FOURIER COEFFICIENTS OF THETA FUNCTIONS AT CUSPS OTHER THAN INFINITY

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1. INTRODUCTION

In this paper, we use the adelic theory to study the Fourier coefficients of twisted theta functions at different cusps. The main results are Theorems 7.1.5 and 7.2.2.

The theory of theta functions has a long history, going back to Jacobi, and has a broad range of applications throughout various branches of mathematics. See, for example, [Mumford].

In number theory, theta functions may be used to study representation numbers of quadratic forms (see [Iwaniec, Chapter 11]), or to shed light on the Shimura correspondence ([Shimura]) between modular forms of integral weights and those of half integral weights (see [Shintani], [Katok-Sarnak], [Waldspurger]).

Moreover, in 1976, Serre and Stark ([Serre-Stark]) answered a question of Shimura, by showing that theta functions actually span the space of all modular forms of weight $\frac{1}{2}$ for $\Gamma_0(N)$.

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An important development in the theory of theta functions was their interpretation in terms of a representation of the metaplectic group. This point of view goes back to [Weil]. It leads naturally to a vast generalization of the Shimura correspondence, known as the theta correspondence, as well as a local analogue in the representation theory of groups over local fields. (See the survey article [Prasad], and its references.) For $GL(2)$, the theory was significantly explicated by [Gelbart], using explicit results of [Kubota]. The results of Serre and Stark were explained from this point of view by Gelbart and Piatetski-Shapiro [Gelbart-PS].

The theory of Fourier expansions of automorphic forms at various cusps goes back to [Roelcke] and [Maass]. We briefly review the concept. Our treatment follows [Iwaniec].

To define Fourier coefficients at a cusp $a$ one must choose a suitable “scaling” matrix. This notion is actually relative to a choice of discrete group $\Gamma$. Assume for simplicity that $-I \in \Gamma$. Then a scaling matrix for $a$ relative to $\Gamma$ is a matrix $\sigma \in SL(2, \mathbb{R})$ which maps $\infty$ to $a$ and conjugates the stabilizer of $a$ in $\Gamma$ to the stabilizer of $\infty$ in $SL(2, \mathbb{Z})$. Having fixed such a matrix $\sigma$ to define Fourier coefficients at $a$ of a modular form $f$ for the group $\Gamma$, one studies the function $f\mid_{\frac{1}{2}}^{Hol} \sigma$, where $f\mid_{\frac{1}{2}}^{Hol} \sigma$ is the weight one half slash operator corresponding to $\sigma$. Its definition is reviewed in Section 2.2. In general, the function $f\mid_{\frac{1}{2}}^{Hol} \sigma$ has a Fourier expansion supported on some additive shift of the integers. That is

$$f\mid_{\frac{1}{2}}^{Hol} \sigma(z) = \sum_{n=0}^{\infty} A_f(\sigma, n + \kappa_f, a)e^{2\pi i (n + \kappa_f, a)},$$

for some constant $\kappa_a \in \mathbb{Q} \cap [0, 1)$ and coefficients $A_f(\sigma, n + \kappa_f, a), (n \geq 0)$. We refer to the coefficients $A_f(\sigma, n + \kappa_f, a)$ appearing in this expansion as the Fourier coefficients of $f$ at $a$, defined relative to $\sigma$. Their dependence on the choice of $\sigma$ is addressed by lemma 2.3.1. Note that a modular form for $\Gamma$ is also a modular form for any subgroup of $\Gamma$. Replacing $\Gamma$ by a subgroup may alter which matrices $\sigma$ are considered suitable to be scaling matrices.

The constant $\kappa_f, a$ is called the cusp parameter of $f$ at $a$. It actually depends only on the multiplier system of $f$ (defined as in [Iwaniec §2.6, §2.7]). As a byproduct of our computations, we show that $\kappa_f, a$ is 0 for all cusps $a$ and all of the theta functions which we consider. That is, the Fourier expansion of a theta function at every cusp is supported on the integers. In fact, it is supported on the squares.

This paper was prompted by some numerical computations of Dorian Goldfeld and Paul Gunnells. Let $\chi_2$ and $\chi_3$ be the unique nontrivial quadratic Dirichlet characters modulo 4 and 3 respectively, let $\chi = \chi_2\chi_3$ (a Dirichlet character modulo 12) and let

$$\theta_\chi(z) = \sum_{n=1}^{\infty} \chi(n)e^{2\pi in^2z}.$$

Then $\theta_\chi$ is a modular form of weight $1/2$ and level 576. Goldfeld and Gunnells computed the Fourier coefficients of $\theta_\chi$ and discovered that (for suitable scaling matrices) the sequence of its Fourier coefficients at every other cusp was simply a scalar multiple of the sequence of the Fourier coefficients at infinity. For the cusps which can be transported to $\infty$ by a Fricke involution, this is expected, in view of the results of [Asai]. (See also [Kojima].) For the other cusps, the result is more surprising. But the phenomenon also suggests an explanation: this theta function must correspond to an element of the Weil representation which is fixed, up to scalars, by a group which is larger than $\Gamma_0(576)$, and acts transitively on the cusps. In Theorem 7.1.5 we prove this.

A remark is in order. Recall that for integral $k$, the weight $k$ slash operators define a right action of $SL(2, \mathbb{R})$ on the vector space of modular forms of weight $k$. For half-integral weight modular
forms, this is not the case. Rather, we have
\[
\begin{vmatrix}
\frac{1}{2} & \text{Hol} & 1 \\
\frac{1}{2} & \sigma_1 & \text{Hol} \\
\frac{1}{2} & \sigma_2 & \text{Hol}
\end{vmatrix}
= \pm \begin{vmatrix}
\frac{1}{2} & (\sigma_1 \sigma_2)
\end{vmatrix},
\]
with the occasional minus sign resulting from branch cuts in the square root. Thus the “larger group” that we define is actually a subgroup of a covering group.

The result can be recast classically as follows. Let
\[
\Gamma^{(24)} = \left( \begin{array}{c}
1 \\
24
\end{array} \right) \ SL(2, \mathbb{Z}) \left( \begin{array}{c}
1 \\
24^{-1}
\end{array} \right).
\]
Then \( \Gamma^{(24)} \) acts transitively on the cusps and contains a scaling matrix for each of them relative to \( \Gamma_0(576) \). Moreover, there is a function \( \zeta \) mapping \( \Gamma^{(24)} \) into the group of 24th roots of 1 such that \( \theta\chi \left|_{\text{Hol}} \right. \sigma = \zeta(\sigma)\theta \chi \) for each \( \sigma \in \Gamma^{(24)} \). The function \( \zeta \) is not a homomorphism, but it does satisfy \( \zeta(\sigma_1)\zeta(\sigma_2) = \pm \zeta(\sigma_1\sigma_2) \).

Theorem \[22\] extends this to the twisted theta functions of higher levels. Here again, we were led by numerical computations and a conjecture of Goldfeld and Gunnells. We first state the Goldfeld-Gunnells conjecture for the special case we refer to as the five twist. Let \( \chi_5 \) be the unique primitive quadratic character modulo 5. Set
\[
\theta_{\chi_5}(z) = \sum_{n=1}^{\infty} \chi_5(n)\chi(n)e^{2\pi in^2z}.
\]
Then the conjecture of Goldfeld and Gunnells, motivated by their numerical computations, is as follows.

**Conjecture 1.0.1** (Goldfeld-Gunnells). Let
\[
a = \frac{2\sin(\frac{4\pi}{5})}{\sqrt{5}} = \sqrt{(10 - 2\sqrt{5})/5} \approx 0.52573, \quad b = \frac{2\sin(\frac{2\pi}{5})}{\sqrt{5}} = \sqrt{(10 + 2\sqrt{5})/5} \approx 0.85065.
\]
Let \( a = u/w \in \mathbb{Q} \), and let \( \sigma_a \in SL(2, \mathbb{R}) \) be any scaling matrix for \( a \). If \( 5 \not| w \) or \( 25 \not| w \) then
\[
|A_{\theta_{\chi_5}}(\sigma_a, n^2)| = \begin{cases} 
1, & \text{if } 5 \not| n, \\
0, & \text{if } 5 | n.
\end{cases}
\]
On the other hand, if \( 5 || w \) then either
\[
|A_{\theta_{\chi_5}}(\sigma_a, n^2)| = \begin{cases} 
a, & \text{if } 5 \not| n, \\
2b, & \text{if } 5 | n,
\end{cases}
\]
or
\[
|A_{\theta_{\chi_5}}(\sigma_a, n^2)| = \begin{cases} 
b, & \text{if } 5 \not| n, \\
2a, & \text{if } 5 | n,
\end{cases}
\]

**Remark 1.0.2.** The fact that \( |A_{\theta_{\chi_5}}(\sigma_a, n^2)| \) is independent of the choice of \( \sigma_a \) is an easy consequence of Lemma \[23\].

This conjecture was generalized to several larger primes \( p \) by Gunnells, who replaced \( \chi_5 \) by a Dirichlet character \( \chi_p \) modulo \( p \) which factors through the squaring map \( (\mathbb{Z}/p\mathbb{Z})^\times \to (\mathbb{Z}/p\mathbb{Z})^\times \). For \( a = u/w \), with \( p \not| w \) or \( p^2 \not| w \), the extension is direct. For \( p || w \), Gunnells predicts that \( |A_{\theta_{\chi_p}}(\sigma_a, n^2)| \) depends only on the image of \( n^2 \) in \( \mathbb{Z}/p\mathbb{Z} \), so that the full sequence \( \{|A_{\theta_{\chi_p}}(\sigma_a, n^2)|\}_{n=1}^{\infty} \) is determined by a subsequence of length \( (p+1)/2 \). Moreover, as \( a \) ranges over cusps, only \( p-1 \) distinct sequences should appear, each consisting of zeros of explicit integer polynomials. These \( p-1 \) sequences come
in two classes. For each class there is an element corresponding to zero, and then a cycle of \((p-1)/2\) other values. As \(n\) runs through the nonzero squares modulo \(p\), the sequence \(\{A_{\theta_{\chi_p}}(\sigma_a, n^2)\}\) runs through this cycle. It may start at any point in the cycle and this accounts for the total of \(p-1\) possibilities.

For specific \(p\) and \(a\), the Goldfeld-Gunnells and Gunnells conjectures can be checked using theorem \[\text{7.2.2}\]. In this theorem, we again produce a larger group which acts transitively on the cusps. It no longer fixes the one-dimensional space spanned by our element of the Weil representation. Instead, it fixes a finite dimensional space containing it, and this permits us to obtain explicit results concerning the Fourier coefficients at all the cusps.

As an example, we consider the case \(p = 5\). In this case, the group we consider is

\[
\Gamma^{(120)} = \begin{pmatrix} 1 & 0 \\ 120 & 1 \end{pmatrix} \text{SL}(2, \mathbb{Z}) \begin{pmatrix} 1 & 0 \\ 120^{-1} & 1 \end{pmatrix}.
\]

As a conjugate of \(\text{SL}(2, \mathbb{Z})\) this group certainly acts transitively on the cusps, and it is easily verified that it provides a scaling matrix for every cusp relative to \(\Gamma_0(14400)\). Now, let \(V\) be the three dimensional complex vector space spanned by \(\theta_{\chi_5}, \theta_{\chi_{5^2}}\) and \(\theta_{\chi_{5^3}}\), where

\[
\theta_{\chi_{5^k}}(z) = \sum_{n=1}^{\infty} \chi(5n) e^{2\pi i (5n)^2 z}.
\]

Then, Theorem \[\text{7.2.2}\] gives the following.

**Theorem 1.0.3.** For each \(\sigma \in \Gamma^{(120)}\), there is a matrix \(M(\sigma^{-1}) \in \text{GL}_3(\mathbb{C})\) such that

\[
(1.0.4) \quad \begin{pmatrix} \theta_{\chi_5} & \text{Hol} & \theta_{\chi_{5^2}} & \text{Hol} & \theta_{\chi_{5^3}} & \text{Hol} \end{pmatrix} \begin{pmatrix} \theta_{\chi_5} & \theta_{\chi_{5^2}} & \theta_{\chi_{5^3}} \end{pmatrix} = \begin{pmatrix} \theta_{\chi_5} \cdot \theta_{\chi_{5^2}} \cdot \theta_{\chi_{5^3}} \end{pmatrix} \cdot M(\sigma^{-1}).
\]

Like \(\zeta\), the function \(M : \Gamma^{(120)} \to \text{GL}_3(\mathbb{C})\) defined implicitly by \(1.0.4\) is not a homomorphism, but satisfies \(M(\sigma_1)M(\sigma_2) = \pm M(\sigma_1\sigma_2)\). It is perhaps better understood by working with a metaplectic covering groups. This point of view is presented in the body of the paper.

Now, the Fourier coefficients \(A_{\theta_{\chi_p}}(\sigma, m)\) may be recovered from \(1.0.4\), provided one can compute \(M(\sigma^{-1})\) explicitly. Specifically, if \(\begin{pmatrix} 1 & c_1 & c_2 \\ 0 & c_2 & c_3 \end{pmatrix}\) is the first column of \(M(\sigma^{-1})\) then \(A_{\theta_{\chi_p}}(\sigma, m) = 0\) for \(m\) not a square, and

\[
(1.0.5) \quad A_{\theta_{\chi_p}}(\sigma, n^2) = \chi(n) \cdot \begin{cases} \chi_5(n)c_1 + c_2, & 5 \mid n, \\ c_2 + c_3, & 5 \nmid n. \end{cases}
\]

In order to compute \(M(\sigma)\) explicitly, it is helpful to work one prime at a time. If \(p\) is a prime, let

\[
K_p^{(120)} = \begin{pmatrix} 1 & 0 \\ 120 & 1 \end{pmatrix} \text{SL}(2, \mathbb{Z}_p) \begin{pmatrix} 1 & 0 \\ 120^{-1} & 1 \end{pmatrix}.
\]

Clearly, \(\Gamma^{(120)} < K_p^{(120)}\) for each \(p\).

We first describe a function \(M_5 : K_5^{(120)} \to \text{GL}_3(\mathbb{C})\). Then \(M\) will be the product of \(M_5\), similar but simpler contributions from 2 and 3, and some additional factors. In defining and computing \(M_p\), for \(p = 2, 3, 5\), we make use of the fact (see Lemma \[3.2.1]\) that \(\text{SL}(2, \mathbb{Z}_p)\) is generated by the elements

\[
\left\{ \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbb{Z}_p \right\}.
\]

Hence \(K_p^{(120)}\) is generated by the conjugates of these elements.
To describe $M_5 : K_5^{(120)} \to \text{GL}_3(\mathbb{C})$. It is convenient to introduce the notation
\[
\cos_5(x) = \cos(-\{x\}_5), \quad \sin_5(x) = \sin(-\{x\}_5).
\]
so that
\[
\cos_5(2\pi x) = \frac{1}{2}(e_5(x) + e_5(-x)), \quad \sin_5(x) = -\frac{1}{2i}(e_5(x) - e_5(-x)),
\]
where $\{\}$ and $e_5$ are defined as in section 3.1.

Then
\[
M_5 \left( \begin{array}{c} 1 \\ \frac{a}{5} \\ 1 \end{array} \right) = \left( \begin{array}{ccc} \cos_5(\frac{2\pi a}{5}) & -i \sin_5(\frac{2\pi a}{5}) & 0 \\ -i \sin_5(\frac{2\pi a}{5}) & \cos_5(\frac{2\pi a}{5}) & 0 \\ i \sin_5(\frac{2\pi a}{5}) & 1 - \cos_5(\frac{2\pi a}{5}) & 1 \end{array} \right), \quad (a \in \mathbb{Z}_5),
\]
\[
M_5 \left( \begin{array}{c} a \\ a^{-1} \end{array} \right) = \left( \begin{array}{c} \chi_5(a) \\ 1 \\ 1 \end{array} \right), \quad (a \in \mathbb{Z}_5^\times)
\]
\[
M_5 \left( \begin{array}{c} -5 \\ \frac{1}{5} \\ \frac{1}{\sqrt{5}} \end{array} \right) = \left( \begin{array}{c} -1 \\ \sqrt{5} \\ \frac{1}{\sqrt{5}} \end{array} \right)
\]

$M_5(\gamma_1 \gamma_2) = \beta_5(\gamma_1, \gamma_2)M_5(\gamma_1)M_5(\gamma_2), \quad (\gamma_1, \gamma_2 \in K_5^{(120)})$,
where $\beta_v : \text{SL}(2, \mathbb{Q}_v) \times \text{SL}(2, \mathbb{Q}_v) \to \{\pm 1\}$ is defined for each place $v$ as in section 3.1.

For $p = 2$, we define a function $M_2 : K_2^{(120)} \mu_8$ (the eighth roots of unity) by $M_2(\sigma) = \xi_2(\sigma, 1)$, where $\xi_2$ is defined as in theorem 4.4.8. The function $M_2$ satisfies $M_2(\sigma_1)M_2(\sigma_2) = M_2(\sigma_1\sigma_2)\beta_2(\sigma_1, \sigma_2)$.

Similarly, $M_3 : K_3^{(120)} \to \mu_6$ is defined by $M_3(\sigma) = \xi_3(\sigma, 1)$ with $\xi_3$ as in Theorem 4.4.7 and satisfies
\[
M_3(\gamma_1 \gamma_2) = M_3(\gamma_1)M_3(\gamma_2)\beta_3(\gamma_1, \gamma_2), \quad (\gamma_1, \gamma_2 \in K_3^{(120)}).
\]

**Theorem 1.0.6.** For any $\sigma \in \Gamma^{(120)}$, the equation (1.0.4) holds with
\[
M(\sigma) = s_\lambda(\sigma)\beta_\infty(\sigma^{-1}, \sigma)M_2(\sigma)M_3(\sigma)M_5(\sigma),
\]
where $s_\lambda(\sigma) \in \{\pm 1\}$ is defined in section 5.1.

We now describe briefly how these results may be used to verify the Goldfeld-Gunnells conjecture regarding $\theta_{\chi_5}$. First, since we are only interested in the absolute values of the Fourier coefficients, we may replace $M(\sigma^{-1})$ by $M_5(\sigma^{-1})$ in (1.0.4). As a scaling matrix for the cusp $u/w$, where $u, w \in \mathbb{Z}, \gcd(u, w) = 1$ and $w \mid 14400$, we choose the matrix
\[
\sigma := \left( \begin{array}{c} u/w, 120 \\ w, 120 \end{array} \right),
\]
where $r, s \in \mathbb{Z}$ satisfy $120us - rw = \gcd(w, 120)$. First suppose that $25 \nmid w$. Then we may write
\[
\sigma^{-1} = \left( \begin{array}{c} 1 \\ \frac{-s}{[120, w]} \\ 1 \end{array} \right) \left( \begin{array}{c} \frac{1}{5} \\ \frac{1}{24, w_1} \end{array} \right) \left( \begin{array}{c} 1 \\ \frac{u}{w} \end{array} \right),
\]
where $w_1 = w/(w, 5)$, and we may use
\[
M_5 \left( \begin{array}{c} 1 \\ \frac{-s}{[120, w]} \\ 1 \end{array} \right) M_3 \left( \begin{array}{c} 1 \\ \frac{1}{5} \\ \frac{1}{24, w_1} \end{array} \right) M_5 \left( \begin{array}{c} 1 \\ \frac{u}{w} \end{array} \right),
\]
in lieu of $M(\sigma^{-1})$ since the two are equal up to a sign. By a computation which is essentially a local analogue of Lemma 2.3.1, the matrix $M_5 \left( \begin{array}{c} 1 \\ \frac{-s}{[120, w]} \\ 1 \end{array} \right)$ multiplies each coefficient by a root of
unity without changing the absolute value, so it may be omitted. The product of the remaining terms in (1.0.7) is
\[
\begin{pmatrix}
-\chi_5([24, w_1]) \cos(2\pi u/w) & -\chi_5([24, w_1]) i \sin(2\pi u/w) & 0 \\
\sqrt{3} \sin(2\pi u/w) & \frac{1}{\sqrt{3}}(1 - \cos(2\pi u/w)) & \frac{1}{\sqrt{3}} \\
\sqrt{5} i \sin(2\pi u/w) & \sqrt{5} \cos(2\pi u/w) & 0
\end{pmatrix},
\]

Let \( [c_1 \ c_2 \ c_3] \) be the first column of this matrix. Then equation (1.0.5) holds up to roots of unity. In the case \( 5 \nmid w \), we have \( c_2 = c_3 = 0 \) and \( c_1 \) is a root of unity, yielding the Goldfeld-Gunnells conjecture in this case. The case \( 5 | w \) reduces to checking that
\[| \pm \cos(2\pi/5) \pm i \sin(2\pi/5)/\sqrt{5} | = 2 \sin(4\pi/5), \quad | \pm \cos(4\pi/5) \pm i \sin(4\pi/5)/\sqrt{5} | = 2 \sin(2\pi/5)\]
which is straightforward.

In the case when \( 25 | w \), the scaling matrix is of the form
\[
\begin{pmatrix}
a & b \\
25c & d
\end{pmatrix} = \begin{pmatrix}1 & b/d \\
0 & 1\end{pmatrix} \begin{pmatrix}-d^{-1} & 0 \\
0 & -d\end{pmatrix} \begin{pmatrix}0 & 1/5 \\
-5 & 0\end{pmatrix} \begin{pmatrix}1 & c/d \\
0 & 1\end{pmatrix} \begin{pmatrix}0 & 1/5 \\
-5 & 0\end{pmatrix},
\]
with \( 5 \nmid d \). But then \( M_5 \left( \begin{smallmatrix} 1 & c/d \\ 0 & 1 \end{smallmatrix} \right) \) and \( M_5 \left( \begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix} \right) \) are trivial, so up to a root of unity the last three terms simply cancel, leaving only \( M_5 \left( \begin{smallmatrix} -d^{-1} & 0 \\ 0 & -d \end{smallmatrix} \right) \), and the Goldfeld-Gunnells conjecture follows in this case as well.

The organization of the paper is as follows. In Section 2 we review the classical theory of the theta functions which we shall study. In Sections 3 and 4 we develop the relevant theory of local metaplectic groups and Weil representations, including the explicit formulae which are crucial for our aims in this paper. In Section 5 we review the relevant notions regarding the global metaplectic group, in Section 6 we define adelic theta functions corresponding to the classical objects reviewed in Section 2. The main theorems are proved in Section 7.

It may be noted that in this paper we have restricted attention to theta series attached to Dirichlet characters which are even at every prime except for 2 and 3. However, it seems that the method extends to other characters in a natural way. We hope to return to this in future work.

We would like to thank Dorian Goldfeld and Paul Gunnells for stimulating this research, and for sharing the results of their computations, which provided a perfect (and much needed!) method of checking the formulae which came out of our work. The work was undertaken during a special semester at ICERM and we would like to thank ICERM for providing a fantastic working environment. JH was supported by NSF Grant DMS-1001792 and gratefully thanks the NSF for the support.

2. The Classical Theory

2.1. The Scaling Matrices. Let \( \mathcal{H} \) denote the upper half plane. We shall make use of the classical action of \( SL(2, \mathbb{R}) \) on \( \mathcal{H} \) by fractional linear transformations:
\[
\begin{pmatrix} a & b \\
c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}, \quad \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad z \in \mathcal{H}.
\]

Let \( \mathfrak{a} \) be a cusp for \( \Gamma_0(N) \). Write \( \Gamma_{\mathfrak{a}} \) for the stabilizer of \( \mathfrak{a} \) in \( \Gamma_0(N) \). Choose \( \gamma \in SL(2, \mathbb{Z}) \) with \( \gamma \infty = \mathfrak{a} \). Then
\[\gamma^{-1} \Gamma_{\mathfrak{a}} \gamma \subset \Gamma_\infty = \langle (-1 \ -1), \ (1 \ 1) \rangle.\]
Moreover \( \gamma^{-1} \Gamma_{\mathfrak{a}} \gamma \) contains \( (-1 \ -1) \). It follows that
\[\gamma^{-1} \Gamma_{\mathfrak{a}} \gamma = \langle (-1 \ -1), \ (1 \ m_{\mathfrak{a}}) \rangle.\]
for a unique positive integer \( m_a \) called the width of \( a \) (relative to \( \Gamma_0(N) \)). Note that the matrix
\[
g_a = \gamma \left( \begin{array}{cc} 1 & m_a \\ 1 & 1 \end{array} \right) \gamma^{-1}
\]
is independent of the choice of \( \gamma \), as the elements \( \left( \begin{array}{cc} 1 & m_a \\ 1 & 1 \end{array} \right) \) and \( \left( \begin{array}{cc} 1 & -m_a \\ 1 & 1 \end{array} \right) \) are not conjugate in \( SL(2,\mathbb{R}) \).
Hence we have \( \Gamma_a = \langle -I, g_a \rangle \).
Now choose \( \sigma \in SL(2,\mathbb{R}) \) such that
\[
\sigma \cdot \infty = a \quad \sigma \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \sigma^{-1} = g_a.
\]
Following [Iwaniec], we refer to \( \sigma \) as a “scaling matrix” for \( a \). Clearly, \( \sigma \) is unique up to an element of the centralizer of \( \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right) \) in \( SL(2,\mathbb{R}) \), which is the subgroup
\[
\{ (\varepsilon \varepsilon^t) : \varepsilon \in \{ \pm 1 \}, t \in \mathbb{R} \}.
\]
If \( a = \infty \) then \( m_a = 1 \), \( g_a = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right) \), and one can take \( \sigma \) to be the identity.
If \( a = \frac{u}{w} \in \mathbb{Q} \), then
\[
m_a = \frac{N}{(w^2,N)}, \quad g_a = I + \frac{N}{(w^2,N)} \cdot \left( \begin{array}{cc} -uw & u^2 \\ -w^2 & uw \end{array} \right).
\]
As for the scaling matrix, a common choice is
\[
\sigma^0 = \left( \begin{array}{cc} \frac{w}{w} & \sqrt{[w^2,N]} \\ \sqrt{[w^2,N]} & \frac{w}{w} \sqrt{[w^2,N]}^{-1} \end{array} \right),
\]
where \([m,n]\) is the least common multiple of \( m \) and \( n \). If \( N = M^2 \), this simplifies to
\[
\sigma^0 = \left( \begin{array}{cc} \frac{w}{w} [w, M] & \frac{w}{w} [w, M] \cdot \frac{w}{w} [w, M]^{-1} \end{array} \right).
\]
This choice, however, is not suitable for our purpose. Instead, we would like to have our scaling matrix in a particular conjugate of \( SL(2,\mathbb{Z}) \).
For any integer \( M \), let
\[
\Gamma(M) := \left( \begin{array}{cc} 1 & 0 \\ M & 1 \end{array} \right) SL(2,\mathbb{Z}) \left( \begin{array}{cc} 1 & 0 \\ M & 1 \end{array} \right)^{-1}.
\]
Lemma 2.1.2. Let \( M \geq 1 \), and let \( a = \frac{u}{w} \in \mathbb{Q} \) be a cusp for \( \Gamma_0(M^2) \). Then there exists a scaling matrix \( \sigma \) of \( a \) that lies in \( \Gamma(M) \). Explicitly, choose \( r', s' \in \mathbb{Z} \) with
\[
u M s' - r' w = (M, w),
\]
then we may take the scaling matrix
\[
\sigma = \left( \begin{array}{cc} 1 & 0 \\ M & 1 \end{array} \right) \left( \begin{array}{cc} [w, M] & 0 \\ 0 & [w, M] \end{array} \right) \left( \begin{array}{cc} 1 & r' \\ M & s' \end{array} \right) = \left( \begin{array}{cc} [w, M] & 0 \\ 0 & [w, M] \end{array} \right) \left( \begin{array}{cc} 1 & r' \\ M & s' \end{array} \right) \in \Gamma(M).
\]
Proof. This follows from the direct computation. \( \Box \)
Remark 2.1.4. To illustrate the relation between the usual choice of scaling matrices and our choice, we have
\[
\sigma = \sigma^0 \left( \begin{array}{cc} 1 & \frac{w r'}{M a [M,w]} \\ \frac{w r'}{M a [M,w]} & 1 \end{array} \right).
\]
In particular, the condition \( \sigma \in \Gamma(M) \) determine \( r' \) uniquely modulo \( uM / \gcd(w, M) \) so the quantity \( \frac{w r'}{M a [M,w]} \) is determined modulo \( 1/M \).
Lemma 2.1.5. Under the notations in Lemma 2.1.2, we have the decompositions

\[(2.1.6) \quad \sigma^{-1} = \begin{pmatrix} 1 & 1 \\ -w & M \end{pmatrix} \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix}, \]

\[(2.1.7) \quad \sigma^{-1} = \begin{pmatrix} 1 & 1 \\ -w & M \end{pmatrix} \begin{pmatrix} 1 & -w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & M^{-1} \\ -M & 0 \end{pmatrix}. \]

Proof. Direct calculation. \[\square\]

Remark 2.1.8. Let \( p \mid M \) and

\[ K_p^{(M)} := \begin{pmatrix} 1 \\ M \end{pmatrix} SL(2, \mathbb{Z}_p) \begin{pmatrix} 1 \\ M^{-1} \end{pmatrix}. \]

The merit of the above lemma is that it gives an explicit decomposition of \( \sigma^{-1} \) within the group \( K_p^{(M)} \). More precisely, if \( v_p(w) \leq v_p(M) \), then each matrix in (2.1.6) is an element of \( K_p^{(M)} \), while if \( v_p(w) = 2 \) while \( v_p(M) = 1 \), each matrix in (2.1.7) is an element of \( K_p^{(M)} \).

2.2. The Slash Operators. We define

\[ j : SL(2, \mathbb{R}) \times \mathcal{H} \to \mathcal{H}, \quad j \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \right) = cz + d, \quad \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad z \in \mathcal{H} \right). \]

It satisfies the cocycle condition

\[ j(\gamma_1 \gamma_2, z) = j(\gamma_1, \gamma_2 \cdot z) j(\gamma_2, z). \]

We also define \( j(\gamma, z) = j(\gamma, z)/|j(\gamma, z)| \). Observe that

\[ j \left( \begin{pmatrix} x \frac{1}{2} & xy \frac{1}{2} \\ y^{-\frac{1}{2}} & y^{-\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, i \right) = j \left( \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, i \right) = e^{i \theta}, \]

for all \( x, y, \theta \in \mathbb{R} \) with \( y > 0 \).

For \( g \in SL(2, \mathbb{R}) \) and \( f : \mathcal{H} \to \mathbb{C} \) define

\[ \left( f \bigg| \frac{1}{2} \right)^{Hol} g \right) (z) = j(g, z)^{-\frac{1}{2}} f(g \cdot z), \]

\[ \left( f \bigg| \frac{1}{2} \right)^{Maa} g \right) (z) = j(g, z)^{-\frac{1}{2}} f(g \cdot z). \]

Here, the square roots are defined to be the principal value, having an argument in \( (-\pi/2, \pi/2) \). If \( \mu(f) \) is the function defined by \( [\mu(f)](z) = f(z) \Im(z)^{\frac{1}{2}} \), then for every \( g \in SL(2, \mathbb{R}) \) we have

\[ \mu \left( f \bigg| \frac{1}{2} \right)^{Hol} g \right) = \mu(f) \bigg| \frac{1}{2} \right)^{Maa} g. \]

Observe that although \( j \) and \( j \) are cocycles, their square roots are not, because of the discontinuity of the principal value.
2.3. Modular Forms of Weight 1/2. Let \( f \) be a modular form of weight \( \frac{1}{2} \) and multiplier \( \vartheta \) for \( \Gamma_0(N) \), as in [Iwaniec] §2.6, §2.7. Thus
\[
f(\frac{1}{2}f) = \vartheta(\gamma) \chi_c(d) \varepsilon_{ad}, \quad (\forall \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_0(N)).
\]
At every cusp \( \alpha \), let \( \kappa_{f, \alpha} \) be the unique (rational) solution to \( e^{2\pi i \kappa} = \vartheta(\alpha) \) in \( [0, 1) \). Then the function \( e^{-2\pi i \kappa \alpha} f(\frac{1}{2}f) \sigma \) is periodic of period 1 (see [Iwaniec], Page 43), and the Fourier coefficients of \( f \) at \( \alpha \) relative to \( \sigma \) are the coefficients \( A_f(\sigma, n + \kappa_{f, \alpha}) \) in its Fourier expansion:
\[
\left( f \left| \frac{1}{2} \right| \sigma \right)(z) = \sum_{n=0}^{\infty} A_f(\sigma, n + \kappa_{f, \alpha}) e^{2\pi i (n + \kappa_{f, \alpha})z}.
\]

Lemma 2.3.1. We have
\[
A_f \left( \sigma \left( \begin{array}{c} \varepsilon \\ \varepsilon \end{array} \right), \ n + \kappa_{f, \alpha} \right) = A_f \left( \sigma, \ n + \kappa_{f, \alpha} \right) \vartheta \left( \begin{array}{c} \varepsilon \\ \varepsilon \end{array} \right)
\]
\[
A_f \left( \sigma \left( \begin{array}{c} 1 \\ t \\ 1 \end{array} \right), \ n + \kappa_{f, \alpha} \right) = A_f \left( \sigma, \ n + \kappa_{f, \alpha} \right) e^{2\pi i (n + \kappa_{f, \alpha})t}.
\]

Proof. This is obvious; for example, the latter comes immediately from
\[
\sum_{n=0}^{\infty} A_f \left( \sigma \left( \begin{array}{c} 1 \\ t \\ 1 \end{array} \right), \ n + \kappa_{f, \alpha} \right) e^{2\pi i (n + \kappa_{f, \alpha})z} = \left( f \left| \frac{1}{2} \right| \sigma \left( \begin{array}{c} 1 \\ t \\ 1 \end{array} \right) \right)(z)
\]
\[
= \left( f \left| \frac{1}{2} \right| \sigma \right)(z + t) = \sum_{n=0}^{\infty} A_f \left( \sigma, \ n + \kappa_{f, \alpha} \right) e^{2\pi i (n + \kappa_{f, \alpha})(z + t)}. \quad \square
\]

2.4. The Classical Theta Functions. Let \( \chi \) (mod \( N \)) be an even Dirichlet character, then the classical theta function
\[
\theta_\chi(z) = \sum_{n=1}^{\infty} \chi(n) e^{2\pi in^2}, \quad (z \in \mathcal{H})
\]
(Cf. [Iwaniec] §10.5) is a cusp form of weight 1/2 and level 4N², and we have
\[
\theta_\chi(\gamma z) = \chi(d) \chi_c(d) \varepsilon_d^{-1} (ez + d)^{1/2} \theta_\chi(z), \quad \left( \gamma = \left( \begin{array}{c} a & b \\ c & d \end{array} \right) \in \Gamma_0(4N^2) \right),
\]
where
\[
\varepsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \pmod{4}; \\ i, & \text{if } d \equiv 3 \pmod{4}, \end{cases}
\]
\( \chi_t \) denotes the primitive character corresponding to the field extension \( \mathbb{Q}(\sqrt{t})/\mathbb{Q} \)
1 and the square root takes the principal value. In other words, we have
\[
\left( \theta_\chi \left| \frac{1}{2} \right| \gamma \right)(z) = \chi(d) \chi_c(d) \varepsilon_d^{-1} \theta_\chi(z) \quad (\gamma \in \Gamma_0(4N^2)).
\]
In this paper, we consider the special cases that \( N = 12 \) or \( N = 12p \) for some prime \( p \), and study its Fourier coefficients at different cusps. The main results are Theorems 7.1.5 and 7.2.2.

1For example, if \( t \) is a perfect square, then \( \chi_t = 1 \); if \( t \) is not a perfect square, and let \( D \) be the discriminant of \( \mathbb{Q}(\sqrt{t})/\mathbb{Q} \), then \( \chi_t \) is the primitive quadratic character of conductor \(|D|\) given by
\[
\chi_t = \left( \frac{D}{.} \right).
\]

9
3. LOCAL METAPLECTIC GROUPS AND WEIL REPRESENTATIONS

3.1. LOCAL METAPLECTIC GROUPS. Let $\mathbb{Q}_v$ be one of the completions of $\mathbb{Q}$. Thus $\mathbb{Q}_v = \mathbb{Q}_\infty = \mathbb{R}$ or $\mathbb{Q}_v$, is the $p$-adic numbers $\mathbb{Q}_p$ for some prime $p$. The Hilbert symbol on $\mathbb{Q}_v$ will be denoted by $(\cdot, \cdot)_v$, or, when $v$ is fixed, by $(\cdot, \cdot)$. As in [Gelbart], we define the cocycle

$$\beta_v : SL(2, \mathbb{Q}_v) \times SL(2, \mathbb{Q}_v) \to \{\pm 1\},$$

$$(g_1, g_2) \mapsto \left( x(g_1), x(g_2) \right)_v \left( -x(g_1)x(g_2), x(g_1g_2) \right)_v s(g_1)s(g_2)s(g_1g_2),$$

where

$$x \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c, & \text{if } c \neq 0; \\ d, & \text{if } c = 0, \end{cases} \quad s \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} (c,d)_v, & \text{if } v < \infty, \text{ } cd \neq 0 \text{ and } \text{ord}(c) \text{ is odd;} \\ 1, & \text{if otherwise.} \end{cases}$$

In particular, in the Borel subgroup it simplifies to

$$\beta_v \begin{pmatrix} a_1 & b_1 \\ d_1 & d_2 \end{pmatrix}, \quad \begin{pmatrix} a_2 & b_2 \\ c & d \end{pmatrix} = (a_1, d_2)_v,$$

and over $SL(2, \mathbb{R})$ it simplifies to

$$(3.1.1) \quad \beta_\infty(g_1, g_2) = \left( x(g_1), x(g_2) \right)_\infty \left( -x(g_1)x(g_2), x(g_1g_2) \right)_\infty.$$  

We may then define a double cover of $SL(2, \mathbb{Q}_v)$, denoted $\tilde{SL}(2, \mathbb{Q}_v)$ and consisting of the set $SL(2, \mathbb{Q}_v) \times \{\pm 1\}$ equipped with the operation

$$(g_1, \zeta_1)(g_2, \zeta_2) := \left( g_1g_2, \beta_v(g_1, g_2)\zeta_1\zeta_2 \right), \quad (g_1, g_2 \in SL(2, \mathbb{Q}_v), \zeta_1, \zeta_2 \in \{\pm 1\}).$$

The function $pr : \tilde{SL}(2, \mathbb{Q}_v) \to SL(2, \mathbb{Q}_v)$ given by $pr(g, \zeta) = g$ is a homomorphism.

3.2. GENERATORS FOR $SL(2, \mathbb{Z}_p)$ AND ITS PREIMAGE.

**Lemma 3.2.1.** The group $K_p = SL(2, \mathbb{Z}_p)$ is generated by

$$(3.2.2) \quad \left\{ \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} x & 1 \\ 1 & 1 \end{pmatrix} \mid x \in \mathbb{Z}_p \right\}$$

**Proof.** Write $H_p$ for the subgroup of $SL(2, \mathbb{Z}_p)$ generated by (3.2.2). Then $H_p$ contains $(1 \ x \ 1 \ 1)$ for each $x$. Since

$$\begin{pmatrix} 1 & x-1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -x \\ 1 & -x \end{pmatrix} = \begin{pmatrix} x & 1 \\ -1 & x \end{pmatrix},$$

for all $x \in \mathbb{Z}_p^\times$, it follows that $H_p$ contains all diagonal, and hence all upper or lower triangular elements of $K_p$. Now consider an arbitrary element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $K_p$. Either $c$ or $d$ is a unit. If $d$ is a unit then $-b/d \in \mathbb{Z}_p$ and $\begin{pmatrix} 1 & -b/d \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is upper triangular, and hence in $H_p$, which then implies that $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in H_p$. If $c$ is a unit then $a/c$ and $d/c$ are in $\mathbb{Z}_p$ and

$$\begin{pmatrix} 1 & -(a/c) \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -d/c \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} c & -1 \\ c^{-1} & 1 \end{pmatrix} \in H_p,$$

which completes the proof. \qed
3.3. The Real Metaplectic Group. In our study of the real metaplectic group \( \tilde{SL}(2, \mathbb{R}) \), the basic tools are the metaplectic analogue of the classical Iwasawa decomposition and the corresponding slash operator. Recall that

\[
SL(2, \mathbb{R}) = B^+(\mathbb{R}) \times SO_2(\mathbb{R}),
\]

where

\[
B^+(\mathbb{R}) = \left\{ \begin{pmatrix} y^2 & xy^{-1} \\ xy^{-1} & y^2 \end{pmatrix} : y \in (0, \infty), x \in \mathbb{R} \right\} \leq SL(2, \mathbb{R}).
\]

Hence our discussion starts with the metaplectic preimage of \( SO_2(\mathbb{R}) \).

For \( \theta \in \mathbb{R} \) define

\[
\kappa_\theta = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \in SO_2(\mathbb{R}) \subset SL(2, \mathbb{R}),
\]

and

\[
\tilde{\kappa}_\theta = \begin{pmatrix} \kappa_{2\theta} \\ \zeta(\theta) \end{pmatrix},
\]

where \( \zeta \) is the unique function \( \mathbb{R} \to \pm 1 \) which is periodic mod \( 2\pi \) and satisfies

\[
\zeta(\theta) = \begin{cases} 
1, & \text{if } -\pi/2 \leq \theta < \pi/2; \\
-1, & \text{if } \pi/2 \leq \theta < 3\pi/2.
\end{cases}
\]

**Lemma 3.3.1.** The function \( \theta \mapsto \tilde{\kappa}_\theta \) is a homomorphism.

**Proof.** Using \((3.1.1)\), one checks that

\[
\beta_\infty(\kappa_{2\theta_1}, \kappa_{2\theta_2}) = -1 \iff \frac{\zeta(\theta_1 + \theta_2)}{\zeta(\theta_1)\zeta(\theta_2)} = -1
\]
on a case-by-case basis. \(\square\)

We denote the set of the images \( \tilde{\kappa}_\theta \) by \( \tilde{K} \).

**Remark 3.3.2.** If we define \( \sqrt{z} \) for \( z \in \mathbb{C}^\times \) so that \( \text{Arg}(\sqrt{z}) \in [-\pi/2, \pi/2) \) for all \( z \in \mathbb{C}^\times \), then the function \( S^1 \to S^1 \times \{\pm 1\} \) defined by

\[
e^{i\theta} \mapsto \tilde{\kappa}_\theta \mapsto \left( e^{2i\theta}, \zeta(\theta) \right).
\]

is the restriction of

\[
z \mapsto \left( z^2, \frac{\sqrt{z^2}}{z} \right)
\]

Observe that \( \sqrt{z} \) is the principal value of the square root of \( z \) except when \( z \in (-\infty, 0) \).

Next we discuss the metaplectic phenomena over \( B^+(\mathbb{R}) \).

**Lemma 3.3.3.** The cocycle \( \beta_\infty \) is trivial on \( B^+(\mathbb{R}) \times SL(2, \mathbb{R}) \) and \( SL(2, \mathbb{R}) \times B^+(\mathbb{R}) \).

**Proof.** Let

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}), \quad g_0 = \begin{pmatrix} y^{1/2} & xy^{-1/2} \\ y^{-1/2} & y^{-1/2} \end{pmatrix} \in B^+(\mathbb{R}),
\]

then we have

\[
gg_0 = \begin{pmatrix} * & * \\ cy^{1/2} & (cx + d)y^{-1/2} \end{pmatrix}, \quad g_0g = \begin{pmatrix} * & * \\ cy^{-1/2} & dy^{-1/2} \end{pmatrix}.
\]

In particular, we have

\[
x(g_0) = y^{1/2} > 0, \quad x(g_0g) = x(g_0)x(g), \quad \text{sgn} x(g_0) = \text{sgn} x(g).
\]
Hence by definition we have
\[
\beta_{\infty}(g,g_0) = \left( x(g), x(g_0) \right)_\infty \left( -x(g)x(g_0), x(gg_0) \right)_\infty = \left( -x(g), x(gg_0) \right)_\infty = 1,
\]
\[
\beta_{\infty}(g_0,g) = \left( x(g_0), x(g) \right)_\infty \left( -x(g_0)x(g), x(g_0g) \right)_\infty = \left( -x(g), x(g) \right)_\infty = 1.
\]

**Remark 3.3.4.** By the lemma, we have the injective homomorphism
\[
B^+(\mathbb{R}) \hookrightarrow \tilde{SL}(2,\mathbb{R}), \quad b \mapsto (b,1)
\]
which splits the covering map. We henceforth identify \( B^+(\mathbb{R}) \) with its image in \( \tilde{SL}(2,\mathbb{R}) \).

Now we are ready to introduce the Iwasawa decomposition over \( \tilde{SL}(2,\mathbb{R}) \).

**Lemma 3.3.5.** Each element \( \tilde{g} \) of \( \tilde{SL}(2,\mathbb{R}) \) has a unique expression as \( \tilde{g} = b\tilde{\kappa} \) with \( b \in B^+(\mathbb{R}) \) and \( \tilde{\kappa} \in \tilde{K} \).

**Proof.** This follows immediately from Lemma 3.3.4 and the analogous statement for \( SL(2,\mathbb{R}) \).

Based on the above lemma, we are able to define functions \( b : \tilde{SL}(2,\mathbb{R}) \rightarrow B^+(\mathbb{R}) \) and \( \theta : SL(2,\mathbb{R}) \rightarrow \mathbb{R}/2\pi\mathbb{Z} \) by
\[
\tilde{g} = b(g)\tilde{\kappa}_{\theta(g)}, \quad (\tilde{g} \in \tilde{SL}(2,\mathbb{R})).
\]

**Lemma 3.3.6.** For \( z = x + iy \in \mathcal{H} \), define
\[
b_z = \left( \begin{array}{cc} y^2 & xy^{-\frac{1}{2}} \\ y^{-\frac{1}{2}} & x \end{array} \right) \in B^+(\mathbb{R}).
\]

Then we have
\[
b(\tilde{g}) = b_{pr(\tilde{g})} \cdot i.
\]

**Proof.** Clearly \( \tilde{g} \cdot z := pr(\tilde{g}) \cdot z \) is an action of \( \tilde{SL}(2,\mathbb{R}) \) on \( \mathcal{H} \). The stabilizer of \( i \) is the preimage of \( SO_2(\mathbb{R}) \), which is \( \tilde{K} \). Hence \( \tilde{g} : i = b(\tilde{g}) \cdot i \). And for each \( z \in \mathcal{H} \), the matrix \( b_z \) can be described as the unique element of \( B^+(\mathbb{R}) \) mapping \( i \) to \( z \).

In what follows, we shall continue to use the action of \( \tilde{SL}(2,\mathbb{R}) \) on \( \mathcal{H} \) by \( \tilde{g} \cdot z := pr(\tilde{g}) \cdot z \).

**Corollary 3.3.7.** For any \( \tilde{g} \in \tilde{SL}(2,\mathbb{R}) \) and \( z \in \mathcal{H} \) there exists \( \theta(\tilde{g},z) \in \mathbb{R}/2\pi\mathbb{Z} \) such that \( \tilde{g} \cdot b_z = b_{\tilde{g}z}\tilde{\kappa}_{\theta(\tilde{g},z)} \).

Specifically, \( \theta(\tilde{g},z) = \theta(\tilde{gb}_z) \), where the latter is defined using the Iwasawa decomposition as above. It is immediate from the definitions that the function \( \theta : \tilde{SL}(2,\mathbb{R}) \times \mathcal{H} \rightarrow \mathbb{R}/2\pi\mathbb{Z} \) is a cocycle, i.e.,
\[
\theta(\tilde{g}_1\tilde{g}_2,z) = \theta(\tilde{g}_1,\tilde{g}_2 \cdot z) + \theta(\tilde{g}_2,z).
\]

Lastly, we discuss the slash operator of \( \tilde{SL}(2,\mathbb{R}) \) upon the functions over \( \mathcal{H} \). Define \( \tilde{j} : \tilde{SL}(2,\mathbb{R}) \times \mathcal{H} \rightarrow S^1 \) by \( \tilde{j}(\tilde{g},z) = e^{i\theta(\tilde{g},z)} \). Clearly, \( \tilde{j} \) is a cocycle, since \( \theta \) is.

**Lemma 3.3.8.** The cocycle \( \tilde{j}(\tilde{g},z) \) is always a square root of \( j(pr(\tilde{g}),z) \), namely
\[
\tilde{j}(\tilde{g},z)^2 = j(pr(\tilde{g}),z).
\]

**Proof.** Write \( \tilde{gb}_z = b_{\tilde{g}z}\tilde{\kappa}_{\theta(\tilde{g},z)} \). Then \( pr(\tilde{g})b_z = b_{\tilde{g}z}\tilde{\kappa}_{\theta(\tilde{g},z)} \). But
\[
j(pr(\tilde{g}),z) = \frac{j(pr(\tilde{g})b_z,i)}{j(b_z,i)},
\]
and \( j(b_z,i) = 1 \), so we get
\[
j(pr(\tilde{g}),z) = j(pr(\tilde{g})b_z,i) = e^{i\theta(\tilde{g},z)} = \tilde{j}(\tilde{g},z)^2.
\]
Clearly $\tilde{j}(\tilde{g}, z)$ is the principal value of the square root of $j(pr(\tilde{g}), z)$ if and only if $\theta(\tilde{g}, z) \in (-\pi/2, \pi/2)$.

For $f : \mathcal{H} \to \mathbb{C}$, now we define the slash operator

$$(f|\sim \tilde{g})(z) = \tilde{j}(\tilde{g}, z)^{-1} f(g \cdot z) = \tilde{j}(\tilde{g}, z)^{-1} f(pr(\tilde{g}) \cdot z) \quad (\tilde{g} \in SL(2, \mathbb{R})).$$

**Lemma 3.3.9.** The slash operator $|\sim$ gives a well-defined right action of $SL(2, \mathbb{R})$ on the space of all functions $\mathcal{H} \to \mathbb{C}$.

**Proof.** This follows immediately from the fact that $\tilde{j}$ is a cocycle. \hfill $\square$

**Lemma 3.3.10.** Let $f : \mathcal{H} \to \mathbb{C}$ and $g \in SL(2, \mathbb{R})$. Then we have

$$(3.3.11) \quad (f|\sim (g, 1)) = \left(f\big|^{\text{Maa}}_{\frac{1}{2}} g\right).$$

**Proof.** Clearly $\tilde{j}(\sim (g, 1), z)$ is the principal value of the square root of $j(g, z)$ if and only if $\theta((g, 1), z) \in (-\pi/2, \pi/2)$, which in turn is equivalent to $\zeta(\theta((g, 1), z)) = 1$.

Now in $SL(2, \mathbb{R})$ we have

$$(g, 1) b_z = (g, 1) (b_z, 1) = (gb_z, 1),$$

so by definition $\zeta(\theta((g, 1), z)) = 1$. This confirms that $\tilde{j}(\sim (g, 1), z)$ is the principal value of the square root of $j(g, z)$. Hence

$$(f|\sim (g, 1)) = \tilde{j}(\sim (g, 1), z) f(g \cdot z) = j(g, z)^{-\frac{1}{2}} f(g \cdot z) = \left(f\big|^{\text{Maa}}_{\frac{1}{2}} g\right). \quad \square$$

As an example, let $\chi \pmod{N}$ be an even Dirichlet character and consider the classical theta function $\theta_X$. Define

$$\theta_X^{\text{Maa}}(x + iy) = y^{\frac{1}{2}} \theta_X(x + iy).$$

Then for $\gamma \in \Gamma_0(4N^2)$ we have

$$\theta_X^{\text{Maa}}(\gamma z) = \chi(d) \chi_c(d) \varepsilon_d^{-1} j(\gamma, z)^{1/2} \theta_X^{\text{Maa}}(z).$$

Hence

$$(\theta_X^{\text{Maa}}|\sim (\gamma, 1))(z) = \left(\theta_X^{\text{Maa}}\big|^{\text{Maa}}_{\frac{1}{2}} \gamma\right)(z) = \chi(d) \chi_c(d) \varepsilon_d^{-1} \theta_X^{\text{Maa}}(z).$$

### 4. Local Weil Representations

**4.1. Local Weil Representation.** The Bruhat-Schwartz space of $\mathbb{Q}_v$ will be denoted $\mathcal{S}(\mathbb{Q}_v)$. It is the Schwartz space when $v = \infty$ and the space of all locally constant compactly supported functions when $v$ is a prime. Following [Gelbart-PS] we consider the family of representations $r^{\psi_v}$ of $SL(2, \mathbb{Q}_v)$ on $\mathcal{S}(\mathbb{Q}_v)$, indexed by the nontrivial characters $\psi_v$ of $\mathbb{Q}_v$, and defined by

$$r^{\psi_v} \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) \varphi(x) = \gamma(\psi_v) \hat{\varphi}(x)$$

$$r^{\psi_v} \left( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1 \right) \varphi(x) = \psi_v(bx^2) \varphi(x)$$

$$r^{\psi_v} \left( \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, 1 \right) \varphi(x) = |a|^{\frac{1}{2}} \gamma(\psi_v) \gamma(\psi_v,a) \varphi(ax)$$

$$[r^{\psi_v}(I_2, \zeta) \varphi] = \zeta \cdot \varphi,$$
where the Fourier transform is given by

$$\hat{\varphi}(x) = \alpha(\psi_v) \int_{Q_v} \varphi(y)\psi_v(2xy) \, dy,$$

$dy$ is the standard Haar measure over $Q_v$, $\alpha(\psi_v)$ is the normalization factor such that $\hat{\varphi}(x) = \varphi(-x)$, and

$$\gamma(\psi_v,a) = \begin{cases} e^{\frac{2\pi i}{a}}, & \text{if } v = \infty \text{ and } \psi_\infty(x) = e^{2\pi ix}; \\ \lim_{m \to -\infty} \alpha(\psi_v,a) \int_{p^nZ_p} \psi_v(ay^2) \, dy, & \text{if } v \text{ is a prime.} \end{cases}$$

**Remark 4.1.1.** The constant $\gamma(\psi_v,a)$ is an eighth root of one. This is obvious when $v = \infty$ and a result of Weil otherwise. (Cf. [Gelbart, Page 36].)

Now we would like to explicitly describe the local Weil representation with respect to the additive character $e_v(x) = \begin{cases} e^{2\pi ix}, & \text{if } v = \infty; \\ e^{-2\pi i(x)_p}, & \text{if } v = p, \end{cases}$ where, for every prime $p$, we denote by

$$\{\cdot\}_p : Q_p \to Q, \quad \sum_{n=-N}^{\infty} a_np^n \mapsto \sum_{n=-N}^{-1} a_np^n$$

the “$p$-adic fractional part” of $Q_p$.

**Proposition 4.1.2.** Let $a \in Q_v^\times$. Then we have

$$\alpha(e_v,a) = |2a|_v^{1/2}.$$

**Proof.** If $v$ is a finite place, say at $p$, then as the test function we take $\varphi(x) = 1_{Z_p}(x)$. By definition we have

$$\hat{\varphi}(x) = \alpha(e_{p,a}) \int_{Z_p} e_{p}(2axy) \, dy = \alpha(\psi_{p,a}) 1_{(2a)^{-1}Z_p}(x),$$

so

$$\hat{\varphi}(x) = \alpha(e_{p,a})^2 \int_{(2a)^{-1}Z_p} e_{p}(2axy) \, dy = \frac{\alpha(e_{p,a})^2}{|2a|_p} 1_{Z_p}(x).$$

Hence by definition we have $\alpha(e_{p,a}) = |2a|_p^{1/2}$.

Now consider the case $v = \infty$. As the test function we take $\varphi(x) = e^{-\pi x^2}$, then

$$\hat{\varphi}(x) = \alpha(e_{\infty,a}) \int_{-\infty}^{\infty} e^{-\pi y^2 + 4\pi axy} \, dy = \alpha(e_{\infty,a}) e^{-4\pi^2 a^2 x^2},$$

$$\hat{\varphi}(x) = \alpha(e_{\infty,a})^2 \int_{-\infty}^{\infty} e^{-4\pi a^2 y^2 + 4a\pi xy} \, dy = \frac{\alpha(e_{\infty,a})^2}{|2a|} e^{-\pi x^2}.$$

Hence again we have $\alpha(e_{\infty,a}) = |2a|^{1/2}$. \qed

Next we evaluate $\gamma(e_{p,a})$.

---

2 Note that the formula for $\gamma(\psi_v,a)$ on Page 36 of [Gelbart] contains a typo.
Proposition 4.1.3. Let \( a \in \mathbb{Z}_p \setminus \{0\} \), say with the decomposition \( a = \alpha p^r \) for some \( r \in \mathbb{Z} \) and \( \alpha \in \mathbb{Z}_p^\times \). Then we have

\[
\gamma(e_{p,a}) = \begin{cases} 
\frac{1+i}{\sqrt{2}} \varepsilon_{-\alpha} \left( \frac{2}{-\alpha} \right)^r & \text{if } p = 2; \\
\left( -\frac{\alpha}{p} \right) & \text{if } p \equiv 1 \pmod{4} \text{ and } 2 \nmid r; \\
i \left( -\frac{\alpha}{p} \right) & \text{if } p \equiv 3 \pmod{4} \text{ and } 2 \nmid r; \\
1 & \text{if } 2 \mid p, 2 \mid r,
\end{cases}
\]

where, for the factor \( \varepsilon_{-\alpha} \) and for the Kronecker symbols involving \( \alpha \), we adopt the convention they are evaluated with respect to some \( \alpha^* \in \mathbb{Z} \) such that \( |\alpha - \alpha^*|_p \) is sufficiently small.

Proof. We have, for \( m \gg 1 \), that

\[
\gamma(e_{p,a}) = \alpha(e_{p,a}) \int_{p^{-m} \mathbb{Z}_p} e_p(a y^2) \, dy = |2a|^{1/2} \int_{p^{-m} \mathbb{Z}_p} e(-\{ay^2\}_p) \, dy = |2a|^{1/2} p^m \int_{\mathbb{Z}_p} e \left( -\left\{ \frac{ay^2}{p^{2m}} \right\}_p \right) \, dy \\
= |2|^{1/2} p^{m-r/2} \int_{\mathbb{Z}_p} e \left( -\left\{ \frac{\alpha y^2}{p^{2m-r}} \right\}_p \right) \, dy = |2|^{1/2} p^{r/2-m} \sum_{y \in \mathbb{Z}_p/p^{2m-r} \mathbb{Z}_p} e \left( -\frac{\alpha y^2}{p^{2m-r}} \right) \\
= |2|^{1/2} p^{r/2-m} \sum_{y \in \mathbb{Z}/p^{2m-r} \mathbb{Z}} e \left( -\frac{\alpha y^2}{p^{2m-r}} \right).
\]

To evaluate the inner quadratic Gauss sum, we quote the following famous result of Gauss

\[
\sum_{n=0}^{c-1} e \left( \frac{an^2}{c} \right) = \begin{cases} 
\varepsilon_c \left( \frac{a}{c} \right) \sqrt{c}, & \text{if } 2 \nmid c; \\
\varepsilon_{-a}^{-1}(1+i) \left( \frac{c}{a} \right) \sqrt{c}, & \text{if } a \text{ is odd}, 4|c; \\
0, & \text{if } c \equiv 2 \pmod{4}.
\end{cases}
\]

We start with the case that \( p = 2 \). Since we have assumed \( m \gg 1 \), the quadratic Gauss sum becomes

\[
\sum_{y \in \mathbb{Z}/2^{2m-r} \mathbb{Z}} e \left( -\frac{\alpha y^2}{2^{2m-r}} \right) = \varepsilon_{-\alpha}^{-1}(1+i) \left( \frac{2^{2m-r}}{-\alpha} \right) 2^{m-r/2} = \varepsilon_{-\alpha}^{-1}(1+i) \left( \frac{2}{-\alpha} \right)^r 2^{m-r/2}.
\]

Hence we have

\[
\gamma(e_{2,a}) = \frac{1+i}{\sqrt{2}} \varepsilon_{-\alpha}^{-1} \left( \frac{2}{-\alpha} \right)^r.
\]

Now we assume that \( p \neq 2 \), then the above result on quadratic Gauss sums shows that

\[
\gamma(e_{p,a}) = \varepsilon_{p^{2m-r}} \left( \frac{-\alpha}{p^{2m-r}} \right) = \varepsilon_{p^{2m-r}} \left( \frac{-\alpha}{p} \right)^r.
\]

Since \( p^2 \equiv 1 \pmod{4} \), we have

\[
\varepsilon_{p^{2m-r}} = \varepsilon_{p^r} = \begin{cases} 
1, & \text{if } p \equiv 1 \pmod{4} \text{ or } 2|r; \\
i, & \text{if } p \equiv 3 \pmod{4} \text{ and } 2 \nmid r.
\end{cases}
\]
Hence we have
\[ \gamma(e_{p,a}) = \begin{cases} \frac{-\alpha}{p}, & \text{if } p \equiv 1 \pmod{4} \text{ and } 2 \nmid r; \\ i \left( \frac{-\alpha}{p} \right), & \text{if } p \equiv 3 \pmod{4} \text{ and } 2 \nmid r; \\ 1, & \text{if } 2 | r. \end{cases} \]

4.2. The Real Weil Representation. In this section, we consider the real Weil representation of $\widetilde{SL}(2, \mathbb{R})$.

Lemma 4.2.1. Let $\phi_\infty^0(x) = e^{-2\pi x^2}$. Then
\[ r^{e_\infty}(\kappa_\theta)\phi_\infty^0 = e^{-i\theta}\phi_\infty^0, \quad (\forall \theta \in \mathbb{R}) \]

Proof. We recall that
\[ \kappa_\theta = (\kappa_{2\theta}, \zeta(\theta)) = \begin{pmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, \zeta(\theta). \]

It is straightforward to verify that the lemma is valid if $\theta$ is of form $n\pi/2$ for some $n \in \mathbb{Z}$, so we may assume henceforth that $\sin 2\theta \neq 0$.

By direct computations we have
\[
\begin{pmatrix} 1 & \cos 2\theta/\sin 2\theta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sin 2\theta & -\sin 2\theta \\ -\sin 2\theta & -1 \end{pmatrix} \begin{pmatrix} 1 & \cos 2\theta/\sin 2\theta \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta - \sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{pmatrix}, -\varepsilon,
\]

where for simplicity we write $\varepsilon = \text{sgn} \sin 2\theta$. Write
\[
f(x) = r^{e_\infty} \begin{pmatrix} 1 & \cos 2\theta/\sin 2\theta \\ 1 & 1 \end{pmatrix} \phi_\infty^0(x)
= e_\infty \frac{\cos 2\theta}{\sin 2\theta} x^2 \phi_\infty^0(x) = \exp \left( -2\pi x^2 (1 - i \cot 2\theta) \right),
\]
then
\[
r^{e_\infty}(\kappa_\theta)\phi_\infty^0(x) = -\varepsilon \zeta(\theta) e_\infty \begin{pmatrix} 1 & \cos 2\theta/\sin 2\theta \\ 1 & 1 \end{pmatrix} r^{e_\infty} \begin{pmatrix} -1/\sin 2\theta & 0 \\ 0 & -\sin 2\theta \end{pmatrix} \phi_\infty^0(x)
= -\varepsilon \zeta(\theta) e_\infty \begin{pmatrix} 1 & \cos 2\theta/\sin 2\theta \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1/\sin 2\theta & 0 \\ 0 & -\sin 2\theta \end{pmatrix} \begin{pmatrix} 1 & \cos 2\theta/\sin 2\theta \\ 1 & 1 \end{pmatrix} f(x)
= -\frac{\varepsilon \zeta(\theta)}{|\sin 2\theta|^{1/2}} e_\infty (x^2 \cot 2\theta) \gamma(e_\infty)^2 \gamma(e_\infty, -1/\sin 2\theta) \hat{f} \left( -\frac{x}{\sin 2\theta} \right)
= -\frac{\varepsilon \zeta(\theta)}{|\sin 2\theta|^{1/4}} e^{(2\pi |z|)^{1/2} + 2\pi ivx^2 \cot 2\theta} \hat{f} \left( -\frac{x}{\sin 2\theta} \right)
= \frac{\varepsilon \zeta(\theta)}{|\sin 2\theta|^{1/2}} e^{-2\pi |z|^{1/2} + 2\pi ivx^2 \cot 2\theta} \hat{f} \left( -\frac{x}{\sin 2\theta} \right).
\]

Now recall that, if $\varphi_z(x) = e^{-2\pi x^2}$ for some $z \in \mathbb{C}$ with $\text{Re } z > 0$, then
\[
\hat{\varphi}_z(x) = \frac{1}{\sqrt{\pi}} e^{-2\pi x^2 z}.
\]
In our case, we have
\[
z = 1 + i \cot 2\theta = \frac{\sin 2\theta - i \cos 2\theta}{\sin 2\theta} = \frac{1}{|\sin 2\theta|} e^{2i\theta - \frac{\pi i}{2}},
\]
and, according to our convention,
\[
\frac{1}{z} = (\sin 2\theta)^2 (1 + i \cot 2\theta), \quad \sqrt{z} = \frac{\zeta(\theta)}{\sqrt{|\sin 2\theta|}} e^{i\theta - \frac{\pi i}{4}},
\]
so
\[
\hat{f}(x) = \zeta(\theta) |\sin 2\theta|^{1/2} e^{\frac{\pi i}{4} - i\theta} e^{-2\pi x^2 (\sin 2\theta)^2 (1 + i \cot 2\theta)}.
\]
Hence
\[
\frac{e^{i\kappa \phi_0}(\kappa \theta)}{\phi_0(x)} = \frac{\zeta(\theta)}{|\sin 2\theta|^{1/2}} e^{-\frac{\pi i}{4} + 2\pi i x^2 \cot 2\theta} \hat{f} \left( -\frac{x}{\sin 2\theta} \right) = e^{-i\theta} e^{-2\pi x^2} = e^{-i\theta} \phi_0(x).
\]

4.3. The Nonarchimedean Weil Representation. I. We denote the characteristic function of \(Z_p\) by \(\phi_p^0\).

**Lemma 4.3.1.** If \(p > 2\) and \(a \in Z_p^\times\), then \(r^e_{p,a}(SL(2, Z_p))\) fixes \(\phi_p^0\).

**Proof.** By lemma [3.2.1](#), it suffices to check the assertion on the elements of the set \((3.2.2)\), which is straightforward. \(\square\)

Take \(\mu : Z_p^\times \to \mathbb{C}^\times\) a nontrivial character, and define \(\phi_p^\mu = \mu \cdot 1_{Z_p^\times} : Q_p^\times \to \mathbb{C}\).

**Proposition 4.3.2.** Let \(p > 2\), and let
\[
f = \min \{m \geq 1 \mid \mu(1 + pfZ_p) = 1\}.
\]
Then
\[
r^e_{p} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \phi_p^\mu(y) = \frac{\mu^{-1}(2)\tau(\mu)}{p^f} \phi_p^{\mu^{-1}} \left( ypf \right),
\]
where
\[
\tau(\mu) = \sum_{(p,i)=1}^{p^f-1} \mu(i) e_p \left( \frac{i}{pf} \right).
\]

**Proof.** One may decompose \(\phi_p^\mu\) as a linear combination of characteristic functions:
\[
\phi_p^\mu = \sum_{(p,i)=1}^{p^f-1} \mu(i) 1_{i+pfZ_p},
\]
then
\[
r^e_{p} \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \phi_p^\mu(y) = \frac{1}{p^f} \sum_{(p,i)=1}^{p^f-1} \mu(i) e_p(2iy) 1_{p^{-f}Z_p}(y) = \frac{\mu^{-1}(2)}{p^f} \sum_{(p,i)=1}^{p^f-1} \mu(i) e_p(iy) 1_{p^{-f}Z_p}(y),
\]
where we have applied our previous results that
\[
\gamma(e_p) = 1, \quad \alpha(e_p) = 1.
\]
Obviously the right hand side vanishes if \( y \notin p^{-f} \mathbb{Z}_p \). Moreover, if \( y \notin p^{-f} \mathbb{Z}_p^\times \), then \( i \mapsto e_p(iy) \) is constant on \( 1 + p^f \mathbb{Z} \), which causes the sum against \( \mu \) to vanish. Thus the support of \( r^{e_p} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_p^\mu \) is precisely \( p^{-f} \mathbb{Z}_p^\times \). Further, a change of variables in \( i \) shows that
\[
r^{e_p} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_p^\mu (yy') = \mu(y')^{-1} r^{e_p} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_p^\mu (y).
\]
It follows that the function
\[
y \mapsto r^{e_p} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_p^\mu \left( \frac{y}{p^f} \right)
\]
is a scalar multiple of \( \phi_p^{\mu^{-1}} \), and the scalar is easily seen to be the value at \( \mu^{-1}(2) \tau(\mu)/p^f \).

**Corollary 4.3.3.** With notation as before, we have
\[
r^{e_p} \begin{pmatrix} 0 & p^{-f} \\ -p^f & 0 \end{pmatrix} \phi_p^\mu = \left( \frac{-1}{p} \right)^f \frac{\tau(\mu)\mu^{-1}(2)}{p^{f/2} \gamma(e_{p,p-1})} \phi_p^{\mu^{-1}}.
\]

**Proof.** By definition we have
\[
\beta_p \left( \begin{pmatrix} p^{-f} & 0 \\ 0 & p^f \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) = (p^f,-1)_p(p^f,-p^f) = (p^f,p^f)_p = \left( \frac{-1}{p} \right)^f,
\]
so in \( \widetilde{SL}(2,\mathbb{Q}_p) \) we have
\[
\left( \begin{pmatrix} p^{-f} & 0 \\ 0 & p^f \end{pmatrix}, 1 \right) \left( \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1 \right) = \left( \begin{pmatrix} 0 & p^{-f} \\ -p^f & 0 \end{pmatrix}, \frac{-1}{p} \right)^f.
\]
Hence it follows immediately from Proposition 4.3.2 and the definition of \( r^{e_p} \) on diagonal elements that
\[
r^{e_p} \begin{pmatrix} 0 & p^{-f} \\ -p^f & 0 \end{pmatrix} \phi_p^\mu = \left( \frac{-1}{p} \right)^f r^{e_p} \begin{pmatrix} p^{-f} & 0 \\ 0 & p^f \end{pmatrix} r^{e_p} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \phi_p^\mu
\]
\[
= \left( \frac{-1}{p} \right)^f \frac{\tau(\mu)\mu^{-1}(2)}{p^{f/2} \gamma(e_{p,p-1})} \phi_p^{\mu^{-1}}.
\]

Note that \( \mu \) factors through \( (\mathbb{Z}/p^f\mathbb{Z})^\times \) and that \( \tau(\mu) \) is the Gauss sum of the (primitive) Dirichlet character mod \( p^f \) which it induces. Therefore, \( \frac{\mu^{-1}(2)\tau(\mu)}{\gamma(e_{p,p-1})} \) is a root of unity.

### 4.4. The Nonarchimedean Weil Representation. II

Let \( p \) be a prime, \( M \geq 1 \), and
\[
K_p^{(M)} := \begin{pmatrix} 1 & 0 \\ M & 1 \end{pmatrix} SL(2,\mathbb{Z}_p) \begin{pmatrix} 1 & 0 \\ M^{-1} & 1 \end{pmatrix}.
\]
Note that \( K_p^{(M)} \) depends only on the \( p \)-adic valuation of \( M \). By Lemma 3.2.1, the group \( K_p^{(M)} \) is generated by
\[
\left\{ \begin{pmatrix} -M & 1/M \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 1 & x/M \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{Z}_p \right\}.
\]
In addition, let \( \widetilde{K}_p^{(M)} \) denote the preimage of \( K_p^{(M)} \) in \( \widetilde{SL}(2,\mathbb{Q}_p) \).

In this section, for each prime \( p \) and for suitable values of \( M \), we study a finite dimensional subspace of \( \mathcal{S}(\mathbb{Q}_p) \) which is invariant under the action of \( \widetilde{K}_p^{(M)} \). Specifically, when \( p = 2 \) we consider \( \widetilde{K}_p^{(8)} \); for odd \( p \) we consider \( \widetilde{K}_p^{(p)} \). As our discussions and conclusions change dramatically according to whether \( p > 3 \), \( p = 3 \) or \( p = 2 \). Hence we will consider these three cases separately.
4.4.1. The case \( p > 3 \). In order to work explicitly we introduce some notation from elementary linear algebra. If \( V \) is a complex vector space of finite dimension \( n \), \( B = (\beta_1, \ldots, \beta_n) \) is an ordered basis for \( V \), and \( v \) is a vector in \( V \) then we write \( [v]_B \) for the coordinates of \( v \) relative to \( B \). Thus

\[
[v]_B = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \iff \sum_{i=1}^{n} x_i \beta_i = v.
\]

By identifying \( B \) with the row vector \( [\beta_1 \ldots \beta_n] \), we may write this succinctly as \( B \cdot [v]_B = v \). Similarly, if \( T : V \rightarrow V \) is a linear operator, then \( [T]_B \in \text{Mat}_{n \times n}(\mathbb{C}) \) is the matrix satisfying

\[
[T]_B[v]_B = [Tv]_B \quad (\forall v \in V).
\]

Finally, if \( B_1 \) and \( B_2 \) are two ordered bases of the same space, then \( B_1 c_{B_2} \) is the change-of-basis matrix satisfying

\[
B_1 c_{B_2}[v]_{B_2} = [v]_{B_1}, \quad (\forall v \in V).
\]

Now let \( p > 3 \) and consider the \( \frac{p^2+1}{2} \)-dimensional vector space

\[
V = \text{Span} \left( \left\{ \phi_{p}^{x^2} \mid \xi : (\mathbb{Z}/p\mathbb{Z})^{\times} \rightarrow \mathbb{C} \right\} \cup \{1_{p\mathbb{Z}_p}\} \right)
\]

For our later discussions, it is convenient to construct two other bases for \( V \).

For the first basis of \( V \), recall that if \( p \) is odd then \( x \in \mathbb{Z}_p \) is a square if and only if it is a square modulo \( p \). Now let \( \mathbbm{1}_0 \) denote the characteristic function of \( \{ x \in \mathbb{Z}_p : x^2 \equiv 1 (\text{mod } p) \} \). For example, we have

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1_{p\mathbb{Z}_p}, \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1_{1+p\mathbb{Z}_p} + 1_{-1+p\mathbb{Z}_p}.
\]

Obviously

\[
B_1 = \begin{bmatrix} 1 & \ldots & 1 \\ 1 & \ldots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \ldots & 1 \\ \frac{p-1}{2} & \ldots & \frac{p-1}{2} \\ 0 & \ldots & 0 \end{bmatrix}
\]

forms a basis for \( V \).

For the second basis of \( V \), write \( \mu_{\frac{p-1}{2}} \) for the set of \( (p-1)/2 \)th roots of unity. If \( \psi \) is any Dirichlet character mod \( p \), then the image of \( \psi^2 \) is in \( \mu_{\frac{p-1}{2}} \). We parametrize the set of such characters. Fix a generator \( g \) for \( (\mathbb{Z}/p\mathbb{Z})^{\times} \), and define the character

\[
\psi_j : (\mathbb{Z}/p\mathbb{Z})^{\times} \rightarrow \mu_{\frac{p-1}{2}}, \quad g \mapsto e \left( \frac{2j}{p-1} \right).
\]

Now, our second ordered basis is

\[
B_2 = (\phi_{p}^{\psi_1}, \ldots, \phi_{p}^{\psi_{\frac{p-3}{2}}}, \phi_{p}^{\psi_{\frac{p-1}{2}}}, \phi_{p}, \mathbbm{1}_0).
\]

**Proposition 4.4.1.** The Weil representation \( r_{p}^e \left( \tilde{K}_p^{(p)} \right) \) of the group \( \tilde{K}_p^{(p)} \) preserves the vector space \( V \). Moreover, we have

\[
(4.4.2) \quad \begin{bmatrix} e_p \left( \frac{p}{a} \right) & e_p \left( \frac{2p}{a} \right) \\ r_p \begin{bmatrix} 1 & \frac{a}{p} \\ \frac{p}{a} & 1 \end{bmatrix} & \cdots & e_p \left( \frac{(p-1)}{2} \frac{a}{p} \right) \end{bmatrix}_{V, B_1} = \begin{bmatrix} e_p \left( \frac{1}{p} \right) \\ e_p \left( \frac{4p}{a} \right) \\ \vdots \\ e_p \left( \frac{(p-1)}{2} \frac{a}{p} \right) \\ 1 \end{bmatrix} (a \in \mathbb{Z}_p),
\]

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The key observation is that the function $V$ action of the Weil representation upon the space nontrivial Dirichlet character modulo 3 is an odd quadratic character, so our interest is not in the $\psi$

To prove (4.4.4), note that (4.4.2) follows immediately from the definitions of $r_p$ and $1_{[B]}$. The identity (4.4.3) follows immediately from the definitions of $r_p$ and the elements of $B_2$, along with the fact that for $a \in \mathbb{Z}_p^*$ we have

$$|a|_p = \gamma(e_p) = \gamma(e_p,a) = 1.$$ [4.4.1]

To prove (4.4.4), note that $\psi_j = \psi_{p^{-1} - j}$. Therefore, by Corollary 4.3.3 we have

$$\tau(\psi_j) = \frac{\sqrt{p}}{p^{1/2}\gamma(e_p, p^{-1})} \phi_p.$$ [4.4.5]

Further, direct calculation shows that

$$\gamma(e_p, p^{-1}) = \frac{1}{\varepsilon_p \left( \frac{-1}{p} \right)}$$

and that

$$\phi_p = \frac{\sqrt{p}}{\gamma(e_p, p^{-1})} \phi_p = \frac{1}{\varepsilon_p \sqrt{p}} \phi_p.$$ [4.4.6]

The identity (4.4.4) follows. The rest of the proposition follows from these explicit results, since the elements studied generate $K_p$. $\square$

**Definition 4.4.5.** Set $\phi_{p,B_1}(\gamma) = [r_p(\gamma)]_{V,B_1}$, where $V$ and $B_1$ are defined as above.

For later use we record the change of basis matrix from $B_2$ to $B_1$. It is given by

$$B_1 c_{B_2} = \begin{pmatrix} C_0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$ [4.4.7]

where $C_0$ is a $\frac{p-1}{2} \times \frac{p-3}{2}$ block with $i,j$ entry equal to $\psi_j(i)$. The 1 and 0 in the first row of $B_1 c_{B_2}$ indicate a column of all ones and a column of all zeros. The 0 below $C_0$ is a row of 0’s.

4.4.2. The special case $p = 3$. The case $p = 3$ is different from the above case in that the only nontrivial Dirichlet character modulo 3 is an odd quadratic character, so our interest is not in the action of the Weil representation upon the space $V$ constructed above but upon this odd quadratic character. The key observation is that the function

$$\mathbb{Z}_3^* \rightarrow \mathbb{C}^*, \quad x \mapsto e_3 \left( \frac{a x^2}{3} \right)$$

is constant for every $a \in \mathbb{Z}_3$. 20
**Theorem 4.4.7.** Let \( \chi_3 \) be the unique nontrivial quadratic Dirichlet character modulo 3. Then the span of \( \phi_3^{\chi_3} \) is fixed by \( \overline{K}_3^{(3)} \), and there is a character \( \xi_3 : \overline{K}_3^{(3)} \to \mathbb{C}^\times \) such that

\[
\eta_3^{\chi_3}(k) \phi_3^{\chi_3} = \xi_3(k) \cdot \phi_3^{\chi_3} \quad (\forall k \in \overline{K}_3^{(3)}).
\]

On elementary generators, it is given by

\[
\xi_3 \left( \begin{pmatrix} 1 & a \\ 3 & 1 \end{pmatrix}, 1 \right) = e_3 \left( \frac{a}{3} \right), \quad (a \in \mathbb{Z}_3)
\]

\[
\xi_3 \left( \begin{pmatrix} a & a^{-1} \\ 1 & 1 \end{pmatrix}, 1 \right) = \chi_3(a), \quad (a \in \mathbb{Z}_3^\times)
\]

\[
\xi_3 \left( \begin{pmatrix} -3 & 1/3 \\ 1 & 1 \end{pmatrix}, 1 \right) = 1, \quad \xi_3(I_2, -1) = -1.
\]

**Proof.** The fact that \( \left( \begin{pmatrix} 1 & a \\ 3 & 1 \end{pmatrix}, 1 \right) \) acts by \( e_3 \left( \frac{a}{3} \right) \) follows from the fact that \( x^2 \equiv 1 \pmod{3} \) for all \( x \in \mathbb{Z}_3^\times \). To study the effect of \( \left( \begin{pmatrix} a & a^{-1} \\ 1 & 1 \end{pmatrix}, 1 \right) \), one only has to invoke the definition and check that \( \gamma(e_3) = \gamma(e_3,a) = 1 \) by Proposition (4.1.3). The action of \( \xi_3 \left( \begin{pmatrix} -3 & 1/3 \\ 1 & 1 \end{pmatrix}, 1 \right) \) is given in Corollary 4.3.3 with

\[
\tau(\chi_3) = -i\sqrt{3}, \quad \chi_3(2) = -1, \quad \gamma(e_3) = 1, \quad \gamma(e_3,1/3) = -i. \quad \square
\]

**4.4.3. The case \( p = 2 \).**

**Theorem 4.4.8.** Let \( \chi_2 \) denote the unique nontrivial quadratic Dirichlet character modulo 4. Then the span of \( \phi_2^{\chi_2} \) is fixed by \( \overline{K}_2^{(8)} \), and there is a character \( \xi_2 : \overline{K}_2^{(8)} \to \mathbb{C}^\times \) such that

\[
\eta_2^{\chi_2}(k) \phi_2^{\chi_2} = \xi_2(k) \cdot \phi_2^{\chi_2} \quad (\forall k \in \overline{K}_2^{(8)}).
\]

On elementary generators, it is given by

\[
\xi_2 \left( \begin{pmatrix} 1 & a \\ 8 & 1 \end{pmatrix}, 1 \right) = e_2 \left( \frac{a}{8} \right), \quad (a \in \mathbb{Z}_2)
\]

\[
\xi_2 \left( \begin{pmatrix} a & a^{-1} \\ 1 & 1 \end{pmatrix}, 1 \right) = -i \varepsilon_a \chi_2(a), \quad (a \in \mathbb{Z}_2^\times)
\]

\[
\xi_2 \left( \begin{pmatrix} -8 & 1/8 \\ 1 & 1 \end{pmatrix}, 1 \right) = \frac{1+i}{\sqrt{2}}, \quad \xi_2(I_2, -1) = -1.
\]

**Proof.** The fact that \( \left( \begin{pmatrix} 1 & a \\ 8 & 1 \end{pmatrix}, 1 \right) \) acts by \( e_2 \left( \frac{a}{8} \right) \) follows from the fact that \( x^2 \equiv 1 \pmod{8} \) for all \( x \in \mathbb{Z}_2^\times \).

Next, \( \left( \begin{pmatrix} a & a^{-1} \\ 1 & 1 \end{pmatrix}, 1 \right) \) acts by \( \frac{\gamma(e_2)}{\gamma(e_2,a)} \chi_2(a) \), and it follows from the computation of \( \gamma(e_2,a) \) given earlier that for \( a \in \mathbb{Z}_2^\times \), this is equal to \( -i \varepsilon_a \chi_2(a) \).

Lastly, we have

\[
\beta_2 \left( \begin{pmatrix} -1 & 1 \\ 1/8 & 1 \end{pmatrix}, \begin{pmatrix} 8 \\ 1/8 \end{pmatrix} \right) = 1.
\]

Since

\[
\left[ \eta_2^{e_2} \left( \begin{pmatrix} 8 \\ 1/8 \end{pmatrix} \right) \phi_2^\chi \right] (x) = \left| 8 \right|^{1/2} \frac{\gamma(e_2)}{\gamma(e_2,8)} \phi_2^\chi (8x) = \frac{1}{2\sqrt{2}} \phi_2^\chi (8x),
\]

\[21\]
we have
\[
\left[r_2^{e_2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right] \phi_2^\beta(x) = \frac{1 - i}{4\sqrt{2}} \int_{\mathbb{Q}_2} \phi_2^\beta(8y)e_2(2xy) \, dy
\]
\[
= \frac{1 - i}{4\sqrt{2}} \int_{\mathbb{Q}_2} \left( 1 \frac{z_2}(y) + \frac{z_2}(y) - 1 \frac{z_2}(y) - 1 \frac{z_2}(y) \right) e_2(2xy) \, dy
\]
\[
= \frac{1 - i}{4\sqrt{2}} \left( e_2\left(\frac{x}{4}\right) + e_2\left(\frac{5x}{4}\right) - e_2\left(\frac{3x}{4}\right) - e_2\left(\frac{7x}{4}\right) \right) \int_{\mathbb{Q}_2} e_2(2xy) \, dy
\]
\[
= \frac{1 - i}{4\sqrt{2}} e_2\left(\frac{x}{4}\right) e_2\left(\frac{x}{2}\right) \left( 1 - e_2\left(\frac{x}{2}\right) \right) \left( 1 + e_2(x) \right)
\]
\[
= \frac{1 - i}{2\sqrt{2}} e_2\left(\frac{x}{4}\right) e_2\left(\frac{x}{2}\right) \left( 1 - e_2\left(\frac{x}{2}\right) \right)
\]
\[
= \frac{1 - i}{\sqrt{2}} e_2\left(\frac{x}{4}\right) e_2\left(\frac{x}{2}\right) = \frac{1 - i}{\sqrt{2}} e_2\left(\frac{x}{4}\right) e_2\left(\frac{x}{2}\right)
\]
\[
= \frac{1 - i}{\sqrt{2}} e_2\left(\frac{x}{4}\right) e_2\left(\frac{x}{2}\right) = \frac{1 - i}{\sqrt{2}} e_2\left(\frac{x}{4}\right) e_2\left(\frac{x}{2}\right)
\]
\[
= \frac{1 - i}{\sqrt{2}} \left( 1_{1+4z_2}(x) - 1_{3+4z_2}(x) \right) = -\frac{1 + i}{\sqrt{2}} \phi_2^\beta(x). \quad \Box
\]

5. Global Metaplectic Group and Weil Representation

5.1. Global Metaplectic Group. If \( g = \{g_v\}_v, h = \{h_v\}_v \in SL(2, \mathbb{A}) \), then \( \beta_v(g_v, h_v) = 1 \) for all but finitely many \( v \) (see [Gelbart, Proposition 2.8]). Set
\[
\beta(g, h) = \prod_v \beta_v(g_v, h_v).
\]

Here \( v \) runs over the places on \( \mathbb{Q} \). Then \( \widetilde{SL}(2, \mathbb{A}) \) is defined as \( SL(2, \mathbb{A}) \times \{ \pm 1 \} \) equipped with the product
\[
(g_1, \zeta_1)(g_2, \zeta_2) := (g_1g_2, \beta(g_1, g_2)\zeta_1\zeta_2), \quad (g_1, g_2 \in SL(2, \mathbb{A}), \; \zeta_1, \zeta_2 \in \{ \pm 1 \}).
\]

For each \( v \), we have the embedding \( i_v : SL(2, \mathbb{Q}_v) \hookrightarrow SL(2, \mathbb{A}) \). The definition is that for \( g_v \in SL(2, \mathbb{Q}_v) \) and \( w \) a place of \( \mathbb{Q} \), the component \( i_v(g_v)_w \) of \( i_v(g_v) \) at \( w \) is \( g_v \) if \( v = w \) and the identity \( I_2 \) otherwise. Now, for all \( w \), the cocycle \( \beta_w \) is trivial on \( \{ I_2 \} \times SL(2, \mathbb{Q}_w) \) and \( SL(2, \mathbb{Q}_w) \times \{ I_2 \} \), which implies that the restriction of the global cocycle \( \beta \) to the image of \( SL(2, \mathbb{Q}_v) \times SL(2, \mathbb{Q}_v) \) in \( SL(2, \mathbb{A}) \) is precisely the local cocycle \( \beta_v \). It follows that \( i_v \) extends to an embedding \( \tilde{i}_v : \widetilde{SL}(2, \mathbb{Q}_v) \hookrightarrow \widetilde{SL}(2, \mathbb{A}) \) defined by
\[
\tilde{i}_v(g, \zeta) = (i_v(g), \zeta), \quad (g \in SL(2, \mathbb{Q}_v), \; \zeta \in \{ \pm 1 \}).
\]

We shall also make use of the embedding
\[
i_{\text{diag}} : SL(2, \mathbb{Q}) \hookrightarrow \widetilde{SL}(2, \mathbb{A}), \quad \gamma \mapsto \left( \gamma, s_\mathbb{A}(\gamma) \right)
\]
as described in [Gelbart] Page 23, where
\[
s_\mathbb{A} = \prod_v s_v.
\]

We also let
\[
i_t(\gamma) = i_\infty(\gamma^{-1}, 1)i_{\text{diag}}(\gamma) \in \widetilde{SL}(2, \mathbb{A}), \quad (\gamma \in SL(2, \mathbb{Q})).
\]
Observe that \( i_f \) is not a homomorphism. Rather, it satisfies
\[
(5.1.1) \quad i_f(\gamma_1)i_f(\gamma_2) = i_f(\gamma_1 \gamma_2) \cdot (I_2, \beta_\infty(\gamma_2^{-1}, \gamma_1^{-1})), \quad (\gamma_1, \gamma_2 \in SL(2, \mathbb{Q})).
\]
Indeed, \( i_f(\gamma_1) \) commutes with \( \tilde{\iota}_\infty(\gamma_2) \) since either one or the other of them has the identity matrix at each place. Hence
\[
i_f(\gamma_1)i_f(\gamma_2) = i_f(\gamma_1)\tilde{\iota}_\infty(\gamma_2^{-1}, 1)i_{\text{diag}}(\gamma_2) = \tilde{\iota}_\infty(\gamma_2^{-1}, 1)i_f(\gamma_1)i_{\text{diag}}(\gamma_2)
\]
\[
= \tilde{\iota}_\infty(\gamma_2^{-1}, 1)\tilde{\iota}_\infty(\gamma_1^{-1}, 1)i_{\text{diag}}(\gamma_1)i_{\text{diag}}(\gamma_2)
\]
\[
= \tilde{\iota}_\infty(\gamma_2^{-1}, 1, \beta_\infty(\gamma_2^{-1}, \gamma_1^{-1}))i_{\text{diag}}(\gamma_1)(\gamma_2)
\]
\[
= \tilde{\iota}_\infty(\gamma_2^{-1}, \gamma_1^{-1}, 1)i_{\text{diag}}(\gamma_1\gamma_2)(I_2, \beta_\infty(\gamma_2^{-1}, \gamma_1^{-1})).
\]
By [Gelbart, Proposition 2.8], the restriction of \( \tilde{\iota}_p \) to \( SL(2, \mathbb{Z}_p) \) is a homomorphism for \( p > 2 \). (Cf. [Gelbart, Page 19].) It follows that the inclusion
\[
i_S : \prod_{v \in S} SL(2, \mathbb{Q}_v) \times \prod_{p \notin S} SL(2, \mathbb{Z}_p) \hookrightarrow SL(2, \mathbb{A})
\]
extends to a homomorphism
\[
\tilde{i}_S : \prod_{v \in S} \tilde{SL}(2, \mathbb{Q}_v) \times \prod_{p \notin S} SL(2, \mathbb{Z}_p) \hookrightarrow \tilde{SL}(2, \mathbb{A})
\]
for any finite set \( S \) of places of \( \mathbb{Q} \) which contain \( \{2, \infty\} \). The kernel of this homomorphism is
\[
\ker \tilde{i}_S = \left\{ (I_2, \varepsilon_v)_{v \in S} \times (I_2)_{p \notin S} : \prod_{v \in S} \varepsilon_v = 1 \right\}.
\]
Moreover
\[
\tilde{SL}(2, \mathbb{A}) = \bigcup_S \text{im} \tilde{i}_S,
\]
with the union ranging over finite sets \( S \) of places of \( \mathbb{Q} \) which contain \( \{2, \infty\} \).

5.2. **Global Weil Representation.** The adelic Bruhat-Schwartz space \( S(\mathbb{A}) \) consists of all finite linear combinations of functions \( \bigotimes_{v} \varphi_v \), where \( \varphi_v \) is in \( S(\mathbb{Q}_v) \) for all \( v \) and \( \varphi_p = \phi_p^0 \) is the characteristic function of \( \mathbb{Z}_p \) for all but finitely many primes \( p \).

For any finite set \( S \) of places of \( \mathbb{Q} \), the injection
\[
\bigotimes_{v \in S} \varphi_v \mapsto \bigotimes_{v \in S} \varphi_v \otimes \bigotimes_{p \notin S} \phi_p^0,
\]
sends \( \bigotimes_{v \in S} S(\mathbb{Q}_v) \) to a subspace of \( S(\mathbb{A}) \). The action of \( \tilde{SL}(2, \mathbb{Q}_v) \) on \( S(\mathbb{Q}_v) \) induces an action on \( S(\mathbb{A}) \) for all \( v \). Moreover, the action of \( \{I_2\} \times \{\pm 1\} \subset \tilde{SL}(2, \mathbb{Q}_v) \) is the same (scalar multiplication) for all \( v \). By Lemma [4.3.1] \( SL(2, \mathbb{Z}_p) \) fixes \( \phi_p^0 \) for all but finitely many \( p \). To be precise, if
\[
\psi(\{x_v\}) = \prod_v \varepsilon_v(ax_v)
\]
for some \( a \in \mathbb{Q}^\times \), then \( SL(2, \mathbb{Z}_p) \) fixes \( \phi_p^0 \) for all \( p > 2 \) such that \( a \in \mathbb{Z}_p^\times \).

Take \( S \) a finite set of places containing \( \{2, \infty\} \), then the formula
\[
\left[ r_{S}^\psi \left( i_S ((g_v, \zeta_v)_{v \in S} \times (k_p, 1)_{p \notin S}) \right) \right] \cdot \left[ \bigotimes_{v \in S} \varphi_v \otimes \bigotimes_{p \notin S} \phi_p^0 \right] = \left( \prod_v \zeta_v \right) \cdot \bigotimes_v [r_{\psi_v(\{g_v\})}: \varphi_v \otimes \bigotimes_{p \notin S} \phi_p^0]
\]
gives a well-defined action of \( \tilde{SL}(2, \mathbb{A}) \) on \( S(\mathbb{A}) \).
6. The Adelic Theta Functions

For any $\varphi \in S(\mathbb{A})$ and character $\psi : \mathbb{Q}\backslash \mathbb{A} \to \mathbb{C}$, define

$$\Theta^\psi_{ad}(\varphi; \bar{g}) := \sum_{\xi \in \mathbb{Q}} \left[r^\psi(\bar{g}).\varphi\right](\xi).$$

It follows from [Gelbart, Proposition 2.33] that

$$\Theta^\psi_{ad}(\varphi; i_{\text{diag}}(\gamma)\bar{g}) = \Theta^\psi_{ad}(\varphi; \bar{g}), \quad \left(\forall \varphi \in S(\mathbb{A}), \ \bar{g} \in \widetilde{SL}(2, \mathbb{A}), \ \gamma \in SL(2, \mathbb{Q})\right).$$

The function $\Theta^\psi_{ad}$ is then an intertwining map from the representation $r^\psi$ to the representation of $\widetilde{SL}(2, \mathbb{A})$ on automorphic forms by right translation, namely

$$(6.0.1) \quad \Theta^\psi_{ad}(\varphi; g_1g_2) = \Theta^\psi_{ad}(r^\psi(g_2)\varphi; g_1), \quad (g_1, g_2 \in \widetilde{SL}(2, \mathbb{A}), \ \varphi \in S(\mathbb{A})).$$

We now construct an adelic theta function corresponding to the classical theta functions $\theta_\chi$. The first step is to define the corresponding element of $S(\mathbb{A})$. Recall that

$$\phi_0^0(u) = e^{-2\pi u^2}, \quad \phi_p^0 = 1_{\mathbb{Z}_p},$$

and that, if $\mu$ is any character of $\mathbb{Z}_p^\times$, define $\phi_\mu^p = \mu \cdot 1_{\mathbb{Z}_p^\times} : \mathbb{Q}_p^\times \to \mathbb{C}$.

Let $\chi \ (\text{mod} \ M)$ be an even Dirichlet character. One may write $\chi$ uniquely

$$\chi(m) = \prod_{p | M} \chi_p(m),$$

where $\chi_p$ is a Dirichlet character modulo $p^{\nu_p(M)}$ with $p^{\nu_p(M)} \parallel M$. Now, $(\mathbb{Z}/p^{\nu_p(M)}\mathbb{Z})^\times$ is identified with $\mathbb{Z}_p^\times/(1 + p^{\nu_p(M)}\mathbb{Z}_p)$, so we may regard $\chi_p$ as a character of $\mathbb{Z}_p^\times$ which is trivial on $1 + p^{\nu_p(M)}\mathbb{Z}_p$.

Now let

$$\phi_\chi^\nu(v) = \begin{cases} \phi_\nu^0, & \text{if } v = \infty \text{ or } v \nmid M; \\ \phi_\nu^p, & \text{if } v = p \mid M, \end{cases} \quad \phi_\chi(x) = \prod_v \phi_\chi^v(x_v), \quad \{x_v\}_v \in \mathbb{A} = \mathbb{A}_\mathbb{Q}$$

This is an element of the adelic Bruhat-Schwartz space $S(\mathbb{A})$.

**Lemma 6.0.2.** Let $z \in \mathcal{H}$. Then we have

$$\Theta^e_{ad}(\phi_\chi^\nu; i_{\infty}(b_z)) = \theta_\chi^\nu_{\text{Ma}}(z).$$

**Proof.** For $y > 0$, we write

$$\phi_\chi^\nu(x) = r^e \left(i_{\infty}\left(\begin{pmatrix} y & \frac{1}{2} \\ \frac{1}{2} & y^{-\frac{1}{2}} \end{pmatrix}\right)\right) \phi_\chi(x),$$

then by definition

$$\Theta^e_{ad}(\phi_\chi^\nu; i_{\infty}(b_z)) = \sum_{\xi \in \mathbb{Q}} r^e \left(i_{\infty}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \phi_\chi^\nu\left(\begin{pmatrix} 1 & y \frac{1}{2} \\ 0 & y^{-\frac{1}{2}} \end{pmatrix}\right)\right) (\xi) = \sum_{\xi \in \mathbb{Q}} e_{\infty}(x\xi^2) \phi_\chi^\nu(\xi).$$
Corollary 6.0.4. If \( H \) then we have
\[
\phi^\chi_y(x) = r^\infty \left( \left( \begin{array}{cc}
\frac{1}{2} & iy \\
y & \frac{1}{2}
\end{array} \right) \right) \phi^\chi_\infty(x_\infty) \cdot \prod_p \phi^\chi_p(x_p)
\]
so
\[
\Theta^e_{ad}(\phi^\chi; \iota(\eta)) = \frac{1}{2} \sum_{\xi \in \mathbb{Q}} e^{\infty}(x\xi^2)\phi^\chi_y(\xi) = \frac{1}{2} \sum_{\xi \in \mathbb{Q}} e^{2\pi ix\xi^2} e^{-2\pi y\xi^2} \prod_p \phi^\chi_p(\xi).
\]

For every \( \xi \in \mathbb{Q} \), direct computations show that
\[
\prod_p \phi^\chi_p(\xi) = \begin{cases} 0, & \text{if } \xi \notin \mathbb{Z}; \\ \chi(\xi), & \text{if } \xi \in \mathbb{Z}. \end{cases}
\]
Hence we have
\[
\Theta^e_{ad}(\phi^\chi; \iota(\eta)) = \begin{cases} 0, & \text{if } \xi \notin \mathbb{Z}; \\ \chi(\xi), & \text{if } \xi \in \mathbb{Z}. \end{cases}
\]

Theorem 6.0.3. Let \( e = \prod_v e_v \). Then
\[
\Theta^e_{ad}(\phi^\chi; \iota(\eta)) = (\theta^\text{Maa}_{\chi}(\eta)) (i), \quad (\forall \eta_\infty \in \hat{SL}(2, \mathbb{R})).
\]
Proof. Let \( \eta_\infty \in \hat{SL}(2, \mathbb{R}) \), then we have shown the Iwasawa decomposition
\[
\hat{\eta}_\infty = \left( \begin{array}{cc}
y/2 & xy^{-1/2} \\
y^{-1/2} & y_1/2
\end{array} \right) \kappa = b_x + iy \kappa
\]
for some \( x \in \mathbb{R}, y > 0 \) and \( \theta \in \mathbb{R} \). In particular, we have \( \hat{j}(\eta_\infty, i) = e^{i\theta} \). Hence
\[
\Theta^e_{ad}(\phi^\chi, \iota(\eta_\infty)) = \sum_{\xi \in \mathbb{Q}} \left[ r^e(i\eta_\infty) \phi^\chi(x) \right] (\xi) = \sum_{\xi \in \mathbb{Q}} \left[ r^e(i\eta_\infty(b_x + iy)) r^{e(\kappa)} \phi^\chi(x) \right] (\xi)
\]
\[
= e^{-i\theta} \sum_{\xi \in \mathbb{Q}} \left[ r^e(i\eta_\infty(b_x + iy)) \phi^\chi(x) \right] (\xi) = e^{-i\theta} \theta^\text{Maa}_{\chi}(x + iy)
\]
\[
= \theta^\text{Maa}_{\chi}(\eta_\infty \cdot i) = (\theta^\text{Maa}_{\chi}(\eta_\infty))(i).
\]

Corollary 6.0.4. If \( \gamma \in SL(2, \mathbb{Q}) \) then
\[
\left( \theta^\text{Maa}_{\chi} \right)_{\chi}^{\gamma} (z) = \Theta^e_{ad} (r^e(i\eta(\gamma^{-1})), \phi^\chi; \iota(\eta)) (z).
\]
Proof. By equation (3.3.11),
\[
\left( \theta^\text{Maa}_{\chi} \right)_{\chi}^{\gamma} (z) = \left( \theta^\text{Maa}_{\chi} \right)^{(\gamma, 1)} (z) = \Theta^e_{ad} (\phi^\chi; \iota(\gamma b_z, 1)).
\]
But \( \Theta^e_{ad} \) is invariant on the left by \( i_{\text{diag}}(\gamma^{-1}) \), so this is equal to
\[
\Theta^e_{ad} (\phi^\chi; \iota(\gamma^{-1})) = \Theta^e_{ad} (\phi^\chi; \iota((b_z, 1))) \iota(\gamma^{-1}).
\]
Applying (6.0.1) completes the proof.

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7. Fourier Coefficients of Classical Theta Functions

7.1. The First Twist.

**Theorem 7.1.1.** Let $\chi_2$ and $\chi_3$ be the nontrivial quadratic Dirichlet characters modulo 4 and 3 respectively. Let $\chi = \chi_2\chi_3$ and let $\theta_\chi$ be the associated classical theta function of level 576. Define $\xi_2 : \hat{K}_2 \rightarrow \mathbb{C}^\times$ as in Theorem 4.4.8 and $\xi_3 : \hat{K}_3 \rightarrow \mathbb{C}^\times$ as in Theorem 4.4.7 and write

$$\xi : \Gamma^{(24)} \rightarrow \mathbb{C}^\times \quad \gamma \mapsto \xi_2(\gamma,1)\xi_3(\gamma,1)s_\chi(\gamma)\beta_\infty(\gamma^{-1},\gamma).$$

Then for all $\sigma \in \Gamma^{(24)}$ we have

$$\theta_\chi\bigg|_{\frac{1}{2}} \sigma = \xi(\sigma^{-1})\theta_\chi.$$

**Remark 7.1.2.** Note that $\xi$ is not a homomorphism. In fact, $\xi$ is the composition of a homomorphism with $i_\Gamma$. But as in (5.1.1), $i_\Gamma$ is not a homomorphism.

**Proof.** By Corollary 4.3.4

$$(\theta_\chi \big|_{\frac{1}{2}})^{\text{Maa}}(z) = \Theta^e_{\text{ad}}(r^e(i_\Gamma(\sigma^{-1})),\phi^\chi; i_\infty(b_2)).$$

Using Theorems 4.4.7 and 4.4.8 as well as Lemma 4.3.1 and the definitions of $s_\chi, i_\Gamma$ and $\xi$ yields,

$$\Theta^e_{\text{ad}}(r^e(i_\Gamma(\sigma^{-1})),\phi^\chi; i_\infty(b_2)) = \xi(\sigma^{-1})\Theta^e_{\text{ad}}(\phi^\chi; i_\infty(b_2)) = \xi(\sigma^{-1})\theta_\chi^{\text{Maa}}(z).$$

Multiplying by $\text{Im}(z)^{-\frac{1}{4}}$ gives the result. \qed

**Corollary 7.1.3.** Let $a$ be a cusp for $\Gamma_0(576)$ and $\sigma \in \Gamma^{(24)}$ a scaling matrix for $a$. Then we have

$$A_{\theta_\chi}(\sigma, n) = \xi(\sigma^{-1})A_{\theta_\chi}(I_2, n).$$

**Proof.** The existence of a scaling matrix $\sigma \in \Gamma^{(24)}$ is guaranteed by Lemma 2.1.2 and the rest follows immediately from the theorem. \qed

**Theorem 7.1.5.** Under the notations of Theorem 7.1.1 let $a = u/w$ be a cusp of $\Gamma_0(576)$. Then we have

$$A_{\theta_\chi}(\sigma^0_a, n) = \begin{cases} \xi_2(\sigma_a^{-1})\xi_3(\sigma_a^{-1})s_2(\sigma_a^{-1})s_3(\sigma_a^{-1})e\left(-\frac{n^2wr}{24u[24,w]}\right)\chi(m), & \text{if } n = m^2 \geq 1, \\ 0, & \text{if otherwise}, \end{cases}$$

where we choose $r, s \in \mathbb{Z}$ such that $24us - wr = (24, w)$, and the scaling matrices $\sigma^0_a$ and $\sigma_a$ are as given in (2.1.1) and (2.1.3) respectively.

**Proof.** By Corollary 7.1.3 we have

$$A_{\theta_\chi}(\sigma, n) = \xi_2(\sigma_a^{-1})\xi_3(\sigma_a^{-1})s_\chi(\sigma_a^{-1})\beta_\infty(\sigma_a,\sigma_a^{-1})A_{\theta_\chi}(I_2, n).$$

To begin with, we note that

$$\beta_\infty(\sigma_a,\sigma_a^{-1}) = ([24, w], [24, w]) = 1.$$ 

Further, since $w|576$, we have $s_p(\sigma_a,\sigma_a^{-1}) = 1$ for every $p \geq 5$. Hence

$$A_{\theta_\chi}(\sigma, n) = \xi_2(\sigma_a^{-1})\xi_3(\sigma_a^{-1})s_2(\sigma_a^{-1})s_3(\sigma_a^{-1})A_{\theta_\chi}(I_2, n).$$

Hence the theorem follows immediately as we recall that

$$A_{\theta_\chi}(\sigma^0_a, n) = e\left(-\frac{nwr}{24u[24,w]}\right)A_{\theta_\chi}(\sigma, n).$$

\qed
7.2. The Higher Twists. Now we consider the higher twists of the classical theta functions. In this section, we let \( p \geq 5 \).

**Theorem 7.2.1.** Let \( \psi \pmod{p} \) be an even Dirichlet character and \( \gamma \in \Gamma^{(24p)} \). Then we have

\[
\theta_{\chi, \psi}^{\text{Ma}} \mid_{\frac{1}{2}} \gamma(z) = \xi(\gamma^{-1}) \Theta_{\text{ad}}^e \left( \prod_{v \neq 2, 3, p} \phi_v^\gamma \cdot i_\infty(b_z) \right),
\]

where \( \xi \) is as defined in Theorem 7.1.1.

**Proof.** We have

\[
\theta_{\chi, \psi}^{\text{Ma}} \mid_{\frac{1}{2}} \gamma(z) = \Theta_{\text{ad}}^e \left( \prod_{v \neq 2, 3, p} \phi_v^\gamma \cdot i_\infty(b_z) \right) = \Theta_{\text{ad}}^e \left( \prod_{v \neq 2, 3, p} \phi_v^\gamma \cdot i_\infty(b_z) \right).
\]

**Theorem 7.2.2.** Fix a generator for \((\mathbb{Z}/p\mathbb{Z})^\times\), and adopt the notations from Section 4.1.1. Further, we define \( \chi \) and \( \xi \) as in Theorem 7.1.1. Then for \( \sigma \in \Gamma^{(24p)} \),

\[
A_{\theta_{\chi, \psi}}(\sigma, n) = 2\xi(\sigma^{-1}) \chi(n) \cdot \ell \cdot \mathcal{G}_{B_1, p}(\sigma^{-1}) \cdot B_1 c B_2 \cdot e_j,
\]

where \( i(n) \in \{1, \ldots, \frac{p-1}{2}\} \) is \( \frac{p+1}{2} \) if \( p \mid n \) and otherwise is the unique element of \( \{1, \ldots, \frac{p-1}{2}\} \) satisfying \( i(n)^2 \equiv n^2 \pmod{p} \).

**Proof.** Arguing as before, we find that

\[
\left( \theta_{\chi, \psi}^{\text{Ma}} \right)^{\text{Ma}} \mid_{\frac{1}{2}} \sigma(z) = \xi(\sigma^{-1}) \Theta_{\text{ad}}^e \left( \prod_{v \neq 2, 3, p} \phi_v^\gamma \cdot i_\infty(b_z) \right).
\]

Note that \( e_j = [\phi_p^\gamma]_{B_2} \). Thus

\[
\left[ r_{\sigma^{-1}}(1), \phi_p^\gamma \right]_{B_1} = \left[ r_{\sigma^{-1}}(1) \right]_{B_1} = \mathcal{G}_{B_1, p}(\sigma^{-1}) \cdot B_1 c B_2 \cdot e_j.
\]

Thus

\[
\ell \cdot \mathcal{G}_{B_1, p}(\sigma^{-1}) \cdot B_1 c B_2 \cdot e_j
\]

is the coefficient of \( \left( \theta_{\chi, \psi}^{\text{Ma}} \right)^{\text{Ma}} \mid_{\frac{1}{2}} \sigma \) in the expansion of \( [r_{\sigma^{-1}}(1), \phi_p^\gamma] \) in terms of \( B_1 \). For \( k \in \{0, \ldots, \frac{p-1}{2}\} \), let

\[
\theta_{\chi, \psi}^{[p]} \mid_{\frac{1}{2}} \sigma(z) = \Theta_{\text{ad}}^e \left( \prod_{v \neq 2, 3, p} \phi_v^\gamma \cdot i_\infty(b_z) \right).
\]

Let \( \theta_{\chi, \psi}^{[p]} = [\theta_{\chi, \psi}^{[p]}]_{\chi(1)}, \ldots, [\theta_{\chi, \psi}^{[p]}]_{\chi(\frac{p-1}{2})} \). Then by linearity of \( \Theta_{\text{ad}}^e \) in the first argument, one obtains

\[
\left( \theta_{\chi, \psi}^{\text{Ma}} \right)^{\text{Ma}} \mid_{\frac{1}{2}} \sigma(z) = \xi(\sigma^{-1}) \cdot \theta_{\chi, \psi}^{[p]} \cdot \mathcal{G}_{B_1, p}(\sigma^{-1}) \cdot B_1 c B_2 \cdot e_j.
\]
Now, following the proof of Lemma 6.0.2, one readily checks that
\[
\theta_{\chi^p[k]}(z) = \sum_{n=-\infty}^{\infty} \frac{1}{k(n)} \chi(n) e^{-2\pi in^2z}.
\]
Thus
\[
A_{\theta_{\chi^p[k]}}(I^2, n) = \begin{cases} 
2\chi(n), & \text{if } k \equiv i \pmod{n}, \\
0, & \text{if } k \not\equiv i \pmod{n}.
\end{cases}
\]
From here, the result follows easily. □

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