Transverse Bending and Vibration Characteristics of a Simply Supported Discontinuous Beam under Axial Loads

Heng Liu*
School of Energy and Power Engineering, Beihang University, Beijing, China

*Corresponding author e-mail: h-liu@foxmail.com

Abstract. An abrupt change of stiffness may occur at the interface of discontinuous beams, and the transverse stiffness and vibration characteristics of this part of the beam can be significantly different from those of the remaining beam. In this paper, the relationship between the internal force and the displacement of discontinuous beams is analyzed based on the Timoshenko beam theory, and the equations for the deflection curve and free vibration of a simply supported beam under axial load are analytically derived. The normal function of the transverse vibration equation is obtained by separating variables, while the endpoint displacement is derived using the method of transfer matrix. The deflection and free vibration frequency of the discontinuous beam under different axial loads and connection stiffness were calculated. The results show that with an increase of stiffness, the deflection and free vibration frequency decreases and increases exponentially respectively. Moreover, the free vibration frequency of each order decreases with an increase of axial load, and the lower order frequency is more dependent on the axial load.

1. Introduction
The mechanical structures of rocket body and gas turbine shaft can be simplified to a structural system consisting of multi-section beams and connecting structures. The structural system can be divided into continuous structural system and discontinuous structural system [1]. The bending stiffness of continuous beams can be easily obtained by material mechanics method, while the abruptly changed stiffness on discontinuous beam must be considered.

The solutions for deformation and free vibration of continuous Euler-Bernoulli’s beams and Timoshenko beams under axial loads have been studied by many researchers [2~4]. Chenaghlo [5, 6] proposed a semi-rigid connection model of jointing system based on experimental results. Gao et al. [7] introduced spring elements to express weakened stiffness of a connection structure and modified his stiffness model by experimental data. Zhang et al. [8, 9] simplified a connection structure into a mass-free hinge-bending spring to express weakened stiffness in study of a tie rod rotor. Liu, et al. [10~12] established a three-dimensional finite element model of a tie rod rotor to determine its deflection and vibration frequency. Although the experimental method and FEM can solve engineering problems, it is still necessary to study the bending and vibration behaviour of discontinuous beams theoretically.

This paper presents an analytical method to simulate the transverse bending and free vibration characteristics of a simply supported discontinuous beam under axial load. Based on the Timoshenko
beam theory, both an approximate equation of deflection curve and a partial differential equation of transverse vibration are derived, and the corresponding solutions for stiffness and frequency are given.

2. Bending stiffness of discontinuous beam

A simply supported beam with length $l$ has a connection structure at distance $m$ from the left end point. The length of the connection structure is much shorter than the length of the beam. The connection structure could be simplified as a torsional spring with stiffness $k_t$. A transverse force $F$ is applied at distance $a$ and an axial force $F_N$ is applied at the right end of the beam, as shown in Fig. 1.

The beam has a total deflection $w_d$. The deflection and slope caused by bending and shearing are $w_b$, $\theta_b$, $w_s$, $\theta_s$ respectively.

![Figure 1. Bending deformation of discontinuous beam](image)

These parameters have the following relationships

$$w_d = w_b + w_s, \quad w'_d = w'_b + w'_s$$

(1)

$\theta_1$ and $\theta_2$ representing the slope of ends at $m$ is obtained as

$$\theta_1 - \theta_2 = \frac{M(m)}{k_t}$$

(2)

The approximate differential equation for simply supported discontinuous beam with axial and transverse loads is derived as [13]

$$M + F_N w_d = -EI \frac{d^2 \theta_b}{dx^2}$$

(3)

Where $E$ is Young’s modulus and $I_z$ is sectional moment of inertia. By substituting of equations (1) into (3), the general solution is of the form

$$w_d = \begin{cases} 
A_1 \cos \kappa x + A_2 \sin \kappa x - \frac{F_{b} x}{F_{N} l} & x \in [0, a) \\
B_1 \cos \kappa x + B_2 \sin \kappa x - \frac{F_{a} (l - x)}{F_{N} l} & x \in [a, m] \\
C_1 \cos \kappa x + C_2 \sin \kappa x - \frac{F_{a} (l - x)}{F_{N} l} & x \in (m, l] 
\end{cases}$$

(4)

Where
\[ \kappa^2 = \frac{F_N}{EI_z} \]

Because of the requirement of continuity at \( m \), the following conditions should be satisfied:

\[
\begin{align*}
    w_b(0) &= w_b(l) = 0, \quad w_b(a^-) = w_b(a^+), \quad w_b(m^-) = w_b(m^+), \\
    w'_b(m^-) - w'_b(m^+) &= \theta_1 - \theta_2, \quad w'_b(a^-) = w'_b(a^+) 
\end{align*}
\]

Unknown in equation (4) is derived as

\[
\begin{align*}
    A_1 &= 0, \quad A_2 = \frac{Fa \sin[k(l-m)]}{lk\kappa \sin \kappa l} (l-x) + \frac{F \sin \kappa b}{\kappa \sin \kappa l} \left( \frac{1}{F_N} + \frac{f_s}{GA} \right), \\
    B_1 &= \frac{F \sin \kappa a}{\kappa} \left( \frac{1}{F_N} + \frac{f_s}{GA} \right), \quad B_2 = \frac{F a \sin[k(l-m)]}{lk\kappa \sin \kappa l} (l-x) - \frac{F \sin \kappa a \cos \kappa l}{\kappa \sin \kappa l} \left( \frac{1}{F_N} + \frac{f_s}{GA} \right), \\
    C_1 &= \frac{F a \sin \kappa m}{lk\kappa} (l-x) + \frac{F \sin \kappa a}{\kappa} \left( \frac{1}{F_N} + \frac{f_s}{GA} \right), \\
    C_2 &= \frac{F a (l-x) \sin \kappa m \cos \kappa l}{lk\kappa \sin \kappa l} - \frac{F \sin \kappa a \cos \kappa l}{\kappa \sin \kappa l} \left( \frac{1}{F_N} + \frac{f_s}{GA} \right) 
\end{align*}
\]

Where \( f_s \) is shear coefficient, \( G \) is shear modulus and \( A \) is sectional area.

By substituting of equations (7) into (4) the stiffness can be derived.

3. Vibration of discontinuous beam

A beam segment under bending moment \( M \), shearing force \( F_s \) and axial force \( F_N \) is illustrate as Fig 1. The transverse displacement of the beam \( w(x, t) \) can be expressed as

\[
\begin{align*}
    \frac{\partial F_s}{\partial x} \, dx - \rho A \frac{\partial^2 W}{\partial t^2} \, dx &= 0 \\
    - \frac{\partial M}{\partial x} \, dx + F_s \, dx - \rho l \frac{\partial^2 \theta_b}{\partial t^2} \, dx + F_N \frac{\partial W}{\partial x} \, dx &= 0 
\end{align*}
\]

Where \( \rho \) is density of material and the second order term of \( dx \) is omitted.
The relationship among bending moment, shearing force and rotation angle is as follows

\[ M = -EI_z \frac{d^2 \theta_b}{dx^2}, \quad F_s = \frac{1}{f_s} \left( \frac{\partial w}{\partial x} - \theta_b \right)GA \]  

(9)

By substituting of equations (9) into (8) and eliminating \( \theta_b \), the result is

\[ EI_z \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho f_s \left( \frac{f_s E}{G} + 1 \right) \frac{\partial^4 w}{\partial x^4 \partial t^2} + \frac{\rho^2 I_z f_s}{G} \frac{\partial^4 w}{\partial t^4} + F_N \frac{\partial^2 w}{\partial x^2} = 0 \]  

(10)

Although the solution for continuous beam has been given in [14], due to discontinuous at \( m \), it is difficult to solve by analytic method. Therefore, transfer matrix method is used to solve the problem.

Separation of variables is assumed in the forms

\[ w(x, t) = X(x) e^{j\omega t} \]  

(11)

Where \( j \) is the square root of -1.

By substituting of equations (11) into (10) one obtains

\[ X^{(4)} + (q + r + t)X^{(2)} - (s - rt)X = 0 \]  

(12)

\[ q = \frac{F_N}{EI_z}, \quad r = \frac{\rho\omega^2}{E}, \quad s = \frac{\rho A\omega^2}{EI_z}, \quad t = \frac{f_s\omega^2}{G} \]  

(13)

Let

\[ p = \sqrt{(q + r)^2 + 2\omega(l(q - r) + t^2 + 4s) \]  

(14)

The eigenvalues of its characteristic equation are

\[ \lambda_{1,2} = \pm k_1, \quad \lambda_{3,4} = \pm jk_2 \]  

(15)

\[ k_1 = \frac{\sqrt{2}}{2} \sqrt{p - (q + r + t)}, \quad k_2 = \frac{\sqrt{2}}{2} \sqrt{p + (q + r + t)} \]  

(16)

The general solution of equation (11) can be expressed as

\[ X(x) = A \sin k_1 x + B \sinh k_1 x + C \cos k_1 x + D \sin k_2 x \]  

(17)

The coefficient \( A, B, C, D \) can be determined during the operating of transfer matrix method.

Define vector \( \mathbf{W} = [w, \theta, M, F_s] \) representing deflection, slope, bending moment and shearing force of a station. Suppose there are two stations on each beam segment, expressed by \( i \) and \( i+1 \) respectively. The transfer matrix of the beam segment is obtained as

\[ \mathbf{W}_{i+1} = \mathbf{T} \mathbf{W}_i \]  

(18)

And the elements of matrix \( \mathbf{T} \) are
\[ T_{11} = \frac{k_2 \cosh k_1 l_i + k_1^2 \cos k_2 l_i}{k_1^2 + k_2^2}, \quad T_{12} = \frac{k_2^3 \sinh k_1 l_i + k_1^3 \sin k_2 l_i}{(k_1^2 + k_2^2)k_1k_2}, \quad T_{13} = -\frac{\cosh k_1 l_i + \cos k_2 l_i}{k_1^2 + k_2^2}, \]
\[ T_{14} = \frac{-k_2 \sinh k_1 l_i + k_1 \sin k_2 l_i}{EI_z(k_1^2 + k_2^2)k_1k_2}, \quad T_{21} = \frac{(k_2 \sinh k_1 l_i - k_1 \sin k_2 l_i)k_1k_2}{k_1^2 + k_2^2}, \quad T_{22} = T_{11}, \]
\[ T_{23} = -\frac{k_1 \sinh k_1 l_i - k_2 \sin k_2 l_i}{EI_z(k_1^2 + k_2^2)k_1k_2}, \quad T_{24} = T_{13}, \quad T_{31} = -\frac{EI_z(-\cosh k_1 l_i + \cos k_2 l_i)k_1^2 k_2^2}{k_1^2 + k_2^2}, \]
\[ T_{32} = \frac{EI_z(-k_2 \sinh k_1 l_i + k_1 \sin k_2 l_i)k_1k_2}{k_1^2 + k_2^2}, \quad T_{33} = \frac{k_1^2 \cosh k_1 l_i + k_2^2 \cos k_2 l_i}{k_1^2 + k_2^2}, \]
\[ T_{34} = \frac{k_1 \sinh k_1 l_i + k_2 \sin k_2 l_i}{EI_z(k_1^2 + k_2^2)k_1k_2}, \quad T_{41} = -\frac{EI_z(-k_1 \sinh k_1 l_i + k_2 \sin k_2 l_i)k_1^2 k_2^2}{k_1^2 + k_2^2}, \quad T_{42} = T_{31}, \]
\[ T_{43} = \frac{k_1^3 \sinh k_1 l_i - k_2^3 \sin k_2 l_i}{k_1^2 + k_2^2}, \quad T_{44} = T_{33}. \]

The transfer matrix of torsional spring is

\[ \begin{bmatrix} w \\ \theta \\ M \\ F_z \end{bmatrix}_R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1/k_i & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ \theta \\ M \\ F_z \end{bmatrix}_L \] (20)

Rewritten as

\[ \mathbf{W}_R = \mathbf{K} \mathbf{W}_L \] (21)

The displacement relationship at both ends of the beam segment is

\[ \mathbf{W}_0 = \mathbf{T}_n \mathbf{T}_{n-1} \cdots \mathbf{T}_2 \mathbf{T}_1 \mathbf{W}_0 \] (22)

Which could be solved by using the conditions of

\[ w(0,t) = w(l,t) = 0, \quad w''(0,t) = w''(l,t) = 0 \] (23)

4. Deflection and vibration frequency calculation

The basic structural parameters and material properties of a discontinuous beam are as follows: total length \( l=1 \text{m} \), outer radius \( R=0.0205 \text{m} \), inner radius \( r=0.016 \text{m} \), Young’s modulus \( E=2\times10^{11} \), mass density \( \rho=7800 \text{kg/m}^3 \). The deflection and free vibration frequency of a discontinuous beam are calculated by using the above method.
4.1. Deflection calculation

Figure 3. The influence of axial load and torsion spring stiffness on maximum beam deformation

Figure 4. The first third order frequency of the same connection stiffness

Figure 5. Frequency versus connection stiffness with different axial loads (logarithmic coordinates): (a) first order frequency; (b) third order frequency
The beam is under an axial load of $F_N=5\text{kN}$ and a transverse load of $F=20\text{kN}$. Calculate bending deflection of the beam with different torsional spring stiffness while $a=0.2\text{m}$ and $m=0.4\text{m}$. The results are shown in Fig. 3. Comparing with the beam without axial loads, the transverse force has great influence on the deflection of discontinuous beam when the torsion spring stiffness is small, and the deflection differences and the influence of axial load on deflection decrease gradually with the exponential increase of torsion spring stiffness.

4.2. Free vibration frequency calculation
Calculate free vibration frequency while $m=0.4\text{m}$. The results are shown as Fig. 4 and Fig. 5. When the stiffness of torsion spring is the same, the frequency of each order decreases linearly with the increase of the axial loads, and the trend of the second order frequency is relatively small, which is related to the fact that the $m$ is located at the midpoint of the beam. With the increase of torsion spring stiffness, the first and third order frequency increase rapidly. When the torsion spring stiffness reaches a certain degree, the frequency increase slows down and tends to be stable.

Compared with Fig. 5(a) and Fig. 5(b), the trend of the third order frequency is similar to that of the first order frequency, but the frequency differences under corresponding axial loads are relatively small, which indicates that the influence of the axial loads on lower order frequency is greater, and the influence decreases gradually with the increase of the frequency order.

5. Conclusion
Based on the Timoshenko beam theory, the deflection equation and the free vibration differential equation of a simply supported discontinuous beam under axial loads are established. The equations are subsequently solved by the undetermined coefficient method and the transfer matrix method respectively. The free vibration frequency equation in typical cases is given.

The results show that under the same axial load and transverse force, the smaller the connection stiffness is, the more significant the influence of transverse force on the deflection of the beam is. With an increase in the connection stiffness, the influence of transverse force on stiffness decreases, while the influence of axial force on low order frequency is greater, demonstrating that the relationship between connection stiffness and frequency is exponential. A critical stiffness exists, below which both the deflection and free vibration frequency vary exponentially. After this critical stiffness is reached, both values tend to stay stable.

References
[1] W. D. Pilkey, W. Kang, Exact Stiffness Matrix for a Beam Element with Axial Force and Shear Deformation, Finite Elem. Anal. Des. 22 1 (1996) 1-13
[2] J. S. Wu, B. H. Chang, Free Vibration of Axial-loaded Multi-step Timoshenko Beam Carrying Arbitrary Concentrated Elements Using Continuous-Mass Transfer Matrix Method, Eur. J. Mech. A-Solid 38 3 (2013) 20-37
[3] W. H. Liu, D. S. Liu, Natural Frequencies of a Restrained Timoshenko Beam with a Tip Body at Its Free End, J. Sound Vib. 128 1 (1989) 167-173
[4] H. Y. Lin, Dynamic Analysis of a Multi-span Uniform Beam Carrying a Number of Various Concentrated Elements, J. Sound Vib. 309 1-2 (2008) 262-275
[5] M. R. Chenaghlou, H. Nooshin, Axial Force-bending Moment Interaction in a Jointing System Part I: Experimental Study, J. Constr. Steel. Res. 113 (2015) 261-276
[6] M. R. Chenaghlou, H. Nooshin, Axial Force-bending Moment Interaction in a Jointing System: Part II: Analytical Study, J. Constr. Steel. Res. 113 (2015) 277-285
[7] J. Gao, Q. Yuan, P. Li, Y. Liu, Methods for Modeling Stiffness and Model Updating for the Rod-fastened Rotor of Gas Turbine, J. Xi’an Jiaotong Univ. 47 5 (2013) 18-23
[8] Q. Yuan, R. Gao, Z. Feng, J. Wang. Analysis of Dynamic Characteristics of Gas Turbine Rotor Considering Contact Effects and Pre-Tightening Force, Proceedings of ASME Turbo Expo 2008: Power for Land, Sea and Air. (ASME, Berlin, 2008)
[9] R. Gao, Q. Yuan, J. Gao, A Study of a Finite Element Model for a Gas Turbine Tie-rod Rotor and Its Critical Speed Calculation. J. Eng. Therm. Energ. Power. 24 3 (2009) 305-308

[10] S. Liu, Y. Ma, D. Zhang, J. Hong. Studies on Dynamic Characteristics of the Joint in the Aero-engine Rotor System, Mech. Syst. Signal. Pr. 29 5 (2012) 120-136

[11] H. Li, H. Liu, L. Yu, Determination of Preload Force of Circumferential Distributed Rod Fastening Rotor, J. Aerospace Power. 26 12 (2011) 2791-2797

[12] P. Li, Q. Yuan, J. Gao, L. Tan, et.al, Investigation on Stiffness of Circumferential Distributed Tie-rod Rotor with Curvic Couplings of Trochal Disk, J. Aerospace Power. 28 7(2013) 1618-1623

[13] S. Timoshenko, *Vibration Problems in Engineering*, (D. Van Nostrand, New York, 1937)

[14] L. Wang, M. Yu, Effect of Axial Force on the Lateral Vibration Characteristics of Timoshenko Beam Under Free Boundary Condition, J. Ordnance Equip. Eng. 39 (2018) 36-39