Counting generalized Schröder paths

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Abstract

A Schröder path is a lattice path from $(0, 0)$ to $(2n, 0)$ with steps $(1, 1), (1, -1)$ and $(2, 0)$ that never goes below the $x$–axis. A small Schröder path is a Schröder path with no $(2, 0)$ steps on the $x$–axis. In this paper, a 3-variable generating function $R_L(x, y, z)$ is given for Schröder paths and small Schröder paths respectively. As corollaries, we obtain the generating functions for several kinds of generalized Schröder paths counted according to the order in a unified way.

Keywords: Schröder path, Narayana number, generating function

1 Introduction

In this paper, we will consider the following sets of steps for lattice paths:

\[ S_1 = \{(1, 1), (1, -1)\}, \]
\[ S_2 = \{(r, r), (r, -r) | r \in \mathbb{N}^+\}, \]
\[ S_3 = \{(1, 1), (1, -1), (2, 0)\}, \]
\[ S_4 = \{(r, r), (r, -r), (2r, 0) | r \in \mathbb{N}^+\}, \]

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$S_5 = \{(1,1), (1,-1), (2r,0)| r \in \mathbb{N}^+\}$,
$S_6 = \{(r,r), (r,-r), (2,0)| r \in \mathbb{N}^+\}$,

where $(r,r)$, $(r,-r)$, $(2r,0)$ are called up steps, down steps and horizontal steps respectively.

For a given set $S$ of steps, let $L_S(n)$ denote the set of lattice paths from $(0,0)$ to $(2n,0)$ with steps in $S$, and never go below the $x-$axis. Let $A_S(n)$ denote the subset of $L_S(n)$ whose member paths have no horizontal steps on the $x-$axis. We denote by $L_S = \bigcup_{n \geq 1} L_S(n)$ and $A_S = \bigcup_{n \geq 1} A_S(n)$. Then $L_{S_1}(n)$, $L_{S_3}(n)$ and $A_{S_3}(n)$ are the sets of Dyck paths, Schröder paths and small Schröder paths of order $n$ respectively.

It is well known that $|L_{S_1}(n)|$ is the $n$th Catalan number (A000108 in [8]), $|L_{S_3}(n)|$ is the $n$th large Schröder number (A006318), and $|A_{S_3}(n)|$ is the $n$th small Schröder number (A001003). Define a peak in a Dyck path to be a vertex between an up step and a down step. Then the number of Dyck paths of order $n$ with $k$ peaks is the well known Narayana number (A001263)

$$N(n,k) = \frac{1}{n} \binom{n}{k} \binom{n}{k-1}.$$

The $n$th Narayana polynomial is defined as $N_n(y) = \sum_{1 \leq k \leq n} N(n,k)y^k$ for $n \geq 1$ with $N_0(y) = 1$. In [11], Sulanke gave the generating function for the Narayana polynomial as

$$\sum_{n \geq 0} N_n(y)x^n = (1 + (1-y)x - \sqrt{1-2(1+y)x + (1-y)^2x^2})/(2x). \quad (1.1)$$

Let

$$P_{S_i}(x) = 1 + \sum_{n \geq 1} |L_{S_i}(n)|x^n$$

and

$$Q_{S_i}(x) = 1 + \sum_{n \geq 1} |A_{S_i}(n)|x^n$$

denote the generating functions for $|L_{S_i}(n)|$ and $|A_{S_i}(n)|$ respectively. As one type of generalization of Dyck paths, $L_{S_2}(n)$ has been studied by several authors. The generating function $P_{S_2}(x)$ is given in [7] and [1] with different methods as

$$P_{S_2}(x) = (1 + 3x - \sqrt{1 - 10x + 9x^2})/(8x). \quad (1.2)$$
Moreover, Coker [1] and Sulanke [11] expressed $|L_{S_2}(n)|$ as a combination of Narayana numbers, and Woan [12] gave a three-term recurrence for $|L_{S_2}(n)|$. For other types of generalization of Dyck paths, readers can refer to [6] and [9].

Comparing to the above results about generalization of Dyck paths, generalization of Schröder paths has been rarely studied until Kung and Miler [7] gave the generating functions $P_{S_i}(x)$ ($4 \leq i \leq 6$). Later, Huh and Park [5] expressed $|A_{S_4}(n)|$ as a combination of Narayana numbers.

Note that we can also obtain Equation (1.2) by considering the number of runs of Dyck paths. Here a run in a lattice path is defined to be a vertex between two consecutive steps of the same kind. Let $R(n, k, S_1)$ denote the number of lattice paths in $L_{S_1}(n)$ with $k$ runs. Since a Dyck path of order $n$ with $k$ peaks has $2n - 2k$ runs, we obtain from Equation (1.1) that

$$1 + \sum_{n,k \geq 1} R(n, k, S_1)x^n y^k = 1 + \sum_{n,k \geq 1} N(n, k)x^n y^{2n-2k}$$

$$= (1 + (y^2 - 1)x - \sqrt{1 - 2(1 + y^2)x + (1 - y^2)^2x^2})/(2xy^2).$$

Then Equation (1.2) is derived from Equation (1.3) by setting $y = 2$.

Motivated by the above observation, we study the number of runs for Schröder paths according to the following two types: a run is diagonal if it is the joint of two up steps or two down steps, and a run is horizontal if it is the joint of two horizontal steps.

For a Schröder path $P$, let $dr(P)$, $hr(P)$ and $\text{order}(P)$ denote the number of diagonal runs, the number of horizontal runs and the order of $P$ respectively. Then the generating function $R_{L}(x, y, z)$ is defined for $L \subseteq L_{S_3}$ as

$$R_{L}(x, y, z) = 1 + \sum_{P \in L} x^{\text{order}(P)} y^{dr(P)} z^{hr(P)}.$$ 

In this paper, we give $R_{L}(x, y, z)$ for $L = L_{S_3}$ and $L = A_{S_3}$. As corollaries, we obtain the generating functions $P_{S_i}(x)$ and $Q_{S_i}(x)$ for $4 \leq i \leq 6$ in a unified way.

### 2 The case for Schröder paths

In the following, we use $U$, $D$ and $H$ to denote the steps $(1,1)$, $(1,-1)$ and $(2,0)$ respectively. For a lattice $P$ and a step $s$, the insertion of $s$ at a vertex
Figure 1: An example of the insertion of an $H$ step.

$v$ of $P$ is defined as following: decompose $P$ into two parts at $v$ as $P = P_1P_2$, where $P_1$ maybe empty. Then we connect the initial vertex of $s$ to the end vertex of $P_1$, and connect the end vertex of $s$ to the initial vertex of $P_2$. See Figure 1 for an example.

Given $P \in L_{S_1}(n)$ with $k$ peaks, let $V$ denote the set of vertices of $P$ other than runs. We then insert $m$ $H$ steps to $P$ as following:

1. We firstly choose $i$ vertices from $V$, and insert an $H$ step at each chosen vertex. In this step, we have $\binom{2k+1}{i}$ choices, and each insertion has no effect to the number of runs.

2. For the lattice path obtained after step (1), we choose $j$ vertices from its runs, and insert an $H$ step at each chosen vertex. In this step, we have $\binom{2n-2k}{j}$ choices, and the number of diagonal runs will decrease by $j$ after insertion.

3. For the lattice path obtained after step (2), we insert the remaining $m - i - j$ $H$ steps immediately after the $i + j$ $H$ steps that have been inserted. In this step, we have $\binom{m-i-j}{i+j}$ choices, and the number of horizontal runs will increase by $m - i - j$ after insertion.

Let $\text{Ins}_m(P)$ denote the set of all Schröder paths obtained from $P$ by the above insertion. Then we have

$$|\text{Ins}_m(P)| = \binom{2k+1}{i} \binom{2n-2k}{j} \binom{m-i-j}{i+j}. \quad (2.1)$$

Moreover, we have $\text{order}(P') = m+n$, $\text{dr}(P') = 2n - 2k - j$, $\text{hr}(P') = m - i - j$ for each $P' \in \text{Ins}_m(P)$.

On the other hand, let $H L_{S_3}$ denote the subset of $L_{S_3}$ whose member paths consisting of $H$ steps only. Let $U L_{S_3}$ denote the subset of $L_{S_3}$ whose member paths have at least one $U$ step. It is obvious that each path of $U L_{S_3}$
can be obtained uniquely from a Dyck path by inserting some $H$ steps as
above. Thus we have

$$L_{S_3} = H L_{S_3} \cup U L_{S_3} = H L_{S_3} \cup \bigcup_{P \in L_{S_1}, m \geq 0} \text{Ins}_m(P).$$

Summarizing the above discussion, we then obtain the following result.

**Theorem 2.1.**

$$R_{L_{S_3}}(x, y, z) = \frac{1 - xz + x}{1 - xz} (1 + (1 - w)u - \sqrt{1 - 2(1 + w)u + (1 - w)^2u^2})/(2u),$$

where $u = x \left( \frac{y + (1 - xz)}{1 - xz} \right)^2$ and $w = \left( \frac{1 - xz + x}{x + y(1 - xz)} \right)^2$.

**Proof.** By Equation (2.1), we have

$$R_{L_{S_3}}(x, y, z) = 1 + \sum_{P \in HL_{S_3}} x^{\text{order}(P)} y^{\text{dr}(P)} z^{\text{hr}(P)} + \sum_{P \in UL_{S_3}} x^{\text{order}(P)} y^{\text{dr}(P)} z^{\text{hr}(P)}$$

$$= 1 + \frac{x}{1 - xz} + \sum_{n, k \geq 1} N(n, k) \binom{2k + 1}{i} \binom{2n - 2k}{j} \left( \binom{m - i - j}{i + j} \right) x^{m + n} y^{2n - 2k - j} z^{m - i - j}$$

$$= 1 - xz + x \frac{1}{1 - xz} + \sum_{n, k \geq 1} N(n, k) x^n y^{2n - 2k}$$

$$+ \sum_{0 \leq i \leq 2k + 1} \sum_{0 \leq j \leq 2n - 2k} \binom{2k + 1}{i} \binom{2n - 2k}{j} x^{i + j} y^{-j} \sum_{m \geq i + j} \left( \binom{m - i - j}{m - i - j} \right) (-xz)^{m - i - j}$$

$$= 1 - xz + x \frac{1}{1 - xz} + \sum_{n, k \geq 1} N(n, k) x^n y^{2n - 2k}$$

$$\sum_{0 \leq i \leq 2k + 1} \sum_{0 \leq j \leq 2n - 2k} \binom{2k + 1}{i} \binom{2n - 2k}{j} x^{i + j} y^{-j} (1 - xz)^{-i - j}$$

$$= 1 - xz + x \frac{1}{1 - xz} + \sum_{n, k \geq 1} N(n, k) x^n y^{2n - 2k} \left( 1 + \frac{x}{y(1 - xz)} \right)^{2n - 2k} \left( 1 + \frac{x}{1 - xz} \right)^{2k + 1}$$

$$= 1 - xz + x \frac{1}{1 - xz} + \sum_{n, k \geq 1} N(n, k) x^n y^{2n - 2k} \left( 1 + \frac{x}{y(1 - xz)} \right)^{2n - 2k} \left( 1 + \frac{x}{1 - xz} \right)^{2k + 1}$$
\[ \frac{1 - xz + x}{1 - xz} \sum_{n \geq 0} N_n(w)u^n, \]

then Theorem 2.1 is derived from Equation (1.1). \[ \square \]

The generating functions \( P_{S_i}(x) \) for \( 4 \leq i \leq 6 \) were derived by Kung and Mier [7]. Here we can obtain them as a direct corollary of the above result.

**Corollary 2.2.** \([7]\)

\[
\begin{align*}
P_{S_4}(x) &= \frac{(1 - x)(1 + x - 4x^2) - (1 - x)^2\sqrt{1 - 12x + 16x^2}}{2x(2 - 3x)^2}, \\
P_{S_5}(x) &= \frac{2x - 1 + \sqrt{1 - 8x + 12x^2} - 4x^3}{2x(x - 1)}, \\
P_{S_6}(x) &= \frac{1 + 2x - x^2 - \sqrt{(1 - x)(1 - 11x + 7x^2 - x^3)}}{2x(x - 2)^2}.
\end{align*}
\]

**Proof.** We use a bijection given by Huh and Park [5]. Let \( \bar{L}_{S_3}(n) \) denote the set of Schröder paths of order \( n \) whose runs are colored in either black or white, and other vertices are colored in black only. For \( P \in \bar{L}_{S_3}(n) \), let \( \phi(P) \) denote the lattice path obtained from \( P \) as following: delete all white vertices of \( P \), and then connect adjacent black vertices with line segments. See [5, Figure 8] for an example. It is obvious that \( \phi \) is a bijection from \( \bar{L}_{S_3}(n) \) to \( L_{S_4}(n) \), which implies that

\[ P_{S_i}(x) = R_{L_{S_3}}(x, y, z) = \frac{(1 - x)(1 + x - 4x^2) - (1 - x)^2\sqrt{1 - 12x + 16x^2}}{2x(2 - 3x)^2}. \]

Similarly, we can obtain \( P_{S_5}(x) \) and \( P_{S_6}(x) \) by setting the pair \((y, z)\) to be \((1, 2)\) and \((2, 1)\) in \( R_{L_{S_3}}(x, y, z) \) respectively. \[ \square \]

Using the techniques in [4] ([Chapter VI]), Kung and Miler [7] gave the asymptotic formula for \( |L_{S_4}(n)| \). The asymptotic formulas for \( |L_{S_5}(n)| \) and \( |L_{S_6}(n)| \) can be obtained from Corollary 2.2 in a similar way:

\[
\begin{align*}
|L_{S_4}(n)| &\sim \frac{\beta_1}{\alpha_1^4 \sqrt{\pi n^3}}, \\
|L_{S_5}(n)| &\sim \frac{\beta_2}{\alpha_2^4 \sqrt{\pi n^3}}, \\
|L_{S_6}(n)| &\sim \frac{\beta_3}{\alpha_3^4 \sqrt{\pi n^3}}.
\end{align*}
\]
where $\alpha_i$ and $\beta_i$ are defined as following:

1. $\alpha_1 = \frac{3 - \sqrt{5}}{8}$ is the root of equation $f_1(x) = 1 - 12x + 16x^2 = 0$, and $\beta_1 = \frac{(1-\alpha_1)^2-\sqrt{-\alpha_1f_1'(\alpha_1)}}{4\alpha_1(2-3\alpha_1)^2} = \frac{35-15\sqrt{5}+10\sqrt{6}}{4}$;

2. $\alpha_2 = 0.16243 \cdots$ is the root of equation $f_2(x) = 1 - 8x + 12x^2 - 4x^3 = 0$, and $\beta_2 = \frac{\sqrt{-\alpha_2f_2'(\alpha_2)}}{4\alpha_2(1-\alpha_2)} = 1.55669 \cdots$;

3. $\alpha_3 = 0.09678 \cdots$ is the root of equation $f_3(x) = (1-x)(1-11x+7x^2-x^3)$, and $\beta_3 = \frac{\sqrt{-\alpha_3f_3'(\alpha_3)}}{4\alpha_3(2-\alpha_3)^2} = 0.68998 \cdots$.

Theorem 2.1 can also be used to study colored Schröder paths. For instance, let $a(n)$ denote the number of Schröder paths of order $n$ with their horizontal runs colored in one of three given colors. Then we obtain from Theorem 2.1 that

\[
1 + \sum_{n \geq 1} a(n)x^n = R_{LS_3}(x, 1, 3) = \frac{3x - 1 + \sqrt{1 - 10x + 25x^2 - 16x^3}}{2x(2x - 1)}.
\]  

The coefficients of the above function appear as sequence A186338 in OEIS, and is related to sequence A091866.

For the definition of pyramid and pyramid weight, see [2, Definition 2.1]. Let $T(n, k)$ denote the number of Dyck paths of order $n$ that have pyramid weight $k$. Combing Equation 2.2 with a result of Denise and Simion ([2, Theorem 2.3]), we then obtain the following result.

**Corollary 2.3.**

\[
a(n) = \sum_{k=0}^{n} T(n, k)2^k.
\]

### 3 The case for small Schröder paths

A lattice path in $A_{S_3}$ is said to be *primitive* if it does not intersect the $x$–axis except at $(0, 0)$ and $(2n, 0)$. Let $PA_{S_3}$ denote the set of all primitive paths
in $A_{S_3}$. Since every path in $A_{S_3}$ can be decomposed uniquely into a sequence of paths in $P A_{S_3}$, we have

$$R_{A_{S_3}}(x, y, z) = \frac{1}{1 - R_{P A_{S_3}}(x, y, z)}, \quad (3.1)$$

where we use $\bar{R}_L(x, y, z)$ to denote the function $R_L(x, y, z) - 1$ for a given set $L$ of lattice paths.

We now consider the generating function $\bar{R}_{P A_{S_3}}(x, y, z)$. Note that the set $UL_{S_3}$ can be partitioned as $UL_{S_3} = \bigcup_{i=1}^{4} U_i$, where

1. $U_1 = \{P \mid P \text{ starts with } U \text{ and ends with } D\}$;
2. $U_2 = \{P \mid P \text{ starts with } H \text{ and ends with } D\}$;
3. $U_3 = \{P \mid P \text{ starts with } U \text{ and ends with } H\}$;
4. $U_4 = \{P \mid P \text{ starts and ends with } H\}$.

As shown in Section 2, each path $P \in UL_{S_3}$ can be obtained uniquely from a Dyck path $P'$ by inserting some $H$ steps, and we have the following fact:

1. if it is not allowed to insert at either the initial vertex or the end vertex of $P'$, then $P \in U_1$;
2. if it is required to insert at the initial vertex of $P'$, and not allowed to insert at the end vertex, then $P \in U_2$;
3. if it is required to insert at the end vertex of $P'$, and not allowed to insert at the initial vertex, then $P \in U_3$;
4. if it is required to insert at both the initial vertex and the end vertex of $P'$, then $P \in U_4$.

Based on the above observation, we can obtain the following result after some calculation.

**Lemma 3.1.**

$$R_{U_1}(x, y, z) = \frac{(1 - xz)^2}{(1 - xz + x)^2} R_{UL_{S_3}}(x, y, z),$$
\( \overline{R}_{U_2}(x, y, z) = \overline{R}_{U_3}(x, y, z) = \frac{x(1 - xz)}{(1 - xz + x)^2} \overline{R}_{ULS_3}(x, y, z), \)

\( \overline{R}_{U_4}(x, y, z) = \frac{x^2}{(1 - xz + x)^2} \overline{R}_{ULS_3}(x, y, z). \)

**Proof.** The proof of the above result is almost the same as that of Theorem 2.1. Here we take \( \overline{R}_{U_2}(x, y, z) \) as an example. By the definition of \( U_2 \), we have

\[
\overline{R}_{U_2}(x, y, z) = \sum_{n,k \geq 1} \sum_{i,j \geq 0} \sum_{m \geq 1} \overline{N}(n, k) \left( \frac{2k - 1}{i} \right) \left( \frac{2n - 2k}{j} \right) \left( \frac{m - i - j - 1}{i + j + 1} \right) x^{m+n} y^{2n-2k-j} z^{m-i-j-1}.
\]

Now we can obtain \( \overline{R}_{U_2}(x, y, z) \) as a direct corollary of the above result.

**Theorem 3.2.**

\[
\overline{R}_{A_{S_3}}(x, y, z) = \frac{1}{2} + \frac{-1 + zx + (1 - z)x^2 + (1 - xz)\sqrt{1 - 2(1+w)u + (1-w)^2u^2}}{2(y^2z - 2y + 1)x^2 - 2y^2x - 2x},
\]

where \( u = x \left( \frac{y(1-xz)}{x+y(1-xz)} \right)^2 \) and \( w = \left( \frac{1-xz+x}{x+y(1-xz)} \right)^2 \).

**Proof.** Since \( P_{A_{S_3}} = \{UPD \mid P \text{ is empty or } P \in L_{S_3} \} \), we obtain from Lemma 3.1 that

\[
\overline{R}_{P_{A_{S_3}}}(x, y, z) = x + x \overline{R}_{ULS_3} + x y \overline{R}_{U_1} + x y \overline{R}_{U_2} + x y \overline{R}_{U_3} + x \overline{R}_{U_4}
\]

\[
= \frac{x(1 - xz + x)}{1 - xz} + \frac{x}{w} \overline{R}_{ULS_3}(x, y, z)
\]

\[
= \frac{x(1 - xz + x)}{1 - xz} + \frac{x}{w} \left( R_{LS_3}(x, y, z) - \frac{1 - xz + x}{1 - xz} \right). \tag{3.2}
\]
Combining Equation (3.1), Equation (3.2) and Theorem 2.1 together, we then obtain Theorem 3.2.

Setting the pair \((y, z)\) to be \((2, 2)\), \((1, 2)\) and \((2, 1)\) respectively in Theorem 3.2, we then obtain the generating functions \(Q_{S_i}(x)\) for \(4 \leq i \leq 6\).

Corollary 3.3.

\[
Q_{S_4}(x) = 1 + 4x - \sqrt{1 - 12x + 16x^2},
\]
\[
Q_{S_5}(x) = \frac{-1 + \sqrt{1 - 8x + 12x^2 - 4x^3}}{2x(x - 2)},
\]
\[
Q_{S_6}(x) = \frac{-1 - 4x + x^2 + \sqrt{(1 - x)(1 - 11x + 7x^2 - x^3)}}{2x(x - 5)}.
\]

Expanding the above functions, we have
\[
Q_{S_4}(x) = 1 + x + 6x^2 + 41x^3 + 306x^4 + 2426x^5 + 20076x^6 + \cdots,
\]
\[
Q_{S_5}(x) = 1 + x + 3x^2 + 12x^3 + 53x^4 + 248x^5 + 1209x^6 + \cdots,
\]
\[
Q_{S_6}(x) = 1 + x + 6x^2 + 40x^3 + 293x^4 + 2286x^5 + 18637x^6 + \cdots.
\]

The coefficients of \(Q_{S_4}(x)\) appear as sequence A078009 in OEIS. The generating functions \(Q_{S_5}(x)\) and \(Q_{S_6}(x)\), to our knowledge, have not been studied before. From Corollary 3.3, we can obtain the following asymptotic formulas:

\[
|A_{S_4}(n)| \sim \frac{\gamma_1}{\alpha_1^4 \sqrt{\pi n^3}},
\]
\[
|A_{S_5}(n)| \sim \frac{\gamma_2}{\alpha_2^4 \sqrt{\pi n^3}},
\]
\[
|A_{S_6}(n)| \sim \frac{\gamma_3}{\alpha_3^4 \sqrt{\pi n^3}},
\]

where \(\alpha_i\) and \(f_i\) are the same as those in Section 2, and \(\gamma_i\) is defined as following:

\[
\gamma_1 = \frac{\sqrt{-\alpha_1 f'_1(\alpha_1)}}{20\alpha_1} = \frac{\sqrt{10 + 6\sqrt{5}}}{10},
\]
\[
\gamma_2 = \frac{\sqrt{-\alpha_2 f'_2(\alpha_2)}}{4\alpha_2(2 - \alpha_2)} = 0.70954 \cdots.
\]
\[ \gamma_3 = \frac{\sqrt{-\alpha_3 f_3'(\alpha_3)}}{4\alpha_3(5 - \alpha_3)} = 0.50971 \cdots. \]

It is well known (see, for example, [3, 10]) that \(|L_{S_i}(n)| = 2|A_{S_i}(n)|\). Comparing the asymptotic formulas of \(|L_{S_i}(n)| \text{ and } |A_{S_i}(n)| \text{ for } 4 \leq i \leq 6\), we have the following analogue.

**Corollary 3.4.**

\[
\begin{align*}
\lim_{n \to \infty} \frac{|L_{S_4}(n)|}{|A_{S_4}(n)|} &= \frac{5(1 - \alpha_1)^2}{(2 - 3\alpha_1)^2} = 1.39320 \cdots, \\
\lim_{n \to \infty} \frac{|L_{S_5}(n)|}{|A_{S_5}(n)|} &= \frac{2 - \alpha_2}{1 - \alpha_2} = 2.19393 \cdots, \\
\lim_{n \to \infty} \frac{|L_{S_6}(n)|}{|A_{S_6}(n)|} &= \frac{5 - \alpha_3}{(2 - \alpha_3)^2} = 1.35364 \cdots.
\end{align*}
\]

By giving a bijection between 5-colored Dyck paths and \(|A_{S_4}(n)|\), Huh and Park [5] gave the following expression for \(|A_{S_i}(n)|\), which we can also prove here with generating function.

**Corollary 3.5.** [5]

\[ |A_{S_i}(n)| = \sum_{k=1}^{n} N(n, k)5^{n-k}. \]

**Proof.** By Equation (1.1), we have

\[ 1 + \sum_{n,k \geq 1} N(n, k)5^{n-k}x^n = \sum_{n \geq 0} N_n(\frac{1}{5})(5x)^n = \frac{1 + 4x - \sqrt{1 - 12x + 16x^2}}{10x}. \]

Then Corollary 3.5 is derived from Corollary 3.3.

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