On superlinear Fractional p-Laplacian in $\mathbb{R}^n$

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Abstract

In this article we are interested in the following fractional $p$-Laplacian equation in $\mathbb{R}^n$:

$$(-\Delta)^s_p u + V(x)u^{p-2}u = f(x,u) \text{ in } \mathbb{R}^n,$$

where $p \geq 2$, $0 < s < 1$, $n \geq 2$ and subcritical $p$-superlinear nonlinearity. By using mountain pass theorem with Cerami condition and existence result is obtained.

1. Introduction

Recently, a great attention has been focused on the study of problem involving fractional and non-local operators. This type of problem arises in many different applications, such as, continuum mechanics, phase transition phenomena, population dynamics and game theory, as they are the typical outcome of stochastically stabilization of Lévy processes, see [4, 6, 7, 20] and the references therein. The literature on non-local operators and their applications is very interesting and quite large, we refer the interested reader to [3, 5, 7, 8, 13, 14, 16, 17, 22] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the interested reader to [11].

In this paper, our aim is to show the existence of weak solution for the following class of equations:

$$(1.1) (-\Delta)^s_p u + V(x)u^{p-2}u = f(x,u) \text{ in } \mathbb{R}^n,$$

where $p \geq 2$, $0 < s < 1$, $n \geq 2$ and $(-\Delta)^s_p$ is the fractional $p$-laplacian defined by

$$(1.2) (-\Delta)^s_p u(x) = 2 \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^n \setminus B_{\varepsilon}(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{n+sp}} dy, \quad x \in \mathbb{R}^n.$$

In order to obtain the existence of weak solution for (1.1), let us recall some result related to the fractional Sobolev space $W^{s,p}(\mathbb{R}^n)$. First of all, define the Gagliardo seminorm by

$$[u]_{s,p} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy \right)^{1/p}.$$
where $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function. Now, consider that the fractional Sobolev space given by

$$W^{s,p}(\mathbb{R}^n) = \{ u \in L^p(\mathbb{R}^n) : u \text{ is measurable and } [u]_{s,p} < \infty \}$$

is assumed to be endowed with norm

$$\|u\|_{s,p} = \left( [u]_{s,p} + \|u\|_p^p \right)^{1/p},$$

where the fractional critical exponent is defined by

$$p^*_s = \begin{cases} \frac{np}{n-sp}, & \text{if } sp < n; \\ \infty, & \text{if } sp \geq n \end{cases}$$

Moreover we consider the fractional Sobolev space with potential

$$X^s := \left\{ u \in W^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^pdx < \infty \right\}$$

endowed with the norm

$$\|u\|_{X^s} = \left( [u]_{s,p}^p + \|V(x)^{1/p}u\|_p^p \right)^{1/p}.$$ 

Now, we make the following assumptions on the functions $V$ and $f$.

(V) $V \in C(\mathbb{R}^n)$, $\inf_{\mathbb{R}^n} V(x) \geq V_0 > 0$ and $\mu(x \in \mathbb{R}^n : V(x) \leq M) < +\infty$, $\forall M > 0$.

(f1) $f \in C(\mathbb{R}^n \times \mathbb{R})$ and satisfies

\begin{equation}
\lim_{|t| \to \infty} \frac{f(x,t)}{|t|^{q-1}} = 0, \quad \lim_{|t| \to \infty} \frac{f(x,t)t}{|t|^p} = +\infty
\end{equation}

uniformly in $x \in \mathbb{R}^n$ for some $q \in (p, p^*_s)$, where $F(x,t) = \int_0^t f(x,s)ds$.

(f2) $f(x,t) = o(|t|^{p-2}t)$ as $|t| \to 0$, uniformly in $x \in \mathbb{R}^n$.

(f3) There exists $\theta \geq 1$ such that $\theta F(x,t) \geq F(x,\sigma t)$ for $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $\sigma \in [0,1]$, where

$$F(x,t) = f(x,t)t - pF(x,t).$$

By condition (f1) and (f2), for any $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ such that

\begin{equation}
|F(x,t)| \leq \frac{\epsilon}{p} |t|^p + \frac{C_\epsilon}{q} |t|^q.
\end{equation}

Consequently, the energy functional $I : X^s(\mathbb{R}^n) \to \mathbb{R}$,

\begin{equation}
I(u) = \frac{1}{p} \|u\|_{X^s}^p - \int_{\mathbb{R}^n} F(x,u)dx
\end{equation}

is well defined and of class $C^1$. The derivative of $I$ is given by

$$I'(u)v = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} dxdy + \int_{\mathbb{R}^n} V(x)uvdx - \int_{\mathbb{R}^n} f(x,u)vdx.$$ 

for $v \in X^s$. Therefore, the critical points of $I$ are weak solutions of (1.1).

Now we are ready to state our main result.
Theorem 1.1. Suppose that the conditions $(V), (f_1) - (f_3)$ hold. Then, the problem (1.1) has a nontrivial solution.

2. Preliminaries and notation

To study the fractional problem (1.1), the so-called fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$ with $0 < s < 1$ are expedient. If $1 < p < \infty$, as usual, the norm is defined through

$$
\|u\|_{s,p} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dxdy + \int_{\mathbb{R}^n} |u(x)|^p dx.
$$

We recall the Sobolev embedding theorem.

Theorem 2.1. [11] Let $s \in (0, 1)$ and $p \in [1, +\infty)$ be such that $sp < n$. Then there exists a positive constant $C = C(n, p, s)$ such that

$$
\|u\|_{L^p} \leq C \left( \int_{\mathbb{R}^n} |x - y|^{n+sp} dxdy \right),
$$

where $p_s^* = \frac{np}{n-sp}$ is the so-called “fractional critical exponent”. Consequently, the space $W^{s,p}(\mathbb{R}^n)$ is continuously embedded in $L^q(\mathbb{R}^n)$ for any $q \in [p, p_s^*]$. Moreover, the embedding $W^{s,p}(\mathbb{R}^n) \hookrightarrow L^q_{\text{loc}}(\mathbb{R}^n)$ is compact for $q \in [p, p_s^*)$.

Now consider the space $X^s$ defined by

$$
X^s := \left\{ u \in W^{s,p}(\mathbb{R}^n) : \int_{\mathbb{R}^n} V(x)|u|^p dx < \infty \right\},
$$

endowed with the norm

$$
\|u\|_{X^s} = \left( \int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) + u(y)|^p}{|x - y|^{n+sp}} dxdy + \int_{\mathbb{R}^n} V(x)|u(x)|^p dx \right)^{1/p}.
$$

From $(V_1)$, Theorem 2.1 and Hölder inequality, we have $X^s \hookrightarrow L^q(\mathbb{R}^n)$ for $p \leq q \leq p_s^*$. Moreover, the following compactness result holds. It was proved in [9] in the case $p = 2$. For the general case, the proof is similar. We give it here for reader’s convenience.

Lemma 2.1. Suppose that $(V_1)$ hold. Then $X^s \hookrightarrow L^q(\mathbb{R}^n)$ is compact for $q \in [p, p_s^*)$

Proof. Let $\{u_n\} \subset X^s$ be a bounded sequence of $X^s$ such that $u_n \rightharpoonup 0$ in $X^s$. Then, by Theorem 2.1, $u_n \to 0$ in $L^q_{\text{loc}}(\mathbb{R}^n)$ for $p \leq q < p_s^*$. We claim that

$$
(2.1) \quad u_n \to 0 \text{ strongly in } L^p(\mathbb{R}^n).
$$

To prove (2.1), we only need to show that for any $\epsilon > 0$, there exists $R > 0$ such that

$$
\int_{\mathbb{R}^n \setminus B_R} |u_n(x)|^p dx < \epsilon.
$$
Set
\[ B_R = \{ x \in \mathbb{R}^n : |x| < R \}, \]
\[ A(R, M) = \{ x \in \mathbb{R}^n \setminus B_R : V(x) \geq M \}, \]
\[ B(R, M) = \{ x \in \mathbb{R}^n \setminus B_R : V(x) < M \}, \]
then
\[ \int_{A(R, M)} |u_n(x)|^p dx \leq \int_{\mathbb{R}^n} \frac{V(x)}{M} |u_n(x)|^p dx \leq \frac{\|u_n\|_{X^s}^p}{M}. \]
Now choose \( \sigma \in (1, \frac{p^*}{p}) \) such that \( \frac{1}{\sigma} + \frac{1}{\sigma'} = 1 \), then we have
\[ \int_{B(R, M)} |u_n(x)|^p dx \leq \left( \int_{B(R, M)} |u_n(x)|^{p\sigma} dx \right)^{1/\sigma} (\mu(B(R, M)))^{1/\sigma'} \leq C \|u_n\|_{X^s}^p (\mu(B(R, M)))^{1/\sigma'}. \]
Since \( \|u_n\|_{X^s} \) is bounded and condition \((V_1)\) holds, we may choose \( R, M \) large enough such that \( \frac{\|u_n\|_{X^s}^p}{M} \) and \( \mu(B(R, M)) \) are small enough. Hence, \( \forall \epsilon > 0 \), we have
\[ \int_{\mathbb{R}^n \setminus B_R} |u_n(x)|^p dx = \int_{A(R, M)} |u_n(x)|^p dx + \int_{B(R, M)} |u_n(x)|^p dx < \epsilon, \]
from which (2.1) follows.
To prove the lemma for general exponent \( q \), we use an interpolation argument. Let \( u_n \rightharpoonup 0 \) in \( X^s \), we have just proved that \( u_n \rightarrow 0 \) in \( L^p(\mathbb{R}^n) \). That is
\[ \int_{\mathbb{R}^n} |u_n(x)|^p dx \rightarrow 0. \]
Moreover, because the embedding \( X^s \hookrightarrow L^{p^*}(\mathbb{R}^n) \) is continuous and \( \{ u_n \} \) is bounded in \( X^s \), we also have
\[ \sup_n \int_{\mathbb{R}^n} |u_n(x)|^{p^*} dx < \infty. \]
Since \( q \in (p, p^*_s) \), there is a number \( \lambda \in (0, 1) \) such that
\[ \frac{1}{q} = \frac{\lambda}{p} + \frac{1 - \lambda}{p^*_s}. \]
Then by Hölder inequality
\[ \int_{\mathbb{R}^n} |u_n(x)|^q dx = \int_{\mathbb{R}^n} |u_n(x)|^{\lambda q} |u_n(x)|^{(1-\lambda)q} dx \leq \|u_n\|_{L^p}^{\lambda q} \|u_n\|_{L^{p^*_s}}^{(1-\lambda)q} \rightarrow 0. \]
This implies \( u_n \rightarrow 0 \) in \( L^q(\mathbb{R}^n) \). \( \square \)
The dual space of \((X^s, \| \cdot \|_{X^s})\) is denoted by \(((X^s)^*, \| \cdot \|_*)\). We rephrase variationally the fractional p-Laplacian as the nonlinear operator \(A : X^s \to (X^s)^*\) defined for all \(u, v \in X^s\) by

\[
\langle A(u), v \rangle = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|u|^{p-2}v \, dx.
\]

It can be seen that, if \(u\) is smooth enough, this definition coincides with that of (1.2). A weak solution of problem (1.1) is a function \(u \in X^s\) such that

\[
(A(u), v) = \int_{\mathbb{R}^n} f(x, u)v \, dx
\]

for all \(v \in X^s\). Clearly, for all \(u \in X^s\)

\[
\langle A(u), u \rangle = \|u\|_{X^s}^p, \quad \|A(u)\|_* \leq \|u\|_{X^s}^{p-1},
\]

and \(A\) satisfies the following properties.

**Lemma 2.2.** For any \(u, v \in X^s\), it holds that

\[
\langle A(u) - A(v), u - v \rangle \geq (\|u\|_{X^s}^{p-1} - \|v\|_{X^s}^{p-1})(\|u\|_{X^s} - \|v\|_{X^s})
\]

**Proof.** By direct computation, we have

\[
\langle A(u) - A(v), u - v \rangle = \langle A(u), u - v \rangle - \langle A(v), u - v \rangle
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u(x) - u(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|u|^{p-2}u \, dx
\]

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|v|^{p-2}v \, dx
\]

\[
= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy
\]

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy
\]

\[
+ \int_{\mathbb{R}^n} V(x)|u|^{p} \, dx - \int_{\mathbb{R}^n} V(x)|u|^{p-2}u \, dx
\]

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} \, dx \, dy
\]

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy
\]

\[
= \|u\|_{X^s}^p + \|v\|_{X^s}^p
\]

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} \, dx \, dy
\]

\[
- \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy
\]

\[
- \int_{\mathbb{R}^n} V(x)|u|^{p-2}u \, dx
\]

\[
- \int_{\mathbb{R}^n} V(x)|v|^{p-2}v \, dx.
\]
By Hölder inequality, it holds that
\[ \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-1}|v(x) - v(y)|}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|u|^{p-1}|v| \, dx \leq \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{p}} + \left( \int_{\mathbb{R}^n} V(x)|u|^p \right)^{\frac{p-1}{p}} \left( \int_{\mathbb{R}^n} V(x)|v|^p \right)^{\frac{1}{p}}. \]

Using the following inequality
\[(a + b)^{\beta}(c + d)^{1-\beta} \geq a^\beta c^{1-\beta} + b^\beta d^{1-\beta}\]
which holds for any \( \beta \in (0, 1) \) and \( a > 0, b > 0, c > 0, d > 0 \), set \( \beta = \frac{p-1}{p} \) and
\[
\begin{align*}
a &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
b &= \int_{\mathbb{R}^n} V(x)|u|^p \, dx \\
c &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \\
d &= \int_{\mathbb{R}^n} V(x)|v|^p \, dx
\end{align*}
\]
we can deduce that
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|u|^{p-2}uv \, dx \leq \|u\|^{p-1}_{X^s} \|v\|_{X^s}.
\]
Similarly, we can obtain
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x) - v(y)|^{p-2}(v(x) - v(y))(u(x) - u(y))}{|x - y|^{n+sp}} \, dx \, dy + \int_{\mathbb{R}^n} V(x)|v|^{p-2}uv \, dx \leq \|v\|^{p-1}_{X^s} \|u\|_{X^s}.
\]
Therefore we have
\[
\langle A(u) - A(v), u - v \rangle \geq \|u\|^{p}_{X^s} + \|v\|^{p}_{X^s} - \|u\|^{p-1}_{X^s} \|v\|_{X^s} - \|v\|^{p-1}_{X^s} \|u\|_{X^s} = (\|u\|^{p}_{X^s} - \|v\|^{p-1}_{X^s})(\|u\|_{X^s} - \|v\|_{X^s}).
\]
\[\square\]

**Lemma 2.3.** If \( u_n \to u \) and \( \langle A(u_n), u_n - u \rangle \to 0 \), then \( u_n \to u \) in \( X^s \)

**Proof.** Since \( X^s \) is a reflexive Banach space, it is isometrically isomorphic to a locally uniformly convex space. So as it was proved in [10], weak convergence and norm convergence imply strong convergence. Therefore we only need to show that \( \|u_n\|_{X^s} \to \|u\|_{X^s} \).

We note that
\[
\lim_{n \to \infty} \langle A(u_n) - A(u), u_n - u \rangle = \lim_{n \to \infty} (\langle A(u_n), u_n - u \rangle - \langle A(u), u_n - u \rangle) = 0
\]
By Lemma 2.2 we have
\[ \langle A(u_n) - A(u), u_n - u \rangle \geq (\|u_n\|_{X_s}^{p-1} - \|u\|_{X_s}^{p-1})(\|u_n\|_{X_s} - \|u\|_{X_s}) \geq 0. \]
Hence \( \|u_n\|_{X_s} \to \|u\|_{X_s} \) as \( n \to \infty \) and the assertion follows. \( \square \)

3. PROOF OF THEOREM 1.1

In this section we consider the functional \( I : X^s \to \mathbb{R} \) defined by
\[ I(u) = \frac{1}{p} \|u\|_{X_s}^p - \int_{\mathbb{R}^n} F(x, u) dx. \]
I is well defined and of class \( C^1 \). The derivative of \( I \) is given by
\[ \langle I'(u), v \rangle = \langle A(u), v \rangle - \int_{\mathbb{R}^n} f(x, u)v dx, \]
for \( v \in X^s \). Therefore, the critical points of \( I \) are weak solutions of (1.1).

Lemma 3.1. Suppose that \((V_1), (f_1), (f_2)\) and \((f_3)\) hold, then \( I \) satisfies the Cerami condition (C).

Proof. Let \( \{u_k\} \) be a sequence in \( X^s \) satisfying
\[ I(u_k) \to c, \quad (1 + \|u_k\|_{X_s})I'(u_k) \to 0. \]
We claim that \( \{u_k\} \) is bounded in \( X^s \). Otherwise, if \( \|u_k\|_{X_s} \to \infty \), we consider \( w_k = \frac{u_k}{\|u_k\|_{X_s}} \). Then, up to subsequence, we have
\[ w_k \rightharpoonup w \text{ in } X^s, \]
\[ w_k \to w \text{ in } L^q(\mathbb{R}^n) \text{ for } p \leq q \leq p^*_s \]
\[ w_k(x) \to w(x) \quad \text{a.e. } x \in \mathbb{R}^n, \]
as \( k \to \infty \). We first consider the case that \( w \neq 0 \) in \( X^s \). Since \( I'(u_k)u_k \to 0 \), we have
\[ \|u_k\|_{X_s}^p - \int_{\mathbb{R}^n} f(x, u_k)u_k dx \to 0. \]
By dividing the left-hand side of (3.3) with \( \|u_k\|_{X_s}^p \), we get
\[ \left| \int_{\mathbb{R}^n} \frac{f(x, u_k)u_k}{\|u_k\|_{X_s}^p} \right| \leq 1. \]
On the other hand, by Fatou’s Lemma and condition \((f_1)\) we have
\[ \int_{\mathbb{R}^n} \frac{f(x, u_k)u_k}{\|u_k\|_{X_s}^p} dx = \int_{\{w_k \neq 0\}} |w_k|^p \frac{f(x, u_k)u_k}{|u_k|^p} dx \to \infty, \]
this contradicts to (3.4).
If \( w = 0 \) in \( X^s \), consider
\[ \gamma_k : [0, 1] \to \mathbb{R}, \]
\[ t \to I(tu_k). \]
By the continuity of $\gamma_k$, we choose a sequence $\{t_k\} \in [0,1]$ such that

$$I(t_k u_k) = \max_{t \in [0,1]} I(t u_k).$$

For any $\lambda > 0$, let $v_k = (2p\lambda)^{1/p} u_k = \frac{(2p\lambda)^{1/p} u_k}{\|u_k\|_{X^s}}$, then

$$v_k \to 0 \text{ in } L^q(\mathbb{R}^n) \text{ for } q \in [p, p^*_s).$$

We claim that

$$\lim_{n \to \infty} \int_{\mathbb{R}^n} F(x, v_k) dx = 0. \tag{3.6}$$

In fact, by (1.4) and (3.5) for $k$ large enough we have

$$\left| \int_{\mathbb{R}^n} F(x, v_k) dx \right| \leq \frac{\varepsilon}{p} \|v_k\|_{L^p}^p + \frac{C \varepsilon}{q} \|v_k\|_{L^q}^q \to 0.$$}

Since $\|u_k\|_{X^s} \to \infty$, for $k$ large enough we have

$$\frac{(2p\lambda)^{1/p}}{\|u_k\|_{X^s}} \in (0,1).$$

Hence, for $k$ large enough, (3.6) gives

$$I(t_k u_k) \geq I(v_k) = \frac{1}{p} \|v_k\|_{X^s}^p - \int_{\mathbb{R}^n} F(x, v_k) dx$$

that is

$$I(t_k u_k) \to +\infty. \tag{3.7}$$

Since $I(0) = 0$, $I(u_k) \to c$, we have $t_k \in (0,1)$. By the definition of $t_k$,

$$\langle I'(t_k u_k), t_k u_k \rangle = 0. \tag{3.8}$$

From (3.7), (3.8), we have

$$I(t_k u_k) - \frac{1}{p} \langle I'(t_k u_k), t_k u_k \rangle = \int_{\mathbb{R}^n} \left( \frac{1}{p} f(x, t_k u_k) t_k u_k - F(x, t_k u_k) \right) dx \to \infty.$$

By $(f_3)$, there exists $\theta \geq 1$ such that

$$\int_{\mathbb{R}^n} \left( \frac{1}{p} f(x, u_k) u_k - F(x, u_k) \right) dx \geq \frac{1}{\theta} \int_{\mathbb{R}^n} \left( \frac{1}{p} f(x, t_k u_k) t_k u_k - F(x, t_k u_k) \right) dx \to \infty. \tag{3.9}$$

On the other hand,

$$\int_{\mathbb{R}^n} \left( \frac{1}{p} f(x, u_k) u_k - F(x, u_k) \right) dx = I(u_k) - \frac{1}{p} \langle I'(u_k), u_k \rangle \to c. \tag{3.10}$$

(3.9) contradicts (3.10). Hence $\{u_k\}$ is bounded in $X^s$, therefore up to a subsequence we may assume that $u_k \to u$ in $X^s$ and

$$u_k \to u \text{ in } L^q(\mathbb{R}^n). \tag{3.11}$$
By the boundedness of \( \{u_k\} \) in \( L^p(\mathbb{R}^n) \), we have

\[
(3.12) \quad \Lambda_1 = \sup_k \int_{\mathbb{R}^n} |u_k|^p dx < \infty.
\]

By Hölder inequality we also have

\[
\int_{\mathbb{R}^n} |u_k|^{p-1} u dx \leq 2 \left( \int_{\mathbb{R}^n} |u_k|^{(p-1)p'} dx \right)^{1/p'} \|u\|_{L^p} \equiv 2\|u\|_{L^p} \left( \int_{\mathbb{R}^n} |u_k|^p dx \right)^{1/p'}.
\]

Applying (3.12), we deduce

\[
\Lambda_2 = \sup_k \int_{\mathbb{R}^n} |u_k|^{p-1} u dx < \infty.
\]

Similarly,

\[
\Lambda_3 = \sup_k \int_{\mathbb{R}^n} |u_k|^{p-1} u_k dx < \infty.
\]

Now, by \((f_1)\) and \((f_2)\), for any \( \epsilon > 0 \) there exists \( C_\epsilon > 0 \) such that

\[
|f(x,t)| \leq \epsilon |t|^{p-1} + C_\epsilon |t|^{q-1}, \quad \text{for all } (x,t) \in \mathbb{R}^n \times \mathbb{R}.
\]

Then using Hölder inequality we have

\[
\int_{\mathbb{R}^n} (f(x,u_k) - f(x,u))(u_k - u)dx \\
\leq \int_{\mathbb{R}^n} \left[ \epsilon (|u_k|^{p-1} + |u|^{p-1}) + C_\epsilon (|u_k|^{q-1} + |u|^{q-1}) \right] |u_k - u| d\!
\leq \epsilon \int_{\mathbb{R}^n} (|u_k|^p + |u|^p + |u_k|^{p-1}|u| + |u|^{p-1}|u_k|) dx \\
+ C_\epsilon \left( \int_{\mathbb{R}^n} |u_k|^{q-1} |u_k - u| dx + \int_{\mathbb{R}^n} |u|^{q-1} |u_k - u| dx \right) \\
\leq \epsilon \left( \Lambda_1 + \Lambda_2 + \Lambda_3 + \int_{\mathbb{R}^n} |u|^p dx \right) \\
+ 2C_\epsilon \left( \sup_k \|u_k|^{q-1}\|_{L^{q'}} + \|u|^{q-1}\|_{L^{q'}} \right) \|u_k - u\|_{L^q} \\
(3.13) \\
\leq \epsilon \left( \Lambda_1 + \Lambda_2 + \Lambda_3 + \int_{\mathbb{R}^n} |u|^p dx \right) \\
+ 2C_\epsilon \left( \sup_k \|u_k|^{q-1}\|_{L^{q'}} + \|u|^{q-1}\|_{L^{q'}} \right) \|u_k - u\|_{L^q} \\
\leq \epsilon \left( \Lambda_1 + \Lambda_2 + \Lambda_3 + \int_{\mathbb{R}^n} |u|^p dx \right) \\
+ 2C_\epsilon \left( \sup_k \|u_k|^{q-1}\|_{L^{q'}} + \|u|^{q-1}\|_{L^{q'}} \right) \|u_k - u\|_{L^q} \\
\leq \epsilon \left( \Lambda_1 + \Lambda_2 + \Lambda_3 + \int_{\mathbb{R}^n} |u|^p dx \right) \\
+ 2C_\epsilon \left( \sup_k \|u_k|^{q-1}\|_{L^{q'}} + \|u|^{q-1}\|_{L^{q'}} \right) \|u_k - u\|_{L^q} \\
\to 0.
\]

Since \( \{u_k\} \) is bounded in \( L^q(\mathbb{R}^n) \), it follows that \( \{|u_k|^{q-1}\} \) is bounded in \( L'^{(q')}(\mathbb{R}^n) \). That is

\[
\sup_k \|u_k|^{q-1}\|_{L'^{(q')}} < \infty.
\]

Therefore we can deduce from (3.11) and (3.13) that

\[
\int_{\mathbb{R}^n} (f(x,u_k) - f(x,u))(u_k - u)dx \to 0.
\]
Note that $I'(u_k) \to 0$ and we have

$$\langle A(u_k) - A(u), u_k - u \rangle = \langle I'(u_k) - I'(u), u_k - u \rangle + \int_{\mathbb{R}^n} (f(x, u_k) - f(x, u))(u_k - u) \, dx \to 0.$$ 

Therefore, by Lemma 2.3 we obtain $u_n \to u$ in $X^s$. The proof is complete. □

**Proof of Theorem 1.1**

We check that $I$ has the mountain pass geometry. By (1.4), we have

$$|F(x, t)| \leq \frac{\epsilon}{p} |t|^p + \frac{C_\epsilon}{q} |t|^q,$$

so

$$I(u) \geq \frac{1}{p} \|u\|_{X^s}^p - \frac{\epsilon}{p} \|u\|_{L^p}^p - \frac{C_\epsilon}{q} \|u\|_{L^q}^q$$

$$\geq \left( \frac{1}{p} - \frac{\epsilon C_p}{p} \right) \|u\|_{X^s}^p - \frac{C_p C_q}{q} \|u\|_{X^s}^q.$$

Let $\epsilon > 0$ small enough such that $\frac{1}{p} - \frac{\epsilon C_p}{p} > 0$ and $\|u\|_{X^s} = \rho$. Since $q > p$, taking $\rho$ small enough such that

$$\frac{1}{p} - \frac{\epsilon C_p}{p} - \frac{C_p C_q}{q} \rho^{q-p} > 0.$$

Therefore

$$I(u) \geq \rho^p \left( \frac{1}{p} - \frac{\epsilon C_p}{p} - \frac{C_p C_q}{q} \rho^{q-p} \right) = \beta > 0.$$

Now we note that by $(f_1)$

$$\lim_{|t| \to \infty} \frac{f(x, t)t}{|t|^p} = +\infty \quad \text{implies that} \quad \lim_{|t| \to \infty} \frac{F(x, t)}{|t|^p} = +\infty.$$

So, for any $\epsilon > 0$, there exists $M > 0$ such that

$$F(x, t) > \frac{|t|^p}{\epsilon}, \quad \text{for all} \quad |t| > M.$$

Let $c(\epsilon) = \frac{M^p}{\epsilon}$, then

$$F(x, t) > \frac{|t|^p}{\epsilon} - \frac{M^p}{\epsilon}.$$

Next, for $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} \frac{F(x, t\varphi)}{|t|^p} \geq \frac{1}{\epsilon} \int_{\mathbb{R}^n} |\varphi|^p \, dx - \frac{M^p}{\epsilon |t|^p} \int_{\text{supp}(\varphi)} \, dx.$$

This implies

$$\lim_{|t| \to \infty} \int_{\mathbb{R}^n} \frac{F(x, t\varphi)}{|t|^p} \, dx \geq \frac{1}{\epsilon} \int_{\mathbb{R}^n} |\varphi|^p \, dx,$$

for all $\epsilon > 0$. Since $\epsilon$ is arbitrary, by (3.14) we get

$$\lim_{|t| \to \infty} \int_{\mathbb{R}^n} \frac{F(x, t\varphi)}{|t|^p} \, dx = +\infty.$$
Consequently,
\[
\frac{I(t\varphi)}{|t|^p} = \frac{1}{p} \|\varphi\|_{X^p}^p - \int_{\mathbb{R}^n} \frac{F(x, t\varphi)}{|t|^p} dx \to -\infty, \quad \text{as } |t| \to \infty.
\]

Hence, let \( t_0 \) be big enough and \( e = t_0 \varphi \), then we have \( I(e) < 0 \).

Therefore, since by Lemma 3.1, \( I \) satisfies the Cerami condition and has mountain pass geometry, using the Mountain pass Lemma the proof of Theorem is complete. \( \square \)

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