Factorization of differential expansion for non-rectangular representations

A. Morozov

ITEP, Moscow 117218, Russia
Institute for Information Transmission Problems, Moscow 127994, Russia
National Research Nuclear University MEPhI, Moscow 115409, Russia

ABSTRACT

Factorization of the differential expansion (DE) coefficients for colored HOMFLY-PT polynomials of antiparallel double braids, discovered in [1] in the case of rectangular representations $R$, is extended to the first non-rectangular representations $R = [2, 1]$ and $R = [3, 1]$. This increases chances that such factorization will take place for generic $R$, thus fixing the shape of the DE. We illustrate the power of the method by conjecturing the DE-induced expression for double-braid polynomials for all $R = [r, 1]$. In variance with rectangular case, the knowledge for double braids is not fully sufficient to deduce the exclusive Racah matrix $\bar{S}$ – the entries in the sectors with non-trivial multiplicities sum up and remain unseparated. Still a considerable piece of the matrix is extracted directly and its other elements can be found by solving the unitarity constraints.

1 Introduction

Wilson loop averages in 3d Chern-Simons theory [2]

$$H^K_R = \langle Tr_R P \exp \left( \oint A \right) \rangle$$

are exactly calculable and provide an important set of examples for non-perturbative quantum field theory (QFT). At this moment the calculations can be most effectively performed with the help of the modified Reshetikhin-Turaev method [3]-[12], where the answers are certain combinations of various Racah matrices $(6j, 9j, \ldots)$-symbols). The problem is, however, that these matrices are not known for most representations $R$, and their direct evaluation is far beyond modern computer capacities. Also, most valuable is not the answer for particular representation $R$, but its analytic dependence on the representation – and here even the absolute success of calculation for particular $R$ (which we are very far from) would not be sufficient. Still, there is a considerable progress in the field during the last decade, and it reveals the interesting properties of the quantities $H^K_R$, which in no way follow from their definition (1) – and imply the existence of some complementarity (dual) descriptions, which still remain to be found.

The puzzling properties of $H^K_R$ include:

- $H^K_R$ are polynomials in two non-perturbative variables, $q = \exp \left( \frac{2\pi i}{g^2 + N} \right)$ and $A = q^N$, which are made from the coupling constant and parameter of the gauge group $SL(N)$. This is a long known fact, coming from identification [2] of $H^K_R$ with the HOMFLY-PT polynomials [13], but its exact meaning in QFT remains obscure.
- Coefficients of the polynomials are integer, what is explained in alternative Khovanov-Rozansky approach [14], however, its QFT interpretation is still unavailable. Moreover, this approach is not yet developed for non-trivial representations $R$, and it does not explain the more delicate integrality properties [15, 16] of Ooguri-Vafa sums $\sum_R H_R$ and their loop expansions.
- Knots can be glued from simpler components, and this provides a new description of knot polynomials – in terms of some effective gauge invariant field theory. This line of reasoning is so far developed [10]-[12] for a family of arborescent knots [17], which are distinguished because in this case the problem is reduced to just two types of "exclusive" Racah matrices $S$ and $\bar{S}$ (which, however, depend on representation $R$ and are still very non-trivial to calculate).
- Vogel’s universality [18] works perfectly well for knot polynomials [19, 20]: while particular dimensions in the $E_8$-sector of representation theory are transcendental in the $u, v, w$-variables, they combine into Vieta-like rational combinations in the expressions for $H^K_R$ for $R$ which are descendants of the adjoint representation. Perhaps, this is not too surprising, because universality was actually inspired by knot-theory considerations – and it is now getting clear that knot theory distinguishes a "healthy" part of representation theory, which
includes the exclusive Racah matrices $S$ and $\tilde{S}$ (they are currently known in the universal form for the adjoint representation itself [20]).

- As functions of $R$, HOMFLY-PT polynomials satisfy non-trivial difference equations [21, 22] – so far fully described only for symmetric representations $R = [r]$ and for particular knots, where the full $r$-dependence is known from [23, 24].
- The HOMFLY-PT polynomials possess a non-trivial structure of differential expansion (DE) [23], [25]-[27] – which probably reflects the duality between Reshetikhin-Turaev and Khovanov-Rozansky approaches, though both are not yet formulated in such a way that this statement can be made explicit.

At this moment DE is deepest structure, found in knot polynomials, and the present paper is devoted to a new progress in its investigation.

DE controls the dependence of knot polynomials on representation $R$ and can be used to characterize (and further – classify) the complexity of knots – already the DE defect of [27] (occasionally equal to minus one plus degree of the fundamental Alexander polynomial) seems to be a much better characteristic than the usual minimal-crossing number. DE is quite a powerful tool – originally it was used in [23, 24] to find the ”exclusive” Racah matrices $S$ and $\tilde{S}$ for all symmetric and antisymmetric representations $R = [r], [1^*]$, and recently the method was extended in [1] to generic rectangular $R$ (though the calculation is not yet completed in generic case).

This power comes from promoting the observation about available hardly-calculated examples for particular $R$ to all $R$ of a certain type – and thus obtaining (conjecturing) the statements far beyond the domain of direct calculability. The main problem with DE at this moment, after its tremendous success with rectangular $R$, is the lasting difficulty for non-rectangular $R$, beginning already at the simplest level of $R = [2, 1]$, see [28, 29] and [12] for a number of previous attempts. With the new knowledge and insight from [30, 31] we now manage to resolve this [2, 1]-problem – and this is what the present paper is about.

Namely, we suggest, that DE for $R = [2, 1]$ involves two new structures, as compared to the case of symmetric $R$, which we denote $F_{[2,1]}$ and $\tilde{F}_{[2,1]}$, and non of them is reduced to the previously known $F_{[3]}$ – contrary to what was assumed so far. Spectacularly, this conjecture appears consistent with another conjecture – about factorization of DE for double braids [1], and this looks absolutely non-trivial and extremely restrictive, leaving practically no doubts in validity of the both, at least for $R = [2, 1]$. We also provide a simple example of the defect-zero knot 9_{46} which is beyond the double-braid family.

Once the case of $R = [2,1]$ is understood, the way is open to other non-rectangular representations, though it is neither fully straightforward nor easy. Still as an illustration that now this is doable, we provide DE for the next representation $R = [3, 1]$.

2 Differential expansion and its factorization for double braids

In this letter we assume the familiarity with the summary [31] of the recent developments about DE for rectangular $R$ and provide only the new details, needed beyond rectangular representations. The short list of abbreviations, used in the theory of knot polynomials include $\{x\} = x - x^{-1}$, the quantum numbers $[n] = \frac{q^n}{q}$ and the ”differentials” $D_n = \{A q^n\}$.

Differential expansion, as we currently understand it, decomposes colored HOMFLY-PT polynomial and separate representation (color) and braid dependencies in the following way:

$$H_R^K = \sum_{\lambda} C_R^K(q) \cdot Z_R^K(A, q) \cdot F^K_{\lambda}(A, q)$$

Combinatorial coefficients $C(q)$ can contain denominators, made from $q$-numbers. The Laurent-polynomial $Z$-factors depend on the defect of the knot $K$, or, more precisely, $F^K_{\lambda}$ contain knot-independent $D$-factors, which can be absorbed into $Z$. In this paper we consider the knots with defect zero. All the $F$-factors also are Laurent polynomials, moreover, for the three distinguished knots

$$F^{\text{unknot}}_{\lambda} = 0, \quad F^{4_1}_{\lambda} = 1, \quad F^{3_1}_{\lambda} = (-)^\alpha q^\beta A^\gamma$$

with some $\lambda$-dependent integers $\alpha, \beta, \gamma$.

For rectangular $R = [r\sigma]$ the sum goes over $\lambda$, which are sub-diagrams of $R$. For non-rectangular $R$ there are additional (”anomalous”) contributions, which can not be associated just with the sub-diagrams $\lambda \subset R$ and which depend also on the eigenvalues, not directly associated with sub-diagrams $\mu \subset \lambda$. These additional contributions, which we denote by tilde, however, come with additional factors of $\{q\}^4$. Our notation is not too
informative, what reflects the lack of a true understanding of the phenomenon – the goal of the present paper is mostly to put it to light and describe, not yet to fully explain.

**Double braid** of type, relevant for our purposes in this paper is shown in the picture:

\[ \begin{align*}
   &2n \\
   &\vdots \\
   &2m \\
\end{align*} \]

**Twist knots** form a particular subset of this two-parametric family with \( n = 2 \) and arbitrary \( m \). For rectangular \( R \) in this case every \( F_\lambda^{(m)} \) is a linear combination of powers of the eigenvalues \( \Lambda_m^2 \), which are labeled by sub-diagrams \( \mu \subset \lambda \). As an amusing side remark, all \( F_\lambda \) seem to be polynomials with positive integer coefficients – this emphasizes their relation to superpolynomials, first suggested in the original [23].

For double braids \( F_{\lambda}^{(m,n)} \) are bilinear combinations of \( \Lambda_m^2 \Lambda_n^2 \). However, **factorization** of HOMFLY-PT polynomials for **double braids**, discovered in [1], reduces them to those for **twist knots** – but only if both are realized by their differential expansions: factorization states that

\[
F_{\lambda}^{(m,n)} \sim F_{\lambda}^{(m)} F_{\lambda}^{(n)}
\]

In this paper we extend this conjecture from rectangular representations \( R \) to the simplest non-rectangular \( R = [2, 1] \) and \( R = [3, 1] \).

### 3 Representation \( R = [1, 1] \)

As a by-now-elementary starting point, we remind the DE formulas from [23] and [1] for the antisymmetric representation \( R = [1, 1] \):

\[
H_{[1,1]}^{(m,n)} = 1 + Z_{[1,1]}^{[1]} \cdot \frac{F_{[1]}^{(m)} F_{[1]}^{(n)}}{F_{[1]}^{(1)} F_{[1]}^{(-1)}} + Z_{[1,1]}^{[1]} \cdot \frac{F_{[1]}^{(m)} F_{[1]}^{(n)}}{F_{[1]}^{(1)} F_{[1]}^{(-1)}}
\]

with the \( Z \)-factors

\[
Z_{[1,1]}^{[1]} = D_1 D_{-3} + D_1 D_{-1} = [2] D_1 D_{-2}, \quad Z_{[1,1]}^{[1]} = D_1 D_2 D_{-2} D_{-3}
\]

Underlined is the differential, which is omitted from the differential expansion for knots with defects greater than zero, see [27]. The two \( F \)-functions can be found in the list (9) below.

### 4 Representation \( R = [2, 1] \)

Contributing to the DE (2) in this case are seven different Young diagrams from \( R \otimes R = [2, 1] \otimes [2, 1] \), of which five are naturally labeled by sub-diagrams \( \lambda \) of \( R = [2, 1] \) itself:
These five "regular" diagrams are shown in the left column. Somewhat miraculously, the two "anomalous" diagrams in the right column have identical dimensions and Casimir eigenvalues – thus it is not a big surprise that they provide a single (rather than two separate) contributions to the differential expansion. Note that double-braid factorization is bilinear and thus very sensitive to the difference between single and two separate contributions. As to the multiplicity, in the case of $R = [2, 1]$ it appears only for $\lambda = [1]$. Surprisingly or not, it does not show up in the differential expansion.

Making use of the results of [32] for the $[2, 1]$ Racah matrices and of [29] for the evolution of the $[2, 1]$-colored twisted knots, by a tedious trial-and-error attempt, we discover the following differential expansion, which nicely fits all the known results for double-braid $[2, 1]$-colored HOMFLY-PT from [33]:

$$
H_{[2,1]}^{(m,n)} = 1 + Z_{[1]}^{[2,1]} \cdot \frac{F^{(m)}_{[1]} F^{(n)}_{[1]}}{F^{(1)}_{[1]} F^{(-1)}_{[1]}} + \left[ \frac{3}{2} \right] Z_{[2]}^{[2,1]} \cdot \frac{F^{(m)}_{[2]} F^{(n)}_{[2]}}{F^{(2)}_{[2]} F^{(-1)}_{[1]}} + \left[ \frac{3}{2} \right] Z_{[1,1]}^{[1,1]} \cdot \frac{F^{(m)}_{[1,1]} F^{(n)}_{[1,1]}}{F^{(1)}_{[1,1]} F^{(-1)}_{[1,1]}} + Z_{[2,1]}^{[2,1]} \cdot \frac{F^{(m)}_{[2,1]} F^{(n)}_{[2,1]}}{F^{(1)}_{[2,1]} F^{(-1)}_{[2,1]}} + \\
+ \tilde{Z}_{[2]}^{[2,1]} \cdot \frac{\tilde{F}^{(m)}_{[2]} \tilde{F}^{(n)}_{[2]}}{\tilde{F}^{(2)}_{[2]} F^{(-1)}_{[2,1]}} \right) \right)\right) (7)
$$

with the knot-independent $Z$-factors

$$
Z_{[1]}^{[2,1]} = D_3 D_{-3} + D_2 D_0 + D_0 D_{-2}, \quad Z_{[2]}^{[2,1]} = D_3 D_2 D_0 D_{-2}, \quad Z_{[1,1]}^{[1,1]} = D_2 D_0 D_{-2} D_{-3} \\
Z_{[2,1]}^{[2,1]} = D_3 D_2 D_1 D_{-1} D_{-2} D_{-3}, \quad \tilde{Z}_{[2,1]}^{[2,1]} = -[3]^2 (q)^4 D_2 D_{-2} \right)\right) (8)
$$

The first three $Z$-factors are known since [23] from DE in symmetric representations, the last two are new. The
knot-dependent $F$-factors for twist family are

\[ F_{[1]}^{(m)} = A \cdot \left( \frac{A_0^{2m}}{D_0} - \frac{A_2^{2m}}{D_0} \right) \]

\[ F_{[2]}^{(m)} = q \cdot A^2 \cdot \left( \frac{A_0^{2m}}{D_1 D_0} - \frac{A_2^{2m}}{D_2 D_0} + \frac{A_4^{2m}}{D_2 D_1} \right) \]

\[ F_{[1,1]}^{(m)} = q^{-1} \cdot A^2 \cdot \left( \frac{A_0^{2m}}{D_0 D_{-1}} - \frac{A_2^{2m}}{D_2 D_{-1}} + \frac{A_4^{2m}}{D_2 D_{-2}} \right) \]

\[ F_{[2,1]}^{(m)} = A^3 \cdot \left( \frac{A_0^{2m}}{D_1 D_0 D_{-1}} - \frac{[4]}{2} \frac{A_2^{2m}}{D_2 D_0 D_{-2}} - \frac{1}{[2]^2 q^2} \right) \cdot \left( \frac{D_1 A_2^{2m}}{D_2 D_1 D_0} - \frac{2 A_2^{2m}}{D_0} + \frac{D_3 A_4^{2m}}{D_3 D_1 D_0 D_{-1}} \right) \]

(9)

with $A_0 = 0$, $A_1 = A^2$, $A_4 = A^4$, $A_2 = q^4 A^4$, $A_{11} = q^{-4} A^4$, $A_{21} = A^6$. Underlined is the eigenvalue, associated with the two "anomalous" diagrams in the second column. Wherever possible we put tildes over dimensions rather than representation labels (subscripts) to make them better visible. Note that $F_{[2,1]}^{(m)}$ does not depend on $A_{21}$.

5 Comments and checks

The formula (7) differs from most previous suggestions about the differential expansion for $H_{[2,1]}$, but this time it works nicely for all available twist and double-braid answers. A few additional checks/comments are now in order.

5.1. It is easy to see, that the Alexander polynomial $A^K(q) = H^K(A = 1, q)$ is strongly affected by the last term with $\tilde{Z}_{[2,1]}$ and $\tilde{F}_{[2,1]}$ in (7), and this provides an additional non-trivial check of our formulas – because is can be performed for arbitrarily large values of $m$ and $n$, where alternative answers for the $[21]$-colored polynomials are not available. The thing to check is that the Alexander polynomial an arbitrary single-hook representation $\tilde{R}$ satisfies [34]

\[ A_{\tilde{R}}(q) = A_{\tilde{R}[1]}(q^{[R]}) \quad \text{for single – hook } R = [p, 1^q] \quad (10) \]

5.2. As explained in [1], the differential expansion for double braids contains important information about the Racah matrix $\tilde{S}$, which allows to fully extract it for arbitrary rectangular representation (the practical obstacle there is incomplete knowledge of the $F_\lambda$-functions for Young diagrams $\lambda$ with more than two columns). However, for non-rectangular diagrams such extraction is not fully possible: contributions from representations with multiplicities are summed and additional effort is needed to separate them. In the case of $\tilde{R} = [21]$ this looks as follows. Just as in [1], one can easily deduce all the elements of the matrix $\tilde{S}_{ab}$ from

\[ H_{[2,1]}^{(m,n)} = \sum_{a,b=0}^5 \sqrt{d_a d_b} d_{[2,1]} \cdot \tilde{S}_{ab} \cdot A_a^{2m} A_b^{2n} \quad (11) \]

where indices $a, b$ run over the set $\{0, 1, 2, 2, 11, 21\}$ and the corresponding dimensions are:

\[ d_0 = 1, \quad d_1 = \frac{D_1 D_{-1}}{[q]^2}, \quad d_2 = \frac{D_2 D_1 D_{-1} D_{-2}}{[2]^2 [q]^4}, \quad d_3 = \frac{D_2 D_1 D_{-1} D_{-2}}{[2]^2 [q]^4}, \quad d_{11} = \frac{D_1 D_2 D_{-1} D_{-3}}{[2]^2 [q]^4}, \quad d_{21} = \frac{D_3 D_2 D_{-1} D_{-3}}{[3]^2 [q]^6} \]

\[ d_3 = \frac{D_3 D_2 D_1 D_{-1}}{[3]^2 [2]^2 [q]^6}, \quad d_3 = \frac{D_3 D_2 D_1 D_{-1} D_{-2}}{[3]^2 [2]^2 [q]^6}, \quad d_{31} = \frac{D_3 D_2 D_1 D_{-1} D_{-3}}{[4]^2 [2]^2 [q]^8} \quad (12) \]

(dimensions in the second line will matter in sec.6 below). Multiplicities matter when $a$ or $b$ equals 1. Also the "extra" diagrams with the eigenvalue $\tilde{A}_2$ appear twice. In the standard notation the labeling is different:

\[
\begin{array}{ccccccccc}
\text{present paper} & 0 & 1 & 2p & 2 & 2m & 21 \\
\text{previous} [32, 10, 12] & 1 & 7 & 10 & 6 & 2 \& 3 & 4 & 5
\end{array}
\]

(13)
Accordingly we have the following expressions for the matrix elements $\tilde{S}_{ij} = \tilde{S}_{0a}$ through those of the $10 \times 10$ symmetric unitary Racah matrix $\tilde{S}_{ij}$, which was first calculated in [32] and then re-deduced by the two different evolution-based methods in [12] and [1]:

\[
\begin{align*}
S_{0,0} &= \tilde{S}_{11} & S_{0,1} &= \sum_{j=7}^{10} \tilde{S}_{ij} & S_{0,2} &= \tilde{S}_{16} & S_{0,3} &= \tilde{S}_{12} + \tilde{S}_{13} & S_{0,4} &= \tilde{S}_{14} & S_{0,21} &= \tilde{S}_{15} \\
S_{1,0} &= \sum_{i,j=7}^{10} \tilde{S}_{ij} & S_{1,1} &= \sum_{i,j=7}^{10} \tilde{S}_{ij} & S_{1,2} &= \sum_{i,j=7}^{10} \tilde{S}_{ij} + \tilde{S}_{13} & S_{1,3} &= \sum_{i,j=7}^{10} \tilde{S}_{ij} + \tilde{S}_{14} & S_{1,21} &= \sum_{i,j=7}^{10} \tilde{S}_{ij} + \tilde{S}_{15} \\
S_{2,0} &= \tilde{S}_{63} & S_{2,1} &= \sum_{j=7}^{10} \tilde{S}_{6j} & S_{2,2} &= \tilde{S}_{66} & S_{2,3} &= \tilde{S}_{62} + \tilde{S}_{63} & S_{2,21} &= \sum_{i,j=7}^{10} \tilde{S}_{ij} + \tilde{S}_{15} \\
S_{2,2} &= \tilde{S}_{21} + \tilde{S}_{23} & S_{2,3} &= \sum_{i,j=7}^{10} (\tilde{S}_{2j} + \tilde{S}_{3j}) & S_{2,23} &= \tilde{S}_{22} + \tilde{S}_{23} + \tilde{S}_{32} + \tilde{S}_{33} & S_{2,21} &= \tilde{S}_{24} + \tilde{S}_{25} + \tilde{S}_{35} \\
S_{11,0} &= \tilde{S}_{41} & S_{11,1} &= \sum_{i,j=7}^{10} \tilde{S}_{4j} & S_{11,2} &= \tilde{S}_{46} & S_{11,3} &= \tilde{S}_{42} + \tilde{S}_{43} & S_{11,21} &= \tilde{S}_{44} \\
S_{21,0} &= \tilde{S}_{51} & S_{21,1} &= \sum_{j=7}^{10} \tilde{S}_{5j} & S_{21,2} &= \tilde{S}_{56} & S_{21,3} &= \tilde{S}_{52} + \tilde{S}_{53} & S_{21,21} &= \tilde{S}_{54} \\
\end{align*}
\]

Note that the $6 \times 6$ matrix $\tilde{S}$ in (11) is symmetric, but not unitary. The lacking elements of the $10 \times 10$ unitary $\tilde{S}$ can be restored by solving the unitarity constraints, what once again reproduces the result of [32].

5.3. For other defect-zero [27] knots $\mathcal{K}^{(i)}$ (when Alexander polynomial in the fundamental representation is of degree one in $q^{\pm 2}$, i.e. contains only three-terms) we expect the differential expansion with the same Z-factors (8), which depend on representation, but not on the knot, and with different, knot-dependent $F$-factors, i.e. the expectation is that

\[
\mathcal{H}^{(i)}_{[2,1]} = 1 + Z^{[1]}_{[2,1]} \cdot F^{(i)}_{[1]} + \frac{[3]}{[2]} \cdot Z^{[2]}_{[2,1]} \cdot F^{(i)}_{[2]} + \frac{[3]}{[2]} \cdot Z^{[1]}_{[2,1]} \cdot F^{(i)}_{[1,1]} + Z^{[2]}_{[2,1]} \cdot F^{(i)}_{[2,1]} + Z^{[1]}_{[2,1]} \cdot F^{(i)}_{[2,1]} \quad (14)
\]

with polynomial $F^{(i)}_{[3]}$. Moreover, $F^{(i)}_{[1]}$ and $F^{(i)}_{[1,1]} (A, q) = F^{(i)}_{[2]} (A, -q^{-1})$ are defined from the expansions of simpler colored HOMFLY-PT polynomials

\[
\begin{align*}
\mathcal{H}^{(i)}_{[2,1]} &= 1 + D_1 D_{-1} \cdot F^{(i)}_{[1]} \\
\mathcal{H}^{(i)}_{[2,1]} &= 1 + [2] D_2 D_{-1} \cdot F^{(i)}_{[1]} + D_3 D_2 D_0 D_{-1} \cdot F^{(i)}_{[2]} \\
\end{align*}
\]

Indeed, for the simplest defect-zero knot, which is not a double braid, $\mathcal{K} = 9_{46}$, we get:

\[
\begin{align*}
F^{9_{46}}_{[1]} &= A^2 \cdot (A^2 + 1) \\
F^{9_{46}}_{[2]} &= A^4 \cdot (q^8 A^4 + [2] q^5 A^2 + 1) \\
F^{9_{46}}_{[3]} &= q^4 A^6 \cdot (q^4 A^2 + 1) \cdot (q^8 A^4 + [2] q^5 A^2 - q^6 + q^2 + 1) \\
F^{9_{46}}_{[2,1]} &= A^6 \cdot \left( A^6 + \frac{[6]}{[2]} \cdot A^4 + [3] \left( 1 + [3] \{q\}^2 + \frac{[6]}{[2]} \{q\}^4 \right) \cdot A^2 + (1 + [3] \{q\}^2) \right) \\
\tilde{F}^{9_{46}}_{[2,1]} &= A^6 \cdot \left( [q] \cdot A^6 - \frac{[8]}{[2]} \{q\}^2 \cdot A^4 + \frac{[6]}{[2]} \{3\} \{5\} \{q\}^2 \cdot A^2 - \frac{[18]}{[9]} \right) \\
\end{align*}
\]

$F_{[3]}$ does not contribute to the expansion (14) of $\mathcal{H}_{[2,1]}$ in representation $[2, 1]$, it is provided here for comparison and for future use. 

5.4. For knots with non-zero defect a slightly weaker form of the DE can be expected, with underlined differentials omitted from the Z-factors in (8). For example, for the knot $6_2$ with defect one

\[
\mathcal{H}^{6_2}_{[2,1]} = 1 + Z^{[1]}_{[2,1]} \cdot F^{6_2}_{[1]} + \frac{[3]}{[2]} D_2 D_{-2} \left( D_3 \cdot C^{6_2}_{[2]} + D_{-3} \cdot C^{6_2}_{[1,1]} \right) + D_3 D_2 D_{-2} D_{-3} \cdot C^{6_2}_{[2,1]} - [3] \{q\}^4 D_2 D_{-2} \cdot \tilde{G}^{6_2}_{[2,1]} \\
\]

with

\[
\begin{align*}
C^{6_2}_{[1]} &= \frac{[6]}{[3]} \cdot A^2 \\
C^{6_2}_{[2]} &= q^2 A^4 \left( D_2 + \frac{[6]}{[3]} \{q\}^2 D_0 \right) - [2] q A^3 \{q\}^2 \\
C^{6_2}_{[2,1]} &= A^6 D_0^2 + \{q\}^2 A^4 \left( \frac{[6]}{[2]} A^2 - \frac{[4]}{[2]} \{4\} \{2\} + \{3\} \right) + A^2 + \frac{[6]}{[2]} \{5\} \{q\}^2 \cdot A^2 + \frac{[6]}{[2]} \} \\
\tilde{G}^{6_2}_{[2,1]} &= A^4 \left( [3] A^4 - [3] (3 + 2 \frac{[4]}{[2]} \{3\} \{q\}^2 \right) \cdot A^2 + \frac{[6]}{[2]} \} \quad (17)
\end{align*}
\]
6 Representation $R = [3, 1]$

This time the Young diagrams in the product $R \otimes \bar{R} = [3, 1] \otimes [3, 1]$ are:

$\lambda = [3, 1]$

$\lambda = [3]$

$\lambda = [2, 1]$

$\lambda = [2, 1]$

$\lambda = [1, 1]$

$\lambda = [1, 1]$

$\lambda = [1]$

$\lambda = [1]$

$\lambda = \emptyset$

The last two diagrams in the right column are exactly the same as in the case of $R = [2, 1]$ (with one full line added at the bottom, which does not affect dimensions and Casimirs) and therefore should provide the same
The first two diagrams in the right column again have coincident dimensions and Casimir eigenvalues and provide another unified contribution $\tilde{\mathcal{F}}_{[3,1]}$:

$$
\mathcal{H}^{(m,n)}_{[3,1]} = 1 + Z^{[3,1]} [3] \cdot \frac{F^{(m)}_{[3]} F^{(n)}_{[1]}}{F^{(1)}_{[1]} F^{(-1)}_{[1]}} + Z^{[2,1]} [3] \cdot \frac{F^{(m)}_{[2]} F^{(n)}_{[1]}}{F^{(2)}_{[1]} F^{(-1)}_{[1]}} + \frac{[4]}{[2]} Z^{[1,1]} [3] \cdot \frac{F^{(m)}_{[1]} F^{(n)}_{[1]}}{F^{(1)}_{[1]} F^{(-1)}_{[1]}} + 
+ \frac{[4]}{[3]} Z^{[3,1]} [3] \cdot \frac{\tilde{F}^{(m)}_{[3]} \tilde{F}^{(n)}_{[3]}}{\tilde{F}^{(3)}_{[3]} \tilde{F}^{(-3)}_{[3]}} + \frac{[4]}{[2]} Z^{[2,1]} [3] \cdot \frac{\tilde{F}^{(m)}_{[2]} \tilde{F}^{(n)}_{[2]}}{\tilde{F}^{(2)}_{[2]} \tilde{F}^{(-2)}_{[2]}} + \frac{[4]}{[3]} Z^{[1,1]} [3] \cdot \frac{\tilde{F}^{(m)}_{[1]} \tilde{F}^{(n)}_{[1]}}{\tilde{F}^{(1)}_{[1]} \tilde{F}^{(-1)}_{[1]}} + 
+ \frac{[4]}{[3]} \tilde{Z}^{[2,1]} [3] \cdot \frac{\bar{\tilde{F}}^{(m)}_{[2]} \bar{\tilde{F}}^{(n)}_{[2]}}{\bar{\tilde{F}}^{(2)}_{[2]} \bar{\tilde{F}}^{(-2)}_{[2]}} + \tilde{Z}^{[3,1]} [3] \cdot \frac{\bar{\tilde{F}}^{(m)}_{[3]} \bar{\tilde{F}}^{(n)}_{[3]}}{\bar{\tilde{F}}^{(3)}_{[3]} \bar{\tilde{F}}^{(-3)}_{[3]}} + \frac{[4]}{[3]} \tilde{Z}^{[1,1]} [3] \cdot \frac{\bar{\tilde{F}}^{(m)}_{[1]} \bar{\tilde{F}}^{(n)}_{[1]}}{\bar{\tilde{F}}^{(1)}_{[1]} \bar{\tilde{F}}^{(-1)}_{[1]}}
$$

with the $Z$-factors

$$
Z^{[1]}_{[3,1]} = D_5 D_{-3} + D_3 D_1 + D_2 D_{-2} + D_0 D_{-2}, \quad Z^{[2]}_{[3,1]} = D_4 D_0 \cdot \left( D_5 D_{-2} + \frac{[6]}{[3]} D_4 D_2 + D_2 D_{-1} \right),
$$

$$
Z^{[1,1]}_{[3,1]} = D_5 D_0 D_{-2} D_{-3}, \quad Z^{[2,1]}_{[3,1]} = D_5 D_3 D_1 D_0 D_{-2}, \quad Z^{[2,1]}_{[3,1]} = D_4 D_3 D_1 D_{-1} D_{-2} D_{-3},
$$

$$
Z^{[3,1]}_{[3,1]} = D_5 D_4 D_3 D_2 D_0 D_{-2} D_{-3}, \quad Z^{[2,1]}_{[3,1]} = -[4]^2[2][q]^4 D_3 D_{-2}, \quad \tilde{Z}^{[3,1]}_{[3,1]} = -[4]^2[2][q]^4 D_4 D_3 D_0 D_{-2}
$$

They are deduced/guessed from the DE structure and factorization conjecture, used is also the knowledge [12] of the [3, 1]-HOMFLY of the three 3-strand twist knots $3_1$, $4_1$ and $5_2$. Underlined are the differentials, which should be omitted in the case of knots with non-vanishing defects. Important insight from (17) is that $D_1 D_{-1}$ should be obligatory present in $Z^{[2,1]}_{[3,1]}$ – to be eliminated in the DE for knots with non-vanishing defects. Additional $F$-functions, which did not appear in the list (9), are:

$$
F^{(m)}_{[3]} = q^2 A^4 \cdot \left( \frac{A^{2m}_{[3]} D_{2} D_1 D_{0} D_{-1}}{D_{2} D_1 D_{0} D_{-2}} - \frac{[4]}{[3]} \cdot \frac{A^{2m}_{[3]} D_{1} D_{0} D_{-2}}{D_{2} D_1 D_{0} D_{-2}} + \frac{[4]}{[2]} \cdot \frac{A^{2m}_{[3]} D_{3} D_{0} D_{-2}}{D_{2} D_1 D_{0} D_{-2}} + \frac{[4]}{[3]} \cdot \frac{A^{2m}_{[3]} D_{4} D_{0} D_{-2}}{D_{2} D_1 D_{0} D_{-2}} \right)
$$

(20)

and

$$
\tilde{F}^{(m)}_{[3,1]} = q^2 A^4 \cdot \left\{ \frac{A^{2m}_{[3]} D_{2} D_1 D_{0} D_{-1}}{D_{2} D_1 D_{0} D_{-2}} - \frac{[4]}{[3]} \cdot \frac{A^{2m}_{[3]} D_{1} D_{0} D_{-2}}{D_{2} D_1 D_{0} D_{-2}} + \frac{[4]}{[2]} \cdot \frac{A^{2m}_{[3]} D_{3} D_{0} D_{-2}}{D_{2} D_1 D_{0} D_{-2}} + \frac{[4]}{[3]} \cdot \frac{A^{2m}_{[3]} D_{4} D_{0} D_{-2}}{D_{2} D_1 D_{0} D_{-2}} \right\}
$$

(21)

The first two are contributing in the case of rectangular representations and are known from [1], the last one is peculiar for non-rectangular case and is new. Note that, like $\tilde{F}_{[2,1]}$ was independent of $A_{21}$, this $\tilde{F}_{[3,1]}$ does not depend on $A_{31}$.

The relevant eigenvalues are

$$
\begin{align*}
\Lambda_0 &= 1, \quad \Lambda_1^2 = A^2, \quad \Lambda_2^2 = q^4 A^4, \quad \Lambda_3^2 = \frac{q^2}{2} A^4, \quad \Lambda_4^2 = \frac{q^4}{3} A^4, \\
\Lambda_5^2 &= A^6, \quad \Lambda_6^2 = q^6 A^6, \quad \Lambda_7^2 = q^{12} A^6, \quad \Lambda_{31}^2 = q^8 A^8
\end{align*}
$$

(22)

Underlined are the two eigenvalues, associated with the two pairs of "anomalous" Young diagrams. We remind that for each integer $m$ all $F^{(m)}_{[3]}$ are polynomials, moreover, they drastically simplify to (3) for $m = 0, \pm 1$. Factorization conjecture for double braids is checked by the reduction property (10) of Alexander polynomials, which is applicable to the case of $R = [3, 1]$.

As in the case of $R = [2, 1]$, one can easily deduce the $9 \times 9$ matrix $\mathcal{S}_{ab}$, but it is not unitary, because its elements are actually averaged over the multiplicity spaces. The elements of the first line are made from dimensions,
\[
\bar{S}_{0b} = \eta_b \frac{\sqrt{d_0 d_b}}{d_{[3,1]}}
\]  
(23)

with \(\eta_b = 2\) rather than 1 for \(b = 1, 2, 3\) – this fact is actually used in the derivation of \(\hat{F}_{[3,1]}\). These \(\eta_b\) (actually, \(\eta^{[3,1]}\)) take into account the pairwise "degeneracy" of the "anomalous diagrams 2 and 3 and the multiplicity in the channels 1 and 2. Note that in the case of \(R = [2, 1]\) the two factors \(\eta^{[2,1]} = \eta_2^{[2,1]} = 2\) were also non-trivial, but \(\eta_2^{[2,1]} = 1 \neq \eta_2^{[3,1]} = 2\). Even in the simplest case of the first line instead of unitarity we have

\[
\sum_{b=0}^8 \frac{1}{\eta_b} \bar{S}_{0b}^2 = 1 \iff \sum_{b=0}^8 \eta_b \cdot d_b = d_{[3,1]}^2
\]  
(24)

with non-trivial \(\eta\)-weights. Extraction of the elements of the bigger unitary exclusive Racah matrix \(\bar{S}\) requires additional effort. Due to peculiar properties (many vanishing entries) of \(\bar{S}\), in addition to (24) there are two more elementary sum rules, involving not only the first, but also the last lines in \(\bar{S}\): \(\sum_{b=0}^8 \bar{S}_{21,b}^2 = 1\) and \(\sum_{b=0}^8 \bar{S}_{b,21}^2 = 0\) in the case of \(R = [2, 1]\) and \(\sum_{b=0}^8 \bar{S}_{21,b}^2 = 1\) and \(\sum_{b=0}^8 \bar{S}_{b,21}^2 = 0\) in the case of \(R = [3, 1]\) (note that the first sums do not contain \(\eta_b\)-factors). Like elements in the first line, those in the last are also fully factorized. Since \(\bar{S}\) is symmetric, the same is true about the first and the last rows. These boundary elements can be directly identified (modulo factors 2 and 0) with the corresponding elements of the unitary \(\bar{S}\). Like in the case of \(R = [2, 1]\), all other elements can be restored by solving the unitarity constraints. Then they can be compared with the inclusive ones, found for the case of \(R = [3, 1]\) in the second paper of [12].

7 Other representation \(R = [r, 1]\)

It is now straightforward to describe the shape of the DE for other non-rectangular representations, at least for the entire family \(R = [r, 1]\):

\[
\mathcal{N}^{(m,n)}_{[r,1]} = 1 + \sum_{i=1}^{r} \left[ \frac{1}{[r]} \cdot \frac{[r+1]!}{[i]! [r-i]!} \cdot \frac{D_{r+i-2} D_{r-i-2}}{D_{r-1}} \cdot \left( D_{r-2 D_{i-1}} - [r+1][i] \cdot \{q\}^2 \cdot \frac{F^{(m)}_{[k]} F^{(n)}_{[k]}}{F^{(1)}_{[k]} F^{(-1)}_{[k]}} + \cdots \right) \right] - \{q\}^4 \sum_{i=2}^{r} \left[ \frac{[r+1]!}{[r] [r-i]! [i-2]!} \cdot \frac{D_{r+i-2} D_{r-i-2}}{D_{r-1}} \cdot D_{r-2} \cdot \frac{F^{(m)}_{[k]} F^{(n)}_{[k]}}{F^{(1)}_{[k]} F^{(-1)}_{[k]}} \right]
\]  
(25)

where \(F\)-functions are defined in eq.(37) of [1] (see also [31] for a more profound description in terms of shifted skew-characters):

\[
F^{(m)}_{[a+b+1]} = \left( q^{\frac{a+b}{2}} A \right)^{a+b+1} \left\{ \frac{1}{\{A q^a\} \cdots \{A q^b\}} + \sum_{i=0}^{a} \sum_{j=0}^{b} \frac{(-)^{i+j+1} (q^{-j} A)^{2m+i+j+1}}{\{A q^{a+i+1}\} \cdots \{A q^{b+j+1}\} \cdot \{A q^a\} \cdots \{A q^b\} \cdot \{A q^{a+i}\} \cdots \{A q^{b+j}\} \cdot \{A q^{a+i+1}\} \cdots \{A q^{b+j+1}\}} \right\}
\]  
(26)

and \(\bar{F}\)-functions can be restored recursively in \(r\) from the conditions (23), as explained in the previous section, – see eq.(31) below. As usual, they vanish for the unknot \((m = 0)\), turn to unity for the figure-eight knot \(4_1 (m = -1)\) and coincide with the ordinary \(F_{[k,1]}\) for the trefoil \(3_1 (m = 1)\):

\[
\bar{F}^{(-1)}_{[k,1]} = F^{(-1)}_{[k,1]} = 1
\]
\[
\bar{F}^{(0)}_{[k,1]} = F^{(0)}_{[k,1]} = 0
\]
\[
\bar{F}^{(1)}_{[k,1]} = F^{(1)}_{[k,1]} = (-)^{k+1} \cdot A^{2k+2} \cdot q^{(k-2)(k+1)}
\]  
(27)

However, for other \(m\) there is no coincidence: \(\bar{F}^{(m)}_{[k,1]} \neq F^{(m)}_{[k,1]}\). Note that \(D\)-factorials \(D_n! = \prod_{i=0}^{n} D_i\) are defined as products of \(n + 1\) differentials, beginning from \(i = 0\).
The unitary Racah matrix $\bar{S}$ is expressed as follows:

$$\bar{S} = \bar{S}_{\text{sym}}(\bar{S})$$

In these terms

$$\bar{S}_{\text{sym}} = \begin{pmatrix} \Lambda_1^2 & q^{2i(i-1)} A^{2i} \\ \Lambda_1^2 & q^{2(i+1)(i-2)} A^{2i+2} \end{pmatrix}$$

In these terms

$$\bar{F}^{(m)}_{[k,1]} = q^{(k+1)(k-2)} A^{k+1} + \left\{ \frac{\Lambda_1^2m \cdot D_{k+1}}{\prod_{l=1}^{k-1} D_l} - \frac{\Lambda_1^2m \cdot (D_k + [k-1]D_{-2})}{\prod_{l=1}^{k} D_l} \right\}$$

Eq. (25) successfully reproduces the only available answers in all representations inside the double-braid family – those for the trefoil $3_1$.

### 8 Towards exclusive Racah matrix $\bar{S}$ for $R = [r, 1]$

As already mentioned, the suggestion of [1] was to extract exclusive Racah matrices from the $\Lambda$-expansion of the antiparallel double braids, see (11) in sec.5 above:

$$H^{(m,n)}_{R} = \sum_{a,b} \frac{\sqrt{d_{a}d_{b}}}{d_{R}} \cdot \bar{S}_{ab} \cdot \Lambda_{a}^{2m} \Lambda_{b}^{2n}$$

This is an example of the evolution-based [25] approach to Racah calculus, proposed and successfully used in [12]. Factorization of differential expansion for double braids, which allows to efficiently calculate $H^{(m,n)}_{R}$, opens spectacular possibilities for the case of exclusive Racah $\bar{S}$. The only remaining problem is that for non-
rectangular representations eq.(32) contains the "averaged" matrix $\bar{S}$, which is non-unitary and smaller than
the unitary matrix $\bar{S}$ itself. We now describe a way to restore $\bar{S}$ from the known $\bar{S}$. Basically, it uses unitarity
constraints to find the lacking elements of the bigger matrix. With known $\bar{S}$, the number of conditions for
$\bar{S}$ seems sufficient. However, one could expect that this approach is impractical, because it involves solving a
number of quadratic relations, involving polynomials of complexity, which fastly increases with $r$. Fortunately,
things turn to be a little simpler. In the by-now-standard case of $R = [2, 1]$ the testable properties of $\bar{S}$
allow a procedure, which involves solving just linear equations and taking a few square roots of the factorized
expressions. For $r > 2$ the situation is more involved, still calculation looks practically possible.

The matrix $\bar{S}$, which can be directly read from (32), has dimension $3r \times 3r$ and is symmetric, but not unitary.
The unitary Racah matrix $\bar{S}$ has the lines/columns $i$ doubled and $i$ with $0 < i < r$ quadrupled – see [10] for
detailed explanations and comments. In result this symmetric and unitary matrix has the size $(7r-4) \times (7r-4)$.
If we keep the order of columns, chosen in [10] for the case of $R = [2, 1]$ (with second and the third columns
permuted to make the matrix symmetric), then the $(7r-4) \times (7r-4)$ symmetric and unitary $\bar{S}$ is expressed
through the $3r \times 3r$ symmetric, but not unitary, $\bar{S}$ as follows:
\[
\tilde{S} = \begin{pmatrix}
0 & \tilde{S}_{0,0} & \frac{1}{2} \tilde{S}_{0,j} & \frac{1}{2} \tilde{S}_{0,j} & j = 2, \ldots, r \\
\tilde{j}, 1 & \tilde{j}, 2 & \tilde{j}, 1, \tilde{j}, 2 & j = 1, \ldots, r - 1 \\
\tilde{j}, 2 & \tilde{j}, 2 & \frac{1}{2} \tilde{S}_{i,j} & \frac{1}{2} \tilde{S}_{i,j} & \tilde{S}_{i,j} - x_{ij} - x_{ij} & x_{ij} - Y_{ij} - y_{ij} & \frac{1}{2} \tilde{S}_{i,j} - x_{ij} - x_{ij} & x_{ij} + y_{ij} & Y_{ij} - y_{ij} & j = 1, \ldots, r - 1 \\
\end{pmatrix}
\]

Remaining parameters \(x, u, v, z\) are not defined from the double-braid evolution. However, they are unambiguously (up to inessential signs) dictated by the unitarity of the matrix \(\tilde{S}\). Calculation are greatly simplified (almost reduced to just linear equations, with most quadratic left only for the checks of consistency) because of the special property of the distinguished line/column with \(r1\), associated with the "biggest" diagram in \(R \otimes \tilde{R}\) – for this reason it is written separately from all other \(i1\). Numerous zeroes in this case (and also in the very first line) are implied by the three identities, already familiar from the end of sec.6:

\[
\sum_{b=0}^{3r-1} \frac{1}{\eta_b} \tilde{S}_{0,b}^2 = 1 \iff \sum_{b=0}^{3r-1} \eta_b \cdot d_b = d_{r,1}^2 \\
\sum_{b=0}^{3r-1} \tilde{S}_{r1,b}^2 = 1, \quad \sum_{b=0}^{3r-1} \frac{1}{\eta_b} \tilde{S}_{0,b} \cdot \tilde{S}_{r1,b} = 0
\]

(33)

The fact that the entries \(\pm v\) in the last two columns for the lines \(i1\) and \(r\) differ only by a sign follow from additional properties of \(\tilde{S}\).

Orthogonality to the two distinguished lines \(0\) and \(r1\), provides linear equations for parameters \(x\) and \(u\) and most of \(z\), which for \(r = 2\) fix them unambiguously. After that \(v\)'s can be defined by taking the square roots

\[
v_i = \sqrt{\frac{1}{2} \left( 1 - \sum_{j=1}^{7r-6} \tilde{S}_{i,j}^2 \right)}
\]

(34)

– this is the only place where the sign ambiguity occurs. Finally, orthogonality to these lines with known \(v\)'s provide linear equations for the remaining \(y, Y\) and \(z\). For \(r \geq 3\) some bilinear relations also need to be used.

Once \(\tilde{S}\) is known, the second exclusive \(S\) (which is unitary, but not symmetric) can be found as its diagonalization matrix from

\[
S = T^{-1} \tilde{S} T^{-1} (35)
\]

with the known diagonal \(T\) and \(\tilde{T}\), see [10] and eq.(2) in [1].
9 Conclusion

This paper describes a new important progress in the study of colored knot polynomials. The form of the differential expansion is finally fixed for the family of twist knots in the case of the simplest non-rectangular representations $R = [2, 1], R = [3, 1]$ and, conjecturally, $R = [r, 1]$, thus complementing the recent results of [1, 31] for rectangular $R$. The crucial feature of the differential expansion is its factorization for the antiparallel double braid family – and the main result of this paper is that it continues to hold for non-rectangular $R$. As a byproduct we get $[3, 1]$-colored HOMFLY-PT polynomials for the infinite double-parametric double-braid family of defect-zero knots – of which only three examples (for the 3-strand $K^{(0)} = 3_1, 4_1, 5_2$) were known so far. Infinitely many new (double-braid, but not twist, known since [29]) are also added in the better-studied $[2, 1]$ case. Conjecture (25) for $R = [r, 1]$ provides much more new results and has other far-going implications.

The three immediate next questions to address are the derivation of general formulas for the exclusive Racah matrices $\bar{S}$ and $S$ as functions of $r$, the search for equations a la [21, 22] in $r$ and extension from the cases of rectangular $R = [r^s]$ and non-rectangular $R = [r, 1]$, tamed respectively in [1, 31] and in the present paper, to generic representations $R$.

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