Beliefs and Expertise in Sequential Decision Making

Daewon Seo, Ravi Kiran Raman, Joong Bum Rhim, Vivek K Goyal, Lav R. Varshney

Abstract

This work explores a sequential decision making problem with agents having diverse expertise and mismatched beliefs. We consider an $N$-agent sequential binary hypothesis test in which each agent sequentially makes a decision based not only on a private observation, but also on previous agents' decisions. In addition, the agents have their own beliefs instead of the true prior, and have varying expertise in terms of the noise variance in the private signal. We focus on the risk of the last-acting agent, where precedent agents are selfish. Thus, we call this advisor(s)-advisee sequential decision making.

We first derive the optimal decision rule by recursive belief update and conclude, counterintuitively, that beliefs deviating from the true prior could be optimal in this setting. The impact of diverse noise levels (which means diverse expertise levels) in the two-agent case is also considered and the analytical properties of the optimal belief curves are given. These curves, for certain cases, resemble probability weighting functions from cumulative prospect theory, and so we also discuss the choice of Prelec weighting functions as an approximation for the optimal beliefs, and the possible psychophysical optimality of human beliefs. Next, we consider an advisor selection problem wherein the advisee of a certain belief chooses an advisor from a set of candidates with varying beliefs. We characterize the decision region for choosing such an advisor and argue that an advisee with beliefs varying from the true prior often ends up selecting a suboptimal advisor, indicating the need for a social planner. We close with a discussion on the implications of the study toward designing artificial intelligence systems for augmenting human intelligence.

Index Terms

social learning, sequential binary hypothesis test, cumulative prospect theory, augmented intelligence

I. INTRODUCTION

Team decision making typically involves individual decisions that are influenced by private observations and the opinions of the rest of the team. The social learning setting is one such context where decisions of individual agents are influenced by preceding agents in the team [3], [4]. We consider the setting in which individual agents

This paper was presented in part at the IEEE International Symposium on Information Theory (ISIT 2013) [1] and at the IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP 2018) [2].

J. B. Rhim is with GroupM, New York, NY 10007 USA (e-mail: jbrhim@alum.mit.edu).

V. K. Goyal is with the Department of Electrical and Computer Engineering, Boston University, Boston, MA 02215 USA (e-mail: v.goyal@ieee.org).

D. Seo, R. K. Raman, and L. R. Varshney are with the Coordinated Science Laboratory, and the Department of Electrical and Computer Engineering, University of Illinois at Urbana-Champaign, Urbana, IL 61801 USA (e-mail: {dseo9, rraman10, varshney}@illinois.edu).
are selfish and aim to minimize their perceived Bayes risk, according to their beliefs as reinforced by the decisions of preceding agents. Social learning, also referred to as observational learning, has been widely studied and we provide a non-exhaustive listing of some of the relevant works.

Aspects of conformism and “herding” were studied in [5]–[7], where an incorrect decision may cascade for the rest of the agents once agents at the beginning make incorrect decisions. The concept of herding is a consequence of boundedly informative private signals [8]. For example, assume the private signals are binary and give true or false information, each with positive probability. It can happen that a couple of the first agents receive false private signals and thus choose wrong actions. Then, the effect of these actions on the beliefs of subsequent agents can be so great as to cause them to ignore their private signals and follow their precedent agents. The private signals are bounded so that they are not strong enough to overcome the effect of the wrong actions. Further convergence properties of actions taken under social learning have been explored under imperfect information [9]. The notion of sequential social learning has been generalized to learning from neighbors in networks [10], and explored in generality [11]. Social learning has also been explored under quantization of priors [12], and distributed detection with symmetric fusion [13].

This paper differs from the literature in the sense that we consider unbounded private signals so that there is no herding behavior. In addition, we focus largely on the effects of prior probability and private signal strength. Unlike sequential decision making [14] where all agents know the true prior, we assume agents may have beliefs that do not match the true prior. Further, private signal strengths of agents could be different, i.e., noise variances are not necessarily identical, which translates to varying expertise. The decision making of individual agents is also different in that the agents perform locally Bayes optimal decisions, i.e., decisions that minimize their individual Bayes risk. This is different from the context of collectively optimizing the team’s risk [13] or decision making that maximizes a personal reward [15].

We study a sequential binary hypothesis test in the social learning framework, termed social teaching, and characterize optimal beliefs of agents that minimize the Bayes risk of the last-acting agent. As such, we often refer to agents as advisor(s)-advisee, where the advisee is the last-acting agent. In general, it counterintuitively turns out that agents using beliefs that do not match the true prior are optimal, i.e., each agent has a perceived belief of the prior. For instance, in the two-agent system with equal expertise levels, the optimal advisor in the social teaching context is one who is open-minded, i.e., overweights the belief for small prior, and underweights when it is large. On the other hand, the corresponding optimal advisee is one who is closed-minded and behaves in the opposite way to the advisor. We describe analytical characteristics of the optimal beliefs and also show how the nature of such behaviors of agents change when expertise levels differ (which is characterized by the observational noise in the private signals).

We are ultimately interested in the Bayes risk of the advisee (i.e., the last-acting agent), and thus it is important that the advisee uses the correct set of advisors for the task. To this end, we consider team selection for such sequential hypothesis testing, and characterize the criterion used for advisor selection. Through this study we observe that self-organized teams may results in suboptimal compositions, emphasizing the importance of a social planner who is aware of the true prior.
Human actions are typically affected by individual perceptions of the underlying context. Cumulative prospect theory [16]–[18] seeks to provide a psychological understanding of human behaviors under risk. It introduces the notion of probability reweighting functions to explain boundedly rational human behaviors. Among reweighting functions, the Prelec reweighting function [19] has significant empirical support and satisfies a majority of the axioms of prospect theory. Interestingly, the Prelec function spans a class of open- and closed-minded beliefs, that are observed to be optimal in certain cases of social teaching, and hence one might expect these human cognitive biases to emerge as the information-theoretically optimal choices for social learning. However, we discuss that it does not capture all behavioral patterns for optimal beliefs of agents in the case of diverse expertise levels.

In the era of AI (Artificial Intelligence or Augmented Intelligence), a sequential decision making model captures the nature of collaboration in human-AI teams with either the AI system advising the human who makes the final decision or less typically a human advising an AI system that makes the final decision [20, p. 56]. Examples of this kind include AI-assisted physicians or chess players (called centaur chess), and of the second kind, human-in-the-loop AI systems such as crowdsourcing systems and collaborative filtering mechanisms. Our work shows the interesting conclusion that a team of suboptimal human-AI could beat the team of individually optimal human-AI, if it is well-composed.

The rest of this paper is organized as follows. Sec. II describes the sequential binary hypothesis testing problem. Sec. III proposes a recursive belief update equation that transforms the sequential decision making problem into a single-agent decision making problem. Sec. IV shows the optimal beliefs that minimize the advisee’s risk and Sec. V evaluates them for Gaussian likelihoods. Sec. VI considers a two-agent team construction problem. Sec. VII concludes.

This sequential decision making problem with identical expertise agents was first presented in [1] and in particular two-agent systems with diverse expertise were investigated in [2]. This paper integrates and generalizes our previous results, and also, significantly improves analytic understanding on optimal beliefs in sequential decision making. In addition, we provide a novel interpretation of Prelec-like beliefs in terms of AI-human collaboration systems.

II. PROBLEM DESCRIPTION

Consider an $N$-agent sequential decision making problem, as illustrated in Fig. 1. The underlying hypothesis, $H \in \{0, 1\}$, is a binary signal with prior $\mathbb{P}(H = 0) = p_0$ and $\mathbb{P}(H = 1) = 1 - p_0$. There are $N$ agents that sequentially...
detect the state in a predetermined order. The \( n \)th agent has a private signal \( Y_n \) generated according to the likelihood \( f_{Y_n|H} \), which is not necessarily identical for all \( n \). Let the decision made by the \( n \)th agent be \( \hat{H}_n \). In addition to the private signal, the \( n \)th agent also observes the decisions made by preceding agents, \( \{\hat{H}_1, \ldots, \hat{H}_{n-1}\} \), which is referred to as the public signal and is used in the agent’s decision making.

However, the \( n \)th agent believes the prior probability of the null hypothesis is \( q_n \in [0, 1] \) as against the true prior probability \( p_0 \). We call this the belief of the agent in order to distinguish it from the prior. Agent \( n \) is also aware of her own likelihoods \( f_{Y_n|H} \) that define her private signal. However, she also perceives the likelihoods and beliefs of the other agents to be the same as hers, i.e., she thinks \( f_{Y_j|H} = f_{Y_n|H}, q_j = q_n \) for all \( j \neq n \), even though they could be different and unknown to her. We assume that the likelihood ratio of each agent is an increasing function in \( y \), i.e., for all agents \( L_n(y_n) \) is an increasing function in \( y_n \), where

\[
L_n(y_n) := \frac{f_{Y_n|H}(y_n|1)}{f_{Y_n|H}(y_n|0)}.
\]

Our performance analysis focuses on the last agent (\( N \)th agent, Norah) and her decision \( \hat{H}_N \). Upon observing her private signal \( Y_N \) and the \((N-1)\) preceding decisions, she determines her decision rule. The relative importance of correct decisions and errors can be abstracted as a cost function. For simplicity, we assume correct decisions have zero cost and use the shorthand notations \( c_{10} = c(1, 0) \) as the cost for false alarm or Type I error (choosing \( \hat{H} = 1 \) when \( H = 0 \), and \( c_{01} = c(0, 1) \) as the cost for missed detection or Type II error (choosing \( \hat{H} = 0 \) when \( H = 1 \)). In addition, we assume that agents have the same costs; they are a team in the sense of Radner [21]. Then the Bayes risk is

\[
R_N = c_{10}p_0p_{\hat{H}_N|H}(1|0) + c_{01}(1 - p_0)p_{\hat{H}_N|H}(0|1).
\]

As \( \hat{H}_n \) depends on the previous decisions, the computation of (1) also depends on \( \{\hat{H}_1, \ldots, \hat{H}_{N-1}\} \), and the Bayes risk can be expanded as

\[
R_N = \sum_{\hat{h}_1, \ldots, \hat{h}_{N-1}} c_{10}p_0p_{\hat{H}_N, \hat{H}_{N-1}, \ldots, \hat{h}_1|H}(1, \hat{h}_{N-1}, \ldots, \hat{h}_1|0) + c_{01}(1 - p_0)p_{\hat{H}_N, \hat{H}_{N-1}, \ldots, \hat{h}_1|H}(0, \hat{h}_{N-1}, \ldots, \hat{h}_1|1).
\]

We determine the optimal set of beliefs of the agents \( \{q_n^*\}_{n=1}^N \) that minimize (2).

In our model, the \( n \)th agent minimizes her perceived Bayes risk, which is the Bayes risk with prior probability \( p_0 \) replaced by her belief \( q_n \). In other words, for all \( n = 1, \ldots, N \), the \( n \)th agent adopts the decision rule that minimizes her perceived Bayes risk \( R_n \), and her decision is revealed to other agents as a public signal. The decisions \( \{\hat{H}_1, \ldots, \hat{H}_{n-1}\} \) of the earlier-acting agents reveal information about \( H \) and thus should be incorporated into the decision-making process by agent \( n \). As mentioned earlier, since she believes \( q_n \) is the true prior, she aggregates information under the assumption that \( q_1 = q_2 = \cdots = q_n \).

It is important to note that every agent is selfish and rational; the agents do not adjust their decision rules for Norah’s sake. The novelty in the model (and hence in the conclusions) comes from agent \( n \) having the limitation of using a private initial belief \( q_n \) in place of the true prior probability \( p_0 \).
In this problem statement section, let us also formally introduce the Prelec reweighting function from cumulative prospect-theoretic models of human behavior. It spans a family of open- and closed-minded beliefs (will be discussed later) and thus the optimal beliefs that emerge in following sections could be approximated by a function in the Prelec family.

**Definition 1 ([19]):** For $\alpha, \beta > 0$, the Prelec reweighting function $w : [0, 1] \mapsto [0, 1]$ is

$$w(p; \alpha, \beta) = \exp(-\beta(-\log p)^\alpha).$$

The function satisfies several properties such as:

1) $w(p; \alpha, \beta)$ is strictly increasing;
2) has a unique fixed point $w(p; \alpha, \beta) = p$ at $p^* = \exp(-\exp(\log \beta/(1 - \alpha)))$; and
3) spans a class of open-minded beliefs when $\alpha < 1$, i.e., overweights (underweights) small (high) probability, and vice versa when $\alpha > 1$.

A more generic form, termed composite Prelec weighting function, has been defined in [22].

Throughout the paper, we use $f$ for continuous probability density functions and $p$ for discrete probability mass functions. All logarithms are natural logarithms. We use $\mathcal{N}(\mu, \sigma^2)$ to denote a Gaussian distribution with mean $\mu$ and variance $\sigma^2$, and $\phi(x; \mu, \sigma^2)$ to denote its density function, i.e.,

$$\phi(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

Also in the case of the standard Gaussian, $\phi(x) = \phi(x; 0, 1)$ for simplicity. $Q(x)$ is defined as the complementary cumulative distribution function of the standard Gaussian,

$$Q(x) = \int_x^\infty \phi(t) dt.$$

**III. BELIEF UPDATE AND SEQUENTIAL DECISION MAKING**

Our model assumes unbounded private signals. Thus, unlike in [5], [6], it is always possible that a subsequent agent may not follow previous decisions; that is, herding happens with arbitrarily low probability. We now discuss using both a decision history and private signals for Bayesian binary hypothesis testing. The decision rule can be interpreted as each agent updating her posterior belief based on the decision history and then applying a likelihood ratio test to her private signal.

**A. Alexis, the First Agent**

Since Alexis has no prior decision history, she follows usual binary hypothesis testing. She uses the following likelihood ratio test with her prior belief $q_1$, with ties broken arbitrarily:

$$L_1(y_1) = \frac{f_{Y_1|H}(y_1|1)}{f_{Y_1|H}(y_1|0)} \frac{\hat{H}_1=1}{\hat{H}_1=0} \frac{c_{10}q_1}{c_{01}(1 - q_1)}.$$

Since we assume the likelihood ratio is increasing in $y_1$, the rule simplifies to comparing the private signal with an appropriate decision threshold:

$$y_1 \begin{cases} \hat{H}_1=1 & \lambda_1(q_1), \\ \hat{H}_1=0 \end{cases}$$

(4)
where \( \lambda_1(q_1) \) denotes the decision threshold \( \lambda \) that satisfies

\[
\mathcal{L}_1(\lambda) = \frac{f_{Y_1|H}(\lambda|1)}{f_{Y_1|H}(\lambda|0)} = \frac{c_{10} q_1}{c_{01}(1 - q_1)}.
\] (5)

Note that in assuming \( \mathcal{L}_n \) is monotonically increasing, there is at most one solution to (5). Then, the decision \( \hat{H}_1 \) made by Alexis, according to the likelihood ratio test, is revealed to other agents.

B. Blake, the Second Agent

Blake observes Alexis’s decision \( \hat{H}_1 = \hat{h}_1 \) and evaluates the likelihood ratio for \( \hat{H}_1, Y_2 \), using his prior belief \( q_2 \) as

\[
\frac{f_{Y_2,\hat{H}_1|H}(y_2, \hat{h}_1|1)}{f_{Y_2,\hat{H}_1|H}(y_2, \hat{h}_1|0)} \overset{\hat{H}_2=1}{\gtrless} \frac{c_{10} q_2}{c_{01}(1 - q_2)}.
\] (6)

The private signals \( Y_1 \) and \( Y_2 \) are independent conditioned on \( H \), so \( \hat{H}_1 \) and \( Y_2 \) are also independent conditioned on \( H \). Hence, the left side of (6) is

\[
f_{Y_2,\hat{H}_1|H}(y_2, \hat{h}_1|h) = f_{Y_2|H}(y_2|h)p_{\hat{H}_1|H}(\hat{h}_1|h).
\]

So we can rewrite (6) as

\[
\frac{f_{Y_2|H}(y_2|1)}{f_{Y_2|H}(y_2|0)} \overset{\hat{H}_2=1}{\gtrless} \frac{c_{10} q_2}{c_{01}(1 - q_2)} \frac{p_{\hat{H}_1|H}(\hat{h}_1[2]|0)}{p_{\hat{H}_1|H}(\hat{h}_1[1]|2)}.
\] (7)

The likelihood ratio test (7) can be interpreted as Blake updating his prior belief upon observing Alexis’s decision \( \hat{H}_1 \). Combined with \( q_2 \), his prior belief is updated according to \( p_{\hat{H}_1|H}(\hat{h}_1[2]|2) \), from \( q_2 \) to \( q_2^\wedge \):

\[
\frac{q_2^\wedge}{1 - q_2^\wedge} = \frac{q_2}{1 - q_2} \frac{p_{\hat{H}_1|H}(\hat{h}_1[0]|2)}{p_{\hat{H}_1|H}(\hat{h}_1[1]|2)}.
\] (8)

The updated belief is

\[
q_2^\wedge = \frac{q_2p_{\hat{H}_1|H}(\hat{h}_1[0]|2)}{q_2p_{\hat{H}_1|H}(\hat{h}_1[0]|2) + (1 - q_2)p_{\hat{H}_1|H}(\hat{h}_1[1]|2)} = \frac{p_{\hat{H}_1,\hat{H}}(\hat{h}_1, 0)[2]}{p_{\hat{H}_1,H}(\hat{h}_1, 0)[2] + p_{\hat{H}_1,H}(\hat{h}_1, 1)[2]} = p_{H|\hat{H}_1}(0|\hat{h}_1)[2].
\] (9)

It should be noted that the true \( p_{\hat{H}_1|H}(\hat{h}_1|h) \) is given by

\[
p_{\hat{H}_1|H}(0|h) = p_{\hat{H}_1,0|H}(0|h) = \mathbb{P}Y_1 \leq \lambda_1(q_1)|H = h = \int_{-\infty}^{\lambda_1(q_1)} f_{Y_1|H}(y|h)dy,
\]

\[
p_{\hat{H}_1|H}(1|h) = \int_{\lambda_1(q_1)}^{\infty} f_{Y_1|H}(y|h)dy.
\]

1The subscript \([2]\) in the term \( p_{\hat{H}_1|H}(\hat{h}_1[2]|h) \) indicates the value of \( p_{\hat{H}_1|H}(\hat{h}_1|h) \) that Blake (the second agent) thinks. We specify this because Blake does not know Alexis’s belief \( q_1 \). Thus, he interprets her decision based on his belief \( q_2 \). The value is different from the true value of \( p_{\hat{H}_1|H}(\hat{h}_1|h) = p_{\hat{H}_1|H}(\hat{h}_1[1]|h) \). Of course, it will also be different from what Chuck, the third agent, perceives, which is denoted by \( p_{\hat{H}_1|H}(\hat{h}_1|h)[3] \). This will be explained in the next subsection.
But Blake evaluates Alexis’s decision $\hat{H}_1$ as if it were made based on $q_2$ and the likelihood $f_{Y_2|H}(\cdot)$, as against $q_1, f_{Y_1|H}(\cdot)$ respectively. Thus the probability $p_{\hat{H}_1|H}(\hat{h}_1|h)$ is computed based on $\lambda_2(q_2)$, instead of $\lambda_1(q_1)$:

$$p_{\hat{H}_1|H}(0|h)[2] = \int_{-\infty}^{\lambda_2(q_2)} f_{Y_2|H}(y|h)dy,$$

$$p_{\hat{H}_1|H}(1|h)[2] = \int_{\lambda_2(q_2)}^{\infty} f_{Y_2|H}(y|h)dy.$$  \hspace{1cm} (10a)

An interesting observation is that Alexis’s belief $q_1$ does not affect Blake’s belief update as observed in (9) and (10). That is, for any belief $q_1$ that Alexis might hold, Blake, who does not know this belief, presumes that the conditional probabilities are computed according to (10) and updates his belief as in (9) which depends only on Blake’s initial belief and Alexis’s decision.

However, Alexis’s prior belief implicitly affects Blake’s performance since her biased belief changes the resulting decisions whose probabilities are embedded in the probability of Blake’s decision:

$$p_{\hat{H}_2|H}(\hat{h}_2|h) = \sum_{\hat{h}_1 \in \{0,1\}} p_{\hat{H}_2,\hat{h}_1|H}(\hat{h}_2, \hat{h}_1|h)$$

$$= p_{\hat{H}_2|H}(\hat{h}_2|0, h)[2] \times p_{\hat{H}_1|H}(0|h)[1] + p_{\hat{H}_2|H}(\hat{h}_2|1, h)[2] \times p_{\hat{H}_1|H}(1|h)[1].$$

Thus, Alexis’s biased belief changes the probability of not only her decision but also of Blake’s decision.

C. Chuck, the Third Agent

Chuck’s detection process is similar to Blake’s. He observes both Alexis’s and Blake’s decisions and also updates his prior belief $q_3$ like in (8):

$$\frac{q_3^{AB}}{1 - q_3^{AB}} = \frac{q_3}{1 - q_3} \frac{p_{\hat{H}_2, \hat{H}_1|H}(\hat{h}_2, \hat{h}_1)[3]}{p_{\hat{H}_2, \hat{H}_1|H}(\hat{h}_2, \hat{h}_1)[3]} = \left(\frac{q_3}{1 - q_3} \frac{p_{\hat{H}_1|H}(\hat{h}_1)[3]}{p_{\hat{H}_1|H}(\hat{h}_1)[3]} \right) \frac{p_{\hat{H}_2|H}(\hat{h}_2, \hat{h}_1)[0]}{p_{\hat{H}_2|H}(\hat{h}_2, \hat{h}_1)[3]}.$$  \hspace{1cm} (11)

Note that $\hat{H}_1$ and $\hat{H}_2$ are not conditionally independent given $H$ as Blake’s decision $\hat{H}_2$ depends on Alexis’s decision $\hat{H}_1$.

Chuck’s belief update can be understood as a two-step process. The first step is to update his belief according to Alexis’s decision:

$$\frac{q_3^A}{1 - q_3^A} = \frac{q_3}{1 - q_3} \frac{p_{\hat{H}_1|H}(\hat{h}_1)[3]}{p_{\hat{H}_1|H}(\hat{h}_1)[3]}.$$  \hspace{1cm} (12)

The second step is to update it from $q_3^A$ based on Blake’s decision:

$$\frac{q_3^{AB}}{1 - q_3^{AB}} = \frac{q_3^A}{1 - q_3^A} \frac{p_{\hat{H}_2|\hat{H}_1, H}(\hat{h}_2|\hat{h}_1, 0)[3]}{p_{\hat{H}_2|\hat{H}_1, H}(\hat{h}_2|\hat{h}_1, 1)[3]}.$$  \hspace{1cm} (13)

Again, Chuck is not aware of neither Alexis’s nor Blake’s prior beliefs or likelihoods. Thus, Chuck computes all probabilities based on his own belief $q_3$ and likelihood $f_{Y_3|H}$, which is indicated by the subscript [3] in (12) and (13).
Details of computations of (12) and (13) are as follows:

\[ p_{\hat{B}_1|H}(0|h)[3] = \int_{-\infty}^{\lambda_3(q_3)} f_{Y_3|H}(y|h)dy, \]

\[ p_{\hat{B}_1|H}(1|h)[3] = \int_{\lambda_3(q_3)}^{\infty} f_{Y_3|H}(y|h)dy. \]

Similar to Blake (8), Chuck computes \( q_3^A \) for \( \hat{H}_1 = 0 \) and \( \hat{H}_1 = 1 \) respectively as:

\[ q_3^0 = \frac{q_3}{q_3 + (1 - q_3) \int_{-\infty}^{\lambda_3(q_3)} f_{Y_3|H}(y|1)dy}, \]  \hspace{1cm} (14a)

\[ q_3^1 = \frac{q_3}{q_3 + (1 - q_3) \int_{-\infty}^{\lambda_3(q_3)} f_{Y_3|H}(y|0)dy}. \]  \hspace{1cm} (14b)

Then,

\[ p_{\hat{B}_2|\hat{H}_1, H}(0|\hat{h}_1, h)[3] = \int_{-\infty}^{\lambda_3(q_3^0)} f_{Y_3|H}(y|h)dy, \]  \hspace{1cm} (15a)

\[ p_{\hat{B}_2|\hat{H}_1, H}(1|\hat{h}_1, h)[3] = \int_{\lambda_3(q_3^0)}^{\infty} f_{Y_3|H}(y|h)dy. \]  \hspace{1cm} (15b)

Even though the value of \( \hat{h}_1 \) does not appear in (15), it is implicit in \( q_3^A \) and affects the computation results. Chuck’s updated belief \( q_3^{AB} \) is obtained by substituting (14) and (15) in (13).

D. Norah, the Nth Agent

Norah, the Nth agent, observes \( Y_N \) and \{\( \hat{H}_1, \ldots, \hat{H}_{N-1} \}\). Paralleling the arguments in the preceding subsections, her prior belief update is a function of \( q_N \) as well as \{\( \hat{H}_1, \ldots, \hat{H}_{N-1} \}\), but not of \{\( q_1, \ldots, q_{N-1} \)\}. Generalizing (11), we have

\[ \frac{q_{N}^{AB\cdots M}}{1 - q_{N}^{AB\cdots M}} = \frac{q_N}{1 - q_N} \frac{p_{\hat{B}_1|H}(\hat{h}_1)[N]}{\prod_{n=2}^{N-1} p_{\hat{B}_n|\hat{B}_{n-1}, \ldots, \hat{B}_1, H}(\hat{h}_n|\hat{h}_{n-1}, \ldots, \hat{h}_1)[N]} \]  \hspace{1cm} (16)

Combining all observations, we obtain the following theorem. Define the prior belief update function, \( U_n \) as

\[ q_{N}^{AB\cdots M} = U_N(q_N, \hat{h}_1, \hat{h}_2, \ldots, \hat{h}_{N-1}). \]

Theorem 1: The function \( U_n \) has the following recurrence relation:

- For \( n = 1 \), \( U_1(q) = q \).
- For \( n > 1 \),

\[ U_n(q, \hat{h}_1, \ldots, \hat{h}_{n-2}, 0) = \frac{\hat{q}}{\hat{q} + (1 - \hat{q}) \int_{-\infty}^{\lambda_3(q_{n+1})} f_{Y_{n+1}|H}(y|1)dy}, \]  \hspace{1cm} (17a)

\[ U_n(q, \hat{h}_1, \ldots, \hat{h}_{n-2}, 1) = \frac{\hat{q}}{\hat{q} + (1 - \hat{q}) \int_{\lambda_3(q_{n+1})}^{\infty} f_{Y_{n+1}|H}(y|0)dy}. \]  \hspace{1cm} (17b)

where \( \hat{q} = U_{n-1}(q, \hat{h}_1, \ldots, \hat{h}_{n-2}) \).
Updated belief \((q^n_{ABC})\) — for each possible combination of Alexis’s, Blake’s, and Chuck’s decisions \([\hat{h}_1, \hat{h}_2, \hat{h}_3]\) when \(c_{10} = c_{01} = 1\) and private signals are distorted by additive Gaussian noise with variance 1 (left panel) and 0.5 (right panel), respectively. The updated belief is mostly dependent on Chuck’s decision; the top four curves are for \(\hat{h}_3 = 0\) and the bottom four curves are for \(\hat{h}_3 = 1\).

Fig. 2 depicts the function \(U_4(q_4, \hat{h}_1, \hat{h}_2, \hat{h}_3)\) for \(N = 4\) for eight possible combinations of Alexis’s, Blake’s, and Chuck’s decisions \([\hat{h}_1, \hat{h}_2, \hat{h}_3]\). An interesting property of \(U_n\) is that the updated belief is much more dependent on the most recent decision \(\hat{h}_{n-1}\) than on the earlier decisions \(\hat{h}_1, \ldots, \hat{h}_{n-2}\). This is especially the case when the \((n-1)\)th agent has not followed precedent. This is because the \(n\)th agent rationally concludes that the \((n-1)\)th agent observed strong evidence to justify a deviation from precedent. For example, if the decision history of the first five agents is \([0, 0, 0, 0, 1]\) then the sixth agent takes the last decision 1 seriously even though the first four agents chose 0. A reversal of an arbitrarily long precedent sequence may occur because we assume unbounded private signals; if private signals are bounded [5], [6], then the influence of the precedent can reach a point where agents cannot receive a signal strong enough to justify a decision running counter to precedent. Another interesting point is that smaller noise variance changes beliefs more. It is clear from (17), but also reasonable that when the variance is smaller (i.e., agents with more expertise), the \(n\)th agent trusts and is more inclined towards previous decisions. Note even though the prior updates of Norah in Fig. 2 do not depend on \(\{q_1, \ldots, q_{N-1}\}\) and their corresponding likelihoods, the probability of prior decisions depends on them and implicitly, so does Norah’s decision.

As we can see in Fig. 2 the dominant previous decision for agent \(n\) is the decision of agent \((n-1)\). We can prove that observing the \((n-1)\)th agent’s decision 0 (or decision 1), the \(n\)th agent’s updated belief becomes larger (or smaller), which in turn implies that the decision threshold of \(n\)th agent becomes larger (or smaller) so that she is more likely to declare decision 0 (or 1) as well.

**Theorem 2:** Suppose that noises are independent and additive, and have continuous densities. Fix some prior decisions \(\{\hat{h}_1, \ldots, \hat{h}_{n-2}\}\) and let \(\hat{q}_n, \hat{q}_n^0, \hat{q}_n^1\) denote the posterior beliefs of the \(n\)th agent given the \(n-2\) decisions only, the \(n-2\) decisions and \(\hat{h}_{n-1} = 0\), and the \(n-2\) decisions and \(\hat{h}_{n-1} = 1\). Then,

\[
\hat{q}_n^1 < \hat{q}_n < \hat{q}_n^0.
\]
Proof: We know that \( \tilde{q}_n, \tilde{q}_n^0, \tilde{q}_n^1 \) differ only by the last multiplicative term of (16). Since \( \frac{q}{1-q} \) is monotone increasing, the statement is equivalent to showing:

\[
\frac{\int_{\lambda_n(\tilde{q}_n)}^{\infty} f_{Y_n|H}(y|0)dy}{\int_{\lambda_n(\tilde{q}_n)}^{\infty} f_{Y_n|H}(y|1)dy} < 1 < \frac{\int_{\lambda_n(\tilde{q}_n)}^{\infty} f_{Y_n|H}(y|0)dy}{\int_{\lambda_n(\tilde{q}_n)}^{\infty} f_{Y_n|H}(y|1)dy}.
\]

Since the noise is independent and additive, \( f_{Y_n|H}(y|1) = f_{Y_n|H}(y-1|0) \) so the term on the left side

\[
\frac{\int_{\lambda_n(\tilde{q}_n)}^{\infty} f_{Y_n|H}(y|0)dy}{\int_{\lambda_n(\tilde{q}_n)-1}^{\infty} f_{Y_n|H}(y|0)dy} \leq \frac{\int_{\lambda_n(\tilde{q}_n)}^{\infty} f_{Y_n|H}(y|0)dy}{\int_{\lambda_n(\tilde{q}_n)-1}^{\infty} f_{Y_n|H}(y|0)dy} < 1.
\]

The right inequality can be shown similarly.

Considering the complicated relationships that individual decisions have on the evolution of posterior beliefs, it is also important to verify if the posterior belief evolution preserves the ordering, given the same set of decisions. That is, given two posterior beliefs \( q < q' \) and the same sequence of following \( d \) decisions, then it is important to characterize the likelihoods for which the ordering is preserved in the resulting posterior beliefs, given the sequence of decisions, which is described in the following theorem.

**Theorem 3:** Suppose that noise is independent and additive, and has a continuous density. Consider two beliefs \( q_L < q_R \). Then, for any given later-acting decisions \( d \), updated belief satisfies \( q_L^d < q_R^d \) if and only if

\[
\begin{align*}
g_1(q) & := \frac{q}{1-q} \int_{\lambda_n(q)}^{\infty} f_{Y_n|H}(y|0)dy, \\
g_2(q) & := \frac{q}{1-q} \int_{\lambda_n(q)}^{\infty} f_{Y_n|H}(y|1)dy.
\end{align*}
\]

are both increasing in \( q \).

Proof: Note that once observing decision 0, beliefs are updated as

\[
\begin{align*}
\frac{q_L^0}{1-q_L^0} & = \frac{q_L}{1-q_L} \int_{\lambda_n(q_L)}^{\infty} f_{Y_n|H}(y|0)dy, \\
\frac{q_R^0}{1-q_R^0} & = \frac{q_R}{1-q_R} \int_{\lambda_n(q_R)}^{\infty} f_{Y_n|H}(y|0)dy,
\end{align*}
\]

so that the condition for which order \( q_L^0 < q_R^0 \) is preserved is (18). Similarly, (19) can be shown by updating after decision 1.

Let us state some properties of Mills ratio (23), (24), which is about Gaussian distribution, and we will see that \( g_1(q), g_2(q) \) are both increasing if likelihoods are Gaussian.

**Lemma 1** (24): Define \( \eta(x) := \phi(x)/Q(x) \), the inverse of Mills ratio. Then, for any \( x \in \mathbb{R} \), it is true that

\( 0 < \eta'(x) < 1 \) and \( \eta''(x) > 0 \).

**Corollary 1:** Consider a Gaussian likelihood, i.e., \( Y_n = H + Z_n \), where \( Z_n \) are independent and identically drawn from \( N(0, \sigma^2) \), for some \( \sigma^2 > 0 \). Then the functions \( g_1(q), g_2(q) \) are both increasing functions of \( q \).

Proof: Let us consider \( g_2(q) \) first. For the Gaussian hypothesis test, we know that the threshold for the likelihood ratio test is given by

\[
\lambda_n(q) = \frac{1}{2} + \sigma^2 \log \left( \frac{c_{10}q}{c_{01}(1-q)} \right).
\]
Then, we have

\[ g_2(q) = \frac{q}{1 - q} \frac{Q\left(\frac{\lambda_n(q)}{\sigma}\right)}{Q\left(\frac{\lambda_n(q) - 1}{\sigma}\right)}. \]

Letting \( x := \log \frac{c_{10}}{c_{01}(1-q)} \), it is sufficient to show that

\[ \tilde{g}(x) := \log \left( \frac{c_{10}}{c_{01}} g_2(q) \right) = x + \log \left( Q\left(\sigma x + \frac{1}{2\sigma}\right) \right) - \log \left( Q\left(\sigma x - \frac{1}{2\sigma}\right) \right), \]

is increasing in \( x \) since \( c_{10}, c_{01} \) are positive constants, \( \log(\cdot) \) is a monotonically increasing function, and \( x \) is a strictly increasing function of \( q \).

The first derivative of \( \tilde{g} \) is given by

\[ \tilde{g}'(x) = 1 - \eta \left( \sigma x + \frac{1}{2\sigma} \right) + \eta \left( \sigma x - \frac{1}{2\sigma} \right). \tag{20} \]

Since \( \eta(\cdot) \) is a continuous function, using the mean value theorem, there exists \( y \in (\sigma x - \frac{1}{2\sigma}, \sigma x + \frac{1}{2\sigma}) \), such that

\[ \eta \left( \sigma x + \frac{1}{2\sigma} \right) - \eta \left( \sigma x - \frac{1}{2\sigma} \right) = \sigma \eta'(y) \frac{1}{\sigma} = \eta'(y). \tag{21} \]

From the first property of Lem. 1 \( 0 < \eta'(y) < 1 \), we have

\[ \eta \left( \sigma x + \frac{1}{2\sigma} \right) - \eta \left( \sigma x - \frac{1}{2\sigma} \right) < 1. \]

Thus, from (20), it follows that \( \tilde{g}'(x) > 0 \) for all \( x \), indicating that \( \tilde{g}(\cdot) \) is an increasing function of \( x \). This in turn implies that \( g_2(\cdot) \) is also an increasing function.

To prove the result for \( g_1 \), it is sufficient to observe that by the symmetry of error probabilities:

\[ g_1(q) = \frac{1}{g_2(1-q)}. \]

IV. Optimal Belief

We described the posterior belief evolution and decision making model in Sec. III. In this section, we investigate the set of prior beliefs that minimize the Bayes risk. We consider the case of two agents for analytical tractability although the broad nature of the arguments extend to multi-agent systems. Note that the Bayes risk of the system with \( N = 2 \) is the same as Blake’s Bayes risk because his decision is adopted as the final decision.

Let us recapitulate the computation of Blake’s Bayes risk. Alexis chooses her decision threshold as \( \lambda_1 := \lambda_1(q_1) \). Her probabilities of error are given by

\[ P_{e,1}^I = p_{R_1|H}(1|0) = \int_{-\infty}^{\lambda_1} f_{Y_1|H}(y|0) dy, \]

\[ P_{e,1}^II = p_{R_1|H}(0|1) = \int_{-\infty}^{\lambda_1} f_{Y_1|H}(y|1) dy. \]
Blake however presumes Alexis uses the decision threshold $\lambda_1, q_2 := \lambda_2(q_2)$ and computes her probabilities of error accordingly$^3$

\[
P^I_{e,1,2} = p_{\hat{H}_1|H}(1|0,2) = \int_{\lambda_1,2}^{\infty} f_{Y_2|H}(y|0)dy,
\]

\[
P^H_{e,1,2} = p_{\hat{H}_1|H}(0|1,2) = \int_{-\infty}^{\lambda_1,2} f_{Y_2|H}(y|1)dy.
\]

When Alexis decides $\hat{H}_1 = 0$, Blake updates his belief $q_2$ to the posterior $q_2^0$:

\[
\frac{q_2^0}{1-q_2^0} = \frac{q_2}{1-q_2} \frac{1-P^I_{e,1,2}}{P^H_{e,1,2}} \quad \Rightarrow \quad q_2^0 = \frac{q_2(1-P^I_{e,1,2})}{q_2(1-P^I_{e,1,2}) + (1-q_2)P^H_{e,1,2}}.
\] (22)

his decision threshold is $\lambda_2^0 := \lambda_2(q_2^0)$, and the probabilities of error are

\[
P^I_{0,e,2} = p_{\hat{H}_2|\hat{H}_1,H}(1|0,0) = \int_{\lambda_2^0}^{\infty} f_{Y_2|H}(y|0)dy,
\]

\[
P^H_{0,e,2} = p_{\hat{H}_2|\hat{H}_1,H}(0|1,0) = \int_{-\infty}^{\lambda_2^0} f_{Y_2|H}(y|0)dy.
\]

Likewise, when Alexis decides $\hat{H}_1 = 1$, Blake updates his belief $q_2$ to the posterior $q_2^1$:

\[
\frac{q_2^1}{1-q_2^1} = \frac{q_2}{1-q_2} \frac{1-P^I_{e,1,2}}{P^H_{e,1,2}} \quad \Rightarrow \quad q_2^1 = \frac{q_2P^I_{e,1,2}}{q_2P^I_{e,1,2} + (1-q_2)(1-P^H_{e,1,2})}.
\] (23)

his decision threshold is $\lambda_2^1 := \lambda_2(q_2^1)$, and the probabilities of error are

\[
P^I_{1,e,2} = p_{\hat{H}_2|\hat{H}_1,H}(0|1,0) = \int_{\lambda_2^1}^{\infty} f_{Y_2|H}(y|0)dy,
\]

\[
P^H_{1,e,2} = p_{\hat{H}_2|\hat{H}_1,H}(1|0,1) = \int_{-\infty}^{\lambda_2^1} f_{Y_2|H}(y|1)dy.
\]

Now we compute the system’s Bayes risk (or Blake’s Bayes risk) $R_2$:

\[
R_2 = c_{10}p_{\hat{H}_2,H}(1,0) + c_{01}p_{\hat{H}_2,H}(0,1)
\]

\[
= c_{10} \sum_{\hat{h}_1 \in \{0,1\}} p_{\hat{H}_2|\hat{H}_1,H}(1|\hat{h}_1,0)p_{\hat{H}_1,H}(\hat{h}_1,0)p_H(0) + c_{01} \sum_{\hat{h}_1 \in \{0,1\}} p_{\hat{H}_2|\hat{H}_1,H}(0|\hat{h}_1,0)p_{\hat{H}_1,H}(\hat{h}_1,1)p_H(1)
\]

\[
= c_{10} [P^I_{0,e,2}(1-P^I_{e,1}) + P^I_{e,2}P^I_{e,1}] p_0 + c_{01} [P^I_{0,e,2}P^I_{e,1} + P^H_{e,2}(1-P^H_{e,1})] (1-p_0).
\] (24)

Note that the Bayes risk $R_2$ in (24) is a function of $q_1$ and $q_2$. One might think that $R_2$ is minimum at $q_1 = q_2 = p_0$ as Alexis makes the best decision for the true prior and Blake does not misunderstand her decision. Surprisingly, however, this turns out to not be true. We prove this by studying Alexis’s optimal belief $q_1^*$ that minimizes $R_2$.

**Theorem 4:** Alexis’s and Blake’s optimal beliefs $q_1^*, q_2^*$ that minimize $R_2$ satisfy

\[
\frac{q_1^*}{1-q_1^*} = \frac{p_0(P^I_{e,2} - P^I_{e,1})}{(1-p_0)(P^H_{e,2} - P^H_{e,1})}.
\] (25)

Before proceeding to the proof, note that error probability terms in the right-side are dependent on $q_2$, but not on $q_1$. Furthermore, the value of $(P^I_{e,2} - P^I_{e,1})/(P^H_{e,2} - P^H_{e,1})$ is generally not 1, i.e., in general $q_1 = q_2 = p_0$ is not

$^3$Recall that the subscript $[2]$ denotes the quantity ‘seen by’ Blake.
the optimal belief. For example, for the additive Gaussian noise model considered in the next section, the ratio is not equal to 1 except when $p_0 = c_{01}/(c_{10} + c_{01})$.

**Proof:** Let us consider the first derivative of (24) with respect to $q_1$:

$$\frac{\partial R_2}{\partial q_1} = c_{10}p_0(P_{q_1}^{|H} - P_{q_1}^{|\bar{H}}) \frac{\partial P_{q_1}^{|H}}{\partial q_1} + c_{01}(1 - p_0)(P_{q_1}^{|\bar{H}} - P_{q_1}^{|H}) \frac{\partial P_{q_1}^{|\bar{H}}}{\partial q_1}.
$$

We want to find $q_1$ that minimizes $R_2$, i.e., $q_1$ makes the first derivative zero. Using

$$\frac{dP_{q_1}^{|H}}{dq_1} = \frac{dP_{q_1}^{|H}}{dq_1} = -f_{Y|H}(\lambda_1) \frac{d\lambda_1}{dq_1}.
$$

this occurs when

$$c_{10}p_0(P_{q_1}^{|H} - P_{q_1}^{|\bar{H}})f_{Y|H}(\lambda_1) = c_{01}(1 - p_0)(P_{q_1}^{|\bar{H}} - P_{q_1}^{|H})f_{Y|H}(\lambda_1).$$

Note that $\lambda_1 = \lambda_1(q_1)$ is a solution to (4).

$$\frac{f_{Y|H}(\lambda_1|1)}{f_{Y|H}(\lambda_1|0)} = \frac{c_{10}q_1}{c_{01}(1 - q_1)}.$$

Equating (26) and (27) completes the proof.

The theorem considers general continuous likelihoods $\{f_{Y|H}\}$ with the monotonicity assumption on $\lambda(q)$. It is interesting to evaluate the optimal beliefs in the case of Gaussian likelihoods (i.e., additive Gaussian noise) and obtain insights into optimality in the sequential decision making problem.

V. GAUSSIAN LIKELIHOODS

We now focus on Gaussian likelihoods and study their optimal beliefs in this section. Suppose the $n$th agent receives the signal $Y_n = H + Z_n$, where $Z_n$ is an independent additive Gaussian noise with zero mean and variance $\sigma_n^2 > 0$. Note that smaller noise variance implies the agent is more likely to infer correctly and so has more *expertise*. Thus in this context, the received signal probability densities for $H = h$ are

$$f_{Y_n|H}(y_n|h) = \phi(y_n; h, \sigma_n^2).$$

For a belief $q_n$, the decision threshold is then determined by the likelihood ratio test,

$$L_n(y_n) = \frac{f_{Y_n|H}(y_n|1)}{f_{Y_n|H}(y_n|0)} \frac{c_{10}q_n}{c_{01}(1 - q_n)},$$

that simplifies to the following simple threshold condition for Gaussian likelihoods:

$$y_n \overset{\hat{H}_1}{\overset{\hat{H}_0}{\gtrless}} \lambda_n(q_n) = \frac{1}{2} + \sigma_n^2 \log \left( \frac{c_{10}q_n}{c_{01}(1 - q_n)} \right).$$

Here the index $n$ represents the $n$th agent in the system, as the belief and variance of the agent varies along the chain.

Using the recursive update in Sec. III and decision threshold (28), it is possible to obtain the Bayes risk of Blake (i.e., $N = 2$) for given beliefs $q_1, q_2$. Fig. 3 depicts Blake’s Bayes risk for $q_1, q_2 \in [0, 1]$, and explicitly shows that
Fig. 3: The Bayes risk for $q_1, q_2 \in [0, 1]$ with $p_0 = 0.3$, $c_{10} = c_{01} = 1$, and additive standard Gaussian noise. The pair of optimal beliefs (▲) yields $R_2 = 0.2186$, while the true prior (●) yields $R_2 = 0.2214$.

Fig. 4: The trend of the optimal beliefs for varying $p_0$ for $N = 2$ (Alexis, Blake). Left panel: $c_{10} = c_{01} = 1$, right panel: $c_{10} = 1, c_{01} = 3$. $Z_1, Z_2$ are both standard Gaussian.

knowing true prior probability is not optimal. The social teaching problem with Bayes costs $c_{10} = c_{01} = 1$, prior $p_0 = 0.3$, and additive Gaussian noise with zero mean and unit variance results in a Bayes risk that is minimum when Alexis’s belief is 0.38 and Blake’s belief is 0.23, shown in the figure (triangle) and is compared to the true prior (circle).

Figs. 4 and 5 show the trend of optimal belief pair that minimizes the advisee’s Bayes risk, when all agents have the same expertise (i.e., same noise levels) for the case of two and three agents respectively. We can observe several common characteristics. First, the advisors (i.e., non-terminal agents: Alexis for $N = 2$ and Alexis and Blake for $N = 3$) overweight their beliefs if $p_0$ is small and underweight it if $p_0$ is large. We call this open-minded behavior as it enhances the prior belief of unlikely outcomes. Second, the advisee (i.e., Blake for $N = 2$ and Chuck for $N = 3$) underweights the belief if $p_0$ is small and overweights it if $p_0$ is large, implicitly compensating for the biases of the advisors. Such behavior is referred to as being closed-minded as it represents a cautious outlook to the
Fig. 5: The trend of the optimal beliefs for varying $p_0$ for $N = 3$ (Alexis, Blake, and Chuck). Left panel: $c_{10} = c_{01} = 1$, right panel: $c_{10} = 1, c_{01} = 3$. $Z_1, Z_2, Z_3$ are standard Gaussian.

decision-making problem. Lastly, there is a unique, non-trivial prior, $p_0 \in (0,1)$, where all agents’ optimal beliefs are identical to the true prior.

However, the case of diverse expertise of agents results in a very different behavior of optimal beliefs, especially when the advisee has more expertise. The optimal beliefs for $N = 2$ and the case of the advisor having more expertise, and that of the advisee having more expertise respectively are shown in Figs. 6 and 7. As can be observed, the optimal belief curves are markedly different when the advisee has more expertise, and we now derive some analytical properties of $q^*_1, q^*_2$.

**Theorem 5:** For any $\sigma_1^2$ and $\sigma_2^2$, $q^*_1$ and $q^*_2$ satisfy:

1) for $p_0 \in (0,1)$, $q^*_1 \leq p_0$ if and only if $q^*_2 \geq \frac{c_{01}}{c_{01}+c_{10}}$, with equality for $q^*_2 = \frac{c_{01}}{c_{01}+c_{10}}$.

2) $p_0 = q^*_1 = q^*_2$ if and only if $p_0 \in \left\{0, \frac{c_{01}}{c_{01}+c_{10}}, 1\right\}$.

**Proof:** Given in App. A.

Thm. 5 highlights the fact that if the advisee believes the null hypothesis is more likely, then the ideal advisor underweights the prior, and vice versa. Additionally, for $p_0$ near zero (near one) the optimal advisor overweights (underweights) the prior.

In particular, let us consider two cases separately. First, let the advisor have more expertise, i.e., $\sigma_1^2 < \sigma_2^2$. Then the curves for optimal beliefs and the corresponding Bayes risk are as shown in Fig. 6. The behavior here is similar to the case with equal expertise, indicating that the additional expertise of the advisor does not alter the overall behaviors of beliefs, as the advisee is unaware of this improved expertise.

On the other hand, when the advisee has more expertise, i.e., $\sigma_1^2 > \sigma_2^2$, we notice that the nature of curves changes, as shown in Fig. 7. The behavior of the ideal agents indicates that when the advisor has significantly less expertise than the advisee, the advisee stays open-minded. In addition, $q^*_1$ has multiple crossings with $p_0$, i.e., $q^*_2 = \frac{c_{01}}{c_{01}+c_{10}}$.

As expected, the ideal advisor is open-minded for near-deterministic priors ($p_0$ close to zero or one). However,
when the prior uncertainty in the hypotheses is high ($p_0$ near $1/2$), we note that the ideal advisee with more expertise favors the less likely hypothesis. This can be attributed to the fact that the advisee stays open-minded to the less likely hypothesis when the advisor with less expertise is more likely to make errors. To further understand the nature of such an advisor, we characterize the crossings of the optimal belief curve with the prior $q^* = p_0$.

**Theorem 6:** The set of all $p_0$ such that $q^*_1 = p_0$, $q^*_2 = \frac{c_{01}}{c_{01}+c_{10}}$ is given by the solutions to

\[ e^x = \frac{1 - \beta Q(-\alpha + \sigma_1 x)}{1 - \beta Q(-\alpha - \sigma_1 x)}, \tag{29} \]

where

\[ x = \log \left( \frac{c_{10}p_0}{c_{01}(1-p_0)} \right), \quad \alpha = \frac{1}{2\sigma_1}, \quad \beta = 1 - \frac{Q \left( 1/2\sigma_2 \right)}{Q \left( -1/2\sigma_2 \right)}. \]

**Proof:** Given in App. B. 

We note that $p^* = \frac{c_{01}}{c_{01}+c_{10}}$ is always a solution to (29). The case of multiple solutions to (29) is of particular interest and a sufficient condition is given in the following corollary.
Corollary 2: If
\[ \frac{2\beta \sigma_1 \phi(\alpha)}{1 - \beta Q(-\alpha)} > 1, \] (30)
then, (29) has at least 3 solutions in \((0, 1)\).

Proof: Since \(x\) is a monotonic function of \(p_0\), it is sufficient to show that (29) has at least 3 solutions in \(x\).

From the symmetry in (29), since \(x = 0\) is always a root, it suffices to show the existence of at least one more root in \(x > 0\). First note the ranges of variables, \(x \in (-\infty, \infty), \alpha \in (0, \infty), \beta \in (0, 1)\).

Letting \(r(x)\) be the right side of (29), since \(0 \leq Q(\cdot) \leq 1\), we have
\[ 1 - \beta \leq r(x) := \frac{1 - \beta Q(-\alpha + \sigma_1 x)}{1 - \beta Q(-\alpha - \sigma_1 x)} \leq \frac{1}{1 - \beta}, \]
indicating that \(r(x) \in [1 - \beta, \frac{1}{1 - \beta}]\). However, note that \(e^x\) monotonically increases in \((1, \infty)\) for \(x > 0\). Since \(e^x, r(x)\) coincide at \(x = 0\), it follows that they cross at least once on \((0, \infty)\) and also on \((-\infty, 0)\), if \(r'(x) = \frac{d}{dx} e^x\) at \(x = 0\) by the intermediate value theorem. Thus, the sufficient condition follows:
\[ r'(0) = \frac{2\sigma_1 \phi(\alpha; 0, 1)}{1 - \beta Q(-\alpha)} > 1 = \frac{d}{dx} e^x \bigg|_{x=0}. \]

Fig. 8: Contour plot of (30) with values for various \(\sigma_1, \sigma_2\). The red dotted contour shows the contour that results in 1 so that the area below it satisfies (30) and therefore has multiple solutions to (29).

Cor. 2 provides a sufficient condition on the expertise of agents under which there exists multiple crossings of the curves \(q_1^*(p_0)\) and \(p_0\). The range of standard deviations of the additive Gaussian noise of the advisor and advisee that satisfy the sufficient condition of Cor. 2 is shown in Fig. 8. Note from the figure that the area below the red dotted contour in Fig. 8 has multiple solutions to \(q_1^* = p_0\), i.e., when the advisee has comparatively more expertise than the advisor.

This is important as the crossings indicate a change in the perceived bias of the advisor and also indicates the regions of in which the advisor overweights the unlikely hypothesis as in Fig. 7.
A. Approximation by Prelec Family

From Figs. 4, 5, and 6, we observe that the optimal belief curves are similar in form to the Prelec reweighting functions. Considering the fact that the Prelec reweighting functions represent a mathematical model of human cognitive biases, one might wonder whether people are (approximately) naturally optimal for social teaching.

To investigate this hypothesis, we approximate the optimal belief curves $q^*_n$ by the Prelec function and study the resulting increase in the Bayes risk. We first restrict to the Prelec family whose fixed point is identical to

$$p^* = \frac{c_{10}}{c_{01} + c_{10}},$$

and then find best parameters $(\alpha_n, \beta_n)$ in the minimax absolute error sense, i.e.,

$$(\alpha_n, \beta_n) = \arg\min_{\alpha, \beta} \max_{w(p^*; \alpha, \beta)=p^*} \|q^*_n(\cdot) - w(\cdot; \alpha, \beta)\|_{\infty}.$$

Let the Prelec function approximations be $(q_{1, \text{Pre}}, q_{2, \text{Pre}})$ and let the resulting Bayes risk be $R_{2, \text{Pre}}$.

The Prelec approximations for the two-agent case are shown in dotted curves in Figs. 6 and 7. When the advisor has more expertise as in Fig. 6, the Prelec function approximates the optimal beliefs well and the Bayes risk does not increase by much. To evaluate the loss from the approximation, consider the set of correct beliefs $q_1 = q_2 = p_0$, that result in a Bayes risk of $R_{2, \text{corr}}$. The maximal loss in terms of Bayes risk from using the correct beliefs is $\max_{p_0} (R_{2, \text{corr}} - R_{2, \text{min}}) \approx 0.0039$. On the other hand, the maximal loss from the best Prelec approximation is $\approx 0.0009$. This indicates that the natural cognitive biases of humans are effective for social teaching when the advisor has more expertise.

On the other hand, when the advisee has more expertise as in Fig. 7 the Prelec approximation does not accurately mimic the optimal behavior of agents. Recall that the Prelec function is always increasing and has only one crossing with unit slope line in $(0, 1)$. Therefore, the Prelec function fails to account for all the variations in the optimal belief. Moreover, while the additional loss of Bayes risk by the Prelec fitting is $\approx 0.0187$, the loss from using the correct beliefs, $p_0 = q_1 = q_2$, is $\approx 0.0060$. This indicates that even though the Prelec weighting functions serve as good approximations with expert advisors, they do not model the optimal behavior in the case of poor advisors. These results suggest that human behavior models originating from cumulative prospect theory [16] are better-matched to advisors having more expertise rather than less, and in that sense are better suited to function in the first context.

The result sheds some light on AI-human collaboration frameworks [20]. In many AI-human joint teams, a human agent makes the final decision based on the advice of an AI component as depicted in Fig. 9a, but the opposite structure of Fig. 9b is also possible. Considering the fact that the cognitive biases of the human agent are approximated by the Prelec reweighting functions, the results in this section indicate that an AI assistant with more expertise could be an effective advisor to the human decision-maker. In particular, an open-minded AI advisor and a closed-minded human advisee with appropriate Prelec reweighted beliefs function efficiently together, as in Fig. 6. However, an AI component with less expertise might not prove to be a good advisor to the human advisee who does not have beliefs that mimic the optimal behavior in Fig. 7 and so perhaps counterintuitively, the architecture of Fig. 9b should be adopted, with the AI agent having less expertise making the global decision.
VI. TEAM CONSTRUCTION CRITERION

Having studied the mathematical conditions for optimal reweighting of prior probabilities, we now investigate team selection for social learning. Naturally, a social planner who is aware of the context \( p_0 \) can pick the optimal agent pairs to minimize Bayes risk. However, it is not clear if agents are capable of organizing themselves into ideal teams in the absence of contextual knowledge. Thus, we now identify the criterion for the advisee to identify the optimal advisors among a set of given advisors.

**Theorem 7:** Consider two advisors with \( q_1 < q_{1'} \). Let \( \lambda_1, \lambda_{1'} \) be the decision thresholds of the respective advisors. Then, the advisor with belief \( q_1 \) is the optimal choice if and only if

\[
\frac{P_1 [Y_1 \in [\lambda_1, \lambda_{1'}], Y_2 \in [\lambda_1^0, \lambda_{1'}^0]]}{P_0 [Y_1 \in [\lambda_1, \lambda_{1'}], Y_2 \in [\lambda_1^0, \lambda_{1'}^0]]} \geq \frac{c_{10}p_0}{c_{01}(1 - p_0)}.
\]

(31)

**Proof:** Given in App. C.

Thus selecting an ideal advisor requires a social planner who is aware of the context \( p_0 \). Without this, the advisee selects an advisor according to his personal belief \( q_2 \). That is, the advisee verifies condition (31) by replacing \( p_0 \) by \( q_2 \). Such a choice of advisor might not always conform to the optimal choice when the belief of the advisee deviates significantly from the prior. To illustrate, we consider the problem of choosing between two advisors with beliefs \( q_1(p_0) = q_1^*(p_0) \) and \( q_{1'}(p_0) = p_0 \). Let \( q(p_0, q_2) \) be the belief of the optimal advisor choice for a given pair \( (p_0, q_2) \). We identify the region of correct selection by shading, \( S = \{(p_0, q_2) : q(p_0, q_2) = q(q_2, q_2)\} \).

First, when expertise levels are equal, the region in which the advisee picks the correct advisor is shown in Fig. 10a. We note that the correct region is relatively small and does not include \( q_2^* \). In particular, the advisee with...
optimal belief chooses the wrong advisor always, whereas a suboptimal advisee with beliefs in the shaded region picks the correct one.

On the other hand, when the advisee has more expertise than the advisor, the corresponding region is as shown in Fig. 10b. Here we note that the advisee with optimal belief picks the correct advisor always.

Thus, we note that knowledge of the mathematically optimal beliefs does not guarantee selection of the right advisor. Further, we also observe that the diversity of expertise levels may increase the feasibility of selecting the right advisor when the advisee has optimal belief.

We also explore the optimal choice of advisor for the given optimal advisee in the absence of knowledge of the prior probability. From (25), the belief of the optimal advisor, $\tilde{q}_1$ chosen by an advisee, in the absence of context (prior probability $p_0$) satisfies

$$\tilde{q}_1 = \frac{p_0 \left( P_{e,2}^{I} \right)^{\frac{1}{2}} - P_{e,2}^{I}}{1 - p_0 \left( P_{e,2}^{I} \right)^{\frac{1}{2}} - P_{e,2}^{II}}.$$  \hspace{1cm} (32)

The advisee’s behavior with belief $q_2^*$ is as shown in Fig. 10c. We note that the advisor chosen by the advisee differs from the optimal choice. Further, it is also evident that this choice consequently results in an increased Bayes risk. Such behavior in team selection highlights the significance of context and thus a social planner for identifying the right team.

The result again provides some insight into human-AI collaborations when the human agent picks an AI advisor, given a choice among different agents. In particular, consider the AI-human team where the human, who has a Prelec-weighted belief, chooses one of two possible AI advisors—one that has the optimal belief $q_1^*$ and the other that is aware of the true prior $p_0$. In case the human agent has lesser expertise, and a closed-minded Prelec belief as in Fig. 10a, she unfortunately picks the AI advisor with $q_1 = p_0$ and the team becomes suboptimal. However, if the human agent has more expertise, and an open-minded Prelec belief, she picks the optimal AI component $q_1 = q_1^*$
and therefore can make the optimal decision as in Fig. 10b. Thus it is evident that optimal team organization is feasible when the human has more expertise and the appropriate open-minded belief.

VII. CONCLUSION

We discussed the sequential social learning problem with individual biased beliefs. Unlike previous works on herding, we focused on the Bayes risk of the last-acting agent. We first derived the optimal belief update rule for general likelihoods and evaluated for Gaussian likelihoods. Counterintuitively, optimal beliefs that yield minimum Bayes risk are in general different from the true prior. Under equal expertise levels, we observed that optimal advisors have open-minded beliefs, that is, overweight small priors and underweight large priors, while the optimal advisee has closed-minded belief. However, the trend may change depending on varying expertise levels such that especially when the advisee has much more expertise, optimal belief of the advisee is inverted as she becomes open-minded.

We also showed that the Prelec reweighting function from cumulative prospect theory approximates the behavior of the optimal beliefs under specific levels of expertise, however, when the advisee has much more expertise, it fails to capture all the behavioral traits of the optimal beliefs.

Finally, we considered the ability of agents to organize themselves into optimal teams and showed that in the absence of a social planner, the advisee can get paired with the wrong advisor when the individual belief deviates significantly from the underlying prior value. The setup arises from the consideration of AI and it tells us without knowing the true prior, our human-machine team construction could be misorganized.

APPENDIX A

PROOF OF THEOREM 5

Let us prove Thm. 5 starting with the premise that \( q_1^* \geq p_0 \). First, from (25), we have

\[
q_1^* \geq p_0 \iff \frac{P_{e,2}^{I_1} - P_{e,2}^{I_0}}{P_{e,2}^{I_1} - P_{e,2}^{I_0}} \geq -1.
\]  

(33)

To study the ratio in (33), consider the Type I vs. Type II error curve for binary hypothesis testing under additive Gaussian noise. This is shown in Fig. 11, and as seen here is a convex function [25]. Note that on the curve, the Type I and Type II error probabilities, \((P_{e,2}^{I_0}, P_{e,2}^{I_1})\), are the points on the curve that have tangents with slope matching \(-\left(\frac{c_{10}q}{c_{01}(1-q)}\right)\), where \(q\) is the corresponding prior probability, and \(\sigma^2\) is the variance of the additive Gaussian noise.

First, from Thm. 2 we know that \( q_0^2 \geq q_1^2 \) which in turn implies that \( \lambda_0^2 \geq \lambda_1^2 \). This in turn indicates that

\[
P_{e,2}^{I_0} = Q\left(\frac{\lambda_0^2}{\sigma^2}\right) \leq Q\left(\frac{\lambda_1^2}{\sigma^2}\right) = P_{e,2}^{I_1}.
\]

Similarly, \( P_{e,2}^{I_0} \geq P_{e,2}^{I_1} \), and thus, as shown in the figure, the point \( B_0 = (P_{e,2}^{I_0}, P_{e,2}^{I_0}) \) lies to the left of \( B_1 = (P_{e,2}^{I_1}, P_{e,2}^{I_1}) \).

Further, since \( B_1 \) lies on the curve, so does the point \( \bar{B}_1 = (P_{e,2}^{I_1}, P_{e,2}^{I_1}) \) as it caters to the error probabilities corresponding to the probability of the null hypothesis \( P_H = 0 = 1 - q_{1}^1 \). Thus, the line \( \bar{B}_1 \bar{B}_1 \) has a slope of \(-1\).

\footnote{It is also called Receiver Operating Characteristic (ROC) curve [25], [26] when the curve is vertically inverted.}
Fig. 11: The point $B_0$ always exists between points $B_1$ and $\bar{B}_1$.

Note that the condition (33) translates to the slope of the line $\overline{B_0B_1}$ is greater than $-1$. Observe that if $\bar{B}_1$ lies to the right of $B_1$ then it implies that the slope of $\overline{B_0B_1}$ is less than $-1$, violating (33). Similarly, if $B_0$ lies to the left of $\bar{B}_1$, then again the (33) is violated.

On the other hand, if $B_0$ lies between $\bar{B}_1$ and $B_1$, then we know that the slope of $\overline{B_0B_1}$ is greater than that of $\overline{B_1\bar{B}_1}$, therein satisfying (33). Thus, (33) is true if and only if the point $B_0$ lies between the two points $B_1$ and $\bar{B}_1$.

From the convexity of the curve and comparing coordinates of $B_0$ and $\bar{B}_1$, we have

$$q_1^* \geq p_0 \iff P_{e,2}^{II} \geq P_{e,2}^{I} \text{ and } P_{e,2}^{II} \leq P_{e,2}^{I}$$

$$\iff Q\left(\frac{\lambda_2^0}{\sigma_2}\right) \geq 1 - Q\left(\frac{\lambda_1^1}{\sigma_2}\right) \text{ and } Q\left(\frac{\lambda_2^1}{\sigma_2}\right) \geq 1 - Q\left(\frac{\lambda_2^0}{\sigma_2}\right)$$

$$\iff \lambda_2^0 + \lambda_2^1 \leq 1$$

$$\iff 2\lambda_{1,[2]} + \sigma_2^2 \log \left(\frac{P_{e,1,[2]}^{II} \left(1 - P_{e,1,[2]}^{II}\right)}{P_{e,1,[2]}^{II} \left(1 - P_{e,1,[2]}^{I}\right)}\right) \leq 1,$$

(34)

where (a) follows from the false alarm and missed detection probabilities in terms of the $Q$-function of the standard Gaussian random variable; (b) follows from the fact that the $Q$-function is monotonically decreasing and that $1 - Q(x) = Q(-x)$; and (c) follows from (22), (23), and $\lambda_{1,[2]} = \lambda_2(q_2)$.

From (28), we have

$$\lambda_{1,[2]} = \frac{1}{2} + \sigma_2^2 \log \left(\frac{c_{10}q_2^*}{c_{01}(1 - q_2^*)}\right).$$

Substituting in (34), we have

$$q_1^* \geq p_0 \iff 2\log \left(\frac{c_{10}q_2^*}{c_{01}(1 - q_2^*)}\right) \leq \log \left(\frac{P_{e,1,[2]}^{II} \left(1 - P_{e,1,[2]}^{II}\right)}{P_{e,1,[2]}^{II} \left(1 - P_{e,1,[2]}^{I}\right)}\right).$$
Letting $x := \log \left( \frac{c_{10}q_1^2}{c_{01}(1-q_1^2)} \right) = \frac{1}{\sigma^2} (\lambda_2 - \frac{1}{2})$ and using $Q(\cdot)$ representation of error probabilities, we have

$$q_1^* \geq p_0 \iff 2x \leq \log \left( \frac{Q \left( \sigma_2x - \frac{1}{2\sigma^2} \right) Q \left( -\sigma_2x + \frac{1}{2\sigma^2} \right)}{Q \left( \sigma_2x + \frac{1}{2\sigma^2} \right) Q \left( -\sigma_2x - \frac{1}{2\sigma^2} \right)} \right).$$

(35)

From Cor. 1, we know that the function $g(x) = x + \log \left( \frac{Q \left( \sigma x + \frac{1}{2\sigma^2} \right)}{Q \left( \sigma x - \frac{1}{2\sigma^2} \right)} \right)$ is an increasing function of $x$. Thus, reformulating (35) using $g(\cdot)$,

$$q_1^* \geq p_0 \iff \hat{g}(x) \leq \hat{g}(-x) \iff x \leq 0 \iff q_2^* \leq \frac{c_{01}}{c_{01} + c_{10}}.$$

The condition for equality follows from observing the condition for equality at all the inequalities, proving the first part of the result.

The second part follows directly from the first, taking into account the trivial cases of $p_0 \in \{0, 1\}.$

**Appendix B**

**Proof of Theorem 6**

We will consider the case of $c_{01} = c_{10} = 1$ for convenience. The proof extends directly by a simple scaling argument.

The optimal belief of worker two satisfies $\partial R_2 / \partial q_2 = 0$. Thus, differentiating (24) with respect to $q_2$ and rearranging,

$$p_0 \left[ (1 - P_{e,1}^I) f_{Y^2|H}(\lambda_2^0|0) \frac{\partial \lambda_2^0}{\partial q_2} + P_{e,1}^I f_{Y^2|H}(\lambda_2^1|0) \frac{\partial \lambda_2^1}{\partial q_2} \right] = (1 - p_0) \left[ P_{e,1}^{II} f_{Y^2|H}(\lambda_2^0|1) \frac{\partial \lambda_2^0}{\partial q_2} + (1 - P_{e,1}^{II}) f_{Y^2|H}(\lambda_2^1|1) \frac{\partial \lambda_2^1}{\partial q_2} \right].$$

Let $x = \log \left( \frac{p_0}{1-p_0} \right)$. For $q_2^* = 1/2$ and $q_1^* = p_0$, we have

$$\lambda_1^* = \frac{1}{2} + \sigma^2 x \quad \text{and} \quad \lambda_{1,[2]} = \frac{1}{2}.$$

It implies $P_{e,1,[2]} = P_{e,1,[2]}^{II} = Q(1/2\sigma_2)$. Then,

$$\mathcal{L}(\lambda_2^0) = f_{Y^2|H}(\lambda_2^0|1) f_{Y^2|H}(\lambda_2^0|0) = \frac{q_2}{1 - q_2} \frac{1 - P_{e,1,[2]}^I}{P_{e,1,[2]}^I} = \frac{Q(-1/2\sigma_2)}{Q(1/2\sigma_2)} = \frac{1}{c},$$

$$\mathcal{L}(\lambda_2^1) = f_{Y^2|H}(\lambda_2^1|1) f_{Y^2|H}(\lambda_2^1|0) = \frac{q_2}{1 - q_2} \frac{P_{e,1,[2]}^I}{1 - P_{e,1,[2]}^{II}} = \frac{Q(1/2\sigma_2)}{Q(-1/2\sigma_2)} = c.$$

Equivalently, this implies that

$$\lambda_2^0 = \frac{1}{2} + \sigma^2 \log \left( \frac{1}{c} \right), \quad \lambda_2^1 = \frac{1}{2} - \sigma^2 \log \left( \frac{1}{c} \right).$$
Thus, $\lambda_2^0 + \lambda_2^1 = 1$, and so,

$$f_{Y_2|H}(\lambda_2^1|1) = \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(\lambda_2^1-1)^2}{2\sigma_2^2}\right)$$

$$= \frac{1}{\sqrt{2\pi}\sigma_2} \exp\left(-\frac{(\lambda_2^0)^2}{2\sigma_2^2}\right)$$

$$= f_{Y_2|H}(\lambda_2^0|0).$$

Similarly, we also have

$$f_{Y_2|H}(\lambda_2^1|0) = f_{Y_2|H}(\lambda_2^0|1).$$

Further, from (22) and (23), we have

$$\frac{d\lambda_0^0}{dq_2} = \frac{d\lambda_0^1}{d\lambda_1^1} \frac{d\lambda_1^1}{dq_2} = \left[1 + \frac{\sigma_2^2\phi\left(\frac{\lambda_1^1}{\sigma_2}\right)}{1 - P^I_{c,1}} - \frac{\sigma_2^2\phi\left(\frac{\lambda_1^0}{\sigma_2}\right)}{1 - P^II_{c,1}}\right] \frac{d\lambda_1^1}{dq_2},$$

$$\frac{d\lambda_0^2}{dq_2} = \frac{d\lambda_1^2}{d\lambda_1^1} \frac{d\lambda_1^1}{dq_2} = \left[1 - \frac{\sigma_2^2\phi\left(\frac{\lambda_1^1}{\sigma_2}\right)}{1 - P^I_{c,1}} + \frac{\sigma_2^2\phi\left(\frac{\lambda_1^0}{\sigma_2}\right)}{1 - P^II_{c,1}}\right] \frac{d\lambda_1^1}{dq_2}.$$

When, $\lambda_1^1 = \frac{1}{2}$, $P^I_{c,1} = P^II_{c,1} = Q\left(\frac{1}{2\sigma_2}\right)$, and $\phi\left(\frac{\lambda_1^1}{\sigma_2}\right) = \phi\left(\frac{\lambda_1^0}{\sigma_2}\right)$. Thus, $\frac{d\lambda_0^0}{dq_2} = \frac{d\lambda_0^2}{dq_2}$.

Using these, the values of prior for which $q^*_1 = p_0$, $q^*_2 = 1/2$ are given by

$$p_0 = \frac{Q\left(\frac{1}{2\sigma_2}\right) Q\left(\frac{1}{2\sigma_2} - \sigma_1 x\right) + Q\left(\frac{1}{2\sigma_2}\right) Q\left(\frac{1}{2\sigma_2} + \sigma_1 x\right)}{Q\left(\frac{1}{2\sigma_2}\right) Q\left(\frac{1}{2\sigma_2} - \sigma_1 x\right) + Q\left(\frac{1}{2\sigma_2}\right) Q\left(\frac{1}{2\sigma_2} + \sigma_1 x\right)}, \quad (36)$$

Using the definitions of $x, \alpha, \beta$ in (36), and the fact that $Q(-y) = 1 - Q(y)$, the result follows.

**APPENDIX C**

**PROOF OF THEOREM 7**

From (1), we note that the Bayes risk for social learning with beliefs $(q_1, q_2)$ is

$$R_2(q_1, q_2) = c_{10}p_0\left[P^I_{c,2}(1 - P^I_{c,1}) + P^II_{c,2}(1 - P^I_{c,1})\right] + c_{11}(1 - p_0)\left[P^II_{c,2}(1 - P^I_{c,1}) + P^II_{c,2}(1 - P^I_{c,1})\right].$$

Then, the difference in Bayes risk between the two choices of advisors is given by

$$\Delta R_2 = R_2(q_1, q_2) - R_2(q_1', q_2)
= c_{10}p_0\left(P^I_{c,2}(1 - P^I_{c,1}) - P^I_{c,2}(1 - P^I_{c,1})\right) + c_{11}(1 - p_0)(P^II_{c,2}(1 - P^I_{c,1}) - P^II_{c,2}(1 - P^I_{c,1})). \quad (37)$$

Since $q_1 < q_1'$, the decision thresholds satisfy $\lambda_1 < \lambda_1'$. Thus, from (37) and independence of $Y_1, Y_2$ given $H$, we see that $\Delta R_2 \leq 0$ if and only if (31) holds.

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