On surfaces with \( p_g = q = 1 \) and non-ruled bicanonical involution

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Abstract. This paper classifies surfaces \( S \) of general type with \( p_g = q = 1 \) having an involution \( i \) such that \( S/i \) has non-negative Kodaira dimension and that the bicanonical map of \( S \) factors through the double cover induced by \( i \).

It is shown that \( S/i \) is regular and either: a) the Albanese fibration of \( S \) is of genus 2 or b) \( S \) has no genus 2 fibration and \( S/i \) is birational to a \( K3 \) surface. For case a) a list of possibilities and examples are given. An example for case b) with \( K^2 = 6 \) is also constructed.

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1. Introduction

Let \( S \) be a smooth irreducible projective surface of general type. The pluricanonical map \( \phi_n \) of \( S \) is the map given by the linear system \([nK_S]\), where \( K_S \) is the canonical divisor of \( S \). For minimal surfaces \( S \), \( \phi_n \) is a birational morphism if \( n \geq 5 \) (cf. [4, Chapter VII, Theorem (5.2)]). The bicanonical map

\[ \phi_2 : S \longrightarrow \mathbb{P}^{K^2_S + \chi(S) - 1} \]

is a morphism if \( p_g(S) \geq 1 \) (this result is due to various authors, see [7] for more details). This paper focuses on the study of surfaces \( S \) with \( p_g(S) = q(S) = 1 \) having an involution \( i \) such that the Kodaira dimension of \( S/i \) is non-negative and \( \phi_2 \) is composed with \( i \), i.e. it factors through the double cover \( p : S \to S/i \).

There is an instance where the bicanonical map is necessarily composed with an involution: suppose that \( S \) has a fibration of genus 2, i.e. it has a morphism \( f \) from \( S \) to a curve such that a general fibre \( F \) of \( f \) is irreducible of genus 2. The

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system $|2K_S|$ cuts out on $F$ a subseries of the bicanonical series of $F$, which is composed with the hyperelliptic involution of $F$, and then $\phi_2$ is composed with an involution. This is the so called standard case of non-birationality of the bicanonical map.

By the results of Bombieri, [2], improved later by Reider, [22], a minimal surface $S$ satisfying $K^2 > 9$ and $\phi_2$ non-birational necessarily presents the standard case of non-birationality of the bicanonical map.

The non-standard case of non-birationality of the bicanonical map, i.e. the case where $\phi_2$ is non-birational and the surface has no genus 2 fibration, has been studied by several authors.

Du Val, [16], classified the regular surfaces $S$ of general type with $p_g \geq 3$, whose general canonical curve is smooth and hyperelliptic. Of course, for these surfaces, the bicanonical map is composed with an involution $i$ such that $S/i$ is rational. The families of surfaces exhibited by Du Val, presenting the non-standard case, are nowadays called the Du Val examples.

Other authors have later studied the non-standard case: the articles [8, 10, 12, 13, 25] and [3] treat the cases $\chi(O_S) > 1$ or $q(S) \geq 2$ (cf. the expository paper [11] for more information on this problem).

Xiao Gang, [25], presented a list of possibilities for the non-standard case of non-birationality of the bicanonical morphism $\phi_2$. For the case when $\phi_2$ has degree 2 and the bicanonical image is a ruled surface, Theorem 2 of [25] extended Du Val’s list to $p_g(S) \geq 1$ and added two extra families (this result is still valid assuming only that $\phi_2$ is composed with an involution such that the quotient surface is a ruled surface). Recently G. Borrelli [3] excluded these two families, confirming that the only possibilities for this instance are the Du Val examples.

For irregular surfaces the following holds (see [25, Theorems 1, 3], [8, Theorem A], [12, Theorem 1.1], [13]):

Suppose that $S$ is a smooth minimal irregular surface of general type having non-birational bicanonical map. If $p_g(S) \geq 2$ and $S$ has no genus 2 fibration, then only the following (effective) possibilities occur:

- $p_g(S) = q(S) = 2, K_S^2 = 4$;
- $p_g(S) = q(S) = 3, K_S^2 = 6$.

In both cases $\phi_2$ is composed with an involution $i$ such that $\text{Kod}(S/i) = 2$.

This paper completes this result classifying the minimal surfaces $S$ with $p_g(S) = q(S) = 1$ such that $\phi_2$ is composed with an involution $i$ satisfying $\text{Kod}(S/i) \geq 0$.

The main result is the following:

**Theorem 1.1.** Let $S$ be a smooth minimal irregular surface of general type with an involution $i$ such that $\text{Kod}(S/i) \geq 0$ and the bicanonical map $\phi_2$ of $S$ is composed with $i$. If $p_g(S) = q(S) = 1$, then only the following possibilities can occur:
a) $S/i$ is regular, the Albanese fibration of $S$ has genus 2 and
   (i) $\text{Kod}(S/i) = 2$, $\chi(S/i) = 2$, $K_S^2 = 2$, $\deg(\phi_2) = 8$, or
   (ii) $\text{Kod}(S/i) = 1$, $\chi(S/i) = 2$, $2 \leq K_S^2 \leq 4$, $\deg(\phi_2) \geq 4$, or
   (iii) $S/i$ is birational to a K3 surface, $3 \leq K_S^2 \leq 6$, $\deg(\phi_2) = 4$;

b) $S$ has no genus 2 fibration and $S/i$ is birational to a K3 surface.

Moreover, there are examples for (i), (ii) with $K_S^2 = 4$, (iii) with $K_S^2 = 3, 4$ or 5 and for b) with $K_S^2 = 6$ and $\phi_2$ of degree 2.

**Remark 1.2.** Examples for (iii) were given by Catanese in [6]. The other examples will be presented in Section 5.

Note that surfaces of general type with $p_g = q = 1$ and $K^2 = 3$ or 8 were also studied by Polizzi in [19] and [20].

In the example in Section 5 for case b) of Theorem 1.1, $S$ has $p_g = q = 1$ and $K^2 = 6$. This seems to be the first construction of a surface with these invariants. This example contradicts a result of Xiao Gang. More precisely, the list of possibilities in [25] rules out the case where $S$ has no genus 2 fibration, $p_g(S) = q(S) = 1$ and $S/i$ is birational to a K3 surface. In Lemma 7 of [25] it is written that $R$ has only negligible singularities, but the possibility $\chi(K_{\tilde{P}} + \tilde{\delta}) < 0$ in formula (3) of page 727 was overlooked. In fact we will see that $R$ (in our notation) can have a non-negligible singularity.

An important technical tool that will be used several times is the **canonical resolution** of singularities of a surface. This is a resolution of singularities as described in [4].

The paper is organized as follows. Section 2 studies some general properties of a surface of general type $S$ with an involution $i$. Section 3 states some properties of surfaces with $p_g = q = 1$. Section 4 contains the proof of Theorem 1.1. Crucial ingredients for this proof are the existence of the Albanese fibration of $S$ and the formulas of Section 2. In Section 5 examples for Theorem 1.1 are obtained, via the construction of branch curves with appropriate singularities. The Computational Algebra System *Magma* is used to perform the necessary calculations (visit http://magma.maths.usyd.edu.au/magma for more information about Magma).

**Notation and conventions.** We work over the complex numbers; all varieties are assumed to be projective algebraic. We do not distinguish between line bundles and divisors on a smooth variety. Linear equivalence is denoted by $\equiv$. A **nodal curve** or $(-2)$-curve $C$ on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $C^2 = -2$. Given a surface $X$, Kod$(X)$ means the **Kodaira dimension** of $X$. We say that a curve singularity is **negligible** if it is either a double point or a triple point which resolves to at most a double point after one blow-up. A $(n, n)$ **point**, or **point of type** $(n, n)$, is a point of multiplicity $n$ with an infinitely near point also of multiplicity $n$. An **involution** of a surface $S$ is an automorphism of $S$ of order 2. We say that a map is composed with an involution $i$ of $S$ if it factors through the map $S \to S/i$. The rest of the notation is standard in Algebraic Geometry.
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2. Generalities on involutions

Let \( S \) be a smooth minimal surface of general type with an involution \( i \). As \( S \) is minimal of general type, this involution is biregular. The fixed locus of \( i \) is the union of a smooth curve \( R'' \) (possibly empty) and of \( t \geq 0 \) isolated points \( P_1, \ldots, P_t \). Let \( S/i \) be the quotient of \( S \) by \( i \) and \( p : S \to S/i \) be the projection onto the quotient. The surface \( S/i \) has nodes at the points \( Q_i := p(P_i), i = 1, \ldots, t \), and is smooth elsewhere. If \( R'' \neq \emptyset \), the image via \( p \) of \( R'' \) is a smooth curve \( B'' \) not containing the singular points \( Q_i, i = 1, \ldots, t \). Let now \( h : V \to S \) be the blow-up of \( S \) at \( P_1, \ldots, P_t \) and set \( R' = h^*R'' \). The involution \( i \) induces a biregular involution \( \tilde{i} \) on \( V \) whose fixed locus is \( R := R' + \sum_1^t h^{-1}(P_i) \). The quotient \( W := V/\tilde{i} \) is smooth and one has a commutative diagram:

\[
\begin{array}{ccc}
V & \xrightarrow{h} & S \\
\downarrow \pi & & \downarrow p \\
W & \xrightarrow{g} & S/i
\end{array}
\]

where \( \pi : V \to W \) is the projection onto the quotient and \( g : W \to S/i \) is the minimal desingularization map. Notice that

\[
A_i := g^{-1}(Q_i), \quad i = 1, \ldots, t,
\]

are \((-2)\)-curves and \( \pi^*(A_i) = 2 \cdot h^{-1}(P_i) \). Set \( B' := g^*(B'') \). Because \( \pi \) is a double cover with branch locus \( B' + \sum_1^t A_i \), there exists a line bundle \( L \) on \( W \) such that

\[
2L \equiv B := B' + \sum_1^t A_i.
\]

It is well known that (cf. [4, Chapter V, Section 22]):

\[
\begin{align*}
p_g(S) &= p_g(V) = p_g(W) + h^0(W, O_W(K_W + L)), \\
q(S) &= q(V) = q(W) + h^1(W, O_W(K_W + L)),
\end{align*}
\]

and

\[
K_S^2 - t = K_V^2 = 2(K_W + L)^2,
\]

\[
\chi(O_S) = \chi(O_V) = 2\chi(O_W) + \frac{1}{2} L(K_W + L).
\]

(2.1)
Furthermore, from the papers [12] and [9], if $S$ is a smooth minimal surface of general type with an involution $i$, then

$$\chi(\mathcal{O}_W(2K_W + L)) = h^0(W, \mathcal{O}_W(2K_W + L)), \tag{2.2}$$

$$\chi(\mathcal{O}_W) - \chi(\mathcal{O}_S) = K_W(K_W + L) - h^0(W, \mathcal{O}_W(2K_W + L)) \tag{2.3}$$

and the bicanonical map

$$\phi_2$$ is composed with $i$ if and only if $h^0(W, \mathcal{O}_W(2K_W + L)) = 0. \tag{2.4}$

From formulas (2.1) and (2.3) one obtains the number $t$ of nodes of $S/i$:

$$t = K_S^2 + 6\chi(\mathcal{O}_W) - 2\chi(\mathcal{O}_S) - 2h^0(W, \mathcal{O}_W(2K_W + L)). \tag{2.5}$$

Let $P$ be a minimal model of the resolution $W$ of $S/i$ and $\rho : W \to P$ be the natural projection. Denote by $\overline{B}$ the projection $\rho(B)$ and by $\delta$ the “projection” of $L$.

**Remark 2.1.** Resolving the singularities of $\overline{B}$ we obtain exceptional divisors $E_i$ and numbers $r_i \in 2\mathbb{N}^+$ such that $E_i^2 = -1$, $K_W = \rho^*(K_P) + \sum E_i$ and $B = \rho^*(\overline{B}) - \sum r_i E_i$.

**Proposition 2.2.** With the previous notations, the bicanonical map $\phi_2$ is composed with $i$ if and only if

$$\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2).$$

**Proof.** From formulas (2.3), (2.4) and Remark 2.1 we get

$$\chi(\mathcal{O}_P) - \chi(\mathcal{O}_S) = \frac{1}{2} K_W(2K_W + 2L)$$

$$= \frac{1}{2} \left( \rho^*(K_P) + \sum E_i \right) \left( 2 \rho^*(K_P + \delta) + \sum (2 - r_i) E_i \right)$$

$$= K_P(K_P + \delta) + \frac{1}{2} \sum (r_i - 2).$$

\[ \square \]

3. **Surfaces with $p_g = q = 1$ and an involution**

Let $S$ be a minimal smooth projective surface of general type satisfying $p_g(S) = q(S) = 1$.

Note that then $2 \leq K_S^2 \leq 9$: we have $K_S^2 \leq 9\chi(\mathcal{O}_S)$ by the Myiaoka-Yau inequality (see [4, Chapter VII, Theorem (4.1)]) and $K_S^2 \geq 2p_g$ for an irregular surface (see [14]).
Furthermore, if the bicanonical map of $S$ is not birational, then $K_S^2 \neq 9$. In fact, by [12], if $K_S^2 = 9$ and $\phi_2$ is not birational, then $S$ has a genus 2 fibration, while Théorème 2.2 of [24] implies that if $S$ has a genus 2 fibration and $p_g(S) = q(S) = 1$, then $K_S^2 \leq 6$.

Since $q(S) = 1$ the Albanese variety of $S$ is an elliptic curve $E$ and the Albanese map is a connected fibration (see e. g. [1] or [4]).

Suppose that $S$ has an involution $i$. Then $i$ preserves the Albanese fibration (because $q(S) = 1$) and so we have a commutative diagram

$$
\begin{array}{c}
V \xrightarrow{h} S \xrightarrow{p} E \\
\downarrow \pi \downarrow \downarrow \\
W \xrightarrow{\pi} S/i \xrightarrow{\rho} \Delta
\end{array}
$$

where $\Delta$ is a curve of genus $\leq 1$. Denote by

$$f_A : W \to \Delta$$

the fibration induced by the Albanese fibration of $S$.

Recall that

$$\rho : W \to P$$

is the projection of $W$ onto its minimal model $P$ and

$$\overline{B} := \rho(B),$$

where $B := B' + \sum_1^t A_i \subset W$ is the branch locus of $\pi$.

Let

$$\overline{B}' := \rho(B'), \quad \overline{A}_i = \rho(A_i).$$

When $\overline{B}$ has only negligible singularities, the map $\rho$ contracts only exceptional curves contained in fibres of $f_A$. In fact, there exists otherwise a $(-1)$-curves $J \subset W$ such that $JB = 2$ and so $\pi^*(J)$ is a rational curve transverse to the fibres of the (genus 1 base) Albanese fibration of $S$, which is impossible. Moreover, $\rho$ contracts no curve meeting $\sum A_i$, because $h : V \to S$ is the contraction of isolated $(-1)$-curves. Therefore the singularities of $\overline{B}$ are exactly the singularities of $\overline{B}'$, i.e. $\overline{B}' \cap \sum \overline{A}_i = \emptyset$. In this case the image of $f_A$ on $P$ will be denoted by $f_A$.

If $\Delta \cong \mathbb{P}^1$, then the double cover $E \to \Delta$ is ramified over 4 points $p_j$ of $\Delta$, thus the branch locus $B' + \sum_1^t A_i$ is contained in 4 fibres

$$F^j_A := f_A^*(p_j), \quad j = 1, \ldots, 4,$$

of the fibration $f_A$. Hence, by Zariski’s Lemma (see e. g. [4]), the irreducible components $B'_i$ of $B'$ satisfy $B'^2_i \leq 0$. If $\overline{B}$ has only negligible singularities, then also $\overline{B}'^2 \leq 0$. As $\pi^*(F^j_A)$ is of multiplicity 2, each component of $F^j_A$ which is not a component of the branch locus $B' + \sum_1^t A_i$ must be of even multiplicity.
4. The classification theorem

In this section we will prove Theorem 1.1. We will freely use the notation and results of Sections 2 and 3.

Proof of Theorem 1.1. Since $p_g(P) \leq p_g(S) = 1$, then $\chi(\mathcal{O}_P) \leq 2 - q(P) \leq 2$. Proposition 2.2 gives $\chi(\mathcal{O}_P) \geq 1$, because $K_P$ is nef (i.e. $K_PC \geq 0$ for every curve $C$). So from Proposition 2.2 and the classification of surfaces (see e.g. [1] or [4]) only the following cases can occur:

1. $P$ is of general type;
2. $P$ is a surface with Kodaira dimension 1;
3. $P$ is an Enriques surface, $\mathcal{B}$ has only negligible singularities;
4. $P$ is a K3 surface, $\mathcal{B}$ has a 4-uple or $(3, 3)$ point, and possibly negligible singularities.

We will show that: case 3 does not occur, in cases 1 and 2 the Albanese fibration has genus 2 and only in case 4 the Albanese fibration can have genus $\neq 2$.

Each of cases 1, ..., 4 will be studied separately. We start by considering:

Case 1. As $P$ is of general type, $K_P^2 \geq 1$ and $K_P$ is nef, Proposition 2.2 gives $\chi(\mathcal{O}_P) = 2$, $K_P^2 = 1$, $K_P\delta = 0$ and $\mathcal{B}$ has only negligible singularities. The equality $K_P\overline{B} = K_P2\delta = 0$ implies $\overline{B}^2 < 0$ when $B' \neq 0$. In the notation of Remark 2.1 one has $K_W \equiv \rho^*(K_P) + \sum E_i$ and $B' = \rho^*(\mathcal{B}') - 2\sum E_i$. So

$$K_S^2 = K_V^2 + t = \frac{1}{4}(2K_V)^2 + t = \frac{1}{4}\pi^*(2K_W + B)^2 + t$$

$$= \frac{1}{2}(2K_W + B)^2 + t = \frac{1}{2}(2K_W + B')^2 = \frac{1}{2}(2K_P + \mathcal{B}')^2 = \frac{1}{2}(4 + \overline{B}'^2).$$

Since $K_S^2 \geq 2p_g(S)$ for an irregular surface (see [14]), $\overline{B}'^2 < 0$ is impossible, hence $B' = 0$ and $K_S^2 = 2$. By [5] minimal surfaces of general type with $p_g = q = 1$ and $K^2 = 2$ have Albanese fibration of genus 2. This is case (i) of Theorem 1.1. We will see in Section 5 an example for this case.

Finally the fact that $\deg(\phi_2) = 8$ follows immediately because $\phi_2$ is a morphism onto $\mathbb{P}^2$ and $(2K_S)^2 = 8$.

Next we exclude:

Case 3. Using the notation of Remark 2.1 of Section 2, we can write $K_W \equiv \rho^*(K_P) + \sum E_i$ and $2L \equiv \rho^*(2\delta) - 2\sum E_i$, for some exceptional divisors $E_i$. Hence

$$L(K_W + L) = \frac{1}{2}L(2K_W + 2L)$$

$$= \frac{1}{2}(\rho^*(\delta) - \sum E_i)(2\rho^*(K_P) + \rho^*(2\delta)) = \frac{1}{2}\delta(2K_P + 2\delta) = \delta^2.$$
and then, from (2.1), \( \delta^2 = -2 \). Now (2.4) and (2.5) imply \( t = K^2_S + 4 \), thus

\[
\overline{B}'^2 = \overline{B}^2 + 2t = (2\delta)^2 + 2t = -8 + 2t = 2K^2_S > 0.
\]

This is a contradiction because we have seen that \( \overline{B}'^2 \leq 0 \) when \( \overline{B}' \) has only negligible singularities. Thus case 3) does not occur.

Now we focus on:

**Case 2.** Since we are assuming that \( \text{Kod}(P) = 1 \), \( P \) has an elliptic fibration (i.e. a morphism \( f_e : P \to C \) where \( C \) is a curve and the general fibre of \( f_e \) is a smooth connected elliptic curve). Then \( K_P \) is numerically equivalent to a rational multiple of a fibre of \( f_e \) (see e.g. [1] or [4]). As \( K_P \delta \geq 0 \), Proposition 2.2, together with \( \chi(O_P) \leq 2 \), yield \( K_P \delta = 0 \) or 1.

Denote by \( F_e \) (respectively \( F_A \)) a general fibre of \( f_e \) (respectively \( f_A \)) and let \( \overline{F_A} := \rho(F_A) \). If \( K_P \delta = 0 \), then \( F_e \overline{B} = 0 \), which implies that the fibration \( f_e \) lifts to an elliptic fibration on \( S \). This is impossible because \( S \) is a surface of general type. So \( K_P \delta = 1 \) and, since \( p_g(P) \leq p_g(S) = 1 \), the only possibility allowed by Proposition 2.2 is

\[
p_g(P) = 1, \quad q(P) = 0 \quad \text{and} \quad \overline{B} \text{ has only negligible singularities.}
\]

Now \( q(P) = 0 \) implies that the elliptic fibration \( f_e \) has a rational base, thus the canonical bundle formula (see e.g. [4, Chapter V, Section 12]) gives \( K_P = \sum (m_i - 1) F_i \), where \( m_i F_i \) are the multiple fibres of \( f_e \). From

\[
2 = 2\delta K_P = \overline{B}' K_P = \overline{B}' \sum (m_i - 1) F_i, \quad \overline{B}' F_i \equiv 0 \pmod{2}
\]

we get

\[
K_P = \frac{1}{2} F_e.
\]

Since \( \overline{B} \) has only negligible singularities, \( \overline{B}'^2 \leq 0 \) and then

\[
2K^2_S = (2K_W + B')^2 = \rho^* \left( 2K_P + \overline{B}' \right)^2 = 8 + \overline{B}'^2 \leq 8. \quad (4.1)
\]

Therefore \( 2 \leq K^2_S \leq 4 \). If \( K^2_S = 2 \), then the Albanese fibration of \( S \) is of genus 2, by [5]. So, to prove statement a), (ii) of Theorem 1.1, we must show that for \( K^2_S = 3 \) or 4 the Albanese fibration of \( S \) has genus 2. We will study each of these cases separately.

First we consider

* **\( K^2_S = 4 \)**

Let \( \overline{F^i} := \rho(F^i_A) \), \( i = 1, \ldots, 4 \).
Claim 4.1. If $f_A$ is not a genus 2 fibration then

$$\overline{F_A^j} = 2\overline{B'},$$

for some $j \in \{1, \ldots, 4\}$.

**Proof.** By formula (4.1) $\overline{B'}^2 = 0$, and so $\overline{B'}$ contains the support of $x \geq 1$ of the $\overline{F_A^i}$’s. The facts $K_P F_A > 0$ (because $g(F_A) \geq 2$) and $K_P \overline{B'} = 2$ imply $x = 1$, i.e. $\overline{F_A^j} = k\overline{B'}$, for some $j \in \{1, \ldots, 4\}$ and $k \in \mathbb{N}^+$. If $k = 1$ then $F_A K_P = 2$, thus $F_A$ is of genus 2 and $S$ is as in case (ii) of Theorem 1.1.

Suppose now $k \geq 2$. Then each irreducible component of the divisor

$$D := \overline{F_A^1} + \ldots + \overline{F_A^4}$$

whose support is not in $\sum_{1}^{14} \overline{\Gamma_i}$ is of multiplicity greater than 1. The fibration $\overline{f_A}$ gives a cover $F_e \to \mathbb{P}^1$ of degree $\overline{F_A}F_e$, for a general fibre $F_e$ of the elliptic fibration $f_e$. The Hurwitz formula (see e.g. [17]) says that the ramification degree $r$ of this cover is $2\overline{F_A}F_e$. Let $p_1, \ldots, p_n$ be the points in $F_e \cap D$ and $\alpha_i$ be the intersection number of $F_e$ and $D$ at $p_i$. Of course $F_e D = 4\overline{F_A}F_e = \sum_{1}^{n} \alpha_i$ and then $F_e \cap \sum \overline{\Gamma_i} = \emptyset$ implies $\alpha_i \geq 2$, $i = 1, \ldots, n$. We have

$$2\overline{F_A}F_e = r = \sum_{1}^{n} (\alpha_i - 1) = \sum_{1}^{n} \alpha_i - n = 4\overline{F_A}F_e - n,$$

i.e. $n \geq 2\overline{F_A}F_e$. The only possibility is $n = 2\overline{F_A}F_e$ and $\alpha_i = 2 \forall i$, which means that every component $\Gamma$ of $D$ such that $\Gamma F_e \neq 0$ is exactly of multiplicity 2. In particular an irreducible component of $\overline{B'}$ is of multiplicity 2, thus $k = 2$, i.e. $\overline{F_A^j} = 2\overline{B'}$.

**Claim 4.2.** There is a smooth rational curve $C$ contained in a fibre $F_C$ of the elliptic fibration $f_e$, and not contained in fibres of $\overline{f_A}$, such that

$$m := \overline{C} \sum_{1}^{t} \overline{A_i} \leq 3,$$  \hspace{1cm} (4.2)

where $\overline{C}$ is the strict transform of $C$ in $W$.

**Proof.** Since $\overline{A_i} F_e = \overline{A_i} 2K_P = 0$, then each $\overline{A_i}$ is contained in a fibre of $f_e$, and in particular the elliptic fibration $f_e$ has reducible fibres. Denote by $C$ an irreducible component of a reducible fibre $F_C$ of $f_e$, by $\xi$ the multiplicity of $C$ in $F_C$ and by $\overline{C}$ the strict transform of $C$ in $W$. If the intersection number of $C$ and the support of $F_C - \xi C$ is greater than 3 then, from the configurations of singular fibres of an elliptic fibration (see e.g. [4, Chapter V, Section 7]), $F_C$ must be of type $I_0^*$,
i.e. it has the following configuration: it is the union of four disjoint $(-2)$-curves $\theta_1, \ldots, \theta_4$ with a $(-2)$-curve $\theta$, with multiplicity 2, such that $\theta \theta_i = 1$, $i = 1, \ldots, 4$.

So if $\sum_{i=1}^t A_i > 3$, the fibre $F_C$ containing $C$ is of type $I_0^*$ with $\sum_{i=1}^t A_i = 4$. Since the number of nodes of $S/i$ is $t = K_S^2 + 10 = 14 \not\equiv 0 \pmod{4}$, there must be a reducible fibre such that for every component $C \not\subseteq \sum_{i=1}^t A_i$, $\sum_{i=1}^t A_i \leq 3$.

As $f_e \neq f_A$ and the $A_i$'s are contained in fibres of $f_A$, we can choose $C$ not contained in fibres of $f_A$.

Let $C$ be as in Claim 4.2 and consider the resolution $\tilde{\nu} : V \rightarrow \nu$ of the singularities of $\pi^*(\widehat{C})$. Let $G \subset V$ be the strict transform of $\pi^*(\widehat{C})$. Notice that $G$ has multiplicity 1, because $C$ transverse to the fibres of $f_A$ implies $C \not\subseteq B$. Recall that $E$ denotes the basis of the Albanese fibration of $S$.

Claim 4.3. The Albanese fibration of $\tilde{\nu}$ induces a cover $G : E$ with ramification degree

$$r := K_{\tilde{\nu}} G + G^2.$$

Proof. Let $G_1, \ldots, G_h$ be the connected (hence smooth) components of $G$. The curve $C$ is not contained in fibres of $\overline{f_A}$, thus $G$ is not contained in fibres of the Albanese fibration of $\tilde{\nu}$. This fibration induces a cover $G_i : E$ with ramification degree, from the Hurwitz formula,

$$r_i = 2g(G_i) - 2 = K_{\tilde{\nu}} G_i + G_i^2.$$ 

This way we have a cover $G : E$ with ramification degree

$$r = \sum r_i = K_{\tilde{\nu}} (G_1 + \cdots + G_h) + \left( G_1^2 + \cdots + G_h^2 \right) = K_{\tilde{\nu}} G + G^2. \quad \square$$

We are finally in position to show that $g(F_A) = 2$.

Let $n := \widehat{C} B'$. We have

$$2K_V \pi^*(\widehat{C}) = \pi^*(2K_W + B' + \sum A_i) \pi^*(\widehat{C})$$
$$= 2(2K_W + B' + \sum A_i) \widehat{C} = 4K_W \widehat{C} + 2(B' + \sum A_i) \widehat{C}$$
$$= 4(-2 - \widehat{C}^2) + 2(n + m) = -8 - 2\pi^*(\widehat{C})^2 + 2(n + m),$$

i.e.

$$K_V \pi^*(\widehat{C}) + \pi^*(\widehat{C})^2 = n + m - 4.$$

Suppose that $g(F_A) \neq 2$. Let $\Lambda \subset V$ be the double Albanese fibre induced by $F_A^J = 2B'$ (as in Claim 4.1) and $\overline{\Lambda} \subset \overline{V}$ be the total transform of $\Lambda$. From

$$G\overline{\Lambda} = \pi^*(\widehat{C}) \Lambda \geq \pi^*(\widehat{C}) \pi^*(B') = 2n$$

we conclude that $g(F_A) = 2$. 

\begin{flushright} \square \end{flushright}
one has \( r \geq n \). Then
\[
n + m - 4 = K_V \pi^*(\mathcal{C}) + \pi^*(\mathcal{C})^2 \geq K_{\tilde{V}} G + G^2 = r \geq n
\]
and so \( m \geq 4 \), which contradicts Claim 4.2.

So if \( K_S^2 = 4 \), then the Albanese fibration of \( S \) is of genus 2.

We will now consider the possibility
\[ K_S^2 = 3 \]

In this case a general Albanese fibre \( \Lambda \) has genus 2 or 3 (see [7]). Suppose then \( g(\Lambda) = 3 \). Surfaces \( S \) with \( K_S^2 = g(\Lambda) = 3 \) are studied in detail in [7]. There (see also [18]) it is shown that the relative canonical map \( \gamma \), given by \( |K_S + n\Lambda| \) for some \( n \), is a morphism.

We know that \( K_p \overline{B'} = 2 \) and \( \overline{B'}^2 = -2 \), by (4.1). We have already seen that \( \overline{B} \) has only negligible singularities (which means \( r_i = 2 \forall i \), in the notation of Remark 2.1) and then \( \rho \) contracts no curve meeting \( \sum A_i \). Let \( R' \) be the support of \( \pi^*(B') \).

**Claim 4.4.** We have
\[
K_V R' = 1.
\]

**Proof.**
\[
2K_V \cdot 2R' = \pi^*(2K_W + B)\pi^*(B') = 2(2K_W + B)B'
\]
\[
= 2(2K_W + B')B' = 2 \left( 2\rho^*(K_p) + \rho^*(\overline{B'}) \right) \left( \rho^*(\overline{B'}) - \sum 2E_i \right)
\]
\[
= 2(2K_p + \overline{B'})\overline{B'} = 2(4 - 2) = 4,
\]
thus \( K_V R' = 1 \).

As the map
\[
\gamma \circ h : V \to \gamma(S)
\]
is a birational morphism, \( \gamma \circ h(R') \) is a line (plus possibly some isolated points). This way there exists a smooth rational curve \( \beta \subset B' \) such that
\[
K_V \tilde{\beta} = 1,
\]
where \( \tilde{\beta} \subset R' \) is the support of \( \pi^*(\beta) \). The adjunction formula gives \( \tilde{\beta}^2 = -3 \), thus \( \beta^2 = -6 \). Notice that \( \tilde{\beta} \) is the only component of \( R' \) which is not contracted by the map \( \gamma \circ h \).

Let
\[
\alpha := B' - \beta \subset W,
\]
\[
\overline{\beta} := \rho(\beta), \overline{\alpha} := \rho(\alpha) \subset P.
\]
When \( \alpha \) is non-empty, the support of \( \pi^*(\alpha) \) is an union of \((-2)\)-curves, since it is contracted by \( \gamma \circ h \). Equivalently \( \alpha \) is a disjoint union of \((-4)\)-curves.
Claim 4.5. We have

\[ K_W^2 \geq -2. \]

Proof. Consider the Chern number \( c_2 \) and the second Betti number \( b_2 \). It is well known that, for a surface \( X \),

\[ c_2(X) = 12\chi(O_X) - K_X^2, \quad b_2(X) = c_2(X) - 2 + 4q(X). \]

Therefore

\[ b_2(W) = 22 - K_W^2, \quad b_2(V) = b_2(S) + t = 11 + 13 = 24. \]

The inequality \( K_W^2 \geq -2 \) follows from the fact \( b_2(V) \geq b_2(W) \).

From Claim 4.5, we conclude that the resolution of \( B' \) blows-up at most two double points, thus

\[ B'^2 \geq -2 + 2(-4) = -10 = \beta^2 + (-4). \]

This implies that \( \alpha \) is a smooth \((-4)\)-curve when \( \alpha \neq 0 \).

Claim 4.6. Only the following possibilities can occur:

- \( \overline{\beta} \) has one double point and no other singularity, or
- \( \overline{\alpha}, \overline{\beta} \) are smooth, \( \overline{\alpha}\overline{\beta} = 2 \).

Proof. Recall that \( B' = \alpha + \beta \) is contained in fibres of \( f_A \) and, since \( B' \) has only negligible singularities, then also \( \overline{B'} = \overline{\alpha} + \overline{\beta} \) is contained in fibres of \( \overline{f_A} \). In particular \( \overline{\alpha}^2, \overline{\beta}^2 \leq 0 \).

If \( \overline{\alpha} \) is singular, then it has arithmetic genus \( p_a(\overline{\alpha}) = 1 \) and \( \overline{\alpha}^2 = 0 \). But then \( \overline{\alpha} \) has the same support of a fibre of \( \overline{f_A} \), which is a contradiction because \( \overline{f_A} \) is not elliptic. Therefore \( \overline{\alpha} \) is smooth.

Since \( K_P\overline{\alpha} \geq 0, K_P\overline{B'} = 2 \) implies \( K_P\overline{\beta} \leq 2 \). We know that \( \beta \) is a smooth rational curve and \( \beta^2 = -6 \), thus \( K_W\beta = 4 \). If \( \overline{\beta} \) is smooth, then one must have \( \overline{\alpha}\overline{\beta} > 1 \). From Claim 4.5 the only possibility in this case is \( \overline{\alpha}\overline{\beta} = 2 \). If \( \overline{\beta} \) is singular, then \( \overline{\beta}^2 \leq 0 \) implies that \( \overline{\beta} \) has one ordinary double point and no other singularity.

Let \( D := \overline{\beta} \) if \( \overline{\beta} \) is singular. Otherwise let \( D := \overline{\alpha} + \overline{\beta} \).

The 2-connected divisor \( \widetilde{D} := \frac{1}{2}(\rho \circ \pi)^*(D) \) has arithmetic genus \( p_a(\widetilde{D}) = 1 \). We know that \( (K_V + n\Lambda)\widetilde{D} = 1 \) (because \( K_V R' = 1 \)) and that \( \widetilde{D} \) contains a component \( A \) such that \( (K_V + n\Lambda)A = 0 \) (because \( D \) has at least one negligible singularity). These two facts imply, from [10, Proposition A.5, (ii)], that the relative canonical map \( \gamma \) has a base point in \( \widetilde{D} \). As mentioned above, \( \gamma \) is a morphism, which is a contradiction.
Finally the assertion about deg(φ₂) in Case 2.: we have proved that S has a genus 2 fibration, so it has an hyperelliptic involution j. The bicanonical map φ₂ factors through both i and j, thus deg(φ₂) ≥ 4.

This finishes the proof of case a), (ii) of Theorem 1.1.

We end the proof of Theorem 1.1 with Case a), (iii): A surface of general type with a genus 2 fibration and pg = q = 1 satisfies K² ≤ 6 (see [24]). Denote by j the map such that φ₂ = j ◦ i. The bicanonical map φ₂ factors through both i and j, thus deg(φ₂) ≥ 4.

This follows from [24, page 66] that, if the genus 2 fibration of S has a rational basis, then K² ≥ 3. It is shown in [19] that, in these conditions, deg(φ₂) = 2. We then conclude that the genus 2 fibration of S is the Albanese fibration.

Examples for case a), (iii) with K² = 3, 4 or 5 were given by Catanese in [6]. The existence of the other cases is proved in the next section.

5. Examples

In this section we will construct smooth minimal surfaces of general type S with pg(S) = q(S) = 1 having an involution i such that the bicanonical map φ₂ of S is composed with i and:

1) K² = 6, g = 3, deg(φ₂) = 2, S/i is birational to a K3 surface;
2) K² = 4, g = 2, Kod(S/i) = 1;
3) K² = 2, g = 2, Kod(S/i) = 2,

where g denotes the genus of the Albanese fibration of S.

Example 5.1. In [23] Todorov gives the following construction of a surface of general type S with pg(S) = q(S) = 1, q = 0 and K² = 8. Consider a Kummer surface Q in P³, i.e. a quartic having as singularities only 16 nodes (ordinary double points). Let G ⊂ Q be the intersection of Q with a general quadric, Ř be the minimal resolution of Q and ˜G ⊂ Ř be the pullback of G. The surface S is the minimal model of the double cover π : V → Ř ramified over ˜G + ∑₁⁻¹ Aᵢ, where Aᵢ ⊂ Ř, i = 1, . . . , 16, are the (−2)-curves which contract to the nodes of Q.

It follows from the double cover formulas (cf. [4, Chapter V, Section 22]) that the imposition of a quadruple point to the branch locus decreases K² by 2 and the Euler characteristic χ by 1.

We will see that we can impose a quadruple point to the branch locus of the Todorov construction, thus obtaining S with K² = 6. In this case I claim that pg(S) = q(S) = 1. In fact, let W be the surface Ř blown-up at the quadruple point, E be the corresponding (−1)-curve, B be the branch locus and L be the line bundle such that 2L ≡ B. From formula (2.3) in Section 2, one has
We will see that \( \deg(\phi_2) = 2 \), hence \( \phi_2(S) \) is a \( K3 \) surface and so \( S \) has no genus 2 fibration.

First we need to obtain an equation of a Kummer surface. The Computational Algebra System Magma has a direct way to do this, but I prefer to do it using a beautiful construction that I learned from Miles Reid.

We want a quartic surface \( Q \in \mathbb{P}^3 \) whose singularities are exactly 16 nodes. Projecting from one of the nodes to \( \mathbb{P}^2 \), one realizes the “Kummer” surface as a double cover \( \psi: X \rightarrow \mathbb{P}^2 \) with branch locus the union of 6 lines \( L_i \) (see [17, page 774]), each one tangent to a conic \( C \) (the image of the projection point) at a point \( p_i \). The surface \( X \) contains 15 nodes (from the intersection of the lines) and two \((−2)\)-curves (the pullback \( \psi^*(C) \)) disjoint from these nodes. To obtain a Kummer surface we have just to contract one of these curves.

Denote also by \( L_i \) the defining polynomial of each line \( L_i \). An equation for \( X \) is \( z^2 = L_1 \cdots L_6 \) in the weighted projective space \( \mathbb{P}(3, 1, 1, 1, 1, 1) \), with coordinates \((z, x_1, x_2, x_3)\). We will see that this equation can be written in the form \( AB+DE = 0 \), where the system \( A = B = D = E = 0 \) has only the trivial solution and \( B, E \) are the defining polynomials of one of the \((−2)\)-curves in \( \psi^*(C) \). Now consider the surface \( X' \) given by \( Bs = D, Es = −A \) in the space \( \mathbb{P}(3, 1, 1, 1, 1, 1) \) with coordinates \((z, s, x_1, x_2, x_3)\). There is a morphism \( X \rightarrow X' \) which restricts to an isomorphism \( X - \{B = E = 0\} \rightarrow X' - \{[0 : 1 : 0 : 0 : 0]\} \)

and which contracts the curve \( \{B = E = 0\} \) to the point \([0 : 1 : 0 : 0 : 0]\). This is an example of unprojection (see [21]).

The variable \( z \) appears isolated in the equations of \( X' \), therefore eliminating \( z \) we obtain the equation of the Kummer \( Q \) in \( \mathbb{P}^3 \) with variables \((s, x_1, x_2, x_3)\). All this calculations will be done using Magma.

In what follows a line preceded by > is an input line, something preceded by // is a comment. A \ at the end of a line means continuation in the next line. The other lines are output ones.

```
> K<e>:=CyclotomicField(6);//e denotes the 6th root of unity.
> //We choose a conic C with equation x1x3-x2^2=0 and fix the
> //p_i's: (1:1:1), (e^2:e:1), (e^4:e^2:1), (e^6:e^3:1),
> // (e^8:e^4:1), (e^10:e^5:1).
> R<z,s,x1,x2,x3>:=PolynomialRing(K,[3,1,1,1,1]);
```
> g:=&*[e^(2*i)*x1-2*e^i*x2+x3:i in [0..5]];  
> //g is the product of the defining polynomials  
> //of the tangent lines L_i to C at p_i.  
> X:=z^2-g;  
> X eq (z+x1^3-x3^3)*(z-x1^3+x3^3)+4*(x1*x3-4*x2^2)^2;  
> (-x1*x3+x2^2);//The decomposition AB+DE.  
true  
> i:=Ideal([s*(z-x1^3+x3^3)-4*(x1*x3-4*x2^2)^2,  
> s*(x1*x3-x2^2)-(z+x1^3-x3^3)]);  
> j:=EliminationIdeal(i,1);  
> j;  
Ideal of Graded Polynomial ring of rank 5 over K  
Lexicographical Order Variables: z, s, x1, x2, x3  
Variable weights:31111 Basis:  
[-1/2*s^2*x1*x3+1/2*s^2*x2^2+s*x1^3-s*x3^3+2*x1^2*x3^2-  
16*x1*x2^2*x3+32*x2^4]  
> 2*Basis(j)[1];  
-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-  
32*x1*x2^2*x3+64*x2^4  
> //This is the equation of the Kummer Q.

We want to find a quadric $H$ such that $H \cap Q$ is a reduced curve $B'$ having an ordinary quadruple point $pt$ as only singularity. Since the computer is not fast enough while working with more than 5 or 6 variables, we first need to think what the most probable case is.

Like we have seen in Section 3, the branch locus $B' + \sum_{1}^{16} A_i$ is contained in 4 fibres $F_A^1, \ldots, F_A^4$ of a fibration $f_A$ of $W$, where $W$ is the resolution of $Q$ blown-up at $pt$ and the $A_i$'s are the $(-2)$-curves which contract to the nodes of $Q$.

Of course we have a quadric intersecting $Q$ at a curve with a quadruple point $pt$ : the tangent space $T$ to $Q$ at $pt$ counted twice. But this one is double, so we need to find an irreducible one (and these two induce $f_A$), the curve $B'$. These curves $2T$ and $B'$ are good candidates for $F_A^1$ and $F_A^2$ (in the notation of Sections 3 and 4). If this configuration exists, then the 16 nodes must be contained in the other two fibres, $F_A^3$ and $F_A^4$. These fibres are divisible by 2, because $F_A^1 = 2T$, and are double outside the nodes. Since in a $K3$ surface only 0, 8 or 16 nodes can have sum divisible by 2, it is reasonable to try the following configuration: each of $F_A^3$ and $F_A^4$ contain 8 nodes with sum divisible by 2 and is double outside the nodes.

It is well known (see e. g. [17]) that the Kummer surface $Q$ has 16 double hyperplane sections $T_i$ such that each one contains 6 nodes of $Q$ and that any two of them intersect in 2 nodes. The sum of the 8 nodes contained in

$$N := (T_1 \cup T_2) \setminus (T_1 \cap T_2)$$

is divisible by 2. Magma will give 3 generators $h_1, h_2, h_3$ for the linear system of quadrics through these nodes.
K<e>:=CyclotomicField(6);
P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-
32*x1*x2^2*x3+64*x2^4;
Q:=Scheme(P3,F); /* The Kummer */
S:=SingularSubscheme(Q);
T1:=Scheme(P3,x1-2*x2+x3);
T2:=Scheme(P3,s);
N:=Difference(T1 join T2 meet S, T1 meet T2);
s:=SetToSequence(RationalPoints(N));
// s is the sequence of the 8 nodes.
L:=LinearSystem(P3,2);
// This will give the h_i's:
LinearSystem(L,[P3!s[i] : ii n [1..8]]);

Now we want to find a quadric \( H \) in the form \( h_1 + bh_2 + ch_3 \), for some \( b, c \) (or, less probably, in the form \( bh_2 + ch_3 \)) such that the projection of \( H \cap Q \) to \( \mathbb{P}^2 \) (by elimination) is a curve with a quadruple point. To find a quadruple point we just have to impose the annulation of the derivatives up to order 3 and ask Magma to do the rest.

This last command gives the points of \( S \), as well as the necessary equations to define the field extensions where they belong. There are various solutions. One of them gives the desired quadruple point. The confirmation is as follows:
ON SURFACES WITH $p_g = q = 1$ AND NON-RULED BICANONICAL INVOLUTION

```plaintext
> R<x>:=PolynomialRing(Rationals());
> K<r13>:=ext<Rationals()|x^4 + x^3 + 1/4*x^2 + 3/32>;
> P3<s,x1,x2,x3>:=ProjectiveSpace(K,3);
> F:=-s^2*x1*x3+s^2*x2^2+2*s*x1^3-2*s*x3^3+4*x1^2*x3^2-
  32*x1*x2^2*x3+64*x2^4;
> b:=64/55*r13^3-272/55*r13^2-96/55*r13-46/55;
> c:=-2176/605*r13^3+448/605*r13^2+624/605*r13-361/605;
> H:=(s*x1-2*x1^2-4*x1*x2-2*x1*x3+8*x2^2+4*x2*x3+2*x3^2)+
  b*(s*x2-2*x1^2-4*x1*x2+4*x2*x3+2*x3^2)+
  c*(s*x3-2*x1^2-4*x1*x2+2*x1*x3-8*x2^2+4*x2*x3+2*x3^2);
> Q:=Scheme(P3,F);
> C:=Scheme(Q,H);
> IsReduced(C);
false
> RC:=ReducedSubscheme(C);
> HasSingularPointsOverExtension(RC);
false
> pt:=Representative(SingularPoints(RC));
> pt in SingularSubscheme(Q); // pt is not a node of Q.
false
> T:=DefiningPolynomial(TangentSpace(Q,pt));
> T2:=Scheme(Q,T^2);
> #RationalPoints(T2 meet C);
1
> pt in RationalPoints(T2 meet C);
true
> HasPointsOverExtension(T2 meet C);
false
```

This way $T$ and $C$ generate a pencil with a quadruple base point and the curve $B'$ is a general element of this pencil.

Finally, it remains to be shown that the degree of the bicanonical map $\phi_2$ is 2. As $(2K_S)^2 = 24$, it suffices to show that $\phi_2(S)$ is of degree 12. Since, in the notation of diagram (3.1), $h^*|2K_S| = \pi^*|2K_W + B'|$ then $\phi_2(S)$ is the image of $W$ via the map $\tau : W \to \phi_2(S)$ given by $|2K_W + B'|$. The projection of this linear system on $Q$ is the linear system of the quadrics whose intersection with $Q$ has a double point at $pt$. In order to easily write this linear system, we will translate the point $pt$ to the origin (in affine coordinates).

```plaintext
> QA:=AffinePatch(Scheme(P3,F),4);
> p:=Representative(RationalPoints(AffinePatch(Cluster(pt),4)));
> A3<x,y,z>:=Ambient(QA);
> psi:=map<A3->A3|[x-p[1],y-p[2],z-p[3]]>;Q0:=psi(QA);
> FA:=DefiningPolynomial(Q0);
> j:=Evaluate(Derivative(FA,A3,i),Origin(A3));i in [1,2,3];
> J:=LinearSystem(A3,[j[1]*x+j[2]*y+j[3]*z,x^2,x*y,x*z,y^2,y*z,
```
Example 5.2. Here we will construct a surface of general type $S$, with $p_g = q = 1$ and $K^2 = 4$, as the minimal model of a double cover of a surface $W$ such that $	ext{Kod}(W) = p_g(W) = 1$ and $q(W) = 0$.

Step 1. Construction of $W$.

Consider five distinct lines $L_1, \ldots, L_5 \subset \mathbb{P}^2$ meeting in one point $p_0$. Let $p_1 \in L_4$, $p_2, p_3 \in L_5$ be points distinct from $p_0$. Choose three distinct non-degenerate conics, $C_1, C_2, C_3$, tangent to $L_4$ at $p_1$ and passing through $p_2, p_3$. Define

$$ D := L_1 + \ldots + L_4 + C_1 + C_2. $$

Denote by $p_4, \ldots, p_{15}$ the 12 nodes of $D$ contained in $L_1 + L_2 + L_3$. To resolve the $(3, 3)$ point of $D$ at $p_1$ we must do two blow-ups: one at $p_1$ and other at an infinitely near point $p_1'$. Let $\mu : X \to \mathbb{P}^2$ be the blow-up with centers $p_0, p_1, p_1', p_2, \ldots, p_{15}$ and $E_0, E_1, E_1', E_2, \ldots, E_{15}$ be the corresponding exceptional divisors (with self-intersection $-1$). Consider

$$ D' := \mu^*(D) - 4E_0 - 2E_1 - 4E_1' - 2 \sum_{i=2}^{15} E_i. $$

Let $\psi : \tilde{X} \to X$ be the double cover of $X$ with branch locus $D'$. The surface $\tilde{X}$ is the canonical resolution of the double cover of $\mathbb{P}^2$ ramified over $D$. Let $W$ be the minimal model of $\tilde{X}$ and $\nu$ be the corresponding morphism.

$$ \begin{array}{ccc}
\tilde{X} & \overset{\nu}{\longrightarrow} & W \\
\downarrow \psi & & \downarrow \\
X & \overset{\mu}{\longrightarrow} & \mathbb{P}^2.
\end{array} $$

Notice that $\nu$ contracts two $(-1)$-curves contained in $(\mu \circ \psi)^*(L_4)$.

We have $K_X = -\mu^*(3L) + E_1' + \sum_{i=0}^{15} E_i$, where $L$ denotes a general line of $\mathbb{P}^2$. Hence, using the double cover formulas (cf. (2.1)),

$$ K_{\tilde{X}} \equiv \psi^* \left( K_X + \frac{1}{2} D' \right) \equiv \psi^* (\mu^*(L) - E_0 - E_1') \equiv \psi^* (\widehat{L}_4 + (E_1 - E_1') + E_1'), $$

where $\widehat{L}_4 \subset X$ is the strict transform of $L_4$. Since $\widehat{L}_4$ and $E_1 - E_1'$ are $(-2)$-curves contained in the branch locus $D'$, then $\frac{1}{2} \psi^* (\widehat{L}_4)$ and $\frac{1}{2} \psi^* (E_1 - E_1')$ are $(-1)$-curves in $\tilde{X}$, thus

$$ K_W \equiv \nu(\psi^*(E_1')). $$
The divisor $2\nu(\psi^*(E'_1)) \equiv 2K_W$ is a (double) fibre of the elliptic fibration of $W$ induced by the pencil of lines through $p_0$. So $p_g(W) = 1$ and $W$ has Kodaira dimension 1.

From (2.1) one has

$$\chi(O_W) = 2 + \frac{1}{8} D'(2K_X + D') = 2 + \frac{1}{8} (28 - 28) = 2.$$

**Step 2.** The branch locus in $W$.

Since the strict transforms $\hat{L}_1, \ldots, \hat{L}_4 \subset X$ are in the branch locus $D'$, then there are curves $l_1, \ldots, l_4 \subset \tilde{X}$ such that

$$(\mu \circ \psi)^*(L_1 + \cdots + L_4) = 2l_1 + \cdots + 2l_4 + 4\psi^*(E_0) + \psi^*(E_1 - E'_1)$$

$$+ 2\psi^*(E'_1) + \sum_{4}^{15} A_i,$$

where each $A_i := \psi^*(E_i)$ is a $(-2)$-curve. But also $E_1 - E'_1$ is in the branch locus, thus $\psi^*(E_1 - E'_1) \equiv 0 \pmod{2}$ and then

$$\sum_{4}^{15} A_i \equiv 0 \pmod{2}.$$

The strict transform $\hat{L}_5$ is a $(-2)$-curve which do not intersect $D'$ thus

$$\psi^*(\hat{L}_5) = A_{16} + A_{17},$$

with $A_{16}, A_{17}$ disjoint $(-2)$-curves.

Denote by $\hat{C}_3 \subset X$ the strict transform of the conic $C_3$. We have

$$(\mu \circ \psi)^*(C_3 + L_4 + L_5) = \psi^*(\hat{C}_3) + 2l_4 + A_{16} + A_{17}$$

$$+ 2\psi^*(E_0 + \cdots + E_3) + 2\psi^*(E'_1) \equiv 0 \pmod{2}.$$

With this we conclude that

$$\psi^*(\hat{C}_3) + \sum_{4}^{17} A_i \equiv 0 \pmod{2}.$$

Notice that $F \cdot \nu(\psi^*(\hat{C}_3)) = 4$ for a fibre $F$ of the elliptic fibration of $W$, thus $K_W \cdot \nu(\psi^*(\hat{C}_3)) = 2$. 
**Step 3.** Construction of $S$.

Let $\pi : V \to W$ be the double cover with branch locus

$$B := \nu \left( \psi^* (\widehat{C}_3) + \sum_{i=4}^{17} A_i \right)$$

and $S$ be the minimal model of $V$. From the double cover formulas (2.1) we obtain

$$2K_V^2 = (2K_W + B)^2 = 4K_W^2 + 4K_W B + B^2 = 4 \cdot 0 + 4 \cdot 2 + (-28) = -20$$

and, by contraction of the $(-1)$-curves $\frac{1}{2} \pi^*(\nu(A_i))$,

$$K_S^2 = K_V^2 + 14 = -10 + 14 = 4.$$ 

Let $L := \frac{1}{2} B$. Formulas (2.1) give

$$\chi(O_S) = 2 \chi(O_W) + \frac{1}{2} L(K_W + L) = 4 - 3 = 1.$$ 

Using now formula (2.3) we obtain $h^0(W, O_W(2K_W + L)) = 0$, which means that the bicanonical map of $V$ factors through $\pi$.

Because $K_W$ is effective then also $h^0(W, O_W(K_W + L)) = 0$ and

$$p_g(S) = p_g(W) + h^0(W, O_W(K_W + L)) = 1.$$ 

Hence $q(S) = 1$ and then, as we noticed in the beginning of Section 4, the curve $\nu \left( \psi^*(\widehat{C}_3) \right)$ is contained in the fibration of $W$ which induces the Albanese fibration of $S$. As $\nu \left( \psi^*(\widehat{C}_3) \right)^2 = 0$, we conclude that the Albanese fibration of $S$ is the one induced by the pencil $|\widehat{C}_3|$. It is of genus 2 because $\widehat{C}_3 D' = \widehat{C}_3(L_1 + L_2 + L_3) = 6$.

**Example 5.3.** Now we will obtain a surface of general type $S$, with $p_g = q = 1$ and $K^2 = 2$, as the minimal model of a double cover of a surface of general type $W$ such that $K_W^2 = p_g(W) = 1$ and $q(W) = 0$.

**Step 1.** Construction of $W$.

Let $p_0, \ldots, p_3 \in \mathbb{P}^2$ be distinct points and $L_i$ be the line through $p_0$ and $p_i$, $i = 1, 2, 3$. For each $j \in \{1, 2, 3\}$ let $C_j$ be the conic through $p_1, p_2, p_3$ tangent to the $L_i$’s except for $L_j$. Denote by $D$ a general element of the linear system generated by $3C_1 + 2L_1, 3C_2 + 2L_2$ and $3C_3 + 2L_3$. The singularities of $D$ are a $(3, 3)$-point at $p_i$, tangent to $L_i$, $i = 1, 2, 3$, and a double point at $p_0$. Let $L_4$ be a line through $p_0$ transverse to $D$.

Denote by $W'$ the canonical resolution of the double cover of $\mathbb{P}^2$ with branch locus

$$D + L_1 + \ldots + L_4$$
and by $W$ the minimal model of $W'$. The formulas of [4, Chapter V, Section 22] give $\chi(W) = 2$ and $K^2_W = 1$ (notice that the map $W' \to W$ contracts three $(-1)$-curves contained in the pullback of $L_1 + L_2 + L_3$). Since $K^2 \geq 2p_g$ for an irregular surface ([14]), $W$ is regular and then $p_g(W) = \chi(W) - 1 = 1$.

**Step 2.** The branch locus in $W$.

The pencil of lines through $p_0$ induces a (genus 2) fibration of $W$. Let $F_i$ be the fibre induced by $L_i$, $i = 1, \ldots, 4$. The fibre $F_4$ is the union of six disjoint $(-2)$-curves (corresponding to the nodes of $D - p_0$) with a double component (the strict transform of $L_4$). Each $F_i$, $i = 1, 2, 3$, is the union of two $(-2)$-curves with a double component (cf. [24, Section 1]). Thus $F_1 + \cdots + F_4$ contain disjoint $(-2)$-curves $A_1, \ldots, A_{12}$ such that

$$\sum_{i=1}^{12} A_i \equiv 0 \pmod{2}.$$

**Step 3.** Construction of $S$.

Let $V$ be the double cover of $W$ with branch locus $\sum_{i=1}^{12} A_i$ and $S$ be the minimal model of $V$. From (2.1) we obtain $\chi(O_S) = 1$ and $K^2_V = -10$. The $A_i$’s lift to $(-1)$-curves in $V$, thus $K^2_S = -10 + 12 = 2$. We have $1 = p_g(W) \leq p_g(S)$, hence $q(S) \neq 0$ and then $2 = K^2_S \geq 2p_g(S)$. So $p_g(S) = q(S) = 1$.

The genus 2 fibration of $W$ induces the Albanese fibration of $S$.

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