A note on modular Terwilliger algebras of association schemes

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Abstract
Let $p$ denote a prime number. In this note, we focus on the modular Terwilliger algebras of association schemes defined in Hanaki (Graphs Combin, 2021, https://doi.org/10.1007/s00373-021-02363-0). We define the primary module of a modular Terwilliger algebra of an association scheme and determine all its composition factors up to isomorphism. We then characterize the $p'$-valenced association schemes in terms of numerous properties of their modular Terwilliger algebras. We conclude our investigation with a few corollaries and questions on the modular Terwilliger algebras of association schemes.

Keywords Association scheme · Modular Terwilliger algebra · $p'$-valenced scheme

Mathematics Subject Classification 05E30 primary · 05E10 secondary

1 Introduction
In this present note, we study a class of associative algebras which are associated to the association schemes of finite valency (or schemes as we shall simply say) in a similar way as the subconstituent algebras (also known as Terwilliger algebras) defined by Terwilliger in Terwilliger (1992) are associated to the distance regular graphs. While the original Terwilliger algebras were defined only for the distance regular graphs that are equivalent to the symmetric schemes of class two, Terwilliger algebras can also be defined for arbitrary schemes. Terwilliger algebras of schemes are finite-dimensional semisimple $\mathbb{C}$-algebras. In Hanaki (2021), Hanaki introduced the Terwilliger algebras of schemes over an arbitrary commutative unital ring. In this present note, we are interested in the Terwilliger algebras of schemes over a field of positive characteristic.
Following Hanaki (2021), we call these algebras the modular Terwilliger algebras of schemes.

The primary module of a Terwilliger algebra of a scheme $S$ is an irreducible module of this algebra. It closely relates to the structure of $S$ (see Egge (2000); Terwilliger (1992, 1993a, b)). We define the primary module for a modular Terwilliger algebra of $S$ (see Definition 3.2). So it is natural to study the primary module of a modular Terwilliger algebra of $S$. As the first main result of this note, in Theorem 3.15, we determine all composition factors of the primary module of a modular Terwilliger algebra of $S$ up to isomorphism.

The $p'$-valenced schemes enjoy many algebraic properties. For example, according to (Hanaki 2021, Theorem 3.4), a modular Terwilliger algebra of $S$ is semisimple only if $S$ is a $p'$-valenced scheme. This listed example motivates us to characterize the $p'$-valenced schemes in terms of some properties of their modular Terwilliger algebras. As the second main result of this note, in Theorem 4.21, we characterize the $p'$-valenced schemes in terms of numerous properties of their modular Terwilliger algebras.

The organization of this note is as follows. The basic notation and preliminaries are given in Section 2. Two main results are proved in Sections 3 and 4, respectively.

## 2 Basic notation and preliminaries

For a general background on association schemes, one may refer to Bannai and Ito (1984), Zieschang (1996), or Zieschang (2005).

### 2.1 General conventions

Throughout the present note, fix a field $\mathbb{F}$ of positive characteristic $p$ and a nonempty finite set $X$. Let $\mathbb{N}$ be the set of all natural numbers. Set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. If $a, b \in \mathbb{N}_0$, put $[a, b] = \{c \in \mathbb{N}_0 : a \leq c \leq b\}$. For a nonempty subset $Y$ of an $\mathbb{F}$-linear space, write $\langle Y \rangle_\mathbb{F}$ for the $\mathbb{F}$-linear space generated by $Y$. By convention, $\langle \emptyset \rangle_\mathbb{F}$ is the zero space. The addition, the multiplication, and the scalar multiplication of matrices in this note are the usual matrix operations. A scheme means an association scheme on $X$. All modules are finitely generated left modules.

### 2.2 Schemes

Let $S = \{R_0, R_1, \ldots, R_d\}$ be a partition of the Cartesian product $X \times X$. Then $S$ is a scheme of class $d$ if the following conditions hold:

1. $R_0 = \{(x, x) : x \in X\};$
2. For any $i \in [0, d]$, there is $i' \in [0, d]$ such that $\{(x, y) : (y, x) \in R_i\} = R_i' \in S;$
3. For any $i, j, \ell \in [0, d]$ and $(x, y), (\tilde{x}, \tilde{y}) \in R_\ell$, the following equality holds: $|\{\tilde{z} \in X : (x, \tilde{z}) \in R_i, (\tilde{z}, y) \in R_j\}| = |\{\tilde{z} \in X : (\tilde{x}, \tilde{z}) \in R_i, (z, \tilde{y}) \in R_j\}|.$
Throughout the whole note, \( S = \{ R_0, R_1, \ldots, R_d \} \) is a fixed scheme of class \( d \). By (iii), for any \( i, j, \ell \in [0, d] \) and \( (x, y) \in R_i \), there exists a constant \( p^\ell_{ij} \in \mathbb{N}_0 \) such that \( p^\ell_{ij} = |\{ z \in X : (x, z) \in R_i, (z, y) \in R_j \}| \). Let \( i \in [0, d] \) and \( x, y \in X \). Set \( xR_i = \{ z \in X : (x, z) \in R_i \} \). Write \( k_i \) for \( p^0_{ii} \). The number \( k_i \) is called the valency of \( R_i \). Observe that \( |xR_i| = |yR_i| = k_i \). Hence \( k_i > 0 \) since \( R_i \neq \emptyset \). If \( n \in \mathbb{N}_0 \), set \( S_n = \{ j \in [0, d] : p^n | k_j \) and \( p^{n+1} \nmid k_j \} \). Call \( S \) a \( p \)’-valenced scheme if \( S_0 = [0, d] \). Put \( O_\emptyset(S) = \{ R_j \in S : k_j = 1 \} \). Note that \( R_0 \in O_\emptyset(S) \). We list a lemma as follows.

**Lemma 2.1** (Bannai and Ito 1984, Proposition 2.2 (vi)) Let \( i, j, \ell \in [0, d] \). Then
\[
k_\ell p^\ell_{ij} = k_i p^\ell_{j\ell} = k_j p^\ell_{i\ell}.
\]

### 2.3 Algebras

Let \( \mathbb{Z} \) be the integer ring and \( \mathbb{F}_p \) be the prime subfield of \( \mathbb{F} \). Given \( a \in \mathbb{Z} \), let \( \overline{a} \) be the image of \( a \) under the unital ring homomorphism from \( \mathbb{Z} \) to \( \mathbb{F}_p \).

Let \( A \) be a finite-dimensional associative unital \( \mathbb{F} \)-algebra and \( B \) be a two-sided ideal of \( A \). Call \( B \) a nilpotent ideal of \( A \) if there is \( m \in \mathbb{N} \) such that the product of any \( m \) elements of \( B \) is the zero element of \( A \). Let \( \text{Rad}(A) \) be the Jacobson radical of \( A \). Recall that \( \text{Rad}(A) \) is the sum of all nilpotent two-sided ideals of \( A \). Let \( U \) be a nonzero \( A \)-module and \( V, W \) be \( A \)-modules. Call \( U \) an irreducible \( A \)-module if \( U \) has no nonzero proper \( A \)-submodule. Call \( U \) an indecomposable \( A \)-module if \( U \) is not a direct sum of two nonzero \( A \)-submodules. Let \( \text{Ann}_A(V) = \{ \hat{a} \in A : \hat{a} \hat{v} = 0 \forall \hat{v} \in V \} \). Observe that \( \text{Ann}_A(V) \) is a two-sided ideal of \( A \). If \( V \) is an irreducible \( A \)-module, it is known that \( \text{Rad}(A) \subseteq \text{Ann}_A(V) \). If \( W \) is an \( A \)-submodule of \( V \), \( V/W \) denotes the quotient \( A \)-module of \( V \) with respect to \( W \). A composition series of \( U \) of length \( n \) is an \( A \)-submodule series \( U_n \subset U_{n-1} \subset \cdots \subset U_1 \subset U_0 = U \) of \( U \), where \( U_n \) is the zero module and \( U_{q-1}/U_q \) is an irreducible \( A \)-module for every \( q \in [1, n] \). Call \( U_{q-1}/U_q \) a composition factor of \( U \) for every \( q \in [1, n] \). By the Jordan-Hölder Theorem, all composition factors of \( U \) are independent of the choice of composition series of \( U \) up to permutation and isomorphism. So the length of a composition series of \( U \) is an invariant of \( U \). Call this invariant the composition length of \( U \).

### 2.4 Modular Terwilliger algebras of schemes

Let \( \mathbb{F}^X \) be the \( \mathbb{F} \)-linear space of \( \mathbb{F} \)-column vectors whose coordinates are labeled by the elements of \( X \). Let \( \mathbf{1} \) and \( \mathbf{0} \) be the all-one column vector and the all-zero column vector in \( \mathbb{F}^X \), respectively. Let \( M_X(\mathbb{F}) \) be the full matrix algebra of \( \mathbb{F} \)-square matrices whose rows and columns are labeled by the elements of \( X \). Let \( I, J, O \) denote the identity matrix, the all-one matrix, and the all-zero matrix in \( M_X(\mathbb{F}) \), respectively. If \( Z \in M_X(\mathbb{F}) \), let \( Z^t \) denote the transpose of \( Z \). Let \( y, z \in X \) and \( E_{yz} \) denote the \( (0, 1) \)-matrix in \( M_X(\mathbb{F}) \) whose unique nonzero entry is the \((y, z)\)-entry.

Let \( i, j \in [0, d] \). The adjacenty \( \mathbb{F} \)-matrix with respect to \( R_i \), denoted by \( A_i \), is the matrix \( \sum_{(x, y) \in R_i} E_{xy} \). The dual \( \mathbb{F} \)-idempotent with respect to \( y \) and \( R_i \), denoted by \( E_i^*(y) \), is the matrix \( \sum_{z \in y R_i} E_{yz} \). Let \( \delta_{\alpha \beta} \) denote the Kronecker delta of integers \( \alpha \) and \( \beta \).
Lemma 2.2 Let $i, j, \ell \in [0, d]$. 

(i) (Hanaki 2021, Lemma 3.2) $E_i^* A_j E_\ell^* 1 = p_{ij}^\ell E_i^* 1$. In particular, $E_i^* A_j E_\ell^* J = p_{ij}^\ell E_i^* J$.

(ii) (Jiang 2020, Lemma 3.2) If $E_i^* A_j E_\ell^* \neq 0$ and $\min\{k_i, k_\ell\} = 1$, then $E_i^* A_j E_\ell^* = E_i^* J E_\ell^*$.

(iii) Every element of $T$ is a finite $F$-linear combination of elements of $\bigcup_{m \in \mathbb{N}_0} T_m$.

**Proof** For (iii), $A_0, A_1, \ldots, A_d, E_0^*, E_1^*, \ldots, E_d^*$ belong to $\bigcup_{m \in \mathbb{N}_0} T_m$ by (2), (3), (6). Notice that $\bigcup_{m \in \mathbb{N}_0} T_m$ is a unital $F$-subalgebra of $T$ by (6). So $T = \bigcup_{m \in \mathbb{N}_0} T_m$ by the definition of $T$. (iii) thus follows as $T = \bigcup_{m \in \mathbb{N}_0} T_m$ and $\dim_F T \in \mathbb{N}$. \qed

### 3 Primary modules of modular Terwilliger algebras of schemes

In this section, we define the primary module of $T$ and study its basic properties. In particular, we determine all its composition factors up to isomorphism. For our purpose, notice that $M_X(F)$ acts on $F^X$ by left multiplication. We first list a lemma.
Lemma 3.1 The following statements hold:

(i) \{E^*_h 1 : i \in [0, d]\} is an $F$-linearly independent subset of $R^X$ of cardinality $d + 1$.

(ii) \{\langle E^*_h 1 : i \in [0, d]\rangle\} is a $T$-module under the left multiplication action of $T$.

(iii) Assume that $n \in \mathbb{N}$. Then \{\langle E^*_i 1 : i \in [0, d], p^n | k_i\rangle\} is a $T$-module under the left multiplication action of $T$.

Proof For (i), for every $h \in [0, d]$, notice that $E^*_h 1 \neq 0$ by the definitions of $E^*_h$ and $1$. Suppose that there exist $c_0, c_1, \ldots, c_d \in F$ such that $\bigcup^{d}_{i=0}(c_i) \cap (F \setminus \{0\}) \neq \emptyset$ and $\sum^{d}_{i=0} c_i E^*_i 1 = 0$. So there is $j \in [0, d]$ such that $c_j \neq 0$. By (3), observe that $c_j E^*_j 1 = E^*_j (\sum^{d}_{i=0} c_i E^*_i 1) = E^*_j 0 = 0$. Since $E^*_j 1 \neq 0$, we thus have $c_j = 0$, which contradicts the inequality $c_j \neq 0$. So \{\langle E^*_i 1 : i \in [0, d]\rangle\} is an $F$-linearly independent subset of $R^X$. We also note that $|\langle E^*_i 1 : i \in [0, d]\rangle| = d + 1$ by (3). (i) thus follows.

Let $a, b, c \in [0, d]$. For (ii), by (3) and Lemma 2.2 (i), $E^*_a A_b E^*_c E^*_h 1 = \delta_{ch} p^a_{hb} E^*_a 1$ for every $h \in [0, d]$. In particular, notice that $E^*_a A_b E^*_c E^*_h 1 \in \langle \langle E^*_i 1 : i \in [0, d]\rangle\rangle_F$ for every $h \in [0, d]$. Hence, since $\langle \langle E^*_i 1 : i \in [0, d]\rangle\rangle_F$ is an $F$-linear space and $a, b, c$ are chosen from $[0, d]$ arbitrarily, (ii) thus follows from Lemma 2.2 (iii) and (6).

For (iii), notice that (iii) is trivial if $\langle \langle E^*_i 1 : i \in [0, d]\rangle, p^n | k_i\rangle\rangle_F = \{0\}$. We thus assume further that $\langle \langle E^*_i 1 : i \in [0, d]\rangle, p^n | k_i\rangle\rangle_F \neq \{0\}$. For every $h \in [0, d]$ and $p^n | k_h$, we claim that $E^*_a A_b E^*_c E^*_h 1 \in \langle \langle E^*_i 1 : i \in [0, d]\rangle, p^n | k_i\rangle\rangle_F$. Suppose that there is $\ell \in [0, d]$ such that $E^*_a A_b E^*_c E^*_h 1 \notin \langle \langle E^*_i 1 : i \in [0, d]\rangle, p^n | k_i\rangle\rangle_F$ and $p^n | k_\ell$. So $E^*_a A_b E^*_c E^*_h 1 \neq 0$. So $E^*_a A_b E^*_c E^*_h 1 = p^a_{\ell b'} E^*_a 1 \notin \langle \langle E^*_i 1 : i \in [0, d]\rangle, p^n | k_i\rangle\rangle_F$ by (3) and Lemma 2.2 (i). We thus have $p^a | p^a_{\ell b'}$ and $p^n | k_\ell$. Since $p^n | k_\ell$, $p^n | k_\ell$, and $k_\ell p^a_{\ell b'} = k_\ell p^a_{\ell b'}$ by Lemma 2.1, observe that $p | p^a_{\ell b'}$, which contradicts the fact $p \nmid p^a_{\ell b'}$. The desired claim follows. As $\langle \langle E^*_i 1 : i \in [0, d]\rangle, p^n | k_i\rangle\rangle_F$ is an $F$-linear space and $a, b, c$ are chosen from $[0, d]$ arbitrarily, (iii) is proved by combining Lemma 2.2 (iii), (6), and the proven claim. \(\square\)

We are now ready to define the primary module of $T$.

Definition 3.2 The $T$-module in Lemma 3.1 (ii) is similar to the primary module of a Terwilliger algebra of $S$ (see (Terwilliger 1992, Lemma 3.6)). Call the $T$-module in Lemma 3.1 (ii) the primary module of $T$ and denote it by $W_0$. Let $n \in \mathbb{N}$. Let $W_n$ denote the $T$-module in Lemma 3.1 (iii). For every $m \in \mathbb{N}_0$, note that $W_{m+1}$ is a $T$-submodule of $W_m$. Notice that $W_1 \subset W_0$. Notice that $\dim_F W_0 = d + 1$ by Lemma 3.1 (i).

Lemma 3.3 Let $n \in \mathbb{N}_0$.

(i) If $S_m = \emptyset$ for every $n < m \in \mathbb{N}$, then $W_m = \{0\}$ for every $n < m \in \mathbb{N}$.

(ii) $W_n / W_{n+1}$ has an $F$-basis $\{E^*_i 1 + W_{n+1} : i \in S_n\}$ of cardinality $|S_n|$.

Proof By Definition 3.2 and the definition of $S_n$, $W_m = \langle \langle E^*_i 1 : i \in \bigcup_{m \leq q \in \mathbb{N}} S_q\rangle\rangle_F$ for every $n < m \in \mathbb{N}$. By hypotheses, $W_m = \{0\}$ for every $n < m \in \mathbb{N}$. (i) is shown. (ii) is proved by combining Lemma 3.1 (i), Definition 3.2, the definition of $S_n$. \(\square\)

The following lemma contains more properties of the objects in Definition 3.2.
Lemma 3.4  The following statements hold:

(i)  \( W_1 \) is the unique maximal \( T \)-submodule of \( W_0 \).

(ii) \( W_0 \) is an indecomposable \( T \)-module.

(iii) \( W_0/W_1 \) is an irreducible \( T \)-module.

(iv) \( W_0 \) is an irreducible \( T \)-module if and only if \( S \) is a \( p' \)-valenced scheme.

(v) Let \( n \in \mathbb{N}_0 \) and \( U \) denote a \( T \)-submodule of \( W_n/W_{n+1} \). Then there do not exist \( T \)-submodules \( V, W \) of \( W_{n+1} \) such that \( W \subset V \) and \( U \cong V/W \) as \( T \)-modules.

Proof  For (i), by the definitions of \( W_0 \) and \( W_1 \), \( W_1 \) is a proper \( T \)-submodule of \( W_0 \). Let \( M \) denote a maximal \( T \)-submodule of \( W_0 \). According to the definition of \( W_0 \), we pick \( \sum_{i=0}^d c_i E_i^* 1 \in M \), where \( c_i \in \mathbb{F} \) for every \( i \in [0, d] \). For every \( i \in [0, d] \), we claim that \( c_i = 0 \) if \( p \nmid k_i \). Suppose that there exists \( j \in [0, d] \) such that \( p \nmid k_j \) and \( c_j \neq 0 \). Let \( \ell \in [0, d] \). Notice that \( E_\ell^* J E_j^* \in T \) by (2) and the definition of \( T \). Since \( M \) is a \( T \)-submodule of \( W_0 \), notice that \( E_\ell^* J E_j^* (\sum_{i=0}^d c_i E_i^* 1) = c_j E_\ell^* J E_j^* 1 = c_j k_j E_\ell^* 1 \in M \) by (3) and (5). Hence \( E_j^* 1 \in M \) as \( c_j \neq 0 \) and \( p \nmid k_j \). Since \( \ell \) is chosen from \([0, d]\) arbitrarily and \( M \) is a maximal \( T \)-submodule of \( W_0 \), we thus have \( W_0 \nsubseteq M \subset W_0 \), which is a contradiction. So the desired claim follows. Since \( \sum_{i=0}^d c_i E_i^* 1 \) is chosen from \( M \) arbitrarily, by the proven claim and the definition of \( W_1 \), observe that \( M \subseteq W_1 \subset W_0 \), which implies that \( M = W_1 \) since \( M \) is a maximal \( T \)-submodule of \( W_0 \). Since \( M \) is chosen from the set of all maximal \( T \)-submodules of \( W_0 \) arbitrarily, \( W_1 \) is the unique maximal \( T \)-submodule of \( W_0 \). (i) is proved.

For (ii), \( W_0 \neq \{0\} \) since \( \dim \mathbb{F} W_0 = d + 1 > 0 \). Suppose that there exist nonzero \( T \)-modules \( N_1 \) and \( N_2 \) such that \( W_0 = N_1 \oplus N_2 \). Hence \( W_0 \) has at least two distinct maximal \( T \)-submodules, which contradicts (i). Therefore \( W_0 \) is an indecomposable \( T \)-module, (ii) is shown.

For (iii), note that (iii) is from (i).

For (iv), by (i) and (iii), \( W_0 \) is an irreducible \( T \)-module if and only if \( W_1 = \{0\} \). By the definition of \( W_1 \), note that \( W_1 = \{0\} \) if and only if \( S \) is a \( p' \)-valenced scheme. (iv) thus follows.

For (v), there is no loss to assume further that \( U \neq \{0 + W_{n+1}\} \). By Lemma 3.3 (ii), pick \( \sum_{u \in S_n} e_u E_u^* 1 + W_{n+1} \in U \setminus \{0 + W_{n+1}\} \), where \( e_u \in \mathbb{F} \) for every \( u \in S_n \). So there is \( v \in S_n \) such that \( e_v \neq 0 \). As \( U \) is a \( T \)-submodule of \( W_n/W_{n+1} \), by (3) and Lemma 3.3 (ii), \( 0 + W_{n+1} \neq e_v E_v^* 1 + W_{n+1} = E_v^* (\sum_{u \in S_n} e_u E_u^* 1 + W_{n+1}) \in U \). Suppose that there exist \( T \)-submodules \( V, W \) of \( W_{n+1} \) such that \( W \subset V \) and \( U \cong V/W \) as \( T \)-modules. Let \( \phi \) be a \( T \)-isomorphism from \( U \) and \( V/W \). Since \( \phi \) is injective and \( 0 + W_{n+1} \neq e_v E_v^* 1 + W_{n+1} \in U \), notice that \( 0 + W \neq \phi(e_v E_v^* 1 + W_{n+1}) \in V/W \). Since \( v \in S_n \), by the definitions of \( S_n \) and \( W_{n+1} \), (3) tells us that \( E_v^* \in \text{Ann}_T(W_{n+1}) \). In particular, as \( W \subset V \subseteq W_{n+1} \), notice that \( E_v^* \phi(e_v E_v^* 1 + W_{n+1}) = 0 + W \). As \( \phi \) is a \( T \)-isomorphism, by (3) again, we thus can deduce that

\[
0 + W \neq \phi(e_v E_v^* 1 + W_{n+1}) = \phi(E_v^* (e_v E_v^* 1 + W_{n+1})) = E_v^* \phi(e_v E_v^* 1 + W_{n+1}) = 0 + W,
\]

which is impossible. So there are not \( T \)-submodules \( V, W \) of \( W_{n+1} \) such that \( W \subset V \) and \( U \cong V/W \) as \( T \)-modules. (v) is proved. \( \square \)
For further discussion, we need the following conditions and lemma.

**Condition 3.5** Let $\sim$ denote a binary relation on $[0, d]$, where, for any $i, j \in [0, d]$, $i \sim j$ if and only if the following conditions hold:

(i) There are $m \in \mathbb{N}_0$ and sequence $i_0, j_0, \ell_0, i_1, j_1, \ell_1, \ldots, i_m, j_m, \ell_m$ of $[0, d]$ such that $i_0 = i$, $\ell_m = j$, $p \uparrow \prod_{a=0}^{m} p_{i_a,j_a}^{\ell_a}$, and $\ell_{b-1} = i_b$ for every $b \in [1, m]$;

(ii) There are $n \in \mathbb{N}_0$ and sequence $r_0, s_0, t_0, r_1, s_1, t_1, \ldots, r_n, s_n, t_n$ of $[0, d]$ such that $r_0 = j$, $t_n = i$, $p \uparrow \prod_{c=0}^{n} p_{r_c,s_c}^{t_c}$, and $t_{e-1} = r_e$ for every $e \in [1, n]$.  

**Example 3.6** Let us illustrate the definition of $\sim$ by examples. Assume that $p > 2$ and $S$ is the scheme of order 12, No. 21 in Hanaki (2020). Notice that $S = \{ R_0, R_1, R_2, R_3, R_4 \}$, where $3' = p_{4,4} = p_3^4 = 4$. So $3 \sim 4$ as the sequence $3, 4$ satisfies Condition 3.5 (i) and the sequence $4, 3$ satisfies Condition 3.5 (ii). Note that $3 \sim 3$ as the sequence $3, 4, 4, 3$ satisfies Condition 3.5 (i) and (ii).

**Lemma 3.7** The binary relation $\sim$ is an equivalence relation on $[0, d]$.

**Proof** Let $i, j, \ell \in [0, d]$. Notice that $0' = 0$ and $p_{i0}^j = 1$ by the definition of $p_{i0}^j$. Hence the sequence $i, 0, i$ satisfies Condition 3.5 (i) and (ii), which implies that $i \sim i$. Since $i$ is chosen from $[0, d]$ arbitrarily, $\sim$ is reflexive.

Assume that $i \sim j$. According to Condition 3.5 (i) and (ii), we have the following two facts:

(i) There are $m_1 \in \mathbb{N}_0$ and sequence $i_0, j_0, \ell_0, i_1, j_1, \ell_1, \ldots, i_{m_1}, j_{m_1}, \ell_{m_1}$ of $[0, d]$ such that $i_0 = i$, $\ell_{m_1} = j$, $p \uparrow \prod_{a=0}^{m_1} p_{i_a,j_a}^{\ell_a}$, and $\ell_{b-1} = i_b$ for every $b \in [1, m_1]$.

(ii) There are $n_1 \in \mathbb{N}_0$ and sequence $r_0, s_0, t_0, r_1, s_1, t_1, \ldots, r_{n_1}, s_{n_1}, t_{n_1}$ of $[0, d]$ such that $r_0 = j$, $t_{n_1} = i$, $p \uparrow \prod_{c=0}^{n_1} p_{r_c,s_c}^{t_c}$, and $t_{e-1} = r_e$ for every $e \in [1, n_1]$.

We thus have $j \sim i$ as Condition 3.5 (i) follows from (ii) and Condition 3.5 (ii) follows from (i). So $\sim$ is symmetric.

Assume further that $j \sim \ell$. According to Condition 3.5 (i) and (ii), we have the following two facts:

(iii) There are $m_2 \in \mathbb{N}_0$ and sequence $u_0, v_0, w_0, u_1, v_1, w_1, \ldots, u_{m_2}, v_{m_2}, w_{m_2}$ of $[0, d]$ such that $u_0 = j$, $w_{m_2} = \ell$, $p \uparrow \prod_{f=0}^{m_2} p_{u_f,v_f}^{w_f}$, and $w_{g-1} = u_g$ for every $g \in [1, m_2]$.

(iv) There are $n_2 \in \mathbb{N}_0$ and sequence $x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{n_2}, y_{n_2}, z_{n_2}$ of $[0, d]$ such that $x_0 = \ell$, $z_{n_2} = j$, $p \uparrow \prod_{h=0}^{n_2} p_{x_h,y_h}^{z_h}$, and $z_{k-1} = x_k$ for every $k \in [1, n_2]$.

Set $i_{m_1+f+1} = u_f, j_{m_1+f+1} = v_f$, and $\ell_{m_1+f+1} = w_f$ for every $f \in [0, m_2]$. We also put $x_{n_2+c+1} = r_c, y_{n_2+c+1} = s_c$, and $z_{n_2+c+1} = t_c$ for every $c \in [0, n_1]$. By (i) and (iii), observe that $i_0 = i$, $\ell_{m_1+m_2+1} = \ell$, $p \uparrow \prod_{r=0}^{m_1+m_2+1} p_{i_r,j_r}^{\ell_r}$, and $\ell_{s-1} = i_s$ for every $s \in [1, m_1+m_2+1]$. According to (ii) and (iv), observe that $x_0 = \ell$, $z_{n_1+n_2+1} = i$, $p \uparrow \prod_{a=0}^{n_1+n_2+1} p_{x_a,y_a}^{z_a}$, and $z_{v-1} = x_v$ for every $v \in [1, n_1+n_2+1]$. We thus have $i \sim \ell$ as the sequence $i_0, j_0, \ell_0, i_1, j_1, \ell_1, \ldots, i_{m_1+m_2+1}, j_{m_1+m_2+1}, \ell_{m_1+m_2+1}$ satisfies Condition 3.5 (i) and the sequence $x_0, y_0, z_0, x_1, y_1, z_1, \ldots, x_{n_1+n_2+1}, y_{n_1+n_2+1}, z_{n_1+n_2+1}$ satisfies Condition 3.5 (ii). Therefore $\sim$ is transitive. The desired lemma thus follows from the definition of an equivalence relation on a set.  

$\square$
For further discussion, we also need the following notation and lemma.

**Notation 3.8** Let \( n \in \mathbb{N}_0 \) and \( \sim_n \) denote a binary relation on \( S_n \), where, for any \( i, j \in S_n \), \( i \sim_n j \) if and only if \( i \sim j \). As \( S_n \subseteq [0, d] \), by Lemma 3.7, notice that \( \sim_n \) is an equivalence relation on \( S_n \). If \( S_n \neq \emptyset \), let \( Q_n \) be the quotient set of \( S_n \) with respect to \( \sim_n \). If \( S_n = \emptyset \), set \( Q_n = \emptyset \).

**Lemma 3.9** Assume that \( n \in \mathbb{N}_0 \), \( Q_n \neq \emptyset \), and \( C \in Q_n \).

(i) \( \emptyset \neq C \subseteq S_n \).

(ii) \( \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) is a nonzero \( T \)-submodule of \( W_n/W_{n+1} \).

(iii) \( \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) is an irreducible \( T \)-submodule of \( W_n/W_{n+1} \).

**Proof** As \( Q_n \neq \emptyset \), by the definition of \( Q_n \), \( Q_n \) is a partition of \( S_n \). (i) thus follows.

By (i) and Lemma 3.3 (ii), \( \langle 0 + W_{n+1} \rangle \neq \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \subseteq W_n/W_{n+1} \). Let \( a, b, c \in [0, d] \). We claim that \( E_a^* A_b E_c^* (E_i^* + W_{n+1}) \in \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) for every \( h \in C \). We suppose that \( E_a^* A_b E_c^* (E_i^* + W_{n+1}) \notin \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) and \( j \in C \). Therefore \( E_a^* A_b E_c^* (E_j^* + W_{n+1}) \neq 0 + W_{n+1} \). So \( E_a^* A_b E_c^* E_j^* 1 \notin W_{n+1} \), which implies that \( c = j \) and

\[
0 \neq p_{ jb} a E_a^* 1 = E_a^* A_b E_c^* E_j^* 1 \notin W_{n+1}
\]

by (3) and Lemma 2.2 (i). So we have \( p \upharpoonright p_{jb} a \). Notice that \( E_a^* 1 + W_{n+1} \in W_n/W_{n+1} \) as \( E_a^* A_b E_c^* (E_i^* + W_{n+1}) \in W_n/W_{n+1} \), (7) holds, and \( p \upharpoonright p_{jb} a \). Moreover, notice that \( E_a^* 1 \notin W_{n+1} \) as (7) holds. Therefore \( a \in S_n \) by Lemmas 3.3 (ii) and 3.1 (i). Since \( j \in C \), by (i), notice that \( j \in S_n \). As \( a, j \in S_n \), \( p \upharpoonright p_{ab} a \), and \( k_a p_{ab} a = k_j p_{ab} j \) by Lemma 2.1, notice that \( p \upharpoonright p_{ab} j \) by the definition of \( S_n \). So we have \( j \sim a \) and \( j \sim a \) since the sequence \( a, j \), \( a \) satisfies Condition 3.5 (i) and the sequence \( a, b, j \) satisfies Condition 3.5 (ii). As \( j \in C, j \sim a, a, \) and \( C \) is an equivalence class of \( S_n \) with respect to \( \sim_n \), notice that \( a \in C \). So \( E_a^* A_b E_c^* (E_j^* + W_{n+1}) \in \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) by (7). We have a contradiction as \( E_a^* A_b E_c^* (E_j^* + W_{n+1}) \notin \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \). The desired claim thus follows. As we have known that \( \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) is a nonzero \( \mathbb{F} \)-linear subspace of \( W_n/W_{n+1} \) and \( a, b, c \) are chosen from \([0, d]\) arbitrarily, (ii) is proved by combining Lemma 2.2 (iii), (6), and the proven claim.

For (iii), we first suppose that the \( T \)-module \( \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) in (ii) has a nonzero proper \( T \)-submodule \( U \). Pick \( \sum_{i \in C} c_i E_i^* 1 + W_{n+1} \in U \setminus \{0\} \), where \( c_i \in \mathbb{F} \) for every \( i \in C \). So there exists \( k \in C \) such that \( c_k \neq 0 \). Let \( \ell \in C \). Notice that \( \ell \sim k \) and \( \ell \sim k \) as \( C \) is an equivalence class of \( S_n \) with respect to \( \sim_n \). By Condition 3.5 (i), there are \( m \in \mathbb{N}_0 \) and sequence \( i_0, j_0, \ell_0, i_1, j_1, \ell_1, \ldots, i_m, j_m, \ell_m \) of \([0, d]\) such that \( i_0 = \ell, \ell_m = k, p \upharpoonright \prod_{e=0}^m p_{e \ell_e} e \), and \( \ell_{f-1} = i_f \) for every \( f \in [1, m] \). Let \( \gamma \)

\[
\prod_{e=0}^m p_{e \ell_e} e
\]

Since \( c_k \neq 0 \) and \( p \upharpoonright \gamma \), notice that \( c_k \gamma \neq 0 \). Since \( U \) is a \( T \)-submodule of the \( T \)-module \( \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) in (ii), by (6), (3), and Lemma 2.2 (i), notice that \( \prod_{e=0}^m (E_{i_e}^* A_{j_e} E_{\ell_e}) (\sum_{i \in C} c_i E_i^* 1 + W_{n+1}) = c_k \gamma \langle E_i^* 1 + W_{n+1} + U \rangle \). We thus have \( E_i^* 1 + W_{n+1} + U \) as \( c_k \gamma \neq 0 \). Since \( \ell \) is chosen from \( C \) arbitrarily, notice that \( \langle E_i^* 1 + W_{n+1} + U \rangle \subseteq U \subseteq \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \), which is impossible. So the \( T \)-module \( \langle \{ E_i^* + W_{n+1} : i \in C \} \rangle_F \) in (ii) has no nonzero proper \( T \)-submodule. (iii) thus follows.

\( \square \)
We now can introduce the following notation.

**Notation 3.10** Assume that $n \in \mathbb{N}_0$, $Q_n \neq \emptyset$, and $C \in Q_n$. Let $Irr_n(C)$ denote the irreducible $T$-submodule of $W_n/W_{n+1}$ in Lemma 3.9 (iii). By combining Lemmas 3.9 (i), (iii), and 3.3 (ii), notice that $Irr_n(C)$ has an $\mathbb{F}$-basis $\{E_i^*1 + W_{n+1} : i \in C\}$ of cardinality $|C|$. Write $B_n(C)$ for $\{E_i^*1 + W_{n+1} : i \in C\}$.

We need the following three lemmas to deduce the main result of this section.

**Lemma 3.11** Assume that $n_m \in \mathbb{N}_0$, $Q_{n_m} \neq \emptyset$, and $C_m \in Q_{n_m}$ for every $m \in \{1, 2\}$. Then $Irr_n(C_1) \cong Irr_n(C_2)$ as $T$-modules if and only if $n_1 = n_2$ and $C_1 = C_2$.

**Proof** If $n_1 = n_2$ and $C_1 = C_2$, $Irr_n(C_1) = Irr_n(C_2)$ by the definitions of $Irr_n(C_1)$ and $Irr_n(C_2)$. So $Irr_n(C_1) \cong Irr_n(C_2)$ as $T$-modules. Conversely, we assume that $Irr_n(C_1) \cong Irr_n(C_2)$ as $T$-modules. By the definitions of $Irr_n(C_1)$ and $Irr_n(C_2)$, $Irr_n(C_m)$ is an irreducible $T$-submodule of $W_{n_m}/W_{n_{m+1}}$ for every $m \in \{1, 2\}$.

If $n_1 < n_2$, by the Correspondence Theorem for Modules, there is a $T$-submodule $U$ of $W_{n_2}$ such that $W_{n_2+1} \cup U \subseteq W_{n_1+1}$ and $U/W_{n_2+1} = Irr_n(C_2)$ as $T$-modules. We thus have a contradiction by Lemma 3.4 (v). If $n_1 > n_2$, according to the Correspondence Theorem for Modules again, there exists a $T$-submodule $V$ of $W_{n_1}$ such that $W_{n_1+1} \subseteq V \subseteq W_{n_2+1}$ and $V/W_{n_1+1} = Irr_n(C_1) \cong Irr_n(C_2)$ as $T$-modules. We also have a contradiction by Lemma 3.4 (v). So $n_1 = n_2$.

Set $n = n_1 = n_2$. Let $\phi$ denote a $T$-isomorphism from $Irr_n(C_1)$ to $Irr_n(C_2)$. Pick $i \in C_1$. Since $B_n(C_1)$ is an $\mathbb{F}$-basis of $Irr_n(C_1)$, $0 + W_{n+1} \neq E_i^*1 + W_{n+1} \in B_n(C_1)$. As $\phi$ is injective, we thus have $0 + W_{n+1} \neq \phi(E_i^*1 + W_{n+1}) \in B_n(C_2)$. Moreover, as $B_n(C_2)$ is an $\mathbb{F}$-basis of $Irr_n(C_2)$, $\phi(E_i^*1 + W_{n+1})$ is an $\mathbb{F}$-linear combination of the elements of $B_n(C_2)$. Suppose that $C_1 \neq C_2$. Since $C_1$ and $C_2$ are distinct equivalence classes of $S_n$ with respect to $\sim_n$, notice that $C_1 \cap C_2 = \emptyset$ and $i \notin C_2$. By (3), we thus have $E_i^*(E_i^*1 + W_{n+1}) = 0 + W_{n+1}$ for every $E_i^*1 + W_{n+1} \in B_n(C_2)$. As (3) holds and $\phi$ is a $T$-isomorphism, we thus can deduce that

$$0 + W_{n+1} \neq \phi(E_i^*1 + W_{n+1}) = \phi(E_i^*(E_i^*1 + W_{n+1})) = E_i^*\phi(E_i^*1 + W_{n+1}) = 0 + W_{n+1},$$

which is a contradiction. We thus have $C_1 = C_2$. The proof is now complete. $\square$

**Lemma 3.12** Let $n \in \mathbb{N}_0$.

(i) $W_n/W_{n+1} = \{0 + W_{n+1}\}$ if and only if $Q_n = \emptyset$.

(ii) If $Q_n \neq \emptyset$, then $W_n/W_{n+1} = \bigoplus_{C \in Q_n} Irr_n(C)$.

(iii) If $Q_n$ contains precisely $m$ elements $C_1, C_2, \ldots, C_m$, then there is a $T$-submodule series $W_{n+1} = U_n \cup U_{n-1} \subseteq \cdots \subseteq U_1 \subseteq U_0 = W_n$ of $W_n$ such that $U_q - U_{q-1}/U_q \cong Irr_n(C_q)$ as $T$-modules for every $q \in \{1, m\}$.

**Proof** For (i), by the definition of $Q_n$, observe that $Q_n = \emptyset$ if and only if $S_n = \emptyset$. We also have $\dim_{\mathbb{F}} W_n/W_{n+1} = |S_n|$ by Lemma 3.3 (ii). So $W_n/W_{n+1} = \{0 + W_{n+1}\}$ if and only if $Q_n = \emptyset$. The proof of (i) is now complete.
For (ii), as \( Q_n \neq \emptyset \), the definition of \( Q_n \) tells us that \( Q_n \) is a partition of \( S_n \). By Lemmas 3.9 (i) and 3.3 (ii), we thus have \( \bigcup_{i \in Q_n} B_n(C) = \{ E_i^* 1 + W_{n+1} : i \in S_n \} \), where \( B_n(C^{(1)}) \cap B_n(C^{(2)}) = \emptyset \) if \( C^{(1)}, C^{(2)} \in Q_n \) and \( C^{(1)} \neq C^{(2)} \). By Lemma 3.3 (ii) again, we thus can deduce that \( W_n/W_{n+1} = \bigoplus_{C \in Q_n} (B_n(C)) \) is the proof of (ii) is now complete.

For (iii), we set \( V_q = \text{Irr}_n(C_q) \oplus \text{Irr}_n(C_{q+1}) \oplus \cdots \oplus \text{Irr}_n(C_m) \) for every \( q \in [1, m] \). By (ii), observe that \( W_n/W_{n+1} = V_1 \). By the Correspondence Theorem for Modules, there exists a \( T \)-submodule series \( W_{n+1} = U_m \subset U_{m-1} \subset \cdots \subset U_1 \subset U_0 = W_n \) of \( W_n \) such that \( U_{q-1}/U_m = V_q \) for every \( q \in [1, m] \). In particular, \( U_{m-1}/U_m = \text{Irr}_n(C_m) \) as \( T \)-modules. For every \( r \in [1, m-1] \), by the Third Isomorphism Theorem, note that \( U_{r-1}/U_r \cong (U_{r-1}/U_m)/(U_r/U_m) \) for every \( r \in [1, m-1] \), by the Third Isomorphism Theorem, note that \( U_{r-1}/U_r \cong (U_{r-1}/U_m)/(U_r/U_m) = V_r/V_{r+1} \cong \text{Irr}_n(C_r) \) as \( T \)-modules. For every \( r \in [1, m-1] \), by the Third Isomorphism Theorem, note that \( U_{r-1}/U_r \cong (U_{r-1}/U_m)/(U_r/U_m) = V_r/V_{r+1} \cong \text{Irr}_n(C_r) \) as \( T \)-modules. The proof of (iii) is now complete. \( \square \)

**Remark 3.13** We have \( |Q_0| = 1 \) and \( Q_0 = \{ S_0 \} \) by Lemmas 3.4 (iii) and 3.12 (ii).

**Lemma 3.14** Let \( \epsilon = \max \{ m \in \mathbb{N}_0 : \exists i \in [0, d], \ p^m | k_i \} \). Then there exists a \( T \)-submodule series \( \{ 0 \} = W_{\epsilon+1} \subset W_\epsilon \subset W_{\epsilon-1} \subset \cdots \subset W_1 \subset W_0 \) of \( W_0 \), where, for every \( n \in [0, \epsilon], \)

\[
W_n/W_{n+1} = \left\{ \begin{array}{ll}
\bigoplus_{C \in Q_n} \text{Irr}_n(C), & \text{if } Q_n \neq \emptyset, \\
\{ 0 \} + W_{n+1}], & \text{if } Q_n = \emptyset.
\end{array} \right.
\]

**Proof** By the definitions of \( \epsilon \) and \( S_i \), observe that \( S_\epsilon \neq \emptyset = S_q \) for every \( \epsilon < q \in \mathbb{N} \). So \( W_{\epsilon+1} = \{ 0 \} \subset W_\epsilon \) by Lemma 3.3 (i) and (ii). So the desired \( T \)-module series follows from Definition 3.12. The desired equality is from Lemma 3.12 (i) and (ii). \( \square \)

We are now ready to deduce the main result of this section.

**Theorem 3.15** Let \( \epsilon = \max \{ m \in \mathbb{N}_0 : \exists i \in [0, d], \ p^m | k_i \} \). Let \( Q \) denote the set \( \{ n \in [0, \epsilon] : Q_n \neq \emptyset \} \).

(i) \( \bigcup_{q \in Q} \bigcup_{C \in Q_q} \{ \text{Irr}_q(C) \} \) is the set of all composition factors of \( W_0 \) with respect to a composition series of \( W_0 \). Furthermore, \( \bigcup_{q \in Q} \bigcup_{C \in Q_q} \{ \text{Irr}_q(C) \} \) is a complete set of representatives of isomorphic classes of all composition factors of \( W_0 \).

(ii) \( \sum_{q \in Q} |Q_q| \) equals the composition length of \( W_0 \). Furthermore, \( \sum_{q \in Q} |Q_q| \) equals the number of isomorphic classes of all composition factors of \( W_0 \).

**Proof** For (i), by Lemmas 3.14 and 3.12 (iii), there exists a composition series of \( W_0 \) whose successive subquotients are precisely the elements of \( \bigcup_{q \in Q} \bigcup_{C \in Q_q} \{ \text{Irr}_q(C) \} \). So the first statement of (i) holds. By Lemma 3.11, any two distinct elements of \( \bigcup_{q \in Q} \bigcup_{C \in Q_q} \{ \text{Irr}_q(C) \} \) are not isomorphic to each other as \( T \)-modules. So the second statement of (i) is from the first one and the Jordan-Hölder Theorem. (i) is shown.

For (ii), by the first statement of (i) and the definition of the composition length of \( W_0 \), the composition length of \( W_0 \) is \( |\bigcup_{q \in Q} \bigcup_{C \in Q_q} \{ \text{Irr}_q(C) \}|. \) By Lemma 3.11,

\[
\bigcup_{q \in Q} \bigcup_{C \in Q_q} \{ \text{Irr}_q(C) \} = \sum_{q \in Q} \sum_{C \in Q_q} 1 = \sum_{q \in Q} |Q_q|.
\]

\( \square \) Springer
The first statement of (ii) thus follows. The second statement of (ii) comes from the second statement of (i) and (8). (ii) is proved. □

**Remark 3.16** Let \( A \) be a finite-dimensional associative unital \( \mathbb{F} \)-algebra. Call an \( A \)-module a multiplicity free \( A \)-module if its composition length equals the number of isomorphic classes of all its composition factors. By Lemma 3.4 (ii) and Theorem 3.15 (ii), notice that \( W_0 \) is an indecomposable multiplicity free \( T \)-module.

**Example 3.17** Let us illustrate Theorem 3.15 by an example. Assume that \( p = 2 \) and \( S \) is the scheme in Example 3.6. Notice that \( k_0 = k_1 = 1, k_2 = 2, k_3 = k_4 = 4, \) \( S_0 = \{0, 1\}, S_1 = \{2\}, S_2 = \{3, 4\}, \epsilon = 2, \) and \( Q = \{0, 1, 2\} \). By Remark 3.13, note that \( Q_0 = \{(0, 1)\} \) and \( Q_1 = \{\{2\}\} \). Let \( i, j \in \{0, 4\} \). For every \( \ell \in \{2, 4\} \), \( 2 \nmid p^i_j \) if and only if \( i = \ell \) and \( j \in \{0, 1\} \), which implies that \( Q_2 = \{(3), \{4\}\} \) and \( \langle (E_1^* 1) \rangle \) is a \( 1 \)-submodule of \( W_0 \) for every \( \ell \in \{2, 4\} \). According to Theorem 3.15 (i), notice that \( \{\text{Irr}_0((0, 1)), \text{Irr}_1(\{2\}), \text{Irr}_2(\{3\}), \text{Irr}_2(\{4\})\} \) is a complete set of representatives of isomorphic classes of all composition factors of \( W_0 \). By Theorem 3.15 (ii), note that the composition length of \( W_0 \) is four.

The following corollary is an application of Lemmas 3.12 and 3.14.

**Corollary 3.18** The following statements are equivalent:

(i) For any \( T \)-submodules \( U \) and \( V \) of \( W_0 \), we have either \( U \subseteq V \) or \( V \subseteq U \);
(ii) For every \( n \in \mathbb{N}_0 \), we have either \( W_{n+1} = W_n \) or \( W_{n+1} \) is the unique maximal \( T \)-submodule of \( W_n \).

**Proof** We prove (ii) by (i). Let \( q \in \mathbb{N}_0 \). Assume further that \( W_q \neq W_{q+1} \). Therefore \( W_{q+1} \subset W_q \) by Definition 3.2. Suppose that \( W_{q+1} \) is not a maximal \( T \)-submodule of \( W_q \). The nonzero \( T \)-module \( W_q/W_{q+1} \) is not an irreducible \( T \)-module. By combining Lemma 3.12 (i), (ii), and the Correspondence Theorem for Modules, notice that \( W_q \) has at least two distinct maximal \( T \)-submodules. Let \( M \) and \( N \) be distinct maximal \( T \)-submodules of \( W_q \). By (i), notice that either \( M \subseteq N \subseteq W_0 \) or \( N \subseteq M \subseteq W_0 \). We thus have a contradiction since \( M \) and \( N \) are distinct maximal \( T \)-submodules of \( W_q \). So \( W_{q+1} \) is a maximal \( T \)-submodule of \( W_q \). By (i), we can also get a similar contradiction if we suppose that the maximal \( T \)-submodule \( W_{q+1} \) is not the unique maximal \( T \)-submodule of \( W_q \). As \( q \) is chosen from \( \mathbb{N}_0 \) arbitrarily, (ii) thus follows.

We prove (i) by (ii). Let \( \epsilon = \max\{m \in \mathbb{N}_0 : \exists i \in \{0, d\}, p^m \nmid k_i\} \). Let \( W \) denote a nonzero \( T \)-submodule of \( W_0 \). We have \( \{0\} = W_{\epsilon+1} \subset W_\epsilon \subset \cdots \subset W_1 \subset W_0 \) by Lemma 3.14. So there exists \( r \in \{0, \epsilon\} \) such that \( W \subseteq W_r \) and \( W \nsubseteq W_{r+1} \). Hence \( W_r \neq W_{r+1} \), which implies that \( W_{r+1} \) is the unique maximal \( T \)-submodule of \( W_r \) by (ii). Suppose that \( W \neq W_r \). Then \( W \subseteq W_{r+1} \) since \( W \subset W_r \) and \( W_{r+1} \) is the unique maximal \( T \)-submodule of \( W_r \). We thus have a contradiction since \( W \nsubseteq W_{r+1} \).

Therefore \( W = W_r \). As \( W \) is an arbitrarily chosen nonzero \( T \)-submodule of \( W_0 \), every \( T \)-submodule of \( W_0 \) is one of the \( T \)-modules \( W_0, W_1, \ldots, W_{\epsilon+1}. \) (i) thus follows from Lemma 3.14. □

We end this section with the following remarks.
Remark 3.19 Let $A$ be a finite-dimensional associative unital $\mathbb{F}$-algebra. Call an $A$-module a uniserial $A$-module if, for any $A$-submodules $U$ and $V$ of this $A$-module, we have either $U \subseteq V$ or $V \subseteq U$. Corollary 3.18 describes a criterion to determine whether $W_0$ is a uniserial $T$-module.

Remark 3.20 Observe that $W_0$ is a uniserial $T$-module if $d = 1$. In general, $W_0$ may not be a uniserial $T$-module. In Example 3.17, observe that $W_1$ is a direct sum of its $T$-submodules $\langle\{E^*_2 1\}\rangle_{\mathbb{F}}, \langle\{E^*_3 1\}\rangle_{\mathbb{F}}, \langle\{E^*_4 1\}\rangle_{\mathbb{F}}$. Hence $W_0$ of this case is not a uniserial $T$-module by Lemma 3.1 (i) and the definition of a uniserial $T$-module.

4 Some characterizations of $p'$-valenced schemes

In this section, we characterize the $p'$-valenced schemes in terms of the properties of their modular Terwilliger algebras. We recall the notations in Definition 3.2 and Notation 3.10. By (2), $E^*_i J E^*_j \in T$ for any $i, j \in [0, d]$. We first present a lemma.

Lemma 4.1 The following statements hold:

(i) $\{E^*_i J E^*_j : i, j \in [0, d]\}$ is an $\mathbb{F}$-linearly independent set of cardinality $(d + 1)^2$.

(ii) $\langle\{E^*_i J E^*_j : i, j \in [0, d]\}\rangle_{\mathbb{F}}$ is a two-sided ideal of $T$.

(iii) $\langle\{E^*_i J E^*_j : i, j \in [0, d], p \mid k_i k_j\}\rangle_{\mathbb{F}}$ is a two-sided ideal of $T$.

(iv) Assume that $\ell \in [0, d]$. Then $\langle\{E^*_i J E^*_\ell : i \in [0, d]\}\rangle_{\mathbb{F}}$ is a $T$-module under the left multiplication action of $T$.

Proof For (i), suppose that $\sum_{i=0}^{d} \sum_{j=0}^{d} c_{ij} E^*_i J E^*_j = 0$, where $\sum_{i=0}^{d} \sum_{j=0}^{d} (c_{ij}) \subseteq \mathbb{F}$ and $(\bigcup_{i=0}^{d} \bigcup_{j=0}^{d} (c_{ij})) \cap (\mathbb{F} \setminus \{0\}) \neq \emptyset$. So there exist $g, h \in [0, d]$ such that $c_{gh} \neq 0$. Hence $c_{gh} E^*_g J E^*_h = E^*_s (\sum_{i=0}^{d} \sum_{j=0}^{d} c_{ij} E^*_i J E^*_j) E^*_h = E^*_s O E^*_h = O$ by (3). By (4), we thus deduce that $c_{gh} = 0$, which contradicts the inequality $c_{gh} \neq 0$. Therefore $\{E^*_i J E^*_j : i, j \in [0, d]\}$ is an $\mathbb{F}$-linearly independent set. According to (3) and (4) again, we also note that $\langle\{E^*_i J E^*_j : i, j \in [0, d]\}\rangle = (d + 1)^2$.

Let $a, b, c \in [0, d]$. For (ii), according to (1), (3), and Lemma 2.2 (i), observe that $E^*_a J E^*_v E^*_a E^*_b E^*_c E^*_v = (E^*_a A_{vb} E^*_a E^*_b E^*_c E^*_v)' = (\delta_{av} p_{vb} E^*_a J E^*_v)' = \delta_{av} p_{vb} E^*_a J E^*_v$ and $E^*_a E^*_b E^*_c E^*_a J E^*_v = \delta_{cua} p_{ab} E^*_a J E^*_v$ for any $u, v \in [0, d]$. In particular, $E^*_a A_{vb} E^*_a J E^*_v$ and $E^*_a J E^*_v E^*_a A_{vb} E^*_c$ are contained in $\langle\{E^*_i J E^*_j : i, j \in [0, d]\}\rangle_{\mathbb{F}}$ for any $u, v \in [0, d]$. Since $\langle\{E^*_i J E^*_j : i, j \in [0, d]\}\rangle_{\mathbb{F}}$ is an $\mathbb{F}$-linear space and $a, b, c$ are chosen from $[0, d]$ arbitrarily, (ii) thus follows from Lemma 2.2 (iii) and (6).

For (iii), notice that (iii) is trivial if $\langle\{E^*_i J E^*_j : i, j \in [0, d], p \mid k_i k_j\}\rangle_{\mathbb{F}} = \{O\}$. We thus assume further that $\langle\{E^*_i J E^*_j : i, j \in [0, d], p \mid k_i k_j\}\rangle_{\mathbb{F}} \neq \{O\}$. Let $r, s \in [0, d]$ and $p \mid k_r k_s$. Observe that $p \mid k_r k_s$ if $p \mid p_{rs}^c$. Otherwise, suppose that $p \nmid p_{rs}^c$ and $p \nmid k_r k_s$. As $p \mid k_r k_s$ and $p \mid k_r k_s$, observe that $p \mid k_s k_r$. As $p \mid k_s k_r$ and $p \mid k_r k_s$, and $p \mid k_s k_r$ by Lemma 2.1, we thus have $p \mid p_{rs}^c$, which contradicts the assumption $p \mid p_{rs}^c$. So we have $E^*_c A_{vb} E^*_a E^*_c E^*_v E^*_s = \delta_{as} p_{sb} E^*_a J E^*_v \in \langle\{E^*_i J E^*_j : i, j \in [0, d], p \mid k_i k_j\}\rangle_{\mathbb{F}}$ by (3) and Lemma 2.2 (i). So $E^*_r J E^*_s E^*_a E^*_b E^*_c E^*_v = \langle\{E^*_i J E^*_j : i, j \in [0, d], p \mid k_i k_j\}\rangle_{\mathbb{F}}$ as $E^*_r J E^*_s E^*_a E^*_b E^*_c = (E^*_a A_{vb} E^*_a E^*_c E^*_v)' = \delta_{as} p_{sb} E^*_a J E^*_v$ by (1). Notice that $p \mid k_3 k_5$ if $p \nmid p_{r3}^b$. Otherwise, suppose that $p \mid p_{r3}^b$, and $p \mid k_a k_s$. Since $p \mid k_r k_s$ and $p \mid k_a k_s$,
note that $p \nmid k_a$ and $p \mid k_r$. Since $k_ap_{rb}^a = k_rp_{ab}^r$, by Lemma 2.1, we thus have $p \mid p_{rb}^a$, which contradicts the assumption $p \nmid p_{rb}^a$. By (3) and Lemma 2.2 (i), we thus have $E_a^*A_bE_c^*E_i^*JE_j^* = \delta_{cr}p_{rb}^aE_c^*E_i^*JE_j^* \in \langle \{E_i^*JE_j^*: i, j \in [0, d], p \nmid k_ik_j\}\rangle$. We thus deduce that $E_i^*JE_j^* = \sum_{i,j}E_c^*E_i^*E_j^* \in \langle \{E_i^*JE_j^*: i, j \in [0, d], p \nmid k_ik_j\}\rangle$ for any $u, v \in [0, d]$ and $p \nmid k_a, k_b$. Since $\langle \{E_i^*JE_j^*: i, j \in [0, d], p \nmid k_ik_j\}\rangle$ is an $F$-linear space and $a, b, c$ are chosen from $[0, d]$ arbitrarily, (iii) thus follows from Lemma 2.2 (ii) and (6).

For (iv), by (3) and Lemma 2.2 (i), $E_a^*A_bE_c^*E_i^*JE_j^* = \delta_{cr}p_{rb}^aE_c^*E_i^*JE_j^*$ for every $u \in [0, d]$. Since $\langle \{E_i^*JE_j^*: i \in [0, d]\}\rangle$ is an $F$-linear space and $a, b, c$ are chosen from $[0, d]$ arbitrarily, (iv) thus follows from Lemma 2.2 (iii) and (6).  

The following notation will be heavily used in the following discussion.

**Notation 4.2** Denote the two-sided ideals of $T$ in Lemma 4.1 (ii) and (iii) by $B_0$ and $B_1$, respectively. So $B_0$ is a $T$-module under the left multiplication action of $T$. Let $\tau B_0$ denote this $T$-module. Notice that $T$ itself is also a $T$-module under the left multiplication action of $T$. Let $\tau T$ denote this $T$-module. Let $\ell \in [0, d]$. Denote the $T$-module in Lemma 4.1 (iv) by $M_\ell$. Observe that $B_1 \subseteq B_0 \subseteq T$, $M_\ell \subseteq \tau B_0 \subseteq \tau T$, $\dim_F B_0 = \dim_F \tau B_0 = (d + 1)^2$, $\dim_F M_\ell = d + 1$, and $B_0$ is an $F$-subalgebra of $T$.

The following lemma summarizes some properties of the objects in Notation 4.2.

**Lemma 4.3** The following statements hold:

(i) $B_1$ is the unique maximal two-sided ideal of the $F$-subalgebra $B_0$ of $T$.
(ii) The matrix product of any three elements of $B_1$ is $O$.
(iii) $B_1$ is a nilpotent two-sided ideal of $T$. In particular, $B_1 \subseteq \text{Rad}(T)$.
(iv) Assume that $\ell \in [0, d]$. Then $\tau M_\ell \cong W_0$ as $T$-modules.
(v) $\tau B_0$ is isomorphic to a direct sum of $d + 1$ copies of $W_0$ as $T$-modules.

**Proof** For (i), by the definitions of $B_0$ and $B_1$, $B_1$ is a proper two-sided ideal of the $F$-subalgebra $B_0$ of $T$. Let $M$ be a maximal two-sided ideal of the $F$-subalgebra $B_0$ of $T$. According to the definition of $B_0$, pick $\sum_{i=0}^d \sum_{j=0}^d c_{ij}E_i^*JE_j^* \in M$, where $c_{ij} \in F$ for any $i, j \in [0, d]$. For any $i, j \in [0, d]$, we claim that $c_{ij} = 0$ if $p \nmid k_ik_j$. Suppose that there exist $a, b \in [0, d]$ such that $p \nmid k_ik_j$. Let $a, c \in [0, d]$. Observe that $E_c^*JE_a^*, E_b^*JE_c^* \in B_0$ by the definition of $B_0$. Since $M$ is a two-sided ideal of the $F$-subalgebra $B_0$ of $T$, $c_abk_bk_aE_c^*JE_e^* = E_c^*JE_a^*(\sum_{i=0}^d \sum_{j=0}^d c_{ij}E_i^*JE_j^*)E_b^*JE_e^* \in M$ by (3) and (5). Hence $E_c^*JE_e^* \in M$ as $c_ab \neq 0$ and $p \nmid k_ik_j$, $E_b^*JE_c^*$ are chosen from $[0, d]$ arbitrarily and $M$ is a maximal two-sided ideal of the $F$-subalgebra $B_0$ of $T$, we thus have $B_0 \subseteq M \subseteq B_0$, which is impossible. Hence the desired claim follows. Since $\sum_{i=0}^d \sum_{j=0}^d c_{ij}E_i^*JE_j^*$ is chosen from $M$ arbitrarily, by the proven claim and the definition of $B_1$, notice that $M \subseteq B_1 \subseteq B_0$, which implies that $M = B_1$ as $M$ is a maximal two-sided ideal of the $F$-subalgebra $B_0$ of $T$. As $M$ is an arbitrarily chosen maximal two-sided ideal of the $F$-subalgebra $B_0$ of $T$, $B_1$ is the unique maximal two-sided ideal of the $F$-subalgebra $B_0$ of $T$. (i) is proved.  

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For (ii), notice that (ii) is trivial if $B_1 = \{O\}$. We assume further that $B_1 \neq \{O\}$. Let $g, h, r, s, u, v \in [0, d]$, where $p \mid k_gk_h$, $p \mid k_rk_s$, and $p \mid k uk_v$. We thus deduce that

$$E^*_gJE^*_rJ^*E^*_uJE^*_v = \delta_{hr}\delta_{su}E^*_gJE^*_rJ^*E^*_uJE^*_v = \delta_{hr}\delta_{su}k_rk_sE^*_gJE^*_v = O \quad (9)$$

by (3) and (5). As $g, h, r, s, u, v$ are chosen from $[0, d]$ arbitrarily, (ii) thus follows from (9) and the definition of $B_1$.

For (iii), as $B_1$ is a two-sided ideal of $T$, (iii) is shown by (ii).

For (iv), by the definitions of $W_0$ and $M_\ell$, let $\phi$ be the $F$-linear homomorphism from $W_0$ to $M_\ell$ that sends every $E^*_i 1$ to $E^*_iJE^*_*$. By Lemmas 3.1 (i) and 4.1 (i), $\phi$ is an $F$-linear isomorphism. Observe that $\phi$ is also a $T$-isomorphism by combining the definition of $W_0$, the definition of $M_\ell$, Lemma 2.2 (i), (iii), and (6). We are done.

For (v), by the definition of $T B_0$ and Lemma 4.1 (i), notice that $T B_0 = \bigoplus_{i=0}^d M_i$. (v) thus follows from (iv).

**Remark 4.4** In general, the matrix product of any two elements of $B_1$ may not be $O$. Assume that $S$ is not a $p'$-valenced scheme. Then there exists $R_i \in S$ such that $p \mid k_i$. Notice that $O \neq E^*_iJE^*_* = \left((E^*_iJ E^*_{O_0} + E^*_0JE^*_*)^2 \right) \in B_1$ by (3), (4), (5).

**Remark 4.5** The containment in Lemma 4.3 (iii) may be strict (see (Hanaki 2021, 5.1)). The containment in Lemma 4.3 (iii) may become equality (see (Jiang 2020, Theorems B and C)).

The following lemma contains some characterizations of the $p'$-valenced schemes.

**Lemma 4.6** The following statements are equivalent:

(i) $S$ is a $p'$-valenced scheme;

(ii) The $F$-subalgebra $B_0$ of $T$ is unital. Its identity element is central in $T$;

(iii) There exists a two-sided ideal $D$ of $T$ such that $T$ is a direct sum of $B_0$ and $D$;

(iv) The $F$-subalgebra $B_0$ of $T$ is isomorphic to a full matrix algebra over a division $F$-algebra as $F$-algebras.

**Proof** We prove (ii) by (i). Set $e_{B_0} = \sum_{i=0}^d k_i^{-1} E^*_iJE^*_i \in B_0$ by (i) and the definition of $B_0$. By (3) and (5), $E^*_aJE^*_bE_{B_0} = E^*_bE^*_aJE^*_b = E^*_aJE^*_b$ for any $a, b \in [0, d]$. So $e_{B_0}$ is the identity element of the $F$-subalgebra $B_0$ of $T$ by the definition of $B_0$. By combining (3), (1), Lemmas 2.2 (i), and 2.1, notice that

$$E^*_a A_b E^*_c e_{B_0} = k_c^{-1} p^{-1}_{cb} E^*_a JE^*_c = k_a^{-1} p^{-1}_{ab} E^*_a JE^*_c = (E^*_c A_b E^*_a e_{B_0})' = e_{B_0} E^*_a A_b E^*_c$$

for any $a, b, c \in [0, d]$. So $e_{B_0}$ is a central element of $T$ by Lemma 2.2 (iii) and (6). The proof of (ii) is now complete.

We prove (iii) by (ii). By (ii), there exists $f_{B_0} \in B_0$ such that $f_{B_0}$ is the identity element of the $F$-subalgebra $B_0$ of $T$ and $f_{B_0}$ is a central element of $T$. Let $D$ denote $\{(I - f_{B_0})Z : Z \in T\}$. As $f_{B_0}$ is a central element of $T$, notice that $D$ is a two-sided ideal of $T$ by the definition of $D$. As $f_{B_0}$ is the identity element of the $F$-subalgebra
$B_0$ of $T$, notice that $f_{B_0}^2 = f_{B_0}, (I - f_{B_0})^2 = I - f_{B_0}$, and $f_{B_0}(I - f_{B_0}) = O$, which implies that $B_0 \cap D = \{O\}$ by the definition of $D$. As $B_0$ is a two-sided ideal of $T$ and $Z = f_{B_0}Z + (I - f_{B_0})Z$ for every $Z \in T$, we thus get that $T$ is a direct sum of $B_0$ and $D$. The proof of (iii) is now complete.

We prove (i) by (iii). Since $I \in T$ and $B_0$ is a two-sided ideal of $T$, by (iii), notice that the $F$-subalgebra $B_0$ of $T$ is a unital $F$-algebra. By the definition of $B_0$, assume that $\sum_{i=0}^d \sum_{j=0}^d c_{ij} E_i^* J E_j^*$ is the identity element of the $F$-subalgebra $B_0$ of $T$, where $c_{ij} \in F$ for any $i, j \in [0, d]$. Suppose that $S$ is not a $p'$-valenced scheme. Then there exists $\ell \in [0, d]$ such that $\ell \mid k_{\ell}$. Moreover, observe that $O \neq E_{\ell}^* J E_{\ell}^* \in B_0$ by (4).

So (iv) follows from the Artin-Wedderburn Theorem.

We prove (iv) by (i). Notice that $B_1 = \{O\}$ by (i) and the definition of $B_1$. As (i) implies (ii), by Lemma 4.3 (i), the $F$-subalgebra $B_0$ of $T$ is a simple unital $F$-algebra.

We prove (i) by (iv). By (iv), observe that the $F$-subalgebra $B_0$ of $T$ is a simple unital $F$-algebra. So it has no nonzero proper two-sided ideal. Hence $B_1 = \{O\}$ by Lemma 4.3 (i). (i) thus follows from the definition of $B_1$.}

For further discussion, we need the following two lemmas.

**Lemma 4.7** Let $n \in \mathbb{N}_0$ and $i_m, j_m, \ell_m \in [0, d]$ for every $m \in [0, n]$.

(i) If $\min \bigcup_{m=0}^n \{k_{i_m}, k_{\ell_m}\} = 1$, then $\prod_{m=0}^n (E_{i_m}^* A_{j_m} E_{\ell_m}^*) \in \langle \{E_{i_0}^* J E_{\ell_0}^*\} \rangle_F$.

(ii) If $k_{i_0} = 1$ or $k_{\ell_0} = 1$, then $\prod_{m=0}^n (E_{i_m}^* A_{j_m} E_{\ell_m}^*) \in \langle \{E_{i_0}^* J E_{\ell_0}^*\} \rangle_F$.

**Proof** We may assume further that $\prod_{m=0}^n (E_{i_m}^* A_{j_m} E_{\ell_m}^*) \neq O$. For (i), note that there is $q \in [0, n]$ such that $\min \{k_{i_q}, k_{\ell_q}\} = 1$. Moreover, $E_{i_q}^* A_{j_q} E_{\ell_q}^* \neq O$ by (6) and the fact $\prod_{m=0}^n (E_{i_m}^* A_{j_m} E_{\ell_m}^*) \neq O$. We thus have $E_{i_q}^* A_{j_q} E_{\ell_q}^* = E_{i_q}^* J E_{\ell_q}^*$ by Lemma 2.2 (ii). (i) is shown by combining the equality $E_{i_q}^* A_{j_q} E_{\ell_q}^* = E_{i_q}^* J E_{\ell_q}^*$, (6), (3), (1), and Lemma 2.2 (i). As (ii) is a special case of (i), (ii) is proved by (i).\[\square\]

**Lemma 4.8** If $Z \in \text{Ann}_T(W_0)$, then $ZE_i^* = O$ for every $R_i \in O_{\theta}(S)$.

**Proof** Suppose that there exists $R_j \in O_{\theta}(S)$ such that $ZE_j^* \neq O$. We thus deduce that $O \neq ZE_j^* = IZE_j^* = \sum_{i=0}^d E_i^* Z E_j^*$ by (2). So there exists $\ell \in [0, d]$ such that $E_{\ell}^* Z E_j^* \neq O$. Since $k_j = 1, Z \in \text{Ann}_T(W_0)$, and $\text{Ann}_T(W_0)$ is a two-sided ideal of $T$, by (3), Lemmas 2.2 (iii), 4.7 (ii), we thus have $E_{\ell}^* Z E_j^* = c_{\ell j} Z E_{\ell}^* J E_j^* \in \text{Ann}_T(W_0)$, where $0 \neq c_{\ell j} \in F$. Since $k_j = 1$ and $E_{\ell}^* 1 \in W_0$ by the definition of $W_0$, by (3) and (5), we thus have $E_{\ell}^* Z E_j^* E_{\ell}^* 1 = c_{\ell j} Z E_{\ell}^* J E_j^* 1 = c_{\ell j} Z E_{\ell}^* 1 \neq 0$, which contradicts the definition of $\text{Ann}_T(W_0)$. The desired lemma thus follows. \[\square\]

The following lemma includes some characterizations of the $p'$-valenced schemes.

**Lemma 4.9** The following statements are equivalent:

(i) $S$ is a $p'$-valenced scheme;
(ii) If \( Z \in \text{Ann}_T(W_0) \), then \( E_i^*Z = ZE_i^* = 0 \) for every \( R_i \in O_\theta(S) \);

(iii) If \( \tilde{Z} \in \text{Rad}(T) \), then \( E_i^*\tilde{Z} = \tilde{Z}E_i^* = 0 \) for every \( R_i \in O_\theta(S) \).

**Proof** We prove (ii) by (i). By Lemma 4.8, it is enough to check that \( E_i^*Z = O \) for every \( R_i \in O_\theta(S) \). Suppose that there is \( R_j \in O_\theta(S) \) such that \( E_j^*Z \neq O \). We thus have \( O \neq E_j^*Z = E_j^*Z1 = \sum_{i=0}^d E_j^*Z^i E_i^* \) by (2). So there exists \( \ell \in [0, d] \) such that \( E_j^*ZE_\ell^* \neq O \). Since \( k_j = 1, Z \in \text{Ann}_T(W_0) \), and \( \text{Ann}_T(W_0) \) is a two-sided ideal of \( T \), by (3), Lemmas 2.2 (iii), 4.7 (ii), we thus have \( E_j^*ZE_\ell^* = c_{j \ell}E_j^*J E_\ell^* \in \text{Ann}_T(W_0) \)

where \( 0 \neq c_{j \ell}E_\ell^* \in \mathbb{F} \). Since \( E_\ell^*1 \in W_0 \) by the definition of \( W_0 \), according to (3) and (5), we thus have \( E_j^*ZE_\ell^* E_\ell^*1 = c_{j \ell}E_j^*J E_\ell^*1 = c_{j \ell}k_\ell E_j^*1 \).

By (i), notice that \( p \nmid k_\ell \) and \( E_j^*ZE_\ell^* E_\ell^*1 = c_{j \ell}k_\ell E_j^*1 \neq 0 \). The inequality \( E_j^*ZE_\ell^* E_\ell^*1 \neq 0 \) contradicts the definition of \( \text{Ann}_T(W_0) \). (ii) thus follows.

We prove (iii) by (i). Observe that \( \text{Rad}(T) \subseteq \text{Ann}_T(W_0) \) by (i) and Lemma 3.4 (iv). Observe that (ii) holds since (i) holds. (iii) thus follows from the containment \( \text{Rad}(T) \subseteq \text{Ann}_T(W_0) \) and (ii).

We prove (i) by (ii) or (iii). Suppose that (i) does not hold. There exists \( a \in [0, d] \) such that \( p \nmid k_a \). Notice that \( O \neq E_0^*E_0^*J E_a^* = E_0^*J E_a^* \in \text{Rad}(T) \) by combining (3), (4), the definition of \( B_1 \), and Lemma 4.3 (iii). By (3) and (5), \( E_0^*J E_a^* E_0^*1 = 0 \) for every \( b \in [0, d] \). So \( E_0^*J E_a^* \in \text{Ann}_T(W_0) \) by the definitions of \( W_0 \) and \( \text{Ann}_T(W_0) \).

Hence we have \( O \neq E_0^*E_0^*J E_a^* = E_0^*J E_a^* \in \text{Rad}(T) \cap \text{Ann}_T(W_0) \), which contradicts (ii) and (iii) as \( R_0 \in O_\theta(S) \). Therefore (i) follows if (ii) or (iii) holds. \( \square \)

**Remark 4.10** By (Hanaki 2021, Theorem 3.4), the equality \( \text{Rad}(T) = \{ O \} \) holds only if \( S \) is a \( p' \)-valenced scheme. Since \( \text{Rad}(T) = \{ O \} \) implies that Lemma 4.9 (iii) holds, this result can also be verified by Lemma 4.9. In general, \( \text{Rad}(T) \) may not be \( \{ O \} \) even if \( S \) is a \( p' \)-valenced scheme (see (Hanaki 2021, 5.1)).

**Example 4.11** In general, notice that \( \text{Rad}(T) \) may not equal \( \text{Ann}_T(W_0) \) even if \( S \) is a \( p' \)-valenced scheme. Let us illustrate this fact by a counterexample. Assume that \( p > 2 \) and \( S \) is the scheme of order 5, No. 2 in Hanaki (2020). Observe that \( S = \{ R_0, R_1, R_2 \}, \) where \( k_0 = 1 \) and \( k_1 = k_2 = 2 \). Hence \( S \) is a \( p' \)-valenced scheme. By computation, \( O \neq E_1^*A_1 E_2^* - E_1^*A_2 E_2^* \in \text{Ann}_T(W_0) \). However, \( \text{Rad}(T) = \{ O \} \) by (Jiang 2020, Theorem B).

The following lemma describes a characterization of the \( p' \)-valenced schemes.

**Lemma 4.12** The following statements are equivalent:

(i) \( S \) is a \( p' \)-valenced scheme;

(ii) For every decomposition of \( TT \) into a direct sum of indecomposable \( T \)-modules,

there exist exactly \( d + 1 \) indecomposable direct summands isomorphic to \( W_0 \) as \( T \)-modules.

**Proof** We prove (ii) by (i). By (i) and Lemma 4.6 (iii), there is a two-sided ideal \( D \) of \( T \) such that \( T \) is a direct sum of \( B_0 \) and \( D \). Hence \( D \) is a \( T \)-module under the left multiplication action of \( T \). Use \( TD \) to denote this \( T \)-module. Hence \( T = TB_0 \oplus TD \).

Notice that \( W_0 \) is not isomorphic to a direct summand of \( TD \) for every decomposition
of $TD$ into a direct sum of $T$-modules. Otherwise, we suppose that there indeed exist $T$-submodules $M$ and $N$ of $TD$ such that $TD = M \oplus N$ and $M \cong W_0$ as $T$-modules.

By (i) and Lemma 4.6 (ii), the $F$-subalgebra $B_0$ of $T$ is unital. Moreover, its identity element $f_{B_0}$ is a central element of $T$. Since $T$ is a direct sum of the two-sided ideals $B_0$ and $D$, notice that $I - f_{B_0}$ is also the identity element of the $F$-subalgebra $D$ of $T$.

By Lemma 4.3 (v) and the definition of $W_0$, we thus deduce that $f_{B_0}E_i^*1 = E_i^*1$ for every $i \in [0, d]$. As $TD = M \oplus N$ and $M \cong W_0$ as $T$-modules, by the definition of $W_0$ again, we can also deduce that $(I - f_{B_0})E_i^*1 = E_i^*1$ for every $i \in [0, d]$. Since $f_{B_0}(I - f_{B_0}) = O, 0 = f_{B_0}(I - f_{B_0})E_i^*1 = E_i^*1 \neq 0$ for every $i \in [0, d]$, which is a contradiction. Since $TD = B_0 \oplus TD$ and Lemmas 4.3 (v), 3.4 (ii) hold, we thus observe that there exists a decomposition of $TD$ into a direct sum of indecomposable $T$-modules such that exactly $d + 1$ indecomposable direct summands are isomorphic to $W_0$ as $T$-modules. (ii) thus follows from the Krull-Schmidt Theorem.

For any given $T$-submodule $U$ of $TD$, claim that $U \subseteq B_0$ if $U \cong W_0$ as $T$-modules. Assume that $U \cong W_0$ as $T$-modules. Let $\phi$ denote a $T$-isomorphism from $W_0$ to $U$. Notice that $W_0$ has an $F$-basis $\{E_i^*1 : i \in [0, d]\}$ by the definition of $W_0$ and Lemma 3.1 (i). Since $\phi$ is a $T$-isomorphism and $1 = \sum_{i=0}^d E_i^*1 \in W_0$ by (2), we thus get that $\{E_i^*\phi(1) : i \in [0, d]\}$ is an $F$-basis of $U$. Since $k_0 = 1$, $\phi(1) \in U$, and $U \subseteq TD$, $E_0^*\phi(1) \in \langle \{E_i^*JE_i^* : i \in [0, d]\}\rangle_F$ by combining (3), Lemmas 2.2 (iii), and 4.7 (ii). Hence $E_0^*\phi(1) \in B_0$ by the definition of $B_0$. As $\phi$ is a $T$-isomorphism and $k_0 = 1$, by (5), notice that $E_0^*\phi(1) = \phi(E_i^*1) = \phi(E_i^*JE_i^*1) = E_i^*JE_i^*\phi(1)$ for every $i \in [0, d]$. As $E_i^*\phi(1) \in B_0$ and $B_0$ is a two-sided ideal of $T$, we thus deduce that $E_i^*\phi(1) \in B_0$ for every $i \in [0, d]$. The desired claim thus follows as $B_0$ contains an $F$-basis of $U$.

We prove (i) by (ii). By (ii), there exist $T$-submodules $V$ and $W$ of $TD$ such that $TD = V \oplus W$ and $V$ is isomorphic to a direct sum of $d + 1$ copies of $W_0$ as $T$-modules. Therefore $V \subseteq B_0$ by the Krull-Schmidt Theorem and the proven claim. Moreover, as $\dim F W_0 = d + 1$ and $\dim F B_0 = (d + 1)^2$, we also observe that $\dim F V = \dim F B_0$.

We have $V = B_0$ and $TD = B_0 \oplus W$. So there exist $f_V \in B_0$ and $f_W \in W$ such that $I = f_V + f_W$. As $B_0$ is a two-sided ideal of $T$ and $W$ is a $T$-submodule of $TD$, notice that $Zf_W \in B_0 \cap W$ for every $Z \in B_0$. As $TD = B_0 \oplus W$, $Zf_W = O$ for every $Z \in B_0$. Hence $Zf_V = Z$ for every $Z \in B_0$. Suppose that (i) does not hold. Then there exists $j \in [0, d]$ such that $p \mid k_j$. Since $f_V \in B_0$, by the definition of $B_0$, notice that $f_V \in \langle \{E_i^*JE^*_\ell : i, \ell \in [0, d]\}\rangle_F$. Since $E_j^*JE^*_\ell \in B_0$, we thus deduce that $O = E_j^*JE_i^*f_V = E_j^*JE_i^* \neq O$ by combining (3), (4), (5), and the known result that $Zf_V = Z$ for every $Z \in B_0$. So we have a contradiction. (i) thus follows.

For further discussion, we recall the following definition and present a lemma.

**Definition 4.13** Let $U$ be a $T$-module. Let $\text{Hom}_F(U, F)$ denote the $F$-linear space generated by all linear functionals from $U$ to $F$. By Lemma 2.2 (iii), (6), and (1), $Z' \in T$ for every $Z \in T$. Let $T$ act on $\text{Hom}_F(U, F)$ by setting $(Z\psi)(\hat{u}) = \psi(Z'\hat{u})$ for any $Z \in T$, $\psi \in \text{Hom}_F(U, F)$, and $\hat{u} \in U$. So $\text{Hom}_F(U, F)$ is a $T$-module under the defined action of $T$. Call this $T$-module the contragredient $T$-module of $U$. Let $U^0$ denote the contragredient $T$-module of $U$. Notice that $\dim F U^0 = \dim F U$. Call $U$ a self-contragredient $T$-module if $U \cong U^0$ as $T$-modules.
Lemma 4.14 If \( n \in \mathbb{N}_0 \), \( Q_n \neq \emptyset \), and \( C \in Q_n \), then \( Irr_n(C) \) is a self-contragredient \( T \)-module.

**Proof** Note that \( C \subseteq S_n \) by Lemma 3.9 (i). By the definition of \( S_n \), for every \( h \in S_n \), there is \( q_h \in \mathbb{N} \) such that \( k_h = p^n q_h \) and \( p \uparrow q_h \). By Notation 3.10, recall that \( Irr_n(C) \) has an \( \mathbb{F} \)-basis \( \{ E_i^* 1 + W_{n+1} : i \in C \} \) of cardinality \( |C| \). In particular, observe that \( \{ q_i^{-1} E_i^* 1 + W_{n+1} : i \in C \} \) is also an \( \mathbb{F} \)-basis of \( Irr_n(C) \) and \( \dim \mathbb{F} \, Irr_n(C)^c = |C| \).

For every \( i \in C \), let \( \psi_i \) denote the linear functional from \( Irr_n(C) \) to \( \mathbb{F} \) that sends \( E_j^* 1 + W_{n+1} \) to \( \delta_{ij} \) for every \( j \in C \). Since \( \{ q_i^{-1} E_i^* 1 + W_{n+1} : i \in C \} \) is an \( \mathbb{F} \)-basis of \( Irr_n(C) \), notice that \( \{ \psi_i : i \in C \} \) is an \( \mathbb{F} \)-basis of \( Irr_n(C)^c \). Let \( \Phi \) be the \( \mathbb{F} \)-linear isomorphism from \( Irr_n(C) \) to \( Irr_n(C)^c \) that sends \( q_i^{-1} E_i^* 1 + W_{n+1} \) to \( \psi_i \) for every \( i \in C \). By Definition 4.13, it is enough to check that \( \Phi \) is a \( T \)-isomorphism. Let \( a, b, c \in \{0, d\} \). We list three cases to prove that \( \Phi \) preserves the action of \( E_a^* A_b E_c^* \).

Case 1: \( a \notin C \).

As \( a \notin C \) and \( \{ E_i^* 1 + W_{n+1} : i \in C \} \) is an \( \mathbb{F} \)-basis of \( Irr_n(C) \), by (3), \( E_a^* A_b E_c^* \psi_i \) is the zero element of \( Irr_n(C)^c \) for every \( i \in C \). Notice that \( E_a^* A_b E_c^* (q_i^{-1} E_i^* 1 + W_{n+1}) \) equals \( 0 + W_{n+1} \) for every \( i \in C \). Otherwise, suppose that there is \( \ell \in C \) such that \( E_a^* A_b E_c^* (q_i^{-1} E_i^* 1 + W_{n+1}) \neq 0 + W_{n+1} \). Since \( Irr_n(C) \) is a \( T \)-submodule of \( W_n / W_{n+1} \) with an \( \mathbb{F} \)-basis \( \{ E_i^* 1 + W_{n+1} : i \in C \} \), by combining (3), Lemmas 2.2 (i), 3.1 (i), \( a \in C \), which is absurd as \( a \notin C \). Hence \( \Phi \) preserves the action of \( E_a^* A_b E_c^* \).

Case 2: \( c \notin C \).

As \( c \notin C \), by (3), \( E_a^* A_b E_c^* (q_i^{-1} E_i^* 1 + W_{n+1}) = 0 + W_{n+1} \) for every \( i \in C \). Observe that \( E_a^* A_b E_c^* (E_i^* 1 + W_{n+1}) = 0 + W_{n+1} \) for every \( i \in C \). Otherwise, suppose that there exists \( c \in C \) such that \( E_a^* A_b E_c^* (E_i^* 1 + W_{n+1}) \neq 0 + W_{n+1} \). Since \( Irr_n(C) \) is a \( T \)-submodule of \( W_n / W_{n+1} \) with an \( \mathbb{F} \)-basis \( \{ E_i^* 1 + W_{n+1} : i \in C \} \), by combining (3), Lemmas 2.2 (i), 3.1 (i), \( c \in C \), which is absurd as \( c \notin C \). As \( E_i^* 1 + W_{n+1} : i \in C \) is an \( \mathbb{F} \)-basis of \( Irr_n(C) \) and (1) holds, we thus notice that \( E_a^* A_b E_c^* \psi_i \) is the zero element of \( Irr_n(C)^c \) for every \( i \in C \). So \( \Phi \) preserves the action of \( E_a^* A_b E_c^* \) by the definition of \( \Phi \) and the fact that \( \{ q_i^{-1} E_i^* 1 + W_{n+1} : i \in C \} \) is an \( \mathbb{F} \)-basis of \( Irr_n(C) \).

Case 3: \( a, c \in C \).

Let \( v \in C \). Since \( k_a = p^n q_a \) and \( k_v = p^n q_v \), notice that \( q_a p^{a'}_{vb'} = q_v p^{a'}_{ab} \) by Lemma 2.1. As \( a \in C \), by combining (3), Lemma 2.2 (i), and the definition of \( \Phi \), we thus have

\[
\Phi(E_a^* A_b E_c^* (q_v^{-1} E_v^* 1 + W_{n+1})) = \Phi(\delta_{cv} q_v^{-1} p^{a'}_{vb'} (E_a^* 1 + W_{n+1})) = \Phi(\delta_{cv} q_a^{-1} p^{a'}_{ab} (E_a^* 1 + W_{n+1})) = \delta_{cv} p^{a'}_{ab} \psi_a.
\]

By the definition of \( \Phi \) again, we also have \( E_a^* A_b E_c^* (q_v^{-1} E_v^* 1 + W_{n+1}) = E_a^* A_b E_c^* \psi_v \). Let \( w \in C \). As \( c \in C \), by combining (3), (1), and Lemma 2.2 (i),
notice that

$$\delta_{cv} \overline{p_{ab}}^v \psi_a(E_w^* 1 + W_{n+1}) = \delta_{cv} \delta_{aw} \overline{p_{wb}}^v = \delta_{cv} \delta_{aw} \overline{p_{wb}}^v$$

which implies that $\delta_{cv} \overline{p_{ab}}^v \psi_a = E_a^* A_b E_c^* \psi_v$ since $\{E_i^* 1 + W_{n+1} : i \in C\}$ is an $F$-basis of $\text{Irr}_{n}(C)$ and $v$ is chosen from $C$ arbitrarily. We thus deduce that

$$\Phi(E_a^* A_b E_c^* (\overline{q_v}^{-1} E_v^* 1 + W_{n+1})) = \delta_{cv} \overline{p_{ab}}^v \psi_a = E_a^* A_b E_c^* \psi_v$$

Therefore $\Phi$ preserves the action of $E_a^* A_b E_c^*$ since $\{\overline{q_v}^{-1} E_v^* 1 + W_{n+1} : i \in C\}$ is an $F$-basis of $\text{Irr}_{n}(C)$ and $v$ is chosen from $C$ arbitrarily.

By Cases 1, 2, 3, $\Phi$ preserves the action of $E_a^* A_b E_c^*$. As $a, b, c$ are chosen from $[0, d]$ arbitrarily, by Lemma 2.2 (iii) and (6), we thus obtain that $\Phi$ is a $T$-isomorphism. The desired lemma thus follows. \(\square\)

The following lemma gives a characterization of the $p'$-valenced schemes.

**Lemma 4.15** The following statements are equivalent:

(i) $S$ is a $p'$-valenced scheme;

(ii) $W_0$ is a self-contragredient $T$-module.

**Proof** We prove (ii) by (i). By (i), note that $S_0 = [0, d]$. By combining Lemmas 3.4 (iii), (iv), 3.12 (ii), Remark 3.13, $W_0 = \text{Irr}_0([0, d])$. So (ii) is from Lemma 4.14.

We prove (i) by (ii). Since $\dim F W_0^c = \dim F W_0 = d + 1$, let $\{\psi_i : i \in [0, d]\}$ be an $F$-basis of $W_0^c$. By (ii), let $\Phi$ denote a $T$-isomorphism from $W_0$ to $W_0^c$. So there exist $c_0, c_1, \ldots, c_d \in F$ such that $\Phi(E_0^* 1) = \sum_{i=0}^{d} c_i \psi_i$. Suppose that (i) does not hold. Then there is $j \in [0, d]$ such that $p \mid k_j$. Notice that $E_j^* J \in T$ by (2). By (3) and (5), $J E_j^* E_0^* 1 = 0$ for every $i \in [0, d]$. As $\{E_i^* 1 : i \in [0, d]\}$ is an $F$-basis of $W_0$, by (1), we thus get that $E_j^* J \psi_i$ is the zero element of $W_0^c$ for every $i \in [0, d]$. Since $\Phi$ is a $T$-isomorphism, $\Phi(E_j^* J E_0^* 1) = E_j^* J \Phi(E_0^* 1) = \sum_{i=0}^{d} c_i E_j^* J \psi_i$, which implies that $E_j^* J E_0^* 1 = 0$. This is absurd as $E_j^* J E_0^* 1 = E_j^* 1 \neq 0$ by (5). (i) thus follows. \(\square\)

For our purpose, we also introduce the following notation and list four lemmas.

**Notation 4.16** Let $U$ denote a $T$-module. Define $E_i^* U = \{E_i^* \hat{u} : \hat{u} \in U\}$ for every $i \in [0, d]$. For every $i \in [0, d]$, note that $E_i^* U$ is an $F$-linear subspace of $U$. Therefore $\dim F U \geq \dim F E_i^* U$ for every $i \in [0, d]$.

**Lemma 4.17** The following statements are equivalent:

(i) $U \cong W_0/W_1$ as $T$-modules;

(ii) $U$ is an irreducible $T$-module satisfying $\dim F E_i^* U > 0$ for some $R_i \in O_\theta(S)$.  

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Proof We prove (ii) by (i). By (i) and Lemma 3.4 (iii), $U$ is an irreducible $T$-module. Note that $\{E_j^*1 + W_1 : j \in S_0\}$ is an $F$-basis of $W_0/W_1$ by Lemma 3.3 (ii). As $0 \in S_0$, we thus have $\dim_F E_0^*(W_0/W_1) = 1 > 0$ by (3) and the definition of $E_0^*(W_0/W_1)$. So $\dim_F E_0^*U = 1 > 0$ by (i) and the definition of $E_0^*U$. (ii) follows as $R_0 \in O_\theta(S)$.

We prove (i) by (ii). As $U$ is an irreducible $T$-module, $U$ is a $T$-module generated by a single element. So $T T/V \cong U$ as $T$-modules for some $T$-submodule $V$ of $T T$. As $\dim_F E_i^*U > 0$ for some $R_i \in O_\theta(S)$ and $T T/V \cong U$ as $T$-modules, according to the definition of $E_i^*U$, there exists $Z \in T$ such that $E_i^*Z + V \neq O + V$. So $E_i^*Z + V = E_i^*ZI + V = \sum_0^d E_i^*Z E_j^* + V \neq O + V$ by (2). Hence there exists $c \in [0, d]$ such that $E_i^*Z E_c^* + V \neq O + V$. Notice that $k_i = 1 = R_i \in O_\theta(S)$. We thus have $E_i^*Z E_c^* + V = c_i E_i^*Z E_c^* + V \neq O + V$ and $0 \neq c_i E_i Z \in F$ by combining (3), Lemmas 2.2 (iii), 4.7 (ii). In particular, $\langle\{E_i^*J E_c^* + V : j \in [0, d]\}\rangle_F \neq \{O + V\}$.

We claim that $\langle\{E_i^*J E_c^* + V : j \in [0, d]\}\rangle_F = T T/V$. As $T T/V \cong U$ as $T$-modules and $U$ is an irreducible $T$-module, we thus have $\langle\{E_i^*J E_c^* + V : j \in [0, d]\}\rangle_F \neq \{O + V\}$, the desired claim follows if we verify that $\langle\{E_i^*J E_c^* + V : j \in [0, d]\}\rangle_F$ is a $T$-submodule of $T T/V$. Let $a, b, c \in [0, d]$. For every $h \in [0, d]$, notice that $E_i^*A_b E_i^*E_h^*E_c^* + V = \delta_{cb} p_h^*\overline{e_i^*A_h} E_i^*E_c^* + V$ by (3) and Lemma 2.2 (i). So $\langle\{E_i^*A_b E_i^*E_h^*E_c^* + V : j \in [0, d]\}\rangle_F$ is an $F$-linear space and $a, b, c$ are chosen from $[0, d]$ arbitrarily, we thus get that $\langle\{E_i^*J E_c^* + V : j \in [0, d]\}\rangle_F$ is a $T$-submodule of $T T/V$ by Lemma 2.2 (iii) and (6).

The desired claim thus follows. By the definition of $M_\ell$ and the proven claim, there exists an obvious surjective $T$-homomorphism from $M_\ell$ to $T T/V$. So there is a surjective $T$-homomorphism from $W_0$ to $T T/V$ by Lemma 4.3 (iv). As $U$ is an irreducible $T$-module and $T T/V \cong U$ as $T$-modules, by Lemma 3.4 (i), $W_0/W_1 \cong T T/V \cong U$ as $T$-modules. (i) is proved. □

Lemma 4.18 The following statements are equivalent:

(i) $S$ is a $p'$-valanced scheme;

(ii) $U$ is an irreducible $T$-module satisfying $\dim_F E_i^*U > 0$ for some $R_i \in O_\theta(S)$ if and only if $U \cong W_0$ as $T$-modules.

Proof We prove (ii) by (i). According to (i) and Lemma 3.4 (iv), observe that $W_0$ is an irreducible $T$-module. So $W_1 = \{0\}$ by Lemma 3.4 (i). Therefore (ii) follows from Lemma 4.17. We prove (i) by (ii). By (ii) and Lemma 4.17, we have $W_0 \cong W_0/W_1$ as $T$-modules. So $W_0$ is an irreducible $T$-module by Lemma 3.4 (iii). Therefore (i) follows from 3.4 (iv). □

Lemma 4.19 If $S$ is a $p'$-valanced scheme, the following statements are equivalent:

(i) $U \cong W_0$ as $T$-modules;

(ii) $U$ is a $T$-module satisfying $\dim_F U = d+1 \geq \dim_F E_i^*U > 0$ for some $R_i \in O_\theta(S)$.

Proof We prove (ii) by (i). We recall that $W_0$ has an $F$-basis $\{E_j^*1 : j \in [0, d]\}$ and $\dim_F W_0 = d+1$. By (3) and the definition of $E_0^*W_0$, note that $\dim_F E_0^*W_0 = 1 > 0$. So we have $\dim_F U = d+1 \geq \dim_F E_0^*U = 1 > 0$ by (i) and the definition of $E_0^*U$. (ii) thus follows as $R_0 \in O_\theta(S)$.
Theorem 4.21 The following statements are equivalent:

We are now ready to present the main result of this section.

We prove (i) by (ii). By (ii), notice that \( \dim_{\mathbb{F}} E_i^* U > 0 \) for some \( R_i \in O_{\emptyset}(S) \). Pick an element of an \( \mathbb{F} \)-basis of \( E_i^* U \). Let \( M \) denote the \( T \)-submodule of \( U \) generated by this chosen element. So there exists a \( T \)-submodule \( V \) of \( \tau T \) such that \( \tau T / V \cong M \) as \( T \)-modules. Since \( M \) contains an element of an \( \mathbb{F} \)-basis of \( E_i^* U \) and \( \tau T / V \cong M \) as \( T \)-modules, according to the definition of \( E_i^* U \), notice that there is \( Z \in T \) such that \( E_i^*(Z+V) = E_i^*Z + V \neq O + V \). So \( E_i^*Z + V = E_i^*Z1 + V = \sum_{j=0}^d E_i^*E_j^* + V \neq O + V \) by (2). Hence there exists \( \ell \in [0, d] \) such that \( E_i^*E_j^* + V \neq O + V \). Notice that \( k_1 = 1 \) as \( R_i \in O_{\emptyset}(S) \). We thus have \( E_i^*E_j^* + V = c_{i\ell}Z E_i^*E_j^* + V \neq O + V \) and \( 0 \neq c_{i\ell}Z \in \mathbb{F} \) by combining (3), Lemmas 2.2 (iii), 4.7 (ii). In particular, notice that \( O + V \neq E_i^*E_j^* + V \in \{ E_i^*E_j^* + V : j \in [0, d] \} \mathbb{F} \).

We claim that \( \langle \{ E_i^*E_j^* + V : j \in [0, d] \} \rangle_{\mathbb{F}} = \tau T / V \cong M = U \). As we have known that \( \langle \{ E_i^*E_j^* + V : j \in [0, d] \} \rangle_{\mathbb{F}} \subseteq \tau T / V \cong M \subseteq U \), the desired claim thus follows if we can verify that \( \dim_{\mathbb{F}} \langle \{ E_i^*E_j^* + V : j \in [0, d] \} \rangle_{\mathbb{F}} = \dim_{\mathbb{F}} U \).

We now suppose that \( \sum_{j=0}^d c_j E_i^*E_j^* + V = O + V \), where \( \bigcup_{j=0}^d \{ c_j \} \subseteq \mathbb{F} \) and \( (\bigcup_{j=0}^d \{ c_j \}) \cap (\mathbb{F} \setminus \{ 0 \}) \neq \emptyset \). Therefore there exists \( a \in [0, d] \) such that \( c_a \neq 0 \). According to (3), observe that \( c_a E_a^*E_i^* + V = E_a^* (\sum_{j=0}^d c_j E_j^*E_i^* + V) = E_a^*(O + V) = O + V \), which implies that \( E_a^*E_i^* + V = O + V \) as \( c_a \neq 0 \). As \( S \) is a \( p' \)-valenced scheme, notice that \( p \nmid k_a \). By (3) and (5), we thus have \( k_a E_a^*E_i^* + V = E_i^*E_a^*(E_a^*E_i^* + V) = O + V \) and \( E_a^*E_i^* + V = O + V \), which contradicts the inequality \( E_i^*E_j^* + V \neq O + V \). Hence we deduce that \( \dim_{\mathbb{F}} \langle \{ E_i^*E_j^* + V : j \in [0, d] \} \rangle_{\mathbb{F}} = d + 1 \). The desired claim thus follows as \( \dim_{\mathbb{F}} U = d + 1 \) by (ii).

By the definition of \( M_\ell \) and the proven claim, there exists an obvious surjective \( T \)-homomorphism from \( M_\ell \) to \( \tau T / V \). So there exists a surjective \( T \)-homomorphism from \( W_0 \) to \( \tau T / V \) by Lemma 4.3 (iv). According to the proven claim and (ii), notice that \( \tau T / V \cong U \) as \( T \)-modules and \( \dim_{\mathbb{F}} U = \dim_{\mathbb{F}} W_0 = d + 1 \). (i) thus follows. □

Lemma 4.20 Assume that the following statements are equivalent:

(i) \( U \cong W_0 \) as \( T \)-modules;
(ii) \( U \) is a \( T \)-module satisfying \( \dim_{\mathbb{F}} U = d + 1 \geq \dim_{\mathbb{F}} E_i^* U > 0 \) for some \( R_i \in O_{\emptyset}(S) \).

Then \( S \) is a \( p' \)-valenced scheme.

Proof Recall that \( W_0 \) has an \( \mathbb{F} \)-basis \( \{ E_j^* U : j \in [0, d] \} \). For every \( j \in [0, d] \), let \( \psi_j \) denote the linear functional from \( W_0 \) to \( \mathbb{F} \) that sends \( E_j^* U \) to \( \delta_{j\ell} \) for every \( \ell \in [0, d] \). Notice that \( \{ \psi_j : j \in [0, d] \} \) is an \( \mathbb{F} \)-basis of \( W_0^\circ \) and \( \dim_{\mathbb{F}} W_0^\circ = d + 1 \). According to the definition of \( E_0^* W_0^\circ \), (1), and (3), notice that \( E_0^* W_0^\circ = \langle \{ \psi_0 \} \rangle_{\mathbb{F}} \). So we have \( \dim_{\mathbb{F}} W_0^\circ = d + 1 \geq \dim_{\mathbb{F}} E_0^* W_0^\circ = 1 > 0 \). Since \( R_0 \in O_{\emptyset}(S) \) and (ii) implies (i), \( W_0 \) is a self-contragredient \( T \)-module. So \( S \) is a \( p' \)-valenced scheme by Lemma 4.15. □

We are now ready to present the main result of this section.

Theorem 4.21 The following statements are equivalent:

(i) \( S \) is a \( p' \)-valenced scheme;
(ii) The \( \mathbb{F} \)-subalgebra \( B_0 \) of \( T \) is unital. Its identity element is central in \( T \);
There exists a two-sided ideal \( D \) of \( T \) such that \( T \) is a direct sum of \( B_0 \) and \( D \);

The \( F \)-subalgebra \( B_0 \) of \( T \) is isomorphic to a full matrix algebra over a division \( F \)-algebra as \( F \)-algebras;

If \( Z \in \text{Ann}_T(W_0) \), then \( E_i^*Z = ZE_i^* = O \) for every \( R_i \in O_\theta(S) \);

If \( \tilde{Z} \in \text{Rad}(T) \), then \( E_i^*\tilde{Z} = \tilde{Z}E_i^* = O \) for every \( R_i \in O_\theta(S) \);

For every decomposition of \( T \) into a direct sum of indecomposable \( T \)-modules, there exist exactly \( d + 1 \) indecomposable direct summands isomorphic to \( W_0 \) as \( T \)-modules;

\( W_0 \) is an irreducible \( T \)-module;

\( W_0 \) is a self-contragredient \( T \)-module;

\( U \) is a \( T \)-module satisfying \( \dim_F E_j^*U > 0 \) for some \( R_j \in O_\theta(S) \) if and only if \( U \cong W_0 \) as \( T \)-modules.

Proof (i) is equivalent to all the other listed statements by combining Lemmas 4.6, 4.9, 4.12, 3.4 (iv), 4.15, 4.18, 4.19, and 4.20. Hence the desired theorem follows.

Let \( FG \) be the group algebra of a finite group \( G \) over \( F \). Note that \( FG \) itself is an \( FG \)-module under the left multiplication action of \( FG \). Denote this \( FG \)-module by \( FGFG \). It is known that the following statements are equivalent:

(i) The principal block algebra of \( FG \) is a simple unital \( F \)-algebra;

(ii) The Jacobson radical of \( FG \) is the zero ideal;

(iii) For every decomposition of \( FGFG \) into a direct sum of indecomposable \( FG \)-modules, there exists exactly one indecomposable direct summand isomorphic to the trivial \( FG \)-module as \( FG \)-modules.

The following corollary tells us that similar statements about \( T \) are also equivalent. Its proof comes from the Artin-Wedderburn Theorem and Theorem 4.21.

Corollary 4.22 The following statements are equivalent:

(i) The \( F \)-subalgebra \( B_0 \) of \( T \) is a simple unital \( F \)-algebra;

(ii) If \( Z \in \text{Rad}(T) \), then \( E_i^*Z = ZE_i^* = O \) for every \( R_i \in O_\theta(S) \);

(iii) For every decomposition of \( T \) into a direct sum of indecomposable \( T \)-modules, there exist exactly \( d + 1 \) indecomposable direct summands isomorphic to \( W_0 \) as \( T \)-modules.

We close this note by proposing two questions motivated by our main results.

Every composition factor of \( W_0 \) is an irreducible self-contragredient \( T \)-module by Theorem 3.15 and Lemma 4.14. This fact motivates us to ask the following question.

Question 4.23 Can one determine all irreducible self-contragredient \( T \)-modules up to isomorphism?

We call an indecomposable \( T \)-module a semiprimary \( T \)-module if this \( T \)-module satisfies Lemma 4.19 (ii). It is obvious that \( W_0 \) and \( W_0^\circ \) are semiprimary \( T \)-modules. By Theorem 4.21, note that a semiprimary \( T \)-module may not be isomorphic to \( W_0 \) as \( T \)-modules. So the notion of a semiprimary \( T \)-module generalizes the notion of the
primary \( T \)-module. The definition of a semiprimary \( T \)-module motivates us to ask the following question.

**Question 4.24** Can one determine all semiprimary \( T \)-modules up to isomorphism?

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