ON HIGHER SPINS WITH A STRONG $Sp(2, R)$ CONDITION

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Abstract

We report on an analysis of the Vasiliev construction for minimal bosonic higher-spin master fields with oscillators that are vectors of $SO(D - 1, 2)$ and doublets of $Sp(2, R)$. We show that, if the original master field equations are supplemented with a strong $Sp(2, R)$ projection of the 0-form while letting the 1-form adjust to the resulting Weyl curvatures, the linearized on-shell constraints exhibit both the proper mass terms and a geometric gauge symmetry with unconstrained, traceful parameters. We also address some of the subtleties related to the strong projection and the prospects for obtaining a finite curvature expansion.

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Contents

1 INTRODUCTION 3

2 HIGHER SPINS AND TENSIONLESS LIMITS 6

3 THE OFF-SHELL THEORY 8
   3.1 General Set-Up 8
   3.2 Oscillators and Master Fields 9
   3.3 Master Constraints 10
   3.4 Off-Shell Adjoint and Twisted-Adjoint Representations 11
   3.5 Weak-Field Expansion and Linearized Field Equations 12

4 THE ON-SHELL THEORY 15
   4.1 On-Shell Projection 15
   4.2 On-Shell Adjoint and Twisted-Adjoint Representations 17
      4.2.1 Dressing Functions 18
   4.3 Linearized Field Equations and Spectrum of the Model 19
      4.3.1 Φ Constraint and Role of the Dressing Functions 19
      4.3.2 F Constraint and Mixing Phenomenon 20
      4.3.3 Spectrum of the Model and Group Theoretical Interpretation 21
   4.4 Compensator Form of the Linearized Gauge-Field Equations 22

5 TOWARDS A FINITE CURVATURE EXPANSION 25
   5.1 On the Structure of the Interactions 26
   5.2 A Simpler U(1) Analog 27
      5.2.1 Analytic U(1) Projector 28
      5.2.2 Distributional ⋆-Inverse Function and Normalizable Projector 31
   5.3 Proposals for Finite Curvature Expansion Schemes 35
      5.3.1 Minimal Expansion Scheme 35
      5.3.2 Modified Expansion Scheme 35
A consistent interacting higher-spin (HS) gauge theory can be viewed as a generalization of Einstein gravity by the inclusion of infinitely many massless HS fields, together with finitely many lower-spin fields including one or more scalars [1]. There are many reasons, if not a rigorous proof, that point to the inevitable presence of infinite towers of fields in these systems.

Although HS gauge theories were studied for a long time for their own sake \(^{1}\) (see, for instance, [3, 4, 5] for reviews and more references to the original literature), the well-known recent advances on holography in AdS have shifted to some extent the emphasis toward seeking connections with String/M Theory. Indeed, tensionless strings should lie behind massless higher spins in AdS, as proposed in [6], with the leading Regge trajectory becoming massless at some critical value of the tension [7], and interesting work in support of the latter proposal was recently done in [8]. The theory of massless HS fields has a long history, and the literature has been growing considerably in recent years. Therefore, rather than attempting to review the subject, we shall stress that consistent, fully interacting and supersymmetric HS gauge theories have not been constructed yet in dimensions beyond four (although the linearized field equations [9] and certain cubic couplings [10] are known in 5D, and to a lesser extent in 7D [7]). This was also the case for bosonic HS gauge theories, until Vasiliev [11] recently proposed a set of nonlinear field equations for totally symmetric tensors in arbitrary dimensions. The new ingredient is the use of an \(Sp(2, R)\) doublet of oscillators valued in the vector representation of \(SO(D − 1, 2)\) [12], called \(Y^{\lambda \Lambda}\) in the following, as opposed to the bosonic spinor oscillators [13] used in the original construction of the 4D theory [1].

The two formulations\(^{2}\) rest on the same sequence of minimal HS Lie algebra extensions of \(SO(D − 1, 2)\) [34, 15]. Not surprisingly, the associative nature of the oscillator \(*\)-product algebra, discussed in Section 3.2, is reflected in the possibility of extending the HS field equations to matrix-valued master fields associated to the classical algebras, although here we shall actually confine our attention to the minimal HS gauge theory, where the matrices are one-dimensional. We would like to emphasize, however, that the matrix extensions have the flavor of the Chan-Paton generalizations of open strings [14], and their consistency actually rests of the same key features of the matrix algebras, which suggests a natural link of the Vasiliev equations to open, rather than to closed strings, in the tensionless limit. More support for this view will be presented in [15].

The Vasiliev construction [11] requires, as in [1], a pair of master fields, a 1-form \(\hat{A}\) and a 0-form \(\hat{\Phi}\). It rests on a set of integrable constraints on the exterior derivatives \(d\hat{A}\) and \(d\hat{\Phi}\) that, when combined with Lorentz-trace conditions on the component fields, give the equations of motion within the framework of unfolded dynamics [16], whereby a given field is described via an infinity of distinct, albeit related, tensor fields. Notice that here all curvatures are constrained, so that the condition on the 2-form curvature embodies generalized torsion constraints and identifies

\(^{1}\)A notable exception is [2], where a connection between massless HS gauge theories and the eleven-dimensional supermembrane was proposed.

\(^{2}\)The precise relation between the vector and spinor-oscillator formulations of the Vasiliev equations is yet to be determined.
generalized Riemann curvatures with certain components of $\hat{\Phi}$, while the constraint on the 1-form curvature subsumes the Bianchi identities as well as the very definition of an unfolded scalar field. The presence of trace parts allows in this vector construction both an “off-shell” formulation, that defines the curvatures, and an “on-shell” formulation, where suitable trace conditions turn these definitions into dynamical equations. This is in contrast with the original 4D spinor construction [1], that is only in “on-shell” form, since the special properties of $SL(2,C)$ multi-spinors make its 0-form components inevitably traceless.

Aside from leading to an expansion of the 0-form master field in terms of traceful, and thus off-shell, Riemann tensors, the vector oscillators also introduce an $Sp(2,R)$ redundancy. The crucial issue is therefore how to incorporate the trace conditions for on-shell Riemann tensors and scalar-field derivatives into suitable $Sp(2,R)$-invariance conditions on the master fields. Here, and in a more extensive paper that we hope to complete soon [17], we would like to propose that the Vasiliev equations of [11] be supplemented with the strong $Sp(2,R)$-invariance condition

$$\hat{K}_{ij} \star \hat{\Phi} = 0 ,$$

where the $\hat{K}_{ij}$ are fully interacting $Sp(2,R)$ generators, while subjecting $\hat{A}$ only to the weak condition

$$\hat{D}\hat{K}_{ij} = 0 .$$

Whereas one could consider alternative forms of the strong projection condition (1.1), involving various higher-order constructs of the $Sp(2,R)$ generators, the above linear form is motivated to some extent by the Lagrangian formulation of tensionless strings, or rather, string bits [15], as well as by related previous works of Bars and others on two-time physics and the use of Moyal products in String Field Theory [12, 18]. There is one crucial difference, however, in that the Vasiliev equations (1.1) involve an additional set of vector oscillators, called $Z_i^A$ in the following, whose origin in the tensionless limit will be made more transparent in [15]. Further support for (1.1) is provided by previous constructions of linearized 5D and 7D HS gauge theories based on commuting spinor oscillators [4, 19], where the linearized trace conditions are naturally incorporated into strong $U(1)$ and $SU(2)$ projections of the corresponding Weyl 0-forms. As we shall see, the projection (1.1) imposes trace conditions leading to correct mass-terms in the linearized field equations for the Weyl tensors and the scalar field, while the resulting gauge field equations exhibit a mixing phenomenon. To wit, the Einstein metric arises in an admixture with the scalar field, that can be resolved by a Weyl rescaling.

The strong $Sp(2,R)$ projection (1.1) introduces non-polynomial redefinitions of the linearized zero-form $\Phi$ via “dressing functions”, similar to those that first appeared in the spinorial constructions of [4, 19], corresponding to a projector $M$, via $\Phi = M \star C$ as discussed in Section 4.3. It turns out that this projector is singular in the sense that $M \star M$ is divergent, and thus $M$ is not normalizable [20, 34]. This poses a potential obstruction to a well-defined curvature expansion of the Vasiliev equations based on the vector oscillators. We have started to address this issue, and drawing on the similar, if simpler, $U(1)$ projection of [4, 20], we discuss how, in principle, one could extract nonetheless a finite curvature expansion. At the present time, however, it cannot be fully excluded that the vector-oscillator formulation of [11] contains a pathology, so that consistent interactions would only exist in certain low dimensions fixed by
the HS algebra isomorphisms to spinor-oscillator realizations outlined in Section 2, although this is unlikely. In fact, the presence of such limitations would be in line with the suggestion that the Vasiliev equations be somehow related to String Field Theory, and might therefore retain the notion of a critical dimension, but no such constraints are visible in the tensionless limit of free String Field Theory, that rests on a contracted form of the Virasoro algebra where the central charge has disappeared altogether. A final word on the matter would require a more thorough investigation of the actual interactions present in the $Sp(2, R)$ system, to which we plan to return soon [17].

An additional result discussed here concerns the existence of a direct link between the linearized equations and the geometric formulation with traceful gauge fields and parameters of [21, 22]. The strong condition (1.1) on $\hat{\Phi}$, together with the weak condition (1.2) on $\hat{A}$, implies indeed that the spin-$s$ gauge field equations embodied in the Vasiliev equations take the local compensator form
\[
\mathcal{F}_{a_1...a_s} = \nabla_{(a_1} \nabla_{a_2} \nabla_{a_3} \alpha_{a_4...a_s)} + \text{AdS covariantizations},
\]
where $\mathcal{F}$ denotes the Fronsdal operator [23] and $\alpha$ is a spin-$(s - 3)$ compensator, that here often differs from its definition in [21, 22] by an overall normalization. This result reflects the link, first discussed by Bekaert and Boulanger [24], between the Freedman-de Wit connections [25] and the compensator equations of [21, 22]. These non-Lagrangian compensator equations are equivalent, in their turn, to the non-local geometric equations of [21], where the Fronsdal operator is replaced by a higher-spin curvature.

By and large, we believe that a thorough understanding of HS gauge theories can be instrumental in approaching String Theory at a deeper conceptual level, since the HS symmetry already implies a far-reaching extension of the familiar notion of spacetime. In particular, the unfolded formulation [16] embodies an unusually large extension of diffeomorphism invariance, in that it is intrinsically independent of the notion of spacetime coordinates, and therefore lends itself to provide a natural geometric basis for translation-like gauge symmetries, somehow in the spirit of the discussion of supertranslations in [26].

The plan of this article is as follows. We begin in Section 2 by relating the HS symmetries to tensionless limits of strings and branes in AdS, and continue in Section 3 with the off-shell definitions of the curvatures, to move on, in Section 4, to the on-shell theory and the linearized field equations, including their compensator form. Finally, in Section 5 we examine the singular projector and other related non-polynomial objects, primarily for the simpler 5D $U(1)$ case, and suggest how one could arrive at a finite curvature expansion.

This article covers part of the lectures delivered by one of us (P.S.) at the First Solvay Workshop on Higher-Spin Gauge Theories. A group of students was expected to edit the complete lectures and to co-author the resulting manuscript with the speaker, as was the case for other Solvay talks, but this arrangement turned out not to be possible in this case. We thus decided to write together this contribution, combining it with remarks made by the other authors at other Meetings and with some more recent findings, in order to make some of the results that were presented in Brussels available. Hopefully, we shall soon discuss extensively these results in a more complete paper [17], but a fully satisfactory analysis cannot forego the need for a better grasp of the non-linear interactions in the actual $SP(2, R)$ setting. The remaining part of the
lectures of P.S. was devoted to the relation between bosonic string bits in the low-tension limit, minimal HS algebras and master equations. Their content will be briefly mentioned in the next section, but will be published elsewhere [15], together with the more recent results referred to above, that were obtained in collaboration with J. Engquist.

2 HIGHER SPINS AND TENSIONLESS LIMITS

HS symmetries naturally arise in the tensionless limits of strings and branes in AdS, and there is a growing literature on this very interesting subject, for instance [27, 28, 29, 12, 6, 18, 7, 30, 22, 31, 32, 8, 33]. The weakly coupled description of a p-brane with small tension is a system consisting of discrete degrees of freedom, that can be referred to as “bits”, each carrying finite energy and momentum [29]. In the simplest case of a bit propagating in AdS\(_D\), one finds the locally Sp\(_(2, R)\)-invariant Lagrangian

\[ S = \int DY^{iA} Y_{iA} , \] (2.1)

where \(Y^{iA} (i = 1, 2)\) are coordinates and momenta in the \((D + 1)\)-dimensional embedding space with signature \(\eta_{AB} = (- + \cdots +)\) and \(DY^{iA} = dY^{iA} + \Lambda^{ij} Y^A_j\) denotes the Sp\(_(2, R)\)-covariant derivative along the world line. The local Sp\(_(2, R)\) symmetry, with generators

\[ K_{ij} = \frac{1}{2} Y^A_i Y^A_j , \] (2.2)

embodies the p-brane \(\tau\)-diffeomorphisms and the constraints associated with the embedding of the conical limit of AdS\(_D\) in the target space. Multi-bit states with fixed numbers of bits are postulated to be exact states in the theory, in analogy with the matrix-model interpretation of the discretized membrane in flat space. A more detailed discussion of the open-string-like quantum mechanics of such systems will be presented in [15].

A crucial result is that the state space of a single bit,

\[ \mathcal{H}_{1\text{-bit}} = \{ |\Psi\rangle : K_{ij} * |\Psi\rangle = 0 \} , \] (2.3)

where \(*\) denotes the non-commutative and associative oscillator product, coincides with the CPT self-conjugate scalar singleton \([18, 34, 15]\)

\[ \mathcal{H}_{1\text{-bit}} = D(\epsilon_0; \{0\}) \oplus \tilde{D}(\epsilon_0; \{0\}) , \quad \epsilon_0 = \frac{1}{2} (D - 3) , \] (2.4)

where \(D(E_0; S_0)\) and \(\tilde{D}(E_0; S_0)\) denote\(^3\) lowest and highest weight spaces of the SO\((D - 1, 2)\) generated by

\[ M_{AB} = \frac{1}{2} Y^A_i Y^B_i . \] (2.5)

\(^3\)We denote Sl\((N)\) highest weights by the number of boxes in the rows of the corresponding Young tableaux, \((m_1, \ldots, m_n) \equiv (m_1, \ldots, m_n, 0, \ldots, 0)\), and SO\((D - 1, T)\) highest weights by \(\{m_1, \ldots, m_n\} \equiv \{m_1, \ldots, m_n, 0, \ldots, 0\}\), for \(N = D + 1\) or \(D\), and \(T = 1\) or \(2\). The values of \(N\) and \(T\) will often be left implicit, but at times, for clarity, we shall indicate them by a subscript.
The naively defined norm of the 1-bit states diverges, since the lowest/highest weight states $|\Omega_{\pm}\rangle$ are "squeezed" $Y_{1}^{+}$-oscillator excitations [34, 15]. As we shall see, a related issue arises in the corresponding HS gauge theory, when it is subject to the strong $Sp(2, R)$-invariance condition (1.1).

The single-bit Hilbert space is irreducible under the combined action of $SO(D - 1, 2)$ and the discrete involution $\pi$ defined by $\pi(|\Omega_{\pm}\rangle) = |\Omega_{\mp}\rangle$, and by

$$\pi(P_a) = -P_a, \quad \pi(M_{ab}) = M_{ab}, \quad (2.6)$$

where $P_a = V_a^A V_B^B M_{AB}$ and $M_{ab} = V_a^A V_b^B M_{AB}$ are the (A)dS translation and rotation generators, and the tangent-space Lorentz index $a$ is defined by $X_a = V_a^A X_A$ and $X = V^A X_A$, where $X_A$ is any $SO(D - 1, 2)$ vector and $(V_a^A, V^A)$ is a quasi-orthogonal embedding matrix subject to the conditions

$$V_a^A V_b^B \eta_{AB} = \eta_{ab}, \quad V_a^A V_A = 0, \quad V^A V_A = -1. \quad (2.7)$$

The AdS generators, however, do not act transitively on the singleton weight spaces. The smallest Lie algebra with this property is the minimal HS extension $h_{00}(D - 1, 2)$ of $SO(D - 1, 2)$ defined by

$$h_{00}(D - 1, 2) = Env_1(SO(D - 1, 2))/I, \quad (2.8)$$

where $Env_1(SO(D - 1, 2))$ is the subalgebra of $Env(SO(D - 1, 2))$ elements that are odd under the $\tau$-map

$$\tau(M_{AB}) = -M_{AB}, \quad (2.9)$$

that may be regarded as an analog of the transposition of matrices and acts as an anti-involution on the enveloping algebra,

$$\tau(P \ast Q) = \tau(Q) \ast \tau(P), \quad P, Q \in Env(SO(D - 1, 2)), \quad (2.10)$$

and $I$ is the subalgebra of $Env_1(SO(D - 1, 2))$ given by the annihilating ideal of the singleton

$$I = \{ P \in Env_1(SO(D - 1, 2)) : P \ast |\Psi\rangle = 0 \text{ for all } |\Psi\rangle \in D(\epsilon_0; \{0\}) \}. \quad (2.11)$$

The minimal HS algebra has the following key features [34, 15]:

1. It admits a decomposition into levels labelled by irreducible finite-dimensional $SO(D - 1, 2)$ representations $\{2 \ell + 1, 2 \ell + 1\}$ for $\ell = 0, 1, 2, \ldots$, with the 0-th level identified as the $SO(D - 1, 2)$ subalgebra.

2. It acts transitively on the scalar singleton weight spaces.

3. It is a minimal extension of $SO(D - 1, 2)$, in the sense that if $SO(D - 1, 2) \subseteq \mathcal{L} \subseteq h_{00}(D - 1, 2)$ and $\mathcal{L}$ is a Lie algebra, then either $\mathcal{L} = SO(D - 1, 2)$ or $\mathcal{L} = h_{00}(D - 1, 2)$.

\footnote{This algebra is denoted by $hu(1/sp(2)[2, D - 1])$ in [11].}
One can show that these features determine uniquely the algebra, independently of the specific choice of realization [15]. In particular, in \( D = 4, 5, 7 \) this implies the isomorphisms [34, 15]

\[
ho_0(3, 2) \simeq hs(4) , \quad ho_0(4, 2) \simeq hs(2, 2) , \quad ho_0(6, 2) \simeq hs(8^*) ,
\]

where the right-hand sides denote the spinor-oscillator realizations.

From the space-time point of view, the dynamics of tensionless extended objects involves processes where multi-bit states interact by creation and annihilation of pairs of bits. Roughly speaking, unlike ordinary multi-particle states, multi-bit states have an extended nature that should reflect itself in a prescription for assigning weights \( \bar{\sigma} \) to the amplitudes in such a way that “hard” processes involving many simultaneous collisions be suppressed with respect to “soft” processes. This prescription leads to a \((1+1)\)-dimensional topological \( Sp \)-gauged \( \sigma \)-model \( \text{` a la Cattaneo-Felder} \) [39], whose associated Batalin-Vilkovisky master equation have a structure that might be related to the Vasiliev equations [15].

The 2-bit states are of particular interest, since it is natural to expect, in many ways, that their classical self-interactions form a consistent truncation of the classical limit of the full theory. The 2-bit Hilbert space consists of the symmetric (S) and anti-symmetric (A) products of two singletons, that decompose into massless one-particle states in AdS with even and odd spin [37, 34, 15]

\[
D(\epsilon_0; \{0\}) \otimes D(\epsilon_0; \{0\}) = \left[ \bigoplus_{s=0,2,\ldots} D(s + 2\epsilon_0; \{s\}) \right]_S \oplus \left[ \bigoplus_{s=1,3,\ldots} D(s + 2\epsilon_0; \{s\}) \right]_A .
\]  

The symmetric part actually coincides with the spectrum of the minimal bosonic HS gauge theory in \( D \) dimensions, to which we now turn our attention.

### 3 THE OFF-SHELL THEORY

#### 3.1 General Set-Up

The HS gauge theory based on the minimal bosonic HS algebra \( ho_0(D - 1, 2) \), given in (2.8), is defined by a set of constraints on the curvatures of an adjoint one form \( \hat{A} \) and a twisted-adjoint zero form \( \hat{\Phi} \). The off-shell master fields are defined by expansions in the \( Y_{\hat{A}} \) oscillators, obeying a weak \( Sp(2, R) \) invariance condition that will be defined in eq. (3.17) below. The constraints reduce drastically the number of independent component fields without implying any on-shell field equations. The independent fields are a real scalar field, arising in the master 0-form, a metric vielbein and an infinite tower of symmetric rank-\( s \) tensors for \( s = 4, 6, \ldots \), arising in the master 1-form. The remaining components are auxiliary fields: in \( \hat{A}_\mu \) one finds the Lorentz connection and its HS counterparts, that at the linearized level reduce to the Freedman-de Wit connections in a suitable gauge to be discussed below; in \( \hat{\Phi} \) one finds the derivatives of the scalar field, the spin 2 Riemann tensor, its higher-spin generalizations, and all their derivatives. Therefore, the minimal theory is a HS generalization of Einstein gravity, with infinitely many
bosonic fields but no fermions, and without an internal gauge group. We have already stressed that the minimal model admits generalizations with internal symmetry groups, that enter in way highly reminiscent of Chan-Paton groups [14] for open strings, but here we shall confine our attention to the minimal case.

The curvature constraints define a Cartan integrable system, a very interesting construction first introduced in supergravity, in its simplest non-conventional setting with 1-forms and 3-forms, by D’Auria and Fré [40]. Any such system is gauge invariant by virtue of its integrability, and is also manifestly diffeomorphism invariant since it is formulated entirely in terms of differential forms. The introduction of twisted-adjoint zero forms, however, was a key contribution of Vasiliev [16], that resulted in the emergence of the present-day unfolded formulation. The full HS gauge theory \textit{a priori} does not refer to any particular space-time manifold, but the introduction of a $D$-dimensional bosonic spacetime $\mathcal{M}_D$ yields a weak-field expansion in terms of recognizable tensor equations. An illustration of the general nature of this setting is provided in [26], where 4D superspace formulations are constructed directly in unfolded form picking a superspace as the base manifold.

A key technical point of Vasiliev’s construction is an internal noncommutative $\mathbb{Z}$-space $\mathcal{M}_Z$, that, from the space-time viewpoint, may be regarded as a tool for obtaining a highly non-linear integrable system with 0-forms on a commutative space-time $\mathcal{M}_D$ [1] from a simple integrable system on a non-commutative extended space. Although apparently \textit{ad hoc}, this procedure has a rather precise meaning within the BRST formulation of the phase-space covariant treatment of bits \textit{à la} Kontsevich-Cattaneo-Felder [39], as will be discussed in [15].

### 3.2 Oscillators and Master Fields

Following Vasiliev [11], we work with bosonic oscillators $Y^A_i$ and $Z^i_A$, where $A = -1, 0, 1, \ldots, D-1$ labels an $SO(D - 1, 2)$ vector and $i = 1, 2$ labels an $Sp(2, R)$ doublet. The oscillators obey the associative \textit{⋆}-product algebra

\begin{align}
Y^A_i \star Y^B_j &= Y^A_i Y^B_j + i\epsilon_{ij} \eta^{AB} , & Y^A_i \star Z^B_j &= Y^A_i Z^B_j - i\epsilon_{ij} \eta^{AB} , \\
Z^A_i \star Y^B_j &= Z^A_i Y^B_j + i\epsilon_{ij} \eta^{AB} , & Z^A_i \star Z^B_j &= Z^A_i Z^B_j - i\epsilon_{ij} \eta^{AB} ,
\end{align}

where the products on the right-hand sides are Weyl ordered, \textit{i.e.} symmetrized, and obey the conditions

\begin{equation}
(Y^A_A)^\dagger = Y^A_A \quad \text{and} \quad (Z^A_A)^\dagger = -Z^A_A.
\end{equation}

It follows that the \textit{⋆}-product of two Weyl-ordered polynomials $\hat{f}$ and $\hat{g}$ can be defined by the integral

\begin{equation}
\hat{f}(Y, Z) \star \hat{g}(Y, Z) = \int \frac{d^{2(D+1)}S \, d^{2(D+1)}T}{(2\pi)^{2(D+1)}} \hat{f}(Y + S, Z + S) \hat{g}(Y + T, Z - T) e^{iT^A S_A} ,
\end{equation}

where $S$ and $T$ are real unbounded integration variables.
The master fields of the minimal model are a 1-form and a 0-form

\[ \hat{A} = dx^\mu \hat{A}_\mu(x, Y, Z) + dZ^i \hat{A}_{iA}(x, Y, Z) , \quad \hat{\Phi} = \hat{\Phi}(x, Y, Z) , \]  

subject to the conditions defining the adjoint and twisted-adjoint representations,

\[ \tau(\hat{A}) = -\hat{A} , \quad \hat{A}^\dagger = -\hat{A} , \quad \tau(\hat{\Phi}) = \pi(\hat{\Phi}) , \quad \hat{\Phi}^\dagger = \pi(\hat{\Phi}) , \]

where \( \pi \) and \( \tau \), defined in (2.6) and (2.9), act on the oscillators as

\[ \pi(\hat{f}(x^\mu, Y^i_a, Y^{i_A}, Z^i_a, Z^{i_A})) = \hat{f}(x^\mu, Y^i_{-a}, -Y^{a_i}, Z^i_{-a}, -Z^{a_i}) , \]

(3.7)

\[ \tau(\hat{f}(x^\mu, Y^i_{iA}, Z^i_{iA})) = \hat{f}(x^\mu, iY^i_{-A}, -iZ^i_{-A}) . \]

(3.8)

The \( \pi \)-map can be generated by the \( \ast \)-product with the hermitian and \( \tau \)-invariant oscillator construct

\[ \kappa = e^{iZ^i Y_i} = (\kappa)\dagger = \tau(\kappa) , \]

(3.9)

such that

\[ \kappa \ast \hat{f}(Y^i, Z^i) = \pi(\hat{f}(Y^i, Z^i)) \ast \kappa = \kappa \hat{f}(Z^i, Y^i) . \]

(3.10)

Strictly speaking, \( \kappa \) lies outside the domain of “arbitrary polynomials”, for which the integral representation (3.4) of the \( \ast \)-product is obviously well-defined. There is, however, no ambiguity in (3.10), in the sense that expanding \( \kappa \) in a power series and applying (3.4) term-wise, or making use of the standard representation of the Dirac \( \delta \) function, one is led to the same result. This implies, in particular, that \( \kappa \ast \kappa = 1 \), so that \( \kappa \ast \hat{f} \ast \kappa = \pi(\hat{f}) \), although the form (3.10) will be most useful in the formal treatment of the master constraints. It is also worth stressing that the \( n \)-th order curvature corrections, to be discussed later, contain \( n \) insertions of exponentials, of the type

\[ \cdots \ast \kappa(t_1) \ast \cdots \ast \kappa(t_n) \cdots , \quad \kappa(t) = e^{itZ^i Y_i} , \quad t_i \in [0, 1] , \]

(3.11)

that are thus well-defined\(^5\), and can be expanded term-wise, as was done for instance to investigate the second-order scalar corrections to the stress energy tensor in [41], or the scalar self-couplings in [42].

### 3.3 Master Constraints

The off-shell minimal bosonic HS gauge theory is defined by

\[ \hat{F} = \frac{i}{2} dZ^i \wedge dZ_i \hat{\Phi} \ast \kappa , \quad \hat{D}\hat{\Phi} = 0 , \]

(3.12)

\(^5\)We are grateful to M. Vasiliev for an extensive discussion on these points during the Brussels Workshop.
where $Z^i = V^A Z^i_A$, the curvature and the covariant derivative are given by

$$
\hat{F} = d\hat{A} + \hat{A} \wedge \hat{A}, \quad \hat{D}\Phi = d\hat{\Phi} + [\hat{A}, \hat{\Phi}]_\pi ,
$$

(3.13)

and the $\pi$-twisted commutator is defined as

$$
[\hat{f}, \hat{g}]_\pi = \hat{f} \ast \hat{g} - \hat{g} \ast \pi(\hat{f}) .
$$

(3.14)

The integrability of the constraints implies their invariance under the general gauge transformations

$$
\delta \hat{\epsilon} \hat{A} = \hat{D}\hat{\epsilon} , \quad \delta \hat{\epsilon} \hat{\Phi} = -[\hat{\epsilon}, \hat{\Phi}]_\pi ,
$$

(3.15)

where the covariant derivative of an adjoint element is defined by

$$
D\hat{\epsilon} \equiv d\hat{\epsilon} + \hat{A} \ast \hat{\epsilon} - \hat{\epsilon} \ast \hat{A} ;
$$

ii) the invariance of the master fields under global $Sp(2, R)$ gauge transformations with $\hat{\epsilon}(\lambda) = \frac{i}{2} \lambda^{ij} \hat{K}_{ij}$, where the $\lambda^{ij}$ are constant parameters, and [11]

$$
\hat{K}_{ij} = K_{ij} + \frac{1}{2} \left( \hat{S}^A_{(i} \ast \hat{S}^A_{j)} - Z^A_i Z^A_j \right) , \quad \hat{S}^A_i \equiv Z^A_i - 2i \hat{A}^A_i ,
$$

(3.16)

where $K_{ij}$ are the $Sp(2, R)$ generators of the linearized theory, defined in (2.2).

The $Sp(2, R)$ invariance conditions can equivalently be written in the form

$$
[\hat{K}_{ij}, \hat{\Phi}]_\pi = 0 , \quad D\hat{K}_{ij} = 0 .
$$

(3.17)

These conditions remove all component fields that are not singlets under $Sp(2, R)$ transformations. We also stress that the $\hat{S} \ast \hat{S}$ terms play a crucial role in the $Sp(2, R)$ generators $\hat{K}_{ij}$ of (3.16): without them all $Sp(2, R)$ indices originating from the oscillator expansion would transform canonically but, as shown in [11], they guarantee that the same holds for the doublet index of $\hat{A}_{A_i}$.

### 3.4 Off-Shell Adjoint and Twisted-Adjoint Representations

The gauge transformations (3.15) are based on a rigid Lie algebra that we shall denote by $ho(D - 1, 2)$, and that can be defined considering $x$- and $Z$-independent gauge parameters. Consequently, this algebra is defined by [11]

$$
ho(D - 1, 2) = \left\{ Q(Y) : \tau(Q) = Q^\dagger = -Q , \quad [K_{ij}, Q]_\ast = 0 \right\} ,
$$

(3.18)

with Lie bracket $[Q, Q']_\ast$, and where the linearized $Sp(2, R)$ generators are defined in (2.2). The $\tau$-condition implies that an element $Q$ admits the level decomposition $Q = \sum_{\ell=0}^\infty Q_\ell$, where $Q_\ell(\lambda^Y) = \lambda^{4\ell+2} Q_\ell(Y)$ and $[K_{ij}, Q_\ell]_\ast = 0$. As shown in [11], the $Sp(2, R)$-invariance condition implies that $Q_\ell(Y)$ has a $Y$-expansion in terms of (traceless) $SO(D - 1, 2)$ tensors combining
into single $SL(D + 1)$ tensors with highest weights corresponding to Young tableaux of type $(2\ell + 1, 2\ell + 1)$:

\[ Q_\ell = \frac{1}{2\ell} \sum_{m=0}^{2\ell+1} Q^{(2\ell+1,m)}_{a_1 \ldots a_2 \ell+1 b_1 \ldots b_m} M^{a_1 b_1} \ldots M^{a_m b_m} P^{a_{m+1}} \ldots P^{a_{2\ell+1}} , \]

(3.19)

where the products are Weyl ordered and the tensors in the second expression are labelled by highest weights of $SL(D)$. To reiterate, the algebra $ho(D - 1, 2)$ is a reducible HS extension of $SO(D - 1, 2)$, whose $\ell$-th level generators fill a finite-dimensional reducible representation of $SO(D - 1, 2)$. In the next section we shall discuss how to truncate the trace parts at each level in $ho(D - 1, 2)$ to define a subalgebra of direct relevance for the on-shell theory.

The 0-form master field $\Phi$, defined in (3.22) below, belongs to the twisted-adjoint representation $T[ho(D - 1, 2)]$ of $ho(D - 1, 2)$ associated with the rigid covariantization terms in $\hat{D}_\mu \Phi|_{Z=0}$. Consequently,

\[ T[ho(D - 1, 2)] = \{ R(Y) : \tau(R) = R^\dagger = \pi(R) , \quad [K_{ij}, R]^\dagger = 0 \} , \]

(3.20)

on which $ho(D - 1, 2)$ acts via the $\pi$-twisted commutator $\delta \Phi = -[\epsilon, R]_\pi$. A twisted-adjoint element admits the level decomposition $R = \sum_{\ell=-1}^{\infty} R_\ell$, where $R_\ell$ has an expansion in terms of $SL(D)$ irreps as ($\ell \geq -1$)

\[ R_\ell = \sum_{k=0}^{\infty} \frac{1}{k!} R^{(2\ell+2+k, 2\ell+2)}_{a_1 \ldots a_{2\ell+2+k} b_1 \ldots b_{2\ell+2}} M^{a_1 b_1} \ldots M^{a_{2\ell+2+k} b_{2\ell+2}} P^{a_{2\ell+3}} \ldots P^{a_{2\ell+2+k}} . \]

(3.21)

It can be shown that each level forms a separate, infinite dimensional reducible $SO(D - 1, 2)$ representation, that includes an infinity of trace parts that will be eliminated in the on-shell formulation. In particular, $R_{-1}$ has an expansion in terms of $SL(D)$ tensors carrying the same highest weights as an off-shell scalar field and its derivatives, while each of the $R_\ell$, $\ell \geq 0$, consists of $SL(D)$ tensors corresponding to an off-shell spin-$(2\ell + 2)$ Riemann tensor and its derivatives.

### 3.5 Weak-Field Expansion and Linearized Field Equations

The master constraints (3.12) can be analyzed by a weak-field expansion, which will sometimes be referred to as a perturbative expansion, in which the scalar field, the HS gauge fields and all curvatures (including the spin-two curvature) are indeed treated as weak. One can then start from the initial conditions

\[ A_\mu = \hat{A}_\mu|_{Z=0} , \quad \Phi = \hat{\Phi}|_{Z=0} , \]

(3.22)

fix a suitable gauge and solve for the $Z$-dependence of $\hat{A}$ and $\hat{\Phi}$ order by order in $\Phi$ integrating the constraints

\[ \hat{D}_i \hat{\Phi} = 0 , \quad \hat{F}_{ij} = -i\epsilon_{ij} \hat{\Phi} \ast \kappa , \quad \hat{F}_{i\mu} = 0 , \]

(3.23)
thus obtaining, schematically,

\[ \hat{\Phi} = \hat{\Phi}(\Phi), \quad \hat{A}_\mu = \hat{A}_\mu(\Phi, A_\mu), \quad \hat{A}_i = \hat{A}_i(\Phi). \] (3.24)

Substituting these solutions in the remaining constraints evaluated at \( Z = 0 \) gives

\[ \hat{F}_{\mu\nu}|_{Z=0} = 0, \quad \hat{D}_\mu \hat{\Phi}|_{Z=0} = 0, \] (3.25)

that describe the full non-linear off-shell HS gauge theory in ordinary spacetime, i.e. the full set of non-linear constraints that define its curvatures without implying any dynamical equations.

A few subtleties are involved in establishing the integrability in spacetime of (3.25) in perturbation theory. One begins as usual by observing that integrability holds to lowest order in \( \Phi \), and proceeds by assuming that all the constraints hold for all \( x^\mu \) and \( Z \) to \( n \)-th order in \( \Phi \). One can then show that the constraint \( (\hat{D}_i \hat{\Phi})^{(n+1)} = 0 \) is an integrable partial differential equation in \( Z \) for \( \hat{\Phi}^{(n+1)} \), whose solution obeys \( \partial_i \hat{F}^{(n+1)}_{\mu\nu} = 0 \), which in its turn implies that \( (\hat{D}_\mu \hat{\Phi})^{(n+1)} = 0 \) for all \( Z \) if \( (\hat{D}_\mu \hat{\Phi})^{(n+1)}|_{Z=0} = 0 \). Proceeding in this fashion, one can obtain \( \hat{A}^{(n+1)}_i \), and then \( \hat{A}^{(n+1)}_\mu \), via integration in \( Z \), to show that \( \partial_i \hat{F}^{(n+1)}_{\mu\nu} = 0 \), so that if \( \hat{F}^{(n+1)}_{\mu\nu}|_{Z=0} = 0 \), then \( \hat{F}^{(n+1)}_{\mu\nu} = 0 \) for all \( Z \). It thus follows, by induction, that once the constraints (3.25) on \( \hat{F}_{\mu\nu} \) and \( \hat{\Phi} \) are imposed at \( Z = 0 \), they hold for all \( Z \), and are manifestly integrable in spacetime, since \( Z \) can be treated as a parameter. Hence, their restriction to \( Z = 0 \) is also manifestly integrable in spacetime, simply because the exterior derivative \( dx^\mu \partial_\mu \) does not affect the restriction to \( Z = 0 \).

Having obtained (3.25) in a \( \Phi \)-expansion, one can write

\[ A_\mu = e_\mu + \omega_\mu + W_\mu, \] (3.26)

where \( e_\mu = \frac{1}{2} e_\mu^a P_a \) and \( \omega_\mu = \frac{1}{2} \omega_\mu^{ab} M_{ab} \), and where \( W_\mu \) contains the HS gauge fields residing at levels \( \ell \geq 1 \). We would like to stress that until now \( \mu \) has been treated as a formal curved index with no definite intrinsic properties. Treating \( e_\mu \) and \( \omega_\mu \) as strong fields and referring \( \mu \) explicitly to a \( D \)-dimensional bosonic spacetime clearly builds a perturbative expansion that preserves local Lorentz invariance and \( D \)-dimensional diffeomorphism invariance. It would be interesting to investigate to what extent the higher-spin geometry could be made more manifest going beyond this choice.

To first order in the weak fields \( \Phi \) and \( W_\mu \), the constraints (3.25) reduce to

\[ \mathcal{R} + \mathcal{F} = i e^a \wedge e^b \frac{\partial^2 \Phi}{\partial Y_a \partial Y_b}|_{Y^i=0}, \] (3.27)

\[ \nabla \Phi + \frac{1}{2i} e^a \{ P_a, \Phi \}_\pi = 0, \] (3.28)

where \( \mathcal{R} \) is the \( SO(D - 1, 2) \)-valued curvature of \( E \equiv e + \omega \) defined by \( \mathcal{R} = dE + [E, E]_\pi \), and \( \mathcal{F} \) is the linearized \( SO(D - 1, 2) \) covariant curvature defined by \( \mathcal{F} = dW + \{ E, W \}_\pi \), and \( \nabla \Phi = d\Phi + [\omega, \Phi]_\pi \) is the Lorentz covariant derivative of \( \Phi \). Since each level of the adjoint
and twisted-adjoint master fields forms a separate representation of $SO(D - 1, 2)$, the linearized constraints split into independent sets for the individual levels:

$$\ell = 0 : \; \mathcal{R} = -8 i e^a \wedge e^b \Phi^{(2,2)}_{ab,cd} M^{cd} ,$$

$$\ell \geq 1 : \; \mathcal{F}_\ell = i e^a \wedge e^b \frac{\partial^2 \Phi^{(2\ell+2,2\ell+2)}_\ell}{\partial Y^a_i \partial Y^b_i} ,$$

$$\ell \geq -1 : \; \nabla \Phi_\ell + \frac{1}{2i} e^a \left\{ P_a, \Phi_\ell \right\}_* = 0 .$$

(3.29)

(3.30)

(3.31)

Note that $\mathcal{F}$ is to be expanded as in (3.19), and $\Phi_\ell$ as in (3.21). Furthermore, in order to compute the star anticommutator in the last equation, one must use the $Y$-expansion of all generators involved, and double contractions contribute to this term. Note also, for example, that even if the star commutator occurs in $\mathcal{R}$, using the commutation relation between the AdS generators is not sufficient, and one must also recall relations such as $[M_{ab}, P_c] = 0$, that follow from the oscillator realization of these generators.

The first of (3.31) contains the usual torsion constraint, and identifies $\Phi^{(2,2)}_{ab,cd}$ with the $SO(D - 1, 2)$-covariantized Riemann curvature of $e_{\mu}^a$. As $\Phi^{(2,2)}_{ab,cd}$ is traceful, these equations describe off-shell AdS gravity: the trace of (3.29) simply determines the trace part of $\Phi^{(2,2)}_{ab,cd}$, rather than giving rise to the Einstein equation. This generalizes to the higher levels, and a detailed analysis of (3.30) reveals that [17]:

i) the gauge parameter $\epsilon_\ell$ contains Stückelberg-type shift symmetries;

ii) $W_\ell$ contains pure gauge parts and auxiliary gauge fields that can be eliminated using shift symmetries or constraints on torsion-like components of $\mathcal{F}_\ell$, respectively;

iii) the remaining independent components of $W_\ell$ correspond to the fully symmetric tensors

$$\phi^{(s)}_{a_1...a_s} \equiv e_{(a_1} \mu W^{(s-1)}_{\mu, a_2...a_s)} , \quad s = 2\ell + 2 ;$$

(3.32)

iv) the system is off-shell: the remaining non-torsion-like components of $\mathcal{F}_\ell$ vanish identically, with the only exception of the $s$-th one, that defines the generalized (traceful) Riemann tensor of spin $s = 2\ell + 2$,

$$R^{(s,s)}_{a_1...a_s, b_1...b_s} \equiv e^\mu_{(a_1} e^\nu(b_1} \mathcal{F}^{(s-1,s-1)}_{\mu, a_2...a_s)} b_2...b_s) = 4s^2 \phi^{(s,s)}_{a_1...a_s, b_1...b_s} ,$$

(3.33)

where the identification follows from (3.27) and the Riemann tensor is built from $s$ derivatives of $\phi^{(s)}$.

Turning to the $\Phi_\ell$-constraint (3.31), one can show that its component form reads ($s = 2\ell + 2$, $\ell \geq -1, k \geq 0$):

$$\nabla_\mu \Phi^{(s+k,s)}_{a_1...a_{s+k}, b_1...b_s} = \frac{i}{4} (s+k+2) \Phi^{(s+k+1,s)}_{a_1...a_{s+k}, b_1...b_s} + i \frac{(k+1)(s+k)}{s+k+1} \eta_{(a_1} \phi^{(s+k-1,s)}_{a_2...a_{s+k}), b_1...b_s} .$$

(3.34)

14
where we have indicated a Young projection to the tableaux with highest weight \((s + k, s)\). Symmetrizing \(\mu\) and \(a_1 \ldots a_k\) shows that \(\Phi^{(s + k + 1, s)}\) \((k \geq 0)\) are auxiliary fields, expressible in terms of derivatives of \(\Phi^{(s, s)}\). The \(\Phi^{(s, s)}\) components are generalized (traceful) Riemann tensors given by (3.33) for \(\ell \geq 0\), and an independent scalar field for \(\ell = -1\),

\[
\phi \equiv \Phi^{(0,0)} .
\]

The remaining components of (3.34), given by the \((s + k, s + 1)\) and \((s + k, s, 1)\) projections, are Bianchi identities. Hence, no on-shell conditions are hidden in (3.34).

In particular, combining the \(k = 0, 1\) components of (3.34) for \(s = 0\), one finds

\[
\left( \nabla^2 + \frac{D}{2} \right) \phi = -\frac{3}{8} \eta^{ab} \Phi^{(2,0)}_{ab} ,
\]

which, as stated above, determines the trace part \(\eta^{ab} \Phi^{(2)}_{ab}\) of the auxiliary field \(\Phi^{(2)}\) rather than putting the scalar field \(\phi\) on-shell. Similarly, the \(s \geq 2\) and \(k = 0, 1\) components of (3.34) yield

\[
\nabla^2 \Phi^{(s,s)}_{a_1 \ldots a_s, b_1 \ldots b_s} + \frac{(D + s)}{2} \Phi^{(s,s)}_{a_1 \ldots a_s, b_1 \ldots b_s} = \frac{(s + 2)(s + 3)}{16} \eta^{\mu\nu} \Phi^{(s+2,s)}_{\mu\nu \ a_1 \ldots a_s, b_1 \ldots b_s} ,
\]

that determine the trace parts of the auxiliary fields \(\Phi^{(s+2,s)}\), rather than leading to the usual Klein-Gordon-like equations satisfied by on-shell Weyl tensors.

In summary, the constraints (3.12) describe an off-shell HS multiplet with independent field content given by a tower of real and symmetric rank-

\(s\) \(SL(D)\) tensors with \(s = 0, 2, 4, \ldots\),

\[
\phi, e^a_\mu, \phi^{(s)}_{a_1 \ldots a_s} \quad (s = 4, 6, \ldots) ,
\]

where the scalar field \(\phi\) is given in (3.35), the vielbein \(e^a_\mu\) is defined in (3.26), and the real symmetric HS tensor fields are given in (3.32).

## 4 THE ON-SHELL THEORY

### 4.1 On-Shell Projection

It should be appreciated that, if the trace parts in (3.36) and (3.37) were simply dropped, the resulting masses would not coincide with the proper values for a conformally coupled scalar and on-shell spin-

\(s\) Weyl tensors in \(D\) dimensions. Hence, the trace parts contained in \(\Phi\) must be carefully eliminated, and cannot be simply set equal to zero. To this end, as discussed in the Introduction, motivated by the arguments based on \(Sp(2, R)\)-gauged noncommutative phase spaces arising in the context of tensionless strings and two-time physics, we would like to propose the on-shell projection

\[
\tilde{K}_{ij} \ast \tilde{\Phi} = 0 ,
\]

(4.1)
where $\hat{K}_{ij}$ is defined in (3.16), be adjoined to the master constraints of [11]

$$
\hat{F} = \frac{i}{2} dZ^i \wedge dZ_i \hat{\Phi} \star \kappa , \quad \hat{D} \hat{\Phi} = 0 .
$$

(4.2)

Although one can verify that the combined constraints (4.1) and (4.2) remain formally integrable at the full non-linear level, the strong $Sp(2, R)$ projection (4.1) should be treated with great care, since its perturbative solution involves non-polynomial functions of the oscillators associated with projectors whose products can introduce divergences in higher orders of the perturbative expansion unless they are properly treated [34]. It is important to stress, however, that these singularities draw their origin from the curvature expansion, and in Section 5 we shall describe how a finite curvature expansion might be defined.

We can thus begin by exploring the effects of (4.1) on the linearized field equations and, as we shall see, the correct masses emerge in a fashion which is highly reminiscent of what happens in spinor formulations. There is a further subtlety, however. According to the analysis in the previous section, if the internal indices of the gauge fields in $\hat{A}$ were also taken to be traceless, the linearized 2-form constraint (3.30) would reduce to the Fronsdal equations [23], that in the index-free notation of [21] read

$$
\mathcal{F} = 0 ,
$$

(4.3)

where the Fronsdal operator is

$$
\mathcal{F} \equiv \nabla^2 \phi - \nabla \nabla \cdot \phi + \nabla \nabla \phi' - \frac{1}{L^2} \{ [(3 - D - s)(2 - s) - s]\phi + 2g\phi' \} ,
$$

(4.4)

and where “primes” denote traces taken using the AdS metric $g$ of radius $L$. We would like to stress that this formulation is based on *doubly traceless* gauge fields, and is invariant under gauge transformations with traceless parameters. However, the constrained gauge fields of this conventional formulation should be contrasted with those present in (4.2), namely the metric and the HS gauge fields collected in (3.32). These fields do contain trace parts, and enforcing the 0-form projection as in (4.1), as we shall see, actually leaves the gauge fields free to adjust themselves to their constrained sources in the projected Weyl 0-form $\hat{\Phi}$, thus extending the conventional Fronsdal formulation based on (4.3) and (4.4) to the geometric formulation of [21]. This is due to the fact that, once the 0-forms on the right-hand side of (3.30) are constrained to be traceless Weyl tensors, certain trace parts of the spin-$s$ gauge fields can be expressed in terms of gradients of traceful rank-$(s - 3)$ symmetric tensors $\alpha_{a_1...a_{s-3}}$. As we shall see in detail in subsection 4.4, up to the overall normalization of $\alpha$, that is not chosen in a consistent fashion throughout this paper, these will enter the physical spin-$s$ field equations embodied in the Vasiliev constraints precisely as in [21, 22]. In the index-free notation of [21], the complete linearized field equations in an AdS background would thus read

$$
\mathcal{F} = 3 \nabla \nabla \nabla \alpha - \frac{4}{L^2} g \nabla \alpha ,
$$

(4.5)

where $\mathcal{F}$ denotes again the Fronsdal operator. Notice, however, that *now the gauge field $\phi$ is traceful and $\alpha$ plays the role of a compensator for the traceful gauge transformations*,

$$
\delta \phi = \nabla \Lambda , \quad \delta \alpha = \Lambda' ,
$$

(4.6)
with $\Lambda'$ the trace of the gauge parameter $\Lambda$. When combined with the Bianchi identity, eq. (4.5) implies that \[ \phi'' = 4 \nabla \cdot \alpha + \nabla \alpha', \] so that once $\alpha$ is gauged away using $\Lambda'$, eq. (4.5) reduces to the Fronsdal form. It will be interesting to explore the role of the additional gauge symmetry in the interactions of the model.

To reiterate, the system (4.1) and (4.2) provides a realization à la Cartan of an on-shell HS gauge theory that embodies the non-local geometric equations of [21], with traceful gauge fields and parameters, rather than the more conventional Fronsdal form [23].

### 4.2 On-Shell Adjoint and Twisted-Adjoint Representations

It is important to stress that in our proposal both the on-shell and off-shell systems contain a gauge field in the adjoint representation of $ho(D-1,2)$, while in the on-shell system the 0-form obeys an additional constraint, the strong $Sp(2,R)$ projection condition (4.1). The on-shell twisted-adjoint representation is thus defined for $D \geq 4$ by\footnote{In $D = 3$ one can show that $S$ is actually a constant [34, 17].}

\[ T_0[h\mathfrak{o}(D-1,2)] = \{ S \in T[h\mathfrak{o}(D-1,2)] : K_{ij} \star S = 0, \right S \star K_{ij} = 0 \}, \]

where the linearized $Sp(2,R)$ generators $K_{ij}$ are defined in (2.2). The part of $A_\mu$ that annihilates the twisted-adjoint representation can thus be removed from the rigid covariantizations in $\hat{D}_\mu \hat{\Phi}|_{Z=0}$. It forms an $Sp(2,R)$-invariant ideal $I(K) \subset h\mathfrak{o}(D-1,2)$ consisting of the elements generated by left or right $\star$-multiplication by $K_{ij}$, i.e.

\[ I(K) = \{ K^{ij} \star f_{kl} : \tau(f_{kl}) = (f_{ij})^\dagger = -f_{ij}, [K_{ij}, f^{kl}]_\star = 4i\delta^{(k}(l)_{i]} \} \]

Factoring out the ideal $I(K)$ from the Lie bracket $[Q, Q']_\star$, one is led to the minimal bosonic HS algebra

\[ h\mathfrak{o}(D-1,2)/I(K) \simeq h\mathfrak{o}_0(D-1,2) \equiv Env_1(SO(D-1,2))/T, \]

defined in (2.8). The isomorphism follows from the uniqueness of the minimal algebra [15], and from the fact that $h\mathfrak{o}(D-1,2)/I(K)$ shares its key properties [11, 34, 15]. Thus, at the linearized level, the gauging of $h\mathfrak{o}_0(D-1,2)$ gives rise, for each even spin $s$, to the canonical frame fields \[ A_{\mu,a_1...a_{s-1},b_1...b_k}^{(s-1,k)} = \left\{ A_{\mu,a_1...a_{s-1},b_1...b_k}^{(s-1,k)} \right\}_{k=0}^{s-1}, \]

required to describe massless spin-$s$ degrees of freedom in the conventional Fronsdal form.

We would like to stress, however, that at the non-linear level the full on-shell constraints (4.1) and (4.2) make use of the larger, reducible set of off-shell gauge fields valued in $h\mathfrak{o}(D-1,2) = h\mathfrak{o}_0(D-1,2) \oplus I(K)$. Hence, while the linearized compensator form (4.5) can be simply gauge fixed to the conventional Fronsdal form, interesting subtleties might well arise at the nonlinear...
level. It would thus be interesting to compare the interactions defined by (4.1) and (4.2), to be extracted using the prescription of Section 5.3, with those resulting from the formulation in [11].

The minimal algebra allows matrix extensions that come in three varieties, corresponding to the three infinite families of classical Lie algebras [34]. We have already stressed that these enter in a fashion highly reminiscent of how Chan-Paton factors enter open strings [14], and in this respect the presently known HS gauge theories appear more directly related to open than to closed strings. The minimal bosonic HS gauge theory has in fact a clear open-string analog, the $O(1)$ bosonic model described in the review paper of Schwarz in [14]. More support for this view will be presented in [15].

### 4.2.1 Dressing Functions

The strong $Sp(2, R)$-invariance condition on the twisted-adjoint representation $T_0[ho(D - 1, 2)]$, $K_{ij} \ast S = 0$, or equivalently $K_I \ast S = 0$, where $I$ is a triplet index, can be formally solved letting [34]

$$S = M \ast R \, , \quad (4.12)$$

where $R \in T[ho(D - 1, 2)]/I(K)$ and $M$ is a function of $K^2 = K^I K_I$ that is analytic at the origin and satisfies

$$K_{ij} \ast M = 0 \, , \quad \tau(M) = M^\dagger = M \, . \quad (4.13)$$

This and the normalization $M(0) = 1$ imply that

$$M(K^2) = \sum_{p=0}^{\infty} \frac{(-4 K^2)^p}{p!} \frac{\Gamma(D/2)}{\Gamma(D/2 + 2p)} = \Gamma \left( \frac{D}{2} \right) \frac{J_{\frac{D}{2} - 1}(4 \sqrt{K^2})}{(2 \sqrt{K^2})^{\frac{D}{2} - 1}} \, , \quad (4.14)$$

where $J$ is a Bessel function and $\Gamma$ is the Euler $\Gamma$ function. Actually, $M$ belongs to a class of dressing functions

$$F(N; K^2) = \sum_{p=0}^{\infty} \frac{(-4 K^2)^p}{p!} \frac{\Gamma(\frac{1}{2}(N + D))}{\Gamma(\frac{1}{2}(N + D + 2p))} = \Gamma (\nu + 1) \frac{J_\nu(4 \sqrt{K^2})}{(2 \sqrt{K^2})^\nu} \, , \quad (4.15)$$

related to Bessel functions of order $\nu = \frac{N + D - 2}{2}$ and argument $4 \sqrt{K^2}$, and in particular

$$M(K^2) = F(0; K^2) \, . \quad (4.16)$$

The dressing functions arise in the explicit level decomposition of the twisted-adjoint element $S$ in (4.12). In fact, for $D \geq 4$ one finds

$$S = \sum_{\ell = -1}^{\infty} S_\ell \, , \quad S_\ell = \sum_{q=0}^{\ell+1} S_{\ell,q} \, , \quad (4.17)$$
where the expansion of $S_{\ell,q}$ is given by ($s = 2\ell + 2$) [17]

$$S_{\ell,q} = \sum_{k=0}^{\infty} \frac{d_{s,k,q}}{k!} \mathcal{S}_{(s+k,s-2q)}^{(s+k,s-2q)} \eta_{b_1 b_2} \cdots \eta_{b_{2q-1} b_{2q}} F(2(s+k); K^2) \times$$

$$\times M^{a_1 b_1} \cdots M^{a_s b_s} P^{a_{s+1}} \cdots P^{a_{s+k}}. \quad (4.18)$$

The coefficients $d_{s,k,q}$ with $q \geq 1$ are fixed by the requirement that all Lorentz tensors arise from the decomposition of AdS tensors, $S_{\ell,s+k} \in S_{\ell,s+k+1}$, while $d_{s,k,0}$ can be set equal to one by a choice of normalization [17]. The Lorentz tensors arising in $S_{\ell,q}$ with $q \geq 1$ are simply combinations of those in $S_{\ell,0}$, that constitute the various levels of the on-shell twisted-adjoint representation ($s = 2\ell + 2$):

$$S_{\ell,0} : \left\{ S_{(s+k,s)}^{a_1 \cdots a_{s+k,b_1 \cdots b_s}} (k = 0, 1, 2, \ldots) \right\}. \quad (4.19)$$

The $\ell$-th level forms an irreducible multiplet within the twisted-adjoint representation of $SO(D-1,2)$. In showing this explicitly, the key point is that the translations, generated by $\delta_\xi = \xi^a \{ P_a, \cdots \}$, do not mix different levels [17]. Thus, $S_{-1,0}$ affords an expansion in terms of Lorentz tensors carrying the same highest weights as an on-shell scalar field and its derivatives, while the $S_{\ell,0}$, $\ell \geq 0$, correspond to on-shell spin-$(2\ell + 2)$ Weyl tensors and their derivatives.

### 4.3 Linearized Field Equations and Spectrum of the Model

In order to obtain the linearized field equations, it suffices to consider the initial conditions at $Z = 0$ to lowest order in the 0-form

$$\hat{A}_\mu \big|_{Z=0} = A_\mu \in ho(D-1,2), \quad \hat{\Phi} \big|_{Z=0} = M \star C + O(C^2), \quad (4.20)$$

where $M \star C \in T_0[ho(D-1,2)]$. The first $C$ correction to $\hat{A}_\mu$ is then obtained integrating in $Z$ the constraint $\hat{F}_{\mu
u} = 0$. Expanding also in the higher-spin gauge fields as discussed below (3.26) and fixing suitable gauges [11, 17], one finds that the Vasiliev equations reduce to [17]

$$\mathcal{R} + \mathcal{F} = i \frac{\partial^2 (C \star M)}{\partial Y^\mu \partial Y^\nu} \bigg|_{Y^i = 0}, \quad (4.21)$$

$$\nabla (C \star M) + \frac{1}{2i} e^a \{ P_a, C \star M \}_* = 0. \quad (4.22)$$

### 4.3.1 $\Phi$ Constraint and Role of the Dressing Functions

The expansion of the master field $C \star M$ involves the dressing functions $F(N; K^2)$, as in (4.18). These, in their turn, play a crucial role in obtaining the appropriate field equations already at

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7As we shall see in the next subsection, this mixing has a sizable effect on the free field equations.

8The strong $Sp(2, \mathbb{R})$ condition induces higher-order $C$ corrections to the initial condition on $\hat{\Phi}$ [17].
the linearized level, since they determine the trace parts of the auxiliary fields $\Phi^{(s+2,s)}$ in (3.37). Using these traces in the iterated form of (4.22), one can show that the field equations for the scalar $\phi = C|_{Y=0}$ and the Weyl tensors $C^{(s,s)}$ are finally
\[
(\nabla^2 - m_0^2) \phi = 0, \quad (\nabla^2 - m_s^2) C^{(s,s)}_{a_1...a_s,b_1...b_s} = 0, \quad m_s^2 = -\frac{1}{2} (s + D - 3),
\]
and contain the proper mass terms. The details of the computations leading to this result will be given in [17], but the effect on the scalar equation is simple to see and can be spelled out in detail here. Indeed, the relevant contributions to $K_{ij} \delta * \phi$ can be traced to the action of $K_{ij}$ on $(\phi + \frac{1}{2} \Phi^{(2,0)}_{ab} P^a P^b)$, that leads, after expanding the $*$-products, to a term proportional to $K_{ij} \left( \phi + \frac{1}{4} \eta^{ab} \Phi^{(2,0)}_{ab} \right)$. This disappears precisely if $\eta^{ab} \Phi^{(2,0)}_{ab} = -4 \phi$, which leads, via eq. (3.36), to the correct scalar mass term.

Once the mass terms are obtained, it is straightforward to characterize the group theoretical content of the spectrum, as we shall do in subsection 4.3.3.

4.3.2 $F$ Constraint and Mixing Phenomenon

Since on shell and after gauge fixing $F \in ho(D-1,2)/I(K)$, its expansion takes the form (3.19), in which the (traceful) $SL(D)$ representations $Q^{(2\ell+1,m)}$ must be replaced by the (traceless) $SO(D-1,1)$ representations $Q^{(2\ell+1,m)}$. Using this expansion for $F$, and the expansions (4.17) and (4.18) for $C* M$, an analysis of the $F$ constraint (4.21) shows that the physical fields at level $\ell$ present themselves in an admixture with lower levels. Still, the integrability of the constraints ensures that diagonalization is possible, and we have verified this explicitly for the spin-2 field equation. Indeed, starting from (4.21) one finds
\[
R_{\mu\nu} + \frac{1}{4} (D-1) g_{\mu\nu} = -\frac{16}{D} \nabla_{(\mu} \nabla_{\nu)} \phi + \frac{8(D-1)}{D} g_{\mu\nu} \phi ,
\]
where $R_{\mu\nu}$ denotes the spin-2 Ricci tensor for the metric $g_{\mu\nu} = e_{\mu}^a e_{\nu}^a$ and $\phi$ is the physical scalar. Using the scalar field equation (4.23), one can show that the rescaled metric
\[
\tilde{g}_{\mu\nu} = e^{-2u} g_{\mu\nu} , \quad \text{with} \quad u = \frac{16}{3(D-2)} \phi ,
\]
is the Einstein metric obeying
\[
\tilde{R}_{\mu\nu} + \frac{1}{4} (D-1) \tilde{g}_{\mu\nu} = 0 .
\]
It is natural to expect that this result extend to higher orders, and that the end result be equivalent to some generalization of the Weyl transformation (4.25) at the level of master fields. Barring this mixing problem, the Fronsdal operators at higher levels can be sorted out from the constraint on $F$ in (4.21) noting that the $SO(D-1,2)$-covariant derivatives do not mix $SO(D-1,2)$ irreps, which allows one to consistently restrict the internal indices at level $\ell$ to the $SO(D-1,2)$ irrep with weight $\{2\ell+1,2\ell+1\}$. Decomposing this irrep into Lorentz tensors and using cohomological methods [43, 17] then reveals that the curvatures are properly on-shell, although the precise form of the resulting field equations, as we have anticipated, requires a more careful elimination of the ideal gauge fields, that will be discussed in subsection 4.4.
4.3.3 Spectrum of the Model and Group Theoretical Interpretation

Although, as we have seen, the linearized field equations for the HS fields arising from the $F$ constraint exhibit a mixing phenomenon, the physical degrees of freedom and the group theoretical interpretation of the resulting spectrum can be deduced from the mass-shell conditions (4.23) for the scalar field $\phi$ and the Weyl tensors. This is due to the fact that all local degrees of freedom enter the system via the 0-form sources. Thus, the mode expansions of a gauge-fixed symmetric tensor and its Weyl tensor are based on the same lowest-weight space $D(E_0; S_0)$, and simply differ in the embedding conditions for the irreducible Lorentz representation, that we label by $J$.

In order to determine the lowest-weight spaces that arise in the spectrum, one can thus treat $AdS_D$ as the coset space $SO(D-1,2)/SO(D-1,1)$ and perform a standard harmonic analysis [46] of (4.23), expanding $C^{(s,s)}$ in terms of all $SO(D-1,2)$ irreps that contain the $SO(D-1,1)$ irrep $\{s, s\}$. It follows that these irreps have the lowest weight $\{E_0, S_0\}$, where $S_0 \equiv \{s_1, s_2\}$ (recall that we are suppressing the zeros, as usual) with $E_0 \geq s \geq s_1 \geq s_2 \geq 0$. Taking into account the Bianchi identity satisfied by $\Phi^{(s,s)}$, one can then show that $s_2 = 0$ [17], and therefore

$$S_0 = \{s\} . \quad (4.27)$$

Next, one can use the standard formula that relates the Laplacian $\nabla^2$ on $AdS_D$ acting on the $SO(D-1,2)$ irreps described above to the difference of the second order Casimir operators for the $SO(D-1,2)$ irrep $\{E_0, \{s\}\}$ and the $SO(D-1,1)$ irrep $\{s, s\}$, thus obtaining the characteristic equation

$$\frac{1}{4} \left( C_2[SO(D-1,2)|E_0; \{s\}] - C_2[SO(D-1,1)|\{s, s\}] \right) + \frac{1}{2} (s + D - 3) = 0 . \quad (4.28)$$

The well-known formula for the second-order Casimir operators involved here then leads to

$$\frac{1}{4} \left( [E_0(E_0 - D + 1) + s(s + D - 3)] - [2s(s + D - 3)] \right) + \frac{1}{2} (s + D - 3) = 0 , \quad (4.29)$$

with the end result that

$$E_0 = \frac{D - 1}{2} \pm \left( \frac{s + D - 5}{2} \right) = \left\{ \begin{array}{l} \frac{s + D - 3}{2 - s} \\ \frac{s + D - 3}{2} \end{array} \right. . \quad (4.30)$$

The root $E_0 = 2 - s$ is ruled out by unitarity, except for $D = 4$ and $s = 0$, when both $E_0 = 2$ and $E_0 = 1$ are allowed. These two values correspond to Neumann and Dirichlet boundary conditions on the scalar field, respectively. Hence, the spectrum for $D \geq 5$ is given by

$$S_D = \bigoplus_{s=0,2,4,\ldots} D(s + D - 3, \{s\}_{D-1}) . \quad (4.31)$$

while the theory in $D = 4$ admits two possible spectra, namely $S_4$ and

$$S_4' = D(2,0) \bigoplus \bigoplus_{s=2,4,\ldots} D(s + 1, s) . \quad (4.32)$$
In $D = 3$ the twisted-adjoint representation is one-dimensional, and hence \[ S_3 = R. \] (4.33)

As discussed in Section 2, the spectrum (4.31) fills indeed a unitary and irreducible representation of $h_0(D - 1, 2)$ isomorphic to the symmetric product of two scalar singlets \[ \text{[34, 15]} \]. The alternative 4D spectrum $S'_4$ is also a unitary and irreducible representation of $h_0(3, 2)$, given by the anti-symmetric product of two spinor singletons $D(1, 1)$. These arise most directly in the spinor oscillator realization of $h_0(3, 2)$, often referred to as $hs(4)$ (see (2.12)).

### 4.4 Compensator Form of the Linearized Gauge-Field Equations

Leaving aside the mixing problem, and considering for simplicity the flat limit, one is thus faced with the linearized curvature constraints ($k = 0, \ldots, s - 1$)

\[
\mathcal{F}^{(s-1,k)}_{\mu\nu,a(s-1),b(k)} = 2\partial_{[\mu} W^{(s-1,k)}_{\nu],a(s-1),b(k)} + 2c_{s,k} W_{[\mu|a(s-1),|b(k)}^{(s-1,k+1)}} + \delta_{k,s-1} C^{(s,s)}_{[\mu|a(s-1),|b(s-1)}, \] (4.34)

where $c_{s,k}$ is a constant, whose precise value is inconsequential for our purposes here, and the gauge fields $W^{(s-1,k)}_{\mu}$ are traceful, while the 0-form $C^{(s,s)}$ on the right-hand side is the traceless spin-$s$ Weyl tensor. The trace of (4.34) in a pair of internal indices generates a homogeneous set of cohomological equations of the type considered by Dubois-Violette and Henneaux [44], and the analysis that follows is in fact a combination of the strongly projected Vasiliev equations with the results of Bekaert and Boulanger [24] on the link between the Freedman-de Wit connections and the compensator equations of [21, 22]. Let us begin by discussing some preliminaries and then turn to the cases of spin 3 and 4. These exhibit all the essential features of the general case, that will be discussed in [17].

The scheme for eliminating auxiliary fields parallels the discussion of the off-shell case. Thus, after fixing the Stückelberg-like shift symmetries, one is led to identify the independent spin-$s$ gauge field, a totally symmetric rank-$s$ tensor $\phi_{a(s)} \equiv \phi_{a_1...a_s}$, via

\[
W^{(s-1,0)}_{\mu,a(s-1)} = \phi_{a(s-1)} \mu, \] (4.35)

where the hooked Young projection can be eliminated by a gauge-fixing condition. The constraints on $\mathcal{F}^{(s-1,k-1)}_{\mu\nu}$, for $k = 1, \ldots, s - 1$, determine the Freedman-de Wit connections, or generalized Christoffel symbols

\[
W^{(s-1,k)}_{\mu,a(s-1),b(k)} \equiv \gamma_{s,k} \partial_{b(k)} \phi_{(s-1) a(s-1) \mu}, \] (4.36)

where the subscript $(s - 1, k)$ defines the Young projection for the right-hand side and the $\gamma_{s,k}$ are constants whose actual values are immaterial for the current discussion. The remaining

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9In this section we use the notation in which $a(k)$ denotes a symmetric set of $k$ Lorentz vector indices.
gauge transformations are then, effectively,
\[ \delta_\Lambda \phi_{a(s)} = s \partial_{(a_1} \Lambda_{a_2 \ldots a_s)} , \]
and are of course accompanied by gauge-preserving shift parameters \( e^{(s-1,k)} \). It is important to note that the gauge-fixed frame fields \( W^{(s-1,k)} (k = 0, 1, \ldots, s - 1) \) are irreducible \( SL(D) \) tensors of type \((s,k)\), since their \((s-1,k+1)\) and \((s-1,k,1)\) projections vanish as a result of too many antisymmetrizations. On the other hand, the traces \( W^{(s-1,k)}_{\mu, a(s-1), b(k-2)} \) for \( k = 2, \ldots, s - 1 \) are reducible \( SL(D) \) tensors, comprising \((s,k-2)\) and \((s-1,k-1)\) projections (the remaining \((s-1,k-2,1)\) projection again vanishes due to too many antisymmetrizations). In particular, the \((s-1,k-1)\) part does not contain the d'Alembertian and takes the form
\[ W^{(s-1,k)}_{\mu, a(s-1), b(k-2)} = \partial_{b(k-1)} \hat{j}_{a(s-1)} , \]
where the constructs \( \hat{j}_{a(s-1)} \) are linear combinations of \( \partial \cdot \phi_{a(s-1)} \) and \( \partial_a \phi'_{a(s-2)} \). The above observations imply that the trace of (4.34) in a pair of internal indices
\[ 2\partial_{\mu} W^{(s-1,k)}_{\nu, a(s-1), b(k-2)} c + 2c_{s,k} W^{(s-1,k+1)}_{\mu, a(s-1), b(k+1)} c = 0 , \]
\[ k = 2, \ldots, s - 1 , \]
results in a homogeneous cohomological problem for \( W^{(s-1,k)}_{\mu, a(s-1), b(k-2)} c \) for \( k = 2, \ldots, s - 1 \), whose integration gives rise to the compensators as "exact" forms.

For the sake of clarity, let us begin by examining the spin-3 case. Here the relevant trace part, given by
\[ W^{(2,2)}_{\mu, a \beta c} = \frac{\gamma_{3,2}}{3} K_{\mu, ab} , \quad \text{where} \quad K_{\mu, ab} = \Box \phi_{a \beta c} - 2\partial_{(a} \partial \cdot \phi_b)_{\mu} + \partial_a \partial_b \phi'_{\mu} , \]
and where \( \phi' \) denotes the trace of \( \phi \) and \( \gamma_{3,2} \) is a constant whose actual value plays no role in our argument, obeys the constraint
\[ \partial_{[\mu} W^{(2,2)}_{\nu] \alpha, ab \beta} c = 0 . \]
Hence, \( K \) can be related to a symmetric rank-2 tensor:
\[ K_{\mu, ab} = \partial_{[\mu} \beta_{ab}] , \]
where \( \beta_{a \beta} \) is constrained since the \((2,1)\) projection of (4.42), equivalent to anti-symmetrizing on \( \mu \) and \( a \), implies that
\[ \partial_{[\mu} (\partial \cdot \phi_{a] \beta} - \partial_{\beta} \phi'_{a}) = \partial_{[\mu} \beta_{a] \beta} , \]
whose solution reads
\[ \beta_{ab} = \partial \cdot \phi_{ab} - 2\partial_{(a} \phi_{b)} + 3\partial_a \partial_b \alpha . \]
Notice that the last term is a homogeneous solution parameterized by an unconstrained scalar function \( \alpha \) and involves two derivatives. Plugging this expression for \( \beta_{ab} \) into the \((3,0)\) projection of (4.42) finally gives
\[ F_{abc} \equiv \Box \phi_{abc} - 3\partial_{(a} \partial \cdot \phi_{bc)} + 3\partial_{(a} \partial_b \phi'_{c)} = 3\partial^3 \phi_{abc} \alpha , \]
the flat-space version of the spin-3 compensator equation (1.3).

The origin of the triple gradient of the compensator can be made more transparent by an alternative derivation, that has also the virtue of generalizing more simply to higher spins. To this end, one can analyze the consequences of the invariance under the gauge transformation

$$\delta_\Lambda \phi_{abc} = 3\partial_{(a} \Lambda_{bc)} ,$$

that affects the left-hand side of (4.42) according to

$$\delta_\Lambda K_{\mu,ab} = \partial_\mu \tilde{\Lambda}_{ab} , \quad \tilde{\Lambda}_{ab} = \Box \Lambda_{ab} - 2\partial_{(a} \partial \cdot \Lambda_{bc)} + \partial^2_{ab} \Lambda' .$$

Gauge invariance thus implies that the homogeneous solution $\beta_{ab}$ must transform as

$$\delta_\Lambda \beta_{ab} = \tilde{\Lambda}_{ab} .$$

From $\delta_\Lambda \partial \cdot \phi_{ab} = \Box \Lambda_{ab} + 2\partial_{(a} \partial \cdot \Lambda_{bc)}$ and $\delta_\Lambda \partial_{(a} \phi_{bc)}' = \partial_a \partial_b \Lambda' + 2\partial_{(a} \partial \cdot \Lambda_{bc)}$, one can see that $\beta_{ab}$ is a linear combination of $\partial \cdot \phi_{ab}$, $\partial_{(a} \phi_{bc)}'$ and of a term $\partial^2_{ab} \alpha$, with $\alpha$ an independent field transforming as

$$\delta_\Lambda \alpha = \Lambda' .$$

This in its turn implies that the symmetric part of (4.42) is a gauge covariant equation built from $\Box \phi_{abc}$, $\partial_{(a} \partial \cdot \phi_{bc)}$, $\partial^2_{ab} \alpha$ and $\partial^3_{abc} \alpha$, and uniquely leads to the compensator equation (4.45).

The direct integration for spin 4 is slightly more involved, since two traced curvature constraints,

$$\partial_\mu W_{\nu_{1,a(3)},bc}^{(3,3)} = 0 ,$$

$$\partial_\mu W_{\nu_{1,a(3)},c}^{(3,2)} + c_{4,2} W_{[\mu,a(3)],\nu}^{(3,3)\ c} = 0 ,$$

have to be dealt with in order to obtain the compensator equation. The first constraint implies that

$$W_{\mu,a(3),bc}^{(3,3)} = \partial_\mu \beta_{a(3),b}^{(3,1)} ,$$

where the explicit form of the Freedman-de Wit connection is

$$\frac{4}{\gamma_{4,3}} W_{\mu,a(3),bc}^{(3,3)} = \partial_b \Box \phi_{a(3)} - 2\partial_a \Box \phi_{a(2)\mu} - \gamma_{4,3}^2 \partial^2_{ab} \partial \cdot \phi_{a(2)\mu} - \gamma_{4,3}^3 \partial^3 a_{(2)} \partial \cdot \phi_{a(2)\mu} + \gamma_{4,3}^3 \partial a_{(3)} \phi_{a\mu} ,$$

with $\gamma_{4,3}$ a constant whose actual value plays no role in our argument. The $(3,2)$ projection does not contain any d’Alembertians, and can thus be identified with the double gradient of $J_{a(3)}$, a construct of $\partial \cdot \phi_{a(3)}$ and $\partial a_{(3)} \phi_{a(2)}$. It can be integrated once, with the result that

$$\beta_{a(3),b}^{(3,1)} = \partial_b J_{a(3)} + \gamma_{4,3}^1 \alpha_{a(3)} ,$$

where the last term, with $\alpha_{a(3)}$ an unconstrained vector field, is the general solution of the homogeneous equation, i.e. its cohomologically exact part in the language of Dubois-Violette and
Henneaux [44]. Eq. (4.54) implies that the second curvature constraint can be written in the form

$$\partial_\mu W^{(3,2)}_{\alpha(3),c} + c_{4,2} \partial_\mu \beta^{(3,1)}_{\alpha(3),[\nu]} = 0$$

and integrating this equation one finds

$$W^{(3,2)}_{\mu,a(3),c} + c_{4,2} \beta^{(3,1)}_{a(3),\mu} = \partial_\mu \beta^{(3)}_{a(3)}$$

where the homogeneous term $\beta^{(3)}$ is constrained, since a derivative can be pulled out from the $(3,1)$ projection of the left-hand side. Thus, using

$$\frac{\partial^3_{a(3)} \alpha_b}{(3,1)} = -\partial_b \frac{\partial^2_{a(2)} \alpha_a}{(3,1)}$$

one finds that

$$\beta^{(3)}_{a(3)} = \tilde{J}_{a(3)} - c_{4,2} \frac{\partial^2_{a(2)} \alpha_a}{(3,1)}$$

where $\tilde{J}_{a(3)}$ is another construct of the first derivatives of $\phi$, and the homogeneous solution has been absorbed into a shift of $\alpha_a$ by a gradient. Finally, the $(4,0)$ projection of (4.56) reads

$$W^{(3,2)}_{a,a(3),c} = \partial_a \tilde{J}_{a(3)} - c_{4,2} \partial^3_{a(3)} \alpha_a$$

and gauge invariance implies that, up to an overall rescaling of $\alpha_a$, this is the flat-space compensator equation (1.3) for spin 4.

In summary, a careful analysis of the Vasiliev equations shows that, if the gauge fields are left free to fluctuate and adjust themselves to constrained Weyl 0-form sources, one is led, via the results of [24], to the compensator equations of [21, 22] rather than to the conventional Fronsdal formulation.

## 5 TOWARDS A FINITE CURVATURE EXPANSION

In this section we state the basic problem one is confronted with in the perturbative analysis of the strong $Sp(2, R)$-invariance condition, and discuss how finite results could be extracted from the non-linear Vasiliev equations in this setting. These observations rest on properties of the $\star$-products of the singular projector and other related non-polynomial objects, that we shall illustrate with reference to a simpler but similar case, the strong $U(1)$ condition that plays a role in the 5D vector-like construction [4, 20] based on spinor oscillators [13]. Whereas these novel possibilities give a concrete hope that a finite construction be at reach, a final word on the issue can not forego a better understanding of the interactions in the actual $Sp(2, R)$ setting.

We intend to return to these points in [17].
5.1 On the Structure of the Interactions

In the previous section, we found that the strong $Sp(2, R)$ condition (4.1) led to a linearized zero-form master field $\Phi$ with an expansion involving non-polynomial dressing functions $F(N; K^2)$, according to (4.18). In particular, the master-field projector $M$, introduced in (4.12) and such that $\Phi = M \ast C$,

$$M(K^2) = F(0; K^2),$$

is the dressing function of the scalar field $\phi$ component of $\Phi$. The dressing functions are proportional to $J_\nu(\xi)/(\xi/2)^\nu$, where $J_\nu$ is a Bessel function of order $\nu = (N + D - 2)/2$ and of argument $\xi = 4\sqrt{K^2}$, and hence satisfy the Laplace-type differential equation

$$\left( \frac{\xi}{d^2/d\xi^2} + (2\nu + 1) \frac{d}{d\xi} + \xi \right) F(N; \xi^2/16) = 0.$$

The solution of this equation that is analytic at the origin can be given the real integral representation

$$F(N; K^2) = N_\nu \int_{-1}^{1} ds \left(1 - s^2\right)^{\nu-\frac{1}{2}} \cos \left(4\sqrt{K^2} s \right),$$

where the normalization, fixed by $F(N; 0) = 1$, is given by

$$N_\nu = \frac{1}{B \left(\nu + \frac{1}{2}, \frac{1}{2} \right)} = \frac{\Gamma(\nu + 1)}{\Gamma \left(\nu + \frac{1}{2} \right) \Gamma \left(\frac{1}{2} \right)},$$

with $B$ the Euler $B$-function. Alternatively, in terms of the variable $z = K^2$, the differential equation takes the form

$$\left( \frac{z}{4} \frac{d^2}{dz^2} + \frac{\nu + 1}{4} \frac{d}{dz} + 1 \right) F(N; z) = 0,$$

which is also of the Laplace type, and leads directly to the Cauchy integral representation

$$F(N; K^2) = \Gamma(\nu + 1) \oint_{\gamma} dt \frac{\exp \left(\frac{t - 16K^2}{4} \right)}{2\pi i t^{\nu+1}},$$

where the Hankel contour $\gamma$ encircles the negative real axis and the origin.

In building the perturbative expansion of the strongly projected zero form, one also encounters an additional non-polynomial object, that we shall denote by $G(K^2)$ and plays the role of a $\ast$-inverse of the $Sp(2, R)$ Casimir operator. It is the solution of

$$K I \ast K I \ast G \ast H = H,$$

where $H$ belongs to a certain class of functions not containing $M(K^2)$ [17]. This element instead obeys

$$K I \ast K I \ast G \ast M = 0,$$
in accordance with the associativity of the $\star$-product algebra.

Dealing with the above non-polynomial operators, all of which can be given contour-integral representations, requires pushing the $\star$-product algebra beyond its standard range of applicability. The integral representations can be turned into forms containing exponents linear in $K_1$, suitable for $\star$-product compositions, at the price of further parametric integrals, but one then discovers the presence of a one-dimensional curve of singularities in the parametric planes, leading to an actual divergence in $M \star M$ \[34\].

Let us consider in more detail the structure of the interactions. At the $n$-th order they take the form

$$\hat{O}_1 \star M \star \hat{O}_2 \star M \star \hat{O}_3 \star \cdots \star M \star \hat{O}_{n+1},$$

(5.9)

where the $\hat{O}_s$, $s = 1, \ldots, n + 1$ are built from components of the master fields $A_\mu$ and $C$ defined in (4.20), the element $\kappa(t)$ defined in (3.11), and the $\star$-inverse functions $G(K^2)$. One possible strategy for evaluating this expression would be to group the projectors together, rewriting it as

$$\hat{O}_1 \star M \star M \star \cdots \star M \star [\hat{O}_2]_0 \star [\hat{O}_3]_0 \star \cdots \star [\hat{O}_n]_0 \star \hat{O}_{n+1},$$

(5.10)

where $[\hat{O}]_0$ denotes the $Sp(2, R)$-singlet, or neutral, projection\[10\] of $\hat{O}$. This procedure clearly leads to divergent compositions that appear to spoil the curvature expansion. However, (5.10) is only an intermediate step in the evaluation of (5.9), and the singularities may well cancel in the final Weyl-ordered form of (5.9). Therefore, one should exploit the associativity of the $\star$-product algebra to find a strategy for evaluating (5.9) that may avoid intermediate singular expressions altogether. To achieve this, one may first $\star$-multiply the projectors with adjacent operators $\hat{O}_s$, and then consider further compositions of the resulting constructs. It is quite possible that this rearrangement of the order of compositions result in interactions that are actually completely free of divergences, as we shall discuss further below.

In principle, one may instead consider a modified curvature expansion scheme, based on an additional $Sp(2, R)$ projector, which we shall denote by $\Delta(K^2)$, that is of distributional nature and has finite compositions with both the analytic projector and itself \[17\]. We shall return to a somewhat more detailed discussion of possible expansion schemes in Section 5.3, while we next turn to the analysis of a simpler $U(1)$ analog.

### 5.2 A Simpler $U(1)$ Analog

In this section we examine a strong $U(1)$ projection that is simpler than the actual $Sp(2, R)$ case but exhibits similar features. The key simplification is that the corresponding non-polynomial objects admit one-dimensional parametric representations, directly suitable for $\star$-product compositions, that only contain isolated singularities in the parameter planes. Drawing on these results, in Section 5.3 we shall finally discuss two possible prescriptions for the curvature expansion.

\[10\] In general, any polynomial in oscillators can be expressed as a finite sum of $(2k + 1)$-plets. The neutral part is the singlet, $k = 0$, component. A properly refined form of eq. (5.10) applies also to more general structures arising in the $\Phi$-expansion, where the inserted operators can carry a number of doublet indices.
5.2.1 Analytic \( U(1) \) Projector

The \( Sp(2,R) \) dressing functions \( F(N;K^2) \) are closely related to others occurring in the 5D and 7D constructions of [4] and [19]. These are based on bosonic Dirac-spinor oscillators [13] \( y_\alpha \) and \( \bar{y}^\beta \) obeying

\[
y_\alpha \star \bar{y}^\beta = y_\alpha \bar{y}^\beta + \delta_\alpha^\beta ,
\]

and providing realizations of the minimal bosonic higher-spin algebras \( h_{0\alpha}(4,2) \) and \( h_{0\alpha}(6,2) \), at times referred to as \( hs(2,2) \) and \( hs(8*) \), via on-shell master fields subject to \( U(1) \) and \( SU(2) \) conditions, respectively. In particular, the 5D \( U(1) \) generator is given by [4]

\[
x = \bar{y} y .
\]

One can show that for \( 2n \)-dimensional Dirac spinor oscillators

\[
x \star f(x) = \left( x - 2n \frac{d}{dx} - x \frac{d^2}{dx^2} \right) f(x) ,
\]

where \( n = 2 \) in \( D = 5 \). The strong \( U(1) \)-invariance condition defining the 5D Weyl zero form [4] can be formally solved introducing a projector \( m(x) \) analytic at the origin and such that [20]

\[
x \star m(x) = 0 .
\]

The analytic projector \( m \) belongs to a class of \( U(1) \) dressing functions \( f(N;x) \), that arise in the component-field expansion of the linearized Weyl master zero form [4], and which obey

\[
\left( x \frac{d^2}{dx^2} + (2\nu + 1) \frac{d}{dx} - x \right) f(N;x) = 0 ,
\]

with \( \nu = n + N - \frac{1}{2} \), so that

\[
m(x) = f(0;x) .
\]

The solution of (5.15) that is analytic at the origin can be expressed in terms of the modified Bessel function of index \( \nu \) and argument \( x \) as \( I_\nu(x)/x^\nu \), and admits the real integral representation [45]

\[
f(N;x) = \mathcal{N}_\nu \int_{-1}^{1} ds \ (1 - s^2)^{\nu - \frac{1}{2}} e^{sx} ,
\]

where the normalization \( \mathcal{N}_\nu \), determined by the condition that \( f(N;0) = 1 \), is as in (5.4).

In order to compose \( m \) with other objects, it is useful to recall the \( \star \)-product formula

\[
e^{sx} \star e^{s'x} = \frac{e^{t(s,s')x}}{(1 + ss')^{2n}} , \quad \text{with} \quad t(s,s') = \frac{s + s'}{1 + ss'} ,
\]

that holds for arbitrary values of \( s, s' \in \mathbb{C} \) such that \( 1 + ss' \neq 0 \), and to note that the projective transformation \( t(s,s') \) has the form of the relativistic addition of velocities, so that \( t(\pm 1,s') = t(s,\pm 1) = \pm 1 \). This can be used to show that, for any complex number \( \lambda \neq \pm 1 \),

\[
e^{\lambda x} \star m(x) = \frac{1}{(1 - \lambda^2)^n} m(x) ,
\]

28
and hence that the $\star$-product of a pair of analytic projectors results in the singular expression

$$m(x) \star m(x) = \mathcal{N} m(x) \int_{-1}^{1} \frac{ds}{1 - s^2} ,$$

(5.20)

where $\mathcal{N} = \mathcal{N}_{n^{1/2}}$. The logarithmic nature of the singularity is consistent with power counting in the double-integral expression

$$m(x) \star m(x) = \mathcal{N}^2 \int_{-1}^{1} ds \int_{-1}^{1} ds' \frac{(1 - s^2)^{n-1}(1 - s'^2)^{n-1}}{(1 + ss')^{2n}} e^{(s,s')x}$$

(5.21)

where the integrand diverges as $\epsilon^{-2}$ as $s \sim 1 - \epsilon$ and $s' \sim -1 + \epsilon$.

As discussed in Section 5.1, the above singularity need not arise in the curvature expansion of a HS theory based on the spinor oscillators. Indeed, if one first composes the $m$'s with other operators, expands in terms of dressing functions, and continues by composing these, one encounters

$$f(N; x) \star f(N'; x) = \mathcal{N}_\nu \mathcal{N}_{\nu'} \sum_{k=0}^{N'} \left( \frac{2N'}{2k} \right) B \left( k + \frac{1}{2}, N + N' \right) I_{N,N';k}(x) ,$$

(5.22)

with $\nu = n + N - \frac{1}{2}$, $\nu' = n + N' - \frac{1}{2}$ and

$$I_{N,N';k}(x) = \int_{-1}^{1} dt \ t^{2k}(1 - t^2)^{\nu'-\frac{1}{2}} \ 2F1 \left( k + \frac{1}{2}, 2N'; N + N' + k + \frac{1}{2}, t^2 \right) e^{tx} ,$$

(5.23)

where $2F1$ is the hypergeometric function. This type of expression is free of divergences for $N + N' \geq 1$, while it is apparently equal to the singular expression $\mathcal{N} \Gamma(0) m(x)$ if $N = N' = 0$, but in fact, using $2F1(0, \beta; \gamma; z) = 1$, one can see that

$$f(N; x) \star m(x) = \mathcal{N}_\nu \ B \left( N, \frac{1}{2} \right) m(x) .$$

(5.24)

More generally, multiple compositions $f(N_1; x) \star \cdots \star f(N_n; x)$ are finite for $N_1 + \cdots + N_n \geq 1$, as can be seen by power counting. Therefore, the finiteness of the interactions is guaranteed provided the case $N_1 = \cdots = N_n = 0$ never presents itself, a relatively mild condition that could well hold for the interactions 11.

It is interesting to examine more closely the nature of the singularity in (5.20).

Problems with Regularization of the Singular Projector

The singularities of (5.20) can be removed by a cutoff procedure, at the price of violating $U(1)$ invariance. For instance, given the regularization

$$m_\epsilon(x) = \mathcal{N} \int_{-1+\epsilon}^{1-\epsilon} ds \ e^{sx} (1 - s^2)^{n-1} ,$$

(5.25)

\[\text{11This corresponds, roughly speaking, to constraints on scalar-field non-derivative self interactions, which might be related to the fact that all such couplings actually vanish in the 4D spinor-formulation as found in [42].}\]
one can use (5.13) to show that
\[ x \star m_\epsilon(x) = 2N [1 - (1 - \epsilon)^2]^n \sinh[(1 - \epsilon)x], \] 
(5.26)

where \( \sinh[(1 - \epsilon)x] \) belongs to the ideal, since \( \sinh[\lambda x] \star m(x) = 0 \) for all \( \lambda \). However, while the resulting violation may naively appear to be small, it is actually sizeable, since \( \sinh[(1 - \epsilon)x] \) and \( m_\epsilon(x) \) have a very singular composition, with the end result that an anomalous finite violation of \( U(1) \) invariance can emerge in more complicated expressions, so that for instance
\[ x \star m_\epsilon(x) \star m_\epsilon(x) = N^2 \int_0^1 ds \, (1 - s^2)^{n-1} \sinh(sx) + \text{evanescent}. \] 
(5.27)

This anomaly can equivalently be computed first composing
\[ m_\epsilon(x) \star m_\epsilon(x) = N^2 \int_0^{(1-\epsilon,1-\epsilon)} du \int_{-u}^u dt \, (1 - t^2)^{n-1} e^{tx}, \] 
(5.28)

and then expanding in \( \epsilon \), which yields
\[ m_\epsilon(x) \star m_\epsilon(x) = N \log \left( \frac{2 - \epsilon}{\epsilon} \right) m(x) + A^{(2)}(x) + \text{evanescent}, \] 
(5.29)

where the divergent part is proportional to \( m(x) \), while the finite, anomalous, part
\[ A^{(2)}(x) = N^2 \int_0^1 du \, \log \left( \frac{1 + u}{1 - u} \right) (1 - u^2)^{n-1} \sinh ux, \] 
(5.30)

belongs to the ideal. One can verify that its composition with \( x \) indeed agrees with the right-hand side of (5.28), in compliance with associativity. Continuing in this fashion, one would find that higher products of \( m_\epsilon(x) \) with itself keep producing anomalous finite terms together with logarithmic singularities, all of which are proportional to \( m(x) \), together with tails of evanescent terms that vanish like powers or powers times logarithms when the cutoff is removed [17]:
\[ m_\epsilon(x) \star \cdots \star m_\epsilon(x) = \sum_{l=-p+1}^{-1} \left( \log \frac{1}{\epsilon} \right)^l c_l^{(p)} m(x) + F^{(p)}(x) + \sum_{k \geq 1, l \geq 0} \epsilon^k \left( \log \frac{1}{\epsilon} \right)^l E_{k,l}^{(p)}(x), \] 
(5.31)

with \( F^{(p)}(x) = C^{(0)} m(x) + A^{(p)}(x) \), where \( A^{(p)}(x) \) represents the anomaly.

We would like to stress that the integral representations should be treated with some care, as can be illustrated considering
\[ m_\epsilon(x) \star m(x) = N \log \left( \frac{2 - \epsilon}{\epsilon} \right) m(x), \] 
(5.32)

and its generalization
\[ m(x) \star m_\epsilon(x) \star \cdots \star m_\epsilon(x) = \left( N \log \frac{2 - \epsilon}{\epsilon} \right)^p m(x). \] 
(5.33)
Comparing (5.33) with (5.29), it should be clear that one should perform the parametric integrals prior to expanding in $\epsilon$, since one would otherwise encounter ill-defined compositions of “bare” $m$’s.

In order to appreciate the meaning of the anomaly, let us consider a tentative Cartan integrable system\footnote{These considerations would apply, in their spirit, to the 5D spinor construction, whose completion into an integrable non-linear system is still to be obtained.} on the direct product of spacetime with an internal noncommutative $z$-space, containing a strongly $U(1)$-projected 0-form master field $\hat{\Phi}$ with a formal perturbative expansion in $m \star C$. At the $n$-th order, one could use the $U(1)$ analog of the rearrangement in (5.10) to bring all projectors together, and then attempt to regulate the resulting divergent $\star$-product compositions, for instance replacing each $m$ by an $m_\epsilon$. This induces a violation of the constraints in $z$-space. Therefore, in order to preserve the Cartan integrability in spacetime, one could use unrestricted $z$-expansions for the full master fields, at the price of introducing “spurious” space-time fields that one would have to remove by some form of consistent truncation. To analyze this, one first observes that the structure of the $\epsilon$-dependence in (5.31) implies that, the Cartan integrability condition satisfied by the finite part of the regulated interactions can not contain any contributions from singular times evanescent terms. Thus, the finite part in itself constitutes a consistent set of interactions, albeit for all the space-time fields, including the spurious ones.

In the absence of anomalies it would be consistent to set to zero all the spurious fields, thus obtaining a well-defined theory for the original space-time master fields. However, as the earlier calculations show, these anomalies arise inevitably and jeopardize these kinds of constructions, in that they must cancel in the final form of the interactions, along lines similar to those discussed in Section 5.1 and indicated below (5.22).

5.2.2 Distributional $\star$-Inverse Function and Normalizable Projector

As we have seen, the $U(1)$ projection condition $x \star m(x) = 0$ corresponds to the Laplace-type differential equation determined by (5.13), which admits two solutions that are ordinary functions of $x$, so that the singular projector $m(x)$ is the unique one that is also analytic at $x = 0$. Interestingly, there also exist solutions that are distributions in $x$. Their Laplace transforms involve $e^{sx}$ with an imaginary parameter $s$, which improves their $\star$-product composition properties, and in particular allows for a normalizable projector. A related distribution, with similar properties, provides a $U(1)$ analog of the $\star$-inverse of the $Sp(2, R)$ Casimir defined in (5.7).

To describe these objects, it is convenient to observe that, if $\gamma$ is a path in the complex $s$-plane, the function

$$ p_\gamma(x) = \int_\gamma ds \; (1 - s^2)^{n-1} \; e^{sx} \tag{5.34} $$

obeys

$$ x \star p_\gamma = \left[(1 - s^2)^n \; e^{sx}\right]_{\partial \gamma} \tag{5.35} $$

For instance, the projector $m(x) \sim I_\nu(x)/x^\nu$, with $\nu = n - \frac{1}{2}$, obtains taking for $\gamma$ the unit interval, while including boundaries also at $\pm\infty$ yields in general linear combinations of $I_\nu(x)/x^\nu$.
and \( K_\nu(x)/x^\nu \). The latter is not analytic at \( x = 0 \), however, and as such it would not seem to play any role in the dressing of the linearized zero-form master field. An additional possibility is provided by the standard representation of Dirac’s \( \delta \) function,

\[
\delta(x) = \int_{-\infty}^{\infty} \frac{dt}{2\pi} e^{itx-\epsilon t^2} \tag{5.36}
\]

with \( \epsilon \to 0^+ \), which suggests that boundaries at \( \pm i \infty \) might give rise to projectors that are distributions in \( x \). To examine this more carefully, let us consider

\[
d(x) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} (1-s^2)^{n-1} e^{xs+\epsilon s^2} . \tag{5.37}
\]

Treating \( d(x) \) as a distribution acting on test functions \( f(x) \) such that \( f^{(2k)}(0) \) fall off fast enough with \( k \), one can indeed expand it as

\[
d(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} (-1)^k \delta^{(2k)}(x) . \tag{5.38}
\]

One can now verify that \( x \star d(x) = 0 \) holds in the sense that

\[
\int_{-\infty}^{\infty} dx f(x) (x \star d(x)) = \sum_{k=0}^{n-1} \binom{n-1}{k} \left[ (xf(x))^{(2k)} + 2n f^{(2k+1)}(x) - (xf(x))^{(2k+2)} \right] \bigg|_{x=0} = 0 .
\]

We can now look more closely at the composition properties of \( d(x) \), beginning with

\[
d(x) \star m(x) = \mathcal{N} \int_{-i\infty}^{i\infty} \frac{ds}{\pi i} \int_{-1}^{1} ds' \frac{(1-s^2)^{n-1}(1-s'^2)^{n-1}}{(1+ss')^{2n}} e^{t(s,s')x+\epsilon s^2} , \tag{5.40}
\]

that can be obtained from (5.18). Notice that the denominator is no longer singular, since \( 1 + ss' \neq 0 \) for \( (s,s') \in iR \times [-1,1] \), and after a projective change of integration variable, \( t(s,s') \to t \) at fixed \( s \), one then finds

\[
d(x) \star m(x) = \mathcal{N} \int_{-i\infty}^{i\infty} \frac{ds}{\pi i} e^{\epsilon s^2} \int_{\gamma_s} dt \frac{(1-t^2)^{n-1}}{t^2} e^{tx} . \tag{5.41}
\]

The original real contour has been mapped to \( \gamma_s \), a circular arc from \( t = -1 \) to \( t = +1 \) crossing the imaginary axis at \( t = s \), but can be deformed back to the real interval \([-1,1]\) using Cauchy’s theorem, so that

\[
d(x) \star m(x) = m(x) \left( \int_{-i\infty}^{i\infty} \frac{ds}{\pi i} e^{\epsilon s^2} \right) , \tag{5.42}
\]

to be compared with (5.20). The integral over \( s \) is now convergent, and is actually equal to 1, and therefore

\[
d(x) \star m(x) = m(x) . \tag{5.43}
\]
The iteration of this formula then yields
\[
\underbrace{d(x) \star \cdots \star d(x)}_{p \text{ times}} \star m(x) = m(x) . \tag{5.44}
\]
This indicates that \(d(x)\) is in fact \textit{normalizable}, that is \cite{17}
\[
d(x) \star d(x) = d(x) , \tag{5.45}
\]
as can be seen by a direct computation using a prescription in which the singularity at \(ss' + 1 = 0\)
is avoided by small horizontal displacements of the contour that are eventually sent to zero, as in eqs. (5.54) and (5.55) below.

Let us next turn to the \(U(1)\) analog of the \(\star\)-inverse of the \(Sp(2, R)\) Casimir. This element, which we shall refer to as \(g(x)\), is defined by

i) the inverse property
\[
x \star g(x) \star h(x) = h(x) , \tag{5.46}
\]
for an arbitrary ideal function \(h(x)\). This milder form is the counterpart of the condition for a distributional solution \(d\) in (5.39).

ii) the twisted reality condition
\[
(g(x))^\dagger = \sigma g(-x) , \tag{5.47}
\]
where \(\sigma\) can be either \(+1\) or \(-1\);

iii) the orthogonality relation
\[
m(x) \star g(x) = 0 . \tag{5.48}
\]
The two first conditions suffice to ensure that, given a linearized master 0-form obeying \(x \star \Phi = 0\) and \(\tau(\Phi) = \Phi^\dagger = \pi(\Phi)\) \cite{4}, one can construct a full master 0-form \(\hat{\Phi}\) that satisfies

1) a strong \(U(1)\)-invariance condition
\[
\hat{x} \star \hat{\Phi} = 0 , \tag{5.49}
\]
where \(\hat{x}\) is a full (non-linear) version of the \(U(1)\) generator, with a \(\Phi\)-expansion given by
\[
\hat{x} = x + \sum_{n=1}^\infty \hat{x}_{(n)} ; \tag{5.50}
\]

2) the twisted \(\tau\) and reality conditions
\[
\tau(\hat{\Phi}) = \hat{\Phi}^\dagger = \pi(\hat{\Phi}) . \tag{5.51}
\]
The perturbative expression for the full zero form is then given by
\[ \hat{\Phi} = (g(x) \star \hat{x})^{-1} \star \Phi \star (\pi(\hat{x}) \star g(x))^{-1}, \] (5.52)
provided that (5.46) applies to \( g(x) \star x \star \hat{\Phi}_{(n)} \), in which case we note that the inverse elements in the above formula can be expanded in a geometric series using \( g(x) \star \hat{x} = 1 + \sum_{n=1}^{\infty} g \star \hat{x}_{(n)}. \)

To solve the conditions on \( g(x) \) in the above formula can be expanded in a geometric series using \( d \). In summary, we have found that distributions can be used to construct a normalizable projector these manipulations, it follows that \( x \star g \) with \( \epsilon, \eta \), where
\[ g(x) \triangleq \int_{-\infty}^{0} ds \, (1 - s^2)^{n-1} \sinh(sx) \, e^{\epsilon s^2} , \] (5.53)
where \( \epsilon \to 0^+ \), \( \alpha \) is a constant (to be fixed below), and we have assumed that the boundary terms in (5.35) at \( \pm i \infty \) drop out when used in (5.46). These boundary terms play a role, however, in verifying associativity in \( x \star g(x) \star m(x) = 0 \). Assuming that it is legitimate to set \( \epsilon = 0 \) during these manipulations, it follows that \( x \star g(x) \star m(x) = (1 - \alpha)m(x) \) which fixes \( \alpha = 1. \)

In summary, we have found that distributions can be used to construct a normalizable projector \( d(x) \) as well as a \( \star \)-inverse \( g(x) \) of \( x \), given by
\[ d(x) = \left( \int_{-i\infty+\eta}^{0} + \int_{i\infty-\eta}^{0} \right) ds \, (1 - s^2)^{n-1} \, e^{sx + \epsilon s^2} , \] (5.54)
\[ g(x) = \frac{1}{2} \left( \int_{-i\infty+\eta}^{0} + \int_{i\infty-\eta}^{0} \right) ds \, (1 - s^2)^{n-1} \, e^{sx + \epsilon s^2} , \] (5.55)
with \( \epsilon, \eta \to 0^+ \), where \( \eta \) is a prescription for avoiding singularities in \( d(x) \star d(x) \) and \( g(x) \star g(x) \). These distributional objects can be expanded in terms of elementary distributions using the standard result
\[ \int_{0}^{\infty} dt \, e^{ixt} = \pi \delta(x) + i PP \left( \frac{1}{x} \right) , \] (5.56)
with \( PP \) the principal part. Furthermore,
\[ \tau(d(x)) = \pi(\delta(x)) = \delta(x) , \quad \pi(g(x)) = \tau(g(x)) = -g(x) , \] (5.57)
and both elements are hermitian, \( i.e. \)
\[ (d(x))^\dagger = d(x) , \quad (g(x))^\dagger = g(x) , \] (5.58)
so that \( \sigma = -1 \) in (iii).

Finally, the generalization of \( m(x) \star g(x) = 0 \) to arbitrary \( U(1) \) dressing functions reads
\[ f(N; x) \star g(x) = \frac{i N}{2} \sum_{k=0}^{N-1} (-1)^k \left( \begin{array}{c} 2N \\ 2k + 1 \end{array} \right) B(k + 1, N - k) I_{N; k}(x) , \] (5.59)
where $\nu = n + N - \frac{1}{2}$ and

$$I_{N,k}(x) = \int_{-1+i\delta}^{1+i\delta} \frac{dt}{2} t^{2k+1} (1 - t^2)^{\nu - \frac{1}{2}} F_1(k + 1, 2N; N + 1; 1 - t^2) e^{tx}, \quad (5.60)$$

where $\delta \to 0^+$ is a prescription for how to encircle poles at $t = 0$, and we note that there are no inverse powers of $x$ in this expression.

5.3 Proposals for Finite Curvature Expansion Schemes

We would like to conclude by summarizing our current understanding of how a finite curvature expansion could be obtained from the Vasiliev equations (4.2) supplemented with the strong $Sp(2, R)$ projection condition (4.1). The material presently at our disposal suggests two plausible such schemes, that we have partly anticipated, and we shall refer to as minimal and modified. We hope to report conclusively on the fate of these schemes in [17].

5.3.1 Minimal Expansion Scheme

In this scheme, which is the most natural one, one solves the internal constraints (3.23) and the strong projection condition (4.1) by an expansion in the linearized Weyl zero form $\Phi = M(K^2) \star C$. This object can be written using dressing functions $f(N; K^2)$, as in (4.18). We anticipate that the $\star$-products of the $Sp(2, R)$ dressing functions obey an analog of (5.22). The expansion also requires a $\star$-inverse function $G(K^2)$ obeying (5.7) and a suitable set of boundary conditions, analogous to those discussed in the $U(1)$ case. We expect this $\star$-inverse to be also distributional and to satisfy an $Sp(2, R)$ analog of (5.59).

Let us consider an $n$-th order interaction of the form (5.9). Since the $\tilde{O}_s$ are either arbitrary polynomials or their $\star$-products with a $\star$-inverse function, the products $M \star \tilde{O}_s$ yield either dressing functions or their products with the $\star$-inverse function, respectively. Here we note that, anticipating an $Sp(2, R)$ analog of (5.59), all parametric integrals associated with (distributional) $\star$-inverse functions would not give inverse powers of $K^2$. The resulting form of (5.9) could be written as a multiple integral over a set of parameters $s_i \in [-1, 1]$ where the integrand contains factors $(1 - s_i^2)^{\nu_i - \frac{1}{2}}$, where $\nu_i \geq (D - 2)/2$, and a set $\star$-products involving exponentials of the form $e^{s_i x}$. These $\star$-products give rise to divergent powers of $(1 + s_is_j)$, and the idea is that all of these would be cancelled by the remaining part of the integrand, resulting in a well-defined interaction devoid of singularities.

As mentioned earlier, however, while the singularities in the $U(1)$ case are isolated points in the parametric planes, in the $Sp(2, R)$ case one has to deal with lines of singularities, and this requires further attention before coming to a definite conclusion.

5.3.2 Modified Expansion Scheme

In this scheme, which is less natural than the previous one, albeit still a logical possibility, one first solves the internal constraints (3.23) and the strong projection condition (4.1) by an
expansion in $\Phi = \Delta(K^2) \ast C$, where the distributional $Sp(2, R)$ projector obeys $K_I \ast \Delta(K^2) = 0$ and is given by \[ \Delta(K^2) = \int_{-\infty}^{\infty} \frac{ds}{2\pi} (1 - s^2)^{D-3} \cos(4\sqrt{K^2}s) . \] (5.61)

Starting from the above distributional master fields, and using the rearrangement in (5.10) to bring all $\Delta$'s together, one can define analytic master fields $\hat{A}_\mu^I[A, \Phi], \hat{A}_i^I[\Phi]$ and $\hat{\Phi}'[\Phi]$ by replacing the resulting single $\Delta$ by an $M$. The so constructed analytical master fields obey

\begin{align*}
\hat{F}'^\mu &\equiv d\hat{A}' + \hat{A} \ast \hat{A}' = \frac{i}{2} dZ_i^i \wedge dZ_i \hat{\Phi}' \ast \kappa , \quad (5.62) \\
\hat{D}\hat{\Phi}' &\equiv d\hat{\Phi}' + [\hat{A}, \hat{\Phi}'][\pi] = 0 , \quad (5.63)
\end{align*}

where it is immaterial which of the two master fields in each of the covariantizations that is primed, since the rearrangement in (5.10) implies that, schematically,

\[ \hat{U}^{(n)} \ast \hat{V}^{(p)} = \hat{U}'^{(n)} \ast \hat{V}'^{(p)} . \] (5.64)

Following steps similar to those outlined below eq. (3.25) \[17\], one can show the perturbative integrability of the analytic constraints, eqs. (5.62)-(5.63). As a consequence, the evaluation of (5.62) and (5.63) at $Z = 0$, that is

\[ \hat{F}^\mu_{\nu}\big|_{Z=0} = 0 , \quad \hat{D}_\mu \hat{\Phi}'\big|_{Z=0} = 0 , \quad (5.65)\]

give a set of integrable constraints on spacetime that describe a perturbative expansion of the full field equations.

The main difference between the minimal and modified schemes is that the latter involves additional parametric integrals associated with representations of distributional projectors. This might result in undesirable negative powers of $K^2$ (see (5.56)). Therefore, we are presently inclined to believe that the minimal scheme will be the one that ultimately leads to a finite curvature expansion of the $D$-dimensional Vasiliev equations with supplementary $Sp(2, R)$ invariance conditions.

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