Abstract

Fractal structures are observed in the universe in two very different ways. Firstly, in the gas forming the cold interstellar medium in scales from $10^{-4}$ pc till 100 pc. Secondly, the galaxy distribution has been observed to be fractal in scales up to hundreds of Mpc. We give here a short review of the statistical mechanical (and field theoretical) approach developed by us for the cold interstellar medium (ISM) and large structure of the universe. We consider a non-relativistic self-gravitating gas in thermal equilibrium at temperature $T$ inside a volume $V$. The statistical mechanics of such system has special features and, as is known, the thermodynamical limit does not exist in its customary form. Moreover, the treatments through microcanonical, canonical and grand canonical ensembles yield different results. We present here for the first time the equation of state for the self-gravitating gas in the canonical ensemble. We find that it has the form

$$p = \left[\frac{N}{V}\right] f(\eta),$$

where $p$ is the pressure, $N$ is the number of particles and $\eta \equiv \frac{Gm^2 N}{V^{2/3}T}$. The $N \to \infty$ and $V \to \infty$ limit exists keeping $\eta$ fixed. We compute the function $f(\eta)$ using Monte Carlo simulations and for small $\eta$, analytically. We compute the thermodynamic quantities of the system as free energy, entropy, chemical potential, specific heat, compressibility and speed of sound. We reproduce the well-known gravitational phase transition associated to the Jeans’ instability. Namely, a gaseous phase for $\eta < \eta_c$ and a condensed phase for $\eta > \eta_c$. Moreover, we derive the precise behaviour of the physical quantities near the transition. In particular, the pressure vanishes as $p \sim (\eta_c - \eta)^B$ with $B \sim 0.2$ and $\eta_c \sim 1.6$ and the energy fluctuations diverge as $\sim (\eta_c - \eta)^{B-1}$. The speed of sound decreases monotonically with $\eta$ and approaches the value $\sqrt{T/6}$ at the transition.
I. STATISTICAL MECHANICS OF THE SELF-GRAVITATING GAS

Physical systems at thermal equilibrium are usually homogeneous. This is the case for gases with short range intermolecular forces (and in absence of external fields). When long range interactions as the gravitational force are present even the ground state is often inhomogeneous. In this case, each element of the substance is acted on by very strong forces due to distant particles of the gas. Hence regions near to and far from the boundary of the volume occupied by the gas will be in very different conditions and as a result the homogeneity of the gas is destroyed \[1\]. This basic inhomogeneity suggests that fractal structures can arise in a self-interacting gravitational gas \[2–5,8\].

Let us review very briefly our recent work \[2–5\] on the statistical properties of a self-interacting gravitational gas in thermal equilibrium. We discussed two relevant astrophysical applications of such gas: the cold interstellar medium (ISM) and the galaxy distributions.

In the grand canonical ensemble, we showed that the self-gravitating gas is exactly equivalent to a field theory of a single scalar field \(\phi(\vec{x})\) with exponential self-interaction. We analyzed this field theory perturbatively and non-perturbatively through the renormalization group approach. We showed scaling behaviour (critical) for a continuous range of the temperature and of the other physical parameters. We derive in this framework the scaling relation

\[ M(R) \sim R^{d_H} \]

for the mass on a region of size \(R\), and

\[ \Delta v \sim R^q \]

for the velocity dispersion where \(q = \frac{1}{2}(d_H - 1)\). For the density-density correlations we find a power-law behaviour for large distances

\[ \sim |\vec{r}_1 - \vec{r}_2|^{2d_H-6}. \]

The fractal dimension \(d_H\) turns to be related with the critical exponent \(\nu\) of the correlation length by

\[ d_H = 1/\nu. \]

Mean field theory yields for the scaling exponents \(\nu = 1/2, \ d_H = 2\) and \(q = 1/2\). Such values are compatible with the present ISM observational data: \(1.4 \leq d_H \leq 2, \ 0.3 \leq q \leq 0.6\).

We developed in ref. \[4\] a field theoretical approach to the galaxy distribution. We consider a gas of self-gravitating masses on the Friedman-Robertson-Walker background, in quasi-thermal equilibrium. We derive the galaxy correlations using renormalization group methods. We find that the connected \(N\)-points density correlator \(C(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)\) scales as

\[ r_1^{N(D-3)} \]

when \(r_1 >> r_i, \ 2 \leq i \leq N\). There are no free parameters in this theory.

Our study of the statistical mechanics of a self-gravitating system indicates that gravity provides a dynamical mechanism to produce fractal structure.

This paper is organized as follows. In section II we summarize the main properties of the ISM, in section III we review the relevant aspects of the large scale structure of the universe, in sec. IV we develop the statistical mechanics of the self-gravitating gas in the canonical ensemble. Discussion and remarks are presented in section V.
II. THE INTERSTELLAR MEDIUM

The interstellar medium (ISM) is a gas essentially formed by atomic (HI) and molecular (H$_2$) hydrogen, distributed in cold ($T \sim 5 - 50K$) clouds, in a very inhomogeneous and fragmented structure. These clouds are confined in the galactic plane and in particular along the spiral arms. They are distributed in a hierarchy of structures, of observed masses from $10^{-2} M_\odot$ to $10^6 M_\odot$. The morphology and kinematics of these structures are traced by radio astronomical observations of the HI hyperfine line at the wavelength of 21cm, and of the rotational lines of the CO molecule (the fundamental line being at 2.6mm in wavelength), and many other less abundant molecules. Structures have been measured directly in emission from 0.01pc to 100pc, and there is some evidence in VLBI (very long based interferometry) HI absorption of structures as low as $10^{-4} pc = 20$ AU ($3 \times 10^{14}$ cm). The mean density of structures is roughly inversely proportional to their sizes, and vary between 10 and $10^5$ atoms/cm$^3$ (significantly above the mean density of the ISM which is about $0.1$ atoms/cm$^3$ or $1.6 \times 10^{-25}$ g/cm$^3$). Observations of the ISM revealed remarkable relations between the mass, the radius and velocity dispersion of the various regions, as first noticed by Larson [6], and since then confirmed by many other independent observations (see for example ref. [7]). From a compilation of well established samples of data for many different types of molecular clouds of maximum linear dimension (size) $R$, total mass $M$ and internal velocity dispersion $\Delta v$ in each region:

$$M(R) \sim R^{d_H}, \quad \Delta v \sim R^q,$$  \hspace{1cm} (2.1)

over a large range of cloud sizes, with $10^{-4} - 10^{-2} pc \leq R \leq 100 pc$,

$$1.4 \leq d_H \leq 2, \quad 0.3 \leq q \leq 0.6.$$  \hspace{1cm} (2.2)

These scaling relations indicate a hierarchical structure for the molecular clouds which is independent of the scale over the above cited range; above 100 pc in size, corresponding to giant molecular clouds, larger structures will be destroyed by galactic shear.

These relations appear to be universal, the exponents $d_H, \ q$ are almost constant over all scales of the Galaxy, and whatever be the observed molecule or element. These properties of interstellar cold gas are supported first at all from observations (and for many different tracers of cloud structures: dark globules using $^{13}$CO, since the more abundant isotopic species $^{12}$CO is highly optically thick, dark cloud cores using $HCN$ or $CS$ as density tracers, giant molecular clouds using $^{12}$CO, HI to trace more diffuse gas, and even cold dust emission in the far-infrared). Nearby molecular clouds are observed to be fragmented and self-similar in projection over a range of scales and densities of at least $10^4$, and perhaps up to $10^6$.

The physical origin as well as the interpretation of the scaling relations (2.1) are not theoretically understood. The theoretical derivation of these relations has been the subject of many proposals and controversial discussions. It is not our aim here to account for all the proposed models of the ISM and we refer the reader to refs. [7] for a review.

The physics of the ISM is complex, especially when we consider the violent perturbations brought by star formation. Energy is then poured into the ISM either mechanically through supernovae explosions, stellar winds, bipolar gas flows, etc. or radiatively through star light, heating or ionising the medium, directly or through heated dust. Relative velocities
between the various fragments of the ISM exceed their internal thermal speeds, shock fronts develop and are highly dissipative; radiative cooling is very efficient, so that globally the ISM might be considered isothermal on large-scales. Whatever the diversity of the processes, the universality of the scaling relations suggests a common mechanism underlying the physics.

We propose that self-gravity is the main force at the origin of the structures, that can be perturbed locally by heating sources. Observations are compatible with virialised structures at all scales. Moreover, it has been suggested that the molecular clouds ensemble is in isothermal equilibrium with the cosmic background radiation at $T \sim 3K$ in the outer parts of galaxies, devoid of any star and heating sources [8]. This colder isothermal medium might represent the ideal frame to understand the role of self-gravity in shaping the hierarchical structures. Our aim is to show that the scaling laws obtained are then quite stable to perturbations.

Till now, no theoretical derivation of the scaling laws eq.(2.1) has been provided in which the values of the exponents are obtained from the theory (and not just taken from outside or as a starting input or hypothesis).

The aim of our work [2–5] is to develop a theory of the cold ISM. A first step in this goal was to provide a theoretical derivation of the scaling laws eq.(2.1), in which the values of the exponents $d_H$, $q$ are obtained from the theory [2–5]. For this purpose, we implemented for the ISM the powerful tool of field theory and the Wilson’s approach to critical phenomena [9].

### III. GALAXY DISTRIBUTIONS

One obvious feature of galaxy and cluster distributions in the sky is their hierarchical property: galaxies gather in groups, that are embedded in clusters, then in superclusters, and so on [35,18].

The knowledge of the galaxy and cluster correlations allows a more precise characterization of their distributions. Unfortunately, the most widely spread two point correlation function $\xi(r)$ in galaxy distributions studies, is based on the assumption that the Universe reaches homogeneity on a scale smaller than the sample size. $\xi(r)$ is defined as

$$\xi(r) = \frac{< n(r_i).n(r_i + r) >}{< n >^2} - 1$$

where $n(r)$ is the number density of galaxies, and $< ... >$ is the volume average (over $d^3r_i$). The length $r_0$ is defined by $\xi(r_0) = 1$. The function $\xi(r)$ has a power-law behaviour of slope $-\gamma$ for $r < r_0$, then it turns down to zero rather quickly at the statistical limit of the sample. This rapid fall leads to an over-estimate of the small-scale $\gamma$. One finds the slope $\gamma$, the same for galaxies and clusters, of $\approx 1.7$ (e.g. [30]). It has been shown in refs. [19] and [20] that the homogeneity hypothesis could perturb significantly the results.

Pietronero [32] introduces the conditional density

$$\Gamma(r) = \frac{< n(r_i).n(r_i + r) >}{< n >}$$
which is the average density around an occupied point. For a fractal medium, where the mass depends on the size as
\[ M(r) \propto r^D \]
\(D\) being the fractal (Haussdorf) dimension, the conditional density behaves as
\[ \Gamma(r) \propto r^{D-3} \]
This is exactly the statistical analysis used for the interstellar clouds, since the ISM astronomers have not adopted from the start any large-scale homogeneity assumption (cf. [31]).

The fact that for a fractal the correlation \(\xi(r)\) can be highly misleading is readily seen since
\[ \xi(r) = \frac{\Gamma(r)}{\langle n \rangle} - 1 \]
and for a fractal structure the average density of the sample \(\langle n \rangle\) is a decreasing function of the sample length scale. In the general use of \(\xi(r)\), \(\langle n \rangle\) is taken for a constant, and we can see that
\[ D = 3 - \gamma . \]
If for very small scales, both \(\xi(r)\) and \(\Gamma(r)\) have the same power-law behaviour, with the same slope \(-\gamma\), then the slope appears to steepen for \(\xi(r)\) when approaching the length \(r_0\). This explains why with a correct statistical analysis [38], the actual \(\gamma \approx 1 - 1.5\) is smaller than that obtained using \(\xi(r)\). This also explains why the amplitude of \(\xi(r)\) and \(r_0\) increases with the sample size, and for clusters as well.

This scale-invariance has suggested very early the idea of fractal models for the clustering hierarchy of galaxies [24,28]. Since then, many authors have shown that a fractal distribution indeed reproduces quite well the aspect of galaxy catalogs, for example by simulating a fractal and observing it, as with a telescope [34,37].

There is some ambiguity in the definition of the two-point correlation function \(\xi(r)\) above, since it depends on the assumed scale beyond which the universe is homogeneous; indeed it includes a normalisation by the average density of the universe, which, if the homogeneity scale is not reached, depends on the size of the galaxy sample. Once \(\xi(r)\) is defined, one can always determine a length \(r_0\) where \(\xi(r_0) = 1\) [21]. For galaxies, the most frequently reported value is \(r_0 \approx 5h^{-1}\) Mpc (where \(h = H_0/100\) km s\(^{-1}\)Mpc\(^{-1}\)), but it has been shown to increase with the distance limits of galaxy catalogs [22], \(r_0\) is called ‘correlation length’ in the galaxy literature. [The notion of correlation length \(\xi_0\) is usually different in physics, where \(\xi_0\) characterizes the exponential decay of correlations \(\sim e^{-r/\xi_0}\). For power decaying correlations, it is said that the correlation length is infinite].

The same problem occurs for the two-point correlation function of galaxy clusters; the corresponding \(\xi(r)\) has the same power law as galaxies, their length \(r_0\) has been reported to be about \(r_0 \approx 25h^{-1}\) Mpc, and their correlation amplitude is therefore about 15 times higher than that of galaxies [33]. The latter is difficult to understand, unless there is a considerable difference between galaxies belonging to clusters and field galaxies (or morphological segregation). The other obvious explanation is that the normalizing average density of the universe was then chosen lower.

This statistical analysis of the galaxy catalogs has been criticized in refs. [32,20,20], who stress the uncomfortable dependence of \(\xi(r)\) and of the length \(r_0\) upon the finite size
of the catalogs, and on the *a priori* assumed value of the large-scale homogeneity cut-off. A way to circumvent these problems is to deal instead with the average density as a function of size. It has been shown that the galaxy distribution behaves as a pure self-similar fractal over scales up to \( \approx 100h^{-1}\) Mpc, the deepest scale to which the data are statistically robust \[38\]. This is more consistent with the observation of contrasted large-scale structures, such as superclusters, large voids or great walls of galaxies of \( \approx 200h^{-1}\) Mpc \[23\] After a proper statistical analysis of all available catalogs (CfA, SSRS, IRAS, APM, LEDA, etc.. for galaxies, and Abell and ACO for clusters) Pietronero et al \[32,38\] state that the transition to homogeneity might not yet have been reached up to the deepest scales probed until now. At best, this point is quite controversial, and the large-scale homogeneity transition is not yet well known.

Isotropy and homogeneity are expected at very large scales from the Cosmological Principle (e.g. \[30\]). However, this does not imply local or mid-scale homogeneity (e.g. \[25,38\]). A fractal structure can be locally isotropic, but inhomogeneous. The main observational evidence in favor of the Cosmological Principle is the remarkable isotropy of the cosmic background radiation (e.g. \[36\]), that provides information about the Universe at the matter/radiation decoupling. There must therefore exist a transition between the small-scale fractality to large-scale homogeneity. This transition is certainly smooth, and might correspond to the transition from linear perturbations to the non-linear gravitational collapse of structures. The present catalogs do not yet see the transition since they do not look up sufficiently back in time. It can be noticed that some recent surveys begin to see a different power-law behavior at large scales (\(\lambda \approx 200 - 400h^{-1}\) Mpc, e.g. \[27\]).

There are several approaches to understand non-linear clustering, and therefore the distribution of galaxies, in an infinite gravitating system. Numerical simulations have been widely used, in the hope to trace back from the observations the initial mass spectrum of fluctuations, and to test postulated cosmologies such as CDM and related variants (cf \[29\]). [That is numerically solving Newton’s equations of motion of self-gravitating particles]. This approach has not yet yielded definite results, especially since the physics of the multiple-phase universe is not well known. Also numerical limitations (restricted dynamical range due to the softening and limited volume) have often masked the expected self-similar behavior.

We presented in \[4\] a new approach based on field theory and the renormalisation group to understand the clustering behaviour of a self-gravitating expanding universe. We also consider the thermodynamics properties of the system, assuming quasi-equilibrium for the range of scales concerned with the non-linear regime and virialisation. Using statistical field theory, the renormalisation group and the finite-size scaling ideas, we determined the scaling behaviour. The small-scale fractal universe can be considered critical with large density fluctuations developing at any scale. We derived the corresponding critical exponents which yielded the fractal dimension \(D\). It is very close to those measured on galaxy catalogs through statistical methods based on the average density as function of size; these methods reveal in particular a fractal dimension \(D \approx 1.5 - 2\) \[25,38\]. This fractal dimension is strikingly close to that observed for the interstellar medium or ISM (e.g. \[17\]) We showed in ref. \[4\] that the theoretical framework based on self-gravity that we have developed for the ISM \[25\] is also a dynamical mechanism leading to the fractal structure of the universe. This theory is powerfully predictive without any free parameter. It allowed to compute the
N-points density correlations without any extra assumption \[4\].

IV. THERMODYNAMICS OF THE SELF-GRAVITATING GAS: THE CANONICAL ENSEMBLE

We investigate in this section a gas formed by \( N \) non-relativistic particles with mass \( m \) interacting only through Newtonian gravity and which are in thermal equilibrium at temperature \( T \equiv \beta^{-1} \). We shall work in the canonical ensemble assuming the gas being on a cubic box of side \( L \).

The partition function of the system can be written as

\[
Z = \int \ldots \int \prod_{l=1}^{N} \frac{d^3p_l}{(2\pi)^3} \frac{d^3q_l}{(2\pi)^3} e^{-\beta H_N} \tag{4.1}
\]

where

\[
H_N = \sum_{l=1}^{N} \frac{p_l^2}{2m} - G \frac{m^2}{\sum_{1 \leq l < j \leq N} 1} \frac{1}{|\vec{q}_l - \vec{q}_j|} \tag{4.2}
\]

\( G \) is Newton’s gravitational constant.

Computing the integrals over the momenta \( p_l, (1 \leq l \leq N) \)

\[
\int \frac{d^3p}{(2\pi)^3} e^{-\frac{\beta p^2}{2m}} = \left( \frac{m}{2\pi \beta} \right)^{3/2}
\]

yields

\[
Z = \left( \frac{m}{2\pi \beta} \right)^{\frac{3N}{2}} \int_0^L \ldots \int_0^L \prod_{l=1}^{N} \frac{d^3q_l}{(2\pi)^3} e^{\beta G m^2 \sum_{1 \leq l < j \leq N} \frac{1}{|\vec{q}_l - \vec{q}_j|}} \tag{4.3}
\]

We make now explicit the volume dependence introducing the new variables \( \vec{r}_l, 1 \leq l \leq N \) as

\[
\vec{q}_l = L \vec{r}_l \quad , \quad \vec{r}_l = (x_l, y_l, z_l) ,
\]

\[0 \leq x_l, y_l, z_l \leq 1 \quad . \tag{4.4}\]

The partition function takes then the form,

\[
Z = \left( \frac{mTL^2}{2\pi} \right)^{\frac{3N}{2}} \int_0^1 \ldots \int_0^1 \prod_{l=1}^{N} d^3r_l \ e^\eta u(\vec{r}_1, \ldots, \vec{r}_N) \tag{4.5}
\]

where we introduced the variable \( \eta \),

\[
\eta \equiv \frac{G m^2 N}{L T} \tag{4.6}
\]

and
\[ u(\vec{r}_1, \ldots, \vec{r}_N) \equiv \frac{1}{N} \sum_{1 \leq l < j \leq N} \frac{1}{|\vec{r}_l - \vec{r}_j|} \]

Recall that
\[ U \equiv -\frac{G m^2 N}{L} u(\vec{r}_1, \ldots, \vec{r}_N) \quad (4.7) \]
is the potential energy of the gas.

The free energy takes then the form,
\[ F = -T \log Z = -3NT \log \left( \sqrt{\frac{mT}{2\pi}} \right) - T \Phi_N(\eta) \quad (4.8) \]

Here
\[ \Phi_N(\eta) = \log \int_0^1 \ldots \int_0^1 \prod_{l=1}^N d^3\vec{r}_l \ e^{\eta u(\vec{r}_1, \ldots, \vec{r}_N)} , \quad (4.9) \]
The derivative of the function \( \Phi_N(\eta) \) will be computed by Monte Carlo simulations and, in the weak field limit \( \eta \ll 1 \), it will be calculated analytically.

We get for the pressure of the gas,
\[ p = -\left( \frac{\partial F}{\partial V} \right)_T = \frac{NT}{V} - \frac{\eta T}{3V} \Phi_N'(\eta) . \quad (4.10) \]
[Here, \( V \equiv L^3 \) stands for the volume of the box]. We see from eq.(4.9) that \( \Phi_N(\eta) \) increases with \( \eta \). Therefore, the second term in eq.(4.10) is a negative correction to the perfect gas pressure \( \frac{NT}{V} \).

The mean value of the potential energy \( U \) can be written from eq.(4.7) as
\[ \langle U \rangle = -T \eta \Phi_N'(\eta) \quad (4.11) \]
Combining eqs.(4.10) and (4.11) yields the virial theorem,
\[ \frac{pV}{NT} = 1 + \frac{\langle U \rangle}{3NT} , \]
or more explicitly
\[ \frac{pV}{NT} = 1 - \frac{\eta}{3N} \Phi_N'(\eta) \quad (4.12) \]
where,
\[ \Phi_N'(\eta) = e^{-\Phi_N(\eta)} \int_0^1 \ldots \int_0^1 \prod_{l=1}^N d^3\vec{r}_l \ u(\vec{r}_1, \ldots, \vec{r}_N) \ e^{\eta u(\vec{r}_1, \ldots, \vec{r}_N)} \]
\[ = \frac{1}{2} (N - 1) e^{-\Phi_N(\eta)} \int_0^1 \ldots \int_0^1 \prod_{l=1}^N d^3\vec{r}_l \ \frac{1}{|\vec{r}_1 - \vec{r}_2|} e^{\eta u(\vec{r}_1, \ldots, \vec{r}_N)} \quad (4.13) \]
This formula indicates that $\Phi'_N(\eta)$ is of order $N$ for large $N$. Monte Carlo simulations as well as analytic calculations for small $\eta$ show that this is indeed the case. In conclusion, we can write the equation of state of the self-gravitating gas as

$$\frac{pV}{NT} = f(\eta),$$

(4.14)

where the function

$$f(\eta) \equiv 1 - \frac{\eta}{3N} \Phi'_N(\eta),$$

is independent of $N$ for large $N$ and fixed $\eta$. [In practice, Monte Carlo simulations show that $f(\eta)$ is independent of $N$ for $N > 100$].

We get in addition,\n
$$\langle U \rangle = -3NT \left[ 1 - f(\eta) \right].$$

In the dilute limit, $\eta \to 0$ and we find the perfect gas value

$$f(0) = 1.$$

Equating eqs.(4.12) and (4.14) yields,

$$\Phi_N(\eta) = 3N \int_0^\eta dx \frac{1 - f(x)}{x}.$$

Relevant thermodynamic quantities can be expressed in terms of the function $f(\eta)$. We find for the free energy from eq.(4.8),

$$F = -3NT \log \left( \sqrt{\frac{mT}{2\pi}} \right) - 3NT \int_0^\eta dx \frac{1 - f(x)}{x}.$$  (4.15)

We find for the total energy

$$E = -3NT \left[ \frac{1}{2} - f(\eta) \right],$$

for the chemical potential,

$$\mu = \left( \frac{\partial F}{\partial N} \right)_{T,V} = -3T \log \left( \sqrt{\frac{mT}{2\pi}} \right) - 3T [1 - f(\eta)] - 3T \int_0^\eta dx \frac{1 - f(x)}{x}$$

and for the entropy

$$S = - \left( \frac{\partial F}{\partial T} \right)_V$$

$$= -\frac{3}{2} N + 3N \log \left( \sqrt{\frac{mT}{2\pi}} \right) + 3N \int_0^\eta dx \frac{1 - f(x)}{x} + 3N f(\eta).$$  (4.16)

The specific heat at constant volume takes the form [1],

$$c_V = \frac{T}{N} \left( \frac{\partial S}{\partial T} \right)_V.$$
\[ f(\eta) - \eta f'(\eta) - \frac{1}{2} \]  
(4.17)

where we used eq. (4.16). This quantity is also related to the fluctuations of the potential energy \((\Delta U)^2\) and it is positive defined,

\[ c_V = \frac{3}{2} + (\Delta U)^2. \]

Here,

\[ (\Delta U)^2 \equiv \frac{< U^2 > - < U >^2}{NT^2} = 3 [f(\eta) - \eta f'(\eta) - 1]. \]  
(4.18)

The specific heat at constant pressure is given by [1]

\[ c_P = c_V - T \left( \frac{\partial p}{\partial T} \right)_V. \]

and then,

\[ c_P = c_V + \frac{[f(\eta) - \eta f'(\eta)]^2}{f(\eta) + \frac{1}{2} \eta f'(\eta)} = -\frac{3}{2} + \frac{4 f(\eta) (f(\eta) - \eta f'(\eta))}{f(\eta) + \frac{1}{2} \eta f'(\eta)}. \]  
(4.19)

The isotherm \((K_T)\) and adiabatic \((K_S)\) compressibilities take the form

\[
K_T = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_T = \frac{V}{NT} \frac{1}{f(\eta) + \frac{1}{2} \eta f'(\eta)},
\]

\[
K_S = -\frac{1}{V} \left( \frac{\partial V}{\partial p} \right)_S = \frac{c_V}{c_P} K_T.
\]

The speed of sound \(v_s\) can be written here as [16]

\[
v_s^2 = -\frac{c_P V^2}{c_V N} \left( \frac{\partial p}{\partial V} \right)_T = \frac{V^2}{N} \left[ \frac{T}{c_V} \left( \frac{\partial p}{\partial T} \right)_V \right]^2 - \left( \frac{\partial p}{\partial V} \right)_T \].

Therefore,

\[
\frac{v_s^2}{T} = \frac{3 [f(\eta) - \eta f'(\eta)]^2}{f(\eta) - \eta f'(\eta) - \frac{1}{2}} + f(\eta) + \frac{1}{3} \eta f'(\eta). \]

(4.20)

We see that the large \(N\) limit of the self-gravitating gas is special. Energy, free energy and entropy are extensive in the sense that they are proportional to the number of particles \(N\) (for fixed \(\eta\)). They all depend on the variable \(\eta = \frac{GM^2 N}{L^2 T}\) which is to be kept fixed for the thermodynamic limit \((N \to \infty \text{ and } V \to \infty)\) to exist. Notice that \(\eta\) contains the ratio \(N/L = N V^{-1/3}\) and it is not an intensive variable in the usual sense. Here, the presence of long-range gravitational situations calls for a new type of variables in the thermodynamic limit.
A. Short-distance cutoff

At short distance the particle interaction for the self-gravitating gas in physical situations is not gravitational. Its exact nature depends on the problem under consideration (opacity limit, Van der Waals forces for molecules etc.). We shall just assume a repulsive short distance potential. That is,

\[ v_a(r) = -\frac{1}{r} \text{ for } r \geq a \]
\[ v_a(r) = +\frac{1}{a} \text{ for } r \leq a \]  \hspace{1cm} (4.21)

where \( r \equiv |\vec{r}_i - \vec{r}_j| \) stands for the distance between the particles and \( a \ll 1 \) is the short distance cut-off.

The presence of the repulsive short-distance interaction prevents the collapse (here unphysical) of the self-gravitating gas. In the situations we are interested to describe (interstellar medium, galaxy distributions) the collapse situation is unphysical.

B. The diluted regime: \( \eta << 1 \)

We can obtain the thermodynamic quantities as a series in powers of \( \eta \) just expanding the exponent in the integrand of \( \Phi_N(\eta) \) \[\text{eq.(4.13)}\].

To first order in \( \eta \) we get,

\[ \Phi_N(\eta) = \eta \int_0^1 \cdots \int_0^1 \prod_{l=1}^N d^3r_l \ u(\vec{r}_1, \ldots, \vec{r}_N) + \mathcal{O}(\eta^2) \]
\[ = \frac{1}{2} \eta (N - 1) \int_0^1 \int_0^1 \frac{d^3r_1 d^3r_2}{|\vec{r}_1 - \vec{r}_2|} + \mathcal{O}(a^2) + \mathcal{O}(\eta^2) \]
\[ = 3(N - 1) b_0 \eta + \mathcal{O}(a^2) + \mathcal{O}(\eta^2) \]  \hspace{1cm} (4.22)

where the coefficient \( b_0 \) is just a pure number. For the cubic geometry chosen, it takes the value

\[ b_0 = \frac{4}{3} \int_0^1 (1 - x) dx \int_0^1 (1 - y) dy \int_0^1 \frac{(1 - z) dz}{\sqrt{x^2 + y^2 + z^2}} = 0.31372 \ldots . \]

To first order in \( \eta \) we see that the cutoff effect is negligible \( \sim \mathcal{O}(a^2) \).

We therefore find in the low density limit using eqs.(4.12), (4.14) and (4.22)

\[ \frac{pV}{NT} = f(\eta) = 1 - b_0 \eta + \mathcal{O}(\eta^2) , \]  \hspace{1cm} (4.23)

in the large \( N \) limit.

Furthermore, the speed of sound approaches linearly in \( \eta \) to its perfect gas value,

\[ \frac{v_s^2}{T} \equiv \frac{\eta v_0}{3} = \frac{5}{3} - \frac{4}{3} b_0 \eta + \mathcal{O}(\eta^2) . \]

We used here eqs.(4.20) and (4.23).
C. Monte Carlo calculations

We have applied the standard Metropolis algorithm to the self-gravitating gas in a cube of size \( L \) at temperature \( T \). We computed in this way the pressure, the potential energy fluctuations and the average particle distance as functions of \( \eta \).

Two different phases show up: for \( \eta < \eta_c \) we have a non-perfect gas and for \( \eta > \eta_c \) it is a condensed system with negative pressure. The transition between the two phases is very sharp. We can say that the phase transition is of first order since there is a jump in the entropy. However, it is an unusual phase transition since the pressure is negative in the denser phase. In particular, the phases cannot coexist since the pressure has opposite sign in the two phases. This phase transition is associated with the Jeans instability.

We plot in figs. 1-3 \( f(\eta) = pV/[NT] \), \((\Delta U)^2\) and the speed of sound squared \( v_s^2/T \) as functions of \( \eta \).

We find that for small \( \eta \), the Monte Carlo results for \( pV/[NT] \) well reproduce the analytical formula (4.23). \( pV/[NT] \) monotonically decreases with \( \eta \). When \( \eta \) gets close to the value \( \eta_c \sim 1.6 \) a phase transition suddenly happens and \( pV/[NT] \) becomes large and negative. \(< r >\) monotonically decreases with \( \eta \) too. Near \( \eta_c , < r > \) has a sharp decrease.

In the Monte Carlo simulations the phase transition to the condensed phase happens for \( \eta = \eta_T \) slightly below \( \eta_c \). For \( \eta_T < \eta < \eta_c \), the gaseous phase may only exist as a metastable state. We find that \( \eta_c - \eta_T \) decreases with the number \( N \) of particles in the simulation. For example, \( \eta_c - \eta_T \sim 0.2 \) for \( N = 2000 \).

Both, the values of \( pV/[NT] \) and \(< r >\) in the condensed phase depend on the cutoff \( a \). The Monte Carlo results for \( \eta > \eta_c \) can be approximated as

\[
\frac{pV}{NT} = f(\eta) \simeq 1 - \frac{\eta}{Ka} , \quad < r > \simeq 1.5a .
\]

where \( K \sim 30 \). Therefore, the latent heat of the transition is

\[ q \simeq T \left[ 1 - \frac{\eta}{K a} \right] < 0 . \]

The behaviour of \( pV/[NT] \) near \( \eta_c \) in the gaseous phase can be well reproduced by

\[
\frac{pV}{NT} = f(\eta) ^ {\eta=\eta_c} A(\eta_c - \eta)^B
\]

where \( A \simeq 0.68, \eta_c \simeq 1.59 \) and \( B \simeq 0.22 \).

In addition, the behaviour of \( (\Delta U)^2 \) in the same region is well reproduced by

\[
(\Delta U)^2 ^ {\eta=\eta_c} C(\eta_c - \eta)^{B-1}
\]

with \( C \simeq 0.2 \) and with the exponent \( B - 1 \) as it must be since \( (\Delta U)^2 \) grows as \( f'(\eta) \) [see eq.(4.18)]. [Notice that for finite \( N \), \( (\Delta U)^2 \) will be finite albeit very large at the phase transition].

We thus find a critical region just below \( \eta_c \) where the energy fluctuations tend to infinity as \( \eta \uparrow \eta_c \).
\(v_s^2, c_V\) and \(K_S\) turn to be positive in the whole interval \(0 \leq \eta \leq \eta_c\) while \(c_P\) and \(K_T\) are positive for \(0 \leq \eta \leq \eta_0\) and change sign (and diverge) at the point \(\eta_0 < \eta_c\) defined by

\[
f(\eta_0) + \frac{1}{3} \eta_0 f'(\eta_0) = 0 .
\]

We find \(\eta_0 \simeq 1.49\) from the Monte Carlo simulations. We find that \(c_P\) and \(K_T\) are negative for \(\eta_0 < \eta < \eta_c\).

The specific heat behaviour near the transition follows from eqs. (4.17), (4.18) and (4.24),

\[
c_P \eta \bigg|_{\eta = \eta_c} = -\frac{3}{2} + O \left( (\eta_c - \eta)^B \right),
\]

\[
c_V \eta \bigg|_{\eta = \eta_c} = 3 A B \eta_c (\eta_c - \eta)^{B-1} - \frac{3}{2} + O \left( (\eta_c - \eta)^B \right).
\]

That is, while \(c_P\) tends to a negative value, \(c_V\) grows without bound when \(\eta \uparrow \eta_c\). The usual thermodynamic inequalities [1] forbidding such negative values do not apply here due to the inhomogeneity of the self-gravitating gas at thermal equilibrium.

We find for the speed of sound near the phase transition from eqs. (4.20) and (4.24),

\[
\frac{v_s^2}{T} \eta \bigg|_{\eta = \eta_c} = \frac{1}{6} + O \left( (\eta_c - \eta)^B \right).
\]

That is, the speed of sound tends to a constant value when \(\eta \uparrow \eta_c\).

In the condensed phase, \(v_s^2\) becomes negative indicating that there is no sound propagation in such state.

We verified that the Monte Carlo results in the gaseous phase \((\eta < \eta_c)\) are cutoff independent for \(0.001 \geq a \geq 0.0\).

**D. Particle Distribution**

The particle distribution at thermal equilibrium obtained through the Monte Carlo simulations is inhomogenous both in the gaseous and condensed phases.

In the gaseous phase we find from the Monte Carlo values of the particle density distribution that the mass \(\mathcal{M}\) within a volume \(V = R^3\) scales as

\[
\mathcal{M} = \mathcal{C} \, R^D \quad (4.26)
\]

where \(\mathcal{C}\) is a \(R\) independent constant and \(D\) takes values in the range,

\[D = 1.9 - 2.2 .\]

near the phase transition. That is for \(1.4 \leq \eta \leq \eta_c\). For smaller \(\eta\), \(D\) increases towards the value \(D = 3\) for \(\eta \to 0\). That is, the distribution becomes uniform in the perfect gas limit, as expected.

The exponent \(D\) found here suggest the presence of a fractal distribution near the the critical point \(pV/[NT] = 0^+\) for a very large number \(N\) of particles.
V. DISCUSSION

We present here a set of new results for the self-gravitating thermal gas obtained by Monte Carlo and analytic methods. In particular, they confirm the general picture of the thermal self-gravitating gas. Namely, a gaseous phase for higher temperature and lower density and a condensed phase for lower temperature and higher density \[10,13,14\]. Actually, we find more appropriate to characterize the phases by the sign of the pressure: positive pressure in the gaseous phase and negative pressure in the condensed phase. The pressure plays here the rôle of order parameter.

The parameter \(\eta\) [introduced in eq. (4.6)] can be related to the Jeans length of the system

\[
d_J = \sqrt{\frac{3T}{m}} \frac{1}{\sqrt{Gm\rho}},
\]

where \(\rho \equiv N/V\) stands for the number volume density. Combining eqs. (4.6) and (5.1) yields

\[
\eta = 3 \left( \frac{L}{d_J} \right)^2.
\]

We see that the phase transition takes place for \(d_J \sim L\). [The precise numerical value of the proportionality coefficient depends on the geometry]. For \(d_J > L\) we find the gaseous phase and for \(d_J < L\) the system condenses as expected.

Contrary to mean field treatments \[12,10\], we do not assume here an equation of state but we obtain the equation of state for the canonical ensemble [see eq. (4.14)]. We find at the same time that the relevant variable is here \(\eta = \frac{Gm^2N}{[V^{1/3}T]}\). The relevance of the ratio \(Gm^2/[V^{1/3}T]\) has been noticed on dimensionality grounds \[10\]. However, dimensionality arguments alone cannot single out the crucial factor \(N\) in the variable \(\eta\).

The crucial point is that the thermodynamic limit exist if we let \(N \to \infty\) and \(V \to \infty\) keeping \(\eta\) fixed. Notice that \(\eta\) contains the ratio \(N/V^{-1/3}\) and not \(N/V\). This means that in this thermodynamic limit \(V\) grows as \(N^3\) and thus the volume density \(\rho = N/V\) decreases as \(\sim N^{-2}\). \(\eta\) is to be kept fixed for a thermodynamic limit to exist in the same way as the temperature. \(pV\), the energy \(E\), the free energy, the entropy are functions of \(\eta\) and \(T\) times \(N\). The chemical potential, specific heat, etc. are just functions of \(\eta\) and \(T\).

The divergent growth of the energy fluctuations \((\Delta U)^2\) near the phase transition has been previously noticed \[10,13,14\]. We find here the precise behaviour of \((\Delta U)^2\) for \(\eta \uparrow \eta_c\) using Monte Carlo methods [eq. (4.25)].

In refs. \[2,3\] we worked in the grand canonical ensemble around the point where log \(\mathcal{Z}_{GC} = 0\). This precisely corresponds to \(pV/[NT] = 0\) which is the critical point in the canonical treatment given here. The presence of a critical region where scaling holds supports the previous work in the grand canonical ensemble \[2,3\].

VI. ACKNOWLEDGEMENTS

One of us (H J de V) thanks M. Picco for useful discussions on Monte Carlo methods.
REFERENCES

[1] L. D. Landau and E. M. Lifchitz, Physique Statistique, 4ème édition, Mir-Ellipses, 1996.
[2] H. J. de Vega, N. Sánchez and F. Combes, Nature, 383, 56 (1996).
[3] H. J. de Vega, N. Sánchez and F. Combes, Phys. Rev. D54, 6008 (1996).
[4] H. J. de Vega, N. Sánchez and F. Combes, Ap. J. 500, 8 (1998).
[5] H. J. de Vega, N. Sánchez and F. Combes, in ‘Current Topics in Astrofundamental Physics: Primordial Cosmology’, NATO ASI at Erice, N. Sánchez and A. Zichichi editors, vol 511, Kluwer, 1998.
[6] R. B. Larson, M.N.R.A.S. 194, 809 (1981)
[7] J. M. Scalo, in ‘Interstellar Processes’, D.J. Hollenbach and H.A. Thronson Eds., D. Reidel Pub. Co, p. 349 (1987).
[8] D. Pfenniger, F. Combes, L. Martinet, A&A 285, 79 (1994)
D. Pfenniger, F. Combes, A&A 285, 94 (1994)
[9] Wilson K.G., Kogut, J., Phys. Rep. 12, 75 (1974). K. G. Wilson, Rev. Mod. Phys. 47, 773 (1975) and Rev. Mod. Phys. 55, 583 (1983).
Phase transitions and Critical Phenomena vol. 6, C. Domb & M. S. Green, Academic Press, 1976. J. J. Binney, N. J. Dowrick, A. J. Fisher and M. E. J. Newman, The Theory of Critical Phenomena, Oxford Science Publication, 1992.
[10] See for example, W. C. Saslaw, ‘Gravitational Physics of stellar and galactic systems’, Cambridge Univ. Press, 1987.
[11] See for example, H. Stanley in Fractals and Disordered Systems, A. Bunde and S. Havlin editors, Springer Verlag, 1991.
[12] S. Chandrasekhar, ‘An Introduction to the Study of Stellar Structure’, Chicago Univ. Press, 1939.
[13] D. Lynden-Bell and R. M. Lynden-Bell, Mon. Not. astr. Soc. 181, 405 (1977).
[14] T. Padmanabhan, Phys. Rep. 188, 285 (1990).
[15] H. J. de Vega, N. Sánchez, B. Semelin and F. Combes, in preparation.
[16] L. Landau and E. Lifchitz, Mécanique des Fluides, Eds. MIR, Moscou 1971.
[17] S.C. Kleiner, R.L. Dickman, ApJ 286, 255 (1984), 295, 466 (1985) and 312, 837 (1987)
[18] Abell G.O.: 1958, ApJS 3, 211
[19] Coleman P.H., Pietronero L., Sanders R.H.: 1988, A&A 200, L32
[20] Coleman P.H., Pietronero L.: 1992, Phys. Rep. 231, 311
[21] Davis M.A., Peebles P.J.E.: 1983, ApJ 267, 465. Hamilton A.J.S.: 1993, ApJ 417,19.
[22] Davis M.A., Meiksin M.A., Strauss L.N., da Costa and Yahil A.: 1988, ApJ 333, L9
[23] de Lapparent V., Geller M.J., Huchra J.P.: 1986, ApJ 302, L1. Geller M.J., Huchra J.P.: 1989, Science 246, 897
[24] de Vaucouleurs G.: 1960, ApJ 131, 585. de Vaucouleurs G.: 1970, Science 167, 1203
[25] Di Nella H.,Montuori M.,Paturel G., Pietronero L.,Sylos Labini F. 1996, A&A 308,L33
[26] Einasto J.: 1989, in ‘Astronomy, cosmology and fundamental physics’, Proc. of the 3rd ESO-CERN Symposium, Dordrecht, Kluwer, p. 231
[27] Lin H. et al: 1996, ApJ 471, 617
[28] Mandelbrot B.B.: 1975, ‘Les objets fractals’, Paris, Flammarion Mandelbrot B.B.: 1982, ‘The fractal geometry of nature’, New York: Freeman
[29] Ostriker J.P.: 1993, ARAA 31, 689
[30] Peebles P.J.E.: 1980, ‘The Large-scale structure of the Universe’, Princeton Univ. Press.
Peebles P.J.E.: 1993, ‘Principles of physical cosmology’ Princeton Univ. Press

[31] Pfenniger D., Combes F.: 1994, A&A 285, 94

[32] Pietronero L.: 1987, Physica A, 144, 257 Pietronero L., Montuori M., Sylos Labini F.: 1997, in ‘Critical Dialogs in Cosmology’, astro-ph/9611197

[33] Postman M., Geller M.J., Huchra J.P.: 1986, AJ 91, 1267. Postman M., Huchra J.P., Geller M.J.: 1992, ApJ 384, 404

[34] Scott E.L., Shane S.D., Swanson M.D.: 1954, ApJ 119, 91

[35] Shapley H.: 1934, MNRAS 94, 791

[36] Smoot G., et al: 1992, ApJ 396, L1

[37] Soneira R.M., Peebles P.J.E.: 1978, AJ 83, 845

[38] F. Sylos Labini, M. Montuori, L. Pietronero, Phys.Rept. 293, (1998) 61-226.
FIG. 1. $f(\eta) = \frac{pV}{N\tau}$ as a function of $\eta$ in the gaseous phase from Monte Carlo simulations.
\[(\Delta U)^2 \equiv \langle U \rangle^2 - \langle U \rangle^2 \]

Recall that \(c_v = \frac{3}{2} + (\Delta U)^2\). From Monte Carlo simulations, \(\frac{\varepsilon}{\bar{\varepsilon}} = \frac{\lambda}{\bar{\lambda}}\) as a function of \(\eta\) in the gaseous phase.
FIG. 3. The speed of sound squared divided by the temperature, $v_s^2/T$, as a function of $\eta$ in the gaseous phase from Monte Carlo simulations. Notice that $v_s^2/T$ takes the value 1/6 at the critical point $\eta = \eta_c$. 