UNIVERSAL COVERING SPACES, A FOOTNOTE

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Abstract. We give a short proof that, for nice $X$, the based fundamental groupoid of $X$ with topology induced by the compact open topology on the space of paths, is indeed the universal covering space of $X$.

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Standard textbook construction of the universal covering space of $X$ usually goes as follows. Pick a point $x_0 \in X$ and consider the set $P(X,x_0)$ of all paths in $X$ starting at $x_0$. Define an equivalence relation on $P(X,x_0)$ by letting $\alpha \sim \alpha'$ if $\alpha(1) = \alpha'(1)$ and $\alpha \simeq \alpha'(\text{rel} \partial I)$. Endow the set of equivalence classes $P(X,x_0)/\sim$ with the topology generated by subsets of the form $\langle \alpha, U \rangle$, where $\alpha \in P(X,x_0)$, $U$ is an open neighbourhood of $\alpha(1)$ and $\langle \alpha, U \rangle$ consists of equivalence classes of paths obtained by concatenating $\alpha$ to a path in $P(U,\alpha(1))$. It then requires certain amount of work (see for example [4, pp. 82-83]) to show that the resulting space is the universal covering space of $X$ (provided $X$ satisfies suitable connectivity assumptions).

However, there is another natural way to topologize $P(X,x_0)/\sim$, namely by taking the quotient topology of the standard compact-open topology on the space of paths. It raises an obvious question: are the two topologies equal? A short answer is: not in general, but they coincide when $X$ admits a universal covering space.

So why are various textbooks not using quotient of the compact-open topology, which seems to be more standard and easier to define? Indeed, Spanier [4, p.82] states that ‘We could start with the compact-open topology on $P(X,x_0)$ and use the quotient topology on the set of equivalence classes, but it seems no simpler than merely topologizing the set of equivalence classes directly, as is done below.’ One can have long discussions about more or less efficient proofs, but most later authors followed Spanier’s approach without further explanation which lead to an unanticipated consequence. I was asked on a number of occasions, even by topologists, why is the compact-open topology inadequate for the construction of universal covering spaces. Similar questions pop-out on various forums so it appears that there is a widespread conviction that Spanier’s approach is not just a matter of convenience but that compact-open topology is actually ‘wrong’.

Although one can find in the literature statements that the two topologies coincide (see notes at the end), they are always buried among more general results. That’s why I decided to write a short direct proof that for nice spaces universal coverings can be constructed by taking quotient topology of the compact-open topology on the space of based paths. The proof is reasonably self-contained in the sense that it relies only on standard results from general topology (as in Dugundji [I], in particular for his notation related to compact-open topology) and on some results on covering spaces that are usually proved before the construction of universal coverings (as in Spanier [4, Chapter II]).

Let $X$ be a path-connected and locally path-connected space. We will first describe the based fundamental groupoid of $X$ as a topological space. Choose $x_0 \in X$ and denote as $P(X,x_0)$ the space of based continuous paths $\alpha: (I,0) \to (X,x_0)$, endowed with the compact-open topology. The evaluation map $ev: P(X,x_0) \to X$ given as $ev(\alpha) := \alpha(1)$ is continuous (cf. [I, XII.2.4(2)]), and surjective (because $X$ is path-connected). It is also an open map by the following argument. Every point in $ev\left(\bigcap_{i=1}^{n}(A_i,V_i)\right)$ is of the form $\alpha(1)$ for some $\alpha \in \bigcap_{i=1}^{n}(A_i,V_i)$. If $V$ is the path-component of $\bigcap_{i=1}^{n}(V_i)$ containing $\alpha(1)$, then $V$ is open in $X$ because $X$ is locally path-connected. It is easy to modify the final segment of $\alpha$ to obtain a path in $\bigcap_{i=1}^{n}(A_i,V_i)$ ending in any given point of $V$ (cf. [4, II.4.4]), therefore $\alpha(1)$ is an interior point of $ev\left(\bigcap_{i=1}^{n}(A_i,V_i)\right)$.

Let $\sim$ be the equivalence relation on $P(X,x_0)$ defined by $\alpha \sim \alpha'$ if $\alpha(1) = \alpha'(1)$ and $\alpha \simeq \alpha'(\text{rel} \partial I)$. The fundamental groupoid of $X$ based at $x_0$ is defined as the quotient topological space

$$\Pi(X,x_0) := P(X,x_0)/\sim.$$
Since $ev$ preserves $\sim$, we obtain by \texttt{[1] VI.4.3} the induced map $p: \Pi(X,x_0) \to X$ as in the diagram
\[
P(X,x_0) \xrightarrow{q} \Pi(X,x_0) \\
\downarrow ev \quad \downarrow p \\
\quad X
\]
The map $p$ is continuous, open (by \texttt{[1] VI.3.2(2)}) and surjective.

We now describe an action of the fundamental group $\pi_1(X,x_0)$ on the based fundamental groupoid. Let $g \in \pi_1(X,x_0)$ be represented by a loop $\gamma$, and let $\alpha \in P(X,x_0)$. Then $[\beta] \mapsto [\beta]^g := [\gamma \cdot \beta]$ defines a continuous self-map of $\Pi(X,x_0)$ (it is clearly continuous on the level of paths, continuity on $\Pi(X,x_0)$ then follows by \texttt{[1] VI.4.3}). Thus we obtain a right action of $\pi_1(X,x_0)$ on $\Pi(X,x_0)$ by homeomorphisms. To see that the action is free, assume $[\alpha]^g = [\alpha]^g'$ and represent $g, g' \in \pi_1(X,x_0)$ by loops $\gamma, \gamma'$. Then $\gamma \cdot \alpha \simeq \gamma' \cdot \alpha (\text{rel } \partial I)$ implies $\gamma \simeq \gamma' (\text{rel } \partial I)$ by \texttt{[1] XIX.1.7}, therefore $g = g'$.

If $\alpha, \alpha' \in P(X,x_0)$ are such that $\alpha(1) = \alpha'(1)$, then $\alpha' \sim \alpha \cdot \overline{\alpha} \cdot \alpha$, and thus $[\alpha'] = [\alpha][\alpha']$. It follows that the orbits of the action of $\pi_1(X,x_0)$ on $\Pi(X,x_0)$ coincide with the fibres of $p$. In other words, $X$ is homeomorphic to the quotient space of $\Pi(X,x_0)$ by the action of $\pi_1(X,x_0)$ (recall that $p$ is open, hence a quotient map).

Before proceeding, let us recall that $X$ is said to be \textit{semi-locally simply connected} if every point in $x \in X$ has a neighbourhood $U$, such that every loop in $U$ can be contracted to a point in $X$. One can equivalently say that every two paths in $U$ that have common endpoints are homotopic in $X$ relative to endpoints, or that inclusion induces a trivial homomorphism $\pi_1(U,x) \to \pi_1(X,x)$.

**THEOREM:** \textit{If $X$ is path-connected, locally path-connected and semi-locally simply connected, then $p: \Pi(X,x_0) \to X$ is the universal covering of $X$.}

**Proof.** Given any $\alpha \in P(X,x_0)$, choose a path-connected open neighbourhood $U$ of $\alpha(1)$ as in the above definition of semi-locally simple connectedness. For every $g \in \pi_1(X,x_0)$ define
\[U_g := \{[\alpha \cdot \beta]^g \mid \beta \in P(U,\alpha(1))\} \subset \Pi(X,x_0).\]

By the assumptions on $U$, if $\beta(1) = \beta'(1)$ for $\beta, \beta' \in P(U,\alpha(1))$, then $[\alpha \cdot \beta] = [\alpha \cdot \beta']$, therefore the restriction $p|_{U_g}: U_g \to U$ is a bijection. It also follows that $U_g$ and $U_g'$ are disjoint, unless $g = g'$, so we have that
\[p^{-1}(U) = \bigsqcup_{g \in \pi_1(X,x_0)} U_g.\]

Finally, observe that $U_g$ is an open subset of $\Pi(X,x_0)$ because $q^{-1}(U_g) = \{[\alpha \cdot \beta] \mid \beta \in P(U,\alpha(1))\}$ is clearly open in of $P(X,x_0)$. It follows that the restriction $p|_{U_g}: U_g \to U$ is a continuous, open bijection, hence a homeomorphism. Therefore $U$ is evenly covered by the sets $U_g$ and $p: \Pi(X,x_0) \to X$ is a covering projection.

It is now easy to show that $p$ is indeed the universal covering projection. Let $r: \tilde{X} \to X$ be a covering projection with $\tilde{X}$ path-connected and choose some $\tilde{x}_0 \in r^{-1}(x_0)$. Then unique path-lifting property of $r$ determines a continuous map $l: P(X,x_0) \to P(\tilde{X},\tilde{x}_0)$, which in turn induces a continuous map $\tilde{l}: \Pi(X,x_0) \to \tilde{X}$, such that $r \circ \tilde{l} = p$.

Alternatively, we may consider the final part of the exact sequence of the fibration $p$
\[1 \to \pi_1(\Pi(X,x_0), [x_0]) \longrightarrow \pi_1(X,x_0) \xrightarrow{\partial} p^{-1}(x_0) \to *,\]
where $c_{x_0}$ is the constant path at $x_0$ and $\partial(g) = [c_{x_0}]^g$. Since $\pi_1(X,x_0)$ acts freely, $\partial$ is injective, therefore $\Pi(X,x_0)$ is simply connected and thus universal by \texttt{[4] II.5.7].}
Short two-step summary:

(1) If $X$ is path-connected and locally path-connected, then the evaluation $P(X, x_0) \to X$ induces a continuous, open surjection $p: \Pi(X, x_0) \to X$, whose fibres are precisely the orbits of the action of $\pi_1(X, x_0)$ on $\Pi(X, x_0)$.

(2) Assume $U$ is an open, path-connected subset of $X$, such that every loop in $U$ can be contracted to a point in $X$. Then $p^{-1}(U) = \bigsqcup \{U_g \mid g \in \pi_1(X, x_0)\}$, where $U_g := \{[\alpha \cdot \beta]_g \mid \beta \in P(U, \alpha(1))\}$ and $p|_{U_g}: U_g \to U$ is a homeomorphism for every $g \in \pi_1(X, x_0)$. So if $X$ is semi-locally simply connected, then it is evenly covered by $p: \Pi(X, x_0) \to X$ with fibre $\pi_1(X, x_0)$, therefore $p$ is the universal covering projection.

Sub-footnotes:

(1) Spanier’s book is a standard reference for the construction of universal coverings, but the original idea is much older. I was able to date it at least to H. Weyl’s monograph on Riemann surfaces from 1913. Here is an excerpt from a 1955 translation with an extraordinary modern description of the universal covering surface.

The universal covering surface may be defined as follows. If $v_0$ is a fixed point of $\tilde{F}$ then every curve $\gamma$ starting at $v_0$ defines a “point of $\tilde{F}$” which we say lies over the endpoint of $\gamma$. Two such curves $\gamma, \gamma'$ define the same point of $\tilde{F}$ if and only if on every perfect covering surface over $\tilde{F}$ every pair of curves which start at the same point have traces $\gamma, \gamma'$ end at the same point. Let $\gamma_0$ be a curve on $\tilde{F}$ from $v_0$ to $v$ which defines the point $\tilde{v}$ on $\tilde{F}$, and let $U$ be a neighbourhood of $v$ on $\tilde{F}$. If I attach to $\gamma_0$ all possible curves $\gamma$ in $U$ which start at $v$, then I say that the points of $\tilde{F}$ defined by all these curves $\gamma_0 + \gamma$ form a “neighborhood” $\tilde{U}$ of $\tilde{v}$. Since $\tilde{U}$ is simply connected there is just one point of $\tilde{U}$ over each point of $U$; hence our concept of “neighborhood” satisfies the conditions stated in §4. (see [6] pp. 58-59)

(2) Various topologies on the based fundamental groupoid were thoroughly studied in the recent years. In that context the topology used by Spanier was called whisker topology, while the quotient of the compact-open topology was called CO-topology (see Virk and Zastrow [5]). In particular, Fisher and Zastrow [2] Lemma 2.1 proved that whisker topology is always stronger than CO-topology and that the two agree for spaces that are locally path-connected and semi-locally simply connected. However, there are locally path-connected spaces (e.g., the ‘Hawaiian earring’) for which whisker topology is strictly stronger than CO-topology.

(3) It can be occasionally useful to know that the fact that $ev: P(X, x_0) \to X$ is an open map, that played a crucial role in our argument, can be extended to arbitrary Hurewicz fibrations. Indeed, McAuley [3] Cor. 1.1 proved that $X$ is locally path-connected if, and only if, every Hurewicz fibration with base $X$ is an open map.

References

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