Note on quantitative homogenization results
for parabolic systems in $\mathbb{R}^d$ *

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Abstract
In $L_2(\mathbb{R}^d; C^n)$, we consider a semigroup $e^{-tA_\varepsilon}$, $t \geq 0$, generated by a matrix elliptic second order differential operator $A_\varepsilon \geq 0$. Coefficients of $A_\varepsilon$ depend on $x/\varepsilon$ and oscillate rapidly as $\varepsilon \to 0$. Approximations for $e^{-tA_\varepsilon}$ were obtained by T. A. Suslina (2004, 2010) via the spectral method and by V. V. Zhikov and S. E. Pastukhova (2006) via the shift method. In the present note, we give another short proof based on the contour integral representation for the semigroup and approximations for the resolvent with two-parametric error estimates obtained by T. A. Suslina (2015).

Key words: homogenization, convergence rates, parabolic systems, Trotter-Kato theorem.

Introduction
The subject of this note is quantitative estimates in periodic homogenization, i.e., approximations for the corresponding resolving operator in the uniform operator topology. There are several approaches to obtaining results of such type, see [BSu, CDaGr, Sh, ZhPas3].

In introduction, let us consider the simplest periodic operator $A_\varepsilon = -\text{div} g(\varepsilon^{-1} x) \nabla$, $\varepsilon > 0$, acting in $L_2(\mathbb{R}^d)$. Here $g$ is a periodic positive definite matrix valued function such that $g, g^{-1} \in L_\infty$. Let $u_\varepsilon$ be the solution of the equation $A_\varepsilon u_\varepsilon + u_\varepsilon = F$, where $F \in L_2(\mathbb{R}^d)$. The homogenization problem is to describe the behavior of the solution $u_\varepsilon$ in the small period limit $\varepsilon \to 0$. The classical answer is that $u_\varepsilon \to u_0$ in the $L_2$-norm, where the limit function $u_0$ is the solution of the equation of the same type $A^0 u_0 + u_0 = F$ but with the so-called effective operator $A^0 = -\text{div} g^0 \nabla$ with the constant matrix $g^0$.

By using the spectral method, M. Sh. Birman and T. A. Suslina [BSu] prove that $\|u_\varepsilon - u_0\|_{L_2} \leq C\varepsilon \|F\|_{L_2}$. This estimate can be rewritten as approximation for the resolvent $(A_\varepsilon + I)^{-1}$ in the uniform operator topology. Approximations for the semigroup $e^{-tA_\varepsilon}$, $t \geq 0$, were obtained in [Su1, Su2], and [ZhPas2] via the spectral and shift methods, respectively:

$$\|e^{-tA_\varepsilon} - e^{-tA^0}\|_{L_2(\mathbb{R}^d)} \leq C\varepsilon(t + \varepsilon^2)^{-1/2}.$$  (1)

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Since the point zero is the lower edge of the spectra for $A_ε$ and $A^0$, estimate (1) can be treated as a stabilization result for $t \to \infty$. Later in [MSu], it was observed that the quantitative results for parabolic problems can be derived from the corresponding elliptic results with the help of the identity $e^{-tA_ε} = \frac{1}{2\pi i} \int_{\gamma} e^{-\zeta} (A_ε - \zeta I)^{-1} d\zeta$, where $\gamma \subset \mathbb{C}$ is a contour enclosing the spectrum of $A_ε$ in a positive direction. But in [MSu] only problems in a bounded domain $\mathcal{O} \subset \mathbb{R}^d$ were studied and operators under consideration were positive definite. In the case of the Dirichlet boundary condition, it was obtained that
\[
\|e^{-tA_{D,\varepsilon}} - e^{-tA^0_{D}}\|_{L_2(\mathcal{O}) \to L_2(\mathcal{O})} \leq C\varepsilon(t + \varepsilon^2)^{-1/2}e^{-ct}.
\] (2)

The unique conceptual difference between (1) and (2) is the behaviour at $t \to 0$. While (2) contains exponentially decaying factor $e^{-ct}$, we cannot speak on stabilization for $A_{D,\varepsilon}$. Indeed, for the difference of the exponentials we have the rough estimate $\|e^{-tA_{D,\varepsilon}} - e^{-tA^0_{D}}\| \leq 2e^{-c_*t}$, where $c_* > 0$ is a common lower bound for $A_{D,\varepsilon}$ and $A^0_{D}$. In (2), the constant $c$ is such that $0 < c < c_*$ and the constant $C$ depends on our choice of $c$ and grows as $c \to c_*$. This caused by the used Cauchy integral representation and the behaviour of the error estimate in approximation the resolvent $\left((A_{D,\varepsilon} - \zeta I)^{-1}\right)$ for small fixed $|\zeta|$. According to the results of [Su1], the estimate for the resolvent $\left((A_ε - \zeta I)^{-1}\right)$ has different behaviour with respect to $\zeta$ compared to the one for $\left((A_{D,\varepsilon} - \zeta I)^{-1}\right)$.

The goal of the present note is to show how parabolic results from [Su1, ZhPas2, Su3] can be derived from approximations for $\left((A_ε - \zeta I)^{-1}\right)$ in $(L_2 \to L_2)$- and $(L_2 \to H^1)$-norms from [Su4].

The difference between methods of the present paper and [MSu] consists of choosing the contour $\gamma$ depending on time $t$. This idea is inspired by the proof of [INZ Lemma 1].

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1 Preliminaries. Known results

Let $\Gamma \subset \mathbb{R}^d$ be a lattice, and let $\Omega$ be the cell of the lattice $\Gamma$. By $H^1_{\text{per}}(\Omega)$ we denote the subspace of matrix valued functions from $H^1(\Omega)$ whose $\Gamma$-periodic extension belongs to $H^1_{\text{loc}}(\mathbb{R}^d)$. For any $\Gamma$-periodic matrix valued function $f$ we use the notation $f^\varepsilon(x) := f(\varepsilon^{-1}x)$, $\varepsilon > 0$. By $[f^\varepsilon]$ we denote the operator of multiplication by the matrix valued function $f^\varepsilon(x)$.

In $L_2(\mathbb{R}; \mathbb{C}^n)$, we consider matrix elliptic second order differential operator $A_ε$, $\varepsilon > 0$, formally given by the expression $A_ε = b(D)^*g(x)b(D)$. Here $g$ is a $\Gamma$-periodic $(m \times m)$-matrix valued function, $g(x) > 0$, $g, g^{-1} \in L_\infty$, and $b(D) = \sum_{l=1}^d b_l D_l$ is a first order differential operator with constant $(m \times n)$-matrix valued coefficients $b_l$, $l = 1, \ldots, d$. The coefficients of matrices $g(x)$ and $b_l$, $l = 1, \ldots, d$, are in general complex. Suppose that $m \geq n$ and that the symbol $b(\xi) = \sum_{l=1}^d b_l \xi_l$ satisfies the full rank condition $\text{rank} b(\xi) = n$, $0 \neq \xi \in \mathbb{R}^d$. Or, equivalently, there exist constants $\alpha_0, \alpha_1$ such that
\[
\alpha_0 1_n \leq \langle b(\theta)^*b(\theta) \rangle \leq \alpha_1 1_n, \quad \theta \in S^{d-1}, \quad 0 < \alpha_0 \leq \alpha_1 < \infty.
\]

Under the above assumptions, the operator $A_ε$ is self-adjoint, non-negative and strongly elliptic. The precise definition is given via the corresponding quadratic form on $H^1(\mathbb{R}^d; \mathbb{C}^n)$.
The simplest example of the operator under consideration is the acoustics operator $A_\varepsilon = -\text{div} g^\varepsilon(x) \nabla$. The operator of elasticity theory also can be written as $b(D)^* g^\varepsilon(x) b(D)$, see details in [BSu, Chapter 5].

The coefficients of the operator $A_\varepsilon$ oscillate rapidly as $\varepsilon \to 0$. The limit behaviour of its resolvent or the semigroup $e^{-tA_\varepsilon}$ is given by the corresponding function of the so-called effective operator $A^0 = b(D)^* g^0 b(D)$ with the constant matrix $g^0$. The definition of $g^0$ is given in terms of the $\Gamma$-periodic $(n \times m)$-matrix valued function $\Lambda$:

$$g^0 = |\Omega|^{-1} \int_\Omega g(x)(b(D)\Lambda(x) + 1_m) \, dx,$$

where $\Lambda \in H^1_{\text{per}}(\Omega)$ is the weak solution of the cell problem

$$b(D)^* g(x)(b(D)\Lambda(x) + 1_m) = 0, \quad \int_\Omega \Lambda(x) \, dx = 0.$$

By $S_\varepsilon$ we denote the Steklov smoothing operator acting in $L_2(\mathbb{R}^d; \mathbb{C}^m)$ by the rule

$$(S_\varepsilon u)(x) = |\Omega|^{-1} \int_\Omega u(x - \varepsilon z) \, dz.$$

According to [ZhPas1, Lemma 1.2], for any $\Gamma$-periodic function $f$ in $\mathbb{R}^d$ such that $f \in L_2(\Omega)$, the operator $[f^\varepsilon]S_\varepsilon$ is continuous in $L_2(\mathbb{R}^d)$, and $\|[f^\varepsilon]S_\varepsilon\|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq |\Omega|^{-1/2}\|f\|_{L_2(\Omega)}$. By using this statement and inclusion $\Lambda \in H^1_{\text{per}}(\Omega)$, one can show that the so-called corrector

$$K(\varepsilon; \zeta) := [A^\varepsilon]S_\varepsilon b(D)(A^0 - \zeta I)^{-1}$$

acts continuously from $L_2(\mathbb{R}^d; \mathbb{C}^m)$ to $H^1(\mathbb{R}^d; \mathbb{C}^m)$, and $\|K(\varepsilon; \zeta)\|_{L_2 \rightarrow H^1} = O(\varepsilon^{-1})$ for fixed $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$. The $(L_2 \rightarrow H^1)$-continuity of the operator $[10]$ below can be checked by using the same arguments.

The following result was obtained in [Su4] Theorems 2.2 and 2.4.

**Theorem 1 (Su4).** Let the above assumptions be satisfied. Let $\zeta \in \mathbb{C} \setminus \mathbb{R}_+$, $\phi = \arg \zeta$. Denote

$$c(\phi) = \begin{cases} |\sin \phi|^{-1}, & \phi \in (0, \pi/2) \cup (3\pi/2, 2\pi), \\ 1, & \phi \in [\pi/2, 3\pi/2]. \end{cases}$$

Then for $\varepsilon > 0$ we have

$$\| (A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_1 c(\phi)^2 |\zeta|^{-1/2} \varepsilon. \quad (6)$$

Let $K(\varepsilon; \zeta)$ be the corrector [10]. Then for $\varepsilon > 0$ we have

$$\| D((A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon K(\varepsilon; \zeta)) \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_2 c(\phi)^2 \varepsilon, \quad (7)$$

$$\| (A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1} - \varepsilon K(\varepsilon; \zeta) \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_3 c(\phi)^2 |\zeta|^{-1/2} \varepsilon. \quad (8)$$

The constant $C_1$ depends only on $\alpha_0$, $\alpha_1$, $\|g\|_{L_\infty}$, $\|g^{-1}\|_{L_\infty}$, and parameters of the lattice $\Gamma$. The constants $C_2$ and $C_3$ depend on the same parameters and also on $m$ and $d$.

The aim of the present paper is to give another proof for the following theorem.

**Theorem 2 (Su4, ZhPas2, Su3).** Under the above assumptions, for $\varepsilon > 0$ and $t \geq 0$ we have

$$\| e^{-tA_\varepsilon} - e^{-tA^0} \|_{L_2(\mathbb{R}^d) \rightarrow L_2(\mathbb{R}^d)} \leq C_4 \varepsilon (t + \varepsilon^2)^{-1/2}, \quad t \geq 0. \quad (9)$$
Denote
\[ K(\varepsilon; t) := [\Lambda^t] S_\varepsilon b(D) e^{-tA_0}. \]  
Then for \( \varepsilon > 0 \) and \( t > 0 \) we have
\[
\| D(e^{-tA_\varepsilon} - e^{-tA_0} - \varepsilon K(\varepsilon; t)) \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_5\varepsilon t^{-1},
\]
\[
\| e^{-tA_\varepsilon} - e^{-tA_0} - \varepsilon K(\varepsilon; t) \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_6\varepsilon t^{-1/2}.
\]
The constant \( C_4 \) depends only on \( \alpha_0, \alpha_1, \| g\|_{L^\infty}, \| g^{-1}\|_{L^\infty}, \) and parameters of the lattice \( \Gamma \). The constants \( C_5 \) and \( C_6 \) depend on the same parameters and also on \( m \) and \( d \).

**Remark 3.** The estimate (9) was announced in [Su1, Theorem 1] and proven in [Su2, Theorem 7.1] and [ZhPas2, Theorem 1.1]. For the scalar elliptic operator \( A_\varepsilon = -\text{div} g^\varepsilon (x) \nabla \), one has \( \Lambda \in L^\infty \) and it is possible to replace the smoothing operator \( S_\varepsilon \) in the corrector by the identity operator. In this case, estimate (11) was obtained in [ZhPas2, Theorem 1.3]. For the matrix elliptic operator, \((L_2 \to H^1)\)-approximation for \( e^{-tA_\varepsilon} \) was proven in [Su3, Theorem 11.4].

## 2 New proof

Using the Riesz–Dunford functional calculus, we can represent the operator exponential \( e^{-tA_\varepsilon} \) as an integral over a suitable contour in a complex plane enclosing the spectrum \( \sigma(A_\varepsilon) \subset [0, \infty) \) in a positive direction:
\[
e^{-tA_\varepsilon} = \frac{1}{2\pi i} \int_{\gamma} e^{-\zeta t} (A_\varepsilon - \zeta I)^{-1} d\zeta.
\]
One can choose \( \gamma = \gamma_1 \cup \gamma_2 \) with
\[
\gamma_1 = \{ \zeta \in \mathbb{C} : \zeta = e^{i\phi}, \pi/4 \leq \phi \leq 7\pi/4 \},
\]
\[
\gamma_2 = \{ \zeta \in \mathbb{C} : \zeta = re^{i\pi/4}, r \geq 1 \} \cup \{ \zeta \in \mathbb{C} : \zeta = re^{i7\pi/4}, r \geq 1 \}.
\]
But we take the contour depending on \( t > 0 \), shrinking the contour \( \gamma \) in \( t \) times: \( \gamma_t = t^{-1}\gamma = \{ \zeta \in \mathbb{C} : \zeta = t^{-1}\eta, \eta \in \gamma \} \). Applying these arguments to operators \( e^{-tA_\varepsilon} \) and \( e^{-tA_0} \) and changing variable, we get
\[
e^{-tA_\varepsilon} - e^{-tA_0} = \frac{1}{2\pi i} \int_{\gamma_t} e^{-\zeta t} ((A_\varepsilon - \zeta I)^{-1} - (A^0 - \zeta I)^{-1}) d\zeta
\]
\[
= -\frac{1}{2\pi it} \int_{\gamma} e^{-\eta t} ((A_\varepsilon - t^{-1}\eta I)^{-1} - (A^0 - t^{-1}\eta I)^{-1}) d\eta.
\]
Using (10) and taking into account that \( c(\phi_t) \leq 2^{1/2} \), where \( \eta \in \gamma \) and \( \phi_t := \arg(t^{-1}\eta) \), for \( t > 0 \) we have
\[
\| e^{-tA_\varepsilon} - e^{-tA_0} \|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)} \leq C_1 t^{-1/2} \int_{\gamma} e^{-\eta t} |t^{-1}\eta|^{-1/2} |d\eta|
\]
\[
\leq \pi^{-1} C_1 e^{-t^{-1/2}} \left( 3\pi/2 + 2 \int_1^\infty e^{-r} dr \right) \leq \tilde{C}_4 e^{-t^{-1/2}},
\]
where \( \tilde{C}_4 := C_1 (3/2 + 2\pi^{-1} e^{-1}) \). Obviously, for \( t \geq 0 \) the left-hand side of (14) does not exceed 2. Since \( \min\{2; \tilde{C}_4 e^{-t^{-1/2}} \} \leq C_4 e(t + \varepsilon^2)^{-1/2} \), where \( C_4 = 2^{1/2} \max\{2; \tilde{C}_4 \} \), we arrive at the estimate (9).
By using the contour integral representation for the exponential \( e^{-tA_0} \) and the fact that the operator \( [Λ^ε]S_εb(D) \) is closed, we see that the operators (4) and (10) satisfy the identity
\[
K(ε; t) = -\frac{1}{2\pi i} \int_{γ_t} e^{-ζt} K(ε; ζ) dζ.
\] (15)

Similarly to the proof of estimate (9), relations (7), (13), and (15) imply estimate (11) with the constant \( C_5 := C_2(3/2 + 2π^{-1}e^{-1}) \). Estimate (12) follows from (8) on the same way, \( C_6 := C_3(3/2 + 2π^{-1}e^{-1}) \).

3 Discussion

Since we derive the parabolic estimates from the elliptic ones, the achievement of the present paper can be interpreted as a quantitative Trotter-Kato like result in homogenization context. For derivation of hyperbolic results from elliptic ones, see the preprint [M].

References

[BSu] M. Sh. Birman and T. A. Suslina, Second order periodic differential operators. Threshold properties and homogenization, Algebra i Analiz 15 (2003), no. 5, 1-108; English transl., St. Petersburg Math. J. 15 (2004), no. 5, 639–714.

[CDaGr] D. Cioranescu, A. Damlamian, and G. Griso, The periodic unfolding method. Theory and applications to partial differential problems, Springer, Series in Contemporary Mathematics, vol. 3, 2018.

[INZ] T. Ichinose, H. Neidhardt, and V. A. Zagrebnov, Trotter-Kato product formula and fractional powers of self-adjoint generators, J. Funct. Anal. 207 (2004), 33–57.

[M] Yu. Meshkova, Variations on the theme of the Trotter-Kato theorem for homogenization of periodic hyperbolic systems, arXiv:1904.02781 (2019).

[MSu] Yu. M. Meshkova and T. A. Suslina, Homogenization of initial boundary value problems for parabolic systems with periodic coefficients, Applicable Analysis 95:8 (2016), 1736–1775.

[Sh] Z. Shen, Periodic homogenization of elliptic systems, Advances in Partial Differential Equations, Oper. Theory Adv. Appl., vol. 296, Birkhäuser/Springer, 2018.

[Su1] T. A. Suslina, On homogenization of periodic parabolic systems, Funktsional. Analiz i ego Prilozhen. 38 (2004), no. 4, 86–90; English transl., Funct. Anal. Appl. 38 (2004), no. 4, 309–312.

[Su2] T. A. Suslina, Homogenization of a periodic parabolic Cauchy problem, Nonlinear equations and spectral theory, 201–233, Amer. Math. Soc. Transl. (2), vol. 220, Amer. Math. Soc., Providence, RI, 2007.

[Su3] T. A. Suslina, Homogenization of a periodic parabolic Cauchy problem in the Sobolev space \( H^1(\mathbb{R}^d) \), Math. Model. Nat. Phenom. 5 (2010), no. 4, 390–447.

[Su4] T. A. Suslina, Homogenization of elliptic operators with periodic coefficients in dependence of the spectral parameter, Algebra i Analiz 27 (2015), no. 4, 87–166; English transl., St. Petersburg Math. J. 27 (2016), no. 4, 651–708.
[ZhPas1] V. V. Zhikov and S. E. Pastukhova, *On operator estimates for some problems in homogenization theory*, Russ. J. Math. Phys. 12 (2005), no. 4, 515-524.

[ZhPas2] V. V. Zhikov and S. E. Pastukhova, *Estimates of homogenization for a parabolic equation with periodic coefficients*, Russ. J. Math. Phys. 13 (2006), no. 2, 224–237.

[ZhPas3] V. V. Zhikov and S. E. Pastukhova, *Operator estimates in homogenization theory*, Uspekhi Matem. Nauk 71 (429) (2016), no. 3, 27–122; English transl., Russian Math. Surveys 71 (2016), no. 3, 417–511.