ON THE $C^{1,1}$ REGULARITY OF GEODESICS IN THE SPACE OF K"{A}HLER METRICS

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Abstract. We prove that any two K"{a}hler potentials on a compact K"{a}hler manifold can be connected by a geodesic segment of $C^{1,1}$ regularity. This follows from an a priori interior real Hessian bound for solutions of the nondegenerate complex Monge-Amp"{e}re equation, which is independent of a positive lower bound for the right hand side.

1. Introduction

Let $(X^m, g)$ be a compact $m$-dimensional K"{a}hler manifold without boundary. Write $\omega = \sqrt{-1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ for the K"{a}hler form of $g$. Write $H_\omega$ for the space of K"{a}hler metrics cohomologous to $\omega$, which we identify with smooth functions $\varphi$ with $\omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0$, modulo constants. The space $H_\omega$ can be endowed with the structure of an infinite dimensional Riemannian manifold [20, 29, 14]. Chen [7] showed that any two potentials $\varphi_0, \varphi_1 \in H_\omega$ can be joined by a weak geodesic segment $\{\varphi_t\}_{t \in [0,1]}$. The purpose of this note is to explain how the arguments of our earlier paper [10] on the Monge-Amp"{e}re equation imply a $C^{1,1}$ regularity result for these geodesics.

We first recall the equivalence of the geodesic equation with the homogeneous complex Monge-Amp"{e}re equation [29, 14]. Given a smooth family of K"{a}hler potentials $\{\varphi_t\}_{t \in [0,1]}$, we let $\Sigma = \{z \in \mathbb{C} \mid 1 \leq |z| \leq e\}$ and we define a function $\Phi$ on $X \times \Sigma$ by $\Phi(x, z) = \varphi_{\log |z|}(x)$. Then, as noted in [29, 14], the path $\{\varphi_t\}$ is a geodesic connecting $\varphi_0$ and $\varphi_1$ if and only if $\Phi$ solves the homogeneous complex Monge-Amp"{e}re equation
\begin{equation}
(\pi^* \omega + \sqrt{-1} \partial \bar{\partial} \Phi)^m+1 = 0,
\end{equation}
with boundary data $\Phi(x, 1) = \varphi_0(x), \Phi(x, e) = \varphi_1(x), x \in X$. From (1.1) it also follows easily that $\pi^* \omega + \sqrt{-1} \partial \bar{\partial} \Phi \geq 0$ on $X \times \Sigma$, where $\pi : X \times \Sigma \to X$ is the projection. Generalizing this fact, we say that a bounded function $\Phi$ on $X \times \Sigma$ is a weak geodesic connecting $\varphi_0$ and $\varphi_1$ if $\pi^* \omega + \sqrt{-1} \partial \bar{\partial} \Phi \geq 0$ weakly on $X \times \Sigma$ and $\Phi$ solves (1.1) in the sense of Bedford-Taylor [1] with the same boundary data as above. Chen [7] proved that there is a unique such weak geodesic $\Phi$, and that the quantities $\sup_{X \times \Sigma} |\Phi|, \sup_{X \times \Sigma} |\nabla \Phi|, \sup_{X \times \Sigma} \Delta \Phi$ and $\sup_{\partial (X \times \Sigma)} |\nabla^2 \Phi|$ are all bounded (see also [1], [6], [16], [33]), so in particular $\Phi$ is in $C^{1,\alpha}(X \times \Sigma)$ for all $0 < \alpha < 1$. Here and in the following, when we say that a function belongs to a certain function space...
on a manifold with boundary, we mean that it has the stated regularity up to (and including) the boundary.

It was expected that the real Hessian of $\Phi$ is bounded in the interior (namely that $\sup_{X \times \Sigma} |\nabla^2 \Phi| \leq C$). This would imply that geodesics are $C^{1,1}$. Some progress towards this was made recently: B/suppress locki [4] proved that $\Phi$ is in $C^{1,1}(X \times \Sigma)$ provided that $(X, g)$ has nonnegative bisectional curvature, while Berman [2] proved that the restrictions to $X$ given by $\varphi_t = \Phi(\cdot, e^t), 0 \leq t \leq 1$, are all in $C^{1,1}(X)$ provided that $[\omega] = c_1(L)$ for some holomorphic line bundle $L$ over $X$. However, the problem of proving the full $C^{1,1}$ regularity of weak geodesics remained open in general. Our main result resolves this completely.

**Theorem 1.1.** Given any compact Kähler manifold $(X, g)$ and any two Kähler potentials on it, the weak geodesic $\Phi$ connecting them belongs to $C^{1,1}(X \times \Sigma)$.

This result should be contrasted with the negative results of Lempert-Vivas [19], Darvas-Lempert [13] and Darvas [12], which show that in general the weak geodesic satisfies that $\sqrt{-1} \partial \bar{\partial} \Phi \notin C^0(X \times \Sigma)$. See also [3, 8, 9, 15, 21, 22, 25, 27, 28] and references therein for other recent developments on issues related to weak geodesics and their regularity. Our result can also be used to simplify part of the arguments in [3], and possibly to extend the results in [28] to more general manifolds.

Theorem 1.1 is a consequence of the following general interior estimate for the complex Monge-Ampère equation. Let $(M^n, g)$ now be a compact $n$-dimensional Kähler manifold with possibly nonempty boundary $\partial M$. Again, write $\omega$ for the Kähler form of $g$. Suppose that $\varphi \in C^\infty(M, \mathbb{R})$ satisfies $\omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0$ and solves the complex Monge-Ampère equation

$$(1.2) \quad (\omega + \sqrt{-1} \partial \bar{\partial} \varphi)^n = e^F \omega^n,$$

for some smooth function $F$.

We have the following interior a priori $C^{1,1}$ estimate, which is proved in Section 2.

**Theorem 1.2.** Let $\varphi$ solve the complex Monge-Ampère equation (1.2). Then there exists a constant $C$, depending only on $(M, g)$, on upper bounds on $\|\varphi\|_{C^1(M, g)}$, $\sup_M \Delta_g \varphi$, $\sup_{\partial M} |\nabla^2 \varphi|_g$ (if $\partial M \neq \emptyset$), $\sup_M |\nabla F|_g$, and on a lower bound on $\nabla^2 F$ (with respect to $g$) such that

$$\sup_M |\nabla^2 \varphi|_g \leq C.$$

Crucially, our estimate is independent of $\inf_M F$, and so it can be applied to the homogeneous complex Monge-Ampère equation to obtain regularity for geodesics in the space of Kähler metrics. This is explained in Section 3.

More generally, combining Theorem 1.2 with [4, Theorem 1.4] and [5, Theorem B] we immediately obtain the following improvement of these results:
**Corollary 1.3.** Let \((M, g)\) be a compact Kähler manifold with nonempty boundary, which we assume is weakly pseudoconcave. Given a smooth function \(\varphi_0\) on \(M\) with \(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_0 > 0\), there exists a unique solution \(\varphi \in C^{1,1}(M)\) of

\[
(\omega + \sqrt{-1}\partial\bar{\partial}\varphi)^n = 0, \text{ on } M, \quad \varphi = \varphi_0, \text{ on } \partial M.
\]

As alluded to above, the proof of Theorem 1.2 is essentially contained in our earlier paper [10] where we solved the Monge-Ampère equation on compact almost-complex manifolds (see also [26]). Indeed, the reader may easily verify, combining the arguments in [10] with the small modifications in this paper, we also obtain that the second order estimate in the general almost-complex setting of [10, Proposition 5.1] does not depend on \(\inf_M F\). This includes the Hermitian case, when \(J\) is integrable but \(d\omega \neq 0\), which was studied for example in [17, 32]. The reader who is interested in the most general setup is referred to [10].

The arguments of [10] are long and intricate because of technicalities that arise when the complex structure is not integrable, and/or when \(d\omega \neq 0\), and because there we do not have a bound on the complex Hessian of \(\varphi\). The aim of this note is to give a self-contained proof of Theorem 1.2 which is substantially simpler and shorter than the arguments of [10].

We discuss briefly the idea of the proof. If one follows the approach of Blocki [4], a difficulty arises when applying the maximum principle to a quantity involving the largest eigenvalue of the real Hessian, given by

\[
\sup_{|V|_g = 1} \nabla^2 \varphi(V, V).
\]

A perturbation argument is needed to make this smooth near a maximum point \(x_0\). In [4] this is done by taking a unit vector \(V_0\) which maximizes (1.3) at \(x_0\), extending it smoothly to a unit vector \(V\) in a neighborhood of \(x_0\), and considering the smooth quantity \(\nabla^2 \varphi(V, V)\) near \(x_0\). To deal with bad terms involving the trace of \(g_\varphi\) with respect to \(g_\varphi + \varphi g_\varphi\), one is forced to consider the logarithm of this quantity, which introduces bad third order terms that cannot be controlled, unless \(g\) has nonnegative bisectional curvature in which case there is no need to take the logarithm [3]. Instead, we use a different perturbation argument as in [10, 30, 31], replacing (1.3) near \(x_0\) with the largest eigenvalue function of a small perturbation of \(\nabla^2 \varphi\), so that its largest eigenspace is 1-dimensional and therefore the largest eigenvalue varies smoothly near \(x_0\). Crucially, this gives new good third order terms (the terms in the first sum on the right hand side of (2.6)), which are shown, via a series of delicate calculations and estimates, to be just barely enough to control the bad terms.

**Remark 1.4.** In a follow up paper [11], we will investigate the regularity of weak geodesic rays which arise from test configurations, as in the work of Phong-Sturm [22, 23, 24, 25] who established \(C^{1,\alpha}\) regularity in this setting.
Acknowledgments. The authors thank J. Song for pointing out a simplification of the proof given in an earlier version of this paper, and M. Păun for interesting discussions. We also thank the referee for some helpful comments. The first-named author would like to thank his advisor G. Tian for encouragement and support. The second-named author was partially supported by NSF grant DMS-1610278, and the third-named author by NSF grant DMS-1406164. This work was completed while the second-named author was visiting the Yau Mathematical Sciences Center at Tsinghua University in Beijing, which he would like to thank for the hospitality.

2. $C^{1,1}$ bound for the complex Monge-Ampère equation

In this section we give the proof of Theorem 1.2. We follow the approach that we introduced recently in [10], taking a little more care to make sure that all the estimates are independent of inf$_M F$. Moreover, we make many simplifications, due to the fact that we assume we are in the Kähler case, and also because we allow our estimates to depend on sup$_M \Delta_g \varphi$ (since this quantity is known to be bounded a priori in the settings of Theorem 1.1 and Corollary 1.3 due to previous works mentioned in the introduction), while one of the main points of [10] is that such an estimate on the Laplacian is not available in the almost-complex case, and our real Hessian bound there does not use it (but of course it implies it).

**Proof of Theorem 1.2.** Up to modifying $\varphi$ by adding a constant to it, we will assume that sup$_M \varphi = 0$. We apply the maximum principle to the quantity

$$Q = \log \lambda_1(\nabla^2 \varphi) + h(|\partial \varphi|^2_g) - A\varphi,$$

where $\lambda_1(\nabla^2 \varphi)$ is the largest eigenvalue of the real Hessian $\nabla^2 \varphi$ (with respect to the metric $g$). The function $h$ is given by

$$h(s) = -\frac{1}{2} \log(1 + \sup_M |\partial \varphi|^2_g - s),$$

and $A > 1$ is a constant to be determined (which will be uniform, in the sense that it will depend only on the background data and on the quantities in the statement of Theorem 1.2). Note that $h(|\partial \varphi|^2_g)$ is uniformly bounded, and

$$\frac{1}{2} \geq h' \geq \frac{1}{2 + 2 \sup_M |\partial \varphi|^2_g} > 0, \quad \text{and} \quad h'' = 2(h')^2,$$

where we are evaluating $h$ and its derivatives at $|\partial \varphi|^2_g$. Observe that since

$$|\nabla^2 \varphi|_g \leq C \lambda_1(\nabla^2 \varphi) + C,$$

a simple consequence of the fact that $\omega + \sqrt{-1} \partial \bar{\partial} \varphi > 0$, it suffices to bound $Q$ from above on $M$. Note also that $Q$ is a continuous function on its domain $\{ \lambda_1(\nabla^2 \varphi) > 0 \}$, and achieves a maximum at a point $x_0 \in M$ with $\lambda_1(\nabla^2 \varphi(x_0)) > 0$, which we may assume is not on $\partial M$ (otherwise we are
done). On the other hand, $Q$ may not be smooth, since the eigenspace associated to $\lambda_1$ may have dimension strictly larger than 1. Because of this, we use a perturbation argument, as in [10, 30, 31].

Fix holomorphic normal coordinates $(z^1, \ldots, z^n)$ for $g$ centered at $x_0$, and let $z^j = x^{2j-1} + \sqrt{-1}x^{2j}$, so that $(x^1, \ldots, x^{2n})$ are real coordinates near $x_0$. In what follows, we will use Latin letters $i, j, k, \ldots$ for “complex” indices ranging from 1 to $n$, and Greek letters $\alpha, \beta, \ldots$ for “real” indices ranging from 1 to $2n$. We define

$$\tilde{g}_{ij} = g_{ij} + \varphi_{ij}$$

and we may assume that at $x_0$ the matrix $[\tilde{g}_{ij}]$ is diagonal with

$$\tilde{g}_{11} \geq \tilde{g}_{22} \geq \cdots \geq \tilde{g}_{nn}.$$ 

Let $V_1$ be a unit vector (with respect to $g$) corresponding to the largest eigenvalue $\lambda_1$ of $\nabla^2 \varphi$, so that at $x_0$,

$$\nabla^2 \varphi(V_1, V_1) = \lambda_1.$$ 

We extend $V_1$ to an orthonormal basis $V_1, \ldots, V_{2n}$ of eigenvectors of $\nabla^2 \varphi$ with respect to $g$ at $x_0$, corresponding to eigenvalues $\lambda_1(\nabla^2 \varphi) \geq \lambda_2(\nabla^2 \varphi) \geq \cdots \geq \lambda_{2n}(\nabla^2 \varphi)$. Denote by $\{V_\beta\}_{\beta=1}^{2n}$ the components of the vector $V_\beta$ at $x_0$, with respect to the coordinates $x^1, \ldots, x^{2n}$. Extend $V_1, V_2, \ldots, V_{2n}$ to be vector fields in a neighborhood of $x_0$ by taking the components to be constant, noting that the $V_\beta$ may only be eigenvectors for $\nabla^2 \varphi$ at $x_0$.

To avoid the inconvenient situation where $\lambda_1(\nabla^2 \varphi) = \lambda_2(\nabla^2 \varphi)$, define near $x_0$ a smooth symmetric semipositive definite section $B = (B_{\alpha\beta})$ of $T^*M \otimes T^*M$ by

$$B = B_{\alpha\beta} dx^\alpha \otimes dz^\beta = \sum_{\alpha, \beta} (\delta_{\alpha\beta} - V^\alpha_\beta V^\beta_1) dx^\alpha \otimes dx^\beta,$$

and a local endomorphism $\Phi = (\Phi^\alpha_\beta)$ of $TM$ by

$$(2.4) \quad \Phi^\alpha_\beta \equiv g^{\alpha\gamma} \nabla^2_{\gamma\beta} \varphi - g^{\alpha\gamma} B_{\gamma\beta}.$$ 

We now consider $\lambda_1(\Phi)$, which is smooth and satisfies $\lambda_1(\Phi) \leq \lambda_1(\nabla^2 \varphi)$ in a neighborhood of $x_0$ and $\lambda_1(\Phi) = \lambda_1(\nabla^2 \varphi)$ at $x_0$. The vector fields $V_1, \ldots, V_{2n}$ are still eigenvectors for $\Phi$ at $x_0$, with eigenvalues $\lambda_1(\Phi) > \lambda_2(\Phi) > \cdots > \lambda_{2n}(\Phi)$. In what follows we will often write $\lambda_\alpha$ for $\lambda_\alpha(\Phi)$.

Define a new perturbed smooth quantity $\tilde{Q}$ in a neighborhood of $x_0$ by

$$\tilde{Q} = \log \lambda_1(\Phi) + h(|\partial \varphi|_g^2) - A \varphi,$$

which still attains a maximum at $x_0$.

Let $\Delta_{\tilde{g}} = \tilde{g}^{ij} \partial_i \partial_j$ be the (complex) Laplacian of $\tilde{g}$, where we are writing $\partial_i, \partial_j$ for $\partial / \partial x^i, \partial / \partial z^j$. Later we may also write $\varphi_i$ for $\partial_i \varphi$ etc. We will compute $\Delta_{\tilde{g}} \tilde{Q}$ at $x_0$. In the computation, we may and do assume without loss of
Lemma 2.1. At $c > 0$ for a uniform generality that $\lambda_1 \gg 1$ at $x_0$. Note that by assumption $\Delta g \varphi \leq C$, so $\tilde{g}_{ii}$ and $\varphi_{ii}$ are uniformly bounded from above at $x_0$ and hence also

$$\tilde{g}_{ii} \geq c, \quad \text{for } i = 1, 2, \ldots, n,$$  

(2.5) for a uniform $c > 0$.

**Lemma 2.1.** At $x_0$, we have

$$0 \geq \Delta_{\tilde{g}} \hat{Q} \geq 2 \sum_{\alpha > 1} \frac{\tilde{g}_{ii} |\partial_i (\varphi V_{\alpha V_1})|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \frac{\tilde{g}_{ij} \tilde{g}_{k\ell} |V_i (\tilde{g}_{j\ell})|^2}{\lambda_1} - \frac{\tilde{g}_{ii} |\partial_i (\varphi V_{\alpha V_1})|^2}{\lambda_1^2}$$  

(2.6) + $h' \sum_k \tilde{g}_{ii} (|\varphi_{ik}|^2 + |\varphi_{i\ell}|^2) + h'' \tilde{g}_{ii} |\partial_i |\varphi|^2|^2$  

+ $(A-C) \sum_i \tilde{g}_{ii} - An.$

where

$$\varphi_{\alpha\beta} = \nabla^2_{\alpha\beta} \varphi, \quad \varphi V_{\alpha V_\beta} = \varphi_{\gamma\delta} V_\gamma V_\delta = \nabla^2 \varphi (V_{\alpha V_\beta}).$$

**Proof.** Compute

$$\Delta_{\tilde{g}} \hat{Q} = \frac{\Delta_{\tilde{g}} (\lambda_1)}{\lambda_1} - \frac{\tilde{g}_{ii} |\partial_i (\lambda_1)|^2}{\lambda_1^2} + h' \Delta_{\tilde{g}} (|\partial_i |\varphi|^2)^2 + h'' \tilde{g}_{ii} |\partial_i |\varphi|^2|^2 - A \Delta_{\tilde{g}} \varphi.$$  

We now prove a lower bound for $\Delta_{\tilde{g}} (\lambda_1)$. Using the fact that $\hat{g}$ is diagonal at $x_0$ and the coordinates are normal for $g$,

$$\Delta_{\tilde{g}} (\lambda_1) = \tilde{g}_{ii} \lambda_1 \alpha\beta, \gamma\delta |\partial_i (\Phi_\gamma) |\partial_i (\Phi_\delta) + \tilde{g}_{ii} \lambda_1 \alpha\beta, \partial_\alpha \partial_\beta (\Phi_\gamma)$$  

(2.8) + $\tilde{g}_{ii} \lambda_1 \alpha\beta, \gamma\delta |\partial_i (\varphi_{\alpha\beta}) |\partial_i (\varphi_{\gamma\delta}) + \tilde{g}_{ii} \lambda_1 \alpha\beta, \partial_\alpha \partial_\beta (\varphi_{\gamma\delta}) + \tilde{g}_{ii} \lambda_1 \alpha\beta, \varphi_{\gamma\delta} |\partial_i (\varphi_{\alpha\beta}) |\partial_i (\varphi_{\gamma\delta})$  

- $\tilde{g}_{ii} \lambda_1 \alpha\beta, \partial_\beta \partial_i (\varphi_{\gamma\delta}) - C \lambda_1 \sum_i \tilde{g}_{ii}.$

Here we used the elementary formulas (see [10] Lemma 5.2]), holding at $x_0$,

$$\lambda_1^{\alpha\beta} := \frac{\partial \lambda_1}{\partial \Phi_\alpha_\beta} = V_1 V_1^{\alpha\beta}$$  

(2.9) $$\lambda_1^{\alpha\beta, \gamma\delta} := \frac{\partial^2 \lambda_1}{\partial \Phi_\alpha_\beta \partial \Phi_{\gamma\delta}} = \sum_{\mu>1} \frac{V_\alpha V_\mu V_\gamma V_\delta + V_\alpha V_\mu V_1 V_1^{\gamma\delta}}{\lambda_1 - \lambda_\mu}.$$

Since the Christoffel symbols of the connection of $g$ vanish at $x_0$ and the components of $V_1$ are constant in our coordinate system, a short calculation
shows that
\[ \tilde{g}^{ij} \partial_i \partial_j (\varphi V_1 V_1) \geq \tilde{g}^{ij} V_1 V_1 (\partial_i \partial_j \varphi) - C \lambda_1 \sum_i \tilde{g}^{ii}, \]
(2.10)
\[ \geq \tilde{g}^{ij} V_1 V_1 (\tilde{g}^{ii}) - C' \lambda_1 \sum_i \tilde{g}^{ii}, \]
where we used that \( |\partial \varphi|_g \) is uniformly bounded, and that we may assume without loss of generality that \( \lambda_1(x_0) \) is large. Applying \( V_1 V_1 \) to the logarithm of (1.2),
\[ \log \det \tilde{g} = \log \det g + F, \]
(2.11)
we obtain
\[ \tilde{g}^{ij} V_1 V_1 (\tilde{g}^{ii}) = \tilde{g}^{ij} \frac{\partial_i (\varphi V_1 V_1)}{\lambda_1 - \lambda_\alpha} + \tilde{g}^{ij} |V_1 (\tilde{g}_{ij})|^2 - C \lambda_1 \sum_i \tilde{g}^{ii}. \]
(2.12)
Next, at \( x_0 \),
\[ \Delta \tilde{g} (|\partial \varphi|_g^2) = \sum_k \tilde{g}^{ii} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) + 2 \text{Re} \left( \sum_k \varphi_k F_k \right) \]
(2.14)
\[ \geq \sum_k \tilde{g}^{ii} (|\varphi_{ik}|^2 + |\varphi_{ik}|^2) - C \sum_i \tilde{g}^{ii}, \]
where to obtain the first line we have applied \( \partial_k \) to (2.11). Lastly, we observe that
\[ \Delta \tilde{g} \varphi = n - \sum_i \tilde{g}^{ii}. \]
(2.15)
The result then follows by combining (2.7), the first equation of (2.9), (2.13), (2.14) and (2.15).

We need to deal with the negative third term
\[ -\frac{\tilde{g}^{ij} |\partial_i (\varphi V_1 V_1)|^2}{\lambda_1^2} \]
(2.16)
on the right hand side of (2.6).
Define a local \((1,0)\) vector field by
\[ W_1 := \frac{1}{\sqrt{2}} (V_1 - \sqrt{-1} J V_1), \]
where \( J \) is the complex structure. We write at \( x_0 \),
\[ W_1 = \sum_{q=1}^n \nu_q \partial_q, \quad \sum_{q=1}^n |\nu_q|^2 = 1, \]
(2.17)
for complex numbers $\nu_1, \ldots, \nu_n$, where the second equation follows from the fact that $W_1$ is $g$-unit at $x_0$.

Next, define $\mu_2, \mu_3, \ldots, \mu_{2n} \in \mathbb{R}$ by

$$
\tag{2.18} JV_1 = \sum_{\alpha > 1} \mu_\alpha V_\alpha, \quad \sum_{\alpha > 1} \mu_\alpha^2 = 1, \quad \text{at } x_0,
$$

noting that at $x_0$ the vector $JV_1$ is $g$-unit and $g$-orthogonal to $V_1$.

Since we are using complex coordinates, the complex structure $J$ in our real coordinates $(x_1, \ldots, x_{2n})$ has constant coefficients, and so do the vector fields $JV_1$ and $W_1$.

Then we have:

**Lemma 2.2.** There is a uniform constant $C \geq 1$ such that if $0 < \varepsilon < 1/2$ and $\lambda_1(x_0) \geq C/\varepsilon^2$, then at $x_0$ we have

$$
\sum_i \frac{\hat{g}^i |\partial_i (\varphi V_1)_1|^2}{\lambda_1^2} \leq 2(h')^2 \hat{g}^i |\partial_i \varphi|_g^2 \lambda_1^2 + 4\varepsilon A^2 \hat{g}^i |\varphi|^2
$$

$$
+ 2 \sum_{\alpha > 1} \frac{\hat{g}^i |\partial_i (\varphi_{V_1 V_\alpha})|^2}{\lambda_1 (\lambda_1 - \lambda_\alpha)} + \frac{\hat{g}^p \hat{g}^q |V_1(\tilde{g}_{pq})|^2}{\lambda_1} + \sum_i \hat{g}^i.
$$

**Proof.** Writing $E$ for an “error” term which is uniformly bounded $|E| \leq C$, we use the definition of $W_1$, the $\mu_\alpha$ and the $\nu_q$ to obtain at $x_0$,

$$
\partial_i (\varphi V_1) = \sqrt{2} \partial_i \varphi V_1 + \sqrt{-1} \partial_i (\varphi V_1 JV_1)
$$

$$
= \sqrt{2} W_1 \partial_i (V_1(\varphi)) - \sqrt{-1} JV_1 \partial_i (V_1(\varphi)) + E
$$

$$
= \sqrt{2} \sum_q \nu_q V_1(\tilde{g}_{pq}) - \sqrt{-1} \sum_{\alpha > 1} \mu_\alpha \partial_i (\varphi_{V_1 V_\alpha}) + E,
$$

where we used that $V_1, JV_1$ and $W_1$ have constant coefficients, and that the coordinates are normal at $x_0$, which imply that $(\nabla V_1 W_1)(x_0) = 0$ and so on, which shows that all the commutations of 3 derivatives above give error...
terms which only involve $\partial \varphi$, hence are uniformly bounded. Therefore
\begin{equation}
(1 - 2\varepsilon) \sum_{i} \frac{\tilde{g}^{ii} |\partial_i (\varphi_{V_1 V_1})|^2}{\lambda_1^2}
= (1 - 2\varepsilon) \sum_{i} \frac{\tilde{g}^{ii} |\sqrt{2} \sum_q \varphi_{V_1} (\tilde{g}_{\varphi}) - \sqrt{-1} \sum_{\alpha > 1} \mu_{\alpha} \partial_i (\varphi_{V_1 V_{\alpha}}) + E|^2}{\lambda_1^2}
\leq (1 - \varepsilon) \sum_{i} \frac{\tilde{g}^{ii} |\sqrt{2} \sum_q \varphi_{V_1} (\tilde{g}_{\varphi}) - \sqrt{-1} \sum_{\alpha > 1} \mu_{\alpha} \partial_i (\varphi_{V_1 V_{\alpha}}) |^2}{\lambda_1^2} \frac{C}{\varepsilon \lambda_1^2} \sum_{i} \tilde{g}^{ii}
\leq (1 - \varepsilon) \left( 1 + \frac{1}{\varepsilon} \right) \sum_{i} \frac{2 \tilde{g}^{ii} |\sqrt{2} \sum_q \varphi_{V_1} (\tilde{g}_{\varphi}) |^2}{\lambda_1^2} + (1 - \varepsilon^2) \sum_{i} \frac{\tilde{g}^{ii} |\sum_{\alpha > 1} \mu_{\alpha} \partial_i (\varphi_{V_1 V_{\alpha}}) |^2}{\lambda_1^2}
+ \sum_{i} \tilde{g}^{ii},
\end{equation}

using that $\lambda_1 \geq C/\varepsilon^2 \geq \sqrt{C/\varepsilon}$. Using this together with (2.21) and the fact that $|\nu_q| \leq 1$, and $\lambda_1 \geq C/\varepsilon$,
\begin{equation}
(1 - \varepsilon) \left( 1 + \frac{1}{\varepsilon} \right) \sum_{i} \frac{2 \tilde{g}^{ii} |\sqrt{2} \sum_q \varphi_{V_1} (\tilde{g}_{\varphi}) |^2}{\lambda_1^2} \leq \frac{C}{\varepsilon \lambda_1} \sum_{i} \sum_{q} \frac{\tilde{g}^{ii} \tilde{g}^{qq} |\varphi_{V_1} (\tilde{g}_{\varphi}) |^2}{\lambda_1}
\leq \sum_{i} \sum_{q} \frac{\tilde{g}^{ii} \tilde{g}^{qq} |\varphi_{V_1} (\tilde{g}_{\varphi}) |^2}{\lambda_1}.
\end{equation}

Next, since $\sum_{\alpha > 1} \mu_{\alpha}^2 = 1$,
\begin{equation}
\left| \sum_{\alpha > 1} \mu_{\alpha} \partial_i (\varphi_{V_1 V_1}) \right|^2 \leq \left( \sum_{\alpha > 1} (\lambda_1 - \mu_{\alpha}) \mu_{\alpha}^2 \right) \left( \sum_{\alpha > 1} |\partial_i (\varphi_{V_1 V_1}) |^2 \right) / \lambda_1
= \left( \lambda_1 - \sum_{\alpha > 1} \mu_{\alpha} \mu_{\alpha}^2 \right) \left( \sum_{\alpha > 1} |\partial_i (\varphi_{V_1 V_1}) |^2 \right) / \lambda_1.
\end{equation}

But by the definition of $W_1$, the $\lambda_{\alpha}$ and $\mu_{\alpha}$,
\begin{equation}
0 < \tilde{g}(W_1, W_1) = 1 + \frac{1}{2} (\varphi_{V_1 V_1} + \varphi_{V_1 V_1}) = 1 + \frac{1}{2} \left( \lambda_1 + \sum_{\alpha > 1} \mu_{\alpha} \mu_{\alpha}^2 \right),
\end{equation}
so
\begin{equation}
\lambda_1 - \sum_{\alpha > 1} \mu_{\alpha} \mu_{\alpha}^2 \leq 2 \lambda_1 + 2 \leq (2 + 2 \varepsilon^2) \lambda_1,
\end{equation}
as long as $\lambda_1 \geq 1/\varepsilon^2$. Hence
\begin{equation}
(1 - \varepsilon^2) \left( \lambda_1 - \sum_{\alpha > 1} \mu_{\alpha} \mu_{\alpha}^2 \right) \leq 2 (1 - \varepsilon^2) (1 + \varepsilon^2) \lambda_1 \leq 2 \lambda_1,
\end{equation}
Finally, using $d\tilde{Q} = 0$ at $x_0$,

$$
2\varepsilon \sum_i \tilde{g}^{\tilde{i}} |\tilde{r}_i (\varphi_{\tilde{V}_i})|^2 = 2\varepsilon \sum_i \tilde{g}^{\tilde{i}} |A \varphi_i + h' \partial_i |\partial \varphi|_g|^2 \\
\leq 4\varepsilon A^2 \sum_i \tilde{g}^{\tilde{i}} |\varphi_i|^2 + 4\varepsilon (h')^2 \sum_i \tilde{g}^{\tilde{i}} |\partial_i |\partial \varphi|_g|^2.
$$

(2.24)

Combining (2.20), (2.21), (2.22), (2.23) and (2.24) completes the proof of the lemma.

We now complete the proof of Theorem 1.2. Combining (2.6) with Lemma 2.2 we have

$$
0 \geq -4\varepsilon A^2 \tilde{g}^{\tilde{i}} |\varphi_i|^2 - 2(h')^2 \sum_i \tilde{g}^{\tilde{i}} |\partial_i |\partial \varphi|_g|^2 \\
+ h' \sum_k \tilde{g}^{\tilde{i}} (|\varphi_{ik}|^2 + |\varphi_{\tilde{k}i}|^2) + h'' \tilde{g}^{\tilde{i}} |\partial_i |\partial \varphi|_g|^2 \\
+ (A - C_0) \sum_i \tilde{g}^{\tilde{i}} - A n,
$$

as long as $\varepsilon < \frac{1}{2}$ and $\lambda_1(x_0) \geq C/\varepsilon^2$. Recalling that $h'' = 2(h')^2$, and choosing $A = C_0 + 2$ and

$$
\varepsilon = \frac{1}{4A^2 (\sup_M |\partial \varphi|_g^2 + 1)},
$$

we may now assume without loss of generality that $\lambda_1(x_0) \geq C/\varepsilon^2$, and we conclude that at $x_0$ we have

$$
\sum_i \tilde{g}^{\tilde{i}} + h' \sum_k \tilde{g}^{\tilde{i}} (|\varphi_{ik}|^2 + |\varphi_{\tilde{k}i}|^2) \leq C,
$$

which implies that $\lambda_1(x_0) \leq C$, completing the proof of Theorem 1.2.

3. Proof of Theorem 1.1

With the notation of the introduction, $\Phi$ is the limit of $\Phi_\varepsilon - |z|^2$ as $\varepsilon \to 0$ (in $C^{1,\alpha}(X \times \Sigma)$, for any $0 < \alpha < 1$), where for $\varepsilon > 0$ the smooth functions $\Phi_\varepsilon$ solve the “$\varepsilon$-geodesic” equation

$$
(\pi^* \omega + \sqrt{-1} dz \wedge d\bar{z} + \sqrt{-1} \partial \bar{\partial} \Phi_\varepsilon)^{m+1} = \varepsilon \pi^* \omega^m \wedge \sqrt{-1} dz \wedge d\bar{z},
$$

with the same boundary conditions as $\Phi$. Chen [7] (see also [4, 5, 6, 16]) has proved uniform bounds, independent of $\varepsilon$, on $\sup_{X \times \Sigma} |\Phi_\varepsilon|$, $\sup_{X \times \Sigma} |\nabla \Phi_\varepsilon|$, $\sup_{X \times \Sigma} |\Delta \Phi_\varepsilon|$ and $\sup_{\partial(X \times \Sigma)} |\nabla^2 \Phi_\varepsilon|$, where we are using the reference Kähler metric $\pi^* \omega + \sqrt{-1} dz \wedge d\bar{z}$ on $X \times \Sigma$. Theorem 1.2 thus applies to (3.1) and we obtain

$$
\sup_{X \times \Sigma} |\nabla^2 \Phi_\varepsilon| \leq C,
$$

where $C$ does not depend on $\varepsilon$. Letting $\varepsilon \to 0$ we conclude that $|\nabla^2 \Phi|$ is in $L^\infty(X \times \Sigma)$, and so $\Phi$ is in $C^{1,1}(X \times \Sigma)$. 
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