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Abstract. We obtain some necessary and sufficient conditions for the boundedness of the composition operators between weighted Bergman spaces of logarithmic weights. In terms of the conditions for the boundedness, we compute the essential norm of the composition operators.

1. Introduction

1.1. Logarithmic weights and modified counting functions

For $-1 < \gamma < \infty$, $\delta \leq 0$ and $0 < p < \infty$, we define the weighted Bergman space $A^p_{\gamma, \delta}$ as consisting of holomorphic functions $f$ on the unit disc $D = \{ z : |z| < 1 \}$ of the complex plane $C$ for which

$$\left\| f \right\|_{A^p_{\gamma, \delta}}^p := \int_D |f(z)|^p \omega_{\gamma, \delta}(z) \, dA(z) < \infty,$$

where the weight is defined by

$$\omega_{\gamma, \delta}(z) = \left( \log \frac{1}{|z|} \right)^\gamma \left( \log \left( 1 - \frac{1}{\log |z|} \right) \right)^\delta$$

and $dA$ is the Lebesgue measure on $D$ normalized to be $A(D) = 1$. It is same as the space of holomorphic functions $f$ satisfying

$$\int_D |f(z)|^p (1 - |z|)^\gamma \left( \log \frac{1}{1 - |z|^2} \right)^\delta \, dA(z) < \infty.$$

When $\gamma = 0$, $\delta = 0$ the space becomes the Bergman space $A^p$, and when $\delta = 0$ it is the weighted Bergman space $A^p_{\gamma}$. 

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For holomorphic self-maps \( \varphi \) of \( \mathbb{D} \), \( 0 \leq r < 1 \), \( 0 \leq \gamma < \infty \), \( \delta \leq 0 \) and \( a \in \mathbb{D} \setminus \{ \varphi(0) \} \), we define \( N_{\varphi, \gamma, \delta} \) as

\[
N_{\varphi, \gamma, \delta}(r, a) := \sum \left( \log \frac{r}{|z_j(a)|} \right)^\gamma \left( \log \left( 1 + \frac{1}{\log |z_j(a)|} \right) \right)^\delta
\]

with \( |z_j(a)| < r \), counting multiplicities, and

\[
N_{\varphi, \gamma, \delta}(a) = N_{\varphi, \gamma, \delta}(1, a) := \sum \omega_{\gamma, \delta}(z_j(a)).
\]

We consider \( N_{\varphi, \gamma, \delta}(r, a) \) to be defined on the space \( \mathbb{D} \setminus \{ \varphi(0) \} \) and \( N_{\varphi, \gamma, \delta}(r, a) = 0 \) if \( a \) is not in \( \varphi(r\mathbb{D}) \) where \( r\mathbb{D} = \{ z \in \mathbb{D} : |z| < r \} \).

When \( \delta = 0 \), \( N_{\varphi, \gamma, \delta}(r, a) \) coincides with the generalized Nevanlinna counting function \( N_{\varphi, \gamma} \) introduced by J. H. Shapiro ([5]) as, for \( a \in \mathbb{D} \setminus \{ \varphi(0) \} \), \( 0 \leq r < 1 \) and \( \gamma \geq 0 \),

\[
N_{\varphi, \gamma}(r, a) := \sum_{z \in \varphi^{-1}(a), |z| < r} \left( \log \frac{r}{|z|} \right)^\gamma,
\]

\[
N_{\varphi, \gamma}(a) = N_{\varphi, \gamma}(1, a) = \sum_{z \in \varphi^{-1}(a)} \left( \log \frac{1}{|z|} \right)^\gamma.
\]

### 1.2. Boundedeness and essential norm of composition operator

Any holomorphic self-map \( \varphi \) of \( \mathbb{D} \) induces the composition operator \( C_\varphi \) on holomorphic function spaces as \( C_\varphi f(z) = f(\varphi(z)) \), \( z \in \mathbb{D} \). The linear operator \( C_\varphi \) is bounded on \( A^p \) by Littlewood’s Subordination Theorem (see [1]). Concerning \( C_\varphi \) between different Bergman spaces, W. Smith ([6]) characterized the condition on \( \varphi \) that makes \( C_\varphi \) between weighted Bergman spaces bounded:

**Theorem 1.1** ([6], Theorem 3.1 and Theorem 4.3). Let \( 0 < p \leq q \), \( -1 < \alpha, \beta < \infty \). Then \( C_\varphi : A^p_\alpha \to A^q_\beta \) is bounded if and only if

\[
N_{\varphi, \alpha + 2}(a) = O\left( \left( \log \frac{1}{|a|} \right)^{(\alpha + 2)/p} \right) \quad (|a| \to 1^-).
\]

Later, F. Pérez-González, J. Rättyä and D. Vukotić established more equivalences.

**Theorem 1.2** ([4], Theorem 1). For \( 0 < p \leq q < \infty \), \( -1 < \alpha < \infty \) and \( -1 < \beta < \infty \). Then the following statements are equivalent: for \( 0 \leq s < \infty \)

1. \( C_\varphi : A^p_\alpha \to A^q_\beta \) is bounded;
2. \( N_{\varphi, \beta + 2}(z) = O\left( \log \frac{1}{|z|} \right)^{(\alpha + 2)/p} \quad (|z| \to 1^-) \);
3. \( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'(\varphi(z))|^{\alpha + 2 + s} |\varphi'(z)|^s (1 - |z|^2)^{s + \beta} dA(z) < \infty \);
4. \( \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |\varphi'(\varphi(z))|^{\alpha + 2 + s} (1 - |\varphi(z)|^2)^s (1 - |z|^2)^{3 + \beta} dA(z) < \infty \).
Theorem 1.3 ([4], Theorem 5). Let $1 < p \leq q < \infty$, $-1 \leq \alpha < \infty$ and $-1 < \beta < \infty$. If $C_{\varphi} : A_{\alpha}^{p} \to A_{\beta}^{q}$ is bounded, then the following quantities are comparable: for $0 \leq s < \infty$

$$A = ||C_{\varphi}||_{\infty};$$

$$B = \limsup_{|z| \to 1} \frac{N_{\varphi, \beta+2}(z)}{q\alpha+2};$$

$$C = \limsup_{|a| \to 1} \int_{D} |\varphi''(\varphi(z))|^{s+2} \frac{|\varphi'(z)|^{2}}{1-|z|^{2}} dA(z);$$

$$D = \limsup_{|a| \to 1} \int_{D} |\varphi''(\varphi(z))|^{\frac{s+2}{q}+1} \frac{|\varphi'(z)|^{2}}{1-|z|^{2}} dA(z).$$

1.3. Main results of this paper

Our purpose of this paper is to extend the results of Theorem 1.2 and Theorem 1.3 up to $A_{\omega_{\gamma_{1}, \delta_{1}}}^{p}$. That is, we give equivalent characterizations that provide the boundedness of composition operator $C_{\varphi}$ from one $A_{\omega_{\gamma_{1}, \delta_{1}}}^{p}$ to another, and obtain parallel equivalences for the essential norm of the composition operator. The followings are our main results.

Theorem 1.4. Let $0 < p \leq q < \infty$, $-1 < \gamma_{1}, \gamma_{2} < \infty$ and $\delta_{1}, \delta_{2} \leq 0$. Then there exists $s = s(\omega_{\gamma_{1}, \delta_{1}}) > 1$ with $2 + \gamma_{1} - \delta_{1} < s < \infty$ such that the following conditions are equivalent:

1. $C_{\varphi}$ maps $A_{\omega_{\gamma_{1}, \delta_{1}}}^{p}$ boundedly into $A_{\omega_{\gamma_{2}, \delta_{2}}}^{q}$;

2. $N_{\varphi, \gamma_{2}+2, \delta_{2}}(a) = O\left(\left(\frac{1}{|a|}\right)^{2} \omega_{\gamma_{1}, \delta_{1}}(a)^{q/p}\right)$ ($|a| \to 1^{-}$);

3. $\sup_{a \in D} \left(\frac{1}{\log |a|}^2 \omega_{\gamma_{1}, \delta_{1}}(a)^{q/p}\right) \int_{D} \frac{(1-|a|^{2})^{q/p}}{1-\bar{z}a^{q/p+2}} N_{\varphi, \gamma_{2}+2, \delta_{2}}(z) dA(z) < \infty$;

4. $\sup_{a \in D} \int_{D} |\varphi''(\varphi(z))|^{\frac{2+\gamma_{1}}{p}+2} \left(\log \left(1 - \frac{1}{\log |z|}\right)\right)^{\frac{4\gamma_{1}}{p}} N_{\varphi, \gamma_{2}+2, \delta_{2}}(z) dA(z) < \infty$;

5. $\sup_{a \in D} \int_{D} |\varphi''(\varphi(z))|^{\frac{2+\gamma_{1}}{p}+2} |\varphi'(z)|^{2} \left(\log \left(1 - \frac{1}{\log |\varphi(z)|}\right)\right)^{\frac{2}{p+2}} \omega_{\gamma_{2}+2, \delta_{2}}(z) dA(z) < \infty.$
Theorem 1.5. For $1 < p \leq q < \infty$, $-1 < \gamma_1, \gamma_2 < \infty$ and $\delta_1, \delta_2 \leq 0$, if $C_\varphi : A^q_{\varphi, \gamma_1, \delta_1} \rightarrow A^q_{\varphi, \gamma_2, \delta_2}$ is bounded, then there exists $s = s(\omega_{\gamma_1, \delta_1}) > 1$ with $2 + \gamma_1 - \delta_1 < s < \infty$ such that the following quantities are comparable:

$A = ||C_\varphi||_{q/p}$;

$B = \limsup_{|z| \rightarrow 1} \frac{1}{\omega_{\gamma_1, \delta_1}(z)} \left( \frac{1}{\log |z|} \right)^2 \int_{|a|}^{\frac{1}{|a|}} \frac{(1 - |a|^2)^{q/p}}{|1 - \bar{a}z|^{q/p + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(z) dA(z)$;

$C = \limsup_{|a| \rightarrow 1} \int_{|a|}^{\frac{1}{|a|}} \frac{1}{\omega_{\gamma_1, \delta_1}(a)} \left( \frac{1}{\log |z|} \right)^2 \int_{|a|}^{\frac{1}{|a|}} \frac{(1 - |a|^2)^{q/p}}{|1 - \bar{a}z|^{q/p + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(z) dA(z)$;

$D = \limsup_{|a| \rightarrow 1} \int_{|a|}^{\frac{1}{|a|}} \frac{1}{\omega_{\gamma_1, \delta_1}(a)} \left( \frac{1}{\log |z|} \right)^2 \int_{|a|}^{\frac{1}{|a|}} \frac{(1 - |a|^2)^{q/p}}{|1 - \bar{a}z|^{q/p + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(z) dA(z)$;

$E = \limsup_{|a| \rightarrow 1} \int_{|a|}^{\frac{1}{|a|}} \frac{1}{\omega_{\gamma_1, \delta_1}(a)} \left( \frac{1}{\log |z|} \right)^2 \int_{|a|}^{\frac{1}{|a|}} \frac{(1 - |a|^2)^{q/p}}{|1 - \bar{a}z|^{q/p + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(z) dA(z)$.

We may compare the case of $\delta_1 = 0$, $\delta_2 = 0$ and $\gamma_1 = \alpha$, $\gamma_2 = \beta$ in Theorem 1.4 and Theorem 1.5 to Theorem 1.2 and Theorem 1.3, respectively. The authors recently find a nice work of J. A. Peláez and J. Rättyä ([3]) wherein parts of Theorem 1.4 and Theorem 1.5 are included under a wide scope and different approach.

1.4. Contents of this paper

In Section 2, we introduce some properties for the modified Nevanlinna counting function and the weighted Bergman space of logarithmic weight. In Section 3, we prove some necessary and sufficient conditions for the boundedness of the composition operator. In Section 4, we compute the essential norm. All the functions $f$ under consideration are assumed to be holomorphic on $D$. Moreover, $\varphi$ always denotes a holomorphic self map of $D$. Also throughout this paper, the symbols " $\lesssim$ " means that the left hand side is bounded above by a constant multiple of the right hand side, where the constant is positive and independent of $f$. " $\gtrsim$ " means analogously. The symbol " $\approx$ " means " $\lesssim$ " and " $\gtrsim$ " simultaneously. We are to abbreviate $\omega_{\gamma, \delta}$ as $\omega$, $\omega_{\gamma_1, \delta_1}$ as $\omega_1$, and $\omega_{\gamma_2, \delta_2}$ as $\omega_2$.

2. Background contents for $N_{\varphi, \gamma, \delta}$ and $A^p_{\omega}$

In this section we introduce some useful tools for our main theorems. See [2], for proofs.
2.1. Subharmonic mean value property

For the generalized counting function \( N_{\varphi,\gamma} \), the subharmonic mean value property appeared in [5]. Similar result holds for \( N_{\varphi,\gamma,\delta} \).

**Lemma A** ([2], Theorem 2.1). Let \( 1 \leq \gamma < \infty \) and \( \delta \leq 0 \). If \( \varphi \) is a holomorphic self-map of \( \mathbb{D} \) and \( \Delta \) is a disc in \( \mathbb{D} \) not containing \( \varphi(0) \) with center \( a \), then

\[
N_{\varphi,\gamma,\delta}(a) \leq \frac{1}{|\Delta|} \int_{\Delta} N_{\varphi,\gamma,\delta}(u) \, dA(u),
\]

where \( |\Delta| \) is the normalized area measure of \( \Delta : |\Delta| = \int \chi_\Delta(z) \, dA(z) \).

2.2. Change of a variable formula

**Lemma B** ([2], Lemma 2.3). If \( g \) is a non-negative measurable function on \( \mathbb{D} \), then

\[
\int_{\mathbb{D}} (g \circ \varphi)(z) |\varphi'(z)|^2 \omega(z) \, dA(z) = \int_{\mathbb{D}} g(u) N_{\varphi,\gamma,\delta}(u) \, dA(u).
\]

**Lemma C** ([2], Lemma 2.4). For a holomorphic self-map \( \varphi \) of \( \mathbb{D} \) and \( a \in \mathbb{D} \) we have

\[
(N_{\varphi,\gamma,\delta}) \circ \varphi \circ a = N_{\varphi \circ \varphi,\gamma,\delta}.
\]

2.3. Quantities compared to the norm

**Lemma D** ([2], Lemma 3.2). For a fixed \( r_0 \in [0,1) \),

\[
||f||_{A^p_\varphi} \approx \int_{\mathbb{D} \setminus r_0 \mathbb{D}} |f(z)|^p \omega(z) \, dA(z).
\]

**Lemma E** ([2], Lemma 3.3). Let \( 0 < p < \infty \), \( -1 < \gamma < \infty \) and \( \delta \leq 0 \). If \( f \in A^p_\varphi \), then

\[
|f(z)| \lesssim \left( \left( \log \frac{1}{|z|} \right)^2 \omega(z) \right)^{-\frac{1}{2}} ||f||_{A^p_\varphi}
\]

for \( z \in \mathbb{D} \) with \( |z| \geq \frac{1}{2} \).

**Lemma F** ([2], Lemma 3.4). Let \( \delta \leq 0 \), \( -1 < \gamma < \infty \), and \( \beta > \gamma - \delta \). Then for \( a \in \mathbb{D} \) with \( |a| \geq \frac{1}{2} \),

\[
\int_{\mathbb{D}} \frac{1}{|1 - az|^2} \omega(z) \, dA(z) \lesssim \frac{1}{(1 - |a|)^{\gamma - \delta}} \left( \log \left( \frac{1}{|a|} \right) \right)^{\delta}.
\]

**Lemma G** ([2], Theorem 3.6). Let \( 0 < p < \infty \), \( -1 < \gamma < \infty \) and \( \delta \leq 0 \). Then for \( r_0 \in [0,1) \), \( f \in A^p_\varphi \) if and only if

\[
\int_{\mathbb{D} \setminus r_0 \mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left( \log \frac{1}{|z|} \right)^2 \omega(z) \, dA(z) < \infty.
\]

**Lemma H.** Let \( 0 < p < \infty \), \( -1 < \gamma < \infty \) and \( \delta \leq 0 \). Then

\[
||f||_{A^p_\varphi} \approx |f(0)|^p + \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left( \log \frac{1}{|z|} \right)^2 \omega(z) \, dA(z).
\]

**Proof.** See the proof of Lemma G in [2].

\( \square \)
3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4. Let \( D(\lambda, \delta) = \{ w : |\varphi_\lambda(w)| < \delta \} \) be the pseudohyperbolic disk with center \( \lambda \) and radius \( \delta \).

**Lemma 3.1.**

\[
\log \left( 1 - \frac{1}{\log x} \right) \approx \log \left( 1 - \frac{1}{1 - x} \right), \quad \frac{1}{2} \leq x < 1.
\]

**Proof.** From

\[
1 - x \leq \log \left( 1 - \frac{1}{x} \right) \leq (\log 4)(1 - x), \quad \frac{1}{2} \leq x < 1,
\]

by letting \( c = \log 4 \), we have

\[
c \log \left( 1 - \frac{1}{\log x} \right) \geq \log \frac{c}{\log 1 - x}.
\]

On the other hand,

\[
\log \left( 1 - \frac{1}{\log x} \right) \leq \log \left( 1 + \frac{1}{1 - x} \right) \leq \log \frac{1}{1 - x} + 2 \log \left( 1 - \frac{1}{1 - x} \right). \quad \square
\]

**Lemma 3.2.** Let \( |a| > \frac{1}{2} \). Then

\[
(1) \quad \log \left( 1 - \frac{1}{\log |w|} \right) \approx \log \left( 1 - \frac{1}{\log |a|} \right), \quad w \in D(a, 1/2).
\]

**Proof.** Let \( w \in D(a, 1/2) \) be \( w = \frac{a - z}{1 - az} \) with \( |z| < \frac{1}{2} \). Then

\[
\frac{2|a| - 1}{2 - |a|} \leq |w| \leq \frac{1 + 2|a|}{2 + |a|},
\]

so that

\[
\frac{1 - |a|}{2 + |a|} \leq 1 - |w| \leq \frac{3(1 - |a|)}{2 - |a|}.
\]

Thus \( 1 - |w| \approx 1 - |a| \), whence the equivalence (1) follows from Lemma 3.1. \( \square \)

**Lemma 3.3.** Let \( 1 \leq \gamma < \infty, \alpha, \delta \leq 0 \) and \( 0 < m, t < \infty \) with \( m - t > -\alpha \). Then

\[
(2) \quad N_{\varphi, \gamma, \delta}(a) = O \left( \omega_{t, \alpha}(a) \right) \quad (|a| \to 1^-)
\]

if and only if

\[
(3) \quad \sup_{a \in D} \frac{1}{\omega_{t, \alpha}(a)} \int_D \frac{(1 - |a|^2)^m}{|1 - az|^{n+2}} N_{\varphi, \gamma, \delta}(z) dA(z) < \infty.
\]

In particular,

\[
(4) \quad \limsup_{|a| \to 1} \frac{N_{\varphi, \gamma, \delta}(a)}{\omega_{t, \alpha}(a)} \approx \limsup_{|a| \to 1} \frac{1}{\omega_{t, \alpha}(a)} \int_D \frac{(1 - |a|^2)^m}{|1 - az|^{n+2}} N_{\varphi, \gamma, \delta}(z) dA(z).
\]
Proof. Suppose that (2) is satisfied. Then there exists $r$ such that

$$N_{\varphi, \gamma, \delta}(a) \lesssim \omega_{t, \alpha}(a) \quad \text{for} \quad r \leq |a| < 1.$$ 

Thus, taking $m - t > -\alpha$, by Lemma F

$$\int_{D \setminus rD} \frac{(1 - |a|^2)^m}{|1 - \bar{a}z|^{m+2}} N_{\varphi, \gamma, \delta}(z)dA(z) \lesssim \int_{D \setminus rD} \frac{|a|^2}{|1 - \bar{a}z|^{m+2}} \omega_{t, \alpha}(z)dA(z) \lesssim (1 - |a|^2)^m \int_{D} \frac{1}{|1 - \bar{a}z|^{m+2}} \omega_{t, \alpha}(a)dA(z) \lesssim (1 - |a|^2)^m \frac{1}{(1 - |a|^2)^{m-t}} \left[ \log \left( \frac{1}{\log |a|} \right) \right]^\alpha = \omega_{t, \alpha}(a).$$

On the other hand

$$\frac{1}{\omega_{t, \alpha}(a)} \int_{D \setminus rD} \frac{(1 - |a|^2)^m}{|1 - \bar{a}z|^{m+2}} N_{\varphi, \gamma, \delta}(z)dA(z) = (1 - |a|^2)^{m-t} \left[ \log \left( \frac{1}{\log |a|} \right) \right]^{-\alpha} \int_{D \setminus rD} \frac{N_{\varphi, \gamma, \delta}(z)}{|1 - \bar{a}z|^{m+2}}dA(z) < \infty.$$ 

Therefore

$$\sup_{a \in D} \frac{1}{\omega_{t, \alpha}(a)} \int_{D \setminus rD} \frac{(1 - |a|^2)^m}{|1 - \bar{a}z|^{m+2}} N_{\varphi, \gamma, \delta}(z)dA(z) < \infty.$$ 

Conversely, suppose that (3) is satisfied. Then by Lemmas A and C

$$N_{\varphi, \gamma, \delta}(a) = N_{\varphi \circ \varphi, \gamma, \delta}(0) \leq 4 \int_{D(a, 1/2)} N_{\varphi \circ \varphi, \gamma, \delta}(u) dA(u)$$

$$= 4 \int_{D(a, 1/2)} N_{\varphi, \gamma, \delta}(\varphi_a(z)) \varphi'_a(z)^2 dA(z)$$

$$= 4 \int_{D(a, 1/2)} N_{\varphi, \gamma, \delta}(z) \varphi'_a(z)^2 dA(z)$$

$$= 4 \int_{D(a, 1/2)} N_{\varphi, \gamma, \delta}(z) (1 - |a|^2)^2 \frac{1}{|1 - \bar{a}z|^4} dA(z),$$

so that by the fact $1 - |a| \approx |1 - \bar{a}z|$ for $a$ in the pseudohyperbolic disk $D(z, 1/2)$, we have

$$N_{\varphi, \gamma, \delta}(a) \lesssim \int_{D(a, 1/2)} N_{\varphi, \gamma, \delta}(z) (1 - |a|^2)^{2m-2} \frac{1}{|1 - \bar{a}z|^{4m-2}} dA(z)$$

$$\lesssim \int_{D} \frac{(1 - |a|^2)^m}{|1 - \bar{a}z|^{m+2}} N_{\varphi, \gamma, \delta}(z) dA(z) \lesssim \omega_{t, \alpha}(a) \quad \text{as} \quad |a| \to 1^-.$$
In particular, by (8), we have
\[
\limsup_{|a| \to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)} \leq \limsup_{|a| \to 1} \frac{1}{\omega_{t,\alpha}(a)} \int_D \frac{(1 - |a|^2)^m}{|1 - az|^{m+2}} N_{\varphi,\gamma,\delta}(z) dA(z).
\]
To prove the inverse inequality, putting
\[
B = \limsup_{|a| \to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)},
\]
then given \( \varepsilon > 0 \), there exists \( r_\varepsilon \in (0, 1) \) such that \( \frac{N_{\varphi,\gamma,\delta}(z)}{\omega_{t,\alpha}(z)} \leq B + \varepsilon \) for all \( |z| \geq r_\varepsilon \). Therefore, by (5)
\[
\int_D \frac{(1 - |a|^2)^m}{|1 - az|^{m+2}} N_{\varphi,\gamma,\delta}(z) dA(z)
= \int_{r_\varepsilon D} + \int_{D \setminus r_\varepsilon D} \frac{(1 - |a|^2)^m}{|1 - az|^{m+2}} N_{\varphi,\gamma,\delta}(z) dA(z)
\leq \frac{(1 - |a|^2)^m}{(1 - r_\varepsilon)^{m+2}} \int_{r_\varepsilon D} N_{\varphi,\gamma,\delta}(z) dA(z) + (B + \varepsilon) \int_{D \setminus r_\varepsilon D} \frac{(1 - |a|^2)^m}{|1 - az|^{m+2}} \omega_{t,\alpha}(z) dA(z)
\leq \frac{(1 - |a|^2)^m}{(1 - r_\varepsilon)^{m+2}} \int_{r_\varepsilon D} N_{\varphi,\gamma,\delta}(z) dA(z) + (B + \varepsilon) \omega_{t,\alpha}(a),
\]
and it follows that by (6)
\[
\limsup_{|a| \to 1} \frac{1}{\omega_{t,\alpha}(a)} \int_D \frac{(1 - |a|^2)^m}{|1 - az|^{m+2}} N_{\varphi,\gamma,\delta}(z) dA(z) \leq \limsup_{|a| \to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)}.
\]
The proof is complete. \( \square \)

**Lemma 3.4.** Let \( 1 \leq \gamma < \infty \), \( \alpha, \delta \leq 0 \) and \( -1 < t < \infty \). Then (2) is equivalent to
\[
\sup_{a \in D} \int_D |\varphi_a'(z)|^{2+t} \left[ \log \left( 1 - \frac{1}{\log |z|} \right) \right]^{-\alpha} N_{\varphi,\gamma,\delta}(z) dA(z) < \infty.
\]
Furthermore,
\[
\limsup_{|a| \to 1} \frac{N_{\varphi,\gamma,\delta}(a)}{\omega_{t,\alpha}(a)} \approx \limsup_{|a| \to 1} \int_D |\varphi_a'(z)|^{2+t} \left[ \log \left( 1 - \frac{1}{\log |z|} \right) \right]^{-\alpha} N_{\varphi,\gamma,\delta}(z) dA(z).
\]

**Proof.** The proof uses arguments similar to those in Lemma 3.3. By (2), there exists \( r \) such that
\[ N_{\varphi,\gamma,\delta}(a) \leq \omega_{t,\alpha}(a) \quad \text{for} \quad r \leq |a| < 1. \]
Thus, we have
\[
\int_{D \setminus r D} |\varphi_a'(z)|^{2+t} \left[ \log \left( 1 - \frac{1}{\log |z|} \right) \right]^{-\alpha} N_{\varphi,\gamma,\delta}(z) dA(z)
\]
\[
\lesssim \int_{D \setminus r} |\varphi'_a(z)|^{2+t} \left[ \log \left( 1 - \frac{1}{\log |z|} \right) \right]^{\alpha} \omega_{\gamma,\delta}(z) dA(z)
\]
\[
= \int_{D \setminus r} \frac{(1 - |a|^2)^{2+t}}{|1 - \bar{a}z|^{4+2t}} \left( \log \frac{1}{|z|} \right)^t dA(z)
\]
\[
\lesssim \int_{D} \frac{(1 - |a|^2)^{2+t}}{|1 - \bar{a}z|^{4+2t}} (1 - |z|)^t dA(z) \lesssim 1,
\]
and obviously
\[
\int_{r} |\varphi'_a(z)|^{2+t} \left[ \log \left( 1 - \frac{1}{\log |z|} \right) \right]^{\alpha} N_{\varphi,\gamma,\delta}(z) dA(z) < \infty,
\]
so that (9) is satisfied.

Conversely, by (7)
\[
N_{\varphi,\gamma,\delta}(a) \lesssim 4 \int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z) |\varphi'_a(z)|^{2+t} \left( \frac{|1 - \bar{a}z|^2}{1 - |a|^2} \right)^t dA(z)
\]
\[
\lesssim (1 - |a|^2)^t \int_{D(a,1/2)} N_{\varphi,\gamma,\delta}(z) |\varphi'_a(z)|^{2+t} dA(z).
\]
Hence, for \(|a| > \frac{1}{2}\), (1) and the condition (9) yield
\[
N_{\varphi,\gamma,\delta}(a) \lesssim (1 - |a|^2)^t \int_{D(a,1/2)} |\varphi'_a(z)|^{2+t} \left[ \log \left( 1 - \frac{1}{\log |z|} \right) \right]^{\alpha} N_{\varphi,\gamma,\delta}(z) dA(z)
\]
\[
\lesssim (1 - |a|^2)^t,
\]
thus (2) is satisfied. The proof of (10) follows from a similar approach used in Lemma 3.3. The proof is complete. \(\square\)

**Proof of Theorem 1.4.** (1) \(\iff\) (2) follows from Theorem 1.1 in [2]. When \(\gamma = \gamma_2 + 2, \delta = \delta_2, t = \frac{(2 + \gamma_1)q}{p}, \alpha = \frac{3q}{p}\) and \(m = sq/p\) with \(2 + \gamma_1 - \delta_1 < s < \infty\),
(2) \(\iff\) (3) follows from Lemma 3.3. (2) \(\iff\) (4) follows from Lemma 3.4.
(3) \(\iff\) (4) follows from the change of variables formula, Lemma B. \(\square\)

**4. Proof of Theorem 1.5**

For the same indices as Theorem 1.4, (4) and (10) ensure that \(B \approx C\) and \(B \approx D\), respectively. We are enough to prove \(A \approx B\). For \(a \in \mathbb{D}\) with \(|a| > \frac{1}{2}\),
consider the test function
\[
k_a(z) = \frac{(1 - |a|)^{-\gamma_1}}{|1 - \bar{a}z|^{2+2\gamma_1}} \left[ \log \left( 1 - \frac{1}{\log |a|} \right) \right]^{\frac{\delta_1}{p}}, \quad z \in \mathbb{D},
\]
which is by Lemma F,
\[
|k_a|_{A^p_{\gamma_1}}^p \lesssim 1
\]
and \( k_a \to 0 \) uniformly in compact subsets of \( D \) as \(|a| \to 1\). If \( K : A^p_{\omega_1} \to A^q_{\omega_2} \) is compact, then by Lemma D

\[
\|C \varphi - K\| \geq \limsup_{|a| \to 1} \|C \varphi(k_a) - K k_a\|_{A^q_{\omega_2}}
\]

\[
\geq \limsup_{|a| \to 1} \|C \varphi(k_a)\|_{A^q_{\omega_2}} - \limsup_{|a| \to 1} \|K k_a\|_{A^q_{\omega_2}}
\]

\[
= \limsup_{|a| \to 1} \|C \varphi(k_a)\|_{A^q_{\omega_2}}.
\]

By Lemmas H and B,

\[
\|C \varphi(k_a)\|_{A^q_{\omega_2}} \gtrsim \int_D |(k_a \circ \varphi)(z)|^{q-2} |(k_a \circ \varphi)'(z)|^2 \left( \log \frac{1}{|z|} \right)^2 \omega_2(z) \, dA(z)
\]

\[
= \int_D |k_a(u)|^{q-2} |k_a'(u)|^2 N_{\varphi, \gamma_2 + 2, \delta_2}(u) \, dA(u).
\]

Inserting the test function and its derivative, the last integral equals

\[
(\frac{\gamma_1 + 2 - 2\delta_1}{p})^2 |a|^2 (1 - |a|^2)^{-\frac{2\delta_1}{p}} \left[ \log \left( 1 - \frac{1}{\log |a|} \right) \right]^{-\frac{2\delta_2}{p}}
\]

\[
\times \int_D \frac{1}{|1 - \bar{a}u|^{\frac{\gamma_1 + 2 - 2\delta_1}{p} + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(u) \, dA(u).
\]

The change of variables \( u = \varphi_a(z) \) gives

\[
\int_D \frac{1}{|1 - \bar{a}u|^{\frac{\gamma_1 + 2 - 2\delta_1}{p} + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(u) \, dA(u)
\]

\[
= \frac{1}{(1 - |a|^2)^2} \int_D \frac{1}{|1 - \bar{a} \varphi_a(z)|^{\frac{\gamma_1 + 2 - 2\delta_1}{p} + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(\varphi_a(z)) \, dA(z).
\]

Since \(|1 - \bar{a} \varphi_a(z)| \leq 2(1 - |a|^2)\) if \(|z| \leq \frac{1}{2}\), we have

\[
\frac{1}{(1 - |a|^2)^2} \int_D \frac{1}{|1 - \bar{a} \varphi_a(z)|^{\frac{\gamma_1 + 2 - 2\delta_1}{p} + 2}} N_{\varphi, \gamma_2 + 2, \delta_2}(\varphi_a(z)) \, dA(z)
\]

\[
\gtrsim \frac{1}{(1 - |a|^2)^{\gamma_1 + 2 - \delta_1 + \delta_2}} \int_D N_{\varphi, \gamma_2 + 2, \delta_2}(\varphi_a(z)) \, dA(z).
\]

Collecting these up, it now follows that

\[
\|C \varphi(k_a)\|_{A^q_{\omega_2}} \gtrsim \frac{|a|^2}{(1 - |a|^2)^{\gamma_1 + 2} + \delta_2} \left[ \log \left( 1 - \frac{1}{\log |a|} \right) \right]^{-\delta_2 \gamma/p}
\]

\[
\times \int_D N_{\varphi, \gamma_2 + 2, \delta_2}(\varphi_a(z)) \, dA(z).
\]

Now applying Lemmas C and A, we obtain

\[
\|C \varphi(k_a)\|_{A^q_{\omega_2}} \gtrsim \frac{|a|^2}{(1 - |a|^2)^{\gamma_1 + 2} + \delta_2} \left[ \log \left( 1 - \frac{1}{\log |a|} \right) \right]^{-\delta_2 \gamma/p}
\]
Then there exists $A$

By (11), we get

$$\liminf_{|z| \to 1} \frac{N_{\varphi, r_{2} + 2, \delta_{2} + 2}}{(\log \frac{1}{|z|})^{2} \omega_{1}(z)} \leq B > 0.$$  

Then there exists $r_{0} \in (0, 1)$ such that

$$N_{\varphi, r_{2} + 2, \delta_{2} + 2} \left(\frac{2}{\log \frac{1}{|z|}} \omega_{1}(z)\right)^{q/p} \leq 2B$$

for $|z| \geq r_{0}$. For a holomorphic function $f(z) = \sum_{k=0}^{\infty} a_{k} z^{k}$ on $D$, let

$$T_{n} f(z) = \sum_{k=0}^{n} a_{k} z^{k}, \quad R_{n} f(z) = \sum_{k=n+1}^{\infty} a_{k} z^{k}.$$  

Then $T_{n} : A_{p_{1}}^{\infty} \to A_{q_{2}}^{\infty}$ is compact, and

$$\|C_{\varphi}\|_{C_{c}} = \|C_{\varphi}(T_{n} + R_{n})\|_{C_{c}} \leq \|C_{\varphi}T_{n}\|_{C_{c}} + \|C_{\varphi}R_{n}\|_{C_{c}} = \|C_{\varphi}R_{n}\|_{C_{c}} \leq \|C_{\varphi}R_{n}\|.$$  

Thus $||C_{\varphi}||_{c} \leq \liminf_{n \to \infty} ||C_{\varphi}R_{n}||$. Since $(R_{n} f \circ \varphi)(0) \to 0$ as $n \to \infty$, hence, by Lemmas H, B and (12),

$$||C_{\varphi}||_{q}^{2} \leq \liminf_{n \to \infty} ||C_{\varphi}R_{n}||^{2} = \liminf_{n \to \infty} \sup_{||f||_{A_{p_{1}}^{\infty}} \leq 1} ||C_{\varphi}R_{n} f||_{A_{q_{2}}}^{2} \approx \liminf_{n \to \infty} \sup_{||f||_{A_{p_{1}}^{\infty}} \leq 1} \int_{D} \left| (R_{n} f \circ \varphi)(z) \right|^{q-2} \left| (R_{n} f \circ \varphi)'(z) \right|^{2} \left( \log \frac{1}{|z|} \right)^{2} \omega_{2}(z) dA(z)$$

$$= \liminf_{n \to \infty} \sup_{||f||_{A_{p_{1}}^{\infty}} \leq 1} \int_{D} \left| R_{n} f(w) \right|^{q-2} \left| R_{n} f'(w) \right|^{2} N_{\varphi, r_{2} + 2, \delta_{2} + 2} dA(w)$$
\[
\lesssim B \lim_{n \to \infty} \inf \sup_{\|f\|_{A^p_{\omega_1}} \leq 1} \int_{\mathbb{D}} |R_n f(w)|^{q-2} |R_n f'(w)|^2 \left( \frac{1}{|w|} \right)^2 \omega_1(w)^{q/p} \ dA(w).
\]

From Lemma E, we have
\[
|R_n f(z)| \lesssim \left( \frac{1}{|z|} \right)^2 \omega_1(z)^{-\frac{q}{p}} \|R_n f\|_{A^p_{\omega_1}}
\]
\[
\lesssim \left( \frac{1}{|z|} \right)^2 \omega_1(z)^{-\frac{q}{p}} \|f\|_{A^p_{\omega_1}},
\]
and by Lemma H, we obtain
\[
\|C_{\phi}\|_q^q \lesssim B \lim_{n \to \infty} \inf \sup_{\|f\|_{A^p_{\omega_1}} \leq 1} \|f\|_{A^p_{\omega_1}}^{q-p} \int_{\mathbb{D}} |R_n f(w)|^{q-2} |R_n f'(w)|^2 \left( \frac{1}{|w|} \right)^2 \omega_1(w) \ dA(w)
\]
\[
\approx B \lim_{n \to \infty} \inf \sup_{\|f\|_{A^p_{\omega_1}} \leq 1} \|R_n f\|_{A^p_{\omega_1}}^{q-p}
\]
\[
\lesssim B \sup_{\|f\|_{A^p_{\omega_1}} \leq 1} \|f\|_{A^p_{\omega_1}}^{q-p} = B.
\]

The proof is complete.

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