OVERLAP BETWEEN USUAL AND MODIFIED BETHE VECTORS

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We consider the overlap of Bethe vectors of the XXX spin chain with a diagonal twist and the modified Bethe vectors with a general twist. We find a determinant representation for this overlap under one additional condition on the twist parameters. Such objects arise in the calculations of nonequilibrium physics.

Keywords: modified algebraic Bethe ansatz, scalar product, twisted boundary condition

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1. Introduction

Integrable quantum spin chains are a powerful tool for studying nonequilibrium physics. They have parameters that can be considered time-dependent. A simple example is an external magnetic field given by a Heaviside function that turns the field on and off. One can also consider the change in the parameters that define the boundary conditions. For example, a chain of \( N \) spins can be closed periodically or in the form of the Möbius strip. In the general case, the classification of integrable boundary parameters follows from the quantum inverse scattering method and related quantum algebras. In particular, the twisting of isotropic closed spin chains is given by an arbitrary invertible matrix.

To study the behavior of quantum integrable models when the boundary conditions change, we should solve the eigenproblem for any integrable boundary parameters and calculate the overlap between the states corresponding to different values of these parameters. For this, we use the quantum inverse scattering method [1]–[3]. In this method, the eigenvectors of quantum Hamiltonians are constructed using the algebraic Bethe ansatz, and the roots of the Bethe equations parameterize the spectrum. A modified version of the Bethe ansatz is needed in the case of so-called “off-diagonal boundaries” [4]. The spectrum of such models is given by modified Bethe equations, which contain an inhomogeneous term [5], [6]. In this case, there is an additional restriction on the number of Bethe equations.

In a recent series of papers, we studied the scalar products of modified Bethe vectors [7]–[9]. We found that the well-known formulas of the standard algebraic Bethe ansatz smoothly transform into a modified form, which has the same structure. This applies both to scalar products of vectors of a general form [12] and to scalar products containing the eigenvectors of a modified transfer matrix [12]–[13]. In this paper,
we consider the overlap (scalar product) of two vectors corresponding to different transfer matrices. One of the vectors corresponds to a transfer matrix with a diagonal twist. It can be constructed within the framework of the standard algebraic Bethe ansatz. The second vector is a modified Bethe vector that corresponds to a transfer matrix with a twist of the general form. To construct this vector, the modified Bethe ansatz must be used. We show that when one additional condition is imposed on the twist parameters, such an overlap has a compact determinant representation. This opens a possibility to study quenches in the XXX chains of an arbitrary spin.

This paper is organized as follows. In Sec. 2, we recall basic notions of the quantum inverse scattering method and introduce a special notation used below. In Sec. 3, we introduce a modified transfer matrix and construct modified Bethe vectors. Section 4 is devoted to a modified Izergin determinant [7]. There, we derive a new representation that contains a set of arbitrary complex numbers. In Sec. 5, we consider the overlap between Bethe vectors with diagonal twist and modified Bethe vectors of the general type. We show that under one additional constraint on the twist parameters, this overlap has a determinant representation. The Appendix contains some auxiliary formulas for the modified Izergin determinant and proofs of some propositions used in the calculations.

2. Quantum inverse scattering method

To formulate the problem in the framework of the quantum inverse scattering method, we consider a monodromy matrix

\[ T(u) = \begin{pmatrix} t_{11}(u) & t_{12}(u) \\ t_{21}(u) & t_{22}(u) \end{pmatrix}. \] (2.1)

The matrix elements \( t_{kl}(u) \) act in a Hilbert space \( \mathcal{H} \) and depend on a complex parameter \( u \). The commutation relations between \( t_{kl}(u) \) are defined by the RTT relation

\[ R_{12}(u - v)T_1(u)T_2(v) = T_2(v)T_1(u)R_{12}(u - v). \] (2.2)

Here, \( T_1(u) = T(u) \otimes 1 \) and \( T_2(u) = 1 \otimes T(u) \), where \( 1 \) is the identity matrix in \( \mathbb{C}^2 \). The R-matrix \( R_{12}(u) \) acts in \( \mathbb{C}^2 \otimes \mathbb{C}^2 \) and has the form

\[ R(u) = \frac{uc}{2}(1 \otimes 1) + P, \] (2.3)

where \( P \) is the permutation operator such that \( P x \otimes y = y \otimes x \) for any \( x, y \in \mathbb{C}^2 \), and \( c \) is a constant. R-matrix (2.3) has the property that

\[ [R_{12}(u - v), K_1K_2] = 0 \] (2.4)

for any \( 2 \times 2 \) matrix \( K \). This property implies that the matrix \( KT(u) \) satisfies RTT relation (2.2).

The trace of the monodromy matrix \( \text{tr} T(u) = t_{11}(u) + t_{22}(u) \) is called the transfer matrix. Due to RTT relation (2.2),

\[ [\text{tr} T(u), \text{tr} T(v)] = 0 \] (2.5)

for any complex \( u \) and \( v \). This property of the transfer matrix allows us to consider it as a generating function of the integrals of motion of a quantum model. Due to the cyclic property of the trace, this model has periodic boundary conditions. The trace of the twisted monodromy matrix \( \text{tr}(KT(u)) \) can also be used to generate integrals of motion. However, they satisfy boundary conditions of a more general form.

We define a highest-weight representation by \( V(\lambda_1(u), \lambda_2(u)) \), where \( \lambda_i(u) \) are some complex-valued functions, and the highest-weight vector \( |0\rangle \in \mathcal{H} \) is defined by

\[ t_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle, \quad t_{21}(u)|0\rangle = 0. \] (2.6)
The action of the operator \( t_{12}(u) \) on \( |0\) is free. A state obtained by the successive action of the \( t_{12} \) operators on the highest-weight vector is called the Bethe vector:

\[
|\Phi(\bar{u})\rangle = t_{12}(u_1) \ldots t_{12}(u_n)|0\rangle. \tag{2.7}
\]

Here, \( n = 0, 1, \ldots \) and \( \bar{u} = \{u_1, \ldots, u_n\} \). If the parameters \( \bar{u} \) are generic complex numbers, then \( |\Phi(\bar{u})\rangle \) is called an off-shell Bethe vector. Under certain constraints on the parameters \( \bar{u} \) (see Sec. 3), vector (2.7) becomes an eigenvector of the transfer matrix. Then it is called the on-shell Bethe vector.

To study scalar products, we also introduce the dual vector \( \langle 0| \in \mathcal{H}^* \) defined by

\[
\langle 0|t_{12}(u) = \lambda_i(u)|0\rangle, \quad \langle 0|t_{12}(u) = 0 \tag{2.8}
\]

and the condition \( \langle 0|0\rangle = 1 \). The functions \( \lambda_i(u) \) are the same as in (2.6). Dual Bethe vectors are constructed by the successive action of the \( t_{21} \) operators on \( \langle 0| \):

\[
\langle \Phi(\bar{u})| = \langle 0|t_{21}(u_1) \ldots t_{21}(u_n). \tag{2.9}
\]

**Notation.** To simplify formulas, we introduce three rational functions

\[
g(u, v) = \frac{c}{u-v}, \quad f(u, v) = 1 + g(u, v) = \frac{u-v+c}{u-v}, \tag{2.10}
\]

where \( c \) is the constant entering the \( R \)-matrix. It is easy to see that these functions have the properties

\[
\chi(u, v)|_{c \to -c} = \chi(v, u), \quad \chi(-u, -v) = \chi(v, u), \quad \chi(u-c, v) = \chi(u, v+c),
\]

where \( \chi \) is any of the three functions. Besides, we have

\[
g(u, v-c) = \frac{1}{h(u, v)}, \quad h(u, v+c) = \frac{1}{g(u, v)}, \quad f(u, v+c) = \frac{1}{f(v, u)}. \tag{2.11}
\]

Below, we deal with sets of complex parameters. We use a bar to denote them, for example, \( \bar{u} = \{u_1, \ldots, u_n\} \). The notation \( \bar{u} \pm c \) means that \( \pm c \) is added to all the arguments of the set \( \bar{u} \). The notation \( \bar{u}_k \) refers to the subset \( \bar{u}_k = \bar{u} \setminus u_k \).

To make the formulas more compact, we use a shorthand notation for the products of rational function (2.10), the operators \( t_{kl}(u) \) in (2.1), and their vacuum eigenvalues \( \lambda_i(u) \) in (2.6). Namely, if a function (operator) depends on a set of variables, then this means the product over the corresponding set. For example, if \( \bar{u} = \{u_1, \ldots, u_n\} \), then

\[
t_{kl}(\bar{u}) = \prod_{j=1}^n t_{kl}(u_j), \quad \lambda_i(\bar{u}) = \prod_{j=1}^n \lambda_i(u_j), \tag{2.12}
\]

\[
f(z, \bar{u}) = \prod_{j=1}^n f(z, u_j), \quad f(\bar{u}_k, u_k) = \prod_{j=1}^n f(u_j, u_k)
\]

and so on. We note that RTT relation (2.2) implies \( [t_{kl}(u), t_{kl}(v)] = 0 \). Thus, the first product in (2.12) is well defined. In what follows, we extend this convention to the products of matrix elements of the twisted monodromy matrix.
The notation \( f(\bar{u}, \bar{v}) \) means the double product over the sets \( \bar{u} \) and \( \bar{v} \). By definition, any product over the empty set is equal to 1. A double product is equal to 1 if at least one of the sets is empty.

Finally, for any set of complex parameters \( \bar{u} = \{u_1, \ldots, u_n\} \) such that \( n \geq 2 \), we also introduce special products of the \( g \)-functions

\[
\Delta(\bar{u}) = \prod_{1 \leq k < j \leq n} g(u_j, u_k), \quad \Delta'(\bar{u}) = \prod_{1 \leq k < j \leq n} g(u_k, u_j).
\]

It is easy to see that \( \Delta(\bar{u}) = (-1)^{n(n-1)/2} \Delta'(\bar{u}) \). For \( n = 0, 1 \), we set \( \Delta(\bar{u}) = \Delta'(\bar{u}) = 1 \) by definition.

### 3. Twisted monodromy matrix

We consider two twisted monodromy matrices \( K^{(\ell)}T(u) \), \( \ell = 1, 2 \). The first corresponds to the twist matrix \( K^{(1)} = \text{diag}(1, \alpha) \), where \( \alpha \) is a complex parameter. Then the twisted transfer matrix has the form

\[
t^{(1)}(u) = t_{11}(u) + \alpha t_{22}(u).
\]

We call it the diagonal transfer matrix. It generates a model with quasiperiodic boundary conditions. The eigenvectors (resp. dual eigenvectors) of the diagonal transfer matrix have form (2.7) (resp. (2.9)):

\[
|\Phi^{(1)}(\bar{u})\rangle = t_{12}(\bar{u})|0\rangle, \quad \langle \Phi^{(1)}(\bar{u})| = \langle 0|t_{21}(\bar{u}).
\]

We add an extra superscript to the vectors to emphasize that they refer to the transfer matrix \( t^{(1)}(u) \).

If the parameters \( \bar{u} = \{u_1, \ldots, u_n\} \) are arbitrary complex numbers, then vectors (3.2) are off-shell. If these parameters satisfy the twisted Bethe equations

\[
\lambda_1(u_j)f(\bar{u}_j, u_j) = \alpha \lambda_2(u_j)f(\bar{u}_j, \bar{u}_j), \quad j = 1, \ldots, n,
\]

then

\[
t^{(1)}(v)|\Phi^{(1)}(\bar{u})\rangle = \Lambda^{(1)}(v|\bar{u})|\Phi^{(1)}(\bar{u})\rangle, \quad \langle \Phi^{(1)}(\bar{u})|t^{(1)}(v) = \Lambda^{(1)}(v|\bar{u})\langle \Phi^{(1)}(\bar{u}),
\]

where

\[
\Lambda^{(1)}(v|\bar{u}) = \lambda_1(v)f(\bar{u}, v) + \alpha \lambda_2(v)f(v, \bar{u}).
\]

The vectors \( |\Phi^{(1)}(\bar{u})\rangle \) and \( \langle \Phi^{(1)}(\bar{u})| \) are said to be on-shell if conditions (3.3) are satisfied. In the case of the XXX spin-1/2 chain, the number of parameters \( \{u_1, \ldots, u_n\} \) does not exceed the length of the chain.

The second deformation corresponds to the twist matrix of the general form

\[
K^{(2)} = \begin{pmatrix} \bar{\kappa} & \kappa^+ \\ \kappa^- & \kappa \end{pmatrix},
\]

where the entries of \( K^{(2)} \) are arbitrary complex numbers. This twist produces models with nondiagonal boundary conditions. The standard procedure of the algebraic Bethe ansatz must be modified in this case [8], [14]. We represent the twist matrix in the form \( K^{(2)} = BDA \) where

\[
A = \sqrt{\mu} \begin{pmatrix} 1 & \rho_2 \\ \rho_1 & \kappa^+ \end{pmatrix}, \quad B = \sqrt{\mu} \begin{pmatrix} 1 & \rho_1 \\ \rho_2 & \kappa^+ \end{pmatrix}, \quad D = \begin{pmatrix} \bar{\kappa} - \rho_1 & 0 \\ 0 & \kappa - \rho_2 \end{pmatrix}.
\]
Here,
\[ \mu = \frac{1}{1 - \rho_1 \rho_2 / \kappa^+ \kappa^-}, \]
and the parameters \( \rho \) satisfy the condition
\[ \rho_1 \rho_2 - (\kappa \rho_1 + \kappa \rho_2) + \kappa^+ \kappa^- = 0. \]

Then the twisted transfer matrix is defined as
\[ t^{(2)}(u) = \text{tr}(K^{(2)}T(u)) = \text{tr}(\mathcal{D} \mathcal{T}(u)), \]
where \( \mathcal{T}(u) \) is the modified monodromy matrix
\[ \mathcal{T}(u) = AT(u)B = \begin{pmatrix} \nu_{11}(u) & \nu_{12}(u) \\ \nu_{21}(u) & \nu_{22}(u) \end{pmatrix}. \]

The entries of \( \mathcal{T}(u) \) have the following expressions in terms of the initial monodromy matrix elements:
\begin{align}
\nu_{11}(z) &= \mu \left( t_{11}(z) + \frac{\rho_2}{\kappa^+} t_{12}(z) + \frac{\rho_2}{\kappa^-} t_{21}(z) + \frac{\rho_3^2}{\kappa^+ \kappa^-} t_{22}(z) \right), \\
\nu_{22}(z) &= \mu \left( t_{22}(z) + \frac{\rho_1}{\kappa^+} t_{12}(z) + \frac{\rho_1}{\kappa^-} t_{21}(z) + \frac{\rho_2^2}{\kappa^+ \kappa^-} t_{11}(z) \right), \\
\nu_{12}(z) &= \mu \left( t_{12}(z) + \frac{\rho_1}{\kappa^-} t_{11}(z) + \frac{\rho_2}{\kappa^+} t_{22}(z) + \frac{\rho_1 \rho_2}{(\kappa^-)^2} t_{21}(z) \right), \\
\nu_{21}(z) &= \mu \left( t_{21}(z) + \frac{\rho_1}{\kappa^-} t_{11}(z) + \frac{\rho_2}{\kappa^+} t_{22}(z) + \frac{\rho_1 \rho_2}{(\kappa^-)^2} t_{12}(z) \right). 
\end{align}

The modified Bethe vectors (resp. the dual modified Bethe vectors) related to twisted transfer matrix (3.5) can be constructed from the new entries \( \nu_{12}(z) \) (resp. \( \nu_{21}(z) \)). They are given by
\[ |\Phi^{(2)}(\bar{u})\rangle = \nu_{12}(\bar{u})|0\rangle, \quad \langle \Phi^{(2)}(\bar{u})| = \langle 0|\nu_{21}(\bar{u}). \]

Here, we use the convention on the shorthand notation for the products of the operators \( \nu_{ij}(u) \).

If \( \bar{u} = \{u_1, \ldots, u_N\} \) is a set of arbitrary complex numbers, then vectors (3.11) are off-shell. However, if the set \( \bar{u} \) satisfies the inhomogeneous Bethe equations
\[ (\kappa - \rho_1) \lambda_1(u_j) f(\bar{u}, u_j) = (\kappa - \rho_2) \lambda_2(u_j) f(u_j, \bar{u}) + (\rho_1 + \rho_2) \lambda_1(u_j) \lambda_2(u_j) g(u_j, \bar{u}) \]
for \( j = 1, \ldots, N \), then they are eigenvectors of twisted transfer matrix (3.5),
\[ t^{(2)}(v) |\Phi^{(2)}(\bar{u})\rangle = \Lambda^{(2)}(v|\bar{u}) |\Phi^{(2)}(\bar{u})\rangle, \quad \langle \Phi^{(2)}(\bar{u})| t^{(2)}(v) = \Lambda^{(2)}(v|\bar{u}) \langle \Phi^{(2)}(\bar{u})|, \]
with the eigenvalue
\[ \Lambda^{(2)}(v|\bar{u}) = (\kappa - \rho_1) \lambda_1(v) f(\bar{u}, v) + (\kappa - \rho_2) \lambda_2(v) f(v, \bar{u}) + (\rho_1 + \rho_2) \lambda_1(v) \lambda_2(v) g(v, \bar{u}). \]

In this case, we say that the vectors \( |\Phi^{(2)}(\bar{u})\rangle \) and \( \langle \Phi^{(2)}(\bar{u})| \) are modified on-shell vectors.
The number of parameters \{u_1, \ldots, u_N\} depends on the chain length and spin. In particular, in the XXX spin-1/2 chain, \(N\) coincides with the number of sites. In the case of higher spins, \(N\) exceeds the chain length (see [8] for more details).

The main goal of this paper is to calculate the overlap
\[
S^{n,N}(\tilde{v}, \tilde{u}) = \langle \Phi^{(1)}(\tilde{v}) \rangle^{(2)}(\tilde{u}),
\]
(3.13)
where \(\langle \Phi^{(1)}(\tilde{v}) \rangle\) and \(|\Phi^{(2)}(\tilde{u})\rangle\) are the eigenvectors of the corresponding transfer matrices, \(n = \# \tilde{v}, N = \# \tilde{u}\).

Then according to Fermi's Golden rule for the transition between two twists, we obtain
\[
\Gamma_{1 \rightarrow 2} = \frac{2\pi}{\hbar} \left| \langle \Phi^{(1)}(\tilde{v}) \rangle |(H^{(1)} - H^{(2)})|\Phi^{(2)}(\tilde{u})\rangle \right|^2 \rho^{(2)}(\tilde{u}).
\]
(3.14)
Here, \(H^{(1)}\) and \(H^{(2)}\) are the Hamiltonians corresponding to the respective transfer matrices \(t^{(1)}\) and \(t^{(2)}\), and \(\rho^{(2)}(\tilde{u})\) is a density of the resulting states.

In the case of the XXX spin-1/2 chains, we can express the Hamiltonians in terms of the twisted transfer matrices using the relation
\[
H^{(1)} - H^{(2)} = 2\epsilon \frac{d}{dz} \left( \ln t^{(1)}(z) - \ln t^{(2)}(z) \right)_{z=0}.
\]
(3.15)
Thus, we find
\[
\Gamma_{1 \rightarrow 2} = \frac{2\pi}{\hbar} \left| 2\epsilon \frac{d}{dz} \ln \frac{\Lambda^{(1)}(z)|\tilde{v}\rangle}{\Lambda^{(2)}(z)|\tilde{u}\rangle} \right|^2 |S^{n,N}(\tilde{v}, \tilde{u})|^2 \rho^{(2)}(\tilde{u}).
\]
(3.16)

4. Modified Izergin determinant

In various formulas for the scalar products, a modified Izergin determinant (MID) arises.

**Definition 1.** Let \(\tilde{u} = \{u_1, \ldots, u_n\}\), \(\tilde{v} = \{v_1, \ldots, v_m\}\), and \(z\) be arbitrary complex numbers. Then the MID \(K^{(z)}_{n,m}(\tilde{u}|\tilde{v})\) is defined by

\[
K^{(z)}_{n,m}(\tilde{u}|\tilde{v}) = \det_m \left( -z\delta_{jk} + \frac{f(u_j, v_j) f(v_j, \tilde{v}_j)}{h(v_j, v_k)} \right). \tag{4.1}
\]

Alternatively, the MID can be represented as

\[
K^{(z)}_{n,m}(\tilde{u}|\tilde{v}) = (1 - z)^{m-n} \det_n \left( \delta_{jk} f(u_j, \tilde{v}) - z \frac{f(u_j, \tilde{v}_j)}{h(u_j, u_k)} \right). \tag{4.2}
\]

The proof of the equivalence of representations (4.1) and (4.2) can be found in proposition 4.1 in [15].

For \(m = n\) and \(z = 1\), the MID turns into the usual Izergin determinant [16], equal to the partition function of the six-vertex model with the domain-wall boundary condition [12], [16].

We also use the conjugate MID

\[
K^{(z)}_{n,m}(\tilde{u}|\tilde{v}) = K^{(z)}_{n,m}(\tilde{u}|\tilde{v})_{c \rightarrow -c} = \det_m \left( -z\delta_{jk} + \frac{f(v_j, \tilde{u}) f(\tilde{v}_j, v_j)}{h(v_j, v_k)} \right). \tag{4.3}
\]

Equivalently, it can be defined as

\[
K^{(z)}_{n,m}(\tilde{u}|\tilde{v}) = (1 - z)^{m-n} \det_n \left( \delta_{jk} f(\tilde{v}, u_j) - z \frac{f(\tilde{v}_j, u_j)}{h(u_k, u_j)} \right). \tag{4.4}
\]

There also exist representations for the MID that contain additional parameters.
Proposition 4.1. Let \( \tilde{u} = \{u_1, \ldots, u_n\}, \tilde{\eta} = \{\eta_1, \ldots, \eta_n\}, \tilde{v} = \{v_1, \ldots, v_m\}, \) and \( z \) be arbitrary complex numbers. Then

\[
K_{n,m}^{(z)}(\tilde{u}|\tilde{v}) = (1 - z)^{m-n} \Delta'(\tilde{u}) \Delta(\tilde{\eta}) \det_n \left( \frac{f(u_j, v_l)}{g(u_j, \eta_k)} - z h(u_j, \eta_k) \right). \tag{4.5}
\]

Proof. Let \( \tilde{u} = \{u_1, \ldots, u_n\} \) be pairwise distinct complex numbers. We consider the \( n \times n \) matrix \( W \) with the elements

\[
W_{jk} = \frac{g(u_j, \tilde{u}_j)}{g(u_j, \eta_k)}, \tag{4.6}
\]

where \( \tilde{\eta} = \{\eta_1, \ldots, \eta_n\} \) are arbitrary pairwise distinct complex numbers. The entries \( W_{jk} \) are proportional to the Cauchy matrix \( g(u_j, \eta_k) \). Thus,

\[
det W = \Delta'(\tilde{u}) \Delta(\tilde{\eta}) \det_n g(u_j, \eta_k) = \Delta(\tilde{\eta}). \tag{4.7}
\]

Hence, the determinant of \( W \) exists and is nonvanishing. We transform representation (4.2) as

\[
K_{n,m}^{(z)}(\tilde{u}|\tilde{v}) = \frac{(1 - z)^{m-n}}{\det_n W} \det_n f(u_j, \tilde{v}) W_{jk} - \sum_{l=1}^{n} \frac{f(u_j, \tilde{u}_j) W_{lk}}{h(u_j, u_l)} W_{lk}. \tag{4.8}
\]

The sum over \( l \) is easily computable. Indeed, let

\[
G_{jk} = \sum_{l=1}^{n} \frac{W_{lk}}{h(u_j, u_l)} = \sum_{l=1}^{n} \frac{g(u_l, \eta_k)}{g(u_l, \tilde{u}_l)} \frac{g(u_l, \tilde{u}_l)}{g(u_l, \tilde{\eta})}.	ag{4.9}
\]

Then we have

\[
\frac{1}{2\pi i} \oint_{|z|=R} \frac{g(z, \eta_k)}{g(z, \tilde{\eta})} dz = 0 = G_{jk} - \frac{h(u_j, \tilde{u}_k)}{h(u_j, \tilde{\eta})}. \tag{4.10}
\]

Substituting this in (4.8), we obtain

\[
K_{n,m}^{(z)}(\tilde{u}|\tilde{v}) = \frac{(1 - z)^{m-n}}{\Delta(\tilde{\eta})} \det_n \left( f(u_j, \tilde{v}) \frac{g(u_j, \tilde{u}_j)}{g(u_j, \eta_k)} - z g(u_j, \tilde{u}_j) h(u_j, \eta_k) \right). \tag{4.11}
\]

Extracting the products \( g(u_j, \tilde{u}_j) \), we arrive at (4.5). It remains to note that the limits \( u_j = u_k \) and \( \eta_j = \eta_k \) \((j, k = 1, \ldots, n)\) are well defined. Indeed, in this case, the prefactor \( \Delta'(\tilde{u}) \Delta(\tilde{\eta}) \) has a pole, while the determinant vanishes. Therefore, representation (4.5) is valid for any complex \( \tilde{u} \) and \( \tilde{\eta} \).

Similarly, we can prove the following representation for the MID:

\[
K_{n,m}^{(z)}(\tilde{u}|\tilde{v}) = \Delta'(\tilde{u}) \Delta(\tilde{\eta}) \det_m \left( f(\tilde{u}, v_j) h(u_j, \eta_k) - \frac{z}{g(u_j, \eta_k)} \right). \tag{4.11}
\]

Here, \( \tilde{\eta} = \{\eta_1, \ldots, \eta_m\} \) is a set of arbitrary complex numbers.

Corollary 4.1. Let \( \tilde{u} = \{u_1, \ldots, u_n\}, \tilde{\eta} = \{\eta_1, \ldots, \eta_n\}, \) and \( z \) be arbitrary complex numbers. Then

\[
\Delta'(\tilde{u}) \Delta(\tilde{\eta}) \det_n \left( \frac{1}{g(u_j, \eta_k)} - z h(u_j, \eta_k) \right) = (1 - z)^n. \tag{4.12}
\]

Proof. Setting \( \tilde{v} = \varnothing \) in (4.5) we obtain

\[
\Delta'(\tilde{u}) \Delta(\tilde{\eta}) \det_n \left( \frac{1}{g(u_j, \eta_k)} - z h(u_j, \eta_k) \right) = (1 - z)^n K_{n,0}^{(z)}(\tilde{u}|\varnothing). \tag{4.13}
\]

Then (4.12) follows from (4.1).
5. Overlap

We now proceed to the calculation of the overlap in Eq. (3.13),

\[ S^{n,N}(\bar{v}, \bar{u}) = \langle 0| t_{21}(\bar{v}) \nu_{12}(\bar{u})|0 \rangle. \]  

(5.1)

We recall that \( n = \# \bar{v} \) and \( N = \# \bar{u} \). Ultimately, we are interested in the case where the parameters \( \bar{v} \) and \( \bar{u} \) respectively satisfy Eqs. (3.3) and (3.12). However, in this paper, we consider a more general case when the parameters \( \bar{u} \) are arbitrary complex numbers. In particular, they can satisfy system (3.12). Moreover, at the first stage of calculations, we do not impose any restrictions on the parameters \( \bar{v} \). We only require that \( n \leq N \), because otherwise the scalar product (5.1) vanishes.

5.1. Scalar product of off-shell vectors. It follows from (3.9) that the vector \( \mu^{-N}\Phi^{(2)}(\bar{u}) \) is independent of \( \kappa^+ \). Therefore, the scalar product \( \mu^{-N} S^{n,N}(\bar{v}, \bar{u}) \) is also independent of \( \kappa^+ \). On the other hand, it follows from (3.10) that

\[ t_{21}(v) = \lim_{\kappa^+ \to \infty} \nu_{21}(v). \]  

(5.2)

Thus,

\[ \langle 0| t_{21}(\bar{v}) \nu_{12}(\bar{u})|0 \rangle = \mu^N \lim_{\kappa^+ \to \infty} \langle 0| \nu_{21}(\bar{v}) \nu_{12}(\bar{u})|0 \rangle, \]

(5.3)

where we used \( \mu \to 1 \) as \( \kappa^+ \to \infty \). In its turn, a formula for the scalar product \( \langle 0| \nu_{21}(\bar{v}) \nu_{12}(\bar{u})|0 \rangle \) was derived in [7]:

\[ \langle 0| \nu_{21}(\bar{v}) \nu_{12}(\bar{u})|0 \rangle = \mu^{N+n} \left( \frac{\rho_1}{\kappa} \right)^{N-n} \sum_{\{\bar{u}_1, \bar{u}_2\} \vdash \bar{u}} \left( \frac{\rho_1}{\rho_2} \right)^{n_{II} - N_{II}} \lambda_2(\bar{v}_1) \lambda_2(\bar{u}_1) \lambda_1(\bar{v}_1) \lambda_1(\bar{u}_1) \times \]

\[ \times f(\bar{v}_1, \bar{v}_1, \bar{u}_1) K^{(1/\mu)}_{N_{II}, n_{II}}(\bar{u}_1|\bar{v}_1) \mathcal{K}^{(1)}_{N_{II}, n_{II}}(\bar{u}_1|\bar{v}_1). \]

(5.4)

Here, the sum is taken over all possible partitions \( \{\bar{v}_1, \bar{v}_1\} \vdash \bar{v} \) and \( \{\bar{u}_1, \bar{u}_1\} \vdash \bar{u} \) without restrictions on the cardinalities of the subsets \( N_{II} = \# \bar{u}_{1,II} \) and \( n_{II} = \# \bar{v}_{1,II} \).

Taking the limit \( \kappa^+ \to \infty \) in (5.4) corresponds to the limit \( \mu \to 1 \). Thus, we obtain

\[ S^{n,N}(\bar{v}, \bar{u}) = \mu^N \left( \frac{\rho_1}{\kappa} \right)^{N-n} \sum_{\{\bar{u}_1, \bar{u}_2\} \vdash \bar{u}} \left( \frac{\rho_1}{\rho_2} \right)^{n_{II} - N_{II}} \lambda_2(\bar{v}_1) \lambda_2(\bar{u}_1) \lambda_1(\bar{v}_1) \lambda_1(\bar{u}_1) \times \]

\[ \times f(\bar{v}_1, \bar{v}_1, \bar{u}_1) f(\bar{u}_1, \bar{u}_1) K^{(1)}_{N_{II}, n_{II}}(\bar{u}_1|\bar{v}_1) \mathcal{K}^{(1)}_{N_{II}, n_{II}}(\bar{u}_1|\bar{v}_1). \]

The sum is taken over the partitions described above. Formally, we still have no restrictions on the cardinalities of the subsets. However, at least one of the MIDs in this equation vanishes if \( N_{II} < n_{II} \) or \( N_{II} < n_{II} \) (see (4.1), (4.4)).

5.2. Scalar product with an on-shell Bethe vector. Up to now, both vectors were off-shell. Let \( \langle \Phi^{(1)}(\bar{v}) \rangle \) be an on-shell Bethe vector, which means that the set \( \bar{v} \) satisfies Bethe equations (3.3). Now we can use the Bethe equations in the form

\[ \lambda_1(\bar{v}_{II}) f(\bar{v}_1, \bar{v}_1) = \alpha^{n_{II}} \lambda_2(\bar{v}_1) f(\bar{v}_1, \bar{v}_1). \]  

(5.5)

Then we obtain

\[ S^{n,N}(\bar{v}, \bar{u}) = \mu^N \left( \frac{\rho_1}{\kappa} \right)^{N-n} \sum_{\{\bar{u}_1, \bar{u}_2\} \vdash \bar{u}} \left( \frac{\rho_2}{\rho_1} \right)^{N_{II}} \lambda_2(\bar{u}_1) \lambda_1(\bar{u}_1) f(\bar{u}_1, \bar{u}_1) G(\bar{u}_1, \bar{u}_1), \]

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where
\[
G(\bar{u}_i, \bar{u}_{ii}) = \sum_{\{\bar{u}_i, \bar{u}_{ii}\}} \left( \frac{\alpha \rho_1}{\rho_2} \right)^{n_{ii}} f(\bar{v}_i, \bar{v}_i) K^{(1)}_{N_{ii}, n_{ii}}(\bar{u}_i, \bar{v}_i) \overline{K}^{(1)}_{N_{ii}, n_{ii}}(\bar{u}_i, \bar{v}_i).
\]

We impose an additional constraint on the twist parameters: \( \alpha = -\rho_2/\rho_1 \). Then the sum over partitions in (5.6) can be explicitly computed via (A.5):
\[
G(\bar{u}_i, \bar{u}_{ii}) |_{\alpha = -\rho_2/\rho_1} = (-1)^n f(\bar{v}, \bar{u}_i) K^{(1)}_{N, n}(\{\bar{u}_i - c, \bar{u}_{ii}\} | \bar{v}).
\]

Accordingly, the scalar product takes the form
\[
S^{n,N}(\bar{v}, \bar{u}) = (-1)^n \mu^N \left( \frac{\rho_1}{\kappa} \right)^{N-n} \lambda_2(\bar{v}) \times \sum_{\{\bar{u}_i, \bar{u}_{ii}\}} \left( \frac{\rho_2}{\rho_1} \right)^{N_{ii}} \lambda_2(\bar{u}_i) \lambda_1(\bar{v}) f(\bar{v}, \bar{u}_i) f(\bar{v}, \bar{u}_i) K^{(1)}_{N, n}(\{\bar{u}_i - c, \bar{u}_{ii}\} | \bar{v}).
\]

The sum over partitions of the set \( \bar{u} \) in (5.8) is computed via Proposition A.3. For this, it is enough to use the MID representation in form (4.5),
\[
K^{(1)}_{N, n}(\bar{u} | \bar{v}) = \lim_{z \to 1} (1 - z)^{n-N} \Delta(\bar{u}) \Delta'(\bar{\eta}) \det_N \left( \frac{f(w_j, \bar{v})}{g(w_j, \bar{\eta}_k)} - zh(w_j, \bar{\eta}_k) \right)
\]
where \( \bar{w} = \{\bar{u}_i - c, \bar{u}_{ii}\} \) for any fixed partition \( \{\bar{u}_i, \bar{u}_{ii}\} \). Then we satisfy the condition of Proposition A.3. We also recall that \( \bar{\eta} = \{\eta_1, \ldots, \eta_N\} \) are arbitrary complex numbers. Hence, we find
\[
S^{n,N}(\bar{v}, \bar{u}) = (-1)^n \mu^N \left( \frac{\kappa}{\rho_1} \right)^n \lambda_2(\bar{v}) \Delta'(\bar{\eta}) \Delta(\bar{u}) \lim_{z \to 1} (1 - z)^{n-N} \det_N \mathcal{N}_{jk}(z),
\]
where
\[
\mathcal{N}_{jk}(z) = \frac{(-1)^{N-1} \lambda_1(u_j)(h(\bar{\eta}_k, u_j) - z f(\bar{v}, u_j)) + \lambda_2(u_j)(f(u_j, \bar{v}) - z h(u_j, \bar{\eta}_k))}{g(u_j, \bar{\eta}_k)}
\]
and
\[
\lambda_{\ell}(u) = \frac{\rho_{\ell}}{\kappa} \lambda(u), \quad \ell = 1, 2.
\]

5.3. How to take the limit \( z \to 1 \). To take the limit \( z \to 1 \) in Eq. (5.10), we set \( \eta_k = v_k \) for \( k = 1, \ldots, n \) in matrix (5.11). Then
\[
S^{n,N}(\bar{v}, \bar{u}) = (-1)^n \mu^N \left( \frac{\kappa}{\rho_1} \right)^n \lambda_2(\bar{v}) \Delta'(\bar{\eta}) g(\bar{v}, \bar{\eta}) \Delta(\bar{u}) \times \lim_{z \to 1} (1 - z)^{n-N} \det_N \mathcal{N}_{jk}(z),
\]
where here and hereafter \( \bar{\eta} = \{\eta_{n+1}, \ldots, \eta_N\} \). The matrix \( \mathcal{N}_{jk}(z) \) now consists of two parts:
\[
\mathcal{N}_{jk}(z) = \mathcal{N}_{jk}^{(1)}(z), \quad k = 1, \ldots, n,
\]
\[
\mathcal{N}_{jk}(z) = \mathcal{N}_{jk}^{(2)}(z), \quad k = n + 1, \ldots, N.
\]
Here,
\[
\Lambda_{jk}^{(1)}(z) = (-1)^{N-1} \hat{\lambda}_1(u_j) h(\bar{v}, u_j) \left( \frac{h(\bar{v}, u_j)}{h(v_k, u_j)} - z \frac{g(v_k, u_j)}{g(\bar{v}, u_j)} \right) + \\
+ \hat{\lambda}_2(u_j) h(u_j, \bar{v}) \left( \frac{g(u_j, v_k)}{g(u_j, \bar{v})} - z \frac{h(u_j, \bar{v})}{h(u_j, v_k)} \right), \quad k = 1, \ldots, n, \tag{5.15}
\]
\[
\Lambda_{jk}^{(2)}(z) = (-1)^N \hat{\lambda}_1(u_j) h(\bar{v}, u_j) \left( \frac{z}{g(\bar{v}, u_j)} - h(\bar{v}, u_j) \right) + \\
+ \hat{\lambda}_2(u_j) h(u_j, \bar{v}) \left( \frac{1}{g(u_j, \bar{v})} - zh(u_j, \bar{v}) \right), \quad k = n + 1, \ldots, N. \tag{5.16}
\]

We can now transform \( \det_N \Lambda_{jk}(z) \) using Corollary A.1 (see, in particular, (A.25)). Then
\[
\det_N \Lambda(z) = (z - 1)^{N-n} \det \Lambda(z), \tag{5.17}
\]
where \( \Lambda_{jk}(z) = \Lambda_{jk}^{(1)}(z) \) for \( k = 1, \ldots, n \), and
\[
\Lambda_{jk}(z) = (-1)^{n+1} \hat{\lambda}_1(u_j) h(\bar{v}, u_j) - \hat{\lambda}_2(u_j) h(u_j, \bar{v}) h(\bar{v}, \bar{v}), \quad k = n + 1, \ldots, N. \tag{5.18}
\]
Substituting this result in (5.13) yields
\[
S^{n,N}(\bar{v}, \bar{u}) = (-\mu)^N \left( \frac{\kappa - \rho}{\rho_1} \right)^n \lambda_2(\bar{v}) \Delta'(\bar{v}) \Delta'(\bar{v}) g(\bar{v}, \bar{v}) \Delta(\bar{v}) \det \Lambda_{jk}(1). \tag{5.19}
\]
In this representation, the parameters \( \bar{\eta} = \{\eta_{n+1}, \ldots, \eta_N\} \) remain arbitrary complex numbers.

### 5.4. Particular case.
We did not impose any restrictions on the parameters \( \bar{u} \) in representation (5.19). Therefore, the vector \( |\Phi^{(2)}(\bar{u})\rangle \), generally speaking, is off-shell. We consider the particular case \( \alpha = 1 \). Then the dual vector \( \langle \Phi^{(1)}(\bar{v}) | \) is an eigenvector of the ordinary transfer matrix \( t_{11}(z) + t_{22}(z) \). On the other hand, we find \( \rho_1 = -\rho_2 \) from the condition \( \alpha = -\rho_2/\rho_1 \). Then the inhomogeneous term vanishes in the modified Bethe equations (3.12). We obtain
\[
\hat{\lambda}_1(u_j) = (-1)^N \frac{\kappa + \rho}{\kappa - \rho} \frac{h(u_j, \bar{v})}{h(u_j, u_j)}, \quad j = 1, \ldots, N, \tag{5.20}
\]
where \( \rho = \rho_1 = -\rho_2 \). At the same time, the Bethe vectors \( |\Phi^{(2)}(\bar{u})\rangle \) remain modified in this particular case. They are still given by (3.11).

We now require the vector \( |\Phi^{(2)}(\bar{u})\rangle \) to be an on-shell modified Bethe vector. For this, it suffices to substitute conditions (5.20) in representation (5.19). After simple algebra, we obtain
\[
S^{n,N}(\bar{v}, \bar{u}) = (-\mu)^N \left( \frac{\rho}{\kappa^2} \right)^{N-n} \lambda_2(\bar{v}) \lambda_2(\bar{u}) h(\bar{v}, \bar{u}) \Delta'(\bar{v}) \Delta'(\bar{v}) g(\bar{v}, \bar{v}) \Delta(\bar{v}) \det \Omega_{jk},
\]
where the matrix \( \Omega \) consists of two parts,
\[
\Omega_{jk} = \frac{g(u_j, v_k)}{g(u_j, \bar{v})} - \frac{h(u_j, \bar{v})}{h(u_j, v_k)} - V_j \left( \frac{h(\bar{v}, u_j)}{h(v_k, u_j)} - \frac{g(v_k, u_j)}{g(\bar{v}, u_j)} \right), \quad k = 1, \ldots, n,
\]
\[
\Omega_{jk} = (-1)^{N+n+1} \frac{V_j}{g(u_j, \bar{v})} - h(u_j, \bar{v}), \quad k = n + 1, \ldots, N,
\]
with
\[
V_j = \frac{\kappa + \rho}{\kappa - \rho} \frac{h(u_j, \bar{v})}{h(u_j, u_j)}.
\]
It is easy to see that in the particular case \( n = N \), we reproduce a determinant representation for the scalar product of the usual on-shell and twisted on-shell Bethe vectors [13].
6. Conclusion

We have considered overlaps of the Bethe vectors with a diagonal twist and the modified Bethe vectors. We have shown that under one additional condition on the twist parameters, such an overlap has a determinant representation. At the same time, the modified Bethe vector can remain an off-shell vector. In this sense, the result obtained is analogous to the scalar product of on-shell and off-shell Bethe vectors [13].

However, despite this similarity, the resulting determinant representation has an entirely new structure. Namely, it contains a set of arbitrary parameters. This representation was obtained thanks to new formulas for the MID. These parameters can be chosen in the most convenient way depending on the specific task. Such formulas can find applications in models with a higher symmetry rank [17].

The resulting determinant representation for the overlap remains valid if the modified Bethe vector is an eigenvector of the modified transfer matrix. In that case, the parameters of the modified vector satisfy inhomogeneous Bethe equations (3.12). We have considered only one specific case where the inhomogeneous term in the Bethe equations disappears. The more general case requires further study.

One of the apparent directions for further research is the application of the results obtained to nonequilibrium physics. For this, it is necessary to consider the overlaps in the thermodynamic limit and calculate the density of states. Another area to consider is overlaps under diagonal and off-diagonal boundary conditions in the models on a segment. It is natural to expect that, in this case, the results can be expressed in terms of the modified Tsuchiya determinant [18]. We hope that these formulas are easy to obtain using the method of reducing overlaps to a system of linear equations [19].

Appendix: Properties of the MID

A.1. Transformations of the MID. In this Appendix, we list several properties of the MID. A more detailed list together with the proofs can be found in [7].

The MID and the conjugate one are related by

\[ R_{n,m}^{(z)}(\bar{u}|\bar{v}) = (1 - z)^{m-n}K_{m,n}^{(z)}(\bar{v}|\bar{u}). \]  

(A.1)

The MID has the following property under the shift of one of the parameter sets:

\[ K_{n,m}^{(z)}(\bar{u} - c|\bar{v}) = \frac{(-z)^{n}(1 - z)^{m-n}}{f(\bar{v}, \bar{u})}K_{m,n}^{(1/z)}(\bar{v}|\bar{u}), \]  

(A.2)

\[ R_{n,m}^{(z)}(\bar{u} + c|\bar{v}) = \frac{(-z)^{n}(1 - z)^{m-n}}{f(\bar{v}, \bar{u})}R_{m,n}^{(1/z)}(\bar{v}|\bar{u}). \]  

(A.3)

Some bilinear combinations of the MIDs reduce to a new MID.

**Proposition A.1.** Let \( \xi, u, \) and \( v \) be sets of arbitrary complex numbers such that \#\( \xi \) = \( l \), \#\( u \) = \( n \) and \#\( v \) = \( m \). Then

\[ \sum_{\{\xi, \xi_i\} \vdash \xi} z^{l_1}_{1} K_{n,l_1}^{(z_1)}(u|\xi_1)K_{m,l_1}^{(z_2)}(v|\xi_{1i})f(\xi_1, \xi_1)f(u, \xi_{1i}) = K_{n+m,l}^{(z_1 z_2)}(\{u, v\} | \xi). \]  

(A.4)

Here, \( l_1 = \#\xi_1 \) and \( l_1 = \#\xi_{1i} \). The sum is taken over all partitions \( \{\xi_1, \xi_{1i}\} \vdash \xi \). There are no restrictions on the cardinalities of the subsets.

Replacing \( K_{n,l_1}^{(z_1)}(u|\xi_1) \) in (A.4) with the conjugate MID via (A.1) and (A.2), we obtain

\[ \sum_{\{\xi, \xi_i\} \vdash \xi} \left( \frac{z_{22}}{z_{11}} \right)^{l_1} R_{n,l_1}^{(z_1)}(u|\xi_1)K_{m,l_1}^{(z_2)}(v|\xi_{1i})f(\xi_1, \xi_1)f(u, \xi_{1i}) = f(\xi, u)K_{n+m,l}^{(z_1 z_2)}(\{u - c, v\} | \xi). \]
Setting $z_1 = z_2 = 1$ here yields

$$
\sum_{\{\xi, \xi_i\}} (-1)^h \left( \frac{d^h}{d\bar{\xi}^h} \right) R_{n+i}^{(1)}(\bar{u}, \bar{\xi}) K^{(1)}_{m+j} \left( \bar{v}, \xi_i \right) f(\xi_i, \bar{\xi}) F_{n+m,i}(\bar{u} - c, \bar{v}) \bar{\xi}.
$$

(A.5)

A.2. Other determinants related to the MID.

Proposition A.2. Let $\bar{u} = \{u_1, \ldots, u_N\}$, $\bar{\eta} = \{\eta_1, \ldots, \eta_N\}$, and $z$ be arbitrary complex numbers. Let $F_k^{(1)}(u)$ and $F_k^{(2)}(u)$, $k = 1, \ldots, N$, be two sets of functions

$$
F_k^{(1)}(u) = \phi_1(u) \left( \frac{z}{g(\bar{\eta}_k, u)} - h(\bar{\eta}_k, u) \right) + \phi_2(u) \left( \frac{1}{g(u, \bar{\eta}_k)} - zh(u, \bar{\eta}_k) \right),
$$

$$
F_k^{(2)}(u) = \left( -1 \right)^{N-1} \phi_1(u) \left( \frac{1}{g(u, \bar{\eta}_k)} - \phi_2(u) h(u, \bar{\eta}_k),
$$

(A.6)

where $\phi_\ell(z)$ ($\ell = 1, 2$) are two arbitrary functions. We compose two $N \times N$ matrices $\hat{F}^{(1)}$ and $\hat{F}^{(2)}$ with the entries

$$
\hat{F}_{jk}^{(1)} = F_k^{(1)}(u_j), \quad \hat{F}_{jk}^{(2)} = F_k^{(2)}(u_j).
$$

(A.7)

Then

$$
\det \hat{F}^{(1)} = (z - 1)^N \det \hat{F}^{(2)}.
$$

(A.8)

Proof. Obviously, both determinants can be presented in the form

$$
\det \hat{F}^{(\ell)} = \sum_{\{\xi, \xi_i\}} X^{(\ell)}(\bar{u}, \bar{\eta}_i) \phi_1(\bar{u}) \phi_2(\bar{\eta}_i), \quad \ell = 1, 2,
$$

(A.9)

where the coefficients $X^{(\ell)}(\bar{u}, \bar{\eta}_i)$ are independent of $\phi_1$ and $\phi_2$. In this equation, we used the shorthand notation for the products of the functions $\phi_\ell(u)$, $\ell = 1, 2$. Because $\phi_\ell$ are arbitrary functions, Eq. (A.8) holds if and only if

$$
X^{(1)}(\bar{u}, \bar{\eta}_i) = (z - 1)^N X^{(2)}(\bar{u}, \bar{\eta}_i)
$$

(A.10)

for an arbitrary partition $\{\bar{\eta}_i, \bar{\eta}_i\} \vdash \bar{u}$. However, it is easy to see that without a loss of generality, it suffices to prove (A.10) for $\bar{u}_i = \{u_1, \ldots, u_p\}$ and $\bar{\eta}_i = \{u_{p+1}, \ldots, u_N\}$. Here, $p$ is an arbitrary integer from the set $\{0, 1, \ldots, N\}$.

To obtain the coefficients $X^{(\ell)}(\bar{u}, \bar{\eta}_i)$, we should set $\phi_2(u_j) = 0$ for $j = 1, \ldots, p$ and $\phi_1(u_j) = 0$ for $j = p + 1, \ldots, N$ in (A.6) and (A.6). We obtain

$$
X^{(\ell)}(\bar{\eta}_i, \bar{\eta}_i) = \det \Phi^{(\ell)}_N, \quad \ell = 1, 2,
$$

(A.11)

where

$$
\Phi^{(1)}_{jk} = \left( \frac{z}{g(\bar{\eta}_j, \bar{u}_j)} - h(\bar{\eta}_j, \bar{u}_j) \right), \quad j = 1, \ldots, p,
$$

$$
\Phi^{(1)}_{jk} = \left( \frac{1}{g(u_j, \bar{\eta}_k)} - zh(u_j, \bar{\eta}_k) \right), \quad j = p + 1, \ldots, N,
$$

(A.12)

and

$$
\Phi^{(2)}_{jk} = \left( -1 \right)^{N-1} \frac{1}{g(u_j, \bar{\eta}_k)} \left( -1 \right)^{N-1} \frac{1}{g(u_j, \bar{\eta}_k)} - \phi_2(u_j, \bar{\eta}_k), \quad j = p + 1, \ldots, N.
$$

(A.13)
Let $u_j = u'_j + c$ for $j = 1, \ldots, p$, and $u_j = u'_j$ for $j = p + 1, \ldots, N$. Using $1/g(u_j, \bar{\eta}_k) = h(u'_j, \bar{\eta}_k)$ and $h(\bar{\eta}_k, u_j) = (-1)^{N-1} / g(u'_j, \bar{\eta}_k)$ (see (2.11)) for $k = 1, \ldots, p$, we obtain

$$X^{(\ell)}(\bar{u}_1, \bar{u}_n) = (-1)^{(\ell+1)(N-1)} \det_{N-\ell} \tilde{\Phi}(\ell), \quad \ell = 1, 2,$$

(A.14)

where

$$\tilde{\Phi}_{jk}(1) = \frac{1}{g(u'_j, \bar{\eta}_k)} - z h(u'_j, \bar{\eta}_k), \quad \tilde{\Phi}_{jk}^{(2)} = h(u'_j, \bar{\eta}_k), \quad j, k = 1, \ldots, N.$$  

(A.15)

It is easy to see that $\det N \tilde{\Phi}^{(2)}$ reduces to the Cauchy determinant:

$$\det_{N} \tilde{\Phi}^{(2)} = h(\bar{u}', \bar{\eta}) \det_{N} \left( \frac{1}{h(u'_j, \bar{\eta}_k)} \right) = \frac{1}{\Delta(\bar{\eta}) \Delta'(\bar{u}')}.$$  

(A.16)

The determinant of $\tilde{\Phi}^{(1)}$ is computed in Corollary 4.1. Due to (4.12), we have

$$\det_{N} \tilde{\Phi}^{(1)} = \frac{(1-z)^N}{\Delta(\bar{\eta}) \Delta'(\bar{u}')}.$$  

(A.17)

Comparing this equation with (A.16) and using (A.14), we see that

$$X^{(1)}(\bar{u}_1, \bar{u}_n) = (z - 1)^N X^{(2)}(\bar{u}_1, \bar{u}_n).$$  

(A.18)

The proposition is proved.

**Corollary A.1.** Let $0 \leq n \leq N$, and let $\bar{u} = \{u_1, \ldots, u_N\}$, $\bar{\eta} = \{\eta_{n+1}, \ldots, \eta_N\}$, and $z$ be arbitrary complex numbers ($\bar{\eta} = \emptyset$ for $n = N$). We compose two $N \times N$ matrices $\tilde{F}^{(01)}$ and $\tilde{F}^{(02)}$ with the entries

$$\tilde{F}_{jk}^{(01)} = \begin{cases} F_k^{(0)}(u_j), & k = 1, \ldots, n, \\ F_k^{(1)}(u_j), & k = n + 1, \ldots, N, \end{cases} \quad \tilde{F}_{jk}^{(02)} = \begin{cases} F_k^{(0)}(u_j), & k = 1, \ldots, n, \\ F_k^{(2)}(u_j), & k = n + 1, \ldots, N, \end{cases}$$

where $F_k^{(1)}(u)$ and $F_k^{(2)}(u)$ are given by (A.6), while $F_k^{(0)}(u), k = 1, \ldots, n$, are arbitrary functions. Then

$$\det_{N} \tilde{F}^{(01)} = (z - 1)^{N-n} \det_{N} \tilde{F}^{(02)}.$$  

(A.19)

**Proof.** Developing the determinant $\det \tilde{F}^{(01)}$ with respect to the first $n$ columns, we obtain

$$\det_{N} \tilde{F}^{(01)} = \sum_{\{u_{i1}, u_{1i}\} \vdash \bar{u}} (-1)^{P_{n1}} \det_{N-n} (F_k^{(0)}(u_j)) \det_{N-n} (F_k^{(1)}(u_j)_{\bar{u}}),$$

(A.20)

where the sum is taken over partitions $\{u_{i1}, u_{1i}\} \vdash \bar{u}$ such that $\# u_i = n$ and $\# \bar{u}_i = N - n$. The notation $u'_j$ (resp. $u''_j$) means the $j$th element of the subset $\bar{u}_i$ (resp. $\bar{u}_i$). Finally, $P_{n1}$ is the parity of the partition $\{u_{i1}, u_{1i}\} \vdash \bar{u}$.

Due to Proposition A.2,

$$\det_{N-n} (F_k^{(1)}(u_j)) = (z - 1)^{N-n} \det_{N-n} (F_k^{(2)}(u_j))$$

(A.21)

for an arbitrary subset $\bar{u}_i$. Hence,

$$\det_{N} \tilde{F}^{(01)} = (z - 1)^{N-n} \sum_{\{u_{i1}, u_{1i}\} \vdash \bar{u}} (-1)^{P_{n1}} \det_{N-n} (F_k^{(0)}(u_j)) \det_{N-n} (F_k^{(2)}(u_j)).$$

(A.22)

Taking the sum over partitions, we arrive at (A.19).
In particular, setting

\[ \phi_1(u) = (-1)^N \tilde{\lambda}_1(u) h(\bar{v}, u), \quad \phi_2(u) = \tilde{\lambda}_2(u) h(u, \bar{v}), \quad \text{(A.23)} \]

we obtain

\[ F_k^{(1)}(u_j) = N_{jk}^{(2)}, \quad k = n + 1, \ldots, N, \quad \text{(A.24)} \]

where \( N_{jk}^{(2)} \) is given by \((5.16)\). By Corollary A.1, we obtain

\[ \det_F^{(01)} = (z - 1)^{N-n} \det_F^{(02)}, \quad \text{(A.25)} \]

where

\[ \begin{aligned}
F_{jk}^{(01)} &= \begin{cases} F_k^{(0)}(u_j), & k = 1, \ldots, n, \\
N_{jk}^{(2)}, & k = n + 1, \ldots, N, \end{cases} \\
F_{jk}^{(02)} &= \begin{cases} F_k^{(0)}(u_j), & k = 1, \ldots, n, \\
(-1)^{n+1} \tilde{\lambda}_1(u_j) \frac{h(\bar{v}, u_j)}{g(u_j, \eta_k)} - \tilde{\lambda}_2(u_j) h(u, \bar{v}) h(u, \eta_k), & k = n + 1, \ldots, N, \end{cases}
\end{aligned} \quad \text{(A.26)} \]

and \( F_k^{(0)}(u), k = 1, \ldots, n, \) are arbitrary functions.

**A.3. Summation formula.** Let a function \( H(\bar{u}) \) of \( N \) variables \( \bar{u} = \{u_1, \ldots, u_N\} \) be defined as

\[ H(\bar{u}) = \Delta(\bar{u}) \det_F \Phi_k(u_j), \quad \text{(A.27)} \]

where \( \Phi_k(u) \) is a set of functions of one variable.

**Proposition A.3.** Let \( \phi_1(u) \) and \( \phi_2(u) \) be one-variable functions. Then

\[ \sum_{\{\bar{u}_i, \bar{u}_{ii}\}=\bar{u}} \phi_1(\bar{u}_i)\phi_2(\bar{u}_{ii})f(\bar{u}_{ii}, \bar{u}_i)H(\{\bar{u}_i - c, \bar{u}_{ii}\}) = \Delta(\bar{u}) \det_N (\phi_1(u_j)\Phi_k(u_j - c) + \phi_2(u_j)\Phi_k(u_j)). \quad \text{(A.28)} \]

The sum is taken over all possible partitions \( \{\bar{u}_i, \bar{u}_{ii}\} \vdash \bar{u} \), and we use the shorthand notation for the products of the functions \( \phi_\ell(u), \ell = 1, 2 \).

**Proof** is similar to the one of Proposition A.2. The right-hand side of \((A.28)\) can be presented as

\[ \Delta(\bar{u}) \det_N (\phi_1(u_j)\Phi_k(u_j - c) + \phi_2(u_j)\Phi_k(u_j)) = \sum_{\{\bar{u}_i, \bar{u}_{ii}\}=\bar{u}} X(\bar{u}_i, \bar{u}_{ii})\phi_1(\bar{u}_i)\phi_2(\bar{u}_{ii}), \]

where the coefficients \( X(\bar{u}_i, \bar{u}_{ii}) \) are independent of \( \phi_1 \) and \( \phi_2 \). Because \( \phi_\ell \) are arbitrary functions, Eq. \((A.28)\) holds if and only if

\[ X(\bar{u}_i, \bar{u}_{ii}) = f(\bar{u}_{ii}, \bar{u}_i)H(\{\bar{u}_i - c, \bar{u}_{ii}\}) \quad \text{(A.29)} \]

for an arbitrary partition \( \{\bar{u}_i, \bar{u}_{ii}\} \vdash \bar{u} \). Due to the symmetry of \((A.28)\) over \( \bar{u} \), it suffices to prove \((A.29)\) for \( \bar{u}_i = \{u_1, \ldots, u_p\} \) and \( \bar{u}_{ii} = \{u_{p+1}, \ldots, u_N\} \), where \( p \) is an arbitrary integer from the set \( \{0, 1, \ldots, N\} \).

To obtain the coefficients \( X(\bar{u}_i, \bar{u}_{ii}) \), we should set \( \phi_2(u_j) = 0 \) for \( j = 1, \ldots, p \) and \( \phi_1(u_j) = 0 \) for \( j = p + 1, \ldots, N \) in \((A.28)\). We obtain

\[ X(\bar{u}_i, \bar{u}_{ii}) = \Delta(\bar{u}_i)\Delta(\bar{u}_{ii})g(\bar{u}_{ii}, \bar{u}_i) \det F, \quad \text{(A.30)} \]
where
\[ \Phi_{jk} = \Phi_k(u_j - c), \quad j = 1, \ldots, p, \]
\[ \Phi_{jk} = \Phi_k(u_j), \quad j = p + 1, \ldots, N. \tag{A.31} \]

Let \( u_j = u_j' + c \) for \( j = 1, \ldots, p \), and \( u_j = u_j' \) for \( j = p + 1, \ldots, N \). Using \( g(x, y + c) = -1/h(y, x) \) for any \( x \) and \( y \), we obtain
\[ \Delta(\bar{u}_i)\Delta(\bar{u}_{ii})g(\bar{u}_{ii}, \bar{u}_i) = (-1)^{p(N-p)} \frac{\Delta(u'_i)\Delta(u''_i)}{h(u'_i, u''_i)} = \frac{\Delta(u'')}{h(u'_i, u''_i)}. \tag{A.32} \]

Then (A.30) takes the form
\[ X(\bar{u}, \bar{u}_{ii}) = \frac{\Delta(u'')}{h(u'_i, u''_i)} \det \Phi_k(u_j') = \frac{H(u')}{f(u'_i, u''_i)}. \tag{A.33} \]

Returning to the original variables and using \( f(x - c, y) = 1/f(y, x) \) for any \( x \) and \( y \), we arrive at (A.29).

We deal with a particular case of the sum in (A.28) in Eq. (5.8). Indeed, let
\[ \phi_1(u) = \lambda_1(u)f(\bar{v}, u), \quad \phi_2(u) = \frac{\rho_2}{\rho_1}\lambda_2(u), \quad \Phi_k(u) = \frac{f(u, \bar{v})}{g(u, \bar{\eta}_k)} - zh(u, \bar{\eta}_k), \]
where \( \bar{\eta} = \{\eta_1, \ldots, \eta_N\} \) are arbitrary complex numbers. Then
\[ \Delta'(\bar{\eta})H(\bar{u}) = (1 - z)^{N-n} K_{N,n}^{(z)}(\bar{u}|\bar{v}). \tag{A.34} \]

Hence, due to Proposition A.3, we have
\[ \sum_{(\bar{u}_i, \bar{u}_{ii})} \left( \frac{\rho_2}{\rho_1} \right)^{N_{ii}} \lambda_2(\bar{u}_{ii})\lambda_1(\bar{u}_i)f(\bar{v}, \bar{u}_i)f(\bar{v}, \bar{u}_i)K_{N,n}^{(1)}(\{\bar{u}_i - c, \bar{u}_{ii}\}) = \]
\[ = (1 - z)^{N-n} \Delta'(\bar{\eta})\Delta(\bar{u}) \det \frac{M_{jk}}{N}. \tag{A.35} \]

where
\[ M_{jk} = \lambda_1(u_j)f(\bar{v}, u_j) \left( \frac{f(u_j - c, \bar{v})}{g(u_j - c, \bar{\eta}_k)} - zh(u_j - c, \bar{\eta}_k) \right) + \]
\[ + \frac{\rho_2}{\rho_1} \lambda_2(u_j) \left( \frac{f(u_j, \bar{v})}{g(u_j, \bar{\eta}_k)} - zh(u_j, \bar{\eta}_k) \right). \tag{A.36} \]

Using Eqs. (2.11), we easily transform this result to representation (5.10).

Conflicts of interest. The authors declare no conflicts of interest.

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