INTRODUCTION

The behaviour of an oscillator may be controlled by the frequency time–dependence. For example, one can kick the oscillator frequency by short pulses and this kicking produces an excitation of the parametric oscillator. The amplitude of the oscillator vibrations and its energy may increase due to the external influence expressed as the frequency time–dependence. Also the statistical properties of the oscillator state may be changed due to the action of external forces. The aim of the talk is to discuss the exact solution of the time–dependent Schrödinger equation for a damped quantum oscillator subject to a periodical frequency delta–kicks describing squeezed states which are expressed in terms of Chebyshev polynomials. The cases of strong and weak damping are investigated in the frame of Caldirola–Kanai model [1], [2].

The problem of quantum oscillator with a time–dependent frequency was solved in Refs. [3]–[11]. It was shown that the wave function and, consequently, all physical characteristics of the oscillator can be expressed in terms of the solution of the classical equation of motion

\[ \dddot{\varepsilon}(t) + 2\gamma \dot{\varepsilon}(t) + \omega^2(t)\varepsilon(t) = 0, \]  

with initial conditions

\[ \varepsilon(0) = 1, \]  
\[ \dot{\varepsilon}(0) = i\Omega(0). \]  

where \( \Omega(0) = \Omega \) will be defined below. The remaining problem is to find explicit
DIFFERENT REGIMES OF DAMPING

Here we consider the case of a periodically kicked oscillator, where the frequency depends on time as follows

\[ \omega^2(t) = \omega_0^2 - 2\kappa \sum_{k=1}^{N-1} \delta(t - k\tau), \]

where \( \omega_0 \) is constant part of frequency, \( \delta \) is Dirac delta–function, \( \gamma \) is the damping coefficient, and \( \kappa \) is the force of delta–kicks. We consider the damping in the frame of Caldirola–Kanai model, and take into account three cases:

(i) undamped case \((\gamma = 0)\);
(ii) the case of weak damping \((\omega_0 > \gamma)\);
(iii) the case of strong damping \((\omega_0 < \gamma)\).

The undamped case was considered in [10]; following [10] we have the equation for function \( \varepsilon(t) \)

\[ \ddot{\varepsilon}(t) + 2\gamma \dot{\varepsilon}(t) + \omega_0^2 \varepsilon(t) - 2\kappa \sum_{k=1}^{N-1} \delta(t - k\tau) = 0. \] (3)

It is obvious, due to substitutions \( t \) by \( x \), \( \varepsilon \) by \( \psi \), and \( \omega_0^2/2 \) by \( E \), that if the damping is absent this equation coincides with the equation for the wave function of a quantum particle of unit mass in a Kronig–Penney potential (the sequence of \( \delta \)–potentials). For every interval of time \((k - 1)\tau < t < k\tau\) the solution for the classical equation of motion is given by

\[ \varepsilon_k(t) = A_k e^{\mu_1 t} + B_k e^{\mu_2 t}, \quad k = 0, 1, \ldots, N, \] (4)

\( \mu_1 \) and \( \mu_2 \) are complex numbers. Due to continuity conditions we have

\[ \varepsilon_{k-1}(k\tau) = \varepsilon_k(k\tau), \quad \dot{\varepsilon}_k(k\tau) = \dot{\varepsilon}_{k-1}(k\tau) = 2\kappa \varepsilon_{k-1}(k\tau). \] (5)

Formulae (5) are obtained by integrating Eq. (3) over the infinitely small time interval \( n\tau - 0 < t < n\tau + 0 \). The coefficients \( A_k \) and \( B_k \) must satisfy the relations which can be expressed in the matrix form

\[ \begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} 1 - 2\kappa / D & -2\kappa e^{D\tau k} \\ 0 & 1 - 2\kappa / D \end{pmatrix} \begin{pmatrix} A_{k-1} \\ B_{k-1} \end{pmatrix}, \] (6)
where \( D = \mu_2 - \mu_1 \). After the sequence of \( \delta \)-kicks the coefficients \( A_n, B_n \) are connected with the initial ones \( A_0, B_0 \) through the equation

\[
\begin{pmatrix}
A_n \\
B_n
\end{pmatrix} = S^{(n)} \begin{pmatrix}
A_0 \\
B_0
\end{pmatrix}, \quad S^{(n)} = T^{-(N-1)}(MT)^n,
\]

with matrices \( T \) and \( M \) given by

\[
T = \begin{pmatrix}
e^{-D\tau/2} & 0 \\
0 & e^{D\tau/2}
\end{pmatrix}, \quad M = \begin{pmatrix}1 - \frac{2\kappa}{D} & -\frac{2\kappa}{D} \\
\frac{2\kappa}{D} & 1 + \frac{2\kappa}{D} \end{pmatrix}.
\]

Thus the elements of the matrix \( S^{(n)} \) are of the form

\[
S^{(n)}_{11} = (1 - \frac{2\kappa}{D})U_{n-1}(\chi/2)e^{D\tau(n-2)/2} - U_{n-2}(\chi/2)e^{D\tau(n-1)/2},
\]

\[
S^{(n)}_{12} = -\frac{2\kappa}{D}U_{n-1}(\chi/2)e^{D\tau/2},
\]

\[
S^{(n)}_{21} = \frac{2\kappa}{D}U_{n-1}(\chi/2)e^{-D\tau/2},
\]

\[
S^{(n)}_{22} = (1 + \frac{2\kappa}{D})U_{n-1}(\chi/2)e^{-D(n-2)\tau/2} - U_{n-2}(\chi/2)e^{-D(n-1)\tau/2}.
\]

where \( U_{n-1}, U_{n-2} \) are Chebyshev polynomials of the second kind defined by the expression:

\[
U_n(\cos \varphi) = \frac{\sin(n + 1)\varphi}{\sin \varphi};
\]

with argument \( \chi/2 = \frac{1}{2} \text{Tr } MT \).

If at the initial moment of time the quantum oscillator was in a coherent state the parametric excitation will transform it into a squeezed correlated state with coordinate variances \( \sigma_x(t) = \frac{n}{2m\Omega} |\epsilon|^2 \), and squeezing coefficient \( K = \frac{\sigma_x(t)}{\sigma_x(0)} = |\epsilon|^2 \). Thus after the sequence of \( \delta \)-kicks one has

\[
\sigma_x(t) = |A_n|^2 \exp(\mu_1 + \mu_1^*)t + |B_n|^2 \exp(\mu_2 + \mu_2^*)t + B_n A_n^* \exp(\mu_2 + \mu_1^*)t + A_n B_n^* \exp(\mu_1 + \mu_2^*)t.
\]

In the case of zero damping ( \( \gamma = 0 \) )

\[
\mu_1 = i\omega_0, \quad \mu_2 = -i\omega_0,
\]

\[
\cos \varphi = \frac{\chi}{2\omega_0} = \cos \omega_0 \tau + \frac{\kappa}{\omega_0} \sin \Omega \tau, \quad \Omega = \sqrt{\omega_0^2 + \kappa^2}.
\]
and from initial conditions (2) one has \( A_0 = 1, \ B_0 = 0. \) The explicit expression for squeezing coefficient is

\[
K = U_{n-1}^2 + U_{n-2}^2 + \frac{2\kappa}{\omega_0} U_{n-1}^2 \sin 2\omega_0[t - (n - 1)\tau] - \chi U_{n-1}U_{n-2} \\
+ \frac{4\kappa^2}{\omega_0^2} U_{n-1}^2 (\sin \omega_0[t - (n - 1)\tau])^2 - \frac{2\kappa}{\omega_0} U_{n-1}U_{n-2} \sin 2\omega_0[t - (n - 1/2)\tau]. \tag{10}
\]

In the case of weak damping the squeezing coefficient is determined by Eq. (9) with following parameters

\[
A_0 = 1 - i\gamma/2\Omega, \\
B_0 = \frac{i\gamma}{2\Omega}, \\
\Omega = (\omega_0^2 - \gamma^2)^{1/2}, \\
\frac{\chi}{2} = \cos \Omega \tau + \frac{\kappa}{\Omega} \sin \Omega \tau, \\
\mu_1 = -\gamma + i(\omega_0^2 - \gamma^2)^{1/2}, \\
\mu_2 = -\gamma - i(\omega_0^2 - \gamma^2)^{1/2}. \tag{11}
\]

One has the squeezing coefficient

\[
K = e^{-2\gamma t} \{ K(\gamma = 0) + \frac{\gamma}{\Omega} \frac{2\kappa}{\Omega} U_{n-1}^2 \cos 2\Omega \tau + \frac{2\kappa^2}{\Omega^2} U_{n-1}^2 \sin 2\Omega \tau - \frac{2\kappa}{\Omega} U_{n-1}U_{n-2} \cos \Omega \tau \\
+ (1 - \frac{\kappa^2}{\Omega^2}) U_{n-1}^2 \sin 2\Omega(t - \tau(n - 2)) + U_{n-2}^2 \sin 2\Omega(t - \tau(n - 1)) \\
- \frac{2\kappa}{\Omega} U_{n-1}^2 \cos 2\Omega(t - (n - 2)\tau) + 2U_{n-1}U_{n-2} \sin 2\Omega(t - (n - 3/2)\tau) \}
+ \frac{\gamma^2}{2\Omega^2} [(1 + \frac{\kappa^2}{\Omega^2}) U_{n-1}^2 + U_{n-2}^2 - \chi U_{n-1}U_{n-2} + \frac{2\kappa}{\Omega} U_{n-1}^2 (\sin 2\Omega \tau - \frac{\kappa}{\Omega} \cos 2\Omega \tau) \\
- \frac{2\kappa}{\Omega} U_{n-1}U_{n-2} \sin \Omega \tau + \frac{2\kappa}{\Omega} U_{n-1}^2 \sin 2\Omega(t - (n - 1)\tau) \\
- \frac{2\kappa^2}{\Omega^2} U_{n-1}^2 \cos 2\Omega(t - (n - 1)\tau) - \frac{2\kappa}{\Omega} U_{n-1}U_{n-2} \sin 2\Omega(t - (n - 1/2)\tau) \\
+ \frac{\kappa^2}{\Omega^2} U_{n-1}^2 \cos 2\Omega(t - n\tau) - (1 - \frac{\kappa^2}{\Omega^2}) U_{n-1}^2 \cos 2\Omega(t - (n - 2)\tau) \\
- \frac{2\kappa}{\Omega} U_{n-1}^2 \sin 2\Omega(t - (n - 2)\tau) - U_{n-2}^2 \cos 2\Omega(t - (n - 1)\tau) \\
+ 2U_{n-1}U_{n-2} \cos 2\Omega(t - (n - 3/2)\tau) \} \}. \tag{12}
\]
The squeezing phenomenon appears when the squeezing coefficient starts to be less than 1. The force of delta–kicks $\kappa$ plays the main role in appearing of the squeezing phenomenon at initial moments of time as can be seen from the previous formula, with time increasing the damping begins to play the main role through the exponential function. Let us mention for simplicity the expression for squeezing coefficient in the case of one delta–kick of frequency at the moment of time $t = 0$

$$K(t = 0) = e^{-2\gamma t}[K(\gamma = 0) + \frac{\gamma}{\Omega}(\sin 2\Omega t + \frac{4}{\Omega}(\kappa + \frac{\gamma}{4})\sin^2 \Omega t)].$$

In the case of strong damping one has the following expressions for the parameters

$$A_0 = 1/2 + i/2 + \gamma/2\Omega,$$

$$B_0 = 1/2 - i/2 - \gamma/2\Omega,$$

$$\Omega = (\gamma^2 - \omega_0^2)^{1/2},$$

$$\mu_1 = -\gamma + (\gamma^2 - \omega_0^2)^{1/2},$$

$$\mu_2 = -\gamma - (\gamma^2 - \omega_0^2)^{1/2},$$

$$\frac{\chi}{2} = \cosh \Omega \tau + \frac{\kappa}{\Omega} \sinh \Omega \tau.$$

Thus we have considered the parametric excitation of damped oscillator in the frame of Caldirola–Kanai model and discussed the influence of different regimes of damping on the squeezing coefficient which describes squeezing phenomenon in the system. The parametric excitation is chosen in the form of periodical $\delta$–kicks of frequency and the formulae for squeezing coefficient are expressed through the Chebyshev polynomials. It is necessary to add that different aspects of the damped oscillator problem in the frame of Caldirola–Kanai model were considered in Refs. [12]–[17].

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