Rank one discrete valuations of $k((X_1, \ldots, X_n))$

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Abstract

In this paper we study the rank one discrete valuations of $k((X_1, \ldots, X_n))$ whose center in $k[X_1, \ldots, X_n]$ is the maximal ideal $(X_1, \ldots, X_n)$. In sections 2 to 6 we give a construction of a system of parametric equations describing such valuations. This amounts to finding a parameter and a field of coefficients. We devote section 2 to finding an element of value 1, that is, a parameter. The field of coefficients is the residue field of the valuation, and it is given in section 5.

The constructions given in these sections are not effective in the general case, because we need either to use the Zorn’s lemma or to know explicitly a section $\sigma$ of the natural homomorphism $R_v \to \Delta_v$ between the ring and the residue field of the valuation $v$.

However, as a consequence of this construction, in section 7, we prove that $k((X_1, \ldots, X_n))$ can be embedded into a field $L((Y_1, \ldots, Y_n))$, where the “extended valuation” is as close as possible to the usual order function.

Key words: Valuation theory, local uniformization, formal power rings, completions.

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1 Terminology and preliminaries

Let $k$ be an algebraically closed field of characteristic 0, $R_n = k[X_1, \ldots, X_n]$, $M_n = (X_1, \ldots, X_n)$ its maximal ideal and $K_n = k((X_1, \ldots, X_n))$ its quotient field. Let $v$ be a rank-one discrete valuation of $K_n|k$, $R_v$ the valuation ring, $m_v$ the maximal ideal and $\Delta_v$ the residue field of $v$. The center of $v$ in $R_n$ is $m_v \cap R_n$. Throughout this paper “discrete valuation of $K_n|k$” means “rank-one discrete valuation of $K_n|k$ whose center in $R_n$ is the maximal ideal $M_n$”.

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dimension of $v$ is the transcendence degree of $\Delta_v$ over $k$. In order to simplify the redaction we shall assume, without loss of generality, that the group of $v$ is $\mathbb{Z}$.

Let $\hat{K}_n$ be the completion of $K_n$ with respect to $v$, $\hat{v}$ the extension of $v$ to $\hat{K}_n$, $R_{\hat{v}}$, $m_{\hat{v}}$ and $\Delta_{\hat{v}}$ the ring, maximal ideal and the residue field of $\hat{v}$, respectively (see [4] for more details). We know that $\Delta_v$ and $\Delta_{\hat{v}}$ are isomorphic ([3]). Let $\sigma : \Delta_{\hat{v}} \to R_{\hat{v}}$ be a $k$–section of the natural homomorphism $R_{\hat{v}} \to \Delta_{\hat{v}}$, $\theta \in R_{\hat{v}}$ an element of value 1 and $t$ an indeterminate. We consider the $k$–isomorphism

$$\Phi = \Phi_{\sigma, \theta} : \Delta_{\hat{v}}[t] \to R_{\hat{v}}$$

given by

$$\Phi \left( \sum \alpha_i t^i \right) = \sum \sigma(\alpha_i) \theta^i,$$

and denote also by $\Phi$ its extension to the quotient fields. We have a $k$–isomorphism $\Phi^{-1}$ which, when composed with the usual order function on $\Delta_{\hat{v}}((t))$, gives the valuation $\hat{v}$. This is the situation we will consider throughout this paper, and we will freely use it without new explicit references.

We shall use two basic transformations in order to find an element of value 1 and construct the residue field:

(1) **Monoidal transformation:**

$$k[[X_1, \ldots, X_n]] \to k[[Y_1, \ldots, Y_n]]$$

$$X_1 \mapsto Y_1$$

$$X_2 \mapsto Y_1 Y_2$$

$$X_i \mapsto Y_i, \ i = 3, \ldots, n.$$  

with $v(X_2) > v(X_1)$.

(2) **Change of coordinates:**

$$k[[X_1, \ldots, X_n]] \to L[[Y_1, \ldots, Y_n]]$$

$$X_1 \mapsto Y_1$$

$$X_i \mapsto Y_i + c_i Y_1, \ i = 2, \ldots, n,$$

where $c_i \in R_{\hat{v}} \setminus m_{\hat{v}}$ and $L$ is an extension field of $k$.

For both transformations we have the following facts:

(a) The transformations are one to one: In the case of the monoidal transformations this property is well known. In the other case it is a consequence of [6] (corollary 2, page 137).
(b) New variables $Y_i$ lie in $R_v^\sigma$, so we can put $(\Phi_{\sigma,\theta})^{-1}(Y_i) = \sum a_{i,j}t^j$.

(c) Let $\varphi : K_n \to \Delta_v((t))$ be the restriction of $(\Phi_{\sigma,\theta})^{-1}$ to $K_n$. Let us denote by $\varphi' : L_n = L((Y_1, \ldots , Y_n)) \to \Delta_v((t))$ the natural extension of $\varphi$ to the field $L_n$. Then $v = \nu_t \circ \varphi'_{K_n}$, with $\nu_t$ the usual order function over $\Delta_v((t))$. Therefore, if $\varphi'$ is injective we can extend the valuation $v$ to the field $L_n$ and the extension is $v' = \nu_t \circ \varphi'$.

From now on transformation will mean monoidal transformation, change of coordinates, variables interchanges or finite compositions of these.

2 Construction of an element of value 1

Remember that we are assuming that the group of $v$ is $\mathbb{Z}$, so there exists an element $u \in K$ such that $v(u) = 1$.

Lemma 1 Let $\alpha_i = v(X_i)$ for all $i = 1, \ldots , n$. By a finite number of monoidal transformations we can find $n$ elements $Y_1, \ldots , Y_n \in \bar{K}_n$ such that $v(Y_i) = \alpha = \gcd\{\alpha_1, \ldots , \alpha_n\}$.

PROOF. We can suppose that $v(X_1) = \alpha_1 = \min\{\alpha_i|1 \leq i \leq n\}$ and consider the following two steps:

Step 1.- If there exists $n_i \in \mathbb{Z}$ such that $\alpha_i = n_i\alpha_1$ for all $i = 2, \ldots , n$, then for each $i$ we apply $n_i - 1$ monoidal transformations

$$k[[X_1, \ldots , X_n]] \longrightarrow k[[Y_1, \ldots , Y_n]]$$

$$X_i \mapsto Y_1Y_i$$

$$X_j \mapsto Y_j, j \neq i.$$

Trivially $v(Y_i) = \alpha_1$ for all $i = 1, \ldots , n$.

Step 2.- Assume there exists $i$, with $2 \leq i \leq n$, such that $v(X_1) = \alpha_1$ does not divide to $v(X_i) = \alpha_i$. We can suppose that $i = 2$ with no loss of generality and then $\alpha_2 = q\alpha_1 + r$. So we apply $q$ times the monoidal transformation

$$k[[X_1, \ldots , X_n]] \longrightarrow k[[Y_1, \ldots , Y_n]]$$

$$X_2 \mapsto Y_1Y_2$$

$$X_i \mapsto Y_i, i \neq 2$$

to obtain a new ring $k[[Y_1, \ldots , Y_n]]$ where $v(Y_2) = r > 0$ and $Y_2$ is the element of minimum value.
As the values of the variables are greater than zero, in a finite number of steps 2 we come to the situation of step 1. In fact, this algorithm is equivalent to the “euclidean algorithm” to compute the greatest common divisor of $\alpha_1, \ldots, \alpha_n$. $\blacksquare$

**Theorem 2** We can construct an element of value 1 applying a finite number of monoidal transformations and changes of coordinates.

**PROOF.** We call $Y_{1,r}, \ldots, Y_{n,r}$ the elements found after $r$ transformations.

We can suppose that we have applied the previous lemma to obtain $Y_{1,1}, \ldots, Y_{n,1}$ such that $v(Y_{i,1}) = \alpha$ for all $i = 1, \ldots, n$. Let us prove that there exists $c_i \in R_0 \setminus m_0$ for each $i = 2, \ldots, n$ such that $\hat{v}(Y_{i,1} - c_i Y_{1,1}) > \alpha$. We can take

$$(\Phi_{\sigma,\theta})^{-1}(Y_{i,1}) = \sum_{j \geq a} a_{i,j} t^j = \omega_i(t), \ a_{i,j} \in \Delta_0, \ a_{i,\alpha} \neq 0,$$

and so it suffices taking $b_i = a_{i,\alpha}/a_{1,\alpha}$ and $c_i = \sigma(b_i)$.

The following two steps defines a procedure to obtain an element of value 1:

**Step 1.-** We apply the coordinate change

$$k[Y_{1,1}, \ldots, Y_{n,1}] \longrightarrow L[Y_{1,2}, \ldots, Y_{n,2}]$$

$$Y_{1,1} \longmapsto Y_{1,2}$$

$$Y_{i,1} \longmapsto Y_{i,2} + c_i Y_{1,2}, \ i = 2, \ldots, n.$$  

With this transformation the values of the new variables are not equal to $\hat{v}(Y_{1,2})$.

**Step 2.-** We apply lemma 1 to equalize the values of elements and go to step 1. Obviously, the minimum of the values of the elements does not increase, because the greater common divisor of the values does not exceed the minimum of the values. Moreover the first variable does not change.

If we obtain an element of value 1 then we are finished.

We have to show that the procedure produces an element of value 1 in a finite number of transformations. The only way for the process to be infinite is that, in step 2, the minimum of the values of the elements does not decrease. This means that, in step 1, the value of the first variable divides the values of the new variables.
The composition of steps 1 and 2 is the transformation
\[ k\{Y_{1,r}, \ldots, Y_{n,r}\} \rightarrow L\{Y_{1,r+1}, \ldots, Y_{n,r+1}\} \]
\[ Y_{1,r} \mapsto Y_{1,r+1} \]
\[ Y_{i,r} \mapsto Y_{i,r+1} + c_i Y_{1,r+1}^{m_i}, \quad i = 2, \ldots, n. \]

If we use steps 1 and 2 infinitely many times, we have an infinite sequence of transformations
\[ k\{Y_1, \ldots, Y_n\} \rightarrow L\{Y_1, \ldots, Y_n\} \]
\[ Y_1 \mapsto Y_{1,j} \]
\[ Y_i \mapsto Y_{i,j} + \sum_{k=1}^{j} c_{i,k} Y_{1,j}^{m_i,k}, \quad i = 2, \ldots, n. \]

Then we can obtain an infinite sequence of variables
\[ Y_{1,j} = Y_{1,j} \]
\[ Y_{i,j} = Y_i - \sum_{k=1}^{j} c_{i,k} Y_{1,j}^{m_i,k}, \quad i = 2, \ldots, n, \]
with \( \hat{v}(Y_{i,j}) > \hat{v}(Y_{i,j-1}) \) for all \( i, j \). So any sequence of partial sums of the series
\[ Y_i - \sum_{k=1}^{\infty} c_{i,k} Y_{1,j}^{m_i,k}, \quad \forall i = 2, \ldots, n \]
have strictly increasing values. Then these series converge to zero in \( R_{\hat{v}} \), so
\[ Y_i = \sum_{k=1}^{\infty} c_{i,k} Y_{1,j}^{m_i,k}, \quad \forall i = 2, \ldots, n. \]

Let \( f(Y_1, \ldots, Y_n) \in K_n \), then
\[ v(f) = \hat{v}\left( f\left( \sum_{k=1}^{\infty} c_{2,k} Y_{1}^{m_{2,k}}, \ldots, \sum_{k=1}^{\infty} c_{n,k} Y_{1}^{m_{n,k}} \right) \right) = m \cdot v(Y_1). \]

In this situation, the group of \( v \) is \( v(Y_1) \cdot \mathbb{Z} \) (see [1]) but as the group is assumed to be \( \mathbb{Z} \), \( \hat{v}(Y_1) = 1 \). \( \square \)

**Example 3** Let us consider the embedding
\[ \Psi : \mathbb{C}[X_1, X_2, X_3] \rightarrow \mathbb{C}(T_2, T_3)[t] \]
\[ X_1 \mapsto t^2 \]
\[ X_2 \mapsto T_2 t^4 + T_2 t^6 \]
\[ X_3 \mapsto T_3 t^5 \]
with $t, T_2$ and $T_3$ variables over $\mathbb{C}$. We are going to denote its extension to the quotient fields by $\Psi$ as well. The composition of this injective homomorphism with the order function in $t$ gives a discrete valuation of $\mathbb{C}(X_1, X_2, X_3)|\mathbb{C}$, $v = \nu_t \circ \Psi$. If we apply the procedure given in this section we construct the following element of value 1:

$$\frac{X_3 - c_2X_1}{X_1^2},$$

where $c_2 \in R_v \setminus m_v$ such that $\Psi(c_2) = T_2 + t \cdot f$, with $f \in \mathbb{C}(T_2, T_3)[t]$. In this case we can take $c_2 = \frac{X_2}{X_1^2 + X_1}$.

**Remark 4** We need to know some elements $c_i \in R_v \setminus m_v$ (or $b_i \in \Delta_v$ and $\sigma : \Delta_v \rightarrow R_v$) such that $\hat{v}(Y_{i,1} - c_iY_{1,1}) > \alpha$ for each $i = 2, \ldots, n$ in order to apply the procedure described in the proof of theorem 2. Let $\Delta$ be a field, if the valuation is given as a composition of an injective homomorphism

$$\Psi : k[[X_1, \ldots, X_n]] \rightarrow \Delta[t],$$

$$X_i \mapsto \sum_{j \geq 1} a_{i,j}t^j$$

with the usual order function of $\Delta((t))$, $v = \nu_t \circ \Psi$, then we can find the $c_i$’s using the coefficients $a_{i,j} \in \Delta$ of $\Psi(X_i)$.

### 3 Transcendental and algebraic elements of $\Delta_v$

In the following sections we give a procedure to construct the residue field $\Delta_v$ of a discrete valuation of $K_n|k$, as a transcendental extension of $k$.

Before the description of the procedure we have to do the following remark about the $k$–section $\sigma$.

**Remark 5** We are going to check all the variables searching those residues which generate the extension $k \subset \Delta_v$. Hence we will have to move between $R_v$ and $\Delta_v$ by the $k$–section $\sigma$ and the natural homomorphism $\Delta_v \rightarrow R_v$. We can do the following considerations:

1) Let us consider the diagram
where $\mathbb{F}$ and $\mathbb{F}'$ are subfields of $R_\hat{v}$ and $\Delta_\hat{v}$ respectively. Let $\omega \in R_\hat{v}$ an element such that $\hat{v}(\omega) = 0$. The question is: if $\omega + m_\hat{v}$ is transcendental over $\mathbb{F}'$, is $\sigma(\omega + m_\hat{v})$ transcendental over $\mathbb{F}$? What happens in the algebraic case?

So we suppose $\omega + m_\hat{v}$ to be transcendental over $\mathbb{F}'$. Let $f(X) \in \mathbb{F}[X]$ be a non-zero polynomial. Let us put

$$f(X) = \sum_{i=0}^{n} \sigma(a'_i)X^i, \quad a'_i \in \mathbb{F}' .$$

Then

$$f(\sigma(\omega + m_\hat{v})) = \sum_{i=0}^{n} \sigma(a'_i)\sigma(\omega + m_\hat{v})^i = \sigma \left( \sum_{i=0}^{n} a'_i(\omega + m_\hat{v})^i \right) \neq 0$$

because $\omega + m_\hat{v}$ is transcendental over $\mathbb{F}'$. So we have proved that $\sigma(\omega + m_\hat{v})$ is transcendental over $\mathbb{F}$ if $\omega + m_\hat{v}$ is transcendental over $\mathbb{F}'$

2) In the algebraic case let us consider the next diagram:

Let $\alpha + m_\hat{v} \in \Delta_\hat{v}$ be an algebraic element over $\mathbb{F}'$, with $\hat{v}(\alpha) = 0$ (i.e. $\alpha + m_\hat{v} \neq 0$). Let

$$f(X) = X^m + \beta_1X^{m-1} + \cdots + \beta_m \in \mathbb{F}'[X]$$

be its minimal polynomial over $\mathbb{F}'$. Let us take the polynomial

$$f(X) = X^m + b_1X^{m-1} + \cdots + b_m \in \mathbb{F}[X], \quad \text{with } b_i = \sigma(\beta_i).$$
By Hensel’s Lemma ([6], corollary 1, page 279) we know that there exists \( a \in R_{\bar{v}} \) such that \( a \) is a simple root of \( f(X) \) \( y \varphi(a) = \alpha + m_{\bar{v}} \). As \( \varphi \sigma = \text{id} \), \( f(X) \) is the minimal polynomial of \( a \), so we can extend \( \sigma : F'[\alpha + m_{\bar{v}}] \rightarrow F[a] \). Then we have

\[
\begin{array}{c}
R_{\bar{v}} & \xrightarrow{\sigma} & \Delta_{\bar{v}} \\
\downarrow & & \downarrow \\
F(a) & \xrightarrow{\varphi} & F'(\alpha + m_{\bar{v}}) \\
\downarrow & & \downarrow \\
F & \xrightarrow{\sigma} & F' \\
\downarrow & & \downarrow \\
k & \xrightarrow{id} & k \\
\end{array}
\]

Let us consider the set

\[
\Omega = \{(F_1, \sigma_1) | F_1 \supset F \text{ and } \sigma_1 \text{ extends } \sigma\}
\]

partially ordered by

\[
(F_1, \sigma_1) < (F_2, \sigma_2) \iff F_1 \subset F_2 \text{ and } \sigma_2|_{F_1} = \sigma_1.
\]

By Zorn’s Lemma there exists a maximal element \((L, \sigma') \in \Omega\), and again by Hensel’s Lemma ([6], corollary 2, page 280) we have \( \varphi(L) = \Delta_{\bar{v}} \). So we can extend \( \sigma \) to a \( k \)-section \( \sigma' \) of \( \varphi \) in such a way that \( a = \sigma'(\alpha + m_{\bar{v}}) \) is an algebraic element over \( F \).

3) Hence we have showed that if \( \omega + m_{\bar{v}} \in \Delta_{\bar{v}}, \tilde{v}(\omega) = 0 \), is a transcendental (resp. algebraic) element over \( F' \), there exists a \( k \)-section of \( \varphi \) which extends \( \sigma \) and \( \sigma(\omega + m_{\bar{v}}) \) is transcendental (resp. algebraic) over \( F \).

\[
\begin{array}{c}
R_{\bar{v}} & \xrightarrow{\sigma'} & \Delta_{\bar{v}} \\
\downarrow & & \downarrow \\
F & \xrightarrow{\sigma} & F' \\
\downarrow & & \downarrow \\
k & \xrightarrow{id} & k \\
\end{array}
\]
4 A first transcendental residue.

We devote this section to finding a first transcendental residue of $\Delta_v$ over $k$. Note that this preliminary transformations construct the residue field in the case $n = 2$.

Lemma 6 There exists a finite number of monoidal transformations and changes of coordinates that constructs $n$ elements $Y_1, \ldots, Y_n$ such that $v(Y_i) = v(Y_1) = \alpha$ and the residue $Y_2/Y_1 + m_v$ is not in $k$.

PROOF. We can suppose that we have applied lemma 1 to obtain $Y_1, \ldots, Y_n$ such that $v(Y_i) = \alpha$ for all $i = 1, \ldots, n$.

In this situation $v(Y_i/Y_j) = 0$, so $0 \neq (Y_i/Y_j) + m_v \in \Delta_v$. If this residue lies in $k$ then there exists $a_{i,1} \in k$ such that

$$\frac{Y_i}{Y_j} + m_v = a_{i,1} + m_v,$$

so

$$\frac{Y_i}{Y_j} - a_{i,1} = \frac{Y_i - a_{i,1}Y_j}{Y_j} \in m_v,$$

and then

$$v \left( \frac{Y_i - a_{i,1}Y_j}{Y_j} \right) > 0.$$

So we have $v(Y_i - a_{i,1}Y_j) = \alpha_1 > \alpha$. If $\alpha$ divides to $\alpha_1$ then $\alpha_1 = r_1\alpha$ with $r_1 \geq 2$ and

$$v \left( \frac{Y_i - a_{i,1}Y_j}{Y_j^{r_1}} \right) = 0.$$

If the residue of this element lies too in $k$, then exist $a_{i,r_1} \in k$ such that

$$v(Y_i - a_{i,1}Y_j - a_{i,r_1}Y_j^{r_1}) = \alpha_2 > \alpha_1.$$

If $\alpha$ divides to $\alpha_2$ then $\alpha_2 = r_2\alpha$ with $r_2 > r_1$ and we can repeat this operation.

The above procedure is finite for some pair $(i, j)$. We know ([1]) that any discrete valuation of $k((X_1, X_2))$ has dimension 1, so the restriction, $v'$, of our valuation $v$ to the field $k((X_1, X_2))$ is a valuation with dimension 1, and the dimension of $v$ is greater or equal than 1, because a transcendental residue of $v'$ over $k$ is a transcendental residue of $v$ too. If the procedure never ends for all $(i, j)$ then all the residues of $v$ are in $k$, so the dimension of $v$ is 0 and there is a contradiction. So we can suppose that the above procedure ends for $(1, 2)$ by reordering the variables if necessary.
Hence there exists a first transcendental residue. We can apply the above procedure to the variables $Y_1, Y_2$, and so we have the transformations:

$$Z_i = Y_i, \ i \neq 2$$

$$Z_2 = Y_2 - \sum_{i=1}^{s_2} a_{2,i} Y_1^i,$$

such that one of the following two situation occurs:

a) $v(Y_1)$ divides $v(Z_2)$ and the residue of $Z_2/Y_1^r$ is not in $k$ with $v(Z_2) = r \cdot v(Y_1)$.

b) $v(Y_1)$ does not divide $v(Z_2)$.

In case a), we make the transformation

$$Z_2 = Y_2 - \sum_{i=1}^{s_2} a_{2,i} Y_1^i,$$

and apply lemma 1 to obtain elements with the same values. We note these elements by $Y_1, \ldots, Y_n$ again in order not to complicate the notation. So, after this procedure, we have a transcendental element $u_2 = \sigma(Y_2/Y_1 + m_\nu)$ over $k$.

In case b) we make the same transformation and go back to the beginning of the proof.

Anyway this procedure stops, because the value of the variables are greater or equal than 1.

Then we can suppose that, after a finite number of transformations, we have $n$ elements $Y_1, \ldots, Y_n$ such that $v(Y_1) = v(Y_1) = \alpha$ and the residue $Y_2/Y_1 + m_\nu$ is not in $k$. □

**Example 7** Let $v = \nu \circ \Psi$ the discrete valuation of $\mathbb{C}((X_1, X_2))|\mathbb{C}$ defined by the embedding

$$\Psi : \mathbb{C}[X_1, X_2] \longrightarrow \mathbb{C}(u)[t]$$

$$X_1 \longmapsto t$$

$$X_2 \longmapsto t + t^3 + \sum_{i \geq 1} u^i t^{i+3}$$

with $u$ and $t$ independent variables over $\mathbb{C}$.

The residue $X_2/X_1 + m_\nu = 1 + m_\nu$, because $v(X_2 - X_1) = 3 > 1$. So we have

$$v \left( \frac{X_2 - X_1}{X_1^3} \right) = 0.$$
The residue
\[
\frac{X_2 - X_1}{X_1^3} + m_v = 1 + m_v
\]
too, because \(v(X_2 - X_1 - X_1^3) = 4 > 3\). So we have
\[
v \left( \frac{X_2 - X_1 - X_1^3}{X_1^4} \right) = 0.
\]
As \(\Psi((X_2 - X_1 - X_1^3)/X_1^4) = u\) and \(u\) is transcendental over \(\mathbb{C}\), then
\[
\frac{X_2 - X_1 - X_1^3}{X_1^4} + m_v \notin \mathbb{C}
\]
and this is a first transcendental residue of \(\Delta_v\) over \(\mathbb{C}\).

In this situation we can do the transformation
\[
\mathbb{C}[X_1, X_2] \rightarrow \mathbb{C}[Y_1, Y_2]
\]
\[
X_1 \mapsto Y_1
\]
\[
X_2 \mapsto Y_2Y_1^3 + Y_1 + Y_1^3
\]
to obtain elements \(\{Y_1, Y_2\}\) such that \(\Psi(Y_1) = t\) and \(\Psi(Y_2) = \sum_{i \geq 1} u^i t^i\). So \(v(Y_2) = v(Y_1) = 1\) and the residue
\[
\frac{Y_2}{Y_1} + m_v = \frac{X_2 - X_1 - X_1^3}{X_1^4} + m_v
\]
is not in \(\mathbb{C}\).

In this example the extension of the valuation \(v\) to the field \(\mathbb{C}((Y_1, Y_2))\) is the usual order function. Theorem 9 says that, for \(n = 2\), we always have this.

We end up the section with some specific arguments for the case \(n = 2\).

The proof of the following lemma is straightforward from ([1], theorem 2.4):

**Lemma 8** Let \(v\) be a discrete valuation of \(K_n|k\). If \(v\) is such that \(v(f_r) = r\alpha\) for all forms \(f_r\) of degree \(r\) with respect to the usual degree, then the group of \(v\) is \(\alpha \cdot \mathbb{Z}\).

So we have

**Theorem 9** In the case \(n = 2\), the extension of the valuation \(v\) to the field \(k((Y_1, Y_2))\) is the usual order function.
PROOF. After a finite number of transformations we are in the situation of the end of the previous proof. Obviously, if \( n = 2 \), \( k((Y_1, Y_2)) \subset R_c \) so \( v \) can be extended to a valuation \( v' \) over \( k((Y_1, Y_2)) \) such that \( \Delta_{v'} = \Delta_v = \Delta_\hat{v} \). We denote the extension by \( v \) for simplifying. Let \( \sigma : \Delta_\hat{v} \to R_\hat{v} \) a \( k \)-section of \( R_\hat{v} \to \Delta_\hat{v} \), \( u_2 = \sigma(Y_2/Y_1 + m_v) \), \( h \neq 0 \) a form of degree \( r \) and \( \gamma = Y_2 - u_2Y_1 \). From the construction procedure of \( u_2 \) we know that \( \hat{v}(\gamma) > \alpha \) (remember \( \alpha = v(Y_1) \)). Then

\[
h(Y_1, Y_2) = h(Y_1, u_2Y_1 + \gamma) = Y_1^r h(1, u_2) + \gamma',
\]

where \( \gamma' \) is such that \( v(\gamma') > r\alpha \). As \( u_2 \notin k \), \( u_2 \) is transcendental over \( k \), so \( h(1, u_2) \neq 0 \) and \( v(h) = r\alpha \). By the previous lemma, the group of \( v \) is \( \alpha \cdot \mathbb{Z} \), so \( \alpha = 1 \) and \( v \) is the usual order function. \( \square \)

5 The general case

Let us move to the general case. Assume that \( n > 2 \) and suppose we have applied the procedure of the lemma 6 to find \( Y_1, \ldots, Y_n \in \hat{K} \) such that

a) The value of these elements are \( \alpha \in \mathbb{Z} \).

b) The residue of \( Y_2/Y_1 \) is transcendental over \( k \).

This section and the next one describe the transformations that we have to do in order to construct the residue field of \( v \).

Remark 10 Let \( \Delta_2 = k(Y_2/Y_1 + m_v) \) a purely transcendental extension of \( k \) of transcendence degree 1. Let \( \sigma_2 : \Delta_2 \to k(Y_2/Y_1) \) defined by

\[
\sigma_2 \left( \frac{Y_2}{Y_1} + m_v \right) = \frac{Y_2}{Y_1} = u_2.
\]

We know that there exists a \( k \)-section \( \sigma \) which extends \( \sigma_2 \) in the sense of the remark 5.

Remark 11 Let us suppose that the residue of \( Y_3/Y_1 \) is algebraic over \( \Delta_2 \), and let \( u_{3,1} \) be its image by \( \sigma \). Then \( v(Y_3 - u_{3,1}Y_1) = \alpha_1 > \alpha \). If \( \alpha \) divides to \( \alpha_1 \) then there exists \( u_{3,r} \in \text{im}(\sigma) \) and \( r > 1 \) such that \( v(Y_3 - u_{3,1}Y_1 - u_{3,r}Y_1^r) = \alpha_2 > \alpha_1 \).

Let us suppose that \( u_{3,r} \) is algebraic over \( \Delta_2 \) too and \( \alpha \) divides to \( \alpha_2 \). Then we can find ourselves in one of the three situations shown in the following items.

(Situation 1) After a finite number of transformations, we obtain a value \( \alpha_s \)
such that it is not divided by $\alpha$. Then we make the transformation

$$Z_3 = Y_3 - \sum_{j=1}^{s} u_{3,j} Y_1^j,$$

with $u_{3,j}$ algebraic over $\Delta_2$ for all $j = 1, \ldots, s$. So we have to apply transformations to find elements with the same values and begin with all the procedure described in this section. When this occurs, the values of the elements decrease, so we can suppose that after a finite number of transformations we have reached a strictly minimal value. In fact this value should be 1, because we are assuming that the values group is $\mathbb{Z}$. We shall denote these elements by $Y_1, \ldots, Y_n$ in order not to complicate the notation. So we can suppose that this situation will never occur again for any variable.

(Situation 2) After a finite number of steps, we have a transcendental residue of $\Delta_2$. Let us denote this residue by $u_3$. This means

$$Z_3 = Y_3 - \sum_{j=1}^{s_3} u_{3,j} Y_1^j,$$

where the elements $\{u_{3,j}\}_{j=1}^{s_3}$ are algebraic over $\Delta_2$ and $u_3 = \sigma(Z_3/Y_1^{\widehat{v}(Z_3)} + m_v)$ is transcendental over $\Delta_2$. We shall note $\Delta_3 = k(u_2, \{u_{3,j}\}_{j=1}^{s_3}, u_3)$.

In this situation, if $n = 3$ we can apply monoidal transformations to obtain elements with the same values. We will denote these elements again by $\{Y_1, Y_2, Y_3\}$. The extension of the valuation $v$ to the field $L((Y_1, Y_2, Y_3))$ with $L = k(\{u_{3,j}\}_{j=1}^{s_3})$, is the usual order function, for analogy with the case $n = 2$ (theorem 9).

(Situation 3) All the residues obtained are algebraic elements. Then we take $\Delta_3 = \Delta_2(\{u_{3,j}\}_{j\geq 1})$, an algebraic extension of $\Delta_2$.

Remark 12 Let us suppose that we have repeated the previous construction with each element $Y_4, \ldots, Y_{i-1}$, so we have a field

$$\Delta_{i-1} = k(u_2, \zeta_3, \ldots, \zeta_{i-1}) \subset \sigma(\Delta_v),$$

where each $\zeta_k$ is:

- either $\{\{u_{k,j}\}_{j=1}^{s_k}, u_k\}$ if $\{u_{k,j}\}_{j=1}^{s_k}$ are algebraic over $\Delta_{k-1}$ and $u_k = \sigma((Z_k/Y_1^{\widehat{v}(Z_k)} + m_\zeta)$ is a transcendental element over $\Delta_{k-1}$ (i.e. situation 2),

- or $\Delta_{k-1} \subset \Delta_{k-1}(\{u_{k,j}\}_{j\geq 1})$ is an algebraic extension (i.e. situation 3).

So we have two possible situations concerning variable $Y_i$:
1) There exists a transformation

\[ Z_i = Y_i - \sum_{j=1}^{s_i} u_{i,j} Y_1^j, \]

where the elements \( u_{i,j} \) are algebraic over \( \Delta_{i-1} \) and \( u_i = \sigma((Z_i/Y_1^{\bar{v}(Z_i)} + m_{\bar{v}}) \) is a transcendental element over \( \Delta_{i-1} \). So we have the transcendental extension

\[ \Delta_{i-1} \subset \Delta_{i-1}(\{u_{i,j}\}_{j=1}^{s_i}, u_i) = \Delta_i. \]

2) All the elements \( u_{i,j} \) we have constructed are algebraic over \( \Delta_{i-1} \), so we have the algebraic extension

\[ \Delta_{i-1} \subset \Delta_{i-1}(\{u_{i,j}\}_{j \geq 1}) = \Delta_i. \]

Remark 13 We have given a procedure to construct elements \( \{Y_1, \ldots, Y_n\} \) such that they satisfy these important properties:

1) After reordering if necessary, we can suppose that the first \( m \) elements give us all the transcendental residues over \( k \), i.e. the residue of each \( Y_i/Y_1 \) is transcendental over \( \Delta_{i-1} \) with \( i = 2, \ldots, m \). So the rest of variables \( Y_{m+1}, \ldots, Y_n \) are such that we enter in the situation of previous item 2).

2) With the usual notations, the extension

\[ \Delta_m \subset \Delta_m(\{u_{i,j}\}_{j \geq 1}), \ i = m + 1, \ldots, n \]

is algebraic.

Theorem 14 The residue field of \( v \) is

\[ \Delta_n = k \left( u_2, \{u_{3,j}\}_{j=1}^{s_3}, u_3, \ldots, \{u_{m,j}\}_{j=1}^{s_m}, u_m \right) \left( \{u_{m+1,j}\}_{j \geq 1}, \ldots, \{u_{n,j}\}_{j \geq 1} \right), \]

and the transcendence degree of \( \Delta_n \) over \( k \) is \( m - 1 \).

PROOF. In this section we have given a construction by writing the elements \( Y_i \) depending on \( Y_1 \) and some transcendental and algebraic residues. So we have constructed a map

\[ \varphi' : L_n((Y_1, \ldots, Y_n)) \rightarrow \Delta_n((t)) \]

\[ Y_1 \mapsto t \]

\[ Y_i \mapsto u_i t, \ i = 2, \ldots, m \]

\[ Y_k \mapsto \sum_{j \geq 1} u_{k,j} t^j, \ u_{k,1} \neq 0, \ k = m + 1, \ldots, n. \]
This map is not injective in the general case, but we know that \( v = \nu_t \circ \varphi|_{K_n} \).
So the residue field of \( v \) is equal to the residue field of \( \nu_t \), i.e. \( \Delta_n \).  

A straightforward consequence of this theorem is the following well-known result

**Corollary 15** The usual order function over \( K_n \) has dimension \( n - 1 \), i.e. the transcendence degree of its residue field over \( k \) is \( n - 1 \).

**PROOF.** Let \( \nu \) be the usual order function over \( K_n \). All the residues \( X_i/X_1 + m_\nu \) are transcendental over \( k(X_2/X_1 + m_\nu, \ldots, X_i-1/X_1 + m_\nu) \): if this were not the case, there would exist \( u_i \in \sigma(\Delta_\nu) \) such that \( \nu(X_i - u_iX_1) > 1 \) and \( \nu \) would not be an order function. So \( \Delta_\nu = k(X_2/X_1, \ldots, X_n/X_1) \).  

6 Explicit construction of the residue field: an example

In order to compute explicitly the residue field of a valuation we need to construct a section \( \sigma : \Delta_{\hat{v}} \rightarrow R_{\hat{v}} \) as in remark 5. This procedure is not constructive in general. As in section 1, if the valuation is given as a composition \( v = \nu_t \circ \Psi \), where \( \Psi : k[[X_1, \ldots, X_n]] \rightarrow \Delta[[t]] \) is an injective homomorphism and \( \nu_t \) is the order function in \( \Delta[[t]] \), then we can construct \( \sigma \) using the coefficients \( a_{i,j} \in \Delta \) of \( \Psi(X_i) = \sum_{j \geq 1} a_{i,j}t^j \).

(Of course, explicit does not mean effective because we are working with the series \( \sum_{j \geq 1} a_{i,j}t^j \) and this input is not finite).

**Example 16** Let us consider the embedding

\[
\Psi : \mathbb{C}[[X_1, X_2, X_3, X_4, X_5]] \rightarrow \Delta[[t]]
\]

\[
X_1 \rightarrow t \\
X_2 \rightarrow T_2t \\
X_3 \rightarrow T_2^2t + T_2t^2 + T_3t^3 \\
X_4 \rightarrow T_2^3t + T_2^2t^2 + T_3t^3 + T_4t^4 \\
X_5 \rightarrow T_2t \sum_{j \geq 1}(T_4^{1/p}t)^j,
\]

with \( t, T_2, T_3 \) and \( T_4 \) variables over \( \mathbb{C} \), \( p \in \mathbb{Z} \) prime and \( \Delta \) is a field such that \( \mathbb{C}(T_4)/(T_2, T_3) \subseteq \Delta \). \( \mathbb{C}(T_4) \) is the algebraic closure of \( \mathbb{C}(T_4) \). We are going to denote its extension to the quotient fields by \( \Psi \). The composition of this injective homomorphism with the order function in \( t \) gives a discrete valuation
of $\mathbb{C}((X_1, X_2, X_3, X_4, X_5))/\mathbb{C}$, $v = \nu \circ \Psi$. The residues of $X_i/X_1$ are not in $\mathbb{C}$ for $i = 2, 3, 4, 5$.

Let us put $u_2 = \sigma(X_2/X_1 + m_v)$, a transcendental element over $\mathbb{C}$. By remark 5 we know how to construct $\sigma$ step by step, so let take us $u_2 = X_2/X_1$ and $\Delta_2 = \mathbb{C}(u_2)$.

The residue $X_3/X_1 + m_v$ is algebraic over $\mathbb{C}(u_2)$, in fact

$$\frac{X_3}{X_1} + m_v = \frac{X_2^2}{X_1^2} + m_v.$$

So we can take $u_{3,1} = \sigma((X_3/X_1) + m_v) = u_2^2$. The value of $X_3 - u_{3,1}X_1$ is 2, therefore we have to see if the residue

$$\frac{X_3 - u_{3,1}X_1}{X_1^2} + m_v$$

is algebraic over $\mathbb{C}(u_2)$. We have that

$$\frac{X_3 - u_{3,1}X_1}{X_1^2} + m_v = \frac{X_2}{X_1} + m_v,$$

so it is algebraic and we can take $u_{3,2} = u_2$. Now $v(X_3 - u_{3,1}X_1 - u_{3,2}X_1^2) = 3$ and we have to check if

$$\frac{X_3 - u_{3,1}X_1 - u_{3,2}X_1^2}{X_1^3} + m_v$$

is algebraic over $\Delta_2$. In this case, as

$$\Psi \left( \frac{X_3 - u_{3,1}X_1 - u_{3,2}X_1^2}{X_1^3} + m_v \right) = T_3,$$

this residue is transcendental. So we take

$$u_3 = \sigma \left( \frac{X_3 - u_{3,1}X_1 - u_{3,2}X_1^2}{X_1^3} + m_v \right) = \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^4}.$$

Let us take $\Delta_3 = \mathbb{C}(u_2, u_3)$.

We have to apply this procedure to $X_4$. The residue $X_4/X_1 + m_v$ is algebraic over $\Delta_3$ because

$$\frac{X_4}{X_1} + m_v = \frac{X_2^3}{X_1^3} + m_v,$$

so we can take $u_{4,1} = \sigma((X_4/X_1) + m_v) = u_2^3 \in \Delta_3$. Now $v(X_4 - u_{4,1}X_1) = 2$, and we have to check what happens with the residue

$$\frac{X_4 - u_{4,1}X_1}{X_1^2} + m_v.$$
As
\[
\frac{X_4 - u_{4,1}X_1}{X_1^2} + m_v = \frac{X_2}{X_2^2} + m_v,
\]
it holds
\[
u_{4,2} = \sigma \left( \frac{X_4 - u_{4,1}X_1}{X_1^2} + m_v \right) = u_2^2.
\]
Clearly \(v(X_4 - u_{4,1}X_1 - u_{4,2}X_1^2) = 3\) and
\[
\frac{X_4 - u_{4,1}X_1 - u_{4,2}X_1^2}{X_1^3} + m_v = \frac{X_1X_4 - X_3^2 - X_2^2X_2}{X_1^4} + m_v,
\]
therefore
\[
u_{4,3} = \sigma \left( \frac{X_4 - u_{4,1}X_1 - u_{4,2}X_1^2}{X_1^3} + m_v \right) = u_3.
\]
The following residue is transcendental because \(v(X_4 - u_{4,1}X_1 - u_{4,2}X_1^2 - u_{4,3}X_1^3) = 4\) and
\[
\psi \left( \frac{X_4 - u_{4,1}X_1 - u_{4,2}X_1^2 - u_{4,3}X_1^3}{X_1^4} \right) = T_4.
\]
Then we can take
\[
u_4 = \sigma \left( \frac{X_4 - u_{4,1}X_1 - u_{4,2}X_1^2 - u_{4,3}X_1^3}{X_1^4} + m_v \right) =
\]
\[
= \frac{X_2^2X_4 - X_3^2 - X_2^2X_2^2 - X_1^2X_3 - X_2^2X_2 - X_2X_2}{X_1^6}.
\]
So \(\Delta_4 = \mathbb{C}(u_2, u_3, u_4)\).

With the variable \(X_5\) we obtain the next algebraic residues
\[
u_{5,j} = \sigma \left( \frac{X_5 - u_{5,1}X_1 - \cdots - u_{5,j-1}X_1^{j-1}}{X_1^j} + m_v \right) = u_4^{1/p^j}
\]
for all \(j \geq 1\). So we have \(\Delta_5 = \mathbb{C}(u_2, u_3, u_4)(\{u_4^{1/p^j}\}_{j\geq1})\), an algebraic extension of \(\Delta_4\).

Then the residue field of \(v\) is
\[
\Delta_v = \mathbb{C} \left( \frac{X_2}{X_1} + m_v, \frac{X_1X_3 - X_2^2 - X_1^2X_2}{X_1^3} + m_v, \right)
\]
\[
\frac{X_1^2X_4 - X_3^2 - X_1^2X_2^2 - X_1^2X_3 - X_1^2X_2 - X_2X_2}{X_1^6} \left( \left\{ \left( \frac{X_2}{X_1} + m_v \right)^{1/p^j} \right\}_{j\geq1} \right).
\]
In this case, by the transformation

\[
\begin{align*}
X_1 &\to Y_1 \\
X_2 &\to Y_2 \\
X_3 &\to Y_1^2 Y_3 + u_{3,1}Y_1 + u_{3,2}Y_1^2 \\
X_4 &\to Y_1^3 Y_4 + u_{4,1}Y_1 + u_{4,2}Y_1^2 + u_{4,3}Y_1^3 \\
X_5 &\to Y_5,
\end{align*}
\]

we can extend the valuation \( v \) to a discrete valuation \( v' = \nu_t \Psi' \) of \( \mathbb{C}(Y_1, Y_2, Y_3, Y_4, Y_5) \), with the injective homomorphism

\[
\Psi' : \mathbb{C}[Y_1, Y_2, Y_3, Y_4, Y_5] \to \Delta[t]
\]

\[
\begin{align*}
Y_1 &\mapsto t \\
Y_i &\mapsto T_i t, \ i = 2, 3, 4 \\
Y_5 &\mapsto \sum_{j \geq 1} (T_4^{1/p})^j.
\end{align*}
\]

The restriction \( v'_{|\mathbb{C}(Y_1, Y_2, Y_3, Y_4)} \) is the usual order function. This is not the general case because \( \Psi' \) may not be injective.

7 Rank one discrete valuations and order functions

We can summarize the constructions of previous sections in the following theorem which generalize the results of [1,2]

**Theorem 17** Let \( v \) be a discrete valuation of \( K_n|k \), then

(1) If the dimension of \( v \) is \( n - 1 \), we can embed \( k[X_1, \ldots, X_n] \) into a ring \( L[Y_1, \ldots, Y_n] \), where \( L \subset \sigma(\Delta_v) \) and the extended valuation of \( v \) over the field \( L((Y_1, \ldots, Y_n)) \) is the usual order function.

(2) If the dimension of \( v \) is \( m - 1 < n - 1 \), we can embed \( k[X_1, \ldots, X_n] \) into a ring \( L[Y_1, \ldots, Y_m] \), where \( L \subset \sigma(\Delta_v) \) and the restriction into \( L((Y_1, \ldots, Y_m)) \) of the “extended valuation” of \( v \) over \( L((Y_1, \ldots, Y_n)) \) is the usual order function.
PROOF. We have the following map:

$$\varphi' : L_n((Y_1, \ldots, Y_n)) \rightarrow \Delta_n((t))$$

$$Y_1 \mapsto t$$

$$Y_i \mapsto u_i t, \ i = 2, \ldots, m$$

$$Y_k \mapsto \sum_{j \geq 1} u_{k,j} t^j, \ u_{k,1} \neq 0, \ k = m + 1, \ldots, n,$$

where $m - 1$ is the dimension of $v$. Let us prove the theorem:

(1) In the case $m = n$, $\varphi'(Y_i) = u_i t$ for all $i = 2, \ldots, n$. Let $\nu_i$ be the usual order function over $\Delta_n((t))$. The homomorphism $\varphi'$ is injective and the valuation $v' = \nu_t \circ \varphi'$ of $L((Y_1, \ldots, Y_n))$ is the usual order function over this field. Obviously $v'$ extends $\nu$.

(2) If $m < n$ we can consider the elements $W_k = Y_k - \sum_{j \geq 1} u_{k,j} Y_1^j$. Hence we have $L((Y_1, \ldots, Y_n)) = L((Y_1, \ldots, Y_m, W_{m+1}, \ldots, W_n))$. We define the discrete valuation of rank $n - m + 1$ over $L((Y_1, \ldots, Y_n))$:

$$v'(Y_1) = \ldots = v'(Y_m) = (0, \ldots, 0, 1),$$

$$v'(W_{m+1}) = (0, \ldots, 1, 0), \ldots, v'(W_n) = (1, 0, \ldots, 0).$$

The restriction of this valuation to $K_n$ is a rank one discrete valuation, because the value of any element is in $0 \times \cdots \times 0 \times \mathbb{Z}$. In fact $v'(f) = (0, \ldots, 0, v(f))$ for all $f \in K_n$, so $v'$ “extends” $\nu$ in this sense. Obviously $v'_{L((Y_1, \ldots, Y_m))}$ is the usual order function. We want note that this ideal $(W_{m+1}, \ldots, W_n)$ is the implicit ideal of $v$ that appears in some works of M. Spivakovsky ([5]).

For the case of valuations of dimension $n - 1$, we can combine corollary 15 and assertion 1 of the previous theorem:

**Corollary 18** Let $v$ be a discrete valuation of $K_n|k$. The following conditions are equivalent:

1) The transcendence degree of $\Delta_0^v$ over $k$ is $n - 1$.

2) There exists a finite sequence of monoidal transformations and coordinates changes which take $v$ into an order function.
Example 19 Let us consider the homomorphism

\[ \Psi : \mathbb{C}[X_1, X_2, X_3, X_4, X_5] \to \Delta[t] \]

\[ X_1 \mapsto t \]
\[ X_2 \mapsto T_2 t \]
\[ X_3 \mapsto T_2^2 t + T_2 t^2 + T_3 t^3 \]
\[ X_4 \mapsto T_2^2 t + T_2^2 t^2 + T_3 t^3 + T_4 t^4 \]
\[ X_5 \mapsto T_2 t \left( \sum_{j \geq 1} a_j(T_4 t)^j \right) , \]

with \( a_j \in \mathbb{C} \) such that \( \Psi \) is injective (we can take \( \sum_{j \geq 1} a_j(T_4 t)^j = e^{T_4 t} - 1 \)). Then the residue field of this valuation (see example 16) is

\[ \Delta_v = \mathbb{C} \left( \frac{X_2}{X_1} + m_v, \frac{X_1X_3 - X_2^2 - X_1^2 X_2}{X_1^2} + m_v, \right. \]
\[ \left. \frac{X_1^2 X_4 - X_3^2 - X_1^2 X_2^2 - X_1^2 X_3 - X_1^2 X_2}{X_1^2} + m_v \right) . \]

By the transformation (see example 16)

\[ X_1 \to Y_1 \]
\[ X_2 \to Y_2 \]
\[ X_3 \to Y_1^2 Y_3 + u_{3,1} Y_1 + u_{3,2} Y_1^2 \]
\[ X_4 \to Y_1^3 Y_4 + u_{4,1} Y_1 + u_{4,2} Y_1^2 + u_{4,3} Y_1^3 \]
\[ X_5 \to Y_2 Y_5, \]

we obtain a new field \( \mathbb{C}((Y_1, Y_2, Y_3, Y_4, Y_5)) \), but we cannot extend \( \nu \) to this field because the homomorphism

\[ \Psi' : \mathbb{C}[Y_1, Y_2, Y_3, Y_4, Y_5] \to \Delta[t] \]

\[ Y_1 \mapsto t \]
\[ Y_i \mapsto T_i t, \ i = 2, 3, 4 \]
\[ Y_5 \mapsto \sum_{j \geq 1} a_j(T_4 t)^j \]

is not injective. Then let us take \( W_5 = Y_5 - \sum_{j \geq 1} a_j(Y_4)^j \) (because we can consider \( T_4 Y_1 = Y_4 \)). Then \( \mathbb{C}((Y_1, Y_2, Y_3, Y_4, Y_5)) = \mathbb{C}((Y_1, Y_2, Y_3, Y_4, W_5)) \) and the discrete valuation of rank 2 defined by \( \nu'(Y_i) = (0, 1) \) for \( i = 1, \ldots, 4 \) and \( \nu'(W_5) = (1, 0) \) is such that for all \( f \in \mathbb{C}((X_1, X_2, X_3, X_4, X_5)) \) we have \( \nu'(f) = (0, \nu(f)) \) and \( \nu'_{\mathbb{C}((Y_1, Y_2, Y_3, Y_4))} \) is the usual order function.
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