Numerical method for an inverse dynamical problem for composite beams

Antonino Morassi¹, Gen Nakamura², Kenji Shirota³, Mourad Sini⁴

¹ Department of Georesources and Territory, University of Udine, 33100 Udine, Italy
² Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan
³ Domain of Mathematical Sciences, Ibaraki University, Ibaraki 310-8512, Japan
⁴ Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences, Altenbergerstrasse 69, A-4040 Linz, Austria

E-mail: ³shirota@mx.ibaraki.ac.jp

Abstract. In this paper we present a numerical method for an inverse problem of nondestructive testing for a composite system formed by the connection of a steel beam and a reinforced concrete beam. The small vibrations of the composite beam are governed in space by two second order and two fourth order differential operators, which are coupled in the lower order terms by two coefficients which express the shearing and axial stiffness of the connection. Our inverse problem is to determine these stiffness coefficients by using Neumann type boundary data measured at one end of the beam and transversal displacements given in an interior portion of the beam axis. We recast the inverse problem as a constrained variational issue and an iterated projected gradient method is proposed for the numerical solution of the minimizing problem. Suitable clip-off and mollifier operators are introduced in order to describe the constrained conditions. The effectiveness of method and the sensitivity of the results to errors in the measured data are tested on the basis of an extensive series of numerical experiments.

1. Introduction

In this paper a reconstruction technique for coefficient identification in a steel-concrete composite beam is presented. A composite beam is obtained by connecting two beams, a metallic one and a reinforced concrete beam, by means of small metallic elements, called connectors, which are welded on the top flange of the metallic beam and immersed in the concrete, in order to hinder sliding on the concrete-steel interface, see Figure 1. The infinitesimal free vibrations of a steel-concrete composite beam of length $L$ are governed by the following system of partial differential equations

$$
\begin{align*}
\rho_1 u_{1,tt} &= (a_1 u_{1,x})_x + k (u_2 - u_1 + v_2 e_s) & \text{in } (0, L) \times (0, T), \\
\rho_2 u_{2,tt} &= (a_2 u_{2,x})_x - k (u_2 - u_1 + v_2 e_s) & \text{in } (0, L) \times (0, T), \\
\rho_1 v_{1,tt} &= -(j_1 v_{1,xx})_{xx} + \left( \frac{ke_2^2}{6} (2v_{1,x} + v_{2,x}) \right)_{xx} - \mu (v_1 - v_2) & \text{in } (0, L) \times (0, T), \\
\rho_2 v_{2,tt} &= -(j_2 v_{2,xx})_{xx} + \left( \frac{ke_2^2}{6} (2v_{2,x} + v_{1,x}) \right)_{xx} + (k(u_2 - u_1 + v_{2,x} e_s) e_s)_{x} + \mu (v_1 - v_2) & \text{in } (0, L) \times (0, T), \\
\end{align*}
$$

(1)
A diagnostic method based on dynamic data has been proposed for the simpler situation in which measurements taken at the boundary and on some interior portion of the beam axis. In [9] a problem interesting for applications consists in estimating the coefficients inaccessibility of the connection from the exterior makes direct inspection difficult, an inverse deterioration of the connection, causing a decreasing of the stiffness coefficient \( k \) of its integrity. In particular, typical damages occurring in real steel-concrete systems involve a define the mechanical properties of the connection and they contain direct information on the stiffness of the connection between the concrete and the steel beam. These two coefficients see [6]. Here, the symbols ”, \( t \)” and ”, \( x \)” stand for \( \frac{\partial}{\partial t} \) and \( \frac{\partial}{\partial x} \), respectively. Under the assumption that the system is at rest at \( t = 0 \), that is \[ u_1|_{t \leq 0} = u_2|_{t \leq 0} = v_1|_{t \leq 0} = v_2|_{t \leq 0} = 0, \quad x \in (0, L), \]

we shall concern with the following Dirichlet boundary conditions at \( x = 0 \) and \( x = L \):

\[
\begin{align*}
\begin{aligned}
  u_1|_{x=L} &= u_2|_{x=L} = v_1|_{x=L} = v_2|_{x=L} = v_{1,x}|_{x=L} = v_{2,x}|_{x=L} = 0 & t \in (0, T), \\
  u_1|_{x=0} &= \overline{u}_1(t), \quad u_2|_{x=0} = \overline{u}_2(t) & t \in (0, T), \\
  v_1|_{x=0} &= \overline{v}_1(t), \quad v_2|_{x=0} = \overline{v}_2(t) & t \in (0, T), \\
  v_{1,x}|_{x=0} &= \overline{\varphi}_1(t), \quad v_{2,x}|_{x=0} = \overline{\varphi}_2(t) & t \in (0, T).
\end{aligned}
\end{align*}
\]

Hereinafter, the quantities related to the concrete beam (the upper one in Figure 1) and the steel beam (the lower one) will be denoted by indices \( i = 1, 2 \), respectively. The functions \( u_i = u_i(x, t) \) and \( v_i = v_i(x, t) \) denote the longitudinal and transversal displacement, respectively, of the cross-section of abscissa \( x \), evaluated at the moment of time \( t \). In equations (1), the quantities \( j_i = E_i I_i \) and \( \alpha_i = E_i A_i \) are the flexural and the axial stiffness of the cross-section, respectively, where \( I_i \) and \( A_i \) are the moment of inertia and the area of the transversal cross-section. The positive constant \( E_i, \ i=1,2, \) is the Young modulus of the material of the \( i \)th beam. The function \( \rho_i = \rho_i(x), \ \rho_i > 0, \) is the linear mass density of the \( i \)th beam, \( i=1,2 \). Finally, the constant \( e_s \) is the half-height of the steel beam and \( e_c \equiv e - e_s \), where \( e \) is the (constant) distance between the axes of the two beams forming the system.

The two positive quantities \( k = k(x), \ \mu = \mu(x) \) express respectively the shearing and axial stiffness of the connection between the concrete and the steel beam. These two coefficients define the mechanical properties of the connection and they contain direct information on its integrity. In particular, typical damages occurring in real steel-concrete systems involve a deterioration of the connection, causing a decreasing of the stiffness coefficient \( k \) and \( \mu \). Since the inaccessibility of the connection from the exterior makes direct inspection difficult, an inverse problem interesting for applications consists in estimating the coefficients \( k, \ \mu \) from suitable nondestructive techniques.

The main goal of this research is to reconstruct these unknown coefficients from dynamical measurements taken at the boundary and on some interior portion of the beam axis. In [9] a diagnostic method based on dynamic data has been proposed for the simpler situation in which

![Figure 1. Steel-concrete composite beam with free left end and clamped right end: longitudinal view (a) and transversal cross-section (b).](image-url)
the coupling between bending and longitudinal motions is neglected. In this case, by formally taking \(v_1 = v_2 = 0\) in the previous model, the system (1) simplifies into a two-velocity dynamical system. For this reduced problem it was proved that the shearing stiffness coefficient \(k\) can be uniquely determined from the measurement of the frequency response function of the composite system taken at one end of the beam. The strategy of the reconstruction procedure is based on a transformation of the equations governing the free longitudinal vibrations to an equivalent first order system and, subsequently, on the use of the progressive waves approach to reduce the local reconstruction of \(k\) to the resolution of a system of non-linear Volterra integral equations. Finally, an iterative use of a layer stripping technique allows for a reconstruction, step by step, of the coefficient \(k\) on the whole interval \([0, L]\). We refer to [3] for an interesting application of the Boundary Control Method to solve this inverse problem when measurements are taken at both the ends of the beam.

However, in the engineering applications, see, for example, [8], it is important to examine the full complete coupled system (1), which includes two fourth order and two second order differential operators coupled on a term of low order. Unfortunately, it seems rather involved to extend the techniques presented in [9] and [3] to this general case.

In this paper we present a numerical method for reconstructing the unknown stiffness coefficients \(k, \mu\) in the full complete coupled system (1) by using a different approach. We try to identify the coefficients \(k, \mu\) by using dynamical measurements taken at the free left end and on an open interval \(I, I \subseteq [0, L]\). We assume that the Neumann type boundary data \(\overline{Q}(t) = (\overline{N_1}(t), \overline{N_2}, \overline{T_1}(t), \overline{T_2}(t), \overline{M_1}(t), \overline{M_2}(t))^T\) are given at \(x = 0\) for any \(t \in (0, T)\), where

\[
\begin{align*}
\quad a_1u_{1,x}|_{x=0} &= \overline{N_1}, \quad a_2u_{2,x}|_{x=0} = \overline{N_2}, \\
-(j_1v_{1,xx})|_{x=0} &= \frac{kc^2}{6}(2v_{1,x} + v_{2,x})|_{x=0} = \overline{T_1}, \\
-(j_2v_{2,xx})|_{x=0} &= \frac{kc^2}{6}(2v_{2,x} + v_{1,x}) + k(u_2 - u_1 + v_{2,x}e_s)|_{x=0} = \overline{T_2}, \\
-j_1v_{1,xx}|_{x=0} &= \overline{M_1}, \quad -j_2v_{2,xx}|_{x=0} = \overline{M_2}.
\end{align*}
\]

Moreover, the transversal displacements \(v_i = \overline{v_i}, i = 1, 2\), are assumed to be given in \(I\) for any \(t \in (0, T)\). In order to determine the unknowns \(k\) and \(\mu\), we adopt the variational method. We introduce a cost functional with two variables by using given measured data, and then, a minimizing problem with some constrained condition is produced. We propose an iterated method based on the projected gradient method to solve our minimizing problem. In our iteration, we use the analytical expressions of the first partial derivatives of the cost function presented in our theoretical paper [10]. The clip-off and Friedrichs’s mollifier operators are employed in order that all updated coefficients in our method satisfy the constrained condition. Moreover, we discuss the effectiveness of our method by some numerical experiments.

2. Variational approach and numerical algorithm

In this section, we shall present a variational approach to the inverse problem of determining \(k, \mu\) and we shall introduce an iterated numerical algorithm based on a projected gradient method in order to solve the minimization problem.

2.1. Variational approach and minimizing problem

To simplify the notation, we rewrite the dynamical system (1) governing the infinitesimal vibrations \(w = (u_1, u_2, v_1, v_2)^T\) of a composite beam in the following compact form:

\[
\begin{align*}
 Cw_{\mu} - A_{k,\mu}w &= 0 \quad &\text{in} \ (0, L) \times (0, T), \\
 w|_{t=0} = 0, &\quad w|_{t=0} = 0 \quad &\text{in} \ (0, L), \\
 Dw|_{x=0} = \overline{U}, &\quad Dw|_{x=L} = 0 \quad &\text{on} \ (0, T),
\end{align*}
\]

(2)
The following notation will be useful in the sequel:

\[
A_{k,\mu} w = \begin{pmatrix}
(a_1 u_1, x) + k(u_2 - u_1 + v_2, x) e_s \\
(a_2 u_2, x) - k(u_2 - u_1 + v_2, x) e_s \\
-(j_1 v_1, x), x + \left( \frac{ke^2}{6}(2v_1, x + v_2, x) \right) - \mu(v_1 - v_2) \\
-(j_2 v_2, x), x + \left( \frac{ke^2}{6}(2v_2, x + v_1, x) \right) + (k(u_2 - u_1 + v_2, x) e_s, x) + \mu(v_1 - v_2)
\end{pmatrix}
\]

and \(D\) is the operator given by

\[
Dw = (u_1, u_2, v_1, v_2, v_1, x, v_2, x)^T
\]

The coefficients \(\rho_i, a_i, \) and \(j_i (i = 1, 2)\) are assumed to be positive and regular in \([0, L]\). More precisely, for \(i = 1, 2\), we assume

\[
\begin{align*}
\rho_i & \in C^0[0, L], \quad \rho_i(x) \geq \rho_{i0} > 0 \text{ in } (0, L), \\
a_i & \in C^1[0, L], \quad a_i(x) \geq a_{i0} > 0 \text{ in } (0, L), \\
j_i & \in C^2([0, L]), \quad j_i(x) \geq j_{i0} > 0 \text{ in } (0, L),
\end{align*}
\]

where \(\rho_{i0}, a_{i0}\) and \(j_{i0}\) are given constants. Concerning the stiffness coefficients of the connection, we shall assume

\[
\begin{align*}
k & \in C^1[0, L], \quad k(x) \geq 0 \text{ in } [0, L], \\
\mu & \in C^0[0, L], \quad \mu(x) \geq 0 \text{ in } [0, L].
\end{align*}
\]

The Dirichlet boundary data \(U(t) = (\varphi_1(t), \varphi_2(t), \varphi_1(t), \varphi_2(t), \varphi_1(t), \varphi_2(t))^T\) are assumed to be such that

\[
U(t) \in (C^3([0, T]))^6 \text{ with } U_{j,x'}(t)|_{t=0} = 0, \quad i = 0, 1, 2, 3, \quad j = 1, \ldots, 6,
\]

where \(f_{,t} := \frac{\partial f(t)}{\partial t}\) for any integer number \(i, \) with \(i \geq 1.\)

Under the above assumptions, the initial-boundary value problem (2) has a unique solution

\[
w \in \left( H^2((0, T); H^1(0, L)) \right)^2 \times \left( H^2((0, T); H^2(0, L)) \right)^2,
\]

see [10]. Moreover, we have also that

\[
w \in \left( H^1((0, T); H^2(0, L)) \right)^2 \times \left( H^1((0, T); H^4(0, L)) \right)^2.
\]

Let us denote by \(C\) the set of pairs of coefficient \((k, \mu)\) such that

\[
C = \left\{ (k, \mu) \in C^1[0, L] \times C^0[0, L] \mid k(x) \geq 0, \ \mu \geq 0 \text{ for } x \in [0, L] \right\}.
\]

The following notation will be useful in the sequel:

\[
\begin{align*}
w[k, \mu] & := w[k, \mu](x, t) \text{ solution of } (2) \text{ for } k = \tilde{k} \text{ and } \mu = \tilde{\mu} \text{ for given } (\tilde{k}, \tilde{\mu}) \in C, \\
Q[k, \mu](t) & := \text{Neumann boundary data at } x = 0 \text{ obtained by using } w[k, \mu],
\end{align*}
\]

\[
\begin{align*}
\bar{Q}(t) := Q[k, \mu](t), \quad \varphi_i(x, t) := v_i[k, \mu](x, t), \quad i = 1, 2, \ x \in [0, L], \ t \in [0, T],
\end{align*}
\]
for some \((k, \mu) \in C\). We define a functional of two variables as follows

\[ J(\tilde{k}, \tilde{\mu}) := J_1(\tilde{k}, \tilde{\mu}) + J_2(\tilde{k}, \tilde{\mu}), \]

where

\[
J_1(\tilde{k}, \tilde{\mu}) = \int_0^T \left( Q[\tilde{k}, \tilde{\mu}](t) - \overline{Q}(t) \right)^2 dt,
\]

\[
J_2(\tilde{k}, \tilde{\mu}) = \int_0^T \int_1^2 \sum_{i=1}^2 (v_i[\tilde{k}, \tilde{\mu}](x, t) - \overline{\tau_i}(x, t))^2 dx dt.
\]

Since we know from (6) that \(Q[\tilde{k}, \tilde{\mu}]\) and \(v_i[\tilde{k}, \tilde{\mu}]\) belong to \((H^1(0, T))^6\) and \(H^1((0, T); H^4(0, L))\), respectively, for \((\tilde{k}, \tilde{\mu}) \in C\), the cost functional \(J\) is well-defined. Moreover, the functional \(J(\tilde{k}, \tilde{\mu})\) attains its global minimum when \(\tilde{k} = k\) and \(\tilde{\mu} = \mu\) in \([0, L]\). Therefore, one reasonably expects to recover information on the unknown coefficients \(k\) and \(\mu\) on the interval \([0, L]\) by minimizing \(J\) on the set \(C\).

### 2.2. An iterated method and numerical algorithm

In this section we shall introduce a numerical method for finding a minimum of the cost functional \(J\) on \(C\), see [5]. In order to employ this minimization algorithm, we need to know the concrete expression of the gradient \((\partial_k J(\tilde{k}, \tilde{\mu}), \partial_\mu J(\tilde{k}, \tilde{\mu}))^T\) of \(J\), whose components are defined as

\[
\frac{d}{dx} J(k + \epsilon \delta k, \mu) \bigg|_{\epsilon = 0} = < \partial_k J(\tilde{k}, \tilde{\mu}), \delta k >, \quad \frac{d}{d\gamma} J(k, \mu + \gamma \delta \mu) \bigg|_{\gamma = 0} = < \partial_\mu J(\tilde{k}, \tilde{\mu}), \delta \mu >. \tag{7}
\]

Under the above assumptions on the boundary data and coefficients, the Gateaux partial derivatives of \(J\) exist and it is possible to obtain their concrete expressions. Let us define the two bilinear forms

\[
a_k^{(1)}(\tilde{w}, m) = \int_0^L k(\tilde{w}_2 - \tilde{u}_1 + \tilde{v}_2, e_s)(g_2 - g_1 + h_2, e_s) \, dx
\]

\[
+ \int_0^L \frac{k_\mu e^2}{6}(2\tilde{v}_1, h_1, x + 2\tilde{v}_2, h_2, x + \tilde{v}_2, h_1, x + \tilde{v}_1, h_2, x) \, dx,
\tag{8}
\]

\[
a_k^{(2)}(\tilde{w}, m) = \int_0^L \mu(\tilde{v}_1 - \tilde{v}_2)(h_1 - h_2) \, dx
\tag{9}
\]

where \(\tilde{w} = (\tilde{u}_1, \tilde{u}_2, \tilde{v}_1, \tilde{v}_2)^T\) and \(m = (g_1, g_2, h_1, h_2)^T\). Then, we have the following theorem.

**Theorem 1.** Let the coefficients of problem (2) satisfy (3), (4) and let the assumptions (5) on the boundary data be satisfied.

Let \(\tilde{k}, \tilde{\mu}, \delta k, \delta \mu \in C\). For any \(\epsilon, \gamma \in \mathbb{R}_+ := [0, \infty)\) such that \(\sqrt{\epsilon^2 + \gamma^2} \to 0\) we have

\[
J(\tilde{k} + \epsilon \delta k, \tilde{\mu} + \gamma \delta \mu) - J(\tilde{k}, \tilde{\mu}) = \epsilon \partial_k J(\tilde{k}, \tilde{\mu})\delta k + \gamma \partial_\mu J(\tilde{k}, \tilde{\mu})\delta \mu + o(\sqrt{\epsilon^2 + \gamma^2}), \tag{10}
\]

where \(\lim_{\epsilon \to 0^+} \frac{a(\epsilon)}{\epsilon} = 0\).

The Gateaux partial derivatives \(\partial_k J(\tilde{k}, \tilde{\mu}), \partial_\mu J(\tilde{k}, \tilde{\mu})\) of the functional \(J\), evaluated at the point \((\tilde{k}, \tilde{\mu})\) with respect to the first and the second variable respectively, are given by

\[
\partial_k J(\tilde{k}, \tilde{\mu})\delta k = \int_0^T a_k^{(1)}(w[k, \tilde{\mu}], V) \, dt
\]
where the bilinear forms $\alpha^{(1)}_{\delta k}(\cdot, \cdot)$ and $\alpha^{(2)}_{\delta \mu}(\cdot, \cdot)$ are defined in (8) and (9), respectively.

Here, $W \in (H^2(0, L))^2 \times (H^1(0, L))^2$ is the strong solution of the elliptic problem

\[
\begin{align*}
A_{\mu \tilde{k}} W &= 0 \quad x \in (0, L), \\
D W |_{x=0} &= 2Q[k, \tilde{\mu}](0, T) - \overline{\partial \Omega}(T), \\
D W |_{x=L} &= 0 \quad t \in (0, T).
\end{align*}
\]

The function $V \in (C^0([0, T]; H^1(0, L))^2 \times (C^0([0, T]; H^2(0, L))^2)$, with $V_{t, t} \in (C^0([0, T]; L^2(0, L))^2 \times (C^0([0, T]; L^2(0, L))^2)$, is the weak solution of the abstract hyperbolic problem

\[
\begin{align*}
CV_{tt} - A_{\mu \tilde{k}} V &= 0 \quad (x, t) \in (0, L) \times (0, T), \\
V|_{t=T} &= W, \quad V|_{t=T} = 0 \quad x \in (0, L), \\
D V |_{x=0} &= 2Q[k, \tilde{\mu}](0, t) - \overline{\partial \Omega}(t) \quad t \in (0, T), \\
D V |_{x=L} &= 0 \quad t \in (0, T).
\end{align*}
\]

The function $U^{(k)} := U^{(k)}_{\delta k} \in (C^0([0, T]; H^1_0(0, L))^2 \times (C^0([0, T]; H^2_0(0, L))^2)$, with $U^{(k)}_{t, t} \in (C^0([0, T]; L^2(0, L))^2 \times (C^0([0, T]; L^2(0, L))^2)$, is the weak solution of the abstract hyperbolic problem

\[
\begin{align*}
CU^{(k)}_{tt} - A_{k, \tilde{\mu}} U^{(k)} &= F(\delta k)w[k, \tilde{\mu}] \quad (x, t) \in (0, L) \times (0, T), \\
U^{(k)} |_{t=0} &= 0, \quad U^{(k)}_{t} |_{t=0} = 0 \quad x \in (0, L), \\
D U^{(k)} |_{x=0} &= 0, \quad D U^{(k)} |_{x=L} = 0 \quad t \in (0, T).
\end{align*}
\]

The function $U^{(\mu)} := U^{(\mu)}_{\delta \mu} \in (C^0([0, T]; H^1_0(0, L))^2 \times (C^0([0, T]; H^2_0(0, L))^2)$, with $U^{(\mu)}_{t, t} \in (C^0([0, T]; L^2(0, L))^2 \times (C^0([0, T]; L^2(0, L))^2)$, is the weak solution of the abstract hyperbolic problem

\[
\begin{align*}
CU^{(\mu)}_{tt} - A_{k, \tilde{\mu}} U^{(\mu)} &= G(\delta \mu)w[k, \tilde{\mu}] \quad (x, t) \in (0, L) \times (0, T), \\
U^{(\mu)} |_{t=0} &= 0, \quad U^{(\mu)}_{t} |_{t=0} = 0 \quad x \in (0, L), \\
D U^{(\mu)} |_{x=0} &= 0, \quad D U^{(\mu)} |_{x=L} = 0 \quad t \in (0, T).
\end{align*}
\]

The function $Z \in (C^0([0, T]; H^1_0(0, L))^2 \times (C^0([0, T]; H^2_0(0, L))^2$, with $Z_{t, t} \in (C^0([0, T]; L^2(0, L))^2 \times (C^0([0, T]; L^2(0, L))^2)$, is the weak solution of the abstract hyperbolic problem

\[
\begin{align*}
CZ_{tt} - A_{k, \tilde{\mu}} Z &= 2(0, 0, (v_1[k, \tilde{\mu}] - \overline{v_1}) \chi_I, (v_2[k, \tilde{\mu}] - \overline{v_2}) \chi_I)^T, \\
Z |_{t=T} &= 0, \quad Z_{t} |_{t=T} = 0 \quad x \in (0, L), \\
D Z |_{x=0} &= 0, \quad D Z |_{x=L} = 0 \quad t \in (0, T),
\end{align*}
\]

where $\chi_I$ is the characteristic function of the interval $I$, i.e. $\chi_I(x) = 1$ for $x \in I$ and $\chi_I(x) = 0$ for $x \in [0, L] \setminus I$.
The operators \(F(\delta k)\) and \(G(\delta \mu)\) appearing on the right hand side of (15) and (16), respectively, are defined as follows:

\[
F(\delta k)w[\tilde{k}, \tilde{\mu}] = \begin{pmatrix}
\delta k(u_2 - u_1 + v_{2,x}e_s) \\
-\delta k(u_2 - u_1 + v_{2,x}e_s) \\
\frac{\delta k e^2}{6} (2v_{1,x} + v_{2,x})_x \\
\frac{\delta k e^2}{6} (2v_{2,x} + v_{1,x})_x + (\delta k(u_2 - u_1 + v_{2,x}e_s))_x
\end{pmatrix},
\]

\[
G(\delta \mu)w[\tilde{k}, \tilde{\mu}] = \begin{pmatrix}
0 \\
0 \\
-\delta \mu(v_1 - v_2) \\
\delta \mu(v_1 - v_2)
\end{pmatrix}.
\]

For a proof of the Theorem 1 we refer to [10]. From (7) and (10), we have

\[
\partial_1 J(\tilde{k}, \tilde{\mu}) \delta k = < \partial_{k}J(\tilde{k}, \tilde{\mu}), \delta k > ,
\]

\[
\partial_2 J(\tilde{k}, \tilde{\mu}) \delta \mu = < \partial_{\mu}J(\tilde{k}, \tilde{\mu}), \delta \mu > ,
\]

where the expressions on the left hand side of (18) and (19) have been determined in Theorem 1, see equations (11) and (12). By integrating by parts the third term of (11), we have

\[
\int_0^T < Z, F(\delta k)w[\tilde{k}, \tilde{\mu}] > dt = -\int_0^T \alpha^{(1)}_{\delta k}(w[\tilde{k}, \tilde{\mu}], Z) dt.
\]

(20)

Concerning the second term of (11), by the boundedness of the linear functional \(C^1[0, L] \ni K \rightarrow C\partial_k U_{\delta k}^{(k)}(T), W \in \mathbb{R}\) and the Sobolev imbedding theorem, we can write

\[
< d_k, \delta k > = < C\partial_k U_{\delta k}^{(k)}(T), W >
\]

(21)

with some \(d_k \in H^2(0, L)^* = H_0^{-2}(0, L)\). Therefore, by inserting (20) and (21) in (11), we have

\[
\partial_1 J(\tilde{k}, \tilde{\mu}) \delta k = \int_0^T \alpha^{(1)}_{\delta k}(w[\tilde{k}, \tilde{\mu}], V - Z) dt + < d_k, \delta k >
\]

(22)

and we can get the expression of \(\partial_k J(\tilde{k}, \tilde{\mu})\) as follows

\[
\partial_k J(\tilde{k}, \tilde{\mu}) = \int_0^T (u_2 - u_1 + v_{2,x}e_s) \{ (p_2 - p_1 + q_{2,x}e_s) - (z_2 - z_1 + Z_{2,x}e_s) \} dt
\]

\[
+ \int_0^T \frac{e^2}{6} \{ (2v_{1,x} + v_{2,x})(q_{1,x} - Z_{1,x}) + (2v_{2,x} + v_{1,x})(q_{2,x} - Z_{2,x}) \} dt
\]

(23)

where \(V = (p_1, p_2, q_1, q_2)^T\) and \(Z = (z_1, z_2, Z_1, Z_2)^T\).

By proceeding in a similar way for the partial derivative of \(J\) with respect to the variable \(\mu\), we have

\[
\partial_\mu J(\tilde{k}, \tilde{\mu}) = \int_0^T (v_1 - v_2) \{ (q_1 - q_2) - (Z_1 - Z_2) \} dt + d_{\mu},
\]

(24)

where \(d_{\mu} \in H^1(0, L)^* = H_0^{-1}(0, L)\) is obtained by using again the boundedness of the linear functional \(C^0[0, L] \ni M \rightarrow C\partial_k U_{\delta k}^{(\mu)}(T), W \in \mathbb{R}\) and the Sobolev imbedding theorem.
Since $L^2(0, L)$ is dense both in $H^{-1}_0(0, L)$ and in $H^{-2}_0(0, L)$, we can approximate $d_\mu$ and $d_k$ by $L^2$ functions. Therefore, we have

\[
< d_k, K > = \lim_{i'_{2h_k} \rightarrow d_k} (h_k, i_2 K)_{L^2} \quad \forall \ K \in H^2(0, L),
\]

\[
< d_\mu, M > = \lim_{i'_{h_\mu} \rightarrow d_\mu} (h_\mu, i_1 M)_{L^2} \quad \forall \ M \in H^1(0, L),
\]

where $i_j : H^j(0, L) \rightarrow L^2(0, L)$ and $i'_j : L^2(0, L) \rightarrow H^{-j}_0(0, L)$ are continuous, injective embeddings, $j = 1, 2$. Hence, in our numerical procedure, we try to find the approximations $\hat{d}_k$ and $\hat{d}_\mu$ by solving the weak equations

\[
\int_0^L K \hat{d}_k \, dx = < C \partial_t U^{(k)}_K (T), W >, \tag{25}
\]

\[
\int_0^L M \hat{d}_\mu \, dx = < C \partial_t U^{(\mu)}_M (T), W >, \tag{26}
\]

on the finite dimensional subspaces of $H^2(0, L)$ and $H^1(0, L)$, respectively. By using these approximations, we define the partial derivatives $\partial_k J(\tilde{k}, \tilde{\mu})$ and $\partial_\mu J(\tilde{k}, \tilde{\mu})$ such that

\[
\partial_k J(\tilde{k}, \tilde{\mu}) = \int_0^T \left( u_2 - u_1 + v_{2, x} e_s \right) \left( (p_2 - p_1 + q_{2, x} e_s) - (z_2 - z_1 + Z_{2, x} e_s) \right) \, dt
\]

\[
+ \int_0^T \frac{e_x^2}{6} \left( (2 v_{1, x} + v_{2, x}) (q_{1, x} - Z_{1, x}) + (2 v_{2, x} + v_{1, x}) (q_{2, x} - Z_{2, x}) \right) \, dt \tag{27}
\]

\[
+ \hat{d}_k,
\]

\[
\partial_\mu J(\tilde{k}, \tilde{\mu}) = \int_0^T \left( v_1 - v_2 \right) \left( (q_1 - q_2) - (Z_1 - Z_2) \right) \, dt + \hat{d}_\mu. \tag{28}
\]

Then, $\partial_k J(\tilde{k}, \tilde{\mu})$ and $\partial_\mu J(\tilde{k}, \tilde{\mu})$ belong to $H^1(0, L)$, namely, these approximations are continuous functions on $[0, L]$.

Now, to implement the projected gradient method, we need to introduce a projection operator. Throughout this part, beside conditions (4), we assume that the coefficients $\tilde{k}$ and $\tilde{\mu}$ are bounded from above, i.e.

\[
0 \leq \tilde{k}(x) \leq \overline{k}, \quad 0 \leq \tilde{\mu}(x) \leq \overline{\mu} \quad \text{for } x \in [0, L], \tag{29}
\]

where $\overline{k}$ and $\overline{\mu}$ are given positive constants. Moreover, we define a convex subset $C_{\overline{k}, \overline{\mu}}$ of $C$ as follows

\[
C_{\overline{k}, \overline{\mu}} = \left\{ (k, \mu) \in C \mid 0 \leq \tilde{k}(x) \leq \overline{k}, \quad 0 \leq \tilde{\mu}(x) \leq \overline{\mu} \text{ for } x \in [0, L] \right\}. \]

Let $CL_k$ and $CL_\mu$ be the clip-off operators such that

\[
(C_{L_k} \tilde{k})(x) = \begin{cases} 0 & \tilde{k}(x) < 0, \\ \tilde{k}(x) & 0 \leq \tilde{k}(x) \leq \overline{k}, \\ \overline{k} & \tilde{k}(x) > \overline{k}, \end{cases} \quad (C_{L_\mu} \tilde{\mu})(x) = \begin{cases} 0 & \tilde{\mu}(x) < 0, \\ \tilde{\mu}(x) & 0 \leq \tilde{\mu}(x) \leq \overline{\mu}, \\ \overline{\mu} & \tilde{\mu}(x) > \overline{\mu}. \end{cases}
\]

Since $(\tilde{k}, \tilde{\mu}) \in (C^0[0, L])^2$, we know that $CL_k \tilde{k}$ and $CL_\mu \tilde{\mu}$ satisfy the constrained condition (29). In order to keep the smoothness of the coefficients, we employ Friedrichs's mollifier operators
$S_{k,\varepsilon} : C^0[0, L] \to C^1[0, L]$ and $S_{\mu,\varepsilon} : C^0[0, L] \to C^0[0, L]$ defined as follows

$$
(S_{k,\varepsilon} f) = \int_{\mathbb{R}} s_{k,\varepsilon}(y - x) f(y) \, dy,
$$

$$
(S_{\mu,\varepsilon} g) = \int_{\mathbb{R}} s_{\mu,\varepsilon}(y - x) g(y) \, dy,
$$

for $f \in C^0[0, L]$ and $g \in C^0[0, L]$, where $s_{k,\varepsilon}(x) = s_k(x/\varepsilon_k)/\varepsilon_k$ ($\varepsilon_k > 0$) and $s_{\mu,\varepsilon}(x) = s_{\mu}(x/\varepsilon_{\mu})/\varepsilon_{\mu}$ ($\varepsilon_{\mu} > 0$). The functions $s_k$ and $s_{\mu}$ are the nonnegative $C^1$ and $C^0$ functions over $\mathbb{R}$, respectively, such that

$$
supp s_k = [-c_k, c_k], \quad \int_{\mathbb{R}} s_k(x) \, dx = 1,$$

$$
supp s_{\mu} = [-c_{\mu}, c_{\mu}], \quad \int_{\mathbb{R}} s_{\mu}(x) \, dx = 1,$$

where $c_k$ and $c_{\mu}$ are given positive constants.

Moreover, we shall introduce the operators $P_{k,\varepsilon} = S_{k,\varepsilon} \circ CL_k$ and $P_{\mu,\varepsilon} = S_{\mu,\varepsilon} \circ CL_{\mu}$, such that $(P_{k,\varepsilon} f, P_{\mu,\varepsilon} g) \in C_{k,\varepsilon}$ if $(f, g) \in (C^0[0, L])^2$. Finally, the iterative descent procedure is defined as follows

$$
(k_{l+1}, \mu_{l+1}) = (k_l, \mu_l) - \alpha_l \left( s^{(k)}_{l+1} / s^{(\mu)}_{l+1} \right),
$$

for $l = 0, 1, 2, \ldots$, where

$$
s^{(k)}_l = k_l - P_{k,\varepsilon}\left(k_l - \frac{\partial_k J(k_l, \mu_l)}{\|\partial_k J(k_l, \mu_l)\|}\right),
$$

$$
s^{(\mu)}_l = \mu_l - P_{\mu,\varepsilon}\left(\mu_l - \frac{\partial_\mu J(k_l, \mu_l)}{\|\partial_\mu J(k_l, \mu_l)\|}\right).
$$

Here, $\alpha_l$ is suitable step size which satisfy $0 < \alpha_l \leq 1$.

From the above definition of the minimization procedure and the convexity of subset $C_{k,\varepsilon}$, all updated coefficients belong to $C_{k,\varepsilon}$ whenever the initial guess $(k_0, \mu_0) \in C_{k,\varepsilon}$. Moreover, if the kernel functions are smooth, for example $s_k, s_{\mu} \in C^2[0, L]$, then the updated coefficients can be estimated as follows:

$$
\|k_l\|_{C^2} \leq k \left(1 + \frac{2c_k}{\varepsilon_k} \|s_k^I\|_{C^0} + \frac{2c_k}{\varepsilon_k} \|s_k''\|_{C^0}\right), \quad \|\mu_l\|_{C^2} \leq \mu \left(1 + \frac{2c_\mu}{\varepsilon_\mu} \|s_{\mu}^I\|_{C^0} + \frac{2c_\mu}{\varepsilon_\mu^2} \|s_{\mu}''\|_{C^0}\right)
$$

for $l \geq 1$. Therefore, we can expect that the estimation of the unknown coefficients $k, \mu$ strictly depends on the values of the parameters $\varepsilon_k$ and $\varepsilon_{\mu}$.

We can summarize the algorithm of our variational approach to the inverse problem for composite beams as follows.
Numerical algorithm for coefficient identification

(i) Set an initial guess \((\tilde{k}_0, \tilde{\mu}_0) \in \mathbb{C}_{E,P}\).

(ii) For \(l = 0, 1, 2, \ldots:\)

(a) Solve the hyperbolic problem (2) with \((k, \mu) = (\tilde{k}_l, \tilde{\mu}_l)\) to find \(w, v_{i,x}, i = 1, 2, \) and \(Q\).

(b) Solve the elliptic problem (13) with \((k_l, \tilde{\mu}_l)\) to get \(W\).

(c) Solve the hyperbolic problem (14) with \((k_l, \tilde{\mu}_l)\) to find \(V\) and \(q_{i,x}, i = 1, 2\).

(d) Solve the hyperbolic problem (17) to find \(Z\) and \(Z_{i,z}, i = 1, 2\).

(e) Solve the weak equations (25) and (26) to obtain the functions \(\hat{d}_k\) and \(\hat{d}_\mu\).

(f) Calculate the approximated derivatives \(\hat{\partial}_k J(\tilde{k}_l, \tilde{\mu}_l)\) and \(\hat{\partial}_\mu J(\tilde{k}_l, \tilde{\mu}_l)\) by (27) and (28), respectively.

(g) Set the search direction \(s_l^{(k)}\) and \(s_l^{(\mu)}\) by (31) and (32), respectively.

(h) Get the step size \(\alpha_l\) by using the line search algorithm.

(i) Update the coefficients by (30).

(j) If the updated coefficients satisfy the condition

\[
\left| \frac{J(\tilde{k}_{l+1}, \tilde{\mu}_{l+1}) - J(\tilde{k}_l, \tilde{\mu}_l)}{J(\tilde{k}_{l+1}, \tilde{\mu}_{l+1})} \right| < \varepsilon,
\]

for a small given control parameter \(\varepsilon\), then stop the iterations.

3. Numerical experiments

In this section we shall present some numerical results obtained by applying our approach to the identification of the stiffness coefficients \(k\) and \(\mu\) of a composite beam. In the following applications we shall refer to the specimen T1PR of steel-concrete beam analyzed in the paper [6]. In particular, the case of damaged composite beam having initially uniform elastic and inertial properties will be closely examined. This case is simple but rather meaningful for applications.

3.1. Example and calculation settings

The physical constants of the reference configuration of the undamaged beam are summarized in Table 1. Here "\(d\)" means the physical dimension.

To implement the identification algorithm, it is useful to introduce non-dimensional values for the geometrical and mechanical quantities, that is

\[
L = 1.0, \quad \rho_i = \frac{\rho_i^d}{\rho_i^d}, \quad a_i = \frac{E_i^d A_i^d}{\eta_i^d (\eta_i^d)^2}, \quad j_i = \frac{E_i^d r_i^d}{\eta_i^d (\eta_i^d)^2 (L_i^d)^2},
\]

\[
e_s = \frac{e_s^d}{L_i^d}, \quad e_c = \frac{e_c^d}{L_i^d}, \quad T = \frac{\eta_i^d}{L_i^d} L_i^d,
\]

for \(i = 1, 2\), where

\[
\rho_i^d = \rho_i^d + \rho_i^d \text{ [kg m}^{-1}], \quad \eta_i^d = \frac{E_i^d A_i^d}{\rho_i^d} \text{ [m s}^{-1}], \quad \eta_i^d = \frac{\eta_i^d + \eta_i^d}{2} \text{ [m s}^{-1}],
\]

where \(\eta_i^d\) is the velocity of the longitudinal elastic waves propagating through the \(i\)th beam.

We assume that our target coefficients \(k\) and \(\mu\) for the damaged configuration of the composite beam are defined by

\[
k(x) = \frac{n_p L_i^d K(L_i^d x)}{\eta_i^d (\eta_i^d)^2}, \quad \mu(x) = \frac{n_p L_i^d E_i(L_i^d x) A_i(L_i^d x)}{e_c^d \eta_i^d (\eta_i^d)^2}
\]

(33)
Table 1. Physical constants of the steel-concrete beam T1PR [6].

| Physical constants | Value                           |
|--------------------|---------------------------------|
| $L^d$              | 3.50 m                          |
| $A_1^d$            | $3.00 \times 10^{-2}$ m$^2$     |
| $I_1^d$            | $9.00 \times 10^{-6}$ m$^4$     |
| $\rho_1^d$        | 73.19 kg m$^{-1}$               |
| $E_1^d$            | $4.2292 \times 10^{10}$ N m$^{-2}$ |
| $A_2^d$            | $1.64 \times 10^{-3}$ m$^2$     |
| $I_2^d$            | $5.41 \times 10^{-6}$ m$^4$     |
| $\rho_2^d$        | 12.90 kg m$^{-1}$               |
| $E_2^d$            | $2.1 \times 10^{11}$ N m$^{-2}$ |
| $e_s^d$            | 0.07 m                          |
| $e_c^d$            | 0.03 m                          |

for $0 \leq x \leq L$ (see Figure 2), where

$$K(x^d) = \begin{cases} 
1.77525 + 0.59175 \cos 4\pi (x^d - 1.5) \times 10^8 \text{[N m}^{-1}] & (1.5 < x^d < 2.0) \\
2.36700 \times 10^8 \text{[N m}^{-1}] & (\text{otherwise})
\end{cases}$$

$$\frac{E_c(x^d)A_c(x^d)}{e_c^d} = \begin{cases} 
6.44250 + 2.14750 \cos 4\pi (x^d - 1.5) \times 10^8 \text{[N m}^{-1}] & (1.5 < x^d < 2.0) \\
8.59000 \times 10^8 \text{[N m}^{-1}] & (\text{otherwise})
\end{cases}$$

for $0 \leq x^d \leq L^d$, and where $n_p = 16$ is the number of connectors of the beam, see [6]. The present case corresponds to a composite beam with a localized damage at mid-span. Actually, it has been shown that real damages in the connection of a steel-concrete beam can be described as a rather abrupt reduction of the stiffness coefficients $k$ and $\mu$ in the damaged region [6].

Concerning the boundary data, in our experiments both the longitudinal displacements $u_i(0, t)$ and the rotations $v_{i,x}(0, t)$ of the left end of the beam, $i = 1, 2$, are assumed to be fixed, whereas the transversal displacements $v_i(x = 0, t)$, $i = 1, 2$, are assigned at the same cross-section, see Figure 1. Therefore, the vector $\overline{U}(t)$ of the Dirichlet boundary data is given by

$$\overline{U}(t) = \left(0, 0, \frac{C_v B_5(t)}{L^d}, \frac{C_v B_5(t)}{L^d}, 0, 0\right)^T,$$

where $C_v = 1.5 \times 10^{-2}$ [m]. Here $B_5$ is a 5th order normalized B-spline function such that

$$B_5(t) = (\tilde{t}_5 - \tilde{t}_0) \sum_{j=0}^5 \frac{(\tilde{t}_j - t)_+}{\omega_5^j(\tilde{t}_j)},$$

with $\omega_5^j(\tilde{t}_j)$ being the $j$th coordinate of the control net $\{\tilde{t}_j\}_0^5$. The total damage $B(t)$ is then given by

$$B(t) = B_5(t) \sum_{i=1}^2 \sum_{j=0}^5 \frac{A_i^d A_j^d}{a_i^d a_j^d},$$

where $A_i^d$, $I_i^d$, $\rho_i^d$, and $E_i^d$ are the physical constants of the $i$th layer, and $a_i^d$ is the cross-sectional area of the $i$th layer. The function $F_i(t)$ is defined as

$$F_i(t) = F_i^d(t) \sum_{i=1}^2 \sum_{j=0}^5 \frac{A_i^d A_j^d}{a_i^d a_j^d},$$

where $F_i^d(t)$ is the strain energy density of the $i$th layer. The total strain energy $\mathcal{E}(t)$ is then given by

$$\mathcal{E}(t) = \mathcal{E}_d(t) \sum_{i=1}^2 \sum_{j=0}^5 \frac{A_i^d A_j^d}{a_i^d a_j^d},$$

where $\mathcal{E}_d(t)$ is the strain energy density of the damaged region.
where \( \tilde{t}_j = \eta^d (8.0 j \times 10^{-5}) / L^d \) \((j = 0, 1, ..., 5)\), and

\[
t_+ = \begin{cases} 
    t & (t \geq 0) \\
    0 & (t < 0)
\end{cases}, \quad \omega_5(t) = \prod_{j=0}^{5} (t - \tilde{t}_j).
\]

Then, \( B_5 \in C^3(\mathbb{R}) \) and \( B_5 \equiv 0 \) on \((-\infty, \tilde{t}_0)\) and \((\tilde{t}_5, +\infty)\), see [11]. Therefore, the coefficients and the Dirichlet boundary data satisfy the assumptions of Theorem 1.

The Neumann boundary data and the interior measured data are obtained by solving numerically the hyperbolic problem (2) for the damaged beam with \( U \) given as above. To solve numerically the direct problem, we make use of the Newmark method for time integration, see, for example, [2], and we introduce linear spline functions and cubic Hermite functions for approximating \( u_i \) and \( v_i \) in space, respectively, \( i = 1, 2 \), see [7]. Here, the intervals \([0, L]\) and \([0, T]\) are divided into 1200 and 7200 equally spaced sub-intervals, respectively. We use the numerical integration to get the mass and the stiffness matrices in this calculation, and then, we only use the values of coefficient functions at the quadrature points on each element. We denote by \( \hat{Q} = (\hat{N}_1, \hat{N}_2, \hat{T}_1, \hat{T}_2, \hat{M}_1, \hat{M}_2) \) and \( \hat{v}_i, i = 1, 2 \), the calculated Neumann data at \( x = 0 \) and the transversal displacements of the beams, respectively.

In order to solve numerically the initial-boundary value problems in our inverse analysis, we also use the Newmark method for time integration with linear elements for approximating \( u_i \) and cubic Hermite elements for approximating \( v_i \) in space. The intervals \([0, L]\) and \([0, T]\) are divided coarsely into 600 and 3600 equally spaced, respectively. The values of updated coefficients on the quadrature points are employed to determine the stiffness matrix at each step. Then, these quadrature points do not coincide with the points for generating the measured data. The second terms of the search directions are interpolated by the piecewise linear function before clipping and mollifying. We make use of the vertices and the quadrature points on each element to interpolate the second terms. To do approximately the mollifying, we use the Gauss-Legendre formula. Therefore, our target coefficients are identified approximately on a subspace of function space for calculating \( \{\hat{Q}, \hat{v}_1, \hat{v}_2\} \). Here we notice that the updated coefficients depend on the finite elements for the inverse analysis, because we employ the interpolation technique with the vertices and the quadrature points on the elements. On the other hand, the exact coefficients do not depend on the finite elements for calculating the measured data, because we only use its values.

The upper bounds for the unknown coefficients appearing in (29) are supposed to coincide with the undamaged values \( K \equiv 2.3670 \times 10^8 \text{ Nm}^{-1} \) and \( E_c A_c / c_d^e \equiv 8.5900 \times 10^8 \text{ Nm}^{-1} \). The
kernel functions for the mollifier are chosen as the normalized cubic B-spline function such that

$$s_k(x) = s_\mu(x) = 4 \sum_{j=0}^{4} \frac{(j - 2 - x)^3}{\omega_4'(j - 2)}, \quad \omega_4(x) = \prod_{j=0}^{4} (x - j + 2).$$

Then, we know that $\text{supp} \ s_k = \text{supp} \ s_\mu = [-2, 2]$ and $s_k, s_\mu \in C^2(\mathbb{R})$, see [11]. Moreover, we fix the parameter of the convergence criterion as $\varepsilon = 1.0 \times 10^{-4}$. Finally, we employ the Armijo criterion [1] to get the step size $\alpha_l$ at every iteration.

### 3.2. Sensitivity of the identification to measurement errors

In a first series of experiments, we show the effectiveness of our algorithm when the measured data are affected by some error. We assume that the interior data $v_i, i = 1, 2$, are measured in the whole interval $I = (0, L)$ and the mollifier parameters are given as $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-2}$. The length of observation time is set as $T_d = 4.0 \times 10^{-3}$ s. Moreover, we assume that both the Neumann and the interior data include the Gaussian random error, namely, $\overline{Q}(t) = \overline{Q}(t) + Q^\delta(t)$ and $\overline{u}_i(x, t) = \hat{v}_i(x, t) + v_i^\delta(x, t)$ for $(x, t) \in [0, L] \times (0, T), i = 1, 2$. Here, the functions $Q^\delta(t)$ and $v_i^\delta(x, t)$ depend on the Gaussian distribution with the mean value 0 and the variance $(\delta \cdot \text{maximum value of each data})/100)^2$. The initial guess in the minimization procedure is set as $k_0 \equiv k$ and $\mu_0 \equiv \mu$ in $[0, L]$. For errors with $\delta = 1.0\%$ the convergence criterion is satisfied after 65 steps and the iteration stops. Figures 3 (a,b) show the graph of identified coefficients $\tilde{k}_{65}$ and $\tilde{\mu}_{65}$, respectively, and a good agreement with the exact values can be observed. These results seem to suggest that the minimization algorithm is effective for small measurements errors.

![Figure 3](image.png)

**Figure 3.** Calculated coefficients for $I = [0, L]$ with 1.0% measurement error.

The identified coefficients $\tilde{k}$ and $\tilde{\mu}$ obtained for errors $\delta = 5.0\%$ and $\delta = 10.0\%$ are shown in Figures 4 (a,b) and 5 (a,b), respectively. It can be seen that the accuracy of the identified coefficients becomes worse when the error level increases. However, these results can be still considered acceptable for a practical localization of damages in the connection of composite beams.
3.3. The effect of different choices of the mollifier parameters

In the above experiments, the kernel functions were chosen as cubic B-spline function. Therefore, from our previous discussion, we can expect that the choice of the parameters $\epsilon_k$ and $\epsilon_\mu$ has an influence on the stability of the identification with respect to measurement errors. To check this point, a series of experiments was carried out by changing the values of these parameters. We suppose that the measured data have 5.0% Gaussian random error and we take $T^d = 4.0 \times 10^{-3}$ s. The initial guess is set as $\tilde{k}_0 \equiv \overline{k}$, $\tilde{\mu}_0 \equiv \overline{\mu}$ and the interior measured data are supposed to be given on the whole interval $I = (0, L)$. Figures 6, 7 and 8 show the identified results with $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-3}$, $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-4}$, and $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-8}$, respectively. From these results it emerges that the calculated coefficients are rather unstable when the parameters $\epsilon_k$ and $\epsilon_\mu$ decrease too much. In particular, the accuracy of the identification of the coefficient $\tilde{k}$ is less accurate, especially near at $x = 0$. Although the choice $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-2}$ seems to guarantee a stable identification procedure, further work must be done in order to find an optimal choice of the mollifier parameters.

3.4. Experiments concerning the choice of the initial guess

The proposed minimization procedure is based on a gradient type method and, therefore, it is well known that the choice of the initial guess is important for the convergence process. The
Figure 6. Calculated coefficients for $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-3}$ with 5.0% measurement error.

Figure 7. Calculated coefficients for $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-4}$ with 5.0% measurement error.

Figure 8. Calculated coefficients for $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-8}$ with 5.0% measurement error.

Influence of the initial guess throughout the numerical procedure has been evaluated in a series of numerical experiments by considering the error level on the measured data equal to 5.0%, the mollifier parameters set as $\epsilon_k = \epsilon_\mu = 1.0 \times 10^{-2}$ and the length of the observation time equal to $4.0 \times 10^{-3}$ s. Several choices of the initial guess have been considered. When, for example, the
starting coefficients are of parabolic type

\[ \tilde{k}_0 = \bar{k} (1.0 + 2.4x(x - 1.0)) , \quad \tilde{\mu}_0 = \bar{\mu} (1.0 + 2.4x(x - 1.0)) , \quad (34) \]

the optimal coefficients \( \tilde{k}_{61} \) and \( \tilde{\mu}_{61} \) calculated at the 61st iteration are shown in Figures 9 (a,b). The accuracy of these identified results is worse near at \( x = 0 \) and within the damaged region, but discrepancies still are acceptable.

Figures 10 (a,b) show the identified coefficients \( \tilde{k}_{90} \) and \( \tilde{\mu}_{90} \) for the initial guess such that

\[ \tilde{k}_0 \equiv 0.4\bar{k} , \quad \tilde{\mu}_0 \equiv 0.4\bar{\mu} . \quad (35) \]

The accuracy of these results is not good near the boundary and the damaged region, but differences between computed and exact coefficients are again acceptable. Comparing the present results of the experiments with those obtained in Figure 4, we can not confirm big difference of the accuracy of the identification. However the number of iterations is not the same if the initial guess is changed. Therefore, we can conclude that the choice of the initial guess has an influence on the speed of convergence of the iterative procedure.

\[ \begin{align*}
\text{(a) Identified } \tilde{k}_{61} \\
\text{(b) Identified } \tilde{\mu}_{61}
\end{align*} \]

Figure 9. Calculated coefficients for the initial guess (34) with 5.0% measurement error.

\[ \begin{align*}
\text{(a) Identified } \tilde{k}_{90} \\
\text{(b) Identified } \tilde{\mu}_{90}
\end{align*} \]

Figure 10. Calculated coefficients for the initial guess (35) with 5.0% measurement error.
3.5. Experiments concerning the choice of the interval \( I \)

In several real applications, interior measurements are more difficult to make with respect to boundary measurements. Therefore, in order to discuss how the selection and the size of the interval \( I \) affect the results of identification, we have applied the variational algorithm for different choices of the interval \( I \). Here, in particular, we present the results of identification for \( I = (0.25, 0.75) \), \( I = (0, 0.25) \).

In the first case the interval \( I \) includes the damaged region of the beam, namely, \( I = (0.25, 0.75) \). The initial guess and the error level of measured data are chosen as \((\tilde{k}_0, \tilde{\mu}_0) = (\bar{k}, \bar{\mu})\) and \(\delta = 1.0\%\), respectively. We suppose \( T^d = 4.0 \times 10^{-3} \) s. The identified values obtained after 96 iterations are shown in Figures 11 (a,b). We can deduce that the accuracy of the identification is almost the same of the case in which the interior measurements were taken on the whole interval \((0, L)\).

![Image](a) Identified \( \tilde{k}_{96} \)

![Image](b) Identified \( \tilde{\mu}_{96} \)

Figure 11. Calculated coefficients for \( I = (0.25, 0.75) \) with 1.0\% measurement error.

In the second case, the interval \( I = (0, 0.25) \) does not include the damaged region of the beam. By proceeding as before, after 95 iterations we obtain the optimal coefficients shown in Figures 12 (a,b). From these results, we can not confirm the difference of the accuracy of the identification by choosing the interval \( I \).

![Image](a) Identified \( \tilde{k}_{95} \)

![Image](b) Identified \( \tilde{\mu}_{95} \)

Figure 12. Calculated coefficients for \( I = (0, 0.25) \) with 1.0\% measurement error.

In the above experiments the length of the observation time is such that \( T^d \cdot \min\{\bar{\eta}_1, \bar{\eta}_2\} > 4.0L^d \). This means that our boundary measured data have a potential to identify the coefficients...
(33) without inner measurements. In order to check this point, an experiment with $I = \emptyset$ has been carried out. The optimal solution obtained after 88 iterations is shown in Figures 13 (a,b) and the identified coefficients are in good agreement with the exact ones. This result suggests that the lack of interior measurements can be partially compensated by the choice of a sufficiently large observation time $T^d$.

In order to further investigate on the influence of the length of the observation time and the size of the interval $I$, we set $T^d = 2.0 \times 10^{-3}$ s and we consider two cases corresponding to $I = (0.25, 0.75)$ and $I = (0, 0.25)$. From the Figures 14 (a,b) one can see that the identification procedure is effective for a short observation time if the interval $I$ includes the damaged region. However, if the interval $I$ does not include the damaged region, from the Figures 15 (a,b), we can see that the identification of $\tilde{\mu}$ is failed. Moreover, in order to confirm the influence on the size of the interval, we show the results of identification for $I = (0.49, 0.51)$. The identified coefficients can be obtained after 75 iterations as shown in Figures 16 (a,b). Then, from the Figures 14 and 16, we can confirm that the accuracy of the identification becomes worse when the size of the interval $I$ decreases. Therefore, we can deduce that the selection of the interval is important for the case of short observation time in order to obtain high accuracy result.
4. Concluding remarks
This paper dealt with a numerical method for solving the inverse problem of estimating the stiffness coefficients of the connection of a steel-concrete composite beam by using dynamic data. The inverse problem is formulated as a constrained variational problem and an iterated method based on the projected gradient technique is proposed for minimizing the cost functional with two variables. Suitable maps combined clip-off and mollifier operators are introduced in order to describe the constrained conditions. The effectiveness of method and the sensitivity of the results to errors in the measured data are tested on the basis of an extensive series of numerical experiments.

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