HOMOTHETICAL SURFACES IN THREE DIMENSIONAL
PSUEDO-GALILEAN SPACE

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ABSTRACT. A homothetical surface arises as a graph of a function \( z = \varphi_1(v_1)\varphi_2(v_2) \).
In this paper, we study the homothetical surfaces in three dimensional pseudo-
Galilean space \( G^3_1 \) satisfying the conditions \( \Delta x_i = \lambda x_i \), where \( \Delta \) is the
Laplacian with respect to second fundamental form. In particular, we show
the non-existence of any such type of surface in \( G^3_1 \).

1. INTRODUCTION

An Euclidean submanifold is said to be of finite Chen-type if its coordinate
function are the finite sum of eigenfunctions of its Laplacian. B.Y. Chen posed
the problem of classifying the finite type of surfaces in 3-dimensional Euclidean
spaces \( E^3 \). The notion of finite type can be extended to any smooth function on a
submanifold of a Euclidean space or any ambient space.

Let \( x : M \to E^m \) be an isometric immersion of a connected \( n \)-dimensional
manifold in the \( m \)-dimensional Euclidean space \( E^m \). Denote by \( H \) and \( \Delta \) the mean
curvature and the Laplacian of \( M \) with respect to the Riemannian metric on \( M \)
induced from that of \( E^m \), respectively. Takahashi \[18\] proved that the submanifold
in \( E^m \) satisfying \( \Delta x = \lambda x \), that is, all the coordinate functions are eigenfunctions of
the Laplacian with the same eigenvalue \( \lambda \in \mathbb{R} \), are either the minimal submanifolds
of \( E^m \) or the minimal submanifolds of hypersphere \( S^{m-1} \) in \( E^m \).

As an extension of Takahashi theorem, in \[12\] Garay studied hypersurfaces in \( E^m \)
whose coordinate functions are eigenfunctions of the Laplacian, but not necessary
according to the same eigenvalue. He considered hypersurfaces in \( E^m \) satisfying the
condition
\[
\Delta x = Ax,
\]
where \( A \in \text{Mat}(m, \mathbb{R}) \) is an \( m \times m \)-diagonal matrix and proved that such hyper-
surfaces are minimal (\( H = 0 \)) in \( E^m \) and open pieces of either round hypersurfaces
or generalized right spherical cylinders.

Related to this, Dillen, Pas and Verstraelen \[10\] investigated surfaces in \( E^3 \) whose
immersions satisfy the condition
\[
\Delta x = Ax + B,
\]
where \( A \in \text{Mat}(3, \mathbb{R}) \) is a \( 3 \times 3 \) real matrix and \( B \in \mathbb{R}^3 \). In other words, each
coordinate function is of \( 1 \)-type in the sense of Chen \[8\]. For the Lorentzian
version of surfaces satisfying \( \Delta x = Ax + B \), Alias, Ferrandez and Lucas \[1\] proved
that the only such surfaces are minimal surfaces and open pieces of Lorentz circular
cylinders, hyperbolic cylinders, Lorentz hyperbolic cylinders, hyperbolic spaces or pseudo-spheres.

The notion of an isometric immersion \( x \) is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the manifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Baikoussis and Verstraelen \cite{4} studied the helicoidal surfaces in \( \mathbb{E}^3 \). Choi \cite{9} completely classified the surface of revolution in the three dimensional Minkowski space \( \mathbb{E}^3_1 \) satisfying the condition

\[
\Delta G = AG.
\]

Yoon \cite{19} classified the translation surfaces in the three dimensional Galilean space under the condition

\[
\Delta x^i = \lambda^i x^i,
\]

where \( \lambda^i \in \mathbb{R} \). The authors in \cite{5} \cite{13} classified translation surfaces and surface of revolution, respectively in three dimensional spaces satisfying

\[
\Delta^{III} r_i = \mu_i r_i.
\]

Yanhua Yu and Huili Liu \cite{20} and the authors in \cite{15} studied the homothetical minimal homothetical surfaces in 3-dimensional Euclidean and Minkowski spaces. Mohammed Bekkar and Bendehiba Senoussi \cite{6} classified the homothetical surfaces in 3-dimensional Euclidean and Lorentzian spaces satisfying

\[
\Delta r_i = \lambda_i r_i.
\]

Aydin, Ögrenmiş and Ergüt \cite{3} investigated the homothetical surfaces in pseudo-Galilean space with null Gaussian and mean curvature. Karacan, Yoon and Bukcu \cite{14} classified translation surfaces of type-1 satisfying \( \Delta^J x_i = \lambda_i x_i \), \( J = 1, 2 \) and \( \Delta^{III} x_i = \lambda_i x_i \). Recently Ali Cakmak, Murat Kemal Karacan, Sezai Kiziltug and Dae Won Yoon \cite{7} studied the translation surfaces in the 3-dimensional Galilean space satisfying

\[
\Delta^{II} x_i = \lambda_i x_i.
\]

The main aim of this paper is to prove the non existence of homothetical surface in three-dimensional pseudo-Galilean space \( G^3_3 \) in terms of the position vector field and the Laplacian operator.

2. Preliminaries

The pseudo-Galilean space \( G^3_3 \) is a Cayley-Klein space defined from a 3-dimensional projective space \( P\mathbb{R}^3 \) with the absolute figure that consists of an ordered triplet \( \{\omega, f, I\} \), where \( \omega \) is the ideal(absolute) plane, \( f \) the line(absolute line) in \( \omega \) and \( I \) the fixed hyperbolic involution of points of \( f \). We introduce homogeneous coordinates in \( G^3_3 \) in such a way that the absolute plane \( \omega \) is given by \( x_0 = 0 \), the absolute line \( f \) by \( x_0 = x_1 = 0 \) and the hyperbolic involution by \( (0:0:x_2:x_3) \rightarrow (0:0:x_3,x_2) \). In affine coordinates defined by \( (0:x_1:x_2:x_3) \rightarrow (1:x:y:z) \), distance between points \( Q_i = (x_i,y_i,z_i), i = 1,2 \) is defined by: \( (11) \) \cite{11} \cite{17}

\[
d(Q_1,Q_2) = \begin{cases} |x_2 - x_1|, & x_1 \neq x_2, \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}, & x_1 = x_2. \end{cases}
\]
The group of motions of $\mathbb{G}^3_3$ is a six parameter group given (in affine coordinates) by

$$\mathbf{x} = a + x,$$
$$\mathbf{y} = b + cx + y \cosh \theta + z \sinh \theta,$$
$$\mathbf{z} = d + ex + y \sinh \theta + z \cosh \theta.$$  

The pseudo-Galilean scalar product of two vectors $Q_1 = (x_1, x_2, x_3)$ and $Q_2 = (y_1, y_2, y_3)$ is defined as

$$Q_1 \cdot Q_2 = \begin{cases} x_1 y_1 & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0, \\ x_2 y_2 - x_3 y_3 & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

In pseudo-Galilean space a vector $Q = (x_1, x_2, x_3)$ is called isotropic (non-isotropic) if $x_1 = 0(x_1 \neq 0)$. All unit non-isotropic vectors are of the form $(1, x_2, x_3)$. The isotropic vector $Q = (0, x_2, x_3)$ is called spacelike, timelike and lightlike if $x_2^2 - x_3^2 > 0$, $x_2^2 - x_3^2 < 0$ and $x_2 = \pm x_3$, respectively. The pseudo-Galilean cross product of $Q_1$ and $Q_2$ in $\mathbb{G}^3_3$ is given by

$$Q_1 \times Q_2 = \begin{vmatrix} 0 & -e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}$$

where $e_2$ and $e_3$ are the standard basis.

Let $\mathbf{M}$, be a $C^r$, $r \geq 1$ surface in pseudo-Galilean space $\mathbb{G}^3_3$ parameterized by

$$\mathbf{x}(v_1, v_2) = (x(v_1, v_2), y(v_1, v_2), z(v_1, v_2)).$$

From now onward set $x_i = \frac{\partial x}{\partial v_i}$, $i = 1, 2$, similarly for $y(v_1, v_2)$ and $z(v_1, v_2)$. The surface $\mathbf{M}$ has the following first fundamental form

$$\mathbf{I} = \begin{pmatrix} ds_1^2 & 0 \\ 0 & ds_2^2 \end{pmatrix},$$

where $ds_1^2 = (g_{11}dv_1^2 + g_{22}dv_2^2)$, $ds_2^2 = (h_{11}dv_1^2 + 2h_{12}dv_1dv_2 + h_{22}dv_2^2)$, $g_{ij} = x_i \cdot x_j$ and $h_{ij} = \hat{x}_i \cdot \hat{x}_j$ stands for derivatives of the first coordinate function $x(v_1, v_2)$ with respect to $v_1, v_2$ and for the Euclidean scalar product of the projections $\hat{x}_k$ of the vectors $x_k$ onto the $yz$-plane, respectively. A surface is called admissible if it has no Euclidean tangent planes. Therefore, for an admissible surface either $g_1 \neq 0$, or $g_2 \neq 0$, holds. An admissible surface can always be expressed as

$$z = \varphi(v_1, v_2).$$

The vector $\mathbf{N}$ defines a normal vector to a surface and is given by

$$\mathbf{N} = \frac{1}{W}(0, -x_2z_1 + x_1z_2, x_1y_2 - x_2y_1),$$

where $W = \sqrt{(x_1y_2 - x_2y_1)^2 - (x_1z_2 - x_2z_1)^2}$ and $\mathbf{N} \cdot \mathbf{N} = \epsilon = \pm 1$.

Hence two types of admissible surfaces can be distinguished: spacelike having timelike unit normal ($\epsilon = -1$) and timelike having spacelike unit normal ($\epsilon = 1$). The Gaussian $\mathbf{K}$ and the mean curvature $\mathbf{H}$ are $C^{r-2}$ functions, $r \geq 2$, defined by

$$\mathbf{K} = -\frac{LN - M^2}{W^2}, \quad \mathbf{H} = -\frac{g_2L - 2g_1g_2M + g_1^2N}{2W^2},$$

where

$$L_{ij} = \epsilon \frac{x_1x_{ij} - x_{ij}x_1}{x_1}, \quad x_1 = g_1 \neq 0.$$
We will use $L_{ij}, i, j = 1, 2$ for $L, M, N$ for more convenience.

It is well known in terms of local coordinates \{v_1, v_2\} of $M$ the Laplacian operator $\Delta^{II}$, with respect to second fundamental form on $M$ is defined by \[ \Delta^{II}x = \frac{-1}{\sqrt{LN - M^2}} \left[ \frac{\partial}{\partial v_1} \left( \frac{Nx_{v_1} - Mx_{v_2}}{\sqrt{LN - M^2}} \right) - \frac{\partial}{\partial v_2} \left( \frac{Mx_{v_1} - Lx_{v_2}}{\sqrt{LN - M^2}} \right) \right], \]
where the second fundamental form is non-degenerate or $LN - M^2 \neq 0$.

3. Homothetical surfaces in $\mathbb{G}_3^1$

A surface $M$ in the pseudo-Galilean space $\mathbb{G}_3^1$ is called a homothetical(or factorable) surface if it can be locally written as

\begin{align*}
(3.1) & \quad x(v_1, v_2) = (v_1, v_2, \varphi_1(v_1)\varphi_2(v_2)) \\
(3.2) & \quad x(v_1, v_3) = (v_1, \varphi_1(v_1)\varphi_3(v_3), v_3) \\
(3.3) & \quad x(v_2, v_3) = (\varphi_2(v_2)\varphi_3(v_3), v_2, v_3),
\end{align*}

where $\varphi_i$ are $C^r, r \geq 1$ smooth functions. The surfaces given by (3.1), (3.2) and (3.3) are called as the homothetical surfaces of the first, the second and the third type, respectively. We have a complete classification result of Null Gaussian homothetical surfaces in the following theorem:

**Theorem 3.1.**\[^{[3]}\] Let $M^2$ be a factorable(or homothetical) surface with null Gaussian curvature in $\mathbb{G}_3^1$. If $M^2$ is a factorable surface of the first type (respectively the second type and the third type), then either

(a) at least one of $\varphi_1, \varphi_2$ (respectively $\varphi_1, \varphi_3$ and $\varphi_2, \varphi_3$) is a constant function, or

(b) $\varphi_i(v_i) = c_i e^{d_i v_i}$, where $c_i, d_i \in \mathbb{R} \setminus \{0\}, i \in \{1, 2\}$, (respectively $i \in \{1, 3\}$ and $i \in \{2, 3\}$) or

(c) $\varphi_i(v_i) = \left[ (1 - m_i) n_i v_i + \lambda_i \right]^{-1}$, where $m_i \neq 0, 1, m_i \in \mathbb{R}$ and $m_i m_j = 1$, $n_i \in \mathbb{R} \setminus \{0\}$ and $\lambda_i \in \mathbb{R}, i \in \{1, 3\}$ (respectively $i \in \{1, 3\}$ and $i \in \{2, 3\}$).

Conversely, the factorable surfaces satisfying the above cases have null Gaussian curvature.

We see that the first and the second type homothetical surfaces have up to a sign similar second fundamental form, we will discuss only first and third type homothetical surfaces \[^{[2]} [3].\]

Noting that we shall be considering the surfaces not falling under the ambit of theorem 3.1.

**Definition 3.2.** A surface in three dimensional pseudo-Galilean space $\mathbb{G}_3^1$ is called $\Pi$–harmonic if it satisfies the condition $\Delta^{II}x = 0$.

The main results in this paper are:

**Theorem 3.3.** There are no $\Pi$–harmonic homothetical surface of the type first in $\mathbb{G}_3^1$. 
**Theorem 3.4.** Let $M$ be a non-II-harmonic homothetical surface of type first with non-degenerate second fundamental form given by (3.1) in $G^3$. Then there exists no such surface satisfying $\Delta^2 x_i = \lambda_i x_i$, where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$.

**Theorem 3.5.** There are no II-harmonic homothetical surfaces of the type third in $G^3$.

**Theorem 3.6.** Let $M$ be a non-II-harmonic homothetical surface of type third with non-degenerate second fundamental form given by (3.3) in $G^3$. Then there exists no such surface satisfying $\Delta^2 x_i = \lambda_i x_i$, where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$.

4. Homothetical Surfaces of Type First Satisfying $\Delta^2 x_i = \lambda_i x_i$

**Proof of theorem (3.3) and (3.4):**

Let $x$ be a homothetical surfaces of the type first with non-degenerate second fundamental form in $G^3$ satisfying the condition

$$\Delta^2 x_i = \lambda_i x_i,$$

where $\lambda_i \in \mathbb{R}$, $i = 1, 2, 3$ and

$$\Delta^2 x_i = (\Delta^2 x_1, \Delta^2 x_2, \Delta^2 x_3),$$

where

$$x_1 = v_1, \quad x_2 = v_2, \quad x_3 = \varphi_1(v_1)\varphi_2(v_2).$$

Now for the homothetical surface given by (3.1), the coefficients of the second fundamental form are given by

$$L = -\frac{\epsilon}{W}\varphi_1^2, \quad M = -\frac{\epsilon}{W}\varphi_1^2, \quad N = -\frac{\epsilon}{W}\varphi_1^2,$$

where $W = \sqrt{1 - (\varphi_1^2)^2} \neq 0$.

The Gaussian curvature $K$ is given by

$$K = -\frac{\epsilon W}{W^2}(\varphi_1^2\varphi_2^2 - \varphi_1^2 \varphi_2^2).$$

Since the surface is non-degenerate everywhere, we have

$$D = \varphi_1^2\varphi_2^2 - \varphi_1^2 \varphi_2^2 \neq 0, \forall v_1, v_2 \in I.$$

The Laplacian operator of $x_i$, $i = 1, 2, 3$ with the help of (3.2) turns out to be

$$\Delta^2 v_1 = -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varphi_1^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\epsilon \varphi_1 \varphi_2}{\sqrt{D}} \right) \right],$$

$$\Delta^2 v_2 = -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varphi_1^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\epsilon \varphi_1 \varphi_2}{\sqrt{D}} \right) \right],$$

and

$$\Delta^2 \varphi_1 \varphi_2 = \varphi_1^2 \varphi_2^2 \left( \frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varphi_1^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\epsilon \varphi_1 \varphi_2}{\sqrt{D}} \right) \right] \right) +$$

$$f \varphi_2 \left( \frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varphi_1^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\epsilon \varphi_1 \varphi_2}{\sqrt{D}} \right) \right] \right) + 2\epsilon \sqrt{1 - \varphi_1^2 \varphi_2^2}.$$ (4.6)
Combining (4.1), (4.5) and (4.6), we can write

$$\Delta'^I x = \left\{ \begin{array}{l}
-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_1'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_1^2'}{\sqrt{D}} \right) \right], \\
-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_2'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_2^2'}{\sqrt{D}} \right) \right], \\
\varphi_1'^2 \varphi_2 \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_1'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_1^2'}{\sqrt{D}} \right) \right] \right) + \\
\varphi_1^2 \varphi_2' \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_2'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_2^2'}{\sqrt{D}} \right) \right] \right) + 2\varepsilon \sqrt{1 - \varphi_1^2 \varphi_2^2}.
\end{array} \right\}
$$

(4.7)

Since M satisfies (4.11), equation (4.7) gives rise to the following differential equations

$$-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_1'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_1^2'}{\sqrt{D}} \right) \right] = \lambda_1 v_1,$n

(4.8)

$$-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_2'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_2^2'}{\sqrt{D}} \right) \right] = \lambda_2 v_2,$n

(4.9)

$$\varphi_1'^2 \varphi_2 \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_1'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_1^2'}{\sqrt{D}} \right) \right] \right) + \varphi_1^2 \varphi_2' \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_1} \left( \frac{-\varepsilon \varphi_2'^2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_2} \left( \frac{\varepsilon \varphi_2^2'}{\sqrt{D}} \right) \right] \right) + 2\varepsilon \sqrt{1 - \varphi_1^2 \varphi_2^2} = \lambda_3 \varphi_1 \varphi_2.
$$

(4.10)

This means that M is at most of 3-types. On combining (4.8), (4.9) and (4.10), we get

$$\varphi_1^2 \varphi_2 \lambda_1 v_1 + \varphi_1 \varphi_2^2 \lambda_2 v_2 + 2\varepsilon \sqrt{1 - \varphi_1^2 \varphi_2^2} = \lambda_3 \varphi_1 \varphi_2.
$$

(4.11)

Since $\varphi_1 \varphi_2 \neq 0$, (4.11) can be written as

$$\varphi_1^2 \varphi_1 \lambda_1 v_1 + \varphi_1 \varphi_2^2 \lambda_2 v_2 + 2\varepsilon \sqrt{1 - (\varphi_1 \varphi_2)^2} = \lambda_3 \varphi_1 \varphi_2.
$$

(4.12)

According to the choices of constants $\lambda_1$, $\lambda_2$ and $\lambda_3$, we discuss all the possible cases of $\lambda_i$, $i \in \{1, 2, 3\}$.

**Case 1**: Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (4.12), get

$$2\varepsilon \sqrt{1 - \varphi_1^2 \varphi_2^2} = 0, \text{ or } W = 0,$n

which is a contradiction to our assumption. Hence there exists no II-harmonic homothetical surfaces of type first in $G_1^I$. This completes the proof of the theorem (3.3).

**Case 2**: Let $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 \neq 0$, from (4.12), we get

$$2\varepsilon \sqrt{1 - \varphi_1 \varphi_2^2} = \lambda_3 \varphi_1 \varphi_2.
$$

(4.13)

From (4.13), we obtain

$$\frac{4}{\varphi_1^2} - 4\varphi_2^2 - \lambda_3 \varphi_2^2 = 0.$$

Since $\varphi_1$ and $\varphi_2$ are functions of two independent variables. The above equation can be written as
\[
\frac{4}{\varphi_1^2} = c, \quad 4\varphi_2^2 + \lambda_3^2\varphi_2^2 = c,
\]
where $c \in \mathbb{R} \setminus 0$. Thus, we get
\[
\varphi_1(v_1) = \pm \frac{2}{\sqrt{c}}, \quad \varphi_2(v_2) = \pm \frac{\sqrt{c}\tan\left(\frac{1}{2}\lambda_3(v_2 + 2c_1)\right)}{\lambda_3\sqrt{1 + \tan\left(\frac{1}{2}\lambda_3(v_2 + 2c_1)\right)^2}}.
\]
In this case the surface may be parameterized as
\[
(4.15) \quad \mathbf{x}(v_1, v_2) = \left( v_1, v_2 \left( \pm \frac{2}{\sqrt{c}} \right) \left( \pm \frac{\sqrt{c}\tan\left(\frac{1}{2}\lambda_3(v_2 + 2c_1)\right)}{\lambda_3\sqrt{1 + \tan\left(\frac{1}{2}\lambda_3(v_2 + 2c_1)\right)^2}} \right) \right).
\]
We observe that the parameterization in (4.15) is a contradiction to non-degenerate property as well as to the part (a) of theorem (3.1).

**Case 3:** Let $\lambda_1 = 0, \lambda_2 \neq 0$ and $\lambda_3 \neq 0$, from (4.12), we get
\[
(4.16) \quad \frac{\varphi_2'}{\varphi_2} \lambda_2 v_2 + 2\epsilon \sqrt{1 - \varphi_1^2\varphi_2^2} = \lambda_3.
\]
From (4.16), we obtain
\[
(4.17) \quad \frac{4}{\varphi_1^2} = c, \quad (\varphi_2')^2 + (\varphi_2'\lambda_2 v_2 - \lambda_3\varphi_2)^2 = c,
\]
where $c \in \mathbb{R} \setminus 0$. Since the first equation in (4.17) is constant, so regardless of the second equation of (4.17), it gives rise to a contradiction to the property of being non-degenerate. Therefore there exists no parameterization in this case.

**Case 4:** Let $\lambda_1 \neq 0, \lambda_2 = 0$ and $\lambda_3 = 0$, from (4.12), we have
\[
\frac{\varphi_1'}{\varphi_1} v_1 + 2\epsilon \sqrt{1 - \varphi_1^2\varphi_2^2} = 0.
\]
Squaring and adjusting the like terms in above equation, we get
\[
(4.18) \quad (\varphi_1'v_1)^2 = \frac{4}{\varphi_2^2} - 4\varphi_1^2 \left( \frac{\varphi_1'}{\varphi_2} \right)^2.
\]
Differentiating (4.18), with respect to $v_1$, we get
\[
(4.19) \quad \frac{\lambda_1^2}{4} \left( \frac{\varphi_1''}{\varphi_1} v_1^2 + \frac{\varphi_1'}{\varphi_1} v_1 \right) + \left( \frac{\varphi_2'}{\varphi_2} \right)^2 = 0.
\]
since $\varphi_1$ and $\varphi_2$ are functions of two independent variables, we may write
\[
\frac{\lambda_1^2}{4} \left( \frac{\varphi_1''}{\varphi_1} v_1^2 + \frac{\varphi_1'}{\varphi_1} v_1 \right) = -c, \quad \left( \frac{\varphi_2'}{\varphi_2} \right)^2 = c
\]
or
\[
(4.20) \quad \left( \frac{\varphi_1''}{\varphi_1} v_1^2 + \frac{\varphi_1'}{\varphi_1} v_1 \right) = \tilde{c}, \quad \left( \frac{\varphi_2'}{\varphi_2} \right)^2 = c,
\]
where $\tilde{c} = -c\frac{1}{\lambda_1^2}, c \in \mathbb{R}$. If $c = 0$, then the second equation of (4.20) implies $\varphi_2$ =constant which leads to a contradictions. Therefore for $c \in \mathbb{R} \setminus 0$, we have
\[
(4.21) \quad \varphi_1(v_1) = c_1 \cos \sqrt{\tilde{c}} \log(v_1) + c_2 \sin \sqrt{\tilde{c}} \log(v_1), \quad \varphi_2(v_2) = c_3 e^{\pm \sqrt{\tilde{c}} v_2},
\]
where \( c_i \in \mathbb{R}, i \in \{1, 2, 3\} \). In this case the surface may be parameterized as

\[
x(v_1, v_2) = \left( v_1, v_2, \left( c_1 \cos \sqrt{c} \log(v_1) + c_2 \sin \sqrt{c} \log(v_1) \right) \left( c_3 e^{\pm \sqrt{c} v_2} \right) \right).
\]

We observe that the second equation of (4.21) is a contradiction to non-degenerate property with respect to the part (b) of the theorem (3.1). Therefore there exists no parameterization in this case.

**Case 5:** Let \( \lambda_1 = 0, \lambda_2 \neq 0 \) and \( \lambda_3 = 0 \), from (4.12), we get

\[
\frac{\varphi'_2}{\varphi_2} \lambda_2 v_2 + \frac{2\epsilon \sqrt{1 - \varphi_1^2 \varphi_2^2}}{\varphi_1 \varphi_2} = 0.
\]

From above equation, we obtain

(4.22) \[
\frac{4}{\varphi_1^2} - 4(\varphi'_2)^2 - (\varphi'_2 \lambda_2 v_2)^2 = 0.
\]

Since \( \varphi_1 \) and \( \varphi_2 \) are functions of independent variables, we can write

(4.23) \[
(4 + \lambda^2 v_2^2)(\varphi_2^2) = c, \quad \frac{4}{\varphi_1^2} = c.
\]

where \( c \in \mathbb{R} \setminus 0 \). From (4.23), we obtain

(4.24) \[
\varphi_2(v_2) = \pm \frac{\sqrt{c}}{\lambda} \sinh^{-1} \left( \frac{\lambda v_2}{2} \right), \quad \varphi_1(v_1) = \pm \frac{2}{\sqrt{c}}.
\]

Therefore the surface may be parameterized in the form

(4.25) \[
x(v_1, v_2) = \left( v_1, v_2, \left( \pm \frac{2}{\sqrt{c}} \right) \left( \pm \frac{\sqrt{c}}{\lambda} \sinh^{-1} \left( \frac{\lambda v_2}{2} \right) \right) \right).
\]

We can easily find out that the parameterization in (4.25) gives rise to a similar type of contradiction as in case 2.

**Case 6:** Let \( \lambda_1 \neq 0, \lambda_2 \neq 0 \) and \( \lambda_3 \neq 0 \), from (4.12), we get

(4.26) \[
\frac{\varphi'_1}{\varphi_1} \lambda_1 v_1 + \frac{\varphi'_2}{\varphi_2} \lambda_2 v_2 + \frac{2\epsilon \sqrt{1 - \varphi_1^2 \varphi_2^2}}{\varphi_1 \varphi_2} = 0.
\]

There is no suitable solution of (4.26). Hence there exists no parameterization in this case also.

**Case 7:** Let \( \lambda_1 \neq 0, \lambda_2 = 0 \) and \( \lambda_3 \neq 0 \), from (4.12), we have

(4.27) \[
\frac{\varphi'_1}{\varphi_1} \lambda_1 v_1 + \frac{2\epsilon \sqrt{1 - \varphi_1^2 \varphi_2^2}}{\varphi_1 \varphi_2} = \lambda_3.
\]

Differentiating (4.27), with respect to \( v_2 \), we get

(4.28) \[
\frac{2\epsilon \sqrt{1 - \varphi_1^2 \varphi_2^2}}{\varphi_1 \varphi_2} = c,
\]

where \( c \in \mathbb{R} \). If \( c = 0 \), then it is a contradiction to \( W \neq 0 \). So assume \( c \neq 0 \). Squaring and adjusting the like terms in (4.28), we get

(4.29) \[
\frac{1}{(\varphi_1)^2} = \varphi_2^2 \left( c + \left( \frac{\varphi'_2}{\varphi_2} \right)^2 \right).
\]
Since \( \varphi_1 \) and \( \varphi_2 \) are functions of two independent variables, we can write

\[
\left( \frac{\varphi_1'}{\varphi_1} \right)^2 = c_1, \quad \varphi_2^2 \left( c + \left( \frac{\varphi_2'}{\varphi_2} \right)^2 \right) = c_1,
\]

where \( c_1 \in \mathbb{R} \setminus 0 \). In any way regardless of the solution of second equation of (4.30), the first equation in (4.30) will cause a contradiction to non-degenerate property. Hence there exists parameterization in this case.

**Case 8:** Let \( \lambda_1 \neq 0, \lambda_2 \neq 0 \) and \( \lambda_3 \neq 0 \), from (4.12), we have

\[
(4.31) \quad \frac{\varphi_1'}{\varphi_1} \lambda_1 v_1 + \frac{\varphi_2'}{\varphi_2} \lambda_2 v_2 + \frac{2 \epsilon \sqrt{1 - \varphi_1'^2 \varphi_2'^2}}{\varphi_1 \varphi_2} = \lambda_3.
\]

Differentiating (4.31) with respect to \( v_1 \) and \( v_2 \), we get

\[
(4.32) \quad \frac{2 \epsilon \sqrt{1 - \varphi_1'^2 \varphi_2'^2}}{\varphi_1 \varphi_2} = c.
\]

If \( c = 0 \), then it is a contradiction to \( W \neq 0 \). Suppose \( c \neq 0 \), from (4.31), we obtain

\[
(4.33) \quad \frac{\varphi_1'}{\varphi_1} \lambda_1 v_1 + \frac{\varphi_2'}{\varphi_2} \lambda_2 v_2 + c = \lambda_3.
\]

Since \( \varphi_1 \) and \( \varphi_2 \) are functions of two independent variables, we can write

\[
(4.34) \quad \frac{\varphi_1'}{\varphi_1} \lambda_1 v_1 = c_1, \quad \lambda_3 - \frac{\varphi_2'}{\varphi_2} \lambda_2 v_2 + c = c_1.
\]

We can easily see that the solutions of (4.34) does not satisfy (4.31). This completes the proof of the theorem (3.4).

5. **Homothetical surfaces of type third satisfying** \( \Delta^{II} x_i = \lambda_i x_i \)

**Proof of theorem (3.5) and (3.6).**

Let \( x \) be a homothetical surfaces of the type third with non-degenerate second fundamental form in \( G_3 \) satisfying the condition

\[
(5.1) \quad \Delta^{II} x_i = \lambda_i x_i,
\]

where \( \lambda_i \in \mathbb{R}, i = 1, 2, 3 \) and

\[
\Delta^{II} x_i = (\Delta^{II} x_1, \Delta^{II} x_2, \Delta^{II} x_3),
\]

where

\[
x_1 = \varphi_1(v_2)\varphi_2(v_3), \quad x_2 = v_2, \quad x_3 = v_3.
\]

For the homothetical surface given by (5.3), the coefficients of the second fundamental form are given by

\[
L = -\frac{\epsilon}{W} \varphi_1'' \varphi_2, \quad M = -\frac{\epsilon}{W} \varphi_1' \varphi_2', \quad N = -\frac{\epsilon}{W} \varphi_1' \varphi_2'',
\]

where \( W = \sqrt{(\varphi_1' \varphi_2')^2 - (\varphi_1 \varphi_2'')^2} \neq 0 \).

The Gaussian curvature \( K \) is given by

\[
K = \frac{-\epsilon}{W^2} (\varphi_1 \varphi_2 \varphi_1'' \varphi_2'' - \varphi_1' \varphi_2'')^2.
\]

Since the surface is non-degenerate, we have

\[
D = \varphi_1 \varphi_2 \varphi_1'' \varphi_2'' - \varphi_1' \varphi_2'')^2 \neq 0, \forall v_2, v_3 \in I.
\]
In this case, the Laplacian operator of $x_i, i = 1, 2, 3$ with the help of (2.2) turns out to be

$$\Delta^{II} \varphi_1 \varphi_2 = \varphi'_1 \varphi_2 \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right] \right) + \varphi'_1 \varphi'_2 \left( \frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi''_1 \varphi_2}{\sqrt{D}} \right) \right] \right) + 2\epsilon \sqrt{\varphi'_1^2 \varphi'_2^2 - \varphi''_1 \varphi'_2^2},$$

(5.3)

$$\Delta^{II} v_2 = -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right],$$

(5.4)

and

$$\Delta^{II} v_3 = -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right].$$

(5.5)

Combining (5.3), (5.4) and (5.5), we get

$$\Delta^{II} x = \left\{ \begin{array}{l}
\varphi'_1 \varphi_2 \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right] \right) + \varphi'_1 \varphi'_2 \left( \frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi''_1 \varphi_2}{\sqrt{D}} \right) \right] \right) + 2\epsilon \sqrt{\varphi'_1^2 \varphi'_2^2 - \varphi''_1 \varphi'_2^2}, \\
-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right], \\
-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right], \\
\end{array} \right\},$$

(5.6)

Since $M$ satisfies (5.1) and $\lambda_1, \lambda_2, \lambda_3$ are three different scalars. For the sake of convenience, set $\lambda_1 = \tilde{\lambda}_3, \lambda_2 = \tilde{\lambda}_1, \lambda_3 = \tilde{\lambda}_2$, i.e., we have

$$\Delta^{II} x_1 = \tilde{\lambda}_3 x_1, \quad \Delta^{II} x_2 = \tilde{\lambda}_1 x_2, \quad \Delta^{II} x_3 = \tilde{\lambda}_2 x_3.$$

Thus (5.6) gives rise to the following differential equations

$$\varphi'_1 \varphi_2 \left( -\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right] \right) + \varphi'_1 \varphi'_2 \left( \frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi''_1 \varphi_2}{\sqrt{D}} \right) \right] \right) + 2\epsilon \sqrt{\varphi'_1^2 \varphi'_2^2 - \varphi''_1 \varphi'_2^2} = \tilde{\lambda}_3 \varphi_1 \varphi_2,$$

(5.7)

$$-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right] = \tilde{\lambda}_2 v_3,$$

(5.8)

$$-\frac{W}{\sqrt{D}} \left[ \frac{\partial}{\partial v_2} \left( \frac{-\epsilon \varphi'_1 \varphi''_2}{\sqrt{D}} \right) + \frac{\partial}{\partial v_3} \left( \frac{\epsilon \varphi'_1 \varphi'_2}{\sqrt{D}} \right) \right] = \tilde{\lambda}_1 v_2.$$n

(5.9)

This means that $M$ is at most of 3-types. On combining (5.7), (5.8) and (5.9), we get

$$\varphi'_1 \varphi_2 \tilde{\lambda}_1 v_2 + \varphi'_1 \varphi'_2 \tilde{\lambda}_2 v_3 + 2\epsilon \sqrt{\varphi'_1^2 \varphi'_2^2 - \varphi''_1 \varphi'_2^2} = \tilde{\lambda}_3 \varphi_1 \varphi_2.$$
Since $\phi(5.12)$
write.
which is a contradiction to the non-vanishing assumption of $W$. Hence there exists no II-harmonic homothetical surfaces of the type third in $G_3^3$. This completes the proof of the theorem (3.5).

Case 1: Let $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 = 0$, from (5.10), we get
\[
2e\sqrt{\phi_1^2 \phi_2^2 - \phi_1^2 \phi_2^2} = 0,
\]
which is a contradiction to the non-vanishing assumption of $W$. Hence there exists no II-harmonic homothetical surfaces of the type third in $G_3^3$. This completes the proof of the theorem (3.5).

Case 2: Let $\lambda_1 = 0$, $\lambda_2 = 0$ and $\lambda_3 \neq 0$, from (5.10), we get
\[
2e\sqrt{\phi_1^2 \phi_2^2 - \phi_1^2 \phi_2^2} = \lambda_3.
\]
From above equation, we derive
\[
\left\{ \left( \frac{\phi'}{\phi_1} \right)^2 - \frac{\lambda_3}{8} \right\} - \left\{ \left( \frac{\phi'}{\phi_2} \right)^2 + \frac{\lambda_3}{8} \right\} = 0.
\]
Since $\phi_1$ and $\phi_2$ are functions of two independent variables, from (5.11), we can write.
\[
\left( \frac{\phi'}{\phi_1} \right)^2 - \frac{\lambda_3}{8} = c, \quad \left( \frac{\phi'}{\phi_2} \right)^2 + \frac{\lambda_3}{8} = c,
\]
where $c \in \mathbb{R}$. Hence we get
\[
\phi_1(v_2) = c_1 e^{\pm \frac{1}{8} \sqrt{8c_1+8c_2}} \phi_2(v_3) = c_1 e^{\pm \frac{1}{8} \sqrt{8c_1-8c_2}}, \quad c > \frac{\lambda_3}{8},
\]
where $c_1 \in \mathbb{R}$.

Case 3: Let $\lambda_1 = 0$, $\lambda_2 \neq 0$ and $\lambda_3 \neq 0$, from (5.10), we get
\[
\phi_2 \lambda_2 v_3 + 2e\sqrt{\phi_1^2 \phi_2^2 - \phi_1^2 \phi_2^2} = \lambda_3.
\]
From above equation, we obtain
\[
4 \left\{ \left( \frac{\phi'}{\phi_1} \right)^2 - \left( \frac{\phi'}{\phi_2} \right)^2 \right\} = \left\{ \lambda_3 - \frac{\phi'}{\phi_2} \lambda_2 v_3 \right\}^2
\]
or
\[
4 \left( \frac{\phi'}{\phi_1} \right)^2 - \frac{\lambda_3^2}{2} - 4 \left( \frac{\phi'}{\phi_2} \right)^2 - \left( \frac{\phi'}{\phi_2} \lambda_2 v_3 \right)^2 + 2\lambda_2 \lambda_3 v_3 \left( \frac{\phi'}{\phi_2} \right) = 0.
\]
Since $\phi_1$ and $\phi_2$ are functions of two independent variables, we can write (5.15) as
\[
4 \left( \frac{\phi'}{\phi_1} \right)^2 - \frac{\lambda_3^2}{2} = c, \quad 4 \left( \frac{\phi'}{\phi_2} \right)^2 + \left( \frac{\phi'}{\phi_2} \lambda_2 v_3 \right)^2 - 2\lambda_2 \lambda_3 v_3 \left( \frac{\phi'}{\phi_2} \right) = c,
\]
where \( c \in \mathbb{R} \). Thus we have
\[
\varphi_1(v_2) = c_1 e^{\pm i \sqrt{\lambda_1^2 + cv_2}},
\]
\[
\varphi_2(v_3) = \frac{1}{\lambda_2^2} (4 + \lambda_2^2 v_3^2) \frac{\lambda_3}{2} (n^2 v_3 + nm) \frac{1}{\lambda_2^2} e^{\pm \lambda_2 \lambda_3 \tanh^{-1} \left( \frac{\lambda_2 \lambda_3 v_3}{nm} \right) + cv_2}.
\]

(5.17)

where
\[
m = \sqrt{4c + (\lambda_2 \lambda_3^2 + \lambda_2^2 \lambda_3) v_3^2}, \quad n = \sqrt{\lambda_2 \lambda_3^2 + \lambda_2^2 \lambda_3} \text{ and } c_1, c_2 \in \mathbb{R}.
\]

**Case 4:** Let \( \lambda_1 \neq 0 \), \( \lambda_2 = 0 \) and \( \lambda_3 = 0 \), from (5.10), we get
\[
\frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 + \frac{2\epsilon \sqrt{\varphi'_2 \varphi_2^2 - \varphi_1^2 \varphi'_2^2}}{\varphi_1 \varphi_2} = 0.
\]

The above equation can be rewritten as
\[
\left( \frac{\varphi'_1}{\varphi_1} \right)^2 (4 - \tilde{\lambda}_1^2 v_2^2) - 4 \left( \frac{\varphi'_2}{\varphi_2} \right)^2 = 0.
\]

We can write (5.18) in the following form:
\[
\left( \frac{\varphi'_1}{\varphi_1} \right)^2 (4 - \tilde{\lambda}_1^2 v_2^2) = c, \quad 4 \left( \frac{\varphi'_2}{\varphi_2} \right)^2 = c,
\]

where \( c \in \mathbb{R} \). If \( c = 0 \), then from the second part of above equation we obtain \( \varphi_2 \) = constant which leads to a contradiction. Thus for \( c \in \mathbb{R} \setminus 0 \), we have
\[
\varphi_1(v_2) = \frac{c_1}{\lambda_1} e^{\pm \sqrt{c} \sin^{-1} \left( \frac{\lambda_3 v_3}{nm} \right)}, \quad \varphi_2(v_3) = c_2 e^{\pm \sqrt{c} \sinh^{-1} \left( \frac{\lambda_3 v_3}{nm} \right)}.
\]

where \( c_1, c_2 \) are non-zero constants.

**Case 5:** Let \( \lambda_1 = 0 \), \( \lambda_2 \neq 0 \) and \( \lambda_3 = 0 \), from (5.10), we get
\[
\frac{\varphi'_2}{\varphi_2} \tilde{\lambda}_2 v_3 + \frac{2\epsilon \sqrt{\varphi'_1 \varphi_1^2 - \varphi_2^2 \varphi'_1^2}}{\varphi_1 \varphi_2} = 0.
\]

On the similar lines as in case 4, we can easily obtain
\[
\varphi_1(v_2) = c_1 e^{\pm \sqrt{c} v_2}, \quad \varphi_2(v_3) = \frac{c_2}{\lambda_2} e^{\pm \sqrt{c} \sinh^{-1} \left( \frac{\lambda_3 v_3}{nm} \right)}.
\]

where \( c_1, c_2 \in \mathbb{R} \setminus 0 \). From (5.13) and (5.14), (5.10) and (5.17), (5.19), (5.20), we see that there exists at least one \( \varphi_i, i \in \{1, 2\} \) of the similar form as in part (b) of the theorem (3.1) leading to a contradiction to the non-degenerate property. Therefore there exists no required parameterization from case 2 to case 5.

**Case 6:** Let \( \lambda_1 \neq 0 \), \( \lambda_2 \neq 0 \) and \( \lambda_3 = 0 \), from (5.10), we get
\[
\frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 + \frac{\varphi'_2}{\varphi_2} \tilde{\lambda}_2 v_3 + \frac{2\epsilon \sqrt{\varphi'_1 \varphi_1^2 - \varphi_2^2 \varphi'_1^2}}{\varphi_1 \varphi_2} = 0.
\]

There exists no non-trivial analytic solution of (5.21) other than \( \varphi_1 = \varphi_2 = \text{constant} \), but this assumption again causes a contradiction.
**Case 7:** Let $\tilde{\lambda}_1 \neq 0$, $\tilde{\lambda}_2 = 0$ and $\tilde{\lambda}_3 \neq 0$, from (5.10), we get

$$
(5.22) \quad \frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 + \frac{2\epsilon \sqrt{\varphi'_1^2 \varphi_2^2 - \varphi'_1^2 \varphi'_2^2}}{\varphi_1 \varphi_2} = \lambda_3.
$$

Squaring and adjusting the like terms, we have

$$
\left( \frac{\varphi'_1}{\varphi_1} \right)^2 - \left( \frac{\varphi'_2}{\varphi_2} \right)^2 = \frac{1}{4} \left( \lambda_3 - \frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 \right)^2
$$

or

$$
\left( \frac{\varphi'_1}{\varphi_1} \right)^2 - \frac{1}{4} \left( \lambda_3 - \frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 \right)^2 = \left( \frac{\varphi'_2}{\varphi_2} \right)^2.
$$

The above equation can be written as

$$
(5.23) \quad \left( \frac{\varphi'_1}{\varphi_1} \right)^2 = c, \quad \left( \frac{\varphi'_1}{\varphi_1} \right)^2 - \frac{1}{4} \left( \lambda_3 - \frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 \right)^2 = c,
$$

where $c \in \mathbb{R}$. If $c = 0$, from the first equation of (5.23), $\varphi_2 = \text{constant}$ leads to a contradiction. Suppose $c \neq 0$, from the first equation of (5.23), we obtain

$$
\varphi_2 = \epsilon \sqrt{c \varphi_3}
$$

which is a contradiction to part (b) of theorem (3.1).

**Case 8:** Let $\tilde{\lambda}_1 \neq 0$, $\tilde{\lambda}_2 \neq 0$ and $\tilde{\lambda}_3 = \neq 0$, from (5.10), we get

$$
(5.24) \quad \frac{\varphi'_1}{\varphi_1} \tilde{\lambda}_1 v_2 + \frac{\varphi'_2}{\varphi_2} \tilde{\lambda}_2 v_3 + \frac{2\epsilon \sqrt{\varphi'_1^2 \varphi_2^2 - \varphi'_1^2 \varphi'_2^2}}{\varphi_1 \varphi_2} = \tilde{\lambda}_3.
$$

Following the similar steps as in case 8 of (4.31), we arrive at similar types of contradictions. Therefore there exists no parameterization in this case also. This completes the proof of the theorem (3.6).

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