MULTIDIMENSIONAL GENERALIZATION OF KASNER SOLUTION

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Abstract

Full generalization of Kasner metric for the case of \(n+1\) dimensions and \(m \leq n+1\) essential variables is obtained. Any solution is defined by the corresponding constant matrix of Kasner parameters. This parameters form in euclidian space Casner hyper-spheres and are connected by additional conditions. General properties of obtained solutions are analised.
1 General solution

In present paper we’ll find full multidimensional generalization of Kasner solution, which in common representation has the following kind:

\[ ds^2 = dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2, \]  

(1)

which describe homogeneous anisotropic 4-D vacuum space-time. Here \( p_1, p_2, p_3 \) — real parameters, satisfying to the following conditions:

\[ p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1. \]

(2)

Original form of Kasner solution, given in [1], is more general and has the following kind:

\[ ds^2 = t^{2p_0} dt^2 - t^{2p_1} dx^2 - t^{2p_2} dy^2 - t^{2p_3} dz^2 \]

(3)

with conditions on parameters \( p_1 + p_2 + p_3 = 1 + p_0, \) \( p_1^2 + p_2^2 + p_3^2 = (1 + p_0)^2 \) In general, there is freedom in choosing of value of parameter \( p_0 \) by the coordinate transformations. Another commonly used form of this solution have been proposed by Narlikar and Karmarkar [2]. 4-D Casner solution is suitable for describing of early epoch of the Universe [3].

Multidimensional generalization of metric (1) for the case of \( n+1 \)-D homogeneous anisotropic space-time has been found in [4]. It has the following kind:

\[ ds^2 = dt^2 - \sum_{k=1}^{n} t^{2p_k} (dx^k)^2, \]

(4)

with the conditions on parameters \( p_k \):

\[ \sum_{k=1}^{n} p_k = \sum_{k=1}^{n} p_k^2 = 1. \]

(5)

We’ll find generalization of the metric (4), when it depends on degrees of another coordinates (all or some parts) with their own Casner parameters. Such space will not be homogeneous now, but it will admits the system of \( n+1 \)-orthogonal hypersurface [5], which in choosen coordinate system will be coordinate hypersurfaces \( x^i = const \).

Multidimensional metric we can write in the following symmetric form:

\[ ds^2 = e^{2\alpha_0} dt^2 - \sum_{k=1}^{n} e^{2\alpha_k} (dx^k)^2, \]

(6)

where \( \{\alpha_0, \ldots, \alpha_n\} \) are, in general, functions of (without loss of generality) first \( m \) coordinates \( \{t, x^1, \ldots, x^m\} \). Multidimensional vacuum Einstein equations \( n+1 R_{ij} = 0 \) have the following kind:

\[ R_{ii} = \sum_{k}^t \varepsilon_k e^{-2\alpha_k} (\alpha_{i,kk} + \alpha_{i,k} \times \]

(7)
\[ 
\left( \sum_{p} \alpha_{p,k} - \alpha_{k,k} \right) + \varepsilon_i e^{-2\alpha_i} \left( \sum_{s} \alpha_{s,ii} + \alpha_{s,i} (\alpha_{s,i} - \alpha_{i,i}) \right) = 0; \\
i = 0, m, \quad k = 0, n \ (k \neq i), \quad p = 0, n \ (p \neq k), \\
s = 0, n \ (s \neq i).
\]

\[ 
R_{ij} = e^{-\alpha_i - \alpha_j} \left( - \sum_k '' \alpha_{k,ij} + \alpha_{i,j} \sum_k '' \alpha_{k,i} \right) + \\
\alpha_{j,i} \sum_k '' \alpha_{k,j} - \alpha_{i,j} \sum_k '' \alpha_{k,i} \alpha_{k,j} \right) = 0. \\
k = 0, n \ (k \neq i, j)
\]

where symbols """ and """ means, that summating is made by all set of corresponding index, excluding one or two values of this set correspondingly. We are using the following abbreviations:

\[ 
\alpha_{i,j} \equiv \frac{\partial \alpha_i}{\partial x^j}; \quad \alpha_{i,jk} \equiv \frac{\partial^2 \alpha_i}{\partial x^j \partial x^k}; \\
\varepsilon_k = \begin{cases} 
+1, & k = 0; \\
-1, & k \neq 0.
\end{cases}
\]

Obviously, that all partial derivatives \( \partial / \partial k \) with \( k > m \) vanishes in eq. (7)-(8). For we are interested by generalization of the solution (4), then it is necessary to specify kind of the functions \( \alpha_k \) by the following way:

\[ 
\alpha_k = \alpha_{ks} \ln x_s 
\]

where \( \alpha_{ks} \) — is a \( n \times m \) constant matrix of Casner parameters. In (8) there is summating by repeating index \( s = 0, m \). Substituting this kind of functions \( \alpha_k \) into system (7)-(8) and taking into account that \( \alpha_{k,i} = \alpha_{ki}/x^i, \quad \alpha_{k,ii} = -\alpha_{ki}/(x^i)^2, \quad \alpha_{k,ij} = 0 \) we can get the following system of equation:

\[ 
\sum_k ' \varepsilon_k \alpha_{ik} \left( -1 + \sum_p ' \alpha_{pk} - \alpha_{kk} \right) + \\
\frac{\varepsilon_i}{(x^i)^2 e^{2\alpha_i}} \left( - \sum_s = 0 ' \alpha_{si} + \alpha_{si}(\alpha_{si} - \alpha_{ii}) \right) = 0; \\
\]

\[ 
e^{-\alpha_i - \alpha_j} \left( \alpha_{ij} \sum_k '' \alpha_{ki} + \alpha_{ji} \sum_k '' \alpha_{k,j} - \sum_k '' \alpha_{ki} \alpha_{kj} \right) \]

\[ = 0.\]

3
Diagonal equations (10) gives the restrictions on Casner parameters for each variable, that can be put to the form:

\[
\sum_{p=0}^{n} \alpha_{pk} = 1 + 2\alpha_{kk}; \quad \sum_{p=0}^{n} (\alpha_{pk})^2 = 1 + 2\alpha_{kk} + 2\alpha_{kk}^2;
\]

\[ (k = 0, m) \]  \hspace{1cm} (12)

Nondiagonal equations (11) give supplement cross restrictions on different pair of Casner parameters:

\[
\sum_{p=0}^{n} \alpha_{pi}\alpha_{pj} = \alpha_{ij}(2\alpha_{ii} + 1) + \alpha_{ji}(2\alpha_{jj} + 1) - 2\alpha_{ij}\alpha_{ji};
\]

\[ (i, j = 0, m, i \neq j) \]  \hspace{1cm} (13)

By the suitable coordinate transformation parameters \(\alpha_{ii}\) can be transformed into any real value. Particularly, if \(\alpha_{ii} = 0\), we come to the common conditions as in (4) parameters \(\alpha_{ij}\). To simplify conditions (13) we take \(\alpha_{ii} = -1/2\). Then (12) — (13) take the following form:

\[
\sum_{p=0}^{n} \alpha_{pk} = 0; \quad \sum_{p=0}^{n} \alpha_{pk}^2 = \frac{1}{2};
\]

\[ \sum_{p=0}^{n} \alpha_{pi}\alpha_{pj} = -2\alpha_{ij}\alpha_{ji}. \]  \hspace{1cm} (14)

So, our generalised solution has the following form:

\[
ds^2 = \prod_{s=0}^{m} (x^s)^{2\alpha_{0s}} dt^2 - \sum_{k=1}^{n} \prod_{s=0}^{m} (x^s)^{2\alpha_{ks}} (dx^k)^2 \]

with conditions (14)-(15).

2 Dimension of manifold of Kasner parameters

It is easy to note firstly, that any generalized Kasner solution (16) characterized by the its own matrix of parameters:

\[
\begin{pmatrix}
-1/2 & \alpha_{02} & \cdots & \alpha_{0m} \\
\alpha_{10} & -1/2 & \cdots & \alpha_{1m} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{n0} & \alpha_{n1} & \cdots & \alpha_{nm}
\end{pmatrix}
\]

\[ \]  \hspace{1cm} (17)

where number of lines is equal to \(n + 1\) — dimension of space-time, and number of columns is equal to the \(m\) — number of variables, which multydimensional metric depend on. Secondly, to every solution some surface in euclidian \(mn\)-dimensional space of parameters \(\{\alpha_{ks}\}\) corresponds. This surface is determined by the equations (14)-(15). Its dimension can be
easily founded by taking difference between number of parameters — $mn$ and the number of eq. (14)-(15) — $m(m + 3)/2$. Result $m(2n - m - 3)/2$ can be written in the form of nonequality, which is a sequence of demanding, that dimension of the surface will be more or equal to zero (or system (14)-(15) will be simultaneous):

$$n \geq \frac{m + 3}{2}.$$  \hspace{1cm} (18)

Then we at once get the following infinite table, showing relation between dimension of space-time and number of variables in metric with dimension of parametric surface of parameters subspace:

| $n \setminus m$ | 1 | 2 | 3 | 4 | 5 | $\cdots$ |
|-----------------|---|---|---|---|---|---------|
| 0               | - | - | - | - | - | $\cdots$ |
| 1               | - | - | - | - | - | $\cdots$ |
| 2               | 0 | - | - | - | - | $\cdots$ |
| 3               | 1 | 1 | 0 | - | - | $\cdots$ |
| 4               | 2 | 3 | 3 | 2 | 0 | $\cdots$ |
| 5               | 3 | 5 | 6 | 6 | 5 | $\cdots$ |
| $\vdots$        | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

Here in left coloumn put the dimension $n$ of $n + 1$-dimensional space-time is laid off, in top horizontal line — the number of coordinates in metric, in table — corresponding dimension of subspace of parameters. Region of the table with empty cells corresponds to the obvious fact, that number of variables, which metric depend on, can not be more then dimension of space-time. Symbol "-" means, that for given values of $n$ and $m$ dimension of subspace of parameters is negative, i.e. there is no solution of eq. (14)-(15). For the cases $n = 0, 1, 2$ there is no solutions, because for $n < 3$ there is no exist nontrivial vacuum solutions of Einstein equations. For $n = 3$, as it can been seen from the table, there is no solutions with $m = 4$. As will be shown in sec.4 case with $m = 3$ reduces to original Kasner solution (1) with special values of parameters. For another cases solutions with arbitrary $m$ and $n$ for $m \leq n + 1$ are exist.

3 Parametrization of Kasner hyperspherssfe

Let us consider two first equations (14). For every variable $x^k$ this two equations formally determine hypersphere $S_{n-1}$ in $n + 1$-dimensional euclidian space of parameters (we remember that in our parametrization $\alpha_{ii} = -1/2$, so we have in fact hypersphere $S_{n-2}$ in $n + 1$-dimensional euclidian space of parameters). Take for example $k = 0$ and denote $\alpha_{k0} = p_k$ — the components of radius-vector of Kasner parameters. So we can describe our hypersphere $S_{n-1}$ by one vector equation

$$\vec{p} = p_0 \vec{e}_0 + p_1 \vec{e}_1 + \ldots + p_n \vec{e}_n$$  \hspace{1cm} (19)
with equations on coordinates $p_i$:

$$\begin{align*}
  &p_0 + p_1 + \ldots + p_n = 0; \\
  &p_0^2 + p_1^2 + \ldots + p_n^2 = 1/2.
\end{align*} \tag{20}$$

From equations (20) one can see, that radius-vector describes intersection of sphere $S_n$ with radius $1/\sqrt{2}$ and plane with normal vector oriented along bissectriss of $n + 1$-dimensional coordinate angle. This intersection will be hypersphere $S_{n-1}$, which we’ll call Kasner hypersphere.

Its parametric description can be obtained by the following way:

$$\vec{p} = \mathcal{O}(n + 1)\vec{p}' \tag{21}$$

where $\vec{p}'$ — radius-vector with euclidian norm $|\vec{p}'| = 1/\sqrt{2}$, lied in plane $p_n = 0$, which can be written in the following form:

$$\vec{p}' = \frac{1}{\sqrt{2}}\vec{n} = \frac{1}{\sqrt{2}} \sum_{k=0}^{n} \vec{e}_k \cos \theta_{k-1} \times \prod_{s=k}^{n-1} \sin \theta_s |_{\theta_{n-1}=\pi/2} \tag{22}$$

Here $\{\theta_0, \ldots, \theta_{n-1}\}$ — set of spherical angles of $n+1$-dimensional spherical coordinate system with unit radius. By $\mathcal{O}(n + 1)$ in (21) denoted $(n + 1) \times (n + 1)$ orthogonal matrix, which transform unit normal vector of plane $p_n = 0$ with coordinates $(0, \ldots, 1)$ into unit normal vector of plane $p_0 + \ldots + p_n = 0$ with coordinates $(1/\sqrt{n+1}, \ldots, 1/\sqrt{n+1})$. This matrix can be put to the following form:

$$\begin{pmatrix}
  s_0, & 0, & 0, & \ldots, & 0, & \frac{1}{\sqrt{n+1}} \\
  a_0, & s_1, & 0, & \ldots, & 0, & \frac{1}{\sqrt{n+1}} \\
  a_0, & a_1, & s_2, & 0, & \ldots, & \frac{1}{\sqrt{n+1}} \\
  a_0, & a_1, & a_2, & s_3, & \ldots, & \frac{1}{\sqrt{n+1}} \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_0, & a_1, & \ldots, & a_{n-2}, & s_{n-1}, & \frac{1}{\sqrt{n+1}} \\
  a_0, & a_1, & \ldots, & a_{n-2}, & -s_{n-1}, & \frac{1}{\sqrt{n+1}}
\end{pmatrix} \tag{23}$$

where $a_i = -1/\sqrt{(n - i)(n - i + 1)}$, 

\[\vspace{0.5cm}\]

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\[ s_i = \sqrt{(n-i)/(n-i+1)}. \] We give here matrixes for \( n = 3, 4: \]

\[
\mathcal{O}(4) = \begin{pmatrix} \frac{\sqrt{3}}{2}, & 0, & 0, & \frac{1}{2} \\
-\frac{1}{2\sqrt{3}}, & \sqrt{\frac{2}{3}}, & 0, & \frac{1}{2} \\
-\frac{1}{2\sqrt{3}}, & -\frac{1}{\sqrt{6}}, & -\frac{1}{\sqrt{2}}, & \frac{1}{2} \\
-\frac{1}{2\sqrt{3}}, & -\frac{1}{\sqrt{6}}, & -\frac{1}{\sqrt{2}}, & \frac{1}{2} \end{pmatrix} ;
\]

(24)

\[
\mathcal{O}(5) = \begin{pmatrix} \frac{\sqrt{5}}{2}, & 0, & 0, & 0, & \frac{1}{\sqrt{5}} \\
-\frac{1}{2\sqrt{5}}, & \frac{\sqrt{3}}{2}, & 0, & 0, & \frac{1}{\sqrt{5}} \\
-\frac{1}{2\sqrt{5}}, & -\frac{1}{\sqrt{6}}, & \frac{1}{\sqrt{3}}, & 0, & \frac{1}{\sqrt{5}} \\
-\frac{1}{2\sqrt{5}}, & -\frac{1}{\sqrt{6}}, & -\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{5}} \\
-\frac{1}{2\sqrt{5}}, & -\frac{1}{\sqrt{6}}, & -\frac{1}{\sqrt{2}}, & \frac{1}{\sqrt{5}} \end{pmatrix} ;
\]

(25)

For the case of \( m \) variables which metric depend on we’ll have parametric space in the form of direct production of \( m \) copies of Kasner hyperspheres: \( S_{n-1} \otimes S_{n-1} \cdots \otimes S_{n-1} \) with supplement cross conditions, given by the equations (15), which connect pairs of parameters from different Kasner hyperspheres.

### 4 4-dimensional generalization of Kasner solution.

Consider in details 4-dimensional generalization of Kasner solutions. In all above formulae we should put \( n = 3. \) For the case \( m = 1 \) we go to the metric (1) with 1-dimensional parametric space. Using the formalism of parametrization given in the previous section we can get the following radius-vector \( \vec{p}: \)

\[
\vec{p} = \begin{pmatrix} -\frac{1}{2} \\
\frac{1}{6} - \frac{\sqrt{2}}{3} \cot \theta_0 \\
\frac{1}{6} + \frac{1}{3\sqrt{2}} \cot \theta_0 + \frac{1}{2} \sqrt{\frac{1}{3} - \frac{2}{3} \cot^2 \theta_0} \\
\frac{1}{6} + \frac{1}{3\sqrt{2}} \cot \theta_0 - \frac{1}{2} \sqrt{\frac{1}{3} - \frac{2}{3} \cot^2 \theta_0} \end{pmatrix} ;
\]

(26)
where parameter $\theta_0$ satisfy to the following conditions: $\frac{-1}{\sqrt{2}} \leq \cot \theta_0 \leq \frac{1}{\sqrt{2}}$.

For the case $m = 2$ we have two Kasner hypherspheres. One of them can be parametrized by the same way as (26). Parametrization of another hyphersphere obtained from (23) by the interchange of $p_0$ and $p_1$ and by replacement $\theta_0 \to \theta_1$. Additional cross condition give the following equations on parameters $\theta_0$ and $\theta_1$:

$$-1 + 2(\xi + \eta) + 5\xi\eta + 3\sqrt{1 - \xi^2}\sqrt{1 - \eta^2} = 0.$$  \hspace{1cm} (27)

where $\xi = \sqrt{2} \cot \theta_0$, $\eta = \sqrt{2} \cot \theta_1$. So, dimension of parametric space is 1 in accordance with table in sec.2.

For $m = 3$ we have third Kasner hyphersurface, which parametrization can be obtained from the second one by interchanging $p_1$ and $p_2$ and by replacement $\theta_1 \to \theta_2$. Cross conditions give else two equations, additionally to (27). So we have three equations on three parameters $\theta_0, \theta_1, \theta_2$ and then dimension of parametric space is 0. Roots of this system of equations have been founded with the help of special computer program. There is three sets of solutions:

$$\begin{align*}
\xi &= -1; \quad \eta = -1; \quad \mu = -1/2; \\
\xi &= -1/2; \quad \eta = 1/2; \quad \mu = 1/2; \\
\xi &= 1/2; \quad \eta = -1/2; \quad \mu = -1;
\end{align*}$$  \hspace{1cm} (28)

where $\mu = \sqrt{2} \cot \theta_2$. It is turn out, that all this solutions reduces by coordinate transformations to the case $m = 1$. In common parametrization all three cases reduces to one (within redenotions of variables):

$$ds^2 = dt^2 - t^{4/3}(dx^2 + dy^2) - t^{-2/3}dz^2.$$  \hspace{1cm} (29)

So, new 4-D solution obtained only in the case $m = 2$ and can be put to the following form:

$$ds^2 = x^{4p_0}dt^2 - t^{4p_1}dx^2 - t^{4p_2}x^{4p_3}dy^2 - t^{4p_3}x^{4p_3}dz^2$$  \hspace{1cm} (30)

with

$$\vec{p} = \begin{pmatrix}
0 \\
1/6 - \xi/3 \\
1/6(1 + \xi) + (1/2\sqrt{3})\sqrt{1 - \xi^2} \\
1/6(1 + \xi) - (1/2\sqrt{3})\sqrt{1 - \xi^2} \\
1/6 - \eta/3
\end{pmatrix} \quad \text{and} \quad \vec{q} = \begin{pmatrix}
0 \\
(1 + \eta)/6 + (1/2\sqrt{3})\sqrt{1 - \eta^2} \\
(1 + \eta)/6 - (1/2\sqrt{3})\sqrt{1 - \eta^2}
\end{pmatrix}$$

and with equation (27) on parameters $\xi$ and $\eta$. 


5 Conclusion

In conclusion we make some remarks:
1. The signature of multidimensional interval (16) has no significance.
2. There are some intersections of obtained solution with one obtained earlier [6].
3. 4-D solution (30), that have been discussed in last section belong to a class of 4-D metric, that admits abelian $G(2)$ group of isometry. Moreover its Killing vectors $\partial_y, \partial_z$ are normal to hypersurface. There are some general theorem relating to this class of metric [6]. This metric can be related to the static acially-symmetric class. One particular solution that corresponds to $\xi = 1/2, \eta = -1/2$ belong to the class $BIII$ in classification of vacuum degenerated (type D) static gravitational fields, proposed by Ehlers and Kundt (see ref. in [6] in §16.6.2).
4. The question about physical application of obtained solution arises. In accordance with method given in [7] this solution provided by the extracoordinates cylindrisity could describe static or evolving anisotropic mixture of several perfect fluid.

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