BRANCHING RANDOM WALKS AND MULTI-TYPE CONTACT-PROCESSES
ON THE PERCOLATION CLUSTER OF \( \mathbb{Z}^d \)

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ABSTRACT. In this paper we prove that under the assumption of quasi-transitivity, if a branching random walk on \( \mathbb{Z}^d \) survives locally (at arbitrarily large times there are individuals alive at the origin), then so does the same process when restricted to the infinite percolation cluster \( C_\infty \) of a supercritical Bernoulli percolation. When no more than \( k \) individuals per site are allowed, we obtain the \( k \)-type contact process, which can be derived from the branching random walk by killing all particles that are born at a site where already \( k \) individuals are present. We prove that local survival of the branching random walk on \( \mathbb{Z}^d \) also implies that for \( k \) sufficiently large the associated \( k \)-type contact process survives on \( C_\infty \). This implies that the strong critical parameters of the branching random walk on \( \mathbb{Z}^d \) and on \( C_\infty \) coincide and that their common value is the limit of the sequence of strong critical parameters of the associated \( k \)-type contact processes. These results are extended to a family of restrained branching random walks, that is branching random walks where the success of the reproduction trials decreases with the size of the population in the target site.

Keywords: branching random walk, contact process, percolation cluster, critical parameters, approximation.
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1. Introduction

The branching random walk is a process which serves as a model for a population living in a spatially structured environment (the vertices of a graph \((X, \mathcal{E}(X))\)). Each individual lives in a vertex, breeds and dies at random times and each offspring is placed (according to some rule) in one of the neighbouring vertices. Since for the branching random walk (BRW in short) there is no bound on the number of individuals allowed per site, it is natural to consider a modification of the process, namely the multitype contact process, where, for some \( k \in \mathbb{N} \), no more than \( k \) particles per site are allowed (if \( k = 1 \) one gets the usual contact process). The multitype contact processes are more realistic models, indeed instead of thinking of the vertices of the graph as small portions of the ecosystem where individuals may pile up indefinitely (like in the BRW), here each vertex can host at most \( k \) individuals. This is in particular true for patchy habitats (each vertex represents a patch of soil) or in host-symbionts interactions (each vertex represents a host on top of which symbionts may live), see for instance [3, 4, 6].

The need for more realistic models also brings random environment into consideration. BRWs in random environment has been studied by many authors (see for instance [13, 16, 18, 23, 24, 27]). In many cases the random environment is a random choice of the reproduction law of the process (in some cases there is no death). In our case we put the randomness into the underlying graph. When choosing \((X, \mathcal{E}(X))\), \( \mathbb{Z}^d \) is perhaps the first choice that comes to mind but other graphs are reasonable options. In particular the BRW and the contact process have been studied also on trees ([19, 20, 21, 25, 28, 32]) and on random graphs as Galton-Watson trees ([29]). Although \( \mathbb{Z}^d \) has clear properties of regularity, which make it a nice case to study, random graphs are believed to serve as a better model for real-life structures and social networks. It is therefore of interest to investigate the behaviour of stochastic processes on random graphs, which possibly retain some regularity properties which make them treatable. An example is the small world, which is the space model in [15] and [5], where each vertex has the same number of neighbours. The percolation cluster of \( \mathbb{Z}^d \) given
by a supercritical Bernoulli percolation, which we denote by $C_\infty$, has no such regularity, but has a “stochastic” regularity and its geometry, if viewed at a large scale, does not differ too much from $\mathbb{Z}^d$ (for instance it is true that, for large $N$, in many $N$-boxes of $\mathbb{Z}^d \cap C_\infty$, there are open paths crossing in each direction and these paths connect to crossing paths in neighbouring boxes, see [17] Chapter 7]). Indeed $C_\infty$ shares many stochastic properties with $\mathbb{Z}^d$: the simple random walk is recurrent in $d = 1, 2$, transient in $d \geq 3$ and the transition probabilities have the same space-time asymptotics as those of $\mathbb{Z}^d$ (with different constants, [11]); two walkers collide infinitely many often in $d = 1, 2$ and finitely many times in $d \geq 3$ (see [2]); the voter model clusters in $d = 1, 2$ and coexists in $d \geq 3$ (see [3]); just to mention a few facts.

The aim of this paper is to compare the critical parameters of the BRW and of the multitype contact process on the infinite percolation cluster $C_\infty$ with the corresponding ones on $\mathbb{Z}^d$ (from now on we tacitly assume that the infinite cluster exists almost surely, that is that the underlying Bernoulli percolation is supercritical). In order to define these parameters, let us give a formal definition of the processes involved.

Let $(X, \mathcal{E}(X))$ be a graph and $\mu : X \times X \rightarrow [0, +\infty)$ such that $\mu(x, y) > 0$ if and only if $(x, y) \in \mathcal{E}(X)$. We require that there exists $K < +\infty$ such that $\zeta(x) := \sum_{y \in X} \mu(x, y) \leq K$ for all $x \in X$. Given $\lambda > 0$, the $\lambda$-branching random walk ($\lambda$-BRW or, when $\lambda$ is not relevant, BRW) is the continuous-time Markov process $\{\eta_t\}_{t \geq 0}$, with configuration space $\mathbb{N}^X$, where each existing particle at $x$ has an exponential lifespan of parameter 1 and, during its life, breeds at the arrival times of a Poisson process of parameter $\lambda \zeta(x)$ and then chooses to send its offspring to $y$ with probability $\mu(x, y)/\zeta(x)$. Thus we associate to $\mu$ a family of BRWs, indexed by $\lambda$. With a slight abuse of notation, we will say that $(X, \mu)$ is a BRW $(\mu(x, y)$ represents the rate at which existing particles at $x$ breed in $y)$. The BRW is called irreducible if and only if the underlying graph is connected. Clearly, any BRW on $\mathbb{Z}^d$ or $C_\infty$ is irreducible; we note that in their graph structure we possibly admit loops, that is, every vertex might be a neighbour of itself (thus allowing reproduction from a vertex onto itself). If $(Y, \mathcal{E}(Y))$ is a subgraph of $(X, \mathcal{E}(X))$, we denote by $\mu_{Y}(x, y)$ the map $\mu \cdot 1_{\mathcal{E}(Y)}$. The associated BRW $(Y, \mu_{Y})$, indexed by $\lambda$, is called the restriction of $(X, \mu)$ to $Y$ and, to avoid cumbersome notation, we denote it by $(Y, \mu)$.

Two critical parameters are associated to the continuous-time BRW: the weak (or global) survival critical parameter $\lambda_w$ and the strong (or local) survival one $\lambda_s$. They are defined as

$$\lambda_w(x_0) := \inf \{ \lambda > 0 : \mathbb{P}^{x_0} (\exists t : \eta_t = 0) < 1 \}$$
$$\lambda_s(x_0) := \inf \{ \lambda > 0 : \mathbb{P}^{x_0} (\exists \bar{t} : \eta_{\bar{t}}(x_0) = 0, \forall t \geq \bar{t}) < 1 \},$$

where $x_0$ is a fixed vertex, 0 is the configuration with no particles at all sites and $\mathbb{P}^{x_0}$ is the law of the process which starts with one individual in $x_0$. Note that, if the BRW is irreducible, then these values do not depend on the choice of $x_0$ nor on the initial configuration, provided that this configuration is finite (that is, it has only a finite number of individuals). When there is no dependence on $x_0$, we simply write $\lambda_w$ and $\lambda_s$. These parameters depend on $(X, \mu)$: when we need to stress this dependence, we write $\lambda_w(X, \mu)$ and $\lambda_s(X, \mu)$. We refer the reader to Section 3 for how to compute the explicit value of these parameters.

Given $(X, \mu)$ and a nonincreasing function $c : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, the restrained branching random walk (briefly, RBRW) $(X, \mu, c)$ is the continuous-time Markov process $\{\eta_t\}_{t \geq 0}$, with configuration space $\mathbb{N}^X$, where each existing particle at $x$ has an exponential lifespan of parameter 1 and, during its life, breeds, as the BRW, at rate $c(0)/\zeta(x)$, then chooses to send its offspring to $y$ with probability $\mu(x, y)/\zeta(x)$ and the reproduction is successful with probability $c(\eta(y))/c(0)$. For the RBRW the rate of successful reproductions from $x$ to $y$, namely $\mu(x, y)c(\eta(y))$ depends on the configuration (for a formal introduction to RBRWs, see [7]).

Restrainted branching random walks have been introduced in [7] in order to provide processes where the natural competition for resources in an environmental patch is taken into account (since $c$ is nonincreasing, the more individuals are present at a vertex, the more difficult it is for new
individuals to be born there). If we imagine that the vertex can host at most \( N \) individuals, a natural example of \( c \) is represented by the logistic growth \( c_N(i) = \lambda(1 - i/N)I_{[0,N]}(i) \). A more general choice where the parameter \( N \) represents the strength of the competition between individuals (the smaller \( N \), the stronger the competition), is given by fixing a nonincreasing \( \tilde{c} \) and letting \( c_N(\cdot) := \tilde{c}(\cdot/N) \).

The usual BRWs and multitype contact processes can be seen as particular cases of RBRWs: if \( c \equiv \lambda \) the associated RBBRW is the \( \lambda \)-BRW; if \( c = \lambda I_{[0,k-1]} \) we call the corresponding RBRW \( k \)-type contact process and we denote it by \( \{\eta_i^k\}_{i \geq 0} \). The critical parameters of the \( k \)-type contact process are denoted by \( \lambda^k_c \) and \( \lambda^k_w \).

The order relations between all these critical values are shown in Figure 1 where we avoid indicating the dependence on \( \mu \).

\[ \begin{array}{c}
\lambda_w(Z^d) \quad \lambda^k_w(Z^d) \quad \lambda_s(Z^d) \quad \lambda^k_s(Z^d) \\
\lambda_c(Z^d) \quad \lambda^k_c(Z^d) \quad \lambda_c(\infty) \quad \lambda^k_c(\infty) \\
\end{array} \]

**Figure 1.** Order relation between critical values (\( a \rightarrow b \) means \( a \geq b \)).

It has already been proven in [11] that if \( \mu \) is quasi-transitive on \( X \) (a property of regularity, see Definition 2.1) then \( \lambda^k_s(X,\mu) \xrightarrow{k \to \infty} \lambda_s(X,\mu) \) and, if \( \mu \) is translation invariant on \( Z^d \) then \( \lambda^k_w(Z^d,\mu) \xrightarrow{k \to \infty} \lambda_w(Z^d,\mu) \). Analogous results for discrete-time processes can be found in [33] and recently some progress has been made for discrete-time BRWs on Cayley graphs of finitely generated groups (see [26]).

When considering BRWs and multitype contact processes on \( C_\infty \) two natural questions arise. Firstly, we wonder whether the critical parameters of the BRW on \( C_\infty \) can be deduced from the ones of the BRW on \( Z^d \); secondly, whether the parameters of the \( k \)-type contact process converge to the corresponding ones of the BRW. Note that even if the BRW \( (Z^d,\mu) \) has good properties of regularity, like quasi-transitivity, its restriction to \( C_\infty \) has none of these properties and the aforementioned questions are not trivial.

Our main result answers both questions regarding \( \lambda_s \); for quasi-transitive BRWs on \( Z^d \) the strong critical parameter coincides with the one on \( C_\infty \) (this result was actually already in [11] Theorem 7.1] but here we provide a different proof which can be extended to answer the second question). Moreover the sequence of the strong critical parameters of \( k \)-type contact processes restricted to \( C_\infty \) converge to the one of the BRW on \( Z^d \). We note that here we consider only continuous-time processes but analogous results hold for discrete-time BRWs as well.

**Theorem 1.1.** Let \((Z^d,\mu)\) be a quasi-transitive BRW and \( C_\infty \subseteq Z^d \) be the infinite cluster of a supercritical Bernoulli percolation. Then

1. \( \lambda_s(C_\infty,\mu) = \lambda_s(Z^d,\mu) \) a.s. with respect to the realization of \( C_\infty \);
2. \( \lim_{k \to \infty} \lambda^k_s(C_\infty,\mu) = \lim_{k \to \infty} \lambda^k_s(Z^d,\mu) = \lambda_s(Z^d,\mu) \) a.s. with respect to the realization of \( C_\infty \).

The result for the weak critical parameter can be obtained when \( \lambda_w(Z^d,\mu) = \lambda_s(Z^d,\mu) \), which is for instance true when \( \mu \) is quasi-transitive and symmetric (see Section 2).

**Theorem 1.2.** Let \((Z^d,\mu)\) be a quasi-transitive BRW such that \( \lambda_w(Z^d,\mu) = \lambda_s(Z^d,\mu) \) and let \( C_\infty \subseteq Z^d \) be the infinite cluster of a supercritical Bernoulli percolation. Then, a.s. with respect to the realization of \( C_\infty \), \( \lim_{k \to \infty} \lambda^k_w(C_\infty,\mu) = \lim_{k \to \infty} \lambda^k_w(Z^d,\mu) = \lambda_w(C_\infty,\mu) = \lambda_w(Z^d,\mu) \).

The fact that whenever a quasi-transitive BRW on \( Z^d \) is locally supercritical (i.e. \( \lambda > \lambda_s \)), so are the \( k \)-type contact processes restricted to \( C_\infty \), whenever \( k \) is sufficiently large, also holds
for families of RBRWs, where \( c_N(z) := c(z/N) \) and \( c \) is a given nonnegative function such that \( \lim_{z \to 0^+} c(z) = c(0) > \lambda_s(Z^d, \mu) \).

**Theorem 1.3.** Let \((Z^d, \mu)\) be a quasi-transitive BRW. Let \( c \) be a nonnegative, nonincreasing function such that \( \lim_{z \to 0^+} c(z) = c(0) > \lambda_s(Z^d, \mu) \) and let \( c_N(z) := c(z/N) \). Consider the RBRWs \((Z^d, \mu, c_N)\) and \((C_\infty, \mu, c_N)\): they both survive locally whenever \( N \) is sufficiently large.

As an application, we have that [6, Theorem 1.2] can be refined, here is the improved statement.

**Corollary 1.4.** Let \( \mu(x, x) = \alpha \) and \( \mu(x, y) = \beta/2d \) for all \( x \in Z^d \) and \( y \) such that \(|x - y| = 1\), where \( \alpha \geq 0 \) and \( \beta > 0 \). Consider the RBRW \((C_\infty, \mu, c_N)\) where \( c_N(i) = (1 - i/N)1_{[0, N]}(i) \). Then

1. For all \( N > 0 \), the process dies out if \( \alpha + \beta \leq 1 \).
2. If \( \alpha + \beta > 1 \) then the process survives locally provided that \( N \) is sufficiently large.

To compare with [6, Theorem 1], we recall that the extinction phase, that is Corollary 1.4(1), was already stated as [6 Theorem 1.1]; to ensure survival when \( \alpha + \beta > 1 \) and \( N \) is large, [6, Theorem 1.2] requires that the parameter of the underlying Bernoulli percolation is sufficiently close to 1. This request has now been proven unnecessary, since it suffices that the Bernoulli percolation is supercritical.

### 2. Basic definitions and preliminaries

Explicit characterizations of the critical parameters are possible. For the strong critical parameter we have \( \lambda_s(x) = 1/\limsup_{n \to \infty} \sqrt{\mu^{(n)}(x, x)} \) (see [10, Theorem 4.1], [12, Theorem 3.2(1)]) where \( \mu^{(n)}(x, y) \) are recursively defined by \( \mu^{(n+1)}(x, y) = \sum_{w \in X} \mu^{(n)}(x, w) \mu(w, y) \) and \( \mu^{(0)}(x, y) = \delta_{xy} \).

As for \( \lambda_w(x) \), it is characterized in terms of solutions of certain equations in Banach spaces (see [10, Theorem 4.2]); moreover, \( \lambda_w(x) \geq 1/\liminf_{n \to \infty} \sqrt{\sum_{y \in X} \mu^{(n)}(x, y)} \) ([10, Theorem 4.3], [12, Theorem 3.2(2)]). The last inequality becomes an equality in a certain class of BRWs which contains quasi-transitive BRWs (see [10, Proposition 4.5], [12, Theorem 3.2(3)]). The definition of quasi-transitive BRW is the following.

**Definition 2.1.** \((X, \mu)\) is a quasi-transitive BRW if and only if there exists a finite set of vertices \( \{x_1, \ldots, x_r\} \) such that for every \( x \in X \) there exists a bijection \( f : X \to X \) such that \( f(x_j) = x \) for some \( j \) and \( \mu \) is \( f \)-invariant, that is \( \mu(w, z) = \mu(f(w), f(z)) \) for all \( w, z \).

Note that if \( f \) is a bijection such that \( \mu = f \)-invariant, then \( f \) is an automorphism of the graph \((X, E(X))\). In many cases \( \lambda_s \) coincides with \( \lambda_w \). For quasi-transitive and symmetric BRWs (that is, \( \mu(x, y) = \mu(y, x) \) for all \( x, y \)) it is known that \( \lambda_s = \lambda_w \) is equivalent to amenability ([12, Theorem 3.2]) which is essentially based on [10] and [32, Theorem 2.4]). Amenability is a slow growth condition: see [32, Section 1], for the definition of amenable graph and [12, Section 2], where \( m_{xy} \) stands for \( \mu(x, y) \), for the definition of amenable BRW. It is easy to prove that a quasi-transitive BRW is amenable if and only if the underlying graph is amenable. Examples of amenable graphs are \( Z^d \) along with its subgraphs. Therefore, every quasi-transitive and symmetric BRW on \( Z^d \) or \( C_\infty \) has \( \lambda_s = \lambda_w \).

A more general sufficient condition is the following, where symmetry is replaced by reversibility (i.e. the existence of measure \( \nu \) on \( X \) such that \( \nu(x) \mu(x, y) = \nu(y) \mu(y, x) \) for all \( x, y \)). It is a slight generalization of [9, Proposition 2.1] and easily extends to discrete-time non-oriented BRWs.

**Theorem 2.2.** Let \((X, \mu)\) be a continuous-time BRW and let \( x_0 \in X \). Suppose that there exists a measure \( \nu \) on \( X \) and \( \{c_n\}_{n \in \mathbb{N}} \) such that, for all \( n \in \mathbb{N} \)

\[
\begin{align*}
&\nu(y)/\nu(x_0) \leq c_n, \quad \forall y \in B(x_0, n) \\
&\nu(x)\mu(x, y) = \nu(y)\mu(y, x), \quad \forall x, y \in X,
\end{align*}
\]
Lemma 2.3. Let matrices $M \in \mathbb{C}$ be irreducible BRWs with a reversibility measure $\nu$. The assumptions of Theorem 2.2 are for instance satisfied, on subexponentially growing graphs, by noting that the irreducibility assumptions which were present in [31, 11, 33] are here dropped. In [31, Theorem 6.8], we restate here both the lemma and the approximation theorem. It is worth on nonnegative matrices and their convergence parameters, which in its original form can be found theorems which have been proven in a weaker form in [11, Theorem 3.1] for continuous-time BRWs, the family, indexed by $\mathbb{N}$, without requiring the whole matrix $M$ to be irreducible. The following theorem is the application

\[ \frac{1}{\lambda_s(x_0)} \leq \frac{1}{\lambda_w(x_0)} \leq \liminf_n \sqrt[n]{\sum_{y \in \mathbb{N}} \mu^{(n)}(x_0, y)} = \liminf_n \sqrt[n]{\sum_{y \in \mathbb{N}} \mu^{(n)}(x_0, y) \frac{\nu(x)}{c_n |B(x_0, n)|}} \leq \frac{1}{\lambda_s(x_0)}. \]

The condition $\sqrt[2]{|B(x_0, n)|} \to 1$ is usually called subexponential growth. Examples of subexponentially growing graphs are euclidean lattices $\mathbb{Z}^d$ or $d$-dimensional combs (see [8] for the definition). The assumptions of Theorem 2.2 are for instance satisfied, on subexponentially growing graphs, by irreducible BRWs with a reversibility measure $\nu$ such that $\nu(x) \leq C$ for all $x \in X$ and for some $C > 0$.

One of the tools in the proof of our results is the fact that if the BRW survives locally on a graph $X$, it also survives locally on suitable large subsets $X_n \subset X$. This follows from the spatial approximation theorem which have been proven in a weaker form in [11] Theorem 3.1 for continuous-time BRWs and in a stronger form in [33] Theorem 5.2 for discrete-time BRWs. The proofs rely on a lemma on nonnegative matrices and their convergence parameters, which in its original form can be found in [31] Theorem 6.8. We restate here both the lemma and the approximation theorem. It is worth noting that the irreducibility assumptions which were present in [31] [11] [33] are here dropped.

Given a nonnegative matrix $M = (m_{xy})_{x,y \in X}$, let $R(x, y) := 1/\limsup_{n \to \infty} \sqrt[m]{(m(n))}$ be the family, indexed by $x$ and $y$, of the convergence parameters, which are the entries of the $n$-th power matrix $M^n$. Note that, as recalled earlier in this section, $\lambda_s(X)$ coincides with the convergence parameter $R(x, x')$ of the matrix $(m_{xy})_{x,y \in X}$. Given a sequence of sets $\{X_n\}_{n \in \mathbb{N}}$ let $\liminf_{n \to \infty} X_n := \bigcap_{n \in \mathbb{N}} X_n$. Let $\{X_n\}_{n \in \mathbb{N}}$ be a general sequence of subsets of $X$ such that $\liminf_{n \to \infty} X_n = X$ and suppose that $M = (m_{xy})_{x,y \in X}$ is a nonnegative matrix. Consider a sequence of nonnegative matrices $M_n = (m(n))_{x,y \in X}$ such that $0 \leq m(n)_{xy} \leq m_{xy}$ for all $x, y \in X$, $n \in \mathbb{N}$ and $\liminf_{n \to \infty} m(n)_{xy} = m_{xy}$ for all $x, y \in X$. Then for all $x \in X$ we have $\lambda_n R(x_0, x_0) \to R(x_0, x_0)$ (a $R(x_0, x_0)$ being a convergence parameter of the matrix $M_n$).

Clearly, if $M$ is irreducible then $R(x, y) = R$ does not depend on $x, y \in X$ and for all $x \in X$ we have $\lambda_n R(x_0, x_0) \to R$. One can repeat the proof of [33] Theorem 5.2], noting that, since $R(x_0, x_0)$ depends only on the values of the irreducible class of $[x_0]$ then $\lambda_n R(x_0, x_0) \to R(x_0, x_0)$ without requiring the whole matrix $M$ to be irreducible. The following theorem is the application
of Lemma 2.3 to the spatial approximation of continuous-time BRWs (an analogous result holds for discrete-time BRWs, see [33, Theorem 5.2], where we can drop the irreducibility assumption).

**Theorem 2.4.** Let \( (X, \mu) \) be a continuous-time BRW and let us consider a sequence of continuous-time BRWs \( \{(X_n, \mu_n)\}_{n \in \mathbb{N}} \) such that \( \liminf_{n \to \infty} X_n = X \). Let us suppose that \( \mu_n(x, y) \leq \mu(x, y) \) for all \( x, y \in X_n \), \( n \in \mathbb{N} \) and \( \mu_n(x, y) \to \mu(x, y) \) as \( n \to \infty \) for all \( x, y \in X \). Then \( \lambda_s(X_n, \mu_n) \geq \lambda_s(X, \mu) \) and \( \lambda_s(X_n, \mu_n) \to \lambda_s(X, \mu) \) as \( n \to \infty \).

3. PROOFS AND APPLICATIONS

Before proving our main results, we need to prove some preparatory lemmas. The first lemma gives a useful expression for the expected value of the progeny living at time \( t \) at vertex \( y \) of a particle which was at \( x \) at time 0. Its proof, which can be found in [22, Section 3], is based on the construction of the process by means of its generator as done in [33]. The key to the proof is the fact that the expected value is the solution of a system of differential equations.

**Lemma 3.1.** For any \( \lambda \)-BRW on a graph we have that

\[
E(\eta_t(y)|\eta_0 = \delta_x) = e^{-t} \sum_{n=0}^{\infty} \mu^{(n)}(x, y) \frac{(\lambda t)^n}{n!}.
\]

The expected number of descendants of generation \( n \) (of a particle at \( x \) at time 0) is

\[
e^{-t} \mu^{(n)}(x, y) \frac{(\lambda t)^n}{n!},
\]

and the expected number of descendants of generation \( n \) (of a particle at \( x \) at time 0), along a path \( \gamma = (\gamma_0, \ldots, \gamma_n) \) is

\[
e^{-t} \prod_{i=0}^{n-1} \mu(\gamma_i, \gamma_{i+1}) \frac{(\lambda t)^n}{n!}.
\]

The following lemma shows that whenever a BRW on \( \mathbb{Z}^d \) survives locally (that is, \( \lambda > \lambda_s(\mathbb{Z}^d, \mu) \)), it also survives locally if restricted to boxes of sufficiently large radius. We denote by \( B(m) = [-m, m]^d \) the box centered at 0 and by \( x + B(m) \) its translate centered at \( x \).

**Lemma 3.2.** Let \( \mu \) be a BRW on \( \mathbb{Z}^d \). Then for all \( \lambda > \lambda_s(\mathbb{Z}^d, \mu) \) and for all \( x \in \mathbb{Z}^d \), there exists \( m(x) \in \mathbb{N} \) such that for all \( m \geq m(x) \), \( \lambda > \lambda_s(x + B(m), \mu) \). Moreover, if \( \mu \) is quasi-transitive, then there exists \( m_0 \) such that for all \( m \geq m_0 \), \( \lambda > \sup_{x \in \mathbb{Z}^d} \lambda_s(x + B(m), \mu) \).

**Proof.** Let \( X = \mathbb{Z}^d \), \( X_n := (x + B(n)) \) and \( \mu_n := \mu \cdot 1_{X_n \times X_n} \). By Theorem 2.4 there exists \( m \) such that \( \lambda > \lambda_s(X, \mu) \) for all \( n \geq m \).

If \( \mu \) is quasi-transitive, there exists a finite set of vertices \( \{x_1, \ldots, x_r\} \) as in Definition 2.4. It is clear that \( \lambda_s(A, \mu) = \lambda_s(f(A), \mu) \) for all \( A \subset \mathbb{Z}^d \) and for every automorphism \( f \) such that \( \mu \) is \( f \)-invariant. Given \( \lambda > \lambda_s(\mathbb{Z}^d, \mu) \), for every \( i \) there exists \( m_i \) such that \( \lambda > \lambda_s(x_i + B(m_i), \mu) \). Take \( m \geq m_0 := \max_{i=1,\ldots,r} m_i \); by monotonicity \( \lambda_s(x_i + B(m), \mu) \geq \lambda_s(x_i + B(m), \mu) \) for all \( i \). Thus \( \lambda > \max_{i=1,\ldots,r} \lambda_s(x_i + B(m), \mu) \). Let \( x \in \mathbb{Z}^d \) and \( f \) as in Definition 2.4 such that \( f(x_j) = x \) for some \( j \). Then \( \lambda_s(x + B(m), \mu) = \lambda_s(f(x + B(m)), \mu) = \lambda_s(x_j + B(m), \mu) \) and \( \max_{i=1,\ldots,r} \lambda_s(x_i + B(m), \mu) = \sup_{x \in \mathbb{Z}^d} \lambda_s(x + B(m), \mu) \).

The following lemma states that for any \( \lambda \)-BRW on a graph \( X \), with \( \lambda > \lambda_s(x) \) the expected value of the number of particles in a given site, grows exponentially in time.

**Lemma 3.3.** Let \( \mu \) be a BRW on a graph \( X \), \( x \in X \) and \( \lambda > \lambda_s(x) \). Let \( \{\eta_t\}_{t \geq 0} \) be the associated \( \lambda \)-BRW. Then there exists \( \varepsilon = \varepsilon(x, X), C = C(x, X) \) such that

\[
E(\eta_t(x)|\eta_0 = \delta_x) \geq C e^{\varepsilon t}, \quad \forall t \geq 0.
\]
Proof. We prove (3.2) for all \( t \geq t_1 \) for some \( t_1 \); the assertion then follows by replacing \( C \) with \( \min(C, C_1) \), where \( C_1 = \min_{t \in [0, t_1]} e^{-ct}E(\eta_t(x)|\eta_0 = \delta_x) \) which exists and it is strictly positive by continuity (since \( t \mapsto E(\eta_t(x)|\eta_0 = \delta_x) \) is a solution of a differential equation).

Since \( \lambda > \lambda_s(x) \), then \( \lambda \sqrt{\mu(n)(x, x)} > 1 \) for some \( n \). Therefore there exist \( n_0 \geq 1 \) and \( \varepsilon_1 > 0 \) such that \( \mu(n_0)(x, x) > \left( \frac{1 + \varepsilon_1}{\lambda} \right)^{n_0} \). By the supermultiplicativity of the sequence \( \mu(n)(x, x) \), for all \( r \in \mathbb{N} \),

\[
\mu^{(n_0 r)}(x, x) \geq \left( \frac{1 + \varepsilon_1}{\lambda} \right)^{n_0 r}.
\]

Recalling Lemma 3.1 we get

\[
\mathbb{E}(\eta_t(x)|\eta_0 = \delta_x) \geq e^{-t} \sum_{r \geq 0} \frac{((1 + \varepsilon_1)t)^{n_0 r}}{(n_0 r)!}.
\]

Let \( \tilde{\lambda} := 1 + \varepsilon_1 \). We can write a lower bound for the summands in the previous series:

\[
\frac{((\tilde{\lambda} t)^{n_0 r})}{(n_0 r)!} \geq \frac{\tilde{\lambda} t - 1}{(\tilde{\lambda} t)^{n_0} - 1} \cdot \left\{ \frac{(\tilde{\lambda} t)^{n_0 r}}{(n_0 r)!} + \frac{(\tilde{\lambda} t)^{n_0 r + 1}}{(n_0 r + 1)!} + \cdots + \frac{(\tilde{\lambda} t)^{n_0 (r+1) - 1}}{(n_0 (r+1) - 1)!} \right\}
\]

whence, for all \( t \geq t_1 \) and for some \( t_1 > 0 \), the following holds

\[
\mathbb{E}(\eta_t(x)|\eta_0 = \delta_x) \geq e^{-t} \cdot \frac{\tilde{\lambda} t - 1}{(\tilde{\lambda} t)^{n_0} - 1} \cdot e^{\tilde{\lambda} t} \geq \frac{\tilde{\lambda} t - 1}{(\tilde{\lambda} t)^{n_0} - 1} \cdot e^{\tilde{\lambda} t} \geq e^{\varepsilon_1 t/2}.
\]

The following lemma states that, for the BRW on \( \mathbb{Z}^d \), given two vertices \( x \) and \( y \) (also at a large distance), the expected progeny at \( y \) of a particle at \( x \), can be made arbitrarily large, after a sufficiently large time, even if the process is restricted to a large box centered at \( x \) plus a fixed path from \( x \) to \( y \) (see Figure 2). The idea of the proof is that the BRW can stay inside the box until the expected number of particles at \( x \) is large, and then move along the path towards \( y \).

\[\text{Figure 2. The portion of } \mathbb{Z}^d \text{ where we restrict the BRW.}\]

**Lemma 3.4.** Let \( \mu \) be a BRW on \( \mathbb{Z}^d \), \( x \in \mathbb{Z}^d \), \( \lambda > \lambda_s(\mathbb{Z}^d, \mu) \). Fix \( M, \delta > 0 \) and choose \( m \) such that \( \lambda > \lambda_s(x + B(m), \mu) \). Then there exists \( T = T(x, m, M, \delta) \) such that

\[
\mathbb{E}(\eta_t(y)|\eta_0 = \delta_x) \geq 1 + \delta,
\]

for all \( t \geq T \), \( \gamma \) path of length \( l \leq M \) with \( \gamma_0 = x \), \( \gamma_l = y \), where \( \{\eta_t\}_{t \geq 0} \) is the BRW restricted to \( (x + B(m)) \cup \gamma \). Moreover, if \( \mu \) is quasi-transitive, we can choose \( m \) and \( T \) independent of \( x \) such that (3.3) holds for all \( x \in \mathbb{Z}^d \).

**Proof.** Fix \( t_2 > 0 \). We use the Markov property of the BRW (and the superimposition with respect to the initial condition) and apply Lemma 3.1

\[
\mathbb{E}(\eta_{t_1 + t_2}(y)|\eta_0 = \delta_x) \geq \mathbb{E}(\eta_{t_1}(x)|\eta_0 = \delta_x) \cdot e^{-t_2} \prod_{i=0}^{l-1} \mu(\gamma_i, \gamma_{i+1}) \frac{(\lambda t_2)^i}{i!} \geq \mathbb{E}(\eta_{t_1}(x)|\eta_0 = \delta_x) \cdot e^{-t_2} \frac{(\lambda t_2 \alpha)^i}{i!} \geq \mathbb{E}(\eta_{t_1}(x)|\eta_0 = \delta_x) \cdot \delta,
\]

where \( \alpha = \min \lambda \).
where $0 < \alpha = \alpha(x, M) = \min \{ \mu(\gamma_i, \gamma_{i+1}^\prime): i = 0, \ldots, l^\prime - 1, \gamma^\prime \text{ path of length } l^\prime \leq M, \gamma_i^\prime = x \}$ and $0 < \varepsilon = \varepsilon(t_2, m, M) = \min \{ e^{-t_2} (\lambda t_2 \alpha^2)/l!: l \leq M \}$. Since $\bar{\eta}$ restricted to $x + B(m)$ survives locally, by Lemma 3.3

$$\mathbb{E}(\bar{\eta}_{t_1+t_2}(y)|\bar{\eta}_0 = \delta_x) \geq C e^{t_1} \cdot \varepsilon \geq 1 + \delta,$$

for all sufficiently large $t_1$ depending on $x, M$ and $\delta$. Fix $t_1$ and define $T(x, m, M, \delta) := t_1 + t_2$.

If $\mu$ is quasi-transitive, take $\{x_1, \ldots, x_r\}$ and $m_i$ as in the proof of Lemma 3.2. Take $m := \max_{i=1, \ldots, r} m_i$ and $T := \max_{i=1, \ldots, r} T(x_i, m, M, \delta)$ and the proof is complete. \hfill \Box

In the next lemma we prove that given $x$, $y$ and $y^\prime$, if we start the process with $l$ particles at $x$, after a sufficiently large time, with arbitrarily large probability, we will have $l$ particles both at $y$ and at $y^\prime$, even if we restrict the process to a large box centered at $x$ plus a fixed path from $x$ to $y$ and a fixed path from $x$ to $y^\prime$ (see Figure 3). The proof relies on Lemma 3.4 and the central limit theorem.

**Figure 3.** From $\ell$ individuals at $x$ to $\ell$ individuals at $y$ and $y^\prime$.

**Lemma 3.5.** Let $\mu$ be a BRW on $\mathbb{Z}^d$ and let $x$, $\lambda$ and $m$ as in Lemma 3.4. Fix $M, \varepsilon > 0$. Then choosing $T = T(x, m, M, 1)$ as in Lemma 3.4, for all $t \geq T$ there exists $\ell(\varepsilon, x, m, M, t) \in \mathbb{N}$ and

$$\mathbb{P}(\bar{\eta}_t(y) \geq \ell, \bar{\eta}_t(y^\prime) \geq \ell|\bar{\eta}_0(x) = \ell) > 1 - \varepsilon,$$

for all $\ell \geq \ell(\varepsilon, x, t)$, $1 \leq \ell \leq M$, $\gamma$, $\gamma^\prime$ paths of length $l$ and $l^\prime$ from $x$ to $y$ and to $y^\prime$ respectively, where $\bar{\eta}_t$ is the BRW restricted to $(x + B(m)) \cup \gamma \cup \gamma^\prime$. Moreover, if $\mu$ is quasi-transitive, we can choose $m$ and $T$ independent of $x$ and $\ell(\varepsilon, m, M)$ such that (3.4) holds for all $x \in \mathbb{Z}^d$ when $t = T$.

**Proof.** By monotonicity it suffices to prove the result with the event $(\bar{\eta}_0 = \ell \delta_x)$ in place of $(\bar{\eta}_0(x) = \ell)$.

Let $X = (x + B(m)) \cup \gamma \cup \gamma^\prime$. Let us denote by $\{\xi_t\}_{t \geq 0}$ the BRW, restricted to $X$, starting from $\xi_0 = \delta_x$. By Lemma 3.4 there exists $T$ such that $\mathbb{E}(\xi_t(x)|\xi_0 = \delta_x) > 2$ for all $t \geq T$, $z = y, y^\prime$. A realization of our process is $\bar{\eta}_t = \sum_{j=1}^l \xi_{t,j}$ where $\{\xi_{t,j}(y)\}_{j \in \mathbb{N}}$ is an iid family of copies of $\{\xi_t\}_{t \geq 0}$. Fix $z \in \{y, y^\prime\}$. Since $\xi_{t,j}$ is stochastically dominated by a continuous time branching process with birth rate $\lambda \sup_{w} \sum_v \mu(w, v) < +\infty$, it is clear that $\text{Var}(\xi_{t,j}(z)) := \sigma_{t,z}^2 < +\infty$ (note that the variance depends on $x$). Thus the Central Limit Theorem, if $\ell$ is sufficiently large,

$$\varepsilon \geq \frac{1}{4} \left| \frac{\sum_{j=1}^l \xi_{t,j}(z) \geq s|\xi_0 = \delta_x} - 1 + \phi \left( \frac{s - \ell \mathbb{E}(\xi_t(z)|\xi_0 = \delta_x)}{\sqrt{\ell \sigma_{t,z}}} \right) \right|$$

uniformly with respect to $s \in \mathbb{R}$, where $\phi$ is the cumulative distribution function of the standard normal. Whence there exists $\ell(\varepsilon, x, m, M, z, t)$ such that, for all $\ell \geq \ell(\varepsilon, x, m, M, z, t)$,

$$\mathbb{P}(\bar{\eta}_t(z) \geq \ell|\xi_0 = \delta_x) \geq 1 - \phi \left( \frac{\sqrt{\ell(1 - \mathbb{E}(\bar{\eta}_t(y)|\bar{\eta}_0 = \delta_x)) \sigma_{t,z}}}{\sigma_{t,z}} \right) - \frac{\varepsilon}{4} \geq 1 - \frac{\varepsilon}{2},$$

since $\sqrt{\ell(1 - \mathbb{E}(\bar{\eta}_t(y)|\bar{\eta}_0 = \delta_x)) / \sigma_{t,z}} \to -\infty$ as $\ell \to +\infty$. Take $\ell(\varepsilon, x, m, M, t) := \ell(\varepsilon, x, m, M, y, t) \lor \ell(\varepsilon, x, m, M, y^\prime, t)$. Hence, (3.4) follows.

If $\mu$ is quasi-transitive, take $\{x_i\}_{i=1}^r$ and $\{m_i\}_{i=1}^r$ as in the proof of Lemma 3.4. It suffices to choose $m := \max_{i=1, \ldots, r} m_i$ and $T := \max_{i=1, \ldots, r} T(x_i, m, M)$. \hfill \Box
We say that a subset $A$ of $\mathbb{Z}^d$ is contained in $\mathcal{C}_\infty$ if all the vertices are connected to $\mathcal{C}_\infty$ and all the edges $(x, y)$, with $x, y \in A$, are open. The following is a lemma on the geometry of $\mathcal{C}_\infty$ which states that $\mathcal{C}_\infty$ contains a bi-infinite open path where one can find large boxes at bounded distance from each other.

**Lemma 3.6.** For every $m \in \mathbb{N}$ there exists $M = M(m) > 0$ such that the infinite percolation cluster $\mathcal{C}_\infty$ contains a pairwise disjoint family $\{B_j\}_{j=-\infty}^{+\infty}$ with the following properties:

1. there exists $\{x_j\}_{j=-\infty}^{+\infty}$, $x_j \in \mathbb{Z}^d$ for all $j$, and $B_j = x_j + B(m)$ for all $j$;
2. there is a family of open paths $\{\pi_j\}_{j=-\infty}^{+\infty}$ such that $x_j \xrightarrow{\pi_j} x_{j+1}$, and $|\pi_j| \leq M$ for all $j$.

**Proof.** For every $N \in \mathbb{N} \setminus \{0\}$, we define the $N$-partition of $\mathbb{Z}^d$ as the collection $\{2N\mathbb{Z} + B(N) : x \in \mathbb{Z}^d\}$.

We use [30, Proposition 4.1] which holds also for $d = 2$ according to [14, Proposition 11]. In order to achieve in [14, Proposition 11] the same generality of [30, Proposition 4.1] one has to take into account also a general family of events $\{V_T\}_T$ (indexed on the boxes of the collection of the $N$-partitions as $N \in \mathbb{N} \setminus \{0\}$) satisfying equation (4.4) of [30]. This can be easily done by noting that the inequality (4.25) of [30] still holds in the case $d = 2$. From now on, when we refer to [30, Proposition 4.1] we mean this “enhanced” version which holds for $d \geq 2$.

We define $V_T := \{\text{there exists a seed } x_T + B(N^{1/2d}) \subseteq \Gamma\}$ where by seed we mean a box with no close edges in the percolation process (to avoid a cumbersome notation, we omit the integer part symbol $\lceil \cdot \rceil$ in the side length). Note that $V_T$ is measurable with respect to the $\sigma$-algebra of the percolation process restricted to $\Gamma$, thus independent from the rest of the process. Given a box $\Gamma$ of side length $N$, by partitioning it into disjoint boxes of side length $N^{1/2d}$ we obtain the following upper estimate $\Phi[(V_T)] \leq (1 - p)(N/N^{1/2d})^d = (1 - p)N^{-1/2} \to 0$ as $n \to \infty$, where $\Phi$ is the law of the Bernoulli percolation on $\mathbb{Z}^d$ with parameter $p$. This implies that $\{V_T\}_T$ satisfies equation (4.4) of [30].

We consider $N > m^{2d} \vee 23 (N \geq 24$ is required in [14, 30]). Thus the seed $x_T + B(N^{1/2d}) \subseteq \Gamma$, when it exists, contains an open path of length $N^{1/2} - 1 > N^{1/2}/10$, hence it is connected to the crossing cluster in $\Gamma$ by construction of the renormalized process (see [30, Section 4.2] or [14, Section 5]). Moreover it contains a translated box $x_T + B(m)$. By [30, Proposition 4.1], given a supercritical Bernoulli percolation on $\mathbb{Z}^d$, for all sufficiently large $N$, there exists an open cluster of boxes in the “macroscopic” renormalized graph (see [30] for details on the definition of occupied box). This implies the existence of an infinite cluster (in the original “microscopic” percolation) which contains a seed no smaller than the box $B(N^{1/2d})$ in each occupied box of the “macroscopic” cluster (see Figure 1 where the grayed boxes are occupied).

By uniqueness, this infinite microscopic cluster coincides with $\mathcal{C}_\infty$. Clearly, by construction, the centers of the seeds in two adjacent occupied “macroscopic” boxes are connected (in $\mathcal{C}_\infty$) by a path contained into these two boxes; clearly, the length of such a path is bounded from above by $M := 2N^d$. Since the percolation cluster in the renormalized “macroscopic” process contains a bi-infinite self-avoiding path of open boxes the proof is complete. □

**Proof of Theorem 1.1.** Even though (1) follows easily from (2) and the diagram in Figure 1 we prove it separately in order to introduce the key idea, which will be used later to prove (2), in a simpler case. (1) Since $\mathcal{C}_\infty$ is a subgraph of $\mathbb{Z}^d$, we have that $\lambda_\lambda(\mathcal{C}_\infty, \mu) \geq \lambda_\lambda(\mathbb{Z}^d, \mu)$. Take $\lambda > \lambda_\lambda(\mathbb{Z}^d, \mu)$: our goal is to prove that $\lambda > \lambda_\lambda(\mathcal{C}_\infty, \mu)$. By Lemma 3.2 we know that there exists (a smallest) $m$ such that $\lambda > \lambda_\lambda(x + B(m), \mu)$ for all $x \in \mathbb{Z}^d$. Let $M$, $\{x_j\}_{j=-\infty}^{+\infty}$, $\{\pi_j\}_{j=-\infty}^{+\infty}$ as in Lemma 3.6. By Lemma 3.5 and by monotonicity, for all $\varepsilon > 0$ there exist $T$ and $\ell$ such that $\mathbb{P}(\widetilde{T}_T(x_{j}) \geq \ell, \widetilde{T}_T(x_{j+1}) \geq \ell \widetilde{T}_\mu(x_j) = \ell) > 1 - \varepsilon$, where $\{\widetilde{T}_T\}_{T \geq 0}$ is the BRW restricted to $A = \bigcup_{j=-\infty}^{+\infty}(x_j + B(m)) \cup \pi_j$ (which, by Lemma 3.6, is a subset of $\mathcal{C}_\infty$ which exists whenever the cluster is infinite).
We construct a process \( \{\xi_t\}_{t \geq 0} \) on \( \mathcal{A} \), by iteration of independent copies of \( \{\tilde{\eta}_t\}_{t \geq 0} \) on time intervals \([nT, (n+1)T]\) and we associate it with a percolation process \( \varrho \) on \( \mathbb{Z} \times \bar{\mathbb{N}} \) (\( \mathbb{Z} \) representing space and \( \bar{\mathbb{N}} \) representing time), where \( \bar{\mathbb{N}} \) is the oriented graph on \( \mathbb{N} \) where all edges are of the type \((n,n+1)\). We index the family of copies needed as \( \{\tilde{\eta}_{(i,j)}\}_{i,j \in \mathbb{Z}, j \in \bar{\mathbb{N}}} \) and use \( \tilde{\eta}_{(i,j), t} \) when also the dependence on time has to be stressed; moreover \( \tilde{\eta}_{(i,j), 0} = \ell \delta_{x_i} \) for all \( i, j \). The construction will be made in such a way that \( \tilde{\eta}_t \) stochastically dominates \( \xi_t \) for all \( t \geq 0 \) and, whenever in the percolation process \( \varrho \) we have that \((0,0) \xrightarrow{\varrho} (j,n)\), then \( \xi_{nT}(x_j) \geq \ell \).

Let us begin our iterative construction with its first step. Start \( \{\tilde{\eta}_{(0,0), t}\}_{t \geq 0} \) and let \( \xi_t = \tilde{\eta}_{(0,0), t} \) for \( t \in [0, T] \). In the percolation process, the edge \((0,0) \xrightarrow{\varrho} (j,1), j = \pm 1\), is open if \( \tilde{\eta}_{(0,0), T}(x_j) \geq \ell \). Now suppose that we constructed \( \{\xi_t\}_{t \geq 0} \) for \( t \in [0, nT]\); to construct it for \( t \in (nT, (n+1)T]\), we put \( \xi_t = \sum_{h \in [-n,n]: \xi_{nT}(x_h) \geq \ell} \tilde{\eta}_{(h,n), t-nT} \) for all \( t \in (nT, (n+1)T]\). In the percolation \( \varrho \), for all \((i,n)\) such that there is an open path \((0,0) \xrightarrow{\varrho} (i,n)\), we connect \((i,n) \xrightarrow{\varrho} (j,n+1), j = i \pm 1, i \in \mathbb{Z}\) if \( \tilde{\eta}_{(i,n), T}(x_j) \geq \ell \).

In order to show that, by choosing \( \ell \) sufficiently large, with positive probability there is an open path in the percolation \( \varrho \), from \((0,0)\) to \((0,n)\) for infinitely many \( n \) (which means that at arbitrarily large times there are at least \( \ell \) individuals at \( x_0 \) in the original process), we need a comparison with a one-dependent oriented percolation \( \varrho_2 \) on \( \mathbb{Z} \times \bar{\mathbb{N}} \). This new percolation \( \varrho_2 \) is obtained by “enlarging” \( \varrho \) in the following way: for all \((i,n) \in \mathbb{Z} \times \bar{\mathbb{N}} \) we connect \((i,n) \xrightarrow{\varrho_2} (j,n+1), j = i \pm 1, i \in \mathbb{Z}\) if \( \tilde{\eta}_{(i,n), T}(x_j) \geq \ell \). Note that \( \varrho \) differs from \( \varrho_2 \) simply in the fact that in \( \varrho \) the opening procedure takes place only from sites already connected to \((0,0)\). By induction on \( n \), this coupled construction implies that there exists a \( \varrho_2 \)-open path from \((0,0)\) to \((i,n)\) if and only if there exists a \( \varrho \)-open path from \((0,0)\) to \((i,n)\) by Lemma 3.5 for all \( \varepsilon > 0 \), by choosing \( \ell \) sufficiently large, we have that for \( \varrho_2 \) the probability of opening all edges from \((i,n)\) is at least \( 1 - \varepsilon \). Let us choose \( \varepsilon \) such that the one-dependent percolation \( \varrho_2 \) dominates a supercritical independent (oriented) Bernoulli percolation. It is well-known that the infinite Bernoulli percolation cluster in the cone \( \{(i,j): j \geq |i|\} \) contains infinitely many sites of type \((0,n)\) a.s. Hence, by coupling, there is a positive probability that the one-dependent infinite percolation cluster contains infinitely many sites of type \((0,n)\) as well.

The first claim follows since the \( \lambda \)-BRW on \( C_\infty \) stochastically dominates \( \{\tilde{\eta}_t\}_{t \geq 0} \), which in turn dominates \( \{\xi_t\}_{t \geq 0} \), and by comparison with \( \varrho_2 \) we know that \( \xi_{nT}(x_0) \geq \ell \) for infinitely many \( n \in \mathbb{N} \).

(2) Let us now consider the \( k \)-type contact process \( \{\eta_{(x)}^k\}_{t \geq 0} \). Take \( \lambda > \lambda_c(\mathbb{Z}^d, \mu) \), as in the previous step, and \( \mathcal{A} \) (along with \( \{x_j\}_{j=-\infty}^\infty \) and \( \{\pi\}_{j=-\infty}^\infty \)) given by Lemma 3.6 as before. Consider
survives locally for all
\begin{align*}
T & \text{ time } 0:
\ell, N_n \text{ for all total progeny has reached size } \bar{\xi},
\text{ reach sites at distance larger than } \bar{\xi}\text{ on the event (} \xi \text{ was constructed from } \bar{\xi} \text{).}
\end{align*}

If follows easily from Theorem 1.1, the hypothesis \( \lambda > \lambda_{(C_\infty, \mu)} \) for all \( k \) sufficiently large. To this aim it is enough to prove that for the above fixed \( \lambda \), \( \{ \bar{\eta}^k_t \}_{t \geq 0} \) survives locally for all \( k \) sufficiently large.

Fix \( \varepsilon > 0 \) and let \( T \) and \( \ell \) be given by Lemma 3.5, such that
\begin{align*}
P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell| \bar{\eta}_0 = \ell \delta_x) > 1 - \varepsilon.
\end{align*}

Let \( N_T \) be the total progeny up to time \( T \) (including the initial particles), in the BRW \( (A, \mu) \), starting from \( \ell \) individuals at site \( x \). Define \( N_T \) as the total number of individuals ever born (including the initial particles), up to time \( T \), in a branching process with rate \( \lambda K \), starting with \( \ell \) individuals at time \( 0 \): \( N_T \) stochastically dominates \( N_T^x \) for all \( x \in A \). We have
\begin{align*}
P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell, N_T \leq n| \bar{\eta}_0 = \ell \delta_x)
& \geq P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell| \bar{\eta}_0 = \ell \delta_x) + P(N_T \leq n| \bar{\eta}_0 = \ell \delta_x) - 1
& \geq P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell| \bar{\eta}_0 = \ell \delta_x) + P(N_T \leq n) - 1 > 1 - 2\varepsilon,
\end{align*}

for all \( n \geq \bar{n} \) where \( \bar{n} \) satisfies \( P(N_T \leq n) > 1 - \varepsilon \) (\( \bar{n} \) is independent of \( x \)).

Define an auxiliary process \( \{ \bar{\xi}_t \}_{t \geq 0} \) obtained from \( \{ \bar{\eta}_t \}_{t \geq 0} \) by killing all newborns after that the total progeny has reached size \( \bar{n} \). This implies that, in the process \( \{ \bar{\eta}_t \}_{t \in [0, T]} \), the progeny does not reach sites at distance larger than \( \bar{n} \) from the \( \ell \) ancestors, nor it goes beyond the \( \bar{n} \)-th generation. In particular, when started from \( \ell \delta_x \), the processes \( \{ \bar{\eta}_t \}_{t \in [0, T]} \) and \( \{ \bar{\xi}_t \}_{t \geq 0} \) coincide, up to time \( T \), on the event \( (N_T \leq \bar{n}) \). Thus
\begin{align*}
P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell| \bar{\eta}_0 = \ell \delta_x) & \geq P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell, N_T \leq n| \bar{\eta}_0 = \ell \delta_x)
& = P(\bar{\eta}_T(y) \geq \ell, \bar{\eta}_T(y') \geq \ell, N_T \leq n| \bar{\eta}_0 = \ell \delta_x) > 1 - 2\varepsilon.
\end{align*}

The percolation construction of Step (1) can be repeated by using iid copies of \( \{ \bar{\eta}_t \}_{t \in [0, T]} \) instead of \( \{ \bar{\eta}_{t,i,j} \}_{i \in \mathbb{Z}, j \in \mathbb{N}} \). Call \( \{ \xi_t \}_{t \geq 0} \) the corresponding process constructed from these copies as \( \{ \xi_t \}_{t \geq 0} \) was constructed from \( \{ \bar{\eta}_{t,i,j} \}_{i \in \mathbb{Z}, j \in \mathbb{N}} \). As in Step (1), by choosing \( \varepsilon \) sufficiently small, we have that \( \xi_{t,T}(x_0) \geq \ell \) for infinitely many \( n \in \mathbb{N} \).

Let \( H \) be the number of paths in \( \mathbb{Z}^d \) of length \( \bar{n} \), containing the origin: \( H \) is an upper bound for the number of such paths in \( C_\infty \) or in \( A \). It is easy to show that \( \xi_t(x) \leq H \bar{n} \) for all \( t \) and \( x \). Thus if we take \( k \geq H \bar{n} \), then \( \bar{\eta}_T^k \) stochastically dominates \( \xi_T \). The supercriticality of the percolation on \( \mathbb{Z} \times \bar{\mathbb{N}} \), associated to \( \bar{\xi} \), implies that \( \{ \bar{\eta}_T^k \}_{t \geq 0} \) survives locally. The claim follows since \( \{ \bar{\eta}_T^k \}_{t \geq 0} \) stochastically dominates \( \{ \bar{\eta}_T \}_{t \geq 0} \).

\begin{proof}[Proof of Theorem 1.2] If follows easily from Theorem 1.1 the hypothesis \( \lambda_s(\mathbb{Z}^d, \mu) = \lambda_w(\mathbb{Z}^d, \mu) \) and the diagram shown in Figure 1.
\end{proof}

\begin{proof}[Proof of Theorem 1.3] Let \( \varepsilon > 0 \) such that \( c(0) - \varepsilon > \lambda_s(\mathbb{Z}^d, \mu) \). By the assumptions on \( c \), there exists \( \delta > 0 \) such that \( c(z) > c(0) - \varepsilon \) for all \( z \in [0, \delta] \). From Theorem 1.1 we know that there exists

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{A realization of the cluster in the percolation \( \varrho \) (left) and \( \varrho_2 \) (right).
}
\end{figure}
such that the \( k \)-type contact process \((C_\infty, \mu)\) associated with \( \lambda := c(0) - \varepsilon \) survives locally, for all \( k \geq k_0 \). Moreover, there exists \( N_0 \) such that \( \delta N > k_0 - 1 \) for all \( N \geq N_0 \). Since \( c_N(i) \geq \lambda \mathbb{1}_{[0,k-1]}(i) \) for all \( i \in \mathbb{N} \) (see Figure 6), by coupling we have local survival for the RBRWs \((C_\infty, \mu, c_N)\) for all \( N \geq N_0 \).

\[
\begin{align*}
  c(0) &= c_{N_0}(0) \\
  \lambda &= \\
  \lambda_s &= \\
  k_0 - 1 &= \delta N_0
\end{align*}
\]

**Figure 6.** Comparison between \( c_{N_0} \) (dashed) and the \( k_0 \)-type contact process (thick).

Proof of Corollary 1.4. (1) It suffices to note that the total number of individuals is dominated by the total number of particles in a continuous-time branching process with breeding parameter \( \alpha + \beta \) (for the details see [6, Theorem 1.1]).

(2) Note that \( \mu \) is translation invariant, hence quasi-transitive. The claim follows from Theorem 1.3 since \( \lambda_s(\mathbb{Z}^d, \mu) = (\alpha + \beta)^{-1} \) and \( c(0) = 1 \). □

References

[1] M. Barlow, Random walks on supercritical percolation clusters, Ann. Probab. 32 (2004), 3024–3084.
[2] M. Barlow, Y. Peres, P. Sousi, Collisions of random walks, Ann. Inst. Henri Poincaré Probab. Stat. 48 (2012), no. 4, 922–946.
[3] L. Belhadji, N. Lanchier, Individual versus cluster recoveries within a spatially structured population, Ann. Appl. Probab. 16 (2006), no.1, 403–422.
[4] L. Belhadji, D. Bertacchi, F. Zucca, A self-regulating and patch subdivided population, Adv. Appl. Probab. 42 n.3 (2010), 899–912.
[5] D. Bertacchi, D. Borrello, The small world effect on the coalescing time of random walks, Stochastic Process. Appl. 121 no.5 (2011), 925–956.
[6] D. Bertacchi, N. Lanchier, F. Zucca, Contact and voter processes on the infinite percolation cluster as models of host-symbiont interactions, Ann. Appl. Probab. 21 n. 4 (2011), 1215–1252.
[7] D. Bertacchi, G. Posta, F. Zucca, Ecological equilibrium for restrained random walks, Ann. Appl. Probab. 17 n. 4 (2007), 1117–1137.
[8] D. Bertacchi, F. Zucca, Uniform asymptotic estimates of transition probabilities on combs, J. Aust. Math. Soc. 75 n. 3 (2003), 325–353.
[9] D. Bertacchi, F. Zucca, Critical behaviors and critical values of branching random walks on multigraphs, J. Appl. Probab. 45 (2008), 481–497.
[10] D. Bertacchi, F. Zucca, Characterization of the critical values of branching random walks on weighted graphs through infinite-type branching processes, J. Stat. Phys. 134 n. 1 (2009), 53–65.
[11] D. Bertacchi, F. Zucca, Approximating critical parameters of branching random walks, J. Appl. Probab. 46 (2009), 463–478.
[12] D. Bertacchi, F. Zucca, Strong local survival of branching random walks is not monotone, Adv. Appl. Probab. 46 no.2 (2014).
[13] F. Comets, M.V. Menshikov, S.Yu. Popov, One-dimensional branching random walk in random environment: A classification, Markov Process. Related Fields 4 (1998), 465–477.
[14] O. Couronné, R.J. Messikh, Surface order large deviations for 2D FK-percolation and Potts models, Stochastic Process. Appl. 113 n. 1, (2004), 81–99.
[15] R. Durrett, P. Jung, Two phase transitions for the contact process on small worlds, Stochastic Process. Appl. 117 n. 12, (2007), 1910–1927.
[16] N. Gantert, S. Müller, S.Yu. Popov, M. Vachkovskaia, Survival of branching random walks in random environment, J. Theoret. Probab. 23 (2010), no. 4, 1002–1014.
[17] G. Grimmett, Percolation. Second edition. Springer-Verlag, Berlin, 1999.
[18] F. den Hollander, M.V. Menshikov, S.Yu. Popov, A note on transience versus recurrence for a branching random walk in random environment, J. Stat. Phys. 95 (1999), 587–614.
[19] I. Hueter, S.P. Lalley, Anisotropic branching random walks on homogeneous trees, Probab. Theory Related Fields 116, (2000), n.1, 57–88.
[20] T.M. Liggett, Branching random walks and contact processes on homogeneous trees, Probab. Theory Related Fields 106, (1996), n.4, 495–519.
[21] T.M. Liggett, Branching random walks on finite trees, Perplexing problems in probability, 315–330, Progr. Probab., 44, Birkhäuser Boston, Boston, MA, 1999.
[22] T.M. Liggett, F. Spitzer, Ergodic theorems for coupled random walks and other systems with locally interacting components, Z. Wahrscheinlichkeitsthe., 56, 443–468.
[23] F.P. Machado, S.Yu. Popov, One-dimensional branching random walk in a Markovian random environment, J. Appl. Probab. 37 (2000), 1157–1163.
[24] F.P. Machado, S.Yu. Popov, Branching random walk in random environment on trees, Stoch. Proc. Appl. 106 (2003), 95–106.
[25] N. Madras, R. Schinazi, Branching random walks on trees, Stoch. Proc. Appl. 42, (1992), n.2, 255–267.
[26] S. Müller, Interacting growth processes and invariant percolation, [arXiv:1304.3556].
[27] S. Müller, A criterion for transience of multidimensional branching random walk in random environment, Electron. J. Probab. 13 (2008).
[28] R. Pemantle, The contact process on trees, Ann. Prob. 20, (1992), 2089–2116.
[29] R. Pemantle, A.M. Stacey, The branching random walk and contact process on Galton–Watson and nonhomogeneous trees, Ann. Prob. 29, (2001), n.4, 1563–1590.
[30] A. Pisztora, Surface order large deviations for Ising, Potts and percolation models, Probab. Theory Related Fields 104 (1996), no. 4, 427–466.
[31] E. Seneta, Non-negative matrices and Markov chains, Springer Series in Statistics, Springer, New York, 2006.
[32] A.M. Stacey, Branching random walks on quasi-transitive graphs, Combin. Probab. Comput. 12, (2003), n.3 345–358.
[33] F. Zucca, Survival, extinction and approximation of discrete-time branching random walks, J. Stat. Phys., 142 n.4 (2011), 726–753.

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