Einstein’s $R^{00}$ equation for nonrelativistic sources derived from Einstein’s inertial motion and the Newtonian law for relative acceleration

Christoph Schmidt

ETH Zurich, Institute for Theoretical Physics, 8093 Zurich, Switzerland

(Dated: March 22, 2018)

With Einstein’s inertial motion (freefalling and nonrotating relative to gyroscopes), geodesics for nonrelativistic particles can intersect repeatedly, allowing one to compute the space-time curvature $R^{00}$ exactly. Einstein’s $R^{00}$ for strong gravitational fields and for relativistic source-matter is identical with the Newtonian expression for the relative radial acceleration of neighbouring freefalling test-particles, spherically averaged. — Einstein’s field equations follow from Newtonian experiments, local Lorentz-covariance, and energy-momentum conservation combined with the Bianchi identity.

PACS numbers: 04.20.-q, 04.20.Cv

Up to now, a rigorous derivation of Einstein’s field equations for general relativity has been lacking: Wald [1] writes “a clue is provided”, “the correspondence suggests the field equation”. Weinberg [2] takes the ”weak static limit”, makes a “guess”, and argues with ”number of derivatives”. Misner, Thorne, and Wheeler [3] give ”Six Routes to Einstein’s field equations”, among which they recommend (1) “model geometrodynamics after electrodynamics”, (2) ”take the variational principle with only a scalar linear in second derivatives of the metric and no higher derivatives”.

In contrast, we give a rigorous derivation of Einstein’s field equations for general relativity.

The crucial input is Einstein’s concept of inertial motion (freefalling and nonrotating relative to gyroscopes): the worldlines of freefalling nonrelativistic test-particles, geodesics, can intersect repeatedly. Two stones released one after the other from rest at the North Pole freefalling into a vertical well through the center of the Earth to the South Pole and back through the center of the Earth to the North Pole: Einstein’s geodesics cross repeatedly, violating the spacetime analogue of Euclid’s axiom of parallels, evidence that space-time is curved.

In our decisive first step, we prove that the exact space-time curvature encoded in $R^{00} (P)$ (curvature-side of Einstein’s equation) is fully determined by measuring test-particles which are quasi-static (or non-relativistic) relative to an observer with worldline through $P$ and $\bar{u}_{\text{obs}} (P) = \bar{e}_{0} (P)$. It is superfluous to re-measure $R^{00}$ with relativistic test-particles. — Further, we prove that $R^{00}$ is identical with the Newtonian expression for the relative radial acceleration of neighboring freefalling nonrelativistic test-particles, spherically averaged, for gravitational fields of arbitrary strength and for arbitrary (relativistic) source-matter. — Hats over indices denote Local Ortho-Normal Bases (LONB) following Misner, Thorne, and Wheeler, and we use their sign conventions [3].

The explicit expression for Einstein’s curvature $R^{00}$ in general coordinates in terms of Christoffel connection coefficients $(\Gamma_{\beta\gamma}^\alpha)_{\tau} = \Gamma_{\beta\gamma}^\alpha$ has 106 terms, utterly unconstructive. Newtonian relative acceleration in general Lagrangian 3-coordinates (e.g. comoving with the wind) has the same number of 106 unconstructive terms.

The expressions for Einstein’s $R^{00} (P)$ and the Newtonian relative acceleration are extremely simple and explicitly identical with the following choices: (1) We work with Local Ortho-Normal Bases (LONBs) in Cartan’s method. (2) We use a primary observer (non-inertial or inertial) with worldline through $P$, with $\bar{u}_{\text{obs}} = \bar{e}_{0}$, and with his spatial LONBs $\bar{e}_{j}$ along his worldline. (3) We use the primary observer’s spacetime slicing $\Sigma_{t}$ by radial 4-geodesics starting Lorentz-orthogonal to his worldline, and $t = \text{time measured on his worldline}$. (4) Most crucial: for measuring relative accelerations of neighboring test-particles, we need auxiliary observers with LONBs radially parallel (at a given time) to the primary observer’s LONBs (to avoid unnecessary extra terms). Therefore the Ricci connection coefficients for radial displacements from the primary observer’s worldline vanish, auxiliary observers with radially parallel LONBs at given time

$$\Leftrightarrow \left[ [\omega^-registration_free}^{\hat{\alpha}_{i}^{\hat{\beta}}} x_{j}^{\gamma} \right]_{r=0} = 0, \quad (1)$$

for $(a, b, ...)$ = spacetime indices, $(i, j, ...)$ = 3-space indices. Eq. (1) is a special case of Newtonian observers at relative rest at a given time. (5) We use Riemann normal 3-coordinates centered on the primary observer on the slices of fixed time: $r$-coordinate lines are radial geodesics, $r \equiv$ geodesic radial distance, $(\theta, \phi)_{P} \equiv$ starting angles of radial geodesics to $P$, and $(x, y, z)_{P}$ with the standard connection to spherical coordinates.

In this first paper, we treat only inertial primary observers. Hence the Ricci connection coefficients for displacements along the primary observer’s worldline vanish,

inertial primary observer $\Leftrightarrow \left[ [\omega^-registration_free}^{\hat{\alpha}_{i}^{\hat{\beta}}} x_{j}^{\gamma} \right]_{r=0} = 0. \quad (2)$

With Eqs. (1) [2], all Ricci connection coefficients vanish on the worldline of the primary inertial observer. Our auxiliary observers cannot be inertial, unless spacetime is flat.
The result: the expression for Einstein’s $R^0_0$ in terms of quasi-static (or non-relativistic) test-particles, for gravitational fields of arbitrary strength, and for arbitrary (relativistic) source-matter, is exactly and explicitly identical with the Newtonian expression, and this expression is exactly linear in the gravitational field, inertial primary observer, radially parallel LONBs:

$$R^0_0 = \text{div} \vec{E}_g = \text{div} \vec{g}. \quad (3)$$

In our exact operational definition in arbitrary (3+1)-spacetimes, the gravito-electric field $\vec{E}_g = \vec{g}$ is the acceleration of quasi-static (or non-relativistic) freefalling test-particles, measured by the chosen observer. But this $\vec{E}_g$ remains exactly valid for relativistic test-particles in the equations of motion and in curvature calculations.

In a companion paper, we shall show in detail that for non-inertial observers and for quasi-static (or non-relativistic) test-particles, (1) the exact explicit expression for Einstein’s $R^0_0$ and the 19th-century Newtonian expression for relative acceleration of neighbouring freefalling particles, spherically averaged, are identical, if one uses Einstein’s equivalence of fictitious forces and gravitational forces ($\vec{E}_g$, $\vec{B}_g$), which has been demonstrated explicitly in [4], and (2) that the two identical expressions are nonlinear in the gravitational fields.

In the second (trivial) step for deriving Einstein’s $R^0_0$ equation, we put non-relativistic source-matter on the matter-side of Einstein’s equation: it follows that Einstein’s $R^0_0$ equation for nonrelativistic source-matter and for gravitational fields of arbitrary strength is exactly identical with the Newtonian equation for the relative radial acceleration of neighbouring freefalling test-particles, spherically averaged.

In a third, well-known step, given in textbooks, one derives the general Einstein equations from Einstein’s $R^0_0$ equation for nonrelativistic source-matter by using local Lorentz covariance and energy-momentum conservation combined with the Bianchi identity.

These three steps complete our rigorous derivation of Einstein’s field equations for general relativity. — Additional results in [4].

The tools needed in this paper are: (1) our exact operational definition of the gravito-electric field $\vec{E}_g$, (2) the Ricci connection coefficients for a Lorentz boost of LONBs under a displacement in time, $(\omega^j_0)\delta_0$, and (3) our identity $E_i^{(s)} = -\omega_{i\beta}^0\delta_0^\beta$.

The gravito-electric field $\vec{E}_g$ measured by any local observer (with his LONBs along his worldline) is given by our exact and general operational definition in arbitrary (3+1)-spacetimes, Eq. (4), which is probably new. — In contrast to the literature, we use no perturbation theory on a background geometry, no weak gravitational fields. — $\vec{E}_g$ is defined as the measured acceleration of quasistatic freefalling test-particles analogous to the operational definition of the ordinary electric field, where we replace the particle’s charge by its rest mass $m$,

$$m^{-1} \frac{d}{dt} p_i = E_i^{(s)} = \vec{g},$$

for freefalling, quasistatic test-particles. Local time-intervals $dt$ are measured on the observer’s wristwatch. The measured 3-momentum is $p_i$ with respect to the observer’s LONB. For a freefalling test-particle, quasistatic relative to the observer, the measured gravitational acceleration relative to the observer is $\vec{a}^{(\text{quasistatic})} = \vec{g} = \vec{E}_g$, measured by Galilei.

The LONB-components $p^\alpha$ are directly measurable. This is in stark contrast to coordinate-basis components $p^\alpha$, which are not measurable before one has obtained $g_{\alpha\beta}$ by solving Einstein’s equations for the specific problem at hand.

LONBs off the observer’s worldline are not needed in Eq. (4), because a particle released from rest (or quasistatic state) will still be on the observer’s worldline after an infinitesimal time $\delta t$, since $\delta s \propto (\delta t)^2 = 0$, while $\delta v \propto \delta t \neq 0$.

For a freefalling observer, $\vec{E}_g = \vec{g}$ is zero on his worldline: Einstein’s “happiest thought of my life”.

Gravito-electric fields $\vec{E}_g$ of arbitrary strength can be measured exactly with freefalling test-particles which are quasistatic relative to the observer, Eq. (4). But this same measured $\vec{E}_g$ is exactly valid for relativistic test-particles in the equations of motion.

The gravito-magnetic field $\vec{B}_g$ has been postulated by Heaviside in 1893 [2]. Our exact operational definition of $\vec{B}_g$ is given in [4].

The term “weak gravitational fields” for local discussions is often used in textbooks. But “weak gravity” is meaningless locally, because the gravitational field $\vec{g}$ and the gravitational tidal field $R^0_0$ are not dimensionless.

Cartan’s method with LONB-connection coefficients is unavoidable for our computation of curvature from measurements by non-inertial observers. But Cartan’s LONB method is not taught in almost all graduate programs in general relativity in the USA, and most researchers have never used Cartan’s method to solve a problem. Therefore we introduce elements of Cartan’s method.

Ricci’s LONB-connection coefficients are illustrated by an airplane on the shortest path (geodesic) from Zurich to Chicago and the Local Ortho-Normal Bases (LONBs) chosen to be in the directions “East” and “North”. These LONBs rotate relative to the geodesic (relative to parallel transport) with a rotation angle $\delta \alpha$ per measured path length $\delta s$, i.e. with the rotation rate $\omega = (\delta \alpha / \delta s)$.

For infinitesimal displacements $\delta \vec{D}$ in any direction, the rotation angle $\delta \alpha$ of LONBs is given by a linear map encoded by the Ricci rotation coefficients $\omega_{i\beta}$,

$$\delta \alpha = \omega_{i\beta} \delta D_i.$$
The Ricci rotation coefficients are also called connection coefficients, because they connect the LONBs at infinitesimally neighboring points by a rotation relative to the infinitesimal geodesic between these points.

Cartan’s LONB connection coefficients use displacements in the coordinates,

\[ \delta \alpha = \omega, \delta D^\gamma. \]

In three spatial dimensions, the rotation of LONBs relative to the geodesic from \( P \) to \( Q \) must be given by a rotation matrix. For a rotation in the \( (\bar{e}_x, \bar{e}_y) \)-plane,

\[ \begin{pmatrix} \bar{e}_x \\ \bar{e}_y \end{pmatrix}_P = \begin{pmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix} \begin{pmatrix} \bar{e}_x \\ \bar{e}_y \end{pmatrix}_Q. \]

For infinitesimal displacements, hence infinitesimal rotations (first derivatives in \( \alpha \)), the rotation matrix is,

\[ \begin{pmatrix} \bar{e}_x \\ \bar{e}_y \end{pmatrix}_Q = \begin{pmatrix} 1 + \alpha & 0 \\ 0 & 1 + \alpha \end{pmatrix} \begin{pmatrix} \bar{e}_x \\ \bar{e}_y \end{pmatrix}_P. \]

The infinitesimal LONB-rotation matrix \( \delta R_{ij} \) is given by the linear map from the infinitesimal coordinate-displacement vector \( D^\gamma \),

\[ \delta R_{ij} = (\omega_{ij})_\gamma \delta D^\gamma, \]

\[ \omega_{i2} = -\omega_{2i} = \alpha \bar{e}_2 = \text{rotation angle in } \bar{e}_2 \text{-plane.} \]

The \( (\omega_{ij})_\gamma \) are the Ricci connection coefficients.

In \((1+1)\)-spacetime, the Lorentz transformation of the chosen LONBs relative to a given displacement geodesic is a Lorentz boost \( L^a_b \),

\[ \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}_Q = \begin{pmatrix} \cosh \chi & \sinh \chi \\ \sinh \chi & \cosh \chi \end{pmatrix} \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}_P, \]

with \( \tanh \chi \equiv v/c, \) with \( \chi \) called “rapidity”, and \( \chi \) additive for successive Lorentz boosts in the same spatial direction. For infinitesimal displacements, the infinitesimal Lorentz boost \( L^a_b \) is,

\[ \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}_Q = \begin{pmatrix} 1 + \chi & 0 \\ 0 & 1 + \chi \end{pmatrix} \begin{pmatrix} \bar{e}_1 \\ \bar{e}_2 \end{pmatrix}_P. \]

In \((3+1)\)-spacetime, and with two lower indices, \( \omega^a_{\bar{b} \bar{c}} \) is antisymmetric for Lorentz boosts (and for rotations),

\[ \delta L^a_{\bar{b} \bar{c}} = (\omega^a_{\bar{b} \bar{c}})_\gamma \delta D^\gamma, \]

\[ (\omega^a_{\bar{b} \bar{c}})_\gamma = -(\omega^a_{\bar{c} \bar{b}})_\gamma = (\chi^a_{\bar{b}})_\gamma. \]

For a displacement in observer-time, the exact Ricci connection coefficients \( (\omega^a_{\bar{b} \bar{c}})_0 \) of general relativity can be measured in quasistatic experiments. But these Ricci connection coefficients predict the motion of relativistic particles with the equations of motion.

Our gravito-electric field \( \vec{E} \) is identical with minus the Ricci Lorentz-boost coefficients for a displacement in time,

\[ E^a_i = - (\omega^a_{i \bar{0}})_0. \quad (5) \]

The proof: from the point of view of the observer with his LONBs along his worldline, the gravitational acceleration \( g_i = a_i^{(\text{free-particle})} \) of freefalling quasistatic test-particles (starting on the observer’s worldline) is by definition identical to the exact gravitoelectric field \( E_i \) of general relativity, Eq. \( (\text{4}) \). — But from the point of view of freefalling test-particles, the acceleration of the quasistatic observer with his LONBs is by definition identical to the exact Ricci LONB-boost coefficients \( (\omega^a_{i \bar{0}})_0 \).

\[ E^a_i = \left[ (a_i)^{(\text{rel.to.obs.})}_{\text{particle}} \right]_{\text{quasistatic}} = g_i = - \left[ (a_i)^{(\text{rel.to.obs.})}_{\text{observer}} \right]_{\text{quasistatic}} = - (\omega^a_{i \bar{0}})_0. \]

Galilei measured Ricci connection coefficients of general relativity: \( (\omega^a_{i \bar{0}})_0 = \delta^a_i \) \((9.1 \text{ m/s}^2)\) for LONBs in directions East, North, vertical.

Our general, exact definition of the gravitomagnetic field, \( \vec{B}/2 = \Omega^{(\text{gyroscope})} \), is discussed in \( [\text{1}] \). The Ricci connection coefficients \( (\omega^a_{i \bar{0}})_0 \) equal minus the precession rate of gyroscopes (comoving with the observer),

\[ (\omega^a_{i \bar{0}})_0 = - \Omega^{(\text{gyro})}_{ij} \equiv - \bar{e}_i \omega_{jk} \Omega^{(\text{gyro})}_{kj}. \]

These exact Ricci connection coefficients of general relativity were measured by Foucault in 1853.

In striking contrast, Christoffel connection coefficients (for coordinate bases), \( \Gamma^a_{\beta \gamma} \equiv (\Gamma^a_{\beta \gamma})_{(\text{obs})} \), have no direct physical-geometric meaning, and they cannot be known, until the metric fields \( g_{\mu \nu}(x) \) have been obtained by solving Einstein’s equations for a given problem.

We write Christoffel connection coefficients with a bracket: inside the bracket are the coordinate-basis transformation-indices \((a, \beta)\), outside the bracket is the coordinate-displacement index \( \gamma \).

For curvature computations there are two methods, (1) the standard method with coordinate bases and Christoffel connections \( (\Gamma^a_{\beta \gamma})_{(\text{obs})} \), (2) Cartan’s method with Local Ortho-Normal Bases and LONB-connections \( (\omega^a_{\bar{b} \gamma})_0 \).

For a primary non-inertial observer, Cartan’s method is strongly preferred, because a radially parallel LONB-vector \( \vec{e}_0(P) \) off the primary observer’s worldline, which is highly convenient for measuring relative radial acceleration, does not point in the same direction as the natural coordinate-basis vector \( \vec{e}_0(P) = \partial t \) for a rotating or non-freefalling observer.

Cartan’s curvature equation gives the Riemann curvature 2-form \( R^a_{b \gamma} \) with 2-form components \( (R^a_{b \gamma})_\delta \). 2-form components are antisymmetric covariant components in a coordinate basis, denoted by Greek letters. — For an inertial primary observer and with our LONBs radially parallel, all of Cartan’s LONB connection coefficients \( (\omega^a_{\bar{b} \gamma})_0 \) vanish on the worldline of the primary observer, Eqs. \( (\text{1}) \ (\text{2}) \). Therefore, in Cartan’s curvature equation, the term bilinear in the connection, the wedge product (antisymmetric in the suppressed coordinate-basis displacement-indices) \( [\omega^a_{\bar{c} \gamma} \wedge \omega^c_{\bar{b} \gamma}] \) vanishes. Hence
Cartan’s curvature 2-form $\mathcal{R}^{\hat{a}}_{\hat{b}}$ is equal to the exterior derivative $d$ of the LONB-connection 1-form $\omega^{\hat{a}}_{\hat{b}}$ in notation free of form-components,

$$\mathcal{R}^{\hat{a}}_{\hat{b}} = d\omega^{\hat{a}}_{\hat{b}}, \quad (6)$$

where $d$ denotes the antisymmetric ordinary partial derivative, and $(\hat{a}, \hat{b})$ are the Lorentz-transformation indices of the LONBs.

Writing explicitly the antisymmetric 2-form component indices $[\mu, \nu]$ (plaqueette indices) on the left-hand-side and the antisymmetric pair of derivative-index and displacement-index on the right-hand side, Eq. (6) reads,

$$(\mathcal{R}^{\hat{a}}_{\hat{b}})_{\mu\nu} = (d\omega^{\hat{a}}_{\hat{b}})_{\mu\nu} \equiv \partial_{\mu}(\omega^{\hat{a}}_{\hat{b}})_{\nu} - [\mu \leftrightarrow \nu]. \quad (7)$$

An instructive elementary derivation of the curvature equations (6, 7) for 2-space is given in [4].

Eqs. (6, 7) give our crucial curvature result for general relativity:

- inertial primary observer, radially parallel LONBs:
  $$R^{\hat{a}}_{\hat{b}} = (\mathcal{R}^{\hat{a}}_{\hat{b}})_{0i} =$$
  $$= -\partial_{i}(\omega^{\hat{a}}_{\hat{b}})_{0} = \text{div} \tilde{E}_{g} = \partial_{i} < a_{r} >_{\text{avg.average}}. \quad (8)$$

The last expression states that Einstein’s exact $R^{\hat{a}}_{\hat{b}}$ curvature is identical with the Newtonian relative acceleration of free-falling test-particles, spherically averaged for gravitational fields of arbitrary strength and for arbitrary source-matter (e.g. relativistic). — It is superfluous to re-measure or re-compute $R^{\hat{a}}_{\hat{b}}$ with relativistic test-particles.

The second step, the derivation of Einstein’s $R^{\hat{a}}_{\hat{b}}$ equation for non-relativistic sources is now trivial: we write the sources on the right-hand-side of the equation,

- inertial primary observer, radially parallel LONBs, nonrelativistic sources:
  $$R^{\hat{a}}_{\hat{b}} = (\mathcal{R}^{\hat{a}}_{\hat{b}})_{0i} =$$
  $$= \partial_{i} < a_{r} >_{\text{avg.average}} \quad \text{Newton}$$
  $$= \text{div} \tilde{E}_{g} \quad \text{Gauss}$$
  $$= -4\pi G_{N}\rho_{\text{mass}}. \quad (9)$$

But for non-inertial primary observers, Einstein’s $R^{\hat{a}}_{\hat{b}}$ equation and the Newtonian relative acceleration equation are both non-linear in the gravitational fields and identical, if one uses Einstein’s equivalence of gravitational forces and fictitious forces [4].

For a superficial reader, Gauss’s law in general relativity, Eq. (9), is “nothing new”. However: (1) Our law of Eq. (9) is derived rigorously, and it is exactly linear for inertial primary observers. We have not used the usual approximation of linearized gravity. The exact law is non-linear for non-inertial primary observers. (2) Our law of Eq. (9) only holds for auxiliary observers with LONBs parallel along radial geodesics to the LONBs of the primary observer at a given time. (3) Our law of Eq. (9) does not hold for the Local Inertial Frame (LIF) and the Local Inertial Coordinate Systems (LICS) around $P_{0}$ (used in textbooks), where the basis vectors are parallel along geodesics radiating out from one point $P_{0}$ in all spacetime directions. Our law of Eq. (9) cannot hold in a LIF, because (with curvature) LONBs cannot be parallel on all three sides of the (geodesic) triangle: (i) from $P_{0}$ along the worldline of the primary inertial observer, (ii) from $P_{0}$ along the worldline of an inertial particle with nonzero velocity relative to the primary observer, (iii) from the primary to the auxiliary observer at a fixed time $t = t_{0} + \delta t$. (4) Our law of Eq. (9) only holds for our exact operational definition of $E_{g}$ in Eq. (8), which is probably new.

The third step, the derivation of Einstein’s equations starting from Einstein’s $R^{\hat{a}}_{\hat{b}}$ equation for nonrelativistic sources, Eq. (9), is well known and described in textbooks: one uses local Lorentz covariance and energy-momentum conservation combined with the contracted Bianchi identity. This completes our rigorous and simple derivation of Einstein’s field equations of general relativity,

$$G^{\hat{a}\hat{b}} = 8\pi G_{N}T^{\hat{a}\hat{b}}. \quad (10)$$