A REFINED GREEN’S FUNCTION ESTIMATE OF 
THE TIME MEASURABLE PARABOLIC OPERATORS 
WITH CONIC DOMAINS 

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ABSTRACT. We present a refined Green’s function estimate of the 
time measurable parabolic operators on conic domains that in-
volves mixed weights consisting of appropriate powers of the dis-
tance to the vertex and of the distance to the boundary. 

1. Introduction 

In recent years we have been interested in the stochastic heat diffu-
sion occurring in wedge shaped subdomains of $\mathbb{R}^2$, which are probably 
simplest non-smooth Lipschitz domains. In the literature there exist al-
most fully developed regularity results for the stochastic heat diffusion 
on $C^1$ domains, but when it comes to Lipschitz domains the results are 
quite unsatisfactory and very little is known. To fill in the gap between 
the theory for $C^1$ domains and the theory for Lipschitz domains, the 
 wedge domains are what we decided to pay attention first. 

Along the way, we set the theme that the angle around the vertex 
affects regularity of the temperature when the boundary temper ature 
is controlled. We believe that our previous work [4] captured such 
relation in a certain way. Based on this work, in [3] we proceeded to 
construct a theory on the stochastic diffusion in polygonal domains. 
The main tool of our results was an estimate on Green’s function for 
the heat operator with the wedge domains obtained in [5]. Looking 
back, what we feel sorry about is that the estimate only involves the 
weight of powers of the distance to the vertex. “only” means that 

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it could be better or much better if the estimate also involves weight of the distance to the boundary. Having weight depending only on the distance to the vertex in the estimate did not yield satisfactory boundary regularity of the solution and caused quite a bit of trouble when we constructed a global regularity theory for polygonal domains.

Aiming more natural and hopefully complete theory for polygonal domains, we imagined a refined Green’s function estimate that involves both the distance to the vertex and the distance to the boundary. This paper is about this improvement task.

The main contents of this paper are as follows. In Section 2, we introduce a Green’s function estimate of the time measurable parabolic operator

\[ \mathcal{L} = \frac{\partial}{\partial t} - \sum_{i,j=1}^{d} a_{ij}(t) D_{ij} \]

defined on a conic domain \( \mathcal{D} \subset \mathbb{R}^d \) with a vertex at the origin. We prove an estimate of the type

\[ G(t, s, x, y) \leq N e^{-\sigma \frac{|x-y|^2}{t-s}} \left( \frac{|x|}{\sqrt{t-s}} \wedge 1 \right)^{\beta_1} \left( \frac{|y|}{\sqrt{t-s}} \wedge 1 \right)^{\beta_2} \times \left( \frac{\rho(x)}{\sqrt{t-s}} \wedge 1 \right) \left( \frac{\rho(y)}{\sqrt{t-s}} \wedge 1 \right), \quad \beta_1, \beta_2 \geq 0, \quad (1.1) \]

where \( \rho(x) := \text{dist}(x, \partial\mathcal{D}) \). The ranges of \( \beta_1 \) and \( \beta_2 \) are determined by \( \mathcal{D} \) and \( \mathcal{L} \) and described in Remark 2.2. Note that estimate (1.1) involves both the distance to the vertex and the distance to the boundary, and gives a subtle decay rate as \( x, y \) approach the boundary or the origin.

In Sections 3 and 4, we obtain some critical upper bounds of \( \beta_1, \beta_2 \) for the operator \( \mathcal{L} \).

In this paper we use the following notations:

- We use := to denote a definition.
- \( \alpha \wedge \beta = \min\{\alpha, \beta\} \), \( \alpha \vee \beta = \max\{\alpha, \beta\} \)
- \( N(\cdots) \) means a constant depending only on what are indicated.
- \( D_{ij} u = \frac{\partial^2 u}{\partial x_j \partial x_i} \)

and

- \( B_R(x) = \{y \in \mathbb{R}^d \mid |y-x| < R\} \)
- \( B_R^D(x) = B_R(x) \cap \mathcal{D} \)
- \( Q_R(t, x) = (t-R^2, t] \times B_R(x) \)
- \( Q^D_R(t, x) = (t-R^2, t] \times (B_R(x) \cap \mathcal{D}) \).

Also, we will frequently use the following sets of functions (see [6]).

- \( \mathcal{V}(Q_R(t_0, x_0)) \) : the set of functions \( u \) defined at least on \( Q_R(t_0, x_0) \) and satisfying

\[ \sup_{t \in (t_0-R^2, t_0]} \|u(t, \cdot)\|_{L^2(B_R(x_0))} + \|\nabla u\|_{L^2(Q_R(t_0, x_0))} < \infty. \]
- $\mathcal{V}_{\text{loc}}(Q_R(t_0, x_0))$: the set of functions $u$ defined at least on $Q_R(t_0, x_0)$ and satisfying

$$u \in \mathcal{V}(Q_r(t_0, x_0)), \ \forall r \in (0, R).$$

- $\mathcal{V}(Q_R^D(t_0, x_0))$: the set of functions $u$ defined at least on $Q_R^D(t_0, x_0)$ and satisfying

$$\sup_{t \in (t_0-R^2, t_0]} \|u(t, \cdot)\|_{L_2(B_R^D(x_0))} + \|\nabla u\|_{L_2(Q_R^D(t_0, x_0))} < \infty.$$

- $\mathcal{V}_{\text{loc}}(Q_R^D(t_0, x_0))$: the set of functions $u$ defined at least on $Q_R^D(t_0, x_0)$ and satisfying

$$u \in \mathcal{V}(Q_r^D(t_0, x_0)), \ \forall r \in (0, R).$$

2. Main result

We define our conic domain in $\mathbb{R}^d$ by

$$\mathcal{D} = \left\{ x \in \mathbb{R}^d \setminus \{0\} \bigg| \frac{x}{|x|} \in \mathcal{M} \right\},$$

where $\mathcal{M}$ is a connected open subset in the sphere $S^{d-1} = \{ \xi \in \mathbb{R}^d \mid |\xi| = 1 \}$ which has $C^2$ boundary. Here, $C^2$ boundary means that for any fixed point $p \in S^{d-1} \setminus \mathcal{D}$ and the stereographic projection of $S^{d-1} \setminus \{p\}$ onto the tangent hyperplane at $-p$, the antipode of $p$, the image of $\mathcal{D}$ has $C^2$ boundary in the hyperplane.

![Figure 1. Cases of $d = 2$ and $d = 3$](image-url)
For example, when \( d = 2 \), for each fixed angle \( \kappa \in (0, 2\pi) \) we can consider
\[
\mathcal{D} = \mathcal{D}_\kappa = \left\{ (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid r \in (0, \infty), \ -\frac{\kappa}{2} < \theta < \frac{\kappa}{2} \right\}.
\]
(2.1)

In this paper we consider the Green's function of the operator
\[
\mathcal{L} = \frac{\partial}{\partial t} - \sum_{i,j} a_{ij}(t) D_{ij}
\]
(2.2)

with the domain \( \mathcal{D} \). We assume that the diffusion coefficients \( a_{ij}, i, j = 1, \ldots, d \), are real valued measurable functions of \( t \), \( a_{ij} = a_{ji}, i, j = 1, \ldots, d \), and satisfy the uniform parabolicity condition, i.e. there exists a constant \( \nu \in (0, 1] \) such that for any \( t \in \mathbb{R} \) and \( \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \),
\[
\nu|\xi|^2 \leq \sum_{i,j} a_{ij}(t) \xi_i \xi_j \leq \nu^{-1}|\xi|^2.
\]
(2.3)

We denote the Green’s function by \( G(t, s, x, y) \). By the definition of Green’s function \( G \) is nonnegative and, for any fixed \( s \in \mathbb{R} \) and \( y \in \mathcal{D} \), the function \( v = G(\cdot, s, \cdot, y) \) satisfies
\[
\mathcal{L} v = 0 \text{ in } (s, \infty) \times \mathcal{D} ; \quad v = 0 \text{ on } (s, \infty) \times \partial \mathcal{D} ; \quad v(t, \cdot) = 0 \text{ for } t < s.
\]

Also, in this paper we use the notations \( \rho_0(x) = |x|, \rho(x) = \text{dist}(x, \partial \mathcal{D}) \) and
\[
R_{t,x} := \frac{|x|}{\sqrt{t}} \wedge 1 = \frac{\rho_0(x)}{\sqrt{t}} \wedge 1, \quad J_{t,x} := \frac{\rho(x)}{\sqrt{t}} \wedge 1.
\]

Remark 2.1. Since \( \frac{a}{a+1} \leq a \wedge 1 \leq 2 \cdot \frac{a}{a+1} \) for any \( a \geq 0 \), we can also define \( R_{t,x} \) and \( J_{t,x} \) by
\[
R_{t,x} := \frac{\rho_0(x)}{\rho_0(x) + \sqrt{t}}, \quad J_{t,x} := \frac{\rho(x)}{\rho(x) + \sqrt{t}}.
\]

From the probabilistic point of view related to a Brownian motion killed at the boundary of \( \partial \mathcal{D} \), \( G \) is essentially a transition probability and bounded by a constant multiple of Gaussian density function:
\[
0 \leq G(t, s, x, y) \leq N \frac{1}{(t-s)^{d/2}} e^{-\sigma \frac{|x-y|^2}{t-s}}, \quad t > s, \quad x, y \in \mathcal{D},
\]
(2.4)
where the constants \( N, \sigma > 0 \) depend only on space dimension \( d \) and \( \nu \) in the assumption (2.3).

Having further information of the domain, the right hand side of (2.4) can be refined. Especially, for our conic domains \( \mathcal{D} \), one can pursue the following type of estimate
\[
G(t, s, x, y) \leq N \frac{1}{(t-s)^{d/2}} R_{t-s,x}^\lambda R_{t-s,y} e^{-\sigma \frac{|x-y|^2}{t-s}}, \quad t > s, \quad x, y \in \mathcal{D}
\]
for some positive constants $\lambda^+, \lambda^-$. Since $R_{t,x}$ is less than equal to 1, this estimate is sharper as we find bigger $\lambda^+, \lambda^-$ satisfying the estimate.

**Remark 2.2.** As in [6, Section 2], the critical upper bound $\lambda^+_c > 0$ of $\lambda^+$ can be characterized by the supremum of all $\lambda$ such that for some constant $K_0 = K_0(\mathcal{L}, \mathcal{M}, \lambda)$ it holds that

$$|u(t, x)| \leq K_0 \left( \frac{|x|}{R} \right)^\lambda \sup_{Q^D_{R/2}(t_0, 0)} |u|, \quad \forall (t, x) \in Q^D_{R/2}(t_0, 0)$$

(2.5)

for any $t_0 > 0, R > 0$, and $u$ belonging to $\mathcal{V}_{loc}(Q^D_{R}(t_0, 0))$ and satisfying

$$\mathcal{L}u = 0 \text{ in } Q^D_{R}(t_0, 0); \quad u(t, x) = 0 \text{ for } x \in \partial D.$$ 

The value of $\lambda^+_c$ does not change if one replaces $\frac{2}{4}$ in (2.5) by any number in $(1/2, 1)$ (see [6, Lemma 2.2]).

Moreover, the critical upper bound $\lambda^-_c > 0$ of $\lambda^-$ is characterized by the supremum of $\lambda$ with above property for the operator

$$\hat{\mathcal{L}} = \frac{\partial}{\partial t} - \sum_{i,j} a_{ij}(-t) D_{ij}. \quad (2.6)$$

Both $\lambda^+_c$ and $\lambda^-_c$ will definitely depend on $\mathcal{M} = \mathcal{D} \cap S^{d-1}$. Especially when $\mathcal{D} = \mathcal{D}_\kappa$ in (2.1), $\lambda^+_c$ and $\lambda^-_c$ will depend on the opening angle $\kappa$.

If in addition $\mathcal{L}$ is the heat operator, $\mathcal{L} = \frac{\partial}{\partial t} - \Delta_x$, then

$$\lambda^+_c = \lambda^-_c = \frac{\pi}{\kappa}.$$ 

See Section 2 of [6] and Section 3 of this paper for details.

The following lemma is, we think, the most updated estimate of $G$ among the ones involving $R_{t,x}$ only.

**Lemma 2.3.** Let $\lambda^+ \in (0, \lambda^+_c)$, $\lambda^- \in (0, \lambda^-_c)$, and denote $K^+_0 := K_0(\mathcal{L}, \mathcal{M}, \lambda^+)$ and $K^-_0 := K_0(\mathcal{L}, \mathcal{M}, \lambda^-)$. Then there exist positive constants $N = N(\mathcal{M}, \nu, \lambda^+, \lambda^+_0)$ and $\sigma = \sigma(\nu)$ such that

$$G(t, s, x, y) \leq \frac{N}{(t-s)^{d/2}} R_{t-s,x}^{\lambda^+} R_{t-s,y}^{\lambda^-} e^{-\sigma \frac{|x-y|^2}{t-s}}$$

(2.7)

and

$$|\nabla G(t, s, x, y)| \leq \frac{N}{(t-s)^{(d+1)/2}} R_{t-s,x}^{\lambda^+-1} R_{t-s,y}^{\lambda^-} e^{-\sigma \frac{|x-y|^2}{t-s}}$$

for any $t > s, x, y \in \mathcal{D}$.

**Proof.** See [6, Theorem 3.10]. We only remark that in [6] the dependency of $N$ on $K^+_0$ is taken for granted and omitted. By inspecting the proof of [6, Theorem 3.10] one can check that constant $N$ actually depends also on $K^-_0$. \qed
Remark 2.4. In fact, [6] has the estimates of the derivatives of $G$ up to the second order that contain Lemma 2.3 as a part. We refer to Theorem 3.10 of [6]. Yet, the estimates involve $R_{t,x}$ only.

Remark 2.5. Despite the beauty in estimate (2.7), we note that the right hand side of (2.7) does not go to zero as $x$ or $y$ approaches boundary of $D$, meaning that the estimate is not sharp enough in terms of the boundary behavior of the Green’s function.

On the other hand, for any domain satisfying, for instance, the uniform exterior ball condition, the corresponding Green’s function of $L$ is bounded by the constant multiple of

$$
rac{1}{(t-s)^{d/2}} J_{t-s,x} J_{t-s,y} e^{-\sigma|x-y|^2/(t-s)}
$$

which is now forcing the degeneracy of the Green’s function at the boundary (see e.g. [2]).

Of course, our domains, for instance, like $D_\kappa$ in (2.1) does not satisfy the uniform exterior ball condition if $\kappa > \pi$. However, for any $\kappa$, $D_\kappa$ is mostly flat except a small neighborhood of the vertex and we hoped a refined estimate that involves both $R_{t,x}$ and $J_{t,x}$ together. After all, we settled down with the following theorem, which is the refined estimate we mentioned in the introduction and is the main result of this paper.

Theorem 2.6. Let $\lambda^+ \in (0, \lambda^+_c)$, $\lambda^- \in (0, \lambda^-_c)$, and denote $K^+_0 := K_0(L, M, \lambda^+)$ and $K^-_0 := K_0(\hat{L}, \hat{M}, \lambda^-)$. Then there exist positive constants $N = N(M, \nu, \lambda^\pm, K^\pm_0)$ and $\sigma = \sigma(\nu)$ such that

$$
G(t, s, x, y) \leq \frac{N}{(t-s)^{d/2}} R_{t-s,x}^{\lambda^+_0-1} R_{t-s,y}^{\lambda^-_0-1} J_{t-s,x} J_{t-s,y} e^{-\sigma|x-y|^2/(t-s)}
$$

(2.8)

for any $t > s, x, y \in D$.

Remark 2.7. Obviously estimate (2.8) is sharper than estimate (2.7) since $J_{t,x} \leq R_{t,x}$. Moreover, estimate (2.8) gives delicate boundary behavior of Green’s function.

Remark 2.8. The strategy of our proof of Theorem 2.6 is inspired by [2] and [7] although the details are quite different.

In the proof of Theorem 2.6 we will use the following two lemmas from [6].

Lemma 2.9 (Proposition 3.2 of [6]). Let $u$ belong to $\mathcal{V}(Q_R(t_0, x_0))$ and satisfy $Lu = 0$ in $Q_R(t_0, x_0)$, then

$$
|\nabla u(t, x)| \leq \frac{N}{R} \sup_{Q_R(t_0, x_0)} |u|, \quad \forall (t, x) \in Q_{R/2}(t_0, x_0),
$$

where $N$ is a constant.
where the constant $N$ depends only on $\nu$ and $d$.

**Lemma 2.10** (Proposition 3.4 of [6]). There exists a sufficiently small $\delta_0$ such that the following holds for any $\delta \in (0, \delta_0)$: Let $x_0 \in D$, $\rho(x_0) < \delta|x_0|$, and $R \leq \frac{|x_0|}{2}$. Then if $u$ belongs to $V(Q^D_R(t_0,x_0))$ and satisfies $Lu = 0$ in $Q^D_R(t_0,x_0)$ and $u(t,x) = 0$ for $x \in \partial D$, then

$$|\nabla u(t,x)| \leq \frac{N}{R} \sup_{Q^D_R(t_0,x_0)} |u|, \quad \forall (t,x) \in Q^D_{R/8}(t_0,x_0),$$

where the constant $N$ depends only on $\mathcal{M}, \nu, \delta$.

**Proof of Theorem 2.6.**

1. First, we fix $s \in \mathbb{R}$, $y \in D$. We show that there exist positive constants $N = N(\mathcal{M}, \nu, \lambda^\pm, K_0^\pm)$ and $\sigma = \sigma(\nu)$ such that for any $t \in (s, \infty)$ and $x \in D$,

$$G(t,s,x,y) \leq \frac{N}{(t-s)^{d/2}} J_{l-s,x} R^{\lambda^+ - 1}_{t-s,x} R^{\lambda^-}_{t-s,y} e^{-\sigma \frac{|x-y|^2}{t-s}}. \quad (2.9)$$

For given $t \in (s, \infty)$, we consider the following two cases of $x \in D$.

**Figure 2.** Two cases of $x$

- **Case** $\rho(x) \geq \frac{1}{2} \left( |x| \wedge \sqrt{t-s} \right)$.

In this case, by assumption we have

$$2 \frac{\rho(x)}{\sqrt{t-s}} \geq \left( \frac{|x|}{\sqrt{t-s}} \wedge 1 \right).$$

Therefore,

$$R_{l-s,x} = \frac{|x|}{\sqrt{t-s}} \wedge 1 \leq 2 \frac{\rho(x)}{\sqrt{t-s}} \wedge 2 = 2 \left( \frac{\rho(x)}{\sqrt{t-s}} \wedge 1 \right). \quad (2.10)$$

Then, using Lemma 2.3, we immediately get (2.9).
- Case $\rho(x) < \frac{1}{2} (|x| \wedge \sqrt{t-s})$; the point close to the boundary.

For such point $x \in D$, there exists $x_0 \in \partial D$ such that $|x-x_0| = \rho(x)$. For this $x_0 \in \partial D$, $G(t,s,x_0,y) = 0$ and there exists $\theta \in (0,1)$ such that

$$G(t,s,x,y) = G(t,s,x,y) - G(t,s,x_0,y)$$

$$= |x-x_0| |\nabla_x G(t,s,\bar{x},y)|$$

$$= \rho(x) |\nabla_x G(t,s,\bar{x},y)|,$$  \hspace{1cm} (2.11)

where $\bar{x} = (1-\theta)x + \theta x_0 \in D$.

To estimate the gradient part, we make use of Lemma 2.3. Now, since

$$|x| \geq |x| - \theta |x-x_0| \geq |x| - \rho(x) > \frac{1}{2} |x|,$$

$$|x| - |x| + \theta |x-x_0| \leq |x| + \rho(x) < 2|x|,$$

we note that

$$\frac{1}{2} R_{t-s,x} \leq R_{t-s,\bar{x}} \leq 2R_{t-s,x}.$$

In addition, the inequalities

$$|x-y| \leq |x-y| + |\bar{x} - x| \leq |x-y| + |x-x_0| \leq |x-y| + \sqrt{t-s}$$

give

$$-|x-y|^2 \leq -\frac{1}{2} |x-y|^2 + t-s.$$  

Hence, $|\nabla_x G(t,s,\bar{x},y)|$ is bounded by

$$N' \frac{1}{(t-s)^{d+1/2}} R_{t-s,x}^{\lambda_+} R_{t-s,y}^{\lambda_-} e^{-\sigma'(\|x-y\|^2)},$$

where $N' = N'(\mathcal{M}, \nu, \lambda^\pm, K_0^+) > 0$ and $\sigma' = \sigma'(\nu) > 0$. This, (2.11), and $\rho(x) \leq \sqrt{t-s}$ lead us to (2.9) again.

2. Now, we consider the operator $\hat{\mathcal{L}}$ defined in (2.6). Let $\hat{G}$ denote the Green’s function for $\hat{\mathcal{L}}$ with the same domain $\mathcal{D}$. Note that the diffusion coefficients $a_{ij}(-t)$, $i,j = 1, \ldots, d$, also satisfy the uniform parabolicity condition (2.3) with the same $\nu$. Since for any $s \in \mathbb{R}$ and $y \in \mathcal{D}$, $\hat{\mathcal{L}}\hat{G}(\cdot,s,\cdot,y) = 0$ on $(s,\infty) \times \mathcal{D}$ and $\hat{G}(\cdot,s,\cdot,y) = 0$ on $(s,\infty) \times \partial \mathcal{D}$, we can repeat the argument in Step 1 literally by line.

Hence, denoting the critical upper bounds of $\lambda$ for the operator $\hat{\mathcal{L}}$ by $\hat{\lambda}_+^\pm$, $\hat{\lambda}_-^\pm$ and noting that $\hat{\lambda}_c^+ = \lambda_c^-$, $\hat{\lambda}_c^- = \lambda_c^+$ by Remark 2.2 with the same constants $N, \sigma$ in (2.9), we obtain that

$$\hat{G}(t,s,x,y) \leq \frac{N}{(t-s)^{d+2}} J_{t-s,x} R_{t-s,y}^{\lambda_-^+} R_{t-s,y}^{\lambda_-^+} e^{-\sigma'(\|x-y\|^2)}$$  \hspace{1cm} (2.12)

for any $t > s$ and $x,y \in \mathcal{D}$. Note that the locations of $\lambda^+, \lambda^-$ in (2.12) in comparison with the locations of them in (2.9). This is simply because $\lambda^- \in (0,\hat{\lambda}_c^+)$ and $\lambda^+ \in (0,\hat{\lambda}_c^-)$. 

3. Next, using the result of Step 2 and the following identity

\[ G(-s, -t, y, x) = \hat{G}(t, s, x, y) \quad \text{or} \quad G(t, s, x, y) = \hat{G}(-s, -t, y, x), \quad t > s \]

which is due to a duality argument (see (3.12) of [6] for the detail), we observe that with the same constants \( N, \sigma \) in (2.9) we have

\[
G(t, s, x, y) \leq \frac{N}{(t-s)^{d/2}} J_{t-s,y} R_{t-s,y}^{\lambda^+ - 1} R_{t-s,x}^{\lambda^+} e^{-\frac{\sigma |x-y|^2}{t-s}}
\]

for any \( t > s \) and \( x, y \in D \).

4. Finally to finish the proof of (2.8) we repeat the argument in Step 1.

For the points \( x \) away from the boundary the argument is the same. Indeed, if \( \rho(x) \geq \frac{1}{2} (|x| \wedge \sqrt{t-s}) \), then (2.10) and (2.13) certainly give (2.8).

Therefore, for the rest of the proof, we may assume

\[
\rho(x) < \frac{1}{2} (|x| \wedge \sqrt{t-s}).
\]

In this case we first show

\[
|\nabla_x G(t, s, x, y)| \leq \frac{N}{(t-s)^{d/2}} J_{t-s,y} R_{t-s,y}^{\lambda^+ - 1} R_{t-s,x}^{\lambda^+} e^{-\frac{\sigma |x-y|^2}{t-s}}. \quad (2.14)
\]

For this, we fix \((s, y)\) and set

\[ u(t, x) = G(t, s, x, y). \]

Take \( \delta \in (0, \delta_0 \wedge 1/2) \), where \( \delta_0 \) is from Lemma 2.10 which depends only on \( \mathcal{M} \). We consider the following two cases.

- **Case** \( \rho(x) \geq \delta |x| \). Put \( R = \frac{\delta}{2} (|x| \wedge \sqrt{t-s}) \) which is less than \( \frac{1}{2} \rho(x) \) so that \( \bar{B}_R(x) \subset D \). Since \( u \) belongs to \( \mathcal{V}(Q_R(t, x)) \) and satisfies \( \mathcal{L}u = 0 \) in \( Q_R(t, x) \), by Lemma 2.9 we get

\[
|\nabla_x u(t, x)| \leq \frac{N}{R} \sup_{Q_R(t, x)} |u|.
\]

We note that for \((r, z) \in Q_R(t, x)\),

\[
0 \leq t-r \leq \frac{t-s}{4}, \quad \frac{3}{4}(t-s) \leq r-s \leq t-s,
\]

\[
|z| \leq |x| + R \leq 2|x|, \quad |z| \geq |x| - R \geq \frac{1}{2} |x|
\]

and
\[ |z - y| \geq |x - y| - R \geq |x - y| - \sqrt{t - s}, \]
\[ - |z - y|^2 \leq - \frac{1}{2} |x - y|^2 + (t - s), \]
\[ - \frac{|z - y|^2}{r - s} \leq - \frac{1}{2} \frac{|x - y|^2}{t - s} + \frac{4}{3}. \]

Hence, using (2.13) we get
\[ |u(r, z)| \leq \frac{N}{(r - s)^{d/2}} R^\lambda_{r-s,z} J_{r-s,y} R^\lambda_{r-s,y} e^{-\sigma \frac{|z - y|^2}{r - s}} \]
\[ \leq \frac{N}{(t - s)^{d/2}} R^\lambda_{t-s,x} J_{t-s,y} R^\lambda_{t-s,y} e^{-\sigma' \frac{|z - y|^2}{t - s}}. \]

Consequently, we have
\[ |\nabla_x u(t, x)| \leq \frac{N}{R} \sup_{Q_R(t,x)} |u| \]
\[ \leq \frac{N}{|x| \wedge \sqrt{t - s}} \frac{1}{(t - s)^{d/2}} R^\lambda_{t-s,x} R^\lambda_{t-s,y} J_{t-s,y} R^\lambda_{t-s,y} e^{-\sigma' \frac{|z - y|^2}{t - s}} \]
\[ = \frac{N}{(t - s)^{(d+1)/2}} J_{t-s,y} R^\lambda_{t-s,x} R^\lambda_{t-s,y} e^{-\sigma' \frac{|z - y|^2}{t - s}}, \]
and thus (2.14) is proved.

- **Case** $\rho(x) \leq \delta |x|$. In this case, we put $R = \frac{1}{2} (|x| \wedge \sqrt{t - s})$. Since $u$ belongs to $\mathcal{V}(Q^D_R(t, x))$ and satisfies $Lu = 0$ in $Q^D_R(t, x)$, and $u(t, x) = 0$ for $x \in \partial D$, we can apply Lemma 2.10 and have
\[ |\nabla_x u(t, x)| \leq \frac{N}{R} \sup_{Q^D_R(t,x)} |u|, \]

Similarly as before, we again obtain (2.14).

Finally, by (2.11), the computations below (2.11), and (2.14), we obtain (2.8). This ends the proof. \[ \square \]

3. **On the critical upper bounds $\lambda_c^\pm$**

In this section we discuss some detailed informations of the critical upper bounds $\lambda_c^+$ and $\lambda_c^-$, whose characterizations are given in Remark 2.2.

We first introduce some known results on $\lambda_c^\pm$. The following statements are the 3rd, the 8th, and the 7th in Theorem 2.4 of [6]:
• If $\mathcal{L} = \mathcal{L}_0 := \frac{\partial}{\partial \tau} - \Delta_x$, then

$$\lambda_c^\pm(\mathcal{L}_0, \mathcal{D}) = -\frac{d - 2}{2} + \sqrt{\Lambda + \frac{(d - 2)^2}{4}}, \quad (3.1)$$

where $\Lambda$ is the first eigenvalue of Laplace-Beltrami operator with the Dirichlet condition on domain $\mathcal{M} = \mathcal{D} \cap S^{d-1}$, where $S^{d-1}$ is the sphere with radius 1 in $\mathbb{R}^d$.

• Suppose that $(a_{ij})_{d \times d}$ is a constant matrix. Then

$$\lambda_c^\pm(\mathcal{L}, \mathcal{D}) = \lambda_c^\pm(\mathcal{L}_0, \tilde{\mathcal{D}}) = -\frac{d - 2}{2} + \sqrt{\tilde{\Lambda} + \frac{(d - 2)^2}{4}}, \quad (3.2)$$

where $\tilde{\Lambda}$ is the first eigenvalue of the Dirichlet boundary value problem to Beltrami-Laplacian in the domain $\tilde{\mathcal{M}} = \tilde{\mathcal{D}} \cap S^{d-1}$ while cone $\tilde{\mathcal{D}}$ is the image of $\mathcal{D}$ under the change of variables $x \to y$ that reduces $(a_{ij})_{d \times d}$ to the canonical form $(\delta_{ij})_{d \times d}$ with the Kronecker delta $\delta_{ij}$, $i, j = 1, \ldots, d$.

• For the general operator $\mathcal{L} = \frac{\partial}{\partial \tau} - \sum_{i,j=1}^d a_{ij}(t)D_{ij}$ in (2.2), we have

$$\lambda_c^\pm \geq -\frac{d}{2} + \nu \sqrt{\Lambda + \frac{(d - 2)^2}{4}}, \quad (3.3)$$

where $\nu$ is the uniform parabolicity constant in (2.3).

Remark 3.1. One big difference between (3.2) and (3.3) is that “$d$” appears in (3.3) in place of “$d - 2$”. This actually causes a big gap between (3.2) and (3.3). To demonstrate this, let $d = 2$, $\mathcal{D} = \mathcal{D}_\kappa$ in (2.1), and $\mathcal{L} = \mathcal{L}_0 = \frac{\partial}{\partial \tau} - (D_{x_1}x_1 + D_{x_2}x_2)$. Then we can easily find $\Lambda$ in (3.1), which is the same as $\tilde{\Lambda}$ in (3.2). To find $\Lambda$, we just need to find the smallest eigenvalue $\lambda > 0$ and its eigenfunction $\phi = \phi(\theta)$ satisfying

$$-\phi'' = \lambda \phi, \quad -\frac{\kappa}{2} < \theta < \frac{\kappa}{2}, \quad ; \quad \phi \left(\frac{\kappa}{2}\right) = \phi \left(-\frac{\kappa}{2}\right) = 0,$$

which yields $\phi(\theta) = \cos(\sqrt{\Lambda}\theta)$ and $\cos\left(\sqrt{\Lambda}\frac{\kappa}{2}\right) = 0$. Hence, the eigenvalues satisfy $\sqrt{\Lambda}\frac{\kappa}{2} = \pi/2 + k\pi, k = 0, 1, 2, \ldots$, and thus $\Lambda = \pi^2/\kappa^2$.

In this example, if for instance $\kappa = \pi$, then (3.3) yields, as we can take $\nu = 1$, a trivial information $\lambda_c^\pm \geq 0$, whereas (3.2) gives $\lambda_c^\pm = 1$.

In this section we improve (3.3). In particular, we will replace $d$ in (3.3) by $d - 2$. We assume that the coefficients $a_{ij}(t), i, j = 1, \ldots, d$, satisfy $a_{ij}(t) = a_{ji}(t)$, and there exist constants $\nu_1, \nu_2 > 0$ such that for
any $t \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$,

$$
\nu_1 |\xi|^2 \leq \sum_{i,j} a_{ij}(t)\xi_i \xi_j \leq \nu_2 |\xi|^2.
$$

(3.4)

The condition (2.3) is a special case of this condition: $\nu_1 = \nu, \nu_2 = \nu^{-1}$.

**Theorem 3.2.** Let $\nu_1, \nu_2$ be the uniform parabolicity constants in (3.4). If

$$
\lambda < -\frac{d-2}{2} + \sqrt{\frac{\nu_1}{\nu_2}} \sqrt[4]{\Lambda + \frac{(d-2)^2}{4}},
$$

then there exists a positive constant $K_0 = K_0(\nu_1, \nu_2, \mathcal{M}, \lambda)$ such that

$$
|u(t, x)| \leq K_0 \left( \frac{|x|}{R} \right)^\lambda \sup_{Q_{\frac{R}{2}}(t_0, 0)} |u|, \quad \forall (t, x) \in Q_{\frac{R}{2}}(t_0, 0)
$$

for any $t_0 > 0$, $R > 0$, and $u$ belonging to $\mathcal{V}_{\text{loc}}(Q_{\frac{R}{2}}(t_0, 0))$ and satisfying

$$
\mathcal{L}u = 0 \quad \text{in} \quad Q_{\frac{R}{2}}(t_0, 0) \quad ; \quad u(t, x) = 0 \quad \text{for} \quad x \in \partial D.
$$

In particular, we have

$$
\lambda_c^\pm \geq -\frac{d-2}{2} + \sqrt{\frac{\nu_1}{\nu_2}} \sqrt[4]{\Lambda + \frac{(d-2)^2}{4}}.
$$

(3.5)

Note that if $\nu \leq \nu_1 \leq \nu_2 \leq \nu^{-1}$, the right hand side of (3.5) is quite bigger than that of (3.5). Indeed,

$$
\left(-\frac{d-2}{2} + \sqrt{\frac{\nu_1}{\nu_2}} \sqrt[4]{\Lambda + \frac{(d-2)^2}{4}}\right) - \left(-\frac{d}{2} + \nu \sqrt[4]{\Lambda + \frac{(d-2)^2}{4}}\right)
$$

$$
= 1 + \left(\sqrt{\frac{\nu_1}{\nu_2}} - \nu\right) \sqrt[4]{\Lambda + \frac{(d-2)^2}{4}} \geq 1.
$$

To prove the above theorem, we start with the following lemma which is a slight modificaiton of Lemma A.1 of [6].

**Lemma 3.3.** Let $\mu^2 < \frac{\nu_2}{\nu_1} \left(\Lambda + \frac{(d-2)^2}{4}\right)$ and $0 < \epsilon_1 < \epsilon_2 \leq 1$. Then there exists a constant $N$ depending only on $\mu, \epsilon_1, \epsilon_2, \nu_1, \nu_2, \mathcal{M}$ such that

$$
\int_{Q_{\frac{R}{2}}(t_0, 0)} |x|^{2\mu} |\nabla u|^2 dx dt + \int_{Q_{\frac{R}{2}}(t_0, 0)} |x|^{2\mu-2} |u|^2 dx dt \leq NR^{2\mu-2} \int_{Q_{\frac{R}{2}}(t_0, 0)} |u|^2 dx dt
$$

for any $R > 0$ and any function $u$ belonging to $\mathcal{V}_{\text{loc}}(Q_{\frac{R}{2}}(t_0, 0))$ and satisfying $\mathcal{L}u = 0$ in $Q_{\frac{R}{2}}(t_0, 0)$, $u = 0$ on $\mathbb{R} \times \partial D$. 


Proof. The proof of this lemma is almost the same as that of Lemma A.1 of [6]. The only difference is that we use condition (3.4) instead of condition (2.3).

Proof of Theorem 3.2.
1. Refering to Remark 2.2, we note that it is enough to show that for any \( \mu \in \mathbb{R} \) satisfying \( \mu^2 < \frac{\nu_1}{\nu_2} \left( \Lambda + \left( \frac{d-2}{4} \right)^2 \right) \), there exists a constant \( N \) depending only on \( M, \mu, \nu_1, \nu_2 \) such that

\[
|u(t,x)| \leq N \left( \frac{|x|}{R} \right)^{-\frac{d-2}{2} + \mu} \sup_{Q_{\frac{R}{2}}^D(t_0,0)} |u|, \quad \forall (t,x) \in Q_{R/2}^D(t_0,0)
\]

for any \( t_0 > 0, R > 0 \), and \( u \) belonging to \( \mathcal{V}_{loc}(Q_{R}^D(t_0,0)) \) and satisfying \( \mathcal{L}u = 0 \) in \( Q_{R}^D(t_0,0) ; \quad u(t,x) = 0 \) for \( x \in \partial D \).

Also, we note that we may assume \( t_0 = 0, R = 1 \).

2. Take any function \( u \) satisfying the conditions in Step 1 with \( t_0 = 0, R = 1 \) and take any \( (t,x) \in Q_{1/2}^D(0,0) \). Let us denote

\[
r = |x| \left( < \frac{1}{2} \right), \quad D_r = (t - r^2/4, t] \times (B_{\frac{1}{2}r}(0) \setminus B_{\frac{1}{4}r}(0)).
\]

Then as in the proof of statement 7 of Theorem 2.4 in [6], we have

\[
|u(t,x)|^2 \leq N r^{-d-2} \int_{D_r} |u(\tau,y)|^2 dy d\tau \\
\leq N r^{-d+2 \mu} \int_{D_r} |y|^{-2\mu-2} |u(\tau,y)|^2 dy d\tau. \tag{3.7}
\]

The last inequality in (3.7) holds since for the points \( y \) in \( D_r \), \( |y| \) are comparable with \( r \).

Now, we define a time-changed function of \( u \):

\[
v(s,y) := u(t + r^2 s, y).
\]

This function is well defined at least on \( Q_1^P(0,0) \) due to \( t+r^2 s \in (-1,0] \) for \( s \in (-1,0] \). Moreover, \( v \) belongs to \( \mathcal{V}_{loc}(Q_1^P(0,0)) \) and satisfies

\[
\tilde{\mathcal{L}}v = 0 \quad \text{in} \quad Q_1^P(0,0) ; \quad v = 0 \quad \text{on} \quad \mathbb{R} \times \partial D,
\]

where \( \tilde{\mathcal{L}} = \frac{\partial}{\partial s} - \sum_{i,j} r^2 a_{ij}(s) D_{ij} \). We note that

\[
r^2 \nu_1 |\xi|^2 \leq \sum_{i,j} r^2 a_{ij}(s) \xi_i \xi_j \leq r^2 \nu_2 |\xi|^2
\]

is the uniform parabolicity condition for \( \tilde{\mathcal{L}} \) and the ratio \( \frac{r^2 \nu_1}{r^2 \nu_2} \) is the same as \( \frac{\nu_1}{\nu_2} \) and hence we can apply Lemma 3.3 for \( \tilde{\mathcal{L}} \) and \( v \). Having this in mind, we continue with (3.7) as below.
Since 
\[(t + r^2 s, y) \in D_r \implies (s, y) \in (-1/4, 0] \times B_{\frac{1}{2}r}(0)\]
and \((-1/4, 0] \times B_{\frac{1}{2}r}(0) \subset Q_{\frac{3}{4}}^D(0, 0),\) the last quantity in (3.7) is bounded by
\[N r^{-d+2+2\mu} \int_{Q_{\frac{3}{4}}^D(0, 0)} |y|^{-2\mu-2} |v(s, y)|^2 dy ds. \tag{3.8}\]
Then we apply Lemma 3.3 with \(\epsilon_1 = \frac{3}{4}, \epsilon_2 = \frac{7}{8}\) and see
\[\int_{Q_{\frac{3}{4}}^D(0, 0)} |y|^{-2\mu-2} |v(s, y)|^2 dy ds \leq N \int_{Q_{\frac{3}{4}}^D(0, 0)} |v(s, y)|^2 dy ds \leq N \sup_{Q_{\frac{3}{4}}^D(0, 0)} |v|^2 \leq N \sup_{Q_{\frac{3}{4}}^D(0, 0)} |u|^2, \tag{3.9}\]
where the last quantity in (3.9) follows the observation \(t + r^2 s \in (-\left(\frac{7}{8}\right)^2, 0]\) for any \(s \in (-\left(\frac{7}{8}\right)^2, 0].\)

All the constants \(N\) in this Step 2 depend only on \(M, \mu,\) and \(d.\) Hence, (3.7), (3.8) and (3.9) yield (3.6), and the claim in Step 1 is proved. \(\square\)

**Remark 3.4.** For instance, let \(d = 3\) and for any fixed \(\kappa \in (0, 2\pi)\) take
\[D = D_\kappa = \left\{ (r \sin \theta \cos \phi, r \sin \theta \sin \phi, r \cos \theta) \in \mathbb{R}^3 \mid \right.\]
\[r \in (0, \infty), \, 0 \leq \theta < \frac{\kappa}{2}, \, 0 < \phi \leq 2\pi \right\}.\]
Then the first eigenvalue \(\Lambda\) of Laplace-Beltrami operator with the Dirichlet condition on domain \(D_\kappa \cap S^2\) satisfies
\[\frac{1}{2|\log(\cos(\kappa/4))|} \leq \Lambda \leq \frac{4j_0^2}{\kappa^2} \tag{3.10}\]
where \(j_0 \approx 2.4048\) is the first zero of the Bessel function \(J_0\) (see [11]). Hence, using (3.10) and Theorem 3.2 we can obtain rough lower bounds of \(\lambda_\pm\).

### 4. Evaluation of \(\lambda_\pm\) when \(d = 2\)

Finding the exact values of \(\lambda_\pm\) are very difficult in general. In Section 3 we presented a decent estimation of them from below. In this section we attempt to evaluate \(\lambda_\pm\) when \(d = 2\) and the diffusion coefficients \(a_{ij}, i, j = 1, 2,\) in our operator \(L\) are constants.
As $a_{12} = a_{21}$, we can set

$$A := (a_{ij})_{2 \times 2} := \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

By [2.3], matrix $A$ is positive-definite and the eigenvalues are greater than equal to $\nu$ and in particular there is a symmetric matrix $B$ such that $A = B^2$.

For any fixed $\kappa \in (0, 2\pi)$ and $\alpha \in [0, 2\pi)$ we denote

$$D_{\kappa, \alpha} := \left\{ x = (r \cos \theta, r \sin \theta) \in \mathbb{R}^2 \mid r \in (0, \infty), -\frac{\kappa}{2} + \alpha < \theta < \frac{\kappa}{2} + \alpha \right\},$$

calling $\kappa$ the central angle of the domain $D_{\kappa, \alpha}$.

We consider the operator

$$\mathcal{L} = \frac{\partial}{\partial t} - (aD_{x_1x_1} + b(D_{x_1x_2} + D_{x_2x_1}) + cD_{x_2x_2})$$

with the conic (angular) domain $D_{\kappa, \alpha}$.

Below arctan is a map from $\mathbb{R} \to (-\pi/2, \pi/2)$.

**Proposition 4.1.** For $\mathcal{L}$ and $D_{\kappa, \alpha}$ defined above, we have

$$\lambda_c^\pm (\mathcal{L}, D_{\kappa, \alpha}) = \frac{\pi}{\tilde{\kappa}},$$

where

$$\tilde{\kappa} = \pi - \arctan \left( \frac{\bar{c} \cot(\kappa/2) + \bar{b}}{\sqrt{\det(A)}} \right) - \arctan \left( \frac{\bar{c} \cot(\kappa/2) - \bar{b}}{\sqrt{\det(A)}} \right) \quad (4.1)$$

with constants $\bar{a}, \bar{b}$ from the relation

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{pmatrix} = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \quad (4.2)$$

In particular,

(i) if $\kappa = \pi$, then $\tilde{\kappa} = \pi$;

(ii) if $\alpha = 0$ and $b = 0$, then $\tilde{\kappa}$ is determined by the relation

$$\tan \left( \frac{\tilde{\kappa}}{2} \right) = \sqrt{\frac{\bar{a}}{\bar{c}}} \tan \left( \frac{\kappa}{2} \right) \quad (4.3)$$

for $\kappa \in (0, 2\pi) \setminus \{ \pi \}$.

**Proof.** 1. We first consider the operator

$$\mathcal{L}_0 := \frac{\partial}{\partial t} - \Delta_x$$

with domain $D_{\kappa, \alpha}$. In this case we note $\tilde{\kappa} = \kappa$ and, as in Remark 3.1 we again have

$$\lambda_c^+ = \lambda_c^- = \sqrt{\Lambda} = \frac{\pi}{\kappa}.$$

Indeed, the eigenvalue/eigenfunction problem

$$-\phi''(\theta) = \lambda \phi(\theta), \quad \theta \in \left(-\frac{\kappa}{2} + \alpha, -\frac{\kappa}{2} + \alpha\right); \quad \phi\left(-\frac{\kappa}{2} + \alpha\right) = \phi\left(\frac{\kappa}{2} + \alpha\right) = 0$$

leads us to have $$\phi(\theta) = \cos\left(\sqrt{\lambda}(\theta - \alpha)\right)$$ and $$\cos\left(\sqrt{\lambda} \frac{\kappa}{2}\right) = 0$$. Hence, the first eigenvalue $$\Lambda$$ again satisfies $$\sqrt{\Lambda} \kappa/2 = \pi/2$$. Thus we have

$$\lambda^\pm_c(\mathcal{L}_0, \mathcal{D}_{\kappa,\alpha}) = \sqrt{\Lambda} = \frac{\pi}{\kappa}.$$ 

2. General case. Having (3.2) and the accompanied explanation in mind, we take a symmetric matrix $$B$$ satisfying $$A = B^2$$. The change of variables $$x = By$$ transforms the operator $$aD_{x_1x_1} + bD_{x_1x_2} + bD_{x_2x_1} + cD_{x_2x_2}$$ into $$\Delta_y = D_{y_1y_1} + D_{y_2y_2}$$ in $$y$$-coordinates, that is, putting $$v(t, y) = u(t, By)$$, we obtain

$$(aD_{11}u + bD_{12}u + bD_{21}u + cD_{22}u)(t, By) = \Delta_y v(t, y), \quad (t, y) \in \mathbb{R} \times \tilde{D},$$

where $$\tilde{D}$$ is the image of $$\mathcal{D}_{\kappa,\alpha}$$ under a linear transformation defined by

$$\tilde{D} := B^{-1} \mathcal{D}_{\kappa,\alpha} := \{B^{-1}x : x \in \mathcal{D}_{\kappa,\alpha}\}.$$ 

We note that $$\tilde{D}$$ is also a conic (angular) domain with a certain central angle $$\tilde{\kappa}$$. In fact, we can use (3.2) and Step 1 to have

$$\lambda^\pm_c(\mathcal{L}, \mathcal{D}_{\kappa,\alpha}) = \lambda^\pm_c(\mathcal{L}_0, \tilde{D}) = \frac{\pi}{\tilde{\kappa}}.$$ 

Let us verify the formula for $$\tilde{\kappa}$$. We first note

$$\frac{\tilde{\kappa}}{2\pi} = \frac{|\tilde{D} \cap B_1(0)|_\ell}{|B_1(0)|_\ell} \quad \text{and hence} \quad \tilde{\kappa} = 2 \cdot |\tilde{D} \cap B_1(0)|_\ell,$$

where $$|E|_\ell$$ denotes the Lebesgue measure of $$E \subset \mathbb{R}^2$$. By the relation $$y = B^{-1}x$$, we then have
\[
\left| \mathcal{D} \cap B_1(0) \right|_\ell = \int_{\{y \in \mathcal{D} : |y| \leq 1\}} dy
\]
\[
= \frac{1}{\det(B)} \int_{\{x \in \mathcal{D} : |B^{-1}x| \leq 1\}} dx
\]
\[
= \frac{1}{\sqrt{\det(A)}} \int_{-\kappa/2+\alpha}^{\kappa/2+\alpha} \int_0^{1/B_{\theta} v_\theta} r dr d\theta
\]
\[
= \frac{1}{2\sqrt{\det(A)}} \int_{-\kappa/2+\alpha}^{\kappa/2+\alpha} \frac{1}{\sqrt{\det(A)}} \int_0^{1/B_{\theta} v_\theta} \frac{1}{v_\theta^T A^{-1} v_\theta} d\theta
\]
where \( v_\theta := \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix} \). Now, a direct calculation based on translation, symmetry, and change of variable gives

\[
\left| \mathcal{D} \cap B_1(0) \right|_\ell
\]
\[
= \frac{1}{2\sqrt{\det(A)}} \left( \int_0^{\kappa/2} \frac{1}{v_{\theta}^T \bar{A}^{-1} v_{\theta}} d\theta + \int_{-\kappa/2}^0 \frac{1}{v_{\theta}^T \bar{A}^{-1} v_{\theta}} d\theta \right)
\]
\[
= \sqrt{\det(A)} \left( \frac{1}{2} \int_0^{\kappa/2} \frac{1}{\bar{c} \cot^2 \theta - 2\bar{b} \cot \theta + \bar{a}} + \frac{1}{\bar{c} \cot^2 \theta + 2\bar{b} \cot \theta + \bar{a}} \right) \cdot \frac{1}{\sin^2 \theta} d\theta
\]
\[
= \sqrt{\det(A)} \left( \frac{1}{2} \int_{\cot(\kappa/2)}^{\infty} \frac{1}{\bar{c} t^2 - 2\bar{b} t + \bar{a}} + \frac{1}{\bar{c} t^2 + 2\bar{b} t + \bar{a}} dt \right)
\]
\[
= \frac{1}{2} \left( \pi - \arctan \left( \frac{\bar{c} \cot(\kappa/2) - \bar{b}}{\sqrt{\det(A)}} \right) - \arctan \left( \frac{\bar{c} \cot(\kappa/2) + \bar{b}}{\sqrt{\det(A)}} \right) \right),
\]
where

\[
\bar{A} = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{b} & \bar{c} \end{pmatrix}
\]

with \( \bar{a}, \bar{b}, \) and \( \bar{c} \) defined in (3.2). Hence, we obtain (4.1) for \( \tilde{\kappa} \) and the proof is done. \( \square \)

Remark 4.2. Let us consider the simple but essential case of \( b = 0 \) and \( \alpha = 0 \), i.e., \( \mathcal{L} \) with \( A = \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} \) and domain \( \mathcal{D}_\kappa \). Then, from (4.3), we observe that the ratio \( r := \frac{2}{\kappa} \) of the diffusion constants, rather than the exact values of \( a \) and \( c \), along with \( \kappa \) decides \( \tilde{\kappa} \) and hence the values
\[ \lambda^\pm. \] We also note that for \( \kappa \in (0, \pi) \)
\[ \tilde{\kappa} \to \pi^- \text{ as } r \to \infty; \quad \tilde{\kappa} \to 0^+ \text{ as } r \to 0^+ \]
and for \( \kappa \in (\pi, 2\pi) \)
\[ \tilde{\kappa} \to \pi^+ \text{ as } r \to \infty; \quad \tilde{\kappa} \to 2\pi^- \text{ as } r \to 0^+. \]
In particular, if \( \kappa \in (0, \pi) \), or domain \( \mathcal{D}_\kappa \) is convex, and the diffusion constant to \( x_2 \) direction is relatively much larger than the the diffusion constant to \( x_1 \) direction, then \( \lambda^\pm \) are much bigger than 1 and hence Green's function estimate (2.8) gives better decay near the vertex since \( R_{t,x} \leq 1 \).

References

[1] C. Betz, G.A. Cámara, and H. Gzyl, *Bounds for the first eigenvalue of a spherical cap*, Appl. Math. Optim. 10, no. 1, 193–202 (1983).

[2] Sungwon Cho, *Two-sided global estimates of the Green's function of parabolic equations*, Potential Anal. 25, no. 4, 387–398 (2006).

[3] Petru A. Cioica, Kyeong-Hun Kim, and Kijung Lee, *On the regularity of the stochastic heat equation on polygonal domains in \( \mathbb{R}^2 \)*, J. Differential Equations 267, 6447–6479 (2019).

[4] Petru A. Cioica, Kyeong-Hun Kim, Kijung Lee, and Felix Lindner, *An \( L_p \)-estimate for the stochastic heat equation on an angular domain in \( \mathbb{R}^2 \)*, Stoch. Partial Differ. Equ. Anal. Comput. 6, no. 1, 45–72 (2018).

[5] Vladimir A. Kozlov, *Asymptotics of the Green function and Poisson kernels of a mixed parabolic problem in a cone. II.*, Z. Anal. Anwendungen 10, no. 1, 27–42, (in Russian) (1991).

[6] Vladimir A. Kozlov, Alexander Nazarov, *The Dirichlet problem for non-divergence parabolic equations with discontinuous in time coefficients in a wedge*, Math. Nachr. 287, no. 10, 1142–1165 (2014).

[7] Riahi, L. *Comparison of Green functions and harmonic measures for parabolic operators*. Potential Anal. 23, no. 4, 381–402 (2005).