ANOTHER APPROACH ON POWER SUMS

CHRISTOPH MUSCHIELOK

Abstract. We show that explicit forms for certain polynomials $\psi_m^{(a)}(n)$ with the property

$$\psi_m^{(a+1)}(n) = \sum_{\nu=1}^n \psi_m^{(a)}(\nu)$$

can be found (here, $a,m,n \in \mathbb{N}_0$). We use these polynomials as a basis to express the monomials $n^m$. Once the expansion coefficients are determined, we can express the $m$-th power sums $S_m^{(a)}(n)$ of any order $a$,

$$S_m^{(a)}(n) = \sum_{\nu_1=1}^n \cdots \sum_{\nu_m=1}^n \nu_1^a \cdots \nu_m^a,$$

in a very convenient way by exploiting the summation property of the $\psi_m^{(a)}$,

$$S_m^{(a)}(n) = \sum_k c_{mk} \psi_k^{(a)}(n).$$

1. Introduction

Power sums have been studied for a long time, for example by Nicomachus of Gerasa, who demonstrated that the sum of cubes is the square of a triangular number, that is

$$S_3^{(1)}(n) = \sum_{\nu=1}^n \nu^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

This was known to Faulhaber who worked on power sums in the 17th century. In his work *Academia Algebræ* he presents formulas for the sums of odd powers up to the 17th. For odd powers, the power sum can be represented by a polynomial of the triangular numbers. This was shown by Jacobi shortly over 200 years later. Jacob Bernoulli found a closed form for power sums in 1713 (Summae potestatum) which can be written

$$S_m^{(1)}(n) = \sum_{\nu=1}^n \nu^m = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k n^{m-k+1},$$

with the binomial coefficient $\binom{m+1}{k}$ and the Bernoulli numbers $B_k$ ($B_1 = 1/2$). This work was published posthumously in 1713.

In the following, we define polynomials $\psi_m(n)$ which are related to some pyramidal numbers and use them to express a power $n^m$. The coefficients of this expansion can be used together with a generalization of the $\psi_m(n)$ which we also give here, to yield the value of any power sum.

2. Definition and Properties of the $\psi_m$ Polynomials

Definition 1. We define the polynomial sequences

$$\psi_m(n) = n + (m - 1)(n - 1)B_{m-1,n-1},$$
where \( m, n \in \mathbb{N} \) are natural numbers and
\[
B_{n,b} = \frac{(a+b)!}{a!b!}
\]
is the binomial coefficient.

The \( \psi_m(n) \) yield the sequences of the \( m \)-th powers of \( n \) for \( m < 4 \). That is, the quadratic and cubic numbers, \( n^2 \) and \( n^3 \), for \( m = 2 \) and \( m = 3 \), and the “linear numbers” \( n \) for \( m = 1 \). This is of no particular interest, we could probably find some families of sequences, for which this is true. To see, why this could indeed be interesting, we need to introduce further sequences.

**Definition 2.** On top of the \( \psi_m \), we recursively define the polynomial sequences \( \psi_m^{(a)} \) for \( a \in \mathbb{N}_0 \),
\[
(4) \quad \psi_m^{(a)}(n) = \psi_m^{(a)}(n-1) + \psi_m^{(a-1)}(n).
\]

Notationwise, we use the convention \( \psi_m^{(0)} \equiv \psi_m \).

**Lemma 1.** It immediately follows from Eq. \((4)\), that \( \psi_m^{(a)}(n) \) is given by the sum of all \( \psi_m^{(a-1)}(\nu) \) for \( 1 \nu \leq n \):
\[
(5) \quad \psi_m^{(a)} = \sum_{\nu=1}^{n} \psi_m^{(a-1)}(\nu).
\]

The \( \psi_m^{(a)}(n) \) are the polynomial series of \( a \)-th order with respect to the sequence \( \psi_m(n) \).

**Proof of Lemma 2.** We use Eq. \((4)\) time after time to rewrite the \( \psi_m^{(a)} \) term until we are left with Eq. \((5)\):
\[
\begin{align*}
\psi_m^{(a)}(n) &= \psi_m^{(a)}(n-1) + \psi_m^{(a-1)}(n), \\
&= \psi_m^{(a)}(n-2) + \psi_m^{(a-1)}(n-1) + \psi_m^{(a-1)}(n), \\
&\vdots \\
&= \psi_m^{(a)}(n-k) + \sum_{\nu=n-k+1}^{n} \psi_m^{(a-1)}(\nu), \\
\psi_m^{(a)}(n) &= \sum_{\nu=1}^{n} \psi_m^{(a-1)}(\nu). \quad \Box
\end{align*}
\]

We can state a lemma about linear combinations of functions which show the recursive property of Eq. \((4)\).

**Lemma 2.** The recursive property Eq. \((4)\) and its reformulation as the sum in Eq. \((5)\) hold for any sequence which is a certain type of linear combination of the \( \psi_m^{(a)} \). Let the coefficients \( c_1, c_2 \in \mathbb{K} \) be elements of some field.
\[
(6) \quad f^{(a)}(n) = c_1 \psi_m^{(a)}(n) + c_2 \psi_{m'}^{(a)}(n) \implies f^{(a)}(n) = f^{(a)}(n-1) + f^{(a-1)}(n).
\]

**Proof of Lemma 3.** Insert the recursive property into the \( \psi \)-terms of Eq. \((6)\) and evaluate:
\[
\begin{align*}
(7) \quad f^{(a)}(n) &= c_1 \left[ \psi_m^{(a)}(n-1) + \psi_m^{(a-1)}(n) \right] + c_2 \left[ \psi_{m'}^{(a)}(n-1) + \psi_{m'}^{(a-1)}(n) \right], \\
&= c_1 \psi_m^{(a)}(n-1) + c_2 \psi_{m'}^{(a)}(n-1) + c_1 \psi_m^{(a-1)}(n) + c_2 \psi_{m'}^{(a-1)}(n), \\
(8) \quad f^{(a)}(n) &= f^{(a)}(n-1) + f^{(a-1)}(n). \quad \Box
\end{align*}
\]
3. Linking the $\psi_m$ Polynomials to Power Sums

Why bother with all of this? Notice, how the $\psi_m(n)$ are just the $m$-th powers of $n$. If we can find a closed form for $\psi_m^{(a)}(n)$, we automatically have the $a$-th power sum

$$S_m^{(a)}(n) = \sum_{\nu_a=1}^{n} \sum_{\nu_{a-1}=1}^{\nu_a} \cdots \sum_{\nu_1=1}^{\nu_2} \nu^m.$$  \hfill (9)

However, this holds only for $m < 4$. Fortunately, we can take the nice property of Eq. (4) or Eq. (5) with us.

**Notational Convention.** For multiple summations with common ultimate summation boundaries in which the intermediate upper summation boundary of each sum is given by the index of the next sum, as in Eq. (9), we want to introduce the notation

$$\sum_{\nu_n=1}^{n} \nu^m,$$

where the multi-index notation $\nu_n$ means $(\nu_a, \nu_{a-1}, \ldots, \nu_1)$.

Therefore, we want to find an expansion of the power $n^m$ in terms of the $\psi_m(n)$,

$$n^m = \sum_k c_{mk} \psi_k(n),$$  \hfill (10)

so that we can rewrite the power sums of $a$-th order as

$$\sum_{\nu_n=1}^{n} \nu^m = \sum_{\nu=1}^{n} \sum_k c_{mk} \psi_m(\nu),$$  \hfill (11)

$$= \sum_k c_{mk} \sum_{\nu=1}^{n} \psi_m(\nu),$$  \hfill (12)

$$= \sum_k c_{mk} \psi_m^{(a)}(n).$$  \hfill (13)

Notice, how all the $a$ sums over the $\nu_i$ are swallowed by the basis functions $\psi_k$ by multiple use of Eq. (5) and turn them into $\psi_k^{(a)}$.

The important bit is: if we have a closed form for the $\psi_m^{(a)}(n)$, once we know the set of $c_{mk}$, we have the value for any $S_m^{(a)}(n)$. In the following, we first show that indeed one can find such a closed form and that we can easily obtain the values of the expansion coefficients.

4. Finding a Closed Form for the Series Polynomials

In the following, it is our goal to find an expression for the $\psi_m^{(a)}(n)$. It turns out, that a good starting point for this is to realize, that a similar identity to Eq. (4), holds for the binomial coefficient:

$$B_{a,b} = B_{a-1,b} + B_{b,a-1}. $$  \hfill (14)

This is just what we see in Pascal’s triangle and we can show this by a few simple steps of algebra, after inserting the definition for each symbol. With the same argument with which we proved Lemma [4] we may write

$$B_{a,b} = \sum_{\beta=1}^{b} B_{a-1,\beta}.$$  \hfill (15)
Due to this identity, we can already find a closed form for the $\psi_m^{(a)}(n)$ in terms of binomial coefficients as we show in the rest of this section. Let us first rewrite the expression for $\psi_m(n)$ as a linear combination of binomial coefficients.

(16) \[ \psi_m^{(a)}(n) = n + (m - 1)(n - 1)B_{m-1,n-1} , \]

(17) \[ = B_{1,n-1} + m(m - 1)B_{m,n-2} . \]

Cancelling the factor $n - 1$ from the binomial coefficient $B_{m-1,n-1}$, so that we can make a $B_{m,n-2}$ out of the second summand, leads to the term formally not being defined for $n = 1$. We have to make sure, that the limit of the series at $n = 1$ has still a defined value.

\[
\lim_{n \to 1} B_{m,n-2} = \lim_{n \to 1} \frac{(n + m - 2)!}{m!(n - 2)!} ,
\]

\[
= \lim_{n \to 1} (n + m - 2)! \frac{(n - 1)}{m!(n - 1)!} ,
\]

(18) \[ = \lim_{n \to 1} B_{m,n-2} = 0 . \]

**Lemma 3.** The elements of the series of $a$-th order $\psi_m^{(a)}(n)$ have the closed form

(19) \[ \psi_m^{(a)}(n) = B_{a+1,n-1} + \frac{m(m - 1)}{m + a} (n - 1)B_{m+a-1,n-1} . \]

**Proof of Lemma 3.** We will use a proof via induction and begin from Eq. (5) for $a = 1$ and insert the binomial coefficient representation for $\psi_m(n)$.

(20) \[ \psi_m^{(1)}(n) = \sum_{\nu=1}^{n} \psi_m(n) , \]

(21) \[ = \sum_{\nu=1}^{n} [B_{1,\nu-1} + m(m - 1)B_{m,\nu-2}] , \]

(22) \[ = \sum_{\nu=1}^{n} B_{1,\nu-1} + m(m - 1) \sum_{\nu=1}^{n} B_{m,\nu-2} , \]

(23) \[ = B_{2,n-1} + m(m - 1)B_{m+1,n-2} , \]

(24) \[ = B_{2,n-1} + \frac{m(m - 1)}{m + 1} (n - 1)B_{m,n-1} . \]

This is just the form given by Eq. (19) for $a = 1$, so that we have a valid start for the induction. Now suppose, that Eq. (19) holds for any $a$. We have to prove, that it holds also for $a + 1$:

(25) \[ \psi_m^{(a+1)}(n) = \sum_{\nu=1}^{n} \psi_m^{(a)}(\nu) , \]

(26) \[ = \sum_{\nu=1}^{n} \left[ B_{a+1,\nu-1} + \frac{m(m - 1)}{m + a} (\nu - 1)B_{m+a-1,\nu-1} \right] , \]

(27) \[ = \sum_{\nu=1}^{n} B_{a+1,\nu-1} + \frac{m(m - 1)}{m + a} \sum_{\nu=1}^{n} (\nu - 1)B_{m+a-1,\nu-1} , \]

(28) \[ = B_{a+2,n-1} + m(m - 1) \sum_{\nu=1}^{n} B_{m-a,\nu-2} , \]

(29) \[ = B_{a+2,n-1} + m(m - 1)B_{m+a,n-2} , \]

(30) \[ = B_{a+2,n-1} + \frac{m(m - 1)}{m + a + 1} (n - 1)B_{m+a,n-1} . \]
This is just Eq. (19) for \(a + 1\) substituted for \(a\). Thus, the \(\psi_m^{(a)}(n)\) have indeed the proposed closed form. \(\square\)

5. Coefficients of the Monomial Expansion of \(\psi_m\)

With the expression for the \(\psi_m^{(a)}(n)\) ready, what remains to do is to find the coefficients \(c_{mk}\). Before we tackle this problem, let us write the \(\psi_m(n)\) in terms of powers of \(n\), at first, that is we expand it in terms of the monomials

\[
\psi_m(n) = \sum_{i=0}^{m} a_{mi} n^i.
\]

We can find the coefficients \(a_{mi}\) in Eq. (31) using Vieta’s formulas.[2] For this, we rewrite \(\psi_m(n)\) as

\[
\psi_m(n) = n + \frac{1}{(m-2)!} f_m(n),
\]

with \(f_m(n)\) given by

\[
f_m(n) = \prod_{k=-1}^{m-2} (n + k) = \prod_{i=1}^{m} (n - \alpha_i) = \sum_{k=0}^{m} b_{m,m-k} n^k,
\]

where the \(\alpha_i\) are of course the integer roots of this polynomial:

\[
\alpha_i = -(i - 2), 1 \leq i \leq m.
\]

The coefficients \(b_{m,m-k}\) are connected to our coefficients of interest \(a_{mi}\) by

\[
a_{m,m-k} = \frac{b_{m,m-k}}{(m-2)!} + \delta_{1k}, \quad m \geq 2,
\]

where we use the Kronecker symbol \(\delta_{1k}\) to account for the additional term \(n\) of \(\psi_m\) with respect to \(f_m\). We discuss the cases \(m = 1\) and \(m = 0\) later. Until then, we consider everything under the condition \(m \geq 2\).

Corrolary 1. By inserting the roots \(\alpha_i\) into Vieta’s formulas, it can be verified that the expansion coefficients of \(f_m(n)\) for the lowest orders are given by the following expressions:

\[
b_{m,1} = -(m-2)!, \quad b_{m,0} = 0.
\]

Together with \(b_{mm} = 1\), we find for the actual expansion coefficients \(a_{mk}\) of \(\psi_m\)

\[
a_{m0} = 0, \quad a_{m1} = 0, \quad a_{mm} = \frac{1}{(m-2)!}.
\]

As we want to consider only classical polynomials, we write, subsuming the results for the \(b_{mi}\), for the expansion coefficient \(a_{mk}\)

\[
a_{mk} = 0, \quad \text{if } k < 2 \vee k > m,
\]

Therefore, the non-zero values for the \(a_{mk}\) are those with \(2 \leq k \leq m\).

We turn now to the remaining cases \(m = 0\) and \(m = 1\). For the latter, we already mentioned, that \(\psi_1(n) = n\). Thus, its single expansion coefficient is \(a_{11} = 1\). Power sums of \(a\)-th order of \(n\) are given by the \(n\)-th simplicial \(a\)-polytopic number[5]

\[
\sigma_a(n) = \frac{(n+a-1)!}{(n-1)! a!} = B_{a,n-1}.
\]
This is qualitatively different to the case of the $\psi_{m}$ with $m \geq 2$. For those, we have in general a linear combination of two binomial coefficients, whereas for $m = 1$ we can express it also as a single binomial coefficient.

The case $m = 0$ is not included in how we defined the $\psi_{m}$ here. However, it can be reduced to $m = 1$: clearly, $n^{0} = 1$, so that any power sum of $a$-th order can be reduced to a power sum of $n$ of $(a - 1)$-th order. In the following, we will restrict ourselves to $m \geq 2$.

6. Recursive Definition of the $\psi_{m}$-Expansion Coefficients of $n^{\mu}$

Expressing $\psi_{m}$ in terms of the monomials $\{n^{k}\}_{k=2}^{m}$ and expressing $n^{\mu}$ in terms of the $\{\psi_{k}\}_{k=2}^{m}$, are transformations between a pair of dual bases. Thus, the expansion coefficients must build mutually inverse square matrices $A_{m} = (a_{\mu\kappa})_{2 \leq \mu, \kappa \leq m}$ and $C_{m} = (c_{\mu\kappa})_{2 \leq \mu, \kappa \leq m}$, such that

$$
\sum_{\kappa=2}^{m} a_{\mu\kappa} c_{\kappa\mu'} = \delta_{\mu\mu'}, (2 \leq \mu, \mu' \leq m).
$$

(43)

Then, we can solve this for $c_{\mu\mu'}$, the first term in the sum in Eq. (43) which is non-zero, to obtain a recursive expression for these coefficients.

$$
c_{\mu\mu'} = \frac{1}{a_{\mu\mu}} \left( \delta_{\mu\mu'} - \sum_{\kappa=2}^{\mu-1} a_{\mu\kappa} c_{\kappa\mu'} \right),
$$

(44)

$$
= (\mu - 2)! \left( \delta_{\mu\mu'} - \sum_{\kappa=2}^{\mu-1} a_{\mu\kappa} c_{\kappa\mu'} \right).
$$

(45)

For clarity, we truncated the upper summation boundary to explicitly include only non-zero values, $\mu' \leq \kappa$, for $c_{\mu\mu'}$. Alternatively, we can write

$$
c_{ml} = \begin{cases} 
1/a_{mm} = (m - 2)! & \text{if } l = m, \\
-(m - 2)! \sum_{k=2}^{m-1} a_{mk} c_{kl} & \text{if } 2 \leq l \leq m - 1, \\
0 & \text{else.}
\end{cases}
$$

(46)

This solves our problem: we now have a closed expression for the $\psi_{m}(n)$ and also the transformation coefficients $c_{mk}$. Of course, we can also build the matrix $A_{m}$ and calculate its inverse.

As an example, we give the matrix $C_{8} = (c_{\mu\kappa})_{2 \leq \mu, \kappa \leq 8}$ with explicit values up to $\mu = 8, \kappa = 8$. This matrix includes the $C_{\mu}$ matrices, $\mu < 8$, as square submatrices, which are obtained by truncating $C_{8}$ at the appropriate row and column:

$$
C_{8} = (c_{\mu\kappa})_{2 \leq \mu, \kappa \leq 8} = \begin{pmatrix}
1 & 0 & 1 & -2 & 2 & \cdots & (0)_{\mu<\kappa} \\
1 & 5 & 10 & 6 \\
1 & 40 & 54 & 24 \\
0 & 336 & -336 & 120 \\
1 & 462 & -1764 & 3024 & -2400 & 720
\end{pmatrix}
$$

(47)

The coefficients $c_{mk}$ within each row seem to be of alternating sign, where the highest-order non-zero coefficient $c_{mm}$ on the diagonal always has positive sign. Furthermore, the coefficient $c_{m2}$ apparently is 0 for odd orders $m$ and 1 for even orders. Looking at the distribution of the absolute values of the coefficients $|c_{mk}|$ for a set order $m$, it seems as if it assumes a maximum value for some $k < m$. 
7. Various Power Sums

With this, we can write down any power sum $S_m^{(a)}(n)$. Exemplarily, we want to give expressions for some of the better known of them. It is easy for $m = 2$ and $m = 3$:

$$S_2^{(1)}(n) = \sum_{\nu=1}^{n} \nu^2 = \psi_2^{(1)}(n) = \left[1 + \frac{2}{3}(n-1)\right] B_{2,n-1} = \frac{1}{6} n(n+1)(2n+1).$$

$$S_3^{(1)}(n) = \sum_{\nu=1}^{n} \nu^3 = \psi_3^{(1)}(n) = B_{2,n-1} + \frac{3}{2}(n-1)B_{3,n-1} = \left[\frac{n(n+1)}{2}\right]^2.$$

We reproduce Nicomachus’s formula for $m = 3$. For $m = 2$ we find the square pyramidal numbers.[6] The expressions become more difficult starting with $m = 4$:

$$S_4^{(1)}(n) = \sum_{\nu=1}^{n} \nu^4 = 2\psi_4^{(1)}(n) - 2\psi_3^{(1)}(n) + \psi_2^{(1)}(n),$$

$$= B_{2,n-1} + 4(n-1)B_{4,n-1} - 3(n-1)B_{3,n-1} + \frac{2}{3}(n-1)B_{2,n-1},$$

$$= 4(n-1)B_{4,n-1} - 3(n-1)B_{3,n-1} + \frac{2n+1}{3}B_{2,n-1},$$

$$= \frac{1}{30} n(6n^4 + 15n^3 + 10n^2 - 1).$$

$$S_5^{(1)}(n) = 5\psi_5^{(1)}(n) - 10\psi_4^{(1)}(n) + 6\psi_3^{(1)}(n),$$

$$= B_{2,n-1} + 20(n-1)B_{5,n-1} - 24(n-1)B_{4,n-1} + \frac{15}{2}(n-1)B_{3,n-1},$$

$$= \frac{1}{12} n^2(n+1)[2n^2 + 2n - 1].$$

As a final example, we furthermore give the expression for $S_8^{(2)}(n)$:

$$S_8^{(2)}(n) = 720\psi_8^{(2)}(n) - 2400\psi_7^{(2)}(n) + 3024\psi_6^{(2)}$$

$$- 1764\psi_5^{(2)} + 462\psi_4^{(2)} - 42\psi_3^{(2)} + \psi_2^{(2)}$$

$$= (n-1) \left(4032B_{9,n-1} - 11200B_{8,n-1} + 11340B_{7,n-1}\right)$$

$$- 5040B_{6,n-1} + 924B_{5,n-1} - \frac{252}{5}B_{4,n-1}$$

$$+ \left[1 + \frac{1}{2}(n-1)\right] B_{3,n-1}$$

$$= \frac{1}{180} n(n+1)(n+2)(2n^2 + 4n - 1)(n^4 + 4n^3 + n^2 - 6n + 3).$$

8. Final Considerations

Finally, we want to show how the expansion coefficients $c_{mk}$ are related to the Bernoulli numbers $B_i$, as part of our results is equivalent to Bernoulli’s ($a = 1$). To
put this into context, we expand the $\psi_k^{(1)}(n)$ and set equal to Bernoulli’s formula:

\begin{equation}
\sum_{k=2}^{m} c_{mk} \psi_k^{(1)}(n) = \frac{1}{m+1} \sum_{l=1}^{m+1} \binom{m+1}{l} B_{m-l+1} n^l,
\end{equation}

\begin{equation}
\sum_{k=2}^{m} c_{mk} \sum_{j=1}^{k+1} g_{kj} n^j = \frac{1}{m+1} \sum_{l=1}^{m+1} \binom{m+1}{l} B_{m-l+1} n^l,
\end{equation}

\begin{equation}
\sum_{j=1}^{m+1} \left( \sum_{k=2}^{m} c_{mk} g_{kj} \right) n^j = \frac{1}{m+1} \sum_{l=1}^{m+1} \binom{m+1}{l} B_{m-l+1} n^l.
\end{equation}

We require that the expansion coefficients of $\psi_k^{(1)}(n)$, $g_{kj} = 0$ if $j > k$. Similar to what we already have used before, we then can decouple the upper summation boundary and switch the summation order. We may identify the product of coefficients with the coefficients in Bernoulli’s expansion:

\begin{equation}
\sum_{k=2}^{m} c_{mk} g_{kj} = \frac{1}{m+1} \binom{m+1}{j} B_{m-j+1},
\end{equation}

which we can solve for the Bernoulli number $B_{m-j+1}$:

\begin{equation}
B_{m-j+1} = \frac{(m-j+1)! j!}{m!} \sum_{k=2}^{m} c_{mk} g_{kj}.
\end{equation}

We can thus express the Bernoulli numbers in terms of the $c_{mk}$ and the coefficients $g_{kj}$ of the $\psi_k^{(1)}$.

In the end, we want to state explicitly the main advantage of expressing the $S_m^{(a)}(n)$ in terms of the $\psi_k^{(a)}$ instead of using a single polynomial at once: instead of putting the complexity of the summation procedure into the coefficients, we shift it into the basis functions. The recursive trait of those basis functions $\psi_k^{(a)}(n)$ then makes for an elegant generalization for more complex sums ($a > 1$).

\textbf{References}

[1] J. Bernoulli. *Ars conjectandi, opus posthumum. Accedit Tractatus de seriebus infinitis, et epistola gallicé scripta de ludo pilae reticularis*. Basiliae, impensis Thurnisiorum, fratrum, 1713.

[2] I. N. Bronstein et al. *Taschenbuch der Mathematik*. German. 7th ed. Verlag Harri Deutsch, Frankfurt am Main, 2008. ISBN: 978-3-8171-2017-8.

[3] C. G. Jacobi. “De usu legitimo formule summatorii Maclaurinianae”. In: J. Reine Angew. Math. 1834.12 (1834), pp. 263–272. DOI: doi:10.1515/crll.1834.12.1834.12.263 URL: https://doi.org/10.1515/crll.1834.12.263

[4] D. E. Knuth. “Johann Faulhaber and sums of powers”. In: Math. Comp. 61.203 (1993), pp. 277–294. ISSN: 0025-5718, 1088-6842. DOI: 10.1090/S0025-5718-1993-1197512-7 URL: https://www.ams.org/mcom/1993-61-203/S0025-5718-1993-1197512-7/ (visited on 03/04/2022).

[5] *Simplicial polytopic numbers - OEIS*. URL: http://oeis.org/wiki/Simplicial_polytopic_numbers (visited on 02/22/2022).

[6] *Square Pyramidal Numbers - OEIS*. URL: http://oeis.org/A000330 (visited on 03/04/2022).

\textbf{Department of Chemistry, Technical University of Munich, 85748 Garching, Germany}

\textbf{Email address: c.muschielok(at)tum.de}