Analytical estimates of the locations of phase transition points in the ground state for the bimodal Ising spin glass model in two dimensions

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We analytically estimate the locations of phase transition points in the ground state for the $\pm J$ random bond Ising model with asymmetric bond distributions on the square lattice. We propose and study the percolation transitions for two types of bond shared by two non-frustrated plaquettes. The present method indirectly treats the sizes of clusters of correlated spins for the ferromagnetic and spin glass orders. We find two transition points. The first transition point is the phase transition point for the ferromagnetic order, and the location is obtained as $p_c^{(1)} \approx 0.89539954$ as the solution of $[p^2 + 3(1 - p)^2]p^2 - \frac{1}{2} = 0$. The second transition point is the phase transition point for the spin glass order, and the location is obtained as $p_c^{(2)} = \frac{1}{4}(2 + \sqrt{2(\sqrt{5} - 1)}) \approx 0.89307569$. Here, $p$ is the ferromagnetic bond concentration, and $1 - p$ is the antiferromagnetic bond concentration. The obtained locations are reasonably close to the previously estimated locations. This study suggests the presence of the intermediate phase between $p_c^{(1)}$ and $p_c^{(2)}$; however, since the present method produces remarkable values but has no mathematical proof for accuracy yet, no conclusions are drawn in this article about the presence of the intermediate phase.

§1. Introduction

The establishment of reliable theories of spin glasses has been one of the most challenging problems in statistical physics for years. In this article, our main interests are to make the determination of the structure of the phase diagram for spin glasses, and to clarify the properties of the phases.

Figure 1 shows a schematic phase diagram for the $\pm J$ random bond Ising model on the square lattice. In this article, we do not mention the problem of the presence or absence of the spin glass phase at a finite temperature for the square lattice. In this article, we mention the locations of the phase transition points $p_c^{(1)}$ and $p_c^{(2)}$ at zero temperature for the square lattice and mention the intermediate phase between $p_c^{(1)}$ and $p_c^{(2)}$. The exact locations of $p_c^{(1)}$ and $p_c^{(2)}$ have not been analytically obtained. One of the aims of this article is to analytically estimate the two locations.

We examine the bond configurations for the exchange interaction. We propose and study the percolation transitions for two types of bond shared by two non-frustrated plaquettes. The present method indirectly treats the sizes of clusters of correlated spins for the ferromagnetic and spin glass orders. A similar scheme has already been proposed by Sadiq et al. Sadiq et al. investigated the plaquette percolation for frustrated and non-frustrated plaquettes by computer simulation, and argued a connection with statistical properties at zero temperature. We show in this article that the present percolation and the plaquette percolation yield different results. Recently, Miyazaki proposed an analytical method that estimates the location of the phase transition point at zero temperature by analyzing the
number of frustrated plaquettes. Miyazaki’s method estimates one location for the square lattice. On the other hand, e.g. by computing the domain wall energy, it is possible to numerically estimate two locations for the square lattice, although there is a problem with the agreement or disagreement of the two locations. In this study, the two locations of the phase transition points are analytically estimated.

Maynard and Ramma proposed that a highly correlated state, called the random antiphase state (RAS), is present near the ferromagnetic phase in the low temperature region, but Morgenstern claimed that a transfer-matrix calculation indicates the absence of the RAS. Kawashima and Rieger investigated the ground state by an exact numerical polynomial method, estimated the two locations of the phase transition points, and concluded that the intermediate phase is absent in the ground state. The locations estimated in this study are remarkably close to the locations estimated by Kawashima and Rieger. On the other hand, the present study suggests the presence of the intermediate phase. We discuss the properties of phases by using the present bonds, and give another interpretation of previous numerical results. One of the aims of this article is to clarify the properties of the phases at zero temperature.

In order to calculate the locations, the approximate approach, regarding the present bonds as random bonds, is applied. By comparing the obtained locations with the previously estimated locations, we show the validity of the present results. Lower bounds of the locations with no approximation by using the present bonds are also obtained. Most of the present results are results for the square lattice. Application of the present scheme to the model on a simple cubic lattice is also mentioned.

This article is organized as follows. We explain the present model and the concepts of frustration and bond percolation in Sect. These concepts are used in this study. The analytical estimates of the locations of the phase transition points are given in Sect. The properties of phases are discussed in Sect. In Sect. another interpretation of previous numerical results is given. The concluding remarks for
Phase transitions for the bimodal Ising spin glass model

Fig. 2. Examples of plaquettes. The solid lines represent the ferromagnetic bonds, and the dotted lines represent the antiferromagnetic bonds. (a) A non-frustrated plaquette. (b) A frustrated plaquette.

this article are in Sect. 6.

§2. The model and the concepts of frustration and bond percolation

The Hamiltonian \( H \) for the \( \pm J \) random bond Ising model is given by

\[
H = -\sum_{\langle ij \rangle} J_{ij} S_i S_j, \tag{2.1}
\]

where \( \langle ij \rangle \) denotes nearest-neighbor pairs, \( S_i \) is a state of the spin at site \( i \), and \( S_i = \pm 1 \). \( J_{ij} \) is a strength of the exchange interaction between the spins at sites \( i \) and \( j \), and is a quenched variable. The value of \( J_{ij} \) is given with the distribution \( P(J_{ij}) \), and \( P(J_{ij}) \) is given by

\[
P(J_{ij}) = p \delta_{J_{ij},1} + (1 - p) \delta_{J_{ij},-1}, \tag{2.2}
\]

where \( \delta \) is the Kronecker delta. When \( J_{ij} = 1 \), the interaction is also called the ferromagnetic bond. When \( J_{ij} = -1 \), the interaction is also called the antiferromagnetic bond. A schematic phase diagram for this model on the square lattice is shown in Fig. 1.

Figure 2 shows examples of plaquettes. In Fig. 2 (a), there are ways of choosing the orientations of the spins on the four sites without frustration. On the other hand, in Fig. 2 (b), there is no way of choosing the orientations of the spins on the four sites without frustrating at least one bond. This frustration effect is measured with the frustration function \( \Phi \), given by

\[
\Phi = \prod_{\{c\}} J_{ij}, \tag{2.3}
\]

where \( \{c\} \) is a set of closed contours. When there is a frustrated plaquette, \( \Phi \) gives \(-1\) and gives \(1\) otherwise. For example, \( \Phi \) of Fig. 2 (a) gives \(1\), and \( \Phi \) of Fig. 2 (b) gives \(-1\).

In the bond percolation, bonds for generating clusters are placed on the edges of the lattice, and one of the clusters is percolated at the threshold fraction \( P_c \). For example, in Fig. 3, the clusters for the p1 bonds are not percolated, and the cluster for the p2 bonds is percolated. The details of Fig. 3 and the p1 and p2 bonds are given in Sect. 4.
§3. Analytical estimates of the locations of the phase transition points

We analytically estimate the locations of the phase transition points at zero temperature for the square lattice. The first step is to extract and analyze the non-frustration network, since it is considered that the local energy on the non-frustration network tends to be lower than that on the frustration network. In addition, we consider effective bonds that describe phase transitions. Thus, it is speculated that such a bond is shared by two non-frustrated plaquettes. This bond has two types: we call them p1 and p2 bonds. We call the ferromagnetic bond, which is shared by two non-frustrated plaquettes, the p1 bond. We call the bond that includes ferromagnetic and antiferromagnetic bonds and that is shared by two non-frustrated plaquettes the p2 bond.

We expect that the sizes of the clusters for the p1 bonds correspond to the sizes of clusters of correlated spins for the ferromagnetic order and expect that the sizes of the clusters for the p2 bonds correspond to the sizes of clusters of correlated spins for the spin glass order. The correlation for the ferromagnetic order at sites $i$ and $j$ is given by $\langle S_i S_j \rangle_\beta J$, where $\langle \rangle_\beta$ denotes the thermal average and $[ ]_J$ denotes the configurational average for the exchange interactions. The correlation for the spin glass order at sites $i$ and $j$ is given by $\langle (S_i S_j)_2 \rangle_\beta J$.

The present method has no mathematical proof for accuracy yet; on the other hand, from the definitions of the present bonds in this section and the arguments for phases by using the present bonds in Sect.4 connections between the sizes of clusters of correlated spins for the orders and the sizes of the clusters for the present bonds are inferred. The degrees of these connections are related to the accuracies of the present results. The validity of the present bonds is shown from the percolation thresholds in this section.

Figure 3 shows examples of four bonds. The solid lines represent the ferromagnetic bonds, the dotted lines represent the antiferromagnetic bonds, the dashed lines represent the p1 bonds, and the short dashed lines represent the p2 bonds. As shown in the figure, the p1 and p2 bonds are placed on the edges of the lattice. We analytically estimate the percolation thresholds for the p1 and p2 bonds on the square lattice.

Figure 4 shows two plaquettes that are separated into three parts a, b, and c for analytical calculation. The p1 and p2 bonds are placed on the edge $AB$ of Fig.4. By generalizing these placements, we calculate the probabilities for placing the present
Phase transitions for the bimodal Ising spin glass model

Fig. 4. Two plaquettes that are separated into three parts a, b, and c for analytical calculation. The p1 and p2 bonds are placed on the edge AB.

bonds on an edge. Part a of Fig. 4 consists of three bonds. The probability $P_+^{(a)}$ that the frustration function $\Phi^{(a)}$ for part a gives 1 is given by

$$P_+^{(a)} = \binom{3}{0} p^3 + \binom{3}{2} p(1-p)^2 = p^3 + 3p(1-p)^2,$$

where

$$\binom{l}{m} = \frac{l!}{m!(l-m)!}.$$

The probability $P_-^{(a)}$ that the frustration function $\Phi^{(a)}$ gives $-1$ is given by

$$P_-^{(a)} = \binom{3}{1} p^2(1-p) + \binom{3}{3} (1-p)^3 = 3p^2(1-p) + (1-p)^3.$$

Part b of Fig. 4 consists of a bond. Similarly, $P_+^{(b)}$ and $P_-^{(b)}$ are respectively given by

$$P_+^{(b)} = p \quad \text{and} \quad P_-^{(b)} = 1 - p.$$

In addition, $P_+^{(c)}$ and $P_-^{(c)}$ are respectively

$$P_+^{(c)} = P_+^{(a)} \quad \text{and} \quad P_-^{(c)} = P_-^{(a)}.$$

The probability $P^{(p1\text{ bond})}$ for placing the p1 bond on an edge is obtained as

$$P^{(p1\text{ bond})} = P_+^{(a)} P_+^{(b)} P_+^{(c)} = p^3[p^2 + 3(1-p)^2]^2.$$

At the percolation transition point, there is a relation:

$$P^{(p1\text{ bond})} = P^{(p1\text{ bond})}_c,$$

where $P^{(p1\text{ bond})}_c$ is the percolation threshold for the p1 bond. Calculation of $P^{(p1\text{ bond})}_c$ is not trivial; thus, here we approximate the $P^{(p1\text{ bond})}_c$ by regarding the p1 bond as a random bond. The percolation threshold of the random bond for the square lattice is $\frac{1}{2}$; therefore, we obtain $P^{(p1\text{ bond})}_c \approx \frac{1}{2}$. Therefore, by solving the equation

$$p^3[p^2 + 3(1-p)^2]^2 = \frac{1}{2},$$

we obtain the location of the phase transition point $p_c^{(1)}$ for the ferromagnetic order as

$$p_c^{(1)} \approx 0.89539954.$$

The obtained location is consistent with previous estimates such as 0.8955(1), 0.896(1), 0.88(2), 0.89(2) and 0.89(1) and the obtained location is
C. Yamaguchi

smaller than but marginally consistent with 0.8969, while the obtained location is inconsistent with 0.897(1), 0.8969(1) and 0.885(1).

The probability $P_{c(p2 bond)}$ for placing the p2 bond on an edge is obtained as:

$$P_{c(p2 bond)} = P_{c(a)} + P_{c(b)} + P_{c(c)} + P_{c(a)} - P_{c(b)} - P_{c(c)} - P_{c(a)} = p_3 \left[ p_2 + 3(1-p)^2 \right]^2 + (1-p)^3 [3p^2 + (1-p)^2]^2. \tag{3.7}$$

At the percolation transition point, there is a relation:

$$P_{c(p2 bond)} = P_{c(p2 bond)}, \tag{3.8}$$

where $P_{c(p2 bond)}$ is the percolation threshold for the p2 bond. Calculation of $P_{c(p2 bond)}$ is not trivial; thus, here we approximate the $P_{c(p2 bond)}$ by regarding the p2 bond as a random bond. The percolation threshold of the random bond for the square lattice is $1/2$, therefore, we obtain $P_{c(p2 bond)} \approx 1/2$. Therefore, by solving the equation

$$p_3 \left[ p_2 + 3(1-p)^2 \right]^2 + (1-p)^3 [3p^2 + (1-p)^2]^2 = 1/2,$$

we obtain the location of the phase transition point $p_{c(2)}$ for the spin glass order as

$$p_{c(2)} = \frac{1}{4}[2 + \sqrt{2(\sqrt{5} - 1)}] \approx 0.89307569.$$

The obtained location is consistent with a previous estimate such as 0.894(2), and the obtained location is larger than but marginally consistent with 0.85 and 0.870, while the obtained location is inconsistent with 0.896(2), 0.86(2) and 0.854(2).

As described above, the obtained locations of $p_{c(1)}$ and $p_{c(2)}$ are reasonably close to the previously estimated locations, although the approximate approach by regarding the present bonds as random bonds is applied. Thus, it is considered that this approximate approach is valid in this study. We derive the lower bounds of the locations with no approximation by using the present bonds. The present bonds are slightly more complicated than the random bonds, so the exact values of $P_{c(p1 bond)}$ and $P_{c(p2 bond)}$ can be slightly larger than $1/2$: $P_{c(p1 bond)} > 1/2$ and $P_{c(p2 bond)} > 1/2$. Then, by using Eqs. (3.5), (3.6), (3.7), and (3.8), we obtain the lower bounds for $p_{c(1)}$ and $p_{c(2)}$ with no approximation as $0.8953995 \cdots < p_{c(1)}$ (no approximation) and $0.8930756 \cdots < p_{c(2)}$ (no approximation). From one of these inequalities, it is shown that the present percolation and the plaquette percolation yield different results, since the plaquette percolation gives $p_{c(2)} = 0.86(2)$.

We consider application of the present scheme to the model on a simple cubic lattice. We consider a bond shared by four plaquettes. It is considered that the $q_4$ bond shared by four non-frustrated plaquettes is effective, but it is supposed that the $q_3$ bond shared by three non-frustrated plaquettes and a frustrated plaquette is also effective. The percolation threshold of the random bond for the simple cubic lattice is about 0.248. Therefore, the equation for the ferromagnetic phase transition is given by $4zp^4(1-p)[p^2 + 3(1-p)^2]^2[3p^2 + (1-p)^2] + p^5[3p^2 + 3(1-p)^2]^4 = 0.24881$, where $0 \leq x \leq 1$. In this equation, the effect for the ferromagnetic $q_4$ bond
is fully counted. When \( x = 0 \), the effect for the ferromagnetic \( q_3 \) bond is not counted, and the result is given by \( p_c^{(x=0)} \approx 0.88495 \). When \( x = 1 \), the effect for the ferromagnetic \( q_3 \) bond is fully counted, and the result is given by \( p_c^{(x=1)} \approx 0.69479 \). The previous numerical estimates for \( p_c \) are 0.7747(25) and 0.778(5) [26]. Compared to the previous numerical estimates, the value of \( x \) takes a value for \( 0 < x < 1 \), and therefore the ferromagnetic \( q_3 \) bond is also related to the ferromagnetic phase transition.

§ 4. The properties of phases

We now discuss the properties of phases at zero temperature for the square lattice. We call the antiferromagnetic bond, which is shared by two non-frustrated plaquettes, the \( p_3 \) bond. The probability \( P^{(p_3 \text{ bond})} \) for placing the \( p_3 \) bond on an edge is obtained as

\[
P^{(p_3 \text{ bond})} = P^{(a)} - P^{(b)} - P^{(c)} = (1 - p)^3 [3p^2 + (1 - p)^2]^2. \tag{4.1}
\]

One can see that \( P^{(p_3 \text{ bond})} > 0 \) for \( \frac{1}{2} \leq p < 1 \). Therefore, non-frustration networks always include the \( p_3 \) bonds for \( \frac{1}{2} \leq p < 1 \) if the system size is large enough. There is no percolation transition for clusters that consist only of the \( p_3 \) bonds when \( \frac{1}{2} \leq p \leq 1 \). The \( p_3 \) bond works as the antiferromagnetic interaction and breaks the ferromagnetism. The \( p_1 \) and \( p_2 \) bonds are defined in Sect. 3.

We now discuss the ferromagnetic phase. The ferromagnetic correlation length of spins on the percolated \( p_1 \) cluster in the ground state is equal to the system length. Then, the ferromagnetic phase appears. Next, we discuss the intermediate phase.

Figure shows a relationship for the \( p_1 \) and \( p_2 \) clusters. As shown in Fig. 5, the sizes of the \( p_2 \) clusters are always larger than those of the \( p_1 \) clusters, since the \( p_2 \) bond includes the \( p_1 \) and \( p_3 \) bonds. The intermediate phase appears when the \( p_1 \) cluster is not percolated and the \( p_2 \) cluster is percolated, as illustrated in Fig. 6. If the \( p_1 \) cluster is not percolated, then the \( p_3 \) bonds break the ferromagnetism, and the ferromagnetism does not appear. In the intermediate phase, since the \( p_2 \) cluster is percolated, the non-frustration network is percolated, and therefore the spins are in highly correlated states. Highly correlated states are a feature of the RAS. Thus, the nature of the intermediate phase is the RAS.

According to the locations obtained in Sect. 3, the width of the intermediate
phase is estimated as
\[ p_c^{(1)} - p_c^{(2)} \approx 0.002 \, 323 \, 85. \]
Thus, the width is very narrow.

The probability \( P_O \) of the occurrence of the antiferromagnetic bonds on the non-frustration network is obtained as
\[ P_O = \frac{(1-p)^3[3p^2+(1-p)^2]}{p^3[p^2+3(1-p)^2]+(1-p)^3[3p^2+(1-p)^2]} \]
by solving
\[ P_O = \frac{p(p^3 \text{ bond})}{p(p^2 \text{ bond})} \]
By using the values at \( p = p_c^{(1)} \) and \( p = p_c^{(2)} \), the range of \( P_O \) for the intermediate phase is estimated as
\[ 0.01318 < P_O < 0.01414. \]
Therefore, the antiferromagnetic bonds contribute to the non-frustration network at low probability.

We also estimate the required system length \( L_R \) for investigating the intermediate phase. In order to flip a cluster of spins in one dimension by using the p3 bonds, at least two p3 bonds are needed if the periodic boundary condition is implemented. Therefore, the required system length \( L_R \) is given by
\[ L_R > \frac{151}{0.01318}. \]
By considering the range of the \( P_O \) for the intermediate phase, the length is obtained as
\[ L_R > 151. \]
This inequality limits the system lengths that should be investigated. If the system length \( L \) is shorter than 151, the RAS does not appear because of the effect of finite system size, and the spin states indicate the ferromagnetic phase. If \( L > 151 \), the RAS appears. The required system sizes for investigating the intermediate phase are large.

We now discuss the spin glass phase. In the spin glass phase, the p1 and p2 clusters are not percolated, and thus the non-frustration network is not percolated. Therefore, the spin glass phase is different from the other phases, and the spins are not in highly correlated states.

§5. Another interpretation of previous numerical results

We now give another interpretation of the previous numerical results. As described in Sect.1, Kawashima and Rieger\(^9\) concluded that the intermediate phase is absent by using an exact numerical method. According to Kawashima and Rieger\(^9\), the location of \( p_c^{(1)} \) is estimated as 0.896(1), and the location of \( p_c^{(2)} \) is estimated as 0.894(2). The difference is estimated as 0.002(2). According to Amoruso and Hartmann\(^1\) by using an exact numerical method, the location of \( p_c^{(1)} \) is estimated as 0.897(1), and the location of \( p_c^{(2)} \) is estimated as 0.896(2). The difference is estimated as 0.001(2). On the other hand, in this study, as described in Sect.4, the location of \( p_c^{(1)} \) is estimated as about 0.8954, and the location of \( p_c^{(2)} \) is estimated as about 0.8931. The difference is estimated as about 0.0023, as also described in Sect.4. Therefore, the difference for the present results is consistent with the differences for the previous results. If the intermediate phase is absent, the difference gives zero. Therefore, both interpretations of the presence and the absence are available when
the previous results and the present result are compared. Thus, the precisions of the previous results are not sufficient for determining the phase.

In addition, the required system length \( L_R \) for investigating the intermediate phase is estimated as \( L_R > 151 \), as described in Sect.4. The largest system length \( L_{\text{Max}} \) investigated by Kawashima and Rieger is \( L_{\text{Max}} = 32 \), and the length investigated by Amoruso and Hartmann is \( L_{\text{Max}} = 700 \). However, in the result by Amoruso and Hartmann, the data for \( L < 151 \) of the system length \( L \) were also used for estimating the locations of the phase transition points. Specifically, the data for \( L = 80, 100, 120, \) and \( 140 \) were used in addition to the data for \( L > 151 \). According to the estimated required system length, the data for \( L < 151 \) should not be used. Thus, the system lengths investigated in the previous results are not sufficient for determining the phase.

§6. Concluding remarks

We analytically estimated the locations of phase transition points in the ground state for the \( \pm J \) random bond Ising model with asymmetric bond distributions on the square lattice. By comparing the obtained locations of transition points with the previously estimated locations, we showed the validity of the present results. We confirmed that the present percolation transitions are very effectively related to the phase transitions at zero temperature. This indicates that the sizes of the clusters for the present bonds correspond to the sizes of clusters of correlated spins for the ferromagnetic and spin glass orders.

We also mentioned application of the present scheme to the model on the simple cubic lattice. It seems that the simple cubic lattice case is more complicated than the square lattice case.

We also discussed the properties of phases at zero temperature for the square lattice. The present study suggests that the intermediate phase between the ferromagnetic and spin glass phases is present and also suggests that the nature of the intermediate phase is the random antiphase state. Our arguments for the presence of the intermediate phase are summarized as follows: the antiferromagnetic bonds usually break the non-frustration network, but at low probability the antiferromagnetic bonds contribute to the non-frustration network, so that the intermediate phase is present with a very narrow width.

Also, it was argued that the precisions of previous numerical results are not sufficient for determining the phase when the previous numerical results and the present result are compared. In addition, the required system length for investigating the intermediate phase was estimated and, by using the estimated required system length, it was argued that the system lengths investigated in the previous results are not sufficient for determining the phase.

No conclusions, however, are drawn on the presence of the intermediate phase in this article, since the present method produces remarkable values but has no mathematical proof for accuracy yet. More investigations with exact numerical methods are required to clarify the accuracies of the present percolations and the presence or absence of the intermediate phase. These investigations are tasks for the future.
In this study, the first step was to extract and analyze the non-frustration network, but this step contradicts the description of Ref. [5], i.e., “Given some configuration of positive and negative bonds, the first step must be to extract and analyze the frustration network”. Given some configuration of ferromagnetic and antiferromagnetic bonds, the first step must be to extract and analyze the non-frustration network, although the effects of frustration are significant, since it is considered that the local energy on the non-frustration network tends to be lower than that on the frustration network.

The present bonds are not derived from the Fortuin-Kasteleyn (FK) representation[27][28] known as a representation for well studied spin clusters. The bonds for the FK representation depend on spin configurations; on the other hand, the present bonds do not depend on spin configurations.

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