THE CODEGREE TURÁN DENSITY OF TIGHT CYCLES
MINUS ONE EDGE

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Abstract. Given $\alpha > 0$ and an integer $\ell \geq 5$, we prove that every sufficiently large 3-uniform hypergraph $H$ on $n$ vertices in which every two vertices are contained in at least $\alpha n$ edges contains a copy of $C^{-}_\ell$, a tight cycle on $\ell$ vertices minus one edge. This improves a previous result by Balogh, Clemen, and Lidický.

§1. Introduction

A $k$-uniform hypergraph $H$ consists of a vertex set $V(H)$ together with a set of edges $E(H) \subseteq V(H)^{(k)} = \{S \subseteq V(H) : |S| = k\}$. Throughout this note, if not stated otherwise, by hypergraph we always mean a 3-uniform hypergraph. Given a hypergraph $F$, the extremal number of $F$ for $n$ vertices, $\text{ex}(n, F)$, is the maximum number of edges an $n$-vertex hypergraph can have without containing a copy of $F$. Determining the value of $\text{ex}(n, F)$, or the Turán density $\pi(F) = \lim_{n \to \infty} \frac{\text{ex}(n, F)}{|H(n, 3)|}$, is one of the core problems in combinatorics. In particular, the problem of determining the Turán density of the complete 3-uniform hypergraph on four vertices, i.e., $\pi(K_4^{(3)})$, was asked by Turán in 1941 [12] and Erdős [4] offered 1000$ for its resolution. Despite receiving a lot of attention (see for instance the survey by Keevash [7]) this problem, and even the seemingly simpler problem of determining $\pi(K_4^{(3)-})$, where $K_4^{(3)-}$ is the $K_4^{(3)}$ minus one edge, remain open.

Several variations of this type of problem have been considered, see for instance [1, 6, 11] and the references therein. The one that we are concerned with in this note asks how large the minimum codegree of an $F$-free hypergraph can be. Given a hypergraph $H$ and $S \subseteq V$ we define the degree $d(S)$ of $S$ (in $H$) as the number of edges containing $S$, i.e., $d(S) = |\{e \in E(H) : S \subseteq e\}|$. If $S = \{v\}$ or $S = \{u, v\}$ (and $H$ is 3-uniform), we omit the parentheses and speak of $d(v)$ or $d(uv)$ as the degree of $v$ or codegree of $u$ and $v$, respectively. We further

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write \( \delta(H) = \delta_1(H) = \min_{v \in V(H)} d(v) \) and \( \delta_2(H) = \min_{u \in V(H)} d(uv) \) for the minimum degree and the minimum codegree of \( H \), respectively.

Given a hypergraph \( F \) and \( n \in \mathbb{N} \), Mubayi and Zhao [10] introduced the codegree Turán number \( \text{ex}_2(n, F) \) of \( n \) and \( F \) as the maximum \( d \) such that there is an \( F \)-free hypergraph \( H \) on \( n \) vertices with \( \delta_2(H) \geq d \). Moreover, they defined the codegree Turán density of \( F \) as

\[
\gamma(F) := \lim_{n \to \infty} \frac{\text{ex}_2(n, F)}{n}
\]

and proved that this limit always exists. It is not hard to see that

\[
\gamma(F) \leq \pi(F).
\]

The codegree Turán density is known only for a few (non-trivial) hypergraphs (and blow-ups of these), see the table in [1]. The first result that determined \( \gamma(F) \) exactly is due to Mubayi [8] who showed that \( \gamma(\mathbb{F}) = 1/2 \), where \( \mathbb{F} \) denotes the ‘Fano plane’. Later, using a computer assisted proof, Falgas-Ravry, Pikhurko, Vaughan, and Volec [5] proved that \( \gamma(K_4^{(3)^-}) = 1/4 \). As far as we know, the only other hypergraph for which the codegree Turán density is known is \( F_{3,2} \), a hypergraph with vertex set \([5] \) and edges 123, 124, 125, and 345. The problem of determining the codegree Turán density of \( K_4^{(3)^-} \) remains open, and Czygrinow and Nagle [2] conjectured that \( \gamma(K_4^{(3)}) = 1/2 \). For more results concerning \( \pi(F), \gamma(F), \) and other variations of the Turán density see [1].

Given an integer \( \ell \geq 3 \), a tight cycle \( C_\ell \) is a hypergraph with vertex set \( \{v_1, \ldots, v_\ell\} \) and edge set \( \{v_i v_{i+1} v_{i+2} : i \in \mathbb{Z}/\ell\mathbb{Z}\} \). Moreover, we define \( C_\ell^- \) as \( C_\ell \) minus one edge. In this note we prove that the Turán codegree density of \( C_\ell^- \) is zero for every \( \ell \geq 5 \).

**Theorem 1.1.** Let \( \ell \geq 5 \) be an integer. Then \( \gamma(C_\ell^-) = 0 \).

The previously known best upper bound was given by Balogh, Clemen, and Lidický [1] who used flag algebras to prove that \( \gamma(C_\ell^-) \leq 0.136 \).

§2. Proof of Theorem 1.1

For singletons, pairs, and triples we may omit the set parentheses and commas. For a hypergraph \( H = (V, E) \) and \( v \in V \), the link of \( v \) (in \( H \)) is the graph \( L_v = (V \setminus v, \{e \setminus v : v \in e \in E\}) \). For \( x, y \in V \), the neighbourhood of \( x \) and \( y \) (in \( H \)) is the set \( N(xy) = \{z \in V : xyz \in E\} \). For positive integers \( \ell, k \) and a hypergraph \( F \) on \( k \) vertices, denote the \( \ell \)-blow-up of \( F \) by \( F(\ell) \). This is the
A $k$-partite hypergraph $F(\ell) = (V, E)$ with $V = V_1 \cup \ldots \cup V_k$, $|V_i| = \ell$ for $1 \leq i \leq k$, and $E = \{v_i, v_{i+1}, v_{i+2} : v_i \in V_i \text{ and } i \in \mathbb{Z}\}$. (Since $i + 2 \equiv i \pmod{k}$, the indices repeat every $k$ terms.)

In their seminal paper, Mubayi and Zhao [10] proved the following supersaturation result for the codegree Turán density.

**Proposition 2.1 (Mubayi and Zhao [10]).** For every hypergraph $F$ and $\varepsilon > 0$, there are $n_0$ and $\delta > 0$ such that every hypergraph $H$ on $n \geq n_0$ vertices with $\delta_2(H) \geq (\gamma(F) + \varepsilon)n$ contains at least $\delta n^\varepsilon$ copies of $F$. Consequently, for every positive integer $\ell$, $\gamma(F) = \gamma(F(\ell))$.

**Proof**. We begin by noting that it is enough to show that $\gamma(C_k^-) = 0$.

Indeed, we shall prove by induction that $\gamma(C_k^-) = 0$ for every $k \geq 5$. For $k = 6$, the result follows since $C_6^-$ is a subgraph of $C_3^2$. Hence, by Proposition 2.1, we have $\gamma(C_6^-) \leq \gamma(C_3^2) = \gamma(C_3) = 0$. For $k = 7$, note that $C_7^-$ is a subgraph of $C_3^2$. To see that, let $v_1, \ldots, v_5$ be the vertices of a $C_5^-$ with edge set $\{v_i, v_{i+1}, v_{i+2} : i \neq 3\}$, where the indices are taken modulo 5. Now add one copy $v'_2$ of $v_2$ and one copy $v'_3$ of $v_3$. Then $v_1v_2v_3v_4v_5v'_2$ is the cyclic ordering of a $C_7^-$ with the missing edge being $v'_2v_5v'_3$. Therefore, if $\gamma(C_5^-) = 0$, then, by Proposition 2.1, we have $\gamma(C_7^-) = 0$. Finally, for $k \geq 8$, $\gamma(C_k^-) = 0$ follows by induction using the same argument and observing that $C_k^-$ is a subgraph of $C_{k-3}^2$.

**Proof of Theorem 1.1.** Given $\varepsilon \in (0, 1)$, consider a hypergraph $H = (V, E)$ on $n \geq \left(\frac{2}{\varepsilon}\right)^{5/\varepsilon^2 + 2}$ vertices with $\delta_2(H) \geq \varepsilon n$. We claim that $H$ contains a copy of a $C_5^-$.

Given $v, b \in V$, $S \subseteq V$, and $P \subseteq (V \setminus S)^2$, we say that $(v, S, b, P)$ is a nice picture if it satisfies the following:

(i) $S \subseteq N_{L_v}(b)$, where $N_{L_v}(b)$ is the neighborhood of $b$ in the link $L_v$.

(ii) For every vertex $u \in S$ and ordered pair $(x, y) \in P$, the sequence $ubxy$ is a path of length 3 in $L_v$.

Note that if $(v, S, b, P)$ is a nice picture and there exists $u \in S$ and $(x, y) \in P$ such that $uxy \in E$, then $ubvx$ is a copy of $C_5^-$ (with the missing edge being $yub$).

To find such a copy of $C_5^-$ in $H$, we are going to construct a sequence of nested sets $S_t \subseteq S_{t-1} \subseteq \ldots \subseteq S_0$, where $t = 5/\varepsilon^2 + 1$, and nice pictures $(v_i, S_i, b_i, P_i)$ satisfying $v_i \in S_{i-1}$, $|S_i| \geq \left(\frac{\varepsilon}{2}\right)^{i+1}n \geq 1$ and $|P_i| \geq \varepsilon^2n^2/5$ for $1 \leq i \leq t$. Suppose that such a sequence exists. Then by the pigeonhole principle, there exist two indices $i, j \in [t]$ such that $P_i \cap P_j \neq \emptyset$ and $i < j$. Let $(x, y)$ be an element of $P_i \cap P_j$. Hence, we obtain a nice picture $(v_j, S_i, b_i, P_i)$, $v_j \in S_i$ and $(x, y) \in P_i$ such that $v_jxy \in E$ (since $xy$ is an edge in $L_{v_j}$). Consequently, $v_jb_i,v_3xy$ is a copy of $C_5^-$ in $H$. 

\[ \]
It remains to prove that the sequence described above always exists. We construct it recursively. Let $S_0 \subseteq V$ be an arbitrary subset of size $\varepsilon n/2$. Suppose that we already constructed nice pictures $(v_i, S_i, b_i, P_i)$ for $1 \leq i < k \leq t$ and now we want to construct $(v_k, S_k, b_k, P_k)$. Pick $v_k \in S_{k-1}$ arbitrarily. The minimum codegree of $H$ implies that $\delta(L_{v_k}) \geq \varepsilon n$ and thus for every $u \in S_{k-1}$, we have that $d_{L_{v_k}}(u) \geq \varepsilon n$. Observe that

$$\sum_{b \in V \setminus v_k} |N_{L_{v_k}}(b) \cap S_{k-1}| = \sum_{u \in S_{k-1} \setminus v_k} d_{L_{v_k}}(u) \geq \varepsilon n (|S_{k-1}| - 1) \geq \left(\frac{\varepsilon}{2}\right)^{k+1} n^2$$

and therefore, by an averaging argument there is a vertex $b_k \in V \setminus v_k$ such that the subset $S_k := N_{L_{v_k}}(b_k) \cap S_{k-1} \subseteq S_{k-1}$ is of size at least $|S_k| \geq \left(\frac{\varepsilon}{2}\right)^{k+1} n$. Let $P_k$ be all the pairs $(x, y) \in (V \setminus S_k)^2$ such that for every vertex $v \in S_k$, the sequence $v, b_k, x, y$ forms a path of length 3 in $L_{v_k}$. Since $|S_k| \leq \varepsilon n/2$ and $\delta(L_{v_k}) \geq \varepsilon n$, it is easy to see that $|P_k| \geq \varepsilon^2 n^2/5$. That is to say $(v_k, S_k, b_k, P_k)$ is a nice picture satisfying the desired conditions.

\[\square\]

§3. Concluding remarks

A famous result by Erdős [3] asserts that a hypergraph $F$ satisfies $\pi(F) = 0$ if $F$ is tripartite (i.e., $V(F) = X_1 \cup X_2 \cup X_3$ and for every $e \in E(F)$ we have $|e \cap X_i| = 1$ for every $i \in [3]$). Note that if $H$ is tripartite, then every subgraph of $H$ is tripartite as well and there are tripartite hypergraphs $H$ with $|E(H)| = \frac{2}{5}(|V(H)|^3)$. Therefore, if $F$ is not tripartite, then $\pi(F) \geq 2/9$. In other words, Erdős’ result implies that there are no Turán densities in the interval $(0, 2/9)$. It would be interesting to understand the behaviour of the codegree Turán density in the range close to zero.

**Question 3.1.** Is it true that for every $\xi \in (0, 1]$, there exists a hypergraph $F$ such that

$$0 < \gamma(F) \leq \xi$$
Mubayi and Zhao \cite{mubayi2007} answered this question affirmatively if we consider the codegree Turán density of a family of hypergraphs instead of a single hypergraph.

Since $C_5^-$ is not tripartite, we have that $\pi(C_5^-) \geq 2/9$. The following construction attributed to Mubayi and Rödl (see e.g. \cite{balogh2021}) provides a better lower bound. Let $H = (V, E)$ be a $C_5^-$-free hypergraph on $n$ vertices. Define a hypergraph $\tilde{H}$ on $3n$ vertices with $V(\tilde{H}) = V_1 \cup V_2 \cup V_3$ such that $\tilde{H}[V_i] = H$ for every $i \in [3]$ plus all edges of the form $e = \{v_1, v_2, v_3\}$ with $v_i \in V_i$. Then, it is easy to check that $\tilde{H}$ is also $C_5^-$-free. We may recursively repeat this construction starting with $H$ being a single edge and obtain an arbitrarily large $C_5^-$-free hypergraph with density $1/4 - o(1)$. In fact, those hypergraphs are $C_{\ell}^-$-free for every $\ell$ not divisible by three. The following is a generalisation of a conjecture in \cite{mubayi2011}.

**Conjecture 3.2.** If $\ell \geq 5$ is not divisible by three, then $\pi(C_{\ell}^-) = \frac{1}{4}$.

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