Self-Localization of Asynchronous Wireless Nodes with Parameter Uncertainties
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Abstract—We investigate a wireless network localization scenario in which the need for synchronized nodes is avoided. It consists of a set of fixed anchor nodes transmitting according to a given sequence and a self-localizing receiver node. The setup can accommodate additional nodes with unknown positions participating in the sequence. We propose a localization method which is robust with respect to uncertainty of the anchor positions and other system parameters. Further, we investigate the Cramér-Rao bound for the considered problem and show through numerical simulations that the proposed method attains the bound.

Index Terms—Wireless sensor networks, maximum a posteriori estimators, Cramér-Rao bound, anchor uncertainty.

I. INTRODUCTION

The rapid emergence of wireless sensor network (WSN) applications, often requiring accurate knowledge of the node positions, has created remarkable research interest for the WSN localization field [1]. In many application scenarios. In the literature, the problem of sensor network localization in the presence of anchor position uncertainty has been posed as a maximum likelihood (ML) estimation problem, cf. [2]. However, due to the highly nonlinear likelihood function, a closed-form expression of the ML estimator cannot be derived. Therefore the solution has been approached using semi-definite programming in [2] and second-order cone programming in [3]. Further, in [4], the authors studied the joint network localization and time synchronization problem with inaccurate anchors, pointing out the strong interdependence between the timing and positioning aspects. All the above mentioned works assume that the timing measurement noise variance is known.

In this letter, we consider the problem of self-localization in sensor networks with uncertainty in anchor position, without assuming knowledge of the timing noise level. More importantly, we propose a system configuration which allows for a receiver node to perform self-localization even without time synchronization of the anchor nodes. In such a configuration the transceiving anchor nodes transmit signals in a predetermined sequence, one after the other, with a turn-around delay which we assume is not perfectly known at the receiver. By passively listening to the transmissions and exploiting available prior knowledge, the receiver can localize itself as well as the transceivers that participate in the sequence. This setup can accommodate the self-localization of an indefinite number of receiver nodes.

We propose an iterative maximum a posteriori (MAP) estimator that effectively copes with uncertainty in the deployment of anchors and with tolerance of the turn-around delay of the asynchronous transceivers. It also provides delay estimates which may be useful for hardware calibration. Moreover, we study the fundamental performance bounds for the problem considered by deriving the hybrid Cramér-Rao bound. Finally, we evaluate the performance of the proposed estimator by numerical simulations of an ultra-wideband sensor network setup as an application example [5].

Notation: $\|x\|_W = \sqrt{x^\top W x}$ is the weighted norm, where $W$ is positive definite.

II. PROBLEM FORMULATION

We consider a wireless network of $N-1$ transceiving nodes operating asynchronously with local clocks. They transmit signals according to a known sequence, denoted $\mathcal{T}$, set across the network [6]; when the next node in the sequence receives a signal, it transmits in return after a certain delay. On this basis, the goal is to achieve self-localization of the $N$th node, which is a passive receiver that knows $\mathcal{T}$.

The signals are assumed to have a resolvable temporal signature that allows for timing events, e.g., pulses, symbol boundaries, etc. and the propagation velocity $c$ is known. Let $x_i \in \mathbb{R}^d$ denote the position of node $i$, where $d = 2$ or 3, and $\rho_{i,j} \triangleq \|x_i - x_j\|_2$ denote the range between nodes $i$ and $j$. Then the observed time interval between a pair of signals received at node $N$, involving transceiving nodes $i$ and $j$, is modeled by

$$y^{(i,j)} = \frac{1}{c} \rho_{i,j} + \delta_j + \frac{1}{c} \rho_{j,N} - \frac{1}{c} \rho_{i,N} + w^{(i,j)},$$

(1)

where $\delta_j$ denotes the turn-around delay at node $j$, cf. [6], [7]. The delay $\delta_j$ is generated without any common time reference between the nodes, and is therefore asynchronous. The noise $w^{(i,j)}$ arises from three uncorrelated timing measurements, one at node $j$ and two at node $N$, and is modeled as zero-mean Gaussian with unknown variance $E[(w^{(i,j)})^2] = \sigma^2$. The next observed time interval

$$y^{(j,k)} = \frac{1}{c} \rho_{j,k} + \delta_k + \frac{1}{c} \rho_{k,N} - \frac{1}{c} \rho_{j,N} + w^{(j,k)},$$

(2)

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uses one timing measurement from the previous observation. Modeling the timing noise variance across nodes equally we have the correlation $E[w_i^{(j)}w_j^{(k)}] = \sigma^2 / 3$ for all consecutive observations.

Prior knowledge of the positions will be modeled as $x_i \sim \mathcal{N}(\mu_i, \Sigma_i)$, where $\mu_i$ and $\Sigma_i$ are known, $\forall i$. Setting $\Sigma_i^{-1} = 0$, leads to a noninformative prior, $p(x_i) \propto 1 [8]$. Further, to avoid signal collisions it is necessary that the delays exceed $\rho_{\text{max}}/c$, where $\rho_{\text{max}}$ is the maximum range between any pair of transceivers and can easily be ensured in any bounded localization scenario. Whilst the delay in each transceiver node may be set to some nominal value $\mu_d$, the actual delay $\delta_j$ will deviate due to hardware imperfections. We model this as $\delta_j \sim \mathcal{N}($mean, $\sigma^2_j)$, with a given $\sigma^2_j$. For the unknown noise variance, we assume a noninformative prior $p(\sigma^2) \propto 1/\sigma^2 [8]$.

For notational simplicity we write $\theta \triangleq [x_1 \cdots x_N]^T \in \mathbb{R}^{dN}$ and $\delta \triangleq [\delta_1 \cdots \delta_{N-1}]^T \in \mathbb{R}^{N-1}$. The goal is to estimate $\theta$, $\delta$ and $\sigma^2$ from a set of $M$ observations $(y_m)_{m=1}^M$.

III. MAP ESTIMATOR

For a given sequence $T$, the pair of nodes involved in each observed time interval $(y_m)_{m=1}^M$ is known. In vector form, the observation model is

$$y = c^{-1}Hg(\theta) + w \in \mathbb{R}^M,$$ \hspace{1cm} (3)

where $\theta \triangleq [\theta^T \delta^T]^T \in \mathbb{R}^T$, and $T = dN + N - 1$. The noise follows $w \sim \mathcal{N}(0, \sigma^2Q)$, where $[Q_{i,i}, i = 1, [Q_{i,j}, i \neq j]$ such that $|i - j| = 1$ and $[Q_{i,i}, i = 0$ otherwise. The nonlinear mapping $g(\theta) = [\rho^T(\theta) \cdot \delta^T]^T \in \mathbb{R}^{N(N-1)/2+N-1}$ contains all the pairwise ranges and delays, and $H$ is determined by the transmission sequence $T$, cf. [1]. We aim to find the maximum a posteriori (MAP) estimator, i.e., the maximizer of $p(\theta, \sigma^2|y)$.

A. Concentrated cost function

Using Bayes’ Rule, the MAP estimate can be computed by maximizing $J(\theta, \sigma^2) = \ln p(y|\theta, \sigma^2) + \ln p(\sigma^2) + \ln p(\theta)$. Further, define

$$J_1(\theta, \sigma^2) \triangleq \ln p(y|\theta, \sigma^2) + \ln p(\sigma^2)$$

$$= -\frac{M+2}{2} \ln \sigma^2 - \frac{1}{2\sigma^2}||y - c^{-1}Hg(\theta)||_Q^{-1} + K_1,$$ \hspace{1cm} (4)

where $K_1$ is a constant. Similarly,

$$J_2(\theta) \triangleq \ln p(\theta) = -\frac{1}{2}||\theta - \mu||_P^{-1} + K_2,$$ \hspace{1cm} (5)

where $K_2$ is a constant. Maximizing $J_1$ with respect to $\sigma^2$ yields the estimate $\hat{\sigma}^2 = \{y - c^{-1}Hg(\hat{\theta})\}_Q^{-1} / (M+2)$. Inserting this back into $J_1$, and combining with $J_2$, results in a concentrated cost function. The MAP estimator is then given by

$$\hat{\theta} = \arg\min_{\theta \in \mathbb{R}^T} V(\theta),$$ \hspace{1cm} (6)

where

$$V(\theta) \triangleq \frac{1}{2} \ln ||y - c^{-1}Hg(\theta)||_Q^{-2} + \frac{1}{2} ||\theta - \mu||_P^{-2},$$ \hspace{1cm} (7)

$\beta = 1/(M+2)$, $\mu = [\mu_1^T \cdots \mu_N^T \mu_{d,1}^T \cdots \mu_{d,N}^T \mu_{d,1}^T \cdots \mu_{d,1}^T]^T$ and $P^{-1} = \text{diag}(P_1^{-1}, \cdots, P_N^{-1}, \sigma_\delta^{-2}I_{N-1})$.

B. Iterative solution

The optimization problem $\theta$ does not lend itself to a closed-form solution. A standard method for solving (6) is gradient descent. However, its performance is heavily dependent on the user-defined step length. Instead we propose an iterative solution by first exploiting the linearization around an initial estimate $\hat{\theta}$, i.e., $g(\theta) \approx g(\hat{\theta}) + \Gamma(\hat{\theta})\theta$, where $\Gamma(\hat{\theta})$ is the Jacobian of $g(\theta)$. Then we may write a cost function which approximates (7) as

$$V_\ell(\hat{\theta}) \triangleq \frac{1}{2} \ln ||\hat{y}_\ell - G_\ell \hat{\theta}||_Q^{-2} + \frac{1}{2} ||\hat{\mu}_\ell - \hat{\theta}||_P^{-2},$$ \hspace{1cm} (8)

where we introduce $g_\ell = g(\hat{\theta})$, $\Gamma_\ell = \Gamma(\hat{\theta})$, $\hat{y}_\ell = y - c^{-1}Hg_\ell$, $G_\ell = c^{-1}HG_\ell$ and $\hat{\mu}_\ell = \mu - \hat{\theta}$. We aim to iteratively updating the estimate $\hat{\theta}$ by finding the optimal increments $\hat{\theta}$. To this end, we use a fixed-point iteration. First, we note that the gradient of $V_\ell(\hat{\theta})$ equals

$$\partial_\theta V_\ell = \alpha(\hat{\theta})G_\ell^TQ^{-1}(G_\ell\hat{y}_\ell - \hat{\mu}_\ell) + \beta P^{-1}(\hat{\theta} - \hat{\mu}_\ell),$$

where $\alpha(\hat{\theta}) \triangleq 1/||\hat{y}_\ell - G_\ell \hat{\theta}||_Q^{-2}$. Then, we hold $\alpha(\hat{\theta})$ fixed and solve $\partial_\theta V_\ell = 0$, thus obtaining the following fixed-point iteration

$$\hat{\theta} := \left(\alpha(\hat{\theta})G_\ell^TQ^{-1}G_\ell + \beta P^{-1}\right)^{-1}$$

$\times (\alpha(\hat{\theta})G_\ell^TQ^{-1}\hat{y}_\ell + \beta P^{-1}\hat{\mu}_\ell),$ \hspace{1cm} (9)

By iteratively applying (9), starting with the zero increment $\hat{\theta} = 0$, we converge to a stationary point. The analytical convergence properties are difficult to derive and are beyond the scope of this letter. However, in Section IV we show the convergence properties by numerical evaluation in a practical scenario. Further, we provide a practical method for obtaining an initial estimate $\hat{\theta}_0$. The iterative estimator is summarized in Algorithm 1, where $\varepsilon$ is the convergence threshold.

Algorithm 1 Iterative MAP estimator

1: Input: $y$, $\mu$, $P$, $c$ and $\hat{\theta}_0$
2: Set $\ell := 0$ and $\beta = 1/(M+2)$
3: repeat
4: \hspace{0.5cm} $\hat{y}_\ell = y - c^{-1}Hg(\hat{\theta}_\ell)$
5: \hspace{0.5cm} $G_\ell = c^{-1}HG_\ell$
6: \hspace{0.5cm} $\mu_\ell = \mu - \hat{\theta}_\ell$
7: \hspace{0.5cm} $\hat{\mu}_\ell := 0$
8: \hspace{0.5cm} $\hat{\theta} := 0$
9: \hspace{0.5cm} Repeat (8) until convergence
10: \hspace{0.5cm} $\hat{\theta}_{\ell+1} := \hat{\theta} + \hat{\theta}_\ell$, $\ell := \ell + 1$
11: until $||\hat{\theta}_\ell - \hat{\theta}_{\ell-1}||_2 < \varepsilon$
12: Output: $\hat{\theta}_\ell$

IV. CRAMÉR-RAO BOUND

Let $\eta \triangleq [\theta^T \delta^T \sigma^2]^T \in \mathbb{R}^{T+1}$ and $\hat{\eta}$ be any estimator that is conditionally unbiased with respect to the deterministic parameters. Then its mean square error (MSE) matrix is constrained by the hybrid Cramér-Rao bound (HCRB) [9], $C_{\eta} \geq J_\eta^{-1}$, where $J_\eta = J_\eta^D + J_\eta^H \in \mathbb{R}^{(T+1) \times (T+1)}$. $J_\eta^D$.
Here $J^D_{\eta} = E_{\tilde{\eta}}[J^D(\eta)]$ is the expected Fisher information matrix, where $\tilde{\eta}$ denotes the subset of parameters that are modeled as random quantities and

$$
J^D(\eta)_{i,j} = \frac{\partial^2 g}{\partial \theta_i \partial \theta_j} H^T Q^{-1} H \frac{\partial g}{\partial \theta_j} + \frac{M}{2\sigma^4} \frac{\partial \sigma^2}{\partial \theta_i} \frac{\partial \sigma^2}{\partial \theta_j},
$$

as given in [10]. As the expectation does not have a closed form solution, we evaluate it by Monte Carlo simulation. If a subset of node positions, $\theta_a$, and the noise level, $\sigma^2$, are treated as deterministic and unknown parameters, while the remaining parameters, $\theta_r$ and $\delta$, are random Gaussian then the prior information matrix is given by

$$
J^P_\theta = \begin{bmatrix}
P_{\theta_a}^{-1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \sigma^2 I_{N-1} & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \in \mathbb{R}^{(T+1) \times (T+1)},
$$

where $P_{\theta_a}$ is the covariance matrix of $\theta_a$. This division between deterministic and random parameters may be useful to study practical configurations where we lack prior knowledge on the position of a subset of nodes.

V. NUMERICAL RESULTS

To evaluate the performance of the MAP estimator proposed in Section III we provide numerical simulation results in an ultra wideband wireless sensor network scenario [5]. In particular, we compare the HCRB derived in Section IV with the root mean square error (RMSE) of the MAP estimates.

A. Setup

We consider a network of $N = 6$ nodes consisting of one self-localizing passive receiver, without prior knowledge of its position, and $N_a = 4$ anchors with $P_{\theta_a} = \sigma^2 \tilde{I}_{N_a}$. In addition, we consider one auxiliary node, which is a transceiver participating in the sequence with a noninformative position prior. Therefore, the number of unknown-position nodes is $N_u = 2$. The simulation has been performed using the topology shown in Fig. 1 and setting $\mu_\delta = 10^{-6}$ s. In each realization, the positions of the anchors and the delays have been randomly generated according to their prior distributions. Furthermore, we construct the sequence $T$ to ensure that all pairwise combinations appear at least once. E.g., $T = \{1, 2, 3, 1, 4, 1, 5, 2, 3\ldots\}$.

The average RMSE of the position and delay estimates are given by

$$
\text{RMSE}_\xi \triangleq \frac{1}{N_\xi} \sqrt{\text{tr}\{C_\xi\}},
$$

where $\xi$ can be either $\theta_a$, in which case $N_\xi = N_a$, or $\delta$, and $N_\xi = N - 1$. Here $C_\xi$ is the MSE matrix of $\xi$. We estimated the RMSE from $10^4$ Monte Carlo iterations.

We initialize the MAP estimator of Algorithm 1 with $\hat{\theta}_0 = \mu = [\mu_1^T \ldots \mu_N^T \mu_a^T \mu_\delta^T \tilde{I}_{N-1}^T]^T$, where $\mu$ is an arbitrary position initialization of the auxiliary node, with $\mu \neq \mu_a$. The self-localizing node’s initial position estimate $\hat{\mu}$ is the centroid of the anchor positions, i.e. $\hat{\mu} = \frac{1}{N_a} \sum_{i=1}^{N_a} \mu_i$. Further, we set $\varepsilon = 10^{-4}$. The same tolerance is used for Step 8 in Algorithm 1.

B. Results

Fig. 1 shows the topology of the network, together with the error ellipses produced by the MAP estimator and given by the HCRB. It can be seen that the proposed estimator attains the bound. Next, we provide an evaluation of the position estimates as a function of timing noise level $\sigma_a$. In Fig. 2 we vary the uncertainty of the anchor positions which is parameterized by $\sigma_a$. In a low noise scenario, we achieve an accuracy of the same order of magnitude as the anchor position prior. In Fig. 3 we vary the uncertainty of the delays set by $\sigma_\delta$. The results show that the proposed position-estimation method is robust with respect to deviations from the nominal delay. Similarly, Fig. 4 shows the robustness of the delay estimator. We tested a range of deviations that spans two orders of magnitude. Furthermore, the extended simulation setup in Fig. 5 shows that the proposed method can localize 10 auxiliary nodes which participate in the transmission sequence and are not collocated. However, the MAP does not attain the
In this letter, we have considered a wireless network localization scenario with asynchronous nodes where the entire network topology can be estimated by a self-localizing receiver node. We have proposed a MAP estimator and compared its performance with the HCRB that we derived for the problem. The simulation results show that the estimator attains the HCRB in all scenarios, in particular when nodes are close to anchors as can be seen in Fig. 5. Moreover, Fig. 6 shows the convergence behavior of the estimator, the average is 5.12 iterations.

Finally, we repeated the simulations in the scenario in Fig. 1 assigning a more conservative prior with respect to the actual variability of the anchor positions. In particular, we defined $\sigma_a' = 10\sigma_a$, where $\sigma_a'$ is used by Algorithm 1 and $\sigma_a = 0.2$ m is used to generate the random anchor positions. We obtained $\text{RMSE}_{\theta_a} = 0.37$ m and $\text{HCRB}_{\theta_a} = 0.26$ m, indicating inherent robustness to such model mismatches.

**VI. Conclusion**

In this letter, we have considered a wireless network localization scenario with asynchronous nodes where the entire network topology can be estimated by a self-localizing receiver node. We have proposed a MAP estimator and compared its performance with the HCRB that we derived for the problem. The simulation results show that the estimator attains the bound and is robust with respect to uncertainty of the anchor positions, delays and noise level. In addition to the anchor nodes, we found that the setup allows for the localization of auxiliary nodes participating in the transmission sequence.

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