Characterization of Minimum Cycle Basis in Weighted Partial 2-trees

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Abstract
For a weighted outerplanar graph, the set of lex short cycles is known to be a minimum cycle basis [Inf. Process. Lett. 110 (2010) 970-974]. In this work, we show that the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees (graphs of treewidth two) which is a superclass of outerplanar graphs.

1 Introduction
A cycle basis is a compact description of the set of all cycles of a graph and has various applications including the analysis of electrical networks [6]. Let \( G = (V(G), E(G)) \) be an edge weighted graph and let \( m = |E(G)| \) and \( n = |V(G)| \). A cycle is a connected graph in which the degree of every vertex is two. An incidence vector \( x \), indexed by \( E(G) \) is associated with every cycle \( C \) in \( G \), where for every edge \( e \in E(G) \), \( x_e \) is 1 if \( e \in E(C) \) and 0 otherwise. The cycle space of \( G \) is the vector space over \( \mathbb{F}_2^m \) spanned by the incidence vectors of cycles in \( G \). A cycle basis of \( G \) is a minimum set of cycles whose incidence vectors span the cycle space of \( G \). The weight of a cycle \( C \) is the sum of the weights of the edges in \( C \). A cycle basis \( B \) of \( G \) is a minimum cycle basis (MCB) if the sum of the weights of the cycles in \( B \) is minimum. A minimum cycle basis of \( G \) is denoted by \( MCB(G) \).

Motivation: For a weighted graph \( G \), Horton has identified a set \( \mathcal{H} \) of \( O(mn) \) cycles and has shown that a minimum cycle basis of \( G \) is a subset of \( \mathcal{H} \) [5]. Liu and Lu have shown that the set of lex short cycles (defined later) is a minimum cycle basis in weighted outerplanar graphs [8]. We generalize this result for partial 2-trees which is a superclass of outerplanar graphs.

Our contribution: The following are the main results in this work.

Theorem 1.1 Let \( G \) be a weighted partial 2-tree on \( n \) vertices and \( m \) edges. Then the number of lex short cycles in \( G \) is \( m - n + 1 \).

Theorem 1.2 For a weighted partial 2-tree \( G \), the set of lex short cycles is a minimum cycle basis.

Related work: The characterization of graphs using cycle basis was initiated by MacLane [9]. In particular, MacLane showed that a graph \( G \) is planar if and only if \( G \) contains a cycle basis \( B \) such that each edge in \( G \) appears in at most two cycles of \( B \). However, he referred to a cycle basis as a complete independent set of cycles. Formally, the concept of cycle space in graphs was introduced in [3] after four decades. Later, it was characterized that a planar 3-connected graph \( G \) is a Halin graph if and only if \( G \) has a planar basis \( B \) such that each cycle in \( B \) has an external edge [12]. There after, it was shown that every 2-connected outerplanar graph has a unique MCB [7]. Subsequently, it was proven that Halin graphs that are not necklaces have a unique MCB [11].

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The first polynomial time algorithm for finding an MCB was given by Horton [5]. Since then, many improvements have taken place on algorithms related to minimum cycle basis and its variants. A detailed survey of various algorithms, characterizations and the complexity status of cycle basis and its variants was compiled by Kavitha et al. [6]. The current best algorithm for MCB runs in $O(m^2 n/ \log n)$ time and is due to Amaldi et al. [1].

**Graph preliminaries:** In this paper, we consider only simple, finite, connected, undirected and weighted graphs. We refer [13] for standard graph theoretic terminologies. Let $G$ be an edge weighted graph. Let $X \subseteq V(G)$. $G - X$ denotes the graph obtained after deleting the set of vertices in $X$ from $G$. $G[X]$ denotes the subgraph induced by vertices in $X$. $X$ is a **vertex separator** if $G - X$ is disconnected. A **component** of $G$ is a maximal connected subgraph. $K_2$ denotes a cycle on 3 vertices and $K_3$ denotes an edge. $K_{2,3}$ is a complete bipartite graph $(V_1, V_2)$ such that $|V_1| = 2$, $|V_2| = 3$. A graph is **planar** if it can be drawn on the plane without any edge crossings. A planar graph is **outerplanar** if it can be drawn on the plane such that all of its vertices lie on the boundary of its exterior region. A 2-tree is defined inductively as follows: $K_3$ is a 2-tree; if $G'$ is a 2-tree and $G = G' \cup \{v\}$ is such that $N_G(v)$ forms a $K_2$ in $G$, then $G$ is a 2-tree. A graph is a **partial 2-tree** if it is a subgraph of a 2-tree. Alternatively, a graph of treewidth (defined in [10]) two is a **partial 2-tree**. An $H$-**subdivision** (or subdivision of $H$) is a graph obtained from a graph $H$ by replacing edges with pairwise internally vertex disjoint paths.

## 2 MCB in Weighted Partial 2-trees

For a weighted partial 2-tree $G$ associated with a weight function $w : E(G) \rightarrow \mathbb{N}$, we show that the set of lex short cycles (defined below) in $G$ is an MCB($G$). The notion of lex shortest path and lex short cycle is presented from [4]. For a totally ordered set $S$, min$(S)$ denotes the minimum element in $S$. For a graph $G$, let $V(G)$ be a totally ordered set. A path $P(u, v)$ between two distinct vertices $u$ and $v$ is **lex shortest path** if for all the paths $P'$ between $u$ and $v$ other than $P$, exactly one of the following three conditions hold: 1) $w(P') > w(P)$ 2) $w(P') = w(P)$ and $|E(P')| > |E(P)|$ 3) $w(P') = w(P)$, $|E(P')| = |E(P)|$ and $\min(V(P') \setminus V(P)) > \min(V(P) \setminus V(P'))$, where $w(P) = \Sigma_{e \in E(P)} w(e)$. The lex shortest path between any two vertices $u$ and $v$ is unique and is denoted by lsp$(u, v)$. A cycle $C$ is **lex short** if for every two vertices $u$ and $v$ in $C$, lsp$(u, v) \subseteq C$. The set of lex short cycles of $G$ is denoted by LSC$(G)$. For a subgraph $G_1$ of $G$, the total order of $V(G_1)$ is the order induced by the total order of $V(G)$. We use lsp$_{G_1}(x, y)$ to denote the lex shortest path between vertices $x$ and $y$ in $G_1$. We use the following lemmas from the literature.

**Lemma 2.1 ([4])** For a simple weighted graph $G$, LSC$(G)$ contains an MCB($G$).

**Lemma 2.2 ([8])** For a simple weighted outerplanar graph $G$, $|\text{LSC}(G)| = m - n + 1$.

We present the following lemmas and theorems that are required to prove our main result.

**Lemma 2.3** Let $G$ be a partial 2-tree and $\{u, v\}$ be a vertex separator in $G$. Let $P$ be the lex shortest path between $u$ and $v$. There exist one component $H$ in $G - \{u, v\}$ such that $V(P) \cap V(H) = \emptyset$ and $E(P) \cap E(H) = \emptyset$.

**Proof** If $P = (u, v)$, then none of the components in $G - \{u, v\}$ contain $V(P)$ and $E(P)$. If $P = (u, x, v)$, then no component in $G - \{u, v\}$ contain $E(P)$ and exactly one component in $G - \{u, v\}$ contains $x$. If $P$ is not captured by these two cases, then $P$ has at least three edges. If $|E(P)| \geq 3$, then exactly one component in $G - \{u, v\}$ that contains $P - \{u, v\}$. Since $\{u, v\}$ is a vertex separator in $G$, the number of components in $G - \{u, v\}$ is at least two. Therefore, there exist a component $H$ in $G - \{u, v\}$ such that $V(P) \cap V(H) = \emptyset$ and $E(P) \cap E(H) = \emptyset$. $\square$

**Lemma 2.4** Let $G$ be a partial 2-tree that is not outerplanar. Then there exists a $K_{2,3}(\{u, v\}, \{x, y, z\})$-subdivision in $G$ such that $G - \{u, v\}$ contains at least three components.
Proof A graph is outerplanar if and only if it contains no subgraph that is a subdivision of $K_4$ or $K_{2,3}$ [2]. Since a partial 2-tree does not contain a subdivision of $K_4$, a partial 2-tree is outerplanar if and only if it does not contain a subdivision of $K_{2,3}$. Consider a $K_{2,3}\{u, v\}, \{x, y, z\}$-subdivision in $G$. Assume that $G - \{u, v\}$ has at most two components. Then there exist a path in $G - \{u, v\}$ between two vertices in $\{x, y, z\}$ which does not go through the other vertex. Without loss of generality, we assume that $x$ and $y$ are those two vertices and $z$ is the other vertex. Such a path between $x$ and $y$ is shown as a dotted path in Figure 1. It follows that there are six internal vertex disjoint paths in $G$, namely $P(x, u), P(x, v), P(y, u), P(y, v), P(x, y)$ and $P(u, v)$ via $z$. Thus, there is a $K_4$-subdivision on the vertex set $\{u, v, x, y\}$ in $G$. This is a contradiction that $G$ is a partial 2-tree. Therefore, $\{u, v\}$ is a vertex separator in $G$ whose removal gives at least three components. □

Figure 1: A $K_4$-subdivision on the vertex set $\{u, v, x, y\}$

Lemma 2.5 Let $G'$ be a weighted subgraph of a weighted graph $G$. Let $P$ be a path and $C$ be a cycle contained both in $G$ and $G'$.
(a) If $P$ in $G$ is lex shortest, then $P$ in $G'$ is lex shortest.
(b) If $C \in LSC(G)$, then $C \in LSC(G')$.

Proof Suppose if the path $P$ in $G'$ is not lex shortest, then the path $P$ in $G$ would not be lex shortest. Hence, the $P$ in $G'$ is lex shortest.

Recall that $C$ is a lex short cycle if for every $x, y \in V(C)$, lsp$(x, y)$ is contained in $C$. Since $C$ is in $G'$ and for every $x, y \in C$, lsp$(x, y)$ is same as lsp$_{G'}(x, y)$, $C$ is a lex short cycle in $G'$. □

Lemma 2.6 The intersection of two lex shortest paths is either empty or a lex shortest path.

Proof Consider two lex shortest paths lsp$(x, y)$ and lsp$(u, v)$ in $G$. Let $G' = (V(G'), E(G'))$, where $V(G') = V(lsp(x, y)) \cap V(lsp(u, v))$, $E(G') = E(lsp(x, y)) \cap E(lsp(u, v))$. Suppose $V(G') \neq \emptyset$ and $G'$ is not a path, then we have at least two maximal paths $P(a, b)$ and $P(a', b')$ which are common to both lsp$(x, y)$ and lsp$(u, v)$, where $b \neq a'$. Consequently, the path $P_1(b, a')$ contained in $lsp(x, y)$ and the path $P_2(b, a')$ contained in $lsp(u, v)$ are different. Since a subpath of a lex shortest path is a lex shortest path, both $P_1$ and $P_2$ are lex shortest paths between $b$ and $a'$; a contradiction to the fact that for any two vertices in a graph, there is a unique lex shortest path. □

We decompose a weighted partial 2-tree $G$ that is not outerplanar into two weighted partial 2-trees $G_1$ and $G_2$ in such a way that $LSC(G)$ is equal to the disjoint union of $LSC(G_1)$ and $LSC(G_2)$. From Lemma 2.4 there exist two vertices $u, v \in V(G)$ such that $G - \{u, v\}$ is disconnected and has at least three components. Let $P$ be the lex shortest path between $u$ and $v$ in $G$. By Lemma 2.3 there exist a component $H$ in $G - \{u, v\}$ such that $V(P) \cap V(H) = E(P) \cap E(H) = \emptyset$. The operation $\text{decomp}(G, u, v)$ decomposes $G$ into $G_1$ and $G_2$, where $V(G_1) = V(H) \cup V(P)$, $E(G_1) = E(H) \cup E(P) \cup \{x, y\} | x \in V(H), y \in \{u, v\}$ and $(x, y) \in E(G)$, $G_2 = G[V(G) \setminus V(H)]$. An example is shown in Figure 2 to illustrate the operation $\text{decomp}(G, u, v)$.

We use the following notation for the rest of the paper. $G$ is a weighted partial 2-tree that is not outerplanar. $\{u, v\}$ is a vertex separator that disconnects $G$ into at least three components. $H$ is a component in $G - \{u, v\}$ such that $V(lsp(u, v)) \cap V(H) = \emptyset$ and $E(lsp(u, v)) \cap E(H) = \emptyset$. $G_1$ and $G_2$ are the graphs obtained from the operation $\text{decomp}(G, u, v)$.

From the definition of $\text{decomp}(G, u, v)$, we have the following two observations.
Observation 1 For $x, y \in V(lsp(u, v))$, $lsp(x, y)$ is in $G_1$.

Proof This observation follows, since every subpath of a lex shortest path is a lex shortest path. □

Observation 2 For $x \in V(G_i)$ and $y \in V(G_i) \setminus V(lsp(u, v))$, every path $P(x, y)$ in $G$ such that the internal vertices of $P$ are in $V(G_i) \setminus \{u, v\}$, is present in $G_i$.

Proof Assume that $P(x, y)$ is not in $G_i$. In the path $P$ from $x$ to $y$, let $(a, b)$ be the first edge such that $(a, b) \notin E(P)$. Clearly, $b \notin V(G_i)$. It follows that $a$ is an intermediate vertex in $P$ and $a \in \{u, v\}$; a contradiction to the premise that no intermediate vertex in $P$ belong to $\{u, v\}$. Hence, the observation. □

Lemma 2.7 For $i \in \{1, 2\}$, for every two vertices $x, y \in V(G_i)$, $lsp(x, y)$ is in $G_i$.

Proof If no vertex is common in $lsp(x, y)$ and $lsp(u, v)$, then from Observation 2 $lsp(x, y)$ is in $G_i$. If at least one vertex is common in $lsp(x, y)$ and $lsp(u, v)$, then due to Lemma 2.6 $lsp(x, y) \cap lsp(u, v)$ is $lsp(a, b)$ for some $a, b \in V(lsp(u, v)) \cap V(lsp(x, y))$. The $lsp(x, y)$ can be viewed as a union of three paths $P(x, a)$, $P(a, b)$ and $P(b, y)$. From Observation 1 $P(a, b)$ is contained in $G_i$. If $x = a$, then trivially $P(x, a)$ appears in $G_i$. Also, if $y = b$, then clearly $P(b, y)$ appears in $G_i$. If $x \neq a$, then $x \notin V(lsp(u, v))$. Similarly, if $y \neq b$, then $y \notin V(lsp(u, v))$. From Observation 2 it follows that both $P(x, a)$ and $P(b, y)$ appear in $G_i$. These observations imply that $lsp(x, y)$ is in $G_i$. □

Corollary 2.8 For $i \in \{1, 2\}$, if there is a cycle $C$ in $LSC(G_i)$, then $C$ is in $LSC(G)$.

Proof From Lemmas 2.7 and 2.9(a), for every $x, y \in V(G_i)$, $lsp_G(x, y)$ and $lsp(x, y)$ are same. Since $C \in LSC(G_i)$, for every $x, y \in V(C)$, $lsp_G(x, y)$ is contained in $C$. Consequently, for every $x, y \in V(C)$, $lsp(x, y)$ is contained in $C$. Hence $C \in LSC(G)$. □

Theorem 2.9 $LSC(G) = LSC(G_1) \cup LSC(G_2)$.

Proof Since $E(G_1) \cap E(G_2) = E(lsp(u, v))$, $LSC(G_1)$ and $LSC(G_2)$ are disjoint. We now prove that $LSC(G) \subseteq LSC(G_1) \cup LSC(G_2)$. Let $C \in LSC(G)$. If $C$ contains at most one vertex from $\{u, v\}$, then $C$ is contained either in $G_1$ or $G_2$, because $\{u, v\}$ is a vertex separator. Consider the other case when $C$ contains both $u$ and $v$. Since $C \in LSC(G)$, $C$ contains $lsp(u, v)$. Note that $lsp(u, v)$ is contained both in $G_1$ and $G_2$. Observe that $E(C) \setminus E(lsp(u, v))$ belongs to $E(G_i)$ for some $i \in \{1, 2\}$, because $E(G_1) \cap E(G_2) = E(lsp(u, v))$. Hence, $C$ is either in $G_1$ or $G_2$. In both of the cases, by Lemma 2.9(b), $C$ is either in $LSC(G_1)$ or $LSC(G_2)$. Therefore, $LSC(G) \subseteq LSC(G_1) \cup LSC(G_2)$. From Corollary 2.8 $LSC(G_1) \cup LSC(G_2) \subseteq LSC(G)$. Hence, $LSC(G) = LSC(G_1) \cup LSC(G_2)$. □

Lemma 2.10 The number of $K_{2,3}$-subdivisions in each of $G_1$ and $G_2$ is less than the number of $K_{2,3}$-subdivisions in $G$.
Proof Recall that there is a $K_{2,3}((u,v),\{x,y,z\})$-subdivision in $G$, and $G_1$ and $G_2$ are obtained from $\text{decomp}(G,u,v)$. Without loss of generality, assume that $x \in V(H)$. Then at most one vertex from $\{y,z\}$ is in $G_1$. Further, observe that $x$ is not in $G_2$. Therefore, no $K_{2,3}((u,v),\{x,y,z\})$-subdivision exist in $G_1$ and $G_2$. Thus the lemma holds. □

Proof of Theorem 1.1

Proof We use induction on the number of $K_{2,3}$-subdivisions in $G$. If the number of $K_{2,3}$-subdivisions in $G$ is zero, then $G$ is outerplanar, since $G$ is a partial 2-tree. From Lemma 2.2, $|\text{LSC}(G)| = m - n + 1$ when $G$ is outerplanar. Hence base case is true. If $G$ is not an outerplanar graph, then there exists a $K_{2,3}((u,v),\{x,y,z\})$-subdivision in $G$. From Lemma 2.4, $G - (u,v)$ is disconnected and contains at least three components. Let $P$ be the lsp($u,v$) in $G$ and $k = |V(P)|$. We apply $\text{decomp}(G,u,v)$ to obtain $G_1$ and $G_2$ from $G$. For $i \in \{1,2\}$, $m_i$ and $n_i$ indicate $|E(G_i)|$ and $|V(G_i)|$, respectively. Now, we can apply induction hypothesis due to Lemma 2.10. By induction hypothesis, it follows that $|\text{LSC}(G_i)| = m_i - n_i + 1$ for $i \in \{1,2\}$. As $P$ is present in $G_1$ and $G_2$, it follows that $n_1 + n_2 = n + k$ and $m_1 + m_2 = m + k - 1$. From Theorem 2.9, $\text{LSC}(G) = \text{LSC}(G_1) \cup \text{LSC}(G_2)$. Hence $|\text{LSC}(G)| = |\text{LSC}(G_1)| + |\text{LSC}(G_2)| = m_1 - n_1 + 1 + m_2 - n_2 + 1 = m - n + 1$. Therefore, $|\text{LSC}(G)| = m - n + 1$. □

Proof of Theorem 1.2

Proof For a simple weighted graph $G$, from Lemma 2.1, an $\text{MCB}(G) \subseteq \text{LSC}(G)$. The cardinality of any cycle basis in a graph is known to be $m - n + 1$. For a weighted partial 2-tree $G$, by Theorem 1.1, we have $|\text{LSC}(G)| = m - n + 1$. Therefore, the set of lex short cycles is a minimum cycle basis in weighted partial 2-trees. □

Figure 3: For the wheel graph shown, if $b \gg a$, then the set of all triangles and the exterior face are lex short cycles.

We present a family of partial 3-trees for which the set of lex short cycles is not a cycle basis, whose construction is as follows: Let $G_n = K_1 + C_{n-1}$ be a wheel graph on $n$ vertices, where $n \geq 4$. A wheel graph on 9 vertices is depicted in Figure 3. Note that $G_n$ is planar. For every edge $e \in E(G_n)$, assign $w(e) = a$ if $e$ is in external face, otherwise $w(e) = b$, where $a,b \in \mathbb{N}$ and $b \gg a$. Since every face in $G_n$ is a lex short cycle, $|\text{LSC}(G_n)| = m - n + 2$ by Euler’s formula. Therefore, $\text{LSC}(G_n)$ can not be a cycle basis.

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