A high order projection method for low Mach number flows

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Abstract

The high order projection hybrid finite volume/finite element method presented in [BFSVC14] and [BFTVC18] is extended to solve compressible low Mach number flows. The main differences with the incompressible method are highlighted. Detailed analysis of the flux term is presented aiming at correcting the spurious oscillations originated by changes in density. As a result a new upwind term is added to the numerical flux function. Moreover, LADER methodology is applied so a second order in space and time scheme is obtained. The mass conservation equation is used in order to compute the extrapolated and half-step in time evolved values of density. The proposed modifications on the methodology are endorsed by the study of the scalar advection-diffusion-reaction equation with time and space dependent coefficients. The formal order of accuracy of the schemes is assessed for various test problems.

Keywords: compressible low Mach number flows, projection method, finite volume methods, finite element methods, high order schemes.
1 Introduction

Low Mach number flows are involved in a wide variety of natural phenomena and industrial processes. Therefore, an increasing number of researchers have focused, during the last decades, on better understanding their behaviour. Classical approaches to solve this kind of flows comprise mixed and discontinuous Galerkin finite element methods (see [BF91], [HFB86], [CKS00], [CS01] and [TD17]) and finite volume methods (see [Tor09] and references therein).

It has been shown that, among low Mach number flows, the difference compressible/incompressible deeply changes the physics and also the numerics. While for compressible flows pressure is directly related to other flow variables as density and energy via a state equation, for incompressible flows pressure acts as a Lagrange multiplier that adapts itself to ensure that the velocity satisfies the incompressibility condition. In order to handle the latter situation, the typical explicit stage of finite volume methods has to be complemented with the so-called projection stage where a pressure correction is computed in order to get a divergence-free velocity. Many papers exist in the literature devoted to introduce and analyse projection finite volume methods for incompressible Navier-Stokes equations (see, for instance, [BFSVC14], [BFTVC18], [BDA+02] or [Col90]). These projection methods, initially introduced for incompressible flows can be easily adapted to low-Mach number flows. In fact, the main difference is that the divergence-free condition for the velocity is replaced by an equation prescribing the divergence of the linear momentum density which is a conservative variable. In order to get stability, staggered grids have been used to discretize velocity and pressure. While this can be done straightforwardly in the context of structured meshes, the adaptation to unstructured meshes is more challenging (see [BDDVC98], [GGHL08], [GLL12], [PBH04], [TA07]).

The scope of this paper is to extend the high order projection hybrid FV/FE method introduced in [BFSVC14] and [BFTVC18] to simulate incompressible flows, to the case of compressible low Mach number flows. To this end we will assume that temperature and composition of the mixture are known (maybe by solving energy and species balance equations), so that the system of equations to be solved reduces to mass and momentum conservation equations and equation of state.

Starting from a 3D tetrahedral finite element mesh of the computational domain, the equation of the transport-diffusion stage is discretized by a finite volume method associated with a dual finite volume mesh where the
nodes of the volumes are the barycentre of the faces of the initial tetrahedra. These volumes, which allow us for an easy implementation of flux boundary conditions, have already been used, among others, for the 2D shallow water equation (see [BDDVC98]), for solving conservative and non conservative systems in 2D and 3D (see [THD09] and [DHC¹¹]) and for DG schemes employed to solve compressible Navier-Stokes equations (see [TD17]). Further, for time discretization we use the explicit Euler scheme.

One of the main challenges regarding compressible Navier-Stokes equations is the time and space dependency of density. The use of classical upwind numerical flux functions like the ones involved in the $Q$-scheme of van Leer or the Rusanov scheme may produce spurious oscillations on the solution. In order to get insight in study their causes we simplify the problem by posing the scalar advection equation with variable advection coefficient. Analysing the flux term we observe that this bad behaviour may be caused by a lack of upwinding related to the spatial derivative of the normal flux with respect to the density (see [BLVC17] for a detailed analysis on multi-component Euler equations). Therefore, we propose a modification of the flux function aiming to ensure the stability of the resulting scheme.

Seeking to attain a second order in time and space scheme we apply LADER methodology, a modification of ADER methods (see [BFTVC18], [TMN01] and [Tor09]). This fully discrete approach relies on non-linear reconstructions and the resolution of generalized Riemann problems to produce schemes arbitrarily accurate in both space and time. To develop a high order numerical scheme for the compressible Navier-Stokes equations we will perform the non-linear reconstructions not only in the conservative unknowns but also in the density. Moreover, we construct, analyse and asses LADER scheme for the scalar advection-diffusion-reaction equation with variable coefficients to confirm the necessity of reconstructing the advection coefficient.

Concerning the projection stage, the pressure correction is computed by continuous piecewise linear finite elements associated with the initial tetrahedral mesh. The derivative of the density with respect to time is needed to define the source term involved in the projection stage. Therefore, a pre-projection stage, in which the state equation is used to compute the density from the provided mixture temperature and composition, has been introduced.

The use of the staggered meshes together with a simple specific way of passing the information from the transport-diffusion stage to the projection one and vice versa leads to a stable scheme. The former is done by re-
defining the conservative variable (i.e. the momentum density) constant per
tetrahedron. Conversely, the finite element pressure correction is redefined
to constant values on the faces of the finite volumes and then used in the
transport-diffusion stage.

The outline of the paper is as follows. In Section 2 we recall the system
of equations modelling low Mach number flows. Section 3 is devoted to the
discretization of the equations and the description of the algorithm stages.
In Section 4, the transport-diffusion stage is analysed highlighting the main
differences with the incompressible case. A modification on the flux function
is proposed to avoid the spurious oscillations due to the space dependency
of density. Moreover, LADER methodology is applied providing a second
order accuracy scheme. The computation of the source term for the pro-
jection stage is detailed in Section 5. The projection and post-projection
stages are depicted in Section 6. Finally, the treatment of the boundary
conditions is described in Section 7 and some of the numerical results ob-
tained are shown in Section 8. The appendix includes the resolution of the
scalar advection-diffusion-reaction equation with time and space dependent
advection coefficient.

2 Governing equations

In this section, the system of equations to be solved is introduced. The model
for compressible low Mach number flows is recalled following [Ber05]. The
underlying assumption is that the Mach number, \( M \), is sufficiently small so
that the pressure \( p \) (N/m\(^2\)) can be decomposed into a constant function \( \pi \)
and a small perturbation \( \pi \),

\[
p (x, y, z, t) = \pi (t) + \pi (x, y, z, t), \quad \frac{\pi}{\pi} = O (M^{-2}), \tag{1}
\]

where \( \pi (t) \) is a data. The perturbation will be neglected in the state equation
but it has to be retained in the momentum equation.

Assuming that the mass fractions of the species and the temperature of
the mixture are given, the system of equations to be solved becomes

\[
\frac{\partial \rho}{\partial t} + \text{div} \mathbf{w}_u = 0, \tag{2}
\]

\[
\frac{\partial \mathbf{w}_u}{\partial t} + \text{div} \mathcal{F}^{\mathbf{w}_u} (\mathbf{w}_u, \rho) + \text{grad} \pi - \text{div} \tau = 0, \tag{3}
\]
\[
\tau = \mu (\text{grad} \mathbf{u} + \text{grad} \mathbf{u}^T) - \frac{2}{3} \mu \text{div} \mathbf{u},
\]
\[
\rho = \frac{\pi}{R \theta},
\]
(4)
(5)

where standard notations are used:

- \( \rho \) is density (kg/m\(^3\)),
- \( \mathbf{u} = (u_1, u_2, u_3)^T \) is velocity (m/s),
- \( \mathbf{w}_u := \rho \mathbf{u} \) is the vector of conservative variables related to velocity (kg/m\(^2\)s),
- \( \mathcal{F}^\mathbf{w}_u \) is the flux tensor:
  \[
  \mathcal{F}_i^\mathbf{w}_u(\mathbf{w}_u, \rho) = \frac{1}{\rho} \mathbf{w}_{u,i} \mathbf{w}_u = u_i \mathbf{w}_u, \quad i = 1, 2, 3,
  \]
- \( \tau \) is the viscous part of the Cauchy stress tensor (Pa),
- \( \mu \) is the dynamic viscosity (kg/ms),
- \( \theta \) is the temperature (K),
- \( R \) is the gas constant (J/kgK),
- \( \mathcal{R} \) the universal gas constant (8.314 J/molK), \( \mathcal{M}_l \) the molar mass of the \( l \)-th species, \( y_l \) (g/mol) its mass fraction and \( N_e \) the number of species of the mixture.

- \( f_u \) is a generic source term used, for instance, for manufactured test problems (N/m\(^3\)).

In case the data provided is not the temperature but the specific enthalpy, \( h \) (J/kg), we can recover the temperature by taking into account the standard enthalpy formation, \( h_0 \), and the specific heat at constant pressure, \( c_\pi \) (J/kgK):

\[
h(\theta) = h_0 + \int_{\theta_0}^{\theta} c_\pi(r)dr,
\]
(6)

where \( \theta_0 \) denotes the standard temperature, usually 273.15 K.
3 Numerical discretization

The numerical discretization of the complete system is performed by extending the projection method first put forward in [BFSVC14]. The methodology proposed decouples the computation of the linear momentum density (from now on, the conservative velocity) and the pressure perturbation. At each time step, equation (3) is solved with a finite volume method (FVM) and so an approximation of $w_u$ is obtained. Next, the projection step is applied to system (2)-(3). The pressure correction is provided by a piecewise linear finite element method (FEM). In the post-projection step, an approximation of $w_u$ satisfying the divergence condition, (2), is obtained.

We start by considering a two-stage in time discretization scheme: in order to get the solution at time $t^{n+1}$, we use the previously obtained approximations $W^n_u$ of conservative velocities $w_u(x, y, z, t^n)$, $U^n$ of velocity $u(x, y, z, t^n)$, $\rho^n$ of density $\rho(x, y, z, t^n)$ and $\pi^n$ of pressure perturbation $\pi(x, y, z, t^n)$, and compute $W^{n+1}_u$, $\rho^{n+1}$ and $\pi^{n+1}$ from the following system of equations:

\[
\frac{1}{\Delta t} \left( \tilde{W}^{n+1}_u - W^n_u \right) + \text{div} \mathcal{F}W_u (W^n_u, \rho^n) + \text{grad} \pi^n - \text{div} \tau^n = 0, \quad (7)
\]

\[
\rho^{n+1} = \frac{\pi}{\mathcal{R} \theta^{n+1} \sum_{i=1}^{N_x} Y_i^{n+1}} \cdot \frac{M_i}{N_x}, \quad (8)
\]

\[
\frac{1}{\Delta t} \left( W^{n+1}_u - \tilde{W}^{n+1}_u \right) + \text{grad} (\pi^{n+1} - \pi^n) = 0, \quad (9)
\]

\[
\text{div } W^{n+1}_u = Q^{n+1}, \quad (10)
\]

where $Y^{n+1}$ and $\theta^{n+1}$ are the evaluations of the given functions $y(x, y, z, t^{n+1})$ and $\theta(x, y, z, t^{n+1})$, respectively, and $Q^{n+1}$ is an approximation of $q(x, y, z, t^{n+1}) = -\partial_t \rho(x, y, z, t^{n+1})$. The field $\tilde{W}^{n+1}_u$ computed at this stage does not necessarily satisfy the divergence constraint at $t^{n+1}$. Let us notice that $q(x, y, z, t^{n+1})$ could also be provided so that $Q^{n+1}$, would be its evaluation. The procedure to determine the solution of the above system is similar to the already presented in [BFSVC14] and [BFTVC18] for the incompressible case. Firstly, we solve equation (7) to obtain an intermediate approach of the conservative variables $\tilde{W}^{n+1}_u$. Let us notice that, in order to get $Q^{n+1}$, to accomplish the projection stage, we need to compute the density. Herein, we set a new stage, the pre-projection stage, where equation (8) is used to obtain $\rho^{n+1}$. Once we
get the approximation of the density, we calculate its time derivative to approximate $Q^{n+1}$ and apply a finite element method to estimate the pressure correction. Summarizing, the global algorithm involves four stages:

- **Transport-diffusion stage:** equation (7) is solved through a FVM.
- **Pre-projection stage:** density is computed from mass fractions of species and temperature using (8). Next, $Q^{n+1}$ is obtained. If we are given the enthalpy, the temperature is recovered by solving (6).
- **Projection stage:** a FEM is applied to (9)-(10) in order to determine the pressure correction, $\delta^{n+1} := \pi^{n+1} - \pi^n$.
- **Post-projection stage:** the conservative velocities are updated by using the pressure correction and relation (9).

The following sections are devoted to the description of the previous stages highlighting the main differences with the incompressible method.

### 3.1 A dual finite volume mesh

For the space discretization we consider a 3D unstructured tetrahedral finite element mesh \( \{T_k, i = 1, \ldots, nel\} \). From this mesh we build a dual finite volume mesh as introduced in [BFSVC14], [BFTVC18] and [BDDVC98]. The nodes, to be denoted by \( \{N_i, i = 1, \ldots, nvol\} \), are the barycenters of the faces of the initial tetrahedra. In Figure 1 node $N_i$ is the barycenter of the face defined by vertices $V_1, V_2$ and $V_3$. This is why we will call this finite volume of face-type.

The notation employed is as follows:

- Each interior node $N_i$ has as neighbouring nodes the set $K_i$ consisting of the barycentres of the faces of the two tetrahedra of the initial mesh to which it belongs.
- Each finite volume is denoted by $C_i$. We denote by $\Gamma_i$ its boundary and $\tilde{\eta}_i$ its outward unit normal.
- The face $\Gamma_{ij}$ is the interface between cells $C_i$ and $C_j$. $N_{ij}$ is the barycentre of the face.
Figure 1: Interior (left) and boundary (right) finite volumes of the face-type.

- The boundary of $C_i$ is denoted by $\Gamma_i = \bigcup_{N_j \in \mathcal{K}_i} \Gamma_{ij}$.

- $|C_i|$ is the volume of $C_i$.

- Finally, $\tilde{\eta}_{ij}$ represents the outward unit normal vector to $\Gamma_{ij}$. We define $\eta_{ij} := \tilde{\eta}_{ij} ||\eta_{ij}||$, where, $||\eta_{ij}|| := \text{area}(\Gamma_{ij})$.

4 Transport-diffusion stage

Within the transport diffusion stage a finite volume method is applied in order to provide a first approximation of the conservative variables related to the velocity. Their discrete approximations are taken to be constant per finite volume, as it represents an integral average. Integrating equation (7) on $C_i$ and applying Gauss’ theorem we get

$$\frac{|C_i|}{\Delta t} \left( \tilde{W}_{u,i}^{n+1} - W_{u,i}^n \right) + \int_{\Gamma_i} \mathcal{F}^{W_u}(W_u^n, \rho^n) \tilde{\eta}_i dS + \int_{C_i} \nabla \pi^n dV - \int_{\Gamma_i} \tau^n \tilde{\eta}_i dS = 0.$$  \hspace{1cm} (11)
The above integrals can be computed analogously to the ones related to the incompressible model (see [BFSVC14] and [BFTVC18]) although the time and space dependency of density and the compressibility condition produce several changes on advection and viscous terms. More importantly, when willing to obtain a second order in space and time scheme using LADER methodology, special attention must be paid to the density approximation.

The former concerns also appear when solving a scalar advection equation with a time and space dependent advection coefficient. Therefore, this simplified model has been used to develop the numerical schemes presented in this paper (see A for further details). Subsequently, they have been extended to solve compressible Navier-Stokes equations.

4.1 Numerical flux

We start by approximating the flux term in (11). To this end we define the global normal flux on $\Gamma_i$ as $Z(W^n, \rho^n, \tilde{n}_i) := F_{wa}(W^n, \rho^n) \tilde{n}_i$. Next, we split $\Gamma_i$ into the cell interfaces $\Gamma_{ij}$, namely

$$\int_{\Gamma_i} F_{wa}(W^n, \rho^n) \tilde{n}_i \, dS = \sum_{N_j \in K_i} \int_{\Gamma_{ij}} Z(W^n, \rho^n, \tilde{n}_{ij}) \, dS. \quad (12)$$

Then, in order to get a stable discretization, the integral on $\Gamma_{ij}$ is approximated by an upwind scheme using a numerical flux function $\phi_u$. We use Rusanov scheme (see [Rus62]),

$$\phi_u(W^n_{u,i}, W^n_{u,j}, \rho^n_{i}, \rho^n_{j}, \eta_{ij}) = \frac{1}{2} \left( Z(W^n_{u,i}, \rho^n_{i}, \eta_{ij}) + Z(W^n_{u,j}, \rho^n_{j}, \eta_{ij}) \right) - \frac{1}{2} \alpha_{RS,ij}^W (W^n_{u,j} - W^n_{u,i}) \quad (13)$$

with

$$\alpha_{RS,ij}^W(W^n_{u,i}, W^n_{u,j}, \rho^n_{i}, \rho^n_{j}, \eta_{ij}) := \max \left\{ 2 \left| U^n_i \cdot \eta_{ij} \right|, 2 \left| U^n_j \cdot \eta_{ij} \right| \right\} \quad (14)$$

the so-called Rusanov coefficient. Therefore, equation (11) can be rewritten as

$$\frac{1}{\Delta t} (\tilde{W}^{n+1}_{u,i} - W^n_{u,i}) + \frac{1}{|C_i|} \sum_{N_j \in K_i} \phi_u(W^n_{u,i}, W^n_{u,j}, \rho^n_{i}, \rho^n_{j}, \eta_{ij})$$
where $\varphi_u$ denotes a diffusion flux function to be detailed in Section 4.2. For constant density this scheme corresponds to the proposed in [BFTVC18] to solve incompressible flows. However, it may produce spurious oscillations on the solutions for compressible low Mach number flows. We come now to propose a modification of it trying to avoid this undesirable behaviour.

### 4.1.1 Artificial viscosity related to density

As it is well known, the numerical flux of Rusanov scheme splits into a centred approximation of the whole flux and a numerical viscosity needed for the stability of the scheme. This upwind term is built with the Jacobian matrix of the flux, namely,

\[
\frac{\partial F_{w u}(w_u, \rho)}{\partial w_u} \text{div} w_u. \tag{16}
\]

For incompressible flows, it corresponds with the spatial derivative of the flux. However, for compressible flows, the flux also depends on the spatial variable through the density:

\[
\text{div} F_{w u}(w_u, \rho) = \frac{\partial F_{w u}(w_u, \rho)}{\partial w_u} \text{div} w_u + \frac{\partial F_{w u}(w_u, \rho)}{\partial \rho} \text{grad} \rho. \tag{17}
\]

Therefore, when applying (14) we are taking into account just the numerical viscosity related to the first term on the right hand side of (17) whereas the second term is considered only in the centred part of the flux. This lack of upwinding may produce spurious oscillations on the solution of the compressible model. Consequently, the bad behaviour of (15) can be corrected by adding a new artificial viscosity term to get an upwind discretization of

\[
\frac{\partial F_{w u}(w_u, \rho)}{\partial \rho} \text{grad} \rho. \tag{18}
\]

To this end, we present two different approaches extending the treatment first put forward in [BLVC17] for the unidimensional case. The first one consists in subtracting the integral in the control volume of (18) from both sides of (15). Next, a centred discretization is used to compute the integral
introduced in the left-hand side, while the one inserted in the right-hand side is approximated in an upwind form. The second approach incorporates the upwind term into the Rusanov scheme by considering an approximation of the derivative of the normal flux with respect to the density.

**Remark 1.** For an incompressible flow the density gradient is zero and so is \( (18) \).

**Approach 1** Denoting

\[
V(W_u, \rho) := \frac{\partial F_{wu}(W_u, \rho)}{\partial \rho} \text{grad} \rho = -\frac{1}{\rho^2} W_u \otimes W_u \text{grad} \rho,
\]

\[
G(W_u, \rho) := -\frac{\partial F_{wu}(W_u, \rho)}{\partial \rho} \text{grad} \rho = \frac{1}{\rho^2} W_u \otimes W_u \text{grad} \rho
\]

and incorporating both terms in the discrete equation (15) we get

\[
\frac{1}{\Delta t} (\tilde{\mu}^{n+1}_u - \mu^n_u) + \frac{1}{|C_i|} \sum_{N_j \in K_i} \phi_u (W_u^n, W_j^n, \rho_i^n, \rho_j^n, \eta_{ij})
\]

\[
\quad + \frac{1}{|C_i|} \int_{C_i} \text{grad} \pi^n dV - \frac{1}{|C_i|} \sum_{N_j \in K_i} \varphi_u (U_i^n, U_j^n, \eta_{ij})
\]

\[
\quad - \frac{1}{|C_i|} \int_{C_i} V(W_u^n, \rho^n) dV = \frac{1}{|C_i|} \int_{C_i} G(W_u^n, \rho^n) dV.
\]

(21)

To compute these new terms, we propose to divide each interior finite volume into six sub-tetrahedra denoted by \( C_{ij} \). Each of them has, as basis, a face, \( \Gamma_{ij} \), of the original finite volume, \( C_i \), and the barycenter of the finite volume, that is, the node \( N_i \), as opposite vertex. Similarly, boundary volumes are split into three sub-tetrahedra. Let us denote \( d_{ij} \) the distance between \( \Gamma_{ij} \) and \( N_i \). We notice that \( d_{ij} = d_{ji} \) due to the construction of the staggered mesh and \( |C_{ij}| = \frac{1}{3} d_{ij} \| \eta_{ij} \| \). In Figure 2 the 2D sub-triangles corresponding to the 3D sub-tetrahedra are depicted. We observe that in 2D we divide each interior cell into four sub-triangles instead of the six sub-tetrahedra needed in 3D.

Using this new structure, we can split the integral on the finite volume as the sum of the integrals on the sub-tetrahedra, \( C_{ij} \):

\[
\frac{1}{|C_i|} \int_{C_i} V(W_u^n, \rho^n) dV = \frac{1}{|C_i|} \sum_{N_j \in K_i} \int_{C_{ij}} V(W_u^n, \rho^n) dV.
\]

(22)
Figure 2: Subtriangles related to face $\Gamma_{ij}$ (shadowed in red) for a bi-dimensional grid.

Next, we denote $V^n_i$ an approximation of (22),

$$V^n_i := \frac{1}{|C_i|} \sum_{N_j \in K_i} V^n_{i,ij},$$

(23)

where

$$V^n_{i,ij} \approx \int_{C_{ij}} V(W^n_u, \rho^n) \, dV.$$  
(24)

To approximate these new integrals we consider the conservative variables to be constant by sub-tetrahedra and set its value as the average velocity at the two nodes related to the face. On the other hand, the density gradient is approximated by its orthogonal part to be consistent with the approximation of the divergence of velocities on the viscous term of Rusanov’s flux (see [BVC94] and [VC94]). Therefore, we obtain

$$V^n_{i,ij} = -|C_{ij}| \frac{1}{4} \left[ (W^n_{u,i} + W^n_{u,j}) \otimes (W^n_{u,i} + W^n_{u,j}) \right] \frac{1}{4} \left( \frac{\rho^n_j - \rho^n_i}{\rho^n_j + \rho^n_i} \right)^2 \frac{\rho^n_j - \rho^n_i}{\rho^n_j + \rho^n_i} d_{ij} \eta_{ij}$$

$$= -\frac{1}{3} (W^n_{u,i} + W^n_{u,j}) (W^n_{u,i} + W^n_{u,j}) \cdot \eta_{ij} \frac{\rho^n_j - \rho^n_i}{\rho^n_j + \rho^n_i}.$$  
(25)
Substituting (25) in (23) we get

\[
V_n^i = -\frac{1}{3 |C_i|} \sum_{N_j \in K_i} \left[ (W_{u,i}^n + W_{u,j}^n) \left( W_{u,i}^n + W_{u,j}^n \right) \cdot \eta_{ij} \frac{\rho_j^n - \rho_i^n}{(\rho_i^n + \rho_j^n)^2} \right].
\]

(26)

Let us denote \( G_n^i \) the upwind approximation of \( \frac{1}{|C_i|} \int_{C_i} \mathbf{G} (W_{u}^n, \rho^n) \, dV \). Then, taking into account the procedure to upwind source terms when using Rusanov flux, we obtain

\[
G_n^i = \frac{1}{|C_i|} \sum_{N_j \in K_i} \psi_{i,j} (W_{u,i}^n, W_{u,j}^n, \rho_i^n, \rho_j^n, \eta_{ij}) = -\frac{1}{|C_i|} \sum_{N_j \in K_i} \left[ 1 - \text{sign} (\tilde{\alpha}_{RS,i,j}^n) \right] V_{i,ij}^n
\]

\[
= \frac{1}{3 |C_i|} \sum_{N_j \in K_i} \left[ 1 - \text{sign} (\tilde{\alpha}_{RS,i,j}^n) \right] (W_{u,i}^n + W_{u,j}^n) \left( W_{u,i}^n + W_{u,j}^n \right) \cdot \eta_{ij} \frac{\rho_j^n - \rho_i^n}{(\rho_i^n + \rho_j^n)^2},
\]

(27)

where we have denoted by \( \text{sign} (\tilde{\alpha}_{RS,i,j}^n) \) the sign of the eigenvalue considered in Rusanov constant. Moreover, we have introduced the mappings

\[
\psi_{i,j} = \left[ 1 - \text{sign} (\tilde{\alpha}_{RS,i,j}^n) \right] V_{i,ij}^n
\]

(28)
in order to extend the procedure to upwind source terms presented in [VC94] and [BLVC17] to the upwind of \( \mathbf{G} \).

Substituting (23) and (27) in (21) it results

\[
\frac{1}{\Delta t} \left( \tilde{W}_{u,i}^{n+1} - W_{u,i}^n \right) + \frac{1}{|C_i|} \sum_{N_j \in K_i} \phi_u (W_{u,i}^n, W_{u,j}^n, \rho_i^n, \rho_j^n, \eta_{ij})
\]

\[
\quad + \frac{1}{|C_i|} \int_{C_i} \text{grad} \pi^n \, dV - \frac{1}{|C_i|} \sum_{N_j \in K_i} \varphi_u (U_i^n, U_j^n, \eta_{ij})
\]

\[
- \frac{1}{|C_i|} \sum_{N_j \in K_i} V_{i,ij}^n = -\frac{1}{|C_i|} \sum_{N_j \in K_i} \left[ 1 - \text{sign} (\tilde{\alpha}_{RS,i,j}^n) \right] V_{i,ij}^n.
\]

(29)

Hence, the discretized equation reads

\[
\frac{1}{\Delta t} \left( \tilde{W}_{u,i}^{n+1} - W_{u,i}^n \right) + \frac{1}{|C_i|} \sum_{N_j \in K_i} \phi_u (W_{u,i}^n, W_{u,j}^n, \rho_i^n, \rho_j^n, \eta_{ij})
\]

\[
\quad + \frac{1}{|C_i|} \int_{C_i} \text{grad} \pi^n \, dV - \frac{1}{|C_i|} \sum_{N_j \in K_i} \varphi_u (U_i^n, U_j^n, \eta_{ij})
\]

\[
- \frac{1}{|C_i|} \sum_{N_j \in K_i} V_{i,ij}^n.
\]
\[ + \frac{1}{|C_i|} \int_{C_i} \text{grad} \pi^n dV - \frac{1}{|C_i|} \sum_{N_j \in K_i} \varphi_u \left( U^n_i, U^n_j, \eta_{ij} \right) - \frac{1}{|C_i|} \sum_{N_j \in K_i} \text{sign} \left( \hat{a}^{w,u,n}_{RS} \right) V^n_{i,j} = 0 . \tag{30} \]

**Approach 2** Operating with the increment of the normal flux, we get

\[ \Delta Z \left( w_u, \rho, \eta \right) = A \Delta w_u + R \Delta \rho \tag{31} \]

where \( A \) denotes the jacobian matrix of the normal flux and

\[ R = \frac{\partial Z \left( w_u, \rho, \eta \right)}{\partial \rho} . \tag{32} \]

Analysing expression (31), we notice that the first term is related to the upwind term of Rusanov scheme, whereas the second term,

\[ \frac{\partial F^{w_u} \left( w_u, \rho \right) \eta}{\partial \rho} \Delta \rho = - w_u \left( w_u \cdot \eta \right) \rho^{-2} \Delta \rho, \tag{33} \]

is not included. Thus, we propose a modification of the scheme consisting in incorporating the term

\[ \left[ - W^n_u \left( W^n_u \cdot \eta \right) \left( \rho^n \right)^{-2} \Delta \rho^n \right]_{i,j} \approx - \left[ \frac{1}{2} \left( W^n_{u,i} + W^n_{u,j} \right) \right] \left[ \frac{1}{2} \left( W^n_{u,i} + W^n_{u,j} \right) \cdot \eta_{ij} \right] \left[ 1 \right] \left[ 1 \right] \left[ \frac{1}{2} \left( \rho^n_i + \rho^n_j \right) \right]^{-2} \left( \rho^n_j - \rho^n_i \right) \]

\[ = - \left[ \left( W^n_{u,i} + W^n_{u,j} \right) \left[ ( W^n_{u,i} + W^n_{u,j} ) \cdot \eta_{ij} \right] \left( \rho^n_i + \rho^n_j \right)^{-2} \left( \rho^n_j - \rho^n_i \right) \right] \quad (34) \]

to the flux function. Since the approximation must be consistent, we multiply the above expression by a constant \( \beta = \frac{1}{3} \). Furthermore, we use the sign of the eigenvalue considered in the Rusanov constant, \( \hat{a}^{w,u,n}_{RS,i,j} \), to account for the sense of the flux. Therefore, the term reads

\[ - \beta \text{sign} \left( \hat{a}^{w,u,n}_{RS,i,j} \right) \left( W^n_{u,i} + W^n_{u,j} \right) \left( W^n_{u,i} + W^n_{u,j} \right) \cdot \eta_{ij} \left( \rho^n_i + \rho^n_j \right)^{-2} \left( \rho^n_j - \rho^n_i \right) \tag{35} \]

and the new flux function, to be replaced in (15), becomes

\[ \varphi_u \left( W^n_{u,i}, W^n_{u,j}, \rho^n_i, \rho^n_j, \eta_{ij} \right) \]
\[
Z(W^n_{u, i}, \rho^n_i, \eta_{ij}) + Z(W^n_{u, j}, \rho^n_j, \eta_{ij}) - \frac{1}{2} \alpha_{RS, ij} (W^n_j - W^n_i)
\]

\[-\frac{1}{3} \text{sign}(\hat{\alpha}^{w_{u, n}}_{RS, ij}) (W^n_{u, i} + W^n_{u, j}) (W^n_{u, i} + W^n_{u, j}) \cdot \eta_{ij} (\rho^n_i + \rho^n_j)^{-2} (\rho^n_j - \rho^n_i).\]

\[(36)\]

**Remark 2.** Both approaches produce the same numerical scheme.

### 4.1.2 LADER methodology

By using the flux function (36) we would obtain a first order scheme both in space and time. To achieve a second order scheme we extend LADER techniques to solve equation (7). This method was first proposed in [BFTVC18] as a modification of ADER methodology (see [TMN01] and [Tor09]) and makes its extension to solve the three-dimensional problem easier.

We follow the steps described in [BFTVC18] for solving the incompressible Navier-Stokes equations taking into account the dependency of density in time and space. Accordingly, the conservative variables are extrapolated and the mid-point rule is applied.

As a novelty, we propose to compute particular evolved values of density at the neighbouring of faces, \(\Gamma_{ij}\). This additional step is essential to obtain a second order scheme. Moreover, the variables involved in the approximation of the last term added to the flux, (35), must also be evolved. We come now to detail the new computations to be performed at each step of the extended method:

**Step 1.** ENO-based reconstruction. Reconstruction of the data in terms of first degree polynomials is considered. For each finite volume we define four polynomials, each one of them at the neighbourhood of one of the boundary faces. Focusing on a face \(\Gamma_{ij}\) and on the discretization of a scalar variable, \(W\), its two related reconstruction polynomials for the conservative velocity are

\[
p^i_{ij}(N) = W_i + (N - N_i) (\nabla W)^i_{ij}, \quad p^j_{ij}(N) = W_j + (N - N_j) (\nabla W)^j_{ij}.
\]

\[(37)\]
The slopes are adaptively chosen as follows:

\[

(\nabla W)^i_{ij} = \begin{cases} 
(\nabla W)_{T_{ijL}}, & \text{if } \left| (\nabla W)_{T_{ijL}} \cdot (N_{ij} - N_i) \right| \leq \left| (\nabla W)_{T_{ij}} \cdot (N_{ij} - N_i) \right|, \\
(\nabla W)_{T_{ij}}, & \text{otherwise};
\end{cases}
\]

\[

(\nabla W)^j_{ij} = \begin{cases} 
(\nabla W)_{T_{ijR}}, & \text{if } \left| (\nabla W)_{T_{ijR}} \cdot (N_{ij} - N_j) \right| \leq \left| (\nabla W)_{T_{ij}} \cdot (N_{ij} - N_j) \right|, \\
(\nabla W)_{T_{ij}}, & \text{otherwise};
\end{cases}
\]

where \((\nabla W)_{T_{ij}}\) is the gradient of \(W\) at the auxiliary tetrahedra that intersects the face (see, on the 2D representation in Figure 3, the triangle with green contour).

**Step 2.** Computation of the boundary extrapolated values at the barycenter of faces, \(N_{ij}\):

\[

W_{i,N_{ij}} = p^i_{ij}(N_{ij}) = W_i + (N_{ij} - N_i) (\nabla W)^i_{ij},
\]

\[

W_{j,N_{ij}} = p^j_{ij}(N_{ij}) = W_j + (N_{ij} - N_j) (\nabla W)^j_{ij}.
\]

**Step 3.** Computation of the variables involved in the flux term with second order of accuracy using the mid-point rule. Taylor series expansion in time and Cauchy-Kovalevskaya procedure are applied to locally approximate the conservative variables at time \(\frac{\Delta t}{2}\). This methodology accounts for the contribution of the advection and diffusion terms to
the time evolution of the flux term at the momentum conservation equation. The resulting evolved variables read

\[
\vec{W}_{iN} = W_{u,iN} - \frac{\Delta t}{2L_{ij}} \left[ Z(W_{u,iN}, \eta_{ij}) + Z(W_{u,jN}, \eta_{ij}) \right]
+ \mu \frac{\Delta t}{2L_{ij}^2} \left[ (\text{grad } W_u + \text{grad } W_u^T)_{ij} \eta_{ij} + (\text{grad } W_u + \text{grad } W_u^T)_{ij} \eta_{ij} \right. \\
\left. - \frac{2}{3} \left( \text{div } W_{u,iN} \eta_{ij} + \text{div } W_{u,jN} \eta_{ij} \right) \right],
\]

\[\tag{40}\]

\[
\vec{W}_{jN} = W_{u,jN} - \frac{\Delta t}{2L_{ij}} \left[ Z(W_{u,iN}, \eta_{ij}) + Z(W_{u,jN}, \eta_{ij}) \right]
+ \mu \frac{\Delta t}{2L_{ij}^2} \left[ (\text{grad } W_u + \text{grad } W_u^T)_{ij} \eta_{ij} + (\text{grad } W_u + \text{grad } W_u^T)_{ij} \eta_{ij} \right. \\
\left. - \frac{2}{3} \left( \text{div } W_{u,iN} \eta_{ij} + \text{div } W_{u,jN} \eta_{ij} \right) \right].
\]

\[\tag{41}\]

We have denoted \(L_{ij} = \min \left\{ |C_i|, |C_j| \right\} \) with \(S(C_i)\) the area of the surface of cell \(C_i\).

Two different options can be considered concerning the evolved variables related to conservative velocities. The first one corresponds to the previous definition of the evolved variables, (40)-(41). Meanwhile, the second one neglects the contribution of the diffusion term:

\[
\vec{W}_{iN} = W_{u,iN} - \frac{\Delta t}{2L_{ij}} \left[ Z(W_{u,iN}, \eta_{ij}) + Z(W_{u,jN}, \eta_{ij}) \right],
\]

\[\tag{42}\]

\[
\vec{W}_{jN} = W_{u,jN} - \frac{\Delta t}{2L_{ij}} \left[ Z(W_{u,iN}, \eta_{ij}) + Z(W_{u,jN}, \eta_{ij}) \right].
\]

\[\tag{43}\]

Regarding the densities, we propose to consider the mass conservation equation in order to apply Cauchy-Kovalevskaya procedure. Therefore, we define

\[
\vec{\rho}_{iN} := \rho_{iN} - \frac{\Delta t}{4L_{ij}} \|\eta_{ij}\| \left( \text{div } W_{u,iN} + \text{div } W_{u,jN} \right),
\]

\[\tag{44}\]

\[
\vec{\rho}_{jN} := \rho_{jN} - \frac{\Delta t}{4L_{ij}} \|\eta_{ij}\| \left( \text{div } W_{u,iN} + \text{div } W_{u,jN} \right).
\]

\[\tag{45}\]
Step 4. Computation of the numerical flux using (36):

\[
\phi_u \left( W_{u,i}^n, W_{u,j}^n, \bar{p}_i^n, \bar{p}_j^n, \eta_{ij} \right) = 
+ \frac{1}{2} \left[ Z \left( W_{u,i}^n, \bar{p}_i^n, \eta_{ij} \right) + Z \left( W_{u,j}^n, \bar{p}_j^n, \eta_{ij} \right) \right] 
- \frac{1}{2} \alpha_{RS,ij} \left( W_{u,j}^n - W_{u,i}^n \right) 
- \frac{1}{3} \text{sign} \left( \frac{\omega_{RS,n}}{\alpha_{RS,ij}} \right) \left( \bar{p}_i^n + \bar{p}_j^n \right)^{-2} \left( W_{u,i}^n + W_{u,j}^n \right) 
\left( W_{u,i}^n + W_{u,j}^n \right) \cdot \eta_{ij} \left( \bar{p}_j^n - \bar{p}_i^n \right) . \tag{46}
\]

4.2 Viscous term

In this section we describe the computation of the viscous term. First, applying Gauss’ theorem we relate the volume integral of the diffusion term with a surface integral over the boundary, \( \Gamma_i \). Next, this integral is split into the integrals over the cell interfaces \( \Gamma_{ij} \). Thus, the viscous term of the momentum conservation equation reads

\[
\int_{C_i} \text{div} \tau^n dV = \sum_{N_j \in K_i} \int_{\Gamma_{ij}} \tau^n \tilde{\eta}_{ij} dS
= \sum_{N_j \in K_i} \int_{\Gamma_{ij}} \mu \left[ \text{grad} U^n + (\text{grad} U^n)^T \right] \tilde{\eta}_{ij} dS, \tag{47}
\]

where a new divergence term has appeared with respect to the incompressible model. We define the numerical diffusion function as

\[
\int_{\Gamma_{ij}} \varphi_u \left( U_i^n, U_j^n, \eta_{ij} \right) \approx \mu \left[ \text{grad} U^n + (\text{grad} U^n)^T \right] - \frac{2}{3} \text{div} U^n] \tilde{\eta}_{ij} dS. \tag{48}
\]

Accounting for the dual mesh structure, we compute the spatial derivatives on the auxiliary tetrahedra \( T_{ij} \) through a Galerkin approach. Hence,

\[
\varphi_u \left( U_i^n, U_j^n, \eta_{ij} \right) = \mu \left( \text{grad} U^n \right)_{T_{ij}} \eta_{ij} + \mu \left( \text{grad} U^n \right)_{T_{ij}} \eta_{ij} - \frac{2}{3} \mu \left( \text{div} U^n \right)_{T_{ij}} \eta_{ij}. \tag{49}
\]

In order to attain a second order scheme we apply LADER methodology also to the diffusion term (see [BFTVC18]). That is, we construct evolved variables \( \overline{U}_i^n \) which only have the diffusion term as contribution for the half in time evolution:
1. The gradients of the original variables are computed at each auxiliary tetrahedra of the FE mesh, \( T_{ij} \). The value of the gradient at each node, \( N_i \), is obtained as the average of the values on the two tetrahedra containing the node, \( T_{ijL} \) (green filled triangle in Figure 3) and \( T_{ij} \). Taking into account the viscosity coefficients and the turbulent kinetic energy term, we introduce the auxiliary variable:

\[
D^n_i := \frac{\mu}{2} \left[ \left( \text{grad} \ U^n + (\text{grad} \ U^n)^T - \frac{2}{3} \text{div} (U^n) \right)_{T_{ijL}} \right.
+ \left. \left( \text{grad} \ U^n + (\text{grad} \ U^n)^T - \frac{2}{3} \text{div} (U^n) \right)_{T_{ij}} \right].
\] (50)

2. The divergence is computed as the average of the trace of the gradients of \( V^n \) obtained on the auxiliary tetrahedra:

\[
\bar{U}_i^n = U_i^n + \frac{\Delta t}{4} \text{tr} \left( (\text{grad} D^n)_{T_{ijL}} + (\text{grad} D^n)_{T_{ij}} \right).
\] (51)

3. The diffusion function \( \psi_u \) is evaluated on the evolved variables:

\[
\psi_u \left( \bar{U}_i^n, \bar{U}_j^n, \eta_{ij} \right) = \mu \left( \text{grad} \bar{U}^n \right)_{T_{ij}} \eta_{ij} + \mu \left( \text{grad} \bar{U}^n \right)^T_{T_{ij}} \eta_{ij}
- \frac{2}{3} \mu \left( \text{div} \bar{U}^n \right)_{T_{ij}} \eta_{ij}.
\] (52)

### 4.3 Pressure term

For the integral of the pressure gradient, we follow [BFSVC14]. We split the boundary \( \Gamma_i \) into the cell interfaces \( \Gamma_{ij} \) using Gauss’ theorem and we compute the pressure as the arithmetic mean of its values at the three vertices of face \( \Gamma_{ij} \) and the barycentre of the tetrahedra to which the face belongs. Then, the corresponding approximation of the integral is given by

\[
\int_{\Gamma_{ij}} \pi^n \tilde{n}_{ij} dS \approx \left[ \frac{5}{12} (\pi^n(V_1) + \pi^n(V_2)) + \frac{1}{12} (\pi^n(V_3) + \pi^n(V_4)) \right] \eta_{ij}.
\] (53)

### 5 Pre-projection stage

The novel pre-projection stage is devoted to the computation of the source term of equation (10), \( Q_i^{n+1} \). Since we are assuming that we are given the
mass fraction of the species and the temperature, we compute the density, \( \rho^{n+1}_i \), by substituting \( \mathbf{Y}^{n+1}_i \) and \( \theta^{n+1}_i \) in the state equation, namely,

\[
\rho^{n+1}_i = \frac{\pi^{n+1}_i}{R \theta^{n+1}_i \sum_{l=1}^{N_i} \mathbf{Y}_{l,i} \mathbf{M}_l}.
\]  

(54)

Finally, the source term of the projection stage results

\[
Q^{n+1}_i = \frac{\rho^n_i - \rho^{n+1}_i}{\Delta t}.
\]  

(55)

**Remark 3.** Let us notice that if an analytical expression of the density is known then, we can directly evaluate the term

\[
q(x, y, z, t^n) = \frac{\partial \rho(x, y, z, t^n)}{\partial t}
\]  

(56)

so that the exact value of \( Q^{n+1}_i \) is provided to the mass conservation equation.

**Remark 4.** If the supplied data is no longer the temperature but the enthalpy, we apply Newton’s method to equation (6),

\[
H^{n+1}_i = h_0 + \int_{\theta_0}^{\theta^{n+1}_i} c_\pi(r) \, dr,
\]  

(57)

to obtain the value of the temperature at node \( N_i \) at the new time step, \( \theta^{n+1}_i \).

### 6 Projection and post-projection stages

Following the methodology already introduced in [BFSVC14] for incompressible flows, equations (9)-(10) are solved via a finite element method. Firstly, we obtain the weak problem:

Find \( \delta^{n+1} \in V_0 := \{ z \in H^1(\Omega) : \int_\Omega z dV = 0 \} \) verifying

\[
\int_{\Omega} \nabla \delta^{n+1} \cdot \nabla z \, dV = \frac{1}{\Delta t} \int_{\Omega} \mathbf{W}^{n+1}_u \cdot \nabla z \, dV + \frac{1}{\Delta t} \int_{\Omega} Q^{n+1} z \, dV
\]  

\[
- \frac{1}{\Delta t} \int_{\partial \Omega} G^{n+1} z \, dA
\]  

(58)
for all $z \in V_0$. 

The strong formulation corresponds to the Laplace problem

$$\Delta \delta^{n+1} = \frac{1}{\Delta t} \left( \text{div} \tilde{W}_u^{n+1} - Q^{n+1} \right) \quad \text{in } \Omega,$$

$$\frac{\partial \delta^{n+1}}{\partial \eta} = \frac{1}{\Delta t} \left( \tilde{W}_u^{n+1} \cdot \eta - G^{n+1} \right) \quad \text{in } \Gamma.$$

Finally, at the post-projection stage, $W_u^{n+1}$ is calculated by substituting $\delta^{n+1}$ in

$$W_{u,i}^{n+1} = \tilde{W}_{u,i}^{n+1} + \Delta t \text{ grad } \delta_{i}^{n+1}. \quad (61)$$

### 7 Boundary conditions

The boundary conditions were defined following [BFSVC14]:

- **Dirichlet boundary conditions for inviscid fluids:** the normal component of the conservative variable is set at the boundary nodes.

- **Dirichlet boundary conditions for viscous fluids:** the value of the conservative variable is imposed at the boundary nodes.

In the manufactured tests designed to analyse the order of accuracy of the numerical discretizations, it is a usual practice to impose the values of the exact solution at the boundary nodes. This practice avoids that the accuracy of the method can be affected by the treatment of the boundary conditions. From the mathematical point of view, it is like considering Dirichlet boundary conditions.

### 8 Numerical results

Two analytical test problems will be presented willing to assesse the performance of the methodology. Firstly, we recall the function spaces defined by

$$L^2(\Omega) = \left\{ g : \Omega \to \mathbb{R} \mid g \text{ is measurable and } \int_\Omega |g|^2 \, dV < \infty \right\}, \quad (62)$$
\[ \ell^2([a,b]) = \left\{ g : [a, b] \rightarrow \mathbb{R} \mid g \text{ is measurable and } \int_a^b |g|^2 \, dt < \infty \right\} \]  
(63)

\[ \ell^2(L^2(\Omega), [a,b]) = \left\{ g : \Omega \times [a, b] \rightarrow \mathbb{R} \mid g \text{ is measurable and } \int_a^b \int_{\Omega} |g|^2 \, dV \, dt < \infty \right\} \]  
(64)

with the corresponding norms

\[ \|g\|_{L^2(\Omega)} = \left( \int_{\Omega} |g|^2 \, dV \right)^{\frac{1}{2}}, \]  
(65)

\[ \|g\|_{\ell^2([a,b])} = \left( \int_a^b |g|^2 \, dt \right)^{\frac{1}{2}}, \]  
(66)

\[ \|g\|_{\ell^2(L^2(\Omega))} = \left[ \int_a^b \left( \int_{\Omega} |g|^2 \, dV \right) \, dt \right]^\frac{1}{2}. \]  
(67)

Then, the errors are computed as follows:

\[ E(\pi)_{M_i} = \|\pi - \pi_{M_i}\|_{\ell^2(L^2(\Omega))}, \quad E(\mathbf{w}_u)_{M_i} = \|\mathbf{w}_u - \mathbf{w}_{uM_i}\|_{\ell^2(L^2(\Omega))}, \]  
(68)

\[ o_{\pi_{M_i}/M_j} = \frac{\ln \left( \frac{E(\pi)_{M_i}}{E(\pi)_{M_j}} \right)}{\ln \left( \frac{h_{M_i}}{h_{M_j}} \right)}, \quad o_{\mathbf{w}_{uM_i}/M_j} = \frac{\ln \left( \frac{E(\mathbf{w}_u)_{M_i}}{E(\mathbf{w}_u)_{M_j}} \right)}{\ln \left( \frac{h_{M_i}}{h_{M_j}} \right)}, \]  
(69)

where \( \pi_{M_i} \) and \( \mathbf{w}_{uM_i} \) denote, respectively, the pressure and the conservative velocity obtained for mesh \( M_i \) whereas \( \pi \) and \( \mathbf{w}_u \) are the respective exact values.

Regarding the estimation of the time step a CFL number is set. Thus, at each time iteration we determine a local value for the time step at each cell \( C_i \),

\[ \Delta t_{C_i} = \frac{\text{CFL} \, L_i^2}{2 |\mathbf{U}_i| \, L_i + 2\mu}, \]  
(70)

with \( L_i := \frac{|C_i|}{S(C_i)} \). The global time step at each time iteration, \( \Delta t \), is chosen as the minimum of the time steps obtained at each cell.
8.1 Test 1. Euler flow

As a first test, we consider the computational domain $\Omega = [0, 1]^3$ and define the flow as

\begin{align*}
\rho(x, y, z, t) &= \cos(t) + x + 1, \\
\pi(x, y, z, t) &= 1, \\
\mathbf{u}(x, y, z, t) &= \left( \frac{x \sin(t) + 1}{\cos(t) + x + 1}, 0, 0 \right)^T, \\
y(x, y, z, t) &= 1, \\
\theta(x, y, z, t) &= \frac{10^3}{\cos(t) + x + 1}
\end{align*}

(71) \quad (72) \quad (73) \quad (74) \quad (75)

with $\mu = 0$ and

\begin{align*}
f_{u_1} &= f_{u_2} = 0, \\
f_{u_3} &= x \cos(t) - \frac{(x \sin(t) + 1)^2}{(x + \cos(t) + 1)^2} + \frac{(2 \sin(t)(x \sin(t) + 1))}{x + \cos(t) + 1}
\end{align*}

(76) \quad (77)

the source terms related to the momentum equation. Let us notice that we have assumed the mixture to be homogeneous, therefore we only consider one species, so that it is easier to apply the method of manufactured solutions. Moreover, since the mass fractions of the species are supposed to be known, the performance of the method can be analysed independently on the number of species considered. Dirichlet boundary conditions are set on the boundary.

| Mesh | $N$ | Elements | Vertices | Nodes | $V_h^m$ (m$^3$) | $V_h^M$ (m$^3$) |
|------|-----|----------|----------|-------|----------------|----------------|
| $M_1$ | 4   | 384      | 125      | 864   | $6.51E - 04$   | $1.30E - 03$   |
| $M_2$ | 8   | 3072     | 729      | 6528  | $8.14E - 05$   | $1.63E - 04$   |
| $M_3$ | 16  | 24576    | 4913     | 50688 | $1.02E - 05$   | $2.03E - 05$   |

Table 1: Test 1. Euler flow. Mesh features.

The numerical simulations were run on the meshes defined in Table 1. We have denoted by $N + 1$ the number of nodes along the edges of the
domain, $h = 1/N$, $V^m_h$ the minimum volume of the finite volumes and $V^M_h$ the maximum volume of the finite volumes.

We present the results obtained at time $t_{end} = 1$ and for a $CFL = 1$. In Table 2, the errors and the convergence rates are depicted. We have consider three different schemes:

- The first order scheme constructed from (15) by substituting (36), (49) and (53).
- LADER scheme without the evolution of the density, $\overline{\rho_{iN_{ij}}}$, which results from assembling (15), (46) neglecting the evolution of the density, (52) and (53).
- LADER scheme built gathering together (15), (46) with de density values taken from (44)-(45), (52) and (53).

| Method          | Variable | $E_{M_1}$ | $E_{M_2}$ | $E_{M_3}$ | $\sigma_{M_1/M_2}$ | $\sigma_{M_2/M_3}$ |
|-----------------|----------|-----------|-----------|-----------|---------------------|---------------------|
| Order 1         | $\pi$    | $3.73E-03$| $1.44E-03$| $5.29E-04$| 1.37                | 1.44                |
|                 | $w_u$    | $7.30E-03$| $4.05E-03$| $2.13E-03$| 0.85                | 0.93                |
| LADER           | $\pi$    | $4.13E-03$| $1.60E-03$| $6.15E-04$| 1.37                | 1.38                |
| without $\overline{\rho_{iN_{ij}}}$ | $w_u$    | $6.11E-03$| $3.23E-03$| $1.71E-03$| 0.92                | 0.91                |
| LADER           | $\pi$    | $4.64E-04$| $1.93E-04$| $9.21E-05$| 1.26                | 1.07                |
|                 | $w_u$    | $6.31E-04$| $1.62E-04$| $4.15E-05$| 1.97                | 1.96                |

Table 2: Test 1. Euler flow. Observed errors and convergence rates. $CFL = 1$.

We can conclude that, as we have analysed in A for the unidimensional advection-diffusion-reaction equation, applying LADER methodology to compute the density is crucial to achieve a second order scheme.

**Remark 5.** If the new viscosity term, (35), is not considered, spurious oscillations arise when applying LADER methodology.
8.2 Test 2. Navier-Stokes flow

The flow of the second academic test reads

\[ \rho(x, y, z, t) = \sin(\pi yt) + 2, \quad (78) \]
\[ \pi(x, y, z, t) = \exp(xyz)\cos(t), \quad (79) \]
\[ \mathbf{u}(x, y, z, t) = ((\cos(\pi xt))^2, \exp(-2\pi yt), -\cos(\pi xy t))^T, \quad (80) \]
\[ y(x, y, z, t) = 1, \quad (81) \]
\[ \theta(x, y, z, t) = \frac{10^3}{\sin(\pi yt) + 2}. \quad (82) \]

Then, the source terms for the momentum equations result

\[ f_{u_1} = \pi y \cos(\pi tx)^2 \cos(\pi ty) - 4\pi t \sin(\pi tx) \cos(\pi tx)^3 (\sin(\pi ty) + 2) \]
\[ - (2\mu(2\pi^2 t^2 \cos(\pi tx)^2 - 2\pi^2 t^2 \sin(\pi tx)^2))/3 + 2\pi^2 t^2 \mu \cos(\pi tx)^2 \]
\[ -2\pi^2 t^2 \mu \sin(\pi tx)^2 + \pi t \exp(-2\pi ty) \cos(\pi tx)^2 \cos(\pi ty) + yz \exp(xyz) \cos(t) \]
\[ -2\pi x \sin(\pi tx) \cos(\pi tx)(\sin(\pi ty) + 2) - 2\pi t \exp(-2\pi ty) \cos(\pi tx)^2 (\sin(\pi ty) + 2), \quad (83) \]

\[ f_{u_2} = \pi t \exp(-4\pi ty) \cos(\pi ty) + \pi y \exp(-2\pi ty) \cos(\pi ty) \]
\[ - (4\pi^2 t^2 \mu \exp(-2\pi ty))/3 - 4\pi t \exp(-4\pi ty)(\sin(\pi ty) + 2) + xz \exp(xyz) \cos(t) \]
\[ -2\pi y \exp(-2\pi ty)(\sin(\pi ty) + 2) - 2\pi t \sin(\pi tx) \exp(-2\pi ty) \cos(\pi tx)(\sin(\pi ty) + 2), \quad (84) \]

\[ f_{u_3} = 2\pi t \exp(-2\pi ty) \cos(\pi tx y)(\sin(\pi ty) + 2) - \pi y \cos(\pi tx y) \cos(\pi ty) \]
\[ + \pi xy \sin(\pi tx y)(\sin(\pi ty) + 2) - \pi t \exp(-2\pi ty) \cos(\pi tx y) \cos(\pi ty) - \pi^2 t^2 x^2 \mu \cos(\pi tx y) \]
\[ -\pi^2 t^2 y^2 \mu \cos(\pi tx y) + 2\pi t \sin(\pi tx) \cos(\pi tx y) \cos(\pi tx)(\sin(\pi ty) + 2) + xz \exp(xyz) \cos(t) \]
\[ + \pi tx \exp(-2\pi ty) \sin(\pi tx y)(\sin(\pi ty) + 2) + \pi ty \sin(\pi tx y) \cos(\pi tx)^2 (\sin(\pi ty) + 2). \quad (85) \]

We have taken \( \mu = 10^{-2} \). Besides, in order to satisfy the mass conservation equation we also define a source term for it:

\[ f_{\rho} = \pi y \cos(\pi ty) + \pi t \exp(-2\pi ty) \cos(\pi ty) \]
\[ -2\pi t \exp(-2\pi ty)(\sin(\pi ty) + 2) - \pi t \sin(2\pi tx)(\sin(\pi ty) + 2). \quad (86) \]

We consider the computational domain used in the previous test as well as the three uniform meshes introduced above. Computations were carried out until time \( t_{\text{end}} = 1 \) with \( CFL = 1 \).
The obtained numerical results are shown in Table 3. To analyse the performance of the proposed methodology we have run several simulations slightly modifying the proposed schemes.

| Method | Variable | $E_{M_1}$ | $E_{M_2}$ | $E_{M_3}$ | $o_{M_1/M_2}$ | $o_{M_2/M_3}$ |
|--------|----------|-----------|-----------|-----------|---------------|---------------|
| Order 1 | $\pi$    | $3.32E-01$ | $1.52E-01$ | $6.76E-02$ | $1.13$        | $1.17$        |
|        | $w_u$    | $1.45E-01$ | $7.80E-02$ | $4.27E-02$ | $0.89$        | $0.87$        |
| Order 1 | $\pi$    | $3.33E-01$ | $1.52E-01$ | $6.76E-02$ | $1.13$        | $1.17$        |
| $(\partial_t \rho)_{exact}$ | $w_u$    | $1.45E-01$ | $7.80E-02$ | $4.27E-02$ | $0.89$        | $0.87$        |
| LADER  | $\pi$    | $8.65E-02$ | $1.72E-02$ | $4.40E-03$ | $2.33$        | $1.97$        |
|        | $w_u$    | $7.43E-02$ | $1.76E-02$ | $4.33E-03$ | $2.08$        | $2.02$        |
| LADER  | $\pi$    | $8.55E-02$ | $1.66E-02$ | $3.77E-03$ | $2.36$        | $2.14$        |
| $(\partial_t \rho)_{exact}$ | $w_u$    | $7.40E-02$ | $1.75E-02$ | $4.30E-03$ | $2.08$        | $2.02$        |
| LADER  | $\pi$    | $1.02E-01$ | $2.68E-02$ | $1.06E-02$ | $1.93$        | $1.33$        |
| $\rho^{n+\frac{1}{2}\Delta t}_{\text{exact}}$ | $w_u$    | $8.29E-02$ | $2.78E-02$ | $1.23E-02$ | $1.58$        | $1.18$        |
| LADER  | $\pi$    | $8.81E-02$ | $1.74E-02$ | $4.43E-03$ | $2.34$        | $1.98$        |
| $\overline{\rho^{iN_i}}_{\text{exact}}$ | $w_u$    | $7.44E-02$ | $1.76E-02$ | $4.33E-03$ | $2.08$        | $2.02$        |

Table 3: Test 2. Navier-Stokes flow. Observed errors and convergence rates. $CFL = 1$.

On the one hand, rows denoted by Order 1 $(\partial_t \rho)_{exact}$ and LADER $(\partial_t \rho)_{exact}$ provide the results obtained when the exact value of the time derivative of the density is imposed in the computation of the pressure correction. We observe that the errors obtained are close to the ones corresponding to the simulations in which the former derivative is approximated.

On the other hand, in LADER $\rho^{n+\frac{1}{2}\Delta t}_{\text{exact}}$ we assume that the exact values of the density at half in time steps are known and so, instead of approximating $\overline{\rho^{iN_i}}_{\text{exact}}$, we set $\rho^{n+\frac{1}{2}\Delta t}$. The decrease in the accuracy of the method compared with LADER scheme can be explained by the fact that when applying LADER methodology we are not only evolving the densities half in
time but we also extrapolate them in a vicinity of the face. In fact, the errors obtained when applying LADER almost match the ones corresponding with LADER $\tilde{p}_{iN_j \text{exact}}$. Within this last simulation, we calculate the exact gradient of the density to obtain the extrapolation polynomials. Next, we compute the exact value of the density at the half in time step. Therefore, we confirm that coupling both techniques is necessary to attain the order of accuracy sought. This conclusion is reinforced by the studies presented in A for the advection-diffusion-reaction equation with a time and space dependent advection coefficient.

9 Summary and conclusions

In this paper we have developed a projection hybrid high order FV/FE method to solve compressible low Mach number flows. It has been shown that the space dependence of density may produce spurious oscillations on the solution of the momentum equation. Therefore, two different approaches have been presented aiming to correct this behaviour. As a result, the numerical flux function has been modified by adding a new upwind term. Willing to obtain a high order scheme, LADER methodology has also been used. In order to get insight on the effects that the variable density has on the accuracy of the scheme, the unidimensional advection-diffusion-reaction equation with space and time dependent advection coefficient has been examined in the appendix. The corresponding accuracy analysis together with the empirical convergence rate studies reveal the necessity of reconstructing and evolving both the conservative variable and the advection coefficient to attain a second order scheme. These conclusions were confirmed by the numerical results obtained for the three-dimensional problem.

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A Advection-diffusion-reaction equation with variable advection coefficient

The numerical analysis of high order finite volume schemes is too complex in three dimensions. Consequently, a usual practice is to set a scalar simplified problem which retains the main difficulties of the three-dimensional model but eases the theoretical study of the developed methods. In this Appendix, we will apply this technique to analyse the main difficulties that compressible Navier-Stokes equations entail.
Time and space dependency of density constitute one of the leading challenges of modeling compressible low Mach number flows. This problem can be conveyed to the scalar advection equation by assuming a time and space dependent advection coefficient. Moreover, an arbitrary source term can be introduced. As a result we obtain the advection-diffusion-reaction equation with variable coefficients,

$$\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} (\lambda q) (x,t) = \frac{\partial}{\partial x} (\alpha \frac{\partial}{\partial x} q) (x,t) + s(x,t,q).$$

(87)

Let us remark that the corresponding analysis for the advection-diffusion-reaction equation with constant advection coefficient can be found in [BFTVC18] and [BTVC16].

In order to facilitate the theoretical study of advection equations with variable advection coefficient, we focus on the following scalar advection equation

$$\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} (\lambda q) (x,t) = 0.$$  

(88)

Two main issues must be taken into account with respect to the advection equation with constant coefficient:

- A new numerical viscosity related to the spatial derivative of $\lambda(x,t)$ should be included. The main difficulties introduced due to the time and space dependency of the advection coefficient have been put forward in [BLVC17] for the multi-component Euler equations. In this article, we follow the methodology already proposed to solve the nonlinear advection equation.

- To build a second order in time and space scheme using LADER methodology, extrapolation and half in time evolution of $\lambda(x,t)$ need to be performed.

In what follows, we will further develop the points above.

### A.1 Advection term with time and space dependent coefficient

The use of Rusanov scheme to approximate the numerical flux function may not work properly. Let us remark that Rusanov flux is divided into two
terms:

\[
\phi(q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}) = \frac{1}{2} \left( \lambda^n_j q^n_j + \lambda^n_{j+1} q^n_{j+1} \right) - \frac{1}{2} \alpha_{RS} (q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}) \left( q^n_{j+1} - q^n_j \right)
\]

(89)

with

\[
\alpha_{RS} (q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}) = \max \left\{ |\lambda^n_j|, |\lambda^n_{j+1}| \right\}
\]

(90)

the Rusanov coefficient. The first term corresponds to a centred approximation of the flux. The second one is supposed to introduce the numerical viscosity needed for the stability of the scheme. However, splitting the spatial derivative of the flux into two terms,

\[
\partial_x (\lambda q) (x, t) = \partial_q (\lambda q) (x, t) \partial_x q(x, t) + \partial_\lambda (\lambda q) (x, t) \partial_x \lambda(x, t)
\]

\[= \lambda(x, t) \partial_x q(x, t) + q(x, t) \partial_x \lambda(x, t),\]

(91)

we notice that Rusanov flux only adds the artificial viscosity related to the first one. Therefore, spurious oscillations may be produced. To correct this lack of upwinding, we propose to introduce a new artificial viscosity term, namely,

\[- [\partial_\lambda (\lambda q) \Delta \lambda]_{j+\frac{1}{2}} \approx - \frac{1}{2} \text{sign} (\bar{\alpha}_{RS}) q^n_{j+\frac{1}{2}} (\lambda^n_{j+1} - \lambda^n_j)
\]

(92)

with \(\bar{\alpha}_{RS}\) the value of the eigenvalue used to compute \(\alpha_{RS}\), that is, \(|\bar{\alpha}_{RS}| = \alpha_{RS}\). Then, the numerical flux on the boundary \(x_{j+\frac{1}{2}}\) reads,

\[
\phi(q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}) = \frac{1}{2} \left( \lambda^n_j q^n_j + \lambda^n_{j+1} q^n_{j+1} \right) - \frac{1}{2} \alpha_{RS} (q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1}) \left( q^n_{j+1} - q^n_j \right)
\]

\[- \frac{1}{2} \text{sign} (\bar{\alpha}_{RS} (q^n_j, q^n_{j+1}, \lambda^n_j, \lambda^n_{j+1})) q^n_{j+\frac{1}{2}} (\lambda^n_{j+1} - \lambda^n_j).
\]

(93)

A.2 LADER scheme for advection equation with time and space dependent advection coefficient

To attain a second order of accuracy scheme for equation (88) we apply LADER methodology. Initially, the polynomial reconstruction is performed
on the conservative variable obtaining the following extrapolated values at
the neighbouring of the boundaries of a cell $C_j$:

\[
q_{j-1}^n = q_j^n + \frac{1}{2} \left( q_{j+1}^n - q_{j-2}^n \right),
\]

\[
q_j^L = q_j^n - \frac{1}{2} \left( q_{j+1}^n - q_{j}^n \right),
\]

\[
q_{j+1}^R = q_j^n + \frac{1}{2} \left( q_{j+2}^n - q_{j+1}^n \right),
\]

\[
q_{j+1}^L = q_j^n - \frac{1}{2} \left( q_{j+2}^n - q_{j+1}^n \right),
\]

where we have considered the approximations of the spatial derivatives of
$q(x, t)$ at time $t^n$ related to the auxiliary elements of each volume
$C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$:

\[
T_{j-1} = [x_j, x_{j+1}], \quad T_{jj} = [x_{j-1}, x_j]
\]

(see Figure 4). Next, the resolution of the generalised Riemann problem
entails the computation of the half in time evolved values,

\[
\overline{q}_{j-1}^n = q_j^n - \frac{\Delta t}{2\Delta x} \left( \lambda_{j-1}^n q_j^n - \lambda_{j-1}^n q_{j-1}^n \right),
\]

\[
\overline{q}_j^L = q_j^n - \frac{\Delta t}{2\Delta x} \left( \lambda_j^n q_j^n - \lambda_{j-1}^n q_{j-1}^n \right),
\]

\[
\overline{q}_j^R = q_j^n - \frac{\Delta t}{2\Delta x} \left( \lambda_{j+1}^n q_{j+1}^n - \lambda_j^n q_j^n \right),
\]
\[ q_{j+1}^{n} = q_{j+1}^{n} \left( \frac{\Delta t}{2\Delta x} \right) \cdot \left( \lambda_{j+1}^{n} q_{j+1}^{n} - \lambda_{j}^{n} q_{j}^{n} \right). \]  \tag{102} 

Furthermore, by extrapolating the advection coefficient and calculating its half in time evolution, we get

\[ \overline{\lambda}_{j-1R}^{n} = \lambda_{j-1R}^{n} + \frac{\Delta t}{2} \partial_t \lambda_{j-1R}^{n} = \lambda_{j-1R}^{n} + \frac{1}{2} \left( \lambda_{j-1}^{n} - \lambda_{j-2}^{n} \right) + \frac{\Delta t}{2} \partial_t \lambda_{j-1R}^{n}, \] \tag{103} 

\[ \overline{\lambda}_{jL}^{n} = \lambda_{jL}^{n} + \frac{\Delta t}{2} \partial_t \lambda_{jL}^{n} = \lambda_{jL}^{n} - \frac{1}{2} \left( \lambda_{j+1}^{n} - \lambda_{j}^{n} \right) + \frac{\Delta t}{2} \partial_t \lambda_{jL}^{n}, \] \tag{104} 

\[ \overline{\lambda}_{jR}^{n} = \lambda_{jR}^{n} + \frac{\Delta t}{2} \partial_t \lambda_{jR}^{n} = \lambda_{jR}^{n} + \frac{1}{2} \left( \lambda_{j}^{n} - \lambda_{j-1}^{n} \right) + \frac{\Delta t}{2} \partial_t \lambda_{jR}^{n}, \] \tag{105} 

\[ \overline{\lambda}_{j+1L}^{n} = \lambda_{j+1L}^{n} + \frac{\Delta t}{2} \partial_t \lambda_{j+1L}^{n} = \lambda_{j+1L}^{n} - \frac{1}{2} \left( \lambda_{j+2}^{n} - \lambda_{j+1}^{n} \right) + \frac{\Delta t}{2} \partial_t \lambda_{j+1L}^{n}. \] \tag{106} 

Substituting (99)-(106) in the numerical flux function (93), LADER scheme for the advection equation, (88), finally reads

\[ q_{j+1}^{n} = q_{j}^{n} - \frac{\Delta t}{2\Delta x} \left\{ \left( \lambda_{jR}^{n} q_{jR}^{n} + \overline{\lambda}_{j+1L}^{n} q_{j+1L}^{n} \right) - \overline{\alpha}_{RS,i+\frac{1}{2}}^{n} \left( q_{j+1L}^{n} - \overline{q}_{jR}^{n} \right) \right\} \]

\[ -\text{sign} \left( \overline{\alpha}_{RS,i+\frac{1}{2}}^{n} \right) \overline{q}_{j+\frac{1}{2}}^{n} \left( \overline{\lambda}_{j+1L}^{n} - \overline{\lambda}_{jR}^{n} \right) \] \[ -\overline{\alpha}_{RS,i-\frac{1}{2}}^{n} \left( \overline{q}_{jL}^{n} - \overline{q}_{j-1R}^{n} \right) - \text{sign} \left( \overline{\alpha}_{RS,i-\frac{1}{2}}^{n} \right) \overline{q}_{j-\frac{1}{2}}^{n} \left( \overline{\lambda}_{jL}^{n} - \overline{\lambda}_{j-1R}^{n} \right) \] \tag{107} 

where

\[ \overline{\alpha}_{RS,i-\frac{1}{2}}^{n} = \max \left\{ \left| \overline{\lambda}_{j-1R}^{n} \right|, \left| \overline{\lambda}_{jL}^{n} \right| \right\}, \]

\[ \overline{\alpha}_{RS,i+\frac{1}{2}}^{n} = \max \left\{ \left| \overline{\lambda}_{jR}^{n} \right|, \left| \overline{\lambda}_{j+1L}^{n} \right| \right\}. \]

### A.3 LADER scheme for the advection-diffusion-reaction equation with time and space dependent advection coefficient

The construction of a numerical scheme to solve the advection-diffusion-reaction equation with time and space dependent advection and diffusion
coefficients requires for a modification of the generalized Riemann problem to be solved. As a result, the approximation of the evolved variables (99)-(102) accounts for the diffusion and reaction terms,

\[
\bar{q}^n_{j-1R} = q^n_{j-1R} - \frac{\Delta t}{2\Delta x} \left( \lambda^n_j q^n_j - \lambda^n_{j-1} q^n_{j-1} \right) + \frac{\Delta t}{2\Delta x^2} \left[ \alpha^n_j (q^n_{j+1} - q^n_j) - \alpha^n_{j-1} (q^n_{j-1} - q^n_{j-2}) \right] + \frac{\Delta t}{2} s^n_{j-\frac{1}{2}},
\]

(108)

\[
\bar{q}^n_{jL} = q^n_{jL} - \frac{\Delta t}{2\Delta x} \left( \lambda^n_j q^n_j - \lambda^n_{j-1} q^n_{j-1} \right) + \frac{\Delta t}{2\Delta x^2} \left[ \alpha^n_j (q^n_{j+1} - q^n_j) - \alpha^n_{j-1} (q^n_{j-1} - q^n_{j-2}) \right] + \frac{\Delta t}{2} s^n_{j+\frac{1}{2}},
\]

(109)

\[
\bar{q}^n_{jR} = q^n_{jR} - \frac{\Delta t}{2\Delta x} \left( \lambda^n_{j+1} q^n_{j+1} - \lambda^n_j q^n_j \right) + \frac{\Delta t}{2\Delta x^2} \left[ \alpha^n_{j+1} (q^n_{j+2} - q^n_{j+1}) - \alpha^n_j (q^n_j - q^n_{j-1}) \right] + \frac{\Delta t}{2} s^n_{j+\frac{1}{2}},
\]

(110)

\[
\bar{q}^n_{j+1L} = q^n_{j+1L} - \frac{\Delta t}{2\Delta x} \left( \lambda^n_{j+1} q^n_{j+1} - \lambda^n_j q^n_j \right) + \frac{\Delta t}{2\Delta x^2} \left[ \alpha^n_{j+1} (q^n_{j+2} - q^n_{j+1}) - \alpha^n_j (q^n_j - q^n_{j-1}) \right] + \frac{\Delta t}{2} s^n_{j+\frac{1}{2}},
\]

(111)

where \( s^n_{j+\frac{1}{2}} = s \left( \frac{1}{2} (x_j + x_{j+1}), t^n, q^n_{j+\frac{1}{2}} \right) \), \( s^n_{j-\frac{1}{2}} = s \left( \frac{1}{2} (x_{j-1} + x_j), t^n, q^n_{j-\frac{1}{2}} \right) \). These new approximations are substituted in the flux function (93) providing the expression of the numerical flux. The diffusion and reaction terms are computed by the mid-point rule. Finally, LADER scheme for equation (87) reads

\[
q^n_{j+1} = q^n_j - \frac{\Delta t}{2\Delta x} \left\{ \left[ \frac{\lambda^n_{j+1L} \bar{q}^n_{j+1L} + \bar{q}^n_{j+1L} \bar{q}^n_{j+1L}}{\lambda^n_{j+1L} + \lambda^n_{j+1L}} \right] - \frac{\alpha^n_{j+1L}}{\alpha^n_{j+1L}} (\bar{q}^n_{j+1L} - \bar{q}^n_{j+1L}) \right\} - \frac{\Delta t}{2\Delta x^2} \left[ \left[ \alpha^n_{j+\frac{1}{2}} + \frac{\Delta t}{2} \partial \alpha^n_{j+\frac{1}{2}} \right] \left[ q^n_{j+1} - q^n_j \right] + \frac{\Delta t}{2\Delta x^2} \left( \alpha^n_{j+\frac{1}{2}} (q^n_{j+2} - q^n_{j+1}) - 2\alpha^n_{j+\frac{1}{2}} (q^n_{j+1} - q^n_j) + \alpha^n_{j-\frac{1}{2}} (q^n_j - q^n_{j-1}) \right) + \frac{\Delta t}{2} \left( s^n_{j+1} - s^n_j \right) \right] \right\}
\]
We start by computing the local truncation error related to dependent advection coefficient, LADER scheme for the advection equation with a time and space A.4 Accuracy analysis

\[
\begin{align*}
    &\left[\alpha_{j-\frac{1}{2}}^{n} + \frac{\Delta t}{2} \partial_t \alpha_{j-\frac{1}{2}}^{n}\right] \left[q_{j-1}^{n} - q_{j}^{n} + \frac{\Delta t}{2\Delta x^2} \left(-\alpha_{j+\frac{1}{2}}^{n+1} (q_{j+1}^{n} - q_{j}^{n}) + 2\alpha_{j-\frac{1}{2}}^{n} (q_{j}^{n} - q_{j-1}^{n}) - \alpha_{j-\frac{1}{2}}^{n} (q_{j-1}^{n} - q_{j-2}^{n}) + \frac{\Delta t}{2} (s_{j-1}^{n+1} - s_{j}^{n})\right)\right] \\
    &+ s_{j}^{n+\frac{1}{2}} \\
\end{align*}
\]

with \( s_{j}^{n} = s(x_j, t^n, q_j^n) \) and \( s_{j}^{n+\frac{1}{2}} = s(x_j, t^n + \frac{\Delta t}{2}, q_j^{n+\frac{1}{2}}) \).

**A.4 Accuracy analysis**

**Lemma 1.** LADER scheme for the advection equation with a time and space dependent advection coefficient, (107), is second order accurate in space and time.

**Proof.** We start by computing the local truncation error related to \( \overline{\lambda}_{jR}^{n} \overline{q}_{jR}^{n} + \overline{\lambda}_{j+1L}^{n} \overline{q}_{j+1L}^{n} \):

\[
2\lambda(x_j, t^n)q(x_j, t^n) + \partial_x (\lambda(x_j, t^n)q(x_j, t^n)) \Delta x \\
+ \frac{(\Delta x)^2}{2} \left(-\lambda(x_j, t^n)\partial_x^2 q(x_j, t^n) - q(x_j, t^n)\partial_x^2 \lambda(x_j, t^n) + \partial_x \lambda(x_j, t^n)\partial_x q(x_j, t^n)\right) \\
- \Delta t \left[\lambda(x_j, t^n)\partial_x (\lambda(x_j, t^n)q(x_j, t^n)) - q(x_j, t^n)\partial_t \lambda(x_j+\frac{1}{2}, t^n)\right] \\
- \frac{\Delta t \Delta x}{2} \left[\lambda(x_j, t^n)\partial_x^2 (\lambda(x_j, t^n)q(x_j, t^n)) + \partial_x \lambda(x_j, t^n)\partial_x (\lambda(x_j, t^n)q(x_j, t^n)) \\
- \partial_t \lambda(x_j+\frac{1}{2}, t^n)\partial_x q(x_j, t^n)\right] - \frac{(\Delta t)^2}{2} \partial_t \lambda(x_j+\frac{1}{2}, t^n)\partial_x (\lambda(x_j, t^n)q(x_j, t^n)) + \mathcal{O}(\Delta x^3).
\]

Similarly, the local truncation error of \( \overline{\lambda}_{j-1R}^{n} \overline{q}_{j-1R}^{n} + \overline{\lambda}_{jL}^{n} \overline{q}_{jL}^{n} \) results

\[
2\lambda(x_j, t^n)q(x_j, t^n) - \partial_x (\lambda(x_j, t^n)q(x_j, t^n)) \Delta x \\
+ \frac{(\Delta x)^2}{2} \left(-\lambda(x_j, t^n)\partial_x^2 q(x_j, t^n) - q(x_j, t^n)\partial_x^2 \lambda(x_j, t^n) + \partial_x \lambda(x_j, t^n)\partial_x q(x_j, t^n)\right) \\
- \Delta t \left[\lambda(x_j, t^n)\partial_x (\lambda(x_j, t^n)q(x_j, t^n)) - q(x_j, t^n)\partial_t \lambda(x_j-\frac{1}{2}, t^n)\right] \\
+ \frac{\Delta t \Delta x}{2} \left[\lambda(x_j, t^n)\partial_x^2 (\lambda(x_j, t^n)q(x_j, t^n)) + \partial_x \lambda(x_j, t^n)\partial_x (\lambda(x_j, t^n)q(x_j, t^n)) \\
- \partial_t \lambda(x_j-\frac{1}{2}, t^n)\partial_x q(x_j, t^n)\right] - \frac{(\Delta t)^2}{2} \partial_t \lambda(x_j-\frac{1}{2}, t^n)\partial_x (\lambda(x_j, t^n)q(x_j, t^n)) + \mathcal{O}(\Delta x^3).
\]
Finally, it is verified that \( \alpha_{\alpha, i}^{n+1/2} (\lambda_j^{n+1} - \lambda_j^n), \alpha_{\alpha, i}^{n-1/2} (\lambda_j^n - \lambda_j^{n-1}) \), \( \bar{\lambda}_j^n \) and \( \bar{\lambda}_j^{n-1} \) have a local truncation error of \( \mathcal{O}(\Delta x^3) \). Hence,

\[
\tau^n = \partial_t q(x_j, t^n) + \partial_x (q(x_j, t^n)\lambda(x_j, t^n)) + \frac{\Delta t}{2\Delta x} q(x_j, t^n) \left( \partial_t \lambda(x_j^{1/2}, t^n) - \partial_t \lambda(x_j^{-1/2}, t^n) \right) + \frac{\Delta t}{4} \left( \partial_t \lambda(x_j^{1/2}, t^n) + \partial_t \lambda(x_j^{-1/2}, t^n) \right) \partial_x q(x_j, t^n)
\]

Moreover, applying the Taylor series expansion in space of \( \lambda(x_j^{1/2}, t^n) \) and \( \lambda(x_j^{-1/2}, t^n) \) we obtain

\[
\frac{\Delta t}{2\Delta x} q(x_j, t^n) \left( \partial_t \lambda(x_j^{1/2}, t^n) - \partial_t \lambda(x_j^{-1/2}, t^n) \right) = \frac{\Delta t}{2} q(x_j, t^n) \partial_t (\partial_x \lambda(x_j, t^n)) + \mathcal{O}(\Delta x \Delta t),
\]

\[
\frac{\Delta t}{4} \left( \partial_t \lambda(x_j^{1/2}, t^n) + \partial_t \lambda(x_j^{-1/2}, t^n) \right) \partial_x q(x_j, t^n) = \frac{\Delta t}{2} \partial_t \lambda(x_j, t^n) \partial_x q(x_j, t^n) + \mathcal{O}(\Delta x \Delta t).
\]

Therefore,

\[
\tau^n = \partial_t q(x_j, t^n) + \partial_x (q(x_j, t^n)\lambda(x_j, t^n)) + \frac{\Delta t}{2} \partial_x^{(2)} q(x_j, t^n)
\]

\[
\quad + \frac{\Delta t}{2} q(x_j, t^n) \partial_t (\partial_x \lambda(x_j, t^n)) + \frac{\Delta t}{2} \partial_t \lambda(x_j, t^n) \partial_x q(x_j, t^n)
\]

\[
\quad - \frac{\Delta t}{2} \left[ \lambda(x_j, t^n) \partial_x^{(2)} (\lambda(x_j, t^n) q(x_j, t^n)) + \partial_x \lambda(x_j, t^n) \partial_x (\lambda(x_j, t^n) q(x_j, t^n)) \right]
\]

\[
\quad + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta t \Delta x).
\]

Therefore,

\[
\tau^n = \partial_t q(x_j, t^n) + \partial_x (q(x_j, t^n)\lambda(x_j, t^n)) + \frac{\Delta t}{2} \partial_x^{(2)} q(x_j, t^n)
\]

\[
\quad + \frac{\Delta t}{2} q(x_j, t^n) \partial_t (\partial_x \lambda(x_j, t^n)) + \frac{\Delta t}{2} \partial_t \lambda(x_j, t^n) \partial_x q(x_j, t^n)
\]

\[
\quad - \frac{\Delta t}{2} \left[ \lambda(x_j, t^n) \partial_x^{(2)} (\lambda(x_j, t^n) q(x_j, t^n)) + \partial_x \lambda(x_j, t^n) \partial_x (\lambda(x_j, t^n) q(x_j, t^n)) \right]
\]

\[
\quad + \mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta t \Delta x),
\]

where we have taken into account that the time derivative equation (88) reads

\[
\partial_t^{(2)} q + \partial_x q \partial_t \lambda + q \partial_t (\partial_x \lambda) - \partial_x \lambda \partial_x (\lambda q) - \lambda \partial_x^{(2)} (\lambda q) = 0.
\]

\[\square\]

**Remark 6.** Let us notice that, when applying LADER methodology to the advection equation with a space and time dependent advection coefficient,
we must consider the extrapolated and evolved values for the two variables involved in equation (88), \( \lambda \) and \( q \). Otherwise, the order of the resulting scheme is lower than two.

We will give the main idea to the proof of the previous statement. If we do not evolve in time the advection coefficient then we will not have any time derivative of it in the final local truncation error expression analogous to (114). On the other hand, equation (115) still holds. Since this relation is no more useful to cancel the terms obtained at the last step of the truncation error analysis, the first order terms can not be removed. As a result, we obtain a first order scheme. A similar argument can be given if \( q \) is not properly evolved.

**Lemma 2.** LADER scheme for the advection-diffusion-reaction equation with time and space dependent advection and diffusion coefficients is second order accurate in space and time.

**Proof.** The result is derived from Lemma 1 following the calculus presented in [BFTVC18] for the advection-diffusion-reaction equation with constant advection coefficient.

\[ \square \]

### A.5 Numerical results

To assess the accuracy of LADER scheme (107), we present several analytical tests including space and time dependent advection coefficients. In fact, the method of manufactured solutions is applied in order to obtain a generalized source term assuming a given expression for the exact solution. Dirichlet boundary conditions are considered in all the presented tests.

#### A.5.1 Test A1

As first test, we assume a space dependent advection coefficient of the form \( \lambda(x, t) = x + 2 \) and an exact solution given by \( q(x, t) = e^{-2(x-t)^2-t} \). Applying the method of manufactured solutions to compute the source term of the corresponding advection equation, the problem to be solved reads

\[
\frac{\partial}{\partial t} q(x, t) + \frac{\partial}{\partial x} [\lambda(x, t)q(x, t)] = s(x, t), \\
q(x, 0) = e^{-2x^2},
\]

with

\[
s(x, t) = 4(x - t)(-1 + x)e^{-2(x-t)^2-t}.
\]
The simulations were run considering six uniform meshes on the computational domain $\Omega = [0, 2]$.

| Cells | $\text{Err}_{L^1}$ | $O_{L^1}$ | $\text{Err}_{L^2}$ | $O_{L^2}$ | $\text{Err}_{L^\infty}$ | $O_{L^\infty}$ |
|-------|---------------------|-----------|---------------------|-----------|---------------------|-----------|
| 8     | $8.87E - 02$        |           | $8.71E - 02$        |           | $1.13E - 01$        |           |
| 16    | $5.36E - 02$        | 0.73      | $5.26E - 02$        | 0.73      | $6.98E - 02$        | 0.69      |
| 32    | $2.95E - 02$        | 0.86      | $2.88E - 02$        | 0.87      | $3.87E - 02$        | 0.85      |
| 64    | $1.55E - 02$        | 0.93      | $1.50E - 02$        | 0.94      | $2.03E - 02$        | 0.93      |
| 128   | $7.98E - 03$        | 0.96      | $7.69E - 03$        | 0.97      | $1.04E - 02$        | 0.96      |
| 256   | $4.04E - 03$        | 0.98      | $3.89E - 03$        | 0.98      | $5.28E - 03$        | 0.98      |
| 512   | $2.03E - 03$        | 0.99      | $1.96E - 03$        | 0.99      | $2.65E - 03$        | 0.99      |

Table 4: Test A1. Errors and convergence rates obtained by using the first order scheme. $\Omega = [0, 2]$, $t_{\text{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\text{Err}_{L^1}$ | $O_{L^1}$ | $\text{Err}_{L^2}$ | $O_{L^2}$ | $\text{Err}_{L^\infty}$ | $O_{L^\infty}$ |
|-------|---------------------|-----------|---------------------|-----------|---------------------|-----------|
| 8     | $8.34E - 02$        |           | $8.20E - 02$        |           | $1.07E - 01$        |           |
| 16    | $5.13E - 02$        | 0.70      | $5.03E - 02$        | 0.71      | $6.65E - 02$        | 0.68      |
| 32    | $2.88E - 02$        | 0.83      | $2.79E - 02$        | 0.85      | $3.73E - 02$        | 0.83      |
| 64    | $1.53E - 02$        | 0.91      | $1.47E - 02$        | 0.92      | $1.98E - 02$        | 0.92      |
| 128   | $7.92E - 03$        | 0.95      | $7.55E - 03$        | 0.96      | $1.02E - 02$        | 0.96      |
| 256   | $4.02E - 03$        | 0.98      | $3.83E - 03$        | 0.98      | $5.17E - 03$        | 0.98      |
| 512   | $2.03E - 03$        | 0.99      | $1.93E - 03$        | 0.99      | $2.60E - 03$        | 0.99      |

Table 5: Test A1. Errors and convergence rates obtained by using the first order scheme without the new numerical viscosity term. $\Omega = [0, 2]$, $t_{\text{end}} = 1$, $c = c_M = 0.5$.

Table 4 reports the errors and convergence rates obtained using the first order scheme with the numerical flux function defined by (93). The attained orders match the theoretically expected ones. In table 5, we present the
results obtained by applying the first order scheme without the extra viscosity term in the flux, that is, assuming the numerical flux function being given by (89). We observe that they are not highly affected by the lack of the upwind term. This fact is consistent with the truncation error analysis performed which proves that the new term is of first order and so it is needed only when applying high order schemes.

| Cells | $\text{Err}_{L^1}$ | $\mathcal{O}_{L^1}$ | $\text{Err}_{L^2}$ | $\mathcal{O}_{L^2}$ | $\text{Err}_{L^\infty}$ | $\mathcal{O}_{L^\infty}$ |
|-------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 8     | $5.73E - 02$        | 5.80 $E - 02$       | 8.73 $E - 02$       |                     |                     |
| 16    | $2.10E - 02$        | 1.45 $E - 02$       | 1.52 $E - 02$       | 3.54 $E - 02$       | 1.30 $E - 02$       |                     |
| 32    | $6.58E - 03$        | 1.67 $E - 03$       | 1.66 $E - 03$       | 1.20 $E - 02$       | 1.56 $E - 02$       |                     |
| 64    | $2.57E - 03$        | 1.36 $E - 03$       | 1.30 $E - 03$       | 8.93 $E - 03$       | 0.43 $E - 03$       |                     |
| 128   | $1.44E - 03$        | 0.83 $E - 03$       | 0.75 $E - 03$       | 8.08 $E - 03$       | 0.14 $E - 03$       |                     |
| 256   | $7.62E - 04$        | 0.92 $E - 04$       | 0.74 $E - 04$       | 7.71 $E - 03$       | 0.07 $E - 03$       |                     |
| 512   | $3.92E - 04$        | 0.96 $E - 04$       | 0.69 $E - 04$       | 7.54 $E - 03$       | 0.03 $E - 03$       |                     |

Table 6: Test A1. Errors and convergence rates obtained by using LADER scheme without the new numerical viscosity term. $\Omega = [0, 2]$, $t_{\text{end}} = 1$, $c = c_M = 0.5$.

| Cells | $\text{Err}_{L^1}$ | $\mathcal{O}_{L^1}$ | $\text{Err}_{L^2}$ | $\mathcal{O}_{L^2}$ | $\text{Err}_{L^\infty}$ | $\mathcal{O}_{L^\infty}$ |
|-------|---------------------|---------------------|---------------------|---------------------|---------------------|---------------------|
| 8     | $5.05E - 02$        | 5.59 $E - 02$       | 8.69 $E - 02$       |                     |                     |
| 16    | $1.73E - 02$        | 1.55 $E - 02$       | 1.53 $E - 02$       | 3.43 $E - 02$       | 1.34 $E - 02$       |                     |
| 32    | $5.17E - 03$        | 1.74 $E - 03$       | 1.71 $E - 03$       | 1.17 $E - 02$       | 1.55 $E - 02$       |                     |
| 64    | $1.27E - 03$        | 2.02 $E - 03$       | 1.94 $E - 03$       | 3.48 $E - 03$       | 1.75 $E - 03$       |                     |
| 128   | $3.16E - 04$        | 2.01 $E - 04$       | 1.98 $E - 04$       | 9.63 $E - 04$       | 1.85 $E - 04$       |                     |
| 256   | $7.88E - 05$        | 2.00 $E - 05$       | 1.99 $E - 05$       | 2.57 $E - 04$       | 1.91 $E - 04$       |                     |
| 512   | $1.97E - 05$        | 2.00 $E - 05$       | 1.99 $E - 05$       | 6.63 $E - 05$       | 1.95 $E - 05$       |                     |

Table 7: Test A1. Errors and convergence rates obtained by using LADER scheme. $\Omega = [0, 2]$, $t_{\text{end}} = 1$, $c = c_M = 0.5$. 
The requirement of introducing the new artificial viscosity term related to the advection coefficient is evidenced in Tables 6 and 7. The first of them was obtained by applying LADER scheme to the flux function which neglects the new numerical viscosity term, (89). Neglecting this new upwind term in the numerical flux function, LADER methodology does not achieve second order of accuracy. Moreover, spurious oscillations arise. On the other hand, Table 7 confirms that scheme (112) reaches the expected second order of accuracy.

Finally, the results presented in Table 8 ratify that second order can not be reached if we do not apply LADER methodology to the advection coefficient. That is, if we consider the values of \( \lambda \) at the previous time step neglecting its extrapolation to the cell boundaries and their half in time evolution. Furthermore, instability occurs.

| Cells | \( \text{Err}_{L^1} \) | \( O_{L^1} \) | \( \text{Err}_{L^2} \) | \( O_{L^2} \) | \( \text{Err}_{L^\infty} \) | \( O_{L^\infty} \) |
|-------|----------------|-------|----------------|-------|----------------|-------|
| 8     | 2.69E - 02    | O_{L^1} | 2.83E - 02    | O_{L^2} | 3.82E - 02    | O_{L^\infty} |
| 16    | 4.86E - 03    | 1.75  | 5.60E - 03    | 1.65  | 8.76E - 03    | 1.48  |
| 32    | 1.95E - 03    | 1.33  | 1.76E - 03    | 1.59  | 2.39E - 03    | 1.64  |
| 64    | 1.81E - 03    | 1.00  | 1.62E - 03    | 1.18  | 2.15E - 03    | 1.53  |
| 128   | 1.18E - 03    | 0.68  | 1.05E - 03    | 0.62  | 1.35E - 03    | 0.67  |
| 256   | 6.64E - 04    | 0.83  | 5.90E - 04    | 0.83  | 7.49E - 04    | 0.85  |
| 512   | 3.50E - 04    | 0.92  | 3.12E - 04    | 0.92  | 3.93E - 04    | 0.93  |

Table 8: Test A1. Errors and convergence rates obtained by using LADER scheme without applying LADER methodology to the advection coefficient. \( \Omega = [0, 2], t_{\text{end}} = 1, c = c_M = 0.5. \)

### A.5.2 Test A2

The second proposed test with a variable advection coefficient reads

\[
\partial_t q(x,t) + \partial_x [\lambda(x,t)q(x,t)] = s(x,y),
\]

\[
q(x,0) = e^{-2x^2}, \quad (118)
\]
where
\[ \lambda(x, t) = x + t^2 + 2, \ s(x, y) = 4(x - t) \left(-1 - x - t^2\right) e^{-2(x-t)^2-t}. \] (119)

Let us notice that the advection coefficient considered is time and space dependent. Furthermore, we have included a generalized source term so that the exact solution is
\[ q(x, t) = e^{-2(x-t)^2-t}. \] (120)

| Cells | Err$_{L^1}$ | $O_{L^1}$ | Err$_{L^2}$ | $O_{L^2}$ | Err$_{L^\infty}$ | $O_{L^\infty}$ |
|-------|-------------|-----------|-------------|-----------|----------------|-----------|
| 8     | 1.23E$-01$ |           | 1.20E$-01$ |           | 1.44E$-01$     |           |
| 16    | 8.35E$-02$ | 0.56      | 7.47E$-02$ | 0.68      | 8.62E$-02$     | 0.74      |
| 32    | 4.77E$-02$ | 0.81      | 4.14E$-02$ | 0.85      | 4.74E$-02$     | 0.86      |
| 64    | 2.55E$-02$ | 0.91      | 2.18E$-02$ | 0.93      | 2.49E$-02$     | 0.93      |
| 128   | 1.32E$-02$ | 0.95      | 1.12E$-02$ | 0.96      | 1.27E$-02$     | 0.97      |
| 256   | 6.71E$-03$ | 0.97      | 5.66E$-03$ | 0.98      | 6.45E$-03$     | 0.98      |
| 512   | 3.38E$-03$ | 0.99      | 2.85E$-03$ | 0.99      | 3.24E$-03$     | 0.99      |

Table 9: Test A2. Errors and convergence rates obtained by using a first order scheme. $\Omega = [0, 2]$, $t_{end} = 1$, $c = c_M = 0.5$.

| Cells | Err$_{L^1}$ | $O_{L^1}$ | Err$_{L^2}$ | $O_{L^2}$ | Err$_{L^\infty}$ | $O_{L^\infty}$ |
|-------|-------------|-----------|-------------|-----------|----------------|-----------|
| 8     | 6.82E$-02$ |           | 6.66E$-02$ |           | 9.31E$-02$     |           |
| 16    | 2.44E$-02$ | 1.48      | 2.27E$-02$ | 1.55      | 3.62E$-02$     | 1.36      |
| 32    | 6.59E$-03$ | 1.89      | 6.56E$-03$ | 1.79      | 1.20E$-02$     | 1.59      |
| 64    | 1.83E$-03$ | 1.85      | 1.67E$-03$ | 1.97      | 3.50E$-03$     | 1.78      |
| 128   | 6.23E$-04$ | 1.56      | 5.11E$-04$ | 1.71      | 9.66E$-04$     | 1.86      |
| 256   | 2.39E$-04$ | 1.38      | 2.02E$-04$ | 1.34      | 2.82E$-04$     | 1.78      |
| 512   | 1.06E$-04$ | 1.17      | 8.90E$-05$ | 1.18      | 1.18E$-04$     | 1.26      |

Table 10: Test A2. Errors and convergence rates obtained by using LADER scheme with an ENO-based reconstruction. $\Omega = [0, 2]$, $t_{end} = 1$, $c = c_M = 0.5$. 
The results obtained using the first order scheme are depicted in Table 9. Regarding LADER scheme, this test case requires ENO reconstruction on the conservative variable to avoid instabilities. Moreover, the test case is set to push the capabilities of the numerical scheme to the limit. This leads to a decrease on the attained order of accuracy (see Table 10).

A.6 Test A3

As last test we propose the following initial value problem on the advection-diffusion-reaction equation with variable advection and diffusion coefficients

\[
\partial_t q(x,t) + \partial_x \left[ \lambda(x,t)q(x,t) \right] + \partial_x \left[ \alpha(x,t)\partial_x q(x,t) \right] = s(x,t),
\]

\[
q(x,0) = e^{-2x^2}, \quad (121)
\]

where

\[
\lambda(x,t) = 2 + x + t^2, \quad (122)
\]

\[
\alpha(x,t) = 10^{-5} e^{x(t-1)^2}, \quad (123)
\]

\[
s(x,t) = -e^{-2(x-t)^2-t} + 4e^{-2(x-t)^2-t}(x-t)(-1 - x - t^2) + e^{-2(x-t)^2-t}
\]

\[
+ 10^{-5}(t-1)^2 e^{x(t-1)^2}(-4(x-t)e^{-2(x-t)^2-t})
\]

\[
+ 10^{-5} e^{x(t-1)^2} (-4 + 16(x-t)^2)e^{-2(x-t)^2-t}. \quad (124)
\]

Herein, the exact solution reads

\[
q(x,t) = e^{-2(x-t)^2-t}. \quad (125)
\]

The results obtained using the first order scheme and LADER methodology with ENO reconstruction are shown in Tables 11 and 12. They reinforce the conclusions already got for the previous tests. In Table 13 we present the errors and order obtained for LADER methodology when neglecting the artificial viscosity term related to the gradient of the advection coefficient. We can observe that spurious oscillations arise if this term is not included. Hence, in order to avoid instabilities, an ENO-based reconstruction needs to be performed resulting in a decrease of the order of accuracy. Although, the magnitude of the obtained errors is lower than for the first order scheme.
Table 11: Test A3. Errors and convergence rates obtained by using the first order scheme. $\Omega = [0, 2]$, $t_{end} = 1$, $c = c_M = 0.5$.

| Cells | $\text{Err}_{L^1}$ | $O_{L^1}$ | $\text{Err}_{L^2}$ | $O_{L^2}$ | $\text{Err}_{L^\infty}$ | $O_{L^\infty}$ |
|-------|---------------------|-----------|---------------------|-----------|---------------------|-----------|
| 8     | $1.40E - 01$        |           | $1.35E - 01$        |           | $1.61E - 01$        |           |
| 16    | $9.46E - 02$        | 0.56      | $8.43E - 02$        | 0.67      | $9.65E - 02$        | 0.74      |
| 32    | $5.37E - 02$        | 0.82      | $4.64E - 02$        | 0.86      | $5.28E - 02$        | 0.87      |
| 64    | $2.86E - 02$        | 0.91      | $2.43E - 02$        | 0.93      | $2.76E - 02$        | 0.94      |
| 128   | $1.47E - 02$        | 0.95      | $1.24E - 02$        | 0.97      | $1.41E - 02$        | 0.97      |
| 256   | $7.50E - 03$        | 0.98      | $6.30E - 03$        | 0.98      | $7.14E - 03$        | 0.98      |

Table 12: Test 3. Errors and convergence rates obtained by using LADER scheme with an ENO-base reconstruction. $\Omega = [0, 2]$, $t_{end} = 1$, $c = c_M = 0.5$.

| Cells | $\text{Err}_{L^1}$ | $O_{L^1}$ | $\text{Err}_{L^2}$ | $O_{L^2}$ | $\text{Err}_{L^\infty}$ | $O_{L^\infty}$ |
|-------|---------------------|-----------|---------------------|-----------|---------------------|-----------|
| 8     | $6.86E - 02$        |           | $6.67E - 02$        |           | $9.31E - 02$        |           |
| 16    | $2.44E - 02$        | 1.49      | $2.27E - 02$        | 1.56      | $3.62E - 02$        | 1.36      |
| 32    | $6.57E - 03$        | 1.89      | $6.56E - 03$        | 1.79      | $1.20E - 02$        | 1.59      |
| 64    | $1.83E - 03$        | 1.84      | $1.67E - 03$        | 1.97      | $3.50E - 03$        | 1.78      |
| 128   | $6.23E - 04$        | 1.56      | $5.11E - 04$        | 1.71      | $9.66E - 04$        | 1.86      |
| 256   | $2.38E - 04$        | 1.38      | $2.02E - 04$        | 1.34      | $3.48E - 04$        | 1.47      |
Table 13: Test A3. Errors and convergence rates obtained by using LADER scheme without the new numerical viscosity term. Ω = [0, 2], \( t_{end} = 1 \), \( c = c_M = 0.5 \).

| Cells | \( \text{Err}_{L^1} \) | \( \mathcal{O}_{L^1} \) | \( \text{Err}_{L^2} \) | \( \mathcal{O}_{L^2} \) | \( \text{Err}_{L^\infty} \) | \( \mathcal{O}_{L^\infty} \) |
|-------|----------------|----------------|----------------|----------------|----------------|----------------|
| 8     | 7.80E−02       | 7.49E−02       | 9.51E−02       |                 |                 |                 |
| 16    | 3.18E−02       | 2.68E−02       | 3.88E−02       | 1.29           |                 |                 |
| 32    | 1.12E−02       | 1.50           | 9.57E−03       | 1.48           | 1.59E−02       | 1.28           |
| 64    | 5.29E−03       | 1.08           | 4.25E−03       | 1.17           | 7.00E−03       | 1.19           |
| 128   | 2.59E−03       | 1.03           | 2.15E−03       | 0.99           | 3.22E−03       | 1.12           |
| 256   | 1.30E−03       | 1.00           | 1.09E−03       | 0.98           | 1.53E−03       | 1.07           |