Generalized Misner-Sharp Energy in $f(R)$ Gravity

Rong-Gen Cai, Li-Ming Cao, Ya-Peng Hu, and Nobuyoshi Ohta

1 Key Laboratory of Frontiers in Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, P.O. Box 2735, Beijing 100190, China
2 Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan
3 Graduate School of the Chinese Academy of Sciences, Beijing 100039, China

We study generalized Misner-Sharp energy in $f(R)$ gravity in a spherically symmetric spacetime. We find that unlike the cases of Einstein gravity and Gauss-Bonnet gravity, the existence of the generalized Misner-Sharp energy depends on a constraint condition in the $f(R)$ gravity. When the constraint condition is satisfied, one can define a generalized Misner-Sharp energy, but it cannot always be written in an explicit quasi-local form. However, such a form can be obtained in a FRW universe and for static spherically symmetric solutions with constant scalar curvature. In the FRW universe, the generalized Misner-Sharp energy is nothing but the total matter energy inside a sphere with radius $r$, which acts as the boundary of a finite region under consideration. The case of scalar-tensor gravity is also briefly discussed.

PACS numbers: 04.20.Cv, 04.50.+h, 04.70.Dy

*Electronic address: cairg@itp.ac.cn
†Electronic address: caolm@itp.ac.cn
‡Electronic address: yapenghu@itp.ac.cn
§Electronic address: ohtan@phys.kindai.ac.jp
I. INTRODUCTION

A gravitational field has certainly an associated energy. However, it is a rather difficult task to define energy for a gravitational field in general relativity. A local energy density of gravitational field does not make any sense because the energy-momentum pseudo-tensor of gravitational field, which explicitly depends on metric and its first derivative, will vanish due to the strong equivalence principle at any point of spacetime in a locally flat coordinate \[1, 2, 3\]. In general relativity, however, there exist two well-known definitions of total energy; one is the Arnowitt-Deser-Misner (ADM) energy \(E_{ADM}\) at spatial infinity \[4\], and the other is the Bondi-Sachs (BS) energy \(E_{BS}\) at null infinity \[5\] describing an isolated system in an asymptotically flat spacetime.

Due to the absence of the local energy density of gravitational field, it is tempting to define some meaningful quasi-local energy, which is defined on a boundary of a given region in spacetime. Indeed, it is possible to properly define such quasi-local energies. Some useful definitions for quasi-local energy exist in the literature, for instance, Brown-York energy \[6\], Misner-Sharp energy \[7\], Hawking-Hayward energy \[8, 9\] and Chen-Nester energy \[10\], etc. A nice review on this issue can be found in \[3\]. In this article, we focus on the Misner-Sharp energy.

The Misner-Sharp energy \(E\) is defined in a spherically symmetric spacetime. Various properties of the Misner-Sharp energy are discussed in some detail by Hayward in \[11, 12\]. For example, the following properties are established. In the Newtonian limit of a perfect fluid, the Misner-Sharp energy \(E\) yields the Newtonian mass to leading order and the Newtonian kinetic and potential energy in the next order. For test particles, the corresponding Hajicek energy is conserved and has the behavior appropriate to energy in the Newtonian and special-relativistic limits. In the small-sphere limit, the leading term in \(E\) is the product of volume and the energy density of the matter. In vacuo, the Misner-Sharp energy \(E\) reduces to the Schwarzschild energy. At null and spatial infinity, \(E\) reduces to the BS and ADM energies, respectively. In particular, it is shown that the conserved Kodama current produces the conserved charge \(E\).

In a four-dimensional, spherically-symmetric spacetime with metric

\[
ds^2 = h_{ab}dx^a dx^b + r^2(x)d\Omega_2^2, \tag{1.1}
\]

where \(a = 0, 1\), \(x^a\) is the coordinate on a two-dimensional spacetime \((M^2, h_{ab})\) and \(d\Omega_2^2\)
denotes the line element for a two-dimensional sphere with unit radius, the Misner-Sharp energy $E$ can be defined as

$$E(r) = \frac{r}{2G} \left(1 - h^{ab} \partial_a r \partial_b r \right).$$

(1.2)

With this energy, the Einstein equations can be rewritten as

$$dE = A \Psi_a dx^a + W dV,$$

(1.3)

where $A = 4\pi r^2$ is the area of the sphere with radius $r$ and $V = 4\pi r^3/3$ is its volume, $W$ is called work density defined as $W = -h^{ab}T_{ab}/2$ and $\Psi$ energy supply vector, $\Psi_a = T_a \cdot \partial r + W \partial r_a$, with $T_{ab}$ being the projection of the four-dimensional energy-momentum tensor $T_{\mu \nu}$ of matter in the normal direction of the 2-dimensional sphere. The form (1.3) is called “unified first law” [13, 14]. Projecting this form along a trapping horizon, one is able to arrive at the first law of thermodynamics for dynamical black hole

$$\langle dE, \xi \rangle = \frac{\kappa}{8\pi G} \langle dA, \xi \rangle + W \langle dV, \xi \rangle,$$

(1.4)

where $\xi$ is a projecting vector and $\kappa = \frac{1}{2\sqrt{-h}} \partial_a (\sqrt{-h} h^{ab} \partial_b r)$ is surface gravity on the trapping horizon. Defining $\delta Q = \langle A \Psi, \xi \rangle = T dS$, we can derive entropy formula associated with apparent horizon in various gravity theories [15, 16, 17]. Indeed, the Minser-Sharp energy plays an important role in connection between the Einstein equations and first law of thermodynamics in FRW cosmological setup [15, 16, 17, 18, 19] and black hole setup [20].

Note that the original form (1.2) for the Misner-Sharp energy is applicable for Einstein gravity without cosmological constant in four dimensions, thus it is tempting to give corresponding forms for the case with a cosmological constant and/or in other gravity theories. Indeed, a generalized form is given for Gauss-Bonnet gravity and more general Lovelock gravity in [21, 22]. In particular, we would like to mention here that Gong and Wang in [23] introduce a modified Misner-Sharp energy and discuss its relation to horizon thermodynamics. With the generalized Misner-Sharp energy, it is shown that the Clausius relation $\delta Q = T dS$ indeed gives correct entropy formula for Lovelock gravity [15, 17].

Recently, a kind of modified gravity theories, $f(R)$, whose Lagrangian is a function of curvature scalar $R$, has attracted a lot of attention. A main motivation is to explain the observed accelerated expansion of the universe without introducing the exotic dark energy with a large negative pressure. For a review on $f(R)$ gravity, see [24]. Of course, $f(R)$
gravity is a simple generalization of Einstein gravity; when \( f(R) = R \), it goes back to Einstein theory. However, \( f(R) \) is quite different from another generalization of Einstein gravity, Lovelock gravity. The equations of motion of the latter do not contain more than second-order derivatives, while the equations of motion for the former do. In addition, let us notice that in some sense, the \( f(R) \) gravity is quite similar to scalar-tensor gravity, a generalization of Einstein gravity again.

In this paper we are mainly concerned with the question whether there exists a similar Misner-Sharp energy for \( f(R) \) gravity in a spherically symmetric spacetime. For this goal, we will take two methods, which are basically equivalent, in fact. One is called integration method, and the other is conserved charge method associated with the Kodama current. The integration method is introduced in a previous paper of ours [17] for the case of radiation matter in Lovelock gravity. We find that existence of a generalized Misner-Sharp energy is not trivial for \( f(R) \) gravity. Its existence depends on a constraint. Once the constraint is satisfied, we could have a generalized Misner-Sharp energy. Otherwise, the answer is negative. The same situation happens for the scalar-tensor gravity theory.

The organization of the paper is as follows. In Sec. II, as a warm-up exercise, we derive the generalized Misner-Sharp energy in Gauss-Bonnet gravity by using the integration method and by generalizing the discussion in [17] to more general matter content. In Sec. III, we discuss the generalized Misner-Sharp energy in \( f(R) \) gravity by the integration method and conserved charge method, respectively. Sec. IV is devoted to investigating some special cases, homogeneous and isotropic FRW cosmology and static spherically symmetric case. In these cases the generalized Misner-Sharp energy has a simple form. The conclusion and some discussions are given in Sec. V. In the appendix, we briefly discuss the generalized Misner-Sharp energy for scalar-tensor gravity in a FRW universe.

II. GENERALIZED MISNER-SHARP ENERGY IN GAUSS-BONNET GRAVITY: INTEGRATION METHOD

The equations of motion of Gauss-Bonnet gravity can be written down as

\[
G_{\mu\nu} + \alpha H_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu},
\]  

(2.1)
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}, \]
\[ H_{\mu\nu} = 2(R R_{\mu\nu} - 2 R_{\mu\alpha} R_{\nu}\alpha - 2 R^{\alpha\beta} R_{\mu\alpha\beta \gamma} + R_{\mu}^{\alpha\beta\gamma} R_{\nu\alpha\beta\gamma}) - \frac{1}{2} g_{\mu\nu} L_{GB}, \]  
(2.2)

and \( \alpha \) is a coupling constant with dimension of length squared. The Gauss-Bonnet term is
\[ L_{GB} = R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}. \]

Consider an \( n \)-dimensional spherically symmetric spacetime of metric in the double-null form
\[ ds^2 = -2 e^{-\varphi(u,v)} du dv + r^2(u,v) \gamma_{ij} dz^i dz^j, \]  
(2.3)

where \( \gamma_{ij} \) is the metric on an \((n-2)\)-dimensional constant curvature space \( K^{n-2} \) with its sectional curvature \( k = \pm 1, 0 \), and the two-dimensional spacetime spanned by two null coordinates \((u,v)\) and its metric are denoted as \((M^2, h_{ab})\). Thus, the equations of gravitational field (2.1) can be written explicitly as
\[ -\frac{8\pi G}{n-2} T_{uv} = \left( r_{,uu} + \varphi_{,u} r_{,u} \right) \left[ 1 + \frac{2\tilde{\alpha}}{r^2} (k + 2e^\varphi r_{,u} r_{,v}) \right], \]
\[ -\frac{8\pi G}{n-2} T_{vv} = \left( r_{,vv} + \varphi_{,v} r_{,v} \right) \left[ 1 + \frac{2\tilde{\alpha}}{r^2} (k + 2e^\varphi r_{,u} r_{,v}) \right], \]
\[ \frac{8\pi G}{n-2} r^2 T_{uv} = r r_{,uv} + (n-3) r_{,u} r_{,v} + \frac{n-3}{2} k e^{-\varphi} + \frac{\tilde{\alpha}}{2r^2} (n-5) k^2 e^{-\varphi} + 4 r r_{,uv} (k + 2e^\varphi r_{,u} r_{,v}) \]
\[ + 4(n-5) r_{,u} r_{,v} (k + e^\varphi r_{,u} r_{,v}) - \frac{n-1}{2} \tilde{\Lambda} r e^{-\varphi}, \]  
(2.4)

where \( \tilde{\alpha} = (n-3)(n-4)\alpha, \tilde{\Lambda} = 2\Lambda/[(n-1)(n-2)] \).

The essential point of the integration method is that, similar to the case of Einstein gravity (1.3), one assumes the equations (2.4) of gravitational field can be cast into the form
\[ dE_{eff} = A \Psi_a dx^a + W dV, \]  
(2.5)

where \( A = V_{n-2}^{k} r^{n-2} \) and \( V = V_{n-2}^{k} r^{n-1}/(n-1) \) are area and volume of the \((n-2)\)-dimensional space with radius \( r \), and energy supply vector \( \Psi \) and energy density \( W \) are defined on \((M^2, h_{ab})\) as in the case of Einstein gravity. The right hand side in (2.5) can be explicitly expressed as
\[ A \Psi_a dx^a + W dV = A(u,v) du + B(u,v) dv, \]  
(2.6)

where
\[ A(u,v) = V_{n-2}^{k} r^{n-2} e^\varphi (r_{,u} T_{uv} - r_{,v} T_{uu}), \]  
(2.7)
\[ B(u,v) = V_{n-2}^{k} r^{n-2} e^\varphi (r_{,v} T_{uv} - r_{,u} T_{uu}). \]  
(2.8)
With the equations in (2.4), we can express $A$ and $B$ in terms of geometric quantities as

\[
A(u,v) = \frac{V_k}{8\pi G} e^{\varphi}(n-2)r^{n-4} \left\{ \frac{e^{-\varphi}}{2r^2} \left[ -(n-1)\tilde{\Lambda}r^4 + (n-3) r^2 (k + 2e^{\varphi} r_u r_w) 
+ (n-5)\tilde{\alpha}r (k + 2e^{\varphi} r_u r_w) 
+ rr_{uv} \left[ 1 + \frac{2\tilde{\alpha}}{r^2} (k + 2e^{\varphi} r_u r_w) \right] \right] 
\right\},
\]

\[
B(u,v) = \frac{V_k}{8\pi G} e^{\varphi}(n-2)r^{n-4} \left\{ \frac{e^{-\varphi}}{2r^2} \left[ -(n-1)\tilde{\Lambda}r^4 + (n-3) r^2 (k + 2e^{\varphi} r_u r_w) 
+ (n-5)\tilde{\alpha}r (k + 2e^{\varphi} r_u r_w) 
+ rr_{uv} \left[ 1 + \frac{2\tilde{\alpha}}{r^2} (k + 2e^{\varphi} r_u r_w) \right] \right] 
\right\},
\]

(2.9)

Now we try to derive the generalized Misner-Sharp energy by integrating the equation (2.5). Clearly, if it is integrable, the following integrable condition has to be satisfied

\[
\frac{\partial A(u,v)}{\partial v} = \frac{\partial B(u,v)}{\partial u}.
\]

(2.10)

It is easy to check that $A$ and $B$ given in (2.9) indeed satisfy the integrable condition (2.10). Thus directly integrating (2.5) gives the generalizing Misner-Sharp energy

\[
E_{eff} = \int A(u,v)du + \int \left[ B(u,v) - \frac{\partial}{\partial v} \int A(u,v)du \right] dv 
= \frac{(n-2)V_k}{16\pi G} \left[ -\tilde{\Lambda}r^2 + (k + 2e^{\varphi} r_u r_w) + \tilde{\alpha}r^{-2} (k + 2e^{\varphi} r_u r_w)^2 \right].
\]

(2.11)

Note that here the second term in the first line of (2.11) in fact vanishes and we have fixed an integration constant so that $E_{eff}$ reduces to the Misner-Sharp energy in Einstein gravity when $\tilde{\alpha} = 0$. In addition, the generalized Misner-Sharp energy can be rewritten in a covariant form

\[
E_{eff} = \frac{(n-2)V_k}{16\pi G} \left[ -\tilde{\Lambda}r^2 + (k - h^{ab}D_a r D_br) + \tilde{\alpha}r^{-2} (k - h^{ab}D_a r D_br)^2 \right].
\]

(2.12)

This is the generalized Misner-Sharp energy given by Maeda and Nozawa in [22] through Kodama conserved charge method.

III. GENERALIZED MISNER-SHARP ENERGY IN $f(R)$ GRAVITY: THE GENERAL CASE

In this section, we first try to derive the generalized Misner-Sharp energy in $f(R)$ gravity by using the integration method. Then we consider the conserved charge method. Here we
consider the four-dimensional case with spherical symmetry, and the line element is
\[ ds^2 = -2e^{-\varphi(u,v)}du dv + r^2(u,v)(d\theta^2 + \sin^2 \theta d\phi^2). \]  
(3.1)

The action of the \( f(R) \) gravity in the metric formalism is
\[ S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) + S_{\text{matter}}, \]  
(3.2)

Varying the action with respect to metric yields equations of gravitational field
\[ f_R R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - \nabla_\mu \nabla_\nu f_R + g_{\mu\nu} \Box f_R = 8\pi G T_{\mu\nu}, \]  
(3.3)

where \( f_R = df(R)/dR \), and \( T_{\mu\nu} \) is the energy-momentum tensor for matter field from \( S_{\text{matter}} \).

Note that the field equations also can be rewritten in the form
\[ G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{1}{f_R} \left[ \frac{1}{2} g_{\mu\nu} (f - R f_R) + \nabla_\mu \nabla_\nu f_R - g_{\mu\nu} \Box f_R \right] + \frac{8\pi G}{f_R} T_{\mu\nu}. \]  
(3.4)

In this case, the right hand side can be regarded as an effective energy-momentum tensor.

A. Integration method

In the metric (3.1), the field equations (3.3) can be explicitly expressed as
\[ 8\pi G T_{uu} = -2f_R \left( \frac{\varphi_u r_u}{r} + \frac{r_{uu}}{r} \right) - f_R r_{uu} - \varphi_u f_R, \]
\[ 8\pi G T_{vv} = -2f_R \left( \frac{\varphi_v r_v}{r} + \frac{r_{vv}}{r} \right) - f_R r_{vv} - \varphi_v f_R, \]
\[ 8\pi G T_{uv} = f_R \varphi_{uv} - 2f_R \frac{r_{uv}}{r} + \frac{1}{2} f e^{-\varphi} + f_R r_{uv} + \frac{2r_u f_R r_v + 2r_v f_R r_u}{r}. \]  
(3.5)

In this case, following the method discussed in the previous section, we obtain
\[ A(u, v) = 4\pi r^2 e^\varphi (r_u T_{uv} - r_v T_{uu}) \]
\[ = \frac{r^2 e^\varphi}{2G} \left[ r_u \left( f_R \varphi_{uv} - 2f_R \frac{r_{uv}}{r} + \frac{1}{2} f e^{-\varphi} + f_R r_{uv} + \frac{2r_u f_R r_v + 2r_v f_R r_u}{r} \right) \right. \]
\[ + r_v \left( 2f_R \frac{\varphi_u r_u}{r} + \frac{r_{uu}}{r} + f_R r_{uu} + \varphi_u f_R \right) \right], \]
\[ B(u, v) = 4\pi r^2 e^\varphi (r_v T_{uv} - r_u T_{vv}) \]
\[ = \frac{r^2 e^\varphi}{2G} \left[ r_v \left( f_R \varphi_{uv} - f_R \frac{2r_{uv}}{r} + \frac{1}{2} f e^{-\varphi} + f_R r_{uv} + \frac{2r_u f_R r_v + 2r_v f_R r_u}{r} \right) \right. \]
\[ + r_u \left( 2f_R \frac{\varphi_v r_v}{r} + \frac{r_{vv}}{r} + f_R r_{vv} + \varphi_v f_R \right) \right]. \]  
(3.6)
Checking the integrable condition, however, unlike the case of Gauss-Bonnet gravity, we find that it is not always satisfied for the $f(R)$ gravity:

$$\frac{\partial A(u,v)}{\partial v} - \frac{\partial B(u,v)}{\partial u} = -r^2 e^\varphi [(\varphi_{,uu} r_{,u} + r_{,uu}) (f_{R,vv} + \varphi_{,v} f_{R,v}) - (\varphi_{,v} r_{,v} + r_{,vv})] / 2G.$$  \hfill (3.7)

If the right hand side of the above equation vanishes, in principle, one is able to obtain a generalized Misner-Sharp energy by integrating (2.5). On the other hand, if the integrable condition is not satisfied, one is not able to rewrite the form $Adu + Bdv$ as a total differential form, which implies that generalized Misner-Sharp energy does not exist in this case. Now we assume that the integrable condition is satisfied, that is to say, the right hand side of the equation (3.7) vanishes. Thus, we can obtain the generalized Misner-Sharp energy for the $f(R)$ gravity as

$$E_{\text{eff}} = \int A(u,v) du + \int \left[ B(u,v) - \frac{\partial}{\partial v} \int A(u,v) du \right] dv$$

$$= \frac{r}{2G} \left[ (1 + 2e^\varphi r_{,uu}) f_R + \frac{1}{6} r^2 (f - f_R R) + re^\varphi (f_{R,u} r_{,v} + f_{R,v} r_{,uu}) \right]$$

$$- \frac{1}{2G} \int [f_{R,u} e^\varphi (r^2 r_{,v})_{,u} + f_{R,v} (r - \frac{1}{6} r^3 R) + f_{R,v} r^2 (r_{,u} e^\varphi)_{,u}] du$$

$$= \frac{r}{2G} \left[ (1 - h^{ab} \partial_a r \partial_b r) f_R + \frac{1}{6} r^2 (f - f_R R) - rh^{ab} \partial_a f_{R,b} \right]$$

$$- \frac{1}{2G} \int \left[ f_{R,u} e^\varphi (r^2 r_{,v})_{,u} + f_{R,v} (r - \frac{1}{6} r^3 R) + f_{R,v} r^2 (r_{,u} e^\varphi)_{,u} \right] du,$$ \hfill (3.8)

where we have used $R = 2[\frac{1}{r} + e^\varphi (2 \frac{r^2 r_{,v}}{r^2} - \varphi_{,uv} + 4 \frac{r_{,uv}}{r})]$ and $f_{,u} = f_{R,R,u}$. We see that $E_{\text{eff}}$ reduces to the Misner-Sharp energy in the Einstein gravity when $f_R = 1$. Unfortunately, we see from (3.8) that due to the existence of the integration in (3.8), we cannot arrive at an explicit quasi-local energy for the general case. In the next section, however, we will show that the integration can be carried out in some special cases. Before that, we will first obtain the same result using the conserved charge method in the next subsection.

**B. Conserved charge method**

In a spherically symmetric spacetime, one can define a Kodama vector. The energy-momentum tensor together with the Kodama vector can lead to a conserved current, whose corresponding conserved charge is just the Misner-Sharp energy in Einstein gravity. In Gauss-Bonnet gravity, Maeda and Nozawa \[22\] obtain the generalized Misner-Sharp energy...
with help of this method. In this section we would like to see whether the conserved current method leads to a generalized Misner-Sharp energy for the \( f(R) \) gravity.

For a spherically symmetric spacetime, one can define the Kodama vector as \[ K^\mu = -\epsilon^{\mu\nu} \nabla_\nu r, \] (3.9)
where \( \epsilon_{\mu\nu} = \epsilon_{ab}(dx^a)_\mu(dx^b)_\nu \), and \( \epsilon_{ab} \) is the volume element of \((M^2, h_{ab})\). For the spherically symmetric spacetime (3.1), we have

\[ K^\mu = e^\phi r_v \left( \frac{\partial}{\partial u} \right)^\mu - e^\phi r_u \left( \frac{\partial}{\partial v} \right)^\mu. \] (3.10)

Conservation of the energy-momentum tensor for matter fields \( T_{\mu\nu} \) in (3.3) guarantees that the left hand side of the equation (3.3) is also divergence-free, which can be easily checked by using the identity

\[ (\Box \nabla_\nu - \nabla_\nu \Box) F = R_{\mu\nu} \nabla^\mu F, \] (3.11)
where \( F \) is an arbitrary scalar function. With the Kodama vector, define an energy current as

\[ J^\mu = -T^\mu_\nu K^\nu. \] (3.12)
However, we find that unlike in the cases of Einstein gravity and Gauss-Bonnet gravity [22], the energy current defined in (3.12) is not always divergence-free for the \( f(R) \) gravity except the case with condition

\[ \nabla_\mu \nabla_\nu f_R \nabla^\mu K^\nu = 0. \] (3.13)
Namely, if the constraint equation (3.13) is satisfied, the energy current is divergence-free

\[ \nabla_\mu J^\mu = 0. \] (3.14)
In this case, we can define an associated conserved charge

\[ Q_J = \int_\Sigma J^\mu d\Sigma_\mu, \] (3.15)
where \( \Sigma \) is some hypersurface and \( d\Sigma_\mu = \sqrt{-g} dx^\nu dx^\lambda dx^\rho \delta_{\mu\nu\lambda} \) is a directed surface line element on \( \Sigma \). By using the line element in (3.1) and equations in (3.5), we obtain

\[ Q_J = \int_\Sigma J^\mu d\Sigma_\mu = \int_\Sigma J^\mu d\Sigma_\mu \]
\[ = \frac{r}{2G} \left[ (1 - h^{ab} \partial_a r \partial_b r) f_R + \frac{1}{6} r^2 (f - f_R R) - rh^{ab} \partial_a f_R \partial_b r \right] 
- \frac{1}{2G} \int \left[ f_{R, u} e^\phi (r^2 r_v),_u + f_{R, v} e^\phi (r - \frac{1}{6} r^3 R) + f_{R, v} r^2 (r_u e^\phi),_u \right] du, \] (3.16)
where we have chosen the hypersurface $\Sigma$ with a given $v$. One can immediately see that the charge $Q_J$ is precisely the generalized Misner-Sharp energy $E_{\text{eff}}$ given by the integration method in (3.8). Again, this is not a satisfying situation since we cannot express the generalized Misner-Sharp energy in a true quasi-local form.

IV. GENERALIZED MISNER-SHARP ENERGY IN F(R) GRAVITY: SPECIAL CASES

The existence of the integration in (3.8) is painful. An interesting question is whether it will be absent in some special cases. The answer is positive. We will here discuss two special cases. One is the homogeneous and isotropic FRW universe and the other is the static spherically symmetric spacetime with constant scalar curvature.

A. FRW Universe

Consider the metric

$$ds^2 = -dt^2 + e^{2\psi(t, \rho)}d\rho^2 + r^2(t, \rho)(d\theta^2 + \sin^2 \theta d\phi^2).$$

(4.1)

In this metric, the Kodama vector is

$$K^a = e^{-\psi}r_o \left( \frac{\partial}{\partial t} \right)^a - e^{-\psi}r_t \left( \frac{\partial}{\partial \rho} \right)^a.$$  

(4.2)

Following the same procedure, we can rewrite the equation in (2.6) as

$$A\Psi_d dx^a + WdV = A(t, \rho)dt + B(t, \rho)d\rho.$$  

(4.3)

where

$$A(t, \rho) = 4\pi r^2 e^{-2\psi} (T_{t\rho} r_{\rho} - T_{\rho \rho} r_t),$$

$$B(t, \rho) = 4\pi r^2 (T_{t\rho} r_{\rho} - T_{\rho \rho} r_t).$$  

(4.4)
With the equations of gravitational field of the $f(R)$ gravity, $A$ and $B$ can be expressed in terms of geometric quantities. One can then arrive at the generalized Misner-Sharp energy

$$E_{\text{eff}} = \int B(t, \rho) d\rho + \int [A(t, \rho) - \frac{\partial}{\partial t} \int B(t, \rho) d\rho] dt$$

$$= \frac{r}{2G} \left[ (1 - h^{ab} \partial_a r \partial_b r) f_R + \frac{r}{6} (f - f_R R) - r h^{ab} \partial_a f_R \partial_b r \right]$$

$$+ \frac{1}{2G} \int \left\{ f_{R,\rho} \left[ (-e^{-2\psi} r^2 e \psi_{,\rho} + e^{-2\psi} r^2 e_{,\rho \rho} - r^2 e_{,t} \psi_{,t}) - r (1 + r^2 - e^{-2\psi} r^2) + \frac{1}{6} r^3 R \right]$$

$$+ r^2 f_{R, t} (\psi_{,t} r_{,\rho} - r_{,t \rho}) \right\} d\rho,$$  \hspace{1cm} (4.5)

where the integrable condition is assumed to be satisfied

$$\frac{\partial A(t, \rho)}{\partial \rho} - \frac{\partial B(t, \rho)}{\partial t} = 0.$$  \hspace{1cm} (4.6)

Now we express the generalized Misner-Sharp energy in a FRW metric

$$ds^2 = -d\tau^2 + \frac{\rho^2(t)d\rho^2}{1-k\rho^2} + r^2(t, \rho)(d\theta^2 + \sin^2 \theta d\phi^2),$$  \hspace{1cm} (4.7)

where $r(t, \rho) \equiv a(t) \rho$, and $e^{\psi(t, \rho)} = \frac{a(t)}{\sqrt{1-k\rho^2}}$ corresponding to \[4.1\]. Because in the FRW universe, the Ricci scalar $R = 6(\frac{k}{a^2} + \frac{a^2}{a^2} + \frac{a}{a})$ just depends on time, we can check that the integrand in the final step in \[4.5\] exactly vanishes. Thus, the generalized Misner-Sharp energy $E_{\text{eff}}$ in this case can be explicitly expressed as

$$E_{\text{eff}} = \frac{r}{2G} \left[ (1 - h^{ab} \partial_a r \partial_b r) f_R + \frac{1}{6} r^2 (f - f_R R) - r h^{ab} \partial_a f_R \partial_b r \right]$$

$$= \frac{r^3}{2G} \left( \frac{1}{r_A^2} f_R + \frac{1}{6} (f - f_R R) + H f_{R, t} \right),$$  \hspace{1cm} (4.8)

where $r_A = 1/\sqrt{H^2 + \frac{k}{a^2}}$, which in fact, is the location of apparent horizon of the FRW universe.

### B. Static spherically symmetric case

The general line element of a static spherically symmetric spacetime can be written down as

$$ds^2 = -\lambda(r) d\tau^2 + g(r) dr^2 + r^2 d\Omega^2_2,$$  \hspace{1cm} (4.9)

where $\lambda$ and $g$ are two functions of the radial coordinate. In this case, the Kodama vector is

$$K^\mu = \frac{1}{\sqrt{g\lambda}} \left( \frac{\partial}{\partial t} \right)^\mu.$$  \hspace{1cm} (4.10)
Using the static spherically symmetric metric, we can easily check that the constraint (3.13) is naturally satisfied

\[ \nabla_\mu \nabla_\nu f_R \nabla^\mu K^\nu = 0. \tag{4.11} \]

Following the same procedure, we can rewrite the equation in (2.6) as

\[ dE_{\text{eff}} = A(t, r) dt + B(t, r) dr, \tag{4.12} \]

where

\[ A(t, r) = \frac{4\pi r^2}{g} (T_{tr} r_r - T_{rr} r_t) = 0, \tag{4.13} \]

\[ B(t, r) = \frac{4\pi r^2}{\lambda} (T_{tr} r_r - T_{rr} r_t) \]
\[ = \frac{r^2}{2G} \left( \frac{1}{2} (f - f_R R) + \frac{1}{r^2} (1 + \frac{rg'}{g^2} - \frac{1}{g}) f_R + f_{R,r} \left( \frac{g'}{2g^2} - \frac{2}{rg} \right) - \frac{1}{g} f_{R,rr} \right), \tag{4.14} \]

where a prime denotes the derivative with respect to \( r \). Integrating (4.12) gives the generalized Misner-Sharp energy

\[ E_{\text{eff}} = \int B(t, r) dr = \frac{r}{2G} \left( (1 - h^a_b \partial_a r \partial_b r) f_R + \frac{r^2}{6} (f - f_R R) - r h^a_b \partial_a f_R \partial_b r \right) \]
\[ - \frac{1}{2G} \int dr \left( \frac{r^2 g'}{2g^2} + r - \frac{r}{g} - \frac{1}{6} r^3 R \right) f_{R,r}. \tag{4.15} \]

Clearly the integral in (4.15) will be absent in two cases, one is \( \frac{r^2 g'}{2g^2} + r - \frac{r}{g} - \frac{1}{6} r^3 R = 0 \), the other is \( f_{R,r} = 0 \). We here consider the latter case. The trivial case with \( f(R) = R \) naturally satisfies the condition. In this case, \( f_R = 1 \), and (4.15) gives the Misner-Sharp energy. A little nontrivial case is that the solution is a constant curvature one with scalar curvature \( R = R_0 = \text{const.} \). In that case, \( f_{R,r} = 0 \), and (4.15) reduces to

\[ E_{\text{eff}} = \frac{r}{2G} \left( (1 - h^a_b \partial_a r \partial_b r) f_R + \frac{r^2}{6} (f - f_R R) \right). \tag{4.16} \]

Note that here \( R, f_R \) and \( f \) are all constants. Compare to the Misner-Sharp energy (1.2) and the (2.12), we can see clearly that this expression is nothing but the generalized Misner-Sharp energy with a cosmological constant. Here the effective Newtonian constant is \( G/f_R \) and the effective cosmological constant \( \Lambda = -(f - R f_R)/(2f_R) \).

V. CONCLUSION AND DISCUSSION

The Misner-Sharp quasi-local energy plays a key role in understanding the “unified first law”, the relation between the first law of thermodynamics and dynamical equations of
gravitational field and thermodynamics of apparent horizon in FRW universe, etc. In this paper we studied the generalized Misner-Sharp energy in $f(R)$ gravity by two approaches. One is the integration method and the other is the conserved charge method. It turns out that in general we cannot arrive at an explicit expression for the generalized Misner-Sharp energy in a quasi-local form [see (3.8) and (4.5)], even assuming the integrable condition (2.10) is satisfied. This situation is quite different from the cases of Einstein gravity and Gauss-Bonnet gravity. This is certainly related to the fact that for the $f(R)$ gravity, the energy current (3.12) is not always divergence-free, while it does in Einstein and Gauss-Bonnet gravities. The existence of the conserved current requires (3.13) is satisfied.

Some remarks on our results are in order.

(1) The relation between the two methods to derive the generalized Misner-Sharp energy. We obtained the same generalized Misner-Sharp energy by employing two methods: integration and conserved charge methods. At first glance, these two methods looks different, but in fact, they are equivalent. First let us notice that the constraint equation (3.7) has a relation to the one (3.13):

$$\frac{\partial A(u, v)}{\partial v} - \frac{\partial B(u, v)}{\partial u} = -e^{-\phi}r^2 \nabla_{\mu} \nabla_{\nu} f R \nabla_{\mu} K^v/2. \quad (5.1)$$

Namely these two integrable conditions are equivalent. Second, substituting the conserved current in (3.12) into (3.15), we can write the associated charge

$$Q_J = \int_{\Sigma} J^\mu d\Sigma_{\mu},$$

$$= \int 4\pi r^2 e^\phi (r_{uv} T_{wu} - r_{vw} T_{wu}) du, \quad (5.2)$$

where the integrand is precisely $A(u, v)$ in (3.6). Thus, we have finished our proof of the equivalence of the two methods.

In addition, the useful components of $K_{\mu\nu} \equiv \nabla_{\mu} K_{\nu}$ and $F_{\mu\nu} \equiv \nabla_{\mu} \nabla_{\nu} F$ in coordinates $(t, \rho, \theta, \phi)$ are

$$K_{tt} = e^{-\psi}(\psi_t r_{t\rho} - r_{t\rho}), \quad K_{t\rho} = -e^{\psi} r_{t\rho}, \quad K_{\rho t} = -e^{\psi} r_{\rho t}, \quad K_{\rho\rho} = e^{\psi}(\psi_{t\rho} r_{\rho t} - r_{t\rho}),$$

$$F_{tt} = F_{\rho t}, \quad F_{t\rho} = F_{\rho t}, \quad F_{\rho\rho} = e^{-2\psi} [\psi_{t\rho} e^{-2\psi} F_{\rho t} + \psi_{\rho t} F_{\rho t}], \quad (5.3)$$

With the help of those quantities, we can easily check that the constraint equation (3.13) is satisfied for the FRW universe.
(2) The meaning of the generalized Misner-Sharp energy in FRW universe. To see clearly this, let us write down the Friedmann equations of the \( f(R) \) gravity

\[
H^2 + \frac{k}{a^2} = \frac{1}{6f_R} [(f_R R - f) - 6H \partial_t f_R + 16\pi G \tilde{\rho}],
\]

\[
\dot{H} - \frac{k}{a^2} = \frac{1}{2f_R} [H \partial_t f_R - \partial_t \partial_t f_R - 8\pi G (\tilde{\rho} + \tilde{p})],
\]

(5.4)

where \( \tilde{\rho} \) and \( \tilde{p} \) are energy density and pressure of the ideal fluid in the universe. With the first line in (5.4), we can easily see that the generalized Misner-Sharp energy in (4.8) can be rewritten as

\[
E_{eff} = \tilde{\rho} V,
\]

(5.5)

where \( V = 4\pi r^3/3 \) is the volume of a sphere with radius \( r \). Therefore, in fact, the generalized Misner-Sharp energy in the FRW universe is nothing but the total matter energy within a sphere with radius \( r \).

(3) Thermodynamics of apparent horizon in the \( f(R) \) gravity. On the apparent horizon of a FRW universe, the energy crossing the apparent horizon within time interval \( dt \) is [18, 30]

\[
\delta Q = dE_{eff}|_{r_A} = A(\tilde{\rho} + \tilde{p}) H r_A dt.
\]

(5.6)

Note that the horizon entropy of the \( f(R) \) gravity is \( S = \frac{A}{4G} f_R = \pi r_A^2 f_R / G \), while the temperature of the apparent horizon is [18, 27, 28]: \( T = \frac{1}{2\pi r_A} \). Obviously, the usual Clausius relation \( \delta Q = T dS \) does not hold. On the other hand, an internal entropy production is needed to balance the energy conservation, \( \delta Q = T dS + T d_i S \) with

\[
d_i S = \pi r_A [H r_A^3 (H \partial_t f_R - \partial_t \partial_t f_R) - \partial_t f_R r_A]/G.
\]

(5.7)

This is an effect of the non-equilibrium thermodynamics of spacetime [29, 30, 31, 32].

(4) The case of scalar-tensor gravity. Indeed the \( f(R) \) gravity is quite similar to scalar-tensor gravity theory in some sense [24]. Our conclusion on the \( f(R) \) gravity therefore also holds for scalar-tensor gravity. In particular, the existence of a generalized Misner-Sharp energy has to obey a constraint condition as well for the scalar-tensor theory. However, in the FRW universe, a simple expression for the generalized Misner-Sharp energy can be given, which can be seen in appendix A.
Acknowledgments

YPH thanks D. Orlov for useful discussions. RGC and YPH are supported partially by grants from NSFC, China (No. 10525060, No. 108215504 and No. 10975168) and a grant from MSTC, China (No. 2010CB833004). NO was supported in part by the Grant-in-Aid for Scientific Research Fund of the JSPS No. 20540283, and also by the Japan-U.K. Research Cooperative Program. This work is completed during RGC’s visit to Kinki University, Japan with the support of JSPS invitation fellowship.

APPENDIX A: GENERALIZED MISNER-SHARP ENERGY OF SCALAR-TENSOR THEORY IN FRW UNIVERSE

The Lagrangian of a generic scalar-tensor gravity in 4-dimensional space-time can be written as

\[ L = \frac{1}{16\pi} F(\phi) R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \nabla^2 \phi + V(\phi) + L_m. \]  

where we set Newtonian constant \( G = 1 \), \( F(\phi) \) is an arbitrary positive continuous function of the scalar field \( \phi \), \( V(\phi) \) is its potential, and \( L_m \) denotes the Lagrangian of other matter fields. Varying the associated action with respect to spacetime metric and the scalar field yields equations of motion

\[ F G_{\mu\nu} + g_{\mu\nu} \Box F - \nabla^2_F = 8\pi \left( T^{\phi}_{\mu\nu} + T^m_{\mu\nu} \right), \]  

\[ \Box \phi - V'(\phi) + \frac{1}{16\pi} F'(\phi) R = 0. \]

where \( T^{m}_{\mu\nu} \) is the energy-momentum tensor of matter fields, and \( T^\phi_{\mu\nu} \) is defined as

\[ T^\phi_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi + V(\phi) \right). \]

Note that here \( T^\phi_{\mu\nu} \) is not the energy-momentum tensor of the scalar field. Similar to the case of f(R) gravity, we find that the current \( J^\mu = -T^\mu_{\nu}(m) K^\nu \) is not always divergence-free unless the condition is satisfied

\[ (\nabla_\mu \nabla_\nu F + 8\pi \partial_\mu \phi \partial_\nu \phi) \nabla^\mu K^\nu = 0. \]

However, we can easily check that the condition \( A5 \) can be always satisfied for the FRW universe (4.7) by using (2.8). Some useful components of equations of gravitational field
\( (A2) \) are given by
\[
8\pi T^m_{\mu} = 3F\left(\frac{k}{a^2} + H^2\right) + 3HF - 8\pi \left(\frac{1}{2} \dot{\phi}^2 + V\right), \quad 8\pi T^m_{\nu} = 0,
\]
\[
8\pi T^m_{\rho\rho} = \frac{a^2}{1 - k\rho^2} \left[ - F\left(\frac{k}{a^2} + H^2 + \frac{2\dot{a}}{a}\right) - \ddot{F} - 2HF + 8\pi \left(\frac{-1}{2} \dot{\phi}^2 + V\right)\right].
\]

In this case, corresponding \( A \) and \( B \) in (4.3) for the scalar-tensor theory, respectively, are
\[
A(t, \rho) = \frac{1}{2} H r^3 \left[ F\left(\frac{k}{a^2} + H^2 + \frac{2\ddot{a}}{a}\right) + \ddot{F} + 2HF - 8\pi \left(\frac{-1}{2} \dot{\phi}^2 + V\right)\right],
\]
\[
B(t, \rho) = \frac{1}{2} \rho^2 a^3 \left[ 3F\left(\frac{k}{a^2} + H^2\right) + 3HF - 8\pi \left(\frac{1}{2} \dot{\phi}^2 + V\right)\right].
\]

They obey the integrable condition (4.6). With these quantities we can obtain the generalized Misner-Sharp energy of the scalar-tensor theory in the FRW universe
\[
E_{eff} = \int B(t, \rho) d\rho = \frac{r^3}{2} \left[ F\left(\frac{k}{a^2} + H^2\right) + HF - 8\pi \left(\frac{1}{3} \frac{1}{2} \dot{\phi}^2 + V\right)\right].
\]

Comparing this with the first equation in (A6), one can immediately see that the generalized Misner-Sharp energy is just the total matter energy in a sphere with radius \( r \) in the FRW universe.

---

[1] C. W. Misner, K. S. Thorne and J. A. Wheeler, Gravitation (Freeman, San Francisco 1973); R. M. Wald, General Relativity (Chicago, The University of Chicago Press, 1984); L. D. Landau and E. M. Lifshitz, The Classical Theory of Fields (Beijing, Butterworth-Heinemann, 1999).

[2] C. Moller, Annals Phys. 4 (1958) 347; C. Moller, Annals Phys. 12 (1961) 118;

[3] L. B. Szabados, Living Rev. Rel. 7, 4 (2004).

[4] R. L. Arnowitt, S. Deser and C. W. Misner, Phys. Rev. 116, 1322 (1959); R. L. Arnowitt, S. Deser and C. W. Misner, [arXiv:gr-qc/0405109].

[5] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Proc. Roy. Soc. Lond. A 269 (1962) 21; R. K. Sachs, Proc. Roy. Soc. Lond. A 270, 103 (1962); R. P. Geroch and J. Winicour, J. Math. Phys. 22, 803 (1981).

[6] J. D. Brown and J. W. York, Phys. Rev. D 47, 1407 (1993) [arXiv:gr-qc/9209012].

[7] C. W. Misner and D. H. Sharp, Phys. Rev. 136, B571 (1964).

[8] S. Hawking, J. Math. Phys. 9, 598 (1968).
[9] S. A. Hayward, Phys. Rev. D 49, 831 (1994) [arXiv:gr-qc/9303030].

[10] C. M. Chen and J. M. Nester, Class. Quant. Grav. 16, 1279 (1999) [arXiv:gr-qc/9809020].

[11] S. A. Hayward, Phys. Rev. D 49, 831 (1994) [arXiv:gr-qc/9303030].

[12] S. A. Hayward, Phys. Rev. D 53, 1938 (1996) [arXiv:gr-qc/9408002].

[13] S. A. Hayward, Class. Quant. Grav. 15, 3147 (1998) [arXiv:gr-qc/9710089]; S. Mukohyama and S. A. Hayward, Class. Quant. Grav. 17, 2153 (2000) [arXiv:gr-qc/9905085].

[14] S. A. Hayward, S. Mukohyama and M. C. Ashworth, Phys. Lett. A 256, 347 (1999) [arXiv:gr-qc/9810006].

[15] R. G. Cai and L. M. Cao, Phys. Rev. D 75, 064008 (2007) [arXiv:gr-qc/0611071].

[16] R. G. Cai and L. M. Cao, Nucl. Phys. B 785, 135 (2007) [arXiv:hep-th/0612141]; A. Sheykhi, B. Wang and R. G. Cai, Nucl. Phys. B 779, 1 (2007) [arXiv:hep-th/0701198]; A. Sheykhi, B. Wang and R. G. Cai, Phys. Rev. D 76, 023515 (2007) [arXiv:hep-th/0701261]; R. G. Cai, L. M. Cao and Y. P. Hu, [arXiv:0807.1232] [hep-th]; R. G. Cai, Prog. Theor. Phys. Suppl. 172, 100 (2008) [arXiv:0712.2142 [hep-th]]. X. H. Ge, Phys. Lett. B 651, 49 (2007) [arXiv:hep-th/0703253]; Y. Gong and A. Wang, Phys. Rev. Lett. 99, 211301 (2007) [arXiv:0704.0793 [hep-th]]. S. F. Wu, G. H. Yang and P. M. Zhang, [arXiv:0710.5394] [hep-th]. S. F. Wu, B. Wang and G. H. Yang, Nucl. Phys. B 799, 330 (2008) [arXiv:0711.1209] [hep-th]; S. F. Wu, B. Wang, G. H. Yang and P. M. Zhang, [arXiv:0801.2688] [hep-th]; T. Zhu, J. R. Ren and S. F. Mo, [arXiv:0805.1162] [gr-qc]; M. Akbar, Chin. Phys. Lett. 25, 4199 (2008) [arXiv:0808.0169] [gr-qc]]. K. Bamba and C. Q. Geng, Phys. Lett. B 679, 282 (2009) [arXiv:0901.1509] [hep-th]].

[17] R. G. Cai, L. M. Cao, Y. P. Hu and S. P. Kim, Phys. Rev. D 78, 124012 (2008) [arXiv:0810.2610 [hep-th]].

[18] R. G. Cai and S. P. Kim, JHEP 0502, 050 (2005) [arXiv:hep-th/0501055].

[19] M. Akbar and R. G. Cai, Phys. Rev. D 75, 084003 (2007) [arXiv:hep-th/0609128].

[20] T. Padmanabhan, Class. Quant. Grav. 19, 5387 (2002) [arXiv:gr-qc/0204019]; T. Padmanabhan, Phys. Rept. 406, 49 (2005) [arXiv:gr-qc/0311036]; T. Padmanabhan, Int. J. Mod. Phys. D 15, 1659 (2006) [arXiv:gr-qc/0606061]; A. Paranjape, S. Sarkar and T. Padmanabhan, Phys. Rev. D 74, 104015 (2006) [arXiv:hep-th/0607240]; D. Kothawala, S. Sarkar and T. Padmanabhan, Phys. Lett. B 652, 338 (2007) [arXiv:gr-qc/0701002]; M. Akbar, Chin. Phys. Lett. 24, 1158 (2007) [arXiv:hep-th/0702029]; M. Akbar and A. A. Siddiqui, Phys. Lett. B 656, 217
(2007); R. G. Cai and N. Ohta, arXiv: 0910.2307.

[21] H. Maeda, Phys. Rev. D 73, 104004 (2006) [arXiv:gr-qc/0602109].

[22] H. Maeda and M. Nozawa, Phys. Rev. D 77, 064031 (2008) [arXiv:0709.1199 [hep-th]].

[23] Y. Gong and A. Wang, Phys. Rev. Lett. 99, 211301 (2007) [arXiv:0704.0793 [hep-th]].

[24] T. P. Sotiriou and V. Faraoni, arXiv:0805.1726 [gr-qc]; V. Faraoni, arXiv:0810.2602 [gr-qc]; S. Nojiri and S. D. Odintsov, eConf C0602061, 06 (2006) [Int. J. Geom. Meth. Mod. Phys. 4, 115 (2007)] [arXiv:hep-th/0601213]; S. Nojiri and S. D. Odintsov, arXiv:0807.0685 [hep-th].

[25] H. Kodama, Prog. Theor. Phys. 63, 1217 (1980).

[26] M. Minamitsuji and M. Sasaki, Phys. Rev. D 70, 044021 (2004) [arXiv:gr-qc/0312109].

[27] R. G. Cai, L. M. Cao and Y. P. Hu, Class. Quant. Grav. 26, 155018 (2009) [arXiv:0809.1554 [hep-th]].

[28] R. Li, J. R. Ren and D. F. Shi, Phys. Lett. B 670, 446 (2009) [arXiv:0812.4217 [gr-qc]].

[29] C. Eling, R. Guedens and T. Jacobson, Phys. Rev. Lett. 96, 121301 (2006) [arXiv:gr-qc/0602001]; T. Jacobson, Phys. Rev. Lett. 75, 1260 (1995) [arXiv:gr-qc/9504004].

[30] M. Akbar and R. G. Cai, Phys. Lett. B 635, 7 (2006) [arXiv:hep-th/0602150]; M. Akbar and R. G. Cai, Phys. Lett. B 648, 243 (2007) [arXiv:gr-qc/0612089].

[31] E. Elizalde and P. J. Silva, Phys. Rev. D 78, 061501 (2008) [arXiv:0804.3721 [hep-th]]; S. F. Wu, G. H. Yang and P. M. Zhang, Prog. Theor. Phys. 120, 615 (2008) [arXiv:0805.4044 [hep-th]]; K. Bamba, C. Q. Geng, S. Nojiri and S. D. Odintsov, arXiv:0909.4397 [hep-th].

[32] C. Eling, JHEP 0811, 048 (2008) [arXiv:0806.3165 [hep-th]]; G. Chirco and S. Liberati, arXiv:0909.4194 [gr-qc].