In this paper, we prove existence and uniqueness results for a fractional sequential fractional $q$-Hahn integrodifference equation with nonlocal mixed fractional $q$ and fractional Hahn integral boundary condition, which is a new idea that studies $q$ and Hahn calculus simultaneously.

**Keywords:** fractional $q$-calculus; fractional Hahn calculus; fractional integral boundary value problems; existence

**MSC:** 39A10; 39A13; 39A70

### 1. Introduction

A $q$-difference operator $D_q$ is an important tool in areas of mathematics and applications [1–4] such as orthogonal polynomials problems and mathematical control theories. Basic definitions and properties for $q$-difference calculus were presented by Kac and Cheung [5], Al-Salam [6], Agarwal [7], and Annaby and Mansour [8]. There are many research works widely studying the $q$-difference operators (see [9–23]).

A Hahn difference operator $D_{q,\omega}$ arose from the forward difference operator and the Jackson $q$-difference operator was introduced by Hahn [24] in 1949. Then, the right inverse of $D_{q,\omega}$ presented in terms of Jackson $q$-integral and Nörlund sum was proposed by Aldwoah [25,26] in 2009. The Hahn difference operator can be used in studied of families of orthogonal polynomials and approximation problems (see [27–29]). More research works about Hahn difference calculus can be found in [30–39].

The fractional Hahn difference operators was introduced by Briskhavana and Sitthiwirattham [40] in 2017, and Wang et al. [41] in 2018. The extension of this operator has been used in the study of existence results of solution of boundary value problems [42–45], a generalization of Minkowski’s inequality [46], and impulsive fractional quantum Hahn operator [47,48].

From the literature, we have found that the study of fractional $q$-difference and fractional Hahn difference operators simultaneously have not been studied. Therefore, in this article, we devote ourselves to study the boundary value problem for equations that contain both fractional $q$-difference and Hahn difference operators. Our problem is a nonlocal mixed fractional $q$ and Hahn integral boundary value problem for sequential fractional $q$-Hahn integrodifference equation of the form
We let the notations,

\[ D_q^a D_q^b u(t) = F \left[ t, u(t), \Psi_q^a u(t), \Phi_q^b u(t) \right], \quad t \in I_{q}^{\omega(T)^+}, \]

\[ u(\eta) = \lambda I_q^{\omega} u(\eta), \quad \eta \in (\omega_0, T), \]

\[ u(T) = \mu I_q^{\omega} u(T), \]

where \( I_q^{\omega(T)} := \bigcup_{k=0}^{\infty} I_q^{\omega+k|q|^k}, s \in [\omega_0, T]; I_q^\omega := \{ q^n x : n \in \mathbb{N}_0 \} \cup \{ 0 \}; \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \}; 0 < q < 1; \omega > 0; T > \omega_0; \alpha, \beta, \gamma, \theta, \phi \in (0, 1]; \alpha + \beta \in (1, 2]; \lambda, \mu \in \mathbb{R}^+; F \in C([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) is given function; and for \( \psi \in C([0, T] \times [\omega_0, T], [0, \infty)) \), \( \varphi \in C([\omega_0, T] \times [\omega_0, T], [0, \infty)) \), we define

\[
\Psi_q^a u(t) := \left( \int_q^t \psi(t, s) u(s) \, ds \right)
\]

\[
\Phi_q^b u(t) := \left( \int_q^t \varphi(t, s) u(s) \, ds \right)
\]

This paper is organized as follows. In Section 2, we provide some definitions and lemmas for \( q \)-difference and Hahn difference operators. In Section 3, we prove the existence and uniqueness of a solution to problem (1) by using the Banach fixed point theorem. In the last section, we give an example to illustrate our results.

2. Preliminaries

In this section, we recall the notations, definitions, and lemmas for \( q \) and Hahn difference calculus. For \( q \in (0, 1), \omega > 0 \), we define

\[ [n]_q := \frac{1 - q^n}{1 - q} = q^{n-1} + \ldots + q + 1 \quad \text{and} \quad [n]_q ! := \prod_{k=1}^{n} \frac{1 - q^k}{1 - q}, \quad n \in \mathbb{N}. \]

The \( q \)-analogue of the power function \( (a - b)_q^n \) with \( n \in \mathbb{N}_0 \) is given by

\[ (a - b)_q^0 := 1, \quad (a - b)_q^n := \prod_{k=0}^{n-1} (a - bq^k), \quad a, b \in \mathbb{R}. \]

The \( q, \omega \)-analogue of the power function \( (a - b)_{q, \omega}^n \) with \( n \in \mathbb{N}_0 \) is given by

\[ (a - b)_{q, \omega}^0 := 1, \quad (a - b)_{q, \omega}^n := \prod_{k=0}^{n-1} \left[ a - (bq^k + \omega|q|) \right], \quad a, b \in \mathbb{R}. \]

For \( \alpha \in \mathbb{R} \), the power function is given by

\[ (a - b)_q^\alpha := a^\alpha \prod_{n=0}^{\infty} \frac{1 - \left( \frac{b}{a} \right)^n q^n}{1 - \left( \frac{b}{a} \right)^n q^{a+n}}, \quad a \neq 0, \]

\[ (a - b)_{q, \omega}^\alpha := (a - \omega_0)^\alpha \prod_{n=0}^{\infty} \frac{1 - \left( \frac{b - \omega_0}{a - \omega_0} \right)^n q^n}{1 - \left( \frac{b - \omega_0}{a - \omega_0} \right)^n q^{a+n}} \left( (a - \omega_0) - (b - \omega_0) \right)_q^\alpha, \quad a \neq \omega_0. \]

We let the notations, \( \alpha_q = a^\alpha \), \( (a - \omega_0)_{q, \omega}^\alpha = (a - \omega_0)^\alpha \), and \( (0)_{q, \omega}^\alpha = (\omega_0)_q^\alpha = 0 \) for \( \alpha > 0 \).
The $q$-gamma and $q$-beta functions are defined by
\[
\Gamma_q(x) := \frac{(1 - q)x^{-1}}{(1 - q)^x}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\},
\]
and
\[
B_q(x, s) := \int_0^1 t^{x-1} (1 - qt)^{s-1} \, dq_t = \frac{\Gamma_q(x) \Gamma_q(s)}{\Gamma_q(x + s)},
\]
respectively.

For $k \in \mathbb{N}$, the $q$-analogue and $q, \omega$-analogue of forward jump operator are defined by
\[
\sigma^k_q(t) := q^k t \quad \text{and} \quad \sigma^k_{q,\omega}(t) := q^k t + \omega[k]_q,
\]
respectively. The $q$-analogue and $q, \omega$-analogue of backward jump operator are defined by
\[
\rho^k_q(t) := \frac{t}{q^k}, \quad \text{and} \quad \rho^k_{q,\omega}(t) := \frac{t - \omega[k]_q}{q^k},
\]
respectively.

**Definition 1.** For $q \in (0, 1)$, the $q$-difference of a real function $f$ is defined by
\[
D_qf(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \neq 0 \quad \text{and} \quad D_qf(0) = \lim_{t \to 0} D_qf(t).
\]

Let $f$ be a function defined on the interval $[0, T]$, $q$-integral is defined by
\[
\mathcal{I}_q f(t) = \int_0^t f(s) \, dq_s = (1 - q)^t \sum_{n=0}^{\infty} q^n f(q^n t)
\]
where the infinite series is convergent.

**Definition 2.** For $q \in (0, 1), \omega > 0$ and $f$ defined on an interval $I \subseteq \mathbb{R}$ which contains $\omega_0 := \frac{\omega}{1 - q}$, the Hahn difference of $f$ is defined by
\[
D_{q,\omega}f(t) = \frac{f(qt + \omega) - f(t)}{(qt - 1) + \omega} \quad \text{for} \quad t \neq \omega_0,
\]
and $D_{q,\omega}f(\omega_0) = f'(\omega_0)$.

For $a, b \in I \subseteq \mathbb{R}$ with $a < \omega_0 < b$ and $[k]_q := \frac{1- q^k}{1-q}$, $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, the $q, \omega$-interval is defined by
\[
[a, b]_{q,\omega} := \left\{ q^k a + \omega[k]_q : k \in \mathbb{N}_0 \right\} \cup \left\{ q^k b + \omega[k]_q : k \in \mathbb{N}_0 \right\} \cup \{\omega_0\}
\]
\[
= [a, \omega_0]_{q,\omega} \cup [\omega_0, b]_{q,\omega}
\]
\[
= (a, b)_{q,\omega} \cup \{a, b\} = [a, b]_{q,\omega} \cup \{b\} = (a, b)_{q,\omega} \cup \{a\}.
\]
We note that, for each $s \in [a, b]_{q,\omega}$, the sequence $\{\sigma^k_{q,\omega}(s)\}_{k=0}^{\infty} = \{q^k s + \omega[k]_q\}_{k=0}^{\infty}$ is uniformly convergent to $\omega_0$.

**Definition 3.** Let $I$ be any closed interval of $\mathbb{R}$ that contains $a, b$ and $\omega_0$. Letting $f : I \to \mathbb{R}$ be a given function, $q, \omega$-integral of $f$ from $a$ to $b$ is defined by
\[
\int_a^b f(t) \, dq,\omega t := \int_{\omega_0}^b f(t) \, dq_0,\omega t - \int_{\omega_0}^a f(t) \, dq_0,\omega t.
\]
where \( \int_{\omega_0}^{T} f(t) d_{q,t} := \left[ x(1-q) - \omega \right] \sum_{k=0}^{\infty} q^k f \left( xq^k + \omega[k] \right), \ x \in I, \) and the series converges at \( x = a \) and \( x = b \) where the sum of the right-hand side is called the Jackson–Nörlund sum.

Note that the actual domain of function \( f \) is defined on \([a, b]_{q,\omega} \subset I\).

The following fractional \( q \) integral, fractional Hahn integral, fractional \( q \) difference, and fractional Hahn difference of Riemann–Liouville type are defined.

**Definition 4.** Let \( f \) be defined on \([0, T]\) and \( \alpha \geq 0 \), the fractional \( q \)-integral of the Riemann–Liouville type is defined by

\[
\left( I^\alpha_q f \right)(t) := \frac{1}{\Gammaq(\alpha)} \int_{\omega_0}^{t} (t - qs)^{\frac{\alpha - 1}{q}} f(s) d_{q,s} s
\]

and \( (I^0_q f)(x) = f(x) \).

**Definition 5.** Let \( f \) be defined on \([0, T]_{q,\omega}\) and \( \alpha, \omega > 0, \ q \in (0,1) \), and the fractional Hahn integral, is defined by

\[
I^\alpha_{q,\omega} f(t) := \frac{1}{\Gammaq(\alpha)} \int_{\omega_0}^{t} (t - s_{q,\omega}(s))^\frac{\alpha - 1}{q} f(s) d_{q,\omega,s}
\]

\[
= \frac{[t(1-q) - \omega]}{\Gammaq(\alpha)} \sum_{n=0}^{\infty} q^n \left( t - s_{q,\omega}(t) \right)^\frac{\alpha - 1}{q} f\left( s_{q,\omega}^n(t) \right)
\]

and \( (I^0_{q,\omega} f)(t) = f(t) \).

**Definition 6.** Let \( f \) be defined on \([0, T]\) and \( \alpha \geq 0 \), the fractional \( q \)-derivative of the Riemann–Liouville type of order \( \alpha \), is defined by

\[
\left( D^\alpha_q f \right)(t) := \left( D^N_q I^{N-\alpha}_q f \right)(t)
\]

\[
= \frac{1}{\Gammaq(-\alpha)} \int_{\omega_0}^{t} \left( t - s_q(s) \right)^{-\frac{\alpha - 1}{q}} f(s) d_{q,s},
\]

and \( (D^0_q f)(x) = f(x) \), where \( N \) is the smallest integer that is greater than or equal to \( \alpha \).

**Definition 7.** Let \( f \) be defined on \([0, T]_{q,\omega}\) and \( \alpha, \omega > 0, \ q \in (0,1) \), the fractional Hahn difference of the Riemann–Liouville type of order \( \alpha \) is defined by

\[
D^\alpha_{q,\omega} f(t) := \left( D^N_{q,\omega} I^{N-\alpha}_{q,\omega} f \right)(t)
\]

\[
= \frac{1}{\Gammaq(-\alpha)} \int_{\omega_0}^{t} \left( t - s_{q,\omega}(s) \right)^{-\frac{\alpha - 1}{q}} f(s) d_{q,\omega,s},
\]

and \( D^0_{q,\omega} f(t) = f(t) \), where \( N \) is the smallest integer that is greater than or equal to \( \alpha \).
Lemma 1 ([10]). Letting \( \alpha > 0, q \in (0, 1) \) and \( f : I^T_q \rightarrow \mathbb{R} \),

\[
T^\alpha_q D^\alpha_q f(t) = f(t) + C_1 t^{\alpha-1} + \ldots + C_N t^{\alpha-N},
\]

for some \( C_i \in \mathbb{R}, i = \{1, 2, \ldots, N\} \) and \( N - 1 < \alpha \leq N, N \in \mathbb{N} \).

Lemma 2 ([40]). Letting \( \alpha > 0, q \in (0, 1), \omega > 0 \) and \( f : I^T_{q,\omega} \rightarrow \mathbb{R} \),

\[
T^\alpha_{q,\omega} D^\alpha_{q,\omega} f(t) = f(t) + C_4 (t - \omega_0)^{\alpha-1} + \ldots + C_N (t - \omega_0)^{\alpha-N},
\]

for some \( C_i \in \mathbb{R}, i = \{1, 2, \ldots, N\} \) and \( N - 1 < \alpha \leq N, N \in \mathbb{N} \).

Some auxiliary lemmas used to investigate the solution of the linear variant of (1) are provided as follows.

Lemma 3 ([16]). Let \( \alpha, \beta \geq 0 \) and \( p, q \in (0, 1) \). Then, the following formulas hold:

\[
\int_0^\eta (\eta - qt)^{\alpha-1} q^\alpha dt = \frac{\eta^{\alpha+\beta} B_q (\beta + 1, \alpha)}{\beta^\alpha q^\beta},
\]

\[
\int_0^\eta (\eta - ps)^{\alpha-1} q^\alpha dt d_p s = \frac{\eta^{\alpha+\beta} B_q (\beta + 1, \alpha)}{\beta^\alpha q^\beta}.
\]

Lemma 4 ([40]). Letting \( \alpha, \beta > 0, p, q \in (0, 1) \) and \( \omega > 0 \),

\[
\int_{\omega_0}^t (t - s_q \omega(s))^{\alpha-1} q^\alpha d_q s = (t - \omega_0)^{\alpha+\beta} B_q (\beta + 1, \alpha),
\]

\[
\int_{\omega_0}^t \int_{\omega_0}^x (t - s_q \omega(s))^{\alpha-1} q^\alpha d_q s d_p x = \frac{(t - \omega_0)^{\alpha+\beta} B_q (\beta + 1, \alpha)}{\beta^\alpha q^\beta}.
\]

Employing Lemmas 3 and 4, we obtain the solution of the linear variant of problem (1) as shown in the following lemma.

Lemma 5. Let \( \alpha, \beta, \gamma \in (0, 1), \alpha + \beta \in (1, 2]; 0 < q < 1; \omega > 0; T > \omega_0; \lambda, \mu \in \mathbb{R}^+; h \in C([0, T], \mathbb{R}) \) be a given function. Then, the linear variant problem

\[
D^\alpha_q D^\beta_{q,\omega} u(t) = h(t), \quad t \in I^\omega_q, \quad u(\eta) = \lambda \mathcal{O}_q(\eta), \quad \eta \in (\omega_0, T), \quad u(T) = \mu \mathcal{O}_q(T)
\]

has the unique solution which is in a form

\[
u(t) = \frac{1}{\Gamma_q(\alpha) \Gamma_q(\beta)} \int_{\omega_0}^t \int_{\omega_0}^x (t - s_q \omega(s))^{\beta-1} q^\alpha (x - s_q \omega(s))^{\alpha-1} q^\alpha h(s) d_q s d_{q,\omega} x
\]

\[
+ \left\{ A_T \mathcal{O}_q[h] - A_q \mathcal{O}_T[h] \right\} \frac{1}{\Omega_q(\beta)} \int_{\omega_0}^t (t - s_q \omega(s))^{\beta-1} q^\alpha s^{\alpha-1} q^\alpha d_{q,\omega} s
\]

\[
- \left\{ B_T \mathcal{O}_q[h] - B_q \mathcal{O}_T[h] \right\} \frac{(t - \omega_0)^{\beta-1} \Omega^\alpha}{\Omega}
\]

for \( t \in [\omega_0, T] \), where the functionals \( \mathcal{O}_q[h] \) and \( \mathcal{O}_T[h] \) are defined by
\[
\mathcal{O}_q[h] := -\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{\eta} \int_{0}^{x} (\eta - \sigma_{q,\alpha}(x))^{\frac{\beta-1}{q\alpha}} (x - \sigma_q(s))^{\frac{\alpha-1}{q\alpha}} h(s) d_q s d_q x d_q \omega x
\]
\[
+ \frac{\lambda}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)} \int_{\omega_0}^{\eta} \int_{0}^{x} \int_{0}^{r} (\eta - \sigma_{q,\alpha}(r))^{\frac{\gamma-1}{q\gamma}} (r - \sigma_{q,\alpha}(x))^{\frac{\beta-1}{q\beta}} \times (x - \sigma_q(s))^{\frac{\alpha-1}{q\alpha}} h(s) d_q s d_q x d_q \omega r,
\]

\[
\mathcal{O}_T[h] := -\frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_{\omega_0}^{T} \int_{0}^{x} (T - \sigma_{q,\alpha}(x))^{\frac{\beta-1}{q\beta}} (x - \sigma_q(s))^{\frac{\alpha-1}{q\alpha}} h(s) d_q s d_q x d_q \omega x
\]
\[
+ \frac{\mu}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)} \int_{\omega_0}^{T} \int_{0}^{x} \int_{0}^{r} (T - \sigma_{q}(r))^{\frac{\gamma-1}{q\gamma}} (r - \sigma_{q,\alpha}(x))^{\frac{\beta-1}{q\beta}} \times (x - \sigma_q(s))^{\frac{\alpha-1}{q\alpha}} h(s) d_q s d_q x d_q T r,
\]

and the constants \( A_q, A_T, B_q, B_T, \Omega \) are defined by

\[
A_q := (\eta - \omega_0)^{\beta-1} - \frac{\lambda}{\Gamma_q(\gamma)} \int_{0}^{\eta} (\sigma_{q,\alpha}(s))^{\frac{\gamma-1}{q\gamma}} (s - \omega_0)^{\beta-1} d_q s,
\]

\[
A_T := (T - \omega_0)^{\beta-1} - \frac{\mu}{\Gamma_q(\gamma)} \int_{0}^{T} (\sigma_{q}(s))^{\frac{\gamma-1}{q\gamma}} (s - \omega_0)^{\beta-1} d_q s,
\]

\[
B_q := \frac{1}{\Gamma_q(\beta)} \int_{0}^{\eta} (\eta - \sigma_{q,\alpha}(s))^{\frac{\beta-1}{q\beta}} s^{\alpha-1} d_q \omega s
\]
\[
- \frac{\lambda}{\Gamma_q(\beta)\Gamma_q(\gamma)} \int_{0}^{\eta} \int_{0}^{x} (\eta - \sigma_{q,\alpha}(x))^{\frac{\gamma-1}{q\gamma}} (x - \sigma_q(s))^{\frac{\beta-1}{q\beta}} s^{\alpha-1} d_q \omega s d_q \omega x,
\]

\[
B_T := \frac{1}{\Gamma_q(\beta)} \int_{0}^{T} (T - \sigma_{q,\alpha}(s))^{\frac{\beta-1}{q\beta}} s^{\alpha-1} d_q \omega s
\]
\[
- \frac{\mu}{\Gamma_q(\beta)\Gamma_q(\gamma)} \int_{0}^{T} \int_{0}^{x} (T - \sigma_{q}(x))^{\frac{\gamma-1}{q\gamma}} (x - \sigma_q(s))^{\frac{\beta-1}{q\beta}} s^{\alpha-1} d_q \omega s d_q \omega x,
\]
\[
\Omega := A_T B_q - A_q B_T \neq 0.
\]

**Proof.** Firstly, we take fractional \( q \)-integral of order \( \alpha \) for (2). Then, we have

\[
D_q^{\beta} \sigma_{q,\alpha} u(t) = C_0 t^{\alpha-1} + \frac{(1 - q)t^\alpha}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} q^k (1 - q^{k+1})^{\frac{\alpha-1}{q\alpha}} h\left(c_q^k(t)\right)
\]
\[
= C_0 t^{\alpha-1} + \frac{1}{\Gamma_q(\alpha)} \int_{0}^{t} (t - \sigma_q(s))^{\frac{\alpha-1}{q\alpha}} h(x) d_q s,
\]
for \( t \in I_{[\omega_0,T]} := \{ q^n s + \omega[n] : s \in [\omega_0,T], n \in \mathbb{N}_0 \} \cup \{ \omega_0 \} \).

Taking fractional Hahn integral of order \( \beta \) for (11), we obtain
\[ u(t) = C_1(t - \omega_0)\beta^{-1} + \frac{C_0}{\Gamma_q(\beta)}(1 - q)(t - \omega_0)^\beta \sum_{k=0}^{\infty} q^k \left(1 - q^{k+1}\right)^{\beta-1}q \left(\sigma_{q,t}^h(t)\right)^{a-1} \\
+ \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}(1 - q)^2(t - \omega_0)^\beta \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} q^{h+k} \left(1 - q^{h+k+1}\right)^{\beta-1}q \left(\sigma_{q,h,t}^h(t)\right) \times \\
\left(1 - q^{k+1}\right)^{\alpha-1}q \left(\sigma_{q,h,t}^h(t)\right) h \left(\sigma_{q,h,t}^h(t)\right) \\
= C_1(t - \omega_0)\beta^{-1} + \frac{C_0}{\Gamma_q(\beta)} \int_{\omega_0}^t \left(t - \sigma_{q,t}(\omega)\right)^{\beta-1}q \left(\sigma_{q,t}^h(t)\right) h(\sigma_{q,t}^h(t)) d_q d_q \left(\sigma_{q,t}^h(t)\right) x, \quad (12) \]
for \( t \in [\omega_0, T]. \)

Taking fractional \( q \)-integral of order \( \gamma \) for (12), we have
\[
\mathcal{I}_q^\gamma u(t) = \frac{C_1}{\Gamma_q(\gamma)} \int_{\omega_0}^t \left(t - \sigma_{q,t}(\omega)\right)^{\gamma-1}q \left(\sigma_{q,t}^h(t)\right)^{\beta-1}q \left(\sigma_{q,t}^h(t)\right) h(\sigma_{q,t}^h(t)) d_q d_q \left(\sigma_{q,t}^h(t)\right) x, \quad (13) \]
for \( t \in [0, T]. \)

In addition, we take fractional Hahn integral of order \( \gamma \) for (12) to get
\[
\mathcal{I}_{q,t}^\gamma u(t) = \frac{C_1}{\Gamma_q(\gamma)} \int_{\omega_0}^t \left(t - \sigma_{q,t}(\omega)\right)^{\gamma-1}q \left(\sigma_{q,t}^h(t)\right)^{\beta-1}q \left(\sigma_{q,t}^h(t)\right) h(\sigma_{q,t}^h(t)) d_q d_q \left(\sigma_{q,t}^h(t)\right) x, \quad (14) \]
for \( t \in [\omega_0, T]. \)

Substituting \( t = \eta \) into (12) and (14), and employing the first condition of (2), we have
\[ A_\eta C_1 + B_\eta C_0 = O_\eta[h]. \quad (15) \]

Taking \( t = T \) into (12) and (13), and employing the second condition of (2), we have
\[ A_T C_1 + B_T C_0 = O_T[h]. \quad (16) \]

Solving Equations (15) and (16), we obtain
\[ C_1 = \frac{B_\eta O_T[h] - B_T O_\eta[h]}{\Omega} \quad \text{and} \quad C_0 = \frac{A_T O_\eta[h] - A_\eta O_T[h]}{\Omega} \]
where \( O_\eta[h], O_T[h], A_\eta, A_T, B_\eta, B_T \) and \( \Omega \) are defined by Equations (4)–(10).

Substituting \( C_0 \) and \( C_1 \) into (12), we obtain the solution (3). \( \square \)
3. Existence Results

In this section, the existence and uniqueness result for the mixed $q$-Hahn problem (1) is studied. Let $C = C ([\omega_0, T], \mathbb{R})$ be a Banach space of all function $u$ with the norm defined by

$$\|u\|_C = \max_{t \in [\omega_0, T]} |u(t)|.$$  

The operator $F : C \to C$ is defined by

$$F (u)(t) := \frac{1}{\Gamma_q (\alpha) \Gamma_q (\beta)} \int_{\omega_0}^{\ell} \int_{\omega_0}^{x} (t - \sigma_{q, \omega} (s))^{\beta - 1} q^{\alpha - 1} \times F \left[ s, u(s), \Psi_q (u(s)), \Psi_{q, \omega} (u(s)) \right] \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) \omega (t, s, \Psi_q (u(s))) \omega (t, s, \Psi_{q, \omega} (u(s))) \omega (t, s, u(s)) $$

where

$$\mathbf{A}_q O_q [F_t] = \mathbf{A}_q O_T [F_t]$$

and

$$\mathbf{O}_q [F_t] = \mathbf{O}_T [F_t]$$

and the constants $A_q, A_T, B_q, B_T, \Omega$ are defined by (6)–(10), respectively.

The problem (1) has solution if and only if the operator $F$ has fixed point. We show the proof in the following theorem.

**Theorem 1.** Assume that $F : [0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, $\psi : [0, T] \times [\omega_0, T] \to [0, \infty)$ and $\varphi : [\omega_0, T] \times [\omega_0, T] \to [0, \infty)$ are continuous with $\varphi_0 = \max \{ \varphi(t, s) : (t, s) \in [0, T] \times [\omega_0, T] \}$ and $\varphi_0 = \max \{ \varphi(t, s) : (t, s) \in [0, T] \times [\omega_0, T] \}$. In addition, suppose that the following conditions hold:

(H1) There exist constants $\ell_1, \ell_2, \ell_3 > 0$ such that for each $t \in [0, T]$ and $u, v \in \mathbb{R}$,

$$|F \left[ t, u, \Psi_q (u), \Psi_{q, \omega} (u) \right] - F \left[ t, v, \Psi_q (v), \Psi_{q, \omega} (v) \right]| \leq \ell_1 |u - v| + \ell_2 |\Psi_q (u) - \Psi_q (v)| + \ell_3 |\Psi_{q, \omega} (u) - \Psi_{q, \omega} (v)|.$$  

(H2) $L \Xi < 1$,  

where
\[ \mathcal{L} := \ell_1 + \ell_2 \psi_0 \Gamma_q(\theta + 1) + \ell_3 \psi_0 \Gamma_q(\varphi + 1), \]  
\[ \Xi := \frac{T^a(T - \omega_0)^\beta}{\Gamma_q(a+1) \Gamma_q(\beta + 1)} + \Phi_1 \Theta_T + \Phi_2 \Theta_\eta, \]  
\[ \Phi_1 := \frac{\eta^\beta (\eta - \omega_0)^\beta}{\Gamma_q(a+1) \Gamma_q(\beta + 1)} \left| 1 - \frac{\lambda (\eta - \omega_0)^\gamma}{\Gamma_q(\gamma + 1)} \right|, \]  
\[ \Phi_2 := \frac{T^a(T - \omega_0)^\beta}{\Gamma_q(a+1) \Gamma_q(\beta + 1)} \left| 1 - \frac{\mu T^\gamma}{\Gamma_q(\gamma + 1)} \right|, \]  
\[ \Theta_T := \frac{1}{|\Omega|} \left\{ |A_T| \frac{T^{a-1}(T - \omega_0)^\beta}{\Gamma_q(\beta + 1)} + |B_T|(T - \omega_0)^{\beta - 1} \right\}, \]  
\[ \Theta_\eta := \frac{1}{|\Omega|} \left\{ |A_\eta| \frac{T^{a-1}(T - \omega_0)^\beta}{\Gamma_q(\beta + 1)} + |B_\eta|(T - \omega_0)^{\beta - 1} \right\}. \]

Then, problem (1) has a unique solution.

**Proof.** Firstly, we verify \( \mathcal{F} \) map bounded sets into bounded sets in \( B_L = \{ u \in \mathcal{C} : \|u\|_C \leq L \} \). Let \( K = \max_{t \in [0,T]} |F(t,0,0,0)| \), \( L \) be a constant satisfied with

\[ L \geq \frac{K \Xi}{1 - \Xi'}, \]  
and the notation \( |S(t,u,0)| = \left| F(t,u,\Psi_0 u,\Psi_0 u) - F(t,0,0,0) \right| + |F(t,0,0,0)|. \)

For each \( t \in [0,T] \) and \( u \in B_L \)

\[ \left| \mathcal{O}_T[F_u] \right| \leq \left| \mathcal{O} \right| \left| u \right|_C + \left| K \right| \Phi_1 \]

\[ \leq |\mathcal{L}| L + K \Phi_1. \]  

**Similary,**

\[ \left| \mathcal{O}_T[F_u] \right| \leq |\mathcal{L}| L + K \Phi_2. \]  

From (27) and (28), we find that

\[ |(F_u)(t)| \leq \left| \mathcal{O} \right| \left| u \right|_C + \left| K \right| \Phi_1 \]

\[ \leq |\mathcal{L}| L + K \Phi_2. \]  

(29)
Therefore, we obtain \( |F u|_C \leq L \), which implies that \( F B_L \subset B_L \).

Next, we aim to prove that \( F \) is contraction. Let the notation

\[
\mathcal{H}[u - v](t) = \left| F \left[ t, u(t), \Psi^\phi u(t), Y^\phi_{q,u} u(t) \right] - F \left[ t, v(t), \Psi^\phi v(t), Y^\phi_{q,u} v(t) \right] \right|
\]

for each \( t \in [0, T] \) and \( u, v \in C \). From (18), we find that

\[
\left| \mathcal{O}_\eta[F_u] - \mathcal{O}_\eta[F_v] \right| \\
\leq \left| \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^\eta \int_0^x (\eta - \sigma_{q,u}(x))^{\beta-1} q, (x - \sigma_{q,s}(s))^{\alpha-1} q, \mathcal{H}[u - v](s) ds d_q x \right|
\]

\[
- \frac{\lambda}{\Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)} \int_0^\eta \int_0^x (\eta - \sigma_{q,u}(r))^{\beta-1} q, (r - \sigma_{q,u}(x))^{\alpha-1} q, \mathcal{H}[u - v](r) dr d_q x d_q r
\]

\[
(x - \sigma_{q,s}(s))^{\alpha-1} q, \mathcal{H}[u - v](s) ds d_q x d_q r
\]

\[
\leq (\ell_1 |u - v| + \ell_2 |\Psi^\phi u - \Psi^\phi v| + \ell_3 |Y^\phi_{q,u} u - Y^\phi_{q,u} v|) \times
\]

\[
\left| \frac{\eta^\alpha (\eta - \omega_0)^\beta}{\Gamma_q(\alpha + 1)\Gamma_q(\beta + 1)} - \frac{\lambda\eta^\alpha (\eta - \omega_0)^{\beta + \gamma}}{\Gamma_q(\alpha + 1)\Gamma_q(\beta + 1)\Gamma_q(\gamma + 1)} \right|
\]

\[
\leq (\ell_1 + \ell_2 \psi_0 T^\eta \Gamma_q(\beta + 1) + \ell_3 \psi_0 (T - \omega_0)^\phi) \left| u - v \right| \Phi_1
\]

\[
\leq \mathcal{L} \Phi_1 \left| u - v \right|_C.
\]

Similarly, from (19), we have

\[
\left| \mathcal{O}_T[F_u] - \mathcal{O}_T[F_v] \right| \leq \mathcal{L} \Phi_2 \left| u - v \right|_C.
\]

Next, we find that

\[
\left| (Fu)(t) - (Fv)(t) \right|
\]

\[
\leq \left| \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)} \int_0^\eta \int_0^x (T - \sigma_{q,u}(s))^{\beta-1} q, (x - \sigma_{q,s}(s))^{\alpha-1} q, \mathcal{H}[u - v](s) ds d_q x \right|
\]

\[
+ \left\{ \left| A_T \right| \left| \mathcal{O}_\eta[F_u] - \mathcal{O}_\eta[F_v] \right| + \left| A_\eta \right| \left| \mathcal{O}_T[F_u] - \mathcal{O}_T[F_v] \right| \right\} \frac{1}{\Omega \Gamma_q(\beta)} \int_0^\eta (T - \sigma_{q,u}(s))^{\beta-1} q,\mathcal{H}[u - v](s) ds d_q x
\]

\[
+ \frac{\Phi_2}{\Omega} \left\{ \left| A_\eta \right| \left| \frac{T^\alpha (T - \omega_0)^\beta}{\Gamma_q(\alpha + 1)\Gamma_q(\beta + 1)} + \Phi_1 \left\{ \left| A_T \right| \frac{T^{\alpha - 1} (T - \omega_0)^\beta}{\Gamma_q(\beta + 1)} + \left| B_T \right| (T - \omega_0)^{\beta - 1} \right\} \right| + \left| B_\eta \right| (T - \omega_0)^{\beta - 1} \right\}
\]

\[
\leq \mathcal{L} \Xi \left| u - v \right|_C.
\]

By \((H_2)\), we can conclude that \( F \) is a contraction. From Banach fixed point theorem, \( F \) has a fixed point. Therefore, problem (1) has a unique solution. \( \square \)
4. Example

In this section, we give an example of nonlocal fractional $q$ and Hahn integral boundary value problem for sequential fractional $q$-Hahn integrodifference equation:

$$D_{\frac{1}{2}}^\frac{3}{2} \psi \left( t \right) = \frac{1}{1000e^{t^2}} \left( 1 + \left| u(t) \right| \right) \left[ e^{-\left( \frac{4t}{3} + \frac{7}{3} \right)} \left( u^2 + 2|u| \right) + e^{-\left( \frac{t}{2} + \cos^2 \pi t \right)} \left| \psi_{\frac{3}{2}} u(t) \right| \right], \quad t \in I_{1,2}^\frac{1}{2,2}$$

$$u \left( 5 \right) = \frac{1}{10\pi T_{\frac{3}{2}}^1} u \left( 5 \right),$$

$$u \left( 10 \right) = \frac{1}{20E T_{\frac{1}{2}}^3} u \left( 10 \right),$$

where $\psi(t,s) = e^{-\frac{\alpha t^2}{360}}$ and $\varphi(t,s) = e^{-\frac{\beta t^2}{360}}$.

Here, $\alpha = \frac{1}{360}$, $\beta = \frac{2}{360}$, $\gamma = \frac{1}{360}$, $\theta = \frac{1}{360}$, $\phi = \frac{2}{360}$, $q = \frac{1}{360}$, $\omega = \frac{1}{360}$, $\omega_0 = \frac{1}{360}$, $T = 10$, $\eta = 5$, $\lambda = \frac{1}{360}$, $\mu = \frac{1}{360}$, and $F \left[ t, u(t), \Psi_{\phi,\omega}^u(t), \Psi_{\phi,\omega}^\varphi(t) \right] = \frac{1}{1000e^{t^2}} \left( 1 + \left| u(t) \right| \right) + e^{-\left( \frac{4t}{3} + \frac{7}{3} \right)} \left( u^2 + 2|u| \right) + e^{-\left( \frac{t}{2} + \cos^2 \pi t \right)} \left| \psi_{\frac{3}{2}} u(t) \right| + e^{-\left( \frac{t}{2} + \cos^2 \pi t \right)} \left| \psi_{\frac{3}{2}} u(t) \right| \left| \psi_{\frac{3}{2}} u(t) \right| .

After calculating, we get

$$|A_\eta| \approx 0.7567, \quad |A_T| \approx 0.5984, \quad |B_\eta| \approx 0.9962, \quad |B_T| \approx 1.1816,$$

and $|\Omega| \approx 0.2980$.

For all $t \in [0, 10]$ and $u, v \in \mathbb{R}$, we find that

$$\left| F \left[ t, u, \Psi_{\phi,\omega}^u, \Psi_{\phi,\omega}^\varphi \right] - F \left[ t, v, \Psi_{\phi,\omega}^u, \Psi_{\phi,\omega}^\varphi \right] \right| \leq \frac{1}{1000e^{t^2}} \left| u - v \right| + \frac{1}{1000e^{t^2}} \left| \Psi_{\phi,\omega}^u - \Psi_{\phi,\omega}^\varphi \right| + \frac{1}{1000e^{t^2}} \left| \Psi_{\phi,\omega}^u - \Psi_{\phi,\omega}^\varphi \right| .$$

Thus, $(H_1)$ holds with $\ell_1 = 0.0000475$, $\ell_2 = 0.0000547$, and $\ell_3 = 0.0000498$.

Next, we find that

$$\psi_0 = 0.00125, \quad \varphi_0 = 0.00111, \quad \mathcal{L} = 0.000461, \quad \Phi_1 = 4.9572, \quad \Phi_2 = 12.1191,$$

$$\Theta_T = 4.6218, \quad \Theta_\eta = 4.8705 \quad \text{and} \quad \Xi = 92.4997.$$

Since

$$\mathcal{L} \Xi \approx 0.0426 < 1,$$

we see that the condition $(H_2)$ holds.

Hence, by Theorem 1, problem (31) has a unique solution.

5. Conclusions

We have proved existence and uniqueness results of the sequential fractional $q$-Hahn integrodifference equation with nonlocal mixed fractional $q$ and fractional Hahn integral boundary condition (1) by using the Banach fixed point theorem, and the existence of at least a solution by Schauder’s fixed point theorem. Our problem contains both fractional $q$-difference and fractional Hahn difference operators, which is a new idea.

**Author Contributions:** Conceptualization, T.D., N.P. and T.S.; Formal analysis, T.D., N.P. and T.S.; Funding acquisition, N.P.; Investigation, T.D., N.P. and T.S.; Methodology, T.D., N.P. and T.S.; Writing—original draft, T.D., N.P. and T.S.; Writing—review and editing, T.D., N.P. and T.S. All authors contributed equally to this work. All authors have read and agreed to the published version of the manuscript.
Funding: This research was funded by King Mongkut’s University of Technology North Bangkok. Contract No. KMUTNB-61-GOV-D-64.

Acknowledgments: This research was supported by Chiang Mai University.

Conflicts of Interest: The authors declare that they have no competing interests.

References
1. Jackson, F.H. On \( q \)-difference equations. *Am. J. Math.* 1910, 32, 305–314. [CrossRef]
2. Jackson, F.H. Basic integration. *Q. J. Math.* 1951, 2, 1–16. [CrossRef]
3. Carmichael, R.D. The general theory of linear \( q \)-difference equations. *Am. J. Math.* 1912, 34, 147–168. [CrossRef]
4. Mason, T.E. On properties of the solutions of linear \( q \)-difference equations with entire function coefficients. *Am. J. Math.* 1915, 37, 439–444. [CrossRef]
5. Kac, V.; Cheung, P. Quantum Calculus; Springer: New York, NY, USA, 2002.
6. Al-Salam, W.A. Some fractional \( q \)-integrals and \( q \)-derivatives. *Proc. Edinb. Math. Soc.* 1966, 15, 135–140. [CrossRef]
7. Agarwal, R.P. Certain fractional \( q \)-integrals and \( q \)-derivatives. *Proc. Camb. Philos. Soc.* 1969, 66, 365–370. [CrossRef]
8. Annaby, M.H.; Mansour, Z.S. \( q \)-Fractional Calculus and Equations; Lecture Notes in Mathematics 2056; Springer: Berlin/Heidelberg, Germany, 2012.
9. Atici, F.M.; Eloe, P.W. Fractional \( q \)-calculus on a time scale. *J. Nonlinear Math. Phys.* 2007, 14, 333–344. [CrossRef]
10. Abdeljawad, T.; Benli, B.; Baleanu, D. Generalized \( q \)-Mittag–Leffler function by \( q \)-Caputo fractional linear equations. *Abst. Appl. Anal.* 2012, 2012, 546062.
11. Baleanu, D.; Agarwal, P. Certain inequalities involving the fractional-integral operators. *Abst. Appl. Anal.* 2014, 2014, 371274.
12. Ahmad, B.; Ntouyas, S.K. Boundary value problems for \( q \)-difference inclusions. *Abst. Appl. Anal.* 2011, 2011, 292860. [CrossRef]
13. Ahmad, B.; Nieto, J.J.; Alsaedi, A.; Al-Hutami, H. Existence of solutions for nonlinear fractional \( q \)-difference integral equations with two fractional orders and nonlocal four-point boundary conditions. *J. Frankl. Inst.* 2014, 351, 2890–2909. [CrossRef]
14. Agarwal, R.P.; Wang, G.; Hobiny, A.; Zhang, L.; Ahmad, B. Existence and nonexistence of solutions for nonlinear second order \( q \)-integro-difference equations with non-separated boundary conditions. *J. Nonlinear Sci. Appl.* 2015, 8, 976–985. [CrossRef]
15. Almeida, R.; Martins, N. Existence results for fractional \( q \)-difference equations of order \( \alpha \in ]2, 3] \) with three-point boundary conditions. *Commun. Nonlinear Sci. Numer. Simulat.* 2014, 19, 1675–1685. [CrossRef]
16. Pangarm, N.; Asawasamrit, S.; Tariboon, J.; Ntouyas, S.K. Multi-strip fractional \( q \)-integral boundary value problems for nonlinear fractional \( q \)-difference equations. *Adv. Differ. Equ.* 2014, 2014, 193. [CrossRef]
17. Ferreira, R.A. Nontrivial solutions for fractional \( q \)-difference boundary value problems. *Electron. J. Qual. Theory Differ. Equ.* 2010, 70, 1–10. [CrossRef]
18. Sithijirawirath, T.; Tariboon, J.; Ntouyas, S.K. Three-point boundary value problems of nonlinear second-order \( q \)-difference equations involving different numbers of \( q \). *J. Appl. Math.* 2013, 763786. [CrossRef]
19. Saengngammongkol, T.; Kaewwisetkul, B.; Sithijirawirath, T. Existence results for nonlinear second-order \( q \)-difference equations with \( q \)-integral boundary conditions. *Differ. Equ. Appl.* 2015, 7, 303–311. [CrossRef]
20. Sithijirawirath, T. On nonlocal fractional \( q \)-integral boundary value problems of fractional \( q \)-difference and fractional \( q \)-integrodifference equations involving different numbers of order and \( q \). *Bound. Value Probl.* 2016, 2016, 12. [CrossRef]
21. Patanarprrlert, N.; Sithijirawirath, T. Existence results of sequential derivatives of nonlinear quantum difference equations with a new class of three-point boundary value problems conditions. *J. Comput. Anal. Appl.* 2015, 18, 844–856.
22. Patanarapeelert, N.; Srivannakorn, U.; Sithithawrat, T. On a class of sequential fractional $q$-integrodifference boundary value problems involving different numbers of $q$ in derivatives and integrals. *Adv. Differ. Equ.* 2016, 2016, 146. [CrossRef]

23. Sriphanomwan, U.; Tariboon, J.; Patanarapeelert, N.; Sithithawrat, T. Existence results of nonlocal boundary value problems for nonlinear fractional $q$-integrodifference equations. *J. Nonlinear Funct. Anal.* 2017, 2017, 28.

24. Hahn, W. Über Orthogonalpolynome, die $q$-Differenzenbeziehungen genügen. *Math. Nachr.* 1949, 2, 4–34. [CrossRef]

25. Aldwoah, K.A. Generalized Time Scales and Associated Difference Equations. Ph.D. Thesis, Cairo University, Cairo, Egypt, 2009.

26. Annaby, M.H.; Hamza, A.E.; Aldwoah, K.A. Hahn difference operator and associated Jackson-Nörlund integrals. *J. Optim. Theory Appl.* 2012, 154, 133–153. [CrossRef]

27. Costas-Santos, R.S.; Marcellán, F. Second structure Relation for $q$-semiclassical polynomials of the Hahn Tableau. *J. Math. Anal. Appl.* 2007, 329, 206–228. [CrossRef]

28. Kwon, K.H.; Lee, D.W.; Park, S.B.; Yoo, B.H. Hahn class orthogonal polynomials. *Kyungpook Math. J.* 1998, 38, 259–281.

29. Foupouagnigni, M. Laguerre-Hahn Orthogonal Polynomials with Respect to the Hahn Operator: Fourth-Order Difference Equation for the $r$th Associated and the Laguerre-Freud Equations Recurrence Coefficients. Ph.D. Thesis, National University of Benin, Proto Novo, Benin, 1998.

30. Malinowska, A.B.; Torres, D.F.M. The Hahn quantum variational calculus. *J. Optim. Theory Appl.* 2010, 147, 419–442. [CrossRef]

31. Malinowska, A.B.; Torres, D.F.M. *Quantum Variational Calculus*; Springer Briefs in Electrical and Computer Engineering-Control, Automation and Robotics; Springer: Berlin/Heidelberg, Germany, 2014.

32. Malinowska, A.B.; Martins, N. Generalized transversality conditions for the Hahn quantum variational calculus. *Optimization* 2013, 62, 323–344. [CrossRef]

33. Hamza, A.E.; Ahmed, S.M. Theory of linear Hahn difference equations. *J. Adv. Math.* 2016, 753, 13 of 14.

34. Hamza, A.E.; Ahmed, S.M. Existence and uniqueness of solutions of Hahn difference equations. *Adv. Differ. Equ.* 2013, 2013, 316. [CrossRef]

35. Hamza, A.E.; Makharesh, S.D. Leibniz’ rule and Fubinis theorem associated with Hahn difference operator. *J. Adv. Math.* 2016, 12, 6335–6345.

36. Asawasamrit, S.; Sudprasert, C.; Ntouyas, S.K.; Tariboon, J. Some results on quantum Hahn integral inequalities. *Adv. Differ. Equ.* 2019, 2019, 154. [CrossRef]

37. Tariboon, J.; Ntouyas, S.K.; Sudsutad, W. New concepts of Hahn calculus and impulsive Hahn difference equations. *Adv. Differ. Equ.* 2016, 2016, 255. [CrossRef]

38. Sithithawrat, T. On a nonlocal boundary value problem for nonlinear second-order Hahn difference equation with two different $q$, $\omega$-derivatives. *Adv. Differ. Equ.* 2016, 2016, 116. [CrossRef]

39. Sriphanomwan, U.; Tariboon, J.; Patanarapeelert, N.; Ntouyas, S.K.; Sithithawrat, T. Nonlocal boundary value problems for second-order nonlinear Hahn integro-difference equations with integral boundary conditions. *Adv. Differ. Equ.* 2017, 2017, 170. [CrossRef]

40. Brikshavana, T.; Sithithawrat, T. On fractional Hahn calculus. *Adv. Differ. Equ.* 2017, 2017, 354. [CrossRef]

41. Wang, Y.; Liu, Y.; Hou, C. New concepts of fractional Hahn’s $q$, $\omega$-derivative of Riemann–Liouville type and Caputo type and applications. *Adv. Differ. Equ.* 2018, 2018, 292. [CrossRef]

42. Patanarapeelert, N.; Sithithawrat, T. Existence results for fractional Hahn difference and fractional Hahn integral boundary value problems. *Discrete Dyn. Nat. Soc.* 2017, 2017, 7895186. [CrossRef]

43. Patanarapeelert, N.; Brikshavana, T.; Sithithawrat, T. On nonlocal Dirichlet boundary value problem for sequential Caputo fractional Hahn integrodifference equations. *Bound. Value Probl.* 2018, 2018, 6. [CrossRef]

44. Patanarapeelert, N.; Sithithawrat, T. On nonlocal Robin boundary value problems for Riemann–Liouville fractional Hahn integrodifference equation. *Bound. Value Probl.* 2018, 2018, 46. [CrossRef]

45. Dumrongpokaphan, T.; Patanarapeelert, N.; Sithithawrat, T. Existence results of a coupled system of caputo fractional Hahn difference equations with nonlocal fractional Hahn integral boundary value conditions. *Mathematics* 2019, 7, 15. [CrossRef]

46. Khan, H.; Tunç, C.; Alkhazan, A.; Ameen, B.; Khan, A. A generalization of Minkowski’s inequality by Hahn integral operator. *J. Taibah Univ. Sci.* 2018, 12, 506–513. [CrossRef]
47. Tariboon, J.; Ntouyas, S.K.; Sutthasin, B. Impulsive fractional quantum Hahn difference boundary value problems. *Adv. Differ. Equ.* **2019**, *2019*, 220. [CrossRef]

48. Sitho, S.; Sudprasert, C.; Ntouyas, S.K.; Tariboon, J. Noninstantaneous impulsive fractional quantum Hahn integro-difference boundary value problems. *Mathematics* **2020**, *8*, 671. [CrossRef]