KÄHLER MANIFOLDS WITH POSITIVE FIRST CHERN CLASS
AND MIRRORS OF RIGID CALABI–YAU MANIFOLDS

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Abstract
Recent work on the relation between a special class of Kähler manifolds with positive first Chern class and critical N=2 string vacua with c=9 is reviewed and extended.
1. Introduction

To answer the question whether mirror symmetry [1][2][3] is in fact a symmetry of string theory is nontrivial for two rather different reasons. In the absence of a universal mirror construction there are ‘technical’ difficulties, related to the explicit construction of mirror pairs. More importantly, there might exist a ‘fundamental obstruction’ which forbids the existence of mirrors for particular vacua from first principles and hence might lead to a restriction of the class of string vacua for which mirror symmetry is an allowed operation.

At first sight it appears that such a fundamental obstruction is furnished by a small, but prominent, class of vacua: toroidal orbifolds. Such manifolds lead to so-called rigid vacua, i.e. groundstates which do not have complex deformations. This means the following: Recall that because Calabi–Yau spaces are complex manifolds they admit a Hodge decomposition of their real de Rham cohomology groups. Therefore the Betti numbers $b_i = \dim_{\mathbb{R}} H^{DR}_i(M, \mathbb{R})$, $i = 0, 1, ..., \dim_{\mathbb{R}} M$, split into the Hodge numbers $h^{(p,q)} = \dim_{\mathbb{C}} H^{p,q}(M, \mathbb{C})$, $p, q = 0, ..., \dim_{\mathbb{C}} M$:

$$b_i(M) = \sum_{p+q=i} h^{(p,q)}(M).$$  \hspace{1cm} (1)

Because Calabi–Yau spaces are Kähler the Hodge numbers are symmetric, $h^{(p,q)} = h^{(q,p)}$, and because the first Chern class vanishes it follows that $h^{(p,0)} = 0 = h^{(0,p)}$ for $p = 1, 2$ and $h^{(3,0)} = 1 = h^{(0,3)}$. Hence the cohomology of the internal space consists of only two independent groups, the Kähler sector $H^{(1,1)} \equiv H^{(2,2)}$ and the complex deformation sector $H^{(2,1)} \equiv H^{(1,2)}$ which parametrize the number of antigenerations and generations, respectively, that are observed in low energy physics.

It has been observed in [1][2] in the framework of Landau–Ginzburg vacua and in [3] in the class of Gepner models, that the space of $(2,2)$–vacua features mirror symmetry, i.e. for critical string vacua $V_{\text{crit}}$ with Hodge numbers

$$V_{\text{crit}} : (\chi, h^{(1,1)}, h^{(2,1)})$$  \hspace{1cm} (2)

there exists a mirror partner in which the Hodge numbers are exchanged

$$\Lambda_{\text{crit}} : (-\chi, h^{(2,1)}, h^{(1,1)}),$$  \hspace{1cm} (3)

defining what is called a mirror spectrum of $V_{\text{crit}}$. An explicit construction relating different Landau–Ginzburg theories via a path integral argument that involves fractional transformations of the order parameters has been provided in [2].

Rigid Calabi–Yau vacua without $(2,1)$–cohomology then appear to furnish a fundamental obstruction to mirror symmetry: Since the mirror map exchanges complex deformations and Kähler
deformations of a manifold it would seem that the mirror of a rigid Calabi–Yau manifold cannot be Kähler and hence does not describe a consistent string vacuum. In fact, it appears, using Zumino’s result [4] that $N = 2$ supersymmetry of a $\sigma$–model requires the target manifold to be Kähler, that the mirror vacuum cannot even be $N = 1$ spacetime supersymmetric. It follows that the class of Calabi–Yau manifolds is not the appropriate setting for mirror symmetry and the question arises what the proper framework might be.

This review describes and extends recent work [5] which shows the existence of a new class of manifolds which generalizes the class of Calabi–Yau spaces of complex dimension $D_{\text{crit}}$ in a natural way. The manifolds involved are of complex dimension $(D_{\text{crit}} + 2(Q - 1))$, $Q \in \mathbb{N}$ and have a positive first Chern class which is quantized in multiples of the degree of the manifold. Thus they do not describe, a priori, consistent string groundstates. It was shown in [5] however, that it is possible to derive from these higher dimensional manifolds the spectrum of critical string vacua. This can be done not only for the generations but also for the antigenerations. For particular types of these new manifolds it is also possible to construct the corresponding $D_{\text{crit}}$–dimensional Calabi–Yau manifold directly from the $(D_{\text{crit}} + 2(Q - 1))$–dimensional space.

It should be emphasized that the noncritical manifolds described below do not just provide an alternative realization of the physical modes observed in four dimensions: even though the spectrum of the critical vacuum $V_{\text{crit}}$, parametrized by (generalized, if necessary) cohomology groups, is embedded in the Hodge diamond of the noncritical manifold $M_{D_{\text{crit}} + 2(Q - 1)}$

$$H^{(p, q)}(V_{D_{\text{crit}}}) \subset H^{(p + Q - 1, q + Q - 1)}(M_{D_{\text{crit}} + 2(Q - 1)}),$$

the cohomology groups of the higher dimensional manifold are generically larger than those of the critical groups and hence they contain additional modes, which at present do not have a physical interpretation. The main result of [5] is a geometrical construction which projects out the critical modes.

An important feature of the higher dimensional cohomology is a sort of ‘doubling phenomenon’: It can happen that more than one copy of the complete critical cohomology is embedded in the Hodge diamond of the noncritical manifold. In particular the Hodge–diamond for noncritical manifolds of odd complex dimension will, in general, lead to a middle Betti number of the form

$$b_{D_{\text{crit}} + 2(Q - 1)} = 2n + 2\sum_{i=1}^{D_{\text{crit}} - 1} h^{(D_{\text{crit}} + (Q - 1) - i, (Q - 1) + i)}$$

where $n$ is some integer larger than zero which can be larger than unity. The reason why this can happen is explained by the construction introduced in [5], as will become clear in Section 4.
This new class of manifolds is not in one to one correspondence with the class of Calabi–Yau manifolds as it also contains manifolds which describe string vacua which do not contain massless modes corresponding to antigenerations. It is precisely this new type of manifold that is needed in order to construct mirrors of rigid Calabi–Yau manifolds without generations.

2. Higher Dimensional Manifolds with Quantized Positive First Chern Class

The construction of noncritical manifolds proceeds via the following prescription [5]:

- Fix the central charge \( c \) of the \((2,2)\)–vacuum states and its critical dimension
  \[
  D_{\text{crit}} = c/3. \tag{6}
  \]

- Choose a positive integer \( Q \in \mathbb{N} \).

- Introduce \((D_{\text{crit}} + 2Q)\) complex coordinates \((z_1, \ldots, z_{D_{\text{crit}}+2Q})\), \(z_i \in \mathbb{C}\).

- Define an equivalence relation
  \[
  (z_1, \ldots, z_{D_{\text{crit}}+2Q}) \sim (\lambda^{k_1}z_1, \ldots, \lambda^{D_{\text{crit}}+2Q}z_{D_{\text{crit}}+2Q}) \tag{7}
  \]
  where \( \lambda \in \mathbb{C}^\ast \) is a nonzero complex number and the positive integers \( k_i \in \mathbb{N} \) are the weights of these coordinates. The set of these equivalence classes defines so–called weighted projective spaces, compact manifolds which are denoted by \( \mathbb{P}^{(k_1,\ldots,k_{D_{\text{crit}}+2Q})} \).

- Define hypersurfaces in the ambient weighted projective space by imposing a constraint defined by polynomials \( p \) of degree
  \[
  d = \frac{1}{Q} \sum_i k_i \tag{8}
  \]
  i.e. \( p(\lambda^iz_i) = \lambda^dp(z_i) \).

The family of hypersurfaces embedded in the ambient space as the zero locus of \( p \) will be denoted by

\[
M_{D_{\text{crit}}+2(Q-1)} = \{ p(z_1, \ldots, z_{D_{\text{crit}}+2Q}) = 0 \} \cap \mathbb{P}^{(k_1,\ldots,k_{D_{\text{crit}}+2Q})} = \mathbb{P}^{(k_1,\ldots,k_{D_{\text{crit}}+2Q})} \left[ \frac{1}{Q} \sum_{i=1}^{D_{\text{crit}}+2Q} k_i \right] \tag{9}
\]
and is called a configuration.

The constraint (8) is the essential defined property, which links the degree of the polynomial to the weights of the ambient space and is a rather restrictive condition in that it excludes many types of varieties which are transverse but are not of physical relevance \(^1\). A simple example is the Brieskorn–Pham type hypersurface

\[
\mathbb{P}_{(420,280,210,168,140,120,105)}[840] \ni \left\{ p = \sum_{i=1}^7 z_i^{i+1} = 0 \right\}
\]

which is a transverse, i.e. quasismooth manifold. It is also interesting from a different point of view: An important feature of Calabi–Yau hypersurfaces is that they are automatically what is called well formed, i.e. they do not contain orbifold singularities that are surfaces (in the case of threefolds). More generally this fact translates into the statement that the only resolutions that have to be performed are so–called small resolutions, i.e. the singular sets are of codimension larger than one. The same is true for the higher dimensional manifolds defined above whereas the manifold (10) contains the singular 4–fold \(S = \mathbb{P}_{(210,140,105,84,70,60)}[420]\).

Alternatively, manifolds of the type above may be characterized via a curvature constraint. Because of (8) the first Chern class is given by

\[
c_1(M_{D_{crit}+2(Q-1)}) = (Q - 1) c_1(N)
\]

where \(c_1(N) = dh\) is the first Chern class of the normal bundle \(N\) of the hypersurface \(M_{D_{crit}+2(Q-1)}\) and \(h\) is the pullback of the Kähler form \(H \in H^{(1,1)}(\mathbb{P}_{(k_1,...,k_{D_{crit}+2Q})})\) of the ambient space. Hence the first Chern class is quantized in multiples of the degree of the hypersurface \(M_{D_{crit}+2(Q-1)}\). For \(Q = 1\) the first Chern class vanishes and the manifolds for which (8) holds are Calabi–Yau manifolds, defining consistent groundstates of the supersymmetric closed string. For \(Q > 1\) the first Chern class is nonvanishing and therefore these manifolds cannot possibly describe vacua of the critical string, or so it seems.

It has been shown in [5] however that it is possible to derive from these higher dimensional manifolds the massless spectrum of critical vacua. It is furthermore possible, for certain subclasses of hypersurfaces of type (9), to construct Calabi–Yau manifolds \(M_{CY}\) of dimension \(D_{crit}\) and complex codimension

\[
codim_{\mathbb{C}}(M_{CY}) = Q
\]

\(^1\) A subclass of the manifolds described by (8) has recently also been discussed in [6] and a particular simple manifold in this class, the cubic sevenfold \(\mathbb{P}_8[3]\), has been analyzed in detail in [6][7][8].
directly from these manifolds. The integer $Q$ thus plays a central rôle: the critical dimension is the dimension of the noncritical manifolds offset by twice the coefficient of the first Chern class of the normal bundle of the hypersurface, which involves $Q$. The physical interpretation of the integer $Q$ is that of a total charge associated to the corresponding Landau–Ginzburg theory which determines the codimension of the Calabi–Yau manifold which it describes.

As mentioned in the introduction the class of spaces defined by (9) contains manifolds with no antigenerations and hence it is necessary to have some way other than Calabi–Yau manifolds to represent string groundstates if one wants to compare them with the higher dimensional manifolds. One possible way to do this is to relate them to Landau–Ginzburg theories: manifolds of type (9) can be viewed as a projectivization via a weighted equivalence defined on an affine noncompact hypersurface defined by the same polynomial

$$\mathfrak{C}(k_1,\ldots,k_{D_{\text{crit}}+2Q}) [d] \ni \{p(z_1,\ldots,z_{D_{\text{crit}}+2Q}) = 0\}.$$  \hspace{1cm} (13)

Because the polynomial $p$ is assumed to be transverse in the projective ambient space the affine variety has a very mild singularity: it has an isolated singularity at the origin defining what is called a catastrophe in the mathematics literature.

The complex variables $z_i$ parametrizing the ambient space are to be viewed as the field theoretic limit $\varphi_i(z,\bar{z}) = z_i$ of the lowest components of the order parameters $\Phi_i(z_i,\bar{z}_i,\theta^\pm_i,\bar{\theta}^\pm_i)$, described by chiral $N = 2$ superfields of a 2–dimensional Landau–Ginzburg theory defined by the action

$$\int d^2 z d^2 \theta d^2 \bar{\theta} K(\Phi_i,\bar{\Phi}_i) + \int d^2 z d^2 \theta W(\Phi_i) + c.c.$$ \hspace{1cm} (14)

where $K$ is the Kähler potential and $W$ is the superpotential. It was shown in [9][10] that such Landau–Ginzburg theories are useful for the understanding of string vacua and also that much information about such groundstates is already encoded in the associated catastrophe (13). A crucial piece of information about a vacuum, e.g., is its central charge. Using a result from singularity theory, it is easy to derive that the central charge of the conformal fixed point of the LG theory is

$$c = 3 \sum_{i=1}^{D_{\text{crit}}+2Q} (1 - 2q_i),$$ \hspace{1cm} (15)

where $q_i = k_i/d$ are the U(1) charges of the superfields. It is furthermore possible to derive the massless spectrum of the GSO projected fixed of the LG theory, defining the string vacuum, directly from the catastrophe (13) via a procedure described by Vafa [11].

Even though the manifolds (9) therefore correspond to LG theories of central charge $c = 3D_{\text{crit}}$ they can, however, not be identical to such theories: Consider the case when the critical dimension
of the internal space corresponds to our world, i.e. $D_{\text{crit}} = 3$ and $Q = 2$. The cohomology of $M_5$ leads to the Hodge diamond

\[
\begin{array}{cccccccc}
& & & & & 1 & & \\
& & & & 0 & (1,1) & 0 & \\
& & & 0 & 0 & 0 & 0 & 0 \\
& 0 & (2,2) & 0 & 0 & 0 & & \\
0 & (4,1) & (3,2) & (3,2) & (4,1) & & 0 & \\
0 & 0 & (2,2) & 0 & 0 & & & \\
0 & 0 & 0 & 0 & 0 & & & \\
0 & (1,1) & 0 & & & & & \\
& 0 & 0 & & & & & \\
& & & & & & & 1 \\
\end{array}
\]

where $(p, q)$ denotes the dimension $h^{(p,q)}$ of the cohomology group $H^{(p,q)}(M_5)$.

It is clear from this Hodge diamond that the higher dimensional manifolds will contain more modes than the critical vacuum and hence the relation of the spectrum of the critical vacuum and the cohomology of the noncritical manifolds will be a nontrivial one.

3. Noncritical Manifolds and Critical Vacua

In certain benign situations the subring of monomials of charge unity in the chiral ring describes the generations of the vacuum [12]. For this to hold at all it is important that the GSO projection is the canonical one with respect to the cyclic group $\mathbb{Z}_d$, the order of which is the degree $d$ of the superpotential \(^{2}\). Thus the generations are easily derived for this subclass of theories in (9) because the polynomial ring is identical to the chiral ring of the corresponding Landau–Ginzburg theory. In general a more complicated analysis, involving the singularity structure of the higher dimensional manifolds, will have to be done [14].

It remains to extract the second cohomology. In a Calabi–Yau manifold there are no holomorphic 2–forms and hence all of the second cohomology is in $H^{(1,1)}$. Because of Kodaira’s vanishing theorem the same is true for manifolds with positive first Chern class and therefore for the manifolds under

\(^2\)It does not hold for projections that involve orbifolds with respect to different groups such as those discussed in [13]. This is to be expected as these modified projections can be understood as orbifolds of canonically constructed vacua. The additional moddings generate singularities the resolution of which introduces, in general, additional modes in both sectors, generations and antigenerations.
discussion. At first sight it might appear hopeless to find a construction corresponding to the analysis of (2,1)–forms because of the following example which involves the orbifold of a 3–torus.

Consider the orbifold $T_3^3/Z^3_2$ where the two actions are defined as $(z_1, z_4) \rightarrow (\alpha z_1, \alpha^2 z_4)$, all other coordinates invariant and $(z_1, z_7) \rightarrow (\alpha z_1, \alpha^2 z_7)$, all other invariant. Here $\alpha$ is the third root of unity. The resolution of the singular orbifold leads to a Calabi–Yau manifold with 84 antigenerations and no generations [15]. This is precisely the mirror flipped spectrum of the exactly solvable tensor model $1^9$ of 9 copies of $N = 2$ superconformal minimal models at level $k = 1$ [16] which can be described in terms of the Landau–Ginzburg potential $W = \sum z_i^3$ which belongs to the configuration $C_{(1,1,1,1,1,1,1,1)}[3]$. After imposing the GSO projection by modding out a $Z_3$ symmetry this Landau–Ginzburg theory leads to the same spectrum as the $1^9$ theory.

This Landau–Ginzburg theory clearly is a mirror candidate for the resolved torus orbifold just mentioned and the question arises whether a manifold corresponding to this LG potential can be found. Since the theory does not contain modes corresponding to (1,1)–forms it seems that the manifold cannot be Kähler and hence not projective. Thus it appears that the 7–dimensional manifold $\mathbb{P}_8[3]$ whose polynomial ring is identical to the chiral ring of the LG theory is merely useful as an auxiliary device in order to describe one sector of the critical LG string vacuum. Even though there exists a precise identity between the Hodge numbers in the middle cohomology group of the higher dimensional manifold and the middle dimension of the cohomology of the Calabi–Yau manifold this is not the case for the second cohomology group.

It turns out however, that by looking at the manifolds (9) in a slightly different way it is nevertheless possible to extract the second cohomology in a canonical manner (even if there is none). The way this works is as follows: the manifolds of type (9) will, in general, not be described by smooth spaces but will have singularities which arise from the projective identification. The basic idea now is to associate the existence of antigenerations in a critical string vacuum with the existence of singularities in these higher dimensional noncritical spaces.

Consider again the simple example related to the tensor model $1^9$. Its LG theory is described by $C_{(9)}[3]$ the naive compactification of which leads to

$$\mathbb{P}_8[3] \ni \left\{ \sum_{i=1}^9 z_i^3 = 0 \right\}. \quad (16)$$

Counting monomials leads to the spectrum of 84 generations found previously for the corresponding string vacuum and because this manifold is smooth no antigenerations are expected in this model!
Hence there does not exist a Calabi–Yau manifold that describes the groundstate\footnote{It would seem that a generalization of this 7–dimensional smooth manifold is the infinite class of models $C^{*}_{(1,1,1,1,1,1,1,1+3q)}[3+q]$, but since the manifolds (14) are required to be transverse the only possibility is $q = 0$.}. A second theory in the space of all LG vacua with no antigenerations is

$$(2^6)^{(0,90)} \equiv C^{*}_{(1,1,1,1,1,1,1,1)}[4] \ni \left\{ \sum_{i=1}^{6} z_i^4 + z_7^2 = 0 \right\}$$

with an obviously smooth manifold $\mathbb{P}_{(1,1,1,1,1,1,1,2)}[4]$.

Vacua without antigenerations are rather exceptional however; the generic groundstate will have both sectors, generations and antigenerations. The idea described above to derive the antigenerations works for higher dimensional manifolds corresponding to different types of critical vacua but in the following we will illustrate it with two types of such manifolds. A more detailed analysis can be found in [14].

To be concrete consider the exactly solvable tensor theory $(1 \cdot 16^3)_{A_2 \otimes E_7}$ with 35 generations and 8 antigenerations which corresponds to a Landau–Ginzburg theory belonging to the configuration

$C^{*}_{(2,3,2,3,3,3)}[9]^{(8,35)}$

and which induces, via projectivization, a 5–dimensional weighted hypersurface

$\mathbb{P}_{(2,2,2,3,3,3,3)}[9] \ni \left\{ p = 3 \sum_{i=1}^{3} (y_i^3 x_i + x_i^3) + x_4^3 = 0 \right\}$.  \hspace{1cm} (19)

with orbifold singularities

$$
\mathbb{Z}_3 : \mathbb{P}_3[3] \ni \left\{ p_1 = \sum_{i=1}^{4} x_i^3 = 0 \right\}, \\
\mathbb{Z}_2 : \mathbb{P}_2.
$$

The $\mathbb{Z}_3$–singular set is a smooth cubic surface which supports seven (1,1)–form as can be easily shown. The $\mathbb{Z}_2$–singular set is just the projective plane and therefore adds one further (1,1)–form. Hence the singularities induced on the hypersurface by the singularities of the ambient weighted projective space give rise to a total of eight (1,1)–forms. A simple count leads to the result that the subring of monomials of charge 1 is of dimension 35. Thus we have derived the spectrum of the critical theory from the noncritical manifold (19).

It is furthermore possible, by using the singularity structure of the noncritical manifold to actually construct a Calabi–Yau manifold of critical dimension directly from (19): Recall that the
structure of the singularities of the weighted hypersurface only involved part of the superpotential, namely the cubic polynomial $p_1$ which determined the $\mathbb{Z}_3$–singular set described by a surface. The superpotential thus splits naturally into the two parts

$$p = p_1 + p_2$$

where $p_2$ is the remaining part of the polynomial. The idea now is to consider the product $\mathbb{P}_3[3] \times \mathbb{P}_2$ where the factors are determined by the singular sets of the higher dimensional space and to impose on this 4–dimensional space a constraint described by the remaining part of the polynomial which did not take part in constraining the singularities of this ambient space. In the case at hand this leaves a polynomial of bidegree $(3, 1)$ and hence we are lead to a manifold embedded in

$$\begin{pmatrix} \mathbb{P}_2 & [3 \ 0] \\ \mathbb{P}_3 & [1 \ 3] \end{pmatrix}$$

defined by polynomials

$$p_1 = y_1^3x_1 + y_2^3x_2 + y_3^3x_3$$

$$p_2 = \sum_{i=1}^{4} x_i^3$$

which is precisely the manifold constructed in [17], the exactly solvable model of which was later found in [18]. Thus we have shown how to construct the critical Calabi–Yau manifold from the noncritical manifold (19).

A class of manifolds of a different type which can be discussed in this framework rather naturally is defined by

$$\mathbb{P}_{(2k,K-k,2k,K-k,2k_3,2k_4,2k_5)}[2K]$$

where $K = k + k_3 + k_4 + k_5$. It has been shown in [5] that these higher dimensional spaces lead to manifold embedded in

$$\begin{pmatrix} \mathbb{P}_1 & [2 \ 0] \\ \mathbb{P}_{(k,k_3,k_4,k_5)} & [k \ K] \end{pmatrix},$$

spaces which have which admit [19] a $\sigma$–model description via the mean field Landau–Ginzburg representation of ADE minimal tensor models constructed by Gepner [16].

The geometrical picture that emerges from the constructions above then is the following: embedded in the higher dimensional manifold is a submanifold which is fibered, the base and the fibres being determined by the singular sets of the ambient manifold. The Calabi–Yau manifold itself is a hypersurface embedded in this fibered submanifold.
The examples described so far illustrate the simplest situation that can appear. In more complicated manifolds the singularity structure will consist of hypersurfaces whose fibers and/or base themselves are fibered, leading to an iterative procedure. The submanifold to be considered will, in those cases, be of codimension larger than one and the Calabi–Yau manifold will be described by a submanifold with codimension larger than one as well. To illustrate this point consider the 7–fold \(\mathbb{P}_{(1,1,6,6,2,2,2,2,2)}[8]\) which leads to the \(\mathbb{Z}_2\) fibering \(\mathbb{P}_1 \times \mathbb{P}_{(3,3,1,1,1,1,1)}[4]\) which in turn leads to the \(\mathbb{Z}_3\) fibering \(\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_4[4]\). Following the splits of the potential thus leads to the Calabi–Yau configuration

\[
\begin{bmatrix}
2 & 0 & 0 \\
1 & 1 & 0 \\
0 & 1 & 4
\end{bmatrix} \ni \begin{cases}
p_1 = \sum_{i=1}^{2} x_i^2 y_i = 0 \\
p_2 = \sum_{i=1}^{2} y_i z_i = 0 \\
p_3 = \sum_{j=1}^{5} z_j^4 = 0
\end{cases}
\]

(27)

which is of codimension 3. This example also shows that there are nontrivial relations between these higher dimensional manifolds. The way to see this is via the process of splitting and contraction of Calabi–Yau manifolds introduced in ref. [20]. It can be shown in fact that the Calabi–Yau manifold (27) is an ineffective split of a Calabi–Yau manifold in the class (24). Thus there also exists a corresponding relation between the higher dimensional manifolds.

4. Generalization to Arbitrary Critical Dimensions

Even though the examples discussed in the previous section were all concerned with 6–dimensional Calabi–Yau manifolds and the way they are embedded in the new class of spaces, it should be clear that the ideas presented are not specific to this dimension. A generalization of an infinite series described in [21] is furnished by the doubly infinite series of manifolds

\[
\mathbb{P}_{(m+1,n-1,m+1,n-1,...,m+1,n-1,m+1,...,m+1)}[(m+1)n]
\]

(28)

with \((m + 1)\) pairs of coordinates with weights \((m + 1, n - 1)\) and \((n - m)\) coordinates of weight \((m + 1)\), defined by polynomials

\[
p = \sum_{i=1}^{m+1} (x_i^n + x_i y_i^{m+1}) + \sum_{j=m+2}^{n+1} x_j^n.
\]

(29)

According to the considerations above these \((n + 1)\)–dimensional spaces lead to Calabi–Yau manifolds embedded in

\[
\begin{bmatrix}
m+1 & 0 \\
1 & n
\end{bmatrix}
\]

(30)
via the equations

\[ p_1 = \sum_{i=1}^{n+1} y_i \xi_i, \quad p_2 = \sum_{i=1}^{n+1} x_i^n. \]  \tag{31}

The simplest example is, of course, the case \( n = 2 \) where the higher dimensional manifold is a 3–fold described by

\[ \mathbb{P}_{(2,1,2,1,2)}[4] \ni \left\{ \sum_{i=1}^{2} (z_i^2 + y_i^2) + z_3^2 = 0 \right\} \]  \tag{32}

with a \( \mathbb{Z}_2 \)-singular set isomorphic to the sphere \( \mathbb{P}_2[2] \sim \mathbb{P}_1 \) which contributes one \((1,1)\)–form, the remaining one being provided by the \( \mathbb{P}_1 \) defined by the remaining coordinates. The singularity structure of the 3–fold then relates this space to the complex torus described by the algebraic curve

\[ \begin{pmatrix} \mathbb{P}_1 & 2 & 0 \\ \mathbb{P}_2 & 1 & 2 \end{pmatrix}. \]  \tag{33}

The Landau–Ginzburg theory corresponding to this theory derives from an exactly solvable tensor model \((2^2)_{E^6} \otimes A^2\) described by two \( N = 2 \) superconformal minimal theories at level \( k = 2 \) equipped with the affine D–invariant.

It is of interest to consider the cohomology groups of the 3–fold itself. With the third Chern class \( c_3 = 2h^3 \) the Euler number of the singular space is

\[ \chi_s = \int c_3 = 1 \]  \tag{34}

and hence the Euler number of the resolved manifold is

\[ \tilde{\chi} = 1 - (2/2) + 2 \cdot 2 = 4. \]  \tag{35}

Since the singular set is a sphere its resolution contributes just one \((1,1)\)–form and hence the second Betti number becomes \( b_2 = 2 \). With \( \tilde{\chi} = 2(1 + h^{(1,1)}) - 2h^{(2,1)} \) it follows that

\[ h^{(2,1)} = 1. \]  \tag{36}

The series \((29)\) can be generalized to weighted spaces as is illustrated by the following example leading to a 4–dimensional critical manifold:

\[ p = \sum_{i=1}^{3} (x_i^3 + x_i^2 y_i^2) + \sum_{j=4}^{5} x_i^6 \]  \tag{37}

corresponds to the tensor model \((16^2 \cdot 4)_{E^6 \otimes A^2}\) with central charge \( c = 12 \) and belongs to the configuration

\[ \mathbb{P}_{(6,4,6,4,6,3,3)}[18]. \]  \tag{38}
The critical manifold derived from this 6–fold belongs to the configuration class
\[
\begin{bmatrix}
  3 & 0 \\
  2 & 6
\end{bmatrix}
\] (39)
which is indeed a Calabi–Yau deformation class.

A further infinite class [22] of interest consists of the spaces
\[
\begin{align*}
\mathbb{P}_{(1,1,...,1,2,2,2,2,2,1,1)}[n+1], & \quad n + 1 \text{ even} \\
\mathbb{P}_{(1,1,...,2,n+1,2,2,2,2,2,1,1)}[2(n+1)], & \quad n + 1 \text{ odd},
\end{align*}
\] (40)
of dimension \((n + 1)\). For \((n + 1)\) odd the \(\mathbb{Z}_2\)–singular set is a Calabi–Yau manifolds and the \(\mathbb{Z}_{n+1}\)–singular set consists of two points. Hence the higher dimensional space leads to \(two\) copies of the Calabi–Yau hypersurfaces
\[
\mathbb{P}_n[n+1], \quad n \in \mathbb{N}
\] (41)
embedded in ordinary projective space.

The simplest case is \(n = 2\) for which the resolution of the orbifold singularities of the noncritical 3–fold
\[
\mathbb{P}_{(2,2,2,3,3)}[6]
\] (42)
leads to two independent Hodge numbers \(h^{(1,1)} = 4, h^{(2,1)} = 2\) and hence the Hodge diamond contains \(twice\) the Hodge diamond of the torus, as it must, according to the geometrical picture described above. Similarly \(\mathbb{P}_{(2,2,2,2,2,5,5)}[10]\) leads to two copies of the critical quintic.

The construction is not restricted to the infinite series defined in (29) or its weighted generalization as the next example illustrates. A five–dimensional critical vacuum of higher codimension is obtained by considering the Landau–Ginzburg potential
\[
W = \sum_{j=1}^{2} \left( u_i^3 + u_i v_i^2 \right) + \sum_{i=3}^{5} \left( u_i^3 + u_i w_i^3 \right)
\] (43)
which corresponds to the exactly solvable model \((16^3 \cdot 4^2)_{E_7 \otimes D^2}\). The nine–dimensional noncritical manifold
\[
\mathbb{P}_{(3,2,3,2,3,3,3,3,3)}[9]
\] (44)
leads, via its singularity structure, to the five–dimensional critical manifold
\[
\begin{bmatrix}
  2 & 0 & 0 \\
  0 & 3 & 0 \\
  1 & 1 & 3
\end{bmatrix}
\] . (45)
It is crucial that a polynomial was chosen which is not of Brieskorn–Pham type for the last four coordinates in the noncritical manifold.

Finally, consider the 9-fold
\[ \mathbb{P}_{(5,5,6,6,4,4,4,8,8)}[16] \ni \left\{ \sum_{i=1}^{2} \left( u_i^2 v_i + v_i^2 w_i + w_i^2 x_i + x_i^2 \right) + v_3^2 w_3 + w_3^2 x_3 + x_3^2 = 0 \right\} . \] (46)

The \( \mathbb{Z}_2 \)-fibering leads to the split \( \mathbb{P}_1 \times \mathbb{P}_{(3,3,2,2,4,4,4)} \) which in turn leads to a further \( \mathbb{Z}_2 \) split \( \mathbb{P}_1 \times \mathbb{P}_2 \times \mathbb{P}_{(1,1,1,2,2,2)} \) which finally leads to
\[ \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \ni \begin{cases} p_1 = \sum u_i^2 v_i = 0 \\ p_2 = \sum v_i^2 w_i = 0 \\ p_3 = \sum w_i^2 x_i = 0 \\ p_4 = \sum x_i^2 = 0. \end{cases} \] (47)

Thus the 9-fold fibers iteratively and the splits of the polynomial \( p \) are dictated by the fibering.

5. Cohomology of Noncritical Manifolds

A general relation between the cohomology of the critical vacuum (not described by a Calabi–Yau manifold in general) and the cohomology of the higher dimensional space emerges:
\[ H^{(p,q)}(V_{D_{\text{crit}}}) \subset H^{(p+Q-1,q+Q-1)}(M_{D_{\text{crit}}+2(Q-1)}). \] (48)

The embedding is nontrivial because the cohomology groups of the noncritical manifolds are generically larger than those of the critical vacuum and hence a projection to the critical spectrum is necessary. The construction of [5] described above provides a geometrical framework for such a projection.

One important point regarding the cohomology of the higher dimensional manifolds of type (9) that follows from the considerations of the previous sections is the fact that \( h^{(D_{\text{crit}}+Q-1,Q-1)} \) can exceed unity. Since a Calabi–Yau manifold is defined by the existence of a holomorphic \( (D_{\text{crit}},0) \)-form one might have expected the noncritical spaces to be characterized by the existence of a unique \( (D_{\text{crit}}+(Q-1),(Q-1)) \)-form. This is not true as the example (12) shows. In general the Hodge–diamond for manifolds of odd complex dimension leads to a middle Betti number
\[ b_{D_{\text{crit}}+2(Q-1)} = 2n + 2 \sum_{i=1}^{D_{\text{crit}}-1} h^{(D_{\text{crit}}+(Q-1)-i,(Q-1)+i)} \] (49)
where $n$ is some integer larger than zero that may exceed unity. The reason why $n$ may, and sometimes will, be larger than unity is clear from the geometrical construction reviewed above.

Furthermore, because $h^{(p,p)}(M_{\text{crit}}+2(Q-1))$ is, in general, larger than zero and $h^{(Q,Q)}(M_{\text{crit}}+2Q) > h^{(1,1)}(V_{\text{crit}})$, it follows that there exists more than one ‘remaining’ mode, not accounted for by the generations and antigenerations of the critical vacuum. This happens because the noncritical manifolds almost always have blow-up modes and therefore the cohomology becomes more complicated than that of the smooth spaces $\mathbb{P}^8[3]$ and $\mathbb{P}^{(1,1,1,1,1,1,2)}[4]$ which contain only one additional field. In general, then, we have to expect that not only the dilaton but also other string modes, such as torsion, will play a role in a possible stringy interpretation.

6. Conclusion

It follows from the existence of rigid Calabi–Yau spaces that mirror symmetry cannot be understood in the framework of Kähler manifolds with vanishing first Chern class. To the believer this suggests that beyond the class of such spaces there must exist a space of a new type of noncritical manifolds which contain information about critical vacua, such as the mirrors of these rigid Calabi–Yau manifolds. Mirrors of spaces with both sectors, antigeneration and generations, however, are again of Calabi–Yau type and hence those noncritical manifolds which correspond to such groundstates should make contact with Calabi–Yau manifolds in some manner.

What has been shown in [5] is that the class [3] of higher dimensional Kähler manifolds with positive first Chern class, quantized in a particular way, generalizes the framework of Calabi–Yau vacua in the desired way: For particular types of such noncritical manifolds Calabi–Yau manifolds of critical dimension are embedded algebraically in a fibered submanifold. For string vacua which cannot be described by Kähler manifolds and which are mirror candidates of rigid Calabi–Yau manifolds the higher dimensional manifolds still lead to the spectrum of the critical vacuum and a rationale emerges that explains why a Calabi–Yau representation is not possible in such theories. Thus these manifolds of dimension $c/3 + 2(Q - 1)$ define an appropriate framework in which to discuss mirror symmetry.

There are a number of important consequences that follow from the results of the previous sections. First it should be realized that the relevance of noncritical manifolds suggests the generalization of a conjecture regarding the relation between $(2,2)$ superconformal field theories of central charge $c = 3D$, $D \in \mathbb{N}$, with N=1 spacetime supersymmetry on the one hand and Kähler manifolds
of complex dimension $D$ with vanishing first Chern class on the other. It was suggested by Gepner that this relation is 1–1. It follows from the results above that instead superconformal theories of the above type are in correspondence with Kähler manifolds of dimension $c/3 + 2(Q - 1)$ with a first Chern class quantized in multiples of the degree.

A second consequence is that the ideas of section 3 lead, for a large class of Landau–Ginzburg theories, to a new canonical prescription for the construction of the critical manifold, if it exists, directly from the 2D field theory.

Batyrev [23] has introduced a combinatorical construction of Calabi–Yau mirrors based on toric geometry. This method appears to apply only to manifolds defined by one polynomial in a weighted projective space or products thereof. Because the method used in [23] is not restricted to Calabi–Yau manifolds [24] the constructions described in sections 3 and 4 lead to the possibility of extending Batyrev’s results to Calabi–Yau manifolds of codimension larger than one by proceeding via noncritical manifolds.

As a final remark it should be emphasized that in this framework the role played by the dimension of the manifolds parametrizing the spectrum observed in four dimensions becomes of secondary importance. This is as it should be, at least for an effective theory, which tests only matter content and couplings. It is then, perhaps, not too surprising that via ineffective splittings manifolds of different dimension describe one and the same critical vacuum.

It is clear that the emergence in string theory of manifolds with quantized first Chern class should be understood better. The results described here are a first step in this direction. They indicate that these manifolds are not just auxiliary devices but may be as physical as Calabi–Yau manifolds of critical dimension. In order to probe the structure of these models in more depth it is important to get further insight into the complete spectrum of these theories and to compute the Yukawa couplings of the fields. It is clear from the results presented here that the spectra of the higher dimensional manifolds contain additional modes beyond those that are related to the generations and antigenerations of the critical vacuum and the question arises what physical interpretation these fields have.

A better grasp on the complete spectrum of these spaces should also give insight into a different, if not completely independent, approach toward a deeper understanding of these higher dimensional manifold, which is to attempt the construction of consistent $\sigma$–models defined via these spaces. Control of the complete spectrum will shed light on the precise relation between the $\sigma$–models associated to the noncritical manifolds and critical $\sigma$–models.
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8. References

1. P. Candelas, M. Lynker and R. Schimmrigk, Nucl.Phys. B341(1990)383
2. M. Lynker and R. Schimmrigk, Phys.Lett. B249(1990)237
3. B. R. Greene and R. Plesser, Nucl.Phys. B338(1990)15
4. B. Zumino, Phys.Lett. B87(1979)203
5. R. Schimmrigk, Phys.Rev.Lett. 70(1993)3688
6. P. Candelas, E. Derrick and L. Parkes, CERN–TH. 6931/93 preprint
7. C. Vafa, in Essays in Mirror Symmetry, ed. S.-T. Yau
8. T. Hübsch, unpublished
9. E. Martinec, Phys.Lett. B217(1989)431
10. C. Vafa and N. Warner, Phys.Lett. B218(1989)51
11. C. Vafa, Mod.Phys.Lett. A4(1989)1169
12. P. Candelas, Nucl.Phys. B298(1988)458
13. P. Berglund, B. R. Greene and T. Hübsch, Mod.Phys.Lett. A7(1992)1855
14. R. Schimmrigk, work in progress
15. B. R. Greene, C. Vafa and N. Warner, Nucl.Phys. B324(1989)371
16. D. Gepner, Nucl.Phys. B296(1988)757
17. R. Schimmrigk, Phys.Lett. B193(1987)175
18. D.Gepner, PUPT preprint, December 1987

19. R.Schimmrigk, Phys.Lett. B229(1989)227

20. P.Candelas, A.Dale, C.A.Lütken and R.Schimmrigk, Nucl.Phys. B298(1988)493

21. R.Schimmrigk, in PROCEEDINGS OF THE NATO ARW ON LOW DIMENSIONAL TOPOLOGY AND QUANTUM FIELD THEORY, Cambridge, England, 1992 and the PROCEEDINGS OF THE INTERNATIONAL WORKSHOP ON STRING THEORY, QUANTUM GRAVITY AND THE UNIFICATION OF FUNDAMENTAL INTERACTIONS, Rome, Italy, 1992

22. R.Schimmrigk, in PROCEEDINGS OF THE TEXAS/PASCOS MEETING, Berkeley, CA, 1992

23. V.V.Batyrev, University of Essen preprint Nov. 1992

24. V.V.Batyrev, Duke Math.J. 69(1993)349