Stability Analysis of Compression Member on Elastic Springs

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Abstract. Part of the large parametric study focused on the stability behaviour of compression single members and the member structures with various loadings and boundary conditions is presented here. This part relates to the stability analysis of the compression member on elastic springs. The span of the member L = nℓ, where ℓ is the length of the field which equals to the distance between the neighbouring elastic springs. The members under investigation have seven various numbers of the fields n = 2, 3, 4, 5, 6, 7 and ∞. The members have uniform bending rigidity EI, uniform normal force N and (n – 1) elastic springs with equal spring stiffness Cw. The results are presented in the form of eight diagrams. The curves of diagrams were replaced by the straight lines to be able derived approximate formulae for calculation of the critical force. It is shown that approximate formulae give the values of the critical forces Fcr.a which differ from diagram values Fcr less than 3.3 %, being slightly on the safe side. The values of the critical forces Fcr obtained from the solution of the stability equation are verified by comparisons with results Fcr.IQ of the computer program IQ 100. Excellent agreement is achieved between Fcr.IQ and Fcr. The presented solution of the compression member on elastic springs may be used also for the calculation of the critical force of the compression member on elastic foundation. The solution given in the paper creates scientific background needed in the design of the semi-through bridge trusses.

1. Introduction
The paper describes behaviour of simply supported compression member at its ends and supported by n – 1 intermediate elastic springs. The span of the member L = nℓ, where ℓ is the length of the field which equals to the distance between the neighbouring elastic springs. The investigated members have various numbers of the fields n = 2, 3, 4, 5, 6, 7 and ∞.

The early investigations of the behaviour of the column transversely supported by elastic springs were reported by Engesser [1]. He assumed the equally spaced elastic supports having the same spring stiffness. The problem was later investigated by F. S. Jasinsky [2], S. P. Timoshenko [3], I. G. Bubnov [4], F. Bleich [5], P. F. Papkovitch [6], N. K. Sntiko [7], S. D. Lejtes [8] and others. They used different methods: e.g. Ritz method [2, 6], force method [4], mixed method (unknown quantities were force factors - moments and displacement factors – displacement of the elastic springs) [5], etc. We use the procedure described in [8].

The solution of this problem attracted a lot of famous scientists because it has important application in design of the truss bridges. Where the bridge spans are short, and underslung trusses are not possible, the semi-through trusses may be used. Bracing between the top chords is not possible and
restraint to the compression members has to be provided by U-frames. However for spans where semi-through trusses have been used in the past, plate girder bridges are now very competitive. In the case of semi-through truss bridges, the top chord is supported laterally by the diagonals and behaves as a strut supported on elastic springs. The method of determination of its effective length is given in the appropriate bridge codes, e.g. in Eurocode EN 1993-2 for design of steel bridges. The solutions below are scientific backgrounds for such application of the simply supported compression member on the intermediate elastic springs.

2. Critical force and buckling length of simply supported member on n-1 intermediate elastic springs

Solution of the problem leads to the relation between relative non-dimensional spring stiffness and the dimensionless member parameter \( \varepsilon \).

\[
\bar{C}_w = \frac{C_w \ell^3}{EI}, \quad \varepsilon = \sqrt{\frac{N}{EI \ell}}
\]  

(1)

where

- \( C_w \) is the spring stiffness [kN/m],
- \( EI \) is the bending stiffness of the member [kNm²],
- \( N \) is the compression axial force [kN],
- \( \ell \) is the length of the field, distance between the springs [m],
- \( \bar{C}_w \) is the relative non-dimensional spring stiffness [-],
- \( \varepsilon \) is the dimensionless member parameter [-].

The following stability equation may be used for the number of the fields \( n = 2, 3, 4, \ldots \infty \):

\[
\bar{C}_w = 2 \varepsilon^2 \Phi_n
\]  

(2)

where the function \( \Phi_n \) is

\[
\Phi_n(\varepsilon) = \frac{1 - \cos \varepsilon - \left(1 - \cos \frac{p\pi}{n}\right)}{1 - \cos \varepsilon - \left(1 - \frac{\sin \varepsilon}{\varepsilon}\right)}, \text{ for } n = 1, 2, 3, 4, \ldots, \quad \Phi_\infty(\varepsilon) = \frac{1 - \cos \varepsilon}{1 + \sqrt{\frac{\sin \varepsilon}{\varepsilon}}}, \text{ for } n = \infty
\]  

(3)

where

- \( p \) is the number of the half-sine waves, \( p = 1, 2, 3, 4, \ldots n \).

Results obtained from the large parametric study enable to calculate the following quantities:

\[
\beta = \frac{\pi}{\varepsilon}, \quad \ell_{cr} = \beta \ell, \quad F_{cr} = \frac{\pi^2 EI}{(\beta \ell)^2}, \quad F_E = \frac{\pi^2 EI}{\ell^2}, \quad \frac{F_{cr}}{F_E} = \frac{F_{cr} \ell^2}{\pi^2 EI}, \quad \frac{F_{cr} \ell^2}{\pi^2 EI} = \frac{1}{\beta^2}
\]  

(4)

The results of the parametric study are given in the graphical form in the figures 1, 2, 3, 4, 5, 6, 7, 8. The diagrams show the areas I, II, III, IV, V, VI, VII in which the value of the critical force \( F_{cr} \) is minimal and in which the buckling mode is defined by the number of the half-sine waves \( p = 1, 2, 3, 4, 5, 6, 7 \). The results for the investigated number of the fields \( n = 2, 3, 4, 5, 6, 7 \) are compared in figures 1 – 8 with the results valid for the infinitive number of the fields \( n = \infty \). The case \( n = \infty \) may be used for calculation of the critical force of the compression member on the elastic foundation.
Figure 1. $F_{cr}/F_E$ for $n = 2$ and $\infty$ fields.  
Number of half-sine waves $p = 1, 2$.

Figure 2. $F_{cr}/F_E$ for $n = 3$ and $\infty$ fields.  
Number of half-sine waves $p = 1, 2, 3$.

Figure 3. $F_{cr}/F_E$ for $n = 4$ and $\infty$ fields. 
Number of half-sine waves $p = 1, 2, 3, 4$.

Figure 4. $F_{cr}/F_E$ for $n = 5$ and $\infty$ fields. 
Number of half-sine waves $p = 1, 2, 3, 4, 5$.

Figure 5. $F_{cr}/F_E$ for $n = 6$ and $\infty$ fields. 
Number of half-sine waves $p = 1, 2, 3, 4, 5, 6$.

Figure 6. $F_{cr}/F_E$ for $n = 6$ and $\infty$ fields. 
Detail of diagram beginning.
3. Approximate formulae for calculation of the critical force

Replacing of the curves by the straight lines the following approximate formulae were obtained.

For $n = 2$:

$$F_{cr,a} = \begin{cases} 
0.25 \left( F_E + 1.5 \cdot C_w \cdot l \right) & \text{if } C_w \cdot l < 2 \cdot F_E \\
F_E & \text{if } 2 \cdot F_E \leq C_w \cdot l 
\end{cases} \quad (5)$$

For $n = 3$:

$$F_{cr,a} = \begin{cases} 
\frac{1}{9} F_E + 0.90245 \cdot C_w \cdot l & \text{if } C_w \cdot l < 0.49024 \cdot F_E \\
0.46632 \cdot F_E + 0.17789 \cdot C_w \cdot l & \text{if } 0.49024 \cdot F_E \leq C_w \cdot l < 3 \cdot F_E \\
F_E & \text{if } 3 \cdot F_E \leq C_w \cdot l 
\end{cases} \quad (6)$$

For $n = 4$:

$$F_{cr,a} = \begin{cases} 
\frac{1}{16} F_E + 1.61679 \cdot C_w \cdot l & \text{if } C_w \cdot l < 0.15455 \cdot F_E \\
0.25296 \cdot F_E + 0.38445 \cdot C_w \cdot l & \text{if } 0.15455 \cdot F_E \leq C_w \cdot l < 1.38405 \cdot F_E \\
0.63851 \cdot F_E + 0.10588 \cdot C_w \cdot l & \text{if } 1.38405 \cdot F_E \leq C_w \cdot l < 3.41422 \cdot F_E \\
F_E & \text{if } 3.41422 \cdot F_E \leq C_w \cdot l 
\end{cases} \quad (7)$$

For $n = 5$:

$$F_{cr,a} = \begin{cases} 
\frac{1}{25} F_E + 2.53067 \cdot C_w \cdot l & \text{if } C_w \cdot l < 0.06322 \cdot F_E \\
0.16057 \cdot F_E + 0.62373 \cdot C_w \cdot l & \text{if } 0.06322 \cdot F_E \leq C_w \cdot l < 0.57226 \cdot F_E \\
0.37757 \cdot F_E + 0.24452 \cdot C_w \cdot l & \text{if } 0.57226 \cdot F_E \leq C_w \cdot l < 2.15419 \cdot F_E \\
0.7635 \cdot F_E + 0.06537 \cdot C_w \cdot l & \text{if } 2.15419 \cdot F_E \leq C_w \cdot l < 3.61808 \cdot F_E \\
F_E & \text{if } 3.61808 \cdot F_E \leq C_w \cdot l 
\end{cases} \quad (8)$$
For \( n = 6 \):

\[
F_{cr,a} := \begin{cases} 
\frac{1}{36} F_E + 3.64441 C_w l & \text{if } C_w l \leq 0.03048 F_E \\
0.11122 F_E + 0.90654 C_w l & \text{if } 0.03048 F_E < C_w l \leq 0.27518 F_E \\
0.25407 F_E + 0.38742 C_w l & \text{if } 0.27518 F_E < C_w l \leq 1.1029 F_E \\
0.48964 F_E + 0.17385 C_w l & \text{if } 1.1029 F_E \leq C_w l < 2.6761 F_E \\
0.83519 F_E + 0.04471 C_w l & \text{if } 2.6761 F_E \leq C_w l < 3.68647 F_E \\
F_E & \text{if } 3.68647 F_E \leq C_w l 
\end{cases}
\]

(9)

For \( n = 7 \):

\[
F_{cr,a} = \begin{cases} 
\frac{1}{49} F_E + 4.96477 C_w l & \text{if } C_w l \leq 0.01641 F_E \\
0.08157 F_E + 1.23835 C_w l & \text{if } 0.01641 F_E < C_w l \leq 0.14828 F_E \\
0.18493 F_E + 0.54135 C_w l & \text{if } 0.14828 F_E < C_w l \leq 0.59598 F_E \\
0.33985 F_E + 0.2814 C_w l & \text{if } 0.59598 F_E \leq C_w l < 1.62999 F_E \\
0.58906 F_E + 0.12851 C_w l & \text{if } 1.62999 F_E \leq C_w l < 3.01461 F_E \\
0.88638 F_E + 0.02988 C_w l & \text{if } 3.01461 F_E \leq C_w l < 3.80198 F_E \\
F_E & \text{if } 3.80198 F_E \leq C_w l 
\end{cases}
\]

(10)

4. Comparisons and verifying of the results

The following examples enable to compare the values of the critical force \( F_{cr} \) obtained from diagrams, \( F_{cr,a} \) obtained from approximate formulae and the exact value \( F_{cr,IQ} \) calculated by computer program IQ 100 [9]. The comparisons are given in table 1. All critical forces in table 1 were calculated for the following input values: \( E = 210 \text{ GPa}, \ l = 349.2 \text{ cm}^2, \ f = 5 \text{ m} \). The value of the spring stiffness \( C_w \) was chosen to obtain the maximum difference between curves and straight lines in the diagrams.

| \( n \) | \( C_w \) [kN/m] | \( F_{cr} \) [kN] | \( F_{cr,a} \) [kN] | Difference between \( F_{cr} \) and \( F_{cr,a} \) | \( F_{cr,IQ} \) [kN] |
|---|---|---|---|---|---|
| 2 | 58.66 | 186.73 | 182.374 | 2.73 % | 187.50 |
| 3 | 99.732 | 231.603 | 223.707 | 3.324 % | 231.40 |
| 4 | 134.931 | 260.553 | 256.282 | 1.841 % | 261.09 |
| 5 | 74.505 | 202.652 | 200.398 | 1.447 % | 203.34 |
| 6 | 99.732 | 231.603 | 228.435 | 1.282 % | 231.40 |
| 7 | 134.931 | 260.553 | 257.235 | 0.918 % | 259.62 |

Table 1 shows that the maximum difference between diagram values \( F_{cr} \) and approximate values \( F_{cr,a} \) calculated from the above formulae is – 3.324 %. The \( F_{cr,a} \) values are rather smaller than \( F_{cr} \) being on the safe side. Differences between \( F_{cr} \) and exact \( F_{cr,IQ} \) values are negligible.

It was verified that equation (2) with \( \Phi_{\infty} \) valid for infinitive number of fields \( n = \infty \) defined by (3b) may be used for calculation of the critical force of the compression member on the elastic foundation. In such case the foundation modulus \( k \) [kN/m^2] should be calculated as follows
\[ k = \frac{(n - 1)C_w}{n\ell} \] (11)

5. Conclusions
Stability analysis of the compression member on elastic springs is presented. The members under investigation have seven various numbers of the fields \( n = 2, 3, 4, 5, 6, 7 \) and \( \infty \) (figures 1–8). The members have uniform bending rigidity \( EI \), uniform normal force \( N \) and \( (n - 1) \) elastic springs with equal spring stiffness \( C_w \). The results are presented in the form of eight diagrams. The curves of diagrams were replaced by the straight lines to be able derived approximate formulae (equations 5–10) for calculation of the critical force. It is shown that approximate formulae give the values of the critical forces \( F_{cr,a} \) which differ from diagram values \( F_{cr} \) less than 3.3 \%, being slightly on the safe side (table 1). The values of the critical forces \( F_{cr} \), obtained from the solution of the stability equation (2) are verified by comparisons with values \( F_{cr,IQ} \) of the computer program IQ 100. Excellent agreement is achieved between \( F_{cr,IQ} \) and \( F_{cr} \) (table 1). The presented solution of the compression member on elastic springs may be used also for the calculation of the critical force of the compression member on the elastic foundation. The solution given in the paper creates scientific background needed in the design of the upper chord of the semi-through bridge trusses.

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