Gauge freedom of entropies on $q$-Gaussian measures

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Abstract. A $q$-Gaussian measure is a generalization of a Gaussian measure. This generalization is obtained by replacing the exponential function with the power function of exponent $1/(1-q)$ ($q \neq 1$). The limit case $q = 1$ recovers a Gaussian measure. For $1 \leq q < 3$, the set of all $q$-Gaussian densities over the real line satisfies a certain regularity condition to define information geometric structures such as an entropy and a relative entropy via escort expectations. The ordinary expectation of a random variable is the integral of the random variable with respect to its law. Escort expectations admit us to replace the law to any other measures. A choice of escort expectations on the set of all $q$-Gaussian densities determines an entropy and a relative entropy. One of most important escort expectations on the set of all $q$-Gaussian densities is the $q$-escort expectation since this escort expectation determines the Tsallis entropy and the Tsallis relative entropy.

The phenomenon gauge freedom of entropies is that different escort expectations determine the same entropy, but different relative entropies. In this note, we first introduce a refinement of the $q$-logarithmic function. Then we demonstrate the phenomenon on an open set of all $q$-Gaussian densities over the real line by using the refined $q$-logarithmic functions. We write down the corresponding Riemannian metric.

Keywords: Information geometry · gauge freedom of entropies · refined $q$-logarithmic function · $q$-Gaussian measure

1 $q$-Logarithmic functions and their refinements

1.1 Definitions
For $q \in \mathbb{R}$, we set $\chi_q : (0, \infty) \to (0, \infty)$ by
$$\chi_q(s) := s^q.$$
We define a strictly increasing function $\ln_q : (0, \infty) \to \mathbb{R}$ by

$$\ln_q(t) := \int_1^t \frac{1}{\chi_q(s)} ds$$

and we denote by $\exp_q$ the inverse function of $\ln_q : (0, \infty) \to \ln_q(0, \infty)$. The functions $\ln_q$ and $\exp_q$ are called the $q$-logarithmic function and the $q$-exponential function, respectively. We observe that

$$\frac{d}{dt} \ln_q(t) = \frac{1}{\chi_q(t)} = t^{-q} \quad \text{for } t \in (0, \infty),$$

$$\frac{d}{d\tau} \exp_q(\tau) = \chi_q(\exp_q(\tau)) = \exp_q(\tau)^q \quad \text{for } \tau \in \ln_q(0, \infty).$$

It holds for $q \in \mathbb{R}$ that $\chi_q(1) = 1$ and $\ln_q(1) = 0$.

**Remark 1.** (1) For $q = 1$, we have that

$$\ln_1(t) = \log(t) \quad \text{for } t \in (0, \infty),$$

$$\ln_1(0, \infty) = \mathbb{R},$$

$$\exp_1(\tau) = \exp(\tau) \quad \text{for } \tau \in \mathbb{R}.$$ 

(2) For $q \neq 1$, we have that

$$\ln_q(t) = \frac{t^{1-q} - 1}{1-q} \quad \text{for } t \in (0, \infty),$$

$$\ln_q(0, \infty) = \begin{cases} (-\infty, \frac{1}{q-1}) & \text{if } q > 1, \\ (-\frac{1}{1-q}, \infty) & \text{if } q < 1, \end{cases}$$

$$\exp_q(\tau) = \{1 + (1-q)\tau\}^{\frac{1}{1-q}} \quad \text{for } \tau \in \ln_q(0, \infty).$$

Taking account into the negativity of $\ln_q$ in $(0, 1)$, we introduce a refinement of the $q$-logarithmic function and the $q$-exponential function. For $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, define two functions $\chi_{q,a} : (0, 1) \to (0, \infty)$ and $\ln_{q,a} : (0, 1) \to \mathbb{R}$ respectively by

$$\chi_{q,a}(s) := \chi_q(s) \cdot (-\ln_q(s))^{1-a}, \quad \ln_{q,a}(t) := -\frac{1}{a}(-\ln_q(t))^a.$$ 

It turns out that

$$\frac{d}{ds} \chi_{q,a}(s) = \chi_q'(s)(-\ln_q(s))^{1-a} - (1-a)(-\ln_q(s))^{-a} \quad \text{for } s \in (0, 1),$$

$$\frac{d}{dt} \ln_{q,a}(t) = \frac{1}{\chi_{q,a}(t)} > 0 \quad \text{for } t \in (0, 1). \quad (1.1)$$
Hence the function \( \ln_{q,a} : (0, 1) \rightarrow \mathbb{R} \) is strictly increasing. We denote by \( \exp_{q,a} \) the inverse function of \( \ln_{q,a} : (0, 1) \rightarrow \ln_{q,a}(0, 1) \), which is given by

\[
\exp_{q,a}(\tau) = \exp_q\left(-(-a\tau)^{\frac{1}{a}}\right) \quad \text{for} \ \tau \in \ln_{q,a}(0, 1). \tag{1.2}
\]

The functions \( \ln_{q,a} \) and \( \exp_{q,a} \) are called the \( a \)-refined \( q \)-logarithmic function and the \( a \)-refined \( q \)-exponential function, respectively.

On one hand, it holds for \( q \geq 1 \) that

\[
\ln_{q,a}(0, 1) = \begin{cases} 
(\infty, 0) & \text{if} \ a > 0, \\
(0, \infty) & \text{if} \ a < 0.
\end{cases}
\]

On the other hand, it holds for \( q < 1 \) that

\[
\ln_{q,a}(0, 1) = \begin{cases} 
\left(-\frac{1}{a}(1-q)^{-a}, 0\right) & \text{if} \ a > 0, \\
\left(-\frac{1}{a}(1-q)^{-a}, \infty\right) & \text{if} \ a < 0.
\end{cases}
\]

Remark 2. (1) The refinement of the ordinary logarithmic function, that is the case \( q = 1 \), was introduced by Ishige, Salani and the second named author [3], where they studied the preservation of concavity by the heat flow in Euclidean space.

(2) For a positive function \( \chi : (0, \infty) \rightarrow (0, \infty) \) and \( a \in \mathbb{R} \setminus \{0\} \), the \( \chi \)-logarithmic function \( \ln_\chi : (0, \infty) \rightarrow \mathbb{R} \) and its refinement \( \ln_{\chi,a} : (0, 1) \rightarrow \mathbb{R} \) are respectively defined in the same way as \( \chi_q \).

1.2 Properties

In this section, we give a condition for \( \ln_{q,a} \) to be concave and compute the higher order derivatives of \( \exp_{q,a} \), which will be used to define information geometric structures.

For \( q \in \mathbb{R} \) and \( a \in \mathbb{R} \setminus \{0\} \), define

\[
t_{q,a} := \begin{cases} 
0 & \text{if either} \ q > 0 \text{ or} \ q = 0 \text{ with} \ a - 1 > 0, \\
1 & \text{if} \ q \leq 0 \text{ with} \ a - 1 \leq 0, \\
\frac{1}{\exp_q\left(\frac{1-a}{q}\right)} & \text{otherwise},
\end{cases}
\]

\[
T_{q,a} := \begin{cases} 
0 & \text{if} \ q > 1 \text{ with} \ 1 - a \geq \frac{q}{q-1}, \\
1 & \text{if} \ q \leq 0, \\
\frac{1}{\exp_q\left(\max\left\{0, \frac{1-a}{q}\right\}\right)} & \text{otherwise},
\end{cases}
\]

and set \( I_{q,a} := (t_{q,a}, T_{q,a}) \). Note that \( I_{q,a} \) is nonempty if and only if one of the following three conditions holds:
$\bullet$ $q > 1$ with $1 - a < \frac{q}{q - 1}$;

$\bullet$ $0 < q \leq 1$;

$\bullet$ $q \leq 0$ with $a - 1 > 0$.

**Proposition 1.** Fix $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$. For an interval $I \subset (0, 1)$, the strict concavity of $\ln_{q,a}$ in $I$ is equivalent to the strict convexity of $\exp_{q,a}$ in $\ln_{q,a}(I)$. Moreover, if $I_{q,a} \neq \emptyset$, then $\ln_{q,a}$ is strictly concave in $I_{q,a}$.

**Proof.** Due to Equation (1.1), $\ln_{q,a}$ is strictly increasing in $(0, 1)$ and so is $\exp_{q,a}$ in $\ln_{q,a}(0, 1)$. Fix an interval $I \subset (0, 1)$. For $t_i \in I$, $\tau_i \in \ln_{q,a}(I)$ ($i = 0, 1$) with

$$
\tau_i = \ln_{q,a}(t_i) \quad \text{or equivalently} \quad t_i = \exp_{q,a}(\tau_i)
$$

and $\lambda \in (0, 1)$, it follows from the continuity of $\ln_{q,a}$ that

$$(1 - \lambda)t_0 + \lambda t_1 \in I, \quad (1 - \lambda)\tau_0 + \lambda \tau_1 \in \ln_{q,a}(I).$$

We observe from the monotonicity of $\ln_{q,a}$ and $\exp_{q,a}$ that

$$\ln_{q,a} \left( (1 - \lambda)t_0 + \lambda t_1 \right) > (1 - \lambda) \ln_{q,a}(t_0) + \lambda \ln_{q,a}(t_1)$$

$\Leftrightarrow$ $\ln_{q,a} \left( (1 - \lambda)t_0 + \lambda t_1 \right) > (1 - \lambda)\tau_0 + \lambda \tau_1$

$\Leftrightarrow$ $\exp_{q,a} \left( \ln_{q,a} \left( (1 - \lambda)t_0 + \lambda t_1 \right) \right) > \exp_{q,a} \left( (1 - \lambda)\tau_0 + \lambda \tau_1 \right)$

$\Leftrightarrow$ $(1 - \lambda)t_0 + \lambda t_1 \in \exp_{q,a}(\ln_{q,a}(I))$,

where we used the fact that $\exp_{q,a}$ is the inverse function of $\ln_{q,a}$. This proves the first claim.

Assume $I_{q,a} \neq \emptyset$. A direct calculation provides that

$$
\frac{d^2}{dt^2} \ln_{q,a}(t) = \frac{d}{dt} \frac{1}{\chi_{q,a}(t)} = -\frac{1}{\chi_{q,a}(t)^2} \frac{d}{dt} \chi_{q,a}(t)
$$

$$= -\frac{(-\ln_{q}(t))^{-a}}{\chi_{q,a}(t)^2} \left\{ \chi'_{q}(t) (-\ln_{q}(t)) - (1 - a) \right\}
$$

$$= \frac{(-\ln_{q}(t))^{-a}}{\chi_{q,a}(t)^2} \left\{ qt^{q-1} \ln_{q}(t) + (1 - a) \right\}.
$$

Notice that $(-\ln_{q}(t))^{-a}/\chi_{q,a}(t)^2$ is positive in $t \in I_{q,a}$. In the case $q = 0$, the condition $I_{0,a} \neq \emptyset$ leads to $a - 1 > 0$, consequently

$$
\frac{d^2}{dt^2} \ln_{0,a}(t) = \frac{(-\ln_{0}(t))^{-a}}{\chi_{0,a}(t)^2} (1 - a) < 0.
$$

Since the function given by

$$t^{q-1} \ln_{q}(t) = -\ln_{q} \left( \frac{1}{t} \right) = \begin{cases} 
\log(t) & q = 1, \\
1 - t^{q-1} & q \neq 1
\end{cases}
$$
Lemma 1. For $q > 0$ and $t \in I_{q,a}$ that
\[
\frac{d^2}{dt^2} \ln_{q,a}(t) = \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \{ q t^{q-1} \ln_g(t) + (1-a) \} \\
< \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \left\{ -q \ln_g \left( \frac{1}{t_{q,a}} \right) + (1-a) \right\} \\
= \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \left\{ -q \cdot \max \left\{ 0, \frac{1-a}{q} \right\} + (1-a) \right\} \\
= \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \left\{ \min \{0, a-1\} + (1-a) \right\} \\
\leq 0.
\]

On the other hand, we see that
\[
\frac{d^2}{dt^2} \ln_{q,a}(t) = \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \{ q t^{q-1} \ln_g(t) + (1-a) \} \\
< \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \left\{ -q \ln_g \left( \frac{1}{t_{q,a}} \right) + (1-a) \right\} \\
= \left( -\frac{\ln_g(t)}{\chi_{q,a}(t)^2} \right) \left\{ -q \cdot \frac{1-a}{q} + (1-a) \right\} \\
= 0
\]
for $q < 0$ and $t \in I_{q,a}$. This completes the proof of the second claim. \qed

Lemma 1. For $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, there exists $\{ b_j^n = b_j^n(q,a) \}_{n \in \mathbb{N}, 0 \leq j \leq n-1}$ such that
\[
\frac{d^n}{d\tau^n} \exp_{q,a}(\tau) = \exp_{q,a}(\tau)^{(n-1)(q-1)+q(-a \tau)^{n(a-1)+1} \sum_{j=0}^{n-1} b_j^n(q,a) \cdot (-a \tau)^{\frac{j}{a}}} \quad (a < 0)
\]
for $\tau \in \ln_{q,a}(0,1)$. Moreover, $\{ b_j^n \}_{n \in \mathbb{N}, 0 \leq j \leq n-1}$ satisfies
\[
b_0^0 = 1, \\
b_j^{n+1} = \begin{cases} 
{na(q-1)+1} b_0^n & \text{if } j = 0, \\
{(na+j)(q-1)+1} b_j^n - {n(1-a) - (j-1)} b_{j-1}^n & \text{if } 1 \leq j \leq n-1, \\
(na-1) b_{n-1}^n & \text{if } j = n.
\end{cases}
\]

Proof. We observe that
\[
\frac{d}{d\tau} \exp_{q,a}(\tau) = \chi_{q,a}(\exp_{q,a}(\tau)) \\
= \chi_q \left( \exp_{q,a}(\tau) \right) \cdot \left\{ -\ln_g \left( \exp_{q,a}(\tau) \right) \right\}^{1-a} \\
= \exp_{q,a}(\tau)^a \cdot (-a \tau)^{\frac{1-a}{a}},
\]
where we used Equation (1.2). Thus the lemma holds for $n = 1$. If the lemma holds for $n$, then we compute that
\[
\frac{d^{n+1}}{d\tau^{n+1}} \exp_{q,a}(\tau)
= \frac{d}{d\tau} \left( \exp_{q,a}(\tau)^{(n-1)(q-1)+q} (-a\tau)^{\frac{n(1-a)}{a}} \sum_{j=0}^{n-1} b_j^n \cdot (-a\tau)^{-\frac{j}{a}} \right)
= \left( \frac{d}{d\tau} \exp_{q,a}(\tau)^{(n-1)(q-1)+q} \right) \times (-a\tau)^{\frac{n(1-a)}{a}} \sum_{j=0}^{n-1} b_j^n \cdot (-a\tau)^{-\frac{j}{a}}
+ \exp_{q,a}(\tau)^{(n-1)(q-1)+q} \times \frac{d}{d\tau} \left( (-a\tau)^{\frac{n(1-a)}{a}} \sum_{j=0}^{n-1} b_j^n \cdot (-a\tau)^{-\frac{j}{a}} \right)
= \{(n-1)(q-1)+q\} \exp_{q,a}(\tau)^{(n-1)(q-1)+q-1} \cdot \exp_{q,a}(\tau)^{q} (-a\tau)^{\frac{n(1-a)}{a}}
\times (-a\tau)^{\frac{n(1-a)}{a}} \sum_{j=0}^{n-1} b_j^n \cdot (-a\tau)^{-\frac{j}{a}}
+ \exp_{q,a}(\tau)^{(n-1)(q-1)+q} \times \left\{ -a \sum_{j=0}^{n-1} \frac{n(1-a) - j}{a} b_j^n \cdot (-a\tau)^{\frac{n(1-a)-j}{a}} \right\}
= \exp_{q,a}(\tau)^{(n-1)(q-1)+q} (-a\tau)^{\frac{n(n+1)(1-a)}{a}}
\times \left[ \{(n-1)(q-1)+q\} \sum_{j=0}^{n-1} b_j^n (-a\tau)^{-\frac{j}{a}}
- \exp_{q,a}(\tau)^{1-q} \sum_{j=0}^{n-1} \{n(1-a) - j\} b_j^n (-a\tau)^{-\frac{j+1}{a}} \right].
\]

We deduce from $\exp_{q,a}(\tau)^{1-q} = 1 - (1-q)(-a\tau)^{\frac{1}{a}}$ that
\[
\exp_{q,a}(\tau)^{1-q} \sum_{j=0}^{n-1} \{n(1-a) - j\} b_j^n (-a\tau)^{-\frac{j+1}{a}}
= \sum_{j=0}^{n-1} \{n(1-a) - j\} b_j^n (-a\tau)^{-\frac{j+1}{a}} - (1-q) \sum_{j=0}^{n-1} \{n(1-a) - j\} b_j^n (-a\tau)^{-\frac{j}{a}}.
\]
This completes the proof of the lemma. \qed

Remark 3. For $q \in \mathbb{R}$ and $a \in \mathbb{R} \setminus \{0\}$, we have that
\[
b_0^0 = 1, \quad b_0^1 = a(q-1) + 1, \quad b_0^2 = 2a(q-1) + 1 \{a(q-1) + 1\}, \quad b_0^3 = \{2a(q-1) + 1\} \{a(q-1) + 1\},
b_1^1 = a-1, \quad b_1^2 = (a-1) \{(4a+1)(q-1) + 3\}, \quad b_1^3 = (a-1)(2a-1).
\]
Corollary 1. For \( a \in \mathbb{R} \setminus \{0\} \) and \( n \in \mathbb{N} \), then \( b_0^n(1, a) = 1 \).

Proof. It follows from Lemma 1 that
\[
\begin{align*}
\tag{16} b_0^{n+1}(1, a) &= \{na(1 - 1) + 1\}b_0^n(1, a) = b_0^n(1, a) = \cdots = b_0^1(1, a) = 1. \quad \Box
\end{align*}
\]

Corollary 2. Let \( q \in \mathbb{R} \) and \( n \in \mathbb{N} \). For \( 1 \leq j < n \), then \( b_j^n(q, 1) = 0 \).

Proof. This holds for \( 1 = j < n = 2 \) by Remark 3. For \( n \geq 2 \), if \( b_j^n(q, 1) = 0 \) holds for \( 1 \leq k \leq n - 1 \), then Lemma 1 implies that
\[
\begin{align*}
\tag{17} b_{j+1}^n(q, 1) &= \{(n + j)(q - 1) + 1\}b_{j}^n(q, 1) + (j - 1)b_{j-1}^n(q, 1) = 0
\end{align*}
\]

by the assumption \( b_k^n(q, 1) = 0 \) for \( 1 \leq k \leq n - 1 \). For \( j = 1 \), we have that
\[
\begin{align*}
\tag{18} b_{1+1}^n(q, 1) &= \{(n + 1)(q - 1) + 1\}b_{1}^n(q, 1) + (1 - 1)b_{0}^n(q, 1) = 0. \quad \Box
\end{align*}
\]

2 Escort expectations

The ordinary expectation of a random variable is the integral of the random variable with respect to its law. An introduction to escort expectations admits us to replace the law to any other measures. The escort expectation with respect to a probability measure was first introduced by Naudts [5].

Definition 1. For a measure \( \nu \) on a measurable space \( \Omega \), the escort expectation of a function \( f \in L^1(\nu) \) with respect to \( \nu \) is defined by
\[
\mathbb{E}_\nu[f] := \int_\Omega f(\omega) d\nu(\omega).
\]

In this section, we fix a manifold \( S \) consisting of positive probability densities on a measure space \( (\Omega, \nu) \). We assume that \( S \) is homeomorphic to an open set \( \Xi \) in \( \mathbb{R}^d \) and we denote each element in \( S \) by \( p(\cdot; \xi) \) for \( \xi \in \Xi \). Namely,
\[
\mathcal{S} = \left\{ p(\cdot; \xi) : \Omega \to (0, \infty) \mid \int_\Omega p(\omega; \xi) d\nu(\omega) = 1, \; \xi \in \Xi \right\}.
\]

We moreover require that \( \mathcal{S} \) satisfies a certain regularity condition to define information geometric structures via escort expectations. For the regularity condition, we refer to [1, Chapter 2].

Remark 4. One of manifolds consisting of probability densities on a measure space satisfying the regular condition is a \( q \)-exponential family, which is a generalization to the space of \( q \)-Gaussian densities over \( \mathbb{R} \) for \( 1 \leq q < 3 \).

Take \( c \in (0, \infty] \) such that
\[
c > \sup \{ p(\omega) \mid p \in \mathcal{S}, \; \omega \in \Omega \}
\]
if the above supremum is finite, otherwise \( c := \infty \).
Definition 2. Let $\ell : (0, c) \to \mathbb{R}$ be a differentiable function such that $\ell' > 0$ in $(0, c)$. For $p \in \mathcal{S}$, we define a measure $\nu_{\ell,p}$ on $\Omega$ as the absolutely continuous measure with respect to $\nu$ with Radon–Nikodym derivative
\[
\frac{d\nu_{\ell,p}}{d\nu}(\omega) = \frac{1}{\ell'(p(\omega))}.
\]
Note that $\ell$ is often assumed to be concave such as the logarithmic function. In the case $\ell = \log$, we have that $\frac{d\nu_{\ell,p}}{d\nu}(\omega) = p$.

Definition 3. Fix a differentiable function $\ell : (0, c) \to \mathbb{R}$ such that $\ell' > 0$ in $(0, c)$ and assume that $\ell(r) = \ell \circ r \in L^1(\nu_{\ell,p})$ for $p, r \in \mathcal{S}$.
\[
(2.1)
\]
(1) For $p, r \in \mathcal{S}$, the $\ell$-cross entropy of $p$ with respect to $r$ is defined by
\[
d_\ell(p, r) := -\mathbb{E}_{\nu_{\ell,p}}[\ell(r)].
\]
(2) The $\ell$-entropy of $p \in \mathcal{S}$ is defined by
\[
\text{Ent}_\ell(p) := d_\ell(p, p).
\]
(3) For $p, r \in \mathcal{S}$, the $\ell$-relative entropy of $p$ with respect to $r$ is defined by
\[
D^{(\ell)}(p, r) := -d_\ell(p, p) + d_\ell(p, r).
\]

Remark 5. In general, the $\ell$-entropy does not satisfy nonextensive Shannon–Khinchin axioms [7]. However, if $\mathcal{S}$ is a manifold of all Gaussian densities over Euclidean space and $\ell = \log$, then the $\ell$-entropy coincides with the Boltzmann–Shannon entropy.

A choice of differentiable functions $\ell : (0, c) \to \mathbb{R}$ such that $\ell' > 0$ in $(0, c)$ determines an entropy and a relative entropy on $\mathcal{S}$. However, the converse is not true. This phenomenon is related to gauge freedom, which was proposed by Zhang and Naudts [8] (see also [6]).

In the next section, we demonstrate gauge freedom of entropies on an open set of $q$-Gaussian densities over $\mathbb{R}$ for $1 \leq q < 3$. To be precise, we show that different escort expectations determine the same entropy up to scalar multiple, but different relative entropies, where the entropy satisfies nonextensive Shannon–Khinchin axioms.

3 Gauge freedom of Entropies

3.1 $q$-Gaussian measures

To define $q$-Gaussian measures, we extend $\exp_q$ to the whole of $\mathbb{R}$ by
\[
\text{Reexp}_q(\tau) := \max\{0, 1 + (1 - q)\tau\}^{1 \over 1-q} \quad \text{for } \tau \in \mathbb{R},
\]
where by convention $0^c := \infty$ for $c < 0$. We recall the following improper integral.
Lemma 2. For \( q \in \mathbb{R} \) and \((\mu, \lambda) \in \mathbb{R} \times (0, \infty)\), the improper integral of the function
\[
x \mapsto \text{Rexp}_q(-\lambda(x-\mu)^2)
\]
on \( \mathbb{R} \) converges if and only if \( q < 3 \). For \( q < 3 \),
\[
\sqrt{3-q} \int_{\mathbb{R}} \text{Rexp}_q(-x^2) dx = Z_q := \begin{cases} \sqrt{3-q} B \left( \frac{3-q}{2(q-1)}, \frac{1}{2} \right) & \text{if } q > 1, \\ \sqrt{2\pi} \sqrt{3-q} B \left( \frac{2-q}{1-q}, \frac{1}{2} \right) & \text{if } q < 1, \\ \sqrt{3-q} \frac{1}{1-q} B \left( \frac{2}{1-q}, \frac{1}{2} \right) & \text{if } q = 1, \end{cases}
\]
where \( B(\cdot, \cdot) \) stands for the beta function.

Proof. By the change of variables, it is enough to show the case \((\mu, \lambda) = (0, 1)\). We omit the proof for the case \( q = 1 \), which is well-known.

Assume \( q \neq 1 \). There exist \( c, C, R > 0 \) depending on \( q \) such that
\[
 cx^{\frac{2}{1-q}} \leq \text{Rexp}_q(-x^2) = \left\{ 1 - (1-q)x^2 \right\}^{\frac{1}{1-q}} < Cx^{\frac{2}{1-q}}
\]
for \( x > R \). Since the improper integral of the function
\[
x \mapsto x^{\frac{2}{1-q}}
\]
on \([1, \infty)\) converges if and only if \( 2/(1-q) < -1 \), that is \( q < 3 \), so does the improper integral of the function \( x \mapsto \text{Rexp}_q(-x^2) \) on \( \mathbb{R} \).

For \( 1 < q < 3 \), we observe that
\[
\int_{\mathbb{R}} \text{Rexp}_q(-x^2) dx = 2 \int_0^\infty \left\{ 1 - (1-q)x^2 \right\}^{\frac{1}{1-q}} dx
\]
\[
= \frac{1}{\sqrt{q-1}} \int_0^\infty \left( 1 + r \right)^{\frac{1}{1-q}} r^{-\frac{3}{2}} dr
\]
\[
= \frac{1}{\sqrt{q-1}} B \left( \frac{3-q}{2(q-1)}, \frac{1}{2} \right),
\]
where we used that
\[
B(s-t, t) = \int_0^\infty \frac{r^{s-1}}{(1+r)^t} dr \quad \text{for } s > t > 0.
\]

In the case \( q < 1 \), the support of the function \( x \mapsto \text{Rexp}_q(-x^2) \) on \( \mathbb{R} \) is
\[
\left[ -\frac{1}{\sqrt{1-q}}, \frac{1}{\sqrt{1-q}} \right]
\]
implying that
\[
\int_{\mathbb{R}} \exp_q(-x^2) \, dx = 2 \int_0^{\sqrt{1-q}} [1 - (1-q)x^2]^{\frac{1}{2}} \, dx
= \frac{1}{\sqrt{1-q}} \int_0^1 [1 - r]^{\frac{1}{2}} r^{-\frac{1}{2}} \, dx
= \frac{1}{\sqrt{1-q}} B\left(\frac{2-q}{1-q}, \frac{1}{2}\right). \quad \Box
\]

**Definition 4.** For \( q < 3 \) and \( \xi = (\mu, \sigma) \in \mathbb{R} \times (0, \infty) \), the \( q \)-Gaussian measure with location parameter \( \mu \) and scale parameter \( \sigma \) on \( \mathbb{R} \) is an absolutely continuous probability measure with respect to the one-dimensional Lebesgue measure with Radon–Nikodym derivative
\[
p_q(x; \xi) = p_q(x; \mu, \sigma) := \frac{1}{Z_q \sigma} \exp_q\left( -\frac{1}{3-q} \left( \frac{x-\mu}{\sigma} \right)^2 \right).
\]
We call \( p_q(x; \xi) = p_q(x; \mu, \sigma) \) the \( q \)-Gaussian density with location parameter \( \mu \) and scale parameter \( \sigma \).

A \( q \)-Gaussian density corresponds to a normal (Gaussian) distribution for \( q = 1 \), and a Student \( t \)-distribution for \( 1 < q < 3 \). In the both cases, the support of each \( q \)-Gaussian measure is the whole of \( \mathbb{R} \) and
\[
p_q(x; \xi) = p_q(x; \mu, \sigma) = \frac{1}{Z_q \sigma} \exp\left( -\frac{1}{3-q} \left( \frac{x-\mu}{\sigma} \right)^2 \right).
\]
The set of all \( q \)-Gaussian densities satisfies the regularity condition to define information geometric structures. For example, see [4].

### 3.2 Sufficient conditions for \([2,1]\)

In order to give a rigorous treatment of an escort expectation associated to the \( a \)-refined \( q \)-logarithmic function, we only deal with the case \( 1 \leq q < 3 \). Set
\[
\Sigma_q := \left\{ \sigma > 0 \mid \frac{1}{Z_q \sigma} < 1 \right\}, \quad S_q := \{ p_q(\cdot; \xi) \mid \xi \in \mathbb{R} \times \Sigma_q \}.
\]
It holds for \( \sigma \in \Sigma_q, p \in S_q \) and \( x \in \mathbb{R} \) that
\[
\ln_q \left( \frac{1}{Z_q \sigma} \right) = \ln_q(1) = 0, \quad \ln_q(p(x)) \in (-\infty, 0).
\]

**Definition 5.** For \( 1 \leq q < 3 \) and \( \xi \in \mathbb{R} \times \Sigma_q \), define \( \ell_q(\cdot; \xi) : \mathbb{R} \to (-\infty, 0) \) by
\[
\ell_q(x; \xi) := \ln_q \left( p_q(x; \xi) \right),
\]
which is called the \( q \)-likelihood function of \( p_q(\cdot; \xi) \).
For $1 \leq q < 3$, $a \in \mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R} \times \Sigma_q$, we define a measure $\nu_{q,a;\xi}$ on $\mathbb{R}$ as the absolutely continuous measure with respect to the one-dimensional Lebesgue measure with Radon–Nikodym derivative

$$\frac{d\nu_{q,a;\xi}}{dx}(x) = \frac{1}{\ln_{q,a}(p_q(x;\xi))}.$$  

Since the inverse function of $\ln_{q,a}$ is $\exp_{q,a}$, Lemma 1 in the case $n = 1$ leads to

$$\frac{d\nu_{q,a;\xi}}{dx}(x) = \exp'_{q,a}(\ln_{q,a}(p_q(x;\xi))) = (-\ell_q(x;\xi))^{1-a} \chi_q(p_q(x;\xi)).$$

A direct computation leads to the relation that

$$\ell_q(x;\xi) = \ln_q \left( \frac{1}{Z_q \sigma} \right) + \frac{1}{(Z_q \sigma)^{1-a}(3-q)} \left( \frac{x-\mu}{\sigma} \right)^2,$$

$$\ln_{q,a}(p_q(x;\xi)) = -\frac{1}{a} (-\ell_q(x;\xi))^a.

**Lemma 3.** Let $1 \leq q < 3$, $a \in \mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R} \times \Sigma_q$. Then for $\lambda, \gamma \in \mathbb{R}$ with $\lambda > 0$, $(\lambda + x^2)^\gamma \in L^1(\nu_{q,a;\xi})$ if and only if

either $q = 1$ or $q > 1$ with $\gamma < \frac{1}{2} + \frac{1}{q-1} + a - 1$.

**Proof.** Since the decay rate of $\nu_{1,a;\xi}$ is $o(\exp(-x^2))$ as $x \to \infty$ for $\varepsilon < 2$, the lemma holds for $q = 1$.

Assume $q > 1$. By the change of variables, it is enough to show the case $\xi = (0, 2/Z_q)$. Here we have that $Z_q \sigma = 2$. There exist $c, C, R > 0$ depending on $q$ such that

$$c x^{2(1-a)+2\gamma} \lambda^{2(1-a)+2\gamma} < (-\ell_q(x;\xi))^{1-a} \chi_q(p_q(x;\xi)) \cdot (\lambda + x^2)^\gamma$$

$$= \left\{-\ln_q \left( \frac{1}{2} \right) + \frac{1}{2^{1-q}(3-q)} \frac{x^2}{4} \right\}^{1-a} \cdot \frac{1}{2a} \left( 1 + \frac{q - 1 - Z_q^2 x^2}{3-q} \right)^{\frac{1}{2a}} \cdot (\lambda + x^2)^\gamma$$

$$< C x^{2(1-a)+2\gamma}$$

for $x > R$. This means that $(c + x^2)^\gamma \in L^1(\nu_{q,a;\xi})$ if and only if

$$2(1-a) + \frac{2q}{1-q} + 2\gamma < -1 \iff \gamma < \frac{1}{2} + \frac{1}{q-1} + a - 1. \quad \square$$

Lemma 3 in the case $\gamma = 0$ provides the condition for $(q,a)$ such that $\nu_{q,a;\xi}$ has a finite mass.

**Corollary 3.** Let $1 \leq q < 3$, $a \in \mathbb{R} \setminus \{0\}$ and $\xi \in \mathbb{R} \times \Sigma_q$. Then $1 \in L^1(\nu_{q,a;\xi})$ if and only if

either $q = 1$ or $q > 1$ with $\frac{1}{2} - \frac{1}{q-1} < a$. 

Note that
\[ \frac{1}{2} - \frac{1}{q - 1} < 0 \quad \text{for } 1 < q < 3. \]

**Corollary 4.** Let \( 1 \leq q < 3 \) and \( a \in \mathbb{R} \setminus \{0\} \). Then \( \ln_{q,a}(r) \in L^1(\nu_{q,a};\xi) \) for \( \xi \in \mathbb{R} \times \Sigma_q \) and \( r \in S_q \).

**Proof.** The corollary trivially holds for \( q = 1 \).
Assume \( q > 1 \). We observe from (3.1) that
\[
\ln_{q,a}(p(x;\mu,\sigma)) = -\frac{1}{a} \left\{-\ln_q \left( \frac{1}{Z_q}\sigma \right) + \frac{1}{(Z_q\sigma)^{1-q}(3-q)} \left( \frac{x - \mu}{\sigma} \right)^2 \right\}^a
\]
for \((\mu,\sigma) \in \mathbb{R} \times \Sigma_q\). This with Lemma 3 yields that
\[
\ln_{q,a}(r) \in L^1(\nu_{q,a};\xi) \iff a < \frac{1}{2} + \frac{1}{q - 1} + a - 1, \tag{3.2}
\]
which holds for \( q < 3 \). \( \square \)

Following Definition 3, we define an entropy and a relative entropy on \( S_q \).
Recall the escort expectation of a function \( f \in L^1(\nu_{q,a};\xi) \) with respect to \( \nu_{q,a};\xi \)
is defined by
\[
E_{\nu_{q,a};\xi}[f] = \int_{\mathbb{R}} f(x) d\nu_{q,a;\xi}(x) = \int_{\mathbb{R}} f(x) \exp_{q,a}^\prime (\ln_{q,a}(p_q(x;\xi))) dx.
\]

**Definition 6.** Let \( 1 \leq q < 3 \) and \( a \in \mathbb{R} \setminus \{0\} \). Take \( \xi \in \mathbb{R} \times \Sigma_q \) and set \( p = p_q(\cdot;\xi) \in S_q \).
(1) The \((q,a)\)-cross entropy of \( p \) with respect to \( r \in S_q \) is defined by
\[
d_{q,a}(p, r) := -E_{\nu_{q,a;\xi}}[\ln_{q,a}(r)].
\]
(2) The \((q,a)\)-entropy of \( p \) is defined by
\[
\text{Ent}_{q,a}(p) := d_{q,a}(p, p).
\]
(3) The \((q,a)\)-relative entropy of \( p \) with respect to \( r \in S_q \) is defined by
\[
D_{q,a}(p, r) := -d_{q,a}(p, p) + d_{q,a}(p, r).
\]

**Remark 6.** The domain of the \((q,1)\)-entropy can be extended to the whole of \( q\)-Gaussian densities. The \((q,1)\)-entropy coincides with the Boltzmann–Shannon entropy if \( q = 1 \), and the Tsallis entropy otherwise.

**Theorem 1 (gauge freedom of entropies).** Let \( 1 \leq q < 3 \) and \( a \in \mathbb{R} \setminus \{0\} \).
Then
\[
\text{Ent}_{q,1} = a \text{Ent}_{q,a}, \quad D^{(q,1)} \neq \lambda D^{(q,a)} \quad \text{for } a \neq 1 \text{ and } \lambda \in \mathbb{R}.
\]
Proof. By the definition, we have that
\[
d_{q,a}(p_q(;\xi_0), p_q(;\xi)) = \frac{1}{a} \int_{\mathbb{R}} (-\ell_q(x; \xi))^a \nu_{q,a;\xi_0}(x)
\]
\[
= \frac{1}{a} \int_{\mathbb{R}} (-\ell_q(x; \xi))^a (-\ell_q(x; \xi_0))^{1-a} \chi_q(p_q(x; \xi_0)) dx
\]
for \(\xi_0, \xi \in \mathbb{R} \times \Sigma_q\), which implies that
\[
\text{Ent}_{q,1}(p) = a \text{Ent}_{q,a}(p) = - \int_{\mathbb{R}} \ln q(p(x)) p(x)^q dx
\]
\[
= \begin{cases}
- \int_{\mathbb{R}} \frac{p(x) - p(x)^q}{1 - q} dx & q > 1, \\
- \int_{\mathbb{R}} p(x) \log(p(x)) dx & q = 1
\end{cases}
\]
for \(p \in S_q\).

Recall that \(\Sigma_q = \{\sigma > 0 \mid \sigma > 1/Z_q\}\). Since we observe that
\[
\lim_{\sigma \to \infty} (-\ell_q(x; 0, \sigma))^\alpha = \begin{cases}
\infty & a > 1, \\
1 & a = 1, \\
0 & a < 1, a \neq 0
\end{cases}
\]
we apply the dominated convergence theorem \(a \leq 1\) and the monotone convergence theorem for \(a > 1\) to have
\[
\frac{\lambda D^{(q,a)}(p, p_q(;; 0; \sigma)) - D^{(q,1)}(p, p_q(;; 0; \sigma))}{-\ln q \left( \frac{1}{Z_q \sigma} \right)}
\]
\[
= \frac{-\lambda d_{q,a}(p, p) - d_{q,1}(p, p)}{-\ln q \left( \frac{1}{Z_q \sigma} \right)} + \frac{\lambda d_{q,a}(p, p_q(;; 0; \sigma)) - d_{q,1}(p, p_q(;; 0; \sigma))}{-\ln q \left( \frac{1}{Z_q \sigma} \right)}
\]
\[
\stackrel{\sigma \to \infty}{\longrightarrow} \begin{cases}
\lambda \cdot \infty - c & \text{if } a > 1, \\
(\lambda - 1)c & \text{if } a = 1, \\
-c & \text{if } a < 1, a \neq 0
\end{cases}
\]
for \(p \in S_q\) and \(\lambda \in \mathbb{R}\), where we put \(0 \cdot \infty := 0\) and
\[
c := \int_{\mathbb{R}} \chi_q(p(x)) dx.
\]
This constant \(c\) is obviously positive, and \(c\) is finite due to Lemma 5 in the next section. This ensures that \(D^{(q,a)} \neq \lambda D^{(q,1)}\) for \(a \neq 1\) and \(\lambda \in \mathbb{R}\). \(\square\)

The proof of Theorem 1 immediately gives the following corollary.

**Corollary 5.** Let \(1 \leq q < 3\) and \(a \in \mathbb{R} \setminus \{0\}\). Then
\[
d_{q,1} \neq \lambda d_{q,a} \quad \text{for } a \neq 1 \text{ and } \lambda \in \mathbb{R}.
\]
4 Refined Riemannian metrics

Throughout of this section, we fix $1 \leq q < 3$ and $a \in \mathbb{R} \setminus \{0\}$ such that $I_{q,a} \neq 0$, namely
\[
either q = 1 \quad \text{or} \quad q > 1 \quad \text{with} \quad 1 - a < \frac{q}{q-1}.
\]
In this case, $t_{q,a} = 0$. Set
\[
\Sigma_{q,a} := \left\{ \sigma \in \Sigma_q \mid \frac{1}{Z_q} < T_{q,a} \right\}, \quad \mathcal{S}_{q,a} := \{ p_q(\cdot; \xi) \in \mathcal{S}_q \mid \xi \in \mathbb{R} \times \Sigma_{q,a} \}.
\]
The manifold $\mathcal{S}_{q,a}$ admits information geometric structures.

4.1 Derivatives of $(q, a)$-relative entropy

The $(q, a)$-relative entropy is nondegenerate on $\mathcal{S}_{q,a} \times \mathcal{S}_{q,a}$.

Lemma 4. For $p, r \in \mathcal{S}_{q,a}$, $D^{(q,a)}(p, r) > 0$.

Proof. Proposition [1] yields that $\exp_{q,a}''(\ln_{q,a}(p(x))) > 0$ in $x \in \mathbb{R}$ for $p \in \mathcal{S}_{q,a}$.

The strict convexity of $\exp_{q,a}$ leads to the inequality that
\[
r(x) = \exp_{q,a}(\ln_{q,a}(r(x))) > \exp_{q,a}(\ln_{q,a}(p(x))) + \{ \ln_{q,a}(r(x)) - \ln_{q,a}(p(x)) \} \exp_{q,a}'(\ln_{q,a}(p(x))) = p(x) + \ln_{q,a}(r(x)) \exp_{q,a}'(\ln_{q,a}(p(x))) - \ln_{q,a}(p(x)) \exp_{q,a}'(\ln_{q,a}(p(x)))
\]
for $x \in \mathbb{R}$ and $p, r \in \mathcal{S}_{q,a}$. Integrating this inequality on $\mathbb{R}$ gives
\[
1 > 1 - d_{q,a}(p, r) + d_{q,a}(p, p) = 1 - D^{(q,a)}(p, r). \quad \square
\]

Let us define a function $\rho^{(q,a)}$ on $(x, \xi_1, \xi_2) \in \mathbb{R} \times (\mathbb{R} \times \Sigma_{q,a})^2$ by
\[
\rho^{(q,a)}(x; \xi_1, \xi_2) := \{ \ln_{q,a}(p_q(x; \xi_1)) - \ln_{q,a}(p_q(x; \xi_2)) \} \exp_{q,a}'(\ln_{q,a}(p_q(x; \xi_1))),
\]
which is the integrand of $D^{(q,a)}(p_q(\cdot; \xi_1), p_q(\cdot; \xi_2))$.

Given $\xi_i = (\mu_i, \sigma_i) \in \mathbb{R} \times \Sigma_{q,a}$, it turns out that
\[
\left. \frac{\partial}{\partial s_1} \left. \frac{\partial}{\partial s_2} \rho^{(q,a)}(x; \xi_1, \xi_2) \right|_{(\xi_1, \xi_2)} \right. = -\left. \frac{\partial}{\partial s_2} \ln_{q,a} \left( p_q(x; \xi_2) \right) \cdot \frac{\partial}{\partial s_1} \exp_{q,a}'(\ln_{q,a} \left( p_q(x; \xi_1) \right)) \right|_{(\xi_1, \xi_2)}
\]
\[
= -\left. \frac{\partial}{\partial s_2} \ln_{q,a} \left( p_q(x; \xi_2) \right) \cdot \exp_{q,a}'(\ln_{q,a} \left( p_q(x; \xi_1) \right)) \cdot \exp_{q,a}''(\ln_{q,a} \left( p_q(x; \xi_1) \right)) \right|_{(\xi_1, \xi_2)}
\]
\[
= -\left. \frac{\partial}{\partial s_2} \left\{ \frac{1}{a} (-\ell_q(x; \xi_2))^a \right\} \cdot \frac{\partial}{\partial s_1} \left\{ \frac{1}{a} (-\ell_q(x; \xi_1))^a \right\} \right|_{(\xi_1, \xi_2)}
\]
\[
\times p_q(x; \xi)^{2(1-q)+q} (-\ell_q(x; \xi))^{2(1-a)} \sum_{j=0}^{1} a_j^2 (-\ell_q(x; \xi))^{-j}
\]
\[
= -\sum_{j=0}^{1} a_j^2 \left( \frac{\partial}{\partial s_2} \ell_q(x; \xi_2) \cdot \frac{\partial}{\partial s_1} \ell_q(x; \xi_1) \right) \cdot (-\ell_q(x; \xi))^{-j} p_q(x; \xi)^{2q-1}
\]
for $s_i \in \{\mu_i, \sigma_i\}$, where we used Lemma 1 in the case $n = 2$.

Let us generalize Lemma 3.

**Lemma 5.** Fix $n \in \mathbb{N}$ and $\gamma \geq 0$. Then $\exp_q(-x^2)^{(n-1)(q-1)} q \cdot x^{2\gamma} \in L^1(dx)$ if and only if

$$\text{either } q = 1 \text{ or } q > 1 \text{ with } \gamma < \frac{1}{2} + \frac{1}{q-1} + n - 1.$$ 

**Proof.** The lemma trivially holds for $q = 1$. Assume $q > 1$. There exist $c, C, R > 0$ depending on $q$ such that

$$cx^{\frac{n-1}{2} + \gamma} < \exp_q(-x^2)^{(n-1)(q-1)} q \cdot x^{2\gamma} = \left\{1 - (1 - q)x^2\right\}^{\frac{(n-1)(q-1)}{q-1} + \gamma} \cdot x^{2\gamma}$$

for $x > R$. This yields that $\exp_q(-x^2)^{(n-1)(q-1)} q \cdot x^{2\gamma} \in L^1(dx)$ if and only if

$$\frac{2(n-1)(q-1)}{1-q} + 2\gamma < -1 \iff \gamma < \frac{1}{2} + \frac{1}{q-1} + n - 1. \quad \square$$

**Corollary 6.** For $n \in \mathbb{N}, 0 \leq \gamma \leq n, j \in \mathbb{Z}_{\geq 0}$ and $\xi \in \mathbb{R} \times \Sigma_q$,

$$p_q(x; \xi)^{(n-1)(q-1)} q \cdot x^{2\gamma} \cdot (-\ell_q(x; \xi)^{-1}) \in L^1(dx).$$

**Proof.** Since we have that

$$n < \frac{1}{2} + \frac{1}{q-1} + n - 1 \quad \text{for } 1 < q < 3,$$

we apply Lemme 5 together with the change of variables to have that

$$p_q(x; \xi)^{(n-1)(q-1)} q \cdot x^{2\gamma} \in L^1(dx) \quad \text{for } 0 \leq \gamma \leq n.$$

Moreover, the fact that

$$-\ell_q(x; \xi) \geq -\ln_q \left(\frac{1}{Z_q}\right) > 0$$

completes the proof of the corollary. \quad \square

Combining the computation that

$$\frac{\partial}{\partial \mu} \ell_q(x; \mu, \sigma) = \frac{2}{(3-q)} \cdot \frac{1}{(Z_q)^{1-q} \sigma} \cdot \frac{x - \mu}{\sigma},$$

$$\frac{\partial}{\partial \sigma} \ell_q(x; \mu, \sigma) = -\frac{1}{(Z_q)^{1-q} \sigma} \left\{1 - \left(\frac{x - \mu}{\sigma}\right)^2\right\}$$

(4.1)
with Corollary in the case \( n = 2 \), we conclude that
\[
x \mapsto \frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2} \rho^{(q,a)}(x; \xi_1, \xi_2) \bigg|_{(\xi, \xi)}
\]
is integrable on \( \mathbb{R} \) for \( \xi \in \mathbb{R} \times \Sigma_{q,a} \). Since the function \( x \mapsto \rho^{(q,a)}(x; \xi_1, \xi_2) \) is integrable on \( \mathbb{R} \) for \( (\xi_1, \xi_2) \in (\mathbb{R} \times \Sigma_{q,a})^2 \), the dominated convergence theorem implies that
\[
\frac{\partial}{\partial s_1} \frac{\partial}{\partial s_2} D^{(q,a)}(p_q(\cdot; \xi_1), p_q(\cdot; \xi_2)) \bigg|_{(\xi, \xi)} = -\int_{\mathbb{R}} \frac{\partial}{\partial s_2} \ln_{q,a} (p_q(x; \xi_2)) \cdot \frac{\partial}{\partial s_1} \ln_{q,a} (p_q(x; \xi_1)) \cdot \exp''_{q,a} (\ln_{q,a} (p_q(x; \xi_1))) \, dx \\
= -\int_{\mathbb{R}} \frac{\partial}{\partial s_2} \ell_q(x; \xi_2) \cdot \frac{\partial}{\partial s_1} \ell_q(x; \xi_1) \cdot (-\ell_q(x, \xi))^{-j} \rho_q(x; \xi)^{2j-1} \, dx
\]
for \( s_i \in \{\mu_i, \sigma_i\} \). This quantity evaluated at the diagonal set \( \{(\xi_1, \xi_2) \mid \xi_1 = \xi_2\} \) provides a Riemannian metric on \( \mathcal{S}_{q,a} \).

**Definition 7.** For \( s, t \in \{\mu, \sigma\} \), define a function \( g_{st}^{(q,a)} : \mathbb{R} \times \Sigma_{q,a} \to \mathbb{R} \) by
\[
g_{st}^{(q,a)}(\xi) := \int_{\mathbb{R}} \frac{\partial}{\partial s} \ln_{q,a} (p_q(x; \xi)) \cdot \frac{\partial}{\partial t} \ln_{q,a} (p_q(x; \xi)) \cdot \exp''_{q,a} (\ln_{q,a} (p_q(x; \xi))) \, dx.
\]

**Theorem 2.** For \( \xi \in \mathbb{R} \times \Sigma_{q,a} \) and \( s, t \in \{\mu, \sigma\} \),
\[
g^{(q,a)} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) (p_q(\cdot; \xi)) := g_{st}^{(q,a)}(\xi)
\]
determines a Riemannian metric on \( \mathcal{S}_{q,a} \).

**Proof.** It is enough to show that \( g_{\mu \mu}^{(q,a)}, g_{\sigma \sigma}^{(q,a)} > 0 \) and \( g_{\mu \sigma}^{(q,a)} = 0 \) on \( \mathbb{R} \times \Sigma_{q,a} \).

The positivities of \( g_{\mu \mu}^{(q,a)}, g_{\sigma \sigma}^{(q,a)} \) follows from that of
\[
\frac{\partial}{\partial s} \ln_{q,a} (p_q(x; \xi)) \cdot \frac{\partial}{\partial s} \ln_{q,a} (p_q(x; \xi)) \cdot \exp''_{q,a} (\ln_{q,a} (p_q(x; \xi))) \quad \text{for } s \in \{\mu, \sigma\}.
\]

We derive \( g_{\mu \sigma}^{(q,a)} = 0 \) from the fact that
\[
\frac{\partial}{\partial \mu} \ln_{q,a} (p_q(x; \xi)) \cdot \frac{\partial}{\partial \sigma} \ln_{q,a} (p_q(x; \xi)) \cdot \exp''_{q,a} (\ln_{q,a} (p_q(x; \xi)))
\]
is an odd function in \( x \in \mathbb{R} \) with respect to \( x = \mu \) according to (4.1). \( \square \)
Remark 7. The Riemannian metric $g^{(q,1)}$ coincides with the Fisher metric up to scalar multiple. The third order derivatives of $(q, 1)$-relative entropy on the set of all $q$-Gaussian densities induce a pair of affine connections. The cubic tensor which expresses the difference between the two affine connections is called the Amari–Cencov tensor. In a similar way, a cubic tensor $C^{(q,a)}$ is defined by

$$C^{(q,a)} \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t}, \frac{\partial}{\partial u} \right) (p_q(x; \xi))$$

$$= \int \frac{\partial}{\partial s} \ln(q,a)(p_q(x; \xi)) \cdot \frac{\partial}{\partial t} \ln(q,a)(p_q(x; \xi)) \cdot \frac{\partial}{\partial u} \ln(q,a)(p_q(x; \xi))$$

$$\times \exp_{q,a}^{\mu}(\ln(q,a)(p_q(x; \xi))) \, dx$$

$$= \int \frac{\partial}{\partial s} \left\{ -\frac{1}{a} (-\ell_q(x; \xi))^a \right\} \cdot \frac{\partial}{\partial t} \left\{ -\frac{1}{a} (-\ell_q(x; \xi))^a \right\} \cdot \frac{\partial}{\partial u} \left\{ -\frac{1}{a} (-\ell_q(x; \xi))^a \right\}$$

$$\times p_q(x; \xi)^{3(1-q^2)} + (\ell_q(x; \xi))^{3(1-a)} \sum_{j=0}^2 b_j^2 (-\ell_q(x; \xi))^{-j} \, dx$$

for $s, t, u \in \{\mu, \sigma\}$ and $\xi \in \mathbb{R} \times \Sigma_q,a$. The above improper integral converges due to Corollary 6 in the case $n = 3$.

The Fisher metric (resp. the Amari–Cencov tensor) is a unique invariant quadric (resp. cubic) tensor under Markov embeddings up to scalar multiple (see [2] Chapter 5).

4.2 Expression of the refined Riemann metrics

We compute the exact value of

$$g^{(q,a)}_{\mu \mu}(\xi) = \frac{4}{(3-q)^2} \sum_{j=0}^{1} \frac{b_j^2}{2(1-q)\sigma^2} \int_{\mathbb{R}} \left( \frac{x - \mu}{\sigma} \right)^2 p_q(x; \xi)^{2q-1} (-\ell_q(x, \xi))^3 \, dx$$

$$= \frac{4}{(3-q)^2} \sum_{j=0}^{1} \frac{b_j^2}{2(1-q)\sigma^2} \Phi(q, 2, 1, j; \xi), \quad (4.2)$$

$$g^{(q,a)}_{\nu \nu}(\xi) = \sum_{j=0}^{1} \frac{b_j^2}{2(1-q)\sigma^2} \Phi(q, 2, 1, j; \xi),$$

for $\xi \in \mathbb{R} \times \Sigma_q,a$, where we set

$$\Phi(q, n, k, j; \xi) := \int_{\mathbb{R}} \left( \frac{x - \mu}{\sigma} \right)^2 p_q(x; \xi)^{2q-1} (-\ell_q(x, \xi))^3 \, dx.$$
Lemma 6. For $n \in \mathbb{N}$, $k \in \{0, 1, \ldots, n\}$ and $\xi = (\mu, \sigma) \in \mathbb{R} \times \Sigma_{q,a}$, then
\[
\Phi(q, n, k, 0; \xi) = \begin{cases} \frac{\sigma}{(Zq\sigma)^{(n-1)(q-1)+q}} \left( \frac{3 - q}{q - 1} \right)^{k+\frac{1}{2}} B \left( \frac{3 - q}{2(q-1)} + n - k, \frac{1}{2} + k \right) & \text{if } q > 1, \\ \frac{\sigma}{(2k-1)!!} & \text{if } q = 1, \end{cases}
\]
where by convention $(2 \cdot 0 - 1)!! := 1$.

Proof. We apply the change of variables with
\[
y = \frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \quad \text{if } q = 1, \quad \text{and} \quad y = \frac{q-1}{3-q} \left( \frac{x - \mu}{\sigma} \right)^2 \quad \text{otherwise.}
\]
For $q = 1$, we observe that
\[
\Phi(1, n, k, 0; \xi) = \int_{\mathbb{R}} p_1(x; \xi)^{(n-1)(1-1)+1} \left( \frac{x - \mu}{\sigma} \right)^{2k} dx
\]
\[
= 2 \int_0^\infty \frac{1}{\sqrt{2\pi\sigma}} \exp \left( -\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2 \right) \left( \frac{x - \mu}{\sigma} \right)^{2k} dx
\]
\[
= \frac{2^k}{\sqrt{\pi}} \int_0^\infty e^{-y} y^{k-\frac{1}{2}} dy
\]
\[
= \frac{2^k}{\sqrt{\pi}} \Gamma \left( k + \frac{1}{2} \right) = \frac{2^k (2k-1)!!}{\sqrt{\pi}}
\]
\[
= (2k-1)!!,
\]
where $\Gamma(\cdot)$ stands for the Gamma function, that is
\[
\Gamma(s) := \int_0^\infty e^{-x} x^{s-1} dx \quad \text{for } s > 0.
\]

For $q > 1$, it turns out that
\[
\Phi(q, n, k, 0; \xi)
\]
\[
= \int_{\mathbb{R}} p_q(x; \xi)^{(n-1)(q-1)+q} \left( \frac{x - \mu}{\sigma} \right)^{2k} dx
\]
\[
= 2 \int_0^\infty \frac{1}{(Zq\sigma)^{(n-1)(q-1)+q}} \left[ 1 + \frac{q-1}{3-q} \left( \frac{x - \mu}{\sigma} \right)^2 \right]^{(n-1)(q-1)+q} \left( \frac{x - \mu}{\sigma} \right)^{2k} dx
\]
\[
= \frac{\sigma}{(Zq\sigma)^{(n-1)(q-1)+q}} \left( \frac{3 - q}{q - 1} \right)^{k+\frac{1}{2}} \frac{1}{\frac{1}{3-q} \Gamma \left( \frac{3 - q}{2(q-1)} + n - k, \frac{1}{2} + k \right)}.
\]
Proposition 2. For $a = 1$ and $\xi = (\mu, \sigma) \in \mathbb{R} \times \Sigma_{q,a}$, we have that

$$g_{\mu\mu}^{(q,1)}(\xi) = \frac{1}{\sigma^2}, \quad g_{\sigma\sigma}^{(q,1)}(\xi) = \frac{3 - q}{\sigma^2}.$$

Proof. It follows from Lemma 6 that

$$\Phi(1, 2, 0, 0; \xi) = 1, \quad \Phi(1, 2, 1, 0; \xi) = 1, \quad \Phi(1, 2, 0; \xi) = 3,$$

implying

$$g_{\mu\mu}^{(1,1)}(\xi) = b_0^2(q, 1) \frac{1}{\sigma^2}, \quad g_{\sigma\sigma}^{(1,1)}(\xi) = b_0^2(q, 1) \sum_{j=0}^{1} \frac{1}{\sigma^2}(1 - 2 + 3) = \frac{2}{\sigma^2}.$$

Assume $q > 1$. By the property that

$$B(s + 1, t) = \frac{st}{s + t} B(s, t) \quad \text{for } s, t > 0,$$

we have that

$$\Phi(q, 2, k, 0; \xi) = \frac{\sigma}{(Z_q \sigma)^{(2-1)(q-1)+q}} \left( \frac{3 - q}{q - 1} \right)^{k+\frac{1}{2}} \frac{B \left( \frac{3 - q}{2(q - 1)} + 2 - k, \frac{1}{2} + k \right)}{B \left( \frac{3 - q}{2(q - 1)}, \frac{1}{2} \right)} f_2(k) \frac{(q - 1)^2 f_2(k)}{q},$$

where we set

$$f_2(0) = \left( \frac{3 - q}{2(q - 1)} + 1 \right) \frac{3 - q}{2(q - 1)} = \frac{(q + 1)(3 - q)}{4(q - 1)^2},$$

$$f_2(1) = \frac{3 - q}{2(q - 1)} \cdot \frac{1}{2} = \frac{3 - q}{4(q - 1)},$$

$$f_2(2) = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4}.$$

This leads to that

$$g_{\mu\mu}^{(q,1)}(\xi) = \frac{4}{(3 - q)^2} \frac{b_0^2(q, 1)}{(Z_q \sigma)^{2(1-q)} \sigma^2} \Phi(q, 2, 1, 0; \xi) = \frac{1}{\sigma^2},$$

$$g_{\sigma\sigma}^{(q,1)}(\xi) = \frac{b_0^2(q, 1)}{(Z_q \sigma)^{2(1-q)} \sigma^2} \sum_{k=0}^{2} \binom{2}{k} (-1)^k \Phi(q, 2, k, 0; \xi)$$

$$= \frac{1}{\sigma^2} \sum_{k=0}^{2} \binom{2}{k} \left( (-1)^k \left( \frac{3 - q}{q - 1} \right)^k (q - 1)^2 f_2(k) \right) = \frac{3 - q}{\sigma^2}. \quad \square$$
Fix \( n, j \in \mathbb{N}, k \in \{0, 1, \ldots, n\} \) and \( \xi = (\mu, \sigma) \in \mathbb{R} \times \Sigma_{q,a} \). Let us compute \( \Phi(q, n, k; j; \xi) \) with the use of the residue theorem. Note that
\[
\Phi(q, n, k; j; \mu, \sigma) = \Phi(q, n, k; j; 0, \sigma).
\]

Define a complex valued function \( \phi_{q,n,k,j;\sigma}(z) \) on \( \mathbb{C} \) by
\[
\phi_{q,n,k,j;\sigma}(z) := \left(\frac{z}{\sigma}\right)^{2k} \frac{p_q(z; 0, \sigma)^{(n-1)(q-1)+q}}{(-\ell_q(z; 0, \sigma))^j}
\]
\[
= \left(\frac{z}{\sigma}\right)^{2k} p_q(z; 0, \sigma)^{(n-1)(q-1)+q} \left\{ \frac{z^2 + r(q, \sigma)^2}{(Z_q \sigma)^1 - q(3 - q)\sigma^2} \right\}^{-j},
\]
where we set
\[
r(q, \sigma) := \sqrt{-\ln_q \left( \frac{1}{Z_q \sigma} \right) \cdot (Z_q \sigma)^1 - q(3 - q)\sigma^2}.
\]
The function \( \phi_{q,n,k,j;\sigma} \) has poles of order \( j \) at \( \pm ir(q, \sigma) \). For \( R > r(q, \sigma) \), let \( L_R \) and \( C_R \) be smooth curves in \( \mathbb{C} \) defined respectively by
\[
L_R := \{ z : [-R, R] \to \mathbb{C} \mid z(\theta) = \theta \}, \quad C_R := \{ z : [0, \pi] \to \mathbb{C} \mid z(\theta) = Re^{i\theta} \}.
\]
The residue theorem yields that
\[
\int_{L_R \cup C_R} \phi_{q,n,k,j;\sigma}(z)\,dz = 2\pi i \cdot \text{Res}(\phi_{q,n,k,j;\sigma}; ir(q, \sigma)), \quad (4.3)
\]
where \( \text{Res}(\phi_{q,n,k,j;\sigma}; ir(q, \sigma)) \) stands for the residue of \( \phi_{q,n,k,j;\sigma} \) at \( z = ir(q, \sigma) \).

**Lemma 7.** For \( n, j \in \mathbb{N}, k \in \{0, 1, \ldots, n\} \) and \( (\mu, \sigma) \in \mathbb{R} \times \Sigma_{q,a} \), then
\[
\Phi(q, n, k; j; \mu, \sigma) = 2\pi i \cdot \text{Res}(\phi_{q,n,k,j;\sigma}, ir(q, \sigma)).
\]

**Proof.** If we show that
\[
\lim_{R \to \infty} \int_{C_R} \phi_{q,n,k,j;\sigma}(z)\,dz = 0,
\]
then we have the desired result by letting \( R \to \infty \) in (4.3). Take \( R > r(q, \sigma) \) large enough. We calculate that
\[
\left| \int_{C_R} \phi_{q,n,k,j;\sigma}(z)\,dz \right|
\leq R \int_{0}^{\pi} \left| \phi_{q,n,k,j;\sigma}(Re^{i\theta}) \right| \,d\theta
\leq R \int_{0}^{\pi} \left( \frac{R}{\sigma} \right)^{2k} \left| p_q(Re^{i\theta}; 0, \sigma)^{(n-1)(q-1)+q} \left( Z_q \sigma \right)^1 - q(3 - q)\sigma^2 \right|^{-j} \,d\theta
\leq CR^{2(k-j)+1} \int_{0}^{\pi} \left| \exp_q \left( \frac{-R^2 e^{2i\theta}}{(3 - q)\sigma^2} \right) \right|^{(n-1)(q-1)+q} \,d\theta,
\]
where the constant \( C \) depends on \( q \) and \( \sigma \).

In the case \( q = 1 \), we have that
\[
\left| \exp_t \left( \frac{-R^2 e^{2\theta}}{3 - q \sigma^2} \right)^{(n-1)(1-1)q} \right| = \exp \left( -\frac{R^2 \cos 2\theta}{2\sigma^2} \right),
\]
consequently
\[
\left| \int_{C_R} \phi_{q,n,k,j;\sigma}(z)dz \right| \leq CR^{2(k-j)+1} \int_0^\pi \exp \left( -\frac{R^2 \cos 2\theta}{2\sigma^2} \right) d\theta \xrightarrow{R \to \infty} 0.
\]
In the case \( q > 1 \), we observe that
\[
\left| \exp_q \left( -\frac{R^2 e^{2\theta}}{3 - q \sigma^2} \right)^{(n-1)(q-1)q} \right| = \left| \frac{q - 1}{3 - q} R^{2\theta} \right|^{(n-1)(q-1)q} \leq C' R^{-(2n+1) - \frac{2}{1-q}}.
\]
where the constant \( C' \) depends on \( q \) and \( \sigma \). This yields that
\[
\left| \int_{C_R} \phi_{q,n,k,j;\sigma}(z)dz \right| \leq C \cdot C' R^{2(k-j)+1-2n+1-\frac{2}{1-q}} \cdot \pi.
\]
The right-hand side converges to 0 as \( R \to \infty \) since we have
\[
2(k-j) + 1 - 2n + \frac{2}{1-q} \leq -1 + \frac{2}{1-q} < 0
\]
due to the assumption \( k \leq n \) and \( j \geq 1 \). \( \square \)

**Proposition 3.** For \( \xi = (\mu, \sigma) \in \mathbb{R} \times \Sigma_{q,a} \), then
\[
\begin{align*}
\phi_{q,a}^{(q,a)}(\xi) &= \frac{b_0^q(q,a)}{b_0^q(q,1)\sigma^2} - \frac{4\pi b_2^q(q,a)}{3 - q (Z_q\sigma)^1-q\sigma^2} r(q,\sigma), \\
\phi_{q,a}^{(q,\sigma)}(\xi) &= \frac{(3 - q)b_0^q(q,a)}{b_0^q(q,1)\sigma^2} + \frac{\pi(3 - q)b_2^q(q,1)}{(Z_q\sigma)^1-q} r(q,\sigma) \left( 1 + \left( \frac{r(q,\sigma)}{\sigma} \right)^2 \right)^2.
\end{align*}
\]

**Proof.** It follows from Lemma \( \text{7} \) that
\[
\begin{align*}
\Phi(q, n, k; 1, \xi) &= 2\pi \cdot \text{Res}(\phi_{q,n,k,1;\sigma}, r(q,\sigma)) \\
&= 2\pi \lim_{z \to r(q,\sigma)} \left\{ (z - r(q,\sigma)) \cdot \phi_{q,n,k,1;\sigma}(z) \right\} \\
&= 2\pi \lim_{z \to r(q,\sigma)} \left( \frac{z}{\sigma} \right)^{2k} p_q(z; 0, \sigma)^{(n-1)(q-1)+q} \left( Z_q\sigma \right)^{1-q} \left( 3 - q \sigma^2 \right)^{-1} \\
&= 2\pi \cdot \left( \frac{r(q,\sigma)}{\sigma} \right)^{2k} \left( Z_q\sigma \right)^{1-q} \left( 3 - q \sigma^2 \right)^2 \\
&= (-1)^k \pi \left( Z_q\sigma \right)^{1-q} \left( 3 - q \sigma^2 \right)^{2k-1},
\end{align*}
\]
where we used \( p_q(r(q,\sigma); 0, \sigma) = 1 \). This with Proposition \( \text{2} \) and \( \text{12} \) concludes the proof of the proposition. \( \square \)
Remark 8. In the case $a = 1$, the Riemannian manifold $(S_1, g^{(q,1)})$ has a constant curvature $-1/(3-q)$. This means that all $(S_1, g^{(q,1)})$ for $1 \leq q < 3$ are homothetic to each other. However, Proposition 3 suggests that this homothety may fail for $a \neq 1$.

5 Concluding remarks

In this note, we presented gauge freedom of entropies on the subset $S_q$ of all $q$-Gaussian densities for $1 \leq q < 3$. We showed that a constant multiple of each $(q, a)$-entropy coincides with the Boltzmann–Shannon entropy if $q = 1$, and the Tsallis entropy otherwise. However, any constant multiple of the $(q, a)$-relative entropy differs from the $(q, 1)$-relative entropy for $a \neq 1$. We remark that the $(q, 1)$-relative entropy coincides with the Kullback–Leibler divergence if $q = 1$, and the Tsallis relative entropy of the Csiszár type otherwise.

In information geometry, the Kullback–Leibler divergence projection from observed data to a statistical model attains the maximum likelihood estimator (see [1] Chapter 4). The terminology “maximum” depends on a criterion. It is known that higher-order asymptotic theory of estimation and Bayesian statistics improve the maximum likelihood estimator in another criterion. Ishige, Salani and the second named author showed in [3, Theorem 3.2] that the concavity related to the case $(q, a) = (1, 1/2)$ is the strongest concavity among all admissible concavities preserved by the heat flow in Euclidean space. We expect that the $(1, 1/2)$-relative entropy improves the maximum likelihood estimator.

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