Formulation of the uncertainty relations in terms of the Rényi entropies

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Quantum-mechanical uncertainty relations for position and momentum are expressed in the form of inequalities involving the Rényi entropies. The proof of these inequalities requires the use of the exact expression for the $(p, q)$-norm of the Fourier transformation derived by Babenko and Beckner. Analogous uncertainty relations are derived for angle and angular momentum and also for a pair of complementary observables in $N$-level systems. All these uncertainty relations become more attractive when expressed in terms of the symmetrized Rényi entropies.

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I. INTRODUCTION

The Rényi entropy is a one-parameter extension of the Shannon entropy. There is extensive literature on the applications of the Rényi entropy in many fields from biology, medicine, genetics, linguistics, and economics to electrical engineering, computer science, geophysics, chemistry, and physics. My aim is to describe the limitations on the information characterizing quantum systems, in terms of the Rényi entropies. These limitations have the form of inequalities that have the physical interpretation of the uncertainty relations.

The Rényi entropy has been widely used in the study of quantum systems. In particular, it was used in the analysis of quantum entanglement [12, 13, 14], quantum communication protocols [6, 7], quantum correlations [8], quantum measurement [9], and decoherence [10], multiparticle production in high-energy collisions [11] and spin systems [15, 16], in the study of the quantum-classical correspondence [17, 18], and the localization in phase space [19]. In view of these numerous and successful applications, it seems worthwhile to formulate the quantum-mechanical uncertainty relations for canonically conjugate variables in terms of the Rényi entropies. I do not want to enter here into the discussion (cf. [20, 21]) of a fundamental problem: which Entropy measures of uncertainty is adequate in a fundamental problem: which Entropy measures of uncertainty is adequate for the quantum system, and

\begin{equation}
H_\alpha = \frac{1}{1-\alpha} \ln \left( \sum p_k^\alpha \right).
\end{equation}

Rényi called this quantity “the measure of information of order $\alpha$ associated with the probability distribution $P = (p_1, \ldots, p_n)$.” The Rényi measure of information $H_\alpha$ may also be viewed as a measure of uncertainty since, after all, the uncertainty is the missing information. In the formulation of the uncertainty relations given below, the Rényi entropy will be used as the measures of uncertainties.

In order to simplify the derivations, I use the natural logarithm in the definition (1) of the Rényi entropy. However, all uncertainty relations derived in this paper [Eqs. (7), (25), (44), (26), (31), (40), and (41)] have the same form for all choices of the base of the logarithm because they are homogeneous in $\ln(\ldots)$. Note that the definition of the Rényi entropy is also applicable when the sum has infinitely many terms, provided this infinite sum converges. The Rényi entropy (1) is a nonincreasing function of $\alpha$ [22]. For $\alpha > \beta$ we have $H_\alpha \leq H_\beta$. In the limit, when $\alpha \to 1$ the Rényi entropy is equal (apart from a different base of the logarithm) to the Shannon entropy

\begin{equation}
\lim_{\alpha \to 1} H_\alpha = - \sum p_k \ln p_k.
\end{equation}

According to the probabilistic interpretation of quantum theory, the probability distribution associated with the measurement of a physical variable represented by the operator $A$ is defined as

\begin{equation}
p_k = \text{Tr}\{\rho P_k\},
\end{equation}

where $\rho$ is the density operator describing the state of the quantum system, and $P_k$ is the projection operator corresponding to the $k$th segment of the spectrum of $A$ (the $k$th bin). The uncertainty is the lowest when only one $p_k$ is different from zero — the Rényi entropy reaches then its lowest value: zero.

The probability distributions $p_k^A$ and $p_k^B$ that correspond to different physical variables but to the same state of the system are, in general, correlated. These correlations lead to restrictions on the values of the Rényi entropies $H_\alpha^A$ and $H_\alpha^B$. When these restrictions have the form of an inequality $H_\alpha^A + H_\alpha^B \geq C > 0$, they serve the name of the uncertainty relations because not only do they prohibit the vanishing of both uncertainties for the same state but they also require that one uncertainty must increase when the other decreases.

In the present paper, I derive the inequalities for three pairs of observables: position and momentum (or time
and frequency), angle and angular momentum, and the complementary observables — the analogs of \( x \) and \( p \) — in finite dimensional spaces. These inequalities are generalizations of the entropic uncertainty relations established before for the Shannon entropies 23 24 25. There is some overlap in mathematical derivations especially in the extensive use of \( (p, q) \)-norms] between the results presented in this paper and the earlier works of Maassen and Uffink 26 27 and Rajagopal 28. However, these authors did not express the uncertainty relations in terms of the Rényi entropies and they did not introduce the finite resolutions that characterize all physical measurements.

II. UNCERTAINTY RELATIONS FOR \( x \) AND \( p \)

The probability distributions associated with the measurements of momentum and position of a quantum particle in a pure state (generalization to mixed states will be given in Sec. VI] are

\[
p_k = \int_{k\delta p}^{(k+1)\delta p} dp |\tilde{\psi}(p)|^2, \quad q_l = \int_{l\delta x}^{(l+1)\delta x} dx |\tilde{\psi}(x)|^2, \tag{4}
\]

where I have assumed that the sizes of all bins are the same. The indices \( k \) and \( l \) run from \(-\infty \) to \( \infty \) and the Fourier transform is defined with the physical normalization, i.e.

\[
\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int dx e^{-ipx/\hbar} \psi(x). \tag{5}
\]

From the two probability distributions (4) we may construct the Rényi entropies \( H_{\alpha}^{(p)} \) and \( H_{\beta}^{(x)} \) that measure the uncertainty in momentum and position

\[
H_{\alpha}^{(p)} = \frac{1}{1-\alpha} \ln \left( \sum p_k^\alpha \right), \quad H_{\beta}^{(x)} = \frac{1}{1-\beta} \ln \left( \sum q_l^\beta \right). \tag{6}
\]

I shall prove in the next section that the uncertainty relation restricting the values of \( H_{\alpha}^{(p)} \) and \( H_{\beta}^{(x)} \) has the following form:

\[
H_{\alpha}^{(p)} + H_{\beta}^{(x)} \geq -\frac{1}{2} \left( \frac{\ln \alpha}{1-\alpha} + \frac{\ln \beta}{1-\beta} \right) - \ln \left( \frac{\delta x\delta p}{\pi\hbar} \right), \tag{7}
\]

where the parameters \( \alpha \) and \( \beta \) are assumed to be positive and they are constrained by the relation

\[
\frac{1}{\alpha} + \frac{1}{\beta} = 2. \tag{8}
\]

In the limit, when \( \alpha \to 1 \) and \( \beta \to 1 \), this uncertainty relation reduces to the uncertainty relation for the Shannon entropies

\[
H^{(p)} + H^{(x)} \geq -\ln \left( \frac{\delta x\delta p}{\epsilon\pi\hbar} \right), \tag{9}
\]

that had already been derived some time ago 23.

Note that the relations (7) and (9) are quite different from the standard uncertainty relations. As has been aptly stressed by Peres 29, “The uncertainty relation such as \( \Delta x\Delta p \geq \hbar/2 \) is not a statement about the accuracy of our measuring instruments.” In contrast, both entropic uncertainty relations (7) and (9) do depend on the accuracy of the measurement — they explicitly contain the area of the phase-space \( \delta x\delta p \) determined by the resolution of the measuring instruments. This aspect of the uncertainty relations (7) and (9) can be summarized as follows: the more precisely one wants to localize the particle in the phase space, the larger the sum of the uncertainties in \( x \) and \( p \).

The uncertainty relation (7) is not sharp — its improvement is a challenging open problem. However, it becomes sharper and sharper when the relative size of the phase space area \( \delta x\delta p/(\pi\hbar) \) defined by the experimental resolutions decreases, as it is when we enter deeper and deeper into the quantum regime.

III. PROOF

The proof of the inequality (7) employs the known value of the \( (p, q) \)-norm of the Fourier transformation. The \( (p, q) \)-norm of an operator \( T \) is defined as the smallest number \( k(p, q) \) such that for all \( \psi \)

\[
\|T\psi\|_p \leq k(p, q) \|\psi\|_q, \tag{10}
\]

where the \( p \)-norm (or the \( q \)-norm) of a function is defined in the standard way

\[
\|\psi\|_p = \left( \int_{-\infty}^{\infty} dx |\psi(x)|^p \right)^{1/p}, \tag{11}
\]

and the values of the parameters \( p \) and \( q \) satisfy the conditions

\[
\frac{1}{p} + \frac{1}{q} = 1, \quad p \geq q. \tag{12}
\]

The parameters \( p \) and \( q \) should no be confused with momentum and position.

The \( (p, q) \)-norm of the Fourier transformation has been found for even values of \( p \) by Babenko 30 and for all values of \( p \) by Beckner 31. For the physical normalization 5 of the Fourier transform, the Babenko-Beckner inequality reads

\[
\|\tilde{\psi}\|_p \leq k(p, q) \|\psi\|_q, \tag{13}
\]

where

\[
k(p, q) = \left( \frac{p}{2\pi\hbar} \right)^{-\frac{1}{2p}} \left( \frac{q}{2\pi^2\hbar} \right)^{\frac{1}{2q}}. \tag{14}
\]

Since the function \( \psi \) can be treated as the Fourier transform of \( \overline{\psi} \), the following inequality also holds:

\[
\|\psi\|_p \leq k(p, q) \|\overline{\psi}\|_q. \tag{15}
\]
The inequalities (13) and (15) are saturated by all Gaussian functions.

In terms of the probability densities \( \hat{\rho}(p) = |\hat{\psi}(p)|^2 \) and \( \rho(x) = |\psi(x)|^2 \), the inequalities (13) and (15) read
\[
\left( \int_{-\infty}^{\infty} dp |\hat{\rho}(p)|^\alpha \right)^{\frac{1}{\alpha}} \leq n(\alpha, \beta) \left( \int_{-\infty}^{\infty} dx |\rho(x)|^\beta \right)^{\frac{1}{\beta}}, \tag{16a}
\]
\[
\left( \int_{-\infty}^{\infty} dx |\rho(x)|^\beta \right)^{\frac{1}{\beta}} \leq n(\alpha, \beta) \left( \int_{-\infty}^{\infty} dp |\hat{\rho}(p)|^\alpha \right)^{\frac{1}{\alpha}}, \tag{16b}
\]
where \( \alpha = p/2, \beta = q/2, \alpha \geq \beta \), and
\[
n(\alpha, \beta) = \left( \frac{\alpha}{\pi \hbar} \right)^{-\frac{1}{\alpha}} \left( \frac{\beta}{\pi \hbar} \right)^{\frac{1}{\beta}}. \tag{17}\]

In the first part of the proof I shall use the inequality (16a). In order to relate this inequality to the Rényi entropies (4), I shall first split the full integration ranges into the \( dp \) and \( \delta x \) bins
\[
\int_{-\infty}^{\infty} dp |\hat{\rho}(p)|^\alpha = \sum_{k=-\infty}^{\infty} \int_{k \delta p}^{(k+1)\delta p} dp |\hat{\rho}(p)|^\alpha, \tag{18a}
\]
\[
\int_{-\infty}^{\infty} dx |\rho(x)|^\beta = \sum_{l=-\infty}^{\infty} \int_{l \delta x}^{(l+1)\delta x} dx |\rho(x)|^\beta. \tag{18b}
\]
Next, for each term in these sums I shall use the integral form of the Jensen inequality (32, 33). For convex functions this inequality can be stated as follows: the value of the function at the average point does not exceed the average value of the function. For concave functions it is just the opposite: the average value of the function does not exceed the value of the function at the average point. Since for \( \alpha > 1 \) the function \( f(z) = z^\alpha \) is convex and for \( \beta < 1 \) the function \( g(z) = z^\beta \) is concave, we obtain the following two inequalities:
\[
\frac{1}{\delta p} \int_{k \delta p}^{(k+1)\delta p} dp |\hat{\rho}(p)|^\alpha \leq |\hat{\rho}(p)|^\alpha, \tag{19a}
\]
\[
\frac{1}{\delta x} \int_{l \delta x}^{(l+1)\delta x} dx |\rho(x)|^\beta \leq |\rho(x)|^\beta. \tag{19b}
\]
Therefore, with the use of the definitions of the probabilities (3), we may convert Eqs. (18) into the following inequalities:
\[
(\delta p)^{1-\alpha} \sum_{k=-\infty}^{\infty} p_k^\alpha \leq \int_{-\infty}^{\infty} dp |\hat{\rho}(p)|^\alpha, \tag{20a}
\]
\[
\frac{1}{\delta x} \int_{-\infty}^{\infty} dx |\rho(x)|^\beta \leq (\delta x)^{1-\beta} \sum_{l=-\infty}^{\infty} q_l^\beta. \tag{20b}
\]
These inequalities combined with the Babenko-Beckner result (16a) give
\[
\left( (\delta p)^{1-\alpha} \sum_{k=-\infty}^{\infty} p_k^\alpha \right)^{\frac{1}{\alpha}} \leq n(\alpha, \beta) \left( (\delta x)^{1-\beta} \sum_{l=-\infty}^{\infty} q_l^\beta \right)^{\frac{1}{\beta}}. \tag{21}\]

This inequality does not depend on the choice of units used to measure \( \delta x, \delta p, \) and \( \hbar \) since it can be transformed to the following dimensionless form
\[
\left( \sum_{k=-\infty}^{\infty} p_k^\alpha \right)^{\frac{1}{\alpha}} \leq \gamma \left( \frac{\alpha}{\pi} \right)^{-\frac{1}{\alpha}} \left( \frac{\beta}{\pi} \right)^{\frac{1}{\beta}} \left( \sum_{l=-\infty}^{\infty} q_l^\beta \right)^{\frac{1}{\beta}}. \tag{22}\]

Finally, by taking the logarithm of both sides we obtain the uncertainty relation (20) but only for \( \alpha > \beta \). To extend this result to the values \( \alpha < \beta \), we have to start from the inequality (16b) instead of (16a).

In order to generalize these results to \( n \) dimensions we need the following value of the \((p, q)\)-norm for the \(n\)-dimensional Fourier transform (31)
\[
k_n(p, q) = \left( \frac{p}{2\pi \hbar} \right)^{-\frac{n}{\alpha}} \left( \frac{q}{2\pi \hbar} \right)^{-\frac{n}{\beta}}. \tag{24}\]

The uncertainty relations are then obtained in the same way as in the one-dimensional case and they have the form:
\[
H_n^{(p)} + H_n^{(x)} \geq -\frac{n}{2} \left( \frac{\ln \alpha}{1-\alpha} + \frac{\ln \beta}{1-\beta} \right) - n \ln \left( \frac{\delta x \delta p}{2\pi \hbar} \right). \tag{25}\]

IV. UNCERTAINTY RELATIONS FOR \( \varphi \) AND \( M_z \)

The uncertainty relations in terms of the Rényi entropies can also be formulated for the angle \( \varphi \) and the angular momentum \( M_z \) and they have the form
\[
H_\alpha^{(M_z)} + H_\beta^{(\varphi)} \geq -\frac{\ln \delta \varphi}{2\pi}. \tag{26}\]

The probability distributions \( p_m^{(M_z)} \) and \( p_l^{(\varphi)} \) that are used to calculate these Rényi entropies are defined as follows:
\[
p_m^{(M_z)} = |c_m|^2, \quad p_l^{(\varphi)} = \int_{l \delta \varphi}^{(l+1)\delta \varphi} d\varphi |\psi(\varphi)|^2, \tag{27}\]
where the amplitudes \( c_m \) are the coefficients in the expansion of \( \psi(\varphi) \) into the eigenstates of \( M_z \),
\[
\psi(\varphi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} c_m e^{im\varphi}, \tag{28}\]
and \( \delta \varphi \) is the experimental resolution in the measurement of the angular distribution. In contrast to the uncertainty
For a system composed of two subsystems described by state vectors in the $N$ and $M$ dimensional spaces the bound on the right hand side of the inequality is equal to $\ln(NM) = \ln N + \ln M$ because the dimensionality of the Hilbert space of the composed system is $NM$. The same result is obtained for two totally independent systems of dimensionality $N$ and $M$ because the Rényi entropy is additive for independent probability distributions. This means that the uncertainty relation is already saturated by separable states and allowing for entanglement does not make any difference.

VI. UNCERTAINTY RELATIONS FOR MIXED STATES

The uncertainty relations for the Rényi entropies hold also for all mixed states. This result is not obvious because the Rényi entropy is not a convex function of the probability distributions for all values of $\alpha$. Hence, the terms on the left-hand side of the uncertainty relation \((7)\) may decrease as a result of mixing. However, I shall prove now that the inequalities \((10)\) that were the starting point in the derivations hold also for mixed states. This follows from the integral form of the Minkowski inequalities \((38)\), namely

$$
\left( \int |dV| f + g |^\alpha \right)^{\frac{1}{\alpha}} \leq \left( \int |dV| f |^\alpha \right)^{\frac{1}{\alpha}} + \left( \int |dV| g |^\alpha \right)^{\frac{1}{\alpha}},
$$

\((34a)\)

$$
\left( \int |dV| f |^\beta \right)^{\frac{1}{\beta}} + \left( \int |dV| g |^\beta \right)^{\frac{1}{\beta}} \leq \left( \int |dV| f + g |^\beta \right)^{\frac{1}{\beta}},
$$

\((34b)\)

where $f$ and $g$ are nonnegative functions and the parameters $\alpha$ and $\beta$ satisfy the condition $\alpha > \beta$. Substituting in the first inequality for the functions $f$ and $g$ the weighted densities in momentum space $f = \lambda \rho_1(p)$ and $g = (1 - \lambda) \rho_2(p)$ and in the second inequality the weighted densities in the coordinate space $f = \lambda \rho_1(x)$ and $g = (1 - \lambda) \rho_2(x)$.

V. UNCERTAINTY RELATIONS FOR $N$-LEVEL SYSTEMS

For quantum systems described by vectors in the $N$-dimensional Hilbert space the analog of the uncertainty relation for the Rényi entropies is

$$
\frac{1}{1 - \alpha} \ln \left( \sum_{k=1}^{N} \hat{\rho}_k^\alpha \right) + \frac{1}{1 - \beta} \ln \left( \sum_{l=1}^{N} \hat{\rho}_l^\beta \right) \geq \ln N,
$$

\((31)\)

where $\hat{\rho}_k = |\alpha_k|^2$, $\hat{\rho}_l = |a_l|^2$ and the amplitudes $\alpha_k$ and $a_l$ are connected by the discrete Fourier transformation

$$
\alpha_k = \frac{1}{\sqrt{N}} \sum_{l=1}^{N} \exp(2\pi ik l/N) a_l.
$$

\((32)\)

The complex numbers $\alpha_k$ and $a_l$ can be interpreted as the probability amplitudes to find a particle in the discretized momentum space and position space \((35)\), but they can also be viewed as amplitudes in a general abstract $N$-dimensional Hilbert space. The uncertainty relation \((31)\) is saturated for the states that are localized either in “position space” (only one of the amplitudes $a_l$ is different from zero) or in “momentum space” (only one of the amplitudes $\alpha_k$ is different from zero). Like in the case of the uncertainty relations for the angle and the angular momentum, the bound does not depend on $\alpha$ and $\beta$. The absence of the Planck constant is again due to a cancellation — it would reappear if $l$ and $\kappa$ in \((32)\) are given the physical dimension of length and momentum.

The proof of the uncertainty relation \((31)\) proceeds along similar lines as the proof of \((7)\) but now we invoke a different known inequality — the $(p,q)$-norm of the discrete Fourier transform (cf., for example, Ref. \((36)\))

$$
\|\tilde{a}\|_p \leq N^{\frac{1}{p} - \frac{1}{q}} \|a\|_q,
$$

\((33)\)

Uncertainty relations for $N$-level systems involving the $(p,q)$-norms of the discrete Fourier transform were established in Ref. \((37)\) but they have not been used to derive the uncertainty relations for the Rényi entropies.
and $g = (1 - \lambda)\rho_2(x)$, we obtain
\[
\left( \int_{-\infty}^{\infty} dp (\lambda \tilde{\rho}_1(p) + (1 - \lambda)\tilde{\rho}_2(p))^\beta \right)^{\frac{1}{\beta}} \leq \lambda \left( \int_{-\infty}^{\infty} dp \tilde{\rho}_1(p) \right)^{\frac{1}{\beta}} + (1 - \lambda) \left( \int_{-\infty}^{\infty} dp \tilde{\rho}_2(p) \right)^{\frac{1}{\beta}},
\]
\[
\beta \lambda \left( \int_{-\infty}^{\infty} dx (\lambda \rho_1(x))^\beta \right)^{\frac{1}{\beta}} + (1 - \lambda) \left( \int_{-\infty}^{\infty} dx \rho_2(x) \right)^{\frac{1}{\beta}} \leq \left( \int_{-\infty}^{\infty} dx (\lambda \rho_1(x) + (1 - \lambda)\rho_2(x))^\beta \right)^{\frac{1}{\beta}}.
\]

Comparing these results with the weighted sum of inequalities \((16a)\) for pure states
\[
\lambda \left( \int_{-\infty}^{\infty} dp (\rho_1(p))^\beta \right)^{\frac{1}{\beta}} + (1 - \lambda) \left( \int_{-\infty}^{\infty} dp \rho_2(p) \right)^{\frac{1}{\beta}} \leq n(\alpha, \beta) \lambda \left( \int_{-\infty}^{\infty} dx (\lambda \rho_1(x))^\beta \right)^{\frac{1}{\beta}} + (1 - \lambda) \left( \int_{-\infty}^{\infty} dx \rho_2(x) \right)^{\frac{1}{\beta}}
\]
\[
+ n(\alpha, \beta) \lambda \left( \int_{-\infty}^{\infty} dx (\lambda \rho_1(x) + (1 - \lambda)\rho_2(x))^\beta \right)^{\frac{1}{\beta}},
\]
we extend the inequality \((16a)\) to mixed states
\[
\left( \int_{-\infty}^{\infty} dp (\lambda \tilde{\rho}_1(p) + (1 - \lambda)\tilde{\rho}_2(p))^\alpha \right)^{\frac{1}{\alpha}} \leq n(\alpha, \beta) \left( \int_{-\infty}^{\infty} dx (\lambda \rho_1(x))^\beta \right)^{\frac{1}{\beta}} + (1 - \lambda) \left( \int_{-\infty}^{\infty} dx \rho_2(x) \right)^{\frac{1}{\beta}}
\]
\[
\leq n(\alpha, \beta) \left( \int_{-\infty}^{\infty} dx (\lambda \rho_1(x) + (1 - \lambda)\rho_2(x))^\beta \right)^{\frac{1}{\beta}}.
\]

In the same manner we can extend the inequality \((16b)\) to mixed states. Once we have proven the validity of the inequalities \((16)\) for mixed states, we may proceed as before to prove the validity of the the Rényi uncertainty relations \((7)\) for mixed states. Similar arguments can be invoked to prove also the uncertainty relations \((20)\) and \((21)\) for mixed states.

VII. UNCERTAINTY RELATIONS FOR CONTINUOUS DISTRIBUTIONS

There exist also purely mathematical versions of the uncertainty relations that do not involve the experimental resolutions $\delta x$ and $\delta p$ of the measuring devices. By taking directly the logarithm of the inequality \((16)\), and using the relations between $\alpha$ and $\beta$, we arrive at
\[
\frac{1}{1 - \alpha} \ln \left( \int_{-\infty}^{\infty} d\rho(p) \right)^\alpha + \frac{1}{1 - \beta} \ln \left( \int_{-\infty}^{\infty} dx \rho(x) \right)^\beta \geq -\frac{1}{2(1 - \alpha)} \ln \frac{\alpha}{\pi} - \frac{1}{2(1 - \beta)} \ln \frac{\beta}{\pi}.
\]

On the left-hand side of this inequality we have what might be called the continuous or integral versions of the Rényi entropies. To derive this mathematical inequality, I have dropped $\hbar$ in the definition \((5)\) of the Fourier transform. This inequality has been also recently independently proven by Zozor and Vignat \((39)\). Analogous relations for the continuous Tsallis entropies for $x$ and $p$ were obtained by Rajagopal \((28)\).

In the limit, when $\alpha \to 1$ and $\beta \to 1$, we obtain from the inequality \((35)\) the entropic uncertainty relation in the form
\[
-\int_{-\infty}^{\infty} d\rho(p) \ln \rho(p) - \int_{-\infty}^{\infty} dx \rho(x) \ln \rho(x) \geq \ln(2\pi)
\]
that has been conjectured by Hirschman \((40)\) and later proved by Bialynicki-Birula and Mycielski \((41)\) and by Beckner \((31)\). The inequalities \((35)\) and \((39)\) are saturated by the Gaussian probability distributions.

For wave functions defined over an $n$-dimensional space, the bound on the right-hand side in \((35)\) and \((39)\) is just multiplied by $n$, as in the previous formula \((25)\). Therefore also in this case, like in the finite-dimensional case, the uncertainty relations are already saturated by separable states.

In a similar fashion we can derive the uncertainty relation for $\varphi$ and $M_z$ that does not involve the resolution $\delta \varphi$.
\[
\frac{1}{1 - \alpha} \ln \left( \sum_{m} \rho_m^\alpha \right) + \frac{1}{1 - \beta} \ln \left( \int_{0}^{2\pi} d\varphi (|\rho(\varphi)|)^\beta \right) \geq \ln(2\pi),
\]
where $\rho_m = |c_m|^2$ and $|\rho(\varphi)| = |\psi(\varphi)|^2$. In the limit, when $\alpha \to 1$ and $\beta \to 1$, we obtain the mathematical entropic uncertainty relation for the angle and the angular momentum derived before \((41)\).
\[
-\sum_{m} \rho_m \ln \rho_m - \int_{0}^{2\pi} d\varphi \rho(\varphi) \ln \rho(\varphi) \geq \ln(2\pi).
\]
The inequalities \((40)\) and \((41)\), like their discrete counterpart \((20)\), are saturated when the Fourier series \((25)\) has only one term.

VIII. SYMMETRIZED RÉNYI ENTROPY

In the uncertainty relations for the Rényi entropies the parameters $\alpha$ and $\beta$ appear always in conjugate pairs. This observation suggests the introduction of the symmetrized Rényi entropy $H_s$ defined as follows
\[
H_s = \frac{1}{2} (H_\alpha + H_\beta),
\]
where $\alpha$ and $\beta$ satisfy the conditions \((12)\) and they are related to the parameter $s$ through the formulas
\[
\alpha = \frac{1}{1 - s}, \quad \beta = \frac{1}{1 + s}, \quad -1 \leq s \leq 1.
\]
The symmetrized Rényi entropy $\mathcal{H}_s$ is a symmetric function of $s$ and for $s = 0$ it becomes the Shannon entropy. The uncertainty relations expressed in terms of the symmetrized Rényi entropies have the form

$$\mathcal{H}_s^{(p)} + \mathcal{H}_s^{(x)} \geq \frac{1}{2} \left( \ln(1 - s^2) + \frac{1}{s} \ln \frac{1 + s}{1 - s} \right) - \ln \left( \frac{\delta x \delta p}{\pi \hbar} \right). \quad (44)$$

They are obtained by taking half of the sum of the inequality (7) and the inequality obtained from (7) by interchanging $\alpha$ and $\beta$. The same symmetrization procedure can be applied to all other uncertainty relations derived in this paper. In particular, we obtain

$$\mathcal{H}_s^{(M_1)} + \mathcal{H}_s^{(c)} \geq -\ln \frac{\delta \varphi}{2\pi}. \quad (45)$$

Analogous symmetrized versions of the uncertainty relations for the Tsallis entropies were introduced also by Rajagopal [28].

In contrast to the inequalities that contain the Rényi entropies $H_\alpha$ and $H_\beta$, in the uncertainty relations that contain the symmetrized entropy the same measure of uncertainty is used for both physical variables. This is clearly a desirable feature but it remains to be seen whether the symmetrized Rényi entropy (42) is a useful concept outside the realm of the uncertainty relations.

Different uncertainty relations in which the same measure of uncertainty is used for both variables follow from the fact that the Rényi entropy is a nonincreasing function of $\alpha$. For example, for the position and momentum we obtain

$$H_\beta^{(p)} + H_\beta^{(x)} \geq -\ln \beta - \frac{\beta - 1/2}{1 - \beta} \ln(2\beta - 1) - \ln \left( \frac{\delta x \delta p}{\pi \hbar} \right), \quad (46)$$

where $1 \geq \beta \geq 1/2$.

IX. CONCLUSIONS

I have shown that quantum mechanical uncertainty relations for canonically conjugate variables can be expressed as inequalities involving the Rényi entropies. The simplicity of these relations indicates, in my opinion, that the Rényi entropy is an apt characteristic of the uncertainties in quantum measurements. A significant feature of the uncertainty relations (7), (9), (25), and (26) is the appearance of the resolving power of the measuring apparatus. Since the Rényi entropy is an extension of the Shannon entropy, the new uncertainty relations generalize the entropic uncertainty relations derived before.

The formulation of the uncertainty relations in terms of the Rényi entropies seems to indicate that a symmetrized version of the Rényi entropy (42) might be a useful concept.

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