EXISTENCE OF MAXIMIZERS FOR SOBOLEV-STRICHARTZ INEQUALITIES

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Abstract. We prove the existence of maximizers of Sobolev-Strichartz estimates for a general class of propagators, involving relevant examples, as for instance the wave, Dirac and the hyperbolic Schrödinger flows.

1. Introduction

In the recent years, an increasing interest has been devoted to the problem of existence of maximizers for Strichartz inequalities, and more in general Fourier Restriction Theorems. Recall that, given two Banach spaces $\mathcal{X}$, $\mathcal{Y}$, and a linear and continuous operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, it is customary to define a maximizer for $T$ any $x_0 \in \mathcal{X}$ such that

$$\|x_0\|_\mathcal{X} = 1, \quad \|Tx_0\|_\mathcal{Y} = \sup_{\|x\|_\mathcal{X} = 1} \|Tx\|_\mathcal{Y}. \quad (1.1)$$

Notice that the existence (as well as the definition) of maximizers depends on the specific norm introduced on $\mathcal{X}$ and $\mathcal{Y}$.

In this paper we consider the existence of maximizers for a large class of propagators of the form $e^{ith(D)}$, and we focus our attention to cases in which some Sobolev-Strichartz estimates of the following type hold

$$\left\| e^{ith(D)} f \right\|_{L_x^r L_t^s} \leq C \|f\|_{\dot{H}^s}, \quad s > 0, \quad r > 2. \quad (1.2)$$

where in general $f$ is a vector valued function $f = (f_1, \ldots, f_n) : \mathbb{R}^d \to \mathbb{C}^n$ and

$$\|f\|_{\dot{H}^s}^2 = \sum_{j=1}^n \|f_j\|_{\dot{H}^s}^2. \quad (1.3)$$

Concerning the definition of the norm $L_{t,x}^r$ in the l.h.s. of (1.2) recall that in general for any given norm $|\cdot|_{\mathbb{C}^n}$ on $\mathbb{C}^n$ one can define the corresponding mixed Lebesgue norms $L_t^p L_x^q$ for vector valued functions $F = F(t, x) : \mathbb{R}^{1+d} \to \mathbb{C}^n$ as follows

$$\|F(t, x)\|_{L_t^p L_x^q}^p = \left( \int \left( \int |F(t, x)|_{\mathbb{C}^n}^q \, dx \right)^{\frac{p}{q}} \, dt \right)^{\frac{q}{p}},$$

where $1 \leq p, q \leq \infty$. In the case $p = q = r$ we shall write shortly

$$L_t^r L_x^r = L_{t,x}^r.$$
Remark 1.1. Recall that all the norms on $\mathbb{C}^n$ are equivalent and hence also the corresponding norms induced on the spaces $L^p_t L^q_x$ are equivalent. Due to this fact in (1.2) it is not necessary to specify the norm on $\mathbb{C}^n$ with respect to which we are working with (of course provided that the corresponding constant $C$ is suitably modified). On the other hand the existence of maximizers for the inequality (1.2) could be affected in principle by changing the corresponding norm on $\mathbb{C}^n$.

Estimates of the kind (1.2) with a loss of derivatives with respect to the datum $f$, are natural in some relevant cases, as for instance the wave and Dirac equations. Let us point our attention to the Cauchy problem for the following system

$$\tag{1.4} \begin{cases} i\partial_t u + h(D)u = 0 \\ u(0, x) = f(x), \end{cases}$$

where $u(t, x) = (u_1(t, x), \ldots, u_n(t, x)) : \mathbb{R}^{1+d} \to \mathbb{C}^n$, $f(x) = (f_1(x), \ldots, f_n(x)) : \mathbb{R}^d \to \mathbb{C}^n$. Here we denote by $h(D) = F^{-1}(h(\xi)F)$, where $F$ is the standard Fourier transform; in addition, the symbol $h(\xi) \in \mathcal{M}_{n \times n}(\mathbb{C})$, $n \geq 1$, is assumed to be a matrix-valued function $h(\xi) = (h_{ij}(\xi))_{ij=1,\ldots,n}$. In the sequel, we always make the following abstract assumptions:

(H1) there exists $0 < s < \frac{d}{2}$ such that the problem (1.4) is globally well-posed in $\dot{H}^s$, and the unique solution is given via propagator $u(t, x) = e^{ith(D)}f(x)$;

(H2) the flow $e^{ith(D)}$ is unitary onto $\dot{H}^s$, i.e.

$$\left\|e^{ith(D)}f\right\|_{\dot{H}^s} = \|f\|_{\dot{H}^s}, \quad \forall t \in \mathbb{R},$$

where $s$ is the same as in (H1), and $\|\cdot\|_{\dot{H}^s}$ is defined in (1.3).

In order to introduce our problem, assume for the moment that $h_{i,j}(\xi)$ is homogeneous of some degree $k > 0$ for every $i, j \in \{1, \ldots, n\}$, namely

$$\tag{1.5} h_{i,j}(\xi) : \mathbb{R}^d \to \mathbb{C}, \quad h(\lambda \xi) = \lambda^k h(\xi) \quad \forall \lambda > 0.$$

Then, equation (1.4) is invariant under the scaling $v_\lambda(t, x) = u(\lambda^k t, \lambda x)$; as a consequence, if a Strichartz estimate of the following type holds

$$\tag{1.6} \left\|e^{ith(D)}f\right\|_{L^k_t L^q_x} \leq C\|f\|_{\dot{H}^s},$$

for some $s > 0$ and some constant $C > 0$, then $p$ and $q$ have to satisfy the following scaling condition

$$\tag{1.7} \frac{k}{p} + \frac{d}{q} = \frac{d}{2} - s.$$

Moreover, by the Sobolev embedding $\dot{H}^s \subset L^{\frac{2d}{d-s}}$, for $0 < s < d/2$, and the $\dot{H}^s$-preservation

$$\left\|e^{ith(D)}f\right\|_{L^\infty_t L^{\frac{2d}{d-s}}_x} = \|f\|_{\dot{H}^s},$$

(assumption (H2) above), we get

$$\tag{1.8} \left\|e^{ith(D)}f\right\|_{L^\infty_t L^{\frac{2d}{d-s}}_x} \leq C\|f\|_{\dot{H}^s}, \quad 0 < s < \frac{d}{2},$$

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for some constant $C > 0$. Hence, if an estimate of the type (1.6) holds with $p < q$, then by interpolation with (1.8) we obtain

$$\|e^{i\tau h(D)} f\|_{L^r_{t,x}} \leq C \|f\|_{\dot{H}^s}, \quad 0 < s < \frac{d}{2}, \quad r = \frac{2(k + d)}{d - 2s}. \quad (1.9)$$

The aim of this paper is to prove that, as soon as an estimate as (1.9) holds, with a strictly positive $s > 0$, then the best constant of the estimate is achieved by some maximizing functions, independently on the norm which is fixed on the target $C^n$ (and consequently on the corresponding definition of the $L^r_{t,x}$-norm).

This kind of problem has been recently studied by several authors, who treated separately different propagators. We firstly mention Kunze [10], who proved the existence of maximizers of the $L^6_{t,x}$-Strichartz inequality for the 1D Schrödinger propagator. Later, Foschi [6] succeeded in characterizing the best constant of the inequality and also the shape of the maximizers, for the 1D and 2D-Schrödinger propagators. In the same paper, the author treats the $L^6_{t,x}$ (in 2D) and the $L^4_{t,x}$ (in 3D) Strichartz estimates for the wave equation, which hold to the scale of $\dot{H}^{\frac{1}{2}} \times H^{-\frac{1}{2}}$-initial data. Recently, Bez and Rogers [1] computed the best constant and described the shape of maximizers of the $L^6_{t,x}$-Strichartz estimate for the wave equation, with initial data in the energy space $\dot{H}^1 \times L^2$, in space dimension $d = 5$.

We also mention the papers by Shao [14] for the Schrödinger equation and by Bulut [3] for the wave equation, in which the existence of maximizers for anisotropic Strichartz estimates in spaces $L^p_t L^q_x$, for general couples $(p, q)$, is also proved. Finally, Ramos [12] proves that there exist maximizers for the isotropic Strichartz estimates for the wave equation, at the scale $\dot{H}^{\frac{1}{2}} \times H^{-\frac{1}{2}}$.

Here we give a unified (and simple) proof of the existence of maximizers of (1.9), when $s > 0$, which involves a large class of examples of propagators. Our main result is the following.

**Theorem 1.1.** Let assumptions (H1), (H2) be satisfied for some $0 < s < \frac{d}{2}$, and let $h(\xi)$ satisfy (1.5) for some $k > 0$. Moreover, assume that, for some $2 \leq p < q \leq \infty$

$$\|e^{i\tau h(D)} f\|_{L^p_t L^q_x} \leq C \|f\|_{\dot{H}^s}, \quad \text{so that, for } r = \frac{2(k + d)}{d - 2s}, \quad \text{we also have}$$

$$\|e^{i\tau h(D)} f\|_{L^r_{t,x}} \leq M \|f\|_{\dot{H}^s}. \quad (1.11)$$

with

$$M := \sup_{\|f\|_{\dot{H}^s} = 1} \|e^{i\tau h(D)} f\|_{L^r_{t,x}}. \quad (1.12)$$

Then there exists $f_0 \in \dot{H}^s$ such that

$$\|f_0\|_{\dot{H}^s} = 1, \quad \|e^{i\tau h(D)} f_0\|_{L^r_{t,x}} = M. \quad (1.13)$$

**Remark 1.2.** We underline that in Theorem 1.1 no specific norm is fixed on the target $C^n$ (and hence also on the corresponding $L^r_{t,x}$ norm). Nevertheless we prove in general the existence of maximizers for the Strichartz estimate (1.11). We also underline that in Theorem 1.1 we assume that the norm on $\dot{H}^s$ is the one defined in (1.3). Indeed the Hilbert structure of the norm $\|\cdot\|_{\dot{H}^s}$ is crucial in our argument.
Remark 1.3. Notice that the conditions $2 \leq p < q \leq \infty$ imply that $r > 2$, which will be crucial in the proof of Theorem 1.1.

Remark 1.4. As we see in the sequel, the proof of Theorem 1.1 is quite simple, and it is based on a suitable variant of a classic result by Brézis and Lieb ([2], [11], see Section 2 below). On the other hand, the technique we use does not allow us to consider the case $s = 0$ for which, in fact, some extra ingredients (being typically suitable improvements of Strichartz estimates) are needed, as substitutes of (1.10).

We shall now give some examples of applications of the previous Theorem.

Example 1.1 (Wave propagator). The Strichartz estimates for the wave propagator $e^{it|D|}$ ([8], [9]), in dimension $d \geq 2$, are the following:

$$
(1.14) \quad \left\| e^{it|D|} f \right\|_{L^p_t L^q_x} \leq C \left\| f \right\|_{H^{\frac{d}{2}} \times \frac{1}{2}},
$$

under the admissibility condition

$$
(1.15) \quad \frac{2}{p} + \frac{d-1}{q} = \frac{d-1}{2}, \quad p \geq 2. \quad (p, q) \neq (2, \infty).
$$

In this case, the gap of derivatives $\frac{1}{p} - \frac{1}{q} + \frac{1}{2} \geq 0$ is null only in the case of the energy estimate $(p, q) = (\infty, 2)$. In particular, we have

$$
(1.16) \quad \left\| e^{it|D|} f \right\|_{L^\infty_t L^{\frac{2d+1}{2}}_x} \leq C \left\| f \right\|_{H^{\frac{d+1}{2}}} \quad d \geq 2,
$$

which is in fact the original estimate proved by Strichartz in [15]. More generally, by Sobolev embedding one also obtains that

$$
(1.17) \quad \left\| e^{it|D|} f \right\|_{L^{\frac{2d+1}{d-1}}_t L^2_x} \leq C \left\| f \right\|_{H^{\frac{d+1}{2}}} \quad 0 \leq \sigma < \frac{d-1}{2}, \quad d \geq 2.
$$

Theorem 1.1 gives a short and simple proof of the fact that, for any dimension $d \geq 2$, and $0 \leq \sigma < \frac{d-1}{2}$, the best constant in (1.17) is achieved on some maximizing function $f$.

Example 1.2 (Dirac equation). Consider the (massless) Dirac operator

$$
\mathcal{D} := \frac{1}{i} \sum_{j=1}^{3} \alpha_j \partial_j,
$$

which is defined in dimension $d = 3$. Here $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_4(\mathbb{C})$ are the so called Dirac matrices, which are $4 \times 4$-hermitian matrices, $\alpha_j^t = \overline{\alpha_j}, \ j = 1, 2, 3$. They are defined as

$$
\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}
$$

or equivalently $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$, where $\sigma_j$ is the $j^{th}$ $2 \times 2$-Pauli matrix, $j = 1, 2, 3$.

Since $\mathcal{D}^2 = -\Delta I_{4\times 4}$, the Strichartz estimates for the massless Dirac operator are the same as for the 3D wave equation (see [4]):

$$
(1.18) \quad \left\| e^{it\mathcal{D}} f \right\|_{L^p_t L^q_x} \leq C \left\| f \right\|_{H^{\frac{d}{2}} \times \frac{1}{2}}.
$$
with the admissibility condition

\begin{equation}
\frac{2}{p} + \frac{2}{q} = 1, \quad p > 2.
\end{equation}

In particular we have

\begin{equation}
\|e^{itD}f\|_{L^{p}L^{q}_{t,x}} \leq C\|f\|_{\dot{H}^{\frac{1}{2}+\sigma}}, \quad 0 \leq \sigma < 1.
\end{equation}

Hence also in this case a loss of derivatives with respect to the initial datum is natural in the estimate. Also in this case Theorem 1.1 applies, and it proves that there exist maximizers for (1.20), which at our knowledge is not a known fact.

**Example 1.3 (Hyperbolic Schrödinger equation).** Let us now consider the hyperbolic Schrödinger operator $L := \sum_{j=1}^{m} \partial_{j}^{2} - \sum_{j=m+1}^{d} \partial_{j}^{2}$, for $d \geq 2$ and $1 \leq m < d$. In this case, Strichartz estimates are the same as for the Schrödinger propagator, namely

\begin{equation}
\|e^{itL}f\|_{L^{p}L^{q}_{t,x}} \leq C\|f\|_{\dot{H}^{s}},
\end{equation}

with the admissibility condition

\begin{equation}
\frac{2}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad p \geq 2, \quad (p, q) \neq (2, \infty).
\end{equation}

In particular, one has

\begin{equation}
\|e^{itL}f\|_{L^{p}L^{q}_{t,x}} \leq C\|f\|_{\dot{H}^{s}},
\end{equation}

for any $0 \leq s < \frac{d}{2}$. The only case in which the existence of maximizers for (1.23) is known is $d = 2, s = 0$. Indeed, the result by Rogers and Vargas in [13] contains all the ingredients which are necessary to prove the profile decomposition for bounded sequences in $L^{2}$, with respect to the propagator $e^{itL}$, in dimension $d = 2$; the existence of maximizers follows from this fact by a general argument in the spirit of [14]. It is a matter of fact that Theorem 1.1 applies for any $d \geq 2$ and $0 < s < \frac{d}{2}$, but we remark that it cannot include the case $s = 0$. We finally remark, our argument is rather simple and does not involve any profile decomposition-theorems associated to the propagator.

In the statement of Theorem 1.1, the homogeneity assumption (1.5) is required; in fact, this is just put in order to write the explicit dependence of $r$ on $k, s$ in (1.11). Motivated by the case of the wave equation, in which the homogeneity property does not hold (see example 1.4 below), we now state a more general version of the previous theorem.

Let us first introduce the notations

\begin{equation}
\dot{H}^{s} = \dot{H}^{s_{1}} \times \cdots \times \dot{H}^{s_{n}}, \quad s := (s_{1}, \ldots, s_{n}),
\end{equation}

\begin{equation}
\|f\|_{\dot{H}^{s}}^{2} = \sum_{i=1}^{n} \|f_{i}\|_{\dot{H}^{s_{i}}}^{2},
\end{equation}

where $f = (f_{1}, \ldots, f_{n}) \in \dot{H}^{s}$. Consider equation (1.4); in addition to (H1) and (H2) (where the norm $\dot{H}^{s}$ in this case is the more general one defined in (1.24)) assume:
(H3) there exists $j \in \{1, \ldots, n\}$ and $2 \leq p < q \leq \infty$ such that, if $(u_1, \ldots, u_n)$ solves (1.4) then
\[
\|u_j\|_{L^p_tL^q_x} \leq C_j \|f\|_{H^s};
\]

(H4) if $u = (u_1, \ldots, u_n)$ solves (1.4), then
\[
u_\lambda = \left(\lambda^{\frac{d}{2}} - s_1 u_1(\lambda t, \lambda x), \ldots, \lambda^{\frac{d}{2}} - s_n u_n(\lambda t, \lambda x)\right)
\]
solves (1.4), for any $\lambda > 0$.

Notice that, by a scaling argument, we have
\[
\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s_j.
\]

Moreover, interpolating with the energy estimate one also obtains
\[
\|u_j\|_{L^r_tL^{s_j,x}} \leq M \|f\|_{\dot{H}^s},
\]
where $r = \frac{2(1+d)}{d - 2s_j}$. We have the following result.

**Theorem 1.2.** Assume the operator $h(D)$ is such that (H1), (H2), (H3), (H4) are satisfied for some $s_1, \ldots, s_n \in \mathbb{R}$, $2 \leq p < q \leq \infty$, and $0 < s_j < \frac{d}{2}$, where $j$ is the one in (H3). Consider estimate (1.25) and let
\[
M := \sup_{\|f\|_{\dot{H}^s} = 1} \|u_j\|_{L^r_tL^{s_j,x}}.
\]

Then there exists $f_0 \in \dot{H}^s$ such that
\[
\|f_0\|_{\dot{H}^s} = 1, \quad \|v_j\|_{L^r_tL^{s_j,x}} = M
\]
where $(v_1, \ldots, v_n) = e^{ih(D)}f_0$.

**Remark 1.5.** In other words, Theorem 1.2 states that the problem of existence of maximizers can be solved analogously for Sobolev-Strichartz estimates involving one single component of the solution of equation (1.4). In a completely analogous way, one could treat the case of estimates for a selected group of components of the solution. The proof of Theorem 1.2 is completely identical to the one of Theorem 1.1: indeed it is sufficient to repeat the same argument for the composition operator $T_j(t) = \pi_j \circ e^{ih(D)}$, where $\pi_j : \mathbb{C}^N \to \mathbb{C}$ is the projection onto the $j$-th component. We will omit the details of the proof.

We pass now to show the main application of Theorem 1.2, on solutions of the wave equation.

**Example 1.4 (Wave equation).** Consider now the wave equation
\[
(1.28)
\begin{cases}
\partial^2_t u - \Delta u = 0 & \text{in } \mathbb{R}^{1+d} \\
u(0, x) = u_0(x) & \partial_t \nu(0, x) = u_1(x).
\end{cases}
\]

The vector-variable $V := (u, \partial_t u)^t$ is uniquely given by $V(t, x) = e^{ih(D)}V_0(x)$, where $V_0 = (u_0, u_1)^t$, and $h(D) = \begin{pmatrix} 0 & -i \\ -i \Delta & 0 \end{pmatrix}$. The Strichartz estimates for (1.28) can be immediately deduced by (1.17), writing the solution $u$ as
\[
u(t, \cdot) = \cos(t|D|)u_0(\cdot) + \frac{\sin(t|D|)}{|D|}u_1(\cdot);
in particular, in terms of the vector variable \( V \), denoting by \( f = (u_0, u_1)^t \), one has
\[
\|u\|_\frac{2(d+1)}{d} \leq C\|f\|_{\dot{H}^{\frac{1}{2}+\sigma} \times \dot{H}^{-\frac{1}{2}-\sigma}}, \quad 0 \leq \sigma < \frac{d-1}{2}, \quad d \geq 2.
\]

Foschi ([6]) proved that the best constant in (1.29) is achieved, in the cases \( d = 2, 3, \sigma = 0 \); moreover, he can characterize the shape of the maximizers. Later, Bulut ([3]) proved the same (also the anisotropic version of (1.29), with \( p \neq q \)); she can treat the range \( \frac{1}{2} \leq \sigma < \frac{d-1}{2} \), with \( d \geq 3 \). More recently, Bez and Rogers ([1]) can prove the existence of maximizers for (1.29), and characterize their shape, in the case \( d = 5, \sigma = \frac{1}{2} \). Finally, Ramos ([12]) proves a refinement of (1.29) in the case \( \sigma = 0 \), which in particular implies the existence of maximizers.

Now notice that Theorem 1.2 applies (with the choice \( j = 1 \) in assumption (H3)). It implies that there exist maximizers for (1.29), for any dimension \( d \geq 2 \) and any \( 0 \leq \sigma < \frac{d-1}{2} \).

The rest of the paper is devoted to the proof of Theorem 1.1.

2. PROOF OF THEOREM 1.1

Before starting with the proof of our main theorem, we need to recall two fundamental results which will play a role in the sequel. The first one is a variant of a well known result obtained by Brézis and Lieb in [2], [11].

**Proposition 2.1** ([5]). Let \( \mathcal{H} \) be a Hilbert space and \( T \in \mathcal{L}(\mathcal{H}, L^p(\mathbb{R}^d)) \) for a suitable \( p \in (2, \infty) \). Let \( \{h_n\}_{n \in \mathbb{N}} \subset \mathcal{H} \) such that:
\[
\begin{align*}
(2.1) & \quad \|h_n\|_{\mathcal{H}} = 1; \\
(2.2) & \quad \lim_{n \to \infty} \|Th_n\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(\mathcal{H}, L^p(\mathbb{R}^d))}; \\
(2.3) & \quad h_n \rightharpoonup h \neq 0; \\
(2.4) & \quad T(h_n) \to T(h) \text{ a.e. in } \mathbb{R}^d.
\end{align*}
\]

Then \( h_n \rightharpoonup h \) in \( \mathcal{H} \), in particular \( \|\bar{h}\|_{\mathcal{H}} = 1 \) and \( \|T(\bar{h})\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(\mathcal{H}, L^p(\mathbb{R}^d))}. \)

**Remark 2.1.** The main difference between Proposition 2.1 and Lemma 2.7 in [11] is that we only need to assume weak convergence in the Hilbert space \( \mathcal{H} \) for the maximizing sequence \( h_n \). On the other hand the argument in [11] works for operators defined between general Lebesgue spaces and not necessarily in the Hilbert spaces framework.

Proposition 2.1 has been proved in [5], in the scalar case. The proof in the vector case is completely analogous and will be omitted. The second tool we need is a byproduct of a well known result by Gérard in [7], in which the lack of compactness of the Sobolev embedding \( \dot{H}^s \subset L^{\frac{2d}{d-2s}} \) is classified.

**Proposition 2.2** ([7]). Let \( 0 < s < \frac{d}{2} \), and let \( w_n \in \dot{H}^s \) be a sequence such that
\[
\begin{align*}
(2.5) & \quad \|w_n\|_{\dot{H}^s} = 1, \quad n = 1, 2, \ldots \\
(2.6) & \quad \inf_{n \geq 1} \|w_n\|_{L^{\frac{2d}{d-2s}}} \geq \epsilon > 0.
\end{align*}
\]

Then there exist a sequence of parameters \( \lambda_n > 0 \), a sequence of centers \( x_n \in \mathbb{R}^d \), and a non-zero function \( 0 \neq v \in \dot{H}^s \) such that
\[
(2.7) \quad v_n(x) := \lambda_n^{\frac{d}{2} - s} w_n(\lambda_n(x - x_n)) \rightharpoonup v,
\]

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Proof of Theorem 1.1. Let $M$ be as in (1.12) and let $u_n \in \dot{H}^s$ be a maximizing sequence, i.e.

$$\|u_n\|_{\dot{H}^s} = 1, \quad \lim_{n \to \infty} \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}} = \left\| e^{ith(D)} \right\|_{L^r(\dot{H}^s, L^r_{t,x})} = M,$$

with $r = \frac{2(k+d)}{d-2s}$. Our aim is to prove that, by a suitable remodulation of $u_n$, we can obtain a new maximizing sequence for which Proposition 2.1 applies. In fact, since $u_n$ is uniformly bounded in $\dot{H}^s$, it admits a weak limit, which in principle could be zero.

We are now ready to perform the proof of our main theorem.

### 2.1. Proof of Theorem 1.1

We remark that the condition $s > 0$ is crucial in the previous proposition, which is in fact false in the case $s = 0$.

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2.1. Proof of Theorem 1.1. Let $M$ be as in (1.12) and let $u_n \in \dot{H}^s$ be a maximizing sequence, i.e.

$$\|u_n\|_{\dot{H}^s} = 1, \quad \lim_{n \to \infty} \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}} = \left\| e^{ith(D)} \right\|_{L^r(\dot{H}^s, L^r_{t,x})} = M,$$

with $r = \frac{2(k+d)}{d-2s}$. Our aim is to prove that, by a suitable remodulation of $u_n$, we can obtain a new maximizing sequence for which Proposition 2.1 applies. In fact, since $u_n$ is uniformly bounded in $\dot{H}^s$, it admits a weak limit, which in principle could be zero.

Notice that, by Sobolev embedding,

$$\left\| e^{ith(D)} u_n \right\|_{L^r_{t,x} L^{\frac{2d}{d-2s}}} \leq C \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x} \dot{H}^s} = C \|u_n\|_{\dot{H}^s} = C,$$

for some constant $C > 0$. Moreover, by assumption (1.10), there exist $2 \leq p < q \leq \infty$ such that

$$\left\| e^{ith(D)} u_n \right\|_{L^p_t L^q_x} \leq C\|u_n\|_{\dot{H}^s} = C,$$

for another constant $C > 0$. Hence, by (2.8), (2.10) and the Hölder inequality we can estimate, for $n$ sufficiently large,

$$\frac{M}{2} \leq \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}} \leq \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x} L^{\frac{2d}{d-2s}}} \leq C \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x} L^{\frac{2d}{d-2s}}} \leq C \left\| e^{ith(D)} u_n \right\|_{L^p_{t,x} L^q_x} \leq C \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x} L^{\frac{2d}{d-2s}}} =: \epsilon > 0,$$

(2.11) 

for $n$ sufficiently large. As a consequence, there exists a sequence of times $t_n \in \mathbb{R}$ such that

$$\left\| e^{ith(D)} u_n \right\|_{L^{\frac{2d}{d-2s}}_{t,x}} \geq \left( \frac{M}{2C^2} \right)^{\frac{q}{p}} =: \epsilon > 0,$$

(2.12) 

for any $n$ sufficiently large. Now denote by

$$w_n(x) = e^{ith(D)} u_n(x),$$

and notice that $w_n$ is still a maximizing sequence, i.e. $\left\| e^{ith(D)} w_n \right\|_{L^r_{t,x}} \to M$, as $n \to \infty$. Moreover, we have

$$\left\| w_n \right\|_{\dot{H}^s} = \|u_n\|_{\dot{H}^s} = 1$$

(2.13) 

$$\left\| w_n \right\|_{L^{\frac{2d}{d-2s}}_{t,x}} \geq \frac{\epsilon}{2} > 0;$$

(2.14)
hence, by Proposition 2.2, there exist two sequences $\lambda_n > 0$, $x_n \in \mathbb{R}^d$, and a non-zero function $0 \neq v \in \dot{H}^s$, such that

$$v_n(x) := \lambda_n^{\frac{d}{2} - s} w_n(\lambda_n(x - x_n)) = \lambda_n^{\frac{d}{2} - s} e^{it_n D} u_n(\lambda_n(x - x_n)) \rightarrow v \neq 0$$

weakly in $\dot{H}^s$, as $n \rightarrow \infty$. By scaling, it is easy to see that $v_n$ is still a maximizing sequence, since $\|e^{ith(D)}v_n\|_{L^r_x} = \|e^{ith(D)}u_n\|_{L^r_x}$.

By Proposition 2.1, in order to conclude the proof it is sufficient to prove that (2.4) holds on $v_n$, up to subsequences, with $T = e^{ith(D)}$. Let us fix $t \in \mathbb{R}$; by the continuity of $e^{ith(D)}$ in $\dot{H}^s$, we have

$$e^{ith(D)}v_n \rightharpoonup e^{ith(D)}v$$

weakly in $\dot{H}^s$, as $n \rightarrow \infty$. Then, by the Rellich Theorem, for any $R > 0$

$$e^{ith(D)}v_n \rightarrow e^{ith(D)}v$$

strongly in $L^2(B(R))$ where $B(R) = \{ x \in \mathbb{R}^d : |x| < R \}$, as $n \rightarrow \infty$. Denote by

$$F_n(t) := \int_{|x| < R} |e^{ith(D)}(v_n - v)|^2 dx = \|e^{ith(D)}(v_n - v)\|^2_{L^2(B(R))}.$$

By the Hölder inequality and Sobolev embedding we obtain

$$F_n(t) \leq CR^\frac{d}{2s} \|e^{ith(D)}(v_n - v)\|^2_{H^s} \leq 2CR^\frac{d}{2s}$$

consequently, by (2.16), the Fubini and the Lebesgue Theorems we have that

$$\int_{-R}^R F_n(t) dt = \int_{-R}^R \int_{|x| < R} |e^{ith(D)}(v_n - v)|^2 dx dt \rightarrow 0$$

as $n \rightarrow \infty$; this implies that, up to a subsequence,

$$e^{ith(D)}(v_n - v) \rightarrow 0 \quad \text{a.e. in } B(R) \times (-R, R).$$

The extraction of the subsequence depends on $R$; now repeat the argument on a discrete sequence of radii $R_n$ such that $R_n \rightarrow \infty$, as $n \rightarrow \infty$ and conclude, by a diagonal argument, that there exists a subsequence of $v_{n_k}$ of $v_n$ such that

$$e^{ith(D)}(v_{n_k} - v) \rightarrow 0 \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^d.$$

This, together with (2.15) and Proposition 2.1, concludes the proof.

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EXISTENCE OF MAXIMIZERS FOR SOBOLEV-STRICHARTZ INEQUALITIES

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Abstract. We prove the existence of maximizers of Sobolev-Strichartz estimates for a general class of propagators, involving relevant examples, as for instance the wave, Dirac and the hyperbolic Schrödinger flows.

1. Introduction

In the recent years, an increasing interest has been devoted to the problem of existence of maximizers for Strichartz inequalities, and more in general Fourier Restriction Theorems.

Recall that, given two Banach spaces \((X, \| \cdot \|_X), (Y, \| \cdot \|_Y)\), and a linear and continuous operator \(T \in \mathcal{L}(X, Y)\), it is customary to define maximizer for \(T\) any \(x_0 \in X\) such that

\[
\|x_0\|_X = 1, \quad \|Tx_0\|_Y = \sup_{\|x\|_X = 1} \|Tx\|_Y.
\]

Notice that the existence (as well as the definition) of maximizers depends on the specific norm introduced on \(X\) and \(Y\).

In this paper we consider the existence of maximizers for a large class of propagators of the form \(e^{it\partial^D}\), and we focus our attention to cases in which some Sobolev-Strichartz estimates of the following type hold

\[
\left\| e^{it\partial^D} f \right\|_{L^r_{t,x}} \leq C \left\| f \right\|_{\dot{H}^s}, \quad s > 0, \quad r > 2.
\]

where in general \(f\) is a vector valued function \(f = (f_1, ..., f_n) : \mathbb{R}^d \to \mathbb{C}^n\) and

\[
\|f\|_{\dot{H}^s}^2 = \sum_{j=1}^n \|f_j\|_{\dot{H}^s}^2.
\]

Concerning the definition of the norm \(L^r_{t,x}\) in the l.h.s. of (1.2) recall that in general for any given norm \(\| \cdot \|_{\mathbb{C}^n}\) on \(\mathbb{C}^n\) one can define the corresponding mixed Lebesgue norms \(L^p_t L^q_x\) for vector valued functions \(F = F(t, x) : \mathbb{R}^{1+d} \to \mathbb{C}^n\) as follows

\[
\|F(t, x)\|_{L^p_t L^q_x}^p = \left( \int \left( \int |F(t, x)|_{\mathbb{C}^n}^q dx \right)^{\frac{p}{q}} dt \right)^{\frac{q}{p}},
\]

where \(1 \leq p, q \leq \infty\). In the case \(p = q = r\) we shall write shortly

\(L^r_t L^r_x = L^r_{t,x}\).

2000 Mathematics Subject Classification. 35J10, 35L05.
Key words and phrases. Fourier restriction theorems, Strichartz estimates.
Remark 1.1. Recall that all the norms on $\mathbb{C}^n$ are equivalent and hence also the corresponding norms induced on the spaces $L_t^p L_x^q$ are equivalent. Due to this fact in (1.2) it is not necessary to specify the norm on $\mathbb{C}^n$ with respect to which we are working with (of course provided that the corresponding constant $C$ is suitably modified). On the other hand the existence of maximizers for the inequality (1.2) could be affected in principle by changing the corresponding norm on $\mathbb{C}^n$.

Estimates of the kind (1.2) with a loss of derivatives with respect to the datum $f$, are natural in some relevant cases, as for instance the wave and Dirac equations. Let us point our attention to the Cauchy problem for the following system

$$
(1.4) \begin{cases}
i\partial_t u + h(D)u = 0 \\
u(0, x) = f(x),
\end{cases}
$$

where $u(t, x) = (u_1(t, x), \ldots, u_n(t, x)) : \mathbb{R}^{1+d} \to \mathbb{C}^n$, $f(x) = (f_1(x), \ldots, f_n(x)) : \mathbb{R}^d \to \mathbb{C}^n$. Here we denote by $h(D) = \mathcal{F}^{-1}(h(\xi))\mathcal{F}$, where $\mathcal{F}$ is the standard Fourier transform; in addition, the symbol $h(\xi) \in \mathcal{M}_{n \times n}(\mathbb{C})$, $n \geq 1$, is assumed to be a matrix-valued function $h(\xi) = (h_{ij}(\xi))_{ij=1,..,n}$. In the sequel, we always make the following abstract assumptions:

(H1) there exists $0 < s < \frac{d}{2}$ such that the problem (1.4) is globally well-posed in $\dot{H}^s$, and the unique solution is given via propagator $u(t, x) = e^{ith(D)} f(x)$;

(H2) the flow $e^{ith(D)}$ is unitary onto $\dot{H}^s$, i.e.

$$
\left\| e^{ith(D)} f \right\|_{\dot{H}^s} = \| f \|_{\dot{H}^s}, \quad \forall t \in \mathbb{R},
$$

where $s$ is the same as in (H1), and $\| \cdot \|_{\dot{H}^s}$ is defined in (1.3).

In order to introduce our problem, assume for the moment that $h_{i,j}(\xi)$ is homogenous of some degree $k > 0$ for every $i, j \in \{1,..,n\}$, namely

$$
(1.5) \quad h_{i,j}(\xi) : \mathbb{R}^d \to \mathbb{C}, \quad h(\lambda \xi) = \lambda^k h(\xi) \quad \forall \lambda > 0.
$$

Then, equation (1.4) is invariant under the scaling $\nu_\lambda(t, x) = u(\lambda^k t, \lambda x)$; as a consequence, if a Strichartz estimate of the following type holds

$$
(1.6) \quad \left\| e^{ith(D)} f \right\|_{L_t^p L_x^q} \leq C \| f \|_{\dot{H}^s},
$$

for some $s > 0$ and some constant $C > 0$, then $p$ and $q$ have to satisfy the following scaling condition

$$
(1.7) \quad \frac{k}{p} + \frac{d}{q} = \frac{d}{2} - s.
$$

Moreover, by the Sobolev embedding $\dot{H}^s \subset L^{\frac{2d}{d-2s}}$, for $0 < s < d/2$, and the $\dot{H}^s$-preservation

$$
\left\| e^{ith(D)} f \right\|_{L_t^p \dot{H}^s_x} = \| f \|_{\dot{H}^s},
$$

(assumption (H2) above), we get

$$
(1.8) \quad \left\| e^{ith(D)} f \right\|_{L_t^p L_x^{\frac{2d}{d-2s}}} \leq C \| f \|_{\dot{H}^s}, \quad 0 < s < \frac{d}{2},
$$

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for some constant $C > 0$. Hence, if an estimate of the type (1.6) holds with $p < q$, then by interpolation with (1.8) we obtain

\[(1.9) \quad \left\| e^{ith(D)} f \right\|_{L^r_t L^q_x} \leq C \| f \|_{\dot{H}^s}, \quad 0 < s < \frac{d}{2}, \quad r = \frac{2(k + d)}{d - 2s}.\]

The aim of this paper is to prove that, as soon as an estimate as (1.9) holds, with a strictly positive $s > 0$, then the best constant of the estimate is achieved by some maximizing functions, independently on the norm which is fixed on the target $C^n$ (and consequently on the corresponding definition of the $L^r_t$-norm).

This kind of problem has been recently studied by several authors, who treated separately different propagators. We firstly mention Kunze [10], who proved the existence of maximizers of the $L^6_t$-Strichartz inequality for the 1D Schrödinger propagator. Later, Foschi [6] succeeded in characterizing the best constant of the inequality and also the shape of the maximizers, for the 1D and 2D-Schrödinger propagators. In the same paper, the author treats the $L^6_t$-Strichartz estimate for the wave equation, which hold to the scale of $\dot{H}^{\frac{1}{2}} \times \dot{H}^{-\frac{1}{2}}$-initial data. Recently, Bez and Rogers [1] computed the best constant and described the shape of maximizers of the $L^6_t$-Strichartz estimate for the wave equation, with initial data in the energy space $\dot{H}^1 \times L^2$, in space dimension $d = 5$.

We also mention the papers by Shao [14] for the Schrödinger equation and by Bulut [3] for the wave equation, in which the existence of maximizers for anisotropic Strichartz estimates in spaces $L^p_t L^q_x$, for general couples $(p, q)$, is also proved. Finally, Ramos [12] proves that there exist maximizers for the isotropic Strichartz estimates for the wave equation, at the scale $\dot{H}^{\frac{3}{2}} \times \dot{H}^{-\frac{3}{2}}$.

Here we give a unified (and simple) proof of the existence of maximizers of (1.9), when $s > 0$, which involves a large class of examples of propagators. Our main result is the following.

**Theorem 1.1.** Let assumptions (H1), (H2) be satisfied for some $0 < s < \frac{d}{2}$, and let $h(\xi)$ satisfy (1.5) for some $k > 0$. Moreover, assume that, for some $2 \leq p < q < \infty$

\[(1.10) \quad \left\| e^{ith(D)} f \right\|_{L^p_t L^q_x} \leq C \| f \|_{\dot{H}^s},\]

so that, for $r = \frac{2(k + d)}{d - 2s}$, we also have

\[(1.11) \quad \left\| e^{ith(D)} f \right\|_{L^r_t L^q_x} \leq M \| f \|_{\dot{H}^s},\]

with

\[(1.12) \quad M := \sup_{\| f \|_{\dot{H}^s} = 1} \left\| e^{ith(D)} f \right\|_{L^r_t L^q_x}.\]

Then there exists $f_0 \in \dot{H}^s$ such that

\[(1.13) \quad \| f_0 \|_{\dot{H}^s} = 1, \quad \left\| e^{ith(D)} f_0 \right\|_{L^r_t L^q_x} = M.\]

**Remark 1.2.** We underline that in Theorem 1.1 no specific norm is fixed on the target $C^n$ (and hence also on the corresponding $L^r_t$-norm). Nevertheless we prove in general the existence of maximizers for the Strichartz estimate (1.11). We also underline that in Theorem 1.1 we assume that the norm on $\dot{H}^s$ is the one defined in (1.3). Indeed the Hilbert structure of the norm $\| \cdot \|_{\dot{H}^s}$ is crucial in our argument.
Remark 1.3. Notice that the conditions $2 \leq p < q \leq \infty$ imply that $r > 2$, which will be crucial in the proof of Theorem 1.1.

Remark 1.4. As we see in the sequel, the proof of Theorem 1.1 is quite simple, and it is based on a suitable variant of a classic result by Brézis and Lieb ([2], [11], see Section 2 below). On the other hand, the technique we use does not allow us to consider the case $s = 0$ for which, in fact, some extra ingredients (being typically suitable improvements of Strichartz estimates) are needed, as substitutes of (1.10).

We shall now give some examples of applications of the previous Theorem.

Example 1.1 (Wave propagator). The Strichartz estimates for the wave propagator $e^{it|D|}$ ([8], [9]), in dimension $d \geq 2$, are the following:

\begin{equation}
\bigg\| e^{it|D|} f \bigg\|_{L^p_t L^q_x} \leq C \big\| f \big\|_{H^{\frac{d+1}{2} + \frac{1}{q}}} ,
\end{equation}

under the admissibility condition

\begin{equation}
\frac{2}{p} + \frac{d - 1}{q} = \frac{d - 1}{2}, \quad p \geq 2, \quad (p, q) \neq (2, \infty).
\end{equation}

In this case, the gap of derivatives $\frac{1}{p} - \frac{1}{d} + \frac{1}{2} \geq 0$ is null only in the case of the energy estimate $(p, q) = (\infty, 2)$. In particular, we have

\begin{equation}
\bigg\| e^{it|D|} f \bigg\|_{L^2_t L^2_x} \leq C \big\| f \big\|_{H^{\frac{1}{2}}} , \quad d \geq 2,
\end{equation}

which is in fact the original estimate proved by Strichartz in [15]. More generally, by Sobolev embedding one also obtains that

\begin{equation}
\bigg\| e^{it|D|} f \bigg\|_{L^{\frac{2(d+1)}{d-1}}_t L^\infty_x} \leq C \big\| f \big\|_{H^{\frac{1}{2} + \sigma}}, \quad 0 \leq \sigma < \frac{d - 1}{2}, \quad d \geq 2.
\end{equation}

Theorem 1.1 gives a short and simple proof of the fact that, for any dimension $d \geq 2$, and $0 < \sigma < \frac{d - 1}{2}$, the best constant in (1.17) is achieved on some maximizing function $f$.

Example 1.2 (Dirac equation). Consider the (massless) Dirac operator

$$
\mathcal{D} := \frac{1}{i} \sum_{j=1}^{3} \alpha_j \partial_j,
$$

which is defined in dimension $d = 3$. Here $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{M}_4(\mathbb{C})$ are the so called Dirac matrices, which are $4 \times 4$ hermitian matrices, $\alpha_j^t = \overline{\alpha_j}$, $j = 1, 2, 3$. They are defined as

\[
\begin{align*}
\alpha_1 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
\alpha_2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\alpha_3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},
\end{align*}
\]

or equivalently $\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j^t & 0 \end{pmatrix}$, where $\sigma_j$ is the $j^{th}$ $2 \times 2$ Pauli matrix, $j = 1, 2, 3$.

Since $\mathcal{D}^2 = -\Delta_{4 \times 4}$, the Strichartz estimates for the massless Dirac operator are the same as for the 3D wave equation (see [4]):

\begin{equation}
\bigg\| e^{it\mathcal{D}} f \bigg\|_{L^p_t L^q_x} \leq C \big\| f \big\|_{H^{\frac{1}{2} + \frac{1}{q}}} ,
\end{equation}
with the admissibility condition

\[(1.19) \quad \frac{2}{p} + \frac{2}{q} = 1, \quad p > 2.\]

In particular we have

\[(1.20) \quad \| e^{itD} f \|_{L^p_t L^q_x} \leq C \| f \|_{\dot{H}^{\frac{d}{2}}}, \quad 0 \leq \sigma < 1.\]

Hence also in this case a loss of derivatives with respect to the initial datum is natural in the estimate. Also in this case Theorem 1.1 applies, and it proves that there exist maximizers for (1.20), in the range 0 < \sigma < 1, which at our knowledge is not a known fact.

**Example 1.3 (Hyperbolic Schrödinger equation).** Let us now consider the hyperbolic Schrödinger operator

\[ L := \sum_{j=1}^{m} \partial_j^2 - \sum_{j=m+1}^{d} \partial_j^2, \text{ for } d \geq 2 \text{ and } 1 \leq m < d. \]

In this case, Strichartz estimates are the same as for the Schrödinger propagator, namely

\[(1.21) \quad \| e^{itL} f \|_{L^p_t L^q_x} \leq C \| f \|_{\dot{H}^{s}},\]

with the admissibility condition

\[(1.22) \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2} - s, \quad p \geq 2, \quad (p, q) \neq (2, \infty).\]

In particular, one has

\[(1.23) \quad \| e^{itL} f \|_{L^p_t L^q_x} \leq C \| f \|_{\dot{H}^{s}},\]

for any 0 ≤ s < \frac{d}{2}. The only case in which the existence of maximizers for (1.23) is known is d = 2, s = 0. Indeed, the result by Rogers and Vargas in [13] contains all the ingredients which are necessary to prove the profile decomposition for bounded sequences in L^2, with respect to the propagator \( e^{itL} \), in dimension d = 2; the existence of maximizers follows from this fact by a general argument in the spirit of [14]. It is a matter of fact that Theorem 1.1 applies for any d ≥ 2 and 0 < s < \frac{d}{2}, but we remark that it cannot include the case s = 0. We finally remark, our argument is rather simple and does not involve any profile decomposition-theorems associated to the propagator.

In the statement of Theorem 1.1, the homogeneity assumption (1.5) is required; in fact, this is just put in order to write the explicit dependence of r on k, s in (1.11). Motivated by the case of the wave equation, in which the homogeneity property does not hold (see example 1.4 below), we now state a more general version of the previous theorem.

Let us first introduce the notations

\[ \dot{H}^s = \dot{H}^{s_1} \times \cdots \times \dot{H}^{s_n} \quad s := (s_1, \ldots, s_n), \]

\[(1.24) \quad \| f \|_{\dot{H}^s}^2 = \sum_{i=1}^{n} \| f_i \|_{\dot{H}^{s_i}}^2,\]

where f = (f_1, \ldots, f_n) ∈ \dot{H}^s. Consider equation (1.4); in addition to (H1) and (H2) (where the norm \( \dot{H}^s \) in this case is the more general one defined in (1.24)) assume:
(H3) there exists \( j \in \{1, \ldots, n\} \) and \( 2 \leq p < q \leq \infty \) such that, if \((u_1, \ldots, u_n)\) solves (1.4) then
\[
\|u_j\|_{L^p_t L^q_x} \leq C_j \|f\|_{\dot{H}^s};
\]
(H4) if \( u = (u_1, \ldots, u_n) \) solves (1.4), then
\[
\lambda^{\frac{d}{2} - s_j} u_1(\lambda t, \lambda x), \ldots, \lambda^{\frac{d}{2} - s_n} u_n(\lambda t, \lambda x)
\]
solves (1.4), for any \( \lambda > 0 \).

Notice that, by a scaling argument, we have
\[
\frac{1}{p} + \frac{d}{q} = \frac{d}{2} - s_j.
\]
Moreover, interpolating with the energy estimate one also obtains
(1.25) \[
\|u_j\|_{L^r_t L^s_x} \leq M \|f\|_{\dot{H}^s},
\]
where \( r = \frac{2(1+d)}{d - 2s_j} \). We have the following result.

**Theorem 1.2.** Assume the operator \( h(D) \) is such that (H1), (H2), (H3), (H4) are satisfied for some \( s_1, \ldots, s_n \in \mathbb{R} \), \( 2 \leq p < q \leq \infty \), and \( 0 < s_j < \frac{d}{2} \), where \( j \) is the one in (H3). Consider estimate (1.25) and let
(1.26) \[
M := \sup_{\|f\|_{\dot{H}^s} = 1} \|u_j\|_{L^r_t L^s_x}.
\]
Then there exists \( f_0 \in \dot{H}^s \) such that
(1.27) \[
\|f_0\|_{\dot{H}^s} = 1, \quad \|v_j\|_{L^r_t L^s_x} = M
\]
where \((v_1, \ldots, v_n) = e^{ih(D)} f_0\).

**Remark 1.5.** In other words, Theorem 1.2 states that the problem of existence of maximizers can be solved analogously for Sobolev-Strichartz estimates involving one single component of the solution of equation (1.4). In a completely analogous way, one could treat the case of estimates for a selected group of components of the solution. The proof of Theorem 1.2 is completely identical to the one of Theorem 1.1: indeed it is sufficient to repeat the same argument for the composition operator \( T_j(t) = \pi_j \circ e^{ih(D)} \), where \( \pi_j : \mathbb{C}^N \to \mathbb{C} \) is the projection onto the \( j \)-th component. We will omit the details of the proof.

We pass now to show the main application of Theorem 1.2, on solutions of the wave equation.

**Example 1.4 (Wave equation).** Consider now the wave equation
(1.28) \[
\begin{aligned}
\partial_t^2 u - \Delta u &= 0 \quad \text{in } \mathbb{R}^{1+d} \\
u(0, x) &= u_0(x) \quad \partial_t u(0, x) = u_1(x).
\end{aligned}
\]
The vector-variable \( V := (u, \partial_t u)^t \) is uniquely given by \( V(t, x) = e^{ih(D)} V_0(x) \), where \( V_0 = (u_0, u_1)^t \), and \( h(D) = \begin{pmatrix} 0 & -i \\ -i \Delta & 0 \end{pmatrix} \). The Strichartz estimates for (1.28) can be immediately deduced by (1.17), writing the solution \( u \) as
\[
u(t, \cdot) = \cos(t|D|)u_0(\cdot) + \frac{\sin(t|D|)}{|D|} u_1(\cdot);
\]
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in particular, in terms of the vector variable \( V \), denoting by \( f = (u_0, u_1)^t \), one has

\[
\|u\|_{L^2(d+1/2)^d} \leq C\|f\|_{\dot{H}^{\frac{1}{2}+\sigma} \times \dot{H}^{-\frac{1}{2}+\sigma}}, \quad 0 \leq \sigma < \frac{d-1}{2}, \quad d \geq 2.
\]

Foschi ([6]) proved that the best constant in (1.29) is achieved, in the cases \( d = 2, 3, \sigma = 0 \); moreover, he can characterize the shape of the maximizers. Later, Bulut ([3]) proved the same (also the anisotropic version of (1.29), with \( p \neq q \)); she can treat the range \( \frac{1}{2} \leq \sigma < \frac{d-1}{2}, \) with \( d \geq 3 \). More recently, Bez and Rogers ([1]) can prove the existence of maximizers for (1.29), and characterize their shape, in the case \( d = 5, \sigma = \frac{1}{2} \).

Now notice that Theorem 1.2 applies (with the choice \( j = 1 \) in assumption (H3)). It implies that there exist maximizers for (1.29), for any dimension \( d \geq 2 \) and any \( 0 < \sigma < \frac{d-1}{2} \). We remark that Ramos ([12]) proves a refinement of (1.29) in the case \( \sigma = 0 \), which in particular implies the existence of maximizers, in this specific case of the wave equation.

The rest of the paper is devoted to the proof of Theorem 1.1.

2. PROOF OF THEOREM 1.1

Before starting with the proof of our main theorem, we need to recall two fundamental results which will play a role in the sequel. The first one is a variant of a well known result obtained by Brézis and Lieb in [2], [11].

**Proposition 2.1** ([5]). Let \( H \) be a Hilbert space and \( T \in \mathcal{L}(H, L^p(\mathbb{R}^d)) \) for a suitable \( p \in (2, \infty) \). Let \( \{h_n\}_{n \in \mathbb{N}} \in H \) such that:

\[
\begin{align*}
(2.1) \quad & \|h_n\|_H = 1; \\
(2.2) \quad & \lim_{n \to \infty} \|Th_n\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(H, L^p(\mathbb{R}^d))}; \\
(2.3) \quad & h_n \rightharpoonup \bar{h} \neq 0; \\
(2.4) \quad & T(h_n) \to T(\bar{h}) \text{ a.e. in } \mathbb{R}^d.
\end{align*}
\]

Then \( h_n \to \bar{h} \) in \( H \), in particular \( \|\bar{h}\|_H = 1 \) and \( \|T(\bar{h})\|_{L^p(\mathbb{R}^d)} = \|T\|_{\mathcal{L}(H, L^p(\mathbb{R}^d))} \).

**Remark 2.1.** The main difference between Proposition 2.1 and Lemma 2.7 in [11] is that we only need to assume weak convergence in the Hilbert space \( H \) for the maximizing sequence \( h_n \). On the other hand the argument in [11] works for operators defined between general Lebesgue spaces and not necessarily in the Hilbert spaces framework.

Proposition 2.1 has been proved in [5], in the scalar case. The proof in the vector-case is completely analogous and will be omitted. The second tool we need is a byproduct of a well known result by Gérard in [7], in which the lack of compactness of the Sobolev embedding \( \dot{H}^s \subset L^{\frac{2d}{d-2s}} \) is classified.

**Proposition 2.2** ([7]). Let \( 0 < s < \frac{d}{2} \), and let \( w_n \in \dot{H}^s \) be a sequence such that

\[
\begin{align*}
(2.5) \quad & \|w_n\|_{\dot{H}^s} = 1, \quad n = 1, 2, \ldots \\
(2.6) \quad & \inf_{n \geq 1} \|w_n\|_{L^\infty} \geq \epsilon > 0.
\end{align*}
\]
Then there exist a sequence of parameters $\lambda_n > 0$, a sequence of centers $x_n \in \mathbb{R}^d$, and a non-zero function $0 \neq v \in H^s$ such that

\begin{equation}
(2.7) \quad v_n(x) := \lambda_n^{\frac{2}{p-2}} w_n(\lambda_n(x - x_n)) \rightharpoonup v,
\end{equation}

weakly in $H^s$, as $n \to \infty$.

Remark 2.2. We remark that the condition $s > 0$ is crucial in the previous proposition, which is in fact false in the case $s = 0$.

We are now ready to perform the proof of our main theorem.

2.1. Proof of Theorem 1.1. Let $M$ be as in (1.12) and let $u_n \in \dot{H}^s$ be a maximizing sequence, i.e.,

\begin{equation}
(2.8) \quad \|u_n\|_{\dot{H}^s} = 1, \quad \lim_{n \to \infty} \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}} = \left\| e^{ith(D)} \right\|_{L(H^s, L^r_{t,x})} = M,
\end{equation}

with $r = \frac{2(k+d)}{d-2s}$. Our aim is to prove that, by a suitable remodulation of $u_n$, we can obtain a new maximizing sequence for which Proposition 2.1 applies. In fact, since $u_n$ is uniformly bounded in $\dot{H}^s$, it admits a weak limit, which in principle could be zero.

Notice that, by Sobolev embedding,

\begin{equation}
(2.9) \quad \left\| e^{ith(D)} u_n \right\|_{L^\infty L^{\frac{2d}{d-2s}}_{t,x}} \leq C \left\| e^{ith(D)} u_n \right\|_{L^\infty \dot{H}^{s}} = C \|u_n\|_{\dot{H}^s} = C,
\end{equation}

for some constant $C > 0$. Moreover, by assumption (1.10), there exist $2 \leq p < q \leq \infty$ such that

\begin{equation}
(2.10) \quad \left\| e^{ith(D)} u_n \right\|_{L^p_t L^q_x} \leq C \|u_n\|_{\dot{H}^s} = C,
\end{equation}

for another constant $C > 0$. Hence, by (2.8), (2.10) and the Hölder inequality we can estimate, for $n$ sufficiently large,

\[
\frac{M}{2} \leq \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}} \leq \left\| e^{ith(D)} u_n \right\|_{L^p_t L^q_x} \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}}^{\frac{1}{p}} \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}}^{\frac{1}{q}} \leq C^p \left\| e^{ith(D)} u_n \right\|_{L^p_t L^q_x}^{\frac{1}{p}} \left\| e^{ith(D)} u_n \right\|_{L^r_{t,x}}^{\frac{1}{q}}
\]

(where we have used (1.7)). The last estimate implies that

\begin{equation}
(2.11) \quad \left\| e^{ith(D)} u_n \right\|_{L^\infty L^{\frac{2d}{d-2s}}_{t,x}} \geq \left( \frac{M}{2C^p} \right)^{\frac{1}{\frac{d}{p}}} =: \epsilon > 0,
\end{equation}

for $n$ sufficiently large. As a consequence, there exists a sequence of times $t_n \in \mathbb{R}$ such that

\begin{equation}
(2.12) \quad \left\| e^{it_n h(D)} u_n \right\|_{L^2_t L^{\frac{2d}{d-2s}}_{x}} \geq \frac{\epsilon}{2} > 0,
\end{equation}

for any $n$ sufficiently large. Now denote by

\[
w_n(x) = e^{it_n h(D)} u_n(x),
\]
and notice that \(w_n\) is still a maximizing sequence, i.e. \(\|e^{ith(D)}w_n\|_{L^s_t L^r_x} \to M\), as \(n \to \infty\). Moreover, we have

\[
(2.13) \quad \|w_n\|_{H^s} = \|u_n\|_{H^s} = 1
\]

\[
(2.14) \quad \|w_n\|_{L^{2d}_{t,x}}^2 \geq \frac{\epsilon}{2} > 0;
\]

hence, by Proposition 2.2, there exist two sequences \(\lambda_n > 0, x_n \in \mathbb{R}^d\), and a non-zero function \(v \neq 0 \in H^s\), such that

\[
(2.15) \quad v_n(x) := \lambda_n^{\frac{d}{2} - s} w_n(\lambda_n(x - x_n)) = \lambda_n^{\frac{d}{2} - s} e^{ith(D)} u_n(\lambda_n(x - x_n)) \to v \neq 0
\]

weakly in \(H^s\), as \(n \to \infty\). By scaling, it is easy to see that \(v_n\) is still a maximizing sequence, since \(\|e^{ith(D)}v_n\|_{L^s_t L^r_x} = \|e^{ith(D)}w_n\|_{L^s_t L^r_x}\).

By Proposition 2.1, in order to conclude the proof it is sufficient to prove that (2.4) holds on \(v_n\), up to subsequences, with \(T = e^{ith(D)}\). Let us fix \(t \in \mathbb{R}\); by the continuity of \(e^{ith(D)}\) in \(H^s\), we have

\[
e^{ith(D)}v_n \rightarrow e^{ith(D)}v\]

weakly in \(H^s\), as \(n \to \infty\). Then, by the Rellich Theorem, for any \(R > 0\)

\[
e^{ith(D)}v_n \rightarrow e^{ith(D)}v
\]

strongly in \(L^2(B(R))\) where \(B(R) = \{x \in \mathbb{R}^d : |x| < R\}\), as \(n \to \infty\). Denote by

\[
F_n(t) := \int_{|x| < R} \left| e^{ith(D)}(v_n - v) \right|^2 \, dx = \left\| e^{ith(D)}(v_n - v) \right\|_{L^2(B(R))}^2.
\]

By the Hölder inequality and Sobolev embedding we obtain

\[
F_n(t) \leq CR^{\frac{2s}{2s - d}} \left\| e^{ith(D)}(v_n - v) \right\|_{H^s}^2 \leq 2CR^{\frac{2s}{2s - d}};
\]

consequently, by (2.16), the Fubini and the Lebesgue Theorems we have that

\[
\int_{-R}^{R} F_n(t) \, dt = \int_{-R}^{R} \int_{|x| < R} \left| e^{ith(D)}(v_n - v) \right|^2 \, dx \, dt \to 0
\]

as \(n \to \infty\); this implies that, up to a subsequence,

\[
e^{ith(D)}(v_n - v) \to 0 \quad \text{a.e. in } B(R) \times (-R, R).
\]

The extraction of the subsequence depends on \(R\); now repeat the argument on a discrete sequence of radii \(R_n\), such that \(R_n \to \infty\), as \(n \to \infty\) and conclude, by a diagonal argument, that there exists a subsequence of \(v_{n_k}\) of \(v_n\) such that

\[
e^{ith(D)}(v_{n_k} - v) \to 0 \quad \text{a.e. in } \mathbb{R} \times \mathbb{R}^d.
\]

This, together with (2.15) and Proposition 2.1, concludes the proof.

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