GAUGE FIELDS ON COHERENT SHEAVES

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Abstract. Given a flat gauge field $\nabla$ on a vector bundle $F$ over a manifold $M$ we deduce a necessary and sufficient condition for the field $\nabla + E$, with $E$ an $\text{End}(F)$-valued 1-form, to be a Yang-
Mills field. For each curve of Yang-Mills fields on $F$ starting at $\nabla$, we define a cohomology class of $H^2(M, \mathcal{P})$, with $\mathcal{P}$ the sheaf of $\nabla$-parallel sections of $F$. This cohomology class vanishes when the curve consists of flat fields. We prove the existence of a curve of Yang-Mills fields on a bundle over the torus $T^2$ connecting two vacuum states.

We define holomorphic and meromorphic gauge fields on a co-
herent sheaf and the corresponding Yang-Mills functional. In this setting, we analyze the Aharonov-Bohm effect and the Wong equa-
tion.

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1. Introduction

From the mathematical point of view, a gauge field is a connection on a coherent sheaf, and the field strength is the curvature of the connection [7, 15, 26].

The set of connections on a coherent sheaf $\mathcal{F}$ over the manifold $M$ is an affine space associated to the vector space $\text{Hom}(\mathcal{F}, \mathcal{A}^1 \otimes \mathcal{F})$, where $\mathcal{A}^1$ is the sheaf of 1-forms on $M$. On the other hand, each element $E \in \Gamma(M, \mathcal{A}^1 \otimes \mathcal{E}nd(\mathcal{F}))$ determines a vector $\mathcal{E} \in \text{Hom}(\mathcal{F}, \mathcal{A}^1 \otimes \mathcal{F})$. But this correspondence is not necessarily 1 to 1. The map $E \mapsto \mathcal{E}$ is bijective, if $\mathcal{F}$ is locally free (Lemma 1) or when $\mathcal{F}$ is a reflexive sheaf (Lemma 3) (the reflexive sheaves might be thought as “vector bundles with singularities” [16 page 121]). We prove some properties of the connections on coherent sheaves, as the relation between the curvatures of two connections (Proposition 1). By means of the identification $E \leftrightarrow \mathcal{E}$, these properties give rise in the setting of locally free sheaves to well-known formulas for the connections on vector bundles. However,

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we will rewrite these formulas in a suitable way to prove some results mentioned below.

When the coherent sheaf is a Hermitian vector bundle $F$ over a Riemannian manifold $M$, the metrics on $F$ and $M$ determine a norm for the $\text{End}(F)$-valued 2-forms on $M$. In this situation, one defines the Yang-Mills functional, $\text{YM}$, which associates to each gauge on $F$ compatible with the Hermitian metric the norm of its strength.

Some vector bundles admit flat gauge fields; i.e. connections whose curvature vanishes. In this case the Yang-Mills functional takes its minimum value at those fields, obviously. They are the vacuum states of the theory [15, p. 447]. The Yang-Mills fields are the stationary “points” of $\text{YM}$. The calculus of variations shows that those fields must satisfy the Yang-Mills equation (2.30), which is the corresponding motion equation.

Given a gauge field $\nabla$ on $F$, any other connection on $F$ can be written as $\nabla + E$, where $E$ is an $\text{End}(F)$-valued 1-form on $M$, as we said. Assumed that $\nabla$ is a flat gauge field, we give, in Theorem 1, a necessary and sufficient condition for $\nabla + E$ to be a Yang-Mills field.

The curves in the space of Yang-Mills fields, with initial point at a flat configuration, are possible evolutions of the field from a vacuum state. We consider a curve $\nabla_t$ of Yang-Mills fields, with $\nabla_0$ a flat connection. We prove that such a curve determines an element of the cohomology $H^2(M, \mathcal{P})$, where $\mathcal{P}$ is the sheaf of sections of $F$ parallel with respect to the flat connection $\nabla_0$. If the curve $\nabla_t$ consists of flat gauge fields, then the corresponding cohomology class vanishes (Proposition 7).

On the trivial bundle $T \times \mathbb{C}^2$ over the 2-torus $T$, we construct a curve $\{\nabla_t\}_{t \in [0, 1]}$ of Yang-Mills fields, such that the only flat connections are the corresponding to $t = 0$ and $t = 1$ (Proposition 9). This curve connects two minimums of the Yang-Mills functional by means of stationary points of this functional. In physical terms, the system could evolve from a vacuum state to another one, through a “continuous sequence” of non-vacuum configurations.

As it is well-known a gauge field on a smooth vector bundle allows us to define the parallel transport along smooth curves. In the complex analytic setting, a gauge field should determine a parallel transport along curves whose tangent field is holomorphic. This involves that the connection must satisfy the Leibniz’s rule for the product of holomorphic functions and sections. That is, the connection must be holomorphic. On the other hand, the branes of type $B$ on a complex analytic manifold $X$ are the objects of $D^b(X)$, the bounded derived category of coherent sheaves [2 Sect. 5.4] [3 Sect. 5.3]. Thus, the coherent sheaves are particular membranes and therefore the study of
gauge fields on coherent sheaves is of interest from the point of view of theoretical Physics.

A holomorphic gauge field on a coherent sheaf $\mathcal{F}$ is a splitting of the extension of $\mathcal{F}$ defined by the 1-jet sheaf $\mathcal{j}(\mathcal{F})$ of $\mathcal{F}$. When $\mathcal{F}$ is a locally free sheaf, the existence of such a splitting is equivalent to the vanishing of the Atiyah class of the sheaf $[4]$. For example, the skyscraper sheaves do not admit holomorphic gauge fields (Proposition $[13]$). By contrast, as a consequence of Cartan’s B theorem, we deduce that the holomorphic bundles on a Stein manifold admit holomorphic gauge fields (Proposition $[10]$).

As the cohomology groups of coherent sheaves over a compact complex manifold $X$ have finite dimension, the holomorphic gauge fields on a holomorphic vector bundle (if they exist) form a finite dimensional affine space. Using this fact, we prove that if the Hodge numbers of $X$ satisfy $h^{2,0}(X) \leq h^{1,0}(X)$, then any “generic” holomorphic Yang-Mills field on a Hermitian line bundle is flat (Corollary $[1]$).

Some geometric phases $[6]$ can be considered in the frame of the meromorphic gauge fields on coherent sheaves. In the context of meromorphic connections, we study the Aharonov-Bohm effect $[6, 7]$ and the Wong equation $[5]$. The Aharonov-Bohm effect can be analyzed by considering a meromorphic gauge field on a trivial vector bundle over the manifold $X$, on which the electron is moving. More precisely, a meromorphic flat connection with a logarithmic pole at a point $* \in X$. By means of the connection, one constructs the respective de Rham complex $[19]$, and the Riemann-Hilbert correspondence (equivalently, the parallel transport) determines a representation of $\pi_1(X \setminus \{*\})$. The phase factor acquired by the wave function of the electron, once it has circumvented the solenoid, is determined by that representation. In this setting, the necessary conditions for the respective shift phase to occur clearly appear.

A similar study will be done for the variation of the spin of a particle that is moving in presence of a gauge field. For this purpose, we will transform the Wong equation, which governs the variation of spin, in a parallel transport equation. According to the gauge field singularities and the topological properties of the manifold on which the particle is moving, we will deduce the value of different phase factors.

The article is organized as follows. In the first part of Section $2$ we introduce notations and deduce some properties of the curvature of connections on coherent sheaves to be used below. In Subsection $2.1$ the above mentioned Theorem $[11]$ is proved. Subsection $2.2$ is concerned with curves of Yang-Mills fields starting from a vacuum state, and we
define the elements of $H^2(X, \mathcal{P})$ which they determine. In Subsection 2.3, one constructs a curve of non-flat Yang-Mills fields on the bundle $T \times \mathbb{C}^2 \to T$ that connects two different vacuum states.

In Subsection 3.1, the construction of the first jet bundle $J^1(F)$ of a holomorphic bundle $F$ is revised. We show that the existence of a holomorphic connection on $F$ is equivalent to the vanishing in the group $\text{Ext}^1(F, \Omega^1(F))$ of the extension $0 \to \Omega^1(F) \to J^1(F) \to F \to 0$, with $\Omega^1(F)$ the sheaf of $F$-valued holomorphic 1-forms. We prove also Proposition 10 and Corollary 1, above mentioned. In Subsection 3.2, we generalize, following [9], the definition of holomorphic connection to coherent sheaves, and then define the functional Yang-Mills on these fields in Subsection 3.3. The meromorphic fields are introduced in Subsection 3.4, and in that context the Aharonov-Bohm effect and the Wong equation are analyzed.

2. Yang-Mills fields and flat fields

By $M$ we denote a compact, connected manifold. We set $\mathcal{A}^k$ for the sheaf of differential $k$-forms on $M$, and $\mathcal{A}$ for the sheaf of differentiable functions.

Let $\mathcal{F}$ be a coherent $\mathcal{A}$-module. A connection on $\mathcal{F}$ is a morphism of abelian sheaves

$$(2.1) \quad \nabla : \mathcal{F} \to \mathcal{A}^1 \otimes_{\mathcal{A}} \mathcal{F},$$

satisfying the Leibniz’s rule

$$(2.2) \quad \nabla(g \sigma) = dg \otimes \sigma + g \nabla \sigma,$$

where $g$ and $\sigma$ are sections of $\mathcal{A}$ and $\mathcal{F}$ (resp.) defined on an open subset of $M$.

The connection $\nabla$ can be extended to a morphism

$$(2.3) \quad \nabla^k : \mathcal{A}^k \otimes_{\mathcal{A}} \mathcal{F} \to \mathcal{A}^{k+1} \otimes_{\mathcal{A}} \mathcal{F},$$

in the usual way

$$(2.4) \quad \nabla^k(\beta \otimes \sigma) = d\beta \otimes \sigma + (-1)^k \beta \wedge \nabla \sigma,$$

where $\beta \wedge \nabla \sigma$ is the image of $\beta \otimes \nabla \sigma$ by the natural morphism

$$(2.4) \quad \nabla : \mathcal{A}^k \otimes_{\mathcal{A}} (\mathcal{A}^1 \otimes_{\mathcal{A}} \mathcal{F}) \to \mathcal{A}^{k+1} \otimes_{\mathcal{A}} \mathcal{F}.$$

The curvature of $\nabla$ is the morphism of $\mathcal{A}$-modules

$$(2.5) \quad \nabla^k : \mathcal{F} \to \mathcal{A}^2 \otimes_{\mathcal{A}} \mathcal{F}.$$

It is well known that

$$\nabla^{k+1} \circ \nabla^k(\alpha \otimes \sigma) = \alpha \wedge \nabla^k(\sigma).$$
The connection $\nabla$ is called flat (or integrable) if $K_{\nabla} = 0$.

If $\nabla$ and $\tilde{\nabla}$ are connections on $\mathcal{F}$, then

$$\hat{\nabla} - \nabla \in \text{Hom}_A(\mathcal{F}, A^1 \otimes_A \mathcal{F}) = \Gamma(M, \mathcal{H}om_A(\mathcal{F}, A^1 \otimes_A \mathcal{F})),$$

where $\mathcal{H}om_A(\ldots, \ldots)$ denotes the respective sheaf of germs of $A$-morphisms [20, p 87]. Conversely, given $\mathcal{E} \in \text{Hom}_A(\mathcal{F}, A^1 \otimes_A \mathcal{F})$ and a connection $\nabla$ on $\mathcal{F}$, then $\hat{\nabla} := \nabla + \mathcal{E}$ is a connection on $\mathcal{F}$. On the other hand, $\mathcal{E}$ induces a natural map

$$E^k : A^k \otimes_A \mathcal{F} \rightarrow A^{k+1} \otimes_A \mathcal{F}, \quad \alpha \otimes \sigma \mapsto N((\alpha \otimes (-1)^k \mathcal{E}(\sigma))).$$

Hence, given $\nabla$ and $\mathcal{E}$ one has two morphisms of abelian sheaves $\nabla^1, \mathcal{E}^1 : A^1 \otimes_A \mathcal{F} \rightarrow A^2 \otimes_A \mathcal{F}$, and five morphisms $\mathcal{F} \rightarrow A^2 \otimes_A \mathcal{F}$,

$$K_{\nabla}, \quad K_{\tilde{\nabla}}, \quad \nabla^1 \circ \mathcal{E}, \quad \mathcal{E}^1 \circ \nabla, \quad \mathcal{E}^1 \circ \mathcal{E}.$$

As the curvature of $\hat{\nabla}$ is given by $K_{\hat{\nabla}} = \hat{\nabla}^1 \circ \hat{\nabla}$, we have the following proposition.

**Proposition 1.** The curvature of $\hat{\nabla} = \nabla + \mathcal{E}$ is

$$K_{\hat{\nabla}} = K_{\nabla} + \nabla^1 \circ \mathcal{E} + \mathcal{E}^1 \circ \nabla + \mathcal{E}^1 \circ \mathcal{E}.$$

The presheaf

$$U \mapsto S(U) := A^1(U) \otimes_{A(U)} \text{Hom}_{A(U)}(\mathcal{F}|_U, \mathcal{F}|_U),$$

defines the $A$-module $S^+ = A^1 \otimes_A \mathcal{H}om_A(\mathcal{F}, \mathcal{F})$ [11, p. 231-232]. On the other hand, the $A$-module $\mathcal{H}om_A(\mathcal{F}, A^1 \otimes_A \mathcal{F})$ is the sheaf associated to the presheaf

$$U \mapsto S'(U) := \text{Hom}_{A(U)}(\mathcal{F}|_U, (A^1 \otimes_A \mathcal{F})|_U).$$

Given $C = \alpha \otimes f \in S(U)$, we define $C' \in S'(U)$ as follows. For any open $V \subset U$ and $\sigma \in \mathcal{F}(V)$, we set

$$C'(\sigma) = \alpha|_V \otimes f|_V(\sigma).$$

Thus, the correspondence $C \mapsto C'$ is a morphism of presheaves $\Phi : S \rightarrow S'$, which in turn determines a map between the spaces of sections of the respective sheaves

$$(2.10) \quad E \in \Gamma(M, A^1 \otimes_A \mathcal{H}om_A(\mathcal{F}, \mathcal{F})) \mapsto \mathcal{E} \in \text{Hom}_A(\mathcal{F}, A^1 \otimes_A \mathcal{F}).$$

(Just as the natural morphism $Z \otimes \text{End}(P) \rightarrow \text{Hom}(P, Z \otimes P)$ in the category of $R$-modules is not in general an isomorphism, (2.10) is not an isomorphism either).

If the restriction of $E \in \Gamma(M, A^1 \otimes_A \mathcal{H}om_A(\mathcal{F}, \mathcal{F}))$ to an open subset $V$ of $M$ can be written as

$$(2.11) \quad E|_V = \sum_a \eta_a \otimes e_a,$$
with \( \eta_a \in \mathcal{A}^1(V) \) and \( e_a \in \text{Hom}_{\mathcal{A}|_V}(\mathcal{F}|_V, \mathcal{F}|_V) \), then
\[
(2.12) \quad \mathcal{E}(\sigma) = \sum_a \eta_a \otimes e_a(\sigma),
\]
for \( \sigma \in \mathcal{F}(V) \).

We analyze the form of the operators involved in Proposition 1 when \( \mathcal{E} \) is the image of \( E \) by the map \( (2.10) \). Assumed that \( E|_V \) can be expressed as in \( (2.11) \), by \( (2.6) \),
\[
(\mathcal{E}^1 \circ \mathcal{E})(\sigma) = \sum_{a,b} \eta_a \wedge \eta_b \otimes e_a(e_b(\sigma)).
\]
That is, on \( \mathcal{F}(V) \),
\[
(2.13) \quad \mathcal{E}^1 \circ \mathcal{E} = \sum_{a,b} (\eta_a \wedge \eta_b) \otimes (e_a \circ e_b) =: E \wedge E.
\]

Similarly, by \( (2.3) \),
\[
(2.14) \quad \nabla^1 \circ \mathcal{E} = \sum_a \left( d\eta_a \otimes e_a(. \ ) - \eta_a \wedge \nabla(e_a(. \ )) \right),
\]
on \( \mathcal{F}(V) \).

If, for the sake of simplicity, we assume that \( E = \eta \wedge e \) and that \( \nabla(\sigma) \) can written as \( \alpha \otimes \tau \), then \( \mathcal{E}^1 \circ \nabla(\sigma) = (\eta \wedge \alpha) \otimes \tau \). Thus,
\[
(2.15) \quad \mathcal{E}^1 \circ \nabla = E \wedge \nabla.
\]

**Connections on locally free sheaves.** Let \( F \) be a \( C^\infty \) vector bundle of rank \( m \) over \( M \). By \( \mathcal{F} \) we denote the \( \mathcal{A} \)-module of sections of \( F \). Let \( \{u_i| i = 1, \ldots, n\} \) and \( s = (s_1, \ldots, s_m) \) be a local frames for \( \mathcal{A}^1 \) and \( \mathcal{F} \), respectively, defined on an open neighborhood \( U \) of a point \( x \in M \). Given an element \( C' \) of \( S'(U) \), it can be written on any \( W \subset V \), as
\[
C' = \sum_{a,b} f_{iab}(u_i \otimes s_a) \otimes s^b,
\]
where \( \bar{s} = (s^1, \ldots, s^m) \) is the frame dual of \( s \), and the \( f_{iab} \) are functions on \( W \). Thus, \( C' \) determines the element \( C \in S(W) \)
\[
C = \sum_{a,b} f_{ab} \otimes (s_a \otimes s^b),
\]
where \( f_{ab} \) is the 1-form \( f_{ab} = \sum_i f_{iab} u_i \). Hence, \( \Phi(W) : S(W) \to S'(W) \)
is an isomorphism. That is, \( \Phi_x \) the map between the corresponding stalks is also an isomorphism; i.e. \( S \simeq S' \). Similarly,
\[
(2.16) \quad \mathcal{A}^k \otimes_{\mathcal{A}} \text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{F}) \simeq \text{Hom}_{\mathcal{A}}(\mathcal{F}, \mathcal{A}^k \otimes_{\mathcal{A}} \mathcal{F}).
\]
Thus, in particular, we have the following lemma.
Lemma 1. If $\mathcal{F}$ is a locally free $\mathcal{A}$-module, then the map defined in (2.10) is an isomorphism of vector spaces.

Analogously, the curvature $K_{\nabla}$ of a connection on $\mathcal{F}$ determines a vector of $\Gamma(M, \mathcal{A}^2 \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}))$, which will be denoted $K_{\nabla}$.

As it is well-known, the connection $\nabla$ induces a connection on $\mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$, the sheaf of endomorphisms of $\mathcal{F}$, by

$$(2.17) \quad \nabla(h) = \nabla \circ h - h \circ \nabla,$$

for $h$ section of $\mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$. According to (2.3) this connection defines in a natural way an operator

$$(2.18) \quad \nabla : \mathcal{A}^k \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}) \to \mathcal{A}^{k+1} \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}).$$

On the $k$-form $\mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$-valued $\beta \otimes h$

$$(2.19) \quad \nabla(E) = \mathcal{E}^1 \circ \nabla + \nabla^1 \circ \mathcal{E}.$$

From Proposition 1 together with (2.13) and (2.19), it follows the following proposition.

**Proposition 2.** Let $\mathcal{F}$ be the sheaf of sections of a vector bundle $\mathcal{F}$ over $M$. Given and $E \in \Gamma(M, \mathcal{A}^1 \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}))$ and $\nabla$ a connection on $\mathcal{F}$, then the curvature $\hat{K}$ of the connection $\hat{\nabla} = \nabla + \mathcal{E}$ is

$$\hat{K} = K + \nabla(E) + E \wedge E,$$

where $K$ is the curvature of $\nabla$.

**Remark.** The connection $\hat{\nabla} = \nabla + \mathcal{E}$ on $\mathcal{F}$ defines the corresponding one on $\mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$ in accordance with (2.17). Thus, with the above notations, $\hat{\nabla}(h) = \nabla + [\mathcal{E}, h]$. Furthermore, one has the extension of $\hat{\nabla}$ to a map on $\mathcal{A}^k \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$. This last operator acting on $\mathcal{A}^k \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F})$ is the sum of $\nabla$, and a morphism of sheaves

$$(2.20) \quad \mathcal{A}^k \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}) \to \mathcal{A}^{k+1} \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}), \quad D \mapsto E \wedge D - (-1)^k D \wedge E.$$

In particular, if $E = \eta \otimes e$, then this map takes the form

$$(2.21) \quad \beta \otimes h \in \mathcal{A}^k \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}) \mapsto (\eta \wedge \beta) \otimes [e, h] \in \mathcal{A}^{k+1} \otimes \mathcal{A} \mathcal{E}nd_{\mathcal{A}}(\mathcal{F}).$$

**Hermitian connections.** Let $\mathcal{F} \to M$ be a $C^\infty$ a Hermitian vector bundle of rank $m$ over the manifold $M$. We set $\mathfrak{u}(\mathcal{F})$ for the sheaf of antihermitian endomorphisms of $\mathcal{F}$. Obviously, $\mathfrak{u}(\mathcal{F})$ is a $\mathbb{R}$-module, $\mathbb{R}$ being the sheaf of $\mathbb{R}$-valued $C^\infty$ functions on $M$. 
The metric $\langle \cdot, \cdot \rangle$ on $F$ determines a metric on $u(F)$ which satisfies for all $x \in X$

\begin{equation}
\langle \text{ad}_c(a(x)), b(x) \rangle + \langle a(x), \text{ad}_c(b(x)) \rangle = 0,
\end{equation}

where $a, b, c$ are any sections of $u(F)$ and $a(x), b(x),$ and $c(x)$ are the corresponding vectors in the fibre $u(F)(x)$ of the sheaf $u(F)$ at $x$.

A Hermitian connection $\nabla$ on $F$ is a connection compatible with the Hermitian structure; i.e. such that

\begin{equation}
\langle \nabla \sigma, \tau \rangle + \langle \sigma, \nabla \tau \rangle = d \langle \sigma, \tau \rangle,
\end{equation}

for any sections $\sigma, \tau$ of $F$. In this case, $\nabla$ determines a $\mathbb{R}_X$-morphism

$$ \nabla : A^k \otimes_{\mathbb{R}} u(F) \to A^{k+1} \otimes_{\mathbb{R}} u(F). $$

If furthermore $\nabla$ is integrable (i.e. a flat gauge field), one has the following complex

$$ A^\bullet \otimes_{\mathbb{R}} u(F) : u(F) \xrightarrow{\nabla} A^1 \otimes_{\mathbb{R}} u(F) \xrightarrow{\nabla} A^2 \otimes_{\mathbb{R}} u(F) \to \ldots $$

The Poincaré’s lemma holds for this complex, and hence

$$ H^j \left( A^\bullet \otimes_{\mathbb{R}} u(F) \right) = 0, \quad j > 0. $$

The 0-cohomology is the local system $\mathcal{P} = \text{Ker}(\nabla)$, defined by the parallel sections of $u(F)$.

The spectral sequence $E_2^{ij} = H^j(M, H^i(A^\bullet \otimes_{\mathbb{R}} u(F)))$, converges to the hypercohomology $H^*(X, A^\bullet \otimes_{\mathbb{R}} u(F))$; that is, to the cohomology of the complex

$$ A^\bullet(u(F)) := \Gamma(M, A^\bullet \otimes_{\mathbb{R}} u(F)). $$

Hence,

\begin{proposition}
The cohomology groups of the sheaf $\mathcal{P}$ of parallel sections of $u(F)$ satisfy

$$ H^i(M, \mathcal{P}) = H^i(A^\bullet(u(F))). $$
\end{proposition}

2.1. Yang-Mills fields. We assume now that $M$ is a Riemannian oriented manifold. The map between the sheaf of $A^1$ of $C^\infty$ 1-forms and the sheaf of $\mathcal{F}$ of $C^\infty$ vector fields induced by the metric will be denoted $\lambda \mapsto \lambda^\sharp$. Given $\xi, \zeta$ k-forms, $(\xi, \eta)$ will denote the function defined by the relation

\begin{equation}
\xi \wedge \ast \zeta = (\xi, \zeta) \text{dvol},
\end{equation}

where $\ast$ is the Hodge star operator and dvol the volume form.
Given the vector field \( v \in T \), we set \( \iota_v \) for the inner derivative defined by \( v \). One has the well-known relation

\[
(\lambda \wedge \xi, \zeta) = (\xi, \iota_{\lambda \xi}(\zeta)).
\]

As we said, \( \mathcal{A}^k(u(F)) \) will denote the vector space

\[
\mathcal{A}^k(u(F)) := \Gamma(M, \mathcal{A}^k \otimes \mathbb{R} u(F))
\]

of global sections of \( \mathcal{A}^k \otimes \mathbb{R} u(F) \).

On the space \( \mathcal{A}^k(u(F)) \) one defines the following product

\[
\langle \langle \xi \otimes \sigma, \xi' \otimes \sigma' \rangle \rangle := \int_M (\xi \wedge \star \xi')(\sigma, \sigma'),
\]

where \( \langle \sigma, \sigma' \rangle \) is the function on \( M \) defined by \( \langle \sigma, \sigma' \rangle(x) = \langle \sigma(x), \sigma'(x) \rangle \) (here \( \sigma(x) \) is the corresponding vector of the fibre of \( u(F) \)).

Let \( \nabla \) be an integrable connection compatible with the metric of \( F \). Then one has the complex

\[
\mathcal{A}^0(u(F)) \xrightarrow{\nabla} \mathcal{A}^1(u(F)) \xrightarrow{\nabla} \ldots
\]

**Lemma 2.** The complex (2.27) is elliptic.

**Proof.** Let \( s \) be a unitary local frame of \( F \) and \( \bar{\sigma} \) denote its dual. An element \( D \in \mathcal{A}^k(u(F)) \) can be written

\[
D = \sum_{i,j} D_{ij} (s_i \otimes s^j) = s \cdot D \cdot \bar{s},
\]

where \( D \) is a matrix of \( k \)-forms. If \( A \) is the matrix of \( \nabla \) in the frame \( s \); that is, \( \nabla s = s \cdot A \), and hence \( \nabla \bar{s} = -A \cdot \bar{s} \), then

\[
\nabla(D) = s \cdot (dD + A \wedge D - (-1)^k D \wedge A) \cdot \bar{s}.
\]

That is, the operator \( \nabla \) acting on the elements of \( \mathcal{A}^k(u(F)) \) is

\[
d + A \wedge (\cdot) - (-1)^k (\cdot) \wedge A.
\]

Thus, the principal symbol of the operator \( \nabla \) is equal to the one of the exterior differential operator \( d \).

Hence,

\[
H^i(\mathcal{A}^*(u(F))) = \text{Ker}(\Delta_i),
\]

where \( \Delta_i \) is the corresponding Laplacian operator [29, Chap. IV, Sect. 5]. From Proposition 3, it follows the proposition.

**Proposition 4.** If \( \nabla \) is a connection flat, then with the above notations

\[
H^i(X, \mathcal{P}) = \text{Ker}(\Delta_i).
\]
Given a point \( x \in M \), we denote by \((A^2 \otimes_R u(F))(x)\) the fibre of the sheaf \( A^2 \otimes_R u(F) \) at \( x \). Let \( \nabla \) be a connection \( F \) compatible with the metric \( \langle , \rangle \). \( K_\nabla(x) \) will denote the vector of \((A^2 \otimes_R u(F))(x)\) defined by \( K_\nabla \).

The Yang-Mills functional is the map defined on the space of Hermitian connections by

\[
\nabla \mapsto \text{YM}(\nabla) = ||K_\nabla||^2,
\]

where \( || \cdot || \) is the norm determined by \( \langle\langle , \rangle\rangle \) \cite{15, page 417} \cite{25, page 44} \cite{26, page 357}. The Yang-Mills fields are the \( \nabla \)'s on which \( \text{YM} \) takes stationary values.

If \( \nabla \) is a connection on the Hermitian bundle \( F \) compatible with the metric \( \langle , \rangle \), from (2.23) together with (2.2), it follows that any other connection compatible with this metric has the form \( \hat{\nabla} = \nabla + E \), with \( E \in A^1(u(F)) \).

A “variation” of the connection \( \nabla \) can be written as \( \nabla + \varepsilon E \). By Proposition \( \hat{} \), \( K_\varepsilon := K_\nabla + \varepsilon \nabla(E) + \varepsilon^2 E \wedge E \). Hence,

\[
\left. \frac{d \|K_\varepsilon\|^2}{d\varepsilon} \right|_{\varepsilon=0} = 2\langle\langle \nabla E, K_\nabla \rangle\rangle = 2\langle\langle E, \delta K_\nabla \rangle\rangle,
\]

where \( \delta \) is the adjoint of \( \nabla \), when \( \nabla \) is considered as an operator \( \nabla : A^1(u(F)) \to A^2(u(F)) \).

Thus, \( \nabla \) is a Yang-Mills field iff it satisfies the Yang-Mills equation

\[
(2.30) \quad \delta_\nabla(K_\nabla) = 0.
\]

Given \( E \in A^1(u(F)) \), the morphism of \( \mathcal{A} \)-modules \( \langle 2.20 \rangle \) gives rise to an operator \( \mathcal{A}^k(u(F)) \to \mathcal{A}^{k+1}(u(F)) \). We will determine \( \hat{\delta} \), the formal adjoint operator of

\[
\hat{\nabla} = \nabla + E : A^1(u(F)) \to A^2(u(F)).
\]

Obviously, \( \hat{\delta} = \delta + E^\dagger \), where \( E^\dagger \) is the adjoint of the operator \( E \).

If \( E = \eta \otimes e \in A^1(u(F)) \), by \( \langle 2.21 \rangle \),

\[
(2.31) \quad E(\beta \otimes b) = (\eta \wedge \beta) \otimes [e, b],
\]

for \( \beta \otimes b \in A^1(u(F)) \). Given \( \gamma \otimes c \in A^2(u(F)) \), by \( \langle 2.26 \rangle \)

\[
\langle\langle E(\beta \otimes b), \gamma \otimes c \rangle\rangle = \int_M (\eta \wedge \beta \wedge *(\gamma))(\langle[e, b], c\rangle).
\]

From \( \langle 2.24 \rangle \) together with \( \langle 2.25 \rangle \), it follows

\[
(\eta \wedge \beta \wedge *\gamma = \beta \wedge *(\iota_{\eta b}(\gamma))).
\]
On the other hand, by (2.22)
\[ \langle [e, b], c \rangle = \langle b, [c, e] \rangle. \]

Thus,
\[ \langle \langle E(\beta \otimes b), \gamma \otimes c \rangle \rangle = \int_M \beta \wedge \star(\iota_{\eta \otimes b} (\gamma)) \langle b, [c, e] \rangle = \langle \langle \beta \otimes b, E^\dagger(\gamma \otimes c) \rangle \rangle, \]
where
\[ E^\dagger(\gamma \otimes c) = \iota_{\eta \otimes b} (\gamma) \otimes [c, e]. \]

We have proved the following proposition.

**Proposition 5.** Given \( E = \eta \otimes e \in \mathcal{A}^1(u(F)) \) the adjoint of the operator \( E \) defined in (2.31) is
\[ E^\dagger: \mathcal{A}^2(u(F)) \to \mathcal{A}^1(u(F)), \quad \gamma \otimes c \mapsto \iota_{\eta \otimes b} (\gamma) \otimes [c, e]. \]

As a consequence of (2.30) and Proposition 2, we have the following theorem.

**Theorem 1.** Let \( \nabla \) be a integrable connection on \( F \) compatible with the Hermitian structure and \( E \in \mathcal{A}^1(u(F)) \). Then \( \hat{\nabla} = \nabla + E \) is a Yang-Mills connection iff
\[ \delta(\nabla(E)) + \delta(E \wedge E) + E^\dagger(\nabla(E)) + E^\dagger(E \wedge E) = 0, \]
where \( E^\dagger \) is the operator defined in Proposition 5 and \( \delta \) is the adjoint operator (2.29).

If \( F \) admits integrable connections, on these gauge fields, \( YM \) takes the minimal value. The vacuum states of corresponding Yang-Mills theory are the gauge equivalence classes integrable connections. The space of vacuum states is in bijective correspondence to the set of equivalence classes of the representations of \( \pi_1(M) \) in \( U(m) \) [28].

Not every vector bundle \( F \) on the manifold \( M \) admits an integrable connection. By the Frobenius theorem, the existence of an integrable connection on \( F \) is equivalent to the fact that \( F \) admits a family of local frames, whose domains cover \( M \) and such that the corresponding transition functions are constant; i.e., \( F \) is a flat vector bundle [21, p. 5].

Assumed that \( F \) is a flat vector bundle, in the next section, we will study properties of continuous families \( \{\nabla_t\}_t \) of Yang-Mills fields with initial point at a vacuum state. That is, properties of the evolution of a Yang-Mills field from a vacuum state.
2.2. Curves of Yang-Mills connections. Throughout this section \( \nabla \) will denote a flat Hermitian connection on \( F \). A curve of Hermitian connections on \( F \) with initial point at \( \nabla \) has the form

\[
\nabla_t = \nabla + \mathcal{E}(t), \quad t \geq 0,
\]

with \( \mathcal{E}(t) \in \mathcal{A}^1(\mathfrak{u}(F)) \) and \( \mathcal{E}(0) = 0 \). We define the following vectors of \( \mathcal{A}^1(\mathfrak{u}(F)) \)

\[
E_1 := \lim_{t \to 0} E(t)/t, \quad E_2 := \lim_{t \to 0} (E(t) - tE_1)/t^2.
\]

Thus, \( E(t) = tE_1 + t^2E_2 + O(t^3) \). We set \( C_E := \nabla E_2 + E_1 \wedge E_1 \in \mathcal{A}^2(\mathfrak{u}(F)) \).

**Proposition 6.** If \( \nabla_t \) is a curve of Yang-Mills connections, then

1. \( E_1 \) determines an element \( \lbrack E_1 \rbrack \in H^1(X, \mathcal{P}) \),
2. \( C_E \) defines an element of \( H^2(X, \mathcal{P}) \).

*Proof.* By Theorem 1,

\[
0 = t\delta(\nabla E_1) + t^2\delta(\nabla E_2) + t^2\delta(E_1 \wedge E_1) + t^2E_1^\dagger(\nabla E_1) + O(t^3).
\]

Thus,

\[
\delta(\nabla E_1) = 0, \quad \delta(\nabla E_2) + \delta(E_1 \wedge E_1) + E_1^\dagger(\nabla E_1) = 0.
\]

The condition \( \delta(\nabla E_1) = 0 \) implies

\[
0 = \langle \delta(\nabla E_1), E_1 \rangle = \langle \nabla E_1, \nabla E_1 \rangle;
\]

that is \( \nabla(E_1) = 0 \). From Proposition 3 it follows the first assertion of the proposition.

Consequently, the second equation in \( \text{(2.33)} \) reduces to

\[
\delta(\nabla(E_2) + E_1 \wedge E_1) = 0.
\]

That is, \( \delta(C_E) = 0 \). Furthermore, as \( \nabla \) is flat,

\[
\nabla(C_E) = \nabla^2(E_2) + \nabla(E_1) \wedge E_1 - E_1 \wedge \nabla(E_1) = 0.
\]

From \( \delta(C_E) = 0 \) and \( \nabla(C_E) = 0 \), together with Proposition 4 it follows that \( C_E \) defines an element of \( H^2(X, \mathcal{P}) \). \( \square \)

**Proposition 7.** If \( \nabla_t = \nabla + \mathcal{E}(t) \) is a curve of integrable connections, then \( C_E = 0 \).

*Proof.* As \( \nabla_t \) is flat, by Proposition 2

\[
\nabla E(t) + E(t) \wedge E(t) = 0.
\]

Hence

\[
t\nabla E_1 + t^2(\nabla E_2 + E_1 \wedge E_1) + O(t^3) = 0.
\]

Thus \( \nabla E_1 = 0 \) and \( C_E = 0 \). \( \square \)
Remark. By \( \mathcal{U}(\mathcal{F}) \) we denote the subsheaf of \( \text{Aut}(\mathcal{F}) \) consisting of the unitary automorphisms. Those automorphisms are the gauge transformations that preserve the Hermitian metric. Given \( G \in \Gamma(X, \mathcal{U}(\mathcal{F})) \), the composition \( G \circ \nabla \circ G^{-1} \) defines a new connection gauge equivalent to \( \nabla \). A smooth family \( G_t \) of elements of \( \Gamma(X, \mathcal{U}(\mathcal{F})) \), where \( G_0 \) is the identity, gives rise to a development

\[
G_t = \text{id} + tA_1 + t^2A_2 + O(t^3).
\]

Taking into account that \( G_t^{-1} = \text{id} - tA_1 + t^2(A_1A_1 - A_2) + O(t^3) \), it is straightforward to check that the corresponding \( E_1 \) and \( E_2 \) associated to the curve of connections \( G_t \circ \nabla \circ G_t^{-1} \) are

\[
E_1 = -\nabla(A_1), \quad E_2 = \nabla(A_1A_1) - \nabla(A_2) - A_1\nabla(A_1).
\]

Therefore for this curve \([E_1] = 0\) and of course \( C_E = 0\).

2.3. Curve connecting vacuum states. Let \( T = T^2 \) be the 2-torus \([0, 1] \times [0, 1]/\sim\), with

\[
(x, 0) \sim (x, 1), \quad (0, y) \sim (1, y),
\]

endowed with the metric \( dx^2 + dy^2 \). We denote by \( w = dx \wedge dy \) the corresponding volume form. Let \( F = T \times \mathbb{C}^2 \to T \) be the trivial bundle equipped with the obvious Hermitian structure. On \( F \) we consider the connection \( \nabla \) defined by the exterior derivative. That is, \( \nabla s_i = 0 \), \( i = 1, 2 \), where \( s_1(z) = (z, (1, 0)) \) and \( s_2(z) = (z, (0, 1)) \), for \( z \in T^2 \). Evidently, \( \nabla \) is a connection compatible with the Hermitian structure.

We will construct a new connection on \( F \), \( \hat{\nabla} = \nabla + E \), where \( E \in \mathcal{A}_T^1 \otimes_{\mathcal{A}_T} \mathfrak{su}(F) \) is defined as follows. By \( \sigma_a, a = 1, 2, 3 \) we denote the Pauli matrices. Then \( e_a(z) = (z, i\sigma_a) \), is a section of \( T \times \mathfrak{su}(2) \to T \).

We set

\[
E := \alpha \otimes e_1 + \beta \otimes e_2 + \gamma \otimes e_3,
\]

with \( \alpha, \beta, \gamma \) 1-forms on \( T \). As \([e_a, e_b] = -2\epsilon_{abc}e_c\), then (see (2.13))

\[
E \wedge E = -2((\beta \wedge \gamma) \otimes e_1 + (\gamma \wedge \alpha) \otimes e_2 + (\alpha \wedge \beta) \otimes e_3).
\]

We will assume that \( E \wedge E = 0 \); that is, \( \beta \wedge \gamma = \gamma \wedge \alpha = \alpha \wedge \beta = 0 \).

Since \( d\alpha, d\beta, d\gamma \) 2-form, there exist functions \( h_1, h_2, h_3 \) such that

\[
d\alpha = h_1w, \quad d\beta = h_2w, \quad d\gamma = h_3w.
\]

Hence,

\[
\nabla E = w \otimes \sum_{a=1}^{3} h_a e_a.
\]
By Theorem 1, \( \hat{\nabla} = \nabla + \mathcal{E} \) is a Yang-Mills connection iff
\[
(2.38) \quad \delta(\nabla E) + E_\dagger(\nabla E) = 0.
\]
From (2.37), one obtains
\[
(2.39) \quad \delta(\nabla E) = -\star d\star(\nabla E) = -\sum_a \star dh_a \otimes e_a.
\]
From Proposition 5 together with (2.37), it follows
\[
(2.40) \quad \delta(\nabla E) = 2\left( -\iota_{\gamma}(h_2 w) + \iota_{\beta}(h_3 w) \right) \otimes e_1 + \\
+ \left( -\iota_{\alpha}(h_3 w) + \iota_{\gamma}(h_1 w) \right) \otimes e_2 + \left( -\iota_{\beta}(h_1 w) + \iota_{\alpha}(h_2 w) \right) \otimes e_3.
\]
Hence, \( \hat{\nabla} \) is a Yang-Mills connection if
\[
(2.40) \quad \star dh_1 = 2\left( \iota_{\gamma}(h_2 w) - \iota_{\beta}(h_3 w) \right), \quad \star dh_2 = 2\left( \iota_{\alpha}(h_3 w) - \iota_{\gamma}(h_1 w) \right)
\]
\[
(2.41) \quad \star dh_3 = 2\left( \iota_{\beta}(h_1 w) - \iota_{\alpha}(h_2 w) \right).
\]
If \( \alpha = a_1 dx + a_2 dy, \beta = b_1 dx + b_2 dy, \gamma = c_1 dx + c_2 dy \), then
\[
\iota_{\alpha}(w) = a_1 dy - a_2 dx, \quad \iota_{\beta}(w) = b_1 dy - b_2 dx, \quad \iota_{\gamma}(w) = c_1 dy - c_2 dx.
\]
When \( \gamma = 0 \), but \( \alpha \neq 0 \neq \beta \) then \( h_3 = 0 \) and from (2.40), we deduce
\[
(2.41) \quad \frac{h_2}{h_1} = \frac{b_1}{a_1} = \frac{b_2}{a_2};
\]
that is, \( \beta = \lambda \alpha \), where \( \lambda \) is a non zero constant. Thus, \( E = \alpha \otimes (e_1 + \lambda e_2) \) defines a Yang-Mills field \( \hat{\nabla} \). From (2.37), it follows \( \nabla E \neq 0 \). From Proposition 2 one deduces the following proposition.

**Proposition 8.** With the above notations, the connection \( \hat{\nabla} = \nabla + \mathcal{E} \),
where \( E = \alpha (e_1 + \lambda e_2) \) with \( \alpha \neq 0 \) and \( \lambda \neq 0 \), is a non flat Yang-Mills connection.

**Remark.** If \( \beta = \gamma = 0 \), the equations for the \( h_a \) imply that \( h_1 \) is constant and \( h_2 = h_3 = 0 \), obviously. From (2.36), it follows \( h_1 = 0 \) (otherwise \( w \) would be exact), and by (2.37) \( \nabla E = 0 \). That is, the connection \( \hat{\nabla} \) defined by \( E = \alpha \otimes e_1 \) is flat. Moreover, if \( \alpha \) is not exact, then \( \nabla \) and \( \hat{\nabla} \) are not gauge equivalent (in fact, \( \hat{\nabla} \) defines a non trivial
representation of $\pi_1(T^2)$, but the representation associated to $\nabla$ is the trivial one).

For instance, $\tilde{\alpha} = (\sin \pi x)dy + (\cos \pi y)dx$ is a non closed 1-form on $T$ and the corresponding connection $\tilde{\nabla}$ defined by $E = \tilde{\alpha} \otimes (e_1 + \lambda e_2)$ (with $\lambda \neq 0$) is not flat, by Proposition 8. For $t \in [0, 1]$, we define

$$E_t = t((\sin \pi x)dy + (\cos \pi y)dx) \otimes (e_1 + (1 - t)e_2).$$

We have a curve $\tilde{\nabla}_t = \nabla + E_t$ of connections, where $\tilde{\nabla}_0$ is obviously flat. $E_1$ reduces to $\tilde{\alpha} \otimes e_1$; thus, according to the Remark, $\tilde{\nabla}_1$ is a flat gauge field. Hence, one has a curve on Yang-Mills fields connecting two different vacuum states of the Yang-Mills theory. From Proposition 8, it follows:

**Proposition 9.** The family $E_t$ defined in (2.42) determines a curve $\{\tilde{\nabla}_t\}_{t \in [0, 1]}$ of Yang-Mills fields on $F = T \times \mathbb{C}^2$ satisfying

1. $\tilde{\nabla}_0$ and $\tilde{\nabla}_1$ are gauge inequivalent integrable connections
2. For each $t \in (0, 1)$, $\tilde{\nabla}_t$ is a non flat Yang-Mills connection.

### 3. Yang-Mills fields on coherent sheaves

#### 3.1. Holomorphic gauge fields on vector bundles.

In this section $X$ is a complex analytic manifold, $\mathcal{O}$ will denote the sheaf of holomorphic functions on $X$, and $\Omega^k$ the sheaf of holomorphic $k$-forms. Let $\tilde{\pi} : F \to X$ be a holomorphic vector bundle over $X$. By $F$ will be also denoted the sheaf of their sections. We review the construction of the 1-jet bundle $J^1(F)$ to define the holomorphic gauge fields on $F$ and to show the obstruction to the existence of these fields.

We denote by $\Gamma(q)$ the set of sections of $F$ defined in some neighborhood of the point $q \in X$. In $\Gamma(q)$ one defines the following equivalence relation $\sigma \sim \tau$, iff

$$\sigma(q) = \tau(q) \quad \text{and} \quad \sigma_*|_{T_qX} = \tau_*|_{T_qX},$$

where

$$\sigma_*|_{T_qX} : T_qX \to T_{\sigma(q)}F,$$

is the linear map induced between the tangent spaces.

The equivalence class of $\sigma$ is denoted $j^1_q(\sigma)$. One has $J^1(F)$, the first jet bundle of $F$,

$$J^1(F) = \{j^1_q(\sigma) \mid q \in X, \sigma \in \Gamma(q)\} \xrightarrow{p} X, \quad j^1_q(\sigma) \mapsto q.$$

Given $\sigma$ a section of $F$ defined on $U$, it determines a section $j^1(\sigma)$ of $J^1(F) \to X$ defined by $j^1(\sigma)(q) = j^1_q(\sigma)$. If $f$ is a holomorphic function on $U$, one defines $f \cdot j^1(\sigma) := j^1(\tau)$, where $\tau$ is a section of $F$, such
that, \( \tau(q) = \sigma(q) \) and \( \tau_*|_{T_qX} = f(q)\sigma_*|_{T_qX} \) for all \( q \in U \). If \( \tau' \) satisfies also the preceding conditions, then \( j^1(\tau) = j^1(\tau') \); thus, \( f \cdot j^1(\sigma) \) is well defined. It is easy to check \( (gf) \cdot j^1(\sigma) = g \cdot (f \cdot j^1(\sigma)) \).

On the other hand, we have the epimorphism of \( \mathcal{O} \)-modules.

\[
\pi : J^1(F) \to F, \quad j^1_q(\sigma) \mapsto \sigma(q),
\]

and the corresponding short exact sequence of \( \mathcal{O} \)-modules

\[
0 \to \ker(\pi) \to J^1(F) \xrightarrow{\pi} F \to 0.
\]

If \( \pi(j^1_q(\sigma)) = 0 \), then \( \sigma_*|_{T_qX} \) maps \( T_qX \) into the tangent vector space to the fibre \( \tilde{\pi}^{-1}(q) \) at the point 0. This last space can be identified with the fibre \( F_q \). Thus, \( \sigma_*|_{T_qX} \) defines an element of \( T_q^*X \otimes F_q = \Omega^1_q(F) \), the fibre at \( q \) of the sheaf of \( F \)-valued holomorphic 1-forms. Thus, one has the following short exact sequence

\[
0 \to \Omega^1(F) \xrightarrow{i} J^1(F) \xrightarrow{\pi} F \to 0. \tag{3.1}
\]

The natural map \( j^1 : F \to J^1(F) \) is a morphism of abelian sheaves, but not an arrow of \( \mathcal{O} \)-modules. In fact,

\[
j^1(f \sigma) = i(df \otimes \sigma) + f \cdot j^1(\sigma). \tag{3.2}
\]

As \( \pi \circ j^1 = \text{id} \), \((3.1)\) splits in the category of abelian sheaves

\[
J^1(F) = F \oplus \Omega^1(F) \quad \text{(as abelian sheaves)}. \tag{3.3}
\]

Expressing an element of \( J^1(F) \) as \( \sigma \oplus \alpha \), the product \( f \cdot (\sigma \oplus \alpha) \) corresponds to

\[
j^1(f \sigma) + i(f \alpha) = i(df \otimes \sigma) + f \cdot j^1(\sigma) + i(f \alpha),
\]

hence, by the identification \((3.3)\), one can write

\[
f \cdot (\sigma \oplus \alpha) = f \sigma \oplus (df \otimes \sigma + f \alpha). \tag{3.4}
\]

On the other hand, if \( D \) is a right inverse of \( \pi \) in the category of \( \mathcal{O} \)-modules, \( \pi \circ D = \text{id} \), then

\[
\text{im}(j^1 - D) \subset \ker(\pi) = i(\Omega^1(F)).
\]

For \( \nabla := j^1 - D \) one has by \((3.2)\)

\[
\nabla(f \sigma) = (j^1 - D)(f \sigma) = i(df \otimes \sigma) + f j^1(\sigma) - fD(\sigma) = i(df \otimes \sigma) + f \nabla \sigma.
\]

Thus, if \( D \) is a right inverse of \( \pi \), it defines a morphism of abelian sheaves

\[
\nabla : F \to \Omega^1(F)
\]

satisfying the Leibniz’s rule for the product of holomorphic functions by sections. That is, \( \nabla \) is a holomorphic connection.
Hence, if there exists a holomorphic connection on $F$, then the exact sequence (3.1) splits, and conversely. In other words, the existence of a holomorphic connection on $F$ is equivalent to the vanishing in $\text{Ext}^1(F, \Omega^1(F))$ of the extension defined by (3.1). The element of $\text{Ext}^1(F, \Omega^1(F))$ determined by that exact sequence is called the Atiyah class of $F$, and it is denoted by $a(F)$. In particular, if $F$ is a line bundle, then $a(F) = 0$ whenever the first Chern class $c_1(F)$ vanishes [4, Prop. 12].

Since $F$ is a locally free $\mathcal{O}$-module, for any coherent $\mathcal{O}$-module $\mathcal{G}$ the sheaves $\text{Ext}^q(F, \mathcal{G})$ vanish for all $q > 0$ and $\text{Ext}^0(F, \mathcal{G}) = \mathcal{H}\text{om}(F, \mathcal{G})$ [17, Chap. III, Prop. 6.5]. On the other hand, the local to global spectral sequence $E_{p,q}^p = H_p(X, \mathcal{E}xt^q(\cdot, \cdot))$ abuts to $\text{Ext}^{p+q}(\cdot, \cdot)$. Hence, $\text{Ext}^p(F, \mathcal{G}) = H^p(X, \mathcal{H}\text{om}(F, \mathcal{G}))$. In particular, for $\mathcal{G} = \Omega^1 \otimes \mathcal{O}_F$, one has

$$\text{Ext}^1(F, \Omega^1 \otimes \mathcal{O}_F) = H^1(X, \Omega^1 \otimes \mathcal{E}nd(F)).$$

That is, the Atiyah class $a(F)$ is an element of the first cohomology of $\Omega^1 \otimes \mathcal{E}nd(F)$. By Cartan’s Theorem B [14, p. 243], we have the following proposition.

**Proposition 10.** If $X$ is a Stein manifold, then any holomorphic vector bundle over $X$ admits holomorphic gauge fields.

Let $F$ be a holomorphic vector bundle such that $a(F) = 0$. If $X$ is a compact Kähler manifold and $F$, as $C^\infty$ bundle, is Hermitian, using the $\star$ operator, one can define the corresponding Yang-Mills functional on the set holomorphic gauge fields compatible with the Hermitian structure. Let $\nabla$ be such a field. Any other field in this set is of the form $\nabla + E$, with

$$E \in \Gamma(X, \Omega^1 \otimes \mathcal{H}u(F)),$$

where $\mathcal{H}u(F)$ is the sheaf of holomorphic antihermitean endomorphisms of $F$. Then the Yang-Mills functional gives rise to the map

$$E \in \Gamma(X, \Omega^1 \otimes \mathcal{H}u(F)) \mapsto ||K_{\nabla + \epsilon E}||^2.$$

The curvature $K_{\nabla + \epsilon E} = K_{\nabla} + \epsilon \nabla(E) + O(\epsilon^2)$. Thus

$$\frac{d}{d\epsilon}||K_{\nabla + \epsilon E}||^2 \bigg|_{\epsilon = 0} = 2\langle\langle K_{\nabla}, \nabla(E)\rangle\rangle.$$

Hence, $\nabla$ is a holomorphic Yang-Mills field iff $\langle\langle K_{\nabla}, \nabla(E)\rangle\rangle = 0$, for all $E$.

As $\Omega^k \otimes \mathcal{H}u(F)$ is a coherent sheaf [11, Annex. §4], $H^0(X, \Omega^k \otimes \mathcal{H}u(F))$ is a finite dimensional vector space [12, page 700]. We denote by $r_k := \dim H^0(X, \Omega^k \otimes \mathcal{H}u(F))$. 
Let $\beta_1, \ldots, \beta_{r_1}$ be a basis of $H^0(X, \Omega^1 \otimes \mathfrak{h}u(F))$. Then the condition of being Yang-Mills reduces to the following conditions for the corresponding curvature:

(3.5) \[ \langle\langle K, \nabla(\beta_i) \rangle\rangle = 0, \quad i = 1, \ldots, r_1. \]

On the other hand, $K_\nabla$ is a vector of the finite dimensional space $H^0(X, \Omega^2 \otimes \mathfrak{h}u(F))$, which, according to (3.5), must be orthogonal to the image of the map

(3.6) \[ \nabla : H^0(X, \Omega^1 \otimes \mathfrak{h}u(F)) \to H^0(X, \Omega^2 \otimes \mathfrak{h}u(F)). \]

When $r_2 \leq r_1$, we say that $\nabla$ is generic if the map (3.6) is surjective. Consequently, the curvature of any generic Yang-Mills connection must be zero. One has the following proposition.

**Proposition 11.** Let $F$ be a holomorphic Hermitian vector bundle on the compact Kähler manifold $X$. If $r_2 \leq r_1$, then all the generic holomorphic Yang-Mills fields on $F$ are flat.

When $F$ is a line bundle $r_2 = \dim H^0(X, \Omega^2)$ and $r_1 = \dim H^0(X, \Omega^1)$ are the Hodge numbers $h^{2,0}(X)$ and $h^{1,0}(X)$, respectively.

**Corollary 1.** When $F$ is a line bundle, if $h^{2,0}(X) \leq h^{1,0}(X)$, then any generic holomorphic Yang-Mills connection on $F$ is flat.

Let $L$ be a Hermitian line bundle over the compact Kähler manifold $X$. Let us assume that $L$ is flat, thus the space of holomorphic connections on $L$ is nonempty. Since $H^0(X, L)$ is a finite dimensional vector space, let $s_1, \ldots, s_k$ be a basis of this space. After a normalization, we can assume that the basis is orthonormal, with respect to the product of sections defined by the Hermitian structure of $L$ and the Kähler structure of $X$.

We denote by $\nabla_0$ the holomorphic connection determined by $\nabla_0(s_i) = 0$ for $i = 1, \ldots, k$. If $\nabla$ is other holomorphic connection on $L$, then $\nabla = \nabla_0 + E$, with

$$E \in \text{Hom}_L(L, \Omega^1 \otimes L) = \Gamma(X, \Omega^1).$$

The curvature of $\nabla$ is $K = \nabla_0(E)$.

The form $E$ can be written as a linear combination $E = \sum_i \alpha_i s_i$, with $\alpha_i$ holomorphic 1-form. Hence, $K = \sum_i d(\alpha_i)s_i$. Setting

$$G(d\alpha_1, \ldots, d\alpha_k) := ||K||^2 = \sum_i ||d\alpha_i||^2,$$

the only stationary point of $G$ is $d\alpha_1 = \cdots = d\alpha_k = 0$, which is defined by the connections whose curvature vanishes. That is, we have the following proposition.
Proposition 12. If $L$ is a Hermitian flat line bundle on the compact Kähler manifold $X$, then all the holomorphic Yang-Mills fields on $L$ are flat.

3.2. Holomorphic connections on coherent sheaves. There is another construction of the jet bundle which admits a translation to the case of coherent $\mathcal{O}$-modules. We summarize the development of [9, p. 6]. By $\mathcal{J}$ we denote the ideal sheaf of the diagonal embedding $X \hookrightarrow X \times X$. The structure sheaf of the first infinitesimal neighborhood of the diagonal is $\mathcal{O}_{X \times X}/\mathcal{I}^2$ [13, p. 5]. We set $X^{(1)}$ for the corresponding subscheme. Denoting by $p_1, p_2$ the projections $X \times X \rightarrow X$, one can define on $\mathcal{O}_{X^{(1)}}$ left and right $\mathcal{O}$-module structures via $p_1$ and $p_2$ respectively.

One can think of sections of $\mathcal{O}$ as functions $h(x)$ and the sections of $\mathcal{O}_{X^{(1)}}$ as classes of functions $g(x, y)$ modulo $\mathcal{I}^2$. The right and left $\mathcal{O}$-structures are defined by

$$((g+\mathcal{I}^2) \cdot f)(x, y) = g(x, y)f(y)+\mathcal{I}^2, \ (f \cdot (g+\mathcal{I}^2))(x, y) = f(x)g(x, y)+\mathcal{I}^2.$$ 

We have the $\mathbb{C}$-linear map

$$m : \mathcal{O}_{X^{(1)}} \rightarrow \mathcal{O} \oplus \Omega^1, \ g + \mathcal{I}^2 \mapsto g(x, x) \oplus (dg)(x, x),$$

where $dg$ is the exterior differential of $g$ regarded as a function of $x$. It is easy to check that $m$ is a well defined morphism of abelian sheaves. Moreover,

$$m(f \cdot (g+\mathcal{I}^2)) = fm(g) + \tilde{g}df, \ m((g+\mathcal{I}^2) \cdot f) = m(g)f$$

where $\tilde{g}(x) = g(x, x)$. If we define a left $\mathcal{O}$-module structure on $\mathcal{O} \oplus \Omega^1$ by

$$f \cdot (h \oplus \alpha) := fh \oplus (hdf + f\alpha),$$

then $m$ is an isomorphism of $\mathcal{O}$-modules.

In this way, for the locally free sheaf $F$, the $\mathcal{O}$-module $\mathcal{O}_{X^{(1)}} \otimes_{\mathcal{O}} F = F \oplus \Omega^1(F)$ with $\mathcal{O}$-module structure

$$f \cdot (\sigma \oplus \beta) = f\sigma \oplus (df\sigma + f\beta)$$

is isomorphic to the $\mathcal{O}$-module $J^1(F)$ (see (3.3) and (3.4)).

It is possible to define the sheaf of 1-jets of a coherent sheaf on $X$, generalizing the definition given for vector bundles. Let $\mathcal{F}$ be a coherent $\mathcal{O}$-module, one defines [9] p.6

$$J^1(\mathcal{F}) := \mathcal{O}_{X^{(1)}} \otimes_{\mathcal{O}} \mathcal{F}.$$ 

One has the short exact sequence of $\mathcal{O}$-modules

$$0 \rightarrow \Omega^1 \otimes_{\mathcal{O}} \mathcal{F} \overset{i}{\rightarrow} J^1(\mathcal{F}) \overset{\pi}{\rightarrow} \mathcal{F} \rightarrow 0.$$
Definition 1. A holomorphic connection on the coherent sheaf $\mathcal{F}$ is a right inverse of the morphism $\pi : \mathcal{F}^1(\mathcal{F}) \to \mathcal{F}$

That is, a connection on $\mathcal{F}$ is a splitting of (3.8). Then, as in Subsection 3.1, there is a morphism of $\mathbb{C}_X$-modules

$$\nabla : \mathcal{F} \to \Omega^1_1 \otimes_{\mathcal{O}} \mathcal{F},$$

satisfying the Leibniz’s rule for the product of holomorphic functions on $X$ and sections of $\mathcal{F}$.

All the formulas obtained in Section 2 from (2.1) to (2.8) hold for a holomorphic connection on the coherent sheaf $\mathcal{F}$ over the complex analytic manifold $X$, if we substitute $A$ by $\mathcal{O}$, substitute $A^*$ by the corresponding sheaf of holomorphic forms, and assume that $d$ is the holomorphic exterior differential.

The Atiyah class of a skyscraper sheaf. Let $p$ be a point of $X$. On $P := \{p\}$ we consider the sheaf $\mathcal{O}_P$ defined by $\mathcal{O}_P(P) = \mathbb{C}$. Denoting by $i : P \to X$ the inclusion, the direct image sheaf $i_* \mathcal{O}_P$ is an $\mathcal{O}$-module, where the morphism $\mathcal{O} \to i_* \mathcal{O}_P$ is given by the evaluation at the point $p$. $S := i_* \mathcal{O}_P$ is the skyscraper sheaf on $X$ at $p$ with stalk $\mathbb{C}$. The exact sequence

$$0 \to \mathcal{I}_p|_X \to \mathcal{O} \to S \to 0,$$

with $\mathcal{I}_p|_X$ the ideal sheaf defining $p$, shows that $S$ is coherent sheaf.

Example 1. Let $S$ be the skyscraper sheaf on $X = \mathbb{C}P^1 = \{[x_0 : x_1] | x_i \in \mathbb{C}\}$ at the point $p = [1 : 0]$. One has the following exact sequence of $\mathcal{O}$-modules

$$(3.9) \quad 0 \to \mathcal{O}(-1) \xrightarrow{f} \mathcal{O} \xrightarrow{q} S \to 0,$$

where the morphism $f$ is the multiplication by $x_1$ and $q$ is the evaluation map at the point $p$.

For any $\mathcal{O}$-module $\mathcal{G}$, $\text{Ext}^i(\mathcal{O}, \mathcal{G}) = H^i(X, \mathcal{G}) [17]$ page 234. On the other hand, $\Omega^1_1(S)$ is supported at $p$, since $\Omega^1_1(S)_p \simeq \Omega^1_1 \otimes_{\mathcal{O}_p} \mathbb{C}_p [20]$ page 88, thus, $H^i(X, \Omega^1_1(S)) = 0$, for $i > 0$. Hence, the ext exact sequence associated to (3.9) is

$$\to \text{Hom}(\mathcal{O}, \Omega^1_1(S)) \xrightarrow{\phi} \text{Hom}(\mathcal{O}(-1), \Omega^1_1(S)) \to \text{Ext}^1(S, \Omega^1_1(S)) \to 0.$$

The morphism $\phi$ is defined as follows: Given $\eta \in \text{Hom}(\mathcal{O}, \Omega^1_1(S))$, then $\phi(\eta)(\tau) = \eta(x_1 \tau)$, for any local section $\tau$ of $\mathcal{O}(-1)$. As $x_1 \tau$ vanishes at $p$, then $\phi = 0$ and $\text{Ext}^1(S, \Omega^1_1(S)) \simeq \text{Hom}(\mathcal{O}(-1), \Omega^1_1(S))$.

On the other hand, the skyscraper sheaf $S$ does not admit holomorphic connections. In fact, let $h$ and $\tilde{h}$ be holomorphic functions defined on an open subset $U \subset \mathbb{C}P^1$, which contains the point $p = [1 : 0]$, such
that they coincide at the point \( p \), \( h(p) = \tilde{h}(p) \), but \( \partial h(p) \neq \partial \tilde{h}(p) \). From the definition of the \( \mathcal{O} \)-module structure of \( \mathcal{S} \), it follows that \( h\sigma = \tilde{h}\sigma \) for any section \( \sigma \in \mathcal{S}(U) \). Suppose there was such a connection; from the Leibniz’s rule applied to \( h\sigma \) and \( \tilde{h}\sigma \), we would deduce that \( \partial h \otimes \sigma = \partial \tilde{h} \otimes \sigma \) on \( U \), and this equality is in contradiction with the fact that the differentials do not coincide at the point \( p \). That is, the Atiyah class \( a(\mathcal{S}) \neq 0 \).

The argument above is applicable to skyscraper sheaves on more general manifolds and from it, one deduces the following proposition.

**Proposition 13.** The skyscraper sheaves over a complex analytic manifold do not admit holomorphic connections.

### 3.3. Holomorphic Yang-Mills fields.

Given a coherent sheaf \( \mathcal{G} \) on \( X \), the fibre of \( \mathcal{G} \) at \( x \in X \) will be denoted \( \mathcal{G}(x) \); that is, \( \mathcal{G}(x) = \mathcal{G}_x / \mathfrak{m}_x \mathcal{G}_x \), where \( \mathfrak{m}_x \) is the maximal ideal of \( \mathcal{O}_x \). If \( \tau \) is a section of \( \mathcal{G} \), we set \( \tau(x) \) for the image of \( \tau_x \) in the fibre \( \mathcal{G}(x) \). A Hermitian metric on \( \mathcal{G} \) is a family of Hermitian metrics \( \langle \cdot, \cdot \rangle_x \) on the vector spaces \( \mathcal{G}(x) \), such that for \( \tau \) and \( \tau' \) sections of \( \mathcal{G} \) on an open set \( U \), the map \( x \mapsto \langle \tau(x), \tau'(x) \rangle_x \in \mathbb{C} \) is \( C^\infty \) on \( U \setminus Y \), with \( Y \) the singularity set of \( \mathcal{G} \).

Let \( \nabla \) be a holomorphic connection on the sheaf \( \mathcal{F} \). Any other holomorphic connection is of the form \( \nabla + \mathcal{E} \), with

\[
\mathcal{E} \in \text{Hom}_\mathcal{O}(\mathcal{F}, \Omega^1 \otimes_\mathcal{O} \mathcal{F}) = \Gamma(X, \mathcal{Hom}(\mathcal{F}, \Omega^1 \otimes_\mathcal{O} \mathcal{F})).
\]

The singularity set \( Z \) of \( \mathcal{F} \) is a closed analytic subset of \( X \) with \( \text{codim } Z \geq 1 \) \([27]\). Moreover, \( \mathcal{F}|_{X \setminus Z} \) is a locally free \( \mathcal{O}_{X \setminus Z} \)-module. Thus, on \( X \setminus Z \), \( \mathcal{E} \) determines a unique element \( E \in \Gamma(X \setminus Z, \Omega^1 \otimes_\mathcal{O} \text{End}(\mathcal{F})) \) (see Lemma \([1]\)). We will denote by \( \psi \) the map

\[
\psi : \text{Hom}_\mathcal{O}(\mathcal{F}, \Omega^1 \otimes_\mathcal{O} \mathcal{F}) \rightarrow \Gamma(X \setminus Z, \Omega^1 \otimes_\mathcal{O} \text{End}(\mathcal{F})), \ E \mapsto E.
\]

Similarly, the curvature \( K_\nabla \in \text{Hom}_\mathcal{O}(\mathcal{F}, \Omega^2 \otimes_\mathcal{O} \mathcal{F}) \) and its restriction to \( X \setminus Z \) defines a vector \( K_\nabla \) of the vector space \( \Gamma(X \setminus Z, \Omega^2 \otimes_\mathcal{O} \text{End}(\mathcal{F})) \), (as \( X \setminus Z \) is not compact, that vector space may be infinite dimensional). On \( X \setminus Z \) the curvature of \( \nabla_\epsilon := \nabla + \epsilon \mathcal{E} \) is \( K_\nabla + \epsilon \mathcal{E} = K_\nabla + \epsilon \nabla(E) + O(\epsilon^2) \).

When \( X \) is a compact Kähler manifold and \( \mathcal{F} \) is a holomorphic Hermitian sheaf, we define

\[
||K_\nabla||^2 := \int_{X \setminus Z} \langle K_\nabla, K_\nabla \rangle \ d\text{vol}.
\]

If \( \nabla \) and \( \nabla_\epsilon \) are compatible with the Hermitian structure, then \( E \in \Gamma(X \setminus Z, \Omega^1 \otimes_\mathcal{O} \text{h}(\mathcal{F})) \). We say that \( \nabla \) is a *holomorphic Yang-Mills connection*.


field on $\mathcal{F}$ if

$$\frac{d}{de} \left| |K_{\nabla_e}|^2 \right| = 0, \text{ for all } E \in \text{im}(\psi) \cap \Gamma(X \setminus Z, \Omega^1 \otimes_0 \mathfrak{h}(\mathcal{F})).$$

That is, $\nabla$ is a Yang-Mills connection if

$$\langle K_{\nabla_e}, \nabla E \rangle := \int_{X \setminus Z} \langle K_{\nabla_e}, \nabla E \rangle \, d\text{vol} = 0,$$

where $\nabla E$ is the covariant derivative of $E$.

**Yang-Mills fields on reflexive sheaves.** Let $\mathcal{H}$ be a coherent reflexive sheaf on the complex manifold $X$. Some properties satisfied by $\mathcal{H}$ that we will use are the following.

1. The singularity set $Z$ of $\mathcal{H}$ has codimension greater than 2 [16, Cor. 1.4], and the restriction $\mathcal{H}|_{X \setminus Z}$ is a locally free sheaf.

2. If $Y \subset X$ is a closed subset of $X$ of codimension $\geq 2$, then the restriction map $\Gamma(X, \mathcal{H}) \rightarrow \Gamma(X \setminus Y, \mathcal{H})$ is an isomorphism [18, Prop. 1.11].

3. If $G$ is a coherent sheaf, then $\mathbb{H}om(G, \mathcal{H})$ is also reflexive [21, Chap V, Prop. 4.15].

On the other hand, a coherent sheaf $\mathcal{H}$ is reflexive iff there exists an exact sequence

$$0 \rightarrow \mathcal{H} \rightarrow \mathcal{L} \rightarrow \mathcal{J},$$

with $\mathcal{L}$ locally free and $\mathcal{J}$ torsion-free [16, Prop. 1.1]. The functor $\Omega^k \otimes_0 -$ is exact, since $\Omega^k$ is locally free. Thus, if $\mathcal{H}$ is reflexive, we have the exact sequence

$$0 \rightarrow \Omega^k \otimes_0 \mathcal{H} \rightarrow \Omega^k \otimes_0 \mathcal{L} \rightarrow \Omega^k \otimes_0 \mathcal{J}.$$  

As $\Omega^k \otimes_0 \mathcal{L}$ is locally free and $\Omega^k \otimes_0 \mathcal{J}$ torsion-free, it follows that $\Omega^k \otimes_0 \mathcal{H}$ is also reflexive.

Let $\mathcal{F}$ be a reflexive sheaf on $X$, whose singularity set is denoted by $Z$, and $\mathcal{C} \in \text{Hom}(\mathcal{F}, \Omega^k \otimes_0 \mathcal{F}))$. From (2) together with (3), it follows that $\mathcal{C}$ is determined by its restriction in $\Gamma(X \setminus Z, \mathbb{H}om(\mathcal{F}, \Omega^k \otimes_0 \mathcal{F})).$

According to (1)

$$\Gamma(X \setminus Z, \mathbb{H}om(\mathcal{F}, \Omega^k \otimes_0 \mathcal{F})) \simeq \Gamma(X \setminus Z, \Omega^k \otimes_0 \text{End}(\mathcal{F})).$$

Again (2) and (3) give rise to the isomorphism

$$\Gamma(X \setminus Z, \Omega^k \otimes_0 \text{End}(\mathcal{F})) \simeq \Gamma(X, \Omega^k \otimes_0 \text{End}(\mathcal{F})).$$

Hence, $\mathcal{C}$ determines and element $C$ of the vector space $\Gamma(X, \Omega^1 \otimes_0 \text{End}(\mathcal{F}))$. That is,
Lemma 3. If $\mathcal{F}$ is a reflexive sheaf, then the map

$$\mathcal{C} \in \text{Hom}_0(\mathcal{F}, \Omega^k \otimes \mathcal{F}) \mapsto C \in \Gamma(X, \Omega^k \otimes \mathcal{O}\text{End}(\mathcal{F}))$$

is a bijective correspondence.

Let $\nabla$ be a holomorphic connection on the reflexive sheaf $\mathcal{F}$, whose singularity set is denoted by $Z$. As we said, any other holomorphic connection on $\mathcal{F}$ is of the form $\nabla + \mathcal{E}$, where $\mathcal{E}$ satisfies (3.10). By Lemma 3, $\mathcal{E}$ determines $E \in \Gamma(X, \Omega^1 \otimes \mathcal{O}\text{End}(\mathcal{F}))$. Similarly, the curvature $K_{\nabla}$ is determined by the vector $K_{\nabla}$ of the vector space $\Gamma(X, \Omega^2 \otimes \mathcal{O}\text{End}(\mathcal{F}))$.

Let us assume that $\mathcal{F}$ is a reflexive sheaf equipped with a Hermitian metric, and $\nabla$ a Hermitian connection. By the properties above mentioned,

$$\Gamma(X \setminus Z, \Omega^k \otimes \mathcal{O}\text{hu}(\mathcal{F})) \subset \Gamma(X, X \setminus Z, \Omega^k \otimes \mathcal{O}\text{End}(\mathcal{F})) = \Gamma(X, \Omega^k \otimes \mathcal{O}\text{End}(\mathcal{F})),$$

and these spaces are finite dimensional, since $\Omega^k \otimes \mathcal{O}\text{End}(\mathcal{F})$ is coherent. We will denote $r_k = \dim H^0(X \setminus Z, \Omega^k \otimes \mathcal{O}\text{hu}(\mathcal{F}))$, thus $K_{\nabla}$ is a vector of a vector space with dimension $r_2$.

If $\nabla$ is a Yang-Mills field on the reflexive sheaf $\mathcal{F}$, then $K_{\nabla}$ is a vector of an $r_2$-dimensional Hermitian vector space which satisfies the orthogonality conditions (3.11). When $r_1 \geq r_2$, we say that $\nabla$ is a generic gauge field if the linear map

$$\nabla : \Gamma(X \setminus Z, \Omega^1 \otimes \mathcal{O}\text{hu}(\mathcal{F})) \to \Gamma(X \setminus Z, \Omega^2 \otimes \mathcal{O}\text{hu}(\mathcal{F}))$$

is surjective. Thus, we have the following proposition that generalizes Proposition 11.

**Proposition 14.** If $\mathcal{F}$ is a reflexive sheaf endowed with a Hermitian metric and $\nabla$ is a generic Yang-Mills connection, then $\nabla$ is flat.

If a coherent sheaf is endowed with a holomorphic flat connection, then it is really a locally free sheaf [10, Prop. 2.3, page 184] [19, Prop. 2.2.5]. Hence, if $r_1 \geq r_2$ and $\mathcal{F}$ is reflexive with singularities, then it does not admit a generic Yang-Mills gauge field. In other words, if $r_1 \geq r_2$ and $\mathcal{F}$ is reflexive with singularities, then the kernel of the morphism (3.12) has dimension $> r_1 - r_2$, for any Yang-Mills connection $\nabla$.

### 3.4. Meromorphic gauge fields and geometric phases

Let $\mathcal{F}$ be a coherent $\mathcal{O}$-module on the complex analytic manifold $X$. We denote by $Z$ the singularity space of $\mathcal{F}$ and we set $\mathcal{O}[\ast Z]$ for the sheaf of meromorphic functions on $X$, that are holomorphic on $X \setminus Z$ and have
poles on \(Z\). We set \(\Omega^1[\ast Z]\) for \(\Omega^1 \otimes \mathcal{O}[\ast Z]\) and \(\mathcal{F}^Z := \mathcal{O}[\ast Z] \otimes_{\mathcal{O}} \mathcal{F}\). A connection on \(\mathcal{F}\) meromorphic along \(Z\) is a morphism of \(\mathbb{C}_X\)-modules

\[
\nabla : \mathcal{F}^Z \to \Omega^1 \otimes_{\mathcal{O}} \mathcal{F}^Z,
\]

which satisfies the Leibniz’s rule for the product of functions of \(\mathcal{O}[\ast Z]\) and sections of \(\mathcal{F}^Z\).

Since the restriction of the meromorphic connection (3.13) to the locally free sheaf \(\mathcal{F}|_{X \setminus Z}\) is a holomorphic one, the vanishing of the Atiyah class \(a(\mathcal{F}|_{X \setminus Z})\) is a necessary condition for the existence of such a meromorphic connection.

A meromorphic connection whose singularities are not “wild” is called regular; that is, when the equation for the parallel transport is regular in the Fuchs’ sense is regular. In this case the solutions to this equation have a “moderate” growth. The regular flat connections can be characterised from a topological point of view, as explained in the following paragraph.

Flat connections. The meromorphic connection \(\nabla\) is called flat if for any holomorphic vector fields \(v, v'\) on \(X\)

\[
[\nabla_v, \nabla_{v'}] = \nabla_{[v, v']}.
\]

When \(Z\) is a divisor of \(X\), the flat connection defines a \(D_X\)-module structure on \(\mathcal{F}^Z\) [19 page 140]. A regular flat connection determines a representation of the fundamental group \(\pi_1(X \setminus Z)\), according to the Deligne’s version of the Riemann-Hilbert correspondence [19 page 149]. That representation is defined, by means of the monodromy of parallel transport. We will consider that property of the regular flat connections in a brief analysis of the Aharonov-Bohm effect.

3.4.1. Aharonov-Bohm effect. The Aharonov-Bohm effect is the change in the phase of the wave function of an electron, when it travels in a closed curve around a solenoid \(s\). The magnetic field created by \(s\) is confined within it, thus the potential vector is a closed 1-form outside \(s\).

Let us consider an ideal solenoid indefinitely long with infinitesimal radius, and an electron \(e\), moving in a plane \(\pi\) orthogonal to the solenoid. Even though the confinement of the magnetic field, after completing a closed curve around the solenoid, appears a shift in the phase of the wave function of \(e\). The corresponding factor of phase is \(e^{-i\Phi}\), where \(\Phi\) is the magnetic flux through the solenoid [28 Sec. 4].

A way to deduce this result, in a geometric setting, consists of endowing the plane \(\pi\) with a complex structure, and then to express the potential vector for the magnetic field as a meromorphic 1-form on \(\pi\).
The shift of phase is the integral of this form along the corresponding closed curve.

This effect can also be analyzed in the frame of the meromorphic gauge fields on coherent sheaves as follow. On $X = \mathbb{C}$ we consider the sheaf $\mathcal{G} = \mathcal{O}[\ast Z]$, with $Z = \{0\}$. By means of the meromorphic closed 1-form $\beta = -k \frac{dz}{z}$, with $k$ a constant in $\mathbb{C} \setminus Z$, one may define the meromorphic integrable connection on $\mathcal{G}$

$$\nabla(h) = dh + h \otimes \beta,$$

for $h \in \mathcal{O}[\ast Z]$.

In this case, the equation for the parallel transport is $dh + h\beta = 0$, and it has not a holomorphic solution. Its solution, up to multiplicative constant, is $\tilde{h}(z) = \exp(k \log z)$, where $\log z$ is any branch of the logarithm function. Let us consider the complex of $\mathbb{C}$-modules

$$D^\bullet : 0 \rightarrow D^{-1} := \mathcal{G} \xrightarrow{\nabla} D^0 := \Omega^1 \otimes_\mathcal{O} \mathcal{G} \rightarrow 0.$$ 

This is in fact, the de Rham complex defined by the flat connection $\nabla$ [19, page 103].

The cohomology sheaf $\mathcal{H}^0(D^\bullet)$ is trivial, and $\mathcal{H}^{-1}(D^\bullet)$ is the extension by zero to $\{0\}$ of the local system on $\mathbb{C} \setminus \{0\}$ determined by the multivalued function $\tilde{h}$.

The analytic continuation of $\tilde{h}(z)$ along the curve $C = \{e^{it}\}_{t \in [0, 2\pi]}$ changes $\tilde{h}(z)$ into $\tilde{h}(z)e^{2\pi ki}$. That is, the above mentioned local system is the one defined by the representation of $\pi_1(X \setminus \{0\})$ that assigns to the homotopy class $[C]$ the complex number $e^{2\pi ki}$. Which is the factor of phase that appears in the Aharonov-Bohm effect, when $k = -\frac{\Phi}{2\pi}$.

The crucial points are that $\pi_1(\mathbb{C} \setminus \{0\}) \neq 1$ and that the holomorphic form $\beta|_{\mathbb{C} \setminus \{0\}}$ can not be extended to a holomorphic form on $\mathbb{C}$. Hence, the above result does not always admit of a direct generalization to dimensions greater than 1. This question is analyzed in the following example.

**Example 2.** Let $Z$ be a submanifold of the $n$-dimensional complex simply connected manifold $X$, such that $\text{codim } Z \geq 2$. Thus, $\pi_1(X \setminus Z) = 1$. We assume also that $X$ is $(n - 1)$-complete [11, page 235]. In particular, $X$ is cohomologically $(n - 1)$-complete; i.e. $H^j(X, \mathcal{G}) = 0$ for $j \geq n - 1$ and for any coherent sheaf $\mathcal{G}$ [11, page 250]. Under these hypotheses, the Hatogs’ extension theorem holds on $X$ [8, 24].

We set $\mathcal{B} := \mathcal{O}[\ast Z]$. As above, let us consider a meromorphic flat connection on $\mathcal{B}$ defined by a closed 1-form $\alpha = f\gamma$, with $f \in \mathcal{O}[\ast Z]$ and $\gamma$ a holomorphic 1-form. As $f$ is holomorphic on $X \setminus Z$, by the Hatogs’ extension theorem, $f$ is holomorphic on $X$ and the monodromy
of the parallel transport equation \( dh + h\alpha = 0 \) is trivial. Therefore, an electron moving moving on \( X \) in presence of a magnetic field confined in the submanifold \( Z \) will not undergo any change in its wave function. That is, in this case, there is no “Aharonov-Bohm” effect.

### 3.4.2. Aharonov-Casher effect

The Aharonov-Casher effect admits a similar interpretation. This effect appears when a nonrelativistic neutron \( n \) completes a closed curve around an electrically charged conductor. The conductor is supposed to be ideal; that is, unlimited long, infinitesimally thin and with a uniform and static distribution of electric charge. Thus, we can suppose that the conductor is on the \( z \)-axis of a coordinate system. When the neutron moves on a plane orthogonal to the conductor, describing a closed curve around the wire, the corresponding factor of phase in the spin of \( n \) is \( \exp(i\pi \Lambda \sigma_3) \), where \( \Lambda \) is a constant related with the density of charge in the conductor and \( \sigma_3 \) is the corresponding Pauli matrix [28, 7].

In this case, as potential can be taken a closed meromorphic \( \mathfrak{sl}(3) \)-valued 1-form on the mentioned plane. The phase shift can be obtained by integration this form along the curve.

### 3.4.3. Wong equation

The Wong equations describe the variation of the spin of a particle moving in a gauge field [30]. Let \( p \) be a particle carrying a like spin variable \( I \) with values in the semisimple Lie algebra \( \mathfrak{g} \). Let \( \{\tau_a\}_{a=1,\ldots,r} \) be a basis of \( \mathfrak{g} \) and \( I = \sum_a I^a \tau_a \), then the variation of \( I \) when \( p \) is moving in the \( \mathfrak{g} \)-valued gauge potential \( A = \sum A^a \tau_a \) satisfies

\[
\dot{I}^a + \sum C^a_{bc} A^b(\dot{x}) I^c = 0,
\]

where \( x(t) \) is the curve described by \( p \), and the \( C^a_{bc} \) are the structure constants of \( \mathfrak{g} \) relative to the basis \( \{\tau_a\} \) [5, p. 53], [22].

The above equations can be written as

\[
\frac{dI}{dt} + [A(\dot{x}(t)), I(t)] = 0,
\]

where \( [\cdot,\cdot] \) is the Lie bracket in \( \mathfrak{g} \).

Since \( \mathfrak{g} \) is semisimple, the adjoint representation \( \text{ad} : \mathfrak{g} \to \text{End}(\mathfrak{g}) \) is a monomorphism of algebras. Thus, we will identify a vector \( u \in \mathfrak{g} \) with the linear map \( \text{ad}(u) \). Applying \( \text{ad} \) to (3.14), one obtains the following equation

\[
\frac{dI}{dt} + A(\dot{x}(t)) \circ I(t) - I(t) \circ A(\dot{x}(t)) = 0,
\]

which can be regarded as the parallel transport of the endomorphism \( I \) along the curve \( x(t) \) (see (2.28) for the case \( k = 0 \)). If the gauge field
A is singular, then it is to be expected phase shifts in the variable $I$, when $p$ completes a closed curve.

A geometric setting for analyzing Wong equation is the following. We denote by $V$ the trivial bundle $X \times \mathfrak{g}_C$ over the complex manifold $X$, and will regard $\mathfrak{g}$ as a subalgebra of $\text{End}(\mathfrak{g})$. Then, as the vectors of $\mathfrak{g}$ define endomorphisms of $V$, one can consider connections

$$\nabla : V \to \Omega^1 \otimes_{\mathcal{O}} \mathfrak{g}_C.$$ 

Letting $s_a(x) := (x, \tau_a)$, the set $\{s_a\}$ is a global frame for $V$. Hence,

$$\nabla(s_i) = \sum_{j=1}^{r} \alpha_{ji} \otimes s_j, \quad i = 1, \ldots, r,$$

with $\alpha_{ij}$ 1-forms on $X$.

Let us assume that the particle $p$ is moving along the curve $x(t)$ on the manifold $X$ in presence of the gauge field $\nabla$. As we said, the variation of the variable spin like $I$ is determined by the Wong equation

$$\nabla \dot{x}(t) I = 0,$$

which in turn defines the corresponding holonomy when $x(t)$ is a closed curve. We will consider some particular cases.

1. The shift factor in the variable $I$, after the particle has completely traveled the curve, is given by the respective holonomy. If the connection $\nabla$ is holomorphic and flat, and the loop is null-homotopic, then the shift factor is trivial.
2. Let us assume that $X$ and $Z$ satisfy the properties stated in Example 2. Moreover, we suppose that $\alpha_{ji} = f_{ji} \gamma_{ji}$, with $f_{ji} \in \mathcal{O}[\ast Z]$ and $\gamma_{ji}$ holomorphic form on $X$. By the Hartogs’ extension theorem, $\alpha = (\alpha_{ji})$ is a matrix of holomorphic 1-forms. If $d\alpha + \alpha \wedge \alpha = 0$, then the curvature of the connection vanishes and the shift factor is the identity.
3. If $\nabla$ is a meromorphic regular flat connection, in the sense that the $\alpha_{ij} \in \Omega^1[\ast Z]$, with $Z$ a divisor of $X$, and the parallel differential equation is regular in Fuchs’ sense, then the shift factors are given by a representation of $\pi_1(X \setminus Z)$. This is a consequence of the Deligne’s version of the Riemann-Hilbert correspondence [19, Sect. 5.2.3] [23].

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