Twisted representations of vertex operator algebras and associative algebras  
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Abstract

Let $V$ be a vertex operator algebra and $g$ an automorphism of order $T$. We construct a sequence of associative algebras $A_{g,n}(V)$ with $n \in \frac{1}{T}\mathbb{Z}$ nonnegative such that $A_{g,n}(V)$ is a quotient of $A_{g,n+1/T}(V)$ and a pair of functors between the category of $A_{g,n}(V)$-modules which are not $A_{g,n-1/T}(V)$-modules and the category of admissible $V$-modules. These functors exhibit a bijection between the simple modules in each category. We also show that $V$ is $g$-rational if and only if all $A_{g,n}(V)$ are finite-dimensional semisimple algebras.

1 Introduction

In this paper we continue our study of twisted representations of vertex operator algebras and lay some further foundations of orbifold conformal field theory.

Given a vertex operator algebra $V$ and an automorphism $g$ of finite order $T$, we have constructed an associative algebra $A_g(V)$ in [DLM1] such that there is a bijection between simple $A_g(V)$-modules and irreducible admissible $g$-twisted $V$-modules, generalizing Zhu’s algebra $A(V)$ [Z]. It was proved in [DLM1] that $A_g(V)$ is a finite-dimensional semisimple algebra if $V$ is $g$-rational. But it is not clear whether the semisimplicity of $A_g(V)$ implies the $g$-rationality of $V$. Partially motivated by this in the case $g = 1$ and by the induced module theory for vertex operator algebra we constructed a series of associative algebras $A_n(V)$ for nonnegative integer $n$ in [DLM2] so that $A_{n-1}(V)$ is quotient algebra of $A_n(V)$.
induced from the identity map on $V$ and that there is a one to one correspondence between the simple $A_n(V)$-modules which cannot factor through $A_{n-1}(V)$ and irreducible admissible $V$-modules. Moreover, $V$ is rational if and only if all $A_n(V)$ are finite-dimensional semisimple algebras. In the case $n = 0$ we have Zhu's original algebra $A(V)$.

In this paper we will study the twisted analogues of the algebras $A_n(V)$. In particular, we will construct a series of associative algebras $A_{g,n}(V)$ for nonnegative numbers $n \in \frac{1}{T} \mathbb{Z}$ and show that $A_{g,n-1/T}(V)$ is a natural quotient of $A_{g,n}(V)$. As in the untwisted case, there is a bijection between the simple $A_{g,n}(V)$-modules which cannot factor through $A_{g,n-1/T}(V)$ and the irreducible admissible $g$-twisted $V$-modules. In the case $g = 1$ we recover the algebras $A_n(V)$ and the case $n = 0$ amounts to the algebra $A_g(V)$.

Since most results in this paper are similar to those in [DLM2] where $g = 1$, we refer the reader in a lot of places of this paper to [DLM2] for details. We assume that the reader is familiar with the elementary theory of vertex operator algebras as found in [B], [FLM], [FHL] and the definition of twisted modules for vertex operator algebras and related ones as presented in [DLM1].

This paper is organized as follows: In Section 2 we introduce the associative algebras $A_{g,n}(V)$. In Section 3 we construct the functor $\Omega_n$ from admissible $g$-twisted modules to $A_{g,n}(V)$-modules. We show that the each homogeneous subspaces of of the first $n$ pieces is a module for $A_{g,n}(V)$. The Section 4 is the heart of the paper. In this section we construct the functor $L_n$ from $A_{g,n}(V)$-modules to admissible $g$-twisted $V$-modules. The main strategy is to prove the associativity for twisted vertex operators.

2 The associative algebra $A_{g,n}(V)$

Let $V = (V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra and $g$ be an automorphism of $V$ of order $T$. Then $V$ is a direct sum of eigenspaces of $T$:

$$V = \bigoplus_{r=0}^{T-1} V^r$$

(2.1)

where $V^r = \{v \in V | gv = e^{-2\pi i r/T} v\}$.

Fix $n = l + \frac{r}{T} \in \frac{1}{T} \mathbb{Z}$ with $l$ a nonnegative integer and $0 \leq i \leq T - 1$. For $0 \leq r \leq T - 1$ we define $\delta_i(r) = 1$ if $i \geq r$ and $\delta_i(r) = 0$ if $i < r$. We also set $\delta_i(T) = 1$. Let $O_{g,n}(V)$ be the linear span of all $u \circ_{g,n} v$ and $L(-1)u + L(0)u$ where for homogeneous $u \in V^r$ and $v \in V$,

$$u \circ_{g,n} v = \text{Res}_z Y(u, z)v \frac{(1 + z)^{wtu - 1 + \delta_i(r) + l + r/T}}{z^{2l + \delta_i(r) + \delta_i(T-r)}}.$$  

(2.2)
Define the linear space $A_{g,n}(V)$ to be the quotient $V/O_{g,n}(V)$. Then $A_{g,n}(V)$ is the untwisted associative algebra $A_n(V)$ as defined in [DLM2] if $g = 1$ and is $A_g(V)$ in [DLM1] if $n = 0$.

We also define a second product $*_g,n$ on $V$ for $u$ and $v$ as above:

\[ u*_g,n v = \sum_{m=0}^{l} (-1)^m \binom{m+l}{l} \text{Res}_2 Y(u, z) \frac{(1+z)^{wtu+l}}{z^{l+m+1}} v \quad (2.3) \]

if $r = 0$ and $u*_g,n v = 0$ if $r > 0$. Extend linearly to obtain a bilinear product on $V$.

**Lemma 2.1** If $r \neq 0$ then $V^r \subset O_{g,n}(V)$.

**Proof:** Let $v \in V^r$ be homogeneous. Then $v \circ_{g,n} 1 \in O_{g,n}(V)$. From the definition we know that

\[ v \circ_{g,n} 1 = \sum_{j=0}^{\infty} \binom{wtv - 1 + l + \delta_i(r) + r/T}{j} v_{j-2l-\delta_i(r)-\delta_i(T-r)} 1. \]

Note that $v_k1 = 0$ and $v_{-k-1}1 = \frac{1}{M} L(-1)^k v$ for $k \geq 0$. Using $L(-1)v \equiv -L(0)v$ modulo $O_{g,n}(V)$ we see that $v \circ_{g,n} 1 \equiv \sum_{j=0}^{2l-1+\delta_i(r)+\delta_i(T-r)} a_j r^j v$ where $a_j$ are integers and $a_{2l-1+\delta_i(r)+\delta_i(T-r)} = \pm 1$. Thus $v \circ_{g,n} 1 \equiv cv$ modulo $O_{g,n}$ for a nonzero constant $c$. This shows that $v \in O_{g,n}(V)$. □

**Lemma 2.2** (i) Assume that $u \in V$ is homogeneous, $v \in V$ and $m \geq k \geq 0$. Then

\[ \text{Res}_2 Y(u, z)v \frac{(1+z)^{wtu-l+\delta_i(r)+\frac{k}{T}+k}}{z^{2l+\delta_i(r)+\delta_i(T-r)+m}} \in O_{g,n}(V). \]

(ii) For homogeneous $u, v \in V^0$, $u*_g,n v - v*_g,n u - \text{Res}_2 Y(u, z)v(1+z)^{wtu-l-1} \in O_{g,n}(V)$.

**Proof:** The proof of (i) is similar to that of Lemma 2.1.2 of [Z]. (ii) follows from a result in Lemma 2.1 (iii) of [DLM2] that $u*_g,n v - v*_g,n u - \text{Res}_2 Y(u, z)v(1+z)^{wtu-l-1} \in O_{1,l}(V^0)$ and the containment $O_{1,l}(V^0) \subset O_{g,n}(V)$. □

**Lemma 2.3** (i) $O_{g,n}(V)$ is a 2 sided ideal of $V$ under $*_g,n$.

(ii) If $I = O_{g,n} \cap V^0$ then $I/O_{1,l}(V^0)$ is a two-sided ideal of $A_{1,l}(V^0)$.
Proof: Since $V^r$ ($r > 0$) is a subset of $O_{g,n}$ by Lemma 2.1, we see that $O_{g,n}(V) = I \oplus (\oplus_{r=1}^{T} V^r)$. Clearly $V^0 *_{g,n} V^r \subset V^r$. So (i) and (ii) are equivalent. We prove (ii). Choose $c \in V^0$ homogeneous and $u \in I$. Using Lemma 2.2 (i) and the argument used to prove Proposition 2.3 of [DLM1] we show that both $c *_{g,n} u$ and $u *_{g,n} c$ lie in $O_{g,n}(V)$. □

The first main result is the following:

**Theorem 2.4** (i) The product $*_{g,n}$ induces the structure of an associative algebra on $A_{g,n}(V)$ with identity $1 + O_{g,n}(V)$.

(ii) The linear map

$$\phi : v \mapsto e^L(1)(-1)^{L(0)} v$$

induces an anti-isomorphism $A_{g,n}(V) \to A_{g-1,n}(V)$.

(iii) $\omega + O_{g,n}(V)$ is a central element of $A_{g,n}(V)$.

**Proof:** (i) follows from the result in [DLM2] that $A_{1,l}(V^0)$ is an associative algebra with respect to $*_{g,n}$ and Lemma 2.3 (i). The proof of (ii) is similar to that of Theorem 2.4 (ii) of [DLM1]. (iii) follows from Theorem 2.3 (iii) of [DLM2] which says that $\omega + O_{1,l}(V^0)$ is a central element of $A_{l,1}(V^0)$ and Lemma 2.3 (ii). □

**Proposition 2.5** The identity map on $V$ induces an onto algebra homomorphism from $A_{g,n}(V)$ to $A_{g,n-1/T}(V)$.

**Proof:** If $n = l + i/T$ with $i \geq 1$ then both $A_{g,n}$ and $A_{g,n-1/T}$ are quotients of $A_{l,T}(V^0)$. Otherwise $i = 0$ and $A_{g,n}$ is a quotient algebra of $A_{1,l}(V^0)$ and $A_{g,n-1/T}$ is a quotient algebra of $A_{1,l-1}(V^0)$. By Proposition 2.5 of [DLM2] the identity map induces an epimorphism from $A_{1,l}(V^0)$ to $A_{1,l-1}(V^0)$. So it is enough to show that $O_{g,n}(V) \cap V^0 \subset O_{g,n-1/T}(V) \cap V^0$. But this follows from Lemma 2.2 (i) immediately. □

As in [DLM2], Proposition 2.3 in fact gives us an inverse system $\{A_{g,n}(V)\}$. Denote by $I_{g,n}(V)$ the inverse limit $\lim_{\leftarrow} A_{n}(V)$. Then

$$I_{g}(V) = \{a = (a_n + O_{g,n}(V)) \in \prod_{n \geq 0} A_{g,n}(V) | a_n - a_{n-1/T} \in O_{n-1/T}(V) \}.$$  \hspace{1cm} (2.4)

An interesting problem is to determine $I_{g}(V)$ explicitly and to study the representations of $I_{g}(V)$.
3 The Functor $\Omega_n$

Recall from [DLM1] the Lie algebra $V[g]$

$$V[g] = \mathcal{L}(V, g)/D\mathcal{L}(V, g)$$

where

$$\mathcal{L}(V, g) = \oplus_{r=0}^{T-1} t^{r/T} \mathcal{C}[t, t^{-1}] \otimes V^r$$

and by $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$. In order to write down the Lie bracket we introduce the notation $a(q)$ which is the image of $t^q \otimes a \in \mathcal{L}(V, g)$ in $V[g]$. Let $a \in V^r$, $v \in V^s$ and $m, n \in \mathbb{Z}$. Then

$$[a(m + \frac{r}{T}), b(n + \frac{s}{T})] = \sum_{i=0}^{\infty} \binom{m + \frac{r}{T}}{i} a_i b(m + n + \frac{r+s}{T} - i).$$

In fact $V[g]$ is $\frac{1}{T} \mathbb{Z}$-graded Lie algebra by defining the degree of $a(m)$ to be $\text{wt} v - m - 1$ if $v$ is homogeneous. Denote the homogeneous subspace of degree $m$ by $\hat{V}[g]_m$. In particular, $\hat{V}[g]_0$ is a Lie subalgebra.

By Lemma [2.2] (ii) we have

**Proposition 3.1** Regarded $A_{g,n}(V)$ as a Lie algebra, the map $v(\text{wt} v - 1) \mapsto v + O_{g,n}(V)$ is a well-defined onto Lie algebra homomorphism from $\hat{V}[g]_0$ to $A_{g,n}(V)$.

For a module $W$ for the Lie algebra $V[g]$ and a nonnegative $m \in \frac{1}{T} \mathbb{Z}$ we let $\Omega_m(W)$ denote the space of “$m$-th lowest weight vectors,” that is

$$\Omega_m(W) = \{ u \in W | V[g] \cdot u = 0 \text{ if } k \geq m \}. \tag{3.1}$$

Then $\Omega_m(W)$ is a module for the Lie algebra $V[g]_0$.

Note that for a $g$-twisted weak $V$-module $M$ the map $v(m) \mapsto v_m$ for $v \in V$ and $m \in \frac{1}{T} \mathbb{Z}$ gives a representation of $V[g]$ on $M$ [DLM2]. For a homogeneous $v \in V$ we set $o_p(v) = v_{\text{wt} v - 1 - p}$ on $M$.

**Lemma 3.2** Let $M$ be a weak $V$-module. Then for any homogeneous $u \in V^r$, $v \in V^s$, $p \in \frac{1}{T} + Z$, $q \in \frac{1}{T} + Z$ with $p \geq q \geq -n$ and $p + q \geq 0$ there exists a unique $w_{u,v}^{p,q} \in V^{r+s}$ such that $o_p(u)o_q(v) = o_{p+q}(w_{u,v}^{p,q})$ on $\Omega_m(M)$. In particular if $s = T - r$ and $p = l + \delta_i(T - r) - k - \frac{r}{T} = -q$ for $k = 0, ..., l$

$$w_{u,v}^{p,-p} = \sum_{m=0}^{k} (-1)^m \binom{2l+\delta_i(r)+\delta_i(T-r)-1+m-k}{m} \text{Res}_z Y(u, z) v^m (1 + z)^{\text{wt} u + l - 1 + \delta_i(r) + \frac{r}{T}} z^{2l+\delta_i(r)+\delta_i(T-r)-k+m}. \tag{3.1}$$
The proof is similar to that of Theorem 3.2 of [DLM2]. This lemma is important in constructing admissible $g$-twisted modules from $A_{g,n}(V)$-modules in the next section.

**Theorem 3.3** Suppose that $M$ is a weak $V$-module. Then there is a representation of the associative algebra $A_{g,n}(V)$ on $\Omega_n(M)$ induced by the map $a \mapsto o(a) = a_{\text{wt}a-1}$ for homogeneous $a \in V$.

**Proof:** By Theorem 5.1 of [DLM2], the map $a \mapsto o(a) = a_{\text{wt}a-1}$ for homogeneous $a \in V^0$ induces a representation of $A_{1,l}(V^0)$ on $\Omega_n(M)$. So it is enough to show that $o(a) = 0$ for $a \in O_{g,n}(V) \cap V^0$. It is clear that $o(L(-1)u + L(0)u) = 0$. It remains to show that $o(u \circ_{g,n} v) = 0$ for $u \in V^r$ and $v \in V^{T-r}$.

Recall identity (10) from [DL]: for $p \in \mathbb{Z}$ and $s, t \in \mathbb{Q},$

$$
\sum_{m \geq 0} (-1)^m \binom{p}{m} (u_{p+s-m}v_{t+m} - (-1)^p v_{p+t-m}u_{s+m}) = \sum_{m \geq 0} \binom{s}{m} (u_{p+m}v)_{s+t-m}.
$$

(3.2)

Now take $-p = 2l + \delta_i(r) + \delta_i(T-r)$, $s = wtu - 1 + l + \delta_i(r) + T$, $t = wtv - 1 + l + \delta_i(T-r) + \frac{T-r}{T}$. Then on $\Omega_n(M)$ we have $u_{s+m} = v_{t+m} = 0$ for $m \geq 0$. Thus on $\Omega_n(M),$

$$
0 = \sum_{m \geq 0} \binom{s}{m} (u_{p+m}v)_{s+t-m} \\
= o(\text{Res}_z Y(u, z)v \frac{(1+z)^{wtu-1+\delta_i(r)+l+r/T}}{z^{2l+\delta_i(r)+\delta_i(T-r)}}) \\
= o(u \circ_{g,n} v).
$$

This completes the proof. □

Let $M = \oplus_{m \geq 0, m \in \frac{1}{2} \mathbb{Z}} M(m)$ be an admissible $g$-twisted module with $M(0) \neq 0$.

**Proposition 3.4** The following hold

(i) $\Omega_n(M) \supset \oplus_{i=0}^n M(i)$. If $M$ is simple then $\Omega_n(M) = \oplus_{i=0}^n M(i)$.

(ii) Each $M(p)$ is an $\hat{V}[g]_0$-module and $M(p)$ and $M(q)$ are inequivalent if $p \neq q$ and both $M(p)$ and $M(q)$ are nonzero. If $M$ is simple then each $M(p)$ is an irreducible $\hat{V}[g]_0$-module.

(iii) Assume that $M$ is simple. Then each $M(i)$ for $i = 0, ..., n$ is a simple $A_{g,n}(V)$-module and $M(i)$ and $M(j)$ are inequivalent $A_{g,n}(V)$-modules.

The proof is similar to that of Proposition 3.4 of [DLM2].
4 The functor $L_n$

In Section 3 we have shown how to obtain an $A_{g,n}(V)$-module from an admissible $g$-twisted $V$-module. We show in this section that there is a universal way to construct an admissible $g$-twisted $V$-module from an $A_{g,n}(V)$-module which cannot factor through $A_{g,n-1/T}(V)$. (If it can factor through $A_{g,n-1/T}(V)$ we can consider the same procedure for $A_{g,n-1/T}(V)$.) As in [DLM2], a certain quotient of the universal object is an admissible $g$-twisted $V$-module $L_n(U)$ and $L_n$ defines a functor which is a right inverse to the functor $\Omega_n/\Omega_{n-1/T}$ where $\Omega_n/\Omega_{n-1/T}$ is the quotient functor $M \mapsto \Omega_n(M)/\Omega_{n-1/T}(M)$.

Fix an $A_{g,n}(V)$-module $U$ which cannot factor through $A_{g,n-1/T}(V)$. Then it is a module for $A_{g,n}(V)_{Lie}$ in an obvious way. By Proposition [3.1] we can lift $U$ to a module for the Lie algebra $V[g]_0$, and then to one for $P_n = \oplus_{p>n} V[g]_p \oplus V[g]_0$ by letting $V[g]_p$ act trivially. Define

$$M_n(U) = \text{Ind}^{V[g]}_{P_n}(U) = U(V[g]) \otimes_{U(P_n)} U.$$  \hfill (4.1)

If we give $U$ degree $n$, the $\frac{1}{n}Z$-gradation of $V[g]$ lifts to $M_n(U)$ which thus becomes a $\frac{1}{n}Z$-graded module for $V[g]$. The PBW theorem implies that $M_n(U)(i) = U(V[g])_{i-n}U$.

We define for $v \in V$,

$$Y_{M_n(U)}(v, z) = \sum_{m \in \frac{1}{n}Z} v(m)z^{-m-1} \hfill (4.2)$$

As in [DLM1], $Y_{M(U)}(v, z)$ satisfies all conditions of a weak $g$-twisted $V$-module except the associativity which does not hold on $M_n(U)$ in general. We have to divide out by the desired relations.

Let $W$ be the subspace of $M_n(U)$ spanned linearly by the coefficients of

$$(z_0 + z_2)^{w_1 + l + \delta(r)} \mathcal{T} Y(a, z_0 + z_2)Y(b, z_2)u$$

$$- (z_2 + z_0)^{w_1 + l + \delta(r)} \mathcal{T} Y(a, z_0)b, z_2)u$$  \hfill (4.3)

for any homogeneous $a \in V^r$, $b \in V$, $u \in U$. Set

$$\bar{M}_n(U) = M_n(U)/U(V[g])W.$$  \hfill (4.4)

**Theorem 4.1** The space $\bar{M}_n(U) = \sum_{m \geq 0} \bar{M}_n(U)(m)$ is an admissible $g$-twisted $V$-module with $\bar{M}_n(U)(0) \neq 0$, $\bar{M}_n(U)(n) = U$ and with the following universal property: for any weak $g$-twisted $V$-module $M$ and any $A_{g,n}(V)$-morphism $\phi : U \to \Omega_n(M)$, there is a unique morphism $\phi : \bar{M}_n(U) \to M$ of weak $g$-twisted $V$-modules which extends $\phi$. 

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See Theorem 4.1 of [DLM2] for a similar proof.

Let \( U^* = \text{Hom}_\mathbb{C}(U, \mathbb{C}) \) and let \( U_s \) be the subspace of \( M_n(U)(n) \) spanned by “length” \( s \) vectors

\[
o_{p_1}(a_1) \cdots o_{p_s}(a_s)U
\]

where \( p_1 \geq \cdots \geq p_s, p_1 + \cdots + p_s = 0, p_i \neq 0, p_s \geq -n \) and \( a_i \in V \). Then by PBW theorem \( M_n(U)(n) = \sum_{s \geq 0} U_s \) with \( U_0 = U \) and \( U_s \cap U_t = 0 \) if \( s \neq t \). Motivated by the results in Lemma 3.2 we extend \( U^* \) to \( M_n(U)(n) \) inductively so that

\[
\langle u', o_{p_1}(a_1) \cdots o_{p_s}(a_s)u \rangle = \langle u', o_{p_1+p_2}(w_{a_1,a_2})o_{p_3}(a_3) \cdots o_{p_s}(a_s)u \rangle.
\] (4.5)

where \( o_j(a) = a(w-ta-1-j) \) for homogeneous \( a \in V \). We further extend \( U^* \) to \( M_n(U) \) by letting \( U^* \) annihilate \( \oplus_{i \neq n} M(U)(i) \).

Set

\[
J = \{ v \in M_n(U) | \langle u', xv \rangle = 0 \text{ for all } u' \in U^*, all \ x \in U(V[g]) \}.
\]

We can now state the second main result of this section.

**Theorem 4.2** Space \( L_n(U) = M_n(U)/J \) is an admissible \( V \)-module satisfying \( L_n(U)(0) \neq 0 \) and \( \Omega_n/\Omega_{n-1/T}(L_n(U)) \cong U \). Moreover \( L_n \) defines a functor from the category of \( A_{g,n}(V) \)-modules which cannot factor through \( A_{g,n-1/T}(V) \) to the category of admissible \( V \)-modules such that \( \Omega_n/\Omega_{n-1/T} \circ L_n \) is naturally equivalent to the identity. Moreover, \( L_n(U) \) is a quotient module of \( M_n(U) \).

The proof of this theorem is the most complicated one. Fortunately we can follow the proof of Theorem 4.2 of [DLM2] step by step with suitable modifications. Again we refer the reader to [DLM2] for details.

The analogue of Theorem 4.9 of [DLM2] (whose proof is easy) is the following.

**Theorem 4.3** \( L_n \) and \( \Omega_n/O_{n-1/T} \) are equivalences when restricted to the full subcategories of completely reducible \( A_{g,n}(V) \)-modules whose irreducible components cannot factor through \( A_{g,n-1/T}(V) \) and completely reducible admissible \( g \)-twisted \( V \)-modules respectively. In particular, \( L_n \) and \( \Omega_n/O_{n-1/T} \) induce mutually inverse bijections on the isomorphism classes of simple objects in the category of \( A_{g,n}(V) \)-modules which cannot factor through \( A_{g,n-1/T}(V) \) and admissible \( g \)-twisted \( V \)-modules respectively.

We also have the generalization of Theorem 4.10 of [DLM2] with a similar proof.
Theorem 4.4  Suppose that $V$ is a $g$-rational vertex operator algebra. Then the following hold:

(a) $A_{g,n}(V)$ is a finite-dimensional, semisimple associative algebra.

(b) The functors $L_n$ and $\Omega_n/O_{n-1/T}$ are mutually inverse categorical equivalences between the category of $A_{g,n}(V)$-modules whose irreducible components cannot factor through $A_{g,n-1/T}(V)$ and the category of admissible $g$-twisted $V$-modules.

(c) The functors $L_n, \Omega_n/O_{n-1/T}$ induce mutually inverse categorical equivalences between the category of finite-dimensional $A_{g,n}(V)$-modules whose irreducible components cannot factor through $A_{g,n-1/T}(V)$ and the category of ordinary $g$-twisted $V$-modules.

As in [DLM2] one expects that if $A_{g,0}(V)$ is semisimple then $V$ is $g$-rational. We present some partial results which are applications of $A_{g,n}(V)$-theory.

Theorem 4.5  All $A_{g,n}(V)$ are finite-dimensional semisimple algebras if and only if $V$ is $g$-rational.

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