The Maximum Spectral Radius of Non-Bipartite Graphs Forbidding Short Odd Cycles

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Abstract

It is well-known that eigenvalues of graphs can be used to describe structural properties and parameters of graphs. A theorem of Nosal and Nikiforov states that if $G$ is a triangle-free graph with $m$ edges, then $\lambda(G) \leq \sqrt{m}$, equality holds if and only if $G$ is a complete bipartite graph. Recently, Lin, Ning and Wu [Combin. Probab. Comput. 30 (2021)] proved a generalization for non-bipartite triangle-free graphs. Moreover, Zhai and Shu [Discrete Math. 345 (2022)] presented a further improvement. In this paper, we present an alternative method for proving the improvement by Zhai and Shu. Furthermore, the method can allow us to give a refinement on the result of Zhai and Shu for non-bipartite graphs without short odd cycles.

Mathematics Subject Classifications: 05C50

1 Introduction

The present work can be viewed as the second paper of our previous project [29]. In this paper, we shall use the following standard notation; see e.g., the monograph [9]. We consider only simple and undirected graphs. Let $G$ be a simple graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$. We usually write $n$ and $m$ for the number of vertices and edges respectively. Let $N(v)$ or $N_G(v)$ be the set of neighbors of $v$, and $d(v)$ or $d_G(v)$ be the degree of a vertex $v$ in $G$. For a subset $S \subseteq V(G)$, we write $e(S)$ for the number of edges with two endpoints in $S$. Let $K_{s,t}$ be the complete bipartite graph with parts of sizes $s$ and $t$. We write $C_n$ and $P_n$ for the cycle and path on $n$ vertices respectively. We denote by $t(G)$ the number of triangles in $G$.

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1.1 The classical extremal graph problems

We say that a graph $G$ is $F$-free if it does not contain an isomorphic copy of $F$ as a subgraph. Apparently, every bipartite graph is $C_{2k+1}$-free for every integer $k \geq 1$. The Turán number of a graph $F$ is the maximum number of edges in an $n$-vertex $F$-free graph, and it is usually denoted by $\text{ex}(n, F)$. A graph on $n$ vertices with no subgraph $F$ and with $\text{ex}(n, F)$ edges is called an extremal graph for $F$. As is known to all, the Mantel theorem [36] asserts that if $G$ is an $n$-vertex graph with at least $\lceil \frac{n^2}{4} \rceil$ edges, then either there exist three edges in $G$ that form a triangle or $G = K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$, the balanced complete bipartite graph.

**Theorem 1** (Mantel, 1907). Let $G$ be an $n$-vertex graph. If $G$ is triangle-free, then $e(G) \leq e(K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}) = \lfloor \frac{n^2}{4} \rfloor$, equality holds if and only if $G = K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil}$.

Mantel’s theorem has many interesting applications and generalizations in the literature; see, e.g., [1, pp. 269–273] and [5, pp. 294–301] for standard proofs, [6, 8] for generalizations, and [16, 47] for recent comprehensive surveys. In particular, Mantel’s Theorem 1 was refined in the sense of the following stability form.

**Theorem 2** (Erdős). Let $G$ be an $n$-vertex triangle-free graph. If $G$ is not bipartite, then $e(G) \leq \lfloor \left(\frac{n-1}{2}\right)^2 \rfloor + 1$.

It is said that this stability result attributes to Erdős; see [9, Page 306, Exercise 12.2.7]. The bound in Theorem 2 is best possible and the extremal graphs are not unique. To show that the bound is sharp for all integers $n$, we take two vertex sets $X$ and $Y$ with $|X| = \lfloor \frac{n}{2} \rfloor$ and $|Y| = \lceil \frac{n}{2} \rceil$. We take two vertices $u, v \in Y$ and join them, then we put every edge between $X$ and $Y \setminus \{u, v\}$. We partition $X$ into two parts $X_1$ and $X_2$ arbitrarily (this shows that the extremal graph is not unique), then we connect $u$ to every vertex in $X_1$, and $v$ to every vertex in $X_2$; see Figure 1 This yields a graph $G$ which contains no triangle and $e(G) = \lfloor \frac{n^2}{4} \rfloor - \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \left(\frac{n-1}{2}\right)^2 \rfloor + 1$. Note that $G$ has a cycle $C_5$, so $G$ is not bipartite.

Figure 1: Extremal graphs in Theorem 2.

1.2 The spectral extremal graph problems

There are various matrices that are associated with a graph, such as the adjacency matrix, the incidence matrix, the distance matrix, the Laplacian matrix and signless Laplacian
One of the main problems of algebraic graph theory is to determine the combinatorial properties of a graph that are reflected from the algebraic properties of its associated matrices. Let $G$ be a simple graph on $n$ vertices. The adjacency matrix of $G$ is defined as $A(G) = [a_{ij}]_{n \times n}$ where $a_{ij} = 1$ if two vertices $v_i$ and $v_j$ are adjacent in $G$, and $a_{ij} = 0$ otherwise. We say that $G$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ if these values are eigenvalues of the adjacency matrix $A(G)$. We denote by $\lambda_i(G)$ the $i$-th largest eigenvalue of $G$. Let $\lambda(G)$ be the maximum value in absolute among all eigenvalues of $G$, which is known as the spectral radius of a graph $G$.

There is a rich history on the study of bounding the eigenvalues of a graph in terms of various parameters; see [2] for eigenvalues and expanders, [11, 15] for eigenvalues and diameters, [21] for spectral radius and genus, [3] for spectral radius and cut vertices, [12, 34] for regularity and eigenvalues, [13, 33] for non-regularity and spectral radius, [7] for spectral radius and cliques, [4, 52] for chromatic number and eigenvalues, [35, 17, 40] for independence number and eigenvalues, [12, 34] for regularity and eigenvalues, [13, 33] for non-regularity and spectral radius, [7] for spectral radius and cliques, [4, 52] for chromatic number and eigenvalues, [35, 17, 40] for independence number and eigenvalues, [14, 46] for matching, edge-connectivity and eigenvalues, [18] for spanning trees and eigenvalues, [48, 30] for eigenvalues of outerplanar and planar graphs, and [49] for the Colin de Verdière parameter, excluded minors and the spectral radius.

Let $G$ be a graph on $n$ vertices with $m$ edges. Let $A(G)$ be the adjacency matrix of $G$. It is well-known that

$$\frac{2m}{n} \leq \lambda(G) \leq \sqrt{2m}. \quad (1)$$

Indeed, the lower bound is guaranteed by Rayleigh’s inequality $\lambda(G) \geq e^T A(G)e = \frac{2m}{n}$, where $e = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T \in \mathbb{R}^n$. The upper bound can be seen by invoking the fact that $\lambda(G)^2 \leq \sum_{i=1}^n \lambda_i^2 = \text{tr}(A^2(G)) = \sum_{i=1}^n d_i = 2m$. This upper bound was further improved by Hong [20] as

$$\lambda(G) \leq \sqrt{2m - n + 1}. \quad (2)$$

We recommend the readers to [22] and [37] for further extensions. The classical extremal graph problems usually study the maximum or minimum number of edges that the extremal graphs can have. Correspondingly, we can study the extremal spectral problem. We denote by $\text{ex}_\lambda(n, F)$ the largest eigenvalue of the adjacency matrix in an $n$-vertex graph that contains no copy of $F$, that is,

$$\text{ex}_\lambda(n, F) := \max\{\lambda(G) : |G| = n \text{ and } F \not\subseteq G\}.$$ 

In 1970, Nosal [45] determined the largest spectral radius of a triangle-free graph in terms of the number of edges, which states that if $G$ is a triangle-free graph, then $\lambda(G) \leq \sqrt{m}$. This result improved both inequalities (1) and (2) conditionally. In order to state this result more accurately, we combine with some contributions of Nikiforov’s works [37, 38, 40], which determined the extremal graphs attaining the equality and also provided the spectral version of Theorem 1. Thus we write it as in the following complete form. Note that when we consider the result on a graph with respect to the given number of edges, we shall ignore the possible isolated vertices if there are no confusions.
Theorem 3 (Nosal–Nikiforov). Let $G$ be a graph on $n$ vertices with $m$ edges. If $G$ is triangle-free, then

$$\lambda(G) \leq \sqrt{m},$$

equality holds if and only if $G$ is a complete bipartite graph. Moreover, we have

$$\lambda(G) \leq \lambda(K_{\lfloor n^2/4 \rfloor, \lceil n^2/4 \rceil}),$$

equality holds if and only if $G$ is a balanced complete bipartite graph $K_{\lfloor n^2/4 \rfloor, \lceil n^2/4 \rceil}$.

Theorem 3 implies that if $G$ is a bipartite graph, then $\lambda(G) \leq \sqrt{m}$, equality holds if and only if $G$ is a complete bipartite graph. On the one hand, inequality (3) implies the classical Mantel Theorem 1. Indeed, applying the Rayleigh inequality, we have $\frac{2m}{n} \leq \lambda(G) \leq \sqrt{m}$, which yields $m \leq \lfloor n^2/4 \rfloor$. On the other hand, combining (3) with Mantel’s theorem, we obtain $\lambda(G) \leq \sqrt{m} \leq \sqrt{\lceil n^2/4 \rceil} = \lambda(K_{\lfloor n^2/4 \rfloor, \lceil n^2/4 \rceil})$. So inequality (3) in Theorem 3 can imply inequality (4), which is usually called the spectral Mantel theorem.

Theorem 3 stimulated the developments of two aspects in spectral extremal graph theory. On the one hand, it is natural to consider the spectral extremal problems for graphs with given number of vertices. In view of this perspective, various extensions and generalizations on inequality (4) have been obtained in the literature; see, e.g., [51, 38, 19, 25] for extensions on $K_{r+1}$-free graphs with given order; see [7, 40] for relations between cliques and spectral radius and [41, 10, 27] for surveys. Very recently, Lin, Ning and Wu [31, Theorem 1.4] proved a generalization on (4) for non-bipartite triangle-free graphs and provided a spectral version of Theorem 2; see [29] for an alternative proof and refinement of spectral Turán theorem, and [28] for more stability theorems on spectral graph problems. In addition, Lin and Guo [32] proved an extension of non-bipartite graphs without short odd cycles. This result was also independently proved by Li, Sun and Yu [26, Theorem 1.6] using a different method.

On the other hand, the inequality (3) in Theorem 3 boosted the great interests of studying the maximum spectral radius of graphs in terms of the number of edges, instead of the given number of vertices; see [37] for an extension on $K_{r+1}$-free graphs, [39] for an analogue of $C_4$-free graphs, [53] for further extensions on $K_{2,r+1}$-free graphs, and similar results of $C_5$-free and $C_6$-free graphs as well, [42] for an extension on $B_k$-free graphs, where $B_k$ denotes the book graph consisting of $k$ triangles sharing a common edge, and [31, 54] for refinements on non-bipartite triangle-free graphs. In this paper, we will focus mainly on the extremal spectral problems for graphs with given number of edges, which is becoming increasingly an important and popular topic in recent research on spectral graph theory.

In 2021, Lin, Ning and Wu [31] proved the following improvement on Theorem 3 by using tools from doubly stochastic matrix theory; see [42] for a simpler proof by using elementary numerical inequalities. Let $P_n$ be the path on $n$ vertices, and $C_n$ be the cycle on $n$ vertices. Given two graphs $G$ and $H$, we write $G \cup H$ for the disjoint union of $G$ and $H$. In other words, $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. For
simplicity, we write \( kG \) for the disjoint union of \( k \) copies of \( G \). The blow-up of a graph \( G \) is a new graph obtained from \( G \) by replacing each vertex \( v \in V(G) \) with an independent set \( I_v \), and for two vertices \( u, v \in V(G) \), we add all edges between \( I_u \) and \( I_v \) whenever \( uv \in E(G) \).

**Theorem 4** (Lin–Ning–Wu, 2021). Let \( G \) be a triangle-free graph with \( m \) edges. Then

\[
\lambda_1^2(G) + \lambda_2^2(G) \leq m,
\]
equality holds if and only if \( G \) is a blow-up of a member of \( \mathcal{G} \) in which
\[
\mathcal{G} = \{P_2 \cup K_1, 2P_2 \cup K_1, P_4 \cup K_1, P_5 \cup K_1\}.
\]

A conjecture of Bollobás and Nikiforov [7, Conjecture 1] states that if \( G \) is a \( K_{r+1} \)-free graph with \( m \) edges, then

\[
\lambda_1^2(G) + \lambda_2^2(G) \leq \left(1 - \frac{1}{r}\right)2m.
\]

Theorem 4 confirmed the case \( r = 2 \); see [42, 26] for recent progress. This conjecture of Bollobás and Nikiforov remains open for the case \( r \geq 3 \). We remark here that \( \lambda_1^2(G) + \lambda_2^2(G) \leq m \) does not hold for the \( C_4 \)-free graphs \( G \). Indeed, take \( G = K_{1,m-1} \), the graph obtained from the star \( K_{1,m-1} \) by adding an edge into its independent set. For example, setting \( m = 20 \), we have \( \lambda_1(K_{1,19}^+) \approx 4.425 \) and \( \lambda_2(K_{1,19}^+) = 0.890 \), while \( \lambda_1^2 + \lambda_2^2 \approx 20.372 > 20 \).

With the help of Theorem 4, Lin, Ning and Wu [31, Theorem 1.3] further proved the following refinement on (3) in Nosal’s theorem for non-bipartite triangle-free graphs with given number of edges.

**Theorem 5** (Lin–Ning–Wu, 2021). Let \( G \) be a triangle-free graph with \( m \) edges. If \( G \) is non-bipartite, then

\[
\lambda(G) \leq \sqrt{m-1},
\]
equality holds if and only if \( m = 5 \) and \( G = C_5 \).

In 2022, Zhai and Shu [54] proved a further improvement on Theorem 5. Before stating their result, we need to introduce the extremal graph firstly. For every integer \( m \geq 3 \), we denote by \( \beta(m) \) the largest root of

\[
Z(x) := x^3 - x^2 - (m - 2)x + m - 3.
\]

It is not difficult to show that for \( m \geq 6 \), we have

\[
\sqrt{m - 2} < \beta(m) < \sqrt{m - 1}.
\]

Furthermore, one can verify that \( \lim_{m \to \infty} (\beta(m) - \sqrt{m - 2}) = 0 \). On the other hand, if \( m \) is odd, let \( SK_{2,\frac{m-1}{2}} \) be the graph obtained from \( K_{2,\frac{m-1}{2}} \) by subdividing an edge; see Figure
2 for two different drawings of $SK_{2, \frac{m-1}{2}}$. In particular, for $m = 5$, we have $SK_{2, 2} = C_5$. Clearly, $SK_{2, \frac{m-1}{2}}$ is a triangle-free graph on $n = \frac{m-1}{2} + 3$ vertices with $m$ edges, and it is non-bipartite as it contains a copy of $C_5$. The characteristic polynomial of $SK_{2, \frac{m-1}{2}}$ is

$$\det(xI_n - A(SK_{2, \frac{m-1}{2}})) = x^{\frac{m-1}{2}}(x^2 + x - 1)(x^3 - x^2 - (m - 2)x + m - 3).$$

Therefore, if $m$ is odd, then $\beta(m)$ is the largest eigenvalue of $SK_{2, \frac{m-1}{2}}$.

For convenience, we denote $H(x) := (x^2 + x - 1)Z(x) = x^5 - mx^3 + (2m - 5)x - m + 3$. (6)

So $\beta(m)$ is also the largest root of $H(x)$.

Figure 2: Two drawings of the graph $SK_{2, \frac{m-1}{2}}$.

The improvement of Zhai and Shu [54] on Theorem 5 can be stated as below.

**Theorem 6** (Zhai–Shu, 2022). Let $G$ be a graph of size $m$. If $G$ is triangle-free and non-bipartite, then

$$\lambda(G) \leq \beta(m),$$

equality holds if and only if $m$ is odd and $G = SK_{2, \frac{m-1}{2}}$.

The way that Lin, Ning and Wu [31] proved Theorem 5 is original, and the line of the proof of Zhai and Shu [54] for Theorem 6 is technical. This paper is organized as follows. In Section 2, we shall present an alternative proof of Theorem 6. The present proof is different from the original proof in [54]. Our proof uses and develops the ideas in both [31] and [43], we shall make use of the information of all eigenvalues of graphs, instead of the second largest eigenvalue only. This proof could introduce the main ideas of the approach of our paper, without some technicalities that arise in the other cases, i.e., it can help us to deal with the extremal spectral problem for graphs without short odd cycles. In Section 3, by applying the ideas of the proof of Theorem 6, we will give further refinement on Theorem 6. In Section 4, we will conclude this paper with some possible open problems for interested readers. This paper can be regarded as a supplement of our previous article [29]. Both of these two papers provide extensions and generalizations on the results involving eigenvalues and triangles.
2 Alternative proof of Theorem 6

Recall that Theorem 6 is an improvement on Theorem 5, since \( \beta(m) < \sqrt{m-1} \), where \( \beta(m) \) is the largest root of \( x^3 - x^2 - (m - 2)x + m - 3 = 0 \). The proof of Theorem 5 is succinct and relies on Theorem 4, which implies that if \( G \) is non-bipartite and \( \lambda_2^2(G) + \lambda_2^2(G) \geq m \), then \( G \) contains a triangle. Combining the condition in Theorem 5, we know that if \( G \) satisfies \( \lambda_1(G) \geq \sqrt{m-1} \), then \( \lambda_2(G) < 1 \). This bound on the second largest eigenvalue provided great convenience to characterize the local structure of \( G \). For instance, combining \( \lambda_2(G) < 1 \) with Cauchy’s interlacing theorem, we obtain that the shortest odd cycle of \( G \) is \( C_5 \). However, it is not sufficient to apply Theorem 4 for the proof of Theorem 6. Indeed, if \( G \) is a graph satisfying \( \lambda(G) \geq \beta(m) \), then invoking the fact that \( \lim_{m \to \infty} (\beta(m) - \sqrt{m-2}) = 0 \), we get only that \( \lambda_2(G) < 2 \). Nevertheless, this bound is invalid for our purpose to describe the local structure of \( G \). The original proof of Zhai and Shu [54] for Theorem 6 is innovative and avoided the use of Theorem 4, thus it made more detailed structure analysis of graphs; see [54] for more details.

In what follows, we shall provide an alternative proof of Theorem 6. Our proof grows out partially from the original proof [31] of Theorem 5. To overcome the obstacle mentioned above, we shall make full use of the information of all eigenvalues of graphs, instead of the second largest eigenvalue merely. By applying Cauchy’s interlacing theorem of all eigenvalues, we will find some forbidden induced subgraphs and refine the structure of the desired extremal graph. A key idea relies on the eigenvalue interlacing theorem and a counting lemma [43], which established the relation between eigenvalues and the number of triangles of a graph.

The main steps of the proof can be outlined as below. It introduces the main ideas of the approach of this paper for treating the problem involving short odd cycles.

\( \star \) First of all, applying the forthcoming Lemmas 7, 8 and 9, we will show that \( G \) cannot contain the odd cycle \( C_{2k+1} \) as an induced subgraph for every \( k \geq 3 \), that is, \( C_5 \) is the shortest odd cycle in \( G \); see Claim 10.

\( \star \) Upon more computations, we will prove that more substructures, e.g., the graphs \( H_1, H_2, H_3 \) in Figure 3, are also forbidden as induced subgraphs in \( G \) by applying Lemmas 7, 8 and 9 again; see Claim 11.

\( \star \) Let \( S \) be the set of vertices of a copy of \( C_5 \) in \( G \). Using the above informations of local structure of \( G \), we will show that every vertex outside of \( S \) has exactly two neighbors in \( S \); see Claim 12.

\( \star \) Combining with the previous steps, we will prove that \( G \) is isomorphic to the subdivision of the complete bipartite graph \( K_{a,b} \) by subdividing an edge, where \( a,b \geq 2 \) are integers satisfying \( m = ab + 1 \). Finally, we will show that \( \lambda(SK_{a,b}) \) is at most \( \beta(m) \), equality holds if and only if \( a = 2 \) or \( b = 2 \).
The following lemma is usually referred to as the eigenvalue interlacing theorem, also known as the Cauchy, Poincaré, or Sturm interlacing theorem. It states that the eigenvalues of a principal submatrix of a Hermitian matrix interlace those of the underlying matrix; see, e.g., [55, pp. 52–53] and [56, pp. 269–271]. It is worth noting that this eigenvalue interlacing theorem provides a useful technique to extremal combinatorics and plays a significant role in two breakthrough works [23, 24].

Lemma 7 (Eigenvalue Interlacing Theorem). Let $H$ be an $n \times n$ Hermitian matrix partitioned as

\[ H = \begin{bmatrix} A & B \\ B^* & C \end{bmatrix}, \]

where $A$ is an $m \times m$ principal submatrix of $H$ for some $m \leq n$. Then for every $1 \leq i \leq m$,

\[ \lambda_{n-m+i}(H) \leq \lambda_i(A) \leq \lambda_i(H). \]

Recall that $t(G)$ denotes the number of triangles in $G$. It is well-known that the value of $(i, j)$-entry of $A^k(G)$ is equal to the number of walks of length $k$ in $G$ starting from vertex $v_i$ to $v_j$. Since each triangle of $G$ contributes 6 closed walks of length 3, we can count the number of triangles and obtain

\[ t(G) = \frac{1}{6} \sum_{i=1}^{n} A^3(i, i) = \frac{1}{6} \text{Tr}(A^3) = \frac{1}{6} \sum_{i=1}^{n} \lambda_i^3. \] (7)

The second lemma needed in this paper is a triangle counting lemma in terms of both the eigenvalues and the size of a graph, it could be seen from [43]. This could be viewed as a useful variant of (7) by using $\sum_{i=1}^{n} \lambda_i^2 = \text{Tr}(A^2) = \sum_{i=1}^{n} d_i = 2m$.

Lemma 8. (see [43]) Let $G$ be a graph on $n$ vertices with $m$ edges. If $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are all eigenvalues of $G$, then

\[ t(G) = \frac{1}{6} \sum_{i=2}^{n} (\lambda_1 + \lambda_i) \lambda_i^2 + \frac{1}{3}(\lambda_1^3 - m)\lambda_1. \]

For convenience, we next introduce a function.

Lemma 9. Let $f(x) := (\sqrt{m-2} + x)x^2$. If $a \leq x \leq b \leq 0$, then

\[ f(x) \geq \min\{f(a), f(b)\}. \]

Proof. Since $f(x)$ is monotonically increasing when $x \in (-\infty, -\frac{2}{3}\sqrt{m-2})$, and monotonically decreasing when $x \in [-\frac{2}{3}\sqrt{m-2}, 0]$. Thus the desired statement holds immediately.

It is the time to show an alternative proof of Theorem 6.
Proof of Theorem 6. Suppose that $G$ contains no triangle and $G$ is non-bipartite such that $\lambda(G) \geq \beta(m)$. We shall prove that $m$ is odd and $G = SK_{2, \frac{m-1}{2}}$. Without loss of generality, we may assume that $G$ has the maximum value of spectral radius. First of all, we can see that $G$ must be connected. Otherwise, we can choose $G_1$ and $G_2$ as two different components, where $G_1$ attains the spectral radius of $G$, by identifying two vertices from $G_1$ and $G_2$ respectively, we get a new graph with larger spectral radius, which is a contradiction. It is not hard to verify the desired theorem for $m \leq 10$, since we can consider whether $C_7 \subseteq G$ or $C_9 \subseteq G$ by a standard case analysis. Next, we shall consider the case $m \geq 11$. The proof proceeds by the following three claims.

Claim 10. The shortest odd cycle in $G$ has length 5.

Proof of Claim 10. Since $G$ is non-bipartite, let $s$ be the length of a shortest odd cycle in $G$. Since $G$ is triangle-free and non-bipartite, we have $s \geq 5$ and $\lambda(G) < \sqrt{m}$ by Theorem 3. Moreover, a shortest odd cycle $C_s \subseteq G$ must be an induced odd cycle. By computation, we know that the eigenvalues of $C_s$ are $2 \cos \frac{2\pi k}{s}$, where $k = 0, 1, \ldots, s - 1$. Since $C_s$ is an induced copy in $G$, we know that $A(C_s)$ is a principal submatrix of $A(G)$. Lemma 7 implies that $\lambda_{s-s+1}(G) \leq \lambda_i(C_s) \leq \lambda_i(G)$ for every $i \in [s]$, where $\lambda_i$ means the $i$-th largest eigenvalue. We next show that $s = 5$. We denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of $G$ for simplicity.

Next we will show that $C_7$ is not an induced subgraph of $G$. By the monotonicity of $\cos x$, one can see in the proof that for odd integer $s \geq 7$, $C_s$ can not be an induced subgraph of $G$. Suppose on the contrary that $C_7$ is an induced odd cycle of $G$, then $\lambda_2 \geq \lambda_2(C_7) = 2 \cos \frac{2\pi}{7} \approx 1.246$ and $\lambda_3 \geq \lambda_3(C_7) = 2 \cos \frac{12\pi}{7} \approx 1.246$. Evidently, we get

$$f(\lambda_2) \geq f(1.246) \geq 1.552\sqrt{m-2} + 1.934,$$

and

$$f(\lambda_3) \geq f(1.246) \geq 1.552\sqrt{m-2} + 1.934.$$ 

Our goal is to get a contradiction by applying Lemma 8 and showing $t(G) > 0$. It is not sufficient to obtain $t(G) > 0$ by using the positive eigenvalues of $C_7$ only. Next, we are going to consider the negative eigenvalues of $C_7$. For $i \in \{4, 5, 6, 7\}$, we know that $\lambda_i(C_7) < 0$. The Cauchy interlacing theorem yields $\lambda_{n-3} \leq \lambda_4(C_7) = -0.445$, $\lambda_{n-2} \leq \lambda_5(C_7) = -0.445$, $\lambda_{n-1} \leq \lambda_6(C_7) = -1.801$ and $\lambda_n \leq \lambda_7(C_7) = -1.801$. To apply Lemma 9, we need to find the lower bounds on $\lambda_i$ for each $i \in \{n-3, n-2, n-1, n\}$. We know from (5) that $\lambda_1 \geq \beta(m) > \sqrt{m-2}$, and then $\lambda_n^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_6^2 + \lambda_7^2 < 2m - (m - 2 + 6.744) = m - 4.744$, which implies $-\sqrt{m - 4.744} < \lambda_n \leq -1.801$. By Lemma 9, we get

$$f(\lambda_n) \geq \min\{f(-\sqrt{m - 4.744}), f(-1.801)\} > \sqrt{m-2}$$

There is another way to get a contradiction. We delete an edge within $G_2$, and then add an edge between $G_1$ and $G_2$. This operation will also lead to a new graph with larger spectral radius.
for every $m \geq 11$. Similarly, we have $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < m - 1.501$. Combining with $\lambda_{n-1}^2 \leq \lambda_n^2$, we get $-\sqrt{(m - 1.501)/2} < \lambda_{n-1} \leq -1.801$. By Lemma 9, we obtain

$$f(\lambda_{n-1}) \geq \min\{f(-\sqrt{(m - 1.501)/2}), f(-1.801)\} > \sqrt{m - 2}$$

for every $m \geq 9$. Note that $\sqrt{m} > \lambda_1 \geq \beta(m) > \sqrt{m - 2}$. By Lemma 8, we get

$$t(G) > \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_n) + f(\lambda_{n-1})) - \frac{2}{3}\lambda_1$$

$$> \frac{1}{6}(5.104\sqrt{m - 2} - 4\sqrt{m} + 3.868) > 0.$$ 

This is a contradiction. Similarly, we can prove by applying the monotonicity of $\cos x$ that $C_s$ cannot be an induced subgraph of $G$ for each odd integer $s \geq 7$. Thus we get $s = 5$. 

Let $S = \{u_1, u_2, u_3, u_4, u_5\}$ be the set of vertices of a copy of $C_5$ in $G$. We define the graphs $H_1, H_2$ and $H_3$ as in Figure 3. The eigenvalues of these graphs can be seen in Table 1. To avoid unnecessary calculations, we did not attempt to get the best bound on the size of graph in the proof. Next, we consider the case $m \geq 514$ in the remaining proof.

![Figure 3: Some forbidden induced subgraphs in $G$.](image)

|     | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $\lambda_7$ |
|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $C_7$ | 2           | 1.246       | 1.246       | -0.445      | -0.445      | -1.801      | -1.801      |
| $H_1$ | 2.115       | 1.061       | 0.618       | -0.254      | -1.618      | -1.860      |
| $H_2$ | 2.641       | 1.072       | 0.723       | 0.414       | -0.589      | -1.775      | -2.414      |
| $H_3$ | 2.681       | 1.064       | 0.642       | 0           | 0           | -2          | -2.323      |

Table 1: Eigenvalues of forbidden induced subgraphs.

**Claim 11.** $G$ does not contain any graph of $\{H_1, H_2, H_3\}$ as an induced subgraph.
Proof of Claim 11. Suppose on the contrary that \( G \) contains \( H_i \) as an induced subgraph for some \( i \in \{1, 2, 3\} \). To obtain a contradiction, we shall show \( t(G) > 0 \) by applying Lemma 8. We first consider the case that \( H_1 \) is an induced subgraph in \( G \). The Cauchy interlacing theorem implies \( \lambda_{n-k-i}(G) \leq \lambda_i(H_1) \leq \lambda_i(G) \) for every \( i \in \{1, 2, \ldots, 6\} \). We denote \( \lambda_i = \lambda_i(G) \) for short. Obviously, we have
\[
 f(\lambda_2) \geq f(1) = \sqrt{m-2} + 1,
\]
and
\[
 f(\lambda_3) \geq f(0.618) \geq 0.381\sqrt{m-2} + 0.236.
\]
We next consider the negative eigenvalues of \( G \). The Cauchy interlacing theorem implies \( \lambda_{n-2} \leq \lambda_4(H_1) = -0.254 \) and \( \lambda_{n-1} \leq \lambda_5(H_1) = -1.618 \) and \( \lambda_n \leq \lambda_6(H_1) = -1.860 \). Moreover, we get from (5) that \( \lambda_1 \geq \beta(m) > \sqrt{m-2} \) and \( \lambda_2^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) \leq 2m - (m - 2 + 4.064) = m - 2.064 \), which implies \( -\sqrt{m-2.064} < \lambda_n \leq -1.860 \). By Lemma 9, we have
\[
 f(\lambda_n) \geq \min\{f(-\sqrt{m-2.064}), f(-1.860)\} > 0.031\sqrt{m-2}.
\]
Secondly, since \( \lambda_{n-1}^2 + \lambda_{n-2}^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_5^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < m + 0.553 \) and \( \lambda_{n-1}^2 \leq \lambda_n^2 \), we get \( -\sqrt{(m+0.553)/2} < \lambda_{n-1} \leq -1.618 \). By Lemma 9, we get
\[
 f(\lambda_{n-1}) \geq \min\{f(-\sqrt{(m+0.553)/2}), f(-1.618)\} > 2.617\sqrt{m-2} - 4.235
\]
for every \( m \geq 12 \). Moreover, we have \( -\sqrt{(m+0.618)/3} < \lambda_{n-2} \leq -0.254 \) and then
\[
 f(\lambda_{n-2}) \geq \min\{f(-\sqrt{(m+0.618)/3}), f(-0.254)\} > 0.064\sqrt{m-2} - 0.016
\]
for every \( m \geq 4 \). By Lemma 8, we get that for \( m \geq 514 \),
\[
 t(G) > \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-2}) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3} \lambda_1
\]
\[
 > \frac{1}{6} (4.093\sqrt{m-2} - 4\sqrt{m-2.015}) > 0,
\]
which is a contradiction.

If \( H_2 \) is an induced subgraph of \( G \), then we get similarly that \( \lambda_2 \geq 1, \lambda_3 \geq 0.723 \) and \( \lambda_4 \geq 0.414 \). Then
\[
 f(\lambda_2) \geq f(1) = \sqrt{m-2} + 1,
\]
\[
 f(\lambda_3) \geq f(0.723) \geq 0.522\sqrt{m-2} + 0.377,
\]
and
\[
 f(\lambda_4) \geq f(0.414) \geq 0.171\sqrt{m-2} + 0.07.
\]
The negative eigenvalues of \( H_2 \) imply that \( \lambda_{n-2} \leq -0.589 \), \( \lambda_{n-1} \leq -1.775 \) and \( \lambda_n \leq -2.414 \). Due to \( \lambda_n^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 2 + 5.191) = m - 3.191 \), we get \( -\sqrt{m-3.191} \leq \lambda_n \leq -2.414 \). Lemma 9 gives
\[
 f(\lambda_n) \geq \min\{f(-\sqrt{m-3.191}), f(-2.414)\} > 0.5\sqrt{m-2}
\]
for every \( m \geq 8 \). In addition, we have
\[
-\sqrt{(m - 0.041)/2} \leq \lambda_{n-1} \leq -1.775 \quad \text{and} \quad f(\lambda_{n-1}) \geq \min\{f(-\sqrt{(m - 0.041)/2}), f(-1.775)\} > 2\sqrt{m - 2}
\]
for every \( m \geq 17 \). By Lemma 8, we obtain that for \( m \geq 6 \),
\[
t(G) > \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_4) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3} \lambda_1
\]
\[
> \frac{1}{6}(4.193\sqrt{m - 2} - 4\sqrt{m} + 1.447) > 0,
\]
which is also a contradiction.

If \( H_3 \) is an induced subgraph of \( G \), then we get \( \lambda_2 \geq 2 \) and \( \lambda_3 \geq 0.642 \). Then
\[
f(\lambda_2) \geq f(1) = \sqrt{m - 2} + 1
\]
and
\[
f(\lambda_3) \geq f(0.642) \geq 0.412\sqrt{m - 2} + 0.264.
\]
Moreover, Cauchy’s interlacing theorem gives \( \lambda_{n-1} \leq -2 \) and \( \lambda_n \leq -2.323 \). Since \( \lambda_n^2 \leq 2m - (\lambda_2^2 + \lambda_3^2 + \lambda_4^2 + \lambda_{n-1}^2) < 2m - (m - 2 + 5.412) = m - 3.412 \), we get \( -\sqrt{m - 3.412} < \lambda_n \leq -2.323 \). Then
\[
f(\lambda_n) \geq \min\{f(-\sqrt{m - 3.412}), f(-2.323)\} \geq 0.7\sqrt{m - 2}.
\]
Similarly, we have
\[
-\sqrt{(m + 0.587)/2} < \lambda_{n-1} \leq -2
\]
and
\[
f(\lambda_{n-1}) \geq \min\{f(-\sqrt{(m + 0.587)/2}), f(-2)\} \geq 4\sqrt{m - 2} - 8.
\]
By Lemma 8, we obtain
\[
t(G) > \frac{1}{6}(f(\lambda_2) + f(\lambda_3) + f(\lambda_{n-1}) + f(\lambda_n)) - \frac{2}{3} \lambda_1
\]
\[
> \frac{1}{6}(6.112\sqrt{m - 2} - 4\sqrt{m} - 6.736) > 0,
\]
which is a contradiction. \( \square \)

Let \( N(S) := \cup_{u \in S} N(u) \setminus S \) be the union of neighborhoods of vertices of \( S \). We denote by \( d_S(v) = |N(v) \cap S| \) the number of neighbors of \( v \) in the set \( S \).

**Claim 12.** \( V(G) = S \cup N(S) \) and \( d_S(v) = 2 \) for every \( v \in N(S) \).

**Proof of Claim 12.** First of all, we prove that \( d_S(v) = 2 \) for each vertex \( v \in N(S) \). Without loss of generality, we may assume that \( v \in N(u_1) \). If \( d_S(v) \geq 3 \), then there exists \( i \in [5] \) such that \( \{v, u_i, u_{i+1}\} \) forms a triangle in \( G \), a contradiction. If \( d_S(v) = 1 \), then \( S \cup \{v\} \) induces a copy of \( H_1 \), a contradiction. This implies that \( d_S(v) = 2 \) for every \( v \in N(S) \). Next we prove that \( V(G) = S \cup N(S) \). Otherwise, if there is a vertex \( v' \in V(G) \setminus (S \cup N(S)) \), then \( v' \) has distance at least 2 from \( S \). We may assume
that \(v'vu_1\) is an induced \(P_3\) such that \(v'ut \notin E(G)\) for every \(i \in [5]\). From the above discussion, we know from \(vu_1 \in E(G)\) that \(d_s(v) = 2\). By symmetry, we may assume that \(N_S(v) = \{u_1, u_3\}\). Since \(G\) is triangle-free and \(v'ut \notin E(G)\) for every \(i \in [5]\), we can see that \(\{v', v, u_3, u_4, u_5, u_1\}\) induces a copy of \(H_1\), a contradiction. Thus, we conclude that \(V(G) = S \cup N(S)\) and \(d_s(v) = 2\) for every \(v \in N(S)\).

Since \(m \geq 11\), we can fix a vertex \(v \in N(S)\) and assume that \(N_S(v) = \{u_1, u_3\}\). For each \(w \in V(G) \setminus (S \cup \{v\})\), since \(G\) contains no triangles and no \(H_3\) as an induced subgraph, we know that \(N_S(w) \neq \{u_3, u_5\}\) and \(N_S(w) \neq \{u_4, u_1\}\). It is possible that \(N_S(w) = \{u_1, u_3\}\), \(\{u_2, u_4\}\) or \(\{u_5, u_2\}\). Furthermore, if \(N_S(w) = \{u_1, u_3\}\), then \(wv \notin E(G)\), since \(G\) contains no triangles; if \(N_S(w) = \{u_2, u_4\}\), then \(wv \in E(G)\), since \(G\) contains no induced copy of \(H_2\). We denote \(N_{i,j} = \{w \in V(G) \setminus S : N_S(w) = \{u_i, u_j\}\}\). Note that \(G\) has no induced copy of \(H_3\), there are at least one empty set in \(\{N_{2,4}, N_{5,2}\}\). If \(N_{2,4} = \emptyset\) and \(N_{5,2} = \emptyset\), then \(V(G) \setminus S = N_{1,3}\). Thus \(m\) is odd and \(G = SK_{2, \frac{m-1}{2}}\). Without loss of generality, if \(N_{2,4} \neq \emptyset\), then \(V(G) \setminus S = N_{1,3} \cup N_{2,4}\). Moreover, \(N_{1,3}\) and \(N_{2,4}\) induce a complete bipartite subgraph in \(G\). We denote \(A = N_{1,3} \cup \{u_2, u_4\}\) and \(B = N_{2,4} \cup \{u_3, u_1\}\). Clearly, we have \(|A| = a \geq 2\) and \(|B| = b \geq 2\). Then we observe that \(G\) is isomorphic to the subdivision of the complete bipartite graph \(K_{a,b}\) by subdividing the edge \(u_1u_4\) of \(K_{a,b}\), and \(m = e(G) = ab + 1\). By a direct computation, we get that \(\lambda(G) \leq \beta(m)\), equality holds if and only if \(a = 2\) or \(b = 2\), and thus \(m\) is odd and \(G = SK_{2, \frac{m-1}{2}}\). The detailed computations are stated below. The characteristic polynomial of \(G = \tilde{S}K_{a,b}\)

\[
\text{det}(xI_n - A(SK_{a,b})) = x^{a+b-4}(x^3 - (ab + 1)x^3 + (3ab - 2a - 2b + 1)x - 2ab + 2a + 2b - 2).
\]

Hence \(\lambda(G)\) is the largest root of

\[
F(x) := x^5 - mx^3 + (3m - 2 - 2a - 2b + 1)x - 2m + 2a + 2b - 2.
\]

Recall in (6) that \(\beta(m)\) denotes the largest root of \(H(x)\). We can easily verify that

\[
H(x) - F(x) = (2a + 2\frac{m-1}{a} - m - 3)(x - 1),
\]

which yields \(H(x) \leq F(x)\) for every \(x \geq 1\). Then we get \(H(\lambda(G)) \leq F(\lambda(G)) = 0\), which implies \(\lambda(G) \leq \beta(m)\). This completes the proof.

**Remark.** Theorem 3 asserts that if \(G\) is a graph with \(\lambda(G) \geq \sqrt{m}\), then either \(G\) contains a triangle, or \(G\) is a complete bipartite graph. Very recently, Ning and Zhai [43] proved an elegant spectral counting result, which states that if \(G\) is an \(m\)-edge graph with \(\lambda(G) \geq \sqrt{m}\), then \(G\) has at least \(\lfloor \sqrt{\frac{m-1}{2}} \rfloor\) triangles, unless \(G\) is a complete bipartite graph. Clearly, this saturation result is a generalization of Nosal’s theorem as well as a spectral analogue of a result of Rademacher. A natural question is whether the counting result analogous to Theorem 6 is true. More precisely, if \(G\) is non-bipartite with \(\lambda(G) \geq \beta(m)\), then it seems possible that \(G\) has at least \(\Omega(\sqrt{m})\) triangles, unless \(G = SK_{2, \frac{m-1}{2}}\).
Although we can see from the proof of Theorem 6 that many cases can yield the conclusion that \( G \) has at least \( \Omega(\sqrt{m}) \) triangles, the answer for the above question is surprisingly **negative**. Taking \( G = K_{1,m-1}^+ \) as the graph obtained from the star \( K_{1,m-1} \) by adding an edge into its independent set, we can see that \( G \) is not bipartite and \( \lambda(K_{1,m-1}^+) > \sqrt{m-1} > \beta(m) \), while \( G \) has only one triangle and \( G \neq SK_2 \). Note that the graph \( K_{1,m-1}^+ \) has \( m \) edges and \( m \) vertices. Moreover, we can show that \( \lambda(K_{1,m-1}^+) \) is the largest root of the equation

\[
x^3 - x^2 - (m-1)x + m - 3 = 0.
\]

For \( m = 4, 5, 6, 7, 8 \), we can verify that \( \lambda(K_{1,m-1}^+) > \sqrt{m} \); while for \( m = 9 \), we get \( \lambda(K_{1,8}^+) = 3 = \lambda(K_{1,9}) \). For \( m \geq 11 \), we can check that \( \lambda(K_{1,m-1}^+) < \sqrt{m} \).

### 3 Graphs without short odd cycles

Let \( S_3(K_{a,b}) \) denote the graph obtained from the complete bipartite graph \( K_{a,b} \) by replacing an edge with a five-vertex path \( P_5 \), that is, introducing three new vertices on an edge. Clearly, the shortest odd cycle in \( S_3(K_{a,b}) \) has length seven.

We next consider the further extension of Theorem 6 for graphs with given size and no short odd cycles. For each integer \( m \geq 7 \), we denote by \( \gamma(m) \) the largest root of

\[
L(x) := x^7 - mx^5 + (4m - 14)x^3 - (3m - 14)x - m + 5.
\]

It is not difficult to check that

\[
\sqrt{m-4} < \gamma(m) \leq \sqrt{m-3}.
\]

Indeed, we observe that

\[
L(\sqrt{m-4}) < x(x^6 - mx^4 + (4m - 14)x^2 - (3m - 14)))|_{x=\sqrt{m-4}} = -m + 6 \leq 0,
\]

which leads to \( \sqrt{m-4} < \gamma(m) \). For every \( m \geq 7 \), we have

\[
L(\sqrt{m-3}) = \sqrt{m-3}(m(m-11) + 29) - m + 5 \geq 0,
\]

equality holds only for \( m = 7 \). Combining with \( L'(x) = 7x^6 - 5mx^4 + 3(4m - 14)x^2 - (3m - 14) \geq 0 \) for every \( x \geq \sqrt{m-3} \), we get \( L(x) \geq L(\sqrt{m-3}) \geq 0 \) for every \( x \geq \sqrt{m-3} \), which implies \( \gamma(m) \leq \sqrt{m-3} \).

Moreover, if \( m \) is odd, let \( S_3(K_{2,\frac{m-3}{2}}) \) be the graph obtained from the complete bipartite graph \( K_{2,\frac{m-3}{2}} \) by subdividing an edge into a path of length 4, i.e., putting 3 new vertices on an edge; see Figure 4. In particular, for \( m = 7 \), we have \( S_3(K_{2,2}) = C_7 \). Clearly, \( S_3(K_{2,\frac{m-3}{2}}) \) has \( n = \frac{m-3}{2} + 5 \) vertices and \( m \) edges. Moreover, \( S_3(K_{2,\frac{m-3}{2}}) \) contains no copy of both \( C_3 \) and \( C_5 \), but it has a copy of \( C_7 \) and so it is non-bipartite. Upon computation, the characteristic polynomial of \( S_3(K_{2,\frac{m-3}{2}}) \) is given as

\[
\det(xI_n - A(S_3(K_{2,\frac{m-3}{2}}))) = x^{m-3}(x^7 - mx^5 + (4m - 14)x^3 - (3m - 14)x - m + 5).
\]
Hence, if $m$ is odd, then $\gamma(m)$ is the largest eigenvalue of $S_3(K_{2,\frac{m-1}{2}})$.

Note that the extremal graph $SK_{2,\frac{m-1}{2}}$ in Theorem 6 contains a copy of $C_5$. In this section, we will prove a refinement on Theorem 6. To be more specific, we will determine the largest spectral radius for $C_3$-free and $C_5$-free non-bipartite graphs. To proceed, we need to introduce a lemma.

**Lemma 13.** Let $a, b \geq 2$ and $m$ be integers with $m = ab + 4$. If $G$ is one of the $m$-edge graphs obtained from $S_3(K_{a,b})$ by adding an edge to one vertex, then $\lambda(G) < \gamma(m)$.

**Proof.** We know that $G$ has 7 possible cases. We prove the above case in Figure 5 only, since the other cases can be proved in the same way. By computation, we obtain that $\lambda(G)$ is the largest root of $E(x)$, where $E(x)$ is defined as

$$E(x) := x^8 - (ab + 4)x^6 + (6ab - 2a - 3b + 5)x^4 - (8ab - 5a - 7b + 5)x^2$$
$$- (2ab - 2a - 2b + 2)x + ab - a - b + 1.$$

Note that $m = ab + 4$ and

$$E(x) - xL(x) = (2a - 3)(b - 1)x^4 - (5a - 7)(b - 1)x^2$$
$$- (ab - 2a - 2b + 3)x + ab - a - b + 1.$$

We can verify that $x^2L(x) < E(x)$ for every $x \geq \sqrt{(a - 1)b}$. Since $K_{a-1,b}$ is a subgraph of $G$, we know that $\lambda(G) \geq \lambda(K_{a-1,b}) = \sqrt{(a - 1)b}$. Then $\lambda^2(G)L(\lambda(G)) < E(\lambda(G)) = 0$, which yields $L(\lambda(G)) < 0$ and $\lambda(G) < \gamma(m)$. \qed
The main result of this section is as follows.

**Theorem 14.** Let $G$ be a graph with $m$ edges. If $G$ does not contain any member of $\{C_3, C_5\}$ and $G$ is non-bipartite, then

$$\lambda(G) \leq \gamma(m),$$

equality holds if and only if $m$ is odd and $G = S_3(K_2, \frac{m-3}{2})$.

**Proof.** Assume that $G$ has no $C_3$ and $C_5$, and $G$ is non-bipartite with $\lambda(G) \geq \gamma(m)$, we will show that $m$ is odd and $G = S_3(K_2, \frac{m-3}{2})$. Similar with that in the proof of Theorem 6 in Section 2, it is sufficient to consider the case that $G$ is connected. Since $G$ is non-bipartite, we can assume that $C$ is a shortest odd cycle of $G$. Note that a shortest odd cycle in $G$ must be an induced subgraph. Since $G$ is $C_3$-free and $C_5$-free, we have $s \geq 7$ and $\lambda(G) < \sqrt{m}$ by Theorem 3. In what follows, we shall show that $s = 7$. We denote

$$g(x) := \sqrt{m - 4 + x}.$$  

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $G$ in decreasing order. Since $G$ is non-bipartite and $G$ has an induced odd cycle of length at least 7. For $m \leq 10$, we can do few case analysis whether $C_7 \subseteq G$ or $C_9 \subseteq G$. Thus it is easy to verify the required theorem for $m \leq 10$. Next, we shall consider the case $m \geq 11$ in the proof.

**Claim 15.** A shortest odd cycle in $G$ is $C_7$.

**Proof of Claim 15.** Assume that $C_9$ is an induced odd cycle in $G$, the Cauchy interlacing theorem implies $\lambda_{i-9+1}(G) \leq \lambda_i(C_9) \leq \lambda_i(G)$ for every $i \in \{1, 2, \ldots, 9\}$. From the following Table 2, we can see that $\lambda_2, \lambda_3 \geq 1.532$. Then

$$g(\lambda_2), g(\lambda_3) \geq g(1.532) \geq 2.347\sqrt{m - 4} + 3.596.$$  

Moreover, we have $\lambda_4, \lambda_5 \geq 0.347$, which implies

$$g(\lambda_4), g(\lambda_5) \geq g(0.347) \geq 0.12\sqrt{m - 4} + 0.041.$$  

We next consider the negative eigenvalues of $G$. Note from (9) that $\lambda_1 \geq \gamma(m) > \sqrt{m - 4}$. Since $\lambda_{n-3} \leq \lambda_{n-2} \leq \lambda_7(C_9) = -1$ and $\lambda_n \leq \lambda_{n-1} \leq \lambda_8(C_9) = -1.879$, we have $\lambda_n^2 \leq 2m - (\sum_{i=1}^5 \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) \leq 2m - (m - 4 + 10.465) = m - 6.465$. Thus $-\sqrt{m - 6.465} \leq \lambda_n \leq -1.879$ and then

$$g(\lambda_n) \geq \min\{g(-\sqrt{m - 6.465}), g(-1.879)\} > 0.8\sqrt{m - 4},$$

where the last inequality holds for every $m \geq 11$. Similarly, we have $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\sum_{i=1}^5 \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < m - 2.934$, which together with $\lambda_{n-1}^2 \leq \lambda_n^2$ yields $-\sqrt{(m - 2.934)/2} \leq \lambda_{n-1} \leq -1.879$. Then for every $m \geq 11$, we have

$$g(\lambda_{n-1}) \geq \min\{g(-\sqrt{(m - 2.934)/2}), g(-1.879)\} > 0.9\sqrt{m - 4}.$$
Moreover, we can similarly get $-\sqrt{(m - 1.934)/3} < \lambda_{n-2} \leq -1$ and
\[ g(\lambda_{n-2}) \geq \min\{g(-\sqrt{(m - 1.934)/3}), g(-1)\} \geq \sqrt{m - 4} - 1. \]
The inequality $-\sqrt{(m - 0.934)/4} < \lambda_{n-3} \leq -1$ implies
\[ g(\lambda_{n-3}) \geq \min\{g(-\sqrt{(m - 0.934)/4}), g(-1)\} \geq \sqrt{m - 4} - 1. \]
Owing to $\sqrt{m} \geq \lambda(G) \geq \gamma(m) > \sqrt{m - 4}$, by Lemma 8, we obtain
\[
t(G) > \frac{1}{6} \left( \sum_{i=2}^{5} g(\lambda_i) + g(\lambda_{n-5+i}) \right) - \frac{4}{3} \lambda_1(G) > \frac{1}{6} (8.634\sqrt{m - 4} - 8\sqrt{m} + 5.274) > 0,
\]
which is a contradiction. Therefore, the odd cycle $C_9$ can not be an induced subgraph in $G$. Similarly, we can show by using the monotonicity of cos $x$ that $C_s$ is not an induced subgraph of $G$ for each $s \geq 11$. Consequently, we get $s = 7$. \hfill \Box

From Claim 15, we denote by $S = \{u_1, u_2, \ldots, u_7\}$ the set of vertices of a copy of $C_7$ in $G$. Next, we shall show that the following graphs are forbidden induced subgraphs in $G$, and compute their eigenvalues; see Figure 6 and Table 2.

![Figure 6: Some forbidden induced subgraphs in $G$.](image)
Our proof needs some tedious calculations similar with that in Claim 29(4) (2022), #P4.2 18. This is slightly different from the proof of Theorem 6 in Section 2, and makes the forthcoming proof more complicated.

**Proof of Claim 16.** Our proof needs some tedious calculations similar with that in Claim 15. Suppose on the contrary that $G$ contains $T_i$ as an induced subgraph for some $i \in \{1, 2, 3, 4, 5, 6\}$. To obtain a contradiction, we shall show $t(G) > 0$ by applying Lemma 8. If $T_i$ is an induced subgraph of $G$, then Cauchy’s interlacing theorem gives $\lambda_{n+1}(G) \leq \lambda_i(G)$ for every $i \in \{1, 2, \ldots, 9\}$. In particular, we have $\lambda_2 \geq 1.568$, $\lambda_3 \geq 1.247$ and $\lambda_4 \geq 0.288$. Then

\[
g(\lambda_2) \geq g(1.568) \geq 2.458\sqrt{m - 4} + 3.855,
\]

\[
g(\lambda_3) \geq g(1.247) \geq 0.555\sqrt{m - 4} + 1.939,
\]

and

\[
g(\lambda_4) \geq g(0.288) \geq 0.082\sqrt{m - 4} + 0.023.
\]

In addition, the negative eigenvalues of $T_i$ imply that $\lambda_{n-3} \leq -0.445$, $\lambda_{n-2} \leq -0.919$, $\lambda_{n-1} \leq -1.801$ and $\lambda_n \leq -2.161$. We know from (9) that $\lambda_1 \geq \gamma(m) > \sqrt{m - 4}$, which yields $\lambda_n^2 \leq 2m - (\sum_{i=1}^4 \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 8.382) = m - 4.382$. Then $-\sqrt{m - 4.382} < \lambda_n \leq -2.161$ and

\[
g(\lambda_n) \geq \min\{g(-\sqrt{m - 4.382}), g(-2.161)\} > 0.15\sqrt{m - 4}.
\]

Since $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\sum_{i=1}^4 \lambda_i^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 5.139) = m - 1.139$ and $\lambda_{n-1}^2 \leq \lambda_n^2$, we get $-\sqrt{(m - 1.139)/2} < \lambda_{n-1} \leq -1.801$ and

\[
g(\lambda_{n-1}) \geq \min\{g(-\sqrt{(m - 1.139)/2}), g(-1.801)\} \geq 3.243\sqrt{m - 4} - 5.841.
\]

Similarly, we have $-\sqrt{(m - 0.294)/3} < \lambda_{n-2} \leq -0.919$ and

\[
g(\lambda_{n-2}) \geq \min\{g(-\sqrt{(m - 0.294)/3}), g(-0.919)\} \geq 0.844\sqrt{m - 4} - 0.776.
\]

Table 2: Eigenvalues of graphs $C_9$ and $T_i$ for $i \in \{1, 2, \ldots, 6\}$.

|   | $\lambda_1$ | $\lambda_2$ | $\lambda_3$ | $\lambda_4$ | $\lambda_5$ | $\lambda_6$ | $\lambda_7$ | $\lambda_8$ | $\lambda_9$ |
|---|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $C_9$ | 2           | 1.532      | 1.532      | 0.347      | 0.347       | -1          | -1          | -1.879      | -1.879      |
| $T_1$ | 2.223       | 1.568      | 1.247      | 0.288      | 0           | -0.445      | -0.919      | -1.801      | -2.161      |
| $T_2$ | 2.573       | 1.453      | 1.441      | 0.566      | -0.358      | -0.485      | -0.795      | -1.871      | -2.523      |
| $T_3$ | 2.579       | 1.618      | 1.373      | 0          | 0           | -0.451      | -0.618      | -2          | -2.501      |
| $T_4$ | 2.503       | 1.813      | 1.264      | 0          | 0           | -0.470      | -0.576      | -2.191      | -2.342      |
| $T_5$ | 2.414       | 1.508      | 1.247      | 0.679      | -0.414      | -0.445      | -0.825      | -1.801      | -2.362      |
| $T_6$ | 2.124       | 1.540      | 1.247      | 0.807      | -0.337      | -0.445      | -1.101      | -1.801      | -2.032      |
By Lemma 8, we have
\[
t(G) > \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_4) + g(\lambda_n) + g(\lambda_{n-1}) + g(\lambda_{n-2})) - \frac{4}{3}\lambda_1(G)
\]
\[
> \frac{1}{6}(8.332\sqrt{m-4} - 8\sqrt{m} - 0.8) > 0,
\]
which is a contradiction.

If \( T_2 \) is an induced subgraph of \( G \), then Cauchy’s interlacing theorem implies \( \lambda_{n-9+i}(G) \lesssim \lambda_i(T_2) \lesssim \lambda_i(G) \) for every \( i \in \{1, 2, \ldots, 9\} \). Since \( \lambda_2 \geq 1.453, \lambda_3 \geq 1.441 \) and \( \lambda_4 \geq 0.566 \), we obtain
\[
g(\lambda_2) \geq g(1.453) \geq 2.111\sqrt{m-4} + 3.067,
\]
\[
g(\lambda_3) \geq g(1.441) \geq 2.076\sqrt{m-4} + 2.992,
\]
and
\[
g(\lambda_4) \geq g(0.566) \geq 0.320\sqrt{m-4} + 0.181.
\]
On the other hand, the negative eigenvalues of \( T_2 \) can imply that \( \lambda_{n-4} \leq -0.358, \lambda_{n-3} \leq -0.485, \lambda_{n-2} \leq -0.795, \lambda_{n-1} \leq -1.871 \) and \( \lambda_n \leq -2.523 \). Due to \( \lambda_1 > \sqrt{m-4} \), then we have \( \lambda_2^2 \leq 2m - (\sum_{i=1}^{4}\lambda_i^2 + \lambda_{n-4}^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 9.004) \leq m - 5.004 \), which yields \(-\sqrt{m-5.004} < \lambda_n \leq -2.523 \). Consequently, we get
\[
g(\lambda_n) \geq \min\{g(-\sqrt{m-5.004}), g(-2.523)\} > 0.4\sqrt{m-4}.
\]
Moreover, since \( \lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\sum_{i=1}^{4}\lambda_i^2 + \lambda_{n-4}^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 5.503) = m - 1.503 \) and \( \lambda_{n-1}^2 \leq \lambda_n^2 \), we get \(-\sqrt{(m-1.503)/2} < \lambda_{n-1} \leq -1.871 \) and
\[
g(\lambda_{n-1}) \geq \min\{g(-\sqrt{(m-1.503)/2}), g(-1.871)\} \geq 3.5\sqrt{m-4} - 6.549.
\]
Similarly, we have \(-\sqrt{(m-0.871)/3} < \lambda_{n-2} \leq -0.795 \) and
\[
g(\lambda_{n-2}) \geq \min\{g(-\sqrt{(m-0.871)/3}), g(-0.795)\} \geq 0.632\sqrt{m-4} - 0.502.
\]
By Lemma 8, we have
\[
t(G) > \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_4) + g(\lambda_n) + g(\lambda_{n-1}) + g(\lambda_{n-2})) - \frac{4}{3}\lambda_1(G)
\]
\[
> \frac{1}{6}(9.039\sqrt{m-4} - 8\sqrt{m} - 0.811) > 0,
\]
which is a contradiction.

If \( T_3 \) is an induced subgraph of \( G \), then \( \lambda_2 \geq 1.618 \) and \( \lambda_3 \geq 1.373 \). We get
\[
g(\lambda_2) \geq g(1.618) \geq 2.617\sqrt{m-4} + 4.235,
\]
and
\[
g(\lambda_3) \geq g(1.373) \geq 1.885\sqrt{m-4} + 2.588.
\]
The negative eigenvalues of $T_3$ can give that $\lambda_{n-3} \leq -0.451$, $\lambda_{n-2} \leq -0.618$, $\lambda_{n-1} \leq -2$ and $\lambda_n \leq -2.501$. Since $\lambda_n^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 9.088) = m - 5.088$, then $-\sqrt{m - 5.088} < \lambda_n \leq -2.501$ and

$$g(\lambda_n) \geq \min\{g(-\sqrt{m - 5.088}), g(-2.501)\} > 0.5\sqrt{m - 4}.$$ 

Since $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 5.088) = m - 1.088$ and $\lambda_{n-1}^2 \leq \lambda_n^2$, we get $-\sqrt{(m - 1.088)/2} < \lambda_{n-1} \leq -2$ and

$$g(\lambda_{n-1}) \geq \min\{g(-\sqrt{(m - 1.088)/2}), g(-2)\} \geq 4\sqrt{m - 4} - 8.$$

By Lemma 8, we have

$$t(G) > \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_n) + g(\lambda_{n-1})) - \frac{4}{3}\lambda_1(G)$$

$$> \frac{1}{6}(9.002\sqrt{m - 4} - 8\sqrt{m - 1.177}) > 0,$$

which is a contradiction.

If $T_4$ is an induced subgraph of $G$, then we get from Cauchy’s interlacing theorem that $\lambda_2 \geq 1.813$ and $\lambda_3 \geq 1.264$. Then

$$g(\lambda_2) \geq g(1.813) \geq 3.286\sqrt{m - 4} + 5.959,$$

and

$$g(\lambda_3) \geq g(1.264) \geq 1.597\sqrt{m - 4} + 2.019.$$ 

Moreover, we have $\lambda_{n-3} \leq -0.470$, $\lambda_{n-2} \leq -0.576$, $\lambda_{n-1} \leq -2.191$ and $\lambda_n \leq -2.342$. Since $\lambda_n^2 \leq 2m - (\sum_{i=1}^{n-3} \lambda_i + \lambda_2 + \lambda_{n-1}) < 2m - (m - 4 + 10.237) = m - 6.237$, we get $-\sqrt{m - 6.237} < \lambda_n \leq -2.342$ and

$$g(\lambda_n) \geq \min\{g(-\sqrt{m - 6.237}), g(-2.342)\} \geq \sqrt{m - 4}.$$ 

Since $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\sum_{i=1}^{n-3} \lambda_i + \lambda_{n-3} + \lambda_{n-2}) < 2m - (m - 4 + 5.437) = m - 1.437$ and $\lambda_{n-1}^2 \leq \lambda_n^2$, we get $-\sqrt{(m - 1.437)/2} < \lambda_{n-1} \leq -2.191$ and

$$g(\lambda_{n-1}) \geq \min\{g(-\sqrt{(m - 1.437)/2}), g(-2.191)\} \geq 4.8\sqrt{m - 4} - 10.517.$$  

By Lemma 8, we obtain

$$t(G) > \frac{1}{6}(g(\lambda_2) + g(\lambda_3) + g(\lambda_n) + g(\lambda_{n-1})) - \frac{4}{3}\lambda_1(G)$$

$$> \frac{1}{6}(10.683\sqrt{m - 4} - 8\sqrt{m - 2.539}) > 0,$$

which is a contradiction.
If $T_5$ is an induced subgraph of $G$, then Cauchy’s interlacing theorem implies $\lambda_2 \geq 1.508$, $\lambda_3 \geq 1.247$ and $\lambda_4 \geq 0.679$. Then

\[
g(\lambda_2) \geq g(1.508) \geq 2.274\sqrt{m-4} + 3.429, \\
g(\lambda_3) \geq g(1.247) \geq 1.555\sqrt{m-4} + 1.939,
\]

and

\[
g(\lambda_4) \geq g(0.679) \geq 0.461\sqrt{m-4} + 0.313.
\]

The negative eigenvalues of $T_5$ imply that $\lambda_{n-4} \leq -0.414$, $\lambda_{n-3} \leq -0.445$, $\lambda_{n-2} \leq -0.825$, $\lambda_{n-1} \leq -1.801$ and $\lambda_n \leq -2.362$. Since $\lambda_n^2 \leq 2m - (\sum_{i=1}^4 \lambda_i + \lambda_{n-i}) < 2m - (m - 4 + 8.583) = m - 4.583$, we get $-\sqrt{m - 4.583} < \lambda_n \leq -2.362$ and

\[
g(\lambda_n) \geq \min\{g(-\sqrt{m - 4.583}), g(-2.362)\} \geq 0.25\sqrt{m-4}.
\]

Since $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\sum_{i=1}^4 \lambda_i^2 + \lambda_{n-4}^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 5.34) = m - 1.34$, and $\lambda_{n-1}^2 \leq \lambda_n^2$, we have $-\sqrt{(m - 1.34)/2} < \lambda_{n-1} \leq -1.801$ and $\lambda_{n-1} \geq \min\{g(-\sqrt{(m - 1.34)/2}), g(-1.801)\} \geq 3.243\sqrt{m-4} - 5.841.

Similarly, we can get $-\sqrt{(m - 0.659)/3} < \lambda_{n-2} \leq -0.825$ and $\lambda_{n-2} \geq \min\{g(-\sqrt{(m - 0.659)/3}), g(-0.825)\} \geq 0.68\sqrt{m-4} - 0.561.

By Lemma 8, we obtain

\[
t(G) > \frac{1}{6} (g(\lambda_2) + g(\lambda_3) + g(\lambda_4) + g(\lambda_n) + g(\lambda_{n-1}) + g(\lambda_{n-2})) - \frac{4}{3} \lambda_1(G) \\
> \frac{1}{6} (8.463\sqrt{m-4} - 8\sqrt{m} - 0.721) > 0,
\]

which is a contradiction.

If $T_6$ is an induced subgraph of $G$, then Cauchy’s interlacing theorem implies that $\lambda_2 \geq 1.540$, $\lambda_3 \geq 1.247$ and $\lambda_4 \geq 0.807$. Then

\[
g(\lambda_2) \geq g(1.540) \geq 2.371\sqrt{m-4} + 3.652, \\
g(\lambda_3) \geq g(1.247) \geq 1.555\sqrt{m-4} + 1.939,
\]

and

\[
g(\lambda_4) \geq g(0.807) \geq 0.651\sqrt{m-4} + 0.525.
\]

The negative eigenvalues of $T_6$ yield that $\lambda_{n-4} \leq -0.337$, $\lambda_{n-3} \leq -0.445$, $\lambda_{n-2} \leq -1.101$, $\lambda_{n-1} \leq -1.801$ and $\lambda_n \leq -2.032$. Since $\lambda_n^2 \leq 2m - (\sum_{i=1}^4 \lambda_i^2 + \lambda_{n-1}^2) < 2m - (m - 4 + 9.345) = m - 5.345$, we get $-\sqrt{m - 5.345} < \lambda_n \leq -2.032$ and $\lambda_n \geq \min\{g(-\sqrt{m - 5.345}), g(-2.032)\} \geq 0.65\sqrt{m-4}.$
Since $\lambda_{n-1}^2 + \lambda_n^2 \leq 2m - (\sum_{i=1}^4 \lambda_i^2 + \lambda_{n-4}^2 + \lambda_{n-3}^2 + \lambda_{n-2}^2) < 2m - (m - 4 + 6.101) = m - 2.101$ and $\lambda_n^2 \leq \lambda_n^2$, we get $-\sqrt{(m - 2.101)/2} < \lambda_{n-1} \leq -1.801$ and

$$g(\lambda_{n-1}) \geq \min\{g(-\sqrt{(m - 2.101)/2}), g(-1.801)\} \geq 3.243 \sqrt{m - 4} - 5.841.$$  

Similarly, we can get $-\sqrt{(m - 0.889)/3} < \lambda_{n-2} \leq -1.101$ and

$$g(\lambda_{n-2}) \geq \min\{g(-\sqrt{(m - 0.889)/3}), g(-1.101)\} \geq 1.212 \sqrt{m - 4} - 1.334.$$  

By Lemma 8, we obtain

$$t(G) > \frac{1}{6} (g(\lambda_2) + g(\lambda_3) + g(\lambda_4) + g(\lambda_n) + g(\lambda_{n-1}) + g(\lambda_{n-2})) - \frac{4}{3} \lambda_1(G)$$  

$$> \frac{1}{6} (9.682 \sqrt{m - 4} - 8 \sqrt{m - 1.059}) > 0,$$

which is a contradiction. \hfill \Box

**Claim 17.** $V(G) = S \cup N(S)$ and $d_S(v) \in \{1, 2\}$ for each vertex $v \in N(S)$.  

**Proof of Claim 17.** For each $v \in N(S)$, without loss of generality, we may assume that $v \in N(u_1)$. If $d_S(v) \geq 3$, then we can find either a $C_3$ or $C_5$ in $G$, a contradiction. This implies that $d_S(v) \in \{1, 2\}$ for every $v \in N(S)$. Next we prove that $V(G) = S \cup N(S)$. Otherwise, if there is a vertex $v' \in V(G) \setminus (S \cup N(S))$, then $v'$ has distance at least 2 from $S$. Let $v'uv_1$ be a path of $G$ such that $v'uv_i \notin E(G)$ for every $i \in [7]$. Since $d_S(v) \in \{1, 2\}$ and $G$ is both $C_3$-free and $C_5$-free, we know by symmetry that either $N_S(v) = \{u_1\}$ or $N_S(v) = \{u_1, u_3\}$. If $N_S(v) = \{u_1\}$, then $\{v', v\} \cup S$ induces a copy of $T_6$, a contradiction. If $N_S(v) = \{u_1, u_3\}$, then $\{v', v\} \cup S$ induces a copy of $T_3$, which is a contradiction. Thus, we conclude that $V(G) = S \cup N(S)$ and $d_S(v) \in \{1, 2\}$ for every $v \in N(S)$. \hfill \Box

From Claim 17, we assume that $V(G) \setminus S = V_1 \cup V_2$, where $V_i = \{v \in N(S) : d_S(v) = i\}$ for every $i = 1, 2$. Since $T_1$ is not an induced subgraph of $G$, we get $0 \leq |V_1| \leq 1$. Since $m \geq 11$, we get $V_2 \neq \emptyset$. We can fix a vertex $v \in V_2$ and assume that $N_S(v) = \{u_1, u_3\}$. For each $w \in V_2$, since $G$ contains no triangles and no $T_3$ as induced subgraphs, we know that $N_S(w) \neq \{u_3, u_5\}$ and $N_S(w) \neq \{u_6, u_1\}$. Similarly, since $G$ contains no pentagons and $T_3$ as induced subgraphs, we get $N_S(w) \neq \{u_4, u_6\}$ and $N_S(w) \neq \{u_5, u_7\}$. Therefore, it is possible that $N_S(w) = \{u_1, u_3\}, \{u_2, u_1\}$ or $\{u_7, u_2\}$. Furthermore, if $N_S(w) = \{u_1, u_3\}$, then $uw \notin E(G)$, since $G$ contains no triangles; if $N_S(w) = \{u_2, u_4\}$, then $uw \in E(G)$, since $G$ contains no induced copy of $T_2$. We denote $N_{i,j} = \{w \in V(G) \setminus S : N_S(w) = \{u_i, u_j\}\}$. Note that $G$ has no induced copy of $T_3$, there are at least one empty set in \{N_{2,4}, N_{7,2}\}.

**Case 1.** If both $N_{2,4} = \emptyset$ and $N_{7,2} = \emptyset$, then $V_2 = N_{1,3}$ and $V(G) \setminus S = N_{1,3} \cup V_1$. If $|V_1| = 0$, then $m$ is odd and $G = S_3(K_{2, m/2})$, as desired. If $|V_1| = 1$, then $m$ is even and $G$ is a graph obtained from $S_3(K_{2, m/2})$ by hanging an edge to one vertex. By setting $a = 2$ and $b = \frac{m}{2}$ in Lemma 13, we know that $\lambda(G) < \gamma(m)$. 

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Recall in (8) that \( \gamma(m) \) denotes the largest root of \( L(x) \). We can easily verify that \( L(x) \leq Q(x) \) for every \( x \geq 1 \), so we get \( L(\lambda(G)) \leq Q(\lambda(G)) = 0 \), which implies \( \lambda(G) \leq \gamma(m) \), equality holds if and only if \( a = 2 \) or \( b = 2 \), and thus \( m \) is odd and \( G = S_3(K_{2, m-3}) \). If \( |V_1| = 1 \), then \( m = ab + 4 \) and \( G \) is obtained from \( S_3(K_{a,b}) \) by hanging an edge to one vertex. By Lemma 13 again, we get \( \lambda(G) < \gamma(m) \). This completes the proof.

\[ Q(x) := x^7 - (ab + 3)x^5 + (5ab - 2a - 2b + 2)x^3 + (-5ab + 4a + 4b - 3)x - 2ab + 2a + 2b - 2. \]

Case 2. Without loss of generality, we may assume that \( N_{2,4} \neq \emptyset \), then \( V(G) \setminus S = N_{1,3} \cup N_{2,4} \cup V_1 \). Moreover, \( N_{1,3} \) and \( N_{2,4} \) induce a complete bipartite subgraph in \( G \). We denote \( A = N_{1,3} \cup \{u_2, u_4\} \) and \( B = N_{2,4} \cup \{u_3, u_1\} \). Clearly, we have \( |A| = a \geq 2 \) and \( |B| = b \geq 2 \). If \( |V_1| = 0 \), then \( G \) is isomorphic to the subdivision of \( K_{a,b} \) by replacing the edge \( u_1u_4 \) of \( K_{a,b} \) with a path of length 4, and \( m = ab + 3 \). Note that \( \lambda(S_3(K_{a,b})) \) is the largest spectral radius of non-bipartite graphs with no copy of \( S_3 \) of \( C_3, C_5 \), and so far as to \( C_9 \) by more careful computations. From this evidence, we propose the following conjecture for interested readers. Let \( S_{2k-1}(K_{s,t}) \) denote the graph obtained from the complete bipartite graph \( K_{s,t} \) by replacing an edge with a path \( P_{2k+1} \) on \( 2k + 1 \) vertices, that is, introducing \( 2k - 1 \) new vertices on an edge. Clearly, the odd girth of \( S_{2k-1}(K_{s,t}) \) is \( 2k + 3 \).

**Conjecture 18.** Let \( G \) be a graph with \( m \) edges. If \( G \) does not contain any member of \( \{C_3, C_5, \ldots, C_{2k+1}\} \) and \( G \) is non-bipartite, then

\[ \lambda(G) \leq \lambda(S_{2k-1}(K_{2, m-2k+1})), \]

equality holds if and only if \( m \) is odd and \( G = S_{2k-1}(K_{2, m-2k+1}) \).

Let \( B_k \) be the book graph, i.e., the graph obtained from \( k \) triangles by sharing a common edge. In particular, we have \( B_1 = K_3 \) and \( B_2 = K_4 \), the 4-vertex complete graph minus an edge. In 2021, Zhai, Lin and Shu [53, Conjecture 5.2] made the following conjecture: Let \( m \) be large enough and \( G \) be a \( B_k \)-free graph with \( m \) edges. Then

\[ \lambda(G) \leq \sqrt{m}, \]

equality holds if and only if \( G \) is a complete bipartite graph.

Soon after, Nikiforov [42] confirmed Zhai–Lin–Shu’s conjecture by showing the following stronger theorem. Let \( bk(G) \) denote the booksize of \( G \), that is, the maximum number
of triangles with a common edge in $G$. Nikiforov [42] proved that if $G$ is a graph with $m$ edges and $\lambda(G) \geq \sqrt{m}$, then
\[ bk(G) > \frac{1}{12} \sqrt{m}, \]
unless $G$ is a complete bipartite graph (with possibly some isolated vertices). Since $B_2$ contains both $C_3$ and $C_4$ as a subgraph, the result of Nikiforov generalized Theorem 3.

We conclude this paper with the following problem and conjecture that the lower bound $bk(G) \geq c\sqrt{m}$ is true for some constant $c > 0$.

**Conjecture 19.** If $G$ is a graph with $m$ edges and $\lambda(G) \geq \sqrt{m}$, then
\[ bk(G) \geq c\sqrt{m} \]
for some constant $c > 0$, unless $G$ is a complete bipartite graph.

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