NOTES ON (THE BIRMAK–KREĬN–VISHIK THEORY ON) SELFADJOINT EXTENSIONS OF SEMIBOUNDED SYMMETRIC OPERATORS

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ABSTRACT. We give an explicit and versatile parametrization of all positive selfadjoint extensions of a densely defined, closed, positive operator. In addition, we identify the Friedrichs extension by specifying the parameter to which it corresponds.

1. Preliminaries

Consider a closed, densely defined, symmetric operator $S$ on a Hilbert space $H$. When dealing with such an operator, the main problem is to extend it to a selfadjoint one. A complete result to this problem was given by J. von Neumann.

A second and more difficult problem is to find all the semibounded selfadjoint extensions of a given semibounded symmetric operator $S$. For such an operator, the existence of a semibounded selfadjoint extension having the same maximal semibound was solved by Friedrichs. The important step in this direction was done by M.G. Kreĭn [6], and immediately after that by M.S. Birman and M.I. Vishik, and this is what is called the Birman–Kreĭn–Vishik theory.

A possible approach involving quadratic forms on Hilbert spaces was recently pointed out by A. Alonso and B. Simon [1].

Stimulated by this kind of investigations, the aim of this paper is to give a new and easy to handle parametrization of the set of all semibounded selfadjoint extensions and, simultaneously, new proofs to classical results are obtained.

In the following, block-matrix representations are used with respect to appropriate orthogonal decompositions of Hilbert spaces. As a starting point, we need only two results concerning completing matrix contractions. The first one is the Sz.-Nagy–Foiaș Lemma [8].

Lemma 1.1. Let $T = \begin{bmatrix} T_1 & X \\ 0 & T_2 \end{bmatrix}$. Then $T$ is a contraction if and only if $T_1$, $T_2$ are contractions and $X = D_{T_1}C D_{T_2}$ with $C$ a contraction $D_{T_2} \to D_{T_1}^*$.

Here, for a given contraction $C$: $\mathcal{H}_1 \to \mathcal{H}_2$ and Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $D_C = (I_{\mathcal{H}_1} - C^* C)^{1/2}$, the defect operator of $C$, and by $D_C = D_{C\mathcal{H}_1}$, the defect space of $C$.

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Note of the second named author: This is a manuscript that was circulated as the first part of the preprint "Two papers on selfadjoint extensions of symmetric semibounded operators", INCREST Preprint Series, July 1981, Bucharest, Romania, but never published. In this \LaTeX{} typeset version, only typos and a few inappropriate formulations have been corrected, with respect to the original manuscript. I decided to post it on arXiv since, taking into account recent articles, the results are still of current interest. Tiberiu Constantinescu died in 2005.
The second one is a recently obtained result of Gr. Arsene and A. Gheondea [2]. Let 
\( A_t = [A \ B] \) and 
\( A_c = \begin{bmatrix} A \\ C \end{bmatrix} \) be two contractions. Then, by Lemma 1.1 there exist unique contractions \( \Gamma_1 \) valued in \( D_{A^*} \) and \( \Gamma_2 \) defined in \( D_A \) such that 
\( A_t = [A \ D_A^* \Gamma_1] \) and 
\( A_c = \begin{bmatrix} A \\ \Gamma_2 D_A \end{bmatrix} \).

**Lemma 1.2.** There exists a bijective correspondence between the sets 
\( \{ T = \begin{bmatrix} A & B \\ C & X \end{bmatrix} \mid T \text{ is a contraction} \} \) and 
\( \{ \Gamma : D_{\Gamma_1} \to D_{\Gamma_2} \mid \Gamma \text{ is a contraction} \} \) given by the formula

\[
T(\Gamma) = \begin{bmatrix} A & D_A^* \Gamma_1 \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_1 + D_{\Gamma_2^*} \Gamma_D \Gamma_1 \end{bmatrix}.
\]

We make use of the old idea of M.G. Kreǐn to introduce a special kind of Cayley transform so we first make some preparations.

## 2. The Cayley Transform

Let us consider a closed, densely defined, symmetric operator \( S_0 \) acting on a Hilbert space \( \mathcal{H} \). Suppose also that it is bounded from below, i.e. there exists \( a \in \mathbb{R} \) such that

\[
\langle S_0 h, h \rangle \geq a \| h \|^2, \text{ for all } h \in \text{Dom}(S_0).
\]

If \( m(S_0) \) is the largest real number \( a \) such that (2.1) holds true then for every \( m_0 \leq m(S_0) \) the operator \( S = S_0 - m_0 I \) is positive and it is easy to check that, in order to find all selfadjoint extensions of \( S_0 \) bounded from below by \( m_0 \), we have to find all positive selfadjoint extensions of \( S_\), [6].

Consider a densely defined positive operator \( S \), that is, (2.1) holds with \( a = 0 \). Then \( S \) is closable and hence, without restricting the generality, we can assume that \( S \) is closed. For every \( h \in \text{Dom}(S) \), here and in throughout \( \text{Dom}(S) \) denotes the domain of \( S \), we get

\[
\| (I + S) h \|^2 = \| h \|^2 + 2 \langle Sh, h \rangle + \| Sh \|^2 \\
\geq \| h \|^2 - 2 \langle Sh, h \rangle + \| Sh \|^2 = \| (I - S) h \|^2.
\]

It follows that \( I + S \) is one-to-one and \( \text{Ran}(I + S) \), the range of \( I + S \), is closed. These all enable us to define the operator

\[
C(S) = T : \text{Ran}(I + S) \to \mathcal{H}, \quad T = (I - S)(I + S)^{-1},
\]

and this is what we call the Cayley transform of \( S \). By means of (2.2) one can easily prove that \( T : \text{Dom}(T) \to \mathcal{H} \) is a contraction, hence bounded. Moreover, \( T \) is symmetric, since

\[
\langle T(I + S) h, (I + S) g \rangle = \langle (I - S) h, (I - S) g \rangle \\
= \langle h, g \rangle - \langle Sh, g \rangle + \langle h, Sg \rangle - \langle Sh, Sg \rangle \\
= \langle h, g \rangle - \langle h, Sg \rangle + \langle Sh, g \rangle - \langle Sh, Sg \rangle \\
= \langle (I + S) h, (I - S) g \rangle = \langle (I + S) h, T(I + S) g \rangle, \quad h, g \in \text{Dom}(S).
\]

Since

\[
(I + T)(I + S) h = (I + S) h + (I - S) h = 2 h, \quad h \in \text{Dom}(S),
\]

\( I + T \) is one-to-one, and \( \text{Ran}(I + T) \) is dense in \( \mathcal{H} \).
Conversely, suppose that $T$ is a symmetric contraction with $\text{Dom}(T)$ closed and such that $\text{Ran}(I + T) = (I + T)\text{Dom}(T) = \mathcal{H}$. Then $I + T$ is one-to-one, hence one can introduce the operator
\begin{equation}
C^{-1}(T) = S : \text{Ran}(I + T) \to \mathcal{H}, \quad S = (I - T)(I + T)^{-1},
\end{equation}
and from the assumptions on $T$ one can prove easily that $S$ is a positive closed and densely defined operator on $\mathcal{H}$. We have proven

Lemma 2.1 (M.G. Krein). The Cayley transform is bijective between the set of all positive closed and densely defined operators $S$ on $\mathcal{H}$ and the set of all symmetric contractions $T : \text{Dom}(T) \subseteq \mathcal{H} \to \mathcal{H}$, with $\text{Dom}(T)$ closed and $(I + T)\text{Dom}(T) = \mathcal{H}$.

The following result is also essential for our approach.

Lemma 2.2 (M.G. Krein). For a given positive, densely defined, and closed operator $S$ on $\mathcal{H}$, the Cayley transform \((2.3)\) is bijective between the sets $\mathcal{S}$ and $\mathcal{T}$
\[ \mathcal{S} = \{ \tilde{S} \mid \tilde{S} \text{ is positive selfadjoint, } \tilde{S}|\text{Dom}(S) = S \}, \]
\[ \mathcal{T} = \{ \tilde{T} \mid \tilde{T} \text{ is a symmetric contraction on } \mathcal{H}, \tilde{T}|\text{Dom}(T) = T \}. \]

Proof. It is a classical result, see [5], that a positive operator $R$ is selfadjoint if and only if $(I + R)\text{Dom}(R) = \mathcal{H}$. Finally, use Lemma 2.1 \(\square\)

Definition 2.3 (\cite{6, 7}). Suppose $R_1$ and $R_2$ are two positive selfadjoint operators on $\mathcal{H}$. Then $R_1 \leq R_2$ means
\begin{equation}
\text{Dom}(R_2^{1/2}) \subseteq \text{Dom}(R_1^{1/2}) \quad \text{and} \quad \|R_1^{1/2}\xi\| \leq \|R_2^{1/2}\xi\| \quad \text{for all } \xi \in \text{Dom}(R_2^{1/2}).
\end{equation}

Lemma 2.4. If $R_1$ and $R_2$ are two positive selfadjoint operators on $\mathcal{H}$ then $R_1 \leq R_2$ if and only if $C(R_1) \geq C(R_2)$.

Proof. We use essentially the following result from [5], VI.2, Theorem 2.21,
\begin{equation}
R_1 \leq R_2 \text{ if and only if } (I + R_1)^{-1} \geq (I + R_2)^{-1},
\end{equation}
the order from the right hand side being the usual one for bounded selfadjoint operators.

Suppose $R_1 \leq R_2$. What we have to prove is that for every $\xi \in \mathcal{H}$ it holds
\begin{equation}
\langle (I - R_1)(I + R_1)^{-1}\xi, \xi \rangle \geq \langle (I - R_2)(I + R_2)^{-1}\xi, \xi \rangle.
\end{equation}
To this end, for a fixed $\xi \in \mathcal{H}$ there exist two uniquely determined vectors $h \in \text{Dom}(R_1)$ and $g \in \text{Dom}(R_2)$ such that
\begin{equation}
(I + R_1)h = \xi = (I + R_2)g,
\end{equation}
therefore \((2.7)\) is equivalent with
\[ \langle (I - R_1)h, (I + R_1)h \rangle \geq \langle (I - R_2)g, (I + R_2)g \rangle, \]
and this holds if and only if
\begin{equation}
\|h\|^2 - \|R_1 h\|^2 \geq \|g\|^2 - \|R_2 g\|^2.
\end{equation}
Making use of \((2.8)\), it follows that
\[ \|(I + R_1)h\|^2 = \|(I + R_2)g\|^2, \]
hence
\begin{equation}
\|R_2g\|^2 - \|R_1h\|^2 = 2(\langle R_1h, h \rangle - \langle R_2g, g \rangle) + (\|h\|^2 - \|g\|^2).
\end{equation}
From (2.10) we get that (2.9) holds if and only if
\[\langle (I + R_1)h, h \rangle \geq \langle (I + R_2)g, g \rangle,\]
and using again (2.8) we conclude that (2.7) is equivalent to
\begin{equation}
\langle \xi, (I + R_1)^{-1}\xi \rangle \geq \langle \xi, (I + R_2)^{-1}\xi \rangle.
\end{equation}
Since, in (2.11), \(\xi \in \mathcal{H}\) is arbitrary, we have proven that \((I + R_1)^{-1} \geq (I + R_2)^{-1}\), hence the direct implication in (2.6) is proven. The converse implication in (2.6) follows as well, since all implications from above are reversible.

\textbf{Definition 2.5 ([7])}. Suppose \((R_n)_{n \in \mathbb{N}}\) and \(R\) are selfadjoint operators on the same Hilbert space \(\mathcal{H}\). The sequence \((R_n)_{n \in \mathbb{N}}\) converges in the strong resolvent sense to \(R\) if for every \(\lambda \in \mathbb{C} \setminus \mathbb{R}\)
\[\lambda I - R_n)^{-1}\xi \xrightarrow[n \to \infty]{} (I - R)^{-1}\xi, \quad \xi \in \mathcal{H}.
\]

\textbf{Lemma 2.6}. With notation as in Lemma 2.2, the mapping \(\mathcal{T} \ni \tilde{T} \mapsto C^{-1}(\tilde{T}) \in \mathcal{S}\) defined at (2.4) is sequentially continuous when considering on \(\mathcal{T}\) the norm convergence and on \(\mathcal{S}\) the strong resolvent convergence.

\textit{Proof}. Suppose \((\tilde{T}_n)_{n \in \mathbb{N}}\) is a sequence of operators in \(\mathcal{T}\) such that \(\tilde{T}_n \to \tilde{T} \in \mathcal{T}\) as \(n \to \infty\). Since
\[\langle I + C^{-1}(\tilde{T}) \rangle^{-1} = \frac{1}{2}(I + \tilde{T}),\]
it follows that
\begin{equation}
(I + C^{-1}(\tilde{T}_n))^{-1}\xi \to (I + C^{-1}(\tilde{T}))^{-1}\xi, \quad \xi \in \mathcal{H}.
\end{equation}
For every \(h \in \text{Dom}(C^{-1}(\tilde{T}))\) let the sequence \((h_n)_{n \in \mathbb{N}}\), with elements in \(\text{Dom}(C^{-1}(\tilde{T}))\), be defined by
\[h_n = (I + C^{-1}(\tilde{T}_n))^{-1}(I + C^{-1}(\tilde{T}))h, \quad n \in \mathbb{N}.\]
By means of (2.12) we get
\begin{equation}
h_n \xrightarrow[n \to \infty]{} (I + C^{-1}(\tilde{T}))^{-1}(I + C^{-1}(\tilde{T}))h = h,
\end{equation}
and, moreover,
\begin{equation}
C^{-1}(\tilde{T}_n)h_n = C^{-1}(\tilde{T}_n)(I + C^{-1}(\tilde{T}_n))^{-1}(I + C^{-1}(\tilde{T}))h = h + C^{-1}(\tilde{T})h - h_n \xrightarrow[n \to \infty]{} C^{-1}(\tilde{T})h.
\end{equation}
From (2.13), (2.14), and [7], VIII. 26, it follows that the sequence \((C^{-1}(\tilde{T}_n))_{n \in \mathbb{N}}\) converges in the strong resolvent sense to \(C^{-1}(\tilde{T})\). \(\Box\)
3. The Main Theorem

Let $T: \text{Dom}(T) \to \mathcal{H}$ be a symmetric contraction with $\text{Dom}(T)$ a closed subspace of $\mathcal{H}$ and consider the set, cf. [6],

\[ \mathcal{B}(T) = \{ \tilde{T} \in \mathcal{L}(\mathcal{H}) \mid \tilde{T} \text{ is a selfadjoint contraction}, \tilde{T}|\text{Dom}(T) = T \}; \]

where $\mathcal{L}(\mathcal{H})$ denotes the algebra of all bounded linear operators $\mathcal{H} \to \mathcal{H}$. We present our argument for the fundamental result of M.G. Krein in [6].

**Theorem 3.1.** $\mathcal{B}(T) \neq \emptyset$ and there exist $\tilde{T}_{-1}$ and $\tilde{T}_1$ in $\mathcal{B}(T)$, with $\tilde{T}_{-1} \leq \tilde{T}_1$, such that, if $B \in \mathcal{L}(\mathcal{H})$ then

\[ B \in \mathcal{B}(T) \text{ if and only if } B = B^* \text{ and } \tilde{T}_{-1} \leq B \leq \tilde{T}_1. \]

**Proof.** The operator $T$ may be regarded as follows

\[ T: \text{Dom}(T) \to \mathcal{H} \oplus \text{Dom}(T), \]

hence, by means of Lemma 1.2, $T = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & X \end{bmatrix}$ with $A: \text{Dom}(T) \to \text{Dom}(T)$ a symmetric contraction and $\Gamma_2: \mathcal{D}_A \to \mathcal{H} \oplus \text{Dom}(T)$ a contraction, $A$ and $\Gamma_2$ being uniquely determined by $T$.

We search now for $\tilde{T} \in \mathcal{B}(T)$. Since $\tilde{T}$ is selfadjoint and $\tilde{T}|\text{Dom}(T) = T$, it must be of the following form

\[ \tilde{T} = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & X \end{bmatrix} \]

with $X: \mathcal{H} \oplus \text{Dom}(T) \to \mathcal{H} \oplus \text{Dom}(T)$ selfadjoint. Since $\tilde{T}$ must be a contraction as well, by Lemma 1.2 and (3.1), we get

\[ \tilde{T} = \tilde{T}(\Gamma) = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_2^* + D_{\Gamma_2^*} \Gamma D_{\Gamma_2^*} \end{bmatrix}, \]

with $\Gamma: \mathcal{D}_{\Gamma_2^*} \to \mathcal{D}_{\Gamma_2^*}$ a selfadjoint contraction. The existence of at least one $\Gamma$ (for instance, $\Gamma = 0$) proves that $\mathcal{B}(T) \neq \emptyset$. If $\mathcal{D}_{\Gamma_2^*} = \{0\}$ we take $\tilde{T}_{-1} = \tilde{T}_1 = \tilde{T}(0)$, that is, $\mathcal{B}(T)$ has a single element. If $\mathcal{D}_{\Gamma_2^*} \neq \{0\}$ define

\[ \tilde{T}_1 = \tilde{T}(I) = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & \Gamma_2^* + D_{\Gamma_2^*} \end{bmatrix} = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & I - \Gamma_2(I + A)^* \Gamma_2^* \end{bmatrix}, \]

\[ \tilde{T}_{-1} = \tilde{T}(-I) = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & -\Gamma_2 A^* \Gamma_2^* - D_{\Gamma_2^*} \end{bmatrix} = \begin{bmatrix} A & D_A \Gamma_2^* \\ \Gamma_2 D_A & \Gamma_2(I - A)^* \Gamma_2^* - I \end{bmatrix}. \]

It is clear that $\tilde{T}_{-1} \leq \tilde{T}(\Gamma) \leq \tilde{T}_1$ for any selfadjoint contraction $\Gamma: \mathcal{D}_{\Gamma_2^*} \to \mathcal{D}_{\Gamma_2^*}$.

Conversely, suppose $B \in \mathcal{L}(\mathcal{H})$ is selfadjoint and

\[ \tilde{T}_{-1} \leq B \leq \tilde{T}_1. \]
With respect to the decomposition $H = \text{Dom}(T) \oplus (\mathcal{H} \ominus \text{Dom}(T))$, where $C: \text{Dom}(T) \to \text{Dom}(T)$ is a selfadjoint contraction, $\Delta: D(C) \to \mathcal{H} \ominus \text{Dom}(T)$ is a contraction, and $\Gamma': D_{\Delta'} \to D_{\Delta'}$ is a selfadjoint contraction, all uniquely associated to $B$. If one makes use of (3.5) on $\text{Dom}(T)$ it follows that $C = A$. Now (3.5) can be written as

$$B - \Gamma = \begin{bmatrix} 0 & D_A \Delta - \Gamma^2 \Delta \end{bmatrix} \geq 0,$$

(3.6)

$$\tilde{T}_1 - B = \begin{bmatrix} 0 & D_A \Delta - \Gamma^2 \Delta \end{bmatrix} \geq 0.$$

(3.7)

At this point we have to recall that, for a given direct sum decomposition of $\mathcal{H}$, if one considers the operator $\begin{bmatrix} 0 & M \end{bmatrix}$ then

$$\begin{bmatrix} 0 & M \end{bmatrix} = 0 \text{ if and only if } M = 0 \text{ and } N \geq 0.$$

(3.8)

If one applies (3.8) to (3.6) and (3.7) it follows that $\Delta = \Gamma_2$ and, since (3.2) is proven to be the general form of operators from $\mathcal{B}(T)$, we conclude that $B \in \mathcal{B}(T)$.

For a given symmetric contraction $T$, with notation as in (3.2), we consider the set

$$\mathcal{C}(T) = \{ \Gamma \in \mathcal{L}(D_{\Gamma_2}) \mid \Gamma \text{ is a selfadjoint contraction} \}.$$  

(3.9)

**Proposition 3.2.** Given $T$ a symmetric contraction, with notation as in (3.2), the mapping $\mathcal{C}(T) \ni \Gamma \mapsto \tilde{T}(\Gamma) \in \mathcal{B}(T)$ as in (3.2) is continuous, where $\mathcal{C}(T)$ and $\mathcal{B}(T)$ carry the operator norm topologies, and such that, for any $\Gamma', \Gamma'' \in \mathcal{C}(T)$,

$$\Gamma' \leq \Gamma'' \text{ if and only if } \tilde{T}(\Gamma') \leq \tilde{T}(\Gamma'').$$

**Proof.** If $\Gamma', \Gamma'' \in \mathcal{C}(T)$ then

$$\tilde{T}(\Gamma') - \tilde{T}(\Gamma'') = \begin{bmatrix} 0 & 0 & D_{\Gamma_2}(\Gamma' - \Gamma'') \end{bmatrix},$$

(3.10)

and, considering the natural order relation for bounded selfadjoint operators, by (3.8) it now follows that $\Gamma' \leq \Gamma''$ if and only if $\tilde{T}(\Gamma') \leq \tilde{T}(\Gamma'')$.

From (3.10) we get

$$\|\tilde{T}(\Gamma') - \tilde{T}(\Gamma'')\| = \|D_{\Gamma_2}(\Gamma' - \Gamma'')\|,$$

(3.11)

hence the continuity of the mapping $\mathcal{C}(T) \ni \Gamma \mapsto \tilde{T}(\Gamma) \in \mathcal{B}(T)$ is clear. \hfill $\square$

Consider now a positive, densely defined, and closed operator $S$ in $\mathcal{H}$ and let $T = C(S)$ defined as in (2.3). We associate to $S$ the sets $\mathcal{S}$ and $\mathcal{T}$ as in Lemma 2.2 and clearly $\mathcal{T} = \mathcal{B}(T)$. By means of Lemma 2.2 and the proof of Theorem 3.1 we have obtained a bijective mapping $\mathcal{C}(T) \ni \Gamma \mapsto C^{-1}(\tilde{T}(\Gamma)) \in \mathcal{S}$.  

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On $\mathcal{C}(T)$ we consider the natural order relation for bounded selfadjoint operators and the norm topology and on $\mathcal{S}$ we consider the order relation as in Definition 2.5 and the strong resolvent convergence. From what we have proven until now and the considerations from the preceding section we get

**Theorem 3.3.** Given a positive, densely defined, and closed operator $S$ in $\mathcal{H}$, with notation as before, the bijective mapping $\mathcal{C}(T) \ni \Gamma \mapsto C^{-1}(\tilde{T}(\Gamma)) = \tilde{S}(\Gamma) \in \mathcal{S}$ is sequentially continuous and non-increasing, that is, for any $\Gamma', \Gamma'' \in \mathcal{C}(T)$ we have

$$\Gamma' \leq \Gamma'' \text{ if and only if } \tilde{S}(\Gamma') \geq \tilde{S}(\Gamma'').$$

Moreover, there exist two positive selfadjoint extensions of $S$, $\tilde{S}_K = \tilde{S}(I) \leq \tilde{S}(-I) = \tilde{S}_F$ such that, a positive selfadjoint operator $R$ on $\mathcal{H}$ belongs to $\mathcal{S}$ if and only if $\tilde{S}_K \leq R \leq \tilde{S}_F$.

Consider $\Gamma \in \mathcal{C}(T)$. Then $\text{Dom}(\tilde{S}(\Gamma)) = \text{Ran}(I + \tilde{T}(\Gamma))$. Since $\tilde{T}(\Gamma)$ as in (3.2) is defined as a $2 \times 2$ block matrix corresponding to the decomposition $\mathcal{H} = \text{Ran}(I + S) \oplus \text{Ker}(I + S^*)$ and $\text{Dom}(S) = \text{Ran}(I + T)$, we get

$$\text{Dom}(\tilde{S}(\Gamma)) = \text{Dom}(S) + \left[ \frac{D_A \Gamma_2^*}{I - \Gamma_2 A \Gamma_2^* + D_{\Gamma_2} T_{\Gamma_2}} \right] \text{Ker}(I + S^*),$$

and for the moment this is all we can say about $\text{Dom}(\tilde{S}(\Gamma))$. The next section will improve the above formula, see Proposition 4.3.

### 4. Special Semibounded Selfadjoint Extensions

We have obtained in Theorem 3.1 two remarkable positive selfadjoint extensions of $S$, $\tilde{S}_K = \tilde{S}(I)$ and $\tilde{S}_F = \tilde{S}(-I)$. According to the general theory, [1], [5], $\tilde{S}_F$ must be the Friedrichs extension. $\tilde{S}_K$ was called in [1] the Kreın extension.

Let us denote by $F$ the Friedrichs extension of a positive, densely defined, closed operator $S$ in $\mathcal{H}$. Then, [5], [7],

$$\text{Dom}(F) = \{ \xi \in \text{Dom}(S^*) \mid \text{there exists } (\xi_n)_{n \in \mathbb{N}} \text{ in } \text{Dom}(S), \xi_n \xrightarrow{\|\cdot\|} \xi \text{ and } \langle (\xi_n - \xi_m), S(\xi_n - \xi_m) \rangle \xrightarrow{m,n \to \infty} 0 \},$$

and

$$F\xi = S^*\xi, \quad \xi \in \text{Dom}(F).$$

As before, we consider

$$\text{C}(S) = T = \left[ \begin{array}{c} A \\ \Gamma_2 D_A \end{array} \right]: \text{Dom}(T) \to \text{Dom}(T) \oplus \mathcal{H} \ominus \text{Dom}(T),$$

where $A: \text{Dom}(T) \to \text{Dom}(T)$ is a symmetric contraction and $\Gamma_2: \mathcal{D}_A \to \mathcal{H} \ominus \text{Dom}(T)$ is a contraction, uniquely determined by $T$ and hence by $S$.

**Lemma 4.1.** $\text{Ran}(I + A) \subseteq \text{Ran}(I + T)^* \subseteq \text{Ran}((I + A)^{1/2})$, where the identity operator $I: \text{Dom}(T) \to \text{Dom}(T)$ is identified with the embedding operator $: \text{Dom}(T) \to \text{Dom}(T) \subseteq \mathcal{H} = \text{Dom}(T) \oplus (\mathcal{H} \ominus \text{Dom}(T)).$
Proof. From \((4.3)\) it follows
\[
(4.4) \quad I + T = \begin{bmatrix} I + A & \Gamma_2 D_A \end{bmatrix}, \quad (I + T)^* = [I + A \ D_A \Gamma_2^*].
\]
Therefore, \(\text{Ran}(I + A) \subset \text{Ran}(I + T)^*\) and, one the other hand,
\[
(I + T)^*(I + T) = (I + A)^2 + D_A\Gamma_2\Gamma_2 D_A
\leq (I + A)^2 + \Gamma_2^2 = (I + A)^2 + (I - A^2)
= 2(I + A) = 2(I + A)^{1/2}(I + A)^{1/2}.
\]
Making use of Theorem 1 in \([4]\) we get \(\text{Ran}(I + T)^* \subset \text{Ran}(I + A)^{1/2}\). \(\square\)

Lemma 4.2. Suppose \(\xi \in \text{Dom}(S^*)\). Then \(\xi \in \text{Dom}(F)\) if and only if there exists a sequence \((\eta_n)_{n \in \mathbb{N}}\) of vectors in \(\text{Dom}(T) = \text{Ran}(I + S)\) such that
\[
(I + T)\eta_n \xrightarrow{n \to \infty} \xi \quad \text{and} \quad (I + A)^{1/2}(\eta_n - \eta_m) \xrightarrow{m,n \to \infty} 0.
\]

Proof. By means of \((4.1)\) and using \(\text{Dom}(S) = \text{Ran}(I + T)\) and \(\text{Dom}(T) = \text{Ran}(I + S)\), an arbitrary vector \(\xi \in \text{Dom}(S^*)\) belongs to \(\text{Dom}(F)\) if and only if there exists a sequence \((\eta_n)_{n \in \mathbb{N}}\) of vectors in \(\text{Ran}(I + S)\) such that
\[
(4.5) \quad (I + T)\eta_n \xrightarrow{n \to \infty} \xi,
\]
and
\[
(4.6) \quad \langle (I + T)(\eta_n - \eta_m), (I - T)(\eta_n - \eta_m) \rangle \xrightarrow{m,n \to \infty} 0.
\]
Let us observe that \((4.5)\) yields
\[
(4.7) \quad \langle (I + T)(\eta_n - \eta_m), (I + T)(\eta_n - \eta_m) \rangle \xrightarrow{m,n \to \infty} 0.
\]
Finally, letting \(P\) denote the orthogonal projection of \(\mathcal{H}\) onto \(\text{Dom}(T)\),
\[
\| (I + A)^{1/2}(\eta_n - \eta_m) \|^2 = \langle (I + A)(\eta_n - \eta_m), (\eta_n - \eta_m) \rangle
= \langle P(I + T)(\eta_n - \eta_m), (\eta_n - \eta_m) \rangle
= \langle (I + T)(\eta_n - \eta_m), P(\eta_n - \eta_m) \rangle
= \langle (I + T)(\eta_n - \eta_m), (\eta_n - \eta_m) \rangle \xrightarrow{m,n \to \infty} 0,
\]
where the convergence is obtained by adding the quantities in \((4.6)\) and \((4.7)\). We have proven that \((I + A)^{1/2}(\eta_n - \eta_m) \xrightarrow{m,n \to \infty} 0\). \(\square\)

Proposition 4.3. \(\tilde{S}_F = F\).

Proof. Since both operators \(\tilde{S}_F\) and \(F\) are selfadjoint extensions of \(S\), hence maximal symmetric, it is sufficient to prove \(\text{Dom}(\tilde{S}_F) \subset \text{Dom}(F)\).

To this end, let \(\xi \in \text{Dom}(\tilde{S}_F) = \text{Ran}(I + \tilde{T}(-I))\). There exists \(\eta \in \mathcal{H}\) such that \(\xi = (I + \tilde{T}(-I))\eta\) and, since \(I + T\) is one-to-one it follows that \(\text{Ran}(I + T)^*\) is dense in \(\text{Dom}(T)\),
hence, by Lemma 4.1, Ran((I + A)^{1/2}) is dense in Dom(T). Again by Lemma 4.1 there exists a sequence \((\eta_n)_{n \in \mathbb{N}}\) of vectors in Dom(T) such that

\[
(I + A)^{1/2} \eta_n \xrightarrow{n \to \infty} \langle I + A \rangle^{-1/2}(I + T)^* \eta,
\]

hence

\[
(I + A) \eta_n \xrightarrow{n \to \infty} (I + T)^* \eta.
\]

Applying the bounded operator \((I - A)^{1/2}\) to (4.8) we get

\[
D_A \eta_n = (I - A)^{1/2}(I + A)^{-1/2} \eta_n \xrightarrow{n \to \infty} (I - A)^{1/2}(I + A)^{-1/2}(I + T)^* \eta,
\]

and, since, considering \(P\) the orthogonal projection of \(\mathcal{H}\) onto Dom(T) and using (4.4), we have

\[
(I - A)^{1/2}(I + A)^{-1/2}(I + T)^* \eta = (I - A)^{1/2}(I + A)^{-1/2}((I + A)P \eta + (I - A^2)^{1/2} \Gamma_2^*(I - P) \eta)
\]

\[
= (I - A)^{1/2}(I + A)^{1/2} P \eta + (I - A) \Gamma_2^*(I - P) \eta
\]

\[
= D_A P \eta + (I - A) \Gamma_2^*(I - P) \eta,
\]

it follows that

\[
(I + T) \eta_n = \left[ \begin{array}{c} (I + A) \eta_n \\ \Gamma_2 D_A \eta_n \end{array} \right] \xrightarrow{n \to \infty} \left[ \begin{array}{c} (I + A) P \eta + D_A \Gamma_2^*(I - P) \eta \\ \Gamma_2 D_A P \eta + \Gamma_2(I - A) \Gamma_2^*(I - P) \eta \end{array} \right] = (I + \tilde{T}(-I)) \eta = \xi.
\]

Finally, from (4.8), (4.11), and Lemma 4.2 we obtain \(\xi \in \text{Dom}(F)\).

As a consequence we can determine the domain of an arbitrary positive selfadjoint extension \(\tilde{S}(\Gamma)\) in terms of the domain of the Friedrichs extension and the parameter \(\Gamma\).

**Proposition 4.4.** For every \(\Gamma \in \mathcal{C}(T)\) we have

\[
\text{Dom}(\tilde{S}(\Gamma)) = \text{Dom}(F) + D_{\Gamma_2^*} (I + \Gamma) D_{\Gamma_2} \text{Ker}(I + S^*).
\]

**Proof.** For arbitrary \(\Gamma \in \mathcal{C}(T)\), by (3.2) and (3.4), we have

\[
I + \tilde{T}(\Gamma) = \left[ \begin{array}{cc} I + A & D_A \Gamma_2^* \\ \Gamma_2 D_A & \Gamma_2(I - A)^* \Gamma_2^* + D_{\Gamma_2^*} (I + \Gamma) D_{\Gamma_2} \end{array} \right]
\]

\[
= I + \tilde{T}(-I) + \left[ \begin{array}{cc} 0 & 0 \\ 0 & D_{\Gamma_2^*} (I + \Gamma) D_{\Gamma_2} \end{array} \right],
\]

hence, from Proposition 4.3 we get

\[
\text{Dom}(\tilde{S}(\Gamma)) = \text{Ran}(I + \tilde{T}(\Gamma))
\]

\[
= \text{Ran}(I + \tilde{T}(-I) + D_{\Gamma_2^*} (I + \Gamma) D_{\Gamma_2} (\mathcal{H} \ominus \text{Dom}(T))
\]

\[
= \text{Dom}(F) + D_{\Gamma_2^*} (I + \Gamma) D_{\Gamma_2} \text{Ker}(I + S^*). \quad \square
\]
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