GLSM for Calabi-Yau Manifolds of Berglund-Hubsch Type

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Abstract

In this note we briefly present the results of our computation of special Kähler geometry for polynomial deformations of Berglund-Hübsch type Calabi-Yau manifolds. We also build mirror symmetric Gauge Linear Sigma Model and check that its partition function computed by Supersymmetric localization coincides with exponent of the Kähler potential of the special metric.

1 Special geometry for invertible singularities

Special Kähler geometry [1] is the geometrical structure underlying particular coupling constants in superstring compactifications (such as coupling constants of particles of vector supermultiplets in type IIA/B compactifications). Knowledge of this metric is important for phenomenological questions arising in superstring theories.

In a series of papers [2–5] we showed that geometry of complex moduli of Calabi-Yau manifolds associated with Landau-Ginzburg models can be effectively computed through the computations in the corresponding Landau-Ginzburg model. These results were used to confirm the Refined Swampland Finite Distance conjecture [6–8].

The special Kähler metric (Weil-Peterson metric) has a Kähler potential whose logarithm is a Hermitian pairing of the period integrals of the holomorphic volume form on the Calabi-Yau manifold. It is much easier to compute these data on the Landau-Ginzburg side.

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Invertible singularities  A polynomial $W(x_1, \ldots, x_N)$ in $N$ variables (Landau-Ginzburg superpotential) is called an invertible singularity if it is of the form

$$W_0(x_1, \ldots, x_N) = \sum_{i=1}^{N} \prod_{j=1}^{N} x_j^{M_{ij}}$$  \hspace{1cm} (1)$$

for an invertible matrix $M$ with positive integer coefficients. Such a polynomial is always weighted homogeneous:

$$W_0(\lambda q_i x_1, \ldots, \lambda q_i x_N) = \lambda W_0(x),$$  \hspace{1cm} (2)$$

where $q_i = \sum_j M_{ji}^{-1}$ are positive rational numbers. Let $d$ be the least common denominator of the integers $k_1, \ldots, k_N$ so that $k_i = dq_i$.

Then the equation $W_0(x) = 0$ is well-defined in the weighted projective space

$$\mathbb{P}^{N-1}_{(k_1, \ldots, k_N)} = (\mathbb{C}^N - \{0\})/\mathbb{C}^*,$$  \hspace{1cm} (3)$$

where $\mathbb{C}^*$ acts on coordinates of $\mathbb{C}^N$ with weights $k_1, \ldots, k_N$ correspondingly. Typically these weighted projective spaces are smooth orbifolds. Zero locus $X_0$ of the superpotential $W_0(x)$ is an orbifold as well. In order for $X_0$ to be smooth orbifold (or quasi-smooth variety) the superpotential must be transverse: $dW_0(x) = 0 \equiv x = 0$. By classification of singularities performed by Kreuzer and Skarke \[9\] $W_0(x)$ must be a sum of three basic or atomic types:

$$x^A - \text{point},$$

$$x_1^{A_1} x_2 + x_2^{A_2} x_3 + \cdots + x_n^{A_n} - \text{chain},$$

$$x_1^{A_1} x_2 + x_2^{A_2} x_3 + \cdots + x_n^{A_n} x_1 - \text{loop},$$  \hspace{1cm} (4)$$

where each variable belongs to only one of the blocks.

If $d = \sum_{i \leq N} k_i$ then the quasi-smooth variety $X_0$ is actually a Calabi-Yau variety.

In this note we describe the geometry of the (formal neighbourhood of the) space of polynomial deformations of $W_0(x)$ which describes a certain subspace in the space of deformations of complex structures on the Calabi-Yau manifold $X_0$.

We note that invertible singularities were studied in the context of Berglund-Hübsch-Krawitz mirror symmetry \[11,10\].

The space of states and cohomology  Given an isolated (even transverse) singularity $W_0(x)$ its Jacobi or Milnor ring is a space of infinitesimal deformations modulo infinitesimal coordinate changes

$$\text{Jac}(W_0) = \frac{\mathbb{C}[x_1, \ldots, x_N]}{\langle \partial_{i} W_0, \ldots, \partial_N W_0 \rangle}.$$  \hspace{1cm} (5)$$

Let us denote $\{e_1, \ldots, e_h\}$ to be a set of homogeneous degree $d$ monomials which span a basis of the degree $d$ part of $\text{Jac}(W_0)$, where we denote $e_l = x_1^{S_{l1}} \cdots x_N^{S_{ln}}$ and $l = 1, \ldots, h$. The general polynomial weighted homogeneous deformation of $W_0(x)$ can be expressed as

$$W(x, \phi) = W_0(x) + \sum_{l=1}^{h} \phi_l e_l.$$  \hspace{1cm} (6)$$

This family is a particular deformation of complex structures on $X_\phi = \{W(x, \phi) = 0\}$ and each monomial $e_l$ corresponds to a cohomology class $e_l \to H_{N-3,1}(X_\phi)$.

The Jacobi ring of an invertible singularity is a tensor product of the Jacobi rings of its atomic blocks. The latter ones are described by the following proposition \[9\]:

**Proposition 1.1.** *Milnor rings for three elementary blocks (atomic types) have the following bases:*

1. Point, $W(x) = x^A$,

$$\text{Jac} W_0 = \langle x^a \rangle_{a=0}^{A-2}$$  \hspace{1cm} (7)$$
2. Chain, $x_1^{A_1}x_2 + x_2^{A_2}x_3 + \cdots + x_N^{A_N}$. Then the basis for $\text{Jac} W_0$ is most conveniently written recursively:

$$\text{Jac} W_0 = \bigoplus (x_1^{a_1} \cdots x_N^{a_N}),$$

where 1) either $a_1 \leq A_1 - 2$ and $a_i \leq A_i - 1$ for $i > 1$ or 2) $a_1 = A_1 - 1$, $a_2 = 0$ and $x_3^{a_3} \cdots x_N^{a_N}$ define a basis element of the chain $x_3^{A_3}x_4 + x_4^{A_4}x_5 + \cdots + x_N^{A_N}$.

3. Loop, $x_1^{A_1}x_2 + x_2^{A_2}x_3 + \cdots + x_N^{A_N}x_1$

$$\text{Jac} W_0 = \bigoplus_{k_i=0}^{A_i-1} (x_1^{a_1} \cdots x_N^{a_N}),$$

where $0 \leq a_i < A_i$.

The $\mathbb{Z}_d \subset \mathbb{C}^*$ from the definition of the weighted projective space [3] acts on $\mathbb{C}^N$ and on the Jacobian ring $\text{Jac} W_0$. The invariant part $\text{Jac}(W_0)^{\mathbb{Z}_d}$ decomposes into $\text{Jac}(W_0)^{\mathbb{Z}_d} = \bigoplus_{k=0}^{N-2} \text{Jac}(W_0)^{dk}$, where $\text{Jac}(W_0)^{dk}$ consists of elements of weight $dk$. In what follows we specify to the case $N = 5$. In this case

$$\text{Jac}(W_0)^{Z_d} = (1) \oplus \text{Jac}(W_0)^d \oplus \text{Jac}(W_0)^{2d} \oplus \langle \text{Hess}W_0 \rangle$$

and $\text{Jac}(W_0)^{2d} \simeq \text{Jac}(W_0)^d$ and has a natural dual basis. We denote the monomial basis of $\text{Jac}(W_0)^{Z_d}$ by $\{e_a\}_{a=0}^{1+2h}$, where $e_a = \prod_{j=1}^{5} x_j^{a_j}$.

Period integrals The period integrals of the holomorphic form are

$$\sigma[\gamma_a](\phi) := \int_{\gamma_a} \Omega_{\phi},$$

where $\gamma_a \in H_3(X_0)$ is a cycle dual to the $e_a$ - a basis element of the invariant ring. (For more information on defining such cycles, see [4]). $\Omega_{\phi}$ is a holomorphic volume form on $X_{\phi}$ for each value of $\phi$. It is well-known that they satisfy the Picard-Fuchs partial differential equations and in principle can be computed using the Frobenius method. For the considered case Frobenius-type basis of the periods can be computed following our approach in [2,5] as a power series in $\phi_1, \ldots, \phi_h$ around zero:

$$\sigma_a(\phi) = \sum_{S_{\sigma(a)} = a} \left( \prod_{i=1}^h \phi_i^{m_i} \prod_{i=1}^5 \frac{\Gamma(\sum_j (\sum_{l} m_l S_{ij} + 1) B_{ji})}{\Gamma(\sum_j (\sum_l m_l + B_{ji}))} \right),$$

where $B_{ji} = (M^{-1})_{ji}$. The expression (12) can be rewritten as

$$\sigma_a(\phi) = \sum_{n_1, \ldots, n_5} \left( \prod_{i=1}^5 ((a_j + 1) B_{ji})_{n_i} \right) \sum_{S_{\sigma(a)} = a} \left( \prod_{i=1}^h \phi_i^{m_i} \right),$$

where the raising factorials are $(x)_n = \Gamma(x + n)/\Gamma(x)$.

The formula for the special Kähler potential The exponent of the Kähler potential of the Weil-Peterson metric on the deformation space is a Hermitian expression in the periods. The $W_0(x)$ has a huge discrete group of diagonal symmetries of which $\mathbb{Z}_d$ is a subgroup. The diagonal symmetry group naturally acts on the deformation space with coordinates $\{\phi_s\}_{s=1}^h$. For most invertible singularities the period integrals (12) form one-dimensional representations of the diagonal symmetries group with different weights. This implies that the Hermitian pairing should be diagonal in $|\sigma_a(\phi)|$. Indeed, we show that this is the case for all invertible singularities and the coefficients are products of ratios of Gamma-functions in the normalization (12):

$$e^{-K(\phi, \dot{\phi})} = \sum_{a=0,2d-1} (-1)^{|a|/d} \prod_{i=1}^d \frac{\Gamma((a_j + 1) B_{ji})}{\Gamma(1 - (a_j + 1) B_{ji})} |\sigma_a(\phi)|^2,$$

where $|a| = \sum_j k_j a_j$ is the weight of the monomial $e_a$ and $\prod'$ denotes product over all terms where arguments of Gamma functions are non-integer.
2 Localization and Mirror symmetry

Gauge Linear Sigma Models (GLSM) [12] have supersymmetric backgrounds on the round sphere $S^2$. Corresponding partition function was computed in [13,14]. For the case when the gauge group of the model $G = \prod_{l=1}^{h} U(1)_l$, the chiral matter fields $\Phi_a, a = 1, \ldots, N = h+5$ have $R$-charges $g_a$ and their charges with respect to the gauge group $U(1)$ are denoted by $Q_{la}$ the partition function looks as

$$Z_Y = \sum_{m \in A} \prod_{l=1}^{h} \int_{C_l} \frac{d\tau}{2\pi i} \left( z_l^{-\tau_l + \frac{m_l^h}{m_l}} z_l^{-\tau_l + \frac{m_l}{m_l}} \right) \prod_{a=1}^{h+5} \frac{\Gamma(\frac{g_a}{2} + \sum_{l=1}^{h} Q_{la}(\tau_l - \frac{m_l}{m_l}))}{\Gamma(1 - \frac{g_a}{2} - \sum_{l=1}^{h} Q_{la}(\tau_l + \frac{m_l}{m_l}))}$$

(15)

where $z_l = e^{-(2\pi \tau_l + i \theta_l)}$, $\tau_l, \theta_l$ are Fayet-Iliopoulos parameters and theta angles for our GLSM. The summation by $\bar{m} = \{m_1, \ldots, m_h\}$ goes over the set specified below by (20).

The JKLMR conjecture [15] claims that the partition function computes the exponent of Kähler potential of the quantum corrected metric on the Kähler moduli space of GLSM in the Calabi-Yau case.

Mirror symmetry connects quantum corrected Kähler moduli space of one model with complex structures moduli space of another. For all Calabi-Yau threefolds given by invertible singularities we can construct a fan [20], which defines another toric variety. For all Calabi-Yau threefolds given by invertible singularities we can construct the corresponding mirror symmetric GLSM and compute their partition functions following the method suggested in [16,17]. Then we explicitly check that they coincide with exponents of Kähler potentials of Weil-Petersson metrics on the deformation spaces of the invertible singularities.

To do this, we use Batyrev’s approach to the mirror symmetry [19] in the same way as in [16,18] to construct GLSM for Fermat hypersurface cases.

The idea is as follows. Let Calabi-Yau threefold $X$ be defined as a hypersurface in a weighted projective space $\mathbb{P}^4_{(k_1, \ldots, k_5)}$ and given by zero locus of the polynomial $W(x, \phi)$. Exponents of the monomials that make up the polynomial $W(x, \phi)$ determine the finite set $\tilde{V}_{a} a = 1, \ldots, N$ and, thus, define Batyrev’s polytope $\Delta_X$ [19].

Knowing the set of vectors $\tilde{V}_{a}$ we can construct a fan [20], which defines another toric variety. Then Calabi-Yau manifold $X$, the mirror to $Y$, is defined as a hypersurface in this variety given by the zero locus of the homogeneous polynomial $W_Y$. Using this fan we also find the corresponding GLSM with its gauge group $G = \prod_{l=1}^{h} U(1)_l$ and the set of charges $Q_{la}$ of its chiral multiplets. The charges appear as coefficients of the linear relations between the vectors of the fan and set the weights of the toric variety.

In the considered case we have the following expression for the deformed invertible singularity:

$$W(x, \phi) = \sum_{i=1}^{5} \prod_{j=1}^{5} x_j^{M_{ij}} + \sum_{l=1}^{h} \phi_l \prod_{j=1}^{5} x_j^{S_{ij}},$$

(16)

which defines the Calabi-Yau family $X$. The set of the vectors $\tilde{V}_{a} a = 1, \ldots, N$ in this case is

$$V_{ai} = \begin{cases} 5\delta_{a,i}, & 1 \leq a \leq 5, \\ S_{a-5,i}, & 6 \leq a \leq h. \end{cases}$$

(17)

The corresponding GLSM has the gauge group $G = \prod_{l=1}^{h} U(1)_l$, the vector superfields $\{V_l\}$, $l = 1, \ldots, h$, and $h+5$ chiral superfields $\{\Phi_a\}, a = 1, \ldots, h+5$, interacting with $\{V_l\}$ by integer charges $Q_{la}$. It is convenient [17] to introduce instead of the integer charges $Q_{la}$ another “rational charge matrix” $Q'_{la}$ which satisfies

$$\sum_{a \leq 106} Q'_{la} \tilde{V}_{a} = 0,$$

(18)

which looks as

$$Q'_{la} = \begin{cases} S_{ij} B_{ja}, & 1 \leq a \leq 5, \\ -\delta_{a-5,i}, & a > 5. \end{cases}$$

(19)
\( Q'_{\text{la}} \) is connected with the charge matrix \( Q_{\text{la}} \), whose entries are integers, by a rational invertible \( h \times h \) matrix. In terms of the new rational basis of charges the integrality condition for \( m_l \) modifies as follows:

\[
m_l \in \mathbb{Z} \rightarrow \bar{m}_l Q'_{\text{la}} \in \mathbb{Z}.
\] (20)

Then the generic expression (15) for the partition function specifies in our case to

\[
Z_y = \sum_{\bar{m} \in \Lambda_{\text{cl}}} \int \frac{d^h \tau}{(2\pi i)^h} \frac{1}{z_l^{(\tau - m_l)}(\tau + m_l)} \prod_{l=1}^h \frac{\Gamma\left(\frac{q_l}{2} - (\tau - m_l)\right)}{\Gamma\left(1 - \frac{q_l}{2} - (\tau + m_l)\right)} \prod_{i=1}^5 \frac{\Gamma\left(\frac{q_l}{2} + \sum (\tau - m_l) S_{lj} B_{ji}\right)}{\Gamma\left(1 - \frac{q_l}{2} - \sum (\tau + m_l) S_{lj} B_{ji}\right)},
\] (21)

where the summation over \( \bar{m} = m_1, \ldots, m_h \) goes over the specified by (20). We also have to assign the R-charges to the chiral fields in appropriate way [12]. We can choose them be zero besides one \( q_h = 2 \). After this setting we compute the integral for \( |z_l| > 1 \) closing the integration contours to the right half-planes. The result reproduces the formula (14) in the Landau-Ginzburg phase up to the simple coordinate change between Fayet-Iliopoulos parameters of the GLSM and the complex deformation parameters \( \{\phi_l\}_{l=1}^h \) which looks as:

\[
z_l = -\phi_l^{-1}, \quad 1 \leq l \leq h,
\] (22)

which is the mirror map. This gives the realization of the mirror version of the JKLMR conjecture [15] for the considered class of Calabi-Yau families.

## Conclusion

Thus, starting from the model with of Berglund-Hübsch type Calabi-Yau manifolds \( X \) we have constructed the \( \mathcal{N} = (2, 2) \) Gauged Linear Sigma Model with the manifold of supersymmetric vacua \( Y \), which is the mirror for \( X \). Having computed Special geometry on the moduli space of complex structures on \( X \) and using Batyrev’s approach [19] to Mirror symmetry we have checked the JKLMR conjecture [15]. A detailed description of the procedure for obtaining these results will be the subject of another article.

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## References

[1] P. Candelas, X. C. De La Ossa, P. S. Green, and L. Parkes, *A Pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nucl. Phys. B359 (1991) 21–74.

[2] K. Aleshkin and A. Belavin, *A new approach for computing the geometry of the moduli spaces for a Calabi–Yau manifold*, J. Phys. A51 (2018), no. 5 055403, [arXiv:1706.05342](https://arxiv.org/abs/1706.05342).

[3] K. Aleshkin and A. Belavin, *Special geometry on the moduli space for the two-moduli non-Fermat Calabi–Yau*, Phys. Lett. B776 (2018) 139–144, [arXiv:1708.08362](https://arxiv.org/abs/1708.08362).

[4] K. Aleshkin and A. Belavin, *Special geometry on the 101 dimensional moduli space of the quintic threefold*, JHEP 03 (2018) 018, [arXiv:1710.11609](https://arxiv.org/abs/1710.11609).

[5] K. Aleshkin and A. Belavin, *Exact Computation of the Special Geometry for Calabi-Yau Hypersurfaces of Fermat Type*, JETP Lett. 108, no. 10, 705 (2018) [arXiv:1805.02772 [hep-th]](https://arxiv.org/abs/1805.02772).

[6] S. Katmadas and A. Tomasiello, *Gauged supergravities from M-theory reductions*, JHEP 1804, 048 (2018), [arXiv:1712.06608 [hep-th]](https://arxiv.org/abs/1712.06608).
[7] R. Blumenhagen, D. Kläwer, L. Schlechter and F. Wolf, *The Refined Swampland Distance Conjecture in Calabi-Yau Moduli Spaces*, JHEP **1806**, 052 (2018), [arXiv:1803.04989 [hep-th]].

[8] R. Blumenhagen, *Large Field Inflation/Quintessence and the Refined Swampland Distance Conjecture*, PoS CORFU **2017**, 175 (2018) [arXiv:1804.10504].

[9] M. Kreuzer and H. Skarke, *On the classification of quasihomogeneous functions*, Commun. Math. Phys. **150**, 137 (1992) [arXiv:hep-th/9202039].

[10] P. Berglund and T. Hubsch, *A Generalized Construction of Mirror Manifolds*. Nucl. Phys. B393 (1993) 377-391; [hep-th/9201014].

[11] P. Berglund and T. Hubsch, *A Generalized Construction of Calabi-Yau Models and Mirror Symmetry*, SciPost Phys. **4**, no. 2, 009 (2018) [arXiv:1611.10300].

[12] E. Witten, *Phases of N=2 theories in two-dimensions*, Nucl. Phys. B **403**, 159 (1993), [hep-th/9301042].

[13] F. Benini and S. Cremonesi, *Partition Functions of \( N = (2, 2) \) Gauge Theories on \( S^2 \) and Vortices*, Commun. Math. Phys. **334** (2015), no. 3 1483–1527, [arXiv:1206.2356].

[14] N. Doroud, J. Gomis, B. Le Floch, and S. Lee, *Exact Results in D=2 Supersymmetric Gauge Theories*, JHEP **05** (2013) 093, [arXiv:1206.2605].

[15] H. Jockers, V. Kumar, J. M. Lapan, D. R. Morrison, and M. Romo, *Two-Sphere Partition Functions and Gromov-Witten Invariants*, Commun. Math. Phys. **325** (2014) 1139–1170, [arXiv:1208.6244].

[16] K. Aleshkin, A. Belavin and A. Litvinov, *Two-Sphere Partition Functions and Kähler Potentials on CY Moduli Spaces*, Pisma Zh. Eksp. Teor. Fiz. **108**, no. 10, 725 (2018) [JETP Lett. **108**, no. 10, 710 (2018)].

[17] K. Aleshkin, A. Belavin, and A. Litvinov, *JKLMR conjecture and Batyrev construction*, Journal of Statistical Mechanics: Theory and Experiment **2019** (mar, 2019) 034003.

[18] A. Belavin and B. Eremin, *Partition functions of \( N = (2, 2) \) supersymmetric sigma models and Special geometry for the two-moduli non-Fermat Calabi-Yau manifold*, Theoretical and Mathematical Physics **201** (2019) 1606-1613, [arXiv:1907.11102 [hep-th]].

[19] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties*, J. Alg. Geom. **3**, 493 (1994), [arXiv:alg-geom/9310003].

[20] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil and E. Zaslow, *Mirror symmetry*, pages 101-142, AMS, Clay Mathematical Institute, 2003.