WALL-CROSSING IN GENUS ZERO K-THEORETIC LANDAU-GINZBURG THEORY

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Abstract. For a Fermat quasi-homogeneous polynomial $W$, we study a family of K-theoretic quantum invariants parametrized by a positive rational number $\epsilon$. We prove a wall-crossing formula by showing the generating functions lie on the Lagrangian cone of the permutation-equivariant K-theoretic FJRW theory of $W$.

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1. Introduction

Landau-Ginzburg/Calabi-Yau correspondence (LG/CY) arises from a variation of the GIT quotient in Witten’s gauged linear sigma model (GLSM) [18]. LG/CY correspondence describes a relationship between sigma models based on Calabi-Yau hypersurfaces in weighted projective spaces and the Landau-Ginzburg model of the defining equation of the Calabi-Yau. The mathematical A-model on the Calabi-Yau side is given by Gromov-Witten...
LG/CY correspondence was proved by Chiodo-Iritani-Ruan in genus zero for Calabi-Yau Fermat polynomials [1]. They proved a mirror theorem relating the FJRW theory of a Fermat polynomial to a hypergeometric series. The analogous mirror theorem for GW theory was proved by Givental [7] and Lian-Liu-Yau [13]. The LG/CY correspondence follows from analytic continuation on the global Kähler moduli space.

Ross-Ruan [15] obtained a new geometric interpretation of the Landau-Ginzburg mirror theorem of Chiodo-Iritani-Ruan [1] by proving a wall-crossing formula that relates the generating functions of GLSM introduced by Fan-Jarvis-Ruan [6], where a mathematically rigorous definition of GLSM is provided. For hypersurface, the GLSM is a one-dimensional family of cohomological field theories (CohFTs) parametrized by the nonzero rational numbers. The family of CohFTs which arise over the positive rational numbers corresponds to the geometric phase. Wall-crossing formulas in the geometric phase have been studied by Ciocan-Fontanine-Kim [2], [3] and [4]. The family of CohFTs which arise over the negative rational numbers corresponds to the Landau-Ginzburg phase. Ross-Ruan [15] proved wall-crossing formulas analogous to Ciocan-Fontanine-Kim for a Fermat polynomial.

The goal of this paper is to prove an analog of Ross-Ruan’s result in K-theory. K-theoretic Gromov-Witten invariants was defined by A. Givental and Y.-P. Lee [9], [12] as holomorphic Euler characteristics of vector bundles on the moduli space of stable maps. A new enriched version of the theory called permutation-equivariant K-theoretic Gromov-Witten theory has been introduced by Givental [8]. The permutation-equivariant theory takes into account the $S_n$ action on the moduli spaces permuting the marked points. It fits better in the framework of mirror symmetry. Givental [8] proved that certain $q$-hypergeometric series associated to toric manifolds lie on the Lagrangian cone of the permutation-equivariant K-theoretic Gromov-Witten theory. A “quantum Lefschetz” type theorem for an arbitrary smooth projective variety and a K-theoretic mirror theorem for toric fibrations were proved by Tonita [16]. A permutation-equivariant K-theoretic analogy of Ciocan-Fontanine-Kim [3] was proved by Tseng-You [17].

The definition of K-theoretic FJRW theory and the K-theoretic wall-crossing formula in genus zero between FJRW and GW of hypersurfaces has been worked out by Jérémie Guéré [10]. More precisely, Guéré proved his result using mirror symmetry as in the original LG/CY correspondence of Chiodo-Iritani-Ruan [1]. Our result, on the other hand, follows the approach of Ross-Ruan [15].

we now give a more detailed overview of our result.
For each $\epsilon \in \mathbb{Q}$, we consider the CohFT arises from the negative rational number $-\epsilon$ in the Landau-Ginzburg phase. The moduli space $\mathcal{R}^d_{k,\epsilon}$ parametrizes $\epsilon$-stable pairs $(C, L)$, where $C$ is a Hassett-stable rational orbifold curve with $m$ orbifold marked points $x_1, \ldots, x_m$ and $n$ smooth light points $y_1, \ldots, y_n$, and $L$ is an orbifold line bundle which satisfies

$$L^\otimes d \cong \omega_{\log}(-\sum d_i y_i).$$

The vector $\vec{k} = (k_1, \ldots, k_m)$ records the multiplicities of the line bundles at the orbifold marked points.

For a Fermat polynomial $W$, the K-theoretic invariant is defined by

$$\langle \phi_{\vec{k}} \mathbb{L}^j_1 \cdots \phi_{\vec{n}} \mathbb{L}^m | \phi_{\ell_1} \cdots \phi_{\ell_n} \rangle_{W, \epsilon, S^m \times S^n}^{W,\epsilon, \mathbb{L}^j_1 \cdots \mathbb{L}^m} = d \cdot \chi(\mathcal{R}^d_{k,\epsilon}, \mathcal{O}_k \otimes \mathcal{O}_l),$$

where $\mathcal{O}_k$ is called the virtual structure sheaf, $\phi_i$ are elements of the so-called narrow state space $K'_W$, $j_i$ are nonnegative integers and $\mathbb{L}_i$ are line bundles over $\mathcal{R}_{k,\epsilon}$ corresponding to the pullback of the $i$-th tautological line bundles over the Hassett moduli space $[11]$ via forgetful maps. We write

$$\langle \phi_{\vec{k}} \mathbb{L}^j_1 \cdots \phi_{\vec{n}} \mathbb{L}^m | \phi_{\ell_1} \cdots \phi_{\ell_n} \rangle_{W, \epsilon, S^m \times S^n}^{W,\epsilon, \mathbb{L}^j_1 \cdots \mathbb{L}^m}$$

for the permutation-equivariant invariants.

Genus zero wall-crossing formulas are naturally stated via generating functions of permutation-equivariant K-theoretic invariants. Let $t(q)$ be a Laurent polynomial in $q$ with coefficients in $K'_W \otimes \Lambda$, where $\Lambda$ is a $\lambda$-algebra, and $u := \sum u^k \phi_k \in K'_W$. For a positive rational number $\epsilon$, the big $J^{\epsilon}$-function is defined as

$$J^{\epsilon}(t, u, q) := (1 - q)\phi_0 \sum_{a_i \geq 0, i \in \text{nar}} \prod \frac{u^k \phi_i}{1 - q} \prod_{j=1}^N \prod_{0 \leq b < q_j + \sum a_i q_j} \prod_{(b) = (q_j + \sum a_i q_j)} (1 - q^b)$$

$$+ t(1/q) + \sum_{k} \sum_{m,n} \phi^k \frac{\phi_k}{1 - q^L_1} \cdot t(\mathbb{L})^m | \mathbb{L}^n \rangle_{W, \epsilon, S^m \times S^n}^{W,\epsilon, \mathbb{L}^j_1 \cdots \mathbb{L}^m}.$$

Theorem 2.3 shows that the $J^{\epsilon}$-function lies on the Lagrangian cone of the permutation-equivariant K-theoretic FJRW theory of $W$.

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2. K-theoretic Weighted FJRW Theory

2.1. Moduli Spaces.
Definition 2.1 ([15] Definition 1.1). For a positive rational number $\epsilon$, a $(d, \epsilon)$-stable rational curve is a rational connected orbifold curve $C$ with at worst nodal singularity, together with $m$ distinct orbifold marked points $x_1, \ldots, x_m$ and $n$ (not necessarily distinct) smooth light marked points $y_1, \ldots, y_n$ satisfies the following

- all nodes and orbifold marked points $x_i$ have cyclic isotropy $\mu_d$ and the orbifold structure is trivial away from nodes and orbifold markings;
- $\text{mult}_{z}(\epsilon \sum [y_i]) \leq 1$ at every points $z$ in $C$;
- $\omega_{\log}(\epsilon [y_i]) := \omega_C(\sum [x_i] + \epsilon [y_i])$ is ample.

We write $\overline{M}_{0,m+en}^d$ for the moduli space of $(d, \epsilon)$-stable curves.

Definition 2.2 ([15] Definition 1.2). For $\vec{l} := (l_1, \ldots, l_n)$ with $0 \leq l_i \leq d - 1$, an $\vec{l}$-twisted $d$-spin structure on a $(d, \epsilon)$-stable curve consists of an orbifold line bundle $L$ and an isomorphism

$$L^\otimes d \xrightarrow{\kappa} \omega_{\log}(-\sum l_i[y_i])$$

We write $\mathcal{R}^d_{m,\vec{l}}$ for the moduli space of $(d, \epsilon)$-stable curves with $\vec{l}$-twisted $d$-spin structure.

The restriction of $L$ to an orbifold point $x$ is a character of $\mu_d$, that is, multiplication by $e^{2\pi i k/d}$, for some $0 \leq k < d$. The multiplicity of $L$ at $x$ is defined by

$$\text{mult}_x L := k.$$

Definition 2.3. For $\vec{k} = (k_1, \ldots, k_m)$ where $k_i \in \{0, 1, \ldots, d - 1\}$, we write $\mathcal{R}^d_{\vec{k},\vec{l}}$ for the component of $\mathcal{R}^d_{m,\vec{l}}$ indexed by the multiplicities of the line bundle at the orbifold points $x_i$:

$$\text{mult}_{x_i} L = k_i + 1 \pmod d.$$

The forgetful maps to Hassett moduli spaces [11]

$$\theta : \mathcal{R}^d_{\vec{k},\vec{l}} \to \overline{M}_{0,m+en}$$
are defined by forgetting the line bundle $L$ and the orbifold structure on $C$. We define line bundles $L_i$ on $R_{k,\epsilon}^d$ by the pullback of the cotangent line bundle at the $i$-th marked point via $\theta$.

We write $R_{k,\epsilon}^d$ for the graph spaces parametrizing the objects of $R_{k,\epsilon}^d$ with degree one maps $f : C \to \mathbb{P}^1$. Namely, there is an irreducible component $\hat{C}$ of $C$ such that the map restricts to an isomorphism on $\hat{C}$ and contracts the remaining components $C \setminus \hat{C}$. The third condition in Definition 2.1 changes to $\omega_{\log}(\epsilon \sum [y_i])$ is ample on $C \setminus \hat{C}$. We also use $\theta$ to denote the forgetful maps 

\[ \theta : R_{k,\epsilon}^d \to \overline{M}_{0,m+n}(\mathbb{P}^1, 1), \]

and define line bundles $L_i$ on $R_{k,\epsilon}^d$ by the pullback of the cotangent line bundle at the $i$-th marked point via $\theta$.

2.2. State Space. Let $W(x_1, \ldots, x_N)$ be a quasi-homogeneous polynomial for which there exist charges $(q_1, \ldots, q_N) \in \mathbb{Q}^N$ such that for any $\lambda \in \mathbb{C}^*$:

\[ W(\lambda^{q_1}x_1, \ldots, \lambda^{q_N}x_N) = \lambda W(x_1, \ldots, x_N). \]

We define $d$ and $w_i$ for $i \in \{1, \ldots, N\}$ to be the unique positive integers such that $q_i = w_i/d$ for $i \in \{1, \ldots, N\}$ and $\gcd(w_1, \ldots, w_N, d) = 1$. We define $q = \sum q_j$. We also assume that $q_j$ are uniquely determined from $W$ and the affine variety defined by $W$ is singular only at the origin. Equivalently, the hypersurface

\[ X_W = \{W = 0\} \subset W\mathbb{P}^{N-1} \]

defined by $W$ in weighted projective space is nonsingular.

We will only use a small part of the state space and in our situation it can be simplified to the following: We define the extended narrow state space associated to $W$ to be

\[ K_W := \mathbb{Q}^d \]

and write $\{\phi_k\}_{k=0}^{d-1}$ for the basis of $K_W$. Consider the set

\[ \text{nar} := \{k : \text{ for all } j, \langle q_j(k+1) \rangle \neq 0\}, \]

The narrow sector is defined by restricting $K_W$ to the vectors indexed by nar:

\[ K'_W := \oplus_{k \in \text{nar}} \mathbb{Q}\phi_k \]

There is a perfect pairing on the narrow sector defined by

\[ (\phi_i, \phi_j)_W := \delta_{i+j,d-2}. \]

We write $\{\phi^i\}$ for the dual basis of $\{\phi_i\}$ under this pairing.
2.3. **Virtual Structure Sheaf.** For the rest of the paper, we assume that $W$ is Fermat, that is, $w_j|d$ for all $j$. As in [15], we define the integers $s_{ij}$ and $l_{ij}$ by

$$l_i := s_{ij} \frac{d}{w_j} + l_{ij}, \text{ for some } 0 \leq l_{ij} < \frac{d}{w_j}.$$  

We define

$$L_j := L^{w_j} \otimes \mathcal{O}(\sum_i s_{ij}[y_i])$$

By concavity [15, Lemma 1.5], $R^1\pi_*L_j$ is a vector bundle when $k \in \text{narrow}$. We write

$$W_{k,\ell} := \bigoplus (R^1\pi_*L_j)'.$$

The virtual structure sheaf is defined by

$$\mathcal{O}_{k,\ell}^\text{vir} := e^K(W_{k,\ell}) \in K(R^d_{k,\ell}) \otimes \mathbb{Q}$$

where the K-theoretic Euler class of a bundle $V$ is defined by

$$e^K(V) := \sum_k (-1)^k \bigwedge^k V^*.$$  

The virtual structure sheaf is the K-theoretic counterpart of the Witten class in the cohomological FJRW theory.

2.4. **Invariants.** The K-theoretic weighted FJRW invariants are defined as follows

$$\langle \phi_{k_1} \mathbb{L}^{j_1}, \ldots, \phi_{k_m} \mathbb{L}^{j_m} | \phi_{l_1}, \ldots, \phi_{l_n} \rangle_{W,\epsilon} := d \cdot \chi(R^d_{k,\ell} / (\mathbb{R}^d_{k,\ell} \otimes (\otimes_i^m \mathbb{L}_i^{\otimes j_i}))),$$

where $j_i$ are nonnegative integers and $\mathbb{L}_i$ are tautological line bundles over $\mathbb{R}^d_{k,\ell}$ corresponding to the $i$-th orbifold marked points defined in Section 2.1.

The invariants are defined to vanish if any of the $k_i, l_i$ are not narrow or if the underlying moduli space does not exist.

Similarly, the permutation-equivariant version of the invariants

$$\langle \phi_{k_1} \mathbb{L}^{j_1}, \ldots, \phi_{k_m} \mathbb{L}^{j_m} | \phi_{l_1}, \ldots, \phi_{l_n} \rangle_{W,\epsilon, S} := d \cdot \chi(R^d_{k,\ell} / (S_m \times S_n) / (\mathbb{R}^d_{k,\ell} \otimes (\otimes_i^m \mathbb{L}_i^{\otimes j_i})))$$

are defined by the K-theoretic push forward of $\mathcal{O}_{k,\ell}^\text{vir} \otimes (\otimes_i^m \mathbb{L}_i^{\otimes j_i})$ along the projection

$$\pi : R^d_{k,\ell} / (S_m \times S_n) \to [pt].$$

Let $\Lambda$ be a $\lambda$-algebra, that is, an algebra over $\mathbb{Q}$ equipped with abstract Adams operations

$$\Psi^k : \Lambda \to \Lambda, \quad k = 1, 2, \ldots.$$  

Ring homomorphisms $\Psi^k$ satisfy

$$\Psi^r \Psi^s = \Psi^{rs} \quad \text{and} \quad \Psi^1 = \text{id}.$$
We assume that \( \Lambda \) includes the algebra of symmetric polynomials in a given number of variables and \( \Lambda \) has a maximal ideal \( \Lambda_+ \) with the corresponding \( \Lambda_+ \)-adic topology.

We use double bracket notation to denote generating series
\[
\langle \phi_{k_1}, \mathbb{L}^{j_1}, \ldots, \phi_{k_l}, \mathbb{L}^{j_l} \rangle_{W,\epsilon}^{q}(t, u) := \sum_{m,n} \left( \phi_{k_1}, \mathbb{L}^{j_1}, \ldots, \phi_{k_l}, \mathbb{L}^{j_l}, t(\mathbb{L})^m \right) u^n \big|_{t=m,n}
\]
where \( t(q) \) is a Laurent polynomial in \( q \) with coefficients in \( K'_W \otimes \Lambda \),
\[
u := \sum u^k \phi_k \in K'_W, \quad \text{and} \quad t(\mathbb{L})^m := t(\mathbb{L}_1), \ldots, t(\mathbb{L}_m).
\]

2.5. The \( J^\epsilon \) Functions. The permutation-equivariant K-theoretic big \( J^\epsilon \)-function is the following generating function
\[
J^\epsilon(t, u, q) := (1 - q)\phi_0 \sum_{a_i \geq 0, \text{in} \mathbb{Z}} \frac{1}{\prod_i \left( 1 - \frac{u^i}{1} \right)} \prod_{j=1}^{N} \left( \frac{1}{1 - \frac{b_j}{1-q}} \right) \prod_{(b) = (0)} (1 - q^b) \]
\[
+ t(1/q) + \sum_k \phi^k \langle \frac{\phi_k}{1 - q \mathbb{L}_1} \rangle_1,
\]
where the multiplication on \( K'_W \) is defined by
\[
\phi_i \cdot \phi_j := \phi_{i+j \mod d}.
\]

When \( \epsilon > 1 \), we have the big \( J \)-function
\[
J^\infty(t, 1/q) = (1 - 1/q)\phi_0 + t(q) + \sum_k \phi^k \langle \frac{\phi_k}{1 - q \mathbb{L}_1/q} \rangle_1^\infty(t).
\]

We write \( \mathcal{L}_{\infty} \) for the range of the big \( J \)-function in permutation-equivariant quantum K-theory of Landau-Ginzburg model.

We write \( \mathcal{K} \) for the space of rational functions of \( q \) with coefficients from \( K'_W \otimes \Lambda \). The space \( \mathcal{K} \) is equipped with a symplectic form
\[
\Omega(f, g) := -[\text{Res}_{q=0} + \text{Res}_{q=\infty}] (f(q^{-1}) \cdot g(q))_W \frac{dq}{q}.
\]

It can be decomposed into the direct sum
\[
\mathcal{K} = \mathcal{K}_+ \oplus \mathcal{K}_-,
\]
where \( \mathcal{K}_+ \) is the subspace of Laurent polynomials in \( q \) and \( \mathcal{K}_- \) is the complementary subspace of rational functions of \( q \) regular at \( q = 0 \) and vanishing at \( q = \infty \).

Now we state the main result of our paper:

**Theorem 2.4.** For all \( \epsilon > 0 \), \( J^\epsilon(t, u, 1/q) \) is a \( \mathcal{K}[[u]] \)-valued point of \( \mathcal{L}_{\infty} \), in other words, \( J^\epsilon(t, u, 1/q) \) is a formal series of the form
\[
(1 - 1/q)\phi_0 + \hat{t}(q) + \sum_k \phi^k \langle \frac{\phi_k}{1 - q \mathbb{L}_1/q} \rangle_1^\infty(t),
\]
for some \( \dot{t}(q) = t(q) + O(u) \in \mathcal{K}[[u]] \).

3. Proof of Theorem 2.4

The proof of Theorem 2.4 will follow from Lemma 3.1, Lemma 3.2 and the fact that the \( J^\epsilon \)-function satisfies properties (1), (2), and (3) of Lemma 3.2, by the definition of the \( J^\epsilon \)-function. The technique we use in the proof of Lemma 3.1 and Lemma 3.2 is called K-theoretic virtual localization (cf., for example, [14, Theorem 3.3]).

Lemma 3.1. For every \( \epsilon > 0 \), the series

\[
(\partial_{u^r} J^\epsilon(t, u, q), \partial_{t^s} J^\epsilon(t, u, 1/q))
\]

has no pole at roots of unity in \( q \) for all narrow \( r, s \).

Proof. Consider the \( \mathbb{C}^* \)-action on \( \mathbb{P}^1 \)

\[
\lambda[z_0, z_1] := [\lambda z_0, z_1], \quad \lambda \in \mathbb{C}^*.
\]

We consider elements \( p_0, p_\infty \in K_\mathbb{C}^*(\mathbb{P}^1) \) defined by the restriction to the fixed points:

\[
p_0|_0 = q, \quad p_0|_\infty = 1, \quad \text{and} \quad p_\infty|_0 = 1, \quad p_\infty|_\infty = 1/q.
\]

The \( \mathbb{C}^* \)-action on \( \mathbb{P}^1 \) naturally induces an action on \( R_{k,\ell}^{G,d} \) and \( O_{k,\ell}^{G,\text{vir}} \). Consider the equivariant series

\[
\frac{1}{d} \sum (t(L)^m, \phi_s|u^n, \phi_r|ev^*_{m+1}(p_\infty) \otimes ev^*_{m+n+2}(p_0))_{m+1,n+1} := \sum \lambda(R_{(k,s),\ell}(i,r) \otimes t(1/L)^n \otimes u^n \otimes ev_{m+1}(p_\infty) \otimes ev_{m+n+2}(p_0))
\]

where \( ev_i \) are equivariant evaluation maps

\[
ev_i : R_{(k,s),\ell}(i,r) \to \mathbb{P}^1.
\]

By definition, the equivariant series \( \text{(2)} \) has no pole at roots of unity in \( q \). Therefore it remains to prove the claim: \( \text{(1)} = \text{(2)} \). This is done via \( \mathbb{C}^* \)-localization calculation.

In the fixed loci, marked points and nodes of the curves are mapped to 0 and \( \infty \) via \( f \). Invariants vanish when restricted to loci where \( f(x_{m+1}) = 0 \) or \( f(y_{n+1}) = \infty \). We denote a fixed loci by

\[
t : F_{\tilde{k}_0,\tilde{k}_\infty}^{\tilde{k}_0,\tilde{k}_\infty} \to R_{(k,s),\ell}(i,r),
\]

where \( \tilde{k}_0,\tilde{k}_\infty \) is a splitting of the vector \( \tilde{k} \) into subvectors of lengths \( m_0, m_\infty \) over 0 and \( \infty \); \( \tilde{l}_0,\tilde{l}_\infty \) is a splitting of the vector \( \tilde{l} \) into subvectors of lengths
n₀, n∞ over 0 and ∞. By localization formula, the equivariant series (2) equals

\[ \sum_F \chi \left( F, \frac{\iota^* (O_{vir}^{\text{vir}}(\vec{k}, s) \otimes u^\ell) \otimes ev^*_{m+1}(p_\infty) \otimes ev^*_{m+n+2}(p_0))}{e^K_{C^*}(N_F)} \right) \]

where the T-equivariant K-theoretic Euler class of a bundle \(V\) is defined by

\[ e^K_T(V) := \text{tr}_{\lambda \in T} \left( \sum_k (-1)^k \Lambda^k V^* \right) \]

and

\[ t(1/L)^k = t(1/L_1)^{k_1} \otimes \cdots \otimes t(1/L_m)^{k_m} \]

with \(t(q)_k\) the coefficient of \(\phi_k\) in \(t(q)_L\), similar for the notation \(u^\ell\).

A fixed locus is called stable if it has a node over both 0 and ∞. For stable fixed loci, we have

\[ F^{k_\infty, i_\infty}_{k_0, i_0} \cong R^d_{(k_0, k), \epsilon(i_0, r)} \times R^d_{(k_\infty, s, d-2-k), \epsilon(i_\infty)} \]

where \(k\) is uniquely determined. Following the analysis of stable terms in [15, Lemma 2.1], we have

\[ \iota^* (W_G^{(k, s), \epsilon(i, r)}) \cong W_{(k_0, k), \epsilon(i_0, r)} \oplus W_{(k_\infty, s, d-2-k), \epsilon(i_\infty)} \]

Hence

\[ \iota^* e^K (W^{G}_{(k, s), \epsilon(i, r)}) = e^K (W_{(k_0, k), \epsilon(i_0, r)}) \otimes e^K (W_{(k_\infty, s, d-2-k), \epsilon(i_\infty)}) \]

Then we compute the contribution of \(C^\text{\text{\text{-equivariant K-theoretic Euler class}}}\) of the normal bundle \(N_F\). The contribution from smoothing the node at 0 is

\[ 1 - (q/L)^{1/d} \]

In addition there are \(d - 1\) contributions from ghost automorphisms corresponding to the node at 0, they are

\[ 1 - \zeta^k (q/L)^{1/d}, \quad 1 \leq k \leq d - 1, \]

for a primitive \(d\)-th root of unity. Recall that these contributions are in the denominator of (3), using the identity

\[ \sum_{k=1}^{d-1} \frac{1}{1 - \zeta^k x} + \frac{1}{1 - x} = \frac{d}{1 - x^d} \]

and adding these contribution up, we have contribution

\[ \frac{1 - q/L}{d} \]

in the denominator of (3). Similarly, the contribution from smoothing the node at ∞ and the contributions from ghost automorphisms add up to

\[ \frac{1 - 1/(qL)}{d}. \]
The contribution from deforming the map to $\mathbb{P}^1$ is

$$(1 - q)(1 - 1/q).$$

Combining everything together, for stable fixed loci, we have the contribution of stable terms to \((3)\) is equal to

\[(4)\]

\[
\langle t(1/L)^k, \frac{\phi_k}{1 - qL_{m_0+1}^n}, \phi_r \rangle_{m_0+1, n_0+1, 1},
\]

\[
\langle t(1/L)^{k_0}, \frac{\phi_{s}}{1 - qL_{m_0+1}}|u^0, \phi_r \rangle_{m_0+1, n_0+1, 1},
\]

The first factor of \((4)\) corresponds to the coefficient of $\phi^k$ in the stable terms of $\partial_u \mathcal{J}_{t, u, q}^\epsilon(t, u, q)$ and the second factor corresponds to the coefficient of $\phi_k$ in the stable terms of $\partial_{v^0} \mathcal{J}_{t, u, q}^\epsilon(t, u, 1/q)$.

Unstable terms appear in the following two cases. The first case is when $m_\infty = n_\infty = 0$. Its contribution to \((3)\) is equal to

\[(5)\]

\[
\langle t(1/L)^{k_0}, \frac{\phi_s}{1 - qL_{m_0+1}^n}|u^0, \phi_r \rangle_{m_0+1, n_0+1, 1},
\]

where the second factor 1 corresponds to the coefficient of $\phi_s$ in the unstable terms of $\partial_{v^0} \mathcal{J}_{t, u, q}^\epsilon(t, u, 1/q)$.

The second case is when $m_0 = 0$ and $n_0 + 1 \leq 1/\epsilon$.

Following the analysis of unstable terms in [15, Lemma 2.1], we have

$$\iota^* W^G \cong W_\infty \oplus \left( \bigoplus_j \left( \Omega^1 \pi_* \mathcal{L}_j \right)^\vee \right),$$

and Čech representatives of cohomology $H^1((\hat{C}, \mathcal{L}_j)_{\hat{C}})$ are

$$\left\{ \frac{1}{x_0^{d, b} x_1^{q_j + (q_j r) + \sum |q_j (\tilde{l}_0)_i|} - b} \bigg| 0 < b < q_j + \langle q_j r \rangle + \sum |q_j (\tilde{l}_0)_i|, \langle b \rangle = \langle q_j + q_j r + \sum q_j (\tilde{l}_0)_i \rangle \right\}$$

where $x_0, x_1$ are the orbifold coordinates on $\hat{C}$, related to the coarse coordinates on $\mathbb{P}^1$ by

$$z_0 = x_0, z_1 = x_1.$$

Hence, we have

$$e^K \left( \bigoplus_j \left( \Omega^1 \pi_* \mathcal{L}_j \right)^\vee \right) = \prod_{j=1}^N \prod_{0 < b < q_j + (q_j r) + \sum |q_j (\tilde{l}_0)_i| \langle b \rangle = \langle q_j + q_j r + \sum q_j (\tilde{l}_0)_i \rangle} (1 - q^b).$$

The contribution from deforming the $n_0 + 1$ smooth marked points is

$$(1 - q)^{n_0+1}.$$
Hence, the contribution from the second case of the unstable loci is

\[
\left( \frac{u_0^j}{(1 - q)^{g_0}} \prod_{j=1}^{N} \prod_{\substack{0 < b < q_j + (q_j r) + \Sigma_i (q_i l_0_i) \geq 0 \leq q_j + (q_j r) + \Sigma_i (q_i l_0_i) \leq j}} (1 - q^b) \right) \cdot \langle t(1/L) \rangle_{N, \phi}^{\hat{t}_m} \cdot \phi_s \cdot \phi_{k} \cdot 1 - 1/(qL_{m+2}). \]

where

\[ k = r + \sum (\tilde{l}_0)_i \mod d. \]

The factor

\[
\left( \frac{u_0^j}{(1 - q)^{g_0}} \prod_{j=1}^{N} \prod_{\substack{0 < b < q_j + (q_j r) + \Sigma_i (q_i l_0_i) \geq 0 \leq q_j + (q_j r) + \Sigma_i (q_i l_0_i) \leq j}} (1 - q^b) \right)
\]

corresponds to the coefficient of \( \phi_k \) in the unstable terms of \( \partial_{u^r} \mathcal{F}(t, u, z) \).

Adding all the contributions from stable terms (4) and unstable terms (5), (6) together proves (1) = (2), hence proves the lemma.

\[ \Box \]

**Lemma 3.2.** Suppose \( F(t, u, q) \in \mathcal{K}[[u]] \) has the form

\[ F(t, u, q) = (1 - q^a) t(1/q) + f(u, 1/q) + \hat{F}(t, u, q) \]

satisfies

(1): \( f(u, q) \) is a Laurent polynomial in \( q \) with coefficient in \( K_{W}^t[[u]] \) and satisfies \( f(0, q) = 0 \),

(2): \( \hat{F}(t, u, q) \in \mathcal{K}_{-}[[u]] \) only has terms of degree at least 2 in \( t, u \) and is of the form

\[ \sum f_{\xi, m, n, j, s} t^m u^j \frac{(\xi q)^j}{(1 - \xi q)^{j+1}} \phi^s, \]

where \( \xi \) is a root of unity, \( t^m = \sum (t_j^k)^{m_j} \) and similar for \( u^n \).

(3): \( F(t, 0, 1/q) \in \mathcal{L}_{S_{\infty}} \).

Then \( F(t, u, 1/q) \in \mathcal{L}_{S_{\infty}} \) if and only if the series

\[ \langle \partial_{u^r} F(t, u, q), \partial_{t_0^r} F(t, u, 1/q) \rangle \]

has no pole at roots of unity in \( q \) for all narrow \( r, s \).

**Proof.** Suppose \( F(t, u, q) \) lies on \( \mathcal{L}_{S_{\infty}} \) and satisfies properties (1), (2) and (3). Therefore, \( F \) has the form

\[ F(t, u, q) = (1 - q^a) t(1/q) + \sum_k \phi_k \frac{\phi_k}{1 - q^L} \langle t \rangle_{1} \]

where

\[ \hat{f}(q) = t(q) + f(u, q). \]
Following the same localization process as Lemma 3.1, we have

$$\frac{1}{d} \langle \partial_u \hat{t}(L), \phi_s \rangle \text{ev}_1^*(p_0) \otimes \text{ev}_2^*(p_\infty) \rangle_{G, \infty} = (\partial_u F(t, u, q), \partial_{\hat{t}} F(t, u, 1/q))$$

has no pole at roots of unity in $q$ for all narrow $r, s$.

Now suppose $F$ satisfies properties (1), (2), (3) and the series (7) has no pole at roots of unity for all narrow $r, s$. We want to show that $F \in L_{S_\infty}$. Fix a root of unity $\xi$, assume we know $f_{\xi, \hat{t}, j, s}'$ for all $(\hat{t}, \hat{t})$.

The coefficient of $t^{\hat{m} u_{\hat{n}}(1-q)}$ in $(\partial_u F(t, u, q), \partial_{\hat{t}} F(t, u, 1/q))$ is the sum of the leading term $(n' + 1) f_{\hat{m}, \hat{n}, (\hat{m}, r), s}'$ and terms that are determined by induction and $f(u, q)$. This summation is zero since (7) has no pole at roots of unity in $q$. Therefore, $f_{\xi, \hat{t}, j, s}'$ is recursively determined. This implies that $F(t, u, q)$ is recursively determined from $F(t, 0, q)$. Therefore $F(t, u, q)$ and the series

$$(1 - q) \phi_0 + \hat{t}(1/q) + \sum_k \phi_k \langle \phi_k, \frac{1}{1 - qL} \rangle_{1}(\hat{t})$$

agree on the restriction $u = 0$ and satisfy the same recursion relation. Hence

$$F(t, u, q) = (1 - q) \phi_0 + \hat{t}(1/q) + \sum_k \phi_k \langle \phi_k, \frac{1}{1 - qL} \rangle_{1}(\hat{t})$$

lies on $L_{S_\infty}$. $\square$

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