Poisson brackets of hydrodynamic type and their generalizations.

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In this paper, we consider Hamiltonian structures of hydrodynamic type and some of their generalizations. In particular, we discuss the questions concerning the structure and special forms of the corresponding Poisson brackets and the connection of such structures with the theory of integration of systems of hydrodynamic type.

I. INTRODUCTION

This work will be devoted to Hamiltonian structures, which currently play an important role in many areas of mathematics and mathematical physics. Namely, we will mainly consider here Hamiltonian structures of hydrodynamic type and some of their important generalizations. Traditionally, the Hamiltonian structures of the hydrodynamic type are associated with the Poisson brackets arising in hydrodynamics and representing the brackets for the corresponding hydrodynamic densities. Brackets of this type represent, as a rule, expressions of the first order in spatial derivatives and can be written in the following general form

\[ \{U^\nu(x), U^\mu(y)\} = \{U^\nu(x), U^\mu(y)\} = g^{\nu\mu}(U(x)) \delta_x(x-y) + \delta^{\nu\mu}(U(x)) U^\lambda_\nu(x) \delta(x-y), \]

where \( U(x) \) is a complete set of hydrodynamic densities in the problem under consideration.

Brackets \( \{1\} \) in hydrodynamics are usually associated with Hamiltonians of hydrodynamic type, i.e. Hamiltonians of the form

\[ H = \int P_H(U(x)) \, d^nx \ , \quad (1.2) \]

where \( n \) is the dimension of the problem under consideration. It is easy to see that the Hamiltonians \( \{2\} \) in the Hamiltonian structure \( \{1\} \) correspond to systems of the form

\[ U^\nu_i = V^\nu_{\bar{\nu}}(U(x)) \, U^\mu_{x^i} , \quad (1.3) \]

where \( \nu, \mu = 1, \ldots, N, \quad i = 1, \ldots, n \)

The bracket \( \{1\} \) and the system \( \{3\} \) are written in the most general form, which does not reflect any specific hydrodynamic variables \( U(x) \). It is easy to see that the presented form is invariant with respect to any “point” change of variables \( U = U(U) \) with appropriate transformations of the quantities \( g^{\nu\mu}(U) \), \( \delta^{\nu\mu}(U) \) and \( V^\nu_{\bar{\nu}}(U) \). At the same time, certainly, in real hydrodynamics, each variable, as a rule, has its own special physical meaning, and the corresponding brackets \( \{1\} \) and systems \( \{3\} \) have an additional structure associated with this fact. As is well known \( \{1\} \), in the simplest case of a barotropic flow of an ideal fluid, the Poisson brackets of the fluid density \( \rho(x) \) and its velocity components \( v^i(x) \) can be written as

\[ \{\rho(x), \rho(y)\} = 0 , \quad \{v^i(x), \rho(y)\} = \nabla_x \delta(x-y) , \quad \{v^i(x), v^k(y)\} = \frac{1}{\rho(x)} \left( \frac{\partial v^k}{\partial x^i} - \frac{\partial v^i}{\partial x^k} \right) \delta(x-y) , \quad (1.4) \]

and the Hamiltonian corresponding to such a flow has the form

\[ H = \int \left( \frac{\rho v^2}{2} + \epsilon(\rho) \right) \, d^3x \]

An extremely important property of the bracket \( \{1\} \) is that the condition

\[ \text{rot } v(x) = 0 \quad (1.5) \]

is conserved for any Hamiltonian fluid dynamics. Determining in this case the flow potential \( \Phi(x) \) according to the standard formula

\[ v(x) = \nabla \Phi(x) , \]

it is easy to check that the variables \( \rho(x) \) and \( \Phi(x) \) define the canonical Poisson bracket

\[ \{\rho(x), \rho(y)\} = 0 , \quad \{\Phi(x), \Phi(y)\} = 0 , \quad \{\Phi(x), \rho(y)\} = \delta(x-y) \quad (1.6) \]

It can also be noted that, in their physical meaning, the variables \( \rho(x) \) can be classified as action-type variables, and the variables \( \Phi(x) \) as angular (phase) variables. It is also well known that the introduction of canonical variables for the bracket \( \{1\} \) in the more general case of vortex flows is more complicated and is related to the definition of Clebsch variables for such flows (see \( \{2\} \)).

Hamiltonian structures \( \{1\} \) also arise for more general case of non-barotropic flows, as well as for the equations...
of magnetohydrodynamics (3). It can be shown that, both in the barotropic and in the more general non-barotropic case, the introduction of canonical variables (Clebsch variables) for Hamiltonian structures is related to the representation of the corresponding equations in the form of a constrained Lagrangian system (see 13). Moreover, this approach turns out to be fruitful also in the description of non-entropic flows of a classical fluid, as well as superfluidity (6). It can also be noted that in the latter case, the Lagrangian and Hamiltonian approaches often turn out to be an important component not only in describing certain aspects of the dynamics of a superfluid fluid, but also in establishing the equations of such dynamics in general (see, for example. 11).

In the general case, as is well known, the construction of canonical variables for Hamiltonian structures in hydrodynamics is a very important problem associated with the description of many features of the corresponding flows, including their topological features (see 12 13). Below we will show in fact that in many cases for brackets of hydrodynamic type, it seems natural to have a more extended definition of the canonical form. Moreover, in addition to the canonical form of the bracket (II.1), another (diagonal) form of this bracket also turns out to be extremely important in the study of the corresponding Hamiltonian systems. The latter circumstance will be most obvious in the case of one spatial dimension, where the theory of such brackets (and their generalizations) represents the basis of the theory of integrable systems of hydrodynamic type. As is also well known, other structures of Poisson brackets (II.1) are often also important, in particular, their Lie algebraic structure (see eg. 14 16).

Generalizations of Hamiltonian structures of hydrodynamic type include structures containing simultaneously hydrodynamic and phase variables, structures combining hydrodynamic and Lie-algebraic parts, structures containing higher derivatives and nonlocal additives, etc. A huge variety of extremely important structures of this type were considered by I.E. Dzyaloshinsky and G.E. Volovik in the work 17. In particular, as was shown in 17, a huge variety of applications of Hamiltonian structures generalizing structures of hydrodynamic type include the description of elastic dynamics of crystals with impurities and defects, the description of the dynamics of liquid crystals, the dynamics of magnets of various types, as well as spin glasses, etc. This work is dedicated to the 90th anniversary of I.E. Dzyaloshinsky.

II. ONE-DIMENSIONAL HAMILTONIAN STRUCTURES OF HYDRODYNAMIC TYPE

In the case of one spatial variable, systems of hydrodynamic type have the form

$$U_t^\nu = V_\mu^\nu(U) U_x^\mu \quad (II.1)$$

The matrix $V_\mu^\nu(U)$ represents a linear transformation matrix on the tangent space of a manifold with coordinates $U$, in particular, it has the appropriate transformation law under the point transformations $\bar{U} = \bar{U}(U)$.

A system (II.1) is hyperbolic in some range of values of $U$ if at each point of this domain all eigenvalues of $V_\mu^\nu(U)$ are real, and the corresponding eigenvectors form a basis in the tangent space. A system (II.1) is called strictly hyperbolic in some domain if at each point of this domain the eigenvalues of $V_\mu^\nu(U)$ are real and pairwise are different.

In the case of two-component systems (that is, $U = (U^1, U^2)$, each strictly hyperbolic system (II.1) can be reduced to diagonal form

$$R_\mu^\nu = v^\nu(R) R_x^\mu \quad (II.2)$$

with a real change of variables $R = R(U)$. In the case of $N \geq 3$, however, such a reduction, generally speaking, is not always possible. In general, a strictly hyperbolic system (II.1) can be locally reduced to a diagonal form (using a real change of coordinates) if the corresponding Hantjes tensor is identically zero. In coordinate form, the components of the Hantjes tensor can be written as

$$H_\mu^\nu(U) =
= V_\sigma^\rho(U) V_\tau^\nu(U) N_{\mu\lambda}(U) - V_\sigma^\nu(U) N_{\tau\lambda}(U) V_\rho^\mu(U) -
- V_\sigma^\nu(U) N_{\mu\tau}(U) V_\rho^\lambda(U) + N_{\mu\tau}(U) V_\rho^\sigma(U) V_\sigma^\nu(U) V_\rho^\lambda(U),$$

where $N_{\mu\lambda}(U)$ is the Nijenhuis tensor of the operator $V_\mu^\nu(U)$:

$$N_{\mu\lambda}(U) = V_\mu^\sigma(U) \frac{\partial V_\lambda^\nu}{\partial U^\sigma} - V_\lambda^\sigma(U) \frac{\partial V_\mu^\nu}{\partial U^\sigma} +
+ V_\lambda^\rho(U) \frac{\partial V_\sigma^\nu}{\partial U^\mu} - V_\sigma^\nu(U) \frac{\partial V_\lambda^\rho}{\partial U^\mu}.$$

In the most general case, functions $R(U)$, satisfying, by virtue of the system (II.1), the equation

$$R_t = v(U) R_x$$

for some function $v(U)$, are called the Riemann invariants of the system (II.1). In the general case, system (II.1) may have a certain set of independent Riemann invariants, the number of which, however, is insufficient for the complete diagonalization of the system.

The Hamiltonian theory of systems (II.1) is connected, first of all, with local one-dimensional Poisson brackets of hydrodynamic type, introduced by B.A. Dubrovin and S.P. Novikov (18). The Dubrovin-Novikov bracket on the space of fields $(U^1(x), \ldots, U^N(x))$ has the form

$$\{U^\mu(x), U^\nu(y)\} =
= g^{\nu\mu}(U) \delta'(x - y) + b_\lambda^{\nu\mu}(U) U_x^\lambda \delta(x - y) \quad (II.3)$$

As was shown by B.A. Dubrovin and S.P. Novikov, the expression (II.3) with a nondegenerate tensor $g^{\nu\mu}(U)$
defines a Poisson bracket on the space of fields $U(x)$ if
and only if:

1. The tensor $g^{\nu\mu}(U)$ defines a symmetric pseudo-Riemannian metric of zero curvature with upper indices on the space of parameters $(U^1, \ldots, U^N)$;

2. The values
\[ \Gamma^\nu_{\mu\lambda} = -g_{\nu\sigma} b^{\sigma}_{\lambda} , \]
where $g^{\nu\sigma}(U)g_{\nu\mu}(U) = \delta^\nu_{\nu}$, represent the Christoffel symbols for corresponding metric $g_{\nu\mu}(U)$.

As follows from the above statements, any Dubrovin-Novikov bracket can be written in the canonical form (II.16)
\[ \{ n^\nu(x), n^\mu(y) \} = \epsilon^\nu \delta^{\nu\mu} \delta(x-y) , \quad \epsilon^\nu = \pm 1 \quad (II.4) \]
after transition to the flat coordinates $n^\nu = n^\nu(U)$ of the metric $g_{\nu\mu}(U)$. It is natural to introduce the group of point canonical transformations for the bracket (II.5), which, as is easy to see, coincides with the corresponding group $O(K, N-K)$.

The functionals
\[ N^\nu = \int_{-\infty}^{+\infty} n^\nu(x) \, dx \]
represent annihilators of the Dubrovin-Novikov bracket, while the functional
\[ P = \int_{-\infty}^{+\infty} \frac{1}{2} \sum_{\nu=1}^{N} \epsilon^\nu (n^\nu)^2(x) \, dx \]
represents the momentum functional for the bracket (II.3).

It is often also convenient to write the canonical form of the Dubrovin-Novikov bracket in a more general form
\[ \{ n^\nu(x), n^\mu(y) \} = \eta^{\nu\mu} \delta(x-y) , \]
where $\eta^{\nu\mu}$ is an arbitrary constant (non-degenerate) symmetric matrix. It is easy to see that the group of (point) canonical transformations of the Poisson bracket is extended in this case to $GL_N(\mathbb{R})$.

As one can see, if $N$ is even, $N = 2K$, and the metric $g_{\nu\mu}(U)$ has the signature $(K, K)$, we can actually choose the flat coordinates $(a^1, \ldots, a^K, b^1, \ldots, b^K)$ in such a way that the corresponding nonzero Poisson brackets take the form
\[ \{ a^\alpha(x), b^\beta(y) \} = \delta^{\alpha\beta} \delta(x-y) \]

In this case, making a real nonlocal transformation
\[ \Phi^\alpha(x) = \int a^\alpha(x) \, dx \]
we get the Poisson bracket in canonical form
\[ \{ b^\alpha(x), b^\beta(y) \} = 0 , \quad \{ \Phi^\alpha(x), \Phi^\beta(y) \} = 0 , \]
\[ \{ \Phi^\alpha(x), \Phi^\beta(y) \} = \delta^{\alpha\beta} \delta(x-y) \]

To some extent, a special role is also played by the Poisson brackets, which are linear in the coordinates $U(x)$:
\[ \{ U^\nu(x), U^\mu(y) \} = \left( b^\nu_{\lambda} + b^\nu_{\lambda} \right) U^\lambda + g_{\nu\mu} \delta(x-y) + b^\lambda_{\chi} U^\lambda \delta(x-y) \quad (II.5) \]
\[ b^\lambda_{\chi} = \text{const} , \quad g_{\nu\mu} = \text{const} \]

The coordinates $U(x)$ in this case are, as a rule, naturally related to the problem under consideration, and the brackets (II.5) themselves are described by Lie-algebraic structures. In the case of one space variable, the classification of the corresponding Lie algebras, as well as admissible cocycles on them, was constructed in [19], where, in particular, the connection of such brackets with the theory of Frobenius and quasi-Frobenius algebras was discovered.

The bracket (II.3) also has two other important forms on the space of fields $U(x)$. The first of them is the “Liouville” form ([16, 18]), having the form
\[ \{ U^\nu(x), U^\mu(y) \} = \left( \gamma^\nu_{\mu}(U) + \gamma^\nu_{\mu}(U) \right) \delta(x-y) + \frac{\partial \gamma^\nu_{\mu}}{\partial U^\lambda} U^\lambda \delta(x-y) \]
for some functions $\gamma^\nu_{\mu}(U)$.

The “Liouville” form of the Dubrovin-Novikov bracket is also called physical and corresponds to the case when the integrals of coordinates $U^\nu$ commute with each other.

Another important form of the Dubrovin-Novikov bracket is the diagonal form. It corresponds to the case when the coordinates $U^\nu$ are orthogonal coordinates for the metric $g_{\nu\mu}(U)$, and, accordingly, the tensor $g^{\nu\mu}(U)$ in the definition (II.3) has a diagonal form. This form of the Dubrovin-Novikov bracket is closely related to the theory of integrable systems of hydrodynamic type.

S.P. Novikov hypothesized that all systems of hydrodynamic type that can be reduced to the form (II.2) and are Hamiltonian with respect to some bracket (II.3) are integrable. This hypothesis was proved by S.P. Tsarev in the work [20], where a new method (“generalized hodograph method”) for integrating such systems was proposed.

The construction of solutions of the system (II.2) by Tsarev’s method consists in finding systems (commuting flows) that commute with it, which have the same (diagonal) form with the characteristic velocities $w^\nu(\mathbb{R})$ satisfying the system of equations
\[ \frac{\partial_{\nu} w^\nu}{w^\mu - w^\nu} = \frac{\partial_{\nu} v^\nu}{v^\mu - v^\nu} , \quad \mu \neq \nu \quad (II.6) \]
system (II.2), determined from the algebraic system form

\[ R \]

generates a solution \( U \) of the system (II.6), determined from the algebraic system

\[ w^\nu(R) = t v^\nu(R) + x , \quad \nu = 1, \ldots, N \]

The system (II.6) on the functions \( w^\nu(R) \) is an overdetermined system of linear equations with variable coefficients, the compatibility conditions for which have the form

\[
\partial_\lambda \left( \frac{\partial_\mu w^\nu}{v^\lambda - v^\nu} \right) = \partial_\mu \left( \frac{\partial_\lambda w^\nu}{v^\lambda - v^\nu} \right) \quad (II.7)
\]

\[ \nu \neq \mu , \quad \mu \neq \lambda , \quad \lambda \neq \nu \]

As can be shown (see [20, 21]), the fulfillment of conditions (II.7) ensures in the general position the completeness of solutions of the system (II.6), sufficient for the (local) solution of the general Cauchy problem for the corresponding system (II.2).

As was shown in the work [20], the requirement that system (II.2) is Hamiltonian with respect to any Dubrovin-Novikov bracket entails the relations (II.7) and, thus, allows integration of this system by the Tsarev method. In fact, the set of diagonal systems satisfying the condition (II.7) is much wider than the set of the same systems that are Hamiltonian with respect to a local Poisson bracket; therefore, all diagonal systems satisfying the conditions (II.7) were named by Tsarev semi-Hamiltonian. As it turned out later, the class of “semi-Hamiltonian” systems also includes systems that are Hamiltonian with respect to generalizations of the Dubrovin-Novikov bracket - the weakly nonlocal Mokhov-Ferapontov bracket and the Ferapontov bracket. Let us give the appropriate definitions here:

A Mokhov-Ferapontov bracket ([24]) on the space of fields \( U(x) \) is a bracket that can be represented in the form

\[
\{ U^\nu(x), U^\mu(y) \} = g_{\nu\mu}(U) \delta'(x-y) + b^\nu_\lambda(U) U^\lambda_x \delta(x-y) + \frac{1}{2} c \ U^\nu \ \text{sgn}(x-y) \ U^\mu_y
\]

(II.8)

A general Ferapontov bracket ([23-26]) on the space of fields \( U(x) \) is a bracket of the form

\[
\{ U^\nu(x), U^\mu(y) \} = g_{\nu\mu}(U) \delta'(x-y) + b^\nu_\lambda(U) U^\lambda_x \delta(x-y) + \frac{1}{2} \sum_{k=1}^{g} e_k \ w_{(k)\lambda}^\nu(U) U^\lambda_x \ \text{sgn}(x-y) \ w_{(k)\delta}^\mu(U) U^\delta_y
\]

(II.9)

\[ e_k = \pm 1 \]

Similarly to the previously formulated conditions on the functions \( g_{\nu\mu}(U) \) and \( b^\nu_\lambda(U) \) for the Dubrovin-Novikov bracket, one can formulate conditions on the coefficients in the expressions (II.8) and (II.9), under which they define the Poisson brackets on the space of fields \( U(x) \). Namely, the expression (II.8) with a nondegenerate tensor \( g_{\nu\mu}(U) \) defines a Poisson bracket on the space of fields \( U(x) \) if and only if ([23]):

1) The tensor \( g_{\nu\mu}(U) \) defines a symmetric pseudo-Riemannian metric of constant curvature \( c \) (with upper indices) on the space of parameters \( (U^1, \ldots, U^N) \);

2) The values

\[
\Gamma^\nu_{\nu\gamma} = - g_{\nu\lambda} b^\nu_\gamma
\]

represent the Christoffel symbols for the corresponding metric \( g_{\nu\mu}(U) \).

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\Gamma^\nu_{\nu\gamma} = - g_{\nu\lambda} b^\nu_\gamma
\]

represent the Christoffel symbols for the corresponding metric \( g_{\nu\mu}(U) \).

3) The affinors \( w_{(k)\lambda}^\nu(U) \) and the curvature tensor of the metric \( R^\nu_{\mu\lambda}(U) \) satisfy the conditions

\[
g_{\nu\tau} w_{(k)\lambda}^\nu(U) = g_{\mu\tau} w_{(k)\lambda}^\mu(U) , \quad \nabla_\nu w_{(k)\lambda}^\mu(U) = \nabla_\lambda w_{(k)\nu}^\mu(U)
\]

(II.10)

\[
R^\nu_{\mu\lambda} = \sum_{k=1}^{g} e_k \ (w_{(k)\mu}^\nu(U) w_{(k)\lambda}^\nu(U) - w_{(k)\lambda}^\nu(U) w_{(k)\mu}^\nu(U))
\]

(II.11)

\[ [w_k, w_{k'}] = 0 \]

(commutativity).

E.V. Ferapontov also pointed out that the relations (II.10)-(II.11) represent the Gauss - Codazzi relations for a submanifold \( \mathcal{M}^N \) with flat normal connection in a pseudo-Euclidean space \( E^{N+g} \). In this situation, the tensor \( g_{\nu\mu} \) plays the role of the first quadratic form on \( \mathcal{M}^N \), and \( w_{(k)} \) - the role of the Weingarten operators corresponding to “parallel” fields of unit normals \( n_k \) ([23-26]). In addition, E.V. Ferapontov showed that the bracket (II.9) can be considered as a result of the Dirac restriction of the Dubrovin-Novikov bracket, defined in the space \( E^{N+g} \), on the submanifold \( \mathcal{M}^N \) ([24]).

The canonical form of the Mokhov-Ferapontov bracket, written in the densities of annihilators, as well as the expression for the momentum functional for this bracket were proposed in [27].
Canonical forms (as well as “canonical functionals”) of general weakly nonlocal Ferapontov brackets were investigated in [28]. By definition, the bracket (II.9) has the canonical form in the variables $n = n(U)$ if the relations

$$\{n^\nu(x), n^\mu(y)\} = \left(\epsilon^\nu \delta^{\nu\mu} - \sum_{k=0}^{g} e_k f^\nu_{(k)}(n) f^\mu_{(k)}(n)\right) \delta'(x - y) - \sum_{k=0}^{g} e_k \left(f^\nu_{(k)}(n)\right)_x f^\mu_{(k)}(n) \delta(x - y) + \frac{1}{2} \sum_{k=0}^{g} e_k \left(f^\nu_{(k)}(n)\right)_x \text{sgn}(x - y) \left(f^\mu_{(k)}(n)\right)_y,$$

($\epsilon^\nu = \pm 1$), hold for some functions $f^\nu_{(k)}(n)$, such that $f^\nu_{(0)}(0, \ldots, 0) = 0$.

It can be noted here that the canonical form of the Ferapontov bracket in the general case is not uniquely defined. It is most natural to associate the canonical form of the bracket (II.9) with some fixed point $U_0$ on the variety of parameters $U$. It is natural to associate with the point $U_0$ the space of “loops” starting and ending at the point $U_0$, i.e. the space $\mathcal{L}_{U_0}$ of maps

$$\mathbb{R} \to \{U\},$$

such that

$$U(x) \to U_0, \quad x \to \pm \infty.$$  

From the geometric point of view, the construction of the canonical coordinates corresponding to the point $U_0$ is related to the above-mentioned embedding of the variety of parameters $\mathcal{M}^N$ into pseudo-Euclidean space $E^{N+g}$, defined by the bracket (II.9). Namely, consider the corresponding embedding

$$\mathcal{M}^N \to E^{N+g}$$

and choose a (pseudo) Euclidean coordinate system in $E^{N+g}$ starting at the point $U_0 \in \mathcal{M}^N \subset E^{N+g}$ so that the first $N$ coordinates are tangent to the submanifold $\mathcal{M}^N$ at the point $U_0$, and the remaining $g$ are orthogonal to $\mathcal{M}^N$ at this point. The restriction of the first $N$ coordinates in $E^{N+g}$ to $\mathcal{M}^N$ define then the set of canonical coordinates $n^\nu(U)$, corresponding to the point $U_0$. The restriction of the remaining $g$ coordinates to the submanifold $\mathcal{M}^N$ defines $g$ additional functions $h^k(U)$, also playing an important role.

The values of the (vector) functions $f_{(k)}(n)$ in the above canonical form coincide with the values of the projections of the basic parallel normal fields to $\mathcal{M}^N$ in the space $E^{N+g}$ onto the plane tangent to $\mathcal{M}^N$ at the point $U_0$. In particular, for the Mokhov-Ferapontov bracket, corresponding to the case of a (pseudo) sphere in $E^{N+1}$, we have $f^\nu = \sqrt{|c|} n^\nu$.

As can be shown (see [28]), the corresponding functionals

$$N^\nu = \int_{-\infty}^{+\infty} n^\nu(x) \, dx$$

define the annihilators of the bracket (II.9) on the space $\mathcal{L}_{U_0}$. The functionals

$$H^k = \int_{-\infty}^{+\infty} h^k(x) \, dx$$

represent the Hamiltonians, generating the flows

$$U^\nu_t = w^\nu_{(k)}(U) \, U^\mu_x$$

on the space $\mathcal{L}_{U_0}$ according to the bracket (II.9).

Every bracket (II.9) with a nondegenerate tensor $g^{\nu\mu}(U)$ also has a local momentum functional $P$ on the spaces $\mathcal{L}_{U_0}$ having the form

$$P = \frac{1}{2} \int_{-\infty}^{+\infty} \left(\sum_{\nu=1}^{N} \epsilon^\nu n^\nu n^\nu + \sum_{k=1}^{g} e_k h^k h^k\right) \, dx,$$

where the functions $n(U)$ and $h(U)$ correspond to the point $U_0$ (29).

In [28], there was also proposed a Liouville (physical) form of a general Ferapontov bracket, which has the form

$$\{U^\nu(x), U^\mu(y)\} = \left(\gamma^{\nu\mu} + \gamma^{\nu\mu} - \sum_{k=1}^{g} e_k f^\nu_{(k)} f^\mu_{(k)}\right) \delta'(x - y) + \left(\frac{\partial \gamma^{\nu\mu}}{\partial U^\lambda} U^\lambda_x - \sum_{k=1}^{g} e_k \left(f^\nu_{(k)}\right)_x f^\mu_{(k)}\right) \delta(x - y) + \frac{1}{2} \sum_{k=1}^{g} e_k \left(f^\nu_{(k)}\right)_x \text{sgn}(x - y) \left(f^\mu_{(k)}\right)_y$$

with some functions $\gamma^{\nu\mu}(U)$ and $f^\nu_{(k)}(U)$.

The bracket (II.9) has the physical form in coordinates $U^\nu$ if and only if the integrals

$$J^\nu = \int_{-\infty}^{+\infty} U^\nu(x) \, dx$$

generate a set of local commuting flows according to the bracket (II.9).

The Ferapontov bracket (II.9) is the most general (one-dimensional) weakly nonlocal hydrodynamic bracket. On the other hand, the Ferapontov brackets, as a rule, are associated only with integrable systems of hydrodynamic type, since, in reality, they contain integrable structures within themselves. According to the hypothesis of E.V. Ferapontov (28), all diagonalizable semi-Hamiltonian systems are actually Hamiltonian with respect to brackets (II.9), if we admit the presence of an infinite number of terms in their non-local part. This hypothesis was, in
particular, confirmed in the work for a fairly wide class of semi-Hamiltonian systems. This hypothesis was also considered in the works and . It must be said, however, that in the most general formulation a rigorous proof of this hypothesis has not yet been obtained.

At the same time, the brackets and are associated with a much broader class of systems that do not imply integrability, and are, in this sense, more common in real applications.

Both in the case of the Dubrovin-Novikov bracket and in the case of the weakly nonlocal brackets of Mokhov-Ferapontov and Ferapontov, the diagonal form of the metric is most closely related to the theory of integration of systems . With all this, the problem of diagonal coordinates for flat metrics has been, as is known, a classical problem in differential geometry for many years. It should be noted here that recently very important new achievements related to the integrability of systems . With all this, the problem of Poisson brackets also served as a stimulating factor in considering this classical problem.

It must be said, however, that the diagonal form of the Poisson brackets is, certainly, not the only one important when considering the integrability of systems of hydrodynamic type. So, for example, the canonical coordinates of the brackets play the most important role when considering bi-Hamiltonian structures (see ), which are present in the overwhelming majority of integrable hierarchies. Recall here that the basis of a bi-Hamiltonian structure is represented by a pair of compatible Poisson hierarchies. As we saw above, in the case of brackets of hydrodynamic type, such functionals are related precisely to the canonical coordinates of the bracket. It can be added that the canonical coordinates for the entire pencil represent functions for all densities of the Hamiltonians of the hierarchy under consideration that generate higher flows.

It is also impossible not to mention here the important role of compatible local Hamiltonian structures for the quantum field theory discovered by B.A. Dubrovin in the 1990s. Namely, as was shown by B.A. Dubrovin, special pairs of compatible Dubrovin-Novikov brackets give a classification of topological field theories that arise as approximations in various field theory models. Thus, a pair of matched Dubrovin-Novikov brackets underlies the definition of Frobenius manifolds, parametrizing solutions of the WDVV equation describing topological theories. Moreover, the further development of the theory of Frobenius manifolds was the theory of “weakly dispersive” deformations of such Hamiltonian structures, which give corrections to topological theories .

In addition, certainly, in many examples the most important role is played by the coordinates at which the Poisson bracket becomes linear in the fields, where the Lie algebraic aspects of the problem under consideration are clearly manifested.

Another important generalization of local brackets of hydrodynamic type are inhomogeneous Poisson brackets, which in the general case have the form

\[ \{ U^\nu(x), U^\mu(y) \} = g^{\nu\mu}(U(x)) \delta^c(x - y) + \left[ b^{\nu\mu}(U(x)) U^\lambda_x + h^{\nu\mu}(U) \right] \delta(x - y) \]  

The brackets represent actually pairs of compatible brackets, since both their hydrodynamic part and the finite-dimensional part , define independent Poisson brackets on the space of fields , compatible with each other. It can be seen, in addition, that the tensor must also define a simple finite-dimensional bracket on a space with coordinates .

Brackets , as it is easy to see, are associated primarily with inhomogeneous systems of hydrodynamic type

\[ U^\nu_l = V^{\nu\mu}(U) U^\mu_x + f^\nu(U) \]

on the space of fields .

The theory of brackets is also quite non-trivial and contains many interesting geometric and algebraic structures. Thus, in particular, for one-dimensional Poisson brackets

\[ \{ n^\nu(x), n^\mu(y) \} = \eta^{\nu\mu} \delta^c(x - y) + \left[ c^{\nu\mu}_x n^\lambda + d^{\nu\mu} \right] \delta(x - y) \]

the following statement is true :

If the metric is nondegenerate, then, after a coordinate transformation , the bracket can be written in the form

\[ n^\nu(x), n^\mu(x) \}

where the coefficients , are constant, represent the structure constants of a semisimple Lie algebra with Killing metric and is an arbitrary cocycle on this algebra.

An important example of a system associated with the bracket is the n-waves problem

\[ M_t - \varphi(M_x) = [M, \varphi(M)] \]

where is an matrix with zero trace (possibly with additional symmetries), \( \varphi(M) = \)
we have further generalizations. So, in particular, in the work \[42\] inhomogeneous Poisson brackets corresponding to metrics of constant curvature (Mokhov - Ferapontov brackets) were considered.

In conclusion of this chapter, it can be noted that the brackets (II.13) as well as (II.8) - (II.9), (II.14), represent special classes of more general local one-dimensional field-theoretic brackets

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{ij}^k(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y)$$

and weakly non-local brackets

$$\{\varphi^i(x), \varphi^j(y)\} = \sum_{k \geq 0} B_{ij}^k(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y) +$$

\[\frac{1}{2} \sum_{k, s = 1}^g \kappa_{ks} S^i_{ks}(\varphi, \varphi_x, \ldots) \text{sgn}(x - y) S^j_{ks}(\varphi, \varphi_y, \ldots),\]

where \(\kappa_{ks}\) is some arbitrary (constant) quadratic form, \(i = 1, \ldots, n\), \(\varphi(x) = (\varphi^1(x), \ldots, \varphi^n(x))\).

As in the case of brackets of hydrodynamic type, brackets (II.16) can be associated with a very wide variety of systems of evolutionary type, while brackets (II.17) appear, as a rule, in integrable hierarchies (see [28, 43]).

### III. HAMILTONIAN STRUCTURES AND THE THEORY OF SLOW MODULATIONS

The development of the theory of integration of one-dimensional systems of hydrodynamic type actually took place in close connection with the development of the theory of slow modulations (Whitham’s method) of multiphase solutions of integrable hierarchies. As was first shown by G. Whitham (\[42\]), the averaged equations describing the evolution of slowly modulated parameters of single-phase solutions of the KdV equation are a hydrodynamic-type system reduced to a diagonal form. As was later shown in \[43\], this remarkable property is also inherent in the equations of slow modulations (Whitham’s equations) for multiphase KdV solutions. It can be immediately noted here that the construction of multiphase solutions for integrable hierarchies is closely related to the methods of algebraic geometry in the theory of the inverse scattering problem (see \[46, 52\]). As a consequence of this, the methods of the theory of slow modulations for integrable systems are also closely related to algebraic-geometric constructions. In particular, it was the methods of algebraic geometry that made it possible to establish the diagonalizability of Whitham’s equations for multiphase solutions of KdV in the paper \[45\]. Being a universal approach in the theory of integrable systems, the methods of algebraic geometry make it possible actually to establish a similar fact for most hierarchies integrable by the method of the inverse scattering problem. Thus, systems of equations for slow modulations for integrable hierarchies represent the most important class of diagonalizable systems of hydrodynamic type.

As we have already said, integration of diagonal systems of hydrodynamic type usually requires their Hamiltonian nature. In some cases, the Hamiltonian nature of systems of equations can be related to their Lagrangian formalism, which, as a rule, allows us to determine the corresponding Hamiltonian structure in canonical form. Along with the procedure for constructing equations of slow modulations, Whitham (\[44\]) also proposed a procedure for averaging local Lagrangians (also in the presence of “pseudophases”) and obtaining a local Lagrangian formalism for the “averaged system”. It must be said, however, that not all interesting systems of evolutionary type have local Lagrangian structures, and the most common for such systems is, as a rule, the presence of a more general Hamiltonian formalism with the Poisson bracket (II.16) (or (II.17)) and the Hamiltonian

$$H = \int_{-\infty}^{+\infty} h(\varphi, \varphi_x, \ldots) \, dx \quad (\text{III.2})$$

To construct Hamiltonian structures for the equations of slow modulations in the general case, B.A. Dubrovin and S.P. Novikov (\[10, 18\]) proposed a procedure of “averaging” of local Poisson brackets (II.16), which gives a bracket of the form (III.3) for the Whitham system. Let us give here a brief description of this procedure.

Namely, in Whitham’s method, we assume that the system (III.1) has a finite-parameter family of quasiperiodic solutions

$$\varphi^i(x, t) = \Phi^i(k(U) x + \omega(U) t + \theta_0, U) \quad (\text{III.3})$$

where \(\theta = (\theta^1, \ldots, \theta^m)\), \(k(U) = (k^1(U), \ldots, k^m(U))\), \(\omega(U) = (\omega^1(U), \ldots, \omega^m(U))\) and \(\Phi(\theta, U)\) define a family of \(2\pi\)-periodic in all \(\theta^i\) functions that depend on additional parameters \(U = (U^1, \ldots, U^N)\).

In the Whitham method, we stretch both the coordinates \(x\) and \(t\): \(X = \epsilon x, T = \epsilon t, (\epsilon \to 0)\), and consider the parameters \(U\) as functions of “slow” coordinates \(X\) and \(T\).

The method of B.A. Dubrovin and S.P. Novikov is based on the existence of \(N\) (equal to the number of parameters \(U^\nu\) on the family of \(m\)-phase solutions of (III.1)) local integrals

$$I^\nu = \int \mathcal{P}^\nu(\varphi, \varphi_x, \ldots) \, dx \quad (\text{III.4})$$
commuting with the Hamiltonian and with each other
\[ \{ I^\nu, H \} = 0 , \quad \{ I^\nu, I^\mu \} = 0 , \quad (\text{III.4}) \]
and can be described as follows:

Let us calculate the pairwise Poisson brackets of the densities \( P^\nu \), having the form
\[
\{ P^\nu(x), P^\mu(y) \} = \sum_{k \geq 0} A_k^\nu(\varphi, \varphi_x, \ldots) \delta^{(k)}(x - y) ,
\]
where
\[ A_k^\nu(\varphi, \varphi_x, \ldots) = \partial_x Q^{\mu}(\varphi, \varphi_x, \ldots) \]
according to (\text{III.3}). The corresponding Dubrovin - Novikov bracket on the space of functions \( U(X) \) has the form:
\[
\{ U^\nu(X), U^\mu(Y) \} = \langle A^\nu(\mu)(U) \rangle \delta(X - Y) + \partial_x Q^{\nu}(\varphi, \varphi_x, \ldots)
\]
where \( \langle \ldots \rangle \) means the averaging over the family of \( m \)-phase solutions of (\text{III.1}) given by the formula
\[
\langle F \rangle = \frac{1}{(2\pi)^m} \int_{0}^{2\pi} \cdots \int_{0}^{2\pi} F(\Phi, k^\nu(U) \Phi^{\nu}, \ldots) d^m \theta.
\]

In this case, the Dubrovin - Novikov method requires certain conditions of completeness and regularity of the corresponding family of \( m \)-phase solutions of (\text{III.1}) (see e.g. [53]).

Note here that the Dubrovin - Novikov procedure also admits a generalization to the case of weakly nonlocal Hamiltonian structures, which makes it possible to obtain the brackets (\text{III.9}) for the Whitham system by averaging the brackets (\text{III.7}) for the original system (\text{III.1}). On the whole, this procedure is quite convenient both for the case of integrable hierarchies and in a more general situation.

The representation of averaged Poisson brackets in diagonal form for integrable hierarchies, as well as the representation of the corresponding Whitham systems in this form, is associated with algebraic-geometric methods of the inverse scattering problem (see e.g. [18, 21, 55, 56]). In more detail, as in the case of diagonalization of the Whitham systems themselves, the diagonal coordinates for the averaged Poisson brackets for such hierarchies are related to the branch points of the Riemann surfaces that determine the corresponding \( m \)-phase solutions of the original system. As noted above, the diagonal form of such brackets is most closely related to the procedure for integrating the corresponding systems of hydrodynamic type. It can also be noted that the construction of the most interesting solutions to averaged equations is also based on algebraic-geometric methods (see e.g. [57, 68]).

Let us also focus here on the canonical forms of the averaged brackets.

One of the features of the canonical form of averaged Poisson brackets (see e.g. [44, 53, 69, 70]) is that all the functions \( k^\nu(U) \) are part of the canonical coordinates of the averaged bracket and thus represent a part of the flat coordinates for the corresponding metric \( g^{\nu\mu}(U) \) satisfying the relations
\[
\{ k^\nu(X), k^\mu(Y) \} = 0 \quad (\text{III.7})
\]
and can be described as follows:

In addition, it can be shown that the Poisson brackets of \( k^\nu(X) \) with the densities \( U^\nu(Y) \) can be written as
\[
\{ k^\nu(X), U^\nu(Y) \} = \left( \omega^\nu(X) \delta(X - Y) \right)_X \quad (\text{III.8})
\]
As was shown in [71], the relations (III.7) - (III.8) also allow us to modify the Dubrovin-Novikov procedure by reducing the required number of commuting integrals of the original system to $N - m$ and representing the averaged bracket in coordinates $(k^1, \ldots, k^m, U^1, \ldots, U^{N-m})$ in the form (III.7) - (III.8) and

$$\{U^\nu(X), U^\mu(Y)\} = \langle A^{\mu\nu}_1(U) \rangle \delta'(X - Y) + \frac{\partial(Q^\nu\mu)}{\partial U^\lambda} U^\lambda_X \delta(X - Y), \quad \nu, \mu = 1, \ldots, N - m$$

(III.9)

The representation of the averaged bracket in the form (III.7) - (III.9) is actually connected with another important (equivalent) representation of the system of slow modulation equations, separating the phase evolution equations

$$S^\alpha_T = \omega^\alpha(S_X, U)$$

($S^\alpha_T \equiv k^\alpha$) and the transport equations, which can be written as the balance equations for a part of the conservation laws

$$U^\nu_X = \langle Q^\nu \rangle_X, \quad \nu = 1, \ldots, N - m$$

The representation of a bracket in the form (III.7) - (III.9) is especially convenient in the case of several spatial variables. In the next chapter, we will discuss in more detail the “multi-dimensional” Poisson brackets.

**IV. MULTI-DIMENSIONAL HYDRODYNAMIC BRACKETS AND THEIR GENERALIZATIONS**

We now turn to more general multi-dimensional brackets of the form (I.1):

$$\{U^\nu(x), U^\mu(y)\} = g^\nu\mu(x) \delta(x - y) + b^\nu\mu_{\lambda}(x) U^\lambda_x \delta(x - y),$$

the most complete theory of which was constructed in the works [10, 72, 73].

We consider here the case of non-degenerate brackets (I.1), namely, we require that all tensors $g^\nu\mu(x)$ are non-degenerate

$$\det g^\nu\mu(x) \neq 0, \quad i = 1, \ldots, n$$

As in the one-dimensional case, here we can also assert that all tensors $g^\nu\mu(U)$ represent flat metrics on the space of parameters $U$, while the values

$$\Gamma^\nu_{\mu\lambda} = - g^\nu\sigma b^\sigma_{\mu\lambda}$$

give the corresponding Christoffel symbols. In general, to determine a correct Poisson bracket, the coefficients $g^\nu\mu(U)$ and $b^\nu\mu_{\lambda}(U)$ must satisfy a number of nontrivial relations (see [40, 72]), in particular, all the expressions

$$\{U^\nu(x), U^\mu(y)\}^{(i)} = g^\nu\mu(x) \delta(x - y) + b^\nu\mu_{\lambda}(x) U^\lambda_x \delta(x - y)$$

define in this case one-dimensional Poisson brackets compatible with each other.

By analogy with the one-dimensional case, one can define the canonical form of a non-degenerate Poisson bracket (I.1) as a constant bracket

$$\{n^\nu(x), n^\mu(y)\} = \eta^\nu\mu \delta_{x^i}(x - y),$$

where all $\eta^\nu\mu = \text{const}.$

It is easy to see that all the functionals

$$N^\nu = \int n^\nu(x) d^n x$$

represent in this case annihilators of the bracket (I.1).

In contrast to the one-dimensional case, however, the nondegeneracy of the bracket (I.1) here in the general case turns out to be insufficient for the possibility of its reduction to the canonical form by means of some point change of coordinates, and some additional general conditions are required. In particular (see [73]), a non-degenerate bracket (I.1) can be reduced to a constant form in some coordinates $n = n(U)$, if at least one of the metrics $g^\nu\mu_{\lambda}(U)$ forms non-singular pairs with all other metrics, that is, the roots of any of the equations

$$\det \left(g^\nu\mu_{\lambda}(U) - \lambda g^\nu\mu(U)\right) = 0, \quad i \neq i_0$$
are different from each other.

The above condition is generic, however, it is often violated in important examples. In particular, the Poisson brackets, corresponding to the Lie algebras of vector fields in $\mathbb{R}^n$ ($N = n, n \geq 2$):

$$\{p^i(x), p^j(y)\} = p^i(x) \delta_{x^i}(x - y) - p^j(y) \delta_{y^j}(y - x)$$

and describing $n$-dimensional hydrodynamics, cannot be reduced to a constant form. The same is true, in fact, for a number of other important examples of hydrodynamic brackets.

In the most general case, we can assert, however (see [40, 72, 73]), that any non-degenerate bracket (1.1) can be reduced to a linear (non-uniform) form

$$\{U^\nu(x), U^\mu(y)\} = (t^\mu_\lambda + b^\mu_{\nu\lambda}) U^\lambda \delta_{x^\lambda}(x - y) + b^\mu_{\nu\lambda} U^\lambda \delta(x - y), \quad (IV.1)$$

$$t^\mu_\lambda = \text{const}, \quad b^\mu_{\nu\lambda} = \text{const}$$

using a point change of coordinates.

Thus, it can be seen that the theory of non-degenerate Poisson brackets in the case of several space variables is connected in the most general case with the theory of infinite-dimensional Lie algebras. A number of important classification results related to the theory of multi-dimensional brackets (IV.1) and the corresponding Lie-algebraic structures were obtained in the work [72]. On the whole, however, the complete problem of classification of such brackets has not yet been finally solved.

It must be said that the diagonal form of the Poisson bracket in the case of many spatial variables no longer plays the important role that it plays in the one-dimensional case. As follows from the results of [73], nevertheless, in the case of two spatial variables, both metrics $g^{\mu\nu}(U)$ and $g^{\mu\nu2}(U)$ can be reduced to diagonal form by a transformation of coordinates, if they form a non-singular pair. In the case $n \geq 3$, generally speaking, the simultaneous reduction of all metrics $g^{\mu\nu}(U)$ to the diagonal form is impossible.

As in the one-dimensional case, one can define the Liouville form for multi-dimensional brackets (1.2), that has the form

$$\{U^\nu(x), U^\mu(y)\} = (\gamma^{\mu\nu}(U) + \gamma^{\mu\nu}(U)) \delta_{x^\lambda}(x - y) + (\gamma^{\mu\nu}(U))_{x^\lambda} \delta(x - y)$$

and corresponds to the situation when all the functionals

$$l^\nu = \int U^\nu(x) \, d^n x$$

commute with each other.

In conclusion, we would like to consider another generalization of brackets of hydrodynamic type, namely, brackets containing phase variables

$$\{S^\alpha(x), S^\beta(y)\} = 0 \quad (S^\alpha(x), U^\nu(y)) = \omega^{\alpha\nu}(U, S_x) \delta(x - y) \quad (IV.2)$$

$$\{U^\nu(x), U^\mu(y)\} = g^{\rho\mu\nu}(U, S_x) \delta_{x^\rho}(x - y) + b^\nu_{\rho\mu}(U, S_x) U^\lambda \delta_{x^\lambda}(x - y) + f^{\rho\mu\nu\lambda}(U, S_x) S^s_{x^s, x^\lambda} \delta(x - y)$$

As already indicated above, brackets of this type arise, for example, when averaging local Hamiltonian structures in the multi-dimensional case. In fact, such brackets are also encountered in many other cases, where the variables $S(x)$ can have very different meanings (classical or quantum phases, phases of the order parameter, etc.). In particular, brackets of this type are repeatedly encountered in the work [17]. Even more general brackets of this type can also include brackets where the phase variables do not commute with each other, but correspond to a certain Lie algebraic structure (see (IV.2)).

As noted above, brackets of the described type can also be found in “pure” hydrodynamics, for example, when describing potential flows. Quite often, the densities of phase variables do not commute with each other, but brackets of this type can also include brackets where the phase variables are expressions of the hydrodynamic part of the bracket (IV.2) has a Liouville form.

We will be interested here in the general structure of brackets (IV.2), and in particular, in the possibility of reducing such brackets to some canonical forms close to those considered above. From the physical point of view, the phase variables $S(x)$ are distinguished, so it is natural to actually consider changes of coordinates that preserve the variables $S(x)$ and transforming only the remaining coordinates $U(x)$. It must be said, however, that the values $S^s_x$ (“superfluid velocities”) are already variables of the hydrodynamic type and can naturally be used in transformations of the “hydrodynamic variables”

$$U^\nu \rightarrow \tilde{U}^\nu(S_x, U)$$

Here we consider in a sense a non-degenerate case when the frequency matrix $\omega^{\alpha\nu}$ has the full rank

$$\text{rank} ||\omega^{\alpha\nu}(S_x, U)|| = m$$

$(N - m \geq m)$. As can be shown (74), in this case it is always possible to switch to new hydrodynamic variables

$$U \rightarrow (Q(S_x, U), N(S_x, U))$$

$$Q_\alpha(x) = Q_\alpha(S_{x^1}, \ldots, S_{x^m}, U(x)) \quad \alpha = 1, \ldots, m$$

$$N^l(x) = N^l(S_{x^1}, \ldots, S_{x^m}, U(x)) \quad l = 1, \ldots, N - 2m.$$
in which the bracket (IV.2) becomes

\[ \{ S^\alpha(x), S^\beta(y) \} = 0 , \]

\[ \{ S^\alpha(x), Q_\beta(y) \} = \delta^\alpha_\beta \delta(x - y) , \]

\[ \{ S^\alpha(x), N^I(y) \} = 0 , \] (IV.3)

\[ \{ Q_\alpha(x), Q_\beta(y) \} = J_{\alpha\beta}[S_x, N](x, y) , \]

\[ \{ Q_\alpha(x), N^I(y) \} = J^I_\alpha[S_x, N](x, y) , \]

\[ \{ N^I(x), N^J(y) \} = J^{ij}[S_x, N](x, y) \]

The commutators \( J_{\alpha\beta}[S_x, N](x, y) \), \( J^I_\alpha[S_x, N](x, y) \) and \( J^{ij}[S_x, N](x, y) \) are given by expressions of hydrodynamic type similar to those presented in (IV.2), here, however, they depend only on the variables \( S(x) \) and \( N(x) \). It is also not difficult to show that the functionals \( J^{ij}[S_x, N](x, y) \) define a Poisson bracket on the space of fields \( N(x) \) for any fixed values of \( S(x) \).

As can be easily seen, the variables \( Q_\alpha(x) \) and \( N^I(x) \) are defined up to the transformations

\[ Q_\alpha(x) \to Q_\alpha(x) + f_\alpha(S_x, N(x)) , \]

\[ N^I(x) \to N^I(S_x, N(x)) \]

where

\[ \det \left| \frac{\partial N^I}{\partial N^2} \right| \neq 0 \]

Quite often, in fact, we are interested in the situation when the variables \( N^I(x) \) do not appear (\( N = 2m \)), and the bracket (IV.2) is reduced to the form

\[ \{ S^\alpha(x), S^\beta(y) \} = 0 , \]

\[ \{ S^\alpha(x), Q_\beta(y) \} = \delta^\alpha_\beta \delta(x - y) , \] (IV.4)

\[ \{ Q_\alpha(x), Q_\beta(y) \} = \Omega_{\alpha\beta}^i(S_x) \delta_x(x - y) + \Gamma^ij_{\alpha\beta\gamma}(S_x) S_\gamma^i(x) \delta(x - y) \]

\[ (\Gamma^ij_{\alpha\beta\gamma} = \Gamma^{ji}_{\alpha\beta\gamma}) \]

The Jacobi identities

\[ \{ \{ Q_\alpha(x), Q_\beta(y) \}, Q_\gamma(z) \} + c.p. = 0 \]

now give the relations

\[ \frac{\delta J_{\alpha\beta}(S_x)(x, y)}{\delta S^\gamma(z)} + \frac{\delta J_{\beta\gamma}(S_x)(y, z)}{\delta S^\alpha(x)} + \frac{\delta J_{\gamma\alpha}(S_x)(z, x)}{\delta S^\beta(y)} + \cdots = 0 \]

for the functionals \( J_{\alpha\beta}(S_x)(x, y) \), which mean that the 2-form

\[ \int J_{\alpha\beta}[S_x](x, y) S^\alpha(x) \wedge S^\beta(y) \, d^n x \, d^n y \]

is closed on the space of fields (\( S^1(x), \ldots, S^n(x) \)).

The brackets (IV.3) belong to the general class of brackets, named in [75] as “variationally admissible.” The variationally admissible form of Poisson brackets is directly related to a possibility of a Lagrangian description of the corresponding dynamical systems and, as was shown in [75], such brackets generally lead to a nontrivial Lagrangian representation of Hamiltonian systems, where the Lagrange functional is in fact a 1-form, which has nontrivial topological properties.

In our case, we need to remember, in fact, that when reducing the bracket (IV.3) to the canonical form, we are limited only to the transformations of the “hydrodynamic type” presented above. As it was shown in [74], any bracket (IV.3) can be locally reduced to the canonical form

\[ \{ S^\alpha(x), S^\beta(y) \} = 0 , \]

\[ \{ S^\alpha(x), Q_\beta(y) \} = \delta^\alpha_\beta \delta(x - y) , \]

\[ \{ Q_\alpha(x), Q_\beta(y) \} = 0 \]

using the transformation

\[ Q_\alpha(x) \to Q_\alpha(x) + f_\alpha(S_x) \]

As a consequence, in this case the corresponding Hamiltonian system can also be written in Lagrangian form with the Lagrangian of the “hydrodynamic type”

\[ \delta \int \left[ Q_\alpha(X) S^n_i - \langle P_H \rangle (S_x, Q(X)) \right] \, d^n x \, dt = 0 \]

As for the general brackets (IV.3), here we can also raise the question of further reducing them to their canonical form and, in particular, of separating brackets for the variables \( (S(x), U(x)) \) and \( N(x) \). In fact, such a possibility often arises in special examples and, in particular, in the theory of slow modulations for multidimensional systems. It can be shown, however, that in the most general case it is impossible to reduce the brackets (IV.3) to such a canonical form using a transformation of the “hydrodynamic type” ([74]).

In conclusion, we note also that the brackets (IV.2) often have a natural continuation to the extended phase space, in which the variables \( v^n_i = S^n_i \) can be considered completely independent. Extensions of this type can to some extent be naturally called the “vortizations” of the brackets (IV.2). Such continuations naturally arise not only, for example, in hydrodynamics, during the transition from potential to vortex flows, but also, for example, in the description of the motion of superfluids carrying quantum vortex structures inside their volume, etc. Important examples of such “vortizations” of the brackets like (IV.2) are given in the work [77].
V. CONCLUSION

This paper provides a brief overview of the Poisson brackets of hydrodynamic type and their special generalizations. We consider questions related to various forms of such brackets and, in particular, to representations generalizing the canonical forms of Poisson brackets in the situation under consideration. The connection between brackets of hydrodynamic type and the theory of Lie algebras in the cases of one and several spatial variables is considered. The connection of the considered structures with the theory of integration of systems of hydrodynamic type in the one-dimensional case is described in particular detail.

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