On Subgame Perfect Equilibria in Turn-Based Reachability Timed Games

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Abstract. We study multiplayer turn-based timed games with reachability objectives. In particular, we are interested in the notion of subgame perfect equilibrium (SPE). We prove that deciding the constrained existence of an SPE in this setting is EXPTIME-complete.

Keywords: multiplayer turn-based timed games · reachability objectives · subgame perfect equilibria · constrained existence problem

1 Introduction

Games In the context of reactive systems, two-player zero-sum games played on graphs are commonly used to model the purely antagonistic interactions between a system and its environment \(^{18}\). The system and the environment are the two players of a game played on a graph whose vertices represent the configurations. Finding how the system can ensure the achievement of his objective amounts to finding, if it exists, a winning strategy for the system.

When modeling complex systems with several agents whose objectives are not necessarily antagonistic, the two-player zero-sum framework is too restrictive and we rather rely on multiplayer non zero-sum games. In this setting, the notion of winning strategy is replaced by various notions of equilibria including the famous concept of Nash equilibrium (NE) \(^{16}\). When considering games played on graphs, the notion of subgame perfect equilibrium (SPE) is often preferred to the classical Nash equilibrium \(^{17}\). Indeed, Nash equilibrium does not take into account the sequential structure of the game and may allow irrational behaviors in some subgames.

Timed games Timed automata \(^{19}\) is now a well established model for complex systems including real time features. Timed automata have been naturally extended into two-player zero-sum timed games \(^{2,11,14,15}\). Multiplayer non zero-sum extensions have also been considered \(^{17}\). In these models both time and multiplayer aspects coexist. In this non zero-sum timed framework, the main focus has been on NE, and, to our knowledge, not on SPE.

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Main contributions and organization of the paper. In this paper, we consider multiplayer, non zero-sum, turn-based timed games with reachability objectives together with the concept of SPE. We focus on the constrained existence problem (for SPE): given a timed game, we want to decide whether there exists an SPE where some players have to win and some other ones have to lose. The main result of this paper is a proof that the SPE constrained existence problem is EXPTIME-complete for reachability timed games. Let us notice that the NE constrained existence problem for reachability timed games is also EXPTIME-complete [7]. This may look surprising as often, there is a complexity jump when going from NE to SPE, for example the constrained existence problem on qualitative reachability game is NP-complete for NE [12] and PSPACE-complete for SPE [8]. Intuitively, the complexity jump is avoided because the exponential blow up due to the passage from SPE to NE is somehow absorbed by the classical exponential blow up due to the classical region graph used for the analysis of timed system.

In order to obtain an EXPTIME algorithm, we proceed in different steps. In the first step, we prove that the game variant of the classical region graph is a good abstraction for the SPE constrained existence problem. In fact, we identify conditions on bisimulations under which the study of SPE of a given (potentially infinite game) can be reduced to the study of its quotient. This is done in Section 3 for (untimed) games with general objectives. In Section 4, we then focus on (untimed) finite reachability game and provide an EXPTIME algorithm to solve the constrained existence problem. Proving this result may look surprising, as we already know from [8] that this problem is indeed PSPACE-complete for (untimed) finite games. However the PSPACE algorithm provided in [8] did not allow us to obtain the EXPTIME algorithm for timed games. The latter EXPTIME algorithm is discussed in Section 5.

Related works There are many results on SPEs played on graphs, we refer the reader to [10] for a survey and an extended bibliography. Here we focus on the results directly related to our contributions. The constrained existence of SPEs is studied in finite multiplayer turn-based games with different kinds of objectives, for example: (qualitative) reachability and safety objectives [8], $\omega$-regular winning conditions [20], quantitative reachability objectives [9]... In [5], they prove that the constrained existence problem for Nash equilibria in concurrent timed games with reachability objectives is EXPTIME-complete. This same problem in the same setting is studied in [7] with others qualitative objectives.

2 Preliminaries

Transition systems, bisimulations and quotients A transition system is a tuple $T = (\Sigma, V, E)$ where (i) $\Sigma$ is a finite alphabet; (ii) $V$ a set of states (also called vertices) and (iii) $E \subseteq V \times \Sigma \times V$ a set of transitions (also called edges). To ease the notation, an edge $(v_1, a, v_2) \in E$ is sometimes denoted by $v_1 \xrightarrow{a} v_2$. Notice that $V$ may be uncountable. We said that the transition system is finite if
$V$ and $E$ are finite.

Given two transition systems on the same alphabet $T_1 = (\Sigma, V_1, E_1)$ and $T_2 = (\Sigma, V_2, E_2)$, a simulation of $T_1$ by $T_2$ is a binary relation $R \subseteq V_1 \times V_2$ which satisfies the following conditions: (i) \( \forall v_1, v'_1 \in V_1, \forall v_2 \in V_2 \text{ and } \forall a \in \Sigma: ((v_1, v_2) \in R \text{ and } v_1 \xrightarrow{a} v'_1) \Rightarrow (\exists v'_2 \in V_2, v_2 \xrightarrow{a} v'_2 \text{ and } (v'_1, v'_2) \in R) \) and (ii) for each $v_1 \in V_1$ there exists $v_2 \in V_2$ such that $(v_1, v_2) \in R$. We say that $T_2$ simulates $T_1$. It implies that any transition $v_1 \xrightarrow{a} v'_1$ in $T_1$ is simulated by a corresponding transition $v_2 \xrightarrow{a} v'_2$ in $T_2$.

Given two transition systems on the same alphabet $T_1 = (\Sigma, V_1, E_1)$ and $T_2 = (\Sigma, V_2, E_2)$, a bisimulation between $T_1$ and $T_2$ is a binary relation $R \subseteq V_1 \times V_2$ such that $R$ is a simulation of $T_1$ by $T_2$ and the converse relation $R^{-1}$ is a simulation of $T_2$ by $T_1$ where $R^{-1} = \{(v_2, v_1) \in V_2 \times V_1 \mid (v_1, v_2) \in R\}$. When $R$ is a bisimulation between two transition systems, we write $\beta$ instead of $R$. If $T = (\Sigma, V, E)$ is a transition system, a bisimulation on $V \times V$ is called a bisimulation on $T$.

Given a transition system $T = (\Sigma, V, E)$ and an equivalence relation $\sim$ on $V$, we define the quotient of $T$ by $\sim$, denoted by $\bar{T} = (\bar{\Sigma}, \bar{V}, \bar{E})$, as follows: (i) $\bar{V} = \{[v]_\sim \mid v \in V\}$ where $[v]_\sim = \{v' \in V \mid v \sim v'\}$ and (ii) $[v_1]_\sim \xrightarrow{a} [v_2]_\sim$ if and only if there exist $v'_1 \in [v_1]_\sim$ and $v'_2 \in [v_2]_\sim$ such that $v'_1 \xrightarrow{a} v'_2$. When clear from the context which equivalence relation is used, we write $[v]$ instead of $[v]_\sim$.

Given a transition system $T = (\Sigma, V, E)$, a bisimulation $\sim$ on $T$ which is also an equivalence relation is called a bisimulation equivalence. In this context, the following result holds.

**Lemma 1.** Given a transition system $T$ and a bisimulation equivalence $\sim$, there exists a bisimulation $\sim_q$ between $T$ and its quotient $\bar{T}$. This bisimulation is given by the function $\sim_q: V \rightarrow \bar{V} : v \mapsto [v]_\sim$.

**Turn-based games**

Arenas, plays and histories An arena $A = (\Sigma, V, E, \Pi, (V_i)_{i \in \Pi})$ is a tuple where

(i) $T = (\Sigma, V, E)$ is a transition system such that for each $v \in V$, there exists $a \in \Sigma$ and $v' \in V$ such that $(v, a, v') \in E$; (ii) $\Pi = \{1, \ldots, n\}$ is a finite set of players and (iii) $(V_i)_{i \in \Pi}$ is a partition of players between the players. An arena is finite if its transition system $T$ is finite.

A play in $A$ is an infinite path in its transition system, i.e., $\rho = \rho_0 \rho_1 \ldots \in V^\omega$ is a play if for each $i \in \mathbb{N}$, there exists $a \in \Sigma$ such that $(\rho_i, a, \rho_{i+1}) \in E$. A history $h$ in $A$ can be defined in the same way but $h = h_0 \ldots h_k \in V^*$ for some $k \in \mathbb{N}$ is a finite path in the transition system. We denote the set of plays by Plays and the set of histories by Hist. When it is necessary, we use the notation Plays$_A$ and Hist$_A$ to recall the underlying arena $A$. Moreover, the set Hist, is the set of histories such that their last vertex $v$ is a vertex of Player $i$, i.e., $v \in V_i$. A play (resp. a history) in $(G, v_0)$ is then a play (resp. a history) in $G$ starting in $v_0$. 


The set of such plays (resp. histories) is denoted by Plays($v_0$) (resp. Hist($v_0$)). We also use the notation Hist($v_0$) when these histories end in a vertex $v \in V_i$.

Given a play $\rho \in$ Plays and $k \in \mathbb{N}$, its suffix $\rho_k \rho_{k+1} \ldots$ is denoted by $\rho_{\geq k}$. We denote by Succ($v$) = \{ $v' | (v, a, v') \in E$ for some $a \in \Sigma$ \} the set of successors of $v$, for $v \in V$, and by Succ* the transitive closure of Succ. Given a play $\rho = \rho_0 \rho_1 \ldots$, the set Occ($\rho$) = \{ $v \in V$ | $\exists k, \rho_k = v$ \} is the set of vertices visited along $\rho$.

**Remark 1.** When we consider a play in an arena $A = (\Sigma, V, E, \Pi, (V_i)_{i \in \Pi})$, we do not care about the alphabet letter associated with each edge of the play. It is the reason why two different infinite paths in $T = (\Sigma, V, E) \ v_0 \xrightarrow{a} v_1 \xrightarrow{a} \ldots \xrightarrow{a} v_n \xrightarrow{b} \ldots \xrightarrow{b} v_0 \xrightarrow{b} \ldots$ correspond to only one play $\rho = v_0 v_1 \ldots v_n \ldots$ in $A$. The same phenomenon appears with finite paths and histories. We explain later why this is not a problem for our purpose.

**Multiplayer turn-based game** An (initialized multiplayer Boolean turn-based) game is a tuple $(G, v_0) = (A, (g_i)_{i \in \Pi})$ such that: (i) $A = (\Sigma, V, E, \Pi, (V_i)_{i \in \Pi})$ is an arena; (ii) $v_0 \in V$ is the initial vertex and (iii) for each $i \in \Pi$, $g_i :$ Plays $\rightarrow \{0, 1\}$ is a gain function for Player $i$. In this setting, each player $i \in \Pi$ is equipped with a set $\Omega_i \subseteq$ Plays that we call the objective of Player $i$. Thus, for each $i \in \Pi$, for each $\rho \in$ Plays: $g_i(\rho) = 1$ if and only if $\rho \in \Omega_i$. If $g_i(\rho) = 1$ (resp. = 0), we say that Player $i$ wins (resp. loses) along $\rho$. In the sequel of this document, we refer to the notion of initialized multiplayer Boolean turn-based game by the term “game”. For each $\rho \in$ Plays, we write $g(\rho) = p$ for some $p \in \{0, 1\}^{\Pi_1}$ to depict $g_i(\rho) = p_i$ for each $i \in \Pi$.

**Strategies and outcomes** Given a game $(G, v_0)$, a strategy of Player $i$ is a function $\sigma_i :$ Hist($v_0$) $\rightarrow V$ with the constraint that for each $hv \in$ Hist($v_0$), $\sigma_i(hv) \in$ Succ($v$). A play $\rho = \rho_0 \rho_1 \ldots$ is consistent with $\sigma_i$ if for each $\rho_k \in V_i$, $\rho_{k+1} = \sigma_i(\rho_0 \ldots \rho_k)$. A strategy profile $\sigma = (\sigma_i)_{i \in \Pi}$ is a tuple of strategies, one for each player. Given a game $(G, v_0)$ and a strategy profile $\sigma$, there exists a unique play from $v_0$ consistent with each strategy $\sigma_i$. We call this play the outcome of $\sigma$ and denote it by $\langle \sigma \rangle_{v_0}$.

**Remark 2.** We follow up Remark 1. The objectives we consider are of the form $\Omega \subseteq$ Plays. These objectives only depend on the sequence of visited states along a play (for example: visiting infinitely often a given state) regardless the sequence of visited alphabet letters. This is why defining the strategy of a player by a choice of the next vertex instead of a couple of an alphabet letter and a vertex is not a problem. Actually, in all this paper one may consider that the alphabet is $\Sigma = \{a\}$. The reason why we allow alphabet on edge is to be able to consider synchronous products of (timed) automata [19]. In this way, we could consider wider class of objectives (see Section 5.4).

**Subgame perfect equilibria** In the multiplayer game setting, the solution concepts usually studied are equilibria (see [13]). We here recall the concepts of Nash equilibrium and subgame perfect equilibrium.
Let $\sigma = (\sigma_i)_{i \in I}$ be a strategy profile in a game $(\mathcal{G}, v_0)$. When we highlight the role of Player $i$, we denote $\sigma$ by $(\sigma_i, \sigma_{-i})$ where $\sigma_{-i}$ is the profile $(\sigma_j)_{j \in I \setminus \{i\}}$. A strategy $\sigma'_i \neq \sigma_i$ is a deviating strategy of Player $i$, and it is a profitable deviation for him if $g_i((\sigma)_{-i} v_0) < g_i((\sigma'_i, \sigma_{-i}) v_0)$. A strategy profile $\sigma$ in a game $(\mathcal{G}, v_0)$ is a Nash equilibrium (NE) if no player has an incentive to deviate unilaterally from his strategy, i.e., no player has a profitable deviation.

A refinement of NE is the concept of subgame perfect equilibrium (SPE) which is a strategy profile being an NE in each subgame. Formally, given a game $(\mathcal{G}, v_0) = (A, (g_i)_{i \in I})$ and a history $hv \in \text{Hist}(v_0)$, the game $(\mathcal{G}_{hv}, v)$ is called a subgame of $(\mathcal{G}, v_0)$ such that $\mathcal{G}_{hv} = (A, (g_i)_{i \in I})$ and $g_i(h\rho) = g_i(h\rho)$ for all $i \in I$ and $\rho \in V^\omega$. Notice that $(\mathcal{G}, v_0)$ is subgame of itself. Moreover if $\sigma_i$ is a strategy for Player $i$ in $(\mathcal{G}, v_0)$, then $\sigma_{ih}$ denotes the strategy in $(\mathcal{G}_{hv}, v)$ such that for all histories $h' \in \text{Hist}_i(v)$, $\sigma_{ih}(h') = \sigma_i(\text{hh'})$. Similarly, from a strategy profile $\sigma$ in $(\mathcal{G}, v_0)$, we derive the strategy profile $\sigma_{ih}$ in $(\mathcal{G}_{hv}, v)$. Let $(\mathcal{G}, v_0)$ be a game, following this formalism, a strategy profile $\sigma$ is a subgame perfect equilibrium in $(\mathcal{G}, v_0)$ if for all $hv \in \text{Hist}(v_0)$, $\sigma_{ih}$ is an NE in $(\mathcal{G}_{hv}, v)$.

**Studied problem** Given a game $(\mathcal{G}, v_0)$, several SPEs may coexist. It is the reason why we are interested in the constrained existence of an SPE in this game: some players have to win and some other ones have to lose. The related decision problem is the following one:

**Definition 1 (Constrained existence problem).** Given a game $(\mathcal{G}, v_0)$ and two gain profiles $x, y \in \{0, 1\}^{\lvert I \rvert}$, does there exist an SPE $\sigma$ in $(\mathcal{G}, v_0)$ such that $x \leq g((\sigma)_{-i} v_0) \leq y$.

### 3 Speaker Perfect Equilibria in a Game and its Quotient

In this section, we first define the concept of bisimulation between games (resp. bisimulation on a game). Then, we explain how given such bisimulations we can obtain a new game, called the quotient game, thanks to a quotient of the initial game. Finally, we prove that if there exists an SPE in a game with a given gain profile, there exists an SPE in its associated quotient game with the same gain profile, and vice versa.

#### 3.1 Game Bisimulation

We extend the notion of bisimulation between transition systems (resp. on a transition system) to the one of bisimulation between games (resp. on a game). In this paper, by bisimulation between games (resp. on a game) we mean:

**Definition 2 (Game bisimulation).** Given two games $(\mathcal{G}, v_0) = (A, (g_i)_{i \in I})$ and $(\mathcal{G}', v'_0) = (A', (g'_i)_{i \in I})$ with the same alphabet and the same set of players, we say that $\sim \subseteq V \times V'$ is a bisimulation between $(\mathcal{G}, v_0)$ and $(\mathcal{G}', v'_0)$ if (i) $\sim$ is a bisimulation between $T = (\Sigma, V, E)$ and $T' = (\Sigma, V', E')$ and (ii) $v_0 \sim v'_0$. In the same way, if $\sim \subseteq V \times V$ we say that $\sim$ is a bisimulation on $(\mathcal{G}, v_0)$ if $\sim$ is a bisimulation on $T = (\Sigma, V, E)$. 

The notion of bisimulation equivalence on a transition system is extended in the same way to games. In the rest of this document, we use the following notations: (1) If \( \sim \subseteq V \times V' \) is a bisimulation between \((G, v_0) = (A, (g_i)_{i \in N})\) and \((G', v'_0) = (A', (g'_i)_{i \in N})\), for each \( \rho \in \text{Plays}_A \) and for all \( \rho' \in \text{Plays}_{A'} \), we write \( \rho \sim \rho' \) if and only if for each \( n \in N \): \( \rho_n \sim \rho'_n \). (2) If \( \sim \subseteq V \times V \) is a bisimulation on \((G, v_0) = (A, (g_i)_{i \in N})\), for each \( \rho \in \text{Plays}_A \) and for all \( \rho' \in \text{Plays}_A \), we write \( \rho \sim \rho' \) if and only if for each \( n \in N \): \( \rho_n \sim \rho'_n \). (3) Notations 1 and 2 can be naturally adapted to histories. \(^3\)

A natural property that should be satisfied by a bisimulation on a game is the respect of the vertices partition. It means that if a vertex bisimulates an other vertex, then these vertices should be owned by the same player.

**Definition 3** (\( \sim \) respects the partition). Given a game \((G, v_0) = (A, (g_i)_{i \in N})\) and a bisimulation \( \sim \) on \((G, v_0) \), we say that \( \sim \) respects the partition if for all \( v, v' \in V \) such that \( v \sim v' \), if \( v \in V_i \) then \( v' \in V_i \).

### 3.2 Quotient game

Given a game \((G, v_0)\) and a bisimulation equivalence \( \sim \) on it which respects the partition, one may consider its associated quotient game \((\tilde{G}, [v_0])\) such that its transition system is defined as the quotient of the transition system of \((G, v_0)\).

**Definition 4** (Quotient game). Given a game \((G, v_0) = (A, (g_i)_{i \in N})\) such that \( A = (\Sigma, V, E, \Pi, (V_i)_{i \in N}) \), if \( \sim \) is a bisimulation equivalence on \((G, v_0) \) which respects the partition, the associated quotient game \((\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in N})\) is defined as follows: (i) \( \tilde{A} = (\tilde{\Sigma}, \tilde{V}, \tilde{E}, (\tilde{V}_i)_{i \in N}) \) is such that \( \tilde{T} = (\tilde{\Sigma}, \tilde{V}, \tilde{E}) \) is the quotient of \( T \) and, for each \( i \in \Pi \), \( [v] \in \tilde{V}_i \) if and only if \( v \in V_i \) and (ii) for each \( i \in \Pi \), \( \tilde{g}_i : \text{Plays}_A \to \{0, 1\} \) is the gain function of Player \( i \).

In order to preserve some equivalent properties between a game and its quotient game, the equivalence relation on the game should respect the gain functions in both games. It means that if we consider two bisimulated plays either both in the game itself or one in the game and the other one in its quotient game, the gain profile of these plays should be equal.

**Definition 5** (\( \sim \) respects the gain functions). Given an initialized game \((G, v_0) = (A, (g_i)_{i \in N})\) such that \( A = (\Sigma, V, E, \Pi, (V_i)_{i \in N}) \) and a bisimulation equivalence \( \sim \) on \((G, v_0) \), we say that \( \sim \) respects the gain functions if the following properties hold: (i) for each \( \rho \) and \( \rho' \) in \( \text{Plays}_A \), if \( \rho \sim \rho' \) then \( g(\rho) = g(\rho') \) and (ii) for each \( \rho \in \text{Plays}_A \) and \( \tilde{\rho} \in \text{Plays}_{\tilde{A}} \), if \( \rho \sim \tilde{\rho} \) then \( g(\rho) = \tilde{g}(\tilde{\rho}) \).

\(^3\) Once again, with this convention it is possible that two plays (or histories) such that \( \rho \sim \rho' \) do not preserve the sequence of alphabet letters as it should be when we classically consider bisimulated paths in two bisimulated transitions systems. Remark \(^2\) explains why it is not a problem for us.
3.3 Existence of SPE

The aim of this section is to prove that, if there exists an SPE in a game equipped with a bisimulation equivalence which respects the partition and the gain functions, there exists an SPE in its associated quotient game with the same gain profile, and vice versa.

Theorem 1. Let \((G, v_0) = (A, (g_i)_{i \in \pi})\) be a game and \((\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in \pi})\) its associated quotient game where \(\sim\) is a bisimulation equivalence on \((G, v_0)\). If \(\sim\) respects the partition and the gain functions, we have that: there exists an SPE \(\sigma\) in \((G, v_0)\) such that \(g(\langle \sigma \rangle v_0) = p\) if and only if there exists an SPE \(\tau\) in \((\tilde{G}, [v_0])\) such that \(\tilde{g}(\langle \tau \rangle [v_0]) = p\).

The key idea is to prove that: if there exists an SPE in a game equipped with a bisimulation equivalence, there exists an SPE in this game which is uniform and with the same gain profile. If \(\sigma_i\) is an uniform strategy, each time we consider two histories \(h \sim h'\), the choices of Player \(i\) taking into account \(h\) or \(h'\) are in the same equivalence class.

Definition 6. Let \((G, v_0)\) be a game and \(\sim\) a bisimulation on it, we say that the strategy \(\sigma_i\) is uniform if for all \(h, h' \in \text{Hist}_i(v_0)\) such that \(h \sim h'\), we have that \(\sigma_i(h) \sim \sigma_i(h')\). A strategy profile \(\sigma\) is uniform if for all \(i \in \Pi\), \(\sigma_i\) is uniform.

Proposition 1. Let \((G, v_0) = (A, (g_i)_{i \in \pi})\) be a game and \(\sim\) be a bisimulation equivalence on \((G, v_0)\) which respects the partition and such that for each \(p\) and \(p'\) in \(\text{Plays}\), if \(p \sim p'\) then \(g(p) = g(p')\), there exists an SPE \(\sigma\) in \((G, v_0)\) such that \(g(\langle \sigma \rangle v_0) = p\) if and only if there exists an SPE \(\tau\) in \((G, v_0)\) which is uniform and such that \(\tilde{g}(\langle \tau \rangle [v_0]) = p\).

4 Reachability games

In this section we focus on a particular kind of game called reachability game. In these games, each player has a subset of vertices that he wants to reach. First, we formally define the concepts of reachability games and reachability quotient games. Then, we provide an algorithm which solves the constrained existence problem in finite reachability games in time complexity at most exponential in the number of players and polynomial in the size of the transition system of the game.

4.1 Reachability games and quotient reachability games

Definition 7. A reachability game \((G, v_0) = (A, (g_i)_{i \in \pi}, (F_i)_{i \in \Pi})\) is a game where each player \(i \in \Pi\) is equipped with a target set \(F_i\) that he wants to reach. Formally, the objective of Player \(i\) is \(\Omega_i = \{\rho \in \text{Plays} \mid \text{Occ}(\rho) \cap F_i \neq \emptyset\}\) where \(F_i \subseteq V\). This is a reachability objective.
Given a reachability game \((G, v_0) = (A, (g_i)_{i \in I}, (F_i)_{i \in I})\) and a bisimulation equivalence \(\sim\) on this game which respects the partition, one may consider its quotient game \((\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in I}, (\tilde{F}_i)_{i \in I})\) where for each \(i \in I\), \(\tilde{F}_i \subseteq \tilde{V}\).

In attempts to ensure the respect of the gain functions by \(\sim\), we add a natural property on \(\sim\) (see Definition 8) and define the sets \(\tilde{F}_i\) in a proper way. In the rest of this paper, we assume that this property is satisfied and that the quotient game of a reachability game is defined as in Definition 9.

**Definition 8 \(\sim\) respects the target sets.** Let \((G, v_0)\) be a reachability game and \(\sim\) be a bisimulation equivalence on this game, we say that \(\sim\) respects the target sets if for all \(v \in V\) and for all \(v' \in V\) such that \(v \sim v': v \in F_i \iff v' \in F_i\).

**Definition 9 (Reachability quotient game).** Given a reachability game \((G, v_0) = (A, (g_i)_{i \in I}, (F_i)_{i \in I})\) and a bisimulation equivalence \(\sim\) on this game which respects the partition and the target sets, its quotient game is the reachability game \((\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in I}, (\tilde{F}_i)_{i \in I})\) where \(\tilde{F}_i = \{[v]_{\sim} | v \in F_i\}\) for each \(i \in I\). We call this game the reachability quotient game.

**Lemma 2.** Let \((G, v_0)\) be a reachability game and let \(\sim\) be a bisimulation equivalence which respects the target sets on this game, \(\sim\) respects the gain functions.

### 4.2 Complexity results

It is proved that the constrained existence problem is PSPACE-complete in finite reachability game [8]. Our final purpose is to obtain an EXPTIME algorithm for the constrained existence problem on reachability timed games (see Section 5). Naively applying the PSPACE algorithm of [8] to the region games would lead to an EXPSPACE algorithm. That is why we provide here an alternative EXPTIME algorithm to solve the constrained existence problem on (untimed) finite games. This new algorithm will have the advantage to have a running time at most exponential only in the number of players (and polynomial in the size of its transition system). This feature will be crucial to obtain the EXPTIME algorithm on timed games.

**Theorem 2.** Given a finite reachability game \((G, v_0)\), the constrained existence problem can be solved by an algorithm whose time complexity is at most exponential in \(|I|\) and polynomial in the size of its transition system.

This approach follows the proof for quantitative reachability games in [9]. This latter proof relies on two key ingredients: (i) the extended game of a reachability game and (ii) an SPE outcome characterization based on a fixpoint computation of a labeling function of the states. Those two key ingredients will be defined below. Further technical details can be found in [9] for the quantitative case.
Extended game Let $(G, v_0)$ be finite a reachability game, its associated extended game $(X, x_0) = (X, (g^i_X)_{i \in I}, (F^i_X)_{i \in I})$ is the reachability game such that the vertices are enriched with the set of players that have already visited their target sets along a history. The arena $X = (\Sigma, V^X, E^X, \Pi, (V^i_X)_{i \in I})$ is defined as follows: (i) $V^X = V \times 2^I$; (ii) $((v, I), a, (v', I')) \in E^X$ if and only if $(v, a, v') \in E$ and $I' = I \cup \{ i \in I \mid v' \in F_i \}$; (iii) $(v, I) \in V^i_X$ if and only if $v \in V_i$; (iv) $(v, I) \in F^i_X$ if and only if $i \in I$ and $(v) x_0 = (v_0, I_0)$ where $I_0 = \{ i \in I \mid v_0 \in F_i \}$.

The construction of $(X, x_0)$ from $(G, v_0)$ causes an exponential blow-up of the number of states. The main idea of this construction is that if you consider a play $\rho = (v_0, I_0)(v_1, I_1)\ldots(v_n, I_n)\ldots \in \text{Plays}_X(x_0)$, the set $I_0$ means that each player $i \in I_n$ has visited his target set along $\rho_0\ldots\rho_n$. The important points are that there is a one-to-one correspondance between plays in $\text{Plays}_X(v_0)$ and $\text{Plays}_X(x_0)$ and that the gain profiles of two corresponding plays beginning in the initial vertices are equal. From these observations, we have:

**Proposition 2.** Let $(G, v_0)$ be a reachability game and $(X, x_0)$ be its associated extended game, let $p \in \{0, 1\}^{|I|}$ be a gain profile, there exists an SPE $\sigma$ in $(G, v_0)$ with gain profile $p$ if and only if there exists an SPE $\tau$ in $(X, x_0)$ with gain profile $p$.

In the rest of this section, we will write $v \in V^X$ (instead of $(u, I)$) and we depict by $I(v)$ the set $I$ of the players who have already visited their target set.

Outcome characterization Once this extended game is build, we want a way to decide whether a play in this game corresponds to the outcome of an SPE or not: we want an SPE outcome characterization. The vertices of the extended game are labeled thanks to a labeling function $\lambda^* : V^X \rightarrow \{0, 1\}$. For a vertex $v \in V^X$ such that $v \in V_i^X$, the value 1 imposes that Player $i$ should reach his target set if he follows an SPE from $v$ and the value 0 does not impose any constraint on the gain of Player $i$ from $v$.

The labeling function $\lambda^*$ is obtained thanks to an iterative procedure such that each step $k$ of the iteration provides a $\lambda^k$-labeling function. This procedure is based on the notion of $\lambda$-consistent play: that is a play which satisfies the constraints given by $\lambda$ all along it.

**Definition 10.** Let $\lambda : V^X \rightarrow \{0, 1\}$ be a labeling function and $\rho \in \text{Plays}_X$, we say that $\rho$ is $\lambda$-consistent if for each $i \in I$ and for each $n \in \mathbb{N}$ such that $\rho_0 \in V_i^X : g^i_X(\rho_{\geq n}) \geq \lambda(\rho_0)$. We write $\rho \models \lambda$.

The iterative computation of the sequence $(\lambda^k)_{k \in \mathbb{N}}$ works as follows: (i) at step 0, for each $v \in V^X$, $\lambda^0(v) = 0$, (ii) at step $k + 1$, for each $v \in V^X$, by assuming that $v \in V_i^X$, $\lambda^{k+1}(v) = \max_{v' \in \text{Succ}(v)} \min\{g^i_X(\rho) \mid \rho \in \text{Plays}_X(v') \land \rho \models \lambda^k\}$ and (iii) we stop when we find $n \in \mathbb{N}$ such that for each $v \in V^X$, $\lambda^{n+1}(v) = \lambda^n(v)$. The least natural number $k^*$ which satisfies (iii) is called the fixpoint of $(\lambda^k)_{k \in \mathbb{N}}$ and $\lambda^*$ is defined as $\lambda^{k^*}$. The following lemma states that this natural number exists and so that the iterative procedure stops.
Lemma 3. The sequence \((\lambda^k)_{k \in \mathbb{N}}\) reaches a fixpoint in \(k^* \in \mathbb{N}\). Moreover, \(k^*\) is at most equal to \(|V| \cdot 2^{|H|}\).

Proof (Proof sketch). In the initialization step, all the vertex values are equal to 0. Then at each iteration, (i) if the value of a vertex was equal to 1 in the previous step, then it stays equal to 1 all along the procedure and (ii) if the value of the vertex was equal to 0 then it either stays equal to 0 (for this iteration step) or it becomes equal to 1 (for all the next steps thanks to (i)). At each step, at least one vertex value changes and when no value changes the procedure has reached a fixpoint which corresponds to the values of \(\lambda^*\). Thus, it means that \(\lambda^*\) is obtained in at most \(|V| \times 2^{|H|}\) steps.

As claimed in the following proposition, the labeling function \(\lambda^*\) exactly characterizes the set of SPE outcomes. The proof is quite the same as for the quantitative setting \([9]\).

Proposition 3. Let \((\mathcal{X}, x_0)\) be the extended game of a finite reachability game \((\mathcal{G}, v_0)\) and let \(\rho^X \in \text{Plays}^X(x_0)\) be a play, there exists an SPE \(\sigma\) with outcome \(\rho^X\) in \((\mathcal{X}, x_0)\) if and only if \(\rho^X\) is \(\lambda^*\)-consistent.

Complexity Proposition 3 allows us to prove Theorem 2. Indeed, we only have to find a play in the extended game which is \(\lambda^*\)-consistent and with a gain profile which satisfies the constrained given by the decision problem.

Proof (Proof sketch of Theorem 2). Let \((\mathcal{G}, v_0) = (A, (g_i)_{i \in \Pi}, (F_i)_{i \in \Pi})\) be a reachability game and let \((\mathcal{X}, x_0) = (X, (g_i^X)_{i \in \Pi}, (F_i^X)_{i \in \Pi})\) be its associated extended game. The game \((\mathcal{X}, x_0)\) is build from \((\mathcal{G}, v_0)\) in time at most exponential in the number of players and polynomial in the size of the transition system of \(A\).

The proof will be organised in three steps whose respective proofs will rely on the previous step(s): (i) given a gain profile \(p \in \{0, 1\}^{|H|}\), given \(\mathcal{L}^k = \{\lambda^k(v) \mid v \in V^X\} \) for some \(k \in \mathbb{N}\) and given some \(v \in V^X\), we show that we can decide in the required complexity the existence of a play which is \(\lambda^k\)-consistent, beginning in \(v\) and with gain profile \(p\); (ii) given \(\mathcal{L}^k\) for some \(k \in \mathbb{N}\), we show that the computation of \(\mathcal{L}^{k+1}\) can be performed within the required complexity; and finally (iii) given \(x, y \in \{0, 1\}^{|H|}\), we show that the existence of a \(\lambda^*\)-consistent play beginning in \(x_0\) with a gain profile \(p\) such that \(x \leq p \leq y\) can be decided within the required complexity.

- Proof of (i): Given \(\mathcal{L}^k, v \in V^X\) and \(p \in \{0, 1\}^{|H|}\), we want to know if there exists a play \(\rho \in \text{Plays}_X(v)\) which is \(\lambda^k\)-consistent and with gain profile \(p\).

If a play \(\rho\) is such that \(g^X(\rho) = p\), then for each \(i \in \Pi\) such that \(p_i = 1\), the condition of being a \(\lambda^k\)-consistent play is satisfied. For those such that \(p_i = 0\), for each \(n \in \mathbb{N}\) such that \(p_n \in V^X\), \(g^X(\rho_n) = 0\) should be greater than \(\lambda^k(\rho_n)\). This condition is satisfied if and only if for each \(\rho_n \in V^X\), \(\lambda^k(\rho_n) \neq 1\). Thus, we remove from \((\mathcal{X}, x_0)\) all vertices (and all related edges) \(v \in V^X\) such that \(\lambda^k(v) = 1\), for each player \(i\) such that \(p_i = 0\). Then, we only have to check if there exists a play \(\rho\) which begin in \(v\) and with gain
profile \( p \) in this modified extended reachability game. This can be done in \( O(|\Pi| \cdot (|V^X| + |E^X|)) \) (Lemma 23), thus this procedure runs in time at most exponential in the number of players and polynomial in the size of the transition system of \( A \).

- **Proof of (ii):** Given \( \mathcal{L}^k \), we want to compute \( \mathcal{L}^{k+1} \). For each \( v \in V^X \), \( \lambda^{k+1}(v) = \max_{v' \in \text{Succ}(v)} \min \{ \lambda^X(v') \mid \rho \in \Pi \text{ and } \rho \models \lambda^k \} \) (by assuming that \( v \in V^X \)). Thus for each \( v' \in \text{Succ}(v) \), we have to compute \( \min = \min \{ g_i^X(\rho) \mid \rho \in \Pi \text{ and } \rho \models \lambda^k \} \). But \( \min = 0 \) if and only if there exists \( \rho \in \Pi \text{ and } \rho \models \lambda^k \). Thus for each \( v' \in \text{Succ}(v) \), we have to compute \( \min \{ g_i^X(\rho) \mid \rho \in \Pi \text{ and } \rho \models \lambda^k \} \).

On Subgame Perfect Equilibria in Turn-Based Reachability Timed Games

5 Application to Timed Games

In this section, we are interested in models which are enriched with clocks and clock guards in order to consider time elapsing. Timed automata are well known among such models. We recall some of their classical concepts, then we explain how (turn-based) timed games derive from timed automata.

5.1 Timed automata and timed games

In this section, we use the following notations. The set \( C = \{c_1, \ldots, c_k\} \) denotes a set of \( k \) clocks. A **clock valuation** is a function \( \nu : C \to \mathbb{R}^+ \). The set of clock valuation is depicted by \( C_{V} \). Given a clock valuation \( \nu \), for \( i \in \{1, \ldots, k\} \), we sometimes write \( \nu_i \) instead of \( \nu(c_i) \). Given a clock valuation \( \nu \) and \( d \in \mathbb{R}^+ \), \( \nu + d \) denotes the clock valuation \( \nu + d : C \to \mathbb{R}^+ \) such that \( (\nu + d)(c_i) = \nu(c_i) + d \) for each \( c_i \in C \).

A **guard** is any finite conjunction of expressions of the form \( c_i \circ x \) where \( c_i \) is a clock, \( x \in \mathbb{N} \) is a natural number and \( \circ \) is one of the symbols \( \{\leq, <, =, >, \geq\} \). We denote by \( G \) the set of guards. Let \( g \) be a guard and \( \nu \) be a clock valuation, notation \( \nu \models g \) means that \( (\nu_1, \ldots, \nu_k) \) satisfies \( g \). A **reset**
\( Y \in 2^C \) indicates which clocks are reset to 0. We denote by \( Y \leftarrow 0|\nu \) the valuation \( \nu' \) such that for each \( c \in Y \), \( \nu'(c) = 0 \) and for each \( c \in C \backslash Y \), \( \nu'(c) = \nu(c) \).

A timed automaton (TA) is a tuple \((A, \ell_0) = (\Sigma, L, \rightarrow, C)\) where: (i) \( \Sigma \) is a finite alphabet; (ii) \( L \) is a finite set of locations; (iii) \( C \) is a finite set of clocks; (iv) \( \rightarrow \subseteq L \times \Sigma \times G \times 2^C \times L \) a finite set of transitions; and (v) \( \ell_0 \in L \) an initial location. Additionaly, we may equipped a timed automaton with a set of players and partition the locations between them. It results in a players partitioned timed automaton.

**Definition 11** ((Reachability) Players partitioned timed automaton).

A players partitioned timed automaton (PPTA) \((A, \ell_0) = (\Sigma, L, \rightarrow, C, \Pi, (L_i)_{i \in \Pi})\) is a timed automaton equipped with: (i) \( \Pi \) a finite set of players and (ii) \((L_i)_{i \in \Pi}\) a partition of the locations between the players.

If \((A, \ell_0)\) is equipped with a target set \( \text{Goal}_i \subseteq L \) for each player \( i \in \Pi \), we call it a reachability PPTA.

The semantic of a timed automaton \((A, \ell_0)\) is given by its associated transition system \( T_A = (\Sigma, V, E) \) where: (i) \( V = L \times C_V \) is a set of vertices of the form \((\ell, \nu)\) where \( \ell \) is a location and \( \nu : C \rightarrow \mathbb{R}^+ \) is a clock valuation; and (ii) \( E \subseteq V \times \Sigma \times V \) is such that \(((\ell, \nu), a, (\ell', \nu')) \in E \) if \((\ell, a, g, Y, \ell') \in \rightarrow \) for some \( g \in G \) and some \( Y \in 2^C \). and there exists \( d \in \mathbb{R}^+ \) such that: (1) for each \( x \in X \backslash Y \): \( \nu'(x) = \nu(x) + d \) (time elapsing); (2) for each \( x \in Y \): \( \nu'(x) = 0 \) (clocks resetting); (3) \( \nu + d \models g \) (respect of the guard).

In the same way, the semantic of a PPTA \((A, \ell_0)\) is given by its associated game \((G_A, \nu_0)\).

**Definition 12** ((Reachability) Timed games \( G_A \)). Let \((A, \ell_0) = (\Sigma, L, \rightarrow, C, \Pi, (L_i)_{i \in \Pi})\) be a PPTA, its associated game \((G_A, \nu_0) = (A_A, (g_i)_{i \in \Pi})\), called timed game, is such that: (i) \( A_A = (\Sigma, V, E, \Pi, (V_i)_{i \in \Pi}) \) where \( T_A = (\Sigma, V, E) \) is the associated transition system of \((A, \ell_0)\) and, for each \( i \in \Pi \), \((\ell, \nu) \in V_i \) if and only if \( \ell \in L_i \); (ii) for each \( i \in \Pi \), \( g_i : \text{Plays}_{A_A} \rightarrow \{0, 1\} \) is a gain function; (iii) \( \nu_0 = (\ell_0, 0) \) where 0 is the clock valuation such that for all \( c \in C \), \( 0(c) = 0 \).

If \((A, \ell_0)\) is a reachability PPTA, its associated timed game is a reachability game \((G_A, \nu_0) = (A_A, (g_i)_{i \in \Pi}, (F_i)_{i \in \Pi})\) such that for each \( i \in \Pi \), \((\ell, \nu) \in F_i \) if and only if \( \ell \in \text{Goal}_i \). We call this game a reachability timed game.

Thus, in a timed game, when it is the turn of Player \( i \) to play, if the play is in location \( \ell \), he has to choose a delay \( d \in \mathbb{R}^+ \) and a next location \( \ell' \) such that \((\ell, a, g, Y, \ell') \in \rightarrow \) for some \( a \in \Sigma \), \( g \in G \) and \( Y \in 2^C \). If the choice of \( d \) respects the guard \( g \), then the choice of Player \( i \) is valid: the clock valuation evolves according to the past clock valuation, \( d \) and \( Y \) and location \( \ell' \) is reached. Then, the play continues.

### 5.2 Regions and region games

In this section, we consider a bisimulation equivalence on \( T_A \) (the classical time-abstract bisimulation from [19]) which allows us to solve the constrained existence in the quotient of the original timed game (the region game). All along
this section we use the following notations. We denote by \( x_i \) the maximum value in the guard for clock \( c_i \). For all, positive number \( d \in \mathbb{R}^+ \), \( \lfloor d \rfloor \) is the integral part of \( d \) and \( \overline{d} \) is fractional part of \( d \).

**Definition 13 (≈ and region).**

- Two clock valuations \( \nu \) and \( \nu' \) are equivalent (written \( \nu \approx \nu' \)) iff: (i) \( \lfloor \nu_i \rfloor = \lfloor \nu'_i \rfloor \) or \( \nu_i, \nu'_i > x_i \), for all \( i \in \{1, \ldots, k\} \); (ii) \( \overline{\nu_i} = 0 \) iff \( \overline{\nu'_i} \), for all \( i \in \{1, \ldots, k\} \) with \( \nu_i \leq x_i \) and \( \nu'_i \leq \nu_i \); (iii) \( \nu_j \leq \nu_j \) iff \( \nu'_j \leq \nu'_j \) for all \( i \neq j \in \{1, \ldots, k\} \) with \( \nu_j \leq x_j \) and \( \nu_i \leq x_i \).

- We extend the equivalence relation to the states \( (\approx \subseteq V \times V) \) : \( (\ell, \nu) \approx (\ell', \nu') \) iff \( \ell = \ell' \) and \( \nu \approx \nu' \).

- A region \( r \) is an equivalence class for some \( v \in V \) : \( r = [v]_{\approx} \).

This equivalence relation on clocks and its extension to states of \( T_A \) is usual and the following result is well known [19].

**Lemma 4 ([19]).** Let \( (A, \ell_0) \) be a TA, \( \approx \subseteq V \times V \) is a bisimulation equivalence on \( T_A \).

It means that if \( (G_A, v_0) \) is a (reachability) timed game, \( \approx \) is a bisimulation equivalence on it. Moreover, it respects the partition. Thus, we can consider the (reachability) quotient game of this game. We call this game the (reachability) region game. Notice that \( \approx \) respects the target sets, so the reachability quotient game is defined as in Definition 9.

**Definition 14 ((Reachability) region game).** Let \( (G_A, v_0) \) be a (reachability) timed game and \( \approx \subseteq V \times V \) be the bisimulation equivalence defined in Definition 13, its associated (reachability) region game is its associated (reachability) quotient game \( (\tilde{G}_A, [v_0]) \).

We recall [19] that the size of \( \tilde{T}_A \), i.e., its number of states (regions) and edges, is in \( O((|V| + |\rightarrow|) \cdot 2^{\delta(A)}) \) where \( \delta(A) \) is the binary encoding of the constants (guards and costs) appearing in \( A \). Thus \( |\tilde{T}_A| \) is in \( O(2^{|A|}) \) where \( |A| \) takes into account the locations, edges and constants of \( A \). From this follows the following lemma.

**Lemma 5.** The (reachability) region game \( (\tilde{G}_A, [v_0]) \) is a finite (reachability) game.

Finally, in light of the construction of the reachability region game, the bisimulation equivalence \( \approx \) respects the gain functions of the reachability timed game and of the reachability region game.

**Lemma 6.** Given \( (G_A, v_0) = (A_A, (g_i)_{i \in I}, (F_i)_{i \in I}) \) be a reachability timed game and \( (\tilde{G}_A, [v_0]) = (\tilde{A}_A, (\tilde{g}_i)_{i \in I}, (\tilde{F}_i)_{i \in I}) \) its associated region game, \( \approx \) respects the gain functions.
Remark 3. Let \( A = (\Sigma, L, \rightarrow, C) \) be a timed automaton, \( T_A = (\Sigma, V, E) \) be its associated transition system and \( \approx \) be the bisimulation equivalence on \( T_A \) as defined in Definition 13, we have that \( ((\ell, \nu), a, (\ell', \nu')) \in E \) if and only if there exist \( g \in G, Y \subseteq 2^C \) and \( d \in \mathbb{R}^+ \) such that \( ((\ell, a, g, Y, \ell') \in \rightarrow, \nu' = [Y \leftarrow 0](\nu + d)) \) and \( \nu + d \models g \). Thus, we abstract the notion of time elapsing in the edges of the transition system.

Then, since \( \approx \) is a bisimulation equivalence on \( T_A \), for all \( ((\ell_1, \nu_1), a, (\ell_2, \nu_2)) \in E \) and for all \( (\ell_2, \nu_2) \in V \) such that \( (\ell_1, \nu_1) \approx (\ell_2, \nu_2) \), there exists \( (\ell_2, \nu_2') \in E \) such that \( ((\ell_2, \nu_2), a, (\ell_2', \nu_2')) \in E \) and \( (\ell_2', \nu_2') \approx (\ell_2', \nu_2') \). The time elapsing between \( \nu_1 \) and \( \nu_1' \) is not necessarily the same as between \( \nu_2 \) and \( \nu_2' \). Thus, \( \approx \) is a timed abstract bisimulation in the classical way [19].

5.3 Complexity results

Theorem 3. Given a reachability PPTA \( (A, \ell_0) \) and \( x, y \in \{0, 1\}^{|\Pi|} \), the constrained existence problem in reachability timed games is EXPTIME-complete.

The EXPTIME-hardness is due to a reduction from countdown games and is inspired by the one provided in [7, Section 6.3.3]. Thus, we only prove the EXPTIME-easiness.

Proof (EXPTIME-easiness). Given a PPTA \( (A, \ell_0) \) with target sets \( (\text{Goal}_i)_{i \in \Pi} \) and given \( x, y \in \{0, 1\}^{|\Pi|} \). Thanks to Theorem 1, it is equivalent to solve this problem in the reachability region game. Moreover, the size of the reachability region game is exponential, because its transition system \( \tilde{T}_A \) is exponential in the size of \( A \), but not in the number of players. Then, since the reachability region game is a finite reachability game (Lemma 5), we can apply Theorem 2. It causes an exponential blow-up in the number of players but is polynomial in the size of transition system \( \tilde{T}_A \). Thus, this entire procedure runs in (simple) exponential time in the size of the PPTA \( (A, \ell_0) \).

Notice that, since there always exists an SPE in a finite reachability game [20], there always exists an SPE in the region game and so in the reachability timed game (Theorem 4).

5.4 Time-bounded reachability, Zenoness and other extensions

In this paper, we focus on (qualitative) reachability timed games, and ignore the effect of Zeno behaviors.\(^4\) Nevertheless we believe that our approach is rather robust and can be extended to richer objectives and take into account Zeno behaviors. In the following paragraphs, we try to briefly explain how this could be achieve.

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\(^4\) A run \( \rho = (\ell_0, \nu_0) \xrightarrow{d_1, a_1} (\ell_1, \nu_1) \xrightarrow{d_2, a_2} \ldots \) in a timed automaton is said timed-divergent if the sequence \((\sum_{j \leq i} d_j)\), diverges. A timed automaton is non-Zeno if any finite run can be extended into a time-divergent run [6].
Time-bounded reachability. A natural extension of our framework would be to equip the objective of each player with a time-bound. Player $i$ aims at visiting $F_i$ within $TB_i$ time-units. We believe that this time-bound variant of our constrained problem is decidable. Indeed, for each player, his time-bound reachability objective can easily be encoded via a deterministic timed automaton (on finite timed words) $A_i$. Given a timed game $G_b$ equipped with a timed-bounded objective for each player (described via $A_i$), we could, via standard product construction build a new reachability timed game (without time-bound) $G$. Solving the constrained existence problem (with time-bound) in $G_b$ is equivalent to solving the constrained existence problem (of Definition 1) in $G$ (the constrained being encoded in the $A_i$’s). This approach could extend to any property that can be expressed via a deterministic timed automaton.

Towards $\omega$-regular objectives. Let us briefly explain how our approach could be adapted to prove the decidability of the constrained existence problem for timed games with $\omega$-regular objectives. For the sake of clarity, we here focus on parity objective. First, let us notice that the results of Section 3 (including Theorem 1) apply to a general class of games, including infinite games with classical $\omega$-regular objectives such as parity. An algorithm to decide the constrained existence problem (Definition 1) on parity on finite games can be found in [20] via translation into tree automata. Equipped with these two tools, we believe that we could adapt the definitions and results of Section 5 to obtain the decidability of the constrained existence problem for parity timed games. Notice that, in order to obtain our complexity results for finite reachability games, we use other simpler tools than tree automata.

About Zenoness. In the present paper, we allow a player to win (or to prevent other players to win) even if his strategy is responsible of Zeno behaviors. In [1], the authors propose an elegant approach to blame a player that would prevent divergence of time. The main idea is to transform the $\omega$-regular objective of each player into another one which will make him lose if he blocks the time. We believe that this idea could be exploited in our framework in order to prevent from winning a “blocking time player”.

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A Proofs of Section

For the sake of clarity, we denote \(\widetilde{\text{Plays}}\), \(\widetilde{\text{Hist}}\) and \(\widetilde{\text{Hist}}_i\) by \(\text{Plays}\), \(\text{Hist}\) and \(\text{Hist}_i\) respectively.

A.1 Proof of Proposition 1

In this section, when we consider a history \(h = h_0 \ldots h_n\) for some \(n \in \mathbb{N}\), the length of \(h\), denoted by \(|h|\), is its number of vertices.

This section is devoted to prove Proposition 1. Let \((G, v_0) = (A, (g_i)_{i \in \pi})\) be a game and \(\sim\) be a bisimulation equivalence on \((G, v_0)\) which respects the partition and such that for each \(\rho\) and \(\rho'\) in \(\text{Plays}\), if \(\rho \sim \rho'\) then \(g(\rho) = g(\rho')\).

If there exists an SPE \(\tau\) in \((G, v_0)\) which is uniform and such that \(g(\langle \tau \rangle_{v_0}) = p\), clearly there exists an SPE \(\sigma\) in \((G, v_0)\) such that \(g(\langle \sigma \rangle_{v_0}) = p\).

The difficult part is the other implication: if there exists an SPE \(\sigma\) in \((G, v_0)\) such that \(g(\langle \sigma \rangle_{v_0}) = p\), then there exists an SPE \(\tau\) in \((G, v_0)\) which is uniform and such that \(g(\langle \tau \rangle_{v_0}) = p\). Let us prove it.

Let \(\sigma\) be an SPE in \((G, v_0)\) such that \(g(\langle \sigma \rangle_{v_0}) = p\). In order to build \(\tau\), we need some additional material and notations that we explain below.

- for each \(h \in \text{Hist}(v_0)\): \([h] = \{h' \in \text{Hist}(v_0) | h \sim h'\};
- \(C^n = \{[h] | h \in \text{Hist}(v_0) \land |h| = n\};
- \(\mathcal{R} : \bigcup_{n \in \mathbb{N}} C^n \rightarrow \text{Hist}(v_0) \cup \{\bot\}\) which allow us to indentify a witness for each class;
- \(P : \text{Hist}(v_0) \rightarrow \{0, 1\}\)

Inductive construction of \(\mathcal{R}\) and \(P\) The first step, is to choose in a proper way a witness to each each class \([h]\). We proceed by induction on the length of histories. Moreover, we claim that the following properties are satisfied all along the inductive construction.

**Invariant 1:** For each \(hv \in \text{Hist}(v_0)\) such that \(\mathcal{R}([hv]) \neq \bot\),

\[ hv \sim \mathcal{R}([hv]). \]

**Invariant 2:** For each \(hv \in \text{Hist}(v_0)\) such that \(\mathcal{R}([hv]) \neq \bot\) and \(|hv| > 1\),

\[ \mathcal{R}([hv]) = \mathcal{R}([h]) \text{ Last}(\mathcal{R}([hv])) \]

**Invariant 3:** For each \(hv \in \text{Hist}(v_0)\) such that \(\mathcal{R}([hv]) \neq \bot\),

\[ \mathcal{R}([hv]) = \mathcal{R}([h']) \] \(\leq\) \mathcal{R}([hv])< h'\langle \sigma_{|h'}\rangle_{v'}

for some \(h'v'\) such that \(P(h'v') = 1\).

---

\(5\) If \(h = h_0 \ldots h_n\) for some \(n \in \mathbb{N}\), \(\text{Last}(h) = h_n\).
Before beginning the induction, we initialize $P$ and $R$ in the following way: for all $C \in \bigcup_{n \in \mathbb{N}} C^n$, $R(C) = \perp$ and for all $h \in \text{Hist}(v_0)$, $P(h) = 0$.

- For $n = 1$ : $C^1 = \{v_0\}$, we define $P(v_0) = 1$. Then, for each $h$ such that $v_0 \leq h < (\sigma)_v$, we define $R([h]) = h$. Thus, invariant 3 is satisfied with $h'v' = v_0$ and for each $v_0 < hv < (\sigma)_v$, $R([hv])$ is defined in this step and satisfies Invariant 2. Since $R([h]) = h$ for each witness defined in this step, Invariant 1 is satisfied too.

- Let us assume that these two invariant are satisfied after step $k$, and let us prove it remains true after step $k + 1$.

  - In this step, we first define $R$ for each $C \in C^{k+1}$ such that $R(C) = \perp$. We know that for all $h_1v_1, h_2v_2 \in C$, $h_1 \sim h_2$ and $R([h_1]) = R([h_2])$ are already defined (i.e., $\neq \perp$). Moreover, by Invariant 1, $h_1 \sim R([h_1])$, let $h = R([h_1])$, by bisimulation $\sim$, there exists $v \in V$ such that $h_1v_1 \sim hv$. We define $P(hv) = 1$ and $R(C) = hv$. Then, for all $h_2v_2 \in C$, $h_2v_2 \sim h_1v_1 \sim hv$ this implies that $h_2v_2 \sim R([h_2v_2])$ (Inv 1 ok). For all $h_2v_2 \in C$, $R([h_2v_2]) = R(C) = hv = R([h_1])v = R([h_2])v = R([h_2])\text{Last}(R([h_2v_2]))$ (Inv 2 ok). Moreover, for all $h_2v_2 \in C$, $R([h_2v_2]) = hv$ and $P(hv) = 1$, since $hv \leq hv < h(\sigma_{i,h})$, (Inv 3 ok).

Now, we extend the construction of $R$ and $P$ from $hv$ in the following way: for all $h'v' \in \text{Hist}(v_0)$ such that $hv < h'v' < h(\sigma_{i,h})$, we define $R([h'v']) = h'v'$. * 

Now, we have to prove that the invariants remains satisfied for all these new defined classes.

- $\forall h'v' \in [h'v '] : h'v' \sim h'v' = R([h'v']) = R([h'v'])$ (Inv 1 ok);
- $\forall h'v' \in [h'v ']$, we have that $h'v' \sim h'v'$ thus: $R([h'v']) = R([h'v']) = R([h'v']) \text{Last}(R([h'v']))$ (by construction *). Thus, since $h' \sim h'v' \sim R([h'v']) = R([h])$ (Inv 2 ok).
- $\forall h'v' \in [h'v']$, we have by construction * that $P(hv') = 1$ and $hv < h'v' < h(\sigma_{i,h})v$. Since $h'v' = R([h'v'])$ and $R([h'v']) = R([h'v']) (h'v' \sim h'v')$, we are done (Inv 3 ok).

**Construction of $\tau$** To build the uniform strategy profile $\tau$, we proceed as follows: for all $n \in \mathbb{N}$, for all $C \in C^n$, for all $h \in C$, by assuming that $\text{Last}(h) \in V_i$:

- If $R([h]) = h$ ($h$ is a witness, thus we want to follow $\sigma$): $\tau_i(h) = \sigma_i(h)$;
- If $R([h]) \neq h$ (we simulate $\sigma$): we know by Invariant 1 that $h \sim R([h])$, thus in particular $\text{Last}(h) \sim \text{Last}(R([h]))$, by bisimulation $\sim$, there exists $x \in V$ such that $\text{Last}(h)x \sim \text{Last}(R([h]))\sigma_i(R([h]))$. Thus, we define $\tau_i(h) = x$.

We state now, some properties about $\tau$ and $\sigma$. First, we define $\text{Wit} = \{ h \in \text{Hist}(v_0) \mid \exists C \in \bigcup_{n \in \mathbb{N}} C^n \text{ st. } R(C) = h \}$.

**Lemma 7.** For all $h \in \text{Hist}(v_0)$ such that $h \in \text{Wit}$ and $\text{Last}(h) \in V_i$: $\tau_i(h) = \sigma_i(h)$.

**Proof.** This assertion is true due to the construction of $\tau$. 
Lemma 8. For all $h, h' \in \text{Hist}(v_0)$ such that $h \sim h'$: $\tau_i(h) \sim \tau_i(h')$ by assuming that $\text{Last}(h) \in V_i$.

Notice that, since $\sim$ respects the partition, if $\text{Last}(h) \in V_i$ then $\text{Last}(h') \in V_i$, and vice versa.

Proof. Let $h, h' \in \text{Hist}(v_0)$ such that $h \sim h'$ and $\text{Last}(h) \in V_i$ for some $i \in \Pi$ then $\text{Last}(h') \in V_i$. We have that $R(\lceil h \rceil) = R(\lceil h' \rceil)$. By construction, $\tau_i(h) \sim \tau_i(h')$. By transitivity, we have: $\tau_i(h) \sim \tau_i(h')$. \qed

Lemma 9. For all $h \in \text{Wit}$, $h\tau_i(h) \in \text{Wit}$ (by assuming that $\text{Last}(h) \in V_i$ for some $i \in \Pi$).

Proof. Let $h \in \text{Wit}$, such that $\text{Last}(h) \in V_i$ for some $i \in \Pi$. Since $h \in \text{Wit}$, by Invariant 3, there exists $h'v' \in \text{Hist}(v_0)$ such that:

$$h'v' \leq h < h'(\sigma_{|h'|})v'.$$

Thus, we have that

$$h'v' \leq h\sigma_i(h) < h'(\sigma_{|h'|})v'.$$

It follows by construction of $R$, that $h\sigma_i(h) \in \text{Wit}$. Moreover, $h \in \text{Wit}$ implies that $\tau_i(h) = \sigma_i(h)$ (by Lemma 7). Thus, $h\tau_i(h) \in \text{Wit}$. \qed

Lemma 10. For all $hv \in \text{Wit}$, $\langle \sigma_{|h|} \rangle_v = \langle \tau_{|h|} \rangle_v$.

Proof. Let $hv \in \text{Wit}$, let $\rho = \langle \sigma_{|h|} \rangle_v$ and let $\overline{\rho} = \langle \tau_{|h|} \rangle_v$. Let us prove by induction that for all $n \in \mathbb{N}$:

1. $\rho_n = \overline{\rho}_n$;
2. $h\rho_0 \ldots \rho_n \in \text{Wit}$.

For $n = 0$, $\rho_0 = v$ and $\overline{\rho}_0 = v$. And by hypothesis, $hv \in \text{Wit}$. Let us assume that both assertions are satisfied for all $n$ such that $n \leq k$. Let us prove that it remains true for $n = k + 1$. By assuming that $\overline{\rho}_k \in V_i$,

1. $\overline{\rho}_{k+1} = \tau_i(h\overline{\rho}_0 \ldots \overline{\rho}_k) = \sigma_i(h\rho_0 \ldots \rho_k)$ By IH, $\overline{\rho}_0 \ldots \overline{\rho}_k = \rho_0 \ldots \rho_k$
   $= \sigma_i(h\rho_0 \ldots \rho_k)$ By IH, $h\rho_0 \ldots \rho_k \in \text{Wit}$ and by Lemma 7
   $= \rho_{k+1}$.

2. By IH, $h\rho_0 \ldots \rho_k \in \text{Wit}$, moreover we have that, by Lemma 7

$$h\rho_0 \ldots \rho_k \rho_{k+1} = h\rho_0 \ldots \rho_k \sigma_i(h\rho_0 \ldots \rho_k) = h\rho_0 \ldots \rho_k \tau_i(h\rho_0 \ldots \rho_k)$$

And by Lemma 9 we can conclude that $h\rho_0 \ldots \rho_k \tau_i(h\rho_0 \ldots \rho_k) \in \text{Wit}$. \qed
Proof that $\tau$ is an uniform SPE with gain profile $p$ There is still to prove that $\tau$ is an uniform SPE in $⟨G,v_0⟩$ such that $g(⟨\tau⟩_{v_0}) = p$. By Lemma 8, $\tau$ is uniform, let us prove this is an SPE with the gain profile $p$.

Proof. First, since $v_0 ∈ \text{Wit}$ and by Lemma 10 we have that $⟨\sigma⟩_{v_0} = ⟨\tau⟩_{v_0}$. Thus, in particular, $g(⟨\tau⟩_{v_0}) = g(⟨\sigma⟩_{v_0}) = p$.

By absurdum, let us assume that $\tau$ is not an SPE in $⟨G,v_0⟩$. It means that there exist $hv ∈ \text{Hist}(v_0)$, $i ∈ Π$ and a strategy $τ'_i$ of Player $i$ in $⟨G,h,v⟩$ such that $τ'_i$ is a profitable deviation of $τ_i|_{h}$, i.e.,

$$g_i(h⟨τ⟩_{hv}) < g_i(h⟨τ'_i,τ_{-i}|_{h}⟩_{hv}).$$

Let $h'v' = R([hv]) = R([h]) \text{Last}(R([hv]))$ by Invariant 2.

First step: Let $ρ = ⟨τ⟩_{hv}$ and $ρ' = ⟨τ'⟩_{hv'}$, let us prove by induction that for all $n ∈ \mathbb{N}$:

1. $ρ_n ∼ ρ'_n$;
2. $h'ρ_0′ ⋯ ρ'_n ∈ \text{Wit}$.

For $n = 0$, we have that $ρ_0 = v$ and $ρ'_0 = v'$, thus $v ∼ v'$ since $hv ∼ h'v'$. Moreover, $h'v' ∈ \text{Wit}$ by hypothesis. Let us assume that these two properties are satisfied for all $n$ such that $n ≤ k$, let us prove they remain true for $n = k + 1$. Let us assume that $ρ_k ∈ V_j$ for some $j ∈ Π$, since $∼$ respects the partition and due to the fact that $ρ_k ∼ ρ'_{k}$ by IH, we have that $ρ'_k ∈ V_j$.

1. $ρ_{k+1} = τ_j(hρ_0 ⋯ ρ_k) \sim τ_j(h'ρ'_0 ⋯ ρ'_k)$ By IH, $hρ_0 ⋯ ρ_k ∼ h'ρ'_0 ⋯ ρ'_k$ and by Lemma 8

$$= ρ'_{k+1}$$

2. $h'ρ'_0 ⋯ ρ'_{k+1} = h'ρ'_0 ⋯ ρ'_kτ_i(h'ρ'_0 ⋯ ρ'_k), h'ρ'_0 ⋯ ρ'_k ∈ \text{Wit}$ by IH, thus by Lemma 8 $h'ρ'_0 ⋯ ρ'_kτ_i(h'ρ'_0 ⋯ ρ'_k) ∈ \text{Wit}$.

It allows us to state by (1) that $hρ ∼ h'ρ'$, thus by hypothesis on $∼$, we have that

$$g(h⟨τ⟩_{hv}) = g(h'⟨τ⟩_{hv'}).$$

By (2) and Lemma 7 we have that $⟨τ⟩_{hv'} = ⟨σ⟩_{hv'}$ and thus:

$$g(h'⟨τ⟩_{hv'}) = g(h'⟨σ⟩_{hv'}).$$

Second step: Let $ρ = ⟨τ'_i,τ_{-i}|_{h}⟩_{v'}$, we will build a strategy $\tilde{τ}_i$ in $⟨G,h',v'⟩$ such that $ρ ∼ ⟨\tilde{τ}_i,τ_{-i}|_{h}⟩_{v'}$. Let $p ∈ \text{Hist}_i(v')$ and let us assume that $p = p_0 ⋯ p_m$ for some $m ∈ \mathbb{N}$. 


Let us assume that these two properties are true for all \( n \) that they remain true for \( \sigma \) true for all \( n \).

By IH, \( \tilde{\pi}_0 \) respects the partition. If \( \tilde{\pi}_0 = \rho_0 \) then since \( \tilde{\pi}_0 = \rho_0 \), we have that \( \langle \tilde{\pi}_1, \tau_{-1|\tilde{\pi}_1} \rangle \) respects the partition. If \( \tilde{\pi}_0 = \rho_0 \), then since \( \tilde{\pi}_0 = \rho_0 \), we have that \( \langle \tilde{\pi}_1, \tau_{-1|\tilde{\pi}_1} \rangle \) respects the partition.

From this we have that \( \tilde{\pi}_0 \sim \rho_0 \) and by construction of \( \tilde{\pi}_1 \) it follows:

\[
\tilde{\pi}_k+1 = \tilde{\pi}_1(\rho_0 \ldots \rho_k)
\]

\[
\sim \tau_i'(\rho_0 \ldots \rho_k)
\]

By IH, \( \tilde{\pi}_0 \sim \rho_0 \) and by construction of \( \tilde{\pi}_1 \) it follows:

\[
\tilde{\pi}_k+1 = \tau_j(h'\rho_0 \ldots \rho_k)
\]

\[
\sim \tau_j(h'\rho_0 \ldots \rho_k)
\]

By IH, \( h'\rho_0 \sim h\rho_0 \) and by Lemma \ref{lem:rho-sim-rho}

From this we have that \( h\rho \sim h'\rho \) and in particular:

\[
g_i(h'(\tilde{\pi}_1, \tau_{-1|\tilde{\pi}_1})) = g_i(h'(\tilde{\pi}_1, \tau_{-1|\tilde{\pi}_1})).
\]

(4)

Third step: From \( \tilde{\pi}_1 \), we build \( \tilde{\sigma}_i \) in \( (G_{j|h'}', v') \) which is a profitable devition of \( \sigma_{i|h'} \) \( v' \)

Let \( \tilde{\pi}_1 \) be the partition.

\[
\tilde{\sigma}_i(p) = \tilde{\pi}_1(p).
\]

\[
\tilde{\sigma}_i(p) \in \text{Wit}.
\]

\[
\tilde{\sigma}_i(p) \notin \text{Wit}.
\]

Let \( \pi = (\tilde{\sigma}_i, \sigma_{-1|\tilde{\pi}_1}) \) and \( \pi' = (\tilde{\sigma}_i, \tau_{-1|\tilde{\pi}_1}) \). Let us prove that for all \( n \in \mathbb{N} \):

1. \( \pi_n = \pi'_n \)
2. \( h'\pi_0 \ldots \pi_n \in \text{Wit} \)

For \( n = 0 \), we have that \( \pi_0 = v' = \pi'_0 \). Moreover, \( h'v' \in \text{Wit} \) by hypothesis. Let us assume that these two properties are true for all \( n \leq k \) and let us prove that they remain true for \( n = k + 1 \).

If \( \pi_k \in V_i \), then by IH, \( \pi_k = \pi'_k \in V_i \).

1. \( \pi_{k+1} = \tilde{\sigma}_i(\pi_0 \ldots \pi_k)
\]

\[
= \tilde{\sigma}_i(\pi'_0 \ldots \pi'_k)
\]

By IH, \( \pi_0 \ldots \pi_k = \pi'_0 \ldots \pi'_k \).
2.

\[ h'\pi_0 \ldots \pi_k \pi_{k+1} = h'\pi_0 \ldots \pi_k \sigma_i(\pi_0 \ldots \pi_k) \]
\[ = R([h'\pi_0 \ldots \pi_k]) \sigma_i(\pi_0 \ldots \pi_k) \quad \text{By IH, } h'\pi_0 \ldots \pi_k \in \text{Wit} \]
\[ \in \text{Wit} \quad \text{By construction of } \sigma_i. \]

- If \( \pi_k \in V_j \) \((j \neq i)\), then by IH, \( \pi_k = \pi'_k \in V_j \).

1.

\[ \pi_{k+1} = \sigma_j(h'\pi_0 \ldots \pi_k) \]
\[ = \tau_j(h'\pi_0 \ldots \pi_k) \quad \text{By IH, } h'\pi_0 \ldots \pi_k \in \text{Wit} \]
\[ = \tau_j(h'\pi_0 \ldots \pi'_k) \quad \text{By IH} \]
\[ = \pi'_k. \]

2.

\[ h'\pi_0 \ldots \pi_k \pi_{k+1} = h'\pi_0 \ldots \pi_k \tau_j(h'\pi_0 \ldots \pi_k) \]
\[ = h'\pi_0 \ldots \pi_k \tau_j(h'\pi_0 \ldots \pi_k) \quad \text{By IH, } h'\pi_0 \ldots \pi_k \in \text{Wit} \]
\[ \in \text{Wit} \quad \text{By Lemma} [7] \]

Thus, we can conclude that:

\[ g_i(h'(\tilde{\sigma}_i, \sigma_{-i} h')\nu') = g_i(h'(\tilde{\sigma}_i, \tau_{-i} h')\nu'). \quad (5) \]

Now, we want to prove that \( \pi' = \langle \tilde{\sigma}_i, \tau_{-i} h' \rangle \nu' \sim \tilde{\rho} = \langle \tilde{\tau}_i, \tau_{-i} h \rangle \nu \). Let us recall, that from the second step, we know that \( \tilde{\rho} \sim \rho = \langle \tau'_1, \tau_{-i} h \rangle \nu \). Let us prove that for all \( n \in \mathbb{N}: \pi'_n \sim \tilde{\rho}_n \).

For \( n = 0: \pi'_0 = \nu' = \tilde{\rho}_0 \). Let us assume that this property is true for all \( n \leq k \) and let us prove that it remains true for \( n = k + 1 \).

- If \( \pi'_k \in V_i \) then, by IH we have that \( \pi'_k \sim \tilde{\rho}_k \) and so \( \tilde{\rho}_k \in V_i \).

\[ \pi'_{k+1} = \tilde{\sigma}_i(\pi'_0 \ldots \pi'_k) \]
\[ = \text{Last}(R([h'\pi'_0 \ldots \pi'_k \tilde{\tau}_i(\pi'_0 \ldots \pi'_k)])) \quad h'\pi'_0 \ldots \pi'_k \in \text{Wit} \]
\[ \sim \tilde{\tau}_i(\pi'_0 \ldots \pi'_k) \]

By IH, we know that \( \pi'_0 \ldots \pi'_k \sim \tilde{\rho}_0 \ldots \tilde{\rho}_k \) and by hypothesis, we have that \( \tilde{\rho}_0 \ldots \tilde{\rho}_k \sim \rho_0 \ldots \rho_k \). It follows from the construction of \( \tilde{\tau}_i \) that \( \tilde{\tau}_i(\pi'_0 \ldots \pi'_k) \sim \tau'_i(\rho_0 \ldots \rho_k) \) and \( \tilde{\tau}_i(\tilde{\rho}_0 \ldots \tilde{\rho}_k) \sim \tau'_i(\rho_0 \ldots \rho_k) \). Thus, by transitivity, \( \pi'_{k+1} \sim \tilde{\tau}_i(\tilde{\rho}_0 \ldots \tilde{\rho}_k) = \tilde{\rho}_{k+1} \).

- If \( \pi'_k \in V_j \) \((j \neq i)\) then as previously \( \tilde{\rho}_k \in V_j \).

\[ \pi'_{k+1} = \tau_j(h'\pi'_0 \ldots \pi'_k) \]
\[ \sim \tau_j(\tilde{\rho}_0 \ldots \tilde{\rho}_k) \quad \text{By IH, } h'\pi'_0 \ldots \pi'_k \sim h'\tilde{\rho}_0 \ldots \tilde{\rho}_k \text{ and by Lemma} [8] \]
\[ = \tilde{\rho}_{k+1}. \]
Thus $h'\pi' \sim h'\hat{\rho}$ and it follows that:

$$g_i(h'\langle \tilde{\sigma}_i, \tau^{-i|h'}\rangle v') = g_i(h'\langle \tilde{\tau}_i, \tau^{-i|h'}\rangle v'). \quad (6)$$

Fourth step: putting all together

By (1), (2), (3), (4) and (6), we can conclude that

$$g_i(h'\langle \sigma_i|v'\rangle v') < g_i(h'\langle \tilde{\sigma}_i, \sigma^{-i|h'}\rangle v').$$

Thus, there exists a profitable deviation of $\sigma_i|v'$ for $i$ in $(G|_{i|h'}, v')$. This is impossible, since $\sigma$ is an SPE in $(G, v_0)$. \qed

A.2 Proof of Theorem 1

In this section we prove Theorem 3. In order to do so, we prove the two implications of the equivalence in two different propositions.

Proposition 4. Let $(G, v_0) = (A, (g_i)_{i \in \pi})$ be a game and $(\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in \pi})$ its associated quotient game where $\sim$ is a bisimulation equivalence on $(G, v_0)$. If $\sim$ respects the partition and the gain functions, we have that: if there exists an SPE $\sigma$ in $(G, v_0)$ such that $g(\langle \sigma \rangle v_0) = p$ for some $p \in \{0, 1\}$, then there exists an SPE $\tau$ in $(\tilde{G}, [v_0])$ such that $\tilde{g}(\langle \tau \rangle [v_0]) = p$.

Proof. Let $(G, v_0) = (A, (g_i)_{i \in \pi})$ be a game and $(\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in \pi})$ its associated quotient game where $\sim$ is a bisimulation equivalence on $(G, v_0)$ which respects the partition and the gain functions. We assume that there exists an SPE $\sigma$ in $(G, v_0)$ such that $g(\langle \sigma \rangle v_0) = p$ for some $p \in \{0, 1\}$.

Without loss of generality, we can assume thanks to Proposition 1 that $\sigma$ is uniform, i.e., for all histories $h, h' \in \text{Hist}(v_0)$ such that $\text{Last}(h) \in V_i \Leftrightarrow \text{Last}(h') \in V_i$, $\sigma_i(h) \sim \sigma_i(h')$.

Let $h \in \text{Hist}([v_0])$ be a history in the quotient game, by bisimulation $\sim_q \subseteq V \times \hat{V}$, there exists $h = h_0 \ldots h_n \in \text{Hist}(v_0)$ such that $h \sim_q \hat{h} = [h_0] \ldots [h_n]$. Let $v \in V$ be the vertex such that $\sigma_i(h) = v$, by assuming that $\text{Last}(h) \in V_i$. We have that $v \sim_q [v]$ and, we define $\tilde{\tau}_i(h) = [v]$.

**CLAIM 1:** $\forall h \in \text{Hist}([v_0]), \forall \tilde{h} \in \tilde{\text{Hist}}([v_0])$ such that $h \sim_q \hat{h}$, if $\text{Last}(h) \in V_i$, $\sigma_i(h) \sim_q \tilde{\tau}_i(h)$. \[35pt\]

**Proof:** Let $\hat{h} \in \tilde{\text{Hist}}([v_0])$ and $h \in \text{Hist}(v_0)$ such that $\text{Last}(h) \in V_i$ for some $i \in \pi$. By construction of $\tau$, there exists $h' \in \text{Hist}(v_0)$ such that $h' \sim_q \hat{h}$ and $\tau_i(h) = [\sigma_i(h')]$. If $h \sim_q \hat{h}$ and $h' \sim_q \hat{h}$, we have that $h \sim h'$. Thus, by uniformity of $\sigma$, $\sigma_i(h) \sim \sigma_i(h')$ and in particular $[\sigma_i(h)] = [\sigma_i(h')]$. In conclusion, $\sigma_i(h) \sim_q [\sigma_i(h)] = [\sigma_i(h')] = \tilde{\tau}_i(h)$. \[35pt\]

Let $\rho = (\langle \sigma \rangle v_0)_{i \in \pi}$. Let us prove that: $\forall n \in \mathbb{N}$ $\rho_n \sim \hat{\rho}$, Thus, $\rho \sim \hat{\rho}$ and since $\sim$ respects the gain functions, $g(\rho) = \tilde{g}(\rho) = p$.

For $n = 0$: $\rho_0 = v_0 \sim_q [v_0] = \hat{\rho}_0$. We assume that this is true for all $n \leq k$ and we prove it remains true for $n = k + 1$. By induction hypothesis,
we have that \( \rho_0 \ldots \rho_k \sim_q \hat{\rho}_0 \ldots \hat{\rho}_k \). By \( \sim_q \), \( \rho_k \in V_i \) if and only if \( \hat{\rho}_k \in \hat{V}_i \). Thus, \( \rho_{k+1} = \sigma_i(\rho_0 \ldots \rho_k) \sim_q \tau_i(\hat{\rho}_0 \ldots \hat{\rho}_k) = \hat{\rho}_{k+1} \) by Claim 1.

To conclude, we have to prove that \( \tau \) is an SPE in \((\hat{G}, [v_0])\). Ad absurdum, we assume that there exists \( \hat{h} \hat{v} \in \text{Hist}([v_0]) \) such that there exists a player \( i \in I \) and a profitable deviation \( \tau'_i \) of \( \tau_i|\hat{h} \) in \((\hat{G}|\hat{h}, \hat{v})\), i.e.,

\[
g_i(\hat{h}(\tau'_i)\hat{v}) < \tilde{g}_i(\hat{h}(\tau_i, \tau_{-i}|\hat{h})\hat{v}).
\] (7)

By bisimulation \( \sim_q \), there exists \( hv \in \text{Hist}(v_0) \) such that \( hv \sim_q \hat{h} \hat{v} \). We prove that Player \( i \) has a profitable deviation of \( \sigma_i|\hat{h} \) in \((\hat{G}|\hat{h}, v)\). From which a contradiction follows since \( \sigma \) has to be an SPE in \((G, v_0)\).

We build the profitable deviation \( \sigma'_i \). Let \( p \in \text{Hist}_i(v) \) a history such that \( p = \nu p_1 p_2 \ldots p_m \) for some \( m \in \mathbb{N} \). By bisimulation \( \sim_q \), there exists a unique \( \tilde{p} \) such that \( p = [\nu][p_1] \ldots [p_m] \) and thus \( p \sim_q \tilde{p} \). Let \( r \in \hat{V} \) be such that \( \tau'_i(\tilde{p}) = r \). By bisimulation \( \sim_q \), there exists \( x \in V \) such that \( \rho_m x \sim_q [\rho_m] r \). We define \( \sigma'_i(p) = x \). In particular, \( \sigma'_i(p) \sim_q \tau'_i(\tilde{p}) \).

Let \( \rho = \langle \sigma'_i, \sigma_{-i}|\hat{h} \rangle_v = \nu \rho_1 \rho_2 \ldots \) and \( \tilde{\rho} = \langle \tau'_i, \tau_{-i}|\hat{h} \rangle_{\hat{v}} = \hat{v} \hat{\rho}_1 \hat{\rho}_2 \ldots \). Let us show by induction that for all \( n, \rho_n \sim_q \tilde{\rho}_n \). It means that \( h\rho \sim_q \hat{h}\tilde{\rho} \) and since \( \sim_q \) respects the gain functions,

\[
g_i(h(\sigma'_i, \sigma_{-i}|\hat{h})v) = g_i(h\rho) = g_i(\hat{h}\tilde{\rho}) = \tilde{g}_i(\hat{h}(\tau'_i, \tau_{-i}|\hat{h})\hat{v}).
\] (8)

For \( n = 0 \): \( \rho_0 = v \sim_q \hat{v} = \hat{\rho}_0 \). Assume that this property is true for all \( n \leq k \) and let us prove it remains true for \( n = k + 1 \).

- **First case:** if \( \rho_k \in V_i \), by IH \( \rho_k \sim_q \hat{\rho}_k \) and thus \( \hat{\rho}_k \in \hat{V}_i \). It follows that:

\[
\rho_{k+1} = \sigma'_i(\rho_0 \ldots \rho_k) \\
\sim_q \tau'_i(\hat{\rho}_0 \ldots \hat{\rho}_k) \quad \text{by construction of } \sigma_i \text{ and } \rho_0 \ldots \rho_k \sim_q \hat{\rho}_0 \ldots \hat{\rho}_k \text{ (IH)} \\
= \hat{\rho}_{k+1}
\]

- **Second case:** if \( \rho_k \in V_j \) with \((j \neq i)\) then as previously \( \hat{\rho}_k \in \hat{V}_j \) and we have:

\[
\rho_{k+1} = \sigma_{j|\hat{h}}(\rho_0 \ldots \rho_k) \\
\sim_q \tau_{j|\hat{h}}(\hat{\rho}_0 \ldots \hat{\rho}_k) \quad \rho_0 \ldots \rho_k \sim_q \hat{\rho}_0 \ldots \hat{\rho}_k \text{ (IH)} \text{ and Claim 1.} \\
= \hat{\rho}_{k+1}
\]

There is still to prove that

\[
\tilde{g}_i(\hat{h}(\sigma_i|\hat{h})v) = g_i(h(\sigma_i|\hat{h})v).
\] (9)
We define $\sigma_0 \sim \tau_0 \in \mathcal{T}$ thus $h(\sigma_0) \sim h(\tau_0)$. The fact that $\sim_q$ respects the gain functions concludes the reasonment.

By (7), (8), and (9), we conclude that $\sigma'$ is a profitable deviation in $(G_{|h}, v)$.

\begin{proposition}
Let $(G, v_0) = (A, (g_i)_{i \in \mathcal{I}})$ be a game and $(\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in \mathcal{I}})$ its associated quotient game where $\sim$ is a bisimulation equivalence on $(G, v_0)$. If $\sim$ respects the partition and the gain functions, we have that: if there exists an SPE $\tau$ in $(\tilde{G}, [v_0])$ such that $\tilde{g}(\langle \tau \rangle_{[v_0]}) = p$ for some $p \in \{0, 1\}^{|\Pi|}$ then there exists an SPE $\sigma$ in $(G, v_0)$ such that $g(\langle \sigma \rangle_{v_0}) = p$.
\end{proposition}

\begin{proof}
Let $(G, v_0) = (A, (g_i)_{i \in \mathcal{I}})$ be a game and $(\tilde{G}, [v_0]) = (\tilde{A}, (\tilde{g}_i)_{i \in \mathcal{I}})$ its associated quotient game where $\sim$ is a bisimulation equivalence on $(G, v_0)$ which respects the partition and the gain functions. We assume that there exists an SPE $\tau$ in $(\tilde{G}, [v_0])$ such that $\tilde{g}(\langle \tau \rangle_{[v_0]}) = p$ for some $p \in \{0, 1\}$.

Let $h \in \text{Hist}(v_0)$ such that $\text{Last}(h) \in V_i$ for some $i \in \mathcal{I}$. Thanks to bisimulation $\sim_q$, there exists a unique $\tilde{h}$ in $\text{Hist}_i([v_0])$ such that $h \sim_q \tilde{h}$ (\star). We have that $\tau_i(\tilde{h}) = \tilde{v}$ for some $\tilde{v} \in V$, thus by $\sim_q$ there exists $v \in V$ such that $hv \sim_q h\tilde{v}$.

We define $\sigma_i(h) = v$.

\begin{claim}
1. \forall h, h' \in \text{Hist}(v_0) \text{ such that } h \sim h': \sigma_i(h) \sim \sigma_i(h') \text{ (if Last}(h) \in V_i).
2. \forall h \in \text{Hist}(v_0), \forall \tilde{h} \in \text{Hist}([v_0]) \text{ such that } h \sim_q \tilde{h}: \sigma_i(h) \sim_q \tau_i(\tilde{h}) \text{ (if Last}(h) \in V_i).
\end{claim}

\begin{proof}
1. By (\star), we have that for all $h \sim h' \in \text{Hist}_i(v_0)$ there exists a unique $\tilde{h}$ such that $h \sim_q \tilde{h}$ and $h' \sim_q \tilde{h}$. It follows by construction of $\sigma$ that $\sigma_i(h) \sim_q \tau_i(\tilde{h})$ and $\sigma_i(h') \sim_q \tau_i(\tilde{h})$ and thus $\sigma_i(h) \sim \sigma_i(h')$. It means that $\sigma$ is uniform.

2. let $h \in \text{Hist}(v_0)$ and $\tilde{h} \in \text{Hist}([v_0])$ be two histories such that Last$(h) \in V_i$ iff $\text{Last}(\tilde{h}) \in V_i$ for some $i \in \mathcal{I}$ and such that $h \sim_q \tilde{h}$. By construction of $\sigma$, there exists $\tilde{g} \in \text{Hist}_i([v_0])$ such that $h \sim_q \tilde{g}$ and $\sigma_i(h) \sim_q \tau_i(\tilde{g})$. But by $\sim_q$, if $h \sim_q \tilde{g}$ and $h \sim_q \tilde{h}$, then $\tilde{g} = \tilde{h}$. It concludes the proof.
\end{proof}

By (2) in Claim 2, we have that $\langle \sigma \rangle_{v_0} \sim_q \langle \tau \rangle_{[v_0]}$. It follows, due to the fact that $\sim_q$ respects the gain functions, that $g(\langle \sigma \rangle_{v_0}) = g(\langle \tau \rangle_{[v_0]}) = p$.

Now, we prove that $\sigma$ is an SPE. Ad absurdum, we assume that there exists $hv \in \text{Hist}(v_0)$, there exists $i \in \mathcal{I}$ and there exists $\sigma'_i$ a profitable deviation of $\sigma_i|_h$ for Player $i$ in $(G_{|h}, v)$, i.e.,

$$g_i(h(\sigma_i|_h) v) < g_i(h(\sigma'_i, \sigma_i|_h) v) \tag{10}$$
Let $\tilde{h}v = [h_0|h_1] \ldots [v]$ with $[h_0] = [u_0]$ we have that $hv \sim_q \tilde{h}v$. By (2) in Claim 2, we have that $h(\sigma_{i|h})v \sim_q \tilde{h}(\tau_{i|h})\tilde{v}$, since $\sim_q$ respects the gain functions, it follows:

$$g_i(h(\sigma_{i|h})v) = \tilde{g}_i(\tilde{h}(\tau_{i|h})\tilde{v})$$  \hspace{1cm} (11)

To obtain the contradiction, we build $\tau'_i$ a profitable deviation of $\tau_{i|h}$ for Player $i$ in $(\tilde{G}_{i|h}, \tilde{v})$.

Let $\rho = (\sigma'_i, \sigma_{i|h})_v$, let $\tilde{p} \in \tilde{\text{Hist}}_i(\tilde{v})$, we define $\tau_i(\tilde{p})$ as follows:

$$\tau'_i(\tilde{p}) = \begin{cases} [\rho_{n+1}] & \text{if } \tilde{p} < [\rho_0][\rho_1] \ldots \text{ and } \text{Last}(\tilde{p}) = [\rho_n]. \\ \text{some } r \in \text{Succ}(\text{Last}(\tilde{p})) & \text{otherwise} \end{cases}$$

Let $\tilde{\rho} = (\tau'_i, \tau_{-i|h})\tilde{v}$ and let us prove that $\rho \sim_q \tilde{\rho}$, i.e., $\forall n \in \mathbb{N}\rho_n \sim_q \tilde{\rho}_n$. We proceed by induction on $n$.

For $n = 0$, $\rho_0 = v \sim_q [v] = \tilde{v} = \tilde{\rho}_0$. Let us assume that this property is true for all $n \leq k$ and let us prove it remains true for $n = k + 1$.

- **First case:** If $\rho_k \in V_i$, then since $\rho_k \sim_q \tilde{\rho}_k$ by IH, $\tilde{\rho}_k \in \tilde{V}_i$. It follows that:

$$\rho_{k+1} \sim_q [\rho_{k+1}] = \tau'_i(\tilde{\rho}_0 \ldots \tilde{\rho}_k) = \tau'_i(\tilde{\rho}_0 \ldots \tilde{\rho}_k)$$

- **Second case:** If $\rho_k \in V_j$ (with $j \neq i$), then as previously $\tilde{\rho}_k \in \tilde{V}_j$. It follows that:

$$\rho_{k+1} = \sigma_{j|h}(\rho_0 \ldots \rho_k) = \sigma_j(\tilde{h}\rho_0 \ldots \tilde{h}\rho_k) \sim_q \tau_j(\tilde{h}\rho_0 \ldots \tilde{h}\rho_k)$$

Thus, $\rho \sim_q \tilde{\rho}$ and so $h\rho \sim_q \tilde{h}\tilde{\rho}$. Since, $\sim_q$ respects the gain functions, we can conclude that:

$$g_i(h(\sigma'_{i}, \sigma_{-i|h})_v) = \tilde{g}_i(\tilde{h}(\tau'_{i}, \tau_{-i|h})\tilde{v}).$$  \hspace{1cm} (12)

By (11), (11) and (12), we can state that:

$$\tilde{g}_i(\tilde{h}(\tau'_{i}, \tau_{i|h})\tilde{v}) = g_i(h(\sigma'_{i}, \sigma_{-i|h})_v) > g_i(h(\sigma_{i|h})_v) = \tilde{g}_i(\tilde{h}(\tau_{i|h})\tilde{v}).$$
B Additional material for Section 5

In [7], Proposition 6.12 asserts that the value problem for timed games with Büchi objectives and only two clocks is EXPTIME-hard. The proof relies on the notion of countdown game [14] which is known to be EXPTIME-complete. When reading the proof of the latter proposition, one can easily be convinced that it is also proved that the value problem for timed games with reachability objectives and only two clocks is EXPTIME-hard. Indeed, the only accepting state is a deadlock with a self-loop (named \( w_\exists \)). Moreover, one can also notice that although the results of [7] concern concurrent games, the proof of Proposition relies on turn-based games.

The proof of Proposition 6.12 can be slightly modified in order to prove that the constrained existence problem in reachability timed games is EXPTIME-hard with two clocks. The problem in the original proof being that Adam does not have a reachability, but a safety objective. Given a countdown game \( C \), we build a reachability timed games by using nearly the same construction as the one presented in the proof of [7, Proposition 6.12]. The difference are the following ones.

- We replace all the guards \( y \neq c_0 \) by the guards \( y < c_0 \).
- We add a winning state for Adam \( w_\forall \).
- From every state belonging to Eve, we add a transition to \( w_\forall \) with guard \( x = 0 \land y > c_0 \).

The proposed transformations does not really affect the behaviors of the timed game, in the sense that it still bisimulates closely the countdown game. The only difference is discussed below. In the original encoding, Eve was winning if and only if she is able to reach \( w_\exists \). This could happen only when the clock \( y \) is equal to \( c_0 \). As the game is zero-sum, Adam was winning when \( w_\exists \) is never reached. In practice, as the timed game of the encoding is strongly non-zeno, in every winning play of Adam, the clock value \( y \) eventually overtakes \( c_0 \). In our new encoding, every winning play of Adam ends up in \( w_\forall \). That is the only difference. This is important, as we can now see the timed game as a reachability time game where both players have a reachability objective. One can be convinced that Eve as a winning strategy (in the original timed game proposed in [7]) if and only if there exists an SPE where only Eve achieves her objective (in the variant of the timed game proposed above).