NON-SQUARE DOUBLY STOCHASTIC MATRICES WITH SMALL SUPPORT

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Abstract. An $n \times m$ non-negative matrix with row sum $m$ and column sum $n$ is called doubly stochastic. We answer the problem of finding doubly stochastic matrices of smallest possible support for every $1 < n \leq m$. Any matrix of minimum support is extremal in the sense of convexity, while examples of extremal matrices that are not of minimum support are given. But when $n,m$ are coprime integers extremal matrices are precisely those of minimum support.

1. Introduction

According to the definition given by Caron, et al. in [Car96] an $n \times m$ matrix $A = (a_{i,j})$ with $a_{i,j} \geq 0$ is called doubly stochastic (with uniform marginals) if

$$\sum_{i=1}^{n} a_{i,j} = n \quad \text{for all } j = 1, \cdots, m$$

$$\sum_{j=1}^{m} a_{i,j} = m \quad \text{for all } i = 1, \cdots, n$$

The set of all $n \times m$ doubly stochastic matrices is denoted by $M(n,m)$. Furthermore, two matrices in $M(n,m)$ are called equivalent if one can be transformed into the other by permuting rows and columns.

We should mention here that the above definition differs slightly from the usual definition for square doubly stochastic matrices. The common definition for $M(n,n)$ requires the matrices to have nonnegative entries and all row and column sums equal to 1. These matrices have been studied extensively, see for example Chap. 2 in [Mar11].

A matrix $M \in M(n,m)$ is called extremal if it cannot be represented as a convex combination of other doubly stochastic matrices different from $M$, that is, $M$ is an extremal element in the convex set $M(n,m)$. For square $n \times n$ matrices, a full characterization of the extremal matrices in $M(n,n)$ is known by a classical result due to G. Birkhoff [Bir46], that we state here, using the notation of [Car96] that we have adopted.

Birkhoff’s Theorem: $M \in M(n,n)$ is extremal if and only if $\frac{1}{n}M$ is a permutation matrix. That is, $M \in M(n,n)$ is extremal if and only if $\frac{1}{n}M$ is equivalent to $I_n$, the identity matrix.

Several types of characterization of the extremal doubly stochastic matrices in $M(n,m)$ exist using either a matrix representation in some normal form, graph theory or faces of polyhedra, just to mention a few. (The interested reader could look at the list presented in the introduction of [Car96]).

Li, et al, in [Li96] have characterized extremal matrices using their support, that is, the set of their nonzero entries. In particular, they proved that a matrix $M \in M(n,m)$ is extremal if and only if its support $\text{supp}(M)$ is unique in the set $\{\text{supp}(A) \mid A \in M(n,m)\}$ (Theorem 1 in [Li96]).

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In addition, the support of a doubly stochastic matrix has attracted the attention of Kolountzakis and Papageorgiou [KP21] in relation with some tiling problems. If one views a \( n \times m \) matrix as a function \( f \) on the product of cyclic groups 

\[ G = \mathbb{Z}_n \times \mathbb{Z}_m \]

then, with the subgroups 

\[ G_1 = \mathbb{Z}_n \times \{0\}, \quad G_2 = \{0\} \times \mathbb{Z}_m, \]

the constant row sum and the constant column sum properties of the matrix are written as

\[
\sum_{g \in G_2} f(x - g) = m, \quad \sum_{g \in G_1} f(x - g) = n,
\]

respectively, valid for all \( x \in G \). In this language one seeks a nonnegative function \( f \) on \( G \), of as small a support as possible, which tiles simultaneously with the set of translates \( G_1 \) as well as \( G_2 \) (see [KP21] for a more precise definition).

These problems, of tiling simultaneously with various subgroups, derive [Kol97] from a classical problem of Steinhaus who asked if there is a subset of the plane which tiles simultaneously with all rotates of the lattice \( \mathbb{Z}^2 \). This problem is still very much open in case one asks for a measurable subset of the plane [KW99] but the answer is known to be affirmative without the measurability requirement [JM02]. Interestingly, in dimension 3 and higher the situation is the exact opposite: no measurable Steinhaus sets exist [KW99, KP02] but we do not know if such sets exist if we drop measurability [JM02]. In [KW99] the problem was first investigated of how to find a function \( f \) (as opposed to indicator function for Steinhaus sets) on the plane which tiles simultaneously with a finite set of rotates of \( \mathbb{Z}^2 \) and whose support has small \( \text{diameter} \). This problem was continued in [KP21] by examining the problem in a more general finite abelian group setting, the prototype of which is to ask for a function \( f \) on \( G \) satisfying (1.1) and has small support.

In [KP21] the quantity \( S(n, m) \) was defined as follows.

**Definition 1.1.** \( S(n, m) = \min \{|\text{supp } A| : A \in \mathcal{M}(n, m)\} \).

The matrices \( M \in \mathcal{M}(n, m) \) with \( |\text{supp } M| = S(n, m) \) are called minimum matrices in \( \mathcal{M}(n, m) \).

It was shown (see Theorem 4.3 and Lemma 4.5 in [KP21]) that \( S(n, kn) = kn \) while \( S(n, kn + 1) = (k + 1)n \). In addition, a question has been raised about the value of \( S(n, kn + r) \) for \( 1 < r < m \). Our main theorem in this short note gives a complete answer to Question 7 in [KP21] and states the following.

**Theorem I.** For all integers \( 1 < n, m \), we have \( S(n, m) = n + m - \gcd(n, m) \)

According to Corollary 2 in [Li96] a matrix \( A \in \mathcal{M}(n, m) \) is not extremal if and only if there exists \( B \in \mathcal{M}(n, m) \) with \( \text{supp } B \subset \text{supp } A \). Hence every minimum matrix in \( \mathcal{M}(n, m) \) is also extremal. This gives an easy way to verify that a matrix is extremal just by looking at the size of its support, if this happens to be minimum. But there are extremal matrices that are not minimum (some examples are given at the end of this note) so the condition on the size of the support is only sufficient. Nevertheless, when \( n, m \) are coprime integers it is also necessary as the next result states.

**Theorem II.** Let \( n, m \) be coprime integers. Then \( M \in \mathcal{M}(n, m) \) is extremal if and only if \( M \) is minimum. That is, \( M \) is extremal if and only if

\[ |\text{supp } M| = n + m - 1 \]
The rest of the paper contains a method to construct minimum matrices in \( M(n, m) \). In addition, a family of examples of extremal matrices whose size of support is one more than the minimum is constructed. Finally, a few more examples of matrices are given as counterexamples to possible generalizations.

2. Main Results

We start with a method to produce minimum doubly stochastic matrices of size \( n \times m \) for all integers \( 1 \leq n \leq m \).

Is it already known (see Proposition 4 in [Car96]), that in the case \( m = kn \) the matrix \( E(n, kn) \in M(n, kn) \) defined as

\[
E(n, kn) = \begin{pmatrix}
\begin{array}{ccc}
\bar{k} \\
\cdots & n \cdots & n \\
\cdots & \cdots & \cdots \\
\cdots & \cdots & \bar{k} \\
\end{array}
\end{pmatrix}
\]

is an extremal matrix of size \( n \times kn \). Furthermore, \( E(n, kn) \) is minimum since it has exactly one element per column. So, \( |\text{supp}(E(n, kn))| = kn = S(n, kn) \).

Assume now that \( n, m \in \mathbb{N} \) are given with \( 1 < n \leq m \). We use the Euclidean algorithm applied to \( n, m \) to produce as many extremal matrices of type (2.1) as the steps of the algorithm. That is, assume that the Euclidean algorithm goes as follows:

\[
m = k_1n + r_1 \\
n = k_2r_1 + r_2 \\
r_1 = k_3r_2 + r_3 \\
\vdots \\
r_{t-2} = k_tr_{t-1} + r_t \\
r_{t-1} = k_{t+1}r_t
\]

Then at every step we produce the matrices \( E(n, k_1n), E(r_1, k_2r_1), \ldots, E(r_t, k_{t+1}r_t) \). We put them together in a block form to make an \( n \times m \) matrix \( F(n, m) \) as follows.

\[
F(n, m) = \begin{pmatrix}
E(n, k_1n) & \cdots & E(r_1, k_2r_1)^T \\
\cdots & \cdots & \cdots \\
E(r_2, k_3r_2) & \cdots & E(r_3, k_4r_3)^T \\
\cdots & \cdots & \cdots \\
E(r_4, k_5r_4) & \cdots & \cdots \\
\cdots & \cdots & \cdots \\
E(r_{t-2}, k_{t+1}r_{t-1}) & \cdots & E(r_{t-1}, k_{t+1}r_{t-1})^T \\
E(r_t, k_{t+1}r_t) & \cdots & B
\end{pmatrix}
\]

were \( B = \begin{pmatrix}
E(r_t, k_{t+1}r_t)^T \\
E(r_t, k_{t+1}r_t)
\end{pmatrix} \) if \( t \) is odd
\( \begin{pmatrix}
E(r_t, k_{t+1}r_t)^T \\
E(r_t, k_{t+1}r_t)
\end{pmatrix} \) if \( t \) is even
To clarify our method we compute $F(8, 27)$. The Euclidean Algorithm for $(8, 27)$ is

$$
27 = 3 \cdot 8 + 3 \\
8 = 2 \cdot 3 + 2 \\
3 = 1 \cdot 2 + 1 \\
2 = 2 \cdot 1
$$

Hence we form the matrices

$$
E(8, 3 \cdot 8) = \begin{bmatrix}
3 \\
8 \cdot 8 \\
\vdots \\
\vdots \\
\vdots \\
\vdots \\
3 \\
8 \cdot 8
\end{bmatrix}, \\
E(3, 2 \cdot 3)^T = \begin{bmatrix}
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3 \\
3
\end{bmatrix}, \\
E(2, 1 \cdot 2) = \begin{bmatrix}
2 \\
2
\end{bmatrix}, \\
E(1, 2 \cdot 1)^T = \begin{bmatrix}
1 \\
1
\end{bmatrix}
$$

Putting them together we get

$$
F(8, 27) = \begin{bmatrix}
8 & 8 & 8 & 3 \\
8 & 8 & 8 & 3 \\
\vdots & \vdots & \vdots & 3 \\
\vdots & \vdots & \vdots & 3 \\
\vdots & \vdots & \vdots & 3 \\
\vdots & \vdots & \vdots & 3 \\
8 & 8 & 8 & 2 & 1 \\
8 & 8 & 8 & 2 & 1
\end{bmatrix}
$$

We are ready now to prove Theorem I that we restate using the matrices $F(n, m)$.

**Lemma 2.1.** The matrices $F(n, m)$ are minimum and thus extremal in $M(n, m)$. In addition,

$$
S(n, m) = |\text{supp } F(n, m)| = n + m - \gcd(n, m).
$$

**Proof.** As we have already observed, every minimum matrix is also extremal. To show that $F(n, m)$ is minimum we induct on the number of steps needed to complete the Euclidean Algorithm. Note that in view of our notation above, this number is $t + 1$. If $t = 0$, that is $m = kn$, the matrix $E(n, m)$ is of minimum support. So our induction begins.

For the inductive step observe that if the Euclidean algorithm starts with $m = k_1n + r_1$ our construction guarantees that $F(n, m)$ is the sum of the following two matrices, whose blocks are associated with the same column partition

$$
A = \begin{pmatrix}
E(n, k_1n) & 0 \\
\vdots & \vdots
\end{pmatrix} \text{ and } B = \begin{pmatrix}
0 & F(r_1, n)^T
\end{pmatrix}
$$

As $F(n, m) = A + B$ its first $k_1n$ columns are those of $A$ i.e. those of $E(n, k_1n)$. But $E(n, k_1n)$ has exactly one non-zero entry in every column which is the least number
Hence we conclude that for every extremal matrix \( M \) we have
\[
F(n, m) = \left( \begin{array}{c|c}
E(n, k_1n) & F(r_1, n)^T \\
\end{array} \right)
\]
is minimum if \( F(r_1, n) \) is minimum in \( M(r_1, n) \). The steps needed in the Euclidean algorithm for \( (r_1, n) \) are one less than those needed for the pair \((n, m)\). Hence the inductive hypothesis implies that \( F(r_1, n) \) is minimum in \( M(r_1, n) \). Therefore, \( F(n, m) \) is minimum in \( M(n, m) \) and \( S(n, m) = |\text{supp} \ M(n, m)| \).

To compute \(|\text{supp} \ F(n, m)|\) we note that in view of (2.2) and the way \( F(n, m) \) is constructed we get
\[
|\text{supp} \ F(n, m)| = |\text{supp} E(n, k_1n)| + |\text{supp} E(r_1, k_2r_1)| + \cdots + |\text{supp} E(r_t, k_{t+1}r_t)|
\]
\[
= k_1n + k_2r_1 + \cdots + k_tr_{t-1} + k_{t+1}r_t
\]
\[
= m - r_1 + n - r_2 + \cdots + r_{t-2} - r_t + r_{t-1}
\]
\[
= m + n - r_t
\]

But the last non zero remainder in the Euclidean Algorithm (that is \( r_t \)) is the greatest common divisor of \((n, m)\). This completes the proof of the theorem.

According to Proposition 4 in [Car96], when \( r = 0 \), that is \( m = kn \), all the extremal matrices in \( M(n, kn) \) are equivalent to \( E(n, kn) \) and thus are minimum. Hence, minimum and extremal matrices coincide in \( M(n, kn) \).

Furthermore, according to Proposition 6 of [Car96], the same holds when \( r = 1 \). That is, if \( m = kn + 1 \), every extremal matrix \( M \) in \( M(n, m) \) satisfies
\[
|\text{supp} \ M| = (k + 1)n = n + m - 1 = S(n, m)
\]
and thus \( M \) is extremal if and only if \( m \) is minimum.

This neat characterization of extremal matrices does not hold for \( r > 1 \) in general. A counterexample is given by the extremal \( 4 \times 6 \) matrix
\[
(2.3) \quad T = \begin{pmatrix}
2 & 2 & 2 \\
2 & 4 & 2 \\
2 & 4 & 2
\end{pmatrix}
\]
whose support contains 9 non zero entries while \( S(4, 6) = 8 \). One can check that the matrix is extremal using, for example, Proposition 2 in [Car96].

Nevertheless, when \( \gcd(n, m) = 1 \), extremal and minimum matrices in \( M(n, m) \) coincide. This is our Theorem II, that we are now ready to prove.

**Proof of Theorem II.** In view of Theorem 5 in [Car96] every extremal matrix \( M \in M(n, m) \) (with \( m = kn + r \) ) is equivalent to the sum of two matrices \( M_B \) and \( M_R \), where every row of \( M_B \) has exactly \( k \) positive entries while \( M_B \) has at most \( r - 1 \) positive entries. Hence every extremal matrix \( M \in M(n, m) \) satisfies
\[
(k + 1)n \leq |\text{supp} \ M| \leq (k + 1)n + (r - 1).
\]
On the other hand, if \( \gcd(n, m) = 1 \) and \( m = kn + r \) we get
\[
S(n, m) = n + m - 1 = (k + 1)n + (r - 1).
\]
We conclude that for every extremal \( M \) we have
\[
S(n, m) \leq |\text{supp} \ M| \leq (k + 1)n + (r - 1) = S(n, m)
\]
Hence \( |\text{supp} \ M| = n + m - 1 \) and the proposition follows.

\[ \blacksquare \]
The matrix $T$ in (2.3) is not the only example of an extremal matrix that is not minimum, but it is of smallest dimensions. Actually, we can produce arbitrarily large extremal non-minimum matrices as the next result states.

**Theorem III.** For every pair of integers $n, m$ that satisfy

$$m = k_1n + d \text{ where } n > d > 1 \text{ and } n = k_2d,$$

there exist an extremal matrix in $M(n, m)$ that is not minimum.

For its proof we will use a characterization of extremal matrices using their associated graphs given by Brualdi [Bru68]. We first define the associated graph $G(A)$ of any $n \times m$ matrix $A = (a_{i,j})$ with $a_{i,j} \geq 0$ as follows. For every row $i$ and every column $j$ we get a node $x_i$ and $y_j$ respectively, for $1 \leq i \leq n$ and $1 \leq j \leq m$. There is an edge joining $x_i$ and $y_j$ if and only if $a_{i,j} > 0$. Then the following theorem holds, see [Bru68] and [Bru76].

**Theorem:** A matrix $M \in M(n, m)$ is extremal if and only if the connected components of $G(M)$ are trees. Equivalently, $G(M)$ has no cycles.

We are now ready to prove Theorem III.

**Proof.** Assume $n, m$ are as above then $\gcd(n, m) = d$ while $S(n, m) = n + m - d$. The Euclidean Algorithm stops in two steps and our method produces

$$F(n, m) = \left( E(n, k_1n) \mid E(d, k_2d)^T \right)$$

which is equivalent to the following matrix in block form

$$X = \begin{bmatrix} B \\ \vdots \\ B \end{bmatrix}$$

where every block $B$ is

$$B = \begin{bmatrix} \begin{array}{ccc} k_1 \\ d & n \cdots n \\ \vdots & & \ddots \\ d & & \end{array} & \begin{array}{ccc} k_1 \\ d & n \cdots n \\ \vdots & & \ddots \\ d & & \end{array} \end{bmatrix}$$

$k_2$ rows

$k_1$\text{k}_2+1 columns

As $d \geq 2$ there exist at least two blocks in the matrix $X$. We replace the first two $B$-blocks in $X$ with the $2k_2 \times 2(k_1k_2 + 1)$ matrix
So we get

\[
Y = \begin{pmatrix} C & B \\ \vdots & \ddots & \ddots & \ddots \\ & & B \\ \end{pmatrix}_\text{d-2 B-blocks}
\]

Observe that the matrix $T$ in (2.3) is a special case of $Y$ when $d = 2 = k_2$ and $k_1 = 1$. Clearly $Y$ is not minimum as

\[
|\text{supp } C| = k_1 + 2 + (k_1 + 1) \cdot (2k_2 - 1) = 2k_2(k_1 + 1) + 1 = 2|\text{supp } B| + 1
\]

and therefore

\[
|\text{supp } Y| = |\text{supp } C| + (d - 2)|\text{supp } B| = d|\text{supp } B| + 1 = |\text{supp } X| + 1
\]

It remains to show that $Y$ is extremal. $Y$ is defined as a direct sum of the block matrices $C$ and $(d - 2)$-copies of $B$. Each one of those blocks contributes to the graph $\mathcal{G}(Y)$ one or more connected components. Clearly those components that are associated with $B$ are trees. (This can be seen either directly from the matrix $B$ or from the fact that $X$ is extremal and $X$ is a direct sum of $d$ blocks, all equal to $B$.)

We conclude that $Y$ is extremal if and only if the associated graph $\mathcal{G}(C)$ of $C$ is a tree. Which is indeed so, as the graph $\mathcal{G}(C)$ is
Hence $Y$ is extremal and the theorem follows.

We conclude this note with a few more examples of matrices that serve as counterexamples to possible generalizations of the results mentioned.

Remark 1. The matrix

$$F = \begin{pmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

is an element of $\mathcal{M}(3, 4)$ with $\left| \text{supp} \ F \right| = 7 = S(3, 4) + 1$ but it is not extremal. Hence it is not the case that any doubly stochastic matrix in $\mathcal{M}(n, m)$ whose support is just one above the minimum support of $\mathcal{M}(n, m)$ must be extremal.

Remark 2. Clearly a possible generalization of Birkhoff’s theorem to non-square doubly stochastic matrices, stating that any two extremal matrices in $\mathcal{M}(n, m)$ are equivalent fails. This can be easily seen as there exist plenty of examples of extremal matrices $A, B \in \mathcal{M}(n, m)$ with $\left| \text{supp} \ A \right| \neq \left| \text{supp} \ B \right|$. As minimum and extremal matrices coincide in $\mathcal{M}(n, n)$, we entertained the idea that, maybe, any two minimum matrices in $\mathcal{M}(n, m)$ are equivalent. (If this were true Birkhoff’s theorem would be a special case.) But this fails too, as the next two minimum matrices in $\mathcal{M}(4, 5)$ prove.

$$\begin{pmatrix} 1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 1 \\ 3 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 4 & 0 & 0 & 0 & 1 \\ 0 & 4 & 0 & 0 & 1 \\ 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & 0 & 4 & 1 \end{pmatrix}$$

Nevertheless, we have not managed, so far, to produce two minimum matrices whose set of entries are equal (counting multiplicities) without being equivalent. We should mention here that the way $\mathcal{F}(n, m)$ are constructed ensures that the entries of $\mathcal{F}(n, m)$ are $\{n, r_1, r_2, \ldots, r_t\}$ (using the notation in (2.2)) appearing with multiplicities $\{k_1n, k_2r_1, k_3r_2, \ldots, k_{t+1}r_t\}$, respectively.
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