Fault-Tolerant Edge-Disjoint Paths — Beyond Uniform Faults

David Adjiashvili\textsuperscript{1}, Felix Hommelsheim\textsuperscript{*2}, Moritz Mühlenthaler\textsuperscript{3}, and Oliver Schaudt\textsuperscript{4}

\textsuperscript{1}Department of Mathematics, ETH Zürich
\textsuperscript{2}Department of Mathematics, TU Dortmund University
\textsuperscript{3}Laboratoire G-SCOP, Grenoble INP, Univ. Grenoble-Alpes
\textsuperscript{4}Department of Mathematics, RWTH Aachen University

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Abstract

The overwhelming majority of survivable (fault-tolerant) network design models assume a uniform fault model. Such a model assumes that every subset of the network resources (edges or vertices) of a given cardinality $k$ may fail. While this approach yields problems with clean combinatorial structure and good algorithms, it often fails to capture the true nature of the scenario set coming from applications. One natural refinement of the uniform model is obtained by partitioning the set of resources into \textit{vulnerable} and \textit{safe} resources. The scenario set contains every subset of at most $k$ faulty resources. This work studies the \textit{Fault-Tolerant Path} (FTP) problem, the counterpart of the Shortest Path problem in this fault model and the \textit{Fault-Tolerant Flow} problem (FTF), the counterpart of the $\ell$-disjoint Shortest $s$-$t$ Path problem. We present complexity results alongside exact and approximation algorithms for both models. We emphasize the vast increase in the complexity of the problem with respect to the uniform analogue, the Edge-Disjoint Paths problem.

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1 Introduction

The Minimum-Cost Edge-Disjoint Path (EDP) problem is a classical network design problem, defined as follows. Given an edge-weighted directed graph \( D = (V, A) \), two terminals \( s, t \in V \) and an integer parameter \( k \in \mathbb{Z}_{\geq 0} \), find \( k \) edge-disjoint paths connecting \( s \) and \( t \) with minimum total cost. EDP is motivated by the following survivable network design problem: what is the connection cost of two nodes in a network, given that any \( k - 1 \) edges can be a-posteriori removed from the graph. The implicit assumption in EDP is that every edge in the graph is equally vulnerable. Unfortunately, this assumption is unrealistic in many applications, hence resulting in overly-conservative solutions. Our goal is to advance the understanding of non-uniform models for network design problems in order to avoid solutions that are too conservative and hence, too costly. To this end we study a natural generalization of the EDP problem called the Fault-Tolerant Path (FTP) problem, in which we consider a subset of the edges to be vulnerable. The problem asks for a minimum-cost subgraph of a given graph that contains an \( s \)-\( t \) path after removing any \( k \) vulnerable edges from the graph. Formally, it is defined as follows.

**Fault-Tolerant Path (FTP)**

**Instance:** edge-weighted directed graph \( D = (V, A) \), two nodes \( s, t \in V \), subset \( M \subseteq A \) of vulnerable edges, and integer \( k \in \mathbb{Z}_{\geq 0} \).

**Task:** Find minimum-cost set \( S \subseteq A \), such that \( S \setminus F \) contains an \( s \)-\( t \) path for every \( F \subseteq M \) with \(|F| \leq k\).

Observe that FTP becomes EDP when \( M = A \). We will also study EDP with a simpler, but still non-uniform, fault model: The problem Fault-Tolerant Flow (FTF) asks for \( \ell \geq 1 \) fault-tolerant disjoint \( s \)-\( t \) paths, assuming that only a single edge can be a-posteriori removed from the graph. The problem is defined as follows.

**Fault-Tolerant Flow (FTF)**

**Instance:** edge-weighted directed graph \( D = (V, A) \), two nodes \( s, t \in V \), set \( M \subseteq A \) of vulnerable arcs, and integer \( \ell \in \mathbb{Z}_{\geq 0} \).

**Task:** Find minimum cost set \( S \subseteq A \), such that \( S \setminus f \) contains \( \ell \) disjoint \( s \)-\( t \) paths for every \( f \in M \).

1.1 Results

A well-known polynomial algorithm for EDP works as follows. Assign unit capacities to all edges in \( G \) and find a minimum-cost \( k \)-flow from \( s \) to \( t \). The integrality property of the Minimum-Cost \( s \)-\( t \) Flow (MCF) problem guarantees that an extreme-point optimal solution is integral, hence it corresponds to a subset of edges. It is then straightforward to verify that this set is an optimal solution of the EDP problem (for a thorough treatment of this method we refer to the book of Schrijver [18]).

The latter algorithm raises two immediate questions concerning FTP. The first question is whether FTP admits a polynomial time algorithm. In this paper we give a negative answer conditioned on \( P \neq NP \), showing that FTP is \( NP \)-hard. In fact, the existence of constant-factor approximation algorithms is unlikely already for the restricted case of directed acyclic graphs. Consequently, it is natural to ask whether polynomial algorithms can be obtained for restricted
variants of FTP to this question. In particular we provide polynomial-time algorithms for arbitrary graphs and $k = 1$, directed acyclic graphs and fixed $k$ and series-parallel graphs. A second question concerns the natural fractional relaxation FRAC-FTP of FTP, in which the task is to find a minimum cost capacity vector $x \in [0,1]^A$ such that for every set $F$ of at most $k$ vulnerable edges, the maximum $s$-$t$ flow in $G_F = (V,A \setminus F)$, capacitated by $x$, is at least one. As we previously observed, one natural relaxation of EDP is the MCF problem. This relaxation admits an integrality gap of one, namely the optimal integral solution value is always equal to the corresponding optimal fractional value. We later show that, in contrast to MCF, the integrality gap of FRAC-FTP is bounded by $k + 1$. Furthermore, we show that this bound is tight, namely that there exists an infinite family of instances with integrality gap arbitrarily close to $k + 1$. This result also leads to a simple $(k + 1)$-approximation algorithm for FTP, which we later combine with an algorithm for the case $k = 1$ to obtain a $k$-approximation algorithm for FTP.

The second variant of the EDP we study is FTF, which asks for $\ell \geq 1$ disjoint $s$-$t$ paths in the presence of non-uniform faults. Note that if we consider uniform faults (every edge is vulnerable), an optimal solution is a minimum-cost $s$-$t$ flow of value $k + \ell$, which can be computed in polynomial time. We show that, again, the presence of non-uniform faults makes the problem much harder. In fact, it is as hard to approximate as FTP, despite the restriction to single-arc faults (the same result holds for FTF on undirected graphs). On the positive side, we give a simple polynomial-time $(\ell + 1)$-approximation algorithm for FTF which computes a MCF with appropriately chosen capacities.

Note that our positive results for FTP imply a polynomial-time algorithm for FTF and $\ell = 1$. Hence it is natural to investigate the dependence of the complexity of FTF on the number $\ell$ of disjoint paths. To this end, we fix $\ell$ and study the corresponding slice Fault-Tolerant $\ell$-Flow of FTF. Our main result in this setting is a 2-approximation algorithm for Fault-Tolerant $\ell$-Flow. In a nutshell, the algorithm first computes minimum-cost $\ell$-flow and then makes the resulting $\ell$ disjoint paths fault tolerant by solving the corresponding augmentation problem. We solve the augmentation problem by reducing it to a shortest path problem; it is basically a dynamic programming algorithm in disguise. The reduction is quite involved: in order to construct the instance of Shortest $s$-$t$-Path, we solve at most $n^{2\ell}$ instances of the Min-cost Strongly Connected Subgraphs problem on $\ell$ terminal pairs, all of which can be done in polynomial time since $\ell$ is fixed.

Given our approximation results, one may wonder whether Fault-Tolerant $\ell$-Flow might admit a polynomial-time algorithm (assuming $P \neq NP$, say). An indication in this direction is that for number of problems with a similar flavor, including robust paths [3], robust matchings [15] or robust spanning trees [2], hardness results were obtained by showing that the corresponding augmentation problems are hard. In the light of our results above this approach does not work for Fault-Tolerant $\ell$-Flow. On the other hand, we show that such a polynomial-time algorithm for Fault-Tolerant $\ell$-Flow implies polynomial-time algorithms for 1-2-connected Directed 2 Steiner Tree and a special case of 2-connected Directed $k$ Steiner Tree. Whether these two problems are NP-hard or not are long-standing open questions.

1.2 Related Work

The shortest path problem is one of the classical problems in combinatorial optimization, and as such, it has received considerable attention also in the context of fault tolerance/robustness, see for example [4, 5, 9, 12, 17, 19, 20]. Considering FTP and FTF, one of the most relevant notions of robustness is bulk-robust, introduced by Adjishvili, Stiller and Zenklusen [3]. Here, we are given a set of failure scenarios, that is, a set of subsets of resources that may fail simultaneously. The task is to find a minimum-cost subset of the resources, such that a desired property (e.g., connectivity of
a graph) is maintained, no matter which failure scenario emerges. Both FTP and FTF are special cases of this model. Adjiashvili, Stiller and Zenklusen considered bulk-robust counterparts of the Shortest Path and Minimum Matroid basis problems. For bulk-robust shortest paths on undirected graphs they give a \( O(k + \log n) \)-approximation algorithm, where \( k \) is the maximum size of a failure scenario. However, not that the running-time of this algorithm is exponential in \( k \). Note that their bulk-robust shortest path problem generalizes FTP, and therefore the same approximation guarantee holds for FTP. Our approximation algorithm for FTP significantly improves on this bound, on both the approximation guarantee and the running-time.

The robustness model used in this paper is natural for various classical combinatorial optimization problems. Of particular interest is the counterpart of the Minimum Spanning Tree problem. This problem is closely related to the Minimum \( k \)-Edge Connected Spanning Subgraph (\( k \)-ECSS) problem, a well-understood robust connection problem. There are numerous results for the unweighted version of \( k \)-ECSS. Gabow, Goemans, Tardos and Williamson [11] developed a polynomial time \( (1 + \frac{c}{k}) \)-approximation algorithm for \( k \)-ECSS, for some fixed constant \( c \). The authors also show that for some constant \( c' < c \), the existence of a polynomial time \( (1 + \frac{c'}{k}) \)-approximation algorithm implies \( P = NP \). An intriguing property of \( k \)-ECSS is that the problem becomes easier to approximate when \( k \) grows. Concretely, while for every fixed \( k \), \( k \)-ECSS is \( NP \)-hard to approximate within some factor \( \alpha_k > 1 \), the latter result asserts that there is function \( \beta(k) \) tending to one as \( k \) tends to infinity such that \( k \)-ECSS is approximable within a factor \( \beta(k) \). This phenomenon was already discovered by Cheriyan and Thurimella [8], who gave algorithms with a weaker approximation guarantee. The more general Generalized Steiner Network problem admits a polynomial 2-approximation algorithm due to Jain [16]. This is also the best known bound for weighted 2-ECSS.

Adjaiashvili, Hommelsheim and Mührenthaler [2] considered the bulk-robust minimum spanning tree problem with non-uniform single-edge failures. Their main result is a 2.523-approximation algorithm for this problem. A problem of a similar flavor but with a uniform fault model is Weighted Robust Matching Augmentation, which was studied by Hommelsheim, Mührenthaler and Schaudt [15]. The task is to find a minimum-cost subgraph, such that after the removal of any single edge, the resulting graph contains a perfect matching. They show that this problem is as hard to approximate as Directed Steiner Forest, which is known to admit no \( \log^{2-\varepsilon} \)-approximation algorithm unless \( NP \subseteq ZTIME(n^{\log(n)}) \) [14]. We will later show that FTF generalizes Weighted Robust Matching Augmentation.

### 1.3 Notation

We mostly consider directed graphs, which we denote by \((V, A)\), where \( V \) is the set of vertices set and \( A \) the set of arcs. Undirected graphs are denoted by \((V, E)\), where \( E \) is a set of edges. An orientation of a set \( E \) of undirected edges is an arc-set that orients each edge \( vw \in E \) as an arc \( vw \) or \( uv \). For some vertex set \( V' \subseteq V \) we denote by \( \delta(V') := \{ vw \in E \mid v \in V', w \in V \setminus V' \} \). For two vertex sets \( X, Y \subseteq V \) we write \( E(X, Y) := \{ xy \in E \mid x \in X \setminus Y, y \in Y \setminus X \} \) for the set of edges joining \( X \) and \( Y \) (the graph should be clear from the context). In a directed graph we simply replace \( E \) by \( A \). In this paper we usually consider edge-weighted graphs and assume throughout that weights are non-negative. The arcs of \( A \) that are not vulnerable are called safe, denoted by \( \overline{M} := A \setminus M \).

For the sake of a clearer presentation, we moved proofs of results marked by (*) to the appendix. A preliminary version of this paper can be found here [1].
1.4 Organization

The remainder of this paper is organized as follows: We present our results on the problem FTP in Section 2 and our results on the problem FTF in Section 3. In Section 2.1, we show that FTF on undirected graphs is a special case of FTF on directed graphs. We study the approximation hardness of FTF in Section 2.2 and we provide some exact polynomial algorithms for special cases in Section 2.3. In Section 2.4 we relate FTP and FRAC-FTP by proving a tight bound on the interagrality gap and show how this result leads to a $k$-approximation algorithm for FTP. In Section 3.1 we prove approximation hardness of FTF. We then give an $(\ell + 1)$-approximation algorithm in Section 3.2, followed by a 2-approximation algorithm for FTF with a fixed flow value $\ell$. Furthermore, in Section 3.3, we relate the complexity of FTF with fixed flow value to other problems of open complexity status. Section 4 concludes the paper and mentions some interesting open problems.

2 Fault-Tolerant Paths

2.1 Directed Versus Undirected Graphs

The classical shortest path problem is set on directed graphs. Assuming non-negative edge-weights, undirected graphs are a special case, since we may replace each undirected edge by two antiparallel directed edges and conclude that any shortest path in the resulting digraph corresponds to a shortest path in the original undirected graph. Here, we show that the same is true for FTP. The main insight is that, even if at most $k$ vulnerable edges may fail, no undirected edge is used in both directions. As a consequence, all our positive results for directed graphs in this section also hold for FTP on undirected graphs.

Proposition 1. Let $X \subseteq E$ be a feasible solution to an instance of FTP on an undirected graph $(V, E)$. Then there is an orientation $X$ of $X$ such that $(V, X - F)$ contains a directed $s$-$t$ path for every $F \subseteq M$ with $|F| \leq k$.

Proof. Let us assume for a contradiction that there is no such orientation. A set $Y$ of (undirected and directed) edges is a partial orientation of $X$ if there is a partition of $X$ into sets $X_1$ and $X_2$, such that $Y = X_1 \cup \bar{X}_2$, where $\bar{X}_2$ is an orientation of $X_2$. Let $Y$ be a partial orientation of $X$ that maximizes the number of directed edges, such that $(V, X - F)$ contains a directed $s$-$t$ path for every $F \subseteq M$ with $|F| \leq k$. By our assumption, there is at least one undirected edge $e = vw$ in $Y$. Furthermore, there are two sets $S_1, S_2 \subseteq V$ of vertices, such that $\{s\} \subseteq S_1, S_2 \subseteq V \setminus \{t\}$, $v \in S_1 \setminus S_2$, and $w \in S_2 \setminus S_1$. Note that $vw \in \delta(S_1)$ and $wv \in \delta(S_2)$.

Since $e$ is needed in both directions for $Y$ to be feasible, there is some $F \subseteq M, |F| \leq k$, such that $X \setminus F$ contains an $s$-$t$ path that must leave $S_1$ via $vw$. Therefore, the cut $\delta(S_1)$ contains at most $k + 1$ edges and all of them except possibly $e$ are vulnerable. The same holds for $\delta(S_2)$ and therefore we have $|\delta(S_1)| = |\delta(S_2)| = k + 1$. From the feasibility of $Y$ and the fact that all edges in $\delta(S_1)$ and $\delta(S_2)$ except possibly $e$ are vulnerable, it follows that $|\delta(S_1 \cap S_2)| \geq k + 1$ and $|\delta(S_1 \cup S_2)| \geq k + 1$. By the submodularity of the cut function $|\delta(\cdot)|$ we have

$$2k + 2 = |\delta(S_1)| + |\delta(S_2)| \geq |\delta(S_1 \cap S_2)| + |\delta(S_1 \cup S_2)| \geq 2k + 2$$

and it follows that

$$|\delta(S_1)| = |\delta(S_2)| = |\delta(S_1 \cap S_2)| = |\delta(S_1 \cup S_2)| = k + 1.$$  \hspace{1cm} (1)
2.2 Complexity of FTP

Our first observation is that FTP generalizes the Directed m-Steiner Tree Problem (m-DST). The input to m-DST is a weighted directed graph $D = (V, A)$, a source node $s \in V$, a collection of terminals $T \subseteq V$ and an integer $m \leq |T|$. The goal is to find a minimum-cost arborescence $X \subseteq A$ rooted at $s$, that contains a directed path from $s$ to some subset of $m$ terminal.

The m-DST is seen to be a special case of FTP as follows. Given an instance $I = (D, s, T, m)$ of m-DST define the following instance of FTP. The graph $D$ is augmented by $|T|$ new zero-cost arcs $A'$ connecting every terminal $u \in T$ to a new node $t$. Finally, we set $M = A'$ and $k = m - 1$. The goal is to find a fault-tolerant path from $s$ to $t$ in the new graph. It is now straightforward to see that a solution $S$ to the FTP instance is feasible if and only if $S \cap A'$ contains a feasible solution to the k-DST problem (we can assume that all arcs in $A'$ are in any solution to the FTP instance).

The latter observation implies an immediate conditional lower bound on the approximability of FTP. Halperin and Krauthgamer [14] showed that m-DST cannot be approximated within a factor $\log^{2-\epsilon} m$ for every $\epsilon > 0$, unless $\text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)})$. As a result we obtain the following.

**Proposition 2.** FTP admits no polynomial-time approximation algorithms with ratio $\log^{2-\epsilon} k$ for every $\epsilon > 0$, unless $\text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)})$.

The reduction above can be easily adapted to obtain a $k^\epsilon$-approximation algorithm for FTP for the special case that $M \subseteq \{ e \in A : t \in e \}$ using the algorithm of Charikar et. al. [6]. In fact, any approximation algorithm with factor $\rho(k)$ for FTP is an approximation algorithm with factor $\rho(m)$ for m-DST. The best known algorithm for the latter problem is due to Charikar et. al. [6]. Their result is an approximation scheme attaining the approximation factor of $m^\epsilon$ for every $\epsilon > 0$.

We end this discussion by showing that FTP contains a more general Steiner problem, which we call Simultaneous Directed m-Steiner Tree (m-SDST), as a special case. An input to m-SDST specifies two arc-weighted graphs $D_1 = (V, A_1, w_1)$ and $D_2 = (V, E_2, w_2)$ on the same set of vertices $V$, a source $s$, a set of terminals $T \subseteq V$ and an integer $m \leq |T|$. The goal is to find a subset $U \subseteq T$ of $m$ terminals and two arborescences $S_1 \subseteq E_1$ and $S_2 \subseteq A_2$ connecting $s$ to $U$ in the respective graphs, so as to minimize $w_1(S_1) + w_2(S_2)$. m-SDST is seen to be a special case of FTP via the following reduction. Given an instance of m-SDST, construct a graph $D = (V', A)$ as follows. Take a disjoint union of $D_1$ and $D_2$, where the direction of every arc in $D_2$ is reversed. Connect every copy of a terminal $u \in T$ in $D_1$ to its corresponding copy in $D_2$ with an additional zero-cost arc $e_u$. Finally, set $M = \{ e_u : u \in T \}$ and $k = m - 1$. A fault-tolerant path connecting the copy of $s$ in $D_1$ to the copy of $s$ in $D_2$ corresponds to a feasible solution to the m-SDST instance with the same cost, and vice-versa.

2.3 Polynomial Special Cases

This section is concerned with tractable restrictions of FTP. Concretely we give polynomial algorithms for arbitrary graphs and $k = 1$ and directed acyclic graphs (DAGs) and fixed $k$ and for Series-parallel graphs. We denote the problem FTP restricted to instances with some fixed $k$ by $k$-FTP.
1-FTP We start by giving the following structural insight.

Lemma 3 (*). Let $X^*$ be an optimal solution to FTP on the instance $(D, M, k)$. The minimum $s$-$t$ flow in the graph $(V, X^*)$ capacitated by the vector $c_e = 1$ if $e \in M$ and $c_e = \infty$, otherwise is at least $k + 1$.

An $s$-$t$ bipath in the graph $D = (V, A)$ is a union of two $s$-$t$ paths $P_1, P_2 \subseteq A$. In the context of 1-FTP, we call a bipath $Q = P_1 \cup P_2$ robust, if it holds that $P_1 \cap P_2 \cap M = \emptyset$. Note that every robust $s$-$t$ bipath $Q$ in $G$ is a feasible solution to the 1-FTP instance. Indeed, consider any vulnerable edge $e \in M$.

Lemma 4 (*). Every feasible solution $S^*$ to an 1-FTP instance contains a robust $s$-$t$ bipath.

We can conclude from the previous discussion and Lemma 4 that all minimal feasible solutions to the 1-FTP instance are robust bipaths. This observations leads to the simple algorithm, which is given in the proof of the following theorem.

Theorem 5 (*). 1-FTP admits a polynomial-time algorithm.

$k$-FTP and Directed Acyclic Graphs Let us first consider the case of a layered graph. The generalization to a directed acyclic graph is done via a standard transformation, which we describe later. Recall that a layered graph $D = (V, A)$ is a graph with a partitioned vertex set $V = V_1 \cup \cdots \cup V_r$ and a set of edges satisfying $A \subseteq \bigcup_{i \in [r-1]} V_i \times V_{i+1}$. We assume without loss of generality that $V_1 = \{s\}$ and $V_r = \{t\}$. For every $i \in [r-1]$ we let $A_i = A \cap V_i \times V_{i+1}$.

Analogously to the algorithm in the previous section, we reduce $k$-FTP to a shortest path problem in a larger graph. The following definition sets the stage for the algorithm.

Definition 6. An $i$-configuration is a vector $d \in \{0, 1, \ldots, k + 1\}^{V_i}$ satisfying $\sum_{v \in V_i} d_v = k + 1$. We let supp$(d) = \{v \in V_i : d_v > 0\}$. For an $i$-configuration $d^1$ and an $(i+1)$-configuration $d^2$ we let $V(d^1, d^2) = \text{supp}(d^1) \cup \text{supp}(d^2)$ and $A(d^1, d^2) = A[V(d^1, d^2)]$.

We say that an $i$-configuration $d^1$ precedes an $(i+1)$-configuration $d^2$ if the following flow problem is feasible. The graph is defined as $H(d^1, d^2) = (V(d^1, d^2), A(d^1, d^2))$. The demand vector $\nu$ and the capacity vector $c$ are given by

$$\nu_u = \begin{cases} -d^1_u & \text{if } u \in \text{supp}(d^1) \\ d^2_u & \text{if } u \in \text{supp}(d^2) \end{cases}$$

and $c_e = \begin{cases} 1 & \text{if } e \in M \\ \infty & \text{if } e \in E \setminus M \end{cases}$.

respectively. If $d^1$ precedes $d^2$ we say that the link $(d^1, d^2)$ exists. Finally, the cost $\ell(d^1, d^2)$ of this link is set to be minimum value $w(A')$ over all $A' \subseteq A(d^1, d^2)$, for which the previous flow problem is feasible, when restricted to the set of edges $A'$.

The algorithm constructs a layered graph $H = (V, A)$ with $r$ layers $V_1, \ldots, V_r$. For every $i \in [r]$ the set of vertices $V_i$ contains all $i$-configurations. Observe that $V_1$ and $V_r$ contain one vertex each, denoted by $c^0$ and $c^r$, respectively. The edges correspond to links between configurations. Every edge is directed from the configuration with the lower index to the one with the higher index. The cost is set according to Definition 6. The following lemma provides the required observation, which immediately leads to a polynomial algorithm.
Lemma 7 (*). Every $c^*$-$c^t$ path $P$ in $H$ corresponds to a fault-tolerant path $S$ with $w(S) \leq \ell(P)$, and vice-versa.

Finally, we observe that the number of configurations is bounded by $O(n^{k+1})$, which implies that $k$-FTP can be solved in polynomial time on layered graphs.

To obtain the same result for directed acyclic graphs we perform the following transformation of the graph. Let $v_1, \ldots, v_n$ be a topological sorting of the vertices in $D$. Replace every edge $e = v_i v_j$ ($i < j$) with a path $p_e = v_i, u_{i+1}, \ldots, u_{j-1}, v_j$ of length $j - i + 1$ by subdividing it sufficiently many times. Set the cost of the first edge on the path to $w'(v_i u_{i+1}) = w(v_i v_j)$ and set the costs of all other edges on the path to zero. In addition, create a new set of faulty edges $M'$, which contains all edges in a path $p_e$ if $e \in M$. It is straightforward to see that the new instance of FTP is equivalent to the original one, while the obtained graph after the transformation is layered. We summarize the result as follows.

Theorem 8. There is a polynomial algorithm for $k$-FTP restricted to instances with a directed acyclic graph.

Series-Parallel Graphs Recall that a graph is called series-parallel (SRP) if it can be composed from a collection of disjoint edges using the series and parallel compositions. The series composition of two SRP graphs with terminals $s, t$ and $s', t'$ respectively, takes the disjoint union of the two graphs, and identifies $t$ with $s'$. The parallel composition takes the disjoint union of the two graphs and identifies $s$ with $s'$ and $t$ with $t'$. Given a SRP graph it is easy to obtain the aforementioned decomposition.

The algorithm we present has linear running time whenever the robustness parameter $k$ is fixed. The algorithm is given as Algorithm 1. In fact, the algorithms computes the optimal solutions $S_{k'}$ for all parameters $0 \leq k' \leq k$. The symbol $\bot$ is returned if the problem is infeasible.

Theorem 9 (*). Algorithm 1 returns an optimal solution to the FTP problem on SRP graphs. The running time of Algorithm 1 is $O(nk)$.

2.4 Integrality Gap and Approximation Algorithms

In this section we study the natural fractional relaxation of FTP. We prove a tight bound on the integrality gap of this relaxation. This results also suggests a simple approximation algorithm for FTP with ratio $k + 1$. We later combine this algorithm with the algorithm for 1-FTP to obtain a $k$-approximation algorithm.

Fractional FTP and Integrality Gap Let us start by introducing the fractional relaxation of FTP, which we denote by FRAC-FTP. The input to FRAC-FTP is identical to the input to FTP. The goal in FRAC-FTP is to find a capacity vector $x : A \rightarrow [0, 1]$ of minimum cost $w(x) = \sum_{e \in A} w_e x_e$ such that for every $F \subseteq M$ of size at most $k$, the maximum $s$-$t$ flow in $D - F$, capacitated by $x$ is at least one. Note that by the Max-Flow Min-Cut Theorem, the latter condition is equivalent to requiring that the minimum $s$-$t$ cut in $D - F$ has capacity of at least one. We will use this fact in the proof of the main theorem in this section.

Observe that by requiring $x \in \{0, 1\}^E$ we obtain FTP, hence FRAC-FTP is indeed a fractional relaxation of FTP.

In the following theorem by 'integrality gap' we mean the maximum ratio between the optimal solution value to an FTP instance, and the optimal value of the corresponding FRAC-FTP instance.
Algorithm 1 : FTP-SeriesParallel\((G, s, t, k)\)

**Input:** \(G = (V, E)\) a series-parallel graph, \(s, t \in V\) and \(M \subseteq E, k \in \mathbb{Z}_{\geq 0}\).

**Ensure:** Optimal solution to FTP for parameters \(0, 1, \ldots, k\).

1:  
if \(E = \{e\} \land e \in M\) then
2:    Return \((\{e\}, \bot, \ldots, \bot)\)
3:  
if \(E = \{e\} \land e \notin M\) then
4:    Return \((\{e\}, \ldots, \{e\})\)

⇒ \(G\) is a composition of \(H_1, H_2\).

5:  \((S_0^1, \ldots, S_k^1) \leftarrow \text{FTP-SeriesParallel}(H_1, M \cap E[H_1], k)\)
6:  \((S_0^2, \ldots, S_k^2) \leftarrow \text{FTP-SeriesParallel}(H_2, M \cap E[H_2], k)\)
7:  if \(G\) is a series composition of \(H_1, H_2\) then
8:      for \(i = 0, \ldots, k\) do
9:          if \(S_i^1 = \bot \lor S_i^2 = \bot\) then
10:             \(S_i \leftarrow \bot\)
11:          else
12:             \(S_i \leftarrow S_i^1 \cup S_i^2\)
13:      end if
14:  end for
15:  if \(G\) is a parallel composition of \(H_1, H_2\) then
16:      \(m_1 \leftarrow \max\{i : S_i^1 \neq \bot\}\)
17:      \(m_2 \leftarrow \max\{i : S_i^2 \neq \bot\}\)
18:      for \(i = 0, \ldots, k\) do
19:          if \(i > m_1 + m_2 + 1\) then
20:              \(S_i \leftarrow \bot\)
21:          else
22:              \(r \leftarrow \arg\min_{-1 \leq j \leq i} \{w(S_j^1) + w(S_{i-j}^2)\}\)
23:              \(S_i \leftarrow S_r^1 \cup S_{i-r}^2\)
24:          end if
25:      end for
26:  end if
27:  Return \((S_0, \ldots, S_k)\)

**Theorem 10**\((\ast)\). The integrality gap of FTP is bounded by \(k + 1\). Furthermore, there exists an infinite family of instances of FTP with integrality gap arbitrarily close to \(k + 1\).

The proof of Theorem 10 implies a simple \((k + 1)\)-approximation algorithm for FTP. This algorithm simply solves the integer minimum-cost flow problem, defined in proof of the theorem, and returns the set of edges corresponding to the support of an optimal integral flow \(z^*\) as the solution. This result is summarized in the following corollary.

**Corollary 11.** There is a polynomial \((k + 1)\)-approximation algorithm for FTP.

**A \(k\)-Approximation Algorithm** In this paragraph we improve the approximation algorithm from the previous paragraph. The new algorithm can be seen as a generalization of the algorithm for 1-FTP to arbitrary FTP instances. The main observation is the following. The reason why the approximation algorithm implied by Theorem 10 gives an approximation ratio of \(k + 1\) is that the capacity of edges in \(A \setminus M\) is set to \(k + 1\), hence, if the flow \(z^*\) uses such edges to their full capacity, the cost incurred is \(k + 1\) times the cost of these edges. This implies that the best possible lower bound on the cost \(w(z^*)\) is \((k + 1)OPT_{FRAC}\), where \(OPT_{FRAC}\) denotes the optimal solution value of the corresponding FRAC-FTP instance. To improve the algorithm we observe that the edges in \(z^*\), which carry a flow of \(k + 1\) are cut-edges in the obtained solution.
To conveniently analyze our new algorithm let us consider a certain canonical flow defined by minimal feasible solutions.

**Definition 12.** Consider an inclusion-wise minimal feasible solution \( S \subseteq A \) of an instance \( I = (D, s, t, M) \) of FTP. A flow \( f^S \) induced by \( S \) is any integral \( s-t \) \((k+1)\)-flow in \( D \) respecting the capacity vector

\[
c^S_e = \begin{cases} 
1 & \text{if } e \in S \cap M \\
1 + k & \text{if } e \in S \setminus M \\
0 & \text{if } e \in A \setminus S.
\end{cases}
\]

To this end consider an optimal solution \( X^* \subseteq A \) to the FTP instance and consider any corresponding induced flow \( f^* \). Define

\[
X_{PAR} = \{e \in X^* : f^*(e) \leq k\} \quad \text{and} \quad X_{BRIDGE} = \{e \in X^* : f^*(e) = k + 1\}.
\]

As we argued before, every edge in \( X_{BRIDGE} \) must be a bridge in \( H = (V, X^*) \) disconnecting \( s \) and \( t \). Let \( u_e \) denote the tail vertex of an edge \( e \in A \). Since every edge \( e \in X_{BRIDGE} \) constitutes an \( s-t \) cut in \( H \), it follows that the vertices in \( U = \{e_u : e \in X_{BRIDGE}\} \cup \{s, t\} \) can be unambiguously ordered according to the order in which they appear on any \( s-t \) path in \( H \), traversed from \( s \) to \( t \). Let \( s = u_1, \ldots, u_q = t \) be this order. Except for \( s \) and \( t \), every vertex in \( U \) constitutes a cut-vertex in \( H \). Divide \( H \) into \( q-1 \) subgraphs \( H^1, \ldots, H^{q-1} \) by letting \( H^i = (V, Y_i) \) contain the union of all \( u_i-u_{i+1} \) paths in \( H \). We observe the following property.

**Proposition 13.** For every \( i \in [q-1] \) the set \( Y_i \subseteq A \) is an optimal solution to the FTP instance \( I_i = (G, u_i, u_{i+1}, M) \).

Consider some \( i \in [q-1] \) and let \( f_i^* \) denote the flow \( f^* \), restricted to edges in \( H^i \). Note that \( f_i^* \) can be viewed as a \( u_i-u_{i+1} \) \((k+1)\)-flow. Exactly one of the following cases can occur. Either \( H^i \) contains a single edge \( e \in A \setminus M \), or

\[
\max_{e \in Y_i} f_i^*(e) \leq k.
\]

In the former case, the edge \( e \) is the shortest \( u_i-u_{i+1} \) path in \((V, A \setminus M)\). In the latter case we can use a slightly updated variant of the algorithm in Corollary 11 to obtain a \( k \)-approximation of the optimal FTP solution on instance \( I_i \). Concretely, the algorithm defines the capacity vector

\[
c'_e = \begin{cases} 
k & \text{if } e \notin M \\
1 & \text{otherwise},
\end{cases}
\]

and finds an integral minimum-cost \( u_i-u_{i+1} \) \((k+1)\)-flow \( Y^* \) in \( D \), and returns the support \( Y \subseteq A \) of the flow as the solution. The existence of the flow \( f^*_i \) guarantees that \( w(Y^*) \leq w(f^*_i) \), while the fact that the maximum capacity in the flow problem is bounded by \( k \) gives \( w(Y) \leq kw(Y^*) \). It follows that this algorithm approximates the optimal solution to the FTP instance \( I_i \) to within a factor \( k \).

To describe the final algorithm it remains use the blueprint of the algorithm for 1-FTP. There is only one slight difference. Instead of finding two edge-disjoint \( u-v \) paths, the new algorithm solves the aforementioned flow problem. We summarize the main result of this section in the following theorem. The proof is omitted, as it is identical to that of Theorem 5, with the exception of the preceding discussion.

**Theorem 14.** There is a polynomial \( k \)-approximation algorithm for FTP.

10
3 Fault-Tolerant Flows

In this section we present our results on the problem FTF. We show that it admits no \( \log^{2-\epsilon} \ell \)-approximation under standard complexity assumptions. We then investigate its complexity for flows of fixed value \( \ell \). Our main result is a polynomial-time algorithm for the corresponding augmentation problem, which we use to obtain a 2-approximation for Fault-Tolerant \( \ell \)-Flow. Finally, we show that a polynomial-time algorithm for Fault-Tolerant \( \ell \)-Flow implies polynomial-time algorithms for two problems whose complexity status is open.

3.1 Approximation Hardness of FTF

We show that FTF is as hard to approximate as Directed Steiner Forest by using an approximation hardness result from [15] for the problem Weighted Robust Matching Augmentation. The problem Weighted Robust Matching Augmentation asks for the cheapest edge-set (assuming non-negative costs) to add to a bipartite graph such that the resulting graph is bipartite and contains a perfect matching after a-posteriori removing any single edge. The idea of our reduction is similar to that of the classical reduction from the Bipartite Maximum Matching problem to the Max s-t Flow problem. Note that since matchings are required to be perfect, we may assume that both parts of the input graph have the same size. We add to the bipartite input graph \((U, W, E)\) on \(n\) vertices a Weighted Robust Matching Augmentation instance two terminal vertices \(s\) and \(t\), and connect \(s\) to each vertex of \(U\) as well as each vertex of \(W\) to \(t\) by an arc. Now we add all possible arcs from \(U\) to \(W\), marking those as vulnerable that correspond to an edge in \(E\). It is readily observed that a fault-tolerant \(n/2\)-flow corresponds to a feasible solution to the given Weighted Robust Matching Augmentation instance (after removing all arcs incident to \(s\) or \(t\)). We thus obtain the following hardness result.

\[\text{Lemma 15} \ (\ast)\text{. A polynomial-time } f(\ell) \text{-approximation algorithm for FTF implies a polynomial-time } f(n/2)\text{-approximation algorithm for Weighted Robust Matching Augmentation, where } n \text{ is the number of vertices in the Weighted Robust Matching Augmentation instance.}\]

We combine Proposition 15 with two results from [15] and [14] to obtain the following approximation hardness result for FTF.

\[\text{Theorem 16} \ (\ast)\text{. FTF admits no polynomial-time } \log^{2-\epsilon}(\ell)\text{-factor approximation algorithm for every } \epsilon > 0, \text{unless } \text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)}).\]

Note that all results presented in this section also hold for the undirected variant of FTF.

3.2 Approximation Algorithms

We first present a simple polynomial-time \((\ell + 1)\)-approximation algorithm for FTF, which is very similar to the \((k + 1)\)-approximation for FTP. The algorithm computes (in polynomial time) a minimum-cost s-t flow of value \(\ell + 1\) on the input graph with the following capacities: each vulnerable arc receives capacity 1 and any other arc capacity \(1 + 1/\ell\). To see that for this choice of capacities we obtain a feasible solution, recall that the value of any s-t cut upper-bounds the value of any s-t flow. Therefore, each s-t cut \(C\) has value at least \(\ell + 1\), so \(C\) contains either at least \(\ell\) safe arcs or at least \(\ell + 1\) arcs. To prove the approximation guarantee, we show that any optimal solution to an FTF instance contains an s-t flow of value \(\ell + 1\) and observe that we over-pay for safe arcs by a factor of at most \((1 + 1/\ell)\). We obtain the following result.

\[\text{Theorem 17} \ (\ast)\text{. FTF admits a polynomial-time } (\ell + 1)\text{-factor approximation algorithm.}\]
Note that we cannot simply use the dynamic programming approach as in the algorithm for 1-FTP to obtain an \( \ell \)-approximation for FTF, since a solution to FTF in general does not have cut vertices, which are essential for the decomposition approach for the \( k \)-approximation for FTP.

We now show that for a fixed number \( \ell \) of disjoint paths, a much better approximation guarantee can be obtained. That is, we give a polynomial-time 2-approximation algorithm for Fault-Tolerant \( \ell \)-Flow (however, its running time is exponential in \( \ell \)). The algorithm first computes a minimum-cost \( s \)-\( t \) flow of value \( \ell \) and then augments it to a feasible solution by solving the following augmentation problem.

\[
\text{Fault-Tolerant } \ell \text{-Flow Augmentation}
\]

**Instance:** arc-weighted directed graph \( D = (V, A) \), two nodes \( s, t \in V \), arc-set \( X_0 \subseteq A \) that contains \( \ell \) disjoint \( s \)-\( t \) paths, and set \( M \subseteq A \) of vulnerable arcs.

**Task:** Find minimum weight set \( S \subseteq A \setminus X_0 \), such that for every \( f \in M \), the set \( (X_0 \cup S) \setminus f \) contains \( \ell \) disjoint \( s \)-\( t \) dipaths.

Our main technical contribution is that Fault-Tolerant \( \ell \)-Flow Augmentation can be solved in polynomial time for fixed \( \ell \). Our algorithm is based on a dynamic programming approach and it involves solving many instances of the problem Directed Steiner Forest, which asks for a cheapest subgraph connecting \( \ell \) given terminal pairs. This problem admits a polynomial-time algorithm for fixed \( \ell \) [10], but it is \( \mathcal{W}[1] \)-hard when parameterized in the number of terminal pairs, so it is likely not fixed-parameter tractable [13]. Roughly speaking, we traverse the \( \ell \) disjoint \( s \)-\( t \) paths computed previously in parallel, proceeding one arc at a time. In order to deal with vulnerable arcs, at each step, we solve an instance of Directed Steiner Forest connecting the \( \ell \) current vertices (one on each path) to \( \ell \) destinations on the same path by using backup paths. That is, we decompose a solution to the augmentation problem into instances of Directed Steiner Forest connected by safe arcs. An optimal decomposition yields an optimal solution to the instance of the augmentation problem. We find an optimal decomposition by dynamic programming. Essentially, we give a reduction to a shortest path problem in a graph that has exponential size in \( \ell \).

Let us fix an instance \( I \) of Fault-Tolerant \( \ell \)-Flow Augmentation on a digraph \( D = (V, A) \) with arc-weights \( c \in \mathbb{Z}_{\geq 0}^A \) and and terminals \( s \) and \( t \). Let \( P_1, P_2, \ldots, P_\ell \) be \( \ell \) disjoint \( s \)-\( t \) paths contained in \( X_0 \). In fact, we assume without loss of generality, that \( X_0 \) is the union of \( P_1, P_2, \ldots, P_\ell \). If \( X_0 \) contains an arc \( e \) that is not on any of the \( \ell \) paths, we remove \( e \) from \( X_0 \) and assign to it weight 0.

We now give the reduction to the shortest path problem. We construct a digraph \( D = (V, A) \); to distinguish it clearly from the graph \( D \) of \( I \), we call the elements in \( V \) \( (A) \) of \( D \) vertices (arcs) and elements of \( V \) \( (A) \) nodes (links). We order the vertices of each path \( P_i \), \( 1 \leq i \leq \ell \), according to their distance to \( s \) on \( P_i \). For two vertices \( x_1, x_2 \) of \( P_i \), we write \( x_1 \preceq x_2 \) if \( x_1 \) is at least as close to \( s \) on \( P_i \) as \( x_2 \). Let us now construct the node set \( V \). We add a node \( v \) to \( V \) for every \( \ell \)-tuple \( v = (x_1, \ldots, x_\ell) \) of vertices in \( V(X_0) \) satisfying \( x_i \in P_i \), for every \( i \in \{1, 2, \ldots, \ell\} \). Note that the corresponding vertices of a node are not necessarily distinct, since the \( \ell \) edge-disjoint paths \( P_1, P_2, \ldots, P_\ell \) may share vertices. We also define a (partial) ordering on the nodes in \( V \). For two nodes \( v_1 = (x_1, \ldots, x_\ell) \) and \( v_2 = (x_1', \ldots, x_\ell') \) we write \( v_1 \preceq v_2 \) if \( x_i \preceq x_i' \) for every \( 1 \leq i \leq \ell \). Additionally, let \( Q_i(x, y) \), \( i \) be the sub-path of \( P_i \) from a vertex \( x \) to a vertex \( y \) of \( P_i \).

We now construct the link set \( A := A_1 \cup A_2 \) of \( D \) as the union of two link-sets \( A_1 \) and \( A_2 \), which we will define next. We add to \( A_1 \) an arc \( xy \), if \( x \) precedes \( y \) and the subpaths of each \( P_i \) from \( x_i \) to \( y_i \) contain no vulnerable arc. That is, we let

\[
A_1 := \{ xy \mid x, y \in V, x \preceq y, Q_i(x_i, y_i) \cap M = \emptyset \text{ for } 1 \leq i \leq \ell \}.
\]
We now define the link set $\mathcal{A}_2$. For two nodes $x, y \in \mathcal{V}$ such that $x$ precedes $y$, if there is some $1 \leq i \leq \ell$, such that $Q_i(x, y_i)$ contains at least one vulnerable arc, then we first need to solve an instance of Directed Steiner Forest on $\ell$ terminal pairs in order to know whether we add the link $xy$ and, if so, at which cost. We construct an instance $I(x, y)$ of Directed Steiner Forest as follows. The terminal pairs are $(x_i, y_i)_{1 \leq i \leq \ell}$. The input graph is given by $D' = (\mathcal{V}, A')$, where $A' = (A \setminus X_0) \cup \bigcup_{1 \leq i \leq \ell} \overleftarrow{Q_i}(x_i, y_i)$, where $\overleftarrow{Q_i}(x_i, y_i)$ are the arcs of $Q_i(x_i, y_i)$ in reversed direction. The arc costs are given by

$$c'_e := \begin{cases} c_e & \text{if } e \in A \setminus X_0, \\ 0 & \text{if } e \in \overleftarrow{Q_i}(x_i, y_i) \text{ for some } i \in \{1, 2, \ldots, \ell\}. \end{cases}$$

That is, for $1 \leq i \leq \ell$, we reverse the path $Q_i(x_i, y_i)$ connecting $x_i$ to $y_i$ and make the corresponding arcs available at zero cost. We then need to connect $x_i$ to $y_i$ without using arcs in $X_0$. Since the number of terminal pairs is at most $\ell$ and thus constant, the Directed Steiner Forest instance $I(v_1, v_2)$ can be solved in polynomial time by the algorithm of Feldman and Ruhl given in [10]. Let $\text{OPT}(I(x, y))$ be the cost of an optimal solution to $I(v, v)$. We add a link $xy$ to $\mathcal{A}_2$ if the computed solution of $I(x, y)$ is strongly connected. This completes the construction of $\mathcal{A}_2$.

For a link $e \in \mathcal{A}$ we let the weight $w_e$ be given by

$$w_e := \begin{cases} 0 & \text{if } e \in \mathcal{A}_1, \\ \text{OPT}(I(x, y)) & \text{if } e \in \mathcal{A}_2. \end{cases}$$

We now argue that a shortest path $P$ from node $s_1 = (s, \ldots, s) \in \mathcal{V}$ to node $t_1 = (t, \ldots, t) \in \mathcal{V}$ in $\mathcal{D}$ corresponds to an optimal solution to $I$. For every link $xy \in P$, we add the optimal solution to $I(x, y)$ computed by the Feldman-Ruhl algorithm to our solution $Y$. A summary is given in Appendix D. The details can be found in Appendix D.

**Theorem 18 (•).** The set $Y$ computed by Algorithm 2 is an optimal solution to the instance $I$ of Fault-Tolerant $\ell$-Flow Augmentation.

Algorithm 2 runs in polynomial time for a fixed number $\ell$ of disjoint $s$-$t$ paths, since it computes at most $n^\ell$ Min-cost Strongly Connected Subgraphs on $\ell$ terminal pairs, which can be done in polynomial time by a result of Feldman and Ruhl [10].

**Theorem 19 (•).** Algorithm 2 runs in time $O(|\mathcal{A}| |\mathcal{V}|^{\ell-2} + |\mathcal{V}|^{6\ell-1} \log |\mathcal{V}|)$.

From theorems 18 and 19 we obtain a polynomial-time 2-approximation algorithm for Fault-Tolerant $\ell$-Flow: Let $\text{OPT}(I)$ be the cost of an optimal solution to an instance $I$ of Fault-Tolerant $\ell$-Flow. The algorithm first computes a minimum-cost $s$-$t$ flow $X_0$ and then runs Algorithm 2 using $X_0$ as initial arc-set. The algorithm returns the union of the arc-sets computed in the two steps. By Theorem 18 we can augment $X_0$ in polynomial time to a feasible solution $X_0 \cup Y$ to $I$. Since we pay at most $\text{OPT}(I)$ for the sets $X_0$ and $Y$, respectively, the total cost is at cost at most $2 \text{OPT}(I)$.

**Corollary 20.** Fault-Tolerant $\ell$-Flow admits a polynomial-time 2-factor approximation algorithm.

### 3.3 Relation to Other Problems of Open Complexity

In the previous section we showed that there is a polynomial-time algorithm for Fault-Tolerant $\ell$-Flow Augmentation, from which we obtained a 2-approximation for Fault-Tolerant $\ell$-Flow. Ideally, one would like to complement such an approximation result with a hardness or hardness-of-approximation result. Since Fault-Tolerant $\ell$-Flow Augmentation admits a polynomial-time algorithm according to Theorem 19, we cannot use the augmentation problem in order to prove
Algorithm 2: Exact algorithm for Fault-Tolerant $\ell$-Flow Augmentation

**Input:** instance $I$ of Fault-Tolerant $\ell$-Flow Augmentation on a digraph $D = (V, A)$

1. Construct the graph $D = (V, A)$
2. Find a shortest path $P$ in $D$ from $(s, \ldots, s)$ to $(t, \ldots, t)$
3. For each link $vw \in P \cap A$ add the arcs of an optimal solution to $I(v, w)$ to $Y$
4. return $Y$

NP-hardness of Fault-Tolerant $\ell$-Flow; an approach that has been used successfully for instance for robust paths [3], robust matchings [15] and robust spanning trees [2]. Hence, there is some hope that Fault-Tolerant $\ell$-Flow might actually be polynomial-time solvable. However, we show that a polynomial-time algorithm for Fault Tolerant 2-Flow implies polynomial-time algorithms for two other problems with unknown complexity status, namely 1-2-connected Directed 2 Steiner Tree and a special case of 2-connected Directed $k$ Steiner Tree.

We will first consider the relation of Fault-Tolerant $\ell$-Flow and 2-connected Directed $k$ Steiner Tree, which asks for two disjoint directed paths connecting a root vertex with each terminal:

| 2-connected Directed $k$ Steiner Tree |
|--------------------------|-------------------|
| **Instance:** | directed graph $D = (V, A)$, cost function $c \in \mathbb{Q}^A$, root $s \in V$, and $k$ terminal vertices $t_1, t_2, \ldots, t_k \in V$ |
| **Task:** | find a minimum-cost set of edges $X \subseteq A$, such that $(V, X)$ contains two edge-disjoint $s$-$t$ paths for each $t \in \{t_1, t_2, \ldots, t_k\}$ |

We will denote the set of terminals by $T := \{t_1, t_2, \ldots, t_k\}$. According to [7], even the complexity of 1-2-connected Directed 2 Steiner Tree is open, which is the following variant of 2-connected Directed $k$ Steiner Tree: we have only two terminals $t_1$ and $t_2$ and aim to find two disjoint $s$-$t_1$ paths and one $s$-$t_2$ path of minimal total cost. Note that 2-connected Directed $k$ Steiner Tree is a generalization of Directed Steiner Tree and therefore does not admit a polynomial-time $\log^{2-\varepsilon} n$ approximation algorithm unless $\text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)})$ [14]. However, there is a big gap between the complexity of Directed Steiner Tree and 2-connected Directed $k$ Steiner Tree if the number $k$ of terminals is fixed. While it is known that Directed Steiner Tree is fixed-parameter tractable when parameterized by the number of terminals (and therefore polynomial-time solvable for constant $k$), it is unknown whether 2-connected Directed $k$ Steiner Tree admits a polynomial-time algorithm even for $k = 2$ (for $k = 1$, an optimal solution is a minimum-cost 2-flow). We now show that a special case of Fault-Tolerant $\ell$-Flow corresponds to 2-connected Directed $k$ Steiner Tree with the additional constraint that every $s$-$T$ cut contains at least $k + 1$ edges.

**Proposition 21 (⋆).** A polynomial-time algorithm for Fault-Tolerant $k$-Flow implies a polynomial-time algorithm for 2-connected Directed $k$ Steiner Tree with the additional constraint that every $s$-$T$ cut contains at least $k + 1$ edges.

Furthermore, we show that 1-2-connected Directed 2 Steiner Tree is a special case of Fault Tolerant 2-Flow.

**Proposition 22 (⋆).** A polynomial-time algorithm for Fault Tolerant 2-Flow implies a polynomial-time algorithm for 1-2-connected Directed 2 Steiner Tree.
4 Conclusions and Future Work

This paper presents two problems, FTP and FTF, which add a non-uniform fault model to the classical edge-disjoint paths problem. In this model, not all $k$-subsets of edges can be removed from the graph after a solution is chosen, but rather a subset of vulnerable edges, which are provided as part of the input. Such an adaptation is natural from the point of view of many application domains. We observed a dramatic increase in the computational complexity due to the fault model with respect to EDP. At the same time we identified several classes of instances admitting a polynomial exact algorithm. These classes include the case $k = 1$, directed acyclic graphs and fixed $k$ and series-parallel graphs. Next, we defined a fractional counterpart of FTP and proved a tight bound on the corresponding integrality gap. This result lead to a $k$-approximation algorithm for FTP. For FTF, our main results are a $(\ell + 1)$-approximation algorithm and a $2$-approximation algorithm for fixed $\ell$.

One of the main tasks that remains is to improve the understanding of the approximability of FTP. In particular, it is interesting to see if the approximation guarantee for FTP can be improved to the approximation guarantees of the best known algorithms for the Steiner Tree problem. It is also interesting to relate FTP to more general problems such a Minimum-Cost Fixed-Charge Network Flow and special cases thereof. The complexity of $k$-FTP in still unknown. It is interesting to see if the methods employed in the current paper for 1-FTP and $k$-FTP on directed acyclic graphs can be extended to $k$-FTP on general graphs. Another intriguing open question is whether Fault-Tolerant $\ell$-Flow is NP-hard, which is open even for $\ell = 2$. We showed that a positive result in this direction implies polynomial-time algorithms for two Steiner problems whose complexity status is open.

References

[1] David Adjiashvili. Fault-tolerant shortest paths-beyond the uniform failure model. arXiv preprint arXiv:1301.6299, 2013.

[2] David Adjiashvili, Felix Hommelsheim, and Moritz Mühlenthaler. Flexible graph connectivity. In International Conference on Integer Programming and Combinatorial Optimization, pages 13–26. Springer, 2020.

[3] David Adjiashvili, Sebastian Stiller, and Rico Zenklusen. Bulk-robust combinatorial optimization. Mathematical Programming, 149(1-2):361–390, 2015.

[4] Hassene Aissi, Cristina Bazgan, and Daniel Vanderpoorten. Approximation complexity of min-max (regret) versions of shortest path, spanning tree, and knapsack. In European Symposium on Algorithms, pages 862–873. Springer, 2005.

[5] Christina Büsing. Recoverable robust shortest path problems. Networks, 59(1), 2012.

[6] Moses Charikar, Chandra Chekuri, To-yat Cheung, Zuo Dai, Ashish Goel, Sudipto Guha, and Ming Li. Approximation algorithms for directed steiner problems. Journal of Algorithms, 33(1):73–91, 1999.

[7] Joseph Cheriyan, Bundit Laekhanukit, Guyslain Naves, and Adrian Vetta. Approximating rooted steiner networks. ACM Transactions on Algorithms (TALG), 11(2):1–22, 2014.

[8] Joseph Cheriyan and Ramakrishna Thurimella. Approximating minimum-size k-connected spanning subgraphs via matching. SIAM Journal on Computing, 30(2):528–560, 2000.
[9] Kedar Dhamdhere, Vineet Goyal, R Ravi, and Mohit Singh. How to pay, come what may: Approximation algorithms for demand-robust covering problems. In 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’05), pages 367–376. IEEE, 2005.

[10] Jon Feldman and Matthias Ruhl. The directed steiner network problem is tractable for a constant number of terminals. SIAM Journal on Computing, 36(2):543–561, 2006.

[11] Harold N Gabow and Suzanne R Gallagher. Iterated rounding algorithms for the smallest $k$-edge connected spanning subgraph. SIAM Journal on Computing, 41(1):61–103, 2012.

[12] Daniel Golovin, Vineet Goyal, Valentin Polishchuk, R Ravi, and Mikko Sysikaski. Improved approximations for two-stage min-cut and shortest path problems under uncertainty. Mathematical Programming, 149(1-2):167–194, 2015.

[13] Jiong Guo, Rolf Niedermeier, and Ondřej Suchý. Parameterized complexity of arc-weighted directed steiner problems. SIAM Journal on Discrete Mathematics, 25(2):583–599, 2011.

[14] Eran Halperin and Robert Krauthgamer. Polylogarithmic inapproximability. In Proceedings of the 35th Annual ACM Symposium on Theory of Computing, pages 585–594, 2003.

[15] Felix Hommelsheim, Moritz Mühlenthaler, and Oliver Schaudt. How to secure matchings against edge failures. In 36th International Symposium on Theoretical Aspects of Computer Science. Schloss Dagstuhl-Leibniz-Zentrum für Informatik, 2019.

[16] Kamal Jain. A factor 2 approximation algorithm for the generalized Steiner network problem. Combinatorica, 21(1):39–60, 2001.

[17] Christina Puhl. Recoverable robust shortest path problems. Preprint, pages 034–2008, 2009.

[18] Alexander Schrijver. Combinatorial optimization: polyhedra and efficiency, volume 24. Springer Science & Business Media, 2003.

[19] Gang Yu and Jian Yang. On the robust shortest path problem. Computers and Operations Research, 25(6):457468, Jun 1998.

[20] Pawe Zieliski. The computational complexity of the relative robust shortest path problem with interval data. European Journal of Operational Research, 158(3):570576, Nov 2004.

A Proofs Omitted from Section 2.3

Proof of Lemma 3. Assume the statement is not true. Then, by the Max-Flow Min-Cut Theorem there is some capacitated cut $\delta(V')$ for some $V' \subseteq V$ with $s \in V'$ and $t \notin V'$ such that $c(\delta(V')) < k+1$. By the definition of $c$, this implies that $\delta(V')$ does not contain safe edges. But then $F := \delta(V')$ is a cut in $(V, X^*)$ of size at most $k$, a contradiction.

Proof of Lemma 4. We assume without loss of generality that $S^*$ is a minimal feasible solution with respect to inclusion. Let $Y \subseteq S^*$ be the set of bridges in $(V, S^*)$. From feasibility of $S^*$, we have $Y \cap M = \emptyset$. Consider any $s$-$t$ path $P$ in $S^*$. Let $u_1, \ldots, u_r$ be be the set of vertices incident to $Y = P \cap Y$. Let $u_i$ and $u_{i+1}$ be such that $u_iu_{i+1} \notin Y$. (if such an edge does not exist, we have $Y = P$, which means that $P$ is a robust $s$-$t$ bipath). Note that $S^*$ must contain two edge-disjoint $u_iu_{i+1}$ paths $L_1, L_2$. Taking as the set $Y$ together with all such pairs of paths $L_1, L_2$ results in a robust bipath.
Proof of Theorem 5. To solve 1-FTP we need to find the minimum cost robust $s$-$t$ bipath. To this end let us define two length functions $\ell_1, \ell_2 : V \times V \rightarrow \mathbb{R}_{\geq 0}$. For two vertices $u, v \in V$ let $\ell_1(u,v)$ denote the shortest path distance from $u$ to $v$ in the graph $(V, A \setminus M)$, and let $\ell_2(u,v)$ denote the cost of the shortest pair of edge-disjoint $u$-$v$ paths in $G$. Clearly, both length functions can be computed in polynomial time (e.g. using flow techniques). Finally, set $\ell(u,v) = \min \{\ell_1(u,v), \ell_2(u,v)\}$. Construct the complete graph on the vertex set $V$ and associate the length function $\ell$ with it. Observe that by definition of $\ell$, any $s$-$t$ path in this graph corresponds to a robust $s$-$t$ bipath with the same cost, and vice versa. It remains to find the shortest $s$-$t$ bipath by performing a single shortest $s$-$t$ path in the new graph. For every edge $uv$ in this shortest path, the optimal bipath contains the shortest $u$-$v$ path in $(V, a \setminus M)$ if $\ell(u,v) = \ell_1(u,v)$, and the shortest pair of $u$-$v$ paths in $G$, otherwise. \hfill \Box

Proof of Lemma 7. Consider first a fault-tolerant path $S \subseteq A$. We construct a corresponding $c^s$-$c^t$ path in $H$ as follows. Consider any $k+1$ $s$-$t$ flow $f^S$, induced by $S$. Let $p^1, \ldots, p^l$ be a path decomposition of $f^S$ and let $1 \leq \rho_1, \ldots, \rho_l \leq k+1$ (with $\sum_{i \in [l]} \rho_i = k+1$) be the corresponding flow values.

Since $D$ is layered, the path $p^j$ contains exactly one vertex $v^j_i$ from $V_i$ and one edge $e^j_i$ from $A_i$ for every $j \in [l]$ and $i \in [r]$. For every $i \in [r]$ define the $i$-configuration $d^i$ with

$$d^i_v = \sum_{j \in [l]: v = v^j_i} \rho_j,$$

if some path $p^j$ contains $v$, and $d^i_v = 0$, otherwise. The fact that $d^i$ is an $i$-configuration follows immediately from the fact that $f^S$ is a $(k+1)$-flow. In addition, for the same reason $d^i$ precedes $d^i_{i+1}$ for every $i \in [r-1]$. From the latter observations and the fact that $d^1 = c^s$ and $d^r = c^t$ it follows that $P = d^1, d^2, \ldots, d^r$ is a $c^s$-$c^t$ path in $H$ with cost $\ell(P) \leq w(S)$.

Consider next an $c^s$-$c^t$ path $P = d^1, \ldots, d^r$ with cost $\ell(P) = \sum_{i=1}^{r-1} \ell(d^i, d^{i+1})$. The cost $\ell(d^i, d^{i+1})$ is realized by some set of edges $R_i \subseteq E(d^i, d^{i+1})$ for every $i \in [r-1]$. From Definition 6, the maximal $s$-$t$ flow in the graph $D' = (V, R)$ is at least $k+1$, where $R = \cup_{i \in [r-1]} R_i$. Next, Lemma 3 guarantees that there exists some feasible solution $S \subseteq R$, the cost of which is at most $\ell(P)$. In the latter claim we used the disjointness of the sets $R_i$, which is due the layered structure of the graph $G$. This concludes the proof of the lemma. \hfill \Box

Proof of Theorem 9. The proof of correctness is by induction on the depth of the recursion in Algorithm 1. Clearly the result returned by Algorithm 1 in lines 1-6 is optimal. Assume next that the algorithm computed correctly all optimal solutions for the subgraphs $H_1, H_2$, namely that for every $i \in [2]$ and $j \in [k]$, the set $S^j_i$ computed in lines 7-8 is an optimal solution to the problem on instance $T^j_i = (H_i, M \cap E[H_i], j)$.

Assume first that $G$ is a series composition of $H_1$ and $H_2$, and let $0 \leq i \leq k$. If either $S^1_i = \bot$ or $S^2_i = \bot$ the problem with parameter $i$ is clearly also infeasible, hence the algorithm works correctly in this case. Furthermore, since $G$ contains a cut vertex (the terminal node, which is in common to $H_1$ and $H_2$), a solution $S$ to the problem is feasible for $G$ if and only if it is a union of two feasible solutions for $H_1$ and $H_2$. From the inductive hypothesis it follows that $S_i$ is computed correctly in line 14.

Assume next that $G$ is a series composition of $H_1$ and $H_2$. Consider any feasible solution $S'$ to the problem on $G$ with parameter $i$. Let $S^1_i'$ and $S^2_i'$ be the restrictions of $S'$ to edges of $H_1$ and $H_2$ respectively, and let $n_1$ and $n_2$ be the maximal integers such that $S^1_i'$ and $S^2_i'$ are robust paths for $H_1$ and $H_2$ with parameters $n_1$ and $n_2$, respectively. Observe that $i \leq n_1 + n_2 + 1$ must hold.
Indeed if this would not be the case, then taking any cut with \( n_1 + 1 \) edges in \( S'_1 \) and another cut with \( n_2 + 1 \) edges in \( S'_2 \) yields a cut with \( n_1 + n_2 + 2 \) edges in \( G \), contradicting the fact that \( S' \) is a robust path with parameter \( i \). We conclude that the algorithm computes \( S_i \) correctly in line 23. Finally note that the union any two robust paths for the graphs \( H_1 \) and \( H_2 \) with parameters \( n_1 \) and \( n_2 \) with \( i \leq n_1 + n_2 + 1 \) yield a feasible solution \( S_i \). It follows that the minimum cost such robust path is obtained as a minimum cost of a union of two solutions for \( H_1 \) and \( H_2 \), with robustness parameters \( j \) and \( i - j - 1 \) for some value of \( j \). To allow \( S_i = S^1_i \) or \( S_i = S^2_i \) we let \( j \) range from \(-1 \) to \( k \) and set \( S^1_{-1} = S^2_{-1} = \emptyset \). This completes the proof of correctness.

To prove the bound on the running time, let \( T(m, k) \) denote the running time of the algorithm on a graph with \( m \) edges and robustness parameter \( k \). We assume that the graph is given by a hierarchical description, according to its decomposition into single edges. The base case obviously takes \( O(k) \) time. Furthermore we assume that the solution \((S_0, \ldots, S_k)\) is stored in a data structure for sets, which uses \( O(1) \) time for generating empty sets and for performing union operations. If the graph is a series composition then the running time satisfies \( T(m, k) \leq T(m', k) + T(m-m', k) + O(k) \) for some \( m' < m \). If the graph is a parallel composition, then \( T(m, k) \) satisfied the same inequality. We assume that the data structure, which stores the sets \( S_i \) also contains the cost of the edges in the set. This value can be easily updates in time \( O(1) \) when the assignment into \( S_i \) is performed. It follows that \( T(m, k) = O(mk) = O(nk) \) as required. \( \square \)

## B Proofs Omitted from Section 2.4

**Proof of Theorem 10.** Consider an instance \( I = (D, M, k) \) of FTP. Let \( x^* \) denote an optimal solution to the corresponding FRAC-FTP instance, and let \( OPT = w(x^*) \) be its cost. Define a vector \( y \in \mathbb{R}^A \) as follows.

\[
y_e = \begin{cases} 
(k + 1)x_e & \text{if } e \not\in M \\
\min\{1, (k + 1)x_e\} & \text{otherwise.}
\end{cases}
\]

Clearly, it holds that \( w(y) \leq (k + 1)OPT \). We claim that every \( s-t \) cut in \( D \) with capacities \( y \) has capacity of at least \( k + 1 \). Consider any such cut \( C \subset A \), represented as the set of edges in the cut. Let \( M' = \{e \in M : x^*_e \geq \frac{1}{k+1}\} \) denote the set of faulty edges attaining high fractional values in \( x^* \). Define \( C' = C \cap M' \). If \( |C'| \geq k + 1 \) we are clearly done. Otherwise, assume \( |C'| \leq k \). In this case consider the failure scenario \( F = C' \). Since \( x^* \) is a feasible solution it must hold that

\[
\sum_{e \in C \setminus C'} x^*_e \geq 1.
\]

Since for every edge \( e \in C \setminus C' \) it holds that \( y_e = (k + 1)x^*_e \) we obtain

\[
\sum_{e \in C \setminus C'} y_e \geq k + 1,
\]

as desired. From our observations it follows that the maximum flow in \( D \) with capacities \( y \) is at least \( k + 1 \). Finally, consider the minimum cost \( (k + 1) \)-flow \( z^* \) in \( D \) with capacities defined by

\[
c_e = \begin{cases} 
k + 1 & \text{if } e \not\in M \\
1 & \text{otherwise.}
\end{cases}
\]

From integrality of \( c \) and the minimum-cost flow problem we can assume that \( z^* \) is integral. Note that \( y_e \leq c_e \) for every \( e \in A \), hence any feasible \( (k+1) \)-flow with capacities \( y \) is also a feasible \( (k+1) \)-flow with capacities \( c \). From the previous observation it holds that \( w(z^*) \leq w(y) \leq (k + 1)OPT \).
From Lemma 3 we know that $z^*$ is a feasible solution to the FTP instance. This concludes the proof of the upper bound of $k + 1$ for the integrality gap.

To prove the same lower bound we provide an infinite family of instances, containing instances with integrality gap arbitrarily close to $k + 1$. Consider a graph with $p \gg k$ parallel edges with unit cost connecting $s$ and $t$, and let $M = A$. The optimal solution to FTP on this instance chooses any subset of $k + 1$ edges. At the same time, the optimal solution to FRAC-FTP assigns a capacity of $\frac{1}{p-k}$ to every edge. This solution is feasible, since in every failure scenario, the number of edges that survive is at least $p - k$, hence the maximum $s$-$t$ flow is at least one. The cost of this solution is $\frac{p}{p-k}$. Taking $p$ to infinity yields instances with integrality gap arbitrarily close to $k + 1$. \qed

C Proofs Omitted from Section 3.1

Proof of Lemma 15. In the following it will be convenient to denote by $\overline{E}$ the edge-set of the bipartite complement of a bipartite graph with edge-set $E$. Let $I = (G, c)$ be an instance of Weighted Robust Matching Augmentation where $G = (U, W, E)$ is a balanced bipartite graph on $n$ vertices and $c \in \mathbb{Z}_{\geq 0}^E$. Our reduction is similar to the classical reduction from the perfect matching problem in bipartite graphs to the Max $s$-$t$ Flow problem. We construct in polynomial-time an instance $I' = (D', c', s, t, M)$ of FTF as follows. To obtain the digraph $D' = (V, A)$, we add to the vertex set of $G$ two new vertices $s$ and $t$ and add all arcs from $s$ to $U$ and from $W$ to $t$. Furthermore, we add all arcs from $U$ to $W$ and consider those that correspond to an edge in $E$ as vulnerable. That is, we let $M := \{uw : u \in U, w \in W, uw \in E\}$. To complete the construction of $I'$, we let $\ell = n/2$, and let the arc-costs $c'$ be given by

$$c'_{uw} = \begin{cases} c_{uw} & \text{if } uw \in E(G), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

For $X \subseteq E \cup \overline{E}$ we write $q(X)$ for the corresponding set of arcs of $D'$. Similarly, for a set $Y \subseteq A$ of arcs we write $q^{-1}(Y)$ for the corresponding set of undirected edges of $G$. Observe that for a feasible solution $X$ to $I$, the arc set $q(X) \cup A_s \cup A_t$ is feasible for $I'$, where $A_s$ (resp., $A_t$) is the set of arcs leaving $s$ (resp., entering $t$). Furthermore, a feasible solution $Y$ to $I'$ corresponds to a feasible solution $q^{-1}(Y \setminus (A_s \cup A_t))$ to $I$. Also note that, by the choice of $c'$, we have that the cost of two corresponding solutions is the same. It follows that since $\ell = n/2$, any polynomial-time $f(\ell)$-factor approximation algorithm for Fault-Tolerant $\ell$-Flow implies a polynomial-time $f(n/2)$-factor approximation algorithm for Weighted Robust Matching Augmentation, where $n = |U + W|$. \qed

Proof of Theorem 16. We give a polynomial-time cost-preserving reduction from Directed Steiner Forest to FTF via Weighted Robust Matching Augmentation. The intermediate reduction step from Directed Steiner Forest to Weighted Robust Matching Augmentation is given in [15, Prop. 18]. Consider an instance $I$ of Directed Steiner Forest on a weighted digraph $D = (V, A)$ on $n$ vertices with $k$ terminal pairs $(s_1, t_1), (s_2, t_2), \ldots, (s_k, t_k)$. According to the reduction given in the proof of [15, Prop. 18], we obtain an instance of Weighted Robust Matching Augmentation on a graph of at most $2(n+k)+2(n-k) = 4n =: n'$ vertices. By the arguments their proof, a $f(n')$-approximation algorithm for Weighted Robust Matching Augmentation yields a $f(4n)$-approximation algorithm for Directed Steiner Forest. We apply Proposition 15 to conclude that an $f(\ell)$-approximation algorithm for FTF yields a $f(2n)$-approximation algorithm for Directed Steiner Forest. According to the result of Halperin and Krauthgamer [14], the problem Directed Steiner Forest admits no polynomial-time $\log^{2 - \varepsilon} n$-approximation algorithm for every $\varepsilon > 0$, unless $\text{NP} \subseteq \text{ZTIME}(n^{\text{polylog}(n)})$. We conclude
that FTF admits no polynomial-time \( \log^{2-\varepsilon}(\ell/2) \)-factor approximation algorithm under the same assumption.

\[ \square \]

## D Proofs Omitted from Section 3.2

**Proof of Theorem 17.** Let \( I \) be an instance of FTF on a digraph \( D = (V, A) \) with weight \( c \in \mathbb{Z}_{\geq 0}^A \), terminals \( s \) and \( t \), vulnerable arcs \( M \) and desired flow value \( \ell \). We consider an instance \( I' = (D, c, s, t, \ell + 1, g) \) of MCF, where the arc capacities \( g \) are given by

\[
g_e := \begin{cases} 
1 & \text{if } e \in M, \text{ and} \\
1 + \frac{1}{\ell_e} & \text{otherwise}
\end{cases}
\]

An optimal solution to \( I' \) can be computed computed in polynomial-time by standard techniques. We saw in the discussion at the beginning of Section 3.2 that the set of arcs of positive flow in a solution to \( I' \) yields a feasible solution to \( I \).

It remains to bound the approximation ratio. Let \( Y^* \) be an optimal solution to \( I \) of cost \( \text{OPT}(I) \). We first show that \( Y^* \) contains \( \ell + 1 \) disjoint \( s \)-\( t \) paths.

**Claim 1.** \( Y^* \) contains an \( s \)-\( t \) flow of value \( \ell + 1 \) with respect to the capacities \( g \).

**Proof.** First observe that in any feasible solution to \( I \), every \( s \)-\( t \) cut contains either at least \( \ell \) safe arcs or at least \( \ell + 1 \) arcs. Now, an \( s \)-\( t \) cut \( Z \) in \( Y^* \) having at least \( \ell \) safe arcs satisfies \( g(Z) \geq (1 + \frac{1}{\ell}) \cdot \ell = \ell + 1 \). On the other hand, an \( s \)-\( t \) cut \( Z' \) in \( Y^* \) containing at least \( \ell + 1 \) arcs satisfies \( g(Z') \geq \ell + 1 \). Hence, each \( s \)-\( t \) cut in \( Y^* \) has capacity at least \( \ell + 1 \). By the max-flow-min-cut theorem there is an \( s \)-\( t \) flow of value at least \( \ell + 1 \).

The theorem now follows from the next claim.

**Claim 2.** An optimal solution to \( I' \) has cost at most \( (\ell + 1) \cdot \text{OPT}(I) \).

**Proof.** Let \( f^* \in \mathbb{Q}^A \) be an optimal \( s \)-\( t \) flow with respect to the capacities \( g \). Furthermore, let \( Y \) be the set of arcs of positive flow, that is \( Y := \{ e \in A \mid f_e^* > 0 \} \). Let \( Y_M = Y \cap M \) be the vulnerable arcs in \( Y \) and let \( Y_S = Y \setminus Y_M \) be the safe arcs. First, we may assume that each arc \( e \in Y \) has flow value at least \( f_e^* \geq 1/\ell \), since each arc has capacity either 1 or \( 1 + \frac{1}{\ell} \). This is true since we could scale the arc capacities \( g \) by a factor \( \ell \), which allows us to compute (in polynomial time) an integral optimal solution with respect to the scaled capacity function, using any augmenting paths algorithm for MCF. In addition, observe that we may pay a factor of at most \( 1 + \frac{1}{\ell} \) too much for each safe arc since the capacity of the safe arc is \( 1 + \frac{1}{\ell} \). Therefore, we may bound the cost of a safe arc \( e \in Y_S \) by \( \ell \cdot (1 + \frac{1}{\ell}) \cdot c_e \cdot f_e \) and the cost of each vulnerable arc \( e \in Y_M \) by \( \ell \cdot c_e \cdot f_e \), where \( f_e \) is the flow-value of arc \( e \) according to the solution \( Y \). Hence, we obtain

\[
c(Y) = c(Y_S) + c(Y_M) \\
\leq \ell \cdot \left( (1 + \frac{1}{\ell}) \cdot \sum_{e \in Y_S} c_e \cdot f_e^* + \sum_{e \in Y_M} c_e \cdot f_e^* \right) \\
\leq \ell \cdot (1 + \frac{1}{\ell}) \cdot \left( \sum_{e \in Y_S} c_e \cdot f_e^* + \sum_{e \in Y_M} c_e \cdot f_e^* \right) \\
\leq (\ell + 1) \cdot \text{OPT}(I)
\]

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where the first inequality follows from the two arguments above and the last inequality follows from Claim 1.

The remainder of this section is devoted to proving Theorem 18. For this purpose we need another auxiliary graph, that we use as a certificate of feasibility. For a graph \( H = (V, A^*) \) such that \( X_0 \subseteq A^* \subseteq A \), we denote the corresponding residual graph by \( D_{X_0}(A^*) = (V, A') \). The arc-set \( A' \) is given by \( A' := \{ uv \in A \mid uv \notin X_0 \} \cup \{ vu \in A \mid vu \notin X_0 \} \). An illustration of this graph is given in Figure 1. We first show that in a feasible solution \( Y \subseteq A \setminus X_0 \), each vulnerable arc in \( X_0 \) is contained in a strongly connected component of \( D_{X_0}(X_0 \cup Y) \).

**Lemma 23.** Let \( Y \subseteq A \setminus X_0 \). Then \( Y \) is a feasible solution to \( I \) if and only if each vulnerable arc \( f \subseteq M \cap X_0 \) is contained in a strongly connected component of \( D_{X_0}(X_0 \cup Y) \).

**Proof of Lemma 23.** We first prove the “if” part, so let \( f = uv \) be a vulnerable arc in \( X_0 \) that is contained in a strongly connected component of \( D_{X_0}(X_0 \cup Y) \). Since \( f \in X_0 \), the arc \( f \) is reversed in \( D_{X_0}(X_0 \cup Y) \) and since \( f \) is on a cycle \( C \) in \( D_{X_0}(X_0 \cup Y) \), there is a path \( P \) from \( u \) to \( v \) in \( D_{X_0}(X_0 \cup Y) \). Let \( P' \) be the path corresponding to \( P \) in \( X_0 \cup Y \). Note that \( P' \) is not a directed path in \( D \) and that an arc \( e \) on \( P' \) is traversed forward if \( e \in P' \cap Y \) and traversed backward if \( e \in P' \cap X_0 \). We partition \( P' \) into two disjoint parts \( P'_{X_0} = P' \cap X_0 \) and \( P'_{Y} = P' \cap Y \). We now argue that \( (X_0 - P'_{X_0} - f) \cup P'_{Y} \) contains \( \ell \) disjoint \( s-t \) paths. Clearly, we have \( (X_0 - P'_{X_0} - f) \cup P'_{Y} \subseteq X_0 \cup Y \). Furthermore, by our assumption that \( X_0 \) is the union of \( \ell \) \( s-t \) edge-disjoint paths, for each vertex \( v \in V - \{ s, t \} \), we have \( \delta^+(v) = \delta^-(v) \) and \( \delta^+(s) = \delta^-(t) = \ell \). Since \( C \) is a cycle in \( D_{X_0}(X_0 \cup Y) \) the degree constraints also hold for \( (X_0 - P'_{X_0} - f) \cup P'_{Y} \). Hence \( (X_0 - P'_{X_0} - f) \cup P'_{Y} \) is the union of \( \ell \) disjoint \( s-t \) paths.

We now prove the “only if” part. Let \( f = uv \in X_0 \) be a vulnerable arc and suppose \( f \) is not contained in a strongly connected component of \( D_{X_0}(X_0 \cup Y) \). Let \( L \subseteq V \) be the set of vertices that are reachable from \( u \) in \( D_{X_0}(X_0 \cup Y) \) and let \( R = V - L \). Note that \( s \in L \), since \( u \) is on some \( s-t \) path in \( X_0 \) and \( t \in R \), since otherwise there is a path from \( u \) to \( v \) in \( D_{X_0}(X_0 \cup Y) \) (since every arc in \( X_0 \) is reversed in \( D_{X_0}(X_0 \cup Y) \)). Let \( L' = \{ x_1, \ldots, x_\ell \} \subseteq L \), \( x_i \in P_i \) for \( 1 \leq i \leq \ell \), be the vertices of \( L \) that are closest to \( t \) in \( X_0 \). We now claim that \( \delta^+(L) \) is a cut of size \( \ell \) in \( X_0 \cup Y \) containing \( f \). Since \( f \) is vulnerable this contradicts the feasibility of \( X_0 \cup Y \). We have \( f \in \delta^+(L) \) in \( X_0 \cup Y \), since otherwise \( f \) is contained in a strongly connected component of \( D_{X_0}(X_0 \cup Y) \). By the construction of \( L \), we have \( Y \cap \delta^+(L) = \emptyset \). Since \( X_0 \) is the union of \( \ell \) disjoint paths, the set \( \delta^+(L) \) has size at most \( \ell \), proving our claim, since this implies that \( X_0 \cup Y \) is not feasible.

**Proof of Theorem 18.** Let \( \mathcal{P} \) be a shortest path in the auxiliary graph \( \mathcal{D} \) and let \( Y \) be the solution computed by Algorithm 2. We first establish the feasibility of \( Y \).

**Claim 1.** The solution \( Y \) computed by Algorithm 2 is feasible.

**Proof.** For a link \( xy \in \mathcal{P} \cap A_2 \), let \( Y(x, y) \) be an optimal solution to the instance \( I(x, y) \) of Directed Steiner Forest. We now argue that \( X_0 \cup Y \) is feasible to the instance \( I \) of Fault-Tolerant \( \ell \)-Flow Augmentation. By Lemma 23, it suffices to show that each vulnerable arc of \( X_0 \) is contained in some strongly connected component of \( D_{X_0}(X_0 \cup Y) \). Consider a vulnerable arc \( x_ix_i \in X_0 \) on a path \( P_i \) contained in \( X_0 \). Two nodes \( x \) and \( y \) containing \( x_i \) and \( y_i \), respectively, cannot be connected by a link in \( A_1 \) of \( \mathcal{D} \), since \( x_ix_i \) is vulnerable. Let \( u = (u_1, u_2, \ldots, u_\ell) \) (resp., \( v = (v_1, v_2, \ldots, v_\ell) \)) be the node on \( \mathcal{P} \) such that \( u_i \) (resp., \( v_i \)) is closest to \( x_i \) (resp., \( y_i \)) on the subpath from \( s \) to \( x_i \) (resp., \( y_i \)) to \( t \) of \( P_i \). These two nodes exist since \( \mathcal{P} \) is a path from \( (s, \ldots, s) \) to \( (t, \ldots, t) \) in \( \mathcal{D} \). If
there is more than one such node, let $u$ be a greatest and $v$ be a smallest such node with respect to the ordering $\leq$. By this choice of $u$ and $v$, we have that $uv \in P \cap A_2$. Therefore, the optimal solution $Y(u, v)$ to $I(u, v)$ has been added to $Y$ by Algorithm 2. Since $Y(u, v)$ connects $x_i$ and $y_i$ in $D_{X_0}(X_0 \cup Y)$, we have that the arc $x_i, y_i$ is contained in a strongly connected component in the residual graph $D_{X_0}(X_0 \cup Y)$. \hfill \Box

Let $Y^*$ be an optimal solution to $I$ of weight $OPT(I)$. We now show that $Y$ computed by Algorithm 2 is optimal. Observe that the weight of $Y$ is equal to $c'(P)$, so it suffices to show that $w(P) \leq OPT(I)$. To prove the inequality, we first introduce a partial ordering of the strong components of $D_{X_0}(X_0 \cup Y^*)$. Using this ordering we can construct a path $P'$ in $D$ from $(s, \ldots, s)$ to $(t, \ldots, t)$ of cost $w(P') = OPT(I)$. We conclude by observing that a shortest path $P$ has cost at most $w(P')$.

We introduce some useful notation. Let $Z$ be a strongly connected component of $D_{X_0}(X_0 \cup Y^*)$ and let $L(Z) = \{i \in \{1, 2, \ldots, \ell\} \mid E(P_i) \cap E(Z) \neq \emptyset\}$ be the set of indices of the paths $P_1, \ldots, P_\ell$ that have at least one edge in common with $Z$ (ignoring orientations). Additionally, for each $i \in L(Z)$, let $s_i(Z)$ be the vertex of $Z$ that is closest to $s$ on $P_i$ and let $S(Z) := \bigcup_{i \in L(Z)} s_i(Z)$. Similarly, for each $i \in L(Z)$ let $t_i(Z)$ be the vertex of $Z$ that is closest to $t$ on $P_i$ and let $T(Z) := \bigcup_{i \in L(Z)} t_i(Z)$.

**Claim 2.** Let $e_i^1, e_i^2 \in P_i \cap M$ be two vulnerable arcs of a path $P_i$, such that their corresponding connected components $Z_1$ and $Z_2$ of $D_{X_0}(X_0 \cup Y^*)$ are disjoint. If $e_i^1$ precedes $e_i^2$ on $P_i$, then $t_i(Z_1) < s_i(Z_2)$ for every $i \in L(Z_1) \cap L(Z_2)$.

**Proof.** Suppose for a contradiction that there is some $i \in L(Z_1) \cap L(Z_2)$, such that $t_i(Z_1) \geq s_i(Z_2)$. By the definition of $D_{X_0}(X_0 \cup Y^*)$, we have that $t_i(Z_1)$ is connected to $s_i(Z_2)$ in $D_{X_0}(X_0 \cup Y^*)$. But this implies that $Z_1$ and $Z_2$ are not disjoint, a contradiction. \hfill \Box

Using this claim we can construct a path $P'$ in $D$ of cost at most $OPT(I)$.

**Claim 3.** There is a path $P'$ from $(s, \ldots, s)$ to $(t, \ldots, t)$ in $D$ of cost at most $OPT(I)$.

**Proof.** We give an algorithm that constructs a path $P'$ from $(s, \ldots, s)$ to $(t, \ldots, t)$ in $D$, such that $P'$ only uses links in $A_2$ that correspond to strongly connected components of $Y^*$ in $D_{X_0}(X_0 \cup Y^*)$. Starting from $s_1 = (s, \ldots, s) \in V$, we perform the following two steps alternatingly until we reach $(t, \ldots, t) \in V$.

1. From the current node $u$, we proceed by greedily taking links of $A_1$ until we reach a node $v = (v_1, v_2, \ldots, v_\ell) \in V$ with the property that each vertex $v_i, 1 \leq i \leq \ell$, is either $t$ or part of some strongly connected component of $D_{X_0}(X_0 \cup Y^*)$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram}
\caption{Illustration of the structure of feasible solutions to Fault-Tolerant $\ell$-Flow Augmentation. Unsafe arcs are red, safe arcs are black. In Fig. 1a: edges of $X_0$ are black and red; edges of $A - X_0$ are light gray and light red. Dashed edges belong to $Y$.}
\end{figure}
2. From the current node $v$, we take a link $vw \in A_2$ to some node $w \in V$, where the link $vw$ corresponds to a strongly connected component $Z$ of $D_{X_0}(X_0 \cup Y^*)$.

First, we observe that Step 1 is well-defined: If at some point we reach a node $v = (v_1, v_2, \ldots, v_\ell)$ with no out-arcs in $A_1$ and there is some $1 \leq i \leq \ell$, such that $v_i$ is not in a strongly connected component of $D_{X_0}(X_0 \cup Y^*)$, then $Y^*$ is not feasible, since $Q_i(v_1, \cdot)$ contains a vulnerable arc but there is no substitute path in $X_0 \cup Y^*$. To show that we obtain an $s_1-t_1$ path by alternating the two steps above it remains to show that whenever we are in Step 2, there is a link $Z$ with no out-arcs in $S$. Hence we have $\text{OPT}(I) = \text{OPT}(I')$.

Therefore, Algorithm 2 runs in time $O(n^\ell \cdot (mn^{4\ell-2} + n^{2\ell-1} \log n))$ using the algorithm from [10] for finding a cost-minimal strongly connected subgraph on $\ell$ terminal pairs. Hence, $D$ can be constructed in $O(mn^{6\ell-2} + n^{6\ell-1} \log n)$. Since $D$ is acyclic, a shortest path from $(s, \ldots, s)$ to $(t, \ldots, t)$ in $D$ can be computed in time $O(|V| + |A|) = O(n^{2\ell})$. Therefore, Algorithm 2 runs in time $O(mn^{6\ell-2} + n^{6\ell-1} \log n)$.

E Proofs Omitted from Section 3.3

Proof of Proposition 21. Consider an instance $I$ of 2-connected Directed $k$ Steiner Tree on a graph $G = (V, A)$ with edge weights $c \in Q^A$, root $s \in V$, and $k$ terminals $T = \{t_1, \ldots, t_k\}$. We construct (in polynomial time) an instance $I' = ((V', A \cup A_t), c', s, t, A)$ of Fault-Tolerant $k$-Flow as follows. We add to $D$ a vertex $t$ and an arc from each terminal to $t$; that is, we add the edge-set $A_t = \{(t_i, t) : t_i \in T\}$. Let $D'$ be the resulting graph. The cost function $c'$ is given by

$$c'_e := \begin{cases} c_e & \text{if } e \in A, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we set $M := A$, that is, all original arcs are unsafe, while the new arcs $A_t$ are safe.

Let $X$ be a feasible solution to $I'$. We have $A_t \subseteq X$, since otherwise $X$ is not feasible. We now show that there are two disjoint $s-t_i$ paths in $(V, X \setminus A_t)$ for every $t_i \in T$. Assume this is not true and there is some $1 \leq i \leq k$, such that there are no two disjoint $s-t_i$ paths in $(V, X)$. By the
max-flow-min-cut theorem there is a cut edge $e \in A$ (or no edge at all) that separates $s$ and $t_i$ in $(V, X)$. But then $Y := \{(A_t \cup \{e\}) \setminus (t_i, t)\}$ is an $s$-$T$ cut in $(V, X)$ of size $k$ containing vulnerable edge, which contradicts the assumption that $X$ is feasible for $I'$. Furthermore, every $s$-$T$ cut in $(V, X \setminus A)$ contains at least $k + 1$ edges since all edges in $A$ are unsafe, since otherwise $X$ is not feasible for $I'$. Finally, observe that there is a one-to-one correspondence between feasible solutions to $I'$ and those of the 2-connected Directed $k$ Steiner Tree instance $I$ with the additional property that every $s$-$T$ cut contains at least $k + 1$ edges.

Proof of Proposition 22. Let $I$ be an instance of 1-2-connected Directed 2 Steiner Tree on a graph $D = (V, A)$ with edge-weights $c \in \mathbb{Q}^E$, root $s \in V$, and terminals $T = \{t_1, t_2\}$. Similar to the proof of Proposition 21 we construct an instance $I'$ of Fault Tolerant 2-Flow as follows. We add to $D$ two vertices $u$ and $t$ and four directed edges $\hat{A} = \{(s, u), (t_1, u), (u, t), (t_2, t)\}$. Let the resulting graph be $D'$. The edge weights $c'$ of $D'$ are given by

$$c'_e := \begin{cases} c_e & \text{if } e \in A(G), \\ 0 & \text{otherwise.} \end{cases}$$

Finally, we set $M := A \cup \{(s, u), (t_1, u)\}$, that is, the edges incident to $t$ are safe while all other edges are unsafe.

Let $X$ be a feasible solution to $I'$. We have $\hat{A} \subseteq X$, since otherwise $X$ is not feasible. We now show that there is at least one $s$-$t_1$ path and there are at least two disjoint $s$-$t_2$ paths in $(V, X \setminus \hat{A})$. Assume first that there is no path from $s$ to $t_1$ in $(V, X \setminus \hat{A})$. But then $\{(s, u), (v, t)\}$ is a cut of size two in $D'$, where $(s, u)$ is a vulnerable edge. This contradicts the feasibility of $X$. Now assume that there are no two disjoint $s$-$t_2$ paths in $(V, X \setminus \hat{A})$. Similar to the proof of Proposition 21 we then have a contradiction to the feasibility of $X$. Finally, observe that there is a one-to-one correspondence between feasible solutions to $I$ and feasible solutions to $I'$.