Boundary Ground Ring in 2D String Theory

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The 2D quantum gravity on a disc, or the non-critical theory of open strings, is known to exhibit an integrable structure, the boundary ground ring, which determines completely the boundary correlation functions. Inspired by the recent progress in boundary Liouville theory, we extend the ground ring relations to the case of non-vanishing boundary Liouville interaction known also as FZZT brane in the context of the 2D string theory. The ring relations yield an over-determined set of functional recurrence equations for the boundary correlation functions. The ring action closes on an infinite array of equally spaced FZZT branes for which we propose a matrix model realization. In this matrix model the boundary ground ring is generated by a pair of complex matrix fields.

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1. Introduction

The world-sheet description of the non-critical string theories is given by the two-dimensional quantum gravity (for reviews see [1-3]). Considered as a theory of coupled Liouville and matter fields, the 2D QG has much larger symmetry than each of its two components. Turning on the gravitational field erases all the information about distances and instead of correlation functions one obtains ‘correlation numbers’. One of the manifestations of this larger symmetry of 2D QG is the so called ‘ground ring’ integrable structure discovered in the early 90’s [4-8]. It has been shown by Witten [6] that for $c = 1$ matter coupled to Liouville theory, there exists a ring of ghost-number zero operators, with respect to which the local observables, represented by ghost-number one vertex operators, form a module. Kutasov, Martinec and Seiberg [9] have pointed out that the action of ground ring leads to a set of recurrence equations for the correlation functions of the primary fields (the closed string ‘tachyons’). The ground ring structure is particularly interesting also by the fact [6,9] that it resembles the integrable structures observed in the matrix models of 2D QG.

Bershadsky and Kutasov [11] extended the ground ring structure to the case of 2D QG with boundary, or $c = 1$ open string theory. In addition to the two bulk ground ring generators, it involves two boundary operators, which generate the ‘boundary ground ring’. In [11], the action of the boundary ground ring is used to find recurrence relations for the correlation functions of boundary operators, or open string ‘tachyons’. The compatibility of these (over determined) relations allows to evaluate all structure constants and all correlation functions, and to reproduce elegantly the results obtained previously by Coulomb gas integrals [12-16].

Originally the ring relations were applied only for the so called ‘resonant’ amplitudes, which are the only non-vanishing ones in a theory with vanishing bulk and boundary cosmological constants. In the full theory, the resonant amplitudes give the residues of the ‘on-mass-shell’ poles of the exact amplitudes. It was conjectured [6] that the perturbations like the cosmological term are described by certain deformations of the ground ring structure, but the exact field-theoretical meaning of this conjecture remained obscure. Only after the impressive developments in Liouville CFT from the last several years [17-21] it has been realized that the ground ring structure is an exact symmetry of the full CFT constants [22]. Indeed, the conformal bootstrap used to derive the exact expressions for the bulk and boundary structure constants in [19-21] is essentially the same procedure as the one used to establish the ground ring relations. The only technical difference is that while the identities obtained in pure Liouville theory follow from the truncated OPE with

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1 Very recently, new evidences about the deep connection between these two integrable structures were put forward in the interesting paper by Seiberg and Shih [10].
degenerate Liouville fields, in 2D QG one considers products of degenerate Liouville and matter fields.

In this paper we consider the boundary ground ring structure in Liouville quantum gravity with non-vanishing boundary cosmological constant, or equivalently, in a 2D string theory in presence of a FZZT brane. Extending the technique developed by Bershadsky and Kutasov [11], we obtain a set of functional recurrence equations for the \( n \)-point boundary correlation functions. Crucial role in our analysis will play an observation made by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [19] about the nature of the degenerate boundary Liouville fields, which we sketch below.

In boundary Liouville CFT there are two cosmological constants \( \mu \) and \( \mu_B \), the bulk and the boundary one. \( \mu_B/\sqrt{\mu} \). The observables are meromorphic functions of \( \mu_B \), with a branch point singularity at \( \mu_B = -\mu_B^0 \), where \( \mu_B^0 \sim \sqrt{\mu} \) is minus the boundary entropy produced by the fluctuations of the Liouville field in the bulk. The branch point can be resolved by introducing a uniformization parameter \( s \)

\[
\mu_B = \mu_B^0 \cosh(\pi b s)
\]  

where \( b \) is the Liouville coupling constant. FZZ observed [19] that the correctly defined lowest degenerate boundary operators introduce shifts of the boundary parameter \( s \rightarrow s \pm i b \) or \( s \rightarrow s \pm ib^{-1} \) at the points of the boundary where they are inserted. Therefore, in order to close the conformal bootstrap, one is led to associate with a boundary operator two independent boundary parameters, \( s_{\text{left}} \) and \( s_{\text{right}} \), labeling the Liouville boundary conditions on both sides. FZZ derived a simple functional equation for the boundary two-point function (see the concluding section of [19]), which involves a shift of one of the two boundary parameters as well as the Liouville charge of the two boundary operators. Liouville fields. The boundary two-point function can be obtained as the unique solution of this equation that satisfies the reflection and duality symmetries.

The functional equation of FZZ can be written as a difference equation with respect to one of the boundary parameters. In [23] it was found that all boundary Liouville structure constants satisfy similar difference equations. Interestingly, these equations have a natural interpretation in the discrete formulation of 2D QG in terms of a gas of loops and lines on a randomly triangulated disc. In the discrete approach the difference equations are satisfied by the \( n \)-point boundary correlation functions with any \( n \), while in pure Liouville theory this is so only for \( n \leq 3 \). Of course, this is so because starting with 4 operators, the contributions of the Liouville and matter sectors do not factorize.

\[\text{\textsuperscript{2}}\] The general validity of this suggestion, which was proven in [19] only for the two-point function, follows from the formula for the three point function derived subsequently in [20].
Our original motivation for this work was to give a continuous derivation of the difference equations for the boundary correlators obtained in [24,23] within the discrete approach. This naturally led us to consider a deformation of the boundary ground ring structure by both bulk and boundary cosmological terms. Using the results of [19,23] and extending the technique developed in [11], we derived a set of functional recurrence equations for the \( n \)-point boundary correlation functions that generalize the recurrence equations of Bershadsky and Kutasov [11] on one hand, and the difference equations obtained in [19] and [24,23] on the other hand. These equations completely determine all boundary correlation functions in 2D QG.

In this paper we consider the realization of 2D string theory on a FZZT brane in which the matter field is a gaussian field with pure Neumann or Dirichlet boundary conditions. The target space of the gaussian field represents the Euclidean time direction \( x \). The generators of the boundary ground ring, being essentially products of Liouville and matter boundary degenerate fields, introduce shifts both in the Liouville boundary parameter \( s \) and in the time direction \( x \). The time shift is equal to the critical distance \( \Delta x = 1 \) associated with the Kosterlitz-Thouless point. In general the ring relations yield functional equations which involve shifts both in the Liouville and matter boundary conditions, but in the simplest case we are considering the correlation functions depend only on the Liouville boundary parameter \( s \).

The minimal configuration of FZZT branes that closes under the action of the boundary ground ring is given by an array of FZZT branes spaced at the critical distance \( \Delta x = 1 \) in the time direction. We propose a matrix model description for such a background based on a variant of the \( \hat{A}_\infty \) matrix model. In this matrix model the boundary ground ring has explicit realization as the ring of the polynomials of of two complex matrix fields.

2. Some preliminaries

2.1. The boundary CFT for the FZZT brane

We consider the non-unitary realization of Euclidean 2D string theory by a Liouville field \( \phi \) and a gaussian matter field \( \chi \) with background charge \( e_0 \). The background charge is normalized so that the conformal anomaly of the matter field is

\[
c = 1 - 6e_0^2.
\]

The disc partition function of string theory with FZZT-type boundary condition [19,23] is defined by by the following effective action, which comprises a bulk and a boundary term:

\[
\mathcal{A}[\chi, \phi] = \int_{\mathcal{M}} \left( \frac{1}{4\pi} [\nabla \phi]^2 + [\nabla \chi]^2 + (Q\phi - ie_0\chi)\hat{R}^\perp + \mu e^{2\phi} \right) + \int_{\partial\mathcal{M}} \left( \frac{1}{2\pi} (Q\phi - ie_0\chi)\hat{K} + \mu_B e^{\phi} \right) + \text{ghosts}
\]
where the couplings $\mu$ and $\mu_B$ are are referred to as bulk and the boundary cosmological constants. The background charges are expressed in terms of the Liouville coupling constant $b$ as

$$Q = \frac{1}{b} + b, \quad e_0 = \frac{1}{b} - b. \quad (2.3)$$

With the choice (2.3) the two background charges satisfy $Q^2 - e_0^2 = 4$, which is equivalent to the balance of the central charge $c_{\text{tot}} \equiv c_\phi + c_\chi + c_{\text{ghosts}} = (1 + 6Q^2) + (1 - 6e_0^2) - 26 = 0$.

We will consider the generic situation when $b$ is not a rational number.

After mapping the disc $\mathcal{M}$ to the upper half plane $\{\text{Im} x \geq 0\}$, the curvature term disappears

$$A[\phi, \chi] = \int_{\text{Im} x \geq 0} d^2 x \left( \frac{1}{4\pi} \left[ (\nabla \phi)^2 + (\nabla \chi)^2 \right] + \mu e^{2b\phi} \right) + \int_{-\infty}^{\infty} dx \mu_B e^{b\phi} + \text{ghosts} \quad (2.4)$$

and the background charges are introduced through the asymptotics of the fields at spatial infinity

$$\phi(x, \bar{x}) \sim -Q \log |x|^2, \quad \chi(x, \bar{x}) \sim -e_0 \log |x|^2. \quad (2.5)$$

The boundary term encodes an inhomogeneous Neumann boundary condition for the Liouville field

$$i(\partial - \bar{\partial})\phi(x, \bar{x}) = 4\pi \mu_B e^{b\phi(x, \bar{x})} \quad (\bar{x} = x), \quad (2.6)$$

which describes the FZZT brane characterized by the the parameter $\mu_B$. The matter field is assumed to satisfy pure Neumann boundary condition:

$$i(\partial - \bar{\partial})\chi(x, \bar{x}) = 0 \quad (\bar{x} = x), \quad (2.7)$$

which is of course Dirichlet boundary condition for the T-dual field $\tilde{\chi}(x, \bar{x})$. In this paper we consider the simplest situation where no matter screening charges are added.

2.2. Bulk and boundary vertex operators (closed and open string tachyons)

The bulk primary fields $V_P^{(\pm)}(x, \bar{x})$, or left/right moving on-mass-shell closed string tachyons, are defined as the bulk vertex operators

$$V_P^{(+)} = \frac{1}{\pi} \gamma(bP) e^{i(e_0 - P)\chi + (Q - P)\phi}$$
$$V_P^{(-)} = \frac{1}{\pi} \gamma(-\frac{1}{b}P) e^{i(e_0 - P)\chi + (Q + P)\phi} \quad (2.8)$$

where $\gamma(x) \equiv \Gamma(x)/\Gamma(1 - x)$. This normalization removes the external ‘leg pole’ factors in the correlation functions and is the one to be used when comparing with the microscopic realizations of 2D QG.
We will be mainly interested in the correlation functions of the boundary fields that can be inserted along the boundary of the world sheet. The boundary primary fields $\mathcal{B}^{(\pm)}_P(x)$, or left/right moving on-mass-shell open string tachyons, are defined as the boundary vertex operators

$$\mathcal{B}^{(+)}_P = \frac{1}{\pi} \Gamma(2bP) \ e^{i\left(\frac{1}{2}\epsilon_0 - P\right)\chi + \left(\frac{1}{4}Q - P\right)\phi}$$

$$\mathcal{B}^{(-)}_P = \frac{1}{\pi} \Gamma\left(-\frac{2}{b}P\right) \ e^{i\left(\frac{1}{2}\epsilon_0 - P\right)\chi + \left(\frac{1}{4}Q + P\right)\phi}.$$  \hspace{1cm} (2.9)

As any CFT boundary operator, the open string tachyon is unambiguously defined only after both left and right boundary conditions are specified, which should be done both for matter and Liouville components.

In a theory in which the ends of the open strings freely propagate in the two-dimensional space-time ($\mu_B = 0$), both fields satisfy pure Neumann boundary conditions. In a theory with FZZT branes, the Liouville left and right boundary conditions are determined by the values $s_1 = s_{\text{left}}$ and $s_2 = s_{\text{right}}$ of the uniformization parameter defined in (1.1). Slightly modifying the notations of FZZ \cite{19}, will denote such an operator as $s_1 [\mathcal{B}^{(\pm)}_P]^{s_2}$. Geometrically the boundary fields create open string states whose ends propagate in two different FZZT branes labeled by $s_1$ and $s_2$.

The physical observables can be represented in two pictures: either as (1,1)-forms and integrated over the the world-sheet (for the bulk fields) and 1-forms integrated over the boundary (for the boundary fields), or as BRST-closed 0-forms:

$$\int d^2x \ \mathcal{V}^{(\pm)}_P(x, \bar{x}) \leftrightarrow c\bar{c} \ \mathcal{V}^{(\pm)}_P(x, \bar{x})$$

$$\int dx \ \mathcal{B}^{(\pm)}(x) \leftrightarrow c \ \mathcal{B}^{(\pm)}(x).$$  \hspace{1cm} (2.10)

Here and in the following $b, c$ and $b, c$ are the reparametrization ghosts respectively in the bulk and on the boundary. The second representation is more appropriate as far as the ground ring structure is concerned.

2.3. Reflection property

The tachyons of opposite chiralities are related by the Liouville bulk and boundary reflection amplitudes (see Appendix A)

$$\mathcal{V}^{(+)}_P = S^{(+)}_P \mathcal{V}^{(-)}_P,$$

$$s_1[\mathcal{B}^{(\pm)}_P]^{s_2} = D^{(+)}_P(s_1, s_2) \ s_1[\mathcal{B}^{(-)}_P]^{s_2}$$  \hspace{1cm} (2.11)

2.4. Physical states in 2D QG

For each value of the momentum there is only one vertex operator that corresponds to a physical state. The physical operators are

$$\mathcal{V}_P = \begin{cases} 
\mathcal{V}^{(+)}_P & (P > 0) \\
\mathcal{V}^{(-)}_P & (P < 0)
\end{cases}, \hspace{1cm} \mathcal{B}_P = \begin{cases} 
\mathcal{B}^{(+)}_P & (P > 0) \\
\mathcal{B}^{(-)}_P & (P < 0)
\end{cases}.$$  \hspace{1cm} (2.12)
The “wrongly dressed” operators are related to the physical ones by the Liouville reflection amplitude.

2.5. Degenerate fields

By degenerate fields in Liouville quantum gravity we will understand the gravitationally dressed degenerate matter fields. In our case these are the on-shell vertex operators (2.12) with degenerate matter components

\[ V_{rs} = V_{r/b-sb}, \quad B_{rs} = B_{\frac{1}{2}(r/b-sb)} \quad (r, s \in \mathbb{N}). \] (2.13)

The fusion rules for these fields are the same as the fusion rules for the degenerate fields in the matter CFT [26]. Note that the Liouville components of these fields are not degenerate Liouville fields.

The operators that span the ground ring are degenerate with respect to the full conformal algebra. Such an operator is constructed by applying a rising Virasoro operator of level \( rs - 1 \) to a product of matter and Liouville degenerate fields with Kac labels \( r, s \). The conformal weight of the product is zero:

\[ \Delta = \left( \frac{r/b - sb}{2} - \frac{e_0^2}{4} + \frac{Q - (r/b + sb)^2}{4} \right) + rs - 1 = 0. \] (2.14)

2.6. Normalization of the bulk and boundary cosmological constants

It is also convenient to redefine the cosmological constants \( \mu \) and \( \mu_B \) according to the normalizations (2.8) and (2.9) of the vertex operators. The new bulk and boundary cosmological constants \( \Lambda \) and \( z \) are defined as the couplings of the operators \( V_{e_0}^{(+)} \) and \( B_{e_0/2}^{(+)} \) respectively:

\[ \mu e^{2b\phi} = \Lambda V_{e_0}^{(+)}, \quad \mu_B e^{b\phi} = z B_{e_0/2}^{(+)}. \] (2.15)

This gives

\[ \frac{\Lambda}{\mu} = \pi \gamma(b^2), \quad \frac{z}{\mu_B} = \frac{\pi}{\Gamma(1 - b^2)}. \] (2.16)

The uniformization map (1.1) now reads

\[ z = M \cosh(\pi bs), \quad M = \sqrt{\Lambda}. \] (2.17)

Here we introduced the constant \( M \) whose physical meaning is that it is equal to minus the boundary entropy due to the fluctuations of the Liouville field.

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3 When comparing with the discrete approach, the cosmological constants \( \Lambda \) and \( z \) are to be identified with the corresponding quantities in the microscopic formulation of the strings defined on Dynkin graphs [27]. The constant \( M \) equals half the loop tension in the loop gas description of 2D string theory [27, 28]. The uniformization parameter \( s \) is defined as in [19]; it is related to the parameter \( \tau \) of [27] by \( \tau = \pi bs \).
2.7. The self-duality property of Liouville quantum gravity

All correlation function in Liouville theory are invariant w.r.t. the substitution \( b \rightarrow \tilde{b} = 1/b, \quad \Lambda \rightarrow \tilde{\Lambda}, \quad M \rightarrow \tilde{M} \)

where

\[
\tilde{\Lambda} = \Lambda^{1/b^2}, \quad \tilde{M} = M^{1/b^2}.
\]

The boundary parameter \( s \) is self-dual. The duality transformation for the correlation functions follows from the one of left and right chiral tachyons

\[
\tilde{V}_P^{(\pm)} = \bar{V}_P^{(\mp)}, \quad s_1 [\tilde{B}_P^{(\pm)}]^{s_2} = \bar{s}_1 [\bar{B}_P^{(\mp)}]^{s_2},
\]

where the bar means complex conjugation.

It is clear from (2.19) that the non-unitary CFT under consideration is not self-dual in the strict sense. The duality transformation interchanges the states of a pair of CFT characterized by matter background charges \( e_0 = \frac{1}{b} - b \) and \( \tilde{e}_0 = b - \frac{1}{b} = -e_0 \). The duality becomes a symmetry (a chirality flip) only when it is accompanied by a complex conjugation.

3. The (bulk) ground ring

We start with a short review of the ground ring structure for a theory defined on the sphere \((\partial M = 0)\). The ground ring operators are obtained by applying raising operators of level \( rs - 1 \) to the product of two degenerate matter and Liouville fields with Kac labels \( r, s \). The resulting operators have conformal weights \( \Delta = \bar{\Delta} = 0 \). The ring is generated by the lowest of two operators \[3\]

\[
a_+ = -|bc - b\partial_x (\phi - i\chi)|^2 e^{-b^{-1}(\phi + i\chi)}
\]

\[
a_- = -|bc - b^{-1}\partial_z (\phi + i\chi)|^2 e^{-b(\phi - i\chi)}.
\]

where \( b, c \) are the reparametrization ghost and anti-ghost fields. The ground ring is spanned on the polynomials \((a_+)^m(a_-)^n\) with \( m, n \in \mathbb{Z}_+ \). In the case of non-rational \( b^2 \) the ground ring contains no other relations and has an infinite number of elements labeled by the integers \( r, s \geq 1 \).

A crucial property of the operators \( a_\pm \) is that their derivatives \( \partial_x a_\pm \) and \( \partial_z a_\pm \) are BRST exact: \( \partial_x a_- = \{Q_{\text{BRST}}, b_- a_-\} \). Therefore, any amplitude that involves \( a_\pm \) and other BRST invariant operators does not depend on the position of \( a_\pm \). This property allows to write recurrence equations for the correlation functions from the OPE of \( a_\pm \) and the other BRST-invariant fields.
3.1. The limit of free fields ($\Lambda = 0$)

The vertex operators (2.8) form a module under the ground ring:

$$a_+ V^{(+)}_P = -V^{(-)}_{P + b}, \quad a_+ V^{(-)}_P = -V^{(+)}_{P - b}. \quad (3.2)$$

and also

$$a_+ V^{(-)}_P = a_+ V^{(+)}_P = 0. \quad (3.3)$$

Both relations (3.2) and (3.3) follow from the free field OPE\footnote{The normalization of the fields is determined by the action (2.4) and is the same as in \cite{19}.} and are true up to commutators with the BRST charge. While the first one survives in the correlation functions, the second one receives non-linear corrections. Due to the contact terms, the last relation should be modified in presence of an integrated vertex operator to

$$a_+ V^{(-)}_P \int d^2 z \ V^{(-)}_{P_1} = V^{(-)}_{P + P_1 + b} \quad (3.4)$$

$$a_- V^{(+)}_P \int d^2 z \ V^{(+)}_{P_1} = V^{(+)}_{P + P_1 - b - 1}. \quad (3.5)$$

These relations imply a set of recurrence equations \cite{12,11}, which determine completely the resonant tachyon amplitudes (see Appendix B).

3.2. $\Lambda \neq 0$

The first relation (3.2) survives the perturbation with $\Lambda \neq 0$, while the second relation (3.3) gets deformed. The deformation is linear in $\Lambda$ and can be calculated perturbatively by considering a first order insertion of the Liouville interaction $-\mu \int e^{2b\phi} = -\Lambda \int V^{(+)}_{e_0}$ and then applying (3.5):

$$a_+ V^{(+)}_P = -\Lambda V^{(+)}_{P - b}. \quad (3.6)$$

This procedure has been justified by various self-consistency checks in Liouville theory \cite{17,15}. By the same argument one finds the action of the operator $a_+$. This time one should consider a linear insertion of the dual Liouville interaction $-\bar{\mu} \int e^{2\phi/b} = -\bar{\Lambda} \int V^{(-)}_{-e_0}$. The result is

$$a_+ V^{(-)}_P = -\bar{\Lambda} V^{(-)}_{P + 1/b}, \quad \bar{\Lambda} = \Lambda^{1/b^2}. \quad (3.7)$$

The deformed ring relations allow to derive an (over determined) set of recurrence equations for the bulk correlation functions (see Appendix B).

Here we consider the simplest situation where the matter field obeys $U(1)$ fusion rules. In general the free-field representation of 2D QG should be completed by adding to the effective action (2.2) the two matter screening charges

$$Q_+ = -\frac{1}{\gamma(-1/b^2)} \int e^{2ix/b} \quad \text{and} \quad Q_- = -\frac{1}{\gamma(-b^2)} \int e^{-2ibx}. \quad (4.8)$$
Then eqns. (3.6) and (3.7) will further deform to

\[ a_- \mathcal{V}_P^{(+)} = -\Lambda \mathcal{V}_{P-b}^{(+)} + \mathcal{V}_{P+b}^{(+)} \]  
\[ a_+ \mathcal{V}_P^{(-)} = -\tilde{\Lambda} \mathcal{V}_{P+1/b}^{(-)} + \mathcal{V}_{P-1/b}^{(-)} \]  

(3.8)

(3.9)

The ground ring structure in presence of screening charges has been considered in \[30,31\].

3.3. The case \( b = 1 \)

The case \( b = 1 \) plays a special role \[3\]. Consider the action of the product \( a_+a_- \):

\[ a_+a_- \mathcal{V}_P^{(+)} = \Lambda \mathcal{V}_{P+\epsilon_0}^{(+)} \]
\[ a_+a_- \mathcal{V}_P^{(-)} = \tilde{\Lambda} \mathcal{V}_{P+\epsilon_0}^{(-)} \]

When \( b = 1 \) we get, since \( \Lambda = \tilde{\Lambda} \) and \( \epsilon_0 = 0 \),

\[ a_+a_- = \Lambda \quad (b = 1). \]  

(3.10)

This equation has a direct interpretation in the profile of the Fermi surface in the corresponding large \( N \) matrix model \[3\]. This analogy can be pursued further to a generic perturbation by bulk tachyons and relate the ground ring structure and the Toda integrable structure in the matrix realization of the \( c = 1 \) string theory \[32-37\].

4. The boundary ground ring

4.1. The boundary ground ring for pure Neumann boundary conditions (\( \mu_B = 0 \))

We first introduce the boundary ground ring for 2D open strings with of pure Neumann boundary conditions for both coordinate fields, \( \phi \) and \( \chi \), following Bershadsky and Kutasov \[11\]. The two generators are defined, following the same logic as in the bulk case, as

\[ A_+ = -[bc - \frac{1}{2} \partial_x (\phi - i\chi)] e^{-\frac{b}{4}b^{-1}(\phi+i\chi)} \]
\[ A_- = -[bc - \frac{1}{2} \partial_x (\phi + i\chi)] e^{-\frac{b}{4}b(\phi-i\chi)}. \]

(4.1)

The two operators are related by a duality transformation combined with complex conjugation:

\[ \tilde{A}_+ = \overline{A}_- , \quad \tilde{A}_- = \overline{A}_+. \]  

(4.2)
They are BRST closed: \( \partial_x A_\pm = \{ Q_{\text{BRST}}, b_{-1} A_\pm \} \) and have \( \Delta = 0 \). The open string tachyons (2.9) form a module with respect to the ring generated by these two operators. The action of the ring on the tachyon modules is generated by the relations

\[
A_+ B_P^{(+)} = B^{(+)}_{P + \frac{1}{2}}, \quad A_- B_P^{(-)} = B^{(-)}_{P - \frac{1}{2}} \tag{4.3}
\]

and

\[
A_+ B_P^{(-)} = A_- B_P^{(+)} = 0. \tag{4.4}
\]

If one exchanges the places of \( A \) and \( V \), there will be minus on the rhs, since \( A \) is odd and \( V \) is even wrt \( x \rightarrow -x \):

\[
B^{(+)}_P A_+ = -B^{(+)}_{P + \frac{1}{2}}, \quad B^{(-)}_P A_- = -B^{(-)}_{P - \frac{1}{2}}. \tag{4.5}
\]

Again, the first relation (4.3) is exact and the second relation (4.4) gets deformed in presence of integrated boundary tachyon fields [11]:

\[
A_- B^{(+)}_{P_1} \int dx B^{(+)}_{P_2} = \frac{1}{\sin 2\pi b P_1} B^{(+)}_{P_1 + P_2 - \frac{1}{2b}}. \tag{4.6}
\]

Since on the boundary the ordering is important, there is a second relation

\[
\int dx B^{(+)}_{P_1} A_- B^{(+)}_{P_2} = \frac{\sin 2\pi b (P_1 + P_2)}{\sin 2\pi b P_1 \sin 2\pi b P_2} B^{(+)}_{P_1 + P_2 - \frac{1}{2b}}. \tag{4.7}
\]

The coefficients in (4.6) and (4.7) are obtained as standard Coulomb integrals (see for example [16]). Similarly, one finds for \( A_+ \)

\[
A_+ B^{(-)}_{P_1} \int dx B^{(-)}_{P_2} = \frac{1}{\sin \frac{2\pi}{b} P_1} B^{(-)}_{P_1 + P_2 + \frac{1}{2}} \tag{4.8}
\]

and

\[
\int dx B^{(-)}_{P_1} A_+ B^{(-)}_{P_2} = \frac{\sin \frac{2\pi}{b} (P_1 + P_2)}{\sin \frac{2\pi}{b} P_2 \sin \frac{2\pi}{b} P_1} B^{(-)}_{P_1 + P_2 + \frac{1}{2}}. \tag{4.9}
\]

Using these relations, Bershadsky and Kutasov obtained recurrence equations for the \( n \)-point open string amplitudes (Appendix C).

4.2. The boundary ground ring in presence of FZZT branes

The ring relations can be generalized to the case of nonzero boundary cosmological constant using the results of the paper by V. Fateev, A. Zamolodchikov and Al. Zamolodchikov [19]. The crucial observation made by the authors of [13] and later confirmed by B. Ponsot and J. Teschner [20] is that a level-\( n \) degenerate boundary Liouville field \( e^{-ab\phi} \) has vanishing null-vector and therefore a truncated OPE with the other primary fields if
either \( s_{\text{left}} - s_{\text{right}} = ibk \) or \( s_{\text{left}} + s_{\text{right}} = ibk \), with \( k = -n/2, -n/2 + 1, \ldots, n/2 \). By the duality property of Liouville theory, the degenerate boundary fields \( e^{-n\phi/b} \) exhibit a similar property with \( b \) replaced by \( 1/b \). No direct proof is supplied for this statement, but it was shown to be consistent with the exact results obtained in boundary Liouville theory. In particular, the above property leads to a pair of functional equations for the Liouville boundary reflection amplitude, whose unique solution coincides with the result of the standard conformal bootstrap.

According to [19], the operators (4.1) should be defined by

\[
\begin{align*}
A_+ & \rightarrow s[A_+]^{s\pm i/b} \\
A_- & \rightarrow s[A_-]^{s\pm ib}
\end{align*}
\]  

(4.10)

and the relations (4.3), (4.6), (4.8) and (4.9) understood as

\[
s_1[B_P^{(-)}]^{s_1\pm ib} = - s_1[B_{P - \frac{\pi}{2}}^{(-)}]^{s_1\pm ib}
\]  

(4.11)

every. As in the bulk case, the relation (4.4) is deformed by a linear insertion of the boundary Liouville interaction. However, in this case the situation is more subtle since the boundary interaction depends on the point it is inserted. We use the fusion rules that follow from (4.6) and (4.7) where one of the operators is the Liouville boundary interaction \(-z(s) \times \int B_{\epsilon_0/2}^{(+)}\). The result is

\[
s_1[A_-]^{s_1\pm ib}[B_P^{(+)}]^{s_2} = C_{\pm} \times s_1[B_{P - \frac{\pi}{2}}^{(+)}]^{s_2}
\]  

(4.12)

with the coefficient \( C_{\pm} \) given by the sum of the contributions of the three possible insertions of \(-z(s) \times \int B_{\epsilon_0}^{(+)}\) with \( s = s_1, s_1 \pm ib \) and \( s_2 \) (fig.1):

\[
C_{\pm} = -M \left( \cosh(\pi bs_1) \frac{\sin 2\pi b(\frac{1}{2}\epsilon_0 + P)}{\sin(\pi be_0) \sin(2\pi bP)} + \frac{\cosh(\pi b(s_1 \pm ib))}{\sin(\pi be_0)} + \frac{\cosh(\pi bs_2)}{\sin(2\pi bP)} \right). \tag{4.13}
\]

The coefficient \( C_{\pm} \) can be nicely written as

\[
C_{\pm} = -M \frac{\cosh[\pi b(s_1 \pm 2iP)] + \cosh(\pi bs_2)}{\sin 2\pi bP} \\
= -M \frac{\cosh \left[ \pi b \left( \frac{s_1 + s_2}{2} \pm iP \right) \right] \cosh \left[ \pi b \left( \frac{s_1 - s_2}{2} \pm iP \right) \right]}{\frac{1}{2} \sin 2\pi bP}. \tag{4.14}
\]

We omit the boundary conditions for the matter field. In our case of gaussian field \( \chi \) with pure Neumann boundary conditions, those are given by the values of the dual field \( \tilde{\chi} \) on both sides. The momentum conservation yields \( \tilde{\chi}_{\text{right}} - \tilde{\chi}_{\text{left}} = \pm b^{\pm 1} \). In calculation expectation values, the final result will not depend on the only residual parameter, the global mode of \( \tilde{\chi} \). However, if screening charges are allowed, the matter boundary conditions matter.
Fig. 1: The three possible insertions of the boundary Liouville interaction.

Similarly one finds for the action of $A_+$ on $B_P^{(-)}$

$$s_1[A_+]^{s_1 \pm i/b} [B_P^{(-)}]^{s_2} = \tilde{C}_\pm \times s_1[B_P^{(+)}]^{s_2}$$  \hspace{1cm} (4.15)

with

$$\tilde{C}_\pm = -M^{1/b^2} \frac{\cosh[\pi (s_1 \pm 2iP)/b]}{\sin 2\pi P} + \cosh(\pi s_2/b)$$

$$= -M^{1/b^2} \frac{\cosh[\pi \left( \frac{s_1 \pm s_2}{2} \pm i\pi P \right)] \cosh \left[ \frac{\pi}{b} \left( \frac{s_1 - s_2}{2} \pm i\pi P \right) \right]}{\frac{1}{2} \sin(2\pi P)}.$$  \hspace{1cm} (4.16)

Note that the fusion coefficients are symmetric with respect to the change the sign of any of the boundary parameters. This is a general property of all correlation functions and comes from the fact that the $s$-space is an orbifold: the points $s$ and $-s$ should be identified.

4.3. Functional equations for the correlation functions of boundary primary fields

Consider a boundary correlation function of the form (fig. 2)

$$W_{P_1,P_2,\ldots}(s_1,s,s_2,s_3,\ldots) = \langle s_1[B_{P_1}]^s[B_{P_2}]^{s_2}[B_{P_3}]^{s_3} \rangle$$  \hspace{1cm} (4.17)

The realization of the physical boundary fields depends on the sign of the momenta and is given by (2.12). The amplitude (4.17) is by construction symmetric in its arguments. It is analytic in the boundary parameters $s,s_1,\ldots$ but not in the target space momenta. By the momentum conservation the correlation function is zero unless $\sum_k (\frac{1}{2}e_0 - P_k) = e_0$.

Equation (2.19) implies the identity

$$W_{P_1,P_2,\ldots} = W_{-P_1,-P_2,\ldots} = W_{-P_1,-P_2,\ldots}.$$  \hspace{1cm} (4.18)
Let us assume for definiteness that $P_1 < 0$ and $P_2, P_3 > 0$. Then the amplitude (4.17) is realized as

$$W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) = \left< s_1 [\mathcal{B}_{P_1}^{(-)}]^s [\mathcal{B}_{P_2}^{(+)}]^{s_2} [\mathcal{B}_{P_3}^{(+)}]^{s_3} \ldots \right> \tag{4.19}$$

By the symmetry (4.18) the amplitude can be written as

$$W_{P_1, P_2, P_3, \ldots} (s_1, s, s_2, s_3, \ldots) = \left< \ldots s_3 [\mathcal{B}_{-P_3}^{(-)}]^{s_2} [\mathcal{B}_{-P_2}^{(-)}]^{s} [\mathcal{B}_{-P_1}^{(+)}]^{s_1} \right>. \tag{4.20}$$

We will use these two representations to derive two independent functional identities for $W$.

Consider the auxiliary correlation function $F$ with an operator $A_-$ inserted in the r.h.s. of (4.17):

$$F = \left< s_1 [\mathcal{B}_{P_1}^{(-)}]^{s} [A_-]^{s \pm ib} [\mathcal{B}_{P_2}^{(+)}]^{s_2} [\mathcal{B}_{P_3}^{(+)}]^{s_3} \ldots \right>. \tag{4.21}$$

Let us assume that $P_1 < -\frac{b}{2}$ and $P_2 > \frac{b}{2}$. Then, evaluating $F$ by (4.11) and by (4.12) and equating the results we get the following functional identities for the correlation functions of three or more fields:

$$\sin (2\pi b P_2) W_{P_1, P_2, P_3, \ldots} (s_1, s \pm ib, s_2, s_3, \ldots) =$$

$$2M \cosh \left[ \pi b \left( \frac{s+s_2}{2} \pm i P_2 \right) \right] \cosh \left[ \pi b \left( \frac{s-s_2}{2} \pm i P_2 \right) \right] W_{P_1 + \frac{b}{2}, P_2 - \frac{b}{2}, P_3, \ldots} (s_1, s, s_2, s_2, \ldots) + W_{P_1 + \frac{b}{2}, P_2 + P_3 - \frac{b}{2}, \ldots} (s_1, s, s_3, \ldots). \tag{4.22}$$

The last term represents a correlation function with one operator less.

A dual equation can be obtained in the same way by evaluating the complex conjugated function $\overline{F}$ using the representation (4.20) and the relations (4.2):

$$\sin \left( \frac{2\pi}{b} P_2 \right) W_{P_1, P_2, P_3, \ldots} (s_1, s \pm \frac{b}{2}, s_2, s_3, \ldots) =$$

$$2M^{1/b^2} \cosh \left[ \frac{\pi}{b} \left( \frac{s+s_2}{2} \pm i P_2 \right) \right] \cosh \left[ \frac{\pi}{b} \left( \frac{s-s_2}{2} \pm i P_2 \right) \right] W_{P_1 + \frac{1}{b}, P_2 - \frac{1}{b}, P_3, \ldots} (s_1, s, s_2, s_2, \ldots) + W_{P_1 + \frac{1}{b}, P_2 + P_3 - \frac{1}{b}, \ldots} (s_1, s, s_3, \ldots). \tag{4.23}$$

\[6 \] Our notations do not distinguish between integrated and non-integrated fields; the rules are the same as in Appendix C.
where we assumed that $P_1 < -\frac{1}{2b}$ and $P_2, P_3 > \frac{1}{2b}$. These identities form a set of (overdetermined) functional recurrence equations that generalize the recurrence equations of [11].

Taking the differences in (4.21) and (4.22) one finds a pair of homogeneous difference equations which do not have analog in the $\mu_B = 0$ limit. We write them in operator form:

\[
\left[ \sin(b \partial_s) - M \sinh(\pi b s) e^{\frac{b}{2}(\partial P_1 - \partial P_2)} \right] W_{P_1, P_2, P_3, ...}(s_1, s, s_2, s_3, ...) = 0 \quad (4.23)
\]

and

\[
\left[ \sin(\frac{1}{b} \partial_s) - M^{1/b^2} \sinh(\frac{\pi}{b} s) e^{\frac{1}{4b}(\partial P_1 - \partial P_2)} \right] W_{P_1, P_2, P_3, ...}(s_1, s, s_2, s_3, ...) = 0. \quad (4.24)
\]

where it is assumed that $P_1 < -\frac{b}{2}$ and $P_2 > \frac{b}{2}$. The difference equations (4.23) and (4.24) have been first obtained in the microscopic formulation of 2D QG as a gas of loops and lines on the randomly triangulated disc in [24,23].

Functional equations for all values of the momenta can be obtained by the following rule [24,23]. If the sign of one or both momenta changes after the shift, one should apply the reflection property with the amplitude $D_p^{(+-)}(s_1, s_2)$ given in Appendix A. The details are explained in [23].

The functional equations (4.21) and (4.22) can be used to evaluate the two- and one-point functions as well as the disc partition function on the FZZT brane. These quantities are difficult to calculate directly in CFT because of the subtleties associated with the residual global conformal symmetry. The calculations are presented in Appendix D.

5. Matrix model description

The old idea of Polchinski that matrix models for non-critical strings describe backgrounds populated by a large number of unstable D-branes was recently given a concrete shape in [38-46].

The boundary state of such a D-brane is a product of the ZZ boundary state [47] for the Liouville field, localized at large $\phi$, and a boundary state for the matter field corresponding to some local boundary condition. For example the standard matrix model for the $c = 1$ string, the Matrix Quantum Mechanics (MQM), describes the case of pure Neumann boundary condition for the matter field (or equivalently pure Dirichlet boundary condition for the dual field).

In this section we will show, using the results of [24,23], that there exists a neat matrix model interpretation of the boundary ground ring relations. The matrix model dual to the world sheet theory (2.2) is an infinite matrix chain, which can be viewed as a special discretization of MQM. The logic is the following. First, as it was shown in
the difference equations (4.23) and (4.24) have straightforward interpretation in the loop gas formulation of non-critical string theories [18,27,28] in which formulation the string path integral is defined microscopically as a sum over planar graphs embedded in a Dynkin diagram of $ADE$ or $\hat{A}\hat{D}\hat{E}$ type. This allows to find the target-space equivalent of a degenerate boundary field. Second, the loop gas formulation can be restated in terms of the $ADE$ and $\hat{A}\hat{D}\hat{E}$ matrix chains studied in [49-54]. The matrix model description of the world sheet theory (2.2) is provided by the $\hat{A}_\infty$ matrix chain.

It is natural to think of the $\hat{A}_\infty$ matrix chain as a marginal perturbation of MQM by a periodic potential $\sim \cos(2\pi x)$ with the effect that each D0-brane decomposes into an array of D-instantons associated with the minima $x \in \mathbb{Z}$ of the potential. The initial state of such a system is an array of FZZT branes placed at the sites of the $\hat{A}_\infty$ Dynkin graph ($N$ branes per site).

5.1. Loop gas and branes

The continuum limit of the loop gas model can be given the following interpretation within the 2D string theory. It describes the evolution of closed strings in presence of an array of FZZT branes spaced at the critical distance $\delta x = 1$ in the Euclidean time direction. The background charge for the matter field is introduced by boosting the array of D-branes with momentum $e_0$. The target space of such a string theory is a direct product of the Liouville direction and the discretized time direction

$$\bar{\chi} = \pi x b, \quad x \in \mathbb{Z},$$

(5.1)

in which only the points of the array are retained. The string world sheet is divided into domains associated with different branes. It is assumed that only jumps $x \rightarrow x \pm 1$ between two nearest-neighbor branes are allowed. This leads to the geometrical picture of a gas of non-intersecting loops, representing the domain walls on the world sheet (fig.3).

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Fig.3: A disc world sheet with boundary in the $x$-th brane.

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7 Another possible matrix model description of such a perturbation is proposed in [55].
Creation of a domain wall costs energy which is proportional to the length of the domain wall and which is taken into account by the loop tension $M$. The statistical weight of each domain wall is $e^{-2M\ell}$ where $\ell = \int_{\text{domain wall}} e^{b\phi}$ is the length of the domain wall. The loop gas describes a sector of the world sheet theory (2.2) containing certain class of degenerate fields (2.13) (for a recent discussion see [23]). Let us remind that the field $\chi$ satisfies Neumann boundary condition and that the dual field $\tilde{\chi}$ satisfies Dirichlet boundary condition and that the lowest degenerate boundary operator $B_{1,2}$ is realized as a magnetic operator with magnetic charge $P = b/2$. The operator $B_{1,2}$ creates a jump of the boundary to an adjacent brane in the $\tilde{\chi}$ space. Geometrically this operator is described by a line starting at some point on the boundary. Similarly the degenerate boundary operator $B_{1,n+1}$ with magnetic charge $P = nb/2$ creates a jump by $n$ steps and is described by $n$ lines starting at given point on the boundary. The correlation function of two such operators is a sum of world sheets as the one shown in fig. 4.

\[ \chi = \frac{\pi x}{b}, \quad x \in \mathbb{Z}. \quad (5.2) \]

The duality symmetry $b \to 1/b$ exchanges the $\chi$ and $\tilde{\chi}$ spaces. In the dual loop gas model the time coordinate $\chi$ is discretized with spacing $\pi/b$. The weight of a domain wall is $e^{-2M\tilde{\ell}}$, where $\tilde{\ell} = \int_{\text{domain wall}} e^{\phi/b} dx$. The dual loop tension $\tilde{M} = M^{1/b^2}$ is the dual boundary cosmological constant for the theory (2.2).
Now let us see what is the meaning of the operators of the boundary ground ring in the loop gas model. Geometrically the degenerate boundary operator with momentum $B_{n+1,1}$ is obtained from $B_{n,1}$ by adding an extra line. This reminds the fusion rules for the matter fields. However, the process of adding a line cannot be interpreted directly as a fusion of two boundary tachyon operators because the Liouville dressing factors obey a different fusion rule. The correct interpretation comes from the fusion rules (4.3) involving one tachyon and one operator of the boundary ground ring. Thus the operators $A_+$ and $A_-$ generating the boundary ground ring have the geometrical meaning of the operations of adding or removing lines. The action of these operators is encoded in the left and right boundary conditions for the matter field:

\[
A_+ = \frac{s\pm i\pi/b}{\chi + \pi/b} [A_+]_\chi, \quad A_- = \frac{s\pm i\pi/b}{\tilde{\chi} - \pi/b} [A_-]_{\tilde{\chi}}.
\]

5.2. A self-dual $\hat{A}_\infty$ matrix chain

The matrix chains associated with Dynkin diagrams \[49\] were originally defined in terms of a set of $N \times N$ hermitian matrices $\Phi_x$ associated with the nodes $x$ of the Dynkin diagram and $N \times N$ complex matrices $A_{\langle x, x' \rangle}$ associated with the links $\langle x, x' \rangle$. All these matrix chains have a loop gas description and therefore a direct Coulomb gas interpretation. The complex matrices $A_{\langle x, x' \rangle}$ represent the one type of degenerate boundary fields (only electric or only magnetic ones).

Each value of the central charge is realized by two such matrix models, related by duality. For example the theory with $b = \frac{p}{p+1}$ and central charge $c = 1 - \frac{6}{p(p+1)}$ is realized either as an $A_p$ chain in the dilute phase or as an $A_{p+1}$ chain in the dense phase. The electric degenerate boundary fields have a good description in the $A_p$ matrix chain while the magnetic degenerate boundary fields have a good description in the $A_{p+1}$ matrix chain.

Remarkably, each such pair of matrix chains related by duality can be formulated as a single self-dual chain of complex matrices by the following simple prescription. The hermitian matrix variables $\Phi_x$ associated with the nodes of the Dynkin diagram should be considered as composite fields $B_x B_x^\dagger$ or $B_x^\dagger B_x$, where $B_x$ are $N \times N$ complex matrices. Here we will consider only the $\hat{A}_\infty$ chain, which provides the matrix model description of the theory (2.2). The case of the rational $ADE$ theories will be studied elsewhere \[56\].

The partition function of the self-dual $\hat{A}_\infty$ matrix chain is defined in terms of the $N \times N$ complex matrices $A_{\langle x, x' \rangle} \equiv A_x$ and $B_x$ ($x \in \mathbb{Z}$):

\[
Z = \int \prod_{x \in \mathbb{Z}} dA_x dA_x^\dagger dB_x dB_x^\dagger \; e^{-S(A,A^\dagger,B,B^\dagger)}
\]

(5.4)
where the simplest action with linear potential for $AA^\dagger$ and $BB^\dagger$ is

$$S(A, A^\dagger, B, B^\dagger) = \sum_{x \in \mathbb{Z}} \text{Tr} \left( 2TA_xA_x^\dagger + 2TB_xB_x^\dagger - B_xB_x^\dagger A_xA_x^\dagger - B_{x+1}B_{x+1}^\dagger A_xA_x^\dagger \right).$$

(5.5)

(Of course this integral has only formal meaning because of the infinite volume of the chain. It can be regularized as the limit $n \to \infty$ of the periodic chain $\tilde{A}_n$.)

Alternatively one can write the partition function as an integral over the positive definite hermitian matrices

$$\Phi_x = B_xB_x^\dagger, \quad \tilde{\Phi}_x = A_xA_x^\dagger$$

and the unitary matrices $U_x$. Let us denote by $[d\Phi]_{>0}$ the flat measure over the positive definite hermitian matrices and by $[dU]_{SU(N)}$ the Haar measure on the group $U(N)$. Then the integral (5.4) can be written as

$$Z = \int \prod_{x \in \mathbb{Z}} [d\Phi_x]_{>0} [d\tilde{\Phi}_x]_{>0} [dU_x]_{SU(N)} e^{-S(\Phi, \tilde{\Phi}, U)}$$

(5.7)

$$S(\Phi, \tilde{\Phi}, U) = \sum_{x \in \mathbb{Z}} \text{Tr} \left( 2T\Phi_x + 2T\tilde{\Phi}_x - \Phi_x \tilde{\Phi}_x - U\Phi_x U^{-1}\tilde{\Phi}_{x+1} \right)$$

(5.8)

The integration over the unitary group can be done explicitly and the partition function can be written as an integral with respect to the eigenvalues $\phi_{x,j}$ and $\tilde{\phi}_{x,j}$ of $\Phi_c$ and $\tilde{\Phi}_x$ respectively. Using the Harish Chandra-Itzykson-Zuber formula we write (5.7) in the form

$$Z = \int_0^\infty \prod_{x,j} d\phi_{x,j} d\tilde{\phi}_{x,j} e^{-2T\phi_{x,j} - 2T\tilde{\phi}_{x,j}} \det_{jk} e^{\phi_{x,j}\tilde{\phi}_{x,k}} \det_{jk} e^{\phi_{x,j}\tilde{\phi}_{x+1,k}}.$$  

(5.9)

We are interested in the ’t Hooft limit $N \to \infty$ with $\kappa \equiv N/T^2$ fixed. Then the ground state of the matrix model is characterized by the classical spectral densities $\rho_x(\phi)$ and $\tilde{\rho}_x(\tilde{\phi})$ of the hermitian matrices (5.6). The ’t Hooft limit exists for $\kappa < \kappa_c$, where $\kappa_c$ is the critical coupling where the planar graph expansion diverges. One can show that in this case $\rho_x(\phi)$ and $\tilde{\rho}_x(\tilde{\phi})$ vanish for $\phi_x, \tilde{\phi}_x \geq T$, up to exponentially small in $N$ terms.

We will be interested only in the scaling limit, which describes the infinitesimal vicinity of the critical point. It is technically simpler to treat the parameters $N$ and $T$ as large but finite. Then for given $T$ the critical point is achieved at $N_c = \kappa_c T^2$. The value $N_c$ plays the role of a cutoff parameter, which we eventually sent at infinity, while the cosmological constant is $\mu \sim N_c - N$.

The matrix chain (5.5) describes the particular case $b = 1$ of the world sheet theory (2.2), when the theory is unitary. A non-zero background charge $e_0 = 1/b - b$ can be
introduced by imposing by hand a $x$-dependent background. Then the spectral densities are solutions of the saddle point equations under the constraints

$$\rho_x(\phi) = e^{i\pi \epsilon_0 x/b} \rho(\phi), \quad \tilde{\rho}_x(\phi) = e^{-i\pi \epsilon_0 bx} \tilde{\rho}(\phi). \quad (5.10)$$

Relation to the standard \(\hat{A}_\infty\) chain [49]

One can formally integrate with respect to the $\tilde{\phi}$-variables in (5.9) neglecting the contributions of infinity. In this way one neglects exponentially small in $N$ terms, which do not influence the genus expansion. The result of the integration is the partition function of the hermitian \(\hat{A}_\infty\) matrix chain [49]

$$Z_{\hat{A}_\infty} = \int_0^\infty \prod_x \prod_{j,k} d\phi_{x,j} e^{-2T \phi_{x,j}} \prod_x \prod_{j,k} \frac{\prod_{j\neq k} (\phi_{x,j} - \phi_{x,k})}{(2T - \phi_{x,j} - \phi_{x+1,k})}.$$  \quad (5.11)

By integrating first with respect to the $\phi$-variables one obtains the same eigenvalue integral for the dual field $\tilde{\Phi}_x$:

$$Z_{\hat{A}_\infty} = \int_0^\infty \prod_x \prod_{j} d\phi_{x,j} e^{-2T \phi_{x,j}} \prod_x \prod_{j,k} \frac{\prod_{j\neq k} (\tilde{\phi}_{x,j} - \tilde{\phi}_{x,k})}{(2T - \tilde{\phi}_{x,j} - \tilde{\phi}_{x+1,k})}.$$  \quad (5.12)

The saddle point equations for these integrals lead to spectral densities $\rho_x(\phi)$ and $\tilde{\rho}_x(\tilde{\phi})$ that vanish for $\phi, \tilde{\phi} > T$, which justifies the fact that we neglected the contributions at infinity.

Relation to Matrix Quantum Mechanics

It is known [51] that the hermitian \(\hat{A}_\infty\) matrix chain (5.11) and MQM share the same $x$-space correlation functions of on-shell closed string tachyons. Thus the \(\hat{A}_\infty\) chain is expected to yield a discretization of the singlet sector of MQM at euclidean time distance $\beta = \pi/2$ in the normalization in which the self-dual distance is $\beta = 2\pi$. Let us see that this is indeed the case.

The singlet sector of MQM is equivalent to a system of $N$ nonrelativistic fermions in upside-down oscillator potential $U(y) = -\frac{1}{2}y^2$, representing the $N$ eigenvalues of a hermitian matrix variable (see the review [4] and the references to the original papers therein). The evolution of each eigenvalue $y$ in MQM is described by a one-particle Hamiltonian $H_0 = -\frac{1}{2}(\partial_y^2 + y^2)$. The heat kernel for this Hamiltonian is given by

$$K_\beta(y, \tilde{y}) \equiv \langle y | e^{-\beta H_0} | \tilde{y} \rangle = \frac{1}{\sqrt{2\pi \sin \beta}} \exp \left( \frac{2y\tilde{y} - \cos \beta (y^2 + \tilde{y}^2)}{2 \sin \beta} \right). \quad (5.13)$$
Therefore the Laplace kernels entering the determinants in (5.9) can be viewed, after a linear change of variables

\[ y = \phi - T, \quad \tilde{y} = \tilde{\phi} - T, \]

as evolution kernels (5.13) with \( \beta = \frac{1}{2} \pi \). Let us denote by \( \mathcal{H} = \bigoplus_{k=1}^{N} H_0 \) the MQM Hamiltonian in the singlet representation and by \( \mathcal{P}_T \) the projector restricting the integration to the semi-infinite interval \([-T, \infty)\):

\[ \Theta_T = \prod_{i=1}^{N} \theta(y_i - T). \]

Then the partition function (5.9) for a periodic chain \( \hat{A}_n \) can be written as the trace

\[ Z_{\hat{A}_n} = \text{Tr} \left( \Theta_T e^{-\frac{i}{2} \pi \mathcal{H}} \right)^{2n}. \] (5.14)

The projector \( \Theta_T \) has the effect of an infinite potential wall placed at distance \( T \) from the origin, which stabilizes the inverse oscillator potential. For given \( T \) the Fermi level grows with the number of the eigenvalues \( N \) and at some critical \( N = N_c \) reaches the top of the potential. The cosmological constant \( \mu \sim N_c - N \) measures the deviation from the critical point.

### 5.3. Disc loop amplitudes in the matrix chain

The classical background of the matrix chain is determined by the (unnormalized) disc loop amplitudes

\[ W_x(z) = \left\langle \text{Tr} \frac{1}{z - B_x B_x^\dagger} \right\rangle = \int d\phi \rho_x(\phi) \frac{1}{z - \phi} \] (5.15)

and

\[ \tilde{W}_x(\tilde{z}) = \left\langle \text{Tr} \frac{1}{\tilde{z} - A_x A_x^\dagger} \right\rangle = \int d\tilde{\phi} \tilde{\rho}_x(\tilde{\phi}) \frac{1}{\tilde{z} - \tilde{\phi}}. \] (5.16)

The loop amplitude \( W_x(z) \) represents a meromorphic function in the spectral parameter \( z \) with a cut along the support \([0, a]\) of the eigenvalue distribution, where \( 0 < a \leq T \). It is determined from the saddle-point equation for the integral (5.11), which can be written in the form of a boundary condition along the cut:

\[ W_x(z + i0) + W_x(z - i0) + W_{x+1}(2T - z) + W_{x-1}(2T - z) = 2T \quad (0 < z < a). \] (5.17)

The dual loop amplitude \( \tilde{W}_x(\tilde{z}) \) has the same properties with \( z, a \) replaced by \( \tilde{z}, \tilde{a} \).
The true ground state does not depend on $x$ and describes the unitary theory with matter central charge $c = 1$. The non-unitary theory with background momentum $e_0$ is obtained by imposing by hands the following constraints on the loop amplitudes:

$$W_x(z) = e^{i e_0 x / b} W(z), \quad \tilde{W}_x(\tilde{z}) = e^{-i e_0 x b} \tilde{W}(\tilde{z}). \quad (5.18)$$

Then the equation (5.17) becomes

$$W(z + i 0) + W(z - i 0) - 2 \cos(\pi / b^2) W(2T - z) = 2T \quad (0 < z < a). \quad (5.19)$$

This equation is readily solved in the scaling limit, when $\frac{N_c - N_v}{N_c}$ and $\frac{|z - T|}{T}$ are considered small. Then the branch point at $z = T$ can be resolved by the parametrization

$$z - T = M \cosh(\pi s / b), \quad M = T - a, \quad (5.20)$$

and (5.19) becomes a difference equation

$$W(s + i b) + W(s - i b) - 2 \cosh \pi / b^2 W(s) = 2T. \quad (5.21)$$

From here we find

$$W(s) = c_1 T + c_2 T^{1-b^2} M^{1/b^2} \cosh(\pi s / b). \quad (5.22)$$

where $c_1$ and $c_2$ are $b$-dependent constants. The solution reproduces, up to a linear transformation, the loop amplitude (D.16) in the world sheet CFT.

In the following we will retain only the scaling parts of $z$ and $W$, assuming a normalization that matches with the result from the continuum CFT:

$$z = M \cosh(\pi s / b), \quad W = \frac{M^{1/b^2}}{b \cos(\pi s / b)} \cosh(\pi s / b). \quad (5.23)$$

In this normalization the eigenvalue density is given by

$$\rho(z) = \frac{W(s - \frac{i}{b}) - W(s + \frac{i}{b})}{2\pi i} = b M^{1/b^2} \sinh(\pi s / b). \quad (5.24)$$

Proceeding in the same way we find the scaling part of the dual loop amplitude (5.16)

$$\tilde{z} = \tilde{M} \cosh(\pi b / s), \quad \tilde{W} = \frac{\tilde{M}^b}{b \cos(\pi b^2)} \cosh(\pi s b), \quad (5.25)$$

where $\tilde{M} = T - \tilde{a}$. 

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8 It is actually sufficient to impose only the second condition; then the first one follows from the correspondence $W \leftrightarrow \tilde{z}, \quad z \leftrightarrow \tilde{W}$. 

21
The loop amplitudes (5.23) and (5.25) are related by a duality transformation
\[
\tilde{W} = -\frac{1}{b \sin \pi b^2} z, \quad \tilde{z} = -\frac{1}{b} \sin \left(\frac{\pi}{b} \right) W.
\] (5.26)
Intuitively, the duality in Liouville gravity is a kind of electric-magnetic duality on the world sheet, whose target-space equivalent is a Fourier transformation for the eigenvalues of the random matrix.

In the planar limit the matrix model is described by a classical Hamiltonian system in which the spectral parameter \( z \) and the resolvent \( W \) form a pair canonically conjugated variables: the coordinate and the momentum. Then the duality make sense of a canonical transformation exchanging the momentum and the coordinate. The meromorphic functions \( W(z) \) and \( \tilde{W}(\tilde{z}) \) give two parametrizations of the complexified classical trajectories of the Hamiltonian system.

The physical sheets of the Riemann surfaces of \( W(z) \) and \( \tilde{W}(\tilde{z}) \) can be viewed, together with the identification (5.26), as two charts defining a complex curve with the topology of a sphere with two punctures. In the rational models of quantum gravity the Riemann surface is given by an algebraic curve. In the case of generic \( b^2 \) the equation of the algebraic curve can be written as a functional equation
\[
W^2(z) + W^2(-z) - 2 \cos \left(\frac{\pi}{b} \right) W(z)W(-z) = b^2 M^2/b^2
\] (5.27)
or its dual
\[
\tilde{W}^2(\tilde{z}) + \tilde{W}^2(-\tilde{z}) - 2 \cos \left(\pi b^2 \right) \tilde{W}(\tilde{z})\tilde{W}(-\tilde{z}) = \frac{1}{b^2} \tilde{M}^{2b^2}.
\] (5.28)
These equations are particular case of more general functional equations that hold before the scaling limit and allow to express the positions \( a \) and \( \tilde{a} \) of the branch points in terms of \( T \) and \( N \). In particular, the non-scaled functional equation for \( \tilde{W}(\tilde{z}) \), whose derivation presented in Appendix E, implies the scaling \( \tilde{M}^{2b^2} \sim \Lambda \).

5.4. Planar graph expansion and mapping to the world-sheet CFT

The perturbative expansion of any observable in the theory with action (5.5) is a sum of planar graphs composed from the vertices

\[
\begin{array}{c}
A_x \quad A_{x+1} \\
\downarrow \quad \downarrow \\
B_x \quad B_{x+1}
\end{array}
\quad \text{and} \quad \begin{array}{c}
A_x \quad A_x \\
\downarrow \quad \downarrow \\
B_x + 1 \quad B_x + 1
\end{array}
\],
\]
where the continuous and dashed oriented lines represent respectively the propagators of the \( A \) and \( B \)-matrices. The planar graphs can be considered as discretized world sheets.
embedded in \( \mathbb{Z} \). The loops made of the \( A_x \)-propagators can be viewed as domain walls separating domains with time coordinates \( x \) and \( x+1 \).

We are interested in the disc amplitudes representing various boundary correlation functions of tachyon operators. The simplest ones are expectation values of the resolvents (5.15) and (5.16), which are the equal to the boundary one-point functions in the world sheet CFT with Neumann and Dirichlet boundary conditions\(^9\).

In the planar limit \( W_x \) is a sum of planar graphs with the topology of a disc, made by \( A \) and \( B \) type lines. The \( A \)-lines can only form loops while the \( B \)-lines can either close in a loop or have its both ends among the external legs. The condition (5.18) is equivalent to associating \( x \)-dependent phase factors with the vertices and faces of the planar graph as explained in section 2 of [27]. For example, the graph shown in fig.5 gives a particular discretization of the world sheet with three domain walls shown in fig.4.

\[ \text{Fig.5 a,b: Planar graphs contributing to the resolvents of } B_x B_x^\dagger \text{ and } A_x A_x^\dagger. \]

The dual amplitude \( \tilde{W}_x \) is a sum of planar graphs such that the \( B \)-lines are always closed and the \( A \)-lines either close or connect two external legs. In this way the complex matrix chain allows to define both the Dirichlet and Neumann boundary conditions for the loop gas in the sense of [28] and [23]. Exchanging the \( A \) and \( B \) lines is equivalent to the duality transformation, which exchanges Dirichlet and Neumann boundary conditions. In the continuum limit the two amplitudes are given, up to rescaling and subtraction of a constant, by the expressions (D.16) and (D.17).

The boundary ground ring is represented in the matrix model as the algebra of the polynomials of the \( A \) and \( B \) matrix variables. For the generators of the algebra the correspondence is

\[
A_+ \to A_x \quad A_- \to B_x \quad 
\overline{A}_+ \to A_{x}^\dagger \quad \overline{A}_- \to B_x^\dagger.
\]

\[ (5.29) \]

\(^9\) Strictly speaking, the Neuman boundary condition is generated by the resolvent of the operator \((A + A_{x}^\dagger)^2\). However it has the same scaling limit as the resolvent of \( A_{x}^\dagger\).
The boundary segments with boundary condition \((\chi, s)\) or \((\bar{\chi}, s)\) are represented by
the resolvents of the fields \(\Phi_x = B_x B_x^\dagger\) and \(\bar{\Phi}_x = A_x A_x^\dagger\) correspondingly:

\[
\begin{align*}
\chi[s] &\rightarrow \frac{1}{z(s) - \Phi_x}
\chi[s] &\rightarrow \frac{1}{z(s) - \bar{\Phi}_x}.
\end{align*}
\]

The degenerate boundary operators \(B_{n,1}\) and \(B_{1,n}\) are created by inserting in the trace
polynomials of \(A\) and \(B\) matrices according to the following dictionary:

\[
\begin{align*}
\chi + n \pi/b &\rightarrow A_{x+n} \ldots A_x & \bar{\chi} + n \pi/b &\rightarrow B_{x+n} \ldots B_x \\
\chi - n \pi/b &\rightarrow A_x^\dagger \ldots A_{x-n}^\dagger & \bar{\chi} - n \pi/b &\rightarrow B_{x-n}^\dagger \ldots B_x^\dagger.
\end{align*}
\]

For example, the boundary two-point function \(D_P(s, s')\) for \(P = \frac{1}{2} e_0 + \frac{n}{2b}\) corresponds
to the following expectation value in the matrix model (Fig. 4):

\[
\begin{align*}
D_{\frac{1}{2} e_0 + \frac{n}{2b}}(s, s') &= \langle \chi + n \pi/b [B_{n+1,1}]^s_x [B_{-n-1,-1}]^{s'}_x \rangle_{\text{disc}} \\
&\sim \langle \text{Tr} A_{x+n} \ldots A_x^\dagger \frac{1}{z(s) - \Phi_x} A_x \ldots A_{x+n} \frac{1}{z(s') - \bar{\Phi}_x + n} \rangle.
\end{align*}
\]

This correlation function was calculated from the matrix model side using the loop gas
technique [24,23] and the result is in agreement with the prediction of the Liouville theory.
Another example is the amplitude of two excited twist operators with momenta \(\pm P_{n+\frac{1}{2},0} = \pm(n + \frac{1}{2})/2b\), which interpolate between Dirichlet and Neumann boundary conditions:

\[
\begin{align*}
D_{\frac{1}{2} e_{n+\frac{1}{2}}}(s, s') &= \langle \chi^s_x [B_{n+1/2}]^s_x [B_{n+1/2}]^{s'}_{\bar{x}} \rangle_{\text{disc}} \\
&\sim \langle \text{Tr} A_{x+n} \ldots A_x^\dagger \frac{1}{z(s) - \Phi_x} A_x \ldots A_{x+n} \frac{1}{z(s') - \bar{\Phi}_x} \rangle.
\end{align*}
\]

This correlation functions has been calculated (for the case \(n = 0\)) in [28] and given a
Liouville CFT interpretation in [24,23].

### 6. Concluding remarks

1. The ring relations and the functional equations derived in this paper can be generalized
to the case of any matter CFT with \(c < 1\). This can be achieved by replacing the \(U(1)\)
fusion rules for the gaussian field with the fusion rules in the corresponding boundary CFT.

2. In the ADE rational string theories, the matrix model realization of the boundary
ground ring can be achieved by reformulating the corresponding ADE matrix chain in
terms of \(A\)-type and \(B\)-type complex matrices.
Consider for example the simplest case of a theory of the $A$-series with central charge $c = 1 - \frac{6}{p(p+1)}$, which corresponds to $b^2 = \frac{4}{p+1}$. The corresponding string theory is described by the critical point of the $A_p$ matrix model or by the tricritical point of the $A_{p-1}$ matrix model. Both models can be incorporated in a single matrix model of the type (5.5) by replacing the Dynkin graph of $\hat{A}_\infty$ with that of $A_p$. The new model is defined in terms of $p - 1$ $A$-matrices labeled by the links of the Dynkin graph and $p$ $B$-matrices labeled by the points of the graph.

3. It is interesting that Dijkgraaf and Vafa [58] proposed a similar matrix representation for the chiral ring for the $c = 1$ string theory compactified at the self-dual radius. Since the boundary operators can be considered as operators in a chiral CFT [59,60], there might be a connection between the Witten’s chiral ring and the boundary ground ring.

4. The action of the boundary ground ring closes on the discrete subset $s = im + in$, $m, n \in \mathbb{Z}$. This subset is relevant for the ZZ brane boundary states [47], as discussed in [39,22,51,10]. One might ask oneself whether in the ZZ brane there is an analog of the boundary correlation functions in the FZZT brane.

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Appendix A. Bulk and boundary reflection amplitudes

The closed and open string tachyons of same momenta but opposite chiralities are related by the bulk and Liouville boundary reflection amplitudes [19]

$$\mathcal{V}_P(\pm) = S_P^{(\pm\pm)} S_P^{(-\pm)}$$

$$[\mathcal{B}_P^{(+)}]_{s_1,s_2} = D_P^{(+-)}(s_1,s_2) [\mathcal{B}_P^{(-)}]_{s_1,s_2}$$

$$[\mathcal{B}_P^{(-)}]_{s_1,s_2} = D_P^{(-+)}(s_1,s_2) [\mathcal{B}_P^{(-)}]_{s_1,s_2}.$$  \hspace{1cm} (A.1)

In the normalizations (2.8) and (2.9)

$$S_P^{(\pm\pm)} = \frac{1}{b^2} \Lambda^{P/b}, \quad S_P^{(-\pm)} = -b^2 \Lambda^{-P/b} \hspace{1cm} (A.3)$$

$$D_P^{(+-)}(s_1,s_2) = \frac{1}{b} \frac{\Lambda^{P/b}}{S_b(2P+b)} \frac{S_b \left( \frac{Q}{2} + P - i\frac{s_1+s_2}{2} \right)}{S_b \left( \frac{Q}{2} - P - i\frac{s_1+s_2}{2} \right)} \frac{S_b \left( \frac{Q}{2} + P - i\frac{s_1-s_2}{2} \right)}{S_b \left( \frac{Q}{2} - P - i\frac{s_1-s_2}{2} \right)}$$  \hspace{1cm} (A.4)
where $\Gamma_b(x)$ is the Double Gamma function (see [13]) and $S_b(x) = \Gamma_b(x)/\Gamma_b(Q-x)$. The amplitude $D_p^{(+-)} = (D_p^{(-+)})^{-1}$ has the symmetries

$$D_p^{(+-)}D_{-p}^{(+-)} = \frac{1}{b^2} \frac{\sin \frac{2\pi}{b} P}{\sin \frac{2\pi}{b} P}, \quad D_p^{(-+)}D_{-p}^{(-+)} = b^2 \frac{\sin \frac{2\pi}{b} b P}{\sin \frac{2\pi}{b} P}$$

(A.5)

and

$$D_{-p}^{(-+)}(s_1, s_2) = \tilde{D}_p^{(+)}(s_1, s_2),$$

(A.6)

where the function $\tilde{D}_p^{(+)}(s_1, s_2)$ is defined by (A.4) with $b$ replaced by $\tilde{b} = 1/b$. The reflection amplitudes for the degenerate momenta $P_{mn} = \frac{1}{2}(m/b - nb)$ are given by rational functions of $z = M \cosh(\pi bs)$ and $\tilde{z} = M^{1/b^2} \cosh(\pi s/b)$. For example,

$$D_{1/2b}^{+-} = \frac{b^2 \sin(\pi b^2) \tilde{z}_1 + \tilde{z}_2}{\sin(\pi b^2) \tilde{z}_1 - \tilde{z}_2}, \quad D_{-1/2b}^{+-} = \frac{-\tilde{z}_1 - \tilde{z}_2}{\tilde{z}_1 - \tilde{z}_2}.$$  

(A.7)

Appendix B. Recurrence relations for the closed string tachyon amplitudes

Using the ring relations (3.2) and (3.6) one can obtain a set of recurrence equations for the correlation functions of the bulk tachyons

$$G(P_1, ..., P_n|P_{n+1}, ..., P_{n+m}) =$$

$$\left\langle \prod_{k=1}^{n} \int \mathcal{V}^{(-)}_{P_k}(0) \right\rangle \left\langle \prod_{j=n+2}^{m+n-1} \int \mathcal{V}^{(+)}_{P_j}(1) \right\rangle$$

(B.1)

which generalize the recurrence relations for the resonance amplitudes obtained in [11]. Here we assume that the neutrality condition is satisfied

$$\sum_{k=1}^{n+m} (e_0 - P_k) = 2e_0.$$  

(B.2)

The auxiliary function

$$F(x, \bar{x}|P_1, ..., P_n|P_{n+1}, ..., P_{n+m}) =$$

$$\left\langle \prod_{k=1}^{n-1} \int \mathcal{V}^{(-)}_{P_k}(0) \right\rangle \left\langle \prod_{j=n+2}^{m+n-1} \int \mathcal{V}^{(+)}_{P_j}(1) \right\rangle$$

(B.3)

does not depend on $x$ and $\bar{x}$. This can be proved by using $\partial_x a_- = \{Q_{\text{BRST}}, b_- a_-\}$ and deforming the contour, commuting $Q_{\text{BRST}}$ with the other operators in (B.3) [11]. Therefore
one can evaluate this function at \( x = 0 \) and \( x = 1 \) by using the fusion rules (3.2) or (3.5) and (3.6). As a result one obtains the recurrence relation

\[
G(P_1, \ldots, P_n|P_{n+1}, \ldots, P_{n+m}) = \Lambda G(P_1, \ldots, P_n + b|P_{n+1} - b, \ldots, P_{n+m}) - \sum_{j=1}^{m-1} G(P_1, \ldots, P_n + b|P_{n+2}, \ldots, P_{n+j} + P_{n+1} - \frac{1}{b}, \ldots, P_{n+m}).
\]  

(B.4)

Similarly, by inserting \( a_+ \) we get the dual recurrence relation

\[
G(P_1, \ldots, P_n|P_{n+1}, \ldots, P_{n+m}) = \tilde{\Lambda} G(P_1, \ldots, P_n + \frac{1}{b}|P_{n+1} - \frac{1}{b}, \ldots, P_{n+m}) - \sum_{k=1}^{n-1} G(P_1, \ldots, P_k + P_n + b, \ldots, P_{n-1}|P_{n+1}, \ldots, P_{n+m}).
\]  

(B.5)

Note that the three-point function \( m + n = 3 \) coincides with the corresponding Liouville three-point function. The latter has been evaluated for generic momenta (i.e. without imposing the neutrality condition (B.2) by using a very similar argument in [22].

**Appendix C. Recurrence relations for the boundary correlation functions with pure Neumann boundary conditions.**

The fusion rules above lead to a system of recursion relations for the disc correlation functions with arbitrary number of boundary states

\[
W(P_1, \ldots, P_n|P_{n+1}, \ldots, P_{n+m}) = 
\left\langle \prod_{k=1}^{n-1} \int \mathcal{B}_{P_k}^{(-)} c\mathcal{B}_{P_n}^{(-)}(0) c\mathcal{B}_{P_{n+1}}^{(+)}(1) \prod_{j=n+2}^{n+m-1} \int \mathcal{B}_{P_j}^{(+)}) c\mathcal{B}_{P_{n+m}}^{(+)}(\infty) \right\rangle.
\]

(C.1)

Here three of the integrations are canceled by the volume of the global \( SL(2, \mathbb{R}) \) conformal symmetry of the upper half plane. Consider the auxiliary function

\[
F(x|P_1, \ldots, P_n|P_{n+1}, \ldots, P_{n+m}) = 
\left\langle \prod_{k=1}^{n-1} \int \mathcal{B}_{P_k}^{(-)} c\mathcal{B}_{P_n + \frac{1}{2}}^{(-)}(0) A_-(x) c\mathcal{B}_{P_{n+1}}^{(+))(1)} \prod_{j=n+2}^{n+m-1} \int \mathcal{B}_{P_j}^{(+)}) c\mathcal{B}_{P_{n+m}}^{(+))(\infty)} \right\rangle.
\]

(C.2)

Since

\[
\partial_x F = \{Q_{BRST}, b_- a_- \}
\]

(C.3)
the function $F$ does not actually depend on $x$ and one can calculate it in two different ways by taking the limits $x \to 0$ and $x \to 1$ and using (4.3) and (4.6) (see [11] for a discussion concerning possible boundary terms). Note that the boundary operators are ordered and the fusion can be performed only with the operators $B_{P_n}^(-)$ and $B_{P_{n+1}}^(+).$ This leads to the recurrence relation

$$W(P_1, ..., P_n|P_{n+1}, ..., P_{n+m}) = - \frac{W(P_1, ..., P_n + \frac{b}{2}, P_{n+1} + P_{n+2} - \frac{b}{2b}, ..., P_{n+m})}{\sin(2\pi b P_{n+1})}$$

(C.4)

or, after shifting the momenta,

$$W(P_1, ..., P_n - \frac{b}{2}, ... , P_{n+m}) = \frac{W(P_1, ..., P_n | P_{n+1} + P_{n+2}, ..., P_{n+m})}{\sin(2\pi b P_{n+1})}$$

(C.5)

and similarly for $A_+,$ which can be easily solved:

$$\langle B_{P_m}^(-) ... B_{P_1}^(-) B_{K_1}^+ ... B_{K_n}^+ \rangle = (-)^{\frac{n(n-1)}{2} + \frac{m(m-1)}{2}} \prod_{j=1}^{n-1} \sin 2\pi b (P_1 + ... + P_j) \prod_{l=1}^{n-1} \sin \frac{2\pi}{b} (K_1 + ... + K_l).$$

Appendix D. Evaluation of the basic boundary correlation functions on the FZZT brane

D.1. Three-point function

For the three-point function

$$W_{P_1, P_2, P_3}(s', s, s'') = \langle s' [B_{P_1}^(-)]^s [B_{P_2}^+]^s' [B_{P_3}^+]^s'' \rangle.$$  

(D.1)

the functional equations (4.21) and (4.22) read (with the assumption that in the first equation $P_1 \leq -\frac{1}{2b}, P_2 \geq \frac{1}{2b}$ and in the second equation $P_1 \leq -\frac{b}{2}, P_2 \geq \frac{b}{2}$)

$$\sin (2\pi b P_2) \ W_{P_1, P_2, P_3}(s', s \pm ib, s'') = $$

$$= 2M \cosh \left[ \pi b \left( \frac{s + s'}{2} \pm iP_2 \right) \right] \cosh \left[ \pi b \left( \frac{s - s'}{2} \pm iP_2 \right) \right] \ W_{P_1 + \frac{b}{2}, P_2 - \frac{b}{2}, P_3}(s', s, s'') \quad (D.2)$$

$$+ D_{P_1 + \frac{b}{2}}(s', s');$$

$$\sin \left( \frac{2\pi}{b} P_2 \right) \ W_{P_1, P_2, P_3}(s', s \pm \frac{i}{b}, s'') =$$

$$= 2M^{1/b^2} \cosh \left[ \frac{\pi}{b} \left( \frac{s + s'}{2} \pm iP_2 \right) \right] \cosh \left[ \frac{\pi}{b} \left( \frac{s - s'}{2} \pm iP_2 \right) \right] \ W_{P_1 + \frac{1}{2b}, P_2 - \frac{1}{2b}, P_3}(s', s, s'')$$

$$+ D_{P_1 + \frac{1}{2b}}(s', s'). \quad (D.3)$$
Their common solution is given by the boundary three-point function in pure Liouville theory (with the restriction $P_1 + P_2 + P_3 = e_0/2$ on the momenta and rescaled to match with our definition of the boundary cosmological constant), which has been found explicitly as an integral of products of double sine functions [20]. If the three momenta are degenerate, that is of the form $(r/b - sb)/2$, the answer is a rational function of trigonometric functions [23].

D.2. Two-point function

The functional equations (4.21) and (4.22) are derived for the correlation functions of three or more operators. The two- and one-point functions in 2D QG are more subtle quantities whose correct normalization requires to tame the residual global conformal symmetry. To avoid facing this difficult technical problem, we will apply a trick: we will use the relation

$$W_{-P,P,\frac{1}{2}e_0}(s', s, s'') = -\frac{D_P(s, s') - D_P(s, s'')}{z(s') - z(s'')}.$$  

(D.5)

In particular, taking $s' = s''$ we have the more familiar relation

$$\frac{\partial D_P(s, s')}{\partial z(s')} = -W_{-P,P,\frac{1}{2}e_0}(s', s, s').$$  

(D.6)

In this case eqn. (D.3) reads

$$\sin \left( \frac{2\pi}{b} P \right) [D_P(s \pm ib, s') - D_P(s \pm ib, s'')] = 2M \cosh \left[ \frac{\pi}{b} \left( \frac{s+s'}{2} \pm iP \right) \right] \cosh \left[ \frac{\pi}{b} \left( \frac{s-s'}{2} \pm iP \right) \right] [D_{P-\frac{b}{2}}(s, s') - D_{P-\frac{b}{2}}(s, s'')]$$

$$+ [(z(s') - z(s''))] D_{P-\frac{b}{2}}(s', s''),$$

where it is assumed that $P > \frac{b}{2}$, so that the shifted momentum has the same sign. This equation implies the following linear difference equation for two-point function:

$$\frac{D_P(s \pm ib, s')}{D_{P-\frac{b}{2}}(s, s')} = \frac{2M \cosh \left[ \frac{\pi b}{2} \left( \frac{s+s'}{2} \pm iP \right) \right] \cosh \left[ \frac{\pi b}{2} \left( \frac{s-s'}{2} \pm iP \right) \right]}{\sin (2\pi b P)}.$$  

(D.7)

The easiest way we know to prove that is to perform a Laplace transformation from the boundary cosmological constant $z(s)$ to the physical length $\ell = \int dx e^{b\phi(x)}$ for each segment of the boundary.
Similarly one obtains, for $P > \frac{1}{2b}$, the dual equation

$$\frac{D_P(s \pm \frac{i}{b}, s')}{D_{P - \frac{1}{2b}}(s, s')} = \frac{2M^{1/b^2} \cosh \left[ \frac{\pi}{b} \left( \frac{s + s'}{2} \pm iP \right) \right] \cosh \left[ \frac{\pi}{b} \left( \frac{s - s'}{2} \pm iP \right) \right]}{\sin \left( \frac{2\pi}{b} P \right)}.$$  \hspace{1cm} (D.8)

The form of equations (D.7) and (D.8) suggests to look for a solution in a factorized form

$$D_P(s, s') = \hat{D}_P(s + s') \hat{D}_P(s - s')$$  \hspace{1cm} (D.9)

where $\hat{D}_P(s)$ should satisfy

$$\frac{\hat{D}_P(s \pm ib)}{D_{P - \frac{1}{2b}}(s)} = 2M^{1/2} \frac{\cosh[\frac{\pi b}{2} (\frac{s}{2} \pm iP)]}{\sqrt{2 \sin(2\pi b P)}} \quad (P > \frac{b}{2}).$$  \hspace{1cm} (D.10)

and

$$\frac{\hat{D}_P(s \pm \frac{i}{b})}{D_{P - \frac{1}{2b}}(s)} = 2M^{1/2b^2} \frac{\cosh[\frac{\pi b}{2} (\frac{s}{2} \pm iP)]}{\sqrt{2 \sin(2\frac{\pi}{b} P)}} \quad (P > \frac{1}{2b}).$$  \hspace{1cm} (D.11)

In addition the two-point function should be real even function of $P, s$ and $s'$. The solution is proportional to the Liouville reflection amplitude and is unique up to normalization:

$$D_P(s_1, s_2) = \frac{b}{|\sin(2\pi P/b)|} D^{(+-)}_{[P]}(s_1, s_2) = \frac{1/b}{|\sin(2\pi P b)|} D^{(-+)}_{[-P]}(s_1, s_2).$$  \hspace{1cm} (D.12)

Here the normalization is fixed by the requirement that the amplitude is self-dual.

**D.3. One-point function**

One can proceed in the similar way to evaluate the boundary one-point function, or the disc loop amplitude

$$W(s) = \langle [B_{e_0/2}]^{s,s} \rangle_{\text{disc}}$$  \hspace{1cm} (D.13)

using its relation with the boundary two-point function

$$D_{e_0/2}(s_1, s_2) = -\frac{W(s_1) - W(s_2)}{z(s_1) - z(s_2)}.$$  \hspace{1cm} (D.14)

The explicit expression of $D_{e_0/2}$ follows from the last formula in (A.7):

$$D_{e_0/2}(s, s') = -b \frac{M^{1/b^2 - 1}}{|\sin(\frac{\pi}{b} e_0)|} \frac{\cosh(\frac{\pi}{b} s) - \cosh(\frac{\pi}{b} s')}{\cosh(\pi b s) - \cosh(\pi b s')}.$$  \hspace{1cm} (D.15)
Comparing (D.15) and (D.14) we get

\[ W(s) = b \frac{M^{1/b^2} \cosh(\frac{\pi}{b} s)}{\sin(\frac{\pi}{b} e_0)}. \]  

(D.16)

Similarly one obtains the dual loop amplitude

\[ \tilde{W}(s) = -\frac{1}{b} M \frac{\cosh(\pi b s)}{\sin(\pi b e_0)}. \]  

(D.17)

D.4. The disc partition function of the FZZT brane

The one-point function is related to the disc partition function \( S(z) \) by

\[ W(s) = -\partial S(\Lambda, s) \partial z, \quad z(s) = M \cosh \pi b s. \]  

(D.18)

Integrating in \( z(s) \) gives for the disc partition function

\[ S(\Lambda, z) = \frac{b^2 M^{1+1/b^2}}{2 \sin(\frac{\pi}{b} e_0)} \left( \frac{\cosh(\pi Q s)}{Q} - \frac{\cosh(\pi e_0 s)}{e_0} \right) \]  

\[ = \frac{M^{1+1/b^2}}{\sin(\frac{\pi}{b} e_0) Q e_0} \left( b \cosh(\pi b s) \cos(\frac{\pi}{b} s) - \frac{1}{b} \sin(\pi b s) \sinh(\frac{\pi}{b} s) \right). \]  

(D.19)

The dual partition function is

\[ \tilde{S}(\Lambda, z) = -\frac{b^{-2} M^{1+1/b^2}}{2 \sin(\pi b e_0)} \left( \frac{\cosh(\pi Q s)}{Q} + \frac{\cosh(\pi e_0 s)}{e_0} \right) \]  

\[ = \frac{M^{1+1/b^2}}{\sin(\pi b e_0) Q e_0} \left( -\frac{1}{b} \cosh(\pi b s) \cos(\frac{\pi}{b} s) + b \sinh(\pi b s) \sinh(\frac{\pi}{b} s) \right). \]  

(D.20)

The two partition functions are related by a Legendre transformation:

\[ \frac{1}{b} \sin(\frac{\pi}{b} e_0) S(s) + b \sin(\pi b e_0) \tilde{S}(s) + \tilde{z}(s) z(s) = 0. \]  

(D.21)

Appendix E. Bilinear equation for the loop amplitude

The Virasoro conditions on the loop amplitude (5.16) are

\[ \tilde{W}_x^2(\tilde{z}) + \oint_{C_1} \frac{\tilde{z}' - \tilde{W}_x(\tilde{z})}{\tilde{z} - \tilde{z}'} \left[ 2T + \tilde{W}_{x+1}(2T - \tilde{z}') + \tilde{W}_{x-1}(2T - \tilde{z}') \right] \frac{d\tilde{z}'}{2\pi i} = 0 \]  

(E.1)

where the contour \( C_1 \) encircles the support of the eigenvalue density \([0, a]\) but not the interval \([2T - a, 2T]\). For amplitudes of the form (5.18) this equation becomes

\[ \tilde{W}^2(\tilde{z}) + \oint_{C_1} \frac{\tilde{z}' - \tilde{W}(\tilde{z})}{\tilde{z} - \tilde{z}'} \left[ 2T + 2 \cos(\pi b^2) \tilde{W}(2T - \tilde{z}') \right] \frac{d\tilde{z}'}{2\pi i} = 0. \]  

(E.2)
The function $\tilde{W}(\tilde{z})$ is meromorphic with a cut along the support $[0, \tilde{a}]$ of the eigenvalue distribution. Equation (E.2) implies the boundary condition along the cut

$$W(\tilde{z} + i0) + W(\tilde{z} - i0) - 2\cos(\pi b^2) W(2T - \tilde{z}) = 2T \quad (0 < \tilde{z} < \tilde{a}). \quad (E.3)$$

The integral equation (E.2) can be turned into a functional equation by the following trick ( [27], sect. 3.2). Replacing $\tilde{z}$ by $2T - \tilde{z}$ in (E.2) and adding the two equations one can turn the integral along the contour $C_1$ into an integral along a contour $C_\infty$, which encircles both intervals $[0, a]$ and $[2T - a, 2T]$ and therefore can be expanded to infinity. Now the integration can be performed by the residue formula using the asymptotics

$$\tilde{W}(\tilde{z}) = N/\tilde{z} + \tilde{W}_1/\tilde{z}^2 + ... \quad (\tilde{z} \to \infty) \quad (E.4)$$

which yields the functional identity:

$$\tilde{W}^2(\tilde{z}) + \tilde{W}^2(2T - \tilde{z}) - 2\cos(\pi b^2)W(\tilde{z})W(2T - \tilde{z}) = 2T \left[ \tilde{W}(\tilde{z}) + W(2T - \tilde{z}) \right] - \frac{4T^2 \tilde{C}}{\tilde{z}(2T - \tilde{z})}. \quad (E.5)$$

Here the constant $\tilde{C} = \tilde{C}(N, T)$ is given by the integral

$$\tilde{C} = \frac{2\cos(\pi b^2)}{T} \oint_{C_1} \frac{d\tilde{z}'}{2\pi i} W(\tilde{z}')\tilde{W}(2T - \tilde{z}'). \quad (E.6)$$

The critical value of $\tilde{C}$ can be found by solving (E.5) for $\tilde{z} = T$, where it becomes algebraic. This gives

$$\tilde{W}(T) = \frac{T}{2\sin^2(\pi b^2)} - \frac{\sqrt{C_c - C}}{\sin(\pi b^2)}, \quad \tilde{C}_c = \frac{T^2}{4\sin^2(\pi b^2)}. \quad (E.7)$$

At the critical point the solution of (E.5) is given by

$$\tilde{z} - T = \frac{T}{\cosh u}, \quad \tilde{W} = A + B\frac{\sinh(1 - b^2)u}{\sinh u}, \quad (E.8)$$

with $A = \frac{T}{2\sin^2(\pi b^2/2)}$ and $B = \frac{T}{\sin(\pi b^2/2)\sin(\pi b^2)}$. The coefficient $B$ is such that $\tilde{W}(\tilde{z}) = 0$ at $\tilde{z} = \infty$. For given $T$ the critical value of $N$ is determined by comparing the large $\tilde{z}$ asymptotics of the solution (E.8) with (E.4):

$$N_c = \frac{1 - b^2}{\sin(\pi b^2)} T^2. \quad (E.9)$$

For $\Lambda \equiv N - N_c << N_c$ the constant $\tilde{C}$ behaves as

$$\tilde{C} = \tilde{C}_c + c_1\Lambda + c_2\Lambda^{1/2}\sqrt{T^2 - 2/b^2} + ...$$

and as $b^2 < 1$, only the first term survives in the scaling limit $\Lambda/T^2 \to 0$. 

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References

[1] P. Ginsparg and G. Moore, “Lectures on 2D gravity and 2D string theory (TASI 1992)”, hep-th/9304011.

[2] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, “2-D Gravity and random matrices,” Phys. Rept. 254 (1995) 1, hep-th/9306153.

[3] J. Polchinski, “What is string theory”, Lectures presented at the 1994 Les Houches Summer School “Fluctuating Geometries in Statistical Mechanics and Field Theory”, hep-th/9411028.

[4] I. Klebanov, “String theory in two-dimensions,” hep-th/9108013.

[5] A. Jevicki, “Developments in 2-d string theory,” hep-th/9309115.

[6] E. Witten, “Ground ring of two-dimensional string theory,” Nucl. Phys. B 373, 187 (1992), hep-th/9108004.

[7] E. Witten and B. Zwiebach, “Algebraic structures and differential geometry in 2D string theory,” Nucl. Phys. B 377, 55 (1992), hep-th/9201056.

[8] I. R. Klebanov and A. M. Polyakov, “Interaction of discrete states in two-dimensional string theory,” Mod. Phys. Lett. A 6, 3273 (1991) hep-th/9109032.

[9] D. Kutasov, E. Martinec, N. Seiberg, “Ground rings and their modules in 2-D gravity with $c \leq 1$ matter”, Phys.Lett. B276 (1992) 437, hep-th/9111048.

[10] N. Seiberg and D. Shih, “Branes, rings and matrix models on minimal (super)string theory”, hep-th/0312170.

[11] M. Bershadsky and D. Kutasov, “Scattering of open and closed strings in (1+1)-dimensions”, Nucl. Phys. B382 (1992) 213, hep-th/9204049.

[12] M. Goulian and B. Li, Phys. Rev. Lett. 66 (1991), 2051.

[13] V. Dotsenko, Mod. Phys. Lett. A6(1991), 3601.

[14] P. Di Francesco, D. Kutasov, “World Sheet and Space Time Physics in Two Dimensional (Super) String Theory”, Nucl.Phys. B375 (1992) 119, hep-th/9109005.

[15] M. Bershadsky and D. Kutasov, Phys. Lett. 274B (1992) 331, hep-th/9110034.

[16] Y. Tanii, S.-I. Yamaguchi, “Two-dimensional quantum gravity on a disc”, Mod. Phys. Lett. A7 (1992) 521, hep-th/9110068; “Disk Amplitudes in Two-Dimensional Open String Theories”, hep-th/9203002.

[17] H. Dorn, H.J. Otto: Two and three point functions in Liouville theory, Nucl. Phys. B429 (1994) 375, hep-th/9403141.

[18] A.B. Zamolodchikov, Al.B. Zamolodchikov: Structure constants and conformal bootstrap in Liouville field theory, Nucl. Phys. B477 (1996) 577, hep-th/9506136.

[19] V. Fateev, A. B. Zamolodchikov and A. B. Zamolodchikov, “Boundary Liouville field theory. I: Boundary state and boundary two-point function,” hep-th/0001012.

[20] B. Ponsot, J. Teschner, “Boundary Liouville Field Theory: Boundary three point function”, Nucl. Phys. B622 (2002) 309, hep-th/0110244.
[21] K. Hosomichi, ”Bulk-Boundary Propagator in Liouville Theory on a Disc”, JHEP 0111 044 (2001), hep-th/0108093.
[22] M. R. Douglas, I. R. Klebanov, D. Kutasov, J. Maldacena, E. Martinec and N. Seiberg, “A new hat for the c = 1 matrix model,” hep-th/0307195.
[23] I. Kostov, B. Ponsot and D. Serban, “Boundary Liouville Theory and 2D Quantum Gravity”, hep-th/0307189.
[24] I. Kostov, “Boundary Correlators in 2D Quantum Gravity: Liouville versus Discrete Approach ”, Nucl. Phys. B658 (2003) 397, hep-th/0212194.
[25] J. Teschner. On the Liouville Three-Point Function. Phys.Lett., B363 (1995) 65.
[26] A. Belavin, A. Polyakov, A. Zamolodchikov, “Infinite conformal symmetry in two-dimensional quantum field theory”, Nucl. Phys. B241, 333 (1984).
[27] I. Kostov, “Strings with discrete target space”, Nucl. Phys. B376 (1992) 539, hep-th/9112059.
[28] V. Kazakov, I. Kostov, “Loop Gas Model for Open Strings”, Nucl. Phys. B386 (1992) 520.
[29] S. Kachru, “Quantum Rings and Recursion Relations in 2D Quantum Gravity”, Mod. Phys. Lett. A7 (1992) 1419, hep-th/9201072.
[30] S. Govindarajan, T. Jayaraman and V. John, Genus Zero Correlation Functions in $c < 1$ String Theory, Phys. Rev. D 48 (1993) 839, hep-th/9208064.
[31] S. Govindarajan, T. Jayaraman and V. John, “Correlation Functions and Multicritical Flows in $c < 1$ String Theory”, Int. J. Mod.Phys. A10 (1995) 477, hep-th/9309040.
[32] R. Dijkgraaf, G. W. Moore and R. Plesser, “The Partition function of 2-D string theory,” Nucl. Phys. B 394, 356 (1993), hep-th/9208031.
[33] C. Imbimbo, S. Mukhi, Nucl. Phys. B449 (1995) 553, hep-th/9505127.
[34] V. Kazakov, I. Kostov, D. Kutasov, “A Matrix Model for the Two Dimensional Black Hole”, Nucl.Phys. B622 (2002) 141, hep-th/0101011.
[35] S. Alexandrov, V. Kazakov, I. Kostov, “Time-dependent backgrounds of 2D string theory”, Nucl.Phys. B640 (2002) 119, hep-th/0205079.
[36] I. Kostov, “String Equation for String Theory on a Circle”, Nucl. Phys. B624 (2002) 146, hep-th/0107247.
[37] I. Kostov, “Integrable flows in $c = 1$ string theory”, J.Phys. A36 (2003) 3153, hep-th/0208034.
[38] J. McGreevy and H. Verlinde, “Strings from tachyons”, JHEP 0312 (2003) 054, hep-th/0304224.
[39] E. J. Martinec, “The annular report on non-critical string theory,” hep-th/0305148.
[40] I. R. Klebanov, J. Maldacena and N. Seiberg, “D-brane decay in two-dimensional string theory,” hep-th/0305159.
[41] J. McGreevy, J. Teschner and H. Verlinde, “Classical and quantum D-branes in 2D string theory,” hep-th/0305194.
[42] V. Schomerus, “Rolling tachyons from Liouville theory,” hep-th/0306026.
[43] S. Y. Alexandrov, V. A. Kazakov and D. Kutasov, “Non-Perturbative Effects in Matrix Models and D-branes,” hep-th/0306177.
[44] A. Sen, “Open-Closed Duality: Lessons from the Matrix Model,” hep-th/0308068.
[45] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, C. Vafa, “Topological Strings and Integrable Hierarchies”, hep-th/0312085.
[46] I. R. Klebanov, J. Maldacena, N. Seiberg, “Unitary and Complex Matrix Models as 1-d Type 0 Strings”, hep-th/0309168.
[47] A. B. Zamolodchikov and A. B. Zamolodchikov, “Liouville field theory on a pseudosphere”, hep-th/0101152.
[48] I. Kostov, “The ADE face models on a fluctuating planar lattice”, Nucl. Phys. B326 (1989) 583.
[49] I. Kostov, “Gauge Invariant Matrix Model for the \(\hat{A} \rightarrow \hat{D} \rightarrow \hat{E}\) Closed Strings”, Phys.Lett. B297 (1992) 74-81, hep-th/9208053.
[50] S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, S. Pakuliak, “Conformal Matrix Models as an Alternative to Conventional Multi-Matrix Models”, Nucl. Phys. B404 (1993) 717, hep-th/9208044.
[51] I. Kostov, “Solvable Statistical Models on Random Lattices”, Proceedings of the Conference on recent developments in statistical mechanics and quantum field theory. (Trieste, 10 - 12 April 1995), Nucl. Phys. B (Proc. Suppl.) 45 A (1996) 13-28, hep-th/9509124.
[52] S. Higuchi and I. Kostov, “Feynman rules for string theories with discrete target space”, Phys. Lett. B357 (1995) 62, hep-th/9506022.
[53] I. Kostov, “Loop space Hamiltonian for \(c \leq 1\) open strings”, Phys.Lett. B349 (1995) 284, hep-th/9501135.
[54] I. Kostov, “Field Theory of Open and Closed Strings with Discrete Target Space”, Phys.Lett. B344 (1995) 135, hep-th/9410164.
[55] Xi Yin, “Matrix Models, Integrable Structures, and T-duality of Type 0 String Theory”, hep-th/0312236.
[56] I. Kostov and V. Petkova, in preparation.
[57] I. Kostov and M. Staudacher, “Strings in discrete and continuous target spaces: a comparison”, Phys. Lett. B305 (1993) 43.
[58] R. Dijkgraaf, C. Vafa, “N=1 Supersymmetry, Deconstruction, and Bosonic Gauge Theories”, hep-th/0302011.
[59] A. Recknagel, V. Schomerus, “Boundary Deformation Theory and Moduli Spaces of D-Branes”, Nucl.Phys. B545 (1999) 233, hep-th/9811237.
[60] V.B. Petkova, J.-B. Zuber, “BCFT: from the boundary to the bulk”, Talk presented at TMR-conference ”Nonperturbative Quantum Effects 2000”, hep-th/0009219.
[61] J. Teschner, “On boundary perturbations in Liouville theory and brane dynamics in noncritical string theories,” hep-th/0308140.