ABEL-JACOBI MAP AND CURVATURE OF THE PULLED BACK METRIC

INDRANIL BISWAS

Abstract. Let $X$ be a compact connected Riemann surface of genus at least two. The Abel-Jacobi map $\varphi : \text{Sym}^d(X) \to \text{Pic}^d(X)$ is an embedding if $d$ is less than the gonality of $X$. We investigate the curvature of the pull-back, by $\varphi$, of the flat metric on $\text{Pic}^d(X)$. In particular, we show that when $d = 1$, the curvature is strictly negative everywhere if $X$ is not hyperelliptic, and when $X$ is hyperelliptic, the curvature is nonpositive with vanishing exactly on the points of $X$ fixed by the hyperelliptic involution.

1. Introduction

Let $X$ be a compact connected Riemann surface of genus $g$, with $g \geq 2$. The gonality of $X$ is defined to be the smallest integer $\gamma_X$ such that there is a nonconstant holomorphic map from $X$ to $\mathbb{CP}^1$ of degree $\gamma_X$. Consider the Abel-Jacobi map $\varphi : \text{Sym}^d(X) \to \text{Pic}^d(X)$

that sends an effective divisor $D$ on $X$ of degree $d$ to the corresponding holomorphic line bundle $\mathcal{O}_X(D)$. If $d < \gamma_X$, then $\varphi$ is an embedding (Lemma 2.1). On the other hand, $\text{Pic}^d(X)$ is equipped with a flat Kähler form, which we will denote by $\omega_0$. So, $\varphi^*\omega_0$ is a Kähler form on $\text{Sym}^d(X)$, whenever $d < \gamma_X$. The Kähler metric $\varphi^*\omega_0$ on $\text{Sym}^d(X)$ is relevant in the study of abelian vortices (see [Ri], [BR] and references therein).

Our aim here is to study the curvature of this Kähler form $\varphi^*\omega_0$ on $\text{Sym}^d(X)$.

Consider the $g$–dimensional vector space $H^0(X, K_X)$ consisting of holomorphic one-forms on $X$. It is equipped with a natural Hermitian structure. Let

$$\mathbb{G} = \text{Gr}(d, H^0(X, K_X))$$

be the Grassmannian parametrizing all $d$ dimensional quotients of $H^0(X, K_X)$. The Hermitian structure on $H^0(X, K_X)$ produces a Hermitian structure on the tautological vector bundle on $\mathbb{G}$ of rank $d$; this tautological bundle on $\mathbb{G}$ of rank $d$ will be denoted by $V$. The Hermitian structure on $H^0(X, K_X)$ also gives a Fubini–Study Kähler form on $\mathbb{G}$.

There is a natural holomorphic map

$$\rho : \text{Sym}^d(X) \to \mathbb{G}$$

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We prove that the holomorphic Hermitian vector bundle \( \rho^* V \rightarrow \text{Sym}^d(X) \) is isomorphic to the holomorphic cotangent bundle \( \Omega_{\text{Sym}^d(X)} \) equipped with the Hermitian structure given by \( \varphi^* \omega_0 \) (Theorem 3.1).

Since the curvature of the holomorphic Hermitian vector bundle \( V \rightarrow G \) is standard, Theorem 3.1 gives a description of the curvature of \( \varphi^* \omega_0 \) in terms of \( \rho \). In particular, we show that when \( d = 1 \), the curvature of \( \varphi^* \omega_0 \) is strictly negative if \( X \) is not hyperelliptic; on the other hand, if \( X \) is hyperelliptic, then the curvature of \( \varphi^* \omega_0 \) vanishes at the \( 2(g+1) \) points of \( X \) fixed by the hyperelliptic involution; the curvature of \( \varphi^* \omega_0 \) is strictly negative outside these \( 2(g+1) \) points (Proposition 3.2).

2. Gonality and flat metric

As before, \( X \) is a compact connected Riemann surface of genus \( g \), with \( g \geq 2 \). For any positive integer \( d \), let \( \text{Sym}^d(X) \) denote the quotient of the Cartesian product \( X^d \) under the natural action of the group of permutations of \( \{1, \cdots, n\} \). This \( \text{Sym}^d(X) \) is a smooth complex projective manifold of dimension \( d \). The component of the Picard group of \( X \) parametrizing the holomorphic line bundles of degree \( d \) will be denoted by \( \text{Pic}^d(X) \).

Let \( X^d \rightarrow \text{Pic}^d(X) \) be the map that sends any \((x_1, \cdots, x_d) \in X^d\) to the line bundle \( \mathcal{O}_X(x_1 + \cdots + x_d) \). It descends to a map

\[
\varphi : \text{Sym}^d(X) \rightarrow \text{Pic}^d(X).
\]  

This map \( \varphi \) is surjective if and only if \( d \geq g \).

Consider the space of all nonconstant holomorphic maps from \( X \) to the complex projective line \( \mathbb{CP}^1 \). More precisely, consider the degree of all such maps. Since \( g \geq 2 \), the degree of any such map is at least two. The gonality of \( X \) is defined to the smallest integer among the degrees of maps in this space [Ei, p. 171]. Equivalently, the gonality of \( X \) is the smallest one among the degrees of holomorphic line bundles \( L \) on \( X \) with \( \dim H^0(X, L) \geq 2 \). The gonality of \( X \) will be denoted by \( \gamma_X \). Note that \( \gamma_X = 2 \) if and only if \( X \) is hyperelliptic. The gonality of a generic compact Riemann surface of genus \( g \) is \( \lceil \frac{2g+3}{2} \rceil \).

**Lemma 2.1.** Assume that \( d < \gamma_X \). Then the map \( \varphi \) in (2.1) is an embedding.

**Proof.** We will first show that \( \varphi \) is injective.

Take any point \( x := \{x_1, \cdots, x_d\} \in \text{Sym}^d(X) \); the points \( x_i \) need not be distinct. The divisor \( \sum_{i=1}^d x_i \) will be denoted by \( D_x \). If \( y := \{y_1, \cdots, y_d\} \in \text{Sym}^d(X) \) is another point such that the line bundles \( \mathcal{O}_X(D_x) \) and \( \mathcal{O}_X(D_y) \) are isomorphic, where \( D_y = \sum_{i=1}^d y_i \), then there is a meromorphic function on \( X \) with pole divisor \( D_y \) and zero divisor \( D_x \). In particular, the degree of this meromorphic function is \( d \). But this contradicts the given condition that \( d < \gamma_X \). Consequently, the map \( \varphi \) is injective.
We need to show that for $x \in \text{Sym}^d(X)$, the differential
\[ d\varphi(x) : T_x\text{Sym}^d(X) \longrightarrow T_{\varphi(x)}\text{Pic}^d(X) = H^1(X, \mathcal{O}_X) \] (2.2)
is injective.

We will quickly recall a description of the tangent bundle $T\text{Sym}^d(X)$.

Let
\[ D \subset \text{Sym}^d(X) \times X \]
be the tautological reduced effective divisor consisting of all $(\{y_1, \cdots, y_d\}, y)$ such that
$y \in \{y_1, \cdots, y_d\}$. The projection of $\text{Sym}^d(X) \times X$ to $\text{Sym}^d(X)$ will be denoted by $p$. Consider the quotient sheaf
\[ \mathcal{O}_{\text{Sym}^d(X)\times X}(D)/\mathcal{O}_{\text{Sym}^d(X)\times X} \longrightarrow \text{Sym}^d(X) \times X. \]

Note that its support is the divisor $D$. The tangent bundle $T\text{Sym}^d(X)$ is the direct image
\[ p_*(\mathcal{O}_{\text{Sym}^d(X)\times X}(D)/\mathcal{O}_{\text{Sym}^d(X)\times X}) \longrightarrow \text{Sym}^d(X). \]

Take any $x := \{x_1, \cdots, x_d\} \in \text{Sym}^d(X)$. Let
\[ 0 \longrightarrow \mathcal{O}_X(-D_x) \longrightarrow \mathcal{O}_X \longrightarrow \tilde{Q}(x) := \mathcal{O}_X/\mathcal{O}_X(-D_x) \longrightarrow 0 \] (2.3)
be the short exact sequence of sheaves on $X$, where $D_x$, as before, is the effective divisor given by $x$. Tensoring the sequence in (2.3) with the line bundle $\mathcal{O}_X(-D_x)^* = \mathcal{O}_X(D_x)$ we obtain the following short exact sequence of sheaves on $X$:
\[ 0 \longrightarrow \text{End}(\mathcal{O}_X(-D_x)) = \mathcal{O}_X \longrightarrow \text{Hom}(\mathcal{O}_X(-D_x), \mathcal{O}_X) = \mathcal{O}_X(D_x) \longrightarrow Q(x) := \text{Hom}(\mathcal{O}_X(-D_x), \tilde{Q}(x)) \longrightarrow 0. \] (2.4)

Let
\[ 0 \longrightarrow H^0(X, \mathcal{O}_X) \overset{\alpha}{\longrightarrow} H^0(X, \mathcal{O}_X(D_x)) \overset{\beta}{\longrightarrow} H^0(X, Q(x)) \overset{\delta_x}{\longrightarrow} H^1(X, \mathcal{O}_X) \] (2.5)
\[ \overset{\nu}{\longrightarrow} H^1(X, \mathcal{O}_X(D_x)) \longrightarrow H^1(X, Q(x)) = 0 \]
be the long exact sequence of cohomologies associated to the short exact sequence of sheaves in (2.5). From the earlier description of $T\text{Sym}^d(X)$ we have the following:
\[ T_x\text{Sym}^d(X) = H^0(X, Q(x)). \] (2.6)

Since $d < \gamma_X$, we have
\[ H^0(X, \mathcal{O}_X(D_x)) = \mathbb{C}. \] (2.7)

Hence the homomorphism $\alpha$ in (2.5) is an isomorphism. This implies that the homomorphism $\delta_x$ in (2.5) is injective. So the exact sequence in (2.5) gives the exact sequence
\[ 0 \longrightarrow T_x\text{Sym}^d(X) = H^0(X, Q(x)) \overset{\delta_x}{\longrightarrow} H^1(X, \mathcal{O}_X). \] (2.8)

The tangent bundle of $\text{Pic}^d(X)$ is the trivial vector bundle over $\text{Pic}^d(X)$ with fiber $H^1(X, \mathcal{O}_X)$. The differential $d\varphi(x)$ in (2.2) coincides with the homomorphism $\delta_x$ in (2.5). Since $\delta_x$ in (2.8) is injective, it follows that $d\varphi(x)$ is injective. \[ \square \]
Let $K_X$ denote the holomorphic cotangent bundle of $X$. The vector space $H^0(X, K_X)$ is equipped with the Hermitian form

$$\langle \theta_1, \theta_2 \rangle := \int_X \theta_1 \wedge \overline{\theta_2} \in \mathbb{C}, \quad \theta_1, \theta_2 \in H^0(X, K_X).$$

This Hermitian form on $H^0(X, K_X)$ produces a Hermitian form on the dual vector space $H^0(X, K_X)^* = H^1(X, \mathcal{O}_X)$; this isomorphism is given by Serre duality. This Hermitian form on $H^1(X, \mathcal{O}_X)$ produces a Kähler structure on $\text{Pic}^d(X)$ which is invariant under the translation action of $\text{Pic}^0(X)$ on $\text{Pic}^d(X)$. This Kähler structure on $\text{Pic}^d(X)$ will be denoted by $\omega_0$.

Now Lemma 2.1 has the following corollary:

**Corollary 2.2.** Assume that $d < \gamma_X$. Then $\varphi^* \omega_0$ is a Kähler structure on $\text{Sym}^d(X)$.

### 3. Mapping to a Grassmannian

We will always assume that $d < \gamma_X$. Since $\gamma_X \leq g$, we have $d < g$.

Let

$$G = \text{Gr}(d, H^0(X, K_X))$$

be the Grassmannian parametrizing all $d$ dimensional quotients of $H^0(X, K_X)$. Let

$$V \rightarrow G$$

be the tautological vector bundle of rank $d$. So $V$ is a quotient of the trivial vector bundle $G \times H^0(X, K_X) \rightarrow G$. Consider the Hermitian form on $H^0(X, K_X)$ defined in (2.9). It produces a Hermitian structure on the trivial vector bundle $G \times H^0(X, K_X) \rightarrow G$. Identifying the quotient $V$ with the orthogonal complement of the kernel of the projection to $V$, we get a Hermitian structure on $V$. Let

$$H_0 : V \otimes \nabla \rightarrow G \times \mathbb{C}$$

be this Hermitian structure on $V$.

Take any $x := \{x_1, \cdots, x_d\} \in \text{Sym}^d(X)$. Consider the short exact sequence of sheaves on $X$

$$0 \rightarrow K_X \otimes \mathcal{O}_X(-D_x) \rightarrow K_X \rightarrow Q'(x) := K_X/(K_X \otimes \mathcal{O}_X(-D_x)) \rightarrow 0,$$

where $D_x$ as before is the divisor on $X$ given by $x$. Let

$$0 \rightarrow H^0(X, K_X \otimes \mathcal{O}_X(-D_x)) \xrightarrow{\nu'} H^0(X, K_X) \xrightarrow{\delta_x'} H^0(X, Q'(x))$$

$$\xrightarrow{\beta'} H^1(X, K_X \otimes \mathcal{O}_X(-D_x)) \xrightarrow{\alpha'} H^1(X, K_X) \rightarrow H^1(X, Q'(x)) = 0$$

be the long exact sequence of cohomologies associated to it. By Serre duality,

$$H^1(X, K_X \otimes \mathcal{O}_X(-D_x)) = H^0(X, \mathcal{O}_X(D_x))^*;$$

hence from (2.7) it follows that $\alpha'$ in (3.5) is an isomorphism. This implies that $\beta'$ in (3.5) is the zero homomorphism, hence $\delta_x'$ is surjective. In other words, $H^0(X, Q'(x))$ is
a quotient of $H^0(X, K_X)$ of dimension $d$. Therefore, $H^0(X, Q'(x))$ gives a point of the Grassmannian $G$ constructed in (3.1).

Let
\[
\rho : \text{Sym}^d(X) \rightarrow G
\]
be the morphism defined by $x \mapsto H^0(X, Q'(x))$.

**Theorem 3.1.**

1. The vector bundle $\rho^*V \rightarrow \text{Sym}^d(X)$, where $V$ and $\rho$ are constructed in (3.2) and (3.6) respectively, is holomorphically identified with the holomorphic cotangent bundle $\Omega_{\text{Sym}^d(X)}$.

2. Using the identification in (1), the Hermitian structure $\rho^*H_0$, where $H_0$ is constructed in (3.3), coincides with the Hermitian structure on $\Omega_{\text{Sym}^d(X)}$ given by $\varphi^*\omega_0$ in Corollary 2.2.

**Proof.** We will show that the homomorphisms $\alpha', \beta', \delta'_x$ and $\nu'$ in (3.5) are duals of the homomorphisms $\alpha, \beta, \delta_x$ and $\nu$ respectively, which are constructed in (2.5). This actually follows from the fact that the complex in (3.4) is dual of the complex in (2.4). We will elaborate this a bit.

By Serre duality, we have
\[
H^1(X, \mathcal{O}_X)^* = H^0(X, K_X) \quad \text{and} \quad H^1(X, \mathcal{O}_X(D_x))^* = H^0(X, K_X \otimes \mathcal{O}_X(-D_x)).
\]
Using these isomorphisms, the homomorphism $\nu'$ in (3.5) is the dual of the homomorphism $\nu$ in (2.5). Therefore, from (3.5) and (2.5) we have
\[
H^0(X, Q(x))^* = H^0(X, Q'(x)).
\]
But $H^0(X, Q(x)) = T_x\text{Sym}^d(X)$ (see (2.6)). On the other hand, $H^0(X, Q'(x))$ is the fiber of $V$ over the point $\rho(x) \in G$. Therefore, the first statement of the theorem follows from (3.3).

The isomorphism $H^1(X, \mathcal{O}_X)^* = H^0(X, K_X)$ in (3.7) is an isometry, because the Hermitian form on $H^1(X, \mathcal{O}_X)^*$ is defined using the Hermitian form on $H^0(X, K_X)$ in (2.9) and this isomorphism. This implies that the isomorphism in (3.8) is an isometry, after $H^0(X, Q(x))$ (respectively, $H^0(X, Q'(x))$) is equipped with the Hermitian structure obtained from the Hermitian structure on $H^1(X, \mathcal{O}_X)$ (respectively, $H^0(X, K_X)$) using (2.5) (respectively, (3.5)). This completes the proof. \[\Box\]

Take $d = 1$. Consider the Kähler form $\varphi^*\omega_0$ on $X$ in Corollary 2.2. Let $\Theta = \varphi^*\omega_0$ be the curvature of $\varphi^*\omega_0$; so $\Theta$ is a smooth function on $X$.

**Proposition 3.2.** The curvature function $\Theta$ is nonpositive.

1. If $X$ is not hyperelliptic, then $\Theta$ is strictly negative everywhere on $X$. 


(2) If $X$ is hyperelliptic, then $\Theta$ is strictly negative everywhere outside the $2(g + 1)$ points fixed by hyperelliptic involution of $X$. The function $\Theta$ vanishes on the $2(g + 1)$ fixed points of the hyperelliptic involution.

Proof. Take a complex vector space $W$ equipped with a Hermitian form. Let $\mathbb{P}(W)$ be the projective space that parametrizes quotients of $W$ of dimension one. The curvature of the Chern connection, [Ko, p. 11, Proposition 4.9], on the tautological line bundle on $\mathbb{P}(W)$ coincides with the Fubini–Study Kähler form on $\mathbb{P}(W)$. In other words, the curvature form is positive. Let $\mu$ denote the curvature of the line bundle $V \rightarrow G = \mathbb{P}(H^0(X, K_X))$ in (3.2) equipped with the Hermitian form $H_0$ constructed in (3.3). Since $\mu$ is positive, $\rho^*\mu$ is nonnegative, and it is strictly positive wherever the differential $d\rho$ is nonzero. Note that the curvature of the Chern connection on the line bundle $\rho^*V$ equipped with the Hermitian structure $\rho^*H_0$ coincides with the pulled back form $\rho^*\mu$.

If $X$ is not hyperelliptic, then $\rho$ is an embedding [ACGH, p. 11–12], [GH, p. 247]. So $\rho^*\mu$ is a positive form on $X$. Since the curvature of the Chern connection on $(\rho^*V, \rho^*H_0)$ coincides with $\rho^*\mu$, from Theorem 3.1 it follows that the curvature form $\Theta \cdot \varphi^*\omega_0$, for the Kähler structure $\varphi^*\omega_0$, coincides with $-\rho^*\mu$. So $\Theta$ is strictly negative everywhere on $X$.

Now take $X$ to be hyperelliptic. Let $\iota : X \rightarrow X$ be the hyperelliptic involution. The map $\rho$ factors as

$$X \rightarrow X/\iota \xrightarrow{\rho'} \mathbb{P}(H^0(X, K_X)),$$

and $\rho'$ is an embedding [ACGH, p. 11]. In particular, the differential $d\rho$ vanishes exactly on the $2(g + 1)$ points of $X$ fixed by the hyperelliptic involution $\iota$. Hence the above argument gives that the function $\Theta$ vanishes exactly on the $2(g + 1)$ points of $X$ fixed by $\iota$, and it strictly negative outside these $2(g + 1)$ points. $\square$

References

[ACGH] E. Arbarello, M. Cornalba, P. A. Griffiths and J. Harris, Geometry of algebraic curves. Vol. I, Grundlehren der Mathematischen Wissenschaften, 267, Springer–Verlag, New York, 1985.

[BR] I. Biswas and N. M. Romão, Moduli of vortices and Grassmann manifolds, Comm. Math. Phys. 320 (2013), 1–20.

[Ei] D. Eisenbud, The geometry of syzygies. A second course in commutative algebra and algebraic geometry, Graduate Texts in Mathematics, Vol. 229, Springer, New York, 2005.

[GH] P. Griffiths and J. Harris, Principles of Algebraic Geometry, John Wiley & Sons Inc., New York, 1978.

[Ko] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Publications of the Math. Society of Japan 15, Iwanami Shoten Publishers and Princeton University Press, 1987.

[Ri] N. A. Rink, Vortices and the Abel-Jacobi map, J. Geom. Phys. 76 (2014), 242–255.