The canonical involution in the space of connections of a $(J^2 = ±1)$-metric manifold

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Abstract

A $(J^2 = ±1)$-metric manifold has an almost complex or almost product structure $J$ and a compatible metric $g$. We show that there exists a canonical involution in the set of connections on such a manifold, which allows to define a projection over the set of connections adapted to $J$. This projection sends the Levi Civita connection onto the first canonical connection. In the almost Hermitian case, it also sends the $\nabla^c$ connection onto the Chern connection, thus applying the line of metric connections defined by $\nabla^c$ and the Levi Civita connections onto the line of canonical connections. Besides, it moves metric connections onto metric connections.

2010 Mathematics Subject Classification: 53C15, 53C05, 53C50, 53C07.

Keywords: $(J^2 = ±1)$-metric manifold, first canonical connection, Chern connection, connection with totally skew-symmetric torsion, canonical connection.

1 Introduction

In the celebrated paper [9] of Gauduchon, connections on almost Hermitian manifolds are studied, focusing on the 1-parameter family of canonical connections, which is defined as

$$\nabla^t = (1 - t)\nabla^0 + t\nabla^c, \quad \forall t \in \mathbb{R},$$

where $\nabla^0$ denotes the first canonical connection and $\nabla^c$ the Chern connection. It is also said that $\nabla^0$ is the orthogonal projection of the Levi Civita connection $\nabla^g$ onto the affine space of Hermitian connections. As is well known, both connections coincide in the Kähler case.

The purpose of this note is threefold: (1) we want to extend the above result to all the $(J^2 = ±1)$-metric manifolds, i.e., manifolds endowed with an almost complex or almost product structure $J$ and a compatible metric $g$, (2) we will show, in the almost Hermitian context, that

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the connection $\nabla^-$ projects onto the Chern connection, and (3) we will prove that the projection moves metric connections onto metric connections.

Connections with totally skew-symmetric torsion are very useful in Physics (see, e.g., [1, 8] and the references therein). In particular, connections $\nabla^+$ and $\nabla^-$ appear in heterotic string theory (see, e.g., [7, 11] and the references therein). Having a non-Kähler manifold of type $G_1$ in the classification of almost Hermitian manifolds of Gray and Hervella [10], there exists a unique Hermitian connection $\nabla^{sk}$ with totally skew-symmetric torsion (see [5, Theor. 5.21]). Then one can define the connections

$$\nabla^X_Y = \nabla^g_X Y + \frac{1}{2}T^{sk}(X, Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

$$\nabla^X_Y = \nabla^g_X Y - \frac{1}{2}T^{sk}(X, Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

where $T^{sk}$ denotes the torsion tensor of $\nabla^{sk}$. We will show that $\nabla^+$ is invariant under the projection and that $\nabla^-$ projects onto the Chern connection.

We will consider smooth manifolds and operators being of class $C^\infty$. As in this introduction, $\mathfrak{X}(M)$ denotes the module of vector fields of a manifold $M$.

2 Canonical connections

We are dealing with all the four geometries: almost Hermitian, almost Norden, almost product Riemannian and almost para-Hermitian, which correspond to the cases

$$(\alpha, \varepsilon) \in \{(-1, 1), (-1, -1), (1, 1), (1, -1)\}$$

in the following

**Definition 2.1 ([6, Defin. 3.1])** Let $M$ be a manifold, $g$ a semi-Riemannian metric on $M$, $J$ a tensor field of type $(1,1)$ and $\alpha, \varepsilon \in \{-1, 1\}$. Then $(J, g)$ is called an $(\alpha, \varepsilon)$-structure on $M$ if

$$J^2 = \alpha \text{Id}, \quad \text{trace } J = 0, \quad g(JX, JY) = \varepsilon g(X, Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

g being a Riemannian metric if $\varepsilon = 1$. Then $(M, J, g)$ is called a $(J^2 = \pm 1)$-metric manifold.

Condition $\text{trace } J = 0$ is a consequence of the other conditions in all the cases unless the $(1, 1)$. We impose it in this case looking for a common treatment of all the four geometric structures. See [6] for a more complete description.

A linear connection $\nabla$ is said to be adapted to the metric $g$ (resp. to $J$) if $\nabla g = 0$ (resp. $\nabla J = 0$). We use the following notation: $\mathcal{C}(M)$ (resp. $\mathcal{C}(M, J)$, $\mathcal{C}(M, g)$, $\mathcal{C}(M, J, g) = \mathcal{C}(M, J) \cap \mathcal{C}(M, g)$) denotes the affine space of linear connections on $M$ (resp. adapted to $J$, to $g$ and
to both $J$ and $g$). In [3] we have studied a lot of distinguished connections defined in such a manifold. Then $\nabla^g \in C(M, J, g)$ if and only if $(M, J, g)$ is of Kähler type. We are interested in the non Kähler type case. Then two adapted connections will be essential in our study: the first canonical connection $\nabla^0$ and the Chern connection $\nabla^c$.

Connection $\nabla^0$ has been introduced in [5] as

**Definition 2.2** Let $(M, J, g)$ be a $(J^2 = \pm 1)$-metric manifold. The first canonical connection of $(M, J, g)$ is the linear connection having the covariant derivative $\nabla^0$ given by

$$\nabla^0_X Y = \nabla_X Y + \frac{(-\alpha)}{2} (\nabla_X J) J Y, \quad \forall X, Y \in \mathfrak{X}(M).$$

The previous one generalizes the classical definition given in the context of almost Hermitian manifolds (see, e.g., [9]).

The Chern connection was firstly introduced in the case of Hermitian manifolds [3]. In [6] we have extended the connection to the almost para-Hermitian case, recovering the connection defined by Cruceanu and one of us in [4]. The following results establish the existence and uniqueness of the Chern connection on a $(J^2 = \pm 1)$-metric manifold with $\alpha \varepsilon = -1$.

**Theorem 2.3 ([6, Theor. 6.3])** Let $(M, J, g)$ be a $(J^2 = \pm 1)$-metric manifold with $\alpha \varepsilon = -1$. Then there exists a unique linear connection on $M$ adapted to $(J, g)$ defined by $(J, g)$ whose torsion tensor $T^c$ satisfies the following condition

$$T^c(JX, JY) = \alpha T^c(X, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

This connection is called the Chern connection of $(M, J, g)$.

**Remark 1** The Chern connection can not be defined in the $\alpha \varepsilon = 1$ context, as we have proved in [6, Remark 6.4]. In [5] we have proved that in the case $\alpha \varepsilon = -1$, the so-called well adapted connection $\nabla^w$ is also a canonical connection, i.e., is a connection in the line defined in [4]. Then this line can be parametrized as $\nabla^s = (1 - s)\nabla^0 + s\nabla^w, \forall s \in \mathbb{R}$. As the first canonical connection and the well adapted connection can be also defined in the case $\alpha \varepsilon = 1$, we have been able to define canonical connections on any $(J^2 = \pm 1)$-metric manifold $(M, J, g)$.

**Remark 2** (1) We have seen in [5, Remark 6.2], assuming $\alpha \varepsilon = -1$, that the Chern connection corresponds to the case $s = 3$ and in [5, Example 6.3] that the Bismut connection $\nabla^b$ (see [2]) to $s = -3$. This connection coincides, if there exists, with the unique adapted connection with totally skew-symmetric torsion. In the almost Hermitian case the well adapted connection ($s = 1$) coincides with the connection of minimal torsion defined by Gauduchon in [9].

(2) In the case $\alpha \varepsilon = 1$, if $J$ is a non-integrable $\alpha$-structure (which is equivalent to $\nabla^0 \neq \nabla^w$ according to [5, Theor. 5.6]), and $(M, J, g)$ is a quasi-Kähler type manifold, then there exists a unique canonical connection with totally skew-symmetric torsion, which is that given by $s = -1$.  

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3 Canonical involution and projection of connections

First of all, we will need the following well known results of Affine Geometry.

• A subset of an affine space is an affine subspace if and only if the line joining any pair of points of the subset is contained in the subset.

• A map between affine spaces is an affine map if and only if it preserves barycentric combinations.

• An involutive affine map $\sigma$ in an affine space defines a projection $\pi = \frac{1}{2}Id + \frac{1}{2}\sigma$ onto the subspace of fixed points of $\sigma$. The map $\pi$ is also an affine map.

Taking into account the above properties one easily checks that $C(M)$, $C(M,J)$, $C(M,g)$, $C(M,J,g)$ and the 1-parameter family of canonical connections are affine spaces. We introduce the following

Definition 3.1 Let $M$ be a manifold endowed with a tensor field $J$ of type $(1,1)$ such that $J^2 = \alpha Id$, where $\alpha \in \{-1,1\}$. The map $J_* : C(M) \rightarrow C(M)$ defined as

$$(J_*(\nabla))_X(Y) = \alpha J(\nabla_X(JY)), \quad \forall X, Y \in \mathfrak{X}(M),$$

is called the canonical involution induced by $J$ in the affine space of connections $C(M)$.

Then we have:

Proposition 3.2 Let $M$ be a manifold endowed with a tensor field $J$ of type $(1,1)$ such that $J^2 = \alpha Id$, where $\alpha \in \{-1,1\}$.

1. The map $J_* : C(M) \rightarrow C(M)$ is an involutive affine isomorphism.

2. $C(M,J) = \{\nabla \in C(M) : J_\ast(\nabla) = \nabla\}$.

3. For each $\nabla \in C(M)$, the connection

$$\pi(\nabla) = \frac{1}{2}\nabla + \frac{1}{2}J_* (\nabla)$$

is $J$-invariant, so defining a projection $\pi : C(M) \rightarrow C(M,J)$.

4. In addition, if $(M, J, g)$ is a $(J^2 = \pm 1)$-metric manifold, then $\pi(\nabla^g) = \nabla^0$.

Proof.
1. Direct calculations show that $J_*(\nabla)$ is a covariant derivative and 

$$J_*(\lambda_1 \nabla^{(1)} + \ldots + \lambda_k \nabla^{(k)}) = \lambda_1 J_*(\nabla^{(1)}) + \ldots + \lambda_k J_*(\nabla^{(k)})$$

when $\lambda_1 + \ldots + \lambda_k = 1$, thus proving $J_*$ is an affine map. Besides, given $X, Y$ vector fields on $M$ one has:

$$(J_*(J_*(\nabla)))_X(Y) = \alpha J((J_*(\nabla))_X(JY)) = \alpha J(\alpha J(\nabla X(JY))) = \alpha^4 \nabla X Y = \nabla X Y,$$

thus proving $J_*$ is an involutive affine isomorphism.

2. Let $\nabla \in \mathcal{C}(M, J)$. Then

$$(J_*(\nabla))_X(Y) = \alpha J(\nabla_X(JY)) = \alpha J^2(\nabla_X Y) = \alpha^2 \nabla_X Y = \nabla_X Y, \quad \forall X, Y \in \mathfrak{X}(M).$$

The reverse: Suppose that $J_*(\nabla) = \nabla$ then

$$\nabla X Y = (J_*(\nabla))_X(Y) = \alpha J(\nabla_X(JY)), \quad \forall X, Y \in \mathfrak{X}(M),$$

therefore

$$J(\nabla_X Y) = J(\alpha J(\nabla_X(JY))) = \alpha J^2(\nabla_X(JY)) = \nabla_X(JY), \quad \forall X, Y \in \mathfrak{X}(M).$$

3. Taking into account the above items one has:

$$J_*(\pi(\nabla)) = J_*(\frac{1}{2} \nabla + \frac{1}{2} J_*(\nabla)) = \frac{1}{2} J_*(\nabla) + \frac{1}{2} J_*(J_*(\nabla)) = \pi(\nabla).$$

4. Let $X, Y$ be vector fields on $M$, a straightforward calculation is enough:

$$\nabla^0_X Y = \nabla^g_X Y + \frac{(-\alpha)}{2}(\nabla^g_X JY) = \nabla^g_X Y + \frac{(-\alpha)}{2}(\nabla^g_X (J^2 Y) - J(\nabla^g_X (JY)))$$

$$\quad = \nabla^g_X Y - \frac{1}{2}(\nabla^g_X Y + \frac{1}{2} \alpha J(\nabla^g_X(JY))) = \frac{1}{2} \nabla^g_X Y + \frac{1}{2}(J_*(\nabla^g))_X Y = (\pi(\nabla^g))_X Y. \quad \square$$

**Remark 3** In the case of having a Kähler type manifold $(M, J, g)$ then $\nabla^0 = \nabla^g$. In the non-Kähler type case, $\nabla^0$ is an adapted connection to $(J, g)$ which is obtained as the projection of $\nabla^g$ to the set $\mathcal{C}(M, J)$. In fact, $\nabla^0 \in \mathcal{C}(M, J, g)$.

**Corollary 3.3** Let $(M, J, g)$ be an almost Hermitian non Kähler manifold of type $\mathcal{G}_1$. Then

$$\pi(\nabla^+) = \nabla^+, \quad \pi(\nabla^-) = \nabla^c.$$

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Proof. Taking into account Remark 2, $\nabla^+ = \nabla^{sk} = \nabla^b$ is the Bismut connection, which belongs to the line of canonical connections. As this line is contained in $\mathcal{C}(M, J)$, one obtains $\pi(\nabla^+) = \nabla^+$.

Observe that $\nabla^g = \frac{1}{2}\nabla^+ + \frac{1}{2}\nabla^-$. Then

$$\nabla^0 = \pi(\nabla^g) = \frac{1}{2}\pi(\nabla^+) + \frac{1}{2}\pi(\nabla^-) = \frac{1}{2}\nabla^b + \frac{1}{2}\pi(\nabla^-).$$

As $\nabla^0$ is the midpoint between $\nabla^b$ and $\nabla^c$, one obtains $\pi(\nabla^-) = \nabla^c$. □

In [11] the authors have studied the plane of connections defined by the line of canonical connections on an almost Hermitian non Kähler type manifold and the Levi Civita connection. This plane has another significant line defined by $\nabla^g, \nabla^+, \nabla^-$, when there exists a connection with totally skew-symmetric torsion. We have seen that the projection $\pi$ applies this line of connections onto the line of canonical connections.

4 Metric connections

Let $(M, J, g)$ be a $(J^2 = \pm 1)$-metric manifold and let $\nabla$ be a metric connection. We are going to prove that $\pi(\nabla)$ is also a metric connection. First of all, we obtain the following new expression of the projection $\pi$.

Lemma 4.1 Let $(M, J, g)$ be a $(J^2 = \pm 1)$-metric manifold. The projection $\pi : \mathcal{C}(M) \to \mathcal{C}(M, J)$ is given by $\pi(\nabla) = \nabla + S_\nabla$, where

$$S_\nabla(X, Y) = \frac{(-\alpha)}{2}(\nabla X)JY, \quad \forall X, Y \in \mathfrak{X}(M).$$

Proof. A direct calculus shows that:

$$\nabla X + S_\nabla(X, Y) = \nabla X + \frac{(-\alpha)}{2}(\nabla X)JY = \nabla X - \frac{1}{2}\nabla X + \frac{\alpha}{2}(J\nabla X(JY))$$

$$= \frac{1}{2}\nabla X + \frac{\alpha}{2}(J\nabla X(JY)) = (\pi(\nabla))\nabla X, \quad \forall X, Y \in \mathfrak{X}(M).$$ □

We need another lemma.

Lemma 4.2 Let $(M, J, g)$ be a $(J^2 = \pm 1)$-metric manifold, $\nabla$ a metric connection and let $S$ be a $(1, 2)$ tensor field. Then the connection $\nabla + S$ is metric if and only if

$$g(S(X, Y), Z) + g(S(X, Z), Y) = 0, \quad \forall X, Y, Z \in \mathfrak{X}(M).$$
Proof. Given $X, Y, Z$ vector fields on $M$, as $\nabla g = 0$, one has:

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(\nabla_X Z, Y).$$

Then $((\nabla + S)_X g)(Y, Z) = 0$ if and only if

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(S(X, Y), Z) + g(\nabla_X Z, Y) + g(S(X, Z), Y),$$

and thus one can conclude the result. □

**Proposition 4.3** Let $(M, J, g)$ be a $(J^2 = \pm 1)$-metric manifold and $\nabla$ a metric connection. Then $\pi(\nabla)$ is also a metric connection.

Proof. In order to apply the above lemma, given $X, Y, Z$ vector fields on $M$ we obtain:

$$g(S_{\nabla}(X, Y), Z) = \frac{(-\alpha)}{2} g((\nabla_X J)JY, Z)$$

$$= \frac{(-\alpha)}{2} (\alpha g(\nabla_X Y, Z) - g(J(\nabla_X (JY)), Z))$$

$$= \frac{(-\alpha)}{2} (\alpha g(\nabla_X Y, Z) - \alpha \varepsilon g(\nabla_X (JY), JZ))$$

$$g(S_{\nabla}(X, Z), Y) = \frac{(-\alpha)}{2} (\alpha g(\nabla_X Z, Y) - \alpha \varepsilon g(\nabla_X (JZ), JY)),$$

and then

$$g(S_{\nabla}(X, Y), Z) + g(S_{\nabla}(X, Z), Y) = -\frac{1}{2} (g(\nabla_X Y, Z) + g(\nabla_X Z, X))$$

$$+ \frac{\varepsilon}{2} (g(\nabla_X (JY), JZ) + g(\nabla_X (JZ), JY))$$

$$= \frac{1}{2} X(g(Y, Z)) + \frac{\varepsilon}{2} X(g(JY, JZ))$$

$$= -\frac{1}{2} X(g(Y, Z)) + \frac{1}{2} X(g(Y, Z)) = 0. \quad □$$

Thus, $\pi(C(M, g)) \subset C(M, g) \cap C(M, J) = C(M, J, g)$, moving metric connections onto connections adapted to $(J, g)$. In the case of the plane of connections considered in [11] in the almost Hermitian context, the plane remains globally invariant under the projection $\pi$, moving all the points to the line of canonical connections. More in general, $\pi$ moves metric connections onto Hermitian connections.

**Acknowledgments.** The authors are grateful to their colleagues L. Ugarte and R. Villacampa for their useful comments.
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