Unitarization of infinite-range forces: graviton-graviton scattering

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Abstract: A method to unitarize the scattering amplitude produced by infinite-range forces is developed and applied to Born terms. In order to apply S-matrix techniques, based on unitarity and analyticity, we first derive an S-matrix free of infrared divergences. This is achieved by removing a divergent phase factor due to the interactions mediated by the massless particles in the crossed channels, a procedure that is related to previous formalisms to treat infrared divergences. We apply this method in detail by unitarizing the Born terms for graviton-graviton scattering in pure gravity and we find a scalar graviton-graviton resonance with vacuum quantum numbers ($J^{PC} = 0^{++}$) that we call the graviball. Remarkably, this resonance is located below the Planck mass but deep in the complex $s$-plane (with $s$ the usual Mandelstam variable), so that its effects along the physical real $s$ axis peak for values significantly lower than this scale. This implies that the corrections to the leading-order amplitude in the gravitational effective field theory are larger than expected from naive dimensional analysis for $s$ around and above the peak position. We argue that the position and width of the graviball are reduced when including extra light fields in the theory. This could lead to phenomenological consequences in scenarios of quantum gravity.
gravity with a large number of such fields or, in general, with a low-energy ultraviolet completion. We also apply this formalism to two non-relativistic potentials with exact known solutions for the scattering amplitudes: Coulomb scattering and an energy-dependent potential obtained from the Coulomb one with a zero at threshold. This latter case shares the same $J = 0$ partial-wave projected Born term as the graviton-graviton case, except for a global factor. We find that the relevant resonance structure of these examples is reproduced by our methods, which represents a strong indication of their robustness.

**KEYWORDS:** Effective Field Theories, Models of Quantum Gravity, Nonperturbative Effects, Scattering Amplitudes

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1 Introduction

Several techniques from quantum field theory have been successfully applied to scattering processes in gravitational theories, see e.g. [1–7] for recent reviews, and references therein. The study of an S-matrix formulation has been a particularly fruitful direction of research where results from black-hole physics and string theory have found a rich niche, e.g. [8–14]. Furthermore, unitarity arguments have been used in the past as a guide to build ultraviolet completions of general relativity above the Planck mass, see e.g. [15] for a recent effort. The purpose of this work and the related [16] is to initiate the exploration of a complementary direction in this program by considering the unitarization of Born amplitudes, the lowest-order graviton-graviton scattering, at energies below the Planck mass. In this program, we will also derive generic results that can be used to unitarize other theories with long-range interaction.

One of the motivations for our study is to clarify whether the scattering in this kinematic region is resonant. Recall that in the analogous case of pure Yang-Mills, there is theoretical evidence that the self-interactions of the gluons generate massive states (glueballs) filling its spectrum [17–20]. These states lead to a new form of matter in full quantum chromodynamics (QCD) and have been identified with resonances measured in experiment [21–25]. Therefore, it is interesting and pertinent to ask whether exotic resonance states of two gravitons (graviballs) could arise because of their gravitational interactions in a quantum theory of gravity. These may even lead to phenomenological consequences at relatively low-energy scales.

In order to address this question, we deal with the quantum formulation of general relativity within the framework of effective field theory (EFT) where gravitational interactions are organized in a derivative (or momentum) expansion [6, 26–30]. This is valid as long as the energies considered are well below the Planck mass $M_P = G^{-1/2} \sim 10^{19}$ GeV, which is the natural cutoff of the gravitational EFT [30–33]. Interestingly, another energy expansion serves as the basis for the EFT of QCD at low energy. This is called chiral perturbation theory (ChPT) [34–36] and it describes the self-interactions of pions and other hadrons with a cutoff $\Lambda \sim 1$ GeV. Since the low-energy limits of QCD and quantum gravity can be treated with similar EFTs, analogies between the two could be expected. With this in mind, one may recall that in QCD there is a state with a mass and width of the order of $\Lambda/2$, the so-called $\sigma$-meson or $f_0(500)$, which is the lightest resonance known in this theory [37]. The presence of this state can be rigorously determined by combining amplitudes calculated in the EFT with S-matrix methods [38–40], or only from the latter ones [41]. In fact, clear resonance peaks associated with the $\sigma$ have been observed experimentally in the two-pion invariant mass distributions of $D^+ \to \pi^- \pi^+ \pi^+$ [42–44] or $J/\Psi \to \omega \pi^+ \pi^-$ [45, 46]. Furthermore, this indicates that perturbative calculations for some processes, like the scalar-isoscalar $\pi \pi$ phase shifts, $\gamma \gamma \to \pi \pi$, $\eta \to 3\pi$ decays, etc, are affected by strong unitarity corrections (see [40, 47, 48] for reviews).

In this paper we apply a combination of EFT and S-matrix methods to the scattering of gravitons at low energies to shed light on the possible presence of graviballs, as discussed by us recently in ref. [16]. Our study is complementary to other analyses that considered...
the scattering of particles with different flavors induced by graviton exchange in the s-channel \[31, 49, 50\]. These works employ the one-loop calculation of the graviton self-energy in the EFT in the presence of \(N\) light degrees of freedom \[28\], and emphasize the importance of nonperturbative effects in quantum gravity in order to restore unitarity at \(s \sim (N G)^{-1}\). Ref. \[50\] also studies the \(J = 2\) resonances that result from the resummed formula of the graviton propagator. However, none of these references address the scattering of particles with the same flavor. In this case, there are contributions from the exchange of gravitons in the crossed channels leading to infrared (IR) divergences in the partial-wave amplitudes (PWAs). These difficulties already appear at the level of the partial-wave projected Born amplitude, and stem from the angular projection in the angular region of forward or backward scattering, depending on whether the graviton exchange is in the \(t\) or \(u\) channel, respectively (see also \[12\]).

IR divergences are a well understood feature of theories with infinite-range interactions \[51–54\]. Their contributions to scattering matrix elements were derived for gravitons in the classic reference \[53\], showing that the techniques used to remove them from physical observables are common to quantum electrodynamics (QED) and a quantum theory of gravitons. It is also possible to tackle the IR divergences so as to end with finite \(S\)-matrix amplitudes. The general procedure for quantum electrodynamics (QED) was elaborated by Kulish and Faddeev in ref. \[54\], and it allows one to define a new unitary \(S\)-matrix operator with well defined matrix elements free of IR divergences. Reference \[55\] provided an explicit extension of the formalism to gravity (see also \[56–58\] for related recent works).

We will use these methods to remove an IR divergent global phase from the \(S\)-matrix, which is common to all partial waves. This phase, conjectured long before by Dalitz for the case of elastic non-relativistic Coulomb scattering \[59\], stems from the resummation of the diagrams that conform the Born series by iterating the one-graviton exchange and it first appeared in ref. \[53\]. The new \(S\)-matrix allows us to implement nonperturbative unitarity methods for partial-wave amplitudes (PWAs) based on \(S\)-matrix theory, so that our formalism can be regarded as an extension to infinite-range interactions of the \(S\)-matrix techniques so popular in QCD. The study of PWAs is typically the most adequate method to impose restrictions of unitarity to amplitudes at low energies. As an illustration of the method developed here, and to clarify the new subtleties generated by the IR divergences, we will introduce a toy model in non-relativistic Quantum Mechanics, that we call Adler-Coulomb (AC) scattering, which has the same partial-wave projected Born amplitudes as graviton-graviton scattering for the same final and initial helicities and that can be solved exactly. We also study with the same techniques the unitarization of pure Coulomb scattering, which presents interesting new aspects, like, for instance, that the unitarization becomes valid for large (instead of low) three-momenta.

By applying the previous \(S\)-matrix techniques to pure gravity, the resulting \(S\)-wave PWA has a resonant pole corresponding to a graviball with vacuum quantum numbers \(J^{PC} = 0^{++}\).\footnote{Bound states of gravitational waves, known as geons, have been studied in classical general relativity in \[60, 61\]. Also the recent work \[62\] discussed the possibility of bound states of systems of gravitons from the effective potential at large distances. These states have no relation with our resonance corresponding to different physical situations.} One important peculiarity of this resonance is that its pole position \(s_P\) (in
the Mandelstam variable $s$) is almost a purely imaginary number whose absolute value lies below $G^{-1}$ if one considers natural assumptions. As a result, the dynamics of graviton-graviton scattering would be driven by this resonance even at energies significantly lower than the ultraviolet cutoff of the theory. This may open up several phenomenological implications to test quantum gravity predictions at energies lower than the cutoff. Namely, resonant two-graviton exchange in $S$-wave could induce rescattering effects in different processes as, for instance, the production of multiple gravitons by some energetic or massive source. As we will discuss, this may be particularly interesting for theories with large number of light degrees of freedom for which this resonance is expected to become narrower and lighter. Our present study derives in detail the results presented in ref. [16] to deal with the IR divergences for infinite-range interactions and the unitarization of the corresponding Born terms. It also largely extends this analysis in several directions. In particular, we also present the calculations for varying space-time dimensions $d$, make a thorough study of the robustness of the methods and exploit many of its applications to gravitational and Coulomb-like scattering.

Finally, we would like to remark that other results to unitarize the high-energy scattering amplitudes through black-hole production have been derived in the self-completeness scenario of gravity [14, 63]. The latter are interpreted as bound states of gravitons with masses $\gg M_P$, and may start to leave traces at energies below the cut-off [64, 65]. Furthermore, microscopic black holes which become lighter with the number $N$ of particle species as $M_P/\sqrt{N}$ are predicted in refs. [65, 66]. Though the precise connection between these results and the graviball is not clear, it is certainly intriguing that both non-perturbative methods agree qualitatively on the onset of non trivial states.

The contents of the manuscript are organized as follows. The formalism needed to calculate the partial-wave projected Born amplitudes for graviton-graviton scattering is derived in section 2. Some extra technical details in this respect are given in the appendix A. Our treatment of infinite-range interactions leading to PWAs free of IR divergences is discussed in section 3. The partial-wave projected Born terms are then implemented in the unitarization method developed in section 4. The section 5 applies this unitarization method to $\pi\pi$ with $J = 0$ and the appearance of the $\sigma$ resonance is treated. Section 6 is then devoted to the discussion of a graviton-graviton scalar resonance or graviball which emerges as a pole in the $J = 0$ PWA and which manifests in the rescattering of two gravitons with these quantum numbers. Subsection 6.1 is dedicated to the robustness of the graviball, where we discuss the importance of the interplay between the fundamental and unitarity cutoffs. In section 7 we apply our method to the exactly soluble AC model and non-relativistic Coulomb scattering. We develop a comparison with the known exact solutions, and the higher-order contributions to the unitarized Born terms are worked out. We dedicate Section 8 to study the PWAs for $d > 4$ because in that case there are no IR divergences and this fact can be used to elaborate a method to estimate quantitatively the pole positions. This method is tested by applying it to recover successfully the exact solution of the AC model and $S$-wave Coulomb scattering and is then applied to the graviball. Concluding remarks, prospects for future work and new directions are gathered in section 9. Some technical material is given in appendix B on PWAs in $d$ dimensions.
2 Partial-wave projection of the graviton-graviton scattering amplitudes

In this work, we focus on the graviton-graviton scattering process

$$|p_1, \lambda_1\rangle|p_2, \lambda_2\rangle \rightarrow |p_3, \lambda_3\rangle|p_4, \lambda_4\rangle. \quad (2.1)$$

The energy of each graviton is $p_0 = |p|$ and the corresponding four-momentum is indicated by $p_i$. Here we denote by $|p, \lambda\rangle$ the one-graviton state of three-momentum $p$ and helicity $\lambda$. These states are normalized as

$$\langle p', \lambda' | p, \lambda \rangle = 2p_0(2\pi)^3 \delta(p' - p)\delta_{\lambda \lambda'}, \quad (2.2)$$

where $p_0 = |p|$ is the energy of the massless graviton. Our definitions for the one- and two-graviton states are given in detail in appendix A, where the relation between the basis states with well-defined three-momentum and total angular momentum is also worked out.

The $S$- and $T$-matrix operators are related by

$$S = I + i(2\pi)^4 \delta^{(4)}(P_f - P_i)T. \quad (2.3)$$

Their matrix elements for two-graviton scattering are expressed in terms of the usual Mandelstam variables $s$, $t$ and $u$, defined as

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2, \quad (2.4)$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2,$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2.$$  

Because of the null mass of a graviton they fulfill that

$$s + t + u = 0. \quad (2.5)$$

The basic building blocks for our study of the graviton-graviton scattering are tree-level or Born amplitudes. Within an EFT calculation of two-graviton scattering these are the lowest-order amplitudes in powers of $G$, where $G$ is the Newton constant. We adapt them from the calculation in ref. [67]. The normalization in this paper for the Born amplitudes only differs from ours by an extra factor $i$. The expressions for the Born amplitudes $F_{\lambda_3 \lambda_4, \lambda_1 \lambda_2} \equiv T^{\text{Born}}_{\lambda_3 \lambda_4, \lambda_1 \lambda_2}$ are given according to the number of gravitons with helicity $-2$:

1. $F_{\lambda_3 \lambda_4, -2 \lambda_1} = 0$ if only one $\lambda_i = -2(2)$ and the others are $2(-2)$.

2. When all helicities are equal then

$$F_{-2-2, -2-2}(s, t, u) = F_{-2-2, 2-2}(s, t, u) = \frac{\kappa^2}{4} \frac{s^4}{stu}, \quad (2.6)$$

with $\kappa^2 = 32\pi G$. 

- 4 -
3. Finally, if there are two positive and two negative helicities the expressions are all of them related by parity and crossing transformations (among themselves and with case 2 above) and can be written as

\[ F_{-22,-22}(s, t, u) = F_{2-2,2-2}(s, t, u) = \frac{\kappa^2}{4} u^4, \]

\[ F_{2-2,2-2}(s, t, u) = F_{2-2,2-2}(s, u, t) = \frac{\kappa^2}{4} \frac{t^4}{stu}, \]

with any other combination having a zero Born scattering amplitude.

### 2.1 Partial-wave amplitudes and unitarity

We now express the previous amplitudes in the basis with well defined angular momentum \( \phi \) with

\[ \phi = 0 \]

where we have defined coordinates such that the final three-momentum is complementary to these studies.

parameter, where the eikonal analysis would require an infinite number of PWAs, e.g. \([2, 7]\). Our approach is used in nuclear physics to disentangle the interference effects between the Coulomb and the strong interactions e.g. in proton-proton or proton-\( \pi^\pm \) interactions. This formalism is used in nuclear physics to disentangle the interference effects between the Coulomb and the strong interactions e.g. in proton-proton or proton-\( \pi^\pm \) interactions. In this regard, one could consider external probes that select these quantum numbers for a two-graviton system, so that they rescatter with a given total angular momentum \( J \), and hence, according to the corresponding subset of PWAs. The analysis from few PWAs may also be relevant when considering the interference effects between the gravitational and other interactions. This formalism is used in nuclear physics to disentangle the interference effects between the Coulomb and the strong interactions e.g. in proton-proton or proton-\( \pi^\pm \) interactions \([70]\). \(^3\)

The unitarity of the \( S \)-matrix, \( SS^\dagger = S^\dagger S = I \) for positive values of \( s \), in the approximation of keeping only two-graviton intermediate states, \(^4\) implies that

\[ S(p', \lambda_1' \lambda_2'|T|pz, \lambda_1 \lambda_2) S - S(p', \lambda_1' \lambda_2'|T|pz, \lambda_1 \lambda_2) S \]

\[ = \frac{i}{64 \pi^2} \sum_{\mu_1, \mu_2} \int dq' S(p', \lambda_1' \lambda_2'|T|q, \mu_1 \mu_2) S S(q, \mu_1 \mu_2|T|pz, \lambda_1 \lambda_2) S, \]

\(^2\)Note that several studies on gravitational scattering have been devoted to processes with large impact parameter, where the eikonal analysis would require an infinite number of PWAs, e.g. \([2, 7]\). Our approach is complementary to these studies.

\(^3\)In the low-energy region only a few strong PWAs enter into play, and they select for the interference effects the corresponding Coulomb PWAs.

\(^4\)The single graviton channel vanishes from helicity conservation and because \( s = 0 \) on-shell.
where a symmetry factor 1/2 has been included in the right-hand side because of the symmetric states used. It is important to note that gravitons have zero mass and there are also other contributions in the right-hand side of the unitarity relation involving a larger number of gravitons in the intermediate states (three and more) even at $Gs \ll 1$. However, these terms are at least two-loop contributions that, in the low-energy EFT of gravity, are $\mathcal{O}((Gs)^3)$, and we will ignore them in the following. This point will be discussed further in section 6.1.

In terms of the graviton-graviton PWAs the two-body unitarity relation of eq. (2.10) becomes

$$
\Im S\langle pJ,\lambda_1^\prime\lambda_2^\prime|T|pJ,\lambda_1\lambda_2\rangle_S = \frac{\pi}{8} \sum_{\mu_1,\mu_2} S\langle pJ,\lambda_1^\prime\lambda_2^\prime|T|pJ,\mu_1\mu_2\rangle_S S\langle pJ,\mu_1\mu_2|T^\dagger|pJ,\lambda_1\lambda_2\rangle_S \theta(s),
$$

(2.11)

where the symbol $\Im$ denotes the imaginary part and $\theta(x)$ is the Heaviside or step function. Here we have also taken into account that graviton-graviton PWAs are symmetric under the exchange between the final and initial states because of time-reversal invariance.

The $S$-matrix in partial waves $\bar{S}^{(J)}(s)$ is given by

$$
\bar{S}^{(J)}(s) = I + \frac{\pi 2^{|\lambda|/4}}{4} T^{(J)}(s),
$$

(2.12)

being a matrix in the space of initial and final helicities for given $s$, and where $|\lambda| = |\lambda_2^\prime - \lambda_1^\prime| = |\lambda_2 - \lambda_1| = 0$ or 4. The factor $2^{|\lambda|/4}$ appears because of the different normalization between the states having equal or different helicities, as discussed at the end of appendix A. From the unitarity relation in PWAs, eq. (2.11), it can be readily obtained that

$$
\bar{S}^{(J)} \bar{S}^{(J)^\dagger} = I.
$$

(2.13)

Since this matrix is of finite order then this last equation also implies that $\bar{S}^{(J)^\dagger} \bar{S}^{(J)} = I$.

Regarding the set of PWAs under consideration, we notice that $|pJM,\lambda\lambda\rangle_S$ only involves even $J$ because under the exchange of the two helicities, cf. eq. (A.12),

$$
|pJM,\lambda\lambda\rangle_S = (-1)^J |pJM,\lambda\lambda\rangle_S,
$$

(2.14)

so that it is zero if $J$ is odd. Furthermore, since

$$
|pJM,\lambda_2\lambda_1\rangle_S = (-1)^J |pJM,\lambda_1\lambda_2\rangle_S,
$$

(2.15)

we do not consider the scattering amplitudes $S\langle pJM,\lambda_2\lambda_1|T|pJM,\lambda_1\lambda_2\rangle_S$ and $S\langle pJM,\lambda_2\lambda_1|T|pJM,\lambda_2\lambda_1\rangle_S$, because they are $(-1)^J S\langle pJM,\lambda_1\lambda_2|T|pJM,\lambda_1\lambda_2\rangle_S$ and $S\langle pJM,\lambda_1\lambda_2|T|pJM,\lambda_1\lambda_2\rangle_S$, respectively. As a result, we dedicate our study to the partial waves

$$
S\langle pJ,22|T|pJ,22\rangle_S \quad \text{(for even $J$)} \quad \text{and} \quad S\langle pJ,2-2|T|pJ,2-2\rangle_S.
$$

(2.16)

Let us also indicate that due to the parity symmetry, $S\langle pJ,2-2|T|pJ,2-2\rangle_S = S\langle pJ,22|T|pJ,22\rangle_S$ and this PWA is not treated separately.
3 Treatment of the infinite-range nature of the interactions

The direct application of the previous PWA formalism to the amplitudes (2.6) and (2.7) is hindered by divergences reflecting the infinite-range character of the gravitational interaction [12]. In this section we provide a treatment and interpretation of this difficulty, based on well known results about the IR properties of the amplitudes of infinite-range interactions [53–55].

Let us first illustrate the issue by considering the PWA of the Born amplitude \(F_{22,22}(s,t,u)\) with \(J=0\), for which one has the angular projection

\[
F_{22,22}^{(0)}(s) = \frac{k^2 s^3}{32 \pi^2} \int_{-1}^{+1} d \cos \theta \frac{1}{tu} = -\frac{k^2 s^2}{16 \pi^2} \int_{-1}^{+1} d \cos \theta \frac{1}{t}.
\]

(3.1)

Here we have used eq. (2.5) and the explicit expression for \(t\) and \(u\) in terms of the scattering angle \(\theta\)

\[
t = -(p - p')^2 = -2p^2(1 - \cos \theta),
\]

\[
u = -(p + p')^2 = -2p^2(1 + \cos \theta).
\]

(3.2)

We follow the notation that \(p = p_1\) is the initial center-of-mass (CM) three-momentum, \(p' = p_3\) is the final one and \(p\) is the common modulus of all CM three-momenta.

The angular integration in eq. (3.1) has a logarithmic divergence in the upper limit of integration for \(\cos \theta \to 1\). This is actually an IR divergence because the four-momentum squared of the \(t\)-channel exchanged graviton, which is equal to \(t\), vanishes for \(\cos \theta \to 1\). As we show in the following, this IR divergence is associated to a virtual soft graviton that connects two external on-shell graviton lines, following the classification of IR divergences of ref. [53]. Weinberg in this reference studies in detail this type of divergences and shows that they can be resummed correcting the \(S\)-matrix by a factor

\[
\exp \left[ \int \mathcal{L} \frac{d^4 q}{(2 \pi)^4} B(q, \mu) \right],
\]

(3.3)

where \(B(q, \mu)\) represents the correction to the processes given by single soft (virtual) graviton exchange and where the IR divergence is regularized by a graviton mass \(\mu\) (3.3). The variable \(\mathcal{L}\) is a cutoff in \(|q|\) that separates the regions of “hard” and “soft” graviton momenta and is chosen to be low enough to justify the iteration of diagrams leading to eq. (3.3).

The integral in this equation gives a diverging imaginary contribution for every pair of either initial or final particles in the scattering process that is independent of their spins.\(^5\) This contribution can be entirely expressed in terms of the Lorentz invariant relative velocity \(\beta_{ab}\) of the two particles involved, \(a\) and \(b\), in the rest frame of either,

\[
\beta_{ab} = \frac{[(p_a p_b)^2 - (m_a m_b)^2]^{1/2}}{p_a p_b},
\]

(3.4)

\(^5\)The divergence of the real part is not of concern here since it does not appear in the Born series but at higher orders of \(G_s\). Its cancellation in inclusive processes is due to the radiation of soft gravitons [53].
where \( p_a \) and \( p_b \) are the four-vectors of the particles with masses \( m_a \) and \( m_b \), respectively. The phase factor of interest that stems from eq. (3.3) for a given pair of initial or final particles is [53]

\[
\exp \left[ -i \frac{G m_a m_b (1 + \beta_{ab}^2)}{\beta_{ab} [1 - \beta_{ab}^2]^{1/2}} \log \frac{\mu}{\Sigma} \right].
\] (3.5)

The total exponential factor \( S_c(s) \) in our case is given by

\[
S_c(s) = \lim_{m \to 0} \exp \left[ -i 2 \frac{G m^2 (1 + \beta^2)}{\beta [1 - \beta^2]^{1/2}} \log \frac{\mu}{\Sigma} \right] = \exp \left[ -i 2 G s \log \frac{\mu}{\Sigma} \right],
\] (3.6)

where we have taken the massless limit replacing \( \beta_{ab} \) by \( \beta = 1 - \frac{2 m^4}{s^2} + \mathcal{O}(m^8) \), and have taken into account that \((p_a + p_b)^2 = s\) for both pairs of either incoming or outgoing gravitons. As we will see below in section 8.1 the IR divergence can also be derived using dimensional regularization (see also [55, 71]).

From the linear term in the expansion of the phase factor \( S_c(s) \) in powers of \( G \), one can read its (divergent) contribution to the Born scattering amplitude,

\[
S_c(s) = 1 - i 2 G s \log \frac{\mu}{\Sigma} + \mathcal{O}(G^2),
\] (3.7)

that gives the following contribution to the PWA

\[
\delta F^{(J)}_{\lambda_1 \lambda_2, \lambda_1} = - \frac{1}{2^{\lambda/4}} \frac{8 G s}{\pi} \log \frac{\mu}{\Sigma},
\] (3.8)

where the factor \( 1/2^{\lambda/4} \) reflects the different normalization of the symmetrized states with \( \lambda = 0 \) or \( \lambda = 4 \), cf. eq. (2.12). As expected, this is precisely the divergence that stems in the \( J = 0 \) partial-wave projection of eq. (3.1), after giving a mass \( \mu \to 0^+ \) to the exchanged graviton,

\[
F^{(0)}_{22,22}(s) = - \frac{\kappa^2 s^2}{16 \pi^2} \int_{-1}^{1} \frac{d \cos \theta}{t - \mu^2} = \frac{\kappa^2 s}{8 \pi^2} \log \left( 1 + \frac{4 \mu^2}{t - \mu^2} \right) \to \frac{\kappa^2 s}{4 \pi^2} \log \frac{2 p}{\mu} = \frac{8 G s}{\pi} \log \frac{2 p}{\mu}. \]

(3.9)

It is important to emphasize that this contribution (as well as the whole phase space factor) is independent of \( J \), being common to all PWAs with the given set of helicities \( \lambda_i \) and \( \lambda'_i \). For instance, for the case \( \lambda = 0 \) we have

\[
F^{(J)}_{22,22}(s) = \frac{2 G s^2}{\pi} \int_{-1}^{1} d \cos \theta \frac{P_J(\cos \theta)}{t - \mu^2},
\] (3.10)

where we have taken into account that \( d^{(J)}_{00}(\theta) = P_J(\cos \theta) \) [72], with \( P_J(\cos \theta) \) the Legendre polynomials. In this equation we can isolate the divergent piece by adding and subtracting \( P_J(1) = 1 \) to the numerator of the integrand. Then,

\[
F^{(J)}_{22,22}(s) = \frac{2 G s^2}{\pi} \int_{-1}^{1} d \cos \theta \frac{P_J(\cos \theta) - 1}{t} + \frac{8 G s}{\pi} \log \frac{2 p}{\mu},
\] (3.11)

where the last term is the diverging contribution of eq. (3.8).
For $\lambda = 4$ and arbitrary $J$ the partial-wave projection is

$$F_{2-2,2-2}(s) = \frac{\kappa^2}{32\pi^2 s} \int_{-1}^{+1} \frac{d\cos \theta}{t} \frac{d_{44}^{(J)}(\theta)u^3}{t} = \kappa^2 \frac{1}{32\pi^2 s} \int_{-1}^{+1} \frac{d\cos \theta}{t} \frac{d_{44}^{(J)}(\theta)u^3 + s^3}{t}. \quad (3.12)$$

In order to isolate the diverging piece we sum and subtract $-s^3d_{44}^{(J)}(0)$ to the numerator of the integrand and, since $d_{44}^{(J)}(0) = 1$ [72], we have

$$F_{2-2,2-2}(s) = \frac{G}{\pi s} \int_{-1}^{+1} \frac{d\cos \theta}{t} \left[ d_{44}^{(J)}(\theta)u^3 + s^3 \right] + \frac{4Gs}{\pi} \log \frac{2p}{\mu}. \quad (3.13)$$

Again, it is clear the appearance of the diverging factor according to eq. (3.8). One can also explicitly calculate the result for the case $\lambda = -4$ which receives a divergent contribution from $S_c(s)$ that is related to the one in $\lambda = 4$ by $\delta F_{2-2,2-2}^{(J)} = (-1)^J \delta F_{2-2,2-2}^{(J)}$. Calculating the corresponding PWA and using the property $d_{44}^{(J)}(\pi - \theta) = (-1)^J d_{44}^{(J)}(\pi - \theta)$ one, indeed, obtains the same divergence as in eq. (3.13) but multiplied by $(-1)^J$.

Given the previous results, we redefine the $S$ matrix in PWAs by extracting the divergent phase $S_c(s)$ in eq. (3.6) from $\tilde{S}^{(J)}$,

$$\tilde{S}^{(J)} = S_c S^{(J)} = \exp \left[ -i2Gs \log \frac{\mu}{Q} \right] S^{(J)}, \quad (3.14)$$

and where the new $S^{(J)}$ includes virtual contributions from gravitons with momenta only greater than $\mathcal{L}$. This global phase is not “physical” (not observable) because it does not contribute to the cross section for the scattering process of eq. (2.1). Related to this, $S_c(s)$ is trivial from the analytical point of view and it does not lead to interesting features in the full $S$ matrix such as the presence of poles corresponding to bound, virtual or resonance states.

Since we removed a global phase, $S^{(J)}$ fulfills also the unitarity relation in PWAs

$$S^{(J)} S^{(J)\dagger} = S^{(J)\dagger} S^{(J)} = I. \quad (3.15)$$

The $S$ matrix $S^{(J)}(s)$ generates a new $T$ matrix in PWAs by taking into account eq. (2.12),

$$T^{(J)}(s) \equiv -i \frac{4 \left( S^{(J)} - I \right)}{\pi 2^{\lambda/4}}. \quad (3.16)$$

Given that $S^{(J)}(s)$ is a unitary matrix, eq. (3.15), $T^{(J)}$ also satisfies a two-body unitarity relation analogous to that in eq. (2.11). Namely,

$$\Im T^{(J)} = \frac{\pi 2^{\lambda/4}}{8} T^{(J)} T^{(J)\dagger} \theta(s). \quad (3.17)$$

This equation can also be recast as

$$\Im \frac{1}{T^{(J)}} = -\frac{\pi 2^{\lambda/4}}{8} \theta(s). \quad (3.18)$$
We will denote by $V^{(J)}$ the partial-wave projected Born amplitudes corresponding to $S^{(J)}$. They are free of the IR divergences and carry an explicit dependence on the IR scale $\mathcal{L}$. They can be obtained from (3.14) by expanding up to $\mathcal{O}(G)$,

$$S^{(J)}(s) = I + i2^{\frac{|J|}{4}}\frac{\pi}{4} V^{(J)}(s) + \mathcal{O}(G^2) = \left[ I + i2^{\frac{|J|}{4}}\frac{\pi}{4} F^{(J)}(s) \right] \left[ 1 + i2Gs \log \frac{\mu}{\mathcal{L}} \right] + \mathcal{O}(G^2),$$

from where,

$$V^{(J)}(s) = F^{(J)}(s) + \frac{1}{2^{\frac{|J|}{4}}\pi} \frac{8Gs}{\log \frac{\mu}{\mathcal{L}}}. \quad (3.20)$$

For instance, from eq. (3.9) one obtains the IR-safe result

$$V^{(0)}_{22,22}(s) = \frac{8Gs}{\pi} \log \frac{2p}{\mathcal{L}}. \quad (3.21)$$

Recall that the parameter $\mathcal{L}$ is used to define the soft gravitons that are resummed to generate $S_c(s)$. A change of this cutoff produces just a rescaling of the $S^{(J)}$-matrix,

$$S_{\mathcal{L}'}^{(J)} = \left( \frac{\mathcal{L}}{\mathcal{L}'} \right)^{2Gs} S^{(J)}. \quad (3.22)$$

Thus, although the dependence on $\mathcal{L}$ is physically spurious, it will show up in a truncated perturbative expansion of $S^{(J)}$. On dimensional grounds it is clear that $\mathcal{L} \propto p$ because at $\mathcal{O}(G)$ this is the only magnitude with dimension of momentum that is available to enter inside the propagators in the Feynman diagrams giving rise to the Born term [53]. Therefore, we define

$$\mathcal{L} = \frac{\sqrt{s}}{a}, \quad (3.23)$$

where one should expect that $a > 1$ so that $\mathcal{L}$ is significantly smaller than $\sqrt{s}$. Thus $\log a$, which is the parameter finally entering in the partial-wave projected Born term, is $\mathcal{O}(1)$ because, once the IR divergences are removed, the scattering amplitudes do not involve any other pathological scale. Indeed, this ambiguity can only be resolved conclusively by the exact solution, or by higher order calculations (which may also open the possibility of other methods, e.g. [73–75]). Along these lines, besides the naturalness criteria, we develop a method for estimating $\log a$ in section 8.6, which is also tested in models where the exact solution is known.\footnote{In ref. [54], Kulish and Faddeev dealt with the IR divergences affecting QED by defining an $S$ matrix which is free of them and with the same physical cross sections. A similar method was explicitly implemented for gravity recently in [55] (see also [5] for a recent review).}

In summary, our final expressions for $V^{(J)}_{22,22,22,2} (s)$ are

$$V^{(J)}_{22,22}(s) = -\frac{2Gs^2}{\pi} \int_{-1}^{+1} \frac{d\cos\theta}{t} P_J(\cos\theta) - \frac{1}{t} + \frac{8Gs}{\pi} \log a, \quad (3.24)$$

$$V^{(J)}_{22,22}(s) = \frac{G}{\pi s} \int_{-1}^{+1} \frac{d\cos\theta}{t} \left[ d_{44}^{(J)}(\theta) u^3 + s^3 \right] + \frac{4Gs}{\pi} \log a. \quad (3.25)$$
For the important particular case \( J = 0 \), analyzed in detail below,

\[
V_{22,22}^{(0)}(s) = \frac{8Gs}{\pi} \log a. \tag{3.26}
\]

The first higher partial wave with \( \lambda_i = 2 \) has \( J = 2 \) and

\[
V_{22,22}^{(2)}(s) = \frac{8Gs}{\pi} \left( \log a - \frac{3}{2} \right). \tag{3.27}
\]

For \( \log a \gtrsim 1 \) we see that the interaction is weaker than for \( J = 0 \), and may even change its attractive nature.

It is notorious that the resulting (modified) partial-wave projected Born scattering amplitudes \( V_{22,22}^{(J)}(s) \) and \( V_{2-2,2-2}^{(J)}(s) \), eqs. (3.24) and (3.25), respectively, are free of any cut in the complex \( s \)-plane. We will show below that this is also the case for other models with infinite-range interactions that we will explore (cf. section 7, and section 7.2). In a sense, the presence of IR divergences eventually simplifies the analytical structure of PWAs. By the same token it also paves the way for the applicability of the unitarization methods, cf. section 4.

Finally, let us point out that the connection between the soft infrared divergences of the one-loop scattering amplitudes and the kinematic divergences in the tree-level PWAs that we have developed and applied for quantum gravity in this work has been independently demonstrated for gauge theories and gravity in ref. [76].

### 3.1 Interpretation of the new \( S \) matrix à la Kulish-Faddeev

In ref. [54], Kulish and Faddeev dealt with the IR divergences affecting QED by defining an \( S \) matrix which is free of them and with the same physical cross sections. A similar method was explicitly implemented for gravity recently in [55] (see also [5] for a recent review). The procedure consists of two steps: i) one solves the asymptotic interacting Hamiltonian which allows one to define a new \( S \) matrix operator by removing an exponential term that is the analogous to the Coulomb phase factor in a non-relativistic treatment; ii) the very same asymptotic behavior of the interacting Hamiltonian leads to a definition of asymptotic states which contain an infinite number of photons.\(^7\) Each extra soft photon added to the particles that make the nominal in/out states, involving asymptotic (charged) particles with fixed momenta, carries a factor \( e \) and corresponds to higher orders in the standard perturbative expansion in QED.

In our case, the definition of \( S^{(J)} \) in eq. (3.14) follows from multiplying the \( S \)-matrix \( \tilde{S}^{(J)} \) by the inverse of \( S_c \). Thus, it is analogous to the redefinition of the \( S \) matrix in ref. [54] described in point i) above. In this reference, the asymptotic (\( t \to \pm \infty \)) interaction Hamiltonian is calculated, so that its associated time evolution operator takes care of the infinitely long-range character of the forces. That is precisely the way in which the standard diverging Coulomb phase arises when considering the non-relativistic treatment. In our analysis we restricted to the iteration of the \( V^{(J)}(s) \) in eq. (3.20), which is connected to

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\(^7\)These states are not generic and an extra modification is required for some processes [5, 77, 78]. Since this is not relevant for our results, we refer the reader to the previous references for discussion.
two-graviton intermediate states, without any emission of extra internal or external (soft) gravitons (similarly as the iteration of a tree-level amplitude in a Born series \[53, 59\]). This is why the second step involved in the general procedure of ref. \[54\] is not applied in our case.

Kulish and Faddeev argue that the new asymptotic operator corresponds to the semiclassical approximation for large distances, for which the interaction gradually decreases, so that their procedure has a rather straightforward physical interpretation. Due to the infinite-range character of an \(1/r\) potential, we have an extra contribution to the asymptotic phase (for \(r \to \infty\)) of the wave function. Let us consider first the Yukawa potential \(\alpha \exp(-\mu r)/r\) and we will study later the limit \(\mu \to 0\). In the semiclassical approximation the contribution to the phase shift goes as

\[
\int_{r_-}^{\infty} dr' \left[ \sqrt{2m \left( E + \frac{\alpha e^{-\mu r'}}{r'} \right)} - \sqrt{2mE} \right] = \frac{m\alpha}{p} \Gamma(0, \mu r) = \frac{m\alpha}{p} (-\gamma_E - \log(\mu r)) + \mathcal{O}(\mu r),
\]

with \(\Gamma(a, x)\) the incomplete Gamma function. This result is the standard extra phase stemming from the infinite-range character of the \(1/r\) potential, reflecting that the asymptotic waves are not plane waves (a wave function renormalization feature). The final result in a complete calculation is independent of \(r_+\). This variable plays the same role as the time \(t_0\) in the Kulish-Faddeev treatment \[54\], identifying \(pt_0/m = r_-\).

The fundamental result of ref. \[54\] is the relation between the Feynman-Dyson \(S\) matrix, \(S_F\), calculated in the standard form with asymptotic dynamics driven by the free Hamiltonian \(H_0\), and the new \(S\) matrix, \(S_K\) (see also \[55\]). This relation has the form

\[
S_K = \lim_{\mu \to 0} e^{i\Phi(\mu)} S_F e^{i\Phi(\mu)},
\]

with \(\Phi(\mu)\) a phase operator that precisely removes the IR divergences associated to the limit \(\mu \to 0\) for the photon mass in the QED case. Let us emphasize that \(S_K\) is the \(S\) matrix with the appropriate asymptotic dynamics and that fulfills the standard requirement in \(S\)-matrix theory, as the existence of the Möller operators \(U_{\pm}\) \[54, 79\]. For the electron-positron scattering this phase adopts the form

\[
\Phi(\mu) = \alpha \beta^{-1} \log \mu t_0,
\]

with \(\beta\) given by eq. (3.4) and calculated employing the electron mass. Then, \(S_K = S_F \exp 2i\alpha\beta^{-1}\), and the last factor is the one analogous in QED for \(S_c^{-1}\), already introduced above in connection with the cancellation of the IR singularities associated with the crossed exchange of soft gravitons when calculating a graviton-graviton PWA. This exponent is also the relativistic generalization of the well-known diverging non-relativistic Coulomb phase \(\alpha m/p \log 2\mu/p\). Therefore, our prescription for determining \(S(J)\) in eq. (3.14), by multiplying by \(S_c^{-1} \equiv \exp 2i\Phi\), amounts to removing the diverging phase in the limit \(\mu \to 0\), with a remaining ambiguity due to the unspecified \(t_0\). This type of ambiguity was already discussed in section 3 and explicitly shown in eq. (3.22); it is also clear that \(S(J)\) carries an
ambiguity related to the choice of the scale and which is encapsulated in the dependence on \( \log a \). To avoid repeating ourselves, we refer to the discussion after eq. (3.23) about circumventing this ambiguity and provide physical values to \( \log a \).

4 Unitarization of the graviton-graviton PWAs

One important consequence of the treatment of IR singularities discussed in the previous section is the disappearance of the left-hand cut (LHC) from the partial-wave-projected Born amplitudes. Indeed, the expressions in eqs. (3.24) and (3.25) are free from any cut. As a consequence, the corresponding \( T \) matrix in PWAs, obtained from the iteration of the Born terms, only has the right-hand cut (RHC) or unitarity cut, cf. eq. (3.17). In a sense, the presence of IR divergences eventually simplifies the analytical structure of PWAs.

Because of the Schwarz reflection principle, the discontinuity of the inverse of a PWA across the RHC is given by the right-hand side of eq. (3.18) multiplied by \( 2i \). This enables a simple parametrization of the PWAs which fulfills exact two-body unitarity. To simplify the notation let us focus first on the scattering with \( \lambda = 0 \). The generalization of the result to \( \lambda = \pm 4 \) is straightforward and will be given afterwards. It is enough to introduce an analytical function \( g(s) \) in the complex \( s \)-plane with only the RHC and whose discontinuity along it is the same as that of \( 1/T^{J}(s) \). To build it, let us consider a once-subtracted dispersion relation (DR) by taking the integration contour \( C \) drawn in figure 1 and apply the Cauchy’s integration theorem to \( g(z)/[(z - s)(z + s_0)] \), where \( z \) is the integration variable and \( s_0 \) is the subtraction point. Namely,

\[
\oint_{C} \frac{g(z)}{(z - s_0)(z + s_0)} \frac{dz}{z} = 2\pi i \left( \frac{g(s)}{s + s_0} - \frac{g(-s_0)}{s + s_0} \right).
\]

By adding one subtraction, the integration along the circle at infinity vanishes and we are
left with the following expression for \( g(s) \),

\[
g(s) = a(s_0) - \frac{s + s_0}{8} \int_0^\infty \frac{ds'}{(s' - s)(s' + s_0)},
\]

(4.2)

where \( a(s_0) \equiv g(-s_0) \) is called the subtraction constant. The integration can be done explicitly and we rewrite \( g(s) \) as

\[
g(s) = a(s_0) + \frac{1}{8} \log \frac{s}{s_0}.
\]

(4.3)

Let us note that the derivation of \( g(s) \) takes into account analyticity associated with the RHC [80, 81], so that it has a branch point singularity at \( s = 0 \) and for physical values of \( s \), in addition to the imaginary part required by two-body unitarity, it also has a real part.

In summary, by construction \( g(s) \) only has the RHC with an imaginary part

\[
\Im g(s) = -\frac{\pi}{8} \theta(s), \quad s \in \mathbb{R}.
\]

(4.4)

We then take the following general expression for a PWA \( T^{(J)}_{\lambda_1 \lambda_2, \lambda_1 \lambda_2}(s) \) [82], which automatically fulfills two-body unitarity, eq. (3.17),

\[
T^{(J)}_{\lambda_1 \lambda_2, \lambda_1 \lambda_2}(s) = \left[ \frac{1}{R^{(J)}_{\lambda_1 \lambda_2, \lambda_1 \lambda_2}(s)} + 2|\lambda|/4 g(s) \right]^{-1}.
\]

(4.5)

Unless strictly necessary, in the following we suppress the helicity subscripts of \( T^{(J)} \) and \( R^{(J)} \) for simplicity in the writing. The function \( R^{(J)}(s) \) has no two-body unitarity cut because this is fully accounted for by \( g(s) \), as it is clear by comparing with eq. (3.18). Following refs. [83, 84] we obtain \( R^{(J)}(s) \) by matching \( T^{(J)}(s) \) in eq. (4.5) to its perturbative calculation order by order within the low-energy EFT for gravity. Noting that the contribution from \( g(s) \) enters at \( \mathcal{O}((Gs)^2) \) in the expansion of \( T^{(J)}(s) \) in powers of \( s \), eq. (4.5), one obtains that

\[
R^{(J)} = V^{(J)} + \mathcal{O}\left((Gs)^2\right).
\]

(4.6)

In general \( R^{(J)}(s) \) comprises LHCs due to multi-graviton exchanges, as well as extra RHCs involving intermediate states with more than 2 gravitons. However, the former contributions are \( \mathcal{O}\left((Gs)^2\right) \) and the latter even one order higher \( \mathcal{O}\left((Gs)^3\right) \), which suggests a diminished role of these additional multi-graviton contributions to the unitarity relation of eq. (3.17). Therefore, they can be included perturbatively in \( R^{(J)} \) by the matching procedure with the scattering amplitude calculated in the corresponding EFT, as discussed above. In particular, the extra RHCs due to multi-graviton states beyond the two-body one would give rise to an imaginary part in \( R^{(J)}(s) \) for \( s > 0 \), in analogy to the concept of an optical potential [85] for two-body scattering accounting for extra coupled channels and leading to inelasticity parameters in the corresponding PWA. Further discussions on the expected

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The \( \log z \) is defined with \( \arg z \in [-\pi, \pi] \). If the latter is chosen between \([0, 2\pi)\) then \( g(s) = a(s_0) + \frac{1}{8} \log \frac{s}{s_0} - \frac{i\pi}{8} \).
suppression of the multi-particle state contributions are given at the end of section 5.2 and in section 6.1. There, we additionally elaborate on the numerical enhancement of the \( J = 0 \) graviton-graviton (and isoscalar \( \pi \pi \)) PWA, and on the numerical suppression of the multi-particle phase space, respectively.

For the present case we have just one pole in \( T^{(J)}(s)^{-1} \) from eq. (4.5) because \( V^{(J)}(s) \) has only one simple zero at \( s \to 0 \), as found in general relativity. Scenarios including higher-energy degrees of freedom can be implemented in the dispersion relation by including more poles in \( R^{(J)} \), called Castillejo-Dalitz-Dyson (CDD) poles after ref. [86].

The issue of determining \textit{a priori} a value for the subtraction constant \( a(s_0) \) at the scale \( s_0 \) has been already discussed in QCD. In the absence of fine tuning, the typical value of \( a(s_0) \) should be of the same order of magnitude as its variation under changes of order 1 in the subtraction point \( s_0 \). Notice that the combination \( a(s_0) + \frac{1}{8} \log \frac{s}{s_0} \) is independent of the value of \( s_0 \). This fixes the running of \( a(s_0) \) with \( s_0 \) as

\[
a(s_0) - a(s'_0) = \frac{1}{8} \log \frac{s_0}{s'_0},
\]

where \( s'_0 \) is another subtraction point. Therefore, according to eq. (4.7), we would have that

\[
|a(s_0)| \sim |a(s_0) - a(s'_0)| = \frac{1}{8} \left| \log \frac{s_0}{s'_0} \right| \ll 1.
\]

Hence, an assessment on the smallness of the previous estimate can be inferred by comparing it with \( V^{(J)}(s)^{-1} \). Namely, from eq. (3.26) we have that

\[
\frac{1}{V_{22,22}^{(0)}} = \frac{\pi}{8 G \log a},
\]

which is much larger than \( 1/8 \) for \( s \ll \pi G^{-1} \).

Another method to choose \( a(s_0) \) is adopting the so-called \textit{natural values} for both \( a(s_0) \) and \( s_0 \) [84]. This is based on comparing \( g(s) \) in eq. (4.3) with the same integral regularized by using a cutoff \( \Lambda^2 \), that we call \( g_c(s) \) [87],

\[
g_c(s) = -\frac{1}{8} \int_0^{\Lambda^2} ds' \frac{\log s' - s}{s' - s} = \frac{1}{8} \log \frac{s}{\Lambda^2} + O \left( s/\Lambda^2 \right).
\]

Identifying \( s_0 \) with the cutoff in eq. (4.3) and matching the expressions for \( g(s) \) and \( g_c(s) \), one concludes that

\[
a \left( \Lambda^2 \right) = 0,
\]

which is the same value as the one obtained above with the other method in eq. (4.8). Therefore, in the following we fix \( a(\Lambda^2) = 0 \) (with \( s_0 = \Lambda^2 \)) and one could vary \( \Lambda \) to tentatively determine the uncertainty in our results.

It is also important to give an estimate for the cutoff \( \Lambda \) that, generally speaking, corresponds to the scale suppressing the higher-dimension operators in the EFT [33, 88].

\footnote{This is equivalent to the estimation of higher-order effects of perturbative calculations in QCD or ChPT obtained by variations of the renormalization scale.}
Combining eqs. (4.5) and (4.6) the unitarized amplitude can be expanded to next-to-leading order as,

\[ T^{(J)} = V^{(J)} \left( 1 - \frac{V^{(J)}}{s} \log \frac{s}{\Lambda^2} \right) + O\left((Gs)^3\right). \] (4.12)

A value of the cutoff \( \Lambda^2 \) can be thus estimated by assuming that the correction has the expected size in the EFT expansion, \( s/\Lambda^2 \). For definiteness, let us take \( V^{(0)}(s) \), eq. (3.26), and then for \( |\Re \log(-s/s_0)| \simeq 1 \), we obtain that

\[ s_0 = \Lambda^2 = \pi (G \log a)^{-1} \sim G^{-1}. \] (4.13)

Let us stress again that the unitarization procedure only resums the higher order corrections needed to reproduce exactly the two-graviton cut. In particular, the next-to-leading order (NLO) contribution in eq. (4.12) displays a dependency in \( \Lambda \) that is cancelled by other terms in \( V^{(J)}(s) \) when matching with the perturbative amplitude at NLO, which is UV-finite in pure gravity [33, 89, 90].

Another method to determine the scale of new physics based on unitarity of PWA is considering the violation of perturbative unitarity [31, 49, 91]. This stems from the fundamental relations eqs. (3.15) and (3.16), which allow one to express the PWA in \( J = 0 \) for \( s > 0 \) in terms of a phase shift \( \delta \) and the inelasticity parameter \( \eta \) \( (0 \leq \eta \leq 1) \) as

\[ T^{(0)}_{22,22} = -\frac{4i}{\pi} \left( \eta e^{2i\delta} - 1 \right). \] (4.14)

This leads to the constraint,

\[ |\Re T^{(0)}_{22,22}| \leq \frac{4}{\pi}, \] (4.15)

which is violated by the Born amplitude at,

\[ s_{pu} = \frac{1}{2} (G \log a)^{-1} \sim G^{-1}. \] (4.16)

This is of the same order although somewhat smaller than the unitarity correction in eq. (4.13). Note, however, that \( s_{pu} \) represents the energy at which unitarity corrections are needed to fulfill the constraint (4.15), but they can still be small compared to the leading order. For instance, \( \Re T^{(0)}_{22,22} \) evaluated with the unitarized amplitude at \( s_{pu} \) is only \( \sim 5\% \) smaller than the leading-order contribution from the Born term, which is consistent with a correction of order \( s_{pu}/\Lambda^2 = 1/(2\pi) \).

It is important to point out that cutoffs derived from unitarity considerations within the EFT do not correspond, necessarily, to the fundamental scale of its ultraviolet completion (see e.g. ref. [49]). For gravity, this scale could be as low as the TeV range in scenarios with extra dimensions [66, 92–95] or, for example, gravitation could be properly described by a quantum field theory with an ultraviolet fixed point [96–99] or with new non-trivial dynamics at energies below \( G^{-1/2} \) [100, 101]. In the following we assume that the fundamental scale of gravity is of the same order as the one derived from unitarity considerations. We will discuss later the implications of other scenarios where this is not the case.
Finally, the PWA in eq. (4.5) are defined in the physical sheet, where the physical region is the positive real axis of $s$ approached from above. The structure of the amplitude in the unphysical Riemann sheet (RS) can be obtained by analytical continuation of eq. (4.5). This is particularly relevant for resonances, which are identified with poles in the second RS reached from the physical region by burrowing down through the RHC [69, 102, 103].

5 Connection with hadronic resonances

After having derived the dispersion relations for graviton-graviton scattering, it seems worth pausing and recalling how the same unitarization formalism predicts some non-trivial features of hadronic resonances. This will give more solid basis to some of the methods above, illustrating in a nontrivial way our approach for unitarizing amplitudes and estimating the natural values of $a(s_0)$ and $s_0 \simeq \Lambda^2$. It also serves as motivation towards the possible presence of similar phenomena in gravity, the focus of section 6.

5.1 PW unitarization of $\pi\pi$ scattering in the chiral limit

The lightest resonance in QCD, the $f_0(500)$ or $\sigma$ meson, has vacuum quantum numbers, $J^{PC} = 0^{++}$. This resonance was studied in pion-pion scattering with a unitarization method similar to the one employed here in the pioneering works of refs. [38, 104]. In the chiral limit the pions become massless and the unitarity loop function is the same as in graviton-graviton scattering, eq. (4.3), except for a factor $1/(2\pi^2)$ which appears in the standard normalization.\textsuperscript{10} We call it $g_\pi(s)$ and its expression is

$$g_\pi(s) = \frac{1}{2\pi} g(s) = \frac{1}{(4\pi)^2} \left( a_\pi(s_0) + \log \frac{-s}{s_0} \right) ,$$

with $a(s_0) \rightarrow a_\pi(s_0)/8$. The $S$-wave isoscalar $\pi\pi$ scattering amplitude in the chiral limit at leading order is $s/f_\pi^2$, with $f_\pi$ the pion decay constant. This tree-level amplitude is analogous to $V^{(0)}(s)$, the $J = 0$ graviton-graviton amplitude in eq. (3.26) after curing the IR divergence. The zero at $s = 0$ in the case of $\pi\pi$ scattering is referred to as an Adler zero and follows from the Goldstone theorem. The unitarized expression for the isoscalar scalar $\pi\pi$ PWA in the chiral limit is [105]

$$T^{(0)}_\pi(s) = \left[ \frac{f_\pi^2}{s} + g_\pi(s) \right]^{-1} .$$

A subtraction constant in $g_\pi(s)$ of natural size can be determined by using a cutoff $\Lambda$ as in eq. (4.10). Thus,

$$a_\pi(s_0) = \log \frac{s_0}{\Lambda^2} .$$

In the following $s_0 = \Lambda^2$, then $a_\pi(\Lambda^2) = 0$ and eq. (5.2) simplifies to

$$T^{(0)}_\pi(s) = \left[ \frac{f_\pi^2}{s} + \frac{1}{(4\pi)^2} \log \frac{-s}{\Lambda^2} \right]^{-1} .$$

\textsuperscript{10}This stems from the normalization used for the two-graviton states, cf. eq. (A.8) in appendix A.
We are now interested in the emergence of resonances corresponding to poles in the second RS of the unitarized amplitude eq. (5.4). As discussed above, the second RS is reached by analytical continuation of \( g(s) \), crossing the unitarity cut from the physical region above the real \( s \) axis (\( \Im s > 0 \)) towards the lower half plane with \( \Im s < 0 \) [69, 102, 106]. Thus,

\[
g_{\pi,II}(s) = g_{\pi}(s) - i \frac{1}{8\pi}. \tag{5.6}
\]

The cutoff in ChPT is usually taken as \( \Lambda \simeq 4\pi f_\pi \) which agrees with the one obtained by studying the size of the leading unitarity contribution, as explained above in connection to eq. (4.13). Indeed, this correction is \( (s/f_\pi^2)^2/(16\pi^2) \), that equated to \( s/\Lambda^2 \) times the leading order, \( s/f_\pi^2 \), leads also to \( \Lambda = 4\pi f_\pi \) [88].

5.2 Postdiction of the \( \sigma \) meson

We concluded in the previous section that the graviton-graviton and \( \pi\pi \) scatterings share a similar structure for their DRs. As a motivation of the possible consequences for graviton-graviton scattering, let us first recall how eq. (5.4) postdicts the existence of the \( \sigma \) meson.

The PWA \( T_\pi^{(0)}(s) \) continued to the second RS has a pole in \( s_\sigma \) satisfying the secular equation

\[
\frac{(4\pi f_\pi)^2}{s_\sigma} + \log \left( -\frac{s_\sigma}{\Lambda^2} \right) - 2i\pi = 0. \tag{5.7}
\]

We can rewrite this expression as

\[
\frac{1}{x} + \log(-x) - 2i\pi = 0, \text{ where } x = \frac{s_\sigma}{\Lambda^2}, \Lambda = 4\pi f_\pi. \tag{5.8}
\]

The solution to this equation can be found easily by an iterative method. We first neglect \( \log(-x) \) and obtain that \( x_1 = -i/(2\pi) \). Then, in a second iteration, instead of neglecting the log term, we keep only its imaginary part, \( \log(-x_1) = i\pi/2 + \ldots \), so that

\[
x \simeq -i \frac{3}{2\pi} = -i\, 0.20. \tag{5.9}
\]

In order to keep the desirable and standard property in \( S \)-matrix theory for massive particles that \( g_{II}(s^*) = g_{II}(s)^* \), one should use instead

\[
g_{*,II}(s) = g_{*,}(s) + i \frac{1}{8\pi} \sqrt{\frac{s - 4\mu^2}{s}} \tag{5.5}
\]

where the \( \sqrt{z} \) is evaluated for \( \arg z \in [0, 2\pi) \), and then the limit \( \mu \to 0 \) has to be taken at the end of the calculation. With this definition \( g_{*,II}(s) \) is real for \( s \in [0, 4\mu^2] \) and, therefore, it satisfies the Schwarz reflection principle. Nonetheless, we do not insist on this point and conform ourselves with the simpler eq. (5.6) because for \( \Im s < 0 \) they give the same result, and the resonance poles lie in complex conjugate positions. As it actually happens if the proper eq. (5.5) were used for \( g_{*,II}(s) \).

Note that this coincidence between the fundamental and unitarity scales in ChPT is rather exceptional and breaks down for other values of the number of colors \( N_c \) in the large-\( N_c \) QCD (see ref. [49] and discussion below).
A direct numerical evaluation of eq. (5.8) gives

\[ x = 0.07 - i 0.20 \]  

(5.10)

This pole is located deep in the complex \(s\)-plane and has a small real part. Therefore, the dynamics of \(S\)-wave \(\pi\pi\) scattering are strongly influenced by this resonance even at energies significantly lower than the cutoff of the EFT. In table 1 we show the poles in the variable \(\sqrt{s_\sigma}\) (which is the one quoted in phenomenology) and units of GeV that are obtained numerically by solving eq. (5.7) using the nominal value of the cutoff \(\Lambda = 4\pi f_\pi\) and the value of \(f_\pi\) in the chiral limit \(f_\pi \approx 86.1\) MeV [107]. We also investigate variations of the results by changing the cutoff by a factor 2 around the nominal value, which is a range exaggerated compared to the actual one constrained by ChPT phenomenology. Nonetheless, the pole position has a very mild dependence on it, changing only by \(\sim 20\%\).

The previous prediction was based on the chiral limit of massless pions. For physical (massive) pions the isoscalar scalar \(\pi\pi\) PWA at leading order in chiral perturbation theory is \((s - m_\pi^2/2)/f_\pi^2\), and the unitarity loop function becomes

\[ g_\pi(s) = \frac{1}{(4\pi)^2} \left( \log \frac{m_\pi^2}{\Lambda^2} + \sigma(s) \log \frac{\sigma(s) + 1}{\sigma(s) - 1} \right), \]  

(5.11)

with \(\sigma(s) = \sqrt{1 - 4m_\pi^2/s}\). The unitarized amplitude is now

\[ T^{(0)}_\pi(s) = \left[ \frac{f_\pi^2}{s - m_\pi^2/2} + g_\pi(s) \right]^{-1}, \]  

(5.12)

and its analytical continuation to the second RS unveils the poles shown for the “physical point” in table 1, where we have used the physical values for the pion mass, \(m_\pi = 139\) MeV, and the decay constant \(f_\pi = 92.4\) MeV. As in the chiral limit, the pole position shows little sensitivity to the value of the cutoff. It is certainly remarkable that these results, completely fixed by chiral dynamics, unitarity and analyticity, agree with the experimental range quoted by the Particle Data Group (PDG), \(\sqrt{s_\sigma} = (400 - 550) - i(200 - 350)\) MeV [37] and, within a 10%, with more sophisticated theoretical determinations based on dispersion relations [41], \(\sqrt{s_\sigma} = 457 \pm 14 - i 279 \pm 11\) MeV. In particular, the pole position evaluated at the physical point and using the nominal cutoff of ChPT, \(\Lambda = 4\pi f_\pi\), is completely consistent with this determination.

Let us now discuss the role of the RHCs of the \(4\pi\) channel and heavier multipion states in the unitarity relation of pion-pion scattering. These contributions are typically neglected.

|               | \(\Lambda/2\) | \(\Lambda\)  | \(2\Lambda\)  |
|---------------|---------------|---------------|---------------|
| Chiral limit  | 0.36 - i0.35  | 0.41 - i0.29  | 0.43 - i0.22  |
| Physical point| 0.41 - i0.33  | 0.46 - i0.26  | 0.47 - i0.18  |

Table 1. Lightest pole positions in the variable \(\sqrt{s_\sigma}\) and in units of GeV of the \(\sigma\)-resonance in QCD and using the unitarized \(S\)-wave \(\pi\pi\) PWA at leading order (Born term) in ChPT. We show results for the chiral limit and the physical point, and for the nominal cutoff of the EFT, \(\Lambda = 4\pi f_\pi\), and variations of factor 2.
when describing the data or even nonperturbative calculations in lattice QCD [108–110]. When included, they have been shown to be small [21]. We start by presenting two reasons why this suppression is not related to the distance of the heavier multipion thresholds to the $2\pi$ one. First, the $4\pi$ cut starts at $\sqrt{s} \sim 4m_{\pi} \sim 0.5 \text{GeV}$ which is right on top of the $\sigma$ contribution. Second, we do not only account for the imaginary part associated to unitarity, cf. eq. (3.17) and the analogous one for $\pi\pi$ scattering, but also for the analytical properties that follow from it. Therefore, heavier multi-pion states would have also an impact in the unitarization process because the functions describing the corresponding RHCs are analytic and have real parts below their thresholds that follow from analyticity.

There are two reasons why the resummation of the two-body RHC is by far much more important than the higher orders from these multi-pions: i) The scalar isoscalar $\pi\pi$ scattering amplitude at LO has a numerical enhancement as its PWA is a factor 6 larger than the isovector vector one [82]. This makes the resummation for the isoscalar scalar case resonant, and generates the $\sigma$ with a natural value for the cutoff (see below for the case of the $\rho(770)$ resonance in the isovector-vector channel); ii) the multi-pion states give corrections to $\pi\pi \to \pi\pi$ scattering that are suppressed by extra powers of $(s/\Lambda)^L$, with $L$ the numbers of loops that increases as $n - 1$ for an intermediate $n$-pion state. In this way, the $4\pi$ channel requires at least three loops. However, the same and higher number of loops that stem from the $2\pi$ channel are not suppressed because of the enhancement mentioned, so that $|g(s) s/f_{\pi}^2| \sim 1$ for $|s| \gtrsim |s_{\sigma}|$. The power-counting suppression [88] can be related to a general argument extracted from [111] on a strong parametric suppression of the multi-particle phase space in the contributions to the unitarity relation of the corresponding RHC, which is developed in detail below in section 6.1 for the graviton-graviton case. In fact, note that the suppression factors apply also to the chiral limit of QCD (with massless pions) and multi-pion states give very small contributions below the cutoff of the theory, despite all the associated thresholds collapse at $s = 0$.

To further appreciate our framework as a way to obtain resonances that are dynamically generated by the degrees of freedom considered, let us briefly discuss what happens when extending this procedure to other PWAs. In particular, in the $\pi\pi$ $P$-wave scattering amplitude one identifies experimentally the $\rho(770)$ resonance. However, unitarizing as before the leading $\pi\pi$ scattering amplitude in this channel one only obtains a pole consistent with the $\rho(770)$ using unnatural values for the cutoff, $\Lambda \sim 1 \text{ TeV}$ [106]. Thus, this resonance cannot be explained dynamically using pions as explicit degrees of freedom, clearly indicating that the $\rho$ and the $\sigma$ are resonances of very different nature. This difference can be examined by varying the number of colors $N_c$ in QCD while letting its coupling constant scale as $1/\sqrt{N_c}$ to ensure a smooth large $N_c$ limit [112, 113]. The meson decay constant scales as $f_{\pi} \sim O(\sqrt{N_c})$ and, therefore, the position of the pole of the $\sigma$ resonance is very sensitive to the value of $N_c$, becoming heavier and broader as $N_c$ increases [82, 114]. On the other hand, fundamental quark-antiquark mesons such as the $\rho$-resonance become stable particles with a definite $O(1 \text{ GeV})$ mass in the large-$N_c$ limit [113, 114]. In summary, while some resonances are naturally produced by unitarizing the dynamics of the low-energy degrees of freedom in the EFT, other states are “elementary” and they encode information about the UV completion. As discussed above in section 4, these states can be added as CDD
poles to the inverse of the amplitude, which requires using additional experimental (or theoretical) information for their positions and residues. This distinction between nature of resonances is also very relevant for gravity. In addition to our results we refer to the related self-completeness scenario advocated in ref. [63] for gravity, in which a tower of black holes is required by unitarity while other “elementary” states would have mass near the cutoff of the theory. See also the related refs. [65, 66].

The unitarized ChPT has found many other successful applications in the strong regime of QCD. A remarkable example concerning the hadronic spectrum is the $\Lambda(1405)$ resonance in $S$-wave meson-baryon scattering with strangeness $-1$ [84, 115, 116]. This resonance was already predicted in ref. [117] but, only after the unitarization of the chiral EFT PWAs, it was understood that it corresponds to two poles [84]. This important topic deserves a separate review in the PDG [37].

6 Resonances of the graviton-graviton scattering: the graviball

We proceed now with the application of eqs. (4.5) and (4.6) to study graviton-graviton scattering in PWAs. The latter were obtained by unitarizing the Born scattering amplitude, properly modified to take care of the IR divergences, as discussed in section 3. Specifically, we consider the emergence of resonances in these PWAs, which correspond to poles in the second RS. Needless to say, we do not find bound-state poles in the physical RS. In terms of the analytical continuation of $g(s)$ for graviton-graviton scattering into the second RS

$$g_{II}(s) = g(s) - i\frac{\pi}{4},$$

the PWAs in this sheet, $T^{(J)}_{II;\lambda_3\lambda_4,\lambda_1\lambda_2}(s)$, read

$$T^{(J)}_{II;22,22}(s) = \left[V^{(J)}_{22,22}(s)^{-1} + g_{II}(s)\right]^{-1},$$
$$T^{(J)}_{II;2-2,2-2}(s) = \left[V^{(J)}_{2-2,2-2}(s)^{-1} + 2g_{II}(s)\right]^{-1}.$$

Let us discuss the pole positions in $J = 0$, for which $V^{(0)}(s)$ is given in eq. (3.21). We suppress from now on the helicity indices because there is a contribution to this PWA only when all the graviton helicities are equal. Indeed, in the other possible configurations with helicities $\pm 2$ in the initial and final states angular momentum conservation requires that $J \geq 4$. The PWA $T^{(0)}_{II}(s)$ has a pole in the second RS at $s_P$, whose position satisfies the secular equation

$$\frac{\pi}{Gs_P \log a} + \log \frac{-s_P}{\Lambda^2} - 2\pi = 0.$$

We rewrite the previous equation in terms of dimensionless variables as

$$\frac{1}{\omega x} + \log(-x) - 2\pi = 0,$$

where

$$x = \frac{s_P}{\Lambda^2}, \quad \omega = \frac{\Lambda^2 G \log a}{\pi}.$$
Figure 2. The moduli squared of $T^{(0)}(s)$ (left) and $1/D^{(0)}(s)$ (right panel) are plotted for $J = 0$ as a function of $s$ in units of $\Lambda^2$. The presence of the resonance pole is clearly appreciated in $|1/D^{(0)}(s)|^2$ for values of $s$ much lower than $|s_P|$. The resonance shape is strongly distorted for $|T^{(0)}(s)|^2$ because of the zero at $s = 0$. The solid(dashed) lines correspond to the full(perturbative) results.

When the cutoff derived from unitarity in eq. (5.3) is chosen this equation becomes the same secular equation as for the scattering of massless pions, eq. (5.8). Hence, we find a resonance with the quantum numbers of the vacuum in the gravitational EFT which is analogous to the $\sigma$ resonance of QCD,

$$s_P = (0.07 - i 0.20) \Lambda^2 \simeq -i \frac{2}{3\pi} \Lambda^2,$$

(6.5)

where $\Lambda^2 = \pi(G \log a)^{-1}$. Following the analogy with glueballs, we will call this resonance the *graviball*.

The pole position $s_P$ is almost purely imaginary, with a real part that is smaller, by approximately a factor 3, than its imaginary part. Let us write $s_P$ as $s_P = (\varkappa - i2/(3\pi))\Lambda^2$, where $\varkappa \Lambda^2$ is the real part of $s_P$. A Laurent-series expansion around the pole at $s_P$ is dominated by the pole-term contribution,

$$\frac{1}{s - s_P} = \frac{1}{s - \varkappa \Lambda^2 + i\frac{2\Lambda^2}{3\pi}},$$

(6.6)

whose modulus squared is

$$\frac{1}{|s - s_P|^2} = \frac{1}{(s - \varkappa \Lambda^2)^2 + \frac{4\Lambda^4}{9\pi^2}}.$$

(6.7)

Remarkably its peak is located at $\varkappa \Lambda^2 \ll \Lambda^2$, because $|\varkappa| \ll 1$, and its strength spreads over a wide region. For instance, the point in $s$ at which its value is half the value at the peak is $s \sim 2\Lambda^2/(3\pi)$. It is also asymmetric in the physical axis for $s > 0$ because its width is much larger than the position of the peak.

The pole-term contribution in eq. (6.6) is distorted in the shape of $|T^{(0)}(s)|^2$ because of the zero at $s = 0$, which implies an energy-dependent driving factor $s^2$. As a result $|T^{(0)}(s)|^2$ has no resonance shape as shown in the left panel of figure 2. This situation is, again, analogous to the $\sigma/f_0(500)$ resonance in pion-pion scattering in QCD [118] because of the Adler zero, cf. eq. (5.2).
The resonance effects manifest themselves more clearly in the modulus squared of the function that drives the final- and initial-state interactions in the corresponding graviton-graviton PWA. For instance, one could think of a situation where multiple gravitons are produced by an energetic source so that a pair of gravitons with quantum numbers \((J, \lambda_{1-4})\) rescatter with certain energy. The function that controls the exchange of graviton-graviton states with such quantum numbers, or from external sources that couple to this system, \(1/D_{\lambda_3 \lambda_4, \lambda_1 \lambda_2}(s)\), is given by \[106\] (also called an Omnès function)

\[
\frac{1}{D_{2\pm,2\pm,2}(s)} = \left[ 1 + 2^{4/4} V_{2\pm,2\pm,2}(s) g(s) \right]^{-1},
\]

where \(\lambda = \lambda_1 - \lambda_2\). Notice also that \(1/D(J) = T(J)/V(J)\).

We show in the right panel of figure 2 the modulus squared of \(1/|D^{(0)}(s)|^2\) as a function of \(s\) in units of \(\Lambda^2\). We clearly observe a resonance shape with a peak driven by the pole \(s_P\) in the complex \(s\)-plane, eq. (5.9). The peak in \(1/|D^{(0)}(s)|^2\) is located at \(s \simeq 0.1\Lambda^2\), which is much smaller than the modulus of \(s_P\), but displaced towards larger values than \(s\Lambda^2\). Of course, in the Laurent expansion of \(1/D(J)(s)\) there are other contributions in addition to just the pole-term one (this is discussed explicitly for the \(\sigma\) resonance in refs. \[118, 119\]).

The value around 1 of \(1/|D^{(0)}(s)|^2\) at the peak, as shown in figure 2, can be understood by identifying the residue of \(1/D^{(0)}(s)\) at \(s_P\). Then, the isolated pole-term contribution to \(1/|D^{(0)}(s)|^2\) is approximately \((4/(9\pi^2))/((s/\Lambda^2)^2 + 4/(9\pi^2))\), whose maximum value is 1.

Therefore, we conclude that the physical effects of the pole could be felt at values quite lower than \(\Lambda^2\) because its resonant peak occurs at \(s \ll \Lambda^2\). Needless to say that a pure perturbative expansion within the EFT of gravity could be severely affected by this fact.\(^{13}\)

To illustrate this point let us consider the first two terms in the expansion of \(1/D^{(0)}(s)\) in powers of \(s\), which we call \(1/D_2^{(0)}(s)\),

\[
\frac{1}{D_2^{(0)}(s)} = 1 - \frac{s}{\Lambda^2} \log \frac{s}{\Lambda^2}.
\]

We compare in figure 2, right panel, the modulus squared of this function with \(1/|D^{(0)}(s)|^2\) in the lower-energy region up to \(s/\Lambda^2 = 0.5\). The former is indicated by the dashed line and the latter by the solid one. For negative \(s\) the series expansion is much better behaved. However, the difference increases rapidly for positive \(s\), where the peak lies, so that it rises from near a 1% at \(s/\Lambda^2 = 0.06\) o more than 100% at \(s/\Lambda^2 \gtrsim 0.2\). In addition, \(|1/D_2^{(0)}(s)|^2\) completely fails to provide the right resonant shape of \(1/|D^{(0)}(s)|^2\) as a function of \(s/\Lambda^2\) above threshold. We will comment more on possible phenomenological consequences in section 9, though we leave a more detailed analysis for future work. For clarification, with the dashed line in the left panel of figure 2 we actually plot \(|V^{(0)}(s)/D_2^{(0)}(s)|^2\).

\(^{13}\)For instance, this has been the case for low-energy \(\pi\pi\) scattering where the isoscalar scalar \(\pi\pi\) phase shifts receives large higher-order corrections which amount around a 40% of the leading result, while they are negligible for the isotensor scalar scattering length \[120\]. The nucleon \(\sigma\) term is also strongly affected by the non-perturbative physics due to the \(\sigma\) resonance because of the pion-scalar form factor through the \(\Delta_\sigma\) term \[121\].
We do not give the pole positions for $J = 2$ because the partial-wave projected Born amplitude given in eq. (3.27) is much weaker than for $J = 0$, so that the poles lie too deep inside the complex $s$-plane to consider them reliable. Given the fact that, as $J$ increases, the corresponding pole positions move further away from the physical $s$ axis than the discussed pole for $J = 0$, we stress the role of the latter in our exploratory purposes of this work. This trend will be also manifest in the AC toy-model of scattering, developed in section 7, with higher $J$ poles moving deeper in the complex plane as $J = \ell$ increases.

6.1 On the robustness of the graviball

In the analysis of the graviball done above we have assumed that the cutoff of gravity is close to the unitarity one. Namely we have taken $\omega = 1$ in eq. (6.4) leading to a resonance in gravity resembling the $\sigma$ of QCD. The difference between the two theories, however, is twofold: first, the IR divergences generated the parameter $\log a$ in our expressions; second, for gravity we do not know the fundamental cutoff of the EFT and the parameter $\omega$ can be significantly different from 1.

Changing $\omega$ only slightly, we find that $s_P$ is rescaled by the same amount, as can be easily deduced from the derivation of eq. (6.5). Increasing $\omega$ from 1 moves the pole closer to the real $s$-axis and the resonance becomes narrower. One can also check that for $\omega \gtrsim 10$ the real and imaginary parts of the pole are of the same order of magnitude and its absolute value monotonically decreases with $\omega$ reaching the limit $x \to 0$ when $\omega \to \infty$.

An attempt to move towards large values of $\omega$ may be to add new ‘light’ degrees of freedom to the problem. In fact, the strength of the interaction effectively increases when adding new fields because their corresponding two-particle cuts will contribute as intermediate states to $g(s)$ [32, 47, 122]. In eq. (4.5) we already noticed a factor 2 multiplying $g(s)$ for $\lambda = 4$ because the two channels combining helicities $\pm 2$ are coupled with each other. At the level of the denominator of the PWA, where $1/V^{(J)}(s)$ enters, this is equivalent to multiplying the coupling $G$ by the same factor and leaving $g(s)$ untouched. Thus, we can parameterize the strength of the gravitational interaction by replacing $\log a$ by $N \log a$ in eqs. (6.3) and (6.4), where $N$ is the number of channels coupled to the two gravitons in $J = 0$, assuming that all the corresponding fields have masses much lower than $G^{-1/2}$, so that the same $g(s)$ results for these new channels. Therefore, a (naive) realization of the case $\omega \gg 1$ would correspond to increasing $N \gg 1$ while keeping the fundamental cutoff of the theory fixed.

However, this might not be the case. As shown in ref. [31] the scale at which perturbative unitarity is violated in the EFT of quantum gravity coupled to $N_S$ scalar, $N_V$ vector and $N_f$ fermion light fields, is $20 (GN)^{-1}$, with $N = \frac{2}{3} N_S + N_f + 4 N_V$. This may indicate a similar reduction of the fundamental scale of gravity. Note also that a similar reduction has been suggested from other non-perturbative arguments based on black-hole physics, see e.g. refs. [14, 65, 122–125]. In this scenario, changes in $\Lambda^2$ due to the coupling of gravity to lighter fields, as those in the Standard Model, would not modify the value of the ratio $x = s_P/\Lambda^2$ within our approach. In absolute terms, this means that the resonance pole becomes narrower and lighter as $N$ increases, though the relation to the cutoff of the theory remains basically untouched. On the other hand, scenarios with $\omega \gg 1$, corresponding to
theories with fundamental cutoff larger than the unitarity cutoff, may be achieved within the so-called self-healing mechanism of ref. [49]. In this case, the broken perturbative unitarity is “healed” by a resummation yielding a unitary non-perturbative result. There are also theories in which gravitation could be properly described by a quantum field theory with an ultraviolet fixed point [96–99]. It would be interesting to explore the role of unitarity and the in these types of theories.

For \( \omega < 1 \) the pole moves further out in the complex \( s \)-plane and for \( \omega \lesssim 0.2 \) its position is \( |x_P| \gtrsim 1 \), where our approach rooted in the EFT becomes more model dependent. Notice that assuming \( \log a \sim O(1) \) then \( \omega < 1 \) is realized in any scenario of (pure) gravity with a fundamental scale \( \Lambda^2 \) lower than \( \pi G^{-1} \). Nonetheless, as discussed above, even in this last case one could restore that the pole position \( s_P \) is below the fundamental cutoff \( \Lambda^2 \) by increasing the number of light degrees of freedom in hypothetical scenarios in which \( \Lambda \) stayed fixed (as in the presence of self-healing) or, at least, decreased with \( N \) less strongly than \( 1/N \).

Finally, we discuss the impact of the RHC of multi-graviton states in the unitarity relation by showing the suppression of their phase space referred to above in the context of \( \pi \pi \) scattering in QCD. Following the argument of ref. [111], dedicated to estimate the different sources of uncertainties that would affect unitarization methods, one finds that the expression for the \( n \)-body phase-space for massless particles with helicity 2 is

\[
\phi_n = \frac{2^{n-1}(Gs)^{n-2}}{(4\pi)^{2n-3}\Gamma(n)\Gamma(n-1)}.
\]  

(6.10)

In the numerator the factor \( (Gs)^{n-2} \) stems from the additional graviton fields in the interactions. This phase space is dimensionless and then one can straightforwardly compare between phase spaces with different number of particles. Taking then \( Gs \) between 0 and 1, the relevant region for the low-energy EFT of quantum gravity we then have a very fast decrease of \( \phi_n \) with \( n \),

\[
\phi_n \leq \frac{32\pi^3}{2^{3n}n^{2n}(n-1)!(n-2)!}.
\]  

(6.11)

where we have taken \( Gs = 1 \). For instance, comparing this formula for \( n = 3 \) and \( n = 2 \) one obtains a relative suppression of \( \sim 1/16\pi^2 \simeq 0.006 \). This is a parametric suppression of multi-body phase space since \( s \) is of the order of the cutoff \( \Lambda^2 \). Along these lines, we could also take \( Gs \to \pi \), since the unitarity cutoff was identified with \( \Lambda^2 = \pi(G\log a)^{-1} \), and \( \log a \simeq 1 \) (as discussed in section 8.6). Even in such a case the suppression for \( \phi_3/\phi_2 \sim 1/16\pi \simeq 0.02 \). Note also that this suppression strengthens as \( n \) increases because of the rapidly growing factors \( (n-1)!(n-2)! \) in the denominator.\(^{14} \) This clearly indicates that multi-particle state contributions in the unitarity relation are strongly suppressed for the energy region below the Planck scale \( G^{-1} \), which supports our claim of treating them perturbatively as higher orders in \( Gs \).

\(^{14} \)Note that in ChPT, and other similar EFTs, one generates factors of \( 1/n! \) by expanding the monomials in the number of pion fields (here \( n_i \) is the number of pions coming from a giving structure inside the monomial). These factors imply that the scattering amplitudes of 2 into \( n \) pions do not grow as \( n! \), so that such growth could compensate the factors \( 1/(n-1)!(n-2)! \) in the unitarity relation.
In summary, multi-particle (>2) contributions to elastic graviton-graviton scattering at energies below the cutoff can be naturally incorporated in the perturbative expansion of \( R^{(J)} \), cf. eq. (4.5), as higher orders in the EFT. In the unexpected event that these were numerically enhanced (becoming rapidly less likely with \( n \), as noticed above), then one should proceed with their resummation along with the 2-body contributions within a coupled channel formalism, investigating not only the impact in the properties of the graviball but also on the more than likely appearance of new poles below the cutoff in the \( 2 \to 2 \) and \( 2 \to n \) scattering amplitudes.

Regarding other sources of higher orders in the derivative low-energy EFT expansion of graviton-graviton scattering, we considered in ref. [16] the monomials involving the product of three and four Riemann tensors,\(^{15}\) respectively. The former monomial (called \( R^3 \)) has six derivatives and the latter (denoted by \( R^4 \)), actually there are three of them [126], have eight derivatives. However, there is no contribution from \( R^3 \) to the scattering amplitude \( F_{22,22} \) [127]. Hence, the only non-vanishing contributions to the graviball pole position stems from the \( R^4 \) terms, and the explicit expression for the contribution to \( F_{22,22}(s,t,u) \) from this source, that we call \( F_{R^4,22,22}(s,t,u) \), is [126]

\[
F_{R^4,22,22}(s,t,u) = \frac{\tilde{\beta} \kappa^2}{\pi} s^4 , \tag{6.12}
\]

This amplitude is purely \( S \) wave and its \( J = 0 \) partial-wave projection, \( V^{(0)}_{R^4,22,22} \), is

\[
V^{(0)}_{R^4,22,22} = \frac{8\tilde{\beta} G}{\pi^2} s^4 = \frac{8\tilde{\beta}}{\pi \log a \Lambda^2} s^4 , \tag{6.13}
\]

where we have used \( \omega = 1 \). By naive-dimensional analysis \( \tilde{\beta} \sim \Lambda^{-6} \) so that, up to \( \mathcal{O}(1) \) factors, eq. (6.13) is finally written as

\[
V^{(0)}_{R^4,22,22} = \frac{32}{\pi \log a} s^4 . \tag{6.14}
\]

The kernel \( R^{(0)} \) in eq. (4.6) acquires an extra term, and it becomes now

\[
R^{(0)}_{22,22}(s) = \frac{8s}{\Lambda^2} + \frac{32s^4}{\pi \Lambda^4} + \ldots \tag{6.15}
\]

with \( \log a = 1 \), and the ellipsis indicates other higher-order terms starting from one-loop contributions. The secular equation that follows from eq. (4.5) with \( g(s) \) in the second RS, cf. eq. (6.1), is

\[
\left( x + \frac{4x^4}{\pi} \right)^{-1} + \log(-x) - i2\pi = 0 , \tag{6.16}
\]

which gives \( s_P = (0.07 - i0.21) \Lambda^2 \). The resulting corrections are very small, around a 3%, which is in agreement with the expectation of a relative uncertainty of order \( |x|^3 \sim 1\% \) in \( s_P \) for an operator with eight derivatives. This is of course much smaller than the leading loop corrections that are expected to stem from the one-loop contributions, with relative uncertainty \( \mathcal{O}(|x|) \) in \( s_P \).

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\(^{15}\)The monomials of two Riemann tensors do not contribute to the \( S \)-matrix of pure gravity, see e.g. [126] and refs. therein.
7 IR divergences and unitarization of the exactly soluble AC model

Let us illustrate the method for the treatment of the IR singularities in section 3 for the crossed-channel exchange of massless force-carriers by applying them to an exactly solvable model in Quantum Mechanics that shares some similarities with gravitational scattering. For this, we introduce a modified Coulomb potential, with a zero in momentum space for $p \to 0$, vanishing as $p^2$. This mimics the behavior of graviton-graviton scattering. This feature is also shared by $S$- and $P$-wave scattering of pions in low energy QCD and in the chiral limit. For the actual pion masses the zero of the $S$-wave shifts by an amount of $\mathcal{O}(m_{\pi}^2)$, with $m_{\pi}$ the pion mass, and it is called an Adler zero [128, 129], cf. (5.12). This is the inspiration for the name Adler-Coulomb (AC) scattering for the toy model that we want to discuss now.

The energy-dependent AC potential is defined as

$$V^{AC}(r) = \frac{\alpha E^2}{r M^2}, \quad (7.1)$$

where $r$ is the interparticle distance, $E = p^2/2m$ is the non-relativistic kinetic energy, $m$ is the reduced mass, $\alpha$ is a dimensionless constant and $M$ is some (ultraviolet) scale introduced for dimensional reasons.

The scattering by an AC potential can be solved exactly because it differs from the pure Coulomb one only by the energy dependent factor $E^2/M^2$. The energy enters parametrically in the differential Schrödinger equation, so that we can use the exact solutions in Quantum Mechanics for the Coulomb scattering and then make the replacement

$$\alpha \to \alpha \frac{E^2}{M^2}. \quad (7.2)$$

The kinematics for non-relativistic scattering is different from that discussed in section 2.1, cf. appendix B. In particular, the relation between the PWA $T_\ell(p^2)$ and the $S$ matrix $S_\ell(p^2)$ is

$$S_\ell = 1 + i \frac{m p}{\pi} T_\ell. \quad (7.3)$$

As a result, the unitarity relation for a PWA $T_\ell(p^2)$ reads

$$\Im T_\ell(p^2) = \frac{m p}{2\pi} |T_\ell(p^2)|^2. \quad (7.4)$$

A detailed account of scattering in partial waves in non-relativistic scattering can be found in ref. [130], whose conventions we used in the previous expression.

Let us first recall that the Coulomb wave function with orbital angular momentum $\ell$ tends asymptotically, for $r \to \infty$, to [70]

$$u_\ell^C(r) \sim A \sin \left( pr - \frac{\ell \pi}{2} + \sigma_\ell^C(p) + \gamma \log 2pr \right), \quad (7.5)$$

\footnote{For definiteness, one can consider the Hermitian operator

$$V^{AC}(r) = -\frac{\alpha}{r M^2}. \quad (7.6)$$}
where $\gamma$ is the Sommerfeld parameter

$$\gamma = \frac{am}{p},$$  \hspace{1cm} (7.6)

and $\sigma^C_\ell(p)$ is the Coulomb phase shifts. On the other hand, for a finite-range potential the asymptotic wave function is

$$u_\ell(r) \sim A \sin \left( pr - \ell \frac{\pi}{2} + \delta_\ell(p) \right).$$  \hspace{1cm} (7.7)

The difference between the asymptotic forms in eqs. (7.5) and (7.7) is clear. In the case of an infinite-range potential the phase shifts are ill-defined because of the $r$ dependence stemming from the contribution $\gamma \log 2pr$ in the phase of $u^C_\ell(r)$. Precisely, the Coulomb phase is defined once the diverging phase $\log 2pr$ for $r \to \infty$ is removed. The removal of this phase corresponds to the multiplication of the Dyson $S$ matrix for finite-range interactions by $S^{-1}$, so as to remove the IR divergences associated to the infinite-range interactions as discussed in section 3.

The Coulomb phases are connected to the Coulomb $S$ matrix, $\mathcal{S}^C_\ell$, by

$$\mathcal{S}^C_\ell(p^2) = e^{2i\sigma^C_\ell(p)} = \frac{\Gamma(1 + \ell - i\gamma)}{\Gamma(1 + \ell + i\gamma)},$$  \hspace{1cm} (7.8)

so that

$$\sigma^C_\ell(p) = \arg \Gamma(1 + \ell - i\gamma).$$  \hspace{1cm} (7.9)

Taking into account the rule of eq. (7.2), it follows that the asymptotic wave function in the AC model is

$$u^{AC}_\ell(r) \sim A \sin \left( pr - \ell \frac{\pi}{2} + \sigma^{AC}_\ell(p) + \frac{p^4}{(2mM)^2} \gamma \log 2pr \right),$$  \hspace{1cm} (7.10)

with $\sigma^{AC}_\ell(p)$ given now by

$$\sigma^{AC}_\ell(p) = \arg \Gamma \left( 1 + \ell - i\frac{p^4}{(2mM)^2} \gamma \right),$$  \hspace{1cm} (7.11)

corresponding to the AC-model $S$ matrix, $\mathcal{S}^{AC}_\ell$, given by

$$\mathcal{S}^{AC}_\ell(p) = e^{2i\sigma^{AC}_\ell(p)} = \frac{\Gamma \left( 1 + \ell - i\frac{p^4}{(2mM)^2} \gamma \right)}{\Gamma \left( 1 + \ell + i\frac{p^4}{(2mM)^2} \gamma \right)}.$$  \hspace{1cm} (7.12)

Let us now try to unveil some of the non-perturbative properties of (7.12) by using unitarity methods. For this, let us consider a version of the AC potential screened for distances $r > R$ (such that $pR \gg 1$ and taking at the end the limit $R \to \infty$),

$$V^{AC}(r) = \frac{E^2}{M^2} \frac{\alpha}{r} \theta(R - r).$$  \hspace{1cm} (7.13)
The previous potential is analogous to introducing a Yukawa potential with a non-vanishing photon mass [131]. The introduction of a sharp IR cutoff is indeed a technique used to solve numerically the Coulomb potential in partial waves [70], because with this finite-range potential one can solve for the wave function \( u_\ell(r) \) (and hence for \( \delta_\ell(p) \)), and match with the asymptotic behavior of the Coulomb wave function \( u_\ell^C(r) \) at a distance \( R \). As a result, the pure Coulomb phase shifts \( \sigma_\ell^C(p) \) stems from the relation

\[
\sigma_\ell^C(p) = \delta_\ell(p) - \gamma \log 2pR. \tag{7.14}
\]

Another useful feature is that one can proceed purely in momentum space and solve for the Lippmann-Schwinger equation in the presence of another short-range potential, e.g. due to strong interactions, get \( \delta_\ell(p) \), remove from it the divergent phase \( \gamma \log 2kR + \sigma_\ell^C(p) \) and finally take the limit \( R \to \infty \). This is the standard procedure in nuclear physics to calculate the strong phase shifts in the presence of a Coulomb potential in momentum space [70].

The Fourier transform of the AC screened potential in eq. (7.13) is

\[
V^{AC}(q^2) = \frac{p^4}{(mM)^2} \frac{\pi \alpha}{q^2} \left( 1 - \cos qR \right). \tag{7.15}
\]

where \( q = p' - p \) is the momentum transferred in the scattering events and \( q^2 = 2p^2(1 - \cos \theta) \). From the form of the asymptotic wave function \( u_\ell^{AC}(r) \) one concludes that the diverging phase \( S_\ell^{AC}(p^2) \) is

\[
S_\ell^{AC}(p^2) = e^{2i \frac{p^4}{(2mM)^2} \gamma \log 2pR}. \tag{7.16}
\]

Then, at \( \mathcal{O}(\alpha) \) it implies the following diverging contribution to any partial-wave projected Born amplitude

\[
\delta F_\ell^{AC}(p^2) = \frac{p^2 \pi \alpha}{2(mM)^2} \log 2pR. \tag{7.17}
\]

This contribution is independent of the PWA, as it should, and has to be removed from the partial-wave projected Born amplitudes. Let us work out explicitly the PWA with \( \ell = 0 \). The projection in \( S \)-wave of \( V^{AC}(q^2) \) is

\[
F_0^{AC}(p^2) = \frac{1}{2} \int_{-1}^{+1} d\cos \theta \frac{p^4 \pi \alpha}{(mM)^2 q^2} \left( 1 - \cos qR \right) = \frac{p^2 \pi \alpha}{2(mM)^2} \left[ \gamma_E - \text{ci}(2pR) + \log(2pR) \right] \tag{7.18}
\]

\[
= \frac{p^2 \pi \alpha}{2(mM)^2} \left[ \gamma_E + \log 2pR \right] + \mathcal{O}(R^{-2}),
\]

where \( \gamma_E \) is the Euler constant and, as expected, it diverges as \( R \to \infty \). After removing the diverging term of eq. (7.17), which is analogous to eq. (3.20) for graviton-graviton scattering, we are left with the final expression for the \( S \)-wave projected Born amplitude, \( V_0^{AC}(p^2) \),

\[
V_0^{AC}(p^2) = \frac{p^2 \pi \gamma_E \alpha}{2(mM)^2}. \tag{7.19}
\]
It is remarkable that the right $S$-wave Born amplitude is obtained in this way, as it can be concluded by comparing with the expansion in powers of $\alpha$ of $\Theta^A_\ell(p^2)$, eq. (7.12),

$$
\Theta^A_0 = 1 + 2\frac{\alpha}{p} \frac{\gamma_E p^4}{(2mM)^2} + \mathcal{O}(\alpha^2) = 1 + i\frac{m p}{\pi} V^A_0(p^2) + \mathcal{O}(\alpha^2) .
$$

(7.20)

Had we proceeded as explained in section 3, we would have taken from ref. [53] the phase factor $S_c$, which for Coulomb scattering reads [53]

$$
S_c = \exp \left[ 2i \gamma \log \frac{L}{\mu} \right] .
$$

(7.21)

Analogously to eq. (3.8), we then have now that for the AC model $\delta F^A_\ell(p^2)$, eq. (17), is replaced by $\log L/\mu$. It is clear that the radius $R$ of the screening is proportional to $1/\mu$, and then we are left again with $L \propto 2p/\mu$, as explained in eq. (3.23). This term has to be added to the partial-wave projection of the AC potential in momentum space with a finite photon mass $\mu$. Its $S$-wave projection reads

$$
F^A_0(p^2) = \frac{p^4 \pi}{(mM)^2} \int_{-1}^{1} d\cos \theta \frac{k^2 - p^2}{k^2 - C} \frac{k}{k^2 - (k^2 - C)} = \frac{p^2 \pi}{(mM)^2} \log \frac{2p}{\mu} .
$$

(7.22)

After summing it with $\delta F^A_0(p^2)$ we are then left with

$$
V^A_0 = \frac{p^2 \pi}{(mM)^2} \log a .
$$

(7.23)

By comparing this expression with the exact result for $V^A_0(p^2)$ in eq. (19) we then identify $\log a \rightarrow \gamma_E/2$ at LO in the $p^2/(2mM)^2$ expansion. This value is somewhat smaller than the natural values we suggested before, though it is derived by comparing perturbative results only. We will soon see that a larger value of $\log a$ is indeed found when matching non-perturbative predictions for the spectroscopy of the model.

### 7.1 Unitarization of the AC scattering and comparison with its exact solution

For the unitarization of $V^A_\ell(p^2)$ in the AC model we use an expression analogous to eq. (4.2) but with a different integrand in the integral of the RHC because the phase-space factor is different for massive particles in non-relativistic scattering. Now we have,

$$
g(p^2) = a(C) - \frac{m(p^2 - C)}{2\pi^2} \int_0^{\infty} dk \frac{k}{(k^2 - p^2)(k^2 - C)}
$$

$$
= a(C) - \frac{m\sqrt{-C}}{2\pi} - \frac{im\sqrt{p^2}}{2\pi} .
$$

(7.24)

The subtraction point $C$ is negative ($C < 0$), so that the associated subtraction constant $a(C)$ is real. The corresponding expression for the unitarized PWAs is then

$$
T^A_\ell(p^2) = \left[ \frac{1}{V^A_\ell(p^2)} + g(p^2) \right]^{-1} .
$$

(7.25)
For the particular case of $\ell = 0$ it becomes

$$T_0^{AC}(p^2) = \left[ \frac{8(mM)^2}{\gamma E e^2 p^2} + a(C) - \frac{m\sqrt{-C}}{2\pi} - \frac{im\sqrt{p^2}}{2\pi} \right]^{-1}.$$  \hspace{1cm} (7.26)

Let us start the comparison with the exact solution of the AC model by considering the spectroscopy. The exact pole positions of $S_\ell^{AC}(p)$ are given by the poles of the function $\Gamma(z)$ in the numerator of eq. (7.12), which occur for $z = -n$, with $n = 0, 1, 2, \ldots (n \in \{0\} \cup \mathbb{N})$. This gives rise to a third-degree equation for $p$ with the solution,

$$p(\nu) = (-i)^{1/3}\lambda(\nu),$$  \hspace{1cm} (7.27)

where

$$\lambda(\nu) = \left[ \frac{4mm^2}{\alpha(1 + \ell + n)} \right]^{1/3},$$  \hspace{1cm} (7.28)

and the pole position depends only on the sum $\nu = n + \ell$. For every $\nu$ there are three solutions because of the cube root, whose explicit expressions are:

$$p_1(\nu) = i\lambda(\nu),$$
$$p_2(\nu) = e^{-i\pi/6}\lambda(\nu),$$
$$p_3(\nu) = -e^{i\pi/6}\lambda(\nu),$$  \hspace{1cm} (7.29)

and the three poles have the same absolute value for their pole positions, $|p_i(\nu)|$, which monotonically increases with $\nu$. The first pole at $p_1$ is a bound state in the physical RS, while the last two poles at $p_2$ and $p_3$ are resonances in the second RS ($\Im p < 0$). Notice that $p_3 = -p_2^*$, because in the variable $p^2$ or energy $E$ these two pole positions are complex conjugate to each other, as required by the Schwarz reflection principle applied to the $S$ matrix of eq. (7.12).

We now turn to consider the poles in the unitarized expression of $T_0^{AC}(p^2)$, eq. (7.26) (recall also (7.3)). Let us first study the case in which

$$a(C) = \frac{m\sqrt{-C}}{2\pi},$$  \hspace{1cm} (7.30)

so that, according to eq. (7.12), we end up with a third-degree secular equation for $p$ without $p^2$ terms, as in the equation for the exact solution:

$$T_0^{AC}(p^2) = \left[ \frac{2(mM)^2}{\gamma E \pi \alpha p^2} - i\frac{m\sqrt{p^2}}{2\pi} \right]^{-1}.$$  \hspace{1cm} (7.31)

Thus, no a priori undetermined parameter remains. We will show in section 7.2 that eq. (7.31) can be directly derived from analyticity and unitarity invoking the Sugawara-Kanazawa theorem [106, 132].

The secular equation for the poles of eq. (7.31) is

$$\frac{2(mM)^2}{\gamma E \pi \alpha p^2} - \frac{im\sqrt{p^2}}{2\pi} = 0,$$  \hspace{1cm} (7.32)
whose solution is

\[ p = \left[ -\frac{4mM^2}{\alpha \gamma_E} \right]^{\frac{1}{3}}. \]  

(7.33)

This expression differs from the exact pole position for \( \nu = 0 \) given in eq. (7.27) by the factor \( \gamma_E^{-1/3} = 1.20 \). It is remarkable that one recovers rather accurately the features and parameters of the resonances of the full theory with such a simple input (the Born amplitude) in the unitarization process. Also notice that, in connection with the general treatment of IR divergences in section 3, one recovers the exact pole positions by taking \( \log a = 1/2 \) in eq. (7.23), instead of \( \gamma E/2 \), as required by the LO expansion of \( V_0^{AC}(p^2) \) in eq. (7.19). We also show below that by including higher-order terms in the expansion of \( V_0^{AC}(p^2) \) in powers of \( \alpha \) one can recover the pole positions in eq. (7.29) for \( \nu = 0 \) with arbitrary precision. From here one also learns that by varying \( \log a \) with respect to its LO value one may reabsorb effectively higher-order corrections and improve the convergence, as well as the predictions that result from the unitarization of the Born term.

Rather than imposing eq. (7.30), which requires the knowledge of the exact solution \( \Theta_0^{AC}(p^2) \), we now determine the poles of the unitarized amplitude by using estimates of \( a(C) \) and the ultraviolet scale \( \Lambda^2 \) based on naturalness. This connects to the treatment we performed in section 4 for the graviton scattering and in section 5 for the \( \sigma \) meson. The first type of estimate for the subtraction constant \( a(C) \) is based in a change of \( O(1) \) in the subtraction point. Proceeding similarly as in eq. (4.7), we find for this case that

\[ a(C_1) - a(C_2) = \frac{m}{2 \pi} \left( \sqrt{-C_1} - \sqrt{-C_2} \right), \]  

(7.34)

where \( \sqrt{-C} \sim \Lambda \) and \( \Lambda \) is the cutoff of the theory. Indeed the value for \( a(C) \) used above in eq. (7.30) to end with the same type of secular equation as in the exact solution is compatible with the estimate in eq. (7.34).

The other technique used in section 4 to estimate the natural size of \( a(C) \) consists of evaluating the unitarity function with a three-momentum cutoff \( \Lambda \), obtaining

\[ g_c(p^2) = -\frac{m\Lambda}{\pi^2} - im\sqrt{p^2} + O \left( \frac{p}{\Lambda} \right). \]  

(7.35)

By taking \( \sqrt{-C} = 2\Lambda/\pi \) we then have that \( a(C) = 0 \). Considering also the first iterated contribution involving the unitarity loop function \( g_c(s) \) by expanding eq. (7.25), we obtain that this contribution is suppressed relative to the Born term by \( p^2\Lambda\alpha \gamma_E/2\pi m M^2 \equiv p^2/\Lambda^2 \), where we have identified \( \Lambda \) with the unitarity cutoff,

\[ \Lambda = \left( \frac{2\pi m M^2}{\alpha \gamma_E} \right)^{1/3}. \]  

(7.36)

The unitarized PWA of eq. (7.25) when using \( g(s) = g_c(s) \) reads

\[ T_0^{AC}(p^2) = \left[ \frac{2(m M)^2}{\pi \gamma E \alpha \pi^2} - \frac{mb\Lambda}{\pi^2} - i\frac{m\sqrt{p^2}}{2\pi} \right]^{-1}. \]  

(7.37)
where \( b = \Lambda / \left( \frac{2 \pi m M^2}{\alpha \gamma_E} \right)^{1/3} \) parametrizes the difference between unitarity and fundamental cutoffs, being analogous to the parameter \( \omega \) of eq. (6.4) but now for three-momentum cutoffs. The secular equation in \( x = p/\Lambda \) for determining the poles of \( T_0^{AC} \) becomes

\[
\frac{1}{x^3} - \frac{b}{x} - i \frac{\pi}{2} = 0.
\]

(7.38)

In table 2 we show the resulting pole positions in units of \( \Lambda = (2\pi m M^2/\alpha \gamma_E)^{1/3} \) of the \( S \)-wave PWA \( T_0^{AC}(p^2) \) for the AC model. The second, third and fourth columns give the values obtained for the unitarized Born-term PWA \( V_0^{AC}(p^2) \), and the exact results are shown in the last column. In the rows 2 and 3 a three-momentum cutoff \( b \Lambda \), with \( b = \gamma_E/2 \), 1/2 and 1, is used in the secular equation (7.38). For the rows 4 and 5 there is no linear term in the new secular equation (7.40) and \( \log a = \gamma_E/2 \), 1/2 and 1. For this case the pole positions vary as \( (2 \log a)^{-1/3} \) with respect to the exact values. There is a third pole at \( p_3 = -p_2^2 \) which is not shown.

| \( b \) | \( \gamma_E/2 \) | 1/2 | 1 | Exact |
|-----|----------------|-----|---|------|
| \( p_1 \) [\( \Lambda \)] | 0.93 | 0.98 | 1.13 | 0.72 |
| \( p_2 \) [\( \Lambda \)] | 0.74 - i 0.37 | 0.73 - i 0.33 | 0.70 - i 0.25 | 0.62 - i 0.36 |
| \( \log a \) | \( \gamma_E/2 \) | 1/2 | 1 | |
| \( p_1 \) [\( \Lambda \)] | 0.86 | 0.72 | 0.57 | 0.72 |
| \( p_2 \) [\( \Lambda \)] | 0.75 - i 0.43 | 0.62 - i 0.36 | 0.49 - i 0.28 | 0.62 - i 0.36 |

Table 2. Lightest pole positions in units of \( \Lambda = (2\pi m M^2/\alpha \gamma_E)^{1/3} \) of the \( S \)-wave PWA \( T_0^{AC}(p^2) \) for the AC model. The second, third and fourth columns give the values obtained for the unitarized Born-term PWA \( V_0^{AC}(p^2) \), and the exact results are shown in the last column. In the rows 2 and 3 a three-momentum cutoff \( b \Lambda \), with \( b = \gamma_E/2 \), 1/2 and 1, is used in the secular equation (7.38). For the rows 4 and 5 there is no linear term in the new secular equation (7.40) and \( \log a = \gamma_E/2 \), 1/2 and 1. For this case the pole positions vary as \( (2 \log a)^{-1/3} \) with respect to the exact values. There is a third pole at \( p_3 = -p_2^2 \) which is not shown.

Let us now consider how the results vary under changes in \( \log a \) around 1. For concreteness, we take the subtraction constant \( a(C) \) as given in eq. (7.30), so that we end with a third-degree secular equation without \( p^2 \) terms. Now the Born term is given by eq. (7.23) and then, from the suppression of the first iterated contribution in powers of \( p^3/\Lambda^3 \), we have the unitarity scale \( \Lambda' \) corresponding to

\[
\Lambda' = \left( \frac{2 \pi m M^2}{\alpha \log a} \right)^{1/3},
\]

(7.39)

and the associated dimensionless variable \( x' = p/\Lambda' \). The new secular equation reads

\[
\frac{1}{x'^3} - i \pi = 0,
\]

(7.40)

and its solution is \( x' = (-i/\pi)^{1/3} \). The pole positions for \( \log a = \gamma_E/2 \), 1/2 and 1 are given in units of \( \Lambda = (2\pi m M^2/\alpha \gamma_E)^{1/3} \) in the rows 5 and 6 and the columns 2, 3 and 4,
respectively, of table 2. They vary as \((\log a)^{-1/3}\) and for \(\log a = 1/2\) they reproduce the exact values, given also in the last column. Let us recall that the value \(\gamma_E/2\) for \(\log a\) is the one derived above for the Born-term amplitude at LO, cf. eq. (7.48).

For higher partial waves, \(\ell \geq 2\), the partial-wave projected Born amplitudes tend to become repulsive and the corresponding poles move deeper in the complex plane when \(\ell\) increases as \((1 + n + \ell)^{1/3}\), cf. eq. (7.27). This behaviour is similar to what we encountered for the graviton-graviton scattering in eq. (3.27). As already noticed, the expansion of \(\mathcal{S}^{AC}_\ell(p^2)\) in powers of \(\alpha\) generates the same results as our calculation for the partial-wave projected Born amplitudes with the replacement of \(\log a\) by \(\gamma_E/2\). For \(\ell = 2\) this yields \(\gamma_E - 3/2 < 0\), for \(\ell = 4\) one has \(\gamma_E - 25/12 < 0\), etc. Nonetheless, the repulsive nature of these interactions is compensated in the AC model when considering the full nonperturbative solutions. In fact, the pattern of the poles found does not change as \(\ell\) increases, for a given value of \(\nu\) one always has a bound state plus two resonance poles corresponding to the same resonance. Within our approach, based on the unitarization of the partial-wave projected Born amplitudes, for \(\nu = 0\) we can reproduce this pattern, even quantitatively, as shown in table 2.

The quantum-mechanical AC model also allows us to illustrate the procedure for including higher orders in the unitarization process. The PWA \(T^{AC}_\ell(p^2)\), corresponding to \(\mathcal{S}^{AC}_\ell(p^2)\), is given in eq. (7.3). From this equation together with eq. (7.25) it also follows that \(V^{AC}_\ell\) is given by

\[
V^{AC}_\ell \left( p^2 \right) = \frac{2i\pi}{mp} \frac{\Gamma \left( 1 + \ell + i \frac{p^2 \alpha}{4mM^2} \right) + \Gamma \left( 1 + \ell - i \frac{p^2 \alpha}{4mM^2} \right)}{\Gamma \left( 1 + \ell + i \frac{p^2 \alpha}{4mM^2} \right) - \Gamma \left( 1 + \ell - i \frac{p^2 \alpha}{4mM^2} \right)},
\]

(7.41)

which is a real function because \(\Gamma(z^*) = \Gamma(z)^*\). It is also clear that \(V^{AC}_\ell\) is a function of \(p^2\) because it is even under the change \(p \rightarrow -p\). Its expansion in powers of \(\delta = p^2 \alpha/(4mM^2(1 + \nu))\) is

\[
V^{AC}_\ell = -\frac{2\pi}{mp} v(\delta),
\]

(7.42)

where

\[
v(\delta) = (1 + \nu) \psi_0(1 + \ell) \delta + \frac{(1 + \nu)^3}{6} \left[ 2 \psi_0(1 + \ell)^3 - \psi_2(1 + \ell) \right] \delta^3
+ \frac{(1 + \nu)^5}{120} \left[ 16 \psi_0(1 + \ell)^5 - 20 \psi_0(1 + \ell)^2 \psi_2(1 + \ell) + \psi_4(1 + \ell) \right] \delta^5 + \mathcal{O}(\delta^7),
\]

(7.43)

and \(\psi_n(z)\) is the polygamma function.\(^\text{17}\) The interest of having used \(\delta\) as an expansion parameter stems from the fact that at the resonance poles characterized by \(\lambda(\nu)\) its value is \(-i\). It is then clear from the expansion in eq. (7.42) that the coefficients grow very fast with \(\nu\) (as odd powers of \((1 + \nu)\)), so that the expansion deteriorates quickly as \(\nu\) increases. This

\(^{17}\)\(\psi_n(z) = d^{n+1} \log \Gamma(z)/dz^{n+1}\). In particular, \(\psi_0(n) = -\gamma_E + \sum_{k=1}^{n-1} 1/k\).
\begin{tabular}{|c|cccc|}
\hline
 & \( N = 1 \) & \( N = 3 \) & \( N = 5 \) & Exact \\
\hline
\( p_1 [\Lambda] \) & i 0.860 & i 0.729 & i 0.717 & i 0.716 \\
\( p_2 [\Lambda] \) & 0.745 - i 0.430 & 0.631 - i 0.364 & 0.621 - i 0.358 & 0.620 - i 0.358 \\
\hline
\end{tabular}

\textbf{Table 3.} Lightest pole positions in units of \( \Lambda = \left( 2\pi m M^2/\alpha \gamma E \right)^{1/3} \) of the \( S \)-wave PWA \( T_{0;N}^{AC}(p^2) \) for the AC model with \( v(\delta) \) calculated at order \( \delta^N \), for \( N = 1, 3 \) and 5. The pole positions are compared with the exact result, eq. (7.27), given in the last column. The convergence in the results as \( N \) increases is clear. There is a third pole at \( p_3 = -p_2^* \) which is not shown.

is why with the unitarization of the AC model we can provide an accurate determination of the three lowest-lying \( S \)-wave poles only (\( \nu = 0 \)). Namely, we look for the poles of \( T_{0;N}^{AC} \),

\begin{equation}
T_{0;N}^{AC} = \left[ \frac{1}{V_{0;N}^{AC} - i \frac{mp}{2\pi}} \right]^{-1}, \tag{7.44}
\end{equation}

that is the PWA that results by using \( V_{0;N}^{AC} \), which in turn is the expansion of \( V_0^{AC} \) up to order \( N \) in odd powers of \( \delta \). The secular equation can be written more easily in terms of \( v(\delta) \) as

\begin{equation}
\frac{1}{v(\delta)} + i = 0, \tag{7.45}
\end{equation}

and \( v(\delta) \) is expanded up to the required order in \( \delta \). We show in table 3 the pole positions for \( p_1 \) and \( p_2 \). It is clear that the pole positions get closer to the exact results as the order increases, which are again reproduced in the last column for convenience.

A priori, the case for the \( \sigma \) resonance is more favourable than the one of the AC model. This is because for the latter \( |x| \simeq 0.7 \) at the pole positions while \( |x| \simeq 0.2 \) for the \( \sigma \) (in the chiral limit of QCD), for which the unitarization methods are known to be accurate to obtain \( s_\sigma \) [109]. As discussed above, this is the same value of \( |x| \) for the graviball when the fundamental cutoff is taken to be \( \pi/G \log a \). Nonetheless, as in the case of the \( \sigma \) with ChPT, it is necessary to unitarize the higher orders in the EFT of gravity to confirm that the results for \( s_P \) of the graviball are also converging. In this regard, the unitarization of the next-to-leading order graviton-graviton scattering amplitude [89] is most significant, and deserves a dedicated study.

\section{7.2 Unitarization of Coulomb scattering}

In the previous sections we have used unitarization methods to unveil non-perturbative information about the scattering of three theories with infinite-range interactions: ChPT in the chiral limit, general relativity and the AC model. For completeness, and also on its own sake, we now study what these methods imply for the most familiar model of this sort: Coulomb scattering. To connect it with some of the techniques above, we can consider it as the non-relativistic limit of, e.g., electron-positron scattering.

The Coulomb potential is

\begin{equation}
V^C(r) = \frac{\alpha}{r}. \tag{7.46}
\end{equation}
To deal firstly with the IR divergences we follow the same logic as for the AC model in eq. (7.13) and consider the screened version for distances \( r > R \),

\[
V^C(r) = \frac{\alpha}{r} \theta(R - r). \tag{7.47}
\]

The Fourier transform, IR-divergent phase and S-wave projected potential are given by the eqs. (7.15)–(7.18), by taking the inverse of the replacement in eq. (7.2), i.e. \( \alpha \to \alpha(M/E)^2 \). After removing the diverging term we are left with the neat S-wave projected Born term,

\[
V_0^C(p^2) = \frac{2\pi\alpha\gamma_E}{p^2}. \tag{7.48}
\]

This result can also be obtained by expanding \( \Theta^C_0(p^2) \), eq. (7.8), to leading order in \( \alpha \), analogously to what was done above for the AC model, cf. eq.(7.20). As discussed in the paragraph following eq.(7.21), the removal of the divergent Coulomb phase-factor of the seminal works [53, 59] changes the S-wave projected Born term \( F^C_0(p^2) = \frac{4\pi\alpha}{p^2} \log \frac{2p}{\mu} \), obtained by employing a photon mass \( \mu \to 0^+ \), to

\[
V_0^C(p^2) = \frac{4\pi\alpha}{p^2} \log a. \tag{7.49}
\]

As in the AC model, for \( \log a = \gamma_E/2 \) we recover eq. (7.48).

Let us move on to the unitarization of \( V_0^C(p^2) \) by considering the PWA multiplied by \( p^2 \), namely, \( T^C_0(p^2)p^2 \). We denote its inverse by

\[
f \left( p^2 \right) \equiv \left( T^C_0 \left( p^2 \right) p^2 \right)^{-1}, \tag{7.50}
\]

because in this way i) the limit for \( f(p^2) \) for \( p^2 \to \infty \) is just given by the inverse of the Born term times \( p^2 \), which is the constant \( (2\pi\alpha\gamma_E)^{-1} \), cf. eq. (7.48). Notice that \( \Theta^C \), eq. (7.8), becomes trivial for \( p^2 \to \infty \); ii) the resulting unitary function has a discontinuity along the RHC equal to \(-im/\pi p\), which vanishes for \( p \to \infty \). The Sugawara-Kanazawa theorem [106, 132] implies that any analytical function \( f(p^2) \) in the complex-\( p^2 \) plane with only a RHC, having a finite limit for \( p^2 \to \infty \pm i\epsilon \), and such that \( f(p^2) \) is bounded by a finite power of \( p^2 \) for \( p^2 \to \infty \), can be represented by the following DR:

\[
f \left( p^2 \right) = \frac{1}{\pi} \int_0^\infty dk^2 \Delta f \left( k^2 \right) \frac{k^2 - p^2}{k^2 - k^2 - p^2}. \tag{7.51}
\]

In this expression, \( \Delta f(k^2) \) is the discontinuity of \( f(p^2) \) along the RHC, \( \Delta f(k^2) = f(k^2 + i\epsilon) - f(k^2 - i\epsilon) \), \( k^2 \geq 0 \), and \( \tilde{f}(\infty) = [f(\infty + i\epsilon) + f(\infty - i\epsilon)]/2 = f(\infty \pm i\epsilon) \) (because there is no LHC). For the case at hand, this translates into the DR representation for \((p^2T^C_0(p^2))^{-1}\),

\[
\frac{1}{p^2T^C_0(p^2)} = \frac{1}{2\pi\alpha\gamma_E} \frac{m}{2\pi^2} \int_0^\infty dk^2 \frac{1}{k(k^2 - p^2)}. \tag{7.52}
\]

The integral is elementary and we end up with

\[
T^C_0 \left( p^2 \right) = \left[ \frac{p^2}{2\pi\alpha\gamma_E} - i\frac{mp}{2\pi} \right]^{-1}. \tag{7.53}
\]
The zeroes of the exact \( T_0^{\text{C}}(p^2) \) would imply poles in \( f(p^2) \) not included in eq. (7.53), which is the unitarization of the LO Born-term PWA. They can be accounted for by calculating \( V_0^{\text{C}}(p^2) \) at higher orders in \( \Lambda/p^2 \), as done below, cf. eq. (7.54).

We can also connect with the DR representation of the PWAs for the AC model, cf. eq. (7.31), by the general rule of substituting \( \alpha \rightarrow \alpha p^4/(2mM)^2 \), as explained when this model was introduced, cf. eq. (7.2). Thus, by implementing this substitution into eq. (7.53), we end with the DR representation for \( T_0^{\text{AC}}(p^2) \) of eq. (7.31).

From eq. (7.53) we can assess the unitarity corrections which enter in the combination \((mp/2\pi)V_0^{\text{C}}(p^2) = ma\gamma_E/p\), suggesting the three-momentum scale \( \Lambda = ma\gamma_E \). It is now clear that the unitarization of Coulomb scattering is valid for \( |p| \gtrsim \Lambda \), because the Born term has a pole at \( p = 0 \). The exact function \( \Re(T_0^{\text{C}}(p^2)|p^2|^{-1} \) wildly oscillates for \( |p| \ll \Lambda \), while for \( |p| \gtrsim \Lambda \) the DR in eq. (7.52) generates a smooth \( T_0^{\text{C}}(p^2) \) that matches with the former. Contrarily, for graviton-graviton scattering and the AC model (cf. eqs. (4.5), (4.6) and (7.25)), one has a zero at \( p = 0 \) and the unitarization is a resummation valid at low momenta, \( |p| \lesssim \Lambda \) (here \( \Lambda \) is the corresponding cutoff scale in those cases).

Let us notice that for pure Coulomb scattering the excited states have less and less binding energy, approaching zero arbitrarily as the degree of excitation increases [70]. Indeed, the Coulomb PWA has an essential singularity at \( p = 0 \), as follows from expanding \( \mathcal{S}_\ell^{\text{C}}(p^2) \) around \( p = 0 \). Therefore, the previous equation concerning the unitarized \( T_0^{\text{C}}(p^2) \), and the resulting unique bound state, is not capturing essential features of Coulomb scattering in partial waves in the limit \( p \ll \Lambda \) and could be discarded as a sensible approximation in such case, but not for \( p \gtrsim ma\gamma_E \). Note that the amplitude in eq. (7.53) has a bound state precisely at \( p = i\nu \) with \( \nu = ma\gamma_E \), corresponding to a binding energy \( \epsilon = \gamma_E ma^2/2 \).

To assess the precision of the method to generate the exact bound state, one can use the knowledge of \( \mathcal{S}_\ell^{\text{C}}(p^2) \), and write down the exact \( T_\ell^{\text{C}}(p^2) \) and \( V_\ell^{\text{C}}(p^2) \). Following closely the procedure explained in section 7.1,

\[
V_\ell^{\text{C}}(p^2) = \frac{2i\pi}{mp} \frac{\Gamma\left(1 + \ell + i\frac{ma}{p}\right) + \Gamma\left(1 + \ell - i\frac{ma}{p}\right)}{\Gamma\left(1 + \ell + i\frac{ma}{p}\right) - \Gamma\left(1 + \ell - i\frac{ma}{p}\right)},
\]

(7.54)

Its expansion in powers of \( \delta = ma/p(1 + \nu) \) is exactly the same as given in eq. (7.42) by introducing the auxiliary function \( v(\delta) \). Hence, the convergence towards the exact pole position \( p = i\alpha \) follows the same pattern as explained with respect to table 3 for the AC model. The difference is that now this discussion is applied to the ground state of the system, i.e. the state with the largest binding energy, while for the AC scattering it was applied to the poles with the smallest \( |p_i| \). For the former the unitarization can be considered as a high-energy procedure while for the latter it is a low-energy one. Since now the absolute units are different to those in table 3, we reproduce the corresponding numbers for Coulomb scattering in table 4 in units of \( \Lambda = ma\gamma_E \) for the first orders from \( N = 1 \) to 15. It is clear that the series converge to the exact result by increasing the order of the expansion.
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|c|c|c|}
\hline
$N$ & 1 & 3 & 5 & 7 & 9 & 11 & 13 & 15 \hline
$p_1$ [\Lambda] & $i1.646$ & $i1.729$ & $i1.738$ & $i1.736$ & $i1.734$ & $i1.733$ & $i1.732$ & $i1.732$ \hline
\end{tabular}
\caption{Deepest binding momentum in units of $\Lambda = m_0 \gamma_E$ of the S-wave PWA $T_C(p^2)$ for Coulomb scattering with $v(\delta)$ calculated at order $\delta^N$, for $N = 1, 3, \ldots, 15$. The pole positions are compared with the exact result $i/\gamma_E$ given in the last column. The (non-monotonic) convergence in the results as $N$ increases is explicitly shown.}
\end{table}

In contrast, for the AC model there is indeed a bound state with finite (nonzero) minimal binding energy, while all the other bound states have increasingly larger binding energies. In addition, we have the resonance poles that also lie further away from zero as $\nu$ increases. As a result, the state with the smallest binding energy and the resonance with the pole positions closest to the physical axis are the ones affecting most the physical low-energy region for scattering. Then, one can interpret the calculated unitarized PWAs as well settled from the low-energy point of view.

8 S-wave scattering for $d > 4$ and the graviball pole position

In this section we exploit the fact that for $d \geq 5$ there are no infrared divergences and study the $J = 0$ AC-model, Coulomb and graviton-graviton scattering as a function of the number $d$ of space-time dimensions. We show that the poles of the scattering amplitudes obtained at $d = 4$ persist for higher dimensions, varying in a smooth continuous way. This property is used below to estimate $\log a$ such that the extrapolation from $d \simeq 5$ to $d = 4$ is as smooth as possible. This can also be interpreted as minimizing higher-order uncertainties introduced by the dependence on $\log a$. This is the essence of the so-called optimized perturbation theory \cite{73–75}. This method is tested successfully with the exactly-solvable AC model and then it is used to obtain an “optimal” value of $\log a$ for the case of gravity.

8.1 Treatment of the infrared divergences from the analytic continuation to $d > 4$

For the sake of self-consistency we start by showing how the soft IR divergences in the projected Born term can be reabsorbed in the exponential factor $S_c(s)$ within dimensional regularization. An important point to notice is that loop contributions with virtual gravitons are IR convergent for $d > 4$. This implies that one can eliminate the cutoff $\mathcal{L}$ from the onset by regulating IR divergences in dimensional regularization and trade it for other scales required by dimensional analysis.\footnote{Furthermore, the one-loop amplitude in pure gravity is free from ultraviolet divergences and all its singularities in $d = 4$ are of IR nature \cite{33, 89, 90}.} Indeed, the exponent in eq. (3.3) is regularized when evaluated in $d = 4 - 2\epsilon$, and one can then take already $\mu = 0$ and $\mathcal{L} = \infty$. Summing the contributions involving a pair of either incoming or outgoing particles with initial momenta
where the normalization of the states has been set as in

\begin{equation}
\int \frac{d^d q}{(2\pi)^d} B(q) = -4\pi Gs^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon)(p_1 \cdot q - i\varepsilon)(p_2 \cdot q + i\varepsilon)}
\end{equation}

\begin{equation}
= 16\pi Gs^2 \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + i\varepsilon)[(p_1 - q)^2 + i\varepsilon][(p_2 + q)^2 + i\varepsilon]}
\end{equation}

where in the second line we have restored the full \( q^2 \) dependence of the propagators to regulate the ultraviolet divergence in the integral in the first line. This leads to the following

IR-divergent phase,

\begin{equation}
S^{dr}_C(s) = \exp \left[ iGs \left( -\frac{1}{\epsilon} + \log \frac{s}{\mu_h^2} + \gamma_E - \log(4\pi) \right) \right].
\end{equation}

In the previous expression we rescaled the dimensions of the Newton’s constant by inserting a renormalization scale \( \mu_h \) and the physical region is approached by taking \( s + i\varepsilon \) so that

\( \log(-s) = \log(s) - i\pi \). Note that the scale \( \mu_h \) introduces an arbitrariness in the finite part related to the fact that we are adding hard-graviton modes in the integral.\(^{20}\)

One then redefines the \( S \)-matrix by subtracting \( S^{dr}_C(s) \), which provides an IR-divergent contribution to all the PWAs at \( \mathcal{O}(Gs) \),

\begin{equation}
\delta F^{(J)}_{\lambda_1^J \lambda_2^J \lambda_1 \lambda_2} = \frac{1}{2^{3\lambda/4}} \frac{4Gs}{\pi} \left( \frac{1}{\epsilon} - \log \frac{s}{\mu_h^2} - \gamma_E + \log(4\pi) \right),
\end{equation}

that should cancel those stemming from the singularity in the forward direction of the angular-projection integrals of the Born term.

Extending the projections in partial waves to arbitrary dimensions, as discussed in detail in appendix B, one obtains

\begin{equation}
S(p' J' \lambda_1^J \lambda_2^J | p J \lambda_1 \lambda_2)_S = \frac{1}{2(2\pi)^d} \left( \frac{s}{4} \right)^{d/2-2} \times \int d\Omega_{d-2} \int_{-1}^{+1} d\cos \theta' (\sin \theta')^{d-4} D^{(J)}_{\lambda_1^J \lambda_2^J}(\hat{n}) S(p' \lambda_1^J \lambda_2^J | p \lambda_1 \lambda_2)_S,
\end{equation}

where the normalization of the states has been set as in \( d = 4 \) (see appendix B). For the case \( J = 0 \) (\( \lambda = \lambda' = 0 \)), \( D^{(J)}_{\lambda^J \lambda'^J}(\hat{n}) = 1 \) and this integral can be easily solved in arbitrary dimensions (see e.g. [133]),

\begin{equation}
F^{(0)}_{22,22}(s) = \frac{2Gs}{\pi^2} \left( \frac{s}{16\pi^2} \right)^{d/2-2} \int d\Omega_{d-2} \int_{-1}^{+1} d\cos \theta' (\sin \theta')^{d-4} \frac{\Gamma(d/2 - 2)}{\Gamma(d - 3)}.
\end{equation}

\(^{19}\)In our notation the Minkowski metric is \( \text{diag}(1, -1, -1, -1) \), with a global minus sign relative to ref. [53].

\(^{20}\)This scheme dependence in a dimensional treatment of the IR divergence can be made more explicit by regulating the hard modes in the first line of eq. (8.1) also in dimensional regularization [71] or with an explicit cutoff [59].
Expanding in $d = 4 - 2\epsilon$,

$$F^{(0)}_{22,22}(s) = \frac{4Gs}{\pi} \left( -\frac{1}{\epsilon} + \log \frac{s}{\mu_f^2} + \gamma_E - \log(4\pi) \right), \quad (8.6)$$

where we have regulated the IR divergence stemming from the forward singularity at a particular scale $\mu_f$. This is summed to the divergent contribution from $S_c^{dr}(s)^{-1}$ in eq. (8.3) up to $O(Gs)$, to give the finite PWA

$$V^{(0)}_{22,22}(s) = 8Gs \log\left(\frac{\mu_h}{\mu_f}\right). \quad (8.7)$$

The log's that depend on the energy of the process cancel along with the singularities because they must arise from the expansion of the dimensionless factor $(s/\mu_i^2)^{-\epsilon}$. Hence, the dependence on energy of the PWA at $d = 4$ can only be linear in $s$, which is consistent with the argument made in eq. (3.23) to fix the scale $\mathcal{L}$ in the calculation with an explicit cutoff. The final coefficient corresponds to $\log a = \log \mu_h/\mu_f$ in that language, which also makes obvious the expectation that $\log a \sim O(1)$.

Extending our analysis to integer dimensions $d > 4$ opens up a new angle to investigate the analytic structure of the graviton-graviton scattering amplitude. In particular, the treatment of the divergent phase in $d = 4$ should not introduce spurious effects and one expects that interesting features in the analytic structure of the PWA, such as the presence of poles, remain in $d > 4$. In this case, the PWA simply corresponds to the one derived from the (IR convergent) $S^{(J)}$ and for $J = 0$ it is equal to eq. (8.5),

$$V^{(0),d}_{22,22}(s) = \frac{\Gamma(d/2 - 2)}{\Gamma(d - 3)} \frac{4Gs}{\pi} \left( \frac{s}{4\pi\mu_f^2} \right)^{\frac{d-4}{2}}, \quad (8.8)$$

where we have expressed the Newton’s constant in $d$-dimensions $G_d$ in terms of the 4-dimensional $G$, as $G = G_d(\mu_f^2)^{\frac{d-4}{2}}$. Note that the extra functional dependence $s^{d/2-2}$ with respect to $d = 4$ does not introduce a left-hand cut in the PWAs for odd-numbered $d$ by taking $p = \sqrt{s}/2$ as the argument. Besides the additional factor $s^{d/2-2}$, the strength of the interaction gets also rapidly diluted with $d > 4$ due to the ratio of the $\Gamma$ functions.

### 8.2 The graviball in $d > 4$

The graviton-graviton $S$-wave projected Born amplitude without IR singularities for $d > 4$ has been worked out in eq. (8.8). Analogously as done for $d = 4$, we can estimate the unitarity cutoff $\Lambda_d$ by taking the first iterated expression of the unitarized formula, which now reads

$$T^{(0),d}_{22,22}(s) = \left[ V^{(0),d}_{22,22}(s)^{-1} + g(s) \right]^{-1}. \quad (8.9)$$

Therefore, the unitarity cutoff $\Lambda^2_d$ is

$$\Lambda^2_d = \left( \frac{4\pi\mu_f^2}{2\pi} \right)^{\frac{d-4}{2}} 2\pi G^{-1} \frac{\Gamma(d - 3)}{\Gamma(d/2 - 2)} \frac{\pi^2}{2}. \quad (8.10)$$
Figure 3. Real and imaginary parts of \( s_P \) (using the left vertical axis) as functions of \( d \) plotted using dashed and solid (black) lines, respectively. The empty and filled triangles show the values for integer \( d \) where, for \( d = 4 \), the value \( \log a = 1 \) has been chosen. We also plot the real and imaginary parts of \( x \) for the graviball \( (x_P, \text{using the right vertical axis}) \) with dashed and solid (gray) lines, respectively. The empty and full squares are the corresponding values for integer \( d \).

To simplify the dependence on \( d \) in this expression we choose units such that \( 2\pi G^{-1} = 1 \) and, for illustrative purposes, we take by now \( 2\mu^2 G = 1 \). Then,

\[
\Lambda_d^2 = \left( \frac{\Gamma(d-3)}{\Gamma(d/2-2)} \right)^{d-2}.
\]

(8.11)

One obtains the secular equation,

\[
(\omega x_d)^{1-d/2} + \log(-x_d) - i2\pi = 0,
\]

(8.12)

where \( \omega = \Lambda^2 / \Lambda_d^2 \) corresponds again to the ratio of the fundamental and unitarity cutoffs. For the nominal case of \( \omega = 1 \) this equation is then the same as eq. (8.15) below for searching the pole of the \( \sigma \). The resulting poles position \( s_P = x_d \Lambda_d^2 \) as a function of \( d \) are shown in figure 3. The empty and filled triangles represent the real and imaginary parts of \( s_P \), respectively, and the empty and filled squares correspond to the real and imaginary parts of \( x \) as a function of \( d \). Notice that the curves for \( s_P \), as a function of \( d \), would evolve to zero if continued to \( d = 4 \) because \( \Lambda_d^2 = 0 \) due to the IR divergence that has not been regulated in this expression. We also show the real and imaginary parts for \( s_P \) at \( d = 4 \) for \( \log a = 1 \). We also observe that the real part of \( s_P \) rapidly increases with \( d \) and it becomes larger than the absolute value of the imaginary part already for \( d \gtrsim 5 \), a feature also observed below for the \( s_\sigma \) in figure 4. Therefore, the noteworthy fact that the peak in the resonance signal of \( 1/|D^0(s)|^2 \) is much smaller than the width, as represented in the right panel of figure 2, is very characteristic of \( d = 4 \). Also note that the positions of the poles for \( d \geq 5 \) are transplanckian (i.e. heavier than \( G^{-1} \)), even though \( |x| < 1 \).

8.3 The \( \sigma \) meson in \( d > 4 \)

Let us now turn to \( \pi\pi \) scattering and the \( \sigma \) meson. In this case there are no IR divergences so that the extrapolation for integer \( d > 4 \) can be connected straightforwardly with the
result at $d = 4$, a point that will require further treatment for the infinite-range interactions, as we treat below.

Making use of eq. (B.16), the LO ChPT scalar isoscalar $\pi\pi$ PWA for $d$ dimensions in the chiral limit is given by

$$V_{0}^{\pi\pi,d}(s) = \frac{s}{f_{\pi}^{2}} \frac{(p/\mu f)^{d-4}}{2^{d-2}\pi^{d-3}} \int d\Omega_{d-1} = \frac{\Gamma(d/2 - 1)}{\Gamma(d - 2)} \frac{s}{f_{\pi}^{2}} \left( \frac{s}{4\pi f_{\pi}^{2}} \right)^{\frac{d}{2} - 2}, \quad (8.13)$$

where we have used that the isoscalar $\pi\pi$ scattering amplitude $S_{\langle p'\mid T\mid p\rangle_{S}}$ is $s/f_{\pi}^{2}$, as derived in ref. [38], and have defined the pion decay constant in $d$-dimensions as $f_{\pi}^{d} = f_{\pi}\mu^{d/2 - 2}$. Compared to eq. (8.5), this comes from a contact interaction and has no divergences in $d = 4$ related to $t$ channel exchange. The unitarized PWA reads as in eq. (5.2) but with $f_{\pi}^{2}/s$ replaced by $1/V_{0}^{\pi\pi,d}(s)$. The unitarity cutoff $\Lambda_{d}$ in this case is

$$\Lambda_{d}^{2} = \left( \pi f_{\pi}^{2} \right)^{\frac{d}{2} - 2}(4\pi f_{\pi})^{2} \frac{\Gamma(d - 2)}{\Gamma(d/2 - 1)} \frac{s}{f_{\pi}^{2}} \left( \frac{s}{4\pi f_{\pi}^{2}} \right)^{\frac{d}{2} - 2}, \quad (8.14)$$

and the secular equation, with $x = s/\Lambda_{d}^{2}$ reads now

$$x^{1 - \frac{d}{2}} + \log(-x) - i2\pi = 0. \quad (8.15)$$

We can simplify the dependence of $\Lambda_{d}^{2}$ (and of $s_{\sigma} = x\Lambda_{d}^{2}$) on $d$ by taking energy units such that $4\pi f_{\pi} = 1$. For illustration, if in these units we choose $\mu_{f}^{2} = 1/\pi$, this scale squared becomes

$$\Lambda_{d}^{2} = \left( \frac{\Gamma(d - 2)}{\Gamma(d/2 - 1)} \frac{s}{f_{\pi}^{2}} \right)^{\frac{d}{2} - 2}. \quad (8.16)$$

The resulting real (empty triangles) and imaginary part (filled triangles) of $s_{\sigma}$ as a function of $d$ is shown in figure 4 using the left vertical axis. We also plot the real and imaginary parts of $x$ using the right vertical axis and the full and empty squares, respectively. Notice that in all cases $x$ is always less than 1, which means that the pole position is lower than the unitarity cutoff of the theory (which is growing with $d$) and for $d = 5\sqrt{|s_{\sigma}|}$ is also below the putative cutoff of QCD. The real part of $s_{\sigma}$ grows with $d$ faster than the absolute value of the imaginary part, so that already for $d \gtrsim 5$ the former becomes larger than the latter.

### 8.4 The $S$-wave amplitude in $d > 4$ for the AC model

Let us consider the $S$-wave projection of the Born term of the AC model for integer dimensions larger than 4. The corresponding LO tree-level PWA in $S$ wave, employing eq. (B.16), is

$$F_{0}^{AC,d}(p^{2}) = \frac{\pi\alpha^{d-6}}{(2\pi)^{d-3}\mu_{f}^{d-4}} \left( \frac{p^{2}}{2mM} \right)^{2} \int d\Omega_{d-2} \int_{-1}^{+1} d\cos\theta' \left( \frac{\sin\theta'}{1 - \cos\theta'} \right)^{d-4} = \frac{\alpha^{d-6}}{\mu_{f}^{d-4}\pi^{d/2 - 3}} \frac{\Gamma(d/2 - 2)}{\Gamma(d - 3)}. \quad (8.17)$$
Figure 4. Real and imaginary parts of \( s_\sigma \) (using the left vertical axis) as functions of \( d \) plotted using dashed and solid (black) lines, respectively. The empty and filled triangles show the values for integer \( d \). We also plot the real and imaginary parts of \( x \) (using the right vertical axis) with dashed and solid gray lines respectively. The empty and full squares are the corresponding values for integer \( d \).

The unitarity cutoff \( \Lambda'_d \) is obtained by multiplying \( F_0^{AC,d}(p^2) \) by \( m_p/2\pi^2 \), from where we deduce that

\[
\Lambda'_d = (\pi \mu_f^2)^{\frac{1}{2}} \frac{8\pi m M^2/\alpha}{\Gamma(d/2-2)} \Gamma(\frac{d}{2}) \Gamma(d-3) \left( \frac{\pi \mu_f^2}{\Gamma(d/2-2)} \right)^{\frac{1}{2}} \left( \frac{\Gamma(\alpha)}{\Gamma(d-3)} \right)^{\frac{1}{2}}. \tag{8.18}
\]

The unitarized expression for the \( S \)-wave is the same as in eq. (7.31) but substituting \( V_0^{AC}(p^2) \) by \( F_0^{AC,d}(p^2) \). The secular equations that follows is

\[
x^{d-1} = -\frac{i}{\pi}, \tag{8.19}
\]

whose solution is explicitly

\[
p^2(d) = \pi \mu_f^2 \left( -i \frac{8m M^2/\alpha}{\Gamma(\frac{d}{2}) \Gamma(d/2-2)} \right)^{\frac{1}{2}} \left( \frac{\pi \mu_f^2}{\Gamma(d/2-2)} \right)^{\frac{1}{2}}. \tag{8.20}
\]

To simplify the dependence on \( d \) we choose units such that \( 2m M^2/\alpha = 1 \) and a value for \( \pi \mu_f^2 = 1 \) for illustrative purposes. The running of \( p^2(d) \) with \( d \) then simplifies to \((-i4\Gamma(d-3)/\Gamma(d/2-2))^{2/(d-1)} \) and \(|p^2(d)| \) is shown by the triangles of figure 5. We also give the exact result for \( d = 4 \). We can observe that this curve is qualitatively similar to the one for the graviball given above in figure 3. We only show the modulus because all the poles are obtained by the roots of \((-i)^{2/(d-1)} \times \) times the latter.

8.5 Unitarization of Coulomb scattering for \( d > 4 \)

Let us consider electron-positron scattering of initial(final) four-momenta \( p_i(p'_i), i = 1, 2 \), whose non-relativistic limit reduces to Coulomb scattering. The Mandelstam variable \( s \) is
given by \( s = (p_1 + p_2)^2 = 4(\mu^2 + p^2) \), where \( p \) is the modulus of the CM three-momentum and \( \mu \) is the mass of an electron. According to ref. [53] the exponent giving rise to the IR-divergent contributions from soft intermediate photons exchanged between external lines belonging to the same initial or final state is

\[
\int \frac{d^2 q}{(2\pi)^2} A(q) = i e^2 \frac{s - 2\mu^2}{(2\pi)^2} \frac{1}{(q^2 + i\varepsilon)(-p_1 q + i\varepsilon)(p_2 q + i\varepsilon)}
\]

\[
\rightarrow i e^2 \frac{s - 2\mu^2}{(2\pi)^2} \frac{1}{(q^2 + i\varepsilon)} \left( (p_1 - q)^2 - \mu^2 + i\varepsilon \right) \left( (p_2 + q)^2 - \mu^2 + i\varepsilon \right)
\]

\[
= - \frac{2\alpha \Gamma(1 + \epsilon) s^{-1-\epsilon}}{(4\pi)^{1-\epsilon} \epsilon} \int_0^1 dt \left( t(t-1) + \mu^2 / s \right)^{-1-\epsilon}.
\]

The roots of the denominator in the integrand are

\[
t_{1,2} = \frac{1}{2} \pm \frac{1}{2} w, \quad w \equiv \frac{p}{\sqrt{\mu^2 + p^2}} = \frac{v}{\sqrt{1 + v^2}}.
\]

Notice that \( w \) is the relativistic velocity while \( v = p/m \) is the non-relativistic expression. This decomposition is appropriate for our discussion because at the end we are interested in the non-relativistic limit. Making a shift in the integration variable \( t \to t - 1/2 \), we are then left with the integral

\[
\int \frac{d^2 q}{(2\pi)^2} A(q) = - \frac{2\alpha \Gamma(1 + \epsilon) s^{-1-\epsilon}}{(4\pi)^{1-\epsilon} \epsilon} \left\{ \int_{-1/2}^{1/2} dt \frac{dt}{t^2 - \frac{w^2}{4}} - \epsilon \int_{-1/2}^{1/2} dt \frac{\log(t^2 - \frac{w^2}{4})}{t^2 - \frac{w^2}{4}} \right\} + \mathcal{O}(\epsilon).
\]

Instead of giving the full expressions for the integration, which is somewhat lengthy, we give its imaginary part in the leading non-relativistic limit, which is the one that actually
enters as the exponent of $S^C_c$. It generates the global phase

$$
S^C_c(p) = \exp \left[ i\gamma \left( -\frac{1}{\epsilon} + \log \frac{p^2}{\mu_h^2} + \gamma_E - \log \pi \right) \right].
$$

(8.23)

Notice that the terms in parentheses are the same as those in eq. (8.2) for gravity once we replace $s$ by $4p^2$ in graviton-graviton scattering. As in this case, we have introduced the renormalization scale $\mu_h$ to keep right the dimensions when varying $d$. As commented above, regarding eq. (8.2), the scale $\mu_h$ introduces an arbitrariness in the finite part of the exponent due to having integrated over harder-photon modes, cf. footnote 20.

Now, we consider the $S$-wave projection of the Coulomb scattering amplitude in $d$ dimensions by employing eq. (B.16). We are then left with

$$
F^C_{0,d}(p^2) = \frac{\pi \alpha p^{d-6}}{(2\pi)^{d-3}} \int d\Omega_{d-2} \int_{-1}^{+1} d\cos \theta' \left( \frac{\sin \theta'}{1 - \cos \theta'} \right)^{d-4} = \frac{\alpha p^{d-6}}{\pi^{d/2-3}} \frac{\Gamma(d/2 - 2)}{\Gamma(d - 3)}.
$$

(8.24)

Expanding in powers of $\epsilon$ we have

$$
F^C_{0,d}(p^2) = \frac{\pi \alpha}{p^2} \left( -\frac{1}{\epsilon} + \log \frac{p^2}{\mu_f^2} + \gamma_E - \log \pi \right).
$$

(8.25)

The sum of $F^C_{0,d}(p^2)$ with $-i\pi(S_c^{-1} - 1)/mp$ expanded up to $O(\alpha)$ (let us recall that $m = \mu/2$, the reduced for the electron-position system) gives the neat $S$-wave projected Born term for Coulomb scattering

$$
V^C_0(p^2) = \frac{2\pi \alpha}{p^2} \log \frac{\mu_h}{\mu_f},
$$

(8.26)

and $\log \mu_h/\mu_f$ corresponds to $\log a^2$, when compared to eq. (7.49).

We also give from eq. (8.24) the $S$-wave Born term for integer dimensions $d \geq 5$, which is not affected by the issue of the IR divergences in the forward scattering. The dimensions can be kept right e.g. by dividing $p^{d-4}$ in this equation by $(m\alpha)^{d-4}$, and we end up with the dimensionless Sommerfeld parameter $\gamma$ to the power $d - 4$. Then,

$$
F^C_{0,d}(p^2) = \frac{\alpha \gamma^{d-4}}{\pi^{d/2-3} p^2} \frac{\Gamma(d/2 - 2)}{\Gamma(d - 3)}.
$$

(8.27)

Of course, any other choice would differ by some coupling that could vary with $d$ (but not with $p$).

We do not study for this case the evolution of the poles in $S$ wave with $d$ because for $d = 5$ the secular equation for $x$,

$$
xd^5 = -\frac{i}{\pi},
$$

(8.28)

has obviously no solution. The reason is the change in the character of the theory, because as $d$ increases the theory can be unitarized as a low-energy EFT, while at $d = 4$ this procedure is valid for $p \gg \Lambda$. 

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8.6 Maximal-stability estimate of \( \log a \)

The value of \( \log a \) comes from the separation of hard and soft modes in the IR divergent part of loop diagrams. As discussed in section 3 this points towards using \( \log a \gtrsim 1 \), though its value cannot be fixed by perturbation theory. We saw in section 7.1 that for the AC model, the results from the Born approximation suggest a slightly smaller value: the LO one, \( \gamma_E/2 \), gets “dressed” by higher-order corrections, so that at the end \( \log a = 1/2 \) is the one reproducing the exact positions of the lightest poles. To get more insight on which values of \( \log a \) may be \textit{a priori} adequate without knowing higher-order contributions (like in our present study for the graviball), we investigate these theories in \( d > 4 \), where the IR divergences eventually disappear, and study their evolution to \( d = 4 \). We first discuss the method applied to the AC-scattering model and show that, indeed, it gives a value \( \log a \approx 1/2 \). Then, we apply the analogous method to the \( J = 0 \) graviton-graviton scattering and obtain the larger result \( \log a \approx 1 \).

For \( d \rightarrow 4 \) the effect of the IR divergences affecting the partial-wave projection of the Born terms in the AC model requires including \( S_{AC}^c(p^2) \). This can be obtained in \( d > 4 \) by using the one for QED, \( S_C^c \), calculated in section 8.5, and applying the substitution \( \alpha \rightarrow \alpha p^4/(2mM^2) \) in eq. (8.23). After replacing \( p^2 \) by its value at the pole positions in \( d \) dimensions \( p^2(d) \) given in eq. (8.20) we are left with the combination

\[
\frac{2}{d-4} + \gamma_E - 4 \log a + \frac{2}{d-1} \log \left( \frac{-i8mM^2/\alpha}{(\pi\mu_f^2)^2} \frac{\Gamma(d-3)}{\Gamma(d/2-2)} \right),
\]

in the exponent of \( S_{AC}^c \), where we have used that \( \log a = \frac{1}{2} \log \mu_h/\mu_f \) for Coulomb scattering. In the following we take units such that \( 2mM^2/\alpha = 1 \) and introduce the variable \( y = \pi\mu_f^2 \).

We now study when the pole term in the previous equation is comparable in size with the other terms. This gives a critical dimension, \( d_c \), below which the pole term (associated to the IR divergences) is prominent and one cannot extrapolate the theory by directly evaluating the \( S \)-wave projection of the Born term in \( d \) dimensions. This is certainly expected to happen at some \( d_c \lesssim 5 \), since for integer dimensions one could in principle apply an essentially analogous unitarization procedure as for \( d = 4 \), cf. section 7. Specifically, the criterion that we will use for the onset of large IR corrections, giving us \( d_c \), is

\[
\frac{2}{d-4} = \left| 4 \log a - \gamma_E - \frac{2}{d-1} \log \left( \frac{-i4\Gamma(d-3)}{y^{3/2}\Gamma(d/2-2)} \right) \right|,
\]

For each value of \( d_c(\log a, y) \) we then quantify the difference with respect to \( d = 4 \) by calculating the relative difference between the unitarity-cutoff scales at \( d_c \) and \( d = 4 \). Namely,

\[
r(\log a, y) = \frac{\left| \Lambda_{d_c}^2 - \Lambda_4^2 \right|}{\Lambda_4^2},
\]

with \( \Lambda_{d_c}^2 \) and \( \Lambda_4^2 \) given in eqs. (8.18) and (7.39), respectively. This definition is based on the fact that the unitarity-cutoff scale \( \Lambda_{d_c}^2 \) controls the strength of the interaction as \( d \) varies. The relative difference is mostly determined by the rapid variation of \( \Lambda_{d_c}^2 \) for \( d \) close to 4,
which is much affected by the IR divergences. The latter, in turn, manifest through the pole term $\Gamma(d/2 - 2)$ in the denominator of eq. (8.18).

We then propose to look for the combination of $\log a$ and $y$ that minimizes the difference $r$ with the constraint $d_c < 5$. This minimization requirement of eq. (8.31) is equivalent to a criterion of maximal smoothness in the theory when extrapolating from $d_c$ to $d = 4$. We plot in figure 6 the relative difference $r$ in the $(\log a, y)$ plane. We indeed observe a continuum of local minima for $\log a \sim 0.4 - 0.6$, which is the value we have already determined by reproducing the lightest pole positions with $\nu = 0$ in the AC-scattering model.\footnote{One actually finds a minimum at $\log a \simeq 1/2$ if one relaxes the criterion in eq. (8.30) by changing slightly the balance between the $2/(d - 4)$ piece and the rest of the contributions in the definition of $d_c$.} Also note that the range of $y$ considered corresponds to $\mu^2 \sim 0.1 - 0.4$ in units of $\Lambda'^2$, which are reasonable values for an IR regularization scale.

To summarize, the method is based on the study of the scattering for $d > 4$ and requiring maximum smoothness when passing from $d_c$ to $d = 4$ to determine $\log a$. This procedure is reminiscent of optimized perturbation theory \cite{73–75} which implements the principle of minimal sensitivity to fix scale ambiguities in perturbation theory to improve its convergence properties. One can also see similarities between the principle of maximum smoothness and some techniques used in statistical mechanics. In our case, by minimizing $r(\log a, y)$ one is reducing the needed number of terms in an expansion of the $S$-wave projected Born term (which fixes $\Lambda'^d$), in powers of $d - 5$, since $d_c$ is typically close to 5, from $d = 5$ to 4. Although we know $V_d^{(0)}(s)$, we pretend to be influenced by just a few terms in its expansion, since higher-order terms are increasingly more sensitive to the singularity at $d = 4$ of $\Gamma(d/2 - 2)$ in eq. (8.8). This is then effectively similar to standard application of dimensional continuation in statistical mechanics \cite{134–136}, where typically only a few terms in the density expansion are available and from which the optimal solution is sought.

Encouraged by the successful application of this method to the AC toy model we apply it now to graviton-graviton scattering. Using the same unitarization procedure for the Born term as in $d = 4$, but with the corresponding $V_{22,22}^{(0),d}(s)$ and $\Lambda'^2_d$ in eqs. (8.8) and (8.10), respectively, one obtains the secular equation in eq. (8.12). Therefore, a resonance also arises from the gravitational interactions in $d > 4$, whose position is given as $s_P(d) = x_d \Lambda'^2_d$. In the following we take $\omega = 1$, which is our benchmark scenario, and we refer to the discussions at the beginning of this section for other values of $\omega$.

We first determine the values of the critical dimension, $d_c$, for which the pole term in the exponent of $S_c(s)$, calculated in eq. (8.1), is comparable to the other contributions. In this expression we replace $s$ by $s_P$ in the $s$-dependent log, so that this exponent adopts the form

$$iGs \left(\frac{2}{d - 4} + \gamma_E - 2 \log a + \frac{2}{d - 2} \log \frac{\Gamma(d - 3)}{2y \Gamma(d/2 - 2)} + \log x_d\right), \quad (8.32)$$

where $y = \mu^2 G$ and we have also taken into account here that $\log \mu_h/\mu_f = \log a$, as deduced after eq. (8.7). Notice that the dependence on $x_d$ in the previous equation is mild because
Figure 6. AC model. We show the relative difference \( r = |\Lambda_{\text{d}}^2 - \Lambda^2| / \Lambda^2 \) as a function of \( \log a \) (x axis) and \( y \) (y axis). Notice the minima for \( \log a \simeq 0.4 - 0.6 \).

Figure 7. Graviton-graviton scattering: the relative difference \( r = |\Lambda_{\text{d}}^2 - \Lambda^2| / \Lambda^2 \) is plotted as a function of \( \log a \) (x axis) and \( y \) (y axis). We restrict our analysis to the values \( 0.05 \leq y \leq 1 \) and \( \log a \leq 2 \).
it is only inside the $\log(x_d)$. Our criterion for the definition of $d_c$ is the one obtained from

$$\frac{2}{d-4} = \left| 2 \log a - \gamma_E - \frac{2}{d-2} \log \frac{\Gamma(d-3)}{2y \Gamma(d/2-2)} - \log x_d \right|,$$

(8.33)

which is analogous to eq. (8.30) for the AC model. Note the change of numerical factor multiplying $\log a$ between the two equations.

Now, as for the AC model, for every $d_c$ we define the “distance” $r$ in terms of the relative difference between the unitarity cutoffs at $d_c$ and $d = 4$,

$$r(\log a, y) = \frac{\left| \Lambda_{d_c}^2 - \Lambda^2 \right|}{\Lambda^2},$$

(8.34)

where $\Lambda_d$ can be found in eq. (8.10), which is dominated by a zero as $d \to 4$ due to the IR divergences. We consider $r$ as a good indicator of the difference between $d_c$ and $d = 4$ because the unitarity cutoff controls the strength of the interactions. We also notice that the difference between these scales at different $d$ has the advantage of being independent of $\omega$, while a similar definition in terms of the difference between pole positions at different dimensions would depend on this parameter.

We then minimize $r$ by adjusting $\log a$ and $y$ with the constraint $d_c < 5$. This is shown in figure 7 where $r$ is plotted in the $(\log a, y)$ plane. A minimum is clearly visible for $\log a \simeq 1$. According to eq. (6.5) this leads to a pole at $d = 4$ whose position is

$$s_P \simeq (0.22 - i 0.63) \ G^{-1},$$

(8.35)

providing further support for the existence of the subplanckian graviball in pure gravity discussed in section 6. It is also important to emphasize that these conclusions are robust with respect to moderate variations of the criteria used to obtain the optimal and minimal $\log a$; namely by e.g. studying alternative “distances” such as $|s_P(d_c) - s_P|/|s_P|$ (for $\omega = 1$) or by changing the balance between pole and finite contributions to define $d_c$ in eq. (8.33).

We also stress that the same method that led us to the right value of $\log a \simeq 1/2$ for the AC model gives a larger value $\log a \simeq 1$ for graviton-graviton scattering.

Finally, it is important to point out that for $\log a \simeq 1$ the unitarity cutoff $\Lambda \simeq \pi G^{-1}$, and a dimensional estimate of the higher-order corrections to $s_P$ gives a typical size $|x| \simeq 2/3 \pi \approx 20\%$. Let us stress that this is the same estimate as for the $\sigma$ meson in QCD, whose pole position $s_{\sigma}$ shows a clear convergent pattern after unitarizing the subleading $\pi\pi$ PWAs [38, 40, 109]. It is, then, of great interest to investigate the unitarization of the NLO results [89, 90] for graviton-graviton scattering in order to test the robustness of the LO prediction and to calculate the NLO correction to $s_P$.

9 Conclusions and outlook

We have developed a unitarization formalism for calculating partial-wave amplitudes in the presence of infinite-range forces. This formalism has been applied to pure general relativity to study graviton-graviton scattering in partial waves by unitarizing the Born terms. We
have also applied it to other exactly solvable potentials, where we were able to retrieve non-trivial non-perturbative information about the scattering process.

In order to deal with the IR divergences associated with infinite-range interactions, we have defined a new $S$ matrix by removing a global phase factor whose presence is well known since the seminal paper by S. Weinberg on IR divergences for QED and in quantum gravity [53]. This factor also emerges by studying the asymptotic dynamics at times $t \to \pm \infty$ and should be removed to construct a well-behaved $S$ matrix, as established by the formalism of P. Kulish and L. Faddeev [54].

Within our prescription, we arrive at the conclusion that the graviton-graviton $J = 0$ partial-wave amplitude has a pole at $s_P = (\kappa - i2/(3\pi))\Lambda^2$, with $\kappa$ quite smaller than the imaginary part $2/(3\pi)$ and where $\Lambda^2 \propto G^{-1}$ is the ultraviolet cutoff scale. This corresponds to a graviton-graviton scalar resonance with vacuum quantum numbers $J^{PC} = 0^{++}$, that we dubbed the graviball. This resonance peaks at a value of $s$ significantly below $\Lambda^2$, because $\kappa \ll 1$. This can be understood in simple terms if we think of the Lorentzian function stemming from the pole, cf. eq. (6.6). For the actual modulus squared of the partial-wave amplitude there is some extra $s$ dependence along the real $s$ axis, but the feature that the peak lies at values of $s$ considerable smaller than $|s_P|$ still holds. The presence of such a peaked resonant structure would generate large corrections for graviton-graviton scattering in $S$-wave within the effective field theory of gravity. Indeed, given the smallness of $\kappa$, the consequences of the graviball would be completely analogous to the well-known phenomenon appearing in the scalar isoscalar meson-meson sector due to the lightest resonance of QCD, the $\sigma$.

The presence of the graviball is robust against the introduction of new ‘light’ degrees of freedom (masses below $G^{-1/2}$). In this case, given $N$ new fields, the position of the pole $s_P$ decreases as $\sim 1/N$. The graviball then becomes correspondingly lighter and narrower, with its resonance effects taking place at lower energies. If $\Lambda^2$ also scales like $\sim 1/N$, as expected from general arguments [14, 31, 122–124], then the relative position between the resonance and the cutoff would not change. On the other hand, in a self-healing scenario in which $\Lambda^2$ remained $\mathcal{O}(G^{-1})$ as $N$ increases the position of the resonance would drift towards energies much lower than the cutoff.

An intrinsic limitation of our scheme is related to the presence of an undetermined quantity $\log a$, coming from an artificial separation of hard and soft scales in loop diagrams. The non-perturbative result should substitute this uncertainty by a number which we expect to be $\mathcal{O}(1)$. We have explored this ansatz, together with the other possible drawbacks of our method, by applying our unitarization prescription to restore two-body unitarity in a quantum mechanical toy-model with a structure resembling general relativity (the AC scattering model). We found that our methods do, indeed, reproduce the first bound and resonance states of the spectrum of this model for natural values of $\log a$. Similarly, we studied the consequences for the unitarization of Coulomb potential. For this case it is interesting to point out that the unitarization method yields an approximation valid at higher energies than the typical scale $\Lambda$ for the problem. In this regime, we also managed to reproduce the existence and energy of the deepest bound state.
Finally, we have presented a first attempt to constrain the values of the \( \log a \) from studying the presence of the resonances at \( d > 4 \), and imposing the value of \( \log a \) that minimizes the effect coming from the IR divergences. Quite remarkably, these studies reproduce the resonance of the toy model, and generate a value \( \log a \approx 1 \) for graviton-graviton scattering. If this is confirmed by other calculations, it would lead to the conclusion that a graviball exists at energies below the nominal cutoff of the theory, \( G^{-1} \).

There are several directions worth exploring in the future. On one hand, it would be important to determine the resonance pole position of the graviball by using the full one and two-loop level graviton-graviton scattering amplitudes in the unitarization process. These are known in the literature with different matter content, e.g. [89, 137–139].

Similarly, string theory amplitudes may be relevant for this purpose, see e.g. [15, 140], as may be other inputs from other proposals to UV complete theories of gravity, e.g. [96–101]. We stress that a very well defined program is to extend our unitarization formalism to coupled channels involving other (massive) particle species. Similarly, even if our methods are not able to cure all the problems with gravitational scattering amplitudes that hampered the application of the \( S \)-matrix theory to gravitational amplitudes [12, 141–143], they may eventually suggest possible new approaches. Finally, it seems pertinent to explore the connection to other non-perturbative approaches to graviton-graviton scattering. In particular, the question whether the bound states investigated in [14, 62, 65] may be accessible by our methods, or if any hint of the graviball may appear in AdS/CFT analysis. Actually, the existence of a prominent scalar resonance in graviton-graviton scattering in ten-dimensional supergravity has also been identified within the \( S \)-matrix bootstrap in ref. [144], after the present paper and ref. [16] were sent for publication. For a recent review on the \( S \)-matrix bootstrap see ref. [145] and references therein.

We would like to mention also that having extended the nonperturbative unitarization methods familiar in hadron physics to infinite-range interactions allows one to apply them to account for the Coulomb contributions in two-body scattering of charged particles, like \( \pi^+ \pi^- \), \( \pi^\pm p \) or \( K^\pm p \), at the same time as the strong interactions are unitarized. This should translate in a more accurate description of the data affected by Coulomb scattering near threshold, as well as in a better determination of the free parameters fitted in those processes. E.g. in \( K^\pm p \) scattering there are many data that exhibit large Coulomb corrections just above threshold that could be treated in an improved manner employing the present approach [146, 147].

Before closing, let us remark that our analysis has dealt with the theoretical aspects of the graviball. However, there may also be observational consequences worth exploring after assuming that the resonance survives in UV-complete models. This may happen in the processes with the highest possible energies in the gravitational degrees of freedom. The natural candidate would be the impact in the studies of primordial gravitational waves and its experimental searches. For instance, the presence of a scalar resonance that strongly couples to graviton-graviton could produce a reduction of tensor perturbations in the background of gravitational waves expected from inflationary models, with the subsequent reduction of expected \( B \)-mode signals in the polarization of the CMB. This may be difficult in standard models of inflation, where the viable tensor-to-scalar ratio imply a hierarchy
between the scale of inflation and $G^{-1}$, e.g. [148]. This result may be altered in theories with more degrees of freedom, where the resonance may happen at parametrically smaller values, while the scale of inflation may remain the same [149]. However, since the cut-off of the theory may also scale with $\sqrt{N}$, this possibility requires more investigation. Also, in theories with large enough number of light degrees of freedom [122, 125, 150] (or with large extra dimensions) the resonance may also modify the gravitational phenomena at scales as low as $\mu$m (corresponding to a UV cutoff around the TeV scale) [66, 92–95, 123, 151]. In these cases, the fundamental cutoff of the theory evolves with $N$ as $\Lambda^2 = G^{-1}/N$. Within our calculation, this implies that $x$ and $\omega$ stay put but then $|s_p| = |x|\Lambda^2$ in absolute terms decreases as $1/N$. This opens interesting phenomenological possibilities, for tests of gravity at the $\mu$m scales and even at the level of collider physics, that are left for future studies.

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A Graviton states and partial-wave amplitudes

To fix notation we first define in detail the one- and two-graviton states that we use. The general procedure that we follow to define our states is similar to that in ref. [69]. The one-graviton states $|p, \lambda\rangle$ result from the action of a standard rotation $R(\hat{p})$ on a state with the same helicity and three-momentum along the $z$ axis:

$$|p, \lambda\rangle = R(\hat{p})|pz, \lambda\rangle.$$ (A.1)

The standard rotation transforms $z$ to $p$, namely, $R(\hat{p})z = \hat{p}$. In our convention we define it explicitly as

$$R(\hat{p}) = R_z(\phi)R_y(\theta),$$

$$p = p(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta),$$ (A.2)

where $\theta$ and $\phi$ are the polar and azimuthal angles for $p$.

The two-graviton states in the center-of-mass frame (CM), $|p, \lambda_1\lambda_2\rangle$, with momentum $p$ and helicities $\lambda_1$ and $\lambda_2$, are defined as

$$|p, \lambda_1\lambda_2\rangle = |p, \lambda_1\rangle - |p, \lambda_2\rangle.$$ (A.3)
It is straightforward to show that they can also be expressed in terms of the standard rotation $R(\hat{p})$ as

$$|p, \lambda_1 \lambda_2\rangle = R(\hat{p})|pz, \lambda_1\rangle - pz, \lambda_2\rangle. \tag{A.4}$$

The expansion of these states into states with well defined total angular momentum $J$ and third component $J_z = M$, proceeds by noticing that the states $|pz, \lambda_1 \lambda_2\rangle$ have $M = \lambda_1 - \lambda_2$. Therefore,

$$|pz, \lambda_1 \lambda_2\rangle = 2\pi \sum_J \sqrt{2J + 1} |pJ, \lambda_1 \lambda_2\rangle. \tag{A.5}$$

where $\lambda = \lambda_1 - \lambda_2$ and $2\pi \sqrt{2J + 1}$ is set as normalization factor, cf. eq. (A.8) below. After acting on both sides of eq. (A.5) with the standard rotation $R(\hat{p})$, we arrive at the expression of the plane-wave states in terms of those in the spherical basis,

$$|p, \lambda_1 \lambda_2\rangle = \sum_M \sqrt{2J + 1} \frac{1}{\sqrt{2^{2J}}} \mathcal{D}^{(J)}_{M\lambda}(\phi, \theta, 0) |pJ, \lambda_1 \lambda_2\rangle. \tag{A.6}$$

Here, $\mathcal{D}^{(J)}_{M\lambda}(\phi, \theta, 0)$ is the rotation matrix specified by the Euler angles corresponding to the standard rotation $R(\hat{p})$ for the irreducible representation of the rotation group with angular momentum $J$ [72]. By inverting this expression, one finds $|pJ, \lambda_1 \lambda_2\rangle$ in terms of the $|p, \lambda_1 \lambda_2\rangle$ states as,

$$|pJ, \lambda_1 \lambda_2\rangle = \frac{\sqrt{2J + 1}}{8\pi^2} \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos \theta \mathcal{D}^{(J)}_{M\lambda}(\phi, \theta, 0)^* |p, \lambda_1 \lambda_2\rangle. \tag{A.7}$$

The normalization is chosen such that

$$\langle p'j'\lambda'_1 \lambda'_2|pj, \lambda_1 \lambda_2\rangle = \frac{2}{\pi} \delta_{\lambda'_1 \lambda_1} \delta_{\lambda'_2 \lambda_2}, \tag{A.8}$$

where a global $(2\pi)^4 \delta^{(4)}(P_f - P_i)$ has been factorized out because of the total four-momentum conservation. In obvious notation, $P_f$ and $P_i$ are the total four-momenta of the (left) final and (right) initial states, respectively.

Now we take explicitly into account the Bose-Einstein symmetric character of the two-graviton states which is indicated in the following by a subscript $S$ in the ket,

$$|p, \lambda_1 \lambda_2\rangle_S = \frac{1}{\sqrt{2}} \left[ |p, \lambda_1 \lambda_2\rangle + |p, \lambda_2 \lambda_1\rangle \right], \tag{A.9}$$

which can also be expressed in terms of the standard rotation $R(\hat{p})$ as

$$|p, \lambda_1 \lambda_2\rangle_S = R(\hat{p})|pz, \lambda_1 \lambda_2\rangle_S. \tag{A.10}$$

Its expansion in the basis with well-defined angular-momentum can be obtained by proceeding analogously to the derivation of eq. (A.6). It reads,

$$|p, \lambda_1 \lambda_2\rangle_S = \sum_{J,M} 2\pi \sqrt{2J + 1} \mathcal{D}^{(J)}_{M\lambda}(\phi, \theta, 0) \frac{1}{\sqrt{2}} \left[ |pJ, \lambda_1 \lambda_2\rangle + (-1)^J |pJ, \lambda_2 \lambda_1\rangle \right]. \tag{A.11}$$
In the following we denote by $|pJM, \lambda_1 \lambda_2\rangle_S$ the states
\begin{equation}
|pJM, \lambda_1 \lambda_2\rangle_S = \frac{1}{\sqrt{2}} \left( |pJM, \lambda_1 \lambda_2\rangle + (-1)^J |pJM, \lambda_2 \lambda_1\rangle \right),
\end{equation}
which is (anti)symmetric under the exchange of the helicities of the two gravitons if $J$ is (odd) even.

It is straightforward to invert eq. (A.11) to find
\begin{equation}
|pJM, \lambda_1 \lambda_2\rangle_S = \frac{\sqrt{2J+1}}{8\pi^2} \int_0^{2\pi} d\phi \int_{-1}^{+1} d\cos \theta \, D_{M\lambda}^{(J)}(\phi, \theta, 0)^* |p, \lambda_1 \lambda_2\rangle_S. \tag{A.13}
\end{equation}

Let us discuss the formal definition of the PWAs for graviton-graviton scattering. The PWA expansion of a two by two scattering process follows from eq. (A.11), which is more conveniently applied when the final three-momentum $p'$ lies in the $xz$ plane with $\phi = 0$, namely $p' = p'_{xz} = (\sin \theta, 0, \cos \theta)$. Thus,
\begin{equation}
S(p', \lambda'_1 \lambda'_2 | p \bar{z}, \lambda_1 \lambda_2)_S = 4\pi^2 \sum_J (2J + 1) \sum_{M=-J}^{J} d^{(J)}_{M\lambda}(\theta) d^{(J)}_{M\lambda}(0) S(pJ, \lambda'_1 \lambda'_2 | pJ, \lambda_1 \lambda_2)_S = \delta_{J\lambda} \tag{A.14}
\end{equation}
and only the Wigner (small) d-matrix $d^{(J)}_{\lambda\lambda'}(\theta)$ enters in the angular projection.\footnote{We have dropped the label $M$ of the third component of total angular momentum because the PWAs do not depend on it due to the Wigner-Eckart theorem applied to rotational symmetry.} We can invert eq. (A.6) by taking into account the orthogonality property of the Wigner d-matrix functions [72],
\begin{equation}
\int_{-1}^{1} d\cos \theta \, d^{(j_1)}_{\lambda\lambda}(\theta) d^{(j_2)}_{\lambda\lambda}(\theta) = \frac{2}{2j_1 + 1} \delta_{j_1 j_2}. \tag{A.15}
\end{equation}
Then, the partial-wave amplitude $\tilde{T}^{(J)}_{\lambda'_1 \lambda'_2, \lambda_1 \lambda_2}(s) \equiv S(pJ, \lambda'_1 \lambda'_2 | pJ, \lambda_1 \lambda_2)_S$ is given by
\begin{equation}
\tilde{T}^{(J)}_{\lambda'_1 \lambda'_2, \lambda_1 \lambda_2}(s) = \frac{1}{8\pi^2} \int_{-1}^{+1} d\cos \theta' \, d^{(J)}_{\lambda\lambda'}(\theta') S(p'_{xz}, \lambda'_1 \lambda'_2 | p\bar{z}, \lambda_1 \lambda_2)_S. \tag{A.16}
\end{equation}

The symmetric states $|pJM, \lambda_1 \lambda_2\rangle_S$ for $\lambda_1 = \lambda_2$ can only have $J =$ even and then $|pJM, \lambda_1 \lambda_1\rangle_S = \sqrt{2} |pJM, \lambda_1 \lambda_1\rangle$. Therefore, these states are normalized to $4/\pi$. However, the symmetric states with $\lambda_1 \neq \lambda_2$ can sustain both $J$ even and odd and they are normalized to $2/\pi$. Because of this different normalization by a factor of 2, the relation between the $S$- and $T$-matrix in PWAs is different, and this is the reason behind the factor $2^{|\lambda|/4}$ in eq. (2.12).
B Partial-wave amplitudes in arbitrary dimensions

In this section we derive the PWAs as a function of the space-time dimensions \(d\) for graviton-graviton and spinless-particle scattering. The needed angular integrations in \(d = 4 - 2\epsilon\) dimensions are taken from ref. [133]. One of them is the area of the unit sphere in \(d - 1\) spatial dimensions \(\Omega_{d-1}\), which reads

\[
\Omega_{d-1} = \frac{(4\pi)^{d/2-1}}{\Gamma(d/2-1)} \frac{\Gamma(d/2)}{\Gamma(d-2)}.
\]

We start the discussion by considering the expansion of the plane-wave two-graviton states in the basis of states with definite \(J\),

\[
|\mathbf{p}, \lambda_1 \lambda_2 \rangle = \sum_{J,M} C_J D_M^{(J)}(\mathbf{p}, 0) |pJM, \lambda_1 \lambda_2 \rangle,
\]

which is analogous to eq. (A.6) but now the coefficient \(C_J\) depends on \(d\). The second argument is set equal to zero because the \(\hat{z}\) axis is always rotated to \(\hat{p}\) by the standard rotation, which is completely specified by \(\hat{p}\) once we set, by convention, a first rotation around the \(\hat{z}\) axis equal to the identity (analogously to \(d = 4\)). For brevity in the notation we remove this zero in the following and keep only \(\hat{p}\) in the argument of the standard rotation.

To write the states with definite \(J\) in terms of the plane-wave ones from eq. (B.2), we use the orthogonality property of the rotation matrices in \(d\) dimensions, which is a consequence of the unitary character of the rotations. It can be written in \(d\) dimensions as,

\[
\int d\Omega_{d-1} D_{M'\lambda}^{(J')} (\hat{p})^* D_{M\lambda}^{(J)} (\hat{p}) = \frac{\Omega_{d-1}}{2J(d) + 1} \delta_{JJ'} \delta_{MM'}.
\]

It is clear that this orthogonality relation requires \(\Omega_{d-1}\) because for \(J = 0\) the rotation matrix is the identity matrix, and then we simply have the volume of the unit sphere in \(d - 1\) dimensions. The factor \(2J + 1\) in \(d = 4\) would become dependent on \(d\) for general \(J\), and this is why we have written it as \(2J(d) + 1\), in order to keep track of this fact for \(J \neq 0\). We do not really need to be more specific about it because it will indeed disappear from the final expression for the calculation of the PWAs in terms of the scattering amplitudes. Also, we are only interested in \(J = 0\), and for this case the coefficient in front of the Kronecker delta functions in the right-hand side of eq. (B.3) is \(\Omega_{d-1}\). Multiplying eq. (B.2) by \(D_{M\lambda}^{(J)}(\hat{p})^*\) and applying the orthogonality relation of eq. (B.3) one can express \(|pJM, \lambda_1 \lambda_2 \rangle\) as

\[
|pJM, \lambda_1 \lambda_2 \rangle = \frac{2J(d) + 1}{\Omega_{d-1} C_J} \int d\Omega_{d-1} D_{M'\lambda}^{(J')} (\hat{p})^* |pJM, \lambda_1 \lambda_2 \rangle.
\]

Regarding the normalization of the states with definite \(J\) we keep the same one as in \(d = 4\), given in eq. (A.8), by fixing properly \(C_J\). This choice guarantees that we do not have to change the unitarity loop function \(g(s)\) because its discontinuity does not change and then we have the same DR in the variable \(s\).

The one-graviton states are normalized such that \(\langle p\lambda'|p\lambda \rangle = 2p(2\pi)^{d-1} \delta(p' - p)\delta_{\lambda'\lambda}\).

The two graviton states, after removing the global factor \((2\pi)^d\delta(d)(P' - P)\) associated with
We also introduce the standard rotation

\[ \langle pJ' M', \lambda_1' \lambda_2' | pJM, \lambda_1 \lambda_2 \rangle = \frac{(2J(d) + 1)2d+1\pi^{-2}}{\Omega_{d-1}C_{d}^{2}p^{-4}} \delta_{\lambda_1' \lambda_1} \delta_{\lambda_2' \lambda_2} \delta_{J'J}, \]

(B.5)

The coefficient \( C_{J} \) is then

\[ C_{J} = \sqrt{\frac{(2J(d) + 1)2(2\pi)^{-1}}{\Omega_{d-1}p^{-4}}} \quad \text{(B.6)} \]

Let us now consider the calculation of the PWAs. By directly implementing eq. (B.2) with the coefficient \( C_{J} \) just calculated, one has

\[ s(p', \lambda_1' \lambda_2' \lambda_1 \lambda_2) = \sum_{J', J, M} \sum_{J', M, J, M} C_{J} C_{J'} D^{(J)}_{M' \lambda'}(\hat{p}')^* S(pJ' M', \lambda_1' \lambda_2' | pJM, \lambda_1 \lambda_2) S \quad \text{(B.7)} \]

\[ = \sum_{J} \frac{(2J(d) + 1)2(2\pi)^{-1}}{\Omega_{d-1}p^{-4}} D^{(J)}_{\lambda \lambda}(\hat{p})^* s(pJ, \lambda_1' \lambda_2' | pJM, \lambda_1 \lambda_2) \quad \text{(B.8)} \]

The next step is to isolate the PWA by using the orthogonality properties of the rotation matrices in \( d \) dimensions, cf. eq. (B.3). Then,

\[ s(pJ, \lambda_1' \lambda_2' \lambda_1 \lambda_2) = \frac{p^{-4}}{2(2\pi)^{-1}} \int d\Omega_{d-1} D^{(J)}_{\lambda \lambda}(\hat{p})^* s(pJ, \lambda_1' \lambda_2' | pJM, \lambda_1 \lambda_2) \quad \text{(B.8)} \]

Let us also mention that if the Born term can be expressed in terms of products of momenta, as in gravity [89], we can then use the same expression as in \( d = 4 \) in eq. (B.8), except for an overall change of dimensions in the coupling.

We now derive the expression for the PWAs with varying dimensions for the scattering of two massive spinless particles. The normalization of the one-particle states is the Lorentz invariant one \( 2E(p)(2\pi)^{-1}\delta(\hat{p}' - \hat{p}) \), where \( E(p) = \sqrt{\mu^2 + p^2} \) and \( \mu \) is the mass of either particle. For simplicity, in the discussion we particularize to equal mass scattering, the one we need here, though its generalization to particles with different masses is straightforward.

We also introduce the standard rotation \( R(\hat{p}) \) that takes \( \hat{z} \) to \( \hat{p} \). With the one-particle states defined such that \( |p\rangle = R(p)|\hat{z}\rangle \), then we also have a relation analogous to eq. (A.10) for a state of two spinless particles, \( |p\rangle \otimes | -p\rangle \),

\[ |p\rangle \otimes | -p\rangle = R(\hat{p})|p\hat{z}\rangle \otimes | -p\hat{z}\rangle \quad \text{(B.9)} \]

In order to simplify the notation in the following we simply denote a two particle state also by \(|p\rangle\), and only use this symbol only referred to this state. Then, a \(|p\hat{z}\rangle\) state is an eigenvector of the third component of the angular-momentum operator with null eigenvalue. Therefore, we can write it in terms of partial-wave states with definite angular momentum \( J \) and third component \( M \), \(|pJM\rangle\), as

\[ |p\hat{z}\rangle = \sum_{J} C_{J} |pJ0\rangle \quad \text{(B.10)} \]
From here on the procedure is completely analogous to the one developed for the graviton-graviton states. Instead of eq. (B.2) we have now

$$|p⟩ = \sum_J \sum_M C_J D^{(J)}_{M0}(\hat{p}) |pJM⟩. \quad (B.11)$$

After using the orthogonality relation between the rotation matrices, eq. (B.3), we can invert the expansion in eq. (B.11) with the result,

$$|pJM⟩ = \frac{2J(d) + 1}{Ω_{d-1} C_J} \int dΩ_{d-1} D^{(J)}_{M0}(\hat{p})^* |p⟩. \quad (B.12)$$

The coefficient $C_J$ is fixed as usual by imposing the normalization of the partial-wave states to be the same as in $d=4$. Namely,

$$\langle pJ′M′|pJM⟩ = \frac{4π\sqrt{s}}{p} \delta_{J′J} δ_{M′M}. \quad (B.13)$$

For the kinematics of two relativistic particles of equal mass $µ$ the normalization of the two-body state $⟨p′|p⟩$ is $2^{d+1}π^{d-2}E(p)/p^{d-3}$. This allows us to fix $C_J$, analogously as it was done above in eq. (B.5), with the result

$$C_J = \sqrt{\frac{(2J(d) + 1)2^{d-2}π^{d-3}}{Ω_{d-1} p^{d-4}}}. \quad (B.14)$$

We are then ready to obtain the expressions for the decomposition in PWAs and the calculation of the latter ones, cf. eqs. (B.7) and (B.8) above, which now read, respectively,

$$s⟨p′|T|p⟩S = \sum_J \frac{(2J(d) + 1)2^{d-2}π^{d-3}}{Ω_{d-1} p^{d-4}} D^{(J)}_{00} (\hat{p})^* S⟨pJ0|T|pJ0⟩S, \quad (B.15)$$

$$s⟨pJ0|T|p⟩S = \frac{p^{d-4}}{2^{d-2}π^{d-3}} \int dΩ_{d-1} D^{(J)}_{00} (\hat{p})^* S⟨p′|T|p⟩S. \quad (B.16)$$

In the case of two non-relativistic spinless particles the total energy of the system as a function of $p$ now reads $p^2/2m$ with $m = µ/2$, the reduced mass of the two particles. The normalization of the different states also changes: the one-particle states have the normalization $2π^3δ(p′ − p)$, and the normalization for the two-particle states in the plane-wave basis is $2π^{d-2}δ(\hat{p}' − \hat{p})/mp^{d-3}$. Regarding the states in the plane-wave basis the normalization is fixed to

$$\langle p′J′M′|pJM⟩ = \frac{π}{mp} δ_{J′J} δ_{M′M}. \quad (B.17)$$

One has the same equations (B.9)–(B.12) as above, and from there one deduces that the coefficient $C_J$ is the same as in eq. (B.14) for this case too. Therefore, the decomposition in PWAs and their calculations is again given by eqs. (B.15) and (B.16), respectively.

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