Relevant Attributes in Formal Contexts

Tom Hanika\textsuperscript{1,2}, Maren Koyda\textsuperscript{1,2}, and Gerd Stumme\textsuperscript{1,2}

\textsuperscript{1} Knowledge & Data Engineering Group, University of Kassel, Germany
\textsuperscript{2} Interdisciplinary Research Center for Information System Design
University of Kassel, Germany

tom.hanika@cs.uni-kassel.de, koyda@cs.uni-kassel.de, stumme@cs.uni-kassel.de

Abstract. Computing conceptual structures, like formal concept lattices, is in the age of massive data sets a challenging task. There are various approaches to deal with this, e.g., random sampling, parallelization, or attribute extraction. A so far not investigated method in the realm of formal concept analysis is attribute selection, as done in machine learning. Building up on this we introduce a method for attribute selection in formal contexts. To this end, we propose the notion of relevant attributes which enables us to define a relative relevance function, reflecting both the order structure of the concept lattice as well as distribution of objects on it. Finally, we overcome computational challenges for computing the relative relevance through an approximation approach based on information entropy.

Keywords: Formal Concept Analysis, Relevant Features, Attribute Selection, Entropy, Label Function

1 Introduction

The increasing number of features (attributes) in data sets poses a challenge for many procedures in the realm of knowledge discovery. In particular, methods employed in formal concept analysis (FCA) become more infeasible for large numbers of attributes. Of peculiar interest there is the construction, visualization and interpretation of formal concept lattices, an algebraic structure usually represented through line or order diagrams.

The data structure used in FCA is a formal context, roughly a data table where every row represents an object associated to attributes described through columns. Contemporary such data sets consist of thousands of rows and columns. Since the computation of all formal concepts is at best possible with polynomial delay \cite{11}, thus sensitive to the output size, it is almost unattainable to be computed even for moderately large sized data sets. The problem for the computation of valid (attribute) implications is even more serious since enumerating them is not possible with polynomial delay \cite{7} (in lexicographic order) and only few algorithms are

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known to compute them [18]. Furthermore, in many applications storage space is limited, e.g., mobile computing or decentralized embedded knowledge systems.

To overcome both the computational infeasibility as well as the storage limitation one is required to select a sub-context resembling the original data set most accurately. This can be done by selecting attributes, objects, or both. In this work we will focus on the identification of relevant attributes. This is, due to the duality of formal contexts, similar to the problem of selecting relevant objects. There are several comparable works, e.g., [15], where the author investigated the applicability of random projections. For supervised machine learning tasks there are even more sophisticated methods utilizing filter approaches, which are based on the distribution of labels [20]. Works more related to FCA resort, e.g., to concept sampling [3] and concept selection [13]. Both approaches, however, either need to compute the whole (possibly large) concept lattice or sample from it.

In this work we overcome this limitation and present a feasible approach for selecting relevant attributes from a formal context using information entropy. To this end we introduce the notion for attribute relevance to the realm of FCA, based on a seminal work by Blum and Langley [2]. In there the authors address a comprehensible theory for selecting most relevant features in supervised machine learning settings. Building up on this we formalize a relative relevance measure in formal contexts in order to identify the most relevant attributes. However, this measure is still prone to the limitation for computing the concept lattice. Finally, we tackle this disadvantage by approximating the relative relevance measure through an information entropy approach. Choosing attributes based on this approximation leads to significantly more relevant selections than random sampling does, which we demonstrate in an empirical experiment.

As for the structure of this paper, in Section 2 we give a short overview over the previous works applied to relevant attribute selection. Subsequently we recall some basic notions from FCA followed by our definitions of relevance and relative relevance of attribute selections and its approximations. In Section 4 we illustrate and evaluate our notions through experiments showing the approximations are significantly superior to random sampling. We conclude our work and give an outlook in Section 5.
attribute distribution with respect to the labels in order to weight an attribute’s importance. Hence, they are more efficient but are likely to select redundant or futile features with respect to an underlying machine learning procedure. A well-known method representing this class is RELIEF [12], which denotes the relevance of all features referring to the class label using a statistical method. An entropy based approach of a filter model was introduced by Koller et al. [14]. There the authors introduced selecting features based on the Kullback-Leibler-distance. All these methods incorporate an underlying notion of attribute relevance. This notion was captured and formalized in the seminal work by Blum and Langley in [2], on which we will base the notion for relevant attributes in formal contexts.

There are some approaches in FCA to face the attribute selection problem. In [15] a procedure based on random projection was developed. Less related are methods employed after computing the formal concept lattice, e.g., concept sampling [3] and concept selection [13]. Those could be compared to methods from [16], as they first compute the concept lattice. More related works originate from granular computing with FCA. A basic idea there is to find information granules based on entropy. To this end the authors of [17] introduced an (object) entropy function for formal contexts, which we will utilize in this work as well. Their approach used the principles of granulation as in [21], which is based on merging attributes to reduce the data set. Since our focus is on selecting attributes, we turn away from this notion in general.

3 Relevant Attributes

Before we start with our definition for relevant attributes of a formal context, we want to recall some basic notions from formal concept analysis. For a thorough introduction we refer the reader to [8]. A formal concept is triple \( \mathbb{K} := (G, M, I) \), where \( G \) and \( M \) are finite sets called *object set* and *attribute set*, respectively. Those are connected through a binary relation \( I \subseteq G \times M \), called *incidence*. If \( (g, m) \in I \) for an object \( g \in G \) and an attribute \( m \in M \), we write \( g \mathbin{I} m \) and say “object \( g \) has attribute \( m \).” On the power set of the objects (power set of the attributes) we introduce two operators \( \cdot' : \mathcal{P}(G) \to \mathcal{P}(M) \), where \( A \mapsto A' := \{ m \in M \mid \forall g \in A: (g, m) \in I \} \) and \( \cdot' : \mathcal{P}(M) \to \mathcal{P}(G) \), where \( B \mapsto B' := \{ g \in G | \forall m \in B: (g, m) \in I \} \). A pair \( (A, B) \) with \( A \subseteq G \) and \( B \subseteq M \) is called *formal concept* of the context \( (G, M, I) \) iff \( A' = B \) and \( B' = A \). For a formal concept \( c = (A, B) \) the set \( A \) is called the *extent* \( (\text{ext}(c)) \) and \( B \) the *intent* \( (\text{int}(c)) \). For two concepts \( (A_1, B_1) \) and \( (A_2, B_2) \) there is a natural partial order given through \( (A_1, B_1) \leq (A_2, B_2) \) iff \( A_1 \subseteq A_2 \). The set of all formal concepts of some formal context \( \mathbb{K} \), denoted by \( \mathfrak{B}(\mathbb{K}) \), together with the just introduced partial order constitutes the *formal concept lattice* \( \mathfrak{B}(\mathbb{K}) := (\mathfrak{B}(\mathbb{K}), \leq) \).

A severe computational problem in FCA is to compute the set of all formal concepts, which resembles the CLIQUE problem [11]. Furthermore, the number of formal concepts in a proper sized real-world data set tends to be very large, e.g., 238710 in the (small) mushroom data set, see Section 4.1. Hence, concept lattices for contemporary sized data sets are hard to grasp and hard to cope with through consecutive measures and metrics. Thus, a need for selecting sub-contexts from data sets or sub-lattices is self-evident. This selection can be conducted in the formal
context as well as in the concept lattice. However, the computational feasible choice is to do this in the formal context. Considering a induced sub-context can be done in general in three different ways: One may consider only a subset \( \hat{G} \subseteq G \), a subset \( \hat{M} \subseteq M \), or a combination of those. Our goal for the rest of this work is to identify relevant attributes in a formal context. The notion for (attribute) relevance shall cover two aspects: the lattice structure and the distribution of objects on it. The task at hand is to choose the most relevant attributes which do both reflect a large part of the lattice structure as well as the distribution of the objects on the concepts. For this we will introduce in the next section a notion for relevant attributes in a formal context. Due to the duality in FCA this can easily be translated to object relevance.

### 3.1 Choosing Attributes

There is plenitude of conceptions for describing the relevance of an attribute in a data set. Apparently, the relevance should depend on the particular machine learning or knowledge discovery procedure. One very influential work in this direction was done by Blum and Langley in \([2]\), where the authors defined the (weak/strong) relevance of an attribute in the realm of labeled data. In particular, for some data set of examples \( D \), described using features from some feature set \( F \), where every \( d \in D \) has the label (distribution) \( \ell(d) \), the authors stated: A feature \( x \in F \) is relevant to a target concept-label if there exists a pair of examples \( a, b \in D \) such that \( a \) and \( b \) only differ in their assignment of \( x \) and \( \ell(a) \neq \ell(b) \). They further expanded their notion calling some attribute \( x \) is weakly relevant iff it is possible to remove a subset of the features (from \( a \) and \( b \)) such that \( x \) becomes relevant.

Since in the realm of formal concept analysis data is commonly unlabeled we may not directly adapt the above notion to formal contexts. However, we may motivate the following approach with it. We cope with the lack of a label function in the following way. First, we identify the data set \( D \) with a formal context \( (G, M, I) \), where the elements of \( G \) are the examples and \( M \) are the features describing the examples. Secondly, a formal concept lattice exhibits essentially two almost independent properties, the order structure and the distribution of objects (attributes) on it, cf. Example 3.1. Thus, a conceptual label function then shall reflect both the order structure as well as the distribution of objects in this structure. To achieve this we propose the following.

**Definition 3.1 (Extent Label Function).** Let \( K := (G, M, I) \) be a formal context and its concept lattice \( B(K) \). The map \( \ell_K : G \rightarrow \mathbb{N}, g \mapsto |\{c \in B(K) \mid g \in \operatorname{ext}(c)\}| \) is called extent label function.

One may define an intent label function analogously. Utilizing the just introduced label function we may now define the notion of relevant attributes in formal contexts.

**Definition 3.2 (Relevance).** Let \( K := (G, M, I) \) be a formal context. We say an attribute \( m \in M \) is relevant to \( g \in G \) if and only if \( \ell_{K(m)}(g) < \ell_K(g) \), where \( K_{(m)} := (G \setminus \{m\}, I \cap G \times (M \setminus \{m\})) \). Furthermore, \( m \) is relevant to a subset \( A \subseteq G \) iff there is a \( g \in A \) such that \( m \) is relevant to \( g \). And, we say \( m \) is relevant to the context \( K \) iff \( m \) is relevant to \( G \).
Obviously, the notion for relevant attributes is related to reducibility.

Lemma 3.3 (Irreducible). For \( m \in M \) in \( K = (G,M,I) \) holds

\[
m \text{ is relevant to } K \iff m \text{ is irreducible}.
\]

Proof. We first show \((\Rightarrow)\). We have to show that the following inequality holds:

\[
[\{c \in \mathfrak{B}(K) \mid g \in \text{ext}(c)\}] \leq [\{c \in \mathfrak{B}(K_{(m)}) \mid g \in \text{ext}(c)\}].
\]

Since \( g \in \text{ext}(c) \) and for any \( c \in \mathfrak{B}(K) \) exists a unique concept \( \hat{c} \in \mathfrak{B}(K_{(m)}) \) with \( \text{int}(\hat{c}) \cup \{m\} = \text{int}(c) \), cf. [8, pg 24], we have that \( g \in (\text{int}(\hat{c}) \cup \{m\})' \subseteq \text{int}(\hat{c})' \). For \((\Leftarrow)\) we employ \[8, \text{Prop. 30}\], i.e., there is a join preserving order embedding \((G,M \setminus m,I \cap (G \times (M \setminus \{m\}))) \rightarrow (G,M,I)\) with \((A,B) \rightarrow (A,A')\). Hence, every extent in \( \mathfrak{B}(K_{(m)}) \) is also an extent in \( \mathfrak{B}(K) \) which implies for all \( g \in G \) that \( \ell_{K_{(m)}}(g) < \ell_{K}(g) \).

The last lemma implies that no clarifiable attributes would be considered as relevant, even if the removal of all attributes that have the same closure would have a huge impact on the structure of the concept lattice. Therefore a meaningful

### Example 3.1

Figure 1 (right) shows a formal context and its concept lattice. The objects from there are abbreviated by their first letter in the following. The extent label function of the objects can easily be read from the lattice and is given by \( \ell_{K}(B) = 2, \ell_{K}(F) = 4, \ell_{K}(D) = 2, \ell_{K}(S) = 3 \). Additionally, one can deduct the relevant attributes. E.g., for attribute \( b \) the equality \( \ell_{K_{(b)}}(D) = \ell_{K}(D) \) holds. In contrast \( \ell_{K_{(b)}}(S) < \ell_{K}(S) \), cf. Figure 2. Hence, attribute \( b \) is not relevant to “Dog” but relevant to “Spike-weed”. Thus, \( b \) is relevant to \( K \).

There are two structural approaches in FCA to identify admissible attributes, namely attribute clarifying and reducibility. Those are based purely on the lattice structure. A formal context \( K := (G,M,I) \) is called attribute clarifying iff for all attributes \( m,n \in M \) with \( m' = n' \) follows that \( m = n \). If there is furthermore no \( m \in M \) and \( X \subseteq M \) with \( m' = X' \) the context is called attribute reduced. Analogously, the terms object clarified and object reduced can be determined. An attribute and object clarified (reduced) context is simply called clarified (reduced). The concept lattice of the clarified/reduced context is isomorphic to the concept lattice of the original context. If one of these properties does not hold for an attribute (or an object) the context can be can clarified/reduced by eliminating all such attributes (objects). Obviously, the notion for relevant attributes is related to reducibility.

**Figure 1.** Sub-contexts of "Living Beings and Water" [8]. The attributes are: \( a: \) needs water to live, \( b: \) lives in water, \( c: \) lives on land, \( d: \) needs chlorophyll to produce food, \( e: \) two seed leaves, \( f: \) one seed leaf, \( g: \) can move around, \( h: \) has limbs, \( i: \) suckles its offspring

|          | a | b | c | d | e | f | g | h | i |
|----------|---|---|---|---|---|---|---|---|---|
| Leach    | x | x |   |   |   |   |   |   |   |
| Dream    | x | x | x |   |   |   |   |   |   |
| Frog     | x | x | x | x |   |   |   |   |   |
| Spike-weed| x | x |   | x |   |   |   |   |   |
| Bean     | x | x | x |   |   |   |   |   |   |

|          | a | b | c | d |
|----------|---|---|---|---|
| Bream    | x | x |   |   |
| Frog     | x | x | x | x |
| Dog      | x | x |   |   |
| Spike-weed| x | x | x |   |
identification of relevant attributes restrains to the identification of meaningful equivalence classes \([x]_K := \{y \in M \mid x' = y'\}\) for all \(y \in M\). Accordingly we consider in the following only clarified contexts. Transferring the relevance of an attribute \(m \in M\) to its equivalence class is an easy task which can be executed if necessary.

So far we are only able to decide for the relevance of an attribute but not discriminate attributes upon their relevancy to the concept lattice. To overcome this limitation we introduce in the following a measure which is able to compare the relevancy of two given attributes in a clarified formal context. We consider the change in the object label distribution \(\{(g, \ell_K(g)) \mid g \in G\}\) going from \(K\) to \(K\{m\}\) as characteristic to the relevance of a relevant attribute \(m\). To examine this characteristic in more detail and to make it graspable via a numeric value we propose the following inequality:

\[
\sum_{g \in G} \ell_K\{m\}(g) < \sum_{g \in G} \ell_K(g)
\]

This approach offers not only the possibility to verify the existence of a change in the object label distribution but also to measure the extent of this change. We may quantify this via

\[
\sum_{g \in G} \ell_K\{m\}(g) / \sum_{g \in G} \ell_K(g) = t(m)
\]

whence \(t(m) < 1\) for all attributes \(m \in M\).

**Definition 3.4 (Relative Relevance).** Let \(K = (G, M, I)\) be a clarified formal context. The attribute \(m \in M\) is relative relevant to \(K\) with

\[
r(m) := 1 - \frac{\sum_{c \in B(K)\{m\}} \left| \{g \in \text{ext}(c) \mid g \in \text{ext}(c) \} \right|}{\sum_{c \in B(K)} \left| \{g \in \text{ext}(c) \mid g \in \text{ext}(c) \} \right|} = 1 - t(m).
\]

The values of \(r(m)\) for an attribute are in \([0, 1]\). We say \(m \in M\) is more relevant to \(K\) than \(n \in M\) iff \(r(n) < r(m)\). Double counting leads to the following proposition.

**Proposition 3.5.** Let \(K = (G, M, I)\) be a formal context. For all \(m \in M\) holds

\[
r(m) = 1 - \frac{\sum_{c \in B(K)\{m\}} \left| \text{ext}(c) \right|}{\sum_{c \in B} \left| \text{ext}(c) \right|}
\]

with \(B(K)\{m\} = \{c \in B\mid (\text{int}(c) \setminus \{m\})' = \text{ext}(c)\}\).

This statement reveals an interesting property of the just defined relative relevance. In fact, an attribute \(m \in M\) is more relevant to an formal context \(K\) if the join preserving sub-lattice, which one does obtain by removing \(m\) from \(K\),
does exhibit a smaller sum of all extent sizes. This will enable us to find proper approximations to the relative relevance in Section 3.2.

Example 3.2. Excluding one attribute from the running example in Figure 1 (right) results in the sub-lattices in Figure 2. The relative relevance of the attributes to the original context is given by \( r(a) = 0, r(b) = 4/11, r(c) = 3/11, \) and \( r(d) = 1/11. \)

By means of \( r(\cdot) \) it is also possible to measure the relative relevance of a set \( N \subseteq M. \) We simply lift 3.5 by
\[
\sum_{c \in \mathcal{B}(K)} \text{ext}(c) / \sum_{c \in \mathcal{B}(K)} |\text{ext}(c)| \text{ with } \mathcal{B}(K)_N = \{ c \in \mathcal{B}(K) \mid (\text{int}(c) \setminus \{N\})' = \text{ext}(c) \}.
\]

Lemma 3.6. Let \( \mathcal{K} = (G, M, I) \) be a formal context and \( S, T \subseteq M \) attribute sets. Then

\begin{enumerate} 
\item \( S \subseteq T \implies r(S) \leq r(T), \) and
\item \( r(S \cup T) \leq r(T) + r(S). \)
\end{enumerate}

Proof. We prove i) by showing \( \sum_{c \in \mathcal{B}(K)} |\text{ext}(c)| > \sum_{c \in \mathcal{B}(K)} |\text{ext}(c)|. \) Since \( \forall c \in \mathcal{B}(K) \)
we have \( (\text{int}(c) \setminus T)' \supseteq (\text{int}(c) \setminus S)' \supseteq \text{ext}(c) \) we obtain \( \mathcal{B}(K)_S \supseteq \mathcal{B}(K)_T, \) as required.

For ii) we will use the identity (*) \( \mathcal{B}(K)_S \cap \mathcal{B}(K)_T = \mathcal{B}(K)_{S \cup T}, \) which follows from \( (\text{int}(c) \setminus S)' = \text{ext}(c) \cap (\text{int}(c) \setminus T)' = \text{ext}(c) \iff (\text{int}(c) \setminus (S \cup T))' = \text{ext}(c) \) for all \( c \in \mathcal{B}(K). \) This equivalence is true since \( \Rightarrow: \)
\[
(\text{int}(c) \setminus (S \cup T))' = ((\text{int}(c) \setminus S) \cap (\text{int}(c) \setminus T))' = (\text{ext}(c) \cup (\text{int}(c) \setminus T))' = \text{ext}(c) \cup \text{ext}(c) = \text{ext}(c)
\]
\( \Leftarrow: \) From \( (\text{int}(c) \setminus (S \cup T))' \supseteq (\text{int}(c) \setminus S)' \) and \( (\text{int}(c) \setminus (S \cup T))' \supseteq (\text{int}(c) \setminus T)' \) we obtain with i) that \( (\text{int}(c) \setminus S)' = \text{ext}(c). \) We now show ii) by proving the inequality \( \sum_{c \in \mathcal{B}(K)_{S \cup T}} |\text{ext}(c)| \leq \sum_{c \in \mathcal{B}(K)_S} |\text{ext}(c)| + \sum_{c \in \mathcal{B}(K)_{S \cup T} \setminus \mathcal{B}(K)_S} |\text{ext}(c)|. \)

Using \( \mathcal{B}(K)_S \setminus \mathcal{B}(K)_{S \cup T} \cup \mathcal{B}(K)_{S \cup T} = \mathcal{B}(K)_S \) where \( \mathcal{B}(K)_S \setminus \mathcal{B}(K)_{S \cup T} \cap \mathcal{B}(K)_{S \cup T} = \emptyset \) we find an equivalent equation employing (*)
\[
\sum_{\mathcal{B} \in \mathcal{B}_X} |\text{ext}(c)| + \sum_{\mathcal{B} \in \mathcal{B}_T} |\text{ext}(c)| + 2 \cdot \sum_{\mathcal{B} \in \mathcal{B}_{S \cup T}} |\text{ext}(c)| \leq \sum_{\mathcal{B} \in \mathcal{B}_S} |\text{ext}(c)| + \sum_{\mathcal{B} \in \mathcal{B}_T} |\text{ext}(c)| + \sum_{\mathcal{B} \in \mathcal{B}_{S \cup T} \setminus \mathcal{B}_S} |\text{ext}(c)| + 2 \cdot \sum_{\mathcal{B} \in \mathcal{B}_{S \cup T} \setminus \mathcal{B}_T} |\text{ext}(c)|
\]
\[
0 \leq \sum_{\mathcal{B} \in \mathcal{B}_{(S \cup T) \setminus (S \cup T)}} |\text{ext}(c)|
\]
where \( \mathcal{B}_X \) is short for \( \mathcal{B}(K)_X. \)

Equipped with the notion for relative relevance and some basic observations we are ready to state the associated computational problem. We imagine that in real-world applications attribute selection is a task to identify a set \( N \subseteq M \) of the most relevant attributes for a given cardinality \( n \in \mathbb{N}, \) i.e., an element from \( \{N \subseteq M \mid |N| = n \land r(N) \text{ maximal}\}. \) We call such a set \( N \) a maximal relevant set.

Problem 3.1 (Relative Relevance Problem (RRP)). Let \( \mathcal{K} = (G, M, I) \) be a formal context and \( n \in \mathbb{N} \) with \( n < |M|. \) Find a subset \( N \subseteq M \) with \( |N| = n \) such that \( r(N) \geq r(X) \) for all \( X \subseteq M \) where \( |X| = n. \)
Solving 3.1 is twofold infeasible. First, as \( n \) increases does the number of possible subset combinations. The determination of a maximal relevant set requires the computation and comparison of \( \binom{|M|}{|N|} \) different relative relevances, which presents itself infeasible. Secondly, does the computation of the relative relevance presume that the set of formal concepts is computed. This states also an intractable problem for large formal contexts, which are the focus for applications of the proposed relevance selection method. To overcome the first limitation we suggest an iterative approach. Instead of testing every subset of size \( n \) we construct \( N \subseteq M \) by first considering all singleton sets \( \{m\} \subseteq M \). Consecutively, in every step \( i \) where \( X \) is the so far constructed set we find \( x \in M \) such that \( r(X \cup \{x\}) \geq r(X \cup \{m\}) \) for all \( m \in M \). This approach requires the computation of only \( \sum_{i=|M|-|N|+1}^{M} i \) different relative relevances and their comparisons, which is simplified \( n \cdot |M| - (n-1) \cdot n/2 \). We call a set obtained through this approach an \textit{iterative maximal relevant set} IMRS. In fact the IMRS does not always correspond to the maximal relevant set. In \((G,M,I)\) where \( G = \{1,2,3,4\} \), \( M = \{a,b,c,d\} \) and \( I = \{(1,a),(1,e),(1,d),(2,a),(2,b),(3,b),(3,e),(4,d)\} \) is \( b \) the most relevant attribute, i.e., \( r(b) > r(x) \) for all \( x \in M \setminus \{b\} \). However, we find \( r(\{a,c\}) > r(\{b,x\}) \) for all \( x \in M \setminus \{b\} \). Hence, the relative relevance of an IMRS indicates a lower bound for the relative relevance of the maximal relevant set.

3.2 Approximating RRP

Motivated by the computational infeasibility of 3.1 we investigate in this section the possibility of approximating RRP, more specifically the IMRS. Approaches for this approximation have to incorporate both aspects of the relative relevance the structure of the concept lattice and the distribution of the objects. Considering the former is not complicated due to [8, Proposition 30], which states that for any \((G,M,I)\) is \( \mathcal{B}((G,N,I \cap (G \times N))) \) join preserving order embeddable into \( \mathcal{B}((G,M,I)) \) for any \( N \subseteq M \). Thus, this aspect can be represented through a quotient \( |\mathcal{B}(\mathbb{K})|_{M,N} / |\mathcal{B}(\mathbb{K})| \), which is a special case of the maximal common sub-graph distance, see [5]. Hence, whenever searching for the largest \( \mathcal{B}((G,N,I \cap (G \times N))) \) the obvious choice is to optimize for large contra-nominal scales in sub-contexts of \((G,M,I)\). For example, when selecting three attributes in Figure 1 (left) the largest join preserving order embeddable lattice would be generated by the set \( \{b,c,d\} \). However, the relative relevance of \( \{b,c,g\} \) is significantly larger, in particular, \( r(\{b,c,d\}) = 17/33 \) and \( r(\{b,c,g\}) = 19/33 \).

Considering the second requirement, the distribution of the objects on the concept lattice, the sizes of the concept extents have to be incorporated. Since they are unknown, unless we compute the concept lattice, we need a proxy for estimating the influence of those. Accordingly, we want to reflect this with the quotient \( E(\mathbb{K}_{M \setminus N}) / E(\mathbb{K}) \), which estimates the change of the object distribution on the concept lattices when selecting a set \( N \subseteq M \). This quotient does employ a mapping \( E : \mathbb{K} \rightarrow \mathbb{R}, \mathbb{K} \mapsto E(\mathbb{K}) \), which is to be found. A natural candidate for this mapping would be information entropy, as introduced by Shannon in [19]. He defined the entropy of a discrete set of probabilities \( p_1, \ldots, p_n \) as \( H = -\sum_{i \in I} p_i \log p_i \). We adapt this formula to the realm of formal contexts as follows.
Definition 3.7. Let $\mathcal{K} = (G, M, I)$ be a formal context. Then the Shannon object information entropy of $\mathcal{K}$ is given as follows.

$$E_{SE}(\mathcal{K}) = \sum_{g \in G} \frac{|g''|}{|G|} \log_2 \left( \frac{|g''|}{|G|} \right)$$

For this entropy function we employ the quotient $|g''|/|G|$, which does reflect the extent sizes of the object concepts of $\mathcal{K}$. Obviously this choice does not consider all concept extents. However, since every extent in a concept lattice is either the extent of a object concept or the intersection of finitely many extents of object concepts we see that Shannon object information entropy does relate to all extents to some degree. We found another candidate for $E$ in the literature [17]. The authors there introduced an entropy function which is roughly speaking the mean distance of the extents of object concepts to the complete set objects.

Definition 3.8. Let $\mathcal{K} = (G, M, I)$ be a formal context. Then the object information entropy of $\mathcal{K}$ is given as follows.

$$E_{OE}(\mathcal{K}) = \sum_{g \in G} \left( 1 - \frac{|g''|}{|G|} \right)$$

We directly observe that this entropy decreases as the number of objects having similar attribute sets increases. Furthermore, we recognize an essential difference for $E_{OE}$ compared to $E_{SE}$. The Shannon object information entropy reflects on the number of necessary bits to encode the formal context. In contrary does the object information entropy reflect on the average number of bits to encode an object from the formal context. To enhance the first grasp of the just introduced functions as well as the relative relevance defined in Definition 3.4 we want to investigate them on well known contextual scales. In particular, the ordinal scale $O_n := ([n], [n], \leq)$, the nominal scale $N_n := ([n], [n], =)$, and the contranominal scale $C_n := ([n], [n], \neq)$, where $[n] := \{1, \ldots, n\}$. Since there is a bijection between the set $\{1, \ldots, n\}$ to the extent sizes $|g''|$ in an ordinal scale we obtain that $E_{SE}(O_n) = -\sum_{i=1}^{n} \frac{1}{n} \log_2 \left( \frac{1}{n} \right)$ and $E_{OE}(O_n) = \frac{1}{n} \sum_{i=1}^{n} \left( 1 - \frac{1}{n} \right) = \frac{1}{n} \frac{n(n+1)}{2} = \frac{n+1}{2}$. The former diverges to $\infty$ whereas the latter converges to $1/2$. Based on the linear structure of $\mathcal{B}(O_n)$ we conclude that the set $\mathcal{B}(O_n) \setminus \mathcal{B}(O_n)_{\{m\}} = \{(m', m'')\}$ for all $m \in M$. So the relative relevance of the attribute $m \in M$ amounts to $r(m) = 1 - \left( \sum_{i=1}^{n} i - |m''| \right) / \sum_{i=1}^{n} i = 2|m''|/(n \cdot (n+1))$.

Both the nominal scale as well as the contranominal scale satisfy $g'' = g$ for all $g \in G$ for different reasons. We conclude that $E_{SE}$ and $E_{OE}$ evaluate respectively equally for $N_n$ and $C_n$. In detail, $E_{SE}(N_n) = E_{SE}(C_n) = -\sum_{g \in G} \frac{1}{n} \log_2 \left( \frac{1}{n} \right) = \log_2(n)$ and $E_{OE}(N) = E_{OE}(C) = \frac{1}{n} \sum_{g \in G} \left( 1 - \frac{1}{n} \right) = \frac{n-1}{n}$. For the relative relevance we observe that $r(m) = r(n)$ for all $m, n \in M$ in the case of the nominal/contranominal scale. This is due to the fact that every attribute is part of the same number of concepts. For the nominal scale holds $r(m) = 1 - \frac{2m-1}{2n}$ for all $m \in M$. Hence, as the number of attributes increases does the relevance of a single attribute converge to zero. The relative relevance of an objects in the case of the contranominal scale is $r(m) = 1 - \frac{\sum_{k=0}^{n} (\binom{n}{k})(n-k) - \sum_{k=0}^{m-1} (\binom{n}{k})(n-k)}{\sum_{k=0}^{n} (\binom{n}{k})(n-k)}$ for all $m \in M$. 

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Figure 3. Relevance of attribute selections through entropy (SE, OE), IMRS (IR), and random selection (RA) for the “Living beings in water” (left) and the zoo context (right).

Example 3.3. Revisiting our running example Figure 1 (right). This context has four objects with $\{B\}'' = \{B, F, S\}$, $\{F\}'' = \{F\}$, $\{D\}'' = \{F, D\}$ and $\{S\}'' = \{S\}$. Its entropies are given by $E_{OE}(K) = \frac{1}{4} \sum_{g \in G} \left(1 - \frac{|g'|}{4}\right) \approx 0.56$ and $E_{SE}(K) \approx 0.45$.

Considering both aspects discussed in this section we now want to introduce a function which shall be capable of approximating RRP.

Definition 3.9. Let $K = (G, M, I)$ and $K_N = (G, N, I \cap (G \times N))$ be formal contexts with $N \subseteq M$. The entropic relevance approximation (ERA) of $N$ is defined as

$$ERA(N) := \frac{|B(K_N)|}{|B(K)|} \cdot \frac{E(K_N)}{E(K)}$$

First, the ERA compares the number of concepts in a given formal context to the number of concepts in a sub-context on $N \subseteq M$. This reflects the structural impact when restricting the attribute set. Secondly, an quotient is evaluated where the entropy of $K_N$ is compared to the entropy of $K$. When using Definition 3.9 for finding a subset $N \subseteq M$ with maximal (entropic) relevance it suffices to compute $N$ such that $|B(K_N)| \cdot E(K_N)$ is minimal. This task is essentially less complicated since we only have to compute $|B(K_N)|$ and $E(K_N)$ for some comparable small formal context $K_N$.

4 Experiments

To assess the ability for approximating relative relevance through Definition 3.9 we carried out several experiments in the following fashion. For all data set we computed the iterative maximal relevant subsets of $M$ of sizes one to seven (or ten) in the obvious manner. We decided for those fixed numbers for two reasons. First, using a relative number, e.g., 10% of all attributes, would still lead to an infeasible computation when the initial formal context is very large. Secondly, formal contexts with up to ten attributes permit a plenitude of research methods that are impracticable for larger contexts, in particular, human evaluation.
Relative Relevance Comparison Mushroom

Relative Relevance Comparison Wiki44k

Figure 4. Relevance of attribute selections through entropy (SE, OE), IMRS (IR), and random selection (RA) for the mushroom (left) and the wiki44k context (right).

Then we computed subsets of $M$ using ERA, for which we used both introduced entropy functions, and their relative relevance. Finally, we sampled subsets of $M$ randomly at least $|M| \cdot 10$ many times and computed their average relative relevance as well as the standard deviation in relative relevance.

4.1 Data Set Description

A total of 2678 formal contexts were considered in this experimental study. From those were 2674 contexts excerpts from the BibSonomy platform\(^3\) as described in [1]. All those contexts are equipped with an attribute set of twelve elements and a varying number of objects. The particular extraction method is described in detail in [4]. For the rest we revisited three data sets well known in the realm of formal concept analysis, i.e., *mushroom*, *zoo*, *water* [6, 8], and additionally a data set *wiki44k* introduced in [9], which is based on a 2014 Wikidata\(^4\) database dump. The well-known *mushroom* data set is a collection of 8124 mushrooms described by 119 (scaled) attributes and exhibits 238710 formal concepts. The *zoo* data set possesses 101 animal descriptions using 43 (scaled) attributes and exhibits 4579 formal concepts. The *water* data set, more formally “Living beings and water”, has eight objects and nine attributes and exhibits 19 formal concepts. Finally, the *wiki44k* has 45021 objects and 101 attribute exhibiting 21923 formal concepts.

4.2 Results

In Figures 3 to 5 we depicted the results of our computations. We observe in all experiments that the relative relevance of the subsets found through the iterative approach are an upper bound for the relative relevance of all subsets computed through entropic relevance approximation or random selection, with respect to the same size of subset. In particular we find IMRS of cardinality seven and above have a relative relevance of at least 0.8. Moreover, the relative relevance of the attribute subsets

\(^3\)https://www.kde.cs.uni-kassel.de/wp-content/uploads/bibsonomy/

\(^4\)https://www.wikidata.org
selected by both ERA versions (SE or OE) exceed the relative relevance of the randomly selected subsets except for the Shannon object information entropy for $|N|=1$ and $|N|=2$ in the zoo context. Principally we find for contexts containing a small number of attributes (Figure 3) a large increase of the distance between the relative relevance of the randomly selected attributes and the attribute sets selected through the entropy approach. This characteristic manifests in the relative relevance of both ERA selections excelling not only the mean relative relevance of randomly chosen attribute sets but also the standard deviation for subset sizes of $|N|=4$ and above. In the case of contexts containing a huge number of attributes this observation can be made for selections with $|N|=1$, already. Furthermore, the interval between the relative relevance of the attribute subsets selected by both ERA versions and the relative relevance of the randomly selected subsets is significantly larger than in the case of contexts with small attribute set sizes. In general we may point out that neither of the entropies seems preferable over the other in terms of performance. In Figure 5 we show the results for the experiment with the 2674 formal contexts from BibSonomy. We plotted for all three methods, ERA-OE/SE and random, the mean distance in relative relevance to the IMRS of the same size together with the standard deviation. We detect a significant difference for randomly chosen and ERA chosen sets with respect to their relative relevance. The deviation for both ERA is bound by 0 and 0.12. In contrast, the relative relevance for randomly selected sets is bound by 0.09 and 0.6.

4.3 Discussion

We found in our investigation that attribute sets obtained through the iterative approach for relative relevance do have a high relevance value. Even though their relative relevance is only an lower bound compared to the maximal relevant set they do
exhibit a relative relevance of 0.8 for attribute set sizes seven and above. We conclude from this that iterative approach is a sufficient solution to the relative relevance problem. Based on this we may deduce that entropic relative approximation is also a good approximation for a solution to the RRP. In particular, in large formal contexts investigated in this work the approximation was even better than in the smaller ones.

5 Conclusion

By defining the relative relevance of attribute sets in formal contexts we introduced a novel notion for attribute selection. This notion respects both the structure of the concept lattice and the distribution of the objects on it. To overcome computational limitations, which arise from the notion of relative relevance, we introduced an approximation based on two different entropy functions adapted to formal contexts. For this we used a combination of two factors. The change in the number of concepts and the change in entropy that arise by the selection of an attribute subset. The experimental evaluation for relative relevance as well as the entropic approximation seem to comply with the theoretical modeling.

We may conclude our work with two open questions. First, even though IMRS seems a good choice for relevant attributes we suspect that computing the maximal relevant set, with respect to RRP, can be achieved more feasible as presented in this work. Secondly, so far our justification for RRP is based on theoretical assumptions and a basic experimental study. We imagine, and are curious, if maximal relevant attribute sets are also employable in supervised machine learning setups. For example, one may perceive the task of adding a new object to a given formal context as instance of such a setup. The question is, how capable is the context to add this object to an already existing concept.

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