Some extremal problems for hereditary properties of graphs

Vladimir Nikiforov*

May 7, 2013

Abstract

Let \( P \) be an infinite hereditary property of graphs. Define

\[
\pi (P) = \lim_{n \to \infty} \left( \frac{n}{2} \right)^{-1} \max \{ e(G) : G \in P \text{ and } v(G) = n \}.
\]

In this note \( \pi (P) \) is determined for every hereditary property \( P \).

The same problem is studied for a more general parameter \( \lambda^{(\alpha)}(G) \), defined for every real number \( \alpha \geq 1 \) and every graph \( G \) as

\[
\lambda^{(\alpha)}(G) = \max_{|x_1|^\alpha + |x_2|^\alpha + \cdots + |x_n|^\alpha = 1} \sum_{\{u,v\} \in E(G)} x_u x_v.
\]

It is known that the limit

\[
\lambda^{(\alpha)}(P) = \lim_{n \to \infty} n^{2/\alpha - 2} \max \{ \lambda^{(\alpha)}(G) : G \in P \text{ and } v(G) = n \}
\]

exists. A key result of the note is the equality

\[
\lambda^{(\alpha)}(P) = \pi (P),
\]

which holds for all \( \alpha > 1 \).

1 Introduction

In this note we study problems stemming from the following one:

What is the maximum number of edges a graph of order \( n \), belonging to some hereditary property \( P \).

Let us recall that a hereditary property is a family of graphs closed under taking induced subgraphs. For example, given a set of graphs \( \mathcal{F} \), the family of all graphs that do not contain any \( F \in \mathcal{F} \) as an induced subgraph is a hereditary property, denoted as \( \text{Her}(\mathcal{F}) \).

It seems that the above classically shaped problem has been disregarded in the rich literature on hereditary properties, so we fill in this gap below.

Writing \( P_n \) for the set of all graphs of order \( n \) in a property \( P \), our problem now reads as: Given a hereditary property \( P \), find

\[
ex (P, n) = \max_{G \in P_n} e(G).
\]

Finding \( ex (P, n) \) exactly seems hopeless for arbitrary \( P \). A more feasible approach has been suggested by Katona, Nemetz and Simonovits in [7] who proved the following fact:

*Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA; email: vnikifrv@memphis.edu
Proposition 1  If $\mathcal{P}$ is a hereditary property, then the sequence

$$\left\{ \text{ex}\left(\mathcal{P}, n\right) \left(\frac{n}{2}\right)^{-1} \right\}_{n=1}^{\infty}$$

is nonincreasing and so the limit

$$\pi(\mathcal{P}) = \lim_{n \to \infty} \text{ex}\left(\mathcal{P}, n\right) \left(\frac{n}{2}\right)^{-1}$$

always exists.

One of the aims of this paper is to establish $\pi(\mathcal{P})$ for every $\mathcal{P}$, but our main interest is in extremal problems about a different graph parameter, denoted by $\lambda^{(\alpha)}(G)$ and defined as follows: for every graph $G$ and every real number $\alpha \geq 1$, let

$$\lambda^{(\alpha)}(G) = \max_{|x_1|^\alpha + \cdots + |x_n|^\alpha = 1} 2 \sum_{\{u,v\} \in E(G)} x_u x_v.$$  

Note first that $\lambda^{(2)}(G)$ is the well-studied spectral radius of $G$, and second, that $\lambda^{(1)}(G)$ is another much studied parameter, known as the Lagrangian of $G$. So $\lambda^{(\alpha)}(G)$ is a common generalization of two parameters that have been widely used in extremal graph theory.

The parameter $\lambda^{(\alpha)}(G)$ has been recently introduced and studied for uniform hypergraphs first, by Keevash, Lenz and Mubayi in [6] and next by the author, in [13]. Here we shall study $\lambda^{(\alpha)}(G)$ in the same setting as the number of edges in [11]. Thus, given a hereditary property $P$, set

$$\lambda^{(\alpha)}(\mathcal{P}, n) = \max_{G \in \mathcal{P}_n} \lambda^{(\alpha)}(G).$$

As with $\text{ex}\left(\mathcal{P}, n\right)$ finding $\lambda^{(\alpha)}(\mathcal{P}, n)$ seems hopeless for arbitrary $\mathcal{P}$. So, to begin with, the following theorem has been proved in [13] as an analog to Proposition 1.

Theorem 2 Let $\alpha \geq 1$. If $\mathcal{P}$ is a hereditary property, then the limit

$$\lambda^{(\alpha)}(\mathcal{P}) = \lim_{n \to \infty} \lambda^{(\alpha)}(\mathcal{P}, n) n^{(2/\alpha)-2}$$

exists.

Thus, a natural question is to find $\lambda^{(\alpha)}(\mathcal{P})$ for every $\mathcal{P}$ and every $\alpha \geq 1$. The main goal of this note to answer this question completely.

It turns out that $\lambda^{(\alpha)}(\mathcal{P})$ and $\pi(\mathcal{P})$ are closely related. For example, results proved in [13] imply that $\lambda^{(\alpha)}(\mathcal{P}) \geq \pi(\mathcal{P})$ for every $\mathcal{P}$ and every $\alpha \geq 1$, moreover, if $\alpha \geq 2$, then $\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P})$. In this note we shall extend this relation to: if $\alpha > 1$, then $\lambda^{(\alpha)}(\mathcal{P}) = \pi(\mathcal{P})$.

2 Main results

For notation and concepts undefined here, the reader is referred to [11].

Note first that every hereditary property $\mathcal{P}$ is trivially characterized by $\mathcal{P} = \text{Her}(\overline{\mathcal{P}})$, where $\overline{\mathcal{P}}$ is the family of all graphs that are not in $\mathcal{P}$; however, typically $\mathcal{P}$ can be given as $\mathcal{P} = \text{Her}(\mathcal{F})$ for some $\mathcal{F}$ that is only a small fraction of $\overline{\mathcal{P}}$. 

2
Recall next that a complete \( r \)-partite graph is a graph whose vertices are split into \( r \) nonempty independent sets so that all edges between vertices of different classes are present. In particular, a 1-partite graph is just a set of independent vertices.

To characterize \( \pi(\mathcal{P}) \) and \( \lambda^{(s)}(\mathcal{P}) \) we shall need two numeric parameters defined for every family of graphs \( \mathcal{F} \). First, let
\[
\omega(\mathcal{F}) = \begin{cases} 
0, & \text{if } \mathcal{F} \text{ contains no cliques;} \\
\min \{ r : K_r \in \mathcal{F} \}, & \text{otherwise}, 
\end{cases}
\]
and second, let
\[
\beta(\mathcal{F}) = \begin{cases} 
0, & \text{if } \mathcal{F} \text{ contains no complete partite graphs;} \\
\min \{ r : \mathcal{F} \text{ contains a complete } r\text{-partite graph} \}, & \text{otherwise}.
\end{cases}
\]

The parameters \( \omega(\mathcal{F}) \) and \( \beta(\mathcal{F}) \) are quite informative about the hereditary property \( \text{Her}(\mathcal{F}) \), as seen first in the following observation.

\textbf{Proposition 3} If the property \( \mathcal{P} = \text{Her}(\mathcal{F}) \) is infinite, then \( \omega(\mathcal{F}) = 0 \) or \( \omega(\mathcal{F}) \geq 2 \) and \( \beta(\mathcal{F}) \geq 2 \).

\textbf{Proof} Suppose that \( \omega(\mathcal{F}) \neq 0 \). If \( \omega(\mathcal{F}) = 1 \), then \( \mathcal{P} \) is empty, so we can suppose that \( \omega(\mathcal{F}) \geq 2 \). This implies that \( \beta(\mathcal{F}) > 0 \), as \( \mathcal{F} \) contains \( K_r \) for some \( r \geq 2 \) and \( K_r \) is a complete \( r \)-partite graph. If \( \beta(\mathcal{F}) = 1 \), then \( \mathcal{F} \) contains a graph \( G \) consisting of isolated vertices, say \( G \) is on \( s \) vertices. If \( \mathcal{P} \) is infinite, choose a member \( G \in \mathcal{P} \) with \( v(G) \geq r(K_r, K_s) \), where \( r(K_r, K_s) \) is the Ramsey number of \( K_r \) vs. \( K_s \). Then either \( G \) contains a \( K_r \) or an independent set on \( s \) vertices, both of which are forbidden. It turns out that \( \beta(\mathcal{F}) \geq 2 \), proving Proposition 3.

Clearly the study of (1) and (2) makes sense only if \( \mathcal{P} \) is infinite and Proposition 3 provides necessary condition for this property of \( \mathcal{P} \). The following theorem completely characterizes \( \pi(\mathcal{P}) \).

\textbf{Theorem 4} Let \( \mathcal{F} \) be a family of graphs. If the property \( \mathcal{P} = \text{Her}(\mathcal{F}) \) is infinite, then
\[
\pi(\mathcal{P}) = \begin{cases} 
1, & \text{if } \omega(\mathcal{F}) = 0; \\
1 - \frac{1}{\beta(\mathcal{F}) - 1}, & \text{otherwise}.
\end{cases}
\]

\textbf{Proof} Indeed, since \( \mathcal{P} \) is infinite, Proposition 3 implies that \( \omega(\mathcal{F}) = 0 \) or \( \omega(\mathcal{F}) \geq 2 \) and \( \beta(\mathcal{F}) \geq 2 \). If \( \omega(\mathcal{F}) = 0 \), then \( K_n \in \mathcal{P}_n \), because all subgraphs of \( K_n \) are complete and do not belong to \( \mathcal{F} \). Therefore,
\[
ex(P, n) = \binom{n}{2},
\]
and so, \( \pi(\mathcal{P}) = 1 \). Assume that \( \omega(\mathcal{F}) \geq 2 \) and \( \beta(\mathcal{F}) \geq 2 \), and set for short \( r = \omega(\mathcal{F}) \geq 2 \) and \( \beta = \beta(\mathcal{F}) \). Next, we shall prove that \( T_{\beta-1}(n) \in \mathcal{P}_n \), where \( T_{\beta-1}(n) \) is the complete \((\beta - 1)\)-partite Turán graph of order \( n \). Indeed all subgraphs of \( T_{\beta-1}(n) \) are complete \( r \)-partite graphs for some \( r \leq \beta - 1 \), so should one of them belong to \( \mathcal{F} \), we would have \( \beta(\mathcal{F}) \leq \beta - 1 = \beta(\mathcal{F}) - 1 \), a contradiction. Therefore,
\[
ex(P, n) \geq e(T_{\beta-1}(n)) = \left( 1 - \frac{1}{\beta - 1} + o(1) \right) \binom{n}{2},
\]
and so
\[
\pi(\mathcal{P}) \geq 1 - \frac{1}{\beta(\mathcal{F}) - 1}.
\]
To finish the proof we shall prove the opposite inequality. Let $F \in \mathcal{F}$ be a complete $\beta$-partite graph, known to exist by the definition of $\beta (F)$ and let $s$ be the maximum of the sizes of its vertex classes.

Now assume that $\varepsilon > 0$ and set $t = r (K_r, K_s)$, where $r (K_r, K_s)$ is the Ramsey number of $K_r$ vs. $K_s$. If $n$ is large enough and $G \in \mathcal{P}_n$ satisfies

$$e (G) > \left( 1 - \frac{1}{\beta (F) - 1} + \varepsilon \right) \left( \frac{n}{2} \right),$$

then by the theorem of Erdős and Stone [3], $G$ contains a subgraph $G_0 = K_\beta (t)$, that is to say, a complete $\beta$-partite graph with $t$ vertices in each vertex class. Since $K_r \in F$, we see that $G_0$ contains no $K_r$, hence each vertex class of $G_0$ contains an independent set of size $s$, and so $G$ contains an induced subgraph $K_\beta (s)$, which in turn contains an induced copy of $F$. Hence, if $n$ is large enough and $G \in \mathcal{P}_n$, then

$$e (G) \left( \frac{n}{2} \right)^{-1} \leq 1 - \frac{1}{\beta (F) - 1} + \varepsilon.$$

This inequality implies that

$$\pi (\mathcal{P}) \leq 1 - \frac{1}{\beta (F) - 1},$$

completing the proof.

We continue now with establishing $\lambda (\alpha) (\mathcal{P})$ for $\alpha > 1$. The proof of our key Theorem 7 relies on several other results, some of which are stated within the proof itself. We give two other before the theorem. The first one follows from a result in [13], but for reader’s sake we reproduce its short proof here.

**Theorem 5** Let $\alpha \geq 1$. If $G$ is a graph with $m$ edges and $n$ vertices, with no $K_{r+1}$, then

$$\lambda (\alpha) (G) \leq \left( 1 - \frac{1}{r} \right)^{1/\alpha} (2m)^{1-1/\alpha}$$

and

$$\lambda (\alpha) (G) \leq \left( 1 - \frac{1}{r} \right) n^{2-2/\alpha}.$$  

**Proof** Indeed, let $x = (x_1, \ldots, x_n)$ be a vector such that $|x_1|^\alpha + \cdots + |x_n|^\alpha = 1$ and

$$\lambda (\alpha) (G) = \sum_{\{u,v\} \in E(G)} x_u x_v.$$

Applying Jensen’s inequality, we see that

$$\lambda (\alpha) (G) = 2 \sum_{\{u,v\} \in E(G)} x_u x_v \leq 2 \sum_{\{u,v\} \in E(G)} |x_u| |x_v|$$

$$\leq (2m)^{1-1/\alpha} \left( 2 \sum_{\{u,v\} \in E(G)} |x_u|^\alpha |x_v|^\alpha \right)^{1/\alpha}.$$

But by the result of Motzkin and Straus [8], we have

$$2 \sum_{\{u,v\} \in E(G)} |x_u|^\alpha |x_v|^\alpha \leq 1 - \frac{1}{r},$$

and inequality [4] follows. Now inequality [5] follows from [4] by Turán’s theorem $2m < (1 - 1/r) n^2$. □

We shall need also the following proposition (Proposition 29, [13]) whose proof we omit.
Proposition 6 Let \( \alpha \leq 1, k > 1 \) and \( G_1 \) and \( G_2 \) be graphs on the same vertex set. If \( G_1 \) and \( G_2 \) differ in at most \( k \) edges, then
\[
\left| \lambda^{(\alpha)}(G_1) - \lambda^{(\alpha)}(G_2) \right| \leq (2k)^{1-1/\alpha}.
\]

Here is the main theorem about \( \lambda^{(\alpha)}(\mathcal{P}) \).

Theorem 7 Let \( \alpha > 1 \) and let \( \mathcal{F} \) be a family of graphs. If the property \( \mathcal{P} = \text{Her}(\mathcal{F}) \) is infinite, then
\[
\lambda^{(\alpha)}(\mathcal{P}) = \begin{cases} 
1, & \text{if } \omega(\mathcal{F}) = 0; \\
1 - \frac{1}{\beta(\mathcal{F}) - 1}, & \text{otherwise}.
\end{cases}
\]

Proof First note the inequality
\[
\lambda^{(\alpha)}(G) \geq 2e(G)/n^{2/\alpha},
\]
which follows by taking \((x_1, \ldots, x_n) = (n^{-1/\alpha}, \ldots, n^{-1/\alpha})\) in [2]. So we see that
\[
\lambda^{(\alpha)}(\mathcal{P}) \geq \pi(\mathcal{P}),
\]
and this, together with Theorem 4 gives \( \lambda^{(\alpha)}(\mathcal{P}) = 1 \) if \( \omega(\mathcal{F}) = 0 \) and
\[
\lambda^{(\alpha)}(\mathcal{P}) \geq 1 - \frac{1}{\beta(\mathcal{F}) - 1}
\]
othersise. To finish the proof we shall prove that
\[
\lambda^{(\alpha)}(\mathcal{P}) \leq 1 - \frac{1}{\beta(\mathcal{F}) - 1}
\]

For the purposes of this proof, write \( k_r(G) \) for the number of \( r \)-cliques of \( G \). Let \( F \in \mathcal{F} \) be a complete \( \beta \)-partite graph, which exists by the definition of \( \beta(\mathcal{F}) \), and let \( s \) be the maximum of the sizes of its vertex classes.

We recall the following particular version of the Removal Lemma, one of the important consequences of the Szemerédi Regularity Lemma (3, 1):

Removal Lemma Let \( r \geq 2 \) and \( \varepsilon > 0 \). There exists \( \delta = \delta(r, \varepsilon) > 0 \) such that if \( G \) is a graph of order \( n \), with \( k_r(G) < \delta n^r \), then there is a graph \( G_0 \subset G \) such that \( e(G_0) / e(G) > \varepsilon n^{2} \) and \( k_r(G_0) = 0 \).

In [11] we have proved the following theorem:

Theorem A For all \( r \geq 2 \), and \( \varepsilon > 0 \) there exists \( \delta = \delta(r, \varepsilon) > 0 \) such that if \( G \) a graph of order \( n \) with \( k_r(G) > \varepsilon n^r \), then \( G \) contains a \( K_r(s) \) with \( s = \lfloor \delta \log n \rfloor \).

Now let \( \varepsilon > 0 \), choose \( \delta = \delta(\beta, \varepsilon) \) as in the Removal Lemma, and set \( t = r(K_r, K_s) \), where \( r(K_r, K_s) \) is the Ramsey number of \( K_r \) vs. \( K_s \). If \( G \in \mathcal{P}_n \), then \( K_t \not\subseteq G \) as otherwise we see as in proof of Theorem 4 that \( G \) contains an induced copy of \( F \). So by Theorem A, if \( n \) is large enough, then \( k_{\beta}(G) \leq \delta n^r \). Now by the Removal Lemma there is a graph \( G_0 \subset G \) such that \( e(G_0) / e(G) > \varepsilon n^2 \) and \( k_{\beta}(G_0) = 0 \).

By Propositions 3 and 5 for \( n \) sufficiently large, we see that
\[
\lambda^{(\alpha)}(G) \leq \lambda^{(\alpha)}(G_0) + (2\varepsilon n)^{2-2/\alpha} \leq \left(1 - \frac{1}{\beta - 1}\right)n^{2-2/\alpha} + (2\varepsilon n)^{2-2/\alpha},
\]
and hence,
\[
\lambda^{(\alpha)}(\mathcal{P}, n) n^{2/\alpha - 2} \leq 1 - \frac{1}{\beta - 1} + (2\varepsilon)^{2-2/\alpha}
\]
Since \(\varepsilon\) can be made arbitrarily small, we see that
\[
\lambda^{(\alpha)}(\mathcal{P}) \leq 1 - \frac{1}{\beta - 1},
\]
completing the proof of Theorem 7. \(\square\)

To complete the picture, we need to determine the dependence of \(\lambda^{(1)}(\mathcal{P})\) on \(\mathcal{P}\). Using the well-known idea of Motzkin and Straus, we come up with the following theorem, whose proof we omit.

**Theorem 8** \(\lambda^{(1)}(\mathcal{P})\)

Let \(\mathcal{P}\) be an infinite hereditary property. Then \(\lambda^{(1)}(\mathcal{P}) = 1\) if \(\mathcal{P}\) contains arbitrary large cliques, or \(\lambda^{(1)}(\mathcal{P}) = 1 - 1/r\), where \(r\) is the size of the largest clique in \(\mathcal{P}\).

### 3 Concluding remarks

In a cycle of papers the author has shown that many classical extremal results like the Erdős-Stone-Bolloabs theorem [2], the Stability Theorem of Erdős [3, 4] and Simonovits [14], and various saturation problems can be strengthened by recasting them for the largest eigenvalue instead of the number of edges; see [12] for overview and references.

The results in the present note and in [13] show that some of these results can be extended further for \(\lambda^{(\alpha)}(G)\) and \(\alpha \geq 1\). A natural challenge here is to reprove systematically all of the above problems by substituting \(\lambda^{(\alpha)}(G)\) for the number of edges.

**Acknowledgement** Thanks are due to Bela Bollobás for useful discussions.

### References

[1] B. Bollobás, *Modern Graph Theory*, Graduate Texts in Mathematics, 184, Springer-Verlag, New York (1998), xiv+394 pp.

[2] B. Bollobás and P. Erdős, On the structure of edge graphs, *J. London Math. Soc.* 5 (1973), 317-321.

[3] P. Erdős, Some recent results on extremal problems in graph theory (Results), in: *Theory of Graphs (Internat. Sympos., Rome, 1966)*, pp. 117–130, Gordon and Breach, New York; Dunod, Paris, 1967.

[4] P. Erdős, On some new inequalities concerning extremal properties of graphs, in: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pp. 77–81, Academic Press, New York, 1968.

[5] P. Erdős, A.H. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* 52 (1946), 1087–1091.

[6] P. Keevash, J. Lenz, and D. Mubayi, Spectral extremal problems for hypergraphs, *preprint available at arXiv:1304.0050*. 

6
[7] G. Katona, T. Nemetz and M. Simonovits, On a problem of Turán in the theory of graphs, *Mat. Lapok* **15** (1964), 228–238.

[8] T. Motzkin and E. Straus, Maxima for graphs and a new proof of a theorem of Turán, *Canad. J. Math.*, **17** (1965), 533-540.

[9] B. Nagle, V. Rodl and M. Schacht, The counting lemma for regular k-uniform hypergraphs, *Random Structures Algorithms* **28** (2006), 113-179.

[10] V. Nikiforov, Some inequalities for the largest eigenvalue of a graph. *Combin. Probab. Comput.* **11** (2002), 179–189.

[11] V. Nikiforov, Graphs with many $r$-cliques have large complete $r$-partite subgraphs, *Bull. London Math. Soc.* **40** (2008), 23-25.

[12] V. Nikiforov, Some new results in extremal graph theory, *Surveys in Combinatorics*, Cambridge University Press, 2011, 141–181.

[13] V. Nikiforov, An analytic theory of extremal hypergraph problems, preprint available at ArXiv.

[14] M. Simonovits, A method for solving extremal problems in graph theory, stability problems, in: *Theory of Graphs (Proc. Colloq., Tihany, 1966)*, pp. 279–319, Academic Press, New York, 1968.

[15] E. Szemerédi, Regular partitions of graphs, *In Colloques Internationaux C.N.R.S. No 260 - Problèmes Combinatoires et Théorie des Graphes, Orsay* (1976), pp. 399-401.

[16] P. Turán, On an extremal problem in graph theory (in Hungarian), *Mat. és Fiz. Lapok* **48** (1941) 436-452.