Reconciliation of stability analysis with simulation in a bus route model

Scott A. Hill

James Franck Institute and Department of Physics, University of Chicago, Chicago, Illinois 60637.

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In a recent paper (Physica A 296, 320 (2001)), T. Nagatani proposes a time-headway model for buses on a bus route, and studies the stability of that model’s homogeneous solutions. While investigating the phase diagram gotten by varying the rate of passenger arrival and the initial spacing of buses, he discovers a discrepancy between the results of simulation and those of linear stability analysis. In this paper, we reconcile this discrepancy by noting the existence of three types of phase diagrams, and present simulational results which confirm the stability analysis.

I. INTRODUCTION

While there has been much interest in the study of automobile traffic among scientists, there have been few corresponding studies of buses. The dynamics of a bus route, while having some similarities with those of general traffic, differ due to the added interaction of buses with passengers at designated bus stops. A good reason for studying the dynamics of bus routes is that they are so often unstable. Buses are initially spaced at regular intervals. However, if one bus is delayed for some reason, it will then find a larger number of passengers waiting for it at subsequent stops, thus delaying it further. Meanwhile, the bus following finds fewer passengers waiting for it, allowing it to go faster until eventually it meets up with the delayed bus. Clusters of three, four, or more buses have been known to form due to this dynamic, resulting in slower and less frequent service.

In reference, Nagatani presents a time-headway model for buses which is an extension of earlier work. Using linear stability analysis, he is able to determine the regions of parameter space in which the homogeneous solution (i.e., with buses spaced evenly apart) is unstable. However, his analysis only agrees qualitatively with results found by direct simulation of the model.

In this paper, we make a slight modification to Nagatani’s model, and discover that one can reconcile the simulation results with that of stability analysis. On doing the analysis, one finds that there are three different types of phase diagrams possible: one which agrees with Nagatani’s theoretical predictions, a second which agrees with his simulation results, and a third which he does not see. We present our own simulational data for these three separate cases, showing quantitative agreement.

II. MODEL

We consider the following model of buses on a bus route. Bus stops are labelled by \( s = 1, 2, \ldots \) where stops \( s \) and \( s + 1 \) are a distance \( L \) apart. There are \( J \) buses, \( j = 1, \ldots, J \), which travel from stop to stop, with bus \( j = J \) in the lead and bus \( j = 1 \) in the rear. Every bus stops at every stop, and buses do not pass one another. The arrival time \( t_{j,s} \) of bus \( j \) at stop \( s \) is given by

\[
t_{j,s} = t_{j,s-1} + \frac{L}{V_{j,s-1}} + \lambda \gamma \Delta t_{j,s-1},
\]

where \( \Delta t_{j,s-1} = t_{j,s-1} - t_{j+1,s-1} \) is the time gap in front of bus \( j \) at stop \( s - 1 \). The penultimate term in Eq. (1) is the time it takes for the bus to travel from stop to stop; \( V_{j,s-1} \) is the average velocity of the bus as it moves. The last term is the time it takes for passengers to board the bus at stop \( s - 1 \); \( \lambda \) is the rate of passengers arriving at the bus stop, so \( \lambda \Delta t_{j,s-1} \) is the number of passengers that have arrived since bus \( j + 1 \) left the stop. The parameter \( \gamma \) is the time it takes one passenger to board the bus, and so \( \lambda \gamma \Delta t_{j,s-1} \) is the amount of time needed to board all the passengers. One could also account for the time it takes riding passengers to leave the bus; Nagatani shows, however, that this is not an important effect, so we will ignore it in our own analysis.

It is reasonable to assume that a driver will try to prevent bunching by slowing down when the gap between his bus and the next is too small. One can model this discretion by writing the average speed \( V_{j,s} \) as a function \( V(\Delta t_{j,s}) \) of the headway, where

\[
V(\Delta t) = v_{\text{min}} + (v_{\text{max}} - v_{\text{min}})\frac{\tanh[\Delta t - t_c] + \tanh t_c}{1 + \tanh t_c}.
\]

The hyperbolic tangent factor acts as a spread-out step function centered at \( t_c \), which is roughly the minimum time gap that a driver prefers to have in front of her. We will set \( t_c = 2 \) in all that follows. The parameter \( v_{\text{max}} \) is the...
speed the bus would travel with an infinite time headway. The parameter \( v_{\text{min}} \) is currently absent from Nagatani’s formulation \[2\], although he did use it in an earlier paper \[4\]. We reintroduce it here for generality.

It is convenient to work, not with the arrival times \( t_{j,s} \), but with the time headways \( \Delta t_{j,s} \). We thus rewrite Eq. \[3\] in terms of the headways:

\[
\Delta t_{j,s} = \Delta t_{j,s-1} + L \left[ \frac{1}{V(\Delta t_{j,s-1})} - \frac{1}{V(\Delta t_{j+1,s-1})} \right] + \lambda \gamma \left[ \Delta t_{j,s-1} - \Delta t_{j+1,s-1} \right].
\]

(3)

III. STABILITY ANALYSIS

We consider the stability of a homogeneous flow of buses; that is, a situation where all buses have the same headway \( \Delta t_0 \). Nagatani \[2\] expands Eq. \[3\] in terms of small deviations from homogeneous flow, and finds that the stability condition for small disturbances at long wavelengths is

\[
\lambda \gamma \frac{V(\Delta t_0)^2}{L} < V'(\Delta t_0) < (\lambda \gamma + 1) \frac{V(\Delta t_0)^2}{L},
\]

(4)

where \( \Delta t_0 \) is the initial, constant spacing between buses, \( V(\Delta t) \) is the velocity function in Eq. \[2\] and \( V'(\Delta t_0) \) is the derivative of that function. We can rewrite this in terms of the passenger arrival rate \( \lambda \gamma \),

\[
\frac{LV'(\Delta t_0)}{V(\Delta t_0)^2} - 1 < \lambda \gamma < \frac{LV'(\Delta t_0)}{V(\Delta t_0)^2},
\]

(5)

where, specifically,

\[
\frac{LV'(\Delta t_0)}{V(\Delta t_0)^2} = \frac{L(v_{\text{max}} - v_{\text{min}})(1 - \tanh t_c)(1 - \tanh^2 \Delta t_0)}{(v_{\text{min}}(1 - \tanh \Delta t_0) + v_{\text{max}}(1 - \tanh t_c) \tanh \Delta t_0)^2}.
\]

(6)

This inequality will allow us to construct a phase diagram (Fig. \[1\]) similar to that of Fig. 8 in Ref. \[2\], where we vary the loading rate \( \lambda \gamma \) and the initial spacing between buses \( \Delta t_0 \).

Depending on the choice of parameters, there are actually three general forms this phase diagram can take. The first, corresponding to Nagatani’s analytical results, occurs when the lower bound on \( \lambda \gamma \) lies entirely below the \( \Delta t_0 \)-axis, making it irrelevant since the loading rate \( \lambda \gamma \) is always non-negative. After a straightforward calculation, one can show that the lower boundary curve reaches its maximum for that value \( \Delta t_0 \) which satisfies

\[
v_{\text{min}}(1 - \tanh \Delta t_0) = v_{\text{max}}(1 - \tanh t_c).
\]

(7)

From this we find that the lower boundary never rises above the axis when

\[
v_{\text{min}}^2 - [v_{\text{min}} - v_{\text{max}}(1 - \tanh t_c)]^2 > L(v_{\text{max}} - v_{\text{min}})(1 - \tanh t_c)
\]

(8)

This type of phase diagram is seen in Figure \[3a\]. When Eq. \[8\] does not hold, then the lower bound in Eq. \[3\] rises above the \( \Delta t_0 \) axis, and one gets a phase diagram more like that in Figure \[3b\].

Notice from Eq. \[7\] that the boundary curve has no maximum when \( v_{\text{min}} = 0 \). Figure \[3\] shows the phase diagram for this case. The similarity between Fig. \[3\] and the simulation results in Fig. 3 of Ref. \[2\] is not unexpected, since Nagatani does not use a minimum velocity in this paper. The puzzle is why the phase diagram of his stability analysis is more similar to Figure \[3a\]; perhaps he used \( v_{\text{min}} \neq 0 \) in his analysis.

IV. SIMULATION

To test our theory, we evaluate Eq. \[2\] iteratively in \( s \). Our initial conditions are

\[
\Delta t_{j,0} = \Delta t_0 \pm 0.1r_j
\]

(9)

where \( r_j \) are random numbers chosen between \(-1\) and \(1\). We use periodic boundary conditions in bus number; so for example \( \Delta t_{N,s} = t_{N,s} - t_{1,s} \). By stop \( s = 1000 \), our simulations each have one of four different outcomes: each outcome is represented by a different symbol in Fig. \[4\], and an example of each can be found in Fig. \[5\]. Many choices of parameters result in two or more buses being clumped together (\( \Delta t_i = 0 \) for at least one value of \( i \)); these are
FIG. 1: Phase diagrams for three different sets of parameter values. The curves are given by $L \frac{V'(\Delta t_0)}{V(\Delta t_0)}$ and $L \frac{V'(\Delta t_0)}{V(\Delta t_0)^2} - 1$, which theory predicts will bound the stable regions of phase space. The symbols represent simulation results: the small dots (·) label those simulation runs which are stable. The crosses (×) are those runs which end up with one or more headways equal to zero. The stars (∗) mark those runs which end in a “kink-jam” phase. The boxes (□), which only appear on the left-hand side of Figure b, mark those runs whose headways are roughly homogeneous but higher than the initial headway $\Delta t_0$. (See Fig. 2 for examples of these four possible outcomes.)

In summary, when one takes into consideration the minimum velocity $v_{\text{min}}$ and the fact that there are three possible phase diagrams for this bus system, one is able to reconcile Nagatani’s simulations with his stability analysis to a high degree.

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FIG. 2: Examples of possible outcomes of our simulation. The plot (a) shows a typical stable result, with some fluctuation about the initial value 1.5 (in this case). Plot (b) shows an example of those runs with many buses bunched together; note that this particular (not uncommon) result is completely unphysical, and arises because we have gone far from equilibrium. Plot (c) shows a typical “kink-jam” phase; such phases are marked by having at least one gradient $|\Delta t_{j,s} - \Delta t_{j,s-1}|$ greater than half the distance between maximum and minimum. The last plot, plot (d), looks similar to the stable case, but the headways have all drifted to be much larger than the initial value of 0.5. To be specific, we classified such runs as “unstable” if no headway came within 0.1 of the initial headway $\Delta t_0$.

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