A MATRIX MODEL FOR QUANTUM $SL_2$

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Abstract. We describe a topological ribbon Hopf algebra whose elements are sequences of matrices. The algebra is a quantum version of $U(sl_2)$.

For each nonzero $t \in \mathbb{C}$ that is not a root of unity, we give a quantum analog $\mathcal{A}_t$ of $U(sl_2)$. The underlying algebra of the model is $\prod_{n=1}^{\infty} M_n(\mathbb{C})$. Consequently, the algebra structure, which comes from matrix multiplication, is independent of the variable $t$.

Define $\mathcal{A}_t$ to be the unital Hopf algebra on $X, Y, K, K^{-1}$, with relations:

$$
\begin{align*}
KX &= t^2XK, \\ KY &= t^{-2}YK, \\
XY - YX &= \frac{K^2 - K^{-2}}{t^2 - t^{-2}}, \\ KK^{-1} &= 1.
\end{align*}
$$

The comultiplication is the algebra morphism given by:

$$
\Delta(X) = X \otimes K + K^{-1} \otimes X, \\
\Delta(Y) = Y \otimes K + K^{-1} \otimes Y,
$$

$$
\Delta(K) = K \otimes K.
$$

The antipode is the antimorphism given by $S(X) = -t^2X$, $S(Y) = -t^{-2}Y$, $S(K) = K^{-1}$, and the counit is the morphism given by $\epsilon(X) = \epsilon(Y) = 0$, and $\epsilon(K) = 1$.

The standard representations $m$, where $m$ is a nonnegative integer, of $\mathcal{A}_t$ have basis $e_i$, where $i$ runs in integer steps from $-m/2$ to $m/2$. Hence as a vector space $m$ has dimension $m + 1$. Recall that

$$
[n] = \frac{t^{2n} - t^{-2n}}{t^2 - t^{-2}},
$$

and $[n]! = [n][n-1] \ldots [1]$.

The action of $\mathcal{A}_t$ is given by

$$
\begin{align*}
X \cdot e_i &= [m/2 + i + 1]e_{i+1} \quad \text{but} \quad X \cdot e_{m/2} = 0, \\
Y \cdot e_i &= [m/2 - i + 1]e_{i-1} \quad \text{but} \quad Y \cdot e_{-m/2} = 0, \\
K \cdot e_i &= t^{2i}e_i.
\end{align*}
$$

The representation $m$ can be seen as a homomorphism

$$
\rho_m : \mathcal{A}_t \to M_{m+1}(\mathbb{C}).
$$

Lemma 1. The homomorphisms $\rho_m : \mathcal{A}_t \to M_{m+1}(\mathbb{C})$ are onto.

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Proof. Using the ordered basis, \( \{ e_{-m/2}, \ldots, e_{m/2} \} \), \( \rho_m(X) \) is the matrix that is zero except on the first subdiagonal, where the entries going from the top to the bottom are 1, [2], [3], \ldots, [m]. Similarly, the matrix \( \rho_m(Y) \) is zero except on the first superdiagonal, where starting from the bottom and going up the entries are 1, [2], [3], \ldots, [m].

The image of \( X^nY^p \) is a matrix with zero entries except on a particular super- or sub-diagonal, whose distance from the diagonal is \( |n-p| \). Starting from the top, the first \( \min\{p, n\} \) entries of that diagonal are zero, and the subsequent entries are all nonzero. Thus there exist linear combinations of the matrices \( \rho_m(X^nY^p) \), with \( p \leq n \), corresponding to each of the elementary matrices whose only nonzero entry lies on the \( n-p \) subdiagonal, or on the diagonal. We are using the pattern of zero and nonzero entries on the \( n-p \) subdiagonal to see this. By a similar analysis of \( \rho_m(Y^pX^n) \) we see that all elementary matrices where the nonzero entry lies on a superdiagonal can be written as a linear combination of the \( \rho_m(Y^pX^n) \). Since \( M_{m+1}(\mathbb{C}) \) is spanned by the elementary matrices, this finishes the proof.

Define the linear functionals \( m^c_{ij} : A_t \rightarrow \mathbb{C} \) by letting \( m^c_{ij}(Z) \) be the \( ij \)-th coefficient of the matrix \( \rho_m(Z) \). Let \( qSL_2 \) be the stable subalgebra of the Hopf algebra dual \( A_t' \) generated by linear functionals \( m^c_{ij} \).

Proposition 1. The linear functionals \( m^c_{ij} \) form a basis for the algebra \( qSL_2 \).

Proof. Since

\[
m \otimes n = \bigoplus_{q=|m-n|}^{m+n} q,
\]

the linear functionals \( m^c_{ij} \) span the algebra \( qSL_2 \). We need to show that they are also linearly independent. The quantum Casimir is given by

\[
C = \frac{(tK - t^{-1}K^{-1})^2}{(t^2 - t^{-2})^2} + YX \in A_t.
\]

Since \( C \) is central in \( A_t \), it acts as scalar multiplication in any irreducible representation. In fact, it acts on \( m \) as \( \lambda_m = \frac{(t^{m+1} - t^{-m-1})^2}{(t^2 - t^{-2})^2} \). Let

\[
C_{m,n} = \frac{C - \lambda_n}{\lambda_m - \lambda_n}.
\]

Notice that \( C_{m,n} \) is zero under \( \rho_n \) and is sent to the identity in \( \rho_m \). The product

\[
D_{m,N} = \prod_{p=1, p \neq m}^{N} C_{m,n}
\]

is an element of \( A_t \) that is sent to 0 in all of the representations from 1 to \( N \), except \( m \) where it is sent to the identity matrix.

If some linear combination \( \sum \alpha_{i,j,n} n^c_{ij} \) is equal to zero, it means that for all \( Z \in A_t \),

\[
\sum_{i,j,n} \alpha_{i,j,n} n^c_{ij}(Z) = 0.
\]
Let $N$ be the largest $n$ for which $\alpha_{i,j,n} \neq 0$. For each $m$ such that $\alpha_{i,j,m} \neq 0$, apply the functional $\sum_{i,j,n} \alpha_{i,j,n} c^i_j$ to $D_{m,N}Z$. Since
$$\sum_{i,j,n} \alpha_{i,j,n} c^i_j(D_{m,N}Z) = 0,$$
it follows that for all $Z \in A_t$, and fixed $m$,
$$\sum_{i,j} \alpha_{i,j,m} c^n_i(Z) = 0.$$

Finally, from lemma 1 the homomorphisms $\rho_m$ are surjective, so the independence of the $m c^i_j$ follows from the independence of the matrix coefficients on $M_{m+1}(\mathbb{C})$. Therefore all the $\alpha_{i,j,m} = 0$.

The product of any two matrix coefficients can be written as a linear combination of matrix coefficients
\begin{equation}
m c^i_j \cdot n c^k_l = \sum_{u,v,p} \gamma_{u,v,p}^{i,j,m,k,l,n}(t) c^u_v \end{equation}
Since the functionals $p c^u_v$ are linearly independent, the coefficients $\gamma_{u,v,p}^{i,j,m,k,l,n}(t)$ are uniquely defined. The $\gamma_{u,v,p}^{i,j,m,k,l,n}(t)$ are versions of the Clebsch-Gordan coefficients. Notice that $|m - n| \leq p \leq m + n$, consequently for each tuple $(i, j, m, k, l, n)$ there are only finitely many $(u, v, p)$ with $\gamma_{u,v,p}^{i,j,m,k,l,n}(t) \neq 0$.

A similar computation can be performed with the analogously defined $m c^i_j$ associated to $Sl_2(\mathbb{C})$. The limit as $t$ approaches 1 of the coefficients $\gamma_{u,v,p}^{i,j,m,k,l,n}(t)$ gives the corresponding quantities for $Sl_2(\mathbb{C})$.

Let
$$\bar{A}_t = M_1(\mathbb{C}) \times M_2(\mathbb{C}) \times M_3(\mathbb{C}) \times \ldots$$
be the Cartesian product of all the matrix algebras over $\mathbb{C}$ given the product topology.

**Proposition 2.** The homomorphism
\begin{equation}
\Theta : A_t \to \bar{A}_t
\end{equation}
given by $\Theta(Z) = (\rho_0(Z), \rho_1(Z), \rho_2(Z), \ldots)$ is injective and its image is dense in $\bar{A}_t$.

**Proof.** The fact that the $\rho_m$ are onto and the existence of the elements $C_{m,n}$ defined by equation (7) can be used to prove that the image of $\Theta$ is dense in $\bar{A}_t$.

A version of the Poincaré-Birkhoff-Witt theorem says that the monomials $K^m X^n Y^p$ form a basis for $A_t$ as a vector space. Using the relation $XY - YX = \frac{K^2 - K^{-2}}{t^2}$, this can be replaced by the basis $Z_{m,n,p}$, with $Z_{m,n,p} = K^m X^n Y^p$ for $n \geq p$ and $Z_{m,n,p} = K^m Y^n X^p$ when $n < p$. In order to prove that the map $\Theta$ is injective, consider an element $\sum \alpha_i Z_{m_i,n_i,p_i} \in A_t$. It is our goal to show that if $\Theta(\sum \alpha_i Z_{m_i,n_i,p_i}) = 0$ then all $\alpha_i$ are zero.

In any representation the image of $Z_{m,n,p}$ is a matrix that is zero off of the super (or sub)-diagonal corresponding to $n - p$. Thus it suffices to consider the sums where
\( n_i - p_i \) is a constant, as long as we only work with the parts of the matrices in the image that lie on the super- or sub-diagonal corresponding to that constant.

Assume that \( n_i \geq p_i \). The argument is similar when \( n_i < p_i \). Suppose that, for \( k \geq 0 \), the image under \( \Theta \) of

\[
\sum \alpha_i K^{m_i} X^{p_i+k} Y^{p_i},
\]

on the \( k \)th subdiagonal is zero. The map \( \Theta \) takes \( K^{m} X^{p+k} Y^{p} \) to a sequence of matrices such that the first \( p \) entries along the \( k \)-th subdiagonal are zero. Let \( p \) be the minimum of the \( p_i \) appearing in (8). The \((p+1)\)-st entry of each \( k \)-th subdiagonal of each matrix in the sequence \( \Theta(\sum \alpha_i K^{m_i} X^{p_i+k} Y^{m_i}) \) is the image under \( \Theta \) of the collection of terms in (8) with \( p_i = p \). All the other terms are mapped to matrices with a zero there. Thus it is enough to show that whenever all the \((p+1)\)-st entries on the \( k \)-th subdiagonal in each entry of \( \Theta(\sum \alpha_i K^{m_i} X^{p+k} Y^{p}) \) are zero, then all \( \alpha_i \) are zero.

Assume that all the \((p+1)\)-st entries on the \( k \)-th subdiagonal of \( \Theta(\sum \alpha_i Z^{m_i,p+k,p}) \) are zero. Make a sequence consisting of the \((p+1)\)-st entries of the \( k \)-th diagonal of the image of \( Z_{0,p+k,p} \). This sequence is:

\[
(0, 0, \ldots, [p+k]! \prod_{r=1}^{p} [k + r], [p+k]! \prod_{r=1}^{p} [k + r + 1], \ldots),
\]

where the first nonzero entry corresponds to the representation \( \rho_{p+k+1} \). Hence, the sequence corresponding to \( Z_{m_i,p+k,p} \) is

\[
(0, 0, \ldots, t^{m_i(p+k)} [p+k]! \prod_{r=1}^{p} [k + r], t^{m_i(p+k-1)} [p+k]! \prod_{r=1}^{p} [k + r + 1], \ldots).
\]

Supposing that we have \( J \) terms in our sum, we can truncate these sequences to get a \( J \times J \) matrix, so that the coefficients \( \alpha_i \) as a column vector, must be in the kernel of that matrix. Notice that the coefficient of the power of \( t \) in each column is the same product of quantized integers. Hence its determinant is a product of quantized integers times the determinant of the matrix,

\[
\begin{pmatrix}
t^{m_1(p+k)} & t^{m_1(p+k-1)} & \cdots \\
t^{m_2(p+k)} & t^{m_2(p+k-1)} & \cdots \\
& \vdots & \ddots
\end{pmatrix},
\]

Factoring out a large power of \( t \) from each row we get the Vandermonde determinant,

\[
\begin{vmatrix}
1 & t^{-m_1} & t^{-2m_1} & \cdots \\
1 & t^{-m_2} & t^{-2m_2} & \cdots \\
& \vdots & \ddots & \ddots
\end{vmatrix},
\]

which is nonzero as long as the \( t^{m_i} \) are not equal to one another. Since \( t \) was chosen specifically not to be a root of unity, all the \( \alpha_i \) must be zero. \( \square \)

The topology induced on \( \mathfrak{A}_t \) by its image under \( \Theta \) is the weak topology from \( qSL_2 \).

That is a sequence \( Z_n \) is Cauchy if for every \( \phi \in qSL_2 \), \( \phi(Z_n) \) is a Cauchy sequence of
complex numbers. Hence $\mathcal{A}_t$ is the completion of $\mathcal{A}_t$ by equivalence classes of Cauchy sequences in the weak topology from $qSL_2$.

Let $e_{i,j}(m) \in \mathcal{A}_t$ be the sequence of matrices that is the zero matrix in every entry, except the $m + 1$-st, where it is the elementary matrix that is all zeroes except for a 1 in the $ij$-th entry. Notice that the $e_{i,j}(m)$ are dual to the $m^c_{i,j}$ in the sense that $m^c_{i,j}(e_{k,l}(p))$ is zero unless the indices are identical, in which case it is one. Also notice that any $A \in \mathcal{A}_t$ can be written uniquely as $\sum_{i,j,m} \alpha_{i,j,m} e_{i,j}(m)$. The infinite sum makes sense!

**Proposition 3.** The algebra $\mathcal{A}_t$ has a structure of a topological ribbon Hopf algebra.

**Proof.** We need to define comultiplication on $\mathcal{A}_t$. Every element of $\mathcal{A}_t \otimes \mathcal{A}_t$ can be written as an infinite sum,

$$\sum_{i,j,m,k,l,n} \tau_{i,j,m,k,l,n} e_{i,j}(m) \otimes e_{k,l}(n) \quad (9)$$

so that no $e_{i,j}(m) \otimes e_{k,l}(n)$ is repeated. There are infinite sums of this form that cannot be decomposed as a finite sum of tensors of elements of $\mathcal{A}_t$. We topologize $\mathcal{A}_t \otimes \mathcal{A}_t$ by saying that a sequence $W_n$ is Cauchy if and only if for every $m^c_{i,j} \otimes n^c_{k,l}$ the sequence $(m^c_{i,j} \otimes n^c_{k,l})(W_n)$ is Cauchy. Let $\mathcal{A}_t \otimes \mathcal{A}_t$ be the completion of $\mathcal{A}_t \otimes \mathcal{A}_t$ by equivalence classes of Cauchy sequences. Notice that every sum of the type like in equation (9) yields an equivalence class of Cauchy sequences in $\mathcal{A}_t \otimes \mathcal{A}_t$ by truncating to get a sequence of partial sums. Conversely, if $Z_n \in \mathcal{A}_t \otimes \mathcal{A}_t$ is Cauchy, by applying the $m^c_{i,j} \otimes n^c_{k,l}$ to the sequence, and taking the limit we get the coefficients of a unique expression of the type (9), and two Cauchy sequences are equivalent if and only if they give rise to the same expression. Hence we can identify $\mathcal{A}_t \otimes \mathcal{A}_t$ with the set of expressions like in equation (9).

In order to define the comultiplication on $\mathcal{A}_t$ with values in $\mathcal{A}_t \otimes \mathcal{A}_t$, take the adjoint of multiplication on $qSL_2$. Use $\langle , \rangle$ to denote evaluation of elements of $\mathcal{A}_t$ on $qSL_2$, and extend this to evaluating elements of $qSL_2 \otimes qSL_2$ on elements of $\mathcal{A}_t \otimes \mathcal{A}_t$ pairwise. Then,

$$\langle m^c_{i,j} \otimes n^c_{k,l}, \Delta(e_{u,v}(q)) \rangle = \langle m^c_{i,j} \cdot n^c_{k,l}, e_{u,v}(q) \rangle = \gamma_{i,j,m,k,l,n}^{u,v,q}.$$ 

Therefore,

$$\Delta(e_{u,v}(q)) = \sum_{i,j,m,k,l,n} \gamma_{i,j,m,k,l,n}^{u,v,q} e_{i,j}(m) \otimes e_{k,l}(n).$$

The sum makes sense for an arbitrary element of $\mathcal{A}_t$ as there are only finitely many nonzero $\gamma_{i,j,m,k,l,n}^{u,v,q}$ for any $e_{i,j}(m) \otimes e_{k,l}(n)$. So one can sum

$$\Delta(\sum_{i,j,m} \alpha_{i,j,m} e_{u,v}(q)) = \sum_{i,j,m} \alpha_{i,j,m} \Delta(e_{i,j}(m)) =$$

$$\sum_{i,j,m} \alpha_{i,j,m} \gamma_{i,j,m,k,l,n}^{u,v,q}(t) e_{i,j}(m) \otimes e_{k,l}(n).$$
Comultiplication is continuous since its composition with every $m \otimes n \otimes c_j \otimes n \otimes c_k$ is continuous.

Let $q = t^4$. The standard formula for the universal $R$-matrix $\mathbb{B}$ in the Jimbo-Drinfeld model of $U_h(sl_2)$ is

$$R = \sum_{n \geq 0} \frac{(q - q^{-1})^n}{[n]} q^{-n(n+1)/2} t^{H \otimes nH} (X^n \otimes Y^n).$$

(10)

Recall that the standard Drinfeld-Jimbo model $\mathbb{B}$ of $U_h(sl_2)$ is generated by $X, Y, H$. If we let $K = t^H$ then the relations (1), (2) for $\mathcal{A}_t$ can be derived from the relations for the Drinfeld-Jimbo model. Consequently, interpret $H$ as the traditional image of $H$ under the standard irreducible representations of $U(sl_2)$. That is, $H$ is the sequence of matrices,

$$(1, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, ...).$$

Taking $t$ raised to this sequence gives the sequence $\Theta(K)$, where $\Theta$ is defined in equation (7). Interpret $X$ and $Y$ as the sequences of matrices coming from the standard representations of $\mathcal{A}_t$, i.e., $\Theta(X)$ and $\Theta(Y)$. The resulting expression (10) makes sense as an element of $\mathcal{A}_t \otimes \mathcal{A}_t$ since in any particular irreducible representation only finitely many terms are nonzero. Thus the $R$ matrix is well defined as an element of $\mathcal{A}_t \otimes \mathcal{A}_t$, and has the desired properties.

**References**

[1] C. Kassel, *Quantum Groups*, Springer-Verlag (1995).