A BIGRADED VERSION OF
THE WEIL ALGEBRA AND OF
THE WEIL HOMOMORPHISM
FOR DONALDSON INVARIANTS

Elementary algebra and cohomology
behind the Baulieu-Singer approach
to Witten’s topological Yang-Mills
quantum field theory

Michel DUBOIS-VIOLETTE

Laboratoire de Physique Théorique
Bâtiment 211, Université Paris XI
91405 ORSAY Cedex
E-mail: flad@qcd.th.u-psud.fr
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Abstract
We describe a bigraded generalization of the Weil algebra, of its
basis and of the characteristic homomorphism which besides ordinary
characteristic classes also maps on Donaldson invariants.

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1Laboratoire associé au C.N.R.S.
Introduction

In [1], a small differential algebra was introduced in connection with gauge fixing à la B.R.S., [2], in Witten’s topological quantum field theory [11]. Furthermore, in the last part of [1], §5, this algebra was identified with a differential subalgebra of the algebra of differential forms on the product $P \times C$ of a principal bundle $P$ over a manifold $M$ with the space $C$ of principal connections of $P$. By a slight “abstraction” one can produce an algebra $\mathfrak{A}$ which is a free bigraded-commutative differential algebra which admits the above quoted differential algebra (of [1]) as homomorphic image. This algebra, which is described in part I of the present paper, is obviously a bigraded version of the Weil algebra, [10] (see also in [4] and [7]) ; it is a contractible algebra. Therefore the various cohomologies of $\mathfrak{A}$ as well as the ones of its images are trivial. It was claimed in [9] that the relevant cohomologies where the basic cohomologies for operations in the sense of H. Cartan [4],[7]. Actually this is also implicit in [1] since there, it is claimed, (in §5), that the relevant cohomology is the de Rham cohomology of $M \times C/\mathcal{G}$ (and not the one of $P \times C$) where $\mathcal{G}$ is the group of gauge transformations of $P$. It is the aim of this paper to show that, by using $\mathfrak{A}$ as generalization of the Weil algebra, one can generalize the Weil homomorphism in such a way that beside the usual characteristic classes, it makes contact with the Donaldson invariants [5]. That there is connection between the usual characteristic classes and the Donaldson invariant is natural in view of the apparences of the Donaldson-Witten cocycles [11] which are bigraded expansions of the Chern-Weil cocycles. It is worth noticing here that $\mathfrak{A}$ is a generalization of the Weil algebra based on the finite dimensional Lie algebra of the structure group of $P$ and it not the one based on the infinite dimensional Lie algebra
of the group of gauge transformations of $P$, [8]; in particular, $\mathfrak{A}$ is finitely generated.

The plan of the paper is the following. In part I we describe the bigraded generalization of the Weil algebra and its basis and we compute its various basic cohomologies. In part II we describe the corresponding appropriate generalization of the characteristic homomorphism. One can remark that the independence of the metric in [1] follows from the more general fact proved here that the characteristic homomorphism does not depend on the chosen connection of the space of connection over $P$ (the structure group being there the gauge group). For a review on topological quantum field theory as well as for detailed references on the subject we refer to the Physics Report [3].

We apologize on the fact that we denote by $\alpha$ or more precisely by $\alpha$ what is almost everywhere in the literature denoted by $c$, e.g. [1,3].
A UNIVERSAL MODEL
Generalisation of the Weil algebra

1 Description of the model

Let \( g \) be a finite dimensional Lie algebra and let \((E_i)\) be a basis of \( g \). Consider eight copies \( g^* \otimes A^i, g^* \otimes F^i, g^* \otimes \alpha^i, g^* \otimes \varphi^i, g^* \otimes \psi^i, g^* \otimes \xi^i, g^* \otimes B^i \) and \( g^* \otimes \beta^i \) of the dual \( g^* \) of \( g \) with dual basis respectively denoted by \((A^i), (F^i), (\alpha^i), (\varphi^i), (\psi^i), (\xi^i), (B^i)\) and \((\beta^i)\).

Let \( \mathfrak{A}(g) \), (or simply \( \mathfrak{A} \)), be the free graded commutative algebra generated by the \( A^i \) and \( \alpha^i \) in degree one, the \( F^i, \varphi^i, \psi^i \) and \( \xi^i \) in degree two and the \( B^i \) and \( \beta^i \) in degree three. On the space \( g \otimes \mathfrak{A} \), there is a natural bracket \([\cdot, \cdot]\) defined by \([X \otimes P, Y \otimes Q] = [X, Y] \otimes P \cdot Q\), for \( X, Y \in g \) and \( P, Q \in \mathfrak{A} \).

For any linear mapping \( R \) of \( \mathfrak{A} \) in itself one defines a linear mapping, again denoted by \( R \), of \( g \otimes \mathfrak{A} \) in itself by \( R(X \otimes P) = X \otimes R(P) \) for \( X \in g \) and \( P \in \mathfrak{A} \). Let us introduce the following elements of \( g \otimes \mathfrak{A} \) associated to the generators of \( \mathfrak{A} \): \( A = E_i \otimes A^i \), \( F = E_i \otimes F^i \), \( \alpha = E_i \otimes \alpha^i \), \( \varphi = E_i \otimes \varphi^i \), \( \psi = E_i \otimes \psi^i \), \( \xi = E_i \otimes \xi^i \), \( B = E_i \otimes B^i \) and \( \beta = E_i \otimes \beta^i \). With the above notations, one has the following lemma.

**LEMMA 1.1** There are unique antiderivations \( d \) and \( \delta \) of \( \mathfrak{A} \) satisfying \( d^2 = 0, \delta^2 = 0 \) and \( d\delta + \delta d = 0 \) such that \( F = dA + \frac{1}{2}[A, A], \varphi = \delta \alpha + \frac{1}{2}[\alpha, \alpha], \psi = \delta A + d\alpha + [A, \alpha], \xi = \delta A, B = \delta F + [\alpha, F] \) and \( \beta = d\varphi + [A, \varphi] \).

**Proof.** By a change of generators, \( \mathfrak{A} \) is the free graded commutative algebra generated by the \( A^i \) and \( \alpha^i \) in degree one, \( dA^i, \delta A^i, d\alpha^i, \delta \alpha^i \) in degree two and
\( \delta dA^i, d\delta \alpha^i \) in degree three. The lemma is thus obvious and \( \mathfrak{A} \) is contractible for \( d \) and for \( \delta \). □

One defines an underlying bigraduation on \( \mathfrak{A} \) by giving to the \( A^i \) the bidegree (1,0), to the \( F^i \) the bidegree (2,0), to the \( \alpha^i \) the bidegree (0,1), to the \( \varphi^i \) the bidegree (0,2), to the \( \psi^i \) and the \( \xi^i \) the bidegree (1,1), to the \( B^i \) the bidegree (2,1) and to the \( \beta^i \) the bidegree (1,2). \( \mathfrak{A} \) is then a bigraded differential algebra with total differential \( d + \delta \), \( d \) being the part of bidegree (1,0) and \( \delta \) the part of bidegree (0,1). Since \( \mathfrak{A} \) is contractible for each of these differentials its cohomologies are trivial (\( H^{k,\ell}(\mathfrak{A}, \mathfrak{d}) = 0 \) if \( k + \ell > 0 \), \( H^{0,0}(\mathfrak{A}, \mathfrak{d}) \cong \mathbb{C} \), etc.).

The interest of this bigraded differential algebra lies in the following universal property which is closely related to the triviality of its cohomology.

**Lemma 1.2** Let \( \Omega \) be a bigraded commutative differential algebra with differential \( d + \delta \), \( d \) of bidegree (1,0) and \( \delta \) of bidegree (0,1). Let \( A_i^i = E_i \otimes A^i_i \) be an element of \( \mathfrak{g} \otimes \mathfrak{A}^i \) and \( \alpha_i^i = E_i \otimes \alpha^i_i \) be an element of \( \mathfrak{g} \otimes \alpha^i \). Then there is a unique homomorphism of bigraded differential algebras \( h : \mathfrak{A} \rightarrow \mathfrak{A} \) such that \( h(A^i_i) = A^i_i \) and \( h(\alpha^i_i) = \alpha^i_i \), (in short \( h(A_i^i) = A^i_i, h(\alpha_i^i) = \alpha_i^i \)).

Notice that \( \mathfrak{A} \) contains two Weil algebras namely \( W_d(\mathfrak{g}) = (\mathfrak{g}^* \otimes \mathfrak{g}^*_d, \mathfrak{d}) \) and \( W_\delta(\mathfrak{g}) = (\mathfrak{g}^* \otimes \mathfrak{g}^*_\mathfrak{d}, \delta) \). The first one, \( W_d(\mathfrak{g}) \), will play in the sequel the role of the Weil algebra (see in part II); its base (i.e. basic subalgebra) consists of the \( \mathcal{I}(\mathcal{F}) \) where \( \mathcal{I} \) runs over all \( ad^*(\mathfrak{g}) \)-invariant elements of \( S\mathfrak{g}^* \) (i.e. invariant polynomials on \( \mathfrak{g} \)). Let \( \mathfrak{B}(\mathfrak{g}) \), (or simply \( \mathfrak{B} \)), be the set of elements of \( \mathfrak{A} \) of the form \( \mathcal{I}(\mathcal{F}, \psi, \varphi, B, \beta) \) where \( \mathcal{I} \) runs over all \( ad^*(\mathfrak{g}) \)-invariant elements of \( S\mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \otimes \mathfrak{g}^* \), (invariant “polynomials” in \( F, \psi, \varphi, B \) and \( \beta \)). \( \mathfrak{B} \) is a bigraded subalgebra of \( \mathfrak{A} \) which is easily shown to be stable by \( d \) and by \( \delta \). For reasons which will become clear in part II, the bigraded differential algebra \( \mathfrak{B} \) will be called the **base of** \( \mathfrak{A} \).
\( \mathfrak{B} \) is in fact the set of elements which are basic for two operations of \( \mathfrak{g} \) in \( \mathfrak{A} \) which we now describe. One defines first an operation of \( \mathfrak{g} \) in \( \mathfrak{A} \), \( X \mapsto i_X \), which extends the usual operation of \( \mathfrak{g} \) in the Weil algebra \( W_d(\mathfrak{g}) \) by setting, for \( X \in \mathfrak{g} \), \( i_X(A) = X \) and \( i_X(\text{other generators})=0 \) which implies \( i_X\delta + \delta i_X = 0 \) and, with \( L_X = i_Xd + di_X \), \( L_X(Z) = [Z, X] \) for all generators \( Z = A, F, \alpha, \varphi, \psi, \xi, B, \beta \). By interchanging the bidegrees, one defines similarly an other operation of \( \mathfrak{g} \) in \( \mathfrak{A} \), \( X \mapsto i'_X \), which extends the one of \( W_\delta(\mathfrak{g}) \).

Namely one sets, for \( X \in \mathfrak{g} \), \( i'_X(\alpha) = X, i'_X(A) = i'_X(F) = i'_X(\varphi) = i'_X(\psi) = i'_X(B) = i'_X(\beta) = 0 \) and \( i'_X(d\alpha) = 0 \) (notice that then \( i'_X(\delta A) = i'_X(\xi) \neq 0 \)); this implies \( i'_Xd + di'_X = 0 \) and, with \( L'_X = i'_X\delta + \delta i'_X \), \( L'_X(Z) = [Z, X] \) for all generators \( Z = A, F, \alpha, \varphi, \psi, \xi, B, \beta \). One verifies that one has:

\[
\mathfrak{B} = \cap\{\ker(i_X) \cap \ker(L_\mathfrak{g}) \cap \ker(i'_{X'}) \cap \ker(L'_{\mathfrak{g}'}), \mathfrak{X}, \mathfrak{Y}, \mathfrak{X}', \mathfrak{Y}' \in \mathfrak{g}\}.
\]

However, in part II, only the first operation will be mapped homorphically on a similar operation of \( \mathfrak{g} \). Nevertheless \( \mathfrak{B} \) will be mapped into a basis of two operations.

In constrast to the case of \( \mathfrak{A} \), the various cohomologies of \( \mathfrak{B} \) are non-trivial and it is the aim of the next sections of part I to compute these cohomologies.

2 The \( d \) and the \( \delta \) cohomologies of \( \mathfrak{B}(\mathfrak{g}) \).

Let \( d_1, d_2, \delta_1 \) and \( \delta_2 \) be the unique antiderivations of \( \mathfrak{A} \) satisfying:

\[
d_1\psi = -B, d_1\varphi = \beta, d_1(\text{other generators}) = 0;
\]

\[
d_2B = [F, \psi], d_2\beta = [F, \varphi], d_2(\text{other generators}) = 0;
\]

\[
\delta_1F = B, \delta_1\psi = -\beta, \delta_1(\text{other generators}) = 0;
\]

\[
\delta_2B = [\varphi, F], \delta_2\beta = -[\varphi, \psi], \delta_2(\text{other generators}) = 0.
\]
\( \mathbb{B} \) is stable by \( d_1, d_2, \delta_1 \) and \( \delta_2 \) and one has the following lemma.

**Lemma 2.1** The restriction of \( d \) to \( \mathbb{B} \) coincides with the restriction of \( d_1 + d_2 \) and the restriction of \( \delta \) to \( \mathbb{B} \) coincides with the restriction of \( \delta_1 + \delta_2 \).

**Proof.** One has \( (d + \delta)(A + \alpha) + \frac{1}{2}[A + \alpha, A + \alpha] = F + \psi + \varphi \) which implies \( (d + \delta)(F + \psi + \varphi) + [A + \alpha, F + \psi + \varphi] = 0 \). By taking the homogeneous components and by using the definitions, one obtains:

\[
\begin{align*}
&dF + [A, F] = 0 \\
&d\psi + [A, \psi] = -B \\
&d\varphi + [A, \varphi] = \beta \\
&dB + [A, B] = -[F, \psi] \\
&d\beta + [A, \beta] = [F, \varphi]
\end{align*}
\]

Thus the lemma would be obvious without the terms \([A, \cdot] \) and the terms \([\alpha, \cdot] \). However if \( \mathcal{I}(F, \psi, \varphi, B, \beta) \in \mathbb{B} \), it follows from the invariance of \( \mathcal{I} \) that one can enter \( d \) in \( \mathcal{I} \) by replacing it everywhere by \( d + [A, \cdot] \) and that one can enter \( \delta \) in \( \mathcal{I} \) by replacing it everywhere by \( \delta + [\alpha, \cdot] \). This implies the result. \( \square \)

**Lemma 2.2** Let \( d'_1 \) and \( \delta'_1 \) be the unique antiderivations of \( \mathfrak{A} \) satisfying:

\[
d'_1 B = -\psi, \quad d'_1 \beta = \varphi, \quad d'_1 (other \ generators) = 0
\]

and

\[
\delta'_1 B = F, \quad \delta'_1 \beta = -\psi, \quad \delta'_1 (other \ generators) = 0.
\]

Then one has on \( \mathbb{B} \): \( dd'_1 + d'_1 d = \text{degree in (} \psi, \varphi, B, \beta \text{)} \) and \( \delta \delta'_1 + \delta'_1 \delta = \text{degree in (} F, \psi, B, \beta \text{)} \).

**Proof.** One has on \( \mathfrak{A} \):

\[
d_1 d'_1 + d'_1 d_1 = \text{degree in(} \psi, \varphi, B, \beta \text{)}, \quad d_2 d'_1 + d'_1 d_2 = 0
\]
\[ \delta_1 \delta'_1 + \delta'_1 \delta_1 = \text{degree} (F, \psi, B, \beta), \quad \delta_2 \delta'_1 + \delta'_1 \delta_2 = 0. \]

Since both sides of these equations are derivations of \( \mathfrak{A} \), it is sufficient to verify that they are true on the generators which is straightforward. The lemma follows then from lemma 2.1. \( \square \)

We are now ready to describe the \( d \) and \( \delta \) cohomologies of \( \mathfrak{B} \). Let \( \mathcal{I}_S(g) \) be the space of invariant polynomials on \( g \) and let \( \mathcal{I}^\bot_S(g) \) be the subspace of homogeneous invariant polynomials of degree \( n \) on \( g \); one has the following result.

**THEOREM 2.1** The \( d \) and the \( \delta \) cohomologies of \( \mathfrak{B} \) are given by:

\[
\begin{align*}
H^{k,\ell}(\mathfrak{B}, d) &= 0 \text{ if } \ell \neq 0 \text{ or if } k \text{ is odd} \\
H^{2n,0}(\mathfrak{B}, d) &= \{ P(F) \mid P \in \mathcal{I}_S(g) \} \simeq \mathcal{I}^\bot_S(g)
\end{align*}
\]

and

\[
\begin{align*}
H^{k,\ell}(\mathfrak{B}, \delta) &= 0 \text{ if } k \neq 0 \text{ or if } \ell \text{ is odd} \\
H^{0,2n}(\mathfrak{B}, \delta) &= \{ P(\varphi) \mid P \in \mathcal{I}_S(g) \} \simeq \mathcal{I}^\bot_S(g).
\end{align*}
\]

**Proof.** By the lemma 2.2, \( d'_1 \), (resp. \( \delta'_1 \)), gives an homotopy for \( d \), (resp. \( \delta \)), for terms which contain \( (\psi, \varphi, B, \beta) \) (resp. \( (F, \psi, B, \beta) \)), so we are left with \( P(F) \), (resp. \( P(\varphi) \)) with \( P \in \mathcal{I}_S(g) \). These cocycles are classically cohomologically independent, (in fact they describe the basic cohomologies of the Weil algebras \( W_d(g) \) and \( W_\delta(g) \)). \( \square \)

**Remark.** One has with obvious notations, for \( P \in \mathcal{I}_S(g) \)

\[ dP(\varphi, \cdots, \varphi) = \]
\[ nP(\varphi, \cdots, \varphi, d\varphi + [A, \varphi]) = nP(\varphi, \cdots, \varphi, \beta) = \delta(-nP(\varphi, \cdots, \varphi, \psi)) \]
and
\[ \delta P(F, \cdots, F) = nP(F, \cdots, F, B) = d(-nP(F, \cdots, F, \psi)) \]
which means that
\[ d(H(\mathcal{B}, \delta)) = 0 \quad \text{and} \quad \delta(H(\mathcal{B}, \delta)) = 0 \]
or equivalently,
\[ H(H(\mathcal{B}, \delta), \delta) = \mathcal{H}(\mathcal{B}, \delta) = \{ \mathcal{P}(\varphi) | \mathcal{P} \in \mathcal{I}_S(\mathfrak{g}) \} \simeq \mathcal{I}_S(\mathfrak{g}) \]
and
\[ H(H(\mathcal{B}, \delta), \delta) = \mathcal{H}(\mathcal{B}, \delta) = \{ \mathcal{P}(\mathfrak{f}) | \mathcal{P} \in \mathcal{I}_S(\mathfrak{g}) \} \simeq \mathcal{I}_S(\mathfrak{g}). \]

Finally, let us notice that one has:
\[ (d + \delta)F + [A + \alpha, F] = B \]
\[ (d + \delta)(F + \psi + \varphi) + [A + \alpha, F + \psi + \varphi] = 0 \]
\[ (d + \delta)\varphi + [A + \alpha, \varphi] = \beta \]
\[ (d + \delta)B + [A + \alpha, B] = -[F, \psi] + [\varphi, F] \]
\[ (d + \delta)\beta + [A + \alpha, \beta] = -[\varphi, \psi] + [F, \varphi]. \]

Therefore one can apply the same method as in the proof of theorem 2.1, by changing \((F, \psi, \varphi, B, \beta)\) into \((F, F + \psi + \varphi, \varphi, B, \beta)\), to prove the following theorem.
THEOREM 2.2 The \(d + \delta\) cohomology of \(\mathcal{B}\) is given by

\[
\begin{cases}
H^m(\mathcal{B}, \delta + \delta) = 0 & \text{if } m \text{ is odd} \\
H^{2n}(\mathcal{B}, \delta + \delta) = \{ P(\mathfrak{F} + \psi + \varphi) | P \in \mathcal{I}^\delta_{\mathfrak{g}}(g) \} \simeq \mathcal{I}^\delta_{\mathfrak{g}}(g)
\end{cases}
\]

The \(d + \delta\) cohomology is of course only graded by the total degree. Notice that \(A + \alpha\) and \(F + \psi + \varphi\) generate a third Weil algebra \(W_{d + \delta}(\mathfrak{g})\) with differential \(d + \delta\).

3 The \(\delta\) cohomology modulo \(d\) of \(\mathcal{B}(\mathfrak{g})\)

The computation of the \(d\) and the \(\delta\) cohomologies of \(\mathcal{B}\) was a necessary step for the computation of the \(\delta\) cohomology modulo \(d\) of \(\mathcal{B}\) which, as will become clear later on, contains all relevant cohomological informations on \(\mathcal{B}\).

THEOREM 3.1 One has \(H^{k,\ell}(\mathcal{B}, \delta \mod (d)) = H^{k,\ell}(\mathcal{B}, \delta \mod (\delta)) = 0\) if \(k + \ell\) is odd and \(H^{k,\ell}(\mathcal{B}, \delta \mod (\delta)) \simeq \mathcal{I}^\delta_{\mathfrak{g}}(g)\) if \(k + \ell = 2n\). More precisely, on \(\mathcal{B}\), a complete system of cohomologically independent \(\delta\) cocycles modulo \(d\) in bidegree \((m, 2n - m)\) which is also a complete system of cohomologically independent \(d\) cocycles modulo \(\delta\) is given by:

\[
\{ \text{term of bidegree} (m, 2n - m) \text{ in } P(F + \psi + \varphi) | P \in \mathcal{I}^\delta_{\mathfrak{g}}(g) \}.
\]

Proof. Let \(Q^{k,\ell}\) be an element of \(Z^{k,\ell}(\mathcal{B}, \delta \mod (\delta))\) and let us denote by \([Q^{k,\ell}]\) its class in \(H^{k,\ell}(\mathcal{B}, \delta \mod (\delta))\). By definition there is a \(Q^{k-1,\ell+1} \in \mathcal{B}^{t-1,\ell+1}\) such that \(\delta Q^{k,\ell} + dQ^{k-1,\ell+1} = 0\). By applying \(d\) one gets \(\delta dQ^{k,\ell} = -d\delta Q^{k,\ell} = 0\) which implies, since \(H^{k+1,\ell}(\mathcal{B}, \delta) = 0\), that there is a \(Q^{k+1,\ell-1} \in \mathcal{B}^{t+1,\ell-1}\) such that \(\delta Q^{k+1,\ell-1} + dQ^{k,\ell} = 0\). If \([Q^{k,\ell}] = 0\), i.e. if \(Q^{k,\ell} = \delta L^{k,\ell-1} + dL^{k-1,\ell}\) with \(L^{k,\ell-1}, L^{k-1,\ell} \in \mathcal{B}\), then one has \(\delta(Q^{k+1,\ell-1} - dL^{k,\ell-1}) = 0\) and therefore, since
$H^{k+1,\ell-1}(\mathcal{B}, \delta) = 0$, there is a $L^{k+1,\ell-2} \in \mathcal{B}$ such that $Q^{k+1,\ell-2} = \delta L^{k+1,\ell-2} + dL^{k,\ell-1}$. Thus $[Q^{k,\ell}] = 0$ implies $[Q^{k+1,\ell-1}] = 0$ and therefore there is a well defined linear mapping $\partial' : H^{k,\ell}(\mathcal{B}, \delta \mod \mathfrak{d}) \to \mathcal{H}^{k+1,\ell-1}(\mathcal{B}, \delta \mod \mathfrak{d})$ defined by $\partial'[Q^{k,\ell}] = [Q^{k+1,\ell-1}]$.

$\partial'$ is injective for $(k, \ell) \neq (2n, 0)$. Indeed, assume $\partial'[Q^{k,\ell}] = 0$ or, equivalently $Q^{k+1,\ell-1} = \delta L^{k+1,\ell-2} + dL^{k,\ell-1}$ with $L^{k+1,\ell-2}, L^{k,\ell-1} \in \mathcal{B}$; then $dQ^{k,\ell} = d\delta L^{k,\ell-1}$ which implies $[Q^{k,\ell}] = 0$ if $(k, \ell) \neq (2n, 0)$ since then $H^{k,\ell}(\mathcal{B}, \mathfrak{d}) = 0$.

$\partial'$ is surjective. Indeed let $Q^{k+1,\ell-1}$ be a $\delta$ cocycle modulo $d$ of $\mathcal{B}$, i.e. there is a $Q^{k,\ell} \in \mathcal{B}^{k,\ell}$ such that $\delta Q^{k+1,\ell-1} + dQ^{k,\ell} = 0$; then $d\delta Q^{k,\ell} = 0$ which implies that $Q^{k,\ell}$ is also a $\delta$ cocycle modulo $d$ of $\mathcal{B}$ since $H^{k,\ell+1}(\mathcal{B}, \mathfrak{d}) = 0$ and therefore one has $[Q^{k+1,\ell-1}] = \partial' J[Q^{k,\ell}]$.

We have proved that $\partial' : H^{k,\ell}(\mathcal{B}, \delta \mod \mathfrak{d}) \to \mathcal{H}^{k+1,\ell-1}(\mathcal{B}, \delta \mod \mathfrak{d})$ is an isomorphism for $(k, \ell) \neq (2n, 0)$, which implies that

$$\mathcal{I}^1_{\delta}(g) = \mathcal{H}^{\alpha,2n}(\mathcal{B}, \delta) \overset{\partial'}{\sim} H^{1,2n-1}(\mathcal{B}, \delta \mod \mathfrak{d}) \overset{\partial'}{\sim} \cdots \overset{\partial'}{\sim} H^{2n-1,1}(\mathcal{B}, \delta \mod \mathfrak{d}) \overset{\partial'}{\sim} \mathcal{H}^{2n,\alpha}(\mathcal{B}, \mathfrak{d})$$

where, for the last isomorphism, we use $H^{2n,0}(\mathcal{B}, \delta \mod \mathfrak{d}) \simeq \mathcal{H}^{2n,\alpha}(\mathcal{B}, \mathfrak{d})$ which is implied by the remark of Section 2.

One can use a similar argument for $H(\mathcal{B}, \mathfrak{d} \mod (\delta))$.

Finally, if $P \in \mathcal{I}^1_{\delta}(g)$, then $(d + \delta)P(F + \psi + \varphi) = 0$ follows from $(d + \delta)(F + \psi + \varphi) = [F + \psi + \varphi, A + \alpha]$ and therefore if one defines $Q^{k,\ell} = \text{term of bidegree } (k, \ell)$ in the expansion of $P(F + \psi + \varphi)$ for $k + \ell = 2n$, one has $\delta Q^{k,\ell} + dQ^{k-1,\ell+1} = 0$, i.e. $[Q^{k,\ell}] = \partial'[Q^{k-1,\ell+1}]$. □
1 General framework

Let $M$ be a smooth finite dimensional manifold and $P \to M$ be a smooth principal bundle over $M$ with structure group $G$ such that $\text{Lie}(G) = \mathfrak{g}$. The gauge group (vertical automorphisms) $G$ of $P$ acts on the affine space $\mathcal{C}$ of connections on $P$. Let $\mathcal{P}(\subset \mathcal{C})$ be a $G$-invariant smooth manifold of connections on $P$. On the data $(\mathcal{P}, G)$ we assume the following regularity condition: The quotient $M = \mathcal{P}/G$, i.e. the set of orbits, is a smooth manifold in such a way that $\mathcal{P} \to M$ is a smooth principal $G$-bundle.

For instance one can take $\mathcal{P}$ to be the space of irreducible connections on $P$ and $G$ to be the group of all vertical automorphisms of $P$, or one can take $\mathcal{P}$ to be the space of all connections on $P$ and $G$ to be the group of pointed gauge transformations, i.e. the group of vertical automorphisms leaving invariant one point, and therefore one fibre, of $P$, (all this with appropriate smooth structure). Another classical example is, for $M$= a compact connected oriented 4-dimensional riemannian manifold and $G$ compact (e.g. $G = SU(2)$), to take $\mathcal{P}$ to be the space of irreducible self-dual connections; in this case $M$ is a finite dimensional manifold. In any case $M$ plays the role of a moduli space of connections on $P$.

The algebra $\Omega(P \times \mathcal{P})$ of differential forms on $P \times \mathcal{P}$ is a bigraded differential algebra with differential $d + \delta$ where $d$ is the exterior differential of $P$ and $\delta$ is the exterior differential of $\mathcal{P}$; $d$ is of bidegree (1,0) and $\delta$ is of bidegree (1,1).
(0,1). Tangent vectors to $P \times P$ split in two parts, $X = X^{1,0} + X^{0,1}$, where $X^{1,0}$ is tangent to $P$ and $X^{0,1}$ is tangent to $P$.

There is a canonical Lie $(G)$-valued one form $\mathcal{A}$ on $P \times P$ of bidegree $(1,0)$, i.e. $\mathcal{A} \in \mathfrak{g} \otimes ^{1,0}(\mathcal{P} \times \mathcal{P})$, defined by $\forall (\xi, a) \in P \times \mathcal{P}, \mathcal{A}(\xi, a)$ is the connection form at $\xi \in P$ of the connection $a \in \mathcal{P}$, $(a$ is a connection on $P)$. The algebra $\Omega(M \times M)$ of differential forms on $M \times M$ is canonically a bigraded differential subalgebra of $\Omega(P \times P)$ which we shall identify as the base of two operations on $\Omega(P \times P)$.

The structure group $G$ of $P$ acts unambiguously on $P \times P$ via the right action on $P$. There is a corresponding operation, in the sense of H. Cartan, of $\mathfrak{g} = \text{Lie}(\mathcal{G})$ in $\Omega(P \times \mathcal{P})$: For $X \in \mathfrak{g}$, $i_X$ denotes the inner antiderivation of $\Omega(P \times P)$ by the vertical vector field of type $(1,0)$ corresponding to $X$ (infinitesimal action of $G$). One has $i_X \mathcal{A} = X$ and $i_X \delta + \delta i_X = 0$ so the corresponding Lie derivation $L_X$ is given by $L_X = i_X d + d i_X$ and $L_X \mathcal{A} = [\mathcal{A}, X]$. There are several possible actions of $\mathcal{G}$ on $P \times \mathcal{P}$: One is the simultaneous action on $P$ and on $\mathcal{P}$, another one is the action on $P$ only. For the sequel (i.e. the characterization of $\Omega(M \times M)$ in $\Omega(P \times \mathcal{P})$), one can use the operation corresponding to any of these two actions. The simultaneous action might seem more natural since, after all, $\mathcal{G}$ is defined by its action on $P$ (vertical automorphisms). Nevertheless we choose the second one because the corresponding operation seems to us easier to describe; in this case, the infinitesimal action of $\mathcal{G}$ corresponds to vertical vector fields of type $(0,1)$.

The Lie algebra $\text{Lie}(\mathcal{G})$ identifies with $\text{Lie}(G)$-valued equivariant functions on $P$, $\mathfrak{k} : \mathcal{P} \to \mathfrak{g} = \text{Lie}(\mathcal{G})$. For $\mathfrak{k} \in \text{Lie}(\mathcal{G})$, $i_{\mathfrak{k}}$ denotes the inner antiderivation of $\Omega(P \times \mathcal{P})$ by the vertical vector field of type $(0,1)$ corresponding to $\mathfrak{k}$. One has $i_{\mathfrak{k}} \mathcal{A} + \mathcal{A} i_{\mathfrak{k}} = 0$ so the corresponding Lie derivative $\mathcal{L}_{\mathfrak{k}}$ is given by $\mathcal{L}_{\mathfrak{k}} = i_{\mathfrak{k}} \delta + \delta i_{\mathfrak{k}}$. Furthermore one has $i_{\mathfrak{k}}(\mathcal{A}) = 0$ and $\mathcal{L}_{\mathfrak{k}}(\mathcal{A}) = \mathcal{A} \mathfrak{k} + [\mathcal{A}, \mathfrak{k}]$.
with obvious notations, (reminding that \(X\) is a \(\text{Lie}(G)\)-valued function).

\(\Omega(M \times M)\) is the set of basic elements of \(\Omega(P \times P)\) for these operations i.e. \(\cap\{\ker(i_X) \cap \ker(L_{X'}) \cap \ker(i_{X'}) \cap \ker(L_X)\}\) \(X, X' \in \text{Lie}(G), \ X, X' \in \text{Lie}(G)\).

### 2 Characteristic homomorphism

It is well known that there exist connections on principal \(G\)-bundles \(P \to M\) for \(G\) compact, (for instance there are such connections associated to choices of riemannian structures on \(M\)). In any case let \(A\) be a connection on \(P\). The corresponding connection form is a \(\text{Lie}(G)\)-valued one-form on \(P\),. Remembering that \(\text{Lie}(G)\) consists of \(\text{Lie}(G)\)-valued functions on \(P\), it follows, by evaluation on \(P\), that this connection form defines a \(\text{Lie}(G)\)-valued one-form \(\alpha\) on \(P\) of bidegree \((0,1)\), i.e. \(\alpha \in g \otimes \mathcal{A}^1(P \times P)\).

One has \(i_X(\alpha) = X\) and \(L_X(\alpha) = [\alpha, X]\) for \(X \in \text{Lie}(G)\) and \(i_X(\alpha) = 0\) and \(L_X(\alpha) = [\alpha, X]\) for \(X \in \text{Lie}(G)\).

By the lemma 1.2 of part I, there is a unique homomorphism of bigraded differential algebras \(h : \mathcal{A} \to (\mathcal{P} \times \mathcal{P})\) such that \(h(A) = A\) and \(h(\alpha) = \alpha\). The following lemma is the justification of the name base of \(\mathcal{A}\) for the subalgebra \(\mathcal{B}\).

**Lemma 2.1** One has: \(h(\mathcal{A}) \cap (\mathcal{M} \times \mathcal{M}) = \langle(\mathcal{B})\rangle\).

**Proof.** One has \(i_X(h(A)) = i_X(A) = X\) for \(X \in \text{Lie}(G)\) and \(i_X(\alpha) = X, i_X(\xi) = i_X(\delta) = \mathcal{L}_X(\mathcal{A}) = X + [\mathcal{A}, X]\), for \(\mathcal{X} \in \text{Lie}(G)\). Therefore, by horizontality, \(h(\mathcal{A}) \cap (\mathcal{M} \times \mathcal{M})\) cannot contain expressions depending on \(h(A), h(\alpha)\) and \(h(\xi)\); thus elements of \(h(\mathcal{A}) \cap (\mathcal{M} \times \mathcal{M})\) are of the form \(h(\mathcal{I}(\mathcal{F}, \psi, \varphi, \mathcal{B}, \beta))\). On the other hand, if \(Z = h(F), h(\psi), h(\varphi), h(B)\) or \(h(\beta)\) one has \(L_X Z = [Z, X]\) for \(X \in \text{Lie}(G)\) and \(\mathcal{L}_X \beta = [\beta, X]\) for
\[ X \in \mathfrak{Lie}(G), \] therefore invariance of \( h(I(F, \psi, \varphi, B, \beta)) \) is equivalent to total \( \text{ad}(g) \)-invariance of \( I \), i.e. \( h(A) \cap (\mathfrak{M} \times \mathcal{M}) = \langle (\mathfrak{B}) \rangle. \)

It follows from this lemma that \( h \) induces homomorphisms of the various cohomologies of \( \mathfrak{B} \) in the corresponding cohomologies of \( \Omega(M \times \mathcal{M}) \), in particular, \( h \) induces an homomorphism \( h^c \) of \( H(\mathfrak{B}, \delta \mod(\delta)) \) in the \( \delta \) cohomology modulo \( d \) of \( \Omega(M \times \mathcal{M}) \). Of course \( h \) depends on \( A \), however one has the following result.

**THEOREM 2.1** The homomorphism \( h^c \) is independent of \( A \).

**Proof.** Let \( A_{\cdot t}, t \in [0, 1] \), be a smooth family of connections on \( P \to \mathcal{M} \) and let \( h_t \) be the corresponding family of homomorphisms of bigraded differential algebras of \( \mathfrak{A} \) in \( \Omega(P \times P) \). One has for \( P \in \mathcal{I}_S(\mathfrak{g}) \):

\[
\frac{d}{dt} h_t(P(F + \psi + \varphi, \cdots, F + \psi + \varphi)) = (d + \delta)(nP(\frac{d}{dt} h_t(\alpha), h_t(F + \psi + \varphi), \cdots, h_t(F + \psi + \varphi))).
\]

On the other hand \(nP(\frac{d}{dt} h_t(\alpha), h_t(F + \psi + \varphi), \cdots, h_t(F + \psi + \varphi)) \) is basic for the operations of \( \text{Lie}(G) = \mathfrak{g} \) and of \( \text{Lie}(G) \) and therefore it is an element of \( \Omega(M \times \mathcal{M}) \) so, by integration one has

\[
h_1(P(F + \psi + \varphi)) - h_0(P(F + \psi + \varphi)) \in (d + \delta)\Omega(M \times \mathcal{M})
\]

which means, in view of the theorem 3.1 of part I that \( h^c_1 - h^c_0 = 0. \)

The homomorphism \( h^c \) will be called the **characteristic homomorphism**.

**Remarks.**

a) The same proof shows that the homomorphism \( \tilde{h}^c \) of \( H(\mathfrak{B}, \delta \mod(\delta)) \)

in the \( d \) cohomology modulo \( \delta \) of \( \Omega(M \times \mathcal{M}) \) is independent of the
choice of $\mathcal{A}$. This also shows that the homomorphism $h_{d+\delta}^c$ of the $d + \delta$ cohomology of $\mathcal{B}$ in the total de Rham cohomology of $M \times M$ does not depend on $\mathcal{A}$. Of course $h^c$, $\tilde{h}^c$ and $h_{d+\delta}^c$ are essentially the same thing in view of theorems 2.2 and 3.1 of part I.

b) Similarly, $h_d^c : H(\mathcal{B}, \delta) \to \mathfrak{H}(\mathfrak{M} \times \mathfrak{M}, \delta)$ is independent of $\mathcal{A}$ because

$$\frac{d}{dt} h_t(P(\varphi)) = \delta(nP(\frac{d}{dt} h_t(\alpha), h_t(\varphi), \cdots, h_t(\varphi)))$$

and

$$nP(\frac{d}{dt} h_t(\alpha), h_t(\varphi), \cdots, h_t(\varphi)) \in \Omega(M \times M).$$

c) Finally, $h_d^c : H(\mathcal{B}, \delta) \to \mathfrak{H}(\mathfrak{M} \times \mathfrak{M}, [\cdot])$ is independent of $\mathcal{A}$ since $h(P(F))$ is already independent of $\mathcal{A}$. In fact $h_d^c$ is the Weil homomorphism for $P \to M$:

$$h_d^c : H(\mathcal{B}, \delta) \simeq \mathfrak{I}(\mathfrak{g}) \xrightarrow{\text{ch}} \mathfrak{H}(\mathfrak{M}) = \mathfrak{H}(\mathfrak{M}, \delta) \subset \mathfrak{H}(\mathfrak{M} \times \mathfrak{M}, [\cdot]),$$

where $\text{ch}$ is the usual Chern character of $P \to M$.

d) The Weil homomorphism $\text{ch} \simeq h_d^c$ is already included in $h^c$ since $h^c \upharpoonright H^{*,0}(\mathcal{B}, \delta \mod(\delta)) \simeq h_d^c$. In the same way,

$$h^c \upharpoonright H^{0,*}(\mathcal{B}, \delta \mod(\delta)) \simeq h_d^c$$

so the characteristic homomorphisms $h^c$ contains all the relevant informations and generalises the Weil homomorphism in an obvious sense.

3 Cartan operations in $\mathfrak{A}(\mathfrak{g})$ and $h(\mathfrak{A}(\mathfrak{g}))$

Let us denote by $\underline{Z}$ the image by $h$ of the generator $Z = A, F, \alpha, \varphi, \psi, \xi, B, \beta$ of $\mathfrak{A}$, i.e. $\underline{Z} = h(Z)$. Then the Cartan operations of $\mathfrak{g} = \mathfrak{Lie}(\mathfrak{g})$ and of
$Lie(G)$ satisfy the following relations: $(X \in g, \mathfrak{X} \in Lie(G))$ $i_X(\mathfrak{A}) = X$ and $i_X(\mathfrak{Z}) = 0$ for the other $Z$, $(\delta X = dX = 0)$, $i_X\delta + \delta i_X = 0$ so $L_X = i_Xd + di_X$ and $L_X(Z) = [Z, X]$ for $Z = A, F, \alpha, \cdot \cdot \cdot, \beta$;

$i_X(\mathfrak{a}) = \mathfrak{X}$, $i_X(\mathfrak{e}) = d\mathfrak{X} + [\mathfrak{A}, \mathfrak{X}]$ and $i_X(\mathfrak{b}) = \mathfrak{o}$ for the other $Z$, $i_Xd + di_X = \mathfrak{o}$, $(\delta \mathfrak{X} = \mathfrak{o})$, so $\mathfrak{L}_X = i_X\delta + \delta i_X$ and then $\mathfrak{L}_X(\mathfrak{A}) = d\mathfrak{X} + [\mathfrak{A}, \mathfrak{X}]$ and $\mathfrak{L}_X(\mathfrak{b}) = [\mathfrak{b}, \mathfrak{X}]$ for $Z = F, \alpha, \cdot \cdot \cdot, \beta$.

One thus see that the operation of $g = Lie(G)$ corresponds under $h$ to the operation $X \mapsto i_X$ of $g$ in $\mathfrak{A}$ defined in part I. In contrast, there is a problem to define an analogue in $\mathfrak{A}$ of the operation of $Lie(G)$. The reason is that $d\mathfrak{X}$ does not mean anything in $\mathfrak{A}$. So, in order to define the analogue of this operation in $\mathfrak{A}$ one has to add generators, for instance one may combine $\mathfrak{A}(g)$ with the Weil-BRS algebra $\mathfrak{A}(g)$ [6] but the result is complicated and the interest of the universal model $(\mathfrak{A}, \mathfrak{B})$ is its simplicity. It is why we refrain to follow this way and defined directly the base $\mathfrak{B}$ of $\mathfrak{A}$ in part I.

However, in contrast to the operation $X \mapsto i_X$, the operation $X \mapsto i'_X$ does not correspond to an operation in $\Omega(P \times P)$ under the homomorphism $h$. Thus the property of $\mathfrak{B}$ to be basic for $i'$ (and not only for $i$) looks accidental in the context used here but it is an indication that there may be another context in which $(\mathfrak{A}, \mathfrak{B})$ plays the role of a universal model.

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