A spanning-tree method for constructing temporal networks

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Abstract

Temporal networks provide a general framework to capture time-varying interactions of complex systems. The interevent time (IET), the time interval between two successive interactions, is a common statistic to characterize the structure of temporal networks. Previous findings support that burstiness, the IET distribution to be heavy-tailed, is ubiquitous in the activity of nodes and links on empirical temporal networks. However, an analytical method for generating such a non-trivial phenomenon is still lacking. Here we propose a spanning-tree method to construct temporal networks with any consistent activity pattern, including bursty and Poissonian activity patterns. Specifically, our method theoretically ensures that the synthetic IET distribution of every single node/link can match with any targeted distribution fulfilling a consistency (necessary and sufficient) condition, and the underlying topology based on which our construction process is carried out can be either static or time-varying. Our method reproduces the observed burstiness in empirical datasets and uncovers the laws of temporal interactions, which is a key determinant to further study dynamical processes in real-world complex systems.

1 Introduction

Temporal networks provide a powerful tool to model real-world complex systems with specific underlying structures and time-varying interactions \cite{1-3}. One important objective in temporal network studies is to detect the activity pattern of nodes/links, and to analyse structural measures concerning temporal aspects. The time interval or waiting time between two consecutive events, i.e., the interevent time (IET), is a pivotal statistic to measure temporal structure because of its profound impact on interaction patterns \cite{4-7} and dynamical processes \cite{8-12} in real-world systems. The empirical data, such as email and mobile communication \cite{13-15}, epidemic spreading \cite{16-18}, and human mobility \cite{19 20}, exhibits non-Poissonian and heterogeneous activity patterns, known as burstiness \cite{4 21}, due to the possible high or low priority of tasks \cite{21 22} or circadian cycles of human activities \cite{14}. Such bursty behaviour is usually characterized by a heavy-tailed distribution of IETs. Especially noteworthy is that the activity of both single nodes and single links presents burstiness \cite{23 25}, which is a salient feature of empirical temporal networks.

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Considering high time consumption and cost for collecting high-resolution data in real systems, it is necessary to develop a method to construct synthetic temporal networks with similar properties to empirical temporal networks. Previous modelling work can be broadly divided into two categories, structure-based and contact-based. Work in the former category constructs temporal networks based on specific static underlying topologies using random walks [26, 27], link dynamics [28], and inhomogeneous Poisson processes [29], while work in the latter category establishes temporal networks with streams of contacts, which are generated by certain realistic mechanisms ruling the activation of nodes and links, such as social appeal [30], individual resource [31], and memory [32]. One well-known model is called the activity-driven model [33], where each node has a fixed activity potential randomly sampled from a probability distribution. At each time step, the snapshot begins with all isolated nodes. Each node tends to be active with probability proportional to its activity potential, and generates contacts with other nodes. However, most prior studies only allow to generate burstiness on nodes or links individually through numerical simulations [26–28, 30–32]. Some papers, which can generate simultaneous burstiness of nodes and links, have strong restrictions on activity patterns [29, 34]. The aim of this study is to reproduce simultaneous bursty behaviour of nodes and links observed in empirical datasets, and to reveal the origin of burstiness theoretically, so that different levels of burstiness or other common activity patterns can be generated on synthetic temporal networks. Nevertheless, relevant research on this topic remains absent.

In this paper, we propose a spanning-tree based method to systematically construct temporal networks on both static and time-varying underlying topologies, and provide an explicit algorithm for the construction process. Specifically, we formulate the construction process as a stochastic process and define the consistency of an activity pattern. We demonstrate the feasibility of the construction process with any consistent activity pattern, including bursty and Poissonian activity patterns, and uncover the origin of observed burstiness in empirical datasets. Finally, we use the algorithm to reproduce the expected bursty or Poissonian behaviour in empirical and synthetic temporal networks.

### 2 Temporal network construction

The activity of individual nodes (links) in a temporal network is a binary-state process, where each node (link) occupies one of two states, active or inactive. In this case, the interevent time of nodes (links) characterizes the time interval between two consecutive activations. We propose a method to construct temporal networks, such that the IET distribution of every individual node (link) fulfills a targeted distribution.

The theoretical basis of our construction is the renewal process, which is a common counting process in probability theory (see Supplementary Information section 1). Mathematically, our goal is to construct a stochastic process according to targeted distributions (i.e. given distributions) to specify the activity of a whole network system, such that the state update of individual nodes (links) follows renewal processes, and the corresponding IET distributions fulfill respective targeted distributions.
2.1 Two-node systems

We first consider a two-node system consisting of two nodes, $x$ and $y$, connected by a link $z$ (Fig. 1a), which is the minimal component of a network system. The targeted probability mass functions (i.e. pmf) of IETs of $x, y, z$ are $F(m), G(m), H(m)$, respectively, where $m$ belongs to $\mathbb{Z}^+$. The renewal processes corresponding to $x, y, z$ are denoted as $\{X_n\}_{n \geq 0}, \{Y_n\}_{n \geq 0}, \{Z_n\}_{n \geq 0}$. We set the random variable $X_n = 1$ ($Y_n = 1$ or $Z_n = 1$) if $x (y, z)$ is active at time $n$, otherwise we set $X_n = 0$ ($Y_n = 0, Z_n = 0$). The initial state of $x, y, z$ is active (i.e. $X_0 = Y_0 = Z_0 = 1$). At each time step, the link is active if and only if the two nodes are all active (i.e. $Z_n = X_n Y_n$).

Given the state at the previous $n$ moments, the conditional probability of node $x$ being active or inactive at time $n$ is

$$
\mathbb{P}(X_n = a | X_{n-1} = w_x^{(n-1)}, \ldots, X_0 = w_x^{(0)}) := p_x(w_x^{(n-1)}, a) = \begin{cases}
\frac{F(n-m)}{\sum_{i=n-m}^{n-1} F(i)}, & a = 1, \\
\frac{\sum_{i=n-m+1}^{n} F(i)}{\sum_{i=n-m}^{n} F(i)}, & a = 0,
\end{cases}
$$

(1)

where $w_x^{(n-1)} = (w_x^{(0)}, \ldots, w_x^{(n-1)})^T \in \{0,1\}^n$ is said to be a trajectory of $x$ until time $n-1$, which records all historical states before time $n$, and $m = \max\{k \leq n : w_x^{(k)} = 1\}$ represents the last activation time of $x$. The same conclusion can be obtained for $y$ and $z$.

We construct a stochastic process $\{S_n\}_{n \geq 0}$, which for arbitrary sets of $t_1, \ldots, t_k \in \mathbb{N}, k \in \mathbb{Z}^+$ satisfies

$$
\mu_{S_{2i-1}, \ldots, S_{2i+k}} = \mu_{X_{t_1}, \ldots, X_{t_k}},
$$

$$
\mu_{S_{2i-1}+1, \ldots, S_{2i+k+1}} = \mu_{Y_{t_1}, \ldots, Y_{t_k}},
$$

$$
\mu_{(S_{2i-1}, S_{2i+1}), \ldots, (S_{2i+k}, S_{2i+k+1})} = \mu_{Z_{t_1}, \ldots, Z_{t_k}},
$$

(2)

and $S_0 = S_1 = 1$, where $\mu_{X_{t_1}, \ldots, X_{t_k}}$ represents the finite dimensional distribution of $\{S_n\}_{n \geq 0}$ at the time slice $(t_1, \ldots, t_k)$. $\{S_n\}_{n \geq 0}$ can be viewed as composing of the following sequence

$$
(S_0, S_1, \ldots, S_{2n}, S_{2n+1}, \ldots) = (X_0, Y_0, \ldots, X_n, Y_n, \ldots),
$$

(3)

and $Z_n = S_{2n} S_{2n+1}$. If an algorithm can be formulated as $\{S_n\}_{n \geq 0}$, the algorithmic IET distribution of $x, y, z$ generated by the algorithm would satisfy the targeted distributions $F, G, H$.

We propose an algorithm for two-node systems, which is executed as follows:

(i) The following elements are chosen: the total simulation time $t_{tot}$ and the targeted probability mass functions $F, G, H$ for $x, y, z$.

(ii) When $t = n + 1, 0 \leq n \leq t_{tot} - 1$, the probabilities $p_1 = p_z(w_x^{(n)}, 1), p_2 = p_x(w_x^{(n)}, 1) - p_z(w_x^{(n)}, 1), p_3 = p_y(w_y^{(n)}, 1) - p_z(w_x^{(n)}, 1), p_4 = 1 + p_z(w_x^{(n)}, 1) - p_x(w_x^{(n)}, 1) - p_y(w_y^{(n)}, 1)$ are determined.

(iii) Two random numbers, $p$ and $q$, are sampled from a uniform distribution in $[0,1]$ independently. If $p < p_x(w_x^{(n)}, 1)$, $x$ is active at time $t$ (i.e. $w_x^{(t)} = 1$). Let $q$ compare with $p_1 / p_x(w_x^{(n)}, 1)$, if $q$ is smaller, $y$ is active (i.e. $w_y^{(t)} = 1$), otherwise $y$ is inactive (i.e. $w_y^{(t)} = 0$). If $p \geq p_x(w_x^{(n)}, 1), x$ is inactive (i.e. $w_x^{(t)} = 0$). Let $q$ compare with $p_3 / (1 - p_x(w_x^{(n)}, 1))$, if $q$ is smaller, $y$ is active (i.e. $w_y^{(t)} = 1$), otherwise $y$ is inactive (i.e. $w_y^{(t)} = 0$).

(iv) The trajectory of $z$ at $t$ is $w_z^{(t)} = w_x^{(t)} w_y^{(t)}$. 

3
Figure 1: Schematic illustration of constructing temporal networks on different underlying topologies. In every time step, each node (link) switches between two states, active (solid circle and line) and inactive (dashed circle and line), and all nodes and links are active when first entering network systems. 

a. The basic unit of a network system is a two-node system with two nodes $x$ and $y$ connected by a link $z$. The activation of every single element is driven by the renewal process. Before the temporal network construction, three probability mass functions $F(\Delta t), G(\Delta t), H(\Delta t)$ are given as the expected distribution of IETs for $x,y,z$. The dynamic of the whole system is established by a stochastic process $\{S_n\}_{n \geq 0}$ coupling the renewal processes of $x,y,z$. The joint activity of $x$ and $y$ is directly specified by $\{S_n\}_{n \geq 0}$. The state of $z$ is active only if $x$ and $y$ are all active, which is indirectly affected by $\{S_n\}_{n \geq 0}$.

b. A tree system is an extension of a two-node system. In a tree system, nodes are divided into two categories, a root, $r$, and leaves, $a_1,a_2,b_1,b_2$. The state of nodes and links in a tree system is updated by sequentially executing the algorithm on each two-node system from the root to the outermost leaves. c. For any underlying topology, a spanning tree is always found. The links in the spanning tree are called trunks (purple lines) and the links outside the spanning tree are called branches (blue lines). The activity of the network system is spanning-tree based. Through the activity of the spanning tree, the state of every single node (trunk) is established. And the state of single branches is active when the nodes at both sides are active. d. The construction process can be integrated with the evolution of networks, such as the Barabási-Albert model [35]. When the underlying topology evolves in time, the activity of the system is determined by its time-varying spanning tree.
Supplementary Information Algorithm 1 outlines the above procedure. We can prove that the stochastic process, \( \{\hat{S}_n\}_{n \geq 0} \), formulating the above algorithm satisfies equation (2) (see Supplementary Information section 2.1 for detailed derivations). However, mathematically, for some targeted distributions, the algorithm cannot be executed properly. The reason is that \( p_i \) \((i = 1, 2, 3, 4)\) in the algorithm may be less than 0, which violates the definition of probability measure. We propose a definition of distribution consistency for the algorithm. The distributions \( F, G, H \) are consistent, if for any \( w_x(n), w_y(n) \in \{0, 1\}^{n+1}, n \geq 0 \), \( p_j \) \((i = 1, 2, 3, 4)\) are in \([0, 1]\). We claim that a two-node system is consistent if the targeted distributions are consistent. When a two-node system is consistent, the algorithm can be executed correctly and the existence of \( \{\hat{S}_n\}_{n \geq 0} \) is theoretically guaranteed (see Supplementary Information section 2.1). In particular, when \( F = G = H \), we say that the system is synchronous, because \( x, y, z \) would have the same trajectory during the construction process, that is, \( x, y, z \) are active or inactive at the same time.

### 2.2 Tree systems

The algorithm for two-node systems can be naturally extended to tree systems, which consist of a number of two-node systems (Fig. 1b). Due to the structure of tree systems, we can randomly select a node as a root \( r \), and classify all other nodes (i.e. leaves) according to their distance from the root. Before the construction process, we give each node (link) a targeted distribution. The activity of each node (link) follows the renewal process induced by the corresponding targeted distribution. At each time step, we first determine the state of \( r \), which is only related to its own trajectory. Then we treat the two-node system as a basic unit, and use the algorithm for two-node systems to update the state of the leaves whose distance from \( r \) is 1 (such as \( a_1 \) and \( a_2 \) in Fig. 1b). Next, the state of the leaves located at the second layer is established (such as \( b_1 \) and \( b_2 \) in Fig. 1b). By analogy, the state of all leaves is updated in order according to their distance from the root. Supplementary Information Algorithm 2 gives a detailed procedure of the construction process on tree systems.

Mathematically, new conditional independence is required for the algorithm. For a pair of two-node systems sharing a node, if the state of the common node is given, the activity of the other two nodes is independent. Two different cases, \((r, a_1, a_2)\) and \((r, a_1, b_1)\), that fulfill the above condition are shown in Fig. 1b. The former case indicates that the activity of the two nodes on the same layer, \( a_1 \) and \( a_2 \), is independent when the state of the upper layered node, \( r \), is known. The latter case suggests that the activity of a node is not affected by the nodes that are more than one layer away.

We claim that a tree system is consistent, if all two-node systems in the tree system are consistent. When a tree system is consistent, the stochastic process formulating the construction algorithm exists (see Supplementary Information section 2.2). Therefore, the algorithm can be executed correctly and the algorithmic IET distribution of every single node (link) fulfills the targeted distribution.

### 2.3 Spanning-tree based construction

For a network system with any static underlying topology, we can always find a spanning tree of the system (Fig. 1c). We call a link a trunk if it is in the spanning tree, otherwise we call it a branch. In particular, the spanning tree of a tree system is itself, and all links are trunks.
In the algorithm for general structured network systems, we first select a spanning tree of the system and input the targeted distribution of each node (trunk). Next, we execute the algorithm for tree systems on the spanning tree. After finishing the state update of all nodes and trunks, the state of each branch is determined by the state of the nodes on both sides of it. In this case, the dynamic of a system is totally determined by its spanning tree, and the algorithmic IET distribution of individual branches is indirectly determined by the targeted distribution of nodes and trunks (see Supplementary Information section 2.4).

We say the system is consistent when its spanning tree system is consistent. Especially, when all two-node systems in the spanning tree are synchronous, the trajectory of all nodes and links (including trunks and branches) is the same. In this case, the whole system is synchronous, and the algorithmic IET distribution of every single node(link) satisfies the same distribution.

3 Ramifications and applications

3.1 Interevent time distribution of synthetic temporal networks

In order to check the validity of our algorithm, we consider the construction of temporal networks based on two common types of static underlying topologies, small-world networks [36] and scale-free networks generated by the Barabási-Albert model [35]. The spanning tree is denoted as $T = (V, E)$, where $V$ ($E$) is a set of nodes (trunks). The targeted IET distribution of node $i$ and trunk $j$ is denoted as $\phi_i(\Delta t, \alpha_i)$ and $\psi_j(\Delta t, \beta_j)$, respectively, where $\alpha_i$ or $\beta_j$ is the exponent of the distribution, and $\Delta t$ represents the interevent time.

Here we consider two important discrete distributions, which are the power-law distribution (i.e. the probability mass function $p(\Delta t, \gamma) \sim \Delta t^{-\gamma}$) and the discrete exponential distribution (i.e. $p(\Delta t, \lambda) = \int_{\Delta t - 1/2}^{\Delta t + 1/2} \lambda e^{-\lambda x} dx$). The former is a typical heavy-tailed distribution which is universally observed in real data and is the main contribution to bursty behaviour [21]. The continuous version of the latter distribution can induce a commonly used counting process, the Poisson process, which is used to quantify the consequence of human activities such as modelling a bank queuing system [37]. The activity pattern driven by the former (latter) distribution is called a double-power-law (double-exponential) coupling pattern.

To detect the activity pattern of synthetic temporal networks holistically, we focus primarily on the algorithmic IET distribution of all nodes (links), $P(\Delta t)$, which is formed by the samples from all nodes (links) during the construction process. We also count the algorithmic distribution of every individual node (link), and we find a strong correlation between the algorithmic distribution of individual elements and the whole network (see Supplementary Information section 3).

We begin our analysis with a double-power-law coupling pattern (i.e. bursty activity pattern). Under the premise of infinite simulation times, we obtain a necessary and sufficient condition for a double-power-law coupling pattern to be consistent (see Supplementary Information section 2.1.2).

We first consider a simple case that all nodes (trunks) have the same power-law exponent, that is, $\forall i \in V, j \in E, \phi_i(\Delta t, \alpha_i) \sim \Delta t^{-\alpha_i}$ and $\psi_j(\Delta t, \beta_j) \sim \Delta t^{-\beta_j}$. As $P(T > \Delta t) \approx \int_{\Delta t}^{+\infty} x^{-\gamma} dx \sim \Delta t^{1-\gamma}$, the exponent of the survival function is approximately $\gamma - 1$. The algorithmic IET distributions of all nodes (links) on scale-free networks and small-world networks are presented in the first row and the second row of Fig. 2a, respectively. Under different underlying topologies
Figure 2: IET distribution of all nodes and all links on different underlying topologies. The construction algorithm is executed on two static underlying topologies, the Barabási-Albert model (first row) and the small-world network (second row). The targeted exponent of every single node (trunks) is the same, denoted as $\alpha_{pmf}$ ($\beta_{pmf}$). Each subfigure shows the algorithmic probability mass function of IETs for all nodes (circles) and all links (squares), and each inset shows the corresponding survival functions. 

a, To study double-power-law coupling patterns, i.e. bursty activity patterns, two pairs of exponents are selected for nodes and trunks, $(\alpha_{pmf}, \beta_{pmf}) = (2.20, 2.10)$ (blue dots) and $(\alpha_{pmf}, \beta_{pmf}) = (1.80, 1.30)$ (green dots). b, To study double-exponential coupling patterns, i.e. Poissonian activity patterns, two pairs of exponents are selected, $(\alpha_{pmf}, \beta_{pmf}) = (2.50, 2.00)$ (blue dots) and $(\alpha_{pmf}, \beta_{pmf}) = (1.80, 1.30)$ (green dots). The results are well approximated by power-law distributions in a and exponential distributions in b. The thick black lines with fitted exponents are plotted for reference. As one can see, the results are sensitive to the targeted exponent but robust to the underlying topology. For numerical simulations, all networks have the same size $N = 10^5$ and average connectivity $k = 6$. Numerical simulations are run 200 independent trials and the total simulation time, $t_{tol}$, is $10^4$ in a and $10^3$ in b for each trial.
and targeted exponents, both nodes and links exhibit the expected bursty behaviour, and the numerical simulations are robust to the underlying topology and the selection of spanning trees (see Supplementary Fig. 1).

We also study a more general case that the targeted exponent of each node (trunk) is sampled from a distribution $\eta_{\text{node}}$ ($\eta_{\text{link}}$) (see Supplementary Fig. 2). In this case, in addition to analysing the algorithmic distribution of all nodes (links), we also count the distribution of the individual node (link) with the maximal or minimal number of activations. All these distributions show burstiness.

For a double-exponential coupling pattern (i.e., Poissonian activity pattern), $\phi$ and $\psi$ are all discrete exponential distributions. The corresponding survival function is $P(T > \Delta t) = \int_{\Delta t}^{+\infty} \lambda e^{-\lambda x} \, dx = e^{-\lambda \Delta t}$, so the exponent of the survival function is the same as that of the probability mass function. A necessary and sufficient condition for the system to be consistent is $0 \leq \alpha_i - \beta_j \leq \ln 2$, for all $i \in \mathcal{V}$, $j \in \mathcal{E}$ (see Supplementary Information section 2.1.2).

We first consider that all nodes (trunks) have the same exponent. Figure 2b shows the algorithmic distributions of all nodes (links) on two underlying topologies. Both the probability mass functions and survival functions are well predicted by exponential distributions, and the simulation results are also robust to the underlying topology and spanning tree selection. When the exponent of each node (trunk) is sampled from a distribution, the algorithmic distributions maintain exponential decay (see Supplementary Fig. 2).

### 3.2 Robustness analysis

As the activity of branches has a significant impact on the activity of a whole network system, we consider the algorithmic IET distribution of every single branch. We label each branch according to the size of the ring in which it is located, and find that the algorithmic distributions are all heavy-tailed (exponential) in double-power-law (double-exponential) coupling patterns (see Supplementary Fig. 3). This is why the algorithmic activity pattern of a whole network system has the same trend as the targeted activity pattern of its spanning tree.

However, when the system is not synchronous, we notice that the deviation of the algorithmic exponent exists between individual branches and individual trunks (see Supplementary Fig. 3). And since the samples generated by branches are also considered when counting the distribution of all links, the fitted exponent for all links deviates from the targeted exponent (Fig. 2). This raises the question of whether the number of branches can greatly affect the algorithmic distribution of all links. In other words, whether the algorithmic distribution is robust to the underlying topology.

In order to answer this question, we study the relationship between the algorithmic distribution of all links and the average degree of underlying topologies. We first consider an extreme case that the underlying topology is well-mixed (see Supplementary Fig. 4). In this case, the number of branches reaches the maximum and the average clustering coefficient is 1. Surprisingly, in both double-power-law and double-exponential coupling patterns, the algorithmic distribution of all links is almost the same as those in Fig. 2. The relative deviation of the fitted exponent is kept within 3%. Next, we study the algorithmic exponent for all links (branches) on random regular networks with various average degrees (Fig. 3a). The rise in the average degree, $k$, increases the proportion of branches in links, so the algorithmic exponent of all links gradually converges to that of all branches. As one can see, there is no significant correlation between the algorithm-
Figure 3: IET distribution of all links is robust to underlying topologies. 

Relationship between the underlying topology and the algorithmic exponent. Random regular graphs with different average degrees, $k$, are utilized to represent different underlying topologies. We plot the algorithmic exponent for all branches (hollow squares) and all links (i.e. all trunks and branches, solid circles). Black dashed lines show the targeted exponent for trunks. When the average degree becomes larger, the proportion of branches in all links increases. As a result, the algorithmic exponent for all links gradually converges to that for all branches. Furthermore, the algorithmic exponents do not correlate significantly with $k$. The largest magnitude of variances is no more than $10^{-3}$, indicating the robustness to the underlying topology.

(b) Schematic illustration of the upper and lower bounds for individual branches. The algorithmic distribution of every single branch (red diamonds) has uniform upper and lower bounds, which are heavy-tailed distributions in a double-power-law coupling pattern (top panel) and exponential distributions in a double-exponential coupling pattern (bottom panel). See also Supplementary Fig. 3 for numerical simulations.

The algorithmic exponent of all branches and the average degree. The largest variance is $\sim 10^{-3}$ in Fig. 3a, indicating that the fluctuation of the algorithmic exponent is stable. The deviation between the targeted and algorithmic exponent is approximately 0.1 on average.

To give a theoretical explanation, we use the distribution of first activation time to represent the algorithmic distribution of individual branches. We prove that the first activation time distribution of any single branch is upper and lower bounded by heavy-tailed (exponential) distributions in double-power-law (double-exponential) coupling patterns (see Supplementary Information section 3 for detailed derivations and see Supplementary Fig. 5). The upper and lower bounds are only related to the targeted distributions but independent of the underlying topology. Therefore, the algorithmic distribution of individual branches has uniform upper and lower bounds (Fig. 3b), and becomes heavy-tailed (exponential) distributions in double-power-law (double-exponential) coupling patterns. This conclusion demonstrates the reason why the algorithmic IET distribution of all links is sensitive to the targeted distributions but robust to the underlying topology and the selection of spanning trees.
3.3 Analysis of aggregated networks

In addition to studying the IET distribution, we analysie the structural measures of aggregated networks (Fig. 4a) with different aggregation times, $t_{agg}$, to evaluate other temporal properties of our synthetic temporal networks. We study two typical measures, the expected number of activations of individual nodes (links) and the node strength distribution of aggregated networks. The former is a micro statistic capturing the frequency of activations of each unit (i.e., a node or a link) in networks, and the latter is a macro statistic showing the structural information of entire networks.

For a node, $i$, with a given distribution $\phi(\Delta t, \alpha_i)$, we denote the expectation of the distribution by $\mu(\alpha_i)$. The total number of activations of node $i$ up to moment $t$ is denoted as $A^{(i)}_t$. Using the elementary renewal theorem [37], we have

$$\frac{E(A^{(i)}_t)}{t} \to \frac{1}{\mu(\alpha_i)} \text{ as } t \to \infty,$$

where $\frac{1}{\infty} = 0$. The corresponding conclusion for links is similar. Equation (4) presents an intuitive conclusion that the average growth rate of activation numbers asymptotically equals the frequency of activations. In a double-power-law coupling pattern, when $\alpha < 2$, the expectation $\mu(\alpha) = \infty$. This suggests that the growth is sublinear, and the rate asymptotically equals 0. In a double-exponential coupling pattern, for any exponent $\alpha$, $\mu(\alpha) < \infty$, thus the growth is linear (see Supplementary Fig. 6).

To analyse the node strength distribution of aggregated networks, we consider an underlying structure, $G$, where the degree distribution of $G$ is denoted as $d(x)$. We set a random variable $X$ whose probability mass function is $d(x)$. The node strength in the aggregated network with the aggregation time $t$ is a random variable $N_t$, and $p_{N_t}(x)$ is the probability density function of $N_t$.  

10
When the given distribution of link is the same with an exponent \( \beta \), we obtain

\[
p_{N_i}(x) \sim \frac{v(\beta)d\left(x^{\beta}\right)}{t}
\]

for sufficiently large \( t \), where \( v(\beta) \) is the expectation of the given distribution. When \( d(x) \) is power-law, the node strength distribution, \( p_{N_i}(x) \), is also power-law and the exponent is the same as that of \( d(x) \). Figure 4b shows the survival function of node strength with different \( t_{agg} \) based on scale-free underlying topologies. As one can see, the results are all power-law distributions with the same exponent as the degree distribution, which suggests that the distribution of node strength is robust to the aggregation time and the given distribution.

Through normalization of the node strength distribution, we can also obtain similar robustness for other underlying topologies with general degree distributions. We first select a sufficiently large moment, \( t_{base} \), and its corresponding aggregated network, \( G_{base} \), is said to be the baseline. For any aggregation time \( t_{agg} \geq t_{base} \), when all links have the same targeted distribution, the proportion of nodes with strength between \( st_{agg}/t_{base} \) and \( (s+1)t_{agg}/t_{base} \) in \( G_{agg} \) is same as that with strength \( s \) in \( G_{base} \) (see Supplementary Information section 5 for detailed derivations). Therefore, the normalized distribution of node strength for the aggregated network \( G_{t_{agg}} \) is the same as that for \( G_{t_{base}} \). We verify the above robustness on the small-world underlying topology under different activity patterns (see Supplementary Fig. 7). The normalized distribution for different aggregation times all collapses onto the node strength of the baseline network. The robustness of the node strength distribution is a salient property of our synthetic temporal networks.

### 3.4 Burstiness on empirical temporal networks

We further demonstrate the reliability of our algorithm by studying the robustness of the node strength distribution and reproducing the IET distribution of all nodes and all links on four empirical datasets collected from SocioPatterns [40]. Each dataset is formed by a collection of contact events with timestamps. The empirical temporal networks and their corresponding underlying topologies are generated by these events (see Supplementary Information section 7 for details).

Since the degree distribution of these underlying topologies is not power-law, we detect the robustness of the normalized distribution of node strength (Fig. 5a). We find that, in each dataset, the normalized distributions of different aggregated networks collapse onto the node strength distribution of the baseline network. This suggests that the formation of temporal networks in real-world systems might be driven by a series of coupled renewal processes. Moreover, we find that the activity pattern of nodes (links) shows burstiness on these empirical temporal networks (Fig. 5b). We fit the empirical IET distributions with power-law distributions. Taking the fitted distributions as targets, we use our algorithm to construct the corresponding synthetic temporal networks. The results in Fig. 5b show that the trend of empirical distributions is well reproduced by our algorithm.

### 3.5 Combination with network evolution

The temporality of a complex system is not only originated from the state update of nodes and links, but also from the evolution of underlying structures. Temporal networks generated
Figure 5: Analysis of four empirical temporal networks. Four empirical temporal networks are assembled from four real-world datasets of different social contexts: a scientific conference in Nice, France (SFHH) [25], a workplace in two different years in France (InVS13, InVS15) [38], and a hospital in Lyon, France (LH10) [39]. The total length (i.e. the number of snapshots), $t_{\text{tot}}$, of the temporal networks from left to right are 3509, 20129, 21536 and 12605. 

a, Normalized distribution of node strength is robust to the aggregation time $t_{\text{agg}}$. The normalized distribution of node strength for different $t_{\text{agg}}$ collapses onto the node strength distribution for the baseline time, $t_{\text{base}}$. The grey dashed lines show the average of the distributions. 

b, Empirical distributions are well fulfilled by algorithmic results. The empirical IET distribution of all nodes (links) are shown by blue circles (squares). We use power-law distributions to fit them, and the thick black lines are the fitted results. Taking the corresponding fitted distributions as targets, we obtain the algorithmic distribution of all nodes (links) plotted by red circles (squares). Numerical simulations for the algorithm are run 100 independent trials. For each trial, the total simulation time is the total length of the dataset.
under time-varying underlying topologies provide a more realistic representation of real-world complex systems. Figure 1d shows an example which combines our algorithm with the Barabási-Albert model. We call it the temporal Barabási-Albert model. The construction algorithm of the temporal Barabási-Albert model is given as follows.

(i) A initial network with $m_0$ nodes is generated and a spanning tree of this network is selected randomly. Each node (link) is set to be active.

(ii) At time step $k$ before the evolution of the underlying topology stops, one adds a new node with $m \leq m_0$ links that connect to $m$ different old nodes with the preferential attachment, and the spanning tree is updated by adding the new node and adding a trunk randomly selected from $m$ new links. The new node and new links are set to be active. Hence, the old nodes connected to the new node are forced to be active. The other nodes and links update their states by the algorithm under a static underlying topology.

(iii) When all nodes enter the network, the algorithm degenerates to the algorithm under a static underlying topology. The construction process ends after running $t$ more steps. The parameter $t$ characterizes the duration time after the evolution of networks is stable.

Figure 6 shows the algorithmic IET distribution of all nodes (links) in the temporal Barabási-Albert model. The expected bursty and Poissonian activity patterns are presented, and the distributions are robust to the duration time, $t$, indicating that the temporal property established during the evolution of networks is still maintained after evolutionary stabilization. We also provide a general procedure of constructing temporal networks with network evolution (see Supplementary Information Algorithm 4).
4 Discussions and conclusions

Temporal networks provide an advanced mathematical tool to study dynamical processes of complex systems [9–12, 18, 41, 42]. In this paper, we propose an analytical method and the corresponding explicit algorithm to construct temporal networks with any consistent activity pattern based on specific underlying topologies. The state of nodes and links in the underlying topology switches between active and inactive over time, determined by the activity pattern. The core mechanism for the construction process is spanning-tree based activation. The structure of spanning trees describes the most central topological information of complex systems. Therefore, the activity of each node (link) in the system is completely determined by the spanning tree. Such a mechanism can effectively reduce complexity, which is widely applied in other research fields, including clustering algorithms [43], the optimal covering problem [44], the consensus problem [45–48], and so on.

By formalizing the construction algorithm as a mathematical model, we theoretically prove that the algorithmic activity pattern of nodes (links) can well fulfill the targeted activity pattern, as long as the target meets specific consistency conditions. To test the effectiveness of our algorithm, we reproduce the expected bursty or Poissonian activity patterns in synthetic and empirical temporal networks. Furthermore, we demonstrate that the IET distribution of all nodes (links) is sensitive to the targeted distributions but robust to the underlying topology and spanning tree selection. Such sensitivity and robustness are two desirable properties of our construction algorithm, because we can obtain expected results, even though the underlying structures and spanning trees are unknown or changing.

In some real-world complex systems, the underlying topology is time-varying. For example, in an online social network, new users create accounts and build new interactions with other users, leading to changes in the network topology. Meanwhile, existing users can switch between online and offline states. With this as a backdrop, we extend our algorithm from static to evolving underlying topologies, and apply it to generate bursty or Poissonian activity patterns on the so-called temporal Barabási-Albert model. This algorithm might be valuable for explaining the formation and activity of the above complex systems.

We also investigate other temporal structures of algorithmic temporal networks. We analyse a microscopic property about the activation frequency of nodes (links), and detect robust behaviour on the node strength distribution of aggregated networks at all temporal scales. Using the burstiness parameter [49], we measure the level of burstiness (see Supplementary Information Tables S1 and S2), where expected bursty or non-bursty behaviour is presented in double-power-law or double-exponential coupling patterns. Moreover, we use the autocorrelation function to measure the temporal correlation of synthetic temporal networks. As a result, temporal networks with bursty or Poissonian activity patterns demonstrate anticipated positive temporal correlations or no temporal correlations (see Supplementary Information section 6 and Supplementary Fig. 8).

In addition to a large number of pairwise activations, group activations are also ubiquitous in real-world complex systems. Hypergraphs are powerful tools to model such group interactions. The interest in exploring the effect of group and higher-order interactions in complex systems has continued to rise in recent years [50–54]. A recent study has shown that high-order interactions have similar bursty behaviour in pairwise interactions in empirical datasets [55]. Therefore, a natural extension of the current work is to model the activity of higher-order links in temporal networks. Of course, constructing temporal networks under hypergraphs may be more com-
plicated than merely considering pairwise links. Further study is essential to model dynamical processes over complex networks.

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Supporting Information

This supplementary information is structured as follows.

In section 1, we introduce the background of renewal processes and propose a variant that will be used to formalize our construction process and analyse the feasibility of construction algorithms.

In section 2, we propose a spanning-tree based algorithm for temporal network construction based on arbitrary static underlying topologies. We formalize the algorithm as a stochastic process and prove the existence of the stochastic process to show the feasibility of the algorithm.

In section 3, we give evidence for the robustness of algorithmic interevent time distributions to different network structures and spanning trees.

In section 4, we extend our algorithm to time-varying underlying topologies. We integrate our algorithm with the evolution of networks to construct temporal networks where the timescale of the evolution of underlying topologies has the same order of magnitude as that of activity dynamics on the networks.

In section 5, we investigate two structural measures of aggregated networks, which are the number of activations and the distribution of node strength. Using renewal theory, we show the robust behaviour of these two statistics at all time scales.

In section 6, we study two temporal statistics to measure the level of burstiness and temporal correlations of temporal networks, respectively. We show a high level of burstiness and positive temporal correlations present in double-power-law coupling patterns, but a low level of burstiness and no temporal correlations in double-exponential coupling patterns.

In section 7, we introduce the content of empirical datasets and the way to pre-processing the empirical data.

1 Renewal process

The activation process of a node or a link in a temporal network can be naturally modeled as a point process of which the interarrival times are independent and identically distributed with a nonnegative distribution. The interarrival time $\xi$ is a random variable representing the time interval between two adjacent arrivals, which is the interevent time (IET) discussed in the main text. Such a point process is called a renewal process.

We set $\xi$ to be lattice with $d = 1$, that is, the activation can only occur at positive integral moments, $\sum_{n=1}^{\infty} P(\xi = n) = 1$. Therefore, the renewal process of a node or a link is a discrete-time stochastic process $\{\Xi_n\}_{n \geq 0}$, where $\Xi_n = 1$ if the node or the link is active at time $n$, otherwise $\Xi_n = 0$. In this case, the interevent time indicates the time interval between two adjacent activations.

2 Temporal network construction based on static networks

We consider the construction of temporal networks based on a static undirected network. We use the probabilistic graphical model to describe the activation process of individual nodes and individual links in a network system at each time step and generate the corresponding stochastic processes to formulate the construction process. The underlying topologies are two-node systems, tree systems, and arbitrary structured systems in that order.
2.1 Two-node systems

The elementary component of a network system is a two-node system with two nodes $x$ and $y$ connected by a link $z$ (see Fig. 1a in the main text). So we first study the construction of temporal networks on the two-node system such that the IET distribution of $x, y, z$ satisfy targeted distributions.

2.1.1 Theoretical analysis

The renewal processes of nodes $x, y$ and link $z$ are denoted as $\{X_n\}_{n \geq 0}, \{Y_n\}_{n \geq 0}$ and $\{Z_n\}_{n \geq 0}$. The initial states of the nodes and the link are all active (i.e. $X_0 = Y_0 = Z_0 = 1$). Unless otherwise specified, the initial state of all stochastic processes is active. For simplicity, we omit the specification of the initial state in the rest of this Supplementary Information.

The targeted IET distribution of node $x$ is a probability mass function $F(n)$,

$$ F(n) = \mathbb{P}(X_0 = 1, X_1 = 0, X_2 = 0, \ldots, X_n = 1), \quad n \in \mathbb{Z}^+, \quad (S1) $$

which represents the probability of the time interval between two consecutive activations. The probability mass function of node $y$ is denoted as $G(\Delta t)$ and $H(\Delta t)$, respectively. The trajectory of node $x$ (or $y$ or $z$) until time $n$ is denoted as $w_x^{(n)} = (w_x^{(0)}, \ldots, w_x^{(n)})^T$ (or $w_y^{(n)} = (w_y^{(0)}, \ldots, w_y^{(n)})^T$ or $w_z^{(n)} = (w_z^{(0)}, \ldots, w_z^{(n)})^T$) recording all $n + 1$ historical states. Link $z$ is active at time $n$ if and only if node $x$ and $y$ are all active at time $n$, which leads to

$$ w_z^{(n)} = w_x^{(n)} \circ w_y^{(n)}, $$

where the operation $\circ$ is the Hadamard product.

The conditional probability that node $x$ is active at time $n$ given the trajectory $w_x^{(n-1)}$ is denoted as $p_x(w_x^{(n-1)}, 1)$. To calculate $p_x(w_x^{(n-1)}, 1)$, we provide two identities based on the properties of renewal processes,

$$ \mathbb{P}(X_n = 1|X_{n-1} = 0, \ldots, X_1 = 0, X_0 = 1) = \frac{\mathbb{P}(X_0 = 1, X_1 = 0, X_2 = 0, \ldots, X_n = 1)}{\mathbb{P}(X_0 = 1, X_1 = 0, \ldots, X_{n-1} = 0)} = \frac{F(n)}{\sum_{i \geq n} F(i)}, $$

$$ \mathbb{P}(X_n = a|X_m = 1, X_{m-1} = w_{m-1}, \ldots, X_1 = w_1, X_0 = 1) = \mathbb{P}(X_{n-m} = a|X_0 = 1), \quad a \in \{0,1\}. $$

The first equation shows the value of the conditional probability that $x$ is active at time $n$ given that $x$ is inactive at all previous moments (except the initial moment), and the second equation illustrates the state of $x$ at the current moment is determined by all historical states from the last activation moment.

We denote the random vector $(X_0, X_1, \ldots, X_n)^T$ by $X^{(n)}$, the condition probability $p_x(w_x^{(n-1)}, a)$ is given as

$$ p_x(w_x^{(n-1)}, a) = \mathbb{P}(X_n = 1|X^{(n-1)} = w_x^{(n-1)}) = \begin{cases} \frac{F(n - m)}{\sum_{i \geq n-m} F(i)}, & a = 1, \\ \frac{\sum_{i \geq n-m+1} F(i)}{\sum_{i \geq n-m} F(i)}, & a = 0, \end{cases} \quad (S2) $$

20
where \( m = \max\{k \leq n : w_x^{(k)} = 1\} \) records the last activation moment of \( x \). The same conclusions can be obtained for node \( y \) and link \( z \).

The construction algorithm of a two-node system is formulated by a stochastic process \( \{S_n\}_{n \geq 0} \) satisfying for arbitrary sets of \( t_1, ..., t_k \in \mathbb{N}, k \in \mathbb{Z}^+ \),

\[
\begin{align*}
\mu_{S_{2n_1},...,S_{2n_k}} &= \mu_{X_{t_1},...,X_{t_k}}, \\
\mu_{S_{2n_1+1},...,S_{2n_k+1}} &= \mu_{Y_{t_1},...,Y_{t_k}}, \\
\mu_{(S_{2n_1},S_{2n_1+1},...,S_{2n_k},S_{2n_k+1})} &= \mu_{Z_{t_1},...,Z_{t_k}},
\end{align*}
\]  

(S3)

and \( S_0 = S_1 = 1 \), where \( \mu_{X_{t_1},...,X_{t_k}} \) represents the finite dimensional distribution of \( \{S_n\}_{n \geq 0} \) at the time slice \( (t_1, ..., t_k) \).

To prove the existence of \( \{S_n\}_{n \geq 0} \), we need to specify all finite dimensional distributions of \( \{S_n\}_{n \geq 0} \). We set \( X_{n+1} \) and \( Y^{(n)} \) are conditionally independent with respect to \( X^{(n)} \), that is,

\[
\begin{align*}
P(X_{n+1}, Y^{(n)} | X^{(n)}) &= P(X_{n+1} | X^{(n)}) \cdot P(Y^{(n)} | X^{(n)}),
\end{align*}
\]  

(S4)

meaning that the state of \( x \) at the current moment is unrelated to the past of \( y \). When \( X_{n+1} = a \) and \( X^{(n)} = w_x^{(n)} \), using Eqs. S3 and S4,

\[
P(S_{2n+2} = a | S^{(2n+1)}) = P(X_{n+1} = a | X^{(n)}, Y^{(n)}) = \frac{P(X_{n+1} = a, Y^{(n)} | X^{(n)}) \cdot P(X^{(n)})}{P(Y^{(n)} | X^{(n)}) \cdot P(X^{(n)})}
= \frac{P(X_{n+1} = a | X^{(n)}) \cdot P(Y^{(n)} | X^{(n)})}{P(Y^{(n)} | X^{(n)})} = P(X_{n+1} = a | X^{(n)})
= p_x(w_x^{(n)}, a),
\]

and

\[
P(S_{2n+3} = b | S^{(2n+2)}) = P(Y_{n+1} = b | X^{(n+1)}, Y^{(n)}) = \frac{P(X_{n+1} = a, Y_{n+1} = b | X^{(n)}, Y^{(n)}) \cdot P(X^{(n)}, Y^{(n)})}{P(X_{n+1} = a | X^{(n)}, Y^{(n)}) \cdot P(X^{(n)}, Y^{(n)})}
= \frac{P(X_{n+1} = a, Y_{n+1} = b | X^{(n)}, Y^{(n)})}{p_x(w_x^{(n)}, a)}.
\]

The numerator \( P(X_{n+1}, Y_{n+1} | X^{(n)}, Y^{(n)}) \) in Eq. (S6) is related to \( w_x^{(n)}, w_y^{(n)} \) and \( w_z^{(n)} \), which can be calculated as

\[
\begin{align*}
P(X_{n+1} = 1, Y_{n+1} = 1 | X^{(n)}, Y^{(n)}) &= p_z(w_z^{(n)}, 1), \quad (S7a) \\
P(X_{n+1} = 1, Y_{n+1} = 0 | X^{(n)}, Y^{(n)}) &= p_x(w_x^{(n)}, 1) - p_z(w_z^{(n)}, 1), \quad (S7b) \\
P(X_{n+1} = 0, Y_{n+1} = 1 | X^{(n)}, Y^{(n)}) &= p_y(w_y^{(n)}, 1) - p_z(w_z^{(n)}, 1), \quad (S7c) \\
P(X_{n+1} = 0, Y_{n+1} = 0 | X^{(n)}, Y^{(n)}) &= 1 + p_z(w_z^{(n)}, 1) - p_x(w_x^{(n)}, 1) - p_y(w_y^{(n)}, 1). \quad (S7d)
\end{align*}
\]

In particular, when \( P(B) = 0 \), we set \( P(A|B) = 0 \). Using Eqs. (S3)-(S7), we can calculate all finite dimensional distributions of \( \{S_n\}_{n \geq 0} \) and verify that they satisfy the suitable consistency
conditions. Therefore, by the Kolmogorov extension theorem \[ \mathbb{P} \], there exists a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that \(\{S_n\}_{n \geq 0}\) is well-defined on this probability space. When \(F = G = H\), we obtain that Eqs. (S7b) and (S7c) are equal to 0, which indicates that the state of \(x, y\) and \(z\) is the same at any time step. In this case, we claim that the system is synchronous.

### 2.1.2 Consistency condition

Mathematically, not all distributions \(F, G, H\) can lead to a well-defined \(\{S_n\}_{n \geq 0}\). The reason is that Eqs. (S7a)-(S7d) need to satisfy the definition of probability measure. Here, we propose a definition of the consistency of distributions \(F, G, H\).

**Definition 1** (Distribution consistency). The distributions \(F(n), G(n)\) and \(H(n)\) are said to be consistent if Eqs. (S7a)-(S7d) all belong to \([0, 1]\) for any possible \(w_x^{(n)}, w_y^{(n)}, n \geq 0\).

By the consistency of the distributions, we define the consistency of a two-node system.

**Definition 2** (Two-node system consistency). The two-node system is said to be consistent if the targeted distribution of the two nodes and the link are consistent.

When a system is consistent, the IET distribution of each node (link) fulfills the targeted distribution. Here, we first propose a necessary condition to verify system consistency with general targeted distributions, and analyse three types of classic distributions, power-law distributions, exponential distributions and poisson distributions.

The necessary condition is that the support set of the distribution \(H\) needs to be a denumerable set, i.e., \(\forall m \geq 1, \exists n \geq m, H(n) > 0\). If not, there exists a positive integer \(m\), such that \(H(m) > 0\) and \(H(n) = 0\), for all \(n > m\). When

\[
    w_z^{(m-1)} = 0^{(m-1)} = (1, 0, ..., 0)'_m-1, \tag{S8}
\]

the identity \(p_z(w_z^{(m-1)}, 1) = 1\) holds, meaning that link \(z\) is almost surely active at time \(m\). However, there is no \(F\) and \(G\) such that \(p_x(w_x^{(m-1)}, 1) = p_y(w_y^{(m-1)}, 1) = 1\) holds for all \(w_x^{(m-1)} \circ w_y^{(m-1)} = 0^{m-1}\). This results in \(p_z(w_z^{(m-1)}, 1) < p_z(w_z^{(m-1)}, 1)\) or \(p_y(w_y^{(m-1)}, 1) < p_y(w_y^{(m-1)}, 1)\) under some trajectories, then Eq. (S7b) or (S7c) is smaller than 0.

When \(F, G\) and \(H\) are all power-law distributions, we assume

\[
    F(n) = C_1n^{-\alpha_1}, \quad G(n) = C_2n^{-\alpha_2}, \quad H(n) = C_3n^{-\beta},
\]

where \(\alpha_1, \alpha_2, \beta\) are the exponent of the distributions larger than 1, and \(C_i = (\sum_{n=1}^\infty n^{-\alpha_i})^{-1}\) \((i = 1, 2, 3)\) are normalization constants. The conditional probability of \(x\) being active for the first time at time \(n\) is

\[
    p_x^{(n)} = p_x(0^{(n-1)}, 1) = \frac{F(n)}{\sum_{i \geq n} F(i)} \approx \frac{C_1 \int_{n-1/2}^{n+1/2} x^{-\alpha_1}dx}{C_1 \int_{n-1/2}^{n+1/2} x^{-\alpha_1}dx} = \frac{(n - \frac{1}{2})^{-\alpha_1+1} - (n + \frac{1}{2})^{-\alpha_1+1}}{(n - \frac{1}{2})^{-\alpha_1+1}} = 1 - \left(\frac{n + \frac{1}{2}}{n - \frac{1}{2}}\right)^{-\alpha_1+1}. \tag{S9}
\]
\( p_x^{(n)} \) is monotonically decreasing with respect to \( n \) and monotonically increasing with respect to \( a_1 \), when \( n \to \infty, p_x^{(n)} \to 0. \) Since \( w_x^{(i)} \geq w_z^{(i)} \) and \( w_y^{(i)} \geq w_z^{(i)} \) for all \( i \geq 0 \), by Eqs. (S7b) and (S7c), the exponent of the nodes need to be not less than that of the link, that is, \( a_1 \geq \beta \) and \( a_2 \geq \beta \). All conclusion are based on the premise that the algorithm runs for infinite rounds, which also holds when the simulation time is finite.

A necessary and sufficient condition for Eq. (S7d) to satisfy the definition of probability measure for all trajectories is

\[
p_x^{(1)} + p_y^{(2)} < 1, \quad p_x^{(1)} + p_y^{(1)} < 1 + p_z^{(1)}.
\]

In particular, when all nodes have the same exponent \( a \), the first condition becomes \( p_x^{(1)} + p_y^{(2)} < 1 \) and the second condition becomes \( 2C_1 < 1 + C_3 \).

The exponential distributions that we discuss are discrete exponential distributions. We assume

\[
F(n) = C_1 \int_{n-1/2}^{n+1/2} \lambda_1 e^{-\lambda_1 x} dx, \quad G(n) = C_2 \int_{n-1/2}^{n+1/2} \lambda_2 e^{-\lambda_2 x} dx, \quad H(n) = C_3 \int_{n-1/2}^{n+1/2} \lambda_3 e^{-\lambda_3 x} dx,
\]

where \( \lambda_i (i = 1, 2, 3) \) are exponents and \( C_i = e^{\lambda_i} (i = 1, 2, 3) \) are normalization constants.

\[
p_x^{(n)} = \frac{C_1 \int_{n-1/2}^{n+1/2} \lambda_1 e^{-\lambda_1 x} dx}{C_1 \int_{n-1/2}^{n+1/2} \lambda_1 e^{-\lambda_1 x} dx} = \frac{e^{-\lambda_1 (n - \frac{1}{2})} - e^{-\lambda_1 (n + \frac{1}{2})}}{e^{-\lambda_1 (n - \frac{1}{2})}} = 1 - e^{-\lambda_1}.
\]

\( p_x^{(n)} \) is monotonically increasing with respect to \( \lambda_1 \). To satisfy the non-negativity of Eqs. (S7a) and (S7b), the exponent of the nodes need to be not less than the link (i.e. \( \lambda_1 \geq \lambda_3, \lambda_2 \geq \lambda_3 \)). To satisfy the non-negativity of Eq. (S7d), the necessary and sufficient condition is \( e^{-\lambda_1} + e^{-\lambda_2} \geq e^{-\lambda_3} \). When two nodes have the same exponent, the conditions simplifies to \( 0 \leq \lambda_{node} - \lambda_{link} \leq \ln 2 \).

Another common discrete distribution defined over \( \mathbb{N} \) is the Poisson distribution, we now prove that when \( F, G \) and \( H \) are all Poisson distributions, the system is inconsistent. Since \( n \geq 1 \), we modify the definition domain of the Poisson distribution, \( P(X = n) = e^{-\lambda} \frac{\lambda^n}{(n-1)!}, \forall n \in \mathbb{Z}^+ \).

\[
p_x^{(n)} = \frac{e^{-\lambda} \frac{\lambda^n}{(n-1)!}}{1 + \sum_{i \geq 1} \frac{1}{(n+i)!/n!}} = \frac{1}{1 + \frac{\lambda}{n} + \frac{\lambda^2}{(n+1)n} + ...}.
\]

\( p_x^{(n)} \) is monotonically decreasing with respect to \( \lambda \) and monotonically increasing with respect to \( n \), when \( n \to \infty, p_x^{(n)} \to 1. \) Therefore, \( \forall \lambda_{node}, \lambda_{link}, \exists K, \) such that \( p_H^{(k)} > p_F^{(1)} \), that is, Eq. (S7b) may less than 0.

For distributions of general form, we need to judge the consistency at each time step in simulations. Once one of the probabilities in Eq. (S7) is less than 0, the construction process stops, and the system is inconsistent.

### 2.1.3 Construction algorithm

Algorithm 1 describes the construction process of temporal networks on two-node systems, the parameter \( t_{tol} \) is the duration time of simulations (i.e. the number of snapshots). In each time
Algorithm 1 Construction based on two-node systems

**Input:** probability mass functions $F(\Delta t)$, $G(\Delta t)$, $H(\Delta t)$ and parameter $t_{tol}$

**Output:** trajectories $\mathbf{w}^{t_{tol}}_x$, $\mathbf{w}^{t_{tol}}_y$, $\mathbf{w}^{t_{tol}}_z$ or 0

1: for $t = 1$ to $t_{tol}$ do
2:   Compute $p(w_x^{(t-1)}, 1)$, $p(w_y^{(t-1)}, 1)$ and $p(w_z^{(t-1)}, 1)$
3:   Compute $p_1 = p(w_x^{(t-1)}, 1) - p(w_z^{(t-1)}, 1)$, $p_2 = p(w_y^{(t-1)}, 1) - p(w_z^{(t-1)}, 1)$ and $p_3 = 1 + p(w_z^{(t-1)}, 1) - p(w_x^{(t-1)}, 1) - p(w_y^{(t-1)}, 1)$
4:   if $p_1 < 0$ or $p_2 < 0$ or $p_3 < 0$ then
5:     return 0
6:     Generate two independent random numbers $p$, $q$ satisfying a uniform distribution in $[0, 1]$
7:     if $p < p(w_x^{(t-1)}, 1)$ then
8:       $w_x^{(t)} = 1$
9:       $r = p(w_z^{(t-1)}, 1)/p(w_x^{(t-1)}, 1)$
10:      if $q < r$ then
11:        $w_y^{(t)} = 1$
12:      else
13:        $w_y^{(t)} = 0$
14:     else
15:       $w_x^{(t)} = 0$
16:       $r = p_2/(1 - p(w_x^{(t-1)}, 1))$
17:       if $q < r$ then
18:         $w_y^{(t)} = 1$
19:       else
20:        $w_y^{(t)} = 0$
21:    $w_z^{(t)} = w_x^{(t)} \cdot w_y^{(t)}$
22: return $\mathbf{w}^{t_{tol}}_x$, $\mathbf{w}^{t_{tol}}_y$, $\mathbf{w}^{t_{tol}}_z$
2.2.1 Theoretical analysis

IET distribution of all single nodes and single links in the system satisfy pre-given distributions. By orderly determining the activation of each two-node system in a tree system, we can guarantee that the IET distribution of all single nodes and single links in the system satisfy pre-given distributions.

2.2 Tree systems

A natural extension of a two-node system is a tree system (see Fig. 1b in the main text). By orderly determining the activation of each two-node system in a tree system, we can guarantee that the IET distribution of all single nodes and single links in the system satisfy pre-given distributions.

2.2.1 Theoretical analysis

A tree system with \( N \) nodes is denoted as \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{0, \ldots, N-1\} \) is a set of nodes and \( \mathcal{E} = \{(i,j) : i,j \in \mathcal{V}\} \) is a set of links. The renewal process of node \( i \) and link \( (m,n) \) is denoted as \( \{O_n^{(i)}\}_{n \geq 0} \) and \( \{E_n^{(m,n)}\}_{n \geq 0} \), respectively. We randomly select a node \( r \) in \( \mathcal{V} \) as the root of the system and classify all nodes according to their distance from the root. All nodes with distance \( l \) from the root compose the layer \( l \). The total number of layers is denoted as \( l_{\text{tol}} \) and the number of nodes in each layer \( l \) is denoted as \( n_l \).

Formally, we define a mapping \( v \),

\[
v : j \rightarrow (l,m),
\]

where \( j \in \mathcal{V} \) is the old number, \( l \) is the distance from \( j \) to \( r \) and \( m \) is the number of \( j \) in the layer \( l \). \( \tilde{\mathcal{V}} := v(\mathcal{V}) \), and \( v(r) = (0,1) \). We define a partial ordering \( \preceq \) on \( \tilde{\mathcal{V}} \), the relation \((i_1,j_1) \preceq (i_2,j_2)\) holds when \( i_1 \leq i_2 \) or \( i_1 = i_2, j_1 \leq j_2 \). Based on the relationship \( \preceq \), we sort the elements of \( \tilde{\mathcal{V}} \). We use \( \vartheta \) to represent this sorting,

\[
\vartheta : \tilde{\mathcal{V}} \rightarrow \{0, \ldots, N-1\}.
\]

The mapping \( \vartheta := \vartheta \circ v \) represents the reordering of \( \mathcal{V} \). We set \( u = \vartheta^{-1}, u(0) = r \).

For \( \mathcal{E} \), we define a mapping \( \hat{\vartheta} \),

\[
\hat{\vartheta} : (x,y) \rightarrow (p,q),
\]

where \( p = \vartheta(x), q = \vartheta(y) \).

We set \( f = \hat{\vartheta}^{-1} \). Similar to the two-node system, we construct a new stochastic process \( \{T_n\}_{n \geq 0} \) that for arbitrary sets of \( t_1, \ldots, t_k \in \mathbb{N}, k \in \mathbb{Z}^+ \) satisfies

\[
\mu_{T_{Nt_1+i} \cdots T_{Nt_k+i}} = \mu_{O_{t_1}^{(i)} \cdots O_{t_k}^{(i)}}, \quad \forall i \in \{0, \ldots, N-1\},
\]

\[
\hat{\mu}(T_{Nt_1+i} \cdots T_{Nt_k+i}) = \mu_{E_{t_1}^{(i)} \cdots E_{t_k}^{(i)}}, \quad \forall (i,j) \in \hat{\vartheta}(\mathcal{E}),
\]

\[
T_k = 1, \quad \forall k \in \{0, \ldots, N-1\},
\]

where \( \mu_{O_{t_1}^{(i)} \cdots O_{t_k}^{(i)}} \) and \( \mu_{E_{t_1}^{(i)} \cdots E_{t_k}^{(i)}} \) are the finite dimensional distributions about node \( u(i) \in \mathcal{V} \) and link \( f(i,j) \in \mathcal{E} \) at the time slice \((t_1, \ldots, t_k)\), respectively.
Equations (S12a)-(S12c) illustrate that the marginal distributions of \( \{T_n\}_{n \geq 0} \) satisfy the corresponding distributions of single nodes or single links. We can simply consider that \( \{T_n\}_{n \geq 0} \) consists of the following sequence

\[
(T_0, ..., T_{N-1}, ..., T_{Nk}, ..., T_{N(k+1)-1}, ...) = (O_0^u(0), ..., O_0^u(N-1), ..., O_k^u(0), ..., O_k^u(N-1), ...),
\]

and \( T_{Nk+i} \cdot T_{Nk+j} = E_k^{f(i,j)} \).

We continue with the previous notation \( T^{(Nk-1)} = (T_0, ..., T_{Nk-1}) \). To prove the existence of \( \{T_n\}_{n \geq 0} \), we specify all finite dimensional distributions of \( \{T_n\}_{n \geq 0} \). We set the following conditional independence, for all \( k \in \mathbb{N} \),

\[
\mathbb{P}(T_{Nk}|T^{(Nk-1)}) = \mathbb{P}(O_k^u(0)|O_{k-1}^u(0), ..., O_0^u(0)), \tag{S13}
\]

and for arbitrary mutually unequal \( i_1, ..., i_n, i, m \in \mathbb{N} \) that satisfy: (1) \((i_c, i) \notin \hat{\delta}(\mathcal{E}), \forall c \in \{1, ..., k\}\), (2) \((i, m), (j, m) \in \hat{\delta}(\mathcal{E})\), we have

\[
\mathbb{P}(T_{Nk+i}|T_{Nk+m}, T_{Nk+i_1}, ..., T_{Nk+i_r}, T^{(Nk-1)}) = \mathbb{P}(O_k^u(i)|O_{k-1}^u(i), ..., O_0^u(i), O_k^u(m), ..., O_0^u(m)). \tag{S14}
\]

Equation (S13) illustrates that the activation of the root \( r \) is only related to its own trajectory, and Eq. (S14) indicates that the activation of node \( i \) only depends on the two-node system in which it is located given the trajectory of another node in the same two-node system.

The corollary of Eq. (S14) is that for arbitrary mutually unequal \( i_1, ..., i_n, i, j, m \in \mathbb{N} \) that satisfy: (1) \((i_c, i), (i_c, j) \notin \hat{\delta}(\mathcal{E}), \forall c \in \{1, ..., k\}\), (2) \((i, m), (j, m) \in \hat{\delta}(\mathcal{E})\),

\[
\mathbb{P}(T_{Nk+i}, T_{Nk+j}|T_{Nk+m}, T_{Nk+i_1}, ..., T_{Nk+i_r}, T^{(Nk-1)}) = \mathbb{P}(O_k^u(i)|O_{k-1}^u(i), ..., O_0^u(i), O_k^u(m), ..., O_0^u(m)) \times \mathbb{P}(O_k^u(j)|O_{k-1}^u(j), ..., O_0^u(j), O_k^u(m), ..., O_0^u(m)). \tag{S15}
\]

Equation (S15) demonstrates the conditional independence about the triplets \((r, a1, b1)\) and \((r, a1, a2)\) showing in Fig. 1b in the main text.

Using Eqs. (S12)-(S14), we can calculate all finite dimensional distributions of \( \{T_n\}_{n \geq 0} \), then the existence of \( \{T_n\}_{n \geq 0} \) can be proved with the Kolmogorov extension theorem [1].

2.2.2 Consistency condition

We propose the definition of the consistency of a tree system.

**Definition 3** (Tree system consistency). A tree system is said to be consistent if all two-node systems in the tree system is consistent.

This definition shows that determining the consistency of a tree system requires judging the consistency of each two-node system.

The general necessary condition is that for the probability mass function \( F_{\text{link}} \) of each link in the system, the support set of \( F_{\text{link}} \) needs to be a denumerable set.

When the targeted probability mass functions are power-law distributions, the necessary and sufficient condition is that each two-node system satisfies \( p_x^{(1)} + p_y^{(2)} < 1, p_x^{(1)} + p_y^{(1)} < 1 + p_z^{(1)} \). When the targeted distributions are discrete exponential distributions, the necessary and sufficient condition is that the difference between the exponents of each node and each link is between 0 to \( \ln 2 \). When the targeted distributions are Poisson distributions, the tree system is inconsistent.
Algorithm 2 Construction based on tree systems

**Input:** tree system $T$, probability mass functions for all nodes and links and parameter $t_{tol}$

**Output:** trajectories of all nodes and links

1. for $t = 1$ to $t_{tol}$ do
2.   Select the root of $T$ and label all other nodes as $(l, m)$ according to the distance from the root $r$
3.   $p_r = p(w^{(l-1)}_{(0,1)}, 1) (l = 0)$
4.   Generate a random number $q$ satisfying a uniform distribution in $[0, 1]$
5.   if $q < p_r$ then
6.     $w^{(l)}_r = 1$
7.   else
8.     $w^{(l)}_r = 0$
9. for $l = 1$ to $t_{tol}$ do
10.   for $m = 1$ to $n_l$ do
11.     if $z = (x, y) \in E$
12.     Search $k$, such that $x = v^{-1}(l - 1, k), y = v^{-1}(l, m), z = (x, y) \in E$
13.     Execute a single loop of Algorithm 1 on the two node system consisting of
14.     the nodes $x, y$ and the link $z$
15. return trajectories of all nodes and links

2.2.3 Construction algorithm

Algorithm 2 shows the construction process of temporal networks based on tree systems. In each time step, the state of the root, $r$, is updated first according to its trajectory. Then the state of leaves is updated sequentially by executing Algorithm 1 on every two-node system in lines 9-13 of Algorithm 2. For each loop in lines 10-13, the state of all nodes on layer $l$ is updated. To update the state of a node $y$ on layer $l$, the node $x$ on layer $l - 1$, directly connecting to $y$, is identified. At this time, the state of $x$ has already been updated. The state of $y$ is updated according to the trajectory of $x$ and $y$.

If the system is consistent, the IET distribution of every single node (link) in synthetic temporal networks satisfies targeted distributions.

2.3 Arbitrary structured systems

For any network structure, we can always find a spanning tree. We call a link a trunk if it is in the spanning tree, otherwise we call it a branch.

The activation of a network system is established by its spanning tree. We first select a spanning tree of the underlying topology. At each time step, we execute Algorithm 2 on the spanning tree, and the state of all nodes and trunks is updated. The state of each branch is determined according to the state of the nodes on both sides of the branch. Algorithm 3 shows the corresponding construction process of temporal networks.

We claim that the system is consistent if and only if its spanning tree system is consistent. If the system is consistent, the IET distribution of every single node and trunk in the spanning tree fulfill pre-given distributions.
distributions, the probability of a result is recorded as condition

**Algorithm 3 Construction based on arbitrary network systems**

**Input:** network \( G \), probability mass functions of all nodes and links in a spanning tree and parameter \( t_{tol} \)

**Output:** trajectories of all nodes and links in the system

1. Select a spanning tree \( T \) of \( G \)
2. Assign a probability mass function to each node and each link in \( T \)
3. for \( t = 1 \) to \( t_{tol} \) do
4. Execute a single loop of Algorithm 2 for \( T \)
5. Update the state of each branch according to the state of the nodes on both sides of the branch

return trajectories of all nodes and links

---

### 2.4 Systems with ring structure

An significant property of a tree system \( T = (\mathcal{V}, \mathcal{E}) \) is that there is no ring in the system, that is, for all \( i_1, \ldots, i_m \in \mathcal{V} \), \( a_{i_1i_2} \times \cdots \times a_{i_mi_1} = 0 \), where \( a_{ij} = 1 \) if \( (i, j) \in \mathcal{E} \), otherwise \( a_{ij} = 0 \). In the sequel, we demonstrate that the activation process of an arbitrary network system is determined when the renewal process of nodes and trunks in a spanning tree is given.

We consider the simplest system with a ring, which consists of three nodes, \( \{x, y, z\} \), and three links, \( \{(x, y), (y, z), (z, x)\} \). The renewal process of nodes \( x, y \) and \( z \) is denoted as \( \{X_n\}_{n \geq 0} \), \( \{Y_n\}_{n \geq 0} \) and \( \{Z_n\}_{n \geq 0} \). At time \( n + 1 \), the probability of \( x \) being active is \( p(w_x^{(n)}, 1) \) and the result is recorded as \( a \). To make the IET distribution of node \( y \) and link \( (x, y) \) fulfill pre-given distributions, the probability of \( y \) being active is

\[
P(X_{n+1} = a, Y_{n+1} = 1|X_n = w_x^{(n)}, Y_n = w_y^{(n)}) = \frac{p(w_x^{(n)}, a)}{p(w_x^{(n)}, a) + p(w_y^{(n)}, b)}
\]

The result is recorded as \( b \). However, when discussing the activation of \( z \), we need to first determine one of the dynamic of \( \{z, (y, z)\} \) and \( \{z, (x, z)\} \). The former leads to the activation of \( z \) with probability

\[
P(Y_{n+1} = b, Z_{n+1} = 1|Y_n = w_y^{(n)}, Z_n = w_z^{(n)}) = \frac{p(w_y^{(n)}, b)}{p(w_x^{(n)}, a) + p(w_y^{(n)}, b)}
\]

(S16)

and the latter leads to the activation with probability

\[
P(X_{n+1} = a, Z_{n+1} = 1|X_n = w_x^{(n)}, Z_n = w_z^{(n)}) = \frac{p(w_x^{(n)}, a)}{p(w_x^{(n)}, a) + p(w_y^{(n)}, b)}
\]

(S17)

These two probabilities are not always the same. Using Eq. (S16) (Eq. (S17)) to obtain the state of \( z \) is equivalent to the select the spanning tree with links \( (x, y), (y, z) \) \( ((x, y), (x, z)) \), and the state of the remaining link (i.e. the branch) is then established.

Furthermore, in general, the stochastic process of the branch is not a renewal process. Without loss of generality, we assume that links \( (x, y), (x, z) \) are in the spanning tree. We denote the stochastic process of link \( (y, z) \) as \( \{S_n\}_{n \geq 0} \). To prove the above assertion, we verify a sufficient condition

\[
P(S_2 = 1|S_1 = 1) \neq P(S_1 = 1)
\]

(S18)
The left-hand side of Eq. (S18) equals

$$
\frac{1}{P(S_1 = 1)} \sum_{x_1, x_2 \in \{0, 1\}} P(X_1 = x_1, Y_1 = 1, Z_1 = 1, X_2 = x_2, Y_2 = 1, Z_2 = 1).
$$  \tag{S19}

The numerator of Eq. (S19) equals

$$
\sum_{x_1 \in \{0, 1\}} P(X_1 = x_1, Y_1 = 1, Z_1 = 1) \sum_{x_2 \in \{0, 1\}} \left( \frac{P(X_2 = x_2, Y_2 = 1 | X_1 = x_1, Y_1 = 1)}{P(X_2 = x_2, Z_2 = 1 | X_1 = x_1, Z_1 = 1)} \right) / P(X_2 = x_2 | X_1 = x_1).
$$  \tag{S20}

In general, Eq. (S20) is not equal to

$$
\sum_{x_1 \in \{0, 1\}} P(X_1 = x_1, Y_1 = 1, Z_1 = 1) \sum_{x_2 \in \{0, 1\}} P(X_1 = x_2, Y_1 = 1, Z_1 = 1) = P(S_1 = 1)^2,
$$

showing that Eq. (S18) holds.

3 Robustness analysis

Although we can only require the algorithmic distribution of nodes and trunks to precisely fulfill targeted distributions, we find that the algorithmic distribution of single branches is approximate exponential distributions or power-law distributions when executing Algorithm 3 on double-exponential coupling patterns or double-power-law coupling patterns. Furthermore, the algorithmic distribution of all nodes and all links is robust to underlying topologies and the selection of spanning trees. Here, we uncover the origin of these phenomena.

3.1 Double-exponential coupling pattern

We begin our analysis with a three-node ring consisting of nodes $x, y, z$ and links $(x, y), (y, z), (z, x)$. We select a spanning tree with links $(x, y)$ and $(y, z)$, and the root is node $x$. The renewal process of nodes $x, y, z$ is denoted as $\{X_n\}_{n \geq 0}, \{Y_n\}_{n \geq 0}, \{Z_n\}_{n \geq 0}$, and the exponent of nodes $x, y, z$ (links $(x, y), (x, z)$) is $\lambda_x, \lambda_y, \lambda_z$ ($\lambda_{xy}, \lambda_{xz}$). For simplicity, we make the following assumption:

**Assumption 1:** $\lambda_x = \lambda_y = \lambda_z = \lambda_1 > 0$ and $\lambda_{xy} = \lambda_{xz} = \lambda_2 > 0$.

Considering the consistency condition for double-exponential coupling patterns, the inequality $0 \leq \lambda_1 - \lambda_2 \leq \ln 2$ holds. The stochastic process of link $(x, z)$ is denoted as $\{S_n\}_{n \geq 0}$.

We define a random variable

$$
\tau_x = \inf\{n \geq 1 : X_n = 1\},
$$

which is a stopping time for node $x$, indicating the first activation time of $x$. Similarly, the stopping times for nodes $y, z$ and link $(x, z)$ are denoted as $\tau_y, \tau_z$ and $\tau_{x}$, respectively. We use the distribution of $\tau_x$ to approximate the algorithmic distribution of branch $(x, z)$.

We first prove the following theorem,

**Theorem 1.** For a three-node ring satisfying Assumption 1,

$$
P(\tau_x > n) = e^{-\mu n},
$$

where $\mu > 0$. 

29
If Theorem 1 holds, the algorithmic IET distribution of the branch is also exponential.

Proof. By the definition of \( \tau_s \), we have

\[
\mathbb{P}(\tau_s > n) = \sum_{a^{(n)} \in c^{(n)} = \emptyset^{(n)}} \sum_{b^{(n)} \in \{0,1\}^{n+1}} \mathbb{P}(X^{(n)} = a^{(n)}, Z^{(n)} = c^{(n)}) \\
= \sum_{a^{(n)} \in c^{(n)} = \emptyset^{(n)}} \sum_{b^{(n)} \in \{0,1\}^{n+1}} \mathbb{P}(X^{(n)} = a^{(n)}, Y^{(n)} = b^{(n)}, Z^{(n)} = c^{(n)}),
\]

where \( a^{(n)} = (a_0, ..., a_n)^T \), \( b^{(n)} = (b_0, ..., b_n)^T \), and \( c^{(n)} = (c_0, ..., c_n)^T \) are the trajectories of \( x, y, z \) with length \( n + 1 \), the definition of \( \emptyset^{(n)} \) is the same as that in Eq. (S8), and the operation \( \circ \) is the Hadamard product.

For each fixed \( a^{(n)}, b^{(n)}, c^{(n)} \), using Eqs. (S13) and (S14), we have

\[
\mathbb{P}(X^{(n)} = a^{(n)}, Y^{(n)} = b^{(n)}, Z^{(n)} = c^{(n)}) = \mathbb{P}(X_n = a_n | X^{(n-1)} = a^{(n-1)}) \times \mathbb{P}(Y_n = b_n | X^{(n-1)} = a^{(n-1)}) \times \mathbb{P}(Z_n = c_n | Y^{(n-1)} = b^{(n-1)}) = c^{(n-1)}.
\]

This gives

\[
\mathbb{P}(\tau_s > n) = \sum_{a^{(n-1)} \in c^{(n-1)} = \emptyset^{(n-1)}} \sum_{b^{(n-1)} \in \{0,1\}^{n}} \mathbb{P}(X^{(n-1)} = a^{(n-1)}, Y^{(n-1)} = b^{(n-1)}, Z^{(n-1)} = c^{(n-1)}) \\
\times \sum_{a_n \times c_n = 0 \ b_n \in \{0,1\}} \sum_{b_n \in \{0,1\}} \left( \mathbb{P}(X_n = a_n | X^{(n-1)} = a^{(n-1)}) \times \mathbb{P}(Y_n = b_n | X^{(n)} = a^{(n)}, Y^{(n-1)} = b^{(n-1)}) \times \mathbb{P}(Z_n = c_n | Y^{(n)} = b^{(n)}, Z^{(n-1)} = c^{(n-1)}) \right).
\]

Using Eqs. (S5)-(S7) and (S10), we have

\[
\sum_{a_n \times c_n = 0 \ b_n \in \{0,1\}} \sum_{b_n \in \{0,1\}} \mathbb{P}(X_n = a_n | X^{(n-1)} = a^{(n-1)}) \times \mathbb{P}(Y_n = b_n | X^{(n)} = a^{(n)}, Y^{(n-1)} = b^{(n-1)}) \times \mathbb{P}(Z_n = b_n | X^{(n)} = c^{(n)}, Z^{(n-1)} = b^{(n-1)}) = e^{-\lambda_1} \times \frac{1 - e^{-\lambda_2} - e^{-\lambda_2}}{1 - e^{-\lambda_1}} + \frac{e^{-\lambda_2} - e^{-\lambda_2}}{e^{-\lambda_1}}.
\]

This leads to

\[
\mathbb{P}(\tau_s > n) = C \mathbb{P}(\tau_s > n - 1).
\]

Since \( \mathbb{P}(\tau_s > 0) = 1 \), we have

\[
\mathbb{P}(\tau_s > n) = C^n.
\]
In this simplest ring, we can also analyse the decay of $\mathbb{P}(\tau_s > n)$ in three cases.

Case 1: When $\lambda_1 = \lambda_2$, we have $C = e^{-\lambda_1}$, which indicates that

$$\mathbb{P}(\tau_s > n) = e^{-\lambda_1 n} = \mathbb{P}(X(n) = 0(n), Y(n) = 0(n), Z(n) = 0(n)).$$

This identity can also be obtained by the synchronization of the system, because all elements have the same targeted distribution.

Case 2: When $\lambda_1 - \lambda_2 = \ln 2$, we have

$$C = \frac{e^{-\lambda_2}}{2} + \frac{(1 - e^{-\lambda_2})e^{-\lambda_2}}{2 - e^{-\lambda_2}} < 1,$$

meaning that the decay of $\mathbb{P}(\tau_s > n)$ is exponential.

Let $C = e^{-\mu}$, we compare the magnitude of $\mu(\lambda_2)$ and $\lambda_2$. Since

$$\frac{d(\mu(\lambda_2) - \lambda_2)}{d\lambda_2} = -\frac{2e^{\lambda_2}}{8e^{2\lambda_2} - 10e^{\lambda_2} + 3} < 0.$$  

We get

$$\mu(\lambda_2) - \lambda_2 > 0, \quad \mu(\lambda_2) \to \lambda_2 \text{ as } \lambda_2 \to \infty.$$  

This indicates that $\mathbb{P}(\tau_s > n)$ is also an exponential distribution, and the corresponding exponent is larger than that of single trunks, $\lambda_2$.

Case 3: When $0 < \lambda_1 - \lambda_2 < \ln 2$, let $x = e^{-\lambda_1}, y = e^{-\lambda_2}$, then

$$\partial_y C(x, y) = \frac{2(x^2 - 2x + y)}{x(x - 1)}.$$  

This gives

$$\max_y C(x, y) = (2 - x)x < 1.$$  

Similar to Case 2, let $C = e^{-\mu}$,

$$\frac{d(\mu(\lambda_2) - \lambda_2)}{d\lambda_2} = \frac{e^{3\lambda_1} + 2e^{2\lambda_2} - 2e^{\lambda_1 + 2\lambda_2}}{e^{3\lambda_1} - e^{2\lambda_2} + 2e^{\lambda_1 + \lambda_2} - 4e^{2\lambda_1 + \lambda_2} + 2e^{\lambda_1 + 2\lambda_2}}. \quad (S24)$$

For each fixed $\lambda_1$, the numerator and denominator of Eq. (S24) are quadratic function with respect to $e^{\lambda_2}$. Considering $e^{\lambda_1} / 2 < e^{\lambda_2} < e^{\lambda_1}$, when $\lambda_1 > \ln(3/2), e^{3\lambda_1} - e^{2\lambda_2} + 2e^{\lambda_1 + \lambda_2} - 4e^{2\lambda_1 + \lambda_2} + 2e^{\lambda_1 + 2\lambda_2} < 0.$

Then

$$\max_{\lambda_2} (\mu(\lambda_2) - \lambda_2) < \max \{\mu(\lambda_1) - \lambda_1, \mu(\lambda_1 - \ln 2) - \lambda_1 + \ln 2\} = 0.$$  

This indicates that the exponent of $\mathbb{P}(\tau_s > n)$ is lower than $\lambda_2$.

For a general ring $\mathcal{R} = (\mathcal{V}, \mathcal{E})$, $\mathcal{V} = \{1, \ldots, m\}$ and $\mathcal{E} = \{(i, j) : i, j \in \mathcal{V}, |i - j| = 1\} \cup \{(1, m)\}$. The renewal process of individual nodes in $\mathcal{V}$ are denoted as $\{O_n^{(1)}\}_{n \geq 0}, \ldots, \{O_n^{(m)}\}_{n \geq 0}$. We add all links into the spanning tree except link $(1, m)$, and we denote the stochastic process of $(1, m)$ by $\{S_n\}_{n \geq 0}$. The first activation time of branch $(1, m)$ is also denote as $\tau_s$. We can proof the following theorem.
Theorem 2. For a general ring with \( m \) nodes, if every single node (trunk) has the same exponent, then

\[
P(\tau_s > n) = e^{-\mu n},
\]

where \( \mu > 0 \).

**Proof.** We denote the exponent of every single node (trunk) as \( \lambda_1 (\lambda_2) \). Let \( O^{(j,n)} \) and \( i^{(k,n)} \) represent \( (O^{(j)}_0, ..., O^{(j)}_n) \) and \( (i^{(k)}_0, ..., i^{(k)}_n) \), respectively. We have

\[
\mathbb{P}(\tau_s > n) = \sum_{i^{(1,n)} \in \{0\}^n} \mathbb{P}(O^{(1,n)} = i^{(1,n)}, \ldots, O^{(m,n)} = i^{(m,n)})
\]

\[
= \sum_{i^{(1,n)} \in \{0\}^n} \sum_{i^{(m,n)} \in \{0\}^{m-1,n}} \mathbb{P}(O^{(1,n)} = i^{(1,n)}, \ldots, O^{(m,n)} = i^{(m,n)}).
\]

For each fixed \( i^{(1,n)}, \ldots, i^{(m,n)} \), similar to Eq. (S21), we have

\[
\mathbb{P}(O^{(1,n)} = i^{(1,n)}, \ldots, O^{(m,n)} = i^{(m,n)}) = \mathbb{P}(O^{(1,n-1)} = i^{(1,n-1)}_1, i^{(2,n-1)}_1, \ldots, i^{(m,n-1)}_1) \\
\times \mathbb{P}(O^{(2,n)} = i^{(2,n)}_2, i^{(2,n)}_1, \ldots, i^{(m,n)}_2, i^{(m,n)}_1) \\
\times \ldots \\
\times \mathbb{P}(O^{(m,n-1)} = i^{(m,n-1)}_m, i^{(m,n)}_1, \ldots, i^{(m,n)}_m, i^{(m,n)}_1) \\
\times \mathbb{P}(O^{(1,n-1)} = i^{(1,n-1)}_1, i^{(2,n-1)}_1, \ldots, i^{(m,n-1)}_1),
\]

meaning that node 1 is the root of the spanning tree. This gives

\[
\mathbb{P}(\tau_s > n) = \sum_{i^{(1,n-1)}_1 \in \{0\}^{n-1}} \sum_{i^{(2,n-1)}_1, \ldots, i^{(m,n-1)}_1 \in \{0\}^n} \mathbb{P}(O^{(1,n-1)} = i^{(1,n-1)}_1, \ldots, O^{(m,n-1)} = i^{(m,n-1)}_1)
\]

\[
\times \sum_{i^{(1,n)}_1 \times i^{(m)}_m = 0} \mathbb{P}(O^{(1,n)} = i^{(1,n)}_1) \mathbb{P}(O^{(2,n)} = i^{(2,n)}_2) \ldots \mathbb{P}(O^{(m,n)} = i^{(m,n)}_m)
\]

\[
\times \ldots \\
\times \mathbb{P}(O^{(m,n)} = i^{(m,n)}_m) \mathbb{P}(O^{(m,n-1)} = i^{(m,n-1)}_m) \ldots \mathbb{P}(O^{(1,n-1)} = i^{(1,n-1)}_1).
\]

Using Eqs. (S5)-(S7) and (S10), we have

\[
\sum_{i^{(1,n)}_1 \times i^{(m)}_m = 0} \mathbb{P}(O^{(1,n)} = i^{(1,n)}_1) \mathbb{P}(O^{(2,n)} = i^{(2,n)}_2) \ldots \mathbb{P}(O^{(m,n)} = i^{(m,n)}_m)
\]

\[
\times \ldots \\
\times \mathbb{P}(O^{(m,n)} = i^{(m,n)}_m) = 2e^{-\lambda_1} - \sum_{i^{(2)}_2, \ldots, i^{(m-1)}_m = 0} \mathbb{P}(O^{(1)}_1 = 0, O^{(2)}_1 = i^{(2)}_2, \ldots, O^{(m-1)}_1 = i^{(m-1)}_m, O^{(m)}_1 = 0)
\]

\[
:= C.
\]

(S27)
This leads to
\[ P(\tau_s > n) = C^n. \]
Since \( P(\tau_s > n) < P(\tau_s > n - 1) < 1 \) for all \( n \geq 2 \), we have \( C < 1 \), showing that \( P(\tau_s > n) \) is an exponential distribution.

Furthermore, the definition of \( \tau_s \) implies that
\[ \{ \tau_1 > n \} \subset \{ \tau_s > n \}, \quad \forall n \in \mathbb{N}, \]
where \( \tau_1 \) is the first activation time of node 1. Since
\[ P(\tau_1 > n) = e^{-\lambda_1 n}. \]
This gives
\[ P(\tau_s > n) \geq e^{-\lambda_1 n}, \]
meaning that the distribution, \( P(\tau_s > n) \), for any single branch has a uniform lower bounded. And we notice that the constant \( C \) in Eq. (S27) is monotonically increasing with respect to the size of the ring, \( m \), indicating that the exponent for a branch located at a large ring is smaller than that of a branch located at a small ring. Therefore, we can find an exponential distribution that is a uniform upper bound for every single branch, and the exponent of the upper bound is related to the targeted exponent of nodes and trunks.

In a word, the distribution of the first activation time of every single branch is uniformly upper and lower bounded by exponential distributions, so the algorithmic distribution of branches can be approximated as exponential distributions.

### 3.2 Double-power-law coupling pattern

In this section, we prove that the distribution of first activation time for every single branch has uniform upper and lower bounds, which are all heavy-tailed.

Similar to double-exponential coupling patterns, for a ring with \( m \) nodes, we also selection all links except link \((1, m)\) as trunks. The exponent of every single node (trunk) are the same, denoted as \( \alpha_1 (\alpha_2) \).

First, we offer the mathematical definition of the heavy-tailed distribution [2].

**Definition 4** (Heavy-tailed distribution). The distribution of a random variable \( X \) is said to have a heavy tail if
\[
\lim_{x \to \infty} e^{\lambda x} P(X > x) = \infty, \quad \forall \lambda > 0.
\]

By Definition 4, we propose the definition of a sequence to be heavy-tailed.

**Definition 5** (Heavy-tailed sequence). A non-negative sequence \( \{a_n\}_{n=1}^{\infty} \) is said to be heavy-tailed if
\[
\lim_{n \to \infty} e^{\lambda n} a_n = \infty, \quad \forall \lambda > 0.
\]

Under the above definitions, we have the following lemmas.
Lemma 1. For a discrete non-negative random variable $X$, let $a_n = P(X > n)$, if
\[ \lim_{n \to \infty} \frac{a_n}{a_{n-1}} = 1, \] (S28)
then $X$ is heavy-tailed.

Proof. Under Eq. (S28), we have
\[ \lim_{n \to \infty} e^{\lambda n} a_n = e^\lambda > 1, \quad \forall \lambda > 0. \]
Therefore,
\[ \lim_{n \to \infty} e^{\lambda n} a_n = \infty, \]
meaning that the distribution of $X$ is heavy-tailed.

One straightforward implication is that the power-law distribution is heavy-tailed.

Lemma 2. For two non-negative sequences $\{a_n\}_{n=1}^\infty$ and $\{b_n\}_{n=1}^\infty$ that satisfy $a_0 = b_0 = 1$ and $a_n \geq b_n$ for all $n \geq 1$, if
\[ \lim_{n \to \infty} \frac{b_n}{b_{n-1}} = 1, \]
then the sequence $\{a_n\}_{n=1}^\infty$ is heavy-tailed.

Proof. To see this, we assume that there exists $\lambda_0 > 0$ such that $\lim_{n \to \infty} e^{\lambda_0} a_n \neq \infty$. Then $\liminf_{n \to \infty} e^{\lambda_0} a_n = c < \infty$, which shows there exists a subsequence $\{a_{n_k}\}_{k=1}^\infty$ such that the decay of $a_{n_k}$ is exponential, and
\[
\ln a_{n_k} := \sum_{i=1}^{n_k} x_i, \quad x_{n_k} = \ln \frac{a_{n_k}}{a_{n_k-1}} \to -\lambda_0 \quad \text{as } n_k \to \infty, \\
\ln b_{n_k} := \sum_{i=1}^{n_k} y_i, \quad y_{n_k} = \ln \frac{b_{n_k}}{b_{n_k-1}} \to 0 \quad \text{as } n_k \to \infty.
\]
Thus there exists $n_j$ such that for all $n_k \geq n_j$, $a_{n_k} < b_{n_k}$, in contradiction with the assumption that $a_n \geq b_n$.

Since
\[ P(\tau_s > n) \geq P(\tau_1 > n) \approx Cn^{-\alpha_1+1}, \]
with the help of lemmas above, we obtain that $P(\tau_s > n)$ is a heavy-tailed distribution and is lower bounded by a power-law distribution with an exponent $\alpha_1$.

We next turn to derive the uniform upper and bounds for branches.

Theorem 3. For a general ring with $m$ nodes, if the exponent of every single node (trunk) is the same, then
\[ A_n \leq P(\tau_s > n) \leq B_n \]
holds for any $m$, where the sequences $A_n$ and $B_n$ are both heavy-tailed.
Proof. Similar to Eq. (S26), for each fixed \(i^{(1,n)}, \ldots, i^{(m)}\), we have

\[
\sum_{i_n^{(1)} \times i_n^{(m)} = 0} \sum_{i_n^{(1)} \ldots i_n^{(n-1)} \in \{0,1\}} \Pr(O_n^{(1)} = i_n^{(1)} | O^{(1,n-1)} = i^{(1,n-1)}) \\
\times \Pr(O_n^{(2)} = i_n^{(2)} | O^{(1,n)} = i^{(1,n)}, O^{(2,n-1)} = i^{(2,n-1)}) \\
\times \ldots \\
\times \Pr(O_n^{(m)} = i_n^{(m)} | O^{(n-1,n)} = i^{(m-1,n)}, O^{(m,n-1)} = i^{(m,n-1)}) \\
= 1 - p_1(i^{(1,n-1)}, 1) + \sum_{i_n^{(2)}, \ldots, i_n^{(m-1)} \in \{0,1\}} \Pr \left( \begin{array}{c}
O_1^{(1)} = 0, O_2^{(2)} = i_2, \\
\ldots, O_{m-1}^{(m-1)} = i^{(m-1)}, O_m^{(m)} = 1
\end{array} \right) \\
= 2(1 - p_1(i^{(1,n-1)}, 1)) - \sum_{i_n^{(2)}, \ldots, i_n^{(m-1)} \in \{0,1\}} \Pr \left( \begin{array}{c}
O_1^{(1)} = 0, O_2^{(2)} = i_2, \\
\ldots, O_{m-1}^{(m-1)} = i^{(m-1)}, O_m^{(m)} = 0
\end{array} \right).
\]

Let

\[
C_n^{(m)} = \sum_{i_n^{(i+1)}, \ldots, i_n^{(m-1)}} \sum_{a^{(i+1), \ldots, i_n^{(m-1)}} = 0^{(n-1)}} \Pr(O^{(1,n-1)} = i^{(1,n-1)}, \ldots, O^{(m,n-1)} = i^{(m,n-1)})(1 - p_1(i^{(1,n-1)}, 1)),
\]

We have

\[
C_n^{(m)} \leq \Pr(\tau_s > n) \leq 2C_n^{(m)},
\]

Since \(\Pr(\tau_s > n)\) is heavy-tailed, using Lemma 2, we have \(2C_n^{(m)}\) is heavy-tailed. Then \(\Pr(\tau_s > n)\) is upper and lower bounded by heavy-tailed distributions. Let \(A_n = \min_m C_n^{(m)}\) and \(B_n = \max_m 2C_n^{(m)}\), \(A_n\) and \(B_n\) are also heavy-tailed and are only related to the targeted exponent of nodes and trunks. \(\square\)

Finally, we discuss the IET distribution of all links. The distribution is represented by a random variable \(E\),

\[
\Pr(E \leq x) = \frac{1}{|E|} \sum_{i \in E} \Pr(E_i \leq x), \tag{S29}
\]

where \(E_i\) is a random variable representing the IET distribution of link \(i\), \(|E|\) is the number of links. By Eq. (S29), we have

\[
\min_{i \in E} \Pr(E_i \leq x) \leq \Pr(E \leq x) \leq \min_{i \in E} \Pr(E_i \leq x),
\]

which means the distribution of all links is upper and lower bounded by the distribution of individual links. When the algorithmic IET distribution of every single link is heavy-tailed (exponential), the distribution of all links is also heavy-tailed (exponential).

4 Integration with network evolution

Another pivotal advantage of our algorithm is the ability to integrate with the evolution of networks. The integration can describe scenarios in which the state of nodes and links in temporal
Algorithm 4 Combination with network evolution

**Input:** initial underlying topology $G$ and parameter $t_{tol}$

**Output:** trajectories of all nodes and links

1: Select a spanning tree $T$ of $G$
2: Assign a probability mass function to each node and trunk in $T$
3: for $t = 1$ to $t_{tol}$ do
   4: Update $G$ according to the evolution of networks
   5: Update $T$
   6: Assign a input distribution to each new node and trunk in $T$ and set them to be active
   7: Set the old nodes which are connected to the new nodes to be active
   8: Execute a single loop of Algorithm 2 for updating the remaining nodes and trunks
   9: Update the remaining branches
return trajectories of all nodes and links

networks switches between active and inactive. Meanwhile, the underlying topology of the temporal networks also evolves over time.

Algorithm 4 demonstrates a unified framework to construct temporal networks with evolving underlying topologies. At each time step, the underlying topology is updated, and so is its spanning tree. Newly added nodes and links are active, and old nodes connected to new nodes are forced to be active. The state of the remaining nodes and links is established by Algorithm 3.

Common network evolution is the sequential addition of new nodes with links connecting to old nodes in network systems [4], which makes the systems grow over time (including the example presented in the main text). In addition, when considering recessionary effects, the number of nodes and links may decrease [5]. For different network evolution, the main difference in Algorithm 4 is the update of the spanning tree. When the number of nodes increases or decreases, the spanning tree needs to add or delete the corresponding number of nodes and their links.

5 Analysis of aggregated networks

Real-world temporal networks exhibit robust heterogeneous behaviour over different time scales. Here, we analyse two statistics, the activation number of a node (or link) and the node strength distribution of networks aggregated over intervals of various lengths $t_{agg}$ (i.e. aggregated networks).

For a node $i$ (trunk $j$) whose targeted IET distribution is $\phi_i(\Delta t, a_i) \ (\psi_j(\Delta t, \beta_j))$, the expectation of $\phi_i (\psi_j)$ is denoted as $\mu(a_i) \ (v(\beta_j))$, where the exponent $a_i$ ($\beta_j$) is obtained by sampling from a distribution $\eta_{node}$ ($\eta_{link}$).

The total activation number of node $i$ up to moment $t$ is denoted as $A^{(i)}_t$. Let

$$A^{(i)}(t) = \mathbb{E}A^{(i)}_t.$$  

Using the elementary renewal theorem [3], we have

$$\frac{A^{(i)}(t)}{t} \rightarrow \frac{1}{\mu(a_i)} \quad \text{as} \ t \rightarrow \infty,$$  \quad (S30)
where \( \frac{1}{\beta} = 0 \). The conclusion for single links is similar.

Equation (S30) illustrates that the growth rate of the number of activations is asymptotically linear when the expectation of distribution is finite, otherwise the rate is sublinear. When the targeted distribution is a discrete exponential distribution, the expectation is then finite, and the rate is linear. When the distribution is power-law and the exponent is less than 2, the expectation is infinite, so the rate is sublinear.

Another important statistic for an aggregated network is the node strength distribution. For a static unweight network \( G \), the degree distribution of \( G \) is denoted as \( d(x) \) and its maximum value is \( k_{\text{max}} \). We set a random variable \( X \) of which the probability mass function is \( d(x) \). Let \( N_t \) denote the strength of a node in the aggregated network generated by \( G \) with the aggregation time, \( t_{\text{agg}} = t \), \( N_t \) is a random variable. We assume that the exponent of all links is sampled from \( \eta_{\text{link}} \), using a limit theorem of renewal theory [3], with probability 1,

\[
\frac{N_t}{t} \to \sum_{i=1}^{k_{\text{max}}} \frac{1}{v(\eta_i)} \quad \text{as} \quad t \to \infty, \tag{S31}
\]

where \( \{\eta_i\}_{1 \leq i \leq k_{\text{max}}} \) is a sequence of independent random variables with a common distribution \( \eta_{\text{link}} \).

As \( X \) and \( \{\eta_i\}_{1 \leq i \leq k_{\text{max}}} \) are independent, when \( t \) is sufficiently large, from Eq. (S31), the distribution of \( N_t \) is given as

\[
\mathbb{P}(N_t \leq s) = \sum_{x=1}^{k_{\text{max}}} \mathbb{P}(X = x) F(x)^{(\frac{s}{t})}, \tag{S32}
\]

where \( F(x) \) is the x-order convolution of the distribution function of the random variable \( 1/v(\eta_1) \).

The Laplace transform of the random variable \( X \) is defined by \( \rho_X(s) = \mathbb{E}e^{-sx} \), then Eq. (S32) is converted into

\[
\rho_{N_t}(s) = \sum_{x=1}^{k_{\text{max}}} \mathbb{P}(X = x) \left[ \rho_{\frac{1}{v(\eta_1)}}(st) \right]^x.
\]

In particular, when a.s. \( \eta_i \) is a constant and equals \( \beta \), Eq. (S32) can be estimated as follows

\[
\mathbb{P}(N_t \leq s) \approx \int_0^{s\rho(\beta)/t} d(x)dx. \tag{S33}
\]

Formally, let \( p_{N_t}(x) \) denote the probability density function of \( N_t \). From Eq. (S33), we have

\[
p_{N_t}(x) = \frac{v(\beta)d(\frac{x\rho(\beta)}{t})}{t}. \tag{S34}
\]

Equation (S34) shows the relationship between the degree distribution of a static network and the node strength distribution of the aggregated network. When \( d(x) = Cx^{-\gamma} \), the node strength distribution is also power-law with the same exponent \( \gamma \) for each aggregation time, indicating the robustness of aggregated networks.

For general degree distributions, we can also obtain a similar robust behaviour of node strength distributions. For two aggregated networks \( G_1, G_2 \) with the aggregation time \( t_1, t_2 \) \((t_1 > t_2)\), we have

\[
\mathbb{P}(X_{t_2}/v(\beta) > s) = F^{(2)}(s) = F^{(1)}(s\frac{t_1}{t_2}) := F^{(1)}(\delta),
\]

37
where \( F^{(i)}(s) \) is the survivor function of the random variable \( X_t/v(\beta) \) \((i = 1, 2)\). When \( s \) in \( F^{(2)} \) changes from \( s_1 \) to \( s_1 + 1 \), \( s \) in \( F^{(1)} \) changes from \( s_1^{(1)} \) to \( (s_1 + 1)^{(1)} \), which means the proportion of nodes with strength between \( s_1^{(1)} \) and \( (s_1 + 1)^{(1)} \) in \( G_1 \) is same as the proportion of nodes with strength \( s_1 \) in \( G_2 \).

The normalization is executed by first setting a baseline time, \( t_{\text{base}} \), corresponding to the aggregated network, \( G_{\text{base}} \). For all \( t > t_{\text{base}} \), let the strength of each node in \( G_t \) be multiplied by \( t_{\text{base}}/t \). Then, the node strength distribution of \( G_{\text{base}} \) and \( G_t \) is the same.

6 Statistics for measuring burstiness and temporal correlations

A statistic called the burstiness parameter, \( B \), of which the magnitude measures the level of burstiness [6], is defined by the coefficient of variation,

\[
B = \frac{\sigma/\mu - 1}{\sigma/\mu + 1},
\]

where \( \mu \) and \( \sigma \) are the mean and standard deviation of IET distributions. When \( \mu \) and \( \sigma \) are finite, the definition is meaningful and \( |B| < 1 \).

As the total simulation time is finite, the empirical burstiness parameters are meaningful in both double-power-law coupling patterns and double-exponential coupling patterns for any exponent. We calculate the empirical burstiness parameter of all nodes and all links in the algorithmic temporal networks in Fig. 2.

In order to compare empirical and theoretical burstiness parameters, we set a cutoff \( \kappa \) for theoretical calculation. The theoretical mean, \( \mu \), and standard deviation, \( \sigma \), for a power-law distribution and a discrete exponential distribution are

\[
\mu = \frac{\sum_{i=1}^{\kappa} i^{-\gamma+1}}{\sum_{i=1}^{\kappa} i^{-\gamma}}, \quad \sigma = \left( \frac{\sum_{i=1}^{\kappa} i^{-\gamma+2}}{\sum_{i=1}^{\kappa} i^{-\gamma}} - \mu^2 \right)^{1/2},
\]

and

\[
\mu = e^{\gamma/2} \sum_{i=1}^{\kappa} i(e^{-\gamma(i-0.5)} - e^{-\gamma(i+0.5)}) \quad \sigma = \left( e^{\gamma/2} \sum_{i=1}^{\kappa} i^2(e^{-\gamma(i-0.5)} - e^{-\gamma(i+0.5)}) - \mu^2 \right)^{1/2},
\]

respectively. Tables S1 and S2 compares the empirical and theoretical \( B \) of nodes and links in double-power-law coupling patterns and in double-exponential coupling patterns, respectively. As one can see, the empirical results are robust to underlying topologies and are well predicted by theoretical results. As expected, both nodes and links show high level of burstiness in double-power-law coupling patterns, but the opposite in double-exponential coupling patterns.

In addition to the burstiness parameter, we investigate the temporal correlation of temporal networks constructed by our algorithm. The autocorrelation function, \( A(\Delta t) \), is a common statistic to appreciate the global activity correlations for temporal networks. For a temporal network \( \mathcal{G} = \{ G_1, ..., G_T \} \) where \( T \) is the number of snapshots, \( A(\Delta t) \) is defined as

\[
A(\Delta t) = \frac{\frac{1}{T-\Delta t} \sum_{i=1}^{T-\Delta t} E(i)E(i+\Delta t) - \mu_1\mu_2}{\sigma_1\sigma_2},
\]

38
where \( E(i) \) denotes the total activation numbers of nodes or links in the snapshot \( i \), \( \mu_1, \sigma_1^2 \) are the sample mean and sample variance of the series \( \{ E(i) \}_{i=1}^{T-\Delta t} \), and \( \mu_2, \sigma_2^2 \) are the sample mean and sample variance of the series \( \{ E(i+\Delta t) \}_{i=1}^{T-\Delta t} \). The parameter \( \Delta t \) determines the time window of the two series. In particular, when \( \Delta t = 1 \), \( A(\Delta t) \) is called the memory coefficient \( 6 \).

By Hölder’s inequality, we obtain that \( |A(\Delta t)| \leq 1 \). The closer \( A(\Delta t) \) is to 0, the less correlated \( \{ E(i) \}_{i=1}^{T-\Delta t} \) and \( \{ E(i+\Delta t) \}_{i=1}^{T-\Delta t} \), and the weaker the autocorrelation of the temporal network. When \( A(\Delta t) \) is close to 1 or -1, A strong positive or negative linear correlation exists between series \( \{ E(i) - \mu_1 \}_{i=1}^{T-\Delta t} \) and series \( \{ E(i+\Delta t) - \mu_2 \}_{i=1}^{T-\Delta t} \).

Supplementary Figure 8 shows the results of \( A(\Delta t) \) in double-power-law coupling patterns and double-exponential coupling patterns. In double-power-law coupling patterns, the activation process is a non-Markovian process and the results of \( A(\Delta t) \) show positive temporal correlations for all time intervals, \( \Delta t \), which is consistent with heterogeneous temporal behaviour discovering in empirical temporal networks. In double-exponential coupling patterns, the results of \( A(\Delta t) \) are almost 0. Actually, Eq. (S10) indicates that the activation process is a discrete time Markov chains, \( \{M_n \}_{n \geq 0} \), with two states \( \{s_0,s_1\} \), where \( s_0 \) represents the element (node or link) is inactive and \( s_1 \) represents the element is active. We set \( s_0 = 0 \) and \( s_1 = 1 \). The corresponding transition probability matrix is given as follows

\[
\begin{pmatrix}
  e^{-\lambda} & 1 - e^{-\lambda} \\
  e^{-\lambda} & 1 - e^{-\lambda}
\end{pmatrix}
\]

where \( \lambda \) is the exponent of the targeted distribution. For all \( m > 1 \), we have

\[
P(M_m = 1) = \sum_{i_1,...,i_{m-1} \in \{0,1\}} P(M_1 = i_1, ..., M_{m-1} = i_{m-1}, M_m = 1)
\]

\[
= \sum_{i_1,...,i_{m-1} \in \{0,1\}} P(M_1 = i_1, ..., M_{m-1} = i_{m-1})P(M_m = 1 | M_{m-1} = i_{m-1})
\]

\[
= 1 - e^{-\lambda}.
\]

Hence, for all \( n, m \geq 1 \), we have

\[
P(M_n = 1, M_{n+m} = 1) = \sum_{i_1,...,i_{n+m-1} \in \{0,1\}} P(M_n = 1, M_{n+1} = i_1, ..., M_{n+m-1} = i_{m-1}, M_{n+m} = 1)
\]

\[
= \sum_{i_1,...,i_{m-1} \in \{0,1\}} P(M_n = 1, ..., M_{n+m-1} = i_{m-1})P(M_{n+m} = 1 | M_{n+m-1} = i_{m-1})
\]

\[
= (1 - e^{-\lambda})^2.
\]

This gives

\[
A(\Delta t) \sim \frac{1}{T-\Delta t} \sum_{i=1}^{T-\Delta t} EM_i M_{i+\Delta t} - \frac{1}{(T-\Delta t)^2} \sum_{i,j=1}^{T-\Delta t} EM_i EM_{i+\Delta t}
\]

\[
= \frac{1}{(T-\Delta t)^2} \sum_{i,j=1}^{T-\Delta t} P(M_i = 1, M_{i+\Delta t} = 1) - \frac{1}{(T-\Delta t)^2} \sum_{i,j=1}^{T-\Delta t} P(M_i = 1)P(M_{i+\Delta t} = 1)
\]

\[
= 0,
\]

meaning that double-exponential coupling patterns are memoryless (or homogenous).
7 Empirical data pre-processing

We use four datasets from different social contexts collected by the SocioPatterns collaboration [http://www.sociopatterns.org]. Each dataset is formed by a collection of contact events, where a contact event is represented by a triplet \((t, i, j)\), indicating an interaction between individual \(i\) and \(j\) occurs at time \(t\). We first generate the underlying topology of each dataset. A link between node \(i\) and node \(j\) exists on the static underlying topology if \(i\) and \(j\) interact at least once. Then we generate the corresponding temporal network of each dataset. All triplets occurring at time \(t\) form the snapshot at time \(t\), and all snapshots form a temporal network.

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**Supplementary Table 1:** Burstiness parameter of double-power-law coupling patterns. The cutoff $\kappa$ is set to be $5 \times 10^3$. The exponent of empirical data and theoretical results is algorithmic fitted exponents and the average of the two algorithmic exponents.

| Dataset          | Object | Exponent | $B$ | Object | Exponent | $B$ |
|------------------|--------|----------|-----|--------|----------|-----|
| Empirical (BA)   | Nodes  | 1.80     | 0.76| Links  | 1.17     | 0.47|
| Empirical (SW)   | Nodes  | 1.80     | 0.76| Links  | 1.16     | 0.48|
| Theoretical      | Nodes  | 1.80     | 0.79| Links  | 1.17     | 0.51|
| Empirical (BA)   | Nodes  | 2.20     | 0.76| Links  | 1.97     | 0.73|
| Empirical (SW)   | Nodes  | 2.20     | 0.77| Links  | 1.99     | 0.74|
| Theoretical      | Nodes  | 2.20     | 0.79| Links  | 1.98     | 0.81|
**Supplementary Table 2**: Burstiness parameter of double-exponential coupling patterns. The cutoff $\kappa$ is set to be $10^3$. The implication of parameters is the same as Table 1.

| Dataset          | Object | Exponent | $B$  | Object | Exponent | $B$  |
|------------------|--------|----------|------|--------|----------|------|
| Empirical (BA)   | Nodes  | 1.80     | -0.42| Links  | 1.21     | -0.29|
| Empirical (SW)   | Nodes  | 1.80     | -0.42| Links  | 1.22     | -0.29|
| Theoretical      | Nodes  | 1.80     | -0.42| Links  | 1.21     | -0.29|
| Empirical (BA)   | Nodes  | 2.50     | -0.55| Links  | 1.86     | -0.43|
| Empirical (SW)   | Nodes  | 2.50     | -0.55| Links  | 1.89     | -0.44|
| Theoretical      | Nodes  | 2.50     | -0.55| Links  | 1.86     | -0.43|
Supplementary Figure 1: IET distribution of all nodes and all links is robust to the selection of spanning trees. The underlying topology is generated by the Barabasi-Albert model. Nodes with different degrees are selected to be the root of a spanning tree to represent the difference in spanning tree selection. The algorithmic fitted exponent does not correlate significantly with the selection of spanning trees under both patterns, and the largest magnitude of variances is no more than $10^{-3}$, indicating the robustness to spanning tree selection.
Supplementary Figure 2: IET distribution when exponents are sampled from specific distributions. The exponent of every single node (trunk) is a random variable $\eta_{node}$ ($\eta_{link}$). **a**, Double-power-law coupling pattern. **b**, Double-exponential coupling pattern. We set $\eta_{node}$ and $\eta_{link}$ to be a uniform $[1.55, 1.95]$ ($[1.40, 1.70]$) random variable and a uniform $[1.05, 1.50]$ ($[1.05, 1.30]$) random variable in double-power-law coupling patterns (double-exponential coupling patterns). The results of nodes (links) are represented by circles (squares). The algorithmic distribution of all nodes (links) is plotted by brown markers, and the distribution of the single element with the maximal (minimal) activation numbers is plotted by blue (green) markers. Parameter values are the same as those in Fig. 2 for numerical simulations.
Supplementary Figure 3: IET distribution of every single element. The algorithmic IET distribution of every single node (link) is represented by a red (green) line, respectively, showing in the first and second columns. The black lines are upper or lower bounds of the algorithmic results, which are power-law distributions in the first row and exponential distributions in the second row generated by simulations. We also select the results of branches located in the largest and smallest ring showing in the third column by diamonds and triangles. The branch located in the largest ring has a smaller fitted exponent than that in the smallest ring, which is consistent with the theoretical result.
Supplementary Figure 4: IET distributions on well-mixed networks. The largest relative deviation of the algorithmic exponent in this figure and Fig. 2 is $|2.06 - 1.97|/(2.06 + 1.97) < 3\%$. For numerical simulations, the size of the network is $N = 400$. Other parameter values are the same as those in Fig. 2.
Supplementary Figure 5: Distribution of first activation time of branches. The underlying topology is a three-node ring. The exponent of every single node in double-power-law and double-exponential coupling patterns is $\alpha = 2$. The exponent of the branch is selected as the minimal, middle, and maximal values satisfying the corresponding consistency conditions. Numerical simulations are run $10^5$ independent trials. For each trial, the simulation ends when the branch is active for the first time.
Supplementary Figure 6: Activation number of single nodes and links. A node and a link are randomly selected from a tree system. The circles and squares represent the results of the node and the link, respectively, and the inset of each subfigure shows the function of the growth rate of the activation number with respect to $t_{agg}$. When the targeted distribution of the node and the link is power-law with exponents $\alpha = 1.9$ and $\beta = 1.2$ (i.e., $\alpha, \beta < 2$), the grow of the activation number is sublinear, and the rate decreases monotonically with respect to $t_{agg}$. When the targeted distributions are discrete exponential distributions with exponents $\alpha = 2.0$ and $\beta = 1.5$, the grow is always linear, and the rate, $k_{growth}$, is a constant, with a theoretical value $k_{growth} = (e^2 \sum_{i=1}^{\infty} i(e^{-\alpha(i-0.5)} - e^{-\alpha(i+0.5)}))^{-1}$. 
Supplementary Figure 7: Robustness of node strength distributions on small-world underlying topologies. The normalized distribution of node strength of aggregated networks is robust to the aggregation time, $t_{agg}$, but is affected by activity patterns. The targeted exponent of single nodes and links is $(\alpha_{pmf}, \beta_{pmf}) = (2.20, 2.10)$ in double-power-law coupling patterns and $(\alpha_{pmf}, \beta_{pmf}) = (2.50, 2.00)$ in double-exponential coupling patterns.
Supplementary Figure 8: Positive or zero temporal correlations in double-power-law or double-exponential coupling patterns. The relationship between the autocorrelation function, $A(\Delta t)$, and the time interval, $\Delta t$, are plotted based on double-power-law and double-exponential coupling patterns. $A(\Delta t)$ is always positive for any $\Delta t$ in double-power-law coupling patterns, while $A(\Delta t)$ is approximately 0 in double-exponential coupling patterns.