The lightweight topology optimization of multi-material structures with displacement constraints

Zong-Jie Dai, Hong-Ling Ye*, Wei-Wei Wang, Yun-Kang Sui

College of Mechanical Engineering and Applied Electronics Technology, Beijing University of Technology, Beijing 100124, China

* Corresponding author, E-mail: yehongl@bjut.edu.cn

Abstract. Multi-material Topology Optimization is a simulation technique based on the principle of the finite element method which is able to determine the optimal distribution of two or more different materials in combination under thermal and mechanical loads. This paper develops a lightweight topology optimization formulation of multi-material structures considering displacement constraints based on independent, continuous and mapping (ICM) method. Furthermore, explicit expression of optimised formulation is derived, approximations of displacement and weight are given by the first and second order Taylor expansion. And the optimization problem is solved by sequential quadratic programming approach. The feasibility and effectiveness of proposed method are demonstrated by numerical examples. It is found that the best transfer path of load is provided using multi-material topology optimization. The results show that a clear topological structure is obtained and the best transfer path of load is provided after multi-material topology optimization. In addition, under the precondition of satisfying the displacement constraint condition, the weight of the optimized structure based on various materials is lighter. The weight of multi-material topology optimization structure decreases with the increase of displacement constraint. And the optimal topological structure of the multi-material is different with the component materials. Besides, the optimization model established by using the structural performance parameters as a constraint is more reliable and more suitable for practical engineering applications.

Keywords: Independent continuous mapping method; Multi-material; Topology optimization;

1. Introduction

Structural topology optimization is a mathematical method that optimizes material layout within a given design space, for a given set of loads, boundary conditions and constraints with the goal of maximizing the performance of the system. It is an effective means to get conceptual design of lightweight structural layout optimization. Topological optimization design of multi-material structure begins with Thomsen [1], then many scholars have conducted extensive researches using different methods. Sigmund and Torquato [2-3] study the multiphase composite materials with extreme thermal expansion by homogenization method. Gao and Zhang [4] compare the effects of volume constraint and mass constraint on the results in topology optimization of multi-material structure through the solid isotropic material with penalization method. Mei [5] investigates the layout optimization of multiphase materials using the evolution of multiple level set functions. Luo and Wang [6-7] propose a piecewise constant level set method to solve the multiphase material placement optimization problem. Bourdin and Chambolle [8] couple specify material interpolation to solve the topological optimization problem of multiphase materials by phase-field method. Zhou and Wang [9-10] solve the topology optimization
problem by introducing the Cahn-Hilliard equation into phase-field method. Based on discrete material optimization method, Stegmann and Lund [11] solve the orientation and selection of orthotropic materials. Blasques and Stolpe [12-13] discuss the synchronization optimization of cross-sectional topologies and lamination methods for laminated beams with eigenfrequency constraint. Xie and Huang [14] solve the multiphase topological optimization problems at multi-scale through the evolutionary structural optimization method. In addition, Long [15-16] introduce a novel concurrent optimization formulation to meet the requirements of lightweight design and various constraints simultaneously. Yin and Ananthasuresh [17] use univariate descriptions of heterogeneous materials to reduce the number of density variables. Zuo and Saitou [18] normalize the density variable and propose the ordered SIMP method.

2. Strategy for solving the optimization formulation of multi-material structure

Differ from the previous researches on the topology optimization of the multi-material structure, we propose a new topology optimization formulation for the multi-material continuous structure following the ICM method. As for the ICM method, "Independent" and "Continuous" mean that topological variables are continuous values and independent at interval (0,1]. "Mapping" refers to the process of mapping and inversion. It enables independent continuous topological variables to approximate discrete topological variables, and completes the process of "discrete-continuous-discrete" of topological variables[19]. The ICM method adopts a special type of topological variables to indicate the "exist-null" of elements, and it implements the approximation from discrete topology optimization formulation to continuous topology optimization formulation by using filter function [20-22].

2.1 Topology optimization formulation of multi-material structure

Following the ICM method, aiming at weight minimization and subjected to the structural displacement constraint, a multi-material structure topology optimization formulation is established with the minimum structure weight as target and the structural displacement as constraint, the formulation is given as

\[
\begin{align*}
\text{min} & \quad W = \sum_{i=1}^{N} w_i \\
\text{s.t.} & \quad u_j \leq u_j^u \\
& \quad 0 < t_i^{(m)} \leq t_i^{(m)} \leq 1 
\end{align*}
\]

(1)

where \( t_i^{(m)} \) denotes the vector of topological variables, superscripts \( m \) indicates the number of candidate solid material phases and \( E^N \) is \( N \)-dimensional euclidean space. \( W \) is the total weight of structure. \( u_j \) and \( u_j^u \) are the concerned displacement in the total displacement vector and the allowable displacement value of concern displacement respectively. \( t_i^{(m)} = 0.01 \) is the lowest value of design variables to ensure numerical non-singularity in processes of finite element analysis. Besides, displacement constraint is expressed by first-order Taylor expansion, approximate expression of the weight function is obtained by the second-order Taylor expansion.

2.2 Strategy for solving the optimization formulation

Take three candidate solid materials as an example. Independent topological variables \( t_i^{(1)} (0 < t_i^{(1)} \leq 1) \), \( t_i^{(2)} (0 < t_i^{(2)} \leq 1) \) and \( t_i^{(3)} (0 < t_i^{(3)} \leq 1) \) are defined to characterize the presence and absence of distinct phase materials. The filter functions \( f_k (t_i^{(1)}) \), \( f_k (t_i^{(2)}) \), \( f_k (t_i^{(3)}) \) and \( f_w (t_i^{(1)}) \), \( f_w (t_i^{(2)}) \), \( f_w (t_i^{(3)}) \) are used to identify the element stiffness matrix and element weight respectively. The physical parameters of an element can be recognized by filter functions as follows
\[
k_i = f_k(t_i^{(3)}) \left\{ f_k(t_i^{(1)}) \left[ k_i^{\|} + f_k(t_i^{(2)}) \left( k_i^{\perp} - k_i^{\|} \right) \right] + k_i^{\mathbb{II}} \right\} \\
\quad (i = 1, 2, \ldots, N) \\
w_i = f_w(t_i^{(3)}) \left\{ f_w(t_i^{(1)}) \left[ w_i^{\|} + f_w(t_i^{(2)}) \left( w_i^{\perp} - w_i^{\|} \right) \right] + w_i^{\mathbb{II}} \right\} \\
\quad (i = 1, 2, \ldots, N)
\]

where \( N \) is the total number of elements in the design domain; superscripts I, II and III indicate different materials.

In order to solve the proposed formulation, the concerned displacement constraint is approximately expressed as explicit functions. To make it easier to express, the design variables are written as

\[
x_i = \frac{1}{f_k(t_i^{(1)})}, y_i = \frac{1}{f_k(t_i^{(2)})}, z_i = \frac{1}{f_k(t_i^{(3)})}
\]

Power functions are applied in this paper, and they are listed as follows

\[
f_k(t_i^{(1)}) = (t_i^{(1)})^\alpha, f_k(t_i^{(2)}) = (t_i^{(2)})^\alpha, f_k(t_i^{(3)}) = (t_i^{(3)})^\alpha
\]

\[
f_w(t_i^{(1)}) = t_i^{(1)}, f_w(t_i^{(2)}) = t_i^{(2)}, f_w(t_i^{(3)}) = t_i^{(3)}
\]

where \( \alpha \) is the exponent of penalization.

The displacement constraints in model can be expressed by first-order Taylor expansion of three kinds of design variables. It is approximated as

\[
u_j \approx \nu_j^{(v)} + \sum_{i=1}^{N} \frac{\partial \nu_j}{\partial x_i} (x_i - x_i^{(v)}) + \sum_{i=1}^{N} \frac{\partial \nu_j}{\partial y_i} (y_i - y_i^{(v)}) + \sum_{i=1}^{N} \frac{\partial \nu_j}{\partial z_i} (z_i - z_i^{(v)})
\]

where the superscript \( v \) denotes the \( v \)-th optimization iteration step.

As for the structural weight, it can be expressed as

\[
W \approx W^{(v)} + \sum_{i=1}^{N} \frac{\partial W}{\partial (t_i^{(1)})} (t_i^{(1)})^3 + \sum_{i=1}^{N} \frac{\partial W}{\partial (t_i^{(2)})} (t_i^{(2)})^2 + \sum_{i=1}^{N} \frac{\partial W}{\partial (t_i^{(3)})} (t_i^{(3)})^2 \\
+ \frac{1}{2} \left\{ \sum_{i=1}^{N} \frac{\partial^2 W}{\partial (t_i^{(1)})^2} (t_i^{(1)})^2 + \sum_{i=1}^{N} \frac{\partial^2 W}{\partial (t_i^{(2)})^2} (t_i^{(2)})^2 + \sum_{i=1}^{N} \frac{\partial^2 W}{\partial (t_i^{(3)})^2} (t_i^{(3)})^2 \right\} \\
+ \frac{1}{2} \left\{ \sum_{i=1}^{N} \frac{\partial^2 W}{\partial (t_i^{(1)})^2} (t_i^{(1)})^2 + \sum_{i=1}^{N} \frac{\partial^2 W}{\partial (t_i^{(2)})^2} (t_i^{(2)})^2 + \sum_{i=1}^{N} \frac{\partial^2 W}{\partial (t_i^{(3)})^2} (t_i^{(3)})^2 \right\}
\]

Then the multi-material structure is rebuilt, and the convergence criterion is chosen as follows

\[
\frac{|W^{(v+1)} - W^{(v)}|}{W^{(v+1)}} \leq \varepsilon
\]

where \( \varepsilon \) is the precision of convergence.

Up till now, the formulation is transformed into a standard quadratic programming problem. Optimum values of design variables are updated by SQP algorithm, which is an effective nonlinear programming method for constrained optimization problem. The specific process is reported in papers [23-24].

3. Numerical examples and discussions

As shown in Fig. 1, the overall dimension of the structure is: length \( L = 120 \text{ mm} \), height \( H = 30 \text{ mm} \) and thickness is 1 mm. In order to obtain the topological optimization results of different elastic modulus ratios, material I is isotropic with Young's modulus \( E_1=1.0 \times 10^8 \text{ MPa} \), material II is isotropic with Young's modulus \( E_2=2.0 \times 10^8 \text{ MPa} \), but the Young's modulus value of material III is variable. Density \( \rho_1 = 1\text{ kg/cm}^3 \),
\( \rho_2 = 1.5 \text{kg/cm}^3, \rho_3 = 2 \text{kg/cm}^3 \). Poisson’s ratio \( \mu = 0.3 \). Load \( F = 1 \text{kN} \). The boundary conditions and the external load for the simply supported beam are shown in Fig. 2. When the structure is fully composed of material III, the reference weight is \( W_0 \).

**Fig. 1** Simply supported beam design domain

Displacement constraint is set to the point where the load is applied and the value is 1.83 mm. The topological optimization results are given in Table 1. and curve of structure weight fraction with different elastic modulus ratio is given in Fig. 2.

**Table 1** Topological optimization results of simply supported beam

| \( E_3:E_2:E_1 \) | Weight fraction | The final topology configuration diagram |
|------------------|-----------------|-----------------------------------------|
| 4:2:1            | 0.383           | ![Material I](image1)                   |
| 5:2:1            | 0.361           | ![Material I](image2)                   |
| 6:2:1            | 0.346           | ![Material I](image3)                   |
| 7:2:1            | 0.329           | ![Material I](image4)                   |
| 8:2:1            | 0.319           | ![Material I](image5)                   |
The results show that the topological optimization results meet the dynamic design requirements of different elastic modulus materials. In the optimised topological configuration of the multi-material structure, the topological form of structure and material arrangement are similar with changing the elastic modulus ratio of three materials in the vicinity of load position, and material III with better mechanical properties is set on the main transfer path of the structure. The weight of the structure is declining with increasing the elastic modulus ratio of three materials.

4. Conclusion
Based on the ICM method, we established the multi-material topology optimization formulation aiming at minimum weight under displacement constraint. Displacement constraint is obtained through the first-order Taylor expansion, approximate expression of weight function is expressed by second-order Taylor expansion and the optimization problem is solved by sequential quadratic programming approach. Two classical numerical examples are given to demonstrate this method. With different properties of the material, the load conditions and the constraint conditions, the topological optimization configurations of the multi-material structure are various. The optimised topological configuration is diverse and varies with satisfying different displacement constraint, but the main force transmission paths is similar, and material III with better mechanical properties is always set between the constrained position and the load-loading region. And increasing the elastic modulus ratio of three materials, the weight of the structure is declining. Besides, the proposed method is promising in allocating the multi-material automatically, and provides a reference for the conceptual design of practical engineering problems.

Acknowledgments
This work was supported by the National Natural Science Foundation of China (11872080, 11172013), Beijing Natural Science Foundation (3192005) and Beijing Education Committee Development Project (SQKM201610005001).

References
[1] Thomsen J. 1992 J. Struct Multidiscip O. 5:108-15.
[2] Sigmund O, Torquato S. 1997 J. Mech Phys Solids. 45: 1037-67.
[3] Ruiz D., Sigmund O. 2018 J. Struct Multidiscip O. 57: 71-82.
[4] Gao T., Zhang W H.2011 J. Int J Numer Methods Eng. 88: 774-96.
[5] Mei Y L, Wang X M.2004 J. Acta Mech Sinica-Prc. 20: 507-18.
[6] Li H, Luo Z, Gao L and Walker P.2018 J. Comput Method Appl M. 328:340-64.
[7] Wu J L, Luo Z, Li H and Zhang N. 2017. J. Comput Method Appl M. 319:414-41.
[8] Feppon F, Michailidis G, Sidebottom M A, Allaire G, Kruck B A and Vermaak N. 2017. J. Struct Multidiscip O. 55:547-68.
[9] Bourdin B., Chambolle A. 2006. J. Solid Mech and Its Appl. 137:207-51.
[10] Wang M Y, Zhou S W. 2004. J. Comput Aid Mol Des. 11:117-38.
[11] Cui M T, Chen H F, Zhou J L, Wang F L. 2017. J. Eng Comput-Germany. 33:871-84.
[12] Stegmann J, Lund E. 2005. J. Int J Numer Meth Eng. 62:2009-27.
[13] Blasques J P. 2014. J. Compos Struct. 111:45-55.
[14] Huang X, Xie Y M, Jia B, Li Q, Zhou S W. 2012. J. Struct Multidiscip O. 46:385-98.
[15] Long K, Wang X, Gu X G. 2017. J. Acta Mech Sinica-Pr. 34:315-26.
[16] Long, K., Wang, X., Gu X G. 2018. J. Eng Optimiz 50:1-17.
[17] Yin L, Ananthasuresh G K. 2001. J. Struct Multidiscip O. 23:49-62.
[18] Zuo W J, Saitou K. 2017. J. Struct Multidiscip O. 55:477-91.
[19] Ye H L, Wang W W, Chen N, Sui Y K. 2017. J. Acta Mech Sinica-Pr. 33:899-911.
[20] Ye H L, Wang W W, Chen N, Sui Y K. 2016. J. Acta Mech Sinica-Pr. 32:649-58.
[21] Sui Y K, Ye H L. 2013. J. Science Press, Beijing (in Chinese)
[22] Sui Y K, Peng X R. 2005. J. Acta Mech Sinica-Pr. 37:190-98 (in Chinese)
[23] Sui Y K, Peng X R. 2017. J. Acta Mech Sinica-Pr. 49:1135-44,A (in Chinese)
[24] Long K., Wang X, Gu X G. 2018. J. Struct Multidiscip O. 57(3):1-13.