$N$-dimensional $sl(2)$-coalgebra spaces with non-constant curvature

A. Ballesteros$^a$∗ A. Enciso$^b$† F.J. Herranz$^a$‡ O. Ragnisco$^c$§

$^a$ Depto. de Física, Universidad de Burgos, 09001 Burgos, Spain
$^b$ Depto. de Física Teórica II, Universidad Complutense, 28040 Madrid, Spain
$^c$ Dip. di Fisica, Università di Roma 3, and Istituto Nazionale di Fisica Nucleare, Via Vasca Navale 84, 00146 Rome, Italy

Abstract

An infinite family of $ND$ spaces endowed with $sl(2)$-coalgebra symmetry is introduced. For all these spaces the geodesic flow is superintegrable, and the explicit form of their common set of integrals is obtained from the underlying $sl(2)$-coalgebra structure. In particular, $ND$ spherically symmetric spaces with Euclidean signature are shown to be $sl(2)$-coalgebra spaces. As a byproduct of this construction we present $ND$ generalizations of the classical Darboux surfaces, thus obtaining remarkable superintegrable $ND$ spaces with non-constant curvature.

PACS: 02.30.Ik 02.40.Ky
KEYWORDS: Integrable systems, geodesic flow, coalgebras, curvature, Darboux spaces.

1 Introduction

An $N$-dimensional ($ND$) Hamiltonian $H^{(N)}$ is called completely integrable if there exists a set of $(N-1)$ globally defined, functionally independent constants of the motion that Poisson-commute with $H^{(N)}$. Whereas completely integrable systems are quite unusual [1], they have long played a central role in our understanding of dynamical systems and the analysis of physical models. Moreover, in case that some additional independent integrals do exist, the system $H^{(N)}$ is called superintegrable [2] (there are different degrees of superintegrability, as we shall point out later). It is well known that superintegrability is strongly related to the separability of the corresponding Hamilton–Jacobi and Schrödinger equations [3] in more than one coordinate systems, and gives a fighting chance (which can be made precise in several contexts) of finding the general solution of the equations of motion by quadratures [4, 5].

In this paper we consider a specific class of (classical) $ND$ Hamiltonian systems: the geodesic flows on $ND$ Riemannian manifolds defined by the corresponding metrics. Contrary to what happens in the constant curvature cases, these kinetic-energy Hamiltonians can exhibit extremely complicated dynamics in arbitrary manifolds, the prime example being the chaotic geodesic flow on Anosov spaces. The

∗angelb@ubu.es
†aenciso@fis.ucm.es
‡fjherranz@ubu.es
§ragnisco@fis.uniroma3.it
complete integrability of a free Hamiltonian on a curved space and the separability of its Hamilton–
Jacobi equation are rather nontrivial properties, and the analysis of such systems is being actively
pursued because of its significant connections with the geometry and topology of the underlying man-
ifold [6, 7].

In physics, curved (pseudo-)Riemannian manifolds (generally, of dimension higher than four) arise
as the natural arena for general relativity, supergravity and superstring theories, and integrable geodesic
flows in arbitrary dimensions are thus becoming increasingly popular in these areas [8]. Particularly,
the case of Kerr–AdS spaces has attracted much attention due to its wealth of applications [9, 10, 11, 12].
In the studies performed so far, the explicit knowledge of the Stäckel–Killing integrals of motion in
Kerr–AdS spaces has already proven to be an essential ingredient in these contexts (see e.g. [13, 14, 15]
and references therein), which suggests that an explicit analysis of integrable geodesic flows on curved
manifolds would certainly meet with interest from this viewpoint. Very recently, an in-depth analysis
of the integrability properties and separability of the Hamilton–Jacobi equation on Kerr–NUT–AdS
spacetimes have been achieved in [16, 17, 18], thus showing the relevance of developing the required
machinery to deal with superintegrable spaces of non-constant curvature.

From a quite different perspective, quantum groups (in an \(sl_z(2)\) Poisson coalgebra version) have
been recently used to generate a family of distinguished \(ND\) hyperbolic spaces whose curvature is
governed by the deformation parameter \(z\) [19]. In these \(sl_z(2)\)-coalgebra spaces the geodesic flow
is completely integrable and the corresponding \((N−1)\) quadratic first integrals (which give rise to
generalized Killing tensors) are explicitly known. Moreover, these flows turn out to be superintegrable,
since the quantum \(sl_z(2)\)-coalgebra symmetry provides an additional set of \((N−2)\) integrals. We stress
that in several interesting situations (such as in the \(N = 2\) case), Lorentzian analogs of these spaces
can be obtained through an analytic continuation method; this procedure has actually been used to
construct a new type of \((1 + 1)\)D integrable deformations of the (Anti-)de Sitter spaces [20].

In this letter we present a class of \(ND\) spaces with Euclidean signature whose geodesic flow is,
by construction, superintegrable. This is achieved by making use of an undeformed Poisson \(sl(2)\)-
coalgebra symmetry. Furthermore, their \((2N−3)\) constants of the motion, which turn out to be
quadratic in the momenta, are given in closed form. In fact, these invariants have the same form for all
the spaces under consideration as a direct consequence of the underlying Poisson coalgebra structure,
so we can talk about “universal” first integrals. As it has been pointed out in [21], spaces of constant
curvature belong to this class of \(sl(2)\)-coalgebra spaces, but the former are only a small subset of the
superintegrable spaces that can be obtained through this construction. Here we shall present four
new significant \(ND\) examples with non-constant scalar curvature: the \(ND\) generalizations of the so-
called (2D) Darboux spaces, which are the only surfaces with non-constant curvature admitting two
functionally independent, quadratic integrals [22, 23, 24].

The paper is organized as follows. In the next section we briefly sketch the construction of generic
\(sl(2)\)-coalgebra spaces and discuss their superintegrability properties; we also show how spherically
symmetric spaces (with non-constant curvature) arise in this approach. In Section 3 we exploit the
\(sl(2)\)-coalgebra symmetry of the 2D Darboux spaces to construct \(ND\) counterparts. Some brief remarks
of global nature are made. Finally, the closing section includes some comments and open problems.

2 \(sl(2)\)-coalgebra spaces and superintegrability

An \(ND\) completely integrable Hamiltonian \(H^{(N)}\) is called \textit{maximally superintegrable} (MS) if there
exists a set of \(2N−2\) functionally independent global first integrals that Poisson-commute with \(H^{(N)}\).
As is well known, at least two different subsets of \(N−1\) constants in involution can be found among
them. In the same way, a system will be called quasi-maximally superintegrable (QMS) if there are \(2N - 3\) independent integrals with the aforementioned properties, i.e., if the system is “one integral away” from being MS.

Let us now consider the \(sl(2)\) Poisson coalgebra generated by the following Lie–Poisson brackets and comultiplication map:

\[
\{J_3, J_+\} = 2J_+, \quad \{J_3, J_-\} = -2J_-, \quad \{J_-, J_+\} = 4J_3, \\
\Delta(J_l) = J_l \otimes 1 + 1 \otimes J_l, \quad l = +, -, 3.
\]

The Casimir function is \(C = J_+J_3 - J_-^2\). Then, the following result holds \[21\]: Let \(\{q, p\} = \{(q_1, \ldots, q_N), (p_1, \ldots, p_N)\}\) be \(N\) pairs of canonical variables. The ND Hamiltonian

\[
H^{(N)} = \mathcal{H}(J_-, J_+, J_3),
\]

with \(\mathcal{H}\) any smooth function and

\[
J_- = \sum_{i=1}^N q_i^2, \quad J_+ = \sum_{i=1}^N \left( p_i^2 + \frac{b_i}{q_i^2} \right) \equiv \mathbf{p}^2 + \sum_{i=1}^N \frac{b_i}{q_i^2}, \quad J_3 = \sum_{i=1}^N q_i p_i \equiv \mathbf{q} \cdot \mathbf{p},
\]

where \(b_i\) are arbitrary real parameters, is a QMS system. The \((2N - 3)\) functionally independent “universal” integrals of motion for \(H^{(N)}\) read

\[
C^{(m)} = \sum_{1 \leq i < j}^m \left\{ (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \right\} + \sum_{i=1}^m b_i,
\]

\[
C^{(m)} = \sum_{N-m+1 \leq i < j}^N \left\{ (q_i p_j - q_j p_i)^2 + \left( b_i \frac{q_j^2}{q_i^2} + b_j \frac{q_i^2}{q_j^2} \right) \right\} + \sum_{i=N-m+1}^N b_i
\]

where \(m = 2, \ldots, N\) and \(C^{(N)} = C(N)\). Moreover, the sets of \(N\) functions \(\{H^{(N)}, C^{(m)}\}\) and \(\{H^{(N)}, C^{(m)}\}\) \((m = 2, \ldots, N)\) are in involution.

The proof of this result is based on the fact that, for any choice of the function \(\mathcal{H}\), the Hamiltonian \(H^{(N)}\) has an \(sl(2)\) Poisson coalgebra symmetry \[19, 21\]; the generators \(\mathfrak{g}\) fulfil the Lie–Poisson brackets of \(sl(2)\) and the integrals \(\mathfrak{I}\) are obtained through the \(m\)-th coproducts of the Casimir \(C\) within an \(m\)-particle symplectic realization of type \(\mathfrak{g}\).

With the previous general result in mind, we shall say that an ND Riemannian manifold is an \(sl(2)\)-coalgebra space if the kinetic energy Hamiltonian \(H^{(N)}_T\) corresponding to geodesic motion on such a space has \(sl(2)\)-coalgebra symmetry, i.e., if \(H^{(N)}_T\) can be written as

\[
H^{(N)}_T = \mathcal{H}_T(\mathcal{J}_-, J_+, J_3) = \mathcal{H}_T \left( \mathbf{q}^2 + \mathbf{p}^2 + \sum_{i=1}^N \frac{b_i}{q_i^2}, \mathbf{q} \cdot \mathbf{p} \right),
\]

where \(\mathcal{H}_T\) is some smooth function on the \(sl(2)\)-coalgebra generators \(\mathfrak{g}\). Since \(H^{(N)}_T\) has to be homogeneous quadratic in the momenta, we are forced to restrict ourselves to the specific \(sl(2)\) symplectic realizations \[\mathfrak{g}\] with all \(b_i = 0\), so that the most general Hamiltonian corresponding to an \(sl(2)\)-coalgebra space reads

\[
\mathcal{H}_T = \mathcal{A}(\mathcal{J}_-) J_+ + \mathcal{B}(\mathcal{J}_-) J_3 = \mathcal{A}(q^2) \mathbf{p}^2 + \mathcal{B}(q^2) (\mathbf{q} \cdot \mathbf{p})^2,
\]

where}
where \( A \) and \( B \) are arbitrary functions. At this point, we stress that for any choice of both functions, \( H_T^{(N)} \) is a QMS Hamiltonian system with integrals given by (4). Thus, an infinite family of ND spaces with QMS geodesic flow is defined by (4) or, equivalently, by the pair of functions \((A, B)\) that will characterize the ND metric.

### 2.1 Spaces of constant curvature

In the previous discussion it is implicit that the pair \((q, p)\) is an arbitrary set of canonically conjugated positions and momenta, for which no \textit{a priori} geometric interpretation is given. This becomes apparent by considering the (simply connected) ND Riemannian spaces with constant sectional curvature \( \kappa \) (the sphere \( S^N \) and the hyperbolic \( H^N \) space), which are distinguished examples of \( \mathfrak{sl}(2) \)-coalgebra spaces. It can be shown [21] that the corresponding Hamiltonians can be written in the following ways (among others), both of them compatible with (6):

\[
H_T^P = \frac{1}{2} (1 + \kappa J_-) J_+ = \frac{1}{2} (1 + \kappa q^2) p^2,
\]

\[
H_T^B = \frac{1}{2} (1 + \kappa J_-) (J_+ + \kappa J_3^2) = \frac{1}{2} (1 + \kappa q^2) (p^2 + \kappa (q \cdot p)^2).
\]

(7)

The associated coordinate systems are classical: in the first case, \( q \) denotes the Poincaré coordinates in \( S^N \) or \( H^N \), coming from the stereographic projection in \( \mathbb{R}^{N+1} \) [25], whereas in the second one \( q \) are the Beltrami coordinates, which are associated with the central projection. We recall that the image of both the stereographic projection (Poincaré coordinates) and central projection (Beltrami coordinates) is the subset of \( \mathbb{R}^N \) determined by \( 1 + \kappa q^2 > 0 \), which means that for \( H^N \) with \( \kappa = -1 \) such an image is the open subset \( q^2 < 1 \). In both cases, we recover the standard Cartesian coordinates in \( \mathbb{R}^N \) when we set \( \kappa = 0 \).

Note that in this language both Hamiltonians can immediately be interpreted as deformations (in terms of the curvature parameter \( \kappa \)) of the motion on Euclidean space \( \mathbb{R}^N \), to which they reduce when \( \kappa = 0 \). This is analogous to the analysis in terms of the quantum parameter \( z \) carried out in [20]. By construction, the above Hamiltonians admit the universal integrals (4), whose concrete geometric realization depends on the interpretation of \((q, p)\) as either Poincaré or Beltrami coordinates. The generalized Killing vectors of these spaces are the Hamiltonian vector fields (in phase space) associated with the latter first integrals. It should be noted that these spaces admit in fact an additional first integral which makes them MS: while they are not strictly the only ones having this property (see [26] and references therein), this final symmetry is not of coalgebraic nature and, when it exists, must be found by \textit{ad hoc} methods.

### 2.2 Spherically symmetric spaces

Any ND spherically symmetric metric of the type

\[
ds^2 = f(|q|)^2 \, dq^2,
\]

(8)

where \(|q| = \sqrt{q^2}, \, dq^2 = \sum_i dq_i^2\) and \( f \) is an arbitrary smooth function, leads to a geodesic motion described by the Hamiltonian

\[
H_T^{(N)} = \frac{p^2}{f(|q|)^2},
\]

(9)
which is clearly of the form (4). Therefore, the metric (3) corresponds to an $sl(2)$-coalgebra space with
\[ A(J-) = \frac{1}{f(\sqrt{J-})^2}, \quad B(J-) = 0, \] (10)
so that for any choice of $f$ its geodesic flow defines a QMS system whose generalized Killing symmetries are the Hamiltonian vector fields associated with (4). It is apparent that these spaces are conformally flat, so that its Weyl tensor vanishes. The scalar curvature $R$ of (3), which is generally non-constant, can be computed to be
\[ R = -(N-1) \left( \frac{(N-4)f'(|q|)^2 + f(|q|) \left( 2f''(|q|) + 2(N-1)|q|^{-1}f'(|q|) \right)}{f(|q|)^4} \right). \] (11)

If we define $N$D spherical coordinates $(r, \theta_1, \ldots, \theta_{N-1})$ as
\[ q_j = r \cos \theta_j \prod_{k=1}^{j-1} \sin \theta_k, \quad q_N = r \prod_{k=1}^{N-1} \sin \theta_k, \]
where $j = 1, \ldots, N-1$, $r = |q|$ and hereafter a product $\prod_{k=1}^{0} = 1$, the metric (3) can be alternatively written as
\[ ds^2 = f(r)^2 (dr^2 + r^2 d\Omega_{N-1}^2). \]

Here
\[ d\Omega_{N-1}^2 = \sum_{j=1}^{N-1} d\theta_j^2 \prod_{k=1}^{j-1} \sin^2 \theta_k, \]
denotes the metric of the unit $(N-1)$-sphere $S^{N-1}$, with $d\Omega_1^2 = d\theta_1^2$. In these coordinates the free Hamiltonian (4) can be equivalently expressed as
\[ H_T^{(N)} = \frac{p_r^2 + r^{-2}L^2}{f(r)^2}, \]
where
\[ L^2 = \sum_{j=1}^{N-1} p_j^2 \prod_{k=1}^{j-1} (\sin \theta_k)^{-2}, \] (12)
is the squared angular momentum and $(p_r, p_\theta_1, \ldots, p_{\theta_{N-1}})$ are the conjugate momenta of $(r, \theta_1, \ldots, \theta_{N-1})$.

Finally, it is also convenient for our purposes to consider the modified spherical system given by $(\rho, \theta_1, \ldots, \theta_{N-1})$, where $\rho = \ln r$. If $p_\rho$ stands for the conjugate momentum of $\rho$, this yields
\[ ds^2 = F(\rho)^2 (d\rho^2 + d\Omega_{N-1}^2), \] (13)
\[ H_T^{(N)} = \frac{p_\rho^2 + L^2}{F(\rho)^2}, \] (14)
where the arbitrary function $F$ is defined as $F(\rho) = r f(r)$. Hence any metric of the form (13) defines an $sl(2)$-coalgebra space given by
\[ A(J-) = \frac{J_-}{F(\ln \sqrt{J_-})^2}, \quad B(J-) = 0, \] (15)
and the geodesic flows on (13) are QMS for any choice of $F$. The scalar curvature (11) now reads

$$R = -(N-1) \left( \frac{(N-4)F'\rho^2 - (N-2)F\rho^2 + 2F\rho F''(\rho)}{F(\rho)^4} \right).$$

(16)

3 Darboux spaces

The (2D) Darboux surfaces are the 2-manifolds with non-constant curvature admitting two quadratic first integrals, so that its geodesic motion is quadratically MS. There are only four types of such spaces [22], which we will represent by $D_i^{(2)} (i = I, II, III, IV)$ following the notation in [23, 24]. In this section, the spaces $D_i^{(2)}$ will be initially described in terms of isothermal coordinates $(u, v)$ with canonically conjugate momenta $(p_u, p_v)$. In [23, 24] it has been explicitly shown that their natural free Hamiltonians can be expressed in these variables as

$$H^{(2)} = \frac{p_u^2 + p_v^2}{F(u)^2},$$

which implies that

$$ds^2 = F(u)^2 (du^2 + dv^2).$$

(17)

(18)

Occasionally we shall need to consider other different isothermal charts of $D_i^{(2)}$ with isothermal coordinates $(\xi, \eta)$ and conjugate momenta $(p_\xi, p_\eta)$.

Immediately, from (18) we realize that the four 2D Darboux spaces $D_i^{(2)}$ are $sl(2)$-coalgebra spaces of the type $(A, 0)$ with $A(J_-)$ given by (15). As a consequence, we can use the underlying $sl(2)$-coalgebra symmetry to define ND, spherically symmetric, conformally flat generalizations $D_i^{(N)}$ of the Darboux surfaces that will be thoroughly described in the following subsections. By construction, the four ND spaces so constructed will have QMS geodesic motions and their $(2N-3)$ independent integrals will be given by (4). It should be highlighted that, at least for the space $D_{III}^{(N)}$ the additional integral giving rise to an ND MS system can be explicitly constructed [26].

3.1 Type I

The Hamiltonian for geodesic motion on $D_I^{(2)}$ is given by

$$H^{(2)}_I = \frac{p_u^2 + p_v^2}{u}.$$

Therefore the corresponding metric reads [23]

$$ds^2 = u(du^2 + dv^2).$$

(19)

The construction of the ND space $D_I^{(N)}$ can be conveniently performed via the substitution

$$u \to \rho = \ln r, \quad dv^2 \to d\Omega^2_{N-1}.$$

(20)

Note that we have used the same letter for the functions $F(u)$ and $F(\rho)$ appearing in the coordinate expressions of the metric (Eqs. (18) and (13)) because upon this substitution they define, in fact, the same function $F : \mathbb{R} \to \mathbb{R}^+$. 

6
Therefore, we find that the generic ND metric (13) is in this case characterized by the function $F(\rho) = \rho^{1/2}$, yielding the following metric for $D_1^{(N)}$:

$$ds^2 = \rho (d\rho^2 + d\Omega_{N-1}^2) = \frac{\ln |\mathbf{q}| d\mathbf{q}^2}{\mathbf{q}^2}. \quad (21)$$

In other words, $D_1^{(N)}$ is the sl(2)-coalgebra space $(\mathcal{A}, 0)$ given by

$$\mathcal{A}(J) = \frac{J}{\ln \sqrt{J}}. \quad (22)$$

This space is certainly not flat; its scalar curvature can be readily computed to be

$$R = \frac{(N - 1) \left(4(N - 2)\rho^2 - N + 6\right)}{4\rho^3}. \quad (23)$$

Some remarks on the global properties of the type I Darboux $N$-manifold are in order. We define $D_1^{(N)}$ to be the exterior of the closed unit ball $\mathcal{M}_+ = \{\mathbf{q} : |\mathbf{q}| > 1\}$, covered with the coordinates $\mathbf{q}$ and endowed with the metric (21). It is not difficult to see that this space is incomplete by integrating its radial geodesics. In fact, the radial motion on $D_1^{(N)}$ is obtained from the Lagrangian $L = r^{-2} \ln r \dot{r}^2 \equiv G(r)^2 \dot{r}^2$, so that a straightforward calculation shows that the radial geodesics are complete at infinity and incomplete at the hypersphere $|\mathbf{q}| = 1$ since the integral $\int G(r) dr$ diverges at infinity but converges at 1. It should be remarked that one can also replace the conformal factor $\ln r$ by its absolute value and define $D_1^{(N)}$ to be the interior of the unit ball $\mathcal{M}_- = \{\mathbf{q} : |\mathbf{q}| < 1\}$ together with the metric (21). The latter manifold is complete at 0 and incomplete at 1.

### 3.2 Type II

In this case the free Hamiltonian reads (24)

$$H_\text{II}^{(2)} = \frac{p_u^2 + p_v^2}{1 + u^{-2}}.$$ 

The QMS ND extension is performed again using the substitution (20). In this case, $F(\rho) = (1 + \rho^{-2})^{1/2}$ and we have

$$H_\text{II}^{(N)} = \frac{p_u^2 + L^2}{1 + \rho^{-2}} = \frac{\mathbf{q}^2}{1 + (\ln |\mathbf{q}|)^{-2}} \mathbf{b}^2.$$ 

Thus, $D_\text{II}^{(N)}$ is the sl(2)-coalgebra space determined by

$$\mathcal{A}(J) = \frac{J}{1 + (\ln \sqrt{J})^{-2}}. \quad (23)$$

7
The metric of $\mathcal{D}_{II}^{(N)}$ is then given by
\[ ds^2 = (1 + \rho^{-2})(d\rho^2 + d\Omega_{N-1}^2) = \frac{1 + (\ln|\mathbf{q}|)^{-2}}{\mathbf{q}^2} d\mathbf{q}^2, \]
with non-constant scalar curvature
\[ R = \frac{(N - 1)[N(\rho^3 + \rho)^2 - 1] - 2\rho^2(\rho^4 + 2\rho^2 + 4)]}{(\rho^2 + 1)^3}. \]

If we set $G(r)^2 = r^{-2}(1 + \ln^{-2} r)$, the same arguments discussed in the previous subsection show that the radial geodesics of $\mathcal{D}_{II}^{(N)}$ are complete at 0, 1 and at infinity. Hence both manifolds $(\mathcal{M}_0, ds^2)$ are complete.

### 3.3 Type III

The 2D free Hamiltonian reads now [24]
\[ H_{III}^{(2)} = e^{2u} \left( \frac{e^{-u}}{1 + e^{u}}(q^2 + p_e^2) \right). \]

In order to obtain the ND spherically symmetric generalization, it suffices to take $F(\rho) = e^{-\rho}(1+e^\rho)^{1/2}$ and apply the map [20], so that the metric of $\mathcal{D}_{III}^{(N)}$ becomes
\[ ds^2 = e^{-2\rho}(1 + e^\rho)(d\rho^2 + d\Omega_{N-1}^2) = \frac{1 + |\mathbf{q}|}{\mathbf{q}^4} d\mathbf{q}^2, \]
and the space is characterized by
\[ A(J_-) = \frac{J_-^2}{1 + \sqrt{J_-}}. \] (24)

The scalar curvature reads (recall that $r = e^\rho$):
\[ R = \frac{r^3(N - 1)[N(3r + 4) - 6(r + 2)]}{4(r + 1)^3}. \]

In this case the radial geodesics are obtained from the Lagrangian $L = r^{-4}(1+r) r^2 \equiv G(r)^2 r^2$. The integral $\int G(r) \, dr$ diverges at 0 but is finite at $\infty$, and therefore $\mathcal{D}_{III}^{(N)} = (\mathbb{R}^N \setminus \{0\}, ds^2)$ is complete at 0 but incomplete at $\infty$ (i.e., free particles in this space escape to infinity in finite time). Again, the whole manifold is conveniently described in terms of the coordinates $\mathbf{q}$.

It should be pointed out that the Hamiltonian $H_{III}^{(2)}$ can be written in a different coordinate system $(\xi, \eta)$ as
\[ H_{III}^{(2)} = \frac{p_\xi^2 + p_\eta^2}{1 + \xi^2 + \eta^2}, \]
which admits the ND coalgebraic generalization
\[ H_{III}^{(N)} = \frac{p^2}{1 + \mathbf{q}^2} = \frac{J_+}{1 + J_-}. \]

The complete manifold $(\mathbb{R}^N \setminus \{0\}, (1 + \mathbf{q}^2) d\mathbf{q}^2)$ was thoroughly studied in [20], showing that it is in fact MS.
3.4 Type IV

The Darboux Hamiltonian of type IV is given by [24]

\[ H^{(2)}_{IV} = \frac{\sin^2 u}{a + \cos u} \left( p_u^2 + p_v^2 \right), \]  

(25)

where \( a \) is a constant. This Hamiltonian admits a QMS ND generalization via the substitution

\[ u \to \rho = \ln r, \quad p_v \to L^2, \]

with \( F(\rho) = \sin^{-1} \rho (a + \cos \rho)^{1/2} \). More precisely, the system has the form

\[ H^{(N)}_{IV} = \frac{\sin^2 \rho}{a + \cos \rho} \left( p_\rho^2 + L^2 \right) = \frac{1}{a + \cos(\ln |q|)} q^2 \sin^2 (\ln |q|) \, d\rho^2, \]

so that the \( sl(2) \)-coalgebra space corresponds to setting

\[ A(J-) = \frac{\frac{1}{2} \sin^2 (\frac{1}{2} \ln J_-)}{a + \cos(\ln |q|)} \]

(26)

and the metric in \( D^{(N)}_{IV} \) is given by

\[ ds^2 = \frac{a + \cos \rho}{\sin^2 \rho} (d\rho^2 + d\Omega^2_{N-1}) = \frac{a + \cos(\ln |q|)}{q^2 \sin^2 (\ln |q|)} \, dq^2. \]

Its scalar curvature is found to be

\[ R = -\frac{N - 1}{32(a + \cos \rho)^2} \left( 64a^2 + 40(N + 1) \cos(\rho)a + 8(3N - 5) \cos(3\rho)a + 15N \right. \]

\[ \left. + 4[8(N - 2)a^2 + 3(N + 2)] \cos(2\rho) + 5(N - 2) \cos(4\rho) - 14 \right). \]

If we take \( a > 1 \), it is obvious from inspection that the metric becomes singular at \( r = 1 \) and \( r = e^\pi \). The radial geodesics are obtained from the Lagrangian \( L = \sin^{-2} \rho (a + \cos \rho)^2 \tilde{\rho}^2 \equiv G(\rho)^2 \tilde{\rho}^2 \). As the integral \( \int G(\rho) \, d\rho \) diverges both at 0 and \( \pi \), it immediately follows that the Riemannian manifold \( D^{(N)}_{IV} = (M, ds^2) \) is complete, \( M \) being the annulus

\[ M = \{ q \in \mathbb{R}^N : 1 < |q| < e^\pi \}. \]

4 Concluding remarks

In the framework here discussed, the notion of \( sl(2) \)-coalgebra spaces arises naturally when analyzing (generalized) symmetries in Riemannian manifolds, and can be rephrased in terms of an \( sl(2) \otimes \cdots \otimes (N) \otimes sl(2) \) dynamical symmetry of the free Hamiltonian on these spaces. As a matter of fact, we have shown that spherically symmetric spaces are \( sl(2) \)-coalgebra ones. We stress that once a non-constant curvature space is identified within the family \( \text{[3]} \), the underlying coalgebra symmetry ensures that this is, by construction, QMS. It should be explicitly mentioned that not every integrable geodesic flow is amenable to the \( sl(2) \)-coalgebraic approach developed in this paper by means of an appropriate
change of variables. For instance, the completely integrable Kerr–NUT–AdS spacetime studied in [16] does not fit within this framework, even after euclideanization.

Moreover, any potential with $sl(2)$-coalgebra symmetry, i.e. given by a function $V(J_-, J_+, J_3)$, can be added to the kinetic energy $H_T$ of an $sl(2)$-coalgebra background space without breaking the superintegrability of the motion. In this respect, we stress that the symplectic realization (3) with arbitrary parameters $b_i$’s would give rise to potential terms of “centrifugal” type. It is well known that the latter terms can be often added to some “basic” potentials (such as the Kepler–Coulomb and the harmonic oscillator potentials) without breaking their superintegrability.

Among the infinite family of $sl(2)$-coalgebra spaces, the four ND Darboux spaces here introduced are from an algebraic viewpoint the closest ones to constant curvature spaces, since they are the only spaces other than $E^N$, $H^N$ and $S^N$ whose geodesic motion can be expected to be (quadratically) MS for all $N$. In the case $N = 2$, this statement is the cornerstone of Koenigs classification [22], whereas in the case of $D^{(N)}_H$ such maximal superintegrability has been recently proven in [26]. The search for the additional independent integral of motion in the three remaining Darboux spaces is currently under investigation, as is the exhaustive analysis of ND versions of the 2D potentials given in [23, 24].

Another interesting problem is the construction of the Lorentzian counterparts of the Riemannian $sl(2)$-coalgebra spaces presented in this letter. We expect that such an extension should be feasible by resorting to an analytic continuation procedure similar to the one used in [20]. In this direction, we believe that an appropriate shift to the Lorentzian signature should not affect the superintegrability properties of the geodesic flows, in the same way that separability is not altered by the standard analytic continuation techniques [15].

Acknowledgements

This work was partially supported by the Spanish MEC and by the Junta de Castilla y León under grants no. FIS2004-07913 and VA013C05 (A.B. and F.J.H.), by the Spanish DGI under grant no. FIS2005-00752 (A.E.) and by the INFN–CICyT (O.R.). Furthermore, A.E. acknowledges the financial support of the Spanish MEC through an FPU scholarship, as well as the hospitality and the partial support of the Physics Department of Roma Tre University.

References

[1] L. Markus, K.R. Meyer, Mem. Amer. Math. Soc. 144 (1974) 1.
[2] V.I. Arnold, V.V. Kozlov, A.I. Neishtadt, Mathematical aspects of classical and celestial mechanics, Springer-Verlag, Berlin, 1997.
[3] E.G. Kalnins, W. Miller, Adv. Math. 51 (1984) 91.
[4] A. Huaux, Ann. Mat. Pura Appl. 108 (1976) 251.
[5] L.A. Pars, A Treatise on Analytical Dynamics, Ox Bow Press, Woodbridge, 1981.
[6] A. Katok, B. Hasselblatt, Introduction to the modern theory of dynamical systems, Cambridge University Press, Cambridge, 1995.
[7] G.P. Paternain, Geodesic flows, Birkhäuser, Boston, 1999.
[8] G.W. Gibbons, H. Lü, D.N. Page, C.N. Pope, J. Geom. Phys. 53 (2005) 49.
[9] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri, Y. Oz, Phys. Rep. 323, (2000) 183.
[10] J.R. David, G. Mandal, S.R. Wadia, Phys. Rep. 369 (2002) 549.
[11] J.P. Gauntlett, D. Martelli, J.F. Sparks, D. Waldram, Adv. Theor. Math. Phys. 8 (2004) 987.
[12] M. Cvetic, H. Lü, D.N. Page, C.N. Pope, Phys. Rev. Lett. 95 (2005) 071101.
[13] Z.-W. Chong, G.W. Gibbons, H. Lü, C.N. Pope, Phys. Lett. B 609 (2005) 124.
[14] M. Vasudevan, K.A. Stevens, Phys. Rev. D 72 (2005) 124008.
[15] M. Vasudevan, Phys. Lett. B 632 (2006) 532.
[16] D.N. Page, D. Kubiznak, M. Vasudevan, P. Krtous, Phys. Rev. Lett. 98 (2007) 061102.
[17] P. Krtous, D. Kubiznak, D.N. Page, V.P. Frolov, JHEP 0702 (2007) 004.
[18] V.P. Frolov, P. Krtous, D. Kubiznak, JHEP 0702 (2007) 005.
[19] O. Ragnisco, A. Ballesteros, F.J. Herranz, F. Musso, SIGMA 3 (2007) 026; math-ph/0611040.
[20] A. Ballesteros, F.J. Herranz, O. Ragnisco, Phys. Lett. B 610 (2005) 107.
[21] A. Ballesteros, F.J. Herranz, J. Phys. A: Math. Theor. 40 (2007) F51.
[22] G. Koenigs, Sur les géodésiques a intégrales quadratiques, in: G. Darboux (Ed.), Leçons sur la théorie généralé des surfaces, Chelsea, New York, 1972, Vol. 4, p. 368.
[23] E.G. Kalnins, J.M. Kress, W. Miller, P. Winternitz, J. Math. Phys. 43 (2002) 970.
[24] E.G. Kalnins, J.M. Kress, W. Miller, P. Winternitz, J. Math. Phys. 44 (2003) 5811.
[25] B. Doubrovine, S. Novikov, A. Fomenko, Géométrie Contemporaine, Méthodes et Applications First Part MIR, Moscow, 1982.
[26] A. Ballesteros, A. Enciso, F.J. Herranz, O. Ragnisco, math-ph/0612080.
[27] M.P. do Carmo, Differential geometry of curves and surfaces, Prentice-Hall, Englewood Cliffs, 1976.