Light-cone continuous-spin field in AdS space

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Abstract

We develop further the general light-cone gauge approach in AdS space and apply it for studying continuous-spin field. For such field, we find light-cone gauge Lagrangian and realization of relativistic symmetries. We find a simple realization of spin operators entering our approach. Generalization of our results to the gauge invariant Lagrangian description is also described. We conjecture that, in the framework of AdS/CFT, the continuous-spin AdS field is dual to light-ray conformal operator. For some particular cases, our continuous-spin field leads to reducible models. We note two reducible models. The first model consists of massive scalar, massless vector, and partial continuous-spin field involving fields of all spins greater than one, while the second model consists of massive vector, massless spin-2 field, and partial continuous-spin field involving all fields of spins greater than two.

Keywords: continuous-spin field; higher-spin field.

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1 Introduction

In view of the aesthetic features, continuous-spin field has attracted some interest in recent time. For review, see Refs. [123]. Extensive list of references on earlier studies of this theme may be found in Refs. [45]. Alternative points of view on the role of continuous-spin field in string theory are presented in Refs. [67]. Interrelation of continuous-spin field and massive higher-spin field is discussed in Ref.[8]. Interacting continuous-spin fields are considered in Refs.[10-12], while various BRST Lagrangian formulations are studied in Refs. [13, 14, 15]. Continuous-spin field in AdS space was investigated in Refs.[16-19]. Other various important aspects of continuous-spin field were discussed in Refs.[21-28].

Continuous-spin field is decomposed into infinite chain of scalar, vector, and tensor fields which consists of every field just once. A similar infinite chain of fields appears in higher-spin gauge field theories in AdS space [29]. Other example of dynamical system involving infinite number of fields is a string theory. Light-cone gauge formulation simplifies considerable superstring action in AdS space [30, 31, 32]. We think that light-cone gauge formulation will simplify study of continuous-spin field and therefore will be useful for better understanding of various aspects of continuous-spin field.

In this paper, we develop further our light-cone gauge formulation of AdS fields in Refs.[33, 34]. Namely, we obtain representation of the 4th-order Casimir operator of the so(d, 2) algebra in terms of spin operators entering light-cone gauge Lagrangian. This allows us to express two constant parameters entering light-cone gauge Lagrangian of continuous-spin field entirely in terms of the eigenvalues of the 2nd- and 4th-order Casimir operator of the so(d, 2) algebra. Such representation for the Lagrangian and a suitable parametrization of eigenvalues of the Casimir operators make the whole study more transparent and straightforward and considerably simplify analysis of classical unitarity and irreducibility of continuous-spin field. We obtain simple representation for spin operators entering our light-cone gauge approach. Also we make conjecture about duality between continuous-spin field and light-ray conformal operator. Interrelations of light-cone gauge and gauge invariant approaches allow us to extend all our light-cone gauge results to the gauge invariant Lagrangian of continuous-spin field in a rather straightforward way. In due course, we discuss two models of continuous-spin fields which, besides partial continuous-spin fields, involve interesting spectrum of low-spin fields.

2 General light-cone gauge approach in $AdS_{d+1}$ space

General light-cone gauge approach in AdS space was developed in Refs.[33, 34]. In this section, first, we review the formulation obtained in Ref.[34] and, second, we present our new result regarding the light-cone gauge representation for 4th-order Casimir operator of the so(d, 2) algebra.

Let $\phi(x)$ be arbitrary spin and type of symmetry bosonic fields. Collecting the fields into a ket-vector $|\phi\rangle$, we present a light-cone gauge action in the following form [33, 34]:

$$S = \int d^{d+1}x \langle \phi | (\Box - \frac{1}{z^2} A) |\phi\rangle, \quad \Box = 2\partial^+ \partial^- + \partial^i \partial^i + \partial_z^2, \quad (2.1)$$

We use metric of $AdS_{d+1}$ space $ds^2 = (-dx^2_0 + dx^2 + dx^2_{d-1} + dz^2)/z^2$, where $z^+$ is considered as a light-cone time. Our conventions for the coordinates and derivatives are as follows: $x^I = x^i, x^d \equiv z, \partial^i = \partial/\partial x^i, \partial_z \equiv \partial/\partial z, \partial^+ \equiv \partial/\partial x^+, \partial^- = \partial/\partial x^+, \partial^+ = \partial/\partial x^+, \partial^0 = \partial/\partial x^0$. Vectors of so(d-1) algebra are decomposed as $X^I = (X^i, X^z), X^I Y^J = X^I Y^J + X^z Y^z$. Also we use the shortcuts, $x^I x^J = x^i x^j + z^2, x^I \partial^J = x^i \partial^j + z \partial_z, \partial^0 \partial^J = \partial^i \partial^j + \partial_z^2$. 

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\[ \langle \phi \rangle = (| \phi \rangle)^\dagger, \] where operator A being independent of space-time coordinates and their derivatives is acting only on spin indices of \(| \phi \rangle\). In general, fields entering the \(| \phi \rangle\) are complex-valued.

The choice of the light-cone gauge spoils the relativistic so(d, 2) symmetries of fields in AdS_{d+1}. Therefore in order to demonstrate that so(d, 2) symmetries are still present we find the Noether charges which generate them. For free fields, Noether charges (or generators) have the following representation in terms of the \(| \phi \rangle\):

\[ G = \int dx^{-} d^{d-1} x (\partial^+ \phi | G_{\text{diff}} | \phi \rangle + \text{h.c.}, \] (2.2)

where \( G_{\text{diff}} \) stands for differential operators acting on \(| \phi \rangle\). These operators are given by

\[
P^i = \partial^i, \quad P^+ = \partial^+ , \quad P^- = -\frac{\partial^i \partial^j}{2\partial^+} + \frac{1}{2z\partial^+} A , \] (2.3)

\[
J^{+-} = x^+ P^- - x^- \partial^+ , \quad J^{ij} = x^i \partial^j - x^j \partial^i + M^{ij} , \] (2.4)

\[
J^{+i} = x^+ \partial^i - x^i \partial^+ , \quad J^{-i} = x^- \partial^i - x^i P^- + M^{-i} , \] (2.5)

\[
D = x^+ P^- + x^- \partial^+ + x^i \partial^i + \frac{d-1}{2} , \] (2.6)

\[
K^+ = -\frac{1}{2} (2x^+ x^- + x^j x^j) \partial^+ + x^+ D , \] (2.7)

\[
K^i = -\frac{1}{2} (2x^+ x^- + x^j x^j) \partial^i + x^i D + M^{ij} x^j + M^{-i} x^+ , \] (2.8)

\[
K^- = -\frac{1}{2} (2x^+ x^- + x^j x^j) P^- + x^- D + \frac{1}{\partial^+} x^i \partial^j M^{ij} - \frac{x^i}{2z\partial^+} [M^{zi}, A] + \frac{1}{\partial^+} B , \] (2.9)

\[
M^{-i} \equiv M^{ij} \frac{\partial^j}{\partial^+} - \frac{1}{2z\partial^+} [M^{zi}, A] , \quad M^{-i} = -M^{i-} , \] (2.10)

where in (2.3)-(2.10) and below, we use operators \( A, B, B^I, M^{IJ} \), which, being independent of space-time coordinates and derivatives are acting on spin indices of the ket-vector \(| \phi \rangle\). Our conventions for commutators of operators (2.3)-(2.10) may be found in relations (A6),(A7) in Ref. [35]. Note that we use the decompositions \( M^{IJ} = M^{zi}, M^{ij}, B^I = B^z, B^i \). The operators \( M^{IJ} \) and \( B^I \) constitute a base of spin operators, while the operators \( A, B \) can be expressed as

\[
A = C_2 + 2B^z + 2M^{zi} M^{zi} + \frac{1}{2} M^{ij} M^{ij} + \frac{d^2 - 1}{4} , \] (2.11)

\[
B = B^z + M^{zi} M^{zi} , \] (2.12)

where \( C_2 \) stands for an eigenvalue of the 2nd-order Casimir operator of the so(d, 2) algebra. The operators \( B^I, M^{IJ} \) satisfy the commutators

\[
[M^{IJ}, M^{KL}] = \delta^{JK} M^{IL} + 3 \text{ terms} , \quad [B^I, M^{JK}] = \delta^{IJ} B^K - \delta^{IK} B^J , \] (2.13)

\[
[B^I, B^J] = \left( C_2 + \frac{1}{2} M^2 + \frac{d^2 - 3d + 4}{2} \right) M^{IJ} - (M^3)^{[I,J]} , \] (2.14)

where, in (2.13),(2.14) and below, we use the notation

\[
M^2 \equiv M^{IJ} M^{IJ} , \quad (M^3)^{[I,J]} \equiv \frac{1}{2} M^{IK} M^{KL} M^{LJ} - (I \leftrightarrow J) , \] (2.15)
\[ M^4 \equiv M^{IJ} M^{JK} M^{KL} M^{LI}. \]  

(2.16)

Note also that we use the following hermicity conjugation rules for the spin operators:

\[ M^{IJ\dagger} = -M^{IJ}, \quad B^{I\dagger} = B^I. \]  

(2.17)

From (2.13), we see that the \( M^{IJ} \) are spin operators of the \( so(d-1) \) algebra, while the operator \( B^I \) transforms as vector operator under transformations of the \( so(d-1) \) algebra. It is the commutators (2.14) that are basic equations of light-cone gauge formulation of relativistic dynamics in AdS\(_{d+1}\). We now ready to formulate our new result in this Section. We find that the 4th-order Casimir operator of the \( so(d,2) \) algebra is expressed in terms of the spin operators \( B^I, M^{IJ} \) as follows

\[ C_4 = B^I B^I - \frac{1}{2} \left( C_2 + \frac{d^2 - 3d + 4}{4} \right) M^2 - \frac{1}{8} (M^2)^2 - \frac{1}{4} M^4, \]  

(2.18)

where we use notation given in (2.15), (2.16). Our conventions for the Casimir operators are given in Appendix. Action (2.1) is invariant under the transformations \( \delta |\phi\rangle = G_{\text{diff}} |\phi\rangle \).

3 Light-cone gauge continuous-spin AdS field

To discuss light-cone gauge continuous-spin field we introduce a ket-vector\footnote{For oscillators and vacuum, we use the rules: \([\bar{\alpha}^I, \alpha^J] = \delta^{IJ}, \quad [\bar{\nu}, \nu] = 1, \quad \alpha^{I\dagger} = \bar{\alpha}^I, \quad \nu^\dagger = \bar{\nu}, \quad \bar{\alpha}^I |0\rangle = 0, \quad \bar{\nu} |0\rangle = 0.}

\[ |\phi\rangle = \sum_{n=0}^{\infty} \frac{\nu^n}{n! \sqrt{n!}} \alpha^{I_1} \ldots \alpha^{I_n} \phi^{I_1 \ldots I_n} (x) |0\rangle, \]  

(3.1)

where fields in (3.1) with \( n = 0 \) and \( n = 1 \) are the respective scalar and vector fields of the \( so(d-1) \) algebra, while field with \( n \geq 2 \) is rank-\( n \) traceless totally symmetric tensor field of the \( so(d-1) \) algebra. All fields in (3.1) are taken to be complex-valued. Ket-vector (3.1) satisfies the algebraic constraints

\[ (N_\alpha - N_\nu) |\phi\rangle = 0, \quad \bar{\alpha}^I \bar{\alpha}^J |\phi\rangle = 0 \quad N_\alpha \equiv \alpha^I \bar{\alpha}^I, \quad N_\nu \equiv \nu \bar{\nu}. \]  

(3.2)

Our aim is to find realization of the spin operators \( B^I, M^{IJ} \) on space of ket-vector \(|\phi\rangle \). Realization of the \( so(d-1) \) algebra spin operator \( M^{IJ} \) on space of \(|\phi\rangle \) (3.1) is well known,

\[ M^{IJ} = \alpha^I \bar{\alpha}^J - \alpha^J \bar{\alpha}^I, \]  

(3.3)

\[ (M^3)^{[I|J] = \left( -\frac{1}{2} M^2 + \frac{(d-3)(d-4)}{2} \right) M^{IJ}}, \quad M^2 \equiv M^{IJ} M^{IJ}. \]  

(3.4)

where, in (3.4), we show useful relation for \( M^{IJ} \). Using (3.4), we see that equations (2.14) and representation for \( C_4 \) (2.18) are considerably simplified as

\[ [B^I, B^J] = (C_2 + M^2 + 2d - 4) M^{IJ}, \]  

(3.5)

\[ C_4 = B^I B^I - \frac{1}{2} (C_2 + d - 2) M^2 - \frac{1}{4} (M^2)^2. \]  

(3.6)
All that is required is to find solution to equations (3.5)-(3.6). Solution to these equations is found to be
\[ B^l = g\bar{a}^l + A^l\bar{g}, \quad A^l \equiv \alpha^l - \alpha^j\bar{\alpha}^j \frac{1}{2N_\alpha + d - 1}\bar{\alpha}^l, \quad (3.7) \]
\[ g = g_v\bar{v}, \quad \bar{g} = v_g\bar{v}, \quad (3.8) \]
\[ g_v = N_v f_v, \quad \bar{g}_v = N_v f_v, \quad N_v = \left( (N_v + 1)(2N_v + d - 1) \right)^{-1/2}, \quad (3.9) \]
\[ f_v f_v = F_v, \quad (3.10) \]
\[ F_v = F_0 - (C_2 + d - 1)N_v(N_v + d - 2) + N_v^2(N_v + d - 2)^2, \quad (3.11) \]
\[ F_0 = C_4, \quad (3.12) \]

where \( F_0 \) (3.11) is a constant, while quantities \( f_v, \bar{f}_v \) (3.9),(3.10) depend on \( N_v \) (3.2). We note that relations (3.7)-(3.11) are obtained from (3.5), while the constant \( F_0 \) (3.12) is fixed by equation (3.6). Thus, from (3.11),(3.12), we see that our light-cone approach allows us to express the \( F_v \) entirely in terms of the Casimir operators. This fact turns out to be very important for our analysis of equation (3.10) and the condition for \( B^l \) in (2.17). Relations (3.7)- (3.12) exhaust all restrictions imposed on \( f_v, \bar{f}_v \) by equations (3.5),(3.6). Remaining restrictions on \( f_v, \bar{f}_v \) can be obtained by using (3.10) and requirement of classical unitarity and irreducibility of continuous-spin field.

**Irreducible classically unitary continuous-spin field.** Hermicity condition for \( B^l \) (2.17) leads to the relation \( f_v^l = \bar{f}_v^l \). This relation and (3.10) imply that \( \Im F_v(n) = 0, F_v(n) \geq 0 \) for all \( n = 0, 1, \ldots, \infty \), where \( F_v(n) \equiv F_v|_{N_v=n} \). Field (3.1) with such \( F_v \) is referred to as classically unitary continuous-spin field. Field (3.1) with \( F_v \) (3.17) that satisfies the restrictions
\[ \Im F_v(n) = 0, \quad F_v(n) > 0, \quad \text{for all} \quad n = 0, 1, \ldots, \infty, \quad (3.13) \]
is referred to as irreducible classically unitary continuous-spin field. To analyze (3.13) we use the Casimir operators (see (A.6),(A.7) in Appendix A) and labels \( p, q \) defined as
\[ E_0 = \frac{d}{2} + p, \quad s = \frac{2 - d}{2} + q, \quad (3.14) \]

where the labels \( p, q \) are complex-valued. It is the labels \( p, q \) that considerably simplify our analysis of equations (3.13) and make our study transparent. Plugging (3.14) into (A.6),(A.7), we find
\[ C_2 = p^2 + q^2 - \frac{(d-2)^2}{4}, \quad (3.15) \]
\[ C_4 = \left( p^2 - \frac{(d-2)^2}{4} \right) \left( q^2 - \frac{(d-2)^2}{4} \right). \quad (3.16) \]

In turn, plugging (3.15), (3.16) into (3.11), we find the following factorized forms of \( F_v \):
\[ F_v = (N_v + \frac{d-2}{2})^2 - p^2 \left( N_v + \frac{d-2}{2} - q^2 \right) \]
\[ = l_p l_p l_q l_{-q}, \quad l_X \equiv N_v + \frac{d-2}{2} + X. \quad (3.17) \]

In view of \( F_v(n) \equiv F_v|_{N_v=n} \), we note that the 1st equation in (3.13) amounts to the equations
\[ \Im(p^2 + q^2) = 0, \quad \Im(p^2 q^2) = 0. \quad (3.18) \]
All non-trivial solutions to restrictions (3.18) are well-known,

\begin{align}
\text{i} & : \Re p = 0, \Re q = 0; \quad \text{ii} : p^* = q; \quad \text{iii} : p^* = -q; \\
\text{iv} & : \Re p = 0, \Im q = 0; \quad \text{v} : \Im p = 0, \Re q = 0; \quad \text{vi} : \Im p = 0, \Im q = 0.
\end{align}

Using (3.17), and $f^\dagger = \bar{f}$, we see that all solutions to equation (3.10) corresponding to the respective cases in (3.19) can be presented as

\begin{align}
\text{i} & : \quad f_v = l_pl_q, \quad \bar{f}_v = \bar{l}_p\bar{l}_q, \quad \Re p = 0, \Re q = 0; \\
\text{ii, iii} & : \quad f_v = \bar{l}_p, \quad \bar{f}_v = \bar{l}_p q, \quad p^* = \pm q; \\
\text{iv} & : \quad f_v = l_p(l_q l_{-q})^{1/2}, \quad \bar{f}_v = \bar{l}_p(\bar{l}_q l_{-q})^{1/2}, \quad \Re p = 0, \Im q = 0; \\
\text{v} & : \quad f_v = l_q(l_p l_{-p})^{1/2}, \quad \bar{f}_v = \bar{l}_q(l_p l_{-p})^{1/2}, \quad \Im p = 0, \Re q = 0; \\
\text{vi} & : \quad f_v = (l_p l_{-p} l_q l_{-q})^{1/2}, \quad \bar{f}_v = (\bar{l}_p\bar{l}_q l_{-q})^{1/2}, \quad \Im p = 0, \Im q = 0.
\end{align}

The $f_v, \bar{f}_v$ in (3.20)-(3.23) are complex-valued, while $f_v, \bar{f}_v$ in (3.24) are real-valued. Therefore solutions in (3.20)-(3.23) are realized on complex-valued fields (3.1), while solution (3.24) can be realized on real-valued fields (3.1). Simple form of $f_v, \bar{f}_v$ (3.20)-(3.23) is a new result in this Section. The $p, q$ satisfy restrictions (3.20)-(3.24). For solutions iv, v, vi (3.22)-(3.24), equations (3.13) impose additional restrictions on $p, q$. We now analyse those additional restrictions in turn.

**Statement 1.** Solutions (3.20),(3.21) describe irreducible classically unitary continuous-spin fields.

**Statement 2.** Solutions (3.22),(3.23) respect equations (3.13) provided the $p, q$, besides restrictions in (3.22)-(3.24), satisfy the following additional restrictions:

\begin{align}
\text{iv} & : \quad q^2 < x_0; \\
\text{v} & : \quad p^2 < x_0; \\
\text{vi-a} & : \quad p^2 < x_0, \quad q^2 < x_0, \quad p^2 \neq q^2; \\
\text{vi-b} & : \quad x_n < p_n^2 < x_{n+1}, \quad x_n < q_n^2 < x_{n+1}, \quad p_n^2 \neq q_n^2, \quad n = 0, 1, \ldots, \infty; \\
\text{vi-c} & : \quad p^2 \neq x_n; \quad p^2 = q^2, \quad n = 0, 1, \ldots, \infty, \\
& \quad x_n \equiv (n + \frac{d - 2}{2})^2.
\end{align}

We see that, for solution in (3.24), there are three classes of additional restrictions (3.27)-(3.29) on the labels $p, q$. The Statements can easily be proved by using (3.10), (3.17). Relations (3.20)-(3.30) provide the complete description of all irreducible classically unitary continuous-spin fields. As reducible case provides interesting field content we now proceed with the discussion of reducible classically unitary continuous-spin fields.

**Reducible classically unitary continuous-spin field.** Continuous-spin field with $F_v$ (3.17) that satisfies the equations

\begin{align}
F_v(n_r) & = 0 \quad \text{for some } n_r \in 0, 1, \ldots, \infty, \\
\Im F_v(n) & = 0, \quad F_v(n) > 0 \quad \text{for all } n = 0, 1, \ldots, \infty \text{ and } n \neq n_r
\end{align}

is referred to as reducible classically unitary continuous-spin field.
Statement 3. Solutions (3.22)-(3.24) respect equations (3.31)-(3.32) provided the $p$, $q$, besides restrictions in (3.22)-(3.24), satisfy the following additional restrictions

\begin{align*}
\text{iv-1:} & \quad q^2 = x_0, \\
\text{v-1:} & \quad p^2 = x_0, \\
\text{vi-1a:} & \quad p^2 = x_0, \quad q^2 < x_1; \\
\text{vi-1b:} & \quad q^2 = x_0, \quad p^2 < x_1; \\
\text{vi-1c:} & \quad p_n^2 = x_n, \quad x_{n-1} < q_n^2 < x_{n+1}, \quad n = 1, \ldots, \infty; \\
\text{vi-1d:} & \quad q_n^2 = x_n, \quad x_{n-1} < p_n^2 < x_{n+1}, \quad n = 1, \ldots, \infty; \\
\text{vi-2a:} & \quad p_n^2 = x_n, \quad q_n^2 = x_{n+1}, \quad n = 0, 1, \ldots, \infty; \\
\text{vi-2b:} & \quad p_n^2 = x_{n+1}, \quad q_n^2 = x_n, \quad n = 0, 1, \ldots, \infty;
\end{align*}

where $x_n$ is given in (3.30). Relations (3.33)-(3.38) are associated with one root of $F_v(n)$ (3.31), while relations (3.39)-(3.40) are associated with two roots of $F_v(n)$ (3.31). The $f_v$, $\tilde{f}_v$ in (3.22)-(3.24) with the additional restrictions in (3.33)-(3.40) describe decoupled fields. Namely, decomposing ket-vector $|\phi\rangle$ (3.1) as

\begin{equation}
|\phi\rangle = |\phi^{0,0}\rangle + |\phi^{1,\infty}\rangle,
\end{equation}

for iv-1, v-1, vi-1a vi-1b; \quad (3.41)

\begin{equation}
|\phi\rangle = |\phi^{0,n}\rangle + |\phi^{n+1,1}\rangle,
\end{equation}

for vi-1c, vi-1d; \quad (3.42)

\begin{equation}
|\phi\rangle = |\phi^{0,n}\rangle + |\phi^{n+1,n+1}\rangle + |\phi^{n+2,\infty}\rangle,
\end{equation}

for vi-2a, vi-2b; \quad (3.43)

\begin{equation}
|\phi^{M,N}\rangle \equiv \sum_{n=M}^{N} \frac{v_n}{n!\sqrt{n!}}a^I_1 \ldots a^I_n \phi^{I_1 \ldots I_n}(x)|0\rangle,
\end{equation}

we can verify that action (2.1) is decomposed into direct sum of actions for fields appearing on r.h.s in (3.41)-(3.43). Field $|\phi^{0,n}\rangle$ in (3.41)-(3.43) is a spin-$n$ massive field, while field $|\phi^{n+1,n+1}\rangle$ in (3.43) is a spin-$(n + 1)$ massless field. The eigenvalue of $C_2$ for these fields is given by $C_2 = p^2 + q^2 - x_0 - x_1$. Square of mass of the spin-$n$ field is given by $m^2 = p^2 + q^2 - x_n - x_{n-1}$ when $n > 0$ and $m^2 = C_2$ when $n = 0$. Relations (3.22)-(3.24) and (3.33)-(3.40) provide the complete description of all reducible classically unitary continuous-spin fields.

Classically non-unitary reducible continuous-spin field and classically unitary irreducible partial continuous-spin field. In (3.42), (3.43), we see appearance of shortened fields $|\phi^{k+1,\infty}\rangle$. We refer to field $|\phi^{k,\infty}\rangle$ (3.44) with $k > 0$ as depth-$k$ partial continuous-spin field. In (3.42)-(3.43), classically unitary irreducible partial continuous-spin fields enter reducible classically unitary field $|\phi\rangle$ (3.1). We note however that classically unitary irreducible partial continuous-spin fields may enter a reducible classically non-unitary field $|\phi\rangle$. It is easy to understand that such cases can be obtained by considering the equations $\exists F_v(n) = 0$ for all $n = 0, 1, \ldots, \infty$ and

\begin{equation}
F_v(k-1) < 0, \quad F_v(k) = 0, \quad F_v(n) > 0, \quad n = k + 1, k + 2, \ldots, \infty,
\end{equation}

$k \geq 0$. The following Statement can easily be proved by using (3.10) and (3.17).

Statement 4. Solutions (3.22)-(3.24) respect equations (3.45) provided the $p$, $q$, besides restrictions in (3.22)-(3.24), satisfy the following additional restrictions:

\begin{equation}
\text{iv-nu-1:} \quad q_k^2 = x_k, \quad \text{for } k = 1, \ldots, \infty;
\end{equation}

\begin{equation}
\text{iv-nu-2:} \quad p_k^2 = x_k, \quad \text{for } k = 1, \ldots, \infty.
\end{equation}
\[ p_k^2 = x_k, \quad k = 1, \ldots, \infty; \]  
\[ \text{vi-nu-2a: } q_n^2 = x_k, \quad k - n > 1, \quad n, k = 0, 1, \ldots, \infty; \]  
\[ \text{vi-nu-2b: } q_n^2 = x_n, \quad k - n > 1, \quad n, k = 0, 1, \ldots, \infty; \]

where \( x_n \) is given in (3.30). Expressions for \( f_v, \bar{f}_v \) in (3.22)-(3.24) with the additional restrictions in (3.46)-(3.49) describe the following decoupled fields:

\[ |\phi\rangle = |\phi^{0, k}\rangle + |\phi^{k+1, \infty}\rangle, \quad \text{for iv-nu-1, vi-nu-1}; \]
\[ |\phi\rangle = |\phi^{0, n}\rangle + |\phi^{n+1, k}\rangle + |\phi^{k+1, \infty}\rangle, \quad \text{for vi-nu-2a, vi-nu-2b}; \]

where \( |\phi^{0, k}\rangle \) and \( |\phi^{0, n}\rangle \) are the respective classically non-unitary spin-\( k \) and unitary spin-\( n \) massive fields, while \( |\phi^{n+1, k}\rangle \) is a spin-\( k \) classically non-unitary partial-massless field. In (3.50), (3.51), the \( |\phi^{k+1, \infty}\rangle \) is the irreducible classically unitary partial continuous-spin field.

The following remarks are in order.

a) For solution (3.20), there are no restrictions on \( \Im p, \Im q \) (3.20). We note that the same happens for light-ray conformal operator. Namely, such operator is realized as unitary representation of the \( \text{so}(d, 2) \) algebra and labelled by conformal dimension \( \Delta = E_0 \) and continuous-spin \( s \) given in (3.14) with \( \Re p = 0, \Re q = 0 \) and no restrictions on \( \Im p, \Im q \). We recall that our solution (3.20) describes irreducible classically unitary continuous-spin field. We then conjecture that, in the framework of AdS/CFT correspondence, our continuous-spin AdS field with (3.20)-(3.23) is simpler than the one in (3.14) with \( \Re p = 0, \Re q = 0 \). Also we expect that continuous-spin AdS fields associated with the solutions (3.21)-(3.23) are also dual to the respective conformal operators having conformal dimension \( \Delta = E_0 \) and \( s \) as in (3.14).

b) Using decomposition (3.43) for \( n = 0, n = 1 \), we note two models of reducible classically unitary continuous-spin field with interesting spectrum for low-spin finite-component fields:

\[ |\phi\rangle = |\phi^{0, 0}\rangle_{\text{massive scalar}} + |\phi^{1, 1}\rangle_{\text{massless vector}} + |\phi^{2, \infty}\rangle_{\text{partial continuous spin}}, \]
\[ |\phi\rangle = |\phi^{0, 1}\rangle_{\text{massive vector}} + |\phi^{2, 2}\rangle_{\text{massless spin-2}} + |\phi^{3, \infty}\rangle_{\text{partial continuous spin}}. \]

Scalar field \( |\phi^{0, 0}\rangle \) (3.52) has mass \( m^2 = 0 \), i.e., \( m^2 \neq m_c^2 \), where \( m_c^2 \) stands for mass of conformal invariant scalar field, \( m_c^2 = (1 - d^2)/4 \). For this reason, we refer to scalar field \( |\phi^{0, 0}\rangle \) (3.52) as a massive field. The massive vector field \( |\phi^{0, 1}\rangle \) (3.53) has mass \( m^2 = 2d \).

c) By changing phases of complex-valued fields in (3.1), the complex-valued \( f_v, \bar{f}_v \) (3.20)-(3.23) can be cast into real-valued form. The real-valued form of \( f_v, \bar{f}_v \) (3.20)-(3.23) is presented as in (3.24), where \( p,q \) are given in (3.20)-(3.23). It is the real-valued form of \( f_v, \bar{f}_v \) (3.20)-(3.23) that we used in Ref. [16]. Our new complex-valued form of \( f_v, \bar{f}_v \) (3.20)-(3.23) is simpler that the one in (3.24). For example, \( f_v, \bar{f}_v \) (3.20) are degree-2 polynomials in \( N_v \) (3.2), while \( f_v, \bar{f}_v \) (3.24) involve a square root of degree-4 polynomials in the \( N_v \). Therefore it seems preferable to study the continuous-spin field by using \( f_v, \bar{f}_v \) given in (3.20)-(3.23). It is the use of complex-valued fields in (3.1) that allows us to introduce simple representations for solutions in (3.20)-(3.23).

Flat space. We use the chance to discuss new representation for operators \( f_v, \bar{f}_v \) entering continuous-spin fields in flat space. Light-cone gauge formulation of massless and massive continuous-spin fields in flat space \( R^{d-1,1} \), \( d \)-arbitrary, was obtained in Refs.[9] [12]. In the flat space \( R^{d-1,1} \), equation for \( f_v, \bar{f}_v \) (3.10) takes the form:

\[ f_v \bar{f}_v = F_v, \quad F_v = \kappa^2 - m^2 N_v (N_v + d - 3), \quad \kappa^2 > 0, \quad m^2 \leq 0, \]
where the mass parameter $m$ and the continuous-spin parameter $\kappa$ are related to 2nd- and 4th-order Casimir operators of the Poincaré algebra (for details see Ref.[12]). Restrictions on the $m$ and $\kappa$ \ref{3.54} are obtained by requiring the classical unitarity and irreducibility. In Ref.[12], we discussed solutions given by $f_\nu = \sqrt{F_\nu}$, $\bar{f}_\nu = \sqrt{F_{\bar{\nu}}}$. Such solution is realized on space of real-valued continuous-spin field. We now note a new solution given by

$$f_\nu = \sqrt{-m^2}(N_\nu + \frac{d-3}{2}) + i\sigma, \quad \bar{f}_\nu = \sqrt{-m^2}(N_{\bar{\nu}} + \frac{d-3}{2}) - i\sigma,$$

$$\sigma \equiv (\kappa^2 + \frac{(d-3)^2}{4}m^2)^{1/2}, \quad \text{for} \quad \sigma^2 \geq 0. \tag{3.55}$$

The new solution \ref{3.55} is realized on space of complex-valued continuous-spin field. Our new solution enters the spin operator of massive continuous-spin field as follows. In expression for $D$ and $\bar{D}$ solution given by \ref{3.54} are obtained by requiring the classical unitarity and irreducibility. In Ref.[12], we discussed the Casimir operators. Second, we discuss extension of our new representation for solutions given in \ref{2.20}-\ref{2.23} to the gauge invariant Lagrangian formulation.

### 4 Gauge invariant action for continuous-spin AdS field

Gauge invariant action for continuous-spin field was obtained in Ref.[16]. The gauge invariant action depends on two parameters. In this section, our aim is twofold. First, motivated by our result for light-cone gauge continuous-spin field we are going to express the two parameters in terms of the Casimir operators. Second, we discuss extension of our new representation for solutions given in \ref{2.20}-\ref{2.23} to the gauge invariant Lagrangian formulation.

Gauge invariant action is formulated in terms of ket-vector given by

$$|\phi\rangle = \sum_{n=0}^{\infty} \frac{\nu^n}{n!\sqrt{n!}} \alpha^{a_1} \cdots \alpha^{a_n} \phi^{a_1 \cdots a_n}(x) |0\rangle. \quad \tag{4.1}$$

We note that, in \ref{4.1}, fields with $n = 0$ and $n = 1$ are the respective scalar and vector fields of the Lorentz algebra $so(d,1)$, while fields with $n \geq 2$ are the totally symmetric tensor fields of the Lorentz algebra $so(d,1)$. Also, we note that, in \ref{4.1}, fields with $n \geq 4$ are considered to be double-traceless, $\phi^{a_1 a_2 a_3 \cdots a_n} = 0$. All fields in \ref{4.1} are taken to be complex-valued.

Gauge invariant action and Lagrangian of continuous-spin field we found can be presented as

$$S = \int d^{d+1}x \, \mathcal{L}, \quad \mathcal{L} = e \langle \phi | E | \phi \rangle, \quad \tag{4.2}$$

$$E \equiv (1 - \frac{1}{4} \alpha^2 \bar{\alpha}^2)(\Box_{AdS} + m_1 + m_2 \alpha^2 \bar{\alpha}^2) - L \bar{L}, \quad \tag{4.3}$$

$$L \equiv \bar{\alpha} D - \frac{1}{2} \alpha \bar{D} \alpha^2 - \bar{e}_1 \Pi^{[1,2]} + \frac{1}{2} \bar{e}_1 \alpha^2, \quad \tag{4.4}$$

$$\bar{L} \equiv \alpha \bar{D} - \frac{1}{2} \alpha^2 \bar{D} \bar{\alpha}^2 - e_1 \Pi^{[1,2]} + \frac{1}{2} e_1 \bar{\alpha}^2, \quad \tag{4.5}$$

$$\Pi^{[1,2]} \equiv 1 - \alpha^2 \frac{1}{2(2N_\alpha + d + 1)} \bar{\alpha}^2, \quad \tag{4.6}$$

\footnote{Our conventions are: $e = \det e^n_m$, where $e^n_m$ is vielbein in AdS space. $\Box_{AdS}$ is the D’Alembert operator in AdS space. Scalar products are given by $\alpha \bar{D} = \alpha^2 D^a, \bar{\alpha} D = \bar{\alpha}^2 D^a$, $\alpha^2 = \alpha^2 \alpha^a, \bar{\alpha}^2 = \bar{\alpha}^2 \bar{\alpha}^a$, $N_\alpha = \alpha^a \bar{\alpha}^a$ where $D^a = e^a_m D_m$ and $D_m$ is a covariant derivative in AdS. For more details of our notation, see Appendix A in Ref.[16].}
where \( \langle \phi \rangle \equiv (|\phi\rangle)^t \), while quantities \( m_1, m_2, e_1, \) and \( \bar{e}_1 \) are defined by the relations

\[
 m_1 = -\mu_0 + N_v(N_v + d - 1) + 2d - 4, \quad m_2 = -1, \quad (4.7)
\]

\[
 e_1 = N_v f_v \bar{v}, \quad \bar{e}_1 = -v N_v \bar{f}_v, \quad N_v \equiv \left( (N_v + 1)(2N_v + d - 1) \right)^{-1/2}, \quad (4.8)
\]

\[
 f_v \bar{f}_v = F_v, \quad f_v^3 = \bar{f}_v, \quad (4.9)
\]

\[
 F_v \equiv \mu_1 - (\mu_0 - d + 3)N_v(N_v + d - 2) + N_v^2(N_v + d - 2)^2, \quad (4.10)
\]

where \( \mu_0, \mu_1 \) stand for constant parameters. The quantity \( F_v \) appearing in (4.10) is the same as the one appearing in light-cone gauge approach in (3.11). Therefore comparing (4.10) and (3.11)-(3.12), we can entirely express the parameters \( \mu_0 \) and \( \mu_1 \) in terms of the eigenvalues of the 2nd and 4th-order Casimir operators

\[
 \mu_0 = C_2 + 2d - 4, \quad \mu_1 = C_4. \quad (4.11)
\]

Using (3.15), (3.16), and (4.11), we can represent \( F_v \) (4.10) as in (3.17). This implies that our whole analysis we carried out for classically (ir)reducible light-cone gauge continuous-spin field in Section 3 is automatically extended to the gauge invariant formulation in this Section. For example, all expressions for \( f_v, \bar{f}_v \) entering (4.7), (4.8) can read from (3.20)-(3.44). Solutions (3.20)-(3.23) are realized on complex-valued fields (4.1), while solution (3.24) can be realized on real-valued field in (4.1). Note that, in Ref. [16], we discuss solutions in the form given in (3.24).

Gauge symmetries are described by using ket-vector of gauge transformation parameters,

\[
 |\xi\rangle = \sum_{n=0}^{\infty} \frac{\Lambda^{n+1}}{n!\sqrt{(n+1)!}} \alpha^{a_1} \cdots \alpha^{a_n} \xi^{a_1 \cdots a_n}(x)|0\rangle. \quad (4.12)
\]

In (4.12), gauge parameters with \( n = 0 \) and \( n = 1 \) are the respective scalar and vector fields of the Lorentz algebra \( so(d, 1) \), while the gauge parameters with \( n \geq 2 \) are totally symmetric traceless tensor fields of the Lorentz \( so(d, 1) \) algebra. Gauge transformation parameters with \( n \geq 2 \) are taken to be traceless, \( \xi^{a_1 a_2 \cdots a_n} = 0 \). Use of ket-vectors \( |\phi\rangle \) and \( |\xi\rangle \) and operator \( e_1, \bar{e}_1 \) (4.8) allows us to write gauge transformations in the following form:

\[
 \delta |\phi\rangle = G |\xi\rangle, \quad G = \alpha D - e_1 - \alpha^2 \frac{1}{2N_\alpha + d - 1} \bar{e}_1. \quad (4.13)
\]

To summarize, in this paper, we developed further the general light-cone formulation in Ref. [33][34] and applied it for studying free continuous-spin field in AdS space. Extension of our approach to interacting continuous-spin AdS field could be of great interest. In this respect we note that, using methods in Ref. [36], we studied interacting vertices of light-cone gauge continuous-spin field in flat space in Refs. [9][12], while, in Ref. [37], we developed method for studying finite-component light-cone gauge AdS fields. We believe therefore that results in this paper and the ones in Refs. [9][12][37] will be helpful for studying interacting continuous-spin AdS fields. For reader convenience we note that various BRST methods for studying interacting finite-component fields may be found in Refs. [38][39]. Other interesting methods for investigation of interacting finite-component fields were developed in Refs. [40][41]. Study of continuous-spin field along the line of group-theoretical methods in Refs. [42] could also be of some interest.

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Appendix A  Casimir operators of the $so(d,2)$ algebra

In this Appendix, we explain our conventions for Casimir operators of the $so(d,2)$ algebra. To this end we start with a manifestly $(d + 2)$-dimensional covariant approach. Generators of the $so(d,2)$ algebra denoted by $J^{AB}$ satisfy the commutators

$$[J^{AB}, J^{CE}] = \eta^{BC} J^{AE} + \text{3 terms}, \quad J^{AB\dagger} = -J^{AB}, \quad \eta^{AB} = (-, +, \ldots, +), \quad (A.1)$$

where vector indices of the $so(d,2)$ algebra take values $A, B, C, E = 0', 0, 1, 2, \ldots, d$. In terms of the $J^{AB}$, the 2nd- and 4th-order Casimir operators of the $so(d,2)$ algebra are defined to be

$$C_2 = \frac{1}{2} J^{AB} J^{BA}, \quad C_4 = \frac{1}{2} C_2^2 + \frac{d(d-1)}{4} C_2 - \frac{1}{4} J^{AB} J^{BC} J^{CE} J^{EA}. \quad (A.2)$$

In Ref.[33], we shown that the first relation in (A.2) allows us to find the relation for the 4th-order Casimir operator $C_4$ (A.2) takes the form given in (2.18). For doing so, we should relate generators in (A.1) with the light-cone generators (2.3)-(2.10). To this end we decompose $(d + 2)$ coordinates $x^A$ as

$$x^A = x^\oplus, x^\ominus, x^a, \quad a = 0, 1, 2, \ldots, d - 1, \quad x^\oplus \equiv \frac{1}{\sqrt{2}}(x^d + x^{0'}) \equiv \frac{1}{\sqrt{2}}(x^d - x^{0'}). \quad (A.3)$$

In the frame of the coordinates $x^\oplus, x^\ominus, x^a$, the $J^{AB}$ and $\eta^{AB}$ (A.1) are represented as

$$J^{AB} = J^{\oplus\oplus} J^{\ominus\ominus}, J^{\ominus\oplus}, J^{\ominus\ominus}, J^{ab}, \quad \eta^{AB} = \eta^{\oplus\oplus} \eta^{\ominus\ominus}, \eta^{\ominus\oplus}, \eta^{ab}, \quad (A.4)$$

where $\eta^{\oplus\oplus} = 1, \eta^{\ominus\ominus} = 1$. Now, using notation of the conformal algebra considered in the base of the algebra $so(d - 1, 1)$ spanned by $J^{ab}$, we identify generators (A.4) as:

$$P^a = J^{\oplus a}, \quad K^a = J^{\ominus a}, \quad D = J^{\ominus\ominus}. \quad (A.5)$$

Generators (A.5) and $J^{ab}$ satisfy the commutators given in (A6),(A7) in Ref.[35], while relation (A8) in Ref.[35] provides the light-cone gauge decomposition of the generators (A.5). Now using (A.2) and relations (2.3)-(2.10), we verified that $C_4$ in (A.2) amounts to $C_4$ in (2.18).

To parametrize eigenvalues of the Casimir operators for totally symmetric representations of the $so(d,2)$ algebra we use labels $E_0, s$. In conformal algebra notation, $E_0 \equiv \Delta$, where $\Delta$ is conformal dimension of a conformal operator, while $s$ is associated with spin. In general, the $E_0$ and $s$ are complex-valued. Eigenvalues of the Casimir operators (A.2) are given by

$$C_2 = E_0(E_0 - d) + s(s + d - 2), \quad (A.6)$$

$$C_4 = (E_0 - 1)(E_0 - d + 1)s(s + d - 2). \quad (A.7)$$

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