On the diameter of Schrijver graphs∗†

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Abstract

For \( k \geq 1 \) and \( n \geq 2 \), the well known Kneser graph \( KG(n, k) \) has all \( k \)-element subsets of an \( n \)-element set as vertices; two such subsets are adjacent if they are disjoint. Schrijver constructed a vertex-critical subgraph \( SG(n, k) \) of \( KG(n, k) \) with the same chromatic number. In this paper, we compute the diameter of the graph \( SG(2k+r, k) \) with \( r \geq 1 \). We obtain an exact value of the diameter of \( SG(2k+r, k) \) when \( r \in \{1, 2\} \) or when \( r \geq k-3 \). For the remained cases, when \( 3 \leq r \leq k-4 \), we obtain that the diameter of \( SG(2k+r, k) \) belongs to the integer interval \([4..k-r-1]\).

Keywords: Schrijver graphs, Diameter of graphs.

1 Introduction

Let \( G \) be a connected graph. Given two vertices \( a, b \in G \), \( \text{dist}(a, b) \), the distance between \( a \) and \( b \), is defined as the length of the shortest path in \( G \) joining \( a \) to \( b \). The diameter of \( G \), that we denote by \( D(G) \), is defined as the maximum distance between any pair of vertices in \( G \).

Let \([n]\) denote the set \( \{1, \ldots, n\} \). For positive integers \( n \geq 2k \), the Kneser graph \( KG(n, k) \) has as vertices the \( k \)-subsets of \([n]\) and two vertices are connected by an edge if they have empty intersection. In a famous paper, Lovász \[6\] showed that its chromatic number \( \chi(KG(n, k)) \) is equal to \( n-2k+2 \). After this result, Schrijver \[8\] proved that the chromatic number remains the same when we consider the subgraph \( KG(n, k)_{2-\text{stab}} \) of \( KG(n, k) \) obtained by restricting the vertex set to the \( k \)-subsets that are \( 2 \)-stable, that is, that do not contain two consecutive elements of \([n]\) (where 1 and \( n \) are considered also to be consecutive). Schrijver \[8\] also proved that the 2-stable Kneser graphs are vertex critical (or \( \chi \)-critical), i.e. the chromatic number of any proper subgraph of \( KG(n, k)_{2-\text{stab}} \) is strictly less than \( n-2k+2 \); for this reason, the 2-stable Kneser graphs are also known as Schrijver graphs. From now on we will use throughout this paper the notation \( SG(n, k) \) to refer to the graph \( KG(n, k)_{2-\text{stab}} \).

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After these general advances, a lot of work has been done concerning properties of Kneser graphs and stable Kneser graphs (see [11] [2] [3] [4] [5] [7] [9] [10] [11] [12] and references therein). Concerning Kneser graphs, its diameter was computed in [12]. Moreover, it is known that the distance between two vertices in Kneser graphs KG(n, k) only depends on the cardinality of their intersection [12]. However, in the case of Schrijver graphs SG(n, k) this does not work in the same way. For example, note that in SG(10, 4) the vertices {1, 3, 5, 7} and {1, 3, 6, 8} are at distance 3, while {1, 3, 6, 8} and {1, 4, 6, 9} are at distance 2. In this paper, we are interested in computing the diameter of Schrijver graphs. As far as we know this parameter has not been studied for such graphs. The main result of this paper is the following theorem:

**Theorem 1.** Let n, k, r be positive integers such that n = 2k + r. Then, the diameter of the Schrijver graph SG(2k + r, k) verifies:

\[
D(SG(2k+r,k)) = \begin{cases} 
2 & \text{if } r \geq 2k - 2, \text{ with } k \geq 2 \quad (\text{Theorem 2}) \\
3 & \text{if } k - 2 \leq r \leq 2k - 3, \text{ with } k \geq 3 \quad (\text{Theorem 3}) \\
4 & \text{if } r = k - 3, \text{ with } k \geq 5 \quad (\text{Corollary 4}) \\
\in [4..k - r + 1] & \text{if } 3 \leq r \leq k - 4, \text{ with } k \geq 5 \quad (\text{Corollary 4 and Theorem 4}) \\
\left\lfloor \frac{3k}{2} \right\rfloor + (k \mod 2) & \text{if } r = 2, \text{ with } k \geq 6 \quad (\text{Theorem 5}) \\
k & \text{if } r = 1 \quad (\text{Observation 7}) 
\end{cases}
\]

The proof of Theorem 1 will follow from Observation 1, Corollary 4 and Theorems 2, 3, 4 and 5 given in the next sections.

## 2 Main results

A subset \( S \subseteq [n] \) is s-stable if any two of its elements are at least "at distance s apart" on the n-cycle, that is, if \( s \leq |i - j| \leq n - s \) for distinct \( i, j \in S \). For \( s, k \geq 2 \) and \( n \geq ks \), the s-stable Kneser graph \( KG(n, k)_{s-stab} \) is the subgraph of \( KG(n, k) \) obtained by restricting the vertex set of \( KG(n, k) \) to the s-stable \( k \)-subsets of \([n]\).

In [11] it was shown (see Proposition 4.3 in [11]) that \( KG(ks + 1, k)_{s-stab} \) is isomorphic to the complement graph of the \((k - 1)\)th power of a cycle \( C_{ks+1} \). Therefore, \( SG(2k + 1, k) \) is isomorphic to a cycle graph \( C_{2k+1} \) and so, we have the following straightforward observation.

**Observation 1.** \( D(SG(2k + 1, k)) = k. \)

From now on, we assume that \( n \geq 2k + 2 \). We denote by \( [n]_2 \) the family of 2-stable subsets of \([n]\) and by \( [n]_2^k \) the family of 2-stable \( k \)-subsets of \([n]\), i.e. \( [n]_2^k = V(SG(n, k)) \). We will always assume w.l.o.g. that any vertex \( v = \{v_1, v_2, \ldots, v_k\} \) in \( SG(n, k) \) is such that \( v_1 < v_2 < \ldots < v_k \). Arithmetic operations will be supposed modulo \( n \) (being \( 0 \equiv n \)).

### 2.1 Distances between vertices

Let \( A = \{a_1, a_2, \ldots, a_k\} \) and \( B = \{b_1, b_2, \ldots, b_k\} \) be two vertices in \( SG(n, k) \) such that \( |A \cap B| = 1 \). W.l.o.g. we assume that \( A \cap B = \{1\} \) and \( a_2 < b_2 \). Note that \( b_2 \geq 4 \). Let \( X = \{2, b_2, b_3, \ldots, b_k\} \) and \( Y = X + 1 = \{3, b_2 + 1, b_3 + 1, \ldots, b_k + 1\} \). It is not hard to see that the set of vertices \( \{A, X, Y, B\} \) induce a \( P_4 \) or a paw (Figure 1).
Lemma 1. Let $A, B \in [n]_2^k$. If $|A \cap B| = k - 1$ then $\text{dist}(A, B) = 2$ and if $|A \cap B| = 1$ then $\text{dist}(A, B) \in \{2, 3\}$.

Assume now that $|A \cap B| = k - 1$. W.l.o.g. let $a_i = b_i$ for $i \in [k - 1]$ and $a_k < b_k$. Then $B + 1$ is adjacent to $A$ and $B$. So, we have:

**Observation 2.** Let $A, B \in [n]_2^k$. If $|A \cap B| = k - 1$ then $\text{dist}(A, B) = 2$ and if $|A \cap B| = 1$ then $\text{dist}(A, B) \in \{2, 3\}$.

Let $A, B \in [n]_2^k$. In order to compute the distance between vertices $A$ and $B$, we consider the subsets of vertices in the cycle $C_n$ with vertex set $[n]$ induced by the elements in $A \cup B$.

Let $X = A \cup B \subseteq [n]$. We denote $\mathcal{X}$ to the family of connected components of the graph induced by $X$ in the $n$-cycle $C_n$. In the same way, $\overline{\mathcal{X}}$ is the family of connected components of the graph induced by $[n] \setminus X$ in $C_n$. Let $\mathcal{P} = \{C \in \overline{\mathcal{X}} : |C|$ is even $\}$ and $\mathcal{I} = \{C \in \overline{\mathcal{X}} : |C|$ is odd $\}$. From these definitions, we have the following simple observations and Lemma 1

**Observation 3.** Let $A, B \in [n]_2^k$ with $A \cap B \neq \emptyset$. Let $X = A \cup B$. Then,

1. $|\mathcal{X}| = |\overline{\mathcal{X}}|$.

2. If $|\overline{\mathcal{X}}| \geq k$ then, $\text{dist}(A, B) = 2$.

3. If $|C| \leq 2$ for every $C \in \mathcal{X}$ then, $\text{dist}(A, B) = 2$.

**Lemma 1.** Let $A, B \in [n]_2^k$ with $A \cap B \neq \emptyset$ and $X = A \cup B$. Then, $\text{dist}(A, B) = 2$ if and only if $\frac{1}{2}|\mathcal{I}| + \frac{1}{2} \sum_{C \in \overline{\mathcal{X}}} |V(C)| \geq k$.

**Proof.** Let $C \in \mathcal{P}$. Note that the maximum cardinality of a 2-stable subset of $V(C)$ is $\frac{1}{2}|V(C)|$. If $C \in \mathcal{I}$, the maximum cardinality of a 2-stable subset of $V(C)$ is $\frac{1}{2}(|V(C)| + 1)$. For each $C \in \overline{\mathcal{X}}$, let $C_2$ be a maximum cardinality 2-stable subset of $C$. It is not hard to see that $Y = \bigcup_{C \in \overline{\mathcal{X}}} C_2$ is a 2-stable subset of $[n] \setminus (A \cup B)$ with cardinality $\sum_{C \in \overline{\mathcal{X}}} \frac{1}{2}|V(C)| + \sum_{C \in \mathcal{I}} \frac{1}{2}(|V(C)| + 1) = \frac{1}{2}|\mathcal{I}| + \frac{1}{2} \sum_{C \in \overline{\mathcal{X}}} |V(C)|$.

Hence, if $\frac{1}{2}|\mathcal{I}| + \frac{1}{2} \sum_{C \in \overline{\mathcal{X}}} |V(C)| \geq k$, there exists a 2-stable $k$-subset $Y'$ of $[n] \setminus X$. Therefore, $Y'$ is adjacent to $A$ and $B$ in $\text{SG}(n, k)$ and $\text{dist}(A, B) = 2$.

In order to prove the converse, it is enough to note that the maximum cardinality of a 2-stable subset in $[n] \setminus X$ is $\frac{1}{2}|\mathcal{I}| + \frac{1}{2} \sum_{C \in \overline{\mathcal{X}}} |V(C)|$. Then, if $\frac{1}{2}|\mathcal{I}| + \frac{1}{2} \sum_{C \in \overline{\mathcal{X}}} |V(C)| < k$, the distance between $A$ and $B$ in $\text{SG}(n, k)$ is at least 3. \qed

Notice that if $2k + 2 \leq n \leq 4k - 3$ (i.e. if $2 \leq r \leq 2k - 3$ and $n = 2k + r$) then $k \geq 3$ and $\text{D}(\text{SG}(n, k)) \geq 3$ since vertices $A = \{1, 4, 6, \ldots, 2k\}$ and $B = \{1, 5, 7, \ldots, 2k + 1\}$ are at distance 3.
3 in $SG(n, k)$. In fact, observe that $[n] \setminus (A \cup B) = \{2, 3\} \cup \{2k + 2, \ldots, n\}$ and then there is no 2-stable $k$-subset in $[n] \setminus (A \cup B)$, i.e. there is no vertex of $SG(n, k)$, adjacent to $A$ and $B$. Finally, notice that the vertices $A$, $\{3, 5, 7, \ldots, 2k + 1\}$, $\{2, 4, 6, \ldots, 2k\}$ and $B$ induce a $P_4$ in $SG(n, k)$. On the other hand, if $n \geq 4k - 2$, then Lemma 1 is enough to assure that $D(SG(n, k)) = 2$.

**Theorem 2.** Let $n, k$ and $r$ be positive integers, with $k \geq 2$ and $n = 2k + r$. $D(SG(n, k)) = 2$ if and only if $r \geq 2k - 2$.

**Proof.** By the preceding discussion, it only remains to show that if $r \geq k - 2$, then $D(SG(n, k)) = 2$. Let $A, B \in [n]_2^k$ such that $A \cup B \neq \emptyset$. If $r \geq 2k - 2$ then $n \geq 4k - 2$ and thus, $|[n] \setminus X| \geq 2k + 2k - 2 - (2k - 1) = 2k - 1$. Hence,

$$\frac{1}{2} |I| + \frac{1}{2} \sum_{C \in \mathcal{X}} |V(C)| \geq k.$$

Therefore, by Lemma 1 the result holds. \hfill \square

Let $A, B \in [n]_2^k$, with $A \neq B$ and $A \cap B \neq \emptyset$, and let $X = A \cup B$. In what follows, we will study the structure of $\mathcal{X}$ and $\overline{\mathcal{X}}$ in order to construct a path between vertices $A$ and $B$ of length as short as possible.

Notice that a connected component $C$ in $\mathcal{X}$ is either a single element in $A \cap B$, or it alternates between vertices of $A$ and $B$. Furthermore, if $A \neq B$ and $A \cap B \neq \emptyset$, then there is at least one component in $\mathcal{X}$ made from a single element in $A \cap B$, and at least one component not containing elements in $A \cap B$. Consider the following example:

**Example 1.** Let $A, B \in [20]_2^7$, where $A = \{2, 8, 10, 12, 15, 18, 20\}$ and $B = \{1, 6, 8, 10, 12, 14, 17\}$. Thus, we have that $A \cap B = \{8, 10, 12\}$; $X = A \cup B = \{1, 2, 6, 8, 10, 12, 14, 15, 17, 18, 20\}$; $A \setminus B = \{2, 15, 18, 20\}$; $B \setminus A = \{1, 6, 14, 17\}$; $\mathcal{X} = \{\{20, 1, 2\}, \{6\}, \{8\}, \{10\}, \{12\}, \{14, 15\}, \{17, 18\}\}$, and $\overline{\mathcal{X}} = \{\{3, 4, 5\}, \{7\}, \{9\}, \{11\}, \{13\}, \{16\}, \{19\}\}$.

Now, we want to construct two sets $A^*, B^* \in [n]_2$ such that $A^* \subset \overline{\mathcal{X}}$, $|A^*| \geq k$, $B \setminus A \subset A^*$, $B^* \subset \overline{B}$, $|B^*| \geq k$, and $A \setminus B \subset B^*$. Once sets $A^*$ and $B^*$ are constructed, we want to find two subsets $A' \subset A^*$ and $B' \subset B^*$ such that $A \cap A' = B \cap B' = \emptyset$ and $|A'| = |B'| = k$. Furthermore, we want $A' \cap B' = \emptyset$ or, if this cannot be achieved, we want the intersection to be as small as possible.

Looking at Example 1, we start with $\{1, 6, 14, 17\} \subset A^*$ and $\{2, 15, 18, 20\} \subset B^*$. Notice that 5, 7, 13 and 16 cannot be in $A^*$, as we want it to be in $[n]_2$. This happens because $\ell \in \{6, 12, 17\}$ is a vertex in components of $\mathcal{X}$ and $\ell - 1$ or $\ell + 1$ is a vertex in components of $\overline{\mathcal{X}}$. Similarly, 3, 16, and 19 cannot be in $B^*$.

With that in mind, we can have $3 \in A^*$ and $4 \in B^*$, but then 5 cannot be in either; 7 and 13 can only be in $B^*$; 16 cannot be in either; 19 can be in $A^*$. Thus far we have $\{1, 3, 6, 14, 17, 19\} \subset A^*$, and $\{2, 4, 7, 13, 15, 18, 20\} \subset B^*$. As the elements in $A \cap B$ are in neither $A^*$ nor $B^*$, we have no restrictions for 9 and 11. This means that we can have them in $A^*$ or in $B^*$ (even in both, if it was necessary for both of them to have at least $k$ elements). As right now $B^*$ already has $k = 7$ elements, we can have 9 and 11 in $A^*$. Now we have $\{1, 3, 6, 9, 11, 14, 17, 19\} \subset A^*$, and $\{2, 4, 7, 13, 15, 18, 20\} \subset B^*$. As both $A^*$ and $B^*$ are in $[n]_2$ and each of them has at least 7 elements, we stop adding elements to them. Finally, we can obtain $A'$ by eliminating any element from $A^*$, say $A' = \{1, 3, 6, 9, 14, 17, 19\}$, and we can have $B' = B^*$. Because of how we build them, the vertices $A, A', B'$, and $B$ induce a $P_4$ in $SG(20, 7)$, which means that $\text{dist}(A, B) \leq 3$. 4
Construction of sets $A^*$ and $B^*$ and an upper bound for $\text{dist}(A, B)$

In order to construct sets $A^*$ and $B^*$ corresponding to Example 1, we care particularly about the length of the connected components in $\overline{X}$ and their relation with the end-vertices of components of $\mathcal{X}$. In particular, being able to use all the elements in a connected component $C \in \overline{X}$ depends on the parity of $|C|$ and on the end-vertices of $C$. From now on, by Theorem 2, we assume that $n = 2k + r$ with $2 \leq r \leq 2k - 3$, that is, $2k + 2 \leq n \leq 4k - 3$ and we assume that $\text{dist}(A, B) \geq 3$.

We say that $\ell \in [n]$ is an end if $\ell \in X$ and $|\{(\ell - 1, \ell + 1) \cap \overline{X}\} \geq 1$. This is, $\ell$ is an endpoint if it is in $X$ and at least one of its neighbors in the cycle is in $\overline{X}$. Furthermore, we say that $\ell$ is an $A$-end if $\ell \in A \setminus B$, a $B$-end if $\ell \in B \setminus A$ and an $H$-end if $\ell \in A \cap B$. Finally, let $e(A), e(B)$ and $e(H)$ be the sets of $A$-ends, $B$-ends and $H$-ends respectively. Notice that $e(H) = A \cap B$, as every vertex in $A \cap B$ must be an end. Let $h = |e(H)|$. Finally, notice that if $A \neq B$ and $A \cap B = \emptyset$ then $e(H) \neq \emptyset$ and $e(A) \cup e(B) \neq \emptyset$ (actually, neither $e(A)$ nor $e(B)$ are empty). In Example 1, $e(A) = \{2, 15, 18, 20\}$, $e(B) = \{6, 14, 17\}$, and $e(H) = \{8, 10, 12\}$. Notice that $|e(A)| = 4$ and $|e(B)| = 3$, which means that, in general, $e(A)$ and $e(B)$ are not necessarily equal. To obtain the necessary relation between $e(A)$ and $e(B)$, it is helpful to study the structure of the components in $\mathcal{X}$.

We say that a connected component $C \in \mathcal{X}$ is an $A$-component if $|C \cap e(A)| \geq 1$ and $|C \cap e(B)| = 0$. Notice that this can happen in two different ways, either $|C| \geq 3$ and $|e(A) \cap C| = 2$ or $|C| = 1$ and $e(A) \cap C = C$. Similarly, we say that a connected component $C \in \mathcal{X}$ is a $B$-component if $|C \cap e(B)| \geq 1$ and $|C \cap e(A)| = 0$. If $C$ is neither an $A$-component nor a $B$-component, we say that $C$ is an $H$-component. Notice that if $C$ is an $H$-component, then $|C| = 1$ if and only if $C \subset A \cap B$. In this case, we say that $C$ is an $H'$-component, otherwise, we say that $C$ is an $H''$-component. In Example 1, $\{1, 2, 20\}$ is an $A$-component, $\{6\}$ is a $B$-component, and $\{8\}$, $\{10\}$, $\{14, 15\}$, and $\{17, 18\}$ are $H$-components, where $\{8\}$, $\{10\}$ and $\{12\}$ are $H'$-components, and $\{14, 15\}$ and $\{17, 18\}$ are $H''$-components. By $n(A)$, $n(B)$, $n(H)$, $n(H')$ and $n(H'')$ we denote the number of $A$-components, $B$-components $H$-components, $H'$-components and $H''$-components respectively. Notice that in Example 1, $n(A) = n(B)$, which is actually true in general as it is shown in Lemma 2.

**Lemma 2.** The number of $A$-components equals the number of $B$-components.

**Proof.** Consider the components in $\mathcal{X}$. If a component $C \in \mathcal{X}$ is an $H$-component, then $C$ has the same number of elements in $A$ and in $B$, i.e. $|C \cap A| = |C \cap B|$. If $C$ is an $A$-component, then $C$ has one extra element in $A$, i.e. $|C \cap A| = |C \cap B| + 1$. Similarly if $C$ is a $B$-component we get $|C \cap A| + 1 = |C \cap B|$. As $|A| = |B| = k$, we get that the number of $A$-components coincides with the number of $B$ components. \qed

Next, we obtain a formula relating $e(A)$ and $e(B)$. Notice that $|e(A)|$ is equal to twice the number of $A$-components of size at least 3, plus the number of $A$-components of size 1, plus the number of $H''$-components. We partition $e(A)$ into $e'(A)$ and $e''(A)$, where $e''(A)$ are the elements in $e(A)$ that are in connected components with exactly one element, and $e'(A)$ are the rest. In the same way, partition $e(B)$ into $e'(B)$ and $e''(B)$.

**Lemma 3.** If $A, B \in [n]_2$ then, $|e'(A)| + 2|e''(A)| = |e'(B)| + 2|e''(B)|$. 

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Proof. Consider the number $2n(A) + n(H'')$. Notice that in $n(H'')$ every element in $e'(A)$ which is in an $H''$-component is counted once. Furthermore, in $2n(A)$, every vertex in $e'(A)$ which is in an $A$-component is counted once, and every vertex in $e''(A)$ is counted twice. As every element in $e(A)$ is either in an $A$-component or in an $H''$-component, this yields: $2n(A) + n(H'') = |e'(A)| + 2|e''(A)|$. In a similar fashion, we get $2n(B) + n(H'') = |e'(B)| + 2|e''(B)|$. Therefore, by applying Lemma 2 we obtain $|e'(A)| + 2|e''(A)| = |e'(B)| + 2|e''(B)|$. □

Next we turn our focus on the connected components in $\overline{X}$. Now, we call block to each element of $\overline{X}$. Consider the following classifications of a block $[i, j]$:

- **Type I**: if $\{i - 1, j + 1\} \subseteq e(H)$.
- **Type II(A)**: if $i - 1 \in e(H)$ and $j + 1 \in e(A)$.
- **Type II(B)**: if $i - 1 \in e(H)$ and $j + 1 \in e(B)$.
- **Type III(A)**: if $i - 1 \in e(A)$ and $j + 1 \in e(H)$.
- **Type III(B)**: if $i - 1 \in e(B)$ and $j + 1 \in e(H)$.
- **Type IV(A)**: if $\{i - 1, j + 1\} \subset e(A)$.
- **Type IV(B)**: if $\{i - 1, j + 1\} \subset e(B)$.
- **Type IV(H)**: if $[i, j]$ is not of the types above, i.e. if $i - 1 \in e(A)$ and $j + 1 \in e(B)$ or vice versa.

Note that every block is of exactly one of the types above. Let $T = \{I, II(A), II(B), III(A), III(B), IV(A), IV(B), IV(H)\}$. We define $n(T)$ as the amount of blocks of type $T$, for $T \in T$. In Example 11, \{9\} and \{11\} are blocks of type I, there are no blocks of type II(A), \{13\} is a block of type II(B), there are no blocks of type III(A), \{7\} is a block of type III(B), \{19\} is a block of type IV(A), there are no blocks of type IV(B), and \{3, 4, 5\} and \{16\} are blocks of type IV(H).

Notice that if $[i, j]$ is a block, then $i - 1$ and $j + 1$ are ends in some components of $X$. In such a case, we say that $i - 1$ and $j + 1$ are connected to $[i, j]$.

There is an important difference between type IV and the rest of the types. If we are trying to form the sets $A^*$ and $B^*$ as was done with Example 11 and $[i, j]$ is a block of type $T \in \{I, II(A), II(B), III(A), III(B)\}$, then we can use every element in $[i, j]$ because we have restrictions in at most one of $\{i, j\}$ for the sets $A^*$ and $B^*$. If $[i, j]$ is a block of type IV(A) (or IV(B)) of even length, then all but one of the elements of $[i, j]$ can be used, because neither $i + 1$ nor $j - 1$ can be in $B^*$ ($A^*$ resp.). If $[i, j]$ is a block of type IV(H) of odd length, then all but one of the elements of $[i, j]$ can be used, as $i + 1$ and $j - 1$ cannot both be in $A^*$ nor both be in $B^*$. To reflect the fact that for a block $[i, j]$ of type $T \in \{IV(A), IV(B), IV(H)\}$ we can only assure the use of $|[i, j]| - 1$ elements, we define

$$m([i, j]) = \begin{cases} 
|[i, j]| & \text{if } [i, j] \text{ is a block of type } T \in \{I, II(A), II(B), III(A), III(B)\}; \\
|[i, j]| - 1 & \text{if } [i, j] \text{ is a block of type } T \in \{IV(A), IV(B), IV(H)\}.
\end{cases}$$

**Lemma 4.** $\sum_{[i,j] \in \overline{X}} m([i, j]) \geq n - 3k + 2h + 2$. 

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Proof. Observe that $\sum_{[i,j] \in \mathcal{E}} m([i,j]) = n - |A \cup B| - (n(IV(A)) + n(IV(B)) + n(IV(H))) = n - (2k - h) - (n(IV(A)) + n(IV(B)) + n(IV(H)))$. Therefore, in order to prove this result, it is enough to show that $(n(IV(A)) + n(IV(B)) + n(IV(H))) \leq k - h - 2$.

Let $\ell \in e(H)$. Notice that $\{\ell - 1, \ell + 1\} \subseteq X$. Furthermore, $\ell$ is connected to two blocks of type $T \in \{I, II(A), II(B), III(A), III(B)\}$. On the other hand, every block of type $I$ is connected to two elements of $e(H)$, while every block of type $T \in \{II(A), II(B), III(A), III(B)\}$ is connected to one element of $e(H)$. Thus, if we count twice the number of elements in $e(H)$, we count every block of type $I$ twice, and every block of type $T \in \{II(A), II(B), III(A), III(B)\}$ once. In other words, we get

$$2|e(H)| = 2n(I) + n(II(A)) + n(II(B)) + n(III(A)) + n(III(B))$$

$$|e(H)| = n(I) + \frac{n(II(A)) + n(II(B)) + n(III(A)) + n(III(B))}{2}.$$

Using that $|e(H)| = h$ we obtain,

$$h = n(I) + \frac{n(II(A)) + n(II(B)) + n(III(A)) + n(III(B))}{2}.$$

Furthermore, if $A \neq B$ and $A \cap B \neq \emptyset$, $n(II(A)) + n(II(B)) + n(III(A)) + n(III(B)) \geq 2$, as there has to be at least one block $[i,j]$ with $i - 1 \in e(H)$ and $j + 1 \in e(A) \cup e(B)$ and at least one block $[i,j]$ with $i - 1 \in e(A) \cup e(B)$ and $j + 1 \in e(H)$. Therefore, we get

$$n(I) + n(II(A)) + n(II(B)) + n(III(A)) + n(III(B)) \geq h + 1.$$

Let us assume that $|X| \leq k - 1$. Otherwise, by Observation 2, we would have that $\text{dist}(A, B) = 2$ which contradicts the initial assumption $\text{dist}(A, B) \geq 3$.

Therefore, $n(IV(A)) + n(IV(B)) + n(IV(H)) = |X| - (n(I) + n(II(A)) + n(II(B)) + n(III(A)) + n(III(B))) \leq k - 1 - (h + 1) = k - h - 2$. \hfill \Box

Lemma 5. $n(II(A)) + n(III(A)) + 2 n(IV(A)) = n(II(B)) + n(III(B)) + 2 n(IV(B))$.

Proof. Notice that every block of type $T \in \{II(A), III(A), IV(H)\}$ uses one endpoint in $e(A)$, and every block of type $T \in IV(A)$ uses two endpoints in $e(A)$. Furthermore, every endpoint in $e'(A)$ gets used once, while every endpoint in $e''(A)$ is used twice. This yields

$$n(II(A)) + n(III(A)) + n(IV(H)) + 2 n(IV(A)) = |e'(A)| + 2|e''(A)|.$$

Similarly,

$$n(II(B)) + n(III(B)) + n(IV(H)) + 2 n(IV(B)) = |e'(B)| + 2|e''(B)|.$$

Therefore, Lemma 3 yields:

$$n(II(A)) + n(III(A)) + n(IV(H)) + 2 n(IV(A)) = n(II(B)) + n(III(B)) + n(IV(H)) + 2 n(IV(B)).$$

Subtracting $n(IV(H))$ from both sides of the equation we obtain the result. \hfill \Box

We are ready now to construct the sets $A^*$ and $B^*$. We want these sets to satisfy:
(C1) \(A^*, B^* \in [n]_2\);

(C2) \(A \setminus B \subseteq B^*, B \setminus A \subseteq A^*\);

(C3) \(A \cap A^* = \emptyset, B \cap B^* = \emptyset\);

(C4) if \([i, j]\) is a block of type \(I, II(A), II(B), III(A)\) or \(III(B)\), then \([i, j] \subseteq A^* \cup B^*\);

(C5) if \([i, j]\) is a block of type \(IV(A), IV(B), IV(II)\), every element of \([i, j]\) except at most one belongs to \(A^* \cup B^*\).

Let us note the relation between the last two items and the definition of \(m([i, j])\). For each block \([i, j]\) at least \(m([i, j])\) elements belong to \(A^* \cup B^*\).

We denote by \(Z(i, j)\) the set of vertices in \([i, j]\) at odd distance of \(i - 1\) in the \(n\)-cycle in clockwise direction and by \(Y(i, j)\) the set of vertices in \([i, j]\) at even distance of \(i - 1\) in the \(n\)-cycle in clockwise direction. Besides, let \(Z'(i, j)\) be the set of vertices in \([i, j]\) at odd distance of \(j + 1\) in the \(n\)-cycle in counterclockwise direction and \(Y'(i, j)\) the set of vertices in \([i, j]\) at even distance of \(j + 1\) in the \(n\)-cycle in counterclockwise direction. For instance, if \([4, 10]\) is a block, then \(Z([4, 10]) = \{4, 6, 8, 10\} = Z'([4, 10])\) and \(Y([4, 10]) = \{5, 7, 9\} = Y'([4, 10])\). On the other hand, if \([4, 9]\) is a block, then \(Z([4, 9]) = \{4, 6, 8\} = Y'([4, 9]), Y([4, 9]) = \{5, 7, 9\} = Z'([4, 9])\).

Note that \(Z(i, j)\) and \(Z'(i, j)\) are not empty. Besides, these sets are 2-stable, i.e. they belong to \([n]_2\). Let us assign elements from the blocks to the sets \(A^*\) and \(B^*\) by the following rules.

\(R1\) If \([i, j]\) is of type I with at least two elements, include \(Z(i, j)\) in \(A^*\) and \(Y(i, j)\) in \(B^*\).

\(R2\) If \([i, j]\) is of type II(A), include \(Z'(i, j)\) in \(A^*\) and \(Y'(i, j)\) in \(B^*\).

\(R3\) If \([i, j]\) is of type II(B), include \(Z'(i, j)\) in \(B^*\) and \(Y'(i, j)\) in \(A^*\).

\(R4\) If \([i, j]\) is of type III(A), include \(Z(i, j)\) in \(A^*\) and \(Y(i, j)\) in \(B^*\).

\(R5\) If \([i, j]\) is of type III(B), include \(Z(i, j)\) in \(B^*\) and \(Y(i, j)\) in \(A^*\).

\(R6\) If \([i, j]\) is of type IV(A), include \(Z(i, j)\) in \(A^*\) and \(Y(i, j) \setminus \{j\}\) in \(B^*\).

\(R7\) If \([i, j]\) is of type IV(B), include \(Z(i, j)\) in \(B^*\) and \(Y(i, j) \setminus \{j\}\) in \(A^*\).

\(R8\) If \([i, j]\) is of type IV(II), with \(i - 1 \in A \setminus B\) and \(j + 1 \in B \setminus A\) include \(Z(i, j) \setminus \{j\}\) in \(A^*\) and \(Y(i, j)\) in \(B^*\). If \(i - 1 \in B \setminus A\) and \(j + 1 \in A \setminus B\) include \(Z(i, j) \setminus \{j\}\) in \(B^*\) and \(Y(i, j)\) in \(A^*\).

Notice that in rules \(R1-R8\) the elements in blocks of the form \([i, i]\) of type I are not assigned. These elements play a key role that we will mention further. Hence, we define the set \(I'\) as the set of such elements, i.e. \(I' = \{i | [i, i] \text{ is a block of type I}\}\).

Consider the sets \(A^*\) and \(B^*\) constructed following the rules above, and also including to \(A^*\) the elements in \(B \setminus A\) and assigning to \(B^*\) the elements in \(A \setminus B\). It is not hard to see that \(A^*\) and \(B^*\) satisfy:

- \(A^*, B^* \in [n]_2\);
- \(A \cap A^* = A^* \cap B^* = B \cap B^* = \emptyset\);
Lemma 6. Let $A^* = |A^* \cap A| = k - h$; for every block of type $T \in \{II(A), III(A), IV(A)\}$, $|A^* \cap T| \geq 1$; for every block of type $T \in \{II(B), III(B), IV(B)\}$, $|B^* \cap T| \geq 1$; and for every block $[i, j]$ of type $I$ with at least two elements, $|A^* \cap I| \geq 1$ and $|B^* \cap I| \geq 1$.

From sets $A^*$ and $B^*$ we can construct two vertices $A' \subset A^*$ and $B' \subset B^*$ in $[n]^k_2$ as follows.

Lemma 6. Let $A, B \in [n]^k_2$ with $|A \cap B| = h$. Then, there exist $A', B' \in [n]^k_2$ such that $|A' \cap B'| \leq h - 1$ and $A \cap A' = B \cap B' = \emptyset$.

Proof. In order to obtain vertices $A' \subset A^*$ and $B' \subset B^*$ such that $|A'| = |B'| = k$, $A \cap A' = B \cap B' = \emptyset$, and $|A' \cap B'| \leq h - 1$, we will use the elements of $I'$.

First, notice that for every element $i \in A \cap B$ the block of the form $[i + 1, j]$ is a block of type $T \in \{I, II(A), II(B)\}$, thus $h = n(I) + n(II(A)) + n(II(B))$. Similarly, for every element $j \in A \cap B$ the block of the form $[i, j - 1]$ is a block of type $T \in \{I, III(A), III(B)\}$, thus $h = n(I) + n(III(A)) + n(III(B))$. Then

$$2h = 2n(I) + n(II(A)) + n(II(B)) + n(III(A)) + n(III(B)).$$

It follows that $n(II(A)) + n(III(A)) + n(II(B)) + n(III(B))$ is even, so let $s \in \mathbb{N}$ such that $2s = n(II(A)) + n(III(A)) + n(II(B)) + n(III(B))$. Then, $2h = 2n(I) + 2s$, which means $n(I) = h - s$.

Furthermore, assume w.l.o.g. that $n(II(A)) + n(III(A)) = s + t \geq n(II(B)) + n(III(B)) = s - t$, for some $t \in \mathbb{N}$. By Lemma 5

$$2n(IV(B)) \geq n(II(A)) + n(III(A)) - n(II(B)) - n(III(B)) = s + t - (s - t) = 2t.$$ 

Thus, $IV(B) \geq t$. Hence, $n(II(A)) + n(III(A)) + n(IV(A)) \geq s$ and $n(II(B)) + n(III(B)) + n(IV(B)) \geq s$. Therefore, from Rules R2 – R7, we have $A^*$ has at least $s$ elements from blocks of types $II(A), III(A)$ and $IV(A)$, and $B^*$ has at least $s$ elements from blocks of types $II(B), III(B)$ and $IV(B)$.

Let $r$ be the amount of blocks $[i, j]$ of type $I$ with at least two elements. Then from previous remarks and rule R1, both $A^*$ and $B^*$ have at least $r$ elements from these blocks. Furthermore, $|I'| = h - s - r$.

Counting again the size of $A^*$, we have

- $A^*$ has $k - h$ elements from $B$;
- $A^*$ has at least $s$ elements from blocks of types $II(A), III(A)$ and $IV(A)$;
- $A^*$ has at least $r$ elements from blocks of types $I$ with at least two elements.

Thus, $|A^*| \geq k - h + s + r = k - (h - s - r)$. This means that if we assign every element in $I'$ to $A^*$, then $|A^*| \geq k$. Similarly, if we assign every element in $I'$ to $B^*$, then $|B^*| \geq k$. This also yields $|A^* \cap B^*| \leq h - s - r$.

Notice that there must exist at least one block of type $T$ for some $T \in \{II(A), II(B), III(A), III(B)\}$, as otherwise only blocks of type $I$ would exist, which implies $A = B$. Hence $s \geq 1$, and $h - s - r \leq h - 1$. Therefore, taking $A' \subset A^*$ and $B' \subset B^*$, with $|A'| = |B'| = k$, we have that $A \cap A' = B \cap B' = \emptyset$ and $|A' \cap B'| \leq |A^* \cap B^*| \leq h - s - r \leq h - 1$ which proves the result.
Theorem 3. Let then, by Theorem 2, we deduce that dist(\(A, B\)) ≤ 3.

Corollary 1. If there are no blocks \([i, i]\) of type I, then dist(\(A, B\)) ≤ 3.

Proof. Notice that if there are no blocks \([i, i]\) of type I in the proof of Lemma 6, then, \(|I'| = h - s - r = 0\) and thus we have that \(|A'| ≥ k\), \(|B'| ≥ k\) and \(A \cap A'^* = A^* \cap B^* = B \cap B^* = \emptyset\), and thus, dist(\(A, B\)) ≤ 3, which implies actually that dist(\(A, B\)) = 3.

Lemma 7. Let \(A, B \in [n]_2^k\) with \(|A \cap B| = h\). Then, dist(\(A, B\)) ≤ 1 + 2h.

Proof. By applying Lemma 6 \(h\) times, we obtain two vertices \(A^{(h)}, B^{(h)} \in [n]_2^k\), with \(A^{(h)} \cap B^{(h)} = \emptyset\). Hence, dist(\(A, B\)) ≤ 1 + 2h.

Corollary 2. Let \(A, B, Y \in [n]_2^k\) with \(|A \cap Y| = h', |Y \cap B| = h''\), and let \(h^* = h' + h''\) with \(h^* ≥ 2\). Then, dist(\(A, B\)) ≤ 2 + 2h^*.

Proof. The result follows by applying Lemma 7 to bound dist(\(A, Y\)) and dist(\(Y, B\)).

Lemmas 4 and 7 and Corollaries 1 and 2 will be used to compute the diameter of \(SG(n, k)\) when \(2k + 2 ≤ n ≤ 4k + 3\) in the next subsections.

2.2 Case 3k - 2 ≤ n ≤ 4k - 3

Let \(n = 2k + r\) with \(k > 2\) and \(k - 2 ≤ r ≤ 2k - 3\). Let us consider the construction of sets \(A^*\) and \(B^*\) given in Section 2.1. Let remark that in the proof of Lemma 6 we may not need to assign every element in \(I'\) to both \(A^*\) and \(B^*\). Note that following the rules R2-R8, from each block \([i, j]\) of type II, III or IV, we have included at least \(m([i, j])\) elements in \(A^* \cup B^*\) such that \(A^* \cap B^* = \emptyset\) and \(A^*, B^* \in [n]_2\). If we do not assign the \(h - s - r\) elements in \(I'\), by Lemma 4 we have assigned at least \(n - 3k + 2h + 2 - (h - s - r)\) elements from blocks to \(A^* \cup B^*\), \(k - h\) elements from \(A\) and \(k - h\) elements from \(B\). This means that before assigning the \(h - s - r\) elements in \(I'\), we have assigned \(n - k + 2 - (h - s - r)\) elements to \(A^* \cup B^*\). Hence, if \(n - k + 2\) is large enough, we may be able to assign the elements in such blocks maintaining \(A^* \cap B^* = \emptyset\).

Assume \(n ≥ 3k - 2\). Then, before assigning the elements in \(I'\), we have assigned at least \(n - k + 2 - (h - s - r) ≥ 2k - h + s + r\) to \(A^* \cup B^*\), at least \(k - h + s + r\) elements to \(A^*\), and at least \(k - h + s + r + a\) elements to \(B^*\). Let \(0 ≤ a ≤ h - s - r\) and assume that we have assigned \(k - h + s + r + a\) elements to \(A^*\). This means that we assigned at least \(2k - h + s + r - (k - h + s + r + a) = k - a\) elements to \(B^*\). If \(h - s - r - a > 0\), assign that many elements from \(I'\) to \(A^*\) and the remaining \(a\) elements to \(B^*\), otherwise, if \(h - s - r - a = 0\), assign every element in \(I'\) to \(B^*\). Then \(|A^*| ≥ k\), \(|B^*| ≥ k\), and \(A^* \cap B^* = \emptyset\). Let \(A' \subset A^*\) and \(B' \subset B^*\), such that \(|A'| = |B'| = k\). Then we have \(A \cap A' \supseteq A' \cap B' \supseteq B' \cap B = \emptyset\), which means that dist(\(A, B\)) ≤ 3. As by hypothesis, \(n < 4k - 2\) then, by Theorem 2, we deduce that dist(\(A, B\)) ≥ 3.

Theorem 3. Let \(n = 2k + r\) with \(k > 2\) and \(k - 2 ≤ r ≤ 2k - 3\). Then, D(\(SG(n, k)\)) = 3.

Notice that in Rules R1 – R8 we have not assigned elements in blocks of type IV(\(H\)) of the form \([i, i]\).

Observation 4. Let \(k > 2\) and \(3k - 2 ≤ n ≤ 4k - 3\). If two vertices \(A, B\) are at distance 3, there exist two vertices \(A', B'\) constructed following the rules R1-R8 such that \(\{A, A', B', B\}\) induce a \(P_4\) in \(SG(n, k)\). Besides, if \([i, i]\) is a block of type IV(\(H\)), \(i \notin A' \cup B'\).
2.3 Case $2k + 2 \leq n \leq 3k - 3$

In this section, we show that $D(SG(2k + r, k)) \leq k - r + 1$ when $2 \leq r \leq k - 3$, or equivalently, $D(SG(3k - 2 - m, k)) \leq 3 + m$ for $1 \leq m \leq k - 4$. To do this, given two vertices $A, B \in [n]_2^k$, we use two different operations that yield sets $\tilde{A}$ and $\tilde{B}$ in $[n + 1]_2^k$. We apply the operations successively until we obtain two vertices $A^p, B^p \in [n + 1]_2^k$ with $\text{dist}(A^p, B^p) \leq 3$ in $SG(n + p, k)$. If $\text{dist}(A^p, B^p) = 2$ in $SG(n + p, k)$, we obtain a vertex $Y \in [n]_2^k$ such that $\text{dist}(A, Y) + \text{dist}(B, Y) \leq m + 3$. If $\text{dist}(A^p, B^p) = 3$ in $SG(n + p, k)$, we obtain two vertices $A', B' \in [n]_2^k$ using Rules R1-R8 such that $A \cap A' = B \cap B' = \emptyset$, and $\text{dist}(A', B') \leq 1 + m$.

Now we can begin describing the operations that yield sets in $[n + 1]_2^k$. The first operation will work by adding an element in a component $C \in \mathcal{X}$, with $|C| \geq 3$; the second operation will work by adding an element to a block $[t, t]$ of type $I$. Because we are going to be talking about distances in Schrijver graphs with different values of $n$, we denote $\text{dist}_n(A, B)$ the distance between $A$ and $B$ in $SG(n, k)$.

Let $A, B \in [n]_2^k$ such that $\text{dist}_n(A, B) \geq 3$ and $X = A \cup B$. From item (iii) in Observation 3, there exist $C \in \mathcal{X}$ such that $|C| \geq 3$. Consider $C = a_1 b_j a_{i+1} \ldots$ (first case) or $C = b_j a_i b_{j+1} \ldots$ (second case). To obtain sets in $[n + 1]_2^k$ we will add an extra element between the second and third elements in $C$, i.e., between $b_j$ and $a_{i+1}$ in the first case, and between $a_i$ and $b_{j+1}$ in the second case. Hence, we assign to $A$ and $B$ (in any case) the following sets in $[n + 1]$, $A^+ = \{a_1, \ldots, a_i, a_{i+1} + 1, \ldots, a_k + 1\}$ and $B^+ = \{b_1, \ldots, b_j, b_{j+1} + 1, \ldots, b_k + 1\}$. Notice that $|A^+| = |B^+| = k$, as we did not increase the amount of elements. Furthermore, the sets are 2-stable, because $A$ and $B$ are 2-stable ($a_1$ and $a_k + 1$ cannot be consecutive, because that would imply that $a_1 = 1$ and $a_k + 1 = n + 1$, and $a_k = n$). Therefore, $A^+, B^+ \in [n + 1]_2^k$.

By adding this new element, we formed a new block. Observe that if $C = a_i b_j a_{i+1} \ldots$, then $a_{i+1} \notin A^+ \cup B^+$, and if $C = b_j a_i b_{j+1} \ldots$, then $b_{j+1} \notin A^+ \cup B^+$. Thus, in the first case, $[a_{i+1}, a_{i+1}]$ is a block of type $IV(H)$ in $\mathcal{X}$ with $X = A^+ \cup B^+$. Analogously, in the second case, $[b_{j+1}, b_{j+1}]$ is a block of type $IV(H)$ in $\mathcal{X}$.

Concerning the inverse operation of operation $+$, for $Y \in [n + 1]_2^k$, we define a set $Y^- = \{y_1^-, \ldots, y_k^-\} \in [n]_2^k$ by deleting the element that we added. In other words, considering $u = a_{i+1}$ if we are in the first case and $u = b_{j+1}$ if we are in the second case, we have that $y_r^- = y_r$ if $y_r < u$ and $y_r^- = y_r - 1$ if $y_r \geq u$.

The following remark is straightforward from the previous definitions.

**Observation 5.** If $Y \cap A^+ \cap B^+ = \emptyset$ then $Y^- \cap A \cap B = \emptyset$.

Notice that $Y^-$ is not 2-stable if and only if $u - 1, u + 1 \in Y$ or if $u - 2, u \in Y$. But if $X = A^+ \cup B^+$, then $\{u - 2, u - 1\}$ is a connected component of $\mathcal{X}$, and $u + 1$ is the first element (in clockwise direction) in a connected component in $\mathcal{X}$.

**Observation 6.** Let $Y \in [n + 1]_2^k$ and suppose that for every element $v \in Y \cap (A^+ \cup B^+)$, $v$ is not the first element of a connected component $C \in \mathcal{X}$. Then $\{u - 2, u + 1\} \cap Y = \emptyset$ and $Y^- \in [n]_2^k$.

Observation 6 is stated in such a convoluted way to make easier the proof of the main theorem in this section, which uses successive applications of $+$, together with a second operation which is defined later in this section.
As Observation 6 assures that \( \{u-2, u+1\} \cap Y = \emptyset \), we can study the relation between \( Y^- \cap A \) and \( Y \cap A^+ \), and similarly with \( B \), to obtain the following.

**Observation 7.** Let \( Y \in [n+1]_2^k \) such that \( u-2 \) and \( u+1 \) are not in \( Y \). If \( Y^- \) is defined as above, then \( Y^- \in [n]_2^k \) and \(|Y \cap A^+|+|Y \cap B^+| = |Y^- \cap A|+|Y^- \cap B| \) if \( u \notin Y \), or \(|Y \cap A^+|+|Y \cap B^+|+1 = |Y^- \cap A|+|Y^- \cap B| \) if \( u \in Y \). Furthermore, if no element \( v \in Y \cap (A^+ \cup B^+) \) is the first element in a connected component \( C \) of \( A^+ \cup B^+ \), then no element \( v \in Y^- \cap (A \cup B) \) is the first element in a connected component \( C \) of \( A \cup B \).

Notice that if \( u-2, u \in A \), and \( Y \cap A^+ = \emptyset \) then \( u-2, u+1 \notin Y \) and \( Y^- \cap A = \emptyset \). On the other hand, if \( u-1 \notin A \), \( Y \cap A^+ = \emptyset \) and \( u \notin Y \), then \( u-1, u \notin Y \) and \( Y^- \cap A = \emptyset \). This yields the following.

**Observation 8.** Let \( Y \in [n+1]_2^k \). If \( Y \cap A^+ = \emptyset \) and \( u \notin Y \), then \( Y^- \in [n]_2^k \) and \( A \cap Y^- = \emptyset \).

Finally, let \( Y_1, Y_2 \in [n+1]_2^k \) and consider \( Y_1^- \cap Y_2^- \). If \( \{u-1, u\} \not\subseteq Y_1 \cup Y_2 \), then \( y \in Y_1 \cup Y_2 \) if and only if \( y^- \in Y_1^- \cap Y_2^- \) (where \( y^- \) is as in the definition of \( Y^- \)). Hence, we have the following

**Observation 9.** Let \( Y_1, Y_2 \in [n+1]_2^k \). If \( u \notin Y_1 \cup Y_2 \), then \( Y_1^- \cap Y_2^- = Y_1 \cap Y_2 \).

We are ready to introduce the second operation. Notice that if \( \text{dist}_n(A, B) \geq 4 \), Corollary \[\square\] implies that there exists a block \([t, t] \) of type \( I \). The operation will work by adding an extra element at position \( t+1 \), thus increasing the size of the block. To be more precise, we define an operation on \( A \) and \( B \), denoted \( \uparrow \), by assigning the following two sets \( A^\uparrow = \{a_1^\uparrow, \ldots, a_k^\uparrow\} \) and \( B^\uparrow = \{b_1^\uparrow, \ldots, b_k^\uparrow\} \) in \([n+1]_2^k \) as follows:

\[
a_i^\uparrow = \begin{cases} a_i & \text{if } a_i \leq t-1; \\
a_i + 1 & \text{if } a_i \geq t+1. \end{cases}
\]

\[
b_i^\uparrow = \begin{cases} b_i & \text{if } b_i \leq t-1; \\
b_i + 1 & \text{if } b_i \geq t+1. \end{cases}
\]

Note that, if \( X = A^\uparrow \cup B^\uparrow \) \((X \subseteq [n+1])\), then \([t, t+1] \) is a block of type \( I \) in \( \overline{X} \).

We define now the inverse operation of operation \( \uparrow \). Given \( Y \in [n+1]_2^k \), we define a set \( Y^\downarrow = \{y_1^\downarrow, \ldots, y_k^\downarrow\} \in [n]^k \) as follows:

\[
y_i^\downarrow = \begin{cases} y_r & \text{if } y_r \leq t; \\
y_r - 1 & \text{if } y_r \geq t+1. \end{cases}
\]

As with the operation \( + \), we care about when \( Y^\downarrow \) is 2-stable, and about the relation between the intersections of \( Y \) with \( A^\uparrow \) and \( B^\uparrow \), and the intersection of \( Y^\downarrow \) with \( A \) and \( B \). Notice that if \( Y \cap \left[A^\uparrow \cup B^\uparrow\right] = \emptyset \), \( Y^\downarrow \in [n]_2^k \). Moreover, we have the following result.

**Lemma 8.** Let \( A, B \in [n]_2^k \) such that there exists a block \([t, t] \) of type \( I \) in \( X \) with \( X = A \cup B \). Consider \( A^\uparrow \) and \( B^\uparrow \) defined as above. Let \( Y \in [n+1]_2^k \) such that \( Y \cap A^\uparrow \cap B^\uparrow = \emptyset \) and \( Y^\downarrow \) defined as above. Then \( Y^\downarrow \in [n]_2^k \), \( Y^\downarrow \cap A \cup B = \emptyset \) and \(|Y \cap A^\uparrow|+|Y \cap B^\uparrow| = |Y^\downarrow \cap A|+|Y^\downarrow \cap B|\).

**Proof.** Consider \([t, t+1] \) the block of type \( I \) defined as above. Since \( Y \in [n+1]_2^k \), \(|Y \cap \{t, t+1\}| \leq 1 \). In addition, from the fact that \( Y \cap A^\uparrow \cap B^\uparrow = \emptyset \), \( t-1 \) and \( t+2 \) are not in \( Y \). Thus \( t-1 \) and \( t+2 \) are not in \( Y^\downarrow \). Therefore, \( Y^\downarrow \in [n]_2^k \), \( Y^\downarrow \cap A \cup B = \emptyset \) and \(|Y \cap A^\uparrow|+|Y \cap B^\uparrow| = |Y^\downarrow \cap A|+|Y^\downarrow \cap B|\). \[\square\]

In particular, the following result follows immediately from Lemma 8 if \( Y \) \( (A^\uparrow \cup B^\uparrow) = \emptyset \).
Corollary 3. Let $\{t, t\}$ be a block of type I in $X$ with $X = A \cup B$. If $\text{dist}_{n+1}(A^\uparrow, B^\uparrow) = 2$ then $\text{dist}_n(A, B) = 2$.

Let $Y_1, Y_2 \in \{n + 1\}^k_2$ and consider $Y_1^\downarrow \cap Y_2^\downarrow$. If $\{t, t + 1\} \not\subset Y_1 \cup Y_2$, then $y \in Y_1 \cap Y_2$ if and only if $y^\downarrow \in Y_1^\downarrow \cap Y_2^\downarrow$ (where $y^\downarrow$ is as in the definition of $Y^\downarrow$). If $\{t, t + 1\} \subset Y_1 \cup Y_2$, then $t \in Y_1^\downarrow \cap Y_2^\downarrow$. Hence we have the following.

Observation 10. If $Y_1, Y_2 \in \{n + 1\}^k_2$, then $|Y_1^\downarrow \cap Y_2^\downarrow| \leq |Y_1 \cap Y_2| + 1$.

Starting from $A^0 = A$ and $B^0 = B$ and by applying repeatedly the operations $+$ and $\uparrow$, we are able to construct two vertices $A^p, B^p \in \{n + p\}^k_2$, with $p \leq m$, such that $\text{dist}_{n+p}(A^p, B^p) \leq 3$. Then, by applying repeatedly the operations $-$ and $\downarrow$, we obtain the following result.

Theorem 4. Let $n = 3k - 2 - m$ with $1 \leq m \leq k - 4$. Then, $\text{D(SG}(n, k)) \leq m + 3$.

Proof. Let $A, B \in \{n\}^k_2$ and assume that $\text{dist}_{n}(A, B) \geq 4$. Corollary 11 and Observation 33 imply that both $+$ and $\uparrow$ can be applied. Let $A^0 = A, B^0 = B$, and for $1 \leq \ell \leq p$ let

$$A^\ell = \begin{cases} (A^{\ell - 1})^+ & \text{if } \ell \text{ is odd;} \\ (A^{\ell - 1})^\uparrow & \text{if } \ell \text{ is even;} \end{cases} \quad \text{and} \quad B^\ell = \begin{cases} (B^{\ell - 1})^+ & \text{if } \ell \text{ is odd;} \\ (B^{\ell - 1})^\downarrow & \text{if } \ell \text{ is even;} \end{cases}$$

where $p$ is the smallest integer such that $\text{dist}_{n+p}(A^p, B^p) \leq 3$. Theorem 3 implies that $p \leq m$.

Let $X^\ell = A^\ell \cup B^\ell$, and $X^\ell$ be the family of connected components of $X^\ell$. We divide the proof in two cases: $\text{dist}_{n+p}(A^p, B^p) = 2$ and $\text{dist}_{n+p}(A^p, B^p) = 3$.

Let $X^\ell = A^\ell \cup B^\ell$, and $X^\ell$ be the family of connected components of $X^\ell$. We divide the proof into two cases: $\text{dist}_{n+p}(A^p, B^p) = 2$ and $\text{dist}_{n+p}(A^p, B^p) = 3$.

If $\text{dist}_{n+p}(A^p, B^p) = 2$, there exists $Y \in \{n + p\}^k_2$ such that $Y \cap (A^p \cup B^p) = \emptyset$. Let $Y^p = Y$, and for $0 \leq \ell \leq p - 1$ let

$$Y^\ell = \begin{cases} (Y^{\ell + 1})^- & \text{if } \ell \text{ is even;} \\ (Y^{\ell + 1})^\downarrow & \text{if } \ell \text{ is odd.} \end{cases}$$

From Corollary 3 it follows that the last operation applied to $A$ and $B$ is $\uparrow$. Thus, $p$ is odd and the sets $A^p$ and $B^p$ are obtained after applying $\frac{p - 1}{2}$ operations $+$ and $\frac{p + 1}{2}$ operations $\uparrow$. Furthermore, Observations 3 5 and 7 imply $Y^{p - 1} \cap A^{p - 1} \cap B^{p - 1} = \emptyset, Y^{p - 1} \in \{n + p - 1\}^k_2$ and $|Y^{p - 1} \cap A^{p - 1}| + |Y^{p - 1} \cap B^{p - 1}| \leq 1$. Furthermore, if $v \in Y^{p - 1} \cap (A^{p - 1} \cap B^{p - 1})$, then $v$ is not the first element in a connected component of $X^{p - 2}$.

Now, the sets $A^{p - 1}, B^{p - 1}$ are obtained from operation $\uparrow$. Since $Y^{p - 1} \cap A^{p - 1} \cap B^{p - 1} = \emptyset$, from Lemma 8 $Y^{p - 2} \in \{n + p - 2\}^k_2$ and $|Y^{p - 2} \cap A^{p - 2}| + |Y^{p - 2} \cap B^{p - 2}| = |Y^{p - 1} \cap A^{p - 1}| + |Y^{p - 1} \cap B^{p - 1}| \leq 1$. Furthermore, if $v \in Y^{p - 2} \cap (A^{p - 2} \cup B^{p - 2})$, then $v$ is not the first element in a connected component of $X^{p - 2}$.

Applying this reasoning repeatedly, we get that for $q \geq 1$, $Y^{p - 2q - 1} \in \{n + p - 2q - 1\}^k_2$ and $|Y^{p - 2q - 1} \cap A^{p - 2q - 1}| + |Y^{p - 2q - 1} \cap B^{p - 2q - 1}| \leq q + 1$, and if $v \in Y^{p - 2q - 1} \cap (A^{p - 2q - 1} \cup B^{p - 2q - 1})$, then $v$ is not a first element in a connected component of $X^{p - 2q - 1}$. Similarly, Lemma 8 implies $Y^{p - 2q - 2} \in \{n + p - 2q - 2\}^k_2, |Y^{p - 2q - 2} \cap A^{p - 2q - 2}| + |Y^{p - 2q - 2} \cap B^{p - 2q - 2}| = |Y^{p - 2q - 1} \cap A^{p - 2q - 1}| + |Y^{p - 2q - 1} \cap B^{p - 2q - 1}| \leq q + 1$ and if $v \in Y^{p - 2q - 2} \cap (A^{p - 2q - 2} \cup B^{p - 2q - 2})$, then $v$ is not the first element in a connected component of $X^{p - 2q - 2}$.

Notice that letting $q = (p - 1)/2$ we have $p - 2q - 1 = 0$. Then, $|Y^0 \cap A^0| + |Y^0 \cap B^0| \leq 1 + (p - 1)/2 = (p + 1)/2$. Then, Corollary 2 implies $\text{dist}_{n}(A, B) \leq 2 + (p + 1)/2 = p + 3 \leq m + 3$.

Assume now that $\text{dist}_{n+p}(A^p, B^p) = 3$. From Observation 4 there exist $(A^p)'(B^p)' \in \{n + p\}^k_2$ constructed following the rules R1-R8 such that $A^p \cap (A^p)' = (A^p)' \cap (B^p)' = B^p \cap (B^p)' = \emptyset$. 

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Let $Y^p_1 = (A^p)'$ and $Y^p_2 = (B^p)'$, and for every $0 \leq \ell \leq p - 1$, $i \in \{1, 2\}$, let

$$Y^{\ell}_i = \begin{cases} \left( Y^{\ell+1}_i \right)^\uparrow & \text{if } \ell \text{ is even;} \\ \left( Y^{\ell+1}_i \right)^\downarrow & \text{if } \ell \text{ is odd.} \end{cases}$$

Let $1 \leq q \leq p$. Note that for every element $u$ added through the $+$ operation, $[u, u]$ is a block of type $IV(H)$ and from Observation 4, $u \notin (A^p)' \cup (B^p)'$. Then Observation 3 and Lemma 8 imply that $Y_1^{p-q} \cap A^{p-q} = \emptyset$, $Y_2^{p-q} \cap B^{p-q} = \emptyset$, and $Y_1^{p-q}, Y_2^{p-q} \in [n + p - q]_2$. This implies that $Y_1^0, Y_2^0 \in [n]_2$, $A \cap Y_1^0 = B \cap Y_2^0 = \emptyset$.

Consider $Y_1^q \cap Y_2^q$. If $q$ is even, then $Y_1^q = \left( Y_1^{q+1} \right)^\downarrow$, $Y_2^q = \left( Y_2^{q+1} \right)^\downarrow$, and the element added in the corresponding $+$ operation is not in $Y_1^{q+1} \cup Y_2^{q+1}$. Hence, Remark 9 implies $|Y_1^q \cap Y_2^q| \leq |Y_1^{q+1} \cap Y_2^{q+1}|$. On the other hand, if $q$ is odd, Remark 10 implies $|Y_1^q \cap Y_2^q| = |Y_1^{q+1} \cap Y_2^{q+1}| + 1$.

Therefore, $|Y_1^0 \cap Y_2^0|$ is bounded by the amount of $\uparrow$ operations applied. But, as the first operation applied is $+$, the number of $\uparrow$ operations is at most $p/2$. Thus, $|Y_1^0 \cap Y_2^0| \leq p/2$ and Lemma 7 implies $\text{dist}_n(Y_1^0, Y_2^0) \leq 1 + 2p/2 = 1 + p$. As $A \cap Y_1^0 = Y_2^0 \cap B = \emptyset$, this means that $\text{dist}_n(A, B) \leq 1 + p + 2 = p + 3 \leq m + 3$ and the result follows.

### 2.3.1 A lower bound for $2k + 2 \leq n \leq 3k - 3$

In this section we will prove that for $n = 2k + r$ with $2 \leq r \leq k - 3$, the diameter of $\text{SG}(n, k)$ verifies

$$\text{D}(\text{SG}(n, k)) \geq 4.$$  

For this, we provide two vertices $A^n_k, B^n_k \in [n]_2$ such that $\text{dist}_n(A^n_k, B^n_k) \geq 4$ as follows.

Let $n = 2k + r$ with $2 \leq r \leq k - 3$ and $t = k - 3 - r$. Let $A^n_k = \{1, 3, 5\} \cup \left( \bigcup_{i=0}^3 \{7 + 2i + 3j\} \right)$ and $B^n_k = \{1, 3, 6\} \cup \left( \bigcup_{i=0}^3 \{8 + 2i\} \right) \cup \left( \bigcup_{j=1}^3 \{8 + 2t + 3j\} \right)$. Note that $A^n_k \cap B^n_k = \{1, 3\}$, every block in $\mathcal{Y}$ has cardinality 1 and $|\mathcal{X}| = k - 1$. Then, $\text{dist}_n(A^n_k, B^n_k) \geq 3$.

Let $A', B' \in [n]_2$ such that $A'$ is a neighbor of $A^n_k$ and $B'$ is a neighbor of $B^n_k$ in $\text{SG}(n, k)$, i.e. $A' \cap A^n_k = B' \cap B^n_k = \emptyset$.

Let $A^* = \bigcup_{i=0}^3 \{7 + 2i - 1\}$ and $A^{**} = \bigcup_{j=1}^3 \{7 + 2t + 3j - 2, 7 + 2t + 3j - 1\}$. Notice that $A^n_k = \{2, 4, n - 1, n\} \cup A^* \cup A^{**}$. Analogously, if $B^* = \bigcup_{i=0}^3 \{8 + 2i - 1\}$ and $B^{**} = \bigcup_{j=1}^3 \{8 + 2t + 3j - 2, 8 + 2t + 3j - 1\}$, $B^n_k = \{2, 4, 5, n\} \cup B^* \cup B^{**}$. Then,

- $A' \subseteq \overline{A^n_k} = \{2, 4, n - 1, n\} \cup A^* \cup A^{**}$,
- $B' \subseteq \overline{B^n_k} = \{2, 4, 5, n\} \cup B^* \cup B^{**}$.

Since $A'$ is a 2-stable set, $|A' \cap (A^* \cup A^{**})| \leq t + 1 + r - 1 = k - 3$. In the same way, $|B' \cap (B^* \cup B^{**})| \leq k - 3$. Therefore, $|A' \cap \{2, 4, n - 1, n\}| \geq 3$ and $|B' \cap \{2, 4, 5, n\}| \geq 3$. These inequalities imply that $A' \cap B' \neq \emptyset$ and then $A'$ and $B'$ are not adjacent in $\text{SG}(n, k)$.

Thus, $\text{dist}_n(A^n_k, B^n_k) \geq 4$ if $2k + 2 \leq n \leq 3k - 3$. 

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Corollary 4. Let \( n = 2k + r \) with \( 2 \leq r \leq k - 3 \). Then \( D(SG(n,k)) \geq 4 \). In particular, \( D(SG(3k-3,k)) = 4 \), for \( k \geq 5 \).

*Proof. The lower follows from previous reasoning. From this lower bound and Theorem 4 we have \( D(SG(3k-3,k)) = 4 \), for \( k \geq 5 \). \qed

3 Case \( n = 2k + 2 \)

In order to study the diameter of \( SG(2k+2,k) \), we start by giving a full description of the graph.

Given a vertex \( A \in SG(2k+2,k) \), let \( \mathcal{A} \) denote the family of connected components of the graph induced by \( [n] \setminus A \) in \( C_n \). Notice that \( \mathcal{A} \) must have \( k \) connected components. This implies that it can have either one connected component of order 3 and the rest of order 1, or two connected components of order 2 and the rest of order 1. Furthermore, in the later case, the shortest path in \( C_n \) between the components of order 2 of \( \mathcal{A} \) (i.e. the minimum length of the two intervals on \( C_n \) separating these two components) contains \( 1 \leq i \leq \lfloor k/2 \rfloor \) elements of \( A \) and \( i-1 \) elements not in \( A \). In this case we say that the components are separated by \( i \) elements of \( A \). Thus, we define the following partition,

\[
\mathcal{B}_3 = \{ A \in SG(2k+2,k) | \mathcal{A} \text{ contains a component of order 3} \}
\]

\[
\mathcal{B}_{2,i} = \{ A \in SG(2k+2,k) | \mathcal{A} \text{ contains two components of order 2 separated by } i \text{ elements of } A \}.
\]

Notice that any vertex \( A \in \mathcal{B}_3 \) is characterized by the position of the first element \( v \) of the component of order 3 in \( \mathcal{A} \). We denote such a vertex as \( A = A_{0,v} \). Thus, \( |\mathcal{B}_3| = 2k+2 \). Similarly, any vertex \( A \in \mathcal{B}_{2,i} \) is characterized by the firsts elements, in the clockwise direction of \( C_n \), \( v \) and \( v + 2i + 1 \) of the connected components of \( \mathcal{A} \) of order 2. Notice that if \( k \) is even and \( i = k \), there are \( k+1 \) such pairs, and that otherwise there are \( 2k+2 \) such pairs. Hence we have

\[
|\mathcal{B}_3| = 2k+2,
\]

\[
|\mathcal{B}_{2,i}| = \begin{cases} k+1 & \text{if } k \text{ is even and } i = k/2, \\ 2k+2 & \text{otherwise}. \end{cases}
\]

We continue by studying the neighborhood of each vertex. Let \( A_0^v = \{ v-1, v+3, v+5, \ldots, v+2k-1 \} \in \mathcal{B}_3 \). If \( B \) is a vertex adjacent to \( A_0^v \), then either \( B \) contains exactly one element in \( \{ v, v+1, v+2 \} \) and every element in \( \{ v+4, v+6, \ldots, v+2k-2, v+2k \} \), or \( B \) contains both \( v \) and \( v+2 \), and \( k-2 \) elements in \( \{ v+4, v+6, \ldots, v+2k-2, v+2k \} \). Thus, the degree of \( A_0^v \) is \( 3 + \binom{k-1}{2} = k+2 \). If \( B \) is contained in \( \{ v, v+2, v+4, v+6, \ldots, v+2k-2, v+2k \} \), then \( B \) is in \( \mathcal{B}_3 \). More precisely, if \( v+2i \) is not in \( B \) for \( 1 \leq i \leq k+1 \), then \( B = A_0^{v+2i-1} \). On the other hand, if \( C = \{ v+1, v+4, v+6, \ldots, v+2k-2, v+2k \} \) is the other vertex adjacent to \( A_0^v \), then \( C \in \mathcal{B}_{2,1} \) because the connected components of order 2 in \( C \) are \( \{ v+2k+1, v \} \) and \( \{ v+2, v+3 \} \). As this is the only neighbor of \( A_0^v \) in \( \mathcal{B}_{2,1} \), we denote \( A_1^v = \{ v, v+2, v+4, v+6, \ldots, v+2k-2, v+2k \} \). Thus

\[
N(A_0^v) = \{ A_0^{v+2i-1} | 1 \leq i \leq k+1 \} \cup \{ A_1^v \}.
\]

Consider now \( A = \{ v-1, v+2, v+4, \ldots, v+2i, v+2i+3, v+2i+5, \ldots, v+2k-1 \} \in \mathcal{B}_{2,i} \), with \( 1 \leq i \leq \lfloor k/2 \rfloor \). The fact that \( \mathcal{A} \) has \( k \) connected components of order at most 2 implies that
any vertex $B$ adjacent to $A$ must intersect $\overline{A}$ in every connected component. Thus that are 4 such vertices, depending on which element of the connected components of order 2 they contain. This means that $|N(A)| = 4$. Furthermore, these neighbors are

$$B_1 = \{v - 2, v + 1, v + 3, \ldots, v + 2i - 1, v + 2i + 4, \ldots, v + 2k - 2\} \in B_{2,i}$$
$$B_2 = \{v, v + 3, v + 5, \ldots, v + 2i + 1, v + 2i + 2, v + 2i + 4, v + 2i + 6, \ldots, v + 2k\} \in B_{2,i}$$
$$B_3 = \{v, v + 3, v + 5, \ldots, v + 2i - 1, v + 2i + 2, v + 2i + 4, v + 2i + 6, \ldots, v + 2k - 2\}$$
$$B_4 = \{v - 2, v + 1, v + 3, \ldots, v + 2i + 1, v + 2i + 4, v + 2i + 6, \ldots, v + 2k\}$$

Notice that $B_3 \in B_3$ if $i = 1$, and $B_3 \in B_{2,i}$ otherwise. Further, notice that $B_4 \in B_{2,i+1}$ if $i \neq [k/2]$. This means that given $1 \leq i \leq [k/2]$, every vertex in $B_{2,i}$ has exactly one neighbor in $B_{2,i+1}$. Thus, for $1 < i \leq [k/2]$, we inductively denote vertex $A_i^u$ as the only neighbor of $A_i^{u-1}$ in $B_{2,i}$. Let $A_i^u = A = \{v - 1, v + 2, v + 4, \ldots, v + 2i - 1, v + 2i + 3, v + 2i + 5, \ldots, v + 2k\}$. Notice that $B_1 = A_i^{u-1}$ and $B_2 = A_i^{u+1}$. Further, notice that when $k$ is even and $i = (k/2) - 1$, we have

$$A_{k/2}^u = B_4$$

$$= \{v - 2, v + 1, v + 3, \ldots, v + 2i + 1, v + 2i + 4, v + 2i + 6, \ldots, v + 2k - 2\}$$

$$= \{v - 2, v + 1, v + 3, \ldots, v + 2i + 1, v + 2i + 4, v + 2i + 6, \ldots, v + 2k - 2\}$$

$$= \{v - 2, v + 1, v + 3, \ldots, v + 2i + 1, v + 2i + 4, v + 2i + 6, \ldots, v + 2k - 2\}$$

$$= \{v + 2k - 2, v + 2, v + 4, \ldots, v + k + 1, v + k + 1, \ldots, v + k + 1, v + k + 1, v + k + 1, \ldots, v + 2k - 1\}$$

$$= A_{k/2}^{u+k+1}.$$

This means that $A_{k/2}^u$ plays the role of $B_4$ for both $A_{(k/2)-1}^u$ and $A_{(k/2)-1}^{u+k+1}$. It also means that if $k$ is even, $i = k/2$ and $A = A_{k/2}^u$, then $B_4 = A_{(k/2)-1}^{u+k+1}$.

We have only left to discuss $B_4$ when $k$ is odd and $i = (k - 1)/2$. Letting $A_{(k-1)/2}^u = A = \{v - 1, v + 2, v + 4, \ldots, v + k - 1, v + k + 1, v + k + 4, \ldots, v + 2k - 1\}$, we have

$$B_4 = \{v - 2, v + 1, v + 3, \ldots, v + 2i + 1, v + 2i + 4, v + 2i + 6, \ldots, v + 2k - 2\}$$

$$= \{v - 2, v + 1, v + 3, \ldots, v + k, v + k + 3, v + k + 5, \ldots, v + 2k - 2\}$$

$$= \{v - 1, v + 1 + k, v + 3 + k, \ldots, v + 2k - 2, v - 2, v + 1, v + 3, \ldots, v + 2k - 2\}$$

$$= \{v + 2k - 2, v + 2, v + 4 + k, \ldots, v + k - 3 + k + 1, v + k - 1 + k + 1, v + 2k - 1 + k + 1\}$$

$$= A_{(k-1)/2}^{u+k+1}.$$

We can now give a full description of $SG(2k + 2, k)$.

**Theorem 5.** The Schrijver graph $SG(2k + 2, k)$ has vertex set $V = V_1 \cup V_2$, where

$$V_1 = \{A_i^u \mid 0 \leq i \leq [k/2] - 1, 0 \leq v \leq 2k + 1\},$$

$$V_2 = \{A_i^{[k/2]} \mid 0 \leq v \leq 2k + 1 \text{ if } k \text{ is odd and } 0 \leq v \leq k \text{ if } k \text{ is even}\}.$$
Proposition 1. If \( k \) is odd, item 3.

The subgraph induced by \( B_3 \) is \( 2k + 2, k \) is the graph obtained as follows. Take the cartesian product \( C_{2k+2} \square P_{[k/2]+1} \). First, add edges between vertices that correspond to one end of \( P_{[k/2]+1} \) and are at odd distance in \( C_{2k+2} \). Next, take pairs of vertices that correspond to the other end of \( P_{[k/2]+1} \) and are at distance \( k+1 \). If \( k \) is odd, add an edge between every pair of such vertices. If \( k \) is even, identify each pair of those vertices. The resulting graph is \( SG(2k+2, k) \).

We can now give find the diameter of \( SG(2k+2, k) \). We begin by giving the diameter of some induced subgraphs.

**Proposition 1.**

1. The diameter of the subgraph of \( SG(2k+2, k) \) induced by \( B_3 \) is 2.

2. If \( k \) is odd, the diameter of the subgraph of \( SG(2k+2, k) \) induced by \( B_{2,(k-1)/2} \) is \((k+1)/2\).

3. If \( k \) is even, the diameter of the subgraph of \( SG(2k+2, k) \) induced by \( B_{2,k/2} \) is \((k/2)\).

Consider now two vertices \( A_i^v, A_j^u \), with \( i \leq j \). Using item 1. of Proposition 1 yields

\[
\text{dist}(A_i^v, A_j^u) \leq \text{dist}(A_i^v, A_0^v) + \text{dist}(A_0^v, A_0^u) + \text{dist}(A_0^u, A_j^u) \\
\leq i + 2 + j.
\]

If \( k \) is odd, item 2. yields

\[
\text{dist}(A_i^v, A_j^u) \leq \text{dist}(A_i^v, A_{(k-1)/2}^v) + \text{dist}(A_{(k-1)/2}^v, A_{(k-1)/2}^u) + \text{dist}(A_{(k-1)/2}^u, A_j^u) \\
\leq \frac{k-1}{2} - i + \frac{k+1}{2} + \frac{k-1}{2} - j \\
= \frac{3k-1}{2} - i - j \\
= \left\lfloor \frac{3k}{2} \right\rfloor - i - j.
\]

If \( k \) is even, item 3. yields

\[
\text{dist}(A_i^v, A_j^u) \leq \text{dist}(A_i^v, A_{k/2}^v) + \text{dist}(A_{k/2}^v, A_{k/2}^u) + \text{dist}(A_{k/2}^u, A_j^u) \\
\leq \frac{k}{2} - i + \frac{k}{2} + \frac{k}{2} - j \\
= \frac{3k}{2} - i - j \\
= \left\lfloor \frac{3k}{2} \right\rfloor - i - j.
\]
Thus, we have

\[ D(SG(2k + 2, k)) \leq \max_{0 \leq i, j \leq \lceil k/2 \rceil} \min \left\{ i + j + 2, \left\lfloor \frac{3k}{2} \right\rfloor - i - j \right\}. \]

As \( i + j + 2 \) and \( \left\lfloor \frac{3k}{2} \right\rfloor - i - j \) are respectively an increasing linear function and a decreasing linear function of \( (i + j) \), which meet at

\[ i + j + 2 = \left\lfloor \frac{3k}{2} \right\rfloor - i - j \]
\[ 2(i + j) = \left\lfloor \frac{3k}{2} \right\rfloor - 2 \]
\[ i + j = \frac{1}{2} \left\lfloor \frac{3k}{2} \right\rfloor - 1. \]

If \( \lfloor 3k/2 \rfloor \) is even, then \((1/2)\lfloor 3k/2 \rfloor = \lfloor 3k/4 \rfloor\), and

\[ D(SG(2k + 2, k)) \leq \max_{0 \leq i, j \leq \lfloor k/2 \rfloor} \min \left\{ i + j + 2, \left\lfloor \frac{3k}{2} \right\rfloor - i - j \right\} = \left\lfloor \frac{3k}{4} \right\rfloor - 1 + \frac{1}{2} = \left\lfloor \frac{3k}{4} \right\rfloor + 1. \]

If \( \lfloor 3k/2 \rfloor \) is odd, then \((1/2)\lfloor 3k/2 \rfloor = \lfloor 3k/4 \rfloor + (1/2)\) is not an integer. This means that, when \( \lfloor 3k/2 \rfloor \) is odd, \( \max_{0 \leq i, j \leq \lfloor k/2 \rfloor} \min \left\{ i + j + 2, \left\lfloor \frac{3k}{2} \right\rfloor - i - j \right\} \) is either \( \lfloor 3k/2 \rfloor - i - j \) with \( i + j = \lfloor 3k/2 \rfloor - 1 \) or \( i + j + 2 \) with \( i + j = \lfloor 3k/2 \rfloor - 1 \). But

\[ \left\lfloor \frac{3k}{2} \right\rfloor - \left( \left\lfloor \frac{3k}{4} \right\rfloor - 1 \right) = \left\lfloor \frac{3k}{2} \right\rfloor - \left\lfloor \frac{3k}{4} \right\rfloor + 1 \]
\[ \leq \left\lfloor \frac{3k}{4} \right\rfloor + 1, \]

and

\[ \left\lfloor \frac{3k}{4} \right\rfloor - 1 + 2 = \left\lfloor \frac{3k}{4} \right\rfloor + 1. \]

Thus, in any case,

\[ D(SG(2k + 2, k)) \leq \max_{0 \leq i, j \leq \lceil k/2 \rceil} \min \left\{ i + j + 2, \left\lfloor \frac{3k}{2} \right\rfloor - i - j \right\} = \left\lfloor \frac{3k}{4} \right\rfloor + 1. \]

When \( k \geq 2 \) is even we need to consider two more upper bounds. Consider vertices \( A_i^v \) and \( A_j^u \) and notice that \( A_0^v \) is either adjacent to \( A_0^u \) or to \( A_0^{u+k+1} \). Thus, let

\[ \ell = \begin{cases} 0 & \text{if } u - v \text{ is odd} \\ k + 1 & \text{if } u - v \text{ is even} \end{cases} \]
and notice that $A_0^v$ is adjacent to $A_0^{u+\ell}$ and that $A_{k/2}^u = A_{k/2}^{u+\ell}$. Thus,

$$\text{dist}(A_i^v, A_j^u) \leq \text{dist}(A_i^v, A_0^v) + \text{dist}(A_0^v, A_0^{u+\ell}) + \text{dist}(A_0^{u+\ell}, A_{k/2}^u) + \text{dist}(A_{k/2}^u, A_j^v)$$

$$\leq i + 1 + \frac{k}{2} + \frac{k}{2} - j$$

$$= k + i + 1 - j.$$

Notice that, if $j - i = k - \lfloor 3k/4 \rfloor - 1$, then

$$k + i + 1 - j \leq k + 1 - \left( k - \left\lfloor \frac{3k}{4} \right\rfloor - 1 \right)$$

$$= \left\lfloor \frac{3k}{4} \right\rfloor.$$

Finally, without loss of generality, $0 \leq j - i \leq k - \lfloor 3k/4 \rfloor - 2$ and $u - v \leq k + 1$. Then, we have

$$\text{dist}(A_i^v, A_j^u) \leq \text{dist}(A_i^v, A_j^v) + \text{dist}(A_j^v, A_j^u)$$

$$\leq j - i + u - v.$$

Notice that, if $v$ and $u$ are chosen so that $\text{dist}(A_{k/2}^v, A_{k/2}^u) = k/2$, then $u - v \leq (k/2) + 1$. Thus, we get

$$j - i + u - v \leq k - \left\lfloor \frac{3k}{4} \right\rfloor - 2 + \frac{k}{2} + 1$$

$$= \left\lfloor \frac{k}{4} \right\rfloor + \frac{k}{2} - 1$$

$$\leq \left\lfloor \frac{k}{4} \right\rfloor + 1 + \frac{k}{2} - 1$$

$$= \left\lfloor \frac{3k}{4} \right\rfloor.$$

Thus, if the previous paths proposed have length at least $\left\lfloor \frac{3k}{4} \right\rfloor + 1$, this last path has length $\left\lfloor \frac{3k}{4} \right\rfloor$. Therefore, we have

$$\text{D}(\text{SG}(2k + 2, k)) \leq \begin{cases} 
\left\lfloor \frac{3k}{4} \right\rfloor & \text{if } k \geq 2 \text{ is even} \\
\left\lfloor \frac{3k}{4} \right\rfloor + 1 & \text{otherwise.}
\end{cases}$$

In order to give two vertices at such distance, let

$$v = \begin{cases} 
\frac{k}{2} & \text{if } k \equiv 0 \pmod{4} \\
\frac{k}{2} + 1 & \text{if } k \equiv 1 \pmod{4} \\
\frac{k}{2} + 1 & \text{if } k \equiv 2 \pmod{4} \\
\frac{k}{2} + 1 & \text{if } k \equiv 3 \pmod{4}
\end{cases}.$$ 

Notice that $\text{dist}(A_0^0, A_0^v) = \text{D}(B_3) = 2$, as $v$ is even, and that $\text{dist}(A_{k/2}^0, A_{k/2}^v) = \text{D}(B_{2, \lfloor k/2 \rfloor}) = \lfloor (k+1)/2 \rfloor$. Take vertices $B_1 = A_{k/2}^0$ and $B_2 = A_{3k/4-1-\lfloor k/2 \rfloor}^v$. Notice that if $\lfloor 3k/4 \rfloor - 1 - \lfloor k/2 \rfloor \geq
0, then $k \geq 3$. It is easy to check that any path that goes from $B_1$ to $B_2$ and does not use vertices from $B_3$ has length at least

$$\left\lfloor \frac{k}{2} \right\rfloor - \left( \left\lfloor \frac{3k}{4} \right\rfloor - 1 - \left\lfloor \frac{k}{2} \right\rfloor \right) + \left\lfloor \frac{(k + 1)/2}{2} \right\rfloor = 2\left\lfloor \frac{k}{2} \right\rfloor + 1 - \left\lfloor \frac{3k}{4} \right\rfloor + \left\lfloor \frac{(k + 1)/2}{2} \right\rfloor$$

$$= \left\lfloor \frac{3k}{2} \right\rfloor - \left\lfloor \frac{3k}{4} \right\rfloor + 1$$

$$\geq \left\lfloor \frac{3k}{4} \right\rfloor + 1.$$  

If $k$ is odd it is also easy to check that the shortest path using vertices from $B_3$ has length

$$\left\lfloor \frac{k}{2} \right\rfloor + \left\lfloor \frac{3k}{4} \right\rfloor - 1 - \left\lfloor \frac{k}{2} \right\rfloor + 2 = \left\lfloor \frac{3k}{4} \right\rfloor + 1.$$

On the other hand, if $k \geq 3$ is even, then a shortest path using vertices from $B_3$ is given by

$$A_0^0, A_{(k/2)-1}^{k+1}, \ldots, A_1^{k+1}, A_0^{k+1}, A_0^v, A_1^v, \ldots, A_{\left\lfloor \frac{3k}{4} \right\rfloor - (k/2)},$$

which has length

$$\frac{k}{2} + 1 + \left\lfloor \frac{3k}{4} \right\rfloor - 1 - \frac{k}{2} = \left\lfloor \frac{3k}{4} \right\rfloor.$$

Hence, we have the following.

**Theorem 6.** Let $k \geq 3$. The diameter of $SG(2k + 2, k)$ is $\left\lfloor \frac{3k}{4} \right\rfloor$ if $k$ is even and $\left\lfloor \frac{3k}{4} \right\rfloor + 1$ if $k$ is odd.

## 4 Conclusions

In this article we found the exact value of $D(SG(2k + r, k))$ when $r \leq 2$ and when $r \geq k - 3$. We give both an upper and a lower bound for the remaining cases, but evidence suggest they can be improved. Finding a lower bound for $3 \leq r \leq k - 4$ that is a function of $k$ would be quite interesting.

Based on Theorem 6 one would think that for other small values of $r$ the bound from Theorem 4 is not tight. Furthermore, the evidence suggest that $D(SG(2k + r, k))$ is a non-increasing function of $r$.

**Conjecture 1.** If $r \geq 1$, then $D(SG(2k + r, k)) \geq D(SG(2k + r + 1, k))$.

If Conjecture 1 is true, then $D(SG(2k + r, k)) \leq \left\lfloor \frac{3k}{4} \right\rfloor - (k \mod 2)$ if $r \geq 2$, which is an improvement on the bound in Theorem 4 when $r \leq \left\lfloor \frac{3k}{4} \right\rfloor$. This would also imply that there are more intervals where the diameter remains constant, as it happens when $k - 2 \leq r \leq 2k - 3$ and when $r \geq 2k - 2$. Finding said intervals would be interesting.

Our results allow us to compute (exactly) the diameter of $SG(n, k)$ for every $2 \leq k \leq 6$ and $n \geq 2k + 1$. We summarize it in Table 1. Notice that Theorem 4 would suggest that $D(SG(n, k)) - D(SG(n + 1, k)) \in \{0, 1\}$, which is the case for $k \leq 5$. But Theorem 6 lets us compute $D(SG(2k + 1, k)) - D(SG(2k + 2, k))$. When $k \geq 6$ we get

$$D(SG(2k + 1, k)) - D(SG(2k + 2, k)) = k - \left( \left\lfloor \frac{3k}{4} \right\rfloor + (k \mod 2) \right)$$

$$= \left\lfloor \frac{k}{4} \right\rfloor - (k \mod 2),$$

$$\geq 2.$$
Table 1: Diameter of $\text{SG}(n,k)$ for $2 \leq k \leq 7$. (*) Computationally computed using SageMath.

This raises the question about whether another such gap could appear as $r$ moves through the interval $[2, k - 3]$. Based on our computations, we propose the following conjecture.

**Conjecture 2.** If $2 \leq r \leq k - 2$, then $D(\text{SG}(2k + r, k)) - D(\text{SG}(2k + r + 1, k)) \in \{0, 1\}$.

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