Interactions of irregular Gaiotto states in Liouville theory

Sang-Kwan Choi\textsuperscript{1}, Dimitri Polyakov\textsuperscript{1,2,a}, Cong Zhang\textsuperscript{1}

\textsuperscript{1} Center for Theoretical Physics, College of Physical Science and Technology, Sichuan University, Chengdu 610064, China
\textsuperscript{2} Institute of Information Transmission Problems (IITP), Bolshoi Karetny per. 19/1, Moscow 127994, Russia

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Abstract We compute the correlation functions of irregular Gaiotto states appearing in the colliding limit of the Liouville theory by using “regularizing” conformal transformations mapping the irregular (coherent) states to regular vertex operators in the Liouville theory. The \(N\)-point correlation functions of the irregular vertex operators of arbitrary ranks are expressed in terms of \(N\)-point correlators of primary fields times the factor that involves regularized higher-rank Schwarzians of the above conformal transformation. In particular, in the case of three-point functions the general answer is expressed in terms of DOZZ (Dorn-Otto-Zamolodchikov-Zamolodchikov) structure constants times exponents of regularized higher-derivative Schwarzians. The explicit examples of the regularization are given for the ranks one and two.

1 Introduction

Irregular (coherent) Gaiotto states emerge in Liouville field theory in the colliding limit, relevant to extensions of the AGT conjecture to Argyres-Douglas type of gauge theories with asymptotic freedom \([1,3,6–8,14,19]\). The irregular states of rank \(N\) are the simultaneous eigenstates of \(N+1\) Virasoro generators:

\[
L_n|U_N\rangle = \rho_n(n)|U_N\rangle
\]

\[N \leq n \leq 2N\] (1)

and are annihilated by higher positive Virasoro generators \((n > 2N)\). This generalizes the definition of primary fields (which technically have rank zero). Just as the regular vertex operators for primary (rank zero) fields in Liouville theory can be expressed as \(V_\alpha = e^{\alpha_0 \phi}\) (where \(\alpha_0\) can be regarded as “electric” charge), the irregular vertex operators for rank \(N\) coherent states can be constructed as

\[
|U_N\rangle = U_N\langle 0| \sum_{\alpha_0} e^{\alpha_0 \phi} \]

where \(\alpha_1, \alpha_2, \ldots\) correspond to coefficients of the multipole expansion and are related to the geometry of the region where the colliding operators are located (e.g. with \(\alpha_1\) being a characteristic size of the region). These coefficients are related to Virasoro eigenvalues \(\rho_n(N)\) according to

\[
\rho_n(N) = \frac{1}{2} \sum_{n=0}^{N} \alpha_n \alpha_{N-n} \] (3)

Computing correlation functions describing interactions of the irregular states in Liouville theory is known to be a hard and tedious problem, especially beyond two-point functions and rank one case \([5,6,9,14,15]\). These correlation functions are important objects as they define the correlators in Argyres-Douglas gauge theories; namely, the \(N\)-point correlators of Argyres-Douglas theories are identified with the \(M\)-point colliding limit of 2d CFT so that \(M = M_1 + M_2 + \cdots + M_N\). Presumably, the correlators of the irregular states are also related to instanton expansions in these asymptotically-free theories, although our understanding of this correspondence is not yet complete.

In this work we address this problem by using the conformal transformations that maps operators for coherent states into regular vertex operators, expressing the interactions of the irregular states in terms of regular correlators in Liouville theory. In particular, since the structure constants of the Liouville theory are known, this conformal transformation makes it possible to express the cubic interactions of the irregular states of arbitrary rank in terms of DOZZ correlators in Liouville theory. The final formula for the correlators, however, is complicated, since it involves the generalized higher-derivative Schwarzians of the conformal transformation that are singular at the insertion points of the correlators and have to be regularized in a rather tedious way. In our work, we limit ourselves performing this regularization for the case of 3-point functions of the rank two irregular
vertex operators. The rest of this paper is organized as follows. In the Sect. 2, we study the behavior of correlators under the conformal transformation mapping the irregular blocks into regular and derive the general result expressing \( N \)-point correlators of irregular vertices of arbitrary ranks in terms of regular conformal blocks. The answer involves the higher-derivative generalized Schwarzians of this conformal transformation that need to be regularized. In the Appendix section we explicitly perform such a regularization for three-point correlators of the rank two states. In the concluding section we discuss the implications of our results.

2 Conformal map: irregular to regular states

Consider a rank \( p \) irregular vertex operator:

\[
W_p(\alpha_0, \alpha_1, \ldots, \alpha_p|\xi) = e^{\sum_{n=0}^{p} a_n \phi^n} : (\xi) \tag{4}
\]

where \( \phi \) is Liouville field and the \( N \)-point correlators

\[
A_N = \left\{ N \prod_{j=1}^N W_{p_j}(\alpha_0^{(j)}, \ldots, \alpha_{p_j}^{(j)}|\xi_j) \right\} \tag{5}
\]

Consider conformal transformation:

\[
f(z) = e^{-i \sum_{j=1}^{N} (z-\xi_j-i\epsilon)^{-1}} \tag{6}
\]

\[
S_{n_1|n_2}(f; z) = \frac{1}{n_1!n_2!} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{m_1 \geq 0, m_2 \geq 0} \sum_{p \geq 0} (-1)^{k_1+m_1+q} 2^{-m_1-m_2} (k_1 + k_2 - 1) \times \frac{\partial^{m_1} B_{n_1|k_1}(f; z) \partial^{m_2} B_{n_2|k_2}(f; z) B_{p|q}(\xi_1, \ldots, \xi_q) }{(s+1)^{k_1+k_2}} \times g_s = 2^{-s-1} (1 + (-1)^s) ; s = 1, \ldots, p + q + 1 \tag{10}
\]

with the small \( \epsilon \) parameter introduced in order to control regularizations. This transformations maps the half-plane to compact Riemann surface with all the the points \( \xi_1, \ldots, \xi_N \) (originally located on the real line) are glued together at zero for \( \epsilon = 0 \) (and are infinitely close to each other if \( \epsilon \) is nonzero). Gluing points together is the trick that will be used below to compare correlation functions before and after conformal transformations. First of all, let us check how the conformal transformation (6) acts on individual irregular vertices. It is straightforward to check that the transformation (6) maps irregular vertex operators (4) into regular. To see this, first consider the infinitesimal transformations of \( W_p \). We have

\[
\delta_{\epsilon} W_p(\xi) = \frac{1}{2} \left[ f \int \frac{dz}{2\pi i} e^{\xi \phi} \phi : (z) W_p(\xi) \right] \tag{11}
\]

This infinitesimal relation is straightforward to integrate (e.g. by imposing a composition constraint). The integrated form (7) for finite conformal transformations \( z \to f(z) \) is then given by

\[
W_p(z) \to \tilde{W}_p(f(z)) = \exp \left\{ a_0 \phi + \sum_{n=1}^{p} \sum_{q=1}^{n} B_{n|q}(f(z); z) a_n \partial^q \phi(f(z)) \right\} \tag{8}
\]

Here

\[
B_{n|q}(f(z); z) = \sum_{n_1|n_2| \ldots |n_q} \frac{\partial^{n_1} \phi \ldots \partial^n \phi}{n_1! \ldots n_q! |r(1) \ldots r(n_q)!|} \tag{9}
\]

are the restricted length \( q \) Bell polynomials in derivatives of \( f \), with the sum taken over the ordered length \( q \) partitions of \( n = n_1 + \ldots + n_q \); with \( 0 < n_1 \leq n_2 \ldots \leq n_q \) and \( r(n_i) \) is multiplicity of element \( n_i \) in the partition (e.g. for \( q = 2 + 3 + 3q(1) = 0, q(2) = 1 \) and \( q(3) = 2 \)). Next, \( S_{n_1|n_2} \) are the generalized rank \( (n_1, n_2) \) Schwarzians of the conformal transformation \( f(z) \), defined according to [23]:

\[
S_{n_1|n_2}(f; z) = \frac{1}{m_1!m_2!} \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} \sum_{m_1 \geq 0, m_2 \geq 0} \sum_{p \geq 0} (-1)^{k_1+m_1+q} 2^{-m_1-m_2} (k_1 + k_2 - 1) \times \frac{\partial^{m_1} B_{n_1|k_1}(f; z) \partial^{m_2} B_{n_2|k_2}(f; z) B_{p|q}(\xi_1, \ldots, \xi_q) }{(s+1)^{k_1+k_2}} \times g_s = 2^{-s-1} (1 + (-1)^s) ; s = 1, \ldots, p + q + 1 \tag{10}
\]

for \( n_1, n_2 \neq 0 \) with the sum over the non-negative numbers \( m_1, m_2 \) and \( p \) taken over all the combinations satisfying

\[
m_1 + m_2 + p = k_1 + k_2
\]

Also, \( S_{0|0}(f; z) = \ln(f'(z)) \), \( S_{0|0} = S_{0|1} = f'(\xi) \) and \( S_{1|1} \) is a usual Schwarzian derivative (up to conventional factor of \( \frac{1}{2} \)). As \( z \to \xi_j \), the coefficients in front of derivatives \( \partial^q \phi \) \( q \neq 0 \) in the exponent of \( \tilde{W}_p(f(z)) \), determined by the length \( q \) Bell polynomials in \( f \), are dropped exponentially as \( e^{-\frac{q}{2}} \), so only the regular part \( \sim a_0 \phi \) survives and the operators become regular in new coordinates. On the other hand, the price paid for the regularity is the appearance of the generalized Schwarzians \( S_{n_1|n_2}(f; z) \) in the transformation law for the irregular vertices. All of these Schwarzians have inverse power behavior in \( \epsilon \) and must be regularized.
as \( \epsilon \to 0 \). The final step is to compute the overlap deformation resulting from contractions of \( T(z) = \frac{1}{2} : \partial \phi \partial \phi : \) with different vertex operators (i.e. each of \( \partial \phi \)'s contracting with different vertex) and to integrate it. This altogether is equivalent to integrating the Ward identities and, in the limit \( \epsilon \to 0 \), the integrated overlap deformation determines the difference between the correlators computed on the half-plane and on the Riemann surface defined by the conformal map \( f(z) \), with the transformation laws (8) for the vertex operators. The relevant infinitesimal overlap transformation of the \( N \)-point correlator is given by

\[
\delta_{\text{overlap}} A_N(\xi_1, ..., \xi_N) = A_N(\xi_1, ..., \xi_N) \\
\sum_{j=1}^{N-1} \sum_{k=1}^{N} p_j \sum_{n_j=1}^{p_j} \alpha_n^{(j)} \alpha_{n_k}^{(k)} \left( n_k \frac{\epsilon(\xi_j)}{(\xi_j - \xi_k)^{n_k+1}} + n_j \frac{\epsilon(\xi_k)}{(\xi_k - \xi_j)^{n_j+1}} \right)
\]

(11)

It is straightforward to integrate it to obtain the contribution of the overlap to the deformation of \( A_N \) under the finite conformal transformation \( z \to f(z) \). The result is

\[
A_N(\xi_1, ..., \xi_N) \to A_N(f(\xi_1), ..., f(\xi_N)) \exp \left\{ N-1 \sum_{j=1}^{N} \sum_{k=1}^{N} p_j \sum_{n_j=1}^{p_j} \alpha_n^{(j)} \alpha_{n_k}^{(k)} \left( \sum_{p_j=0}^{n_j-1} \sum_{p_k=0}^{n_k-1} \left( \frac{B_{n_j+n_k-p_j}\{f(\xi_j); \xi_j\} B_{n_j+n_k-p_k}\{f(\xi_k); \xi_k\}}{(f(\xi_j)-f(\xi_k))^{n_j+n_k-p_j-p_k}} - \frac{1}{(\xi_j-\xi_k)^{n_j+n_k-p_j-p_k}} \right) \right) \right\}
\]

(12)

The relation between the correlators of the irregular and regular states is given by the transformation law (8) divided by the overlap deformation (12). Thus for \( N \)-point correlator of irregular states of arbitrary rank one has:

\[
A_N(\xi_1, ..., \xi_N) = S_N(\xi_1, ..., \xi_N) \times \exp \left\{ \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{p_j=1}^{p_j} \alpha_n^{(j)} \alpha_{n_k}^{(k)} S_{kk}(f(\xi_j); \xi_k) - \sum_{j=1}^{N} \sum_{k=1}^{N} \sum_{p_j=1}^{p_j} \sum_{p_k=1}^{p_k} \alpha_n^{(j)} \alpha_{n_k}^{(k)} \left( \sum_{p_j=0}^{n_j-1} \sum_{p_k=0}^{n_k-1} \left( \frac{B_{n_j+n_k-p_j}\{f(\xi_j); \xi_j\} B_{n_j+n_k-p_k}\{f(\xi_k); \xi_k\}}{(f(\xi_j)-f(\xi_k))^{n_j+n_k-p_j-p_k}} - \frac{1}{(\xi_j-\xi_k)^{n_j+n_k-p_j-p_k}} \right) \right) \right\}
\]

(13)

where \( S_N \) is the \( N \)-point correlator of regular Liouville primaries:

\[
S_N(\xi_1, ..., \xi_N) = \langle e^{\phi(0)}(\xi_1) ... e^{\phi(N)}(\xi_N) \rangle
\]

(14)

This reduces the problem of describing the interactions of irregular states of arbitrary rank in terms of those of the regular states.

In particular, for \( N = 3 \), \( S_3 \) is well-known and related to the regular Liouville S-matrix \([21,22]\). However, the Schwarzians and overlap factors (involving sums over the restricted Bell polynomials of the conformal transformation (6)) are singular at the insertion points of the irregular vertex operators and need to be regularized. In the Appendix section, we show how to perform such a regularization explicitly for the rank two irregular states (\( p_1 = p_2 = p_3 = 2 \)) and for \( N = 3 \). It already turns to be quite cumbersome. It is straightforward to show, in particular, that in a particularly simple case of two-point function of the rank 1 irregular states, this regularization procedure reproduces the inner product calculated at [9,13]. To obtain this product, one has to take the limit \( \xi_1 = 0, \xi_2 \to \infty \) in (5) \((N = 2; p_1 = p_2 = 1)\). In this limit, the overlap factor vanishes, and all the contributions stem from the regularizations of the Schwarzians at \( \xi_1 \) and \( \xi_2 \):

\[
A_2(w_1, w_2) &= e^{\frac{1}{2} \alpha_1^2 S_{111}(f; \xi_1) + \frac{1}{2} \alpha_2^2 S_{111}(f; \xi_2) + \alpha_1 \alpha_2 S_{011}(f; \xi_1) + \alpha_2 \beta_2 S_{011}(f; \xi_2) S_2}
\]

where

\[
S_{011}(f; \xi) = \frac{f''(\xi)}{f'(\xi)}
\]

(15)
of points and for higher ranks are in principle straightforward but require far more tedious calculations.

3 Conclusions

In this work, we analyzed interactions of the irregular states in Liouville theory by using conformal transformations that map the irregular vertex operators into regular. In particular, this allows to express three-point functions of irregulars of arbitrary rank in terms of Liouville structure constants given by DOZZ (Dorn-Otto-Zamolodchikov-Zamolodchikov) formula. The price one pays is the appearance of the objects, such as higher-derivative Schwarzians and overlap factors (sums over restricted Bell polynomials), in the transformation law for the vertices. They are singular at the insertion points and need to be regularized by a rather cumbersome procedure. Interestingly, all these objects appear in the solution of the well-known number theory problem of finding the closed analytic expressions for numbers of restricted partitions [23]. In the current paper, we restrict ourselves to the maximum rank two and the three-point function. In our future work (currently in progress), we hope to be able to develop the algorithm which simplifies the regularization scheme and can be applied to analyze the higher-rank interactions. Irregular states, apart from being relevant to AGT conjecture (extended to Argyres-Douglas class of gauge theories) also irregular states, apart from being relevant to AGT conjecture and can be applied to analyze the higher-rank interactions. Irregular states, apart from being relevant to AGT conjecture and can be applied to analyze the higher-rank interactions.

4. Appendix

In this section, we perform explicit regularizations of the higher-derivative Schwarzians and the overlap factors, for the case of rank 2 irregular states. We limit ourselves to 3-point functions.

4a. Transformation

For each irregular vertex operator of rank 2, the conformal transformation law is

\[ e^{a_1 \phi + a_2 \phi^2} \rightarrow f(z) \]

\[ \exp \left[ a_0 \phi + a_1 B_{11} \partial \phi + a_2 (B_{21} \partial \phi + B_{22} \partial^2 \phi) + \frac{1}{2} a_0^2 S_{00} \right] \]

\[ + a_0 a_1 S_{110} + 2a_0 a_2 S_{210} + \frac{1}{2} a_1^2 S_{111} + 2a_1 a_2 S_{211} + 2a_2^2 S_{212} \]  

\[ , \]

where

\[ S_{00} = \log f', \quad S_{110} = \frac{1}{2} \frac{f'''}{f''}, \quad S_{111} = \frac{1}{6} \left( \frac{f'''}{f''} \right)' - \frac{1}{12} \left( \frac{f''}{f'} \right)^2, \]

\[ S_{210} = \frac{1}{6} \frac{f''}{f'} - \frac{1}{8} \left( \frac{f''}{f'} \right)^2, \quad S_{211} = \frac{1}{8} \left( \frac{f'''}{f''} \right)' - \frac{1}{6} \frac{f''}{f'} \frac{f'''}{f''} \]

\[ + \frac{1}{24} \frac{f^{(4)}}{f'}, \]

\[ S_{212} = \frac{1}{120} \frac{f^{(5)}}{f'} - \frac{1}{24} \frac{f^{(4)} f''}{(f')^2} - \frac{1}{24} \frac{f'''}{f'} \]

\[ + \frac{1}{6} \frac{f^{(3)} f''}{(f')^3} - \frac{5}{32} \frac{f^{(4)}}{(f')^2}, \]

Next, consider the overlap transformation to the 3-point correlator. The overlap contribution between \((V(\xi_1), V(\xi_2))\) is given below, the overlaps between \(V(\xi_1), V(\xi_2)\) and \(V(\xi_2), V(\xi_3)\) are obtained similarly, by replacing variables \(\alpha, \beta, \xi\) accordingly. Denote \(f_1 = f(\xi_1)\) and \(f_2 = f(\xi_2)\). We have:

\[ \text{overlapping} \left( e^{a_0 \phi + a_1 \phi + a_2 \phi^2} (\xi_1) e^{b_0 \phi + b_1 \phi + b_2 \phi^2} (\xi_2) \right) \]

\[ \rightarrow f(z) \exp \left[ -a_0 b_0 \log (f_1 - f_2) - a_1 b_0 \frac{f_1'}{f_1} + f_2' \right] \]

\[ + a_0 a_1 \frac{f_1'}{f_1 - f_2} - a_1 b_1 \left( \frac{f_1'}{f_1} - \frac{f_2'}{f_2} \right)^2 \]

\[ - a_2 b_0 \left( \frac{f_1''}{f_1 - f_2} - \left( \frac{f_1'}{f_1} - \frac{f_2'}{f_2} \right)^2 \right) \]

\[ + a_0 a_2 \left( \frac{f_1''}{f_1 - f_2} + \left( \frac{f_2'}{f_1 - f_2} \right)^2 \right) \]
\[-\alpha_2 \beta_1 \left( \frac{f''}{(f_1 - f_2)^2} - \frac{2(f')^2}{(f_1 - f_2)^3} \right) \]
\[-\alpha_1 \beta_2 \left( \frac{f''}{(f_1 - f_2)^2} \right) + \frac{2(f')^2}{(f_1 - f_2)^3} \]
\[-\alpha_2 \beta_2 \left( \frac{f''}{(f_1 - f_2)^2} - \frac{2(f')^2}{(f_1 - f_2)^3} \right) \]
\[-2 f_2 (f')^2 \left( \frac{f''}{(f_1 - f_2)^2} \right) - \frac{6(f')^2}{(f_1 - f_2)^4} \]
\[+ \alpha_0 \beta_0 \log(\xi_1 - \xi_2) + \frac{\alpha_1 \beta_0}{\xi_1 - \xi_2} \]
\[-\alpha_0 \beta_1 \left( \frac{1}{\xi_1 - \xi_2} + \frac{\alpha_1 \beta_1}{\xi_1 - \xi_2} - \frac{\alpha_2 \beta_0}{\xi_1 - \xi_2} \right) \]
\[-\alpha_0 \beta_2 \left( \frac{1}{\xi_1 - \xi_2} - \frac{2\alpha_2 \beta_1}{\xi_1 - \xi_2} + \frac{2\alpha_1 \beta_2}{\xi_1 - \xi_2} - \frac{6\alpha_2 \beta_2}{(\xi_1 - \xi_2)^3} \right) \]
\[(17)\]

4b. Regularization: 2-point function

We start by regularizing the 2-point function first. We have:

\[A_2 = \left( e^{i\alpha_0 \phi + i\alpha_1 \delta \phi + i\alpha_2 \delta^2 \phi(\xi_1)} e^{i\beta_0 \phi + i\beta_1 \delta \phi + i\beta_2 \delta^2 \phi(\xi_2)} \right) \]
\[f(z) = e^{-i\left( \frac{1}{z - \xi_1} + \frac{1}{z - \xi_2} \right)} \]
\[(18)\]

Regularizing generalized Schwarzians:

For \( e^{i\alpha_0 \phi + i\alpha_1 \delta \phi + i\alpha_2 \delta^2 \phi(\xi_1)} \),

\[S_{00} \sim \log \left( ie^{-i/\xi} \right), \quad S_{11} \sim \frac{i}{2\xi}, \quad S_{111} \sim \frac{7}{12\xi^2} + \frac{1}{\xi^2} \]
\[S_{20} \sim -\frac{7}{24\xi^2} - \frac{i}{2\xi^3}, \quad S_{21} \sim -\frac{1}{4\xi^3} - \frac{1}{\xi^4} \]
\[S_{22} \sim -\frac{23}{480\xi^8} - \frac{19}{12\xi^6} + \frac{1}{2\xi^4} \]
\[(19)\]

For \( e^{i\beta_0 \phi + i\beta_1 \delta \phi + i\beta_2 \delta^2 \phi(\xi_2)} \), \( \xi_{12} \rightarrow -\xi_{12} \).

Regularizing the overlap part:

Denote \( X_i \equiv 1/\xi_i \) and \( F_i \equiv e^{-iX_i} \). Here we omit the free correlator part. Below, one by one, we present the regularized coefficients in front of \( \alpha_i \beta_j; i, j = 0, 1, 2 \) in the exponential. \( \alpha_0 \beta_0 \)

\[-\log \left( F_1 - \frac{1}{F_1} \right) \]
\[(20)\]
\[ -3i \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_3^2 \]
\[ -6 \left( F_1^2 - 1 \right) X_3 \left( 9 \left( F_0^6 + 3 F_1^4 - 3 F_1^2 - 1 \right) \right. \]
\[ \times X_4 + 2i \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_5 \]
\[ + 4 \left( F_1^8 + 26 F_1^6 + 66 F_1^4 + 26 F_1^2 + 1 \right) X_3^2 \]  
\[ \alpha_1 \beta_2 \]
\[ - \frac{F_1^2}{3 \left( F_1^2 - 1 \right)^6} \left[ -3i \left( F_1^2 + 1 \right) \left( F_1^2 - 1 \right)^3 \right. \]
\[ \times X_3^2 - 24i \left( F_1^2 + 1 \right) \left( F_1^2 - 1 \right)^3 X_3 \]
\[ + 18 \left( F_1^4 + 4 F_1^3 + 1 \right) \left( F_1^2 - 1 \right)^2 \]
\[ \times X_3^2 X_3 - 3 \left( F_1^2 - 1 \right)^2 \left( 15i \left( F_1^4 - 1 \right) \right. \]
\[ \times X_6 + 20 \left( F_1^2 - 1 \right)^2 X_5 \]
\[ - 2 \left( F_1^4 + 4 F_1^2 + 1 \right) X_7 \right) + 6 \left( F_1^2 - 1 \right) \]
\[ \times X_2 \left( - \left( F_1^2 - 1 \right)^3 X_3 + 3 \left( F_1^2 - 1 \right) \left( - 3i \left( F_1^4 - 1 \right) \right. \right. \]
\[ + \left( F_1^4 + 4 F_1^2 + 1 \right) X_5 \]
\[ + 3i \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_3^2 \]
\[ + 6 \left( F_1^2 - 1 \right) X_3 \left( 9 \left( F_0^6 + 3 F_1^4 - 3 F_1^2 - 1 \right) \right. \]
\[ \times X_4 + 2i \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_5 \]
\[ - 4 \left( F_1^8 + 26 F_1^6 + 66 F_1^4 + 26 F_1^2 + 1 \right) X_3^2 \]  
\[ \alpha_2 \beta_2 \]
\[ - \frac{F_1^2}{3 \left( F_1^2 - 1 \right)^8} \left[ -3 \left( F_1^2 - 1 \right)^4 \left( F_1^4 + 4 F_1^2 + 1 \right) \right. \]
\[ \times X_2^3 - 24i \left( F_1^2 - 1 \right)^3 \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_3^2 X_3 \]
\[ + 12 \left( F_1^2 - 1 \right)^2 X_3 \left( i \left( F_1^2 + 1 \right) \left( F_1^2 - 1 \right)^3 \right. \]
\[ \times X_3 - 3 \left( F_1^2 - 1 \right) \left( 3 \left( F_0^6 + 3 F_1^4 - 3 F_1^2 - 1 \right) \right. \]
\[ \times X_4 + 2i \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_5 \]
\[ + 3 \left( F_0^8 + 26 F_1^6 + 66 F_1^4 + 26 F_1^2 + 1 \right) X_3^2 \]
\[ + 4i X_2 \left( 24i \left( F_1^4 + 4 F_1^2 + 1 \right) \left( F_1^2 - 1 \right)^4 \right. \]
\[ \times X_2^3 + 3 \left( F_1^2 - 1 \right)^3 \left( 20 \left( F_1^2 + 1 \right) \left( F_1^2 - 1 \right)^2 \right. \]
\[ \times X_5 + i \left( 15 \left( F_0^6 + 3 F_1^4 - 3 F_1^2 - 1 \right) \right. \]
\[ \times X_6 + 2i \left( F_0^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_7 \right) \]
\[ - 6 \left( F_1^2 - 1 \right)^2 X_3 \left( 9 \left( F_0^6 + 10 F_1^4 - 10 F_1^2 - 1 \right) \right. \]
\[ \times X_4 + 2i \left( F_0^6 + 26 F_1^6 + 66 F_1^4 + 26 F_1^2 + 1 \right) X_5 \]  
\[ + 4 \left( F_1^{12} + 56 F_1^{10} + 245 F_1^8 - 245 F_1^6 - 56 F_1^4 - 1 \right) X_3^3 \]
\[ - 2 \left( 28i \left( F_1^2 - 1 \right)^3 \left( F_1^6 + 11 F_1^4 \right. \right.
\[ + 11 F_1^2 + 1 \right) X_3^2 + 6 \left( F_1^2 - 1 \right)^2 \]
\[ \times X_3 \left( 10i F_1^8 X_6 - F_1^8 X_7 + 100i F_1^6 X_5 - 26 F_1^6 X_7 - 66 F_1^4 X_7 \right. \]
\[ - 18i \left( F_1^2 + 1 \right) X_4 - 100i F_1^2 X_6 - 26 F_1^2 X_7 \]
\[ + 26 \left( F_1^2 - 1 \right)^2 \left( F_1^4 + 4 F_1^2 + 1 \right) X_5 \]
\[ - 10i X_6 - X_7 \right) + 3 \left( F_1^2 - 1 \right)^2 \]
\[ \times \left( 27 \left( F_1^2 - 1 \right)^2 \left( F_1^4 + 4 F_1^2 + 1 \right) X_4^2 \right. \]
\[ + 12i \left( F_1^8 + 10 F_1^6 - 10 F_1^2 - 1 \right) X_4 \]
\[ \times X_5 - \left( F_1^2 - 1 \right)^2 \left( 40 \left( F_1^2 - 1 \right)^3 \right. \]
\[ \times X_6 + i \left( 50 \left( F_1^2 + 1 \right) \left( F_1^2 - 1 \right)^2 X_7 \right. \]
\[ + i \left( 14 \left( F_1^6 + 3 F_1^4 - 3 F_1^2 - 1 \right) \right. \]
\[ \times X_8 + i \left( F_1^6 + 11 F_1^4 + 11 F_1^2 + 1 \right) X_9 \right) \}
\[ - \left( F_1^8 + 26 F_1^6 + 66 F_1^4 + 26 F_1^2 + 1 \right) X_3^2 \]
\[ - 6 \left( F_1^2 - 1 \right) X_3 \left( F_1^{10} \left( 6 X_4 + i X_5 + 1 \right) \right. \]
\[ + F_1^8 \left( 150 X_4 + 571 X_5 - 5 \right) \right. \]
\[ + 2 F_1^6 \left( 120 X_4 + 151 i X_5 + 5 \right) \]
\[ + F_1^2 \left( - 150 X_4 + 571 X_5 + 5 \right) - 6 X_4 + i X_5 - 1 \]
\[ \times \left( F_1^{12} + 120 F_1^{10} + 1191 F_1^8 \right. \]
\[ + 2416 F_1^6 + 1191 F_1^4 + 120 F_1^2 + 1 \right) X_4^2 \]  
\[ 4c. \text{ Regularization: 3-point function} \]

Now we generalize the regularization, performed above, to the case of the 3-point correlator of rank 2 operators. The correlator and the conformal transformation, in the limit $\epsilon \to 0$, are given by:

\[ \langle \xi \rangle e^{\alpha_0 \phi + \alpha_1 \beta \phi + \alpha_2 \gamma \phi} \xi \rangle e^{\beta_0 \phi + \beta_1 \beta \phi + \beta_2 \gamma \phi} \]
\[ \times \langle (\xi_2) e^{\gamma_0 \phi + \gamma_1 \beta \phi + \gamma_2 \gamma \phi} \rangle \right) \cdot \]
\[ f(z) = e^{-i\left( \frac{1}{z-\eta_1} + \frac{1}{z-\eta_2} + \frac{1}{z-\eta_3} \right)} \]
Regularization of the generalized Schwarzians

Denoting $\xi_{ab} \equiv \xi_a - \xi_b$ and

$$X_k \equiv \frac{1}{\xi_{12}^k} + \frac{1}{\xi_{13}^k}, \quad Y_k \equiv \frac{1}{\xi_{21}^k} + \frac{1}{\xi_{23}^k}, \quad Z_k \equiv \frac{1}{\xi_{31}^k} + \frac{1}{\xi_{32}^k}$$

(31)

For $e^{\alpha_0 \phi + \alpha_1 \tilde{\phi} + \alpha_2 \tilde{\phi}^2} (\xi_1)$,

$$S_{0|0} \sim \log \left( i e^{-iX_1} \right), \quad S_{1|0} \sim \frac{i}{2} X_2,$$  
$$S_{1|1} \sim \frac{X_4}{2} + \frac{X_2^2}{12} + X_5,$$  
$$S_{2|0} \sim \frac{1}{2} \left( \frac{X_4}{2} + \frac{X_2^2}{12} + i X_3 \right),$$  
$$S_{2|1} \sim \frac{X_5}{6} - \frac{X_2 X_3}{12} - X_3,$$  
$$S_{2|2} \sim \frac{146 X_8}{480} - \frac{23 X_2 X_6}{60} - \frac{2 X_3 X_5}{5}$$
$$-\frac{31 X_2^4}{160} - \frac{20 X_6}{12} + \frac{X_3^2}{12} + X_4 - \frac{X_2^2}{2}$$

(32)

For $e^{\alpha_0 \phi + \beta_1 \tilde{\phi} + \beta_2 \tilde{\phi}^2} (\xi_2)$ and $e^{\gamma_0 \phi + \gamma_1 \tilde{\phi} + \gamma_2 \tilde{\phi}^2} (\xi_3)$, replace $X_k$ with $Y_k$ and $Z_k$, respectively.

Regularization of the overlap part

We show explicitly in the following the overlap contributions between the irregular vertex operators at $\xi_1$ and $\xi_2$. The other contributions are given by simply replacing $X_k$, $Y_k$ with $X_k$, $Z_k$ for $\xi_1$, $\xi_3$ and with $Y_k$, $Z_k$ for $\xi_2$, $\xi_3$.

Denote $D_k \equiv X_k - Y_k$ and $R \equiv e^{iD_1}$. Again, free correlators are omitted.

$\alpha_0 \beta_0$ regularization

$$- \log \left( e^{-iX_1} - e^{-iY_1} \right)$$

(33)

$\alpha_0 \beta_1$ regularization

$$\frac{R}{(1 - R)^7} \left[ i (R - 1)^2 Y_2 - \frac{1}{2} i D_2^2 (R + 1) + D_3 (R - 1) \right]$$

(34)

For $\alpha_1 \beta_0$ regularization, interchange $X_k \leftrightarrow Y_k$.

$\alpha_1 \beta_1$ regularization

$$\frac{R^3}{(1 - R)^6} \left[ \frac{11 D_2^4}{4} - 3 D_2^3 + 6 D_2^2 X_2 \right]$$
$$+ \frac{1}{R^2} \left( D_2^4 + D_2^3 + D_2^2 \left( -X_2 - \frac{1}{2} (i D_3) \right) \right)$$
$$+ D_2 (-3iD_3 - D_4 - X_2 - 4iX_3) - \frac{D_2^3}{2}$$
$$+ 2i D_3 X_2 - 3 D_4 + i D_5 + X_2^2 + 6 X_4$$
$$+ R^2 \left( \frac{D_2^4}{24} + \frac{D_2^3}{2} + \frac{1}{2} i D_2^2 (D_3 + 2iX_2) \right)$$
$$+ D_2 (3iD_3 - D_4 - X_2 - 4iX_3) - \frac{D_2^3}{2}$$
$$- 2i D_3 X_2 - 3 D_4 - i D_5 + X_2^2 + 6 X_4$$
$$+ R \left( \frac{13 D_2^4}{12} + D_2^3 + D_2^2 (-2X_2 + 5iD_3) \right)$$
$$+ D_2 (-6i D_3 - 2D_4 + 4X_2 + 8iX_3)$$
$$- D_2^3 + 4i D_3 X_2 + 12D_4 + 2i D_5 - 4X_2^2 - 24X_4$$
$$+ 1 \left( \frac{13 D_2^4}{12} + D_2^3 + D_2^2 (-2X_2 - 5iD_3) \right)$$
$$+ D_2 (6i D_3 - 2D_4 + 4X_2 - 8iX_3) - D_2^3$$
$$- 4i D_3 X_2 + 12D_4 - 2i D_5 - 4X_2^2 - 24X_4$$
$$+ 6D_2 (D_4 - X_2)$$
$$+ 3 \left( D_2^3 + 2 \left( -3 D_4 + X_2^2 + 6X_4 \right) \right)$$

(35)
\[ +\frac{1}{24} i \left( 3 \left( R^5 + 57 R^4 + 302 R^3 + 302 R^2 + 57 R + 1 \right) D_2^4 \\
+ 32 (R - 1)^2 \left( R^3 + 11 R^2 + 11 R + 1 \right) \right) D_2^3 \]
\[ + 32 (R - 1)^2 \left( R^3 + 11 R^2 + 11 R + 1 \right) \]
\[ D_2^3 + 36 D_3 i (R^5 + 25 R^4 + 40 R^3 - 40 R^2 \\
- 25 R - 1) D_2^3 + 24 i (R - 1)^2 (8 D_3 \left( R^3 + 3 R^2 - 3 R - 1 \right) \]
\[ + 3 D_{4i} \left( R^3 + 11 R^2 + 11 R + 1 \right) \right) D_2 \\
- 12 (R - 1)^2 \left( 3 \left( R^3 + 11 R^2 + 11 R + 1 \right) D_2^3 \\
+ 2 (R - 1) \left( 8 D_4 \left( R^2 - 1 \right) \right) \]
\[ + 3 D_{5i} \left( R^2 + 4 R + 1 \right) \right) \right) (R - 1)^2 ] \]

(36)

For \( \alpha_1 \beta_2 \) regularization, interchange \( X_k \leftrightarrow Y_k \).

\( \alpha_2 \beta_2 \) regularization

\[ - \frac{R}{40320(1 - R)^{12}} \left[ -(R^{10} + 2036 R^9 + 152637 R^8 \\
+ 2203488 R^7 + 9738114 R^6 + 15724248 R^5 \\
+ 9738114 R^4 + 2203488 R^3 + 152637 R^2 + 2036 R + 1 \right) \]
\[ \times D_2^5 - 112 (R - 1)^2 \left( R^8 + 502 R^7 \\
+ 14608 R^6 + 88234 R^5 + 156190 R^4 + 88234 R^3 \\
+ 14608 R^2 + 502 R + 1 \right) D_2^4 \]
\[ + 56 (R - 1) \left( 2 (R - 1) (2 X_2 - 25) R^8 \\
+ 2 (502 X_2 - 1475) R^7 + 8 (3652 X_2 - 2975) R^6 \\
+ 2 (88234 X_2 - 1925) R^5 + 10 (31238 X_2 + 6125) R^4 \\
+ 2 (88234 X_2 - 1925) R^3 \right) \]
\[ + 8 (3652 X_2 - 2975) R^7 + 2 (502 X_2 - 1475) R \\
+ 2 X_2 - 25 \right) - i D_3 \left( R^8 + 1013 R^7 + 47840 R^6 \\
+ 455192 R^5 + 1310354 R^4 + 1310354 R^3 \\
+ 455192 R^2 + 47840 R^2 + 1013 R + 1 \right) D_2^5 \]
\[ + 336 (R - 1)^2 \left( -14 D_{3i} \left( R^8 + 246 R^7 + 4046 R^6 \\
+ 11326 R^5 - 11326 R^5 - 4046 R^2 - 246 R - 1 \right) \right) \]
\[ + D_4 \left( R^8 + 502 R^7 + 14608 R^6 + 88234 R^5 \\
+ 156190 R^4 + 88234 R^3 + 14608 R^2 + 502 R + 1 \right) \]
\[ + 2 (R - 1) \left( (15 X_2 + 4 i X_3 - 20) R^7 \\
+ (1785 X_2 + 988 i X_3 - 460) R^6 \\
+ 9 (1785 X_2 + 1908 i X_3 + 20) R^5 \right) \]
\[ + (18375 X_2 + 62476 i X_3 + 1900) R^4 \\
+ (18375 X_2 + 62476 i X_3 - 1900) R^3 \\
- 9 (1785 X_2 - 1908 i X_3 + 20) R^2 \]
\[-i(14D_6 \left( R^4 + 10R^3 - 10R - 1 \right) + D_7i \left( R^4 + 26R^3 + 66R^2 \right) + 26R + 1 - 2(R - 1)(2 \left( R^3 + 11R^2 + 11R + 1 \right) - X_2^2 - (R - 1)^2(R + 1)X_2^2 + 6 \left( 3R^3 + 11R^2 + 11R + 1 \right) \times X_4 - 4i \left( R^3 + 3R^2 - 3R - 1 \right)X_3) \times X_2 + 2(-iX_3(R - 1)^3 - 9(R + 1)X_4(R - 1)^2 + 6 \left( R^3 + 11R^2 + 11R + 1 \right)X_3^2 + 12 - iR^3X_5 - 36iR^2X_5 + 12iX_5 + 36iRX_5 + 5R^3X_6 + 55R^2X_6 + 55RX_6 + 5X_6)) \right) \times D_3 - 12(R - 1)^2 \left( 2\left( (2X_2 - 25)R^4 + (52X_2 - 50)R^3 + 6(22X_2 + 25)R^2 + (52X_2 - 50)R + 2X_2 - 25 \right)D_4^2 \right. \\
+ 2\left(-14D_5 \left( R^4 + 10R^3 - 10R - 1 \right) \right) D_6 + 84 \left( R^4 + 26R^3 + 66R^2 + 26R + 1 \right) \times (R + 1)X_3 + 12i \left( R^3 + 11R^2 + 11R + 1 \right)X_2X_3 - 27X_4 + 8iX_5)D_4 + 84 \left( R^4 + 26R^3 + 66R^2 + 26R + 1 \right) \times (R - 1)^2 \left( -20(R + 1)X_2(R - 1)^2 + 3 \left( R^3 + 11R^2 + 11R + 1 \right) X_2^2 + 6 \left( R^3 + 11R^2 + 11R + 1 \right) \times X_4 - 4i \left( R^3 + 3R^2 - 3R - 1 \right) X_3) \right) + 2(R - 1) \left( -R^3X_4^2 - 3R^2X_4^2 + 3RX_2^4 \right. \\
+ X_4^2 - 36RX_3X_4^2 - 108R^2X_4X_5^2 + 108RX_4X_5^2 + 36RX_4X_5^2 - 48R^3X_4^3X_2 - 144R^2X_4^3X_2 + 144R^3X_3^3X_2 \times X_4 + 144RX_3^2X_2X_2 - 60R^3X_6X_2 \times -180R^2X_6X_2 + 180RX_6X_2 + 60X_6X_2 + 14D_8R^3 + D_{8i}R^3 + 42D_8R^2 + 11D_{8i}R^2 \\
\left. - 4R^3X_3^2 + 12R^2X_3^2 - 12RX_3^2 + 4X_2^2 - 54RX_3^2X_4^2 + 12R^2X_4^2 + 162RX_4^2 + 54X_4^2 - 14D_8 + D_{9i} - 42D_8R + 11D_{9i}R + 2D_{7}(R + 1) \right) (2X_2 - 25)R^2 + 10(2X_2 + 5)R + 2X_2 - 25 \right) + 2D_6((15X_2 + 4iX_3 - 20)R^3 + (45X_2 + 4iX_3 + 60)R^2 + (-45X_2 + 4iX_3 - 60)R \\
- 15X_2 + 4iX_3 - 20 - 96R^2X_3X_5 - 288R^2X_3X_5 + 288RX_3X_5 + 96X_3X_5 + 80R^3X_6 \\
- 240R^2X_6 + 240RX_6 - 80X_6 - 28R^3X_8 \times - 84R^2X_8 + 84RX_8 + 8X_8) \right) \right) \right) \] 

(37)

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