Transport of interacting electrons through a double barrier in quantum wires

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We generalize the fermionic renormalization group method to describe analytically transport through a double barrier structure in a one-dimensional system. Focusing on the case of weakly interacting electrons, we investigate thoroughly the dependence of the conductance on the strength and the shape of the double barrier for arbitrary temperature $T$. Our approach allows us to systematically analyze the contributions to renormalized scattering amplitudes from different characteristic scales absent in the case of a single impurity, without restricting the consideration to the model of a single resonant level. Both a sequential resonant tunneling for high $T$ and a resonant transmission for $T$ smaller than the resonance width are studied within the unified treatment of transport through strong barriers. For weak barriers, we show that two different regimes are possible. Moderately weak impurities may get strong due to a renormalization by interacting electrons, so that transport is described in terms of theory for initially strong barriers. The renormalization of very weak impurities does not yield any peak in the transmission probability; however, remarkably, the interaction gives rise to a sharp peak in the conductance, provided asymmetry is not too high.

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I. INTRODUCTION

Effects related to the Coulomb interaction between electrons become increasingly prominent in systems of lower spatial dimensionality as their size is made smaller. Recent experimental progress in controlled preparation of nanoscale devices has led to a revival of interest in the transport properties of one-dimensional (1D) quantum wires. Owing to the particular geometry of the Fermi surface, systems of dimensionality one are unique in that the Coulomb correlations in 1D change a noninteracting picture completely and thus play a pivotal role in low-temperature physics.

A remarkable example of a correlated 1D electron phase is the Luttinger-liquid model (for a review see, e.g., Refs. \textsuperscript{}1,2). In this model, arbitrarily weak interactions ruin the conventional Fermi liquid phenomenology by essentially modifying low-energy excitations across the Fermi surface. As a result, the tunneling density of states develops power-law singularities on the Fermi surface. Moreover, interactions between oppositely moving electrons generate charge- and spin-density wave correlations that lead to striking transport properties of a Luttinger liquid in the presence of impurities. In particular, even a single impurity yields a complete pinning of a Luttinger liquid with repulsive interactions, which results in a vanishing conductance at zero temperature. In addition to quantum wires, largely similar ideas apply to edge modes in a Hall bar geometry in the fractional quantum Hall regime, which are thought to behave as spatially separated chiral Luttinger liquids.

Evidence has recently emerged pointing towards the existence of the Luttinger liquid in metallic single-wall carbon nanotubes. The Luttinger liquid behavior was observed via the power-law temperature and bias-voltage dependence of the current through tunneling contacts attached to the nanotubes. Further technological advances have made possible the fabrication of low-resistance contacts between nanotubes and metallic leads (see, e.g., Refs. \textsuperscript{}5,6,7 and references therein). These recent developments have paved the way for systematic transport measurements in Luttinger liquids with impurities.

In this paper, we study electron transport through a double barrier in a 1D liquid. In the 1D geometry, two impurities in effect create a quantum dot inside the system. Resonant tunneling through the two impurities is a particularly attractive setup to investigate the correlated transport in an inhomogeneous Luttinger liquid. Due to the resonant behavior of the current, the interplay of Luttinger-liquid correlations and impurity-induced backscattering should be more easily accessible to transport measurements. Two striking experimental observations have been reported recently. In Ref. \textsuperscript{}8, a resonant structure of the conductance of a semiconductor single-mode quantum wire was attributed to the formation (with reduction of electron density by changing gate voltage) of a single disorder-induced quantum dot. In Ref. \textsuperscript{}9, two barriers were created inside a carbon nanotube in a controlled way with an atomic force microscope. In both cases, the amplitude of a conductance peak $G_p$ as a function of temperature $T$ showed power-law behavior $G_p \propto T^{-\gamma}$ with the exponent $\gamma$ noticeably different from $\gamma = 1$. The latter is the value of $\gamma$ expected in the absence of interactions (for a review, see Refs. \textsuperscript{}10,11) provided $T$ lies in the range $\Gamma \ll T \ll \Delta$, where $\Gamma$ is the width of a resonance in the transmission coefficient and $\Delta$ is the single-particle level spacing. The width of a conductance peak $w$ followed a linear temperature dependence $w \propto T$ in both experiments.

On the theoretical side, resonant tunneling in a Luttinger liquid was studied previously in a number of papers. In particular, the width $\Gamma \propto T^{\alpha_x}$ was shown to shrink with decreasing temperature. The exponent $\alpha_x$ depends on the strength of interaction.
and describes tunneling into the end of a semi-infinite liquid. The dimensionless peak conductance (in units of $e^2/h$) obeys $G_p \sim \Gamma/T$ in the above range of $T$, which indeed leads to a smaller value of $\gamma = 1 - \alpha_e$. The reduced exponent $\gamma$ reported in Ref. 13 was positive (and different for different conductance peaks, in the range $\gamma \sim 0.5 - 0.8$), whereas in Ref. 13 the reported value of $\gamma \simeq -0.7$ was negative. More specifically, in Ref. 13 the conductance as a function of the gate voltage showed certain traces of periodicity characteristic to the Coulomb blockade regime.13,15,21 Surprisingly, both the amplitude $G_p$ and the width $w$ were reported to vanish with decreasing $T$, in sharp contrast to the noninteracting case. While such behavior is known to be possible for very strong repulsive interaction 22,23 the required strength of interaction would then be much larger than expected and indeed reported (see Refs. 16,22,23 and references therein) in carbon nanotubes. Roughly a doubling (or even a larger factor) of the expected 22,23 exponent $\alpha_e$, which is $\alpha_e \sim 0.6 - 1.0$, would be necessary to fit the experimental data. Although the modeling 22,23 of carbon nanotubes as a four-channel Luttinger liquid has its share of complications, the observations 13 appear to present a puzzle.

It was suggested in Ref. 13 that a certain novel mechanism of “correlated tunneling” dominates over the conventional sequential tunneling for $T \gg \Gamma$, leading to a doubling of the exponent $\alpha$ which might explain the experiment. In the subsequent works 22,23 the basic ideas relevant to the resonant tunneling have been questioned. In particular, the lowest-order contribution to the resonance peak conductance $G_p$ for $\Delta \gg T \gg \Gamma$ has been argued 22,23 to come from processes of second order in the end-tunneling density of states. While being a characteristic feature of the cotunneling regime far in the wings of the conductance peak, this, however, disagrees with the sequential tunneling picture inside the conductance peak 16,19. This also poses a problem with the persistence of perfect transmission through symmetric barriers in the case of weak interaction. Moreover, the main suggestion 23 is that taking a finite-range (but falling off fast beyond this range) interaction changes the conventional picture completely, as compared to the standard treatment of a zero-range interaction in the Luttinger liquid model. According to Ref. 24, higher-order tunneling processes in combination with the effects of a non-zero range of interaction dominate the dependence of $G_p$ on $T$ even for $T \gg \Gamma$. A nearly perfect agreement with the experiment data 13 has been claimed (for a range of interaction far smaller than the distance between the barriers). However, no explicit formula for the conductance has been given in Ref. 24, while we see no ground for the low-energy long-distance physics to be essentially different if the radius of interaction is made finite.

In another recent attempt 25 to explain the experiment 13 the observed power law was attributed to the contact resistance. Namely it was noted that the resistance of tunneling contacts to the leads, $R_c$, and the resistance of the quantum dot add up (if one applies Kirchhoff’s law), so that the anomalous $T$ dependence in Ref. 13 might be explained if $R_c \gg G_p^{-1}$, where $G_p$ is understood as the resonance peak conductance of the dot. This is a legitimate proposition, although the measured contact resistance in Ref. 13 was reported to be relatively low 13,24.

It is thus desirable to examine the resonant tunneling in a Luttinger liquid in a broad range of temperature down to $T = 0$ and for various parameters of the barriers. Our purpose in this paper is to analyze transport through a double barrier of arbitrary strength, strong or weak, symmetric or asymmetric, within a general framework of an analytical method applicable to all these situations. There are a variety of techniques to construct the low-energy transport theory 2,27,28. The method we develop here is valid for weak interaction and is based on the renormalization group (RG) approach of Refs. 20,26, which was applied earlier in a variety of contexts 21,32,34. One of the appeals of this kind of theory is that it allows one to treat weak and strong scatterers on an equal footing, which is technically significantly less straightforward in the bosonization method 27. Thus it would be of interest to apply this method to a 1D mesoscopic interacting system with many impurities (previously, the problem of transport in a disordered Luttinger liquid was studied by perturbative in disorder methods based on bosonic field theories in, e.g., Refs. 1,24,35). A double barrier is the simplest “many-impurity” system which exhibits effects essential to the physics of disordered interacting 1D liquids.

Very recently, a generalization of the RG approach (initially developed for a structureless barrier 22) has been proposed in Ref. 33 for the model of a single impurity with an energy-dependent scattering matrix. This model is relevant to the resonant tunneling of weakly interacting electrons through a double barrier and accounts properly for the interaction processes within an energy band of width $\Delta$ around the Fermi level. In particular, a non-monotonic behavior of the conductance peak as a function of $T$, caused by left/right asymmetry of scattering amplitudes, was investigated in Ref. 33 within the single-resonance model. The fermionic RG 33 has been shown to be in accord with the results 16,19 for the resonant tunneling in Luttinger liquids, obtained earlier by different methods.

However, to describe microscopically the spatial structure of a system of two or more impurities, a more systematic analysis is needed. Specifically, one has to develop an approach that would include contributions to the renormalized scattering amplitudes due to interaction processes involving energy transfers larger than $\Delta$. Also, the single-resonance model 13 does not describe contributions from tunneling through multiple levels inside the quantum dot (multi-level quantum dots represent a typical experimental situation). Finally, it would be interesting to study transport through weak impurities (and through strongly asymmetric structures) for which the
bare transmission coefficient exhibits no pronounced resonant structure. All this adds to our motivation to study the resonant tunneling through a double barrier by generalizing the RG approach of Refs. 29,30.

The RG approach enables us to investigate in detail the resonant transport of weakly interacting spinless electrons. Within the fermionic RG approach, we confirm earlier results3,16,19 obtained within bosonic field theories. We examine the conductance through a double barrier for arbitrary strength and an arbitrary shape of the barrier, not restricting ourselves to the model of a single resonant level. In particular, we demonstrate the existence of narrow conductance peaks for two weak impurities, which is in sharp contrast to the noninteracting case. We do not find any trace of the correlated tunnelling mechanism proposed in Refs. 13,24,25. We also clarify the relationship between the RG method29,30 and Hartree-Fock (HF) approaches.

The paper is organized as follows. First, in Sec. II A we briefly outline the fermionic RG approach to transport through a single impurity. In Sec. II B we discuss a HF treatment of transmission through a single impurity and show its inadequacy to the problem. In Sec. III we turn to a double barrier. We start with a perturbative expansion in Sec. III A and derive the RG equation for transport through the double barrier in Sec. III B. We then analyze contributions to scattering amplitudes from different energy scales, compared to the level spacing inside the quantum dot and the resonance width, in Secs. III C–III E. In Sec. III F, we concentrate on the case of weak impurities. Finally, in Sec. IV we calculate the amplitude and the shape of the resonance conductance peaks.

II. SINGLE IMPURITY

A. Renormalization group: Basic results

We begin with a brief description of transport through a single structureless impurity in the spirit of the RG approach29,30. Without interaction, the impurity is characterized by a transmission coefficient \( t_0 \) and reflection coefficients \( r_{L,0} \) and \( r_{R,0} \), from the left and from the right respectively (we put the impurity at the center of coordinates, \( x = 0 \)). Suppose that the energy dependence of the bare scattering matrix can be neglected far from the boundaries of an energy band \((-D_0, D_0)\) around the Fermi level. The energy scale \( D_0 \) serves as the ultraviolet cutoff of RG transformations and, physically, is of the order of \( v_F/d \) (throughout the paper we put \( h = 1 \)) or the Fermi energy \( \epsilon_F \), whichever is smaller. Here \( d \) is the radius of interaction and \( v_F \) is the Fermi velocity. Deep inside the band \((-D_0, D_0)\), we linearize the energy spectrum around the Fermi level. The differential RG equations29,30 read

\[
\frac{\partial t}{\partial \mathcal{L}} = -\alpha t \ , \quad \frac{\partial r_{L,R}}{\partial \mathcal{L}} = \alpha r_{L,R} T \ ,
\]

where \( \mathcal{L} = \ln(D_0/|\epsilon|) \), the energy \( \epsilon \) is measured from the Fermi level, the transmission probability \( T = |t|^2 \), and \( R = 1 - T \). The boundary conditions at \( \mathcal{L} = 0 \), \( t_0 \), \( r_{L,0} \), and \( r_{R,0} \). Throughout the paper we consider spinless electrons, for which the interaction constant is

\[
\alpha = (V_f - V_b)/2\pi v_F \ ,
\]

where \( V_f \) and \( V_b \) are the Fourier transforms of a pair-wise interaction potential yielding forward \( (V_f) \) and backward \( (V_b) \) scattering. The forward scattering does not lead to transitions between two branches of right- and left-movers, whereas the backscattering does. We assume that \( \alpha > 0 \).

Note that, for spinless electrons, the interaction-induced backward scattering and forward scattering relate to each other as direct and exchange processes, so that the backscattering only appears in the combination \( V_f - V_b \) and thus merely redefines parameters of the Luttinger model (formulated29 in terms of forward-scattering amplitudes only). In particular, the backscattering does not lead to any RG flow for \( \alpha \). For spinful electrons this is valid only to one-loop order.12

It is also worth mentioning that for a point interaction \( V_f = V_b \), so that \( \alpha = 0 \), hence for spinless electrons one has to start with a finite-range interaction. However, the RG flow for the scattering matrix1 occurs for \(|\epsilon| \ll v_F/d\) and is governed solely by the constant \( \alpha \). It follows that on low-energy scales one can effectively consider the interaction as local, \( V_{al}(x - x') = 2\pi\alpha v_F \delta(x - x') \), and formally deal exclusively with forward scattering. A non-zero range of interaction for \( k_Fd \gg 1 \) can manifest itself only in the boundary conditions to Eqs. (1) at \(|\epsilon| \sim D_0 = v_F/d\) and therefore does not affect the singular behavior of the renormalized scattering matrix at \( \epsilon \to 0 \). We assume that the Coulomb interaction between electrons is screened by external charges (e.g., by metallic gates, in which case \( d \) is given by the distance to the gates) and that a resulting \( \alpha \ll 1 \). For a treatment of the unscreened Coulomb interaction, see, e.g., Refs. 15,37.

Integration of Eqs. (1) gives29,30

\[
\frac{R}{T} = \frac{R_0}{T_0} \left( \frac{D_0}{|\epsilon|} \right)^{2\alpha} \ .
\]

The phases of the scattering amplitudes are not affected by the renormalization. Equations (1) are equivalent to a one-loop renormalization, so that Eq. (3) is valid to first order in interaction \( \sim O(\alpha) \) in the exponent of the power-law scaling. As follows from Eq. (3), whatever the initial values of \( T_0 \), at \( \alpha > 0 \) they all flow to the fixed point of Eqs. (1) at zero transmission.29,30 \( T = 0 \) at \( \epsilon = 0 \), see Fig. 1. In the limits of a weak impurity (both \( R_0 \ll 1 \) and \( R \ll 1 \)) and a strong tunneling barrier \( (T_0 \ll 1) \), Eq. (3) coincides with the RG results obtained by bosonization29 provided \( \alpha \ll 1 \). Equation (3) gives the transmission probability for electrons with energy \( \epsilon \) at temperature \( T = 0 \). For finite \( T \), the renormalization stops at \( |\epsilon| \sim T \).
regime I shrinks to zero while the scaling exponents in $\alpha$ down. If $\sim$ interaction, renormalization of the transmission coefficient by weak interaction introduces two more energetic scales that characterize the action. The scale $D_0$ yields $-\alpha \propto (1 + 2N_0) - \frac{1}{2} - 1$ and $2 \alpha \propto 2 |(1 + 2N_0)^{1/2} - 1|$, respectively.

To make a connection with experiment, it is worth noting that in the Luttinger liquid model ($V_b = 0$) for $N$ channels with the same Fermi velocity and no interchannel transitions, the constant $\alpha$ is related to the exponent $\alpha_e$ which describes tunneling into the end of the liquid (see Sec. II) as follows:

$$\alpha_e = N^{-1}[(1 + 2N_0)^{1/2} - 1] .$$

E.g., $N = 1$ for spinless and $N = 2$ for spin-degenerate electrons in a quantum wire with a single mode of transverse quantization. $N = 4$ in the Luttinger-liquid model of the armchair carbon nanotube. Note that $\alpha_e = \alpha$ for any $N$ in the limit of weak interaction ($\alpha \to 0$). For a discussion of the tunneling into the end of a multi-channel Luttinger liquid with non-equivalent channels see Ref. 38.

B. Hartree-Fock versus renormalization group

We now turn to a conceptually important point that is highly relevant to our calculation of the resonant tunneling in Sec. III. The basic idea of Refs. 29,30 was to relate the RG transformations to the scattering of an electron off Friedel oscillations created by the impurity screened by other electrons. Indeed, to first order in $\alpha$, when the interaction-induced correction to the density distribution may be neglected, the HF correction to the scattering amplitudes is logarithmically divergent $0 \ln(D_0/|\epsilon|)$. The essence of this singularity is a slow decay of the Friedel oscillations, which behave in 1D, in the absence of interaction, as $|x|^{-1} \sin(2k_F|x| + \phi_F)$. Here $k_F$ is the Fermi wavevector, $|x|$ the distance to the impurity, and $\phi_F$ a constant phase shift. This observation suggests that a HF treatment of the problem would yield a series of leading logarithms $\alpha^n L^n$. In the HF problem, it would be sufficient to solve iteratively in $\alpha$ a nonlinear Schrödinger equation ($H_0 + U_{HF}(\psi - \epsilon)\psi = 0$, where $H_0$ is the noninteracting Hamiltonian. The static self-consistent scattering potential is $U_i + U_{HF}$, where $U_i$ is the bare impurity potential and the nonlocal HF term reads

$$U_{HF}(x,x') = -V(x-x')\rho(x,x')$$

Here $V(x)$ is a potential of pairwise interaction between particles, $\rho(x,x') = \sum_i \psi_i^*(x)\psi_i(x)$ is a sum over all occupied states, and $\rho(x_1, x_1)$ in the Hartree term is the total electron density which includes the Friedel oscillations. Self-consistency requires that both the wave functions $\psi_i$ and the scattering potential $U_{HF}$ be corrected on every step of the HF iterative procedure. In effect, Ref. 31 (see also Ref. 32) suggests that the HF procedure, being carried out in the "leading-log" approximation, would reproduce Eq. (3). In particular, the HF calculation of
second order in $\alpha$ gives a correction $\sim \alpha^2 L^2$ and, according to Ref. 30, this HF correction does coincide with the one obtained from an expansion of Eq. (3) up to terms $\sim O(\alpha^2)$. Following this logic, the RG (4) is essentially an effective method of solving the HF problem self-consistently.

In fact, however, while the HF theory is indeed a "logarithmic theory" in the sense that it can be expanded in a series of leading logarithms $a^n L^n$, the series is very much different from the RG solution (5), even to one-loop order. To demonstrate this, let us represent HF solutions for waves $\psi_\pm(x)$ which are incident on the scatterer from the left (+) and from the right (−) as $\psi_\pm(x) = a_{\pm,\pm}(x)e_+(x) + a_{\pm,\mp}(x)e_-(x)$, where $e_{\pm}(x) = \exp[\pm i(k_F + \epsilon/v_F)x]$, $a_{\mu\nu}(x)$ are amplitudes varying slowly on a scale of $k_F^{-1}$ for $|k_F|x \gg 1$ and satisfying the scattering boundary conditions $a_{\pm,\pm}(\infty) = a_{\pm,\mp}(\infty) = 1$, $a_{\pm,\mp}(\infty) = a_{\pm,\pm}(\infty) = 0$. Separating the slow and fast variables, the HF equations for $\epsilon_F/\epsilon | k_F | x \gg 1$ are written in the leading-log approximation as

\[
\begin{align*}
\partial a_{\pm,\pm}/\partial L_x &= (\alpha/2) \eta_{\pm,\pm}, \\
\partial a_{\pm,\mp}/\partial L_x &= (\alpha/2) \eta^{\mp}_{\pm,\pm},
\end{align*}
\]

(7) where $L_x = \ln|D_0/(v_F/e)|$ and the function $\eta(x)$ describes the envelope $n(x) = |\eta(x)|/2\pi|x|$ of the Friedel oscillations. The latter are given for $|k_F| x \gg 1$ by

\[
\begin{align*}
&\frac{1}{2\pi x} \text{Im} \{\eta(x)e^{2ik_Fx}\} = \\
&\frac{1}{\pi v_F} \sum_{\mu=\pm} \int_{-D_0}^0 \text{Re} \{\langle a^*_{\mu,\pm}a_{\mu,\mp}\rangle e^{-2i(\epsilon_F + \epsilon)x/v_F}\}. 
\end{align*}
\]

(8)

Solutions of Eqs. (7) for $x > 0$ and $x < 0$ are matched onto each other by means of the bare scattering matrix. Equivalently, Eqs. (7) can be cast in the form

\[
\partial \ln \eta/\partial L_x = \alpha I/2, \quad \partial I/\partial L_x = 2\alpha |\eta|^2
\]

(9) with $\eta = a_{\pm,\pm}^* + a_{-\mp}^* + a_{-\pm} + a_{\mp}^*$ and $I = \sum_{\mu\nu} |a_{\mu\nu}|^2$. The function $I(x)$ is a contribution of states close to the Fermi energy to a smooth part of electron density equations. Eqs. (7) can be solved exactly. The scattering amplitudes and the shape of the Friedel oscillations are then found self-consistently. However, the rather intricate HF solution appears to be a qualitatively wrong approximation in our problem of impurity scattering in an interacting 1D liquid. The difference between the RG and HF solutions is already seen for a weak impurity with $R_0 \ll 1$. While the RG gives for $|\epsilon| \gg D_r$ a power-law behavior of the reflection amplitude

\[
r_{L,R} = r_{L,R,0} e^{\alpha L}, \tag{10}
\]

the HF expansion yields a geometric progression

\[
r_{L,R} = r_{L,R,0} \frac{1}{1 - \alpha L}. \tag{11}
\]

Equation (11) can be obtained also by summing up ladder diagrams in the particle-hole channel. Note a pole characteristic to HF approaches.

We thus see that even for weak interaction the HF solution correctly describes the scattering by a screened impurity only for $|\epsilon| \gg D_r$, where it gives a small perturbative correction $\sim \alpha L$. At the next order in $\alpha$, the HF expansion does generate a term $\sim r_{L,R,0} \alpha^2 L^2$ in $r_{L,R}$, but, in contrast to Ref. 30, the numerical coefficient in front of it is a factor of 2 larger than in Eq. (3), which signifies a breakdown of the HF approach beyond the simplest Born approximation.

It is worth noting that in Ref. 32 another HF-type ("non-selfconsistent HF") scheme was employed to substantiate the RG procedure. As mentioned above, single-electron wave functions within the self-consistent HF approach obey $\psi_q = \psi_q^0 + \hat{C}_q^{0} U_{HF}\psi_q$, where $\psi_q^0$ and the Green’s function operator $\hat{C}_q^{0}$ describe noninteracting electrons. Let us write the correction to the transmission amplitude of second order in $\alpha L$ as $\delta\tilde{t}(\epsilon) = -\delta t_0 R_0 (\alpha L)^2$. The self-consistent HF approximation gives $\tilde{C} = T_0 + 1/2$, different from $C = (3T_0 - 1)/2$ following from the RG expansion (3). In Ref. 32 only $U_{HF}$ is renormalized while $\psi_q$ remain unperturbed, i.e., $\psi_q \to \psi_q^0$ in the right-hand side of the above HF equation. This yields $C = 1$, which accidentally coincides with the RG result for a weak impurity. However, for arbitrary $T_0$ this approach can be seen to fail as well, already at order $O(\alpha^2 L^2)$.

It follows that the RG (4) in fact cannot be obtained from the above HF schemes, i.e., the RG is not a method of solving the HF equations. There are other scattering processes, not captured by the HF approximation, that at higher orders in $\alpha$ almost compensate the HF result: a comparison of Eqs. (10), (11) says that the coefficients $c_n$ in the corresponding sums $\sum_n c_n (\alpha L)^n$ are different in the RG and HF expansions by a factor of $n!$.

One can see that the HF approach misses important scattering processes also by noticing that Friedel oscillations in a Luttinger liquid decay in the limit of large $|x| \gg v_F/D_r$ as $|x|^{-1+\alpha}$, more slowly than the repulsive interaction) than in a Fermi liquid. The Bragg reflection by a potential created by the Friedel oscillations can then be easily seen to yield a power-law behavior of the transmission coefficient only if $\alpha$ is put equal to zero in the exponent of the Friedel oscillations. For any $\alpha > 0$ the Bragg reflection leads to an exponential decay of the wave functions, $|\psi| \propto -|x|^\alpha$, instead of a power law required by self-consistency and by Eq. (3).

We can summarize the above observations by stating that, although it is certainly appealing to think of the RG as being associated with scattering off Friedel oscillations, the mechanism of the RG is more complicated. In fact, in a closely related calculation of various correlation functions of a clean Luttinger liquid without impurities the inapplicability of the HF theory was realized long ago. It is the interaction in a Cooper (particle-particle) channel that interferes with the HF interaction. As a result, instead of the summation of a
HF ladder one has to use a much more involved parquet technique.\cite{42,43,45} In the present problem, the parquet summation is equivalent to the RG (I). A rigorous microscopic justification of the RG (I) can be done by using Ward identities in a diagrammatic approach and will be given elsewhere.\cite{26} Our purpose in Sec. (III) is to extend the RG (I) to the case of two impurities.

### III. DOUBLE BARRIER

Consider now two potential barriers located at \( x = 0 \) and \( x = x_0 \) and let the distance \( x_0 \) be much larger than the width of each of them. Clearly, the spatial structure itself yields an energy dependence of the (describing scattering on both impurities) amplitudes \( t(\epsilon) \) and \( r_{L,R}(\epsilon) \), even if such a dependence may be neglected for each of the impurities, as was assumed in the derivation of Eqs. (I). Specifically, without interaction the energy \( \Delta = \pi v_F / x_0 \) gives a period of oscillations in the total scattering amplitudes with changing Fermi level. If the impurities are strong, \( \Delta \) is the level spacing inside a quantum dot formed by the barriers and the period of resonant tunneling oscillations. It follows that an RG description of scattering off a double barrier requires a generalization of the RG\cite{26,28} to the case when the bare amplitudes \( t_0(\epsilon) \) and \( r_{L,R,0}(\epsilon) \) are energy dependent.

A question, however, arises if it is at all possible to construct the RG theory for a compound scatterer in terms of only total \( S \)-matrix, as in Eqs. (I). Put another way, the question is if total scattering amplitudes generated by RG transformations are expressed in terms of themselves only. As we will see below, the answer depends on the parameter \( \Delta / D_0 \). We recall that \( D_0 \) in our problem is the smallest of two energy scales given by \( \epsilon_F \) and \( v_F / d \), where \( d \) is the radius of interaction. If \( \Delta \ll D_0 \), so that there are many resonances within the band \((-D_0,D_0)\), the RG transformations generate more terms than are encoded in the total \( S \)-matrix. This is a generic case we are interested in. On the other hand, if \( \Delta \gg D_0 \) (e.g., when \( d \gg x_0 \)), one would have from the very beginning a model of a single impurity with no spatial structure but with possibly energy dependent scattering amplitudes. In that case, it is sufficient to deal with total amplitudes only (see Secs. (III)(III)). For the case of a single resonance this is a model studied, e.g., in Ref. (I7) (for an exactly solvable case of the Luttinger liquid parameter \( g = 1/2 \)) and Ref. (I3) (for a weak interaction, \( 1 - g \ll 1 \)).

#### A. Perturbative expansion

Building on our knowledge of the single impurity case, we start the derivation of a double barrier RG with a calculation of perturbative in \( \alpha \) corrections to a time-ordered single-particle Green's function \( G_{\mu\nu}(x,x') = -i \langle T \psi_\mu(x) \psi_\nu^\dagger(x') \rangle \), where \( x = (x,t) \) and \( \mu, \nu = \pm \) label two branches of right (\(+\)) and left (\(-\)) movers. The bare Green's function is characterized by bare scattering amplitudes \( t_0(\epsilon) \), \( r_{L,R,0}(\epsilon) \). To first order in \( \alpha \), transforming to the \((x,\epsilon)\)-representation, which is most convenient in the present context, we have a correction to \( G_{\mu\nu}(x_1,x_1;\epsilon) \) (summation over branch indices assumed):

\[
\delta G_{\mu\nu}(x_1,x_1;\epsilon) = \int_{-\infty}^{\infty} dx \ G_{\mu\nu}(x,x_1;\epsilon) \times \sum_{\mu',\nu'}(x) G_{\mu',\nu'}(x_1,x;\epsilon) ,
\]

(12)

where

\[
\Sigma_{\mu\nu}(x) = i \alpha \int_{-D_0}^{D_0} d \epsilon \ G_{\mu\nu}(x,x;\epsilon) .
\]

In the Luttinger liquid model, only forward scattering due to interaction is present: we explicitly assume this in Eqs. (I2, I3) by writing the self-energy that depends on two branch indices only [backward scattering can be straightforwardly incorporated for spinless electrons, see Eq. (I2)]. As explained in Sec. (III), we also assume in Eq. (I2) that the interaction is effectively short-ranged and write the self-energy as a spatially local quantity. Indeed, the perturbative logarithmic correction to \( G_{\mu\nu}(x_1,x_1;\epsilon) \) comes from energies \( |\epsilon| \lesssim v_F / d \), i.e., from spatial scales \( |x| \) larger than \( d \). We start with the case \( d \ll x_0 \) (we will return to the simpler case \( d \gg x_0 \) in Sec. (III)). Moreover, in the following, we neglect the forward scattering of electrons belonging to the same branch: to one-loop order, such processes can be seen to yield only a non-singular renormalization of the bare parameters. The forward-scattering interaction that generates RG transformations similar to Eqs. (I) is that of electrons from different branches.\cite{26} We thus formulate an impurity problem with only non-diagonal couplings \( \Sigma_{+}(x) \) and \( \Sigma_{+}(x) \).

The scattering amplitudes are related to \( G_{\mu\nu}(x_1,x_1;\epsilon) \) for \( \epsilon > 0 \) (we measure \( \epsilon \) from the Fermi level upwards) as

\[
r_{L}(\epsilon) = iv_F G_{-+}(x_1,x_1;\epsilon)e^{-i\epsilon(x_1+x_0)/v_{F}\left|x_1,x_1\to-\infty\right|} ;
\]
\[
r_{R}(\epsilon) = iv_F G_{++}(x_1,x_1;\epsilon)e^{i\epsilon(x_1-x_0)/v_{F}\left|x_1,x_1\to+\infty\right|} ;
\]
\[
t(\epsilon) = iv_F G_{++}(x_1,x_1;\epsilon)e^{-i\epsilon(x_1-x_0)/v_{F}\left|x_1,x_1\to+\infty\right|} ,
\]

(14)

and similarly for \( \epsilon < 0 \) by changing \( \pm \to \mp \) in branch indices of \( G_{\mu\nu} \). We count the phases of the reflection amplitudes from \( x = 0 \). Since the integral over \( x \) in Eq. (I2) involves integration over the interval of \( x \) inside the dot, \( 0 < x < x_0 \), corrections \( \delta t(\epsilon) \), \( \delta r_{L,R}(\epsilon) \) cannot be expressed solely in terms of the bare amplitudes \( t_0(\epsilon) \) and \( r_{L,R,0}(\epsilon) \). A closed set of equations can be written by introducing amplitudes to stay inside the dot

\[
A_{\mu,-\nu}(\epsilon) = iv_F G_{\mu,-\nu}(x,x;\epsilon)e^{2i\mu x} ,
\]

(15)

(we need only nondiagonal amplitudes \( A_{\mu,-\mu} \)), amplitudes to escape from the dot, to the left or to the right,

\[
d^{+}_{\mu}(\epsilon) = iv_F G_{\mu,-\mu}(x_1,x;\epsilon)e^{-i\epsilon(x_1-x_0)}\left|x_1\to\pm\infty\right| ,
\]

(16)
and, similarly, amplitudes to get into the dot from outside

\[ b_{\mu}^\pm(\epsilon) = i\nu_F G_{\pm,\mu}(x, x; \epsilon)e^{i\epsilon(x_1 \mp x_2)}|_{x_1 \to \pm \infty} . \]  

(17)

The amplitudes are constrained by unitarity and, moreover, \( b_{\mu}^\pm(\epsilon) = d_{\mu,\mu}^\pm(\epsilon) \) by time reversal symmetry which we assume throughout the paper. In Eqs. \( 16 \)–\( 17 \), \( x \) lies within the dot. Without interaction, the amplitudes do not depend on \( x \). This property is preserved in the leading-log approximation.

The first-order corrections to \( \delta t(\epsilon) \) and \( r_L(\epsilon) \) read

\[
\delta t(\epsilon) = -\frac{\alpha}{2} \int_{-\Delta_0}^{\Delta_0} \frac{d\epsilon'}{\epsilon - \epsilon'} \left\{ L_+(\epsilon, \epsilon') + \theta(-\epsilon') t(\epsilon)[r_R(\epsilon) r_L^* (\epsilon') \chi_{\epsilon - \epsilon'} + r_L(\epsilon) r_R^*(\epsilon')] \right\},
\]

\[
\delta r_L(\epsilon) = -\frac{\alpha}{2} \int_{-\Delta_0}^{\Delta_0} \frac{d\epsilon'}{\epsilon - \epsilon'} \left\{ L_-(\epsilon, \epsilon') + \theta(\epsilon') r_L(\epsilon') \right\},
\]

and similar equations can be written for corrections to \( r_R(\epsilon) \) and the amplitudes defined in Eqs. \( 15 \)–\( 17 \). Here \( \theta(\epsilon) \) is the step function and

\[ \chi_\epsilon = \exp(2\pi i \epsilon/\Delta) . \]

(20)

The terms \( L_{\mu}(\epsilon, \epsilon') \), which correspond to the integration over \( x \) in \( 12 \) inside the dot, are given by

\[
L_{\mu}(\epsilon, \epsilon') = b_{\mu}^+ A_{+-}(\epsilon') d_{\mu,\mu}(\epsilon)(\chi_{\epsilon - \epsilon'} - 1) + b_{\mu}^- A_{-+}(\epsilon') d_{\mu,\mu}^*(\epsilon)(1 - \chi_{\epsilon - \epsilon'}) .
\]

(21)

The amplitudes to stay inside the dot satisfy the relation \( A_{\mu,\mu}(\epsilon) = -A_{\mu,\mu}^*(\epsilon) \) and it is useful to decompose them as

\[ A_{\mu,\mu}(\epsilon) = \theta(\epsilon) B_{\mu,\mu}(\epsilon) + \theta(-\epsilon) C_{\mu,\mu}(\epsilon) . \]

(22)

We thus arrive at a closed system of perturbative equations written in terms of quantities describing the dot as a whole, without directly referring to parameters characterizing two barriers.

### B. Renormalization group for a double barrier

Generically (singular exceptions are discussed below), the above first-order corrections diverge logarithmically as \( \epsilon \to 0 \), which implies that higher orders of the perturbation theory are important. Naively, one could try to treat Eqs. \( 16 \)–\( 19 \) and other perturbative equations self-consistently, i.e., not as expressions for small corrections \( \delta t(\epsilon) \), etc., but as equations to be solved for \( t(\epsilon) = t_0(\epsilon) + \delta t(\epsilon) \), etc., where \( \delta t(\epsilon) \) is not necessarily small. However, this would not correctly describe the case of strong impurities, either the case of a dot formed by two barriers or even that of a single strong barrier. To see this in the simpler case of a single impurity, take the limit \( x_0 \to 0 \). The terms \( L_\mu(\epsilon, \epsilon') \) and factors \( \chi_\epsilon \) then drop out in Eqs. \( 16 \)–\( 19 \) and the self-consistent equations acquire the form

\[
t(\epsilon) = t_0(\epsilon) + \frac{\alpha}{2} \int_{-\Delta_0}^{\Delta_0} \frac{d\epsilon'}{\epsilon'} \left[ r_R(\epsilon) r_L^* (\epsilon') + r_L(\epsilon) r_R^*(\epsilon') \right],
\]

\[
r_L(\epsilon) = r_{L0}(\epsilon) + \frac{\alpha}{2} \int_{|\epsilon|}^{\Delta_0} \frac{d\epsilon'}{\epsilon'} r_L(\epsilon'),
\]

and

\[
\delta r_L(\epsilon) = \frac{\alpha}{2} \int_{-\Delta_0}^{\Delta_0} \frac{d\epsilon'}{\epsilon'} \left[ t^2(\epsilon) r_R^*(\epsilon') + r_L^2(\epsilon) \right].
\]

As a comparison of Eqs. \( 16 \)–\( 19 \) and \( 20 \)–\( 21 \), shows, the latter do not describe two impurities since they miss the terms \( L_\mu(\epsilon, \epsilon') \) and the factors \( \chi_\epsilon \). Moreover, Eqs. \( 20 \)–\( 21 \) do not describe even a single strong structureless (with no dependence of the bare amplitudes on \( \epsilon \)) impurity. This can be checked straightforwardly, e.g., by putting all reflection coefficients in the integrand of Eq. \( 20 \) equal to unity: in this limit the integration over \( \epsilon' \) in Eq. \( 20 \) gives a geometric progression of logarithms, instead of reproducing Eq. \( 19 \) for \( t(\epsilon) \). Also, plugging in Eq. \( 19 \) into the integrand of Eq. \( 20 \) can be easily seen to give a logarithmic divergence at \( \epsilon \to 0 \), instead of a power law. The logarithmic divergencies are characteristic to HF approaches discussed in Sec. \( 11 \). Equations \( 20 \)–\( 21 \), however, do describe correctly a weak structureless impurity, but only for \( |\epsilon| \gg D_\epsilon \).

We now derive nonperturbative amplitudes for a double barrier using an appropriate RG scheme. To account for the \( \epsilon \) dependence of the bare amplitudes, the derivation of the RG from the perturbative results \( 16 \)–\( 19 \) necessitates introduction of two energies, \( \epsilon \) and \( D \). The latter is a flow parameter in RG transformations, i.e., an ultraviolet cutoff rescaled after tracing over states with energies \( \epsilon' \) in the interval \( |\epsilon'| \in (D, D_0) \). The renormalization stops at \( D = \max(|\epsilon|, T) \).

The essence of the RG procedure \( 20 \)–\( 21 \) is a perturbative treatment of contributions to the renormalized amplitudes at energy \( \epsilon \) from all states with energies \( \epsilon' \) in the interval \( |\epsilon'| \in (D, D_0) \), starting from \( D = D_0/\Lambda \), such that \( \Lambda \gg 1 \) but \( \alpha \ln \Lambda \ll 1 \). The RG equations thus differ from both the HF equations and Eqs. \( 20 \)–\( 21 \) in that all the amplitudes depend on \( D \) and, moreover, the HF-type integration over projected states with energies \( \epsilon' \) only goes over the interval \( |\epsilon'| \in (D, D_0) \) instead of \( (0, D_0) \).

In effect, each step of the RG transformations accounts for the scattering off the Friedel oscillations in a finite spatial region, \( |x|, |x - x_0| \in (v_r/\Lambda D, v_r/D) \). Moreover, the Friedel oscillations are only partly modified, through the (already performed) renormalization of the reflection amplitudes at energies larger than \( D \). At the same time, the scattering matrix at energies smaller than \( D \) is taken at its bare value. This should be contrasted with the HF approach, where the scattering amplitudes are determined by interaction processes on all energy scales on every step of the HF iterations.
The system of one-loop RG equations for a double barrier reads
\[
\frac{\partial t(\epsilon, D)}{\partial L_D} = \hat{I}_\epsilon(\epsilon, D) \left\{ L_+(\epsilon, \epsilon'; D) + \theta(-\epsilon') t(\epsilon, D) [r_R(\epsilon, D) r_R^*(\epsilon', D) \chi_{\epsilon-\epsilon'} + r_L(\epsilon, D) r_L^*(\epsilon', D)] \right\},
\]
\[
\frac{\partial r_L(\epsilon, D)}{\partial L_D} = \hat{I}_\epsilon(\epsilon, D) \left\{ L_-(\epsilon, \epsilon'; D) + \theta(\epsilon') r_L(\epsilon', D) \right\}
\]
and similar equations for other amplitudes. Here \(L_D = \ln(D_0/D)\) and we introduced \(D\) dependent amplitudes \(t(\epsilon, D)\), etc., so that integration of Eqs. (25, 26) over \(D\) acts on all the amplitudes [which should be contrasted with Eqs. (24, 25), where the integration acts on only one amplitude in the products of three]. All amplitudes in \(L_{\nu}(\epsilon, \epsilon')\) are now also functions of \(D\). The integral operator \(\hat{I}_\epsilon(\epsilon, D)\) is defined as
\[
\hat{I}_\epsilon(\epsilon, D) = -\frac{\alpha}{2\ln\Lambda} \int^{A_D}_D \int_{-A_D}^D \frac{d\epsilon'}{\epsilon - \epsilon'} \{ \ldots \},
\]
where \(\Lambda \gg 1\) is restricted by the condition \(\alpha \ln \Lambda \ll 1\). The theory is renormalizable if the action of the operator \(I_\epsilon(\epsilon, D)\) to leading order is independent of \(\Lambda\), which is the case for the present problem within the leading-log approximation. Needless to say, in the limit \(x_0 \to 0\) [i.e., \(L_{\nu}(\epsilon, \epsilon') \to 0\) and \(\chi_{\epsilon-\epsilon'} \to 1\)] Eqs. (24, 25) describe a single impurity.

At finite temperature \(T\), one should substitute the Fermi distribution function \(n_F(\epsilon)\) for the step functions in Eqs. (24, 25) and also in Eq. (22) according to \(\theta(\pm \epsilon) \to n_F (\mp \epsilon)\). The factor \((\epsilon - \epsilon')^{-1}\) in Eq. (26) effectively stops the renormalization at \(D \sim |\epsilon|\), while the factors \(n_F(\pm \epsilon')\) do so at \(D \sim T\), otherwise the renormalization can be carried out down to \(D = 0\). The infrared cutoff at \(D \sim T\) establishes a characteristic spatial scale of \(L_T = v_F T\). Due to the thermal smearing, the Friedel oscillations decay exponentially on a scale of \(L_T\) which can be seen from Eq. (8) if one plugs in \(n_F(\epsilon)\) into the integrand and extends the integration to \(\epsilon > 0\). The absence of renormalization at \(D \ll T\) is closely related to the thermal cutoff of interaction-induced corrections to the conductance in higher dimensions.

The RG equations (25, 26) should be solved with proper boundary conditions at \(D = D_0\), where the renormalization starts from the bare values of the amplitudes (13, 14). For the double-barrier problem, the boundary conditions for the transmission and reflection amplitudes are:
\[
t_0(\epsilon) = \frac{t_1 t_2}{S(\epsilon)}, \quad r_{L0}(\epsilon) = r_1 + \frac{r_2 t_2^2}{S(\epsilon)} \chi_\epsilon, \quad S(\epsilon) = 1 - r_2 r'_1 \chi_\epsilon, \quad (29)
\]
and \(r_{R0}(\epsilon) = -r_1^* t_0(\epsilon)/t_1^* (\epsilon)\) by unitarity. The coefficients \(r_{L2}, r_{R1}\) are the noninteracting reflection amplitudes from the left (right) and \(t_{1,2}\) are the transmission amplitudes of each of the two barriers, respectively. Similarly for other amplitudes:
\[
d_+^+(\epsilon) = \frac{d_+^+(\epsilon)}{r_1^+} = \frac{t_0(\epsilon)}{t_1^+} = \frac{t_0(\epsilon)}{t_1^+} = \frac{t_0(\epsilon)}{t_2^+}, \quad (30)
\]
\[
B_{++}(\epsilon) = \frac{t_0(\epsilon)}{t_1^+} \chi_\epsilon, \quad C_{++}(\epsilon) = -\frac{t_0^* (\epsilon)^{r'-1}}{t_1^+}.
\]
We count the phases of \(r_{L,R}^*\) from \(x = 0\) and the phases of \(r_{1,2}\) are defined for an impurity sitting at \(x = 0\).

C. Separate renormalization of two impurities: \(D \gg \Delta\)

We are now in a position to solve the system of RG equations (24, 25) by integrating out all states with energies \(|\epsilon'| \gtrsim 2 \max|\epsilon, T|\). We begin with the case \(D_0 \gg \Delta\), which is a typical case unless interaction is very long ranged. We proceed in two steps. Let us first integrate over \(D \gg \Delta\). This can be done for arbitrary \(\epsilon\). Specifically, if \(|\epsilon| \gtrsim \Delta\), this will already solve the problem by providing us with fully renormalized amplitudes.

In the more interesting case of \(|\epsilon| \lesssim \Delta\), we will only sum up contributions to the renormalized amplitudes from states with \(|\epsilon'| \lesssim \Delta\) and, as a second step, will have to proceed with renormalization for \(D \lesssim \Delta\).

Since the renormalization for \(D \gg \Delta\) involves many resonant levels, the amplitudes contain slowly varying parts and parts oscillating rapidly with changing \(\epsilon'\) on a scale of \(\Delta\). Integration over \(\epsilon'\) in Eq. (27) allows us to separate the slow and fast variables: as a result, the dependence of the amplitudes on \(D\) will be slow on the scale of \(\Delta\). However, even after that the RG equations look rather cumbersome. To construct the solution to these equations, note that an important parameter \(D/D_{\text{min}}\) is available, where
\[
D_{\text{min}} = \min\{D_{r_1}, D_{r_2}\} \quad (31)
\]
and \(D_{r_{1,2}}\) are defined for each of two barriers by Eq. (4). If both barriers are initially (i.e., at \(D = D_0\)) strong (\(|t_{1,2}| \ll 1\)), then this parameter is small for all \(D < D_0\). However, if one or both of the barriers are initially weak, there is a range of \(D \in (D_{\text{min}}, D_0)\) where at least one barrier still remains weak.

It is useful first to examine some general properties of integrals over \(\epsilon'\) that appear in the course of renormalization. Let us return to the perturbative expansion
We see that the averaging over \( \epsilon' \) involves two types of integrals

\[
\mathcal{I}_1 = \int_{|\epsilon'|}^{D_0 \, d\epsilon'} \frac{1}{\epsilon' \, S(\epsilon')} \, d\epsilon', \quad \mathcal{I}_2 = \int_{|\epsilon'|}^{D_0 \, d\epsilon'} \frac{\chi_{\epsilon'}}{\epsilon' \, S(\epsilon')} \, d\epsilon',
\]

(32)

where \( S(\epsilon) \) is given by Eq. (29) and \( \mathcal{I}_1, \mathcal{I}_2 \) are related by \( \mathcal{I}_2 = (\mathcal{I}_1 - \mathcal{L}) / r_1 r_2 \). The integrals (32) are evaluated in different ways depending on whether at least one of the barriers is weak (so that \(|r_1 r_2| \ll 1\) or both barriers are strong (\(|r_1 r_2| \simeq 1\)). In the former case the integrand of \( \mathcal{I}_1 \) is only slightly modulated, so that one can expand the factor \( S^{-1}(\epsilon') \) and average over harmonics \( \chi_{\epsilon'}(\epsilon') \). Then only zero harmonics give rise to singular (logarithmic) corrections and in the leading-log approximation we have

\[
\mathcal{I}_1 = \mathcal{L}, \quad \mathcal{I}_2 = 0.
\]

(33)

In the opposite case of strong barriers, sharp resonances appear that are described by a Breit-Wigner formula for \( S^{-1}(\epsilon') \) and yield \( |\mathcal{I}_1| \simeq |\mathcal{I}_2|\):

\[
\mathcal{I}_1 = \mathcal{L}/2, \quad \mathcal{I}_2 = -\mathcal{L}/2 r_1 r_2.
\]

(34)

The situation repeats itself in the RG equations (25), (26). The difference in the factor of 1/2 between the values of \( \mathcal{I}_1 \) in Eqs. (33), (34) implies that the renormalization should be carried out differently (see Fig. 2) in the regions \( D \gg D_{\text{min}} \) and \( \Delta \ll D \ll D_{\text{min}} \).

For \( D \gg D_{\text{min}} \), similarly to Eq. (33), after the averaging over \( \epsilon' \) only zero harmonics contribute to the renormalization. The solution at \( D \gg D_{\text{min}} \) is then simple: it has the form of Eqs. (28), (30) with the reflection and transmission amplitudes of each of two barriers renormalized separately (Fig. 2a), according to the RG (11) for a single impurity:

\[
\frac{\partial t_{1,2}(D)}{\partial D} = -\alpha t_{1,2}(D)|r_{1,2}(D)|^2, \quad \frac{\partial r_{1,2}(D)}{\partial D} = \alpha r_{1,2}(D)|t_{1,2}(D)|^2.
\]

(35), (36)

One can check straightforwardly that Eqs. (28), (30) describing the Fabry-Perot resonance with the replacement

\[
t_{1,2} \rightarrow \frac{(D/D_0)^\alpha t_{1,2}}{[|r_{1,2}|^2 + (D/D_0)^{2\alpha}|t_{1,2}|^2]^{1/2}}, \quad r_{1,2}(r_{1,2}') \rightarrow \frac{r_{1,2}(r_{1,2}')}{[|r_{1,2}|^2 + (D/D_0)^{2\alpha}|t_{1,2}|^2]^{1/2}}
\]

(37), (38)
solve Eqs. (28), (30) averaged over harmonics for \( D \gg \max\{D_{\text{min}}, \Delta\} \).

On the other hand, if \( D_{\text{min}} \gg \Delta \), there is an interval of \( D \in (\Delta, D_{\text{min}}) \) in which each of the impurities is strongly reflecting (Fig. 2b), so that the averaged equations are again simplified by summing over resonance poles, similarly to Eq. (31). We get an independent renormalization of \( t_{1,2}(D) \) according to

\[
t_{1,2}(D)/t_{1,2}(D_{\text{min}}) = (D/D_{\text{min}})^{\alpha/2},
\]

(39)

and the scaling exponent is now half that for \( D \gg D_{\text{min}} \). We thus have two solutions given by Eqs. (37), (39) and Eq. (32), respectively, that match onto each other at \( D \sim D_{\text{min}} \). Due to the slow power-law dependence, for \( \alpha \ll 1 \) the matching is exact.

![Fig. 2: Sketch of different stages of the RG procedure for a double barrier for \( D \gg \Delta \gg T \), Eqs. (35) - (38). The Friedel oscillations shown schematically around each of the barriers yield a renormalization of the scattering amplitudes and are renormalized themselves as well. At energy \( \epsilon \), the renormalization comes from spatial scales smaller than \( L_\epsilon = v_F/|\epsilon| \) (these regions are marked by the solid lines). (a) Separate renormalization of two weak barriers. Since there is no energy quantization between the barriers, the Friedel oscillations both inside and outside the dot contribute to the renormalization and the barriers “do not talk to each other”. The scaling exponent \( \alpha \) is the same as for a single barrier. Sec. (b) Separate renormalization of two strong barriers. This figure describes either initially strong barriers or those that are initially weak but both become strongly reflecting due to the renormalization. The inner part of the dot with a discrete energy spectrum does not contribute to the renormalization. The renormalization of each of the barriers is governed by the exponent \( \alpha/2 \), twice as small as for weak barriers.](image)
energies near resonances allows for another formulation which involves \( t(\epsilon) \) and \( r_{L,R}(\epsilon) \) only, we will return to this issue in Sec. III D.

If \( \epsilon \) is close to one of resonant energies, Eqs. (25), (26) can be further simplified by expanding \( \chi \), near the resonance: the renormalized amplitudes for strong barriers take then the form of Breit-Wigner amplitudes with \( D \) dependent widths \( \Gamma_{1,2}(D) = (\Delta/2\pi)|t_{1,2}(D)|^2 \alpha D^\alpha \). Specifically, for initially strong barriers:

\[
\Gamma_{1,2}(D) = \frac{\Delta}{2\pi} \left| t_{1,2} \right|^2 \left( \frac{D_{D_0}}{D} \right)^\alpha, \tag{40}
\]

where \( \left| t_{1,2} \right|^2 \ll 1 \) are the bare transmission probabilities at \( D = D_0 \) and resonant peaks are sharp [i.e., \( \Gamma_{1,2}(D) \ll \Delta \)] for all \( D < D_0 \). If at least one barrier is initially weak, the resonant structure develops only at \( D \ll D_{\text{res}} \). Provided one barrier is initially weak (assume this is the right barrier and \( D_{D_2} \gg \Delta \)), whereas the other is strong, then

\[
\Gamma_1(D) = \frac{\Delta}{2\pi} \left| t_1 \right|^2 \left( \frac{D_{D_{r_1}}}{D} \right)^\alpha, \tag{41}
\]

\[
\Gamma_2(D) = \frac{\Delta}{2\pi} \left( \frac{D}{D_{r_2}} \right)^\alpha. \tag{42}
\]

If \( \Delta \ll D_{D_2} \ll D_{r_1} \ll D_0 \), i.e., both barriers are initially weak, then

\[
\Gamma_1(D) = \frac{\Delta}{2\pi} \left( \frac{DD_2}{D_{D_1}} \right)^\alpha \tag{43}
\]

and \( \Gamma_2(D) \) is given again by Eq. (12).

To summarize this section, substituting \( D \rightarrow |\epsilon| \) in Eqs. (37), (38), (39) and using the Fabry-Perot equations (28), gives the fully renormalized scattering amplitudes for \( |\epsilon| \gtrsim \Delta \) if \( |\epsilon| \ll T \). Also, if \( T \gg \Delta \), substituting \( D \rightarrow T \) solves the problem for arbitrary \( \epsilon \). However, when both \( |\epsilon|, T \ll \Delta \), we should proceed with the renormalization in the range \( D \ll \Delta \).

D. Single resonance: \( D \ll \Delta \)

Let us now consider Eqs. (25), (26) for \( |\epsilon|, |\epsilon'| \ll \Delta \). In this limit, the terms (21) containing the amplitudes \( A_{\mu,\mu'}(\epsilon') \) to stay inside the dot become irrelevant in the RG sense: the phase factors \( \chi_{\epsilon,\epsilon'} \) in Eq. (21) can be expanded about \( \epsilon, \epsilon' = 0 \), which leads to the cancellation of the singular factor \( (\epsilon - \epsilon')^{-1} \) in Eq. (27). As a result, the terms \( L_{\mu}(\epsilon, \epsilon'; D) \) do not contribute to the renormalization at \( D \ll \Delta \). The factors \( \chi_{\epsilon,\epsilon'} \) should also be omitted in the terms of Eqs. (25), (26) that are proportional to \( r_{L,R}^*(\epsilon') \). Thus we are led to a coupled set of RG equations that describe also a single impurity with energy dependent scattering amplitudes: the spatial structure of the double barrier system is of no importance for the renormalization at \( D \ll \Delta \) (see Fig. 3). However, the boundary conditions in the double-barrier case should be written at \( D \sim \Delta \), instead of \( D \sim D_0 \). We obtain for \( |\epsilon|, |\epsilon'| \ll \Delta \), \( D \ll \Delta \):

\[
\frac{\partial t(\epsilon, D)}{\partial \mathcal{E}_D} = -\frac{\alpha}{2} t(\epsilon, D) \left[ r_R(\epsilon, D) \overline{r_R(D)} + r_L(\epsilon, D) \overline{r_L(D)} \right], \tag{44}
\]

\[
\frac{\partial r_L(\epsilon, D)}{\partial \mathcal{E}_D} = -\frac{\alpha}{2} \left[ \overline{r_L(D)} - t^2(\epsilon, D) \overline{r_R(D)} \right], \tag{45}
\]

and similarly for \( r_R(\epsilon, D) \), where the bar over the reflection amplitudes denotes the averaging (27) over \( \epsilon' \).

![FIG. 3: Illustration of the renormalization of a double barrier for \( D \ll \Delta \). Eqs. (44), (45). The Friedel oscillations are shown schematically within the range \( L_\alpha = v_F/|\epsilon| \) outside the dot, where they contribute to the renormalization of the scattering amplitudes at energy \( \epsilon \) (provided \( |\epsilon| \gg T \)). The double barrier for small \( D \) may be considered as a single barrier with an energy dependent scattering matrix, which describes the resonance (sketched by the dotted line) on a single level. The scattering matrix inside the renormalized resonance is discussed in Sec. III E.](image)

We integrate now Eqs. (44), (45) assuming that each of two barriers is characterized by \( D_{r_1,2} \gg \Delta \). This condition means that either the barriers are strong initially at \( D = D_0 \) or get strong in the course of renormalization (35), (36) before \( D \) equals \( \Delta \). We will analyze the case of both or one of \( D_{r_1,2} \) being smaller than \( \Delta \) in Sec. III E.

Thus, if \( D_{r_1,2} \gg \Delta \), we have narrow peaks of resonant transmission in a neighborhood of the Fermi level, even if the bare reflection coefficients are small. Consider first the case of a resonance energy \( \epsilon_0 \) lying exactly on the Fermi level, \( \epsilon_0 = 0 \). Let \( \Gamma(D) \) be a renormalized width of the resonance peak at the Fermi energy (to be found below). If \( D \gg \Gamma(D) \), then \( |\overline{r_L} \overline{r_R}(D)| \simeq 1 \), which allows for a significant simplification of Eqs. (44), (45). It is convenient to introduce phase-shifted amplitudes \( \hat{r}_L = r_L e^{-i\varphi_{r_1}} \), \( \hat{r}_R = r_R e^{-i\varphi_{r_2}} e^{2\pi i(\epsilon_F + \epsilon_0)/\Delta} \).
where \( \varphi_{r_1} \) is the phase of \( r_1 \), etc., in obvious notation. Then we get, by putting the averaged amplitudes far from the resonance \( \tilde{r}_{L,R}(D) = 1 \):

\[
\frac{\partial \tilde{t}(\epsilon,D)}{\partial \epsilon} = -\alpha \tilde{t}(\epsilon,D) \left[ \tilde{r}_L(\epsilon,D) + \tilde{r}_R(\epsilon,D) \right], \tag{46}
\]

\[
\frac{\partial \tilde{r}_L(\epsilon,D)}{\partial \epsilon} = \frac{\alpha}{2} \left[ 1 - \tilde{r}_L^2(\epsilon,D) - \tilde{t}^2(\epsilon,D) \right], \tag{47}
\]

with the following solutions

\[
\tilde{t}(\epsilon,D) = \frac{[u_-^2(D) - u_+^2(D)]^{1/2}}{u_+^2(D) + 2i\epsilon}, \tag{48}
\]

\[
\tilde{r}_L(\epsilon,D) = \frac{u_+(D) + 2i\epsilon}{u_+(D) + 2i\epsilon}, \tag{49}
\]

\[
\tilde{r}_R(\epsilon,D) = -\frac{u_-(D) + 2i\epsilon}{u_+(D) + 2i\epsilon}, \tag{50}
\]

where

\[
u_\pm(D) = \Gamma_\pm(\Delta)/(D/\Delta)^\alpha, \tag{51}
\]

and \( \Gamma_\pm(\Delta) = \Gamma_1(\Delta) \pm \Gamma_2(\Delta) \) should be found by matching onto Eqs. (48)–(50). The width of a resonant tunneling peak \( \Gamma(D) \) is thus given by \( u_+(D) \).

Note that the only condition we have assumed in the above derivation is \( D \gg \Gamma(D) \) with \( D = \max\{|\epsilon|,T\} \), otherwise \( \epsilon \) in Eqs. (48)–(50) may be arbitrary. Thus, Eqs. (48)–(50) give the shape of the \( \epsilon \) dependence of fully renormalized amplitudes for the case of temperature \( T \gg \Gamma(T) \) (with \( T \) substituted for \( D \)). In particular, Eq. (51) says that the width of the resonance behaves as \( T^\alpha \):

\[
\Gamma(T) = \Gamma_+(\Delta)(T/\Delta)^\alpha. \tag{52}
\]

As follows from Eq. (48), while the resonance becomes sharper with decreasing \( T \), the peak value of the transmission amplitude is not renormalized (see Fig. 4), since the \( D \) dependent factors cancel in Eq. (48) at \( \epsilon = 0 \). The absence of renormalization stems from the vanishing of the sum \( \tilde{r}_L(\epsilon,D) + \tilde{r}_R(\epsilon,D) \) in Eq. (48) at \( \epsilon = 0 \), which can be seen from Eqs. (49)–(50).

We recognize Eqs. (49)–(50) as Breit-Wigner solutions that take into account renormalization at \( D \ll \Delta \). On the other hand, we have already obtained Breit-Wigner formulas in Sec. III C where the renormalization has been carried out for \( D \gg \Delta \), for \( \epsilon \) close to a resonance energy. In particular, the results of Sec. III C apply for \( |\epsilon| \ll \Delta \) if \( \epsilon_0 = 0 \). The matching of the two solutions at \( D \sim \Delta \) implies that Eqs. (49)–(50) are in fact valid in a broader range of \( D \), namely for \( D \ll D_{\text{max}} \), provided only that one averages \( r_R(\epsilon',D) \) over \( \epsilon' \) together with the phase factor \( \chi_{\epsilon'} \). It follows that in the case of strong barriers close to resonances the RG equations can be cast in the form (48), (49) containing \( t(\epsilon,D) \) and \( r_{L,R}(\epsilon,D) \) only.

Note also that for \( D_0 \ll \Delta \) the boundary conditions to Eqs. (49), (50) are fixed at \( D = D_0 \), which leads to the change \( \Delta \rightarrow D_0 \) in Eq. (51).

At this point, one might be concerned about a possible contribution to \( t(\epsilon = 0, D) \) from other resonances. Indeed, in the derivation of Eq. (48), which gives no renormalization of \( t(\epsilon = 0, D) \), we approximated \( |r(\epsilon',D)| \) by unity for large \( |\epsilon'|, D \gg \Gamma(\Delta) \). Corrections coming from other resonances are clearly small in the parameter \( \Gamma_{1,2}(D)/\Delta \) but one should check if they might contribute to the renormalization of \( t(\epsilon = 0, D) \). Relaxing the above approximation by allowing for resonant “percolation” of electrons through the barriers at \( D \ll \Delta \) does give a perturbative correction to the RG (16):

\[
\frac{\partial \tilde{t}(\epsilon,D)}{\partial \Delta} = -\alpha \tilde{t}(\epsilon,D) \frac{\pi u_-(D)}{4\Delta} \left[ \tilde{r}_L(\epsilon,D) - \tilde{r}_R(\epsilon,D) \right], \tag{53}
\]

which, in contrast to Eq. (16), does not vanish at \( \epsilon = 0 \) (unless the double barrier is symmetric: the correction is then always zero). However, Eq. (53) tells us that the correction is irrelevant since \( u_-(D) \) itself scales to zero as \( D^\alpha \). We thus conclude that the “single-peak approximation” of Eqs. (16), (17) correctly describes the renormalization of the resonant amplitudes for all \( D \gg \Gamma(D) \).

\[\textbf{E. Inside a peak: } D \ll D_s\]

Now that we have integrated out all \( D \gg u_+(D) \), let us continue with the renormalization for \( D \) inside a resonant tunneling peak. The point at which \( D \) and \( u_+(D) \) become equal to each other yields a new characteristic scale \( D_s \):

\[
D_s = \Gamma_+(\Delta)(D_s/\Delta)^\alpha = \Gamma_+(\Delta)|\Gamma_+(\Delta)/\Delta|^\alpha'. \tag{54}
\]

where \( \Gamma_+(\Delta) \) is obtained from Eqs. (10)–(13) depending on the ratio of \( D_{1,2} \) and \( D_0 \). To leading order in \( \alpha' = \alpha/(1 - \alpha) \) → \( \alpha \). As will be seen below,
the significance of $D_s$ is that the width of the tunneling resonance saturates with decreasing $D_s$. For $D \ll D_s$, the RG equations (Eqs. 11) can be simplified since the scattering amplitudes now depend on a single variable, which is $D = \max \{|\epsilon|, T\}$. The averaged reflection amplitudes $\overline{r_Lr_R(D)}$ coincide then with $r_Lr_R(D)$, and the RG equations can be written in precisely the same form as for a single impurity:

$$\frac{\partial L_D}{\partial L_D} = -\alpha T_D R(D), \quad \frac{\partial r_{L,R}(D)}{\partial L_D} = \alpha r_{L,R}(D)T(D),$$

(55)

with matching conditions at $L_D = \ln(D_0/D_s)$. The difference between the single structureless impurity and the resonance peak is that in the latter case the ultraviolet cutoff is $D_s$. The fact that the scattering amplitudes inside the peak are described by Eqs. (55), was used also in Ref. 38.

In the preceding Secs. 3 and 4, the symmetry of the double barrier, i.e., a possible difference between $t_1$ and $t_2$ was seen to determine the amplitude and the width of a resonance peak but otherwise did not lead to any qualitatively different consequences, as compared to the symmetric case. However, for small $D \ll D_s$ the renormalization of the scattering amplitudes is essentially different depending on whether the barriers are identical to each other or not.

Consider first the symmetric case. Assuming, as in Sec. 4, that the resonance energy $\epsilon_0 = 0$, we get the Breit-Wigner formula with a width given by Eq. (51):

$$\tilde{\epsilon}(D) = \frac{u_+(D)}{u_+(D) + 2i\epsilon},$$

(56)

which turns out to be valid down to $D = 0$. A remarkable consequence of Eq. (56) is that the resonance in the symmetric case is perfect, $T = 1$ at $\epsilon = 0$. While this is trivial for noninteracting electrons, weak interaction in the Luttinger liquid is seen to preserve the perfect transmission, in agreement with the result obtained by a bosonic RG.

So long as inelastic scattering is not taken into account, the perfect transmission at $\epsilon = 0$ is not affected by finite temperature $T$, either, as can be seen from Eq. (56) if one puts $D = T$. However, the width of the resonance does depend on $T$. At $T = 0$, the width is finite and given by $D_s$ (see Fig. 5), which follows from Eq. (55) for $D = |\epsilon|$. For $T \gg D_s$, the width obeys Eq. (52).

The shape of the perfect resonance depends on the parameter $T/D_s$. If $T \ll D_s$, the reflection probability as a function of $|\epsilon|$ behaves near $|\epsilon| = 0$ first as $\epsilon^2$ for $|\epsilon| \ll T$ and then as $|\epsilon|^{2(1-\alpha)}$ for $T \ll |\epsilon| \ll D_s$. For larger energies, the transmission probability falls off with increasing $|\epsilon|$ as

$$T(\epsilon) = (D_s/2|\epsilon|)^{2(1-\alpha)}.$$  

(57)

This lineshape should be contrasted with the Lorentzian which describes the transmission peak for $T \gg D_s$ up to $|\epsilon| \sim T$, at which point a crossover to Eq. (57) occurs.

Let us now turn to the asymmetric case. Inspecting Eqs. (55), we see that a new characteristic scale $D_-$ emerges:

$$D_- = D_s \left( \frac{\Gamma_-(\Delta)}{\Gamma_+(\Delta)} \right)^{1/\alpha},$$

(58)

which coincides with $D_s$ for strongly asymmetric barriers but vanishes for symmetric ones. For $T \gg D_-$ we get the same results for $T(\epsilon)$ as in the symmetric case, only with an overall factor of

$$\lambda = \frac{\Gamma_0^2(\Delta) - \Gamma_0^2(\Delta)}{\Gamma_0^2(\Delta) + \Gamma_0^2(\Delta)} = \frac{4\Gamma_1(\Delta)\Gamma_2(\Delta)}{[\Gamma_1(\Delta) + \Gamma_2(\Delta)]^2}.$$  

(59)

However, for $T \ll D_-$ a new feature in the behavior of $T(\epsilon)$ shows up, namely a power-law falloff with decreasing $|\epsilon|$. The function $T(D)$, as obtained from Eqs. (55), for $D \ll D_s$ reads

$$T(D) = \frac{\lambda(D/D_s)^{2\alpha}}{1 - \lambda[1 - (D/D_s)^{2\alpha}]}.  

(60)

One sees that $T(\epsilon)$ behaves as (Fig. 5)

$$T(\epsilon) = \lambda(|\epsilon|/D_-)^{2\alpha}$$

(61)

in the interval $T \ll |\epsilon| \ll D_-$ and saturates at smaller energies: $T(\epsilon = 0) = \lambda(T/D_-)^{2\alpha}$. Thus, in the limit $T \ll D_-$, the resonant transmission probability as a function of $\epsilon$ exhibits a double-peak structure, see Fig. 5. If, however, the barriers are only slightly asymmetric, the gap near $\epsilon = 0$ develops in a range of $\epsilon$ which is much narrower than the width of the resonance peak. Specifically, $T(\epsilon)$ first grows up with increasing $|\epsilon|$ for $T \ll |\epsilon| \ll D_-$, then there is a plateau with an energy independent transmission for $D_- \ll |\epsilon| \ll D_s$, and $T(\epsilon)$ starts to fall off as $|\epsilon|$ is further increased.

In the above, we analyzed the behavior of $T(\epsilon)$ for the resonance energy $\epsilon_0 = 0$, i.e., when it coincides with the Fermi level. Let us now examine the case $\epsilon_0 \neq 0$. Again, let the barriers first be symmetric. Then, in Eq. (56), the resonant denominator changes to $u_+(D) + 2i(\epsilon - \epsilon_0)$ and $D$ is, as before, max{$\{|\epsilon|, T\}$. The innocent looking shift $\epsilon \rightarrow \epsilon - \epsilon_0$ leads at $T = 0$ to dramatic consequences for transmission at the Fermi energy, $\epsilon = 0$. Namely, $T(\epsilon)$ is now seen to vanish at $\epsilon = 0$ and zero $T$, whatever $\epsilon_0$ unless it is exactly zero. A new characteristic scale $D_1$ becomes relevant at $\epsilon_0 \neq 0$: it is defined by $u_+(D_1) = 2|\epsilon_0|$, which is rewritten as

$$D_1 = D_s(2|\epsilon_0|/D_s)^{1/\alpha}.  

(62)

The significance of the energy $D_1$ is that the width of the gap in the dependence of $T(\epsilon)$ around $\epsilon = 0$ at $T = 0$ is given by $D_1$ for $|\epsilon_0| \lesssim D_s$. Note that $D_1 \ll |\epsilon_0|$ for $|\epsilon_0| \ll D_s$.

The shape of the resonant peak as a function of $\epsilon$ changes in an essential way for $|\epsilon_0| \lesssim D_s$. Specifically, if $T \gg D_1$, the changes are weak; however, for $T \ll D_1$
a range of ε arises, T ≪ |ε| ≪ D₁, within which T(ε) behaves as (Fig. 5b)

\[ T(\epsilon) = (|\epsilon|/D₁)^{2\alpha}. \]  

(63)

The power-law falloff (63) saturates close to the Fermi level as T(ε = 0) = (T/D₁)²α.

We thus see that the width of the resonance in the transmission through a symmetric barrier exactly at the Fermi energy T(ε = 0) as a function of ε₀ vanishes as T → 0. On the other hand, the width of the resonance in the transmission at ε₀ = 0 as a function of ε is finite even at T = 0 and is given by Dᵣ. This peculiar feature is in sharp contrast to the resonant tunneling of noninteracting electrons, for which the two widths are the same.

For asymmetric barriers, T(ε) does not change substantially with increasing |ε₀| as long as D₁ ≪ D₋ and is given by the formulas for symmetric barriers with an overall reduction of the transmission probability by a factor of λ (59) otherwise (see Fig. 5).

F. Weak barriers

In Secs. 11C, 11E, we have assumed that Dᵣ₀ ≫ Δ, which means that even if the impurities are initially (at D = D₀) weak, they get strong in the process of renormalization before D becomes equal to Δ. Under this assumption, we have sharp resonant peaks close to the Fermi level and the bare scattering amplitudes only rescale [according to Eqs. (10)–(13)] parameters in otherwise general formulas for the resonant tunneling. Let us now examine the resonant transmission in the case of at least one barrier being initially so weak that the renormalization does not make it strong at D ∼ Δ.

We begin with the case of both barriers characterized by Dᵣ₁,₂ ≪ Δ. The total reflection coefficient rᵢL(ε, D) as obtained from Eqs. (23), (24) in the limit |rᵢ| ≪ 1 is given by

\[ rᵢL(ε, D) = (r₁ − r₂χₐ)(D₀/D)^α \]  

(64)

with D = max(|ε|, |T|). It is worth noting that, due to the oscillating factor χₐ (20), it is not possible to derive Eq. (64) from Eqs. (23), (24) even in the simplest case of a weak double barrier.

Suppose first that the barrier is symmetric and ε₀ = 0. Then Eq. (64) simplifies to

\[ R(ε, D) = 2[1 − \cos(2πε/Δ)](Dᵣ/D)^{2α}. \]  

(65)

One sees that reflection is enhanced by interaction, but the reflection coefficient is always small, R ≪ 1 for any ε, provided Dᵣ ≪ Δ. No sharp features in the ε dependence of the scattering amplitudes emerge around the Fermi energy (see Fig. 6).

![Diagram](attachment:image.png)

**FIG. 5:** Schematic summary of transmission peak structure for zero temperature and strong barriers: (a) ε₀ = 0, a single peak of T(ε) for symmetric barriers with a width Dᵣ transforms into a double peak with a reduced height and a gap width D₋ < Dᵣ for asymmetric barriers, Eqs. (35), (36); (b) 0 < |ε₀| < Dᵣ, a single peak for symmetric barriers, centered at ε = ε₀, shows a dip at the Fermi energy, ε = 0, with a gap width D₁ < |ε₀|, Eq. (62). For ε₀ ≠ 0 and asymmetric barriers, (a) describes the case of sufficiently small ε₀, namely D₋ ≫ D₁, whereas (b) with a properly rescaled height describes the opposite case. The only effect of finite T ≳ Dᵣ is to fill the gaps at |ε| ≲ T.

![Diagram](attachment:image.png)

**FIG. 6:** Schematic behavior of the transmission coefficient for zero temperature and weak barriers, Eq. (65): (a) for ε₀ = 0, the modulation of T(ε) for both symmetric (solid) and asymmetric (dashed) barriers is strongly increased. However, for symmetric barriers T(ε) remains close to unity for all ε, while there appears a dip of width δ₋ < Dᵣ, Eqs. (67), (68), around ε = 0 for asymmetric barriers; (b) for ε₀ ≠ 0 and symmetric barriers, the bare transmission (dash-dotted) is strongly renormalized (solid) due to interactions and exhibits a gap of width δ₁ < D₁, Eq. (70), around the Fermi level. For ε₀ ≠ 0 and asymmetric barriers, (a) describes the case of sufficiently small ε₀, namely δ₋ ≫ δ₁, whereas (b) with a properly rescaled height describes the opposite case. Similarly to Fig. 5, finite T fills the gaps at |ε| ≲ T.

It is now instructive to introduce a weak asymmetry R₋ = |R₂ − R₁|, such that R₋ ≪ R ≳ R₁,₂. Given that the asymmetry is weak, it can manifests itself only at small energies. Expanding Eq. (65) about ε = 0, we get for |ε| ≪ Δ:

\[ R(ε, D) = \left(\left(\frac{R₋}{2R}\right)^2 + \left(\frac{2πε}{Δ}\right)^2\right) \frac{D₁}{D}^{2α}. \]  

(66)
As can be seen from Eq. (66), asymmetry sets two new characteristic scales of energy: \((R_-/R_\Delta)\Delta\) and a smaller scale

\[
\delta_- = D_r(R_-/2R)^{1/\alpha}.
\]

(67)

Provided that temperature \(T \ll (R_-/R_\Delta)\Delta\), the reflection coefficient starts to grow with approaching the Fermi level at \(|\epsilon| \sim (R_-/R_\Delta)\Delta\). The enhancement of reflection is cut off by temperature before \(R\) becomes of order unity if \(T\) is not too low, specifically if \(\delta_- \ll T\). However, if \(T \ll \delta_-\), then reflection gets strong at \(|\epsilon| \sim \delta_-\). To describe the scattering probabilities at \(|\epsilon| \ll \delta_-\), one should solve Eqs. (55), derived in the same way it was done in Sec. IV.E now with matching onto the perturbative (in \(R_{1,2}\)) solution (64) anywhere in the region \(\delta_- \ll |\epsilon| \ll \Delta\). For \(D \ll (R_-/R_\Delta)\Delta\) the solution reads:

\[
T(D) = 1/ [1 + (\delta_-/D)^{2\alpha}] .
\]

(68)

Thus, the weak double barrier remains slightly reflecting after the renormalization provided that \(T \gg \delta_-\). However, for both \(T, |\epsilon| \ll \delta_-\) the transmission probability is small: within the range \(T \ll |\epsilon| \ll \delta_-\), \(T(\epsilon)\) behaves as (Fig. 3)

\[
T(\epsilon) = (|\epsilon|/\delta_-)^{2\alpha},
\]

(69)

and saturates for smaller \(|\epsilon|\) at \(T(\epsilon = 0) = (T/\delta_-)^{2\alpha}\). Comparing Eqs. (68), (69) with each other, we see that the energy \(\delta_-\) is a counterpart of \(D_-\) for the case of a weak barrier.

Generalizing to \(\epsilon_0 \neq 0\), we have for \(|\epsilon_0| \ll \Delta\) a shift \(\epsilon \rightarrow \epsilon - \epsilon_0\) in Eq. (69), while \(D = \max\{|\epsilon|, T\}\). A new energy scale \(\delta_1 \ll \epsilon_0\) appears, at which the reflection coefficient becomes of order unity in the symmetric case:

\[
\delta_1 = D_r (2\pi |\epsilon_0|/\Delta)^{1/\alpha}.
\]

(70)

analogous to \(D_1\) in Eq. (62) for tunneling barriers. The energy \(\delta_1\) gives the width of the gap in the transmission probability at the Fermi level at \(T = 0\). For \(T \ll |\epsilon| \ll \delta_1\), we get a power-law vanishing of \(T(\epsilon) = (|\epsilon|/\delta_1)^{2\alpha}\) with decreasing \(|\epsilon|\) (see Fig. 3.b) and a saturation for smaller \(|\epsilon|\) at \((T/\delta_1)^{2\alpha}\). A general expression for the scattering probabilities, valid for arbitrary \(D_{r,1,2} \ll \Delta\) and \(|\epsilon|, D \ll \Delta\), can be obtained from Eqs. (65).

\[
\frac{R(\epsilon, D)}{T(\epsilon, D)} = \left[\left(R_{1/2} - R_{1/2}\right)^2\right]^{2\alpha} + \left(R_1 R_2\right)^{1/2} \left(\frac{2\pi}{\Delta}\right)^{2\alpha}\left(\epsilon - \epsilon_0\right)^{2\alpha} .
\]

(71)

Equation (71) reproduces Eqs. (66) – (70) in the corresponding limits.

We conclude that if the barriers are symmetric but \(\epsilon_0\) is nonzero, or if the barriers are asymmetric, the transmission probability vanishes (Fig. 3) at the Fermi level in the limit \(T \rightarrow 0\). We will see in Sec. IV that these features lead to the emergence of a sharp peak in the low-temperature conductance as a function of \(\epsilon_0\) even for two weak impurities, provided only that they are slightly asymmetric.

Finally, when two strongly asymmetric barriers are located nearby, so that at \(D \sim \Delta\) one barrier is strongly reflecting whereas the other is still weak, the effect of the latter on the transmission probability remains small for any \(D\). Let us take the example \(D_{r_1} \gg \Delta\) and \(D_{r_2} \ll \Delta\). Then we get for \(D_{r_1} > D > \Delta\)

\[
T(\epsilon, D) = (D/D_{r_1})^{2\alpha} \left[1 + 2(D_{r_2}/D)^{\alpha} \cos \theta \right] ,
\]

(72)

where \(\theta = 2\pi(\epsilon - \epsilon_0)/\Delta\). One sees that the presence of the weak impurity only leads to a weak modulation with changing \(\epsilon\). For \(D \lesssim \Delta\), the independent renormalization of the weaker impurity is suppressed by reflection from the strong barrier and \(T(D)\) behaves as \((D/D_{r_1})^{2\alpha}\) down to \(D = 0\).

IV. CONDUCTANCE PEAK

The solution to the problem of transmission through a double barrier given in the preceding sections allows us to examine the linear conductance of the system \(G(\epsilon_0, T)\) as a function of temperature \(T\) and the energy distance between the Fermi level and a resonance level \(\epsilon_0\). Recall that we have studied the elastic transmission of interacting electrons, i.e., the energy \(\epsilon\) of an incident electron before and after the transmission is the same, while in the process of scattering off the barrier \(\epsilon\) is not conserved due to interaction with other electrons both in the particle-hole and Cooper channels. At finite \(T\), there are also inelastic processes, characterized by the inelastic scattering length \(L_{in}\). Neglecting the inelastic scattering is legitimate if \(L_{in} \gg L_T \sim v_F/T\), which is satisfied in the present problem for weak interaction \(\alpha \ll 1\). On the other hand, it is worthwhile to note that the very formulation of the scattering problem in the interacting case even for elastic scattering is a delicate issue if one keeps scaling exponents of higher order than linear in \(\alpha\). Also, interaction-induced current-vertex corrections in the Kubo formula in the presence of impurities and interaction are governed by exponents of higher order in \(\alpha\). Here, we avoid these complications by treating the scattering problem within the one-loop approximation, i.e., keeping only first order terms in the exponents. Under these conditions, one can use the Landauer-Büttiker formalism relating the conductance and the transmission probability. The conductance \(G(\epsilon_0, T)\) in units of \(e^2/h\) reads

\[
G(\epsilon_0, T) = \int d\epsilon \; T(\epsilon) \left(-\partial n_F/\partial \epsilon\right) .
\]

(73)

We are interested in the low-temperature regime with \(T \ll \Delta\), otherwise we intend to keep \(T\) arbitrary, i.e., \(T\) may be as small as zero.
Below we analyze various limiting cases. As we have seen in Sec. III the strength of interaction, the strength of the barriers, and the degree of asymmetry set a number of characteristic energy scales which yield a variety of different regimes in the temperature and energy dependence of the transmission probability $T(\epsilon, T)$. By means of Eq. (73), these regimes manifest themselves in the behavior of the conductance $G(\epsilon_0, T)$, which is a directly measurable quantity.

Before turning to the calculation of $G(\epsilon_0, T)$, let us briefly discuss the restrictions and possible implementations of our model. Even though we are dealing with the case of weak interaction, $\alpha \ll 1$, the product $\alpha \ln(D_0/\Delta)$ may be large in small quantum dots, so that the renormalization of scattering amplitudes on scales $D \gg \Delta$ is necessary, as has been done in Sec. III C. Experimentally, as discussed in Sec. II the parameter $\alpha$ is typically not small in carbon nanotubes [in Refs. 6, 7, the value of $\alpha$ extracted according to Eq. (4) from the measured $\alpha_\epsilon \sim 0.6 - 1.0$ is $\alpha \sim 1.3 - 3.0$]. On the other hand, in single-mode semiconductor quantum wires the strength of interaction is typically smaller (in Ref. 12, $\alpha_\epsilon \sim 0.2 - 0.5$ corresponds [Eq. (3)] to $\alpha \sim 0.3 - 0.8$) and can be more easily made small through screening by nearby metallic gates. In weakly interacting wires, our results would be applicable directly. At the same time, our theory captures much of the essential physics of strongly [with $\alpha$ defined in Eq. (2) of order unity] correlated wires too, as compared with the known results obtained by different methods.

As for the impurity strength, both transport and tunneling experiments on quantum wires with imperfections have been so far focused on the case of initially strong inhomogeneities (impurities, artificially created tunneling barriers, or non-adiabatic contacts). Sections III C-III D describe this situation in detail. Moreover, as shown in Secs. III C-III D weak inhomogeneities, which are potentially realizable in a controllable way both in semiconductor quantum wires and carbon nanotubes, are of special interest, since the Coulomb interaction transforms two weak impurities into a quantum dot with a pronounced resonance structure. Finally, a relatively strong asymmetry of the quantum dot appears to be inevitably present in the experimental setups of Refs. 12, 13.

### A. Strong barriers

Consider first the case of strong barriers (more precisely, the bare transmission through the barriers may be high, $T_{1,2} \approx 1$, but we assume that the barriers get strong before the RG flow parameter $D$ equals the single-particle energy spacing inside the dot, $\Delta$), i.e., $D_{\text{min}} = D_0(\min(R_1, R_2))^{1/2} \gg \Delta$. Then we have a sharp peak of the transmission probability centered at $\epsilon = \epsilon_0$ whose width is $\max(D_s, \Gamma(T)) \ll \Delta$, where $D_s$ and $\Gamma(T)$ are defined in Eqs. 54, 57. In other words, the width of the peak in $T(\epsilon, T)$ is $\Gamma(T) = D_s T/D_s \propto \alpha$ for $T \gg D_s$, whereas for smaller $T \ll D_s$ the width is of order $D_s$ and does not depend on $T$.

#### 1. $T \gg D_s$, Sequential tunneling

For $T \gg D_s$, the shape of the conductance peak is given by (see Fig. 4)

$$G(\epsilon_0, T) = \frac{\zeta G_p}{\cosh^2(\epsilon_0/2T)}, \quad (74)$$

where the peak value of the conductance

$$G_p = \pi \lambda T(\Gamma(T))/8T \quad (75)$$

with $\lambda$ defined in Eq. 50, and $\zeta = (\max(|\epsilon_0|, T)/T)^\alpha$. The width of the conductance peak $w$ is of order $T$, as for noninteracting electrons; however, the power-law behavior of $G_p(T)$ is seen to be modified by interaction, in accordance with the results derived in Refs. 16, 19. Note that the scaling of $G_p \propto T^{\alpha - 1}$ is governed by the single-particle density of states $\rho_n(T)$ for tunneling into the end of a Luttinger liquid, namely $G_p \propto \rho_n(T)/T^{\alpha - 1}$.

We recognize Eqs. 51, 53 as the conventional sequential tunneling formulas, but with a $T$ dependent resonance width $\Gamma(T)$. Far in the wings of the resonance the exponential falloff 14, $G(\epsilon_0, T) \sim \lambda T^{-1} \Gamma(|\epsilon_0|) \exp(-|\epsilon_0|/T)$, crosses over into the cotunneling (determined by the processes of fourth order in tunneling amplitudes) power law $G(\epsilon_0, T) = \lambda T^2 / 4 |\epsilon_0|^2$, as usual. The crossover between the sequential tunneling and cotunneling regimes occurs at $|\epsilon_0| \approx T \ln [T/\Gamma(T)]$. In fact, this formula is valid for an arbitrary strength of interaction with $\Gamma(T) \propto \rho_n(T) \propto T^{\alpha_e}$, where $\alpha_e$ (equal to $\alpha$ for a weak interaction) is the end-tunneling exponent 15. It is worthwhile to note that for strong enough interaction (namely for $\alpha_e > 1$) the sequential mechanism of tunneling is effective for the resonance peak for all $T$, down to $T = 0$. Moreover, for $\alpha_e > 1$ the crossover to the cotunneling regime shifts towards larger $|\epsilon_0|$ with increasing strength of interaction.

As demonstrated in Ref. 16, Eqs. 45 can be obtained from a classical kinetic equation 14, 16 for occupation numbers characterizing the state of the quantum dot. We get the same results from the fermionic RG equations. It is worth mentioning that, although Eqs. 44, 45 follow from a classical kinetic equation 14, 16 which involves diagonal elements of the density matrix only, the transmission in our derivation is fully coherent. The fact that in the high-temperature limit it can also be described in terms of the classical kinetic equation merely means that quantum corrections to the kinetic equation may be neglected for small $\Gamma(T)/T$ [whereas the quantum suppression of the tunneling density of states $\rho_n(T)$ may be treated in this equation in a phenomenological way].
which describes the degree of asymmetry. The double-peak structure of the transmission coefficient as a function of \( \epsilon_0 \) for \( T \ll D_- \) translates into a complete vanishing of the conductance peak at \( T \to 0 \). Specifically, for \( T \gg D_- \) the conductance \( G(\epsilon_0, T) \) is given by Eq. (76) for symmetric barriers with an overall factor of \( \lambda \). However, for \( T \ll D_- \) the transmission at the Fermi level falls off with decreasing \( T \) (Fig. 7) and so does the conductance peak (see Fig. 4):

\[
G(\epsilon_0, T) = \frac{\lambda (T/D_s)^{2\alpha}}{(D_-/D_s)^{2\alpha} + (2\epsilon_0/D_s)^2}, \quad (78)
\]

which gives

\[
G_p = \lambda \left( \frac{T}{D_-} \right)^{2\alpha}, \quad w = D_s \frac{\Gamma_-(\Delta)}{\Gamma_+^{(2\alpha)}}, \quad (79)
\]

where \( \Gamma_\pm(\Delta) \) are defined below Eq. (61). Thus, at small \( T \ll D_- \), the height of the conductance peak goes down as \( T \) decreases, whereas the width of the peak does not depend on \( T \) any longer. This kind of behavior of the resonance peak was predicted in Ref. 3: \( G_p(T) \) scales as \( \rho^2(T) \), as for a single impurity. The role of asymmetry was emphasized and expressions similar to Eqs. (79) were obtained in Ref. 33. However, the authors of Ref. 33 do not distinguish between the scales \( D_- \) and \( D_s \) (the latter is denoted \( \tilde{\epsilon} \) there).

### B. Weak barriers

Consider now the case of weak barriers, i.e., \( D_{r,2} \ll \Delta \). Naively one could think that scattering on weak barriers cannot possibly yield a sharp peak of conductance. Indeed, the transmission probability as a function of \( \epsilon \) (Fig. 5) does not have any peak at \( \epsilon = \epsilon_0 \), in contrast to the case of resonant tunneling. At high \( T \gg D_+ \), the conductance \( G(\epsilon_0, T) \) is a weakly oscillating (with a period \( \Delta \)) function of \( \epsilon_0 \). The only difference with the non-interacting case is an enhanced amplitude of the oscillations.

Let us show that in fact the interaction-induced vanishing of \( T(\epsilon) \) at the Fermi energy \( \epsilon = 0 \) for \( T = 0 \) does lead to a narrow Lorentzian peak of \( G(\epsilon_0, T) \) (see Fig. 5), provided that \( T \) is low enough and the barriers are not too asymmetric.

#### 1. Symmetric barriers

Integration (73) of the transmission probability (71) for symmetric barriers yields

\[
G(\epsilon_0, T) = \left[ 1 + \left( \frac{2\pi \epsilon_0}{\Delta} \right)^2 \left( \frac{D_r}{T} \right)^{2\alpha} \right]^{-1}, \quad (80)
\]

which indeed describes a Lorentzian peak (Fig. 5) with the height \( G_p = 1 \) and the width

\[
w = \frac{\Delta}{\pi} \left( \frac{T}{D_r} \right)^{\alpha}. \quad (81)
\]

Let us now turn to low temperatures \( T \ll D_s \), where processes of all orders in the tunneling amplitudes are important. Consider first the symmetric case (Fig. 4). The main contribution to the integral over \( \epsilon \) in Eq. (63) comes from \( |\epsilon| \sim T \ll D_s \), so that the shape of the conductance peak is a Lorentzian:

\[
G(\epsilon_0, T) = \frac{\Gamma^2(T)}{\Gamma^2(T) + 4\epsilon_0^2}. \quad (76)
\]

We see that the height of the peak \( G_p = 1 \) and the width

\[
w = D_s (T/D_s)\alpha \quad (77)
\]

exhibits a power-law temperature dependence with an exponent depending on the strength of interaction. The vanishing of \( w \) as \( T \to 0 \) should be contrasted with the behavior of the peak in \( T(\epsilon, T) \) (Fig. 4), whose width is \( D_s \) for low \( T \). In the limit \( T \to 0 \), the conductance peak becomes infinitely narrow but the resonance at \( \epsilon_0 = 0 \) persists down to \( T = 0 \), in accordance with Ref. 2. We thus confirm the persistence of the perfect resonance at \( T = 0 \) by means of the fermionic RG. For finite \( T \ll D_s \), there is a small correction \( 1 - G_p \sim (T/D_s)^{2(1-\alpha)} \), which comes from the non-perfect transmission for finite \( \epsilon \) at \( \epsilon_0 = 0 \) after the thermal averaging (64).

#### 3. \( T \ll D_s \), Asymmetric barriers

We recall that the renormalization in the asymmetric case is governed by the scale \( D_- \) [defined in Eq. (58)]
Thus, the asymmetry leads (Fig. 8b) to vanishing dependence of \( G \) on temperature \( T \). In the limit \( T \to 0 \), the width of the peak is infinitesimally small. Similarly to the case of resonant tunneling, the resonance at \( \varepsilon_0 = 0 \) remains perfect in the presence of interaction at \( T = 0 \). At finite \( T \), there is a small correction

\[ 1 - G_p \sim (T/\Delta)^2(D_r/T)^{2\alpha}. \] (82)

Equation (82) describes also the high-\( T \) behavior of \( G_p \) in the case of slightly asymmetric barriers (see below).

![FIG. 8: Weak barriers: Qualitative dependence of the conductance \( G \) (in units of \( e^2/h \)) on the resonance energy \( \varepsilon_0 \) for different temperatures \( T \) and weak interaction. (a) Symmetric barriers. The value of \( G(\varepsilon_0 = 0) \) grows [Eq. (82)] with decreasing \( T \) and saturates [Eq. (81)] at \( G_p = 1 \) as \( T \to 0 \). The width of the resonance shrinks to zero, according to \( 1 \) and \( 4 \), similarly to Eq. (79). The resonance peak becomes visible at \( T \ll D_r \) (curves \( T_2 \) and \( T_1 \)). (b) Asymmetric barriers. The dependence of \( G(\varepsilon_0 = 0) \) on \( T \) is non-monotonic: it grows [Eq. (82)] with decreasing \( T \) for large \( T \) and goes down [Eq. (83)] for \( T < \delta_- \). The width of the resonance decreases [Eq. (83), similarly to the symmetric case] with lowering \( T \) and saturates [Eq. (84), curves \( T_2 \) and \( T_1 \)] as \( T \to 0 \). If asymmetry is strong, \( G(\varepsilon_0) \) is only slightly modulated for all \( T \), i.e., exhibits no peak.

2. Asymmetric barriers

Introducing a weak asymmetry \( R_- = |R_1 - R_2| \ll R \approx R_1 R_2 \), we get for \( T < \delta_- \), where \( \delta_- \) is defined in Eq. (67):

\[ G(\varepsilon_0, T) = \frac{R^2(T/D_r)^{2\alpha}}{R^2/4 + R^2(2\pi \varepsilon_0/\Delta)^2}. \] (83)

The height and the width of the peak are

\[ G_p = \left( \frac{T}{\delta_-} \right)^{2\alpha}, \quad w = \frac{\Delta}{2\pi} \frac{R_-}{R}. \] (84)

Thus, the asymmetry leads (Fig. 8b) to vanishing \( G_p \) at \( T \to 0 \) and the width is seen to saturate with decreasing \( T \), similarly to Eq. (79). It is worth noting that the dependence of \( G_p \) on \( T \) is non-monotonic: \( G_p \propto T^{2\alpha} \) grows with increasing \( T \) for \( T \ll \delta_- \), continues to grow in the range \( \delta_- \ll T \ll \Delta \), where the correction behaves similarly to the case of symmetric barriers, \( 1 - G_p \propto T^{2(1-\alpha)} \). The conductance peak is narrow provided the asymmetry is weak, \( R_- \ll R \). If the asymmetry is strong, \( R_- \approx R_1 + R_2 \), the peak is completely destroyed.

V. CONCLUSIONS

In conclusion, we have thoroughly studied transport of weakly interacting spinless electrons through a double barrier. We have described a rich variety of different regimes depending on the strength of the barrier, its shape, and temperature. We have developed a fermionic RG approach to the double barrier problem, which has enabled us to treat on an equal footing both the resonant tunneling and resonant transmission through weak impurities. In the latter case, we have demonstrated how the interaction-induced renormalization in effect creates a quantum dot with tunneling barriers with a pronounced resonance peak structure. Moreover, we have shown that even very weak impurities, for which the renormalized transmission coefficient does not exhibit any peak, may give a sharp peak in the conductance as a function of gate voltage, provided that the double barrier is only slightly asymmetric. In contrast, the resonant structure is shown to be completely destroyed for a strongly asymmetric barrier. All the regimes we have studied may be characterized by three different types of behavior of the conductance peak height \( G_p \) and the peak width \( w \) on temperature \( T \):

(i) for high temperature \( T \gg \Gamma(T) \),

\[ G_p \propto T^{\alpha-1} \text{ and } w \propto T; \]

(ii) for lower \( T \), depending on the shape of the barrier (whether it is symmetric or asymmetric), either

\[ G_p \text{ does not depend on } T \text{ and } w \propto T^\alpha \text{ or} \]

(iii) \( G_p \propto T^{2\alpha} \) and \( w \) is constant.

One can see that none of the regimes (i–iii) supports \( G_p \propto T^{2\alpha-1} \) and \( w \propto T \), as proposed in Ref. [13]. Further experiments would be useful to resolve the puzzle. Including spin and generalizing to the case of several channels (possibly with different Fermi wavevectors) within the framework of the present approach warrant further study.

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In general, the ultraviolet cutoffs in the tracing over states below and above the Fermi energy may be different. However, possible asymmetry only leads to a renormalization of bare couplings in the boundary conditions to Eq. (1). Moreover, the paper deals with physical situations of transport in quantum wires, in which case the cutoffs are symmetric and $D_0$ does not exceed the Fermi energy.

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