Pinning Control of Spatiotemporal Chaos

R. O. Grigoriev, M. C. Cross and H. G. Schuster

Condensed Matter Physics 114-36 and Neural Systems Program 139-74
California Institute of Technology, Pasadena CA 91125
(October 30, 2018)

Linear control theory is used to develop an improved localized control scheme for spatially extended chaotic systems, which is applied to a Coupled Map Lattice as an example. The optimal arrangement of the control sites is shown to depend on the symmetry properties of the system, while their minimal density depends on the strength of noise in the system. The method is shown to work in any region of parameter space and requires a significantly smaller number of controllers compared to the method proposed earlier by Qu and Hu. A nonlinear generalization of the method for a 1-d lattice is also presented.

The first attempt in this direction was undertaken by Hu and Qu. The authors tried to stabilize the homogeneous state by controlling an array of $M$ periodically placed pinning sites $\{i_1, \ldots, i_M\}$ with appropriately chosen control $u^i_m$.

$$z^{t+1}_{i} = F(z^{t}_{i-1}, z^{t}_{i}, z^{t}_{i+1}) + \sum_{m=1}^{M} \delta(i-i_m) u^i_m. \quad (3)$$

This however required a very dense array with distance between controllers $L_p = L/M \leq 3$ in the physically interesting interval of parameters $3.57 < a < 4.0$.

The reason for this is the spatial periodicity of the pinnings. Since the system is spatially uniform, its eigenmodes are just Fourier modes and the pinning sites do not affect the modes whose nodes happen to lie at the pinnings, i.e., modes with periods equal to $2L_p$, $2L_p/2$, $2L_p/3$, etc., provided those are integer. The control scheme worked only when all such modes were stable.

It is however not necessary to destroy the periodicity completely to achieve control: that would complicate the analysis unnecessarily. Instead we add one more pinning site between each of the existing ones. Not all positions are good, but some do solve the problem — previously uncontrollable modes become controllable.

In order to understand how the pinnings should be placed and see whether we achieve improved performance by introducing additional controllers, we have to use a few results of the linear control theory. We will start with linearizing eq. (3) about the homogeneous steady state $z^* = (z^*, \ldots, z^*)$ in both the state vector and control to obtain the following standard equation

$$x^{t+1} = Ax^t + Bu^t, \quad (4)$$

where we denoted the displacement $x = z - z^*$. If we define $\alpha = \partial f(x^*, a^*)/\partial x$, then the $L \times L$ Jacobian $A$ is given by

$$A = \alpha \left( \begin{array}{cccc} 1 - 2\epsilon & \epsilon & 0 & \cdots & \epsilon \\ \epsilon & 1 - 2\epsilon & \epsilon & \cdots & 0 \\ 0 & \epsilon & 1 - 2\epsilon & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \epsilon & 0 & 0 & \cdots & 1 - 2\epsilon \end{array} \right) \quad (5)$$

and the $L \times M$ control matrix $B_{ij} = \delta(j-m)\delta(i-i_m)$ depends on how we place the pinning sites.

If we use synchronous linear feedback $u^t = Kx^t$, equation (4) becomes

$$x^{t+1} = (A - BK)x^t, \quad (6)$$
and the solution $x = 0$ can be made stable by a suitable choice of the feedback gain matrix $K$, if the controllability condition $\text{rank}(C) = L$ is satisfied. The controllability matrix $C$ is defined via

$$C = (B \ AB \cdots A^{L-1}B).$$

One can easily verify that the matrix $B$ calculated for a periodic array of pinning sites does not satisfy the controllability condition and therefore the homogeneous steady state is not controllable. It can be stabilized if the weaker stabilizability condition is satisfied, i.e. all uncontrollable modes are stable. However this imposes excessive restrictions on the pinning density.

The condition for stabilizability can be obtained from the spectrum of eigenvalues of the matrix $B$:

$$\gamma_i = \alpha(1 - 2\epsilon(1 - \cos(k_i))),$$

where $k_1 = 0$, $k_i = k_{i+1} = \pi i/L$ for $i = 2, 4, 6, \cdots$ and, for $L$-even, $K_L = \pi$ and $\alpha = 2 - a$. Specifically, we need $|(a-2)(1-2\epsilon(1-\cos(\pi j/L_p)))| < 1$ for all $j = 1, \cdots, L-2$, such that $L_0/j$ is integer. Using this criterion one can obtain the relation between the minimum coupling, the distance between controllers and parameter $a$ of the local chaotic map for a stabilizable system. For instance, $j = 1$ yields

$$\epsilon = \frac{a - 3}{2(a - 2)(1 - \cos(\pi/L_p))}.$$  

The results are presented in fig. 3. It can be easily verified that they coincide with the numerically obtained results of Hu and Qu.

It is possible however to extend the limits of the control scheme quite substantially by making the system controllable as opposed to stabilizable. This is easily achieved by choosing a different matrix $B$, i.e. placing the pinning sites differently. Doing so will enable us to control the system anywhere in the parameter space at the same time using a smaller density of controllers.

First one has to determine the dimensionality of the matrix $B$, in other words the minimum number of parameters required to control the CML of an arbitrary length. It can be shown that the minimal number of parameters required to control a system with degenerate Jacobian is equal to the greatest multiplicity of its eigenvalues.

Since the system under consideration has parity symmetry, the eigenvalues of its Jacobian are in fact doubly degenerate, so the minimal number of control parameters yielding a controllable system in our case is two, meaning at least two pinning sites are required. One can easily verify that the controllability condition for an $L \times 2$ matrix

$$B_{ij} = \delta(j - 1)\delta(i - i_1) + \delta(j - 2)\delta(i - i_2)$$

is indeed satisfied for a number of arrangements $\{i_1, i_2\}$. The restrictions on the mutual arrangement of the controllers are again given by the condition of controllability:

$L$ should not be a multiple of $|i_2 - i_1|$, otherwise the mode with the period $2|i_2 - i_1|$ becomes uncontrollable.

The next step in the algorithm is to determine the feedback gain $K$. Pole placement techniques based on Ackermann’s method [11] are inapplicable to the problem of controlling spatially extended systems because they are numerically unstable [12] and break down rapidly for problems of order greater than 10.

Instead we use the method of the linear-quadratic (LQ) control theory [3], applicable to the unstable periodic trajectories as well as fixed points. This method is not only numerically stable, but also allows one to optimize the control algorithm to increase convergence speed, and at the same time minimize the strength of control. As we will see below, decreasing control enlarges the basin of attraction, which has very important consequences for the time to achieve control (capture the chaotic trajectory). The optimal solution is obtained by minimizing the cost functional

$$V(x^0) = \sum_{n=0}^{\infty} (x^T Q x + u^T R u),$$

where $Q$ and $R$ are the weight matrices that can be chosen as any positive-definite square matrices.

The minimum of (11) is reached when

$$K = (R + B^T P B)^{-1} B^T P A,$$

where $P$ is the solution to the discrete-time algebraic Ricatti equation

$$P = (Q + A^T P A) - A^T P B (R + B^T P B)^{-1} B^T P A.$$

Numerical simulations show that the CML, can indeed be stabilized by this linear control scheme in a wide range of parameters $a$ and $\epsilon$. The solution for $K$ is presented in figure 3 for $a = 4.0, \epsilon = 0.33$ and $L = 8$ with $Q = I_{8 \times 8}$ and $R = I_{2 \times 2}$. The steady homogeneous state $z^* = 0.75$ has 3 unstable and 5 stable directions and we use 2 pinning sites to control it.
The contribution $-K_m x_i^n$ from the sites $i$ far away from the pinning site $i_m$ is larger, as one would expect: since the feedback is applied indirectly through coupling to the neighbors, the perturbation introduced by the controllers decays with increasing distance to the pinning sites.

Noise limits our ability to locally control arbitrarily large systems with local interactions. We will use a simple illustrative approach to see the effect of noise on the control scheme. The rank of the matrix is given by the number of its nonzero singular values. The singular values of the controllability matrix $\| \gamma_1 \|$ scale roughly as $s_l \sim |\gamma_1|^l$, where $\gamma_1$ is the largest eigenvalue of the Jacobian $A$

$$|\gamma_1| = e^{\lambda_{\text{max}}} = \begin{cases} \sqrt{\alpha}, & \epsilon < 0.5, \\ \alpha(4\epsilon - 1), & \epsilon > 0.5. \end{cases}$$

Assuming that there is an uncertainty in the calculations (due to the uncertainty in the state vector, parameter vector or just numerical roundoff errors) of relative magnitude $\sigma$, we can say that the rank of the controllability matrix can be reliably determined to be equal to the length of the lattice if $s_0/s_L > \sigma$. This gives us the theoretical bounds on the size of the controllable system in the presence of noise:

$$L_{\text{max}}^{(1)} = -\frac{\log(\sigma)}{\lambda_{\text{max}}}.$$  \hspace{1cm} (15)

On the other hand, the perturbation $\delta x_i$ introduced by the controller $i$ affects the dynamics of the remote site $j$ after propagating a distance $\Delta = |i-j|$ in time $\tau = \Delta$, decaying by a factor of $\epsilon$ per iteration, while the noise at site $j$ increases roughly by a factor of $\gamma_1$ per iteration. We therefore need $\delta x_i e^\Delta > \sigma|\gamma_1|^\tau$. Since the maximum distance $\Delta$ to the closest controller is $L/2$ and $\delta x_i \sim 1$, we get another bound, complementing (15)

$$L_{\text{max}}^{(2)} = \frac{2 \log(\sigma)}{\log(\epsilon) - \lambda_{\text{max}}}.$$  \hspace{1cm} (16)

Similar constraints were obtained by Aranson et. al. for the lattices with asymmetric coupling (cf. equation (15) of the ref. [3]).

The maximal length of the system, that can be stabilized by the LQ method with two pinning sites placed next to each other is obtained numerically by choosing the initial condition very close to the fixed point ($|x^0| \ll |\gamma_1|^{-L/2}$) and letting the system evolve under control (12) calculated for $Q = I_{L \times L}$ and $R = I_{2 \times 2}$. This length is quite large even in the presence of noise (fig. [3]) and agrees with the theoretical bounds [13, 14] rather well for such a crude estimate.

The problem of controlling a large 1-dimensional system with the length $L > L_{\text{max}}(\sigma)$ exceeding the maximum allowed for a given noise level can be easily reduced to the problem of controlling a number of smaller systems with the length $L_p < L_{\text{max}}(\sigma)$. We partition the entire lattice $\{ z^t_1, \ldots, z^t_p \}$ into $M = L/L_p$ subdomains $\{ z^t_{(m-1) L_p + 1}, \ldots, z^t_{m L_p} \}$, and control it with an array of pinning sites $i_{m1} = (m-1)L_p + 1$, $i_{m2} = mL_p$, $m = 1, \ldots, M$ positioned periodically at the boundaries of these subdomains.

The stabilization can be achieved by choosing

$$u^t_{m1} = F(z^t_{i_{m2}}, z^t_{i_{m1}}, z^t_{i_{m1} + 1}) - F(z^t_{i_{m1} - 1}, z^t_{i_{m1}}, z^t_{i_{m1} + 1})$$

$$u^t_{m2} = F(z^t_{i_{m2} - 1}, z^t_{i_{m2}}, z^t_{i_{m2} + 1}) - F(z^t_{i_{m2} - 2}, z^t_{i_{m2}}, z^t_{i_{m2} + 1})$$

$$+ \prod_{i=1}^{L_p} \theta(\delta x_i - |x^t_{(m-1) L_p + i}|) \sum_{i=1}^{L_p} K_1 z^t_{i_{m1} - 1} x^t_{(m-1) L_p + i}$$

$$+ \prod_{i=1}^{L_p} \theta(\delta x_i - |x^t_{(m-1) L_p + i}|) \sum_{i=1}^{L_p} K_2 z^t_{i_{m1} - 1} x^t_{(m-1) L_p + i},$$

where $\theta(x)$ is a step-function.

This arrangement effectively carries two functions. We use control (16) to (nonlinearly) decouple the subdomains, simultaneously imposing periodic boundary condition for each subdomain (the first two terms) to make the system controllable. Then we stabilize each subdomain asynchronously by applying a linear (in deviation
placed at the boundaries of subdomains with length $L = 8$. The state of the system was plotted at each 10000-th step.

\[ x_i^t = z_i^t - z^* \] feedback (the last term), inside the neighborhood of the fixed point determined by $\delta x_i$. The linear approximation (1) is only valid if

\[ \delta x_i \ll |K_{mi}|^{-1}, \quad m = 1, \ldots, M \] (18)

and therefore strong feedback significantly decreases the size of the capture region, which makes the capture time vary large. Minimizing the capture time can be achieved by maximizing the feedback strength using the LQ method (12,13).

We demonstrate this approach by stabilizing the homogeneous stationary state of the CML defined by equation (1,2) with $a = 4.0$, $\epsilon = 0.33$. $L = 128$ sites were divided into $M = 16$ subdomains of length $L_p = 8$, each controlled by two pinning sites. The results presented in fig. 4 show the evolution of the system from the initial condition chosen to be a collection of random numbers in the interval [0,1].

Eqs. (14,15) now give the minimal density of pinning sites that yields the controllable fixed point solution. It is indeed seen to be much lower than that given by (1), e.g. $2/L_p = 1/20$ (1/17 from the numerics, see fig. 3) as opposed to $1/L_p = 1/2$ for the choice $a = 4.0$, $\epsilon = 0.4$ and the precision of calculations given by $\sigma = 10^{-14}$.

Although the resulting control scheme becomes nonlinear (and therefore requires full knowledge of the evolution equations), it has the additional benefit, that the capture time is determined by the length $L_p \ll L$ and is typically many orders of magnitude smaller than that obtained for the linear control scheme (obtained by linearizing (17)), which only requires the Jacobian to be known. In fact our computational resources were insufficient to observe even a single capture for $L > 40$ with the linearized control. Generalizing this nonlinear approach to higher-dimensional systems remains a challenge.

To summarize, we have shown that the restrictions on the minimal density of periodically placed single pinning sites obtained by Qu and Hu (1) as a result of numerical simulations can in fact be obtained analytically from the stabilizability condition.

The efficiency of the control scheme can be improved significantly if one uses double pinnings instead of single ones. The homogeneous steady state becomes controllable for any values of the control parameters and the minimal density of pinning sites is reduced substantially. It is shown that the maximal distance between the pinnings depends on the strength of noise in the system and can be estimated analytically.

The appropriately chosen (using the LQ technique) feedback can decrease the capture time for the chaotic trajectory by enlarging the capture region. The introduction of nonlinearity into the control scheme can decrease this time even more significantly by effectively decoupling the large lattice into a number of smaller subdomains.

The authors thank Prof. J.C. Doyle for many fruitful discussions. This work was partially supported by the NSF through grant no. DMR-9013984. H.G.S. thanks C. Koch for the kind hospitality extended to him at Caltech and the Volkswagen Foundation for financial support.

[1] G. Hu and Z. Qu, Phys. Rev. Lett. 72 (1994) 68.
[2] M. Ding et al., Phys. Rev. E 53 (1996) 4334; Y. C. Lai and C. Grebogi, Phys. Rev. E 50 (1994) 1894; J. Warncke, M. Bauer and W. Martienssen, Europhys. Lett. 25 (1994) 323; D. Auerbach, Phys. Rev. Lett. 72 (1994) 1184.
[3] R. A. Katz, T. Galib and J. Cembrola, J. Phys. IV 4 (1994) 1063.
[4] A. Pentek, J. B. Kadkte and Z. Toroczkai, Phys. Lett. A 224 (1996) 85.
[5] P. Colet, R. Roy and K. Weisenfeld, Phys. Rev. E 50 (1994) 3453.
[6] V. Petrov, M. J. Crowley and K. Showalter, Phys. Rev. Lett. 72 (1994) 2955.
[7] F. J. Romeiras, et al., Physica D 58 (1992) 165; V. Petrov et al., Phys. Rev. E 51 (1995) 3988.
[8] K. Kaneko, Prog. Theor. Phys. 72 (1984) 480.
[9] P. Dorato, C. Abdallah and V. Cerrone, Linear-Quadratic Control: An Introduction (Prentice Hall, New Jersey, 1995)
[10] R. O. Grigoriev and H. G. Schuster (in preparation).
[11] E. Barreto and C. Grebogi, Phys. Rev. E 52 (1995) 3553.
[12] J. Kantsky, N. K. Nichols and Van Dooren, Intl. J. Control, 41 (1985), 1129.
[13] I. Aranson, D. Golomb and H. Sompolinsky, Phys. Rev. Lett. 68 (1992) 3495.