Abstract

Often time series are organized into a hierarchy; in this case the forecasts have to satisfy some summing constraints. Forecasts which are independently generated for each time series (base forecasts) do not satisfy such summing constraints. Reconciliation algorithms adjust the base forecast in order to satisfy the summing constraints. We present a novel approach which addresses the reconciliation problem from a Bayesian viewpoint. We first define a prior distribution for the bottom time series, based on the base forecasts for the bottom time series. Then we update it using the information contained in the base forecast for the upper time series. Under the Gaussian assumption, the updating is performed in closed-form. We discuss under which conditions our method is optimal; we study its relation with minT, the current state-of-the-art reconciliation algorithm and we compare the two algorithms experimentally.

1. Introduction

Often time series are organized into a hierarchy. For example, the total visitors of a country can be divided into regions and the visitors of each region can be further divided into sub-regions. The bottom time series of the hierarchy are the most disaggregated time series, while the remaining time series are referred to as the upper time series.

Hierarchical forecasts should satisfy some summing constraints; for instance the sum of the forecasts for the different regions should equal the forecast for the total. The forecasts are incoherent if they do not satisfy the summing constraints. A basic method for generating coherent forecasts is the bottom-up approach: it takes the forecasts for the bottom time series and sums them up in order to produce forecasts for the entire hierarchy. Yet this approach does not leverage the forecasts produced for the upper time series of the hierarchy; such time series are smoother and thus allows estimating more accurately the seasonality or the effect of external covariates.

*Corresponding author

Email addresses: giorgio@idsia.ch (Giorgio Corani), dario.azzimonti@idsia.ch (D. Azzimonti), zaffalon@idsia.ch (M. Zaffalon)

1Istituto Dalle Molle di Studi sull’Intelligenza Artificiale (IDSIA) 
Scuola universitaria professionale della Svizzera italiana (SUPSI) - Università della Svizzera italiana (USI) 
CH-6928 Manno, Switzerland
To overcome this limit, the modern reconciliation methods (Hyndman et al., 2011, 2016; Wickramasuriya et al., 2018) proceed in two steps. First, base forecasts are computed for each time series of the hierarchy, by fitting an independent model for each time series. Then, the base forecasts are adjusted to become coherent. The adjusted forecasts are called reconciled. Besides being coherent, they are often more accurate than the base forecasts, especially when dealing with short time series; in this case the estimation of the models is more uncertain and the reconciliation allows for borrowing statistical strength across time series.

The state-of-the-art approach for reconciliation is constituted by minT algorithm, which minimizes in closed-form the trace of the covariance matrix of the errors of the reconciled forecasts. A non-Gaussian probabilistic reconciliation has been recently proposed by Taieb et al. (2017).

We tackle the reconciliation problem from a novel viewpoint, adopting a Bayesian approach. We first define our prior beliefs about the bottom time series, based on the information contained in the base forecasts for the bottom time series; we assume them to be jointly Gaussian. We then update the prior using the information contained in the base forecasts for the upper time series. We treat such forecasts as noisy observations of linear combinations of the bottom time series. By assuming the noise to be multivariate Gaussian and independent from the bottom time series, we can compute Bayes’ rule in closed form. This is the linear-Gaussian model (Roweis and Ghahramani, 1999), which underlies many algorithms of probabilistic machine learning. The updating equation is similar to a Kalman filter, which is indeed a special case of the linear-Gaussian model. We obtain in this the posterior distribution for the bottom time series, from which we obtain the probability distribution function for the entire hierarchy. Our reconciliation approach is thus probabilistic and it has a closed-form solution.

We discuss under which assumptions our reconciliation is optimal. We then study the similarity between our approach and minT. We show that numerically our Bayesian approach correspond to minT with a specific choice of covariance matrix. While the meaning of frequentist and Bayesian probability is fundamentally different, frequentist and Bayesian procedures yield sometimes corresponding results (Casella and Berger, 1987), and our Bayesian reconciliation constitutes another case of this type. We then experimentally compare our approach and minT in both artificial and real data sets.

Our approach tackles the reconciliation problem in a novel way and for this reason it also opens some research directions. Further improvements in reconciliation can be obtained by borrowing algorithms from the generalization of the Kalman filter (Simon, 2006, Sec. 7). Our framework could also be extended to perform probabilistic reconciliation ensuring the positivity of the reconciled forecasts. This could be achieved by assuming for instance Poisson distributions rather than Gaussian ones, and then computing numerically the resulting posterior distribution (there would be no closed-form solution in this case).

The paper is organized as follows. Section 2 introduces time series reconciliation and the minT algorithm. Section 3 presents our Bayesian reconciliation, proves its optimality and its relation with minT. In Section 4 we study analytically the reconciliation of a simple hierarchy. We present our experiments in Section 5 and the conclusions in Section 6.
2. Time series reconciliation

Fig. 1 shows an example of hierarchy. We could interpret it as the revenues of a company, which are disaggregated first by product (A and B) and then by region (R_1 and R_2). The most disaggregated time series (bottom time series) are shaded (A_{R_1}, A_{R_2}, B_{R_1}, B_{R_2}).

![Diagram of hierarchy](image)

Figure 1: Example of hierarchy; the bottom time series are shaded.

We now introduce our notation. The hierarchy contains m time series, of which n are bottom time series. The vector of observations at time t is \( y_t \in \mathbb{R}^m \). It can be broken down in two parts, namely
\[
y_t = [u_t, b_t]^T; \quad b_t \in \mathbb{R}^n\text{ contains the observations of bottom time series while } u_t \in \mathbb{R}^{m-n}\text{ contains the observations of the upper time series.}
\]

The structure of the hierarchy is represented by the summing matrix \( S \in \mathbb{R}^{m \times n} \) such that:
\[
y_t = S b_t.
\] (1)

The \( S \) matrix of Fig.1 is:
\[
S = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

where the sub-matrix \( A \) encodes which bottom time series should be summed up in order to obtain each upper time series.
The base forecasts refer to the forecast horizon \((t+h)\). The vector of base forecasts for the entire hierarchy at time \(t\) is \(\hat{y}_{t+h} \in \mathbb{R}^m\); we separate base forecasts for bottom time series \((\hat{b}_{t+h} \in \mathbb{R}^n)\) and upper time series \((\hat{u}_{t+h} \in \mathbb{R}^{m-n})\), namely \(\hat{y}_{t+h} = [\hat{b}_{t+h}, \hat{u}_{t+h}]^T\). Vector \(\hat{y}_{t+h}\) contains the point predictions; yet also the variance of the base forecasts are available; they will be used later. If forecasts for different time horizons are needed (e.g., \(h=1,2,3,\ldots\)), the reconciliation is performed independently for each \(h\).

Grouped time series. The hierarchy of Fig.1 considers a unique aggregation sequence: it sums the sales of the same product in different regions, and then it sums the total sales of the two products. Different aggregations are however possible; for instance one could first sum the sales of different products in the same region, and then the total sales of each region. *Grouped* time series allow for multiple aggregation sequences within the same hierarchy, corresponding to the presence of additional nodes within the hierarchy. Such additional aggregations are encoded by adding the corresponding rows to matrix \(S\). Once \(S\) is defined, reconciliation can be performed as usual.

Linear of the reconciliation. It is usually assumed (Hyndman et al., 2011, 2016; Wickramasuriya et al., 2018) that the reconciled bottom forecasts are a linear combination of the forecasts available for the whole hierarchy. The goal is hence to find \(P \in \mathbb{R}^{n \times m}\) such that:

\[
\tilde{b}_{t+h} = P \hat{y}_{t+h},
\]

where \(\tilde{b}_{t+h}\) denotes the reconciled forecasts for the bottom series. The reconciled forecasts for the whole hierarchy are then obtained as

\[
\tilde{y}_{t+h} = S \tilde{b}_{t+h} = SP \hat{y}_{t+h}.
\]

The state-of-the-art approach for estimating \(P\) is MinT (Wickramasuriya et al., 2018), which we briefly review in the following.

2.1. The MinT reconciliation

We denote by \(\hat{e}_{t+h} = y_{t+h} - \tilde{y}_{t+h} \in \mathbb{R}^m\) the vector of forecasts errors at time \(t\). Assuming the base forecasts to be unbiased, the vector of reconciled forecasts \(\tilde{y}_{t+h}\) is unbiased if and only if \(SPS = S\). Wickramasuriya et al. (2018) build a generalized least squares model that minimizes the reconciled forecast error:

\[
\tilde{e}_{t+h} := y_{t+h} - \tilde{y}_{t+h}.
\]

Assuming \(SPS = S\), Wickramasuriya et al. (2018) show that the covariance matrix of the prediction error is:

\[
\text{Var}[\tilde{e}_{t+h}] = SPWP^T S^T,
\]

where \(W = \mathbb{E}[\hat{e} \hat{e}^T]\) is the covariance matrix of the base forecasts errors.

The optimal reconciliation matrix \(P\) which minimizes \(\text{tr}(\text{Var}[\hat{e}]) = \text{tr}(SPWP^T S^T)\) under the constraint \(SPS = S\) is:

\[
P = (S^T W^{-1} S)^{-1} S^T W^{-1}.
\]

The main computational challenge is the estimation of the covariance matrix \(W\). Different estimators of \(W\) are compared by Wickramasuriya et al. (2018); the best results are generally obtained using shrinkage estimator by (Schäfer and Strimmer, 2005), which shrinks the full covariance matrix towards a diagonal matrix.
2.1.1. Reconciliation h-steps ahead

Since the covariances of the residuals depend on the forecast horizon \( h \), it is in principle necessary to estimate \( W \) differently for each \( h \). Yet this implies many practical difficulties. As a workaround, Wickramasuriya et al. (2018) assume \( W(h) = k_h W(1) \), where \( k_h \) is an unknown constant that simplifies when computing Eq. 4, while \( W(1) \) is the covariance matrix of the one-step ahead errors. Hence also the reconciliation for the forecast h-steps ahead is based on \( W(1) \). We will adopt the same approach for our algorithm.

3. Reconciliation via Bayes’ rule

We propose a novel approach to reconciliation adopting a Bayesian viewpoint. We start by defining our prior beliefs about the bottom time series, based on the information \( I \) available up to time \( t \), including the base forecasts for the bottom time series but excluding the base forecasts for the upper time series.

By assuming the bottom time series to be jointly Gaussian, our prior is:

\[
p(B_{t+h} | I) = N \left( B; \hat{b}_{t+h}, \hat{\Sigma}_B \right), \tag{5}
\]

whose mean hence equals the base forecasts for the bottom time series. The matrix \( \hat{\Sigma}_B \) contains in position \([i, j]\) the covariance \( \text{Cov}(B_{t+h,i}, B_{t+h,j} \mid I) \), where \( B_{t+h} \) denotes the vector of bottom time series at time \( t+h \). We discuss in Sec 3.1 how to compute such covariances.

We treat the base forecast \( \hat{u}_{t+h} \) of the upper time series as noisy observations of sums of certain bottom time series. We assume the noise to be multivariate Gaussian with mean zero (the base forecasts are unbiased). We thus have:

\[
\hat{U}_{t+h} = AB_{t+h} + \varepsilon
\]

\[
\varepsilon \sim N \left( 0, \hat{\Sigma}_U \right).
\]

where \( \hat{\Sigma}_U \) is the covariance matrix of the noise affecting the forecasts of the upper time series. Hence:

\[
p(\hat{U}_{t+h} \mid I, B_{t+h}) = N \left( \hat{U}; AB_{t+h}, \hat{\Sigma}_U \right), \tag{6}
\]

We now apply Bayes’ rule in obtain the posterior distribution of the bottom time series:

\[
p(B_{t+h} \mid I, \hat{U}_{t+h}) = \frac{p(B_{t+h} \mid I)p(\hat{U}_{t+h} \mid I, B_{t+h})}{p(\hat{U}_{t+h} \mid I)}
\]

Thanks to the Gaussian assumptions, Bayes’ rule can be computed in closed form. This is called the linear-Gaussian model (Roweis and Ghahramani 1999), and it underlies different algorithms for probabilistic machine learning (Bishop 2006, Chap.8.1.4). The linear-Gaussian model yields the posterior mean of the bottom time series as:

\[
\hat{b}_{t+h} := \mathbb{E}[B_{t+h} \mid \hat{u}_{t+h}] = \hat{b}_{t+h} + \hat{\Sigma}_B A^T (\hat{\Sigma}_U + A \hat{\Sigma}_B A^T)^{-1} (\hat{u}_{t+h} - A \hat{b}_{t+h})
\]
By denoting $G = \hat{\Sigma}_B A^T(\hat{\Sigma}_U + A\hat{\Sigma}_B A^T)^{-1}$, we have the more compact expression:

$$\tilde{b}_{t+h} = \hat{b}_{t+h} + G(\hat{u}_{t+h} - A\hat{b}_{t+h}),$$

which shows that $\tilde{b}_{t+h}$ is a weighted average of the base forecast $\hat{b}_{t+h}$ and the incoherence of the base forecasts ($\hat{u}_{t+h} - A\hat{b}_{t+h}$). The weights of the two terms depend on the variances of the forecasts, as we illustrate later in Sec. 4. Equation (7) has the same structure as the update step of a Kalman filter (Simon [2006], Chap.5), where $G$ is referred to as the gain matrix and $(\hat{u}_{t+h} - A\hat{b}_{t+h})$ is referred to as the innovation. Indeed, the Kalman filter is a special case (Roweis and Ghahramani [1999], Sec.5.4) of the linear-Gaussian model.

The covariance of the reconciled bottom forecasts is:

$$\text{Var}[B_{t+h} | \hat{u}_{t+h}] = \hat{\Sigma}_B - \hat{\Sigma}_B A^T(\hat{\Sigma}_U + A\hat{\Sigma}_B A^T)^{-1}A\hat{\Sigma}_B$$

$$= \hat{\Sigma}_B - G(\hat{\Sigma}_U + A\hat{\Sigma}_B A^T)G^T \quad (8)$$

The reconciled point forecast and the covariance for the entire hierarchy are eventually given by:

$$\hat{y}_{t+h} = S\hat{b}_{t+h}$$

$$\text{Var}[\hat{y}_{t+h}] = S\text{Var}[B_{t+h} | \hat{u}_{t+h}] S^T.$$

3.1. The covariance matrix $\hat{\Sigma}_B$

In the following we show that

$$\text{Cov}(B_{t+h}^i, B_{t+h}^j | I) = \text{Cov}(e_{t+h}^i, e_{t+h}^j),$$

where $e_{t+h}^i$ denotes the vector of residuals of the model fitted on the $i$-th time series, for the forecast horizon $t + h$.

We first demonstrate that $\text{Cov}(e_{t+h}^i, e_{t+h}^j | I) = \text{Cov}(B_{t+h}^i, B_{t+h}^j | I)$:

$$\text{Cov}(e_{t+h}^i, e_{t+h}^j | I) = \mathbb{E}[(B_{t+h}^i - \hat{B}_{t+h}^i)(B_{t+h}^j - \hat{B}_{t+h}^j) | I]$$

$$= \mathbb{E}[B_{t+h}^i B_{t+h}^j | I] - \mathbb{E}[B_{t+h}^i \hat{B}_{t+h}^j | I] - \mathbb{E}[B_{t+h}^i B_{t+h}^j | I] + \mathbb{E}[B_{t+h}^i | I]\mathbb{E}[B_{t+h}^j | I]$$

$$= \mathbb{E}[B_{t+h}^i B_{t+h}^j | I] - \mathbb{E}[B_{t+h}^i | I]\mathbb{E}[B_{t+h}^j | I]$$

$$= \text{Cov}(B_{t+h}^i, B_{t+h}^j | I)$$

where in the second line, $\mathbb{E}[B_{t+h}^i \hat{B}_{t+h}^j | I] = 0$ because $\hat{B}_{t+h}^i$ and $\hat{B}_{t+h}^j$ are independent. By applying the law of total covariance, we now show that $\text{Cov}(e_{t+h}^i, e_{t+h}^j | I) = \text{Cov}(e_{t+h}^i, e_{t+h}^j)$.

$$\text{Cov}(e_{t+h}^i, e_{t+h}^j | I) = \text{Cov}(e_{t+h}^i, e_{t+h}^j) - \text{Cov}(\mathbb{E}[e_{t+h}^i | I], \mathbb{E}[e_{t+h}^j | I])$$

$$= \text{Cov}(e_{t+h}^i, e_{t+h}^j).$$
because the conditional expectations of the errors are independent. Hence
\[
\text{Cov}(B_{t+h}^i, B_{t+h}^j | \mathcal{I}) = \text{Cov}(e_{t+h}^i, e_{t+h}^j | \mathcal{I}) = \text{Cov}(e_{t+h}^i, e_{t+h}^j).
\]

We can thus estimate the covariance matrix $\hat{\Sigma}$ using the covariance of the residuals. The $i$-th diagonal element contains in particular the variance of the residuals of the model fitted on the $i$-th time series. Hence each marginal distribution of our multivariate prior has the same mean and variance of the base forecasts.

3.2. Optimality of the reconciliation

We now show that, under the assumptions stated in Sec.3 (Gaussian prior, forecast of the upper time series affected by a Gaussian noise independent from $B_{t+h}$), our reconciliation minimizes the mean squared error among all possible linear reconciliation methods.

In the Gaussian case, the conditional expectation of $B_{t+h}$ given $\hat{u}_{t+h}$ is also the best linear unbiased predictor of $B_{t+h}$ given $\hat{u}_{t+h}$. Hence:
\[
\tilde{b}_t(h) = \arg \min_{\beta(\hat{u}_{t+h})} \mathbb{E}[\|B_{t+h} - \beta(\hat{u}_{t+h})\|_2^2 | \hat{u}_{t+h}].
\]

(9)

where $\beta(\hat{u}_{t+h})$ is any predictor of $B_{t+h}$ which is a linear function of $\hat{u}_{t+h}$. Our objective is to find the vector of reconciled predictions $\tilde{y}_t(h)$ that is as close as possible to $y_{t+h} = S B_{t+h}$.

**Theorem 1.** Let $\tilde{b}_t(h)$ denote the prediction of the bottom time series reconciled via the Bayesian approach of Eq. (7). The forecast for the whole hierarchy, obtained as $\tilde{y}_t(h) = S \tilde{b}_t(h)$, minimize the expectation of the mean squared error, i.e.
\[
\tilde{y}_t(h) = \arg \min_{\zeta(\hat{u}_{t+h})} \mathbb{E}[\|y_{t+h} - \zeta(\hat{u}_{t+h})\|_2 | \hat{u}_{t+h}],
\]

among all possible reconciled vectors $\zeta(\hat{u}_{t+h}) = S \beta(\hat{u}_{t+h})$.

**Proof 1 (Sketch).** We first prove that $\tilde{b}_t(h)$ is unbiased from which it follows that $\tilde{y}(h)$ is unbiased too. By exploiting the optimality in (9) we obtain the result. We give the complete proof in Appendix A.

4. An illustrating example

Our algorithm yields the reconciliation in closed form and we proved its optimality. We now illustrate the reconciliation of a simple hierarchy, in order to concretely understand how our probabilistic reconciliation works. To the best of our knowledge, it is the first time that reconciliation is illustrated in this didactic way.

We consider the hierarchy of Fig. 2 constituted by two bottom time series ($B_1$ and $B_2$) and an upper time series $U$. In the following for simplicity of notation we drop the indication $(t+h)$ of the horizon to which all the base forecasts refer.

The base forecast for the bottom time series are constituted by the point forecasts $\hat{b}_1$ and $\hat{b}_2$ with variances $\sigma_1^2$ and $\sigma_2^2$. 

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Based on this, our prior beliefs about $B_1$ and $B_2$ are hence:

\[
\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \sim N \left( \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \end{pmatrix}, \begin{pmatrix} \sigma^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma^2 \end{pmatrix} \right).
\]

The summing matrix is:

\[
S = \begin{bmatrix} 1 & 1 \\ \Gamma & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}.
\]

The reconciled forecasts for the bottom time series are:

\[
\tilde{b} = \hat{b} + G(\hat{u} - A\hat{b}),
\]

(10)

where $\hat{u}$ is the base forecast for $U$. Thus if the base forecasts are already coherent (i.e., $\hat{u} = A^T\hat{b}$), they remain unchanged. Otherwise, the adjustment is proportional to the incoherence $(\hat{u} - A^T\hat{b})$.

Since $A = [1 \ 1]$ and $\hat{\Sigma}_U = \sigma_u^2$ we get:

\[
A^T\hat{\Sigma}_B A = \sigma^2 + \sigma^2 + 2\sigma_{1,2},
\]

\[
\hat{\Sigma}_B A^T = [\sigma^2 + \sigma_{1,2}, \sigma^2 + \sigma_{1,2}]^T
\]

\[
\hat{\Sigma}_U + A\hat{\Sigma}_B A^T = \sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2},
\]

where we denote by $\sigma_{1,2}$ the covariance $\text{Cov}(\hat{B}_1, \hat{B}_2)$. Hence:

\[
G = \hat{\Sigma}_B A^T(\hat{\Sigma}_U + A\hat{\Sigma}_B A^T)^{-1} = \frac{1}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}} \begin{bmatrix} \sigma^2 + \sigma_{1,2} \\ \sigma^2 + \sigma^2 + \sigma_{1,2} \end{bmatrix}.
\]

We can expand equation (10) as

\[
\tilde{b} = \frac{\hat{b}_1}{\hat{b}_2} + \frac{\sigma^2 + \sigma_{1,2}}{\sigma^2 + \sigma^2 + \sigma_{1,2}} \frac{\hat{u} - A^T\hat{b}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}
\]

(11)

\[
= \frac{\hat{b}_1 + (\hat{u} - \hat{b}_1 - \hat{b}_2)}{\frac{\sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}} \frac{\sigma^2 + \sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}} \frac{\hat{u} - A^T\hat{b}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}}
\]

\[
= \tilde{b}_1 \left( \frac{1 - \frac{\sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}}{\frac{\sigma^2 + \sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}} \right) + (\hat{u} - \hat{b}_2) \frac{\sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}
\]

\[
= \tilde{b}_1 \left( \frac{1 - \frac{\sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}}{\frac{\sigma^2 + \sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}} \right) + (\hat{u} - \hat{b}_2) \frac{\sigma^2 + \sigma_{1,2}}{\sigma_u^2 + \sigma^2 + \sigma^2 + 2\sigma_{1,2}}
\]

(12)
Equation (11) shows that the size of the adjustment applied to the base forecasts linearly decreases with the variance $\sigma_u^2$ of the upper forecast; if $\sigma_u^2$ is very large, the adjustment tends to zero, as the upper forecast is not trustable. Instead, if $\sigma_u^2 \to 0$ (i.e., the upper forecast is extremely reliable), it is possible to show that $\tilde{b}_1 + \tilde{b}_2 \to \tilde{u}$.

From Equation (12) we see that the reconciled bottom forecast are a linear combination of the base forecasts of the whole hierarchy. In particular $\tilde{b}_1$ is a weighted average of $\hat{b}_1$ and $(\tilde{u} - \tilde{b}_2)$.

Since the reconciled bottom forecasts are a linear combination of the base forecasts of the whole hierarchy, we can explicit the equivalent matrix $P$ for the Bayesian reconciliation. Recall that:

$$\tilde{b} = \begin{bmatrix} \tilde{b}_1 \\ \tilde{b}_2 \end{bmatrix} = P \hat{y} = P \begin{bmatrix} \tilde{u} \\ \hat{b}_1 \\ \hat{b}_2 \end{bmatrix}$$

Hence the $P$ matrix of our algorithm, denoted by $P_{Bayes}$ is:

$$P_{Bayes} = \begin{bmatrix} g_1 & 1 - g_1 & -g_1 \\ g_2 & -g_2 & 1 - g_2 \end{bmatrix}$$

with $g_1$ and $g_2$ have been defined in Eqn. (12).

4.1. The MinT reconciliation

We now explicit the matrix $P$ of MinT for the same example, which is based on the full covariance matrix $W$:

$$W = \begin{bmatrix} \sigma_u^2 & \sigma_{u,1} & \sigma_{u,2} \\ \sigma_{u,1} & \sigma_1^2 & \sigma_{1,2} \\ \sigma_{u,2} & \sigma_{1,2} & \sigma_2^2 \end{bmatrix}$$

where $\sigma_{u,1}$, $\sigma_{u,2}$, $\sigma_{1,2}$ are the covariances between the residuals of the models fitted on the couple of time series $(U, B_1)$, $(U, B_2)$ and $(B_1, B_2)$ respectively. We can now use equation (4) to compute $P_{MinT}$. Let us denote: $g_1^* = \frac{\sigma_1^2 + \sigma_{1,2} - \sigma_{u,1}}{\sigma_1^2 + \sigma_{1,2} + 2\sigma_{u,2} + 2\sigma_{1,2} - 2\sigma_{u,1} - 2\sigma_{u,2}}$ and $g_2^* = \frac{\sigma_2^2 + \sigma_{u,2} - \sigma_{u,1}}{\sigma_2^2 + \sigma_{u,2} + 2\sigma_{1,2} - 2\sigma_{u,1} - 2\sigma_{u,2}}$. The $P$ matrix of minT is eventually:

$$P_{MinT} = \begin{bmatrix} g_1^* & 1 - g_1^* & -g_1^* \\ g_2^* & -g_2^* & 1 - g_2^* \end{bmatrix}$$

Thus the weights of MinT depend also on the cross-covariance between the residuals of the bottom layer and of the upper layers, which instead are not present in our Bayesian algorithm (which indeed assumes them to be independent).

4.2. Relationship with minT

We now prove under which conditions our reconciliation is numerically equivalent to that yielded by minT. Our reconciled bottom time series can be written as:

$$\tilde{b} = \hat{b} + G(\tilde{u} - A\hat{b}) = (I - GA)\hat{b} + G\tilde{u} = [G (I - GA)] \tilde{u} = \tilde{b} = P\tilde{y}$$
Hence the matrix $P$ of the Bayesian reconciliation is $P_{\text{Bayes}} = [G \ (I - GA)]$.

**Theorem 2.** The $P$ matrix of the minT algorithm is equal to the Bayesian reconciliation matrix if the covariance matrix $W$ has the following block-diagonal structure:

$$W = \begin{bmatrix} \hat{\Sigma}_U & 0 \\ 0 & \hat{\Sigma}_B \end{bmatrix}$$

**Proof 2.** Recall that the $P$ matrix for minT is $P_{\text{minT}} = (S^T W^{-1} S)^{-1} S^T W^{-1}$. Since $W$ is block-diagonal, we have that

$$W^{-1} = \begin{bmatrix} \Sigma_U^{-1} & 0 \\ 0 & \Sigma_B^{-1} \end{bmatrix}.$$ 

By substituting $W^{-1}$ in the formula for $P_{\text{minT}}$, recalling that $S = \begin{bmatrix} A \\ I \end{bmatrix}$, we obtain

$$P_{\text{minT}} = (A^T \hat{\Sigma}_U^{-1} A + \hat{\Sigma}_B^{-1})^{-1} S^T W^{-1}$$

$$= (A^T \hat{\Sigma}_U^{-1} A + \hat{\Sigma}_B^{-1})^{-1} \begin{bmatrix} A^T \hat{\Sigma}_U^{-1} & \hat{\Sigma}_B^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} (A^T \hat{\Sigma}_U^{-1} A + \hat{\Sigma}_B^{-1})^{-1} A^T \hat{\Sigma}_U^{-1} (A^T \hat{\Sigma}_U^{-1} A + \hat{\Sigma}_B^{-1})^{-1} \hat{\Sigma}_B^{-1} \\ \hat{\Sigma}_B A^T (\hat{\Sigma}_B + A\hat{\Sigma}_B A^T)^{-1} (\hat{\Sigma}_B - \hat{\Sigma}_B A^T (\hat{\Sigma}_U + A\hat{\Sigma}_B A^T)^{-1} A\hat{\Sigma}_B) \hat{\Sigma}_B^{-1} \end{bmatrix}$$

$$= [G \ I - GA]$$

where the fourth equality follows from an application of the Woodbury identity and of the Woodbury identity for positive definite matrices.

While the meaning of frequentist and Bayesian probability is fundamentally different, frequentist and Bayesian procedures yield sometimes corresponding results (Casella and Berger, 1987), and our Bayesian reconciliation seems another case of this type.

5. Experiments

Given the discussed similarities, we do not expect major differences in terms of performance between minT and our algorithm. Yet we check the empirical effect of the block-diagonal approach both on synthetic and real data sets.

We consider the mean squared error for the $h$-step ahead forecasts, where the average is taken with respect to the $m$ times series of the hierarchy:

$$\text{mse}(h) = \frac{1}{m} [y_{t+h} - \tilde{y}_t(h)]^T [y_{t+h} - \tilde{y}_t(h)].$$
We repeat each experiment several times and eventually we report the median of the mse ratio between minT and our approach:

\[
\text{mse ratio}(h) = \text{median} \left( \frac{\text{mse}(\text{MinT}, h)}{\text{mse}(\text{Bayes}, h)} \right)
\]

Medians larger than 1 imply a better performance of the Bayesian approach and vice versa.

5.1. Artificial data

We consider the structure of Fig. 3. We generate the four bottom time series as AR(1) processes, drawing their parameters from the stationary region. They bottom time series are affected by a correlated Gaussian noise, with covariance:

\[
\Sigma = \begin{bmatrix}
5 & 3 & 2 & 1 \\
3 & 4 & 2 & 1 \\
2 & 2 & 5 & 3 \\
1 & 1 & 3 & 4
\end{bmatrix}
\]

Such covariance matrix enforces the correlation between time series which have the same parent; it has been used already in the experiments of [Wickramasuriya et al. (2018)] We compute the base forecast using the ets algorithm [Hyndman and Khandakar, 2008]. We consider the following values for the length \(T\) of the time series: \(\{10; 100; 1000\}\). For each value of \(T\) we perform 3000 simulations, measuring the mse yielded by both minT and our approach. We consider both the sample estimate of the covariance matrices (sam) and the shrinkage estimate (shr). The results are given in Tab.1

| \(T\)    | minT(sam)/Bayes(sam) | minT(shr)/Bayes(shr) | Bayes(shr)/Base |
|---------|----------------------|----------------------|-----------------|
| 10      | 1.22                 | 1.00                 | 0.93            |
| 100     | 1.01                 | 1.00                 | 0.97            |
| 1000    | 0.99                 | 0.99                 | 0.97            |

Table 1: Median of the mse ratios for the synthetic experiment. For each value of \(T\) we performed 3000 experiments.
If the sample estimate of the covariance is adopted, the Bayesian approach improves significantly the mse; this is the positive effect of the block-diagonalization of the covariance matrix, and it is especially pronounced for small $T$. Yet this advantage is offset once the more advanced shrinkage approach is adopted for estimating the covariance matrix; in this case the performance of minT and of our approach is practically equivalent. For very large $T$ ($T=1000$, while we the data are generated from AR(1) processes), minT has a slight advantage over our approach. As a sanity check, we also report the reduction of mse compared to the base forecasts, which ranges between 3% and 7% depending on $T$ (it decreases with $T$).

5.2. Experiments with real data sets

We consider two real hierarchical data sets, [Hyndman et al. 2018]: \textit{infantgts} and \textit{tourism}. The \textit{infantgts} is available within the \texttt{hts} package while \textit{tourism} is available in raw format from \url{https://robjhyndman.com/publications/MinT/}. We summarize their main properties in Tab. 2.

| data set   | frequency | length | number of TS layers |
|------------|-----------|--------|---------------------|
| \textit{infantgts} | yearly    | 71     | 27                  | 4                        |
| \textit{tourism}    | monthly   | 228    | 555                 | 8                        |

Table 2: Main properties of the grouped time series.

The \textit{infantgts} data set contains infant mortality counts in Australia, disaggregated by sex and by eight different states. Each time series contains 71 yearly observations, covering the period 1933-2003. This a grouped time series, whose hierarchy contains two possible aggregation paths. The bottom layer contains 16 time series (8 states x 2 genders). The first path aggregates the counts of the same sex across states, yielding 2 nodes (one for each sex) in the intermediate level; a further aggregation yields the total. The second path sums males and females in each state, yielding 8 nodes (one for each state) as a first aggregation; a further aggregation yields the total.

The \textit{tourism} data set regards the number of nights spent by Australians away from home. The time series cover the period 1998–2016 with monthly frequency. There are 304 bottom time series, referring to 76 regions and 4 purposes. The first aggregation path sums the purposes in each region, yielding 76 time series (one for each region); such values are further aggregated into zones (27) and states (7), before reaching the total (1). A second path aggregates the bottom time series of the same zone (yielding 108 intermediate time series: 27 zones x 4 purposes), which are then aggregated into states (yielding 28 time series: 7 states x 4 purposes), then by states (yielding 4 time series: 4 purposes). The last aggregation yields the total. Overall the hierarchy contains 555 time series.

We reconcile each hierarchy 50 times: every time we extend the training data of one time step. Every time we reconcile the forecasts referring to 1-step, 2-steps, 3-steps and 4-steps ahead. We consider two algorithms for generating the base forecast: \textit{ets} (state-space exponential smoothing) and \textit{auto-arima}. They are both available from the \texttt{forecast} package [Hyndman and Khandakar, 2008]. In this case we only report the results for the
shrinkage estimates, since the sample covariance yields numerical problems, being not positive definite in most cases.

The results are shown in Tab. 3 and 4. The results obtained with the same forecasting method for different forecast horizons are quite consistent; this is to be expected, since both minT and our method compute the reconciliation using the 1-step ahead covariance matrix. We thus focus our comments on themse ratio averaged over the different forecasting horizons.

| h | forecaster | minT/Bayes | base/Bayes | base/minT |
|---|------------|------------|------------|-----------|
| 1 | ets        | 0.95       | 0.97       | 0.97      |
| 2 | ets        | 1.00       | 0.97       | 0.93      |
| 3 | ets        | 1.00       | 0.98       | 1.00      |
| 4 | ets        | 1.01       | 0.99       | 1.02      |
| avg |   | **0.99**   | **0.98**   | **0.98**  |

| h | forecaster | minT/Bayes | base/Bayes | base/minT |
|---|------------|------------|------------|-----------|
| 1 | arima      | 0.98       | 0.9        | 0.91      |
| 2 | arima      | 1.01       | 1.05       | 0.99      |
| 3 | arima      | 1.03       | 1.04       | 1.03      |
| 4 | arima      | 1.01       | 1.07       | 1.03      |
| avg |   | **1.01**   | **1.02**   | **0.99**  |

Table 3: Median mse ratios on the infantgts data set. The shrinkage estimation of the covariance is adopted for both minT and the Bayesian approach.

On the infantgts data set the performance of our algorithm and minT are equivalent: our algorithm has a slight advantage in the ets case (about 1%) while the vice versa happens in the arima case.

| h | forecaster | minT/Bayes | base/Bayes | base/minT |
|---|------------|------------|------------|-----------|
| 1 | ets        | 0.98       | 0.98       | 0.97      |
| 2 | ets        | 0.97       | 0.96       | 0.97      |
| 3 | ets        | 0.98       | 0.96       | 0.96      |
| 4 | ets        | 0.98       | 0.98       | 0.98      |
| avg |   | **0.98**   | **0.97**   | **0.97**  |

| h | forecaster | minT/Bayes | base/Bayes | base/minT |
|---|------------|------------|------------|-----------|
| 1 | arima      | 0.97       | 1.04       | 1.07      |
| 2 | arima      | 0.96       | 1.02       | 1.05      |
| 3 | arima      | 0.97       | 1.11       | 1.09      |
| 4 | arima      | 0.99       | 1.12       | 1.11      |
| avg |   | **0.97**   | **1.07**   | **1.08**  |

Table 4: Median mse ratios on the tourism data set. The shrinkage estimation of the covariance is adopted for both minT and the Bayesian approach.

On the tourism data set, the median mse ratios are slightly in favor of minT, both in the arima and in the ets case (mse ratio of 0.97 and 0.98, averaging over the different
forecasting horizons). This can be explained considering that the tourism time series are long ($T=227$), and the shrinkage approach can thus precisely estimate the covariances that might exist between the bottom time series and the noise affecting the upper time series. We note that both algorithms yield a higher mse than the base forecast in the arima case.

Reproducibility

We implemented our experiments in R; the code of our experiments is available at [https://github.com/gcorani/hierTs](https://github.com/gcorani/hierTs). For instance the reconciliation experiment for the `infantgts` data set, with $h=1$ and ets as forecaster can be performed as follows:

```r
hierRec(dset="infantgts", fmethod="ets", h=1)
```

An additional parameter `iTest` can be used to control how to split the data between train set and test set. See the website above for more details.

6. Conclusions

We have presented a novel reconciliation method, based on a Bayesian approach to reconciliation. We prove that numerically it yields reconciled forecasts that are equivalent to those of minT under a specific choice of covariance matrix and we discussed under which assumptions it is optimal. Empirically, its point predictions are slightly less accurate than those of minT, especially when dealing with long time series as in the case of tourism ($T=228$). This shows that the block-diagonal approach, though theoretically justified, is slightly less effective compared to the shrinkage of the full covariance matrix, on which minT relies.

The novel viewpoint introduced by our approach could allow for important extension, based for instance on generalizations of the Kalman filter which account for the correlation (Simon, 2006, Chap.7.1) between the noise affecting the upper base forecasts and the bottom time series; or developing robust inference algorithm in spite of an imprecisely estimated covariance matrix (Simon, 2006, Chap.7.2).

We also see the possibility of extending our framework to the case of probabilistic reconciliation with positivity constraints. This could be implemented by assuming e.g. a Poisson distribution rather than a Gaussian one for the bottom time series, and then computing numerically the posterior, i.e. the reconciled bottom forecasts. This could be for instance pursued by adopting a probabilistic programming approach (Carpenter et al., 2017).

References

Athanasopoulos, G., Hyndman, R.J., Kourentzes, N., Petropoulos, F.. Forecasting with temporal hierarchies. European Journal of Operational Research 2017;262(1):60–74.

Bishop, C.M.. Pattern Recognition and Machine Learning. Springer, 2006.

Carpenter, B., Gelman, A., Hoffman, M.D., Lee, D., Goodrich, B., Betancourt, M., Brubaker, M., Guo, J., Li, P., Riddell, A.. Stan: A probabilistic programming language. Journal of Statistical Software 2017;76(1).

Casella, G., Berger, R.L.. Reconciling Bayesian and frequentist evidence in the one-sided testing problem. Journal of the American Statistical Association 1987;82(397):106–111.
Appendix A. Proof of Proposition [1]

Lemma 1. Consider $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^m$, if $\tilde{b} \in \mathbb{R}^m$ minimizes $\beta \rightarrow \|b - \beta\|$, then $\tilde{A}b$ minimizes $\beta \rightarrow \|Ab - \tilde{A}\beta\|$.

Proof 3. Note that the first norm is on $\mathbb{R}^m$ and the second on $\mathbb{R}^n$.

$$\|Ab - \tilde{A}\beta\|_2 = \|A(b - \beta)\|_2 \leq \|A\|_F \|b - \beta\|_2,$$

where $\|\cdot\|_F$ is the Frobenius norm of a matrix. Since $\|A\|_F$ does not depend on $\beta$, the minimizer of $\|b - \beta\|_2$ minimizes $\|Ab - \tilde{A}\beta\|_2$.

Proof 4 (Proposition [1]). The predictor $\tilde{b}_t(h)$ defined in (7) is unbiased, in fact $E[\tilde{b}_t(h)] = E[E[\tilde{b}_t(h) | \tilde{u}] = E[\tilde{b}_t(h)] = E[B_t \beta]$, where we applied two times the law of iterated expectation. It also follows that the base forecasts are unbiased, therefore we have $E[Y_{t+h}] = E[Y_t(h)]$. Moreover

$$E \left[ \|Y_{t+h} - \zeta(\tilde{u})\|^2 | \tilde{u} \right] = E \left[ \|S\tilde{b}_{t+h} - S\beta(\tilde{u})\|^2 | \tilde{u} \right].$$

By lemma [1] we obtain that the minimizer is $S\tilde{b}_t(h)$. 