On the domain of moduli fields

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Abstract: Moduli fields are the Goldstone modes associated with the symmetries broken in the presence of a soliton-like configuration. As such, they are functions of the spacetime coordinates corresponding to the unbroken symmetries. Extending their domain on all of spacetime gives rise to fields which are no longer Goldstones, but rather a combination of them and the theory’s massive modes. This article investigates the fields obtained in this way and shows that they share generalized versions of most of the properties usually associated with Goldstone fields.

After discussing the foundations of their theory, it is demonstrated how these fields can be constructed from and embedded into the standard theory of fluctuations around a soliton-like background. It is shown that the most important features of the moduli fields can be generalized in a way that makes them also applicable to these new fields. This allows for an application of important concepts familiar from moduli fields to a much larger subset of the full theory, also covering a group of configurations involving massive modes.
1 Introduction

This article sets out from the connection between two important results of modern theoretical physics, the theory of fluctuations around topological solitons and Goldstone’s theorem. The semi-classical quantization of topological solitons has established itself as a standard procedure, not only due to its clear, intuitive comprehensibility, but also because of its great successes in the description of real physical systems. Topological solitons can be found in many fields of modern physics, from the effective description of baryons over string theory to solid state and condensed matter physics as well as in cosmology [1]-[9].

As all solitons break symmetries of their underlying theories, the spectra of fluctuations around them contain non-trivial zero modes associated to these broken symmetries. Over the last decades, several approaches to handle these modes have been developed [11]-[16]. While physical results do not depend on the chosen approach, each of them is associated with an own perspective and has the potential to simplify certain computations.

One way to understand these modes is Goldstone’s theorem. This theorem, which links spontaneously broken symmetries to massless excitations, is one of the most consequential and universal results of the second half of the 20th century. Its applications reach from the foundations of pion physics to solid state physics, shaping the low energy dynamics of the systems it is applicable to.

A short review of topological solitons in (1 + 1) dimensions focused on the collective coordinate is given within this introduction, as well as a generalization of the concept to higher dimensions via a detour recapitulating Goldstone’s theorem.

1.1 Motivation & Overview

When considering a system whose translational symmetry is broken due to the presence of an extended object like a soliton or a brane, Goldstone’s theorem applies, giving rise to Goldstone modes, the so-called moduli fields. These fields are supported only on the worldvolume of the configuration, while the fluctuations they induce extend also outside of it, decaying in a manner determined by the configurations energy density.

This article sets out from a simple question: What happens if one attempts to extend the domain of the moduli fields to all of spacetime?

Clearly, the fields obtained by doing so are no longer moduli fields in the sense that they are no longer the theory’s Goldstone modes. Rather, they form a family of configurations consisting of contributions of massless as well as massive modes. To distinguish them from the moduli fields, they will be referred to as warp fields, as they correspond to a warping of the configuration. Their connection with the Goldstone modes will turn out to provide them with a series of interesting properties, which generalize features of their corresponding Goldstones:

Even though they are supported on all of spacetime, their influence on the background configuration is localized by a weight function proportional to the background’s energy density, in the sense that any physical quantity appears weighted with it in the classical as well as the quantum theory.

The warp fields disappear from the action in the limit of vanishing momenta, a property
usually known from Goldstone fields. The warp fields however do not only appear via their
derivatives, but also explicitly in combination with the spacetime coordinates corresponding
to the broken symmetries. This links any (constant) shift of the warp fields to an equivalent
(constant) spatial translation of the system, just as in the case of the moduli fields.
This structure of the action might wrongly suggest that the warp fields are massless, which
would be in disagreement with the uniqueness of the Goldstone modes. This is of course
not the case, and the mass of the massive modes is generated by the modification of the
kinetic operator of the linearized theory. The spectrum of the warp fields can thus be used
to recover the complete spectrum of the full theory.
All Goldstone modes share the property that gauging their associated symmetry allows for
them to be eaten up by the corresponding gauge boson, or by the metric in the case of spacetime symmetries. The warp fields can be absorbed into the metric in a similar way
using a Lagrange multiplier. While the resulting metric is flat and contains no dynamical
degrees of freedom, the Lagrange multiplier takes the role of the degree of freedom.
In the case of solitons, the moduli fields are reduced to collective coordinates, and the
configuration’s (angular) velocities are given by the time derivatives of the collective coor-
dinates. When considering interactions between the soliton and a localized source, causality
implies that the change in the soliton’s velocity should be accompanied by excitations of the non-zero modes, ensuring that the total perturbation spreads out with a finite velocity.
Around the configuration’s center, the warp field allows for a more compact description
of such processes in terms of a localized velocity, which is given by the warp field’s time
derivative.
Finally, the field space spanned by the warp fields is a generalization of the moduli space
spanned by the Goldstone modes. Due to the contributions of the non-zero modes the
metric of the field space is no longer constant, but becomes a function of both spacetime
and the warp fields. This dependence of the field space metric on the warp fields encodes
all interactions, as the theory of the warp field no longer contains a potential.
Thus, the warp fields allow to enlarge the applicability of concepts previously exclusive to
the moduli field sector to a larger subset of field configurations, covering also a subsector
of the non-zero mode sector, revealing new structures of the underlying theory. All these
features are already present in the most simple example of a (1 + 1)-dimensional scalar
soliton. Thus, this article restricts itself to this simplistic setting not only to allow for exact
solutions, but also to avoid the central conceptual points getting lost in the complexity of
higher-dimensional theories.

1.2 Topological solitons in (1 + 1) dimensions

For any classical field configuration to have finite energy, it is necessary that it converges
to some vacuum value for \( |x| \to \infty \). This implies that the field on the boundary of space \( \partial \mathbb{R}^d = S^{d-1} \subset \mathbb{R}^d \) defines a map onto the set of all vacua of the theories potential \( U \), denoted
by \( \mathcal{V} \):

\[
\Phi_\infty : S^{d+1} \to \mathcal{V} \\
p \mapsto \Phi_\infty(p) := \lim_{\lambda \to \infty} \Phi(\lambda p)
\]  

(1.1)
A classical configuration of finite energy, whose restriction to the boundary represents a non-trivial winding, is called a soliton. Here, non-trivial is meant in the sense that there exists no continuous homotopy capable of unraveling the configuration to obtain a constant value on all of $\partial \mathbb{R}^d$. For a much more detailed, precise and complete formulation of this extensive topic, consider e.g. [16]-[19].

Such configurations have a wide range of interesting properties: Due to their winding, they are stable. As their energy is supposed to be finite, they are localized. Being connected to a winding, they appear in discrete numbers. All of these properties motivated the interpretation of solitons as particle-like objects, starting from the early phases of their investigation [13]. In the Skyrme model, they can indeed be used to obtain many properties of baryons [1], [2].

Note that while the discussion of this article focuses on the most simple example of a soliton in $(1+1)$ dimensions, it can be transferred to any setting in which soliton-like configurations are of importance.

While there are different approaches to quantize a system containing solitons, this article will mostly focus on the so-called semi-classical quantization. The quantization using path integrals is discussed e.g. in [20]-[22].

The semi-classical quantization is based on the usual canonical quantization procedure and sets out from the decomposition

$$\Phi(t, x) = \Phi_s(t, x) + \delta \Phi(t, x). \quad (1.2)$$

In this decomposition the soliton is treated as a classical background, while the fluctuation around it, $\delta \Phi$, is quantized. To do so, one requires the knowledge of the eigenfunctions and corresponding eigenvalues of the kinetic operator of $\Omega^2$. This operator can be read off the linearized equation of motion of $\delta \Phi$, which is in general of the form

$$(\partial_t^2 - \Omega^2(x))\delta \Phi(t, x) = 0. \quad (1.3)$$

This approach is however confronted with several intricacies, which can be understood in terms of the system’s zero modes. Every symmetry of the theory is related to a zero eigenfunction of $\Omega^2$, satisfying $\Omega^2 \phi_0 = 0$. These zero modes represent the fluctuations along the symmetries of the system, e.g. $\phi_0 = \nabla \Phi_s$ for the translation of a translationally invariant system.

Treating this mode as a part of $\delta \Phi$ leads to a breakdown of the usual quantization procedures. One point in the quantization where this can be observed is the free propagator. If one would try to follow the standard procedure for its construction, one would obtain

$$\Delta(t, x; t', x') = \lim_{\epsilon \to 0} \int \frac{1}{2\sqrt{\omega_k^2 - i\epsilon}} e^{-i|t' - t|\sqrt{\omega_k^2 - i\epsilon}} f_k(x)f_k^*(x'), \quad (1.4)$$

where the symbol $\int$ is meant as sum over the discrete and integral over the continuous part of the spectrum of $\Omega^2$ [15]. As $\omega_0 = 0$ is part of the discrete spectrum, the propagator constructed in this way has an isolated, discrete singularity and is hence ill-defined.

A second, more technical problem lies in the expansion of the field operator in terms of
creation and annihilation operators. The usual perturbative approach is based on the assumption that the linearized theory describes the system's dynamics to leading order. However, for $\omega_0 = 0$, the fluctuation caused by $\phi_0$ can grow arbitrarily large. Furthermore, the particles corresponding to these modes wouldn’t underlie any time evolution.

Another problem of this perspective arises from its discrepancy with the interpretation of the soliton as an extended particle. Up to this point, the formalism lacks a way to describe the center-of-mass motion as well as rotations of the soliton [22]. The canonical way of solving these issues lies now in the idea to split the fluctuations around the soliton. The non-zero modes are assigned to a field theoretical degree of freedom $\phi$ underlying the constraint

$$\int dx \, \phi(t, x) \Phi'(x) = 0, \quad (1.5)$$

while the zero modes are understood as the manifestation of the mechanical degrees of freedom representing the solitons center-of-mass motion and rotations. Consider again the example of the translational zero modes, which are given by the fluctuation corresponding to a global translation, i.e. $\phi \propto \nabla \Phi$. As they are the result of the theories translational invariance, the corresponding collective coordinates represent the position of the center-of-mass of the soliton. This can be formalized via the Ansatz

$$\Phi(t, x) = \Phi_s(t, \gamma(x - z(t))) + \phi(t, x) = \Phi_s(t, x - z(t)) + \phi(t, x), \quad (1.6)$$

where $\gamma$ denotes the Lorentz factor with respect to $\dot{z}$ and the approximation holds for velocities $\dot{z} \ll 1$. Note that this decomposition can also be performed in the formalism of path integrals, as it is done e.g. in [20] and [21].

This Ansatz yields the correct zero mode for sufficiently small $z(t)$. It is also in perfect agreement with the interpretation of the collective coordinates as the position of an extended particle, as the action of (1.6) is given by

$$S[z(t)] = M_{sol} \int dt \sqrt{1 + \dot{z}^2} \approx \int dt \, M_{sol} + \frac{M_{sol}}{2} \dot{z}^2, \quad (1.7)$$

where $M_s$ denotes the energy of the unperturbed soliton, which, when understanding the soliton as a particle, takes the role of its rest mass.

This collective coordinate allows to handle the problems arising from the zero mode and hence to construct a quantum theory of this system. The nonzero modes can be treated as an usual quantum field by promoting it and its conjugate momentum to operators consisting of creation and annihilation operators. Due to the constraint (1.5) the canonical commutation relations for $\delta \Phi$ and its conjugate momentum $\delta \Pi$ imply that these new field operators satisfy

$$[\hat{\phi}(t, x), \hat{\pi}(t, x')] = i \delta(x - x') - i \frac{f_0(x) f_0(x')}{M_{sol}}, \quad (1.8)$$

This is only consistent with the canonical equations for $\delta \Phi$ and $\delta \Pi$ if the collective coordinate and its conjugate momentum $p$ act as quantum mechanical operators,

$$[\hat{z}, \hat{p}] = i, \quad (1.9)$$
in perfect agreement with their previous interpretation. From here, it follows that the position of the soliton is determined by some wave function $\Psi(t,x) \in L^2(\mathbb{R},dx)$ or, in momentum space, $\tilde{\Psi}(t,p) \in L^2(\mathbb{R},dp)$. This implies that the full quantum theory of the soliton can be formulated on the Fock space $F = L^2 \otimes F_p$, where $F_p$ denotes the Fock space generated by the $\{\hat{a}_k^{\dagger}\}_{k}$-operators from the state of an unexcited soliton $|\text{sol}\rangle$.

So, for the free theory, the dynamics of the system is split into the motion of the center of mass of the soliton and the one of its localized, particle-like excitations. The dynamics along rotational and gauge moduli can be split off in a similar way.

This splitting also translates to the theory of a soliton interacting with an external source. Consider the soliton together with some interaction term $\int dx\, J(t,x)\Phi(t,x)$. Similar as $\delta\Phi$, for any time $t$ the current $J$ can be decomposed in terms of the eigenfunctions of $\Omega^2$

$$J(t,x) = \int_k j^k(t)\phi_k(x-z(t)) = f(t)\phi_0(x-z(t)) + j(t,x-z(t)). \quad (1.10)$$

This splitting translates to the equations of motions for $\phi$ and $z(t)$, yielding

$$M_s\ddot{z}(t) = f(t) = -\frac{d}{dz}V(z) \quad (1.11)$$

with the potential

$$V(z) = \int dx\, J(t,x)\Phi_s(x-z(t)).$$

The consequences of these equations are discussed in [13].

The treatment of higher orders can most easily be discussed in the formulation via path integrals. This approach is given in great detail in [20]-[22].

Recall that there exist several alternatives to this procedure, which all lead to the same physical results, as discussed e.g. in [11] and [12]. The one which will turn out to be most interesting for the bulk of this article is given in [11], as it attempts a treatment of all modes on an equal footing.

Similar as the canonical procedure, it starts from the mode expansion of the fluctuation

$$\delta\Phi(t,x) = \int_k q^k(t) f_k(x) \quad (1.12)$$

The free theory can now fully be described in terms of the new parameters $\{q^k(t)\}_k$.

Inserting (1.12) into the Hamiltonian of the fluctuation $\delta\Phi$ yields

$$H_0[q(t)] = M_{\text{sol}} + \frac{1}{2} \int_k (q^k)^2(t) + \omega_k^2(q^k)^2(t).$$

From the corresponding action one obtains the equations of motion,

$$\ddot{q}^k(t) = -\omega_k^2 q^k(t), \quad (1.13)$$

and the commutation relations satisfied by the fluctuation $\delta\Phi$ and its conjugate momentum lead to their pendant for the parameters $q$:

$$[q^m(t), q^n(t)] \propto i\delta_{mn}, \text{ as well as}$$

$$[q^m(t), q^n(t)] = 0 = [\dot{q}^m(t), \dot{q}^n(t)]. \quad (1.14)$$
The main difference of this approach when compared with the usual introduction of a collective coordinate lies now in the intention to treat all the modes, including the zero mode, on an equal footing. The normalized solutions of (1.13) are given by

$$q^k(t) = \frac{1}{\sqrt{2\omega_k}} (a_k^\dagger e^{i\omega t} + a_k e^{-i\omega t}) \text{ for } k \neq 0$$

$$q^k(t) = q_0 + v_0 t \text{ for } k = 0.$$

The $a_k^{(1)}$-operators satisfy the usual commutation relations, so that the dynamics associated with the non-zero modes is identical to the one obtained via the usual scheme to introduce a collective coordinate. For equal times, also the commutation relation of the collective coordinate is reproduced, as

$$[q(t), p(t)] = i,$$
$$[q(t), q(t)] = 0 = [p(t), p(t')].$$

(1.16)

where $p(t) = M_{sol} v_0$ and $q(t) = q_0$.

In order to analyze the perturbative expansion, one needs to find the propagators of the systems degrees of freedom. As each of the $q^k$ is treated as a mechanical degree of freedom, each of them can be assigned a propagator. These can be found to be

$$\Delta_0(t', t) = -\frac{1}{2} |t' - t|,$$

$$\Delta_k(t', t) = -\frac{i}{2\omega_k} e^{-i\omega_k |t' - t|} \text{ for } k \neq 0.$$

(1.17)

These propagators can then be combined to obtain a propagator for the fluctuation $\delta \Phi$,

$$\Delta(t', x; t, x) = \oint f_k(x') \Delta_k(t', t) f_k(x).$$

(1.18)

From here, the perturbative treatment of the interactions can be developed in the usual way.

These structures can be rediscovered in higher dimensions. For any symmetry broken due to the existence of the soliton, the spectrum of fluctuations obtains one zero mode, which can be treated by the introduction of a corresponding collective coordinate. As it will turn out to be helpful to motivate the bulk of this article, this is discussed in the next subsection from the view of Goldstone’s theorem. That discussion is strongly inspired by the discussion in [23].

1.3 Goldstone’s theorem

Goldstone’s theorem states that every spontaneously broken, continuous symmetry of a given theory manifests itself as a massless excitation in the spectrum of the theory after the symmetry breaking [23]-[26]. This excitation corresponds to a pole of the propagator at $p^2 = 0$, but also in the observation that the energy of this excitation disappears in the case of a vanishing three-momentum, $\lim_{p \to 0} E(p) = 0$, just as all interactions disappear. Such excitations are referred to as Goldstone modes or Goldstone fields, the corresponding
particles are called Goldstone bosons.

This can be seen by considering the effect of a field transformation on the (quantum effective) action. Let the field content transform as

$$\Phi^m(x) \rightarrow \Phi^m(x) + q^a \Delta_a \Phi^m(x),$$

where $m$ labels the different fields of the theory, $\Delta_a \Phi^m(x)$ denotes the action of the transformation’s $a$th generator on the field, and the $\{q^a\}_a$ are some parameters. A generator of the symmetry is spontaneously broken if it leaves the (effective) action invariant, $\delta S[\Phi] = 0$, but not the configuration $\Phi^m_0(x^{n+1}, \ldots x^d)$ around which the field is expanded. Here, $d$ denotes the dimension of the underlying spacetime and $x^0 = t$ as usual. While this configuration can be given by some constant vacuum value, it can also be some soliton-like background. For a proper soliton, this background is a function of all spatial coordinates, but there exist also configurations such a branes or strings, which are constant along certain spatial dimensions.

As the above transformation is a symmetry of the theory it leaves the effective action invariant,

$$0 = \delta S[\Phi] = \int d^d x \frac{\delta S[\Phi]}{\delta \Phi^m(x)} \Delta_a \Phi^m(x),$$

and $\Delta_a \Phi^m(x) \neq 0$ as the symmetry is spontaneously broken. Taking a second functional derivative with respect to the field one obtains

$$0 = \int d^d x_1 \frac{\delta^2 S[\Phi]}{\delta \Phi^m(x_1) \delta \Phi^m(x_2)} \Delta_a \Phi^m(x_1) + \frac{\delta S[\Phi]}{\delta \Phi^m(x_1)} \frac{\delta \Delta_a \Phi^m(x_1)}{\delta \Phi^m(x_2)}.$$  \hspace{1cm} (1.21)

Choosing as a background either a constant vacuum of the theory or a soliton-like configuration $\Phi^m_0(x^{n+1}, \ldots x^d)$, the first functional derivative of the action disappears, and with it the second term in equation (1.21). As $\Phi^m_0$ is only a function of $(x^{n+1}, \ldots x^d)$, the same is true for $\Delta_a \Phi^m_0$, and equation (1.21) can be simplified to

$$0 = \int d^d x_1^{n+1} \ldots d^d x_d \left. \frac{\delta^2 S[\Phi]}{\delta \Phi^m(x_1) \delta \Phi^m(x_2)} \right|_{\Phi = \Phi_0(x_1^{n+1}, \ldots x_d^d)} \Delta_a \Phi^m(x_1^{n+1}, \ldots x_d^d).$$  \hspace{1cm} (1.22)

Now, considering the limit of vanishing momenta, the action reduces to the potential term and its functional derivative to

$$\left. \frac{\delta^2 S[\Phi]}{\delta \Phi^m(x_1) \delta \Phi^m(x_2)} \right|_{\Phi = \Phi_0(x_1^{n+1}, \ldots x_d^d)} \rightarrow - \frac{\partial^2 V}{\partial \Phi^m \partial \Phi^m}(\Phi_0(x_1^{n+1}, \ldots x_d^d)) \delta^{(d)}(x_1 - x_2).$$  \hspace{1cm} (1.23)

Hence (1.22) leads to

$$0 = - \frac{\partial^2 V}{\partial \Phi^m \partial \Phi^m}(\Phi_0(x^{n+1}, \ldots x^d)) \Delta_a \Phi^m(x_1^{n+1}, \ldots x_d^d).$$  \hspace{1cm} (1.24)

Therefore $\Delta_a \Phi^m$ is a zero eigenvector of the mass matrix, which represents the desired massless excitation. As there exists one index $a$ for each broken generator, there exists one
massless excitation for any independent, broken generator. Linearity now implies that also linear combinations of these modes are massless, and it can be seen from equation (1.22) that the general form of such an excitation is given by

$$
\phi^m(x) = q^a(x^0, \ldots , x^n) \Delta_a \Phi^m_0(x^{n+1}, \ldots , x^d),
$$

where the fields $q^a(x^0, \ldots , x^n)$ are usually referred to as Goldstone fields or, in the context of soliton-like configurations, moduli fields. The crucial property of these fields is that they are only a function of the coordinates along which the translational symmetry is not broken. Physically speaking, they are functions of the configurations worldvolume only, leading to the interpretation of them being confined to it.

As an example, consider the Lagrangian

$$
L(\Phi) = \frac{1}{2} \partial_{\mu} \Phi^* \partial^{\mu} \Phi - \frac{\lambda}{4} (\nu^2 - |\Phi|^2)^2.
$$

The vacua of this theory are given by the set of complex constants $a$ satisfying $|a| = \nu$, each of them breaking the $U(1)$-symmetry. Picking the vacuum $a = \nu$ for simplicity, the Goldstone mode can be found to be $\varphi(t, x) = q(t, x) \cdot i \nu$, as the $U(1)$-transformation acts on $\nu$ via $\nu \rightarrow e^{i q(t, x)} \nu = \nu + i \nu q(t, x)$. This Goldstone field can be made manifest in the parametrization

$$
\Phi(t, x) = (\nu + \rho(t, x)) e^{i q(t, x)}.
$$

In terms of the real fields $\rho$ and $\varphi$, the Lagrange density takes the form

$$
L(\rho, \varphi) = \frac{\nu^2}{2} \partial_\mu \varphi \partial^\mu \varphi + \frac{1}{2} \partial_\mu \rho \partial^\mu \rho - \frac{m^2}{2} \rho^2 + \frac{\rho^2}{2} \partial_\mu \varphi \partial^\mu \varphi + \nu \rho \partial_\mu \varphi \partial^\mu \varphi - \frac{\lambda}{4} \rho^4 - \lambda \nu^2 \rho^3,
$$

where $m^2 = 2 \lambda \nu^2$. So, the field $\varphi$ enters only via its derivatives, thereby reproducing all the properties that are to be expected from a Goldstone field.

The field $q(t, x)$ appears as a field on all of spacetime as the chosen vacuum breaks only the internal $U(1)$-symmetry, i.e. is a constant. The $(1+1)$-dimensional soliton considered above however breaks the spatial translation symmetry of the system. The soliton transforms under a translation as $\Phi_s(x) \rightarrow \Phi_s(x - z) \sim \Phi_s(x) - \Phi'_s(x) \cdot z$. Following the previously given arguments, the corresponding Goldstone field $z$ is a function of time, $z(t)$, which is nothing but the collective coordinate.

A straightforward generalization of this example and the focal point of [23] are branes, i.e. configurations which take the form of a soliton along certain spatial dimensions $(x^{n+1}, \ldots , x^d)$, while being constant along the remaining ones, $\Phi_s = \Phi_s(x^{n+1}, \ldots , x^d)$. Therefore, they break the translational symmetry in the $(x^{n+1}, \ldots , x^d)$-directions, so that the Goldstone fields are given by $z^k(x^0, \ldots , x^n)$ and their influence on the brane by $\Phi_s \rightarrow \Phi_s(x^{n+1} -$  

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1When considering spacetime translations, it is not guaranteed that all generators are independent. This is discussed in great detail in [28]-[32].
$z^{n+1}(x^0, \ldots, x^n), \ldots, z^d(x^0, \ldots, x^n)$). In the context of soliton-like configurations, these are the well-known moduli fields.

At this point an important subtlety has to be mentioned. Several works [28]-[32] have shown that in the context of broken spacetime symmetries Goldstone’s theorem is subject to several intricacies. As these are not relevant for the remainder of this article as the simplistic proof above applies, they won’t be further discussed at this point.

The space spanned by the moduli fields is called *moduli space*. Using the language of this section, their action is to leading order given by

$$S^{(2)}[q^a] = \frac{1}{2} \int \int \int \frac{\delta^2 S[\Phi]}{\delta \Phi^m(x_1) \delta \Phi^n(x_2)} \cdot q^a(x_1, \ldots, x^n) \Delta_b \Phi^m(x_1^{n+1}, \ldots, x^n_1) \Delta_b \Phi^n(x_2^{n+1}, \ldots, x^n_2)$$

Without the coefficients $q^a$, this action would be identical to zero due to (1.22). That this is not the case here is owed to the kinetic term of the action. Most relevant physical theories have a kinetic term of the form

$$S_{\text{kin}}[\Phi] = \frac{1}{2} \int \int \Phi^m(x) D(\partial_0, \ldots, \partial_d) \Phi^m(x) = \frac{1}{2} \int \int \Phi^m(x) D_n(\partial_0, \ldots, \partial_n) \Phi^m(x) + \Phi^m(x) D_d(\partial_{n+1}, \ldots, \partial_d) \Phi^m(x),$$

where $D$, $D_n$ and $D_d$ are some hermitian differential operators. The contribution of such a term to the functional derivative of the action is thus given by

$$\frac{\delta^2 S_{\text{kin}}[\Phi]}{\delta \Phi^m(x_1) \delta \Phi^n(x_2)} = D_n(\partial_{x_1^1}, \ldots, \partial_{x_1^n}) \delta^d(x_1 - x_2) + D_d(\partial_{x_2^{n+1}}, \ldots, \partial_{x_2^d}) \delta^d(x_1 - x_2).$$

While the second term is included in (1.22), said equation does not make any statement about the $D_n$ term, as it is trivially zero when applied to the image of the broken generator. Thus, if one considers fluctuations of the form (1.25), one finds

$$S^{(2)}[q^a] = \frac{1}{2} \int \int \Delta_a \Phi^m(x^{n+1}, \ldots, x^d) \Delta_b \Phi^n(x^{n+1}, \ldots, x^d) q^a(x^0, \ldots, x^n) \Delta_b \Phi^m(x^{n+1}, \ldots, x^d) \Delta_b \Phi^n(x^{n+1}, \ldots, x^d)$$

As the expression in the first line depends only on the generators broken by the soliton and not the moduli fields themselves, performing the integrals reduces it to some constant matrix $M_{ab}$, the *moduli space metric*. Thus, the action of the moduli fields can be brought to the compact form

$$S^{(2)}[q^a] = \frac{1}{2} \int \int M_{ab} q^a(x^1, \ldots, x^n) D_n(\partial_0, \ldots, \partial_n) q^b(x^1, \ldots, x^n).$$
In the case of a $(1+1)$-dimensional scalar field, the only moduli field is the collective coordinate $z(t)$, and the moduli space metric has only one component, $M = \int dx (\Phi_s')^2 = M_{\text{sol}}$, yielding the well-known action

$$S^{(2)}[z] = {1 \over 2} \int dt \ M_{\text{sol}} z'^2.$$  \hspace{1cm} (1.29)

A slightly more complicated while also much more insightful example is that of a domain wall, i.e. a configuration which acts as a soliton along one spatial dimension $x^1$, while being constant along the remaining ones, $(x^2, \ldots x^d)$. Thus, the configuration breaks the translational invariance along the $x^1$-direction, and the moduli field becomes a function of $(x^2, \ldots x^d)$. This implies that each point of the wall is displaced along the $x^1$-axis, i.e. the configuration is being sheared [18]. The moduli space metric is in this case proportional to the wall’s tension.

Note that, in the presence of gravity, this situation becomes more involved, as the moduli field is eaten up by the metric, or more precisely, the gravitophoton. A detailed discussion of this feature can be found in [10].

2 Notation and Conventions

For simplicity assume the potential generating the soliton to be symmetric, $V(\Phi) = V(-\Phi)$, with two distinct vacua $\pm \nu$. Further, assume that the soliton carries a topological charge 1, for which it is always possible to find a coordinate frame in which $\Phi_s = \Phi_s(x)$. This allows for the following representation:

$$\Phi_s(x) = \nu \sigma(\mu_m x)$$ \hspace{1cm} (2.1)

Here, $\nu$ corresponds to the dimensionless vacua of the theory as mentioned above, and the factor $\mu_m \propto m_\Phi$ describes the localization of the soliton. $\sigma$ encodes the profile of the soliton, and is in general restricted by $|\sigma(\mu_m x)| < 1$. $\sigma'$ denotes the derivative of $\sigma$ with respect to its dimensionless argument. A general fluctuation of this background will be denoted by $\delta \Phi$, $\Phi(t, x) = \Phi_s(x) + \delta \Phi(t, x)$. A fluctuation generated by non-zero modes only will be denoted by $\phi$.

Throughout this article, two different perspectives on the warp field are explored: One, in which the dynamical degrees of freedom are given by $\phi$ and the collective coordinate, and one in which this role is taken by the warp field $\varphi$. In the first case, the classical dynamics takes place on the phase space of $\phi$, denoted by $\mathcal{P}$. The quantum theory unfolds on the Fock space the field operators act on, $\mathcal{F}$. When using $\varphi$ as degree of freedom, the corresponding phase/Fock space is denoted $\mathcal{P}/\mathcal{F}$.

The eigenfunctions of the $x$-dependent part of the kinetic operator of the linearized theory for $\phi$ will be denoted by $\{f_k\}_k$, so that $\phi$ can be expanded as $\phi = \oint f_k a_k^+ e^{i\omega_k t} f_k + \text{c.c.}$, with some complex coefficients $\{a_k\}_k$. The symbol $\oint$ is meant as sum over the discrete part of the spectrum and integration over the continuous part, if need be containing some normalization factors depending on the normalization of the momentum eigenstates.

To highlight the main conceptual points, most arguments are performed explicitly for the toy model of a linearized $(1+1)$-dimensional scalar field.
3 Classical theory of the warp field

3.1 Incitement: Local interactions

To get a better intuition regarding the physical meaning of the warp field, consider the example of a \((1+1)\)-dimensional soliton together with its collective coordinate, \(\Phi_s(x-z(t))\). While the collective coordinate is only a function of time, the corresponding fluctuation extends over all of spacetime, decaying as \(\Phi'_s\). It is therefore predominantly localized in the same spatial region as the soliton itself. Assume further that this soliton is interacting with some external, localized source, consisting of zero as well as non-zero mode contributions. Such an interaction changes the equation of motion of \(z(t)\), therefore changing \(\dot{z}(t)\). As \(z(t)\) is only a function of time, but the fluctuation induced by it extends over all of space, the information about the excitation seems to spread instantaneously, violating causality. However, there is in fact no problem here. For the external source to be localized, it has to be a superposition of the zero as well as the non-zero modes, so that such an interaction would excite all the involved modes. The resulting excitation can therefore without any problems be localized within the region of the interaction. Afterwards, it can be expected that the excitation spreads out with some finite velocity smaller than 1. This would imply that immediately after the interaction begins, the soliton only moves in the neighborhood in which the interaction is taking place. This excitation then starts spreading with some finite velocity, causing larger and larger parts of the soliton to move. As different regions of the soliton are moving with different velocities, the soliton is warped during this process. This can be formalized by replacing the collective coordinate by a local object,

\[
z(t) \rightarrow \mu_m^{-1} \varphi(t, x),
\]

(3.1)

which is defined with an additional factor of \(\mu_m\) to obtain a dimensionless quantity. This object will from now on be referred to as warp field.

The velocity of the motion in \(x\)-direction in some point \(x_0\) is then given by \(\mu_m^{-1} \dot{\varphi}(t, x_0)\). This velocity can be expected to consist of two contributions: First, a global velocity \(v\), which is the result of the dynamics of the collective coordinate. Second, a local velocity, which will turn out to be fully determined by the dynamics associated with the non-zero modes.

An alternative perspective on such a configuration is based on the soliton’s width. It is well known that providing a soliton (or more general a brane or domain wall respectively) with a sufficient amount of energy will cause a (space)-time dependent perturbation of its thickness [18]. The warp fields can be understood as an attempt to describe this phenomenon in a compact manner.

3.2 Foundations of the classical theory

The action of the warp field can be obtained by inserting the warped soliton into the action of the scalar field. Canceling the terms linear in \(\varphi\) as the soliton is a solution of the equation of motion, one ends up with the action

\[
S[\nu \sigma(\mu_m x - \varphi(t, x))] = S[\Phi_s(x)] + \int d^2x \frac{1}{2} \nu^2 (\sigma(\mu_m x - \varphi))^2 \partial_\mu \varphi \partial^\mu \varphi.
\]

(3.2)
The potential $V(\Phi)$ has been eliminated using Bogomolnyi’s equation $\sqrt{2V(\Phi_s)} = \Phi'_s$ [17]. Note that this equation can also be used to show that the energy density of the unperturbed soliton is given by

$$\epsilon = \mu_m^2 \nu^2 \left( \sigma' \left( \mu_m x \right) \right)^2,$$

which will turn out to be very adjuvant for the interpretation of the theory.

Absorbing the soliton density into the spatial measure, this action can be brought to the much simpler form

$$S[\varphi] = S[\Phi_s] + \frac{1}{2} \int d^2 \omega \varphi(x) \partial_\mu \varphi \partial^\mu \varphi, \quad \text{where}$$

$$d^2 \omega \varphi(x) = \nu^2 \left( \sigma' \left( \mu_m x - \varphi \right) \right)^2 dx dt.$$

Therefore, the action is given by the one of a usual massless scalar field, weighted with the function $\rho(x, \varphi(x)) = \nu^2 \left( \sigma' \left( \mu_m x - \varphi \right) \right)^2$.

In order to allow for a standard perturbative treatment of this theory, one would have to expand the term $\left( \sigma' \left( \mu_m x - \varphi \right) \right)^2$ as a series in $\varphi$. This leads to a kinetic term as well as an infinite number of interaction terms,

$$L_0 = \frac{1}{2} \nu^2 \left( \sigma' \left( \mu_m x \right) \right)^2 \partial_\mu \varphi \partial^\mu \varphi,$$

$$L_{\text{int}} = \frac{(-1)^n}{2n!} \nu^2 \left( \left( \sigma' \left( \mu_m x \right) \right)^2 \right)^{(n)} \partial_\mu \varphi \partial^\mu \varphi.$$

Here, the power $(n)$ refers to the $n^{\text{th}}$ derivative with respect to the dimensionless argument $\mu_m x$. Note that, similar as for a Goldstone field, each of these terms disappears in the limit $\partial \varphi \to 0$, a feature which will be discussed in section 3.6.

This Lagrangian gives rise to the equation of motion,

$$\square \varphi = \frac{\sigma'' \left( \mu_m x - \varphi \right)}{\sigma' \left( \mu_m x - \varphi \right)} \left( \partial_\mu \varphi \partial^\mu \varphi + 2 \mu_m \varphi' \right),$$

and determines the canonical momentum $\varpi$,

$$\varpi(t, x) = \nu^2 \left( \sigma' \left( \mu_m x - \varphi \right) \right)^2 \dot{\varphi}(t, x) = \rho(x, \varphi(t, x)) \dot{\varphi}(t, x).$$

The phase space on which the dynamics of the warp field takes place, spanned by $\varphi$ and $\varpi$, will in the following be denoted by $\mathcal{P}$.

Another important feature of the theory is its energy-momentum tensor, which can be found to be

$$T^\mu_\nu = \nu^2 \left( \sigma' \left( \mu_m x - \varphi \right) \right)^2 \left( \partial_\mu \varphi \partial_\nu \varphi - \delta_\mu^\nu \frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi \right) =$$

$$= \rho(x, \varphi(t, x)) \left( \partial_\mu \varphi \partial_\nu \varphi - \delta_\mu^\nu \frac{1}{2} \partial_\alpha \varphi \partial^\alpha \varphi \right).$$

Note that this expression is again just the usual expression for a massless scalar field multiplied by the weight factor.
3.3 Linearized theory

In order to avoid the technical difficulties arising from the infinite number of interaction terms and prepare for the canonical quantization of this system, it is expedient to focus on the linearized theory. This is a good approximation if \( \varphi \) satisfies the inequality

\[
|\varphi(t, x)| \ll \left| \frac{n!}{\left( \left( \sigma'(\mu_m x) \right)^2 \right)^{\frac{1}{n}}} \right| \quad \forall n \in \mathbb{N},
\]

(3.9)

which can be obtained from the above action. In the case of the \( \Phi^4 \)-theory, the right hand side of this equation is bounded from below by \( \frac{1}{2} \), in the Sine-Gordon model by \( \frac{1}{2} \).

In this approximation the action and equation of motion reduce to

\[
S[\varphi] = \int d^2x \frac{1}{2} \nu^2 (\sigma' (\mu_m x))^2 \partial_\mu \varphi \partial^\mu \varphi, \quad (3.10)
\]

\[
(\partial_t^2 + \mathcal{K}^2) \varphi = 0, \quad \text{where} \quad \mathcal{K}^2 := -\partial_x^2 - 2\mu_m \frac{\sigma''(\mu_m x)}{\sigma'(\mu_m x)} \partial_x \quad (3.11)
\]

The first step towards solving this equation is to investigate the operator \( \mathcal{K}^2 \). While it isn’t hermitian with respect to the measure \( dx \), it is with respect to the due to the linearization \( \varphi \)-independent measure \( d\varsigma := \nu^2 (\sigma')^2 (\mu_m x) dx \). Introducing the compact notation \( \rho(x) \equiv \rho(x, 0) \), this can be brought to the compact form \( d\varsigma := \rho(x) dx \). Hence, the eigenfunctions of \( \mathcal{K}^2 \) form a complete basis, which can be orthonormalized with respect to \( d\varsigma \). These eigenfunctions will be denoted by \( \{g_k\}_k \), the corresponding eigenvalues by \( \lambda_k \). It follows by a straightforward computation that these eigenfunctions are given by \( g_k(x) = \frac{f_k(x)}{\nu\sigma'(\mu_m x)} = \rho^{-1/2}(x) \cdot f_k(x) \), with \( \lambda_k = \omega_k^2 \). Recall that the function \( \{f_k(x)\}_k \) are the eigenfunctions of the operator \( \Omega^2 \) appearing the \( \phi \)-theory. In terms of these eigenfunctions a general solution of (3.11) is of the form

\[
\varphi(t, x) = \sum_{k=0}^{\infty} \alpha_k e^{i\omega_k t} g_k^*(x) + \alpha_k e^{-i\omega_k t} g_k(x) + \mu_m v(t - t_0) + \mu_m \zeta, \quad (3.13)
\]

with some real integration constants \( v \) and \( \zeta \) of mass dimension 0 and –1 respectively. The sum/integral runs over all the non-zero modes while the constant \( \zeta \) represents the zero mode. The term linear in \( t \) is part of the kernel of \( \partial_t^2 \) as well as of \( \mathcal{K}^2 \). As \( \varphi \) isn’t canonically normalized, the additional factor of \( \rho(x) = \nu^2 (\sigma')^2 (\mu_m x) \) causes the contribution arising from the linear term to yield a finite contribution to the action as well as all observables. The expansion (3.13) comes with an important subtlety. As the mass of the soliton is finite, \( \sigma'(\mu_m x) \to 0 \) for \(|x| \to \infty\), so that also \( g_k \to \infty \) in this limit. It is important to notice that this does not necessarily imply that the same holds for \( \varphi \), as it is possible to choose the coefficients \( \{\alpha_k\} \) such that \( \varphi \to 0 \) for \(|x| \to \infty\). This observation is a necessary condition for the linearized theory to apply, as otherwise the condition (3.9) would be violated. This unusual asymptotic behavior of the mode functions will turn out to be of crucial importance for the quantum theory. The restriction (3.9) also affects the term linear in \( t \).
Denoting the lower bound of its right hand side by \( b \), it can be shown that the contribution of this term is agreement with (3.9) if

\[
|v| < \frac{b}{\mu_m |T|}, \tag{3.14}
\]

where \( T \) denotes the considered time interval, which is assumed to be centered around \( t_0 \).

Higher orders can be obtained from standard perturbation theory, using \( v \) as perturbation parameter. Starting from \( \varphi^{(1)} = \mu_m vt \), the resulting perturbative series converges to

\[
\varphi(t, x) = \mu_m (x - \gamma(x - vt)), \tag{3.15}
\]

which reproduces the motion of the soliton for relativistic velocities,

\[
\Phi_s(x - \varphi(t, x)) = \Phi_s(\gamma(x - vt)). \tag{3.16}
\]

### 3.4 Embedding into the full linearized theory

The result (3.16) hints at the interpretation of \( v \) as the classical velocity of the soliton. This observation is consistent with the contribution of the \( v \)-term to the classical energy, which is given by \( \frac{M_{\text{sol}} v^2}{2} \), where \( M_{\text{sol}} = \int d \varsigma (\Phi_s')^2(\varsigma) = \mu_m^2 \int d \varsigma \).

The identification of \( v \) with the soliton’s classical velocity can be extended to recover the full center-of-mass motion of the soliton. Recall that, in the linearized case, the solution to the equation of motion for the collective coordinate is given by \( z(t) = z_0 + v_0 (t - t_0) \).

Identifying the parameter \( \zeta \) with \( z_0 \), it is evident that this is contained in (3.13), confirming that the zero mode contribution of the warp field is equivalent to the collective coordinate.

As the theory of fluctuations \( \delta \Phi \) around the soliton covers all possible fluctuations it must also include the non-zero modes of the warp field. In order to identify these within the \( \phi \)-theory, consider the following map between the phase space of the linearized \( \varphi \)-theory, \( \mathcal{P} \), and the one of the linearized \( \phi \)-theory, \( \mathcal{P} \):

\[
r : \mathcal{P} \to \mathcal{P} \tag{3.17}
\]

\[
(\varphi(x), \omega(x)) \mapsto (r_\varphi[\varphi](x), r_\omega[\omega](x)) \equiv \left(-\rho^{1/2}(x) \cdot \varphi(x), -\rho^{-1/2}(x) \cdot \omega(x)\right) \tag{3.18}
\]

This map is an embedding of the \( \varphi \)-phase space into the \( \phi \)-phase space with two important properties. It maps the mode functions \( \{g_k\}_k \) onto their pendants \( \{f_k\}_k \), leaving the eigenvalue with respect to their respective kinetic operator invariant. And, due to its linearity, the image of any solution of the linearized equation of motion for \( \varphi \) is again a solution of the linearized equations of motion for \( \phi \). This implies in particular that the time evolutions of both theories are equivalent within the constraints imposed by the applicability of the linearized theories.

This is an important result when it comes to the consistency of this analysis. As the fluctuations \( \delta \Phi \) cover all the dynamics around the soliton, they should also include the dynamics of the warp fields. The relevance of the map \( \rho \) lies now in its power to isolate these modes from the spectrum of general fluctuation.
3.5 Nonlinear embedding

The arguments of the last subsection can be generalized to the full, nonlinear theory. As the identification of the zero mode terms with the collective coordinate is already exact, it remains to lift the map between the non-zero modes to the nonlinear theory. This can be done by considering the map $R$, defined by

$$R : \mathcal{P} \rightarrow \mathcal{P}$$

$$\left( \varphi(x), \pi(x) \right) \mapsto \left( R_\varphi(\varphi)(x), R_\pi(\varphi, \pi)(x) \right),$$

(3.20)

where

$$R_\varphi(\varphi)(x) \equiv \nu \sigma(\mu_m x - \varphi(x)) - \nu \sigma(\mu_m x),$$

(3.21)

$$R_\pi(\varphi, \pi)(x) \equiv -\rho^{-1/2}(x, \varphi) \cdot \pi(x).$$

(3.22)

This map is an embedding of the full phase space of the warp field, $\mathcal{P}$, into the one of general fluctuations around the soliton, $\mathcal{P}$. As it is clear that the equations of motion of a general fluctuation must also apply for the ones induced by the warp field, this embedding maps any solution of the $\varphi$-theory onto one of the $\phi$-theory. This map further reveals that the mode functions of $\varphi$ are no longer equivalent to the ones of $\phi$ on the non-linear level. Instead, a plane wave in the $\phi$-picture corresponds to a superposition of $\varphi$-waves and vice versa.

The inverse of this embedding points at an interesting property of the warp field regarding its connection with the Goldstone mode. The inverse of (3.20) is given by

$$R^{-1} : \mathcal{P} \ni \mathcal{D}(R^{-1}) \rightarrow \mathcal{P}$$

$$\left( \phi(x), \pi(x) \right) \mapsto \left( R^{-1}_\varphi(\phi)(x), R^{-1}_\pi(\phi, \pi)(x) \right),$$

(3.23)

with

$$R^{-1}_\varphi(\phi)(x) \equiv -\sum_{n=1}^{\infty} \left( \frac{\sigma^{-1}(n)}{n!} \right) \left. \phi^n(t, x) \right|_{z = \sigma(\mu_m x)}$$

$$R^{-1}_\pi(\phi, \pi)(x) \equiv -\rho^{1/2}(x, R^{-1}_\varphi(\phi)) \cdot \pi(x).$$

(3.24)

Following the arguments above, given a fluctuation in $\mathcal{D}(R^{-1})$, this map allows to identify the warp field configuration causing said fluctuation. Therefore, all configurations in $\mathcal{D}(R^{-1})$ can equivalently be understood as the manifestation of a warp field excitation.

The important question is now which kind of fluctuations form $\mathcal{D}(R^{-1})$, i.e. which part of the phase space is covered by the theory of warp fields. This domain is constrained by the convergence of the series (3.24) or, equivalently, the invertibility of equation (3.22). From the latter one it is straightforward to derive a condition for $\phi$,

$$\left| \frac{\phi(t, x)}{\nu} + \sigma(\mu_m x) \right| < 1.$$

(3.25)
The left hand side being equal to 1 corresponds to the case $\varphi = \pm \infty$.

A discussion of the physical relevance of this inequality and its origin of the warp fields connection with the Goldstone field is given in section 3.7.

Hence, the set of all fluctuations which can be attributed to a warp field $\varphi$ is given by

$$D(R^{-1}) = \{(\phi(x), \pi(x)) \in P|\phi(x)/\nu + \sigma(\mu, x)| < 1\}. \quad (3.26)$$

As $\phi$ is in general time-dependent, it is possible that a certain configuration satisfies (3.25) within some time interval $T$, but ceases to do so for $t \notin T$. A physical interpretation of this property is given in the next subsection.

Given a fixed time interval $T = [t_i, t_f]$, one can define $D_T(R^{-1})$ as the set of configurations which remain within $D(R^{-1})$ during $T$,

$$D_T(R^{-1}) = \{(\phi(x), \pi(x)) \in D(R^{-1})|(\phi(t, x), \pi(t, x)) = (\phi(t_i, x), \pi(t_i, x)) \Rightarrow (\phi(t, x), \pi(t, x)) \in D(R^{-1}) \forall t \in T\}. \quad (3.27)$$

### 3.6 Incompleteness of the field space

The embedding of the theory of the warp field into the full theory of fluctuations has shown that it is possible for configurations to evolve into and out of the sector describable in terms of the warp field. In other words, the dynamical theory of the warp field is classically incomplete. While there exist situations in which such an incompleteness can be cured in the quantum theory (see [33]), this is not the case here. It is rather a necessary consequence of the fact that the warp field describes only a subset of the full theory. To see why this is the case, consider some localized wave package, propagating through the soliton in such a way that it is an element of $D_T$ for some interval $T$ (Fig. 1). Following the standard interpretation of the wave package as a particle, this process can be understood as said particle propagating through the soliton, being deformed while doing so and potentially picking up some phase shift. The notion of the warp field corresponds to an alternative perspective: While the fluctuation is localized in the same area as the soliton, it can equivalently be understood as a warping of the latter. Thus, this process can be described as a wave package hitting the soliton and being continuously absorbed, causing the soliton to warp, a process described by the warp field. After the interval $T$ has passed, the soliton returns to an unexcited state by continuously emitting another wave package.

In other words, $D_T$ is the set of configurations which can equivalently be understood as a warping of the soliton within the time interval $T$, and the warp field is nothing but an effective description for the dynamics of such configurations during $T$.

The existence of such configurations is no surprise. In the context of a domain wall, the warp field can be understood as a perturbation in its thickness. However, it is well-known that such perturbations can very well decay into scalar particles [18]. The notion of the warp field now allows to give a clear characterization of this decay as the configuration reaching the boundary of the phase space of the warp field theory $P$, i.e. $\varphi$ becoming divergent in some point.
Figure 1. Solid line: The soliton together with the wave package before, while and after passing through the soliton.
Dotted line: The unperturbed soliton.

3.7 Localization via the weight factor

The existence of the warp field as an extension of the Goldstone mode is linked to the breaking of the translational symmetry. This symmetry is however approximately restored in regions sufficiently far away from the center of the soliton, as the value of the field converges towards some constant value $\pm \nu$. It can therefore be expected that the physical significance of the warp field is biggest near the soliton’s center, and vanishes sufficiently far away. This is true for the actual Goldstone field, but also for the warp field due to the appearance of the weight factor. Recall that the warp field has been defined with an additional factor of $\mu_m$ to obtain a dimensionless quantity. Taking into account these additional factors, the weight factor becomes

$$\nu^2 \mu_m^2 \left( \sigma'(\mu_m x - \varphi) \right)^2 = \epsilon(x - \mu_m^{-1} \varphi),$$

which is nothing but the energy density of the warped soliton. Thus, the more of the energy making up the soliton is localized in a point, the stronger are the effects induced by the warp field, so that its physical effects are bound to and transmitted through the soliton.

This correlation can also be viewed from another perspective. It is a general feature of Goldstone modes that they dominate the low-energy description of their corresponding theory. Consider now the set of configurations which can be understood as the result of the warp fields existence, i.e. which satisfy (3.25). Near the solitons center symmetry implies that $\sigma(x) \approx 0$, so that any fluctuation whose maximal value is smaller then the vacuum value is included in the theory of the warp field. In the limit $|x| \to \infty$, also $\sigma(x) \to \pm 1$, and the range of fluctuations satisfying (3.25) shrinks exponentially. In other words, the more
the symmetry is broken in a certain neighborhood, the more of the dynamics inside of it can be attributed to the warp field.

3.8 Absence of an explicit mass term from Goldstone’s theorem

The proof of Goldstone’s theorem given in section 1.3 started from the premise that the action is invariant under the considered transformation, (1.20). This is however not the most general transformation law leading to (1.24). Consider some fluctuation \( \Phi_s(x)^m \rightarrow \Phi_s^m(x_1^{1+1}, \ldots x_1^d) + \delta \Phi^m(x_1^0, \ldots x_1^d) \), under which the action transforms as

\[
\delta S[\Phi] = \int \mathrm{d}^d x \frac{\delta S[\Phi]}{\delta \Phi^m(x)} \Big|_{\Phi = \Phi_s} \delta \Phi^m(x) = F[\Phi, q^a, \partial q^a],
\]

where the functional derivative of \( F[\Phi, q^a, \partial q^a] \) with respect to \( \Phi \) vanishes if \( \partial q^a \rightarrow 0 \) when evaluated at the soliton-like background. Due to this property, the right hand side disappears from the final result of the theorem, and the corresponding field disappears from the action.

For the most simple case of a scalar field in \((1 + 1)\) dimensions, this is indeed achieved by fluctuations induced by a warp field,

\[
\delta \Phi(t, x) \propto \Phi_s'(x) \varphi(t, x).
\]

For such fluctuations, one finds

\[
F[\Phi, \varphi, \partial \varphi] = \int \mathrm{d}^2 x \, \Phi'(t, x) \varphi(t, x) \left( \Box \Phi(t, x) + V'(\Phi(t, x)) \right) \quad \text{and}
\]

\[
\frac{\delta F[\Phi, \varphi, \partial \varphi]}{\delta \Phi(t, x)} = - \partial_x \left( \varphi(t, x) \left( \Box \Phi(t, x) + V'(\Phi(t, x)) \right) \right) + \Phi'(t, x) \varphi(t, x) + 2 \Phi'(t, x) \dot{\varphi}(t, x) + \varphi(t, x) \left( V''(\Phi(t, x)) \Phi'(t, x) - \Phi^{(3)}(t, x) \right) + \Phi'(t, x) \Box \varphi(t, x) - 2 \Phi'(t, x) \varphi(t, x).
\]

Picking a static solitonic background \( \Phi = \Phi_s(x) \), the terms in the first line disappear immediately. The same is true for the second line due to Bogmolnyi’s equation, \( V(\Phi_s(x)) = \frac{1}{2} (\Phi_s')^2(x) \). Therefore, the functional derivative of \( F \) when evaluated at a solitonic background becomes

\[
\frac{\delta F[\Phi, \varphi, \partial \varphi]}{\delta \Phi(t, x)} \bigg|_{\Phi = \Phi_s} = \Phi_s'(x) \Box \varphi(t, x) - 2 \Phi_s''(x) \varphi(t, x).
\]

Thus, in the limit of vanishing \( \varphi \)-momenta, this term disappears, and with it the explicit mass term of \( \varphi \).

Note that this only seems to be in contradiction with the fact that the collective coordinate is the only massless excitation of the soliton. The issue is immediately resolved by the observation that in the limit of a vanishing momentum the warp field is reduced to a spatial constant, which is just the representation of the collective coordinate in the \( \varphi \)-theory. In other words, in the limit of vanishing momenta the warp fields becomes identical with the Goldstone mode, naturally obtaining its properties. Indeed, in the limit \( \varphi' \rightarrow 0 \), (3.32) reduces to the action of the collective coordinate.
3.9 Geometric perspective

In the presence of gravity, the moduli fields are eaten up by the metric, as they can be understood as the action of some diffeomorphism on the soliton [27]. As this letter statement is also true for the warp field, it seems only natural that it can also be absorbed in a similar way.

To see that this is indeed the case, consider the coordinates

\[ (\xi^0(t,x), \xi^1(t,x)) = (t, x - \mu_m^{-1} \varphi(t,x)) \]

In terms of these coordinates, the action of the soliton can be rewritten as

\[ S[\Phi] = \int d^2 \xi \sqrt{-g} \left( \frac{1}{2} g^{\mu\nu}(\xi) \partial_\mu \Phi_s(\xi) \partial_\nu \Phi_s(\xi) - V(\Phi_s(\xi(t,x))) \right), \]

where the effective metric is given by

\[ g^{\mu\nu}(\xi) = \eta_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^\mu}(\xi) \frac{\partial x^\beta}{\partial \xi^\nu}(\xi), \tag{3.33} \]

whose determinant is as usual denoted by \( g_\varphi \).

In this new action, all the \( \varphi \)s are combined into \( g^{\mu\nu} \). Hence, it is reasonable to ask whether this can be taken one step further, describing the dynamics completely by \( g^{\mu\nu} \). Formally this corresponds to promoting \( g^{\mu\nu} \) to a degree of freedom, which will be denoted by \( g_{\mu\nu} \) to point out that it is no longer considered a function of the actual degree of freedom \( \varphi \). When doing so, the information about its structure (3.33) has to be implemented via a Lagrange multiplier \( \Lambda \). In 1+1 dimensions, (3.33) is equivalent to \( R = 0 \), where \( R \) is the usual Ricci scalar [35]. Hence, the dynamics of the warp field is fully encoded in the action

\[ S[\Phi_s, g] = \int d^2 \xi \sqrt{-g} (\mathcal{L}_s + \Lambda R), \]

where \( \mathcal{L}_s \) denotes the action of the soliton on the spacetime with metric \( g \). Note that the term multiplied by the Lagrange multiplier is nothing but the Einstein-Hilbert action, formally providing a kinetic term for the metric. This action gives rise to the equation of motion for \( g_{\mu\nu} \),

\[ T^{\mu\nu} = 2(\Lambda G^{\mu\nu} - \nabla^\nu \partial^\mu \Lambda + g^{\mu\nu} \Box \Lambda) \tag{3.34} \]

where \( T^{\mu\nu} \) denotes the energy momentum tensor of the scalar field, \( T^{\mu\nu} = \frac{\delta \sqrt{-g} \mathcal{L}_S}{\delta g^{\mu\nu}} \). Equation (3.34) together with the constraint \( R = 0 \) fully determines the dynamics of the theory, parametrized by \( g_{\mu\nu} \) and \( \Lambda \). Note that the constraint implies in particular that the kinetic term for \( g_{\mu\nu} \), \( G^{\mu\nu} \), is identical to zero, so that the metric is no propagating degree of freedom. Instead, this role is taken by the Lagrange multiplier \( \Lambda \).

More precisely, \( \Lambda \) acts as a dynamical degree of freedom sourced by the soliton’s energy.
density. This can be seen by inserting the energy-momentum tensor of the scalar field into (3.34) and contracting both sides with $g_{\mu\nu}$, yielding

$$\Box g_\Lambda = \text{tr}_g(T) = V(\Phi_s(\xi)) = \frac{1}{2} \tau_s(\xi).$$

Just as in the $\varphi$-theory, the soliton enters only via its energy density. However, while the $\varphi$-theory is plagued by an infinite number of interaction terms, these are no longer present if the system is parametrized in terms of $\Lambda$.

### 3.10 Generalization of the moduli space

The generalization of the moduli fields to warp fields also impacts the structure of the field space. The most significant change is the fact that its metric can no longer be reduced to a constant, as the warp field is a function of the same spacetime coordinates as the broken generators. Instead, one can expect the action to be of the form

$$S[\varphi^a] = \frac{1}{2} \int d^d x \, \rho_{ab}(x^{n+1}, \ldots x^d, \varphi(x)) \partial_{\mu} \varphi^a(x) \partial^{\mu} \varphi^b(x).$$

The matrix density $\rho_{ab}$ is related to the moduli space metric via

$$\mu_m^2 \int dx^{n+1} \ldots \int dx^d \rho_{ab}(x^{n+1}, \ldots x^d, \varphi(x)) \Big|_{\partial_{x^{n+1} \varphi=x=\ldots=\partial_{x^d} \varphi=0}} = M_{ab},$$

as the case $\partial_{x^{n+1}} \varphi = \cdots = \partial_{x^d} \varphi = 0$ corresponds to the pure moduli fields. The best example for this is the $(1+1)$-dimensional setup discussed in this article. Recall that the action of the warp field takes the form

$$S[\varphi] = \frac{1}{2} \int d^d x \, \rho(x, \varphi(t,x)) \partial_{\mu} \varphi(t,x) \partial^{\mu} \varphi(t,x).$$

And indeed, as the weight function $\rho(x, \varphi(t,x))$ is proportional to the energy density, integrating it over space with a constant $\varphi$ yields the mass of the soliton, which is nothing but the 1-dimensional moduli space metric. Thus, the weight factor can equivalently be understood as the manifestation of the non-trivial geometry of the field space. As for most soliton-like configurations the moduli space metric is proportional to the soliton’s mass, tension etc., it can be expected that $\rho_{ab}$ is in general proportional to the configuration’s energy density. Therefore, the warp fields of those configurations naturally appear with a weight factor proportional to their warped energy density. Thus, all the consequences of the presence of the weight function in $(1+1)$ can be immediately transferred to higher-dimensional settings.

This action has the interesting property that it contains no explicit potential term. Instead, the interactions are generated by the field space metric, and can therefore be absorbed in the spacetime metric in the manner described in the last subsection.

### 3.11 Warp fields in different regimes

The region in which the contribution of the warp field to the full dynamical theory is of relevance depends strongly on the localization of the soliton, and therefore indirectly
also on the parameters of the underlying theory. In the simple setting discussed in this article, these parameters are the boundary values the soliton converges towards, the coupling constant and the mass of the scalar field. In general only two out of these three parameters are independent. Taking as an example the case of the $\Phi^4$-theory, these coefficients are connected by the relation $m_\Phi \propto \nu \sqrt{\lambda}$.

As the chosen parametrization does not explicitly depend on the coupling constant $\lambda$, it is most convenient to choose the vacua $\pm \nu$ and the mass of the field $m_\Phi$ as independent parameters. It is also important to recall that the warp field $\varphi$ is obtained from the actual Goldstone mode not only by extending its domain, but by also by multiplying it with a factor of $\mu \propto \frac{1}{m_\Phi}$, so that also $\varphi \propto m_\Phi$.

Consider first the case $m_\Phi \to 0$, implying that also $\mu \to 0$. In this limit, the soliton is widely outstretched, while $\varphi \to 0$. This is in agreement with the origin of the warp field in the breaking of translational symmetry in the presence of the soliton. As the soliton spreads out in this limit, the same is true for the region in which the symmetry is locally broken. However, at the same time the soliton flattens out, so that the symmetry breaking becomes less distinct. Note that, given a fixed value of $\nu$, this limit is equivalent to the one of a weak interaction.

In the limit $m_\Phi \to \infty$, when $\mu \to \infty$, the opposite can be observed. With the soliton becoming increasingly localized, the region affected by the warp field shrinks until the latter one is fully confined to the soliton, which can with increasing precision be approximated as a point particle. Therefore, in the limit of a heavy field or, equivalently, a strong interaction with fixed $\nu$, the warp field becomes identical with the collective coordinate. This can also be observed from (3.25). In the limit of a soliton which is fully localized to a point, (3.25) can only solved by $\phi = 0$, i.e. the non-zero mode contributions to the warp field have to disappear. This also resolves the issue that taking this limit seems to imply $\varphi \to \infty$. As the non-zero mode contributions disappear due to the limit, only the terms representing the collective coordinate remain. Their prefactor of $\mu \propto \frac{1}{m_\Phi}$ factorizes out inside the argument of the soliton, and outside integrates with the weight factor to the soliton’s mass.

The explicit $\nu$-dependence is easily understood: As the energy density of the soliton scales as $\nu^2$, the same is true for all physical effects caused by the warp field.

4 Quantum theory of the warp field

The quantum theory of $\varphi$ is determined by the properties of the operators $\hat{\varphi}$, $\hat{\omega}$ and the Fock space $\mathcal{F}$ they are acting on. Just as in the classical case, it can be embedded into the $\phi$-theory.

Similar to the classical theory, the following discussion is limited to the free theory. The main motivation therefore lies again in the significant technical difficulties of the non-linear theory, which are amplified due to the unusual structure of the Fock space.
4.1 Linearized theory

The operators representing $\varphi$ and $\varpi$ can be obtained from (3.13):

$$\hat{\varphi}(t, x) = \mu_0 \hat{\nu}(t - t_0) + \mu_m \hat{\zeta} + \sum_{k} \hat{\alpha}_k^* e^{i\omega_k t} g_k^*(x) + \hat{\alpha}_k e^{-i\omega_k t} g_k(x)$$
$$\hat{\varpi}(t, x) = iv^2(\sigma')^2(\mu_m x) \cdot \left( \mu_0 \hat{\nu} + \sum_{k} \hat{\alpha}_k^* \omega_k e^{i\omega_k t} g_k^*(x) - \hat{\alpha}_k \omega_k e^{-i\omega_k t} g_k(x) \right)$$ (4.1)

These operators need to satisfy the usual equal time commutation relations,

$$[\hat{\varphi}(t, x), \hat{\varpi}(t, x')] = i\delta(x - x')$$ and
$$[\hat{\varphi}(t, x), \hat{\varphi}(t, x')] = 0 = [\hat{\varpi}(t, x), \hat{\varpi}(t, x')]$$ (4.2)

In terms of the creation/annihilation operators $\{\alpha_k^{(1)}\}_k$ and $\hat{\nu}, \hat{\zeta}$, these relations can be expressed as

$$[\hat{\alpha}_k, \hat{\alpha}_p^\dagger] = \begin{cases} \mathcal{N}_c \delta(k - p), & \text{for } k, p \in \mathcal{I}_c \\ \mathcal{N}_d \delta_{kp}, & \text{for } k, p \in \mathcal{I}_d \end{cases}$$
$$[\hat{\zeta}, \hat{\nu}] = iM_{\text{sol}}^{-1}$$ (4.3)

where $\mathcal{I}_c$ and $\mathcal{I}_d$ denote the continuous and discrete parts of the spectrum. The normalization constants $\mathcal{N}_c$ and $\mathcal{N}_d$ are determined by the chosen normalization of the mode functions. Defining $\hat{p} = M_{\text{sol}}\hat{\nu}$, the last relation is equivalent to the commutation relation satisfied by the collective coordinate and its conjugate momentum.

The Fock space these operators act on is therefore given by

$$\mathcal{F} = L^2(\mathbb{R}, dx) \otimes \mathcal{F}_p, \text{ where } \mathcal{F}_p = \bigoplus_{n=0}^{\infty} S_n(L^2(\mathbb{R}, dx))^\otimes_n$$ (4.4)

denotes the Fock space of particle-like excitations and $L^2(\mathbb{R}, dx)$ corresponds to the collective coordinate. Following the usual naming scheme of particle physics, the particle-like excitations on this Fock space will be named warpions.

The chosen Fock space differs from one of a usual scalar field due to its spatial measure $dx$. This measure is an immediate result of the mode functions $\{g_k(x)\}_k$. Consider the Fock space creation/annihilation operators $A^{(1)}$, which are related to the $\alpha^{(1)}$-operators via

$$A^{(1)}[f] = \sum_{k} f^k \alpha_k^{(1)},$$ (4.5)

where $f^k$ denotes the coefficients of $f$ when expanded in the $\{g_k\}_k$-basis. These operators satisfy the commutation relation

$$[A[f], A^\dagger[g]] = (f, g)_{\mathcal{H}},$$ (4.6)

where $(f, g)_{\mathcal{H}}$ denotes the scalar product on the one particle Hilbert space $\mathcal{H}$ [34]. As $\alpha_k^{(1)} = A^{(1)}[g_k]$, (4.3) and (4.6) cannot be simultaneously realized on the usual $(L^2, dx)$.
But as the mode functions \( \{ g_k \}_k \) are orthonormal with respect to the measure \( d\zeta \), the desired commutation relations are realized on \( \mathcal{H} = (L^2, d\zeta) \). These relations can now be used to obtain an expression for the propagator of the warp field. The non-zero mode contributions can be obtained from the expectation value of the time-ordered product of the non-zero mode terms within \( \hat{\phi} \) with respect to the \( F_\rho \)-vacuum. In terms of the \( \phi \)-propagator, they can be expressed as

\[
\Delta^n_{\phi}((t,x),(t',x')) = \frac{\Delta_{\phi}(\nu \sigma'(\mu_m x))}{(\nu \sigma'(\mu_m x'))}.
\]

The propagator of the zero mode is in great detail discussed in \[11\].

### 4.2 Implications of the probability measure

Similar to the classical theory, the quantum theory of the warp field bears a strong resemblance to the theory of a usual scalar field, with the main differences being the appearance of the weight function \( \rho(x) = \nu^2(\sigma')^2(\mu_m x) \) and the unusual asymptotic behavior of the mode functions. The quantum theory however reveals the true relevance of these functions, namely that they enforce a probability measure which is proportional to the weight function \( \rho \). This is nothing but a particle physics version of the localization behavior observed in the classical theory. While warpion-wave functions are supported on all of spacetime, the associated probability density is localized in the same region as the soliton as it is weighted with \( \rho \).

Due to the incorporation of \( \rho \) in the measure of \( \mathcal{H} \) this localization is also carried over to the expectation values of any observable. Let \( \mathcal{O} \) denote some observable which acts trivially on the \( L^2 \)-space of the zero mode. In terms of the corresponding one-particle Hilbert space observable \( \hat{O} \), the action of this operator on some Fock space state \( |\Psi\rangle = |\Psi_1(x), \Psi_2(x_1, x_2), \ldots\rangle \) is given by

\[
\mathcal{O} |\Psi\rangle = |(\mathcal{O}\Psi)_1(x), (\mathcal{O}\Psi)_2(x_1, x_2), \ldots\rangle,
\]

where

\[
(\mathcal{O}\Psi)_n(x_1, \ldots x_n) = \sum_{i=1}^{n} \hat{O}(x_i) \Psi_n(x_1, \ldots x_n).
\]

The functional dependence of \( \hat{O}(x_i) \) is here meant as the action of \( \hat{O} \) on the \( i \)th argument [34]. Therefore, the expectation value takes the form

\[
\langle \Psi | \mathcal{O} | \Psi \rangle = \langle \Psi_1 | (\mathcal{O}\Psi)_1 \rangle + \langle \Psi_2 | (\mathcal{O}\Psi)_2 \rangle + \ldots
\]

\[
= \int d\zeta(x) \Psi^*_1(x) \hat{O} \Psi_1(x) + \int d\zeta(x_1) \int d\zeta(x_2) \sum_{i=1}^{2} \Psi^*_2(x_1, x_2) \hat{O}(x_i) \Psi_2(x_1, x_2) + \ldots
\]

The important feature of this expression is now that the contribution of each \( n \)-particle state comes with \( n \) integrals, each corresponding to one of the particles. But as the measure of integration is given by \( d\zeta \) and therefore contains the function \( \rho \), the contribution of each warpion to any physical observable is again weighted by the density of the soliton.
4.3 Embedding into the full dynamical theory

Just as in the classical case, the zero mode of the warp field can immediately be identified with the collective coordinate as introduced in [11]. To extend this insight to the non-zero modes, consider the following map between the one particle Hilbert spaces of the \( \varphi \) and the \( \phi \) theory:

\[
\iota : \mathcal{H} \rightarrow \mathbb{H}
\]

\[
\Psi(x) \mapsto \rho^{1/2}(x)\Psi(x)
\]  

(4.10)

This map is an isometry which respects the structure of the mode expansion, in the sense that \( \iota(g_k) = f_k \) and \( a_k = \iota \alpha_k \iota^{-1} \). Note that due to the additional constraints following from the applicability of the linearized theory only a subset of \( \mathcal{H} \) can be considered, so that, in this context, this map has to be understood as an embedding.

Extending this map to the whole Fock space in the canonical way [34], it is clear that the dynamics on \( \mathcal{F}_p \) generated by the \( \{\alpha^\dagger_k\} \) operators is equivalent to the one on the corresponding subset of \( \mathcal{F} \) generated by the \( \{a^\dagger_k\} \) operators.

This equivalence also manifests itself in the propagator. Consider some wave function, which at some time \( t' \) is given \( \psi(t', y) = \int_{k \neq 0} \psi_k(t') g_k(y) \). The action of the propagator on this wave function can then be expressed as

\[
\psi(t, x) = \int \Delta_\varphi((t, x), (t', y))\psi(t', y)d^2\varsigma(y) = \\
= \frac{1}{\nu \sigma'(\mu_{x})} \int \Delta_\phi((t, x), (t', y))(\nu \sigma'(\mu_{y})\psi(t', y))d^2y = \\
= \iota^{-1}\left( \int \Delta_\phi((t, x), (t', y))\iota(\psi)(t', y)dy \right) = \\
= \iota^{-1}(\iota(\psi)(t, x)),
\]  

(4.11)

where \( d^2\varsigma(y) = dt'd\varsigma(y) \).

This does not only confirm the equivalence of the dynamics associated with the nonzero modes in both theories, but also shows that the propagator of the \( \varphi \) theory, in spite of its unusual asymptotic behavior, does not present a problem, but is in agreement with the result of the \( \phi \) theory.

This embedding also leads to an interpretation of the particle-like warpions. It is clear that when considering also higher orders, the one-to-one correspondence of a warpion of momentum \( k \) to a scalar particle of the same momentum breaks down. Instead, a single warpion has to be understood as a localized configuration of scalar particles, which can be prepared for some finite time \( T \), before decaying back into clearly distinguishable scalar particles.

4.4 Local velocity as a quantum theoretical observable

Just as in the classical theory, the local velocity can be introduced via the warp field. As the theory of the warp field corresponds to a subsector of the \( \phi \)-theory, this can also be
done for the full theory. First, observe that equation (3.24) allows to define the non-zero mode contributions of the warp field as an observable of the full dynamical theory via

\[
\hat{\varphi}(t, x) = -\sum_{n=1}^{\infty} \frac{(\sigma^{-1})(n)}{\mu^n n!} \hat{\phi}^n(t, x).
\] (4.12)

As the local velocity \(v_{\text{loc}}\) is given by a series in \(\phi\) and \(\pi\), it can also be represented as an operator on \(F_p\). Taking the time derivative of (4.12) while keeping track of the ordering of \(\phi\) and \(\pi\) then yields

\[
\hat{v}_{\text{loc}} = -\sum_{n=1}^{\infty} \frac{(\sigma^{-1})(n)}{\mu^n n!} (\hat{\phi}^{n-1} \hat{\pi} + \cdots + \hat{\pi} \hat{\phi}^{n-1}),
\] (4.13)

with

\[
\mathcal{D}(\hat{v}_{\text{loc}}(t, x)) = \{ |\Psi\rangle \in \mathcal{F}_\phi : \hat{\varphi}(t, x) |\Psi\rangle \in \mathcal{F}_\phi \}.
\] (4.14)

Note that there are again states which lie within this domain for some finite time \(T\), but cease to do so for \(t \notin T\).

As \(\phi\) and \(\pi\) are generated by \(\{ \hat{a}_k^{(\dagger)} \}_k\) operators, they act solely on \(F_p\) and are independent of the \(L^2\)-piece of the systems dynamics. This confirms that also on the quantum level the velocity of the soliton can be split into a global part, corresponding to the dynamics associated to the zero mode, and a local part, which can be completely described in terms of \(F_p\).

### 4.5 Parallels to the classical theory

Just as in the classical case, the existence of the warp field is bound to the breaking of the translational symmetry. In the classical theory this connection manifests itself in the form of the weight function \(\rho\) in the action, while in the quantum theory the weight function enters via the probability measure of the Fock space of warp particles. In the classical theory this additional factor leads to a localization of the effects of warp field configurations to the same region as the soliton. In the quantum theory, it leads to the expectation value of any observable being dominated by the wave function in said region, whereas the behavior of the wave function far away from the soliton’s center is suppressed. And, as in the classical theory, the weight factor is proportional to the energy density of the soliton.

Just as in the classical theory, it is possible to represent the warp field as well as the local velocity as observables of the full dynamical theory of fluctuations around the warp field. The local velocity is thereby again found to be the result of the dynamics of the non-zero modes, whereas the collective coordinate gives rise to the global velocity.

This is possible as just as in the classical theory, the quantum theory of \(\varphi\) can be shown to be a subset of the general dynamical theory of fluctuations around the soliton. In particular the collective coordinate is fully contained in the warp field, so that its dynamics and operator structure are reproduced.
5 Conclusion & Summary

It has been shown that extending the domain of moduli fields from the worldvolume of their associated soliton-like configuration to all of spacetime gives rise to a well-defined theory. The fields obtained in this way, the warp fields, show generalized versions of the most important features of Goldstone fields. In other words, the introduction of warp fields allows for an extension of concepts characteristic to Goldstone modes to a subsector of the theory also covering a part of the dynamics associated with the non-zero modes.

This theory forms a well-defined subset of the full theory of fluctuations around the corresponding background, in the sense that the phase and Fock space of the warp fields can be embedded into the ones of the full theory. When considering only the linearized theories, the kinetic operator of the warp fields can be used to reconstruct the full spectrum of the fluctuations, as the warp fields are a combination of massless and massive modes. The warp fields contain in particular the moduli fields themselves, and their full dynamics is encoded within their theory.

Just as the moduli fields, the warp fields are localized in a way determined by the soliton’s energy density. This is realized via a change of the spatial measure of the classical as well as the quantum theory, due to which the warp field’s value in any point enters into observables only weighted by the value of the soliton’s energy density in the same point.

The action of the warp fields is of the same structure as the one of the moduli fields, with the central difference being that the field space metric is no longer a constant, but a function of spacetime as well as the warp fields. This implies in particular that the warp fields disappear from the action in the limit $\partial \varphi \to 0$, and that they link any constant shift of the warp fields to a constant translation of the underlying configuration. While the warp fields lack an explicit mass term, their theory is still capable of generating the masses of the non-zero modes due to the modification of its kinetic term induced by the field space metric.

As the warp fields enter the theory in the same way as the moduli fields, they can also be absorbed into the spacetime metric. This can be realized via a Lagrange multiplier, which can take the role of the theory’s dynamical degree of freedom. As the theory of the warp fields is fully determined by the soliton’s energy density, the same is true for this new field, which is sourced by the very same. An important feature of the parametrization of the dynamics in term of the Lagrange multiplier is that its theory naturally appears without self-interactions, leading to a significant simplification of the system’s dynamics.

Finally, the warp field gives rise to the notion of a local velocity, which allows for an elegant description of interactions between the soliton and localized, external sources. Introducing warp fields does not correspond to any new dynamics, it rather provides a new perspective on the well-known theory of fluctuations around soliton-like configurations. This perspective proves itself powerful not only as it allows to widen the understanding of the theory and simplifies the description of certain processes, but also as it allows for the generalization of concepts previously exclusive to moduli fields to a much larger range of configurations, revealing new structures of the underlying theory.
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