The Quantum Hall Effect in Drag: Inter-layer
Friction in Strong Magnetic Fields

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We study the Coulomb drag between two spatially separated electron systems in a strong magnetic field, one of which exhibits the quantum Hall effect. At a fixed temperature, the drag mimics the behavior of $\sigma_{xx}$ in the quantum Hall system, in that it is sharply peaked near the transitions between neighboring plateaux. We assess the impact of critical fluctuations near the transitions, and find that the low temperature behavior of the drag measures an exponent $\eta$ that characterizes anomalous low frequency dissipation; the latter is believed to be present following the work of Chalker.
Coulomb interactions between spatially separated systems of charge carriers lead to a variety of novel phenomena [1]. In particular, density fluctuations in one of the systems exert a frictional force on the other, and consequently a current flowing in one induces a drag current in the other. This effect was first predicted by Pogrebinskii and by Price [2]. A number of recent experiments have successfully observed the Coulomb drag in heterostructures involving a two-dimensional electron system (2DES), separated from another electron [3,4] or hole [5] gas by an insulating barrier sufficiently thick to prevent tunneling. Solomon et al. [3] observed the drag in a three-dimensional electron gas (3DEG) in a GaAs gate electrode, while Gramila et al. [4] and Sivan et al. [5] studied double layer structures at lower temperatures. This experimental progress has been accompanied by a number of theoretical studies [6–9].

In this letter we study the Coulomb drag between two electron systems, one of which is two dimensional and exhibits the quantum Hall effect (QHE) [10], and the other, christened the test system, could (in principle) be in any state feasible in a magnetic field. (We shall concentrate on the cases where it is either an identical layer or is a semi-infinite 3DEG.) As we will illustrate, the Coulomb drag in this situation is a novel probe of the internal dynamics of the 2DES in the QH regime. The geometry we consider is depicted in Figure 1; the test system is situated above the QH layer, across an insulating barrier that suppresses tunneling but permits Coulomb scattering [3,4]. The quantity of interest is the trans-resistivity \( \rho_t \) defined as

\[
\rho_t = \frac{E_u}{j_d},
\]

(0.1)

where \( E_u \) is the parallel electric field induced in U (which is in an open circuit) in response to a current density \( j_d \) established in D. We find that the dependence of \( \rho_t \) on the filling factor \( \nu \) at a fixed temperature is similar to that of the dissipative conductance \( \sigma_{xx} \) in that it peaks at the transition between QHE plateaux and is greatly suppressed deep in the QH phases; this pattern arises independently of whether the test system exhibits the QHE. This is quite intuitive: as with everyday friction, the magnitude of the drag is directly related
to inhomogeneities in the systems—in our problem, long lived density fluctuations. Deep in the QH phases these are suppressed (“incompressibility”) but they grow as the boundary of the phase is approached.

An important consequence of this physics is that the $T$-dependence of $\rho_t$ is sensitive to the low frequency dynamics of the density fluctuations and is different at different filling factors. In particular, for macroscopic systems Zheng and MacDonald [9] have shown that $\rho_t$ involves a convolution of the dissipative density-density response functions Im$\chi(q,\omega)$ of the individual layers (Eq. (1.6)). As a result, $\rho_t$ at low $T$ is sensitive to Im$\chi$ for $\omega < q$, in contrast to dc transport measurements in a single layer that probe $q < \omega$. Most strikingly, for two identical layers we find that while $\rho_t \sim O(T^2)$ in the QH phases, at the critical points $\rho_t \sim T^{2-\eta}$, where $\eta$ is an exponent characteristic of anomalous low-frequency dissipation, first discussed, in the context of the anomalous diffusion at the non-interacting critical point for the IQHE, by Chalker [11]. Such a direct measurement of this exponent, which is not experimentally accessible by other means, could shed considerable light on the nature of the critical fixed points and the issue of the universality of the various transitions in the QH regime.

The paper is organized as follows. First, we discuss a model of a clean system, applicable to quantum wires, where our scenario is realized transparently by means of Fermi’s golden rule. Next, we extend our considerations to disordered systems, particularly focusing on the critical region where the discussion relies on known or conjectured properties of Im$\chi$. In most of the following we shall discuss the IQHE, even though the physics is evidently more general; the extension to the FQHE is discussed at the end of the paper.

**Clean System:** We assume that layer D is free of impurities and is subject to a parabolic confining potential, $\frac{1}{2}m\omega_0^2y^2$. The exactly calculable eigenstates [15] are labeled by the wave-vector in the $x$-direction $k_d$, and the Landau level (LL) index $n_d$; the corresponding energy levels are denoted $E_{n_d,k_d}^{(d)}$. The electron system in layer U is assumed to be a 3DEG; its eigenstates are labeled by $n_u$, $k_u$ and $k_z$, and the energy levels are denoted $E_{n_u,k_z}^{(u)}$. We calculate $\rho_t$ within the Boltzmann transport approach. The current density $j_d$ flowing in D
along the $x$-direction is

$$j_d = \frac{e\hbar}{\pi L_y m^*} \sum_{n_d} \int_{-\infty}^{\infty} dk_d g_{n_d k_d} k_d,$$

where $g_{n_d k_d} \equiv -\hbar v_D (\partial f_{n_d k_d}^0 / \partial E_{n_d k_d}^{(d)}) k_d$ is the deviation of the electron distribution from the equilibrium Fermi distribution $f_{n_d k_d}^0$, $v_D$ is the drift velocity, and $m^* \equiv m(\omega_c/\omega_0)^2$ in terms of the band effective mass $m$ and the cyclotron frequency $\omega_c$. The electric field $E_u$ induced in U is related to the total momentum transfer into this region by

$$E_u = \frac{\hbar L_x L^*_x}{e\pi^2 N_u} \sum_{n_u} \int_0^\infty dk_z \int_{-\infty}^{\infty} dk_u k_u \tau^{-1}(k_u, k_z),$$

where the relaxation rate $\tau^{-1}(k_u, k_z)$ is, to linear order in $g_{n_d k_d}$,

$$\tau^{-1}(k_u, k_z) = \frac{2\pi L_x L^*_x}{\hbar} \sum_{n_u, n_u', n_d} \int_0^\infty dk_z' \int_{-\infty}^{\infty} dk_u' \int_{-\infty}^{\infty} dk_d' \int_{-\infty}^{\infty} dk_d |V_{u u' d d'}|^2 \times \{P(u, d; u', d') - P(u', d'; u, d)\} \delta\left(E_{n_d k_d}^{(d)} + E_{n_u k_z}^{(u)} - E_{n_u' k_z'}^{(u)} - E_{n_d' k_d'}^{(d)}\right),$$

$$P(u, d; u', d') \equiv f_{n_u k_z}^0 (1 - f_{n_u' k_z'}^0) \{g_{n_d k_d} (1 - f_{n_d' k_d'}^0) - g_{n_d' k_d'} f_{n_d k_d}^0\}.$$

Here $N_u$ is the number of electrons in U, and $V_{u u' d d'}$ is the matrix element of the screened Coulomb interaction across the barrier $[10]$.

We now analyze the variation of $E_u$ near the transition between successive plateaux, i.e. when either the density or $B$ is varied so that the chemical potential $\mu$ crosses the bottom of a LL. The interesting region is when $\mu$ lies just above the bottom of the highest occupied LL, where we can ignore scatterings between different Landau levels. For our qualitative purposes it is sufficient to consider the case where the highest occupied levels are the lowest LL in both D and U. The crucial step is to distinguish two different regimes in the position of $\mu$: (a) $k_B T \ll E_F^{(d)}$ and (b) $k_B T \gg E_F^{(d)}$, where $E_F^{(d)} \equiv (\mu - E_{00}^{(d)})$. A bias between the layers is chosen so that in both regimes $E_F^{(u)} \equiv (\mu - E_{00}^{(u)}) \gg k_B T$. Correspondingly, we define $k_F^{(d)} \equiv \sqrt{2m^* E_F^{(d)}}/\hbar$ and $k_F^{(u)} \equiv \sqrt{2m E_F^{(u)}}/\hbar$.

In regime (a), the electron gas in the layer is characterized by a sharp, two points Fermi "surface" $k_d = \pm k_d^{(d)}$. At $T = 0$, $\tau^{-1}(k_u, k_z)$ vanishes (cf. Eq. (1.4)); at small $T$, it is dominated by forward ($k_d \approx k_d^{(d)}$) and backward ($k_d - k_d' \approx \pm 2k_F^{(d)}$) scattering.
processes. The former are negligible, and the latter are \( \sim \exp[-(2k_F^d \ell)^2] \exp[-4k_F^d d] \). For \( E_F^{(d)} > \max\{h^2/2m^*(2\ell)^2, h^2/2m^*(4d)^2\} \), \( \rho_t \) is severely suppressed. As \( E_F^{(d)} \) is reduced below \( \min\{h^2/2m^*(2\ell)^2, h^2/2m^*(4d)^2\} \), \( \rho_t \) becomes more appreciable and approaches its maximal value as \( E_F^{(d)} \) is further reduced, crossing over to regime (b).

In regime (b), where \( \rho_t \) is peaked, we assume the hierarchy of energy scales \( E_F^{(d)} \ll k_B T \ll E_F^{(u)} \). The 2DES becomes an effectively classical gas, where \( k_B T \) replaces \( E_F^{(d)} \) as the typical electron energy. Consequently, one should account for the \( T \)-dependence of \( q_s \), the screening wave-vector in \( D \), as well as the effective width of this layer. The latter is given, roughly, by the effective real-space extent of the electronic states with energy lower than \( E_{00}^{(d)} + k_B T \). We thus obtain the peak trans-resistivity

\[
L^*_z \rho_t(0) \approx \frac{3\pi^5}{16} \frac{\hbar}{e^2} \left( \frac{a_B}{d} \right)^2 \frac{e^2 (k_F^{(u)})^5 \ell^8 (k_B T)^2}{e^4},
\]

where \( a_B \equiv \hbar^2/e^2m \), \( \ell \) is the magnetic length and \( \epsilon \) the background dielectric constant. For \( m \approx 0.07m_e \) and \( \epsilon \sim 10 \) (appropriate to GaAs), \( d = 200 \, \text{Å}, \ell = 100 \, \text{Å}, k_F^{(u)} \sim 1/\ell \) and \( T = 1 \, \text{K} \), we get \( L^*_z \rho_t(0) \sim 10^{-8} \, \Omega\text{cm} \).

**Systems with disorder:** We now turn to realistic systems where, in contrast to the clean example, there are always states at the Fermi level in the bulk. We distinguish two regions: the critical region, where electron delocalization dominates, and the non-critical region.

In the latter, the scattering between the layers and hence the drag may be assisted by hopping among localized states, in addition to the edge-edge processes considered earlier. As with the treatment of \( \sigma_{xx} \) in the framework of variable range hopping \([18]\), these contribute \( \rho_t \sim e^{-(T_0/T)\alpha} \), where the exponent \( \alpha = 1/3, 1/2 \) for the cases of Mott hopping and Coulomb gap physics respectively. We find that for reasonable system sizes this form holds, except at the very lowest temperatures where the range of the hopping exceeds the sample size, and the edge-edge scattering dominates; for a sample of size \( L \sim 10\mu m \), we find that the crossover temperature is at most \( 10^{-4} \, \text{K} \).

**Critical Region:** In this region the system is in the vicinity of a \( T = 0 \) critical point with a finite non-zero \( \sigma_{xx} \). At the latter, \( \rho_t \) will be dominated by the extended states that exist in
the bulk of the sample. This bulk contribution is conveniently evaluated using the result of Zheng and MacDonald [9], who showed that to leading non-trivial order in the interaction $V$ between two 2DESs,

$$\rho_t = \frac{\beta \hbar^2}{\pi n^u n^{(d)} e^2} \int \frac{d^2q}{(2\pi)^2} q^2 |V(q)|^2 \int_0^\infty d\omega \frac{\text{Im}\chi_d(q,\omega)\text{Im}\chi_u(q,\omega)}{4 \sinh^2(\beta \hbar \omega/2)}, \quad (0.6)$$

where $\text{Im}\chi_d(q,\omega)$ are the dissipative density-density response functions of layer D (U) [19].

Two features of Eq. (0.6) are worth noting. First, the leading low $T$ behavior of $\rho_t$ can be obtained by approximating the response functions by their $T = 0$ forms. Second, as $T \to 0$, the $T$ dependence of the denominator forces the relevant range of $\omega$ to vanish, and hence the behavior of $\rho_t$ becomes sensitive to the form of $\text{Im}\chi_d(q,\omega)$ for $\omega < q$.

In applying Eq. (0.6) to the critical region we are faced with the problem that a proper theory of the latter is not available at present. Therefore we take a dual approach. First, for the well studied non-interacting problem, which appears to be relevant to the experiments thus far [20], we show that $\rho_t$ vanishes as a universal power of $T$. Next, we show that this behavior is generic; at any (interacting) critical point $\rho_t$ measures an exponent characteristic of the corresponding universality class. Hence, measuring $\rho_t$ would test the non-interacting theory, as well as serve as a model-independent probe of the universality classes among the QH phase transitions.

For non-interacting electrons Chalker and Daniell [12] have shown that the low frequency response at criticality has the form:

$$\text{Im}\chi_d(q,\omega) = \frac{(dn^{(d)}/d\mu)\omega D(q,\omega) q^2}{[\omega^2 + (D(q,\omega) q^2)^2]^{1/2}}, \quad D(q,\omega) = D \left( \frac{C\omega}{q^2} \right)^{\eta/2}. \quad (0.7)$$

(For $C\omega > q^2$, $D(q,\omega) \approx D = 0.087/\hbar (dn^{(d)}/d\mu)$; $C \approx 60\hbar (dn^{(d)}/d\mu)$ where $dn^{(d)}/d\mu$ is the density of states at the band center, and the universal exponent $\eta \approx 0.38$). On substituting this in Eq. (0.6) we obtain, for two identical layers

$$\rho_t \approx 3.1 \times 10^{-4} \frac{\hbar}{e^2} \left( \frac{k_B T}{\hbar D n^{(d)}} \right)^2 \left( \frac{\epsilon}{e^2 d (dn^{(d)}/d\mu)} \right)^2 \left( \frac{C d^2 k_B T}{\hbar} \right)^{-\eta}. \quad (0.8)$$

Hence, in this scenario, $\rho_t$ in the critical region is parametrically enhanced at low $T$ over non-critical filling factors, and its $T$ dependence directly measures $\eta$ (with a non-universal
prefactor). As noted by Chalker [13], the non-zero value of \( \eta \) reflects the presence of large amplitude fluctuations in the critical eigenstates. Also, as shown in [14], it leads to anomalously slow relaxation of the local density fluctuations. Hence, the enhancement of \( \rho_t \) is entirely in accord with our earlier discussion of the physics. For a sample of mobility \( \sim 10^6 \text{cm}^2/\text{Vs} \), \( n^{(d)} \sim 10^{11} \text{cm}^{-2} \), \( d = 200 \text{Å} \) and \( B = 10T \), we estimate \( \rho_t \approx 10 \text{mΩ} \) at \( T \approx 0.1K \). As with the scaling of \( \sigma_{xx} \), the behavior of \( \rho_t \) becomes non-critical at a crossover temperature \( T^* \approx 1/\xi \) [20].

For completeness we note that for the case where \( U \) is three dimensional, \( \rho_t \sim T^{2-\eta/2} \). Thus, \( \rho_t \) vanishes faster with \( T \) than in Eq. (0.8), and we find that to observe an appreciable sensitivity to \( \eta \), \( T \) has to be reduced to \( \sim 1 \text{ mK} \).

We now consider the situation at an arbitrary interacting critical point, with some unknown values of the localization length exponent \( \nu \xi \) and the dynamic scaling exponent \( z \). Fortunately, \( \text{Im} \chi \) is the correlator of a conserved density and hence by standard renormalization group arguments it has the critical scaling form:

\[
\text{Im} \chi(q, \omega) = q^{d-z} f(\omega/q^z). \tag{0.9}
\]

\( \text{Im} \chi \) is odd in \( \omega \), and away from criticality we expect \( \text{Im} \chi \sim \omega f(q) \) at small \( \omega \) for fixed \( q \). Assuming that the critical fluctuations modify this linear dependence we parametrize the deviation by postulating \( f(x) \sim x^{1-\eta/2} \) for \( x \ll 1 \) [23]. For our problem \( \text{Im} \chi \sim \omega^{1-\eta/2} q^{2-2z+\eta z/2} \) for \( \omega \ll q \); this agrees with the non-interacting form with \( z = 2 \). Substituting this in Eq. (0.6) gives \( \rho_t \sim T^{2-\eta} \) at low \( T \). Hence an enhanced drag at criticality would still measure an exponent characteristic of anomalous low frequency dissipation [24].

Finally, we comment on the FQHE. The qualitative variation of \( \rho_t \) with filling factor at fixed \( T \) derives from variations in the compressibility, which are similar in the IQHE and the FQHE. The behavior of the clean system should be similar, even though one has Luttinger liquids at the edges, for their small \( q \) density-density response is the same as that of the integer Fermi liquids. For disordered samples, some form of variable range hopping should hold deep in the plateaux, where the fractional statistics of the quasiparticles is
likely unimportant. As for the critical region, even less is known than for the IQHE; hence, as already emphasized, we expect measurements of $\rho_t$, and hence of $\eta$, to be particularly illuminating.

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and for $L_z > \lambda_{sc}$, $L_z^* \approx \lambda_{sc}$.

17 We note that near $q = 2k_{F}^{(d)}$ and at finite $T$ there is an algebraic enhancement of the interaction due to the reduction of the screening wave-vector $q_s(q = 2k_{F}^{(d)}) \sim \exp[-(k_{F}^{(d)})^{2}] \ln[8E_{F}^{(d)}/k_{B}T]$. However, this minor enhancement is overwhelmed by the exponentials.

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19 Two comments are in order. First, the reader may be concerned about our using a weak coupling result near a critical point. Our belief is that weak interlayer coupling is, at least practically, irrelevant at the QH critical points. Experiments, such as that by Eisenstein *et. al.* cited in [1], that have tracked the collapse of correlated inter-layer states with separation, indicate that such a regime is indeed experimentally accessible. Second, for the experimentally relevant case of the Coulomb interaction, we follow [3] in trusting that replacing the interaction and the response functions by their screened counterparts is sufficient.
[20] The observed scaling [21] of the width of the transition regions, $\Delta B \sim T^{0.42}$, is consistent with the studies of non-interacting electrons, that yield a localization length exponent $\nu_\xi \approx 7/3$ provided one makes the additional ansatz that the inelastic length diverges as $1/T$ [22]. The simplest assumption is, therefore, that for “fermi liquid” reasons the screened response functions are given correctly by the non-interacting fixed point; for a different interpretation see D. H. Lee, Z. Q. Wang and S. A. Kivelson, Phys. Rev. Lett. 70, 4130 (1993).

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[24] Possibly also at other two-dimensional phase transitions.
FIGURES

FIG. 1.

Schematics of the heterostructure. The shaded area is an insulator, and $\mathbf{B}$ is the magnetic field.