An Ostrogradsky Instability Analysis of Non-minimally Coupled Weyl Connection Gravity Theories

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Abstract. We study the Hamiltonian formalism of the non-minimally coupled Weyl connection gravity (NMCWCG) in order to check whether Ostrogradsky instabilities are present. The Hamiltonian of the NMCWCG theories is obtained by foliating space-time into a real line (representing time) and 3-dimensional space-like hypersurfaces, and by considering the spatial metric and the extrinsic curvature of the hypersurfaces as the canonical coordinates of the theory. Given the fact that the theory we study contains an additional dynamical vector field compared to the usual NMC models, which do not have Ostrogradsky instabilities, we are able to construct an effective theory without these instabilities, by constraining this Weyl field.

1. Introduction

Non-minimally coupled curvature-matter gravity theories (NMC) [1] have interesting features, such as inflationary solutions in cosmology [2], mimicking dark energy and/or dark matter [3–6]. Further interesting properties arise when adding a non-compatible metric, in particular a Weyl connection, to this NMC model. The resulting theory, dubbed as non-minimally coupled Weyl connection gravity (NMCWCG) has been previously studied in Refs. [7, 8]. This paper follows the work done in previous references and focuses on the Hamiltonian of the NMCWCG theories and on searching instabilities, particularly Ostrogradsky instabilities.
On his 1850’s work, Mikhail Ostrogradsky [9] detailed a particular type of instabilities arising on theories described by a Lagrangian depending on higher order derivatives with respect to time. Given that we expect energy-bounded theories, this type of instabilities constrain the field theories since many of them (and, in particular, many gravitation theories) include higher order derivatives. Nevertheless, many of these theories are free from Ostrogradsky instabilities by violating one of the assumptions Ostrogradsky used to prove his theorem. Hence, we seek whether the introduction of a vector field and its dynamics into the NMC model (a theory that does not contain an Ostrogradsky instability) gives rise to this type of instabilities and under which conditions can these instabilities be avoided.

2. Non-minimally Coupled Weyl Connection Gravity

A Weyl connection is a connection that is not compatible with the metric which introduces, through the covariant derivative $D_{\mu}$, a vector field $A_{\mu}$ such that:

$$D_{\lambda}g_{\mu\nu} = A_{\lambda}g_{\mu\nu}.$$  

(2.1)

This covariant derivative can be written as:

$$D_{\lambda}g_{\mu\nu} = \partial_{\lambda}g_{\mu\nu} - \bar{\Gamma}_{\lambda\mu\sigma}g_{\sigma\nu} - \bar{\Gamma}_{\lambda\nu\sigma}g_{\mu\sigma},$$

(2.2)

with connection coefficients given by:

$$\bar{\Gamma}_{\lambda\mu\nu} = \{\lambda_{\mu\nu}\} - \frac{1}{2}g^{\lambda\sigma}(A_{\mu}g_{\nu\sigma} + A_{\nu}g_{\mu\sigma} - A_{\sigma}g_{\mu\nu}),$$

(2.3)

where $\{\lambda_{\mu\nu}\}$ are the usual Christoffel symbols.

We assume that the Weyl vector is a non-abelian field with a field strength given by:

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu} - [A_{\mu}, A_{\nu}],$$

(2.4)

where $[\cdot, \cdot]$ represents the commutator. Naturally, if $A_{\mu}$ is considered to be an abelian field, the second term in the previous expressions vanishes.

The Lagrangian density of the vector field is given, as usual, by:

$$\mathcal{L}_{W}[A_{\mu}, g^{\mu\nu}] = -\frac{1}{4\mu}\text{tr}\{F_{\mu\nu}F^{\mu\nu}\} - V[A],$$

(2.5)

where $\mu$ is equivalent to the electromagnetic permeability, and a potential is admitted.

From Eqs. (2.2) and (2.3), the Riemann tensor can be obtained. Contracting this tensor, we get the Ricci tensor:

$$\mathcal{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{2}g_{\mu\nu}(\nabla_{\lambda} - A_{\lambda})A^{\lambda} - \frac{3}{2}A_{\mu}A_{\nu} - F_{\mu\nu} + \frac{1}{2}E_{\mu\nu},$$

(2.6)
with $R_{\mu\nu}$ the Ricci tensor for the Levi-Civita connection; $\nabla_\mu$ the usual Levi-Civita covariant derivative, $E_{\mu\nu} = \nabla_\mu A_\nu + \nabla_\nu A_\mu + 2 \{ A_\mu, A_\nu \}$ and $\{ \cdot, \cdot \}$ representing the anti-commutator. Finally, the scalar curvature is given by:

$$\bar{R} = R + 3 \nabla^\lambda A_\lambda - \frac{3}{2} A^\lambda A_\lambda,$$

(2.7)

where $R$ is the scalar curvature corresponding to the Levi-Civita connection.

The non-minimally coupled Weyl connection gravity theory (NMCWCG) is a combination of the non-minimally coupled curvature-matter gravity (NMC) and a Weyl connection. Thus, the proposed action for this theory is:

$$S = \int_M \left[ \kappa f_1(\bar{R}) + f_2(\bar{R}) \left( \frac{1}{4\mu} \mathrm{tr} \{ F_{\mu\nu} F^{\mu\nu} \} + V[A] \right) \right] \sqrt{|g|} \; d^4x,$$

(2.8)

where $\kappa = \frac{c^4}{16\pi G}$, $g$ is the determinant of the metric, $f_1$ and $f_2$ are arbitrary functions of the scalar curvature, $\mathcal{L}_m$ is the Lagrangian density of matter fields. This will be the action whose potential Ostrogradsky instabilities we shall analyse.

3. Ostrogradsky’s model and Problem

We start by introducing some basic concepts regarding the structure of the system in order to perform an Ostrogradsky analysis:

**Definition 3.1.** A Lagrangian of the type $L(q_i, \dot{q}_i, \ldots, q_i^{(\alpha_i)})$ is called a high-order Lagrangian if $\alpha_i > 1$ for any $i = 1, \ldots, N$, where $q_i^{(k)}$ represent the $k$-th time derivative of $q_i$; A Lagrangian $L(q_i, \dot{q}_i, \ldots, q_i^{(\alpha_i)})$ such that $\frac{\partial^2 L}{\partial (q_i^{(\alpha_i)})^2} \neq 0$ is said to be non-degenerate.

Hence, Ostrogradsky’s theorem of instabilities follow:

**Theorem 3.1.** Let $L(q_i, \dot{q}_i, \ldots, q_i^{(\alpha_i)})$ be a non-degenerate high-order Lagrangian of a given physical system. Then, the system has unbounded states of energy.

The proof of this statement, which follows the ideas of Ref. [10], starts by reducing the Lagrangian of this system to an equivalent one, but only in first order in the derivatives of the coordinates. Thus, allowing the phase space coordinates to be

$$Q_i^{k_i} = q_i^{(k_i-1)},$$

(3.1)

$$P_i^{k_i} = \frac{\partial L}{\partial (q_i^{(k_i)})} + \sum_{p_i=1}^{\alpha_i-k_i} \left( -\frac{d}{dt} \right) p_i \frac{\partial L}{\partial (q_i^{(p_i+k_i)})},$$

(3.2)

for all $i$ and $k_i = 1, \ldots, \alpha_i - 1$, we can compute the Hamiltonian for the system. Taking into account that by hypothesis, the Lagrangian is non-degenerate, the equation $P_i^{\alpha_i} = \frac{\partial L}{\partial (q_i^{(\alpha_i)})}$ has the solution

$$\dot{Q}_i^{\alpha_i} = q_i^{(\alpha_i)} = \chi_i(Q_j^1, \ldots, Q_j^{\alpha_j}, P_j^{\alpha_j}).$$

(3.3)
Thus, the Hamiltonian is given by:

\[ H(Q, P) = \sum_{i=1}^{N} \left( \sum_{k_i=1}^{\alpha_i-1} P_i^{k_i} \dot{Q}_i^{k_i} + P_i^{\alpha_i} \chi_i \right) - L(Q_1^1, \ldots, Q_1^{\alpha_1}, \chi_1), \quad (3.4) \]

where we see that the dependence of \( H \) on the generalised momenta, \( P_1^1, \ldots, P_1^{\alpha_1-1}, \) appears only linearly. This means that, because this Hamiltonian is a constant of motion, we can choose a configuration of the system that has \( P_1^1, \) for example, as small as we want and, naturally, the energy as negative as we want.

This way to prove Theorem 3.1, follows Ostrogradsky’s own proposal for canonical coordinates. However, there is a more general way to prove this theorem if we instead assume the following coordinate transformation:

\[ Q_0^i = q_i, \quad Q_1^{\beta_i} = \xi_i^{\beta_i}(q_j, \dot{q}_j, \ldots, q_j^{(\theta_i)}) , \quad \theta_{ij} = \min\{\beta_i, \alpha_j - 1\}, \quad (3.5) \]

with \( \chi_i^{\beta_i} \) invertible. Then, by following the ideas of Refs. [11–13], the canonical Hamiltonian is given by:

\[ H_c(Q, P) = \sum_{i=1}^{N} \left( \sum_{\beta_i=0}^{\alpha_i-2} P_i^{\beta_i} Q_i^{\beta_i} + P_i^{\alpha_i} \dot{Q}_i^{\alpha_i} \right) - L(Q, \dot{Q}), \quad (3.6) \]

with,

\[ P_j^{\beta_j} = \frac{\partial}{\partial (Q_j^{\beta_j})} \left( L(Q_i^{\beta_i}, \dot{Q}_i^{\alpha_i}) + \sum_{j=1}^{N} \sum_{\beta_j=0}^{\alpha_i-1} (\dot{Q}_j^{\beta_j} - Q_j^{\beta_j}) \lambda_j^{\beta_j} \right), \quad (3.7) \]

where \( Q_i^{\beta_i}(Q_j^0, \ldots, Q_j^{\beta_j}) = \dot{Q}_i^{\beta_i} \) is obtained by inverting Eq. (3.5) and reintroducing it in the derivative of Eq. (3.5) where \( \lambda_j^{\beta_j} \) are Lagrange multipliers. Once again, Eq. (3.6) contains Ostrogradsky instabilities (the second term inside parenthesis). This second procedure more closely resembles the ideas we shall use when searching for Ostrogradsky instabilities in the NMCWCG model.

4. A Space-time Foliation

The first procedure in order to establish whether the NMCWCG model contains Ostrogradsky instabilities is the foliation2 of space-time into a real line guided by a time-like 4-vector \( t = t^\mu \partial_\mu \) and a family of spatial hypersurfaces \( \Sigma_t \), indexed by time, with spatial metric \( h_{\mu\nu} \) and normal (unit) 4-vector \( n = n^\mu \partial_\mu \). The present section follows closely the Chapter 14 of Ref. [14] as well as the Section 3.3 of Ref. [11]. Both references treated the case \( D_\lambda g_{\mu\nu} = 0 \). We, instead, consider the more general case: \( D_\lambda g_{\mu\nu} \neq 0 \).

1. This comes from the hypothesis that the Lagrangian does not dependent explicitly on time.
2. This foliation is necessary since we require the Hamiltonian of the NMCWCG theory and this formalism depends on singling out time (which we shall denote as \( t \)).
3. In this section we use the notation \( D \) for a generic covariant derivative. Only in the next section we shall recall it as the Weyl connection. For this reason, we shall call \( \hat{R}^\lambda_{\mu\nu\sigma} \) the Riemann tensor corresponding to the connection \( D \).
Firstly, we may relate \( t^\mu \) and \( n^\mu \) as
\[
t^\mu = N n^\mu + N^\mu, \tag{4.1}
\]
where \( N \) and \( \mathbf{N} = N^\mu \partial_\mu \) are the lapse function and shift vector, respectively. The metric tensor can be expressed as:
\[
g_{\mu\nu} = h_{\mu\nu} - n_\mu n_\nu, \tag{4.2}
\]
Notice that we can compute relevant quantities in two different bases for the manifold, \( M = \mathbb{R} \times \Sigma_t \):

(i) The standard basis \( \{ \partial_\mu \mid \mu = 0, \ldots, 3 \} \);
(ii) A basis \( \{ \mathbf{e}_\tilde{a} \mid \tilde{a} = 1, \ldots, 3 \} \) of \( \Sigma_t \) together with the vector \( \mathbf{n} \).

For convenience, elements of the item (ii) are always denoted with a tilde. Thus, basis (ii) can be written in a compact form as:
\[
\{ \mathbf{e}_\tilde{\mu} \mid \tilde{\mu} = 0, \ldots, 3 \} = \{ \mathbf{e}_{\tilde{a}} \mid \tilde{a} = 1, 2, 3 \} \cup \{ \mathbf{n} \}, \quad \text{and} \quad \mathbf{e}_{\tilde{0}} = \mathbf{n}. \tag{4.3}
\]
The definitions we have introduced so far allows us to define the derivative with respect to time as the Lie derivative in the direction of the vector \( \mathbf{t} \):
\[
\frac{d}{dt} \equiv \mathcal{L}_\mathbf{t}. \tag{4.4}
\]
Furthermore, we can define an extrinsic curvature of \( \Sigma_t \), with components\(^4\):
\[
K_{\tilde{a}\tilde{b}} = -h_{\tilde{a}\tilde{b}}' h_{\tilde{b}\tilde{a}}' \mathbf{g} \left( \mathbf{e}_\mu, D_\mu \mathbf{n} \right). \tag{4.5}
\]
We can simplify Eq. (4.5) in two ways: first, by using the definition of the basis (ii); and by computing the covariant derivative of the metric \( D_\tilde{a} \mathbf{g}(\mathbf{e}_{\tilde{a}}, \mathbf{n}) \). These yield:
\[
K_{\tilde{a}\tilde{b}} = -\Gamma_{\tilde{b}\tilde{a}\tilde{0}} = (D_\tilde{a} \mathbf{g})(\mathbf{e}_{\tilde{a}}, \mathbf{n}) - \Gamma_{\tilde{b}\tilde{a}}. \tag{4.6}
\]
In order to complete this description, we must express the scalar curvature of \( M \) written in terms of the scalar curvature of the spatial hypersurfaces \( \Sigma_t \), \( \bar{R} \), the extrinsic curvature, \( K_{\tilde{a}\tilde{b}} \) and its derivatives as well as derivatives of the spatial metric \( h_{\tilde{a}\tilde{b}} \).

4. In Eq. (4.5) we have used the metric as the linear map \( \mathbf{g} : TM \times TM \to TM \), where \( TM \) is the tangent bundle of the space-time manifold \( M \). In the following sections, we use this definition more often.
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Using the definition of the Riemann tensor with respect to the connection coefficients\(^5\),

\[
\hat{\mathcal{R}}^\lambda_{\mu\sigma\nu} = 2\partial_{(\sigma} \Gamma^\lambda_{\nu)\mu} + 2\Gamma^\lambda_{\alpha(\sigma} \Gamma^\alpha_{\nu)\mu}, \tag{4.7}
\]

and by embedding it in \(\Sigma_t\), we obtain

\[
\hat{\mathcal{R}}^a_{\ bcd} = \tilde{\mathcal{R}}^a_{\ bcd} + 2\Gamma^a_{\ [a} \Gamma^a_{\ b]}. \tag{4.8}
\]

Turning Eq. (4.8) into an equivalent covariant expression and contracting to obtain the scalar curvature, we get

\[
\hat{R} = \hat{R} + K^2 - K^{\alpha\beta} K_{\alpha\beta} - 2\hat{R}_{\alpha\beta} n^\alpha n^\beta + K g^{\beta\delta} (D_\beta g)(e_\delta, n) - K^{\alpha\beta} (D_\beta g)(e_\alpha, n), \tag{4.9}
\]

where \(\hat{R}_{\alpha\beta}\) is the Ricci tensor. After simplifying the term \(\hat{R}_{\alpha\beta} n^\alpha n^\beta\), Eq. (4.9) becomes

\[
\hat{R} = \hat{R} + K^2 - 3K^{\alpha\beta} K_{\alpha\beta} - 2h^{\alpha\beta} \mathcal{L}_n K_{\alpha\beta} - 2(D_n h^{\alpha\beta}) K_{\alpha\beta} - K(D_n g)(n, n) - Kh^{\alpha\beta} (D_\beta g)(e_\alpha, n) + K^{\alpha\beta} (D_\beta g)(e_\alpha, n), \tag{4.10}
\]

where \(D_n \equiv n^\nu D_\nu\).

Finally, by using properties of the Lie derivative and definition (4.2), we find that

\[
(L_t h)(\partial_\mu, \partial_\nu) = (D_t h)(\partial_\mu, \partial_\nu) - 2N K_{(\mu\nu)} + 2h_{\alpha(\mu} D_{\nu)} N^\alpha. \tag{4.11}
\]

5. Ostrogradsky Instabilities in the NMCWCG theory

When studying the NMCWCG model, we must first choose the canonical coordinates. Taking into account the study of the previous section, the natural choices for the canonical coordinates are

\[
(q_0)_{\bar{a}\bar{b}} = \hat{h}_{\bar{a}\bar{b}}, \quad (q_1)_{\bar{a}\bar{b}} = K_{\bar{a}\bar{b}}, \tag{5.1}
\]

constrained to Eq. (4.11). Then, we can compute the extended Lagrangian by introducing a Lagrange multiplier to set the constraint:

\[
\mathcal{L}^* = N\sqrt{h} \mathcal{L} + \lambda^{\bar{a}\bar{b}} \left( L_t (q_0)_{\bar{a}\bar{b}} + 2N(q_1)_{\bar{a}\bar{b}} - 2(q_0)_{\bar{a}\bar{b}} \partial_{(\bar{a}} D_{\bar{b})} N^{\bar{a}\bar{b}} \right), \tag{5.2}
\]

where \(\mathcal{L}\) is the Lagrangian of the NMCWCG model, Eq. (2.8), that is the term inside square brackets. After Eq. (5.2), we drop \((q_0)_{\bar{a}\bar{b}}\) and \((q_1)_{\bar{a}\bar{b}}\) in favour of the more intuitive \(h_{\bar{a}\bar{b}}\) and \(K_{\bar{a}\bar{b}}\).6

5. In this definition we have used that, given two tensors \(T_\mu\) and \(S_\mu\), \(T_\mu S_\mu \equiv T_\mu S_\mu - T_\mu S_\mu\). In a later expression we shall use the similar definition \(T_{(\mu S_{\nu)} \equiv T_{\mu} S_{\nu} + T_{\nu} S_{\mu}\).

6. Notice that now \(D_\lambda = D_\lambda\) and \(R^\lambda_{\mu\nu\sigma} = R^\lambda_{\mu\nu\sigma}\), etc.
Before obtaining the canonical momenta, we notice that the existence of the Weyl connection still admits that \( (D_y g)(e_n, n) = A_\gamma g(e_n, n) = 0 \) since \( n \) is a normal vector to \( \Sigma \). Also, we can compute the covariant derivative of \( h^{\mu \nu} \) as

\[
D_n g^{\mu \nu} = -A(n) g^{\mu \nu} \implies D_n h^{\mu \nu} = -A(n) (h^{\mu \nu} - 2n^\mu n^\nu),
\]

where \( A(n) = A_\alpha n^\alpha = A_\tilde{\alpha} \) and, since we only work with spatial components \( \tilde{a} \tilde{b} \), the second term inside parenthesis vanishes. Thus, the scalar curvature \( \bar{R} \) is given by:

\[
\bar{R} = \tilde{R} + K^2 - 3K^\alpha \tilde{\beta} K_{\alpha \beta} - 2h^{\tilde{\alpha} \tilde{b}} L_n K_{\tilde{a} \tilde{b}} + 3KA(n). \tag{5.4}
\]

Now, we proceed with the computation of the Hamiltonian of the NMCWCG model. The conjugate momenta to the canonical coordinates in Eq. (5.1) are\(^7\)

\[
p_0^{\tilde{a} \tilde{b}} = \frac{\partial L^*}{\partial \dot{h}_{\tilde{a} \tilde{b}}} = \lambda^{\tilde{a} \tilde{b}}, \quad p_1^{\tilde{a} \tilde{b}} = \frac{\partial L^*}{\partial \dot{K}_{\tilde{a} \tilde{b}}} = -2\kappa \sqrt{h} \Theta^{\tilde{a} \tilde{b}}, \tag{5.5}
\]

with \( \Theta = f'_1(\tilde{R}) + \kappa f'_2(\tilde{R}) \mathcal{L} \), where the prime represents the derivative with respect to \( \tilde{R} \). Similarly, we find the canonical momenta for the matter fields,

\[
\Pi^{\mu \alpha} = \frac{\partial L^*}{\partial \dot{A}_\alpha^\mu} = N\sqrt{h} f_2(\tilde{R}) \pi^{\mu \alpha}, \tag{5.6a}
\]

\[
\Pi = \frac{\partial L^*}{\partial \dot{\Psi}_m} = N\sqrt{h} f_2(\tilde{R}) \pi_m, \tag{5.6b}
\]

where \( \pi^{\mu \alpha} = \frac{\partial}{\partial A_\alpha^\mu} \left( -\frac{1}{4} F_{\mu \nu} F^{\mu \nu} \right) \) is the usual conjugate momentum found in Yang-Mills theory and, similarly, \( \pi_m = \frac{\partial L^*}{\partial \dot{\Psi}_m} \). The canonical Hamiltonian is given by:

\[
H_c = \int_{\Sigma_t} \left[ p_0^{\tilde{a} \tilde{b}} \dot{h}_{\tilde{a} \tilde{b}} + p_1^{\tilde{a} \tilde{b}} \dot{K}_{\tilde{a} \tilde{b}} + \Pi^{\mu \alpha} \dot{A}_\alpha^\mu + \Pi \dot{\Psi}_m - N\sqrt{h} (\kappa f_1(\tilde{R}) + f_2(\tilde{R}) \mathcal{L}) \right] d^3x. \tag{5.7}
\]

To obtain an explicit expression of the canonical Hamiltonian, we start using the definition of \( p_1^{\tilde{a} \tilde{b}} \), Eq. (5.5), and invert Eq. (5.4). Thus Eq. (5.7) becomes:

\[
H_c = \int_{\Sigma_t} \left[ p_0^{\tilde{a} \tilde{b}} \dot{h}_{\tilde{a} \tilde{b}} + \kappa N\sqrt{h} \Theta \left( \tilde{R} - \tilde{R} - K^2 + 3K^{\tilde{a} \tilde{b}} K_{\tilde{a} \tilde{b}} - 3KA(n) - \frac{2h^{\tilde{a} \tilde{b}}}{N} L_n K_{\tilde{a} \tilde{b}} \right) \right. \\
\left. - N\sqrt{h} (\kappa f_1(\tilde{R}) + f_2(\tilde{R}) \mathcal{L}) \right] d^3x, \tag{5.8}
\]

with

\[
\mathcal{E} = \dot{\Psi}_m \frac{\partial L}{\partial \dot{\Psi}_m} + \dot{A}_\alpha^\mu \frac{\partial L}{\partial \dot{A}_\alpha^\mu} - \mathcal{L}. \tag{5.9}
\]

Grouping together similar terms in Eq. (5.8), we obtain the canonical Hamiltonian as:

\[
H_c = \int_{\Sigma_t} \left[ \kappa N\sqrt{h} \left( \Theta \tilde{R} - f_1(\tilde{R}) - \frac{f_2(\tilde{R})}{\kappa} \mathcal{E} \right) + \frac{N}{6} p_1(\tilde{R} + K^2 - 3K_{\tilde{a} \tilde{b}} + 3KA(n)) \right. \\
\left. + \frac{h^{\tilde{a} \tilde{b}}}{3} p_1(L_n K_{\tilde{a} \tilde{b}}) + p_0^{\tilde{a} \tilde{b}} \left( D\dot{h}_{\tilde{a} \tilde{b}} - 2NK_{\tilde{a} \tilde{b}} + 2h_{\tilde{c}(\tilde{a})} D\dot{\tilde{b}} \tilde{K}_{\tilde{c} \tilde{b}} \mathcal{E} \right) \right] d^3x. \tag{5.10}
\]

\(^7\) In Eq. (5.5) we are using the dot as the time derivative following Eq. (4.4).
The structure of Eq. (5.10) is similar to the one found in General Relativity: 
\[ H_{GR} = \int_{\Sigma} \left( N \mathcal{H} + N^a \mathcal{H}_a \right) d^3x, \]
i.e. it is a sum of constraints. These two terms are the super-Hamiltonian, \( \mathcal{H} \), and the super-momentum, \( \mathcal{H}_a \). Since the super-momentum concerns Lorentz invariance, the Ostrogradsky instabilities appear only on the super-Hamiltonian of the theory, thus, we need only to search for the terms that depend on \( N \) and not in \( N \). The super-Hamiltonian of the NMCWCG model is, therefore,

\[ \mathcal{H} = \kappa \sqrt{h} \left[ \Theta(\bar{R}) - f_1(\bar{R}) - \frac{f_2(\bar{R})}{\kappa} \mathcal{E} \right] + \frac{p_1}{6} (\bar{R} + K^2 - 3K_{\bar{a}\bar{b}}K^{\bar{a}\bar{b}}) - 2p_{\bar{0}}^{\bar{a}\bar{b}}K_{\bar{a}\bar{b}} + \frac{1}{2}p_1KA(n) + p_{\bar{0}}^{\bar{a}\bar{b}}D_hh_{\bar{a}\bar{b}}. \]  
(5.11)

As seen in Section 3, the Ostrogradsky instability appears as oddly powered terms of momenta in the Hamiltonian. By solving second class constraints (see Ref. [11] for further details), \( p_{\bar{0}}^{\bar{a}\bar{b}} = \sqrt{h}K_{\bar{a}\bar{b}} \) and \( p_1^{\bar{a}\bar{b}} = -2\sqrt{h}h^{\bar{a}\bar{b}} \) which means that, when searching for Ostrogradsky instabilities, it is equivalent to look for odd powered terms in the extrinsic curvature. Analysing Eq. (5.11), we find two terms that are linear in the extrinsic/canonical momentum: \( KA(n) \) and \( p_{\bar{0}}^{\bar{a}\bar{b}}D_hh_{\bar{a}\bar{b}} \). These terms give us an expression that constrains the system so to be free from Ostrogradsky instabilities:

\[ \frac{1}{2}p_1KA(n) + p_{\bar{0}}^{\bar{a}\bar{b}}D_hh_{\bar{a}\bar{b}} = 0. \]  
(5.12)

By using Eq. (2.1),

\[ D_hh_{\mu\nu} = A(n)(h_{\mu\nu} + 2n_\mu n_\nu), \]  
(5.13)

which means we can rewrite Eq. (5.12) as:

\[ \left( \frac{1}{2}p_1 + \sqrt{h} \right)KA(n) = 0. \]  
(5.14)

Since the term inside parenthesis does not vanish, because \( p_1 = -6\sqrt{h} \), and the solution \( K = 0 \) is uninteresting, the only possible solution that discards an Ostrogradsky instability is \( A(n) = A\alpha n^\alpha = 0 \). Written in the basis (ii) introduced in Section 4,

\[ A_\alpha = 0, \]  
(5.15)

which constrains the effective theories arising from the NMCWCG model to those where the Weyl vector has only spatial components, \( A_\mu = (0, A_a) \).

### 6. Conclusion

In this work, we search for Ostrogradsky instabilities in the non-minimally coupled Weyl connection gravity (NMCWCG) by studying its Hamiltonian formulation. To achieve this purpose, we foliate space-time into a real line guided by a time-like vector and 3-dimensional spatial Cauchy surfaces. The foliation of space-time allows us to define a spatial metric and an extrinsic curvature which we consider to be the
canonical coordinates of the system, constrained to Eq. (4.11). Using this definition of the canonical coordinates, we study the Hamiltonian formalism.

For the NMCWCG model, we find that the super-Hamiltonian contains linear terms on the conjugate momenta, which would mean that this theory contains Ostrogradsky instabilities. However, since the system contains an arbitrary vector field, we can constrain it in order to avoid the instabilities arising from the problematic terms in the super-Hamiltonian. This is accomplished by imposing the condition $A_0 = 0$ (the Weyl field has no ‘time’ component). This is a relevant result since the difference between the theory studied in this work and the one from Ref. [7] is the existence of dynamics of the Weyl field. Hence, the introduction of this dynamical term in the Lagrangian of the theory induces a constraint on the field which limits the number of consistent effective theories that are free from Ostrogradsky instabilities.

[1] O. Bertolami, C. G. Boehmer, T. Harko and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007) [arXiv:0704.1733 [gr-qc]].
[2] C. Gomes, J. G. Rosa and O. Bertolami, JCAP **1706** (2017) 021 doi:10.1088/1475-7516/2017/06/021 [arXiv:1611.02124 [gr-qc]].
[3] O. Bertolami, P. Frazão and J. Páramos, Phys. Rev. D **81** (2010) 104046 [arXiv:1003.0850 [gr-qc]].
[4] O. Bertolami, P. Frazão and J. Páramos, Phys. Rev. D **83** (2011) 044010 [arXiv:1010.2698 [gr-qc]].
[5] O. Bertolami, P. Frazão and J. Páramos, Phys. Rev. D **86** (2012) 044034 [arXiv:1111.3167 [gr-qc]].
[6] O. Bertolami and J. Páramos, Journal of Cosmology and Astroparticle Physics 1003 (2010), 009 [arXiv:0906.4757]
[7] C. Gomes and O. Bertolami, Class. Quant. Grav. **36** (2019) no. 23, 235016 [arXiv:1812.04976 [gr-qc]].
[8] R. Baptista and O. Bertolami, Class. Quant. Grav. **37** (2020) no. 8, 085011 [arXiv:1911.04983 [gr-qc]].
[9] M. Ostrogradsky; Mem. Ac. St. Petersbourg **VI** 4 (1850) 385
[10] R. F. Woodard, Ostrogradsky’s theorem on Hamiltonian instability, Scholarpedia 10 (2015) 32243, arXiv:1506.02210.
[11] L. Querella, Variational principles and cosmological models in higher order gravity, Ph. D. thesis, University of Liege, 1998, arXiv:gr-qc/9902044
[12] M. Henneaux, C. Teitelboim; Quantization of Gauge Systems; Princeton University Press, 1992
[13] M. Oksanen, Hamiltonian Analysis of Modified Gravitational Theories: Towards a Renormalizable Theory of Gravity, Ph. D. thesis, University of Helsinki, 2013
[14] Øyvind Gron, Sighjorn Hervik; Einstein’s Theory of General Relativity with Modern Applications in Cosmology; Springer, 2007.
[15] E. Gourgoulhon, 3+1 formalism and bases of numerical relativity, arXiv:gr-qc/0703035 [GR-QC].