QUANTITATIVE APPROXIMATIONS OF THE LYAPUNOV EXPONENT OF A RATIONAL FUNCTION OVER VALUED FIELDS

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Abstract. We establish a quantitative approximation formula of the Lyapunov exponent of a rational function of degree more than one over an algebraically closed field of characteristic 0 and complete with respect to a non-trivial and possibly non-archimedean absolute value, in terms of the multipliers of periodic points of the rational function. This quantifies both our former convergence result over general fields and the one-dimensional version of Berteloot–Dupont–Molino’s one over archimedean fields.

1. Introduction

In this article, we establish a quantitative logarithmic equidistribution result for periodic points of a rational function over a more general field than that of complex numbers, using a potential theory on the Berkovich projective line. Let \( K \) be an algebraically closed field of characteristic 0 and complete with respect to a non-trivial and possibly non-archimedean absolute value \( |\cdot| \). We note that \( K \cong \mathbb{C} \) if and only if \( K \) is archimedean. The Berkovich projective line \( \mathbb{P}^1 = \mathbb{P}^1(K) \) over \( K \) is a compactification of the (classical) projective line \( \mathbb{P}^1 = \mathbb{P}^1(K) \) over \( K \) and contains \( \mathbb{P}^1 \) as a dense subset. We also note that \( \mathbb{P}^1 \cong \mathbb{P}^1 \) if and only if \( K \) is archimedean.

Let \( f \in K(z) \) be a rational function over \( K \) of degree \( > 1 \), and let \( \mu_f \) the equilibrium (or canonical) measure of \( f \) on \( \mathbb{P}^1 \). The chordal derivative \( f^\# \) of \( f \) with respect to the normalized chordal metric on \( \mathbb{P}^1 \) extends to a continuous function on \( \mathbb{P}^1 \). The multiplier of a fixed point \( w \in \mathbb{P}^1 \) of \( f^n \) for some \( n \in \mathbb{N} \) is denoted by \( (f^n)'(w) \). For every fixed point \( w \in \mathbb{P}^1 \) of \( f \), we have \( f^\#(w) = |f'(w)| \), and the function \( \log(f^\#) \) on \( \mathbb{P}^1 \) has a logarithmic singularity at each critical point of \( f \) in \( \mathbb{P}^1 \). The Lyapunov exponent of \( f \) with respect to \( \mu_f \) is defined by

\[
L(f) := \int_{\mathbb{P}^1} \log(f^\#) \, d\mu_f,
\]

which is in \( (-\infty, \infty) \).

Our principal result is the following quantitative approximation of \( L(f) \) by the log of the moduli of the multipliers of non-superattracting periodic points of \( f \), the qualitative version (i.e., with no non-trivial order estimates) of which was obtained in [16, Theorem 1] (see also Szpiro–Tucker [23] for the qualitative version for a number field or a function field \( K \)).

\textbf{Theorem 1}. Let \( f \in K(z) \) be a rational function of degree \( d > 1 \) over an algebraically closed field \( K \) of characteristic 0 and complete with respect to a non-trivial...
and possibly non-archimedean absolute value $|\cdot|$. Then
\begin{align*}
L(f) &= \frac{1}{nd^n} \sum_{w \in \mathbb{P}^1 : f^n(w) = w \text{ and } (f^n)'(w) \neq 0} \log |(f^n)'(w)| + O(nd^{-n}) \quad \text{as } n \to \infty.
\end{align*}

In the case that $f$ has at most finitely many attracting but not superattracting periodic points in $\mathbb{P}^1$, Theorem 1 yields the following two kinds of quantitative approximations of $L(f)$ by the multipliers of repelling periodic points of $f$.

**Theorem 2.** Let $f \in K(z)$ be a rational function of degree $d > 1$ over an algebraically closed field $K$ of characteristic 0 and complete with respect to a non-trivial and possibly non-archimedean absolute value $|\cdot|$. For each $n \in \mathbb{N}$, set
\begin{align*}
R(f^n) &:= \{w \in \mathbb{P}^1(\mathbb{C}) : f^n(w) = w \text{ and } |(f^n)'(w)| > 1\}, \\
R^*(f^n) &:= \{w \in R(f^n) : f^j(w) \neq w \text{ for every } j \in \{1, 2, \ldots, n-1\}\}.
\end{align*}
If $f$ has at most finitely many attracting but not superattracting periodic points of $f$, then
\begin{align*}
(1.2') \quad L(f) &= \frac{1}{nd^n} \sum_{w \in R(f^n)} \log |(f^n)'(w)| + O(nd^{-n}) \quad \text{as } n \to \infty, \quad \text{and} \\
(1.2'') \quad L(f) &= \frac{1}{nd^n} \sum_{w \in R^*(f^n)} \log |(f^n)'(w)| + O(nd^{-n/2}) \quad \text{as } n \to \infty.
\end{align*}

For archimedean $K$, the finiteness assumption in Theorem 2 always holds (cf. [15], Theorem 8.6). The qualitative version (i.e., with no non-trivial order estimates) of Theorem 2 for archimedean $K$ also follows from Berteloot–Dupont–Molino [6, Corollary 1.6]; see also Berteloot [5].

A bit surprisingly, the proof of Theorems 1 and 2 are independent of the equidistribution theorem for periodic points of $f$ in $\mathbb{P}^1$ towards the equilibrium measure $\mu_f$, which was due to Lyubich [14, Theorem 3] for archimedean $K$ and due to Favre–Rivera-Letelier [11, Théorème B] for non-archimedean $K$ of characteristic 0.

In Section 2 we prepare a background on dynamics of rational functions over general fields. In Section 3 we show Theorem 1. Let us remark that the proof can be simplified if there are no superattracting periodic points. In Section 4 we show Theorem 2 based on Theorem 1.

## 2. Background

For the foundations of potential theory on $\mathbb{P}^1$, see [3, §5 and §8], [10, §7], [13, §1-§4], [25, Chapter III]. For a potential theoretic study of dynamics on $\mathbb{P}^1$, see [3, §10], [11, §3], [13, §5], [7, Chapitre VIII]. See also [4, 20] including non-archimedean dynamics.

**Chordal metric on $\mathbb{P}^1$.** Let $K$ be an algebraically closed field complete with respect to a non-trivial and possibly non-archimedean absolute value $|\cdot|$. For a while, $K$ has any characteristic. Let $\|\cdot\|$ be the maximal norm (for non-archimedean $K$) or the Euclidean norm (for archimedean $K$) on $K^2$. The origin of $K^2$ is also denoted by 0, and $\pi$ is the canonical projection $K^2 \setminus \{0\} \to \mathbb{P}^1 = \mathbb{P}^1(K)$. Setting the wedge product $(z_0, z_1) \wedge (w_0, w_1) := z_0w_1 - z_1w_0$ on $K^2 \times K^2$, the normalized chordal metric $[z, w]$ on $\mathbb{P}^1(K)$ is the function
\begin{align*}
(2.1) \quad (z, w) &\mapsto [z, w] := |p \wedge q|/(\|p\| \cdot \|q\|)(\leq 1)
\end{align*}
on $\mathbb{P}^1 \times \mathbb{P}^1$, where $p \in \pi^{-1}(z), q \in \pi^{-1}(w)$. Although the topology of the Berkovich projective line $\mathbb{P}^1 = \mathbb{P}^1(K)$, which is a compactification of $\mathbb{P}^1$, is not always metrizable, the relative topology of $\mathbb{P}^1$ coincides with the metric topology on $\mathbb{P}^1$ induced by the normalized chordal metric.

**Hsia kernel on $\mathbb{P}^1$.** Let $\delta_S$ be the Dirac measure on $\mathbb{P}^1 = \mathbb{P}^1(K)$ at $S \in \mathbb{P}^1$. Let $\Omega_{\text{can}}$ be the Dirac measure $\delta_{\mathbb{P}^1_{\text{can}}}$ at the canonical (or Gauss) point $\mathbb{P}^1_{\text{can}} \in \mathbb{P}^1$ for non-archimedean $K$ ([3] §1.2, [11] §2.1) or the Fubini–Study area element $\omega$ on $\mathbb{P}^1$ normalized as $\omega(\mathbb{P}^1) = 1$ for archimedean $K$. For non-archimedean $K$, the normalized chordal metric on $\mathbb{P}^1$ canonically extends to the generalized Hsia kernel $[S, S']_{\text{can}} \text{ with respect to } \mathbb{P}^1_{\text{can}}$ on $\mathbb{P}^1$ (for the construction, see [3] §4.4, [11] §2.1), which vanishes if and only if $S = S' \in \mathbb{P}^1$. For archimedean $K$, $[z, w]_{\text{can}}$ is defined by $[z, w]$, by convention. Let $\Delta$ be the Laplacian on $\mathbb{P}^1$ (for the construction in non-archimedean case, see [3] §5.1, [10] §7.7, [24] §3]) normalized so that for each $S \in \mathbb{P}^1$,

$$\Delta \log[\cdot, S]_{\text{can}} = \delta_S - \Omega_{\text{can}} \text{ on } \mathbb{P}^1$$

(for non-archimedean $K$, see [3] Example 5.19, [11] §2.4; in [3] the opposite sign convention on $\Delta$ is adopted).

**Potential theory on $\mathbb{P}^1$.** A continuous weight $g$ on $\mathbb{P}^1$ is a continuous function on $\mathbb{P}^1$ such that $\mu^g := \Delta g + \Omega_{\text{can}}$ is a probability Radon measure on $\mathbb{P}^1$. For a continuous weight $g$ on $\mathbb{P}^1$, the $g$-potential kernel

$$\Phi_g(S, S') := \log[S, S']_{\text{can}} - g(S) - g(S')$$

(the negative of an Arakelov Green function of $\mu^g$) on $\mathbb{P}^1$ is an upper semicontinuous function on $\mathbb{P}^1 \times \mathbb{P}^1$ and separately continuous in each variables $S, S' \in \mathbb{P}^1$, and introduces the $g$-potential

$$U_{g, \nu}(\cdot) := \int_{\mathbb{P}^1} \Phi_g(\cdot, S)d\nu(S)$$

on $\mathbb{P}^1$ of each Radon measure $\nu$ on $\mathbb{P}^1$. By the Fubini theorem, $\Delta U_{g, \nu} = \nu - \nu(\mathbb{P}^1)\mu^g$ on $\mathbb{P}^1$. The $g$-equilibrium energy $V_g \in [-\infty, +\infty)$ of $\mathbb{P}^1$ is the supremum of the energy functional

$$\nu \mapsto \int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g d(\nu \times \nu)$$

on the space of all probability Radon measures on $\mathbb{P}^1$. Indeed $V_g \in (-\infty, \infty)$ since

$$\int_{\mathbb{P}^1 \times \mathbb{P}^1} \Phi_g d(\Omega_{\text{can}} \times \Omega_{\text{can}}) > -\infty.$$ 

The variational characterization of $\mu^g$ asserts that the above energy functional attains the supremum uniquely at $\nu = \mu^g$. Moreover,

$$U_{g, \mu^g} \equiv V_g \text{ on } \mathbb{P}^1$$

(for non-archimedean $K$, see [3] Theorem 8.67 and Proposition 8.70). A continuous weight $g$ on $\mathbb{P}^1$ is a normalized weight on $\mathbb{P}^1$ if $V_g = 0$. For a continuous weight $g$ on $\mathbb{P}^1$, $V := g + V_g/2$ is a unique normalized weight on $\mathbb{P}^1$ satisfying $\mu^V = \mu^g$.

**Equilibrium measure $\mu_f$.** A rational function $f \in K(z)$ of degree $d > 1$ extends to a continuous, surjective, open, and discrete endomorphism of $\mathbb{P}^1$, preserving $\mathbb{P}^1$ and $\mathbb{P}^1 \setminus \mathbb{P}^1$, respectively, and induces a push-forward $f_*$ and a pullback $f^*$ on the spaces of continuous functions and of Radon measures on $\mathbb{P}^1$ ([3] §9, [11] §2.2). A non-degenerate homogeneous lift $F = (F_0, F_1)$ of (the unextended) $f$ is a homogeneous polynomial endomorphism on $K^2$ such that $\pi \circ F = f \circ \pi$ on $K^2 \setminus \{0\}$ and that $F^{-1}(0) = \{0\}$. The latter condition is equivalent to $\text{Res } F \in K \setminus \{0\}$ (for the definition of the homogeneous resultant $\text{Res } F = \text{Res}(F_0, F_1)$ of $F$, see, e.g., [21] §2.4)). Such $F$ is unique up to multiplication in $K \setminus \{0\}$, and has the algebraic
degree $d$. For every $n \in \mathbb{N}$, $F^n$ is a non-degenerate homogeneous lift of $f^n$, and the function
\begin{equation}
T_{F^n} := \log \| F^n \| - d^n \log \| \cdot \|
\end{equation}
on $K^2 \setminus \{0\}$ descends to $\mathbb{P}^1$ and in turn extends continuously to $\mathbb{P}^1$, satisfying $\Delta T_{F^n} = (f^n)^* \Omega_{can} - d^n \Omega_{can}$ on $\mathbb{P}^1$ (see, e.g., [17] Definition 2.8). The uniform limit $g_F := \lim_{n \to \infty} T_{F^n}/d^n$ on $\mathbb{P}^1$ exists and is indeed a continuous weight on $\mathbb{P}^1$. The equilibrium (or canonical) measure of $f$ is the probability Radon measure
\[ \mu_f := \Delta g_F + \Omega_{can} = \lim_{n \to \infty} d^{-n}(f^n)^* \Omega_{can} \text{ weakly on } \mathbb{P}^1, \]
which is independent of choices of $F$ and satisfies that $f^* \mu_f = d \cdot \mu_f$ and $f \cdot \mu_f = \mu_f$ on $\mathbb{P}^1$ (for non-archimedean $K$, see [10], [8], [11] §3.1). The dynamical Green function $g_F$ of $f$ on $\mathbb{P}^1$ is a unique normalized weight on $\mathbb{P}^1$ such that $\mu^F = \mu_f$. By the energy formula $V_{g_F} = - \langle \log |Res F|/d(d - 1) \rangle$ (due to DeMarco [9] for archimedean $K$ and due to Baker–Runels [2] for $K = \mathbb{C}_v$ associated to each places $v$ of a number field; see [1] Appendix A for a simple proof which works for general $K$) and $Res(cF) = c^{2d} Res F$ for each $c \in K \setminus \{0\}$ (cf. [21] Proposition 2.13(b)), there is a non-degenerate homogeneous lift $F$ of $f$ satisfying $V_{g_F} = 0$, or equivalently, that $g_F = g_f$ on $\mathbb{P}^1$. We note that $U_{g_f} = 0$ on $\mathbb{P}^1$ and that for every $n \in \mathbb{N}$, $\mu_f = \mu_f$ and $g_f = g_f$ on $\mathbb{P}^1$.

**Logarithmic proximity function $\Phi(f^n, Id)_{g_f}$.** For more details of the following, see [17] Proposition 2.9.

**Proposition 2.1.** For rational functions $\phi_i \in K(z)$ of degree $d_i$, $i \in \{1, 2\}$, on $\mathbb{P}^1$ satisfying $\phi_1 \neq \phi_2$ and $\max\{d_1, d_2\} > 0$, the function $z \mapsto [\phi_1(z), \phi_2(z)]$ on $\mathbb{P}^1$ extends continuously to a function $S \mapsto [\phi_1, \phi_2]_{can}(S)$ on $\mathbb{P}^1$.

For each $n \in \mathbb{N}$, we introduce the logarithmic proximity function weighted by $g_f$
\[ \Phi(f^n, Id)_{g_f}(\cdot) := \log [f^n, Id]_{can}(\cdot) - g_f \circ f^n(\cdot) - g_f(\cdot) \]
between $f^n$ and $Id$ on $\mathbb{P}^1$. For a proof of the following, see, e.g., [17] Lemma 2.19.

**Lemma 2.2 (cf. [22] (1.4)).** For every $n \in \mathbb{N}$,
\[ \Phi(f^n, Id)_{g_f}(\cdot) = U_{g_f, [f^n=Id]}(d^n+1)\mu_f + \int_{\mathbb{P}^1} \Phi(f^n, Id)_{g_f} d\mu_f \]
on $\mathbb{P}^1$. Since $U_{g_f} = 0$ on $\mathbb{P}^1$, this is rewritten as
\begin{equation}
\Phi(f^n, Id)_{g_f}(\cdot) = U_{g_f, [f^n=Id]} + \int_{\mathbb{P}^1} \Phi(f^n, Id)_{g_f} d\mu_f \quad \text{on } \mathbb{P}^1.
\end{equation}

**Chordal derivative $f^\#$.** The multiplier of a fixed point $w \in \mathbb{P}^1$ of $f^n$ for some $n \in \mathbb{N}$ is denoted by $(f^n)'(w)$. A fixed point $w \in \mathbb{P}^1$ of $f^n$ for some $n \in \mathbb{N}$ is said to be attracting (resp. repelling) if $|(f^n)'(w)| < 1$ (resp. $|(f^n)'(w)| > 1$).

In the rest of this section, we suppose that $K$ has characteristic 0. Let $C(f)$ be the set of all critical points $c$ of $f$ in $\mathbb{P}^1$, i.e., $f'(c) = 0$, and for each $c \in \mathbb{N}$, let $\text{Fix}(f^n)$ be the set of all fixed points of $f^n$ in $\mathbb{P}^1$. For every $n \in \mathbb{N}$, $f^n$ has $2d^n - 2$ critical points in $\mathbb{P}^1$ if we take into account the multiplicity of each $c \in C(f^n)$. Let $\text{SAT}(f)$ be the set of all superattracting periodic points of $f$ in $\mathbb{P}^1$, i.e., $\text{SAT}(f) = \bigcup_{n \in \mathbb{N}}(\text{Fix}(f^n) \cap C(f^n))$. By $\# C(f) < \infty$ and the chain rule, $\# \text{SAT}(f) < \infty$.

The chordal derivative $f^\#$ is a function
\[ z \mapsto f^\#(z) := \lim_{p^1 \ni w \to z} [f(w), f(z)]/[w, z] \]
on \( \mathbb{P}^1 \). For every non-degenerate homogeneous lift \( F \) of \( f \), there exists a sequence \( \{C_j^F\}_{j=1}^{2d-2} \) in \( K^2 \setminus \{0\} \) such that the Jacobian determinant of \( F \) factors as

\[
\det DF() = \prod_{j=1}^{2d-2} (\cdot \wedge C_j^F) \quad \text{on } K^2.
\]

Setting \( c_j := \pi(C_j^F) \) \((j = 1, 2, \ldots, 2d-2)\), the sequence \( \{c_j\}_{j=1}^{2d-2} \) in \( \mathbb{P}^1 \) is independent of choices of \( F \) up to its permutation, and satisfies that for every \( c \in C(f) \), \( \# \{j \in \{1, 2, \ldots, 2d-2\} : c_j = c\} - 1 \) equals the multiplicity of \( c \). For every \( z \in \mathbb{P}^1 \), by a computation involving Euler’s identity, we have

\[
m'(z) = \frac{1}{|d||F(p)||^2} \left| \det DF(p) \right| \quad \text{for } p \in \pi^{-1}(z)
\]

(cf. [12] Theorem 4.3), which with (2.1) yields the equality \( \log(f') = -\log|d| + \sum_{j=1}^{2d-2} (\log|\cdot, c_j| + \log\|C_j^F\|) - 2T_F|\mathbb{P}^1| \) on \( \mathbb{P}^1 \). The (exp of the) right hand side extends \( f' \) to a continuous function on \( \mathbb{P}^1 \) so that

\[
\log(f') = -\log|d| + \sum_{j=1}^{2d-2} (\log|\cdot, c_j| + \log\|C_j^F\|) - 2T_F \quad \text{on } \mathbb{P}^1,
\]

where the continuous extension of \( f' \) is also denoted by the same \( f' \). The chain rule for \( f' \) on \( \mathbb{P}^1 \) extends to \( \mathbb{P}^1 \).

For completeness, we include a proof of the following.

**Lemma 2.3** ([13] Lemma 3.6). On \( \mathbb{P}^1 \),

\[
\log(f') = L(f) + \sum_{c \in C(f)} \Phi_{gf}(\cdot, c) + 2gf \circ f - 2gf.
\]

Here the sum over \( C(f) \) takes into account the multiplicity of each \( c \in C(f) \).

**Proof.** Let us choose a non-degenerate homogeneous lift \( F \) of \( f \) so that \( g_F = gf \), i.e., \( gf = \lim_{n \to \infty} T_{F^n}/d^n \) on \( \mathbb{P}^1 \).

By the definition of \( \Phi_{gf} \), (2.4) is rewritten as

\[
\log(f') = -\log|d| + \sum_{j=1}^{2d-2} (\Phi_{gf}(\cdot, c_j) + gf(c_j) + \log\|C_j^F\|) - 2T_F + (2d - 2)gf
\]

on \( \mathbb{P}^1 \). We claim that \( gf \circ f - d \cdot gf - T_F \) on \( \mathbb{P}^1 \), which is equivalent to \( -2T_F + (2d - 2)gf = 2gf \circ f - 2gf \) on \( \mathbb{P}^1 \). Indeed, for every \( z \in \mathbb{P}^1 \), by \( gf = g_F = \lim_{n \to \infty} T_{F^n}/d^n \) on \( \mathbb{P}^1 \), we have

\[
gf \circ f(z) - d \cdot gf(z) = \lim_{n \to \infty} \left( \frac{1}{d^n} (\log\|F^n(F(p))\| - d^n \log\|F(p)\|) \right)
\]

\[-d \cdot \lim_{n \to \infty} \left( \frac{1}{d^{n+1}} (\log\|F^{n+1}(p)\| - d^{n+1} \log\|p\|) \right) = -(\log\|F(p)\| - d \cdot \log\|p\|) = -T_F(z),
\]

where \( p \in \pi^{-1}(z) \). Hence \( gf \circ f - d \cdot gf = -T_F \) on \( \mathbb{P}^1 \), which in turn holds on \( \mathbb{P}^1 \) by the continuity of both sides, and the claim holds.

By this claim, (2.4) is rewritten as

\[
\log(f') = -\log|d| + \sum_{j=1}^{2d-2} (\Phi_{gf}(\cdot, c_j) + gf(c_j) + \log\|C_j^F\|) + 2gf \circ f - 2gf \quad \text{on } \mathbb{P}^1.
\]
Integrating both sides in (2.6) against \(d\mu_f\) over \(\mathbb{P}^1\), by \(U_{g_f, \mu_f} \equiv 0\) and \(f_*\mu_f = \mu_f\) on \(\mathbb{P}^1\), we have

\[
L(f) := \int_{\mathbb{P}^1} \log(f^\#)d\mu_f
\]

\[
= -\log|d| + \sum_{j=1}^{2d-2} (U_{g_f, \mu_f}(c_j) + g_f(c_j) + \log\|C^F_j\|) + 2 \int_{\mathbb{P}^1} g_f \circ f d\mu_f - 2 \int_{\mathbb{P}^1} g_f d\mu_f
\]

\[= -\log|d| + \sum_{j=1}^{2d-2} (g_f(c_j) + \log\|C^F_j\|).
\]

This with (2.6) completes the proof of Lemma 2.3.

\[\square\]

Berkovich Julia and Fatou sets \(J(f)\) and \(F(f)\). The exceptional set of (the extended) \(f\) is \(E(f) := \{a \in \mathbb{P}^1 : \# \cup_{n \in \mathbb{N}} f^{-n}(a) < \infty\}\), which agrees with the set of all \(a \in SAT(f)\) such that \(\deg_{f^n(a)} f = d\) for any \(j \in \mathbb{N}\). The Berkovich Julia set of \(f\) is

\[
J(f) := \left\{ S \in \mathbb{P}^1 : \bigcap_{V : \text{open in } \mathbb{P}^1 \text{ and contains } S} \left( \bigcup_{n \in \mathbb{N}} f^n(V) \right) = \mathbb{P}^1 \setminus E(f) \right\}
\]

(cf. [11] Definition 2.8), which is closed in \(\mathbb{P}^1\), and the Berkovich Fatou set of \(f\) is \(F(f) := \mathbb{P}^1 \setminus J(f)\), which is open in \(\mathbb{P}^1\). For archimedean \(K\), these definitions of \(J(f)\) and \(F(f)\) coincide with those of the Julia and Fatou sets of \(f\) in terms of the non-normality and the normality of \(\{f^n : n \in \mathbb{N}\}\), respectively.

A Berkovich Fatou component of \(f\) is a component \(W\) of \(F(f)\), and then \(f(W)\) is also a Berkovich Fatou component of \(f\). A Berkovich Fatou component \(W\) of \(f\) is cyclic under \(f\) if \(f^p(W) = W\) for some \(p \in \mathbb{N}\). For archimedean \(K\), the classification of cyclic (Berkovich) Fatou components into

- immediate attractive basins of either (super)attracting or parabolic cycles, and
- rotation domains, i.e., Siegel disks and Herman rings

is essentially due to Fatou (cf. [13] Theorem 5.2). For non-archimedean \(K\), its counterpart due to Rivera-Letelier (see [11] Proposition 2.16 and its esquisse de démonstration, and also [4] Remark 7.10) asserts that every cyclic Berkovich Fatou component \(W\) of \(f\) is either

- an immediate attractive basin of \(f\) in that \(W\) contains a (super)attracting fixed point \(a\) of \(f^p\) in \(W \cap \mathbb{P}^1\) for some \(p \in \mathbb{N}\) and \(\lim_{n \to \infty} (f^p)^n(w) = a\) for any \(w \in W\), or
- a singular domain of \(f\) in that \(f^p(W) = W\) and \(f^p : W \to W\) is injective for some \(p \in \mathbb{N}\),

and moreover, only one of these two possibilities occurs.

3. Proof of Theorem 1

Let \(K\) be an algebraically closed field of characteristic 0 and complete with respect to a non-trivial and possibly non-archimedean absolute value \(|\cdot|\). Let \(f \in K(z)\) be a rational function over \(K\) of degree \(d > 1\). Set

\[
[f^n = \text{Id}] := \sum_{w \in \mathbb{P}^1 : f^n(w) = w} \delta_w,
\]

where \(\delta_w\) is the Dirac measure at \(w\).
where the sum takes into account the multiplicity of each root \( w \in \mathbb{P}^1 \) of \( f^n = \text{Id} \). Then \([f^n = \text{Id}] / (d^n + 1)\) is a probability Radon measure on \( \mathbb{P}^1 \), and \([f^n = \text{Id}]\) satisfies \( \sup_{n \in \mathbb{N}}([f^n = \text{Id}] | (\text{SAT}(f))) \leq \#\text{SAT}(f) < \infty \).

**Lemma 3.1.** For every \( n \in \mathbb{N} \),

\[
\frac{1}{d^n} \int_{\mathbb{P}^1 \setminus \text{SAT}(f)} \log(f^\#)d[f^n = \text{Id}] = \frac{[f^n = \text{Id}] | (\mathbb{P}^1 \setminus \text{SAT}(f))}{d^n} L(f)
\]

(3.2)

\[
= \frac{1}{nd^n} \sum_{c' \in C(f) \setminus \text{SAT}(f)} \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c') \setminus \text{Fix}(f^n)} \Phi_{g_f}(f^n(w), w)
\]

\[
+ \frac{1}{nd^n} \sum_{c' \in C(f) \cap \text{SAT}(f)} \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c') \setminus \text{Fix}(f^n)} \Phi_{g_f}(f^n(w), w)
\]

(3.3)

\[
- \frac{1}{nd^n} \sum_{c \in C(f^n) \cap \text{Fix}(f^n)} \int_{\text{SAT}(f)} \Phi_{g_f}(c, \cdot )d[f^n = \text{Id}](\cdot)
\]

(3.4)

\[
- \frac{1}{nd^n} \sum_{c \in C(f^n) \cap \text{Fix}(f^n)} \int_{\text{SAT}(f)} \Phi_{g_f}(c, \cdot )d[f^n = \text{Id}](\cdot)
\]

(3.5)

Here the sums over \( C(f) \setminus \text{SAT}(f), C(f) \cap \text{SAT}(f), f^{-j}(c') \setminus \text{Fix}(f^n) \) for each \( c' \in C(f), C(f^n) \setminus \text{Fix}(f^n) \), and \( C(f^n) \cap \text{Fix}(f^n) \) take into account the multiplicity of each \( c' \in C(f) \setminus \text{SAT}(f) \), each \( c' \in C(f) \cap \text{SAT}(f) \), each \( w \in f^{-j}(c') \setminus \text{Fix}(f^n) \) for each \( c' \in C(f) \), each \( c \in C(f^n) \setminus \text{Fix}(f^n) \), and each \( c \in C(f^n) \cap \text{Fix}(f^n) \), respectively.

**Proof.** For every \( n \in \mathbb{N} \) and every \( w \in \text{Fix}(f^n) \setminus C(f^n) \), by (2.3) applied to \( f^n \) and also by \( g_{f^n} = g_f \) on \( \mathbb{P}^1 \), we have

\[
\log(f^n)^\#(w) = L(f^n) + \sum_{c \in C(f^n)} \Phi_{g_f}(w, c).
\]

(2.3)

Integrating both sides in (2.3) against \( d[f^n = \text{Id}] \) over \( \text{Fix}(f^n) \setminus C(f^n) = \text{Fix}(f^n) \setminus \text{SAT}(f) \) and dividing these integrations by \( nd^n \), we have

\[
\frac{1}{nd^n} \int_{\mathbb{P}^1 \setminus \text{SAT}(f)} \log(f^n)^\#d[f^n = \text{Id}]
\]

\[
= \frac{[f^n = \text{Id}] | (\mathbb{P}^1 \setminus \text{SAT}(f))}{nd^n} L(f^n) + \frac{1}{nd^n} \sum_{c \in C(f^n)} U_{g_f, [f^n = \text{Id}]} | (\mathbb{P}^1 \setminus \text{SAT}(f))(c).
\]

(3.6)

By the chain rule, we have

\[
\frac{1}{nd^n} \int_{\mathbb{P}^1 \setminus \text{SAT}(f)} \log(f^n)^\#d[f^n = \text{Id}]
\]

\[
= \frac{1}{d^n} \int_{\mathbb{P}^1 \setminus \text{SAT}(f)} \log(f^\#)d[f^n = \text{Id}],
\]

and by the definition (1.3) of \( L(f) \), the chain rule, and \( f_* \mu_f = \mu_f \) on \( \mathbb{P}^1 \), we have \( L(f^n) = nL(f) \). Hence (3.6) is rewritten as

\[
\frac{1}{d^n} \int_{\mathbb{P}^1 \setminus \text{SAT}(f)} \log(f^\#)d[f^n = \text{Id}]
\]

\[
= \frac{[f^n = \text{Id}] | (\mathbb{P}^1 \setminus \text{SAT}(f))}{d^n} L(f)
\]

\[
= \frac{1}{nd^n} \sum_{c \in C(f^n)} U_{g_f, [f^n = \text{Id}]} | (\mathbb{P}^1 \setminus \text{SAT}(f))(c).
\]
We claim that for every $n \in \mathbb{N}$ and every $c \in C(f^n)$, 

$$U_{g_f,[f^n=Id]((\mathbb{A}\setminus SAT(f)))}(c)$$

\[= \begin{cases} 
\Phi_{g_f}(f^n(c),c) - \int_{SAT(f)} \Phi_{g_f}(c,\cdot)d[f^n=Id](\cdot) & \text{if } c \notin \text{Fix}(f^n) \\
- \int_{SAT(f)\setminus\{c\}} \Phi_{g_f}(c,\cdot)d[f^n=Id](\cdot) & \text{if } c \in \text{Fix}(f^n) 
\end{cases} \]

indeed, using \ref{2.3} and $\Phi(f^n,Id)_{g_f} = \Phi_{g_f}(f^n,Id)$ on $\mathbb{P}^1$, we have

$$U_{g_f,[f^n=Id]((\mathbb{A}\setminus SAT(f)))}(c) = \lim_{p^1,\mathbb{P}^1} U_{g_f,[f^n=Id]((\mathbb{A}\setminus SAT(f)))}(z)$$

\[= \lim_{p^1,\mathbb{P}^1} \left( \Phi_{g_f}(f^n(z),z) - \int_{SAT(f)} \Phi_{g_f}(z,\cdot)d[f^n=Id](\cdot) - \int_{p^1} \Phi(f^n,Id)_{g_f}d\mu_f, \right) \]

and moreover,

$$\lim_{p^1,\mathbb{P}^1} \left( \Phi_{g_f}(f^n(z),z) - \int_{SAT(f)} \Phi_{g_f}(z,\cdot)d[f^n=Id](\cdot) \right)$$

\[= \begin{cases} 
\Phi_{g_f}(f^n(c),c) - \int_{SAT(f)} \Phi_{g_f}(c,\cdot)d[f^n=Id](\cdot) & \text{if } c \notin \text{Fix}(f^n), \\
\lim_{p^1,\mathbb{P}^1} \Phi_{g_f}(f^n(z),z) - \Phi_{g_f}(z,c) - \int_{SAT(f)\setminus\{c\}} \Phi_{g_f}(c,\cdot)d[f^n=Id](\cdot) & \text{if } c \in \text{Fix}(f^n).
\end{cases} \]

In the latter case that $c \in C(f^n) \cap \text{Fix}(f^n)$, the first term is computed as

$$\lim_{p^1,\mathbb{P}^1} (\Phi_{g_f}(f^n(z),z) - \Phi_{g_f}(z,c)) = \lim_{p^1,\mathbb{P}^1} \log \left[ \frac{|f^n(z)c|}{|z-c|} \right]$$

\[= \lim_{p^1,\mathbb{P}^1} \log \left[ \frac{|f^n(z) - f^n(c) + c - z|}{|z-c|} \right] = \log |(f^n)'(c) - 1| = \log |0 - 1| = 0,
\]

where we can assume $c \neq \infty$ by the coordinate change $w \mapsto 1/w$ when $c = \infty$.

Hence the claim holds.

Once \ref{3.7} is at our disposal, we have

\[\sum_{c \in C(f^n)} U_{g_f,[f^n=Id]((\mathbb{A}\setminus SAT(f)))}(c)\]

\[= \sum_{c \in C(f^n)\setminus\text{Fix}(f^n)} \Phi_{g_f}(f^n(c),c) - \sum_{c \in C(f^n)\setminus\text{Fix}(f^n)} \int_{SAT(f)} \Phi_{g_f}(c,\cdot)d[f^n=Id](\cdot) \]

\[= \sum_{c \in C(f^n)\setminus\text{Fix}(f^n)} \int_{SAT(f)\setminus\{c\}} \Phi_{g_f}(c,\cdot)d[f^n=Id](\cdot) \]

\[= -(2d^n - 2) \int_{p^1} \Phi(f^n,Id)_{g_f}d\mu_f, \]

and since $C(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(C(f))$ and $C(f) = (C(f)\setminus SAT(f)) \cup (C(f)\cap SAT(f))$,

$$\sum_{c \in C(f^n)\setminus\text{Fix}(f^n)} \Phi_{g_f}(f^n(c),c)$$

\[= \left( \sum_{c' \in C(f)\setminus SAT(f)} + \sum_{c' \in C(f)\cap SAT(f)} \right) \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c')} \Phi_{g_f}(f^n(w),w). \]

Now the proof of Lemma \ref{3.4} is complete. \hfill \Box
Let us estimate the terms (3.2), (3.3), (3.4), and (3.5).

Lemma 3.2.

\[
\frac{1}{nd^n} \sum_{c' \in C(f) \setminus \text{SAT}(f)} \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c') \setminus \text{Fix}(f^n)} \Phi_{g_f}(f^n(w), w) = O(1) \quad \text{as } n \to \infty.
\]

Here the sums over \(C(f) \setminus \text{SAT}(f)\) and \(f^{-j}(c') \setminus \text{Fix}(f^n)\) for each \(c' \in C(f)\) takes into account the multiplicity of each \(c' \in C(f) \setminus \text{SAT}(f)\) and each \(w \in f^{-j}(c')\) for each \(c' \in C(f)\), respectively.

Proof. For every \(n \in \mathbb{N}\), by the definition of \(\Phi_{g_f}\) and \(C(f^n) = \bigcup_{j=0}^{n-1} f^{-j}(C(f))\),

\[
\frac{2d^n - 2}{nd^n} \cdot 2 \sup_{p_1} |g_f| \geq \frac{1}{nd^n} \sum_{c' \in C(f) \setminus \text{SAT}(f)} \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c') \setminus \text{Fix}(f^n)} \Phi_{g_f}(f^n(w), w) \geq \frac{1}{nd^n} \sum_{c' \in C(f) \setminus \text{SAT}(f)} \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c') \setminus \text{Fix}(f^n)} \log[f^n(w), w] - \frac{2d^n - 2}{nd^n} \cdot 2 \sup_{p_1} |g_f|.
\]

We can fix \(L > 1\) such that \(f : \mathbb{P}^1 \to \mathbb{P}^1\) is \(L\)-Lipschitz continuous with respect to the normalized chordal metric (for non-archimedean \(K\), see, e.g., [21, Theorem 2.14]). Then for every \(c' \in C(f) \setminus \text{SAT}(f)\), every \(j \in \{0, 1, 2, \ldots, n-1\}\), and every \(w \in f^{-j}(c')\),

\[
L^n[f^n(w), w] \geq L^{j}[f^j(f^n(w)), f^j(w)] = [f^n(c'), c'].
\]

Recall the definition of the Berkovich Julia and Fatou sets \(J(f)\) and \(F(f)\) in Section 2. We claim that for every \(c' \in C(f) \setminus \text{SAT}(f)\),

\[
(3.8) \quad \log[f^n(c'), c'] \geq \begin{cases} O(1) & \text{if } c' \in F(f), \\ O(n) & \text{if } c' \in J(f) \end{cases} \quad \text{as } n \to \infty;
\]

indeed, in the former case, if \(\liminf_{s \to \infty} |f^s(c'), c'| = 0\) for some \(c' \in (C(f) \cap F(f)) \setminus \text{SAT}(f)\), then the Berkovich Fatou component \(U\) containing \(c'\) is cyclic under \(f\), i.e., \(f^n(U) = U\) for some \(p \in \mathbb{N}\). Since \(c' \in C(f) \cap U\), \(f^p : U \to U\) is not injective, and by the classification of cyclic Berkovich Fatou components of \(f\) (see Section 2), \(U\) is an immediate attracting basin of an either (super)attracting or parabolic cycle of \(f\). Then since \(c' \notin \text{SAT}(f)\), \(\liminf_{s \to \infty} |f^s(c'), c'| > 0\), which is a contradiction. On the other hand, in the latter case, by (the proof of) Przytycki’s lemma [19, Lemma 1], it holds that for every \(c' \in C(f) \cap J(f)\) and every \(n \in \mathbb{N}\), \([f^n(c'), c'] \geq 1/(2L^n)\). The claim holds.

Now we have

\[
\frac{1}{nd^n} \sum_{c' \in C(f) \setminus \text{SAT}(f)} \sum_{j=0}^{n-1} \sum_{w \in f^{-j}(c') \setminus \text{Fix}(f^n)} \log[f^n(w), w] \geq \frac{1}{nd^n} \sum_{c' \in C(f) \setminus \text{SAT}(f)} \sum_{j=0}^{n-1} d^j \log[f^n(c'), c'] - n \log L \geq \frac{2d - 2}{nd^n} \cdot O(n) \cdot \sum_{j=0}^{n-1} d^j = O(1)
\]

as \(n \to \infty\), and the proof of Lemma 3.2 is complete. \(\square\)

For each point \(w \in \mathbb{P}^1\) and each subset \(S\) in \(\mathbb{P}^1\), we put \([w, S] := \inf_{z \in S} [w, S]\).
Lemma 3.3.

\[
\delta := \inf_{a \in \SAT(f)} \left[ a, \left( \bigcup_{j \geq 0} f^{-j}(C(f)) \right) \setminus \{a\} \right] \in (0, 1].
\]

Proof. For every \( a \in \SAT(f) \), there is \( p \in \mathbb{N} \) such that \( f^p(a) = a \) and \( f^j(a) \neq a \) for every \( j \in \{1, 2, \ldots, p - 1\} \). We can fix an open neighborhood \( U \) of \( a \) in \( \mathbb{P} \) so small that \( f^p(U) \subset U \) by the Taylor expansion of \( f^p \) at \( a \) and that \( f^j(U) (\in \{0, 1, 2, \ldots, p - 1\}) \) are mutually disjoint. Set \( \mathcal{O}_a := \{ f^j(a) : \ell \in \{0, 1, 2, \ldots, p - 1\} \} \) and \( \mathcal{U} := \bigcup_{j=0}^{p-1} f^j(U) \), so that \( f^p(U) \subset \mathcal{U}, \mathcal{U} \cap U = \emptyset \) and \( \mathcal{O}_a \cap U = \{a\} \). Decreasing \( U \) if necessary, we can assume \( f^{-1}(\mathcal{O}_a) \cap \mathcal{U} \subset \mathcal{O}_a \) and \( \mathcal{U} \cap f^{-1}(\mathcal{O}_a) < \infty \) and \( \#C(f) < \infty \), respectively.

If \( [a, \bigcup_{j \geq 0} f^{-j}(C(f)) \setminus \{a\}] = 0 \), then there is \( c \in C(f) \) such that \( (\bigcup_{j \geq 0} f^{-j}(c)) \cap (U \setminus \{a\}) \neq \emptyset \). Then \( c \in \mathcal{U} \cap C(f) \) by \( f^1(U) \subset \mathcal{U} \), and \( c \in \mathcal{O}_a \) by \( \mathcal{U} \cap C(f) \subset \mathcal{O}_a \). Hence by \( f^{-1}(\mathcal{O}_a) \cap \mathcal{U} \subset \mathcal{O}_a \), we have \( \bigcup_{j \geq 0} f^{-j}(c) \cap \mathcal{U} \subset \mathcal{O}_a \), so \( \bigcup_{j \geq 0} f^{-j}(c) \cap \mathcal{U} \subset \mathcal{O}_a \cap U = \{a\} \). This is a contradiction. Hence for every \( a \in \SAT(f), [a, \bigcup_{j \geq 0} f^{-j}(C(f)) \setminus \{a\}] > 0 \), which with \( \#SAT(f) < \infty \) completes the proof. \( \square \)

Lemma 3.4.

\[
\sup_{n \in \mathbb{N}} \sup_{w \in \bigcup_{j=0}^{n-1} f^{-j}(C(f) \cap SAT(f)) \setminus Fix(f^n)} \left| \Phi_{g_f}(f^n(w), w) \right| \leq -\log \delta + 2 \sup_{p^1} |g_f| < \infty.
\]

Proof. We claim \( \inf_{n \in \mathbb{N}} \left( \inf_{w \in \bigcup_{j=0}^{n-1} f^{-j}(C(f) \cap SAT(f)) \setminus Fix(f^n)} [f^n(w), w] \right) \geq \delta \); indeed, if there exist \( n \in \mathbb{N} \) and \( w \in \bigcup_{j=0}^{n-1} f^{-j}(C(f) \cap SAT(f)) \) such that \( f^n(w) \neq w \) and \( [f^n(w), w] < \delta \), then \( a := f^n(w) \in SAT(f) \). Since \( w \in \bigcup_{j \geq 0} f^{-j}(C(f)) \), by the definition (3.9) of \( \delta \), we have \( w = a, \) so \( w = a = f^n(w) \). This is a contradiction.

The claim holds, and the proof is complete by the definition of \( \Phi_{g_f} \). \( \square \)

Lemma 3.5.

\[
\sup_{n \in \mathbb{N}} \left( \sup_{c \in C(f^n) \cap Fix(f^n)} \int_{\SAT(f)} \Phi_{g_f}(c, \cdot) d[f^n = Id](\cdot) \right) \leq -\log \delta + 2 \sup_{p^1} |g_f| \left( \sup_{n \in \mathbb{N}} ([f^n = Id](\SAT(f))) \right) < \infty.
\]

Proof. We claim \( \inf_{n \in \mathbb{N}} \inf_{w \in SAT(f) \cap Fix(f^n)} [c, w] \geq \delta \); indeed, if there exist \( n \in \mathbb{N}, c \in C(f^n) \), and \( w \in SAT(f) \cap Fix(f^n) \) such that \( c \not\in Fix(f^n) \) and \( [c, w] < \delta \), then since \( C(f^n) \subset \bigcup_{j \geq 0} f^{-j}(C(f)) \), by the definition (3.9) of \( \delta \), we have \( c = w, \) so \( c \in Fix(f^n) \). This is a contradiction.

The claim holds. Now the proof is complete by the definition of \( \Phi_{g_f} \) and \( \sup_{n \in \mathbb{N}} ([f^n = Id](\SAT(f))) < \infty \). \( \square \)

Lemma 3.6.

\[
\sup_{n \in \mathbb{N}} \left( \sup_{c \in C(f^n) \cap Fix(f^n)} \int_{\SAT(f) \setminus \{c\}} \Phi_{g_f}(c, \cdot) d[f^n = Id](\cdot) \right) \leq -\log \delta \left( \inf_{\text{distinct } c, c' \in SAT(f)} [c, c'] + 2 \sup_{p^1} |g_f| \right) \times \left( \sup_{n \in \mathbb{N}} ([f^n = Id](\SAT(f))) \right) < \infty.
\]
Proof. Since \(#\text{SAT}(f) < \infty\), we have \(\inf_{c, c' \in \text{SAT}(f)} c, c' \in (0, 1)\). Moreover, by (3.7) for each \(\Phi_{g_j}\), and sup\(n \in \mathbb{N}([f^n = \text{Id}][\text{SAT}(f)]) < \infty\).

By Lemmas 3.1, 3.2, 3.4, 3.5, 3.6, and C(f^n) = \(\bigcup_{j \in \mathbb{N}} f^{-j}(C(f))\), we have

\[
\begin{align*}
(3.10) & \quad \frac{1}{dn} \int_{\mathbb{P}^{1} \setminus \text{SAT}(f)} \log(f^#)d[f^n = \text{Id}] - \frac{[f^n = \text{Id}][\mathbb{P}^{1} \setminus \text{SAT}(f)]}{dn} L(f) \\
& \quad = O(1) + 3 \cdot \frac{2d^n - 2}{nd^n} \cdot O(1) - \frac{2 - 2d^{-n}}{n} \int_{\mathbb{P}^{1}} \Phi(f^n, \text{Id})_{g_j} d\mu_f \\
& \quad = -\frac{2 - 2d^{-n}}{n} \int_{\mathbb{P}^{1}} \Phi(f^n, \text{Id})_{g_j} d\mu_f + O(1) \quad \text{as } n \to \infty.
\end{align*}
\]

Similarly, the following holds.

Lemma 3.7.

\[
\begin{align*}
(3.11) & \quad \frac{1}{dn} \int_{\mathbb{P}^{1} \setminus \text{SAT}(f)} \log(f^#)d[f^n = \text{Id}] - \frac{[f^n = \text{Id}][\mathbb{P}^{1} \setminus \text{SAT}(f)]}{dn} L(f) \\
& \quad = \frac{2d - 2}{d^n} \int_{\mathbb{P}^{1}} \Phi(f^n, \text{Id})_{g_j} d\mu_f + O(nd^{-n}) \quad \text{as } n \to \infty.
\end{align*}
\]

\(\Phi_{g_j}\) for each \(c \in C(f)(\subset C(f^n))\),

\[
\sum_{c \in C(f)} U_{g_j, [f^n = \text{Id}]([\mathbb{P}^{1} \setminus \text{SAT}(f)])(c)}
\]

\[
= \sum_{c \in C(f) \setminus \text{Fix}(f^n)} \Phi_{g_j}(f^n(c), c) - \sum_{c \in C(f) \setminus \text{Fix}(f^n)} \int_{\text{SAT}(f)} \Phi_{g_j}(c, \cdot) d[f^n = \text{Id}](\cdot)
- \sum_{c \in C(f) \setminus \text{Fix}(f^n)} \int_{\text{SAT}(f) \setminus \{c\}} \Phi_{g_j}(c, \cdot) d[f^n = \text{Id}](\cdot) - (2d - 2) \int_{\mathbb{P}^{1}} \Phi(f^n, \text{Id})_{g_j} d\mu_f.
\]
Hence, noting that \((C(f) \setminus SAT(f)) \setminus \text{Fix}(f^n) = C(f) \setminus SAT(f)\), we have
\[
\frac{1}{d^n} \int_{p^{n+1} \setminus SAT(f)} \log(f^n) \, d|f^n = 1| - \frac{[f^n = \text{Id}] \setminus (p^{n+1} \setminus SAT(f))}{d^n} L(f)
\]
\[
= \frac{1}{d^n} \sum_{c \in C(f) \setminus SAT(f)} \Phi_{g_f}(f^n(c), c) + \frac{1}{d^n} \sum_{c \in (C(f) \cap SAT(f)) \setminus \text{Fix}(f^n)} \Phi_{g_f}(f^n(c), c)
\]
\[- \frac{1}{d^n} \sum_{c \in C(f) \setminus \text{Fix}(f^n)} \int_{SAT(f)} \Phi_{g_{f^2}}(c, \cdot) \, d|f^n = 1|\]
\[- \frac{1}{d^n} \sum_{c \in (C(f) \setminus \text{Fix}(f^n)) \setminus \{c\}} \int_{SAT(f) \setminus \{c\}} \Phi_{g_f}(c, \cdot) \, d|f^n = 1|\]
\[- \frac{2d - 2}{d^n} \int_{p^n} \Phi(f^n, \text{Id})_{g_f} \, d\mu_f.
\]
In the right hand side, by (3.10) and the definition of \(\Phi_{g_f}\), the first term is estimated as
\[
\frac{2d - 2}{d^n} \cdot 2 \sup_{p^n} |g_f| \geq \frac{1}{d^n} \sum_{c \in C(f) \setminus SAT(f)} \Phi_{g_f}(f^n(c), c)
\]
\[
\geq \frac{1}{d^n} \sum_{c \in C(f) \setminus SAT(f)} \log(f^n(c), c) - \frac{2d - 2}{d^n} \cdot 2 \sup_{p^n} |g_f|
\]
\[
= \frac{2d - 2}{d^n} \cdot O(n) + O(d^{-n}) = O(nd^{-n}) \quad \text{as } n \to \infty,
\]
and since \(C(f) \subset C(f^n)\), by Lemmas 3.4, 3.5 and 3.6 (or by arguments similar to and simpler than those in their proofs), the second, the third, and the fourth terms have the order \((2d - 2)/d^n \cdot O(1) = O(d^{-n})\) as \(n \to \infty\).

Now the proof is complete. \(\Box\)

By (3.10) and (3.11), we have
\[
\frac{2d - 2}{n} \int_{p^n} \Phi(f^n, \text{Id})_{g_f} \, d\mu_f + O(1)
\]
\[
= \frac{2d - 2}{d^n} \int_{p^n} \Phi(f^n, \text{Id})_{g_f} \, d\mu_f + O(1) = O(d^{-1}) \quad \text{as } n \to \infty,
\]
which with \((0 \neq) (2d - 2)/d^n - (2d - 2)/d^n = O(n^{-1})\) as \(n \to \infty\) yields
\[
\frac{2d - 2}{n} \int_{p^n} \Phi(f^n, \text{Id})_{g_f} \, d\mu_f = n \cdot O(1) = O(n) \quad \text{as } n \to \infty.
\]

Once (3.12) is at our disposal, by (3.11), we have
\[
\frac{2d - 2}{d^n} \int_{p^n \setminus SAT(f)} \log(f^n) \, d|f^n = 1| - \frac{[f^n = \text{Id}] \setminus (p^n \setminus SAT(f))}{d^n} L(f)
\]
\[
= \frac{2d - 2}{d^n} \cdot O(n) + O(d^{-n}) = O(nd^{-n}) \quad \text{as } n \to \infty.
\]

By the chain rule and \(\sup_{n \in \mathbb{N}} ([f^n = \text{Id}] \setminus (SAT(f)) < \infty\), the left hand side is computed as
\[
\frac{1}{d^n} \int_{p^n \setminus SAT(f)} \log(f^n) \, d|f^n = 1| - \frac{[f^n = \text{Id}] \setminus (p^n \setminus SAT(f))}{d^n} L(f)
\]
\[
= \frac{1}{nd^n} \sum_{w \in \text{Fix}(f^n) \setminus SAT(f)} \log(|f^n(w)| - L(f) + O(d^{-n}) \quad \text{as } n \to \infty.
\]
Now the proof of Theorem 1 is complete.

4. Proof of Theorem 2

Let $K$ be an algebraically closed field of characteristic 0 and complete with respect to a non-trivial and possibly non-archimedean absolute value $| \cdot |$. Let $f \in K(z)$ be a rational function over $K$ of degree $d > 1$, and suppose that the set

$$A(f) := \bigcup_{n \in \mathbb{N}} \{ w \in \text{Fix}(f^n) : |f^n(w)| \in (0, 1) \}$$

is finite.

We observe that (i) by the chain rule and the assumption $#A(f) < \infty$, we have

$$0 \geq \sum_{w \in \text{Fix}(f^n) \setminus (R(f^n) \cup \text{SAT}(f))} \log |(f^n)'(w)| = \sum_{w \in \text{Fix}(f^n) \setminus A(f)} \log(f^n)^\#(w)$$

$$= n \cdot \sum_{w \in \text{Fix}(f^n) \setminus A(f)} \log(f^\#)(w) \geq n \cdot \left( \inf_{w \in A(f)} \log(f^\#)(w) \right) \cdot #A(f) = O(n)$$

as $n \to \infty$, that (ii) by an argument based on Bezout’s theorem (cf. [6, §4.2]), for every $n \in \mathbb{N}$,

$$#(R(f^n) \setminus R^*(f^n)) \leq \# \left( \bigcup_{j \in \{1, 2, \ldots, n-1\} : \text{dividing } n} \text{Fix}(f^j) \right) \leq 2nd^{n/2},$$

and that (iii) by the chain rule, for every $n \in \mathbb{N}$ and every $w \in R(f^n)$, we have

$$1 < |(f^n)'(w)| = (f^n)^\#(w) \leq \left( \sup_{w \in \mathbb{P}^1} f^\# \right)^n.$$

By the observation (i) and the quantitative approximation [1,2], we have

$$L(f) = \frac{1}{nd^n} \sum_{w \in \text{Fix}(f^n) \setminus \text{SAT}(f)} \log |(f^n)'(w)| + O(nd^{-n})$$

$$= \frac{1}{nd^n} \left( \sum_{w \in R(f^n)} + \sum_{w \in \text{Fix}(f^n) \setminus (R(f^n) \cup \text{SAT}(f))} \right) \log |(f^n)'(w)| + O(nd^{-n})$$

$$= \frac{1}{nd^n} \sum_{w \in R(f^n)} \log |(f^n)'(w)| + \frac{1}{nd^n} \cdot O(n) + O(nd^{-n})$$

$$= \frac{1}{nd^n} \sum_{w \in R(f^n)} \log |(f^n)'(w)| + O(nd^{-n})$$

as $n \to \infty$,

which shows [1,2]. Once [1,2] is at our disposal, since the observations (ii) and (iii) imply

$$0 \leq \sum_{w \in R(f^n) \setminus R^*(f^n)} \log |(f^n)'(w)|$$

$$\leq \left( n \cdot \log \left( \sup_{w \in \mathbb{P}^1} f^\# \right) \right) \cdot 2nd^{n/2} = O(n^2d^{n/2})$$

as $n \to \infty$,
we have

\[ L(f) = \frac{1}{nd^n} \sum_{w \in R(f^n)} \log |(f^n)'(w) + O(nd^{-n}) | = \frac{1}{nd^n} \left( \sum_{w \in R^+(f^n)} + \sum_{w \in R(f^n) \setminus R^+(f^n)} \right) \log |(f^n)'(w)| + O(nd^{-n}) \]

\[ = \frac{1}{nd^n} \sum_{w \in R^+(f^n)} \log |(f^n)'(w)| + \frac{1}{nd^n} \cdot O(n^2d^{n/2}) + O(nd^{-n}) \]

\[ = \frac{1}{nd^n} \sum_{w \in R^+(f^n)} \log |(f^n)'(w)| + O(nd^{-n/2}) \quad \text{as } n \to \infty, \]

which shows \([12] \). Now the proof of Theorem 2 is complete.

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