A NOTE ON A QUESTION OF R. POL CONCERNING LIGHT MAPS

V. V. Uspenskij

Abstract. Let \( f : X \to Y \) be an onto map between compact spaces such that all point-inverses of \( f \) are zero-dimensional. Let \( A \) be the set of all functions \( u : X \to I = [0, 1] \) such that \( u[f^{-1}(y)] \) is zero-dimensional for all \( y \in Y \). Do almost all maps \( u : X \to I \), in the sense of Baire category, belong to \( A \)? H. Toruńczyk proved that the answer is yes if \( Y \) is countable-dimensional. We extend this result to the case when \( Y \) has property \( C \).

In R. Pol’s article [6] the following question was posed [5, Problem 423]:

Let \( f : X \to Y \) be a continuous map of a compactum \( X \) onto a compactum \( Y \) with \( \dim f^{-1}(y) = 0 \) for all \( y \in Y \). Does there exist a nontrivial continuous function \( u : X \to I \) into the unit interval such that \( u[f^{-1}(y)] \) is zero-dimensional for all \( y \in Y \)?

It was shown in [2] that the answer is positive. However, R. Pol informed me that he actually meant another question: do almost all maps \( u : X \to I \), in the sense of Baire category, have the property considered above? H. Toruńczyk gave a positive answer under the assumption that \( Y \) is countable-dimensional. The aim of the present note is to extend this result to the case when \( Y \) has property \( C \). In the general case the question remains open.

A space \( Y \) is a \( C \)-space, or has property \( C \), if for any sequence \( \{\alpha_n : n \in \omega\} \) of open covers of \( Y \) there exists a sequence \( \{\mu_n : n \in \omega\} \) of disjoint families of open sets in \( Y \) such that each \( \mu_n \) refines \( \alpha_n \) and the union \( \bigcup_{n \in \omega} \mu_n \) is a cover of \( Y \). This notion was first defined about 1973 by W.E. Haver for compact metric spaces and then by D.F. Addis and J.H. Gresham [1] in the general case. Every finite-dimensional paracompact space and every countable-dimensional metric space has property \( C \) [1], [3]. Every normal \( C \)-space is weakly infinite-dimensional [1], [3] (Engelking [3] includes normality in the definition of property \( C \)). R. Pol’s example of a weakly infinite-dimensional compact metric space which is not countable-dimensional ([3], [4]) has property \( C \) and thus distinguishes between property \( C \) and the property of being countable-dimensional. It is an open problem whether for (compact) metric spaces property \( C \) is equivalent to the property of being weakly infinite-dimensional.

All maps are assumed to be continuous. We denote by \( I \) the interval \([0, 1]\). A map \( f : X \to Y \) is \emph{light} if the fibres \( f^{-1}(y) \) are zero-dimensional for all \( y \in Y \). For a compact space \( X \) let \( C(X, I) \) be the space of all maps \( u : X \to I \) with the usual metric, induced by the metric of the Banach space \( C(X) = C(X, R) \).

1991 Mathematics Subject Classification. Primary 54C10. Secondary 54C35, 54E52, 54F45.

Key words and phrases. Selection, zero-dimensional, countable-dimensional, \( Z \)-set, property \( C \).
1. **Theorem.** Let \( f : X \to Y \) be an onto light map between compact spaces. Let \( A \) be the set of all functions \( u : X \to I \) such that \( u[f^{-1}(y)] \) is zero-dimensional for all \( y \in Y \). If \( Y \) has property \( C \), then \( A \) is a dense \( G_δ \)-subset of \( C(X, I) \).

The proof is based on a selection theorem for \( C \)-spaces obtained in [7]. Let \( X \) be a paracompact \( C \)-space. Suppose that to each \( x \in X \) a contractible non-empty subset \( \Phi(x) \) of a space \( Y \) is assigned. Suppose that the multi-valued map \( \Phi \) has the following semi-continuity property: for every compact \( K \subset Y \) the set \( \{ x \in X : K \subset \Phi(x) \} \) is open. Then \( \Phi \) has a continuous selection: there exists a continuous map \( \phi : X \to Y \) such that \( \phi(x) \in \Phi(x) \) for each \( x \in X \) [7, Theorem 1.3]. We shall use a corollary of this theorem involving the notion of a \( Z \)-set. Denote by \( C(X, Y) \) the space of all maps \( f : X \to Y \) in the compact-open topology. Let us say that a closed subset \( F \) of a topological space \( X \) is a \( Z \)-set in \( X \) if for any compact space \( K \) the set \( C(K, X \setminus F) \) is dense in \( C(K, X) \). If \( X \) is a separable metric ANR, this definition agrees with the usual one [4]. For a closed subset \( F \subset X \) to be a \( Z \)-set, it suffices that the identity map of \( X \) be in the closure of the subspace \( C(X, X \setminus F) \) of \( C(X, X) \). If \( F \) is a \( Z \)-set in \( X \) and \( U \) is open in \( X \), then \( F \cap U \) is \( Z \)-set in \( U \). If \( C \) is a convex subset of a Banach space and \( F \) is a \( Z \)-set in \( C \), then \( C \setminus F \) is contractible (see, for example, Proposition 6.4 in [7]). Therefore, the selection theorem formulated above implies

2. **Theorem.** Let \( X \) be a paracompact \( C \)-space. Let \( C \) be a convex subset of a Banach space. Suppose that to each \( x \in X \) a \( Z \)-subset \( Z(x) \) of \( C \) and a convex subset \( U(x) \) of \( C \) are assigned so that the set \( \bigcup_{x \in X} \{ x \} \times Z(x) \) is closed in \( X \times C \) and the set \( \bigcup_{x \in X} \{ x \} \times U(x) \) is open in \( X \times C \). Then there exists a continuous map \( f : X \to C \) such that \( f(x) \in U(x) \setminus Z(x) \) for every \( x \in X \).

It follows from the arguments of [7] that Theorem 2 actually characterizes \( C \)-spaces among paracompact spaces.

3. **Lemma.** Let \( X \) be a convex subset of a locally convex space \( E \). If \( Y \) is a convex dense subspace of \( X \), then any closed \( F \subset X \) which is disjoint from \( Y \) is a \( Z \)-set in \( X \).

*Proof.* It suffices to prove that for any convex symmetric neighbourhood \( V \) of zero in \( E \) and for any compact \( K \subset X \) there exists a map \( f : X \to Y \) such that \( f(x) \in x + V \) for every \( x \in K \). Since \( Y \) is dense in \( X \), we have \( Y + V \supset K \), and by the compactness of \( K \) there exists a finite \( A \subset Y \) such that \( K \subset A + V \). Let \( \{ h_a : a \in A \} \) be a partition of unity subordinated to the cover \( \{ a + V : a \in A \} \) of \( K \). This means that each \( h_a \) is a map from \( K \) to \( I \), \( \sum_{a \in A} h_a = 1 \) and the support \( \text{supp}(h_a) \) of \( h_a \) is contained in \( a + V \) for every \( a \in A \). The partition of unity \( \{ h_a \} \) defines a map of \( K \) into a simplex of dimension \( \text{Card}(A) - 1 \). This map can be extended over \( X \), since a simplex is an absolute retract. It follows that there exists a partition of unity \( \{ H_a : a \in A \} \) on \( X \) such that the restriction of \( H_a \) to \( K \) coincides with \( h_a \) for every \( a \in A \). Define \( f : X \to E \) by \( f(x) = \sum_{a \in A} H_a(x)a \).

The range of \( f \) is contained in the convex hull of \( A \) and hence in \( Y \). Let us show that \( f(x) - x \in V \) for every \( x \in K \). Fix \( x \in K \), and let \( B \) be the set of all \( a \in A \) such that \( h_a(x) > 0 \). If \( a \in B \), then \( x \in \text{supp}(h_a) \subset a + V \). Therefore \( f(x) - x = \sum_{a \in A} h_a(x)(a - x) = \sum_{a \in B} h_a(x)(a - x) \) is a convex combination of points of \( V \) and hence belongs to \( V \).
4. Lemma. Let $X$ be compact, $Y$ be a zero-dimensional closed subspace of $X$. If $F$ is a closed subspace of $C(X, I)$ such that $f(Y)$ is infinite for every $f \in F$, then $F$ is a $Z$-set in $C(X, I)$.

Proof. In virtue of Lemma 3, it suffices to show that the convex set $\{g \in C(X, I) : g(Y) \text{ is finite}\}$ is dense in $C(X, I)$. Fix $f \in C(X, I)$ and $\epsilon > 0$. Since $\dim Y = 0$, there exists a map $h : Y \to I$ with finite range such that $0 \leq f(y) - h(y) < \epsilon$ for every $y \in Y$. Let $k : X \to I$ be an extension of the map $y \mapsto f(y) - h(y)$ ($y \in Y$) over $X$ such that $k(x) < \epsilon$ for every $x \in X$. The function $g = f - k \in C(X)$ is $\epsilon$-close to $f$ and coincides with $h$ on $Y$, hence $g(Y) = h(Y)$ is finite. If the range of $g$ is not contained in $I$, replace $g$ by $rg$, where $r$ is the natural retraction of the real line onto $I$. □

Proof of Theorem 1. Let $f : X \to Y$ be a light map of a compact space $X$ onto a compact $C$-space $Y$. Let $A$ be the set of all maps $u : X \to I$ such that $u[f^{-1}(y)]$ is zero-dimensional for all $y \in Y$. Let $C = C(X, I)$. We must show that $A$ is a dense $G_\delta$-subset of $C$.

For every subset $V \subset I$ let $A_V$ be the set of all maps $u : X \to I$ such that for every $y \in V$ the set $u[f^{-1}(y)]$ does not contain $V$. Fix a countable base $\mathcal{B}$ in $I$. Since a subset of $I$ is zero-dimensional if and only if it does not contain any element of $\mathcal{B}$, we have $A = \bigcap_{V \in \mathcal{B}} A_V$. Thus it suffices to prove that for every $V \in \mathcal{B}$ the set $A_V$ is open and dense in $C$.

We show that for every $V \subset I$ the set $A_V$ is open in $C$. For every $t \in I$ let $B_t$ be the set of all pairs $(y, u)$ in $Y \times C$ such that $t \in u[f^{-1}(y)]$, and let $C_t$ be the set of all triples $(x, y, u)$ in $X \times Y \times C$ such that $f(x) = y$ and $u(x) = t$. Every $B_t$ is closed, since $B_t$ is the image of the closed set $C_t$ under the projection $X \times Y \times C \to Y \times C$ which is a closed map. Similarly, the projection $Y \times C \to C$ is closed and sends the closed set $\bigcap_{t \in V} B_t$ to the complement of $A_V$. Hence $A_V$ is open in $C$.

We prove that $A_V$ is dense in $C$ for every infinite subset $V \subset I$. Fix $h \in C$ and $\epsilon > 0$. We must show that there exists $w \in A_V$ such that $|w(x) - h(x)| < \epsilon$ for every $x \in X$. For every $y \in Y$ let $U(y)$ be the convex set of all $u \in C$ such that $|u(x) - h(x)| < \epsilon$ for every $x \in f^{-1}(y)$, and let $Z(y)$ be the set of all $u \in C$ such that $V \subset u[f^{-1}(y)]$. According to Lemma 4, $Z(y)$ is a $Z$-set in $C$. The set $\bigcup_{y \in Y} \{y\} \times Z(y)$ is closed in $Y \times C$, since it is equal to the closed set $\bigcap_{t \in V} B_t$ considered in the preceding paragraph. The set $\bigcup_{y \in Y} \{y\} \times U(y)$ is open in $Y \times C$, since its complement is equal to the image of the closed subset $\{(x, u) : |u(x) - h(x)| \geq \epsilon\}$ of $X \times C$ under the perfect map $f \times \text{id}_C : X \times C \to Y \times C$. Thus we can apply Theorem 2. In virtue of this theorem, there exists a continuous map $y \mapsto u_y$ from $Y$ to $C$ such that $u_y \in U(y) \setminus Z(y)$ for every $y \in Y$. The map $w : X \to I$ defined by $w(x) = u_{f(x)}(x)$ has the required properties: $w \in A_V$ and $|w(x) - h(x)| < \epsilon$ for every $x \in X$. Indeed, for every $y \in Y$ the map $w$ coincides with $u_y$ on the set $f^{-1}(y)$. Since $u_y \notin Z(y)$, it follows that $w[f^{-1}(y)] = u_y[f^{-1}(y)]$ does not contain $V$. Thus $w \in A_V$. Similarly, for every $x \in X$ we have $|w(x) - h(x)| = |u_{f(x)}(x) - h(x)| < \epsilon$, since $u_{f(x)} \in U(f(x))$. □

References

[1] D.F. Addis and J.H. Gresham, A class of infinite-dimensional spaces. Part I: Dimension theory and Alexandroff’s Problem, Fund. Math. 101 (1978), 195–205.
[2] A.N. Dranishnikov, V.V. Uspenskij, Light maps and extensional dimension, Topology Appl. 80 (1997), 61–99.
[3] R. Engelking, *Theory of dimensions: Finite and infinite*, Heldermann Verlag, Lemgo, 1995.
[4] J. van Mill, *Infinite-dimensional topology: Prerequisites and introduction*, North-Holland, Amsterdam et al., 1989.
[5] J. van Mill and G.M. Reed (eds.), *Open problems in topology*, North-Holland, Amsterdam et al., 1990.
[6] R. Pol, *Questions in dimension theory*, [5], pp. 279–291.
[7] V.V. Uspenskij, *A selection theorem for C-spaces*, Topology Appl. 85 (1998), 351–374.

321 Morton Hall, Department of Mathematics, Ohio University, Athens, Ohio 45701, USA

E-mail address: uspensk@bing.math.ohiou.edu, vvu@uspensky.ras.ru