Lie and Noether point symmetries of a class of quasilinear systems of second-order differential equations

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Abstract

We study the Lie and Noether point symmetries of a class of systems of second-order differential equations with \(n\) independent and \(m\) dependent variables (\(n \times m\) systems). We solve the symmetry conditions in a geometric way and determine the general form of the symmetry vector and of the Noetherian conservation laws. We prove that the point symmetries are generated by the collineations of two (pseudo)metrics, which are defined in the spaces of independent and dependent variables. We demonstrate the general results in two special cases (a) a system of \(m\) coupled Laplace equations and (b) the Klein-Gordon equation of a particle in the context of Generalized Uncertainty Principle. In the second case we determine the complete invariant group of point transformations, and we apply the Lie invariants in order to find invariant solutions of the wave function for a spin-0 particle in the two dimensional hyperbolic space.

Keywords: Lie symmetries; Noether symmetries; Quasilinear systems

1 Introduction

Lie symmetries is a powerful tool for the study of differential equations, because they provide invariant functions which can be used to reduce the order of a differential equation or reduce the number of variables, and possibly lead to the determination of analytic solutions. For differential equations which arise from a variational principle, i.e. follow from a Lagrangian, the Lie point symmetries which in addition leave the action invariant are called Noether point symmetries. Lie point symmetries span a Lie algebra and their specialization Noether point symmetries span a subalgebra. According to Noether’s theorem to each Noether point symmetry there corresponds a conservation law \([1,3]\). Conservation laws play an important role in Classical Mechanics, General Relativity, field theory and in the study of dynamical systems in general \([4,10]\).

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In this work we study the Lie point symmetries and the Noetherian conservation laws of the class of second-order differential equations which follow from the Lagrangian

$$L(x^k, u^C, u^C_{,k}) = \frac{1}{2} \sqrt{g} g^{ij} H_{AB} u^A_i u^B_j - \sqrt{g} V(x^k, u^C)$$  \hspace{1cm} (1)

where $g_{ij} = g_{ij}(x^k)$, $g^{ij} g_{ij} = \delta^i_j$; $H_{AB} = H_{AB}(u^C)$, with $H_{AB} H^{AB} = \delta_A^B$, and $u^A_i = \frac{\partial u^A}{\partial u^C}$. $H_{AB}$ is the metric of the $m$ dependent variables $u^C(x^k)$, $\dim H_{AB} = m$, $g_{ij}$ is the metric of the $n$ independent variables $x^i$, $\dim g_{ij} = n$, and $\delta^i_j$ is the Kronecker delta.

Lagrangian (1) leads to the following system of Euler-Lagrange equations

$$P^A(x^k, u^C, u^C_{,i}, u^C_{,ij}) \equiv g^{ij} u^A_{,ij} + g^{ij} C^A_{BC} u^B_i u^C_j - \Gamma^i u^A_i + F^A(x^k, u^C) = 0,$$  \hspace{1cm} (2)

where $C^A_{BC} = C^A_{BC}(u^D)$, $\Gamma^i = g^{jk} \Gamma^i_{jk} (x^\tau)$ are the connection coefficients of the metrics $H_{AB}$ and $g_{ij}$ respectively, and $F^A = H^{AB} V_B$. The system (2) consists of $m$ equations and depends on $n$ variables; we call it a $n \times m$ system. For $n = 1$ the system (2) reduces to $m$ second-order ordinary differential equations, and for $m = 1$ the system (2) describes a second-order partial differential equation.

The determination of Lie point symmetries of the system (2) consists of two steps: (a) the determination of the conditions which the symmetry vector must satisfy (symmetry conditions), and (b) the solution of these conditions. The first step is formal, however the symmetry conditions which arise can be quite involved. One way to “solve” the system of the symmetry conditions is to write them in geometric form and then use the methods of Differential Geometry to solve them. In this way the determination of the Lie point symmetries of a differential equation is reduced to a problem of Differential Geometry where there is an abundance of known results and methods to work. Indeed the Lie point symmetries of the $1 \times m$ systems of the form (2) have been solved in this manner. Specifically, it has been proved that the Lie point symmetries of a $1 \times m$ system are generated by the elements of the special projective algebra of the space $H_{AB}$ of the dependent variables [11,12], and the Lie point symmetries form the projective group of an affine space $V^{1+m}$ [13,14]. Moreover for a class of singular $1 \times m$ systems in which the Hamiltonian function is vanished the Lie and the Noether point symmetries follow from the conformal algebra of the space of the dependent variables [15].

The geometric approach has also been applied to the case of $n \times 1$ equations of the form (2) and it has been proved that in this case the Lie point symmetries are generated by the elements of the conformal algebra of the space of the independent variables $g_{ij}$ [10,18].

In the following we generalize the above results in the case of the $n \times m$ system (2). In particular, we show that the Lie point symmetry vectors in the space $\{x^i, u^C\}$ follow from the affine collineations of the metric $H_{AB}$ and the conformal Killing vectors of the metric $g_{ij}$. Moreover, there exists a connection between the two algebras if and only if the space $H_{AB}$ admits a gradient homothetic vector. The structure of the paper is as follows.

In section 2 we present the basic definitions concerning the Lie and the Noether point symmetries of differential equations as well as the collinearations of a Riemannian space. The Lie point symmetries of the system (2) are studied in section 3 where we prove that the generic Lie point symmetry vector is generated by the affine algebra and the conformal algebra of the two metrics $H_{AB}$ and $g_{ij}$ respectively. For the Noether point symmetries of Lagrangian (1) we derive the generic form of the Noether vector and of the corresponding

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1The Latin indices $i,j,...$ take the values $1,2,...,n$, and the capital indices $A,B,...$ take the values $1,2,...,m$. 

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coupled Laplace equations. We find that if the system of $m$ coupled Laplace equations (with $n > 2$) admits $(\frac{n+2(n+1)}{2} + m(m+1)$ Lie point symmetries, then the two spaces $H_{AB}$, $g_{ij}$ are flat.

In section 5 we consider the modified Klein-Gordon equation of a particle in the Generalized Uncertainty Principle which is a fourth order partial differential equation. With the use of a Lagrange multiplier we reduce this equation to a system of two second-order partial differential equations of the form of system 2. We study the Lie and the Noether point symmetries of the new system and show that if the $g_{ij}$ space admits an $N$ dimensional Killing algebra then, the Lie and the Noether point symmetries form Lie algebras of dimension $N + 3$ and $N + 2$ respectively. We apply this result in two cases of special interest: (A) the particle lives in the flat Minkowski space-time $M^4$, and (B) the particle lives in a two dimensional hyperbolic sphere. In the latter case, we apply the zero-order invariants of the Lie point symmetries in order to determine invariant solutions of the wave function. Finally, in section 6 we draw our conclusions.

2 Preliminaries

For the convenience of the reader, in this section we discuss briefly the Lie and Noether point symmetries of differential equations and the collineations of Riemannian manifolds.

2.1 Lie point symmetries of differential equations

Geometrically a differential equation (DE) may be considered as a function $H = H(x^i, u^A, u^A, u_{ij}^A)$ in the space $B = B(x^i, u^A, u_{ij}^A)$, where $x^i$ are the independent variables and $u^A$ are the dependent variables. The infinitesimal point transformation

$$
\begin{align*}
\bar{x}^i &= x^i + \varepsilon \xi^i(x^k, u^B), \\
\bar{u}^A &= u^A + \varepsilon \eta^A(x^k, u^B),
\end{align*}
$$

has the infinitesimal symmetry generator

$$
X = \xi^i(x^k, u^B)\partial_{x^i} + \eta^A(x^k, u^B)\partial_{u^A}. 
$$

The generator $X$ of the infinitesimal transformation (3)-(4) is called a Lie point symmetry of the DE $H = 0$ if there exists a function $\kappa$ such that the following condition holds

$$
X^{[2]}(H) = \kappa H 
$$

where

$$
X^{[2]} = X + \eta^A_i \partial_{u^A} + \eta^A_{ij} \partial_{u_{ij}^A} 
$$

is the second-prolongation vector of $X$, in which

$$
\eta^A_i = \eta^A_i + u^B_i \eta^A_B - \xi^i_j u^A_j - u^A_i u^B_j \xi^j_k, 
$$

and

$$
\eta^A_{ij} = \eta^A_{ij} + 2\eta^A_{B(i}u^B_{j)} - \xi^k_i u^A_k + \eta^A_{BC}u^B_i u^C_j - 2\xi^k_i u^B_{ij} u^A_k - \xi^k_{BC} u^B_i u^C_j u^A_k + \eta^A_{B(i}u^B_{j)} u^A_k - 2\xi^k_{BC} u^B_i u^C_j u^A_k.
$$
An application of Lie point symmetries of a DE is that they can be used in order to determine invariant solutions. From the generator \( \xi \) one considers the Lagrange system

\[
\frac{dx^i}{\xi^i} = \frac{du^A}{\eta^A} = \frac{du^A_i}{\eta^A_{ij}} = \frac{du^A_{ij}}{\eta^A_{ij}}
\]

whose solution provides the characteristic functions \( W[0] (x^k, u, u_i) \), \( W[1] (x^k, u, u_i, u_{ij}) \) and \( W[2] (x^k, u, u_i, u_{ij}, u_{ij}) \). The characteristic functions can be applied to reduce the order of the DE or the number of the dependent variables.

Suppose that the DE, \( H = H(x^i, u^A, u^A_i, u^A_{ij}) \), arises from a variational principle, i.e. there exists a Lagrangian function \( L = L(x^k, u^A, u^A_k) \) such that \( H \equiv E(L) = 0 \), where \( E \) is the Euler operator. The Lie point symmetry \( X \) of the DE \( H \) is a Noether point symmetry of \( H \), if the following additional condition is satisfied

\[
X[1] L + LD_i \xi^i = D_i A^i (x^k, u^C)
\]

where \( X[1] \) is the first prolongation of \( X \), and \( D_i \) is the covariant derivative wrt the metric \( g_{ij} \) in the space of variables \( \{x^i\} \), and \( A^i \) is the Noether current. The characteristic property of Noether point symmetries is that the quantity

\[
I^i = \xi^k \left( u^A_k \frac{\partial L}{\partial u^A_i} - \delta^k_i L \right) - \eta^A \frac{\partial L}{\partial u^A_i} + A^i
\]

is a first integral of Lagrange equations, that is, \( D_i I^i = 0 \).

### 2.2 Collineations of Riemannian spaces

A collineation in a Riemannian space is a vector field \( \xi \) which satisfies an equation of the form

\[
\mathcal{L}_\xi A = B
\]

where \( \mathcal{L}_\xi \) is the Lie derivative with respect to the vector field \( \xi \), \( A \) is a geometric object (not necessarily a tensor) defined in terms of the metric and its derivatives and \( B \) is an arbitrary tensor with the same indices as the geometric object \( A \). The collineations of Riemannian spaces have been classified by Katzin et.al.\(^2\).

In the following we are interested in the collineations of the metric tensor, i.e. \( A = \tilde{g}_{\alpha\beta} (z^\gamma) \), and on the affine collineations of the connection coefficients \( \tilde{\Gamma}^\alpha_{\beta\gamma} (z^\delta) \).

#### 2.2.1 Conformal symmetries

The infinitesimal generator \( \xi \) of the point transformation

\[
\tilde{z}^\alpha = z^\alpha + \varepsilon \xi^\alpha (z^\delta)
\]

is called Conformal Killing Vector (CKV) if the Lie derivative of the metric \( \tilde{g}_{\alpha\beta} \) with respect to the vector field \( \xi \) is a multiple of \( \tilde{g}_{\alpha\beta} \). That is, if the following condition holds

\[
\mathcal{L}_\xi \tilde{g}_{\alpha\beta} = 2\psi (z^\gamma) \tilde{g}_{\alpha\beta}
\]

where \( \psi = \frac{1}{\eta} \xi^\gamma \).

\(^2\)The reason that we consider the symbol \( \tilde{g}_{\alpha\beta} (z^\gamma) \) is because in our case we have two metrics in different spaces, that is, the metrics \( H_{AB} \) and \( g_{ij} \).
When $\psi, \gamma \delta = 0$, $\xi$ is a special CKV (sp.CKV), if $\psi =$constant, $\xi$ is a Homothetic Vector (HV) and when $\psi = 0$, $\xi$ is a Killing Vector (KV). A metric $\tilde{g}_{\alpha \beta}$ admits at most one HV. The CKVs of a metric form a Lie algebra, which is called the conformal algebra, $G_{CV}$. Obviously the KVs and the homothetic vector are elements of the conformal algebra $G_{CV}$. If $G_{HV}$ is the algebra of HVs (including the algebra $G_{KV}$ of KVs), then we have

$$G_{KV} \subseteq G_{HV} \subseteq G_{CV}$$  \hspace{1cm} (16)

The maximum dimension of the conformal algebra of an $\ell -$ dimensional metric ($\ell > 2$) is $G_{max} = \frac{\ell}{2} (\ell + 1) (\ell + 2)$, and for $\ell = 2$ the space admits an infinite dimensional conformal group. Moreover, if a space $\tilde{g}_{\alpha \beta}$ of dimension $\dim \tilde{g}_{\alpha \beta} > 2$ admits a conformal algebra of dimension $G_{max}$, then it is conformally flat, that is, that there exists a function $N(z^\gamma)$ such that $\tilde{g}_{\alpha \beta} = N(z^\gamma) \eta_{\alpha \beta}$ where $\eta_{\alpha \beta}$ is a flat metric.

CKVs are important in relativistic physics and the effects of the existence of these vectors can be seen at all levels in General Relativity, that is, geometry, kinematics and dynamics. We continue with the definition of the collineations for the connection coefficients of the metric tensor $\tilde{g}_{\alpha \beta}$.

### 2.2.2 Affine collineations

In a Riemannian space with metric $\tilde{g}_{\alpha \beta}$ and connection coefficients $\tilde{\Gamma}^\alpha_{\beta \gamma}(z^\delta)$, the following identity holds

$$L_\xi \tilde{\Gamma}^\alpha_{\beta \gamma} = \tilde{g}^{\alpha \delta} \left[ (L_\xi \tilde{g}_{\beta \delta})_{,\gamma} + (L_\xi \tilde{g}_{\delta \gamma})_{,\beta} - (L_\xi \tilde{g}_{\beta \gamma})_{,\delta} \right]. \hspace{1cm} (17)$$

If $\xi$ is a HV or KV then from (17) follows that $L_\xi \tilde{\Gamma}^\alpha_{\beta \gamma}$ vanishes, which implies that the connection coefficients $\tilde{\Gamma}^\alpha_{\beta \gamma}$ are invariant under the action of transformation (14). In general the infinitesimal generators which leave invariant the connection coefficients $\tilde{\Gamma}^\alpha_{\beta \gamma}$ are defined by the condition

$$L_\xi \tilde{\Gamma}^\alpha_{\beta \gamma} = 0. \hspace{1cm} (18)$$

and are called Affine collineations (AC).

The geometric property of an AC is that it carries a geodesic into a geodesic and also preserves the affine parameter along each geodesic. The ACs of a Riemannian space form a Lie algebra, which is called the Affine algebra, $G_{AC}$ of the space. Obviously the homothetic algebra $G_{HV}$, is a subalgebra of $G_{AC}$, i.e. $G_{HV} \subseteq G_{AC}$. We shall say that a spacetime admits proper ACs when $\dim G_{HV} < \dim G_{AC}$. Note that the proper CKVs do not satisfy condition (18) therefore proper CKVs are not ACs.

In the case of a flat space, condition (18) becomes

$$\xi^\alpha_{,\beta \gamma} = 0, \hspace{1cm} (19)$$

whose general solution is $\xi^\alpha = A^\alpha_{\beta} z^\beta + B^\alpha$; where $A^\alpha_{\beta}, B^\beta$ are $\ell (\ell + 1)$ constants. Therefore the flat space admits the maximal $\ell (\ell + 1)$ dimensional Affine algebra. The converse is also true, that is, if a Riemannian space with metric $\tilde{g}_{\alpha \beta}$, $\dim \tilde{g}_{\alpha \beta} = \ell$, admits the Affine algebra $G_{AC}$ with $\dim G_{AC} = \ell (\ell + 1)$ then the space is flat. We summarize the above definitions in table 1

\footnote{For the conformal factor of a sp.CKV holds $\psi_{,\delta} = 0$, that is, $\psi_{,i}$ is a gradient KV. A Riemannian space admits a sp.CKV if and only if it admits a gradient KV and a gradient HV \cite{21}.}
### 3 Lie and Noether point symmetries of a class of quasilinear systems of second-order differential equations

The Lie point symmetry condition \((6)\) for the system of equations \((2)\) has the general form

\[
X^{[2]} P^A = \kappa^A_D P^D
\]

where \(\kappa^A_D\) is a tensor. Replacing \((5)\) and \((4)\), for each term of the left-hand side of condition \((20)\) we find

\[
\eta^i_D \frac{\partial P^A}{\partial u^D} = g^{ij} C^A_{BC,D} \eta^j_D (u^C_j u^A_i) + F^A_{ij} \eta^D
\]

\[
(21)
\]

\[
\xi^k \frac{\partial P^A}{\partial x^k} = g^{ij} \xi^k (u^A_{ij}) + g^{ij} \xi^k C^A_{BC} (u^B_j u^C_i) - \Gamma^i \xi^k (u^A_j) + F^A_{ij} \xi^k
\]

\[
(22)
\]

\[
\eta^B_i \frac{\partial P^A}{\partial u^B_j} = 2 g^{ij} C^A_{BC} \eta^B_i (u^C_j) + 2 g^{ij} C^A_{BC} \eta^B_u (u^C_j u^A_i) +
\]

\[
-2 g^{ij} C^A_{BC} \xi^k (u^B_j u^C_i) - 2 g^{ij} C^A_{BC} \xi^k (u^B_j u^D_k u^C_i) +
\]

\[
-\Gamma^i \eta^B_j - \Gamma^i \eta^B_i (u^B_j) + \Gamma^i \xi^B_j (u^A_j) + \Gamma^i \xi^B_i (u^A_j)
\]

\[
(23)
\]

and

\[
\eta^B_i \frac{\partial P^A}{\partial u^B_{ij}} = g^{ij} \eta^A_{ij} + 2 g^{ij} \eta^A_{B(i} (u^B_j) - g^{ij} \xi^k (u^A_{k}) +
\]

\[
+ g^{ij} \eta^A_{BC} (u^B_j u^C_i) - 2 g^{ij} \xi^k (u^A_{ijk}) +
\]

\[
- g^{ij} \xi^k (u^A_{ijk}) + g^{ij} \eta^A_B (u^B_{ij}) +
\]

\[
-2 g^{ij} \xi^B_j (u^A_{ijk}) - g^{ij} \xi^B_i (u^A_{ijk}) + 2 u^B_i u^A_j k
\]

\[
(24)
\]

where indices enclosed in parentheses mean symmetrization, for instance, \(K_{(ij)} = \frac{1}{2} (K_{ij} + K_{ji})\).

We consider the right-hand side of \((16)\) and we introduce new quantities \(\lambda^A_D, \mu^k_D\) by means of the following relation

\[
\kappa^A_D P^D = g^{ij} \lambda^A_B (u^D_{ij}) + g^{ij} \lambda^A_{BC} C^D_{BC} (u^B_j u^C_i) - \Gamma^i \lambda^A_D (u^D_i) + \lambda^A_F P^D +
\]

\[
+ g^{ij} \mu^k_D (u^A_{ijk}) + g^{ij} \mu^k_{BC} C^D_{BC} (u^A_k u^B_j u^C_i) - \Gamma^i \mu^k_D (u^A_{ijk}) + \mu^k_F P^D (u^A_k)
\]

\[
(25)
\]

In order condition \((20)\) to hold identically the coefficients of the terms of the various derivatives of \(u^A\) in the total expression must be equal.

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| Collineation \(\mathcal{L}_\xi \mathbf{A} = \mathbf{B}\) | \(\mathbf{A}\) | \(\mathbf{B}\) |
|---|---|---|
| Killing Vector (KV) | \(\xi_{ij}\) | 0 |
| Homothetic vector (HV) | \(\xi_{ij}\) | 2\(\psi g_{ij}\), \(\psi_i = 0\) |
| Conformal Killing vector (CKV) | \(\xi_{ij}\) | 2\(\psi g_{ij}\), \(\psi_i \neq 0\) |
| Affine Collineation (AC) | \(\Gamma^I_{jk}\) | 0 |

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Table 1: Collineations of a Riemannian space
The key point is to express these conditions in terms of the collineations of the metrics. Substituting this back in (32) we have

\[ u^B_{i;k} u^A_{i} : \xi^i_B = 0 , \]
\[ u^A_{i;k} u^B_{i} : \xi^i_B + \mu^i_B = 0 , \]

which imply that \( \xi^i = \xi^i (x^k) \) and \( \mu^i_B = 0 \); recall that \( n \geq 2 \). For \( n = 1 \) we have the case of ODEs for which the Lie point symmetry conditions are different (see [11]).

Substituting the solution of the system \( (29) \), \( (30) \), in \( (28) \) from the coefficients of the remaining terms we obtain the following symmetry conditions.

Coefficients of \( (u^A_i)^0 \):

\[ g^{ij} \eta^A_{ij} + F^A_{,i} \eta^D_{i} + g^A \xi^i - \Gamma^i \eta^A - \lambda^A_B F^B = 0 . \]  

Coefficients of \( (u^A_i)^1 \):

\[ 0 = - \Gamma^i \xi^k \delta^A_B + \Gamma^k \xi^i \delta^A_B - g^{jk} \xi^i \delta^A_B - \Gamma^i \eta^A_B + + 2 g^{jk} C^A_{BC} \eta^C_k + 2 g^{jk} \eta^A_{Bk} + \lambda^A_B \Gamma^i . \]  

Coefficients of \( (u^A_i)^2 \):

\[ 0 = g^{ij} \xi^k C^A_{BC} + g^{ij} C^A_{BC,D} \eta^D + 2 g^{ij} C^A_{DB} \eta^C + - 2 g^{ij} C^A_{BC} \xi^k + g^{ij} \eta^A_{BC} - \lambda^A_B g^{ij} C^A_{BC} . \]  

Coefficients of \( (u^A_{ij}) \):

\[ (g^{ij} \xi^k - 2 g^{ij(k} \xi^k)) \delta^A_B + g^{ij} (\eta^A_B - \lambda^A_B) = 0 . \]

The solution of the system of equations \( (28)-(31) \) gives the generator of the Lie point symmetry vector \( (20) \).

The key point is to express these conditions in terms of the collineations of the metrics \( g_{ij}, H_{AB} \) and relate the generator \( X \) of the Lie point symmetry to these collineations. Then, in a way, we have geometrized the problem and we may use the well known results of Differential Geometry in order to study the Lie point symmetries of the \( n \times m \) systems of differential equations \( (2) \).

In terms of the Lie derivative equation \( (31) \) is written as follows,

\[ (L \xi g^{ij}) \delta^A_B = - g^{ij} (\eta^A_B - \lambda^A_B) . \]

Because \( \xi^i = \xi^i (x^k) \) the left-hand side of \( (32) \) is independent of \( u^A \), hence

\[ \lambda^A_B = \eta^A_B - 2 \psi (x^k) \delta^A_B . \]  

Substituting this back in \( (32) \) we have

\[ L \xi g_{ij} = 2 \psi (x^k) g_{ij} \]

which means that \( \xi^i (x^k) \) is a CKV of \( g_{ij} \) with conformal factor \( \psi (x^k) \). This implies that \( \xi^i_a = n \psi (x^k) \), where \( ^a \) indicates covariant derivative with respect to the metric \( g_{ij} \).

Replacing \( \lambda^A_B \) from \( (33) \) in the symmetry condition \( (30) \) we find

\[ 0 = C^A_{BC} \left[ g^{ij} \xi^k - 2 g^{ij(k} \xi^k + 2 \psi g^{ij} \right] + + g^{ij} \left[ \eta^A_{BC} + C^A_{BC,D} \eta^D + 2 C^A_{D(B} \eta^D_{C)} - \eta^A_{D} C^D_{BC} \right] \]
\[ = C^A_{BC} (L \xi g^{ij} + 2 \psi g^{ij}) + g^{ij} \left[ \eta^A_{BC} + C^A_{BC,D} \eta^D + 2 C^A_{D(B} \eta^D_{C)} - \eta^A_{D} C^D_{BC} \right] . \]  

For the terms \( u^A_{i;k} u^B_{ij} \), we have the following equations

\[ u^B_{i;k} u^A_{i} : \xi^i_B = 0 , \]  

\[ u^A_{i;k} u^B_{i} : \xi^i_B + \mu^i_B = 0 , \]  

where
But \(C_{BC}^A\) are the connection coefficients of the metric \(H_{AB}\), hence

\[
\mathcal{L}_\xi C_{BC}^A = \eta_{(BC)}^A + C_{BC;D}^A \eta^D + 2C^A_D (\eta^C_C) - \eta_{(D}^A C_{BC)}^D.
\]  

Replacing this back in (35) we find:

\[
\left(\mathcal{L}_\xi g^{ij} + 2\psi g^{ij}\right) C_{BC}^A + g^{ij} \mathcal{L}_\eta C_{BC}^A = 0.
\]  

From (34) the first term vanishes \(^4\), therefore the symmetry condition (36) becomes

\[
\mathcal{L}_\eta C_{BC}^A = 0,
\]  

which means that \(\eta^A\) is an AC of \(H_{AB}\).

Hence, condition (29) becomes

\[
-\Gamma_{,k}^i \xi^k + \Gamma^i_{,k} \xi^k - g^{kj} \xi^i_{,kj} - 2\psi \Gamma^i + 2\psi g^{ij} C_{BC}^A \eta_{,j}^C + 2\psi g^{ij} \eta_{,B}^A = 0.
\]  

Furthermore, because \(\xi^i\) is a CKV of \(g_{ij}\) it holds that

\[
g^{jk} \mathcal{L}_\xi \Gamma^i_{jk} = g^{kj} \xi^i_{,kj} - \Gamma_{,k}^i \xi^k + \Gamma^i_{,k} \xi^k + 2\psi \Gamma^i = (2 - n) \psi^i.
\]  

Hence, equation (38) becomes

\[
\left(\eta_{,A}^B\right)_{,i} = \left(2 - n\right) \frac{\psi}{2} \delta_{,B}^A
\]  

where “\(,\)”, means covariant derivative with respect to the metric \(H_{AB}\). Furthermore the last equation is written,\(^{41}\)

\[
\left(\eta_{A|B}\right)_{,i} = \left(2 - n\right) \frac{\psi}{2} H_{AB} + \Lambda_{AB} \left(u^C\right),
\]  

from where we have that

\[
\eta_{A|B} = \left(2 - n\right) - n \frac{\psi}{2} H_{AB} + \Lambda_{AB} \left(u^C\right),
\]  

in which, \(\Lambda_{AB} = \Lambda_{AB} \left(u^C\right)\) is a second rank tensor defined in the space of the dependent variables. Because \(\eta^A\) is an AC of \(H_{AB}\) it is true that \(\eta_{A|BC} = 0\); this implies that \(\Lambda_{A|BC} = 0\) which means that \(\Lambda_{AB}\) is a Killing tensor of order two of the metric \(H_{AB}\). We conclude that the general form of \(\eta^A \left(x^k, u^C\right)\) is

\[
\eta^A = \left(2 - n\right) - n \frac{\psi}{2} \left(x^k\right) Y^A \left(u^C\right) + Z^A \left(x^k, u^B\right).
\]  

Replacing this in condition (42) we find the constraints

\[
Y_{A|B} = H_{AB}, \quad Z_{A|B} = \Lambda_{AB}
\]  

which mean that vector \(Y^A\) is a proper gradient HV of \(H_{AB}\) and \(Z^A\) is an AC of \(H_{AB}\). Moreover, when \(n > 2\) and the space \(H_{AB}\) does not admit proper gradient HV then from (42) we have that \(\psi \left(x^k\right) = 0\), i.e. \(\xi^i \left(x^k\right)\) is a KV of \(g_{ij}\).

Finally, condition (28) gives the further constraint

\[
\left(\mathcal{L}_\xi F^A + 2\psi F^A\right) + \left(2 - n\right) \frac{\psi}{2} \left(\mathcal{L}_Y F^A + g^{ij} \psi_{;ij} Y^A\right) + \left(\mathcal{L}_Z F^A + g^{ij} Z_{;ij}^A\right) = 0.
\]  

The solution of this system which follows from the symmetry condition (20) leads to the following theorem which is our main result.

\(^{4}\text{Recall that, } \mathcal{L}_g g^{ij} = -2\psi g^{ij}.\)
Theorem 1 The Lie point symmetries of the quasilinear systems of second-order differential equations (2) are generated by the CKVs $\xi^i (x^k)$ of the metric $g_{ij}$ and the ACs $Z^A (x^k, u^C)$ of the metric $H_{AB}$ such that $Z_{A|B} = \Lambda_{AB}$ and $(Z_{A|B})_i = 0$ where $\Lambda_{AB}$ is a Killing tensor of order two for the metric $H_{AB}$ as follows:

(a) If $n > 2$ and the metric $H_{AB}$ admits a proper gradient $HV$ $Y^A (u^C)$, with conformal factor $\bar{\psi}_Y = 1$, the generic Lie point symmetry is

$$X_{L(a)} = \xi^i (x^k) \partial_i + \left[ \frac{2-n}{2} \psi (x^k) \right] Y^A (u^C) + Z^A (x^k, u^C) \partial_A \tag{46}$$

and condition (45) holds.

(b) If $n > 2$, and the metric $H_{AB}$ does not admit a proper gradient $HV$, the generic Lie point symmetry is

$$X_{L(b)} = \xi^i (x^k) \partial_i + Z^A (x^k, u^C) \partial_A \tag{47}$$

and the following condition holds

$$\mathcal{L}_\xi F^A + \mathcal{L}_Z F^A + \Delta g Z^A = 0. \tag{48}$$

(c) If $n = 2$, the generic Lie point symmetry is

$$X_{L(c)} = \xi^i (x^k) \partial_i + Z^A (x^k, u^C) \partial_A \tag{49}$$

and the following condition holds

$$\left( \mathcal{L}_\xi F^A + 2 \psi F^A \right) + \left( \mathcal{L}_Z F^A + \Delta g Z^A \right) = 0. \tag{50}$$

We note that Theorem 1 holds for all the systems of the form (2) i.e. they do not necessarily admit a Lagrangian.

3.1 Noether symmetries

In this section we study the Noether point symmetries of Lagrangian (1). For each term of the Noether condition (11) for the Lagrangian (1) we have

$$D_i A^i = A^i_i + A^i_A (u^A_i), \tag{51}$$

$$\eta^C \frac{\partial L}{\partial u^C} = \left( \frac{1}{2} \sqrt{g} g^{ij} H_{AB} \xi^k_j \right) (u^A_i u^B_j) - \sqrt{g} \xi^k_j +$$

$$\left( \frac{1}{2} \sqrt{g} g^{ij} H_{AB} \xi^k_j \right) (u^A_i u^B_j) - \sqrt{g} \xi^k_j V (u^A_i). \tag{52}$$

Moreover, for the terms of $X^{[1]} L$ we find,

$$\xi^k \frac{\partial L}{\partial x^k} = \left( \frac{1}{2} \sqrt{g} g^{ij} \xi^k \right) H_{AB} (u^A_i u^B_j) - \xi^k \left( \sqrt{g} V \right) (u^A_i). \tag{53}$$

$$\eta^C \frac{\partial L}{\partial u^C} = \left( \frac{1}{2} \sqrt{g} g^{ij} H_{AB,C} \xi^k_j \right) (u^A_i u^B_j) - \sqrt{g} V (u^A_i). \tag{54}$$

\footnote{Where $Z_{A|B}$ means covariant derivative with respect to the metric $H_{AB}$ and $(Z_{A|B})_i = \frac{\partial}{\partial x^i} (Z_{A|B}).$}
and

\[ \eta^C_k \frac{\partial L}{\partial u^C} = (\sqrt{g}g^{ij}H_{AB}\eta^A_i) (u^B_j) - (\sqrt{g}g^{ij}H_{AB}\xi^i_{,C}) (u^B_j u^A_i u^C_k) \]

\[ + (\sqrt{g}g^{ij}H_{AB}\eta^C_j) - (\sqrt{g}g^{ij}H_{AB}\xi^i_{,C}) (u^B_j u^C_k). \tag{55} \]

From the coefficients of the monomial \((u^A_i)^3\) we have \(\xi^i_{,C} = 0\), i.e. \(\xi^i = \xi^i (x^k)\). This should be expected because the Noether point symmetries are Lie point symmetries for which (as we have shown already) \(\xi^i = \xi^i (x^k)\).

Replacing (51)-(55) in the Noether condition (11) and using the Lie derivative we find the following Noether symmetry conditions.

Coefficients of \((u^A_i)^0\):

\[ \sqrt{g} (L_\eta V + L_\xi V + \xi^k_{,A} V) + A^k_{,A} = 0 \tag{56} \]

Coefficients of \((u^A_i)^1\):

\[ \sqrt{g}g^{ij}H_{AB}\eta^A_i - A^i_{,B} = 0 \tag{57} \]

Coefficients of \((u^A_i)^2\):

\[ H_{AB} (L_\xi g^{ij} + \xi^k_{,A} g^{ij}) + g^{ij} (L_\eta H_{AB}) = 0. \tag{58} \]

From Theorem 1 we know that \(\xi^i\) is a CKV of \(g^{ij}\); hence \(\xi^k_{,C} = n\psi (x^k)\). Substituting in (58) we find

\[ H_{AB} (L_\xi g^{ij}) = g^{ij} (L_\eta H_{AB} + n\psi H_{AB}) \tag{59} \]

or equivalently

\[ L_\eta H_{AB} = (2 - n) \psi H_{AB} \tag{60} \]

which implies that

\[ \eta A (x^k, u^C) = \frac{2 - n}{2} \psi (x^k) Y^A (u^K) + K^A (x^k, u^C) \tag{61} \]

where \(Y^A (u^k)\) is a proper gradient HV of \(H_{AB}\) and \(K^A (x^k, u^C)\) is a KV of \(H_{AB}\).

Substituting back in (57) we have

\[ A_{i,A} = \frac{2 - n}{2} \sqrt{g} \psi_i Y_A + K_{A,i} \tag{62} \]

which gives that \(Y_A\) is a gradient HV of \(H_{AB}\), i.e. \(Y_A = Y_A\) and \(K^A = K (x^k, u^C)\). Moreover if \(K^A\) is a non gradient KV of \(H_{AB}\) then from (62) we have that \(K^A = K^A (u^C)\).

Therefore, we may write \(K^A = K^G (x^k, u^C) + K^N_G (u^C)\) where \(K^G, K^N_G\) are the gradient and non gradient KVs respectively. Then from (62) we have for following expression for the Noether vector

\[ A_i = \frac{2 - n}{2} \sqrt{g} \psi_i Y + \sqrt{g} K_{G,i} + \sqrt{g} \Phi_k (x^k). \tag{63} \]

Finally from (58) we have the constraint

\[ L_\xi V + n\psi V + \frac{2 - n}{2} (\psi V_A Y^A + \Delta g \psi Y) + (\Delta g (K_G) + V_A K^A) + \Phi_k = 0. \tag{64} \]

We collect the results in the following theorem.
Theorem 2  The Noether point symmetries of Lagrangian (11) are generated by the CKVs $\xi^i$ of the metric $g_{ij}$ with conformal factor $\psi(x^k)$ and the KVs of the metric $H_{AB}$ as follows:

(a) If $n > 2$ and the metric $H_{AB}$ admits a proper gradient HV with homothetic factor $\tilde{\psi}_Y = 1$, the generic Noether point symmetry is

$$X_{N(a)} = \xi^i (x^k) \partial_i + \left(2 - \frac{n}{2} \psi(x^k)\right) Y^{A} (u^C) + K^A_G (x^k, u^C) + K^A_{\text{NG}} (u^C) \partial_A$$

where $K^A_G$ is a gradient KV/HV of $H_{AB}$, $K^A_{\text{NG}}$ is a non gradient KV/HV of $H_{AB}$ and condition (64) holds.

The corresponding gauge vector field is

$$A^i_{(a)} (x^k, u^C) = \sqrt{g} \left(\frac{2 - n}{2} \sqrt{g} g^{ij} \psi Y + g^{ij} K_{G,j} + \Phi^i (x^k)\right)$$

and the generic Noether conservation current is

$$I^i_{(a)} = \xi^k H^i_{k} - \left(2 - \frac{n}{2} \psi\right) Y^{A} + K^A_G + K^A_{\text{NG}} (u^C) g^{ij} H_{AB} u^A_{j} +$$

$$+ \sqrt{g} \left(2 - \frac{n}{2} \sqrt{g} g^{ij} \psi Y + g^{ij} K_{G,j} + \Phi^i\right).$$

(b) If $n > 2$, and the metric $H_{AB}$ does not admit a proper gradient HV, the generic Noether point symmetry is

$$X_{N(b)} = \xi^i (x^k) \partial_i + \left(K^A_G (x^k, u^C) + K^A_{\text{NG}} (u^C)\right) \partial_A$$

where $K^A_G$ is a gradient KV of $H_{AB}$, $K^A_{\text{NG}}$ is a non gradient KV of $H_{AB}$ and the following condition holds

$$\mathcal{L}_\xi V + (\Delta g (K_G) + V_A K^A) + \Phi^i_k = 0.$$ (69)

The corresponding gauge vector field is

$$A^i_{(b)} (x^k, u^C) = \sqrt{g} \left(g^{ij} K_{G,j} + \Phi^i (x^k)\right)$$

and the generic Noether conservation current is

$$I^i_{(b)} = \xi^k H^i_{k} - (K^A_G + K^A_{\text{NG}}) g^{ij} H_{AB} u^B_{j} + \sqrt{g} \left(g^{ij} K_{G,j} + \Phi^i\right).$$

(c) If $n = 2$, the generic Noether point symmetry is

$$X_{N(c)} = \xi^i (x^k) \partial_i + \left(K^A_G (x^k, u^C) + K^A_{\text{NG}} (u^C)\right) \partial_A$$

where $K^A_G$ is a gradient KV/HV of $H_{AB}$, $K^A_{\text{NG}}$ is a non gradient KV/HV of $H_{AB}$ and the following condition holds

$$\mathcal{L}_\xi V + 2\psi V + (\Delta g (K_G) + V_A K^A) + \Phi^i_k = 0.$$ (73)

The corresponding gauge vector field is

$$A^i_{(c)} (x^k, u^C) = \sqrt{g} \left(g^{ij} K_{G,j} + \Phi^i (x^k)\right)$$

and the generic Noether conservation current is

$$I^i_{(c)} = \xi^k H^i_{k} - (K^A_G + K^A_{\text{NG}}) g^{ij} H_{AB} u^B_{j} + \sqrt{g} \left(g^{ij} K_{G,j} + \Phi^i\right).$$

In all cases the function $\mathcal{H} = \mathcal{H} (x^k, u^C)$, is the Hamiltonian of Lagrangian (11); that is,

$$\mathcal{H}^i_k = \frac{1}{2} \sqrt{g} H_{AB} \left(2g^{ij} u^A_{k,j} - \delta^i_k g^{rs} u^A_{j,r} u^B_{j,s}\right) + \delta^i_k \sqrt{g} V$$

(76)
The vector fields $X_N(a)$, $X_N(b)$ and $X_N(c)$ of Theorem 2 give the generic Noether point symmetry of Lagrangian (1) in a Riemannian space $g_{ij}$.

In the following sections we proceed with the applications of Theorems 1 and 2 in two cases of special interest. Specifically, we study the point symmetries of a system of quasilinear Laplace equations, and the point symmetries of the modified Klein-Gordon equation for a particle in Generalized Uncertainty Principle.

4 System of quasilinear Laplace equations

We assume that the potential $V(x^k, u^C)$ of (1) is zero. Then the Euler-Lagrange equations (2) become

$$g^{ij}u^A_{ij} + g^{ij}C_{BC}^A u^B_{ij} - \Gamma^i u^A_i = 0$$

(77)

and correspond to a system of quasilinear Laplace of dimension $m$. When $n = 1$ the system (77) describes the geodesic equations of a particle with affine parameterization in the space $H_{AB}$, or the wave equation in the space $g_{ij}$ when $m = 1$. Recall that in this work we consider $n \geq 2$. For $n = 1$ we have the case of autoparallel equations see [22].

For this particular case, from Theorems 1 and 2 we have the following corollary.

**Corollary 3** The generic form of the generators of Lie and Noether point symmetries of the system of second-order PDEs (77) are those of Theorems 1 and 2, where the corresponding constraint conditions are as follows:

(a) If $n \geq 2$ and the metric $H_{AB}$ admits a proper gradient HV, the Lie point symmetry constraint condition is $\frac{2-n}{2} \Delta g^A Y^A + \Delta g Z^A = 0$ and the Noether point symmetry condition becomes $\frac{2-n}{2} \Delta g Y^A + \Delta g (K_G) + \Phi^k_k = 0$.

(b,c) If $n > 2$, and the metric $H_{AB}$ does not admit a proper gradient HV, or if $\text{dim } g_{ij} = 2$, the Lie constraint condition is $\Delta g Z^A = 0$ and the Noether symmetry condition becomes $\Delta g K_G = 0$.

We observe that in this particular application the main role is played by the metric $H_{AB}(u^C)$. Therefore, we study two important cases; that is, (a) the space $H_{AB}(u^C)$ is flat, (b) and $H_{AB}(u^C)$ is a space of constant curvature.

4.1 Case a: $H_{AB}$ is flat

We consider a Euclidean space of dimension $m$ in which we employ Cartesian coordinates so that $H_{AB} = \delta_{AB}$. In this case the system (77) takes the simplest form:

$$g^{ij}u^A_{ij} - \Gamma^i u^A_i = 0.$$  

(78)

As we have already remarked, the $m$ dimensional flat space admits an $m(m+1)$ dimensional Lie algebra of ACs. This algebra consists of $m$ linearly independent gradient KVs and $m^2$ proper ACs. We note that the gradient HV and the non gradient KVs (rotation group) of the flat space follow from linear combinations of the proper ACs. In table 2 we give the KVs, the HV and the proper ACs of the one, two and three dimensional flat space.

The general AC for the $m$ dimensional flat space is

$$Z^A = b^A \partial_{u^A} + c^A_B u^B \partial_{u^A}$$

(79)
where the following condition holds

$$2 - n \Delta g \psi = 0, \, \Delta g b^A = 0. \quad (81)$$

From the second condition it follows that the functions $b^A(x^k)$ are solutions of (78) and the conformal factor $\psi(x^k)$ satisfies the Laplacian in the space of the independent variables with metric $g_{ij}$.

The generic Noether symmetry vector for the Lagrangian (1) for the system (78) is the vector field (80) with the constraints (81), where now the constants $c_{AB} = c_{IJK}\delta^I_A\delta^J_B$ with $c_{IJK} \in \mathbb{R}$. That means that the components $c_B^Au^B\partial_{u^A}$ of (80) are HVs of $\delta_{AB}$.

When $\dim H_{AB} = 1$, from Table 2 it follows that the space admits a two dimensional affine algebra; the two vector fields are a gradient KV and a HV. Therefore, what it has been called as "linear/trivial" symmetries of Laplace equation (with $m = 1$) [17][18], are the symmetries which arise from the gradient KVs/HV of the one dimensional space.

Furthermore, we consider $g_{ij}$ to be the flat space metric with dimension $n > 2$, i.e. $g_{ij} = \delta_{ij}$. Then the system (78) takes the simplest form $\delta^{ij}u^A_{ij} = 0$ which corresponds to a system of $m$–Laplace equations. It is well known that the flat space $\delta_{ij}$ admits a $\frac{(n+2)(n+1)}{2}$ dimensional conformal algebra where the proper CKVs are $n$, with the property $\psi_{ij} = 0$, i.e. they are gradient. Furthermore, the case where the two metrics $g_{ij}$, $H_{AB}$ are flat correspond to the case in which equation (78) admits the maximum Lie point symmetries. We conclude with the following corollary.

### Table 2: Affine collineations for flat space of dimension $m$, with $m = 1, 2, 3$

| $m$  | Gradient | Non gradient |
|------|----------|--------------|
| $m = 1$ | $\partial_{u^A}$, $\partial_{u^A}$, $\partial_{u^A}$ | $u^2\partial_{u^A} - u^1\partial_{u^A}$ |
| $m = 2$ | $\partial_{u^A}$, $\partial_{u^A}$, $\partial_{u^A}$ | $u^3\partial_{u^A} - u^2\partial_{u^A}$ |
| $m = 3$ | $\partial_{u^A}$, $\partial_{u^A}$, $\partial_{u^A}$ | $u^4\partial_{u^A} - u^3\partial_{u^A}$ |

where $b^A, c^A_B$ are constants in the space $H_{AB}$ so that $b^A = b^A(x^k)$ and $c^A_B = c^A_B(x^k)$. Moreover from the condition $(Z_{AB})_i = 0$ of Theorem 1, we have that $c^A_B = 0$ and $b^A = b^A(x^k)$. Then from Corollary 3 follows that the generic Lie symmetry vector of the system (78) is

$$X_L = \xi^i (x^k) \partial_i + \left(\frac{2 - n}{2} \psi(x^k)\right) u^A\partial_A + b^A(x^k)\partial_A + c^A_B u^B\partial_{u^A} \quad (80)$$

where the following condition holds

$$2 - n \Delta g \psi = 0, \, \Delta g b^A = 0. \quad (81)$$

Furthermore, we consider $g_{ij}$ to be the flat space metric with dimension $n > 2$, i.e. $g_{ij} = \delta_{ij}$. Then the system (78) takes the simplest form $\delta^{ij}u^A_{ij} = 0$ which corresponds to a system of $m$–Laplace equations. It is well known that the flat space $\delta_{ij}$ admits a $\frac{(n+2)(n+1)}{2}$ dimensional conformal algebra where the proper CKVs are $n$, with the property $\psi_{ij} = 0$, i.e. they are gradient. Furthermore, the case where the two metrics $g_{ij}$, $H_{AB}$ are flat corresponds to the case in which equation (78) admits the maximum Lie point symmetries. We conclude with the following corollary.
Corollary 4 Consider the \( n \times m \) system of second-order system of PDEs (78) with \( n > 2 \) and Lagrangian (1). Then:

(A) If the system (78) is invariant under the action of the group \( \hat{G} \) (Lie point symmetries), then \( \dim \hat{G} \leq \frac{(n+2)(n+1)}{2} + m(m+1) \).

(B) If the Lagrangian (1) with \( V(x^k, u^C) = 0 \), admits Noether point symmetries which form the group \( \hat{G}_N \), then \( \dim \hat{G}_N \leq \frac{(n+2)(n+1)}{2} + m(m+1) \).

In both cases (A) and (B) the equality holds when and only when \( g_{ij}, H_{AB} \) are flat spaces of dimension \( n \), and \( m \) respectively. In this case there exists a coordinate system such that the system (78) becomes

\[
\delta_{ij} u^A_{,ij} = 0.
\]

We would like to remark, that the results of this subsection hold and in the case the flat spaces \( H_{AB} \) or \( g_{ij} \) have Lorentzian signature. What changes in this case is the form of the non gradient KVs of Table 2. Additionally, for \( m = 1 \) Theorem 4 gives the results for the wave equation [25]. Moreover, in the case of the geodesic equations, i.e. \( n = 1 \), the maximum Noether algebra is consistent with that of the geodesic Lagrangian (1). However, when \( n = 1 \), the maximum algebra of Lie point symmetries is different from that of theorem 4 and it is \( (m+1)(m+3) \), which is the projective algebra of the \( m+1 \) flat spacetime [13,14]. That is, in the case of geodesic equations the Lie point symmetries follow from the special projective algebra of the space \( H_{AB} \).

4.2 Case b: \( H_{AB} \) is the metric of a space of constant curvature

We assume now that \( H_{AB} \) is a space of constant curvature \( K \) with \( K \neq 0 \). We choose coordinates so that \( H_{AB} = U(u^C u^C) \delta_{AB} \), and \( U(u^C) = (1 + \frac{K}{4} \delta_{AB} u^A u^B)^{-2} \); in field theory the models which live in that space are called \( \sigma \)–models [24].

In these coordinates, the connection coefficients of the metric \( H_{AB} \) have the following form

\[
C^A_{BC} = -\frac{KU}{2} (u_C \delta^A_B + u_B \delta^A_C - u^A \delta_{BC})
\]

and equation (78) becomes

\[
g^{ij} u^A_{,ij} - \frac{KU}{2} g^{ij} u^B_{,j} u^C_{,i} (u_C \delta^A_B + u_B \delta^A_C - u^A \delta_{BC}) - \Gamma^i u^A_{,i} = 0.
\]

In order to determine the Lie and the Noether point symmetries of the system (83) we have to study the Affine algebra of a space of constant curvature. It is well known that the Affine algebra of a space of constant non-vanishing curvature is the \( SO(n+1) \) Lie algebra of non gradient KVs whose dimension is \( \dim SO(n+1) = \frac{n(n+1)}{2} \) [24]. Therefore the Lie and the Noether point symmetries of (83) follow from theorem 3(b). We have the following result.

Corollary 5 Theorem If the \( n \times m \) system of second-order PDEs (83) with \( n > 2 \) is invariant under the action of the group \( \hat{G} \), then \( \dim \hat{G} \leq \frac{n(n+1)}{2} + \frac{m(m-1)}{2} \). The equality holds when and only when \( g_{ij} \) is the metric of a maximally symmetric space. Moreover, all the Lie point symmetries of (83) are also Noether point symmetries of Lagrangian (1) with \( V(x^k, u^C) = 0 \).

5 The Klein-Gordon equation modified by the Generalized Uncertainty Principle

In this section, we study the Lie and the Noether point symmetries of the modified Klein-Gordon equation of a spin-0 particle modified by the Generalized Uncertainty Principle (GUP) [20,29]. The modified Klein-Gordon
equation is a fourth order PDE which, by means of a Lagrange multiplier, is reduced to a system of two second-order PDEs of the form of the system (2). By applying the results of sections 1 and 2, we prove a corollary concerning the modified Klein-Gordon equation in a Riemannian space $g_{ij}$. We apply the corollary in two cases of physical interest: (A) the underlying manifold is the flat space $M^4$, and (B) the underlying manifold is the two dimensional hyperbolic sphere; in each case we determine the Lie and the Noether point symmetries, and we apply the zero-order invariants in order to determine invariant solutions of the wave function.

5.1 Generalized Uncertainty Principle

The modified structural form of GUP is

$$\Delta X_i \Delta P_j \geq \frac{\hbar}{2} [\delta_{ij}(1 + \beta P^2) + 2\beta P_i P_j]$$

(84)

where the deformed Heisenberg algebra which is found from (84) is

$$[X_i, P_j] = i\hbar [\delta_{ij}(1 + \beta P^2) + 2\beta P_i P_j].$$

(85)

Here $\beta$ is a parameter of deformation defined by $6 \beta = \beta_0/M_P^2 c^2 = \beta_0 \ell_P^2 / \hbar^2$. By keeping $X_i = x_i$ undeformed, the coordinate representation of the momentum operator is $P_i = p_i (1 + \beta p^2)$; $(x, p)$ is the canonical representation satisfying $[x_\alpha, p_\beta] = i\hbar \delta_{\alpha\beta}$.

In the relativistic four vector form, the commutation relation (85) can be written as

$$[X_\mu, P_\nu] = -i\hbar [(1 - \beta (\eta^{\mu\nu} P_\mu P_\nu)) \eta_{\mu\nu} - 2\beta P_\mu P_\nu]$$

(86)

where $\eta_{\mu\nu} = diag(1, -1, -1, -1)$. The corresponding deformed operators in this case are

$$P_\mu = p_\mu (1 - \beta (\eta^{\alpha\gamma} p_\alpha p_\gamma)) , \quad X_\nu = x_\nu,$$

(87)

where $p^\mu = i\hbar \frac{\partial}{\partial x^\mu}$, and $[x_\mu, p_\nu] = -i\hbar \eta_{\mu\nu}$.

Consider a spin-0 particle with rest mass $m$. The Klein-Gordon equation of this particle is

$$\left[\eta^{\mu\nu} P_\mu P_\nu - (mc)^2 \right] \Psi = 0$$

(88)

where $c$ is the speed of light. By substituting $P_\mu$ from (87), we have the modified Klein-Gordon equation

$$\Delta \Psi - 2\beta \hbar^2 \Delta (\Delta \Psi) + V_0 \Psi = 0,$$

(89)

where $V_0 = \left(\frac{mc}{\hbar}\right)^2$; $\Delta$ is the Laplace operator where in $M^4$, $\Delta \equiv \Box$, and the terms $O(\beta^2)$ have been eliminated. Equation (89) is a fourth order PDE; however, with the use of a Lagrange multiplier we can write it as a system of two second-order PDEs.

The action of the modified Klein-Gordon equation (89) is

$$S = \int dx^4 \sqrt{-g} L (\Psi, \mathcal{D}_\sigma \Psi)$$

(90)

where the Lagrangian $L (\Psi, \mathcal{D}_\sigma \Psi)$ of the models is

$$L (\Psi, \mathcal{D}_\sigma \Psi) = \frac{1}{2} \sqrt{-g} g^{\mu\nu} \mathcal{D}_\mu \Psi \mathcal{D}_\nu \Psi - \frac{1}{2} \sqrt{-g} V_0 \Psi^2$$

(91)

6 $M_P$ is the Planck mass, $\ell_P (\approx 10^{-35} \text{ m})$ is the Planck length, $M_P c^2 (\approx 1.2 \times 10^{19} \text{ GeV})$ is the Planck energy.
and the new operator $\mathcal{D}_\mu$ is $\mathcal{D}_\mu = \nabla_\mu + \beta h^2 \nabla_\mu (\Delta)$; $\nabla_\mu$ is the covariant derivative, i.e. $\nabla_\mu \Psi = \Psi_{\mu\nu}$. We introduce the new variable $\Phi = \Delta \Psi$, and the Lagrange multiplier $\lambda$. From the constraint $\frac{\delta S}{\delta \lambda} = 0$ we have that $\lambda = -2\beta h^2 \Phi$, and the action (90) becomes

$$S = \int dx^4 \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \Psi_{\mu\nu} + 2\beta h^2 g^{\mu\nu} \Psi_{,\mu \nu} + \beta h^2 \Phi^2 - \frac{1}{2} V_0 \Phi^2 \right).$$

(92)

Hence, the new Lagrangian is

$$L (\Psi, \Psi_{,\mu}, \Phi, \Phi_{,\mu}) = \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu} \Psi_{,\mu \nu} + 2\beta h^2 g^{\mu\nu} \Psi_{,\mu \nu} \Phi - \beta h^2 \Phi^2 \right)$$

(93)

where $\Phi$ is a new field. We note that the Lagrangian (93) is of the form (1) with

$$H_{AB} = \begin{pmatrix} 1 & 2\beta h^2 \\ 2\beta h^2 & 0 \end{pmatrix}, \quad V (x^k, u^C) = V_0 \Phi^2 - \beta h^2 \Phi^2,$$

(94)

therefore, the previous results apply. We show easily that the Ricci Scalar of the 2-dimensional metric $H_{AB}$ (94) vanishes, hence $H_{AB}$ is the two-dimensional flat metric; furthermore, in this coordinate system the connection coefficients are $C^A_{BC} = 0$. Hence, the Euler-Lagrange equations (2) for Lagrangian (93) are

$$g^{\mu\nu} \Psi_{,\mu \nu} - \Gamma^\mu \Psi_{,\mu} - \Phi = 0$$

(95)

$$2\beta h^2 (g^{\mu\nu} \Phi_{,\mu \nu} - \Gamma^\mu \Phi_{,\mu}) + (V_0 \Phi + \Phi) = 0$$

(96)

where equation (95) is the constraint $\Delta \Phi = \Delta \Psi$.

$H_{AB}$ being a two-dimensional flat space, admits a six-dimensional Affine algebra. In this coordinate system, the two gradient KVs of $H_{AB}$ are $K^1 = \partial_\Phi$, $K^2 = \partial_\Psi$, the non gradient KV is $R = 2\beta h^2 \Psi \partial_\Psi + (\Psi + 2\beta h^2 \Phi) \partial_\Phi$; the gradient HV is $Y^A = \Psi \partial_\Phi + \Phi \partial_\Psi$, and the proper ACs are $A^1 = \Psi \partial_\Phi$, $A^2 = \Phi \partial_\Phi$, $A^3 = \Phi \partial_\Psi$ and $A^4 = \Psi \partial_\Psi$.

By replacing (94), in the constraint equations of Theorems 1 and 2, we get the following result.

**Corollary 6** The dynamical system with Lagrangian (93), which describes the modified Klein-Gordon equation of a particle in GUP, admits as Lie point symmetries the Killing vectors of the space of the independent variables $g_{\mu\nu}$, plus the three vector fields $K_G = b^1 \left( x^k \right) K^1 + b^2 \left( x^k \right) K^2$, $Y^A$ and $Z^A = A^1 + 2\beta h^2 A^3 - V_0 A^4$ where $b^1$, $b^2$ solve the system (95), (96). As far as the Noether point symmetries of Lagrangian (94) are concerned, all the Lie point symmetries of the dynamical system except the $Z^A$ are also Noether point symmetries.

Furthermore, concerning the maximal dimension of the Lie algebra of the system (93)-(96), from corollary 6 follows:

**Corollary 7** If the system (93)-(96) of $n$ independent variables is invariant under a group of one parameter point transformations $G_P$, then $3 \leq \dim G_P \leq \frac{n(n+1)}{2} + 3$; the right equality holds if and only if the space of the independent variables is a maximally symmetric space, and the left equality holds if and only if the space of the independent variables does not admit a Killing vector field.

In the following, we apply theorem 6 in order to determine the Lie and the Noether point symmetries of the system (93), (96), in two cases of special interest: A) the four-dimensional Minkowski spacetime $M^4$, and B) the two-dimensional hyperbolic sphere.

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We remark that equations (93), (96) form a singular perturbation system because $\beta h^2 << 1$. 

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5.2 Case A: The Minkowski spacetime $M^4$

In this case $g_{\mu\nu} = \eta_{\mu\nu}$ and Lagrangian (93) becomes

$$L_{M^4} = \frac{1}{2} \left( (\Psi_{,tt})^2 - (\Psi_{,x})^2 - (\Psi_{,y})^2 - (\Psi_{,z})^2 \right) - \left( \frac{1}{2} V_0 \Psi^2 - \beta \hbar^2 \Phi^2 \right) + 2\beta \hbar^2 \left( \Psi_{,tt} \Phi_{,t} - \Psi_{,xx} \Phi_{,x} - \Psi_{,yy} \Phi_{,y} - \Psi_{,zz} \Phi_{,z} \right).$$

Equations (95)-(96) are

$$\Psi_{,tt} - \Psi_{,xx} - \Psi_{,yy} - \Psi_{,zz} - \Phi = 0$$

(98)

$$2\beta \hbar^2 \left( \Phi_{,tt} - \Phi_{,xx} - \Phi_{,yy} - \Phi_{,zz} \right) + (V_0 \Psi + \Phi) = 0.$$ 

(99)

The $M^4$ spacetime admits a ten dimensional Killing algebra, hence from theorem [6] we have that the system (98)-(99) admits 13 Lie point symmetries, and the Lagrangian (97) admits 12 Noether point symmetries.

Using the zero order invariants of the Lie point symmetry vectors which span the Lie algebra \( \{ \partial_y, \partial_z, \partial_t + cY^A \} \), we find the invariant solutions for the wave function $\Psi(t, x, y, z)$ of the system (98)-(99) to be

$$\Psi(t, x, y, z) = e^{ct} \left( c_1 e^{\mu(\beta \hbar^2)} x + c_2 e^{-\mu(\beta \hbar^2)} x + c_3 e^{\nu(\beta \hbar^2)} x + c_4 e^{-\nu(\beta \hbar^2)} x \right)$$

(100)

where $\mu(\beta \hbar^2) = \frac{1}{2\sqrt{\hbar^2}} \sqrt{4c^2(\beta \hbar^2)^2 + 3\hbar^2(1 - \lambda)}$, $\nu(\beta \hbar^2) = \frac{1}{2\sqrt{\hbar^2}} \sqrt{4c^2(\beta \hbar^2)^2 + \beta \hbar^2(1 + \lambda)}$ and $\lambda = \sqrt{1 - 8V_0 \beta \hbar^2}$. However, symmetries can also be used in order to transform solutions into solutions. The Lie point symmetry $Z^A$ gives us the point transformation

$$\bar{\Psi}(\Psi, \Phi, \varepsilon) = \exp \left( \frac{1 + \lambda}{2\lambda} \right) \left[ (4\Phi \beta \hbar^2 + (1 + \lambda) \Psi) e^{\varepsilon \lambda} - (4\beta \hbar^2 \Phi + (1 - \lambda) \Psi) \right]$$

(101)

$$\bar{\Phi}(\Psi, \Phi, \varepsilon) = \exp \left( \frac{1 + \lambda}{2\lambda} \right) \left[ ((1 - 2V_0 \Psi) e^{\varepsilon \lambda} + (1 + \lambda) \Phi + 2V_0 \Psi) \right]$$

(102)

that is, solution (100) is transformed to the following solution

$$\bar{\Psi}(t, x, y, z) = e^{ct} \left( c'_1 e^{\mu(\beta \hbar^2)} x + c'_2 e^{-\mu(\beta \hbar^2)} x + c'_3 e^{\nu(\beta \hbar^2)} x + c'_4 e^{-\nu(\beta \hbar^2)} x \right)$$

(103)

where now the new constants $c'_{1,2}$ are

$$c'_{(1,2)} = \exp \left( -\frac{1 + \lambda}{2\varepsilon} \right) c_{(1,2)}, \quad c'_{(3,4)} = \exp \left( -\frac{1 - \lambda}{2\varepsilon} \right) c_{(3,4)}.$$  

(104)

5.3 Case B: The Hyperbolic sphere $S^2$

Consider now the two dimensional hyperbolic sphere with line element

$$ds^2 = d\theta^2 - e^{2\theta} d\phi^2.$$  

(105)

In this space, Lagrangian (93) becomes

$$L_S = \frac{e^\theta}{2} \left( (\Psi_{,\theta})^2 - e^{-2\theta} (\Psi_{,\phi})^2 \right) + 2\beta \hbar^2 e^\theta \left( \Psi_{,\theta} \Phi_{,\theta} - e^{-2\theta} \Psi_{,\phi} \Phi_{,\phi} \right) +$$

$$-e^\theta \left( \frac{1}{2} V_0 \Psi^2 - \beta \hbar^2 \Phi^2 \right).$$  

(106)
Hence in the two dimensional hyperbolic sphere (105), the wave function of the particle is computed from the following system of equations

\[ \Psi_{,\theta\theta} - e^{-2\theta} \Psi_{,\phi\phi} + \Psi_{,\theta} - \Phi = 0 \]  
(107)

\[ 2\beta h^2 \left( \Phi_{,\theta\theta} - e^{-2\theta} \Phi_{,\phi\phi} + \Phi_{,\theta} \right) + (V_0 \Psi + \Phi) = 0. \]  
(108)

The two dimensional sphere (105) admits a three dimensional Killing algebra, the $SO(3)$. Hence, from Corollary 6 we have that the system (107)-(108) admits 6 Lie point symmetries, and the Lagrangian (106) admits 5 Noether point symmetries.

The elements of $SO(3)$ algebra in the coordinates of (105) are

\[ X^1 = \partial_\phi, \quad X^2 = \partial_\theta - \phi \partial_\phi \]  
(109)

\[ X^3 = 2\phi \partial_\theta - (\phi^2 + e^{-2\theta}) \partial_\phi. \]  
(110)

From the application of the Lie point symmetry $X^1 + \alpha Y^A$ we have the solution

\[ \Psi_1 (\theta, \phi) = e^{\alpha \phi} e^{-\frac{\theta}{2}} \left[ b_1 K_\mu (\alpha e^{-\theta}) + b_2 I_\mu (\alpha e^{-\theta}) \right] + e^{\alpha \phi} e^{-\frac{\theta}{2}} \left[ b_3 K_\nu (\alpha e^{-\theta}) + b_4 I_\nu (\alpha e^{-\theta}) \right] \]  
(111)

where $\mu = \frac{\sqrt{(\beta^2 - 1) - \lambda}}{2\sqrt{\beta^2}}, \quad \nu = \frac{\sqrt{(\beta^2 - 1) + \lambda}}{2\sqrt{\beta^2}}, \quad \lambda = \sqrt{1 - 8V_0 \beta^2}$, and $I_\mu, K_\nu$ are the modified Bessel functions of the first and second kind respectively.

By using the Lie point symmetry $X^2 + \kappa Y^A$ we find the solution

\[ \Psi_2 (\theta, \phi) = \left( \phi^2 e^{2\theta} + 1 \right)^{-\frac{\gamma}{2}} e^{\kappa \phi} \left[ b_1 P_\kappa (-\mu - \frac{1}{2}) (\phi e^\theta) + b_2 Q_\kappa (-\mu - \frac{1}{2}) (\phi e^\theta) \right] + \left( \phi^2 e^{2\theta} + 1 \right)^{-\frac{\gamma}{2}} \left[ b_3 P_\kappa (-\nu - \frac{1}{2}) (\phi e^\theta) + b_4 Q_\kappa (-\nu - \frac{1}{2}) (\phi e^\theta) \right] \]  
(112)

where $P_\kappa, Q_\kappa$ are the associated Legendre functions of the first and second kind respectively.

Furthermore, from the Lie symmetry $X^3 + \sigma Y^A$ we find the solution

\[ \Psi_3 (\theta, \phi) = \exp \left( \frac{\sigma e^{2\theta}}{\phi^2 e^{2\theta} + 1} \right) \left[ b_1 K_\mu \left( \frac{\sigma e^{2\theta}}{\phi^2 e^{2\theta} + 1} \right) + b_2 I_\mu \left( \frac{\sigma e^{2\theta}}{\phi^2 e^{2\theta} + 1} \right) \right] + \exp \left( \frac{\sigma e^{2\theta}}{\phi^2 e^{2\theta} + 1} \right) \left[ b_3 K_\nu \left( \frac{\sigma e^{2\theta}}{\phi^2 e^{2\theta} + 1} \right) + b_4 I_\nu \left( \frac{\sigma e^{2\theta}}{\phi^2 e^{2\theta} + 1} \right) \right] \]  
(113)

Finally, if we apply the point transformation (101)-(102), which follows from the Lie point symmetry $Z^A$, to the solutions (111)-(113), we find that the new solutions are again (111)-(113), where now the constants $b_{1-4} \to \bar{b}_{1-4}$ are given by the expressions (104).

## 6 Conclusion

The quasilinear systems of second-order differential equations ($n \times m$ systems) describe many important equations of relativistic Physics, both at the classical as well as at the quantum level. Therefore, the study of Lie and the Noether point symmetries of these equations is important in order to establish invariant solutions and find conservation laws.

In this work we followed the geometric approach we have applied in our previous studies for the point symmetries of $1 \times m$ and $n \times 1$ systems in order to generalize it to the case of $n \times m$ systems. We have shown
that for the $n \times m$ systems of the form of (2), with $n \geq 2$, the point symmetries follow from the CKVs and the ACs of the underlying geometries of the $n$ independent and the $m$ dependent variables. This result is consistent with the results concerning the $1 \times m$ and $n \times 1$ systems. Moreover, for the Lagrangian (1) we derived the general form of the Noether symmetry vector and of the Noetherian conservation laws. Specifically, we proved that the Noether point symmetries are generated by the CKVs and the HVs of the metrics $g_{ij}$ and $H_{AB}$ respectively.

We applied the above general results to a system of quasilinear Laplace equations (which contains the geodesic equations and the wave equation as special cases) and determined the Lie and the Noether point symmetries in flat space and in spaces of constant non-vanishing curvature. By using results of Differential Geometry we found the maximum dimension of the Lie algebra of a system of quasilinear Laplace equations. In particular, we showed that if the $n \times m$ system of quasilinear Laplace equations, with $n > 2$, admits $(n+2)(n+1)/2 + m(m+1)$ linear independent Lie point symmetries, then there exists a coordinate system in which this system takes the simplest form $\delta^{ij}u^{A}_{;ij} = 0$.

A further application concerns the Klein-Gordon equation in GUP which is a fourth-order partial differential equation. By using a Lagrange multiplier we reduced this fourth-order equation to a system of two coupled second-order equations in the form of the system (2), in which the main Theorems of this work applies. We showed that the minimum Lie algebra of that system is of dimension three. Furthermore, we determined the Lie and the Noether point symmetries in a Minkowski space and in the 2-dimensional hyperbolic space and subsequently we used the former in order to determine invariant solutions of the wave function of a spin-0 particle.

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