The effect of a radiation-like solid on CMB anisotropies

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Abstract

We compute the power in the lowest multipoles of CMB anisotropies in the presence of a radiation-like solid, a hypothetical new kind of radiation with nonzero shear modulus. If only the ordinary Sachs–Wolfe effect is taken into account, the shear modulus to energy density ratio must be in an absolute value of order $10^{-5}$ or less for the theory to be consistent with observations within cosmic variance. With the integrated Sachs–Wolfe effect switched on, the constraint is relaxed by almost two orders of magnitude.

Keywords: CMB anisotropies, shear modulus, Sachs–Wolfe effect

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1. Introduction

The observed acceleration of the universe is usually interpreted as an effect of dark energy, originating in null oscillations of quantum fields. An alternative explanation is that what accelerates the universe is a solid with negative pressure to energy density ratio $w$. This idea appeared shortly after the effect was discovered [1, 2], and the underlying theory has since been extensively studied [3–12].

Given the attention paid to scenarios in which the solid has negative $w$, it is natural to ask what difference would it make if the value of $w$ of the solid was positive. This question was addressed in [13]. One of the two scenarios explored there was that in a universe filled with radiation there appeared a radiation-like solid, which had, in addition to $w = 1/3$, a constant shear modulus to energy density ratio $\xi$. A possible materialization of such solid could be a Coulomb crystal with a relativistic Fermi gas of moving charges; another, more speculative, possibility could be a network of ‘spring-like’ strings with energy inversely proportional to length. In the latter case $\xi$ would be negative, making the vector perturbations unstable. This, however, would not necessarily make the theory useless, since in the course of inflation typically no vector perturbations are created.
In order to obtain an observable effect of a solid on superhorizon perturbations one must assume that the solid was created with flat internal geometry and nonzero shear stress acting in it. Such solidification cannot take place in pure radiation, whose particles move freely, and hence the concept of shear deformation has no sense for them. It must be linked to some other kind of particle present in the universe, distributed anisotropically from the start of Friedmann expansion. Such a component of the cosmic medium could perhaps be formed in solid inflation [9–12].

In this paper, we investigate how the radiation-like solid, if present in our universe, would manifest itself in the large-angle CMB anisotropies. In section 2 we describe how the superhorizon perturbations evolve in the presence of a solid and determine their magnitude at the moment of recombination; in section 3 we derive a formula for temperature fluctuations observed at present on Earth and calculate the CMB power spectrum for the lowest multipole moments; and in section 4 we discuss the results. A system of units is used in which \( \pi = c = G = 1 \).

2. Evolution of perturbations

2.1. Pure solid

Consider a flat Friedmann–Robertson–Walker–Lemaître universe filled with an isotropic elastic matter and add a small perturbation of the spacetime metric and distribution of matter to it. Following [14], we will use the proper-time comoving gauge for the description of perturbations; thus, we will assume that \( \phi \) (the correction to the \((00)\)-component of the metric tensor) as well as \( \delta x \) (the shift vector of the matter) is zero. Denote the scale parameter by \( a \), the mass density and pressure of the matter by \( \rho \) and \( p \), the compressional modulus of the matter by \( K \) and the shear modulus of the matter by \( \mu \). The condition \( \phi = 0 \) means that the scalar part of the metric is

\[
\delta \eta \delta x = - \delta \eta \delta x,
\]

and from the condition \( \delta x = 0 \) it follows that the scalar part of the energy–momentum tensor is

\[
T^0_0 = \rho + \rho_\phi (3\psi + \mathcal{E}), \quad T^0_i^{(S)} = \rho_\phi B_{,i}, \quad T_i^{(S)} = -p\delta_{ij} - K (3\psi + \mathcal{E}) \delta_{ij} - 2\mu E_{,ij}^T,
\]

where \( \rho_\phi = \rho + p \), \( \mathcal{E} = \Delta E \) and \( E_{,ij}^T \) is the traceless part of the tensor \( E_{,ij} \), \( E_{,ij} = E_{,ij} - \Delta E \delta_{ij}/3 \). In the formulas for \( T^0_0 \) and \( T_i^{(S)} \) we have assumed that the perturbations are adiabatic, that is, the entropy per particle \( S \) is constant throughout the space.

The proper-time comoving gauge allows for a residual transformation \( \delta \eta = \alpha^{-1} \delta t (x) \), where \( \delta t (x) \) is the local shift of the moment at which the time count has started. The function \( \mathcal{E} \) is invariant under such a transformation, and the functions \( B \) and \( \psi \) can be written as

\[
B = B + \chi, \quad \psi = -\mathcal{H} \chi,
\]

where \( B \) is invariant and \( \chi \) transforms as \( \chi \rightarrow \chi + \delta \eta \).

Suppose the solid has flat internal geometry and consider a perturbation of the form of a plane wave with the comoving wave vector \( k \). For \( B \) and \( \mathcal{E} \) as functions of \( \eta \), we have two coupled first-order differential equations coming from equations \( T^\mu_{,\mu} = 0 \) and \( 2G_{00} = T_{00} \),

\[
B' = \left( 3c^2_{50} + \alpha - 1 \right) \mathcal{H} B + c^2_{50} \mathcal{E}, \quad \mathcal{E}' = - \left( k^2 + 3\alpha \mathcal{H}^2 \right) B - \alpha \mathcal{H} \mathcal{E},
\]

where the prime denotes differentiation with respect to \( \eta \), \( \mathcal{H} \) is the rate of expansion, \( \mathcal{H} = a'/a \), and the functions \( \alpha \), \( c^2_{50} \) (auxiliary sound speed squared) and \( c^2_{51} \) (longitudinal...
sound speed squared) are defined as \( \alpha = (3/2)\rho_x/\rho, \)
\( c_{s0}^2 = K/\rho_x \)
and \( c_{\|}^2 = c_{s0}^2 + (4/3)\mu/\rho_x. \)
These equations must be supplemented by the dynamical equations for an unperturbed universe,
\[ a' = \left( \frac{1}{6} \rho a^4 \right)^{1/2}, \quad \rho' = -3\mathcal{H} \rho, \]  
(5)
and an expression for \( K, \)
\[ K = \rho_x \left( \frac{d\rho}{d\rho} \right)_S. \]  
(6)
If the matter has more than one component, it holds that \( \rho = \sum \rho_i \)
and \( K = \sum K_i, \)
and the second equation (5) as well as equation (6) hold for each component separately.
Consider a one-component universe in which the parameters \( \rho \text{ and } \mu \) are constant. By combining the two equation (5) we obtain that \( a \) is a power-like function of \( \eta, \)
\[ a \propto \eta^{2(1+\xi)}, \]
and from equation (6) we find that the parameter \( K = K/\rho \) is constant, too,
\( K = \mu \eta \) \( w_3 \) with \( w_3 = w + 1 \). Equations for \( B \) and \( E \) then combine into an equation for \( B \)
whose solutions are Bessel functions multiplied by a certain power of \( \eta \). In case \( w = 1/3 \), the equations for \( B \) and \( E \) are
\[ B' = 2\eta^{-1} B + c_{\|}^2 E, \quad E' = -\left( k^2 + 6\eta^{-2} \right) B - 2\eta^{-1} E, \]  
(7)
where \( c_{\|}^2 = 1/3 + \xi \); after excluding \( E \), we obtain an equation for \( B, \)
\[ B'' + \left( c_{\|}^2 k^2 + 6\xi \eta^{-2} \right) B = 0, \]  
(8)
with the solution
\[ B = \sqrt{\xi} \left( a_j J_n + a_Y Y_n \right), \]  
(9)
where \( z = c_{\|}^2 kn, n = \sqrt{1/4 - 6\xi}, \) \( J_n \) and \( Y_n \) are Bessel functions of the first and second kind of the argument \( z \).
When describing CMB anisotropies it is convenient to pass to the Newtonian gauge, in which the functions \( B \) and \( E \) are traded for the Newtonian potential \( \phi \) and the scalar part of the shift vector \( \delta x \). Denote the potentials \( \phi \) and \( \psi \) and the perturbation to the mass density \( \delta \rho \) in the Newtonian gauge by \( \Phi, \Psi \) and \( \Delta \). For the functions \( \Psi \) and \( \delta \rho \) we can use expressions coming from the gauge transformation (equations (7.19) and (7.20) in [15]) and for the function \( \Delta \Phi = \Phi - \Psi \) we have an expression following from Einstein equations (equation (7.40) in [15]),
\[ \Psi = \psi + \mathcal{H} (B - E), \quad \Delta \Phi = \frac{1}{2} a^2 \tau^{(2)}, \quad \delta \rho = \delta \rho - \rho' (B - E), \]  
where \( \tau^{(2)} \) is the longitudinal part of the perturbation to \( T_{ij}^{(5)} \) and \( \delta \rho \) is the perturbation to \( T_0^0 \).
By inserting here from equations (2) and (3) we find
\[ \Psi = \mathcal{H} (B - E'), \quad \Delta \Phi = -\mu a^2 E, \quad \delta \rho = \rho_+ (3 \Psi + E), \]  
(10)
and after inserting into the expression for \( \Psi \) from the second equation in (4) and into the expression for \( \Delta \Phi \) from the first equation in (5), we arrive at
\[ \Psi = -k^2 a \mathcal{H}^2 (3HB + E), \quad \Delta \Phi = 6\xi k^2 \mathcal{H}^2 E. \]  
(11)
For \( w = 1/3 \), this yields
\[
\Psi = -2(k\eta)^{-2} \left( 3\eta^{-1}B + E \right), \quad \Delta \Phi = 6\xi (k\eta)^{-2}E, \tag{12}
\]
and by inserting for \( E \) from the first equation in (4), we can write \( \Phi \) and \( \Psi \) in terms of \( B \) only,
\[
\Psi \propto -z^{-2} \left[ \frac{dB}{dz} - (1 - 3\xi)z^{-1}B \right], \quad \Delta \Phi \propto 3\xi z^{-2} \left( \frac{dB}{dz} - 2z^{-1}B \right). \tag{13}
\]

With \( B \) in the form (9), these are expressions for the potentials \( \Phi \) and \( \Psi \) in terms of Bessel functions and their derivatives.

An important special case is superhorizon perturbations, whose wavelength exceeds significantly the size of the sound horizon. The radius of the sound horizon is \( r_s \sim c_{\text{SS}} \eta \), where \( c_s = a \lambda \) is the radius of particle horizon; thus, the variable \( z \) can be written as \( z \sim r/\lambda \), where \( \lambda = \lambda/(2\pi) = ak^{-1} \) is the reduced wavelength, and the condition on the size of the superhorizon perturbations can be written as \( z \ll 1 \). In this limit it holds that
\[
J_n \propto z^n \left( 1 - \frac{z^2}{4n_+} \right), \quad Y_n \propto -z^{-n},
\]
where \( n_+ = n + 1 \), and if we insert this into the expression for \( \Psi \), we obtain
\[
\Psi = A_j \left( \frac{p}{q} z^{-m} + z^{-2-m} \right) - A_y z^{-2-M}, \tag{14}
\]
where \( m = 1/2 - n, M = 1/2 + n \) and the constants \( p \) and \( q \) are defined as
\[
p = \frac{1}{4n_+} (2 - m + 3\xi), \quad q = m - 3\xi. \tag{15}
\]

For \( |\xi| \ll 1 \) it holds that \( p \ll 1/3 \) and \( q \ll 3\xi \), therefore \( |p/q| \gg 1 \) and the term proportional to \( z^{-m} \) can dominate the term proportional to \( z^{-2-m} \) while \( z \) is still small.

Consider now a universe filled with an ideal fluid and suppose that the fluid turned into a solid with the same \( w \) at some moment \( \eta_s \). As before, we pick \( w = 1/3 \). Suppose the solidification was anisotropic, producing a solid with flat internal geometry, and consider perturbations that were superhorizon at the moment \( \eta_s \); in other words, suppose \( z_s \ll 1 \). The first condition means that the equations (4) are valid without modifications for all \( \eta > \eta_s \), and the second condition implies that \( \Psi \) can be written in the form (14) in some interval of \( \eta > \eta_s \). In a fluid, the potentials \( \Phi \) and \( \Psi \) coincide and, if we restrict ourselves to their nondecaying part, they are constant for long-wave perturbations. Denote this constant by \( \Phi(0) \).

If \( |\xi| \) is not too small, \( |\xi| > z_s^2 \), the first term in the brackets in (14) can be neglected at the moment \( \eta_s \) and the matching conditions reduce to
\[
\tilde{A}_j - \tilde{A}_y = \Phi(0), \quad (2 + m)\tilde{A}_j - (2 + M)\tilde{A}_y = -\eta_s \left[ \Psi' \right], \tag{16}
\]
where \( \tilde{A}_j = A_j z_s^{-2-m}, \tilde{A}_y = A_y z_s^{-2-M} \) and the square brackets denote the jump of the function at \( \eta = \eta_s \). To determine \( \left[ \Psi' \right] \), we observe that the density contrast \( \delta = \delta \rho/\rho \) equals \(-2\Phi(0)\) for long-wave perturbations in a universe filled with fluid [15]. After rewriting the last equation in (10) as \( \delta = w_s (3\Phi(0) + E) \) and putting \( w = 1/3 \), we find \( E = -(9/2)\Phi(0) \) and
\[
\eta_s \left[ \Psi' \right] = -6(k\eta_s)^{-2} \left[ B' \right] = -6c_s^2 z_s^{-2} \xi E_s = 27c_s^2 z_s^{-2} \xi E_s. \tag{17}
\]

Our assumption about the values of \( \xi \) guarantees that this is in absolute value much greater than \(|\Phi(0)|\), so that we can neglect the right hand side of the first equation in (16). The solution
is

\[ \dot{A}_J = \dot{A}_Y = r \xi^{-2} \Phi^{(0)}, \quad r = \frac{27}{2n} \xi^{\frac{2}{n}} \xi, \]  

and after inserting this into (14) and introducing a rescaled time \( \tau \) normalized to 1 at the moment \( \eta_s, \eta = \eta / \eta_s = z / z_s \), we obtain

\[ \Psi = P \eta^{-m} + Q \xi^{-2} \left( \eta^{-2-m} - \eta^{-2-M} \right), \]

where \( P = (p/q) r \Phi^{(0)} \) and \( Q = r \Phi^{(0)} \).

2.2. Adding the solid to the rest of matter

Consider a universe filled with nonrelativistic matter (dust) with density \( \rho_m \) and pressure \( p_m = 0 \) and radiation with density \( \rho_r \) and pressure \( \rho_r = \rho_r / 3 \), and suppose that the radiation consists of ordinary radiation (photons and neutrinos) with the density \( \rho_{r_{\text{I}}} \) and zero shear modulus, and of a radiation-like solid with the density \( \rho_{r_{\text{II}}} \) and shear modulus \( \mu \) proportional to \( \rho_{r_{\text{II}}} \). Both ordinary radiation and a radiation-like solid have the same dependence of \( p \) on \( \rho \) and hence of \( \rho \) on \( a \); thus, the densities \( \rho_{r_{\text{I}}} \) and \( \rho_{r_{\text{II}}} \) are at any given moment equal to the same fractions of \( \rho_r \) and \( \mu \) is proportional to \( \rho_r \), with constant \( \xi \).

The shear modulus of ordinary solids is much less than their mass/energy density. On the other hand, one would expect the shear modulus of the hypothetical radiation-like solid to be comparable with mass/energy density, as is the case with random lattices of cosmic strings or domain walls that have both \( \xi_{\text{net}} \) (the net parameter \( \xi \)) equal to 4/15 [16]. If a radiation-like solid has \( \xi \sim \xi_{\text{net}} \), the ratio of its density to the density of ordinary radiation would be of the order \( \xi^2 \), unless \( \xi \) was close to \( \xi_{\text{net}} \) and hence \( \xi^2 \) would be close to 1. However, as we will see, \( \xi^2 \) must be much less than 1 to accommodate observations; therefore, if a radiation-like solid with \( \xi_{\text{net}} \sim 1 \) was present in our universe, its density today would be a tiny fraction of the density of radiation.

For an unperturbed universe with radiation and matter, it makes no difference whether it does or does not contain a radiation-like solid. Let us write down the solution of equation (5) in this case. Denote \( \zeta = \eta / \eta_s \) with \( \eta_s = \eta_{\text{eq}} / (\sqrt{2} - 1) \), where the index 'eq' refers to the moment when the densities \( \rho_r \) and \( \rho_m \) are equal. From the second equation (5), written separately for radiation and matter, we have

\[ \rho_s \propto \alpha^{-4}, \quad \rho_m \propto \alpha^{-3}, \]

and from the first equation (5) with \( \rho = \rho_s + \rho_m \) on the right hand side we obtain

\[ \alpha = a_{\alpha} \zeta (2 + \zeta). \] (19)

Let us now turn to the perturbed universe. For \( B \) and \( E \), we have equation (4) with the derivatives with respect to \( \zeta \) instead of \( \eta \) and with the replacements \( B \to \beta = \eta_{\text{eq}}^{-1} B \), \( H \to \dot{H} = \eta H \) and \( k \to \kappa = k_{\text{eq}} \); and for \( \Phi \) and \( \Psi \) we have equation (11) with an additional replacement \( \xi \to \xi / \rho_{r_{\text{II}}} \). Denote \( \zeta_s = 1 + \zeta \), \( X = \zeta (2 + \zeta) \), \( X_s = 1 + X = \xi^2 \) and \( \dot{X}_s = 1 + (3/4) X \). From the formulas for \( \beta_s \) and \( \beta_m \) we have \( \beta_s / \beta_m = a_{\beta} / a = 1 / X \), so that \( \beta_s / p = 1 / X_s \), \( \beta_s / p = 1 / X_s \) and \( \beta_m / p = (4/3) \dot{X}_s / X_s \). The function \( \alpha \) equals \( (3/2) \beta_s / p \) as before and the functions \( c_{\beta S} \) and \( c_{\beta M} \) now equal \( (1/3) \beta_s / \beta_m \) (because \( K = K_s = \rho_{r_{\text{II}}} / 3 \)) and \( (1/3 + \xi) \beta_s / \beta_m \). Finally, for the function \( \dot{H} \) we have \( \dot{H} = a^{-1} \partial a / \partial \zeta \) with \( a \propto X \). In this way we find
\[ a = \frac{2\dot{X}_+}{X_+}, \quad c_\parallel^2 = \frac{1/3}{X_+}, \quad c_\perp^2 = \frac{1/3 + \xi}{X_+}, \quad \dot{H} = \frac{2\dot{X}_+}{X}, \]

and if we insert this into equations (4) and (11), we arrive at
\[ \frac{d\dot{B}}{d\zeta} = \left( \frac{1}{X_+} + \frac{1 + X/2}{X_+} \right) \frac{2\dot{X}_+}{X} \frac{2\dot{X}_+}{X} + \frac{1/3 + \xi}{X_+} \frac{d\dot{E}}{d\zeta} = -\left( k^2 + \frac{24\dot{X}_+}{X^2} \right) \frac{\dot{B}}{\zeta} - \frac{4\dot{X}_+}{\zeta X} \frac{\dot{E}}{\zeta}, \quad (20) \]

and
\[ \Psi = -8\kappa^{-1} \frac{\dot{X}_+}{X^2} \left( \frac{6\zeta_+}{X} \frac{\dot{B}}{\zeta} + \frac{\dot{E}}{\zeta} \right), \quad \Delta \Phi = \frac{24\zeta}{X^2} \kappa^{-2} \frac{\dot{E}}{\zeta}. \quad (21) \]

A straightforward way to compute the functions \( \Phi \) and \( \Psi \) would be by solving a second-order equation for \( \dot{B} \), similar as in the theory with one-component matter. However, if we are interested in small values of \( \xi \) and \( \kappa \) only, it is preferable to pass to first-order equations for \( \Psi \) and \( \dot{E} \). As we will see, the equations simplify if we factor a certain function of \( \zeta \) out of \( \Psi \).

By differentiating \( \Psi \) and using equations for \( \dot{B} \) and \( \dot{E} \) we find
\[ \frac{d\Psi}{d\zeta} = -\left( 1 + \frac{2\dot{X}_+}{X_+} \right) \frac{2\dot{X}_+}{X} \Psi - 8 \left( \frac{\zeta_+}{X^2} \frac{\dot{E}}{\zeta} - \frac{\dot{X}_+}{X} \frac{\dot{B}}{\zeta} \right), \quad (22) \]

where we have combined the parameters \( \xi \) and \( \kappa \) into a new parameter \( \gamma = 6\zeta \kappa^{-2} \). The coefficient in front of \( \Psi \) is the logarithmic derivative of the function \( \zeta_+ / X^3 \), hence if we define
\[ \Psi = \frac{\zeta_+ F}{X^3}, \quad (23) \]
we obtain
\[ \frac{dF}{d\zeta} = -8 \left( \gamma \frac{\dot{E}}{\zeta} - \frac{\dot{X}_+}{\zeta} \frac{\dot{B}}{\zeta} \right), \quad (24) \]

In addition to that we have
\[ \frac{d\dot{E}}{d\zeta} = \kappa^2 \left( \frac{F}{2X^2} - \frac{\dot{B}}{\zeta} \right), \quad \frac{\dot{B}}{\zeta} = -\frac{1}{6} \left( \kappa^2 \frac{F}{8X_+} + \frac{X}{\zeta_+} \frac{\dot{E}}{\zeta} \right). \quad (25) \]

After inserting for \( \dot{B} \) into the expressions for \( dF/d\zeta \) and \( d\dot{E}/d\zeta \) we arrive at the desired system of first-order equations. We could proceed further and exclude \( \dot{E} \) to obtain an equation of second order for \( F \), but we will not do that since the equation will be not needed in what follows.

2.3. Long-wave limit

Suppose the shear modulus of the radiation-like solid is a small fraction of the total energy density of radiation, \( |\xi| = |\mu|/\rho \ll 1 \), and consider a perturbation that is stretched far beyond the particle horizon at the moment \( \eta_\kappa, \kappa = k\eta_\kappa \ll 1 \). We will be mainly interested in the size of the perturbation at recombination. At that moment, the scale parameter is \( a_{\kappa} = a_{\kappa_0} (2 + \zeta_\kappa) = 3a_{\kappa_0} \), therefore, the value of \( \zeta \) is \( \zeta_\kappa = \eta_\kappa / \eta_\kappa \ll 1 \) and the condition \( \kappa \ll 1 \) means that the perturbation is stretched far beyond the particle horizon at recombination. Note that since the value of \( c_\parallel \) does not fall too much before recombination, such perturbations can be called ‘superhorizon’ in the previous sense ‘stretched far beyond the
sound horizon’. We will calculate the functions $F$ and $E$ for superhorizon perturbations in the leading order in $\xi$ as well as $\kappa$, keeping the parameter $\gamma$ arbitrary.

Suppose the radiation-like solid appeared in the universe at some moment $\eta_s$ deep in the radiation-dominated era, $\eta_s \ll \eta_{eq}$. For $\eta_s < \eta \ll \eta_{eq}$, our $\Psi$ must coincide within a good accuracy with $\Psi$ in the theory with one-component matter with $w = 1/3$, hence at $\eta \ll \eta_{eq}$ our $\Psi$ must be given by equation (18). By using $\tilde{\eta} = \zeta / \zeta_s$ and $\zeta_s = \tilde{\xi}_s \kappa_s$, where $\tilde{\xi}_s$ is the longitudinal sound speed of radiation combined with a radiation-like solid, $\tilde{\xi}_s = 1/3 + \xi$, we find

$$\Psi = \zeta^m \left[ P \zeta^{-m} + \kappa^{-2} \tilde{Q} \left( \zeta^{-2-m} - \xi^{-2-M} \right) \right],$$

where $\tilde{Q} = \tilde{\xi}_s^{-2} Q$. The expression for $\Psi$ in the leading order in $\xi$ is obtained by putting $n = 1/2$, $m = 0$ and $M = 1$ here as well as in the expressions for $p$ and $r$, and $m = 6\xi$ in the expression for $q$. The resulting formulas for $F$ and $\tilde{Q}$ in terms of $\Phi^{(0)}$, which can be interpreted now as the value of $\Phi$ at the beginning of Friedmann expansion. The expressions are $F = \Phi^{(0)}$ and $\tilde{Q} = 3Q = 27\xi\Phi^{(0)}$, so that

$$\Psi = \left[ 1 + \frac{9}{2} \gamma \left( \zeta^{-2} - \zeta_s^{-2} \right) \right] \Phi^{(0)}. \tag{27}$$

Our aim is to extend this expression to the moment of recombination.

Equations for $F$ and $E$ in the leading order in $\xi$ and $\kappa$ are

$$\frac{dF}{d\xi} = -8 \left( \gamma \frac{\xi}{X + X_s} \right) E, \quad E = \text{const.} \tag{28}$$

(The correction to $E$ is of order $\kappa^2$; therefore, even if $\gamma \gg 1$, its contribution to $dF/d\xi$ is of order $\xi$ and can be neglected.) For $\zeta \ll 1$ the second term in the brackets in the expression for $dF/d\xi$ equals $(2/3)\zeta^2$, so that

$$\Psi = \frac{1}{8} \zeta^{-3} F = -\zeta^{-3} \int \left( \gamma + \frac{2}{3} \zeta^2 \right) E d\xi = -\left( \gamma \zeta^{-2} + \frac{2}{9} \right) E + \text{const} \times \zeta^{-3},$$

and by comparing this with equation (27) we rediscover the formula for $E$ in a universe with pure radiation, used already in the matching conditions at the moment of solidification,

$$E = -\frac{9}{2} \Phi^{(0)}. \tag{29}$$

We can also see that the integration constant in $F$ must be chosen in such a way that $F = -8(\gamma (\zeta - \zeta_s) + O(\zeta^3))E$ for $\zeta \rightarrow 0$. By elementary integration we obtain

$$F = -8 \left[ \gamma (\zeta - \zeta_s) + \frac{1}{24} Z \right] E, \quad Z = \zeta^3 \left( \frac{3}{8} \zeta^2 + 3 \zeta + \frac{13}{3} + \frac{1}{\zeta^2} \right). \tag{30}$$

We can now insert for $F$ and $E$ into the expressions for $\Psi$ (equation (23)) and $\Delta \Phi$ (second equation in (21)) to arrive at the approximate formulas

$$\Psi = \frac{\zeta}{X} \left[ 36 \gamma (\zeta - \zeta_s) + \frac{3}{2} Z \right] \Phi^{(0)}, \quad \Delta \Phi = -\frac{18\gamma}{X^2} \Phi^{(0)}. \tag{31}$$

The second part of $\Psi$ is the Newtonian potential of superhorizon perturbations in a universe filled with ideal fluid,
This coincides with the nondecaying part of $\Phi_{id}$ given in equation (7.71) in [15]. The function $\Phi_{id}$ is constant both at $\zeta \ll 1$ (radiation-dominated era) and $\zeta \gg 1$ (matter-dominated era), and its value decreases during the transition between the eras by the factor $\omega_\infty = 9/10$. In approximate calculations, one identifies $\omega_\infty$ with the ratio of the values of $\Phi_{id}$ at recombination and at the beginning of Friedmann expansion [15]. However, the actual ratio is greater. Its value, obtained by inserting $\zeta = 1$ into the expression in front of $\Phi(0)$ in (32), is $\omega = 253/270 \approx 0.94$.

To compute the values of $\Phi$ and $\Psi$ at recombination, we must insert $\zeta = 1$ into the expressions for $\Phi$ and $\Delta \Phi$ in (31). We can neglect $\zeta_\gamma$ since we have assumed $\eta_\gamma \ll \eta_{eq}$, which implies $\eta_\gamma \ll \eta_\eta$ and $\zeta_\gamma = \eta_\gamma/\eta_{eq} \ll 1$. If we return from $\gamma$ back to $\xi \kappa - 2$, we have

$$\Phi_{id,e} = \left(4\xi\kappa^2 + \omega\right)\Phi^{(0)}, \quad \Psi_{id,e} = \left(16\xi\kappa^2 + \omega\right)\Phi^{(0)}.$$

The potentials $\Phi_{id,e}$ and $\Psi_{id,e}$ expressed in terms of the potential $\Phi_{id,te} = \omega\Phi^{(0)}$ are depicted as functions of $\kappa$ in figure 1. The value of the dimensionless shear modulus is $\xi = 1/600$ in the left panel and $\xi = -1/600$ in the right panel. (The values are the same as in figure 1 in [13], where the parameter $b = 6\xi$ was used instead of $\xi$.) When constructing the graphs, we have used a more accurate theory than the one explained in the text, with the initial Newtonian potential modified to $\hat{\Phi}^{(0)} = 1/(2m)(\xi_\gamma/2)^{m}\Phi^{(0)}$. Heavy curves are computed for $\xi_\gamma = 10^{-5}$, which corresponds to shear stress appearing on the scale of nucleosynthesis, and light curves are computed for $\xi_\gamma = 10^{-24}$, which corresponds to shear stress appearing on the grand unified theory (GUT) scale. The curves cross zero in the case $\xi < 0$, therefore we have combined the ordinary and logarithmic scales on the vertical axis in the right panel.

The expressions for $\Phi_{id,e}$ and $\Psi_{id,e}$ were derived under the condition $\kappa \ll 1$. In the figure, we have extrapolated the curves up to $\kappa = 0.5$, where the error is of order 25%. At such $\kappa$, $\Phi_{id,e}$ and $\Psi_{id,e}$ decrease in the exact theory due to the inhomogeneities being suppressed inside the horizon during the first quarter-period of acoustic oscillations. In our approximation, the functions are saturated close to $\kappa = 0.5$, where the right hand side of equations (33) is dominated by the second (constant) term. The first term is small because of small $|\xi|$. As $\kappa$ decreases, the first term becomes dominant and both functions start to rise or fall steeply depending on the sign of $\xi$.

An alternative description of scalar perturbations to the one used here is by means of the curvature perturbation $R$ [17]. It is defined as the three-space curvature in comoving slicing

\[ \Phi_{id} = \frac{3}{2} \frac{\zeta + 2 \Phi(0)}{X^3}. \]
with suppressed factor $4(k/a)^2$, and by using $(3)R = -4(k/a)^2\psi$ (follows from the definition) and $\psi_{com} = HB$ (is obtained by the time shift $\delta\eta = B$), we find that it is given by $R = -HB$. For superhorizon perturbations the first equation (11) yields $HB = -(1/3)E$, and if we insert here the value $E = -(9/2)\Phi^{(0)}$ from equation (29), we obtain $R = -(3/2)\Phi^{(0)}$ irrespective of the value of $|\xi|^2$ as long as it is small. Thus, $R$ is constant in the presence of solid just like in a universe with pure fluid. However, the large-angle CMB anisotropies do not inherit the flat spectrum of primordial perturbations because of that, since they are no longer given solely by $R$.

### 2.4. Perturbations in the presence of the cosmological constant

For the description of the integrated Sachs–Wolfe (SW) effect we need an extended version of the theory, with the cosmological constant (dark energy) added to radiation and matter as the third component of the universe. Thus, we need to consider a universe with the density

$$\rho = \rho_{\text{eq}}(X^{-4} + X^{-3} + x_0^{-3}), \quad x_0 = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3}X_0,$$

(34)

where $X = a/\rho_{\text{eq}}$, $\Omega_m$ and $\Omega_\Lambda$ are the density parameters of matter and the cosmological constant and $X_0$ is the present value of $X$. $x_0 = \rho_{\text{eq}}/\rho_{\text{eq}} = (\rho_{\text{eq}}/\rho_0) = 24800\Omega_m h^2$. By using $\Omega_m = 0.32$ and $h = 0.67$ (Planck values) we find $X_0 = 3560$, and if we insert this, the cited value of $\Omega_m$ and $\Omega_\Lambda = 1 - \Omega_m$ into the definition of $x_0$, we obtain $x_0 = 2770$.

The function $\tilde{\psi}$ can be determined from the first equation (5), after rewriting it as

$$\tilde{\psi} \equiv \frac{1}{X} \frac{d}{d\zeta} \tilde{\eta} \eta = \left(\frac{1}{6} \rho X^2\right)^{1/2}.$$

Clearly, the effect of the cosmological constant is significant only for $X \gtrsim x_0$, and is completely negligible in the first period after the moment $\eta_0$ when $X$ is of order 1. Thus, we can put $X = X$ and $\rho = \rho_{\text{eq}}(X^{-4} + X^{-3})$ during this period to obtain

$$\eta_{\text{eq}} \left(\frac{1}{6} \rho \right)^{1/2} = \frac{4}{9}, \quad \rho_{\text{eq}} = \frac{4}{81} \rho_{\text{eq}}^4,$$

and

$$\tilde{H} = \frac{2\sqrt{\mathcal{Y}}}{X}, \quad \mathcal{Y} = 1 + X + x_0^{-3}X^4.$$

(35)

The theory of perturbations we have developed so far works only if all components of matter are coupled to each other. This is not true even before recombination, because not all the matter is coupled to radiation; baryonic matter is, but dark matter is not. To account for that, one must extend equations for $B$ and $\mathcal{E}$ by terms containing the three-space part of the four-velocity of the dark matter $u_d = a^{-1}\delta X_d$, and add to the theory Euler equation written for dark matter only (equation for $B$ with zero sound speed) to fix $u_d$. Previous equations are valid only under the simplifying assumption that the dark matter moves in the same way as the baryon-radiation plasma, $u_d = 0$.

We are interested in the evolution of perturbations in the period after recombination, when baryonic matter, too, is decoupled from radiation. Radiation then acquires nonzero velocity $u_r = a^{-1}\delta X_r$, but its evolution looks different from that of dark matter. It undergoes free-streaming, described by equation (45) from the next section. As we will see, an approximate theory of free-streaming based on this equation gives not quite as large an effect.
as we would need. Thus, if we do not aim at a complete description of perturbations and just want to estimate the effect of a radiation-like solid on them, we can suppose that the matter filling the universe is fully coupled after recombination and account for the effect of decoupling, as well as for any other effect that is possibly important and was not taken into consideration, by an ad hoc correction of the part of the theory determining the CMB anisotropies in the presence of pure fluid.

In a universe with fully coupled matter, the auxiliary sound speed squared is

\[ c_{S0}^2 = \frac{K_0}{\rho_0} = \frac{1/3}{\rho_{\gamma+} + \rho_\Lambda} = \frac{1/3}{X^+}, \]  

(36)

where \( \hat{X}_+ \) is defined in the same way in terms of \( X \) as \( X_+ \) is defined in terms of \( X \), \( \hat{X}_+ = 1 + (3/4)X \). (The cosmological constant does not contribute to \( c_{S0}^2 \), since \( \rho_\Lambda = 0 \)). Furthermore, for the contribution of the solid to \( c_{S2}^2 \) and the function \( \alpha \) we have

\[ \Delta c_{S2}^2 = \frac{4\xi}{3\rho_\gamma}, \quad \alpha = \frac{3}{2\rho_\gamma} \rho_\gamma = \frac{2}{3\rho_{S0}^2}. \]  

Our starting point will be equations (4) and (11) with the prime replaced by \( \zeta \), \( k \) replaced by \( \kappa \) and tilde attached to \( \Psi \) and \( \Phi \) as well as to \( \xi \) in the expression for \( \Delta \Phi \). First, we find the generalized version of equation (22),

\[ \frac{d\Psi}{d\zeta} = \left[ 1 + \frac{d}{aH^2} \left( a\tilde{H}^2 \right) \right] + \left( \frac{3c_{S0}^2 - \alpha}{\rho_\gamma} \right) \tilde{H} - a\tilde{H}^2 \left( \frac{3}{2\rho_{S0}^2} \tilde{H} \tilde{E} - B \right). \]  

(37)

By using the second equation in (5) along with \( p' = -3HK \) and the first equation (5) along with \( H' = -3(\rho + 3p)a^2 \), we obtain

\[ \frac{1}{a} \frac{d\alpha}{da} = \frac{1}{\rho_\gamma} \frac{d\rho_\gamma}{d\zeta} - \frac{1}{\rho_\gamma} \frac{d\rho}{d\zeta} = 3\tilde{H} \left( \frac{\rho}{\rho_\gamma} - \frac{K}{\rho_\gamma} \right) = \left[ -3(1 + c_{S0}^2) + 2a \right] \tilde{H}, \quad \frac{1}{aH^2} \frac{d\tilde{H}}{d\zeta} = 1 - \alpha, \]

so that the expression in front of \( \Psi \) in equation (37) can be written as

\[ \frac{1}{a\tilde{H}^2} \frac{d\left(a\tilde{H}^3\right)}{d\zeta} + \left( \frac{3c_{S0}^2 - \alpha}{\rho_\gamma} \right) \tilde{H} = -(1 + \alpha)\tilde{H} = -2\tilde{H} + \frac{1}{\tilde{H}} \frac{d\tilde{H}}{d\zeta} = \frac{1}{a^2} \frac{d}{d\zeta} \left( \frac{X^{-2}\tilde{H}}{X} \right). \]

As a result, if we define

\[ \Psi = \frac{\tilde{H}}{2aH^2} F, \]  

(38)

and use \( a\tilde{H}^2 = 8/(3c_{S0}^2X^2) \), we arrive at

\[ \frac{dF}{d\zeta} = -8 \left( \gammaE - \frac{2}{3c_{S0}^2} \tilde{E} \tilde{B} \right). \]  

(39)

This is the generalized version of equation (24). Analogically, for the functions \( E \) and \( B \), we have the generalized version of equations (25),
\[
\frac{d\mathcal{E}}{d\zeta} = \kappa^2 \left( \frac{F}{2 \Lambda^2} - \bar{B} \right), \quad \bar{B} = -\frac{1}{6} \left( \kappa^2 \frac{3c_{S0}^2}{8 F} + \frac{2}{\Lambda^2} \right). \tag{40}
\]

We also find, after writing the modified parameter \( \xi \) as \( \xi = (3/2) \xi c_{S0}^2 \alpha \), that the generalized version of the second equation in (21) is
\[
\Delta \Phi = (24 \xi / \lambda^2) \kappa^{-2} \mathcal{E} = (4 \gamma / \lambda^2) \mathcal{E}. \quad \text{We can see that}
\]

the new equations reduce to the old ones with the replacements \( X \to \lambda \), \( \dot{X} \to \dot{X} \), and \( \xi \to \sqrt{3} \). As for the function \( \lambda (\zeta) \) appearing in them, we do not have an analytical expression for it anymore. It must be calculated numerically from equation (35), rewritten as
\[
d\lambda / d\zeta = 2 \sqrt{3}. \]

In the long-wave limit, we have equations (28) with the replacements listed above for \( F \) and \( \mathcal{E} \) and expression (29) for \( \mathcal{E} \). The resulting generalization of equations (31) is
\[
\Psi = 36 \frac{\sqrt{3}}{\lambda^3} \left( \gamma (\zeta - \zeta_i) + \int \frac{\lambda^2 \dot{X}_+}{6 \sqrt{3}} d\zeta \right) \phi^{(0)}, \quad \Delta \Phi = -\frac{18 \gamma}{\lambda^2} \phi^{(0)}, \tag{41}
\]

The second part of \( \Psi \) is the potential \( \Phi_{id} \) in the presence of the cosmological constant. It can be transformed into the form given in equation (7.69) in [15] by using the identity
\[
\int \frac{\lambda^2 \dot{X}_+}{\sqrt{3}} d\zeta = \frac{1}{4} \left( \frac{\lambda^3}{\sqrt{3}} - 2 \int \lambda^2 d\zeta \right),
\]

which can be obtained by integration by parts. Again, we are restricting ourselves to the nondecaying part of \( \Phi_{id} \). This can be achieved by choosing the integration constant in such a way that the integral is of order \( \zeta^3 \) for \( \zeta \to 0 \); or equivalently, by regarding the integral as definite, starting from \( \zeta = 0 \).

3. Large-angle CMB anisotropies

3.1. Temperature fluctuations

To compute CMB anisotropies, we must know the relative deviation of the temperature from its mean value \( \Delta T = \delta T / T \) at the time of recombination. Since \( \rho \propto T^4 \), it holds that
\[
\Delta T = \frac{1}{4} \delta_r, \tag{42}
\]

where \( \delta_r \) is the density contrast of radiation, \( \delta_r = \bar{\rho}_r / \rho \). Density perturbations of individual components of matter, if independent from each other, are given by a formula analogical to that for the total density perturbation, \( \delta \rho = \rho_i (3 \Psi + \mathcal{E}) \). For the density contrast \( \delta_r \) this yields
\[
\delta_r = 4 \left( \Psi + \frac{1}{3} \mathcal{E} \right). \tag{43}
\]

Instead of \( \Delta T \), we need the effective fluctuation of temperature \( \Delta T_{eq} \), measured by a local observer that is at rest with respect to the unperturbed matter and observes the radiation arriving in the same direction in which it then propagates to the observer on Earth. Let \( \mathbf{v} \) be the local velocity of matter, \( \mathbf{v} = \delta \mathbf{x} \), and \( \mathbf{l} \) be the unit vector in the direction of propagation of radiation. Introduce also the unit vector \( \mathbf{n} \) pointing from Earth towards the place on the sky from which the radiation is coming. The function \( \Delta T_{eq} \) differs from \( \Delta T \) by the Doppler term \( \Delta T_D = \mathbf{v} \cdot \mathbf{l} = -\mathbf{v} \cdot \mathbf{n} \). Suppose the velocity reduces to its scalar part, \( \mathbf{v} = v^{(s)} \), and denote the unit vector in the direction of \( \mathbf{k} \) by \( \mathbf{m} \). By performing the transformation from the proper-time
comoving gauge to the Newtonian gauge, we find $v = -i k E^r = i m \kappa^{-1} \frac{dE}{d\zeta}$ and

$$\Delta T_0 = -i m \cdot n x^{-1} \frac{dE}{d\zeta}.$$  

(44)

Note that the expression for $v$ can be obtained also from the energy conservation law,

$$\left( \delta_i - 4 \Psi \right) + \frac{4}{3} V \cdot v = 0,$$

if one inserts into it from equation (43). The law is the same as in the theory with an ideal fluid, see equation (7.110) in [15], except that $\Phi$ is replaced by $\Psi$.

CMB anisotropies are calculated from the temperature fluctuations $\Delta T_0$ seen today on Earth. (The index '0' denotes the present moment.) To compute $\Delta T_0$ we make use of the fact that the temperature fluctuations of the radiation propagating freely from the surface of last scattering satisfy

$$\Delta \Phi \eta + \Delta \Psi = \frac{\partial}{\partial \tau} \left( \Phi \eta + \Psi \right).$$  

(45)

Again, this is the same equation as in the theory with an ideal fluid, except that one function $\Phi$ on the right hand side is replaced by $\Psi$; see equation (9.20) in [15].

The total temperature fluctuation $\Delta T_0$ is a sum of two terms, one obtained from equation (45) without the right hand side and another one contributed by the right hand side. The mechanisms responsible for these two terms are called the ordinary SW effect and the integrated SW effect. Let us first compute the contribution to $\Delta T_0$ from the ordinary SW effect. If we put the value of $\Phi$ measured on Earth at present equal to zero, we find $\Delta T_0^{SW} = (\Delta T_{eff} + \Phi)_0$; and by inserting here from equations (42), (43) and (44) we obtain

$$\Delta T_0^{SW} = \left( \Phi + \Psi + \frac{1}{3} E - i m \cdot n x^{-1} \frac{dE}{d\zeta} \right)_{re}.$$  

(46)

For superhorizon perturbations the last term is negligible since, as seen from the first equation of (25), it is of order $\kappa \Phi(0)$, while the first three terms are of order $\Phi(0)$. After skipping the former term and inserting for the latter terms from equations (33) and (29), we obtain

$$\Delta T_0^{SW} = \left( 20 \xi \kappa^{-2} + b \right) \Phi(0),$$  

(47)

where $b = 2\omega - 3/2 = 101/270 = 0.37$. For an ideal fluid the first term vanishes and the formula reduces to $\Delta T_0^{SW} = b \Phi(0)$. In approximate calculations, one replaces $b$ by $h_0$, obtained in the limit $\zeta_{re} \gg 1$, $h_0 = 2\omega_{00} - 3/2 = 3/10$; see section 9.5 in [15].

Let us now determine the contribution to $\Delta T_0$ from the integrated SW effect. For any function $f$ describing the perturbation, one must distinguish between the amplitude $\tilde{f}(\eta)$ and the complete wave $f(x, \eta) = \tilde{f}(\eta) e^{i k \cdot x}$. If we denote the sum $\Delta T + \Phi$ as $\tau$, equation (45) can be written as an ordinary differential equation for the function $\tilde{\tau}$,

$$\tilde{\tau} + i k_\parallel \tilde{\tau} = \Phi^r + \Psi^r,$$

where $k_\parallel$ is the projection of the vector $k$ into the direction of propagation of radiation, $k_\parallel = k \cdot l = -k \cdot n$. The solution is

$$\tilde{\tau} e^{i k_\parallel \eta} = \int \left( \Phi^r + \Psi^r \right) e^{i k_\parallel \eta} d\eta,$$
so that
\[ \tau_0 = \tau_{\infty} e^{-ik(\eta - \eta_0)} + \int_{\eta_0}^{\eta} (\Phi' + \Psi') e^{-ik(\eta - \eta')} d\eta. \] (48)

The value of \( \tau \) at present is \( \tau_0 = \tau_{\infty} e^{ikx_0} = \tilde{\tau}_0 \) (we suppose that the observer is located at the origin) and the value of \( \tau \) at recombination is \( \tau_{\infty} = \tau_{\infty} e^{ikx_0} = \tau_{\infty} e^{-ik(\eta - \eta_0)} \). On the other hand, from the definition of \( \tau \), we have \( \tau_0 = \Delta T_0 \) and \( \tau_{\infty} = (\Delta T + \Phi)_{\infty} \); or, if we include the contribution of the Doppler effect to the fluctuation of temperature, \( \tau_{\infty} = (\Delta T_{\text{eff}} + \Phi)_{\infty} \). Thus, equation (48) can be written as \( \Delta T_0 = \Delta T_0^{SW} + \Delta T_0^{SW} \), with \( \Delta T_0^{SW} \) given in equation (46) and
\[ \Delta T_0^{SW} = \int_{\eta_0}^{\eta} (u' + v') e^{ik n(\eta - \eta')} d\eta \Phi^{(0)}, \] (49)

where we have denoted the functions \( \Phi \) and \( \Psi \) with suppressed factor \( \Phi^{(0)} \) as \( u \) and \( v \). With the notation introduced here, we can also repair equation (47): the initial Newtonian potential appearing there must be shifted to the time \( \eta_{\infty} \); \( \Phi^{(0)} \rightarrow e^{ik n(\eta - \eta_0)} \Phi^{(0)} \).

We have described the integrated SW effect in the framework of linearized theory. The full theory accounts also for the fact that the light in the late period propagates through large fluctuations of density (superclusters and supervoids), for which such a description is not applicable. The resulting effect, called the nonlinear integrated SW effect, will not be considered here.

To complete the discussion, let us look how the decoupling of radiation from baryonic matter influences the evolution of the potentials \( \Phi \) and \( \Psi \). Consider first the density contrast of radiation \( \delta_\gamma \). By solving equation (45) without gravitational potentials and averaging the resulting function \( \Delta \xi \) over the directions of \( l \), we obtain
\[ \delta_\gamma = -\beta_\gamma (\eta_{\infty}) \Phi^{(0)}, \] (50)
where \( \beta = 4(3/2 - \omega) = 304/135 = 2.25 \) and \( z = k (\eta - \eta_{\infty}) \); see equation (9.23) in [15]. (We have used the expression for the density contrast of radiation at recombination \( \delta_{\gamma,\text{re}} = -\beta \Phi^{(0)} \).) This is to be compared with the density contrast of radiation in a universe with fully coupled matter,
\[ \delta_\gamma^{(c)} = 4 \left( v - \frac{3}{2} \right) \Phi^{(0)}. \] (51)

Both functions start from \( \delta_{\gamma,\text{re}} = -\beta \Phi^{(0)} \), but then the second function mildly increases in absolute value until it reaches its present value 1.40 \( \delta_{\gamma,\text{re}} \), while the first function has three different regimes depending on the parameter \( z_0 \equiv k \eta_{\gamma} \); for \( z_0 \ll 1 \) it is practically constant, for \( z_0 \sim 1 \) it monotonically decreases in absolute value, and for \( z_0 \gg 1 \) it oscillates with decreasing amplitude.

In a universe with several decoupled components, every component has its own sound speed. However, even then one can introduce an effective common sound speed \( c_{S,\text{eff}} = (\delta p/\delta u)^{1/2} \), whose square appears in the differential equation of second order for \( \Psi \) (or \( \Phi \) if the universe is filled with pure fluid) and in such a way determines the time dependence of both \( \Phi \) and \( \Psi \) [15]. If the perturbations are adiabatic and the matter is fully coupled, the effective sound speed reduces to \( (\delta p/\delta u)^{1/2} \), which is the actual sound speed in pure fluid and the auxiliary sound speed in a matter containing solid component. In general, the effective sound speed is given by
and if we express $\delta_m$ in terms of $\delta^{(c)}$ by using the formula $\delta = (\rho_0/\rho)(3'y + E)$, we obtain

$$c^2_{S,\text{eff}} = \frac{(1/3)\delta^3}{\delta^3 + (1/3)\delta^2} + \frac{(1/3)\delta^3}{\delta^3 + (3/4)\delta^{(c)}X},$$

(52)

Clearly, for fully coupled matter this coincides with the expression (36) for $c^2_{S_0}$. If the radiation is decoupled, the values of $c^2_{S,\text{eff}}$ become smaller, and for large enough wave numbers they can even cross zero and start to oscillate. This causes the function $\Psi$ to decrease a bit slower in the first period after recombination, but the effect is at most 10 to 20% for physically interesting values of parameters.

In addition to reducing the value of $c^2_{S,\text{eff}}$, free-streaming also produces anisotropic stress, hence $\Delta\Phi$ becomes nonzero after recombination, even in the absence of a solid. However, this effect, if estimated by using the same approximate function $\Delta\eta_T(x,\eta)$ as in the computation of $\delta_y$, turns out to be negligible.

### 3.2. Power spectrum

The power spectrum of CMB anisotropies are the coefficients $C_l$ in the expansion of two-point correlation function of $\Delta T_0$,

$$C_l \equiv \left\langle \Delta T_0(n)\Delta T_0(n') \right\rangle = \frac{1}{4\pi} \sum (2l + 1)C_lP_l(\cos \theta),$$

(53)

where the angle brackets denote averaging over different regions in the universe, $P_l$ are Legendre polynomials and $\theta = \arctan(n, n')$. (We will use this definition, although when presenting observational data one usually regards $C_l$ as *dimensional* quantities, defined in terms of $\delta T_0$ rather than $\Delta T_0$.) To compute $C_l$ for small $l$, we need to know $\Delta T_0$ for small $k$. For the time being, we will restrict ourselves to $\Delta T_0$ coming from the ordinary SW effect and postpone the discussion of the integrated SW effect to the next subsection. Thus, we will identify $\Delta T_0$ with $\Delta T_{0}^{\text{SW}}$ of equation (47). What we need to compute $C_l$ is, however, not what appears in that equation. The quantity $\Delta T_0$ on its left hand side is actually the *Fourier coefficient* of the deviation of temperature from the mean value, and the quantity $\Phi(0)$ on its right hand side is the *Fourier coefficient* of the Newtonian potential at the beginning of the Friedmann expansion. In other words, the equation must be read as

$$\Delta T_{0k} = \left(20\xi_k^{-2} + b\right)e^{ikn_0}\Phi_{0k}^{(0)}.$$

(54)

The exponential in front of $\Phi_{0k}^{(0)}$, mentioned already in the discussion after equation (49), takes into account the fact that the right hand side of equation (47) refers to the moment of recombination. We should actually write there $e^{ikn_0} - n_0$, but the time $n_0$ is about 50 times less than the time $\eta_0$, so we have neglected it.

We want to calculate the coefficients $C_l$ in order to establish what values of $\xi$ are allowed by observations. Since our transfer function is not constant but rises with decreasing wave number, we expect $C_l$ to rise with decreasing multipole moment; and $\xi |_{l=1}$ must not be too large in order that this behavior is in agreement with observations within cosmic variance. In addition to $C_0$, one observes also the *bispectrum* $B_{l_1l_2l_3}$, which encodes information about the three-point correlation function of $\Delta T_0$. Nonzero coefficients $B_{l_1l_2l_3}$ mean non-Gaussian probability distribution of $\Delta T_0$; therefore, by measuring the bispectrum, one can obtain an observational upper limit on the parameter of non-Gaussianity $f_{NL}$. In such a way, one
constrains the parameter space of alternative inflationary scenarios producing nonzero $f_{NL}$, such as one-field inflation with a non-standard kinetic term or multifield inflation with a highly nonlinear energy–momentum tensor of one field [18]. The behavior of our transfer function suggests that solidification of a part of radiation in the early universe leads to the enhancement of the coefficients $B_{l},$ for small multipole moments. Thus, the parameter $\xi$ can be constrained also by observational data on the bispectrum.

From inflation, one obtains a spectrum of $\Phi^{(0)}$ which is, within a good accuracy, flat; that is, $\langle \Phi_{k}^{(0)} \Phi_{k'}^{(0)} \rangle = B k^{-3} \delta (k - k')$ with constant $B$. By inserting $\Delta T_0 (n) = \int \Delta T_k (\frac{dk}{2\pi})^2$ into (53) and using the expression for $\langle \Phi_{k}^{(0)} \Phi_{k'}^{(0)} \rangle$, we find

$$C_{l} = \frac{2}{\pi} B \int_{0}^{\infty} \left(20 \xi k^{-2} + b\right) j_{l}^{2} (k \eta_{0}) \frac{dk}{k},$$

where $j_{l}$ is the spherical Bessel function. Next, we pass from $k$ to $\eta = s/\eta_{0}$ to obtain

$$C_{l} = \frac{2}{\pi} B b^{2} \int_{0}^{\infty} \left(20 \xi_{s} s^{-2} + 1\right) \frac{ds}{s},$$

where $\xi_{s} = (20/b)(\eta_{0}/\eta_{s})^2 \xi$. Finally, we compute the integral with the help of the formula

$$\int_{0}^{\infty} s^{-n} j_{l}^{2} \frac{ds}{s} = \frac{\pi}{8 \cdot 2^{n}} \frac{\Gamma (2 + n)}{\Gamma \left( \frac{3 + n}{2} \right)} \frac{\Gamma \left( l - \frac{3}{2} \right)}{\Gamma \left( l + 2 + \frac{n}{2} \right)},$$

to find

$$C_{l} = \left[ \frac{8}{15} \frac{\xi^{2}_{s}}{(l + 3)(l + 2)(l - 1)(l - 2)} + \frac{4}{3} \frac{\xi_{s}}{3 (l + 2)(l - 1)} + 1 \right] C_{l, id},$$

where

$$C_{l, id} = \frac{1}{l (l + 1)} \frac{B b^{2}}{\pi}$$

(57)

After $b$ is replaced by $b_{0}$, the expression for $C_{l, id}$ coincides with that in equation (9.44) in [15].

To complete the theory, we need the numerical value of the coefficient of proportionality between $\xi_{s}$ and $\xi$. By integrating the equation $dX/d\zeta = 2 \sqrt{3}$ from $\zeta = 1$, $X = 3$ to the value of $\zeta$ at which $X = X_{0}$, we obtain $\xi_{0} = 53.4$, and after inserting this into the definition of $\xi_{s}$ for $\eta_{0}/\eta_{s}$, we find

$$\xi_{s} = 1.52 \times 10^{5} \xi_{s}.$$  (58)

The coefficients $C_{0}$, $C_{1}$ and $C_{2}$ are infinite, the first of them even in the case when the cosmic medium is fluid. (This is true for $C_{0}$ unless $\xi_{s} = (15/2)(1 \pm \sqrt{3}/5)$ and for $C_{1}$ unless $\xi_{s} = 10$. However, as will be seen in the following discussion, there is no need to explore these singular cases in detail.) The coefficients become finite if we take into account that the spectrum of perturbations is cut off at some wave number $k_{min}$ given by the duration of inflation; and the coefficient $C_{2}$, as well as the coefficient $C_{0}$ in case $\xi = 0$, become finite also if we introduce a small negative tilt of the primordial spectrum, replacing $B$ by $B e^{\epsilon}$ with $\epsilon > 0$. However, of the three coefficients, we need to consider $C_{2}$ (quadrupole) only. The other two coefficients (monopole and dipole) do not enter the theory, because an individual observer has no clue how much the mean temperature $T_{obs}$ he has measured differs from the true mean temperature $T$, neither can he tell how much his velocity with respect to CMB $V_{obs}$
which he determines by averaging the product \( \Delta T_{\text{obs}} \cos \theta \), differs from his true velocity \( V \). He identifies \( T \) with \( T_{\text{obs}} \) and \( V \) with \( V_{\text{obs}} \); and if we write the coefficients \( C_i \) as

\[
C_i = \left\langle \Delta T_0(n) \int \Delta T_0(n') P(d\Omega) \right\rangle,
\]

we can see that such identification means that the coefficients \( C_0 \) and \( C_1 \) are put equal to zero.

The constant \( c \) appearing in the formula for the renormalized coefficient \( C_2 \),

\[
C_{2r} = \left( c \xi^2 + \frac{1}{3} \xi + 1 \right) C_{2id},
\]

can be written as

\[
c = \frac{4}{75} \left[ e^{-1} \left( 1 - s_{\text{min}}^e \right) + d \right] = \frac{4}{75} \left[ e^{-1} \equiv c_l \text{ if } \epsilon \log \left( 1/s_{\text{min}} \right) \gg 1 \right.
\]

\[
\log \left( 1/s_{\text{min}} \right) + d \equiv c_{\text{II}} \text{ if } \epsilon \log \left( 1/s_{\text{min}} \right) \ll 1,
\]

where the constant \( d \) is given by

\[
d = \log 2 - \frac{1}{2} \psi(6) + \frac{1}{2} \psi(1) + \psi(7/2) = \frac{77}{40} - \log 2 - \gamma = 0.655.
\]

Here \( \psi \) is a digamma function and \( \gamma \) is the Euler–Mascheroni constant. The parameter \( \epsilon \) is the deviation of the scalar spectral index \( n_S \) from 1, \( \epsilon = - n_S - 1 \), so that \( \epsilon = 0.04 \) and \( c_l = 4/3 \) for the observational mean value of \( n_S \), which is 0.96 (again a Planck value). The value of the parameter \( s_{\text{min}} \) depends on the inflationary scenario. It holds that

\[
s_{\text{min}} = k_{\text{min}} h_0 = \frac{n_0}{k_{\text{max}}^0} = \frac{e^{(0) 0}}{\delta_{\text{max}}^0} = \frac{N_{\text{min}} h_0}{NH^{-1}} \approx \frac{N_{\text{min}}}{N},
\]

where the index ‘(0)’ denotes the beginning of Friedmann expansion, \( r_{\text{obs}} \) is the radius of the part of the universe observable today, the index ‘inf’ denotes the beginning of inflation, \( N \) is the number of e-foldings during inflation, \( N_{\text{min}} \) is minimum \( N \) and \( H \) is the Hubble constant during inflation. For inflation on the GUT and Planck scales (new and chaotic), \( N \) is typically about 2000 and \( 10^{7} \div 10^{11} \), respectively, while \( N_{\text{min}} \) has a value between 60 and 70. This yields \( c_{\text{II}} \approx 0.22 \) and 0.67 \( \div 1.16 \), hence \( c_{\text{II}} \ll c_l \) for inflation on the GUT scale and \( c_{\text{II}} \lesssim c_l \) for inflation on the Planck scale. According to (60), the value of \( c \) in both asymptotic regimes is given by the less of the two numbers \( c_l \) and \( c_{\text{II}} \). Thus, in approximate calculations, we can use \( c = c_{\text{II}} \); in other words, we can ignore the tilt of the primordial spectrum and take into account only its cutoff.

The observed values of \( C_i \) must coincide with the theoretical values within cosmic variance,

\[
C_{i,\text{obs}} \in \left( 1 - \delta_i, 1 + \delta_i \right) C_i, \quad \delta_i = \sqrt{\frac{2}{2I + 1}}.
\]

Denote the relative deviation of our \( C_i \) from \( C_{i,\text{id}} \) by \( \delta_i \). After identifying \( C_{i,\text{obs}} \) with \( C_{i,\text{id}} \) we find that \( \xi_3 \) must assume values between \(-4.15 \) and \( 2.30 \) (a consequence of \( 1/(1 + \Delta_{2r}) \gtrsim 1 - \Delta_2 \), if one inserts for \( c \) the value \( c_{\text{II}} \) computed for inflation on GUT scale). Thus, both positive and negative values of \( \xi \) are admissible and \( |\xi_3| \) cannot exceed values of order \( 10^{-5} \).

The identification of \( C_{i,\text{obs}} \) and \( C_{i,\text{id}} \) is more or less acceptable for all multipoles except for the quadrupole, whose observational value lies approximately at the lower limit of the interval allowed by cosmic variance, \( C_{2,\text{obs}} \approx (1 - \Delta_2) C_{2,\text{id}} \). The theory with a solid can provide for that only for \( \xi_3 \) between \(-1.85 \) and \( 0 \) (a consequence of \( \Delta_{2r} \approx 0 \)), which is an unwanted result.
since negative values of $\xi$ are most probably unphysical and were included into the theory only for completeness. To return positive values of $\xi$ into the game, we can use an idea put forward in the cosmology with pure fluid, that the lack of power in the quadrupole comes from a cutoff of the primordial spectrum on the scale $s_{\text{min}} \sim 1$ [19–21]. Such a strong cutoff can be caused either by a short duration of inflation or by jump-like variation of inflationary potential close to the end of inflation. In the presence of a solid, we can use strong cutoff to reconcile positive $\xi$ with observations, while explaining small $C_{2,\text{obs}}$ as before by cosmic variance. Suppose the integral in the expression (55) is cut at $s_{\text{min}}$ and write the power spectrum as $\xi \xi = + \sum_{pq r} C_{l l l l}^{2 * 2 * 2 * 2}$, where $C_{l l l l}$ is the power spectrum in the theory with pure fluid in which no cutoff occurs. After imposing the condition that the ratio $C_{l l l l}^{2 * 2 * 2 * 2}$ equals $\delta^{-1} 2$ for $l = 2$ and $1$ for $l > 2$ within the cosmic variance, we find that the maximum $\xi_0$ is 0.81 (a consequence of $\xi \xi = + \sum_{pq r} C_{l l l l}^{2 * 2 * 2 * 2}$) and the minimum $\xi_0$ is $-3.21$ (a consequence of $\xi \xi = + \sum_{pq r} C_{l l l l}^{2 * 2 * 2 * 2}$). Again, we conclude that $\xi$ can be positive as well as negative and must satisfy $\xi \ll -|1 0 5| 0$.

3.3. Switching on the integrated SW effect

The formula for $\Delta T_{\text{pk}}$ that takes into account the contribution of the integrated SW effect is

$$
\Delta T_{\text{pk}} = \left[ f_k e^{ik \eta e} + \int_{\eta e}^{\eta 0} g'_k e^{ik \eta (\eta - \eta e)} d\eta \right] \Phi_k^{(0)},
$$

(62)

where $f_k = 202k^{-2} + b$ and $g_k = u_k + v_k = (\Phi_k + \Psi_k)/\Phi_k^{(0)}$. Note that we cannot put $\eta = 0$ in the lower limit of the integral, as we did in the argument of the exponential $e^{ik \eta (\eta - \eta e)}$, because the contribution of the solid to the integral would diverge. For superhorizon perturbations, the function $g_k$ can be extracted from expressions (41) for $\Psi$ and $\Delta \Phi$. The result is

$$
g_k = b \left( \xi \eta \eta - g_s + g_{\text{ad}} \right),
$$

(63)

where

$$
g_s = \frac{108}{5} \left( \frac{\sqrt{3}}{\Lambda^3} - 1 \right), \quad g_{\text{ad}} = 12b^{-1} \frac{\sqrt{3} \Lambda^3}{\sqrt{3} \Lambda^3} \int \frac{\Lambda^3}{\sqrt{3} \Lambda^3} d\zeta.
$$

(64)

We have skipped the shift by $\eta_s$ in the expression $\zeta - \eta_s$ appearing in $g_s$, since the functions $g_s$ and $g_{\text{ad}}$ are needed only in the interval $\zeta \geq 1\gg \eta_s$. The expression for $g_k$ is valid if the perturbation is stretched far beyond the sound horizon,

$$
c_{s0} k \eta = \frac{\eta e}{\eta 0} c_{s0} \ll 1,
$$

(65)

where $c_{s0}$ is given in equation (36). The period between the times $\eta e$ and $\eta 0$ consists of two distinctive eras, the early era dominated by matter and the late era dominated by the cosmological constant. They are separated by the time $\eta_e'$ at which the densities $\rho_m$ and $\rho_\Lambda$ coincide, $\eta_e' = 49.6$ for the Planck values of cosmological parameters. Note, however, that the effect of the cosmological constant shows up considerably earlier; for example, the function $g_{\text{ad}}$ switches from the early-time regime to the late-time regime as soon as at $\eta \sim 15$. The function $\zeta = c_{s0} \xi$ rises during the early period from the value $\zeta_{re} = \sqrt{4/39} = 0.32$ to a value close to $\zeta_{\infty} = 2/3$ (the limiting value in a universe without the cosmological constant).
and slightly recedes during the late period. Thus, equation (63) can be safely used for the values of \( s \) substantially smaller than the horizon value \( s_S = (\eta_0/\eta_{\text{ac}}) Z_{\infty}^{-1} = 80.0 \).

We have formulated the condition for long-wave perturbations assuming, as in the computation of \( g_k \), that the matter filling the universe after recombination is fully coupled. In the theory with decoupled radiation, the effective sound speed is smaller and the sound horizon is bigger, hence the value of \( s_S \) is even greater.

Rewrite the expression for \( \Delta T_{0k} \) into a more convenient form,

\[
\Delta T_{0k} = \left( f_k e^{i m n} + \int_{\sigma_0}^{1} \frac{dg_k}{d\sigma} e^{i m n} d\sigma \right) \Phi_k^{(0)},
\]

where \( \sigma = \eta/\eta_0 \) and \( \delta = (1 - \sigma) s \). After inserting this into the mean value \( \langle \Delta T_{0k} (n) \Delta T_{0k} (n') \rangle \) and using the identity

\[
\left\{ e^{i m (n' - n)} \right\}_m = \sum_1 (2 l + 1) j_l (s) j_l (s') P_l (\cos \theta),
\]

where \( \langle \cdot \rangle = \int_{n_0}^{\infty} \frac{d\eta}{4\pi} \), we obtain

\[
C_l = \frac{2}{B} \int_0^{\infty} \left[ f_k j_l (s) + G_l (s) \right] \frac{ds}{s}, \quad G_l = \int_{\sigma_0}^{1} \frac{dg_k}{d\sigma} j_l (s) d\sigma.
\]

(66)

With the function \( f_k \) written as \( f_k = b (\xi_n s^{-2} + 1) \) and the function \( g_k \) given in equations (63) and (64), this yields

\[
C_l = \frac{2}{B} b^2 \int_0^{\infty} \left[ \xi_n s^{-2} (j_l + G_{l,s}) + j_l + G_{l,id} \right] \frac{ds}{s},
\]

(67)

where

\[
G_{l,s} = \int_{\sigma_0}^{1} \frac{dg_k}{d\sigma} j_l (s) d\sigma, \quad G_{l,id} = \int_{\sigma_0}^{1} \frac{dg_k}{d\sigma} j_l (s) d\sigma.
\]

(68)

To compute the coefficients \( C_l \), we need to know the functions

\( j_{l,\text{id}} = j_l + G_{l,\text{id}}, \quad j_{l,s} = j_l + G_{l,s}. \)

Consider first the function \( j_{l,\text{id}} \). It is defined through the derivative of function \( g_{\text{id}} = 2 b^{-1} \Phi_{\text{id}}/\Phi^{(0)} \), whose behavior reflects the division of the interval between \( \eta_{\text{ac}} \) and \( \eta_0 \) into two parts: it decreases from \( 2 b^{-1} \omega_0 \) close to \( 2 b^{-1} \omega_0 \), then it slows down for a while, and then it decreases again to the value \( 2 b^{-1} \times 0.71 \). Thus, its derivative is negative and decreases in absolute value from a finite value close to zero in the early period, to rise again to a finite value in the late period. The behavior of the derivative of \( g_{\text{id}} \) is shown in the left panel of figure 2, where it is depicted by the heavy line denoted as ‘id’. In the early and late periods, the derivative can be approximated by two power-like functions, one with negative power and another one with positive power,

\[
\left( \frac{dg_{\text{id}}}{d\sigma} \right)_I = C_I \left( \frac{\sigma}{\sigma_{\text{ac}}} \right)^{-2}, \quad \left( \frac{dg_{\text{id}}}{d\sigma} \right)_II = C_{II} \sigma^5.
\]

(69)

The constants \( C_I \) and \( C_{II} \) can be calculated by the method of least squares and have values close to \( d g_{\text{id}} / d\sigma \) at \( \sigma = \sigma_{\text{ac}} \) and \( \sigma = 1 \), respectively. In the figure, the two functions defined in equation (69) are depicted by the dashed straight lines and their sum is depicted by the light line.
The behavior of the function $j_{2, id}$ is shown in the right upper panel of figure 2. The thin line represents the function $j_2$, the heavy line represents the function $j_{2, id}$ computed with the help of the late-time contribution to $\frac{d\sigma_{id}}{d\sigma}$, and the dotted line represents the correction to the function $j_{2, id}$ coming from the early-time contribution to $\frac{d\sigma_{id}}{d\sigma}$. The ratio $C_{2, id}/C_{2, id}^{SW}$, computed by using the late-time contribution to $\frac{d\sigma_{id}}{d\sigma}$ only, is 1.20, and with the early-time contribution to $\frac{d\sigma_{id}}{d\sigma}$ included into the calculation, it drops to 0.91. The correct value is about 1.4. As seen from the figure, the value of $C_{2, id}/C_{2, id}^{SW}$ could be raised either by suppressing the early-time contribution to $\frac{d\sigma_{id}}{d\sigma}$ or by enhancing the late-time one. Our discussion of the evolution of perturbations in the theory with decoupled radiation suggests that a certain early-time suppression is necessarily produced by free-streaming. A suppression at the level of 10 to 20%, which we have obtained with the potentials $\Phi$ and $\Psi$ left out of the equation for $\Delta \eta_T(x, l, \eta)$, does not help, but perhaps the effect would become greater after including the potentials in the calculation. Moreover, some late-time enhancement could arise from the nonlinear integrated SW effect. To obtain results that are at least qualitatively correct without complicating the theory too much, we have mimicked the possible modifications of the early-time and late-time contributions to $\frac{d\sigma_{id}}{d\sigma}$ by dropping the former contribution altogether and leaving the latter contribution unchanged.

Let us now discuss the behavior of the function $j_{l, s}$. It is defined through the function $\frac{dg_{id}}{d\sigma}$, depicted in the left panel of figure 2, by the heavy line denoted as ‘$s$’. Except for a small region at the end point, the function $\frac{dg_{id}}{d\sigma}$ is practically identical with the function $\frac{dg_{id}^{(0)}}{d\sigma}$ computed without the cosmological constant. However, its value, and hence its contribution to the function $j_{l, s}$, is negligibly small close to the end point; therefore, we can put it equal to $\frac{dg_{id}^{(0)}}{d\sigma}$ everywhere. The value of $g_{l}^{(0)}$ at $\sigma = \sigma_e$ is 1, therefore we obtain by integration by parts that
The function $g(s,0)$ falls down rapidly with $\sigma$, for example, it decreases from 1 to 0.1 at $\sigma = 3.6 \sigma_e$. As a result, the integral in the expression for $G_{l,s}$ equals approximately the product of the constant

$$\int_{\sigma_e}^{\infty} \frac{g(s,0)}{\sigma} \, d\sigma \approx \frac{3}{5} \sigma_e,$$

and the function $j_i(s)$ evaluated at the lower limit of the integral. The resulting function $j_{l,s}$ is

$$j_{l,s} \approx j_i(s) - j_i \left( (1 - \sigma_e) s \right) + \frac{3}{5} \sigma_e s j'_i \left( (1 - \sigma_e) s \right). \quad (70)$$

The approximate function $j_{2,s}$ is depicted in the right lower panel of figure 2 by the heavy line and the correction to it is depicted in the same panel by the dotted line. The main contribution of $j_{l,s}$ to the coefficient $C_l$ comes from the interval $s \ll 1/\sigma_e$. For such $s$ we have $j_{l,s} \approx (8/5) \sigma_e s j'_i(s)$, with $j'_i(s) = j_i(s) = ls^2/(2l+1)$ if $s \ll 1$ and $s j'_i(s) = sj'_i(s) = \cos(s - \pi l/2)$ if $s \gg 1$. As a result, after completing a semi-oscillation with small amplitude, the function $j_{l,s}$ starts to oscillate uniformly with the amplitude $(8/5) \sigma_e$.

The ratio $C_l/C_{l,SW}$ as a function of $l$, computed for extremal values of $\xi_0$ in the theory with cutoff at $s_{min} = 2$, is depicted in figure 3. The full and dashed lines represent theoretical and observational values, respectively, the shaded strips represent cosmic variance, and the numbers are the values of $\xi$. The main consequence of including the integrated SW effect into the theory is that in the formula for the coefficients $C_l$, the parameter $\xi_0$ becomes multiplied by a function of order $\sigma_e = 1/\zeta_0$. As a result, the interval of admissible values of $\xi_0$ is considerably wider than in the theory without the integrated SW effect, with the limits at $-145$ and 60. This is an enhancement approximately by the factor $\zeta_0^2$, thus, due to the integrated SW effect the maximum and minimum of $\xi$ become greater in absolute value almost by two orders of magnitude.
4. Conclusion

We have calculated the large-angle part of the CMB power spectrum in a universe containing a radiation-like solid with constant shear modulus to energy ratio $\xi$. For that purpose, we had to extend the theory developed in [13] to the case when there is also nonrelativistic matter and the cosmological constant in the universe. Taking into account only the ordinary SW effect, we have confirmed the conclusion of [13] that the parameter $\xi$ must satisfy $|\xi| \lesssim 10^{-5}$ to accommodate observations. After taking the integral SW effect into considerations, we have found that the constraint becomes relaxed almost by two orders of magnitude. We have restricted ourselves to large-angle anisotropies and did not investigate the position and width of acoustic peaks. Obviously, for such small $|\xi|$, they would look practically the same as in a universe filled with pure fluid, unless the theory is modified, say, by endowing the solid with nonzero viscosity. The value of $|\xi|$ could be greater if the solid’s effect was compensated by the tilt of the primordial spectrum, but to assume such canceling of two independent effects seems unreasonable.

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