THE ALMOST HERMITIAN-EINSTEIN FLOW

Chiung-Nan Tsai

Department of Mathematics
Columbia University, New York, NY 10027

Abstract

Let $X$ be a compact Kähler manifold, $E \to X$ a Hermitian vector bundle and $L \to X$ an ample line bundle. We construct a non-linear heat flow corresponding to the almost Hermitian-Einstein equation introduced by N.C. Leung, and prove that the solution exists for a short time. We also construct a potential function $D_k$ for this flow. In particular, $D_k$ decreases along the flow.

1 Introduction and Main Results

The problem of existence and uniqueness of Hermitian-Einstein metrics for Mumford-Takemoto stable bundles has been solved by Uhlenbeck and Yau in [5] using the continuity method and by Donaldson in [1] using the heat equation method. Under a smoothness assumption on the bundles, Leung in [2] and [3] proved a similar result in the case of almost Hermitian-Einstein metrics for Gieseker stable bundles, using a singular perturbation technique and the result obtained by Donaldson-Uhlenbeck-Yau. This motivates us to study the almost Hermitian-Einstein metrics by using the heat equation method.

The setting is the following. Let $E \to X$ be a holomorphic vector bundle over an $n$-dimensional Kähler manifold $(X, \omega)$, and $L$ an ample line bundle over $X$ with $c_1(L) = [\omega]$. The bundle $E$ is called Hermitian when it is equipped with a Hermitian metric $H = H_{\bar{\alpha}\beta}$. Following Leung [2, 3], the metric $H_{\bar{\alpha}\beta}$ is said to be almost-Hermitian-Einstein if it satisfies for large positive $k$ the following equation, called the almost-Hermitian-Einstein equation,

$$\left[\exp\left(\frac{i}{2\pi}F + k\omega I\right)Td(X)\right]^{2n} = \frac{1}{rk(E)}\chi(X, E \otimes L^k)\frac{\omega^n}{n!}I$$

where $I$ is the identity, $F$ is the curvature 2-form on the Hermitian vector bundle $(E, H)$, $Td(X)$ is the harmonic representative with respect to $\omega$ of the Todd class of $X$, $rk(E)$ is the rank of $E$ and $\chi(X, E \otimes L^k)$ is the Euler characteristic of the bundle $E \otimes L^k$.

$$\chi(X, E \otimes L^k) = \sum_{i=0}^{n}(-1)^i\dim H^i(X, E \otimes L^k).$$

The leading terms in the almost Hermitian-Einstein equation correspond to the coefficients of $k^n$, and keeping these terms only gives the well-known Hermitian-Einstein equation

$$\Lambda F = \mu_E I,$$
where \( \mu_E = c_1(E)[\omega]^{n-1}/rkE \) is the slope of \( E \), and \( \Lambda F \equiv \hat{F} = g^{jk}F_{kj} \), with the Kähler form given by \( \omega = \frac{i}{2}g_{kj}dz^k \wedge d\bar{z}^j \). The full almost Hermitian-Einstein equation can be expressed in the following form

\[
\frac{i}{2\pi} F \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{\mu_E}{n!} \omega^n + \sum_{j=1}^{n-1} \frac{1}{k^j} T_{j+1}
\]

with the terms \( T_j \) defined by

\[
T_j = \chi^i_E \frac{\omega^n}{n!} - \sum_{k=0}^{j} (\frac{i}{2\pi} F)^k T^j_{d-k} \frac{\omega^{n-j}}{(n-j)!}.
\]

For our purposes, it is convenient to rewrite (1) also as

\[
\Lambda F = \mu_E I - S(k),
\]

where \( S(k) \) consists of terms involving powers of \( 1/k \) strictly less than \( n \), and is defined so that the equation (3) coincide with the equation (5).

Associated to the almost Hermitian-Einstein equation is the following natural flow of endomorphisms \( h(t) \) of \( E \),

\[
\dot{h}(t)h^{-1}(t) = - (\Lambda F - \mu_E I + S(k)), \quad h(0) = I,
\]

where \( H_0 \) is a fixed Hermitian metric on \( E \), \( H(t) \) is a Hermitian metric evolving with time \( t \), and we have set \( h(t) = H(t)H_0^{-1} \). The primary purpose of this paper is to study this flow, and in particular, to construct a potential function for it.

To construct a potential function, we consider any path \( H(t) \) of Hermitian metrics with \( H(0) = H_0 \), and we introduce the following functional \( D_k \) defined by

\[
D_k(H(t), H_0) = \int_X nR_2 \wedge \omega^{n-1} - \mu_E R_1 \omega^n + \int_0^t tr(\sum_{j=1}^{n-1} \frac{1}{k^j} T_{j+1}h\dot{h}^{-1}).
\]

where \( R_1 = log \det(tr(1H_0^{-1})) \), \( R_2 = \sqrt{-1} \int_0^t tr(F\dot{h}h^{-1})dt \) are the well-known secondary characteristic classes. Then our main results are as follows:

**Theorem 1** (a) The functional \( D_k \) is a potential function on the space of Hermitian metrics in the following sense. If \( h(s, t) \) satisfying \( h(s, 0) = I, h(s, 1) = HH_0^{-1} \), is a smooth deformation between the two paths \( h(0, t)H_0 \) and \( h(1, t)H_0 \) joining two fixed Hermitian metrics \( H_0 \) and \( H \), then \( D_k(h(s, 1)H_0, H_0) \) is independent of \( s \). Thus \( D_k(h(s, 1)H_0, H_0) \) can be considered as a function of just the two end-points \( H_0 \) and \( H \).

(b) The functional \( D_k \) is a Hamiltonian for the flow (6) in the sense that

\[
\frac{d}{dt} D_k(H(t), H_0) = \int_X tr(\Lambda F - \mu_E I + S(k) \dot{h}h^{-1}) \omega^n.
\]

In particular, \( D_k \) is a decreasing function of \( t \) along the flow (6).

(c) For \( k \) large, the flow (6) is parabolic, and hence exists for some interval \( 0 \leq t < T \).
Theorem 2 Let $\mu$ be the moment map, that is, the expression defined by the left hand side of the equation (1). Then $\mu$ satisfies the following evolution equation,

$$\dot{\mu} = \sum \sum_{m=0}^{m-1} \frac{i}{2\pi} C_{l}^{m-1} \omega^{l}(\frac{i}{2\pi} F)^{m-1-l} T_{d} n_{m}(X) \overline{\partial \partial H}(\frac{n!}{\omega^{n}}) \text{(sym)} k^{l}. \quad (9)$$

Acknowledgements. I would like to thank Professor Mu-Tao Wang for many discussions on this work. Special thanks are due to my advisor, Professor D.H. Phong, for suggesting this problem to me and for his constant encouragement. He helps me correct several mistakes and make this paper more readable. This paper will be a part of the my future Ph.D. thesis in the Mathematics Department of Columbia University.

2 Leung’s work

Almost Hermitian-Einstein metrics are natural from several viewpoints. As shown by Leung, their existence is equivalent to the notion of Gieseker stability. They also have a natural interpretation in terms of symplectic geometry. We give a brief review of some of Leung’s work in this section.

Let the tangent vectors to the space of connections at $D_{A}$ be identified with the space of $\text{End}(E)$-valued one forms on $X$. Then in [2, 3], a one-parameter family of gauge-invariant 2-forms on the space of connections on $E$ was defined as follows:

$$\Omega_{k}(D_{A})(B, C) = \int_{X} Tr_{E}[B \wedge \exp(k \omega I + \frac{i}{2\pi} F_{A}) \wedge C]_{\text{sym}} T_{d}(X) \quad (10)$$

where $B$ and $C$ are $\text{End}(E)$-valued one forms on $X$, and $F_{A}$ is the curvature tensor of the connection $D_{A}$. Then, it can be shown that $\Omega_{k}$ is a symplectic form, and the corresponding moment map for the action of the gauge group is

$$\mu(D_{A}) = [\exp(k \omega I + \frac{i}{2\pi} F) T_{d} X]^{(2n)}. \quad (11)$$

Symplectic quotients can then be constructed as level sets of the moment map. The almost Hermitian-Einstein equation (1) can be interpreted then as the equation for $\mu^{-1}(c)$, where $c$ represents the right hand side of (1).

On the other hand, the notion of Gieseker stability of a vector bundle is the following. Let $E$ be a rank $r$ holomorphic vector bundle (or coherent torsion-free sheaf in general) over a projective variety $X$ with ample line bundle $L$. The bundle $E$ is said to be Gieseker stable if for any nontrivial coherent subsheaf $S$ of $E$, we have

$$\frac{\chi(X, S \otimes L^{k})}{rk S} < \frac{\chi(X, E \otimes L^{k})}{rk E} \quad (12)$$

for large enough $k$. Then it is also shown in [2, 3] that if $E$ is an irreducible sufficiently smooth holomorphic vector bundle over a compact Kähler manifold $X$, then $E$ is Gieseker stable if and only if there exists an almost Hermitian-Einstein metric on $E$. 

3
3 Proof of Theorem 1

We begin with the proof of (a). By a basic lemma of Donaldson, the expression
\[ \int_X nR_2 \wedge \omega^{n-1} - \mu E R_1 \omega^n \]
is independent of the paths of Hermitian metrics. See [4], whose methods we adapt below. Thus the two first expressions in the functional \( D_k \) are path-independent, and it suffices to show that each single term in the remaining third expression in [7] is also path-independent. This means that we can simply deal with the following term:
\[ R_3 = \int_0^t \int_X \text{tr}(F^k T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}). \] (13)

We begin by evaluating the second derivative \( R_{3ts} \) of \( R_3 \) with respect to \( r \) and \( s \),
\[
R_{3ts} = \int_X \frac{d}{ds} \left( \text{tr}(F^k T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}) \right)
= \int_X \text{tr}(F_s F^{k-1} T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}) + \int_X \text{tr}(F F_s F^{k-2} T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j})
+ \int_X \text{tr}(F^2 F_s F^{k-3} T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}) + \cdots
+ \int_X \text{tr}(F^{k-1} F_s T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}) + \int_X \text{tr}(F^k T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}) - \int_X \text{tr}(F^k T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}). \] (14)

By the commutation formula
\[ F_s = \bar{\partial} \partial_H (h_s h^{-1}) = -\partial_H \bar{\partial} (h_s h^{-1}) - F h_s h^{-1} + h_s h^{-1} F, \]
we get
\[
R_{3ts} = \int_X \text{tr}\left( (-\partial_H \bar{\partial} (h_s h^{-1}) - F h_s h^{-1} + h_s h^{-1} F) F^{k-1} T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j} \right)
+ \int_X \text{tr}(F (-\partial_H \bar{\partial} (h_s h^{-1}) - F h_s h^{-1} + h_s h^{-1} F) F^{k-2} T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j})
+ \int_X \text{tr}(F^2 (-\partial_H \bar{\partial} (h_s h^{-1}) - F h_s h^{-1} + h_s h^{-1} F) F^{k-3} T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j})
+ \cdots
+ \int_X \text{tr}(F^{k-1} (-\partial_H \bar{\partial} (h_s h^{-1}) - F h_s h^{-1} + h_s h^{-1} F) T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j})
+ \int_X \text{tr}(F^k T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}) - \int_M \text{tr}(F^k T d^{i-k} h_{\cdot h^{-1}} \omega^{n-j}). \]

Here, we can see the second term in each line and the third term in the next line cancel. The underlined terms also cancel, because
\[ \text{tr} AB = \text{tr} BA \]
if \( A \) and \( B \) are even-form-valued matrices of the same size. Hence \( R_{3ts} \) becomes

\[
R_{3ts} = \int_X tr((-\partial_H \bar{\partial}(h_s h^{-1}))F^{k-1}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \\
+ \int_X tr(F(-\partial_H \bar{\partial}(h_s h^{-1}))F^{k-2}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \\
+ \int_X tr(F^2(-\partial_H \bar{\partial}(h_s h^{-1}))F^{k-3}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \\
+ \cdots \\
+ \int_X tr(F^{k-1}(-\partial_H \bar{\partial}(h_s h^{-1}))Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \\
- \int_X tr(F^{k-1}h_s h^{-1}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) + \int_X tr(F^{k}Td^{\bar{i} - k}h_{ts} h^{-1}\omega^{n-j}). \tag{15}
\]

Next, we compute \( \int_X tr\frac{d}{dt}(F^{k}Td^{\bar{i} - k}h_{s} h^{-1}\omega^{n-j}) \). By using this time

\[
F_t = \bar{\partial} \partial_H (h_t h^{-1}) \tag{16}
\]

we get

\[
\int_X tr\frac{d}{dt}(F^{k}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) \\
= \int_X tr(\bar{\partial} \partial_H (h_t h^{-1})F^{k-1}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) + \int_X tr(F \bar{\partial} \partial_H (h_t h^{-1})F^{k-2}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) \\
+ \int_X tr(F^2 \bar{\partial} \partial_H (h_t h^{-1})F^{k-3}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) + \cdots \\
+ \int_X tr(F^{k-1} \bar{\partial} \partial_H (h_t h^{-1})Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) + \int_X tr(F^{k}Td^{\bar{i} - k}h_{ts} h^{-1}\omega^{n-j}) \\
- \int_X tr(F^{k-1} \bar{\partial} \partial_H (h_t h^{-1})Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}). \tag{17}
\]

We can evaluate now \( \int_X tr\frac{d}{ds}(F^{k}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \) and \( \int_M tr\frac{d}{dt}(F^{k}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) \). We notice that the first two terms of \( \int_X tr\frac{d}{ds}(F^{k}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \) and \( \int_M tr\frac{d}{dt}(F^{k}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) \) cancel. Next, by using integration by parts, the Bianchi identity, and the fact that Todd classes are combinations of Chern classes, which are all closed forms, on the rest of the terms of \( \int_X tr\frac{d}{ds}(F^{k}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) \) and \( \int_M tr\frac{d}{dt}(F^{k}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) \), we get

\[
\int_X tr\frac{d}{ds}(F^{k}Td^{\bar{i} - k}h_t h^{-1}\omega^{n-j}) - \int_X tr\frac{d}{dt}(F^{k}Td^{\bar{i} - k}h_s h^{-1}\omega^{n-j}) \\
= \int_X tr(\bar{\partial}(h_s h^{-1})F^{k-1}Td^{\bar{i} - k} \partial_H (h_t h^{-1})\omega^{n-j})) \\
+ \int_X tr(F(\bar{\partial}(h_s h^{-1})F^{k-2}Td^{\bar{i} - k} \partial_H (h_t h^{-1})\omega^{n-j})) \\
+ \cdots \\
+ \int_X tr(F^{k-1}(\bar{\partial}(h_s h^{-1})Td^{\bar{i} - k} \partial_H (h_t h^{-1})\omega^{n-j})) \\
+ \int_X tr((\partial_H (h_t h^{-1})F^{k-1}Td^{\bar{i} - k} \bar{\partial}(h_s h^{-1})\omega^{n-j}))
\]
\[ + \int_X \text{tr}(F(\partial_H(h_s h^{-1}) F^{k-2} T d^{i-k} \partial(h_s h^{-1}) \omega^{n-j})) \\
+ \cdots \\
+ \int_X \text{tr}(F^{k-1}(\partial_H(h_s h^{-1}) T d^{i-k} \partial(h_s h^{-1}) \omega^{n-j})) \]

We claim the sum of these 2k terms is zero. We consider the \(i\)-th term and the \((2k-i+1)\)-th term as a pair. For simplicity, assume \(i = 1\), then the pair

\[ \int_X \text{tr}(\partial(h_s h^{-1}) F^{k-1} T d^{i-k} \partial_H(h_s h^{-1}) \omega^{n-j}) \text{ and } \int_X \text{tr}(F^{k-1}(\partial_H(h_s h^{-1}) T d^{i-k} \partial(h_s h^{-1}) \omega^{n-j})) \]

(18)
cancel, because \(\text{tr}AB = -\text{tr}BA\) if \(A\) and \(B\) are odd-form-valued matrices of the same size.

The same argument works on the other pairs, and this proves the claim. Let \(h(s, t)\) be now a deformation of paths as in the statement of Theorem 1. Then we have

\[ \frac{d}{ds} R_3 = \frac{d}{ds} \int_{t=0}^{1} \int_X \text{tr}(F^k h_t h^{-1} T d^{i-k} \omega^{n-j}) \]
\[ = \int_{t=0}^{1} \int_X \frac{d}{dt} \text{tr}(F^k h_t h^{-1} T d^{i-k} \omega^{n-j}) \]
\[ = \int_{t=0}^{1} \int_X \frac{d}{dt} \text{tr}(F^k h_t h^{-1} T d^{i-k} \omega^{n-j}) \]
\[ = \int_X \text{tr}(F^k h_t h^{-1} T d^{i-k} \omega^{n-j}) |_{t=0}^{1} = 0, \] (19)

since \(h_s = 0\) at \(t = 0\) and 1. This finishes the proof of (a) in Theorem 1.

Next, we prove (b). By the work of Donaldson, we already know that \(\frac{\partial}{\partial t} R_1 = \text{tr}(h h^{-1})\) and \(\frac{\partial}{\partial t} R_2 = \sqrt{-1} \text{tr}(F h h^{-1})\). It follows immediately that

\[ \frac{dD_k}{dt} = \int_X \text{tr}(\Lambda F - \mu E I) \hat{h} h^{-1}) \omega^n + \text{tr}(\sum_{j=1}^{n-1} \frac{1}{k^j} T_{j+1} \hat{h} h^{-1}) \]
\[ = \int_X \text{tr}(\Lambda F - \mu E I + S(k) \hat{h} h^{-1}) \omega^n. \] (20)

In particular, along the flow (6), we have

\[ \frac{dD_k}{dt} = - \int_X |(\Lambda F - \mu E I + S(k))|^2 \omega^n \leq 0, \] (21)

and (b) is proved.

Finally, (c) follows from the fact that the principal symbol of \(- \wedge F\) is actually the same as the symbol of the Laplacian \(\Delta\). Right away we can see the principal symbol of the left hand side of almost Hermitian-Einstein flow is elliptic at the initial time. The short-time existence is then a consequence of the general theory of parabolic equations. Q.E.D.
4 Proof of Theorem 2

To prove Theorem 2, we write the almost Hermitian-Einstein flow in the following form:

\[ \dot{h}h^{-1} = -[\exp(\frac{i}{2\pi}F + k\omega I)Td(X)](2n) + \frac{1}{rk(E)}\chi(X, E \otimes L^k)\omega_n \frac{I_E}{n!}. \]

Recall that the moment map \( \mu \) is given by (11). Thus

\[ \dot{\mu} = d\left[ \sum_m (\frac{i}{2\pi}F + k\omega I)^m Td_{n-m}(X) \right] = \left[ \sum_m (\frac{i}{2\pi}F + k\omega I)^{m-1} \frac{i}{2\pi} \dot{F} Td_{n-m}(X) \right]_{(sym)}. \]

Since \( \dot{F} = \bar{\partial}\partial H(\dot{h}h^{-1}) \) for any flow, we can combine these formulas and get

\[ \dot{\mu} = \left[ \sum_m (\frac{i}{2\pi}F + k\omega I)^{m-1} \frac{i}{2\pi} \bar{\partial}\partial H(\dot{h}h^{-1}) Td_{n-m}(X) \right]_{(sym)} \]

\[ = \left[ \sum_m (\frac{i}{2\pi}F + k\omega I)^{m-1} \frac{i}{2\pi} \bar{\partial}\partial H(\mu \omega_n \frac{I_E}{n!}) Td_{n-m}(X) \right]_{(sym)} \]

\[ = \sum_m \sum_{l=0}^{m-1} \frac{i}{2\pi} C_l^{m-1} \omega^l \left( \frac{i}{2\pi} F \right)^{m-1-l} Td_{n-m}(X) \bar{\partial}\partial H(\mu \frac{I_E}{n!})_{(sym)} k^l. \]

This is the evolution equation for the moment map \( \mu \). We observe that it is a polynomial in \( k \). The highest degree term has coefficient \( \Delta \mu \frac{I_E}{n!} \), but the rest of the terms involve \( F \), which cannot be converted to \( \mu \). Q.E.D.

5 Some explicit formulas

The almost Hermitian-Einstein flow appears to be considerably more complicated than the Hermitian-Einstein, or Donaldson heat flow. For the Donaldson heat flow, as shown by Donaldson [1], important geometric quantities such as the curvature density can be controlled, and the flow exists for all time. But for the almost Hermitian-Einstein equation, the flow for these quantities are more complicated, and a full analysis is still unavailable. In this section, we derive some explicit formulas to illustrate these features. For the sake of simplicity, we consider only the case of \( X \) of dimension 2 (when \( X \) is dimension of 1, there is no difference between the almost Hermitian-Einstein flow and the Hermitian-Einstein flow) and \( E \) of rank 1, where the equation becomes already complicated. In this case, the almost Hermitian-Einstein equation can be written as

\[ \frac{i}{2\pi} F \wedge \omega = \mu \omega^2 + \frac{1}{k} \chi^2 \omega^2 - Td_X - \frac{i}{2\pi} F \wedge Td_X - (\frac{i}{2\pi} F)^2. \]

(24)
If we write down the local coordinate expression of the evolution equation for the metric $h$ on the line bundle $E$, we get:

$$\dot{h}^{-1} = -\hat{F} + \mu + \frac{1}{k}(\chi^2 + \frac{1}{4\pi^2}(F_{ik}F_{ki} - \hat{F}^2) + \frac{1}{8\pi^2}(F_{ik}R_{ki} - \hat{F}\hat{R} - T\tilde{d}_X^2))$$  \hspace{1cm} (25)

where $F$ is the curvature of the bundle $E$, $R$ is the curvature of the base manifold $X$, $\hat{F}$ means $F_{ik}$, $\hat{R}$ is the scalar curvature and $T\tilde{d}_X^2$ is the coefficient of $\frac{\omega^2}{2}$ in the second Todd class $Td^2$. It is known that $\tilde{d}_X^2$ is a degree two polynomial in curvature $R$. In the special case when the base metric $\omega$ is Kähler-Einstein, the flow reduces to

$$\dot{h}^{-1} = -\hat{F} + \mu,$$

the Hermitian-Einstein flow.

Now, let’s derive the evolution equation for $F$:

$$\dot{F} = \bar{\partial}\bar{\partial}h(\dot{h}^{-1})$$

$$= -\bar{\partial}\bar{\partial}h(\hat{F} - \mu - \frac{1}{k}(\chi^2 + \frac{1}{4\pi^2}(\hat{F}^2 - F_{ik}F_{ki}) + \frac{1}{8\pi^2}(\hat{F}\hat{R} - F_{ik}R_{ki} + T\tilde{d}_X^2)))$$

$$= -\bar{\partial}\bar{\partial}h(\hat{F} - \frac{1}{k}(\frac{1}{4\pi^2}(\hat{F}^2 - F_{ik}F_{ki}) + \frac{1}{8\pi^2}(\hat{F}\hat{R} - F_{ik}R_{ki} + T\tilde{d}_X^2))).$$

Hence we have

$$\frac{d}{dt}|F|^2 = 2Re(\hat{F}, F)$$

$$= 2Re(-\bar{\partial}\bar{\partial}h(\hat{F} - \frac{1}{k}(\frac{1}{4\pi^2}(\hat{F}^2 - F_{ik}F_{ki}) + \frac{1}{8\pi^2}(\hat{F}\hat{R} - F_{ik}R_{ki} + T\tilde{d}_X^2)), F).$$

Next we compute the $\frac{1}{k}$ terms:

$$(\bar{\partial}\bar{\partial}h(F_{ik}R_{ki})_{\bar{\imath}\bar{m}}F_{\bar{\jmath}\bar{n}})$$

$$= \nabla_l\nabla_m(F_{ik}R_{ki})F_{\bar{\jmath}\bar{n}}$$

$$= (\nabla_l\nabla_m F_{ik})_{\bar{\jmath}\bar{n}}R_{ki}F_{\bar{\imath}\bar{m}} + \nabla_m F_{ik} \nabla_l R_{ki}F_{\bar{\imath}\bar{m}} + \nabla_l F_{ik} \nabla_m R_{ki}F_{\bar{\imath}\bar{m}}$$

$$+ F_{ik} \nabla_l \nabla_m R_{ki}F_{\bar{\imath}\bar{m}}$$

$$= (\nabla_l \nabla_m F_{mk})_{\bar{\jmath}\bar{n}}R_{ki}F_{\bar{\imath}\bar{m}} + \nabla_m F_{ik} \nabla_l R_{ki}F_{\bar{\imath}\bar{m}} + \nabla_l F_{ik} \nabla_m R_{ki}F_{\bar{\imath}\bar{m}}$$

$$+ F_{ik} \nabla_l \nabla_m R_{ki}F_{\bar{\imath}\bar{m}}$$

$$= (\nabla_l \nabla_m F_{mk} - F_{ak}R_{\bar{m}il}^i - F_{ma}R_{\bar{k}il}^i)R_{ki}F_{\bar{\imath}\bar{m}}$$
implies an inequality
\[ 1_C \leq \partial \partial X - 2 \nabla_i \nabla_m R_{ki} F_{lm} + F_{ik} \nabla_i \nabla_m R_{ki} F_{lm} \]

\[ = (\nabla_i \nabla_m F_{ml}) F_{ki} R_{ki} - F_{ak} R^a_{ml} R_{ki} F_{lm} - F_m a R^a_{ki} R_{ki} F_{lm} \]

\[ + \nabla_m F_{ik} \nabla_i R_{ki} F_{lm} + \nabla_i F_{ik} \nabla_m R_{ki} F_{lm} + F_{ik} \nabla_i \nabla_m R_{ki} F_{lm} \]

\[ = \frac{1}{2} \nabla_i \nabla_k |F_{ml}|^2 R_{ki} - \nabla_i F_{ml} \nabla_k R_{ki} F_{lm} \]

\[ + \nabla_m F_{ik} \nabla_i R_{ki} F_{lm} + \nabla_i F_{ik} \nabla_m R_{ki} F_{lm} \]

\[ - F_{ak} R^a_{ml} R_{ki} F_{lm} - F_m a R^a_{ki} R_{ki} F_{lm} + F_{ik} \nabla_i \nabla_m R_{ki} F_{lm}. \]

Similarly

\[ (\bar{\partial} \partial_H (F_{ik} F_{kl}))_{lm} F_{lm} = \nabla_l \nabla_m |F_{ik}|^2 F_{lm}, \]

\[ (\bar{\partial} \partial_H (\hat{F}^2))_{lm} F_{lm} = \hat{F} \Delta |F|^2 - 2 |\nabla_i F_{ml}|^2 R + 2 \nabla_m F_{ii} \nabla_i \hat{F} F_{lm} - 2 F_m a R^a_{ml} \hat{F} F_{lm}, \]

\[ (\bar{\partial} \partial_H (\hat{F} \hat{R}))_{lm} F_{lm} = \frac{1}{2} \hat{R} \Delta |F|^2 - |\nabla_i F_{ml}|^2 \hat{R} + \nabla_m \hat{F} \nabla_i \hat{R} F_{lm} \]

\[ + \nabla_i \hat{F} \nabla_m \hat{R} F_{lm} + \hat{F} \nabla_i \nabla_m \hat{R} F_{lm} \]

\[ + F_{ai} R^a_{ml} \hat{R} F_{lm} - F_m a R^a_{ml} \hat{R} F_{lm}. \]

If we follow Donaldson’s argument, we can show when \( rk(E) = 1 \), \( \frac{4}{\pi^2} |F|^2 = -2 \text{Re} (\bar{\partial} \partial_H \hat{F}, F) \) implies an inequality

\[ \left( \frac{d}{dt} - \Delta \right) |F|^2 \leq C |F|^2 - 2 |\nabla_i F_{ml}|^2. \]  

(26)

In our case the inequality becomes:

\[ \left( \frac{d}{dt} - \Delta \right) |F|^2 \leq C |F|^2 - 2 |\nabla_i F_{ml}|^2 - \frac{1}{k} \frac{1}{4 \pi^2} (\hat{R} \Delta |F|^2 - F_{ki} \nabla_i \nabla_k |F|^2) \]

\[ - \frac{1}{k} \frac{1}{16 \pi^2} (\hat{R} \Delta |F|^2 - R_{ki} \nabla_i \nabla_k |F|^2) + \frac{1}{k} \frac{1}{8 \pi^2} (-\nabla_i F_{ml} \nabla_k R_{ki} F_{lm}) \]

\[ + 4 |\nabla_i F_{ml}|^2 \hat{R} - 4 \nabla_m \hat{F} \nabla_i \hat{R} F_{lm} + C |F|^3 \]

\[ + |\nabla_i F_{ml}|^2 \hat{R} - \nabla_m \hat{F} \nabla_i \hat{R} F_{lm} - \nabla_i \hat{F} \nabla_m \hat{R} F_{lm} + C |F|^2 + C |F| \]

We summarize this as,

Proposition Let \( X \) be of dimension 2 and \( E \) be of rank 1. We have the following inequality

\[ \frac{d}{dt} |F|^2 + \left( \frac{1}{k} \frac{1}{4 \pi^2} \hat{F} + \frac{1}{k} \frac{1}{16 \pi^2} \hat{R} - 1 \right) \Delta |F|^2 + \frac{1}{k} (-\frac{1}{4 \pi^2} \hat{F}_{ki} - \frac{1}{16 \pi^2} R_{ki}) \nabla_i \nabla_k |F|^2 \]

\[ \leq C |F|^2 - 2 |\nabla_i F_{ml}|^2 + \frac{1}{k} \left( \frac{1}{2 \pi^2} \hat{F} + \frac{1}{8 \pi^2} \hat{R} \right) |\nabla_i F_{ml}|^2 - \frac{1}{2 \pi^2} \nabla_m \hat{F} \nabla_i \hat{F} F_{lm} \]

\[ - \frac{1}{8 \pi^2} \nabla_i F_{ml} \nabla_k R_{ki} + \frac{1}{8 \pi^2} \nabla_m \hat{F} \nabla_i R_{ki} F_{lm} + \frac{1}{8 \pi^2} \nabla_i F_{kl} \nabla_m R_{ki} F_{lm} \]

\[ - \frac{1}{8 \pi^2} \nabla_m \hat{F} \nabla_i \hat{R} F_{lm} - \frac{1}{8 \pi^2} \nabla_i \hat{F} \nabla_m \hat{R} F_{lm} + C |F|^3 + C |F|. \]
The inequality we get is much more complicated than the one, (26), in the case of Hermitian-Einstein flow. However, if we let $k \to \infty$, the inequality reduces to (26)
\[
\left( \frac{d}{dt} - \Delta \right) |F|^2 \leq C |F|^2 - 2 |\nabla_i F_{ml}|^2
\]
as desired.

If we assume that the base metric $g_{ki}$ has a constant scalar curvature, the inequality simplifies to
\[
\begin{align*}
\frac{d}{dt} |F|^2 + \left( \frac{1}{k} \frac{1}{4\pi^2} \hat{F} + \frac{1}{k} \frac{1}{16\pi^2} \hat{R} - 1 \right) \Delta |F|^2 &+ \frac{1}{k} \left( -\frac{1}{4\pi^2} F_{ki} - \frac{1}{16\pi^2} R_{ki} \right) \nabla_i \nabla_k |F|^2 \\
&\leq C |F|^2 - 2 |\nabla_i F_{ml}|^2 + \frac{1}{k} \left( \frac{1}{2\pi^2} \hat{F} + \frac{1}{8\pi^2} \hat{R} \right) |\nabla_i F_{ml}|^2 - \frac{1}{2\pi^2} \nabla_m \hat{F} \nabla_i \hat{F} F_{lm} \\
&\quad - \frac{1}{8\pi^2} \nabla_i F_{ml} \nabla_k F_{lm} + \frac{1}{8\pi^2} \nabla_m F_{ik} \nabla_i R_{ki} F_{lm} + \frac{1}{8\pi^2} \nabla_i F_{ik} \nabla_m R_{ki} F_{lm} \\
&\quad + C |F|^3 + C |F|).
\end{align*}
\]

If we further assume that the base metric $g_{ki}$ is Kähler-Einstein, the inequality simplifies to
\[
\begin{align*}
\frac{d}{dt} |F|^2 + \left( \frac{1}{k} \frac{1}{4\pi^2} \hat{F} - 1 \right) \Delta |F|^2 &+ \frac{1}{k} \left( -\frac{1}{4\pi^2} F_{ki} \right) \nabla_i \nabla_k |F|^2 \\
&\leq C |F|^2 - 2 |\nabla_i F_{ml}|^2 + \frac{1}{k} \left( \frac{1}{2\pi^2} |\nabla_i F_{ml}|^2 \hat{F} - \frac{1}{2\pi^2} \nabla_m \hat{F} \nabla_i \hat{F} F_{lm} \\
&\quad + C |F|^3 + C |F|).
\end{align*}
\]

References

[1] Donaldson, S.K., “Anti Self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles”, Proc. London Math. Soc. 50 (1985) 1-26.

[2] Leung, N., “Differential geometric and symplectic interpretations of stability in the sense of Giseker”, MIT Thesis(1993),

[3] Leung, N., “Einstein type metrics and stability on vector bundles” J. Differential Geom. 45 (1997), 514-546.

[4] Siu, Y.T. “Lectures on Hermitian-Einstein metrics for stable bundles and Kahler-Einstein metrics”, Birkhauser Verlag. (1987)

[5] Uhlenbeck, K. and Yau, S.T., “In the existence of Hermitian-Yang-Mills connections in stable vector bundles”, Comm. Pure Appl. Math. 39(1986),257-293.