Dirac-type tensor equations on a parallelisable manyfolds

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Abstract

The goal of this work is to extend Dirac-type tensor equations to a curved space. We take four 1-forms (a tetrad) as a unique structure, which determines a geometry of space-time.

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In [2] we suggest to take the differential forms \(H, I, K\) (see item 14 of sect. 1 in [1]) as an additional structure on pseudo-Riemannian space. In the current paper, developing this idea, we take four 1-forms \(e^a\) (a tetrad) as a unique structure, which determines a geometry of space-time. A metric tensor is expressed via the tetrad. Hence we arrive at a geometry, which was considered by many authors (see, for example, Møller [3]) as a mathematical model of physical space-time and gravity (according to the Theory of General Relativity the gravity is identified with the metric tensor).

The goal of our work, begining at [1], [2], is to extend Dirac-type tensor equations (see [4]) to a curved space.

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1 A pseudo-Riemannian space

Let $\mathcal{M}$ be a four dimensional differentiable manifold with local coordinates $x^\mu$, $\mu = 0, 1, 2, 3$ and with a metric tensor $g_{\mu\nu} = g_{\nu\mu}$ such that $g_{00} > 0$, $g = \det|g_{\mu\nu}| < 0$ and the signature of matrix $|g_{\mu\nu}|$ is equal to $-2$. The full set of $\{\mathcal{M}, g_{\mu\nu}\}$ is called a pseudo-Riemannian space and is denoted by $\mathcal{V}$. The metric tensor defines the Levi-Civita connection, the curvature tensor, the Ricci tensor, and the scalar curvature

$$\Gamma_{\mu\nu}^\lambda = \frac{1}{2}g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu}),$$

$$R_{\lambda\mu\nu}^\kappa = \partial_\mu \Gamma_{\nu\lambda}^\kappa - \partial_\nu \Gamma_{\mu\lambda}^\kappa + \Gamma_{\mu\eta}^\kappa \Gamma_{\nu\lambda}^\eta - \Gamma_{\nu\eta}^\kappa \Gamma_{\mu\lambda}^\eta,$$

$$R_{\nu\rho} = R_{\mu\nu\rho},$$

$$R = g^{\rho\nu}R_{\rho\nu}$$

with symmetries

$$\Gamma_{\mu\nu}^\lambda = \Gamma_{\nu\mu}^\lambda, \quad R_{\mu\nu\lambda\rho} = R_{\lambda\rho\mu\nu} = R_{[\mu\nu]\lambda\rho}, \quad R_{\mu[\nu\lambda\rho]} = 0, \quad R_{\nu\rho} = R_{\rho\nu}$$

Let $\mathcal{T}_{q}^p$ be the set of all tensor fields of rank $(q, p)$ on $\mathcal{V}$. The covariant derivatives $\nabla_\mu : \mathcal{T}_{q}^p \to \mathcal{T}_{q+1}^p$ are defined via the Levi-Civita connection by the following rules:

1. If $t = t(x), \ x \in \mathcal{V}$ is a scalar function, then

$$\nabla_\mu t = \partial_\mu t.$$

2. If $t^\nu \in \mathcal{T}_1^1$, then

$$\nabla_\mu t^\nu = t^\nu_{;\mu} = \partial_\mu t^\nu + \Gamma_{\mu\nu}^\lambda t^\lambda.$$

3. If $t_\nu \in \mathcal{T}_1^1$, then

$$\nabla_\mu t_\nu = t_{\nu;\mu} = \partial_\mu t_\nu - \Gamma_{\nu\mu}^\lambda t_\lambda.$$

4. If $u = u^\nu_{\lambda_1...\lambda_k} \in \mathcal{T}_k^1$, $v = v^\mu_{\rho_1...\rho_s} \in \mathcal{T}_s^1$, then

$$\nabla_\mu (u \otimes v) = (\nabla_\mu u) \otimes v + u \otimes \nabla_\mu v.$$
With the aid of these rules it is easy to calculate covariant derivatives of arbitrary rank tensor fields. Also we can check the formulas

\[ \nabla_\mu g_{\alpha\beta} = 0, \quad \nabla_\mu g^{\alpha\beta} = 0, \quad \nabla_\mu \delta_\alpha^\beta = 0, \]
\[ \nabla_\alpha (R^{\alpha\beta} - \frac{1}{2}R g^{\alpha\beta}) = 0, \]
\[ (\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) a_\rho = R^\lambda_{\rho\mu\nu} a_\lambda, \]

for any \( a_\rho \in T_1. \)

2 A parallelisable manyfolds

An \( n \)-dimensional differentiable manifolds is called parallelisable if there exist \( n \) linear independent vector or covector fields on it. Let \( M \) be a four dimensional parallelisable manifolds with local coordinates \( x = (x^\mu) \) and

\[ e_\mu^a = e_\mu^a(x), \quad a = 0, 1, 2, 3 \]

be four covector fields on \( M \). This set of four covectors are called a tetrad. The full set \( \{ M, e_\mu^a \} \) is denoted by \( W \). Here and in what follows we use greek indices as tensorial indices and latin indices as nontensorial (tetrad) indices, which enumerate covectors.

Let us take the Minkowski matrix

\[ \eta_{ab} = \eta^{ab} = \text{diag}(1, -1, -1, -1). \]

Then we can define a metric tensor

\[ g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}, \quad (5) \]

such that

\[ g_{\mu\nu} = g_{\nu\mu}, \quad g_{00} > 0, \quad g = \text{det} \| g_{\mu\nu} \| < 0 \]

and the signature of the matrix \( \| g_{\mu\nu} \| \) is equal to \(-2\). The Levi-Civita connection \( \Gamma^{\lambda}_{\mu\nu} \), the curvature tensor \( R^{\mu}_{\nu\lambda\rho} \), the Ricci tensor \( R_{\nu\rho} \), the scalar curvature \( R \), and the covariant derivatives \( \nabla_\mu \) are defined via \( g_{\mu\nu} \) as in sect. 1. All constructions of \[\] (the Clifford product of differential forms, the Spin(1,3) group, Upsilon derivatives \( \Upsilon_\mu \), etc.) are valid in the parallelisable manyfolds \( W \).
We raise and lower latin indices with the aid of the matrix $\eta_{ab} = \eta^{ab}$ and greek indices with the aid of the metric tensor $g_{\mu\nu}$

$$e^{\nu a} = g^{\mu\nu} e_{\mu}^a, \quad e_{\mu a} = \eta_{ab} e_{\mu}^b.$$ 

If we take 1-forms $e^a, e_a \in \Lambda_1$

$$e^a = e_{\mu}^a dx^\mu, \quad e_a = \eta_{ab} e^b,$$

then we see that

$$e^a e^b + e^b e^a = 2 \eta^{ab}. \tag{6}$$

Indeed,

$$e^a e^b + e^b e^a = e_{\mu}^a dx^\mu e^b_{\nu} dx^\nu + e^b_{\nu} dx^\nu e_{\mu}^a dx^\mu =
= e_{\mu}^a e^b_{\nu} (dx^\mu dx^\nu + dx^\nu dx^\mu) = e_{\mu}^a e^b_{\nu} 2g^{\mu\nu} = 2 \eta^{ab}.$$

The transformation

$$e^a \rightarrow \tilde{e}^a = S^{-1} e^a S, \tag{7}$$

where $S \in \text{Spin}(1,3)$, is called a Lorentz rotation of the tetrad. Evidently, formula (6) is invariant under Lorentz rotations of the tetrad, i.e.,

$$e^a e^b + e^b e^a = 2 \eta^{ab} \iff \tilde{e}^a \tilde{e}^b + \tilde{e}^b \tilde{e}^a = 2 \eta^{ab}.$$

In the sequel we use the following lemma.

**Lemma**.

$$e^a U e_a = \begin{cases} 4U & \text{for } U \in \Lambda_0 T^p_q, \\ -2U & \text{for } U \in \Lambda_1 T^p_q, \\ 0 & \text{for } U \in \Lambda_2 T^p_q, \\ 2U & \text{for } U \in \Lambda_3 T^p_q, \\ -4U & \text{for } U \in \Lambda_4 T^p_q. \end{cases}$$

The proof is by direct calculation.

Let us take a tensor $B_\mu \in \Lambda_2 T^1_1$

$$B_\mu = -\frac{1}{4} e^a \wedge \Upsilon_{\mu} e_a. \tag{8}$$

**Theorem**. Under the Lorentz rotation of tetrad (7) the tensor $B_\mu$ transforms as

$$B_\mu \rightarrow \tilde{B}_\mu = S^{-1} B_\mu S - S^{-1} \Upsilon_{\mu} S.$$
Proof. We have \( B_\mu = -\frac{1}{4} e^a \wedge \Upsilon_\mu e_a = -\frac{1}{4} e^a \Upsilon_\mu e_a \). Therefore
\[
-\frac{4}{4} \hat{B}_\mu = e^a \Upsilon_\mu e_a = \hat{S}^{-1} e^a S \Upsilon_\mu (S^{-1} e^a S) = \hat{S}^{-1} e^a S (\Upsilon_\mu S^{-1}) e_a s + \hat{S}^{-1} e^a \Upsilon_\mu e_a S + S^{-1} e^a \Upsilon_\mu e_a S = -4 \hat{S}^{-1} B_\mu S + 4 \hat{S}^{-1} \Upsilon_\mu S + S^{-1} e^a S (\Upsilon_\mu S^{-1}) e_a S.
\]
Here we use the formula \( e^a e_a = 4 \) from the Lemma. It can be checked that \( S \Upsilon_\mu S^{-1} \in \Lambda_2 \setminus_1 \). Consequently from the Lemma we get that
\[
e^a S (\Upsilon_\mu S^{-1}) e_a = 0.
\]
These completes the proof.

Note that the set of 2-forms \( \Lambda_2 \) can be considered as the real Lie algebra of the Lie group \( \text{Spin}(1, 3) \). Hence \( B_\mu \) belong to this Lie algebra. \( B_\mu \) is a tensor with respect to changes of coordinates. But, according to the Th eorem, under Lorentz rotations of the tetrad \( B_\mu \) transforms as a connection.

Now we may define operators \( D_\mu = \Upsilon_\mu - [B_\mu, \cdot] \) acting on tensors from \( \Lambda_\top^p \) and such that
\[
D_\mu e^a = 0, \quad D_\mu e_a = 0, \quad D_\mu (UV) = (D_\mu U)V + U D_\mu V, \quad D_\mu D_\nu - D_\nu D_\mu = 0.
\]
Consider the tensor from \( \Lambda_2 \setminus_2 \)
\[
\frac{1}{2} C_{\mu\nu} = D_\mu B_\nu - D_\nu B_\mu + [B_\mu, B_\nu].
\]
It can be shown that
\[
C_{\mu\nu} = \frac{1}{2} R_{\mu\nu\alpha\beta} dx^\alpha \wedge dx^\beta.
\]
In [1] (see item 14) of sect.1) we define differential forms \( H \in \Lambda_1; I, K \in \Lambda_2; \ell \in \Lambda_4 \), which we call secondary generators of \( \Lambda \). These differential forms are connected with the tetrad \( e^a \) by the following formulas:
\[
H = e^0, \quad I = -e^1 e^2, \quad K = -e^1 e^3, \quad \ell = e^0 e^1 e^2 e^3,
\]
\[
e^0 = H, \quad e^1 = IK\ell H, \quad e^2 = K\ell H, \quad e^3 = -I\ell H.
\]
The formula for \( B_\mu \) from [2]
\[
B_\mu = -\frac{3}{8} H \Upsilon_\mu H + \frac{1}{4} (I \Upsilon_\mu I + K \Upsilon_\mu K)
\]
\[
+ \frac{1}{8} H (I \Upsilon_\mu I + K \Upsilon_\mu K) H - \frac{1}{4} IKH \Upsilon_\mu H K I - \frac{1}{8} (K I \Upsilon_\mu I K + I K \Upsilon_\mu K I)
\]
is equivalent to formula (8).
3 Lagrangians and main equations

Consider the invariant
\[ L_2 = R + 4 \text{Tr}(\delta B), \]
where \( R \) is the scalar curvature, \( B = dx^\mu B_\mu \), and the codifferential \( \delta : \Lambda_k \to \Lambda_{k-1} \) was defined in [5]. It can be checked that the invariant \( L_2 \) doesn’t depend on second derivatives of tetrad components \( e_\mu^a \). Variating the Lagrangian \( L_2 \) with respect to the components of metric tensor \( g_{\mu\nu} \), we get the Einstein tensor
\[
\epsilon \sqrt{-g} (R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R) = \frac{\partial (\sqrt{-g} L_2)}{\partial g_{\mu\nu}} - \partial_\rho \frac{\partial (\sqrt{-g} L_2)}{\partial g_{\mu\nu,\rho}},
\]
where \( g_{\mu\nu,\rho} = \partial_\rho g_{\mu\nu} \), \( \epsilon = 1 \) for \( \mu = \nu \) and \( \epsilon = 2 \) for \( \mu \neq \nu \). Note that we can easily calculate the partial derivatives \( \frac{\partial e_\mu^a}{\partial g^{\alpha\beta}} \), \( \frac{\partial e_\mu^a}{\partial g_{\alpha\beta,\rho}} \) using formulas
\[
\begin{align*}
g_{\alpha\beta} &= e_\alpha^a e_\beta^b \eta_{ab}, \\
\frac{\partial g_{\alpha\beta}}{\partial e_\mu^a} &= \frac{\partial g_{\alpha\beta,\rho}}{\partial e_\mu^a} = \delta_\alpha^\mu e_\beta^a + \delta_\beta^\mu e_\alpha^a, \\
\frac{\partial e_\mu^a}{\partial g_{\alpha\beta}} &= \frac{\partial e_\mu^a}{\partial g_{\alpha\beta,\rho}} = \frac{1}{\delta_\alpha^\mu e_\beta^a + \delta_\beta^\mu e_\alpha^a},
\end{align*}
\]
where \( \rho = 0, 1, 2, 3 \). Finally, we can take the Lagrangian
\[
L = L_0 + L_1 + L_2
\]
\[
= 2 \text{Tr}(e^0(\Psi^* P + P^* \Psi)) + \frac{1}{4} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + (R + 4 \text{Tr}(\delta B)),
\]
where
\[
P = dx^\mu (\mathcal{D}_\mu \Psi + \Psi A_\mu + B_\mu \Psi) N - m \Psi E,
\]
\[
F_{\mu\nu} = \mathcal{D}_\mu A_\nu - \mathcal{D}_\nu A_\mu - [A_\mu, A_\nu].
\]
and the Lagrangians \( L_0, L_1 \) were defined in sect.6 of [1]. Variating the Lagrangian \( L \) with respect to components of \( \Psi, A_\mu \) and w.r.t. components of metric tensor, we arrive at the system of equations
\[
d x^\mu (\mathcal{D}_\mu \Psi + \Psi A_\mu + B_\mu \Psi) N - m \Psi E = 0,
\]
\[
\frac{1}{\sqrt{-g}} \mathcal{D}_\mu (\sqrt{-g} F^{\mu\nu}) - [A_\mu, F^{\mu\nu}] = J^\nu,
\]
\[
R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = -T^{\mu\nu},
\]
where $J^\nu$ are defined in (27) of [1] and $T^{\mu\nu}$ is the energy-momentum tensor

$$\epsilon\sqrt{-g} T^{\mu\nu} = \frac{\partial(\sqrt{-g}(\mathcal{L}_0 + \mathcal{L}_1))}{\partial g_{\mu\nu}} - \partial_\rho \left. \frac{\partial(\sqrt{-g}(\mathcal{L}_0 + \mathcal{L}_1))}{\partial g_{\mu\nu,\rho}} \right|_{\rho = \mu}.$$ 

Let us remark that we may insert two constants into Lagrangian $L = L_0 + c_1 L_1 + c_2 L_2$ in (9) and into equations (10) respectively. Constants $c_1$, $c_2$ depend on physical units and on experimental data.

4 Comparing the Dirac equation with the Dirac-type tensor equation

It is well known that the Dirac equation for the electron has the following form in a curved space (see, for example, [2]):

$$\gamma^c \epsilon^\mu_c (\partial_\mu + ia_\mu - \omega_{\mu ab} \frac{1}{4}[\gamma^a, \gamma^b]) \psi + i m \psi = 0, \quad (11)$$

where $\gamma^a$ are complex valued $4 \times 4$-matrices with the property $\gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab} \mathbf{1}$, $\mathbf{1}$ is identity matrix, and $\omega_{\mu ab} = \omega_{\mu [ab]}$ is a Lorentz connection. Now we show that the Dirac type tensor equation

$$dx^\mu (D_\mu \Psi + a_\mu \Psi I + B_\mu \Psi) + m \Psi HI = 0 \quad (12)$$

can be written in the same form (11). A method of reduction of (12) to (11) was developed in [3] for the case of Minkowski space.

Let us take the idempotent differential form $t = t^2 \in \Lambda^C$

$$t = \frac{1}{4}(1 + H)(1 - iI)$$

and the left ideal

$$\mathcal{I}(t) = \{Ut : U \in \Lambda^C\} \subset \Lambda^C.$$

The exterior forms $t_1, \ldots, t_4 \in \mathcal{I}(t)$

$$t_1 = t, \quad t_2 = Kt, \quad t_3 = -I \ell t, \quad t_4 = -KI \ell t$$

are linear independent and they can be considered as basis elements of $\mathcal{I}(t)$. These differential forms $t_k$ define a map $\gamma : \Lambda^q \rightarrow M(4, C)^q$ by the formula

$$U_{\mu_1 \ldots \mu_p}^{\nu_1 \ldots \nu_q} t_k = \gamma(U_{\mu_1 \ldots \mu_p})^{\nu_1 \ldots \nu_q}_{\mu_1 \ldots \mu_p} n_k.$$
where $M(4,\mathcal{C})\Gamma^p_q$ is the set of all rank $(p, q)$ tensors with values in $4 \times 4$ complex matrices and $\gamma(U)^n_k$ is elements of the matrix $\gamma(U)$ (an upper index enumerates rows and a lower index enumerates columns). It is easily shown that

$$\gamma(UV) = \gamma(U)\gamma(V)$$

for $U \in \Lambda \Gamma^a_b$, $V \in \Lambda \Gamma^c_d$. If we take $dx^\mu = \delta^\mu_\nu dx^\nu \in \Lambda_1 T^1$, then we get

$$dx^\mu t_k = \gamma(dx^\nu)^k_n t_n.$$  

Denoting $\gamma^\mu = \gamma(dx^\mu)$, we see that the equality $dx^\mu dx^\nu + dx^\nu dx^\mu = 2g^{\mu\nu}$ leads to the equality $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}1$. Also we have

$$B_\mu t_p = \gamma(B_\mu)^k_p t_k.$$

Let us multiply (12) by $t$. Then

$$0 = (dx^\mu(D_\mu \Psi + a_\mu \Psi I + B_\mu \Psi) + m\Psi HI)t$$

$$= dx^\mu(D_\mu(\Psi t) + ia_\mu(\Psi t) + B_\mu(\Psi t)) + im(\Psi t)$$

$$= dx^\mu(D_\mu(\psi^k t_k) + ia_\mu(\psi^k t_k) + B_\mu(\psi^p t_p)) + im(\psi^m t_n)$$

$$= (dx^\mu t_k)(\partial_\mu \psi^k + ia_\mu \psi^k + \gamma(B_\mu)^k_p \psi^p) + im(\psi^m t_n)$$

$$= ((\gamma^\mu)^n_k(\partial_\mu \psi^k + ia_\mu \psi^k + \gamma(B_\mu)^k_p \psi^p) + im\psi^n) t_n.$$

As $t_1, \ldots, t_4$ are linear independent, we see that

$$(\gamma^\mu)^n_k(\partial_\mu \psi^k + ia_\mu \psi^k + \gamma(B_\mu)^k_p \psi^p) + im\psi^n = 0, \quad n = 1, \ldots, 4.$$  

These four equations can be written as one equation

$$\gamma^\mu(\partial_\mu + ia_\mu + \gamma(B_\mu))\psi + im\psi = 0,$$  

where $\psi = (\psi^1 \ldots \psi^4)^T$. We may write $B_\mu \in \Lambda_2 \Gamma_1$ as

$$B_\mu = \frac{1}{2} b_{\mu ab} e^a \wedge e^b = \frac{1}{4} b_{\mu ab}(e^a e^b - e^b e^a),$$

where $b_{\mu ab} = b_{\mu[ab]}$. This imply that

$$\gamma(B_\mu) = \frac{1}{4} b_{\mu ab}[\gamma^a, \gamma^b], \quad \gamma^a = \gamma(e^a), \quad \gamma^a \gamma^b + \gamma^b \gamma^a = 2\eta^{ab}1.$$  

Note that $\gamma^\mu = \gamma^c e^\mu_c$ and the eq. (13) can be written in the form

$$\gamma^c e^\mu_c(\partial_\mu + ia_\mu + b_{\mu ab} \frac{1}{4}[\gamma^a, \gamma^b])\psi + im\psi = 0.$$  

Consequently eqs. (11) and (14) are coincide iff a Lorentz connection is defined by the formula $\omega_{\mu ab} = -b_{\mu ab}$.  

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