Non-existence of complete Kähler metric of negatively pinched holomorphic sectional curvature

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Abstract

We prove a theorem which provides a sufficient condition for the non-existence of a complete Kähler–Einstein metric of negative scalar curvature of which holomorphic sectional curvature is negatively pinched: Let \( \Omega \) be a bounded weakly pseudoconvex domain in \( \mathbb{C}^n \) with a Kähler metric \( \omega \) whose holomorphic sectional curvature is negative near the topological boundary of \( \Omega \) (with respect to the relative topology of \( \mathbb{C}^n \)) and \( \omega \) admits quasi-bounded geometry. Then \( \omega \) is uniformly equivalent to the Kobayashi–Royden metric and the following dichotomy holds:

1. \( \omega \) is complete, and \( \omega \) is uniformly equivalent to the complete Kähler–Einstein metric with negative scalar curvature.
2. \( \omega \) is incomplete, and there is no complete Kähler metric with negatively pinched holomorphic sectional curvature. Moreover, \( \Omega \) is Carathéodory incomplete.

Our approach is based on the construction of a Kähler metric of negatively pinched holomorphic sectional curvature and applying the implication of equivalence of invariant metrics inspired by Wu-Yau.

1 Introduction

This following question has been addressed very recently in relation to anti-bisectional curvature in a paper by Khan-Zheng [5]:

**Question 1** When is it the case that a Kähler manifold does not admit a Kähler-Einstein metric with negative holomorphic sectional curvature?

To address this question, it is reasonable to consider a bounded weakly pseudoconvex domain in \( \mathbb{C}^n \) because pseudoconvexity is a necessary and sufficient condition for existence of complete Kähler-Einstein metric of negative scalar curvature on bounded domains in \( \mathbb{C}^n \) by Mok-Yau [8]. In this paper, we show the following:

**Theorem 2** Let \( \Omega \) be a bounded weakly pseudoconvex domain in \( \mathbb{C}^n \) with a Kähler metric \( \omega \) whose holomorphic sectional curvature is negatively pinched near the topological boundary of \( \Omega \) (with respect to relative topology of \( \mathbb{C}^n \)) and \( \omega \) admits quasi-bounded geometry. Then \( \omega \) is uniformly equivalent to the Kobayashi–Royden metric and the following dichotomy holds:

1. \( \omega \) is complete, and \( \omega \) is uniformly equivalent to the complete Kähler–Einstein metric with negative scalar curvature, or
2. \( \omega \) is incomplete, and there is no complete Kähler metric with negatively pinched holomorphic sectional curvature. Moreover, \( \Omega \) is Carathéodory incomplete.

Our approach is by constructing one Kähler metric of negatively pinched holomorphic sectional curvature on \( \Omega \) (not just near the boundary) and applying its consequences of equivalence of invariant metrics based on Wu-Yau’s work [9]. The dichotomy happens because the completeness of the Kähler metric \( \omega \) is not required in the process of construction of the Kähler metric of negatively pinched holomorphic sectional curvature in Theorem 2.

For classes of bounded domains in \( \mathbb{C}^n \) with uniform squeezing properties which must be weakly pseudoconvex, the quasi-bounded geometry of the Bergman metric and the
complete Kähler–Einstein metric of negative scalar curvature are necessary conditions by work of Yau [11, Theorem 2-(b)]. Also, it is known that holomorphic sectional curvature of the Bergman metric is negative near the boundary for smoothly bounded strictly pseudoconvex domains [6], and for the symmetrized bidisk by recent work of Cho-Yuan [3] (also see [12]). On the other hand, there exists an example of a bounded domain in \(\mathbb{C}^n\) with an incomplete Bergman metric, and one can take the product of this domain for higher dimensional cases [1]. Also, there are several known bounded weakly pseudoconvex domains that are not Kobayashi complete which implies Carathéodory incomplete, refer to [4, Chapter 14].

Theorem 2 under the incompleteness of \(\omega\) implies the following:

**Corollary 3** Let \(\Omega\) be a bounded weakly pseudoconvex domain in \(\mathbb{C}^n\) with an incomplete Kähler metric \(\omega\) satisfying all assumptions in Theorem 2. Then there is no complete Kähler metric with negatively pinched holomorphic sectional curvature. In particular, any complete Kähler–Einstein metric of negative scalar curvature does not admit a negatively pinched holomorphic sectional curvature.

Theorem 2 under the incompleteness of \(\omega\) implies the equivalence of two invariant metrics on closed submanifolds of \(\Omega\):

**Corollary 4** Given a bounded weakly pseudoconvex domain \(\Omega\) in \(\mathbb{C}^n\) with a complete Kähler metric \(\omega\) satisfying all assumptions in Theorem 2, any closed complex submanifold \(S\) of \(\Omega\) admits a complete Kähler metric \(g\) with holomorphic sectional curvature negatively pinched. Moreover, \(g\), a unique complete Kähler–Einstein metric \(g^{KE}\) of negative scalar curvature, and the Kobayashi-Royden metric \(g^K\) are uniformly equivalent.

Theorem 2 also has implications for quotient spaces. For quotient spaces, we define a lattice \(\Gamma\) on \(\Omega\) to be a discrete group acting properly discontinuously as biholomorphisms on \(\Omega\).

**Corollary 5** Given a bounded weakly pseudoconvex domain \(\Omega\) in \(\mathbb{C}^n\) with a complete Kähler metric \(\omega\) satisfying all assumptions in Theorem 2, assume that \(\Gamma\) is a torsion-free lattice on \(\Omega\). Then a compact quotient \(N = \Omega/\Gamma\) has to be a projective algebraic variety of general type. A non-compact quotient which has finite volume with respect to \(\omega\) has to be a quasi-projective variety of log-general type.

## 2 Invariant metrics, Quasi-bounded geometry, Holomorphic sectional curvature

Let \(\Omega\) be a domain in \(\mathbb{C}^n\). A pseudometric \(F(z, u) : \Omega \times \mathbb{C}^n \to [0, \infty]\) on a domain \(\Omega\) in \(\mathbb{C}^n\) is called (biholomorphically) invariant if \(F(z, \lambda u) = |\lambda|F(z, u)\) for all \(\lambda \in \mathbb{C}^n\), and \(F(z, u) = F(f(z), f'(z)u)\) for any biholomorphism \(f : \Omega \to \Omega'\). The Kobayashi-Royden metric, Bergman metric, and a Kähler-Einstein metric of negative scalar curvature are examples of invariant metrics on bounded weakly pseudoconvex domains in \(\mathbb{C}^n\).

For a bounded domain \(\Omega\) in \(\mathbb{C}^n\), denote by \(A^2(\Omega)\) the holomorphic functions in \(L^2(\Omega)\). Let \(\{\varphi_j : j \in \mathbb{N}\}\) be an orthonormal basis for \(A^2(\Omega)\) with respect to the \(L^2\)-inner product. The Bergman kernel \(K_\Omega\) associated to \(\Omega\) is given by

\[
K_\Omega(z, \zeta) = \sum_{j=1}^{\infty} \varphi_j(z)\bar{\varphi}_j(\zeta).
\]

Note that \(K_\Omega\) does not depend on the choice of orthonormal basis and gives rise to an invariant metric, the Bergman metric on \(\Omega\):

\[
g_\Omega^B(\xi, \xi) = \sum_{a, \beta=1}^{n} \frac{\partial^2 \log K_\Omega(z, \zeta)}{\partial \overline{z}_a \partial \overline{z}_\beta} \xi_a \overline{\xi}_\beta.
\]

The fundamental property of the Bergman kernel \(K_\Omega\) on a bounded domain \(\Omega \subset \mathbb{C}^n\) is the following transformation law under a biholomorphic map \(\Psi : \Omega_1 \to \Omega_2\) between two bounded domains \(\Omega_1, \Omega_2\) in \(\mathbb{C}^n\):

\[
K_{\Omega_1} \circ \Psi = |\text{det} \Psi'|^{-2} K_{\Omega_2},
\]

here \(\text{det} \Psi'\) is the holomorphic Jacobian of \(\Psi\). This transformation law implies the biholomorphic invariance of the Bergman metric, i.e., the invariant metric.

Let \(\mathbb{D}\) denote the open unit disk in \(\mathbb{C}\). Let \(z \in \Omega\) and \(v \in T_z \Omega\) a tangent vector at \(z\). The Kobayashi-Royden metric is defined by

\[
x_{\Omega}(z; v) = \inf \left\{ \frac{1}{\alpha} : \alpha > 0, f \in \text{Hol}(\mathbb{D}, \Omega), f(0) = z, f'(0) = \alpha v \right\}.
\]

The Carathéodory–Reiffen metric is defined by

\[
\gamma_{\Omega}(z; v) = \sup \{|df(z)v| : f \in \text{Hol}(\Omega, \mathbb{D})\}.
\]

Note that each of these three invariant metrics are also defined on any complex manifolds.

The existence of a complete Kähler-Einstein metric on a bounded pseudoconvex domain was given in the main theorem in [8]. Based on this result, we can always find a unique complete Kähler–Einstein metric of Ricci curvature \(-1\) on a bounded weakly pseudoconvex domain \(\Gamma\), i.e., \(g^{KE}_\Omega\) satisfies \(\kappa^{KE}_\Omega = -\text{Ric}_{g^{KE}_\Omega}\) as a two tensor.

The notion of quasi-bounded geometry was introduced by S.T. Yau and S. Y. Cheng ([2]), and we follow the description
of quasi-bounded geometry from Wu and Yau [9]. We adopt the following formulation. Let \((M, \omega)\) be an \(n\)-dimensional Kähler manifold. Denote by \(B_{C^n}(r)\) the open ball centered at the origin in \(\mathbb{C}^n\) of radius \(r\) with respect to the standard metric \(\omega_{C^n}\).

**Definition 6** An \(n\)-dimensional Kähler manifold \((M, \omega)\) is said to have quasi-bounded geometry if there exist two constants \(r_2 > r_1 > 0\) such that for each point \(p \in M\), there is a domain \(U \subset \mathbb{C}^n\) and a nonsingular holomorphic map \(\psi : U \to M\) satisfying the following properties:

1. \(B_{C^n}(r_1) \subset U \subset B_{C^n}(r_2)\) and \(\psi(0) = p\);
2. there exists a constant \(C > 0\) depending only on \(r_1, r_2, n\) such that
   \[
   C^{-1}\omega_{C^n} \leq \psi^*\omega \leq C\omega_{C^n} \quad \text{on } U;
   \]
3. for each integer \(l \geq 0\), there exists a constant \(A_l\) depending only on \(l, n, r_1, r_2\) such that
   \[
   \sup_{x \in U} \left| \frac{\partial^{(l+1)\mu + l\nu}}{\partial v^{\mu} \partial \overline{v}^{\nu}} g_{\overline{j} \overline{r}} \right| \leq A_l, \quad \text{for all } |\mu| + |\nu| \leq l,
   \]
where \(g_{\overline{j} \overline{r}}\) are the components of \(\psi^*\omega\) on \(U\) in terms of the natural coordinates \((v^1, \cdots, v^n)\), and \(\mu, \nu\) are multiple indices with \(|\mu| = \mu_1 + \cdots + \mu_n\). We choose \(r_1\) the largest possible number for a fixed Kähler metric \(\omega\) and call it the radius of quasi-bounded geometry.

In Definition 6, the larger radius \(r_2\) is needed in the interior Schauder estimates for the Monge-Ampère equation in [9]: Those estimates hold on bounded domains with a uniform upper bound for diameters. For the radius of quasi-bounded geometry \(r_1\), as the injectivity radius in Riemannian geometry is defined for a fixed Riemannian metric, \(r_1\) is also defined for a fixed Kähler metric \(\omega\) and a fixed quasi-coordinate atlas \(\{(U, \psi)\}\) on \(M\). The latter is implicitly indicated in [9, Definition 8]. In this paper, we rigorously state that \(r_1 > 0\) is a radius of quasi-bounded geometry of \((M, \omega)\) associated with the quasi-coordinate atlas on \(M\) and define ‘the’ radius of quasi-bounded geometry of \((M, \omega)\) to be the largest possible number of the above \(r_1\)’s.

Comparing Riemannian geometry, if the injectivity radius of a Riemannian manifold is nonzero, one can rescale it to be arbitrarily large, but such a rescaling usually does not make essential change for a geometric problem. To carry out geometric analysis on manifolds, especially to handle the subtle situation in the case of Riemannian injectivity radius being zero, what we need is one \(r_1 > 0\) or one pair \(r_2 > r_1 > 0\) in [9, Definition 8] that depend only on the curvature bounds and dimension ([9, Theorem 9]). On the other hand, the injective radius is the radius of the geodesic ball for which the exponential map is bijective onto its image, whereas the non-singular holomorphic maps required by the radius of quasi-bounded geometry need not be bijective onto its image.

The following comparison inequality is proven in the Wu-Yau paper. Those authors assume that the Kähler metric is complete, but their proof does not actually use completeness.

**Lemma 7** [9, Lemma 20] Suppose a Kähler manifold \((M, \omega)\) has quasi-bounded geometry. Then the Kobayashi-Royden metric \(\chi_M\) satisfies
\[
\chi_M(x; \xi) \leq C|\xi|_{\omega}, \quad \text{for all } x \in M, \xi \in T'_x M,
\]
where \(C\) depends only on the radius of quasi-bounded geometry of \((M, \omega)\).

A lower bound on the Kobayashi-Royden metric can be obtained from the following Lemma.

**Lemma 8** [9, Lemma 19] Let \((M, \omega)\) be a hermitian manifold such that the holomorphic sectional curvature \(H_\omega\) ≤ \(\kappa\) < 0. Then,
\[
\chi_M(x; \xi) \geq \sqrt{\frac{\kappa}{2}} |\xi|_{\omega}, \quad \text{for all } x \in M, \xi \in T'_x M.
\]

The following Lemma reduces the holomorphic sectional curvature by analyzing the Gaussian curvature.

**Lemma 9** [10, Lemma 4] Let \(M\) be a hermitian manifold with hermitian metric \(g\), and \(t\) be a unit tangent vector to \(M\) at \(p\). Then there exists an imbedded \(1\)-dimensional complex submanifold \(M'\) of \(M\) tangent to \(t\) such that the Gaussian curvature of \(M'\) at \(p\) relative to the induced metric is equal to the holomorphic sectional curvature \(H_\omega(t)\) of \(t\) assigned by \(g\).

For each positive smooth function \(g(z, \overline{z})\) defined in an open set in \(\mathbb{C}\), define a real-valued function \(H(g)\) in the same open set as follows:
\[
H(g) = -\frac{1}{g} \frac{\partial^2 \log g}{\partial z \partial \overline{z}}.
\]

\(H(g)\) is the Gaussian curvature of the metric \(ds^2 = 2g dz d\overline{z}\).

**Proposition 10** [7, p19, Proposition 3.1] For positive smooth functions \(f\) and \(g\),

(a) \(cH(cf) = H(g)\) for all positive numbers \(c\);
(b) \(fgH(fg) = fH(f) + gH(g)\);
(c) \(f + g)^2 H(f + g) \leq f^2 H(f) + g^2 H(g)\);
(d) If \(H(f) \leq -k_1 < 0\) and \(H(g) \geq -k_2 < 0\), then
\[
H(f + g) \leq \frac{-k_1 k_2}{k_1 + k_2}.
\]
3 Proof of Theorem 2

The following proposition is very slightly different from the form written without proof in [6, p. 280].

Proposition 11 Let \( \Omega \) be a bounded weakly pseudoconvex domain in \( \mathbb{C}^n \) with hermitian metrics \( h \) and \( g \). Let \( m \in \mathbb{N} \) and consider a hermitian metric \( g_\Omega := mh + g \). Then

\[
H_{g_\Omega} \leq \left( \frac{1}{1 + mu} \right)^2 \| H_g \| + \left( \frac{u}{1/m + u} \right) \frac{1}{m} H_h , \tag{3.1}
\]

where \( u = \min h(t, t) \) for all \( t \in \mathbb{C}^n \) with \( g(t, t) = 1 \).

**Proof** Let \( t' \) be a unit vector at \( p \in \Omega \) relative to \( g_\Omega = mh + g \). By Lemma 9, there exists a 1-dimensional imbedded complex submanifold \( \Omega' \) tangent to \( t' \) such that

\[
H(mh' + g')(p) = H(mh + g)(t') ,
\]

where \( g' \) and \( h' \) are respectively the induced metrics of \( g \) and \( h \) on \( \Omega \), and therefore, \( g' + mh' \) is the induced metric of \( g + mh \) on \( \Omega' \). Thus, we can regard \( g \) and \( h \) as positive smooth functions defined in an open set in \( \mathbb{C} \), the holomorphic sectional curvatures \( H(mh + g) \) of a hermitian metric \( mh + g \) can be regarded as (2.2). Then for each \( m \in \mathbb{N} \), by Proposition 10,

\[
H_{g_\Omega} = H(mh + g) \leq \left( \frac{mh}{mh + g} \right)^2 H(mh) + \left( \frac{g}{mh + g} \right)^2 H(g) = \left( \frac{u}{u + 1/m} \right)^2 H(h) + \left( \frac{1}{1/m} \right)^2 H(h) \leq \left( \frac{1}{1 + mu} \right)^2 \| H_g \|_\Omega + \left( \frac{u}{u + 1/m} \right)^2 \frac{1}{m} H_h ,
\]

where \( u = \min h(t, t) \) for all \( t \in \mathbb{C}^n \) with \( g(t, t) = 1 \). Here, \( \| H_g \|_\Omega \) means the supremum norm of \( H_g(t) \) (also see p. 280 in [6]). \( \square \)

In the following proof, we refer \( \omega \) and \( g \) refer the same metric in one case viewed as a Kähler two-form and in the other case viewed as a metric tensor.

**Proof of Theorem 2** Consider a Kähler metric of the form

\[
\omega_\Omega := m\omega_p + \omega , \quad m > 0 , \tag{3.2}
\]

where \( \omega_p \) is the Poincaré metric as a two-form of a ball \( D \) in \( \mathbb{C}^n \) with \( \Omega \Subset D \). Since \( \omega_p \) restricted to \( \Omega \) is incomplete, \( \omega_\Omega \) is complete on \( \Omega \) for each \( m > 0 \) if and only if \( \omega \) is complete.

For \( \epsilon > 0 \), let \( C_\epsilon := \{ z \in \Omega : H_\omega > -\epsilon \} \). Then it follows from Proposition 11, on the set \( C_\epsilon \), one has

\[
H_{\omega_\Omega} \leq \left( \frac{1}{1 + mu} \right)^2 \| H_\omega \|_\Omega + \left( \frac{u}{1/m + u} \right) \frac{1}{m} H_{\omega_p} , \tag{3.3}
\]

where \( u = \min g_P(t, t) \) for all \( t \in \mathbb{C}^n \) with \( g_P(t, t) = 1 \). Since the sum of two Kähler metrics with negative upper bounds for holomorphic sectional curvatures has a negative upper bound for the holomorphic sectional curvature (see Lemma 2 in [10]), it is enough to control the quantity of (3.3) on \( C_\epsilon \).

Notice that the quasi-bounded geometry assumption of \( \omega \) forces that \( \| H_\omega \|_\Omega \) is bounded by some constant. Then \( C_\epsilon \) belongs to a compact subset of \( \Omega \) which does not touch \( \partial \Omega \). Notice that \( u \) can be zero on \( C_\epsilon \) only when \( u \) takes (limit) values on \( \partial \Omega \). Then by the hypothesis on the holomorphic sectional curvature of \( \omega \), \( u \) on \( C_\epsilon \) is bounded below by a positive constant. Thus, we can find sufficiently large \( m \gg 0 \) so that the right-hand side of (3.3) becomes uniformly negative, and \( H_{\omega_\Omega} \) has a negative upper bound for \( m \) sufficiently large.

Next, we show that \( \omega_\Omega \) admits quasi-bounded geometry by checking conditions in Definition 6. From the quasi-bounded geometry of \( \omega \), the first requirement in Definition 6 is clearly satisfied. Since the ball \( D \) contains \( \Omega \), the Poincaré metric \( \omega_p \) on \( \Omega \) is merely a weighted hermitian inner product in \( \mathbb{C}^n \); and thus, the second requirement is satisfied trivially. For the last requirement, \( \omega \) satisfies the last requirement by the hypothesis, and \( m\omega_p = m\sqrt{-1}\partial\bar{\partial} \log K_p \) never blows up with any \( k \)-th-order derivative on \( \Omega \), where \( K_p \) is the Bergman kernel of \( D \). This proves that \( \omega_\Omega \) admits the quasi-bounded geometry. Consequently, by Lemma 8 and Lemma 7, \( \omega_\Omega \) is uniformly equivalent to the Kobayashi–Royden metric.

Now, if \( \omega \) is complete, \( \omega_\Omega \) is also uniformly equivalent to the complete Kähler–Einstein metric of negative scalar curvature by Theorem 3 in [9]. Moreover, by quasi-bounded geometry of \( \omega \) and Lemma 7, we have

\[
C^{-1} \chi_\Omega \leq \sqrt{\omega} ,
\]

where \( \chi_\Omega \) denotes the Kobayashi–Royden metric and \( C \) is a universal constant. On the other hand, by the negative upper bound of \( \omega_\Omega \) and Lemma 8, we also have

\[
\sqrt{\omega_\Omega} \leq C \chi_\Omega .
\]

Since \( \omega \leq \omega_\Omega \) from the construction of \( \omega_\Omega \), we obtain in all

\[
C^{-1} \chi_\Omega \leq \sqrt{\omega} \leq \sqrt{\omega_\Omega} \leq C \chi_\Omega .
\]

Hence, \( \omega \) is also uniformly equivalent to the Kobayashi–Royden metric.
Otherwise, $\omega$ is incomplete, and the rest of conclusion follows from Proposition 12 and Corollary 14. □

**Proposition 12** Let $\Omega$ be a bounded weakly pseudoconvex domain in $\mathbb{C}^n$ with an incomplete Kähler metric $\omega$ satisfying all assumptions in Theorem 2. Then there is no complete Kähler metric with negatively pinched holomorphic sectional curvature. The same conclusion holds for the complete Kähler–Einstein metric of negative scalar curvature.

_Proof_ By following the proof of Theorem 2, an incomplete Kähler metric $g_\Omega$ with negatively pinched holomorphic sectional curvature is constructed. If a complete Kähler metric with negatively pinched holomorphic sectional curvature exists, this metric must be uniformly equivalent with the Kobayashi–Royden metric [9, Theorem 2.3], and in particular, the Kobayashi–Royden metric must be complete, so it is a contradiction. In particular, for the complete Kähler–Einstein metric of negative scalar curvature, the holomorphic sectional curvature cannot be negatively pinched. □

**Proof of Corollary 4** It follows from Theorem 2 that the holomorphic sectional curvature of $g_\Omega$ in (3.2) is negatively pinched. Therefore, the second fundamental form of $\mathcal{S}$ with respect to the restriction $g_\Omega|_{\mathcal{S}}$ is bounded. By the decreasing property for holomorphic sectional curvature and the Gauss-Codazzi equation, the holomorphic sectional curvature of $g_\Omega|_{\mathcal{S}}$ is negatively pinched. The conclusion follows from Theorem 2 and Theorem 3 in [9]. □

**Proof of Corollary 5** The proof is the same as the proof of Corollary 2 of [11], as long as we confirm that the complete Kähler–Einstein metric has negative Ricci curvature and has bounded Riemannian sectional curvature. In the case of the complete Kähler–Einstein metric of negative scalar curvature based on Theorem 2, it has quasi-bounded geometry, so it satisfies a stronger condition than the required condition [9, Lemma 31]. □

### 4 Quasi-bounded geometry and Carathéodory–Reiffen metric

Carathéodory completeness refers to the case where the Carathéodory distance is complete as a topological distance [4, Chapter 14]. Even for bounded domains in $\mathbb{C}^n$, information about Carathéodory completeness and Carathéodory metric or distance is a very difficult question. In this section, we will discuss the relationship between quasi-bounded geometry and the Carathéodory metric on Kähler manifolds.

**Lemma 13** Given $(M, \omega)$ a Kähler manifold, assume $\omega$ admits quasi-bounded geometry. Then there exists $C > 0$ such that

$$\gamma_M(p; v) \leq C ||v||_\omega$$

for any $p \in M$, $v \in T_pM$.

**Proof** For each $p \in M$, take any nonsingular $\psi : B_{C^n}(r) \rightarrow M$ by the definition of quasi-bounded geometry that maps 0 to $p$. Then by the contraction (metric decreasing) property of the Carathéodory–Reiffen metric and the fact that $\gamma_{B_r(0)}$ coincides with the Poincaré metric of $B_{C^n}(r)$,

$$\gamma_M(p; v) \leq \gamma_{B_{C^n}(r)}(0; \psi^*v)$$

$$\leq \frac{1}{r}||\psi^*(v)||_{C^n}$$

$$\leq \frac{C}{r}||v||_\omega.$$  

In the last line, we use the requirement (2) of the definition of quasi-bounded geometry. We can replace $\frac{C}{r}$ by $C > 0$ and the proof follows. □

From this Lemma, we get one implication for the Carathéodory incompleteness.

**Corollary 14** Let $M$ be a complex manifold with an incomplete Kähler metric $\omega$ that admits quasi-bounded geometry. Then $M$ is not Carathéodory complete.

### 5 Discussion

It would be very interesting to confirm whether an example corresponding to the class with incomplete Kähler metric $\omega$ in Theorem 2 exists. For example, by Corollary 14, there is no incomplete Kähler metric admitting a quasi-bounded geometry in the case of Carathéodory complete domains.

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**Declarations**

**Conflict of interest** The corresponding author states that there is no conflict of interest.

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