Oblivious Stash Shuffle

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Abstract

This is a companion report to Bittau et al. [1]. We restate and prove security of the Stash Shuffle.

1 Description of the Stash Shuffle

Algorithm 1 The Stash Shuffle algorithm.

1: procedure STASHSHUFFLE(Untrusted arrays in, out, mid)
2: stash ← φ
3: for j ← 0, B − 1 do
4: DISTRIBUTEBUCKET(stash, j, in, mid)
5: DRAINTASH(stash, B, mid)
6: FAIL on ¬stash.Empty()
7: COMPRESS(mid, out)

Algorithm[1] the Stash Shuffle, considers input (in) and output (out) items in B sequential buckets, each holding at most 

\( D \equiv \lceil N/B \rceil \) items, sized to fit in private memory. At a high level, the algorithm first chooses a random output bucket for each input item, and then randomly shuffles each output bucket. It does that in two phases. During the Distribution Phase (lines 2–6), it reads in one input bucket at a time, splits it across output buckets, and stores the split-up but as yet unshuffled, re-encrypted items in an intermediate array (mid) in untrusted memory. During the Compression Phase (line 7), it reads the intermediate array of encrypted items one bucket at a time, shuffles each bucket, and stores it fully-shuffled in the output array.

The algorithm gets its name from the stash, a private structure, whose purpose is to reconcile obliviousness with the variability in item counts distributed across the output buckets. This variability (an inherent result of balls-and-bins properties) must be hidden from external observers, and not reflected in non-private memory. For this, the algorithm caps the number traveling from an input bucket to an output bucket at 

\( C \equiv D/B + \alpha \sqrt{D/B} \)

for a small constant \( \alpha \). If any input bucket distributes more than \( C \) items to an output bucket, overflow items are instead stored in a stash—of size \( S \)—where they queue, waiting to be drained into the chosen output bucket during processing of later input buckets.
Algorithm 2 Distribute one input bucket.

1: procedure DISTRIBUTE_BUCKET(stash, b, Untrusted arrays in, mid)
2:   output ← ∅
3:   targets ← SHUFFLE_TO_BUCKETS(B, D)
4:   for j ← 0, B − 1 do
5:     while ¬output[j].Full() ∧ ¬stash[j].Empty() do
6:       output[j].Push(stash[j].Pop())
7:     for i ← 0, D − 1 do
8:       item ← Decrypt(in[DataIdx(b, i)])
9:       if ¬output[targets[i]].Full() then
10:          output[targets[i]].Push(item)
11:     else
12:        if ¬stash.Full() then
13:           stash[targets[i]].Push(item)
14:        else
15:          FAIL
16:   for j ← 0, B − 1 do
17:     while ¬output[j].Full() do
18:       output[j].Push(dummy)
19:   for i ← 0, C − 1 do
20:     mid[MidIdx(j, i)] ← Encrypt(output[j][i])

Algorithm 2 describes the distribution in more detail, implementing the same logic, but reducing data copies. SHUFFLE_TO_BUCKETS randomly shuffles the D items of the input bucket, and B − 1 bucket separators. The shuffle determines which item will fall into which target bucket, stored in targets (line 3). Then, for every output bucket, as long as there is still room in the maximum C items to output, and there are stashed away items, the output takes items from the stash (lines 4-6). Then the input bucket items are read in from the outside input array, decrypted, and deposited either in the output (if there is still room in the quota C of the target bucket), or in the stash (lines 7-15). Finally, if some output chunks are still not up to the C quota, they are filled with dummy items, encrypted and written out into the intermediate array (lines 16-20). Note that the stash may end up with items left over after all input buckets have been processed, so we drain those items (padding with dummies), filling K extra items per output bucket at the end of the distribution phase (line 5 of Algorithm 1 which is similar to distributing a bucket, except there is no input bucket to distribute). K is set to S/B, that is, the size of the stash divided by the number of buckets.

Algorithm 3 Compress intermediate items.

(L ≡ min(W, B)) is the effective window size, defined to account for pathological cases where W > B.

1: procedure COMPRESS(Untrusted arrays mid, out)
2:   for b ← 0, L − 1 do
3:     IMPORT_INTERMEDIATE(b, mid)
4:   for b ← L, B − 1 do
5:     DRAIN_QUEUE(b − L, mid, out)
6:     IMPORT_INTERMEDIATE(b, mid)
7:   for b ← B − L, B − 1 do
8:     DRAIN_QUEUE(b, mid)

Algorithm 3 shows the compression phase. In this phase, the intermediate items deposited by the distribution phase must be shuffled, and dummy items must be filtered out. To do this, without revealing information about the distribution of (real) items in output buckets, the phase proceeds in a sliding window of W buckets of intermediate items. The window size W is meant to absorb the elasticity of real item counts in each intermediate output bucket due to the Binomial distribution. See

Algorithm 4 Import an intermediate bucket.

1: procedure IMPORT_INTERMEDIATE(b, Untrusted array mid)
2:   bucket ← mid[MidIdx(b, 0..C * B + K − 1)]
3:   Shuffle(bucket)
4:   for i ← 0, C * B + K − 1 do
5:     item ← Decrypt(bucket[i])
6:     if ¬item dummy then
7:       queue.Enqueue(item)
As Algorithm 4 shows, an intermediate bucket is loaded into private memory (C items per input bucket, plus another K items for the final stash drain) in line 2, and shuffled in line 3. Then intermediate items are decrypted, throwing away dummies, and enqueued for export, D items at a time, into the output array in untrusted memory.

The distribution step is constrained by the content of a single bucket D and the stash size (although the latter may be organized on a per-bucket basis, only S/B items must be kept in memory at any one time). The compression step requires keeping CB + K items in memory for the freshly loaded bucket, D(W − 1) items for the buckets previously processed, and Q items as a hedge against overflow. We discuss constraints on these parameters in the next section and offer some concrete choices in Section 5.

2 Security Argument

This section is dedicated to proving our main result, Theorem 2, namely that the output of the Stash Shuffle is statistically close to the uniform distribution on permutations on N items. The shuffle’s obliviousness is established by inspection.

Our proof follows the following steps. First, we define the Buckets Shuffle—an idealized and simplified version of the Stash Shuffle, which operates over unbounded data structures in clear text and never fails. We demonstrate that the output of the Buckets Shuffle is truly uniform. Second, we argue that the output of the real algorithm deviates from the Buckets Shuffle only when it fails, thus bounding the statistical distance between the Stash Shuffle’s output distribution and the uniform distribution. Finally, we bound the probability of failure of the Stash Shuffle, and select its parameters so that the probability is negligible in N.

Algorithm 5 The Buckets Shuffle algorithm.

1: procedure DISTRIBUTEAL(b, arrays in, mid)
2: targets ← SHUFFLETOBUCKETS(B, D) ▷ Same as before
3: for i ← 0, D − 1 do
4: item ← in[DataIdx(b, i)]
5: mid[targets[i]].Push(item)

6: procedure COMPRESSAL(array mid, list out)
7: for i ← 0, B − 1 do
8: bucket ← mid[i]
9: Shuffle(bucket)
10: out.Append(bucket)

11: procedure SHUFFLE(arrays in, out, mid)
12: for j ← 0, B − 1 do
13: DISTRIBUTEAL(j, in, mid)
14: COMPRESSAL(mid, out)

Lemma 2.1. The output of the Buckets Shuffle (the SHUFFLE procedure, Algorithm 5) is uniform.

Proof. First, observe that the shuffle, i.e., the mapping of its input to the output, is independent of the content of the in array. It means, in particular, that if the input is uniformly sampled from the set of all permutations on N elements, the output will be uniformly distributed as well.

Second, we construct a coupling between the shuffle seeded with a uniformly distributed input in1 and an arbitrary in2 as follows. For each assignment of items from the bth bucket of in1 output by SHUFFLETOBUCKETS (line 2), we force identical assignment for the same items in in2 (possibly from different input buckets). After execution of the lines 12–13 the internal state of the two runs of the algorithm (the content of mid) become identical, from which the claim follows.

Lemma 2.2. Statistical distance between the distributions of output of the Stash Shuffle and the Buckets Shuffle is bounded by the probability that the Stash Shuffle fails.
Fix which we use in the following bound:

These two statements are implied by the recurrence (1):

\[ F_1 = 1 - \Pr \left[ \forall 0 \leq i \leq B : \sum_{j=0}^{B-1} x_i^{(j)} \leq S \right]. \]

To bound \( F_1 \) we observe that the distribution \( X_t \) satisfies the following recurrence for all \( i \):

\[
\begin{align*}
X_0 &= 0, \\
X_{i+1} &= \max (0, X_i + \text{Bin}(1/B, D) - C),
\end{align*}
\]

where Bin(\( \cdot \), \( \cdot \)) is the binomial distribution.

Towards bounding the tails of \( X_t \), we define two moment-generating functions as

\[ S_i(t) = E[e^{tX_i}] \quad \text{and} \quad S_{\text{Bin}}(t) = E[e^{t\text{Bin}(1/B,D)}]. \]

These two statements are implied by the recurrence (1):

\[
\begin{align*}
\text{if } X_t &\leq C, \quad \text{then } S_{i+1}(t) \leq S_i(t)S_{\text{Bin}}(t) \leq e^{tC}S_{\text{Bin}}(t), \\
\text{if } X_t &> C, \quad \text{then } S_{i+1}(t) = e^{-tC}S_i(t)S_{\text{Bin}}(t),
\end{align*}
\]

which we use in the following bound:

\[ S_{i+1}(t) = E_{x_i \leftarrow X_i} \left[ E \left[ e^{tX_{i+1}} \mid X_i = x_i \right] \right] \leq \left\{ e^{tC} + e^{-tC}S_i(t) \right\} S_{\text{Bin}}(t). \]

Fix \( t_0 \) so that \( S_{\text{Bin}}(t_0) = 5e^{t_0C} \). By an inductive argument it follows that for all \( i \geq 0 \) and \( t < t_0 \):

\[ S_i(t) \leq S_{\text{Bin}}(t) \cdot \frac{e^{tC}}{1 - e^{-tC}S_{\text{Bin}}(t)} < e^{2tC}. \]

An upper bound on the moment-generating function implies a bound on the tail probability event for any threshold \( \alpha > 0 \):

\[ \Pr[X_t > \alpha] = \Pr[e^{tX_t} > e^{t\alpha}] \leq e^{-t\alpha}S_i(t). \]

Thus, the probability that the size of a single stash[j] exceeds \( \alpha \) is capped by \( e^{t(2C-\alpha)} \), which is minimized for \( t = t_0 \) under the condition that \( \alpha > 2C \).
We note that, under the conditions \( x = 1 + \frac{CB}{D} \), follows by combining Lemmas 2.1 and 2.2, and by collecting the bounds on \( F_1 \):

\[
F_1 \leq \Pr \left[ \forall 0 \leq i \leq B : x_i \leq \frac{S}{B} \right] 
\leq B^2 e^{t_0(2C-S/B)}.
\]

We note that, under the conditions \( x_B \leq \frac{S}{B} \leq K \), all stashes drain, which takes care of the second cause of the shuffle’s failure.

To finish the argument we need to compute a lower bound on \( t_0 \). This is done by using an explicit formula for \( S_{\text{Bin}}(t) = \left[ 1 + (e^t - 1)/B \right] D < e^{D(e^t - 1)/B} \). To solve \( S_{\text{Bin}}(t) < 0.5e^{tC} \) for \( t \), we observe that this is implied by \( t \) satisfying \( D(e^t - 1) < (tC - \ln 2)/B \). The last inequality holds for \( t = CB/D - 1 \) in the regime of interest to us (when \( CB/D = 1 + o(1) \) and \( B \ll D \)).

**Lemma 2.4.** The probability that the compression algorithm fails is bounded by

\[
F_2 \leq B \cdot \left\{ \exp(-2(DW)^2/N) + \exp(-2Q^2/N) \right\}.
\]

assuming that \( L = W \leq B \).

**Proof.** Consider the following shuffle, which is a hybrid between the Stash Shuffle and the Buckets Shuffle. It follows the Buckets Shuffle in the distribution stage, and switches to the Stash Shuffle for the compression step.

Concretely, the compression algorithm of the hybrid shuffle works as follows. It reads one bucket at a time, deposits \( D \) elements, while keeping \( WD + Q \) elements in memory to absorb variability in the fill quotient among the buckets.

Conditional on reaching the compression step, the failure probability of the hybrid shuffle and the Stash Shuffle are identical. Thus, it suffices to analyze the failure probability of the hybrid shuffle, which we do below.

Define the total number of items in the first \( i \) buckets as \( Y_i \), where \( Y_0 = 0 \) and \( Y_B = N \). The probability of the compression’s failing is thus bounded by

\[
F_2 = 1 - \Pr[\forall W \leq i \leq B : 0 \leq Y_i - D(i - W) \leq WD + Q],
\]

\[
\leq \sum_{i=W}^{B} \Pr[Y_i < D(i - W)] + \Pr[Y_i > Di + Q].
\]

We observe that \( Y_i = \text{Bin}(i/B, N) \) (this is where we use the fact that the hybrid’s shuffle distribution stage never fails). Recall that \( D = N/B \), and thus \( E[Y_i] = iD \).

The tails of the binomial distribution are bounded as

\[
\Pr[\text{Bin}(i/B, N) < D(i - W)] \leq \exp(-2(DW)^2/N),
\]

\[
\Pr[\text{Bin}(i/B, N) > Di + Q] \leq \exp(-2Q^2/N),
\]

which implies the claim.

**Theorem 2.5.** The statistical distance between the output of the Stash Shuffle and the uniform distribution is bounded by

\[
B^2 \exp \left\{ (CB/D - 1)(2C - S/B) \right\} + B \cdot \left\{ \exp(-2(DW)^2/N) + \exp(-2Q^2/N) \right\}
\]

assuming \( K \geq S/B > 2C \), \( W \leq B \) and \( e^t < 1 + (tC - \ln 2)/B/D \) where \( t = CB/D - 1 \).

**Proof.** Follows by combining Lemmas 2.3 and 2.4 and by collecting the bounds on \( F_1 \) and \( F_2 \) (Lemmas 2.3 and 2.4).

A simplified, asymptotic statement of Theorem 2 is given by the next corollary. We omit computational assumptions on security of the encryption; the rest of the argument provides unconditional security.

**Corollary 2.6.** There is an oblivious shuffle on \( N \) items with \( N^{1/2+o(1)} \) private memory whose output distribution is distance \( \text{negl}(1/N) = N^{-\omega(1)} \) from the uniform.
Proof. Consider the Stash Shuffle with the following parameters: $B = N^{1/2-\epsilon}$, $D = N^{1/2+\epsilon}$, $C = (1 + \epsilon)N^{2\epsilon}$, $S = N^{1/2+2\epsilon}$, $K = N^{3\epsilon}$, $W = 1$, and $Q = N^{1/2+\epsilon}$. According to Theorem 2, the distance between the shuffle’s output distribution and the uniform is bounded by

$$N^{1-2\epsilon} \exp\left\{-(1 + \epsilon)N^{3\epsilon}\right\} + 2N^{1/2-\epsilon} \exp\left\{-N^{2\epsilon}\right\}.$$ 

(The theorem’s assumptions hold; the only one that requires verification is that $t = CB/D - 1 = \epsilon$ and $e^\epsilon \leq 1 + (\epsilon(1 + \epsilon)N^{2\epsilon} - \ln 2)N^{-2\epsilon}$, which holds for $\epsilon < 1$ and $N^{-2\epsilon} \ln 2 \ll \epsilon^2$).

As long as $1 \gg \epsilon \gg \ln \ln N/\ln N$, the distance will decay faster than a negligible function in $1/N$, and the private memory size will approach $N^{1/2}$. □

3 Sample Parameters

Table 1 lists the Stash Shuffle’s parameters for several select scenarios. Rather than applying the generic bound of our main theorem, we use a tighter estimate of the shuffle’s security level (equivalently, the failure probability of the Stash Shuffle, according to Lemma 2.2). The more precise bounds follow by computing the tail probabilities of the distributions $X_i$ and $Y_i$ in Lemmas 2.3 and 2.4 respectively.

| $N$ | $B$  | $D$  | $C$  | $W$  | $S$   | $Q$   | $\log(\epsilon)$ |
|-----|------|------|------|------|-------|-------|-------------------|
| 10M | 1,000| 10,000| 25   | 2    | 40,000| 18,000| -80.1             |
| 50M | 2,000| 25,000| 30   | 2    | 86,000| 40,000| -81.8             |
| 100M| 3,000| 33,334| 30   | 2    | 117,000| 57,000| -81.9             |
| 200M| 4,400| 45,455| 24   | 2    | 170,000| 73,000| -64.5             |

Table 1: Stash Shuffle parameter scenarios and their security.

References

[1] Andrea Bittau, Úlfar Erlingsson, Petros Maniatis, Ilya Mironov, Ananth Raghunathan, David Lie, Mitch Rudominer, Ushasree Kode, Julien Tinnes, Bernhard Seefeld. “PROCHLO: Strong Privacy for Analytics in the Crowd.” SOSP 2017.