A numerical study of Taylor vortex flow in a finite length tapered annulus

M N Noui–Mehidi¹, ³, N Ohmura² and J Wu¹
¹Energy and Thermofluids Engineering, CSIRO Manufacturing and Infrastructure Technology, PO Box 56, Highett, Victoria 3190, Australia
²Department of Chemical Science and Engineering, Kobe University, Rokkodai, Nada Kobe 657-8501, Japan
E-mail: Nabil.Noui-Mehidi@csiro.au

Abstract. The transient evolution and steady state analysis of Taylor vortex flow in a tapered annulus was conducted by numerical experiments in the case where the inner cylinder was rotated and the outer one fixed. The gap between the cylinders was linearly tapered from a supercritical value at the upper base to the critical value at the lower base. The wavelength adjustment depended on the variation of Reynolds number in the spatially ramped gap. The axisymmetric conservative governing equations were solved by the use of an simplified marker and cell (SMAC) algorithm. A coordinate transformation function allowed us to numerically solve the problem in a rectangular computational domain. The results have shown that Taylor vortices growth was sensitive to the spatial ramp of the gap even with tapering angle values less than one degree. The interaction between the inflow and outflow boundaries could be clearly seen as the taper angle was increased. The investigation of the transient dynamics related to the flow system also revealed a characteristic dependence on the taper angle.

1. Introduction
Taylor vortices have been widely investigated experimentally and theoretically since Taylor [1] showed their existence in the gap between coaxial cylinders. Despite the volumes of work devoted to the study of the effects of the dynamical and geometrical parameters controlling a flow system, the factors determining wavelength selection in a specified geometry are not yet well understood.

A Taylor–Couette flow, as a nonlinear dissipative system, is characterized by transitions that occur to flow states of reduced symmetry with specific wavelengths. The multiplicity of stable flow states was explored in detail by Coles [2] and Snyder [3]. Cannel et al. [4] investigated the wavelength selection in a Taylor–Couette apparatus where a section of the gap was tapered from a supercritical Reynolds number to a subcritical Reynolds number. They found that a sufficiently slow ramp connecting the supercritical region to the subcritical region resulted in the selection of a unique Reynolds number. Ning et al. [5] studied slow spatial ramps of the Reynolds number in a Taylor–Couette system. They found that the range of wave numbers obtained was smaller than the Eckhaus-stable range, and the selected wave number depended on the nature of the ramp.

Previous studies have shown that a gradual axial variation of \( \varepsilon \) (\( \varepsilon = \text{Re}/\text{Re}_c – 1 \), \( \text{Re}_c \) is the critical Reynolds number) could permit a continuous wavelength adjustment mechanism to be effective. Thus,

³ Correspondence should be addressed to MN Noui–Mehidi.
the selection of an \( \varepsilon \)-dependent flow state could be possible. The issue of the existence and nature of such selection processes remains largely unexplored.

The present study is concerned with a numerical simulation of wavelength selection in a Taylor–Couette system where the gap is spatially ramped from a critical state at the lower base to a supercritical state at the upper base. The aim of the numerical experiments is to calculate the velocity and pressure fields for different spatial ramps.

2. Basic equations and boundary conditions

The flow of an incompressible fluid contained in the gap between two coaxial cylinders is investigated in the case where the inner cylinder rotates and the outer remains stationary. The outer cylinder wall is opened with a certain angle resulting in an axially ramped gap, as seen in Figure 1. At the lower base, the inner and outer cylinders radii are respectively \( R_1 \) and \( R_2 \). The taper angle is denoted \( \alpha \).

Consider the velocity components \((u, v, w)\) in the directions defined by the cylindrical coordinates \((r, \theta, z)\) respectively. The non-dimensional governing equations for an axisymmetric flow can be written in conservative form with primitive variables as:

\[
\frac{Du}{Dt} \frac{v^2}{r} + \frac{\partial P}{\partial r} + \frac{1}{Re} \left( \Delta u - \frac{u}{r^2} \right) = 0 \tag{1}
\]

\[
\frac{Dv}{Dt} + \frac{uv}{r} = \frac{1}{Re} \left( \Delta v - \frac{v}{r^2} \right) \tag{2}
\]
\[
\frac{Dw}{Dt} = -\frac{\partial P}{\partial z} + \frac{1}{\text{Re}} \Delta w
\]  
(3)

\[
\frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z} = 0
\]  
(4)

where the operators are given by:

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \frac{1}{r} \frac{\partial (ru)}{\partial r} + \frac{\partial w}{\partial z}
\]  
(5)

\[
\Delta = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{r}{\partial r} \right) + \frac{\partial^2}{\partial z^2}
\]  
(6)

The variables are made dimensionless using the scale \( d \) (\( d = R_2 - R_1 \)), \( \Omega R_1 \) and \( \Omega^{-1} \) for length, speed and time respectively (where \( \Omega \) is the angular velocity of the inner cylinder). The Reynolds number is defined at the lower base of the flow system by:

\[
\text{Re} = \frac{R_1 \Omega d}{\nu}
\]  
(7)

Defining the vertical length of the fluid column by \( L \), the aspect ratio is then given by:

\[
\Gamma = \frac{L}{d}
\]  
(8)

The set of equations (1)–(4) written in the present form needs to fit with the boundaries of the non-constant gap. In order to apply a finite difference approach, as will be discussed in the following section, a rectangular computational domain is more suitable for the numerical calculations. Defining the following new coordinates allows the computational domain to be transformed to a rectangular grid:

\[
\eta = \frac{r - A}{1 + z \cdot \tan(\alpha)} = \frac{r - A}{S}
\]  
(9)

where \( A = R_1/d \).

Applying this coordinate transformation to the set of equations (1)–(4), the following system of equations is obtained:

\[
\frac{\partial u}{\partial t} + \frac{1}{S(\eta S + A)} \frac{\partial \left( \frac{(\eta S + A) u^2}{\partial \eta} \right)}{\partial \eta} + \frac{\partial (uv)}{\partial \xi} - \frac{v^2}{S} \frac{\partial (uv)}{\partial \eta} \frac{1}{\eta S + A} \frac{\partial P}{\partial \eta} = \frac{1}{\text{Re}} \left[ \frac{1}{S(\eta S + A)} \frac{\partial \left( \frac{(\eta S + A) u^2}{\partial \eta} \right)}{\partial \eta} + \frac{2 \eta \cdot \tan(\alpha)}{S} \left( \frac{\partial^2 u}{\partial \xi^2} \frac{\partial \eta}{\partial \xi} \right) + \frac{\eta^2 \cdot \tan^2(\alpha)}{S^2} \left( \frac{\partial^2 u}{\partial \eta^2} \right) - \frac{u}{(\eta S + A)^2} \right]
\]  
(10)
\[
\frac{\partial v}{\partial t} + \frac{1}{S(\eta S + A)} \frac{\partial ((\eta S + A)uv)}{\partial \eta} + \frac{\partial (vw)}{\partial \xi} + \eta \cdot \text{tg} \alpha \frac{\partial (vw)}{\partial \eta} + \frac{uv}{S(\eta S + A)} = \]
\[
\frac{1}{Re} \left[ \frac{1}{S(\eta S + A)} \frac{\partial}{\partial \eta} \left( \frac{(\eta S + A) \frac{\partial v}{\partial \eta}}{S} \right) + \eta \cdot \text{tg} \alpha \frac{\partial^2 v}{\partial \xi^2} + \frac{2 \eta \cdot \text{tg} \alpha \left( \frac{\partial^2 v}{\partial \xi \partial \eta} \right)}{S^2} + \frac{\eta^2 \cdot \text{tg}^2 \alpha \left( \frac{\partial^2 v}{\partial \eta^2} \right)}{(\eta S + A)^2} - \frac{v}{(\eta S + A)^2} \right] \]
\[
\frac{\partial w}{\partial t} + \frac{1}{S(\eta S + A)} \frac{\partial ((\eta S + A)uw)}{\partial \eta} + \frac{\partial (w^2)}{\partial \xi} + \eta \cdot \text{tg} \alpha \frac{\partial (w^2)}{\partial \eta} + \frac{v^2}{S(\eta S + A)} = \frac{\partial P}{S \partial \xi} + \frac{\eta \cdot \text{tg} \alpha \partial P}{\partial \eta} \]
\[
+ \frac{1}{Re} \left[ \frac{1}{S(\eta S + A)} \frac{\partial}{\partial \eta} \left( \frac{(\eta S + A) \frac{\partial w}{\partial \eta}}{S} \right) + \frac{w^2}{S} \left( \frac{\partial^2 w}{\partial \xi \partial \eta} \right) + \eta \cdot \text{tg} \alpha \frac{\partial^2 w}{\partial \eta^2} \right] \]
\[
\frac{\partial u}{\partial \eta} + \frac{S}{(\eta S + A)} u + S \frac{\partial w}{\partial \xi} - u \cdot \text{tg} \alpha \frac{\partial w}{\partial \xi} = 0 \]
\]

where the derivatives with respect to \( \eta \) and \( \xi \) are obtained from:
\[
\frac{\partial}{\partial r} = \frac{1}{S} \frac{\partial}{\partial \eta} \]
\[
\frac{\partial}{\partial z} = \frac{\partial}{\partial \xi} + \frac{\eta \cdot \text{tg} \alpha}{S} \frac{\partial}{\partial \eta} \]

In the set of equations (10)–(13), the taper angle \( \alpha \) allows the axial ramp of the gap width to be fixed. On the other hand, setting \( \alpha \) to zero leads to a constant gap system, which can be used as the reference point for comparison between the different investigated geometries.

The endplates at \( \xi = 0, \Gamma \) are assumed to be attached to the outer fixed cylinder. The boundary conditions at the cylinder walls are given by:
\[
\eta = 0: u = w = 0, \quad v = 1 \]
\[
\eta = 1: u = v = w = 0 \]

### 3. Numerical method

A finite-difference method is used for the integration of the set of equations (10)–(13). This method has been implemented successfully by Ohmura et al. [6] and Noui–Mehidi et al. [7] for the simulation of Taylor vortices in different geometries.

The numerical domain is discretized using a staggered mesh grid, and a Simplified Marker and Cell (SMAC) formulation is used. At each time step \( \Delta t \), the momentum equations are solved for a temporary velocity field \( V^* \) given by:
\[
V^{*n+1} = V^n + \Delta t \left[ - (\nabla P^n - \nabla \cdot V^n + \nabla^2 V^n) \right] / \text{Re} \]

where \( V^n \) is the velocity field at the time step \( n \). Then a scalar potential function \( \phi \) is applied for the Poisson equation:
\[ \nabla^2 \phi = -\text{div}V^n \]  \hspace{1cm} (17)

This Poisson equation is solved using the point SOR method. The final velocity field \( V^{n+1} \) is obtained from:

\[ V^{n+1} = V^n + V' \]  \hspace{1cm} (18)

where \( V' \) is the non-vortex flow defined by \( V' = \text{grad} \phi \). The pressure field is corrected by solving the equation:

\[ P^{n+1} = P^n - \phi / \Delta t \]  \hspace{1cm} (19)

For spatial discretization, finite differences at the second-order are applied, and for time a first-order integration scheme is used. The computational grid in the transformed domain \((\eta, \xi)\) is rectangular. The grid was spatially constant in the radial and axial directions.

4. Numerical results
In the present investigation, the study of the ramp effect on a steady Taylor vortex flow is achieved for the same aspect ratio and Reynolds number. The following results are related to an aspect ratio \( \Gamma = 10.2 \), a Reynolds number \( \text{Re} = 200 \) and a gap ratio \( R_1/R_2=0.72 \). The cases investigated concentrated on very small taper angles \( \alpha \) in the range of 0 to 0.5 degrees.

4.1. Steady state
The final steady state in each particular configuration results in a characteristic flow arrangement. Figure 2 presents the steady solution for different values of taper angle \( \alpha \). The plots correspond to flood contours of different values of the stream function \( \psi \) calculated from the velocity field by:

\[ u = -\frac{1}{(\eta S + A)} \left[ -\frac{\eta \alpha \partial \psi}{\partial \eta} \right], \hspace{1cm} v = \frac{1}{S(\eta S + A)} \frac{\partial \psi}{\partial \xi} \]  \hspace{1cm} (20)

As seen in Figure 2, in the case of a constant gap \( \alpha = 0 \) there are six pairs of steady Taylor vortices arranged symmetrically with regard to the midplane of the fluid column for the aspect ratio of \( \Gamma = 10.2 \). Neitzel [8] has reported that steady state solutions for fractional values of aspect ratios would lead to wavelengths smaller than the critical wavelength in the case of sudden start. One would expect five pairs of vortices for an aspect ratio equal to \( \Gamma_c = 10 \); in the present study the difference between \( \Gamma \) and \( \Gamma_c \) is not an integral number of the wavelength corresponding to the case \( \Gamma = 10 \), thus the resulting steady state leads to smaller wavelengths as \( \Gamma > \Gamma_c \). Neitzel [8] computations have shown the existence of thirteen pairs of vortices for an aspect ratio of 23.35.

When \( \alpha \) increases, the symmetry in the final flow state is not maintained and different vortex arrangements are obtained for each value of \( \alpha \). It should be noted that the analysis of the time sequence calculations has shown that the basic flow, which is unidimensional in the case \( \alpha = 0 \), becomes three-dimensional even at \( \alpha = 0.1 \) degrees. A basic meridional flow takes place in the whole fluid column, upwards with the rotating wall and downwards with the fixed one. The strength of this meridional flow increases as \( \alpha \) increases. This meridional flow is driven by the pressure field resulting from the variation of the centrifugal forces due to the axial variation of the gap.

When \( \alpha \) increases from 0 to 0.5 degrees for the same aspect ratio \( \Gamma = 10.2 \) and \( \text{Re} = 200 \), the number of steady Taylor vortices decreases from six pairs \( \alpha = 0 \) to three pairs \( \alpha = 0.5 \) degrees. For
Figure 2. Steady state flow modes for different values of taper angle $\alpha$ for $\Gamma = 10.2$, $Re = 200$: (a) $\alpha = 0$; (b) $\alpha = 0.1$; (c) $\alpha = 0.2$; (d) $\alpha = 0.3$; (e) $\alpha = 0.4$; and (f) $\alpha = 0.5$.

$\alpha=0.5$ degrees the ratio $Re/Re_c$ varies from 1 at the lower base to 1.09 at the upper one. The experiments of Ahlers et al. [9] have shown that the wavelengths observed for $Re/Re_c=1.09$ could reach up to 2.4 times the gap width in the absence of spatial ramp. In the present study the resulting flow arrangement for $\alpha=0.5$ degrees is probably dictated by the combination of the spatial ramp and the value of the Reynolds number at the upper base of the system.

In Figure 2 the cells with the strongest vorticity are the ones that rotate clockwise identically to the basic meridional flow, i.e. upwards near the rotating wall and downwards near the fixed one. These latter vortices have the largest size in a ramped fluid column compared to the vortices rotating counterclockwise. Wimmer [10] has also observed between rotating conical cylinders that vortices rotating clockwise in a same axial section had the largest size. The extension of this property to non-constant gaps is of interest. On the other hand, the size of the clockwise rotating vortices increases as $\alpha$ increases at the same axial locations. This property can be clearly seen in Figures 2(d) and 2(e) where $\alpha = 0.3$ and 0.4 degrees respectively, and the number of vortices is the same (four pairs).

The experiments of Cannel et al. [4] have shown that for the Taylor–Couette system, inflow and outflow boundaries are not equivalent in their interaction with the spatial ramp, whereas, theoretically, in the amplitude equations all half-cycles behave identically. In the present simulations, this flow property can be seen in Figures 3 and 4, which present the radial variation of the radial component of the velocity at the inflow and outflow boundaries (i.e. as one moves along the inflow/outflow boundary for $z=\zeta=$constant) in the computational domain $(\eta, \zeta)$. In the case of a constant gap ($\alpha = 0$), Figure 3 shows that all inflow and outflow boundaries are respectively equivalent regardless of the
axial position. However, the interaction between inflow and outflow boundaries when a spatial ramp exists can be clearly seen in Figure 4 for a value of $\alpha = 0.3$ degrees.

**Figure 3.** Radial velocity component at the inflow and outflow boundaries in the computational domain $(\eta, \xi)$ for the cylindrical case, $\alpha = 0$, $Re = 200$, $\Gamma = 10.2$.

**Figure 4.** Radial velocity component at the inflow and outflow boundaries in the computational domain $(\eta, \xi)$ for the case of a tapered gap, $\alpha = 0.3$, $Re = 200$, $\Gamma = 10.2$. 
Figure 5. Transient flow in the case of a cylindrical gap with $\alpha = 0$, $Re = 200$ and $\Gamma = 10.2$. The steady state solution is presented on the right hand side.

The radial jets corresponding to the outflow boundaries with the fluid particles moving towards the outer wall, and which are larger at the central region of the gap, become stronger as the gap width increases axially. The maximum of the radial component increases at each outflow boundary in the upward axial direction. On the contrary, in the inflow boundaries (sink regions), the radial velocity magnitude at the centre of the gap decreases as the gap increases to compensate for the increase in the radial velocity at the adjacent outflow boundary.

4.2. Transient flow

After analyzing the steady states obtained in each ramped gap, it is interesting to examine the transient flow when the taper angle $\alpha$ increases. The instantaneous flow patterns are visualized with the absolute value of the stream function $\psi$ at a plane close to the outer wall [11]. Figure 5 presents the transient development of vortices in the fluid column for $\alpha = 0$. The vortices develop from both endplates towards the centre of the column as time increases. The steady state is shown on the right-hand side by flood contours of the streamlines. It can be seen that the development of the vortices is perfectly symmetric with regard to the midplane of the fluid column. At a non-dimensional time equal to 80, the steady state is already reached.

The situation is completely different when the gap spatial ramp exists. Although the first vortices develop at both ends (Figures 6 and 7) for the range of taper angles investigated, the transient development of the vortices is no longer symmetric. In the actual calculations, the finite length of the fluid column (aspect ratio $\Gamma = 10.2$) exhibits the effects of the endplates on the generation and development of the vortices.
In the case of $\alpha = 0.3$ degrees, a characteristic transient behavior is observed, since the flow pattern develops from five pairs of vortices at a time of 60, to a final steady state of four pairs, as seen on the right-hand side of Figure 6. In the time interval 80–120, strong competition between the vortices developed initially at the upper part of the fluid column can be identified. For a value of $\alpha = 0.5$ degrees (Figure 7), the transient development of the flow is not symmetric from the early appearance of the vortices near the endplates. The flow adjustment is continuous until the steady state is reached, as shown on the right-hand side of Figure 7. On the other hand, it can also be seen that the counter-clockwise rotating vortices, initially weaker than the clockwise rotating vortices, gain strength gradually as the flow develops towards the steady state. These observations confirm that the flow adjustment mechanism reveals the selection of a unique $\epsilon$-dependent flow state ($\epsilon = \frac{Re_L}{Re_C} - 1$, where $Re_L$ is the Reynolds number at the upper base and $Re_C$ is the Reynolds number at the lower base), as reported experimentally by Cannel et al. [4].

5. Conclusions

The present study has shown that a sufficiently small ramp of gap width in a Taylor–Couette system can result in completely different flow patterns. The numerical calculations revealed the existence of a unique $\epsilon$-dependent state for the same aspect ratio and Reynolds number. The interaction between inflow and outflow boundaries plays a major role in determining the final steady state and number of vortices in the finite length fluid column.

Figure 6. Transient flow in the case of a tapered gap with $\alpha = 0.3$, $Re = 200$ and $\Gamma = 10.2$. The steady state solution is presented on the right hand side.
Figure 7. Transient flow in the case of a tapered gap with $\alpha = 0.5$, $Re = 200$ and $\Gamma = 10.2$. The steady state solution is presented on the right hand side.

6. References

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