma^\beta_{\alpha\gamma} \) follows similarly. Then we get
\[
Q^\gamma_{\alpha\beta} = 2i\Gamma^\gamma_{\alpha\beta} = -2i\Gamma^\gamma_{\alpha\beta} = -Q^\alpha_{\gamma\beta} \gamma\beta\gamma \) by (2.7). The fourth identity in (2.11) follows from this identity.

For the last identity in (2.11), setting \( X = W_\alpha, Y = W_\beta, Z = W_\gamma \) in (2.8), we get
\[
2h(Q(W_\alpha, W_\beta), W_\gamma) = h([J, J](W_\alpha, W_\gamma), JW_\beta) = i h(J^2[W_\alpha, W_\gamma] + [JW_\alpha, JW_\gamma] - J[JW_\alpha, W_\gamma], JW_\beta) = -2ih([W_\alpha, W_\gamma], W_\beta) + 2h(J[W_\alpha, W_\gamma], W_\beta) = -4ih([W_\alpha, W_\gamma], W_\beta).
\]
By the definition of the torsion tensor and (2.7), the \( T^{(0,1)} M \)-components of \( [W_\alpha, W_\gamma] \) is given by
\[
[W_\alpha, W_\gamma] = \nabla_{W_\alpha} W_\gamma - \nabla_{W_\beta} W_\alpha - \tau(W_\alpha, W_\gamma) = \Gamma^\rho_{\alpha\gamma} W_\rho - \Gamma^\rho_{\gamma\alpha} W_\rho = -\frac{i}{2} Q^\rho_{\gamma\alpha} W_\rho + \frac{i}{2} Q^\rho_{\alpha\gamma} W_\rho \mod W_\rho, T. \tag{2.12}
\]
Therefore, we get
\[
2Q^\rho_{\gamma\alpha} h_\gamma = 2h(Q(W_\alpha, W_\beta), W_\gamma) = -4ih([W_\alpha, W_\gamma], W_\beta) = -2Q^\rho_{\gamma\alpha} h_\beta + 2Q^\rho_{\alpha\gamma} h_\beta.
\]
The last identity of (2.11) holds. \( \square \)

2.2. The Webster torsion, the curvature tensor and the structure equations.

**Lemma 2.1.** (cf. Lemma 1 in [4]) The Webster torsion has following properties:
\[
\tau(T) = 0; \quad \tau(T_{(1,0)} M \subseteq T_{(0,1)} M; \quad \tau(T_{(1,0)} M \subseteq T_{(1,0)} M).
\]  

By Lemma 2.1, we can write \( \tau(W_\alpha) = A^\alpha_{\beta} W_\beta \). Set \( \tau^\alpha := A^\alpha_{\beta} \tau^\beta \). So by (1.4), with respect to \( \{ W_j \} \), we have
\[
Tor(X, X) = 2Re \left( iA_{\alpha\beta} X^\alpha X^\beta \right) = iA_{\alpha\beta} X^\alpha X^\beta - iA_{\alpha\beta} X^\alpha X^\beta \tag{2.13}
\]
for any \( X = X^\alpha W_\alpha \in T^{(1,0)} M \).
The components $R^b_{\alpha\beta\gamma\delta}$ of the curvature tensor $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ is given by $R(W_k, W_l)W_j = R^b_{\alpha\beta\gamma\delta}W_k$. The Ricci tensor is given by

$$Ric(Y, Z) = \text{trace}\{X \rightarrow R(X, Z)Y\},$$

for any $X, Y, Z \in TM$ (cf. p. 299 in [4]). And the scalar curvature is $R = \text{trace}(Ric)$. With respect to a $T^{(1,0)}M$-frame, $R_{\alpha\beta} = R^0_{\alpha\gamma\beta\delta}R_{\gamma\delta}$ (cf. (53) in [4]). The scalar curvature is $R = h^{\alpha\beta}R_{\alpha\beta}$.

**Proposition 2.5.** (cf. (13), (14) and (39) in [4]) With respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, we have the following structure equations:

$$d\theta = -2i\alpha^{\alpha\beta}\theta^{\beta} \wedge \bar{\theta}^{\alpha} = -2i\theta^{\alpha} \wedge \bar{\theta}^{\alpha},$$

$$d\theta^{\alpha} = \theta^{\beta} \wedge \omega^{\alpha}_{\beta} + \theta \wedge \tau^{\alpha} = \theta^{\beta} \wedge \omega^{\alpha}_{\beta} + A^{\beta}_{\alpha} \theta \wedge \bar{\theta}^{\alpha},$$

$$d\omega^{\alpha}_{\beta} - \omega^{\alpha\beta} \wedge \omega^{\beta}_{\gamma} = R^{a}_{\gamma\beta\alpha} \theta^{\gamma} \wedge \bar{\theta}^{\alpha} + \frac{1}{2} R_{\alpha\beta\gamma\delta} \theta^{\gamma} \wedge \bar{\theta}^{\delta} + \frac{1}{2} R^{a}_{\gamma\beta\alpha} \theta^{\gamma} \wedge \bar{\theta}^{\delta} + R_{\alpha\beta\gamma\delta} \theta^{\gamma} \wedge \bar{\theta}^{\delta} - R^{a}_{\alpha\beta\gamma\delta} \theta \wedge \bar{\theta}^{\delta} \wedge \theta^{\gamma},$$

$$R(X, Y)W_a = 2(d\omega^{b}_{\alpha} - \omega^{c}_{\alpha} \wedge \omega^{b}_{\delta})(X, Y)W_b.$$

(2.14)

Here following [4] we use the following definition for exterior product and exterior derivatives

$$\phi \wedge \psi(X, Y) = \frac{1}{2} \left( \phi(X)\psi(Y) - \psi(X)\phi(Y) \right),$$

(2.15)

$$2(d\phi)(X, Y) = X(\phi(Y)) - Y(\phi(X)) - \phi([X, Y]) = (\nabla_X \phi)Y - (\nabla_Y \phi)X + \phi(\tau(X, Y)),$$

for any 1-form $\phi$ and $\psi$. The second identity in (2.14) follows from the orthonormality of $\{W_a\}$.

**Corollary 2.1.** With respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, set $J_{ab} = h_{ac}J^c_b$. We have

$$R^{b}_{a\beta\gamma\delta}W_c = W^{b}_{\beta\gamma\delta} - W^{\alpha}_{\beta\gamma\delta}W_{ca} - \Gamma^{\epsilon}_{\beta\gamma\epsilon} \Gamma^{b}_{\epsilon\delta} - \Gamma^{\epsilon}_{\beta\gamma} \Gamma^{b}_{\epsilon\delta} + \Gamma^{\epsilon}_{\beta\epsilon} \Gamma^{b}_{\gamma\delta} + \Gamma^{\epsilon}_{\beta\gamma} \Gamma^{b}_{\epsilon\delta} + 2\Gamma^{b}_{\epsilon\delta}J_{cd}.\hspace{1cm} (2.16)$$

**Proof.** Note that $h(W_a, JW_b) = h(W_a, J^c_b W_c) = h_{ac}J^c_b = J_{ab}$. By (2.14) and the last identity in (2.14), we have

$$R^{b}_{a\beta\gamma\delta} = 2(d\omega^{b}_{\alpha})(W_c, W_d) - 2\omega_{\alpha} \wedge \omega^{b}_{\delta}(W_c, W_d)$$

$$= (\nabla_{W_c}\omega^{b}_{\alpha})(W_d) - (\nabla_{W_d}\omega^{b}_{\alpha})(W_c) + \omega^{b}_{\alpha}(\tau(W_c, W_d)) - \Gamma^{\epsilon}_{\beta\gamma\epsilon} \Gamma^{b}_{\epsilon\delta} + \Gamma^{\epsilon}_{\beta\epsilon} \Gamma^{b}_{\gamma\delta} + 2\Gamma^{b}_{\epsilon\delta}J_{cd}.$$

(2.17)

**Proposition 2.6.** For the components of the curvature tensor, we have the following commutation relations:

$$R_{\alpha\beta\gamma\delta} = -R_{\alpha\delta\beta\gamma}, \hspace{0.5cm} R_{\alpha\beta\gamma\delta} = -R_{\delta\gamma\alpha\beta}, \hspace{0.5cm} R_{\alpha\beta\gamma\delta} = R_{\alpha\beta\delta\gamma},$$

and their conjugation with respect to a local orthonormal $T^{(1,0)}M$-frame $\{W_j\}$.
Proof. The first identity in (2.17) follows directly by the definition of the curvature tensor. For the last one in (2.17), we refer to Corollary 1 in [4]. For the second identity in (2.17), note that $\nabla h = 0$ implies $Xh(Y,Z) = h(\nabla_XY,Z) + h(Y,\nabla_XZ)$ and $h_{ab}$ are constants. Then

$$R_{\alpha\beta\gamma\delta} = h(\nabla_{W_\alpha} \nabla_{W_\beta} W_\gamma - \nabla_{W_\mu} \nabla_{W_\gamma} W_\alpha - \nabla_{[W_\alpha,W_\beta]} W_\alpha, W_\beta)$$

$$= W_\gamma \left( h(\nabla_{W_\mu} W_\alpha, W_\beta) \right) - h(\nabla_{W_\mu} W_\alpha, \nabla_{W_\gamma} W_\beta) - W_\mu \left( h(\nabla_{W_\gamma} W_\alpha, W_\beta) \right)$$

$$+ h(\nabla_{W_\mu} W_\alpha, \nabla_{W_\gamma} W_\beta) - [W_\gamma, W_\mu](h(W_\alpha, W_\beta)) + h(W_\alpha, \nabla_{[W_\gamma,W_\mu]} W_\beta)$$

$$= W_\gamma W_\mu (h_{\alpha\beta}) - W_\gamma(h(W_\alpha, \nabla_{W_\gamma} W_\beta)) - W_\mu(h(W_\alpha, \nabla_{W_\gamma} W_\beta))$$

$$+ h(W_\alpha, \nabla_{W_\mu} \nabla_{W_\gamma} W_\beta) - W_\mu W_\gamma (h_{\alpha\beta}) + W_\mu(h(W_\alpha, \nabla_{W_\gamma} W_\beta))$$

$$+ W_\gamma (h(W_\alpha, \nabla_{W_\mu} W_\beta)) - h(W_\alpha, \nabla_{W_\gamma} \nabla_{W_\mu} W_\beta) - [W_\gamma, W_\mu](h_{\alpha\beta}) + h(W_\alpha, \nabla_{[W_\gamma,W_\mu]} W_\beta)$$

$$= h(W_\alpha, \nabla_{W_\mu} \nabla_{W_\gamma} W_\beta) - h(W_\alpha, \nabla_{W_\gamma} \nabla_{W_\mu} W_\beta) + h(W_\alpha, \nabla_{[W_\gamma,W_\mu]} W_\beta) = -R_{\beta\alpha\gamma\delta}.$$

Remark 2.3. By Remark 2.1 for the complex extension, it’s easy to see that under the complex conjugation, the Riemannian metric $h$, the almost complex structure $J$, the TWT connection $\nabla$, the torsion tensor $T$, the curvature tensor $R$ and the Tanno tensor $Q$ are preserved, i.e.,

$$h(Z_1, Z_2) = h(Z_1^\perp, Z_2^\perp), \quad JZ_1 = JZ_1^\perp, \quad \nabla Z_1^\perp Z_2 = \nabla Z_1 Z_2^\perp,$$

$$\tau(Z_1, Z_2) = \tau(Z_1^\perp, Z_2^\perp), \quad R(Z_1, Z_2)Z_3 = R(Z_1^\perp, Z_2^\perp)Z_3, \quad Q(Z_1, Z_2) = Q(Z_1, Z_2),$$

for any $Z_1, Z_2, Z_3 \in CHM$. The complex conjugation can be reflected in the indices of the components of $\omega^\alpha_{\beta a}, h_{ab}, J_{\alpha a}, A_{ab}, R_{abcd}$ and their covariant derivatives, e.g.,

$$\overline{\omega}^\beta_{\alpha a} = \omega^\beta_{\alpha a}, \quad \overline{J}_{\alpha a} = J_{\alpha a}, \quad \overline{h}_{\alpha\beta} = h_{\bar{\alpha}\bar{\beta}}.$$

3. The second- and third-order covariant derivatives and their commutation formulae

3.1. The second- and third-order covariant derivatives. The second-order covariant derivative of $u$ is defined as

$$\nabla^2 u(X,Y) := X(Yu) - (\nabla_X Y)u, \quad u_{jk} := \nabla^2 u(W_j, W_k),$$

for any vector fields $X, Y$, and the third-order covariant derivative of $u$ is defined as

$$\nabla^3 u(X, Y, Z) = (\nabla_X \nabla^2 u)(Y, Z) = X(\nabla^2 u(Y, Z)) - \nabla^2 u(\nabla_X Y, Z) - \nabla^2 u(Y, \nabla_X Z),$$

$$u_{jkl} := \nabla^3 u(W_j, W_k, W_l).$$

for any vector fields $X, Y, Z$. By (3.1), for the second-order covariant derivative, we have

$$u_{jk} = \nabla^2 u(W_j, W_k) = W_j W_k u - (\nabla_{W_j} W_k) u = W_j(u_k) - W_k(u_j) - \Gamma_{jk}^l u_l.$$

In particular, by the vanishing of connection coefficients in (2.7) we get

$$u_{\alpha \lambda} = \nabla^2 u(W_\alpha, W_\lambda) = W_\alpha (u_\lambda) - \Gamma_{\alpha\lambda}^\beta u_\beta - \Gamma_{\alpha\lambda}^\gamma u_\gamma = W_\alpha (u_\lambda) - \Gamma_{\alpha\lambda}^\beta u_\beta + \frac{i}{2} Q_{\alpha\lambda}^\beta u_\beta,$$

$$u_{\overline{\alpha} \overline{\lambda}} = W_\alpha (u_\lambda) - \Gamma_{\alpha\lambda}^\beta u_\beta,$$

$$u_{\alpha 0} = W_\alpha (u_0), \quad u_{0\alpha} = T(u_\alpha) - \Gamma_{0\alpha}^\beta u_\beta.$$
In the following, the vanishing of connection coefficients in (2.7), especially,
\[ \Gamma_{\alpha\beta}^\gamma = \Gamma_{\alpha\beta}^\gamma = 0, \]
will be used frequently. By (3.2), for the third-order covariant derivative, we have
\[ u_{abc} = W_a(u_{bc}) - \Gamma^d_{ab}u_{dc} - \Gamma^d_{ac}u_{bd}. \]
In particular, by (2.7), we have
\[ \begin{align*}
u_{\alpha\beta\gamma} &= \Gamma_{\alpha\beta}^\mu u_{\mu\gamma} - \Gamma_{\alpha\beta}^\mu u_{\mu\gamma} - \Gamma_{\alpha\beta}^\mu u_{\mu\gamma}, \\
u_{\alpha\beta\gamma} &= \Gamma_{\alpha\beta}^\mu u_{\mu\gamma} - \Gamma_{\alpha\beta}^\mu u_{\mu\gamma}, \\
u_{\alpha\beta\gamma} &= \Gamma_{\alpha\beta}^\mu u_{\mu\gamma} - \Gamma_{\alpha\beta}^\mu u_{\mu\gamma} - \Gamma_{\alpha\beta}^\mu u_{\mu\gamma}.
\end{align*} \tag{3.4} \]

**Remark 3.1.** (1) In (3.3) and (3.4), we have used \( \Gamma_{0b}^0 = 0, \) \( \Gamma_{\alpha\beta}^\gamma = 0 \) and \( \Gamma_{\alpha\beta}^\gamma = -\frac{1}{2}Q_{\alpha\beta}^\gamma \) repeatedly.

(2) The complex conjugation can also be reflected in the indices of the components of any-order covariant derivative of a real function \( u, \) e.g. \( \overline{u_{\alpha\beta}} = u_{\bar{\alpha}\bar{\beta}}, \quad \overline{u_{\alpha\beta\gamma}} = u_{\bar{\alpha}\bar{\beta}\bar{\gamma}}. \)

**3.2. The sub-Laplacian.** On a contact Riemannian manifold \( M, \) with respect to a local \( T^{(1,0)}M \)-frame \( \{W_j\}, \) we define the sub-Laplacian operator as
\[ \Delta_b u = u_{\alpha\alpha} + u_{\bar{\alpha}\bar{\alpha}}, \]
for \( u \in C_0^\infty(M). \) Furthermore, if \( \{W_j\} \) is an orthonormal \( T^{(1,0)}M \)-frame, we have
\[ \Delta_b u = u_{\alpha\alpha} + u_{\bar{\alpha}\bar{\alpha}}. \tag{3.5} \]
For any functions \( u \in C_0^\infty(M) \) and \( v \in C^\infty(M), \) we define the \( L^2 \) inner product \( (\cdot, \cdot) \) as
\[ (u, v) = \int_M uv dV. \tag{3.6} \]
For any vector field \( X, X^* \) is called the formal adjoint of \( X \) if \( (Xu, v) = (u, X^*v) \) for \( u, v \in C_0^\infty(M). \) And \( \Delta_b \) is hypoelliptic and by a result of [21] has a discrete spectrum
\[ 0 < \lambda_1 < \lambda_2 < \cdots < +\infty. \]

**Lemma 3.1.** We have
\[ W_{\alpha}^* = -W_{\bar{\alpha}} + \Gamma_{\beta\bar{\beta}}^\alpha, \quad W_{\bar{\alpha}}^* = -W_{\alpha} + \Gamma_{\bar{\beta}\beta}^\bar{\alpha}, \quad (iT)^* = iT. \tag{3.7} \]

**Proof.** By (2.14), we have
\[ \begin{align*}
d\theta^n &= (-2i\theta^n \wedge \bar{\theta}^n) = (-2)^n i^n n! \theta^1 \wedge \theta^1 \wedge \cdots \wedge \theta^n \wedge \bar{\theta}^n \\
&= (-2)^n i^n n! (-1)^{n(n-1)/2} \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^1 \wedge \cdots \wedge \theta^n \\
&= (-2)^n i^n n! \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^1 \wedge \cdots \wedge \theta^n.
\end{align*} \]
So the volume form is
\[ dV := \theta \wedge d\theta^n = (-2)^n i^n n! \theta \wedge \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^1 \wedge \cdots \wedge \theta^n. \tag{3.8} \]
For any vector field $X$ and $u \in C_0^\infty(M)$, we get
\[
\int_M X u dV = \int_M v du \wedge i_X dV = -\int_M u dv \wedge i_X dV - \int_M u d(\iota_X dV) + \int_M d(u \iota_X dV) = -\int_M uX dV - \int_M u d(\iota_X dV),
\]
by Stokes’ formula and $0 = \int_M i_X (v du \wedge dV) = \int_M v Xu dV - \int_M v du \wedge i_X dV$. It follows from the structure equation (2.14) that
\[
d\theta^\beta = \theta^\gamma \wedge \omega^\beta_\gamma + \theta^\xi \wedge \omega^\beta_\xi = \Gamma^\beta_\mu \theta^\gamma \wedge \theta^\mu + \Gamma^\beta_\nu \theta^\gamma \wedge \theta^\nu + \Gamma^\beta_\gamma \theta^\xi \wedge \theta^\xi, \quad \text{mod } \theta. \quad (3.10)
\]
Applying (3.8) and (3.10), we get
\[
d(iw_\alpha dV) = (-2)^\alpha n^2 n! d\left( (-1)^\alpha \theta \wedge \theta^1 \wedge \cdots \wedge \theta^n \right)
= (-1)^\alpha (-2)^n n^2 n! \left( \sum_{\beta < \alpha} (-1)^\beta \theta \wedge \theta^1 \wedge \cdots \wedge \theta^\beta \wedge \cdots \wedge \theta^n \wedge \cdots \wedge \theta^n \right)
+ \sum_{\beta > \alpha} (-1)^{\beta - 1} \theta \wedge \theta^1 \wedge \cdots \wedge \theta^\alpha \wedge \cdots \wedge \theta^n \wedge \cdots \wedge \theta^n 
+ \sum_{\beta = 1}^{n-\alpha} (-1)^{n-\beta - 1} \theta \wedge \theta^1 \wedge \cdots \wedge \theta^\alpha \wedge \cdots \wedge \theta^n \wedge \cdots \wedge \theta^n 
= (-2)^n n^2 n! \left( -\Gamma^\beta_\alpha + \Gamma^\beta_\beta - \Gamma^\beta_\alpha + \Gamma^\beta_\alpha \right) \theta \wedge \theta^1 \wedge \cdots \wedge \theta^n 
= \left( -\Gamma^\beta_\alpha + \Gamma^\beta_\alpha - \Gamma^\beta_\alpha + \Gamma^\beta_\alpha \right) dV = \Gamma^\beta_\alpha dV.
\]
The last identity holds because $\Gamma^\beta_\alpha + \Gamma^\beta_\alpha = 0$ by (2.11) and $\Gamma^\beta_\alpha = 0$ by (2.7). For any $u \in C_0^\infty(M)$ and $v \in C_0^\infty(M)$, apply (3.9) with $X = W_\alpha$ to get
\[
(W_\alpha u, v) = \int_M W_\alpha u dv = -\int_M u W_\alpha dv - \int_M u \iota v d(iW_\alpha dV) = \int_M u (-W_\alpha \bar{v} - \Gamma^\beta_\alpha \bar{v}) dV
= \int_M u (-W_\alpha \bar{v} + \Gamma^\beta_\alpha \bar{v}) dV = \left( u, \left( -W_\alpha + \Gamma^\beta_\alpha \right) \bar{v} \right), \quad (3.11)
\]
by $\Gamma^\beta_\alpha = -\Gamma^\beta_\alpha$ in (2.11). The first identity in (3.7) holds. The second identity in (3.7) follows from taking conjugation.

By the structure equation (2.14), $d\theta^\alpha$ doesn’t contain $\theta \wedge \theta^\alpha$ terms and $d\theta^\beta$ doesn’t contain $\theta \wedge \theta^\beta$ terms. So
\[
d(i_T dV) = (-2)^n n^2 n! d\left( \theta^1 \wedge \theta^2 \wedge \cdots \wedge \theta^n \wedge \theta^1 \wedge \cdots \wedge \theta^n \right) = 0. \quad (3.12)
\]
Apply (3.9) with $X = T$ to get
\[
(i_T u, v) = i \int_M T u dv = -i \int_M u T dv = (u, i_T v).
\]
$(iT)^* = iT$ follows.
Corollary 3.1. With respect to an orthonormal $T^{(1,0)}M$-frame $\{W_j\}$, we have
\[ (\Delta_b u, v) = -\sum\alpha (u_\alpha, v_\alpha) + (u_{\bar{\alpha}}, v_{\bar{\alpha}}), \]
for $u \in C^\infty(M)$ and $v \in C^\infty(M)$.

Proof. By the definition of $\Delta_b$, we get
\[ (\Delta_b u, v) = \sum\alpha,\beta (W_\alpha u_\alpha - \Gamma^\alpha_{\alpha\beta} u_\beta, v_\alpha) + \left( W_\alpha u_{\bar{\alpha}} - \tilde{\Gamma}^\alpha_{\alpha\beta} u_{\bar{\beta}}, v_{\bar{\alpha}} \right) \]
\[ = \sum\alpha,\beta (u_\alpha, W^*_\alpha v_\alpha) - \left( \Gamma^\alpha_{\alpha\beta} u_\beta, v_\alpha \right) + \left( u_{\bar{\alpha}}, W^*_\alpha v_\alpha \right) - \left( \tilde{\Gamma}^\alpha_{\alpha\beta} u_{\bar{\beta}}, v_{\bar{\alpha}} \right) \]
\[ = -\sum\alpha,\beta (u_\alpha, u_\beta) + \left( u_\alpha, \Gamma^\alpha_{\alpha\beta} v_\beta \right) - \left( \Gamma^\alpha_{\alpha\beta} u_\beta, v_\alpha \right) + \left( u_{\bar{\alpha}}, \Gamma^\alpha_{\alpha\beta} v_\beta \right) - \left( \tilde{\Gamma}^\alpha_{\alpha\beta} u_{\bar{\beta}}, v_{\bar{\alpha}} \right) \]
\[ = -\sum\alpha (u_\alpha, v_\alpha) + (u_{\bar{\alpha}}, v_{\bar{\alpha}}). \]
\[ \square \]

Corollary 3.2.
\[ (\Delta_b u)_\alpha = u_{\alpha\bar{\beta}} + u_{\bar{\alpha}\beta}. \]  

Proof. By \(3.4\) and Corollary 3.1 we get
\[ u_{\alpha\beta} + u_{\bar{\alpha}\beta} = W_\alpha (u_{\beta\bar{\beta}}) - \Gamma^\mu_{\alpha\beta} u_{\mu\bar{\beta}} - \Gamma^\mu_{\alpha\beta} u_{\bar{\beta}\mu} + W_\alpha (u_{\beta\beta}) - \Gamma^\mu_{\alpha\beta} u_{\bar{\beta}\mu} - \Gamma^\mu_{\alpha\beta} u_{\beta\mu} - \Gamma^\mu_{\bar{\alpha}\beta} u_{\bar{\beta}\mu} \]
\[ = W_\alpha (u_{\beta\bar{\beta}} + u_{\bar{\beta}\beta}) = (\Delta_b u)_\alpha, \]
by the first two identities in \(2.11\). \[\square\]

3.3. The commutation formulae.

Proposition 3.1. For the second-order covariant derivatives of a function $u$, we have the following commutation formulae.
\[ u_{ij} - u_{ji} = -\tau(W_i, W_j)u. \]

Proof. By the definition \(\ref{2.1}\), we get
\[ u_{ij} - u_{ji} = W_i W_j u - \nabla W_i W_j u - W_j W_i u + \nabla W_j W_i u = -\nabla W_i W_j - \nabla W_j W_i - [W_i, W_j]u = -\tau(W_i, W_j)u. \]
\[ \square \]

In particular, Proposition 3.1 \(\ref{3.1}\) and \(\ref{2.1}\) implies that
\[ u_{\alpha\beta} - u_{\beta\alpha} = -\tau(W_\alpha, W_\beta)u = -2d\theta(W_\alpha, W_\beta)Tu = 2ih_{\alpha\beta}u_0, \]
\[ u_{\alpha\beta} = u_{\beta\alpha}, \]
\[ u_{0\alpha} - u_{\alpha 0} = -\tau(T, W_\alpha) = -A^\beta_{\alpha} u_\beta. \]

Following Proposition 9.2 and 9.3 in \cite{4} in the pseudohermitian case, we call the relations between the third-order covariant derivatives of functions $u_{abc}$ and $u_{ach}$ the inner commutation formulae and the relations between $u_{abc}$ and $u_{bca}$ the outer commutation formulae.
Proposition 3.2. (1) We have the inner commutation formulae

\begin{align*}
    u_{\alpha\beta\gamma} &= u_{\alpha\gamma\beta}, \\
    u_{\alpha\beta\gamma} &= u_{\alpha\gamma\beta} - 2ih_{\alpha\beta}u_{\alpha0}. \\
\end{align*}

(3.15)

(2) We have the outer commutation formulae

\begin{align*}
    u_{\alpha\gamma\alpha} &= u_{\gamma\beta\alpha} - 2ih_{\gamma\beta}u_{\alpha0} + u_{\beta}R_{\alpha\gamma\beta}^\gamma + i\frac{3}{2}Q_{\gamma\beta\alpha}^\gamma, \\
    u_{\alpha\beta\alpha} &= u_{\gamma\beta\alpha} + 2i\left(A_{\gamma}^\beta h_{\alpha\gamma} - A_{\beta}^\gamma h_{\alpha\gamma}\right)u_{\beta} - i\frac{3}{2}Q_{\alpha\beta\gamma}^\gamma u_{\beta} - \frac{1}{4}Q_{\alpha\beta\gamma}^\gamma Q_{\alpha\beta\gamma}^\gamma u_{\beta}. \\
\end{align*}

(3.16)

Proof. (1) The first identity of (3.15) follows directly from the second identity in (3.4) and $u_{\alpha\beta} = u_{\beta\alpha}$ in (3.14).

Taking conjugation of the fourth identity in (3.4), we get

\[ u_{\alpha\gamma\beta} = W_{\alpha}(u_{\gamma\beta}) - \Gamma_{\alpha\gamma}^\mu u_{\mu\beta} - \Gamma_{\alpha\gamma}^\mu u_{\beta\mu} - \Gamma_{\alpha\gamma}^\mu u_{\mu\beta}. \]

So by (3.11), (3.3) and (3.14), we get

\begin{align*}
    u_{\alpha\beta\gamma} &= W_{\alpha}(u_{\beta\gamma}) - \Gamma_{\alpha\beta}^\gamma u_{\gamma\mu} - \Gamma_{\alpha\beta}^\gamma u_{\gamma\mu} - \Gamma_{\alpha\beta}^\gamma u_{\mu\gamma} \\
    &= W_{\alpha}\left(u_{\gamma\beta} - 2ih_{\gamma\beta}u_{0} - \Gamma_{\alpha\beta}^\gamma u_{\gamma\mu} - \Gamma_{\alpha\beta}^\gamma u_{\gamma\mu} - \Gamma_{\alpha\beta}^\gamma u_{\mu\gamma}\right) - \Gamma_{\alpha\beta}^\gamma u_{\mu\gamma} \\
    &= W_{\alpha}(u_{\gamma\beta}) - \Gamma_{\alpha\gamma}^\mu u_{\mu\beta} - \Gamma_{\alpha\gamma}^\mu u_{\beta\mu} - \Gamma_{\alpha\gamma}^\mu u_{\mu\beta} - 2i\left(h_{\gamma\beta}W_{\alpha}(u_{0}) - \Gamma_{\alpha\gamma}^\mu u_{\mu\beta} - \Gamma_{\alpha\gamma}^\mu u_{\beta\mu}\right) \\
    &= u_{\alpha\gamma\beta} - 2ih_{\alpha\beta}u_{\alpha0}. \\
\end{align*}

(3.17)

The second identity of (3.15) is proved.

(2) For the first identity in (3.16), note that

\begin{align*}
    du_{\alpha} &= (W_{\beta}u_{\alpha})\theta^\beta + (W_{\beta}u_{\alpha})\theta^\beta + T(u_{\alpha})\theta^\beta \\
    &= \left(u_{\beta\alpha} + \Gamma_{\beta\alpha}^\gamma u_{\gamma\rho} - i\frac{3}{2}Q_{\alpha\beta\gamma}^\gamma u_{\rho}\right)\theta^\beta + (u_{\beta\alpha} + \Gamma_{\beta\alpha}^\gamma u_{\gamma\rho})\theta^\beta + (u_{\alpha0} + \Gamma_{\alpha0}^\gamma u_{0})\theta^\beta \\
    &= u_{\beta\alpha}\theta^\beta + u_{\beta\alpha}\theta^\beta + u_{\alpha0}\theta + u_{\alpha0}\theta^\beta - \frac{i}{2}Q_{\alpha\beta}^\gamma u_{\rho}\theta^\beta \\
\end{align*}

(3.18)

by using (3.3) and $\omega_{\alpha} = \Gamma_{\beta\alpha}^\gamma \theta^\gamma + \Gamma_{\beta\alpha}^\gamma \theta^\gamma + \Gamma_{\alpha0}^\gamma \theta$. Taking exterior differentiation on both sides of (3.18), we get

\begin{align*}
    0 &= du_{\beta\alpha} \wedge \theta^\beta + du_{\beta\alpha} \wedge \theta^\beta + du_{\alpha0} \wedge \theta + u_{\beta\alpha}d\theta^\beta + u_{\beta\alpha}d\theta^\beta + u_{\alpha0}d\theta^\beta \\
    + du_{\beta} \wedge \omega^\beta + u_{\beta}d\omega^\beta - \frac{i}{2}d(Q_{\alpha\beta\gamma}^\gamma u_{\rho})\wedge \theta^\beta - \frac{i}{2}Q_{\alpha\beta\gamma}^\gamma u_{\rho}d\theta^\beta. \\
\end{align*}

(3.19)
Note that
\[ du_{j\alpha} = W_\gamma(u_{\beta\alpha})\theta^\gamma + W_\gamma(u_{\beta\alpha})\theta^\beta + T(u_{j\alpha})\theta, \]
\[ \omega^\beta_\alpha = \Gamma^\beta_{\gamma\alpha}\theta^\gamma + \Gamma^\beta_{\gamma\alpha}\theta^\gamma, \quad \text{mod} \quad \theta, \]
\[ d\theta^\beta = \theta^\gamma \wedge \omega^\gamma_\beta + \theta^\gamma \wedge \omega^\gamma_\beta = \Gamma^\beta_{\gamma\beta}\theta^\gamma \wedge \theta^\alpha + \Gamma^\beta_{\gamma\beta}\theta^\gamma \wedge \theta^\alpha = \Gamma^\beta_{\rho\beta}\theta^\gamma \wedge \theta^\rho + \Gamma^\beta_{\rho\beta}\theta^\gamma \wedge \theta^\rho, \quad \text{mod} \quad \theta, \quad \theta^\gamma \wedge \theta^\rho, \]
\[ d\theta^\beta = \theta^\gamma \wedge \omega^\gamma_\beta + \theta^\gamma \wedge \omega^\gamma_\beta = \Gamma^\beta_{\gamma\beta}\theta^\gamma \wedge \theta^\alpha + \Gamma^\beta_{\gamma\beta}\theta^\gamma \wedge \theta^\alpha = -\Gamma^\beta_{\rho\beta}\theta^\gamma \wedge \theta^\rho + \Gamma^\beta_{\rho\beta}\theta^\gamma \wedge \theta^\rho, \quad \text{mod} \quad \theta, \quad \theta^\gamma \wedge \theta^\rho, \]
\[ d\omega^\beta_\alpha = \omega^\beta_{\mu\mu} + \omega^\gamma_\beta \wedge \omega^\gamma_\beta + R^\alpha_\beta \lambda_\beta \theta^\lambda \wedge \theta^\alpha + \frac{1}{2} R^\beta_\alpha \lambda_\beta \theta^\lambda \wedge \theta^\alpha \]
\[ = \Gamma^\mu_\gamma \Gamma^\beta_{\mu\beta}\theta^\gamma \wedge \theta^\rho + \Gamma^\mu_\gamma \Gamma^\beta_{\mu\beta}\theta^\gamma \wedge \theta^\rho + \Gamma^\mu_\gamma \Gamma^\beta_{\mu\beta}\theta^\gamma \wedge \theta^\rho + R^\beta_\alpha \gamma^\beta \theta^\gamma \wedge \theta^\rho \]
\[ + \Gamma^\mu_\gamma \Gamma^\beta_{\mu\beta}\theta^\gamma \wedge \theta^\rho + \frac{1}{2} R^\beta_\alpha \gamma^\beta \theta^\gamma \wedge \theta^\rho, \quad \text{mod} \quad \theta, \quad \theta^\gamma \wedge \theta^\rho, \quad (3.20) \]
by (2.14) and \( \Gamma^\gamma_{\alpha\beta} = 0 \) in (2.7). Substituting (3.20) to the corresponding terms in (3.19) and comparing the coefficients of \( \theta^\gamma \wedge \theta^\rho \), we get
\[ 0 = -W_\beta u_{j\alpha} + W_\gamma u_{j\alpha} + u_{j\alpha} \Gamma^j_{\rho\gamma} - u_{j\alpha} \Gamma^j_{\rho\gamma} - 2i h_{j\beta} u_{\alpha\gamma} + W_\gamma(u_\beta) \Gamma^\beta_{\rho\gamma} - W_\beta(u_\beta) \Gamma^\beta_{\rho\gamma} \]
\[ + u_\beta \left( R^\alpha_\beta \gamma^\rho + \Gamma^\rho_\gamma \Gamma^\beta_{\rho\gamma} - \Gamma^\rho_\gamma \Gamma^\beta_{\rho\gamma} + \Gamma^\rho_\gamma \Gamma^\beta_{\rho\gamma} \right) + \frac{i}{2} W_\beta \left( Q^\beta_{\alpha\gamma} u_\beta \right) - \frac{i}{2} Q^\beta_{\alpha\beta} u_\beta \Gamma^\beta_{\rho\gamma} \]
\[ = -W_\beta u_{j\alpha} + W_\gamma u_{j\alpha} + u_{j\alpha} \Gamma^j_{\rho\gamma} - u_{j\alpha} \Gamma^j_{\rho\gamma} - 2i h_{j\beta} u_{\alpha\gamma} + u_\beta R^\alpha_\beta \gamma^\rho \]
\[ + \left( W_\gamma(u_\beta) \Gamma^\beta_{\rho\gamma} - u_\beta \Gamma^\beta_{\rho\gamma} + \frac{i}{2} u_\beta Q^\beta_{\alpha\gamma} \Gamma^\beta_{\rho\gamma} \right) - \frac{i}{2} u_\beta Q^\beta_{\alpha\beta} \Gamma^\beta_{\rho\gamma} - \left( W_\beta(u_\beta) \Gamma^\beta_{\rho\gamma} - u_\beta \Gamma^\beta_{\rho\gamma} \right) \]
\[ - \frac{i}{2} u_\beta Q^\beta_{\alpha\beta} \Gamma^\beta_{\rho\gamma} + \frac{i}{2} W_\beta \left( Q^\beta_{\alpha\gamma} u_\beta \right) - \frac{i}{2} Q^\beta_{\alpha\beta} u_\beta \Gamma^\beta_{\rho\gamma}. \quad (3.21) \]
Substitute
\[ W_\gamma(u_\beta) - \Gamma^\beta_{\beta\gamma} u_\mu + \frac{i}{2} Q^\beta_{\beta\gamma} u_\mu = u_{\beta\gamma}, \quad W_\beta(u_\beta) - \Gamma^\beta_{\beta\gamma} u_\mu = u_{\beta\gamma}, \]
by (3.3), into two brackets in (3.21) to get
\[ 0 = -W_\beta u_{j\alpha} + W_\gamma u_{j\alpha} + u_{j\alpha} \Gamma^j_{\rho\gamma} - u_{j\alpha} \Gamma^j_{\rho\gamma} - 2i h_{j\beta} u_{\alpha\gamma} + u_\beta R^\alpha_\beta \gamma^\rho + u_\beta \Gamma^\beta_{\rho\gamma} - u_\beta \Gamma^\beta_{\rho\gamma} \]
\[ - \frac{i}{2} u_\beta Q^\beta_{\alpha\gamma} \Gamma^\beta_{\rho\gamma} - \frac{i}{2} Q^\beta_{\alpha\gamma} u_\beta \Gamma^\beta_{\rho\gamma} + \frac{i}{2} W_\beta(\Gamma^\beta_{\rho\gamma}) u_\beta + \frac{i}{2} Q^\beta_{\alpha\beta} \Gamma^\beta_{\rho\gamma} - \frac{i}{2} Q^\beta_{\alpha\beta} u_\beta \Gamma^\beta_{\rho\gamma} \]
\[ = \left( -W_\beta u_{j\alpha} + \Gamma^\beta_{\rho\gamma} u_\mu + \Gamma^\beta_{\rho\gamma} u_\mu \right) + \left( W_\gamma u_{j\alpha} - u_{j\alpha} \Gamma^\beta_{\rho\gamma} - u_\beta \Gamma^\beta_{\rho\gamma} + \frac{i}{2} Q^\beta_{\alpha\beta} \Gamma^\beta_{\rho\gamma} \right) \]
\[ - \frac{i}{2} Q^\beta_{\alpha\gamma} u_\beta \Gamma^\beta_{\rho\gamma} - 2i h_{j\beta} u_{\alpha\gamma} + u_\beta R^\alpha_\beta \gamma^\rho + \frac{i}{2} Q^\beta_{\alpha\gamma} \Gamma^\beta_{\rho\gamma} + \frac{i}{2} Q^\beta_{\alpha\beta} \Gamma^\beta_{\rho\gamma} \]
\[ + \frac{i}{2} ( W_\beta(\Gamma^\beta_{\rho\gamma} - \Gamma^\beta_{\rho\gamma} Q^\beta_{\alpha\gamma} - \Gamma^\beta_{\rho\gamma} Q^\beta_{\alpha\beta} + \Gamma^\beta_{\rho\gamma} Q^\beta_{\alpha\gamma}) u_\beta \]
\[ = -u_{j\alpha} + u_{j\alpha} - 2i h_{j\beta} u_{\alpha\gamma} + u_\beta R^\alpha_\beta \gamma^\rho + \frac{i}{2} Q^\beta_{\alpha\gamma} \Gamma^\beta_{\rho\gamma} \]
by (2.10), (3.3) and (3.4). The first identity of (3.16) holds.
To prove the second identity in (3.16), we consider the components of $\theta^\gamma \land \theta^\rho$ in (3.19) to get

$$0 = \left( W_\gamma u_{\beta\alpha} + u_{\beta\alpha} \Gamma^\gamma_{\rho\mu} + u_{\beta\alpha} \Gamma^\beta_{\gamma\rho} + u_{\beta\alpha} R_{\gamma\alpha}^\beta_{\rho\mu} + \frac{1}{2} u_{\beta\alpha} \Gamma^\beta_{\gamma\rho} - \frac{i}{2} u_{\beta\alpha} Q^\rho_{\alpha\beta} \Gamma^\beta_{\gamma\rho} \right) \theta^\gamma \land \theta^\rho$$

by using (3.3). Equivalently, we have

$$0 = u_{\gamma\rho} - u_{\rho\gamma} + \frac{1}{2} u_{\beta\alpha} R_{\gamma\alpha}^\beta_{\rho\mu} - \frac{1}{2} u_{\beta\alpha} R_{\gamma\alpha}^\beta_{\rho\gamma} + \frac{1}{4} u_{\beta\alpha} Q^\rho_{\alpha\beta} (Q^\rho_{\alpha\beta} - Q^\rho_{\alpha\beta}),$$

by using (3.3). This together with $R_{\gamma\alpha}^\beta_{\rho\mu} = 2i(A_{\gamma\alpha}^\beta h_{\alpha\rho} - A_{\gamma\alpha}^\beta h_{\alpha\gamma}) - \frac{1}{2} h_{\gamma\beta} h_{\lambda\gamma} Q^\lambda_{\rho\beta\alpha}$ (cf. (43) in [4]) implies the second identity in (3.16).

**Remark 3.2.** Note that when $Q \equiv 0$, the commutative formulae (3.16) is the same as Proposition 9.2 in [4] in pseudohermitian case.

4. **THE BOCHNER-TYPE FORMULA**

By definition, we have $(\nabla_H u)^\alpha = h(\nabla_H u, W_\alpha) = W_\alpha u = u_\alpha$ for any $u \in C^\infty_0(M)$. Thus $\nabla_H u = (\nabla_H u)^\alpha W_\alpha + (\nabla_H u)^\alpha W_\alpha = u_\alpha W_\alpha + u_\alpha W_\alpha$. Consequently, we have $\partial_\alpha u = u_\alpha W_\alpha$ and $\|\partial_\alpha u\|^2 = \|u_\alpha W_\alpha\|^2 = u^2_\alpha u_\lambda$.

**Theorem 4.1.** Under an orthonormal $T^{(1,0)} M$-frame, the Bochner-type formula holds in the following form:

$$\Delta_b(\|\partial_\alpha u\|^2) = 2 \left( u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda} \right) + 4i(u_{\alpha\lambda} u_{\alpha\lambda} - u_{\alpha\lambda} u_{\alpha\lambda}) + 2i(A_{\alpha\beta} u_{\alpha\beta} - A_{\alpha\beta} u_{\alpha\beta}) + 2R_{\alpha\beta} u_{\alpha\beta} + u_{\alpha} (\Delta_b u)_{\alpha} + u_{\alpha} (\Delta_b u),$$

(4.1)

To prove it, we need a lemma.

**Lemma 4.1.** For any $u \in C^\infty_0(M)$,

$$\Delta_b(\|\partial_\alpha u\|^2) = S_1 + S_2,$$

where $S_1 = 2 \left( u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda} \right)$, $S_2 = u_{\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda}$.

**Proof.** We claim that

$$\left(\|\partial_\alpha u\|^2\right)_{\alpha\alpha} = u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda} + u_{\alpha\lambda} u_{\alpha\lambda},$$

(4.3)

Then (4.2) follows directly by taking summation of (4.3) and its conjugation.
By (2.11), we have
\[
\Gamma^\beta_{\alpha\lambda} u_\beta u_\lambda + \Gamma^\beta_{\alpha\lambda} u_\lambda u_\beta = -\Gamma^\lambda_{\alpha\beta} u_\beta u_\lambda + \Gamma^\beta_{\alpha\lambda} u_\lambda u_\beta = 0,
\]
and \(\Gamma^\beta_{\alpha\lambda} u_\beta u_\lambda = -\Gamma^\lambda_{\alpha\beta} u_\beta u_\lambda\). Thus, \(\Gamma^\beta_{\alpha\lambda} u_\beta u_\lambda = 0\). So by (3.3), we get
\[
W_a(\|\delta b u\|^2) = W_a(u_\lambda)u_\lambda + u_\lambda W_a(u_\lambda) = (u_{a\lambda} + \Gamma^\beta_{\alpha\lambda} u_\beta + \Gamma^\beta_{\alpha\lambda} u_\beta) u_\lambda + u_\lambda (u_{a\lambda} + \Gamma^\beta_{\alpha\lambda} u_\beta) = u_{a\lambda} u_\lambda + u_\lambda u_{a\lambda}. \tag{4.4}
\]
Then taking conjugation of (4.4), we get \(W_{\bar{a}}(\|\delta b u\|^2) = u_{\bar{a}\lambda} u_\lambda + u_\lambda u_{\bar{a}\lambda}\). So
\[
(\|\delta b u\|^2)_{\alpha\bar{a}} = W_{\alpha} W_{\bar{a}}(\|\delta b u\|^2) - \Gamma^\gamma_{\alpha\bar{a}} W_{\gamma}(\|\delta b u\|^2)
\]
\[
= W_{\alpha} (u_{\bar{a}\lambda} u_\lambda + u_\lambda u_{\bar{a}\lambda}) - \Gamma^\gamma_{\alpha\bar{a}} (u_{\bar{a}\lambda} u_\lambda + u_\lambda u_{\bar{a}\lambda})
\]
\[
= W_{\alpha} (u_{\bar{a}\lambda} u_\lambda + W_{\alpha} (u_{\bar{a}\lambda} u_\lambda) + W_{\alpha} (u_{\bar{a}\lambda} - \Gamma^\gamma_{\alpha\bar{a}} u_{\bar{a}\lambda})) u_\lambda + (W_{\alpha} (u_{\bar{a}\lambda} - \Gamma^\gamma_{\alpha\bar{a}} u_{\bar{a}\lambda})) u_\lambda
\]
\[
= (u_{a\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma + \Gamma^\gamma_{\alpha\lambda} u_\gamma) u_{\bar{a}\lambda} + (u_{a\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma) u_{\bar{a}\lambda}
\]
\[
+ (u_{a\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma) u_{\bar{a}\lambda} + (u_{a\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma) u_{\bar{a}\lambda}
\]
\[
= u_{a\lambda} u_{\bar{a}\lambda} + u_{a\lambda} u_{\bar{a}\lambda} + u_{a\lambda} u_{\bar{a}\lambda} + u_{a\lambda} u_{\bar{a}\lambda}
\]
\[
= (\Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda}) + (\Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda})
\]
\[
= u_{\bar{a}\lambda} u_{\bar{a}\lambda} + u_{\bar{a}\lambda} u_{\bar{a}\lambda} + u_{\bar{a}\lambda} u_{\bar{a}\lambda} + u_{\bar{a}\lambda} u_{\bar{a}\lambda}
\]
\[
= 0.
\]
where we have used (3.3) and (3.4). The result follows from
\[
\Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} = -\Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} = 0,
\]
and similarly, \(\Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} = 0\). By \(\Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} + \Gamma^\gamma_{\alpha\lambda} u_\gamma u_{\bar{a}\lambda} = 0\) by (2.11). \(\square\)

Proof of theorem 4.7: Note that by (3.13), we have \(\Delta_b u_c = u_{cab} + u_{c\bar{a}a}\). We hope to express third-order covariant derivatives in (4.2) in terms of \(\Delta_b u_c\). To do so, we apply inner and outer commutation formulae to (4.2) to express \(u_{\bar{a}ab}\) and \(u_{\bar{a}a\bar{b}}\) in terms of \(u_{ba\bar{a}}\) and \(u_{b\bar{a}a}\), respectively. By (3.15), we have the following inner commutation formulae.
\[
u_{a\bar{a}\bar{a}} = u_{aa\bar{a}} - 2i h_{\lambda\bar{a}} u_{a\bar{a}} 0, \quad u_{\bar{a}a\lambda} = u_{\bar{a}a\lambda},
\]
\[
u_{\bar{a}a\lambda} = u_{\bar{a}a\lambda} + 2i h_{\alpha\bar{a}} u_{a\lambda} 0, \quad u_{\bar{a}a\lambda} = u_{\bar{a}a\lambda}.
\]
So \(S_2\) in (4.2) becomes
\[
S_2 = u_{\bar{a}} (u_{\bar{a}a\lambda} - 2i h_{\lambda\bar{a}} u_{a\bar{a}} 0) + u_{\lambda} u_{a\lambda\bar{a}} + u_{\lambda} (u_{\bar{a}a\lambda} + 2i h_{\alpha\bar{a}} u_{a\lambda} 0) + u_{\bar{a}} u_{\bar{a}\lambda} 0
\]
\[
= u_{\bar{a}} u_{\bar{a}a\lambda} + u_{\lambda} u_{a\lambda\bar{a}} + u_{\lambda} u_{\bar{a}a\lambda} + u_{\lambda} u_{\bar{a}a\lambda} - 2i u_{a\lambda} u_{a\lambda} 0 + 2i u_{a\lambda} u_{a\lambda} 0 - 2i A_{\alpha\lambda} u_{a\lambda} u_\lambda + 2i A_{\alpha\lambda} u_{a\lambda} u_\lambda, \tag{4.5}
\]
by $u_{\alpha\beta} = u_{\alpha\beta} + A_{\alpha}^{\beta} u_{\beta}$, $u_{\alpha\alpha} = u_{\alpha\alpha} + A_{\alpha}^{\alpha} u_{\alpha}$ in (3.14). The outer commutation formulae (3.10) for $\rho = \alpha$ can be written as

\[
\begin{align*}
    u_{\alpha\lambda\alpha} &= u_{\lambda\alpha\alpha} - 2iu_{0\lambda} + R_{\lambda\beta} u_{\beta} - \frac{i}{2}Q_{\beta\lambda,\alpha}^{\bar{\alpha}} u_{\beta}, \\
    u_{\alpha\lambda\lambda} &= u_{\lambda\alpha\lambda} + 2i(n-1)A_{\lambda}^{\beta} u_{\beta} + \frac{i}{2}Q_{\lambda\beta,\alpha}^{\alpha} u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\lambda} Q_{\alpha\beta}^{\bar{\mu}} u_{\mu}.
\end{align*}
\]

(4.6)

by $Q_{\alpha\lambda,\alpha}^{\beta} = -Q_{\lambda,\alpha}^{\beta}$ in (2.11) and

\[
R_{\alpha}^{\beta} \lambda\alpha = i^{\beta\bar{\alpha}} R_{\alpha\beta}^{\lambda\alpha} = R_{\beta\alpha\lambda}^{\alpha\beta} = R_{\alpha\beta\lambda}^{\alpha\beta} = R_{\lambda\alpha\beta}^{\alpha\beta} = R_{\lambda\alpha\beta}^{\alpha\beta} = R_{\lambda\beta}^{\alpha\beta} = R_{\lambda\beta}^{\alpha\beta},
\]

(4.7)

by using Proposition (2.6) repeatedly. Taking conjugation of (4.6) and noting that $R_{\lambda\beta} = R_{\lambda\beta}^{\alpha\beta} = R_{\lambda\beta}^{\alpha\beta} = R_{\lambda\beta}^{\alpha\beta}$ by (4.7), we get

\[
\begin{align*}
    u_{\alpha\lambda\alpha} &= u_{\lambda\alpha\alpha} + 2iu_{0\lambda} + R_{\beta\lambda} u_{\beta} + \frac{i}{2}Q_{\beta\lambda,\alpha}^{\alpha} u_{\beta}, \\
    u_{\alpha\lambda\lambda} &= u_{\lambda\alpha\lambda} - 2i(n-1)A_{\lambda}^{\beta} u_{\beta} - \frac{i}{2}Q_{\lambda\beta,\alpha}^{\alpha} u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\lambda} Q_{\alpha\beta}^{\bar{\mu}} u_{\mu}.
\end{align*}
\]

(4.8)

Substitute (4.6) and (4.8) to (4.5) to get

\[
\begin{align*}
    S_2 &= u_{\lambda\alpha\lambda} - 2i(n-1)A_{\lambda}^{\beta} u_{\beta} - \frac{i}{2}Q_{\lambda\beta,\alpha}^{\alpha} u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\lambda} Q_{\alpha\beta}^{\bar{\mu}} u_{\mu} \\
    &+ u_{\lambda} u_{\lambda\alpha\alpha} + 2iu_{\lambda} u_{0\lambda} + R_{\beta\lambda\alpha} u_{\beta} + \frac{i}{2}Q_{\beta\lambda,\alpha}^{\alpha} u_{\beta} \\
    &+ u_{\lambda} u_{\lambda\alpha\alpha} + 2i(n-1)A_{\lambda}^{\beta} u_{\beta} + \frac{i}{2}Q_{\lambda\beta,\alpha}^{\alpha} u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\lambda} Q_{\alpha\beta}^{\bar{\mu}} u_{\mu} \\
    &+ u_{\lambda\alpha\alpha} - 2iu_{0\lambda} + R_{\beta\lambda} u_{\beta} - \frac{i}{2}Q_{\lambda\beta,\alpha}^{\alpha} u_{\beta} \\
    &- 2iu_{\alpha\alpha} + 2iu_{\alpha} u_{0\alpha} - 2iA_{\alpha}^{\beta} u_{\alpha} u_{\beta} + 2iA_{\alpha}^{\beta} u_{\alpha} u_{\beta} \\
    &= u_{\lambda}(\Delta_{\lambda} u_{\lambda}) + u_{\lambda}(\Delta_{\lambda} u_{\lambda}) + 4i(u_{\alpha\alpha} - u_{\alpha\alpha}) + 2ni(A_{\alpha}^{\beta} u_{\alpha} u_{\beta} - A_{\alpha}^{\beta} u_{\alpha} u_{\beta}) + 2R_{\alpha\beta} u_{\alpha} u_{\beta} \\
    &+ iQ_{\alpha\beta}^{\lambda} u_{\alpha} u_{\beta} - iQ_{\beta\lambda}^{\alpha} u_{\beta} - \frac{1}{4}Q_{\alpha\beta}^{\lambda} Q_{\alpha\beta}^{\bar{\mu}} u_{\mu}.
\end{align*}
\]

(4.9)

The last identity follows from $u_{\lambda} u_{\lambda\alpha\alpha} + u_{\lambda} u_{\lambda\alpha\alpha} = u_{\lambda}(\Delta_{\lambda} u_{\lambda})$ by (3.13), $A_{\alpha}^{\beta} = h_{\alpha\alpha} A_{\alpha}^{\beta} = A_{\alpha}^{\beta}$ and $Q_{\beta}^{\gamma} = -Q_{\beta}^{\gamma}$ in (2.11). Substituting (4.9) to (4.2), we get (4.1).

Remark 4.1. When $Q \equiv 0$, the Bochner-type formula (4.1) is the same as the pseudohermitian case (see (9.36) in (10) or Theorem 6 in (19)).

5. Two Useful Identities

We need the following lemma to handle the second bracket in the Bochner type formula (4.1).

Lemma 5.1. For any $u \in C^0_0(M)$, we have

\[
\begin{align*}
    \int_M i(u_{\alpha\beta} u_{\alpha\beta} - u_{\alpha\alpha} u_{\alpha}) dV &= \frac{1}{n} \int_M \left( u_{\alpha\beta} u_{\alpha\beta} - u_{\alpha\beta} u_{\alpha\beta} - R_{\alpha\beta} u_{\alpha\beta} - \frac{i}{2}Q_{\alpha\beta,\gamma}^{\alpha} u_{\alpha\beta} \\
    &+ \frac{i}{2}Q_{\alpha\beta,\gamma}^{\bar{\gamma}} u_{\alpha\beta} - \frac{1}{2}Q_{\alpha\beta,\gamma}^{\alpha} u_{\alpha\beta} + Q_{\alpha\beta}^{\gamma} Q_{\alpha\beta}^{\bar{\gamma}} u_{\alpha\beta} \right) dV,
\end{align*}
\]

(5.1)
and
\[ \int_M i(u_\alpha \partial_\alpha - u_\alpha \partial_\alpha) dV = \int_M \left( -\frac{1}{n} \left| \sum_\alpha u_\alpha \partial_\alpha \right|^2 + \frac{1}{2n} (\Delta_b u)^2 + i A_{\alpha \beta} u_\alpha u_\beta - i A_{\alpha \beta} u_\alpha u_\beta \right) dV. \] (5.2)

Remark 5.1. \((5.1)\) and \((5.2)\) are the same as the corresponding identities in the pseudohermitian case when \(Q \equiv 0\) (cf. Lemma 9.1 in [10] or Lemma 4 and Lemma 5 in [13]).

5.1. The Proof of \((5.1)\). By definition we have
\[
\int_M u_{\alpha \beta} \partial_{\alpha \beta} dV = \sum_{\alpha, \beta} \left( u_{\alpha \beta}, W_\alpha(u_\beta) - \Gamma^\gamma_{\alpha \beta} u_\gamma - \Gamma^\gamma_{\alpha \beta} u_\gamma \right) 
= \sum_{\alpha, \beta} \left( -W_\alpha(u_{\alpha \beta}) + \Gamma^\alpha_{\gamma \gamma} u_{\alpha \beta} - \Gamma^\gamma_{\alpha \gamma} u_\alpha, u_\beta \right) 
= \sum_{\alpha, \beta, \gamma} \left( -W_\alpha(W_\alpha(u_\beta) - \Gamma^\gamma_{\alpha \beta} u_\gamma - \Gamma^\gamma_{\alpha \beta} u_\gamma) + \Gamma^\alpha_{\gamma \gamma} u_\alpha, u_\beta \right) - \left( \Gamma^\beta_{\alpha \gamma} u_\alpha, u_\beta \right)
\] (5.3)

by \((3.3)\) and Lemma \((5.1)\). Apply \([W_\alpha, W_\gamma] = \nabla W_\alpha W_\gamma - \nabla W_\alpha W_\gamma - \tau(W_\alpha, W_\gamma) = \Gamma^\gamma_{\alpha \gamma} W_\gamma - \Gamma^\gamma_{\alpha \gamma} W_\gamma = -2h(W_\alpha, JW_\alpha)T = \Gamma^\gamma_{\alpha \gamma} W_\gamma - \Gamma^\gamma_{\alpha \gamma} W_\gamma - 2i\delta_{\alpha \gamma} T\) to get that \(\int_M u_{\alpha \beta} \partial_{\alpha \beta} dV\) equals to
\[
\begin{align*}
\sum_{\alpha, \beta} \left( W_\alpha(W_\alpha(u_\beta), u_\beta) + 2ni \sum_{\beta} (T(u_\beta), u_\beta) + \sum_{\alpha, \beta, \gamma} \left( -\Gamma^\gamma_{\alpha \alpha} W_\gamma(u_\beta) + \Gamma^\gamma_{\alpha \alpha} W_\gamma(u_\beta) \right) + W_\alpha\left( \Gamma^\gamma_{\alpha \beta} u_\gamma + \Gamma^\gamma_{\alpha \beta} u_\gamma \right) + \Gamma^\gamma_{\alpha \gamma} u_\alpha, u_\beta \right) - \sum_{\alpha, \beta, \gamma} \left( \Gamma^\beta_{\alpha \gamma} u_\alpha, u_\beta \right) \\
= -\sum_{\alpha, \beta} \left( W_\alpha(u_{\alpha \beta}) - \sum_{\alpha, \beta, \gamma} \left( W_\alpha(W_\alpha(u_\beta), u_\beta) - \Gamma^\gamma_{\alpha \beta} u_\gamma + \Gamma^\gamma_{\alpha \beta} u_\gamma \right) - \sum_{\alpha, \beta, \gamma} \left( \Gamma^\gamma_{\alpha \gamma} u_\alpha, u_\beta \right) \right)
\end{align*}
\] (5.4)
by using \((3.3)\) repeatedly, where \(\Sigma_1, \Sigma_2\) and \(\Sigma_3\) denote the summation of terms of type \((\ast u_\rho, u_\beta), \quad (\ast u_\rho, u_\beta)\) and \((\ast u_{ab}, u_c)\) respectively. We see that

\[
\Sigma_1 = \sum_{\alpha, \beta, \gamma, \rho} \left( (W_\alpha \Gamma^\rho_{\alpha\beta} - W_\alpha \Gamma^\rho_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} \Gamma^\rho_{\alpha\gamma} + \Gamma^\gamma_{\alpha\beta} \Gamma^\rho_{\alpha\gamma} - \Gamma^\rho_{\alpha\alpha} \Gamma^\rho_{\beta\gamma} + \Gamma^\rho_{\alpha\alpha} \Gamma^\rho_{\beta\gamma} + \Gamma^\rho_{\alpha\beta} \Gamma^\rho_{\alpha\gamma}) \right) u_\rho, u_\beta
\]

\[
= (R^\rho_{\alpha\alpha} u_\rho, u_\beta) - 2ni \sum_{\beta, \rho} \left( \Gamma^\rho_{0\beta} u_\rho, u_\beta \right), \tag{5.5}
\]

by

\[
R^\rho_{\alpha\beta} = W_\alpha \Gamma^\rho_{\alpha\beta} - W_\alpha \Gamma^\rho_{\alpha\beta} - \Gamma^\gamma_{\alpha\beta} \Gamma^\rho_{\alpha\gamma} + \Gamma^\gamma_{\alpha\beta} \Gamma^\rho_{\alpha\gamma} - \Gamma^\rho_{\alpha\alpha} \Gamma^\rho_{\beta\gamma} + \Gamma^\rho_{\alpha\alpha} \Gamma^\rho_{\beta\gamma} + \Gamma^\rho_{\alpha\beta} \Gamma^\rho_{\alpha\gamma} + 2ni \Gamma^\rho_{0\beta}, \tag{5.6}
\]

by \((2.16)\), \(\Gamma^\gamma_{\alpha\beta} = 0\) in \((2.7)\) and \(J_{\alpha\alpha} = \sum_\alpha h(W_\alpha, JW_\alpha) = i \sum_\alpha \delta_{\alpha\alpha} = ni\); and

\[
\Sigma_2 = - \frac{i}{2} \sum_{\alpha, \beta, \gamma, \rho} \left( (W_\alpha Q^\rho_{\beta\alpha} \Gamma^\rho_{\beta\gamma} - \Gamma^\gamma_{\alpha\beta} Q^\rho_{\beta\gamma} + \Gamma^\gamma_{\alpha\beta} Q^\rho_{\beta\gamma} + \Gamma^\rho_{\gamma\alpha} \Gamma^\rho_{\beta\gamma} + \Gamma^\rho_{\gamma\alpha} \Gamma^\rho_{\beta\gamma} + \Gamma^\rho_{\gamma\alpha} \Gamma^\rho_{\beta\gamma} + 2ni \Gamma^\rho_{0\beta} \right) \right) u_\rho, u_\beta
\]

\[
= - \frac{i}{2} \sum_{\alpha, \beta, \rho} (Q^\rho_{\beta\alpha} u_\rho, u_\beta) = - \int_M Q^\beta_{\alpha\alpha} u_\rho u_\beta dV = - \int_M Q^\beta_{\rho\alpha\alpha} u_\rho u_\beta dV = - \sum_{\alpha, \beta, \rho} (Q^\beta_{\beta\alpha\alpha} u_\rho, u_\beta) \quad \text{by \((2.11)\)},
\]

and

\[
\Sigma_3 = \sum_{\alpha, \beta, \gamma} \left( - \Gamma^\gamma_{\alpha\beta} u_{\gamma\gamma} - \Gamma^\gamma_{\alpha\beta} u_{\gamma\beta} + \Gamma^\gamma_{\alpha\beta} u_{\alpha\gamma} + \Gamma^\gamma_{\alpha\beta} u_{\alpha\gamma} + \Gamma^\gamma_{\gamma\alpha} u_{\alpha\beta} - \Gamma^\gamma_{\gamma\alpha} u_{\alpha\gamma}, u_\beta \right)
\]

\[
= \sum_{\alpha, \beta, \gamma} (\Gamma^\gamma_{\alpha\beta} u_{\gamma\beta} + \Gamma^\gamma_{\alpha\beta} u_{\alpha\gamma}, u_\beta) + \sum_{\alpha, \beta, \gamma} (\Gamma^\gamma_{\alpha\beta} u_{\alpha\gamma}, u_\beta) - \sum_{\alpha, \beta, \gamma} (\Gamma^\gamma_{\gamma\alpha} u_{\alpha\gamma}, u_\beta), \tag{5.8}
\]

by using \((2.11)\). We also have

\[
- \sum_{\alpha, \beta} (W_\alpha (u_{\alpha\beta}), u_\beta) = - \sum_{\alpha, \beta} (u_{\alpha\beta}, W_\alpha u_\beta) = \sum_{\alpha, \beta} \left( u_{\alpha\beta}, W_\alpha (u_\beta) - \Gamma^\alpha_{\gamma\gamma} u_\beta \right)
\]

\[
= \sum_{\alpha, \beta} (u_{\alpha\beta}, u_{\alpha\beta}) + \sum_{\alpha, \beta} \left( (\gamma_{\alpha\beta} u_{\gamma\gamma} - \Gamma^\gamma_{\alpha\beta} u_{\gamma\gamma}, u_\beta) \right) = \sum_{\alpha, \beta} \left( u_{\alpha\beta}, u_{\alpha\beta} \right) + \Sigma_4, \tag{5.9}
\]

by Lemma \((3.4)\) and \((3.3)\), where \(\Sigma_4 = \sum_{\alpha, \beta} (\Gamma^\gamma_{\gamma\alpha} u_{\gamma\gamma} - \Gamma^\gamma_{\gamma\alpha} u_{\gamma\beta}, u_\beta)\); and

\[
\sum_{\alpha} \left( T(u_\alpha), u_\alpha \right) - \sum_{\alpha, \rho} (\Gamma^\rho_{0\alpha} u_\rho, u_\alpha) = \sum_{\alpha} (u_{0\alpha}, u_\alpha), \tag{5.10}
\]

holds by \((3.3)\). Note that the first summation in the right hand side of \((5.8)\) is exactly \(\Sigma_4\) by \((2.11)\). Now substituting \((5.5)-(5.10)\) to \((5.4)\), we get \(\int_M u_{\alpha\beta} u_{\alpha\beta} dV\) equals to

\[
\sum_{\alpha, \beta} \left( u_{\alpha\beta}, u_{\alpha\beta} \right) + 2ni \sum_{\alpha} (u_{0\alpha}, u_\alpha) - \sum_{\alpha, \beta} (R^\rho_{\alpha\beta} u_\beta, u_\alpha) - \sum_{\alpha, \beta, \rho} (\gamma^\beta_{\alpha\beta} u_\alpha, u_\beta) - \sum_{\alpha, \beta, \rho} (\Gamma^\rho_{\gamma\alpha} u_{\gamma\gamma}, u_\beta),
\]
Proof. The result follows.

Then (5.12) follows from the last identity, \( Q \).

Lemma 5.2.

ant derivatives. We can transform them into the integral only involving first-order covariant

Comparing with the pseudohermitian case (cf. (9.37) in [10]), the integral formula (5.11)

Substituting (5.12) and its conjugation to (5.11), we get (5.1).

The result follows.

The sum of this identity and its conjugation gives

\[
\int_M u_{\alpha\tilde{\alpha}}u_{\tilde{\alpha}}dV = \frac{1}{2n} \int_M \left( u_{\alpha\beta}u_{\tilde{\alpha}\beta} - u_{\alpha\beta}u_{\tilde{\alpha}\beta} + R_{\alpha\beta}u_{\tilde{\alpha}}u_{\beta} + \Gamma_{\alpha\gamma}u_{\tilde{\alpha}}u_{\gamma\beta} + \Gamma_{\tilde{\alpha}\gamma}u_{\alpha\gamma}u_{\beta} \right) dV,
\]

The result follows.

Comparing with the pseudohermitian case (cf. (9.37) in [10]), the integral formula (5.11) has the extra term \( \int_M \Gamma_{\alpha\gamma}u_{\tilde{\alpha}}u_{\beta} + \Gamma_{\tilde{\alpha}\gamma}u_{\alpha\gamma}u_{\beta} dV \), which is zero by \( \Gamma_{\alpha\gamma} = -\frac{i}{4}Q_{\gamma\alpha} = 0 \) in the pseudohermitian case. Note that this extra term is the integral involving second-order covariant derivatives. We can transform them into the integral only involving first-order covariant derivatives via integration by parts in the following lemma.

Lemma 5.2.

\[
\int_M \Gamma_{\alpha\gamma}^{\beta} u_{\alpha\gamma} u_{\beta} dV = \int_M \left( \frac{i}{2} Q_{\gamma\beta\alpha}^{\gamma} u_{\alpha\gamma} u_{\beta} + \frac{1}{4} Q_{\alpha\gamma}^{\beta} Q_{\beta\gamma}^{\beta} u_{\alpha\gamma} u_{\beta} - \frac{1}{2} Q_{\alpha\gamma}^{\alpha} Q_{\beta\gamma}^{\gamma} u_{\alpha\gamma} u_{\beta} \right) dV.
\]

Proof. By (3.3), (3.11) and (3.14), we have

\[
\int_M \Gamma_{\alpha\gamma}^{\beta} u_{\alpha\gamma} u_{\beta} dV = \frac{i}{2} \int_M Q_{\gamma\beta\alpha}^{\gamma} u_{\alpha\gamma} u_{\beta} dV = \frac{i}{2} \int_M Q_{\gamma\beta\alpha}^{\gamma} u_{\alpha\gamma} u_{\beta} dV
\]

where \( \mathcal{E} = -\Gamma_{\gamma\alpha}^{\beta} Q_{\gamma\alpha}^{\beta} u_{\beta} + \frac{i}{4} Q_{\gamma\alpha}^{\beta} u_{\beta} u_{\beta} - \frac{1}{4} Q_{\gamma\alpha}^{\beta} u_{\beta} u_{\beta} \) by (2.11). By

\[
2Q_{\gamma\beta}^{\alpha} u_{\gamma} = Q_{\gamma\alpha}^{\beta} u_{\beta} - Q_{\gamma\beta}^{\gamma} u_{\gamma} = 0,
\]

by (2.11) and (3.11), it equals to

\[
\frac{i}{2} \int_M \left( -W_{\gamma} Q_{\gamma\beta\alpha}^{\gamma} + \Gamma_{\gamma\alpha}^{\beta} Q_{\gamma\beta\alpha}^{\gamma} + \Gamma_{\gamma\alpha}^{\beta} Q_{\gamma\beta\alpha}^{\gamma} \right) u_{\alpha\gamma} u_{\beta} dV - \frac{1}{4} \int_M \left( Q_{\gamma\beta}^{\alpha} Q_{\gamma\alpha}^{\gamma} + Q_{\gamma\alpha}^{\beta} Q_{\gamma\beta}^{\gamma} \right) u_{\alpha\gamma} u_{\beta} dV
\]

Then (5.12) follows from the last identity, \( Q_{\gamma\alpha}^{\beta} = Q_{\gamma\beta}^{\alpha} - Q_{\gamma\beta}^{\gamma} \) and \( \Gamma_{\gamma\alpha}^{\beta} = -\frac{i}{4}Q_{\gamma\alpha}^{\beta} \) in (2.11). □

Substituting (5.12) and its conjugation to (5.11), we get (5.1).
5.2. The proof of (5.2). Note that
\[
\int_M (\Delta_b u)^2 dV = \int_M \left( \sum_{\alpha} u_{\alpha\bar{\alpha}} + u_{\bar{\alpha}\alpha} \right)^2 dV = \int_M \sum_{\alpha,\beta} (u_{\alpha\bar{\alpha}} u_{\beta\bar{\beta}} + u_{\alpha\bar{\beta}} u_{\bar{\alpha}\beta} + u_{\bar{\alpha}\alpha} u_{\bar{\beta}\beta}) dV
\]
\[
= \int_M \left( 2 \left| \sum_{\alpha} u_{\alpha\bar{\alpha}} \right|^2 + \sum_{\alpha,\beta} (u_{\alpha\bar{\alpha}} + 2i h_{\alpha\bar{\gamma}} u_0) u_{\beta\bar{\beta}} + \sum_{\alpha,\beta} (u_{\alpha\bar{\beta}} - 2i h_{\alpha\bar{\gamma}} u_0) u_{\bar{\alpha}\beta} \right) dV
\]
\[
= \int_M \left( 4 \left| \sum_{\alpha} u_{\alpha\bar{\alpha}} \right|^2 + 2n i \sum_{\alpha} (u_0 u_{\alpha\bar{\alpha}} - u_{\alpha\bar{\alpha}} u_0) \right) dV.
\]
(5.13)

On the other hand, we have
\[
\int_M \sum_{\alpha} u_0 (u_{\alpha\bar{\alpha}} - u_{\bar{\alpha}\alpha}) dV = \sum_{\alpha} (u_0, u_{\alpha\bar{\alpha}} - u_{\bar{\alpha}\alpha}) = \sum_{\alpha,\gamma} (u_0, W_\alpha (u_\alpha) - \Gamma^\gamma_{\alpha\gamma} u_\gamma - W_\alpha (u_\bar{\gamma}) + \Gamma^\gamma_{\alpha\bar{\gamma}} u_\bar{\gamma})
\]
\[
= \sum_{\alpha} (W_\alpha^* u_0, u_\alpha) - \sum_{\alpha,\gamma} (\Gamma^\gamma_{\alpha\alpha} u_0, u_\gamma) = \sum_{\alpha} (W_\alpha^* u_0, u_\alpha) + \sum_{\alpha,\gamma} (\Gamma^\gamma_{\alpha\alpha} u_0, u_\gamma)
\]
\[
= - \sum_{\alpha} (W_\alpha (u_0), u_\alpha) + \sum_{\alpha,\gamma} (\Gamma^\gamma_{\alpha\gamma} u_0, u_\alpha) - (\Gamma^\gamma_{\alpha\alpha} u_0, u_\gamma)
\]
\[
= - \sum_{\alpha} (W_\alpha (u_0), u_\alpha) + \sum_{\alpha,\gamma} (W_\alpha (u_0), u_\alpha)
\]
\[
= - \sum_{\alpha} (u_{0\alpha} + A_{\alpha\beta} u_\beta, u_\alpha) + \sum_{\alpha,\beta} (u_{0\bar{\alpha}} + A_{\bar{\alpha}\beta} u_\beta, u_\alpha)
\]
\[
= \int_M \sum_{\alpha} (u_{\alpha} u_{0\bar{\alpha}} - u_{\bar{\alpha}} u_{0\alpha}) dV + \int_M \sum_{\alpha,\beta} (A_{\alpha\beta} u_\alpha u_\beta - A_{\alpha\beta} u_{\bar{\alpha}} u_{\bar{\beta}}) dV.
\]

Substitute this identity into (5.13) to get
\[
\int_M \left( (\Delta_b u)^2 - 4 \sum_{\alpha} u_{\alpha\bar{\alpha}} \right) dV = 2ni \sum_{\alpha,\beta} \int_M \left( u_{\alpha} u_{0\bar{\alpha}} - u_{\bar{\alpha}} u_{0\alpha} + (A_{\alpha\beta} u_\alpha u_\beta - A_{\alpha\beta} u_{\bar{\alpha}} u_{\bar{\beta}}) \right) dV,
\]
which is equivalent to (5.2).

6. The Proof of Theorem 1.1

Lemma 6.1. For any \( u \in C_0^\infty (M) \), we have
\[
\int_M \left( u_{\alpha} (\Delta_b u)_{\alpha} + u_{\bar{\alpha}} (\Delta_b u)_{\bar{\alpha}} \right) dV = - \int_M (\Delta_b u)^2 dV.
\]

Proof. By (3.3), (3.9) and Lemma 3.1 we get
\[
\int_M u_{\alpha} (\Delta_b u)_{\alpha} dV = \int_M W_\alpha (\Delta_b u)_{\alpha} dV = (W_\alpha (\Delta_b u), u_\alpha) = \left( \Delta_b u, - W_\alpha + \Gamma^\alpha_{\beta\bar{\alpha}} u_\alpha \right)
\]
\[
= (\Delta_b u, - W_\alpha + \Gamma^\alpha_{\beta\alpha} u_\beta) = -(\Delta_b u, u_{\bar{\alpha} \alpha}),
\]
by using (3.3). The summation of this identity and its conjugation gives the result. \qed
Let \( \frac{\partial}{\partial u} \| u \|^2 \in C_0^\infty(M) \). Apply \( f = \| \partial_u \|^2 \in C_0^\infty(M) \) to this identity for \( u \in C_0^\infty(M) \). By the Bochner-type formula (4.1) and Lemma 6.1, we get

\[
0 = \int_M \left( u_{\alpha\beta} \partial_{\alpha\beta} + 2iu_{\alpha\partial u_{\alpha\beta}} + nu_{\alpha\beta} + i\partial_{\alpha\beta} - A_{\alpha\beta} u_{\alpha\beta} + R_{\alpha\beta} u_{\alpha\beta} \right) dV.
\]

Applying (5.1) and (5.2) to get

\[
i \int_M (u_{\alpha\partial u_{\alpha\beta}} - u_{\alpha\beta} u_{\alpha\beta}) dV = Ci \int_M (u_{\alpha\partial u_{\alpha\beta}} - u_{\alpha\beta} u_{\alpha\beta}) dV + (1 - C)i \int_M (u_{\alpha\partial u_{\alpha\beta}} - u_{\alpha\beta} u_{\alpha\beta}) dV
\]

Substituting (6.2) to (6.1), we get

\[
0 = \int_M \left( 1 + \frac{2C}{n} \right) u_{\alpha\beta} u_{\alpha\beta} + \left( 1 - \frac{2C}{n} \right) u_{\alpha\beta} u_{\alpha\beta} - \frac{4(1 - C)}{n} \left( \sum_{\alpha} u_{\alpha\beta} \right)^2
\]

Let \( \left( 1 + \frac{2C}{n} \right) \frac{1}{n} - \frac{4(1 - C)}{n} = 0 \), i.e., \( C = \frac{3n}{4n + 2} \) in (6.3). By using \( u_{\alpha\beta} u_{\alpha\beta} = \sum_{\alpha, \beta} |u_{\alpha\beta}|^2 \geq \frac{1}{n} \sum_{\alpha} |u_{\alpha\beta}|^2 \) (cf. p. 489 in [19]), we get

\[
0 \geq \int_M \left( \frac{2(n - 1)}{2n + 1} R_{\alpha\beta} u_{\alpha\beta} \right) dV + \frac{n - 1}{2n} \left( Q_{\alpha\beta, \gamma} u_{\alpha\beta} - Q_{\alpha\beta, \gamma} u_{\alpha\beta} \right) - \frac{2(n - 1)}{8n} Q_{\alpha\gamma} Q_{\beta\gamma} u_{\alpha\beta} + \frac{3}{2(n - 1)} Q_{\alpha\gamma} Q_{\beta\gamma} u_{\alpha\beta} \right) dV,
\]

(6.4)
for $\Delta_b u = -\lambda_1 u$. We use the following lemma to handle the second term in the right hand side.

**Lemma 6.2.**

$$-2 \int_M \sum_{\alpha} |u_\alpha|^2 dV = \int_M u(\Delta_b u) dV.$$

**Proof.** By Lemma 3.1 and 3.3, we have

$$\int_M \sum_{\alpha} |u_\alpha|^2 dV = \sum_{\alpha} (u_\alpha, u_\alpha) = \sum_{\alpha} (W_\alpha u, u_\alpha) = \sum_{\alpha} (u, W_\alpha^* u_\alpha)$$

$$= -\sum_{\alpha} (u, W_\alpha^* u_\alpha) + \sum_{\alpha, \beta} (u, \Gamma_{\beta\alpha}^\alpha u_\alpha)$$

$$= -\sum_{\alpha} (u, u_{\bar{\alpha}\alpha}) - \sum_{\alpha, \beta} (u, \Gamma_{\alpha\beta}^{\bar{\alpha}} u_\alpha) + (u, \Gamma_{\beta\bar{\alpha}}^\alpha u_\alpha) = -\sum_{\alpha} (u, u_{\bar{\alpha}\alpha}).$$

The summation of this identity and its conjugation gives the result. \qed

Let $X = u_{\bar{\alpha}} W_\alpha \in T^{(1,0)} M$. If (1.6) holds, by the definition of Ricci tensor, and $\text{Tor}$ given by (2.13) and $Q_1$, $Q_2$, $Q_3$ given by (2.9), we get

$$R_{\alpha\bar{\beta}} u_{\bar{\alpha}} u_\beta + i(n + 1)(A_{\alpha\bar{\beta}} u_{\bar{\alpha}} u_\beta - A_{\alpha\bar{\beta}} u_{\bar{\alpha}} u_\beta)$$

$$+ \frac{i}{2} (Q^\gamma_{\alpha\bar{\beta}\gamma} u_{\bar{\alpha}} u_\beta - Q^\gamma_{\alpha\bar{\beta}\gamma} u_{\bar{\alpha}} u_\beta) - \frac{2n + 7}{8(n - 1)} Q_{\alpha\gamma}^\beta Q_{\beta\gamma}^\alpha u_{\bar{\alpha}} u_\beta + \frac{3}{2(n - 1)} Q_{\alpha\gamma}^\beta Q_{\beta\gamma}^\alpha u_{\bar{\alpha}} u_\beta \geq \kappa \sum_{\alpha} |u_\alpha|^2.$$

So by Lemma 6.2, 6.4 and 6.5, we get

$$0 \geq \int_M \left( -\frac{2(n^2 - 1)}{n(2n + 1)} \lambda_1 + \frac{2(n - 1)}{2n + 1} \kappa \right) \sum_{\alpha} |u_\alpha|^2 dV,$$

i.e. $\lambda_1 \geq \frac{\kappa n}{n + 1}$. Theorem 1.2 is proved.

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