ON A JOINT \((m; (q_1, \ldots, q_d))\)-PARTIAL ISOMETRIES AND A JOINT \(m\)-INVERTIBLE \(d\)- TUPLE OF OPERATORS ON A HILBERT SPACE

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Abstract

For \(d \in \mathbb{N}\) with \(d \geq 1\), let \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\) with \(T_j : \mathcal{H} \rightarrow \mathcal{H}\) be a tuple of commuting bounded linear operators. Let \(\alpha = (\alpha_1; \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d, q = (q_1, q_2, \ldots, q_d) \in \mathbb{Z}_+^d\) denote tuples of nonnegative integers respectively, and set \(|\alpha| := \sum_{1 \leq j \leq d} |\alpha_j|, \alpha! := \alpha_1! \ldots \alpha_d!\). Further, define \(T^{\alpha} := T_1^{\alpha_1}T_2^{\alpha_2} \ldots T_d^{\alpha_d}\). A bounded linear \(d\)-tuple of commuting operators \(T = (T_1, T_2, \ldots, T_d)\) acting on a Hilbert space \(\mathcal{H}\) is called an \((m; (q_1, q_2, \ldots, q_d))\)- partial isometry, if

\[
T^q \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^{\alpha} T^{\alpha} \right) = 0.
\]

The aim of the present paper is, firstly we study the concepts of \((m; (q_1, \ldots, q_d))\)-partial isometries on a Hilbert space; secondly, we introduce the notion of \(m\)-invertibility of tuples of operators as a natural generalization of the \(m\)-invertibility in single variable operators.

Keywords. \(m\)-isometric tuple, partial isometry, Left \(m\)-inverse, Right \(m\)-inverse, joint spectrum, joint approximate spectrum.

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1 INTRODUCTION AND TERMINOLOGIES

Let \(\mathcal{H}\) be an infinite dimensional separable complex Hilbert space and denote by \(\mathcal{B}(\mathcal{H})\) the algebra of all bounded linear operators from \(\mathcal{H}\) to \(\mathcal{H}\). For \(T \in \mathcal{B}(\mathcal{H})\) we shall write \(\mathcal{N}(T)\) ,
\( \mathcal{R}(T) \) and \( \mathcal{N}(T)^\perp \) for the null space, the range of \( T \) and the orthogonal complement of \( \mathcal{N}(T) \) respectively. \( I = I_H \) being the identity operator. In what follows \( \mathbb{N}, \mathbb{Z}_+ \) and \( \mathbb{C} \) stands the sets of positive integers, nonnegative integers and complex numbers respectively. Denote by \( \overline{\lambda} \) the complex conjugate of a complex number \( \lambda \) in \( \mathbb{C} \). We shall henceforth shorten \( \lambda I_H - T \) by \( \lambda - T \). The spectrum, the point spectrum, the approximate point spectrum of an operator \( T \) are denoted by \( \sigma(T), \sigma_p(T), \sigma_{ap}(T) \) respectively. \( T^* \) means the adjoint of \( T \).

The study of tuples of commuting operators was the subject of a wide literature carrying out many resemblances with the single case. Some developments toward this subject have been done in \([4], [5], [10], [11], [12], [13], [14], [26], [27] \) and the references therein.

Our aim in this paper is to extend the notions of \( m \)-partial isometries \([24]\) and \((m, q)\)-partial isometries \([21]\) for single variable operators to the tuples of commuting operators defined on a complex Hilbert space.

Some notational explanation is necessary before we begin. For \( d \in \mathbb{N} \ (d \geq 1) \), let \( T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d \) be a tuple of commuting bounded linear operators. Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{Z}_+^d \) denote tuples of nonnegative integers multi-indices) and set \(|\alpha| := \sum_{1 \leq j \leq d} |\alpha_j| \), \( \alpha! := \alpha_1! \ldots \alpha_d! \). Further, define \( T^\alpha := T_1^{\alpha_1} \ldots T_d^{\alpha_d} \) where \( T_j^{\alpha_j} \) denotes the product of \( T_i \) times itself \( \alpha_j \) times .

Now let \( p(z, \overline{z}) \) be a non-commutative complex polynomial in \( z = (z_1, \ldots, z_d) \) and \( \overline{z} = (\overline{z_1}, \ldots, \overline{z_d}) \) given by \( p(z, \overline{z}) = \sum_{\alpha, \beta} a_{\alpha, \beta} z^\alpha \overline{z}^\beta \). If \( T \) denote an \( d \)-tuple of bounded linear operators on a Hilbert space then one can associate with \( p(z, \overline{z}) \) an operator polynomial \( p(T, T^*) \)

\[
p(T, T^*) = \sum_{\alpha, \beta} a_{\alpha, \beta} T^\alpha T^\beta
\]

by replacing \( z \) and \( \overline{z} \) by \( T = (T_1, \ldots, T_d) \) and \( T^* = (T_1^*, \ldots, T_d^*) \) respectively.

One of the most important subclasses, of the algebra of all bounded linear operators acting on a Hilbert space, the class of partial isometries operators. An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be isometry if \( T^* T = I \) and partial isometry if \( TT^* T = T \). In recent years this classes has been generalized, in some sense, to the larger sets of operators so-called \( m \)-isometries and \( m \)-partial isometries. An operator \( T \in \mathcal{B}(\mathcal{H}) \) is said to be \( m \)-isometric for some integer \( m \geq 1 \) if it satisfies the operator equation

\[
\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^{sm-k} T^{m-k} = 0. \tag{1.1}
\]

It is immediate that \( T \) is \( m \)-isometric if and only if

\[
\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \| T^{m-k} x \|^2 = 0 \tag{1.2}
\]
for all $x \in \mathcal{H}$. Major work on $m$-isometries has been done in a long paper consisting of three parts by Agler and Stankus ([1, 2, 3]) and have since then attracted the attention of several other authors (see for example [7], [8], [9], [16]). More recently a generalization of these operators to $m$-partial isometries has been studied in the paper of A.Saddi and the present author in [24] and by the present author in [21].

An operator $T \in \mathcal{B}(\mathcal{H})$ is called an $m$-partial isometry (see [24]) if

$$T \left( T^m T^m - \binom{m}{1} T^{m-1} T^{m-1} + \binom{m}{2} T^{m-2} T^{m-2} - \ldots + (-1)^m I \right) = 0. \quad (1.3)$$

and it is an $(m, q)$-partial isometry for $m \in \mathbb{N}$ and $q \in \mathbb{Z}_+$ (see ([21])) if

$$T^q \left( T^m T^m - \binom{m}{1} T^{m-1} T^{m-1} + \binom{m}{2} T^{m-2} T^{m-2} - \ldots + (-1)^m I \right) = 0. \quad (1.4)$$

Gleason and Richter in [17] extend the notion of $m$-isometric operators to the case of commuting $d$-tuples of bounded linear operators on a Hilbert space. The defining equation for an $m$-isometric tuple $\mathbf{T} = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ reads:

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha x \frac{x}{\alpha} = 0 \quad (1.5)$$

or equivalently

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \| T^\alpha x \|^p = 0 \quad \text{for all } x \in \mathcal{H}. \quad (1.6)$$

Recently, P.H.W.Hoffmann and M.Mackey in [20] introduced the concept of $(m, p)$-isometric tuples on normed space. A tuple of commuting linear operators $\mathbf{T} := (T_1, \ldots, T_d)$ with $T_j : X \rightarrow X$ (normed space) is called an $(m, p)$-isometry (or an $(m, p)$-isometric tuple) if, and only if, for given $m \in \mathbb{N}$ and $p \in (0, \infty)$,

$$\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \| T^\alpha x \|^p = 0 \quad \text{for all } x \in X. \quad (1.7)$$

**Definition 1.1.** Let $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a tuple of operators.

1. If $T_i T_j = T_j T_i$ $1 \leq i, j \leq d$, we say that $\mathbf{T}$ is a commuting tuple.

2. If $T_i T_j = T_j T_i$, $T_i T_j = T_j T_i$ $1 \leq i \neq j \leq d$, we say that $\mathbf{T}$ is a doubly commuting tuple.

**Definition 1.2.** ([18]) A commuting tuple $\mathbf{T} = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ is called:

1. matricially quasinormal if $T_i$ commutes with $T_j T_k$ for all $i, j, k \in \{1, 2, \ldots, d\}$.

2. jointly quasinormal if $T_i$ commutes with $T_j T_j$ for all $i, j \in \{1, 2, \ldots, d\}$ and

3. spherically quasinormal if $T_j$ commutes with $|\mathbf{T}| := \left( \sum_{1 \leq j \leq d} T_j T_j \right)$ for all $j = 1, 2, \ldots, d$. 


If $\mathcal{M}$ is a common invariant subspace of $\mathcal{H}$ for each $T_j \in \mathcal{B}(\mathcal{H})$, then $T_{|\mathcal{M}} = (T_{1|\mathcal{M}}, T_{2|\mathcal{M}}, ..., T_{d|\mathcal{M}})$ denote an $d$-tuple of compressions of $\mathcal{M}$.

The contents of this paper are the following. Introduction and terminologies are described in the first part. The second part is devoted to the study of some basic properties of the class of $(m; (q_1, ..., q_d))$-partial isometries tuples. Several spectral properties of some $(m; (q_1, ..., q_d))$-partial isometries are obtained in section three; concerning the joint point spectrum, the joint approximate spectrum and the spectral radius. In the fourth section we present some results concerning the left $m$-inverses and the right $m$-inverses for tuples of operators.

2 JOINT $(m; (q_1, ..., q_d))$-PARTIAL ISOMETRIES $d$-TUPLE OF OPERATORS

In this Section, we introduce and study some basic properties of an joint $(m; (q_1, ..., q_d))$-partial isometry operators tuples. All of these results are fairly straightforward generalizations of the corresponding single variable results that were proved in [21] and [24].

The notion of a joint $(m; (q_1, ..., q_d))$-partial isometry is a natural higher dimensional generalization of the notion of $(m, q)$-partial isometry.

**Definition 2.1.** Given $m \in \mathbb{N}$ and $q = (q_1, q_2, ..., q_d) \in \mathbb{Z}_+^d$. An commuting operator $d$-tuple $T \in \mathcal{B}(\mathcal{H})^d$ is called an joint $(m; (q_1, ..., q_d))$-partial isometry (or joint $(m; (q_1, ..., q_d))$-partial isometric $d$-tuple) if and only if

$$T^q \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{*\alpha} T^\alpha \right) = 0.$$

**Remark 2.1.**

1. Every $m$-isometric $d$-tuple of operators on $\mathcal{H}$ is a joint $(m; (q_1, q_2, ..., q_d))$-partial isometry $d$-tuple.

2. Every $(m; (q_1, q_2, ..., q_d))$-partial isometry $d$-tuple of operators $T = (T_1, T_2, ..., T_d)$ such that $T$ is entry-wise invertible, $T$ is an $m$-isometric $d$-tuple.

**Remark 2.2.** If $d = 2$, let $T = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2$ be a commuting operator 2-tuple, we have that

(i) $T$ is a joint $(1; (1, 1))$-partial isometry pair if

$$T_1 T_2 \left( I - T_1^* T_1 - T_2^* T_2 \right) = 0.$$

(ii) $T$ is a joint $(2; (1, 1))$-partial isometry pair if

$$T_1 T_2 \left( I - 2 T_1^* T_1 - 2 T_2^* T_2 + T_1^* T_1^2 + T_2^* T_2^2 + 2 T_1^* T_2^* T_1 T_2 \right) = 0.$$
Remark 2.3. Let $T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting operator $d$-tuple. Then $T$ is an joint $(1; (1, 1, \ldots, 1))$-partial isometry if and only if

$$T_1 \ldots T_d \left( I - T_1^* T_1 - T_2^* T_2 - \ldots - T_d^* T_d \right) = 0.$$ 

Example 2.1. Consider $T = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$ and let $T := \left( \frac{1}{\sqrt{d}} T, \frac{1}{\sqrt{d}} T, \ldots, \frac{1}{\sqrt{d}} T \right) \in \mathcal{B}(\mathbb{C}^d)^d$. It is easy to see that $T$ is a joint $(1; (1, 1, \ldots, 1))$-partial isometry $d$-tuple.

Remark 2.4. If $T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d$ be an doubly commuting $d$-tuple of operators on $\mathcal{H}$. Then $T$ is an joint $(1; 1, 1, \ldots, 1)$-partial isometry if and only if $T^* := (T_1^*, T_2^*, \ldots, T_d^*)$ is so.

The following example of a joint $(m; (q_1, \ldots, q_d))$-partial isometry is adopted form [20].

Example 2.2. Let $S \in \mathcal{B}(\mathcal{H})$ be an $(m, q_1)$-partial isometry operator, $d \in \mathbb{N}$ and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in (\mathbb{C}^d, \|\cdot\|_2)$ with

$$\|\lambda\|_2^2 = \sum_{1 \leq j \leq d} |\lambda_j|^2 = 1.$$ 

Then the operator tuple $T = (T_1, T_2, \ldots, T_d)$ with $T_j = \lambda_j S$ for $j = 1, 2, \ldots, d$ is an joint $(m, (q_1, q_2, \ldots, q_d))$-partial isometry $d$-tuple.

In fact, it is clear that $T_i T_j = T_j T_i$ for all $1 \leq i; j \leq d$. Further, by the multinomial expansion, we get

$$\left( |\lambda_1|^2 + |\lambda_2|^2 + \ldots + |\lambda_d|^2 \right)^k = \sum_{\alpha_1 + \alpha_2 + \ldots + \alpha_d = k} \binom{k}{\alpha_1, \alpha_2, \ldots, \alpha_d} \prod_{1 \leq i \leq d} |\lambda_i|^{2\alpha_i}$$

$$= \sum_{|\alpha| = j} \frac{k!}{\alpha!} |\lambda|^2.$$ 

Thus, we have

$$T^q \sum_{0 \leq j \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha T^\alpha = T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \prod_{1 \leq j \leq d} |\lambda_j|^{2\alpha_j} S^{\alpha_j} S^{\alpha_j}$$

$$= \prod_{1 \leq j \leq d} \lambda_j^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} S^k S^k$$

$$= \prod_{1 \leq j \leq d} \lambda_j^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} S^{k} S^k$$

$$= 0.$$ 

Consequently $T$ is an joint $(m; (q_1, \ldots, q_d))$-partial isometry $d$-tuple as required.
The following example shows that the question about joint \((m; (q_1, q_2, \ldots, q_d))\)-partial isometry for \(d\)-tuple is non trivial. There exists a \(d\)-tuple of commuting operators \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\) such that each \(T_j\) is \((m, q_j)\)-partial isometry for \(j = 1, 2, \ldots, d\), but \(T = (T_1, T_2, \ldots, T_d)\) is not an joint \((m; (q_1, q_2, \ldots, q_d))\)-partial isometry.

**Example 2.3.** Let us consider \(\mathcal{H} = \mathbb{C}^3\) and define \(T_1 = \begin{pmatrix} 0 & i & 0 \\ 0 & 0 & i \\ i & 0 & 0 \end{pmatrix}\) and \(T_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\).

It is straightforward that \(T_1\) and \(T_2\) commute. Moreover, \(T_1\) and \(T_2\) are \((2; 1)\)-partial isometry but \((T_1, T_2)\) is not a \((2; (1, 1))\)-partial isometry.

**Lemma 2.1.** Let \(S_d\) be the group of permutation on \(d\) symbols \(\{1, 2, \ldots, d\}\) and let \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\) be an \(d\)-tuple of commuting operators. If \(T\) is an joint \((m; (q_1, q_2, \ldots, q_d))\)-partial isometry, then for every \(\sigma \in S_d\), \(T_{\sigma} := (T_{\sigma(1)}, T_{\sigma(2)}, \ldots, T_{\sigma(d)})\) is an joint \((m; (q_{\sigma(1)}, q_{\sigma(2)}, \ldots, q_{\sigma(d)}))\)-partial isometry.

**Proof.** It follows from the condition that \(\prod_{1 \leq j \leq d} T_j = \prod_{1 \leq j \leq d} T_{\sigma(j)}\) and the identity

\[
\prod_{1 \leq j \leq d} T_j^{q_j} \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \prod_{1 \leq j \leq d} T_j^{*\alpha_j} = 0.
\]

\(\square\)

**Theorem 2.1.** Let \(m \in \mathbb{N}\) and \(q = (q_1, q_2, \ldots, q_d) \in \mathbb{Z}_+^d\). Let \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\) be an commuting \(d\)-tuple operators such that \(\mathcal{N}(T^q)\) is a reducing subspace for \(T_j\) for all \(j = 1, 2, \ldots, d\). Then the following properties are equivalent.

1. \(T\) is an joint \((m; (q_1, \ldots, q_d))\)-partial isometry.
2. \(\sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \|T^\alpha T^* x\|^2 = 0, \text{ for all } x \in \mathcal{H}.

**Proof.** First, assume that \(T\) is an joint \((m; (q_1, q_2, \ldots, q_d))\)-partial isometry. We have that for all \(x \in \mathcal{H}\)

\[
T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha T^* x = 0
\]

\[
\Rightarrow \langle T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha T^* x, x \rangle = 0
\]

\[
\Rightarrow \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \|T^\alpha T^* x\|^2 = 0.
\]

Thus, (2) holds.
To prove the converse, assume that the equality in (2) holds. It follows that,

\[ \langle T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\ast q} x, x \rangle = 0, \ \forall \ x \in \mathcal{H} \]

\[ \implies T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\ast q} x = 0, \ \forall \ x \in \mathcal{H}. \]

Hence,

\[ T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\ast} = 0 \text{ on } R(T^q) = N(T^q)^{\perp}. \]

As \( N(T^q) \) is a reducing subspace for each \( T_j \ (1 \leq j \leq d) \), we have that

\[ T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\ast} = 0 \text{ on } N(T^q) \]

and hence,

\[ T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\ast} = 0. \]

\[ \square \]

The following corollary is an immediate consequence of Theorem 2.1.

**Corollary 2.1.** Let \( m \in \mathbb{N} \) and \( q = (q_1, q_2, \ldots, q_d) \in \mathbb{Z}^d_+ \). Let \( T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d \) be an commuting \( d \)-tuple operators such that \( N(T^q) \) is a reducing subspace for each \( T_j, \ 1 \leq j \leq d \). Then the following properties are equivalent

1. \( T \) is an joint \( (m; q_1, \ldots, q_d) \)-partial isometry.
2. \( T|_{N(T^q)^{\perp}} := (T_1|_{N(T^q)^{\perp}}, T_2|_{N(T^q)^{\perp}}, \ldots, T_d|_{N(T^q)^{\perp}}) \) is an \( m \)-isometric tuple.

**Remark 2.5.** It easy to see that every \( (m; (1, 1, \ldots, 1)) \)-partial isometry \( d \)-tuple of commuting operators is an \( (m; (q_1, \ldots, q_d)) \)-partial isometry \( d \)-tuple.

In the following theorem we show that by imposing certain conditions on \( (m; (q_1, \ldots, q_d)) \)-partial isometry operator it becomes \( m \)-partial isometry.

**Theorem 2.2.** If \( T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d \) is an joint \( (m; (q_1, \ldots, q_d)) \)-partial isometry such that \( N(T_j) = N(T_j^2) \) for each \( j, \ 1 \leq j \leq d \), then \( T \) is an joint \( (m; (1, \ldots, 1)) \)-partial isometry.

**Proof.** By the assumption we have for \( j = 1, \ldots, d \) that \( N(T_j) = N(T_j^n) \) for all positive integer \( n \). It follows that

\[ T^q \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\ast} \right) = 0 \]
implies
\[ \prod_{1 \leq j \leq d} T_j \left( \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^{*\alpha} T^\alpha \right) = 0. \]

The following proposition generalized Proposition 3.1 in [24].

**Proposition 2.1.** If \( T = (T_1, T_2, \ldots, T_d) \in B(H)^d \) is a jointly quasinormal and an joint \((m; (1, \ldots, 1))\)-partial isometry, then \( T \) is a joint \((1; (1, \ldots, 1))\)-partial isometry.

**Proof.** Since \( T = (T_1, T_2, \ldots, T_d) \) is a matricially quasinormal and an joint \((m; (1, \ldots, 1))\)-partial isometry, it follows that
\[ \prod_{1 \leq j \leq d} T_j \left( I - \sum_{1 \leq j \leq d} T_j^* T_j \right)^m = 0. \]

A straightforward computation using this last equation yields that
\[ \prod_{1 \leq j \leq d} T_j \left( I - \sum_{1 \leq j \leq d} T_j^* T_j \right) = 0. \]

The proof is complete.

**Definition 2.2.** Let \( T = (T_1, T_2, \ldots, T_d) \) and \( S = (S_1, S_2, \ldots, S_d) \) are two commuting \( d \)-tuple on of operators on a common Hilbert space \( H \). We said that \( S \) is unitary equivalent to \( T \) if there exists an unitary operator \( V \in B(H) \) such that
\[ S = (S_1, S_2, \ldots, S_d) = (V^* T_1 V, V^* T_2 V, \ldots, V^* T_d V). \]

**Proposition 2.2.** Let \( T = (T_1, T_2, \ldots, T_d) \) and \( S = (S_1, S_2, \ldots, S_d) \in B(H)^d \) are two commuting \( d \)-tuple of operators such that \( S \) is unitary equivalent to \( T \) , then \( T \) is a joint \((m, (q_1, q_2, \ldots, q_d))\)-partial isometry if and only if \( S \) is a joint \((m, (q_1, q_2, \ldots, q_d))\)-partial isometry.

**Proof.** Suppose that \( S \) and \( T \) are unitary equivalent,that is there exists a unitary operator \( V \in B(H) \) such that
\[ S_j = V^* T_j V \quad (1 \leq j \leq d). \]

Since \( T_i T_j = T_j T_i \); it follows that
\[ (V^* T_j V)(V^* T_i V) = (V^* T_i V)(V^* T_j V) \quad \text{for all} \quad 1 \leq i, j \leq d. \]

Using the observations above, we get the following identity
\[ S^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} S^{\alpha} S^{\alpha} = V^* T^q V \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} V^* T^{\alpha} T^{\alpha} V \]
\[ = V^* \left( T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} V^* T^{\alpha} T^{\alpha} \right) V \]
In the proof of the following theorem, we need the following formula

**Remark 2.6.** For \( n, d, k_1, k_2, \ldots, k_d \in \mathbb{N} \) with \( k_1 + \ldots + k_d = n \), \( n \geq 1 \) and \( d \geq 2 \), we have

\[
\binom{n}{k_1 \ldots k_d} = \sum_{1 \leq j \leq d} \binom{n-1}{k_1 \ldots k_j - 1 \ldots k_d}.
\]

**Theorem 2.3.** Let \( T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d \) be an \((m; (q_1, q_2, \ldots, q_d))\)-partial isometry \( d \)-tuple of operators such that \( \mathcal{N}(T^q) \) is an \((m + 1; (q_1, q_2, \ldots, q_d))\)-partial isometry \( d \)-tuple for \( n \in \mathbb{N} \).

Then \( T \) is an \((m + n; (q_1, q_2, \ldots, q_d))\)-partial isometry \( d \)-tuple.

**Proof.** To prove that \( T \) is an \((m + n; (q_1, q_2, \ldots, q_d))\)-partial isometry, it suffices to prove that \( T \) is an \((m + 1; (q_1, q_2, \ldots, q_d))\)-partial isometry.

Indeed, we have

\[
\sum_{0 \leq k \leq m+1} (-1)^k \binom{m+1}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \|T^\alpha T^* q x\|^2 = \|T^q x\|^2 + \sum_{1 \leq k \leq m} (-1)^k \binom{m}{k} + \binom{m}{k-1} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \|T^\alpha T^* q x\|^2 - (-1)^m \sum_{|\alpha| = m+1} \frac{(m+1)!}{\alpha!} \|T^\alpha T^* q x\|^2
\]

\[
= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!(\alpha_1 + \ldots + \alpha_d)}{\alpha_1! \alpha_2! \ldots \alpha_d!} \|T^\alpha T^* q x\|^2
\]

\[
= (-1)^m \sum_{|\alpha| = m+1} \frac{m!(\alpha_1 + \ldots + \alpha_d)}{\alpha_1! \alpha_2! \ldots \alpha_d!} \|T^\alpha T^* q x\|^2
\]

\[
= (-1)^m \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k+1} \frac{k! \alpha_j}{\alpha_1! \alpha_2! \ldots \alpha_d!} \|T^{\alpha_1} \ldots T_{j}^{\alpha_j-1} T_{j+1}^{\alpha_{j+1}} \ldots T_d^{\alpha_d} T_j T^* q x\|^2
\]

\[
= (-1)^m \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \frac{m! \alpha_j}{\alpha_1! \alpha_2! \ldots \alpha_d!} \|T^{\alpha_1} \ldots T_{j}^{\alpha_j-1} T_{j+1}^{\alpha_{j+1}} \ldots T_d^{\alpha_d} T_j T^* q x\|^2
\]

\[
= (-1)^m \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \frac{m! \alpha_j}{\beta!} \|T^\beta T_j T^* q x\|^2
\]

\[
= (-1)^m \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \frac{k!}{\beta!} \|T^\beta T_j T^* q x\|^2
\]

\[
= 0.
\]

This completes the proof. \( \square \)
Proposition 2.3. Let \( T = (T_1, ..., T_d) \in \mathcal{B}(\mathcal{H})^d \) be an commuting \( d \)-tuple of operators such that \( \mathcal{N}(T_i) \) is a reducing subspace for \( T_j \) for all \( j = 1, 2, ..., d \). If \( T \) is a joint \((m + 1; (q_1, q_2, ..., q_d))\)-partial isometry and an joint \((m; (q_1, q_2, ..., q_d))\)-partial isometry on \( \bigcap_{1 \leq j \leq d} \mathcal{R}(T_j) \), then \( T \) is a joint \((m; (q_1, q_2, ..., q_d))\)-partial isometry on \( \mathcal{H} \).

Proof. A simple computation shows that

\[
\left( -1 \right)^k \binom{m + 1}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} ||T^\alpha T^q x||^2 = \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} ||T^\alpha T^q x||^2 - \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} ||T^\alpha_j T^q T^x||^2.
\]

Thus complete the proof by invoking Corollary 2.1. \( \square \)

Proposition 2.4. Let \( T = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d \) be an commuting \((m; (q_1, q_2, ..., q_d))\)-partial isometry \( d \)-tuple. Then \( T \) is an commuting \((m + 1; (q_1, q_2, ..., q_d))\)-partial isometry \( d \)-tuple if and if \( T \) satisfy the following identity

\[
\sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} ||T^\alpha T_j^q x||^2 = 0 \text{ for all } x \in \mathcal{H}. \tag{2.1}
\]

Proof. Assume that \( T \) is an commuting \((m; (q_1, q_2, ..., q_d))\)-partial isometry \( d \)-tuple and an \((m + 1; (q_1, q_2, ..., q_d))\)-partial isometry \( d \)-tuple. In this case, we get

\[
0 = T^q \sum_{0 \leq k \leq m + 1} \left( -1 \right)^k \binom{m + 1}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha T^q = \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha T^q - T^q \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} T^\beta_j T^q T^\beta T_j.
\]

Then, we obtain that

\[
T^q \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} T^\beta_j T^q T^\beta T_j = 0,
\]

and hence

\[
\sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} ||T^\beta T_j^q T^q||^2 = 0 \text{ for all } x \in \mathcal{H}.
\]

Conversely assume that \( T \) is an \((m; q_1, q_2, ..., q_d)\)-partial isometry \( d \)-tuple satisfy (2.1).

From equation (2.1) it follows that

\[
0 = T^q \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\beta| = k} \frac{k!}{\beta!} T^\beta_j T^q T^\beta T_j = T^q \sum_{0 \leq k \leq m} \left( -1 \right)^k \binom{m}{k} \sum_{|\alpha| = k + 1} \frac{(k + 1)!}{\alpha!} T^\alpha T^q.
\]
On the other hand, we have that

\[
T^q \sum_{0 \leq k \leq m+1} (-1)^k \binom{m+1}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\alpha} = T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} T^{\alpha} T^{\alpha} - T^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} \frac{(k+1)!}{\alpha!} T^{\alpha} T^{\alpha} = 0.
\]

The proof is complete. \(\square\)

# 3 SPECTRAL PROPERTIES OF A JOINT

\((m; (q_1, \ldots, q_d))-\text{PARTIAL ISOMETRIES d-TUPLES}\)

Spectral properties of commuting d-tuples received important attention during last decades. Systematic investigations have been carried out to extend known results for single operators to commuting n-tuples. For more details, the interested reader is referred to [6], [10], [11], [12], [13], [14], [25], [27] and the references therein.

First, we recapitulate very briefly the following definitions.

**Definition 3.1.** Let \(T = (T_1, T_2, \ldots, T_d)\) be an d-tuple of operators on a complex Hilbert space \(\mathcal{H}\).

1. A point \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d\) is called a point eigenvalue of \(T\) if there exists a non zero vector \(x \in \mathcal{H}\) such that

\[(T_j - \lambda_j)x = 0 \quad \text{for} \quad j = 1, 2, \ldots, d.\]

Or equivalently if there exists a non-zero vector \(x \in \mathcal{H}\) such that \(x \in \bigcap_{1 \leq j \leq d} \mathcal{N}(T_j - \lambda_j)\), i.e.;

\[\sigma_p(T) = \{\lambda \in \mathbb{C}^d : \bigcap_{1 \leq j \leq d} \mathcal{N}(T_j - \lambda_j) \neq \{0\}\}.
\]

2. The joint point spectrum, denoted by \(\sigma_p(T)\) of \(T\) is the set of all joint eigenvalues of \(T\).

**Definition 3.2.** For a commuting d-tuple \(T = (T_1, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\). A number \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d\) is in the joint approximate point spectrum \(\sigma_{ap}(T)\) if and only if there exists a sequence \((x_n)_n\) such that

\[(T_j - \lambda_j)x_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{for every} \quad j = 1, \ldots, d.
\]

**Lemma 3.1.** ([17]) Let \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\) be a commuting tuples of bounded operators. Then

\[\sigma_{ap}(T) = \left\{\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \mathbb{C}^d : \exists \ (x_n)_n \subset \mathcal{H} \ such \ that \ \lim_{n \to \infty} \sum_{1 \leq j \leq d} \| (T_j - \lambda_j)x_n \| = 0 \right\}.
\]
Definition 3.3. ([27]) The Taylor spectrum of commuting d-tuple \((T_1, ..., T_d) \in \mathcal{B}(\mathcal{H})^d\) is the set of all complex d-tuple \(\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d\) with the property that the translated d-tuple \((T_1 - \lambda_1, ..., T_d - \lambda_d)\) is not invertible. The symbol \(\sigma(T)\) will stand for the Taylor spectrum of \(T\).

Remark 3.1. ([27]) Let \(T = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d\) be a d-tuple of commuting operators on \(\mathcal{H}\). \((\lambda_1, \lambda_2, ..., \lambda_d) \notin \sigma(T)\) if there exist operators \(U_1, ..., U_d, V_1, ..., V_d \in \mathcal{B}(\mathcal{H})\) such that
\[
\sum_{1 \leq k \leq d} U_k(T_k - \lambda_k I) = I \quad \text{and} \quad \sum_{1 \leq k \leq d} (T_k - \lambda_k I)V_k = I.
\]
The spectral radius of \(T\) is
\[
r(T) = \max\{\|\lambda\|_2, \ \lambda \in \sigma(T)\}
\]
where \(\|\lambda\|_2 = \left(\sum_{1 \leq j \leq d} |\lambda_j|^2\right)^{\frac{1}{2}}\).

Proposition 3.1. ([23], Lemma 3.1.1) Let \(T = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d\) be a commuting tuples of bounded operators. Then the following are equivalent:

1. There exists \(\delta > 0\), such that \(\|T_1x\| + ... + \|T_dx\| \geq \delta\|x\|\) for all \(x \in \mathcal{H}\),
2. There exists \(S = (S_1, ..., S_d) \in \mathcal{H}(\mathcal{H})^d\) such that \(S_1T_1 + S_2T_2 + ... + S_dT_d = I_H\),
3. There is no sequence \((x_n)_n \subset \mathcal{H};\|x_n\| = 1\) such that \(\lim_{n \to \infty} \|T_jx_n\| = 0\) for \(j = 1, 2, ..., d\).

In the following results we examine some spectral properties of a joint \((m; (q_1, ..., q_d))-\)partial isometries. That extend the case of single variable \(m\)-partial isometries studied in [24].

We put
\[
\mathcal{B}(\mathbb{C}^d) := \{\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d / \|\lambda\|_2 = \left(\sum_{1 \leq j \leq d} |\lambda_j|^2\right)^{\frac{1}{2}} < 1\}
\]
and
\[
\partial\mathcal{B}(\mathbb{C}^d) := \{\lambda = (\lambda_1, ..., \lambda_d) \in \mathbb{C}^d / \|\lambda\|_2 = \left(\sum_{1 \leq j \leq d} |\lambda_j|^2\right)^{\frac{1}{2}} = 1\}
\]

In ([17], Lemma 3.2), the authors proved that if \(T\) is a \(m\)-isometric tuple, then the joint approximate point spectrum of \(T\) is in the boundary of the unit ball \(\mathcal{B}(\mathbb{C}^d)\). This is not true for an joint \((m; (q_1, q_2, ..., q_d))-\)partial isometry tuple. For example, on \(\mathcal{B}(\mathbb{C}^2)^d\) the operator \(T = (T_0, 0, ..., 0)\) where \(T\) is the matrix operator \(T = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\) with \(|a|^2 = \frac{1 + \sqrt{5}}{2}\) is an \((2; (1, 0, 0, ..., 0))-\)partial isometry. It is clear that with \(\sigma(S) = \{0, a\} \times \{0\} \times ... \times \{0\}\).

However, if in addition assume that \(T_j\) reduces \(\mathcal{N}(T^q)\) for \(1 \leq j \leq d\), we obtain the following result.
Theorem 3.1. Let $\mathbf{T} = (T_1, T_2, ..., T_d) \in \mathcal{B} (\mathcal{H})^d$ be an joint $(m; (q_1, ..., q_d))$-partial isometry of $d$-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each $T_j$ $(1 \leq j \leq d)$. Then $\sigma_{ap}(\mathbf{T}) \subset \partial \mathcal{B}(\mathbb{C}^d) \cup \{0\}$ where

$$[0] := \{ (\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{C}^d : \prod_{1 \leq k \leq d} \lambda_k = 0 \}.$$ 

Proof. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_d) \in \sigma_{ap}(\mathbf{T})$, then there exists a sequence $(x_n)_{n \geq 1} \subset \mathcal{H}$, with $||x_n|| = 1$ such that $(T_j - \lambda_j I)x_n \rightarrow 0$ for all $j = 1, 2, ..., d$. Since for $\alpha_j > 1$,

$$T_j^{\alpha_j} - \lambda_j^{\alpha_j} = (T_j - \lambda_j) \sum_{1 \leq k \leq \alpha_j} \lambda_j^{k-1} T_j^{-\alpha_j-k}$$

By induction, for $\alpha \in \mathbb{Z}_+^d$, we have

$$(T^\alpha - \lambda^\alpha I) = \sum_{1 \leq k \leq d} \left( \prod_{i \leq k} \lambda_i^{\alpha_i} \right) \left( T_j^{\alpha_j} - \lambda_j^{\alpha_j} \right) \prod_{i > k} T_i^{\alpha_i}.$$ 

Since, $\mathcal{R}(\mathbf{T}^q) \subset \mathcal{N}(\mathbf{T}^q)^\perp$ we have from Corollary 2.1 that, for all $n \geq 1$

$$0 = \lambda^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha \mathbf{T}^q x_n, x_n \right)$$

$$= \lambda^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^\alpha \mathbf{T}^q (\mathbf{T}^q - \lambda^q) x_n, x_n \right) + \lambda^{2q} \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} ||T^\alpha x_n||^2$$

$$= \lambda^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} T^\alpha \mathbf{T}^q (\mathbf{T}^q - \lambda^q) x_n, x_n \right)$$

$$+ \lambda^{2q} \left\{ \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \left( ||(T^\alpha - \lambda^\alpha) x_n||^2 + 2 \text{Re} \langle (T^\alpha - \lambda^\alpha) x_n, \lambda^\alpha x_n \rangle + ||\lambda^\alpha||^2 \right) \right\}$$

as $(\mathbf{T}^\alpha - \lambda^\alpha) x_n \rightarrow 0$ as $n \rightarrow \infty$ for all $\alpha \in \mathbb{Z}_+^d$ we obtain that

$$0 = \lambda^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} ||\lambda^\alpha||^2 = \lambda^q (1 - ||\lambda||_2^2)^m,$$

where $|\lambda| = \left( \sum_{1 \leq k \leq d} |\lambda_k|^2 \right)^{\frac{1}{2}}$. Then $\lambda^q = 0$ or $||\lambda||_2 = 1$. This implies that

$$\lambda \in \{ (\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{C}^d : \prod_{1 \leq k \leq d} \lambda_k = 0 \} \text{ or } \lambda \in \partial \mathcal{B}(\mathbb{C}^d).$$

\[ \square \]

Corollary 3.1. If $\mathbf{T} = (T_1, T_2, ..., T_d) \in \mathcal{B} (\mathcal{H})^d$ is an $(m; (q_1, q_2, ..., q_d))$-partial isometry $d$-tuple of operators such that $\mathcal{N}(\mathbf{T}^q)$ is a reducing subspace for each $T_j$ $(1 \leq j \leq d)$. Then $r(\mathbf{T}) = 1$. In particular $\sigma(\mathbf{T}) \subset \partial \mathcal{B}(\mathbb{C}^d)$ or $\sigma(\mathbf{T}) = \mathbb{P}(\mathbb{C}^d)$. 

Consequently we find \( \sigma(\mathbf{T}) = \partial \mathbb{B}(\mathbb{C}^d) \) or \( \sigma(\mathbf{T}) = \overline{\mathbb{B}(\mathbb{C}^d)} \).

We have also, the following properties.

**Proposition 3.2.** Let \( \mathbf{T} = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d \) be an joint \((m; (q_1, q_2, ..., q_d))-partial isometry d-tuple such that \( \mathcal{N}(\mathbf{T}^q) \) is a reducing subspace for \( T_j \) \( (1 \leq j \leq d) \). The following properties hold.

1. If \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_d) \in \sigma_{ap}(\mathbf{T}) \setminus [0] \) then \( \lambda = (\overline{\lambda_1}, \overline{\lambda_2}, ..., \overline{\lambda_d}) \in \sigma_{ap}(\mathbf{T}^*). \)

2. If \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_d) \in \sigma_p(\mathbf{T}) \setminus [0] \) then \( \lambda \in \sigma_p(\mathbf{T}^*). \)

3. Eigenvectors of \( \mathbf{T} \) corresponding to distinct eigenvalues are orthogonal.

**Proof.** 1. Let \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_d) \in \sigma_{ap}(\mathbf{T}) \setminus \{ (\lambda_1, \lambda_2, ..., \lambda_d) \in \mathbb{C}^d : \prod_{1 \leq k \leq d} \lambda_k = 0 \} \), choose a sequence \( (x_n)_n \subset \mathcal{H} \), such that \( \|x_n\| = 1 \) and \( (T_j - \lambda_j)x_n \to 0 \) for all \( j = 1, 2, ..., d \).

On the other hand

\[
\mathbf{T}^{*\alpha}\mathbf{T}^\alpha(\mathbf{T}^q - \lambda^q)x_n = \mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^q x_n - \lambda^q\mathbf{T}^{*\alpha}\mathbf{T}^\alpha x_n \\
= \mathbf{T}^{*\alpha}\mathbf{T}^\alpha\mathbf{T}^q x_n - \lambda^q\mathbf{T}^{*\alpha}(\mathbf{T}^\alpha - \lambda^\alpha)x_n + \lambda^q\mathbf{T}^{*\alpha}\lambda^\alpha x_n \to 0.
\]

Since \( \mathbf{T} \) is an joint \((m; (q_1, q_2, ..., q_d))-partial isometry\), we observe that

\[
\lambda^q \sum_{1 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} (\mathbf{\mathbf{T}^\alpha})^\alpha x_n \to 0.
\]

and hence,

\[
\lambda^q \left( I - \sum_{1 \leq j \leq d} \lambda_j T_j^* \right)^m x_n \to 0,
\]

Using the fact that \( \lambda = (\lambda_1, \lambda_2, ..., \lambda_d) \in \sigma_{ap}(\mathbf{T}) \setminus [0] \), we get

\[
\left( \sum_{1 \leq j \leq d} |\lambda_j|^2 I - \sum_{1 \leq j \leq d} \lambda_j T_j^* \right)^m x_n \to 0,
\]

or equivalently

\[
\left( \sum_{1 \leq j \leq d} \lambda_j (\overline{\lambda_j} - T_j^*) \right)^m x_n \to 0.
\]
We deduce that
\[ \sum_{1 \leq j \leq d} \lambda_j I_{\mathcal{H}}. (\overline{\lambda}_j - T_j^*) \]
is not bounded below and in view of Proposition 3.1, it follows that there exists a
sequence \((x_n)_n \subset \mathcal{H}\) such that \(\|x_n\| = 1\) and
\[ \lim_{n \to \infty} \|(T_j^* - \overline{\lambda}_j)x_n\| = 0 \quad \text{for} \ j = 1, 2, \ldots, d. \]
So we get \((\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_d) \in \sigma_{ap}(T^*)\) and the proof of this implication is over.

2. Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d) \in \sigma_p(T) \setminus \{0\}\), there exists a non zero vector \(x \in \mathcal{H}\) such that
\[ T_jx = \lambda_jx \quad \text{for} \ j = 1, 2, \ldots, d. \]
By using a similar argument as in 1 we show \((T_j^* - \overline{\lambda}_j)I = 0\) for \(j \ (1 \leq j \leq d)\) from which it follows that \(\overline{\lambda} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_d) \in \sigma_p(T^*).\)

3. Let \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_d)\) and \(\mu = (\mu_1, \mu_2, \ldots, \mu_d)\) be distinct eigenvalues of \(T\). Assume that
\[ T_jx = \lambda_jx \quad \text{and} \quad T_jy = \mu_jy \quad \text{for} \ j = 1, 2, \ldots, d. \]
Then
\[
0 = \lambda^q \left\langle \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} T^{*\alpha}T^{\alpha}x, y \right\rangle \\
= \lambda^q \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \langle \lambda, \overline{\mu} \rangle^\alpha \langle x, y \rangle \\
= \left(1 - \sum_{1 \leq j \leq d} \lambda_j \overline{\mu}_j \right)^m \langle x, y \rangle.
\]
where \(\lambda, \overline{\mu} = (\lambda_1 \overline{\mu}_1, \lambda_2 \overline{\mu}_2, \ldots, \lambda_d \overline{\mu}_d).\)

Since \(1 - \sum_{1 \leq j \leq d} \lambda_j \overline{\mu}_j \neq 0\), we obtain that \(\langle x | y \rangle = 0. \)

\[\square\]

**Lemma 3.2.** Let \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\) be an joint \((m; (q_1, q_2, \ldots, q_d))-\)partial isometry such that \(N(T^q)\) is a reducing subspace for \(T_j, j = 1, \ldots, d\). Let \(\lambda = (\lambda_1, \ldots, \lambda_d)\) and \(\mu = (\mu_1, \ldots, \mu_d) \in \sigma_{ap}(T)\) such that \(\lambda - \mu \notin [0].\) If \((x_n)_n\) and \((y_n)_n\) are two sequences of unit vectors in \(\mathcal{H}\) such that
\[ \|(T_j - \lambda_j)x_n\| \to 0 \quad \text{and} \quad \|(T_j - \mu_j)y_n\| \to 0 \quad \text{as} \ n \to \infty \quad \text{for all} \ j = 1, 2, \ldots, d, \]
then we have
\[ \langle x_n | y_n \rangle \to 0 \quad \text{as} \ n \to \infty. \] (3.1)

**Proof.** Assume that \(\mu \notin [0]\). Then from part 1. of Proposition 3.2 we have that \(\|(T_j^* - \overline{\mu}_j)y_n\| \to 0\) as \(n \to \infty, \ j = 1, 2, \ldots, d.\) Hence, for all \(j = 1, 2, \ldots, d\)
\[ (\lambda_j - \mu_j)\langle x_n | y_n \rangle = -\langle (T_j - \lambda_j)x_n | y_n \rangle + \langle x_n | (T_j - \mu_j)^*y_n \rangle \to 0, \ n \to \infty, \]
which implies (3.1) in view of \(\lambda - \mu \notin [0]\) and the proof is complete. \[\square\]
4 JOINT LEFT $m$-INVERSE AND JOINT RIGHT $m$-INVERSE OF TUPLE OF OPERATORS

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be left invertible if there is an operator $S \in \mathcal{B}(\mathcal{H})$ such that $ST = I_{\mathcal{H}}$, where $I_{\mathcal{H}}$ denotes the identity operator. The operator $S$ is called a left inverse of $T$. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be right invertible if there is an operator $R \in \mathcal{B}(\mathcal{H})$ such that $TR = I_{\mathcal{H}}$. The operator $R$ is called a right inverse of $T$.

The left and right $m$-invertibility of operator have been introduced by the present author in [22] and by B.P. Duggal and V. Müller in [15].

Given a positive integer $m$. A bounded linear operator $T$ is called left $m$-invertible (resp. right $m$-invertible) if there exists a bounded linear operator $S$ such that

$$
\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} S^k T^k = 0 \quad (\text{resp.} \quad \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} T^k S^k = 0).
$$

The $m$-invertibility have been extensively studied in the recent paper [19] by C. Gu.

The following definition generalize the definition of left $m$-invertibility and right $m$-invertibility of a single operator to tuple of operators.

**Definition 4.1.** Let $T = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d$ be an commuting $d$-tuple of operators on $\mathcal{H}$, we say that $T$ is a joint left $m$-invertible (resp. joint right $m$-invertible) for some integer $m \geq 1$, if there exists a commuting $d$-tuple operators $S = (S_1, S_2, ..., S_d) \in \mathcal{B}(\mathcal{H})^d$ such that

$$
\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|= k} \frac{k!}{\alpha!} S^\alpha T^\alpha = 0
$$

(resp. $\sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|= k} \frac{k!}{\alpha!} T^\alpha S^\alpha = 0$).

$S$ is called a left (resp. right) $m$-inverse of $T$.

We say that $T = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d$ is $m$-invertible $d$-tuple of commuting operators if it has both a left $m$-inverse and a right $m$-inverse.

An interesting example of a left $m$-invertible commuting tuple operator is that of an $m$-isometric tuple operator.

**Remark 4.1.** It is clear that $S$ is a left $m$-inverse of $T$ if and only if $T^*$ is a left $m$-inverse of $S^*$.

**Example 4.1.** Let $T_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S_1 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$. Then the pair $T = (T_1, T_1)$ is $m$-invertible tuple with $m$-inverse $S = (S_1, S_2)$ in $\mathcal{B}(\mathbb{C}^2)^2$.

**Remark 4.2.** 1. $S = (S_1, S_2, ..., S_d)$ is a joint left inverse (or 1-inverse) of $T = (T_1, T_2, ..., T_d)$ if and only if

$$
S_1 T_1 + S_2 T_2 + ... + S_d T_d = I_{\mathcal{H}}.
$$
Lemma 4.1. \[ \beta_m(S, T) = \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} S^\alpha T^\alpha. \]

II \( T = (T_1, T_2, \ldots, T_d) \) and \( S = (S_1, S_2, \ldots, S_d) \) are commuting \( d \)-tuples of operators, then we have the following equality

\[ \beta_{m+1}(S, T) = -\beta_m(S, T) + \sum_{1 \leq j \leq d} S_j \beta_m(S, T) T_j \]

Proof.

\[
\begin{align*}
\beta_{m+1}(S, T) &= (-1)^{m+1} I_H + \sum_{1 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} S^\alpha T^\alpha + \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} S^\alpha T^\alpha \\
&= \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\alpha|-k} \frac{k!}{\alpha!} S^\alpha T^\alpha + \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|-k} \frac{(k+1)!}{\alpha!} S^\alpha T^\alpha \\
&= -\beta_m(S, T) + \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\alpha|=k+1} \frac{k!(\alpha_1 + \ldots + \alpha_d)}{\alpha_1! \alpha_2! \ldots \alpha_d!} S^\alpha T^\alpha \\
&+ \sum_{|\alpha|=m+1} \frac{m!(\alpha_1 + \ldots + \alpha_d)}{\alpha_1! \alpha_2! \ldots \alpha_d!} S^\alpha T^\alpha \\
&= -\beta_m(S, T) + \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} S_j S^\beta T^\beta T_j \\
&+ \sum_{1 \leq j \leq d} \sum_{|\alpha|=m+1} \frac{m!}{\alpha_1! \alpha_2! \ldots \alpha_d!} S_j S^\alpha T^\alpha T_j \\
&= -\beta_m(S, T) + \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m-1} (-1)^k \binom{m}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} S_j S^\beta T^\beta T_j \\
&+ \sum_{1 \leq j \leq d} \sum_{|\alpha|=m} \frac{m!}{\beta!} S_j S^\beta T^\beta T_j \\
&= -\beta_m(S, T) + \sum_{1 \leq j \leq d} \sum_{0 \leq k \leq m} (-1)^k \binom{m}{k} \sum_{|\beta|=k} \frac{k!}{\beta!} S_j S^\beta T^\beta T_j \\
&= -\beta_m(S, T) + \sum_{1 \leq j \leq d} S_j \beta_m(S, T) T_j.
\]
For \( k, n \in \mathbb{N} \) denote the (descending Pochhammer) symbol by \( n^{(k)} \), i.e.

\[
n^{(k)} = \begin{cases} 
0, & \text{if } n = 0 \\
0 & \text{if } n > 0 \text{ and } k > n \\
\binom{n}{k}k! & \text{if } n > 0 \text{ and } k \leq n.
\end{cases}
\]

**Proposition 4.1.** Let \( S = (S_1, S_2, ..., S_d) \in \mathcal{B}(\mathcal{H})^d \) and \( T = (T_1, T_2, ..., T_d) \in \mathcal{B}(\mathcal{H})^d \) are commuting operators. Then the following properties hold:

1. \[
\sum_{|\alpha|=n} \frac{n!}{\alpha!} S^\alpha T^\alpha = \sum_{0 \leq k \leq n} n^{(k)} \beta_k(S, T), \quad \text{for all } n = 0, 1, ....
\]

2. If \( S \) is an left \( m \)-inverse of \( T \), then

\[
\sum_{|\alpha|=n} \frac{n!}{\alpha!} S^\alpha T^\alpha = \sum_{0 \leq k \leq m-1} n^{(k)} \beta_k(S, T), \quad \text{for all } n = 0, 1, ....
\]

**Proof.**

1. We prove the statement by indication on \( n \). For \( n = 0, 1 \) the statement is true. Suppose that the statement is true for \( n \).

Form the identity

\[
\beta_{n+1}(S, T) = \sum_{0 \leq k \leq n+1} (-1)^{n+1-k} \binom{n+1}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} S^\alpha T^\alpha
\]

it follows that

\[
\sum_{|\alpha|=n+1} \frac{(n+1)!}{\alpha!} S^\alpha T^\alpha = \beta_{n+1}(S, T) - \sum_{0 \leq k \leq n} (-1)^{n+1-k} \binom{n+1}{k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} S^\alpha T^\alpha
\]

By the assumption and similar calculation as in [5] we obtained

\[
\sum_{|\alpha|=n+1} \frac{(n+1)!}{\alpha!} S^\alpha T^\alpha = \beta_{n+1}(S, T) - \sum_{0 \leq k \leq n} (-1)^{n+1-k} \binom{n+1}{k} \sum_{0 \leq j \leq k} k^{(j)} \beta_j(S, T)
\]

\[
= \sum_{0 \leq k \leq n+1} \binom{n+1}{k} \beta_k(S, T).
\]

2. The result follows immediately from the fact that if \( S \) is a left \( m \)-inverse of \( T \) then \( \beta_k(S, T) = 0 \) for all \( k \geq m \) (see Lemma 4.1).
Remark 4.3. If \( T = (T_1, T_2) \in \mathcal{B}(\mathcal{H})^2 \) is an left 2-invertible with the left 2-inverse \( S = (S_1, S_2) \in \mathcal{B}(\mathcal{H})^2 \), then
\[
\sum_{\alpha_1 + \alpha_2 = n} \frac{n!}{\alpha_1! \alpha_2!} S^\alpha T^\alpha = n(S_1 T_1 + S_2 T_2) - (n - 1) I_H
\]

Theorem 4.1. Let \( T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d \). If \( T \) possesses a left \( m \)-inverse \( S = (S_1, S_2, \ldots, S_d) \in \mathcal{B}(\mathcal{H})^d \), then the following statements hold:

1. \( [0] \not\in \sigma_{ap}(T) \),
2. If \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(T) \), then \( \left( \frac{1}{d, \lambda_1}, \ldots, \frac{1}{d, \lambda_d} \right) \in \sigma_{ap}(S) \),
3. If \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_p(T) \), then \( \left( \frac{1}{d, \lambda_1}, \ldots, \frac{1}{d, \lambda_d} \right) \in \sigma_p(S) \).

Proof. (1) Suppose contrary to our claim that \( [0] \subset \sigma_{ap}(T) \) and let \( \lambda = (\lambda_1, \ldots, \lambda_d) \in [0] \). Then there exists a sequence \((x_n)_n \in \mathcal{H}\) such that
\[
\|x_n\| = 1 \quad \text{and} \quad (T_j - \lambda_j)x_n \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty \quad \text{for} \quad j = 1, 2, \ldots, d.
\]
For \( \alpha_j \geq 1 \) we deduce that
\[
(T_j^{\alpha_j} - \alpha_j^{\alpha_j})x_n \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty \quad \text{for} \quad j = 1, 2, \ldots, d
\]
which mean that \((T^\alpha - \lambda^\alpha)x_n \rightarrow 0\) and hence, \(\left(S^\alpha T^\alpha - \lambda^\alpha S^\alpha\right)x_n \rightarrow 0\).

Now, we get
\[
\left(S^\alpha T^\alpha - \lambda^\alpha S^\alpha\right)x_n \rightarrow 0 \quad \Rightarrow \quad \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \left(S^\alpha T^\alpha - \lambda^\alpha S^\alpha\right)x_n \rightarrow 0
\]
\[
\Rightarrow \quad (-1)^m x_n + \sum_{1 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \left(\prod_{1 \leq j \leq d} \lambda_j^{\alpha_j}\right)S^\alpha x_n \rightarrow 0
\]
\[
\Rightarrow \quad x_n \rightarrow 0 \quad \text{as} \quad n \rightarrow 0 \quad \text{(since} \quad \lambda \in [0]),
\]
which is impossible.

(2) Let \( \lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(T) \), then there exists a sequence \((x_n)_n \in \mathcal{H}\) such that
\[
\|x_n\| = 1 \quad \text{and} \quad (T_j - \lambda_j)x_n \rightarrow 0 \quad \text{as} \quad n \rightarrow +\infty \quad \text{for} \quad j = 1, 2, \ldots, d.
\]
\[
\left(S^\alpha T^\alpha - \lambda^\alpha S^\alpha\right)x_n \rightarrow 0 \quad \Rightarrow \quad \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \left(S^\alpha T^\alpha - \lambda^\alpha S^\alpha\right)x_n \rightarrow 0
\]
\[
\Rightarrow \quad \sum_{0 \leq k \leq m} (-1)^{m-k} \binom{m}{k} \sum_{|\alpha| = k} \frac{k!}{\alpha!} \left(\prod_{1 \leq j \leq d} \lambda_j^{\alpha_j}\right)S^\alpha x_n \rightarrow 0
\]
\[
\Rightarrow \quad \left(I_H - \sum_{1 \leq j \leq d} \lambda_j S_j\right)x_n \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty.
\]
On the other hand, we have
\[
\left( I_{\mathcal{H}} - \sum_{1 \leq j \leq d} \lambda_j S_j \right)^m x_n \to 0 \iff \left( \sum_{1 \leq j \leq d} \lambda_j \left( \frac{1}{d} - S_j \right) \right)^m x_n \to 0.
\]

From Proposition 3.1, it follows that there exists a sequence \((x_n)_n \subset \mathcal{H}\) such that \(\|x_n\| = 1\) and
\[
\lim_{n \to \infty} \left\| \left( \frac{1}{d} - S_j \right) x_n \right\| = 0.
\]

From this \(\left( \frac{1}{d} \lambda_j \right)_{1 \leq j \leq d} \in \sigma_{ap}(S)\).

(3) The argument is similar to one given in (2). \(\square\)

The proof of the following theorem is similar to the proof of Theorem 4.1, so we omit it.

**Theorem 4.2.** Let \(T = (T_1, T_2, \ldots, T_d) \in \mathcal{B}(\mathcal{H})^d\). If \(T\) possesses a right \(m\)-inverse \(R = (R_1, R_2, \ldots, R_d) \in \mathcal{B}(\mathcal{H})^d\), then the following statements hold:

1. \([0] \not\subset \sigma_{ap}(R)\).

2. If \(\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_{ap}(R)\), then \(\left( \frac{1}{d} \lambda_1, \ldots, \frac{1}{d} \lambda_d \right) \in \sigma_{ap}(T)\).

3. If \(\lambda = (\lambda_1, \ldots, \lambda_d) \in \sigma_p(R)\), then \(\left( \frac{1}{d} \lambda_1, \ldots, \frac{1}{d} \lambda_d \right) \in \sigma_p(T)\).

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