Light axion-like particles occur in many theories of beyond-Standard-Model physics, and may make up some or all of the universe’s dark matter. One of the ways they can couple to the Standard Model is through the electromagnetic $F_{\mu\nu}\tilde{F}^{\mu\nu}$ portal, and there is a broad experimental program, covering many decades in mass range, aiming to search for axion dark matter via this coupling. In this paper, we derive limits on the absorbed power, and coupling sensitivity, for a broad class of such searches. We find that standard techniques, such as resonant cavities and dielectric haloscopes, can achieve $\mathcal{O}(1)$-optimal axion-mass-averaged signal powers, for given volume and magnetic field. For low-mass (frequency $\ll$ GHz) axions, experiments using static background magnetic fields generally have suppressed sensitivity — we discuss the physics of this limitation, and propose experimental methods to avoid it. We also comment on the detection of other forms of dark matter, including dark photons, as well as the detection of relativistic hidden sector particles.

I. INTRODUCTION

Axion-like particles, in particular the QCD axion, are a well-motivated dark matter (DM) candidate. They occur in many models of beyond-Standard-Model physics, and can naturally be light and weakly-coupled, allowing them to be stable and difficult-to-detect. There are also a number of early-universe production mechanisms, which can produce them in the correct abundance to be the DM [1][3].

A wide range of existing and proposed experiments aim to detect axion DM candidates. These span many decades of mass range, and target a variety of possible couplings to the Standard Model (SM) [5]. In this paper, we will focus on the $aF_{\mu\nu}\tilde{F}^{\mu\nu}$ axion-photon-photon coupling, and address the sensitivity limits on such experiments — how small a DM-SM coupling could we possibly detect, given the dimensions, timescales, sensors etc. available? We choose the $aF_{\mu\nu}\tilde{F}^{\mu\nu}$ coupling partly because, for a generic QCD axion, this coupling must lie within a fairly narrow (logarithmic) range [3][6]; it is also a generic feature of many other axion-like-particle models [7]. In addition, it represents a particularly easily-analysed example of the kind of sensitivity limits we are interested in.

We derive bounds on the power absorbed by axion DM experiments, under fairly general assumptions, in terms of the magnetic field energy maintained inside the experimental volume. We also derive related limits on the achievable sensitivity for such experiments, using the tools of quantum measurement theory. For low-mass (frequency $\ll$ GHz) axions, we review why static-background-field experiments generally have suppressed sensitivity (compared to their scaling at higher frequencies), and point out that this suppression can be alleviated in a number of ways, potentially motivating new experimental concepts.

A. Summary of results

Here, we give a brief summary of our main results. Suppose that dark matter consists of an axion-like particle $a$ of mass $m_a$, with a coupling $\mathcal{L} \supset -\frac{1}{4}g_{a\gamma\gamma}aF_{\mu\nu}\tilde{F}^{\mu\nu}$ to the SM. For non-relativistic axion DM, this acts as an ‘effective current’ $J^{(a)} \simeq g a B$, where $B$ is the magnetic field. If we want to search for axions over a mass range $\Delta m$, then for $g_{a\gamma\gamma}$ small enough that the target system is in the linear response regime, the expected time-averaged absorbed power from the axion effective current, at the least favourable axion mass, satisfies (under certain assumptions)

$$\bar{P} \leq \frac{g_{a\gamma\gamma}^2 \rho_a U_B}{\Delta m}$$  \hspace{1cm} (1)

where $\rho_a$ is the energy density of the axion DM, and $U_B$ is the time-averaged magnetic field energy in the experimental volume (ignoring magnetic fields on very small spatial scales). Here, the expectation value includes integrating over the unknown phases of the axion signal (otherwise, there could be $\mathcal{O}(g_{a\gamma\gamma})$ components of the absorbed power). This bound applies to target systems for which the imaginary part of the response function is non-negative at positive frequencies — that is, the system on average absorbs energy from the axion forcing, rather than emitting. As we discuss below, $\bar{P}$ is, in many circumstances, closely related to the detectability of an axion signal. The $\bar{P} \propto 1/\Delta m$ behaviour corresponds to the power vs bandwidth tradeoff that is a property of many detection schemes; covering a broader axion mass...
range in the same integration time necessarily leads to lower average signal power.

In the $\nu_a \gtrsim \text{GHz}$ regime (where $\nu_a = m_a/(2\pi)$ is the frequency of the axion oscillation), cavity haloscopes such as ADMX \cite{8} and HAYSTAC \cite{9, 10} can attain the bound in equation \ref{eq:1} to $\mathcal{O}(1)$. At higher frequencies ($\nu_a \gtrsim 10\text{GHz}$), dielectric haloscope proposals \cite{11, 12} can also achieve this, taking the experimental volume to be that occupied by the dielectric layers.

However, for $m_a \lesssim L^{-1}$, where $L$ is the length scale of the shielded experimental volume, the EM modes at frequencies $\sim \nu_a$ are naturally in the quasi-static regime. In that case, a static-background-field experiment has

$$\bar{P} \lesssim \pi \frac{g_{a\gamma\gamma}^2 \rho_a U_B}{\Delta m} (m_a L)^2$$ \tag{2}$$

This suppression affects low-frequency ($\nu_a \ll \text{GHz}$) axion DM detection proposals such as ABRA-CADABRA \cite{13} and DM Radio \cite{14}. As discussed below, the scaling of the detectability limit is similarly affected, with the minimum detectable $g_{a\gamma\gamma}$ increased by $\sim (m_a L)^{-1}$.

Even under the assumptions leading to equation \ref{eq:1} this quasi-static suppression is not inevitable. To alleviate it for static background magnetic fields, we would need to enhance the quantum fluctuations of the EM fields that couple to the axion effective current, at frequencies $\sim \nu_a$. Doing this, in an equilibrium setting, requires that the field fluctuations ‘borrow’ energy from some other source, e.g. a circuit component with negative differential resistance. The practicality of such concepts requires further investigation.

The quasi-static suppression can also be alleviated by performing an ‘up-conversion’ experiment, in which the background magnetic field is oscillating at a frequency $\gtrsim L^{-1}$. Up-conversion experiments have been proposed in the optical range \cite{15, 17}, but the relatively small amplitude of achievable optical-frequency fields means that they would have relatively poor sensitivity. Larger magnetic fields are attainable at lower frequencies; in particular, it is routine to obtain magnetic fields of $\sim 0.1\text{T}$ at $\sim \text{GHz}$ frequencies in superconducting (SRF) cavities \cite{18, 19}.

These field strengths were noted in \cite{20}, which proposed a SRF up-conversion experiment \footnote{Microwave up-conversion experiments were also proposed in \cite{21}, but as reviewed in section \ref{sec:up_conversion} errors in their sensitivity calculations made them orders of magnitude too optimistic.}. However, they mainly considered $\nu_a \sim \text{GHz}$, for which static field experiments do not encounter the quasi-static suppression. Consequently, the only benefit of an SRF experiment would be the higher cavity quality factor, which is unlikely to overcome the disadvantages of smaller background magnetic field, higher temperature (due to cooling power requirements), and drive-related noise issues. However, as we point out here, for $\nu_a \ll \text{GHz}$, the lack of quasi-static suppression may make up-conversion more competitive. We investigate this possibility in more detail in a companion paper \cite{22}.

1. Detectability

In the above paragraphs, we discussed the average power absorbed from the axion effective current. It is obvious that, other things being equal, a higher absorbed power makes it easier to detect axion DM. However, in comparing different experiments, other things are often not equal, and more generally, it is useful to have quantitative limits on how small a coupling can be detected.

By using quantum measurement techniques \cite{23}, we could in principle detect almost arbitrarily small $g_{a\gamma\gamma}$, even with a small $U_B$. For example, by placing the target mode in a large-number Fock state, we could Bose-enhance the absorption (and emission) of axions \cite{24}. However, such techniques are often difficult to implement: in axion experiments, the only similar measurement demonstrated so far is HAYSTAC’s squeezed state receiver, which is planned to deliver a factor 2 scan rate improvement in the forthcoming Phase II run \cite{10}.

One standard technique for signal detection is linear, phase-invariant amplification (in particular, it is employed by almost all existing and proposed axion detection experiments at microwave frequencies and below). For different experimental setups, we can place limits on the SNR obtained, in analogy to the absorbed power limit from equation \ref{eq:1}. For example, if a linear amplifier is employed in ‘op-amp’ mode \cite{23}, and is subject to the ‘Standard Quantum Limit’ (SQL) \cite{25}, then the SNR obtained, averaged across a fractionally small axion mass range $\Delta m$, is

$$\text{SNR}^2 \lesssim \pi (g^2 \rho_a U_B)^2 \frac{tQ_a}{\omega_1^2 \gamma \Delta m f(n_T)}$$ \tag{3}$$

$$= \frac{\bar{P}^2}{\pi \max Q_a \Delta m} \frac{Q_a t \Delta m}{f(n_T)}$$ \tag{4}$$

where $Q_a \simeq 10^6$ is the fractional bandwidth of the axion DM signal, $\omega_1 \simeq m_a \pm \omega_B$ is the oscillation frequency of the axion effective current (assuming that the magnetic field oscillation frequency $\omega_B$ is narrow-bandwidth), $\gamma$ is the damping rate for forcing at this frequency, and $f(n_T)$ is a function of the thermal occupation number at $\omega_1$, with $f(n_T) \simeq 3/2$ for $n_T \ll 1$ and $f(n_T) \simeq n_T$ for $n_T \gg 1$. As occurs for the average absorbed power, there is an inverse relationship between the average SNR and $\Delta m$. \footnote{For up-conversion experiments, the converted power in a narrow frequency band is at most half of the value from equation \ref{eq:4} and the SNR$^2$ value is 1/4 of the value from equation \ref{eq:4}.}
Another common setup has a linear amplifier isolated from the target, e.g. using a circulator connected to a cold load \[22\] \[24\], to protect the target system from noise. In this case, the SNR limit is also given by equation \[4\] up to \(O(1)\) numerical factors. Perhaps surprisingly, in both of these cases, the improved sensitivity for \(\omega_1 \ll m_a\) is actually physical, and experiments using ‘down-conversion’ in this way could theoretically achieve improved sensitivities. However, due to a number of practical limitations, including the relatively small \(U_B\) values obtainable for high-frequency magnetic fields, realising such enhancements does not seem to be practical. Additionally, if \(\omega_1 \lesssim L^{-1}\), then the EM fields are naturally in the quasi-static regime, and the SNR is suppressed by \(\sim (\omega_1 L)^2\), similarly to the absorbed power in equation \[2\].

At higher frequencies, detectors other than amplifiers (e.g. photon counters, bolometric detectors, quasiparticle detectors, etc.) become easier to implement. While we could analyse the properties of each individually, it is the case that for a wide range of setups, the sensitivity is bounded by the number of axion quanta absorbed. We can quantify this using quantum measurement theory. The Fundamental Quantum Limit (FQL) \[27\] \[29\] for signal detection is determined by the quantum fluctuations of the EM fields that couple to the axion signal. Using the arguments that lead to equation \[1\], we can constrain the frequency-integrated spectrum of these fluctuations. For general states, we cannot use this to place a bound on detectability (the example of squeezed states etc. shows that there is no such general limit). However, if the sensor interacts with the target via a damping-type interaction, e.g. an absorptive photodetector or bolometer, then its effects are equivalent to a passive load, and the quantum fluctuations of the target EM fields are the same as in an equilibrium state. In these circumstances, the sensitivity to axion DM, over a (fractionally small) mass range \(\Delta m\), is bounded (at the least favourable axion mass) by

\[
P_{\text{det}} \leq \tilde{N}_a = \frac{P_{\text{tot}}}{\omega_1} \tag{5}
\]

where \(P_{\text{det}}\) is the probability of detecting the axion signal. We will refer to this limit as the PQL (‘Passive Quantum Limit’). It has an obvious interpretation in terms of photon counting, for schemes in which axions convert to single photons, but it also applies to other setups, e.g. where a signal consists of multiple quasi-particle excitations. Coherent-state excitations of the target’s EM fields leave their quantum fluctuations unchanged, so do not affect the PQL. As in the SQL case, the \(1/\omega_1\) enhancement for small \(\omega_1\) is physical, but probably not practical.

In we take \(\gamma\) small enough so that the assumptions behind equation \[4\] no longer hold, the maximum SNR from a linear amplifier isolated behind circulator with a cold load saturates to \(\sim \tilde{N}_a\) (as does the SQL op-amp limit). This is as we would expect, since the amplifier acts on the target like a passive load.

As we emphasised above, it is certainly possible to do better than the PQL, by using techniques involving ‘non-classical’ EM field states. One important example is using linear amplifiers with correlated backaction and imprecision noise. By optimising this correlation, we can in theory obtain the ‘quantum limit’ \[23\] \[25\], for which the SNR bound is

\[
\text{SNR} \lesssim \frac{\pi}{2} (g^2 \rho_\alpha U_B)^2 \frac{tQ_a}{\omega_1^3 \gamma^2 (1 + n_T)^2} \tag{6}
\]

where the notation is as for equation \[4\]. Unlike the SQL and PQL, this limit does not involve \(\Delta m\) — a QL-limited experiment is inherently broadband, if we can optimise the amplifier properties across a wide bandwidth. In the quasi-static limit, the SNR is again suppressed by \(\sim (\omega_1 L)^2\). SQUID amplifiers (as proposed for e.g. the ABRACADABRA axion DM detection experiment \[13\]) can, in some circumstances, attain near-QL performance \[30\]. The fact that the QL-limited sensitivity can, in some regimes, be better than the PQL, corresponds to the amplifier back-action enhancing the quantum fluctuations of the target EM fields.

While some other measurement schemes, such as backaction evasion \[23\], do not have such general limits on their sensitivity, we could still analyse their performance given more specific assumptions. In this paper, we will restrict our discussion to the amplifier and PQL limits introduced above. One reason for doing so is that, taken together, they apply to almost all existing and proposed axion DM detection experiments.

As we will discuss, these limits help in understanding what can and cannot enhance an experiment’s sensitivity to axion DM, and in comparing the potential sensitivity of different kinds of experiments.

II. AXION DM INTERACTIONS

We will suppose that dark matter consists of an axion-like particle \(a\), with a coupling to the SM photon. This has Lagrangian\(^3\)

\[
\mathcal{L} \supset \frac{1}{2} (\partial_\mu a)^2 - V(a) - \frac{1}{4} g_{\gamma\gamma a} F_{\mu\nu} \tilde{F}^{\mu\nu} = \frac{1}{2} (\partial_\mu a)^2 - V(a) + g_{a\gamma\gamma} a E \cdot B, \tag{7}
\]

where \(V(a)\) is the potential for the axion — in general, only the mass term \(V(a) = \frac{1}{2} m_a^2 a^2\) will be important for DM.

The \(F \tilde{F}\) term is a total derivative, \(F_{\mu\nu} \tilde{F}^{\mu\nu} = 2 \partial_{(\mu} A_{\nu)} \tilde{F}^{\mu\nu}\), so under integration by parts, the inter-
action term in the Lagrangian is equivalent to
\[ -\frac{1}{4} g a F_{\mu \nu} \tilde{F}^{\mu \nu} \rightarrow \frac{1}{2} g (\partial_a a) A_\mu \tilde{F}^{\mu \nu} \]
\[ = \frac{1}{2} g (A \cdot (\dot{a} B + (\nabla a) \times E) - A_0 \nabla a \cdot B) \]
\[ = -\frac{1}{2} A^\mu J_\mu^{(a)} \]
(8)
(9)
(10)
Note that, for our signature choice, the components of the usual 3-vector potential, which we will denote \( A \), are \(-A_i\). To begin with, we will focus on the case of a spatially-constant (zero-velocity) axion DM field, \( \nabla a = 0 \). This is a good approximation, since the DM is highly non-relativistic, with \( v_{\text{DM}} \sim 10^{-3} \) (we will come back to the consequences of the axion velocity distribution in section [11]). If \( \nabla a = 0 \), then the interaction term is \( \mathcal{L} \supset \frac{1}{2} g a \dot{a} B \cdot A \), and the interaction Hamiltonian density is \( \mathcal{H} \supset \frac{1}{2} g a B \cdot A \).

A. Response dynamics

To analyse the effect of the axion oscillation on the system, we can decompose the EM potential as \( A = A_0 + A_1 \), where \( A_0 \equiv \langle A \rangle \) in the absence of an axion oscillation. In the notation of appendix [A], \( A_1 = \Delta A \). If we assume that \( A_1 \) is small compared to \( A_0 \), then
\[ A \cdot B = A_0 \cdot B + 2 A_1 \cdot B_0 + \nabla \cdot (A_1 \cdot A_0) \]
(11)
so the interaction term can be expanded as
\[ A \cdot B \simeq A_0 \cdot B_0 + 2 A_1 \cdot B_0 + \nabla \cdot (A_1 \cdot A_0) \]
(12)
Thus,
\[ H_{\text{int}} = \frac{1}{2} \int dV g \dot{a} A \cdot B \simeq \int dV g \dot{a} \left( \frac{1}{2} A_0 + A_1 \right) \cdot B_0 \]
(14)
where we take the volume of integration large enough that the divergence term can be neglected. The full Hamiltonian for \( A_1 \) can be written as
\[ H_1 \simeq \int dV \left( E_0 \cdot E_1 + \frac{1}{2} E_1^2 + g \dot{a} A_1 \cdot B_0 + V_{\text{int}}(A_1, \ldots) \right) \]
(15)
where \( V_{\text{int}} \) represents the dynamics of the rest of the system, which couples only to \( A_1 \) (and not to \( E_1 \)).

In general, \( B_0 \) will have some time dependence. To start with, we will assume that the time dependence and spatial profile factorize, \( B_0 = B_0(t) \), as is the case for e.g. a cavity standing mode (we will revisit this in section [11]). We can decompose \( A_1 = A_0 b + A_\perp \), where \( \int dV A_\perp \cdot b = 0 \). Then, writing \( V_b \equiv \int dV b^2 \), our Hamiltonian is
\[ H_1 \simeq V_b \left( E_b^{(0)} E_b + E_b^2 + g \dot{a} A_b B_0 + V(A_b, \ldots) \right) + \ldots \]
(16)
Here, \( E_b \) is the analogous decomposition of the electric field, and the conjugate momentum of \( A_b \) is \(-V_b E_b \), with equal-time commutation relation \([A_b, \dot{E}_b] = -i/V_b\). This is analogous to the Hamiltonian for a driven 1D oscillator,
\[ \dot{H}_{\text{1D}} = \frac{(\dot{p} - p_0)^2}{2M} - g j(t) \dot{x} + V_{\text{int}}(\dot{x}, \ldots) \]
(17)
where \( \dot{x} \equiv A_b, \dot{p} \equiv -E_b V_b, j(t) \equiv \dot{a}(t) B_0(t) V_b, \) and \( M \equiv V_b \).

If we consider a very short \( j(t) \) pulse, turning on and off much faster than the system’s dynamics, then its effect is to impulsively change \( p \) by \( g \int dt j(t) = g J \). Averaging over possible signs of the pulse, the expected energy absorbed is \( \langle W \rangle \simeq (\Delta p)^2/(2M) \). In our case, since \( j \) depends on the time derivative of \( a \), a delta-function \( j \) pulse corresponds to a step function in \( a(t) \), and we have \( \langle W \rangle \simeq \frac{1}{8} g^2 (\Delta a)^2 V_0 B_0^2 \).

This argument tells us the expected energy absorbed by the target from a very fast axion field ‘pulse’. However, as discussed above, we expect axion DM to be a narrow-bandwidth oscillation, with fractional bandwidth \( \sim 10^{-6} \). If \( g \) is small enough that the target is in the linear response regime, then the energy it absorbs from a finite-time \( j(t) \) signal is
\[ \langle W \rangle = \frac{g^2}{2\pi} \int_{-\infty}^{\infty} d\omega \omega \left| j_\omega(\omega) \right|^2 \text{Im} \chi(\omega) \]
(18)
where \( \chi \) is the linear response function for \( x \) (if the dynamics are non-stationary in time, we can consider averaging over all possible starting times). A delta-function pulse has equal power at all frequencies, so
\[ \langle W \rangle = \frac{g^2}{2\pi} J^2 \int_{-\infty}^{\infty} d\omega \omega \text{Im} \chi(\omega) \]
(19)
Equating this to the energy \( g^2 J^2/(2M) \) absorbed from the pulse, we have
\[ \int_{-\infty}^{\infty} d\omega \omega \text{Im} \chi(\omega) = \frac{\pi}{V_b} \]
(20)
This ‘sum rule’ is analogous to the Thomas-Reiche-Kuhn sum rule for ‘oscillator strengths’ in atomic physics [11].

By itself, equation [20] does give us any limit on the response in a specific frequency range, since the integrand could have large cancelling components. However, if \( \omega \text{Im} \chi(\omega) \) is always \( \geq 0 \), then we can bound the absorbed power from any signal. This obviously applies if the target is in its ground state (since a forcing can only add energy), or if \( \text{Im} \chi \) is equivalent to its ground-state form. For a 1D oscillator, the latter is true in any state, up to non-linearities. More generally, if the target is in a mixed state, where the probability of a microstate decreases with increasing energy, then the condition also holds.

These arguments are a generalisation of the pulse-absorption argument from [12], which was used to determine the axion-mass-averaged signal power from a dielectric haloscope.
B. Fluctuation sum rules

To apply the FQL detectability limits discussed in appendix A, we need to understand the fluctuation spectrum of $A_b$. We can relate this to the response function via the Kubo formula, $\text{Im} \chi(\omega) = \frac{i}{2} (S_{A_b A_b}(\omega) - S_{A_b A_b}(-\omega))$, where $S_{A_b A_b}(\omega)$ is the spectral density of $A_b$ fluctuations. Thus, the sum rule in equation 20 implies a corresponding sum rule for $S_{A_b A_b}$,

$$\int_{-\infty}^{\infty} d\omega \omega S_{A_b A_b}(\omega) = \frac{\pi}{V_b}$$

(21)

We can also derive this sum rule directly from the commutation relations of the EM fields. The spectral density of $A_b$ fluctuations (assuming that they are stationary in time) is

$$S_{A_b A_b}(\omega) \equiv \int_{-\infty}^{\infty} dt e^{i\omega t} \langle \dot{A}_b(t) A_b(0) \rangle = F \langle \dot{A}_b(t) A_b(0) \rangle$$

(22)

where $\dot{A}_b(t)$ is the Heisenberg picture operator for the system, in the absence of axion interactions (going forwards, we will drop the hats). Integrating this over $\omega$,

$$\int_{-\infty}^{\infty} d\omega \omega S_{A_b A_b}(\omega) = \int_{-\infty}^{\infty} d\omega \omega F \langle A_b(t) A_b(0) \rangle$$

$$= i \int_{-\infty}^{\infty} d\omega F \langle \dot{A}_b(t) A_b(0) \rangle$$

(23)

(24)

We have

$$E_b = \frac{1}{V_b} \int dV \cdot b = \frac{1}{V_b} \int dV (-\nabla A_0 - \dot{A}) \cdot b = -\dot{A}_b$$

(25)

since $\nabla \cdot b = 0$, so

$$\int_{-\infty}^{\infty} d\omega \omega S_{A_b A_b}(\omega) = -2\pi i \langle E_b(0) A_b(0) \rangle$$

(26)

If the fluctuations of $A_b$ are stationary in time (as for e.g. a coherent state), then $F \langle A_b(t) A_b(0) \rangle$ is real. So, $\langle A_b(0) E_b(0) \rangle$ is imaginary, and consequently, for equal times,

$$\langle E_b A_b \rangle = \frac{1}{2} \langle [E_b, A_b] \rangle = \frac{i}{2V_b}$$

(27)

reproducing equation 21.

As per the previous section, we are usually interested in the fluctuations across some narrow frequency range. In general, there can be contributions to equation 21 from positive and negative $\omega$, leading to cancellations. However, for the ground state of the system, $S_{A_b A_b}(\omega) = 0$ for $\omega < 0$; for any operator $\hat{F}$,

$$\langle 0 | \hat{F}(t) \hat{F}(0) | 0 \rangle = \sum_n e^{-i(\omega_n - \omega)t} | \langle n | \hat{F} | 0 \rangle |^2$$

(28)

So, for the ground state, we obtain the sum rule

$$\int_{0}^{\infty} d\omega \omega S_{A_b A_b}(\omega) = \frac{\pi}{V_b}$$

(29)

The same is true for coherent states, since their fluctuations on top of the c-number expectation value are the same as for the ground state.

C. Effective Hamiltonians

The above derivations relied on the conjugate momentum of $A_b$ being $E_b$, i.e. there being no other terms in the Hamiltonian involving $A_b$. For example, if we were considering a dielectric medium, where the energy density is $\epsilon E^2$, then the conjugate momentum to $A_b$ would be $\epsilon E_b$, and we would have $\int d\omega \omega \dot{A}_b = \frac{i}{\epsilon}$. Thus, if e.g. a resonant cavity is filled with dielectric material, the power it is able to absorb decreases [33]. From above, we know that once all of the dynamics are taken in account, $\int_{-\infty}^{\infty} d\omega \omega \text{Im} \chi \leq \frac{\pi}{V_b}$. This implies that the ‘extra’ fluctuations must be at frequencies above the validity of the effective Hamiltonian.

Similarly, the $\pm\infty$ limits of the $\omega$ integrals above should not be taken literally — at the very least, electroweak physics arises at some energy scale! What we can infer is that, for frequency ranges over which our description of the system is good, $\int_{-\infty}^{\infty} d\omega \omega S(\omega) = \frac{\pi}{V_b}$. This implies that the ‘extra’ fluctuations must be at frequencies above the validity of the effective Hamiltonian.

D. Axion velocity

So far, we have taken the axion velocity to be zero. This will not be strictly true; axion DM in the galaxy is expected to have a virialized velocity distribution, with typical velocity $\sim 10^{-3}$ [34]. As per equation 10, the interaction term is $\mathcal{L} \supset -\frac{1}{2} g A^\mu J_\mu^{(a)}$, where

$$J_\mu^{(a)} = \left( \frac{\nabla a}{\dot{a} B} + (\nabla a) \times E \right)$$

(30)

Compared to the zero-velocity case, the axion velocity term $\nabla a$ results in a coupling to the scalar potential $A_0$, as well as the vector potential $A$. However, we can work in a gauge in which $A_0 = 0$, in which case the extra coupling term is

$$\mathcal{L} \supset A_1 \cdot ((\nabla a) \times E_0)$$

(31)

(after integration by parts). For an axion wave of definite momentum, this corresponds to the $B$ field in the axion rest frame, as expected.

Consequently, we can replace $\dot{a} B_0$ by $\dot{a} B_0 + (\nabla a) \times E_0$ as our forcing term. Since $|\nabla a| \sim v_a |\dot{a}| \sim 10^{-3} |\dot{a}|$, and the attainable (static) magnetic fields in laboratories are significantly larger than attainable electric fields, the $\dot{a} B_0$ term dominates in almost all circumstances of interest.
We have equal-time commutation relations only of time, we have axion masses $m_a$ each with their own time dependence. Writing decomposing $\dot{V}$ is larger, then the axion field is incoherent over times $\gtrsim v_a^{-1} m_a^{-1}$. If the experimental volume is larger, then the axion field is incoherent over distances $\sim v_a^{-1} m_a^{-1}$.

We can treat the spatial variation of the axion field, as well as any time-dependence of the $B_0$ spatial profile, by decomposing $\dot{a}(t)B_0(t)$ into spatially orthogonal modes, each with their own time dependence. Writing

$$\dot{a}(t)B_0(t) = \sum_i \dot{a}_b(t)B_b(t)b_i$$

(32)

where $\int dV b_i \cdot b_j = \delta_{ij} V_b$, and $a_b$ and $B_b$ are functions only of time, we have

$$H_{\text{int}} = g \sum_i \dot{a}_b B_b A_{b_i}$$

We have equal-time commutation relations

$$[A_{b_i}, E_{b_j}] = -\delta_{ij} \frac{i}{V_b}$$

(34)

So, the cross terms in the sum rule vanish, giving

$$\int_{-\infty}^{\infty} d\omega \mathcal{F}(A_b(t_0)A_b(t_0 + t)) = \delta_{ij} \frac{\pi}{V_b}$$

(35)

This is what we would expect from the impulse argument above — over very short timescales, there is no dynamics coupling the spatially orthogonal target modes, so they have independent responses to pulses. We will discuss some of the consequences of this in section III B.

III. PARAMETRICS OF DM DETECTION

The sum rules derived in the previous section can be used to bound the average power absorbed from the axion effective current, in an axion DM detection experiment. The simplest case is when the $B_0$ field is static. Over sufficiently long integration times, so that we resolve the spectral features of the axion signal, the expected time-averaged power absorbed for an axion of mass $m_a$ is

$$P_m \simeq \frac{g^2 B_0^2 v_b^2}{2\pi} \int_{-\infty}^{\infty} d\omega \omega S_{\dot{a}\dot{a}}(\omega) \overline{\chi_i(\omega)}$$

(36)

where we write $\overline{\chi_i} = \text{Im} \tilde{\chi}_i$. Averaging this over different axion masses $m_a$, we can use the fact that, since the axion bandwidth is small, $\delta \omega_a \sim 10^{-6} m_a$, integrating over $m_a$ for fixed $\omega$ is approximately the same as integrating over $m_a$ for fixed $\omega$,

$$\int_0^{\infty} dm \, S_{\dot{a}\dot{a}}(\omega) \simeq \int_0^{\infty} d\omega' \, S_{\dot{a}\dot{a}}(\omega') |_{m=\omega} \simeq \pi \rho_a$$

(37)

Hence,

$$\int_0^{\infty} dm \, P_m \simeq \frac{g^2 B_0^2 v_b^2 \rho_a}{2} \int_{-\infty}^{\infty} d\omega \omega \text{Im} \tilde{\chi}(\omega)$$

(38)

$$= \frac{g^2 \rho_a v_b B_0^2}{2} \pi$$

(39)

Consequently, the absorbed power, integrated over all axion masses, is set by the magnetic field energy in the $B_0$ field (ignoring magnetic fields on very small spatial scales — see appendix B).

If we are interested in looking for an axion within a specific mass range $\Delta m$, then we can average $P_m$ over that mass range,

$$\overline{P} \equiv \frac{1}{\Delta m} \int_{m}^{\Delta m} dm \, P_m \simeq \frac{g^2 \rho_a v_b B_0^2}{2} \frac{\pi}{\Delta m}$$

(40)

where the inequality assumes that $\omega \overline{\chi_i}(\omega) \geq 0$ for all $\omega$. Equality can be obtained if the response function is concentrated into the $\Delta m$ range (we will discuss some of the experimental practicalities of this in section IV). Since $\min_{m} \Delta m \ll \Delta m \leq \bar{P}$, the smallest absorbed power for any axion mass within the range $\Delta m$ is upper-bounded by equation 10. As expected, searching over a smaller axion mass range permits higher conversion powers within that range.

In many cases, instead of operating a single experimental configuration for the whole observation time, we ‘tune’ our experiment by operating it in different configurations, one after the other. The average power for a given axion mass is the appropriately weighted sum of the powers from the different configurations, and the corresponding limits apply.

The equations above apply to the whole experimental apparatus. However, a common experimental setup is to have a conductive shield (e.g. an EM cavity) inside a larger magnetic field. If the relevant dynamics inside and outside the cavity are independent, then we can apply the above arguments to the volume inside the cavity, replacing the total magnetic field energy by the energy inside the cavity.

A. PQL limits

From appendix A if the quantum fluctuations of $A_b$ are stationary, then the $O(g^2)$ formula for the probability of the axion interaction changing the state of the target system is

$$P_{\text{ex}} \simeq \frac{g^2 B_0^2 v_b^2}{\pi} \int_{0}^{\infty} d\omega \, S_{\dot{a}\dot{a}}(\omega) S_{A_b A_b}(\omega)$$

(41)
where \( \tilde{S} \) denotes the symmetrised spectral density, and we assume that \( t_{\text{exp}} \) is much longer than the inverse bandwidth of spectral features. If the fluctuations are equal to those in the ground state, then \( \tilde{S}_{A_b A_b}(\omega) = \tilde{\chi}(\omega) \) for \( \omega > 0 \). In that case,

\[
P_{\text{ex}} \simeq \frac{g^2 B_0^2 V_b^2 t_{\text{exp}}}{\pi} \int_0^{\infty} d\omega \, S_{A_b}(\omega) \text{Im}(\tilde{\chi}(\omega)) \simeq \frac{P_m}{m} \tag{42}
\]

where the latter equality holds since \( S_{A_b} \) is tightly concentrated around \( m \). Hence, \( P_{\text{ex}} \simeq N_a \), the expected number of quanta absorbed. More generally, if \( \tilde{S}_{A_b A_b} \) for the operational state of the detector satisfies \( \int_0^{\infty} d\omega \omega \tilde{S}_{A_b A_b} \simeq \pi/(2V_b) \), then the axion-mass-averaged excitation probability satisfies

\[
\tilde{P}_{\text{ex}} \leq \frac{g^2 \rho_a V_b B_0^2 t_{\text{exp}} \pi}{m \Delta m} \equiv \bar{N}_a \tag{43}
\]

where the average is taken over a fractionally-small axion mass range \( \Delta m \), centred on \( m \). To be confident of identifying or excluding an axion signal, we need \( \bar{N}_a \gtrsim \) few.

In section \[ \text{IV B 2} \], we discuss the circumstances under which we expect \( S_{A_b A_b} \) to satisfy the ground-state sum rule. The most obvious example, in which this limit is achievable, is the case of an absorptive, background-free photon counter. In the presence of noise sources, (such as thermal noise or detector noise), it may not be possible to attain this limit. Conversely, if a detection setup does not satisfy the ground-state sum rule, then the FQL still places limits on its sensitivity, but these will depend on how much the fluctuations exceed the PQL value.

### B. SQL op-amp

If we read out our signal using a phase-invariant, SQL-limited amplifier coupled weakly to the target (i.e. in ‘op-amp’ mode \[ \text{23} \]), then from appendix \[ \text{A 1} \] the SNR from an axion signal satisfies

\[
\text{SNR}^2 \leq (gB_0 V_a)^4 t \int_0^{\infty} d\omega \frac{2\pi}{\bar{S}_n} \left( \frac{S_{A_b} |\tilde{\chi}|^2}{|\tilde{\chi}| + \tilde{\chi}_i + \bar{S}_n} \right)^2 \tag{44}
\]

where \( S_n \) summarises the effects of any additional noise (beyond amplifier backaction, imprecision, and zero-point fluctuations), referred back to \( A_b \). For example, given thermal noise at temperature \( T \), we have \( S_n \geq 2n_T \tilde{\chi}_i \), where \( n_T(\omega) \equiv (\omega/\gamma - 1)^{-1} \[ \text{34} \]. We assume that the spatial profile of \( J(a) \) can be treated as constant in time, to begin with.

If we are interested in a mass range \( \Delta m \gg \delta \omega_a \) (i.e. a fractional mass range \( \gg 10^{-6} \)), then the quantity determining the axion-mass-averaged SNR^2 is

\[
S \equiv \int_{\Delta m} d\omega |\tilde{\chi}|^2 \left( \frac{1}{1 + (1 + 2n_T)\tilde{\chi}_i/|\tilde{\chi}|} \right)^2 \tag{45}
\]

For a single-pole resonator, \( \tilde{\chi}_i = M \omega \gamma |\tilde{\chi}|^2 \). For a more complicated response function, if the minimum distance of a pole of \( \tilde{\chi}(\omega) \) (extended to a complex function) from the real \( \omega \) axis is \( \gamma \) (i.e. the minimum damping rate), then \( \bar{\chi}_i(\omega) \simeq M \omega \gamma |\tilde{\chi}(\omega)|^2 \). If we demand that \( \bar{\chi}_i \) satisfies the sum rule in equation \[ \text{20} \] then \( S \) is maximised by a single-pole response function, for given \( \gamma \). For \( \Delta m \gtrsim (1 + 2n_T)\gamma \), the integrand in equation \[ \text{45} \] is dominated by a bandwidth \((1 + 2n_T)\gamma\), and we have

\[
S \leq \frac{1}{2V_B^2 m^2 \gamma f(n_T)} \tag{46}
\]

where

\[
f(n_T) \simeq \begin{cases} \frac{2}{n_T} & n_T \ll 1 \\ \frac{n_T}{4} & n_T \gg 1 \end{cases} \tag{47}
\]

assuming that \( \gamma \ll m \). To relate this to the SNR, we can for simplicity take the axion signal to have a top-hat spectral form, \( S_{A_b A_b} \equiv \frac{\pi}{\delta \omega_a} \delta_{\omega_a} = \frac{\pi}{\delta \omega_a} \delta_{\omega_a} \). In this case,

\[
\text{SNR}^2 \lesssim \left( gB_0^2 V_b \rho_a^2 \right)^2 \frac{2T Q_a}{\pi m^2} \times \int_{\delta \omega_a} d\omega |\tilde{\chi}|^2 \left( \frac{1}{1 + (1 + 2n_T)\tilde{\chi}_i/|\tilde{\chi}|} \right)^2 \tag{48}
\]

So, averaging this over the \( \Delta m \) axion mass range,

\[
\bar{\text{SNR}}^2 \lesssim \frac{\pi}{4} \left( g B_0^2 V_b \rho_a^2 \right)^2 \frac{T Q_a}{m^3 \Delta m \gamma f(n_T)} \tag{49}
\]

where we assume that \( n_T \) is approximately constant over the range \( \Delta m \) (if not, we can replace it by its minimum value). Writing this in terms of the \( P \) expression from above,

\[
\bar{\text{SNR}}^2 \lesssim \frac{P_{\text{max}}^2 Q_a t \Delta m}{\pi m^3 \gamma f(n_T)} = \frac{P_{\text{max}}^2 Q_a t \Delta m}{\pi m^4 f(n_T)} \tag{50}
\]

where \( Q \equiv \gamma/m \), and \( P_{\text{max}} \) is the quantity from equation \[ \text{40} \]. This inequality can be saturated for an amplifier with the correct coupling to a single-pole resonator. As mentioned above, for each resonant configuration, the bandwidth contributing most of the \( \text{SNR}^2 \) is \((1 + 2n_T)\gamma \), so we need to scan over multiple different configurations to have sensitivity over a wide axion mass range. Conversely, if \( n_T \) is large enough that \( \Delta m \lesssim 2n_T \gamma \), then a single resonant configuration can cover the entire mass range approximately equally, giving

\[
\text{SNR}^2 \lesssim \frac{\pi}{8} (gB_0^2 V_b \rho_a^2)^2 \frac{T Q_a}{m^3 \gamma^2 n_T^2} \tag{51}
\]

\[ \text{5 While the thermal fluctuations of } A_b \text{ are set by } 2n_T \tilde{\chi}_i, \text{ if the amplifier is coupled to other degrees of freedom, then the total effect of thermal noise on the output may be greater.} \]
Optimising axion-mass-averaged sensitivity with an SQL-limited amplifier is also discussed in \([14, 33]\). In their analysis of detection using a flux-to-voltage amplifier (in the quasi-static regime; see section \(\text{IVC}\)), they assume that the amplifier’s coupling does not vary as a function of frequency, as would be required to achieve \(S_{\text{add}} = |\hat{\chi}|\) over the relevant frequency range. Consequently, for them, achieving sensitivity over a \(\sim (1 + 2\nu_f)^\gamma\) bandwidth requires ‘overcoupling’ the amplifier, and degrading the sensitivity on resonance (by \(O(1)\)). These issues only make an \(O(1)\) difference to the SNR achieved; we give the full SQL-limited sensitivity, for completeness.

In some circumstances, we can achieve tighter bounds by looking at the damping in more detail. The power absorbed in response to a monochromatic forcing \(J(t) = J_0 \cos(\omega t)\), is \(\frac{1}{2} J_0^2 \omega \hat{\chi}(\omega)\). This is also equal to the dissipated power, \(P_{\text{diss}} = \gamma U\), where \(\gamma(\omega)\) is the damping rate, and \(U\) is the energy stored. The electric field energy in the \(E_B\) response is \(U_E = \frac{1}{2} V \omega^2 J_0^2 |\hat{\chi}(\omega)|^2\), so

\[
\hat{\chi} = V \omega \gamma U \frac{U}{U_E} |\hat{\chi}|^2
\]

So, if \(U > U_E\), we have a tighter bound than above (for example, in the quasi-static regime, as we discuss in section \(\text{IVC}\)).

The above expressions are only valid when the relevant integration times are long compared to the inverse bandwidth (for example, in the quasi-static regime, as we discuss in section \(\text{IVC}\)).

In contrast to how the fluctuation spectrum can be concentrated across the \(\Delta m/\delta \omega\) at least \(\Delta m\) if we consider a resonator with \(Q\), the width of spectral features. Looking at the integration times are long compared to the inverse bandwidth requires ‘overcoupling’ the amplifier, and degrading the sensitivity on resonance (by \(O(1)\)).

Achieving good SNR in such configurations requires stronger couplings to the target mode, significantly affecting its damping \(\chi\). To find a SNR limit, we can consider the ‘input’ coupling to encompass everything that couples to \(A_b\), writing \(H_{\text{int}} = A_b \hat{F}\). Then, the fluctuation spectrum of \(F\) must be such that \(S_{A_b A_b} = 2 \chi_b \Theta(\omega)\), so \(S_{FF} = \frac{1}{\chi_b} \Theta(\omega)\). Using the analysis from \(\text{[24]}\), this implies that the total noise at the amplifier output is \(S_{A_b A_b} \lesssim 2 \hat{\chi}_i^2\). Consequently,

\[
\text{SNR}^2 \leq \frac{1}{8} \left( \int_0^\infty \frac{d\omega}{2\pi} (2 S_{A_b} \hat{\chi}_i)^2 \right) \approx \frac{1}{8} \left( \frac{1}{\chi_b} \right)^2 \Theta(\omega)
\]

which has the same form as equation \(55\). Taking \(Q \approx \nu t_1\), we again obtain \(\text{SNR} \sim N_{\text{max}}\). An isolated amplifier

in an orthogonal set of spatial profiles. This is necessary if the axion coherence length is smaller than the scale of the experiment (or the extent of the background magnetic field, whichever is smaller), i.e if \(v_{\alpha} \lesssim (v_i L)^{-1} \sim 300 \text{GHz}/\text{meter}/L\) \(\text{[7]}\). The SNRs from these orthogonal spatial profiles will add in quadrature.

C. Isolated amplifier

For the op-amp coupling considered in the previous subsection, the amplifier’s backaction drives the target mode out of equilibrium. Another way to couple an amplifier to the target is to isolate it, so that the overall backaction is simply vacuum noise. This is common at microwave frequencies, where an amplifier is usually isolated behind a circulator, with its backaction absorbed by a cold load \(\text{[23, 26, 36, 37]}\).

The purpose of the circulator-plus-cold-load analyses from \(\text{[22, 33]}\). Similarly to the op-amp case, the axion-mass-averaged SNR \(2\) is maximised, for given \(\gamma\), by a single-pole resonance. This gives an axion-mass-averaged SNR \(2\) of

\[
\text{SNR}^2 \leq \frac{\pi}{4} \left( \frac{g B^2 V_b \rho_a}{m^3 \Delta m \gamma} \right) \approx \frac{1}{2 S_{FF}} \frac{t Q_a}{m^3 \Delta m \gamma}
\]

In contrast to how the fluctuation spectrum can be concentrated into a narrow-bandwidth peak, at the expense of surrounding frequencies, the frequency-averaged fluctuations for each orthogonal spatial mode are fixed by its spatial profile, and cannot be concentrated into one mode at the expense of others. In the impulse picture, if we imagine a set of simultaneous impulses at different spatial points, then causality prevents the energy absorbed from depending on their relative signs etc, whereas the responses to impulses at different times (picking out a specific frequency) can interfere with each other.

For an ideal quantum amplifier, \(S_{A_b A_b} S_{FF} \geq 1\). However, in our case \(S_{FF} = 0\) for \(\omega < 0\), whereas \(S_{II}\) is even, which implies that there must be wasted information in the \(FI\) correlation \(23\). In particular, by feeding back some of the output, the backaction could be made symmetric, decreasing \(S_{FF}\) by a factor \(1/2\). Consequently, in our case, \(S_{A_b A_b} S_{FF} \geq 1/2\). Thus, \(S_{A_b A_b} \geq \frac{1}{2 S_{FF}} \frac{t Q_a}{m^3 \Delta m \gamma} \frac{1}{S_{FF}} \frac{t Q_a}{m^3 \Delta m \gamma}

\text{(56)}\)

We leave such an analysis to future work. This conjecture is of mostly academic interest, since we can work out the SQL limit directly (as done here).

If the spatial profile of \(J(\omega)\) is not constant, then as discussed in section \(\text{IIID}\), we can consider the time variation of the magnitude of the back-action noise is small enough that this has to be true \(23\). For stronger couplings, we would need to condition on the observed output of the amplifier (cf the analysis of backaction evasion in \(25\)). We leave such an analysis to future work. This conjecture is of mostly academic interest, since we can work out the SQL limit directly (as done here).

6 In contrast to how the fluctuation spectrum can be concentrated into a narrow-bandwidth peak, at the expense of surrounding frequencies, the frequency-averaged fluctuations for each orthogonal spatial mode are fixed by its spatial profile, and cannot be concentrated into one mode at the expense of others. In the impulse picture, if we imagine a set of simultaneous impulses at different spatial points, then causality prevents the energy absorbed from depending on their relative signs etc, whereas the responses to impulses at different times (picking out a specific frequency) can interfere with each other.

7 For an ideal quantum amplifier, \(S_{A_b A_b} S_{FF} \geq 1/4\). However, in our case \(S_{FF} = 0\) for \(\omega < 0\), whereas \(S_{II}\) is even, which implies that there must be wasted information in the \(FI\) correlation \(23\). In particular, by feeding back some of the output, the backaction could be made symmetric, decreasing \(S_{FF}\) by a factor 1/2. Consequently, in our case, \(S_{A_b A_b} S_{FF} \geq 1/2\). Thus, \(S_{A_b A_b} \geq \frac{1}{2 S_{FF}} \frac{t Q_a}{m^3 \Delta m \gamma} \frac{1}{S_{FF}} \frac{t Q_a}{m^3 \Delta m \gamma}

\text{(56)}\)
satisfies the PQL assumptions, so this bound necessarily holds.

If the ‘intrinsic’ dissipation in the target is significant compared to the damping from the amplifier, it may not be possible to attain the limit from equation 55. Analysing a circulator plus cold load, as in [22 33], shows that the axion-mass-averaged SNR$^2$ is optimised for $\gamma_{\text{det}} = 2\gamma_{\text{int}}$, which gives a suppression of $8/27$ vs equation 56.

Since the temperature of the back-action noise from the amplifier may be less than the temperature of other parts of the target (see discussions in [14 22 33]), there may no longer be a single equilibrium temperature that describes our target system. As analysed in [14 22 33], if the dissipation in the target system is fixed, it is beneficial to ‘overcouple’ to the amplifier, increasing the dissipation of the overall system. At $T = 0$, this would result in worse sensitivity, but it also reduces the thermal noise reaching the detector, its overall effect is to improve the SNR. From [14 22 33], for $n_T \gg 1$, the resulting SNR$^2$ is suppressed by $\sim 1/n_T$ compared to the $T = 0$ value.

D. Quantum-limited op-amp

As discussed in appendix A 1, if it is possible to optimise the correlations between a linear amplifier’s backaction and imprecision noise, then the minimum added noise, referred back to the measured variable $x$, is $S_{xx}^{\text{add}}(\omega) \geq |\tilde{\chi}_i(\omega)|$ (as opposed to $S_{xx}^{\text{add}} \geq |\tilde{\chi}|$ for uncorrelated noise). Consequently, for an amplifier connected in op-amp mode,

$$\text{SNR}^2 \leq (gB_0V_b)^2 t \int_0^\infty \frac{d\omega}{2\pi} \left( \frac{S_{xx}^{\text{add}}|\tilde{\chi}_i|^2}{2 \tilde{\chi}_i + S_n} \right)^2$$  (57)

Assuming thermal noise, $S_n = 2n_T\tilde{\chi}_i$, gives

$$\text{SNR}^2 \leq \frac{1}{8\pi} (g^2B_0^2V_b^2)^2 t \int_0^\infty d\omega \left( \frac{S_{xx}^{\text{add}}}{\omega \gamma(1 + n_T)} \right)^2$$  (58)

If we take $S_{xx}^{\text{add}}$ to have top-hat form, as above, then

$$\text{SNR}^2 \leq \frac{\pi}{8} (g^2B_0^2V_b\rho_a)^2 t \int_0^\infty d\omega \left( \frac{Q_a}{m_3^3 \gamma^2 (1 + n_T)} \right)^2$$  (59)

This has the same form as the SQL expression in equation 52 when $\Delta m \lesssim (1 + 2n_T)\gamma$; this is as expected, since in both cases, the added noise is dominated by the thermal + ZPF noise. If $\Delta m \gtrsim (1 + 2n_T)\gamma$, then we can improve over the SQL limit by $\sim \Delta m / 2\gamma = Q_a \Delta m / m$ (for $n_T \gg 1$). This is simply the ratio between the sensitivity bandwidth for the SQL-limited amplifier, $\sim (1 + 2n_T)\gamma$, and the broadband $\sim \Delta m$ sensitivity for the QL-limited case. On-resonance, where $\tilde{\chi}$ is purely imaginary, we achieve the QL limit by having uncorrelated back-action and imprecision noise, so we expect the QL and SQL limits to coincide in this case.

As the above limits show, the sensitivity limit for a quantum-limited amplifier is set by how well we can isolate the system from its environment, and so reduce $\gamma$. There is no sensitivity/bandwidth trade-off, as occurs in the SQL and PQL cases, since as $\chi_i$ decreases, the ZPF noise also decreases; a setup that saturates the QL at all frequencies is naturally broadband.

The fact that a QL amplifier can have better sensitivity than the PQL limit, in some regimes, shows that the amplifier must be enhancing the quantum fluctuations of $A_b$ (similarly to how backaction evasion effectively drives an oscillator into a squeezed state [35]).

At higher frequencies ($\gtrsim$ GHz), making use of correlated backaction/imprecision in this way is usually difficult (in particular, it is not compatible with isolation mechanisms such as circulators, as discussed above). However, at lower frequencies, SQUID amplifiers can attain near-QL performance, in some circumstances [30]. We discuss axion detection at low frequencies ($\ll$ GHz) in section IV.C.

E. Up-conversion

We can generalise the power absorption calculations above to a time-dependent magnetic field. In this case,

$$P_m \sim \frac{g^2V_b^2}{(2\pi)^2} \int_{-\infty}^\infty d\omega \omega \tilde{\chi}_i(\omega)(S_{xx}^{\text{add}} * S_{BB})$$  (60)

$$= \frac{g^2V_b^2}{(2\pi)^2} \int_{-\infty}^\infty d\omega S_{xx}^{\text{add}}(\omega)(\tilde{\omega} \tilde{\chi}_i) * S_{BB}$$  (61)

where $S_{BB}$ is the spectral density of $B_0(t)$. So, using equation 57,

$$\int dm P_m \sim \frac{g^2V_b^2\rho_a}{4\pi} \int_{-\infty}^\infty d\omega (\tilde{\omega} \tilde{\chi}_i) * S_{BB}$$  (62)

$$\simeq \frac{g^2V_b\rho_a B_0^2 \pi}{2}$$  (63)

where $\int_{-\infty}^\infty d\omega S_{BB}(\omega) = 2\pi B_0^2$, so $\sqrt{B_0^2}$ is the RMS $B_0$ value. In the case of a static $B_0$ field, this reproduces equation 59.

There are a number of qualitatively distinct cases, depending on the oscillation frequencies of the magnetic field and the axion signal. For a $B_0$ oscillation at frequency $\omega_B$, an axion oscillation at $\omega_a$, we will get a forcing at sum and difference frequencies, $\omega_B + \omega_a$ and $|\omega_B - \omega_a|$. To start with, we will consider the ‘up-conversion’ case where $\omega_B \gg \omega_a$, so that both sum and difference frequencies are $\gg \omega_a$.

The power absorbed from single-frequency axion and $B_0$ oscillations is set by

$$P \simeq \frac{1}{4} g^2V_b^2 B_0^2 \rho_a\omega_B (\tilde{\chi}_i(\omega_B + m_a) + \tilde{\chi}_i(\omega_B - m_a))$$  (64)
(this is valid over times $\gg m_a^{-1}$, so that we resolve the two different frequencies). Averaging this over an axion mass range $\Delta m$, we obtain

$$
P \sim \frac{g^2 \rho \mu B^2 \pi}{\Delta m} \frac{\omega}{m} \frac{n_i}{f(n_T(\omega_1))} \sim \frac{g^2 \rho \mu B^2 \pi}{\Delta m} \frac{\omega N}{T \omega_1^2}$$

which can be saturated by concentrating $\chi_i$ into a $\sim \Delta m$ range either above or below $\omega_B$ (or in both ranges). This only represents half of the total mass-integrated power absorbed. Since axions at $m_a \approx 2\omega_B$ will also excite a target mode at $\sim \omega_B$, the other half of the absorption is at these, much higher, masses.

The PQL detectability is again set by the expected number of quanta absorbed, which is $N_\alpha \approx P\omega_B / \omega$. Compared to a static-field experiment with the same $P$, an up-conversion experiment absorbs fewer but higher-energy quanta. The SQL limit (and parametrically the isolated-amplifier limit) is

$$
\text{SNR}^2 \lesssim \frac{1}{\pi} \frac{P_{\text{max}}}{m} \frac{\Delta m}{\omega_1^2 f(n_T(\omega_1))} \approx \frac{1}{\pi} \frac{P_{\text{max}}}{m} \frac{\Delta m}{\omega B^2} \frac{n_i}{T} \left(\frac{tQ_aQ_1}{m} \right)
$$

where the second equality is for $T \gg \omega_1$. For $n_T \ll 1$, taking $Q$ as large as it can be while still resolving the resonator bandwidth gives $\text{SNR}^2 \sim \left(\frac{P_{\text{max}}}{\omega_1} \right)^2 \sim N^2$, in analogy to equation $\text{SNR}^2 \sim \left(\frac{P_{\text{max}}}{\omega_1} \right)^2 \sim N^2$, in analogy to equation $\text{(64)}$. A QL-limited amplifier could improve on the SQL by the usual $\sim \frac{\Delta m}{\omega B^2}$ factor. However, as we will discuss in section $\text{IV} \text{D}$, up-conversion experiments are most interesting at $\omega_1 \gtrsim \text{GHz}$, and utilising backaction/imprecision correlations is difficult there.

As the limits above indicate, up-conversion experiments have reduced sensitivity, compared to ideal static-field experiments with the same $U_B$. However, as we will discuss in section $\text{IVC}$, for low enough axion masses, it is hard for static-field experiments to attain the power and sensitivity bounds from above, due to being in the quasi-static regime. Consequently, up-conversion experiments can have a parametric advantage in absorbed power, and sensitivity, for low axion masses.

**F. Down-conversion**

We can also consider other frequency combinations. If $\omega_B \ll m_a$, then the situation is basically the same as the static-field case. The remaining distinct case is when $\omega_B \approx m_a$, so that the difference frequency is small, $\omega_s = |\omega_B - m_a| \ll m_a$.

In this case, which we label ‘down-conversion’, it is possible to attain the $P$ bound from equation $\text{(63)}$ by concentrating $\chi_i$ at low frequencies $\sim \omega_s$. Then, $(\omega \chi_i) B_{\epsilon} B_B$ is entirely at frequencies $\sim \omega_s$. Thus, we obtain the same converted power limit as for a static-field experiment.

However, since $N_\alpha = P t / \omega_s$, by taking $\omega_s$ smaller and smaller, we can make the PQL sensitivity better and better (for integration times long enough to resolve $\omega_s$).

Physically, this is the converse of the up-conversion case — we maintain the same converted power, but this corresponds to absorbing more low-energy quanta. As per equation $\text{(66)}$ the SQL-limited SNR is $\sim 1/\omega_s$, for given $U_B$, $Q$, and $T > \omega_s$ (and $\sim \omega_s^{-3/2}$ for $T \lesssim \omega_s$).

Thus, at least in principle, we can improve our sensitivity by making $\omega_s \ll m_a$. However, as we discuss in section $\text{IV} \text{E}$ there are usually serious practical obstacles to obtaining an advantage in this way.

**IV. DM DETECTION EXPERIMENTS**

In the previous section, we analysed how the background magnetic field imposed limits on the power absorption and sensitivity of idealised axion DM detection experiments. In this section, we will investigate the sensitivities of more specific experimental setups, and how they relate to these theoretical limits.

**A. Static background field: power absorption**

From equation $\text{(40)}$, a static-field experiment covering an axion mass range $\Delta m$ must have

$$
\min_{m \in \Delta m} P \leq \frac{g_\gamma^2 \rho m U_B}{\Delta m}
$$

if it is in the linear response regime, and $\chi_i(\omega) \geq 0$ for $\omega > 0$. For experiments with $m_a \gtrsim L^{-1}$, where $L$ is the length scale of the shielded volume, it is easy to see that the usual EM field modes $O(1)$-saturate the sum rules, for slowly-varying $B_0$ profiles. Consequently, the signal power for cavity and dielectric haloscopes is parametrically given by equation $\text{(67)}$. For example, a cavity haloscope has an on-resonance, fully-run-up signal power of

$$
P_{\text{sig}} = C g^2 B_0^2 V Q \rho \frac{\omega}{m}
$$

where $Q$ is the quality factor of the cavity mode, and

$$
C \equiv \frac{\int dV \ E \cdot B_0}{\int dV |B_0|^2} \frac{\int dV |E|^2}{\int dV}
$$

9 From the point of view of the axion field, the $A \cdot B$ term it is interacting with oscillates at a frequency $\sim m_a$, whereas the effective current $g_\gamma B$ oscillates at $\sim \omega_s$ — the energy change of the axion field is therefore greater than the energy gain in the target mode, with the difference made up by the $B_0$ oscillation.
is the normalised overlap between the background magnetic field and the electric field of the mode. Geometrically, $C \leq 1$. If $Q \gg 1$, then the converted power is approximately a Lorentzian function of the axion mass, so if we want to cover a small mass range $\Delta m$ with cavity configurations tuned to different resonant frequencies, we have

$$P \simeq \frac{g^2 B_0^2 V \rho \pi}{\Delta m} \frac{1}{2} C$$

as expected from equation (71). ADMX [8] uses the TM$_{010}$ mode of a cylindrical cavity, which has $C \simeq 0.68$ (ignoring the perturbations from the tuning rods), so is $O(1)$ optimal for its cavity volume. The rest of the sum-rule-determined absorption is into other cavity modes — these will generally be at different frequencies, so will not be useful for searching a small axion mass range (though they may be used to perform simultaneous searches in different mass ranges, e.g. ADMX’s usage of the TM$_{020}$ mode alongside TM$_{010}$ [36]).

For a dielectric haloscope with alternating half-wave layers of refractive indices $n_1, n_2$, the average signal power over an axion mass range $\Delta m$ around the half-wave frequency is

$$P = \frac{4 g^2 B_0^2 V \rho}{\pi} \left( \frac{1}{n_2} - \frac{1}{n_1} \right)^2$$

If we take $n_1 = 1$, $n_2 \gg \infty$, then this is $\sim 8/\pi^2 = 0.81$ of the sum rule limit. The rest of the response function will be at other axion masses. For a half-wave stack, this will mostly be at odd multiples of the half-wave frequency [12]: since $\sum_{n=0}^{\infty} \frac{1}{2n+1} = \frac{\pi^2}{8}$, this corresponds to the fractional contribution calculated above.

At microwave frequencies, it is relatively simple to find low-loss transparent dielectrics, which can almost saturate these bounds — for example, the MADMAX [11, 38] proposal currently aims to use LaAlO$_3$ layers, with permittivity $\epsilon \simeq 24$. However, at higher frequencies, it may be difficult to find suitable dielectrics with large $n$. For example, in the energy range $\sim 0.2 - 0.5$ eV, a potentially practical pair of transparent dielectrics is Si ($n \simeq 3.4$) and Al$_2$O$_3$ ($n \simeq 1.77$). This gives a suppression of $\frac{8}{\pi^2} \left( \frac{1}{n_2} - \frac{1}{n_1} \right)^2 \simeq 0.07$ for the absorbed power, relative to the volume-limited value.

In addition, for a small number of layers, the shielded volume required will be significantly larger than the dielectric-occupied volume, e.g. to allow space for the focusing optics. In that sense, the experiment is even further from optimality. However, such experiments can still be efficient compared to other high-frequency proposals, such as dish antennae [39]. For example, if we consider the Si/SiO$_2$ chirped stacks with reach shown in figure 2 (these have 100 periods each, and area $\sim (18 \text{ cm})^2$, covering an octave in axion mass range), an equivalent reach could be obtained from a $\sim$ meter$^2$ dish antenna in the same background magnetic field. This dish would require a significantly larger volume for the magnetic field, and could only concentrate the emitted photons onto a significantly larger area, complicating detection. If larger volumes of photonic material could be fabricated, or larger refractive index contrasts achieved, the comparison would be even more favourable.

The cavity and dielectric haloscope examples above achieved $P$ values at $O(1)$ of the limit. An example of a target that could theoretically attain the limit almost fully (for a uniform background magnetic field, within a narrow axion mass range) is a uniform plasma, of large extent compared to the axion Compton wavelength (as analysed in [40]). It should be noted that the wire metamaterials proposed in [40] most likely do not have this property, and have power absorption at $O(1)$ of the limit, similarly to a cavity haloscope.

The sum rule limits explain some basic properties of axion conversion rates in experiments; in particular, the power/bandwidth tradeoff, and the saturation of signal power to a limit determined only by the background magnetic field and volume. They also imply less obvious facts. For example, in dielectric haloscopes, one can enhance the conversion rate near a particular frequency by making the dielectric permittivity of some layers $\epsilon < 1$, e.g. by sitting near a resonance (the stack configuration of the molecular absorption proposal in [41] is an example of this). However, the sum rules tell us that this is compensated for by lower powers at other frequencies, and cannot improve the average power over an $O(1)$ axion mass range.

As mentioned in section IV A, static-field experiments for $m_a \ll L^{-1}$ generally suffer a quasi-static suppression, and the maximum power absorbed is suppressed by $\sim (m_a L)^2$. We defer discussion of this regime to section IV C.

B. Detectability

1. Linear amplifiers

Section III showed that, for systems satisfying the PQL assumptions, or for readout via a SQL or isolated amplifier, the theoretical detectability limits are closely related to the limit on $P$. With SQL op-amp readout, it is easy to imagine idealized setups that saturate the SNR bound from equation (50). For example, in a cavity haloscope experiment, we could connect an antenna to a SQL-limited current amplifier. In the case of a linear amplifier isolated behind a circulator and cold load, it is also easy, in principle, to attain the bounds from section III C to $O(1)$ [22].

There have been claims in the literature that simple feedback schemes can give enhanced sensitivities; for example, the digital feedback scheme of [42] (see also [21], which we discuss in section IV D). Since the feedback applied in [12] is entirely coherent-state, and a SQL (or circulator-isolated) readout system seems to be assumed,
the bounds derived above should apply. In particular, the thermal-noise-limited sensitivity cannot be improved by adding a known, coherent signal to the mode. Moreover, attempting to cover multiple frequency ranges in parallel results in worse sensitivity than tuning a narrow resonance over time, as per the results of section III.

2. PQL

In the PQL case, it is also simple to describe setups that can (in principle) attain the \( N_a = P t / m_a \) limit; for example, if we have no added noise (\( T \ll m_a \) etc), and almost all of the converted power is absorbed by a background-free photon counter.

One motivation for discussing the PQL, as opposed to specialising to e.g. absorptive photon counters directly, is that it can also apply to experiments using different kinds of target excitations, such as phonons [43], electron-hole pairs [44], or more complicated quasi-particles [45] (and different detection schemes such as optical homodyne detection). While a detection setup always includes some non-equilibrium components (e.g. amplifiers), in many cases, these are connected to the target system via a damping-type coupling. Examples include absorptive photodetectors, or sensors isolated behind a circulator with a matched load. In these cases, the quantum fluctuations of the target system are as if the sensor were replaced by an equivalent passive load. Consequently, the experiment’s sensitivity should be bounded by the PQL (for example, this applies directly to the case of an SQL amplifier isolated behind a circulator, as discussed above).

As mentioned in section IA, it is possible to violate the PQL by preparing the target in a ‘non-classical’ state, e.g. a Fock state or a squeezed state. It is also the case that some detection schemes push the target into an PQL-violating state; examples include back-action evasions [35], QND photon counting, and (as per the QL discussions above) correlated backaction/imprecision noise.

C. Quasi-static regime

When \( m_a \ll L^{-1} \), it is significantly more difficult to attain the sum-rule bounds with a static-field experiment. This is because the EM fields are naturally in the quasi-static regime at frequencies \( \ll L^{-1} \), and their \( A \) fluctuations are suppressed. The magnitude of the linear response function is similarly suppressed, so the amplifier SNR is also affected.

We can demonstrate this suppression, somewhat heuristically, by considering the fluctuations of the energy in the electromagnetic field. The rate of change of EM field energy is

\[
P_{EM} = - \int dV \; E \cdot J
\]  

(72)

In Lorenz gauge, \( \nabla \cdot A = -\partial_t A_0 \), we have \( \partial_a \partial^a A^\nu = (\partial_1^2 - \nabla^2) A^\nu = J^\nu \). If we are considering very low frequencies, \( \omega \ll L \), then since the \( A^\nu \) and \( J^\nu \) fields are localised on scales \( \sim L \), we have \( \nabla^2 A^\nu \approx -J^\nu \). Hence,

\[
P_{EM} \approx \int dV \; E \cdot (\nabla^2 A)
\]  

(73)

We can ignore the low-frequency condition for a moment, and take the expectation value of equation 73’s RHS, in the ground state. As per section II D, the terms corresponding to spatially orthogonal \( A \) profiles add independently, and have the same sign. Since

\[
- \int dV b \cdot (\nabla^2 b) = \int \frac{d^3k}{(2\pi)^3} k^2 |b_k|^2 \gtrsim L^{-2} V_b
\]  

(74)

we have

\[
- i \int dV \langle E_b \cdot (\nabla^2 A_b) \rangle \gtrsim -i \langle E_b A_b \rangle V_b L^2 = \frac{1}{2L^2}
\]  

(75)

Similarly, if we consider the fluctuations within a particular frequency range, we have

\[
\Delta P_{EM} \gtrsim \frac{V_b}{L^2} \frac{1}{2\pi} \int_{\Delta \omega} d\omega \; \omega \; S_{A_A A_b} (\omega)
\]  

(76)

Hence, if the sum rule were \( O(1) \)-saturated by the low-frequency fluctuations, then the low-frequency fluctuations of \( P_{EM} \) would be \( \frac{1}{L^2} \), in the ground state. Contrast, if the low-frequency EM modes behave like harmonic oscillators, then \( P_{EM} \sim \omega^2 \). Hence, in the latter case, the average value of \( S_{A_A A_b} \) must be suppressed by \( \sim (\omega L)^2 \) from its sum-rule-limited value.

We can gain some physical intuition for this suppression by splitting the quasi-static electric field into a ‘Coulomb’ part \( E_C = -\nabla A_0 \), and a ‘Faraday’ part \( E_F = -\partial_t A \) (again, in Lorenz gauge). The Coulomb part can have large fluctuations (as for an oscillation involving a capacitor), but \( E_C \) has zero integrated overlap with \( B_0 \). Conversely, \( E_F \) is naturally \( \sim (\omega L) B \), where \( B \) is the magnetic field fluctuation. Hence, if the ground state magnetic field fluctuations in a mode carry \( \sim 1 \) quantum of energy, as for a harmonic oscillator, then the \( E_F \) fluctuations will be suppressed, as will the \( A \) fluctuations.

Given this suppression, a static-field experiment searching for an axion over a small mass range \( \Delta m \) around \( m_a \) must have

\[
\text{min} \; P \lesssim \frac{g^2 B_0^2 V_b \rho_a}{\Delta m} (m_a L)^2
\]  

(77)

This scaling can be confirmed by calculations for specific experimental setups. In [69], the response of a small cavity in a constant magnetic field was computed, and in [14] a toroidal magnetic field was considered, both displaying the expected \( \sim (mL)^2 \) suppression.

The suppressed power absorption directly limits the sensitivity of PQL-limited searches. For SQL searches, the \( \sim (mL)^2 \) suppression of the frequency-averaged \( \tilde{\chi}_i \)
value contributes a $\sim (mL)^2$ suppression in SNR$^2$, while another factor comes from $U_E/U \sim (mL)^{-2}$ in equation 53, since the electric field fluctuations are only a $\sim (mL)^2$ fraction of the total energy of a $\omega \approx m$ oscillation. For QL searches, the $U_E/U$ suppression itself suppresses the SNR$^2$ by $\sim (mL)^4$. Consequently, for all of these types of searches, the $g$ sensitivity is suppressed by $\sim (mL)^{-1}$ compared to higher-frequency scaling. A proposal such as [17], which claims to avoid this suppression using a simple capacitive pickup, cannot work [19].

Since it is not possible to saturate the sum rules in the quasi-static regime, some of section III B, and in particular the optimality of a single-pole resonator, does not apply. 14 [33] conducted a detailed analysis of SQL-limited detection in the quasi-static regime, assuming a given axion-induced flux coupled through a pickup loop of given inductance. In these circumstances, the Bode-Fano bounds govern the matching of the receiver to the pickup loop, and one obtains optimum sensitivity from a top-hat transfer function (though a single pole is $O(1)$ as good). The SNR$^2$ limit obtained is parametrically given by equation 50 suppressed by $(mL)^4$.

As well as the resonant approaches of [14] [33], broadband approaches to low-frequency axion DM detection have been proposed, such as ABRACADABRA [13]. This intends to use SQUID amplifiers, which can, in some circumstances, achieve near-QL sensitivity [30]. If a QL-limited broadband experiment could be realised, it would have superior sensitivity to a SQL-limited resonant search, as discussed in section III D (the comparison of resonant to broadband approaches in [14] [33] was based on both being SQL-limited). In practice, achieving the required amplifier properties would likely be very difficult (for a large pickup loop, with a correspondingly large inductance, an amplifier with a matching noise impedance would be required). Figure 1 compares the PQL, SQL and QL sensitivities in the quasi-static regime, illustrating these differences.

1. Evading quasi-static limits

It should be noted that, to escape the ‘quasi-static’ regime where $\nabla^2 A \approx -J$, not all of the dimensions need to be large. For example, the short cylinders proposed in [32] can attain the sum-rule bound to $O(1)$, as could a one-dimensional transmission line with length $\gg m_a^{-1}$.

Even for an experiment with all dimensions $\ll m_a^{-1}$, it is still possible to attain the sum-rule bound if we can enhance the EM energy fluctuations [11]. Conceptually, the simplest way to accomplish this would be to use a material with high magnetic permeability $\mu$ [33]. Then, a magnetic field $B$ has EM energy density $B^2/2$, but the magnetisation $M = (1 - \mu^{-1})B$ contributes a negative energy density $-(1 - \mu^{-1})B^2/2$, giving total energy density $B^2/(2\mu)$. If the total magnetic energy (including the magnetisation term) of the mode has fluctuations $\sim \omega$, then the EM field contribution has fluctuations $\sim \mu\omega$, ‘borrowing’ energy from the spins. If we make $\mu \sim 1/(\omega L)^2$, we can attain the sum-rule bound (at larger $\mu$, the wavelength $\lambda \sim 2\pi/\omega L$ of the EM modes becomes comparable to $L$ [33]).

Circuit elements which can draw energy from e.g. a DC bias current, or a magnetic field bias, can also provide energy for the target mode’s magnetic field to ‘borrow’ from, enhancing its fluctuations. A toy example would be to connect an element with a negative inductance, such as a flux-biased loop [51], in series with the physical pickup inductor. In that case, the energy in the loop fluctuates downwards, as the physical inductor’s magnetic field energy fluctuates upwards.

If such a circuit is operating in a stable, equilibrium configuration, then it cannot beat the usual $P$ bound of equation 67. For example, if we tried to use a negative inductor to cancel the inductance of the physical pickup loop to even higher precision, then the energy of the $E_F$ field due to the changing magnetic field through the pickup loop would become important. The pickup loop necessarily stops behaving like an ideal inductance at some point, in analogy to the wavelength limit on enhancement from permeable materials.

All of these methods are interesting in principle, but would likely encounter practical difficulties. High-permeability materials, in particular, have major noise issues; they are generally lossy, have significant hysteresis and spin noise, and behave pathologically at high (\gg MHz) frequencies [33] [51]. Approaches such as circuits with negative differential resistances (e.g. negative inductors) seem more promising, but may still have noise issues — for example, magnetic flux noise through flux-biased loop [51]. Whether passively stable circuits, which are limited by sum rules, would be easier to implement than more general active approaches, which can evade the sum rules using excited states, is another interesting question. As discussed in the next subsection, another approach to avoiding the quasi-static suppression at low axion masses is via up-conversion.

10 The specific issue in [24] appears to be that they do not take into account the full spatial profile of the excitations’ electric field.

11 This shows that one has to be careful in interpreting the results of papers such as [28] [29], which show that the ‘induced EM fields’ from the axion DM oscillation are suppressed, for background magnetic fields of small extent compared to the axion Compton wavelength. This is true, as defined, but does not have to lead to a smaller converted power, or worse detectability. For example, [35] makes the assumption that “Our detector is composed of a collection of time-independent charges and currents, $\rho_c$ and $J_c$”, which automatically excludes e.g. resonant circuits.
D. Up-conversion

Up-conversion experiments have been proposed at both microwave [20] [21] and optical [15, 17] frequencies. In the case of microwave cavities, it is simple to see that they can attain the sum rule bounds from section III E to O(1). Suppose that we have a narrow-bandwidth axion signal, \( a(t) = a_0 \cos(m_a t) \). By equating \( P_{\text{in}} = -\int dV E \cdot J^{(0)} \) to \( P_{\text{loss}} = \frac{V^2}{Q} \), we can obtain the formula from [20],

\[
P_{\text{sig}} = \frac{1}{8}(g a_0)^2 m_a^2 Q_1 \left( \int dV E_1 \cdot B_0 \right)^2 \frac{\varepsilon}{\varepsilon_1} \frac{\Delta m}{\Delta m} \frac{\rho}{\rho_0} C_{01} V_0
\]

for the fully-rung-up, on-resonance signal power. Here, \( B_0 \) is the amplitude of the oscillating magnetic field, \( B(t) = B_0 \cos(\omega_B t) \). This gives a mass-averaged power of

\[
P \sim \frac{\pi}{8} g^2 B_0^2 \rho \frac{E}{m} \frac{\Delta m}{\Delta m} \frac{\rho}{\rho_0} C_{01} V_0
\]

(78)

Since \( C_{01} \leq 1 \), with equality iff \( E_1 \) and \( B_0 \) have the same profile, this agrees with equation 53. For example, the small-scale up-conversion experiment we discuss in [22], which drives the TE013 mode of a cylindrical cavity, and picks up signals in the TM013 mode, has \( C_{01} \approx 0.19 \). The optical interferometry experiments proposed in [15, 17] are phrased in somewhat different terms, but have the same parametric behaviour, with the orthogonally polarised drive and signal modes having \( C_{01} \approx 1 \).

For \( m_a \gtrsim L^{-1} \), where static-field experiments would be outside the quasi-static regime, an up-conversion experiment with the same volume and RMS magnetic field as a static-field experiment can have parametrically the same \( P \) value. However, as mentioned in section I A the achievable magnetic fields in up-conversion experiments will generally be smaller, being limited by both material properties and cooling systems. The driving required for the oscillating magnetic field will also introduce noise issues [22]. Furthermore, the cooling power required will generally restrict the attainable physical temperature to \( \gtrsim 1 \) K [22]. Consequently, static-field experiments will generally be superior. A potential benefit of up-conversion experiments is that the magnetic field is entirely internal to the cavity, so superconducting cavities with very high quality factors can be used. For a static-field experiment, using an external magnetic field requires either a normal cavity, or a superconducting cavity in a vortex state [22], both of which have worse quality factors. However, this difference will generally not be enough to compensate for the disadvantages listed above.

For \( m_a \lesssim L^{-1} \), static-field experiments are generically in the quasi-static regime, as discussed above, and have coupling sensitivity suppressed by \( \sim (m L)^{-1} \). Up-conversion experiments do not suffer from this suppression, so can scale better at axion frequencies \( \ll \) GHz (for lab-scale experiments). Figure 1 shows a quantitative version of this comparison. For the static-field experiments, we take nominal parameters of \( V = 1 \) m\(^3\), \( B_0 = 4 \) T, and \( T = 10 \) mK. For the nominal SRF experiment, we take \( \sqrt{B_0^2} = 0.2 \) T (at the higher end of values that might be achievable with niobium cavities [22], \( V = 1 \) m\(^3\), and \( T = 1.5 \) K (as discussed in [22], cooling requirements make sub-kelvin temperature impractical). In a realistic experiment, there would likely be some form of factor suppression (e.g. the \( C \approx 0.2^2 \) estimate in [14]; figure 1 displays the volume-wise limits, both so as to set a lower bound on the sensitivity, and since our main point is to illustrate the different scalings.

As investigated in [14, 33], the SQL-limited sensitivity for searches in the quasi-static regime is constant as a function of axion mass. For \( m_a \gtrsim T/Q_1 \), the SQL-limited sensitivity is better, and scales \( \propto m_a^{-1/4} \) in the quasi-static regime. Conversely, the up-conversion sensitivity scales \( \propto m_a^{-1/2} \), so is theoretically better at very low axion masses (of course, additional noise sources, e.g. vibrations, may make practical measurements difficult here [24]). Due to the smaller \( B_0 \) field, the SRF sensitivity is significantly worse at high (\( \sim \) GHz) frequencies, where the quasi-static suppression is not significant. We also plot the static-field sensitivities without the quasi-static suppression (e.g. given a matching circuit of the kind discussed in the previous section), showing that, as expected, both the scaling and absolute sensitivity is superior to up-conversion in these circumstances. For measurement schemes which violate the assumptions of the different limits, e.g. quantum measurement approaches such as backaction evasion, different analyses would apply.

In our companion paper [22], we calculate basic sensitivity estimates for some specific SRF up-conversion setups, attempting to take into account some possible noise sources. Comparing these to the projections from static-field experiments, we find that, with existing technologies, static-field experiments (even in the quasi-static regime) will most likely have better QCD axion reach. At very low axion masses, where the theoretical advantage of up-conversion experiments is greatest, more careful investigation of noise sources would be required to understand whether improving on static-field experiments is plausible. Nevertheless, given the strong motivations for exploring axion parameter space, and the technological developments that may occur, it is important to understand the properties of different experimental approaches.

Comparing SRF setups to optical up-conversion experiments, the latter suffer from two major disadvantages compared to microwave frequencies. The first is that achievable electromagnetic field strengths at optical frequencies are much lower. Taking the optimistic parameters from [17], we can consider a 40m long optical cavity with circulating optical power \( \sim 1 \) MW (for comparison, the circulating power in the LIGO interferometer arms is \( \sim 100 \) kW [53]). This corresponds to a stored magnetic field energy of \( \sim 0.1 \) J. For comparison, the small-scale
SRF experiment discussed in our companion paper has stored energy $\sim 1$ kJ, while the nominal up-conversion experiment in Figure 1 has $\sim 15$ kJ. The other issue is that, for a given signal power, the number of signal photons is much lower, by a factor $\sim eV/(2\pi \text{GHz}) \sim 2 \times 10^5$. As a result, the theoretical sensitivity limits for optical experiments are significantly worse, as illustrated in figure 2 of [22]. Of course, optical experiments may be cheaper or simpler to implement than static-field or SRF experiments.

To illustrate the usefulness of our theoretical limits as a sanity-check, we can also consider the microwave up-conversion experiments proposed in [21]. Taking, as an example, the projections from their figure 8, they envisage a room-temperature experiment, with driven and target modes at $\sim 9$ GHz, in a cavity with quality factor $10^4$. The power dissipated is taken to be 1 W, implying a stored EM power of $2 \times 10^{-7}$ J. They assume an amplifier noise temperature of 50 K, and a factor 10 more expensive than static-field or SRF experiments.

An obvious issue with these proposals is that achievable high-frequency ($\gg$ GHz) magnetic fields are orders of magnitude smaller than static ones, such that even a idealised advantage does not seem practical. Taking the optical-frequency 40 m cavity parameters from the previous section, the $eV/(2\pi \text{GHz}) \sim 2 \times 10^5$ enhancement is not enough to compensate for the decreased magnetic field energy, compared to a nominal static-field experiment with $U_B \sim \text{Tesla}^2 \times \text{meter}^3 \sim \text{MJ}$. Consequently, even with these extreme parameters, no conversion rate advantage would be achieved (even before considering the extra difficulties of detecting microwave photons vs optical photons).

V. DARK PHOTON DM

Apart from spin-0 candidates such as axions, light DM could also consist of oscillations of a vector field. Similarly to spin-0 DM, there are a variety of possible non-thermal production mechanisms, including purely-gravitational production from fluctuations 'stretched' by inflation [53]. The simplest, and (at low vector masses) the least constrained, coupling of such a vector to the SM is the ‘kinetic mixing’ coupling,

$$L \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} \kappa F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A^2$$

(80)

for a ‘dark photon’ $A'$. This is equivalent, after a field redefinition, to a massive vector with a small coupling to the EM current,

$$L \supset -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A'^2 - J_{EM}^\mu (A_\mu + \kappa A'_\mu)$$

(81)

It can be useful to perform a further field redefinition, going to a non-mass-eigenstate basis,

$$\hat{A} = A + \kappa A' \quad \hat{A}' = A' - \kappa A$$

(82)

(to leading order in $\kappa$). Then,

$$L \supset \frac{1}{2} m^2 \hat{A}^2 + \frac{1}{2} \kappa^2 m^2 \hat{A}'^2 + \kappa m^2 \hat{A} \hat{A}' + J_{EM}^\mu \hat{A}$$

(83)

again to leading order in $\kappa$. In the $\kappa \searrow 0$ limit, a shielded volume (e.g. a superconducting cavity) will set $A = A'$ inside it, in the presence of external sources. The leading effect of the $A'$ DM oscillation inside the volume will be via the mass mixing term, $L \supset \kappa m^2 \hat{A}' \hat{A}_\mu$ (from now on, we will drop the tildes).

For a zero-velocity DM field, $A'_0 = 0$, so the interaction term is $-\kappa m^2 \hat{A}' \hat{A}_\mu$. This corresponds to an effective current $J_{A'} = -\kappa m^2 \hat{A}'$, in analogy to the axion effective

E. Down-conversion

As discussed in section [11F] it is theoretically possible to increase $N_a$ (and improve amplifier SNR) by concentrating the target mode’s fluctuations at low frequencies. However, at frequencies $\omega_a \ll L^{-1}$, we are naturally in the quasi-static regime, where the fluctuations are suppressed by $\sim (\omega_a L)^2$. In that case, the down-conversion sensitivity is $N_a \propto \omega_a$, and drops at lower frequencies. Hence, unless we circumvent the quasi-static regime, down-conversion experiments are only of interest for $\nu_a \gg$ GHz, where they could offer an idealised sensitivity up to $\sim \nu_a/\text{GHz}$ times better than static-field experiments with the same $B_0$ amplitude. It is easy to think of toy implementations which can attain this bound. For example, if $\nu_a$ were in the optical regime, we could fill a microwave cavity with a photonic material, with period-icity at optical wavelengths. If a laser at the frequency of periodicity were shone through the cavity, then in the presence of an axion oscillation at almost the same frequency, the axions/laser photons would down-convert to microwave photons.
FIG. 1. Theoretical sensitivity limits for experiments searching for axion DM through its coupling to photons, $L \supset g_{a\gamma\gamma} a E \cdot B$. The red lines correspond to experiments using a static 4 T background magnetic field, with an experimental volume of 1 m$^3$, and an integration time of one year per e-fold in axion mass. The blue lines correspond to up-conversion experiments, with a target frequency of 1 GHz, and taking an average background magnetic field strength of 0.2 T in a volume of 1 m$^3$ (with the same integration time assumptions). The $Q = 10^6$ SQL line assumes a physical temperature of 10 mK, a quality factor of $10^6$, and SQL-limited op-amp readout. The solid line assumes that the experiment operates in the quasi-static regime, with a consequent suppression in sensitivity (see section IV C), while the dot-dashed line shows the full sum-rule-limited sensitivity (e.g. from using an appropriate matching circuit). The $Q = 10^6$ QL line takes the same assumptions, but assumes quantum-limited readout (see section III D). The $Q = 10^{11}$ SRF line corresponds to the isolated-linear-amplifier sensitivity for an up-conversion experiment with physical temperature 1.5 K, and mode quality factor $10^{13}$ [22]. The dashed blue line shows the PQL-limited sensitivity for these parameters — for higher quality factors, linear amplifier readout can approach this [22]. The green diagonal band corresponds to the ‘natural’ range of $g_{a\gamma\gamma}$ values at each QCD axion mass — if we write $g_{a\gamma\gamma} = \frac{\alpha_{EM}}{2\pi f_a} (E/N - 1.92)$ [4], then the upper edge of the band is at $E/N = 5$ [6], and the lower edge at $E/N = 2$ [4]. The gray diagonal lines indicate the KSVZ (upper, $E/N = 0$) and DFSZ (lower, $E/N = 8/3$) models. The gray shaded region corresponds to the parameter space excluded by ADMX [8].

If we have a shielded experimental region, then the sensitivity to $\kappa$ is set by that region’s volume. Unlike in the axion case, the frequency of the forcing term in the interaction Hamiltonian is set by the dark matter only, and does not depend on the target. Correspondingly, the up-conversion and down-conversion scenarios do not apply. The limit on the absorbed power is simply

$$\bar{P} \lesssim \frac{\kappa^2 \rho V m^2 \pi}{\Delta m \frac{\pi}{2}}$$

(84)

giving

$$\tilde{N}_{A'} \lesssim \frac{\kappa^2 \rho V t m \pi}{\Delta m \frac{\pi}{2}}$$

(85)

Note that $\tilde{N}_{A'}$ does not vanish as $m \downarrow 0$, unlike most rates involving dark photons, since the DM field amplitude becomes larger as $m$ becomes smaller (of course, once $t \lesssim m^{-1}$, the expression becomes invalid). Putting in representative numbers,

$$N_{A'} \lesssim (2 \times 10^{-19})^{-2} \frac{\rho}{0.3 \text{ GeV cm}^{-3}} \frac{V}{\text{m}^3} \frac{t}{\text{year}}$$

(86)
for an order-1 mass range. Consequently, for PQL or SQL-limited laboratory searches, $\kappa \sim \text{few} \times 10^{-19}$ is the smallest kinetic mixing we could reasonably hope to detect. If $m \lesssim L^{-1}$, and we are in the quasi-static regime, then $N_{A'} \lesssim \kappa^2 \rho_{A'} V t (m L)^2$.

Almost any axion experiment using a roughly-homogeneous, static background magnetic field will act as a dark photon detector ‘for free’, even in the absence of the magnetic field. In particular, cavity / dielectric haloscopes can again be $O(1)$ optimal for dark photon conversion; existing cavity haloscopes have set stringent limits on dark photon DM in the $\sim \text{GHz}$ frequency range [58], while future proposals will often have sensitivity to dark photon DM significantly before they reach QCD axion sensitivity [59, 60]. There are also many experiments and experimental proposals for which the addition of a strong background magnetic field would be practically difficult, so they can detect dark photons but not axions (e.g. experiments using superconducting phonon detectors, such as [61, 62]).

In section IV A we noted how it is difficult to obtain efficient conversion from dielectric haloscopes at $\sim \text{optical}$ frequencies, due to a lack of suitable low-loss, high-$\eta$ dielectrics. These kinds of difficulties are generic — while achieving $O(1)$ of the $\mathcal{P}$ limit is fairly simple at microwave frequencies, it is more challenging at higher DM masses. In contrast, detecting the converted excitations is often simpler, due to the reduced blackbody noise and more effective single-quantum detectors.

In figure 2, we show some sensitivity limits and projections for higher-mass dark photon DM detection experiments, using different types of target excitations. We compare these to the PQL-limited sensitivities, for the appropriate target volumes and integration times. These illustrate the conversion efficiencies of the different targets and excitation types (all of the schemes assume efficient, almost background-free detection of converted excitations). This can help to identify how and where improved targets can be found, versus where larger volumes or longer integration times would be necessary to improve sensitivity.

At $m_{A'} \gtrsim 10 \text{ eV}$, dark photon DM would have enough energy to ionize atoms, and such absorptions would be visible in WIMP detection experiments. [60] analysed the results of the Xenon10 experiment, and used these to place limits on dark photon DM. The dark photon absorption rate was calculated using the imaginary part of the photon propagator in liquid Xenon [60]. As can
be seen from the PQL bound, LXe is quite an inefficient dark photon absorber at these frequencies — the number of converted quanta is a factor of \( \sim 7000 \) lower than an ideal target occupying the same volume.

For slightly lower masses, dark photons can have enough energy to excite electrons in solid-state materials. Depending on the material, this excitation may be detected through ionisation or scintillation signals [59]. In figure 2, we show background-free, kg-year exposure projections for sapphire and GaAs scintillators, from [59]. As the figure illustrates, these materials are much more efficient at converting dark photon DM, with conversion rates only a factor \( \sim 5 \) lower than ideal.

At lower frequencies, layered dielectrics have been proposed as a way to convert dark photon DM to photons [12], for detection using superconducting devices such as TESs [62–67], nanowires [68] or MKIDs [69–71]. In figure 2, we show a nominal experimental projection using eight different 100-period stacks, each of area \( 71 \text{ cm}^2 \). The material pairings used are Si/Al\(_2\)O\(_3\), Si/SiO\(_2\), and TiO\(_2\)/SiO\(_2\). These material choices give a \( \sim 1/30 \) of the ideal conversion rate, corresponding to the suppression from section IV.A along with a \( \sim 2/3 \) misalignment factor from the dark photon polarization direction [12].

For even smaller dark photon masses, detectors with single-quantum sensitivity are challenging, but it is possible that superconducting technologies such as TESs or nanowires could achieve good enough energy resolution. Given such detectors, polar crystals have been proposed as a target for dark photon conversion [43–59]. Optical phonons in these crystals have gapped dispersion relations, allowing non-relativistic dark photons to convert to low-momentum optical phonons without the need for further momentum-matching. One drawback of this approach is that the optical phonon dispersion relation is not easily tuned, so absorption is concentrated at the resonant frequency set by the optical phonon dispersion relation (with quality factor \( \sim O(100) \) [59]).

In figure 2, we show the sensitivity projections from [59] for 1kg-year integrations with GaAs and sapphire crystals (assuming efficient and background-free detection). The resonant character of the absorption is clearly visible. Averaging across the mass range covered, the GaAs crystals converts \( \sim 250 \) fewer photons than an ideal target, while the sapphire crystal is only a factor \( \sim 25 \) below optimal.

VI. RELATIVISTIC ABSORPTION

For non-relativistic axion DM, we have seen that the maximum absorption rate, averaged over some axion mass range, scales with the magnetic field volume (multiplied by the RMS magnetic field). While coherent absorption can be helpful in terms of absorbing into specific target modes (especially for background rejection purposes), it does not result in parametrically enhanced rates compared to incoherent absorption. For example, the ionisation and electron-excitation techniques discussed in the previous section have the same scaling as dielectric haloscope absorption. However, for absorption of relativistic axions, it is possible to obtain better-than-volume scaling.

If an axion has relativistic momentum \( k \gg m_a \), then its frequency is

\[
\omega = \sqrt{k^2 + m_a^2} = k \left( 1 + \frac{m_a^2}{2k^2} + \ldots \right)
\]

(87)

So, if the axion mass has fractional uncertainty \( O(1) \), then the fractional uncertainty in \( \omega \) is \( \sim m_a^2/(2k^2) \). Taking a given spatial mode in the experimental volume, this has overlap with a range of axion momenta \( \sim \delta k \). Thus, for small enough axion masses, the mode’s EM fluctuations only need to be concentrated into a frequency range \( \sim \delta k \) in order to convert the axion signal over an \( O(1) \) range in \( m_a \). If we can concentrate the fluctuations into this frequency range, this enhances them by \( \sim k/\delta k \) over the \( O(1) \)-averaged value, and so increases the absorption rate. For an experiment on length scales \( \sim L \), we naturally have \( k/\delta k \sim kL \). In contrast, for a non-relativistic axion DM signal, if we are \( O(1) \) uncertain as to \( m_a \), then for each spatial mode we want to absorb into, the frequency uncertainty is \( O(1) \) (for a static field experiment).

This enhancement for relativistic absorption is actually very simple to implement. If we know the approximate direction that the relativistic axions will be coming from, then we can construct a long tube facing in that direction, and fill it with an approximately uniform transverse magnetic field. This is exactly the setup of the CAST experiment [72], which looks for axions produced inside the Sun. If the axion mass is small enough such that \( k_a - k_\gamma = k_a - \omega < L^{-1} \), i.e. \( m_a^2/(2\omega) \lesssim L^{-1} \), then axions travelling down the tube convert coherently to photons, with conversion probability \( \propto (gB)L^2 \). Thus, if \( F_a \sim a^2\omega \) is the axion flux, then rate of converted photons is

\[
\Gamma_\gamma \sim F_a (gB)^2 L^2 \sim (gB)^2 (AL)(L\omega)
\]

(88)

Comparing this to equation [40] for a static-field experiment absorbing non-relativistic axions over an \( O(1) \) mass range, we see that the relativistic case is enhanced by a factor \( \sim L\omega \). This is just the number of wavelengths over which we can build up coherently, as expected. Photons in a mode with a specific momentum naturally have a specific frequency, realising the concentration of the fluctuation spectrum that we wanted.

If \( k_a - k_\gamma > L^{-1} \), then we would need to modify the photon dispersion relation inside the tube to obtain coherent conversion. For example, as done with CAST [73–76], we could introduce some gas into the tube, changing the refractive index for (X-ray) photons. In this case, the maximum possible enhancement, if \( m_a \) is \( O(1) \) uncertain,
is $\sim \omega^2/m_a^2$. For $\omega \sim m_a$, the enhancement disappears, as expected from our analysis of non-relativistic DM.

If we allow the background magnetic field to oscillate in time, we could improve the theoretical detectability further by using the down-conversion ideas discussed above. However, the largest naturally-occurring flux of relativistic axions is from the Sun, and is dominantly at $\sim$ keV energies [77], so creating a suitable background field is not technologically plausible.

For light-shining-through-walls experiments [78–83], the situation is somewhat different, since we can construct our emitter to produce axions at a precise, known frequency. Hence, independent of the relationship between axion wavelength and experiment size, we can concentrate the fluctuations into a small frequency range by making a high-quality-factor receiver [78].

\section{A. Dark photons}

Similarly to the dark matter case, conversion of relativistic dark photons to SM photons is exactly analogous to axion-photon conversion, where the ‘effective’ magnetic field is in the direction of the dark photon’s polarization. For relativistic, transversely-polarized dark photons, this means that CAST gives a conversion rate $\Gamma_a \sim \kappa^2 F_A A L^2$ even without a magnetic field, realising a $\sim kL$ enhancement over the non-relativistic case.

However, in terms of searching for dark photons produced in the Sun, there is a problem. For $m_{A'}$ much less than the plasma frequency in the Sun ($\omega_p \sim 300$ eV in the solar core), the production rate of longitudinally-polarized dark photons from the sun is $\propto m_{A'}^4$, whereas that for transversely-polarized dark photons is $\propto m_{A'}^2$. Consequently, for small dark photon masses, the solar flux is dominated by longitudinal $A'$ [84].

To convert these to SM photons, which are transversely polarised, we would need an anisotropic medium to set the photon polarisation direction. However, such a material will generally modify the dispersion relation of the SM photon as well. This would spoil the momentum match between the dark photon and SM photon, and we would no longer get coherent conversion. While there are theoretically ways to achieve coherent conversion — e.g. by multiplexing many wavelength-sized cavities together with the appropriate phase offsets — the small wavelength of solar dark photons (the peak flux is at $\sim 10$ eV [84]) makes the most obvious approaches impractical.

There are other potential sources of relativistic dark photons, including dark radiation produced in the early universe. If this has a temperature comparable to the CMB, then most of its power is at $\sim$ mm wavelengths, so there would be the potential for $L\omega \sim 10^3$ coherence enhancements in the converted power, from a meter-scale experiment. The polarization of the dark radiation would depend on the early universe production mechanism, and would need to be considered in designing an experiment.

\section{VII. CONCLUSIONS}

In this paper, we have derived limits on the experimental detectability of axion DM, assuming that it couples to the SM through the $aF_{\mu\nu}F^{\mu\nu}$ operator. Similar analyses can be applied to other DM-SM couplings; most directly, to kinetically mixed dark photons, as we considered in section [V]. As another example, suppose that axion-like DM couples to electrons via $\mathcal{L} \supset g_{aee}(\bar{\partial_\gamma a})\bar{e}\gamma^\mu\gamma^5\gamma e$. For slow-moving electrons, the dominant effect is to add a spin-dependent interaction term to the Schrodinger Hamiltonian, $H_{int} \supset g(\nabla a) \cdot \sigma_e$, so the axion field couples like an ‘effective magnetic field’, $B_{eff} = g\nabla a/\mu_e$. Consequently, we encounter problems if we try to detect a DM signal which is too small relative to the vacuum (or thermal) fluctuations of the electromagnetic $B$ field. For example, if we try to increase our sensitivity by packing more and more spins into the same volume, their magnetic fields back-react on each other, and past a certain point, our sensitivity does not improve. The proposed QUAX experiment [85] to search for axion DM through its electron coupling is not very far from this limit, since the saturation magnetisation of its Yttrium Iron Garnet target is $\sim 0.2$ T, which is only an order of magnitude below the $\sim 2$ T external magnetic field setting the Larmor frequency.

Similarly, if DM consists of a vector which couples to the SM $B-L$ current, then its couplings to electrons and protons are the same as those of a dark photon with $\kappa_{eff} = g_{B-L}/e$. So, in circumstances where interactions with neutrons are not important (for example, in dielectric holography detection schemes, where electrons dominate the dielectric response), competition with the EM field’s vacuum fluctuations sets limits on how well we can detect the DM. These correspond to the dark photon limits derived above, since for a shielded volume, the EM current fluctuations are related to the EM field fluctuations. For example, if $e$ were smaller, then the fluctuations of the electron-number current in the shielding would have to be larger in order to cancel out the EM field fluctuations, corresponding to $\kappa_{eff} \propto 1/e$.

In the $g_{aee}$ and $B-L$ examples, there is not a perfect degeneracy with the effects of EM fluctuations, as there is for e.g. a dark photon. Consequently, there are ways to get around the limits set by EM fluctuations. In the $B-L$ case, one could look at the force exerted on neutron-containing objects, which keeps increasing as the number of neutrons increases. For $g_{aee}$, the coupling of $\nabla a$ to protons is not necessarily the same as that of the equivalent magnetic field. Even so, in these cases and more generally, one can still use fluctuation-based (and/or energy absorption) analyses of the type we developed to bound the sensitivity of experimental setups.

As discussed in section [VI] these techniques apply beyond the scenario of DM detection, to other kinds of weakly-coupled particle detection. In particular, quantum measurement theory has been extensively developed by the gravitational wave detection community (see...
systems, could help improve sensitivity. This illustrates how, for Gaussian states, the dependence on the fluctuations of $X$ has a simple interpretation in terms of conjugate variables. However, things do not have to be that simple. For example, if the

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) \]  

(A3)

where $j(t) = j(t) \mathbf{1}_{t_0 < t < t_1}$, and $S_{\Delta X \Delta X}$ is the symmetrised spectral density of $\Delta X$. In the simplest case, where $j(t)$ is a single-frequency oscillation, $j(t) = j_0 \cos(\omega t)$, we have

\[ P_{\exp} \simeq \frac{1}{2} g^2 j_0^2 S_{\Delta X \Delta X}(\omega) t_{\exp} \]  

(A4)

The overlap of $|\psi(t_1)\rangle$ with $|\psi\rangle$ is therefore

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) \]  

(A2)

\[ \rho_{\Delta X \Delta X} \]  

where $D_\Delta \equiv V - \langle \psi | V | \psi \rangle$. Consequently, the ability to distinguish between the presence and absence of a weak signal is set by the fluctuations of the interaction Hamiltonian $[39]$. This is related to the theory of ‘Quantum Cramer-Rao bounds’ [27–29], and is sometimes referred to as an ‘Energetic Quantum Limit’ (PQL) or ‘Fundamental Quantum Limit’ (FQL).

In our cases, the DM signal $j(t)$ will generally be narrow-bandwidth, and it will be more useful to go to a spectral representation. Taking the same assumptions as above,

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) t_{\exp} \]  

(A5)

To gain some intuition for these results, it is helpful to consider the case where our system is a harmonic oscillator, coupled to an external force, $X = \dot{x}$. We will assume that the oscillator has some small coupling to other degrees of freedom, giving it a high quality factor $Q \gg 1$. In the ground state, we have

\[ \tilde{S}_{xx}(\omega) \simeq \frac{Q}{1 + Q^2 (\omega^2 - \omega_0^2)^2} \frac{1}{\omega_0} \frac{1}{M\omega_0} \]  

(A6)

for $\omega$ close to $\omega_0$, where $\omega_0$ is the natural frequency of the oscillator, and $M$ is its mass. Here, $\frac{1}{\omega_0}$ is the squared position uncertainty. By e.g. decreasing $M$ while keeping $\omega_0$ fixed, we increase the position uncertainty, and so decrease the momentum uncertainty. Since the external force changes the moment of the system, having smaller momentum fluctuations helps to detect the forcing.

This illustrates how, for Gaussian states, the dependence on the fluctuations of $X$ has a simple interpretation in terms of conjugate variables. However, things do not have to be that simple. For example, if the

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) t_{\exp} \]  

(A4)

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) \]  

(A3)

where $j(t) = j(t) \mathbf{1}_{t_0 < t < t_1}$, and $S_{\Delta X \Delta X}$ is the symmetrised spectral density of $\Delta X$. In the simplest case, where $j(t)$ is a single-frequency oscillation, $j(t) = j_0 \cos(\omega t)$, we have

\[ P_{\exp} \simeq \frac{1}{2} g^2 j_0^2 S_{\Delta X \Delta X}(\omega) t_{\exp} \]  

(A4)

where $t_{\exp} = t_1 - t_0$, and $P_{\exp} \equiv 1 - |\langle \psi(t_1) | \psi \rangle|^2$ is the probability of changing the detector system’s state. More generally, if we can treat $j(t)$ as a stochastic process, with some power spectral density $S_{jj}$, then in the limit where we evolve for a time long compared to the inverse bandwidth of spectral features in $S_{jj}$,

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) \]  

(A3)

\[ \langle \psi(t_1) | \psi \rangle \simeq 1 - g^2 \int_0^\infty d\omega S_{\Delta X \Delta X}(\omega) \]  

(A2)
harmonic oscillator is initially in a number state, then \( \langle n \hat{x}^2 \rangle = \frac{2n+1}{2m \omega} \) and \( \langle n \hat{p}^2 \rangle = \frac{\hbar^2 (2n + 1)}{2m \omega} \) — both the position and the momentum uncertainties are higher than for the ground state. However, by e.g. measuring the energy of the final state, we can still attain the \( S_{xx} \) bound \cite{30}. Effectively, the rate for absorption (and emission) of quanta due to the forcing is Bose-enhanced by the initial occupation number. This provides an example of how, even for more complicated systems, the \( S_{XX} \) prescription still gives the correct answers.

In the discussion above, we assumed that the detector system was allowed to evolve for a time \( t_1 - t_0 \), and only measured at the end. For many experimental setups, something closer to continuous monitoring is implemented — e.g. in resonant cavity experiments, the output port is connected to an amplifier. However, since we are only concerned with the fluctuations of \( \Delta X \), as long as we include the rest of the system’s dynamics in determining these (including measurements, feedback etc), this does not present a problem (cf the discussion of deferred measurement in \cite{28}).

The above limits were based on knowing precisely which quantum state \( |\psi\rangle \) our system starts in, and precisely how it would evolve from there. Other ‘fluctuations’, due to our uncertainty about the system’s state (e.g. thermal fluctuations) have the opposite effect, making it harder to tell whether a signal is present. In some circumstances, \( j(t) \) itself may be uncertain — for example, the Fourier components for a virialized DM signal are expected to have random amplitudes and phases. If these unknown Fourier amplitudes affect the system’s response to the signal, then the effective SNRs for the independent components generally add in quadrature \cite{91}, rather than linearly, as per the Dicke radiometer formula \cite{92}.

1. Linear amplifiers

The FQL detectability limit discussed above applies to any type of detection system, as long as we properly calculate the quantum fluctuations of \( \Delta X \). However, in many cases, sensors are complicated non-equilibrium devices, and the fluctuations they cause may be difficult to compute. In addition, detection schemes may fail to obtain the FQL. Consequently, it is often helpful to consider the sensitivity limits for more restricted classes of sensors.

A common example of such a sensor, relevant to many axion detection experiments, is a linear amplifier. There is an extensive literature on the quantum theory of linear amplifiers (see e.g. \cite{23} for a review). In many circumstances, it is a good approximation to treat the ‘fluctuations’, both quantum and statistical, of measured and output quantities as Gaussian. Then, the relevant quantities can be summarised as ‘noise’ spectral densities.

We will denote the PSD of backaction noise acting on \( \hat{X} \) as \( S_{FF}(\omega) \), and the output imprecision noise (referred back to \( X \)) as \( S_{XX}^f(\omega) \). The amplifier does not have to be connected ‘directly’ to the \( X \) degree of freedom for this description to make sense, so long as the whole system behaves linearly. A common setup is where a high-power-gain amplifier is coupled weakly to the target system (where ‘weakly’ means that it has a very subdominant effect on the system’s damping). This is referred to as ‘op-amp’ mode \cite{23}. In this scenario, the added noise associated with the amplification process is \cite{23}

\[
S_{XX}^{\text{add}}(\omega) = |\chi(\omega)|^2 S_{FF}(\omega) + S_{XX}^f(\omega) \]

The total output ‘noise’, referred back to \( X \), is \( S_{XX} = S_{XX}^{\text{add}} + S_{XX}^n + S_{XX}^\text{thermal} \), where \( S_{XX}^n(\omega) = \delta(\omega) \) corresponds to the zero-point fluctuations of \( X \), and \( S_{XX}^\text{thermal} \) summarises the other noise contributions (e.g. for thermal noise, \( S_{XX}^\text{thermal}(\omega) = 2n_T(\omega)\chi(\omega) \)). If we are attempting to detect a signal \( j \), whose Fourier components have unknown phases, and integrate for a time \( t \) compared to inverse spectral bandwidths, then the expected SNR\(^2\) is

\[
\text{SNR}^2 = g^4 t \int_0^\infty \frac{d\omega}{2\pi} \left( \frac{|S_{jX}|}{S_{XX}^\text{tot}} \right)^2 \]

To maximise our SNR, we want to reduce \( S_{XX}^{\text{add}} \). For a high-gain amplifier, this is bounded below by \( S_{XX}^{\text{add}}(\omega) = S_{XX}^n(\omega) = |\chi(\omega)|^2 S_{FF}(\omega) \). Achieving this ‘quantum limit’ requires \( S_{FF}(\omega) = \frac{|\chi(\omega)|^2}{S_{XX}(\omega)} \), i.e. that the correlations between the imprecision and backaction noise are set by the phase of the response function.

In many circumstances, it is easier to implement linear amplifiers with uncorrelated imprecision and back-action noise. Following the gravitational wave detection literature \cite{27}, and papers such as \cite{23}, we will refer to this as the ‘Standard Quantum Limit’ (SQL), as opposed to the ‘quantum limit’ (QL) in which correlations are permitted. Generally, it is the case that

\[
S_{XX}^f(\omega) - |S_{FF}(\omega)|^2 \geq \frac{1 + \Delta |S_{XX}^\text{add}(\omega)|}{4} \]

where \( \Delta |z| \equiv (|1 + z|^2 - (1 + |z|^2))/2 \) \cite{23}. This implies that \( S_{XX}^f(\omega) \geq \Delta S_{FF}/4 \). If there are no correlations, then \( S_{XX}^\text{add} \geq |\chi|^2 S_{FF} + \frac{1}{2S_{FF}} \), which is minimised by \( S_{FF} = \frac{1}{|\chi|^2} \), giving \( S_{XX}^\text{add}(\omega) = |\chi(\omega)|^2 \).

These limits apply to detection schemes which are invariant under time translation, usually referred to as ‘phase-invariant’ (i.e. they treat sine and cosine signals in the same way). By varying e.g. the detector coupling in a time-dependent way, sensitivity-enhancing schemes such as backaction evasion can be implemented \cite{23}.

The SNR limits look rather different to the fluctuation-based FQL limits discussed above. However, for equilibrium targets in the linear response regime, \( \chi(\omega) \) is related to \( S_{XX} \) via the fluctuation-dissipation relations. As we will see in the text, this can lead to closely-related FQL and SNR bounds.

In some circumstances, we will want to go beyond the op-amp regime, and couple the detector more strongly to the target system. We discuss this in \cite{III C}.
Appendix B: Appendix: atomic and molecular magnetic fields

In the main text, the ‘background’ $B_0$ field was generally taken to be a smoothed version, not taking into account the large magnetic fields inside molecules, atoms, nuclei, etc. This is justified since, if the electric field associated with signal excitations is slowly-varying in space, then the interaction strength only depends on the magnetic multipole moments of sub-wavelength structures. For example, if we integrate over a volume containing some currents and spins, and the magnetic fields from sources outside the volume are small, then

$$\int dV B = \frac{2}{3}m_{\text{tot}}$$

where $m_{\text{tot}}$ is the total magnetic dipole moment of the matter $\text{[93]}$.

For the $\lesssim\text{eV}$ excitations we considered, the smallest scale of spatial variation is e.g. motions of atoms in a lattice, or molecular vibrations. In particular, these are above the atomic scale. So, the magnetic field strength is, at best, that arising from atomic magnetic dipoles, which give a $\sim$ Tesla field. Thus, we can’t gain an advantage over using a strong, roughly uniform background magnetic field.
[86] H. Miao and Y. Chen, in *Advanced Gravitational Wave Detectors*, edited by D. G. Blair, L. Ju, C. Zhao, and E. J. Howell (Cambridge University Press) pp. 277–297.

[87] H. Miao, Y. Ma, C. Zhao, and Y. Chen, Phys. Rev. Lett. **115**, 211104 (2015), arXiv:1506.00117 [quant-ph].

[88] H. M. Wiseman and G. J. Milburn, *Quantum Measurement and Control* (Cambridge University Press, 2009).

[89] V. B. Braginsky, M. L. Gorodetsky, F. Ya. Khalili, and K. S. Thorne, *Gravitational waves. Proceedings, 3rd Edoardo Amaldi Conference, Pasadena, USA, July 12-16, 1999*. AIP Conf. Proc. **523**, 180 (2000), arXiv:gr-qc/9907057 [gr-qc].

[90] C. M. Caves, K. S. Thorne, R. W. P. Drever, V. D. Sandberg, and M. Zimmermann, Rev. Mod. Phys. **52**, 341 (1980).

[91] C. W. Helstrom, *Information and Control* **10**, 254 (1967).

[92] R. H. Dicke, Rev. Sci. Instrum. **17**, 268 (1946).

[93] J. D. Jackson, *Classical electrodynamics*, 3rd ed. (Wiley, New York, NY, 1999).