Wandering in five-dimensional curved superspace

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\begin{abstract}
This is a brief review of the superspace formulation for five-dimensional $\mathcal{N} = 1$ matter-coupled supergravity recently developed by the authors.
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1 Introduction

Historically, the first attempt to formulate five-dimensional $\mathcal{N} = 1$ (often called $\mathcal{N} = 2$) supergravity in an off-shell superspace setting was made in [1] shortly before its on-shell component formulation was given [2, 3]. Inspired by [2], Howe [4] (see also [5]) proposed a superspace formulation for the minimal multiplet of 5D $\mathcal{N} = 1$ supergravity (“minimal” in the sense of superconformal tensor calculus). After Howe’s work [4], 5D $\mathcal{N} = 1$ curved superspace has been abandoned for 25 years. General matter couplings in 5D $\mathcal{N} = 1$ supergravity have been constructed within on-shell components approaches [6, 7, 8] and within the superconformal tensor calculus [9, 10].

In 2007, we began the program of developing a superspace formulation for 5D $\mathcal{N} = 1$ matter-coupled supergravity. We first elaborated supersymmetric field theory in 5D $\mathcal{N} = 1$ anti-de Sitter superspace which is a maximally symmetric curved background [11]. This was followed by a fully-fledged supergravity formalism developed in a series of papers [12, 13, 14]. In these publications, we not only reproduced the main results of the superconformal tensor approach [9, 10], but also proposed new off-shell supermultiplets and more general supergravity-matter systems. The present note is a brief review of our construction.

Looking back at the 25 year history of 5D $\mathcal{N} = 1$ curved superspace, one can notice a striking historical curiosity. In 1982, Howe had the right superspace setting for pure supergravity – the minimal multiplet [4], which was the starting point of our approach [12, 13]. The same multiplet also occurs within the superconformal tensor calculus [9, 10] by coupling the Weyl multiplet to an Abelian vector multiplet and then gauge fixing some local symmetries (the vector multiplet is one of two compensators required to describe Poincaré supergravity). So why didn’t Howe make use of his formulation to construct Poincaré supergravity and its matter couplings? A partial answer is quite simple. Even in rigid supersymmetry with eight supercharges in diverse dimensions, adequate approaches to generate off-shell supermultiplets and supersymmetric actions appeared only in 1984. They go by the names harmonic superspace [15, 16] and projective superspace [17, 18].

This note is organized as follows. In section 2 we review, following [14], the superspace formulation for the Weyl multiplet of conformal supergravity. Covariant projective supermultiplets and the supersymmetric action principle are introduced in section 3. The same section also contains a few examples of interesting dynamical systems.
2 5D conformal supergravity in superspace

We start by describing the superspace formulation for 5D conformal supergravity \cite{14}. Let \( z^\mathcal{M} = (x^\mathcal{M}, \theta^i) \) be local bosonic \((x)\) and fermionic \((\theta)\) coordinates parametrizing a curved five-dimensional \( \mathcal{N} = 1 \) superspace \( \mathcal{M}^{5|8} \) (\( m = 0, 1, \ldots, 4, \mu = 1, \ldots, 4, \) and \( i = 1, 2 \)). The Grassmann variables \( \theta_i^\mu \) obey the 5D pseudo-Majorana reality condition \( \theta_i^\mu = \theta_i^{\bar{\mu}} = \varepsilon_{\bar{\mu}\bar{\nu}}\varepsilon^{ij}\theta_j^\nu \). The tangent-space group is chosen to be \( \text{SO}(4,1) \times \text{SU}(2) \), and the superspace covariant derivatives \( \mathcal{D}_{\hat{A}} = (\mathcal{D}_{\hat{a}}, \mathcal{D}_{\hat{A}}) \) have the form

\[
\mathcal{D}_{\hat{A}} = E_{\hat{A}} + \frac{1}{2} \Omega_{\hat{A}}^{\hat{b}\hat{c}}(z) M_{\hat{b}\hat{c}} + \Phi_{\hat{A}}^{kl}(z) J_{kl} .
\]

Here \( E_{\hat{A}} = E_{\hat{A}}^\hat{M}(z) \partial_{\hat{M}} \) is the supervielbein, with \( \partial_{\hat{M}} = \partial / \partial z^\mathcal{M} \); \( M_{\hat{b}\hat{c}} \) and \( \Omega_{\hat{A}}^{\hat{b}\hat{c}} \) are the Lorentz generators and connection respectively (both antisymmetric in \( \hat{b}, \hat{c} \)); \( J_{kl} \) and \( \Phi_{\hat{A}}^{kl} \) are respectively the \( \text{SU}(2) \) generator and connection (symmetric in \( k, l \)). The generators of \( \text{SO}(4,1) \times \text{SU}(2) \) act on the covariant derivatives as follows:\footnote{The operation of \((\text{anti})\)symmetrization of \( n \) indices is defined to involve a factor \((n!)^{-1}\).}

\[
[M_{\hat{a}\hat{\beta}}, \mathcal{D}_{\hat{\beta}}] = \varepsilon_{\hat{a}(\hat{\alpha}} \mathcal{D}_{\hat{\beta})}^{\hat{\beta},} , \quad [M_{\hat{a}\hat{b}}, \mathcal{D}_{\hat{c}}] = 2\eta_{\hat{a}\hat{b}} \mathcal{D}_{\hat{c}} , \quad [J^{kl}, \mathcal{D}_{\hat{a}}] = \varepsilon^{k(\hat{\alpha} \mathcal{D}_{\hat{a}})}_{\hat{a}} ,
\]

where \( J^{kl} = \varepsilon^{k(\hat{\alpha} \mathcal{D}_{\hat{a}})}_{\hat{a}} \), and \( M_{\hat{a}\hat{\beta}} = M_{\hat{a}\hat{\beta}} = (\Sigma_{\hat{a}\hat{b}})_{\hat{a} \hat{\beta}} M_{\hat{a}\hat{\beta}} \) and \( (\Sigma_{\hat{a}\hat{b}})_{\hat{a} \hat{\beta}} \) are the spinor Lorentz generators, \( \Sigma_{\hat{a}\hat{b}} = -\frac{1}{2} [\Gamma^{\hat{a}}, \Gamma^{\hat{b}}] \), with \( \Gamma^{\hat{a}} \) the 5D Dirac matrices (see the appendix in \cite{13} for our notation and conventions).

The supergravity gauge group is generated by local transformations of the form

\[
\delta_K \mathcal{D}_{\hat{A}} = [K, \mathcal{D}_{\hat{A}}] , \quad \delta_K U = K U , \quad K = K^\mathcal{C}(z) \mathcal{D}_{\mathcal{C}} + \frac{1}{2} K^\mathcal{C\mathcal{D}}(z) M_{\mathcal{C}\mathcal{D}} + K^{kl}(z) J_{kl} ,
\]

with all the gauge parameters obeying natural reality and symmetry conditions, and otherwise arbitrary. In \( \text{(3)} \) we have also included the transformation rule for a tensor superfield \( U(z) \), with its indices suppressed.

The covariant derivatives obey (anti)commutation relations of the general form

\[
[\mathcal{D}_{\hat{A}}, \mathcal{D}_{\hat{B}}] = T_{\hat{A}\hat{B}}^{\mathcal{C}} \mathcal{D}_{\mathcal{C}} + \frac{1}{2} R_{\hat{A}\hat{B}}^{\mathcal{C}\mathcal{D}} M_{\mathcal{C}\mathcal{D}} + R_{\hat{A}\hat{B}}^{kl} J_{kl} ,
\]

where \( T_{\hat{A}\hat{B}}^{\mathcal{C}} \) is the torsion, and \( R_{\hat{A}\hat{B}}^{\mathcal{C}\mathcal{D}} \) and \( R_{\hat{A}\hat{B}}^{kl} \) are the \( \text{SO}(4,1) \) and \( \text{SU}(2) \) curvature tensors, respectively.

To describe the Weyl multiplet of conformal supergravity \cite{9,10}, the torsion has to be constrained as \cite{14}:

\[
T_{\hat{a}\hat{\beta}}^{ij \hat{c}} = -2i \varepsilon^{ij} (\Gamma_{\hat{a}}^{\hat{\beta}})_{\hat{a}\hat{\beta}} , \quad T_{\hat{a}\hat{\beta}}^{ij \hat{c}} = T_{\hat{a}\hat{\beta}}^{ij \hat{c}} = 0 , \quad T_{\hat{a}\hat{b}}^{\hat{c}} = T_{\hat{a}3(j^k \hat{b}}}^{\hat{c}} = 0 .
\]
With these constraints, it can be shown that the torsion and curvature tensors are expressed in terms of four dimension-1 tensor superfields \( S^{ij}, C_{\hat{a}ij}, X_{\hat{a}b}, \) and \( N_{\hat{a}b}, \) and their covariant derivatives. The superfields \( S^{ij}, C_{\hat{a}ij} \) are symmetric in \( i, j, \) while \( X_{\hat{a}b}, N_{\hat{a}b} \) are antisymmetric in \( \hat{a}, \hat{b}. \) All these tensors are real \( S^{ij} = S_{ij}, C_{\hat{a}ij} = C_{\hat{a}ij}, X_{\hat{a}b} = X_{\hat{a}b}, N_{\hat{a}b} = N_{\hat{a}b}. \)

The covariant derivatives obey the (anti)commutation relations \([14]\):

\[
\{D^i_\alpha, D^j_\beta\} = -2i \varepsilon^{ij}\delta_\alpha \delta_\beta - i \varepsilon_\alpha \varepsilon^{ij} X^{\hat{c}\hat{d}} M_{\hat{c}\hat{d}} + \frac{i}{4} \varepsilon^{ij} \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} (\Gamma_\hat{a})_{\hat{a}\hat{b}} N_{\hat{c}\hat{d}} M_{\hat{c}\hat{d}} \\
- \frac{i}{2} \varepsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} (\Sigma_{\hat{a}\hat{b}})_{\hat{a}\hat{b}} C_{\hat{c}\hat{d}j} M_{\hat{c}\hat{d}} + 4i S^{ij} M_{\hat{a}\hat{d}} + 3i \varepsilon_\alpha \varepsilon^{ij} S^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} \\
- i \varepsilon^{ij} C_{\hat{a}\hat{b}\hat{c}} J_{\hat{a}\hat{b}} - 4i (X_{\hat{a}\hat{b}} + N_{\hat{a}\hat{b}}) J^{ij} ,
\]

\[(D_\hat{a}, D^j_\beta) = \frac{1}{2} (\Gamma_\hat{a})^{\hat{a}\hat{b}\hat{c}} (\Sigma_{\hat{a}\hat{b}})_{\hat{a}\hat{b}} C_{\hat{c}\hat{d}j} M_{\hat{c}\hat{d}} + 4i S^{ij} M_{\hat{a}\hat{d}} + 3i \varepsilon_\alpha \varepsilon^{ij} S^{\hat{a}\hat{b}} J_{\hat{a}\hat{b}} + (\Sigma_{\hat{a}\hat{b}})_{\hat{a}\hat{b}} C_{\hat{c}\hat{d}j} M_{\hat{c}\hat{d}} ,
\]

The dimension-1 components of the torsion, \( S^{ij}, X_{\hat{a}b}, N_{\hat{a}b} \) and \( C_{\hat{a}ij}, \) obey some differential constraints implied by the Bianchi identities \([14]\).

The fact that the supergeometry introduced corresponds to 5D conformal supergravity, manifests itself in the invariance of the constraints \([5]\) under infinitesimal super-Weyl transformations of the form\(^2\)

\[
\delta_\sigma D^i_\alpha = \sigma D^i_\alpha + 4(D^{\hat{a}\hat{b}}_\alpha) M_{\hat{a}\hat{b}} - 6 (D_{\hat{a}\hat{b}} \sigma) J^{k\hat{i}},
\]

\[
\delta_\sigma D_{\hat{a}} = 2\sigma D_{\hat{a}} + i(\Gamma_\hat{a})^{\hat{a}\hat{b}\hat{c}} (D^{k\hat{b}}_\alpha \sigma) D_{\hat{c}k} - 2(D^{k\hat{b}}_\sigma) M_{\hat{a}\hat{b}} + \frac{i}{4} (\Gamma_\hat{a})^{\hat{a}\hat{b}\hat{c}} M_{\hat{a}\hat{b}} D_{\hat{a}} D_{\hat{b}} \sigma) J_{\hat{a}\hat{b}} ,
\]

where the scalar superfield \( \sigma \) is real and unconstrained. The components of the dimension-1 torsion can be seen to transform as follows:

\[
\delta_\sigma S^{ij} = 2\sigma S^{ij}, \quad \delta_\sigma C_{\hat{a}ij} = 2\sigma C_{\hat{a}ij} + i (\Gamma_{\hat{a}})^{\hat{a}\hat{b}\hat{c}} (D^{k\hat{b}}_\alpha D^{j\hat{c}}_\sigma) M_{\hat{a}\hat{b}} ,
\]

\[
\delta_\sigma X_{\hat{a}b} = 2\sigma X_{\hat{a}b} - i (\Sigma_{\hat{a}\hat{b}})^{\hat{a}\hat{b}\hat{c}} D_{\hat{a}} D_{\hat{b}k} \sigma , \quad \delta_\sigma N_{\hat{a}b} = 2\sigma N_{\hat{a}b} - i (\Sigma_{\hat{a}\hat{b}})^{\hat{a}\hat{b}\hat{c}} D_{\hat{a}} D_{\hat{b}k} \sigma .
\]

It follows from here that \( W_{\hat{a}b} := X_{\hat{a}b} - \frac{1}{2} N_{\hat{a}b} \) transforms homogeneously,

\[
\delta_\sigma W_{\hat{a}b} = 2\sigma W_{\hat{a}b} .
\]

Therefore, \( W_{\hat{a}b} \) is a superspace generalization of the Weyl tensor.

\(^2\)The finite form for the super-Weyl transformations has been given in \([19]\).
It turns out that the super-Weyl transformations can be used to gauge away the superfield $C_{a}^{ij}$. Imposing the super-Weyl gauge condition

$$C_{a}^{ij} = 0,$$

(10)
is equivalent to extending the set of constraints by an additional dimension-1 constraint which is $T_{a(\hat{\beta}\hat{\gamma})} = 0$. The resulting superspace geometry provides an alternative description of the Weyl multiplet. Because of (10), the full set of constraints is now invariant under the super-Weyl transformations (7a)–(7b) generated by a constrained parameter $\sigma$. The corresponding constraint is

$$\mathcal{D}_{\hat{a}}^{(i} \mathcal{D}_{\hat{\beta}}^{j)} \sigma - \frac{1}{4} \varepsilon_{\hat{a}\hat{b}} \mathcal{D}_{\hat{\gamma}}^{(i} \mathcal{D}_{\hat{\gamma}}^{j)} \sigma = 0.$$

(11)

Another consequence of (10) in conjunction with the Bianchi identities is that $S^{ij}$ satisfies the equation

$$\mathcal{D}_{i}^{(j} S^{jk)} = 0.$$

(12)

If not specifically mentioned, eq. (10) will be assumed in what follows.

The Weyl multiplet can naturally be coupled to a non-Abelian vector multiplet. This is achieved by introducing gauge-covariant derivatives $\mathcal{D}_{\hat{A}} = \mathcal{D}_{\hat{A}} + \mathcal{V}_{\hat{A}}(z)$, with $\mathcal{V}_{\hat{A}}$ a gauge connection taking its values in the Lie algebra of the gauge group. Then the algebra (4) turns into

$$[\mathcal{D}_{\hat{A}}, \mathcal{D}_{\hat{B}}] = T_{\hat{A}\hat{B}} \mathcal{C}_{\hat{C}} + \frac{1}{2} R_{\hat{A}\hat{B}\hat{C}\hat{D}} M_{\hat{C}\hat{D}} + R_{\hat{A}\hat{B}}^{\hat{C}\hat{D}} J_{\hat{C}\hat{D}} + \mathcal{F}_{\hat{A}\hat{B}}.$$

(13)

An irreducible off-shell vector multiplet emerges if $\mathcal{F}_{\hat{A}\hat{B}}$ is constrained as $\mathcal{F}_{\hat{A}\hat{B}}^{ij} \propto \varepsilon^{ij} \varepsilon_{\hat{a}\hat{b}} \mathcal{W}$ (compare with [5]). The field strength $\mathcal{W}$ possesses the super-Weyl transformation $\delta_{\sigma} \mathcal{W} = 2\sigma \mathcal{W}$ and obeys the following Bianchi identity:

$$\mathcal{D}_{\hat{a}}^{(i} \mathcal{D}_{\hat{\beta}}^{j)} \mathcal{W} - \frac{1}{4} \varepsilon_{\hat{a}\hat{b}} \mathcal{D}_{\hat{\gamma}}^{(i} \mathcal{D}_{\hat{\gamma}}^{j)} \mathcal{W} = 0.$$

(14)

Associated with the vector multiplet is the composite superfield

$$G^{ij} := \text{tr} \left\{ i \mathcal{D}_{\hat{a}}^{(i} \mathcal{W} \mathcal{D}_{\hat{a}}^{j)} \mathcal{W} + \frac{i}{2} \mathcal{W} \mathcal{D}_{\hat{a}}^{ij} \mathcal{W} - 2 S^{ij} \mathcal{W}^{2} \right\}, \quad \mathcal{D}_{i}^{(j} G^{jk)} = 0.$$

(15)

It is characterized by the following fundamental properties:

$$\mathcal{D}_{\hat{a}}^{(i} G^{jk)} = 0, \quad \delta_{\sigma} G^{ij} = 6\sigma G^{ij}.$$

(16)

Let $\mathcal{W} = W Z$, with $Z$ the generator, be the field strength of an Abelian vector multiplet. Then, eq. (14) coincides in form with the constraint (11) obeyed by the super-Weyl parameter. If the vector multiplet is characterized by $W(z) \neq 0$ everywhere in superspace, super-Weyl transformations can be used to impose the gauge $W = 1$. The resulting geometry (13) describes the minimal multiplet of 5D supergravity [4].

4
3 Kinematics and dynamics in curved projective superspace

We have reviewed the geometric description of 5D conformal supergravity in superspace. Let us now turn to a brief discussion of a large family of off-shell supermultiplets coupled to conformal supergravity, which can be used to describe supersymmetric matter. They were introduced in [14] under the name covariant projective supermultiplets. These supermultiplets are a curved-superspace extension of the 5D superconformal projective multiplets [20]. The latter are ordinary projective supermultiplets [18] with respect to the super-Poincaré subgroup of the 5D superconformal group.

It is useful to introduce auxiliary isotwistor coordinates $u_{\hat{i}}^+ \in \mathbb{C}^2 \setminus \{0\}$ in addition to the superspace coordinates $z^\hat{M} = (x^\hat{m}, \theta^\hat{\mu})$. All the coordinates $u_{\hat{i}}^+$ and $z^\hat{M}$ are defined to be inert under the tangent-space group. In particular, the variables $u_{\hat{i}}^+$ do not transform under the local SU(2) group, and hence they are covariantly constant, $D_{\hat{A}} u_{\hat{i}}^+ = 0$. It follows from (6a) that the operators $D_{\hat{A}}^\pm := u_{\hat{i}}^+ D_{\hat{A}}^i$ obey the following algebra (the constraint (10) is not assumed from here until eq. (21) including):

$$\{D_{\hat{A}}^+, D_{\hat{B}}^+\} = -4i \left( X_{\hat{A}\hat{B}} + N_{\hat{A}\hat{B}} \right) J^{++} + 4i S^{++} M_{\hat{A}\hat{B}} - \frac{i}{2} \epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}} (\Sigma_{\hat{a}\hat{b}})_{\hat{A}\hat{B}} C_{\hat{c}\hat{d} + +} M_{\hat{e}} ,$$

where $J^{++} := u_{\hat{i}}^+ u_{\hat{j}}^+ J^{ij}$ and $S^{++} := u_{\hat{i}}^+ u_{\hat{j}}^+ S^{ij}$.

A covariant projective supermultiplet of weight $n$, $Q^{(n)}(z, u^+)$, is defined to be a scalar superfield that lives on $M^{5|8}$, is holomorphic with respect to the isotwistor variables $u_{\hat{i}}^+$ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and is characterized by the following conditions:

(i) it obeys the covariant analyticity constraint

$$D_{\hat{A}}^+ Q^{(n)} = 0 ;$$  \hspace{1cm} (18)

(ii) it is a homogeneous function of $u^+$ of degree $n$, that is,

$$Q^{(n)}(z, cu^+) = c^n Q^{(n)}(z, u^+) , \hspace{1cm} c \in \mathbb{C} \setminus \{0\} ;$$  \hspace{1cm} (19)

(iii) infinitesimal gauge transformations (3) act on $Q^{(n)}$ as follows:

$$\delta_K Q^{(n)} = \left( K^\hat{C} D_{\hat{C}} + K^{ij} J_{ij} \right) Q^{(n)} ,$$

$$K^{ij} J_{ij} Q^{(n)} = -\frac{1}{(u^+ u^-)} \left( K^{++} D^{--} - n K^{+-} \right) Q^{(n)} , \hspace{1cm} K^{++} = K^{ij} u^+_{i} u^-_{j} ,$$  \hspace{1cm} (20)

where $D^{--} = u^{-i} \partial / \partial u^{+i}$. The right-hand side in (20) involves an additional isotwistor, $u^-_{i}$ which is subject to the condition $(u^+ u^-) = u^+ u^- \neq 0$, and is otherwise arbitrary. By
construction, $Q^{(n)}$ is independent of $u^-$, i.e. $\partial Q^{(n)}/\partial u^- = 0$. One can see that $\delta Q^{(n)}$ is also independent of the isotwistor $u^-$, that is $\partial(\delta Q^{(n)})/\partial u^- = 0$, due to (19). It follows from (20) that $J^{++} Q^{(n)} \equiv 0$ which is the integrability condition for the constraint (18). It is important to note that, because of (ii), the isotwistor $u_i^+$ plays the role of homogeneous global coordinates for $\mathbb{C}P^1$ and the covariant projective multiplets live in curved projective superspace $\mathcal{M}^{5|8} \times \mathbb{C}P^1$.

In the case of conformal supergravity, we have to address the issue of how covariant projective multiplets may consistently vary under the super-Weyl transformations. If a weight-$n$ projective superfield $Q^{(n)}$ is chosen to transforms homogeneously, $\delta_\sigma Q^{(n)} \propto \sigma Q^{(n)}$, then its transformation law turns out to be uniquely fixed by the constraint (18) to be

$$\delta_\sigma Q^{(n)} = 3n \sigma Q^{(n)} . \quad (21)$$

Without the assumption of homogeneity, it is easy to construct examples of covariant projective multiplets which do not respect (21). The superfield $S^{++}$ is a particularly important example. Due to eq. (12) (from here on we only consider the geometry with $C^{i,j} = 0$), $S^{++}$ is a projective superfield of weight two, $D_+^\dot{\alpha} S^{++} = 0$. In accordance with (8a), its super-Weyl transformation is inhomogeneous

$$\delta_\sigma S^{++} = 2\sigma S^{++} + \frac{i}{2} (D^+)^2 \sigma , \quad (D^+)^2 := D^+ \hat{\alpha} D^+ \dot{\alpha} . \quad (22)$$

Another important example of weight-two projective multiplet is given by $G^{++} := G^{ij} u_i^+ u_j^+$ with $G^{ij}$ the descendant associated with the Yang-Mills field strength $\mathcal{W}$ defined in (15). It satisfies the constraint $D^+_{\dot{\alpha}} G^{++} = 0$, and possesses the super-Weyl transformation law $\delta_\sigma G^{++} = 6\sigma G^{++} \quad [14]$. If $Q^{(n)}(u^+)$ is a covariant projective multiplet, its complex conjugate $\tilde{Q}^{(n)}(\bar{u}^+)$ is no longer of the same type. However, one can introduce a generalized smile-conjugation, $Q^{(n)} \to \tilde{Q}^{(n)}$, $\tilde{Q}^{(n)}(u^+) \equiv Q^{(n)}(\bar{u}^+ \to \tilde{u}^+) , \quad \tilde{u}^+ = i \sigma_2 u^+ , \quad (23)$

which acts on the space of covariant projective weight-$n$ multiplets, since $D^+_{\dot{\alpha}} \tilde{Q}^{(n)} = (-1)^n(Q^{(n)}) D^+ \hat{\alpha} \tilde{Q}^{(n)}$. One can see that $\tilde{Q}^{(n)} = (-1)^n Q^{(n)}$, and therefore real supermultiplets can be defined for $n$ even.

To define a locally supersymmetric and super-Weyl invariant action, one needs two prerequisites [14]: (i) a Lagrangian $\mathcal{L}^{++}(z, u^+)$ which is a real projective multiplet of
weight two and which possesses the super-Weyl transformation \( \delta_\sigma \mathcal{L}^{++} = 6 \sigma \mathcal{L}^{++} \); (ii) an Abelian vector multiplet with its field strength \( W(z) \) non-vanishing everywhere. The action is:

\[
S(\mathcal{L}^{++}) = \frac{2}{3\pi} \oint (u^+ du^+) \int d^5x d^8\theta \frac{\mathcal{L}^{++} W^4}{(G^{++})^2}, \quad E^{-1} = \text{Ber}(E\hat{A}^M). \tag{24}
\]

Here \( G^{++} := G^{ij} u_i^+ u_j^+ \), where \( G^{ij} \) is the descendant \(15\) associated with \( W \). Note that \( S(\mathcal{L}^{++}) \) is invariant under arbitrary re-scalings \( u_i^+ (t) \rightarrow c(t) u_i^+ (t), \forall c(t) \in \mathbb{C} \setminus \{0\} \), where \( t \) denotes the evolution parameter along the integration contour. The action can be shown to be invariant under supergravity gauge transformations \(3\) and \(20\), see \[14, 13\]. To see that \( S(\mathcal{L}^{++}) \) is invariant under super-Weyl transformations, one has only to note that \( \delta_\sigma E = -2 \sigma E \) and make use of the transformation rules \( \delta_\sigma \mathcal{L}^{++} = 6 \sigma \mathcal{L}^{++}, \delta_\sigma W = 2 \sigma W \) and \( \delta_\sigma G^{++} = 6 \sigma G^{++} \).

The crucial property of \( S(\mathcal{L}^{++}) \) is that it is independent of the concrete choice of \( W \), provided \( \mathcal{L}^{++} \) is independent of such a vector multiplet. Another important feature of the action introduced is that \(24\) provides a natural extension of the action principle in flat projective superspace \[17, 20\].

Since the action \(24\) is super-Weyl invariant, one can choose the super-Weyl gauge \( W = 1 \). Then, the action functional \(24\) takes the form given in \[13\] in the case of the 5D minimal multiplet.

Now we are in a position to give some interesting examples of supergravity-matter systems. Let \( V(z, u^+) \) denote the tropical prepotential \(3\) for the Abelian vector multiplet \( W \) appearing in the action \(24\). The prepotential is a real weight-zero projective multiplet possessing the gauge invariance

\[
\delta V = \lambda + \tilde{\lambda}, \tag{25}
\]

with \( \lambda \) a weight-zero arctic multiplet. A hypermultiplet can be described by an arctic weight-one multiplet \( \Upsilon^+ (z, u^+) \) and its smile-conjugate \( \tilde{\Upsilon}^+ \). Consider a gauge invariant Lagrangian of the form (with the gauge transformation of \( \Upsilon^+ \) being \( \delta \Upsilon^+ = -\xi \lambda \tilde{\Upsilon}^+ \))

\[
\mathcal{L}^{++} = \frac{1}{\kappa^2} \mathcal{V} G^{++} - \tilde{\Upsilon}^+ e^{\mathcal{V}} \Upsilon^+, \tag{26}
\]

with \( \kappa \) the gravitational coupling constant, and \( \xi \) a cosmological constant. It describes Poincaré supergravity if \( \xi = 0 \), and pure gauge supergravity with \( \xi \neq 0 \).

See \[12\] for the definition of covariant arctic and tropical multiplets.
The dynamics of the Yang-Mills supermultiplet can be described by the Lagrangian
\[ \mathcal{L}_{\text{YM}}^{++} = g^{-2} \nabla G^{++}, \]
with \( g \) the coupling constant (compare with the rigid supersymmetric case [21]).

A system of arctic weight-one multiplets \( \Upsilon^+(z, u^+) \) and their smile-conjugates \( \tilde{\Upsilon}^+ \) can be described by the Lagrangian
\[ \mathcal{L}^{++} = i K(\Upsilon^+, \tilde{\Upsilon}^+) , \quad (27) \]
with \( K(\Phi^I, \bar{\Phi}^J) \) a real analytic function of \( n \) complex variables \( \Phi^I \), where \( I = 1, \ldots, n \). For \( \mathcal{L}^{++} \) to be a weight-two real projective superfield, it is sufficient to require
\[ \Phi^I \frac{\partial}{\partial \Phi^I} K(\Phi, \bar{\Phi}) = K(\Phi, \bar{\Phi}) . \quad (28) \]
This is a curved superspace generalization of the general model for superconformal polar multiplets [20] (see also [11]).

Given a system of interacting arctic weight-zero multiplets \( \Upsilon \) and their smile-conjugates \( \tilde{\Upsilon} \), their coupling to supergravity can be described by the Lagrangian
\[ \mathcal{L}^{++} = G^{++} K(\Upsilon, \tilde{\Upsilon}) , \quad (29) \]
with \( K(\Phi^I, \bar{\Phi}^J) \) a real function which is not required to obey any homogeneity condition. The corresponding action is invariant under Kähler transformations of the form
\[ K(\Upsilon, \tilde{\Upsilon}) \to K(\Upsilon, \tilde{\Upsilon}) + \Lambda(\Upsilon) + \bar{\Lambda}(\tilde{\Upsilon}) , \quad (30) \]
with \( \Lambda(\Phi^I) \) a holomorphic function.

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References

[1] P. Breitenlohner and A. Kabelschacht, Nucl. Phys. B 148, 96 (1979).

[2] E. Cremmer, in: S. W. Hawking and M. Roček (Eds.), Supergravity and Superspace, Cambridge Univ. Press, 1981, p. 267.
[3] A. H. Chamseddine and H. Nicolai, Phys. Lett. B 96, 89 (1980).

[4] P. S. Howe, in: M. J. Duff and C. J. Isham (Eds.), Quantum Structure of Space and Time, Cambridge Univ. Press, 1982, p. 239.

[5] P. S. Howe and U. Lindström, Phys. Lett. B 103, 422 (1981).

[6] M. Günaydin, G. Sierra and P. K. Townsend, Nucl. Phys. B 242, 244 (1984); Nucl. Phys. B 253, 573 (1985).

[7] M. Günaydin and M. Zagermann, Nucl. Phys. B 572, 131 (2000) [hep-th/9912027].

[8] A. Ceresole and G. Dall'Agata, Nucl. Phys. B 585, 143 (2000) [hep-th/0004111].

[9] T. Kugo and K. Ohashi, Prog. Theor. Phys. 105, 323 (2001) [hep-ph/0010288]; T. Fujita and K. Ohashi, Prog. Theor. Phys. 106, 221 (2001) [hep-th/0104130].

[10] E. Bergshoeff et al., JHEP 0106, 051 (2001) [hep-th/0104113]; JHEP 0210, 045 (2002) [hep-th/0205230]; Class. Quant. Grav. 21, 3015 (2004) [hep-th/0403045].

[11] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, Nucl. Phys. B 785, 34 (2007), [arXiv:0704.1185].

[12] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, Phys. Lett. B 661, 42 (2008), [arXiv:0710.3440].

[13] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, JHEP 0802, 004 (2008) [arXiv:0712.3102].

[14] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, JHEP 0804, 032 (2008) [arXiv:0802.3953].

[15] A. S. Galperin, E. A. Ivanov, S. N. Kalitsyn, V. Ogievetsky, E. Sokatchev, Class. Quant. Grav. 1, 469 (1984).

[16] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, Harmonic Superspace, Cambridge University Press, 2001.

[17] A. Karlhede, U. Lindström and M. Roček, Phys. Lett. B 147, 297 (1984).

[18] U. Lindström and M. Roček, Commun. Math. Phys. 115, 21 (1988); Commun. Math. Phys. 128, 191 (1990).

[19] S. M. Kuzenko and G. Tartaglino-Mazzucchelli, arXiv:0804.1219 [hep-th].

[20] S. M. Kuzenko, Nucl. Phys. B 745, 176 (2006) [hep-th/0601177].

[21] S. M. Kuzenko and W. D. Linch, III, JHEP 0602, 038 (2006) [hep-th/0507176].