COMPLEX MONGE-AMPÈRE EQUATION FOR MEASURES SUPPORTED ON REAL SUBMANIFOLDS

DUC-VIET VU

ABSTRACT. Let $(X, \omega)$ be a compact $n$-dimensional Kähler manifold on which the integral of $\omega^n$ is 1. Let $K$ be an immersed real $C^3$ submanifold of $X$ such that the tangent space at any point of $K$ is not contained in any complex hyperplane of the (real) tangent space at that point of $X$. Let $\mu$ be a probability measure compactly supported on $K$ with $L^p$ density for some $p > 1$. We prove that the complex Monge-Ampère equation $(dd^c \varphi + \omega)^n = \mu$ has a Hölder continuous solution.

Keywords: Monge-Ampère equation, generic CR submanifold.

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1. INTRODUCTION

Let $X$ be a compact Kähler manifold of dimension $n$ and let $\omega$ be a fixed Kähler form on $X$ so normalized that $\int_X \omega^n = 1$. The aim of this paper is to give a useful explicit class of measures for which the complex Monge-Ampère equation has a Hölder continuous solution. Recall that a real $C^1$ manifold $K$ is said to be immersed in $X$ if there is an injective $C^1$ immersion from $K$ to $X$. In this case we say that $K$ is an immersed $C^1$ submanifold of $X$. An immersed real $C^1$ submanifold $K$ of $X$ is said to be generic CR (or generic for simplicity) in the sense of the Cauchy-Riemann geometry if the tangent space at any point of $K$ is not contained in a complex hyperplane of the tangent space at that point of $X$. Such a submanifold has the real dimension at least $n$. A function $\varphi : X \rightarrow (-\infty, \infty)$ is quasi-p.s.h. if it is locally the sum of a p.s.h. function and a smooth one. A quasi-p.s.h. function is said to be $\omega$-p.s.h. if we have $dd^c \varphi + \omega \geq 0$ in the sense of currents. The following is our main result.

Theorem 1.1. Let $K$ be a generic immersed $C^3$ submanifold of $X$ of real codimension $d > 0$. Let $\mu$ be a probability measure compactly supported on $K$ with $L^p$ density for some $p > 1$.

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Then the Monge-Ampère equation $(dd^c \varphi + \omega)^n = \mu$ has an $\omega$-p.s.h. solution $\varphi$ which is Hölder continuous with Hölder exponent $\alpha$, for any positive number $\alpha < \frac{2(p-1)}{3d(n+1)p}$.

Note that our proof still holds if $K$ is $C^{2,\beta}$ for some $\beta \in (0, 1)$. In this case one just needs to replace the $C^{2,1/2}$ regularity in Section 3 by $C^{2,\beta'}$ one for $\beta' \in (0, \beta)$. For simplicity, we only consider the $C^{3}$ regularity as in Theorem 1.1. Secondly, if the Monge-Ampère equation has a Hölder continuous solution, then that solution is unique up to an additive constant. This is a direct consequence of results in [13, 6].

For a probability measure $\mu$ on $X$, the associated complex Monge-Ampère equation

$$(dd^c \varphi + \omega)^n = \mu$$

has been extensively studied since the fundamental paper [25] of Yau in which he proved that (1.1) has a unique smooth solution if $\mu$ is a (smooth) Riemannian volume form $\text{vol}_X$ of $X$. Later Kołodziej showed that the Monge-Ampère equation admits a unique continuous solution for a larger class of measures $\mu$ which contains $\mu = f \text{vol}_X$ with $f \in L^p(X)$ for $p > 1$, see [14, 13]. For the last measures, he also obtained Hölder regularity of the solution in [15]. The Hölder exponent of that solution is then made precise by Demailly, Dinew, Guedj, Hiep, Kolodziej and Zeriahi in [5] using the regularization method in [4] and the stability theorem in [7]. Moreover, in [11] Hiep obtains the Hölder regularity for $\mu = f \text{vol}_Y$, where $\text{vol}_Y$ is the volume form of a compact real hypersurface $Y$ of $X$ and $f \in L^p(Y)$ for $p > 1$.

Recently, Dinh and Nguyên in [8] show that the class of probability measures $\mu$, for which (1.1) admits a Hölder continuous solution, is exactly the class of probability measures whose super-potentials are Hölder continuous, see Definition 1.3 below. They then recover the aforementioned results in [15, 11, 5]. By [8], we know that if a probability measure $\mu$ having a Hölder continuous super-potential of order $\beta \in (0, 1]$, then the solution of (1.1) is Hölder continuous of order $\beta'$ for any $0 < \beta' < 2\beta/(n+1)$.

For more information on the complex Monge-Ampère equation, the readers may consult the survey [20].

Theorem 1.1 above combined with [8, Pro. 4.4] yields the following nice exponential estimate, see also [22, 9, 12].

**Corollary 1.2.** Let $K$ be a generic immersed $C^3$ submanifold of $X$. Let $\tilde{K}$ be a compact subset of $K$. Then the restriction of the Lebesgue measure on $K$ to $\tilde{K}$ is moderate, that is, there exist two positive constants $\alpha$ and $c$ such that for any $\omega$-p.s.h. function $\varphi$ on $X$ with $\sup_X \varphi = 0$ we have

$$\int_{\tilde{K}} e^{-\alpha \varphi} d\text{vol}_K \leq c.$$ 

Before presenting the idea of the proof of Theorem 1.1, we need to recall some definitions. Let $\mu$ be a probability measure on $X$. Let $\mathcal{C}$ be the set of $\omega$-p.s.h. functions $\varphi$ on $X$ such that $\int_X \varphi \omega^n = 0$. We define the distance $\text{dist}_{L^1}$ on $\mathcal{C}$ by putting

$$\text{dist}_{L^1}(\varphi_1, \varphi_2) := \int_X |\varphi_1 - \varphi_2| \omega^n,$$

for every $\varphi_1, \varphi_2 \in \mathcal{C}$. 
Definition 1.3. The super-potential of $\mu$ (of mean 0) is the function $\mathcal{U} : \mathcal{C} \rightarrow \mathbb{R}$ given by $\mathcal{U}(\varphi) := \int_X \varphi d\mu$. We say that $\mathcal{U}$ is Hölder continuous with Hölder exponent $\alpha \in (0, 1]$ if it is so with respect to the distance $\text{dist}_{L^1}$.

By [8, The. 1.3, Cor. 4.5], Theorem 1.1 is a direct consequence of the following result.

Theorem 1.4. Let $K$ be a generic immersed $C^3$ submanifold of $X$ of real codimension $d > 0$. Let $\tilde{K}$ be a compact subset of $K$ and $1_{\tilde{K}}$ the characteristic function of $\tilde{K}$. Let $\text{vol}_K$ be an arbitrary $C^3$ Riemannian volume form of $K$. Then the super-potential of $1_{\tilde{K}} \text{vol}_K$ is Hölder continuous with Hölder exponent $\alpha$ for any positive number $\alpha < 1/(3d)$.

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$ and let $\partial \mathbb{D}$ be the boundary of $\mathbb{D}$. A $C^1$ analytic disc in $X$ is a $C^1$ map from $\overline{\mathbb{D}}$ to $X$ which is holomorphic on $\mathbb{D}$. For a nonempty arc $I \subset \partial \mathbb{D}$, an analytic disc $f$ is said to be $I$-attached to a subset $K$ of $X$ if $f(I)$ belongs to $K$. When we do not want to mention $I$, we simply say an analytic disc partly attached to $K$. Throughout this paper, for every parameter $\tau$, we will systematically use the notation $\preceq_{\tau}$ or $\preceq$ which means $\leq$ up to a constant depending only on $(\tau, X, K, \omega)$ or on $(X, K, \omega)$ respectively. A similar convention is applied to $\succeq_{\tau}$ and $\succeq$.

The idea of the proof of Theorem 1.4 is as follows. Observe that the codimension $d$ of $K$ is at most equal to $n$. We consider below the case where $d = n$. The other cases can be deduced from it. Let $\varphi_1, \varphi_2 \in \mathcal{C}$ and $\varphi := \varphi_1 - \varphi_2$. To show the Hölder regularity of the super-potential of $\text{vol}_K$, by definition we need to bound the $L^1$-norm of $\varphi$ with respect to $\text{vol}_K$ by a power of the $L^1$-norm of $\varphi$ on $X$. Since one can approximate any $\omega$-p.s.h. function on $X$ by a decreasing smooth ones (see [3]), it is enough to prove the desired property for smooth $\varphi_1, \varphi_2$ with $\varphi_1 \geq \varphi_2$, see Proposition 5.1 and Lemma 5.2. In this case, $\varphi$ is smooth and nonnegative. This reduction is crucial in our proof. Observe that by compactness of $\tilde{K}$, it suffices to estimate

$$\int_{\tilde{K}'} \varphi d\text{vol}_K,$$

for small open subsets $\tilde{K}'$ of $\tilde{K}$. For each point $a \in K$, we will construct a $C^{2,1/2}$-differentiable family $F_{\tau \in Z}$ of analytic discs partly attached to $K$ parameterized by $\tau$ in a compact manifold $Z$ of real dimension $(2n - 2)$ which roughly satisfies the following two properties:

(i) the restriction of $F$ to $\partial \mathbb{D} \times Z$ is a submersion onto an open neighborhood $K'$ of $a$ in $K$, where we consider $F$ as a map from $\overline{\mathbb{D}} \times Z$ to $X$.

(ii) the restriction of $F$ to $\mathbb{D} \times Z$ is a diffeomorphism onto an open subset of $X$.

Put $\tilde{K}' := \tilde{K} \cap K'$ for $a \in \tilde{K}$. These $\tilde{K}'$ covers $\tilde{K}$. By the change of variables theorem and Property (i), we have

\begin{equation}
\int_{\tilde{K}'} \varphi d\text{vol}_K \leq \int_{K'} \varphi d\text{vol}_K \lesssim \int_{\partial \mathbb{D} \times Z} \varphi \circ F.
\end{equation}

Since $\tilde{F}$ is holomorphic on $\mathbb{D}$ and $C^2$ on $\overline{\mathbb{D}}$, observe that $\varphi \circ \tilde{F}$ is the difference of two $C^2$ subharmonic functions on $\overline{\mathbb{D}}$.

Our second step is to bound $\int_{\partial \mathbb{D} \times Z} \varphi \circ \tilde{F}$ by a quantity involving $\int_{\mathbb{D} \times Z} \varphi \circ \tilde{F}$. For this purpose, we will establish a crucial inequality in dimension one which shows that $L^1$-norm on $\partial \mathbb{D}$ of a nonnegative $C^2$ function on $\overline{\mathbb{D}}$ is bounded by a function of its $L^1$-norm on $\mathbb{D}$ and some Hölder norm of its Laplacian on $\mathbb{D}$. The ingredients for the proof of the last
inequality are Riesz’s representation formula and a general interpolation inequality for currents on manifolds with boundary. Note that a version of that interpolation inequality for manifolds without boundary was firstly used by Dinh and Sibony in [10].

The problem will be solved if one is able to bound $\int_{\mathcal{D} \times \mathcal{Z}} \varphi \circ \tilde{F}$ by a constant times $\|\varphi\|_{L^1(X)}$. Taking into account Property (ii), one is tempted to use the change of variables by $\tilde{F}$. However, the Jacobian of $\tilde{F}$ is small near the boundary $\partial \mathcal{D} \times \mathcal{Z}$. This is due to a general fact that any family of analytic discs satisfying Property (i) should degenerate at $\partial \mathcal{D}$ because of its attachment to $K$. So we need a precise control of the Jacobian of $\tilde{F}$ from below and prove some estimates on the integrals of p.s.h. functions and their $dd^c$ on a tubular neighborhood of $K$. These estimates are of independent interest. Consequently, we will get

$$\int_{\mathcal{D} \times \mathcal{Z}} \varphi \circ \tilde{F} \lesssim \left( \int_X \varphi \, d\text{vol}_X \right)^{\alpha_2},$$

for any $\alpha_2 \in (0, 1/n)$. Combining these above inequalities gives the Hölder regularity of the super-potential of $\text{vol}_K$.

The paper is organized as follows. Section 2 is devoted to proving the above mentioned interpolation inequality for currents. In Section 3 we construct the desired family of analytic discs $\tilde{F}$. In Section 4 we present (1.2) and (1.3). Finally, we prove Theorem 1.4 in Section 5. At the beginning of Section 3 we will fix some notations which will be used for the rest of the paper.

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2. Interpolation theory

Let $M$ be a compact smooth manifold of dimension $m$. Fix a partition of unity subordinated to a finite covering of local charts of $M$. For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, let $C^{k,\alpha}(M)$ be the space of $C^k$ functions on $M$ whose partial derivatives of order $k$ are Hölder continuous of order $\alpha$. We endow the last space with the usual norm. For $t \in [0, \infty)$, denote by $C^t(M)$ the space $C^{[t], t-[t]}(M)$ where $[t]$ is the integer part of $t$. Let $\Lambda^lT^*M$ be the $l$th-exterior power of the cotangent vector bundle $T^*M$ for $1 \leq l \leq m$. Let $C^t(M, \Lambda^lT^*M)$ be the set of $l$-differential forms with $C^t$ coefficients. Using the above fixed partition of unity, we can equip $C^t(M, \Lambda^lT^*M)$ with the norm $\| \cdot \|_{C^t}$ which is the maximum of the $C^t$ norms of its coefficients.

Let $T$ be an $l$-current of order $0$, i.e., there is a constant $C$ such that $|\langle T, \Phi \rangle| \leq C\|\Phi\|_{C^0}$ for every smooth $(m-l)$-form $\Phi$. For $t \in [0, \infty)$, define

$$\|T\|_{C^{-t}} := \sup_{\Phi \text{ smooth, } \|\Phi\|_{C^t} = 1} |\langle T, \Phi \rangle|.$$

We will write $\|T\|$ instead of $\|T\|_{C^{-0}}$ which is the usual mass norm of $T$. Dinh and Sibony in [10] proved that for any $t_1, t_2 \in (0, \infty)$ with $t_1 < t_2$, we have

$$\|T\|_{C^{-t_2}} \leq \|T\|_{C^{-t_1}} \leq c \|T\|_{C^{-t_1/t_2}} \|T\|_{C^{-t_2/t_1}},$$

for some constant $c$ independent of $T$. This inequality is very useful when dealing with continuous functionals on differential forms because one can reduce the problem to the
smooth case. In this section, we will establish a generalization of (2.2) for compact smooth manifolds with boundary.

Let \( M \) be a compact smooth manifold of dimension \( m \) with boundary. Cover \( M \) by a finite number of local charts \( U_j \). Take a partition of unity \( \phi_j \) subordinated to this covering. By the aid of these \( \phi_j \), as above we can define the Banach spaces \( C^l(M) \) with the usual norms for \( t \in [0, \infty) \). Denote by \( \text{Int} M \) the interior of \( M \). Let \( C^l(\text{Int} M) \) be the subspace of \( C^l(M) \) of \( f \in C^l(M) \) with compact support in \( \text{Int} M \). Let \( \tilde{C}^t(M) \) be the subspace of \( C^l(M) \) consisting of \( f \) with \( f|_{\partial M} \equiv 0 \). We can also define \( \tilde{C}^t(M, \Lambda T^*M) \) and \( C^l(M, \Lambda T^*M) \) in the same way as above.

Let \( T \) be an \( l \)-current of order \( 0 \) on \( \text{Int} M \). Assume that its mass is finite, that is,

\[
\|T\| := \sup_{\Phi \text{ smooth, } \|\Phi\|_{C^l(\text{Int} M)} = 1} \left| \langle T, \Phi \rangle \right| < \infty.
\]

In our application, \( M \) will be \( \overline{D} \) and \( T \) will be the restriction of a continuous form on \( \mathbb{C} \) to \( \mathbb{D} \). By Riesz’s representation theorem, \( T \) is a differential form whose coefficients are Radon measures on \( M \) with finite total variations. Hence, for any continuous differential form \( \Phi \) on \( \text{Int} M \) with \( \|\Phi\|_{C^0} < \infty \), the value of \( T \) at \( \Phi \) is well-defined. Then the current \( T \) can be extended to be a continuous linear functional on \( \tilde{C}^t(M, \Lambda T^*M) \). Let \( \|T\|_{\tilde{C}^{-t}(M)} \) be the norm of \( T \) as a continuous linear functional on \( \tilde{C}^t(M, \Lambda T^*M) \). As mentioned at the beginning of the section, we will prove the following analogue of (2.2).

**Proposition 2.1.** Let \( T \) be a \( l \)-current of order \( 0 \) on \( \text{Int} M \). Assume that \( T \) has finite mass. Let \( t_0, t_1, t_2 \in [0, \infty) \) with \( t_0 < t_1 < t_2 \). Let \( t_\ast \) be the unique real number for which \( t_1 = t_\ast t_0 + (1 - t_\ast) t_2 \). Then we have

\[
\|T\|_{\tilde{C}^{-t_2}(M)} \leq \|T\|_{\tilde{C}^{-t_1}(M)} \leq C\|T\|_{\tilde{C}^{-t_0}(M)}^\ast \|T\|_{\tilde{C}^{-t_2}(M)}^{1-t_\ast},
\]

for some constant \( C \) independent of \( T \).

The remaining part of this section is devoted to prove the last proposition. Using a partition of unity as above, that proposition is a direct consequence of Corollary 2.9 at the end of this section. We first recall some notations and results from the interpolation theory of Banach spaces and refer to \([17, 23]\) for a general treatment of the theory. Then we compute some interpolation spaces of \( \tilde{C}^t(M) \), see Corollary 2.8 below.

Let \( A_0 \) and \( A_1 \) be two Banach spaces which are continuously embedded to a Hausdorff topological vector space \( A \). Let \( B_0 \) and \( B_1 \) be two Banach spaces which are continuously embedded to a Hausdorff topological vector space \( B \). Let \( T \) be a linear operator from \( A \) to \( B \). Assume that \( T|_{A_j} : A_j \to B_j \) are bounded for \( j = 0, 1 \). The interpolation theory of Banach spaces is to search for Banach subspaces \( A \subset \mathcal{A} \) and \( B \subset \mathcal{B} \) such that the restriction \( T|_{\mathcal{A}} : \mathcal{A} \to \mathcal{B} \) is a bounded linear operator. The spaces \( \mathcal{A} \) and \( \mathcal{B} \) are called interpolation spaces. We will recall below a classical construction of such spaces.

For \( 0 < t < \infty \) and \( a \in A_0 + A_1 \), define

\[
K(t, a; A_0, A_1) := \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}),
\]

where \( a_0 \in A_0, a_1 \in A_1 \). Let \( \alpha \) be a constant in \((0, 1)\). The following class of Banach spaces is of great importance in the interpolation theory.
Definition 2.2. Let \((A_0, A_1)_{\alpha, \infty}\) be the subspace of \(A_0 + A_1\) consisting of \(a \in A_0 + A_1\) for which the following quantity

\[
\|a\|_{(A_0, A_1)_{\alpha, \infty}} := \sup_{t > 0} t^{-\alpha} K(t, a; A_0, A_1)
\]

is finite. The last formula defines a norm on \((A_0, A_1)_{\alpha, \infty}\) which make it to be a Banach space.

The following fundamental theorem explains the role of the space \((A_0, A_1)_{\alpha, \infty}\).

Theorem 2.3. [17, Th. 1.1.6] Let \(A_0, A_1, B_0, B_1\) and \(T\) be as above. Let \(\alpha \in (0, 1)\). Then the restriction \(T_{(A_0, A_1)_{\alpha, \infty}}\) of \(T\) to \((A_0, A_1)_{\alpha, \infty}\) is a bounded linear operator from \((A_0, A_1)_{\alpha, \infty}\) to \((B_0, B_1)_{\alpha, \infty}\) and

\[
\|T\|_{(A_0, A_1)_{\alpha, \infty}} \leq \|T\|_{\alpha, \infty}^{1-\alpha} \|T\|_{A_0}^{\alpha},
\]

where \(\|\cdot\|\) is the norm of bounded linear operators.

Let \(m \in \mathbb{N}^*\) and \(k \in \mathbb{N}\) and \(\alpha \in (0, 1)\). Let \(C^k(\mathbb{R}^m)\) (respectively \(C^{k, \alpha}(\mathbb{R}^m)\)) be the set of \(C^k\) functions (respectively \(C^{k, \alpha}\)) on \(\mathbb{R}^m\). For \(t \in \mathbb{R}^+\), define \(C^t(\mathbb{R}^m) := C^{[t], t-\lfloor t\rfloor}(\mathbb{R}^m)\). Let \(C^t_0(\mathbb{R}^m)\) be the subset of \(C^t(\mathbb{R}^m)\) consisting of elements whose \(C^t\) norms are bounded.

Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^m\) with smooth boundary. Let \(\partial \Omega\) be its boundary. Then \(\overline{\Omega}\) is a smooth compact manifold with boundary which is itself a global chart. We have the Banach spaces \(C^t(\overline{\Omega})\) and \(C^t_0(\overline{\Omega})\) as above. In what follows, we will give a description of the interpolation space

\[
(C^{t_0}(\overline{\Omega}), C^{t_2}(\overline{\Omega}))_{\alpha, \infty}
\]

for \(0 \leq t_0 < t_2 < \infty\). The corresponding interpolation spaces for \(C^t(\overline{\Omega})\) and \(C^t_0(\mathbb{R}^m)\) are already known, see Theorems 2.7.2 and 4.5.2 in [23].

It should be noted that the spaces \((C^{t_0}(\overline{\Omega}), C^{t_2}(\overline{\Omega}))_{\alpha, \infty}\) are easily determined by using the result mentioned above for \(C^t_0(\mathbb{R}^m)\) and the fact that the restriction from \(C^t_0(\mathbb{R}^m)\) to \(C^t(\overline{\Omega})\) is a retraction, see [23, Th. 4.5.1]. Nevertheless, this property is no longer true if we replace \(C^t(\overline{\Omega})\) by \(C^t_0(\overline{\Omega})\) because even the restriction map from \(C^t_0(\mathbb{R}^m)\) to \(C^t(\overline{\Omega})\) is not well-defined. In order to compute (2.7), we will follow the original strategy for \(C^t_0(\mathbb{R}^m)\) in [23], see also [17]. Although, in essence, our below results can be implicitly deduced from [23], we will present them in a simplified and detailed way which is therefore accessible for a wider audience.

The following lemma is well-known but for the reader’s convenience, a complete proof will be given.

Lemma 2.4. For every \(t \in [0, \infty)\), every \(f \in C^t(\overline{\Omega})\) can be extended to be a function \(Ef \in C^t(\mathbb{R}^m)\) such that \(\|Ef\|_{C^t(\mathbb{R}^m)} \leq C \|f\|_{C^t(\overline{\Omega})}\), where \(C\) is a constant independent of \(f\).

Proof. We will use a reflexion argument. By using a partition of unity subordinated to a suitable finite covering of \(\overline{\Omega}\), we can suppose that \(\overline{\Omega} = \mathbb{R}^{m-1} \times \mathbb{R}^+\). Let \(f \in C^t(\mathbb{R}^{m-1} \times \mathbb{R}^+)\). Let \([t]\) be the integer part of \(t\). Let \(a_1, \ldots, a_{[t]+1}\) be real numbers which are chosen later. Define \(Ef := f\) on \(\mathbb{R}^{m-1} \times \mathbb{R}^+\) and

\[
Ef(x_1, \ldots, x_n) := \sum_{k=1}^{[t]+1} a_k f(x_1, \ldots, x_{n-1}, -kx_n)
\]
otherwise. It is easy to see that $Ef$ is continuous on $\mathbb{R}^m$. Now we will choose $a_k$ such that $Ef \in C^l$. If we can do so, we also get $Ef \in C^l$ because $D^{[l]}Ef$ is $C^{l-[l]}$ on $\mathbb{R}^{m-1} \times \mathbb{R}^+$, hence, on the whole $\mathbb{R}^m$ by its defining formula. One only needs to be concerned with the $x_n$-direction. Direct computations show that

$$\frac{\partial^{[l]}_x Ef(x_1, \ldots, x_{n-1}, 0)}{x_n} = \left( \sum_{k=1}^{[l]+1} (-k)^l a_k \right) \frac{\partial^{[l]}_x f(x_1, \ldots, x_{n-1}, 0)}{x_n},$$

for $0 \leq l \leq [l]$. The regularity condition on $Ef$ is equivalent to the linear system

$$\sum_{k=1}^{[l]+1} (-k)^l a_k = 0 \quad \text{for} \quad 0 \leq l \leq [l].$$

Its determinant is a Vandermonde one. Hence the system has a unique solution $(a_1, \ldots, a_{[l]+1})$. When $f|_{\partial \Omega} = 0$, it is clear from the defining formula of $Ef$ that $Ef|_{\partial \Omega} = 0$. The proof is finished. \hfill \Box

**Proposition 2.5.** For every $\alpha \in (0, 1)$ and every $k \in \mathbb{N}^*$, we have

$$(\tilde{C}^0(\Omega), \tilde{C}^k(\Omega))_{\alpha, \infty} \subset \tilde{C}^{\alpha k}(\Omega),$$

where the last inclusion means a continuous inclusion between Banach spaces.

**Proof.** Let $f \in \tilde{C}^{\alpha k}(\Omega)$. Put $t := \alpha k$. We write below $\lesssim$ to indicate $\leq$ up to a constant independent of $(f, e)$. By Lemma 2.4, we can extend $f$ to be a function $F$ in $C^l(\mathbb{R}^m)$ with

$$\|F\|_{C^l(\mathbb{R}^m)} \leq C\|f\|_{\tilde{C}^t(\Omega)},$$

for some constant $C$ independent of $f$ and $F|_{\partial \Omega} = 0$. Let $\mathbb{B}_r$ denotes the ball of radius $r > 0$ centered at 0 in $\mathbb{R}^m$ and $\mathbb{B}_r^+$ denotes the subset of $\mathbb{B}_r$ consisting of $x = (x_1, \ldots, x_n)$ with $x_n \geq 0$. Since $\overline{\Omega}$ is compact, we can cover $\partial \Omega$ by a finite number of small open subsets $\{U_j\}_{1 \leq j \leq N}$ of $\mathbb{R}^m$ such that in each $U_j$, by a suitable change of coordinates $\Psi_j$, we have

$$\Psi_j(\overline{\Omega} \cap U_j) = \mathbb{B}_2^+$$

and $\Psi_j(\partial \Omega \cap U_j) = \mathbb{B}_2^+ \cap \{x_n \geq 0\}$. Without loss of generality, we can suppose that $\Psi_j^{-1}(\mathbb{B}_2^+)$ also covers $\partial \Omega$. Put

$$U_0 := \Omega \setminus \bigcup_{1 \leq j \leq N} \Psi_j^{-1}(\mathbb{B}_2^+).$$

The family $\{U_j\}_{0 \leq j \leq N}$ covers $\overline{\Omega}$. Let $\{\chi_j\}_{0 \leq j \leq N}$ be smooth functions of $\mathbb{R}^m$ such that $0 \leq \chi_j \leq 1$ for $0 \leq j \leq N$, and $\text{supp} \chi_j \subseteq \Psi_j^{-1}(\mathbb{B}_{2\delta/4})$ for $1 \leq j \leq N$ and $\text{supp} \chi_0 \subseteq U_0$, and $\sum_{0 \leq j \leq N} \chi_j = 1$ on $\overline{\Omega}$.

Define $F_j := (\chi_j F) \circ \Psi_j^{-1}$. By the properties of $(\Psi, F)$ mentioned above, we have $F_j|_{x_n=0} = 0$. Let $\chi$ be a nonnegative smooth function on $\mathbb{R}^m$ which is compactly supported on $\mathbb{B}_1$ such that $\int_{\mathbb{R}^m} \chi \, dx = 1$. Taylor’s expansion for $F_j$ gives

$$F_j(x) = F_j(x - y) + DF_j(x - y) y + \cdots + \frac{1}{[l]!} D^{[l]}F_j(x - y) y^{[l]} + R_j(x, y) y^{[l]},$$

where $R_j(x, y)$ is, for $x$ fixed, a $C^{l-[l]}$ linear functional on $(\mathbb{R}^m)^l$ and we have

$$R_j(x, 0) = 0, \quad \|R_j\|_{C^{l-[l]}} \lesssim \|F_j\|_{C^l} \leq C\|f\|_{C^l}.$$ 

Hence, one gets

$$|R_j(x, y)| \lesssim |y|^{l-[l]} \|f\|_{C^l}.$$
Put 
\[ \epsilon_0 := \min\{1/4, \text{dist}(U_0, \partial \Omega)\}. \]

Let \( \epsilon \in (0, \epsilon_0) \). For \( 0 \leq j \leq N \), we define
\[ (2.11) \quad F_{j, \epsilon}(x) := \int_{\mathbb{R}^m} \left[ F_j(x - \epsilon y) + D F_j(x - \epsilon y) (\epsilon y) + \cdots + \frac{1}{[t]!} D^{[t]} F_j(x - \epsilon y) (\epsilon y)^{[t]} \right] \chi(y) dy. \]

Observe that \( F_{0, \epsilon} \) is a smooth function in \( \tilde{C}^\infty(\Omega) \) by the choice of \( \epsilon \) and \( F_{j, \epsilon} \) is smooth on \( \mathbb{R}^m \) and compactly supported on \( \mathbb{B}_{3/2} \) for \( 1 \leq j \leq N \). A property of the convolution implies that \( F_{j, \epsilon} \) converges to \( F_j \) in \( C^0 \)-topology. Precisely, using (2.9), (2.11) and (2.10) yields that
\[ (2.12) \quad |F_{j, \epsilon}(x) - F_j(x)| \leq \epsilon^t \int_{\mathbb{R}^m} |R_j(x, \epsilon y)| \chi(y) dy \leq C \epsilon^t \|f\|_{C^t}, \]
for every \( x \). Let \( \tau \) be a smooth function on \( \mathbb{R} \) compactly supported on \([-2, 2]\) such that \( \tau \equiv 1 \) on \([-3/2, 3/2]\). Define
\[ F'_{j, \epsilon}(x_1, \ldots, x_n) := F_{j, \epsilon}(x_1, \ldots, x_{n-1}, x_n) - \tau(x_n) F_{j, \epsilon}(x_1, \ldots, x_{n-1}, 0), \]
for \( 1 \leq j \leq N \) and we put \( F'_{0, \epsilon} := F_{0, \epsilon} \) for consistence. We immediately see that \( F'_{j, \epsilon} = 0 \) on \( \{x_n = 0\} \) and \( \text{supp} F'_{j, \epsilon} \subset \mathbb{B}_2 \). As a consequence, \( F'_{j, \epsilon} \circ \Psi_j \) is smooth on \( \mathbb{R}^m \) and vanishes on \( \partial \Omega \). We deduce from (2.12) and the fact that \( F_{j\{x_n=0\}} \equiv 0 \) that
\[ (2.13) \quad |F'_{j, \epsilon}(x) - F_j(x)| \leq |F_{j, \epsilon}(x) - F_j(x)| + |F_{j, \epsilon}(x_1, \ldots, x_{n-1}, 0) - F_j(x_1, \ldots, x_{n-1}, 0)| \leq 2C \epsilon^t \|f\|_{C^t}. \]

Define
\[ g_{1, \epsilon} := \sum_{0 \leq j \leq N} F'_{j, \epsilon} \circ \Psi_j|_{\Omega} \in \tilde{C}^\infty(\Omega) \]
and \( g_{0, \epsilon} := f - g_{1, \epsilon} \in \tilde{C}^0(\Omega) \). We have \( f = g_{0, \epsilon} + g_{1, \epsilon} \). In view of (2.6), we have to estimate \( \|g_{0, \epsilon}\|_{\tilde{C}^0(\Omega)} \) and \( \|g_{1, \epsilon}\|_{\tilde{C}^0(\Omega)} \). Since \( f = \sum_{0 \leq j \leq N} F_j \circ \Psi_j \), we have
\[ g_{0, \epsilon} = \sum_{0 \leq j \leq N} (F_j \circ \Psi_j - F'_{j, \epsilon} \circ \Psi_j). \]

Taking into account (2.13), one gets
\[ (2.14) \quad \|g_{0, \epsilon}\|_{\tilde{C}^0(\Omega)} \leq \epsilon^t \|f\|_{C^t}. \]

For \( 0 \leq l \leq [t] \), we define
\[ G_{j,l}(x, y) := D^l F_j(y) + D^{l+1} F_j(y) y + \cdots + \frac{1}{([t] - l)!} D^{[t]} F_j(y)(x - y)^{[t] - l} \]
which is the Taylor expansion up to the \((|t| - l)\) order of \( D^l F_j(x) \) at \( y \). Thus arguing as in (2.10), we get
\[ (2.15) \quad |G_{j,l}(x, y) - D^l F_j(x)| \leq \|f\|_{C^t} |y|^{l-t}. \]
The equality (2.11) can be rewritten as
\[ F_{j, \epsilon}(x) = \epsilon^{-m} \int_{\mathbb{R}^m} \left[ F_j(y') + D F_j(y')(x - y') + \cdots + \frac{1}{[t]!} D^{[t]} F_j(y')(x - y')^{[t]} \right] \chi\left( \frac{x - y'}{\epsilon} \right) dy'. \]
Differentiating the last equality in $x$ for $k'$ times gives

\[ D_x^{k'} F_{j,\varepsilon}(x) = \varepsilon^{-m-k'+l} \sum_{0 \leq l \leq m} \int_{\mathbb{R}^m} G_{j,l}(x, y') \otimes D^{k'-l} \chi(\frac{x - y'}{\varepsilon})dy' \]

\[ = \varepsilon^{-k'+l} \sum_{0 \leq l \leq m} \int_{\mathbb{R}^m} G_{j,l}(x, x - \varepsilon y) \otimes D^{k'-l} \chi(y)dy \]

by a suitable change of coordinates. Since $\int_{\mathbb{R}^m} D_x^{l} \chi(y)dy = 0$ for any $l \geq 1$, we obtain

\[ \int_{\mathbb{R}^m} G_{j,l}(x, x - \varepsilon y) \otimes D^{k'-l} \chi(y)dy = \int_{\mathbb{R}^m} (G_{j,l}(x, x - \varepsilon y) - D^l f_j(x)) \otimes D^{k'-l} \chi(y)dy \]

which is of absolute value $\lesssim \varepsilon^{-l} \|f\|_{C^l}$ by using (2.15) and the fact that $\text{supp}\chi \subset B_1$. Combining (2.16) with (2.17) gives

\[ |D_x^{k'} F_{j,\varepsilon}(x)| \lesssim \varepsilon^{-k'+l} \|f\|_{C^l} \]

which implies that

\[ \|g_{1,\varepsilon}\|_{\hat{C}^k(\Omega)} \lesssim \varepsilon^{-k+l} \|f\|_{C^l} \]

by choosing $k' = k$. Taking into account (2.14), (2.18) and (2.5), one deduces that

\[ \varepsilon^{-ak} K(\varepsilon^k, f; \hat{C}^0(\Omega), \hat{C}^k(\Omega)) \leq \varepsilon^l (\|g_0,\varepsilon\|_{\hat{C}^0} + \varepsilon^k \|g_{1,\varepsilon}\|_{\hat{C}^k}) \lesssim \|f\|_{\hat{C}^l(\Omega)}, \]

for every $\varepsilon \in (0, \varepsilon_0)$. When $\varepsilon \geq \varepsilon_0$, since

\[ f = f + 0 \in \hat{C}^0(\Omega) + \hat{C}^1(\Omega), \]

we have

\[ \varepsilon^{-ak} K(\varepsilon^k, f; \hat{C}^0(\Omega), \hat{C}^k(\Omega)) \leq \varepsilon_0^{-ak} \|f\|_{\hat{C}^0(\Omega)} \leq \varepsilon_0^{-ak} \|f\|_{\hat{C}^0(\Omega)}. \]

Hence, $f \in (\hat{C}^0(\Omega), \hat{C}^k(\Omega))_{a,\infty}$. The proof is finished. \(\square\)

For every $h \in \mathbb{R}^m$ and every a function $g$ on $\mathbb{R}^m$, define the operator

\[ \Delta_h g(x) := g(x + h) - g(x) \]

for every $x \in \mathbb{R}^m$. The following property is crucial for the next proposition.

**Lemma 2.6.** Let $\alpha \in (0, 1)$ and $l$ be an integer $\geq 1$. For $g \in C^\alpha_\ast(\mathbb{R}^m)$, we put

\[ \|g\|_{\alpha,\Delta,l} := \|g\|_{C^\alpha} + \sup_{x,h \in \mathbb{R}^m, h \neq 0} \frac{|\Delta_h g|}{|h|^{\alpha}}. \]

Then the last formula defines a norm on $C^\alpha_\ast(\mathbb{R}^m)$ which is equivalent to its usual $C^\alpha$ norm. More precisely, there exists a positive constant $C_{l,\alpha}$ depending only on $(l, \alpha)$ such that for every $g$, we have

\[ C_{l,\alpha}^{-1} \|g\|_{C^\alpha} \leq \|g\|_{\alpha,\Delta,l} \leq C_{l,\alpha} \|g\|_{C^\alpha}. \]

**Proof.** This is a simplification of Lemma 1.13.4 in [23]. When $l = 1$, the two norms are identical. Consider $l \geq 2$. Observe that it is enough to prove the desired result for $l = 2$ because the general case can easily follow by induction. It is clear that $\|g\|_{\alpha,\Delta,2} \leq 2\|g\|_{C^\alpha}$. We now prove the converse inequality. The key argument is the following formula:

\[ g(x + h) - g(x) = \frac{1}{2} (g(x + 2h) - g(x)) - \frac{g(x + 2h) - 2g(x + h) + g(x)}{2}. \]
Applying the last equality to \((\text{see } [17, \text{Re. 1.3.7}])\), we have the following general formula:

\[
\|g\|_{\alpha^*} \leq 2^{\alpha - 1} \|g\|_{\alpha^*} + \|g\|_{\alpha, \Delta, 2}.
\]

Since \(2^{\alpha - 1} < 1\) we get the desired conclusion. The proof is finished. \(\square\)

**Proposition 2.7.** Let \(k\) be a positive integer and let \(\alpha\) be a real number in \((0, 1)\). Assume that \(\alpha k \in (0, 1)\). Then we have

\[
(\tilde{\mathcal{C}}^{\alpha_k}(\Omega), \tilde{\mathcal{C}}^{k}(\Omega))_{\alpha, \infty} \subset \tilde{\mathcal{C}}^{\alpha k}(\Omega).
\]

**Proof.** Let take an element \(f \in (\tilde{\mathcal{C}}^{\alpha_k}(\Omega), \tilde{\mathcal{C}}^{k}(\Omega))_{\alpha, \infty}\). Suppose that \(f = g_0 + g_1\) with \(g_0 \in \tilde{\mathcal{C}}^{\alpha_k}(\Omega)\) and \(g_1 \in \tilde{\mathcal{C}}^{k}(\Omega)\). We have \(\Delta^k_h f = \Delta^k_h g_0 + \Delta^k_h g_1\). By using Taylor's expansion of \(g_1\), observe that \(\|\Delta^k_h g_1\| \leq C\|h\|^k\|g_1\|_{C^k}\) for some constant \(C\) independent of \((g_1, h)\). On the other hand, \(\|\Delta^k_h g_0\| \leq 2\|g_0\|_{C^k}\). Combining these inequalities gives

\[
\|\Delta^k_h f\| \leq 2\|g\|_{C^k} + C\|h\|^k\|g_1\|_{C^k} \lesssim \|g\|_{C^k} + \|h\|^k\|g_1\|_{C^k},
\]

for every \((g_0, g_1)\) with \(f = g_0 + g_1\). Taking the infimum in the last inequality in \((g_0, g_1)\), we obtain

\[
\|\Delta^k_h f\| \lesssim K(h^k, f; \tilde{\mathcal{C}}^{\alpha_k}(\Omega), \tilde{\mathcal{C}}^{k}(\Omega)) \leq |h|^{\alpha k}\|f\|_{(\tilde{\mathcal{C}}^{\alpha_k}(\Omega), \tilde{\mathcal{C}}^{k}(\Omega))_{\alpha, \infty}}.
\]

As a consequence, one gets

\[
\|f\|_{\alpha_k, \Delta, k} \lesssim \|f\|_{(\tilde{\mathcal{C}}^{\alpha_k}(\Omega), \tilde{\mathcal{C}}^{k}(\Omega))_{\alpha, \infty}}.
\]

By Lemma [2.6] and the hypothesis that \(\alpha k < 1\), we obtain the desired result. The proof is finished. \(\square\)

**Corollary 2.8.** For every \(\alpha \in (0, 1)\), every real nonnegative numbers \(t_1\) and \(t_2\), we have

\[
(\tilde{\mathcal{C}}^{t_1}(\Omega), \tilde{\mathcal{C}}^{t_2}(\Omega))_{\alpha, \infty} \supset \tilde{\mathcal{C}}^{\alpha t_2 + (1-\alpha)t_1}(\Omega).
\]

**Proof.** For simplicity, we give a proof for \(t_2 = k \in \mathbb{N}^*\) and \(t_1 = \beta \in [0, 1)\). The general case can be deduced by using similar arguments. By a consequence of the reiteration theorem (see [17] Re. 1.3.7]), we have the following general formula:

\[
((A_0, A_1)_{\theta, \infty}, A_1)_{\alpha, \infty} = (A_0, A_1)_{(1-\alpha)\theta + \alpha, \infty}.
\]

Applying the last equality to

\[
A_0 = \tilde{\mathcal{C}}^{\alpha_0}(\Omega), \quad A_1 = \tilde{\mathcal{C}}^{h \beta}(\Omega) \quad \text{and} \quad \theta = \beta / k
\]

and using the fact that \((A_0, A_1)_{\theta, \infty} = \tilde{\mathcal{C}}^{\beta}(\Omega)\) (by Proposition [2.7] and [2.5]), we obtain the desired inclusion. The proof is finished. \(\square\)

Since \((\mathbb{R}, \mathbb{R})_{\alpha, \infty} = \mathbb{R}\) for any \(\alpha \in (0, 1)\), applying Theorem [2.3] to

\[
A_0 = \tilde{\mathcal{C}}^{t_0}(\Omega), \quad A_1 = \tilde{\mathcal{C}}^{t_2}(\Omega), \quad B_0 = B_1 = \mathbb{R},
\]

and then using Corollary [2.8], we obtain the following result.
Corollary 2.9. Let $\Omega$ be a bounded open subset of $\mathbb{R}^m$ with smooth boundary. Let $t_0, t_1$ and $t_2$ be three real numbers such that $0 \leq t_0 < t_1 < t_2$. Let $S$ be a bounded linear map from $\mathcal{C}^0(\Omega)$ to $\mathbb{R}$. Then the restriction $S|\mathcal{C}^j(\Omega)$ of $S$ to $\mathcal{C}^j(\Omega)$ for $j = 1$ or $2$ is also a bounded linear map from $\mathcal{C}^j(\Omega)$ to $\mathbb{R}$ and

$$
\|S|\mathcal{C}^1(\Omega)\| \leq c \|S|\mathcal{C}^0(\Omega)\| \|t\| \|S|\mathcal{C}^2(\Omega)\|^{1-t},
$$

where $c$ is a constant independent of $S$ and $t_*$ is the unique real number for which $t_1 = t_0 + (1 - t_*) t_2$.

3. Analytic discs partly attached to a generic submanifold

Firstly we fix some notations which will be valid throughout the rest of the paper. For every Riemannian smooth manifold $Y$, any $a \in Y$ and $r \in \mathbb{R}^+$, we denote by $\mathbb{B}_Y(a, r)$ the ball of radius $r$ centered at $a$ of $Y$ and by $\nu Y$ the Riemannian volume form of $Y$. When $Y = \mathbb{R}^m$ for some $m \in \mathbb{N}$ with the Euclidean metric, we write $\mathbb{B}_m(a, r)$ instead of $\mathbb{B}_Y(a, r)$ and $\mathbb{B}_m$ instead of $\mathbb{B}_m(0, 1)$. In particular, when $Y = \mathbb{C} \simeq \mathbb{R}^2$ and $a = 0$, we put $\mathbb{D}_a := \mathbb{B}_2(a, r)$ and $\mathbb{D} := \mathbb{B}_2(0, 1)$. For every $m \in \mathbb{N}^*$, we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$ via the formula $\mathbb{C}^m = \mathbb{R}^m + i \mathbb{R}^m$.

Let $\partial \mathbb{D}$ be the boundary of $\mathbb{D}$ and $\partial^+ \mathbb{D} := \{ \xi \in \mathbb{D} : \text{Re} \xi \geq 0 \}$. We sometimes identify $\xi \in \mathbb{D}$ with $\theta \in (-\pi, \pi]$ by letting $\xi = e^{i\theta}$. An analytic disc $f$ in $X$ is a holomorphic mapping from $\mathbb{D}$ to $X$ which is continuous up to the boundary $\partial \mathbb{D}$ of $\mathbb{D}$. For an interval $I \subset \partial \mathbb{D}$, $f$ is said to be $I$-attached to a subset $E \subset X$ if $f(I) \subset E$. When $I = \partial^+ \mathbb{D}$, an analytic disc $I$-attached to $E$ is said to be half-attached to $E$.

Let $K$ be a generic immersed $C^3$ submanifold of $X$. Observe that the dimension of $K$ is at least $n$. Throughout the paper, we only consider the case where $\dim K = n$, hence its codimension $d$ equals $n$. This is in fact the most interesting case and the general case will be easily deduced from it. In Section 5, we will explain the necessary modifications to get Theorem 1.4 when $\dim K > n$.

Our goal is to for each $a \in K$ construct a $C^{2,1/2}$-differentiable family of analytic discs partly attached to $K$ which covers an open neighborhood of $a$ in $X$. It should be noted that any family of discs partly attached to $K$ degenerates near $K$ due to its attachment to $K$. Controlling such behaviour around $K$ is actually the key point in this section. We also need that the part of this family lying in $K$ must cover an open neighborhood of $0$ in $K$. Constructing analytic discs is an important tool in Cauchy-Riemann geometry. Generally, one uses a suitable Bishop-type equation together with a choice of initial data depending on situations to obtain the desired result. The reader may also consult [1, 18, 19] and references therein for more information. In what follows, we will apply the same strategy combining with the ideas from [24].

The following local coordinates are frequently used in the Cauchy-Riemann geometry.

Lemma 3.1. Through every point $a$ of $K$, there exist local holomorphic coordinates $(W, z)$ of $X$ around $a$ such that in that local coordinates, the point $a$ is the origin and $K \cap W$ is the graph over $\mathbb{B}_n$, of a $C^3$ map $h$ from $\mathbb{B}_n$ to $\mathbb{R}^n$ which satisfies $D^j h(0) = 0$ with $j = 0, 1, 2$, where $Dh$ denotes the differential of $h$. Moreover, $\|h\|_{C^3}$ is bounded uniformly in $a \in K$.

Proof. The existence of such $h$ with $h(0) = Dh(0) = 0$ is well-known, see [1] for example. In order to obtain the additional property $D^3 h(0) = 0$, one will need to perform a change of coordinates, we refer to [18, Sec. 6.10] for details. The proof is finished. □
From now on, fix an arbitrary point \( a \in K \) and we confine ourselves to the local chart described in Lemma \( \ref{lemma3.1} \). In other words, we will work on \( \mathbb{C}^n \) and

\[ K' := \{ z = x + ih(x) \in \mathbb{C}^n : x \in \mathbb{B}_n \}, \]

where we have \( h(0) = D h(0) = 0 \). For most of the time, the last condition is enough for our purposes, we will only need \( D^2 h(0) = 0 \) in the proof of Proposition \( \ref{proposition4.5} \). The property of \( h \) yields that there is a constant \( c_0 \) for which

\[ |h(x)| \leq c_0 |x|^2, \quad |D h(x)| \leq c_0 |x|, \]

for every \( x \in \mathbb{B}_n \).

In this paragraph, we prepare some useful facts about harmonic functions on the unit disc which will be indispensable for studying Bishop-type equations later. Denote by \( z = x + iy \) the complex variable on \( \mathbb{C} \) and by \( \xi = e^{i \theta} \) the variable on \( \partial \mathbb{D} \). Let \( u_0(\xi) \) be an arbitrary continuous function on \( \partial \mathbb{D} \). Recall that \( u_0 \) can be extended uniquely to be a harmonic function on \( \mathbb{D} \) which is continuous on \( \mathbb{D} \). Since this correspondence is bijective, without stating explicitly, we will freely identify \( u_0 \) with its harmonic extension on \( \mathbb{D} \). We will write \( u_0(z) = u_0(x + iy) \) to indicate the harmonic extension of \( u_0(e^{i \theta}) \). It is well-known that the Cauchy transform of \( u_0 \), given by

\[ C u_0(z) := \frac{1}{2 \pi} \int_{-\pi}^{\pi} u_0(e^{i \theta}) \frac{e^{i \theta} + z}{e^{i \theta} - z} d\theta, \]

is a holomorphic function on \( \mathbb{D} \) whose real part is \( u_0 \). Let \( T u_0 \) be the imaginary part of \( C u_0 \). Decomposing the last formula into the real and imaginary parts, we obtain

\[ u_0(z) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{(1 - |z|^2)}{|e^{i \theta} - z|^2} u_0(e^{i \theta}) d\theta, \]

and

\[ T u_0(z) = \frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{z e^{-i \theta} - \bar{z} e^{i \theta}}{i |e^{i \theta} - z|^2} u_0(e^{i \theta}) d\theta. \]

The function \( T u_0 \) is harmonic on \( \mathbb{D} \) but is not always continuous up to the boundary of \( \mathbb{D} \). Let \( k \) be an arbitrary natural number and let \( \beta \) be an arbitrary number in \( (0, 1) \). A result of Privalov (see [18, Th. 4.12]) implies that if \( u_0 \) belongs to \( C^{k, \beta}(\partial \mathbb{D}) \), then \( T u_0 \) is continuous up to \( \partial \mathbb{D} \) and \( \| T u_0 \|_{C^{k, \beta}(\partial \mathbb{D})} \) is bounded by \( \| u_0 \|_{C^{k, \beta}(\partial \mathbb{D})} \) times a constant independent of \( u_0 \). Hence, the linear self-operator of \( C^{k, \beta}(\partial \mathbb{D}) \) defined by sending \( u_0 \) to the restriction of \( T u_0 \) onto \( \partial \mathbb{D} \) is bounded and called the Hilbert transform. For simplicity, we also denote it by \( T \). For our later purposes, it is convenient to use a modified version \( T_1 \) of \( T \) defined by

\[ T_1 u_0 := T u_0 - T u_0(1). \]

Hence we always have \( T_1 u_0(1) = 0 \) and

\[ \partial_\theta T_1 u_0 - \partial_\theta T u_0 = T \partial_\theta u_0, \]

provided that \( u_0 \in C^{1, \beta}(\partial \mathbb{D}) \) with \( \beta \in (0, 1) \), see [18, p.121] for a proof. The boundedness of \( T \) on \( C^{k, \beta}(\partial \mathbb{D}) \) implies that there is a constant \( C_{k, \beta} > 1 \) such that for any \( v \in C^{k, \beta}(\partial \mathbb{D}) \) we have

\[ \| T_1 v \|_{C^{k, \beta}(\partial \mathbb{D})} \leq C_{k, \beta} \| v \|_{C^{k, \beta}(\partial \mathbb{D})}. \]
Extending $u_0, T_1 u_0$ harmonically to $D$. By construction, the function $f(z) := -T_1 u_0(z) + i u_0(z)$ is holomorphic on $D$ and continuous on $\overline{D}$ provided that $u_0$ is in $C^\beta(\partial D)$ with $0 < \beta < 1$. By \cite[Th. 4.2]{19}, $\|f\|_{C^{k,\beta}(\overline{D})}$ is bounded by $\|f\|_{C^{k,\beta}(\partial D)}$ times a constant depending only on $(k, \beta)$. Since $\|u_0\|_{C^{k,\beta}(\overline{D})} \leq \|f\|_{C^{k,\beta}(\partial D)}$ and $\|f\|_{C^{k,\beta}(\partial D)} \leq (1 + C_{k,\beta}) \|u_0\|_{C^{k,\beta}(\partial D)}$, we have

\begin{equation}
\|u_0\|_{C^{k,\beta}(\overline{D})} \leq C'_{k,\beta} \|u_0\|_{C^{k,\beta}(\partial D)},
\end{equation}

for some constant $C'_{k,\beta}$ depending only on $(k, \beta)$. A direct consequence of the above inequalities is that when $u_0$ is smooth on $\partial D$, the associated holomorphic function $f$ is also smooth on $\overline{D}$.

**Lemma 3.2.** There exist a function $u_0 \in C^\infty(\partial D)$ and two positive constants $(\theta_{u_0}, c_{u_0})$ such that $u_0(e^{i\theta}) = 0$ for $\theta \in [-\theta_{u_0}, \theta_{u_0}] \subset [-\pi/2, \pi/2]$ and $\partial_x u_0(1) = -1$ and $u_0(z) > c_{u_0}(1 - |z|)$ for every $z \in D$.

**Proof.** Let $u$ be a smooth function on $\partial D$ vanishing on $\partial^+ D$. By Poisson’s formula, we have

\begin{equation}
\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - |z|^2}{|e^{i\theta} - z|^2} u(e^{i\theta}) d\theta.
\end{equation}

Differentiating (3.6) gives

\begin{equation}
\partial_x u(1) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{u(e^{i\theta})}{\cos \theta - 1} d\theta.
\end{equation}

Note that the last integral is well-defined because $u$ vanishes on $\partial^+ D$. It is easy to choose a smooth $u$ so that the above integral is equal to $-1$ and $u \equiv 0$ on $\partial^+ \overline{D}$ and $u(e^{i\theta}) > 0$ for $|\theta| \geq 3\pi/2$. The last property and (3.6) show that $u(z) > 0$ for every $z \in D$. We have chosen $u$ with the property that $\partial_x u(1) = -1$ and $u(z) > 0$ for $z \in D$. This implies that $\partial_x u(e^{i\theta}) \leq -1/2$ for every $\theta \in [-\theta_0, \theta_0] \subset (-\pi/2, \pi/2)$ for $\theta_0$ close enough to 1. Since $u$ vanishes on $\partial \overline{D}$, we have

\begin{equation}
0 = \partial_{\theta} u(e^{i\theta}) = -\partial_x u(e^{i\theta}) \sin \theta + \partial_y u(e^{i\theta}) \cos \theta
\end{equation}

which yields

\begin{equation}
\partial_{\theta} u(e^{i\theta}) = \partial_x u(e^{i\theta}) \tan \theta
\end{equation}

for $\theta \in [-\theta_0, \theta_0]$. Let $z = |z| e^{i\theta} \in D$ such that $\theta \in [-\theta_0, \theta_0]$. Taylor’s expansion for $u$ at $e^{i\theta}$ gives

\begin{equation}
\begin{aligned}
u(|z| e^{i\theta}) &= u(e^{i\theta}) + (|z| \cos \theta - \cos \theta) \partial_x u(e^{i\theta}) + (|z| \sin \theta - \sin \theta) \partial_y u(e^{i\theta}) + O((1 - |z|)^2) \\
&= \frac{(|z| - 1) \partial_x u(e^{i\theta})}{\cos \theta} + O((1 - |z|)^2) \quad (\text{by (3.7)}).
\end{aligned}
\end{equation}

By our choice of $\theta_0$, the last equality gives

\begin{equation}
u(|z| e^{i\theta}) \geq (1 - |z|)/2 - \|u\|_{C^2(D)}(1 - |z|)^2 \geq (1 - |z|)/4,
\end{equation}

for $|z| \geq 1 - 1/4\|u\|_{C^2(D)}^{-1}$. When $|z| \leq 1 - 1/4\|u\|_{C^2(D)}^{-1}$, we have $u(z) \geq c$ for some constant $c$ independent of $z$. This combined with the fact that $(1 - |z|) \leq 1$ implies that there is a
positive constant $c'$ for which $u(z) \geq c'(1 - |z|)$ for $|z| \leq 1 - 1/4\|u\|_{C^1_c(\Omega)}^{-1}$. In summary, we can find a positive constant $c'$ for which

$$u(z) \geq c'(1 - |z|),$$

for $z = |z|e^{i\theta} \in \mathbb{D}$ with $\theta \in [-\theta_0, \theta_0]$.

Now let $\Omega$ be a simply connected subdomain of $\mathbb{D}$ with smooth boundary such that $\Omega$ is strictly convex and $\overline{\Omega} \cap \overline{\mathbb{D}} = [e^{-i\theta_0/2}, e^{i\theta_0/2}]$. By Painlevé's theorem (see, for example, [2] Th. 3.1 or [16] Th. 5.3.8), there is a smooth diffeomorphism $\Phi$ from $\mathbb{D}$ to $\overline{\Omega}$ which is a biholomorphism from $\mathbb{D}$ to $\Omega$ and $\Phi(1) = 1$. Define $u_0' := u \circ \Phi$ which is a smooth function on $\overline{\mathbb{D}}$ and harmonic on $\mathbb{D}$. We immediately have $u_0'(z) > 0$ on $\mathbb{D}$.

Since $\Phi(1) = 1$ and $\Phi$ sends $\partial \mathbb{D}$ to $\partial \Omega$, there is a small positive constant $\theta'$ such that $\Phi([e^{-i\theta_0/2}, e^{i\theta_0/2}])$ is contained in $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. This yields $u_0'(e^{i\theta}) = 0$ for $|\theta| \leq \theta_0'$ and $\text{Re}^2 \Phi(e^{i\theta}) + \text{Im}^2 \Phi(e^{i\theta}) = 1$ on $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. Differentiating the last inequality at $\theta' = 0$ gives

$$\text{Re} \Phi(1) \partial_y \text{Re} \Phi(1) + \text{Im} \Phi(1) \partial_y \text{Im} \Phi(1) = 0$$

which combined with $\Phi(1) = 1$ implies that $\partial_y \text{Re} \Phi(1) = 0$. The last equality coupled with the fact that $\Phi$ is holomorphic implies

$$\det D_{(x,y)} \Phi(1) = (\partial_x \text{Re} \Phi(1))^2 + (\partial_y \text{Re} \Phi(1))^2 = (\partial_x \text{Re} \Phi(1))^2.$$

As a result, we have $\partial_x \text{Re} \Phi(1) \neq 0$. On the other hand, since

$$|\Phi(1)|^2 = 1 = \max_{x \in [0,1]} |\Phi(x)|^2,$$

we have

$$0 \leq \partial_x |\Phi(x)|^2|_{x=1} = \text{Re} \Phi(1) \partial_x \text{Re} \Phi(1) + \text{Im} \Phi(1) \partial_x \text{Im} \Phi(1) = \partial_x \text{Re} \Phi(1).$$

Hence, one gets $\partial_x \Phi(1) > 0$. Direct computations gives

$$\partial_x u_0'(1) = \partial_x u(1) \partial_x \text{Re} \Phi(1) + \partial_y u(1) \partial_x \text{Im} \Phi(1) = -\partial_x \text{Re} \Phi(1) < 0.$$

Define $u_0 := u_0'/\partial_x \text{Re} \Phi(1)$. We obtain $\partial_x u_0(1) = -1$ and $u_0(e^{i\theta}) = 0$ for $|\theta| \leq \theta_0'$. It remains to check that

$$(3.10) \quad u_0(z) \geq c''(1 - |z|),$$

for some constant $c'' > 0$. Since $u_0(z) > 0$ and $u(z) > 0$ on $\mathbb{D}$ and $\partial \Omega \cap \overline{\mathbb{D}} = [e^{-i\theta_0/2}, e^{i\theta_0/2}]$, it is enough to check $(3.10)$ for $z$ so that $w = \Phi(z)$ is close to $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. Let $w = \Phi(z) \in \Omega$ close to $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. By our choice of $\Omega$, the axe $\partial \Omega$ is transverse to $\partial \Omega$ at a unique point $w' = \Phi(z')$ for $z' \in \partial \mathbb{D}$. The $C^1$- boundedness of $\Phi^{-1}$ imply that $|w - w'| \geq c_1 |z - z'|$ for some constant $c_1$ independent of $(z, z')$. On the other hand, since $\Omega \subset \mathbb{D}$, we have $|w - w'| \leq 1 - |w|$. Hence,

$$1 - |w| \geq c_1 |z - z'| \geq c_1 (1 - |z|),$$

because $z' \in \partial \mathbb{D}$. Write $w = |w|e^{i\theta_w}$. Note that $\theta_w \in (\theta_0, \theta_0)$ if $w$ is close enough to $[e^{-i\theta_0/2}, e^{i\theta_0/2}]$. We deduce that

$$u_0'(z) = u(\Phi(z)) = u(w) \geq c'(1 - |w|) \geq c'c_1(1 - |z|).$$

Hence, one gets $(3.10)$. The proof is finished. □
We are now ready to introduce the Bishop equation which allows us to construct the promised family of analytic discs. Let $u_0$ be a function described in Lemma 3.2 and $\theta_{u_0}$ be the constant there. Let $\tau_1, \tau_2 \in \mathbb{R}_{n-1} \subset \mathbb{R}^{n-1}$. Define $\tau_1^* := (1, \tau_1) \in \mathbb{R}^n$ and $\tau_2^* := (0, \tau_1) \in \mathbb{R}^n$ and $\tau := (\tau_1, \tau_2)$. Let $t$ be a positive number in $(0, 1)$ which plays a role as a scaling parameter in the equation (3.11) below.

In order to construct an analytic disc partly attached to $K$, it suffices to find a map

$$U : \partial \mathbb{D} \to \mathbb{B}_n \subset \mathbb{R}^n,$$

which is Hölder continuous, satisfying the following Bishop-type equation

(3.11) $$U_{\tau,t}(\xi) = t\tau_2^* - T_1(h(U_{\tau,t}))(\xi) - tT_1 u_0(\xi) \tau_1^*,$$

Indeed, suppose that (3.11) has a solution. For simplicity, we use the same notation $U_{\tau,t}(z)$ to denote the harmonic extension of $U_{\tau,t}(\xi)$ to $\mathbb{D}$. Let $P_{\tau,t}(z)$ be the harmonic extension of $h(U_{\tau,t}(\xi))$ to $\mathbb{D}$. Define

$$F(z, \tau, t) := U_{\tau,t}(z) + iP_{\tau,t}(z) + it u_0(z) \tau_1^*$$

which is a family of analytic discs parametrized by $(\tau, t)$. For any $\xi \in [e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$, the defining formula of $F$ and the fact that $u_0 \equiv 0$ on $[e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$ imply that

$$F(\xi, \tau, t) = U_{\tau,t}(\xi) + iP_{\tau,t}(\xi) = U_{\tau,t}(\xi) + it h(U_{\tau,t}(\xi)) \in K.$$

In other words, $F$ is $[e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}]$-attached to $K$. In what follows, it is convenient to regard $U_{\tau,t}(z)$ as a function of the variable $(z, \tau)$.

**Proposition 3.3.** There are a positive number $t_1 \in (0, 1)$ and a real number $c_1 > 0$ satisfying the following property. For any $t \in (0, t_1]$ and any $\tau \in \mathbb{B}_{n-1}^2$, the equation (3.11) has a unique solution $U_{\tau,t}$ which is $C^{1,1}$ in $(\xi, \tau)$ and such that

(3.12) $$\|D^{j}_{(\xi,\tau)} U_{\tau,t}\|_{C^{1,1} (\partial \mathbb{D})} \leq c_1 t,$$

for any $\tau \in \mathbb{B}_{n-1}^2$ and $j = 0, 1$ or $2$, where $D^{j}_{(\xi,\tau)}$ is the differential with respect to both $(\xi, \tau)$ and $D^{j}_{(\xi,\tau)} := D^{j}_{(\xi,\tau)} \circ D^{j}_{(\xi,\tau)}$.

**Proof.** This is a direct consequence of a general result due to Tumanov, see [19, Th. 4.19] or see [24, Pro. 4.2] for a more simple proof adapted to our present situation. \( \square \)

Let $U_{\tau,t}$ be the unique solution of (3.11). As above we also use $U_{\tau,t}(z)$ to denote its harmonic extension to $\mathbb{D}$. Let $F(z, \tau, t)$ and $P_{\tau,t}$ be as above. Our goal is to study the behaviour of the image of the family $F(\cdot, \tau, t)$ near $K$, or in other words when $z$ is close to $[e^{-i\theta_{u_0}}, e^{i\theta_{u_0}}] \subset \partial \mathbb{D}$.

**Lemma 3.4.** There exists a constant $c_2$ so that for every $t \in (0, t_1]$ and every $(z, \tau) \in \overline{\mathbb{D}} \times \mathbb{B}_{n-1}^2$, we have

(3.13) $$\|D^{j}_{(z,\tau)} U_{\tau,t}(z)\| \leq c_2 t \quad \text{and} \quad \|D^{j}_{(z,\tau)} P_{\tau,t}(z)\| \leq c_2 t^2,$$

for $j = 0, 1, 2$.

**Proof.** In view of (3.5) and (3.12), the first inequality of (3.13) is obvious and for the second one, it is enough to estimate the $C^{1/2}(\partial \mathbb{D})$-norms of $D^{j}_{(\xi,\tau)}P_{\tau,t}(\xi)$ for $j = 0, 1, 2$. Since $P_{\tau,t}(\xi) = h(U_{\tau,t}(\xi))$ on $\partial \mathbb{D}$, we have

$$\partial_{\xi} P_{\tau,t}(\xi) = D h(U_{\tau,t}(\xi)) \partial_{\xi} U_{\tau,t}(\xi).$$
By similar arguments, we also have $|\partial_x^j P_\tau(\xi)| \lesssim t^2$ with $j = 0, 2$. To deal with the other partial derivatives, observe that for $0 \leq j \leq 2$, $D^j_\tau P_{\tau,t}$ is the harmonic extension of $D^j_1 h(U_{\tau,t}(\cdot))$ to $D$. Hence, in order to estimate $D^j_\tau D^k_\tau P_{\tau,t}$ for $0 \leq k, j \leq 2$, we can apply the same reasoning as above. Thus the proof is finished.

**Proposition 3.5.** There are three constants $t_2 \in (0, t_1]$, $\theta_0 \in (0, \theta_{w_0})$ and $\epsilon_0 > 0$ such that for any $\tau_1 \in B_{n-1}$ and $t \in (0, t_2]$ the map $F(\cdot, \tau_1, t) : [e^{-i\theta_0}, e^{i\theta_0}] \times B_{n-1} \to K$ is a diffeomorphism onto its image which contains the graph of $h$ over $B_n(0, t_0)$.

**Proof.** By Cauchy-Riemann equations, we have

$$\partial_y U_{\tau,t}(1) = -t \partial_x u_0(1) \tau_1^* - \partial_x P_{\tau,t}(1) = t \tau_1^* - \partial_x P_{\tau,t}(1).$$

The last term is $O(t^2)$ by Lemma 3.4. Thus the first component of $\partial_y U_{\tau,t}(1)$ is greater than $t/2$ provided that $t \leq t_2$ small enough. A direct computation gives $\partial_y U_{\tau,t}(1) = \partial_0 U_{\tau,t}(1)$. Consequently, the first component of $\partial_y U_{\tau,t}(1)$ is greater than $t/2$ for $t \leq t_2$.

On the other hand, by (3.11), we have $U_{\tau,t}(1) = t \tau_2^*$ which implies $\partial_{\tau_2} U_{\tau,t}(1)$ is a $(n,n-1)$ matrix whose the fist row is $0$ and the other rows form the identity matrix. Combining with the above argument shows that $D^2_{\tau_2,\tau} U_{\tau,t}(1)$ is a nondegenerate matrix. This coupled with the fact that $F(e^{i\theta}, \tau_1, t) = U_{\tau,t}(e^{i\theta})$ for $\theta \in [-\theta_0, \theta_0]$ implies the desired result. The existence of $\epsilon_0$ is obvious. The proof is finished. □

For $a \in \mathbb{C}^n$ and $A \subset \mathbb{C}^n$, $\text{dist}(a, A)$ denotes the distance from $a$ to $A$.

**Proposition 3.6.** There are two constants $t_3 \in (0, t_2]$, $r_0 > 0$ such that for every $t \in (0, t_3)$, the restriction $F_1$ of $F$ to $(B_2(1, r_0) \cap D) \times B_{n-1}$ is a diffeomorphism onto its image and for any $(z, \tau)$, we have

$$\left| \det D F_1(z, \tau, t) \right| \gtrsim t^{2n} \left| 1 - |z| \right|^{n-1}$$

and

$$t(1 - |z|) \lesssim \text{dist} \left( F_1(z, \tau, t), K' \right) \lesssim t(1 - |z|).$$

**Proof.** Let $r_0, t_3$ be two positive small constants to be chosen later. For the moment, we take $r_0$ to be small enough so that if $z = |z| e^{i\theta} \in B_2(1, r_0) \cap D$, then $\theta \in (0, \theta_0)$, thus we have $u_0(e^{i\theta}) = 0$. Fix a constant $t \in (0, t_3]$. Provided that $t_3$ and $r_0$ are small enough we will prove in the order (3.15), (3.14) and finally that $F_1$ is a diffeomorphism. Extend $h$ to be a $C^3$ function on $\mathbb{R}^n$. Let $\Psi : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\Psi(x + iy) := x + iy - ih(x).$$

One can see without difficulty that $\Psi$ is a diffeomorphism sending $K'$ to $B_n$, where we embed

$$\mathbb{R}^n \hookrightarrow \mathbb{R}^n + i\mathbb{R}^n = \mathbb{C}^n.$$

Let $F_1' := \Psi \circ F_1$. We have

$$\text{Im} F_1'(z, \tau, t) = P_{\tau,t}(z) - h(U_{\tau,t}(z)) + tu_0(z)\tau_1^* \quad \text{and} \quad \text{Re} F_1'(z, \tau, t)(z) = U_{\tau,t}(z).$$
By the above property of $\Psi$, it suffices to prove the required property for $(F_1', \mathbb{B}_n)$ in place of $(F_1, K')$. Note that $P_{\tau,t}(z)$ and $h(U_{\tau,t}(z))$ are identical on $\partial \mathbb{D}$. This together with (3.13) yields

\begin{equation}
(3.17) \quad P_{\tau,t}(z) - h(U_{\tau,t}(z)) = t^2(1 - |z|)R_0(z, \tau, t),
\end{equation}

where $R_0(z, \tau, t)$ is $C^1$ in $(z, \tau)$ so that $\|R_0(\cdot, t)\|_{C^1} \lesssim 1$. Remember that $t$ is fixed, so we do not consider it as a variable when taking the $C^1$ norm. On the other hand, by our choice of $u_0$ and Lemma 3.2, one has $u_0(z) \gtrsim (1 - |z|)$. By this and (3.17) and (3.16), we obtain

$\text{dist} \left( F_1(z, \tau, t), K' \right) \gtrsim \text{dist} \left( F_1'(z, \tau, t), \mathbb{R}^n \right) = |\text{Im} F_1'(z, \tau, t)| \gtrsim t(1 - |z|)|\tau^*| - t^2(1 - |z|)$.

Thus if $t$ is sufficiently small, the first inequality of (3.15) follows.

For $t_3$ small enough, $U_{\tau,t}(z) \in \mathbb{B}_n$. Hence, we get

$\text{dist} \left( F_1(z, \tau, t), K' \right) \gtrsim \text{dist} \left( F_1'(z, \tau, t), \mathbb{B}_n \right) \lesssim |\text{Im} F_1'(z, \tau, t)|$.

Write $z = |z|e^{i\theta} \in \mathbb{B}_2(1, r_0) \cap \mathbb{D}$. Hence $\theta \in [-2r_0, 2r_0] \subset (\theta_0, \theta_0)$ if $r_0$ is small enough.

Since $u_0(e^{i\theta}) = 0$, we deduce from (3.8) that

\begin{equation}
(3.18) \quad u_0(z) = (1 - |z|) + \theta(1 - |z|)R_1(z) + (1 - |z|)^2R_2(z),
\end{equation}

where $R_j$ is smooth function with $\|R_j\|_{C^1} \lesssim 1$ for $j = 1, 2$. Put $\epsilon := \max\{2r_0, t\}$. We choose $(t, r_0)$ to be so small that $\epsilon < 1/2$. Put

\begin{equation}
(3.19) \quad T_0(z, \tau, t) := tR_0(z, \tau, t) + (\theta R_1(z) + (1 - |z|)R_2(z))\tau^*_1
\end{equation}

which satisfies

\begin{equation}
(3.20) \quad \|T_0\|_{C^0} \lesssim \epsilon, \quad \|D_\tau T_0\|_{C^0} \lesssim \epsilon
\end{equation}

because $|\theta| \leq 2r_0$ and $1 - |z| \leq r_0$. Combining (3.18), (3.17) and (3.16) gives

\begin{equation}
(3.21) \quad \text{Im} F_1'(z, \tau, t) = t(1 - |z|)[\tau^*_1 + T_0(z, \tau, t)].
\end{equation}

Consequently, using (3.20) we obtain

$|\text{Im} F_1'(z, \tau, t)| \gtrsim t(1 - |z|)$

which proves the second inequality of (3.15).

By (3.11) and the Cauchy-Riemann equations, we have $U_{\tau,t}(1) = t\tau^*_2$ and

$\partial_y U_{\tau,t}(z) = -\partial_x P_{\tau,t}(z) - t\partial_x u_0(z)\tau^*_1$

and

$\partial_z U_{\tau,t}(z) = \partial_y P_{\tau,t}(z) + t\partial_y u_0(z)\tau^*_1$.

Observe that

$\partial_y U_{\tau,t}(e^{i\theta}) = -\partial_x U_{\tau,t}(e^{i\theta}) \sin \theta + \partial_y U_{\tau,t}(e^{i\theta}) \cos \theta$.

These above equalities combined with (3.13) and Taylor's expansion to $U_{\tau,t}(e^{i\theta})$ at $\theta = 0$ gives

\begin{equation}
(3.22) \quad U_{\tau,t}(e^{i\theta}) = t\tau^*_2 + t^2R_3(\theta, \tau, t) + t\theta \tau^*_1 + t\theta^2R_4(\theta)\tau^*_1,
\end{equation}

where

$R_3(\theta, \tau, t) := \int_0^\theta \left[ \partial_y P_{\tau,t}(e^{is}) \cos s - \partial_x P_{\tau,t}(e^{is}) \sin s \right] ds$
which is of $C^1$ norm $\lesssim 1$, and $R_4(\theta)$ is a $C^1$ function satisfying $\|R_4\|_{C^1} \lesssim 1$. Remark that in (3.22), we used the $C^3$ differentiability of $u_0$ and $R_4$ comes from the remainder of the Taylor expansion of $u_0$ at 1 up to the order 2.

Using (3.22), Taylor’s expansion for $\Re F'_1(z, \tau, t)$ at $\tilde{z} = e^{i\theta}$ implies

\begin{equation}
(3.23) \quad \Re F'_1(z, \tau, t) = t\tau^*_2 + t\theta \tau^*_1 + t^2 R_3(\theta, \tau, t) + t\theta^2 R_4(\theta) \tau^*_1 + t(1 - |z|) R_5(z, \tau, t),
\end{equation}

for some $C^1$ function $R_5(z, \tau, t)$ with $\|R_j(\cdot, t)\|_{C^1} \lesssim 1$. Define

\begin{equation}
(3.24) \quad T_j(z, \tau, t) := t R_3(\theta, \tau, t) + \theta^2 R_4(\theta) \tau^*_1 + (1 - |z|) R_5(z, \tau, t),
\end{equation}

which satisfies

\begin{equation}
(3.25) \quad \|D_{\tau, \theta} T_1\|_{C^0} \lesssim \epsilon,
\end{equation}

where we use the polar coordinate $(|z|, \theta)$ for $z$. Combining (3.23), (3.25), (3.21) and (3.20) gives (3.14).

Let $\rho = t\rho_2 + i t\rho_1$ be an arbitrary point in the image of $F'$. This means that

\begin{equation}
(3.26) \quad \rho = F'_1(z_0, \tau^0, t),
\end{equation}

for some $(z_0, \tau^0)$. Let $\theta^0 \in (-\pi/2, \pi/2)$ be the argument of $z_0$. Then $z_0 = |z_0|e^{i\theta^0}$. We will prove that the equation

\begin{equation}
(3.27) \quad F'_1(z, \tau, t) = \rho
\end{equation}

has a unique solution, i.e $F'_1$ is injective. The equation (3.27) is equivalent to the system of the two following equations

\begin{equation}
(3.28) \quad \Re F'_1(z, \tau, t) = t\rho_2
\end{equation}

and

\begin{equation}
(3.29) \quad \Im F'_1(z, \tau, t) = t\rho_1.
\end{equation}

Write $T_j = (T_{j1}, \ldots, T_{jn})$ for $j = 0$ or 1 and $\rho_j = (\rho_{j1}, \ldots, \rho_{jn})$ for $j = 1, 2$. Define

\begin{equation}
\tilde{\rho}_1 := \frac{\rho_1}{1 - |z|}.
\end{equation}

We also write $\tilde{\rho}_1 = (\tilde{\rho}_{11}, \ldots, \tilde{\rho}_{1n})$. Recall that $\tau^*_j = (1, \tau_j)$ for $j = 1$ or 2 and $\tau_j = (\tau_{j1}, \ldots, \tau_{j(n-1)})$. We have

\begin{equation}
(3.30) \quad \tilde{\rho}_1k - \tilde{\rho}_{11} \frac{\rho_{1k}}{\rho_{11}} = 0,
\end{equation}

for $2 \leq k \leq n$. The variable $(\tilde{\rho}_{11}, \theta)$ will be used as a substitute for $z$. If $(z, \tau, t)$ is a solution of (3.27), identifying the first component of (3.21) and (3.29) imply

\begin{equation}
1 + T_{01}(z, \tau, t) = \tilde{\rho}_{11}
\end{equation}

which in turn yields $|\tilde{\rho}_{11} - 1| \lesssim \epsilon$ by (3.20). Hence if $(z, \tau, t)$ is a solution of (3.27), we get

\begin{equation}
(3.31) \quad 1/2 \leq \tilde{\rho}_{11} \leq 3/2.
\end{equation}

By (3.29) again and the fact that $\tau_1 \in \mathbb{H}_{n-1}$, one also gets

\begin{equation}
(3.32) \quad \left| \frac{\rho_{1k}}{\rho_{11}} \right| \approx |\tau_{1(k-1)}| \leq 3/2,
\end{equation}

so
for $2 \leq k \leq n$. Since $z = |z|e^{i\theta}$, we have

$$z = (1 - \frac{\rho_{11}}{\rho_{11}})e^{i\theta}.$$  

From now on, we will consider $T_0, T_1$ as functions of $(\hat{\rho}_{11}, \theta, \tau)$. Define

$$G = (G_1, G_2, G_3): \overline{\mathbb{B}}_{n-1} \times \left[ \frac{1}{2}, \frac{3}{2} \right] \times \mathbb{R}^{n-1} \times [-2r_0, 2r_0] \to \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n-1}$$

by putting

$$G_1(\tau, \hat{\rho}_1, \theta) := \tau_1^* + T_0(\theta, \hat{\rho}_{11}, \tau, t) - \hat{\rho}_1, \quad G_2(\tau, \hat{\rho}_1, \theta) := \tau_2^* + \theta \tau_1^* + T_1(\theta, \hat{\rho}_{11}, \tau, t) - \hat{\rho}_2$$

and

$$G_3(\tau, \hat{\rho}_1, \theta) := (\hat{\rho}_{12} - \hat{\rho}_{11}, \frac{\hat{\rho}_{1k}}{\rho_{11}}, \cdots, \hat{\rho}_{1n} - \hat{\rho}_{11}, \frac{\hat{\rho}_{1k}}{\rho_{11}}).$$

By (3.30), (3.23) and (3.21), resolving the system (3.28)-(3.29) is equivalent to finding $(\tau, \hat{\rho}_1, \theta)$ for which

$$G_1(\tau, \hat{\rho}_1, \theta) = 0.$$  

By (3.26), we know that $a^0 := (\tau_0, \hat{\rho}_{11}^0, \theta^0)$ is a solution of (3.33), where

$$\hat{\rho}_{11}^0 := \frac{\rho_{11}}{1 - |z^0|}.$$  

Suppose that there is another solution $a = (\tau, \hat{\rho}_1, \theta)$ of (3.33). By a direct computation, one gets

$$\partial_{\hat{\rho}_{11}}(1 - |z|) = -\frac{\rho_{11}}{\rho_{11}^2} = -(1 - |z|)\hat{\rho}_{11}^{-1} = O(1 - |z|) \lesssim \epsilon$$

by (3.31). This coupled with (3.19) and (3.24) yields

$$|T_0(a, t) - T_0(a^0, t)| \lesssim \epsilon |a - a^0| + |\theta - \theta^0|.$$  

and

$$|T_1(a, t) - T_1(a^0, t)| \lesssim \epsilon |a - a^0|.$$  

Using (3.35) and identifying the first component of the equation $G_2(\tau, \hat{\rho}_1, \theta) = 0$ imply

$$|\theta - \theta^0| \leq |T_1(a, t) - T_1(a^0, t)| \lesssim \epsilon |a - a^0|.$$  

By doing the same thing for $G_1(\tau, \hat{\rho}_1, \theta) = 0$ and using (3.36), we also obtain

$$|\hat{\rho}_{11} - \hat{\rho}_{11}^0| \leq |T_0(a, t) - T_0(a^0, t)| \lesssim \epsilon |a - a^0|.$$  

Using the last inequality, the equality $G_3(\tau, \hat{\rho}_1, \theta) = 0$ and (3.32), one infers

$$|\hat{\rho}_1 - \hat{\rho}_1^0| \lesssim |\hat{\rho}_{11} - \hat{\rho}_{11}^0| \lesssim \epsilon |a - a^0|.$$  

Similarly, using $G_1(\tau, \hat{\rho}_1, \theta) = 0$ gives

$$|\tau_1 - \tau_1^0| \leq |T_0(a, t) - T_0(a^0, t)| + |\hat{\rho}_1 - \hat{\rho}_1^0| \lesssim \epsilon |a - a^0|.$$  

Finally, using $G_2(\tau, \hat{\rho}_1, \theta) = 0$ gives

$$|\tau_2 - \tau_2^0| \lesssim \epsilon |a - a^0|.$$  

Summing the inequalities from (3.36) to (3.39) and taking into account that

$$|a - a^0| \leq |\tau_2 - \tau_2^0| + |\tau_1 - \tau_1^0| + |\hat{\rho}_1 - \hat{\rho}_1^0| + |\theta - \theta^0|.$$
show that $a = a^0$. This means that (3.33) has a unique solution, or equivalently, so does (3.27) if $r_0$ and $t$ are small enough. The proof is finished. \[\square\]

Let $\Omega$ be a simply connected subdomain of $\mathbb{D}$ with smooth boundary such that $\Omega$ is strictly convex and $\Omega \cap \overline{\mathbb{D}} = [e^{-i\theta_1}, e^{i\theta_1}]$ for some $\theta_1 \in (0, \theta_0)$ and $\overline{\Omega} \subset \mathbb{B}_2(1, r_0)$. By Painlevé's theorem as in the proof of Lemma 3.2, there is a smooth diffeomorphism $\Phi$ from $\overline{\mathbb{D}}$ to $\overline{\Omega}$ which is a biholomorphism from $\mathbb{D}$ to $\Omega$ and $\Phi(1) = 1$.

Define $\tilde{F}(z, \tau, t) := F(\Phi(z), \tau, t)$ which is again a $C^{2,1/2}$ family of analytic discs partly attached to $K$.

**Proposition 3.7.** (i) There are positive constants $\tilde{\theta}_0$ and $\tilde{\epsilon}_0$ so that for every $\tau_1 \in \mathbb{B}_{n-1}$ and $t \in (0, \tilde{t}_3]$, the restriction map $\tilde{F}(. \tau_1, t) : [e^{-i\tilde{\theta}_0}, e^{i\tilde{\theta}_0}] \times \mathbb{B}_{n-1} \to K'$ is a diffeomorphism onto its image which contains the graph of $h$ over $\mathbb{B}_n(0, \tilde{t}_0)$.

(ii) Let $t_3$ be the constant in Proposition 3.6. Then for any $t \in (0, t_3]$, the map $\tilde{F}(. , t)$ is a diffeomorphism from $\mathbb{D} \times \mathbb{B}_{n-1}$ onto its image in $\mathbb{D}^n \subset \mathbb{C}^n$, and for any $(z, \tau)$ we have

\begin{equation}
\label{eq:3.40}
|\det D\tilde{F}(z, \tau, t)| \geq t^{n+1} \operatorname{dist}^{n-1} (\tilde{F}(z, \tau, t), K')
\end{equation}

and

\begin{equation}
\label{eq:3.41}
t(1 - |z|) \lesssim \operatorname{dist} (\tilde{F}(z, \tau, t), K').
\end{equation}

**Proof.** Property (i) is a direct consequence of Propositions 3.5. By the differentiability of $\Phi^{-1}$ on $\overline{\Omega}$, we have $(1 - |\Phi(z)|) \gtrsim 1 - |z|$ for every $z \in \mathbb{D}$. Hence, by (3.15), we get (3.41). The inequality (3.40) follows immediately from the fact that

\[|\det DF_1(z, \tau, t)| \gtrsim t^{n+1} \operatorname{dist}^{n-1} (F_1(z, \tau, t), K')\]

which is in turn implied by (3.15) and (3.14). The proof is finished. \[\square\]

Using the local coordinates of $K$ at the beginning of this section and choosing $t = t_3$, the last proposition can be rephrased as follows.

**Proposition 3.8.** Let $a$ be an arbitrary point of $K$. Then there exist positive constants $\epsilon_a, \tilde{\theta}_a$ and a $C^{2,1/2}$ diffeomorphism $\tilde{F}_a : \mathbb{D} \times \mathbb{B}_{n-1} \to X$ onto its image such that the two following properties hold:

(i) for every $\tau_1 \in \mathbb{B}_{n-1}$, the restriction map $\tilde{F}_a(\cdot , \tau_1) : [e^{-i\tilde{\theta}_a}, e^{i\tilde{\theta}_a}] \times \mathbb{B}_{n-1} \to K'$ is a diffeomorphism onto its image which contains the graph of $h$ over $\mathbb{B}_{K}(a, \tilde{\epsilon}_a)$.

(ii) there is an open relatively compact neighborhood $K'_a$ of $a$ in $K$ such that for any $(z, \tau)$, we have

\begin{equation}
\label{eq:3.42}
|\det D\tilde{F}_a(z, \tau)| \gtrsim \operatorname{dist}^{n-1} (\tilde{F}_a(z, \tau), K'_a)
\end{equation}

and

\begin{equation}
\label{eq:3.43}
(1 - |z|) \lesssim \operatorname{dist} (\tilde{F}_a(z, \tau), K'_a).
\end{equation}

4. Some estimates for p.s.h. functions

In this section, we will prove some key estimates for p.s.h functions and their $dd^c$ on $\mathbb{C}^n$. For a Borel subset $A$ of $\mathbb{R}^m$ with $m \in \mathbb{N}$, denote by $|A|$ the volume of $A$ with respect to the canonical volume form $\operatorname{vol}_{\mathbb{R}^m}$. In what follows, for simplicity, we will write $\int_A f$ instead of $\int_A f \operatorname{vol}_{\mathbb{R}^m}$ for every Borel set $A \subset \mathbb{R}^m$ and every integrable function $f$ on $A$. In particular, this convention is applied to $\mathbb{C}^n = \mathbb{R}^{2n}$. 

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Given the context and the nature of the content, it appears that the document is part of a mathematical exposition, possibly related to complex analysis or differential geometry. The propositions and proofs presented are likely dealing with the properties of certain analytic functions and their diffeomorphisms, possibly within the realm of complex analysis. The notation and terms used suggest a deep dive into the behavior of functions in complex domains, their properties under transformations, and the estimation of volumes or distances related to these functions.
Lemma 4.1. Let \( V \) be an open subset of \( \mathbb{C}^n \) and let \( V_1 \) be a compact subset of \( V \). Let \( \varphi \) be a p.s.h. function on \( V \). Then there exists a constant \( c \) independent of \( \varphi \) such that for any Borel set \( V_2 \subset V_1 \), we have

(4.1) \[
\int_{V_2} |\varphi| \leq c |V_2| \max\{1, -\log |V_2|\} \int_V |\varphi|.
\]

**Proof.** If \( \varphi \equiv 0 \) or \( \int_V |\varphi| = \infty \), then there is nothing to prove. Now suppose \( \varphi \neq 0 \) and \( \int_V |\varphi| < \infty \). Let \( \varphi_1 = \varphi / \int_V |\varphi| \). We have \( \int_V |\varphi_1| = 1 \). As a result, there exist two positive constants \( (c_1, \alpha_1) \) independent of \( \varphi_1 \) for which

(4.2) \[
\int_{V_1} e^{\alpha_1 |\varphi_1|} \leq c_1.
\]

Let \( 1_{V_2} \) be the characteristic function of \( V_2 \). Let \( \mu := |V_2|^{-1} 1_{V_2} vol_{\mathbb{C}^n} \) which is a probability measure supported in \( V_2 \). We have

\[
\int_{V_2} |\varphi_1| = \alpha_1^{-1} \int_{V_2} \log e^{\alpha_1 |\varphi_1|} = \alpha_1^{-1} |V_2| \int_{V_2} \log e^{\alpha_1 |\varphi_1|} d\mu.
\]

This together with the concavity of the log function implies

\[
\int_{V_2} |\varphi_1| \leq \alpha_1^{-1} |V_2| \log \int_{V_2} e^{\alpha_1 |\varphi_1|} d\mu
\]

which, by (4.2), is less than or equal to

\[
\alpha_1^{-1} |V_2| (\log c_1 - \log |V_2|).
\]

Hence (4.1) follows. The proof is finished. \( \square \)

Now let \( h, K' \) be as in Section 3. Let \( \epsilon \) be a real positive number and \( K'_\epsilon \) the compact subset of \( \mathbb{C}^n \) consisting of points of distance \( \leq \epsilon \) to \( K' \). Obviously, the volume of \( K'_\epsilon \) is \( \lesssim \epsilon^n \). Using Lemma 4.1 for \( V_2 = K'_\epsilon \), we get the following.

**Corollary 4.2.** Let \( V \) be an open subset of \( \mathbb{C}^n \) containing \( H_1 \). Let \( \varphi \) be a p.s.h. function on \( V \). Then there is a constant \( c \) independent of \( \varphi \) for which

(4.3) \[
\int_{K'_\epsilon} |\varphi| \leq c \epsilon^n |\log \epsilon| \int_V |\varphi|
\]

for every \( \epsilon \leq 1/2 \).

Now we will give a similar estimate for the mass of \( dd^c \varphi \) on \( K'_\epsilon \). We begin with a general result.

**Lemma 4.3.** Let \( V, V_1, V_2 \) be open subsets of \( \mathbb{C}^n \) such that \( V_2 \subset V_1 \subset V \). Let \( T \) be a closed positive current of bidimension \( (p, p) \) on \( V \) and \( \lambda \) a real number \( > 1 \). Let \( \varphi \) and \( \rho \) be two bounded p.s.h. functions on \( V \). Let \( A \) be a subset of \( V_2 \) and \( a_{\epsilon, \varphi} \) be an upper bound for \( |\varphi| \) on \( V_1 \cap \{ \rho \leq \epsilon \} \) for \( \epsilon > 0 \). Assume that \( \rho \) is bounded by \( 1 \) on \( V \). Then there is a constant \( c \) independent of \( T, A, \rho, \varphi \) such that

(4.4) \[
\int_{A \cap \{\rho \leq \epsilon\}} T \wedge (dd^c \varphi)^p \leq c [\epsilon^{-1} a_{\epsilon, \varphi}]^p \|T\|_{V_1}.
\]

for every \( \epsilon \in (0, 1) \).
Proof. We prove (4.4) by induction on $p$. When $p = 0$, the conclusion is obvious. Suppose (4.4) holds for $p - 1$. We need to prove its validity for $p$. Let $\chi$ be a smooth function compactly supported in some $V' \subset V$ such that $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $V_2$. Let $\epsilon$ be a positive constant. Choose a constant $\lambda' \in (1, \lambda)$. Define
\[
\rho_\epsilon := \max\{\rho, \lambda'\} - \max\{\rho, \epsilon\}
\]
which is the difference of two bounded p.s.h. functions on $V$. Clearly, we have $0 \leq \rho_\epsilon \leq (\lambda' - 1)\epsilon$ and $\rho_\epsilon = (\lambda' - 1)\epsilon$ on $\{\rho \leq \epsilon\}$ and $\rho_\epsilon = 0$ on $\{\rho \geq \lambda'\epsilon\}$. This yields
\[
\int_{\mathbb{A} \cap \{\rho \leq \epsilon\}} T \wedge (dd^c \varphi)^p \leq \epsilon^{-1} \int_V \chi \rho_\epsilon T \wedge (dd^c \varphi)^p \leq \epsilon^{-1} \int_V \chi \rho_\epsilon T \wedge (dd^c \varphi)^p
\]
which is, by integration by parts, equal to
\[
\epsilon^{-1} \int_V \rho_\epsilon \varphi dd^c \chi \wedge T \wedge (dd^c \varphi)^{p-1} + \epsilon^{-1} \int_V \varphi dd^c \rho_\epsilon \wedge T \wedge (dd^c \varphi)^{p-1} + R,
\]
where
\[
R = 2\epsilon^{-1} \int_V \varphi dd^c \chi \wedge d^c \rho_\epsilon \wedge T \wedge (dd^c \varphi)^{p-1}.
\]
Denote by $R_1$ and $R_2$ respectively the first and second terms in (4.6). We are now going to estimate $R_1$, $R$ and finally $R_2$. Let $\omega$ be the canonical Kähler form on $\mathbb{C}^n$. Since $dd^c \chi \leq \omega$ and $|\varphi| \leq 2\epsilon_\varphi \rho_\epsilon$ on $\text{supp} \rho_\epsilon$, we get
\[
R_1 \leq \epsilon^{-1} a_{\lambda', \varphi} \int_{V' \cap \{\rho \leq \lambda'\}} \omega \wedge T \wedge (dd^c \varphi)^{p-1}.
\]
Applying the induction hypothesis to $\omega \wedge T, \lambda' \epsilon$ in place of $T, \epsilon$ implies
\[
R_1 \leq \epsilon^{-1} a_{\lambda', \varphi} \int_{V' \cap \{\rho \leq \lambda'\}} \omega \wedge T \wedge (dd^c \varphi)^{p-1} \leq [\epsilon^{-1} a_{\lambda', \varphi}]^p.
\]
As to $R$, the Cauchy-Schwarz inequality applied to a suitable scalar product gives
\[
|R|^2 \leq \epsilon^{-2} \int_{V'_1} |\varphi| 1_{\{\rho \leq \lambda'\}} d\chi \wedge d^c \chi \wedge T \wedge (dd^c \varphi)^{p-1} \int_{V'_1} |\varphi| d\rho_\epsilon \wedge d^c \rho_\epsilon \wedge T \wedge (dd^c \varphi)^{p-1}
\]
\[
\leq [\epsilon^{-1} a_{\lambda', \varphi}]^{p+1} \int_{V'_1} d\rho_\epsilon \wedge d^c \rho_\epsilon \wedge T \wedge (dd^c \varphi)^{p-1}
\]
by induction hypothesis and the fact that $d\rho_\epsilon \wedge d^c \rho_\epsilon$ is positive and supported on $\{\rho \leq \lambda'\}$. Denote by $R'$ the last integral. Since $\rho_\epsilon$ is the difference of two bounded p.s.h. functions on $V$, so is $\rho_\epsilon^2$. More precisely, since $|\rho| \leq 1$ on $V$ we can find four p.s.h function $\psi_j$ with $1 \leq j \leq 4$ so that they are bounded independent of $\epsilon$ and
\[
\rho_\epsilon^2 = \psi_1 - \psi_2 \quad \text{and} \quad \rho_\epsilon = \psi_3 - \psi_4.
\]
We also have
\[
(dd^c \rho_\epsilon^2 = 2d\rho_\epsilon \wedge d^c \rho_\epsilon + 2d\rho_\epsilon^2 d^c \rho_\epsilon.
\]
Note that each side of the last equality is well-defined. Substituting this to the defining formula of $R'$, then using (4.9), one gets
\[
R' \leq \sum_{j=1}^4 \int_{V' \cap \{\rho \leq \lambda'\}} dd^c \psi_j \wedge T \wedge (dd^c \varphi)^{p-1}
\]
which, by induction hypothesis, is \(\lesssim\)

\[
\left[\varepsilon^{-1}a_{\lambda,\varphi}\right]^{p-1}\sum_{j=1}^{4}\|dd^c\psi_j \wedge T\|_{V_1''},
\]

where \(V_i''\) be a relatively compact subset of \(V_1\) which is open and contains \(\overline{V}_1\). By the Chern-Levine-Nirenberg inequality, the last sum is \(\lesssim\|T\|_{V_1}\). Combining with (4.8), we obtain

\begin{equation}
R \leq \left[\varepsilon^{-1}a_{\lambda,\varphi}\right]^{p}\|T\|_{V_1}.
\end{equation}

Bounding \(R_2\) is done similarly. The proof is finished. \(\square\)

**Lemma 4.4.** Let \(f\) be a real \(C^2\) function on an open set \(V \subset \mathbb{C}^n\). Let \(g(t) := |t| \log(|t|) + 2\) for \(t \in \mathbb{R}\). Let \(\omega\) be the canonical Kähler form on \(\mathbb{C}^n\). Then we have

\[
12dd^c(g \circ f) \geq df \wedge d^c f - 2n\|D^2f\|_{L^\infty(V)} \omega
\]

as currents on \(V\).

**Proof.** By direct computations, one obtains for \(t > 0\),

\[
g'(t) = 1 - \frac{2}{2 + t} + \log(2 + t), \quad g''(t) = \frac{2}{(2 + t)^2} + \frac{1}{2 + t}
\]

and for \(t < 0\),

\[
g'(t) = -1 + \frac{2}{2 - t} - \log(2 - t), \quad g''(t) = \frac{2}{(2 - t)^2} + \frac{1}{2 - t}.
\]

For \(k \geq 3\), we are going to construct a sequence of \(C^2\) convex function \(g_k\) of uniformly bounded \(L^\infty\) norm converging pointwise to \(g\). To this end, we define

\[
q_k(t) := \frac{2}{(2 + |t|)^2} + \frac{1}{2 + |t|} \quad \text{for} \quad t \geq 1/k
\]

and on \([-1/k, 1/k]\), let \(q_k(t)\) be the piece-wise affine function satisfying the two following properties:

(i) \(q_k\) is affine on \([-1/k, 0]\) and on \([0, 1/k]\), \(q_k(0) = 2kg'(1/k) - q_k(1/k) \geq 1\),

(ii) \(q_k\) is continuous on \(\mathbb{R}\).

The value of \(q(0)\) is in fact chosen such that

\[
\int_{-1/k}^{1/k} q_k(t)dt = g'(1/k) - g'(-1/k).
\]

This property ensures the existence of a unique \(C^2\) convex function \(g_k(t)\) on \(\mathbb{R}\) satisfying \(g_k(t) \equiv g(t)\) for \(|t| \geq 1/k\) and \(g_k''(t) = q_k(t)\). One can check that \(g_k\) is uniformly bounded and \(g_k\) converges to \(g\). Hence \(g_k(f)\) converges weakly to \(g(f)\) as currents. On the other hand, direct computations give

\[
g_k''(f) \geq \min\{1/3, 2k \log 2 - 1\} = 1/3, \quad |g_k'(t)| \leq \log 3 + 2 \leq 4
\]

for \(|t| \leq 1\) and

\[
\ddc g_k(f) = g_k''(f)df \wedge d^c f + g_k'(f)\ddc f \geq 12^{-1}(df \wedge d^c f - 2n\|D^2f\|_{L^\infty} \omega).
\]

The proof is finished. \(\square\)
Proposition 4.5. Let \( \varphi \) be a p.s.h. function on an open set \( V \subset \mathbb{C}^n \). Let \( A \) be a generic \( C^3 \) submanifold of dimension \( n \) of \( V \). Let \( A_1 \) be a compact subset of \( A \) and for \( \epsilon > 0 \), let \( A_{1, \epsilon} \) be the set of points in \( \mathbb{C}^n \) of distance \( \leq \epsilon \) to \( A_1 \). Then there is a constant \( c \) independent of \( \varphi, \epsilon \) for which we have

\[
\int_{A_{1, \epsilon}} dd^c \varphi \land \omega^{n-1} \leq c \epsilon^{n-1} \int_V |\varphi|,
\]

where \( \omega \) is the canonical Kähler form of \( \mathbb{C}^n \).

Proof. Let \( \delta \) be a small positive number which will be chosen later. Observe that the question is local. By using a partition of unity and Lemma [3.1], it is enough to prove the desired result for the case where \( A \) is the graph of a \( C^3 \) map \( h \) over \( \mathbb{B}_n(0, 3\delta) \) such that \( h(0) = Dh(0) = D^2 h(0) = 0 \) and \( \|h\|_{C^3} \) is bounded independently of chosen local charts (hence, in particular, independent of \( \delta \)); and \( A_1 \) is the part of the graph over \( \mathbb{B}_n(0, \delta) \). We can assume that

\[
A_{1, \epsilon} = \{ x + iy : x \in \mathbb{B}_n(0, \delta), |y - h(x)| \leq \epsilon \}
\]

and \( V = \mathbb{B}_n(0, 3\delta) + i\mathbb{B}_n \).

Let \( g \) be the function defined in Lemma [4.4]. We write \( z = (z_1, \cdots, z_n) \), \( y = (y_1, \cdots, y_n) \) and \( h = (h_1, \cdots, h_n) \). Since \( |D^2 h| \lesssim \delta \) on \( \mathbb{B}_n(0, 3\delta) \), one has

\[
|D^2 (y_j - h_j(x))| \lesssim \delta
\]

for \( 1 \leq j \leq n \). Using this and Lemma [4.4], we see that the function

\[
\rho(z) := \sum_{j=1}^n g(y_j - h_j(x))
\]

satisfies

\[
dd^c \rho \geq \sum_{j=1}^n \left( \frac{i}{4\pi} dz_j \land d\bar{z}_j - \delta M dz_j \land d\bar{z}_j \right),
\]

for some constant \( M \) independent of \( \delta \). Thus if \( \delta \) is small enough independently of \( \epsilon \), the function \( \rho \) is p.s.h. on \( V \). It is clear that \( A_{1, \epsilon} \subset A_1 \cap \{ \rho \leq 2\epsilon \} \). Let \( \varphi_1(z) := |y - h(x)|^2 \). A direct computation shows that \( \varphi_1 \) is also p.s.h. on \( V \). Note that \( |\varphi_1| \lesssim \epsilon^2 \) on \( \{ \rho \leq 2\epsilon \} \). Now applying Lemma [4.3] to \( (\rho, \varphi_1) \) and to \( T := dd^c \varphi \), we obtain

\[
\int_{A_{1, \epsilon}} dd^c \varphi \land (dd^c \varphi_1)^{n-1} \lesssim \epsilon^{n-1} \|dd^c \varphi\|_{\mathbb{B}_n(0, 2\delta) + i\mathbb{B}_n(0, 1/2)} \lesssim \epsilon^{n-1} \int_V |\varphi|.
\]

The last inequality together with the fact that \( dd^c \varphi_1 \gtrsim \omega \) gives the desired result. The proof is finished. \( \square \)

Note that a similar technique was used by Sibony in [21] when dealing with the extension of positive closed currents (or more generally plurisubharmonic currents) through a CR submanifold. For \( \epsilon \in (0, 1] \), let \( K'_\epsilon \) be as above. The following is just a direct consequence of Proposition 4.5.

Corollary 4.6. Let \( V \) be an open subset of \( \mathbb{C}^n \) containing \( K'_\epsilon \). Let \( \varphi \) be a p.s.h. function on \( V \). Then there is a constant \( c \) independent of \( \varphi, \epsilon \) for which we have

\[
\int_{K'_\epsilon} dd^c \varphi \land \omega^{n-1} \leq c \epsilon^{n-1} \int_V |\varphi|,
\]
where $\omega$ is the canonical Kähler form of $\mathbb{C}^n$.

Now we are going to give some applications of these above estimates to our present problem. Firstly, we prove some auxiliary lemmas. Let $t_3, \epsilon_0$ and $\theta_0$ be the constants in Proposition 3.7. Let $\tilde{F}$ be the family of analytic discs defined there. For simplicity, from now on, we denote $\tilde{F}(z, \tau, t_3)$ by $\tilde{F}(z, \tau)$. Recall that the image of $\tilde{F}$ is contained in $\mathbb{D}^n$. Put $\tilde{\epsilon}_0 := t_3 \epsilon_0$.

**Lemma 4.7.** There exists a positive constant $c_0$ such that for any Borel function $g$ on $\mathbb{D}^n$, we have

$$\int_{B_n(0, e^{\tilde{\epsilon}_0})} |g(x, h(x))| \leq c_0 \int_{[e^{-\omega_0^n}, e^{\epsilon_0^n}] \times \mathbb{B}_{n-1}^2} |g \circ \tilde{F}(e^{i\theta}, \tau)|.$$  

(4.13)

**Proof.** This is a direct consequence of Property $(i)$ of Proposition 3.7 and the change of variables theorem. The proof is finished. $\square$

**Lemma 4.8.** Let $g$ be a Borel function on $\mathbb{D}^n$.

(i) If $n = 1$, then

$$\int_{\mathbb{D} \times \mathbb{B}_{n-1}(0,1)^2} |g \circ \tilde{F}(z, \tau)| \leq c_1 \int_{\mathbb{D}^n} |g(z)|,$$

for some constant $c_1$ independent of $g$.

(ii) Assume $n > 1$. Let $t_0$ and $\delta_0$ be two positive real numbers such that $t_0 + \delta_0 > n - 1 > \delta_0$. Let

$$M_g := \sup_{\epsilon \in (0,1)} \int_{K_0'} |g(z)|$$

and $\lambda_0 := t_0 + \delta_0 - n + 1$. Assume $M_g < \infty$. Then we have

$$\int_{\mathbb{D} \times \mathbb{B}_{n-1}^2} (1 - |z|)^{\lambda_0} |g \circ \tilde{F}(z, \tau)| \leq \frac{2^{t_0} c_1 M_g}{\lambda_0} \left[ \int_{\mathbb{D}^n} |g(z)| \right]^{\lambda_0},$$

(4.14)

for some constant $c_1$ independent of $g, t_0, \delta_0$.

**Proof.** When $n = 1$, the desired inequality is a direct application of the change of variables theorem and (3.40). Consider now $n > 1$. Put $Y := \mathbb{D} \times \mathbb{B}_{n-1}^2$. Let $\epsilon$ be a small positive number which will be chosen later. Set

$$Y_{\epsilon,0} := \{(z, \tau) \in Y : \text{dist} \left( \tilde{F}(z, \tau), K' \right) \geq \epsilon \}$$

and

$$Y_{\epsilon,k} := \{(z, \tau) \in Y : 2^{-k} \epsilon \leq \text{dist} \left( \tilde{F}(z, \tau), K' \right) \leq 2^{-k+1} \epsilon \},$$

for $k \in \mathbb{N}$. It is clear that $Y = \cup_{k=0}^{\infty} Y_{\epsilon,k}$. By definition of $K'_0$, we have

$$\tilde{F}(Y_{\epsilon,k}) \subset H_{2^{-k+1} \epsilon}. $$

(4.14)
Denote by $vol_Y$ the canonical volume form on $Y$. Write

\begin{equation}
\int_Y (1 - |z|)^\delta |g \circ \tilde{F}|\, dvol_Y = \sum_{k=0}^\infty \int_{Y_{\epsilon,k}} (1 - |z|)^\delta |g \circ \tilde{F}|\, dvol_Y
\end{equation}

\begin{equation}
\lesssim \sum_{k=0}^\infty \int_{Y_{\epsilon,k}} (1 - |z|)^\delta |g \circ \tilde{F}(z, \tau)| \frac{|\det D\tilde{F}(z, \tau)|}{\text{dist}^{n-1}(\tilde{F}(z, \tau), K')}\, dvol_Y
\end{equation}

(by \ref{equation:3.40})

\begin{equation}
\lesssim \sum_{k=0}^\infty (2^{-k} \epsilon)^{-n+1+\delta_0} \int_{Y_{\epsilon,k}} |g \circ \tilde{F}| \, dvol_Y,
\end{equation}

by definition of $Y_{\epsilon,k}$, \ref{equation:3.41} and the fact that $-n + 1 + \delta_0 < 0$. By change of variables, the last integral equals

\begin{equation}
\int_{F(Y_{\epsilon,k})} |g|
\end{equation}

which is, for $k \geq 1$, less than or equal to

\begin{equation}
\int_{H_{2^{-k+1},k}} |g| \leq (2^{-k+1} \epsilon)^{\lambda_0} M_g
\end{equation}

by definition of $M_g$ and \ref{equation:4.14}. This coupled with \ref{equation:4.15} yields that

\begin{equation}
\int_Y (1 - |z|)^\delta |g \circ \tilde{F}| \lesssim \epsilon^{-n+1+\delta_0} \int_{D^n} |g| + 2^{\lambda_0} \epsilon^{\lambda_0} M_g \sum_{k=1}^\infty 2^{-k\lambda_0}
\end{equation}

\begin{equation}
\lesssim \epsilon^{-n+1+\delta_0} \int_{D^n} |g| + \frac{2^{\lambda_0} \epsilon^{\lambda_0} M_g}{2^{\lambda_0} - 1}.
\end{equation}

Choose $\epsilon = \|g\|_{L^1(D^n)}^{1/\lambda_0}$. Using \ref{equation:4.16} and the fact that $2^{\lambda_0} \geq 1 + \lambda_0$, we get the desired inequality. The proof is finished. \qed

The following will be crucial for our later purpose.

**Corollary 4.9.** Let $V$ be an open subset of $\mathbb{C}^n$ containing $K'_1 \cup \overline{D}^n$. Let $\varphi$ be a p.s.h. function on $V$. Let $\delta \in (0, 1)$. Define $\gamma := \delta/(n - 1)$ if $n > 1$ and $\gamma = 1$ otherwise. Then we have

\begin{equation}
\int_{D \times B_{n-1}} (1 - |z|)^{\delta} \, dd^c(\varphi \circ \tilde{F})(z, \tau) \lesssim_{\delta} \|\varphi\|_{L^1(V)}.
\end{equation}

Furthermore, we have

\begin{equation}
\int_{\{1 - 2\epsilon \leq |z| \leq 1\} \times B_{n-1}} (1 - |z|) \, dd^c(\varphi \circ \tilde{F})(z, \tau) \lesssim_{\delta} \epsilon^{-\frac{\delta(n-1)}{n-1}} \max\{\|\varphi\|_{L^1(V)}, \|\varphi\|_{L^1(V)}\},
\end{equation}

for every $\epsilon \in (0, 1)$.

**Proof.** Firstly we prove \ref{equation:4.17}. The case where $n = 1$ is clear. Consider $n > 1$. Let $V_1 \subset V$ be an open subset of $V$. Fix a decreasing sequence of smooth p.s.h. functions $\varphi_l$ converging pointwise to $\varphi$ on $V_1$ and $\|\varphi_l\|_{L^1(V_1)} \leq 2\|\varphi\|_{L^1(V)}$. Let $\delta \in (0, 1)$. Since

\begin{equation}
\frac{dd^c \varphi_l}{\pi} = \sum_{1 \leq j, k \leq n} \frac{\partial^2 \varphi_l}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k \geq 0,
\end{equation}

we have

\begin{equation}
\int_{\{1 - 2\epsilon \leq |z| \leq 1\} \times B_{n-1}} (1 - |z|)^{\delta} \, dd^c(\varphi_l \circ \tilde{F})(z, \tau) \lesssim_{\delta} \epsilon^{-\frac{\delta(n-1)}{n-1}} \max\{\|\varphi_l\|_{L^1(V_1)}, \|\varphi_l\|_{L^1(V)}\}.
\end{equation}

By the compactness of $V_1$, we can take a subsequence such that

\begin{equation}
\int_{\{1 - 2\epsilon \leq |z| \leq 1\} \times B_{n-1}} (1 - |z|)^{\delta} \, dd^c(\varphi \circ \tilde{F})(z, \tau) \lesssim_{\delta} \epsilon^{-\frac{\delta(n-1)}{n-1}} \max\{\|\varphi\|_{L^1(V_1)}, \|\varphi\|_{L^1(V)}\}.
\end{equation}

for every $\epsilon \in (0, 1)$. The proof is finished. \qed
using Corollary 4.6, there is a positive constant $c$ independent of $\varphi$ such that for every $j, k, l$ we have

\begin{equation}
\int_{K'} \left| \frac{\partial^2 \varphi_l}{\partial z_j \partial \bar{z}_k} \right| \leq c \epsilon^{n-1} \int_{V_1} |\varphi_l| \leq c \epsilon^{n-1} \|\varphi\|_{L^1(V)}
\end{equation}

which infers that the constant $M_g$, defined in Lemma 4.8 for

$$g := \frac{\partial^2 \varphi_l}{\partial z_j \partial \bar{z}_k}, \quad t_0 = n - 1, \quad \delta_0 = \delta,$$

is finite. Hence applying that lemma to the above mentioned $(g, t_0, \delta_0)$ gives

\begin{equation}
\int_{D \times B^2_{n-1}} (1 - |z|)^{\delta} dd^c(\varphi_l \circ \tilde{F})(z, \tau) \lesssim \|\varphi\|_{L^1(V)}^{\frac{\delta}{L^1(V)}}.
\end{equation}

On the other hand, since $dd^c \varphi_l \circ \tilde{F}$ converges weakly to $dd^c \varphi_l \circ \tilde{F}$ on $D$, we have

\begin{equation}
\liminf_{l \to \infty} \langle dd^c(\varphi_l \circ \tilde{F}(:, \tau)), f \rangle \geq \langle dd^c(\varphi \circ \tilde{F}(:, \tau)), f \rangle,
\end{equation}

for every positive continuous function $f$ on $D$. Letting $l \to \infty$ in (4.20) and then using (4.21) and Fatou's lemma, we get the desired result.

Now we prove (4.18). As above, it is enough to prove it for $\varphi$ smooth. Set $W := \{1 - 2\epsilon \leq |z| \leq 1\} \times B^2_{n-1}$. Let $r$ be a positive constant. Denote by $W_1$ the subset of $W$ containing $(z, \tau)$ with $\text{dist} \left( \tilde{F}(z, \tau), K' \right) \geq r$ and by $W_2$ the complement of $W_1$ in $W$. Let $\epsilon$ be a positive constant in $(0, 1)$. Using (3.40) and the change of variables by $\tilde{F}$ on $W_1$ gives

$$\int_{W_1} (1 - |z|) dd^c(\varphi \circ \tilde{F}) \lesssim \epsilon \int_{W_1} dd^c(\varphi \circ \tilde{F}) \lesssim \epsilon r^{-n+1} \int_{\tilde{F}(W_1)} dd^c \varphi \wedge \omega^{n-1} \lesssim \epsilon r^{-n+1} \|\varphi\|_{L^1(V)}.$$ 

By the proof of Lemma 4.8 applied to $g = \frac{\partial^2 \varphi_l}{\partial z_j \partial \bar{z}_k}$, $t_0 = n - 1$ and $\delta_0 = \delta$, we have

$$\int_{W_2} (1 - |z|) dd^c(\varphi \circ \tilde{F}) \leq \epsilon^{1-\delta} \int_{W_2} (1 - |z|)^{\delta} dd^c(\varphi \circ \tilde{F}) \lesssim \epsilon^{1-\delta} \epsilon r^{-n+1} \|\varphi\|_{L^1(V)}^\gamma$$

by (4.12) and the fact that $\tilde{F}(W_2)$ is contained in $K' \times B^2_{n-1}$. Choose $r := \epsilon^{\frac{\delta}{n-\gamma}}$. Taking the sum of the last two inequalities gives (4.18). The proof is finished.

5. Hölder continuity for super-potentials

Recall that $C$ defined at Introduction is a compact subset of the set of $\omega$-p.s.h. functions on $X$ with respect to $L^1$-topology. Hence there is a positive number $r_0$ such that

$$\|\varphi_0\|_{L^1(X)} \leq r_0 \quad \text{and} \quad \|\max\{\varphi_1, \varphi_2\}\|_{L^1(X)} \leq r_0,$$

for every $\varphi_0, \varphi_1, \varphi_2 \in C$. Let $C'$ be the set of $\omega$-p.s.h. functions $\varphi$ on $X$ such that $\|\varphi\|_{L^1(X)} \leq 2r_0$. In this section, we will finish the proof of Theorem 1.4. In order to do so, we will prove the following which is actually equivalent to Theorem 1.4 by Lemma 5.2 below. Remember that we are still assuming that $\dim K = n$. Let $\tilde{K}$ be the compact subset of $K$ as in Theorem 1.4.
Proposition 5.1. Let $\alpha$ be a positive number strictly less than $1/(3n)$. Then for any $\varphi_1, \varphi_2 \in \mathcal{C}^1$ such that $\varphi_1 \geq \varphi_2$, we have

\begin{equation}
\int_K (\varphi_1 - \varphi_2) dvol_K \leq c \int_X (\varphi_1 - \varphi_2) dvol_X + c \left( \int_X (\varphi_1 - \varphi_2) dvol_X \right)^{\alpha},
\end{equation}

where $c$ is a constant independent of $\varphi_1, \varphi_2$.

Lemma 5.2. Proposition 5.1 implies Theorem 1.4.

Proof. Take $\varphi_1, \varphi_2 \in \mathcal{C}$. Observe that $\max\{\varphi_1, \varphi_2\}, \varphi_1, \varphi_2 \in \mathcal{C}'$ and $\max\{\varphi_1, \varphi_2\} \geq \varphi_j$ for $j = 1, 2$. Hence, we can apply (5.1) to $\max\{\varphi_1, \varphi_2\}, \varphi_j$ with $j = 1, 2$. Using these two inequalities and the fact that

\[|\varphi_1 - \varphi_2| = 2 \max\{\varphi_1, \varphi_2\} - \varphi_1 - \varphi_2\]

gives

\[\|\varphi_1 - \varphi_2\|_{L^1(K \setminus \mathbb{D})} \leq \max\{\|\varphi_1 - \varphi_2\|_{L^1(X)}, \|\varphi_1 - \varphi_2\|_{L^1(X)}\}\]

which means that $1_K \varphi_j$ is Hölder continuous super-potential with Hölder exponent $\alpha$. The proof is finished. \qed

The remaining of this section is devoted to prove Proposition 5.1. By [3], it is enough to prove (5.1) for $\varphi_1, \varphi_2$ smooth. We will firstly show that for any nonnegative $C^2$ function $\varphi$ on $\mathbb{D}$, the integral of $\varphi$ over $\partial \mathbb{D}$ can be bounded by a quantity of the $L^1$-norm of $\varphi$ over $\mathbb{D}$ and some Hölder norm of its Laplacian. This together with the exponent estimates in the last section are the key ingredients in the proof of Proposition 5.1. We will reuse the notations from Section 2 for $M = \mathbb{D}$.

Lemma 5.3. Let $\varphi$ be a nonnegative $C^2$ functions on $\mathbb{D}$. Let $\beta \in (1, 2)$. Then we have

\begin{equation}
\int_{\partial \mathbb{D}} \varphi d\xi \lesssim_{\beta} \|d\varphi\|_{C^{\beta}(\mathbb{D})} + \int_{\mathbb{D}} \varphi.
\end{equation}

Proof. By Riesz’s representation formula, we have

\begin{equation}
\varphi(z) = \int_{-\pi}^{\pi} P(e^{i\theta}, z) \varphi(e^{i\theta}) d\theta + \int_{\{|\eta|<1\}} \log \frac{|z - \eta|}{|1 - \bar{z}\eta|} d\varphi,\end{equation}

for $z \in \mathbb{D}$, where $P(\xi, z)$ is the Poisson kernel given by

\[P(\xi, z) = (2\pi)^{-1}(|\xi|^2 - |z|^2)|\xi - z|^{-2}.

This implies that

\begin{equation}
\int_{\mathbb{D}/2} \varphi(z) = \int_{-\pi}^{\pi} \varphi(e^{i\theta}) d\theta \int_{z \in \mathbb{D}/2} P(e^{i\theta}, z) + \int_{\mathbb{D}} d\varphi(\eta) \int_{z \in \mathbb{D}/2} \log \frac{|z - \eta|}{|1 - \bar{z}\eta|}.
\end{equation}

Set

\[f(\eta) := \int_{\{|\eta|<1/2\}} \log \frac{|z - \eta|}{|1 - \bar{z}\eta|} = \int_{\{|z|<1/2\}} \log |z - \eta| - \int_{\{|z|<1/2\}} \log |1 - \bar{z}\eta|.

Observe that $f(e^{i\theta}) = 0$ because

\[\log \frac{|z - e^{i\theta}|}{|1 - z e^{-i\theta}|} = 0\]
for any $z \in \mathbb{D}$. This means that $f|_{\partial \mathbb{D}} \equiv 0$. We claim that $f$ is indeed in $\tilde{C}^2(\mathbb{D})$ for every $\beta \in (1, 2)$. Since $z \in \mathbb{D}_{1/2}$ and $\eta \in \mathbb{D}$, the function
\[
\int_{\mathbb{D}_{1/2}} \log |1 - z\eta| dxdy
\]
is smooth in $\eta \in \mathbb{D}$. Hence, we only need to take care of $\int_{z \in \mathbb{D}_{1/2}} \log |z - \eta|$. It is clear that
\[
(5.5) \quad \eta \int_{z \in \mathbb{D}_{1/2}} \log |z - \eta| = -\frac{1}{2} \int_{z \in \mathbb{D}_{1/2}} \frac{\overline{z} - \overline{\eta}}{|z - \eta|^2} = -\frac{1}{2} \int_{z \in \mathbb{D}_{1/2}} \frac{1}{z - \eta}.
\]

Let $g$ be the right-hand side of the last equation. We will show that $g \in C^\alpha(\mathbb{D})$ for every $\alpha \in (0, 1)$. If we can do so, then $\eta \int_{z \in \mathbb{D}_{1/2}} \log |z - \eta|$ is bounded by $|\eta - \eta'|^\alpha$ times a constant depending only on $\alpha$. Thus one gets $g \in C^\alpha(\mathbb{D})$.

As explained above, this yields $f \in \tilde{C}^2(\mathbb{D})$. The last property combined with (5.4) gives
\[
(5.7) \quad \int_{-\pi}^\pi v(e^{i\theta})d\theta \int_{z \in \mathbb{D}_{1/2}} P(e^{i\theta}, z) \leq \|v\|_{L^1(\mathbb{D}_{1/2})} + \|dd^c v\|_{\tilde{C}^\beta(\mathbb{D})} \|f\|_{\tilde{C}^\beta(\mathbb{D})}.
\]

By our hypothesis that $v \geq 0$ and the fact that $P(e^{i\theta}, z) \geq 1$ for $z \in \mathbb{D}_{1/2}$, using (5.7), one obtains that
\[
(5.8) \quad \int_{\partial \mathbb{D}} v d\xi \lesssim_\beta \|v\|_{L^1(\mathbb{D}_{1/2})} + \|dd^c v\|_{\tilde{C}^\beta(\mathbb{D})}.
\]

The proof is finished. \qed

**Proposition 5.4.** Let $v$ be a nonnegative $C^2$ function on $\overline{\mathbb{D}}$. Let $\epsilon, \beta_0 \in (0, 1)$ and $\beta \in (1, 2)$. Let $\gamma$ be the unique real number for which $\beta = \gamma \beta_0 + (1 - \gamma)2$. Then we have
\[
(5.9) \quad \int_{\partial \mathbb{D}} v d\xi \lesssim_{(\beta_0, \beta)} \|v\|_{L^1(\mathbb{D})} + \epsilon^{-2(1-\gamma)}\|dd^c v\|_{\tilde{C}^{\gamma - \beta_0}(\mathbb{D})} \|v\|_{L^1(\mathbb{D})} + \|dd^c v\|_{\tilde{C}^{\gamma - \beta_0}(\mathbb{D})} \left( \int_{1 - 2\epsilon \leq |z| \leq 1} (1 - |z|)|dd^c v|^\gamma \right)^{1/\gamma}.
\]

**Proof.** Firstly we will estimate $\|dd^c v\|_{\tilde{C}^{\beta - 2}(\mathbb{D})}$. Let $\chi \in C^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi \equiv 0$ on $[-1, 1]$ and $\chi \equiv 1$ outside $[-2, 2]$. For $\epsilon \in (0, 1)$, put $\chi_\epsilon(z) := \chi(\frac{1 - |z|}{\epsilon})$ for $z \in \mathbb{D}$. We have $\text{supp} \chi_\epsilon \subset \{z : |z| \leq 1 - \epsilon\}$ and $\chi_\epsilon(z) = 1$ for $z$ with $|z| \leq 1 - 2\epsilon$. Let $\Phi$ be a function in $C^2(\mathbb{D})$ with $\|\Phi\|_{C^2} \leq 1$. Since $\Phi \equiv 0$ on $\partial \mathbb{D}$ we have $|\Phi(z)| \leq 1 - |z|$. Decompose
\[
\langle dd^c v, \Phi \rangle = \langle dd^c v, \chi_\epsilon \Phi \rangle + \langle dd^c v, (1 - \chi_\epsilon)\Phi \rangle.
\]
Denote by $I_1$, $I_2$ respectively the first and second terms in the right-hand side of the last equality. By properties of $\Phi$ and $\chi_\epsilon$, one gets

$$|I_2| \leq 2 \int_{1-2\epsilon \leq |z| \leq 1} (1 - |z|) |dd^c v|.$$  

On the other hand, performing an integration by parts gives

$$|I_1| \leq \int_\mathbb{D} |v dd^c (\chi_\epsilon \Phi)| \lesssim \epsilon^{-2} \int_\mathbb{D} |v|.$$  

Hence, we obtain

$$|dd^c v|_{\mathcal{C}^2} \leq \sup_{\{\Phi \in \mathcal{C}^2(\mathbb{D}) : \|\Phi\|_{\mathcal{C}^2} \leq 1\}} |\langle dd^c v, \Phi \rangle| \lesssim \epsilon^{-2} \int_\mathbb{D} |v| + \int_{1-2\epsilon \leq |z| \leq 1} (1 - |z|) |dd^c v|.$$  

Now applying Proposition 2.1 to $dd^c v$ and $M = \overline{\mathbb{D}}$, one gets

$$|dd^c v|_{\mathcal{C}^2} \lesssim |dd^c v|_{\mathcal{C}^2(\mathbb{D})}.$$  

The last inequality combined with (5.2) and (5.10) gives (5.9). The proof is finished. \qed

We are now about to prove the local version of Proposition 5.1. Given a point $a \in K$, a small open neighborhood $K'$ of $a$ in $K$ can be described as in Section 3. Namely, there are a $C^3$ map $h$ from $\mathbb{B}_n$ to $\mathbb{R}^n$ with $h(0) = Dh(0) = 0$ and local holomorphic coordinates in $X$ such that

$$K' := \{x + ih(x) : x \in \mathbb{B}_n\} \subset \mathbb{D}_2^n.$$  

Let $\hat{F}(z, \tau), t_3, \varepsilon_0$ and $\hat{\theta}_0$ be as in Section 4. The couple $(K', \mathbb{D}_2^n)$ is considered as the local counterpart of $(K, X)$. One can replace $\mathbb{D}_2^n$ by any polydisks $\mathbb{D}_r^n$ with $r > 1$ without making any differences in what follows.

Let $\beta_0 \in (0, 1)$. For every positive continuous $(1, 1)$-current $T$ on an open neighborhood of $\overline{\mathbb{D}}$, we have

$$|T|_{\mathcal{C}^{1, \beta_0}(\mathbb{D})} \leq \int_{\mathbb{D}} (1 - |z|)^{\beta_0} T.$$  

Let $\varphi_1$ and $\varphi_2$ be two $C^2$ p.s.h. functions on $\mathbb{D}_2^n$ such that $\varphi_1 \geq \varphi_2$ and $\|\varphi_j\|_{L^1(\mathbb{D}_2^n)} \leq 1$ for $j = 1, 2$. Put $\varphi := \varphi_1 - \varphi_2$ which is $C^2$ and nonnegative. Define

$$g_1(\tau) := \|dd^c (\varphi \circ \hat{F}(\cdot, \tau))\|_{\mathcal{C}^{1, \beta_0}(\mathbb{D})}$$  

which is less than or equal to

$$|dd^c (\varphi_1 \circ \hat{F}(\cdot, \tau))|_{\mathcal{C}^{1, \beta_0}(\mathbb{D})} + |dd^c (\varphi_2 \circ \hat{F}(\cdot, \tau))|_{\mathcal{C}^{1, \beta_0}(\mathbb{D})}.$$  

Since $\hat{F}$ is $C^2$, so is $\varphi_j \circ \hat{F}$ for $j = 1, 2$. Using (5.11) for $T = dd^c (\varphi_j \circ \hat{F}(\cdot, \tau))$ and (4.17), we deduce that the integral of the sum (5.12) with respect to $\tau \in \mathbb{B}_2^{n-1}$ is $\lesssim_{\beta_0} 1$. This implies

$$\int_{\mathbb{B}_2^{n-1}} g_1(\tau) d\tau \lesssim_{\beta_0} 1.$$  

Put

$$g_2(\tau) := \|\varphi \circ \hat{F}(\cdot, \tau)\|_{L^1(\mathbb{D})}.$$
By Corollary 4.2, the function \( \varphi \) satisfy the hypothesis of Lemma 4.3 for \( \delta_0 = 0 \) and \( t_0 = n - 1 + \epsilon \) with \( \epsilon \in (0, 1) \). As a result, we get

\[
\int_{\mathbb{B}^{n-1}_2} g_2(\tau) d\tau \lesssim c \left[ \int_{\mathbb{D}^2} |\varphi| \right]^{\frac{n-1+\epsilon}{n}}.
\]

For \( \epsilon' \in (0, 1) \), we define

\[
g_3(\tau, \epsilon') := \int_{1-2\epsilon' \leq |z| \leq 1} (1 - |z|) d\epsilon' (\varphi_1 \circ \tilde{F}(\cdot, \tau)) + \int_{1-2\epsilon' \leq |z| \leq 1} (1 - |z|) d\epsilon' (\varphi_2 \circ \tilde{F}(\cdot, \tau)).
\]

By (4.18), we have

\[
\int_{\mathbb{B}^{n-1}_2} g_3(\tau, \epsilon') d\tau \lesssim \delta (\epsilon')^{\frac{(n-1)(n-1+\epsilon)}{n-1+\epsilon}},
\]

for any \( \delta \in (0, 1) \).

**Proposition 5.5.** Let \( \varphi_1 \) and \( \varphi_2 \) be two \( C^2 \) p.s.h. functions on \( \mathbb{D}^n_2 \) such that \( \varphi_1 \geq \varphi_2 \) and \( \| \varphi_j \|_{L^1(\mathbb{D}^n_2)} \leq 1 \) for \( j = 1, 2 \). Let \( \varphi := \varphi_1 - \varphi_2 \). Then we have

\[
\int_{\mathbb{B}^{n}(0,\epsilon_0^\prime)} \varphi(x, h(x)) dx \lesssim \| \varphi \|_{L^1(\mathbb{D}^n_2)},
\]

for any \( \delta \in (0, 1) \).

**Proof.** Let \( \epsilon, \epsilon', \beta_0 \in (0, 1) \) and \( \beta \in (1, 2) \). Let \( g_1, g_2, g_3 \) be as above. Applying Lemma 4.7 to \( g = \varphi \) gives

\[
\int_{\mathbb{B}^{n}(0,\epsilon_0^\prime)} |\varphi(x, h(x))| dx \lesssim \int_{\mathbb{B}^{n-1}_2} d\tau \int_{\partial \mathbb{D}^2} |\varphi \circ \tilde{F}(\cdot, \tau)| d\xi.
\]

Put \( \gamma := \frac{2-\beta}{2-\beta_0} \). Applying Proposition 5.4 to \( v = \varphi \circ \tilde{F}(\cdot, \tau) \in C^2 \) shows that the right-hand side of the last inequality is

\[
\lesssim (\beta_0, \beta) \int_{\mathbb{B}^{n-1}_2} g_3 d\tau + (\epsilon')^{-2(1-\gamma)} \int_{\mathbb{B}^{n}_2} g_1^\gamma g_2^{1-\gamma} d\tau + \int_{\mathbb{B}^{n-1}_2} g_1^\gamma g_3^{1-\gamma}(\cdot, \epsilon') d\tau.
\]

The first term of the last sum is

\[
\lesssim \| \varphi \|_{L^1(\mathbb{D}^n_2)}^{\frac{1-\gamma}{1-\epsilon'}}
\]

by (5.14). On the other hand, by the H"older inequality, the second one is \( \lesssim (\epsilon')^{-2(1-\gamma)} \| g_1 \|_{L^1} \| g_2 \|_{L^1}^{-1-\gamma} \) and the third one is \( \lesssim \| g_1 \|_{L^1} \| g_3(\cdot, \epsilon') \|_{L^1}^{-\gamma} \), where the \( L^1 \)-norm is taken over \( \mathbb{B}^{n-1}_2 \). Taking into account (5.13) and (5.14), one obtains

\[
(\epsilon')^{-2(1-\gamma)} \| g_1 \|_{L^1} \| g_2 \|_{L^1}^{-1-\gamma} \lesssim \beta_0 \epsilon' (\epsilon')^{-2(1-\gamma)} \| \varphi \|_{L^1(\mathbb{D}^n_2)}^{\frac{(1-\gamma)(n-1+\epsilon)}{n-1+\epsilon}}.
\]

By (5.13) and (5.15), we have

\[
\| g_1 \|_{L^1} \| g_3(\cdot, \epsilon') \|_{L^1}^{-1-\gamma} \lesssim \beta_0 \epsilon' (\epsilon')^{1-\gamma} \| \varphi \|_{L^1(\mathbb{D}^n_2)}^{\frac{(1-\gamma)(n-1+\epsilon)}{n-1+\epsilon}}.
\]
for every \( \epsilon' \in (0, 1) \). Put

\[
\begin{align*}
    a_1 &:= \frac{\epsilon}{(n - 1 + \epsilon)(3 - \delta(n - 1)/n - 1 + \delta)}, \\
    a_2 &:= \frac{\epsilon(1 - \delta(n - 1)/n - 1 + \delta)(1 - \gamma)}{(n - 1 + \epsilon)(3 - \delta(n - 1)/n - 1 + \delta)}.
\end{align*}
\]

Choose \( \epsilon' := \|\varphi\|_{L^1(\mathbb{D}^n_2)} \). Combining all these above inequalities, we get

\[
\int_{B_n(0, \tilde{\epsilon}_n')} |\varphi(x, h(x))| \, dx \lesssim (\bar{\beta}_0, \beta, \delta, \epsilon) \|\varphi\|_{L^2(\mathbb{D}^n_2)}^2.
\]

Observe that \( a_2 \to \frac{1}{4n} \) as \( \epsilon \to 1, \beta \to 2, \beta_0 \to 0, \delta \to 0 \). Thus, the proof is finished. \( \square \)

End of the proof of Proposition 5.1 in the case where \( \dim K = n \). Given any \( a \in K \), let \( F_a \) and \( \tilde{\epsilon}_a \) be as in Proposition 3.8. Since \( K \) is compact, we can cover it by a finite number of ball \( B_K(a, \tilde{\epsilon}_a) \). Hence, in order to prove (5.1), it is enough to restrict ourselves to local charts. In other words, we are now being in the situation with the model \((K', D^0_2)\) described above. Moreover, by subtracting a suitable common smooth function, we can assume that \( \varphi_1, \varphi_2 \) in (5.1) are \( C^2 \) p.s.h. functions on \( D^0_2 \). Hence, the desired result follows directly from Proposition 5.3. The proof is finished. \( \square \)

We now deal with the case where the dimension of \( K \) is greater than \( n \). Let \( n_K := \dim K > n \). Since \( K \) is generic, we have \( T_a K + JT_a K = T_a X \), where \( a \in K \) and \( J \) is the complex structure of \( X \). We then deduce that \( T_a K \cap JT_a K \) is of even dimension which equals \( 2n_K - 2n \). The codimension \( d \) of \( K \) equals \( 2n - n_K \).

Proposition 5.6. Let \( a \) be a point in \( K \). There exist local \( C^2 \) coordinates \((W, \Psi)\) of \( X \) around \( a \) such that the following properties hold:

(i) \( \Psi : W \to \mathbb{C}^d \times \mathbb{C}^{n_K - n} \) is a \( C^2 \) diffeomorphism onto its image which equals

\[
(\mathbb{B}_d + i\mathbb{B}_d(0, 2)) \times \mathbb{D}^{n_K - n}
\]

and \( \Psi(p) = 0 \) and \( \Psi^{-1}(z_1, z_2) \) is holomorphic in \( z_1 \) for every fixed \( z_2 \in \mathbb{D}^{n_K - n} \),

(ii) there is a \( C^2 \) map \( h(\Re z_1, z_2) \) from \( \mathbb{B}_d \times \mathbb{D}^{n_K - n} \) to \( \mathbb{R}^d \) so that for every \( z_2 \) fixed, \( h(\cdot, z_2) \in \mathbb{C}^d \) and

\[
D^j_{\Re z_1} h(0, z_2) = 0
\]

for \( j = 0 \) or 1 and

\[
\Psi(K \cap W) = \{(z_1, z_2) \in (\mathbb{B}_d + i\mathbb{B}_d) \times \mathbb{D}^{n_K - n} : \Im z_1 = h(\Re z_1, z_2)\}.
\]

Proof. It is well-known that in suitable holomorphic local coordinates, \( K \) is given by

\[
K = \{(z_1, z_2) \in (\mathbb{B}_d + i\mathbb{B}_d) \times \mathbb{D}^{n_K - n} : \Im z_1 = \tilde{h}(\Re z_1, \Re z_2, \Im z_2)\}
\]

where \( \tilde{h} \) is a \( C^3 \) map of uniformly bounded \( C^3 \) norm in \( p \) and \( \tilde{h}(0) = D\tilde{h}(0) = 0 \), see [1]. For \( z_2 \) fixed, we choose the tangent space of the graph of \( \tilde{h}(\cdot, z_2) \) at 0 and its orthogonal subspace as new holomorphic coordinates of \( \mathbb{C}^d \). These new coordinates depend \( C^2 \) on (but in general not holomorphically) on the parameter \( z_2 \). In these new coordinates, one easily see that \( K \) is given by the formula given in the assertion \((ii)\) for some \( C^2 \) map \( h \) with the desired properties. The proof is finished. \( \square \)

Remark 5.7. As in Lemma 3.7, we can obtain furthermore that \( D^2_{\Re z_1} h(0, z_2) = 0 \) and

\[
\|h(\cdot, z_2)\|_{C^3}
\]

is bounded uniformly in \( a = (z_1, z_2) \in \bar{K} \) but in this case we will lose a unit for the regularity in \( z_2 \), i.e. \( \Psi \) and \( h \) are only \( C^1 \) in \( z_2 \).


Thanks to Proposition 5.6, we can consider $K$ locally as a family of generic submanifolds of $\mathbb{C}^d$ of dimension $d$ parameterized by $z_2 \in \mathbb{D}^{nK-n}$. This allows us to reduce the question to the previous case where we already dealt with generic submanifolds of minimal dimension. By compactness of $K$, we can cover it by local charts $W$ as in Proposition 5.6. From now on, we work exclusively on a such local chart. Hence, we can identify $K$ with $\Psi(K \cap W)$. Let $h$ and $\Psi$ be as in that proposition. The map $h$ will be seen as a family of maps of $z_1$ parameterized by $z_2$. For $z_2 \in \mathbb{D}^{nK-n}$, define

$$K'_{z_2} := \{z_1 \in \mathbb{B}_d + i\mathbb{R}^d : \text{Im } z_1 = h(\text{Re } z_1, z_2)\}$$

which is identified with $K'_{z_2} \times \{z_2\} \subset \mathbb{C}^n$. Then $K$ is foliated by $K'_{z_2}$.

We are now going to construct a family of analytic discs partly attached to $K$. The strategy will be almost identical with what we did. Let $u_0$ be a function described in Lemma 3.2 and $\theta u_0$ be the constant there. Let $\tau_1, \tau_2 \in \mathbb{B}_{d-1} \subset \mathbb{R}^{d-1}$. Define $\tau_1^* := (1, \tau_1) \in \mathbb{R}^d$ and $\tau_2^* := (0, \tau_1) \in \mathbb{R}^d$ and $\tau := (\tau_1, \tau_2)$. Let $t$ be a positive number in $(0, 1)$. Consider the following modified version of the equation 3.11:

$$U_{\tau, z_2, t}(\xi) = t\tau_2^* - \tau_1 \left( h(U_{\tau, z_2, t}; z_2) \right)(\xi) - t\tau_1 u_0(\xi) \tau_1^*,$$

where $U : \partial \mathbb{D} \to \mathbb{B}_d$ is Hölder continuous.

Since $h(0, z_2) = D_{\text{Re } z_1} h(0, z_2) = 0$ for every $z_2$, we can use the same reason mentioned in the proof of Proposition 3.3 to show that if $t$ is small enough, the equation (5.17) has a unique solution $U_{\tau, z_2, t}$ in $C^{2,1/2}(\partial \mathbb{D} \times \mathbb{B}_{d-1}^2)$ for $z_2$ fixed so that $U_{\tau, z_2, t} \in C^1$ as a function of $(z, \tau, z_2)$. We use the same notation $U_{\tau, z_2, t}$ to denote the harmonic extension of $U_{\tau, z_2, t}$ to $\mathbb{D}$. Let $P_{\tau, z_2, t}(z)$ be the harmonic extension of $h(U_{\tau, z_2, t}(\xi), z_2)$ to $\mathbb{D}$. Define

$$F(z, \tau, z_2, t) := U_{\tau, z_2, t}(z) + iP_{\tau, z_2, t}(z) + it u_0(z) \tau_1^*$$

which is a family of analytic discs to $\mathbb{C}^d$ parametrized by $(\tau, z_2, t)$. By our choice of $u_0$, we have $F(\xi, \tau, z_2, t) \in K_{z_2}$ for $\xi \in [e^{-i\theta u_0}, e^{i\theta u_0}]$. Now define

$$F'((z, \tau, z_2, t) := (F_{\tau, z_2, t}(z), z_2) \in \mathbb{C}^n$$

which is a family of analytic discs to $X$ partly attached to $K$. Here we used an essential fact that the $\mathcal{O}^2$ coordinates $(z_1, z_2)$ are holomorphic in $z_1$. Proposition 3.6 with $n$ replaced by $d$ implies that for two positive constants $(t, r_0)$ small enough, $F'$ is a diffeomorphism on

$$(\mathbb{B}_2(1, r_0) \cap \mathbb{D}) \times \mathbb{B}_{d-1}^2 \times \mathbb{D}^{nK-n}$$

and its differential satisfies

$$| \det DF'(z, \tau, z_2, t) | \gtrsim t^{d+1} \text{dist}^{d-1} (F'(z, \tau, z_2, t), K'_{z_2}) \gtrsim t^d (1 - |z|)^{d-1}.$$

Now applying the same arguments right before Proposition 3.7 one gets the following.

**Proposition 5.8.** There exists a map $\tilde{F} : \mathbb{D} \times \mathbb{B}_{d-1}^2 \times \mathbb{D}^{nK-n} \to X$ which is a diffeomorphism onto its image such that the following three properties hold:

(i) there are positive constants $\tilde{\theta}_0$ and $\tilde{c}_0$ so that for every $\tau_1 \in \mathbb{B}_{d-1}$ the restriction map $\tilde{F}(\cdot, \tau_1) : [e^{-i\theta}, e^{i\tilde{\theta}}] \times \mathbb{B}_{d-1} \times \mathbb{D}^{nK-n} \to K$ is a diffeomorphism onto its image which contains the graph of $h$ over $\mathbb{B}_d(0, \tilde{c}_0) \times \mathbb{D}^{nK-n}$,

(ii) $\tilde{F}(\cdot, \tau, z_2)$ is an analytic disc to $X$ and

$$| \det DF'(z, \tau, z_2) | \gtrsim \text{dist}^{d-1} (\tilde{F}(z, \tau, z_2, t), K'_{z_2}) \gtrsim (1 - |z|)^{d-1}.$$
Proposition 5.8 and Remark 5.7 allow us to repeat all of arguments in the proof of Theorem 1.4 in the case where \( n_K = n \) for our present situation. Hence, this finishes the proof of Theorem 1.4.

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