Design and Analysis of Time-Invariant SC-LDPC codes with Small Constraint Length

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Abstract

In this paper, we deal with time-invariant low-density parity-check convolutional (LDPCC) codes, which are a subclass of spatially coupled low-density parity-check (SC-LDPC) codes. Classic design approaches usually start from quasi-cyclic (QC) low-density parity-check (LDPC) block codes and exploit suitable unwrapping procedures to obtain LDPCC codes. We show that the direct design of the LDPCC code syndrome former matrix or, equivalently, the symbolic parity-check matrix, leads to codes with smaller syndrome former constraint lengths with respect to the best solutions available in the literature. We provide theoretical lower bounds on the syndrome former constraint length for the most relevant families of LDPCC codes, under constraints on the minimum length of local cycles in their Tanner graphs. We also propose new code design techniques that approach or achieve such theoretical limits.

Index Terms

Constraint length, convolutional codes, girth, LDPC codes, spatially coupled codes, time-invariant codes.

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I. INTRODUCTION

Spatially coupled low-density parity-check (SC-LDPC) codes were firstly introduced by Jiménez Felström and Zigangirov in [2] under the name of low-density parity-check convolutional (LDPCC) codes. By starting from quasi-cyclic low-density parity-check (QC-LDPC) block codes [3] and applying suitable unwrapping techniques [2], [4], [5], LDPCC codes with performance very close to the density evolution bound have been obtained. Furthermore, it has been also shown that SC-LDPC codes asymptotically achieve Shannon’s capacity [6] under belief propagation decoding [7], thanks to the threshold saturation phenomenon. As regards implementation, shift register based circuits are commonly used to encode LDPCC codes [2] whereas sliding window (SW) iterative algorithms based on belief propagation are exploited for decoding [2], [8], [9]. The complexity of the encoding and decoding techniques linearly increases with the product of two parameters of LDPC codes: the syndrome former constraint length and the weight of the columns of the parity-check matrix. For this reason, our target is to design regular codes with small syndrome former constraint length, which is a worthwhile goal from the complexity standpoint. More precisely, in this paper we propose some direct design techniques of the parity-check matrix of LDPC codes without passing through their QC-LDPC block code counterparts, thus avoiding the intermediate conventional step of unwrapping techniques. This approach has already been used in [10], [11] for codes having rates equal to $\frac{a-1}{a}$, with $a$ an integer, and extended in [1] to codes with rate equal to $\frac{a-c}{a}$, but, to the best of our knowledge, a general and exhaustive study is still missing.

Furthermore, the symbolic parity-check matrices of QC-LDPC codes described in the literature [12]–[14] have some fixed entries. We show that the removal of these constraints makes investigations easier and yields significant reductions in terms of constraint length.

The remainder of the paper is organized as follows. In Section II we recall the definitions of time-invariant LDPC codes and introduce representations we use for these codes and their cycles. In Section III some bounds for a wide range of code families and several girth lengths are proposed. In Section IV we introduce three new design methods for as many families of short constraint length, time-invariant LDPC codes. In Section V we perform exhaustive and Montecarlo code searches and compare their results with the theoretical bounds. In Section V-C we show the error rate performance of the proposed codes. Finally, in Section VI we draw some conclusions.
II. Notation and Definitions

Time-invariant LDPCC codes are characterized by semi-infinite parity-check matrices in the form

\[
H = \begin{bmatrix}
H_0 & 0 & 0 & \cdots \\
H_1 & H_0 & 0 & \cdots \\
H_2 & H_1 & H_0 & \cdots \\
\vdots & H_2 & H_1 & \cdots \\
H_{m_h} & \vdots & H_2 & \cdots \\
0 & H_{m_h} & \vdots & \cdots \\
0 & 0 & H_{m_h} & \cdots \\
0 & 0 & 0 & \cdots \\
\vdots & \vdots & \vdots & \cdots 
\end{bmatrix},
\]

where each block \(H_i, i = 0, 1, 2, \ldots, m_h\), is a binary matrix with size \(c \times a\). The syndrome former matrix \(H_s = [H_0^T | H_1^T | H_2^T | \ldots | H_{m_h}^T]\) has \(a\) rows and \((m_h + 1)c\) columns, where \(T\) denotes transposition. Let us introduce the variable \(L_h\), which is defined as the maximum spacing between any two ones appearing in any column of \(H_s^T\). The code is regular in its columns if all the columns of \(H_s^T\) have the same Hamming weight \(w\), otherwise it is irregular.

As evident from (1), \(H\) is obtained by \(H_s^T\) and its replicas, shifted vertically by \(c\) positions each. The code defined by (1) has asymptotic code rate \(R = \frac{a-c}{a}\), syndrome former memory order \(m_h = \lceil \frac{L_h}{c} \rceil - 1\), where \(\lceil x \rceil\) is the smallest integer greater than or equal to \(x\), and syndrome former constraint length \(v_s = (m_h + 1)a = \lceil \frac{L_h}{c} \rceil a\).

Another representation of \(H_s\) which is often used in the literature exploits polynomials \(\in F_2[x]\), where \(F_2[x]\) is the ring of polynomials with coefficients in the binary Galois field \(GF[2]\). In this case, the code is described by a \(c \times a\) matrix with polynomial entries, that is

\[
H(x) = \begin{bmatrix}
h_{0,0}(x) & h_{0,1}(x) & \cdots & h_{0,a-1}(x) \\
h_{1,0}(x) & h_{1,1}(x) & \cdots & h_{1,a-1}(x) \\
\vdots & \vdots & \ddots & \vdots \\
h_{c-1,0}(x) & h_{c-1,1}(x) & \cdots & h_{c-1,a-1}(x)
\end{bmatrix},
\]

where each \(h_{i,j}(x), i = 0, 1, 2, \ldots, c - 1, j = 0, 1, 2, \ldots, a - 1\), is a polynomial \(\in F_2[x]\).

The code representation based on \(H_s\) can be converted into that based on \(H(x)\) through the following procedure. First of all, starting from \(H_s\), the multiset \(I\) containing the sets of indexes (beginning from zero) of the symbols 1 in each row of \(H_s\) must be determined.
Then, the \( j \)-th column of \( H(x) \) is obtained from the set \( I_j \in I, j = 0, 1, 2, \ldots, a - 1 \), as follows:

1. Initialize \( h_{i,j}(x) = 0, i = 0, 1, 2, \ldots, c - 1 \).
2. \( \forall l \in I_j \), compute \( l_d = \lfloor \frac{l}{c} \rfloor \) (where \( \lfloor x \rfloor \) is the integer part of \( x \)), \( l_m = l \mod c \) and add \( x^{ld} \) to \( h_{l_m,j}(x) \).

The syndrome former memory order \( m_h \) has the meaning of the largest difference, in absolute value, between any two exponents of the variable \( x \) appearing in the polynomial entries of \( H(x) \). The highest weight of any polynomial entry of \( H(x) \) defines the type of the code. So, a code containing only monomial or null entries is called here monomial or Type-1 code, a code having also binomial entries is called binomial or Type-2 code and so on. In this paper, up to Type-3 codes are considered in the design of the symbolic parity-check matrix.

The matrix \( H(x) \) can be described by the exponents of the variable \( x \) without loss of information. A simple example of exponent matrix \( P \) for monomial codes is

\[
P = \begin{bmatrix}
p_{0,0} & p_{0,1} & \cdots & p_{0,a-1} \\
p_{1,0} & p_{1,1} & \cdots & p_{1,a-1} \\
\vdots & \vdots & \ddots & \vdots \\
p_{c-1,0} & p_{c-1,1} & \cdots & p_{c-1,a-1}
\end{bmatrix}
\]

Most previous works are devoted to the design of \( H(x) \), through which \( H_s \) is obtained by unwrapping. However, designing \( H(x) \) requires first to choose the form of the polynomials \( h_{i,j}(x) \) (null, monomials, binomials, etc.) and then optimize their exponents. The matrix \( H(x) \) is also used in [15] to find unavoidable cycles and design monomial LDPCC codes free of short local cycles. On the other hand, working with \( H_s \) is advantageous as it allows to perform a single step optimization over all possible choices. As we will see in Sections IV and V, our approach allows to find codes with shorter constraint length with respect to previous approaches.

### A. Local Cycles Characterization

Local cycles are closed loops starting from a node of the Tanner graph associated to a code and returning to the same node by passing only once through any involved edge. Since the parity-check matrix is the bi-adjacency matrix of the Tanner graph of a code, local cycles can be defined over such a matrix as well. This way, we are able to directly relate the syndrome former constraint length of an SC-LDPC code to its local cycles length. The girth of a code is defined as the length of the Tanner graph shortest local cycle. A common constraint in
Tanner graphs of low-density parity-check (LDPC) codes is that their girth must be greater than 4, that is, local cycles with length 4 must not exist in the code parity-check matrix; if this condition is true, the so-called row-column constraint (r.c.c.) is satisfied.

Following an approach similar to that introduced in [10], we describe the matrix $H_s^T$ through a set of integer values representing the differences between each pair of ones in its columns. A similar approach based on differences has also been adopted in [13] for QC-LDPC block codes and their symbolic matrices, with the aim of finding relations between the code girth and the size of the circulant blocks forming the parity-check matrices of these codes. Differently from QC-LDPC block codes, in LDPC codes the symbolic matrix does not translate into a parity-check matrix formed by circulant blocks, therefore an analysis of this kind cannot be performed. We denote the differences as $\delta_{i,j}$, where $i$ is the column index of $H_s^T$ ($i = 0, 1, 2, \ldots, a - 1$) and $j$ is the row index of $H_s^T$ corresponding to the first of the two symbols 1 forming the difference ($j \in [0, 1, 2, \ldots, L_h - 2]$). The index of the second symbol 1 forming the difference is found as $j + \delta_{i,j}$. For each difference we also compute the values of two levels which are relative to the value of the parameter $c$. The starting level is defined as $l_s = j \mod c$, while the ending level is defined as $l_e = (j + \delta_{i,j}) \mod c$. Both levels obviously take values in $\{0, 1, 2 \ldots, c - 1\}$.

Based on this representation of $H_s^T$, it is easy to identify closed loops in the Tanner graph associated to $H$. In fact, a local cycle occurs every time a sum of the type $\delta_{i_1,j_1} \pm \delta_{i_2,j_2} \pm \ldots \pm \delta_{i_l,j_l}$ equals zero, and the length of the cycle is $2l$, with $l$ being an integer $> 1$.

Not all the possible sums or differences of $\delta_{i,j}$ are valid to generate local cycles. In fact, $\delta_{i_1,j_1}$ can be added to $\delta_{i_2,j_2}$ if and only if the starting level of the latter coincides with the ending level of the former. Instead, $\delta_{i_1,j_1}$ can be subtracted to $\delta_{i_2,j_2}$ if and only if their ending levels coincide. In addition, the starting level of the first difference and the ending level of the last difference in $\delta_{i_1,j_1} \pm \delta_{i_2,j_2} \pm \ldots \pm \delta_{i_l,j_l}$ must coincide. Let us denote as $\delta_{i,j}^{(l_s,l_e)}$ the difference $\delta_{i,j}$ with its starting and ending levels. Based on these considerations, for a given $H_s^T$ an efficient numerical procedure can be exploited to find all the local cycles with a given maximum length. Such a procedure has been implemented in software, and has allowed to perform exhaustive (when possible) or Montecarlo (otherwise) analyses of the syndrome former matrices with minimum syndrome former constraint length and free of local cycles up to a given size. Moreover, by studying the cases in which differences can or cannot be summed or subtracted, it is possible to obtain lower bounds on the constraint length which is needed to avoid local cycles up to a given length, as described in the next sections.
B. Symbolic parity-check matrix representation

We say that two parity-check matrices $H_1$ and $H_2$ are isomorphic ($H_1 \cong H_2$) if there is a bijection $f$ so that $(H_1)_{i,j} = 1$ if and only if $(H_2)_{f(i),f(j)} = 1$. If two parity-check matrices are isomorphic, then also their corresponding exponent matrices are isomorphic. Isomorphic parity-check matrices define equivalent codes. Equivalent codes have the same minimum distance, girth and performance.

We give next three lemmas defining useful properties of exponent matrices of type-1 QC-LDPC codes. Their proofs can be found in [12]. These lemmas also hold for LDPCC codes.

**Lemma II.1** Let $P_1$ and $P_2$ be the exponent matrices of the codes $C_1$ and $C_2$. If $P_1$ can be obtained by permuting the rows or the columns of $P_2$, then $C_1 \cong C_2$.

**Lemma II.2** Let $P_1$ and $P_2$ be the exponent matrices of the codes $C_1$ and $C_2$. If $P_1$ can be obtained by adding or subtracting the same constant to all the elements of a row or a column of $P_2$, then $C_1 \cong C_2$.

**Lemma II.3** Let $P_1$ and $P_2$ be the exponent matrices of the codes $C_1$ and $C_2$. Also, let $p$ ($p \to \infty$ in the spatially coupled case) be the size of the circulant permutation matrices in $H_i$ ($i = 1, 2$). Assume that $d \in \{1, 2, ..., p - 1\}$ and $p$ are co-prime. If $(P_2)_{ij} = (d \times (P_1)_{ij}) \mod p$ for $0 \leq i \leq c - 1$ and $0 \leq j \leq a - 1$, then $C_1 \cong C_2$.

The above lemmas permit us to design the exponent matrices of spatially coupled codes with at least a null entry in any row and any column.

1) **ILP model:** We also propose an optimization Integer Linear Programming (ILP) model in order to find the minimum possible $m_h$ for each exponent matrix of monomial codes with $w = c$. We call this method Min-Max; its description is given in Appendix A. This model, instead of performing an exhaustive search, takes benefit of a heuristic optimization approach. This process significantly decreases the search time. The input exponent matrices of our Min-Max model were selected from [12], [14], [16].

III. LOWER BOUNDS ON THE CONSTRAINT LENGTH

We consider some specific families of LDPCC codes and aim at estimating the minimum syndrome former constraint length which is needed to ensure a certain girth $g$.

For the sake of convenience, according to the definition given in Section II, we distinguish between the case of Type-1 codes and Type-$z$ codes, with $z > 1$. For the latter case, though
most of the results obtained are valid for any \( z \), in the following we will focus on the cases \( z = 2 \) and \( z = 3 \), which represent a good trade-off between performance and complexity.

The bounds are described in terms of \( m_h \) when the analysis is performed on \( H(x) \) and in terms of \( L_h \) when \( H_s^T \) is studied. We remind that these parameters are directly related to the syndrome former constraint length \( v_s \). So, the bounds on \( v_s \) can be easily computed as well.

A. Type-1 codes

1) Lower bounds on \( m_h \): Type-1 codes are characterized by a symbolic form of the parity-check matrix containing only monomial or null entries. If the transpose of the syndrome former matrix \( H_s^T \) of a code contains a column with weight \( w > c \), it cannot be a monomial code because the number of entries exceeds the number of available rows. For explicative reasons, in this paper we will focus on \( w = c \).

The following lemma provides a necessary condition on \( m_h \) for the satisfaction of the r.c.c.. Longer girths are studied next.

**Lemma III.1** A Type-1 code having \( g \geq 6 \), \( \forall w = c \), has

\[
m_h \geq \left\lceil \frac{a - 1}{2} \right\rceil. \tag{4}\]

**Proof:** The difference between any two exponents of a column of \( H(x) \) takes values in \([-m_h, m_h]\). Any difference must appear no more than once to meet the r.c.c. Exploiting all values in \([-m_h, m_h]\) to design the matrix columns, we obtain \( a = 2m_h + 1 \), from which (4) is derived.

A very important property of Type-1 codes with parity-check matrix column weight \( w = c = 2 \) is that cycles with length \( 2(2k + 1) \), \( k = 1, 2, \ldots \), cannot exist because of structural characteristics. In fact, a local cycle has to start from and end in the same row of \( H(x) \) but, if the cycle has length \( 2(2k + 1) \), the odd number of involved differences, that is \( 2k + 1 \), always yields the cycle to end in a row other than the one it started from. So, cycles of length 6 cannot exist for \( w = c = 2 \).

In order to find a lower bound on \( m_h \) for \( w = c = 3 \), let us proceed as follows. Let us consider \( H(x) \) and its corresponding exponent matrix \( P \). From the latter we can compute the matrix \( \Delta P \), where \( \Delta p_{0,j} = p_{1,j} - p_{0,j}, \Delta p_{1,j} = p_{2,j} - p_{1,j}, \Delta p_{2,j} = p_{2,j} - p_{0,j} \). Obviously, \( \Delta p_{0,j} + \Delta p_{1,j} = \Delta p_{2,j} \). To satisfy the r.c.c. it must be \( \Delta p_{i,j_1} \neq \Delta p_{i,j_2}, \forall i, \forall j_1 \neq j_2 \). It is
Lemma III.2 The elements of $\Delta P$ for a monomial code with $w = c = 3$ whose Tanner graph has $g \geq 8$ satisfy the following properties:

1) $\Delta p_{i,j} \in [-m_h, m_h], \forall i, j$
2) $\Delta p_{i,j_1} \neq \Delta p_{i,j_2}, \forall i, \forall j_1 \neq j_2$
3) $\Delta p_{0,j_1} + \Delta p_{1,j_2} \neq \Delta p_{2,j_3}, \forall j_1 \neq j_2, j_3, \forall j_2 \neq j_3$

Proof:

1) Any $p_{i,j} \in [0, m_h], \forall i, j$, by definition. Since any $\Delta p_{i,j}$ is a difference between two entries of $P$, it can take values in $[-m_h, m_h]$.
2) In order to satisfy the r.c.c., it must be $p_{i,j_1} - p_{i,j_2} = p_{i,j_2} - p_{i,j_3}, \forall i_1 \neq i_2, \forall j_1 \neq j_2, j_3$, that is, $\Delta p_{i,j_1} \neq \Delta p_{i,j_2}, \forall i, \forall j_1 \neq j_2$.
3) A cycle with length 6 occurs if and only if an equation of the type

$$(p_{1,j_1} - p_{0,j_1}) + (p_{2,j_2} - p_{1,j_2}) = (p_{2,j_3} - p_{0,j_3})$$

is satisfied, for some $j_1, j_2, j_3$, with $j_1 \neq j_2, j_3$ and $j_2 \neq j_3$. Hence, for a given $j_3$, this may occur $\forall j_2 \neq j_3$ and $\forall j_1 \neq j_2, j_3$.

Lemma III.3 A necessary condition to have $g \geq 8$ in monomial codes with $w = c = 3$ is

$$m_h \geq \left\lceil \frac{a(a-1)}{8} \right\rceil.$$  \hspace{1cm} (5)

Proof: Let us consider a generic entry in the third row of $\Delta P$, namely $\Delta p_{2,j}$. In order to satisfy condition 3) of Lemma III.2, $\Delta p_{2,j_3}$ must be different from all the sums between the elements $\Delta p_{0,j_1}, \forall j_1 \neq j_3$ and the elements $\Delta p_{1,j_2}, \forall j_2 \neq j_1, j_3$. Furthermore, for condition 2) of Lemma III.2, $\Delta p_{2,j} = \Delta p_{0,j_1} + \Delta p_{1,j_2}, \forall j_1 \neq j_2, j_3$. Thus, there must be $\sum_{i=0}^{a-2} (a - i - 1) = \binom{a}{2}$ different sums which are also different from $\Delta p_{2,j}$. Since $(\Delta p_{0,j_1} + \Delta p_{1,j_2}) \in [-2m_h, 2m_h]$, the different sums can assume $4m_h + 1$ values (all values except $\Delta p_{2,j}$). It follows that

$$\binom{a}{2} \leq 4m_h,$$

from which (5) is obtained.
Definition III.1 We define a generic difference $\Delta p_j^{(i_1-i_2)} = p_{i_1,j} - p_{i_2,j}$. A difference of differences is defined as

$$\Delta p_j^{(i_1-i_2)} - \Delta p_j^{(i_1-i_3)} = p_{i_1,j_1} - p_{i_2,j_1} - p_{i_1,j_2} + p_{i_2,j_2}, \quad \forall j_1 \neq j_2, i_1 \neq i_2.$$  \hspace{1cm} (7)

Lemma III.4 A necessary and sufficient condition for a monomial LDPCC code with $w = c = 3$ to have girth $g \geq 10$ is that its symbolic parity-check matrix is free of repeated differences of differences.

Proof: For $w = 3$ the following equation holds

$$\Delta p_j^{(i_1-i_2)} + \Delta p_j^{(i_2-i_3)} = \Delta p_j^{(i_1-i_3)}, \quad \forall j, \forall i_3, \forall i_2 \neq i_3, \forall i_1 \neq i_2, i_3. \hspace{1cm} (8)$$

A generic cycle having length 6 can be described as

$$\Delta p_j^{(i_1-i_3)} + \Delta p_j^{(i_2-i_3)} = \Delta p_j^{(i_1-i_3)}, \quad \forall i_3, \forall i_2 \neq i_3, \forall i_1 \neq i_2, i_3, \forall j_3, \forall j_2 \neq j_3, \forall j_1 \neq j_2, j_3. \hspace{1cm} (9)$$

Substituting (8) for $j = j_1$ in (9) we obtain the alternative form

$$\Delta p_j^{(i_1-i_3)} - \Delta p_j^{(i_1-i_3)} = \Delta p_j^{(i_2-i_3)} - \Delta p_j^{(i_2-i_3)}. \hspace{1cm} (10)$$

On the other hand, a cycle having length 8 can be characterized, by definition, as

$$\Delta p_j^{(i_1-i_2)} - \Delta p_j^{(i_1-i_3)} = \Delta p_j^{(i_2-i_3)} - \Delta p_j^{(i_2-i_3)}. \hspace{1cm} (11)$$

From (10) and (11) we see that, avoiding to have coincident differences of differences, we avoid cycles with both lengths 6 and 8. It is also true that (10) and (11) cover all the cases of equalities that can be formed from (7). Therefore the converse holds as well. \hfill \blacksquare

Corollary III.1 For $w = c = 3$, in order to avoid equal differences of differences and achieve $g \geq 10$ we need

$$2m_h \geq 3 \binom{a}{2}. \hspace{1cm} (12)$$

Proof: The absolute value of any difference of differences lies in the range $[1; 2m_h]$. For $w = c = 3$, The total number of differences of differences is $3 \binom{a}{2}$. By imposing that the cardinality of the set of possible values is greater than or equal to the total number of possibilities, we obtain (12). \hfill \blacksquare

The same reasoning can be extended to the case of $w = c = 2$ but, as we have seen before, in such a case cycles with length 10 do not exist. Therefore the following corollary holds.
Corollary III.2 For \( w = c = 2 \), in order to avoid equal differences of differences and achieve \( g \geq 12 \) we need

\[
2m_h \geq \left( \frac{a}{2} \right).
\]  

(13)

Proof: Same as Corollary III.1.■

2) Comparison with QC-LDPC code matrices: As shown in the previous sections, we can describe an LDPCC code with a symbolic parity-check matrix \( H(x) \). Such a representation can also be used for QC-LDPC block codes, the difference being in the procedure that converts \( H(x) \) into the binary parity-check matrix \( H \). For QC-LDPC block codes, the latter involves expansion of each \( H(x) \) entry into a binary circulant block, while for LDPCC codes the procedure described in Section II is used. Nevertheless, we can consider \( H(x) \) matrices designed for QC-LDPC block codes and use them to obtain LDPCC codes. However, as we show next, this approach does not produce optimal results.

Fossorier in [13] provides two bounds for the symbolic matrix of QC-LDPC codes. Both of them relate the minimum size of the circulant permutation blocks forming the code parity-check matrix to the code girth. The symbolic matrices considered in [13] define Type-1 codes and contain all one entries in their first row and column, i.e.,

\[
H(x) = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & x^{p_{1,1}} & \ldots & x^{p_{1,a-1}} \\
\vdots & \vdots & \ddots & \vdots \\
1 & x^{p_{c-1,1}} & \ldots & x^{p_{c-1,a-1}}
\end{bmatrix}.
\]  

(14)

In (14), \( p_{i,j} \) represents the cyclic shift to the right of the elements in each row of an identity matrix, and defines the circulant permutation matrix at position \((i, j)\). This representation has also been used in [12] and [14].

Theorem 2.2 in [13] states that a necessary condition to have \( g \geq 6 \) is \( p_{i,j1} \neq p_{i,j2} \) and \( p_{i1,j} \neq p_{i2,j} \). This theorem is also valid for LDPCC codes and we have used a similar approach to demonstrate Lemma III.1. A corollary of this theorem claims that a necessary condition to have \( g \geq 6 \) in the Tanner graph representation of a \((J, L)\)-regular QC-LDPC code is \( p \geq L \) if \( L \) is odd and \( p > L \) if \( L \) is even \((p \) is the size of the circulant permutation matrices). The bounds on symbolic matrices resulting from this analysis can be extended to LDPCC codes. In fact, for a QC-LDPC code the maximum exponent appearing in \( H(x) \) is equal to \( p - 1 \). Therefore, if we use the symbolic matrix to define an LDPCC code, the condition for \( g = 6 \) becomes \( \max \{m_h\} = p - 1 \). Hence we find that \( m_h \geq a - 1 \) if \( a \) is odd and \( m_h > a - 1 \) if \( a \) is even. By comparing these conditions with (4) we see that having removed the border of
ones in $H(x)$ has allowed us to obtain values of $m_h$ which are about half those needed in the presence of this constraint from [13]. Examples of matrices with the smallest achievable $m_h$, $w = c = 3$ and $g = 6$ with and without the all one border of ones are as follows:

$$H_1(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & x & x^2 & x^4 \\ 1 & x^3 & x & x^2 \end{bmatrix}, \quad (15)$$

$$H_2(x) = \begin{bmatrix} 1 & x^2 & 1 & 1 \\ 1 & 1 & x & x^2 \\ x^2 & x & x & 1 \end{bmatrix}, \quad (16)$$

Indeed, the maximum exponent appearing in $H_1(x)$ is twice that in $H_2(x)$. According to Theorem 2.4 in [13], for $J \geq 3$ and $L \geq 3$, a necessary condition to have $g \geq 8$ is $p_{j_1,l_1} \neq p_{j_2,l_2}$ for $0 < j_1 < j_2$ and $0 < l_1 < l_2$. A corollary of this theorem states that a necessary condition is $p > (J-1)(L-1)$. It is shown in [12] that this bound is usually not tight when $J \geq 4$.

Extending these results to LDPCC codes, we can calculate a necessary condition to avoid local cycles with length both 4 and 6 in codes with the all one border, thus obtaining $m_h \geq (a-1)(c-1)$. Also in this case, the all one border causes an increase in $m_h$, as it is evident from the following example:

$$H_1(x) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & x^6 & x^2 & x^5 \\ 1 & x^3 & x^4 & x \end{bmatrix}, \quad (17)$$

$$H_2(x) = \begin{bmatrix} 1 & x^3 & 1 & x^3 \\ x^3 & x^2 & x^2 & 1 \\ x^2 & 1 & x^3 & x^2 \end{bmatrix}. \quad (18)$$

An intermediate step for the generalization of the codes in [13] is the removal of the constraint of having the first column filled with ones, that is,

$$H(x) = \begin{bmatrix} 1 & 1 & \ldots & 1 \\ x^{p_{1,0}} & x^{p_{1,1}} & \ldots & x^{p_{1,a-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x^{p_{e-1,0}} & x^{p_{e-1,1}} & \ldots & x^{p_{e-1,a-1}} \end{bmatrix}. \quad (19)$$

As we show next, this small modification leads to a reduction in the lower bounds on $m_h$. 
Lemma III.5 A necessary condition for an LDPCC code with $w = c = 3$ in the form (19) to have $g \geq 6$ is $m_h \geq a - 1$, $\forall a$.

Proof: Similar to the proof of Corollary 2.2 in [13].

Lemma III.6 A necessary condition for an LDPCC code with $w = c = 3$ in the form (19) to have $g \geq 8$ is $m_h \geq \frac{a(c-1)}{2} - 1$.

Proof: A local cycle with length 6 follows from equalities having the following form: $p_{j_1,l_1} + p_{j_2,l_2} = p_{j_1,l_2} + p_{j_2,l_3}$. If there exists some $p_{j_1,l_1}^* = p_{j_2,l_1}^*$, then to avoid length-6 cycles there must not be any $p_{j_2,l_2}^d = p_{j_2,l_2}^d$ in the remaining entries of the matrix. Otherwise, either a local cycle with length 6 having form $p_{j_1,l_1}^d + p_{j_1,l_1}^s = p_{j_2,l_2}^d + p_{j_2,l_1}^s$ or a local cycle with length 4 resulting from $p_{j_1,l_1}^d - p_{j_2,l_2}^d = p_{j_1,l_1}^s - p_{j_2,l_1}^s$ would appear. In this case, we would obtain $m_h \geq (a - 1)(c - 1)$, showing that the case studied in [13] is only a particular case of an instance of the above construction. If $p_{j_1,l_1} = p_{j_2,l_1}$ is never verified, we can fill the $(c - 1)a$ possible values with $\frac{(c-1)a}{2}$ couples of exponents (also 0 is admitted), leading to $m_h \geq \frac{a(c-1)}{2} - 1$.

The lower bounds provided by Lemma III.5 and Lemma III.6 remain worse than those obtained with the general structure of $H(x)$, thus confirming that removal of the constraints in (14) is preferable in view of minimizing $m_h$.

B. Type-z codes

In order to meet the condition $g \geq 6$, we must ensure that local cycles with length 4 do not exist. Such short cycles occur when, for some $i_1, j_1, i_2, j_2, j_1 \neq j_2$,

$$\delta_{i_1,j_1} = \delta_{i_2,j_2} \quad \text{and} \quad l_{s_1} = l_{s_2},$$

(20)

where $l_{s_1}$ and $l_{s_2}$ are, respectively, the starting levels of $\delta_{i_1,j_1}$ and $\delta_{i_2,j_2}$. So, in order to avoid cycles with length 4, there must not be any two equal differences starting from the same level. We observe that the two differences may even be in the same column of $H_T^s$.

Lemma III.7 A code with $H_T^s$ having $w = 2$, free of length-4 cycles, has

$$a \leq \sum_{i=0}^{c-1} (L_h - i - 1) = cL_h - \left(\frac{c + 1}{2}\right),$$

(21)

that is

$$L_h \geq \left\lfloor \frac{a + \left(\frac{c+1}{2}\right)}{c} \right\rfloor.$$
Considering that it must be $L_h > c$, we obtain

$$L_h \geq \max \left\{ c + 1, \left\lceil \frac{a + \binom{c+1}{2}}{c} \right\rceil \right\}.$$  \hfill (22)

**Proof:** For column weight $w = 2$, each column of $H_s^T$ only contains one difference $\delta_{i,j}$ and each difference can be used up to $c$ times without incurring cycles with length 4 (by using all the possible $c$ starting levels). For a given $L_h$, the number of possible differences starting from level $l$ is $L_h - l - 1$. Since the differences corresponding to any two of the $a$ columns of $H_s^T$ must be different in value and/or starting level, summing all the contributions we obtain (21), from which (22) is eventually derived.

We can extend (21) to the case of a regular $H_s^T$ with $w > 2$ by considering that, in such a case, each column of $H_s^T$ provides $\binom{w}{2}$ differences that must meet condition (20). Hence (21) becomes

$$a\left( \frac{w}{2} \right) \leq cL_h - \binom{c+1}{2},$$

while (22) becomes

$$L_h \geq \max \left\{ c + 1, \left\lceil \frac{a\left( \frac{w}{2} \right) + \binom{c+1}{2}}{c} \right\rceil \right\}.$$  \hfill (23)

When we have an irregular $H_s^T$ having different columns weights $w_i$, $i = 0, 1, 2, \ldots, a - 1$, each column of $H_s^T$ corresponds to $\binom{w_i}{2}$ differences. Therefore (23) becomes

$$L_h \geq \max \left\{ c + 1, \left\lceil \frac{\sum_{i=0}^{a-1} \binom{w_i}{2} + \binom{c+1}{2}}{c} \right\rceil \right\}.$$  \hfill (24)

In order to find the conditions which permit us to have $g \geq 8$, let us first consider the case with $c = 1$ and $H_s^T$ with column weight $w = 2$. Since the sum of two odd integers is even, the following proposition easily follows.

**Proposition III.1** For $c = 1$ and $w = 2$, if all the $\delta_{i,j}$ are different and odd, then $g \geq 8$.

From Proposition III.1 it follows that, if we wish to minimize $L_h$, we can choose the values of $\delta_{i,j}$ equal to $\{1, 3, 5, \ldots, 2a - 1\}$ and the code will be free of cycles with length smaller than 8.

Another possible choice yielding absence of cycles with length smaller than 8 follows from the fact that, for a given odd integer $x$, summing two values $\in \left[ \frac{x+1}{2}; x \right]$ always gives a result $> x$. Therefore, the following proposition holds.
**Proposition III.2** For \( c = 1 \) and \( w = 2 \), if the \( \delta_{i,j} \) values are equal to \( \{a, a + 1, a + 2, \ldots, 2a - 1\} \), then \( g \geq 8 \).

Based on these propositions, we can prove the following lemma.

**Lemma III.8** For \( c = 1 \) and \( w = 2 \), local cycles having length smaller than 8 can be avoided if and only if

\[
L_h \geq 2a. \tag{25}
\]

**Proof:** From Propositions III.1 and III.2 we see that the maximum value of a difference that is needed to avoid cycles with length smaller than 8 is \( 2a - 1 \). Therefore, we have \( L_h \geq 1 + 2a - 1 = 2a \). In order to prove the converse, let us consider that from the set \([1; y]\) we can select at most \( \frac{y}{2} \) values which may be summed pairwise resulting in other values in the same set. If we choose the values of the differences from the set \([1; 2a - 2]\), we only have \( a - 1 \) values which may be summed pairwise resulting in other values in the same set. Therefore, we can only allocate \( a - 1 \) differences without introducing cycles with length 6, which is not sufficient to cover all the \( a \) rows of \( H_s \). So, \([1, 2a - 1]\) is the smallest set of difference values able to avoid cycles with length 4 and 6, from which (25) follows.

Equation (25) can be extended to the case \( c > 1 \) by considering that, in such a case, each difference value can be repeated up to \( c \) times (by exploiting all the \( c \) available values as starting levels). Therefore, for \( w = 2 \) and \( c > 1 \) we have

\[
L_h \geq \max \left\{ c + 1, \frac{2a}{c} \right\}. \tag{26}
\]

Let us consider larger values of \( w \), i.e., \( w \geq 3 \). For \( c = 1 \), each column of \( H_s^T \) has one or more cycles with length 6, since at least 3 symbols 1 are at the same level (as described in Section II-A).

Instead, for \( w \geq 3 \) and \( c > 1 \) we can follow the same approach used for the case with \( g = 6 \). So, we obtain that to have \( g \geq 8 \) we need

\[
L_h \geq \max \left\{ c + 1, \frac{2a(w)}{c} \right\}. \tag{27}
\]

Finally, when the columns of \( H_s^T \) are irregular with weights \( w_i, i = 0, 1, 2, \ldots, a - 1 \), as done for the case with \( g = 6 \), we can consider that each row of \( H_s \) corresponds to \( \binom{w_i}{2} \) differences and (27) becomes

\[
L_h \geq \max \left\{ c + 1, \left[ \frac{2 \sum_{i=0}^{w-1} \binom{w_i}{2}}{c} \right] \right\}. \tag{28}
\]
IV. Design Methods

In this section we introduce new methods for the design of LDPCC codes with constraint length approaching the bounds described in Section III for Type-1, Type-2 and Type-3 codes. These design approaches have general validity. However, for the sake of simplicity we focus on the case \( w = c = 3 \), which is a common choice for LDPC codes in general and LDPCC codes in particular. A generic code with rate \( \frac{a-3}{a} \) and \( w = 3 \) can be designed as a Type-\( z \) code, with \( 1 \leq z \leq 3 \).

A. Type-1 codes

As we have already demonstrated in Lemma III.1, monomial codes having \( g \geq 6 \) satisfy (4). For any integer value \( k \), a Type-1 code with \( a = 2^k + 1 \) able to achieve this bound is defined by the following symbolic matrix

\[
H_e(x, k) = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
x & x^2 & x^3 & \ldots & x^{k-1} & x^k \\
x^k & x^{k-1} & x^{k-2} & \ldots & x & 1 \\
x & x^2 & x^3 & \ldots & x^{k-1} & x^k \\
& & & & & \\
& & & & & \\
& & & & & \\
1 & 1 & 1 & \ldots & 1 & 1 \\
\end{bmatrix}
\]

when \( k \) is even and by

\[
H_o(x, k) = \begin{bmatrix}
x & x^2 & x^3 & \ldots & x^{k-1} & x^k \\
x & x^2 & x^3 & \ldots & x^{k-1} & x^k \\
x^{k-1} & x^{k-2} & x^{k-3} & \ldots & x & 1 \\
x & x^2 & x^3 & \ldots & x^{k-1} & x^k \\
1 & 1 & 1 & \ldots & 1 & 1 \\
\end{bmatrix}
\]

when \( k \) is odd. Starting from (29) and (30), the removal of one column permits us to also cover the case of \( a = 2k \).

B. Type-2 codes

In the case of Type-2 codes with \( w = c = 3 \), we can jointly use monomial and binomial entries in the same column of the symbolic parity-check matrix. The bound given by (23), expressed in terms of \( m_h \), becomes

\[
m_h \geq \left\lceil \frac{a-1}{3} \right\rceil.
\]

Let \( k \) be an integer and \( H^1_e(x, k), H^2_e(x, k) \) and \( H^3_e(x, k) \) be the following \( 3 \times k \) matrices:

\[
H^1_e(x, k) = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
x + x^k & x^2 + x^{k-1} & \ldots & x^k & x^k + x^{k+2} & 1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 & 1 + x^k & x + x^{k-1} & \ldots & x^{k-2} + x^{k+2}
\end{bmatrix}
\]
\[
H_e^2(x, k) = \begin{bmatrix}
1 & 1 & \ldots & 1 & 1 + x^k & x + x^{k-1} & \ldots & x^{k-\frac{3}{2}} + x^{k+\frac{1}{2}} \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
x + x^k & x^2 + x^{k-1} & \ldots & x^{\frac{k-1}{2}} + x^{\frac{k+1}{2}} & 1 & 1 & \ldots & 1
\end{bmatrix}
\] (33)

\[
H_e^3(x, k) = \begin{bmatrix}
x + x^k & x^2 + x^{k-1} & \ldots & x^{\frac{k-1}{2}} + x^{\frac{k+1}{2}} & 1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 & 1 + x^k & x + x^{k-1} & \ldots & x^{\frac{k-3}{2}} + x^{\frac{k+3}{2}} \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\] (34)

Also, let \(H_o^1(x, k), H_o^2(x, k)\) and \(H_o^3(x, k)\) be the following 3 \(\times k\) symbolic matrices:

\[
H_o^1(x, k) = \begin{bmatrix}
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
x + x^k & x^2 + x^{k-1} & \ldots & x^{\frac{k-1}{2}} + x^{\frac{k+1}{2}} & 1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 & 1 + x^k & x + x^{k-1} & \ldots & x^{\frac{k-3}{2}} + x^{\frac{k+3}{2}}
\end{bmatrix}
\] (35)

\[
H_o^2(x, k) = \begin{bmatrix}
1 & 1 & \ldots & 1 & 1 + x^k & x + x^{k-1} & \ldots & x^{\frac{k-3}{2}} + x^{\frac{k+3}{2}} \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
x + x^k & x^2 + x^{k-1} & \ldots & x^{\frac{k-1}{2}} + x^{\frac{k+1}{2}} & 1 & 1 & \ldots & 1
\end{bmatrix}
\] (36)

\[
H_o^3(x, k) = \begin{bmatrix}
x + x^k & x^2 + x^{k-1} & \ldots & x^{\frac{k-1}{2}} + x^{\frac{k+1}{2}} & 1 & 1 & \ldots & 1 \\
1 & 1 & \ldots & 1 & 1 + x^k & x + x^{k-1} & \ldots & x^{\frac{k-3}{2}} + x^{\frac{k+3}{2}} \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\] (37)

It is easy to verify that each one of the matrices in (32) to (37) results in a code with girth 6. For even and odd values of \(k\), codes with \(a = 3k\) that often achieve the lower bound in (31) are defined by, respectively,

\[
H_e(x, k) = \begin{bmatrix}
H_e^1(x, k) & H_e^2(x, k) & H_e^3(x, k)
\end{bmatrix}
\] (38)

\[
H_o(x, k) = \begin{bmatrix}
H_o^1(x, k) & H_o^2(x, k) & H_o^3(x, k)
\end{bmatrix}
\] (39)

Also in this case, it is possible to remove one or more columns to obtain the required values of \(a\). Finally, let us note that both \(H_e(x, k)\) and \(H_o(x, k)\) define codes with girth 6. This is due to the order of the rows of \(H_e(x, k)\) and \(H_o(x, k)\), which ensures that the supports of any two columns belonging to two different \(H_e(x, k)\)’s or \(H_o(x, k)\)’s do not overlap in more than one position.
Example IV.1 Let $w = c = 3$ and $a = 3k = 12$ (i.e., $k = 4$). Based on (31), $m_h = \left\lfloor \frac{12 - 1}{3} \right\rfloor = 4$. By concatenating the three matrices below, each of size $3 \times 4$, it is possible to construct a syndrome matrix $H_c(x, 4)$ with $m_h = 4$.

$$H^1_c(x, k) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x + x^4 & x^2 + x^3 & 1 & 1 \\ 1 & 1 & 1 + x^4 & x + x^3 \end{bmatrix}, \quad H^2_c(x, k) = \begin{bmatrix} 1 & 1 & 1 + x^4 & x + x^3 \\ 0 & 0 & 0 & 0 \\ x + x^4 & x^2 + x^3 & 1 & 1 \end{bmatrix},$$

$$H^3_c(x, k) = \begin{bmatrix} x + x^4 & x^2 + x^3 & 1 & 1 \\ 1 & 1 & 1 + x^4 & x + x^3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

C. Type-3 codes

Finally, it is possible to use trinomial entries in the columns of the symbolic parity-check matrices. Since $w = 3$, we cannot use more than one trinomial entry per column but it is possible to use every trinomial entry three times, one for each row of the symbolic matrix. For different values of $a$, some groups of trinomials with the smallest possible degrees that permit us to avoid local cycles with length 4 are as follows.

$$\begin{align*}
1 + x + x^4 & \quad 1 + x^3 + x^7 & a = 6 \\
1 + x + x^5 & \quad 1 + x^6 + x^8 & 1 + x^3 + x^{10} & a = 9 \\
1 + x^2 + x^9 & \quad 1 + x^3 + x^8 & 1 + x^4 + x^{10} & 1 + x + x^{12} & a = 12 \\
1 + x^{11} + x^{12} & \quad 1 + x^{13} + x^{15} & 1 + x^7 + x^{10} & 1 + x^5 + x^9 & 1 + x^8 + x^{14} & a = 15 \\
1 + x^6 + x^7 & \quad 1 + x^{13} + x^{15} & 1 + x^{14} + x^{17} & 1 + x^8 + x^{12} & 1 + x^{11} + x^{16} & 1 + x^{10} + x^{19} & a = 18 \\
\vdots
\end{align*}$$

(40)

To see why these trinomials are those with smallest degrees, we address the interested reader to [17], where the authors exploit a combinatorial notion known as Perfect Difference Families (PDF). They show that Type-3 QC-LDPC block codes with $w = 3$, $c = 1$, $a = 3k$ and girth 6 can be constructed by using the entries of each block of a $(v, 3, 1)$ PDF to define a trinomial in the syndrome matrix. In fact, if our target syndrome matrix is only formed by trinomials, PDF’s give us the smallest trinomial degrees. Those reported in (40) and many other groups of trinomials can be found starting from Table I in [18].

We also notice that we can combine trinomial and monomial entries to design codes with $g \geq 6$. In fact, trinomial entries introduce “horizontal” separations whereas monomial entries introduce “vertical” separations. This is also the reason why, in general, trinomial and
binomial entries cannot be combined in the configurations described in Section IV-B. We provide next an example in which we concatenate two matrices having the form (30) and (40) to construct a syndrome matrix with $w = c = 3$, $a = 21$ and girth 6, achieving the smallest possible $m_h$ according to (31):

$$H(x,7)_{odd} = \begin{bmatrix}
1 + x + x^4 & 1 + x^2 + x^7 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 + x + x^4 & 1 + x^2 + x^7 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 + x + x^4 & 1 + x^2 + x^7
\end{bmatrix} \ldots$$

$$\begin{bmatrix}
x^7 & x^7 & x^7 & x^7 & x^7 & x^7 & x^7 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & 1 & x & x^2 & x^3 & x^4 & x^5 & x^6 & x^7 \\
x^8 & x^5 & x^4 & x^3 & x^2 & x & 1 & x^7 & x^6 & x^5 & x^4 & x^3 & x^2 & x & 1
\end{bmatrix}.$$  

V. CODE EXAMPLES AND NUMERICAL RESULTS

In this section we provide some examples of code design and their comparison with analytical bounds. We also assess the codes performance through numerical simulation of coded transmissions.

A. Code design based on exhaustive searches

In Figs. 1–4 we report the bounds on $L_h$ obtained as described in Section III as a function of $a$, for some values of $w$, $g$ and $c$. We compare these bounds with the results obtained through exhaustive searches over all the possible choices of $H_s$.

![Fig. 1](image-url)  

Fig. 1. Bounds on $L_h$ and values found through exhaustive searches as a function of $a$, for $w = 2$, $g = 6$ and some values of $c$.  

Fig. 2. Bounds on $L_h$ and values found through exhaustive searches as a function of $a$, for $w = 3$, $g = 6$ and some values of $c$.

From Fig. 1 we observe that, for the cases with $w = 2$ and $g = 6$, the matching between the theoretical bound and the values found through exhaustive searches is perfect for all the considered values of $c$. Instead, when we have larger column weights of $H_s^T$, i.e., $w > 2$, the theoretical bound may not be achievable in practical terms. This results in Fig. 2 for $w = 3$. However, we also observe that the deviations of the exhaustive search values from the theoretical curves are rather small. The results of exhaustive searches are well matched with the theoretical bounds also for the case with $w = 2$ and $g = 8$, and for $w = 3$ and $g = 8$ as we observe from Fig. 3 and Fig. 4. In the latter case, we note that the gap to the bound increases for increasing values of $c$.

In Fig. 5 we compare the values of $m_h$, for $w = c = 3$ and $g = 6$, achievable through the design methods described in Section IV with the corresponding lower bounds. In Table I we instead provide the results of an exhaustive search for monomial codes with $w = c = 3$ and $g = 8$, and their comparison with the corresponding lower bound.

| CODE RATE | 1/4 | 2/5 | 3/6 | 4/7 |
|----------|-----|-----|-----|-----|
| Exhaustive Search | 3   | 5   | 6   | 8   |
| Bound (5)     | 2   | 3   | 4   | 6   |
Finally, in Table II we show a comparison of the minimum values of $m_h$ found by means of an exhaustive search for monomial codes ($w = c = 2$) with $g = 10$.

**B. Code design based on Montecarlo experiments**

When the girth takes values $g \geq 10$, the values of $m_h$, or, equivalently, $L_h$, are so high that exhaustive searches become unfeasible even for small values of $a$ and $c$. Appendix B contains a brief description of the complexity of the searching algorithms. Therefore, we resort to Montecarlo simulations. This way, we are still able to find symbolic parity-check
Fig. 5. Bounds on $m_h$ and values found through the design method described in Section IV as a function of $a$, for $w = c = 3$, $g = 6$.

### TABLE II

**MINIMUM VALUES OF $m_h$ FOR MONOMIAL CODES WITH $w = c = 2$ AND $g = 12$**

| Code rate | $1/3$ | $2/4$ | $3/5$ | $4/6$ | $5/7$ | $6/8$ | $7/9$ | $8/10$ |
|-----------|-------|-------|-------|-------|-------|-------|-------|-------|
| Exhaustive search | 2 | 3 | 6 | 9 | 13 | 17 | 23 | 25 |
| Bound (13) | 2 | 3 | 5 | 8 | 11 | 14 | 18 | 23 |

### TABLE III

**MINIMUM VALUES OF $m_h$ FOR CODES WITH $w = c = 3$ AND $g = 10$ OBTAINED THROUGH MONTECARLO SIMULATIONS, FOR SEVERAL CODE RATES**

| Code rate | $1/4$ | $2/5$ | $3/6$ | $4/7$ | $5/8$ | $6/9$ |
|-----------|-------|-------|-------|-------|-------|-------|
| Heuristic search | 10 | 19 | 33 | 63 | 124 | 273 |
| Bound (12) | 9 | 15 | 22 | 31 | 42 | 54 |

matrices with smaller constraint length than those reported in previous literature (mostly referred to QC-LDPC block codes), although they are not found exhaustively. In Table III and IV we list the minimum values of $m_h$ that we were able to find for codes with $w = c = 3$ and $g = 10, 12$. For the latter, a bound on $m_h$ has not been found and is left for future works. Furthermore, the results of the ILP model are summarized in Tables V to X for a variety of girths and lengths.
TABLE IV
MINIMUM VALUES OF $m_n$ FOR CODES WITH $w = c = 3$ AND $g = 12$ OBTAINED THROUGH MONTECARLO SIMULATIONS, FOR SEVERAL CODE RATES

| Code rate | 1/4 | 2/5 | 3/6 | 4/7 |
|-----------|-----|-----|-----|-----|
| Heuristic search | 19 | 47 | 159 | 400 |

TABLE V
LOWEST EXponent matrices of Type-1 codes with girth 8 and $w = c = 3$, found through the Min-Max algorithm.

| $r$ | Exponent Matrix |
|-----|-----------------|
| 1   | 1                |
| 2   | 1,10             |
| 3   | 1,10,2          |
| 4   | 1,10,2,3         |
| 5   | 1,10,2,3,4      |
| 6   | 1,10,2,3,4,5  |
| 7   | 1,10,2,3,4,5,6 |
| 8   | 1,10,2,3,4,5,6,7|
| 9   | 1,10,2,3,4,5,6,7,8|
| 10  | 1,10,2,3,4,5,6,7,8,9|
| 11  | 1,10,2,3,4,5,6,7,8,9,10|
| 12  | 1,10,2,3,4,5,6,7,8,9,10,11|
| 13  | 1,10,2,3,4,5,6,7,8,9,10,11,12|
| 14  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13|
| 15  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14|
| 16  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15|
| 17  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16|
| 18  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17|
| 19  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18|
| 20  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19|
| 21  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20|
| 22  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21|
| 23  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22|
| 24  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23|
| 25  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24|
| 26  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25|
| 27  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26|
| 28  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27|
| 29  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28|
| 30  | 1,10,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29|

C. Numerical simulations of transmission

Concerning the bit error rate (BER) performance of the considered codes, as reasonable and expected, there is a trade-off with their constraint lengths. However, LDPC codes with moderately small constraint lengths may still achieve better performance than their block counterparts. For example, using a belief propagation (BP) decoder, we have verified through Montecarlo simulations of binary phase shift keying (BPSK) modulated transmission over the additive white Gaussian noise (AWGN) channel that one of our LDPC codes with
This way, a smaller sliding window can be used also for decoding of the Tanner code. In respect to the WiMax standard [19] LDPC irregular block code with the same rate ($a = 17$, $b = 9$, $g = 8$ and $v_s = 1143$) exhibits a gain of about 0.3 dB at BER = $10^{-5}$ with respect to the WiMax standard [19] LDPC irregular block code with the same rate ($2/3$, maximum weight of the columns $w_{max} = 6$ and length 2304).

We have also compared the performance of our codes with some LDPCCC codes constructed using the technique described in [20], namely Tanner codes, using a BP-based sliding window decoder. Let us consider the code reported in Table V with $a = 17$, $w = c = 3$, $g = 8$ and $v_s = 646$, denoted as $C_1$, and the Tanner code with the same values of $a$, $c$ and $g$, but $v_s = 5185$. By applying the procedures described in Section II-B, an equivalent code of the Tanner code with $v_s = 4641$ has been found and will be called $C_{t1}$ in the following. This way, a smaller sliding window can be used also for decoding of the Tanner code. In order to decode $C_1$ with a sliding window decoder [9], the minimum required window size

### Table VI

| n | m_n | Exponent Matrix |
|---|---|---|
| 5 | 0 | 0.0 0.1 10 0.5 0.9 8 0.4 0.9 5 |
| 6 | 1 | 0.0 0.3 10 6.0 6.0 0.5 0.9 8 0.7 0.3 10 11.0 5.0 0.9 0.7 |
| 7 | 1 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 8 | 3 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 9 | 9 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 10 | 12 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 11 | 3 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 12 | 36 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 13 | 18 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 14 | 53 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 15 | 64 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 16 | 78 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 17 | 87 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 18 | 97 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 19 | 106 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 20 | 116 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 21 | 126 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 22 | 136 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 23 | 146 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 24 | 156 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |
| 25 | 166 | 0.0 0.9 6.0 0.7 0.3 0.7 0.3 10 11.0 5.0 0.9 1.1 13.12 0.9 4.2 |

$w = c = 3$, $a = 9$, $g = 8$ and $v_s = 1143$ exhibits a gain of about 0.3 dB at BER = $10^{-5}$ with respect to the WiMax standard [19] LDPC irregular block code with the same rate ($2/3$, maximum weight of the columns $w_{max} = 6$ and length 2304).

We have also compared the performance of our codes with some LDPCCC codes constructed using the technique described in [20], namely Tanner codes, using a BP-based sliding window decoder. Let us consider the code reported in Table V with $a = 17$, $w = c = 3$, $g = 8$ and $v_s = 646$, denoted as $C_1$, and the Tanner code with the same values of $a$, $c$ and $g$, but $v_s = 5185$. By applying the procedures described in Section II-B, an equivalent code of the Tanner code with $v_s = 4641$ has been found and will be called $C_{t1}$ in the following. This way, a smaller sliding window can be used also for decoding of the Tanner code. In order to decode $C_1$ with a sliding window decoder [9], the minimum required window size
is \( W = m_h + 1 = 38 \), whereas \( C_{t1} \) requires at least \( W = m_h + 1 = 273 \). Their performance comparison, obtained by fixing \( W = 273 \), is shown in Fig. 6, where the BER is plotted as a function of the signal-to-noise ratio per bit \( E_b/N_0 \). In the figure we also report the performance of the codes simulated using a very large (theoretically infinite) window. We notice that \( C_{t1} \) outperforms \( C_1 \) when \( W \rightarrow \infty \), thanks to the better minimum distance properties, but for small window sizes \( C_1 \) significantly outperforms \( C_{t1} \). This happens because, as shown in [21], values of \( W \) roughly 5 to 10 times as large as \( (m_h + 1) \) result in minimal performance degradation, compared to using the full window size. In this case, \( W_{C_1} > 8m_h \), and \( C_1 \)
approaches its full length performance, whereas $W_{C_{t1}} = m_h + 1$, and $C_{t1}$ still incurs some losses due to the small window size.

In order to understand the role of the local cycles length, another code, noted as $C_2$, has been designed with a heuristic search. $C_2$ has $a = 17$, $w = c = 3$, $g = 8$ and $v_s = 1207$; its
TABLE IX
LOWEST EXPONENT MATRICES OF TYPE-I CODES WITH GIRTH 10 AND $w = c = 3$, FOUND THROUGH THE MIN-MAX ALGORITHM.

| $m_A$ | Exponent Matrix |
|-------|-----------------|
| 8, 11 | 6, 11, 0, 9 |
|        | 4, 4, 11, 0, 9 |
| 5, 19 | 19, 19, 0, 17, 0 |
|        | 0, 17, 0, 11 |
|        | 0, 11, 16, 12 |
| 6, 31 | 0, 20, 29, 18, 14, 31 |
|        | 29, 0, 12, 7, 30, 21 |
| 7, 53 | 39, 0, 31, 0, 4, 36, 19 |
|        | 0, 14, 27, 3, 34, 40, 53 |
|        | 41, 63, 0, 30, 8, 43 |
| 8, 76 | 0, 54, 36, 5, 39, 52, 41, 76 |
|        | 51, 45, 16, 76, 76, 53, 50, 0 |
| 9, 127 | 0, 47, 85, 8, 35, 73, 100, 0 |
|        | 27, 6, 1, 0, 70, 117, 127, 19 |
| 10, 222 | 0, 71, 58, 40, 0, 100, 100, 0, 508 |
|        | 34, 222, 177, 0, 145, 57, 17, 153, 50, 217 |
|        | 40, 0, 125, 131, 2, 59, 116, 0, 127, 0 |
| 11, 207 | 0, 60, 0, 0, 0, 0, 0, 100, 249 |
|        | 39, 57, 60, 64, 101, 140, 291, 297, 0, 220 |
|        | 0, 24, 21, 5, 56, 0, 59, 176, 132, 104, 0 |
| 12, 388 | 0, 0, 100, 0, 40, 100, 62, 527, 100, 83, 100 |
|        | 40, 297, 180, 176, 8, 234, 0, 60, 0, 261, 388, 48 |
|        | 100, 290, 299, 87, 236, 0, 83, 0, 551, 278, 221, 7 |

TABLE X
LOWEST EXPONENT MATRICES OF TYPE-I CODES WITH GIRTH 12 AND $w = c = 3$, FOUND THROUGH THE MIN-MAX ALGORITHM.

| $m_A$ | Exponent Matrix |
|-------|-----------------|
| 8, 20 | 2, 20, 0, 20 |
|        | 20, 20, 10, 12 |
| 20, 28 | 28, 42, 1, 3, 42 |
|        | 0, 3, 42, 42, 25 |
| 42, 100 | 0, 100, 0, 100, 100 |
|        | 100, 91, 108, 24, 42, 12 |
| 100, 208 | 37, 18, 108, 22, 76, 9 |
|        | 0, 0, 100, 0, 100, 0 |
| 208, 320 | 0, 5, 220, 130, 172, 216, 76, 27 |
|        | 0, 7, 178, 220, 21, 57, 159 |

![Graph](image.png)

Fig. 6. BER performance of the proposed codes.
The exponent matrix is

\[
P_{C_2} = \begin{bmatrix}
9 & 59 & 30 & 44 & 0 & 55 & 0 & 0 & 65 & 0 & 21 & 0 & 58 & 37 & 24 & 0 & 41 \\
0 & 67 & 26 & 60 & 53 & 0 & 18 & 32 & 0 & 59 & 0 & 57 & 0 & 0 & 0 & 38 & 13 \\
5 & 0 & 0 & 0 & 9 & 55 & 70 & 42 & 27 & 14 & 43 & 16 & 68 & 57 & 56 & 41 & 0
\end{bmatrix}
\]

Considering only its first 7 columns, there are no local cycles with length 8. We notice that \(C_2\) has a very steep curve in the waterfall region when \(W \to \infty\) but its performance is affected by an error floor. However, under sliding window decoding with \(W = 273\), \(C_2\) notably outperforms \(C_{t_1}\) and \(C_1\). Finally, the performance of an array code [22] with \(a = 17\), \(w = c = 3\), \(g = 6\) and \(v_s = 289\) is shown for \(W \to \infty\) and some more values of \(W\). Despite its performance is worse than the other ones when \(W \to \infty\), the low value of its constraint length allows it to achieve good performance for very small window sizes, namely \(W = 38\).

VI. Conclusion

We have studied the design of time-invariant SC-LDPC codes with small constraint length and free of local cycles up to a given length. By directly designing the parity-check matrix of LDPCC codes, we were able to obtain codes with smaller constraint length with respect to those designed by unwrapping QC-LDPC block codes and, for low values of the girth, the smallest possible ones. We have also provided lower bounds on the minimum constraint length which is needed to achieve codes with a fixed minimum length of the local cycles, and shown through new design methods, exhaustive searches and Montecarlo simulations that practical codes exist, which are able to reach or, at least, to approach these bounds.

APPENDIX A

Let us describe the ILP optimization model we propose and consider. As inputs, it takes a big enough penalty \(M\), all the entries of an exponent matrix \(P\), a positive integer \(p \to \infty\) to represent the maximum allowed exponent in \(P\), and the set of relatively prime numbers to \(p\) (this set has cardinality \(\phi(p)\), where, \(\phi\) is the Euler function) and it finds Min-Max of the elements of all the matrices \(P_t^*\)’s, where \(P_t\) is the transformed exponent matrix obtained by applying Lemmas II.2 and II.3 on \(P\). Thus, if \(P^*_t\) is one of the optimal transformed exponent matrices, our model will explicitly find the linear transformations based on Lemmas II.2 and II.3 and, by applying them on \(P\), we achieve new instances of \(P^*_t\). Furthermore, the model finds the maximum syndrome former memory order \(m_h\) in \(P^*_t\) as output; the final \(m_h\) is the minimum possible \(m_h\) of all the transformed matrices \(P_t\).

In the following list, we enumerate the steps of our model:
1. minimize \( Z = \sum_{i=0}^{a-1} \sum_{j=0}^{c-1} x_{ij} \)

s.t.

2. \( b_{ij} = \left( \sum_{g=1}^{\phi(p)} k_g T_g \right) p_{ij} + r_i + c_j \quad i \in \{0, \ldots, a-1\} \) & \( j \in \{0, \ldots, c-1\} \)

3. \( \sum_{g=1}^{\phi(p)} k_g = 1 \)

4. \( p\psi_{ij} \leq b_{ij} \)

5. \( b_{ij} + 0.5 \leq (1 + \psi_{ij}) p \)

6. \( d_{ij} = b_{ij} - p\psi_{ij} \)

7. \( d_{mn} \leq d_{ij} + M (1 - y_{ij}) \quad (m, n) \neq (i, j) \)

8. \( \sum_{i=0}^{a-1} \sum_{j=0}^{c-1} y_{ij} = 1 \)

9. \( x_{ij} \leq My_{ij} \)

10. \( d_{ij} \leq x_{ij} + M (1 - y_{ij}) \)

11. \( k_g, y_{ij} \in \{0, 1\}, 0 \leq T_g, r_i, c_j < p, \) and \( b_{ij}, \psi_{ij}, d_{ij}, x_{ij} \) are integers.

A brief description of the steps of the model is as follows. Each element \( p_{ij} \) of \( P \) is transformed to \( b_{ij} \) by multiplying it to a relatively prime number \( T_g \), as well as by adding two decision variables \( r_i \) and \( c_j \) (Constraint 2). Constraint 3 indicates that just one of the relatively prime number to \( p \) could be selected. Constraints 4 and 5 determine the quotient \( \psi_{ij} \) of element \( b_{ij} \) divided by \( p \). Constraint 6 is the residual of subtracting \( p\psi_{ij} \) from \( b_{ij} \). Two constraints 7 and 8 are used to detect maximum element of the transformed exponent matrix modulo \( p \), where, \( y_{ij} \)’s are identification binary variables. Clearly, just one of these variables can be chosen. Variables \( x_{ij} \)’s in constraints 9 and 10 are created in such a way that just one of them is greater than zero, which is the maximum among all of the elements in \( P^*_T \). This output element is considered to be Min-Max or our desired \( m_h \).

**APPENDIX B**

Let us compute complexity of performing exhaustive analyses of codes with fixed parameters by using their binary or symbolic matrix representation.

**A. Binary matrix**

Let us consider \( H^T_s \) and, for the sake of simplicity, let us suppose that \( L_h \) is an integer multiple of \( c \); so, \( H^T_s \) has size \( L_h \times a \). Considering that any column has weight \( w \), and that all columns must be different in order to satisfy the r.c.c., we can enumerate the number of possible matrices

\[
N = \left( \begin{array}{c}
\frac{L_h}{w} \\
\frac{w}{a}
\end{array} \right),
\]

(42)
The space of possible matrices can be further reduced if we consider that any code has an equivalent code with at least a 1 in the first position of the first column of $H_s^T$, yielding

$$N = \left( \frac{L_h - 1}{w - 1} \right) \left( \frac{L_h}{w} \right).$$  \hfill (43)

Furthermore, any code has an equivalent code with at least a 1 in any of the first $c$ positions of $H_s^T$. Therefore, we have

$$N = \left( \frac{L_h - 1}{w - 1} \right) \left( \sum_{i=1}^{c} \frac{L_h - i}{w - i} \right).$$  \hfill (44)

Finally, any code has $a!$ equivalent codes obtained by permuting the columns of $H_s^T$. This means that the number of possible matrices can be eventually reduced to

$$N = \left( \frac{\sum_{j=1}^{L_h - 1} \left[ \sum_{i=1}^{c} \frac{L_h - i}{w - i} - j \right]}{a} \right).$$  \hfill (45)

**B. Symbolic matrix**

Let us consider an exponent matrix $P$ with size $c \times a$ and such that its $(i,j)$-th element $p_{i,j} \in [0, m_h]$. In general, $N = (m_h + 1)^{ac}$ matrices should be tested. However, if we consider that all the columns of a symbolic matrix must be different, we obtain $N = \binom{(m_h + 1)^{c}}{a}$ possible matrices. By using Lemmas II.1 and II.2 it is possible to show that any code has an equivalent with at least a null value in any column of $P$. Since there can be $m_h^c$ possible columns of $P$ without a null symbol, out of the total $(m_h + 1)^c$ possible columns, we obtain

$$N = \binom{(m_h + 1)^c - m_h^c}{a}.$$  \hfill (46)

Moreover, due to Lemma II.1, any code has an equivalent code with at least one null entry in $p_{1,1}$, reducing again the space of possible matrices to

$$N = (m_h + 1)^{c-1} \binom{(m_h + 1)^c - m_h^c}{a - 1}.$$  \hfill (47)

Finally, exploiting again Lemma II.1, we can consider any $n$-tuple of columns only once (without taking into account its permutations), and obtain

$$N = \binom{\sum_{j=1}^{(m_h + 1)^{c-1}} [(m_h + 1)^c - m_h^c - j]}{a - 1}.$$  \hfill (48)

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