THE BOWDITCH BOUNDARY OF \((G, \mathcal{H})\) WHEN \(G\) IS HYPERBOLIC

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Abstract. In this note we use Yaman’s dynamical characterization of relative hyperbolicity to prove a theorem of Bowditch about relatively hyperbolic pairs \((G, \mathcal{H})\) with \(G\) hyperbolic. Our proof additionally gives a description of the Bowditch boundary of such a pair.

1. Introduction

Let \(G\) be a group. A collection \(\mathcal{H} = \{H_1, \ldots, H_n\}\) of subgroups of \(G\) is said to be almost malnormal if every infinite intersection of the form \(H_i \cap g^{-1}H_jg\) satisfies both \(i = j\) and \(g \in H_i\).

In an extremely influential paper from 1999, recently published in IJAC [Bow12], Bowditch proves the following useful theorem:

**Theorem 1.1.** [Bow12, Theorem 7.11] Let \(G\) be a nonelementary hyperbolic group, and let \(\mathcal{H} = \{H_1, \ldots, H_n\}\) be an almost malnormal collection of proper, quasiconvex subgroups of \(G\). Then \(G\) is hyperbolic relative to \(\mathcal{H}\).

**Remark 1.2.** The converse to this theorem also holds. If \((G, \mathcal{H})\) is any relatively hyperbolic pair, then the collection \(\mathcal{H}\) is almost malnormal by [Osi06, Proposition 2.36]. Moreover the elements of \(\mathcal{H}\) are undistorted in \(G\) [Osi06, Lemma 5.4]. Undistorted subgroups of a hyperbolic group are quasiconvex.

In this note, we give a proof of Theorem 1.1 which differs from Bowditch’s. The strategy we follow is to exploit the dynamical characterization of relative hyperbolicity given by Yaman in [Yam04]. By doing so, we are able to obtain some more information about the pair \((G, \mathcal{H})\). In particular, we obtain an explicit description of its Bowditch boundary \(\partial(G, \mathcal{H})\). Let \(\partial G\) be the Gromov boundary of the group \(G\). If \(H\) is quasiconvex in a hyperbolic group \(G\), its limit set \(\Lambda(H) \subset \partial G\) is homeomorphic to the Gromov boundary \(\partial H\) of \(H\). The next theorem says that \(\partial(G, \mathcal{H})\) is obtained by smashing the limit sets of \(gHg^{-1}\) to points, for \(H \in \mathcal{H}\) and \(g \in G\).

**Theorem 1.3.** Let \(G\) be hyperbolic, and let \(\mathcal{H}\) be an almost malnormal collection of infinite quasi-convex proper subgroups of \(G\). Let \(\mathcal{L}\) be the set of \(G\)-translates of limit sets of elements of \(\mathcal{H}\). The Bowditch boundary \(\partial(G, \mathcal{H})\) is obtained from the Gromov boundary \(\partial G\) as a decomposition space \(\partial G/\mathcal{L}\).

In particular, we can bound the dimension of this space:

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[1] I wrote some version of this note in 2008 and showed it to one or two people, but never published it. A couple of people have asked me about it recently, so I thought I should make it available.
Corollary 1.4. Let $G$ be a hyperbolic group and $\mathcal{H}$ a malnormal collection of infinite quasi-convex proper subgroups. Then $\dim \partial(G, \mathcal{H}) \leq \dim \partial G + 1$.

Proof. This follows from the Addition Theorem of dimension theory, a special case of which says that if a compact metric space $M = A \cup B$, then $\dim(M) \leq \dim(A) + \dim(B) + 1$. By Theorem 1.3, $\partial(G, \mathcal{H})$ can be written as a disjoint union of a countable set (coming from the limit sets of the conjugates of the elements of $\mathcal{H}$) with a subset of $\partial G$. □

At least conjecturally, this proposition gives cohomological information about the pair:

Conjecture 1.5. Let $G$ be torsion-free and hyperbolic relative to $\mathcal{H}$. Let $\text{cd}(G, \mathcal{H})$ be the cohomological dimension of the pair $(G, \mathcal{H})$, and let $\dim$ be topological dimension. Then

(1) \[ \text{cd}(G, \mathcal{H}) = \dim \partial(G, \mathcal{H}) + 1, \]

and more generally,

(2) \[ H^q(G, \mathcal{H}; \mathbb{Z}_G) = \hat{H}^{q-1}(\partial(G, \mathcal{H})) \]

for all integers $q$.

In the absolute setting ($\mathcal{H} = \emptyset$), Equations (1) and (2) are results of Bestvina–Mess [BM91]. In case $G$ is a geometrically finite group of isometries of $\mathbb{H}^n$ for some $n$ and $\mathcal{H}$ is the collection (up to conjugacy) of maximal parabolic subgroups of $G$, Kapovich establishes equations (1) and (2) in [Kap09, Proposition 9.6], and remarks that the proof should extend easily to the case in which all elements of $\mathcal{H}$ are virtually nilpotent. The key step which must be generalized is the existence of an appropriate space for which the Bowditch boundary is a $\mathbb{Z}$–set.

Definition 1.6. Suppose that $M$ is a compact metrizable space with at least 3 points, and let $G$ act on $M$ by homeomorphisms. The action is a convergence group action if the induced action on the space $\Theta^3(M)$ of unordered triples of distinct points in $M$ is properly discontinuous.

An element $g \in G$ is loxodromic if it has infinite order and fixes exactly two points of $M$.

A point $p \in M$ is a bounded parabolic point if $\text{Stab}_G(p)$ contains no loxodromics, and acts cocompactly on $M \setminus \{p\}$.

A point $p \in M$ is a conical limit point if there is a sequence $\{g_i\}$ in $G$ and a pair of points $a \neq b$ in $M$ so that:

(1) $\lim_{i \to \infty} g_i(p) = a$, and

(2) $\lim_{i \to \infty} g_i(x) = b$ for all $x \in M \setminus \{p\}$.

A convergence group action of $G$ on $M$ is geometrically finite if every point in $M$ is either a bounded parabolic point or a conical limit point.

Remark 1.7. If $G$ is countable, $G$ acts on $M$ as a convergence group, and there is no closed $G$-invariant proper subset of $M$, then $M$ is separable. In particular $\partial G$ for $G$ hyperbolic is separable, as is any space admitting a geometrically finite convergence group action by a countable group.

2I believe that (an interpretation of) Kapovich’s proof actually extends to the case in which all peripheral groups have finite $K(x, 1)$’s. So the conjecture is almost certainly true with that hypothesis. It may not be true in greater generality. I don’t know.
Bowditch proved in [Bow98] that if $G$ acts on $M$ as a convergence group and every point of $M$ is a conical limit point, then $G$ is hyperbolic. Conversely, if $G$ is hyperbolic, then $G$ acts as a convergence group on $\partial G$, and every point in $\partial G$ is a conical limit point. For general geometrically finite actions, we have the following result of Yaman:

**Theorem 1.8.** [Yam04, Theorem 0.1] Suppose that $M$ is a non-empty perfect metrizable compact space, and suppose that $G$ acts on $M$ as a geometrically finite convergence group. Let $B \subset M$ be the set of bounded parabolic points. Let \{p_1, \ldots, p_n\} be a set of orbit representatives for the action of $G$ on $B$. For each $i$ let $P_i$ be the stabilizer in $G$ of $p_i$, and let $\mathcal{P} = \{P_1, \ldots, P_n\}$.

$G$ is relatively hyperbolic, relative to $\mathcal{P}$.

**Outline of proof of Theorem 1.1.** We prove Theorem 1.1 by constructing a space $M$ on which $G$ acts as a geometrically finite convergence group, so that the parabolic point stabilizers are all conjugate to elements of $\mathcal{H}$. The space $M$ is a quotient of $\partial G$, constructed as follows. The hypotheses on $\mathcal{H}$ imply that the boundaries $\partial H_i$ embed in $\partial G$ for each $i$, and that $g \partial H_i \cap h \partial H_j$ is empty unless $i = j$ and $g^{-1} h \in H_i$.

Let

$$A = \{g \partial H_i \mid g \in G, \text{ and } H_i \in \mathcal{H}\},$$

and let

$$B = \{x \mid x \in \partial G \setminus \bigcup A\}.$$

The union $\mathcal{C} = A \cup B$ is therefore a decomposition of $\partial G$ into closed sets. We let $M$ be the quotient topological space $\partial G/\mathcal{C} = A \cup B$. There is clearly an action of $G$ on $M$ by homeomorphisms.

We now have a sequence of four claims, which we prove later.

**Claim 1.** $M = A \cup B$ is a perfect metrizable space.

**Claim 2.** $G$ acts as a convergence group on $M$.

**Claim 3.** For $x \in A$, $x$ is a bounded parabolic point, with stabilizer conjugate to an element of $\mathcal{H}$.

**Claim 4.** For $x \in B$, $x$ is a conical limit point.

Given the claims, we may apply Yaman’s theorem 1.8 to conclude that $G$ is relatively hyperbolic, relative to $\mathcal{H}$. \hfill \Box

## 2. Proofs of claims

In what follows we fix some $\delta$-hyperbolic Cayley graph $\Gamma$ of $G$. We’ll use the notation $a \mapsto \bar{a}$ for the map from $\partial G$ to the decomposition space $M$.

### 2.1. Claim 1

We’re going to need some basic point-set topology. What we need is in Hocking and Young [HY88], mostly Chapter 2, Section 16, and Chapter 5, Section 6.

**Definition 2.1.** Given a sequence \{\text{D}_i\} of subsets of a topological space $X$, the lim inf and lim sup of \{\text{D}_i\} are defined to be

$$\liminf D_i = \{x \in X \mid \text{for all open } U \ni x, U \cap D_i \neq \emptyset \text{ for almost all } i\}$$

and

$$\limsup D_i = \{x \in X \mid \text{for all open } U \ni x, U \cap D_i \neq \emptyset \text{ for infinitely many } i\}$$

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The notion of upper semicontinuity for a decomposition of a compact metric space into closed sets can be phrased in terms of Definition 2.1. The following can be extracted from [HY88, section 3-6] and standard metrization theorems.

**Proposition 2.2.** Let $X$ be a compact separable metric space, and let $D$ be a decomposition of $X$ into disjoint closed sets. Let $Y$ be the quotient of $X$ obtained by identifying each element of $D$ to a point. The following are equivalent:

1. $D$ is upper semicontinuous.
2. $Y$ is a compact metric space.
3. For any sequence $\{D_i\}$ of elements of $D$, and any $D \in D$ so that $D \cap \lim \inf D_i$ is nonempty, we have $\lim \sup D_i \subset D$.

**Lemma 2.3.** Let $\{C_i\}$ be a sequence of elements of the decomposition $C = A \cup B$ of $\partial G$, so that no element appears infinitely many times. If $\lim \inf C_i \neq \emptyset$, then $\lim \sup C_i = \lim \inf C_i$ is a single point.

**Proof.** By way of contradiction, assume there are two points in $\lim \inf C_i$. There are therefore points $a_i \in C_i$ limiting on $x$, and $b_i \in C_i$ limiting on $y$. It follows that the $C_i$ must eventually be of the form $g_i \partial H_{j_i}$, for $H_{j_i} \in \mathcal{H}$. Passing to a subsequence (which can only make the $\lim \inf$ bigger) we may assume all the $H_{j_i} = H$ for some fixed $\lambda$-quasiconvex subgroup $H$.

In a proper $\delta$-hyperbolic geodesic space, geodesics between arbitrary points at infinity exist, and triangles formed from such geodesics are $3\delta$–thin. It follows that a geodesic between limit points of a $\lambda$-quasiconvex set lies within $\lambda + 6\delta$ of the quasiconvex set. Let $p$ be a point on a geodesic from $x$ to $y$, and let $\gamma_i$ be a geodesic from $a_i$ to $b_i$. For large enough $i$ the geodesic $\gamma_i$ passes within $6\delta$ of $p$, so the sets $g_i H$ must, for large enough $i$ intersect the $(\lambda + 12\delta)$–ball about $p$. Since this ball is finite and the cosets of $H$ are disjoint, some $g_i H$ must appear infinitely often, contradicting the assumption that no $C_i$ appears infinitely many times.

A similar argument shows that in case $\lim \inf C_i$ is a single point, then $\lim \sup C_i$ cannot be any larger than $\lim \inf C_i$.

**Remark 2.4.** If $\lim \inf C_i$ is empty, then $\lim \sup \{C_i\}$ can be any closed subset of $\partial G$.

**Proof of Claim 4** We first verify condition 3 of Proposition 2.2. Let $\{C_i\}$ be some sequence of elements of the decomposition $C$, and let $D$ be an element of the decomposition so that $D \cap \lim \inf C_i \neq \emptyset$. If no element of $C$ appears infinitely many times in $\{C_i\}$, then Lemma 2.3 implies that $\lim \inf C_i = \lim \sup D_i$ is a single point, so 3 is satisfied almost trivially. We can therefore assume that for some $C \in C$, there are infinitely many $i$ for which $C_i = C$.

In fact, there can only be one such $C$, for otherwise we would have $\lim \inf C_i = \emptyset$. If all but finitely many $C_i$ satisfy $C_i = C$, then $\lim \inf C_i = \lim \sup C_i = C$, and it is easy to see that condition 3 is satisfied.

We may therefore assume that $C_i \neq C$ for infinitely many $i$. Let $\{B_i\}$ be the sequence made up of those $C_i \neq C$. No $B_i$ appears more than finitely many times. Since $\{B_i\}$ is a subsequence of $\{C_i\}$, we have

$$\lim \inf C_i \subset \lim \inf B_i \subset \lim \sup B_i \subset \lim \sup C_i.$$ 

Applying Lemma 2.3 to $\{B_i\}$, we deduce that $\lim \inf B_i = \lim \sup B_i$ is a single point. It follows that $\lim \inf C_i$ is a single point, and so condition 3 is again satisfied trivially.
We’ve shown that $M = \partial G / \mathcal{C}$ is a compact metric space. We now show $M$ is perfect. Let $p \in M$.

Suppose first that $p \in B$, i.e., that the preimage in $\partial G$ is a single point $\tilde{p}$. Because $G$ is nonelementary, $\partial G$ is perfect. Thus there is a sequence of points $x_i \in \partial G \setminus \{\tilde{p}\}$ limiting on $p$. The image of this sequence limits on $p$.

Now suppose that $p \in A$, i.e., the preimage of $p$ in $\partial G$ is equal to $g \partial H$ for some $g \in G$ and some $H \in \mathcal{H}$. Choose any point $x \in \partial G \setminus \partial H$, and any infinite order element $h$ of $gHg^{-1}$. The points $h^i x$ project to distinct points in $M \setminus \{p\}$, limiting on $p$.

2.2. Claim[2] In [Bow99], Bowditch gives a characterization of convergence group actions in terms of collapsing sets. We rephrase Bowditch slightly in what follows.

**Definition 2.5.** Let $G$ act by homeomorphisms on $M$. Suppose that $\{g_i\}$ is a sequence of distinct elements of $G$. Suppose that there exist points $a$ and $b$ (called the attracting and repelling points, respectively) so that whenever $K \subseteq M \setminus \{a\}$ and $L \subseteq M \setminus \{b\}$, the set $\{i \mid g_i K \cap L \neq \emptyset\}$ is finite. Then $\{g_i\}$ is a collapsing sequence.

**Proposition 2.6.** [Bow99 Proposition 1.1] Let $G$, a countable group, act on $M$, a compact Hausdorff space with at least 3 points. Then $G$ acts as a convergence group if and only if every infinite sequence in $G$ contains a subsequence which is collapsing.

**Proof of Claim [2]** We use the characterization of [2.6]. Let $\{\gamma_i\}$ be an infinite sequence in $G$. Since the action of $G$ on $\partial G$ is convergence, there is a collapsing subsequence $\{g_i\}$ of $\{\gamma_i\}$; i.e., there are points $a$ and $b$ in $\partial G$ which are attracting and repelling in the sense of Definition 2.5. We will show that $\{g_i\}$ is also a collapsing sequence for the action of $G$ on $M$, and that the images $\tilde{a}$ and $\tilde{b}$ in $M$ are the attracting and repelling points for this sequence.

Let $K \subseteq M \setminus \{\tilde{a}\}$ and $L \subseteq M \setminus \{\tilde{b}\}$ be compact sets, and let $\tilde{K}$ and $\tilde{L}$ be the preimages of $K$ and $L$ in $\partial G$. We have $\tilde{K} \subset \partial G \setminus \{a\}$ and $\tilde{L} \subset \partial G \setminus \{b\}$, so $\{i \mid g_i \tilde{K} \cap \tilde{L} \neq \emptyset\}$ is finite. But for each $i$, $g_i \tilde{K} \cap \tilde{L} = \cdots = \pi(g_i \tilde{K} \cap \tilde{L})$, so $\{i \mid g_i \tilde{K} \cap \tilde{L} \}$ is also finite.

**Remark 2.7.** In the preceding proof it is possible for $a$ and $b$ to be distinct, but $\tilde{a} = \tilde{b}$.

2.3. Claim[3]

**Proof of Claim [3]** Let $p \in A \subseteq M$ be the image of $g \partial H$ for $g \in G$ and $H \in \mathcal{H}$. Let $P = gHg^{-1}$. Since $H$ is equal to its own commensurator, so is $P$, and $P = \text{Stab}_G(p)$. We must show that $P$ acts cocompactly on $M \setminus \{p\}$. The subgroup $P$ is $\lambda$-quasiconvex in $\Gamma$ (the Cayley graph of $G$) for some $\lambda > 0$. Let $N$ be a closed $R$-neighborhood of $P$ in $\Gamma$ for some large integer $R$, with $R > 2\lambda + 10\delta$. Note that any geodesic from 1 to a point in $\partial H$ stays inside $N$, and any geodesic from 1 to a point in $\partial G \setminus \partial P$ eventually leaves $N$.

Let $C = \{g \in \partial N \mid d(g, 1) \leq 2R + 100\delta\}$. Let $E$ be the set of points $e \in \partial X$ so that there is a geodesic from 1 to $e$ passing through $C$. The set $E$ is compact, and lies entirely in $\partial G \setminus \partial P$. We will show that $PE = \partial G \setminus \partial P$. Let $e \in \partial G \setminus \partial H$, and let $h \in P$ be “coarsely closest” to $e$ in the following sense: If $\{x_i\}$ is a sequence of points in $X$ tending to $e$, then for large enough $i$, we have, for any $h' \in P$, 

...
Figure 1. Bounding the distance from $h$ to $d$.

$d(h, e_i) \leq d(h', e_i) + 4\delta$. Let $\gamma$ be a geodesic ray from $h$ to $e$, and let $d$ be the unique point in $\gamma \cap \partial N$. Since $d \in \partial N$, there is some $h'$ so that $d(h', d) = R$. Let $e'$ be a point on $\gamma$ so that $10R < d(h, e') \leq d(h', e') + 4\delta$, and consider a geodesic triangle made up of that part of $\gamma$ between $h$ and $e'$, some geodesic between $h'$ and $h$, and some geodesic between $h'$ and $e'$. This triangle has a corresponding comparison tripod, as in Figure 1. Since any geodesic from $h'$ to $h$ must stay $R - \lambda > \delta$ away from $\partial N$, the point $\bar{d}$ must lie on the leg of the tripod corresponding to $e'$. Let $\bar{d}'$ be the point on the geodesic from $h'$ to $e'$ which projects to $\bar{d}$ in the comparison tripod. Since $d(h', d) = R$, $d(h', d') \leq R + \delta$. Now notice that $d(h, d) \leq d(h', d') + (e', h)_{h'} - (e', h)_{h'} \leq d(h', d') + 4\delta \leq R + 5\delta$.

But this implies that the geodesic from $1$ to $h^{-1}e$ passes through $C$, and so $h^{-1}e \in E$ and $e \in hE$. Since $e$ was arbitrary in $\partial G \setminus \partial P$, we have $PE = \partial G \setminus \partial P$, and so the action of $P$ on $\partial G \setminus \partial P$ is cocompact. If $E$ is the (compact) image of $E$ in $M$, then $PE = M \setminus \{p\}$, and so $p$ is a bounded parabolic point. □

2.4. Claim

**Lemma 2.8.** For all $R > 0$ there is some $D$, depending only on $R$, $G$, $\mathcal{H}$, and $S$, so that for any $g, g' \in G$, and $H$, $H' \in \mathcal{H}$,

$$\text{diam}(N_R(gH) \cap N_R(g'H')) < D.$$  

($N_R(Z)$ denotes the $R$-neighborhood of $Z$ in the Cayley graph $\Gamma = \Gamma(G, S)$.)

**Lemma 2.9.** There is some $\lambda$ depending only on $G$, $\mathcal{H}$, and $S$, so that if $x$, $y \in gH \cup g\partial H$, then any geodesic from $x$ to $y$ lies in a $\lambda$-neighborhood of $gH$ in $\Gamma$.

**Lemma 2.10.** Let $\gamma : \mathbb{R}_+ \to \Gamma$ be a (unit speed) geodesic ray, so that $x = \lim_{t \to \infty} \gamma(t)$ is not in the limit set of $gH$ for any $g \in G$, $H \in \mathcal{H}$, and so that $\gamma(0) \in G$. Let $C > 0$. There is a sequence of integers $\{n_i\}$ tending to infinity, and a constant $\chi$, so that the following holds, for all $i \in \mathbb{N}$: If $x_i = \gamma(n_i) \in N_C(gH)$ for $g \in G$ and
$H \in \mathcal{H}$, then
\[
\text{diam} \left( N_{C}(gH) \cap \gamma([n_{i}, \infty)) \right) < \chi.
\]

**Proof.** Let $\lambda$ be the quasi-convexity constant from Lemma 2.3. Let $D$ be the constant obtained from Lemma 2.3, setting $R = C + \lambda + 2\delta$, and let $\chi = 2D$.

Let $i \in \mathbb{N}$. If $i = 1$, let $t = 0$; otherwise set $t = n_{i-1} + 1$. We will find $n_{i} \geq t$ satisfying the condition in the statement.

If we can’t use $n_{i} = t_{0}$, then there must be some $gH$ with $g \in G$ and $H \in \mathcal{H}$ satisfying $\gamma(t_{0}) \in N_{C}(gH)$ and
\[
\text{diam} \left( N_{C}(gH) \cap \gamma([t_{0}, \infty)) \right) \geq \chi.
\]
Let $s = \sup \{t \mid \gamma(t) \in N_{C}(gH)\}$. We claim that we can choose
\[
n_{i} = s - \frac{\chi}{2} = s - (D + 4\delta + 2\lambda + 2C).
\]
Clearly we have
\[
\text{diam} \left( N_{C}(gH) \cap \gamma([n_{i}, \infty)) \right) < \chi.
\]
Now suppose that some other $g'H'$ satisfies $x_{i} = \gamma(n_{i}) \in N_{C}(g'H')$ and
\[
\text{diam} \left( N_{C}(g'H') \cap \gamma([n_{i}, \infty)) \right) \geq \chi.
\]
It follows (once one draws the picture) that $\gamma(n_{i})$ and $\gamma(s)$ lie both in the $C + \lambda + 2\delta$ neighborhood of $gH$ and in the $C + \lambda + 2\delta$ neighborhood of $g'H'$. Since $d(\gamma(n_{i}), \gamma(s)) = s - n_{i} = D$, this contradicts Lemma 2.3.

**Proof of Claim 4.** Let $x \in \partial G \setminus \cup A$. We must show that $\bar{x} \in M$ is a conical limit point for the action of $G$ on $M$. Fix some $y \in M \setminus \{x\}$, and let $\gamma$ be a geodesic from $y$ to $x$ in $\Gamma$. Let $C = \lambda + 6\delta$, where $\lambda$ is the constant from Lemma 2.3. Using Lemma 2.10, we can choose a sequence of (inverses of) group elements $\{g_{i}^{-1}\}$ in the image of $\gamma$ so that whenever $x_{i} \in N_{C}(gH)$ for some $g \in G$, $H \in \mathcal{H}$, and $i \in \mathbb{N}$, we have
\[
\text{diam} \left( N_{C}(gH) \cap \gamma([n_{i}, \infty)) \right) < \chi,
\]
for some constant $\chi$ independent of $g$, $H$, and $i$.

Now consider the geodesics $x_{i}\gamma$. They all pass through 1, so we may pick a subsequence $\{x_{i}^{j}\}$ so that the geodesics $x_{i}^{j}\gamma$ converge setwise to a geodesic $\sigma$ running from $b$ to $a$ for some $b, a \in \partial G$. In fact this sequence $\{x_{i}^{j}\}$ will satisfy $\lim_{t_{i} \to \infty} x_{i}^{j}1 = a$ and $\lim_{t_{i} \to \infty} x_{i}^{j}y' = b$ for all $y' \in \partial G \setminus \{x\}$. We will be able to use this sequence to see that $\bar{x}$ is a conical limit point for the action of $G$ on $M$, unless we have $\bar{a} = \bar{b}$ in $M$.

By way of contradiction, we therefore assume that $a$ and $b$ both lie in $g\partial H$ for some $g \in G$, and $H \in \mathcal{H}$. The geodesic $\sigma$ lies in a $\lambda$-neighborhood of $gH$, by Lemma 2.10. Let $R > \chi$. The set $x_{i}\gamma \cap B_{R}(1)$ must eventually be constant, equal to $\sigma_{R} := \sigma \cap B_{R}(1)$. Now $\sigma_{R}$ a geodesic segment of length $2R$ lying entirely inside $N_{C}(gH)$. It follows that, for sufficiently large $i$, $x_{i}^{-1}\sigma_{R} \subseteq \gamma$ lies inside $N_{C}(x_{i}^{-1}gH)$. In particular, if $x_{i}^{-1} = \gamma(t_{i})$, then we have $\gamma([t_{i}, t_{i} + R]) \subseteq N_{C}(x_{i}^{-1}gH)$. $R > \chi$, this contradicts 3.

\[\square\]
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