HYPERBOLIC DISTANCES, NONVANISHING HOLOMORPHIC FUNCTIONS AND KRZYZ’S CONJECTURE

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ABSTRACT. The goal of this paper is to prove the conjecture of Krzyz posed in 1968 that for nonvanishing holomorphic functions $f(z) = c_0 + c_1 z + \ldots$ in the unit disk with $|f(z)| \leq 1$, we have the sharp bound $|c_n| \leq 2/e$ for all $n \geq 1$, with equality only for the function $f(z) = \exp[(z^n - 1)/(z^n + 1)]$ and its rotations. The problem was considered by many researchers, but only partial results have been established. The desired estimate has been proved only for $n \leq 5$.

Our approach is completely different and relies on complex geometry and pluripotential features of convex domains in complex Banach spaces.

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1. Krzyz’s conjecture. Main theorem

Nonvanishing holomorphic functions $f(z) = c_0 + c_1 z + \ldots$ on the unit disk $\Delta = \{z : |z| < 1\}$ (i.e., such that $f(z) \neq 0$ in $\Delta$) form the normal families admitting certain invariance properties, for example, the invariance under action of the Möbius group of conformal self-maps of $\Delta$, complex homogeneity, etc. One of the most interesting examples of such families is the set $B_1 \subset H^\infty$ of holomorphic maps of $\Delta$ into the punctured disk $\Delta^* = \Delta \setminus \{0\}$.

Compactness of $B_1$ in topology of locally uniform convergence on $\Delta$ implies the existence for each $n \geq 1$ the extremal functions $f_0$ maximizing $|c_n(f)|$ on $B_1$. Such functions are nonconstant and must satisfy $|f(e^{i\theta})| = 1$ for almost all $\theta \in [0, 2\pi]$.

The problem of estimating coefficients on $B_1$ was posed by Krzyz [Kz] in 1968. He conjectured that for all $n \geq 1$,

$$|c_n| \leq \frac{2}{e}, \quad (1.1)$$

with equality only for the function

$$\kappa_0(z) := \exp\left(\frac{z - 1}{z + 1}\right) = \frac{1}{e} + \frac{2}{e} z - \frac{2}{3e} z^3 + \ldots \quad (1.2)$$

and its rotations $\epsilon_1 \kappa_0(\epsilon_2 z)$ with $|\epsilon_1| = |\epsilon_2| = 1$. Note that (1.2) provides a holomorphic universal covering map $\Delta \to \Delta^*$ with $f(0) = 1/e$. 

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This fascinating and extremely interesting problem has been investigated by a large
number of mathematicians, however it still remains open. The estimate (1.1) has been proved
only for \( n \leq 5 \) (see [HSZ], [PS], [Sa], [Sz], [Ta]).

The best uniform estimate for all \( n \) given by Horowitz [Ho] is
\[
|c_n| \leq 1 - \frac{1}{3\pi} + \frac{4}{\pi} \sin\left(\frac{1}{12}\right) = 0.999...
\]
(while \( 2/e = 0.7357... \)); it was somewhat improved later. For a more complete history of
this problem we refer e.g., to [Ba], [HSZ], [LS], [Sz].

Our goal is to prove that Krzyz’s conjecture is true for all \( n \geq 1 \):

**Theorem 1.1.** For every \( f(z) = c_0 + c_1z + ... \in B_1 \) and \( n \geq 1 \), we have the sharp bound
(1.1), and the equality occurs only for the function (1.2) and its rotations.

Our approach is completely different and relies on complex geometry and pluripotential
features of convex domains in complex Banach spaces. The underlying idea of the proof is
in fact the same as for Zalcman’s conjecture applied in [Kr4] (and earlier in [Kr3]). It uses
also the important fact that the function (1.2) generates the complex geodesics in a domain
formed by nonvanishing functions on the closed unit disk. Certain results obtained in the
proof of the main theorem have independent interest. Let us mention also that the proof
essentially involves certain specific features of \( H^\infty \).

2. **Preliminaries: Hyperbolic metrics on convex Banach domains**

We first present briefly the basic results on properties of the Kobayashi and Carathéodory
metrics and on complex geodesics on convex domains in complex Banach spaces, which
underly the proof of Theorem 1.1.

2.1. **Equality of metrics.** Let \( D \) be a complex Banach manifold modelled by a Banach
space \( X \). The **Kobayashi metric** \( d_D \) on \( D \) is the largest pseudometric \( d \) on \( D \) that does
not get increased by holomorphic maps \( h : \Delta \to D \) so that for any two points \( x_1, x_2 \in D \), we have
\[
d_D(x_1, x_2) = \inf\{d_\Delta(0, t) : h(0) = x_1, h(t) = x_2\},
\]
where \( d_\Delta \) is the **hyperbolic Poincaré metric** on \( \Delta \) of Gaussian curvature \(-4\), with the
differential form
\[
ds = \lambda_{hyp}(z)|dz| := |dz|/(1 - |z|^2).
\]

The **Carathéodory** distance between \( x_1 \) and \( x_2 \) in \( D \) is
\[
c_D(x_1, x_2) = \sup d_\Delta(f(x_1), f(x_2)),
\]
where the supremum is taken over all holomorphic maps \( f : \Delta \to X \).

The corresponding **differential** (infinitesimal) forms of the Kobayashi and Carathéodory
metrics are defined for the points \( (x, v) \) in the tangent bundle \( TD \) of \( D \), respectively, by
\[
K_D(x, v) = \inf \{r : r > 0, \exists h \in \text{Hol}(\Delta, D), h(0) = x, dh(0)r = v\},
\]
\[
C_D(x, v) = \sup \{|df(x)v| : f \in \text{Hol}(T, \Delta), f(x) = 0\},
\]
where \( \text{Hol}(X, Y) \) denotes the collection of holomorphic maps of a complex manifold \( X \) into
\( Y \). Note that in the general case,
\[
\limsup_{t \to 0, t \neq 0} \frac{d_D(x, x + tv)}{|t|} \leq K_D(x, v).
\]
For general properties of invariant metrics we refer to [Di], [Ko]. A remarkable fact is:

**Proposition 2.1.** If $D$ is a convex domain in complex Banach space, then  
\[ d_D(x_1, x_2) = c_D(x_1, x_2) = \inf \{ d_\Delta(h^{-1}(x_1), h^{-1}(x_2)) : h \in \text{Hol}(\Delta, D) \} \]  
and  
\[ K_D(x, v) = C_D(x, v) \quad \text{for all} \quad (x, v) \in T(D). \]  
In particular, both infinitesimal and global pseudo-distances are logarithmically plurisubharmonic on $D$.

In the case of a bounded domain $D$, both $d_D$ and $c_D$ are distances (metrics), which means that these geometric quantities separate the points in $D$.

The equality of global pseudo-distances on convex domains in $\mathbb{C}^n$ and their representations by (2.3) were established by Lempert [Le]; the coincidence of the infinitesimal metrics for such domains was proved by Royden and Wong [RW]. These results were extended to convex domains in infinite dimensional Banach spaces in Dineen-Timoney-Vigué [DTV].

### 2.2. Pluripotential and curvature properties.

**Proposition 2.1** is rich in corollaries. We shall use several of them.

First recall that the **pluricomplex Green function** $g_D(x, y)$ of a domain $D \subset X$ with pole $y$ is defined by  
\[ g_D(x, y) = \sup u_y(x) \quad (x, y \in D) \]  
and following upper regularization  
\[ v^*(x) = \limsup v(x'). \]

The supremum in (2.5) is taken over all plurisubharmonic functions $u_y(x) : D \to [-\infty, 0)$ such that  
\[ u_y(x) = \log \|x - y\|_X + O(1) \]  
in a neighborhood of the pole $y$; here $\|\cdot\|_X$ denotes the norm on $X$, and the remainder term $O(1)$ is bounded from above (cf. e.g., [Di]). The Green function $g_D(x, y)$ is a maximal plurisubharmonic function on $D \setminus \{y\}$ (unless it is not identically $-\infty$). Proposition 2.1 implies

**Proposition 2.2.** If $D$ is a convex domain in a complex Banach space, then  
\[ g_D(x, y) = \log \tanh d_D(x, y) = \log \tanh c_D(x, y) \]  
for all $x, y \in D$.

The next corollary concerns the curvature properties. There are several generalizations of the smooth Gaussian curvature.

The generalized Gaussian curvature $\kappa_\lambda$ of an upper semicontinuous Finsler (semi)metric $ds = \lambda(t)|dt|$ in a domain $\Omega \subset \mathbb{C}$ is defined by  
\[ \kappa_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \]  
where $\Delta$ is the **generalized Laplacian** defined by  
\[ \Delta \lambda(t) = 4 \liminf_{r \to 0} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \lambda(t + re^{i\theta})d\theta - \lambda(t) \right\}. \]
It is well-known that an upper semicontinuous function $u$ is subharmonic on its domain $D \subset \mathbb{C}$ if and only if $\Delta u(t) \geq 0$ on its domain $D \subset \mathbb{C}$; hence, at the points $t_0$ of local maxima of $\lambda$ with $\lambda(t_0) > -\infty$, we have $\Delta \lambda(t_0) \leq 0$. Note that for $C^2$ functions, $\Delta$ coincides with the usual Laplacian $4\partial^2/\partial z\partial\bar{z}$, and its non-negativity immediately follows from the mean value inequality; for arbitrary subharmonic functions, this is obtained by a standard approximation.

The sectional holomorphic curvature of a Finsler metric on a complex Banach manifold $D$ is defined in a similar way as the supremum of the curvatures $\Delta^u(t)$ over appropriate collections of holomorphic maps from the disk into $D$ for a given tangent direction in the image.

The holomorphic curvature of the Kobayashi metric $K_D(x, v)$ of any complete hyperbolic manifold $D$ satisfies $\kappa_{K_D}(x, v) \geq -4$ at all points $(x, v)$ of the tangent bundle $TD$ of $D$, and for the Carathéodory metric $C_D$ we have $\kappa_{C_D}(x, v) \leq -4$ (see e.g., [Di]). Consequently, at each point, where these metrics are equal, we have the equality

$$\kappa_{K_D}(x, v) = \kappa_{C_D}(x, v) = -4.$$  

(2.9)

By Proposition 2.1, this holds for all convex domains $D$.

It follows from (2.7) that a conformal Finsler metric $ds = \lambda(z)|dz|$ with $\lambda(z) \geq 0$ of generalized Gaussian curvature at most $-K$, $K > 0$, satisfy the inequality

$$\Delta \log \lambda \geq K\lambda^2,$$  

(2.10)

where $\Delta$ is the generalized Laplacian (2.8). We shall use its integral generalization due to Royden [Ro].

A conformal metric $\lambda(z)|dz|$ in a domain $G$ on $\mathbb{C}$ (more generally, on a Riemann surface) has the curvature less than or equal to $K$ in the supporting sense if for each $K' > K$ and each $z_0$ with $\lambda(z_0) > 0$, there is a $C^2$-smooth supporting metric $\tilde{\lambda}$ for $\lambda$ at $z_0$ (i.e., such that $\tilde{\lambda}(z_0) = \lambda(z_0)$ and $\tilde{\lambda}(z) \leq \lambda(z)$ in a neighborhood of $z_0$) with $\kappa_{\tilde{\lambda}}(z_0) \leq K'$ (cf. [Ah], [He]).

A metric $\lambda$ has curvature at most $K$ in the potential sense at $z_0$ if there is a disk $U$ about $z_0$ in which the function

$$\log \lambda + K \text{Pot}_U(\lambda^2),$$

where $\text{Pot}_U$ denotes the logarithmic potential

$$\text{Pot}_U h = \frac{1}{2\pi} \int_U h(\zeta) \log |\zeta - z|d\xi d\eta \quad (\zeta = \xi + i\eta),$$

is subharmonic. One can replace $U$ by any open subset $V \subset U$, because the function $\text{Pot}_U(\lambda^2) - \text{Pot}_V(\lambda^2)$ is harmonic on $U$. Note that having curvature at most $K$ in the potential sense is equivalent to $\lambda$ satisfying (2.10) in the sense of distributions.

**Lemma 2.3.** [Ro] If a conformal metric has curvature at most $K$ in the supporting sense, then it has curvature at most $K$ in the potential sense.

### 2.3. Existence of complex geodesics.

Let $D$ be a Banach domain endowed with a pseudo-distance $\rho$. Following Vesentini (see e.g., [Ve]), a holomorphic map $h : \Delta \to D$ is called complex $\rho$-geodesic if there exist $t_1 \neq t_2$ in $\Delta$ such that

$$d_\Delta(t_1, t_2) = \rho(h(t_1), h(t_2));$$

one says also that the points $h(t_1)$ and $h(t_2)$ can be joined by a complex $\rho$-geodesic.
If $h$ is a complex $c_D$-geodesic, then it is also $d_D$-geodesic, and vice versa, and then the equality (2.3) holds for all points of the disk $h(\Delta)$.

It is important to have the conditions for domains ensuring the existence and uniqueness of complex geodesics. Certain conditions, which will be used here, are given in [Di], [DTV].

Recall that a Banach space $X$ is called the dual of a Banach space $Y$ if $X = Y'$, that is, $X$ is the space of bounded linear functionals $x(y) = <x, y>$ on $Y$. Then $Y$ is called predual of $X$. The weak* topology on $X$ determined by $Y$, $\sigma(X, Y)$, is the topology of pointwise convergence on points of $Y$, i.e., $x_n \in X \rightarrow x \in X$ in $\sigma(X, Y)$ as $n \rightarrow \infty$ if and only if $x_n(y) \rightarrow x(y)$ for all $y \in Y$.

If $X$ has a predual $Y$ then the closure $\overline{X_1}$ of its open unit ball in $\sigma(X, Y)$ is compact.

Proposition 2.4. [Di], [DTV] Let $D$ be a bounded convex domain in a complex Banach space $X$ with predual $Y$. If the closure of $D$ is $\sigma(X, Y)$-compact, then every distinct pair of points in $D$ can be joined by a complex $c_D$-geodesic.

This proposition also has its differential counterpart which provides that under the same assumptions, for any point $x \in D$ and any nonzero vector $v \in X$, there exists at least one complex geodesic $h : \Delta \rightarrow D$ such that $h(0) = x$ and $h'(0)$ is colinear to $v$ (cf. [DTV]).

Note that along a complex geodesic in $D$, the relation (2.2) reduces to the equality.

3. Proof of theorem 1.1

We prove the main theorem in several stages; each stage is of independent interest.

1. Open domain of nonvanishing functions and its holomorphic embedding.

(a) Consider the subsets of $B$ defined by

$$B_r = \{f \in H^\infty(\Delta_{1/r}) : f(z) \neq 0 \text{ on the disk } \Delta_{1/r} = \{|z| < 1/r\}, \quad 0 < r < 1.\} \quad (3.1)$$

Note that $B_r \subset B_{r'}$ if $r < r'$.

Lemma 3.1. Each point of the union

$$B^0 = \bigcup_{r} B_r \quad (3.2)$$

has a neighborhood in $H^\infty(\Delta)$, which entirely belongs to $B^0$. Hence, $B^0$ is a domain in the space $H^\infty(\Delta)$.

Proof. To establish the openness of the union (3.2), it suffices to show that every function $f \in B_r$ has a neighborhood $U(f, \epsilon(r))$ in $H^\infty(\Delta)$, which contains only nonvanishing functions on $\Delta$. The connectedness of $B^0$ follows from widening the sets $B_r$ when $r$ increases.

Assume the contrary. Then there exist a function $f_0 \in B_r$ and the sequences of functions $f_n \in H^\infty(\Delta)$ convergent to $f_0$,

$$\lim_{n \rightarrow \infty} \|f_n - f\|_{H^\infty(\Delta)} = 0 \quad (3.3)$$

and of points $z_n \in \Delta$ convergent to $z_0$, $|z_0| \leq 1$ such that $f_n(z_n) = 0$ $(n = 1, 2, \ldots)$.

In the case $|z_0| < 1$ we immediately reach a contradiction, because then the uniform convergence of $f_n$ on compact sets in $\Delta$ implies $f_0(z_0) = 0$, which is impossible.

The case $|z_0| = 1$ requires other arguments. Since $f_0$ is holomorphic and does not vanish on the closed disk $\overline{\Delta}$,

$$\min_{|z| \leq 1} |f_0(z)| = a > 0.$$
Hence, for each $z_n$,

$$|f_n(z_n) - f_0(z_n)| = |f_0(z_n)| \geq a,$$

and by continuity, there exists a neighborhood $\Delta(z_n, \delta_n) = \{|z - z_n < \delta_n\}$ of $z_n$ in $\Delta$, in which $|f_n(z) - f_0(z)| \geq a/2$ for all $z$. This implies

$$\|f_n - f_0\|_{\mathcal{H}(\Delta)} \geq \max_{\Delta(z_n, \delta_n)} |f_n(z) - f_0(z)| \geq a/2.$$

This inequality must hold for all $n$, contradicting (3.3). Lemma follows.

**Remark.** This lemma does not contradict to existence of a sequence $\{f_n\} \in \mathcal{H}^\infty$, which contains the functions vanishing in $\Delta$ and is convergent to $f_0 \in \mathcal{B}^0$ only uniformly on compact sets in $\Delta$.

Now put

$$\mathcal{B}_1^0 = \mathcal{B}_0^0 \cap \mathcal{B}_1 = \{f \text{ holomorphic on } \overline{\Delta}, \ f(z) \neq 0 \text{ on } \overline{\Delta}; \ \|f_0\|_{\mathcal{H}^\infty} < 1\}. \quad (3.4)$$

By Lemma 3.1, $\mathcal{B}_1^0$ is a domain in $\mathcal{H}^\infty = \mathcal{H}^\infty(\Delta)$ located in the unit ball $\mathcal{H}_1^\infty$ of this space. Note also that

$$\sup_{\mathcal{B}_1^0} |c_n(f)| = \sup_{\mathcal{B}_1} |c_n(f)|,$$

and by the maximum principle, each extremal function $f_0 \in \mathcal{B}_1$ maximizing $|c_n(f)|$ satisfies $\|f_0\|_{\mathcal{H}^\infty} = 1$.

(b) Since any function $f \in \mathcal{B}$ does not vanish in $\Delta$, the function $\log f$ is well-defined in a neighborhood of the origin, taking the principal branch of the logarithmic function, and after holomorphic extension generates a single valued holomorphic function

$$j_f(z) = \log f(z) : \Delta \to \mathbb{C}_- := \{w \in \mathbb{C} : \Re w < 0\}. \quad (3.5)$$

In fact, we lift $f$ to the universal cover

$$\mathbb{C}_- \to \Delta_* = \Delta \setminus \{0\}$$

with the holomorphic universal covering map $\exp$ (cf. Lemma 3.13).

Each such $j_f$ satisfies

$$\sup_{\Delta} (1 - |z|^2)^\alpha \log f(z) \leq \sup_{\Delta} ((1 - |z|^2)^\alpha (\log |f(z)| + |\arg f(z)|) < \infty$$

for any $\alpha > 0$. We embed the set $j\mathcal{B}$ into in the Banach space $\mathcal{B}$ of hyperbolically bounded holomorphic functions on the disk $\Delta$ with norm

$$\|\psi\|_{\mathcal{B}} = \sup_{\Delta} (1 - |z|^2)^2 |\psi(z)|.$$ 

This space is dual to the space $A_1 = A_1(\Delta)$ of integrable holomorphic functions on $\Delta$ with $L_1$-norm, and every continuous linear functional $l$ on $A_1$ can be represented, uniquely, as

$$l(\varphi) = \langle \psi, \varphi \rangle_\Delta := \iint_{\Delta} (1 - |z|^2)^2 \overline{\psi(z)} \varphi(z) dx dy \quad (3.6)$$

with some $\psi \in \mathcal{B}$ (see [Be]).

We want to investigate the geometrical properties of the image $j\mathcal{B}_1^0$. First of all, we have

**Lemma 3.2.** The functions $j_f \in j\mathcal{B}$ fill a convex set in $\mathcal{B}$. Similarly, the subset $j\mathcal{B}_1^0$ is also convex in $\mathcal{B}$.
Proof. For any two distinct points \( \psi_1 = jf_1, \psi_2 = jf_2 \), the points of joining interval \( \psi_t = t\psi_1 + (1-t)\psi_2 \) with \( 0 \leq t \leq 1 \) represent the functions \( jf_t = \log(f_1^{1-t}f_2^t) \) and the product \( f_1^t(z)f_2^{1-t}(z) \neq 0 \) in \( \Delta \) (taking again the principal branch of logarithm). Hence, this interval lies entirely in \( jB \). The proof for \( jB_0^0 \) is similar.

Lemma 3.3. The map \( j \) is a holomorphic embedding of the domain \( B_0^0 \) into the space \( B \) carrying this domain onto a holomorphic Banach manifold modelled by \( B \).

Proof. The map \( j : f \to \log f \) is one-to-one and continuous on \( B_0^0 \). To check its complex holomorphy, observe that for any \( f \in B_0^1, h \in H^\infty(\Delta) \) and sufficiently small \( |t| \) (letting \( j(f) = jf \)),

\[
j(f+th) - j(f) = \log(1 + t \frac{h}{f}) = t \frac{h}{f} + O(t^2),
\]

with uniformly bounded remainder for \( \|h\|_\infty \leq c < \infty \). This means that the directional derivative of \( j \) at \( f \) equals \( h/f \) and also belongs to \( H^\infty(\Delta) \).

In a similar way, one obtains that the inverse map \( j^{-1} : \psi \to e^\psi \) is holomorphic on intersections of a neighborhood of \( \psi \) in \( B \) with complex lines \( \psi + tw \) in \( jB_0^0 \). The lemma is proved.

Holomorphy in Lemma 3.3 is a special case of general results on properties of bounded complex Banach functions (see Lemma 3.12). It implies that both complex structures on \( jB_0^0 \) endowed by norms on \( H^\infty \) and on \( B \) are equivalent.

2. Complex geometry of sets \( jB_1^0 \) and \( B_1^0 \).

As a domain on a complex manifold modelled by \( B \), the set \( jB_1^0 \) admits the invariant Kobayashi and Carathéodory metrics. Our goal is to show that the geometric features of this set are similar to bounded convex domains in Banach spaces.

Proposition 3.4. (i) The Kobayashi and Carathéodory distances on \( jB_1^0 \) and the corresponding differential metrics are equal:

\[
d_{jB_1^0}(\psi_1, \psi_2) = c_{jB_1^0}(\psi_1, \psi_2) = \inf \{ d_\Delta(h^{-1}(\psi_1), h^{-1}(\psi_2)) : h \in Hol(\Delta, jB_1^1) \},
\]

\[
K_{jB_1^0}(\psi, v) = c_{jB_1^0}(\psi, v) \quad \text{for all} \quad (\psi, v) \in T(jB_1^0).
\]

(ii) Every distinct pair of points \( (\psi_1, \psi_2) \) in \( jB_1^0 \) can be joined by a complex \( c_{jB_1^0} \)-geodesic.

Proof. The equality (3.7) follows from the property (ii). We establish this property in two steps.

(a) First take the \( \epsilon \)-blowing up of \( jB_1^0 \), that is, we consider the sets

\[
U_\epsilon = \bigcup_{\psi \in jB_1^0} \{ \omega \in B : \|\omega - \psi\|_B < \epsilon \}, \quad \epsilon > 0.
\]

For these sets, we have

Lemma 3.5. Every set \( U_\epsilon \) is a (bounded) convex domain in \( B \), and its weak*-closure in \( \sigma(B, A_1) \) is compact.

Proof. The openness and connectivity of \( U_\epsilon \) are trivial. Let us check convexity. Take any two distinct points \( \omega_1, \omega_2 \) in \( U_\epsilon \) and consider the line interval

\[
\omega_t = t\omega_1 + (1-t)\omega_2, \quad 0 \leq t \leq 1,
\]

(3.8)
Hence $\phi$ is fixed for all functions $\omega$. Because the functions $h$ applied in (Di) in the proof of Proposition 2.3. the equality (3.9) follows from Lebesgue's theorem on dominant convergence. Lemma follows.

Now, let $\alpha$, $\tau$ be a disk $\Delta$. It suffices to show that for any bounded sequence $\{\omega_n\} \subset B$ we have the equality

$$\lim_{n \to \infty} \int \int_{\Delta} \frac{(1 - |\zeta|^2)^2 \omega_n(\zeta)}{\zeta - z} d\xi d\eta = \int \int_{\Delta} \frac{(1 - |\zeta|^2)^2 \omega(\zeta)}{\zeta - z} d\xi d\eta, \quad z \in \Delta^*, \quad (3.9)$$

because the functions $w_z(\zeta) = 1/(\zeta - z)$ span a dense subset of $A_1(\Delta)$. But if

$$\sup_{\Delta}(1 - |\zeta|^2)^2 |\omega(\zeta)| < M < \infty \quad \text{for all } n,$$

the equality (3.9) follows from Lebesgue’s theorem on dominant convergence. Lemma follows.

(b) We proceed to the proof of Proposition 3.4 and first establish the existence of complex geodesics in domains $U_\epsilon$, $\epsilon < \epsilon_0$. Convexity of these domains allows us to use the arguments applied in [D] in the proof of Proposition 2.3.

Let $\omega_1$ and $\omega_2$ be distinct points in $U_\epsilon$. By Proposition 2.1,

$$d_D(\omega_1, \omega_2) = c_D(\omega_1, \omega_2) = \inf \{d_\Delta(h^{-1}(\omega_1), h^{-1}(\omega_2)) : h \in \text{Hol}(\Delta, U_\epsilon)\};$$

hence there exists the sequences $\{h_n\} \subset \text{Hol}(\Delta, U_\epsilon)$ and $\{r_n\}$, $0 < r_n < 1$, such that $h_n(0) = \omega_1$ and $h_n(r_n) = \omega_2$ for all $n$, $\lim_{n \to \infty} r_n = r < 1$ and $c_{U_\epsilon}(\omega_1, \omega_2) = d_\Delta(0, r)$. Let $h_n(t) = \sum_{m=0}^{\infty} a_{m,n} t^m$ for all $t \in \Delta$ and $n$.

Take a disk $\Delta_R = \{|z| < R\}$ containing $U_\epsilon$. The Cauchy inequalities imply $\|a_{m,n}\|_B \leq R$ for all $n$ and $m$. Passing, if needed, to a subsequence of $\{h_n\}$, one can suppose that for a fixed $m$, the sequence $a_{n,m}$ is weakly* convergent to $a_m \in B$ as $n \to \infty$, that is

$$\lim_{n \to \infty} \langle a_{n,m}, \varphi \rangle_\Delta = \langle a_m, \varphi \rangle_\Delta \quad \text{for any } \varphi \in A_1.$$

Hence $h(t) = \sum_{m=0}^{\infty} a_m t^m$ defines a holomorphic function from $\Delta$ into $B$. Since $a_{n,0} = \zeta$ for all $n$, we have $h(0) = \omega_1$.

Now, let $\alpha$, $0 < \alpha < 1$, and $\varepsilon > 0$ be given. Choose $m_0$ so that

$$r \sum_{m=m_0}^{\infty} \alpha^m < \varepsilon.$$

If $\varphi \in A_1$, $\|\varphi\| = 1$, then

$$\sup_{|t| \leq \alpha} |\langle h_n(t) - h(t), \varphi \rangle_\Delta| \leq \sum_{m=1}^{m_0-1} |\langle a_{n,m} - a_m, \varphi \rangle_\Delta| + 2r \sum_{m=m_0}^{\infty} \alpha^m$$

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for all \( n \), which implies that \( h_n \) is convergent to \( h \) in \( \sigma(B, A_1) \) uniformly on compact subsets of \( \Delta \) as \( n \to \infty \). Since \( \overline{D} \) is \( \sigma(B, A_1) \) compact, \( h(\Delta) \subset \overline{D} \), and since \( h(0) \in D \), it follows that \( h(\Delta) \subset D \). For \( r < r' < 1 \),

\[
\omega_2 = h_n(r_n) = \frac{1}{2\pi i} \int_{|t|=r_n} h_n(t) dt \to \frac{1}{2\pi i} \int_{|t|=r'} h(t) dt = h(r)
\]

(3.10)
as \( n \to \infty \). Hence,

\[
d_T(0, r) = c_D(\omega_1, \omega_2) = c_D(h(0), h(r)),
\]

and \( h \) is simultaneously complex \( c_U \) and \( d_{U, \ast} \) geodesics.

There exists a holomorphic map \( g : \Delta \to U_{\ast} \) such that for any two points \( t_1, t_2 \in \Delta \),

\[
d_\Delta(t_1, t_2) = d_{U, \ast}(g(t_1), g(t_2)) = c_{U, \ast}(g(t_1), g(t_2)),
\]

(3.11)and for any pair \( (t, v), t \in \Delta, v \in \mathbb{C} \),

\[
K_{U, \ast}(g(t), dg(t)v) = \frac{|v|}{1 - |t|^2}.
\]

(3.12)

(c) Let now \( \omega_1 \) and \( \omega_2 \) be two distinct points in \( jB^0_1 \). Choose a decreasing sequence \( \{\epsilon_n\} \) approaching zero and take for every \( n \) a complex geodesic \( h_n = h_{U_{\epsilon_n}} \) joining these points in \( U_{\epsilon_n} \), which was constructed in the previous step. Let \( g_n = g_{U_{\epsilon_n}} \) be the corresponding map \( \Delta \to U_{\epsilon_n} \) which provides the equalities (3.11), (3.12). Since \( d_\Delta \) is conformally invariant, one can take \( g_n \) satisfying \( g_n^{-1}(\omega_1) = 0 \), \( g_n^{-1}(\omega_2) = r_n \in (0, 1) \). Then the inequalities

\[
d_{U_{\epsilon_n}}(\omega_1, \omega_2) \leq d_{U_{\epsilon_m}}(\omega_1, \omega_2) \leq d_{jB^0_1}(\omega_1, \omega_2)
\]

for \( m > n \) imply \( r_n \leq r_m \leq r_* < 1 \), where \( d_\Delta(0, r_*) = d_{jB^0_1}(\omega_1, \omega_2) \). Hence, there exists \( \lim_{n \to \infty} r_n = r' \leq r_* \).

The sequence \( \{g_n\} \) is \( \sigma(B, A_1) \)-compact and similar to (3.10) the weak* limit of \( g_n \) is a function \( g \in \text{Hol}(\Delta, jB^0_1) \) which determines a complex geodesic for both Kobayashi and Carathéodory distances on \( jB^0_1 \) joining the points \( \omega_1 \) and \( \omega_2 \) inside this set. Proposition 3.4 is proved.

An important consequence of Proposition 3.4 is that the initial domain \( B^0_1 \) in \( H^\infty \) has similar complex geometric properties, since the embedding \( j \) is biholomorphic. We present it as

**Proposition 3.6.** (i) The Kobayashi and Carathéodory distances on domain \( B^0_1 \) and the corresponding differential metrics are equal:

\[
d_{B^0_1}(f_1, f_2) = c_{B^0_1}(f_1, f_2) = \inf\{d_\Delta(h^{-1}(f_1), h^{-1}(f_2)) : h \in \text{Hol}(\Delta, B_1)\},
\]

\[
K_{B^0_1}(f, v) = C_{B^0_1}(f, v)
\]

for all \( (f, v) \in T(B^0_1) \).

(ii) Every two points \( f_1, f_2 \) in \( B^0_1 \) can be joined by a complex geodesic.

\[
\sup_{f \in B^0_1} |c_{B^0_1}(f)| = \sup_{f \in B^0_1} |c_{B^0_1}(f)| = M_n \quad (M_n \leq 1),
\]

3. Finsler metric generated by functional \( c_n(f) \).

We proceed to the proof of Theorem 1.1. It will be convenient to regard the free coefficients \( c_0(f) \) also as elements of \( B^0_1 \), which are constant on \( \Delta \). Note that \( 0 < |c_0(f)| < 1 \). Denote

\[
\sup_{f \in B^0_1} |c_n(f)| = |c_n(f)| = M_n \quad (M_n \leq 1),
\]
and consider, for a fix integer \( n > 1 \), the functional
\[
J(f) = \left| \frac{c_n(f)}{M_n} \right|^{1/n} : B_1^0 \to [0, 1).
\] (3.14)

It is logarithmically plurisubharmonic on \( B_1^0 \), taking the values on \([-\infty, 0)\), with \( \log J(f) \to 0 \) as \( f \) tends to the boundary of \( B_1^0 \).

Our goal is to show that \( \log J \) is dominated on \( B_1^0 \) by the pluricomplex Green function of this domain, namely,
\[
\log J(f) \leq g_{B_1^0}(c_0(f), f).
\] (3.15)

This will be established in several steps. The proof is geometric and involves the differential metrics. We construct on each holomorphic disk in \( B_1^0 \) a subharmonic Finsler metric naturally generated by \( J \) and compare this metric with the canonical Kobayashi metric.

The estimate (3.15) trivially holds for the points of the zero-set of our functionals
\[
Z_J = \{ f \in B_1^0 : J(f) = 0 \}. \tag{3.16}
\]

Note that this set contains the disk filled in \( B_1^0 \) by the constant functions \( c_0(f) \), which can be identified with the punctured disk \( \Delta \setminus \{0\} \).

The uniqueness theorem for holomorphic functions (in Banach spaces) implies that this zero-set is nowhere dense on \( B_1^0 \) (in the sense that its complement \( B_1^0 \setminus Z_J \) is open and dense everywhere). This follows also from a theorem from [Kr1] on the existence of special quasiconformal deformations \( \omega \) of the plane, which are conformal on a given set \( E \) of positive two-dimensional Lebesgue’s measure and take, with their derivatives, the prescribed values, see [Kr1, Ch. 4]. One can compose any function \( f \in B_1^0 \) with appropriate deformations of such kind, that are conformal, for example, in the complement of a disk \( \{|z - z_0| < r\} \) located sufficiently far from the origin, and get the composite maps \( \omega \circ f \) whose coefficients \( c_n(\omega \circ f) \) range in a whole neighborhood \( 0 < |w| < \varepsilon \). Then \( \omega \circ f \) provide the points from \( B_1^0 \setminus Z_J \).

Consider first the holomorphic disks \( h : \Delta \to B_1^0 \) with nonconstant holomorphic \( h \) which touch the zero-set (3.16) only at one point. We call such disks distinguished. One can assume that this common point is \( h(0) \).

Let \( D = h(\Delta) \) be such a disk. Take the restriction of \( J(f) \) to \( D \) and consider its root
\[
g(\zeta) = [J \circ h(\zeta)]^{1/n};
\]
this root is an \( n \)-valued function on \( D \), with a single algebraic branch point at \( \zeta = 0 \). Take a single-valued branch of this function in a neighborhood \( U_0 \subset D \) of a point \( \zeta_0 \neq 0 \) and apply the selected branch to pulling back the hyperbolic metric \( \lambda_{hyp} \) on \( \Delta \) to this neighborhood \( U_0 \). Extending this branch analytically, one produces a conformal metric \( ds = \lambda_J(\zeta)|d\zeta| \) on the whole disk \( D \), with
\[
\lambda_J(\zeta) = g^*\lambda_{hyp}(\zeta) = \frac{|g'(\zeta)|}{1 - |g(\zeta)|^2}.
\] (3.17)

This metric does not depend on the choices of an initial branch and of \( U_0 \). It is logarithmically subharmonic on \( D \), and its Gaussian curvature \( \kappa_{\lambda_0} \) equals \(-4\) at noncritical points of the extension of \( g \). This provides that the curvature is less than or equal \(-4\) on \( D \) in both supporting and potential senses and as generalized curvature via (2.8).
Lemma 3.7. On any $d_{B^1_0}$-geodesic disk $D$, the metric (3.17) is dominated by the differential Kobayashi metric $\lambda_{K_{B^1_0}}$,
\[ \lambda_J(\zeta) \leq \lambda_{K_{B^1_0}}(\zeta), \tag{3.18} \]
and if the equality holds here for one value of $\zeta \neq 0$, then it holds identically.

Proof. Consider first the distinguished geodesic disks $D = h(\Delta)$. On such disks, the differential Kobayashi metric $\lambda_{K_{B^1_0}}$ is equal to hyperbolic metric of the unit disk. Since the curvature of $\lambda_J$ is at most $-4$ at noncritical points of $D$ in the supporting sense, the inequality (3.18) follows from the classical Ahlfors-Schwarz lemma (see [Ah], [He], [Mi], [Ro]).

An arbitrary geodesic disk in $B^1_0$ can be strongly (in the norm of $H^\infty$) approximated by distinguished disks, and such approximation preserves the inequality (3.18). This completes the proof.

We must now pass from the inequality (3.18) for infinitesimal metrics to global distances, which requires the reconstruction of the initial functional $J(f)$ from the generated metric. This is rather simple for distinguished geodesic disks.

Lemma 3.8. On any distinguished geodesic disk $h : \Delta \to B^1_0$, we have for each $r < 1$ the equality
\[ \tanh[J(f)] = \tanh^{-1}[\delta(J \circ h(0), J \circ h(r))] = \int_0^r \lambda_J \circ h(t) dt, \tag{3.19} \]
where
\[ \delta(\zeta_1, \zeta_2) = (\zeta_2 - \zeta_1)/(1 - \zeta_1 \zeta_2). \]

Proof. Since any geodesic disk is holomorphically isometric to the hyperbolic plane modelled by $\Delta$, one can write
\[ \tanh^{-1}[\delta(J \circ h(0), J \circ h(r))] = \int_{J \circ h(0)}^{J \circ h(r)} \frac{|dt|}{1 - |t|^2} = \int_0^r \lambda_{J \circ h}(t) |dt| \quad (0 < r < 1), \tag{3.20} \]
which is equivalent to (3.19). Indeed, one can subdivide the hyperbolic interval $[\delta(J \circ h(0), J \circ h(r))]$ into subintervals, taking a finite partition
\[ c_0 < r_1 < \cdots < r_{m-1} < r_m = J \circ h(r) \]
so that on each $[r_{s-1}, r_s]$ the map $J \circ h$ is injective, and apply to these subintervals the equalities similar to (3.20).

Note also that if a geodesic disk $h(\Delta)$ is not distinguished, but does not lie entirely in $Z_J$, then the equality (3.19) holds for a sufficiently small $r < 1$, for which the initial equality (3.20) remains valid. The same holds, in view of (2.6), for any compact subset of the disk $h(\Delta)$ which does not contact the zero set $Z_J$.

Lemmas 3.7 and 3.8, together with Proposition 3.6, imply the desired estimate (3.15) which controls the behavior of $J$ on $B^1_1$. In view of its importance, we present this as a separate lemma.

Lemma 3.9. The inequality (3.15) holds at every point $f \in B^1_1$. 
Proof. The case $J(f) = 0$ is trivial, so we have to establish the inequality (3.15) only for the points $f$ with $J(f) \neq 0$.

Lemmas 3.7 and 3.8 imply that the growth of $J$ on the distinguished geodesic disks is estimated by

$$J(f) = O(d_{B^1}(c_0(f), f)) = O(\|f - c_0(f)\|_{H^\infty}),$$

uniformly on compact subsets of these disks. This estimate provides that $\log J(f)$ is an admissible plurisubharmonic function for comparison with Green’s function $g_B(c_0(f), f)$. Now the maximality of $g_B(c_0(f), f)$ among plurisubharmonic functions with logarithmic growth near the pole $c_0$ implies that $\log J(f)$ is dominated by Green’s function $g_B(c_0(f), f)$.

Further, for any given function $f \in B^1_1$, the point $c_0(f)$ belongs to the zero-set $Z_J$. Using approximation in $B^1_1$, similar to above, one can extend the inequality (3.15) to all complex geodesic disks in $B^1_1$ which touch $Z_J$ at this point. By continuity, the functional $J$ is subharmonic on every such disk, while the relations (2.6), (3.18) and (3.19) preserve the required logarithmic order of the growth of $J$ near its zero set. Lemma follows.

4. Homotopy.

For any $f \in B^1_1$, one can define complex holomorphic homotopy

$$f_t(z) = f(tz) = c_0 + c_t tz + \cdots : \Delta \times \Delta \to \Delta,$$  \hspace{1cm} (3.21)

connecting $f$ with $c_0(f)$ in $B^1_1$. Due to (3.14), our functional $J$ is homogeneous with respect to this isotopy with degree 1 in the following sense:

$$J(f_t) = |t| J(f).$$  \hspace{1cm} (3.22)

We shall need also the following simple fact concerning the homotopy functions.

Lemma 3.10. The pointwise map $t \mapsto f_t$ given by (3.23) determines a holomorphic map $\chi_f : \Delta \to H^\infty$.

This lemma is a rather special case of bounded holomorphic functions in Banach spaces with sup norm, given by the following lemma (cf. [Ea], [Ha], [Kr2]).

Lemma 3.11. Let $E, T$ be open subsets of complex Banach spaces $X, Y$ and $B(E)$ be a Banach space of holomorphic functions on $E$ with sup norm. If $\varphi(x, t)$ is a bounded map $E \times T \to B(E)$ such that $t \mapsto \varphi(x, t)$ is holomorphic for each $x \in E$, then the map $\varphi$ is holomorphic.

We briefly outline the proof. Holomorphy of $\varphi(x, t)$ in $t$ for fix $x$ implies the existence of complex directional derivatives

$$\varphi'(x, t) = \lim_{\zeta \to 0} \frac{\varphi(x, t + \zeta v) - \varphi(x, t)}{\zeta} = \frac{1}{2\pi i} \int_{|\xi| = 1} \frac{\varphi(x, t + \xi v)}{\xi^2} d\xi.$$

On the other hand, the boundedness of $\varphi$ in sup norm provides the uniform estimate

$$\|\varphi(x, t + c\xi v) - \varphi(x, t) - \varphi'(x, t) cv\|_{B(E)} \leq M|c|^2,$$

for sufficiently small $|c|$ and $\|v\|_Y$.

5. Covering maps.

For each $f_t$, there exists, by Proposition 3.4, a complex geodesic in $B^1_1$ joining $f_t$ with $c_0(f)$. It is a holomorphic geodesic disk isometric to the hyperbolic plane $\mathbb{H}^2$, i.e., with the
same hyperbolic geometry as on \( \mathbb{H}^2 = \Delta \). We need to estimate quantitatively the behavior of the distance \( d_{B_1^0}(f_t, c_0) \) when \( t \to 0 \).

**Lemma 3.12.** Every function \( f \in B_1^0 \) admits factorization

\[
f(z) = \kappa_0 \circ \hat{f}(z), \tag{3.23}
\]

where \( \hat{f} \) is a holomorphic map of the disk \( \Delta \) into itself (hence, from \( H_1^\infty \)) and \( \kappa_0 \) is the function (1.2).

**Proof.** Due to a general topological theorem, any map \( f : M \to N \), where \( M, N \) are manifolds, can be lifted to a covering manifold \( \hat{N} \) of \( N \), under appropriate relation between the fundamental group \( \pi_1(M) \) and a normal subgroup of \( \pi_1(N) \) defining the covering \( \hat{N} \) (see, e.g, [Ma]). This construction produces a map \( \hat{f} : M \to \hat{N} \) satisfying

\[
f = p \circ \hat{f}, \tag{3.24}
\]

where \( p \) is a projection \( \hat{N} \to N \). The map \( \hat{f} \) is determined up to composition with the covering transformations of \( \hat{N} \) over \( N \). For holomorphic maps and manifolds the lifted map is also holomorphic.

In our special case, \( \kappa_0 \) is a holomorphic universal covering map \( \Delta \to \Delta_* = \Delta \setminus \{0\} \), and the representation (3.24) provides the equality (3.23) with the corresponding \( \hat{f} \) determined up to covering transformations of the unit disk compatible with the covering map \( \kappa_0 \).

This lemma relates to Lemma 3.2. As a simple corollary of Lemma 3.12 one obtains

**Lemma 3.13.** For any \( f \in B_1^0 \),

\[
|c_1| \leq 2/e, \tag{3.25}
\]

with equality only for the rotations \( e^{i\alpha_1}\kappa_0(e^{i\alpha}z) \) of \( \kappa_0 \).

As was mentioned in the introduction, this bound is known. We reprove it and will use also later the arguments applied in the proof.

**Proof.** One only needs to show that (3.25) holds for each composition of \( \kappa_0 \) with the Möbius (fractional linear) automorphisms \( \gamma \) of of the unit disk \( \Delta \), i.e., that \( \kappa_0 \) (and any its rotation) maximizes \( |c_1| \) among the holomorphic universal covering maps \( \Delta \to \Delta_* \). Then, for any \( f \in B_1^0 \), taking \( \gamma \) with \( \gamma(0) = \hat{f}(0) = a, \gamma(1) = \hat{f}(1) \), where \( \hat{f} \) is the covering map of \( f \) in (3.23), one obtains from Schwarz’ lemma,

\[
|c_1(f)| = |\kappa_0(a)||\hat{f}'(0)| \leq |c_1(k_\gamma(0))| = |\kappa_0(a)||\gamma'(0)| \leq \frac{2}{e}.
\]

Using the rotations about the origin \( z = 0 \), we can restrict ourselves by \( \gamma \) whose compositions with \( \sigma(z) = (z - 1)/(z + 1) \) assumes the form

\[\sigma \circ \gamma(z) = e^{ib} \frac{z - 1}{z + e^{ia}} \] with \( a, b \in [0, 2\pi] \).

We regard here the disk \( \Delta \) as a lune with vertices at the points \( z = 1 \) and \( z = e^{ia} \) and with opening angles equal to \( \pi \). Then

\[
(\kappa_0 \circ \gamma)'(z) = e^{i\sigma \gamma(z)} \frac{A}{(z + e^{ia})^2}, \quad A = e^{ib}(1 + e^{ia}),
\]
and \( c_1(\kappa_0 \circ \gamma) = |(\kappa_0 \circ \gamma)'(0)| = |A| \), with equality only for \( a = 0 \). Lemma follows.

We will denote the Möbius group of \( \Delta \) by \( \text{Mob}(\Delta) \) and put
\[
\gamma^* \kappa_0 := \kappa_0 \circ \gamma.
\]

5. Estimates for the Kobayashi distance on \( B^0_1 \).

**Proposition 3.14.** For any \( f \in B^0_1 \), we have the equality
\[
d_{B^0_1}(f, c_0) = \inf \{ d_{H^\infty}(\widehat{f}, \widehat{c}_0) : \kappa_0 \circ \widehat{f} = f \};
\]
moreover, there exists a map \( \widehat{f}^* (z) = c_0^* + c_1^* z + \ldots \) covering \( f \), on which the infimum in (3.26) is attained, i.e.,
\[
d_{B^0_1}(f, c_0) = d_{H^\infty}(\widehat{f}^*, \widehat{c}_0^*).
\]

**Proof.** It is well-known (and rather simple) that a complex geodesic in the unit ball \( B(0, 1)_X \) of a complex Banach space \( X \), joining its center \( 0 \) with a point \( x \neq 0 \), is a holomorphic isometry \( \Delta \to B(0, 1)_X \) determined by the map
\[
\zeta \mapsto \frac{\zeta x}{\|x\|}
\]
and that the Kobayashi and Carathéodory distances in \( B(0, 1)_X \) between these points are equal to
\[
d_{B(0,1)_X}(0, x) = d_\Delta(0, \|x\|) = \tanh^{-1} \|x\|. \quad (3.28)
\]
We apply this to functions \( \widehat{f}(z) = \widehat{c}_0 + \widehat{c}_1 z + \ldots \) from \( H^\infty_1 \). The corresponding functions
\[
\widehat{g}_f (z) := \frac{\widehat{f}(z) - \widehat{c}_0}{1 - \overline{\widehat{c}_0} \widehat{f}(z)},
\]
also belong to \( H^\infty_1 \), and by (3.28),
\[
d_{H^\infty_1}(\widehat{g}_f, 0) = d_\Delta(\|\widehat{g}_f\|_\infty, 0) = \tanh^{-1}(\|\widehat{f} - \widehat{c}_0\|/(1 - \overline{\widehat{c}_0} \widehat{f})\|_\infty). \quad (3.29)
\]
Since for a fixed \( \widehat{c}_0 \) (regarded again as a constant function on \( \Delta \)), the map
\[
\widehat{g} \mapsto \frac{\widehat{g} + \widehat{c}_0}{1 + (\widehat{c}_0)\overline{\widehat{g}}}
\]
with \( \widehat{g} \) running over the ball \( H^\infty_1 \) is a biholomorphic isometry of this ball, the map
\[
\omega(\zeta) = \frac{\zeta \widehat{g}_f/\|\widehat{g}_f\|_\infty + \widehat{c}_0}{1 + \overline{\widehat{c}_0} \zeta \widehat{g}_f/\|\widehat{g}_f\|_\infty} : \Delta \to H^\infty_1 \quad (3.30)
\]
carries out the complex geodesic \( \zeta \mapsto \zeta \widehat{g}_f/\|\widehat{g}_f\|_\infty \) into a complex geodesic in \( H^\infty_1 \) passing through the points \( \widehat{c}_0 \) and \( \widehat{f} \). The point \( \widehat{f} \) is obtained by (3.30) on the value \( \zeta = \|\widehat{g}_f\|_\infty \).

Now observe that the universal covering map \( \kappa_0 : \Delta \to \Delta^* \) extends by the equality (3.23) to all \( \widehat{f} \in H^\infty_1 \) and this extension induces a holomorphic map of the unit ball \( H^\infty_1 \) into domain \( B^0_1 \). Such maps cannot expand the invariant distances; thus
\[
d_{B^0_1}(f, c_0) = d_{B^0_1}(\kappa_0 \circ \widehat{f}, \kappa_0(\widehat{c}_0)) \leq d_{H^\infty_1}(\widehat{f}, \widehat{c}_0),
\]
and
\[
d_{B^0_1}(f, c_0) \leq \inf_{\widehat{f}} d_{H^\infty_1}(\widehat{f}, \widehat{c}_0), \quad (3.31)
\]
where the infimum is taken over all covers \( \widehat{f} \) of \( f \).
Our goal is to show that in fact one has the equality in (3.31) and that the infimum in (3.31) is attained. We assume, the contrary, i.e., that

\[ d_{B_1^0}(f, c_0) < \inf_{\hat{f} \neq f} d_{H_1^\infty}(\hat{f}, \hat{c}_0), \]

and apply Proposition 3.6. This proposition provides the existence of a complex geodesic \( h : \Delta \to B_1^0 \) joining there the points \( c_0 \) and \( f \), and it follows from the above relations that

\[ d_{B_1^0}(f, c_0) = d_\Delta(\zeta_1, \zeta_2) < \tanh^{-1} \| (f - c_0)/(1 - \overline{c}_0 f) \|_\infty, \]

(3.32)

where \( \zeta_1 = h^{-1}(c_0), \zeta_2 = h^{-1}(f) \). Lifting the map \( h \) by (3.23) to a holomorphic map \( \hat{h} \) of the unit disk into itself, one gets the points \( \hat{h}(\zeta_1), \hat{h}(\zeta_2) \) in \( \Delta \), which lie in the fibers over \( c_0 \) and \( f \), respectively, while the relations (3.32) imply

\[ d_{H_1^\infty}(\hat{h}(\zeta_1), \hat{h}(\zeta_2)) = d_\Delta(\zeta_1, \zeta_2) < \tanh^{-1} \| (f - c_0)/(1 - \overline{c}_0 f) \|_\infty. \]

This inequality contradicts (3.29), which completes the proof of Proposition 3.14.

This proposition provides explicitly some complex geodesics in \( B_1^0 \). Note also that the arguments in the proof above remain in force by replacing there \( \kappa_0 \) by a universal covering map \( \gamma^*\kappa_0 : \Delta \to \Delta_* \) with a fixed \( \gamma \in \text{Mob}(\Delta) \).

We now choose the factor \( \hat{f} \) in (3.23) so that \( \hat{f}(0) = 0 \) and fix such \( \hat{f} \). Accordingly, we must replace \( \kappa_0 \) by \( \gamma^*\kappa_0 = \kappa_0 \circ \gamma \) with \( \gamma \in \text{Mob}(\Delta) \), which satisfies

\[ \gamma(0) = \kappa_0^{-1}(c_0), \]

and taking the point \( \kappa_0^{-1}(c_0) \) in the closure of the fundamental triangle of the Fuchsian group \( \Gamma(\Delta, \Delta_*) \) which uniformizes the punctured disk \( \Delta_* \) in \( D \) (that means \( \Delta_* \) is represented as factor \( D/\Gamma(\Delta, \Delta_*) \) up to conformal equivalence; the desired conformal map is produced by \( \kappa_0 \)). Then the representation (3.23) assumes the form

\[ f(z) = (\gamma^*\kappa_0) \circ \hat{f}(z). \]

(3.33)

Note that \( \gamma \) is determined up to rotations about the origin, which is not essential for dilatation.

As a corollary of Proposition 3.14, one obtains the following result, which we precede by some remarks. Denote again the coefficients of the covering maps \( \hat{f} \) in (3.33) by \( \hat{c}_n \) and note that, in view of the assumption \( \hat{c}_0 = 0 \),

\[ \hat{f}(z) = \hat{c}_1 z + \hat{c}_2 z^2 + \ldots. \]

Accordingly, \( \hat{f}_t(z) = \hat{c}_1 z + \hat{c}_2 z^2 + \ldots \) will denote the covers of homotopies (3.21) for original functions \( f \in B_1^0 \).

It follows from Lemma 3.11 that

\[ \| \hat{f}_t \|_\infty = |\hat{c}_1| |t| + O(t^2) \text{ as } t \to 0, \]

where the estimate of remainder is in \( H^\infty \)-norm (thus uniform for all \( |z| < 1 \)). If \( c_1 = c_2 = \ldots = c_{m-1} = 0 \) (equivalently, for \( \hat{c}_1 = \hat{c}_2 = \ldots = \hat{c}_{m-1} = 0 \)), we have

\[ d_{B_1^0}(f_t, c_0) \leq d_{H_1^\infty}(\hat{f}_t, 0) = |\hat{c}_m| |t|^m + O(|t|^{m+1}); \]

(3.34)

here \( \hat{c}_m \) is the first nonvanishing coefficient.

A consequence of Proposition 3.14 mentioned above is the following
Lemma 3.15. For any function \( f(z) = c_0 + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{B}_1^0 \), \( m \geq 1 \), with \( c_m \neq 0 \) (and \( c_0 \neq 0 \)) we have the sharp asymptotic estimate
\[
d_{\mathcal{B}_1^0}(f_t, c_0) = \inf \{d_{H^\infty}(\widehat{f_t}, 0) = \inf \{ |c_m| : \gamma^* \kappa_0 \circ \widehat{f} = f \} \} t^m + O(|t|^{m+1})
\]
where the infima are taken over all covering maps \( \widehat{f} \) of \( f \) fixing the origin and all \( \gamma \in \Gamma(\Delta, \Delta_*) \), and these infima are attained on some pair \((\widehat{f}, \gamma)\).

In particular, \( \kappa_0(z) = z \), and for this map the equalities (3.35) result in
\[
d_{\mathcal{B}_1^0}(\kappa_0, c_0) = |t| + O(|t|^2), \quad t \to 0.
\]
This equality shows that the holomorphic disk \( \Delta(\kappa_0) \) filled by the homotopy functions \( \kappa_0(z) = \kappa_0(tz) \), \( t \in \Delta \), is a complex geodesic in \( \mathcal{B}_1^0 \).

Accordingly, for \( \kappa_m(z) = \kappa_0(z^m) \), we have
\[
d_{\mathcal{B}_1^0}(\kappa_m, c_0) = |t|^m + O(|t|^{m+1}), \quad t \to 0.
\]
In fact, the remainder terms in the last two equalities can be omitted.

Note also that the covering map \( \widehat{f} \) is a rotation of \( \Delta \) about the origin only for \( f = \kappa_0 \), and for all rotations we have similar results.

6. Finishing the proof of Theorem 1.1.

We can now complete the proof of the main theorem. Let
\[
f^0(z) = c_0^0 + c_1^0 z + c_2^0 z^2 + \ldots
\]
be an extremal function maximizing \( |c_n| \) \((n > 1) \) on \( \mathcal{B}_1 \). Then \( |c_n^0| = M_n \) and \( J(f^0) = 1 \), and for its homotopy functions \( f^0_r(z) = f^0(rz) \) with \( 0 < r < 1 \), which lie in \( \mathcal{B}_1^0 \), we have \( J(f^0_r) = r \).

First we show that any extremal function \( f^0 \) must satisfy
\[
c_1^0 = 0.
\]
Indeed, assuming \( c_1^0 \neq 0 \), one derives from Lemma 3.15 the equalities
\[
d_{\mathcal{B}_1^0}(f^0_r, c_0^0) = |c_1^0| r + O(r^2) = \frac{|c_1^0| r}{|\gamma^* \kappa_0(0)|} + O(r^2), \quad r \to 0,
\]
where \( c_1^0 \) is the first coefficient of a factorizing function \( \widehat{f} \) for \( f^0 \) by (3.33) and \( \gamma \) is the appropriate Möbius automorphism of \( \Delta \), on which the infima in (3.35) are attained. Combining these equalities with the relations (2.2), (2.6), connecting the Kobayashi distance and the Green function \( g_{\mathcal{B}_1^0}(0, f^0_r) \), and with Lemma 3.9, one obtains
\[
rJ(f^0) = r \leq |c_1^0| r + O(r^2),
\]
and therefore,
\[
1 \leq |c_1^0| = \frac{|c_1^0|}{|\gamma^* \kappa_0(0)|}.
\]
By Schwarz’s Lemma and Lemma 3.13, such relations can hold only in the case when \(|c_0^0| = 1\), i.e., for \(f^0 = \kappa_0\) (up to rotations); in addition, we must have the equalities
\[ |c_n^0| = |c_1^0| = |c_1(\gamma^0_0 \kappa_0)| = 2/e. \]
But this is impossible, because in view of Parseval’s equality \(\sum_0^\infty |c_n|^2 = 1\) for the boundary function \(\kappa_0(\epsilon^{\theta} \), \(\theta \in [0, 2\pi]\); in fact, the strict inequality
\[ |c_n(\kappa_0)| < 2/e \]
holds for any \(n > 1\). This contradiction proves the equality (3.37).

Therefore, the extremal functions of \(J\) must be of the form \(f^0(z) = c_0^0 + c_2^0 z^2 + \ldots\); equivalently,
\[ f_t^0(z) = c_0^0 + c_2^0 t^2 z^2 + O(t^3), \quad t \to 0. \] (3.39)
Now the proof of the theorem is continued successively for \(n = 2, 3, \ldots\).

By maximization of the second coefficient \(c_2\), the expansion (3.39) requires to deal with the square functional
\[ J_2(f) = J(f)^2 = |c_2(f)|/M_2 \]
to have homogeneity of degree 2. Its comparison with (3.36) (for \(m = 2\)) provides
\[ J_2(f_\nu^0) = r^2 \leq \frac{|c_2^0|^2 r^2}{(\gamma^0 \kappa_0)'(0)} + O(r^3), \]
which implies, similar to (3.38), the equalities
\[ |c_2^0| = \kappa_0'(0) = 2/e. \]
These equalities yield
\[ f_\theta^0(z) = \kappa_0^0(\theta^2) := \kappa_0(z^2), \]
completing the proof for \(n = 2\).
Let \(n \geq 3\). First, applying the corresponding square functional
\[ J_2(f) = \left( \frac{|c_n(f)|}{M_n} \right)^{2/n}, \]
we derive by the same arguments, as above for the first coefficient \(c_1^0\), that also the second coefficient \(c_2^0\) of any extremal function \(f^0\) for \(c_n\) must vanish. Hence,
\[ f^0(z) = c_0^0 + c_3^0 z^3 + c_4^0 z^4 + \ldots. \]

To complete the proof for \(n = 3\), one must deal with the cubic functional
\[ J_3(f) = \frac{|c_3(f)|}{M_3}. \]
Arguments similar to those applied above provide now the equalities
\[ |c_3^0| = \kappa_0'(0) = 2/e, \]
which can hold only for the function \(f_\theta^0(z) = \kappa_0(z^3)\) and its rotations.

For \(n > 3\), comparing successively the relation (3.36) with the functional
\[ J(f)^{n-1} = (|c_n(f)|/M_n)^{(n-1)/n}, \]
one establishes in the same way that the expansion of any extremal function for \(c_n\) must be of the form
\[ f^0(z) = z + c_n^0 z^n + c_{n+1}^0 z^{n+1} + \ldots. \]
So it suffices to deal now with the functions $f \in \mathcal{B}_1^0$ such that $c_1 = \cdots = c_{n-1} = 0$. For such functions, comparison of the relation (3.36) with the power

$$J(f)^n = |c_n(f)|/M_n,$$

provides immediately that the extremal value of $|c_n|$ is

$$|c_n^0| = \kappa'(0) = 2/e.$$

Therefore, $f^0(z) = \kappa_0(z^n)$, up to rotations. Theorem 1.1 is proved.

References

[Ah] L. Ahlfors, An extension of Schwarz’s lemma, Trans. Amer. Math. Soc. 43 (1938), 359-364.
[Ba] R. W. Barnard, Open problems and conjectures in complex analysis, Computational Methods and Function Theory Proceedings, Valparaiso 1989, St. Ruscheweyh, E. B. Saff, L. C. Salinas, R. S. Varga (eds.), Lecture Notes in Mathematics 1435, Springer, Berlin, 1990, pp. 1-26.
[Be] L. Bers, Automorphic forms and Poincaré series for infinitely generated Fuchsian groups, Amer. J. Math. 87 (1965), 196-214.
[Di] S. Dineen, The Schwarz Lemma, Clarendon Press, Oxford, 1989.
[DTV] S. Dineen, R.M. Timoney and J.P. Vigué, Pseudodistances invariantes sur les domains d’une espace localement convexe, Ann. Scuola. Norm. Sup. Pisa Cl. Sci.(4) 12 (1985), 515-529.
[Ea] C.J. Earle, On quasiconformal extension of the Beurling-Ahlfors type, Contributions to Analysis, Academic Press, New York - London, 1976, pp. 99-105.
[Ha] R.S. Hamilton, Extremal quasiconformal mappings with prescribed boundary values, Trans. Amer. Math. Soc. 138 (1969), 399-406.
[He] M. Heins, A class of conformal metrics, Nagoya Math. J. 21 (1962), 1-60.
[Ho] Ch. Horowitz, Coefficients of nonvanishing functions in $H^\infty$, Israel J. Math. 30 (1978), 285-291.
[HSZ] J. A. Hummel, S. Scheinberg and L. Zalcman, A coefficient problem for bounded nonvanishing functions, J. Anal. Math. 31 (1977), 169-190.
[Ko] S. Kobayashi, Hyperbolic Complex Spaces, Springer, New York, 1998.
[Kr1] S.L. Krushkal, Quasiconformal Mappings and Riemann Surfaces, Wiley, New York, 1979.
[Kr2] S.L. Krushkal, Extension of conformal mappings and hyperbolic metrics, Siberian Math. J. 30 (1989), 730-744.
[Kr3] S.L. Krushkal, Strengthened Moser’s conjecture, geometry of Grunsky inequalities and Fredholm eigenvalues, Central European J. Math. 5(3) (2007), 551-580.
[Kr4] S.L. Krushkal, The Zalcman conjecture and related problems, J. Approx. Theory 131 (2004), 122-136.
[Kz] J. Krzyz, Coefficient problem for bounded nonvanishing functions, Ann. Polon. Math. 70 (1988), 341-361.
[Le] L. Lempert, Holomorphic retracts and intrinsic metrics on convex domains, Anal. Math. 8 (1982), 257-261.
[LS] Z. Lewandowski and J. Szynal, On the Krzyz conjecture and related problems, XVIIIth Rolf Nevanlinna Colloquium (Joensu, 1995), de Gruyter, Berlin, 1996, pp. 257-269.
[Ma] W.S. Massey, A Basic Course in Algebraic Topology, Springer, New York, 1991.
[Mi] D. Minda, The strong form of Ahlfors’ lemma, Rocky Mountain J. Math., 17 (1987), 457-461.
[PS] D. V. Prokhorov and J. Szynal, Coefficient estimates for bounded nonvanishing functions, Bull. Acad. Polon. Sci. Ser. 29 (1981), 223-230.
[Ro] H.L. Royden, The Ahlfors-Schwarz lemma: the case of equality, J. Anal. Math. 46 (1986), 261-270.
[RW] H.L. Royden and P.M. Wong *Carathéodory and Kobayashi metric on convex domains*, preprint.

[Sa] N. Samaris, *A proof of Krzyz’s conjecture for the fifth coefficient*, Complex Variables *48* (2003), 753-766.

[Sz] W. Szapiel, *A new approach to the Krzyz conjecture*, Ann. Univ. Mariae Curie-Skłodowska Sect. A *48* (1994), 169-192.

[Ta] De-Lin Tan, *Estimates of coefficients of bounded nonvanishing analytic functions*, Chinese Ann. Math. Ser. A *14* (1983) 97-104.

[Ve] E. Vesentini, *Complex geodesc and holomorphic mappings*, Sympos. Math. *26* (1982), 211-230.

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