INTRODUCTION TO BLACK HOLE MICROSCOPY

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ABSTRACT

The aim of these notes is both to review the standard understanding of the Hawking effect, and to discuss the modifications to this understanding that might be required by new physics at short distances. The fundamentals of the Unruh effect are reviewed, and then the Hawking effect is explained as a “gravitational Unruh effect”, with particular attention to the state-dependence of this picture. The order of magnitude of deviations from the thermal spectrum of Hawking radiation is estimated under various hypotheses on physics at short distances. The behavior of black hole radiation in a linear model with altered short distance physics—the Unruh model—is discussed in detail.

1. Introduction

When a black hole forms in a collapse process, virtually all features of the original collapsing object are erased. All that survives of its origin is the total mass, angular momentum, and charges. What remains is a pure object, made of nothing but empty spacetime. Thus a non-rotating uncharged black hole is characterized by its Schwarzschild radius

\[ R_S = \frac{2GM}{c^2}. \]

The length \( R_S \) provides the only scale.

If we bring in \( \hbar \) something new happens. The “empty space” that the black hole is made of wakes up and is full of vacuum fluctuations of all fields (including, presumably, the gravitational field itself). And these vacuum fluctuations are propagating in the black hole background. Hawking discovered in 1974 that this black hole vacuum is not stable. He showed that a thermal flux of positive energy at temperature

\[ T_H = \frac{\hbar c}{4\pi R_S} \]

flows away from the vicinity of the black hole. (For a stellar mass black hole this temperature is very low, since the Schwarzschild radius is about a kilometer \((T_H \approx 10^{-7} \text{K} \, (1.5 \text{km}/R_S))\), but the temperature could be very high for a much smaller black hole that might have formed in the early universe.) Moreover, negative energy flows across the horizon into the hole, so total energy in the fixed, static, background is conserved. Presumably the gravitational back-reaction to this process diminishes the mass of the hole, and the process continues until the hole has lost all of its mass.
The fact that the Hawking radiation has a universal, thermal, spectrum can be understood from the observation that there is only one scale in the problem, the Schwarzschild radius \( R_S \). Other than indicating the absorption cross-section of the black hole, the radiation can carry no information. That is, it has maximum entropy, and so must have a thermal spectrum. The temperature corresponds to a wavelength of \( 8\pi^2 R_S \). Since \( \hbar \) is involved, one might think there is a second scale in the problem, the Planck length \( L_P = (\hbar G/c^3)^{1/2} \sim 10^{-33} \text{cm} \). However, \( G \) does not enter into Hawking’s result because the gravitational interaction plays no role. Only when one includes the back-reaction does \( G \) or \( L_P \) enter. Indeed, the lifetime of the hole is of order \( R_S^3/L_P^2 c \). (This is roughly \( 10^{54} \) times the present age of the universe for a stellar mass black hole. For a primordial black hole of mass \( \sim 10^{15} \text{gm} \) however the lifetime is only the present age of the universe.)

Upon further reflection however, it is not so clear that the Planck scale physics is irrelevant to black hole radiation\(^2\),\(^3\),\(^4\),\(^5\). Consider a vacuum fluctuation mode that ends up at infinity populated with Hawking radiation. This mode emerges from the vicinity of the horizon, and must climb out of a very deep well (see Fig. 1). It is tremendously redshifted on the way out. If the mode has a wavelength \( \lambda_{out} \) when observed a time \( t \) after the hole formed, then its initial wavelength \( \lambda_{in} \) must be of order \( \exp(-t/2 R_S) \lambda_{out} \), which is very quickly way below the Planck length. For instance if \( \lambda_{out} \sim R_S \), then \( \lambda_{in} \sim L_P \) when \( t \sim 2R_S \ln(R_S/L_P) \), which is about a millisecond for a solar mass black hole. Hawking’s original calculation assumed that vacuum fluctuations are scale invariant down to such arbitrarily sub-Planckian dimensions. If this assumption is no good, then the true behavior of quantum fields around a black hole could be quite different.

The aim of these notes is both to review the standard understanding of the Hawking effect, and to discuss the modifications to this understanding that might be required by new physics at short distances. In particular, I first review the Unruh effect, namely, the fact that an accelerated observer in Minkowski spacetime sees the vacuum as a thermal state. This is then used to give an elementary explanation of the origin of the Hawking effect, and to argue that, whatever short distance field dy-
namics occurs on sub-Planckian scales, it will give rise to the Hawking effect as long as it produces an ordinary vacuum structure for the outgoing modes near the horizon at scales $\lambda$ satisfying $L_P < \lambda \ll R_s$. The order of magnitude of deviations from the thermal spectrum of Hawking radiation is estimated under various hypotheses. In the remainder, the behavior of a linear model with altered short distance physics—the Unruh model—is discussed. This model consists of a scalar field propagating on a black hole background with non-Lorentz-invariant higher derivatives in the action. The higher derivatives lead to a nonlinear dispersion relation at large wavevectors, but the field equations remain linear.

In the following, unless otherwise indicated, units are adopted for which $\hbar = c = G = k_B = 1$.

2. The Unruh effect

A uniformly accelerated observer in flat spacetime perceives the vacuum state of quantum fields as a thermal state with temperature $T_U = a/2\pi$, where $a$ is the proper acceleration. This is called the Unruh effect$^6$, and $T_U$ is the Unruh temperature. To get a feeling for the magnitude of this temperature, we can put $\hbar$ and $c$ back in:

$$T_U = \frac{\hbar}{c} \frac{a}{2\pi},$$

so $T_U$ is roughly $\hbar$ divided by the time it takes for the velocity to change by the speed of light.

The Unruh effect can be understood as a consequence of the invariance of the vacuum under the symmetries of Minkowski spacetime, translations and boosts, together with the spectral condition that the energy of physical states be nonnegative. I will give two derivations of this effect below, both of which are valid for arbitrary interacting scalar fields in spacetime of any dimension. (The generalization to fields of nonzero spin is essentially trivial.) The second derivation, which uses a path integral method, is much slicker and generalizes to the Hartle-Hawking state for black holes and all other curved but static spacetimes with bifurcate Killing horizons. My reasons for giving the first derivation as well are threefold. First, it is very concrete, second, it illustrates very nicely the dangers of being sloppy with analyticity properties, and third, I have not seen it done in the literature in this much generality in this simple manner. A brief guide to the literature on these matters is provided in subsection below.

The Minkowski line element in two dimensions can be written in both “Cartesian” (Minkowski) and “polar” (Rindler) coordinates:

$$ds^2 = dt^2 - dz^2 = \xi^2 d\eta^2 - d\xi^2$$

where the coordinates are related by

$$t = \xi \sinh \eta, \quad z = \xi \cosh \eta.$$
The coordinates \((\eta, \xi)\) are nonsingular in the ranges \(\xi \in (0, \infty)\) and \(\eta \in (-\infty, \infty)\), and cover the “Rindler wedge” \(x > |t|\) in Minkowski space (see Fig. 2). In the first form of the line element the translation symmetries generated by the Killing vectors \(\partial/\partial t\) and \(\partial/\partial z\) are manifest, and in the second form the boost symmetry generated by the Killing vector \(\partial/\partial \eta\) is manifest. The latter is clearly analogous to rotational symmetry in Euclidean space. The full collection of translation and boost symmetries of Minkowski spacetime is called the Poincaré group.

Fig. 2. Two-dimensional flat spacetime in Minkowski and Rindler coordinates. A hyperbola of constant \(\xi\) is a uniformly accelerated timelike worldline with proper acceleration \(\xi^{-1}\). A boost shifts \(\eta\) and preserves \(\xi\).

2.1. Two-point function and KMS condition

A thermal density matrix \(\rho = Z^{-1} \exp(-\beta H)\) has two identifying properties: First, it is obviously stationary, since it commutes with the Hamiltonian \(H\). Second, because \(\exp(-\beta H)\) coincides with the evolution operator \(\exp(-itH)\) for \(t = -i\beta\), expectation values in the state \(\rho\) possess a certain symmetry under translation by \(-i\beta\) called the KMS condition\(^7\),\(^8\). Let \(\langle A \rangle_\beta\) denote the expectation value \(\text{tr}(\rho A)\), and let \(A_t\) denote the time translation by \(t\) of the operator \(A\). Using cyclicity of the trace we have

\[
\langle A_{-i\beta} B \rangle_\beta = Z^{-1} \text{tr}\left(e^{-\beta H} (e^{\beta H} Ae^{-\beta H}) B\right) = Z^{-1} \text{tr}(e^{-\beta H} BA) = \langle BA \rangle_\beta.
\]

Note that for nice enough operators \(A\) and \(B\), \(\langle A_{-i\tau} B \rangle_\beta\) will be analytic in the strip \(0 < \tau < \beta\). Now let us compare this behavior with that of the two-point function along a uniformly accelerated worldline in the Minkowski vacuum.
If, as is usual, the vacuum state shares the symmetry of Minkowski spacetime, then, in particular, the 2-point function \(G(x, x') = \langle \phi(x)\phi(x') \rangle\) must be a Poincaré invariant function of \(x\) and \(x'\). Thus it must depend on them only through the invariant interval, so one has \(G(x, x') = f((x-x')^2)\) for some function \(f\). Now consider an “observer” traveling along the hyperbolic trajectory \(\xi = a^{-1}\). This worldline has constant proper acceleration \(a\), and \(a\eta\) is the proper time along the world line. Let us examine the 2-point function \(G(\eta, \eta') \equiv G(x(\eta), x(\eta'))\) along this hyperbola. It is clearly stationary, due to the boost invariance of the vacuum, so it can only depend on \(\eta\) and \(\eta'\) through the difference \(\eta - \eta'\). Therefore

\[
G(\eta, \eta') = G(x(\eta - \eta'), x(0)) = f((x(\eta - \eta') - x(0))^2) = f(4a^{-2}\sinh^2((\eta - \eta')/2)),
\]

where the third equality follows from (3). Now, since \(\sinh^2(\eta/2)\) is periodic under translations of \(\eta\) by \(2\pi i\), it appears that \(G(\eta, \eta')\) is periodic under such translations in each argument. In terms of the 2-point function the KMS condition implies \(G(\eta, \eta' + i\beta) = G(\eta', \eta)\), which is not the same as translation invariance in each argument. Does this mean that in fact the 2-point function in the Minkowski vacuum along the accelerated worldline is not thermal? The answer is “no”, because the above “proof” that \(G(\eta, \eta')\) is periodic was bogus. First of all, a Poincaré invariant function of \(x\) and \(x'\) need not depend only on the invariant interval. It can also depend on the invariant step-function \(\theta(x_0 - x_0')\theta((x - x')^2)\). More generally, the analytic properties of the function \(f\) have not been specified, so one cannot conclude from the periodicity of \(\sinh^2(\eta/2)\) that \(f\) itself is periodic. For example, \(f\) might involve the square root, \(\sinh(\eta/2)\), which is anti-periodic. In fact, this is just what happens.

To reveal the analytic behavior of \(G(x, x')\), it is necessary to incorporate the condition that the spacetime momenta of states in the Hilbert space lie inside or on the future light cone. One can show (by inserting a complete set of states between the operators) that this implies there exists an integral representation for the 2-point function of the form

\[
G(x, x') = \int d^n k \theta(k^0) J(k^2) e^{-ik(x-x')} ,
\]

where \(J(k^2)\) is a function of the invariant \(k^2\) that vanishes when \(k\) is spacelike. Now let us evaluate \(G(\eta, \eta')\) along the hyperbolic trajectory. Thanks to boost invariance, it suffices to consider \(\eta' = 0\). Thus we have

\[
G(\eta, 0) = \int d^n k \theta(k^0) J(k^2) e^{-ia^{-1}[k^0\sinh \eta - k^1(\cosh \eta - 1)]} .
\]

Now it still looks as though this is periodic in translation of \(\eta\) by \(i2\pi m\) for any integer \(m\), but closer inspection reveals the difficulty that the integral fails to be convergent.
if the imaginary part of $\sinh \eta$ becomes positive. One has

$$\sinh(\eta - i\theta) = \sinh \eta \cos \theta - i \cosh \eta \sin \theta$$  \hspace{1cm} (12)$$

$$\cosh(\eta - i\theta) = \cosh \eta \cos \theta - i \sinh \eta \sin \theta.$$  \hspace{1cm} (13)

Since $k^0 \geq |k^1|$ the integral is seen to converge for $0 < \theta < \pi$, but one cannot go past $\pi$. To see if the function has an analytic extension beyond $\theta = \pi$ and, if so, to determine the value at $\theta = 2\pi$, one needs a different representation of $G(\eta)$. A suitable representation can be obtained by observing that Lorentz invariance allows us to transform to the frame in which $x - x'$ has only a time component which is given by the invariant norm $[(x - x')^2]^{1/2} = 2a^{-1} \sinh(\eta/2)$. Thus we have

$$G(\eta, 0) = \int d^n k \theta(k^0) J(k^2) e^{-i 2a^{-1} k^0 \sinh(\eta/2)}. \hspace{1cm} (14)$$

In this form it is clear that $\eta - i\theta$ can be extended all the way down to $\theta = 2\pi$. Since $\sinh[(\eta - i2\pi)/2] = -\sinh(\eta/2) = \sinh(-\eta/2)$, we can finally conclude that $G(\eta - i2\pi, 0) = G(-\eta, 0) = G(0, \eta)$, which is the KMS condition (6).

2.2. The vacuum state as a thermal density matrix

The essence of the Unruh effect is the fact that the density matrix describing the Minkowski vacuum, traced over the states in the region $z < 0$, is precisely a Gibbs state for the boost Hamiltonian $H_B$ at a “temperature” $T = 1/2\pi$:

$$Tr_{z<0} |0\rangle \langle 0| = Z^{-1} \exp(-2\pi H_B), \hspace{1cm} (15)$$

$$H_B = \int T_{ab}(\partial/\partial \eta)^a d\Sigma^b \hspace{1cm} (16)$$

This rather amazing fact has been proved in varying degrees of rigor by many different authors. A sloppy path integral argument making it very plausible will be sketched below.

Since the boost Hamiltonian has dimensions of action rather than energy, so does the “temperature”. To determine the local temperature seen by an observer following a given orbit of the Killing field, note from (6) that the norm of the Killing field $\partial/\partial \eta$ on the orbit $\xi = a^{-1}$ is $a^{-1}$, whereas the observer has unit 4-velocity. If the Killing field is scaled by $a$ so as to agree with the unit 4-velocity at $\xi = a^{-1}$, then the boost Hamiltonian (16) and temperature are scaled in the same way. Thus the temperature appropriate to the observer at $\xi = a^{-1}$ is $T = a/2\pi$. Since $a$ is the proper acceleration of this observer, we recover the Unruh temperature defined above. Alternatively, the two-point function defined by (15) along the hyperbola obviously satisfies the KMS condition relative to boost time $\eta$ at temperature $1/2\pi$. When expressed in terms of proper time $a \eta$, this corresponds to the temperature $a/2\pi$. 
One can view the relative coolness of the state at larger values of $\xi$ as being due to a redshift effect—in this case a Doppler shift—as follows. Suppose a uniformly accelerated observer at $\xi_0$ sends some of the thermal radiation he sees to another uniformly accelerated observer at $\xi_1 > \xi_0$. This radiation will suffer a redshift given by the ratio of the norms of the Killing field: say $p$ is the spacetime momentum of the radiation. Then $p \cdot (\partial / \partial \eta)$ is conserved, but the energy locally measured by the uniformly accelerated observer is $p \cdot (\partial / \partial \eta) / |\partial / \partial \eta|$, so that $E_1 / E_0 = |\partial / \partial \eta|_0 / |\partial / \partial \eta|_1$. This is precisely the same as the ratio $T_1 / T_0$ of the locally measured temperatures. At infinity $|\partial / \partial \eta| = \xi$ diverges, so the temperature drops to zero, which is consistent with the vanishing acceleration of the boost orbits at infinity.

The path integral argument to establish (15) goes like this: Let $H$ be the Hamiltonian generating ordinary time translation in Minkowski space. The vacuum $|0\rangle$ is the lowest energy state, and we suppose it has vanishing energy: $H|0\rangle = 0$. If $|\psi\rangle$ is any state with nonzero overlap with the vacuum, then $\exp(-\tau H)|\psi\rangle$ becomes proportional to $|0\rangle$ as $\tau$ goes to infinity. That is, the vacuum wavefunctional $\Psi_0[\phi]$ for a field $\phi$ is proportional to $\langle \phi | \exp(-\tau H) | \psi \rangle$ as $\tau \to \infty$. Now this is just a matrix element of the evolution operator between imaginary times $\tau = -\infty$ and $\tau = 0$, and such matrix elements can be expressed as a path integral in the “lower half” of Euclidean space:

$$\Psi_0[\phi] = \int_{\phi(-\infty)}^{\phi(0)} D\phi \exp(-I)$$

(17)

where $I$ is the Euclidean action.

The key idea in recovering (15) is to look at (17) in terms of the angular “time”-slicing of Euclidean space instead of the constant $\tau$ slicing. (See Fig. 3.) The relevant

Fig. 3. Time slicings of Euclideanized Minkowski space. The horizontal lines are constant $\tau$ surfaces and the radial lines are constant $\theta$ surfaces.

Euclidean metric (restricted to two dimensions for notational convenience) is given by

$$ds^2 = d\tau^2 + d\sigma^2 = \rho^2 d\theta^2 + d\rho^2$$

(18)
Adopting the angular slicing, the path integral (17) is seen to yield an expression for the vacuum wavefunctional as a matrix element of the boost Hamiltonian (16) which coincides with the generator of rotations in Euclidean space:

\[ \langle \phi_L \phi_R | 0 \rangle = N \langle \phi_L | \exp(-\pi H_B) | \phi_R \rangle, \]

(19)

where \( \phi_L \) and \( \phi_R \) are the restrictions of the boundary value \( \phi(0) \) to the left and right half spaces respectively, and a normalization factor \( N \) is included. The Hilbert space \( \mathcal{H}_R \) on which the boost Hamiltonian acts consists of the field configurations on the right half space \( z > 0 \), and is being identified via reflection (really, by reflection composed with CPT\footnote{CPT} with the Hilbert space \( \mathcal{H}_L \) of field configurations on the left half space \( z < 0 \). The entire Hilbert space is \( \mathcal{H} = \mathcal{H}_L \otimes \mathcal{H}_R \), modulo the degrees of freedom at \( z = 0 \). (The boundary conditions at \( z = 0 \) are being completely glossed over here.) Now consider the vacuum expectation value of an operator \( O_R \) that is localized on the right half space:

\[ \langle 0 | O_R | 0 \rangle = \langle 0 | \phi_L \phi_R | \phi_L \phi_R | O_R | \phi_L \phi_R | 0 \rangle \]
\[ = N^2 \langle \phi_L | \exp(-\pi H_B) | \phi_R \rangle \langle \phi_R | O_R | \phi_R \rangle \langle \phi_R | \exp(-\pi H_B) | \phi_L \rangle \]
\[ = N^2 tr(e^{-2\pi H_B} O_R), \]

(21)

where summation over intermediate states is implicit, and (19) was used in the second equality. This shows that, as far as observables located on the right half space are concerned, the vacuum state is given by the thermal density matrix (15). More generally, this holds for observables localized anywhere in the Rindler wedge.

This path integral argument directly generalizes to all static spacetimes with a bifurcate Killing horizon, such as the Schwarzschild and deSitter spacetimes\footnote{Schwarzschild and deSitter}. In the general setting, the state defined by the path integral cannot be called “the” vacuum, but it is a natural state that is invariant under the static Killing symmetry of the background and is nonsingular on the time slice where the boundary values of the field are specified, including bifurcation surface.

### 2.3. Some references

The fact that the Minkowski vacuum is a thermal state for the boost Hamiltonian was proved in axiomatic quantum field theory by Bisognano and Wichmann\footnote{Bisognano and Wichmann}, as a theorem about the action of complex Lorentz transformations on the vacuum. The relevance of this theorem to the Unruh and Hawking effects was recognized by Sewell\footnote{Sewell}, who generalized the framework to curved spacetimes. In completely independent work (as far as I know) the path integral argument has been given by many authors, perhaps the first being Unruh and Weiss\footnote{Unruh and Weiss}. The review articles by Takagi\footnote{Takagi} and by Fulling and Ruijsenaars\footnote{Fulling and Ruijsenaars} cover various aspects of the relation between accel-
eration and temperature in quantum field theory, and contain many other references.

3. Hawking radiation

The Hawking effect can be understood as a kind of “gravitational Unruh effect” as follows. Consider a static, spherically symmetric, asymptotically flat black hole spacetime. This spacetime has a time translation Killing field $\chi$ which is a unit timelike vector at infinity, becomes null on the horizon, and is spacelike within the horizon. An observer at fixed radius is uniformly accelerated, with an acceleration that vanishes at infinity and diverges as the horizon is approached. What can be said about the state of the outgoing modes as viewed from infinity in such a spacetime? It should be consistent with redshifting from the state of the same modes viewed from a vantage point closer in to the black hole, however that of course does not determine the state.

In order to infer the existence of Hawking radiation one must assume some boundary condition on the state of the quantum field. In principle, this should just be the assumption that the initial state was not too pathological and contained matter that would later collapse and form a black hole. However, as discussed in the introduction, to connect this initial condition to the final state one must evolve initially arbitrarily short wavelength field modes through the collapse. Instead let us first consider a boundary condition that does not involve us in the physics of the trans-Planckian domain. Thus let us suppose that the state just outside the horizon, after the black hole has formed, looks like the Minkowski vacuum at very short, but longer than Planckian, distances. If this is the case, then a static observer hovering just outside the horizon with a tremendous acceleration should experience the Unruh effect, and will perceive the state to be thermal at some very high temperature. This thermal state is then redshifted to infinity, as previously discussed in the case of flat spacetime. However a crucial difference now arises: the norm of the Killing field $\chi$ is finite at infinity, rather than divergent. Thus the Unruh radiation survives its trip to infinity, where it arrives with a non-zero temperature. Let us see how this works quantitatively.

For concreteness let us work with a Schwarzschild black hole, whose line element is given in Schwarzschild coordinates by

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

(23)

The static Killing field is $\chi = \partial/\partial t$, and its norm is given by

$$N(r) = (\chi \cdot \chi)^{1/2} = (1 - 2M/r)^{1/2}.$$  

$N$ is also called the “lapse” function, since it determines how much proper time $ds$ corresponds to a given coordinate time $dt$ via $ds = N dt$ at fixed $(r, \theta, \phi)$. The
acceleration of a worldline at constant radius and angles is found to be

\[ a = N^{-1}(1 - N^2)^2 \kappa, \]

where \( \kappa = 1/4M \) is the “surface gravity” of the hole. At the horizon the acceleration diverges and at infinity it vanishes. The Unruh temperature corresponding to the acceleration is \( a/2\pi \). As viewed from infinity this temperature suffers a redshift given (as explained in subsection ) by the ratio \( N(r)/N(\infty) = N(r) \). Thus the Unruh temperature for an observer at \( r \), as viewed from infinity, is

\[ T_{U,\infty}(r) = (1 - N^2)^2 T_H, \]

where \( T_H = \kappa/2\pi = 1/8\pi M \) is the Hawking temperature. As \( r \) approaches the horizon the redshifted Unruh temperature approaches the Hawking temperature, while as \( r \) goes to infinity it vanishes.

Since the value of \( T_{U,\infty}(r) \) in (25) very much depends on \( r \), we appear to have an inconsistency. How can the state viewed from infinity have many different temperatures? It cannot, of course. The inconsistency arises only if we assume that at each radius the static observer experiences the usual Unruh effect, but this assumption requires that along each of these worldlines the 2-point function of the quantum field has the same form as it would have in Minkowski space. We have no basis for making such an assumption, except in the limit of extremely short distances where we expect all states to coincide. Since a higher Unruh temperature arises from a shorter distance probe, this assumption should get better and better as the static world line approaches the horizon. In this limit, \( T_{U,\infty}(r) \) approaches the Hawking temperature, so it is the Hawking temperature that is selected by the Minkowskian boundary condition on the quantum state at very short distances.

There are of course many other states of the quantum field. For example, there is the “Boulware vacuum”, i.e., the Fock vacuum devoid of positive Killing-frequency excitations. In this state the static observers see no particles at any radius. However, this state is not generally believed to be the state that arises when a black hole forms in a collapse process. In fact if one evolves the state of a free quantum field on the background of a collapsing black hole, one sees that the final state is the “Unruh vacuum”. Near the horizon these two states are tremendously different. Whereas the Unruh vacuum looks like the Minkowski vacuum at very short distances, the Boulware vacuum does not. For example, the renormalized expectation value of the stress-energy tensor in the Boulware vacuum blows up as the horizon is approached.

**4. Imposing a cutoff**

Now imagine there is a short distance cutoff of some kind on the validity of ordinary field theory, so one cannot impose the vacuum condition as above in the limit of arbitrarily short distances. What effect would that have on the Hawking
effect? Of course the answer depends on what kind of short distance physics is imagined. Let us suppose first that, whatever the details at the cutoff, the vacuum boundary condition is still satisfied to a fair degree at scales longer than the cutoff $l_c$ but much shorter than the Schwarzschild radius. That is, we assume the out vacuum at scales $\lambda$ satisfying

$$l_c < \lambda \ll R_s.$$  \hfill (26)

One would then expect the Hawking effect to still occur, but with small deviations from the usual Hawking spectrum. The presence of the cutoff presumably limits how close to the horizon the accelerated observer can be assumed to experience the Unruh effect. If the minimum allowed lapse is $N_{\text{min}}$, then we might expect from (25) deviations at least of order $N_{\text{min}}^2$ from the usual Hawking flux, since even if the observer at $N_{\text{min}}$ experiences the usual Unruh effect (which is presumably not quite the case), the redshifted temperature (25) will not agree with $T_H$. The value of $N_{\text{min}}$ would depend on the nature of the cutoff.

4.1. Invariant cutoff

Without breaking local Lorentz invariance, one can imagine a cutoff that limits the acceleration for which the usual Unruh effect takes place. This might be the case if there is a minimum proper time that should be considered along the worldline. If the acceleration is required to be less than $l_c^{-1}$, then $N_{\text{min}} \sim \kappa l_c \approx l_c/R_s$. In this case the above argument suggests deviations of order $(l_c/R_s)^2$ from the usual Hawking effect. This might be relevant, for instance, in string theory.

4.2. non-Lorentz-invariant cutoff

Lorentz invariance of the laws of physics has been confirmed to some precision up to some finite boost factor $\gamma$ relative to the cosmic rest frame. The boosts that play a role in the Hawking effect however grow exponentially without bound in time as $\gamma \sim \exp(t/2R_s)$. Normally physicists do not have confidence extrapolating infinitely beyond their observations, but the case of Lorentz invariance seems to be an exception. Allowing this exception seems to force us into assuming that there are an infinite number of field modes in any finite spatial region, no matter how small. And all of these modes play a role in the Hawking effect as usually formulated. It is therefore worthwhile to ask what would happen to the Hawking radiation in the presence of a non-Lorentz-invariant cutoff. This question will be considered in a detailed model later in these notes. For now, let us just assume as above that the physics produces the out vacuum at scales $\lambda$ satisfying (24) in the free-fall frame of the black hole, that is, in the frame carried in to the black hole by observers who fall freely from far away, starting at rest.

Of course this definition of the preferred frame is ambiguous. In the spherically
symmetric case we can restrict to radial trajectories, and consider only those trajectories that are always infalling, ignoring those that pass through the collapsing matter and later fall back. More generally, the fundamental cutoff theory would be needed to even formulate the nature of the cutoff. But in order to explore the qualitative implications of such a cutoff, it seems reasonable to stick to the spherically symmetric case and to impose the cutoff in the falling frame defined above.

The cutoff for the static observer is Doppler shifted relative to the cutoff for the free-fall observer. The outgoing modes are redshifted while the ingoing modes are blueshifted. Let $k_{c,in}$ and $k_{c,out}$ denote ingoing and outgoing null wave-vectors with the cutoff wavelength, and let $u_{ff}$ and $u_{stat}$ denote the unit 4-velocities of the free-fall and static observers (see Fig. 4). Then the cutoff frequency is given in the free-fall frame by

$$\omega_c = k_{c,in} \cdot u_{ff} = k_{c,out} \cdot u_{ff},$$

while it is given in the static frame by

$$\omega_{c,in}^{stat} = k_{c,in} \cdot u_{stat} \quad \text{and} \quad \omega_{c,out}^{stat} = k_{c,out} \cdot u_{stat}$$

for the in- and out-going modes respectively.

Using these definitions near the horizon where $N \ll 1$ one finds

$$l_{c,out}^{stat} \simeq 2N^{-1}l_c \quad \text{(27)}$$

$$l_{c,in}^{stat} \simeq \frac{1}{2}Nl_c \quad \text{(28)}$$

For the Unruh effect we must require that the acceleration timescale $a^{-1}$ is longer than the cutoff in the accelerated frame. For the outgoing modes, which are relevant for the Hawking effect, this means that $N/\kappa > N^{-1}l_c$. This non-Lorentz-invariant cutoff thus leads to a much larger minimum lapse, $N_{\min} \simeq (l_c/R_S)^{1/2}$. In this case one might expect deviations from the Hawking flux of order $l_c/R_S$, which is still very small for all but Planckian holes.

A quantum field theoretic calculation of the out state at infinity starting from the out vacuum near the horizon at scales (26) in the free-fall frame has also been carried out\(^5\). To do this calculation it is necessary to work with localized wavepackets. Being careful about the errors caused by the inevitable “tails” of the wavepackets, an upper bound of order $(l_P/R_S)^{-1/2}$ for the deviations from the Hawking spectrum was found, which is much larger than the estimate obtained from the Unruh effect.

Fig. 4. The null cutoff wave-vectors and the four-velocities of the freely falling and static observers. The diagram is drawn in the free-fall frame, so the static observer appears to be moving outward at high speed.
argument above. However, this was only an upper bound.

4.3. Horizon fluctuations

So far I have been speaking of a cutoff in terms of a modification of the field theory on a fixed background. On the other hand, taking into account the back reaction and quantum fluctuations in geometry one expects the horizon to fluctuate in some sense. It is interesting to try to estimate the size of these fluctuations and their possible effect on the Hawking radiation. Let us consider the fluctuations in the horizon that would be expected if the horizon is viewed as a system of Planck areas fluctuating about an equilibrium configuration. In equilibrium the typical entropy fluctuation is given by $\delta S \sim 1$. Since $S = A/4$ (where $A$ is the horizon area), we therefore expect $\delta A \sim 1$. Now if we assume the horizon consists of $A$ independent Planck areas with random area fluctuation $\delta a \sim R \delta r / A \sim \delta r / R$ (in four-dimensional spacetime), then by the law of large numbers we have $\delta A \sim A^{1/2} \delta a \sim \delta r$, so $\delta r \sim 1 = l_P$. This reasoning suggests that the horizon is fuzzy on scales of order $l_P$ in the radial coordinate. If this is interpreted to mean that one should not apply the acceleration temperature argument any closer to the horizon than $r = R_s + l_P$, one finds for the minimum lapse $N_{\text{min}} \sim (l_P/R_s)^{1/2}$. This is the same result as one would have obtained by imposing a cutoff at $l_P$ in the free-fall frame as above.

5. Models with a non-Lorentz-invariant cutoff

In the previous section we argued crudely that the Hawking effect will occur independently of short distance physics as long as the out vacuum boundary condition holds near the horizon for wavelengths satisfying $l_c < \lambda \ll R_s$, up to small corrections for large black holes. This leaves us with the following questions. Given a particular theory with some new physics at short distances,

1. Does the abovementioned out vacuum boundary condition hold?

2. Exactly how large are the deviations from the thermal Hawking flux?

3. Are the deviations from the thermal Hawking flux small even at very short wavelengths?

4. Do the deviations for short wavelengths accumulate to make a large difference in any physical quantity, such as the energy flux or energy density?

A class of models in which these questions can be quantitatively addressed with relatively simple methods is obtained if the usual wave equation for the quantum field is modified by the addition of higher derivative terms, while linearity of the field equation is preserved. Such a modification can be covariant, or it can be done
in a manner that violates local Lorentz invariance. I know of no suitable covariant model for ordinary fields, since covariant higher derivatives always lead to negative norm ghosts. However, string theories can be Lorentz-covariant and ghost-free, and it might be feasible and interesting to study Hawking radiation in non-interacting string theory. Attempts in this direction have been made but so far none have been able to compute the spectrum of black hole radiation. On the other hand, several non-covariant models ones have been studied recently. The mother of them all was Unruh’s sonic black hole. Although the subsequent models can all be interpreted in terms of ordinary black holes without reference to fluid flow, I will first describe the sonic black hole model, since it is very interesting and provides good motivation for the other models.

5.1. Sonic black hole

Unruh pointed out in 1981 that perturbations of an irrotational, inviscid, barotropic \((\rho = \rho(p))\) fluid flow behave like a massless field propagating in a curved spacetime whose metric is determined by the background flow. If the flow is supersonic somewhere, then there is a horizon in the effective spacetime geometry, and one has a model of a black hole. Quantizing the fluid perturbations, Unruh argued that, provided the out vacuum boundary condition holds just outside the sonic horizon, the sonic black hole will radiate thermal phonons at the temperature \(\kappa_s/2\pi\), where \(\kappa_s\) is the gradient of the velocity field at the horizon. (Numerically, \(\kappa_s/2\pi \approx 10^{-7}K (\partial v/100\text{m/s}/1\text{mm}).\) This is already interesting, but the real reason Unruh invented this model was in the hope of studying both the consequences of the molecular nature of the fluid and the quantum backreaction. Since the molecular structure has a preferred local frame—the comoving frame of the fluid—the \textit{molecular} fluid model lacks covariance in the fluid metric. That is, the molecular fluid model displays a non-Lorentz-invariant cutoff.

Consider the sonic black hole with liquid Helium-4 as the fluid. The sound field describes the quasiparticle excitations, and satisfies the usual wave equation for long wavelengths. The atomic nature of the fluid begins to show itself in the nonlinearity of the quasiparticle dispersion relation \(\omega(k)\) at shorter wavelengths. This nonlinearity implies that the group velocity is \(k\)-dependent, at first dropping as \(k\) grows. This has important implications for the propagation of wavepackets in the sonic black hole background. In particular, one can infer that an outgoing wavepacket propagated backwards in time toward the sonic horizon will be blueshifted, hence its comoving group velocity will drop. The packet will not quite make it all the way to the horizon, but will stop where its comoving group velocity is equal to the negative of the background flow velocity. This is an unstable situation, and what happens at this stage is remarkable: the packet is blueshifted a bit more, the comoving group velocity drops a bit more, and the packet is swept back away from the horizon by the (time-reversed) flow! This is very different from the behavior of a wavepacket
satisfying the ordinary wave equation, which is to get squeezed more and more up against the black hole horizon, while exponentially blueshifting.

5.2. Unruh model

It is not necessary to deal with liquid Helium to study the behavior just described. All that is essential is the curvature of the dispersion relation, and this can be modeled in linear field theory with higher spatial derivative terms. Since in the case of a fluid the dispersion relation is physically specified in the comoving frame, the comoving time derivative should be left unchanged. Unruh studied the Hawking process in a model of this sort, by numerically integrating the partial differential equation governing the propagation of a wavepacket. Because of the reversal of wavepacket motion at the horizon, it is possible to deduce the final state from an initial condition on ingoing modes far from the horizon, even in a stationary background (see Section 5.3 for further discussion of this issue). Unruh found, to numerical accuracy of his computation, that the usual Hawking flux is emitted from the hole. Subsequently the same and similar models were studied using both analytical approximations, and by first using stationarity to reduce the problem to the solution of ordinary differential equations.

Using the last described method, it was possible to reliably compute deviations from the usual thermal spectrum of Hawking radiation.

Here I will describe the Unruh model and some of its generalizations without further reference to the fluid flow analogy. The model consists of a free, hermitian scalar field propagating in a two dimensional black hole spacetime. The dispersion relation for the field lacks Lorentz invariance, and is specified in the free fall frame of the black hole, that is, the frame carried in from the rest frame at infinity by freely falling trajectories. Let \( u^\alpha \) denote the unit vector field tangent to the infalling worldlines, and let \( s^\alpha \) denote the orthogonal, outward pointing, unit vector, so that \( g^{\alpha\beta} = u^\alpha u^\beta - s^\alpha s^\beta \) (see Fig. 5). The action is assumed to have the form:

\[
S = \frac{1}{2} \int d^2 x \sqrt{-g} g^{\alpha\beta} \mathcal{D}_\alpha \phi \ast \mathcal{D}_\beta \phi, \tag{29}
\]

where the modified differential operator \( \mathcal{D}_\alpha \) is defined by

\[
\begin{align*}
u^\alpha \mathcal{D}_\alpha &= v^\alpha \partial_\alpha, \\
s^\alpha \mathcal{D}_\alpha &= \hat{F}(s^\alpha \partial_\alpha).
\end{align*}
\tag{30}
\tag{31}

The time derivatives in the local free fall frame are thus left unchanged, but the orthogonal spatial derivatives are replaced by \( \hat{F}(s^\alpha \partial_\alpha) \). The function \( \hat{F} \) determines the dispersion relation. For the moment it will be left unspecified. Invariance of the action (29) under constant phase transformations of \( \phi \) guarantees that there is a conserved current for solutions and a conserved “inner product” for pairs of solutions to the equations of motion. However, since \( \mathcal{D}_\alpha \) is not in general a derivation, simple
integration by parts is not allowed in obtaining the equations of motion or the form of the current. We shall obtain these below after further specifying the model.

The black hole line elements we shall consider are static and have the form

\[ ds^2 = dt^2 - (dx - v(x) dt)^2. \]  

This is a generalization of the Lemaître line element for the Schwarzschild spacetime, which is given by \( v(x) = -\sqrt{2M/x} \) (together with the usual angular part). We shall assume \( v < 0 \), \( dv/dx > 0 \), and \( v \to v_o \) as \( x \to \infty \). \( \delta_t \) is a Killing vector, of squared norm \( 1 - v^2 \), and the event horizon is located at \( v = -1 \). The curves given by \( dx - v dt = 0 \) are timelike free fall worldlines which are at rest (tangent to the Killing vector) where \( v = 0 \). Since we assume \( v < 0 \) these are \textit{ingoing} trajectories. \( v \) is their coordinate velocity, \( t \) measures proper time along them, and they are everywhere orthogonal to the constant \( t \) surfaces. (See Fig. 5) We shall refer to the function \( v(x) \) as the \textit{free-fall velocity}. The asymptotically flat region corresponds to \( x \to \infty \).

In terms of the notation above, the orthonormal basis vectors adapted to the free fall frame are given by \( u = \partial_t + v \partial_x \) and \( s = \partial_x \), and and in these coordinates \( g = -1 \). Thus the action (29) becomes

\[ S = \frac{1}{2} \int dt dx \left( |(\partial_t + v \partial_x) \phi|^2 - |\hat{F}(\partial_x) \phi|^2 \right). \]  

(33)

If we further specify that \( \hat{F}(\partial_x) \) is an odd function of \( \partial_x \), then integration by parts yields the field equation

\[ (\partial_t + \partial_x v)(\partial_t + v \partial_x) \phi = \hat{F}^2(\partial_x) \phi. \]  

(34)
The conserved inner product in this case is given by

$$ (\phi, \psi) = i \int dx \left( \phi^* (\partial_t + v \partial_x) \psi - \psi (\partial_t + v \partial_x) \phi^* \right), $$

(35)

where the integral is over a constant $t$ slice and is independent of $t$ if $\phi, \psi$ satisfy the field equation (34). The inner product can of course be evaluated on other slices as well, but it does not take the same simple form on other slices.

The dispersion relation for this model in flat spacetime, or in the local free fall frame, is given by

$$ \omega^2 = F^2(k), $$

(36)

where $F(k) \equiv -i \dot{F}(ik)$. Unruh’s choice for the function $F(k)$ has the property that $F^2(k) = k^2$ for $k \ll k_0$ and $F^2(k) = k_0^2$ for $k \gg k_0$, where $k_0$ is a wavevector characterizing the scale of the new physics. We usually think of $k_0$ as being around the Planck mass. Specifically, he considered the functions

$$ F(k) = k_0 \{ \tanh[(k/k_0)^n] \}^{\frac{1}{n}}. $$

(37)

Of course there are many other modifications one could consider. Perhaps the simplest is given by

$$ F^2(k) = k^2 - k^4/k_0^2. $$

(38)

This dispersion relation has the same behaviour for small $k$ as the one above, but behaves quite differently for large $k$. It is the one which was studied in Ref. 21. These two dispersion relations are plotted in Fig. 6 along with the dispersion relations for the ordinary wave equation and for elementary excitations of liquid helium-4.

![Fig. 6. The dispersion relations for (a) the massless wave equation, (b) the Unruh model (37), (c) the quartic model (38), and (d) liquid helium-4.](image)

5.2.1. Quantization

To quantize the field we assume the field operator $\hat{\phi}(x)$ is self-adjoint and satisfies the equation of motion (34) and the canonical commutation relations. In setting up
the canonical formalism, it is simplest to use the time function and evolution vector for which only first order time derivatives appear in the action. (Otherwise one must introduce extra momenta which are constrained, and then pass to the reduced phase space.) This just means that we define the momenta by

\[ \pi = \delta L/\delta (\partial_t \phi) = (\partial_t + v \partial_x) \phi, \]  

(39)
i.e., \( \pi \) is the time derivative along the free-fall world lines. The equal time canonical commutation relations are then \( [\phi(x), \pi(y)] = i \delta(x, y) \), as usual.

We define an annihilation operator corresponding to an initial data set \( f \) on a surface \( \Sigma \) by

\[ a(f) = (f, \hat{\phi}), \]  

(40)
where the inner product is evaluated on \( \Sigma \). If the data \( f \) is extended to a solution of the field equation then we can evaluate the inner product in (40) on whichever surface we wish. The hermitian adjoint of \( a(f) \) is called the creation operator for \( f \) and it is given by

\[ a^\dagger(f) = -(f^*, \hat{\phi}). \]  

(41)
The commutation relations between these operators follow from the canonical commutation relations satisfied by the field operator. The latter are equivalent to

\[ [a(f), a^\dagger(g)] = (f, g), \]  

(42)
provided this holds for all choices of \( f \) and \( g \). Now it is clear that only if \( f \) has positive, unit norm are the appelations “annihilation” and “creation” appropriate for these operators. From (12) and the definition of the inner product it follows identically that we also have the commutation relations

\[ [a(f), a(g)] = -(f, g^*), \quad [a^\dagger(f), a^\dagger(g)] = -(f^*, g). \]  

(43)

A Hilbert space of “one-particle states” can be defined by choosing a decomposition of the space \( S \) of complex initial data sets (or solutions to the field equation) into a direct sum of the form \( S = S_p \oplus S_p^* \), where all the data sets in \( S_p \) have positive norm and the space \( S_p \) is orthogonal to its conjugate \( S_p^* \). Then all of the annihilation operators for elements of \( S_p \) commute with each other, as do the creation operators. A “vacuum” state \( |\Psi\rangle \) corresponding to \( S_p \) is defined by the condition \( a(f)|\Psi\rangle = 0 \) for all \( f \) in \( S_p \), and a Fock space of multiparticle states is built up by repeated application of the creation operators to \( |\Psi\rangle \).

It is not necessary to construct a specific Fock space in order to study the physics of this system. In fact, any individual positive norm solution \( p \) defines annihilation and creation operators and a number operator \( N(p) = a^\dagger(p)a(p) \). The physical significance of the number operator depends of course on the nature of \( p \).

There are two types of positive norm wavepackets in which we are interested. The first are those corresponding to the quanta of Hawking radiation. These have
positive Killing frequency, that is, they are sums of solutions satisfying $\partial_t \phi = -i\omega \phi$ with $\omega > 0$. It is not obvious that such solutions have positive norm in the inner product (35), and in fact they do not in general. However, using the fact that the Killing frequency is conserved, we know that if a positive Killing frequency wavepacket were to propagate out to infinity (or any other region where $v = 0$), the integral for its norm would be manifestly positive. Since the norm is conserved, this suffices.

The other type of positive norm wavepackets we shall employ are those which correspond to particles as defined by the free-fall observers. These have positive free-fall frequency, that is, they are sums of solutions satisfying $(\partial_t + v\partial_x)\phi = -i\omega' \phi$, with $\omega' > 0$, on some time slice. These have manifestly positive norm, although the free-fall frequency is not conserved.

5.2.2. Computing the particle creation rate

In this subsection we obtain the explicit expression for the particle creation rate in terms of the norm of the negative frequency part of the ingoing wavepacket that corresponds to a given outgoing wavepacket. Let $\psi_{out}$ denote a wavepacket solution of the field equation (34) which at late times is outgoing and localized in the constant velocity region, where it has only positive free-fall frequency components, and only positive Killing frequency components. Propagating this data backwards in time, a part will reflect off the geometry outside the black hole and arrive back in the constant velocity region again as a wavepacket $\psi_r$ with wavevector components at the small negative $k(\omega)$ root with positive free-fall frequency. (If the ordinary two-dimensional wave equation were satisfied, this reflected part would vanish due to conformal invariance. However the field equation (34) lacks conformal invariance at high wavevectors, so in general there will be some reflection, which will be extremely small unless the metric has short wavelength features.) The rest of the wavepacket will reach within a distance of order $k_0^{-1}$ from the horizon, where it will undergo mode conversion and head back out, ending up in the constant velocity region as a combination of positive and negative free fall frequency wavepackets $\psi_+$ and $\psi_-$. The mode conversion process will be explained in the next section.

Since the inner product is time independent we have

$$ (\psi_{out}, \hat{\phi}) = (\psi_r, \hat{\phi}) + (\psi_+, \hat{\phi}) + (\psi_-, \hat{\phi}), $$

(44)

or in terms of annihilation and creation operators,

$$ a(\psi_{out}) = a(\psi_r) + a(\psi_+) - a^\dagger(\psi^-). $$

(45)

We assume that the state of the field at early times is the free-fall vacuum, $|ff\rangle$, which satisfies $a(\psi)|ff\rangle = 0$ for any ingoing positive free-fall frequency wavepacket $\psi$. The particle creation in the packet $\psi_{out}$, characterized by the expectation value of the
number operator, is thus given by the norm of $\psi_-$:

$$N(\psi_{out}) = \langle ff|a^\dagger(\psi_{out})a(\psi_{out})|ff\rangle = -(\psi_-, \psi_-).$$

(46)

6. The Hawking effect and mode conversion

In this section I will first describe the results of the particle production calculation in the Unruh model. These results will then be explained with the help of the WKB approximation. The resulting picture has two essential features: reversal of group velocity without reflection, and “mode conversion” from one branch of the dispersion relation to another. Interestingly, both these phenomena can occur for linear waves in inhomogeneous plasmas\textsuperscript{22,23,24}, and undoubtedly occur in many other settings as well. At the turn-around point the pure WKB approximation breaks down, and partial mode conversion from a positive free-fall frequency to a negative free-fall frequency wave takes place. This mode conversion gives rise to the Hawking effect.

Unruh’s computation\textsuperscript{19} starts with a purely outgoing, low wavevector, positive Killing frequency wavepacket in the asymptotic region where $v(x)$ is constant (see top left graph in Fig. 7 for the wavepacket, top right graph for its power spectrum, taken from Ref. \textsuperscript{19}). Numerically integrating the partial differential equation (34) backwards in time, he finds that this wavepacket bounces off the horizon (middle left graph of Fig. 7) and moves back out to the asymptotic region with only very high wavevector components of both signs (bottom graphs of Fig. 7). The sign of the wavevector coincides with the sign of the free-fall frequency, so this ‘final’ wavepacket (which is actually the initial wavepacket going forward in time) possesses a negative norm component as seen from the power spectrum in the bottom right graph of Fig. 7. Evaluating this norm Unruh obtains the occupation number (46) for the outgoing packet. In all cases, the resulting occupation number agrees, to the accuracy of the computation, with the thermal prediction of Hawking. The computation contains alot more information than just this one number however, since a complete description of the initial wavepacket is obtained. Since the field equation is linear, one can simply fourier analyze the initial and final packets and read off the Bogoliubov coefficients for all wavevectors. Unruh did just this, and again found agreement with the ideal Hawking effect.

These results have since been confirmed in two further calculations. In one approach\textsuperscript{20}, Brout, Massar, Parentani and Spindel (BMPS) introduced an analytical approximation scheme in which the field equation near the horizon is solved by fourier transform. Using this technique, they showed that the spectrum of emitted radiation is thermal in leading order. In another approach, Corley and Jacobson\textsuperscript{21} adopted the dispersion relation (38) and exploited stationarity of the metric to reduce the PDE to a fourth order ODE for each Killing frequency, which was then solved either numerically or, in some cases, analytically. This technique made it possible to
Fig. 7. A wavepacket bouncing off the horizon in the Unruh model, taken from Unruh’s numerical integration results. The dispersion relation is \( \text{(37)} \), with \( k_0 = 512 \) and \( n = 1 \), and the Hawking temperature is \( T = 1.37 \). On the left is the wavepacket at three different times, with time going forward from bottom to top. The horizon is located at about \( x = 0.704 \), and the black hole lies to the right. On the right are power spectra in \( k \) of the initial and final wavepackets.
go beyond leading order and compute the deviations from a perfectly thermal spec-
trum. These deviations are relatively small at low frequencies where the Hawking
radiation is copious, but are so large at larger frequencies that they eventually convert
the exponential tail of the Planck spectrum to an increasing function. Nevertheless,
the total flux associated with these higher frequencies is very small.

6.1. Group velocity reversal and mode conversion

The behavior of a wavepacket propagated back in time can be understood qual-
itatively as follows. Let $\phi = e^{-i(\omega t - kx)}$, substitute into the equation of motion, and
drop terms containing derivatives in the free fall velocity. This is equivalent to using
the WKB approximation, and yields the approximate position dependent dispersion
relation

$$(\omega - v(x)k)^2 = F^2(k).$$

(47)

This is just the dispersion relation in the local free-fall frame, since the free-fall
frequency $\omega'$ is related to the Killing frequency $\omega$ by

$$\omega' = \omega - v(x)k.$$  

(48)

The position-dependent dispersion relation is useful for understanding the motion of
wavepackets that are somewhat peaked in both position and wavevector. A graphical
method we have employed is described below. The same method was used by
BMPS, who also found a Hamiltonian formulation for the wavepacket propagation
using Hamilton-Jacobi theory.

Graphs of both sides of equation (47) are shown in figure 8 for $F(k)$ given by
and for two different values of $v$. As $x$ varies, the slope $-v(x)$ ($= |v(x)|$) of the straight line representing the left hand side of (17) changes, but for a given solution the intercept $\omega$ is fixed since the Killing frequency is conserved. For a given $x$, the intersection points on the graph correspond to the possible wavevectors in this approximation. The coordinate velocity $dx/dt$ of a wavepacket is the group velocity $v_g = d\omega/dk$, which may also be expressed, using (48), as $v_g = d\omega'/dk + v(x)$. Note that $v'_g \equiv d\omega'/dk$ is the group velocity of the packet in the free-fall frame, and corresponds to the slope of the curved line in figure 8.

Now assume the free fall velocity function asymptotes (at large positive $x$) to a value $v_o$ satisfying $-1 < v_o < 0$, and consider a narrow wavepacket located far from the hole, centered about frequency $\omega$, and containing only $k$ values around the smaller positive root (intersection point) shown in the figure. This is an outgoing wavepacket, since $d\omega'/dk > |v|$. Therefore, going backwards in time, the packet moves towards the hole. As $x$ decreases $|v(x)|$ increases, so the slope of the straight line increases, until eventually the straight line becomes tangent to the dispersion curve. At this point $v_g$ drops to zero. If $\omega$ is very small compared to $k_0$, then this stopping point $x_t$ occurs when $v(x)$ is very close to $-1$, that is, just barely outside the horizon.

What happens at the stopping point? It was incorrectly suggested in Ref. 4 that the wavepackets just asymptotically approach limiting position $x_t$ and wavevector $k_t$. However, at the stopping point, the simple picture of a wavepacket peaked around a single wavevector has broken down. The wave packet has a significant spread in wavevectors which fail to satisfy the local dispersion relation. Furthermore, in this region there are nearby solutions to the dispersion relation. Just outside $x_t$, the straight line cuts the dispersion curve in two nearby points which approach each other as $x$ approaches $x_t$. Simple WKB analysis is unable to say what happens. But a supplementary consideration gives the answer. As pointed out by Unruh 18, this is an unstable situation: for $k$ slightly above $k_t$ the group velocity drops below zero (i.e. the comoving group velocity drops below the magnitude of the free-fall velocity) so, backwards in time, the wavepacket tends to move back away from the horizon. Once this begins to happen, $k$ continues to increase as the wavepacket moves further away. Exactly this behavior was found in Unruh’s numerical solution 19 to the PDE.

This is not the end of the story however. There is yet another “nearby” solution to the dispersion relation, on the negative wavevector branch, which is excited as well. This process is an example of “mode conversion” 23 24. The mode conversion is strong if the wavelength and phase velocity of one mode is close to that of another. It appears that this condition for strong conversion is not met, since the negative $k$ solution $k_{t-}$ to (17) at $x_t$ is not very close in magnitude to $k_t$ (see Fig. 8.) However, one must remember that the wavepacket here has a significant spread in wavevectors which fail to satisfy the local dispersion relation. Thus the wavepacket can pick up other modes that fulfill the conversion criterion. That the wavepacket must pick up a negative wavevector piece is clear in the case of the ordinary wave equation,
since there causality implies that the wavepacket strictly vanishes inside the horizon, which cannot be accomplished with only positive wavevectors. In the Unruh model, the wavepacket satisfies approximately the ordinary wave equation until it gets very close to the horizon, so one should expect that a similar amount of mode conversion takes place. Evidently the negative wavevector mode does couple in strongly for sufficiently small \( \omega \), as shown both by Unruh’s solution of the PDE and by the ODE methods applied by BMPS\(^{20}\) and ourselves\(^{21}\). The “converted”, negative wavevector, wavepacket also has a negative group velocity and so also moves, backwards in time, away from the hole. The end result thus consists of two wavepackets, one constructed of large positive \( k \) wavevectors and the other of large negative \( k \) wavevectors, both propagating away from the hole and reaching the asymptotically flat (constant free fall velocity) region. Note that the negative wavevector component will consist of wavevectors of magnitude slightly larger than the positive wavevector component. This is clear from the position of the intersection points in Fig. 8 and is borne out in Unruh’s computation (Fig. 7) as well. The number of created particles in the final, late time, wavepacket is given by the norm of the negative wavevector part of the initial, early time wavepacket.

7. The stationarity puzzle

Particle production via the Hawking effect in the case of a black hole that forms in a collapse process is ordinarily understood to depend critically on the fact that the metric is not static for all times. Although the radiating modes come in from infinity and go back out to infinity, their Killing frequency is not conserved because they propagate through the time dependent part of the spacetime. The way the usual Hawking effect transpires in a strictly stationary spacetime is that the outgoing wavepackets are traced backwards to parts that do not make it back out to infinity, but rather cross the white hole horizon, at which point the Unruh boundary condition on the quantum state is imposed. The piece of the wavepackets that scatters off the curvature and does make it back out to infinity is not associated with particle creation. If, as in the Unruh model, the entire wavepacket turns around and goes back out, remaining for all time in the stationary region of the spacetime, then it would seem that there can be no particle creation at all. So how does the Unruh model yield a nontrivial Hawking flux?

The first answer is that the wavepackets have not been followed all the way out to infinity. Another way to put it is that we have taken the free fall frame, in which the boundary condition is imposed, to be moving towards the black hole at infinity, rather than coinciding with the rest frame of the black hole at infinity. (Technically, this corresponds to the fact that the metric function \( v(x) \) does not vanish at infinity.) But what happens if we take \( v(x) \) to vanish at infinity, and continue to follow the wavepackets backwards in time, not stopping to impose the quantum state boundary
condition until the packets reach infinity—or do something else? Do the wavepackets propagating backwards in time ever reach infinity? Is there any Hawking radiation?

What happens in the Unruh model with the dispersion relation (37) is that the magnitude of the wavevector grows without bound as the wavepacket moves outward where \( v(x) \) is falling to zero. Thus, even though the difference between the free-fall and Killing frames is going to zero, the wavevector is diverging in such a way that the wavepacket always maintains a negative free-fall frequency part of the same, negative, norm. Thus the Hawking effect indeed occurs. From this analysis we see that the Unruh model, while it entails a strict cutoff in free-fall frequency, involves in an essential way arbitrarily high wavevectors, i.e., arbitrarily short wavelengths. Insofar as we wish to explore the consequences of a fundamental short distance cutoff on the Hawking effect, this is an unsatisfactory feature of the model. The outgoing modes emerging from the black hole region still arise from arbitrarily short wavelength modes. Using the dispersion relation (38) on the other hand, the wavevectors are bounded by \( k_0 \). In this case something very strange seems to happen, which remains to be understood: the positive norm piece of the wavepacket goes backwards in time out to infinity at superluminal group velocity, and the velocity of the negative norm piece goes past positive infinity to negative infinity, which seems to result in a propagation “back to the future”.

In search of a more realistic model, it is interesting to go back and consider the behavior of a wavepacket propagated backwards in time using the dispersion relation of liquid helium-4 (see Fig. 4). Up to the reversal of group velocity outside the horizon, the behavior is the same as for the Unruh model. After that (i.e. before that) the packet will go over the first maximum of the dispersion curve, at which point its comoving group velocity changes sign, which only pushes it away from the horizon even faster. Eventually, however, it approaches another turnaround point, near the roton minimum, where the free-fall frequency line becomes tangent once again to the dispersion curve. It seems reasonable to suppose that what happens here is another reversal of direction and further mode conversion: the wavepacket continues along the dispersion curve, falling back towards the horizon, until it runs off the end of the quasi-particle spectrum, where it is presumably unstable and has a wavelength on the order of the interatomic spacing. Or, perhaps, the instability sets in even before this point. It thus appears that there is no way to analyze the helium model fully at the level of quasi-particle field theory. Rather, the many particle dynamics must be directly confronted. It would be very interesting, though perhaps very difficult, to analyze this many body problem.

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