Comment on “Classification of Cosmic Scale Factor via Noether
Gauge Symmetries“ [Int. J. Theor. Phys. 54, 2343 (2015)]

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Abstract

We discuss the relationship between the Noether point symmetries of the geodesic Lagrangian, in a (pseudo)Riemannian manifold, with the elements of the Homothetic algebra of the space. We observe that the classification problem of the Noether symmetries for the geodesic Lagrangian is equivalent with the classification of the Homothetic algebra of the space, which in the case of a Friedmann-Lemaître-Robertson-Walker spacetime is a well-known result in the literature.

Keywords: Noether symmetries; Geodesic equations

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Recently the classification of the Noether symmetries for the geodesic Lagrangian in Friedmann-Lemaître-Robertson-Walker (FRW) spacetimes was performed in [1].

FRW spacetime describes an isotropic universe for which the fundamental line element is

\[ ds^2 = -dt^2 + \frac{a^2(t)}{(1 + \frac{1}{4}Kx^2)^2} \left( dx^2 + dy^2 + dz^2 \right), \]

where \( a(t) \) is the scale factor of the universe, \( K \) is the spatial curvature and \( x^2 = x^2 + y^2 + z^2 \). The spacetime with line element (1) admits a six-dimensional Killing algebra, for arbitrary \( a(t) \). It is the Killing algebra of the three-dimensional Euclidean space for \( K = 0 \), or the Killing algebra of the three-dimensional space of constant curvature, \( K \), with \( K \neq 0 \). The classification of the Killing vectors (KVs), the Homothetic vector (HV) and the Conformal Killing vectors (CKVs) can be found in [2].

The geodesic Lagrangian for a (pseudo)Riemannian manifold is defined as

\[ L(x^i, \dot{x}^i) = \frac{1}{2} g_{ij}(x^k) \dot{x}^i \dot{x}^j \]

in which \( i, j, k = 1, 2, ..., \text{dim} g_{ij} \) and overdot means total derivative with respect to the affine parameter “\( s \)”. The Euler-Lagrange equations which follow from (2) are

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = 0. \]

This system describes the free-fall of a particle in a space with metric \( g_{ij} \), i.e. the geodesic equations.

Consider the infinitesimal one-parameter point transformation in the space of the variables \( \{s, x^i\} \)

\[ \bar{s} = s + \varepsilon \xi \left(s, x^k\right) \]
\[ \bar{x}^i = x^i + \varepsilon \eta^i \left(s, x^k\right) \]

where the generator of the infinitesimal transformation is defined by

\[ X = \frac{\partial \bar{s}}{\partial s} \partial_s + \frac{\partial \bar{x}^i}{\partial s} \partial_i, \]

that is,

\[ X = \xi \left(s, x^k\right) \partial_s + \eta^i \left(s, x^k\right) \partial_i. \]

By definition the generator, \( X \), of the one-parameter point transformation, (4)-(5), of the Action Integral of a Lagrangian \( L = L \left(s, x^k, \dot{x}^k\right) \), which transforms the Action Integral in such a way that the Euler-Lagrange equations are invariant, is called a Noether symmetry [3].

Mathematically, \( X \) is a Noether symmetry if there exists a function \( f^i \) such that [3]

\[ X^{[1]}(L) + L \dot{\xi} = \dot{f}. \]
The corresponding conservation law is

\[ \Phi = \xi \left( x^k \frac{\partial L}{\partial \dot{x}^k} - L \right) - \eta^k \frac{\partial L}{\partial u_k} + f, \]  

(9)

where \( X^{[1]} \) is the first prolongation/extension of \( X \) in the space of variables \( \{ s, x^i, \dot{x}^i \} \).

As far as concerns the Noether (point) symmetries of the geodesic Lagrangian (2), in [4] it has been shown that there exists a unique connection between the Noether symmetries and the Homothetic algebra of the underlying space. Specifically, the Noether symmetries for the Lagrangian (2) are the vector fields

\[ X_1 = \partial_s, \quad X_{KV} = K^I, \quad X_{HV} = 2s\partial_s + H, \]  

(10)

\[ X_{GKV} = sS^i\partial_i, \quad X_{GHV} = t^2\partial_s + s\Omega^i\partial_i, \]  

(11)

in which the \( K^I \) form the Killing algebra of the space \( g_{ij} \), \( H \) is a proper HV of \( g_{ij} \), \( S^i \) is gradient KV and \( \Omega^i \) is a gradient HV. Recall that the space \( g_{ij} \) can a maximum of one HV and the \( S^i, I \) are included in the Killing algebra of \( g_{ij} \). Finally, the vector field \( X_1 \) is the autonomous symmetry and the corresponding conservation law is the Hamiltonian function.

Therefore the problem of the classification of the Noether symmetries for the geodesic Lagrangian (2) is equivalent with the classification problem of the KVs and HV for the metric \( g_{ij} \).

For the line element (1) the geodesic Lagrangian is

\[ L = \frac{1}{2}\dot{t}^2 + \frac{a^2(t)}{1 + \frac{1}{4}Kx^2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right). \]  

(12)

and without loss of generality we can perform a coordinate transformation, \( t \rightarrow \tau \), whereby (12) takes the simpler form

\[ L = R^2(\tau) \left[ -\frac{1}{2}\dot{\tau}^2 + \frac{1}{\left( 1 + \frac{1}{4}x^2 \right)^2} \left( \dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right) \right]. \]  

(13)

The Lie and Noether symmetries for the geodesic Lagrangian, (13), have been given in [4] as an illustrative example for the main results of the paper. The classification is based upon [2]. Hence, under the inverse transformation \( \tau \rightarrow t \), the results of [4] can be used for the Lagrangian, (12).

Furthermore, in [1], the authors claim that the Lagrangian, (12), admits only two Noether symmetries for an arbitrary scale factor, \( a(t) \). However, this is not true because, as we discussed above, the FRW spacetime (1) always admits a six-dimensional Killing algebra. Hence, for an arbitrary functional form of \( a(t) \), the Lagrangian (12) admits seven Noether point symmetries.
Finally we note that the classification of the Lie symmetries for the geodesic equations (3) in a FRW spacetime (1) can be found in [4] and it is equivalent with the classification problem of the special projective algebra of the space [5, 6].

**Remark:** In [1] the Noether symmetries are termed ‘gauge’ symmetries. This is incorrect terminology as the function, $f$, of Noether’s Theorem is not a gauge function. It is a boundary term introduced to allow for the infinitesimal changes in the value of the Action Integral produced by the infinitesimal change in the boundary of the domain caused by the infinitesimal transformation of the variables in the Action Integral. This is clearly stated in Noether’s paper of nearly 100 years ago. Noether’s Theorem really comes in two parts. The first is the equation for the calculation of the symmetries and the second is the formula for a conservation law. The second invokes the Euler-Lagrange equation but not the first. A further remark is that Noether allowed for generalised symmetries and not just point symmetries. It is a pity that writers on a subject connected with Noether’s Theorem do not read the original paper.

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