Nonlinear parameter estimation in damping with Volterra series through harmonic probing

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Abstract. In this work nonlinear system identification procedure, based on Volterra series, is presented to distinguish a damping nonlinearity from stiffness nonlinearity, using measured first and third harmonic amplitude characteristics. First and higher order Volterra kernel synthesis formulations and frequency response functions (FRFs) have been developed in damping and stiffness nonlinearity. The characteristics of first and higher order harmonic amplitudes at various excitation level is studied for the both the nonlinearity. This paper shows comparison study between the Volterra series, Runge-Kutta fourth order in analysis of damping and stiffness nonlinearity in the mechanical systems. Nonlinearity can lead to various system behaviour, like jump phenomenon, stable and unstable region, super harmonic resonances. Using first and higher order harmonic amplitudes, formulated from Volterra series response representation, nonlinear parameters are estimated through the recursive iteration method.

Keywords: Volterra series, System Identification, Cubic nonlinear stiffness, cubic nonlinear damping

1. Introduction

Most of the engineering systems such as rotor bearing elements [1, 2], nonlinear suspension and isolation system [3, 4], aerospace structures, Nano Electro Mechanical systems, Acoustic nonlinearity of an orifice, cochlear amplifier [5], are inherently nonlinear in nature than linear. If the system exhibits linear or nonlinear behaviour cannot be answered easily unless we know the specified range over which system was operated. Nonlinearity can leads to extensive system behaviour\cite{6, 7}, such as, jumps, higher harmonic resonances.

Volterra series represents the input output mapping of physical system, in the form of functional series, which yields a structured and convenient mathematical platform for study of nonlinear systems. Nonlinear system identification and characteristics of first and higher order harmonic amplitudes at different excitation levels for different nonlinear systems are investigated using Volterra series\cite{8}. Recursive iteration algorithm to estimate nonlinear parameter for nonlinear stiffness equation, convergence and error analysis, measurability of higher order harmonics have been presented \cite {9, 10}. Nonlinear bearing stiffness has been studied experimentally and theoretically for rolling element bearings \cite {10}. Mostly the system identification research \cite {1, 8-10} has been limited to, system with stiffness nonlinearity and only few researchers have been studied nonlinear damping \cite {3, 6, 7}. In the present work, Volterra series response is synthesised up to ‘$k$’ number of terms in terms of Frequency response functions (FRFs). Response amplitude for damping nonlinearity in terms of lower order
kernel transform is presented. Characteristics of dynamic behaviour of nonlinear damping are compared with nonlinear stiffness oscillator [10]. Behaviour of nonlinear system has been studied. Nonlinear parameter is estimated based on recursive iteration method

2. Volterra series representation

Volterra series with input excitation \( f(t) \) and output response \( x(t) \), in a form of functional series given by [9]

\[
x(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) + \ldots
\]

\[
x_n(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) f(t - \tau_1) \cdots f(t - \tau_n) d\tau_1 \cdots d\tau_n
\]

(2)

\( h_n(\tau_1, \tau_2, \ldots, \tau_n) \) are \( n \)-th order Volterra kernels. Multidimensional Fourier transforms of Volterra kernel transform with higher order as

\[
H_n(w_1, w_2, w_3, \ldots, w_n) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \tau_2, \tau_3, \ldots, \tau_n) \prod_{i=1}^{n} e^{-jw_i\tau_i} d\tau_1 \cdots d\tau_n
\]

(3)

Equation of motion for nonlinear stiffness and damping oscillator

\[
m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t)
\]

(4)

\[m\ddot{x}(t) + c\dot{x}(t) + c_3x^3(t) + k_1x(t) = f(t)
\]

(5)

For a single input harmonic excitation

\[f(t) = A\cos\omega t = \frac{A}{2}e^{j\omega t} + \frac{A}{2}e^{-j\omega t},\]

(6)

Using Volterra, response from eqn. (2) as

\[
x(t) = \sum_{n=1}^{\infty} \left( \frac{A^n}{2} \right) \sum_{p+s=n} C_{q} H_{p,s}^n(w)e^{jw_p+jw_s}
\]

(7)

Where \( H_{p,q}^n(w) \) the higher are order FRFs given by

\[
H_{p,q}^n(w) = H_n\left(\frac{w_1, \ldots, w_p, -w_s, \ldots, -w}{p \text{ times}} \right) \quad w_{p,s} = (p-s)w
\]

(8)

Volterra series response is synthesised up to ‘\( k \)’ number of terms. Volterra kernels of higher order are reduced in to lower order volterra kernels and response amplitude for the first and third harmonics as

\[X(w) = AH_1(w) + \frac{3}{4} A^3 H_3(w, w, -w) + \frac{5}{8} A^5 H_5(w, w, w, w, -w) + \text{higher order terms}
\]

(9)

\[X(w) = AH_1(w) - \frac{3}{4} A^3 c_jw^3 H_3^3(w)H_1(-w)\]

(10)

\[+ A^3 \left( \frac{9}{16} c_jw^6 H_4^3(w)H_1^3(-w) - \frac{9}{16} c_jw^6 H_4^3(w)H_1^3(-w)H_1(3w) \right)
\]

\[X(3w) = \frac{A^3}{4} H_3^3(w, w, w) + \frac{5}{16} A^5 H_5^3(w, w, w, w, -w) + \text{higher order terms}
\]

(11)

\[X(3w) = \frac{A^3}{4} c_jw^3 H_3^3(w)H_1(3w)\]

(12)
3. Nonlinearity structure identification
One can predict whether the system is linear or nonlinear based on response spectrum. Presence of odd and even harmonics in the response spectrum will indicate that, the system is nonlinear in nature. Response spectrum of stiffness (eqn. (4)) and damping nonlinearity (eqn. (5)), for first and third harmonic amplitudes, with different excitation level and constant excitation frequency is shown in Figure 1. For analysis of single degree of freedom nonlinear system parameters are considered as m=1kg, c₁=0.1Ns/m, c₃=0.1Ns/m, k₁ =1 N/m.

![Figure 1. First and third harmonic in response spectrum (a)System with cubic stiffness nonlinearity, (b) System with cubic damping nonlinearity](image)

3.1. Comparison of dynamic behaviour of stiffness and damping nonlinearity

![Figure 2. Runge-Kutta method First harmonic amplitude (a)System with cubic stiffness nonlinearity, b) System with cubic damping nonlinearity](image)

Figure 2. Shows that the first harmonic amplitude X(ω), in cubic stiffness and cubic damping nonlinearity by Runge-Kutta method for given values c₁=0.1, c₃=0.1, ω =0.33. It is observed that in the Figure 2(a), amplitude undergoes sudden jump near the resonance, this occur with increasing frequency of excitation. Amplitude gradually increases until certain point and then suddenly jumps to a large value and diminishes along the curve to its straight. Amplitude frequency plot is unstable. Higher harmonic peak occurs at one third of fundamental frequency (ω =1/3 ω₀). Figure 2(b) Show
cubic damping nonlinearity in the first harmonic amplitude, there is no jump phenomenon, no unstable region and no super harmonic resonance.

4. Parameter estimation in system with nonlinear cubic damping

4.1. Convergence error analysis

The relative error between the finite term response series approximation and the exact amplitude of the harmonics is [10].

\[ e \approx \left[ X(nr) - k \cdot X(nr) \right] / X(nr) \]  

(13)

![Figure 3](image1)

**Figure 3.** Variation of series approximation error in third harmonic amplitude X (3ω)

For an eqn. (5), Figure 3 show the Volterra approximation error variation in response of third harmonic amplitude X(3ω) in comparison with Runge-Kutta fourth order method under constant excitation amplitude for a single term Volterra series approximation over a frequency range (ω =0.25 to 0.7). It can be seen that the approximation error is very high at natural frequency (ω =0.33), that is 1/3 \(^\text{rd}\) of its fundamental frequency (ω = ω/3=0.33), which is termed as a super harmonic excitation. Error is relatively high at frequency ω =0.33.

4.2. Measurability of higher order harmonics

![Figure 4](image2)

**Figure 4.** Excitation level corresponding to non-linear damping coefficient for a range frequency (a) Absolute measurability index, AMI in X(3ω), (b) Relative measurability index, RMI X(3ω)

Higher order harmonic amplitudes are used to estimate the nonlinear parameters. For weakly nonlinear systems, compared to first harmonic, higher order harmonics are very much smaller, because of vibration and noise. Therefore, it is essential to measure the response spectrum with suitable higher order harmonics. Figure 4(a) show the Absolute Measurability index (AMI) =X(3ω) of third order
harmonic amplitude for a range of frequency, with constant excitation at \( c_3 = 0.1 \). This Figure has been plotted for a given nonlinear eqn. (5) by Runge-Kutta fourth order method.

Relative measurability index (RMI) is defined in terms of strength of relative signal, which is the ratio of harmonics of higher order with respect to the first harmonic amplitude [10].

\[
RMI (m_w) = \frac{X(m_w)}{X(w)} \text{ where } m = 1, 2, 3, \ldots
\]  

(14)

Figure 4(b). Show the RMI variation of third harmonic amplitude \( X(3\omega) \) for a range of frequency. It is shown that peak is max, at \( \omega / 3 = 0.33 \) frequency, is named as peak measurability. This can be suggested that, \( X(3\omega) \) must be measured nearer to the 1/3rd frequency of a given system. RMI in \( X(3\omega) \) Shows in Figure 5 for a different excitation level considering different non-linear parameter \( c_3 \). It can be illustrated that higher order harmonics for weakly nonlinear systems are of much lower amplitude, compare to the fundamental harmonic. It can be seen that the response strength is significantly smaller at one third and near resonant frequency. Better signal/noise ratio is required to measure the third harmonic amplitude.

![Figure 5](image)

**Figure 5.** Relative measurability index, RMI in \( X(3\omega) \) for a different excitation level with corresponding to non-linear damping coefficient (a) \( c_3 = 0.01, 0.02, 0.03 \), (b) \( c_3 = 0.04, 0.05, 0.06 \)

### 4.3. Recursive iteration method

Recursive iteration method is used to estimate non-linear parameters \( c_3 \) and linear parameters \( m, c, k_1 \). The response harmonic series \( X(n \omega) \) is truncated after a finite number of terms, \( k \), and re arranged for first and third harmonic response [10].

\[
H_1(w) = \frac{1}{A} \left[ X(w) - \sum_{i=2}^{k} \sigma_i(w) \right]
\]

(15)

\[
H_3(w, w, w) = \frac{4}{A^3} \left[ X(3w) - \sum_{i=2}^{k} \sigma_i(3w) \right]
\]

(16)

Where Response harmonic amplitude, \( X(mw) \)

\[
X(nw) = \sum_{i=2}^{k} \sigma_i(nw)
\]

(17)

\[
\sigma_i(nw) = 2 \left( \frac{A}{2} \right)^{n+2i-1} c_{i-1} H_i^{2i-1}(w)
\]

(18)

### 4.4. Error estimation in non-linear parameter \( c_3 \)

The non-linear parameter \( c_3 \) will be estimated by single term Volterra series approximation for different excitation level. Error is calculated by comparing estimated value with the obtained value.
with single term Volterra approximation. Figure 6 Show that the error in $c_3$ will be decreases up to excitation frequency ($\omega=0.7$) and then error will increase after the excitation frequency ($\omega =0.7$).

**Figure 6.** Error in First estimation of $c_3$ for a different excitation level

5. Conclusion

Using Volterra series, parameter estimation procedure has been developed for polynomial form of damping and stiffness nonlinearity, considering single degree of freedom system. Analysis show jump phenomenon in nonlinear stiffness at the higher harmonic resonance, and there is no such type of behaviour present in damping nonlinearity. Near the resonant frequency results are not converging for nonlinear damping by Volterra and Runge-Kutta methods, and even there is no jump phenomenon at this point. By measurement of harmonics of higher order, nonlinear parameters are estimated. It is observed that higher order harmonics for weakly nonlinear systems are of much lower amplitude, compare to the fundamental harmonic because of background vibration and noise. Response strength is significantly very small. In order to measure the response amplitude better signal/noise ratio is required. The recursive method is used to estimate nonlinear parameter.

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