Numerical simulation of solitary waves on deep water with constant vorticity

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Abstract. Characteristics of solitary deep water waves on a flow with constant vorticity are investigated by numerical simulation within the framework of fully nonlinear equations of motion (Euler equations) using the method of surface-tracking conformal coordinates. To ensure that solutions observed are stable, soliton formation as a result of disintegration of an initial pulse-like disturbance is modeled. Evidence is obtained that solitary waves with height above a certain threshold are unstable.

1. Introduction
In theoretical studies of water waves it is common to consider the fluid motion as purely irrotational, yet wave dynamics can be strongly affected by interaction with shear currents. In nature shear is generated near the water surface due to the action of surface wind stress, or near the bed of a river or of the sea. An important special case is that of a linear shear flow, i.e. a flow with a uniform vorticity distribution, because in such a system only potential part of velocity field is time-dependent, and usual techniques developed for potential waves can be applied to study its dynamics.

Periodic stationary waves with constant vorticity were studied within a deep water setting in, for example, [1], periodic waves on water of finite depth in [2], and solitary waves on finite depth in [3]. These works describe a spectacular phenomenon of steep bulb-like waves that can propagate along the current. In the present paper we confine our attention to deep water waves propagating against a linear shear current. It is known that long waves propagating against the current are weakly dispersive, and according to weakly nonlinear theory on deep water their evolution is described by Benjamin-Ono equation [4], which has Lorentzian-shaped soliton solutions. The main aim of our work is to ensure stability of obtained solitary waves. To this end, we model soliton formation as a result of disintegration of an initial pulse-like disturbance.

To simulate a two-dimensional flow with constant vorticity one can employ the same efficient technique based upon use of conformal coordinates that was developed for potential waves. Exact equations of motion in conformal variables were derived in [5]. Here we use a slightly different formulation, similar to [6], which was obtained in [7] and applied there by the authors to modelling of dynamics and modulational instability of oscillatory nonlinear waves.
2. Basic equations

Let a two-dimensional ideal fluid flow occur on a current of depth $H_0$ which has a free surface and a constant vorticity. We consider the waves on the water surface which have the lengths much less than $H_0$, thus, for this waves the deep water condition is fulfilled and the fluid layer can be considered to be infinitely deep. Let the ideal incompressible fluid occupy the domain which is bounded by the free surface $F(x,y,t) = 0$ (where $t > 0$ is time) in the $(x,y)$ plane. The fluid flow velocity vector is $\mathbf{v} = \{v_x, v_y, 0\}$. Cartesian velocity components are represented through effective potential $\phi = \phi(x,y,t)$ and its conjugate harmonic function $\Phi = \Phi(x,y,t)$ as

$$\begin{align*}
v_x &= \frac{\partial \phi}{\partial x} - \omega y = \frac{\partial \Phi}{\partial y} - \omega y, \\
v_y &= \frac{\partial \phi}{\partial y} = -\frac{\partial \Phi}{\partial x},
\end{align*}$$

(1)

where $\omega$ is a constant vorticity. Taking into account relations (1) from the condition of incompressibility of the fluid ($\nabla \cdot \mathbf{v} = 0$) it follows that $\phi$ satisfies the Laplace equation:

$$\Delta \phi = 0.$$  

(2)

It is necessary to find the solution of the Laplace equation (2) with the boundary conditions

$$\begin{align*}
\left[ \frac{\partial \phi}{\partial t} + \frac{1}{2} \left( \nabla \phi - \omega y \cdot \mathbf{x}_0 \right)^2 + \omega \left( \Phi - \frac{\omega y^2}{2} \right) + \frac{p}{\rho} + gy = 0 \right] F(x,y,t) = 0,
\end{align*}$$

(3)

$$\begin{align*}
\left[ \frac{\partial F}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial F}{\partial \xi} + \frac{\partial \phi}{\partial y} \frac{\partial F}{\partial \eta} - \omega y \frac{\partial F}{\partial \xi} \right] F(x,y,t) = 0,
\end{align*}$$

(4)

$$\phi \bigg|_{y \to -\infty} = 0, \quad p \bigg|_{F(x,y,t) = 0} = p_0.$$  

(5)

Let an incompressible fluid occupy a domain $G$: $G = \{ -\infty < x < +\infty, -\infty < y \leq f(x,t) \}$ in the $(x,y)$ plane. We consider a complex function $z = z(\xi,\eta,t) = x(\xi,\eta,t) + iy(\xi,\eta,t)$ which is analytic in the domain $G$. We carry out a conformal mapping of the domain $G$ onto the half-plane $-\infty < \xi < +\infty, \quad -\infty < \eta \leq 0$ in the plane $\zeta = \xi + i\eta$ by means of the function $z(\xi,\eta,t) = \xi + i\eta + \tilde{z}(\xi,\eta,t) = \xi + i\eta + \tilde{x}(\xi,\eta,t) + i\tilde{y}(\xi,\eta,t)$. The functions $\tilde{x}(\xi,\eta,t)$, $\tilde{y}(\xi,\eta,t)$ are harmonic in the half-plane $\text{Im} \zeta \leq 0$ and satisfy the Cauchy-Riemann conditions in this half-plane. The functions $y$ and $\tilde{x}$ are connected by the relation $y = H(\tilde{x})$, where $H$ is the Hilbert transform. The Jacobian $J = x_\xi^2 + y_\xi^2 \neq 0$. The functions $\phi(\xi,\eta,t), \Phi(\xi,\eta,t)$ are harmonic in the half-plane $\text{Im} \zeta \leq 0$ and satisfy the Cauchy-Riemann conditions in this half-plane. The “complex potential” $W = \phi + i\Phi$ is an analytic function in the half-plane $\text{Im} \zeta \leq 0$, and $W = \phi + iH(\phi)$ on the real axis. As a result of the conformal mapping, equations (3) and (4) take the form (at $\eta = 0$ and $p_0 = 0$):

$$z \bigg|_{\eta = 0} = -iz \left( 1 + iH \left( \frac{\Phi_\xi - \omega \eta y_\xi}{J} \right) \right) \bigg|_{\eta = 0} = 0,$$

(6)

$$\begin{align*}
\frac{\partial z}{\partial t} + gy + \omega \Phi = \phi \bigg|_{\eta = 0} = \frac{\phi_\xi}{J} \bigg( \frac{\Phi_\xi - \omega \eta y_\xi}{J} \bigg) \bigg|_{\eta = 0} = 0.
\end{align*}$$

(7)

We introduce the new functions $z(\theta,t) = x(\theta,t) + iy(\theta,t)$, $\theta = \xi + i\eta$, $R(\theta,t), V(\theta,t)$, using the formulas [(6)]: $R = \frac{1}{z_{\theta}}$, $V = iz \frac{W_{\theta}}{z_{\theta}}$. The functions $R(\theta,t), V(\theta,t)$ are analytic in the lower half-plane and satisfy the boundary conditions: $V(\theta,t) \to 0$, $|\theta| \to \infty$, $\text{Im} \theta \leq 0$; $R(\theta,t) \to 1$, $|\theta| \to \infty$, $\text{Im} \theta \leq 0$. 


The functions $R(\theta,t), V(\theta,t)$ satisfy the following system of integro-differential equations (see [7]):

\[
\begin{align*}
R_t &= i(R\theta U - RU\theta) \\
V_t &= i(\theta U V - R P(\theta^2 - 2\omega \Im z \Im V)) + g(R-1) + i\omega V
\end{align*}
\]  

(8)

where $U = P[V^* R^* + V R] - 2\omega P(\Im z \cdot \Im R)$, $P$ is the projection operator: $P(f) = 1/(1+iH)f$ and prime denotes the derivative with respect to the complex variable $\theta$.

Equations in conformal variables (6)-(7) or in the Dyachenko variables (8) only contain integro-differential operators that are local in Fourier representation, which allows for very efficient computation using Fast Fourier Transform. In all numerical simulations described here periodic boundary conditions are implied.

3. **Stationary waves**

Let’s consider a solution to (6)-(7) in a form of a stationary wave, that is, a wave propagating with constant phase velocity $c$ without changing its form:

\[
\begin{align*}
x &= \xi + \tilde{x}(\xi - ct) , \quad y = \eta(\xi - ct) , \quad \phi = \tilde{\phi}(\xi - ct) - gb_0 t.
\end{align*}
\]

Substituting (9) into (6)-(7) one can obtain an equation for stationary wave profile [9]:

\[
gy + \left( c + \omega\left( H_{yy} + yx_x \right) \right)^2 - \frac{1}{2} c^2 = gb_0 .
\]

Equations of motion (2)-(5) contain only two dimensional parameters $g$ and $\omega$, and we can eliminate them by changing to appropriate time and length scales

\[
X = x/\lambda_g, Y = y/\lambda_g, T = |\omega| t,
\]

to obtain a dimensionless system. It means that dimensionless stationary wave profiles are universal and can only depend on dimensionless phase velocity $C = c/c_0$ and period $A = \lambda/\lambda_g$. Profiles of solitary waves, which can be considered as waves with infinite period, constitute therefore a one-parametric family of solutions, and all their dimensionless characteristics (say, wave height $a/\lambda_g$ or velocity $c/c_0$) can be found as a function of single parameter.

To solve (10) numerically we have generalized an iterative method from [8] to the case of nonzero vorticity. Let’s rewrite (10) as $gy + M^2 J/2 = q$, $M = (c + \omega (H_{yy} + yx_x))^2$. The iterative method is based on the observation that not only Jacobian $J = x^2_y + y^2_x$ can be computed for a given mapping $z(\theta)$, but also mapping $z(\theta)$ can be “recovered” from its Jacobian by the following manipulations:

\[
h = (1+iH) \ln J/2 , \quad z = \partial_{\xi}^{-1} \exp h .
\]

To obtain a numerical solution of (10) for chosen $\lambda$ and $c$, we initialize the procedure with some seed $J_{[0]} = e^{\cos(2\pi/\lambda)}$, and then perform in a loop:

1. $h_{[n]} = (1+iH) \left( \frac{1}{2} \ln J_{[n]} \right)$
2. $z_{[n]} = \exp h_{[n]}$ and zeroth harmonic of $z_{[n]}$ is set to 1, then $z_{[n]} := \partial_{\xi}^{-1} z_{[n]}$ and zeroth harmonic of $z_{[n]}$ is adjusted so that $\int yx_x \, d\xi = 0$ holds,
3. $q_{[n]} := g \sum_{[n]} y_{[n]} (\lambda/2 - y_{[n]}(0)) / s_{[n]} - 1$, where $s = M(0)f(\lambda/2) / M(\lambda/2)f(0)^{1/2}$.
4. \[ J_{n+1} = -\ln \frac{2(g[y_n] - g[y_{n+1}])}{M[n]} \]

If the sequence \( z[n] \) converges it allows us to obtain the required wave profile with a good precision.

Figure 1 shows example profiles for various wavelengths, with wave height chosen to be close to the limiting value for each given wavelength. It is clear that wave crest becomes more localized with increase of wave period. However, as figure 2 shows, the domain in the parameter space where we were able to get a solution is rather limited. It therefore appears impossible to study much longer \( \lambda > 10^2 \lambda_g \) and steeper waves with that method.

![Figure 1. Example stationary wave profiles for various wavelengths (from the top down, lowest points of all profiles are aligned): 10\( \lambda_g \), 20\( \lambda_g \), 40\( \lambda_g \), 80\( \lambda_g \).](image1)

![Figure 2. Domain in the parameter space where the iterative method for solving stationary wave equation converges. The lower boundary of the domain corresponds to the linear dispersion relation.](image2)

4. **Initial disturbance disintegration**

To observe formation of solitons we modeled evolution of initial pulse-like disturbances

\[ y(x) = \frac{a}{1 + x^2 / d^2} w(x) \]  

within the framework of equations (8). Window function \( w(x) = 1 / 2 + \cos(2\pi x / L) / 2 \) is used here to smooth the jump of derivatives on the ends of the global period. Global period \( L \) is chosen to be at least \( 10^3 \) times greater than \( d \). A pulse with width \( d \) and height \( a \) satisfying \( ad = 4 \lambda_g^2 \) would correspond to a soliton of Benjamin-Ono equation for our system, but we have also run a few simulations with \( d \) and \( a \) not satisfying that constraint.

Figure 3 shows characteristics of solitons as functions of their height. Figure 4 compares exact solitons with solitons of Benjamin-Ono equation having same height and asymptotic. A noteworthy feature of these results is the fact that despite a wide range of initial parameters tested in our simulations, no waves higher than \( \approx 0.42 \lambda_g \) has been formed. Together with the evidence obtained below that there may exist a solution to (10) with greater height, it may indicate that solitary waves of height greater than the \( \approx 0.42 \lambda_g \) threshold are unstable. Other possible explanations include effects of the periodic boundary conditions and insufficient precision of the numerical model.

As we found out by conducting the simulations, even for the initial conditions that are close to the soliton of the Benjamin-Ono equation, it can be difficult to predict the characteristics of the resulting wave. To improve this, we can prescribe the initial conditions as a sum
\[ y = f(x) = \sum_{i} \frac{a_i}{1 + x^2/d_i^2} w(x) \]  

(12)

which allows us to fit both a sharp crest of an exact soliton and its \( y(x) \sim 1/x^2 \) tails. Then for a given vector of parameters \( \{a_i, d_i\} \), having found a conformal parameterization \( z(\xi) \) of the profile (12), we can compute the residual of stationary wave equation (10):

\[ r(a_i, \ldots, d_1, \ldots) = g y + \left( c + \omega \left( x \frac{\partial z}{\partial \xi} + y \frac{\partial z}{\partial \eta} \right) \right)^2 - \frac{c^2}{2} \]

and use it as a criterion \( \int |r|^2 d\xi \rightarrow \min \) for finding optimal \( \{a_i, d_i\} \). Characteristics of the obtained profiles are in good agreement with the results of numerical simulations as is shown at figure 3. We can formally carry out this procedure to obtain profiles even higher than \( 0.42 \lambda_g \). As a numerical experiment with an approximate profile of initial height \( a = 0.47 \lambda_g \) has shown, height and width of such initial disturbance remain almost unchanged for quite a long time (\( \omega t \sim 2 \cdot 10^4 \)), but then the wave quickly collapses.

5. Conclusion

In the present work we have studied the formation of solitons from an initial disturbance and compared their characteristics to the values predicted by weakly nonlinear theory. Results of the simulations indicate that solitary waves on deep water with constant vorticity may have an amplitude threshold above which they become unstable.

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Figure 3. Characteristics of solitons as functions of height, a) width, b) crest curvature, c) velocity. Dashed line – solitons of Benjamin-Ono equation, black markers – solitons obtained from the numerical simulations, solid line – approximate solutions of the stationary wave equation (10).

Figure 4. Comparison between exact solitons (solid lines) and solitons of the Benjamin-Ono equation (dashed lines)