Steppanov ergodic perturbations for nonautonomous evolution equations in Banach spaces

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Abstract
We prove the existence and uniqueness of \( \mu \)-pseudo almost automorphic solutions for a class of semilinear nonautonomous evolution equations of the form:

\[ u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R} \]

where \((A(t))_{t \in \mathbb{R}}\) is a family of closed densely defined linear operators acting on a Banach space \( X \), generating a strongly continuous evolution family that have an exponential dichotomy on \( \mathbb{R} \). The nonlinear term \( f: \mathbb{R} \times X \to X \) is assumed to be only \( \mu \)-pseudo almost automorphic in Stepanov’s sense in \( t \) and Lipschitz continuous with respect to the second variable. To illustrate our theoretical results, we provide an application to a reaction–diffusion equation on \( \mathbb{R} \) with time-dependent parameters.

Keywords Evolution family · Exponential dichotomy · Nonautonomous semilinear evolution equations · \( \mu \)-Pseudo-almost automorphic solutions · Stepanov \( \mu \)-pseudo-almost automorphic functions

Mathematics Subject Classification 46T20 · 47J35 · 34C27 · 35K58
1 Introduction

In this work, we prove the existence and uniqueness of \( \mu \)-pseudo almost automorphic solutions to the following class of semilinear evolution equations:

\[
    u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R},
\]

where \((A(t), D(A(t))), t \in \mathbb{R}\) is a family of closed densely defined linear operators on a Banach space \( X \) and \( f: \mathbb{R} \times X \to X \) is locally integrable in \( t \) (for each \( x \in X \)) and satisfies some suitable conditions with respect to \( x \).

The existence and uniqueness of \( \mu \)-pseudo almost automorphic solutions to evolution equations in Banach spaces has attracted many researchers in the last decades; see [2, 3, 9, 12, 14, 15, 19, 21, 25, 27]. In the parabolic case, i.e., where \((A(t))_{t \in \mathbb{R}}\) are densely defined and satisfy the Acquistapace–Terreni conditions (see [1]), it was shown that Eq. (1.1) has a unique \( \mu \)-pseudo almost automorphic solution provided that the resolvent operator \( R(\omega, A(\cdot)), \) for \( \omega \) large, is almost automorphic and \( f \) is \( \mu \)-pseudo almost automorphic in Stepanov’s sense of order \( 1 < p < +\infty \), see [2, 4]. In [21], the authors proved the existence and uniqueness of weighted pseudo almost automorphic solutions to Eq. (1.1), in the case where in particular \( f \) is weighted pseudo almost automorphic in the strong sense, the Green’s function is bi-almost automorphic and \((A(t))_{t \in \mathbb{R}}\) generates a strongly continuous exponentially stable evolution family \((U(t, s))_{t \geq s}\). A generalization of [21] was given in [27], where the authors proved the existence and uniqueness of weighted pseudo almost automorphic solutions to Eq. (1.1) provided that \((A(t))_{t \in \mathbb{R}}\) generates a strongly continuous evolution family \((U(t, s))_{t \geq s}\) that has an exponential dichotomy on \( \mathbb{R} \) and \( f \) is only weighted pseudo almost automorphic in Stepanov’s sense. Notice that the concept of \( \mu \)-pseudo almost automorphy due to Ezzinbi et al. [7, 11] generalize both notions of pseudo almost automorphy due to Xiao et al. [25] and weighted pseudo almost automorphy due to Diagana; see [10].

Inspired by the above results and under assumptions that \((A(t))_{t \in \mathbb{R}}\) generates a strongly continuous evolution family \((U(t, s))_{t \geq s}\) on \( X \) that has an exponential dichotomy on \( \mathbb{R} \), \( f \) is only \( \mu \)-pseudo almost automorphic in Stepanov’s sense (i.e., in a weaker sense), of order \( 1 \leq p < +\infty \), with respect to \( t \) and satisfies some suitable conditions with respect to the second variable. It is well known that \( \mu \)-pseudo almost automorphic functions in Stepanov’s sense generalize those of \( \mu \)-pseudo almost automorphic functions in the strong sense; see Sect. 2.1. Our strategy concerns, at first, studying the following linear inhomogeneous equation:

\[
    u'(t) = A(t)u(t) + g(t), \quad t \in \mathbb{R},
\]

where \( g \) is \( \mu \)-pseudo almost automorphic in Stepanov’s sense. We show that the associated mild solution given by:

\[
    u(t) = \int_{\mathbb{R}} \Gamma(t, s)g(s)ds, \quad t \in \mathbb{R}
\]
is $\mu$-pseudo almost automorphic of order $1 \leq p < \infty$, where $\Gamma(\cdot, \cdot)$ is the associated Green’s function. Then, by a suitable composition result (see Theorem 2.18), and under the hypothesis of “weak” Lipschitz continuity of $f$ with respect to the second variable (see (H4)), we prove our results of Eq. (1.1) using the Banach fixed point principle.

The rest of this paper is organized as follows. In Sect. 2, we give preliminaries on evolution families and their asymptotic behavior. After that, we recall basic notions of $\mu$-pseudo almost automorphic functions in the classical and in the Stepanov’s senses, respectively. Section 3 is devoted to our main results, and we prove the existence and uniqueness of $\mu$-pseudo almost automorphic solutions to Eqs. (1.1) and (1.2), respectively. Section 4 is devoted to an illustrated application to a class of nonautonomous reaction–diffusion equations on $\mathbb{R}$.

## 2 Preliminaries

Let $A(t) : D(A(t)) \subset X \rightarrow X$, $t \in \mathbb{R}$ be a family of closed linear operators in a Banach space $X$. In general $A(t)$, $t \in \mathbb{R}$ are time-dependent suitable differential operators that correspond to the following non-autonomous Cauchy problem:

$$
\left\{
\begin{array}{l}
u'(t) = A(t)\nu(t), \quad t \geq s \\
u(s) = x \in X.
\end{array}
\right.
$$

(2.1)

A solution (mild) for Eq. (2.1) can be expressed as $u(t) = U(t, s)x$, where $\{U(t, s)\}_{t \geq s}$ is a two-parameter family generated by $(A(t))_{t \in \mathbb{R}}$ on $X$ that is called strongly continuous evolution family, i.e., $\{U(t, s)\}_{t \geq s} \subseteq \mathcal{L}(X)$ such that:

(i) $U(t, r)U(r, s) = U(t, s)$ and $U(t, t) = I$ for all $t \geq r \geq s$ and $t, r, s \in \mathbb{R}$.

(ii) The map $(t, s) \rightarrow U(t, s)x$ is continuous for all $x \in X$, $t \geq s$ and $t, s \in \mathbb{R}$;

see [13, 20, 22] for more details. Unlike semigroups, there is no general theory for existence of a corresponding evolution family. However, we can rely on several existence theorems corresponding to different contexts. In fact, in the hyperbolic case, we refer to [18, 22, 24] and to [1, 17] for the parabolic case. An evolution family $(U(t, s))_{s \leq t}$ on a Banach space $X$ is called an exponential dichotomy (or hyperbolic) in $\mathbb{R}$ if there exists a family of projections $P(t) \in \mathcal{L}(X)$, $t \in \mathbb{R}$, being strongly continuous with respect to $t$, and constants $\delta, M > 0$ such that

(i) $U(t, s)P(s) = P(t)U(t, s)$;

(ii) $U(t, s) : Q(s)X \rightarrow Q(t)X$ is invertible with the inverse $\tilde{U}(t, s)$;

(iii) $\|U(t, s)P(s)\| \leq Me^{-\delta(t-s)}$ and $\|\tilde{U}(t, s)Q(t)\| \leq Me^{-\delta(t-s)}$

for all $t, s \in \mathbb{R}$ with $s \leq t$, where, $Q(t) = I - P(t)$.

Hence, for a given hyperbolic evolution family $(U(t, s))_{s \leq t}$, we define its associated Green’s function by:

$$
\Gamma(t, s) = \begin{cases}
U(t, s)P(s), & t, s \in \mathbb{R}, \ s \leq t \\
\tilde{U}(t, s)Q(s), & t, s \in \mathbb{R}, \ s > t.
\end{cases}
$$
Notice that the exponential dichotomy is a classical concept in the study of long-time behavior of evolution equations. If $P(t) = I$ for $t \in \mathbb{R}$, then $(U(t, s))_{s \leq t}$ is exponentially stable; in such a case, the associated Green’s function is given by

$$\Gamma(t, s) = \begin{cases} U(t, s), & t, s \in \mathbb{R}, s \leq t \\ 0, & t, s \in \mathbb{R}, s > t. \end{cases}$$

For more details, we refer to [13].

### 2.1 $\mu$-Pseudo almost automorphic functions

**Notations.** Let $(X, \| \cdot \|)$ be any Banach space. We denote by $L^p_{\text{loc}}(\mathbb{R}, X)$ with $1 \leq p < \infty$, the space of functions $f : \mathbb{R} \rightarrow X$ measurable such that

$$\left( \int_{[a,b]} \| f(s) \|^p ds \right)^{\frac{1}{p}} < \infty$$

for all $a < b$ in $\mathbb{R}$. $BC(\mathbb{R}, X)$ equipped with the supremum norm is the Banach space of bounded continuous functions from $\mathbb{R}$ into $X$. Let $1 \leq p < \infty$ and $q$ denote its conjugate exponent defined by

$$\frac{1}{p} + \frac{1}{q} = 1.$$

In the following, we give the properties of $\mu$-pseudo almost automorphic functions in the classical sense and in Stepanov’s sense respectively.

**Definition 2.1 (Bohr [6])** A continuous function $f : \mathbb{R} \rightarrow X$ is almost periodic if for every $\varepsilon > 0$, there exists $l_\varepsilon > 0$, such that for every $a \in \mathbb{R}$, there exists $\tau \in [a, a + l_\varepsilon]$ satisfying:

$$\| f(t + \tau) - f(t) \| < \varepsilon \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $\text{AP}(\mathbb{R}, X)$.

**Definition 2.2 (Bochner [5])** A continuous function $f : \mathbb{R} \rightarrow X$ is called almost automorphic if for every sequence $(s_n')_{n \geq 0}$ of real numbers, there exists a subsequence $(s_n)_{n \geq 0} \subset (s_n')_{n \geq 0}$ and a measurable function $g : \mathbb{R} \rightarrow X$, such that

$$g(t) = \lim_{n \to \infty} f(t + s_n) \quad \text{and} \quad f(t) = \lim_{n \to \infty} g(t - s_n) \quad \text{for all } t \in \mathbb{R}.$$

The space of all such functions is denoted by $\text{AA}(\mathbb{R}, X)$.

**Remark 2.3** An almost automorphic function may not be uniformly continuous. Indeed, the real function

$$f(t) = \sin \left( \frac{1}{2 + \cos(t) + \cos(\sqrt{2}t)} \right)$$

for $t \in \mathbb{R}$, belongs to $\text{AA}(\mathbb{R}, \mathbb{R})$, but is not uniformly continuous. Hence, $f$ does not belong to $\text{AP}(\mathbb{R}, \mathbb{R})$. 

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Then, we have the following inclusions:

\[ AP(\mathbb{R}, X) \subset AA(\mathbb{R}, X) \subset BC(\mathbb{R}, X). \]

**Definition 2.4** A continuous function \( F : \mathbb{R} \times \mathbb{R} \rightarrow X \) is said to be bi-almost automorphic if for every sequence \( (s'_n)_{n \geq 0} \) of real numbers, there exist a subsequence \( (s_n)_{n \geq 0} \subset (s'_n)_{n \geq 0} \) and a measurable function \( G : \mathbb{R} \times \mathbb{R} \rightarrow X \), such that

\[
G(t, s) = \lim_{n \to \infty} F(t + s_n, s + s_n) \quad \text{and} \quad F(t, s) = \lim_{n \to \infty} G(t - s_n, s - s_n) \quad \text{for all } t, s \in \mathbb{R}.
\]

The space of all such functions is denoted by \( bAA(\mathbb{R}, X) \).

**Example** [14] \( F(t, s) = \sin(t) \cos(s) \) is bi-almost automorphic function from \( \mathbb{R} \times \mathbb{R} \) to \( \mathbb{R} \) as

\[
F(t + 2\pi, s + 2\pi) = F(t, s) \quad \text{for all } t, s \in \mathbb{R}.
\]

**Definition 2.5** [23] Let \( 1 \leq p < \infty \). A function \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) is said to be bounded in the sense of Stepanov if

\[
\sup_{t \in \mathbb{R}} \left( \int_{[t, t+1]} \| f(s) \|^p \, ds \right)^{\frac{1}{p}} = \sup_{t \in \mathbb{R}} \left( \int_{[0,1]} \| f(t + s) \|^p \, ds \right)^{\frac{1}{p}} < \infty.
\]

The space of all such functions is denoted by \( BS^p(\mathbb{R}, X) \) and it is provided with the following norm:

\[
\| f \|_{BS^p} := \sup_{t \in \mathbb{R}} \left( \int_{[t, t+1]} \| f(s) \|^p \, ds \right)^{\frac{1}{p}} = \sup_{t \in \mathbb{R}} \| f(t + \cdot) \|_{L^p([0,1], X)}.
\]

Then, the following inclusions hold:

\[
BC(\mathbb{R}, X) \subset BS^p(\mathbb{R}, X) \subset L^p_{\text{loc}}(\mathbb{R}, X). \quad (2.2)
\]

Now, we give the definition of almost automorphy in the sense of Stepanov.

**Definition 2.6** [4] Let \( 1 \leq p < \infty \). A function \( f \in L^p_{\text{loc}}(\mathbb{R}, X) \) is said to be almost automorphic in the sense of Stepanov (or \( S^p \)-almost automorphic), if for every sequence \( (\sigma_n)_{n \geq 0} \) of real numbers, there exists a subsequence \( (s_n)_{n \geq 0} \subset (\sigma_n)_{n \geq 0} \) and a measurable function \( g \in L^p_{\text{loc}}(\mathbb{R}, X) \), such that

\[
\lim_n \left( \int_t^{t+1} \| f(s + s_n) - g(s) \|^p \, ds \right)^{\frac{1}{p}} = 0.
\]
and
\[
\lim_n \left( \int_t^{t+1} \| g(s - s_n) - f(s) \| ds \right)^{\frac{1}{p}} \quad \text{for all } t \in \mathbb{R}.
\]

The space of all such functions is denoted by \( \text{AAS}^p(\mathbb{R}, X) \).

**Remark 2.7** [4]

(i) Every almost automorphic function is \( S^p \)-almost automorphic for \( 1 \leq p < \infty \).

(ii) For all \( 1 \leq p_1 \leq p_2 < \infty \), if \( f \) is \( S^{p_2} \)-almost automorphic, then \( f \) is \( S^{p_1} \)-almost automorphic.

In this section, we recall some properties of \( \mu \)-ergodic and \( \mu \)-pseudo almost automorphic functions. In the sequel, we denote by \( \mathcal{B}(\mathbb{R}) \) the Lebesgue \( \sigma \)-field of \( \mathbb{R} \) and by \( \mathcal{M} \) the set of all positive measures \( \mu \) on \( \mathcal{B}(\mathbb{R}) \) satisfying \( \mu(\mathbb{R}) = +\infty \) and \( \mu([a, b]) < +\infty \) for all \( a, b \in \mathbb{R} \) with \( a \leq b \). Moreover, we assume the following hypothesis:

**(M)** For all \( \tau \in \mathbb{R} \), there exist \( \beta > 0 \) and a bounded interval \( I \) such that
\[
\mu(\{a + \tau : a \in A\}) \leq \beta \mu(A) \quad \text{where } A \in \mathcal{B}(\mathbb{R}) \text{ and } A \cap I = \emptyset.
\]

**Definition 2.8** [7] Let \( \mu \in \mathcal{M} \). A continuous bounded function \( f : \mathbb{R} \longrightarrow X \) is called \( \mu \)-ergodic, if
\[
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \| f(t) \| d\mu(t) = 0.
\]

The space of all such functions is denoted by \( \mathcal{E}(\mathbb{R}, X, \mu) \).

**Example** (1) In [26], the author defined the concept of ergodic functions as \( \mu \)-ergodic functions in the particular case where the measure \( \mu \) is the Lebesgue measure.

(2) Let \( \rho : \mathbb{R} \longrightarrow [0, +\infty) \) be a \( \mathcal{B}(\mathbb{R}) \)-measurable function. We define the positive measure \( \mu \) on \( \mathcal{B}(\mathbb{R}) \) by
\[
\mu(A) = \int_A \rho(t) dt \quad \text{for } A \in \mathcal{B}(\mathbb{R}),
\]

with respect to the Lebesgue measure on \( \mathcal{B}(\mathbb{R}) \). The measure \( \mu \) is absolutely continuous with respect to the Lebesgue measure and the function \( \rho \) is its associated Radon–Nikodym derivative of \( \mu \). In this case \( \mu \in \mathcal{M} \) if and only if the function \( \rho \) is locally Lebesgue-integrable on \( \mathbb{R} \) and satisfies
\[
\int_{\mathbb{R}} \rho(t) dt = +\infty.
\]
(3) In [16], the authors considered the space of bounded continuous functions $f : \mathbb{R} \to X$ satisfying

$$
\lim_{r \to +\infty} \frac{1}{2r} \int_{[-r, r]} \| f(t) \| \, dt = 0 \quad \text{and} \quad \lim_{N \to +\infty} \frac{1}{2N + 1} \sum_{n=-N}^{N} \| f(n) \| = 0.
$$

This space coincides with the space of $\mu$-ergodic functions where $\mu$ is defined in $\mathcal{B} (\mathbb{R})$ by the sum $\mu(A) = \mu_1(A) + \mu_2(A)$ with $\mu_1$ is the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ and

$$
\mu_2(A) = \begin{cases} 
\text{card}(A \cap \mathbb{Z}) & \text{if } A \cap \mathbb{Z} \text{ is finite} \\
\infty & \text{if } A \cap \mathbb{Z} \text{ is infinite}.
\end{cases}
$$

**Definition 2.9** [7] Let $\mu \in \mathcal{M}$. A continuous function $f : \mathbb{R} \to X$ is said to be $\mu$-pseudo almost automorphic if $f$ is written in the form:

$$
f = g + \phi,
$$

where $g \in AA (\mathbb{R}, X)$ and $\varphi \in \mathcal{E}(\mathbb{R}, X, \mu)$.

The space of all such functions is denoted by $PAA(\mathbb{R}, X, \mu)$.

**Remark 2.10** Notice that the hypothesis (M) is crucial for the invariance of the space $\mathcal{E} (\mathbb{R}, X, \mu)$ under translation by an element $\tau \in \mathbb{R}$, which is important as assumption in our theoretical results. In particular, if $\mu$ is the Lebesgue measure, the hypothesis (M) holds immediately. Moreover, if the measure $\mu$ is absolutely continuous with respect to the Lebesgue measure with $\rho$ the associated Radon–Nikodym derivative of $\mu$, then (M) holds true under a translation condition with respect to $\rho$; see [7, Remark 3.1] and [7, Theorem 3.5] for more details.

Now, we give the definition and the important properties of $\mu$-$S^p$-pseudo almost automorphic functions.

**Definition 2.11** [11] Let $\mu \in \mathcal{M}$. A function $f \in BS^p (\mathbb{R}, X)$ is said to be $\mu$-ergodic in the sense of Stepanov (or $\mu$-$S^p$-ergodic) if

$$
\lim_{r \to +\infty} \frac{1}{\mu([-r, r])} \int_{[-r, r]} \left( \int_{[t, t+1]} \| f(s) \|^p \, ds \right)^{\frac{1}{p}} \, d\mu(t) = 0. 
$$

(2.3)

The space of all such functions is denoted by $\mathcal{E}^p (\mathbb{R}, X, \mu)$.

**Remark 2.12** Using (2.3), we obtain that $f \in \mathcal{E}^p (\mathbb{R}, X, \mu)$ if and only if $f^b \in \mathcal{E}(\mathbb{R}, L^p ([0, 1], X), \mu)$.

**Proposition 2.13** [11] Let $\mu \in \mathcal{M}$. Then, for all $1 \leq p < \infty$, $(\mathcal{E}^p (\mathbb{R}, X, \mu), \| \cdot \|_{BS^p})$ is a Banach space.

**Proposition 2.14** [11] Let $\mu \in \mathcal{M}$ satisfy (M). Then, the following hold:

(i) $\mathcal{E}^p (\mathbb{R}, X, \mu)$ is translation invariant.

(ii) $\mathcal{E}(\mathbb{R}, X, \mu) \subset \mathcal{E}^p (\mathbb{R}, X, \mu)$.  

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2.2 Uniformly $\mu$-pseudo almost automorphic functions

Definition 2.15 [4] Let $1 \leq p < +\infty$ and $f : \mathbb{R} \times X \to Y$ be a function such that $f(\cdot, x) \in L^p_{\text{loc}}(\mathbb{R}, Y)$ for each $x \in X$. Then, $f \in AASP^U(\mathbb{R} \times X, Y)$ if the following hold:

(i) For each $x \in X$, $f(\cdot, x) \in AAS^p(\mathbb{R}, Y)$.

(ii) $f$ is $S^p$-uniformly continuous with respect to the second argument on each compact subset $K$ in $X$, namely: for all $\varepsilon > 0$, there exists $\delta_{K, \varepsilon}$ such that for all $x_1, x_2 \in K$, we have

$$
\|x_1 - x_2\| \leq \delta_{K, \varepsilon} \implies \left( \int_t^{t+1} \| f(s, x_1) - f(s, x_2) \|^p_Y ds \right)^{\frac{1}{p}} \leq \varepsilon \quad \text{for all } t \in \mathbb{R}.
$$

(2.4)

Definition 2.16 Let $\mu \in \mathcal{M}$. A function $f : \mathbb{R} \times X \to Y$ such that $f(\cdot, x) \in BS^p(\mathbb{R}, Y)$ for each $x \in X$ is said to be $\mu$-$S^p$-ergodic in $t$ with respect to $x$ in $X$ if the following hold:

(i) For all $x \in X$, $f(\cdot, x) \in E^p(\mathbb{R}, Y, \mu)$.

(ii) $f$ is $S^p$-uniformly continuous with respect to the second argument on each compact subset $K$ in $X$.

Denote by $E^pU(\mathbb{R} \times X, Y, \mu)$ the set of all such functions.

Definition 2.17 [4] Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \times X \to Y$ be such that $f(\cdot, x) \in BS^p(\mathbb{R}, Y)$ for each $x \in X$. The function $f$ is $\mu$-$S^p$-almost automorphic if $f$ is written as:

$$
f = g + \varphi,
$$

where $g \in AASP^U(\mathbb{R} \times X, Y)$, and $\varphi \in E^pU(\mathbb{R} \times X, Y, \mu)$.

The space of all such functions is denoted $PAASP^U(\mathbb{R}, X, \mu)$.

Theorem 2.18 [4] Let $\mu \in \mathcal{M}$ and $f : \mathbb{R} \times X \to Y$. Assume that:

(i) $f = g + \varphi \in PAASP^U(\mathbb{R} \times X, Y, \mu)$ with $g \in AASP^U(\mathbb{R} \times X, Y)$ and $\varphi \in E^pU(\mathbb{R} \times X, Y, \mu)$.

(ii) $u = u_1 + u_2 \in PAA(\mathbb{R}, X, \mu)$, where $u_1 \in AA(\mathbb{R}, X)$ and $u_2 \in E^p(\mathbb{R}, X, \mu)$.

(iii) For every bounded subset $B \subset X$, the set $\land := \{ f(\cdot, x) : x \in B \}$ is bounded in $BS^p(\mathbb{R}, X)$.

Then, $f(\cdot, u(\cdot)) \in PAASP(\mathbb{R}, Y, \mu)$.

3 $\mu$-Pseudo almost automorphic solutions of Eq. (1.1)

In this section, we prove the existence and uniqueness of $\mu$-pseudo almost automorphic mild solutions to Eq. (1.1).
Lemma 3.3 Let \( g : \mathbb{R} \rightarrow X \) that satisfies the following variation of constans formula:

\[
    u(t) = U(t, s)u(s) + \int_s^t U(t, r)f(r, u(r))dr \quad \text{for all } t \geq s, \quad t, s \in \mathbb{R}.
\] (3.1)

In the sequel, we assume that:

(H1) The family \( A(t), \ t \in \mathbb{R} \) generates a strongly continuous evolution family \((U(t, s))_{t \geq s}\).

(H2) The evolution family \((U(t, s))_{t \geq s}\) has an exponential dichotomy on \( \mathbb{R} \), with constants \( M \geq 0, \delta > 0 \) and Green’s function \( \Gamma \).

(H3) For each \( x \in X \), \( \Gamma(t, s)x \) for \( t, s \in \mathbb{R} \) is bi-almost automorphic, i.e., for every sequence \((s'_n)_{n \geq 0}\) of real numbers, there exist a subsequence \((s_n)_{n \geq 0} \subset (s'_n)_{n \geq 0}\) and a measurable function \( \tilde{\Gamma} \) with \( \int \tilde{\Gamma}(\cdot, \cdot)x : \mathbb{R} \times \mathbb{R} \rightarrow X \), such that

\[
    \lim_{n \to \infty} \|\Gamma(t + s_n, s + s_n)x - \tilde{\Gamma}(t, s)x\| \quad \text{and} \quad \lim_{n \to \infty} \|\tilde{\Gamma}(t - s_n, s - s_n)x - \Gamma(t, s)x\|
\]

for all \( t, s \in \mathbb{R} \).

Remark 3.2 An explicit example of a strongly bi-almost automorphic Green function, i.e., hypothesis (H3), is given in Sect. 4. Sufficient conditions insuring hypothesis (H3), in the case where \( A(t) = \delta(t)A + \alpha(t), \ t \in \mathbb{R} \) and \( A \) is generator of a strongly continuous semigroup, provided only that \( \delta, \alpha \in AAS^1(\mathbb{R}) \) with \( \inf_{t \in \mathbb{R}} \delta(t) > 0 \) and \( \alpha(t) < 0 \) for all \( t \in \mathbb{R} \), which is a weak condition; see Sect. 4 for more details.

In the interest of establishing our problem, we first study the following linear inhomogeneous evolution equation associated with Eq. (1.1):

\[
    u'(t) = A(t)u(t) + g(t) \quad \text{for all } t \in \mathbb{R},
\] (3.2)

where \( g : \mathbb{R} \rightarrow X \) is locally integrable. We recall that a mild solution to Eq. (3.2) is a continuous function \( u : \mathbb{R} \rightarrow X \) that is given by the following variation of constant formula:

\[
    u(t) = U(t, s)u(s) + \int_s^t U(t, r)g(r)dr \quad \text{for all } t \geq s.
\] (3.3)

The following lemma is needed.

Lemma 3.3 Let \( g \in B^{p}S^p(\mathbb{R}, X) \) for \( 1 \leq p < \infty \). Assume that (H1)–(H2) hold. Then Eq. (3.2) has a unique bounded continuous mild solution given by:

\[
    u(t) = \int_{\mathbb{R}} \Gamma(t, s)g(s)ds, \quad t \in \mathbb{R}.
\] (3.4)

Proof Let us show first that the integral given in formula (3.4) is defined and bounded on \( \mathbb{R} \). We know from the exponential dichotomy of \((U(t, s))_{t \geq s}\) that

\[
    \int_{\mathbb{R}} \Gamma(t, s)g(s)ds = \int_{-\infty}^{t} U(t, \sigma)P(\sigma)g(\sigma)d\sigma - \int_{t}^{\infty} \tilde{U}(t, \sigma)Q(\sigma)g(\sigma)d\sigma, \quad t \in \mathbb{R}.
\]
For $p > 1$, using Hölder’s inequality, we have

\[
\left\| \int \Gamma(t, s) g(s) ds \right\| \leq \int_{-\infty}^{t} \left\| U(t, s) P(s) g(s) \right\| ds + \int_{t}^{\infty} \left\| \tilde{U}(t, s) Q(s) g(s) \right\| ds \\
\leq \int_{-\infty}^{t} M e^{-\delta(t-s)} \left\| g(s) \right\| ds + \int_{t}^{\infty} M e^{-\delta(s-t)} \left\| g(s) \right\| ds \\
\leq \sum_{n \geq 1} \int_{t-n}^{t-n+1} M e^{-\delta(t-s)} \left\| g(s) \right\| ds \\
+ \sum_{n \geq 1} \int_{t+n-1}^{t+n} M e^{-\delta(s-t)} \left\| g(s) \right\| ds \\
\leq M \sum_{n \geq 1} \left( \int_{t-n}^{t-n+1} e^{-q\delta(t-s)} ds \right) \frac{1}{q} \left( \int_{t-n}^{t-n+1} \left\| g(s) \right\|^p ds \right)^{\frac{1}{p}} \\
+ M \sum_{n \geq 1} \left( \int_{t+n-1}^{t+n} e^{-q\delta(s-t)} ds \right) \frac{1}{q} \left( \int_{t+n-1}^{t+n} \left\| g(s) \right\|^p ds \right)^{\frac{1}{p}} \\
\leq 2M \left( \frac{e^{q\delta} - 1}{q\delta} \right)^{\frac{1}{q}} \left\| g \right\|_{BS^p} \sum_{n \geq 1} e^{-\delta n} \\
= 2M \frac{1}{e^{\delta} - 1} \left( \frac{e^{q\delta} - 1}{q\delta} \right)^{\frac{1}{q}} \left\| g \right\|_{BS^p} < \infty.
\]

On the other hand, for $p = 1$, it follows that

\[
\left\| \int \Gamma(t, s) g(s) ds \right\| \leq \int_{-\infty}^{t} \left\| U(t, s) P(s) g(s) \right\| ds + \int_{t}^{\infty} \left\| \tilde{U}(t, s) Q(s) g(s) \right\| ds \\
\leq \sum_{n \geq 1} \int_{t-n}^{t-n+1} M e^{-\delta(t-s)} \left\| g(s) \right\| ds \\
+ \sum_{n \geq 1} \int_{t+n-1}^{t+n} M e^{-\delta(s-t)} \left\| g(s) \right\| ds \\
\leq 2M \frac{e^{\delta} - 1}{\delta} \left\| g \right\|_{BS^1} \sum_{n \geq 1} e^{-\delta n} \\
= \frac{2M}{\delta} \left\| g \right\|_{BS^1} < \infty.
\]

Hence, (3.4) is well defined. Now, the fact that the mild solution of Eq. (3.2) is given by (3.4) can proved as in [4, Theorem 4.2-(i)].

\[ \Box \]

**Theorem 3.4** Let $1 \leq p < \infty$ and $g \in AAS^p(\mathbb{R}, X)$. Assume that (H1)–(H3) are satisfied. Then, Eq. (3.2) has a unique mild solution $u \in AA(\mathbb{R}, X)$ given by (3.4).
\[ u(t) = \int_{\mathbb{R}} \Gamma(t, s)g(s)ds, \quad t \in \mathbb{R}. \]

**Proof** Let \( 1 \leq p < \infty \) and \( g \in AAS^p(\mathbb{R}, X) \). By Lemma 3.3, it is obvious that \( u \) is the unique mild solution to Eq. (3.2) given by (3.4). Now, we show that \( u \in AA(\mathbb{R}, X) \). Let \( k \geq 1 \). Then, for \( p > 1 \), we have

\[
\|u_k(t)\| \leq \int_{t-k}^{t-k+1} \|U(t, s)P(s)g(s)\|ds + \int_{t+k-1}^{t+k+1} \|\tilde{U}(t, s)Q(s)g(s)\|ds \\
\leq \int_{t-k}^{t-k+1} Me^{-\delta(t-s)}\|g(s)\|ds + \int_{t+k-1}^{t+k+1} Me^{\delta(t-s)}\|g(s)\|ds \\
\leq M \left( \int_{t-k}^{t-k+1} e^{-q\delta(t-s)}ds \right)^{\frac{1}{q}} \left( \int_{t-k}^{t-k+1} \|g(s)\|^p ds \right)^{\frac{1}{p}} \\
+ M \left( \int_{t+k-1}^{t+k+1} e^{-q\delta(t-s)}ds \right)^{\frac{1}{q}} \left( \int_{t+k-1}^{t+k+1} \|g(s)\|^p ds \right)^{\frac{1}{p}} \\
\leq 2M \|g\|_{BS^p} \left( \frac{e^{\delta q} - 1}{\delta q} \right)^{\frac{1}{q}} e^{-\delta k} \text{ for all } t \in \mathbb{R}.
\]

In a similar way, for \( p = 1 \), we have

\[
\|u_k(t)\| \leq \int_{t-k}^{t-k+1} \|U(t, s)P(s)g(s)\|ds + \int_{t+k-1}^{t+k+1} \|\tilde{U}(t, s)Q(s)g(s)\|ds \\
\leq 2M \frac{e^\delta - 1}{\delta} \|g\|_{BS^1} e^{-\delta k} \text{ for all } t \in \mathbb{R}.
\]

Since \( \sum_{k \geq 1} e^{-\delta k} = \frac{1}{e^\delta - 1} < \infty \), it follows from Weierstrass theorem that the series \( \sum_{k \geq 1} u_k(t) \) is uniformly convergent on \( \mathbb{R} \). Then, we define

\[ u(t) = \sum_{k \geq 1} u_k(t) \text{ for all } t \in \mathbb{R}. \]

In fact, let \( n \geq 1 \). Then, for \( p > 1 \), we have

\[
\left\| u(t) - \sum_{k=1}^{n} u_k(t) \right\| \\
= \left\| \int_{\mathbb{R}} \Gamma(t, s)g(s)ds - \sum_{k=1}^{n} \int_{t-k}^{t-k+1} U(t, s)P(s)g(s)ds \\
+ \sum_{k=1}^{n} \int_{t+k-1}^{t+k} \tilde{U}(t, s)Q(s)g(s)ds \right\|
\]
\[
\leq \left\| \sum_{k \geq n+1} \int_{t-k}^{t-k+1} U(t, s) P(s) g(s) \, ds \right\| + \left\| \sum_{k \geq n+1} \int_{t+k-1}^{t+k} \tilde{U}(t, s) Q(s) g(s) \, ds \right\|
\leq \sum_{k \geq n+1} \int_{t-k}^{t-k+1} \| U(t, s) P(s) g(s) \| \, ds + \sum_{k \geq n+1} \int_{t+k-1}^{t+k} \| \tilde{U}(t, s) Q(s) g(s) \| \, ds
\leq \sum_{k \geq n+1} \int_{t-k}^{t-k+1} M e^{-\delta(t-s)} \| g(s) \| \, ds + \sum_{k \geq n+1} \int_{t+k-1}^{t+k} M e^{\delta(t-s)} \| g(s) \| \, ds
\leq M \sum_{k \geq n+1} \left( \int_{t-k}^{t-k+1} e^{-q\delta(t-s)} \, ds \right)^{\frac{1}{q}} \left( \int_{t-k}^{t-k+1} \| g(s) \|^p \, ds \right)^{\frac{1}{p}}
+ M \sum_{k \geq n+1} \left( \int_{t+k-1}^{t+k} e^{-q\delta(s-t)} \, ds \right)^{\frac{1}{q}} \left( \int_{t+k-1}^{t+k} \| g(s) \|^p \, ds \right)^{\frac{1}{p}}
\leq 2M \left( \frac{e^{\delta q} - 1}{\delta q} \right)^{\frac{1}{q}} \| g \|_{BS^p} \sum_{k \geq n+1} e^{-\delta k} \to 0 \text{ as } n \to \infty \text{ uniformly in } t \in \mathbb{R}.
\]

Otherwise, for \( p = 1 \), we obtain that

\[
\left\| u(t) - \sum_{k=1}^{n} u_k(t) \right\|
= \left\| \int_{\mathbb{R}} \Gamma(t, s) g(s) \, ds - \sum_{k=1}^{n} \int_{t-k}^{t-k+1} U(t, s) P(s) g(s) \, ds \right. \\
+ \left. \sum_{k=1}^{n} \int_{t+k-1}^{t+k} \tilde{U}(t, s) Q(s) g(s) \, ds \right\|
\leq 2M \frac{e^{\delta} - 1}{\delta} \| g \|_{BS^1} \sum_{k \geq n+1} e^{-\delta k} \to 0 \text{ as } n \to \infty \text{ uniformly in } t \in \mathbb{R}.
\]

To conclude, it suffices to prove that, for all \( k \geq 1 \), \( u_k \) belongs to \( AA(\mathbb{R}, X) \). Let \((s'_n)\) be a sequence of real numbers, as \( g \in AAS^p(\mathbb{R}, X) \) and \( \Gamma \) is bi-almost automorphic, then there exist a subsequence \((s_n) \subset (s'_n)\) and measurable functions \( \tilde{g} \) and \( \tilde{\Gamma} \) such that for all \( t, s \in \mathbb{R} \), we have

\[
\lim_{n \to \infty} \left( \int_{t}^{t+1} \| g(s + s_n) - \tilde{g}(s) \|^p \, ds \right)^{\frac{1}{p}} = 0
\]

\[
\lim_{n \to \infty} \left( \int_{t}^{t+1} \| \tilde{g}(s - s_n) - g(s) \|^p \, ds \right)^{\frac{1}{p}} = 0,
\]

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and for each $x \in X$, we have

$$\lim_{n \to \infty} \|\Gamma(t + s_n, s + s_n)x - \hat{\Gamma}(t, s)x\| = 0; \quad \lim_{n \to \infty} \|\hat{\Gamma}(t - s_n, s - s_n)x - \Gamma(t, s)x\| = 0.$$ 

Let $u_k(t) = \Phi_k(t) - \Psi_k(t)$, where $\Phi_k(t) = \int_{t-k}^{t-k+1} \Gamma(t, s)g(s)ds$ and $\Phi_k(t) = \int_{t+k-1}^{t+k} \Gamma(t, s)g(s)ds$. Thus, we define the measurable function by

$$\tilde{u}_k(t) = \int_{t-k}^{t-k+1} \hat{\Gamma}(t, s)\tilde{g}(s)ds - \int_{t+k-1}^{t+k} \hat{\Gamma}(t, s)\tilde{g}(s)ds$$

$$= \tilde{\Phi}_k(t) - \tilde{\Psi}_k(t),$$

where

$$\tilde{\Phi}_k(t) := \int_{t-k}^{t-k+1} \hat{\Gamma}(t, s)\tilde{g}(s)ds \quad \text{and} \quad \tilde{\Psi}_k(t) := \int_{t+k-1}^{t+k} \hat{\Gamma}(t, s)\tilde{g}(s)ds, \quad t \in \mathbb{R}.$$ 

Therefore, for $p > 1$, we have

$$\|\Phi_k(t+s_n) - \tilde{\Phi}_k(t)\|$$

$$\leq \left\|\int_{t+s_n-k}^{t+s_n-k+1} \Gamma(t+s_n, s)g(s)ds - \int_{t-k}^{t-k+1} \hat{\Gamma}(t, s)\tilde{g}(s)ds\right\|$$

$$\leq \left\|\int_{k-1}^{k} \Gamma(t+s_n, t+s_n-s)g(t+s_n-s)ds - \int_{k-1}^{k} \hat{\Gamma}(t, t-s)\tilde{g}(t-s)ds\right\|$$

$$\leq \int_{k-1}^{k} \|\Gamma(t+s_n, t+s_n-s)g(t+s_n-s) - \hat{\Gamma}(t, t-s)\tilde{g}(t-s)\|ds$$

$$+ \int_{k-1}^{k} \|\Gamma(t+s_n, t+s_n-s)\tilde{g}(t-s) - \hat{\Gamma}(t, t-s)\tilde{g}(t-s)\|ds$$

$$\leq \int_{k-1}^{k} \|\Gamma(t+s_n, t+s_n-s)[g(t+s_n-s) - \tilde{g}(t-s)]\|ds$$

$$+ \int_{k-1}^{k} \|\Gamma(t+s_n, t+s_n-s)\tilde{g}(t-s) - \hat{\Gamma}(t, t-s)\tilde{g}(t-s)\|ds$$

$$\leq M \left(\int_{k-1}^{k} e^{-q\delta s}ds\right)^{\frac{1}{p}} \left(\int_{k-1}^{k} \|g(t+s_n-s) - \tilde{g}(t-s)\|^pds\right)^{\frac{1}{p}}$$

$$+ \int_{k-1}^{k} \|\Gamma(t+s_n, t+s_n-s)\tilde{g}(t-s) - \hat{\Gamma}(t, t-s)\tilde{g}(t-s)\|ds$$

$$= I_1 + I_2,$$
where

\[ I_1 := M \left( \int_{k-1}^{k} e^{-q\delta s} ds \right)^{\frac{1}{p}} \left( \int_{k-1}^{k} \|g(t + s_n - s) - \tilde{g}(t - s)\|^p ds \right)^{\frac{1}{p}} \]

and

\[ I_2 := \int_{k-1}^{k} \|\Gamma(t + s_n, t + s_n - s)\tilde{g}(t - s) - \tilde{\Gamma}(t, t - s)\tilde{g}(t - s)\| ds. \]

As \( g \in AAS^p(\mathbb{R}, X) \), \( I_1 \to 0 \), as \( n \to \infty \) for all \( t \in \mathbb{R} \). Moreover, since

\[
\|\Gamma(t + s_n, t + s_n - s)\tilde{g}(t - s) - \tilde{\Gamma}(t, t - s)\tilde{g}(t - s)\| \\
\leq M e^{-\delta s} \|\tilde{g}(t - s)\| + \|\tilde{\Gamma}(t, t - s)\tilde{g}(t - s)\|,
\]

and by the fact that \( \tilde{g} \) and \( \tilde{\Gamma}(t, \cdot) \) belong to \( L^p_{\text{loc}}(\mathbb{R}, X) \) (this is by definition), using the limit in (H3), it follows in view of the dominated convergence Theorem, that \( I_2 \to 0 \) as \( n \to \infty \) for all \( t \in \mathbb{R} \). Hence,

\[
\lim_{n \to \infty} \|\Phi_k(t + s_n) - \tilde{\Phi}_k(t)\| = 0 \quad \text{for all} \quad t \in \mathbb{R}.
\]

We can show in a similar way that

\[
\lim_{n \to \infty} \|\tilde{\Phi}_k(t - s_n) - \Phi_k(t)\| = 0 \quad \text{for all} \quad t \in \mathbb{R}.
\]

Moreover, using the same argument, for \( p = 1 \), we obtain that

\[
\|\Phi_k(t + s_n) - \tilde{\Phi}_k(t)\| \leq \left| \int_{t + s_n - k}^{t + s_n - k + 1} \Gamma(t + s_n, s)g(s)ds - \int_{t - k}^{t + s_n - k} \tilde{\Gamma}(t, s)\tilde{g}(s)ds \right|
\]

\[
\leq M \int_{k-1}^{k} \|g(t + s_n - s) - \tilde{g}(t - s)\| ds + \int_{k-1}^{k} \|\Gamma(t + s_n, t + s_n - s)\tilde{g}(t - s) - \tilde{\Gamma}(t, t - s)\tilde{g}(t - s)\| ds
\]

\[
= J_1 + I_2,
\]

where

\[
J_1 := M \int_{k-1}^{k} \|g(t + s_n - s) - \tilde{g}(t - s)\| ds.
\]

Then, the result follows from the fact that \( g \in AAS^1(\mathbb{R}, X) \). This proves that \( \Phi_k \in AA(\mathbb{R}, X) \) for each \( k \geq 1 \). In the same way, we prove the result for \( \Psi_k \). We recall that the series \( \sum_{k \geq 1} u_k(t) \) is uniformly convergent on \( \mathbb{R} \), which implies that \( u \in AA(\mathbb{R}, X) \).

\hfill \Box
Theorem 3.5 Let \( \mu \in \mathcal{M} \) satisfy (M). Assume that (H1)–(H3) are satisfied and that \( g \in PAAS^p(\mathbb{R}, X, \mu) \). Then Eq. (3.2) has a unique mild solution \( u \in PA\mu A(\mathbb{R}, X, \mu) \), given by:

\[
u(t) = \int_{\mathbb{R}} \Gamma(t, s) g(s) ds, \quad t \in \mathbb{R}.
\]

Proof Let \( g = \tilde{g} + \varphi \in PAAS^p(\mathbb{R}, X, \mu) \), where \( \tilde{g} \in AAS^p(\mathbb{R}, X) \) and \( \varphi \in \mathcal{E}^p(\mathbb{R}, X, \mu) \). Then, \( u \) has a unique decomposition:

\[
u = u_1 + u_2,
\]

where, for all \( t \in \mathbb{R} \), we have

\[
u_1(t) = \int_{\mathbb{R}} \Gamma(t, s) g(s) ds
\]

and

\[
u_2(t) = \int_{\mathbb{R}} (t, s) \varphi(s) ds
\]

where

\[
u_2^a(t) := \int_{-\infty}^t U(t, s) P(s) \varphi(s) ds \quad \text{and} \quad \nu_2^x(t) := -\int_t^\infty \tilde{U}(t, s) Q(s) \varphi(s) ds.
\]

Using Theorem 3.4, we obtain that \( u_1 \in AA(\mathbb{R}, X) \). Let us prove that \( u_2 \in \mathcal{E}(\mathbb{R}, X, \mu) \). It suffices to show that \( u_2^a, u_2^x \in \mathcal{E}(\mathbb{R}, X, \mu) \). Let \( r > 0 \) and \( p > 1 \), then

\[
\frac{1}{\mu([-r, r])} \int_{-r}^r \|u_2^a(t)\| d\mu(t)
\]

\[
\leq \frac{1}{\mu([-r, r])} \int_{-r}^r \int_{-\infty}^t \|U(t, s) P(s) \varphi(s)\| ds d\mu(t)
\]

\[
\leq \frac{M}{\mu([-r, r])} \int_{-r}^r \int_{-\infty}^t e^{-\delta(t-s)} \|\varphi(s)\| ds d\mu(t)
\]

\[
\leq \frac{M}{\mu([-r, r])} \int_{-r}^r \left( \int_{-\infty}^t e^{-\frac{\delta}{2}q(t-s)} ds \right)^{\frac{1}{q}} \left( \int_{-\infty}^t e^{-\frac{\delta}{2} p(t-s)} \|\varphi(s)\|^p ds \right)^{\frac{1}{p}} d\mu(t)
\]

\[
\leq \frac{M}{\mu([-r, r])} \left( \frac{2}{q \delta} \right)^{\frac{1}{2}} \int_{-r}^r \left( \sum_{k \geq 1} \int_{t-k}^{t+1} e^{-\frac{\delta}{2} p(t-s+k)} \|\varphi(s-k)\|^p ds \right)^{\frac{1}{p}} d\mu(t)
\]

\[
\leq \left( \frac{M}{\mu([-r, r])} \right)^{\frac{1}{q} + \frac{1}{p}} \left( \frac{2}{q \delta} \right)^{\frac{1}{2}} \int_{-r}^r \left( \sum_{k \geq 1} \int_{t-k}^{t+1} e^{-\frac{\delta}{2} p(t-s+k)} \|\varphi(s-k)\|^p ds \right)^{\frac{1}{p}} d\mu(t)
\]
\[ \leq \frac{M}{\mu([-r, r])} \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \left( \sum_{k \geq 1} e^{-\frac{\delta}{q} pk} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{t}^{t+1} \| \varphi(s-k) \|^{p} ds d\mu(t) \right)^{\frac{1}{p}}. \]

As \( \mathcal{E}^{p}(\mathbb{R}, X, \mu) \) is invariant by translation and by \( \varphi \in \mathcal{E}^{p}(\mathbb{R}, X, \mu) \), we have

\[ \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{t}^{t+1} \| \varphi(s-k) \|^{p} ds d\mu(t) = 0 \quad \text{for all } k \geq 1. \]

Since

\[ \left( \sum_{k \geq 1} e^{-\frac{\delta}{q} pk} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{t}^{t+1} \| \varphi(s-k) \|^{p} ds d\mu(t) \right)^{\frac{1}{p}} \leq \sum_{k \geq 1} e^{-\frac{\delta}{q} k} \| \varphi \|_{B^{S^{p}}} , \]

where the series \( \sum_{k \geq 1} e^{-\frac{\delta}{q} k} \) is convergent, it follows by the dominated convergence Theorem that

\[ \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \| u_{2}^{2}(t) \| d\mu(t) = 0. \quad (3.5) \]

Similarly, for \( p = 1 \), we obtain that

\[ \frac{1}{\mu([-r, r])} \int_{-r}^{r} \| u_{2}^{2}(t) \| d\mu(t) \]
\[ \leq \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{-\infty}^{t} \| U(t, s) P(s) \varphi(s) \| ds d\mu(t) \]
\[ \leq \frac{M}{\mu([-r, r])} \int_{-r}^{r} \int_{-\infty}^{t} e^{-\delta(t-s)} \| \varphi(s) \| ds d\mu(t) \]
\[ \leq M \sum_{k \geq 1} e^{-\delta k} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \int_{t}^{t+1} \| \varphi(s-k) \| ds d\mu(t) \to 0 \quad \text{as } r \to \infty. \]

Arguing as above, we show that

\[ \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \| u_{2}^{2}(t) \| d\mu(t) = 0. \quad (3.6) \]

From (3.5) and (3.6), we have

\[ \lim_{r \to \infty} \frac{1}{\mu([-r, r])} \int_{-r}^{r} \| u_{2}(t) \| d\mu(t) = 0. \]

Hence, \( u \in \mathcal{E}(\mathbb{R}, X, \mu) \). \[ \square \]
Now, we turn our to the semilinear equation (1.1). We need the following additional assumption on $f$:

(H4) There exists a nonnegative function $L_f(\cdot) \in BS^p(\mathbb{R}, \mathbb{R})$, for $p \geq 1$, such that

$$
\|f(t, x) - f(t, y)\| \leq L_f(t) \|x - y\| \quad \text{for } t \in \mathbb{R} \text{ and } x, y \in X.
$$

**Theorem 3.6** Let $p \geq 1$ and $\mu \in \mathcal{M}$ satisfy (M). Assume that (H1)–(H4) hold and $f \in PAAS^pU(\mathbb{R} \times X, X, \mu)$ with

$$
\|L_f\|_{BS^p} < \min \left\{ \left( 2M \left( \frac{2}{q\delta} \right)^{\frac{1}{q}} \left( \frac{1}{e^{\delta} - 1} \right)^{\frac{1}{p}} \right) , \left( \frac{2M}{e^{\delta} - 1} \right)^{\frac{1}{p}} \right\}^{-1}.
$$

Then, Eq. (1.1) has a unique mild solution $u \in PAA(\mathbb{R}, X, \mu)$ given by:

$$
u(t) = \int_{\mathbb{R}} \Gamma(t, s) f(s, u(s)) ds, \quad t \in \mathbb{R}.
$$

**Proof** Consider the mapping $F : PAA(\mathbb{R}, X, \mu) \to PAA(\mathbb{R} \times X, \mu)$ defined by

$$(F u)(t) = \int_{-\infty}^{t} U(t, s) P(s) f(s, u(s)) ds - \int_{t}^{\infty} \tilde{U}(t, s) Q(s) f(s, u(s)) ds = (Fu^a)(t) + (Fu^z)(t) \quad \text{for all } t \in \mathbb{R},
$$

where

$$(Fu^a)(t) = \int_{-\infty}^{t} U(t, s) P(s) f(s, u(s)) ds
$$

and

$$(Fu^z)(t) = -\int_{t}^{\infty} \tilde{U}(t, s) Q(s) f(s, u(s)) ds, \quad t \in \mathbb{R}.
$$

By the composition Theorem 2.18, we have that $F(PAA(\mathbb{R}, X, \mu)) \subset PAA(\mathbb{R}, X, \mu)$. Moreover, for $p > 1$, we have

$$
\|(Fu^a)(t) - (Fv^a)(t)\| \leq \int_{-\infty}^{t} \|U(t, s) P(s) f(s, u(s)) - U(t, s) P(s) f(s, v(s))\| ds
$$

$$
\leq M \int_{-\infty}^{t} e^{-\delta(t-s)} \|f(s, u(s)) - f(s, v(s))\| ds
$$

$$
\leq M \left( \int_{-\infty}^{t} e^{-\frac{1}{q}(t-s)} ds \right)^{\frac{1}{q}} \left( \int_{-\infty}^{t} e^{-\frac{1}{p}(t-s)} \|f(s, u(s)) - f(s, v(s))\|^{p} ds \right)^{\frac{1}{p}}
$$

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\[ \begin{align*}
\leq M \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \left( \sum_{k \geq 1} \int_{t-k}^{t-k+1} e^{-\frac{\delta}{2} p(t-s)} L_f^p(s) \parallel u(s) - v(s) \parallel^p ds \right)^{\frac{1}{p}} \\
\leq M \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \left( \sum_{k \geq 1} \int_{t-k}^{t-k+1} e^{-\frac{\delta}{2} p(t-s)} L_f^p ds \right)^{\frac{1}{p}} \parallel u - v \parallel \infty \\
\leq M \parallel L_f \parallel_{BS^p} \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \left( \frac{1}{e^{\frac{\delta}{2}} - 1} \right)^{\frac{1}{p}} \parallel u - v \parallel \infty.
\end{align*} \]

Arguing as above, we have also

\[ \begin{align*}
\parallel (Fu^v)(t) - (Fv^v)(t) \parallel \\
\leq \int_{-\infty}^{t} \parallel \tilde{U}(t, s) Q(s) f(s, u(s)) - \tilde{U}(t, s) Q(s) f(s, v(s)) \parallel ds \\
\leq M \int_{-\infty}^{t} e^{-\delta(s-t)} \parallel f(s, u(s)) - f(s, v(s)) \parallel ds \\
\leq M \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \left( \sum_{k \geq 1} \int_{t-k}^{t-k+1} e^{-\frac{\delta}{2} p(s-t)} L_f^p(s) ds \right)^{\frac{1}{p}} \parallel u - v \parallel \infty \\
\leq M \parallel L_f \parallel_{BS^p} \left( \frac{2}{q \delta} \right)^{\frac{1}{q}} \left( \frac{1}{e^{\frac{\delta}{2}} - 1} \right)^{\frac{1}{p}} \parallel u - v \parallel \infty.
\end{align*} \]

Now, for \( p = 1 \), we obtain that

\[ \begin{align*}
\parallel (Fu^u)(t) - (Fv^u)(t) \parallel \\
\leq \int_{-\infty}^{t} \parallel U(t, s) P(s) f(s, u(s)) - U(t, s) P(s) f(s, v(s)) \parallel ds \\
\leq M \int_{-\infty}^{t} e^{-\delta(t-s)} \parallel f(s, u(s)) - f(s, v(s)) \parallel ds \\
\leq M \sum_{k \geq 1} e^{-\delta k} \int_{t-k}^{t-k+1} L_f(s) ds \parallel u - v \parallel \infty \\
\leq M \parallel L_f \parallel_{BS^1} \left( \frac{1}{e^{\delta} - 1} \right) \parallel u - v \parallel \infty
\end{align*} \]

and that

\[ \begin{align*}
\parallel (Fu^\tilde{v})(t) - (Fv^\tilde{v})(t) \parallel \\
\leq \int_{-\infty}^{t} \parallel \tilde{U}(t, s) Q(s) f(s, u(s)) - \tilde{U}(t, s) Q(s) f(s, v(s)) \parallel ds
\end{align*} \]
\[
\leq M \| L_f \|_{BS^1} \left( \frac{1}{e^{\delta} - 1} \right) \| u - v \|_\infty.
\]

Consequently, we have
\[
\| (F u)(t) - (F v)(t) \| \leq \| L_f \|_{BS^p} \min \left\{ \left( 2M \left( \frac{2}{q \delta} \right)^\frac{1}{q} \left( \frac{1}{e^{\delta} - 1} \right)^\frac{1}{p} \right), \left( \frac{2M}{e^{\delta} - 1} \right)^{\frac{1}{p}} \right\} \| u - v \|_\infty.
\]

Therefore, by the Banach fixed point Theorem, \( F \) has a unique fixed point \( u \in PAA(\mathbb{R}, X, \mu) \) such that \( Fu = u \). This proves the result. \( \square \)

### 4 Application

Let \( \mu \) be a positive measure with a Radon–Nikodym derivative \( \rho \) defined by:
\[
\rho^t = \begin{cases} 
e^t & \text{if } t \leq 0 \\ 1 & \text{if } t > 0. \end{cases}
\]

From [7, Example 3.6], \( \mu \) satisfies the hypothesis \((M)\).

Consider the following reaction–diffusion model, with time-dependent diffusion coefficient and forcing terms, namely:
\[
\frac{\partial v(t, x)}{\partial t} = \delta(t) \frac{\partial^2 v(t, x)}{\partial x^2} + \alpha(t) v(t, x) + f(t, v(t, x)), \quad t \in \mathbb{R}, \ x \in \mathbb{R},
\]

where \( \delta, \alpha : \mathbb{R} \rightarrow \mathbb{R} \) are \( S^1 \)-almost automorphic functions such that there exists \( \delta_0, \alpha_0 > 0 \) satisfying \( \delta(t) > \delta_0 \) and \( \alpha(t) \leq -\alpha_0 < 0 \) for a.e. \( t \in \mathbb{R} \).

Take \( X = L^2(\mathbb{R}) \) i.e., the Lebesgue space equipped with its usual norm denoted by \( \| \cdot \| \), and define the operator
\[
\begin{cases} 
A \phi := \frac{\partial^2 \phi}{\partial x^2} \\
D(A) := H^2(\mathbb{R}) \text{ (i.e., its maximal domain),}
\end{cases}
\]

where \( H^2(\mathbb{R}) \) is the usual Sobolev space of order 2. It is well known that \((A, D(A))\) generates a bounded strongly continuous semigroup \((T(t))_{t \geq 0}\) on \( X \), i.e., \( \| T(t) \| \leq M \).

Now, clearly, the operators
\[
A(t) := \delta(t) A + \alpha(t) \quad \text{with} \quad D(A(t)) = D(A), \ t \in \mathbb{R}
\]

generate the strongly continuous evolution family given by:
\[
U(t, s) = e^\int_s^t \alpha(\tau) d\tau T \left( \int_s^t \delta(\tau) d\tau \right), \quad t \geq s.
\]

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Notice that the formula \( T \left( \int_s^t \delta(\tau) d\tau \right), \ t \geq s \) corresponds to the mild solution operator for the Cauchy problem:

\[
\begin{cases}
u'(t) = A(t)u(t), & t \geq s, \\
u(s) = u_s \in X,
\end{cases}
\]

i.e., \( u(t) = T \left( \int_s^t \delta(\tau) d\tau \right) u_s \) is the corresponding mild solution. This is an immediate consequence of the application of Fourier transform and the Gaussian semigroup explicit formula; see [13]. Hence, from (4.3) and the boundedness of the semigroup \( (T(t))_{t \geq 0} \) on \( X \), we have

\[
\|U(t, s)\phi\| \leq e^{\int_s^t \alpha(\tau) d\tau} \| T \left( \int_s^t \delta(\tau) d\tau \right) \phi\| \\
\leq M e^{-\alpha_0(t-s)} \| \phi\|, \quad t \geq s, \ \phi \in X,
\]

we recall that \( \alpha(t) \leq -\alpha_0 < 0 \) for a.e. \( t \in \mathbb{R} \). That is, the exponential stability of \( (U(t, s))_{t \geq s} \) holds, and so hypotheses (H1) and (H2) are both satisfied. Moreover, the Green’s function associated with our model is given by

\[
\Gamma(t, s) := U(t, s), \quad t \geq s.
\]

To show hypothesis (H3), it suffices to prove that \( (U(t, s))_{t \geq s} \) given by (4.3) is bi-almost automorphic. That is, we have the following result.

**Proposition 4.1** For each \( \phi \in X \), the function \( \Gamma(\cdot, \cdot)\phi \) is bi-almost automorphic.

**Proof** Let \( \delta \in AA(\mathbb{R}, \mathbb{R}) \) and \( \alpha \in AAS^1(\mathbb{R}, \mathbb{R}) \). Then, for every sequence \( (s_k')_{k \geq 0} \) of real numbers, there exist a subsequence \( (s_k)_{k \geq 0} \subset (s_k')_{k \geq 0} \) and measurable functions \( \tilde{\delta} \) and \( \tilde{\alpha} \) such that

\[
\lim_k |\delta(t + s_k) - \tilde{\delta}(t)| = 0 \quad \text{and} \quad \lim_k |\tilde{\delta}(t - s_k) - \delta(t)| = 0,
\]

and

\[
\lim_k \int_t^{t+1} |\alpha(\tau + s_k) - \tilde{\alpha}(\tau)| d\tau = 0 \quad \text{and} \quad \lim_k \int_t^{t+1} |\tilde{\alpha}(\tau - s_k) - \alpha(\tau)| d\tau = 0,
\]

for all \( t \in \mathbb{R} \). Let \( \phi \in X \) and define

\[
\tilde{U}(t, s)\phi = e^{\int_t^s \tilde{\alpha}(\tau) d\tau} T \left( \int_s^t \tilde{\delta}(\tau) d\tau \right) \phi, \quad \text{for all } t \geq s.
\]

Thus, we have

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\[\|U(t + s_k, s + s_k)\phi - \bar{U}(t, s)\phi\| = \left\| e^{t + s_k \alpha} e^{s_k \alpha} T \left( \int_t^{t + s_k} \delta(\tau) \, d\tau \right) \phi - e^{s_k \alpha} e^{t + s_k \alpha} T \left( \int_t^{t + s_k} \delta(\tau) \, d\tau \right) \phi \right\| \]

\[\leq M e^{s_k \alpha} \left| \sum_{i=0}^{t + s_k - t} \int_t^{t + s_k} \alpha(\tau) \, d\tau \right| \phi \rightarrow 0 \quad k \rightarrow \infty.\]

Moreover, since \(\delta \in AA(\mathbb{R}, X)\), it follows by the strong continuity of the semigroup \((T(t))_{t \geq 0}\) that

\[e^{s_k \alpha} e^{t + s_k \alpha} T \left( \int_t^{t + s_k} \delta(\tau) \, d\tau \right) \phi - T \left( \int_t^{t + s_k} \delta(\tau) \, d\tau \right) \phi \rightarrow 0 \quad k \rightarrow \infty,\]

for all \(t, s \in \mathbb{R}\), \(t \geq s\). Then, by (4.4), we obtain that

\[\|U(t + s_k, s + s_k)\phi - \bar{U}(t, s)\phi\| \rightarrow 0 \quad k \rightarrow \infty,\]

for all \(t, s \in \mathbb{R}\), \(t \geq s\). Similarly, we prove that

\[\|\bar{U}(t - s_k, s - s_k)\phi - U(t, s)\phi\| \rightarrow 0 \quad k \rightarrow \infty.\]

Consequently, \(\Gamma(\cdot, \cdot)\phi\) is bi-almost automorphic. \(\square\)

Let \(f : \mathbb{R} \times X \rightarrow X\) be \(\mu\)-pseudo almost automorphic in the sense of Stepanov (of order \(p = 1\)), defined by

\[f(t, \phi)(x) = \left[ a(t) + \left( \arctan(t) - \frac{\pi}{2} \right) \right] h(\phi)(x) \]

\[:= a_0(t), \quad x \in \Omega,\]

where \(h\) is \(L_h\)-Lipschitzian in \(X\), \(a \in AAS^1(\mathbb{R}, \mathbb{R})\) and by the proof in [8, Example 5.5], \(t \mapsto \arctan(t) - \frac{\pi}{2}\) belongs to \(E(\mathbb{R}, \mathbb{R}, \mu)\). Then \(f\) belongs to
$PAAS^1 U(\mathbb{R}, \mathbb{R}, \mu)$ satisfying the Lipschitz continuity condition:

$$\|f(t, \phi) - f(t, \psi)\| \leq L_f(t)\|\phi - \psi\|, \quad \phi, \psi \in X,$$

where

$$L_f(\cdot) = L_h|d_0(\cdot)| \in BS^1(\mathbb{R}, \mathbb{R}^+).$$

Hence, hypothesis (H4) holds. A necessary and sufficient condition to the $L_h$-Lipschitz continuity of $h$ in $X$ is the $L_h$-Lipschitz continuity of the associated scalar function $\xi \mapsto -h(\xi)$.

To establish the existence and uniqueness of $\mu$-pseudo almost automorphic solutions to our model, given by Eq. (4.2), we give its associated abstract form corresponding to Eq. (1.1), namely:

$$u'(t) = A(t)u(t) + f(t, u(t)), \quad t \in \mathbb{R}. \quad (4.5)$$

Consequently, all hypotheses and assumptions of Theorem 3.6 are satisfied. Therefore, we have the following main result.

**Theorem 4.2** Assume $\|L_f\|_{BS^1}$ is small enough. Then, model (4.2) has a unique $\mu$-pseudo almost automorphic (with respect to time $t$) mild solution.

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