Expanding Lie (super)algebras through abelian semigroups

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Abstract

We propose an outgrowth of the expansion method introduced by de Azcárraga et al. [Nucl. Phys. B 662 (2003) 185]. The basic idea consists in considering the direct product between an abelian semigroup $S$ and a Lie algebra $\mathfrak{g}$. General conditions under which relevant subalgebras can systematically be extracted from $S \times \mathfrak{g}$ are given. We show how, for a particular choice of semigroup $S$, the known cases of expanded algebras can be reobtained, while new ones arise from different choices. Concrete examples, including the M algebra and a D’Auria–Fré-like Superalgebra, are considered. Finally, we find explicit, non-trace invariant tensors for these $S$-expanded algebras, which are essential ingredients in, e.g., the formulation of Supergravity theories in arbitrary space-time dimensions.

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I. INTRODUCTION

The rôle played by Lie algebras and their interrelations in physics can hardly be over-emphasized. To mention only one example, the Poincaré algebra may be obtained from the Galilei algebra via a deformation process. This deformation is one of the ways in which different Lie algebras can be related.

The purpose of this paper is to shed new light on the method of expansion of Lie algebras (for a thorough treatment, see the seminal work [1] and references therein; early work on the subject is found in [2]). An Expansion is, in general, an algebra dimension-changing process. For instance, the M algebra [3, 4, 5], with 583 Bosonic generators, can be regarded as an expansion of the orthosymplectic algebra \( \mathfrak{osp}(32|1) \), which possesses only 528. This vantage viewpoint may help better understand fundamental problems related to the geometrical formulation of 11-dimensional Supergravity. Some physical applications of the expansion procedure have been developed in [6, 7, 8, 9, 10, 11, 12, 13, 14].

The approach to be presented here is entirely based on operations performed directly on the algebra generators, and thus differs from the outset with the one found in [1], where the dual Maurer–Cartan formulation is used. As a consequence, the expansion of free differential algebras lies beyond the scope of our analysis.

Finite abelian semigroups play a prominent rôle in our construction. All expansion cases found in [1] may be regarded as coming from one particular choice of semigroup in the present approach, which is, in this sense, more general. Different semigroup choices yield, in general, expanded algebras that cannot be obtained by the methods of [1].

The plan of the paper goes as follows. After some preliminaries in section II, section III introduces the general procedure of abelian semigroup expansion, \( S \)-expansion for short, and shows how the cases found in [1] can be recovered by an appropriate choice of semigroup \( S \). In section IV general conditions are given under which relevant subalgebras can be extracted from an \( S \)-expanded algebra. The case when \( g \) satisfies the Weimar-Woods conditions [15, 16] and the case when \( g \) is a superalgebra are studied. Section V gives three explicit examples of \( S \)-expansions of \( \mathfrak{osp}(32|1) \): (i) the M algebra [3, 4, 5], (ii) a D’Auria–Fré-like Superalgebra [17] and (iii) a new Superalgebra, different from but resembling aspects
of the M algebra, $\mathfrak{osp}(32|1) \times \mathfrak{osp}(32|1)$ and D’Auria–Fré superalgebras. In section [VI], the remaining cases of expanded algebras shown in [1] are seen to also fit within the current scheme. The five-brane Superalgebra [18, 19] is given as an example. Section [VII] deals with the crucial problem of finding invariant tensors for the $S$-expanded algebras. General theorems are proven, allowing for nontrivial invariant tensors to be systematically constructed. We close in section [VIII] with conclusions and an outlook for future work.

II. PRELIMINARIES

Before analyzing the $S$-Expansion procedure itself, it will prove convenient to introduce some basic notation and definitions.

A. Semigroups

Definition II.1 Let $S = \{\lambda_\alpha\}$ be a finite semigroup [32], and let us write the product of $\lambda_{\alpha_1}, \ldots, \lambda_{\alpha_n} \in S$ as

$$\lambda_{\alpha_1} \cdots \lambda_{\alpha_n} = \lambda_{\gamma(\alpha_1, \ldots, \alpha_n)}. \quad (1)$$

The $n$-selector $K_{\alpha_1 \cdots \alpha_n}^\rho$ is defined as

$$K_{\alpha_1 \cdots \alpha_n}^\rho = \begin{cases} 1, & \text{when } \rho = \gamma(\alpha_1, \ldots, \alpha_n) \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Since $S$ is associative, the $n$-selector fulfills the identity

$$K_{\alpha_1 \cdots \alpha_n}^\rho = K_{\alpha_1 \cdots \alpha_{n-1}}^\sigma K_{\alpha_n}^\rho = K_{\alpha_1}^\sigma K_{\alpha_2 \cdots \alpha_n}^\sigma. \quad (3)$$

Using this identity it is always possible to express the $n$-selector in terms of 2-selectors, which encode the information from the multiplication table of $S$.

An interesting way to state the same is that 2-selectors provide a matrix representation for $S$. As a matter of fact, when we write

$$[\lambda_\alpha]_{\mu}^\nu = K_{\mu \alpha}^\nu \quad (4)$$

then we have

$$[\lambda_\alpha]_{\mu}^\sigma [\lambda_\beta]_{\sigma}^\nu = K_{\alpha \beta}^\sigma [\lambda_\sigma]_{\mu}^\nu = [\lambda_{\gamma(\alpha, \beta)}]_{\mu}^\nu. \quad (5)$$
We will restrict ourselves from now on to abelian semigroups, which implies that the $n$-selectors will be completely symmetrical in their lower indices.

The following definition introduces a product between semigroup subsets which will be extensively used throughout the paper.

**Definition II.2** Let $S_p$ and $S_q$ be two subsets of $S$. The product $S_p \cdot S_q$ is defined as

$$S_p \cdot S_q = \{ \lambda_\gamma \text{ such that } \lambda_\gamma = \lambda_\alpha_p \lambda_\alpha_q, \text{ with } \lambda_\alpha_p \in S_p \text{ and } \lambda_\alpha_q \in S_q \} \subset S. \quad (6)$$

In other words, $S_p \cdot S_q \subset S$ is the set which results from the product of every element of $S_p$ with every element of $S_q$. Since $S$ is abelian, $S_p \cdot S_q = S_q \cdot S_p$.

Let us emphasize that, in general, $S_p, S_q$ and $S_p \cdot S_q$ need not be semigroups by themselves. The abelian semigroup $S$ could also be provided with a unique zero element, $0_S$. This element is defined as the one for which

$$0_S \lambda_\alpha = \lambda_\alpha 0_S = 0_S, \quad (7)$$

for each $\lambda_\alpha \in S$.

**B. Reduced Lie Algebras**

The following definition introduces the concept of reduction of Lie algebras.

**Definition II.3** Consider a Lie (super)algebra $\mathfrak{g}$ of the form $\mathfrak{g} = V_0 \oplus V_1$, with $\{T_a\}_0$ being a basis for $V_0$ and $\{T_a\}_1$ a basis for $V_1$. When $[V_0, V_1] \subset V_1$, i.e., when the commutation relations have the general form

$$[T_{a_0}, T_{b_0}] = C_{a_0 b_0}^{c_0} T_{c_0} + C_{a_0 b_0}^{c_1} T_{c_1}, \quad (8)$$

$$[T_{a_0}, T_{b_1}] = C_{a_0 b_1}^{c_1} T_{c_1}, \quad (9)$$

$$[T_{a_1}, T_{b_1}] = C_{a_1 b_1}^{c_0} T_{c_0} + C_{a_1 b_1}^{c_1} T_{c_1}, \quad (10)$$

then it is straightforward to show that the structure constants $C_{a_0 b_0}^{c_0}$ satisfy the Jacobi identity by themselves, and therefore $[T_{a_0}, T_{b_0}] = C_{a_0 b_0}^{c_0} T_{c_0}$ corresponds by itself to a Lie (super)algebra. This algebra, with structure constants $C_{a_0 b_0}^{c_0}$, is called a reduced algebra of $\mathfrak{g}$ and symbolized as $|V_0|$. 
The reduced algebra could be regarded in some way as the “inverse” of an algebra extension, but $V_1$ does not need to be an ideal. Note also that a reduced algebra does not need to be an ideal. Note also that a reduced algebra does not in general correspond to a subalgebra.

III. THE S-EXPANSION PROCEDURE

A. S-Expansion for an Arbitrary Semigroup $S$

The following theorem embodies one of the main results of the paper, the concept of $S$-expanded algebras.

Theorem III.1 Let $S = \{\lambda_\alpha\}$ be an abelian semigroup with 2-selector $K_{\alpha\beta} \gamma$ and $\mathfrak{g}$ a Lie (super)algebra with basis $\{T_A\}$ and structure constants $C_A B C$. Denote a basis element of the direct product $S \times \mathfrak{g}$ by $T_{(A,\alpha)} = \lambda_\alpha T_A$ and consider the induced commutator $[T_{(A,\alpha)}, T_{(B,\beta)}] \equiv \lambda_\alpha \lambda_\beta [T_A, T_B]$. Then, $S \times \mathfrak{g}$ is also a Lie (super)algebra with structure constants

$$C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta} \gamma C_A B C.$$

(11)

Proof. Starting from the form of the induced commutator and using the multiplication law (1) one finds

$$[T_{(A,\alpha)}, T_{(B,\beta)}] = \lambda_\alpha \lambda_\beta [T_A, T_B] = C_A B C \lambda_{(\alpha,\beta)} T_C = C_A B C T_{(C,\gamma(\alpha,\beta))}.$$

The definition of the 2-selector $K_{\alpha\beta} \rho$ [cf. eq. (2)],

$$K_{\alpha\beta} \rho = \begin{cases} 1, & \text{when } \rho = \gamma (\alpha, \beta) \\ 0, & \text{otherwise,} \end{cases}$$

now allows us to write

$$[T_{(A,\alpha)}, T_{(B,\beta)}] = K_{\alpha\beta} \rho C_A B C T_{(C,\rho)}.$$ 

(12)

Therefore, the algebra spanned by $\{T_{(A,\alpha)}\}$ closes and the structure constants read

$$C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = K_{\alpha\beta} \gamma C_A B C.$$

(13)
Since $S$ is abelian, the structure constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)}$ have the same symmetries as $C_{AB}^C$, namely
\begin{equation}
C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = -(-1)^{q(A)q(B)}C_{(B,\beta)(A,\alpha)}^{(C,\gamma)},
\end{equation}
and for this reason, $q(A,\alpha) = q(A)$, where $q(A)$ denotes the degree of $T_A$ (1 for Fermi and 0 for Bose).

In order to show that the structure constants $C_{(A,\alpha)(B,\beta)}^{(C,\gamma)}$ satisfy the Jacobi identity, it suffices to use the properties of the selectors [cf. eq. (3)] and the fact that the structure constants $C_{AB}^C$ satisfy the Jacobi identity themselves. This concludes the proof.

The following definition is a natural outcome of Theorem III.1.

**Definition III.2** Let $S$ be an abelian semigroup and $\mathfrak{g}$ a Lie algebra. The Lie algebra $\mathfrak{S}$ defined by $\mathfrak{S} = S \times \mathfrak{g}$ is called $S$-Expanded algebra of $\mathfrak{g}$.

When the semigroup has a zero element $0_S \in S$, it plays a somewhat peculiar rôle in the $S$-expanded algebra. Let us span $S$ in nonzero elements $\lambda_i$, $i = 0, \ldots, N$ and a zero element $\lambda_{N+1} = 0_S$. Then, the 2-selector satisfies
\begin{align}
K_{i,N+1}^j &= K_{N+1,i}^j = 0, \\
K_{i,N+1}^{N+1} &= K_{N+1,i}^{N+1} = 1, \\
K_{N+1,N+1}^j &= 0, \\
K_{N+1,N+1}^{N+1} &= 1.
\end{align}

Therefore, $\mathfrak{S} = S \times \mathfrak{g}$ can be split as
\begin{align}
\left[ T_{(A,i)}, T_{(B,j)} \right] &= K_{ij}^k C_{AB}^C T_{(C,k)} + K_{ij}^{N+1} C_{AB}^C T_{(C,N+1)}, \\
\left[ T_{(A,N+1)}, T_{(B,j)} \right] &= C_{AB}^C T_{(C,N+1)}, \\
\left[ T_{(A,N+1)}, T_{(B,N+1)} \right] &= C_{AB}^C T_{(C,N+1)}.
\end{align}

Comparing (19)–(21) with (8)–(10), one sees that the commutation relations
\begin{equation}
\left[ T_{(A,i)}, T_{(B,j)} \right] = K_{ij}^k C_{AB}^C T_{(C,k)}
\end{equation}
are those of a reduced Lie algebra of $\mathfrak{S}$ (see Def. III.3). The reduction procedure in this particular case is equivalent to imposing the condition
\begin{equation}
T_{(A,N+1)} = 0_S T_A = 0.
\end{equation}
Notice that in this case the reduction abelianizes large sectors of the algebra; for each \(i\) and \(j\) satisfying \(K_{ij}N+1 = 1\) (i.e., \(\lambda_i\lambda_j = \lambda_{N+1}\)) we have \([T_{(A,i)}, T_{(B,j)}] = 0\).

The above considerations motivate the following definition:

**Definition III.3** Let \(S\) be an abelian semigroup with a zero element \(0_S \in S\), and let \(\mathfrak{g} = S \times \mathfrak{g}\) be an \(S\)-expanded algebra. The algebra obtained by imposing the condition \(0_ST_A = 0\) on \(\mathfrak{g}\) (or a subalgebra of it) is called \(0_S\)-reduced algebra of \(\mathfrak{g}\) (or of the subalgebra).

The algebra \((22)\) appears naturally when the semigroup’s zero matches the (algebra) field’s zero. As we will see in the next section, this is the way Maurer–Cartan forms power-series expanded algebras fit within the present scheme. It is also possible to extract other reduced algebras from \(\mathfrak{g}\); as will be analyzed in Sec. VII the \(0_S\)-reduced algebra turns out to be a particular case of Theorem VII.1.

**B. Maurer–Cartan Forms Power Series Algebra Expansion as an \(S\)-Expansion**

The Maurer–Cartan forms power series algebra expansion method is a powerful procedure which can lead, in stark contrast with contraction, deformation and extension of algebras, to algebras of a dimension higher than the original one. In a nutshell, the idea consists of looking at the algebra \(\mathfrak{g}\) as described by the associated Maurer–Cartan forms on the group manifold and, after rescaling some of the group parameters by a factor \(\lambda\), in expanding the Maurer–Cartan forms as a power series in \(\lambda\). Finally this series is truncated in a way that assures the closure of the algebra. The subject is thoroughly treated by de Azcárraga and Izquierdo in Ref. [20] and de Azcárraga, Izquierdo, Picón and Varela in Ref. [1].

Theorem 1 of Ref. [1] shows that, in the more general case, the expanded Lie algebra has the structure constants

\[
C_{(A,i)(B,j)}^{(C,k)} = \begin{cases} 
0, & \text{when } i + j \neq k \\
C_{AB}^{C}, & \text{when } i + j = k 
\end{cases}
\]

where the parameters \(i, j, k = 0, \ldots, N\) correspond to the order of the expansion, and \(N\) is the truncation order.

These structure constants can also be obtained within the \(S\)-expansion procedure. In order to show this, one must consider the \(0_S\)-reduction of an \(S\)-expanded algebra where \(S\) corresponds to the semigroup defined below.
Definition III.4 Let us define $S_{E}^{(N)}$ as the semigroup of elements

$$S_{E}^{(N)} = \{ \lambda_{\alpha}, \alpha = 0, \ldots, N+1 \},$$

(25)

provided with the multiplication rule

$$\lambda_{\alpha} \lambda_{\beta} = \lambda_{H_{N+1}(\alpha+\beta)},$$

(26)

where $H_{N+1}$ is defined as the function

$$H_{n}(x) = \begin{cases} x, & \text{when } x < n \\ n, & \text{when } x \geq n \end{cases}.$$  

(27)

The 2-selectors for $S_{E}^{(N)}$ read

$$K_{\alpha \beta}^{\gamma} = \delta_{H_{N+1}(\alpha+\beta)},$$

(28)

where $\delta_{\rho}^{\sigma}$ is the Kronecker delta. From eq. (26), we have that $\lambda_{N+1}$ is the zero element in $S_{E}^{(N)}$, i.e., $\lambda_{N+1} = 0_S$.

Using eq. (28), the structure constants for the $S_{E}^{(N)}$-expanded algebra can be written as

$$C_{(A,\alpha)(B,\beta)}^{(C,\gamma)} = \delta_{H_{N+1}(\alpha+\beta)}^{\gamma} C_{AB}^C,$$

(29)

with $\alpha, \beta, \gamma = 0, \ldots, N+1$. When the extra condition $\lambda_{N+1} T_A = 0$ is imposed, eq. (29) reduces to

$$C_{(A,i)(B,j)}^{(C,k)} = \delta_{i+j}^{k} C_{AB}^C,$$

(30)

which exactly matches the structure constants (24).

The above arguments show that the Maurer–Cartan forms power series expansion of an algebra $\mathfrak{g}$, with truncation order $N$, coincides with the $0_S$-reduction of the $S_{E}^{(N)}$-expanded algebra $\mathfrak{g}^{(E)} = S_{E}^{(N)} \times \mathfrak{g}$.

This is of course no coincidence. The set of powers of the rescaling parameter $\lambda$, together with the truncation at order $N$, satisfy precisely the multiplication law of $S_{E}^{(N)}$. As a matter of fact, we have

$$\lambda^{\alpha} \lambda^{\beta} = \lambda^{\alpha+\beta},$$

(31)

and the truncation can be imposed as

$$\lambda^{\alpha} = 0 \text{ when } \alpha > N.$$

(32)
It is for this reason that one must demand $0_{S}T_{A} = 0$ in order to obtain the MC expansion as an $S_{E}$-expansion: in this case the zero of the semigroup is the zero of the field as well.

The $S$-expansion procedure is valid no matter what the structure of the original Lie algebra $\mathfrak{g}$ is, and in this sense it is very general. However, when something about the structure of $\mathfrak{g}$ is known, a lot more can be done. As an example, in the context of MC expansion, the rescaling and truncation can be performed in several ways depending on the structure of $\mathfrak{g}$, leading to several kinds of expanded algebras. Important examples of this are the generalized İnönü–Wigner contraction, or the M algebra as an expansion of $\mathfrak{osp}(32|1)$ (see Refs. [1, 21]). This is also the case in the context of $S$-expansions. As we will show in the next section, when some information about the structure of $\mathfrak{g}$ is available, it is possible to find subalgebras of $\mathfrak{S} = S \times \mathfrak{g}$ and other kinds of reduced algebras. In this way, all the algebras obtained by the MC expansion procedure can be reobtained. New kinds of $S$-expanded algebras can also be obtained by considering semigroups different from $S_{E}$.

### IV. $S$-EXPANSION SUBALGEBRAS

An $S$-expanded algebra has a fairly simple structure. In a way, it reproduces the original algebra $\mathfrak{g}$ in a series of “levels” corresponding to the semigroup elements. Interestingly enough, there are at least two ways of extracting smaller algebras from $S \times \mathfrak{g}$. The first one, described in this section, gives rise to a “resonant subalgebra,” while the second, described in section IV, produces reduced algebras (in the sense of Def. II.3).

#### A. Resonant Subalgebras for an Arbitrary Semigroup $S$

The general problem of finding subalgebras from an $S$-expanded algebra is a nontrivial one, which is met and solved (in a particular setting) in this section (see theorem IV.2 below). In order to provide a solution, one must have some information on the subspace structure of $\mathfrak{g}$. This information is encoded in the following way.

Let $\mathfrak{g} = \bigoplus_{p \in I} V_{p}$ be a decomposition of $\mathfrak{g}$ in subspaces $V_{p}$, where $I$ is a set of indices. For each $p, q \in I$ it is always possible to define $i_{(p,q)} \subset I$ such that

$$[V_{p}, V_{q}] \subset \bigoplus_{r \in i_{(p,q)}} V_{r}. \tag{33}$$
In this way, the subsets $i_{(p,q)}$ store the information on the subspace structure of $\mathfrak{g}$.

As for the abelian semigroup $S$, this can always be decomposed as $S = \bigcup_{p \in I} S_p$, where $S_p \subset S$. In principle, this decomposition is completely arbitrary; however, using the product from Def. II.2, it is sometimes possible to pick up a very particular choice of subset decomposition. This choice is the subject of the following

**Definition IV.1** Let $\mathfrak{g} = \bigoplus_{p \in I} V_p$ be a decomposition of $\mathfrak{g}$ in subspaces, with a structure described by the subsets $i_{(p,q)}$, as in eq. (33). Let $S = \bigcup_{p \in I} S_p$ be a subset decomposition of the abelian semigroup $S$ such that

$$S_p \cdot S_q \subset \bigcap_{r \in i_{(p,q)}} S_r,$$

where the subset product $\cdot$ is the one from Def. II.2. When such subset decomposition $S = \bigcup_{p \in I} S_p$ exists, then we say that this decomposition is in **resonance** with the subspace decomposition of $\mathfrak{g}$, $\mathfrak{g} = \bigoplus_{p \in I} V_p$.

The resonant subset decomposition is crucial in order to systematically extract subalgebras from the $S$-expanded algebra $\mathfrak{G} = S \times \mathfrak{g}$, as is proven in the following

**Theorem IV.2** Let $\mathfrak{g} = \bigoplus_{p \in I} V_p$ be a subspace decomposition of $\mathfrak{g}$, with a structure described by eq. (33), and let $S = \bigcup_{p \in I} S_p$ be a resonant subset decomposition of the abelian semigroup $S$, with the structure given in eq. (34). Define the subspaces of $\mathfrak{G} = S \times \mathfrak{g}$,

$$W_p = S_p \times V_p, \quad p \in I.$$

(35)

Then,

$$\mathfrak{G}_R = \bigoplus_{p \in I} W_p$$

(36)

is a subalgebra of $\mathfrak{G} = S \times \mathfrak{g}$.

**Proof.** Using eqs. (33)–(34), we have

$$[W_p, W_q] \subset (S_p \cdot S_q) \times [V_p, V_q]$$

$$\subset \bigcap_{s \in i_{(p,q)}} S_s \times \bigoplus_{r \in i_{(p,q)}} V_r$$

$$\subset \bigoplus_{r \in i_{(p,q)}} \left[ \bigcap_{s \in i_{(p,q)}} S_s \right] \times V_r$$

(37)
Now, it is clear that for each \( r \in i_{(p,q)} \), one can write

\[
\bigcap_{s \in i_{(p,q)}} S_s \subset S_r. \tag{38}
\]

Then,

\[
[W_p, W_q] \subset \bigoplus_{r \in i_{(p,q)}} S_r \times V_r \tag{39}
\]

and we arrive at

\[
[W_p, W_q] \subset \bigoplus_{r \in i_{(p,q)}} W_r. \tag{40}
\]

Therefore, the algebra closes and \( G_R = \bigoplus_{p \in I} W_p \) is a subalgebra of \( G \).

**Definition IV.3** The algebra \( G_R = \bigoplus_{p \in I} W_p \) obtained in Theorem [IV.2] is called a Resonant Subalgebra of the \( S \)-expanded algebra \( G = S \times g \).

The choice of the name resonance is due to the formal similarity between eqs. (33) and (34); eq. (34) will be also referred to as "resonance condition."

Theorem [IV.2] translates the difficult problem of finding subalgebras from an \( S \)-expanded algebra \( G = S \times g \) into that of finding a resonant partition for the semigroup \( S \). As the examples from section V help make clear, solving the resonance condition (34) turns out to be an easily tractable problem. Theorem [IV.2] can thus be regarded as a useful tool for extracting subalgebras from an \( S \)-expanded algebra.

Using eq. (11) and the resonant subset partition of \( S \) it is possible to find an explicit expression for the structure constants of the resonant subalgebra \( G_R \). Denoting the basis of \( V_p \) by \( \{T_{a_p}\} \), one can write

\[
C_{(a_p, \alpha_p)(b_q, \beta_q)}^{(c_r, \gamma_r)} = K_{\alpha_p \beta_q}^{\gamma_r} C_{a_p b_q}^{c_r} \text{ with } \alpha_p, \beta_q, \gamma_r \text{ such that } \lambda_{\alpha_p} \in S_p, \lambda_{\beta_q} \in S_q, \lambda_{\gamma_r} \in S_r. \tag{41}
\]

An interesting fact is that the \( S \)-expanded algebra "subspace structure" encoded in \( i_{(p,q)} \) is the same as in the original algebra \( g \), as can be observed from eq. (40).

Resonant Subalgebras play a central rôle in the current scheme. It is interesting to notice that most of the cases considered in [1] can be reobtained using the above theorem for \( S = S_E \) [recall eqs. (25)–(26)] and \( 0_S \)-reduction, as we will see in the next section. All remaining cases can be obtained as a more general reduction of a resonant subalgebra (see sec. VI).
**B. ** $S = S_E$ Resonant Subalgebras and MC Expanded Algebras

In this section, some results presented for algebra expansions in Ref. [1] are recovered within the $S$-expansion approach. To get these algebras one must proceed in a three-step fashion:

1. Perform an $S$-expansion using the semigroup $S = S_E$,

2. Find a resonant partition for $S_E$ and construct the resonant subalgebra $\mathfrak{g}_R$,

3. Apply a $0_S$-reduction (or a more general one, see sec. VI) to the resonant subalgebra.

Choosing a different semigroup $S$ or omitting the reduction procedure one finds algebras not contained within the Maurer–Cartan forms power series expansion of Ref. [1]. Such an example is provided in sec. V C.

1. **Case when $g = V_0 \oplus V_1$, with $V_0$ being a Subalgebra and $V_1$ a Symmetric Coset**

Let $g = V_0 \oplus V_1$ be a subspace decomposition of $g$, such that

$$[V_0, V_0] \subset V_0,$$  \hspace{1cm} (42)

$$[V_0, V_1] \subset V_1,$$  \hspace{1cm} (43)

$$[V_1, V_1] \subset V_0.$$  \hspace{1cm} (44)

Let $S_E^{(N)} = S_0 \cup S_1$, with $N$ arbitrary, be a subset decomposition of $S_E$, with

$$S_0 = \left\{ \lambda_{2m}, \text{ with } m = 0, \ldots, \left[ \frac{N}{2} \right] \right\} \cup \{ \lambda_{N+1} \},$$  \hspace{1cm} (45)

$$S_1 = \left\{ \lambda_{2m+1}, \text{ with } m = 0, \ldots, \left[ \frac{N-1}{2} \right] \right\} \cup \{ \lambda_{N+1} \}.$$  \hspace{1cm} (46)

This subset decomposition of $S_E^{(N)}$ satisfies the resonance condition (34), which in this case explicitly reads

$$S_0 \cdot S_0 \subset S_0,$$  \hspace{1cm} (47)

$$S_0 \cdot S_1 \subset S_1,$$  \hspace{1cm} (48)

$$S_1 \cdot S_1 \subset S_0.$$  \hspace{1cm} (49)
Therefore, according to Theorem IV.2, we have that
\[ \mathfrak{G}_R = W_0 \oplus W_1, \] (50)
with
\[ W_0 = S_0 \times V_0, \] (51)
\[ W_1 = S_1 \times V_1, \] (52)
is a resonant subalgebra of \( \mathfrak{G} \).

Using eq. (41), it is straightforward to write the structure constants for the resonant subalgebra,
\[ C_{(\alpha_p, \beta_q)(\beta_p, \gamma_p)}^{(c_r, \gamma_r)} = \delta_{H_{N+1}(\alpha_p+\beta_p)}^{\gamma_r} C_{\alpha_p \beta_q}^{c_r} \]
with
\[ \begin{cases} p, q = 0, 1 \\ \alpha_p, \beta_p, \gamma_p = 2m + p, \\ m = 0, \ldots, \left[ \frac{N-p}{2} \right], \left[ \frac{N+1-p}{2} \right]. \end{cases} \]

Imposing \( \lambda_{N+1} T_A = 0 \), the 0\( _S \)-reduced structure constants are obtained as
\[ C_{(\alpha_p, \beta_q)(\beta_p, \gamma_p)}^{(c_r, \gamma_r)} = \delta_{H_{N+1}(\alpha_p+\beta_p)}^{\gamma_r} C_{\alpha_p \beta_q}^{c_r} \]
with
\[ \begin{cases} p, q = 0, 1 \\ \alpha_p, \beta_p, \gamma_p = 2m + p, \\ m = 0, \ldots, \left[ \frac{N-p}{2} \right]. \end{cases} \] (53)

In order to compare with the MC Expansion, let us observe that, with the notation of [1], the 0\( _S \)-reduction of the \( S_E \)-expanded algebra corresponds to \( \mathcal{G} (N_0, N_1) \) for the symmetric coset case, with
\[ N_0 = 2 \left[ \frac{N}{2} \right], \] (54)
\[ N_1 = 2 \left[ \frac{N-1}{2} \right] + 1. \] (55)

The structure constants (53) correspond to the structure constants (3.31) from Ref. [1] (the notation is slightly different though).

A more intuitive idea of the whole procedure of \( S \)-expansion, resonant subalgebra and \( 0_S \)-reduction can be obtained by means of a diagram, such as the one depicted in Fig. [1].

This diagram corresponds to the case \( \mathfrak{g} = V_0 \oplus V_1 \), with \( V_0 \) a subalgebra and \( V_1 \) a symmetric coset, and the choice \( S = S_{E}^{(3)} \).
FIG. 1: $S_E^{(3)}$-expansion of an algebra $\mathfrak{g} = V_0 \oplus V_1$, where $V_0$ is a subalgebra and $V_1$ a symmetric coset. 

(a) The gray region corresponds to the full $S_E^{(3)}$-expanded algebra, $\mathfrak{G} = S_E^{(3)} \times \mathfrak{g}$. (b) The shaded area here depicts a resonant subalgebra $\mathfrak{G}_R$. (c) The gray region now shows the $0_S$-reduction of the resonant subalgebra $\mathfrak{G}_R$.

The subspaces of $\mathfrak{g}$ are represented on the horizontal axis, while the semigroup elements occupy the vertical one. In this way, the whole $S_E^{(3)}$-expanded algebra $S_E^{(3)} \times \mathfrak{g}$ corresponds to the shaded region in Fig. 1 (a). In Fig. 1 (b), the gray region represents the resonant subalgebra $\mathfrak{G}_R = W_0 \oplus W_1$ with

$$S_0 = \{\lambda_0, \lambda_2, \lambda_4\},$$

$$S_1 = \{\lambda_1, \lambda_3, \lambda_4\}.$$  \hspace{1cm} (56)\hspace{1cm} (57)

Let us observe that each column in the diagram corresponds to a subset of the resonant partition. Finally, Fig. 1 (c) represents the $0_S$-reduced algebra, obtained after imposing $\lambda_4 \times \mathfrak{g} = 0$. This figure actually corresponds to the case $\mathcal{G}(N_0, N_1)$.

As is evident from the above discussion, the case $N = 1$, $\lambda_{N+1} T_A = 0$ reproduces the İnönü–Wigner contraction for $\mathfrak{g} = V_0 \oplus V_1$. More on generalized İnönü–Wigner contractions is presented in section VII.
2. Case when $g$ fulfills the Weimar-Woods Conditions

Let $g = \bigoplus_{p=0}^{n} V_p$ be a subspace decomposition of $g$. In terms of this decomposition, the Weimar-Woods conditions \[15, 16\] on $g$ read
\[
[V_p, V_q] \subset \bigoplus_{r=0}^{H_n(p+q)} V_r.
\]

Let
\[
S_E = \bigcup_{p=0}^{n} S_p
\]
be a subset decomposition of $S_E$, where the subsets $S_p \subset S_E$ are defined by
\[
S_p = \{ \lambda_{\alpha_p}, \text{such that } \alpha_p = p, \ldots, N+1 \}
\]
with $N+1 \geq n$.

This subset decomposition is a resonant one under the semigroup product \[26\], because it satisfies [compare eq. \[61\] with eq. \[58\]]
\[
S_p \cdot S_q = S_{H_n(p+q)} \subset \bigcap_{r=0}^{H_n(p+q)} S_r.
\]

According to Theorem \[IV.2\], the direct sum
\[
\mathfrak{G}_R = \bigoplus_{p=0}^{n} W_p,
\]
with
\[
W_p = S_p \times V_p,
\]
is a resonant subalgebra of $\mathfrak{G}$.

Using eq. \[41\], we get the following structure constants for the resonant subalgebra:
\[
C_{(\alpha_p, \alpha_p)(\beta_q, \beta_q)}^{(c_r, \gamma_r)} = \delta_{H_{N+1}(\alpha_p+\beta_q)}^{\gamma_r} C_{\alpha_p \beta_q}^{c_r} \quad \text{with} \quad \begin{cases} 
p, q, r = 0, \ldots, n \\
\alpha_p, \beta_p, \gamma_p = p, \ldots, N+1
\end{cases}.
\]

Imposing $\lambda_{N+1} T_A = 0$, this becomes
\[
C_{(\alpha_p, \alpha_p)(\beta_q, \beta_q)}^{(c_r, \gamma_r)} = \delta_{H_{N+1}(\alpha_p+\beta_q)}^{\gamma_r} C_{\alpha_p \beta_q}^{c_r} \quad \text{with} \quad \begin{cases} 
p, q, r = 0, \ldots, n \\
\alpha_p, \beta_p, \gamma_p = p, \ldots, N
\end{cases}.
\]
(a) The shaded region shows a $S_E^{(4)}$ resonant subalgebra when $g = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ satisfies the Weimar-Woods conditions. (b) The $0_S$-reduction of this resonant subalgebra removes all sectors of the form $0_S \times g$. This corresponds to the case $G(4,4,4,4)$ in the context of [1].

This $0_S$-reduced algebra corresponds to the case $G(N_0, \ldots, N_n)$ of Theorem 3 from [1] with $N_p = N$ for every $p = 0, \ldots, n$. The structure constants (63) correspond to the ones of eq. (4.8) in Ref. [1] (with a slightly different notation). The more general case,

$$N_{p+1} = \begin{cases} N_p \text{ or} \\ N_p + 1 \end{cases}$$

can be also obtained in the context of an $S$-expansion, from a resonant subalgebra and applying a kind of reduction more general than the $0_S$-reduction (see sec. VI and app. A).

The resonant subalgebra for the Weimar-Woods case with $n = 3$, $N = 4$ and its $0_S$-reduction are shown in Fig. 2 (a) and (b), respectively.

3. Case when $g = V_0 \oplus V_1 \oplus V_2$ is a Superalgebra

A superalgebra $g$ comes naturally split into three subspaces $V_0$, $V_1$ and $V_2$, where $V_1$ corresponds to the Fermionic sector and $V_0 \oplus V_2$ to the Bosonic one, with $V_0$ being a subalgebra.
This subspace structure may be written as

\[
\begin{align*}
[V_0, V_0] &\subset V_0, \quad (64) \\
[V_0, V_1] &\subset V_1, \quad (65) \\
[V_0, V_2] &\subset V_2, \quad (66) \\
[V_1, V_1] &\subset V_0 \oplus V_2, \quad (67) \\
[V_1, V_2] &\subset V_1, \quad (68) \\
[V_2, V_2] &\subset V_0 \oplus V_2. \quad (69)
\end{align*}
\]

Let \( S^{(N)}_E = S_0 \cup S_1 \cup S_2 \) be a subset decomposition of \( S^{(N)}_E \), where the subsets \( S_0, S_1, S_2 \) are given by the general expression

\[
S_p = \left\{ \lambda_{2m+p}, \text{ with } m = 0, \ldots, \left\lfloor \frac{N-p}{2} \right\rfloor \right\} \cup \{ \lambda_{N+1} \}, \quad p = 0, 1, 2. \quad (70)
\]

This subset decomposition is a resonant one, because it satisfies [compare eqs. (71)–(76) with eqs. (64)–(69)]

\[
\begin{align*}
S_0 \cdot S_0 &\subset S_0, \quad (71) \\
S_0 \cdot S_1 &\subset S_1, \quad (72) \\
S_0 \cdot S_2 &\subset S_2, \quad (73) \\
S_1 \cdot S_1 &\subset S_0 \cap S_2, \quad (74) \\
S_1 \cdot S_2 &\subset S_1, \quad (75) \\
S_2 \cdot S_2 &\subset S_0 \cap S_2. \quad (76)
\end{align*}
\]

Theorem IV.2 assures us that \( \mathfrak{g}_R = W_0 \oplus W_1 \oplus W_2 \), with \( W_p = S_p \times V_p \), \( p = 0, 1, 2 \), is a resonant subalgebra.

Using eq. (51), it is possible to write down the structure constants for the resonant subalgebra as

\[
C^{(c_r,\gamma_r)}_{(a_p,\alpha_p)(b_q,\beta_q)}(\alpha_p,\beta_p,\gamma_p) = \delta^{\alpha_p,\beta_p,\gamma_p}_{H_{N+1}^{(N)}} C^{c_r}_{a_p b_q} \quad \text{with} \quad \begin{cases} 
p, q, r = 0, 1, 2, \\
\alpha_p, \beta_p, \gamma_p = 2m + p, \\
m = 0, \ldots, \left\lfloor \frac{N-p}{2} \right\rfloor, \left\lfloor \frac{N+1-p}{2} \right\rfloor. \end{cases} \quad (77)
\]

Imposing \( \lambda_{N+1} T_A = 0 \), the structure constants for the 0 resulted of the resonant
FIG. 3: (a) The shaded area corresponds to an $S_E^{(4)}$ resonant subalgebra of $\mathfrak{g} = S_E^{(4)} \times \mathfrak{g}$ when $\mathfrak{g}$ is a superalgebra. (b) The gray region shows the $0_S$-reduction of the resonant subalgebra $\mathfrak{g}_R$. This corresponds to $\mathcal{G}(4, 3, 4)$ in the context of Ref. [1].

Subalgebra are obtained:

$$C_{(\alpha_p, \beta_p)(\gamma_r, \delta_q)}^{(\epsilon_r, \gamma_r)} = \delta_{H_{N+1}(\alpha_p + \beta_q)} C_{\alpha_p \beta_q}^{\epsilon_r} \text{ with } \begin{cases} p, q, r = 0, 1, 2, \\ \alpha_p, \beta_p, \gamma_p = 2m + p, \\ m = 0, \ldots, \left[\frac{N-p}{2}\right]. \end{cases} \quad (78)$$

This $0_S$-reduced algebra corresponds to the algebra $\mathcal{G}(N_0, N_1, N_2)$ with

$$N_p = 2 \left\lfloor \frac{N - p}{2} \right\rfloor + p, \quad p = 0, 1, 2$$

found in theorem 5 of Ref. [1]. The structure constants (78) match the structure constants (5.6) from Ref. [1].

Fig. 3 (a) shows the resonant subalgebra for the case of superalgebras, and Fig. 3 (b), its corresponding $0_S$-reduction, for the case $N = 4$. 
V. S-EXPANSIONS OF \( \mathfrak{osp}(32\vert 1) \) AND \( d = 11 \) SUPERALGEBRAS

In this section we explore some explicit examples of \( S \)-expansions. In general, every possible choice of abelian semigroup \( S \) and resonant partition will lead to a new \( d = 11 \) superalgebra. Note however that the existence of a resonant partition is not at all guaranteed for an arbitrary semigroup \( S \).

Since our main physical motivation comes from 11-dimensional Supergravity, we shall always take the orthosymplectic superalgebra \( \mathfrak{osp}(32\vert 1) \) as a starting point. A suitable basis is provided by \( \{ P_a, J_{ab}, Z_{abcde}, Q \} \), where \( \{ P_a, J_{ab} \} \) are the anti-de Sitter (AdS) generators, \( Z_{abcde} \) is a 5-index antisymmetric tensor and \( Q \) is a 32-component, Majorana spinor charge. The \( \mathfrak{osp}(32\vert 1) \) (anti)commutation relations explicitly read

\[
[P_a, P_b] = J_{ab}, \\
[J_{ab}, P_c] = \delta^{ab}_{ec} P^e, \\
[J_{ab}, J_{cd}] = \delta^{abf}_{ecd} J^e_f, \\
[P_a, Z_{b_1\cdots b_5}] = -\frac{1}{5!} \varepsilon_{ab_1\cdots b_5c_1\cdots c_5} Z^{c_1\cdots c_5},
\]

\[
[J_{ab}, Z_{c_1\cdots c_5}] = \frac{1}{4!} \delta^{abc_1\cdots e_4}_{d_1\cdots d_5} Z^d e_1\cdots e_4,
\]

\[
Z^{a_1\cdots a_5, b_1\cdots b_5} = \eta^{[a_1\cdots a_5][c_1\cdots c_5]} \varepsilon_{c_1\cdots c_5b_1\cdots b_5} P^e + \delta^{a_1\cdots a_5 e}_{db_1\cdots b_5} J^d_e + \frac{1}{3!3!5!} \varepsilon_{c_1\cdots c_5} \delta^{a_1\cdots a_5 c_4 c_5 c_6}_{d_1 d_2 d_3 b_1\cdots b_5} \eta^{[c_1 c_2 c_3][d_1 d_2 d_3]} Z^{c_7\cdots c_{11}},
\]

\[
[P_a, Q] = -\frac{1}{2} \Gamma_a Q, \\
[J_{ab}, Q] = -\frac{1}{2} \Gamma_{ab} Q, \\
[Z_{abcde}, Q] = -\frac{1}{2} \Gamma_{abcde} Q,
\]

\[
\{Q^\rho, Q^\sigma\} = -\frac{1}{23} \left[ (\Gamma^a C^{-1})^{\rho\sigma} P_a - \frac{1}{2} (\Gamma^{ab} C^{-1})^{\rho\sigma} J_{ab} + \frac{1}{5!} (\Gamma^{abcde} C^{-1})^{\rho\sigma} Z_{abcde} \right],
\]

where \( C_{\rho\sigma} \) is the charge conjugation matrix and \( \Gamma_a \) are Dirac matrices in 11 dimensions.
As a first step towards the $S$-Expansion, the $\mathfrak{osp}(32|1)$ algebra is written as the direct sum of three subspaces:

$$\mathfrak{osp}(32|1) = V_0 \oplus V_1 \oplus V_2,$$

(89)

where $V_0$ corresponds to the Lorentz subalgebra (spanned by $J_{ab}$), $V_1$ to the Fermionic subspace (spanned by $Q$) and $V_2$ to the translations and the M5-brane piece (spanned by $P_a$ and $Z_{abcde}$). The subspace separation (89) satisfies conditions (64)–(69), as can be easily checked.

The $M$ algebra [3, 4, 5] and a Superalgebra similar to those of D’Auria–Fré [17] are rederived in next sections using $S = S_E(N)$ with $N = 2$ and $N = 3$, respectively. As an example of an $S$-expansion with $S \neq S_E$, the case $S = \mathbb{Z}_4$ is considered in section V C, where a new superalgebra resembling aspects of the $M$ algebra, $\mathfrak{osp}(32|1) \times \mathfrak{osp}(32|1)$ and D’Auria–Fré superalgebras is found.

The inclusion of the $M$ algebra among the examples draws from two different but related facts. First, the $S$-Expansion paradigm casts the $M$ algebra as one from a family of superalgebras, all derived from $\mathfrak{osp}(32|1)$ through different choices for the semigroup and different alternatives of reduction, when present at all. This can be relevant from a physical point of view, since all of them share important features. The second reason deals with the construction of Chern–Simons and transgression Lagrangians. As will be shown in sec. VII invariant tensors for resonant subalgebras and $0_S$-reduced algebras thereof are readily available, but this is not the case for general reduced algebras. As such, the fact that the $M$ algebra stems from a $0_S$-reduction is interesting not only because it provides with an invariant tensor derived from $\mathfrak{osp}(32|1)$, but also because it brings about the possibility of considering its direct generalization, namely, the resonant subalgebra from where it was extracted. More on the physical consequences of regarding the $M$ algebra as the $0_S$-reduction of a resonant subalgebra is found in [14].

A. The M Algebra

As treated in detail in [1, 21], the complete $M$ algebra (i.e., including its Lorentz part) can be obtained by means of an MC expansion of $\mathfrak{osp}(32|1)$. Within the present scheme, the $M$ algebra should be recovered via an $S$-expansion with $S = S_E^{(2)}$ followed by a $0_S$-reduction, as explained in section III B.
FIG. 4: The M algebra as an $S_E^{(2)}$-expansion of $osp(32|1)$. (a) A resonant subalgebra of the $S_E^{(2)}$-expanded algebra $\mathfrak{g} = S_E^{(2)} \times osp(32|1)$ is shown in the shaded region. (b) The M algebra itself (gray area) is obtained after $0_3$-reducing the resonant subalgebra.

In order to obtain the M algebra in the context of $S$-expansions, one has to pick $S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\}$, use the resonant partition (70) and impose the condition $\lambda_3 T_A = 0$. This amounts to explicitly writing the structure constants (78) for this case. For the sake of simplicity, let us relabel $J_{ab} = J_{(ab,0)}$, $Q_\alpha = Q_{(\alpha,1)}$, $Z_{ab} = J_{(ab,2)}$, $P_a = P_{(a,2)}$, $Z_{abcde} = Z_{(abcde,2)}$, as shown in Fig. 4. The resulting algebra reads

\[
[P_a, P_b] = 0, \quad (90)
\]
\[
[J^{ab}, P_c] = \delta^{ab}_{ce} P^e, \quad (91)
\]
\[
[J^{ab}, J^{cd}] = \delta^{abf}_{ecd} J^f, \quad (92)
\]
\[
[J^{ab}, Z^{cd}] = \delta^{abf}_{ecd} Z^f, \quad (93)
\]
\[
[Z^{ab}, Z^{cd}] = 0, \quad (94)
\]
\[
[P_a, Z_{b_1 \cdots b_5}] = 0, \quad (95)
\]
\[
[J^{ab}, Z_{c_1 \cdots c_5}] = \frac{1}{4!} \delta^{abc}_{d_1 \cdots d_5} Z^{d_5}_{e_1 \cdots e_4}, \quad (96)
\]
\[
[Z^{ab}, Z_{c_1 \cdots c_5}] = 0, \quad (97)
\]
\[
[Z^{a_1 \cdots a_5}, Z_{b_1 \cdots b_5}] = 0, \quad (98)
\]
\[
[P_a, Q] = 0, \quad (99)
\]
\[
[J_{ab}, Q] = -\frac{1}{2} \Gamma_{ab} Q, \quad (100)
\]
\[
[Z_{ab}, Q] = 0, \quad (101)
\]
\[
[Z_{abcde}, Q] = 0, \quad (102)
\]

\[
\{Q^\alpha, Q^\beta\} = -\frac{1}{2^3} \left[ (\Gamma^\alpha C^{-1})^\rho^\sigma P_a - \frac{1}{2} (\Gamma^{ab} C^{-1})^\rho^\sigma Z_{ab} + \right.
\]
\[
\left. + \frac{1}{5!} (\Gamma^{abcde} C^{-1})^\rho^\sigma Z_{abcde} \right], \quad (103)
\]

Note that the rôle of the 0_S-reduction in the process is that of abelianizing large sectors of the resonant subalgebra.

B. D'Auria–Fré-like Superalgebra

The above example used \( S_E^{(2)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} \) as abelian semigroup to perform the expansion. In this section, the results of choosing instead \( S_E^{(3)} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\} \) while leaving everything else (including the 0_S-reduction) unchanged are examined.

A D'Auria–Fré-like Superalgebra [17], with one extra Fermionic generator as compared with \( \text{osp}(32|1) \) or the M algebra, is obtained by picking the resonant partition (70) and 0_S-reducing the resulting resonant subalgebra. Relabeling generators as \( J_{ab} = J_{(ab,0)}, Q_\alpha = Q_{(\alpha,1)} \), \( Z_{ab} = J_{(ab,2)}, P_a = P_{(a,2)}, Z_{abcde} = Z_{(abcde,2)}, Q'_\alpha = Q_{(\alpha,3)} \), one finds the structure depicted in Fig. 5. While this algebra has the same \textit{structure} (i.e., same number and type of generators, with commutators valued on the same subspaces) as the ones introduced by D’Auria and Fré in [17], the details differ, so it cannot really correspond to any of them [34] (hence the “-like”).

The (anti)commutation relations, which can be read off directly from the structure constants (78) after applying condition (23), bear a strong similarity with those from the M algebra. Only the following three differ:

\[
[P_a, Q] = -\frac{1}{2} \Gamma_a Q', \quad (104)
\]
\[
[Z_{ab}, Q] = -\frac{1}{2} \Gamma_{ab} Q', \quad (105)
\]
\[
[Z_{abcde}, Q] = -\frac{1}{2} \Gamma_{abcde} Q'. \quad (106)
\]
FIG. 5: A D’Auria–Fré-like Superalgebra regarded here as an $S^{(3)}_E$-expansion of $\mathfrak{osp}(32|1)$. (a) A resonant subalgebra of the $S^{(3)}_E$-expanded algebra $\mathfrak{g} = S^{(3)}_E \times \mathfrak{osp}(32|1)$ is shown in the shaded region. (b) A Superalgebra similar to the ones introduced by D’Auria and Fré in [17] is obtained after $0_S$-reducing the resonant subalgebra.

The (anti)commutation relations which directly involve the extra Fermionic generator $Q'$ read

\[ [P_a, Q'] = 0, \]  
\[ [Z_{ab}, Q'] = 0, \]  
\[ [Z_{abcde}, Q'] = 0, \]  
\[ \{Q, Q'\} = 0, \]  
\[ \{Q', Q'\} = 0, \]  
\[ [J_{ab}, Q'] = -\frac{1}{2} \Gamma_{ab} Q'. \]

The extra Fermionic generator $Q'$ is found to (anti)commute with all generators from the algebra but the Lorentz generators (which was to be expected due to its spinor character).
C. Resonant Subalgebra of \( \mathbb{Z}_4 \times \mathfrak{osp}(32|1) \)

Cyclic groups seem especially suitable for an \( S \)-expansion, because, on one hand, they are groups and not only semigroups (and therefore there is no 0 element and no 0\( S \)-reduction), and on the other, because the multiplication law for a cyclic group looks very similar to the multiplication law of the semigroup \( S_E \),

\[
\begin{align*}
S_E^{(N)} : \lambda_\alpha \lambda_\beta &= \lambda_{H_{N+1}(\alpha + \beta)} \\
Z_N : \lambda_\alpha \lambda_\beta &= \lambda_{\text{mod}\,N(\alpha + \beta)}.
\end{align*}
\]

The cyclic group \( \mathbb{Z}_4 \) in particular was chosen for this example because the \( \mathbb{Z}_2 \) case is trivial (the resonant subalgebra is \( \mathfrak{osp}(32|1) \) itself) and \( \mathbb{Z}_3 \) seems to have no resonant partition; therefore \( \mathbb{Z}_4 \) corresponds to the simplest nontrivial case.

Since this example uses a semigroup different from \( S_E \), the algebra obtained does not correspond to a Maurer–Cartan forms power-series expansion.

Given a superalgebra \( \mathfrak{g} = V_0 \oplus V_1 \oplus V_2 \) with the structure (64)–(69), a resonant partition of \( \mathbb{Z}_4 = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3\} \) is given by

\[
\begin{align*}
S_0 &= \{\lambda_0, \lambda_2\}, \\
S_1 &= \{\lambda_1, \lambda_3\}, \\
S_2 &= \{\lambda_0, \lambda_2\}.
\end{align*}
\]

In order to avoid a cluttering of indices, relabel \( J_{ab} = J_{(ab,0)} \), \( Z'_{a_1\ldots a_5} = Z_{(a_1\ldots a_5,0)} \), \( P'_a = P_{(a,0)} \), \( Q_\alpha = Q_{(\alpha, 1)} \), \( Z_{ab} = J_{(ab,2)} \), \( Z_{a_1\ldots a_5} = Z_{(a_1\ldots a_5,2)} \), \( P_a = P_{(a,2)} \) and \( Q'_\alpha = Q_{(\alpha, 3)} \), as shown in Fig. [6].

The full resonant subalgebra can be easily obtained using the structure constants (11); here we shall only quote some of its more interesting sectors.

This algebra has two very conspicuous features: first, it bears some resemblance to the M algebra, for both Fermionic generators \( \mathcal{Q} \) and \( \mathcal{Q}' \) satisfy

\[
\{\mathcal{Q}^\rho, \mathcal{Q}'^\sigma\} = \{\mathcal{Q}^\rho, \mathcal{Q}^\sigma\} = \frac{1}{23} \left[ (\Gamma^a C^{-1})^\rho^\sigma P_a - \frac{1}{2} (\Gamma^{ab} C^{-1})^\rho^\sigma Z_{ab} + \frac{1}{5!} (\Gamma^{abcde} C^{-1})^\rho^\sigma Z_{abcde} \right].
\]

(118)
FIG. 6: A new $d = 11$ Superalgebra, different from but resembling aspects of both the M algebra and the D’Auria–Fré Superalgebras, is obtained directly as a resonant subalgebra (shaded region) of the $S$-expanded algebra $Z_4 \times osp(32|1)$.

Second, the algebra has two “AdS-boosts generators” for the same Lorentz algebra,

\[
\begin{align*}
[P_a, P_b] &= J_{ab}, & [P'_a, P'_b] &= J_{ab}, \\
[J_{ab}, P_c] &= \delta^{ab}_{ec} P^e, & [J_{ab}, P'_e] &= \delta^{ab}_{ec} P'^e, \\
[J_{ab}, J_{cd}] &= \delta^{ab}_{ef} J^{ef}, \\
[P_a, P'_b] &= Z_{ab}.
\end{align*}
\]

The “charges” $Z_{ab}$, $Z_{a_1 \ldots a_5}$ and $Z'_{a_1 \ldots a_5}$ are Lorentz tensors, but they are not abelian,

\[
\begin{align*}
[Z_{ab}, Z_{cd}] &= \delta^{ab}_{ec} J^{ef}, \\
[Z_{a_1 \ldots a_5}, Z'_{b_1 \ldots b_5}] &= \\
&= \eta^{[a_1 \ldots a_5][c_1 \ldots c_5]} \varepsilon_{c_1 \ldots c_5 b_1 \ldots b_5 e} P'^e + \delta^{a_1 \ldots a_5 e} J'_e + \\
&\quad - \frac{1}{3!3!5!} \varepsilon_{c_1 \ldots c_{11}} \delta^{a_1 \ldots a_5 c_4 c_5 e} (c_1 c_2 c_3) [d_1 d_2 d_3] Z'^{c_7 \ldots c_{11}}.
\end{align*}
\]

This algebra also presents a behavior similar to that of the D’Auria–Fré superalgebras; namely, the commutators between the generators $P_a$, $Z_{ab}$, $Z_{a_1 \ldots a_5}$ and a Fermionic generator $Q$ are $Q'$-valued; but in contrast to (107)–(109), their commutator with $Q'$ is $Q$-valued rather than zero. In this regard, the generators $J_{ab}$, $Z'_{a_1 \ldots a_5}$, $P'_a$ have a block-diagonal form on the subspace ($Q$, $Q'$); their commutator with $Q$ is $Q$-valued and the one with $Q'$ is $Q'$-valued.
We have seen several examples showing how, starting from only one original algebra \( \mathfrak{g} \) and using different semigroups, different resonant subalgebras can arise (see Theorem IV.2). This is particularly interesting if one considers the strong similarities between the semigroups considered, which nevertheless lead to different resonant structures.

VI. REDUCED ALGEBRAS OF A RESONANT SUBALGEBRA

In previous sections we have seen how information on the subspace structure of the original algebra \( \mathfrak{g} \) can be used in order to find resonant subalgebras of the \( S \)-expanded algebra \( S \times \mathfrak{g} \). In this section we shall examine how this information can be put to use in a different way, namely, by extracting reduced algebras (in the sense of Def. II.3) from the resonant subalgebra. It is following this path that, e.g., the generalized İnönü–Wigner contraction fits within the present scheme.

The following general theorem provides necessary conditions under which a reduced algebra can be extracted from a resonant subalgebra.

**Theorem VI.1** Let \( \mathfrak{G}_R = \bigoplus_{p \in I} S_p \times V_p \) be a resonant subalgebra of \( \mathfrak{G} = S \times \mathfrak{g} \), i.e., let eqs. (33) and (34) be satisfied. Let \( S_p = \hat{S}_p \cup \check{S}_p \) be a partition of the subsets \( S_p \subset S \) such that

\[
\hat{S}_p \cap \check{S}_q = \emptyset,
\]

(122)

\[
\check{S}_p : \check{S}_q \subset \bigcap_{r \in I(p,q)} \hat{S}_r.
\]

(123)

Conditions (122) and (123) induce the decomposition \( \mathfrak{G}_R = \check{\mathfrak{G}}_R \oplus \hat{\mathfrak{G}}_R \) on the resonant subalgebra, where

\[
\check{\mathfrak{G}}_R = \bigoplus_{p \in I} \check{S}_p \times V_p,
\]

(124)

\[
\hat{\mathfrak{G}}_R = \bigoplus_{p \in I} \hat{S}_p \times V_p.
\]

(125)

When the conditions (122)–(123) hold, then

\[
[\check{\mathfrak{G}}_R, \hat{\mathfrak{G}}_R] \subset \hat{\mathfrak{G}}_R,
\]

(126)

and therefore \( |\check{\mathfrak{G}}_R| \) corresponds to a reduced algebra of \( \mathfrak{G}_R \).
Proof. Let $\hat{W}_p = \hat{S}_p \times V_p$ and $\hat{W}_q = \hat{S}_p \times V_p$. Then, using condition (123), we have

$$\left[\hat{W}_p, \hat{W}_q\right] \subset \left(\hat{S}_p \times \hat{S}_q\right) \times [V_p, V_q]$$

$$\subset \bigcap_{s \in i(p,q)} \hat{S}_s \times \bigoplus_{r \in i(p,q)} V_r$$

$$\subset \bigoplus_{r \in i(p,q)} \left[\bigcap_{s \in i(p,q)} \hat{S}_s\right] \times V_r.$$  

For each $r \in i(p,q)$ we have $\bigcap_{s \in i(p,q)} \hat{S}_s \subset \hat{S}_r$, so that

$$\left[\hat{W}_p, \hat{W}_q\right] \subset \bigoplus_{r \in i(p,q)} \hat{S}_r \times V_r$$

$$\subset \bigoplus_{r \in i(p,q)} \hat{W}_r.$$  

Since $\hat{G}_R = \bigoplus_{p \in I} \hat{W}_p$ and $\hat{\hat{G}}_R = \bigoplus_{p \in I} \hat{W}_p$, we finally find

$$\left[\hat{G}_R, \hat{G}_R\right] \subset \hat{G}_R$$

and therefore $|\hat{G}_R|$ is a reduced algebra of $G_R$.  

Using the structure constants (11) for the resonant subalgebra, it is possible to find the structure constants for the reduced algebra $|\hat{G}_R|$,  

$$C_{(a_p, \alpha_p)(b_q, \beta_q)}^{(c_r, \gamma_r)} = K_{a_p, \beta_q}^{\gamma_r} C_{a_p b_q}^{c_r},$$

with $\alpha_p, \beta_q, \gamma_r$ such that $\lambda_{\alpha_p} \in \hat{S}_p, \lambda_{\beta_q} \in \hat{S}_q, \lambda_{\gamma_r} \in \hat{S}_r.$  

(127)

It might be worth to notice that, when every $S_p \subset S$ of a resonant subalgebra includes the zero element $0_S$, the choice $\hat{S}_p = \{0_S\}$ automatically satisfies conditions (122)–(123). As a consequence, the $0_S$-reduction introduced in Def. 11.3 can be regarded as a particular case of Theorem VI.1.

A. Reduction of Resonant Subalgebras, WW conditions and the IW contraction

Theorem VI.1 above will be useful in order to recover Theorem 3 from Ref. [1] in this context.

Consider the resonant subalgebra from sec. IV B 2 and the following $S_p$ partition, which satisfies (122):

$$\hat{S}_p = \{\lambda_{\alpha_p}, \text{such that } \alpha_p = p, \ldots, N_p\}, \quad (128)$$

$$\hat{S}_p = \{\lambda_{\alpha_p}, \text{such that } \alpha_p = N_p + 1, \ldots, N + 1\}. \quad (129)$$
FIG. 7: (a) $S^{(4)}_E$ resonant subalgebra when $\mathfrak{g} = V_0 \oplus V_1 \oplus V_2 \oplus V_3$ satisfies the Weimar-Woods conditions. (b) One possible reduction of the resonant subalgebra from (a), with $N_0 = 2$, $N_1 = 2$, $N_2 = 3$, $N_3 = 4$. (c) Generalized İnönü–Wigner contraction, corresponding to a different reduction of the same resonant subalgebra, with $N_p = p$, $p = 0, 1, 2, 3$.

In Appendix A it is shown that the reduction condition (123) on (128)–(129) is equivalent to the following requirement on the $N_p$’s:

$$N_{p+1} = \begin{cases} N_p & \text{or} \\ H_{N_p+1}(N_p + 1) \end{cases}.$$ (130)

This condition is exactly the one obtained in Theorem 3 of Ref. [1], requiring that the expansion in the Maurer–Cartan forms closes. In the $S$-expansion context the case $N_{p+1} = N_p = N+1$ for each $p$ corresponds to the resonant subalgebra, and the case $N_{p+1} = N_p = N$ to its $0_S$-reduction. Fig. 7 shows two different reductions for a resonant subalgebra where $\mathfrak{g}$ satisfies the Weimar-Woods conditions.

As stated in Ref. [1], the generalized İnönü–Wigner contraction corresponds to the case $N_p = p$; this means that the generalized İnönü–Wigner contraction does not correspond to a resonant subalgebra but to its reduction. This is an important point, because, as we shall see in sec. VII we have been able to define non-trace invariant tensors for resonant subalgebras and $0_S$-reduced algebras, but not for general reduced algebras.

As an explicit example of the application of Theorem VI.1, the $d = 11$ five-brane Superalgebra is derived as a reduced algebra in the next section.
FIG. 8: (a) Resonant subalgebra of the $S$-expanded algebra $S_E^{(2)} \times \mathfrak{osp}(32|1)$. (b) One particular reduction of this resonant subalgebra reproduces the five-brane Superalgebra.

**B. Five-brane Superalgebra as a Reduced Algebra**

Let us recall the resonant subalgebra used in order to get the M algebra in sec. VA. For this case, the resonant partition $S_E^{(2)} = S_0 \cup S_1 \cup S_2$ corresponds to the one from eq. (70) for the case $N = 2$, i.e.,

\[
S_0 = \{\lambda_0, \lambda_2, \lambda_3\}, \quad (131)
\]
\[
S_1 = \{\lambda_1, \lambda_3\}, \quad (132)
\]
\[
S_2 = \{\lambda_2, \lambda_3\}. \quad (133)
\]

In order to construct a reduced algebra, perform a partition of the sets $S_p$ themselves, $S_p = \hat{S}_p \cup \hat{S}_p$, such as

\[
\hat{S}_0 = \{\lambda_0\}, \quad \hat{S}_0 = \{\lambda_2, \lambda_3\}, \quad (134)
\]
\[
\hat{S}_1 = \{\lambda_1\}, \quad \hat{S}_1 = \{\lambda_3\}, \quad (135)
\]
\[
\hat{S}_2 = \{\lambda_2\}, \quad \hat{S}_2 = \{\lambda_3\}. \quad (136)
\]

It is not hard to see that this partition of $S_p$ satisfies the reduction conditions (122)–(123).
For each \( p, \hat{S}_p \cap \hat{\bar{S}}_p = \emptyset \), and using the multiplication law (26), we have

\[
\hat{S}_0 \cdot \hat{S}_0 \subset \hat{S}_0, \\
\hat{S}_0 \cdot \hat{S}_1 \subset \hat{S}_1, \\
\hat{S}_0 \cdot \hat{S}_2 \subset \hat{S}_2, \\
\hat{S}_1 \cdot \hat{S}_1 \subset \hat{S}_0 \cap \hat{S}_2, \\
\hat{S}_1 \cdot \hat{S}_2 \subset \hat{S}_1, \\
\hat{S}_2 \cdot \hat{S}_2 \subset \hat{S}_0 \cap \hat{S}_2,
\]

[Compare with equations (71)–(76) and (64)–(69)]. Therefore, we have \( \hat{\mathcal{G}}_\text{R} = (\hat{S}_0 \times V_0) \oplus (\hat{S}_1 \times V_1) \oplus (\hat{S}_2 \times V_2) \), which is represented in Fig. 8 and the explicit reduced algebra \( |\hat{\mathcal{G}}_\text{R}| \)

\[
[P_a, P_b] = 0, \quad (137)
\]

\[
[J^{ab}, P_c] = \delta^{ab}_{ec} P^e, \quad (138)
\]

\[
[J^{ab}, J^{cd}] = \delta^{ab}_{ecd} J^e, \quad (139)
\]

\[
[P_a, Z_{b_1\cdots b_5}] = 0, \quad (140)
\]

\[
[J^{ab}, Z_{c_1\cdots c_5}] = \frac{1}{4!} \delta^{abc_1\cdots c_5}_{d_1\cdots d_4} Z^{d_5}_{c_1\cdots c_4}, \quad (141)
\]

\[
[Z^{a_1\cdots a_5}, Z_{b_1\cdots b_5}] = 0, \quad (142)
\]

\[
[P_a, Q] = 0, \quad (143)
\]

\[
[J_{ab}, Q] = -\frac{1}{2} \Gamma_{ab} Q, \quad (144)
\]

\[
[Z_{abcd_e}, Q] = 0, \quad (145)
\]

\[
\{Q^\rho, Q^\sigma\} = -\frac{1}{2^4} \left[ (\Gamma^a C^{-1})^{\rho a} P_a + \frac{1}{5!} (\Gamma^{abcde} C^{-1})^{\rho a} Z_{abcde} \right]
\]

This is the five-brane Superalgebra \[18, 19\].

VII. INVARIANT TENSORS FOR S-EXPANDED ALGEBRAS

Finding all invariant tensors for an arbitrary algebra remains as an open problem until now. This is not only an important mathematical problem, but also a physical one, because an invariant tensor is a key ingredient in the construction of Chern–Simons and Transgression
forms (see, e.g., Refs. [22, 23, 24, 25, 26, 27, 28, 29, 30, 31]), which can be used as gauge Lagrangians for a given symmetry group in an arbitrary odd dimension. The choice of invariant tensor shapes the theory to a great extent.

A standard procedure in order to obtain an invariant tensor is to use the (super)trace in some matrix representation of the generators of the algebra. However, this procedure has an important limitation for $0_S$-reduced algebras and, for this reason, theorems providing nontrivial invariant tensors for $S$-expanded algebras are worth considering.

**Theorem VII.1** Let $S$ be an abelian semigroup, $\mathfrak{g}$ a Lie (super)algebra of basis $\{T_A\}$, and let $\langle T_{A_1} \cdots T_{A_n} \rangle$ be an invariant tensor for $\mathfrak{g}$. Then, the expression

$$\langle T_{(A_1,\alpha_1)} \cdots T_{(A_n,\alpha_n)} \rangle = \alpha_\gamma K_{\alpha_1 \cdots \alpha_n}^{\gamma} \langle T_{A_1} \cdots T_{A_n} \rangle$$

where $\alpha, \gamma$ are arbitrary constants and $K_{\alpha_1 \cdots \alpha_n}^{\gamma}$ is the $n$-selector for $S$, corresponds to an invariant tensor for the $S$-expanded algebra $\mathfrak{g} = S \times \mathfrak{g}$.

**Proof.** The invariance condition for $\langle T_{A_1} \cdots T_{A_n} \rangle$ under $\mathfrak{g}$ reads

$$\sum_{p=1}^n X_{A_0 \cdots A_n}^{(p)} = 0,$$

where

$$X_{A_0 \cdots A_n}^{(p)} = (-1)^{q(A_0)q(A_1) + \cdots + q(A_{p-1})} C_{A_0 A_p}^B T_{A_1} \cdots T_{A_{p-1}} T_{B} T_{A_{p+1}} \cdots T_{A_n}. \quad (148)$$

Define now

$$X_{(A_0,\alpha_0) \cdots (A_n,\alpha_n)}^{(p)} = (-1)^{q(A_0,\alpha_0)q(A_1,\alpha_1) + \cdots + q(A_{p-1},\alpha_{p-1})} C_{(A_0,\alpha_0)(A_p,\alpha_p)}^{(B,\beta)} \times \langle T_{(A_1,\alpha_1)} \cdots T_{(A_{p-1},\alpha_{p-1})} T_{(B,\beta)} T_{(A_{p+1},\alpha_{p+1})} \cdots T_{(A_n,\alpha_n)} \rangle. \quad (149)$$

Using the fact that $q(A, \alpha) = q(A)$ and replacing the expressions (111) for the $S$-expansion structure constants and (146) for $\langle T_{(A_1,\alpha_1)} \cdots T_{(A_n,\alpha_n)} \rangle$, we get

$$X_{(A_0,\alpha_0) \cdots (A_n,\alpha_n)}^{(p)} = \alpha_\gamma K_{\alpha_0 \cdots \alpha_n}^{\gamma} X_{A_0 \cdots A_n}^{(p)}.$$

From (147) one readily concludes that

$$\sum_{p=1}^n X_{(A_0,\alpha_0) \cdots (A_n,\alpha_n)}^{(p)} = 0.$$
Therefore, \( \langle T_{(A_1, \alpha_1)} \cdots T_{(A_n, \alpha_n)} \rangle = \alpha_\gamma K_{\alpha_1 \cdots \alpha_n} \gamma \langle T_{A_1} \cdots T_{A_n} \rangle \) is an invariant tensor for \( \mathfrak{G} = S \times \mathfrak{g} \). □

It is worth to notice that, in general, the expression

\[
\langle T_{(A_1, \alpha_1)} \cdots T_{(A_n, \alpha_n)} \rangle = M \sum_{m=0}^{M} \alpha_{\beta_1}^{\cdots \beta_m} K_{\beta_1 \cdots \beta_m \alpha_1 \cdots \alpha_n} \gamma \langle T_{A_1} \cdots T_{A_n} \rangle ,
\]

where \( M \) is the number of elements of \( S \) and \( \alpha_{\beta_1}^{\cdots \beta_m} \) are arbitrary constants, is also an invariant tensor for \( \mathfrak{G} = S \times \mathfrak{g} \). An example of \((150)\) is provided by the supertrace. As a matter of fact, when the generators \( T_A \) are in some matrix representation, and the generators \( T_{(A, \alpha)} \) in the matrix representation \( T_{(A, \alpha)} = [\lambda_{\alpha}]^{\nu}_{\mu} T_A \), with \( [\lambda_{\alpha}]^{\nu}_{\mu} \) given in eq. (11), we have

\[
\text{STr} \left( T_{(A_1, \alpha_1)} \cdots T_{(A_n, \alpha_n)} \right) = K_{\gamma \alpha_1 \cdots \alpha_n} \gamma \text{Str} \left( T_{A_1} \cdots T_{A_n} \right) ,
\]

where \( \text{STr} \) is the (super)trace for the \( T_{(A, \alpha)} \) generators and \( \text{Str} \) the one for the \( T_A \) generators.

Even though the expression \((150)\) could be regarded as more general than the one from eq. (146), this is not the case. Using only the associativity and closure of the semigroup product, it is always possible to reduce eq. \((150)\) to eq. \((146)\), which in this way turns out to be more “fundamental.”

Given an invariant tensor for an algebra, its components valued on a subalgebra are by themselves an invariant tensor for the subalgebra (if they do not vanish). For the case of resonant subalgebras, and provided all the \( \alpha_\gamma \)’s are different from zero, the invariant tensor for the resonant subalgebra never vanishes. As matter of fact, given a resonant subset partition \( S = \bigcup_{p \in I} S_p \), and denoting the basis of \( V_p \) as \( \{ T_{a_p} \} \), the \( \mathfrak{G}_R \)-valued components of \((146)\) are given by

\[
\langle T_{(a_{p_1}, \alpha_{p_1})} \cdots T_{(a_{p_n}, \alpha_{p_n})} \rangle = \alpha_{p_1} K_{a_{p_1} \cdots a_{p_n}} \gamma \langle T_{a_{p_1}} \cdots T_{a_{p_n}} \rangle , \quad \text{with } \lambda_{a_p} \in S_p
\]

These components form an invariant tensor for the resonant subalgebra \( \mathfrak{G}_R = \bigoplus_{p \in I} S_p \times V_p \). Since \( S \) is closed under the product \((11)\), for every choice of indices \( \alpha_{p_1}, \ldots, \alpha_{p_n} \) there always exists a value of \( \gamma \) such that \( K_{a_{p_1} \cdots a_{p_n}} \gamma = 1 \), and therefore \((152)\) does not vanish (provided that \( \forall \gamma, \alpha_{\gamma} \neq 0 \)).

However, an interesting nontrivial point is that a 0\(_S\)-reduced algebra is not a subalgebra, and therefore, in general the 0\(_S\)-reduced algebra-valued components of expressions \((146)\) or \((152)\) do not lead to an invariant tensor. The following theorem offers a solution by providing a general expression for an invariant tensor for a 0\(_S\)-reduced algebra.
Theorem VII.2 Let $S$ be an abelian semigroup with nonzero elements $\lambda_i$, $i = 0, \ldots, N$, and $\lambda_{N+1} = 0_S$. Let $g$ be a Lie (super)algebra of basis $\{T_A\}$, and let $\langle T_{A_1} \cdots T_{A_n} \rangle$ be an invariant tensor for $g$. The expression
\begin{equation}
\langle T_{(A_1,i_1)} \cdots T_{(A_n,i_n)} \rangle = \alpha_j K_{i_1 \cdots i_n}^j \langle T_{A_1} \cdots T_{A_n} \rangle
\end{equation}
where $\alpha_j$ are arbitrary constants, corresponds to an invariant tensor for the $0_S$-reduced algebra obtained from $\mathcal{G} = S \times g$.

Proof. This theorem is actually a corollary of Theorem VII.1; imposing $\alpha_{N+1} = 0$ in Theorem VII.1 and writing the $i_0 \cdots i_n$ components of eq. (149) one gets
\begin{equation}
\sum_{p=1}^n X_{(A_0,i_0)\cdots(A_n,i_n)}^{(p)} = 0.
\end{equation}

Using the expressions for the $S$-expansion structure constants (11) and for the invariant tensor (153), one finds
\begin{equation}
X_{(A_0,i_0)\cdots(A_n,i_n)}^{(p)} = (-1)^{q(A_0,i_0)+q(A_1,i_1)+\cdots+q(A_{p-1},i_{p-1})} \times
K_{i_0 i_p} K_{i_1 \cdots i_{p-1} 1}^B \frac{C_{A_0 A_p}^B}{A_{i_0}} \alpha_j K_{i_1 \cdots i_{p-1} 1}^j \langle T_{A_1} \cdots T_{p-1} T_{p} T_{p+1} \cdots T_{A_n} \rangle +
+ K_{i_0 i_p} K_{i_1 \cdots i_{p-1} (N+1)}^B \frac{C_{A_0 A_p}^B}{A_{i_0}} \alpha_j K_{i_1 \cdots i_{p-1} (N+1) 1}^j \langle T_{A_1} \cdots T_{p-1} T_{B} T_{p+1} \cdots T_{A_n} \rangle.
\end{equation}

Since
\begin{equation}
\lambda_{i_1} \cdots \lambda_{i_{p-1}} \lambda_{N+1} \lambda_{i_{p+1}} \cdots \lambda_{i_n} = \lambda_{N+1},
\end{equation}
we have
\begin{equation}
K_{i_1 \cdots i_{p-1} (N+1) i_{p+1} \cdots i_n}^j = 0,
\end{equation}
and then,
\begin{equation}
X_{(A_0,i_0)\cdots(A_n,i_n)}^{(p)} = (-1)^{q(A_0,i_0)+q(A_1,i_1)+\cdots+q(A_{p-1},i_{p-1})} K_{i_0 i_p} K_{i_1 \cdots i_{p-1} 1}^B \frac{C_{A_0 A_p}^B}{A_{i_0}} \times
\alpha_j K_{i_1 \cdots i_{p-1} 1}^j \langle T_{A_1} \cdots T_{p-1} T_{B} T_{p+1} \cdots T_{A_n} \rangle.
\end{equation}

But $K_{ij} K_{AB}^C$ are the structure constants [see eq. (22)] of the $0_S$-reduced algebra of $S \times g$, and therefore, from eq. (154) we find that (153) provides an invariant tensor for it. \blacksquare

For the $0_S$-reduction of a resonant subalgebra, the proof is analogous to the one given above, and we have that
\begin{equation}
\langle T_{(a_1,i_1)} \cdots T_{(a_n,i_n)} \rangle = \alpha_j K_{i_1 \cdots i_n}^j \langle T_{i_1} \cdots T_{i_n} \rangle, \text{ such that } \lambda_{i_p} \in S_p,
\end{equation}
Fig. 34
is an invariant tensor for the $0_S$-reduced algebra of $\mathfrak{g}_R = \sum_{p \in I} S_p \times V_p$.

The usefulness of this theorem comes from the fact that, in general, the (super)trace in the adjoint representation for $0_S$-reduced algebras can give only a very small number of components of (153).

As a matter of fact, using the adjoint representation given by the $0_S$-reduced structure constants in eq. (22), one finds

$$\text{STr} \left( T_{(A_1,i_1)} \cdots T_{(A_n,i_n)} \right) = K_{j_1j_2}^{i_2} \cdots K_{j_{n-1}j_n}^{i_{n-1}i_n} \text{Str} \left( T_{A_1} \cdots T_{A_n} \right),$$

and since $\lambda_i \lambda_j = \lambda_{k(i,j)}$ implies that $\lambda_i, \lambda_j \neq 0_S$, one ends up with

$$\text{STr} \left( T_{(A_1,i_1)} \cdots T_{(A_n,i_n)} \right) = K_{j_1 \cdots j_n}^{i_1 \cdots i_n} \text{Str} \left( T_{A_1} \cdots T_{A_n} \right).$$

(157)

In general, this expression has less components than eq. (153). In order to see this, it is useful to analyze the case when there is also an identity element in the semigroup, $\lambda_0 = e$, and each $\lambda_i$ appears only once in each row and each column of the semigroup's multiplication table (i.e., for each $\lambda_i, \lambda_j \neq e$, we have $\lambda_i \lambda_j \neq \lambda_i$ and $\lambda_i \lambda_j \neq \lambda_j$). In this case, $K_{j_1 \cdots j_n}^{i_1 \cdots i_n} = K_{i_1 \cdots i_n}^0$, and the only non-vanishing component of the (super)trace is

$$\text{STr} \left( T_{(A_1,i_1)} \cdots T_{(A_n,i_n)} \right) = K_{i_1 \cdots i_n}^0 \text{Str} \left( T_{A_1} \cdots T_{A_n} \right),$$

(158)

which is clearly smaller than (153). In the expansion case $S = S_E$, we have $K_{i_1 \cdots i_n}^0 = \delta^0_{H_{N+1}(i_1 + \cdots + i_n)} = \delta^0_{i_1 + \cdots + i_n}$ and therefore, the only non-vanishing component of the (super)trace is

$$\text{STr} \left( T_{(A_1,0)} \cdots T_{(A_n,0)} \right) = \text{Str} \left( T_{A_1} \cdots T_{A_n} \right).$$

(159)

The advantage of the invariant tensor (153) as opposed to the (super)trace is now clear; in the case $S_E \times \mathfrak{g}$, the (super)trace only gives a trivial repetition of the invariant tensor of $\mathfrak{g}$, and for a resonant subalgebra, just a piece of it.

One last remark on the invariant tensor (153) is that for the particular case $S = S_E$, since $K_{i_1 \cdots i_n}^j = \delta^j_{H_{N+1}(i_1 + \cdots + i_n)} = \delta^j_{i_1 + \cdots + i_n}$, a topological density or a Chern–Simons form constructed using the invariant tensor (153) coincides with the one from Ref. [1] for the choice $\alpha_\gamma = \lambda^\gamma$, where $\lambda^\gamma$ stands for a power of the expansion parameter of the free differential algebra.
VIII. CONCLUSIONS

We have discussed how one can obtain a bunch of Lie algebras starting from an original one by choosing an abelian semigroup and applying the general theorems IV.2 and VI.1 which give us “resonant subalgebras” and what has been dubbed “reduced algebras.” This procedure is a natural outgrowth of the method of Maurer–Cartan forms power-series expansion presented in Ref. [1], from the point of view of the Lie algebra generators and using an arbitrary abelian semigroup. The $S$-expansion presented here has the feature of being very simple and direct; given a semigroup, one needs only solve the resonance condition (34) in order to get a resonant subalgebra, and the very similar reduction conditions (122)–(123) in order to get a reduced one. These have been solved in several examples in order to show how both theorems work, for general algebra structures as well as for very explicit cases, e.g., $d = 11$ superalgebras. As expected, the $S$-expansion scheme reproduces exactly the results of the Maurer–Cartan forms power-series expansion for a particular choice of semigroup, but it is also possible to get interesting new results using other alternatives, as shown in sec. VC.

The examples of the $S$-expansion procedure have been chosen according to their relevance for the long-term goal of understanding the geometric formulation of 11-dimensional Supergravity. To be able to write a Lagrangian invariant under these symmetries, a key ingredient is an invariant tensor. The theorems given in sec. VII help fill the gap, since they go a long way beyond the simple, and sometimes trivial, invariant tensors obtained from the supertrace. Chern–Simons and Transgression forms appear as a straightforward choice for the construction of a Supergravity Lagrangian in this context [22, 23, 24, 25, 26, 27, 28, 29, 30, 31]. In this sense, theorems IV.2, VI.1, VII.1 and VII.2 provide a very practical “physicist’s toolbox.” These have been used to construct a Lagrangian for the M algebra in 11 dimensions [14] following the techniques developed in Refs. [28, 31].

There are several ways in which this work can be extended. One of them concerns the investigation of the specific properties of the algebras generated from different choices of abelian semigroups; some kind of general classification would be particularly interesting. A first step in this direction would be the construction of the above-mentioned Lagrangians, but of course there are a lot of different possibilities to proceed. A different, and perhaps fruitful path deals with the generalization of the $S$-expansion procedure itself. The conditions of
discreteness and finiteness for the semigroup have been chosen primarily for simplicity, but it seems as though they could be removed in a generalized setting. The abelianity condition, on the other hand, is essential for all of our results to hold, and it is not clear whether it could be relaxed. Removing this requirement, a set with both commuting and anticommuting elements could be considered. If this possibility turns out to be feasible (which is far from trivial; think of the Jacobi identity), it would provide a way to derive superalgebras from ordinary Lie algebras and viceversa.

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APPENDIX A: REDUCTION WHEN $\mathfrak{g}$ SATISFIES THE WEIMAR-WOODS CONDITIONS

In sec. 1V B 2 it was shown that, when $\mathfrak{g}$ satisfies the Weimar-Woods conditions, the partition $S^{(N)}_E = \bigcup_{p=0}^{N} S_p$ with $S_p = \{\lambda_{\alpha_p}, \text{ such that } \alpha_p = p, \ldots, N + 1\}$ is a resonant one. In this appendix we prove that, when each subset $S_p$ is split as $S_p = \tilde{S}_p \cup \hat{S}_p$, with

$$\tilde{S}_p = \{\lambda_{\alpha_p}, \text{ such that } \alpha_p = p, \ldots, N_p\}, \quad (A1)$$

$$\hat{S}_p = \{\lambda_{\alpha_p}, \text{ such that } \alpha_p = N_p + 1, \ldots, N + 1\}, \quad (A2)$$

then the reduction condition (123) from Theorem VI.1 is satisfied when

$$N_{p+1} = \begin{cases} N_p & \text{or} \\ H_{N+1} (N_p + 1) & \end{cases} \quad (A3)$$
Before proceeding, it is worth to notice that the partition \((A1)\text{–}(A2)\) automatically satisfies the following three properties:

\[
N_p \geq N_q \iff \hat{\mathcal{S}}_p \subset \hat{\mathcal{S}}_q, \quad (A4)
\]
\[
\hat{\mathcal{S}}_0 \cdot \hat{\mathcal{S}}_q = \hat{\mathcal{S}}_q, \quad (A5)
\]
\[
\hat{\mathcal{S}}_p \cdot \hat{\mathcal{S}}_q \subset \hat{\mathcal{S}}_x \text{ such that } N_x \leq H_{N+1} \left( N_q + p \right). \quad (A6)
\]

Since \(g\) is assumed to satisfy the Weimar-Woods conditions \([15, 16]\), condition \([123]\) now reads

\[
\hat{\mathcal{S}}_p \cdot \hat{\mathcal{S}}_q \subset \bigcap_{r=0}^{H_n(p+q)} \hat{\mathcal{S}}_r, \quad (A7)
\]

where \(\hat{\mathcal{S}}_p\) and \(\hat{\mathcal{S}}_q\) are given by \((A1)\text{–}(A2)\).

Let us analyze this condition for the particular case \(p = 0\):

\[
\hat{\mathcal{S}}_0 \cdot \hat{\mathcal{S}}_q \subset \bigcap_{r=0}^{q} \hat{\mathcal{S}}_r. \quad (A8)
\]

Using eq. \((A5)\), this turns out to be equivalent to

\[
\hat{\mathcal{S}}_q \subset \bigcap_{r=0}^{q} \hat{\mathcal{S}}_r. \quad (A9)
\]

In this way, we have that for each \(0 \leq r \leq q\), \(\hat{\mathcal{S}}_q \subset \hat{\mathcal{S}}_r\), and using eq. \((A4)\), we get the equivalent condition

\[
\forall r \leq q, N_r \leq N_q. \quad (A10)
\]

This condition and the product \((26)\) now imply that

\[
\bigcap_{r=0}^{H_n(p+q)} \hat{\mathcal{S}}_r = \hat{\mathcal{S}}_{H_n(p+q)}. \]

Using this fact, the condition \((A7)\) takes the form

\[
\hat{\mathcal{S}}_p \cdot \hat{\mathcal{S}}_q \subset \hat{\mathcal{S}}_{H_n(p+q)}. \quad (A11)
\]

Using eq. \((A6)\), one finds that, in order to satisfy this requirement, it is enough to impose that, for each \(\hat{\mathcal{S}}_x\) such that \(N_x \leq H_{N+1} \left( N_q + p \right)\), one has \(\hat{\mathcal{S}}_x \subset \hat{\mathcal{S}}_{H_n(p+q)}\). Alternatively [cf. eq. \((A4)\)], one can write

\[
\forall N_x \leq H_{N+1} \left( N_q + p \right), \quad N_{H_n(p+q)} \leq N_x \quad (A12)
\]
and therefore,
\[ N_{H_n(p+q)} \leq H_{N+1}(N_q + p). \] (A13)

For \( p = 1 \), we have
\[ N_{H_n(p+1)} \leq H_{N+1}(N_q + 1), \]
and using eq. (A10), one finds the inequalities
\[ N_q \leq N_{H_n(q+1)} \leq H_{N+1}(N_q + 1), \]
whose solution is
\[ N_{q+1} = \begin{cases} N_q \text{ or} \\ H_{N+1}(N_q + 1). \end{cases} \]

This solves condition (A7).

Therefore, we have that
\[ |\tilde{\mathcal{O}}_R| = \bigoplus_{p=0}^{n} \tilde{S}_p \times V_p, \] (A14)
with \( \tilde{S}_p = \{ \lambda_{\alpha_p}, \text{such that } \alpha_p = p, \ldots, N_p \} \) and
\[ N_{p+1} = \begin{cases} N_p \text{ or} \\ H_{N+1}(N_p + 1). \end{cases} \] (A15)

is a reduced Lie algebra with structure constants
\[ C_{(a_p,\alpha_p)(b_q,\beta_q)}^{(c_r,\gamma_r)} = K_{\alpha_p\beta_q}^{\gamma_r} C_{a_p b_q}^{c_r}, \text{ with } \alpha_p, \beta_p, \gamma_p = p, \ldots, N_p. \] (A16)

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[32] There does not seem to be a unique, universally-accepted definition of semigroup. Here it is taken to be a set provided with a closed associative product. It does not need to have an identity.
[33] Here \([x]\) denotes the integer part of \(x\).

[34] \textit{Note added:} The algebra here considered and the original ones from D'Auria-Fré correspond to different members of a family of superalgebras introduced in Ref. [8].