The partition function of the extended $r$-reduced Kadomtsev–Petviashvili hierarchy

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Received 28 November 2014, revised 27 March 2015
Accepted for publication 30 March 2015
Published 23 April 2015

Abstract

We derive a particular solution of the extended $r$-reduced KP hierarchy, which is specified by a generalized string equation. The work is a generalization to arbitrary $r \geq 2$ of Buryak’s recent results of a solution to the extended open KdV hierarchy which corresponds to $r = 2$.

Keywords: KP hierarchy, wave function, string equation, tau function

1. Introduction and results

Recently Pandharipande, Solomon and Tessler conjectured$^4$ an integrable equation for descendent integrals for open and closed Riemann surfaces, called the open KdV equation [23] which has attracted significant attention [3, 6, 7, 21]. In particular, Buryak [6, 7] constructed an extension of this equation to a hierarchy of commuting flows, called the extended open KdV hierarchy; he also derived an explicit formula for a particular solution of the extended open KdV hierarchy which satisfies a generalized version of the string equation.

Let $\overline{\mathcal{M}}_{g,n}$ denote the Deligne–Mumford moduli space of stable curves of genus $g$ with $n$ marked points, $L_i$ the $i$th-tautological line bundle over $\overline{\mathcal{M}}_{g,n}$, $i = 1, ..., n$. Denote by $\overline{\mathcal{M}}_{g,a_1,...,a_n}^r (r \geq 2)$ the moduli space of stable $r$-spin curves and by $p: \overline{\mathcal{M}}_{g,a_1,...,a_n}^r \to \overline{\mathcal{M}}_{g,n}$ the forgetful map, where $a_i$ are given integers belonging to $\{0, ..., r-1\}$ satisfying that $2g - 2 - \sum a_i$ is divisible by $r$; here we recall that an $r$-spin curve is a smooth curve $C_{g,x_1,...,x_n}$ together with an $r$-spin structure $L$, which is a line bundle with an identification $L^\otimes r \simeq \omega_C^r (- \sum_{i=1}^n a_i x_i).$ Assuming $H^0(C, L) = 0$ then $V = H^1(C, L)$ is a vector bundle over $\overline{\mathcal{M}}_{g,n}$.

Very recently this open analogue of original Witten’s conjecture [25] has been proved in [8].
\( \overline{M}_{g,n} \) and the celebrated Witten’s class \( C_W(a_1, \ldots, a_n) \) is defined by

\[
C_W(a_1, \ldots, a_n) = \frac{1}{r^g} p_g \left( c_{\text{top}} \left( V^r \right) \right). \tag{1.1}
\]

However, \( H^0(C, L) \) does not vanish identically for all stable curves; see in [1, 9–11, 19] for proper definitions of \( C_W(a_1, \ldots, a_n) \) and for rigorous definitions of \( \overline{M}_{g,n} \).

Recall that Witten’s \( r \)-spin intersection numbers [16, 26] are rational numbers defined as

\[
\left\langle \tau_{a_1,1} \cdots \tau_{a_r,1} \right\rangle_{g,n} = \int_{\overline{M}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \wedge C_W(a_1, \ldots, a_n). \tag{1.2}
\]

Here \( a_i \in \{0, \ldots, r-2\}, a_i = a_i + 1, m_i \geq 0, \psi_i = c_1(L_i) \); e.g. in the case \( r = 3 \),

\[
\left\langle \tau_{1,0} \tau_{2,0} \right\rangle_{0,3} = 1, \quad \left\langle \tau_{1,1} \right\rangle_{1,1} = \frac{1}{12}, \quad \left\langle \tau_{2,1} \tau_{2,3} \right\rangle_{2,2} = \frac{11}{4320}. \tag{1.3}
\]

Due to the dimension reason, \( \left\langle \tau_{a_1,1} \cdots \tau_{a_r,1} \right\rangle_{g,n} \) take zero values unless

\[
a_1 - \frac{1}{r} + \cdots + \frac{a_n - 1}{r} + \frac{r - 2}{r} (g - 1) + m_1 + \cdots + m_n = 3g - 3 + n. \tag{1.4}
\]

Hence we could simply write \( \left\langle \tau_{a_1,1} \cdots \tau_{a_r,1} \right\rangle_{g,n} \) as \( \left\langle \tau_{a_1,1} \cdots \tau_{a_r,1} \right\rangle_{g,n} \). It was conjectured by Witten [26] and proved by Faber, Shadrin, Zvonkine [16] (in a more general setting by Fan, Jarvis, Ruan [17]) that the partition function of Witten’s \( r \)-spin intersection numbers (1.2)

\[
Z^W(T) = \exp \left( \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{m_1, \ldots, m_n \geq 0} \left\langle \tau_{a_1+1,1} \cdots \tau_{a_r+1,1} \right\rangle T_{m_1}^{a_1} \cdots T_{m_n}^{a_n} \right), \tag{1.5}
\]

is a particular tau function of the \( r \)-reduced KP hierarchy. Here the formal variables \( T_{m}^{a} \), \( a = 0, \ldots, r - 2 \), \( m \geq 0 \) are called coupling constants. Up to a multiplicative constant, \( Z^W(T) \) is uniquely determined by the \( r \)-reduced KP hierarchy together with the celebrated string equation

\[
\sum_{m \geq 1} T_m^a \frac{\partial Z^W}{\partial T_m^a} + \frac{1}{2} \sum_{a=0}^{r-2} \sum_{a=0}^{r-2} T_0^{a_1} T^{r-2-a} Z^W = 0, \tag{1.6}
\]

where \( T_m^a = T_m^a - \delta_0^a \delta_{m,1} \).

As far as we know, the open version of these intersection numbers (except \( r = 2 \), the open KdV case) has not been defined; however, we are going to construct a particular solution of the so-called extended \( r \)-reduced KP hierarchy, for which we hope it will play the role of the partition function for open \( r \)-spin structures of type \( A \). In passing, we also present some formulæ which clarify certain series expansions of [7].

To present the result, consider the differential operator \( L \)

\[
L = \partial' + \sum_{a=1}^{r-1} u_a(x) \partial^{r-1-a}, \quad \partial = \partial_x. \tag{1.7}
\]

The \( (r - 1) \)-KdV hierarchy, also called the Gelfand–Dickey hierarchy or the \( r \)-reduced KP hierarchy, is a hierarchy of evolutionary partial differential equations for the operator \( L \) in terms of infinitely many auxiliary parameters \( t_m \) (called customarily ‘times’)
This infinite system of evolution equations for \( L \) (namely for the coefficients \( u_{\alpha}(x; t_1, \ldots) \)) is well-known to be compatible [13]. Here the numbers \( a_m \) are defined by

\[
a_{p+q} = \frac{1}{(p+q)!}, \quad 1 \leq p \leq r, \quad q \geq 0,
\]

and \((p + qr)_r!\) is the \( r \)th generalized double factorial defined by

\[
(p + qr)_r! = (p + qr) \cdot (p + (q-1)r) \cdot \ldots \cdot p
\]

which reduces to the usual double factorial in the case \( r = 2 \); the following convention is used

\[
0! = (-1)! = \ldots = (-r)! = 1.
\]

The \( m = 1 \) equation in (1.8) reads \( \frac{\partial L}{\partial t_m} = [\partial, L] \), which means that the flow along \( t_1 \) simply acts by translation in the variable \( x \); therefore we can (and will) identify \( t_1 \) and \( x \) throughout the paper. Introduce the notations:

\[
t = (t_1, \ldots, t_{r+1}, t_{r+2}, \ldots, t_{2r-1}, \ldots), \quad t_{KP} = (t_1, t_2, t_3, \ldots).
\]

In other words, the infinite vector \( t \) is obtained from \( t_{KP} \) by removing the variables with index which is a multiple of \( r \). Note that if \( m \) is a multiple of \( r \), then the rhs of (1.8) vanishes\(^{5}\) which implies that the operator \( L \) and hence the functions \( u_{\alpha}(t) \) depend only on the reduced set of variables \( t \) (as opposed to the extended set \( t_{KP} \)):

\[
u_{\alpha} = u_{\alpha}(t), \quad \alpha = 1, \ldots, r - 1.
\]

Consider now the flows of the usual wave vector of the \( r \)-reduced KP hierarchy (1.8), but allow them to depend also on the ‘trivial’ flow variables \( t_{ip}, p \in \mathbb{N} \):

\[
\frac{\partial \Psi}{\partial t_m} = a_m \left(L \frac{m}{r}\right) \Psi, \quad m \geq 1.
\]

The compatibility of the equations (1.8), (1.14) is a standard result (see for example [13]):

\[
\partial_{x_n} \partial_{t_m} L = \partial_{x_m} \partial_{t_n} L, \quad \partial_{x_n} \partial_{t_m} \Psi = \partial_{x_m} \partial_{t_n} \Psi, \quad \forall m, n \geq 1.
\]

So equations (1.8) and equations (1.14) together form an integrable system.

**Definition 1.1.** The infinite system of compatible equations (1.8), (1.14) is called the extended \( r \)-reduced KP hierarchy.

The case \( r = 2 \), coincides with the extended open KdV hierarchy [6, 7, 23]. We point out that we are not imposing at this stage an additional spectral equation.

In this paper, we are going to introduce (in theorem 1.2 below) a particular solution of the extended \( r \)-reduced KP hierarchy of Definition 1.1 that satisfies a certain additional constraint in the form of the string equation. It turns out that this solution is closely related to a pair of adjoint ODE problems that are closely related to the Pearcey equation (see (3.29), (3.30)); the approach also should shed light on certain objects that have appeared in the case \( r = 2 \) in [6].

\(^{5}\) Indeed in this case the rhs is the commutator of \( L \) with an integer power of \( L \) itself.
Recall now that the Frobenius manifold corresponding to the $r$-reduced KP hierarchy is the space of miniversal deformations of a simple singularity of type $A_{r-1}$. Let $Z(t)$ denote the partition function \[ Z(t) = \text{partition function} \] of this Frobenius manifold. It is a particular tau function of the $r$-reduced KP hierarchy \[ (1.8) \] which is uniquely determined (up to a multiplicative constant) by the string equation \[ (1.16) \]

$$L_{-1} Z(t) = 0,$$

where

$$L_{-1} := \sum_{p \geq r+1} t_p \frac{\partial}{\partial t_p} - \frac{1}{2} \sum_{k=1}^{r-1} k t_{r-k}, \quad t_p = t_{p, r+1}.$$ \hspace{1cm} (1.17)

It can also be uniquely determined (up to a multiplicative constant) by using only the $W_{A_{r-1}}$-constraints \[ (2, 4, 22) \]. It should be noted that $Z(t)$ in the present normalization of flows coincides (up to an overall multiplicative constant, which is irrelevant) with $Z^W(T)$ under a suitable rescaling of times, namely,

$$Z^W(T) = Z(t), \quad \text{with } T_m = (-1)^{m+mr} (\sqrt{-1})^{m(r-2)+aq} t_{a+1+mr}. \hspace{1cm} (1.18)$$

Our goal is to produce a solution of the extended $r$-reduced KP hierarchy which generalizes the above (see theorem 1.2) and the result of \[ (7) \]. To this end and to introduce the necessary objects, let us denote by $\psi(t_{sr}; z)$ the following wave function associated to $Z(t)$

$$\psi(t_{sr}; z) = \frac{Z(t - \left[z^{-1}\right])}{Z(t)} e^{\xi(t_{sr}; z)}, \hspace{1cm} (1.19)$$

where we have used the shorthand notations

$$t_{sr} - \left[z^{-1}\right] = \left(t_1 - \frac{m_1}{z}, t_2 - \frac{m_2}{z^2}, \ldots\right), \hspace{1cm} (1.20)$$

$$\xi(t_{sr}; z) := \sum_{k \geq 1} \alpha_k t_k z^k, \hspace{1cm} (1.21)$$

$$m_{p+qr} := (p + (q - 1)r) \zeta(r), \quad 1 \leq p \leq r - 1, \quad q \geq 0. \hspace{1cm} (1.22)$$

The unwieldy normalizations of the time flows are chosen to simplify the comparison with existing results later on. The wave function $\psi(t_{sr}; z)$ is the ordinary wave function with the spectral parameter $z$, namely it satisfies

$$L \psi(t_{sr}; z) = z^r \psi(t_{sr}; z). \hspace{1cm} (1.23)$$

While $L$ is independent of the flows $t_1, t_2, t_3, \ldots$, the wave function $\psi$ does depend on them, albeit in a trivial exponential way as dictated by \[ (1.19) \]. In order to emphasize this additional dependence we have used the subscript $t_{sr}$ to indicate the dependence on the complete set of KP flows. Now we are ready to state the result of this paper.

**Theorem 1.2.** Let $\omega = e^{i \pi r/m}$ and set

$$D(z; \Gamma) = \frac{im^{r+1}}{\sqrt{2\pi}} e^{\frac{i \pi r+1}{2}} \exp \left(-\frac{z^{r+1}}{r+1}\right) \int_{w^{r+1}} w^{-1} \exp \left[-\frac{w^{r+1}}{(r+1)r} + \frac{w^r}{r}\right] dw, \hspace{1cm} (1.24)$$

$$\int_{\omega}^{w} \omega^{r+1} e^{-\frac{1}{2} \omega^{r+1}} d\omega = \frac{1}{2} \sqrt{2\pi} r^{r+1} m^r e^{-\frac{1}{2} m^{r+1} r}.$$
where $F = e^{-\pi i R} \cup e^{\pi i R}$ (traversed in the upward direction). It has the following asymptotic expansion

$$D(z; \gamma) \sim 1 + \sum_{k=1}^{\infty} \frac{d_k}{z^{(r+1)k}} = \gamma(z), \quad z \to \infty, \quad z \in S',$$

where $S' = \{ \frac{\pi}{r} - \frac{\pi}{r+1} < \arg(z) < \frac{\pi}{r} - \frac{\pi}{r+1} \}$ and $d_k$ are constants, and the contour has zero index number relative to $w = 0$. Introduce the following function

$$\Psi(t_{kr}) := \frac{1}{2\pi i} \oint_{|z|=\infty} d(z) \psi(t_{kr}; z) \frac{dz}{z},$$

where the formal series $d(z)$ is given in (1.25). Define also the partition function of the extended $r$–reduced KP hierarchy by the formula

$$Z_E(t_{kr}) := Z(t) \Psi(t_{kr}),$$

where $Z(t)$ is the partition function of the $A_{r-1}$ Frobenius manifold satisfying the string equation (1.16). Then we have

$$L_{r-1}^\text{ext} Z_E(t_{kr}) = 0, \quad \Psi(t_{kr})|_{t_0=0} \equiv 1 \quad (\forall t_1),$$

where the operator $L_{r-1}^\text{ext}$ is defined by

$$L_{r-1}^\text{ext} = \sum_{p+r+1} t_p \frac{\partial}{\partial t_p} + \frac{1}{2} \sum_{k=1}^{r-1} t_k t_{r-k} + t_r - \frac{\partial}{\partial t_1}.$$}

Furthermore, the solution $\Psi(t_{kr})$ of (1.14) and such that $Z_E(t_{kr}) := Z(t) \cdot \Psi(t_{kr})$ solves (1.28) is unique up to a multiplicative constant.

Regarding the expectation that $Z_E(t_{KP})$ is a possible candidate of the partition function of open and closed $r$–spin intersections numbers of type $A$, the variable $t_r$ should play the role of the coupling constant corresponding to boundary marked points and $t_m$, $m \geq 2$ their descendants.

The paper is organized as follows: In section 2, we review the tau function of the ($r$-reduced) KP hierarchy and the associated wave function and dual wave function. In Section 3, by applying the string actions we study some properties of the wave function and the dual wave function associated to the partition function of the $r$-spin structures of type $A$. In section 4, we prove theorem 1.2.

### 2. Tau function and wave functions of the $r$-reduced KP hierarchy

Let us consider solutions to the extended $r$-reduced KP hierarchy (1.8), (1.14). It is easy to see that equations (1.8) and (1.14) are decoupled: we can first solve (1.8) and then solve (1.14). Equations (1.8) have a class of well-known solutions corresponding to infinite Grassmannians.

Let $u_1(t), \ldots, u_{r-1}(t)$ be any Grassmannian solution to the $r$-reduced KP hierarchy. It is known that there exists a so-called Sato tau function $\tau(t_{kr})$ of this solution, which satisfies the bilinear identities...
where $\xi(t_{kr} - t_{kr}'; z)$ and $t_{kr} - [z^{-1}], t_{kr} + [z^{-1}]$ are defined as in (1.21), (1.20). The tau function $\tau$ is unique up to a factor of form
\[
\exp \left\{ \sum_{k \geq 1} c_k t_k + c_0 \right\},
\]
where $c_i$ are arbitrary constants.

Substituting the solution $u_1(t), \ldots, u_r(t)$ into equations (1.14) we get a set of evolutionary linear PDEs for $\Psi$. To solve them, let us introduce an auxiliary problem
\[
Ly(t_{kr}; z) = z^r \psi(t_{kr}; z).
\]
Together with the original PDEs (1.14)
\[
\frac{\partial y}{\partial t_m}(t_{kr}; z) = a_m \left( L^m \right)_+ \psi(z; t_{kr}), \quad m \geq 1
\]
$\psi(t_{kr}; z)$ becomes the wave function of the solution with the spectral parameter $z$ (also called the Baker–Akhiezer function). Without loss of generality, we assume it is normalized in the following way
\[
\psi\left(t_{kr}, z\right) \sim \left( 1 + \mathcal{O}\left(z^{-1}\right) \right) e^{z(t_{kr} - z)}, \quad z \to \infty.
\]
Note that the wave function is unique up to an arbitrary asymptotic series $g(z)$ satisfying
\[
g(z) \sim 1 + \sum_{k=1}^{\infty} \frac{g_k}{z^k}, \quad z \to \infty,
\]
where $g_k, k \geq 1$ are constants. The following lemma is well-known.

**Lemma 2.1.** (12, 13) For any Grassmannian solution $u_1(t), \ldots, u_r(t)$ of the $r$-reduced KP hierarchy, let $\tau(t_{kr})$ be any tau-function of this solution. Then
\[
\psi\left(t_{kr}, z\right) = \left( t_{kr} - [z^{-1}] \right)^{\tau(t_{kr}) - \xi(t_{kr} - z^{-1})} e^{z(t_{kr} - z)}.
\]

Similarly to the above construction, let us also recall the dual wave function $\psi^*(t_{kr}; z)$ of a solution $u(t)$. It satisfies the following linear system:
\[
L^* \psi^* = z^r \psi^*,
\]
\[
\frac{\partial \psi^*}{\partial t_m} = -a_m \left( L^m \right)_+ \psi^*.
\]
Here $*$ denotes the formal adjoint operator; the formal adjoint of a (pseudo-)differential operator $P = \sum_j p_j(x) \partial^j$ in a variable $x$ is defined by replacing $\partial_i \to -\partial_i$ and swapping the order of differentiation and multiplication operators, so that $P^* = \sum_j (-\partial_j)^j \circ p_j(x)$. The dual wave function is also normalized such that
In terms of the tau function it is expressed by
\[
\psi^*\left(t_{x}\; ;\; z\right) := \frac{\tau(t_{x} + \left[z^{-1}\right])}{\tau(t_{x})}e^{-\epsilon(t_{x};z)}, \quad z \to \infty.
\] (2.10)

The bilinear identities (2.1) read as follows

\[
\frac{1}{2\pi i} \oint_{z=\infty} \psi\left(t_{x}; z\right) \psi^*\left(t'_{x}; z\right) dz = 0, \quad \forall t_{x}, t'_{x}.
\] (2.12)

Now we take a tau function \(\tau(t)\) of the solution \(u_{1}(t), \ldots, u_{r-1}(t)\) without dependence in \(t_{p},\; p \geq 1\), and take \(g(z)\) any asymptotic series of form (2.6). Then solutions of equations (1.14) are given by

\[
\Psi\left(t_{x}\right) = \frac{1}{2\pi i} \oint_{z=\infty} g(z) \frac{\tau\left(t - \left[z^{-1}\right]\right)}{\tau(t)}e^{\epsilon(t_{x};z)} \frac{dz}{z}.
\] (2.13)

Inspired by Buryak [7] we call

\[
\tau_{\varepsilon}\left(t_{x}\right) := \tau(t) \Psi\left(t_{x}\right)
\] (2.14)

a tau function of the solution \(u_{1}(t), \ldots, u_{r-1}(t), \Psi(t_{x})\) of the extended \(r\)-reduced KP hierarchy.

It is well-known that the \(r\)-reduced KP hierarchy is Miura equivalent to Dubrovin–Zhang’s integrable hierarchy of topological type associated to the \(A_{r-1}\) Frobenius manifold. So we can write the hierarchy (1.8) by using the normal coordinates \(w_{1}, \ldots, w_{r-1}\) [14, 15]

\[
\frac{\partial w_{p}}{\partial \bar{\eta}_{q}} = P_{p}^{q} \frac{\partial h_{\eta_{p}q}}{\partial \eta_{q}}, \quad q \geq 0,
\] (2.15)

where \(P_{p}^{q}\) is a deformation of the Hamiltonian operator \(\eta_{p}\partial_{x}\) with \(\eta_{\alpha\beta} = \delta_{\alpha+\beta,r} \eta_{\alpha\beta}\) denote the variational derivatives, \(h_{\eta_{p}q} = \int h_{\eta_{p}q} dx\) are Hamiltonians whose densities \(h_{\eta_{p}q}\) satisfy the tau-symmetry property

\[
h_{\eta_{p},-1} = w_{p}, \quad \frac{\partial h_{\eta_{p},q-1}}{\partial \bar{w}_{q}} = \frac{\partial h_{\eta_{p},q-1}}{\partial \bar{w}_{q}}, \quad p, \; q \geq 0.
\] (2.16)

In these notations, \(t^{\alpha\beta} = t_{\alpha+\beta r}\), \(p \geq 0\). Here and below free Greek indices always take values \(1, \ldots, r-1\), the Einstein summation convention for repeated Greek indices with one-up and one-down is always assumed, and we use \(\eta_{\alpha\beta}\) and its inverse \(\eta^{\alpha\beta}\) to lower and raise Greek indices. For example, \(w^{\eta} := \eta^{\alpha\beta} w_{\beta}\).

The tau symmetry property allows us to define two-point functions [15] \(\Omega_{\alpha\beta;\eta\eta}^{\alpha\beta;\eta\eta}\) for an arbitrary solution \((w_{1}, \ldots, w_{r-1})\)

\[
\Omega_{\alpha\beta;\eta\eta}(w; w_{1}, w_{2}, \ldots) = \partial_{x}^{-1}\left(\frac{\partial h_{\eta\eta p-1}}{\partial \eta_{p}}\right), \quad p, \; q \geq 0.
\] (2.17)
It also implies the existence of a tau function \( \tau \) of the solution such that

\[
\frac{\partial^2 \log \tau}{\partial x^a \partial y^b} = \Omega_{a,p,b;q}(w(t); w_{x}(t), w_{xx}(t), \ldots).
\]  

(2.18)

We call this tau function the Dubrovin–Zhang tau function associated to the \( A_{r-1} \) Frobenius manifold. Note that \( \Omega_{a,p,b;q} \) are differential polynomials in \( w \), and that \( \log \tau \) admits the genus expansion

\[
\log \tau = \sum_{g=0}^{\infty} F_g := F,
\]

(2.19)

where \( F_g \) are genus \( g \) free energies and \( F \) is the free energy. In the particular example of the \( r \)-reduced KP hierarchy (1.8) that we are considering, the normal coordinates \( u_{kr} \) as well as the Hamiltonian densities \( h_{a,p} \) are given by

\[
w_{kr} = \frac{h_a}{a!},
\]

(2.20)

\[
h_{a,p} = \frac{h_{a+(p+1)r}}{(a+(p+1)r)!},
\]

(2.21)

where

\[
h_k := \text{Res} \ L_k^k, \quad k \geq 1.
\]

(2.22)

In the above formula, Res means taking the coefficient of \( \partial^{-1} \).

For Grassmannian solutions, the Sato tau function and the Dubrovin–Zhang tau function coincide [12, 15] modulo an arbitrary factor of form (2.2).

### 3. Wave potentials, string actions and Pearcey integrals

Recall that the partition function \( Z(t) \) associated to the \( A_{r-1} \) Frobenius manifold is a particular tau function of the \( r \)-reduced KP hierarchy. Up to a multiplicative constant, this tau function is uniquely specified by the string equation

\[
L_{-1} Z = 0,
\]

(3.1)

where \( L_{-1} \) is defined in (1.17). Write

\[
\left\langle \tau_{k_1} \cdots \tau_{k_n} \rightangle := \lim_{t \to 0} \frac{\partial^m \log Z}{\partial t_{k_1} \cdots \partial t_{k_n}}
\]

(3.2)

Then according to (1.4), we know that \( \left\langle \tau_{k_1} \cdots \tau_{k_n} \right\rangle \) vanishes unless there exists a non-negative number \( g \) such that

\[
k_1 \frac{1}{r} + \cdots + k_{m-1} \frac{1}{r} = m + \left( \frac{r-2}{r} - 3 \right)(1-g).
\]

(3.3)

This implies for \( r \geq 3 \), \( Z(x, 0, 0, \ldots) = \text{constant} \). As usual we normalize this constant to be 1. For \( r=2 \), it is known that \( Z(x, 0, 0, \ldots) = \exp \left( x^3/6 \right) \).
We denote by \( u_1(t), \ldots, u_{r-1}(t) \) the solution to the \( r \)-reduced KP hierarchy associated with the tau function \( Z(t) \), in normal coordinates by \( w_1(t), \ldots, w_{r-1}(t) \). The string equation (3.1) implies of the initial data

\[
\frac{\partial^2 \log Z}{\partial t \partial \alpha} \bigg|_{t=0} = \delta_{\alpha, r-1} \cdot x, \quad \alpha = 1, \ldots, r - 1.
\] (3.4)

Here \( \delta_{\alpha, r-1} \) is the Kronecker delta function.

**Lemma 3.1.** The Miura transformation relating \( u \) to \( w \) is triangular and of form

\[
w_\alpha = \frac{1}{r} u_\alpha + M_\alpha(u_1, \ldots, u_{r-1}),
\] (3.5)

where \( M_\alpha \) are differential polynomials in \( u_1, \ldots, u_{r-1} \).

**Proof.** Let us introduce a grading on pseudo-differential operators:

\[
\deg u_\alpha^{(k)} := \alpha + 1 + k, \quad \deg \partial^m := m, \quad k \geq 0, m \in \mathbb{Z}.
\] (3.6)

Then we find that \( L \) and \( L^\perp \) are homogeneous operators with degree

\[
\deg L = r, \quad \deg L^\perp = 1.
\] (3.7)

Recall due to (2.20), (2.22) that

\[
w_\alpha = \frac{1}{\alpha!} \text{Res} L^\perp.
\] (3.8)

So we have

\[
\deg w_\alpha = \alpha + 1.
\] (3.9)

The leading term \( \frac{1}{\alpha!} u_\alpha \) is an easy exercise: one can replace in \( L \) the operator \( \partial \) with a parameter \( \lambda \) and take the usual residue at \( \lambda = \infty \). The lemma is proved. □

The above Lemma implies

\[
u_\alpha \bigg|_{t=0} = r \delta_{\alpha, r-1} \cdot x, \quad \alpha = 1, \ldots, r - 1.
\] (3.10)

In particular, we have

**Lemma 3.2.** The following initial values hold

\[
u_\alpha \big|_{t=0} = 0.
\] (3.11)

Denote by \( \psi(t_{zv}; z) \) and \( \psi^*(t_{zv}; z) \) the wave and the dual wave functions associated to \( Z(t) \), respectively:

\[
\psi(t_{zv}; z) = \frac{Z(t - [z^{-1}])}{Z(t)} e^{z(t_{zv}; z)} \quad \psi^*(t_{zv}; z) = \frac{Z(t + [z^{-1}])}{Z(t)} e^{-z(t_{zv}; z)}.
\] (3.12)
and introduce the following notations
\[
K(t_x; z) := Z(t) \cdot \psi(t_x; z), \quad K^*(t_x; z) := Z(t) \cdot \psi^*(t_x; z), \quad (3.13)
\]
\[
f(x; z) := \psi(t_x; z) \bigg|_{b_z=0}, \quad f^*(x; z) := \psi^*(t_x; z) \bigg|_{t_{z_1}=0}. \quad (3.14)
\]

We call \(K(t_x; z)\) and \(K^*(t_x; z)\) the wave potential and the dual wave potential associated to the partition function \(Z(t)\). Let us consider the string actions on them.

**Proposition 3.3.** The following equalities hold true:
\[
L_{-1}^{\text{ext}} K(t_x; z) = S_z K(t_x; z), \quad (3.15)
\]
\[
L_{-1}^{\text{ext}+} K^*(t_x; z) = -S_z^* K^*(t_x; z). \quad (3.16)
\]

Here \(S_z\) is defined by
\[
S_z = z^{-(r-1)/2} \alpha \partial_z \circ z^{-(r-1)/2} - z = \frac{1}{z^{r-1}} \partial_z - \frac{r-1}{2 z^r} - z, \quad (3.17)
\]
and \(S_z^*\) is the formal adjoint operator of \(S_z\) (the formal adjoint has been already defined just after (2.9)) which has the form
\[
S_z^* = z^{-(r-1)/2} \alpha \partial_z \circ z^{-(r-1)/2} - z = -\frac{1}{z^{r-1}} \partial_z + \frac{r-1}{2 z^r} - z, \quad (3.18)
\]
and \(L_{-1}^{\text{ext}}, L_{-1}^{\text{ext}+}\) are defined by
\[
L_{-1}^{\text{ext}} = \sum_{p \geq r+1} t_p \frac{\partial}{\partial t_{p-r}} + \frac{1}{2} \sum_{k=1}^{r-1} t_{r-k} - t_r - \frac{\partial}{\partial t_1}, \quad (3.19)
\]
\[
L_{-1}^{\text{ext}+} = \sum_{p \geq r+1} t_p \frac{\partial}{\partial t_{p-r}} + \frac{1}{2} \sum_{k=1}^{r-1} t_{r-k} - t_r - \frac{\partial}{\partial t_1}. \quad (3.20)
\]

**Proof.** On one hand, by using the string equation (3.1) we find
\[
L_{-1}^{\text{ext}} K(t_x; z) = \sum_{p \geq r+1} m_p \frac{\partial Z(t - [z^{-1}])}{\partial t_{p-r}} e^{z(t_x; z)} - \frac{r-1}{2 z^r} K(t_x; z) - zK(t_x; z)
\]
\[
+ \sum_{p \geq r+1} a_{p-r} t_p z^{p-r} K(t_x; z)
\]
\[
+ \sum_{p=1}^{r-1} m_{p-r} t_p z^{p-r} K(t_x; z) + t_r K(t_x; z). \quad (3.21)
\]
On another hand from the chain rule it follows that

$$\frac{1}{z^{r+1}} \partial_z K(t_{sr}; z) = \sum_{p \geq r+1} (p-r) m_{p-r} \frac{\partial Z\left(t - \left[z^{-1}\right]\right)}{\partial_{p-r}} z^p (t_{sr}; z) + \sum_{p \geq 1} p \alpha_p t_p z^{p-r} K(t_{sr}; z).$$

Comparing the above two equalities we have

$$L_{-1}^x K(t_{sr}; z) = \frac{1}{z^{r+1}} \partial_z K(t_{sr}; z) - \frac{r-1}{2} z^r K(t_{sr}; z) - zK(t_{sr}; z) = S_1 K(t_{sr}; z).$$

(3.23)

In a similar way we can prove (3.16). The lemma is proved.

**Lemma 3.4.** Let $f, f^*$ as in (3.14). We have for any $k \geq 0$,

$$\partial^k f(x; z) = (-1)^k S^k f(x; z),$$

(3.24)

$$\partial^k f^*(x; z) = \left(S_z^* f^\# \right)(x; z).$$

(3.25)

**Proof.** Note that in the case $r = 2$ we have $Z(x, 0, 0, \ldots) = \exp \left(\frac{x^2}{6}\right)$ and that in the case $r \geq 3$ we have $Z(t_0, 0, 0, \ldots) = 1$. Taking $t_2 = t_3 = \ldots = 0$ in both sides of (3.15), we find

$$f + S_z f = 0,$$

(3.26)

which implies (3.25). Similarly, taking $t_2 = t_3 = \ldots = 0$ in both sides of (3.15), we obtain (3.25).

**Lemma 3.5.** Denote $A(z) = f(0; z)$ and $A^\#(z) = f^\#(0; z)$. Applying the Taylor expansion and lemma 3.4 we have

$$f(x; z) = \sum_{n=0}^{\infty} \frac{\left(-1\right)^n}{n!} S^n_0 A(z) x^n,$$

(3.27)

$$f^\#(x; z) = \sum_{n=0}^{\infty} \frac{1}{n!} S^n_0 A^\#(z) x^n.$$

(3.28)

**Lemma 3.6.** The functions $A(z), A^\#(z)$ satisfy the following ODEs:

$$S_z' A(z) = (-z)' A(z),$$

(3.29)

$$S_z^{*'} A^\#(z) = (-z)' A^\#(z).$$

(3.30)
Proof. Recall that the Lax equations for $\psi(t \omega^*; z)$ and $\psi^*(t \omega^*; z)$ read

$$L\psi = z'\psi, \quad L^*\psi^* = z'\psi^*. \quad (3.31)$$

Taking $t = 0$ in these two equations and using the initial data (3.11) we have

$$\partial_x f(x; z) \bigg|_{x=0} = z'f(0; z), \quad (-1)^{y^*} f^*(x; z) \bigg|_{x=0} = z'f(0; z). \quad (3.32)$$

Finally by employing lemma 3.4 we obtain

$$(-1)^y S^zf(x; z) \bigg|_{x=0} = z'f(0; z), \quad (-1)^y S^zf^*(x; z) \bigg|_{x=0} = z'f^*(0; z). \quad (3.33)$$

The lemma is proved. \qed

Note that (2.5), (2.10) require the (formal) boundary behavior

$$A(z) = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty, \quad (3.34)$$

$$A^*(z) = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty. \quad (3.35)$$

In order to solve the ODEs (3.29), (3.30) let us introduce a change of variable $\eta = z'$ and let

$$H(\eta) = \eta^{r+1} \exp \left( \frac{1}{r+1} \eta \right) = z^{r+1} \exp \left( \frac{z}{r+1} \right). \quad (3.36)$$

A straightforward computation shows that

$$\frac{1}{H} \circ S_z \circ H = r \partial_\eta, \quad (3.37)$$

which allows to transform immediately the ODEs (3.29), (3.30) in a pair of adjoint Pearcey equations. Indeed, let us write $A(z) = H(\eta) \cdot \ell(\eta)$: the above ODE (3.29) becomes

$$\left( r \partial_\eta \right) \ell(\eta) = (-1)^y \eta \ell(\eta). \quad (3.38)$$

Its solutions are well known and are given in the form of a Fourier–Laplace integral representation

$$\ell(\eta) = C_1 \int_r \exp \left( \frac{w^{r+1}}{(r+1)r} - \frac{w}{r} \eta \right) dw, \quad (3.39)$$

where the contour $\Gamma$ is any contour extending to infinity along two different asymptotic directions in the sectors where $\Re(w^{r+1})$ tends to $-\infty$, and $C_1$ is a constant. Applying a formal saddle point method to meet the condition (3.34), we obtain

$$A(z; \Gamma) = \frac{i}{\sqrt{2\pi}} z^{r-1} \exp \left( \frac{z^{r+1}}{(r+1)r} \right) \int_r \exp \left[ \frac{w^{r+1}}{(r+1)r} - \frac{w}{r} \right] dz \quad (3.40)$$

where the contour $\Gamma$ is any contour extending to infinity along two different asymptotic directions $R_j \equiv R_j e^{2\pi i j/(r+1)}, j \equiv 0, \ldots, r \text{ mod } (r + 1)$. The integrals appearing in (3.40) (in the variable $\xi = z^r$) are famously known as Pearcey integrals and generalize the usual integral representation of Airy functions. A relatively straightforward steepest descent analysis shows that if $\Gamma$ is the contour originating at $\infty$ along $R_{-1}$ and ending at $\infty$ along $R_0$ then the function has the asymptotic expansion
\[ A(z; \Gamma) = 1 + \mathcal{O}(1/z), \quad |z| \to \infty, \quad z \in S_{r}, \]  
(3.41)

where \( S_{r} = \{ |\arg(z)| < \frac{\pi}{r} \} \). More precisely the expansion contains only powers of \( z^{-r-1} \) as stated in the following lemma (whose proof is straightforward and thus omitted).

**Lemma 3.7.** \( A(z; \Gamma) \) admits the following asymptotic

\[ A(z; \Gamma) \sim 1 + \sum_{k=1}^{\infty} \frac{a_k}{z^{r+k+1}}, \quad z \to \infty, \quad z \in S_{r}, \]  
(3.42)

where \( a_k \) are constants.

Below we denote by \( a(z) = 1 + \sum_{k=1}^{\infty} \frac{a_k}{z^{r+k+1}} \) the formal asymptotic series of \( A(z; \Gamma) \).

**Lemma 3.8.** With \( \omega = e^{i \frac{\pi}{r}} \), we have

\[ S_{r}^* = \omega^{-1} S_{\omega r}. \]  
(3.43)

**Proof.**

\[ S_{\omega r} = \frac{1}{\omega^r} z^{1-r} - \frac{r-1}{2} \omega^r z^{-1} - \omega z = \omega \left( - \frac{1}{z^{r-1}} \frac{z^{-1}}{2} + \frac{r-1}{2} - z \right) = \omega S_{r}^*. \]  
(3.44)

Lemma 3.8 implies

\[ A^\#(z; \Gamma) = A(\omega z; \Gamma) = \frac{i \omega^{r+1}}{\sqrt{2\pi}} z^{1-r} \exp \left( - \frac{z^{r+1}}{r+1} \right) \int_{\omega^{-r} \Gamma} \exp \left[ - s^{r+1} + \frac{s}{r} z^r \right] ds, \]  
(3.46)

which satisfies

\[ A^\#(z; \Gamma) = 1 + \mathcal{O}(z^{-1}), \quad z \to \infty, \quad z \in \omega S_{r}, \]  
(3.47)

We recall now that there exists a natural bilinear pairing between solutions of two adjoint equations which is called the bilinear concomitant. In the present case, solutions of the two equations can be represented in terms of Fourier–Laplace integrals along two relatively dual bases in a certain homology of contours and the bilinear concomitant is simply a representation of the intersection pairing. For more details (and in more general terms) see [5], section 3. Let us call

\[ B(\tilde{z}; \tilde{\Gamma}) = C \frac{z^{1-r}}{\omega^r} \exp \left( - \frac{z^{r+1}}{r+1} \right) \int_{\tilde{\Gamma}} \exp \left[ - s^{r+1} + \frac{s}{r} \frac{z^r}{\omega^r} \right] ds, \]  
(3.48)

where \( \tilde{\Gamma} \) spans the dual homology of the contours spanned by \( \Gamma^* \). The concomitant reads:

\[ B \left( A(z; \Gamma), B(z; \tilde{\Gamma}) \right) = \int_{\Gamma \times \tilde{\Gamma}} ds \, dw \left( \frac{w^{r+1} - s^{r+1}}{w - s} \exp \left[ - \frac{w^{r+1} - s^{r+1}}{r(r+1)} - \frac{w - s}{r} \eta \right] \right). \]  
(3.49)
Here $\eta = z'$. The result of this integral is the intersection number of $\Gamma$ and $\hat{\Gamma}$. Indeed
\[
\int_{\Gamma \times \hat{\Gamma}} ds \frac{w^r - s^r}{w - s} \exp \left[ \frac{w^{r+1} - s^{r+1}}{r(r + 1)} - \frac{w - s}{r} \right] = 2i \text{link}(\Gamma, \hat{\Gamma}).
\] (3.50)

The bilinear concomitant (3.49) can equivalently be written in terms of a bilinear expression in the derivatives of the functions $A(z; \Gamma)$ and $B(z; \hat{\Gamma})$ by expanding
\[
\sum_{k=0}^{r-1} S^*_z z^{-r-1-k} \left( A\left( wZ; \hat{\Gamma} \right) \right) \hat{S}_z^k A(z; \Gamma) = \text{link}(\Gamma, \hat{\Gamma}) \cdot z^{-r-1}, \quad z \in \mathcal{S}_\Gamma.
\] (3.51)

See section 3 of [5].

We remark that the analytic function $A(z; \Gamma)$ that we have defined has different asymptotic behaviors at $\infty$ as the sector changes. This is called the Stokes phenomenon.

To end this section, let us solve the following ODE problem which will be useful for the next section:
\[
- S^*_z \left( \frac{D(z)}{z} \right) = A(z),
\] (3.52)
with the (formal) asymptotic boundary requirement
\[
D(z) = 1 + \mathcal{O}\left(z^{-1}\right), \quad z \to \infty.
\] (3.53)

Using the representation (3.37) for the differential operator, it is immediate to verify that
\[
D(z; \Gamma) = - i \frac{\omega}{2\pi} \frac{r+1}{r} \exp \left( - \frac{z^{r+1}}{r+1} \right) \int_{w' \Gamma} w^{-1} \exp \left[ - \frac{w^{r+1}}{(r+1)r} + \frac{w}{r} z' \right] dw.
\] (3.54)

(which is (1.24)) indeed solves the equation. Of course the equation (3.52) only determines the solution up to addition of the complementary solution of the associated homogeneous equation. This arbitrariness is reflected in the choice of the contour in (3.54); indeed the integrand has a simple pole at $w = 0$ and thus it is important to specify the index of the contour of integration relative to $w = 0$. Now, the steepest descent analysis shows that (3.54) has the behavior (3.53) provided that the contour can be retracted onto the steepest descent path of the exponential phase. The steepest descent path clearly recedes to infinity (the principal saddle point is at $w = z$). Thus the contour must have index 0 relative to $w = 0$.

Remark 3.10. Another simple argument to determine the index of the integration contour relative to $w = 0$ is that if the contour in (3.54) has index 1, then retracting the contour to the saddle yields the residue at $w = 0$, which adds a multiple of $\sqrt{2\pi z'} \exp \left( \frac{z'}{r+1} \right)$ (the complementary solution), which has necessarily an exponential growth within the sector $S_\Gamma$, and thus violates the boundary condition (3.53).
Applying the saddle point method one can obtain

**Lemma 3.11.** $D(z; \Gamma)$ admits the following asymptotic

$$D(z; \Gamma) \sim d(z) = 1 + \sum_{k=1}^{\infty} \frac{d_k}{(z+1)^k}, \quad z \to \infty, \ z \in \omega^{-1} \Sigma.$$  

(3.55)

4. A particular solution to the extended $r$-reduced KP hierarchy

In this section we finally prove Theorem 1.2. Let us further introduce some useful notations and lemmas.

**Notations**

Denote the vector space of formal series with finitely many terms with positive powers by:

$$\mathcal{A} = \left\{ \sum_{n \geq m} b_n \frac{z^n}{m} \mid m \in \mathbb{Z}, \ b_n \in \mathbb{C} \right\}. \quad (4.1)$$

For any $r \geq 2$, let $\theta = e^{2\pi i / r}$ and let

$$\mathcal{A}^{(k)} := \left\{ b(z) \in \mathcal{A} \mid b(\theta z) = \theta^k b(z), \ k \in \mathbb{Z} \right\}. \quad (4.2)$$

We also use subscripts to denote the leading order term, for example

$$\mathcal{A}^{(0)} = \left\{ 1 + \sum_{n \geq 1} \frac{b_n}{z^{n(r+1)}} \mid b_n \in \mathbb{C} \right\}, \quad \mathcal{A}^{(1)}_{z^n} = \left\{ 3z^n + \sum_{n \geq 1} \frac{b_n}{z^{n(r+1)-m}} \mid b_n \in \mathbb{C} \right\}, \ etc.$$  

Clearly $\mathcal{A}^{(k)} = z^k \mathcal{A}^{(0)}$, $\mathcal{A}^{(k)} = \mathcal{A}^{(k+r+1)}$, $\forall \ k \in \mathbb{Z}$.

**Lemma 4.1.** For any $k \in \mathbb{Z}$, $S_z \mathcal{A}^{(k)} = \mathcal{A}^{(k-r)}$, $S_z^* \mathcal{A}^{(k)} = \mathcal{A}^{(k+r)}$.

**Proof.** It suffices to consider the action of $S_z$ on a monomial $z^k$:

$$S_z z^k = \left( k - \frac{r-1}{2} \right) z^{k-r} - z^{k+1} = z^{k-r} \left( \frac{2k - r + 1}{2} - z^{r+1} \right). \quad (4.3)$$

Similarly for $S_z^*$. \hfill $\square$

In the last section, we have defined a formal series $a(z)$ as the asymptotic expansion of $A(z; \Gamma)$ in $S_z$. 
Lemma 4.2. \( a(z) \) is the unique formal solution in \( A_1 \) to the equation
\[
S_z' a(z) = (-z)^r a(z). \tag{4.4}
\]

Proof. The uniqueness follows from the recursive procedure given by the ODE. \( \square \)

Lemma 4.1 and Lemma 4.2 immediately imply

Lemma 4.3. For any \( n \geq 0 \), \( S_z^n a(z) \in A_{(n)}^{(-1)f_{+z}} \).

Remark 4.4. We know from Lemma 3.5 and the above lemmas that the point in Sato’s (formal) Grassmannian corresponding to \( Z(t) \) is given by
\[
W = \text{Span}_\mathbb{C} \left\{ a(z), S_z a(z), S_z^2 a(z), \ldots \right\}. \tag{4.5}
\]

This is actually well-known for example, in [20] and in [2].

Lemma 4.5. The following identities hold true:
\[
\frac{1}{2\pi i} \oint_{\gamma_{\infty}} S_z^{*m} \left( a(oz) \right) S_z^n a(z) \, dz = 0, \quad \forall \, m, n \geq 0. \tag{4.6}
\]

If \( b(z) \in A_1 \) satisfies that for all \( n \geq 0 \)
\[
\frac{1}{2\pi i} \oint_{\gamma_{\infty}} b(z) S_z^n a(z) \, dz = 0, \tag{4.7}
\]

then \( b(z) = a(oz) \).

Proof. Due to Lemma 3.5 and equation (3.45), the identity (4.6) is nothing but the bilinear identity (2.12) evaluated at \( t \mid_{KP, r=2} = t' \mid_{KP, r=2} = 0 \) and subsequent series expansion in \( x \). Write
\[
b(z) = 1 + \sum_{k=1}^{\infty} b_k \frac{1}{z^k}; \tag{4.8}
\]
then \( b_1, b_2, b_3, \ldots \) can be solved recursively from the relations (4.7) by taking \( n = 1, 2, 3, \ldots \) respectively. \( \square \)

Before proceeding further, let us mention that Lemma 3.9 and Lemma 4.2 imply

Lemma 4.6. The following identities hold true in \( A_1 \):
\[
\sum_{k=0}^{r-1} S_z^{*r-1-k} \left( a(oz) \right) S_z^k a(z) = (-1)^{r-1} r^{r-1}. \tag{4.9}
\]

Remark 4.7. In the case \( r = 2 \), the identity appearing in this lemma was observed in [24] and it was used by Buryak [7] to prove the analog of Lemma 4.5 for \( r = 2 \). We point out here that,
in fact, this identity is a simple consequence of a classical object, the bilinear concomitant of adjoint equations, introduced by Legendre [18]!

**Lemma 4.8.** The formal series $d(z)$ defined in the previous section as the asymptotic expansion of $D(z; \Gamma)$ in $\omega^{-1}\tau$ is the unique formal solution in $A_1$ to the equation

$$-S^*_z \left( \frac{d(z)}{z} \right) = a(\omega z).$$

**(Proof.** The uniqueness of the solution in $A_1$ follows from recursive procedure given by the above ODE. \)

**Example 4.9.** For $r=2$ we have

$$a(z) = 1 - \frac{5}{24z^3} + \frac{385}{1152z^6} - \frac{85085}{82944z^9} + \frac{37182145}{7962624z^{12}} + O(z^{-15}),$$

$$d(z) = 1 + \frac{41}{24z^3} + \frac{9241}{1152z^6} + \frac{5075225}{82944z^9} + \frac{5153008945}{7962624z^{12}} + O(z^{-15}).$$

For $r=3$ we have

$$a(z) = 1 - \frac{7}{12z^4} + \frac{385}{288z^8} - \frac{39655}{10368z^{12}} + \frac{665665}{497664z^{16}} + O(z^{-20}),$$

$$d(z) = 1 + \frac{31}{12z^4} + \frac{4849}{288z^8} + \frac{1785295}{10368z^{12}} + \frac{1200383905}{497664z^{16}} + O(z^{-20}).$$

For $r=4$ we have

$$a(z) = 1 - \frac{9}{8z^5} + \frac{441}{128z^{10}} - \frac{30303}{5120z^{15}} - \frac{25162137}{163840z^{20}} + O(z^{-25}),$$

$$d(z) = 1 + \frac{29}{8z^5} + \frac{3921}{128z^{10}} + \frac{1990803}{5120z^{15}} + \frac{1089687543}{163840z^{20}} + O(z^{-25}).$$

**Theorem 4.10.** Let

$$Z_E(t_{kr}) = Z(t) \Psi(t_{kr}),$$

be the partition function of the extended $r$-reduced KP hierarchy, where $Z(t)$ is the partition function of the $A_{r-1}$ Frobenius manifold and

$$\Psi(t_{kr}) = \frac{1}{2\pi i} \oint_{|z| = \infty} g(z) \Psi(t_{kr}; z) \frac{dz}{z}.$$
then there exists a unique sequence of numbers $g_1, g_2, g_3, \ldots$ such that
\[
L_{-1}^\text{ext} Z_E(t_{kp}) = 0.
\] (4.20)
And these numbers are given by
\[
g_{(r+1)k+p} = 0 \quad (k \geq 0, \quad p = 1, 2, \ldots, r); \quad g_{(r+1)k} = d_k, \quad k \geq 1.
\] (4.21)

**Proof.** Let $K(t_{kp}; z)$ denote the wave potential associated to $Z(t)$. Then by definition
\[
Z_E(t_{kp}) = \frac{1}{2\pi i} \oint_{z=\infty} g(z) K(t_{kp}; z) \frac{dz}{z}.
\] (4.22)
By using Lemma 3.3 we have
\[
0 = L_{-1}^\text{ext} Z_E = \frac{1}{2\pi i} \oint_{z=\infty} g(z) L_{-1}^\text{ext} K(t_{kp}; z) \frac{dz}{z} = \frac{1}{2\pi i} \oint_{z=\infty} g(z) S_t^1 \left( K(t_{kp}; z) \right) \frac{dz}{z}
\]
\[
= \frac{Z(t)}{2\pi i} \oint_{z=\infty} g(z) S_t^1 \left( \psi(t_{kp}; z) \right) \frac{dz}{z}.
\] (4.23)
Taking $t_2 = t_3 = \ldots = 0$ in the above equation we obtain
\[
\frac{1}{2\pi i} \oint_{z=\infty} g(z) S_t f(x; z) \frac{dz}{z} = 0.
\] (4.24)
Substituting (3.27) in (4.24) we have
\[
\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{2\pi i} \oint_{z=\infty} S_t^* \left( \frac{g(z)}{z} \right) S_t^n a(z) \frac{dz}{z} = 0.
\] (4.25)
This is equivalent to
\[
\frac{1}{2\pi i} \oint_{z=\infty} S_t^* \left( \frac{g(z)}{z} \right) S_t^n a(z) \frac{dz}{z} = 0, \quad \forall \ n \geq 0.
\] (4.26)
Due to Lemma 4.5 and the requirement (4.19), we obtain
\[
S_t^* \left( \frac{g(z)}{z} \right) = -a(oz).
\] (4.27)
Lemma 4.8 implies
\[
g(z) = d(z).
\] (4.28)
Now if $g(z) = d(z)$, then we know from Lemma 4.5 that (4.24) holds true. Since the wave function $\psi(t_{kp}; z)$ satisfies the linear evolutionary equations (1.14), we know that (4.24) also implies (4.23). The theorem is proved. \hfill \square
Proposition 4.11. The function $\Psi(t_{kr})$ defined by (4.18) with $g(z) = d(z)$ satisfies
\begin{equation}
\Psi(t_{kr})\big|_{t_j=0} \equiv 1, \quad \forall t_i.
\end{equation}

Proof. Reminding the reader that $t_i = x$, we have for all $x$
\begin{equation}
\Psi(t_{kr})\big|_{t_j=0} = \frac{1}{2\pi i} \oint_{z=\infty} d(z) f(x; z) \frac{dz}{z}.
\end{equation}
\begin{equation}
= \frac{1}{2\pi i} \oint_{z=\infty} d(z) \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} S_n^a(z) x^n \frac{dz}{z} = 1.
\end{equation}

The last equality follows from Lemma 4.8 together with Lemma 4.5 for the terms $n \geq 1$, while for $n = 0$ we use the fact that both $d(z)$ and $a(z)$ are of the form $1 + O(z^{-r-1})$. The proposition is proved.

Theorem 1.2 is then a combination of Theorem 4.10 and Proposition 4.11.

Acknowledgments

We thank Ferenc Balogh and Thomas Bothner for helpful discussions; we also thank an anonymous referee for valuable suggestions. M B acknowledges the support of the Natural Sciences and Engineering Research Council of Canada and the Fonds de recherche du Québec -Nature et technologies. D Y wishes to thank Youjin Zhang and Boris Dubrovin for their advice and helpful discussions. He acknowledges the support of PRIN 2010-11 Grant ‘Geometric and analytic theory of Hamiltonian systems in finite and infinite dimensions’ of Italian Ministry of Universities and Researches, and the support of the Marie Curie IRSES project RIMMP. He also wishes to thank the Centre de Recherches Mathématiques and the Department of Mathematics and Statistics at Concordia University for generous hospitality, where the manuscript was completed.

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