GENERAL ELEPHANTS
FOR THREEFOLD EXTREMAL CONTRACTIONS
WITH ONE-DIMENSIONAL FIBERS: EXCEPTIONAL CASE

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Abstract. Let \((X, C)\) be a germ of a threefold \(X\) with terminal singularities along a connected reduced complete curve \(C\) with a contraction \(f : (X, C) \to (Z, o)\) such that \(C = f^{-1}(o)_{\text{red}}\) and \(-K_X\) is \(f\)-ample. Assume that each irreducible component of \(C\) contains at most one point of index \(> 2\). We prove that a general member \(D \in |-K_X|\) is a normal surface with Du Val singularities.

1. Introduction

The present paper is a continuation of a series of papers on the classification of extremal contractions with one-dimensional fibers (see the survey \cite{MP19} for an introduction). Recall that an extremal curve germ is the analytic germ \((X, C)\) of a threefold \(X\) with terminal singularities along a reduced connected complete curve \(C\) such that there exists a contraction \(f : (X, C) \to (Z, o)\) such that \(C = f^{-1}(o)_{\text{red}}\) and \(-K_X\) is \(f\)-ample. There are three types of extremal curve germs: flipping, divisorial and \(\mathbb{Q}\)-conic bundles, and all of them are important building blocks in the three-dimensional minimal model program.

The first step of the classification is to establish the existence of a “good” member of the anticanonical linear system. This is M. Reid’s so-called “general elephant conjecture” \cite{R87}. In the case of irreducible central curve \(C\) the conjecture has been proved:

**Theorem 1.1** (\cite[Th. 2.2]{KM92}, \cite{MP09}). Let \((X, C)\) be an extremal curve germ with irreducible central curve \(C\). Then a general member \(D \in |-K_X|\) is a normal surface with Du Val singularities.

Moreover, all the possibilities for general members of \(|-K_X|\) have been classified. Firstly, extremal curve germs with irreducible central curve are divided into two classes: semistable and exceptional. Such a germ \((X, C)\) is said to be semistable if for the restriction of the corresponding contraction \(f : (X, C) \to (Z, o)\) to a general member \(D \in |-K_X|\), we have the Stein factorization \(f_D : D \to D' \to f(D)\) with surface \(D'\) having only Du Val singularities of type \(A\) \cite{KM92}.

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Non-semistable extremal curve germs are called exceptional. Semistable extremal curve germs are subdivided into two types: (k1A) and (k2A) while exceptional ones are subdivided into the following types: cD/2, cAx/2, cE/2, cD/3, (IIA), (II′), (IE′), (ID′), (IC), (IIB), (kAD), and (k3A) (see [KM92, MP08a and MP09]).

The result stated in Theorem 1.1 is very important in three-dimensional geometry. For example, the existence of a good member $D \in |-K_X|$ for flipping contractions is a sufficient condition for the existence of flips [K88] and the existence of a good member $D \in |-K_X|$ in the $\mathbb{Q}$-conic bundle case proves Iskovskikh’s conjecture about singularities of the base [P97, MP08a].

Reid’s conjecture also has been proved for arbitrary central curve $C$ in the case of $\mathbb{Q}$-conic bundles over singular base:

**Theorem 1.2 (MP08b).** Let $(X, C)$ be a $\mathbb{Q}$-conic bundle germ and let $f : (X, C) \to (Z, o)$ be the corresponding contraction. Assume that $(Z, o)$ is singular. Then a general member $D \in |-K_X|$ is a normal surface with Du Val singularities. Moreover, for each irreducible component $C_i \subset C$ with two non-Gorenstein points or of types (IC) or (IIB), the dual graph $\Delta(D, C_i)$ has the same form as the irreducible extremal curve germ $(X, C_i)$ (see Theorem 5.1).

Throughout this paper we use the standard notation (IC), (IIB) etc for types of extremal curve germs $(X, C)$ with irreducible central fiber [KM92]. Sometimes, to specify the indices of singular points, we will use subscripts. E.g. $(kAD_{2,m})$ means that the indices of points of $(X, C)$ are 2 and $m$. Some of the subscripts can be omitted if it is not important in the consideration, e.g. $(k2A_2)$ means that $(X, C)$ contains a point of index 2 (and another point of index $> 1$).

According to the classification of birational extremal curve germs the condition [1.3(*)] is equivalent saying that an arbitrary component $C_i \subset C$ of type (k2A) has a point of index 2.

**Corollary 1.4.** Let $(X, C)$ be an extremal curve germ and let $C_i \subset C$ be an irreducible component.

(i) If $C_i$ is of type (IIB), then any other component $C_j \subset C$ is of type (IIA) or (II′).

(ii) If $C_i$ is of type (IC) or (k3A), then any other component $C_j \subset C$ meeting $C_i$ is of type (k1A) or (k2A).

(iii) If $C_i$ is of type (kAD), then any other component $C_j \subset C$ meeting $C_i$ is of type (k1A), (k2A), cD/2, or cAx/2.

(iv) If $C_i$ is of type (k2A_2), then any other component $C_j \subset C$ meeting $C_i$ is of type (k1A), (IC), or (k2A_{n,m}), $n, m \geq 3$.

There are more restrictions on the combinatorics of the components of $C$. This will be treated in a subsequent paper.

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1 This statement is not true as it stands. In the case (IE′) $D$ is always of type $A_7$ and in the case (ID′) $D$ is type $A_9$ or $D_k$, $k \geq 4$ depending on whether the non-Gorenstein point $P \in X$ is of type cA/2 or cAx/2 [MP08a 1.2.4] (see Appendix B for the full list of $\mathbb{Q}$-conic bundle germs with irreducible central fiber).

To correct the original statement, the cases (ID′) with $P$ of type cA/2 and (IE′) should be moved to the group of semistable extremal curve germs, and only the case (ID′) with $P$ of type cAx/2 should remain in the group of exceptional ones.
2. Preliminaries

2.1. Recall that a contraction is a proper surjective morphism \( f : X \to Z \) of normal varieties such that \( f_* \mathcal{O}_X = \mathcal{O}_Z \).

Definition 2.1. Let \((X, C)\) be the analytic germ of a threefold with terminal singularities along a reduced connected complete curve. We say that \((X, C)\) is an extremal curve germ if there is a contraction \( f : (X, C) \to (Z, o) \) such that \( C = f^{-1}(o)_{\text{red}} \) and \(-K_X\) is \( f \)-ample. Furthermore, \( f \) is called flipping if its exceptional locus coincides with \( C \) and divisorial if its exceptional locus is two-dimensional. If \( f \) is not birational, then \( Z \) is a surface and \((X, C)\) is said to be a \( \mathbb{Q} \)-conic bundle germ.

Lemma 2.2. Let \((X, C)\) be an extremal curve germ. Assume that \( C \) is reducible. Then for any proper connected subcurve \( C' \subsetneq C \) the germ \((X, C')\) is a birational extremal curve germ.

Proof. Clearly, there exists a contraction \( f' : X \to Z' \) of \( C' \) over \( Z \) [MS8 Corollary 1.5]. We need to show only that \( f' \) is birational. Assume that \((X, C')\) is a \( \mathbb{Q} \)-conic bundle germ. Then there exists the following commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Z' \\
\downarrow f & & \uparrow f' \\
Z & \xrightarrow{\varphi} & Z
\end{array}
\]

where \( f \) and \( f' \) are \( \mathbb{Q} \)-conic bundles contracting \( C \) and \( C' \), respectively. The image \( \Gamma := f'(C'') \) of the remaining part \( C'' := C - C' \) is a curve on \( Z' \) such that \( \varphi(\Gamma) = f(C) \) is a point, say \( o \in Z \). Hence the fiber \( f'^{-1}(\Gamma) = f^{-1}(o) \) is two-dimensional, a contradiction. \( \square \)

2.2. Recall basic definitions of the \( \ell \)-structure techniques, see [MS8 § 8] for details. Let \((X, P)\) be three-dimensional terminal singularity of index \( m \). Throughout this paper \( \pi : (X^2, P^2) \to (X, P) \) denotes its index-one cover. For any object \( V \) on \( X \) we denote by \( V^2 \) the pull-back of \( V \) on \( X^2 \).

Let \( \mathcal{L} \) be a coherent sheaf on \( X \) without submodules of finite length \( > 0 \). An \( \ell \)-structure of \( \mathcal{L} \) at \( P \) is a coherent sheaf \( \mathcal{L}^\ell \) on \( X^2 \) without submodules of finite length \( > 0 \) with \( \mathbf{\mu}_m \)-action endowed with an isomorphism \((\mathcal{L}^\ell)^{\mathbf{\mu}_m} \cong \mathcal{L}\). An \( \ell \)-basis of \( \mathcal{L} \) at \( P \) is a collection of \( \mathbf{\mu}_m \)-semi-invariants \( s_1^\ell, \ldots, s_r^\ell \in \mathcal{L}^\ell \) generating \( \mathcal{L}^\ell \) as an \( \mathcal{O}_{X^2} \)-module at \( P^2 \). Let \( Y \) be a closed subvariety of \( X \). Note that \( \mathcal{L} \) is an \( \mathcal{O}_Y \)-module if and only if \( \mathcal{L}^\ell \) is an \( \mathcal{O}_{Y^2} \)-module. We say that \( \mathcal{L} \) is \( \ell \)-free \( \mathcal{O}_Y \)-module at \( P \) if \( \mathcal{L}^\ell \) is a free \( \mathcal{O}_{Y^2} \)-module at \( P^2 \). If \( \mathcal{L} \) is \( \ell \)-free \( \mathcal{O}_Y \)-module at \( P \), then an \( \ell \)-basis of \( \mathcal{L} \) at \( P \) is said to be \( \ell \)-free if it is a free \( \mathcal{O}_{Y^2} \)-basis.

Let \( \mathcal{L} \) and \( \mathcal{M} \) be \( \mathcal{O}_Y \)-modules at \( P \) with \( \ell \)-structures \( \mathcal{L} \subset \mathcal{L}^\ell \) and \( \mathcal{M} \subset \mathcal{M}^\ell \). Define the following operations:

- \( \mathcal{L} \oplus \mathcal{M} \subset (\mathcal{L} \oplus \mathcal{M})^\ell \) is an \( \mathcal{O}_Y \)-module at \( P \) with \( \ell \)-structure \( (\mathcal{L} \oplus \mathcal{M})^\ell = \mathcal{L}^\ell \oplus \mathcal{M}^\ell \).

- \( \mathcal{L} \otimes \mathcal{M} \subset (\mathcal{L} \otimes \mathcal{M})^\ell \) is an \( \mathcal{O}_Y \)-module at \( P \) with \( \ell \)-structure \( (\mathcal{L} \otimes \mathcal{M})^\ell = (\mathcal{L}^\ell \otimes_{\mathcal{O}_{X^2}} \mathcal{M}^\ell)/\text{Sat}_{\mathcal{F}_2 \otimes \mathcal{M}^\ell}(0) \),

where \( \text{Sat}_{\mathcal{F}_2} \mathcal{F}_2 \) is the saturation of \( \mathcal{F}_2 \) in \( \mathcal{F}_1 \). These operations satisfy standard properties (see [MS8 8.8.4]). If \( X \) is an analytic threefold with terminal singularities and \( Y \) is a closed subscheme of \( X \), then the above local definitions of \( \oplus \) and \( \otimes \) patch with corresponding operations on \( X \setminus \text{Sing} \). Therefore, they give well-defined operations of global \( \mathcal{O}_Y \)-modules.
Lemma 2.3. Let \((D, C)\) be the germ of a normal Gorenstein surface along a proper reduced connected curve \(C = \cup C_i\), where \(C_i\) are irreducible components. Assume that the following conditions hold:

(i) \(K_D \sim 0\),
(ii) there is a birational contraction \(\varphi : (D, C) \rightarrow (R, o)\) such that \(\varphi^{-1}(o)_{\text{red}} = C\),
(iii) there is a point \(P \in D\) which is not Du Val of type A.

Then \(D\) has only Du Val singularities on \(C\) \(\setminus\{P\}\).

Proof. Assume that there is a point \(Q \in D \setminus \{P\}\) which is not Du Val. If there exists a component \(C_i \subset C\) passing through \(Q\) but not passing through \(P\), we can contract it: \(D \rightarrow D'\) over \(R\). The contraction is crepant, so the image of \(Q\) is again a non-Du Val point. Replace \(D\) with \(D'\). Continuing the process we may assume that \(P, Q\) are connected by some component \(C_i \subset C\). Moreover, by shrinking \(C\) we may assume that \(C_i = C\), i.e. \(C\) is irreducible. Since \(D\) is Gorenstein, the point \(Q \in D\) is not log terminal and the point \(P \in D\) is log terminal only if it is Du Val of type D or E. Hence the pair \((D, C)\) is not log canonical at \(Q\) and not purely log terminal at \(P\) [KM98, Theorem 4.15]. Let \(H\) be a general hyperplane section passing through \(P\). For some \(0 < \epsilon, \delta \ll 1\) the pair \((D, (1 - \epsilon)C + \delta H)\) is not log canonical at \(P\) and \(Q\). Since \(-(K_D + (1 - \epsilon)C + \delta H)\) is \(\varphi\)-ample, this contradicts Shokurov’s connectedness lemma [S92]. □

3. Low index cases

Extremal curve germs of index 2 with arbitrary central curve have been completely classified in [KM92, § 4] and [MP08a, § 12]. As an easy consequence we have the following.

Proposition 3.1. Let \((X, C)\) be an extremal curve germ. Assume that all the singularities of \(X\) are of index 1 or 2, that is, \(2K_X\) is Cartier. Then a general member \(D \in |−K_X|\) is a normal surface with Du Val singularities and \(D\) does not contain any component of \(C\).

Proof. Since the case where \(X\) is Gorenstein is trivial, we assume that \(X\) has at least one point, say \(P\), of index 2. In the birational case there are no other non-Gorenstein points and all the components \(C_i \subset C\) pass through \(P\) [KM92, Prop. 4.6]. By [KM92, Th. 2.2] a general local member \(D \in |−K_{(X, P)}|\) is in fact a general member of \(|−K_X|\) and this \(D\) has only Du Val singularity (at \(P\)) [R87, (6.3)]. For the \(Q\)-conic bundle case we refer to [MP08a, Proof of 12.1] and [MP08b, Corollary 1.4]. □

Proposition 3.2. Let \((X, C)\) be an extremal curve germ. Assume that \(C\) is reducible and \((X, C)\) contains a point \(P\) of one of the types cD/2, cAx/2, cE/2, cD/3. Then one of the following holds.

(i) \(P\) is the only non-Gorenstein point of \(X\), all the components pass through \(P\) and do not meet each other elsewhere, and a general member \(D \in |−K_X|\) is a normal surface with Du Val singularities. Moreover, \(D \cap C = \{P\}\).

(ii) There is a component \(C_i \subset C\) passing through \(P\) such that the germ \((X, C_i)\) is divisorial of type \((kAD)\). Moreover, \((X, P)\) is a singularity of type cD/2 or cAx/2.

Proof. Recall that the intersection points \(C_i \cap C_j\) of different components \(C_i, C_j \subset C\) are non-Gorenstein by [M88, Corollary 1.15], [Ko99, Prop. 4.2] and also by [MP08a, Lemma 4.4.2]. If \(P\) is the only non-Gorenstein point of \(X\), then a general member \(D \in |−K_{(X, P)}|\) is in fact a general member of \(|−K_X|\) [M88, (0.4.14)]. This \(D\) has only Du Val singularity (at \(P\)) [R87, (6.3)]. If there exists a non-Gorenstein point \(Q \in X\) other than \(P\), then we may assume that \(Q\) lies on some component \(C_i \subset C\) passing through \(P\). Thus \((X, C_i)\) is a birational extremal curve germ with two non-Gorenstein points (see Lemma 2.2). According to [KM92, Th. 2.2] and [M07] the germ \((X, C_i)\) is divisorial of type \((kAD)\) and \((X, P)\) is a singularity of type cD/2 or cAx/2. This proves the proposition. □
4. Extension techniques

Theorem 4.1 ([M88 Th. 7.3], [MP08a Prop. 1.3.7]). Let \((X, C \simeq \mathbb{P}^1)\) be an irreducible extremal curve germ satisfying the condition \(\text{(4.1)}\). Then for a general member \(S \in |-2K_X|\) one has \(S \cap C = \{P\}\), where \(P\) is the point of index \(r > 2\) or a smooth point (if \((X, C)\) is of index 2). Moreover, the pair \((X, \frac{1}{r}S)\) is log terminal.

Proposition 4.2 ([KM92 Lemma 2.5], [MP09 Prop. 2.1]). Let \((X, C)\) be an extremal curve germ (\(C\) is not necessarily irreducible) and let \(S \in |-2K_X|\) be a general member. Assume that the set \(\Sigma := S \cap C\) is finite.

(i) If \((X, C)\) is birational, then the natural map
\[
\tau : H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow \omega_{(S, \Sigma)} = H^0(S, \mathcal{O}_S(-K_X))
\]
is surjective, where \(\omega_{(S, \Sigma)}\) is the dualizing sheaf of \((S, \Sigma)\).

(ii) If \((X, C)\) is a \(\mathbb{Q}\)-conic bundle germ over a smooth base surface, then the natural map
\[
\bar{\tau} : H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow \omega_{(S, \Sigma)}/\Omega^2_{(S, \Sigma)}
\]
is surjective, where \(\Omega^2_{(S, \Sigma)}\) is the sheaf of holomorphic 2-forms on \((S, \Sigma)\).

(iii) If \((X, C)\) is a \(\mathbb{Q}\)-conic bundle germ over a smooth base surface and \(\Sigma = \Sigma_1 \amalg \Sigma_2, \Sigma_i \neq \emptyset\), then
\[
\gamma_1 : H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow \omega_{(S, \Sigma_i)}
\]
is surjective.

Proof. For \(\text{[i]}\) we refer to [KM92 Lemma 2.5]. Let us show \(\text{[ii]}\). Note that by the adjunction \(\mathcal{O}_S(K_S) = \mathcal{O}_S(-K_X)\). Let \(f : (X, C) \to (Z, o)\) be the corresponding \(\mathbb{Q}\)-conic bundle contraction and let \(g = f|_S : S \to Z\) be its restriction to \(S\). Since the base surface \(Z\) is smooth, by [MP08a Lemma 4.1] there is a canonical isomorphism
\[
R^1f^*\omega_X \simeq \omega_Z.
\]
Then we apply [MP09 Prop. 2.1] in our situation:
\[
\begin{array}{ccc}
H^0(X, \mathcal{O}_X(-K_X)) & \longrightarrow & H^0(S, \mathcal{O}_S(-K_X)) \\
\uparrow & & \uparrow \downarrow \mathcal{O}^2 \\
f_*\omega_X(S) & \longrightarrow & \omega_{(S, \Sigma)} \\
\downarrow \omega_{(S, \Sigma)}/g^*\omega_{(Z, o)} & \longrightarrow & \omega_{(S, \Sigma)}/\Omega^2_{(S, \Sigma)}
\end{array}
\]
and obtain the surjectivity of \(\tau\).

For \(\text{[iii]}\) we consider the map \(g_i : S_i \to Z\) which is the restriction of \(g\) to \(S_i = (S, \Sigma_i) \subset S\) and the induced exact sequence
\[
f_*\omega_X(S) \longrightarrow \omega_{(S, \Sigma_1)} \oplus \omega_{(S, \Sigma_2)} \longrightarrow \omega_Z \longrightarrow 0
\]
Then we see that \(g^*_2 : \omega_Z \to 0 \oplus \omega_{(S, \Sigma_2)}\) is a splitting homomorphism. Therefore, the homomorphism
\[
f_*\omega_X(S) \longrightarrow \omega_{(S, \Sigma_1)} \oplus (\omega_{(S, \Sigma_2)}/g^*_2\omega_{(Z, o)})
\]
is surjective. \(\square\)

Lemma 4.3. Let \((X, \bar{C})\) be an extremal curve germ with reducible central curve \(\bar{C}\). Suppose \(X\) satisfy the condition \(\text{(4.3)}\) and that there is a component \(C \subset \bar{C}\) of type \((\kappa_1\lambda)\) which meets \(\bar{C} - C\) at a point \(P\) of index 2. Then a general member \(D \in |-K_X|\) does not contain \(C\).
Proof. On each irreducible component $C_i$ of $\bar{C}$ there exist at most one point of index $> 2$. Let \( \{ P_a \}_{a \in A} \) be the collection of such points. For each $C_i$ without points of index $> 2$, choose one general point of $C_i$. Let $\{ P_b \}_{b \in B}$ be the collection of such points. For each $i \in A \cup B$, let $S_i \in | -2K_{(X, P_i)}|$ be a general element on the germ $(X, P_i)$, and set $S = \sum_{i \in A \cup B} S_i$. Then $S$ extends to an element $| -2K_X |$ by [M88] Thm. (7.3)]. A generator $\sigma_b$ of $\Theta_{S, b}(-K_X) \simeq \Theta_{S, b}$ lifts to $s \in H^0(X, \Theta_X(-K_X))$ by $4.2(i)$ if $(X, C)$ is birational and by $A \neq \emptyset$ and $4.2(ii)$ if otherwise. In either case we have $C \not\subset D$. \( \square \)

5. Review of [KM92] § 2

We need some refinement of facts on birational extremal curve germs with irreducible central fiber proved in [KM92] § 2.

5.1. Below, for a normal surface $D$ and a curve $C \subset D$, we use the usual notation of graphs $\Delta(D, C)$ of the minimal resolution of $D$ near $C$: each vertex labeled $\bullet$ corresponds to an irreducible component of $D$ and each $\circ$ corresponds to a component $E_i \subset E$ of the exceptional divisor $E$ on the minimal resolution of $D$. Note that in our situation below $E_i^2 = -2$ for all $E_i$.

Theorem 5.1 ([KM92] Th. 2.2, [M07]). Let $(X, C \simeq \mathbb{P}^1)$ be a birational extremal curve germ and let $D \in | -K_X |$ be a general member. Then $D$ is a normal surface with Du Val singularities. Moreover, either $D \cap C$ is a point or $D \supset C$ and for the graph $\Delta(D, C)$ one and only one of the following possibilities holds:

\[ \begin{align*}
\text{(IC)} & \quad \bullet - \cdots - \circ - \cdots - \circ - \cdots - \circ \\
& \quad m-3 \geq 2 \\
\text{(IIB)} & \quad \circ - \cdots - \circ - \cdots - \circ - \cdots - \bullet \\
& \quad (k1, m=4) \\
\text{(kAD)} & \quad \circ - \cdots - \circ - \cdots - \bullet - \cdots - \circ - \cdots - \circ \\
& \quad (2l-2 \geq 0) \\
\text{(k3A)} & \quad \circ - \cdots - \circ - \cdots - \bullet - \cdots - \circ \\
& \quad (m-1 \geq 2) \\
\text{(k2A)} & \quad \circ - \cdots - \circ - \cdots - \bullet - \cdots - \circ \\
& \quad (km-1 \geq 2) \\
& \quad (ln-1) \\
\end{align*} \]

where $m$ and $k$ are the index and axial multiplicity [M88 1a.5(iii)] of a singular point of $X$, and $n$ and $l$ are the ones for the other non-Gorenstein point (if any).

In the cases (IC), (IIB), (kAD), (k3A), and (k2A) Theorem 5.1 is a consequence of the following.

Theorem 5.2 (cf. [KM92] § 2, [M07]). Let $(X, C)$ be a birational extremal curve germ with irreducible central curve of type (IC), (IIB), (kAD), (k3A), or (k2A). Let $S \in | -2K_X |$ be a general member (so that $S \cap C = \{ P \}$, where $P$ is the point of index $r > 2$). Let $\sigma_S \in H^0(S, \Theta_S(-K_X))$ be a general section. Then for any section $\sigma \in H^0(X, \Theta_X(-K_X))$ such that

\[ \sigma|_S \equiv \sigma_S \mod \Omega^2 \],

(5.1)

(see [1.2]) the divisor $D := \text{div}(\sigma)$ is a normal surface with only Du Val singularities. Furthermore, the configuration $\Delta(D, C)$ is as described in Theorem 5.1.
Below we outline the proof of Theorem 5.2 following [KM92, § 2]. We treat the possibilities (IC), (IIB), (k3A), (kAD), and (k2A2) case by case.

5.2. Case (IC). By [M88, (A.3)] we have the following identification at $P$

$$(X, C) = \left( \mathbb{C}^3_{y_1, y_2, y_4}, \{ y_1^{m-2} - y_2^2 = y_4 = 0 \} \right) / \mu_m(2, m - 2, 1).$$

A general divisor $S \in | - 2K_X|$ is given by $y_1 = \xi(y_2, y_4)$ with $\xi \in (y_2, y_4)^2$ such that $\text{wt}(\xi) \equiv 2 \mod m$. Thus we have

$$S \simeq \mathbb{C}^2_{y_2, y_4} / \mu_m(m - 2, 1), \quad \omega_S = (\mathcal{O}_{S^t, P} dy_2 \wedge dy_4)^{\mu_m}, \quad (5.2)$$

$$\omega_S \otimes \mathbb{C}_P = \mathbb{C} \cdot y_2^{(m-1)/2} dy_2 \wedge dy_4 \oplus \mathbb{C} \cdot y_4 dy_2 \wedge dy_4. \quad (5.3)$$

Furthermore,

$$\text{gr}_C^0 \omega^* = (P^2) = (-1 + \frac{m+1}{2} \cdot 2P^2) \simeq \mathcal{O}_C(-1), \quad (5.4)$$

where

$$\Omega^{-1} := (dy_1 \wedge dy_2 \wedge dy_4)^{-1}$$

is an $\ell$-free $\ell$-basis at $P$. Hence, $H^0(C, \text{gr}_C^0 \omega^*) = 0$ and

$$H^0(X, \mathcal{O}_X(-K_X)) = H^0(X, \mathcal{I}_C \otimes \mathcal{O}_X(-K_X)), \quad (5.5)$$

where $\mathcal{I}_C$ is the defining ideal of $C$ in $X$. Furthermore, by [KM92, (2.10.4)]

$$\text{gr}_C^1 \omega^* = (5P^2) \oplus (0), \quad (5.6)$$

where the $\mu_m$-semi-invariants

$$(y_1^{m-2} - y_2^2) \cdot \Omega^{-1}, \quad y_4 \cdot \Omega^{-1} \quad (5.7)$$

form an $\ell$-free $\ell$-basis at $P$. Therefore,

$$\text{gr}_C^1 \omega^* \simeq \begin{cases} \mathcal{O}_C(-1) \oplus \mathcal{O}_C & \text{if } m \geq 9, \\ \mathcal{O}_C \oplus \mathcal{O}_C & \text{if } m = 7, \\ \mathcal{O}_C(1) \oplus \mathcal{O}_C & \text{if } m = 5. \end{cases}$$

we have natural homomorphisms

$$\delta : H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow \text{gr}_C^1 \omega^* \longrightarrow (\text{gr}_C^1 \omega^*)^2 \otimes \mathbb{C}_P, \quad (5.8)$$

Since $(y_1 - \xi) \cdot (\text{gr}_C^1 \omega^*)^2 \otimes \mathbb{C}_P = 0$, the map $\delta$ factors as

$$\delta : H^0(X, \mathcal{O}_X(-K_X)) \longrightarrow \omega_S \longrightarrow (\text{gr}_C^1 \omega^*)^2 \otimes \mathbb{C}_P. \quad (5.9)$$

As in [MP09, (3.1.1)] we see

$$\Omega_S^2 \subset (m_{S, P} \cdot y_4 + m_{S, P} \cdot y_2^{(m-1)/2}) dy_2 \wedge dy_4 = m_{S, P} \cdot \omega_S,$$

because for arbitrary elements $\phi_1, \phi_2$ of the set of generators

$$\left\{ y_2^m, y_4^m, y_2y_4^2, y_2^{(m-1)/2}y_4 \right\}$$

of the ring $\mathcal{O}_{S^t}^{\mu_m}$ we have

$$d\phi_1 \wedge d\phi_2 \in \left( (y_2, y_4) y_4 + (y_2, y_4) y_2^{(m-1)/2} \right) dy_2 \wedge dy_4.$$

Thus $\delta$ factors further as follows:

$$\delta : H^0(\mathcal{O}_X(-K_X)) \longrightarrow \omega_S / \Omega_S^2 \longrightarrow \omega_S \otimes \mathbb{C}_P \longrightarrow (\text{gr}_C^1 \omega^*)^2 \otimes \mathbb{C}_P,$$

where the last map is a surjection if $m = 5$ and the image is generated by $y_4\Omega^{-1}$ if $m \geq 7$ (see (5.3) and (5.7)). If $m \geq 7$, this implies that the coefficient of $y_4\Omega^{-1}$ in $\sigma_S$ is nonzero. If $m = 5$, then the coefficients of $y_4\Omega^{-1}$ and $(y_1^{m-2} - y_2^2)\Omega^{-1}$ in $\sigma_S$ are independent and the image $\sigma$ of $\sigma$ in
gr_{C}^{1} \omega^{*} \text{ is not contained in } \mathcal{O}_{C}(1). \text{ Hence, } \sigma \text{ is nowhere vanishing and so the singular locus of } D \text{ does not meet } C \setminus \{P\}. \text{ Then we again can take } \sigma_{S} \text{ so that it contains the term } y_{4} \Omega^{-1}. \text{ Therefore, } D \in |-K_{X}| \text{ can be given by the equation } y_{1} + \cdots = 0. \text{ Then [KM92] Computation 2.10.5] shows that } D \text{ is Du Val at } P \text{ and that its graph is as given as type (IC) in [5.1].}

5.3. Case (IIB). Then, by [MSS8 (A.3)], the germ \((X, C)\) at \(P\) can be given as follows

\[(X, C) = \left(\{\phi = 0\} \subset \mathbb{C}^{4}, \{y_{1}^{2} - y_{2}^{3} = y_{3} = y_{4} = 0\}\right) / \mu_{4}(3, 2, 1, 1),\]

\[\phi = y_{1}^{2} - y_{2}^{3} + \psi, \quad \text{wt}(\psi) \equiv 2 \mod 4, \quad \psi(0, 0, y_{3}, y_{4}) \notin (y_{3}, y_{4})^{3}.\]

A general divisor \(S \in |-2K_{X}|\) is given by \(y_{2} = \xi(y_{1}, y_{3}, y_{4})\) with \(\xi \in (y_{1}, y_{3}, y_{4})^{2}\) such that \(\text{wt}(\xi) \equiv 2 \mod 4.\) Thus \(S\) is the quotient of a hypersurface \(\phi(y_{1}, \xi, y_{3}, y_{4}) = 0 \in \mathbb{C}^{3}_{y_{1}, y_{3}, y_{4}}\) by \(\mu_{4}(3, 1, 1).\) We have

\[\omega_{S} = \left(\mathcal{O}_{S_{t}, p_{t}} \frac{dy_{3} \wedge dy_{4}}{y_{1} + \cdots}\right)^{\mu_{4}}, \quad \omega_{S} \otimes \mathbb{C}_{P} = \mathbb{C} \cdot y_{3} \frac{dy_{3} \wedge dy_{4}}{y_{1} + \cdots} + \mathbb{C} \cdot y_{4} \frac{dy_{3} \wedge dy_{4}}{y_{1} + \cdots}.\]  

(5.8)

Furthermore,

\[\text{gr}_{C}^{0} \omega^{*} = (P^{2}) = (-1 + 3P^{2} + 2P^{2}) \simeq \mathcal{O}_{C}(-1),\]  

(5.9)

where

\[\Omega^{-1} := \left(\frac{dy_{2} \wedge dy_{3} \wedge dy_{4}}{\partial \phi / \partial y_{1}}\right)^{-1}\]

is an \(\ell\)-free \(\ell\)-basis at \(P.\) Hence, \(H^{0}(C, \text{gr}_{C}^{0} \omega^{*}) = 0\) and

\[H^{0}(X, \mathcal{O}_{X}(-K_{X})) = H^{0}(X, \mathcal{I}_{C} \otimes \mathcal{O}_{X}(-K_{X}))\]  

(5.10)

where \(\mathcal{I}_{C}\) is the defining ideal of \(C\) in \(X.\) Furthermore, by [KM92 (2.11)],[

\[\text{gr}_{C}^{1} \omega^{*} = (0) \oplus (1) \simeq \mathcal{O}_{C} \oplus \mathcal{O}_{C}(1),\]  

(5.11)

where the \(\mu_{m}\)-invariants

\[y_{3} \cdot \Omega^{-1}, \quad y_{4} \cdot \Omega^{-1}\]  

(5.12)

form an \(\ell\)-free \(\ell\)-basis at \(P.\) As in (IC), we have natural homomorphisms

\[\delta : H^{0}(\mathcal{O}_{X}(-K_{X})) \longrightarrow \omega_{S} \otimes \mathbb{C}_{P} \longrightarrow (\text{gr}_{C}^{1} \omega^{*})^{2} \otimes \mathbb{C}_{P^{2}},\]

where the last homomorphism is an isomorphism (see (5.8) and (5.12)). Thus the coefficients of \(y_{3} \Omega^{-1}\) and \(y_{4} \Omega^{-1}\) in \(\sigma_{S}\) are independent, whence [KM92 Computation 2.11.2] shows that \(D\) is Du Val at \(P,\) and the image \(\bar{\sigma}\) of \(\sigma\) in \(\text{gr}_{C}^{1} \omega^{*}\) is not contained in \(\mathcal{O}_{C}(1).\) Hence \(\bar{\sigma}\) is nowhere vanishing and \(D\) is smooth outside \(P.\) Hence the graph \(\Delta(D, C)\) is as given as type (IIB) in [5.1].

5.4. Case (k3A). The configuration of singular points on \((X, C)\) is the following: a type (IA) point \(P\) of odd index \(m \geq 3,\) a type (IA) point \(Q\) of index 2 and a type (III) point \(R.\) According to [MSS8 (A.3)] and [KM92 (2.12)] we can express

\[(X, C, P) = (\mathbb{C}^{3}_{y_{1}, y_{2}, y_{3}} \text{ (y}_{1}\text{-axis}) , 0) / \mu_{m}(1, (m + 1)/2, -1),\]

\[(X, C, Q) = (\mathbb{C}^{3}_{z_{1}, z_{2}, z_{3}} \text{ (z}_{1}\text{-axis}) , 0) / \mu_{2}(1, 1, 1),\]

\[(X, C, R) = \{\gamma(w_{1}, w_{2}, w_{3}, w_{4}) = 0\}, (w_{1}\text{-axis}), 0\},\]

where \(\gamma \equiv w_{1}w_{3} \mod (w_{2}, w_{3}, w_{4})^{2}.\)
For a general divisor $S \in |-2K_X|$ we have $S \cap C = \{P\}$ and $S$ is given by $y_1 = \xi(y_2, y_3)$ with $\xi \in (y_2, y_3)^2$ such that $\text{wt}(\xi) \equiv 1 \mod m$. Thus
\[
S \simeq \mathbb{C}_{y_2, y_3}/\mu_m \left( \frac{m+1}{2}, -1 \right), \quad \omega_S = \left( \theta_{S^2, P} dy_2 \wedge dy_3 \right)^{\mu_m}, \quad (5.13)
\]
\[
\omega_S \otimes \mathcal{O}_P = \mathbb{C} \cdot y_2 dy_2 \wedge dy_3 \oplus \mathbb{C} \cdot y_3^{(m-1)/2} dy_2 \wedge dy_3. \quad (5.14)
\]
By the proof of [KM92] (2.12.2) we have
\[
\text{gr}^0_{C^\ast} \omega^\ast = \left( -1 + \frac{m+1}{2} P^2 + Q^2 \right) \simeq \mathcal{O}_C(-1), \quad (5.15)
\]
where an $\ell$-free $\ell$-basis at $P$, $Q$ and $R$, respectively, can be written as follows:
\[
\Omega_P^{-1} := (dy_1 \wedge dy_2 \wedge dy_3)^{-1},
\]
\[
\Omega_Q^{-1} := (dz_1 \wedge dz_2 \wedge dz_3)^{-1},
\]
\[
\Omega_R^{-1} := \left( dw_2 \wedge dw_3 \wedge dw_4 \over \partial \gamma / \partial w_1 \right)^{-1}.
\]
Hence, $H^0(C, \text{gr}^0_{C^\ast} \omega^\ast) = 0$ and
\[
H^0(X, \mathcal{O}_X(-K_X)) = H^0 \left( X, \mathcal{I}_C \bar{\otimes} \mathcal{O}_X(-K_X) \right), \quad (5.16)
\]
where $\mathcal{I}_C$ is the defining ideal of $C$ in $X$. Furthermore, as in [KM92] (2.12.4) we can further arrange that
\[
\text{gr}^1_{C^\ast} \omega^\ast = (0) \oplus \left( -1 + \frac{m+3}{2} P^2 \right), \quad (5.17)
\]
where $y_2 \cdot \Omega_P^{-1}$, $z_2 \cdot \Omega_Q^{-1}$, $w_2 \cdot \Omega_R^{-1}$ form an $\ell$-free $\ell$-basis for $(0)$ at $P$, $Q$, and $R$, respectively, and $y_3 \cdot \Omega_P^{-1}$, $z_3 \cdot \Omega_Q^{-1}$, $w_4 \cdot \Omega_R^{-1}$ for $(-1 + (m+3)/2 P^2)$. Moreover,
\[
\gamma \equiv w_1w_3 + c_1w_4^2 + c_2w_2w_4 + c_3w_4^2 \mod (w_3, w_2^2, w_4^2, w_4^2) \cdot \mathcal{I}_C
\]
for some $c_1, c_2, c_3 \in \mathbb{C}$ such that $c_1 \neq 0$ if $m \geq 5$ (see [KM92], (2.12.6) and [M07] Remark 2]) and $(c_1, c_2, c_3) \neq 0$ if $m = 3$ (see [KM92], (2.12.7) and [M07] Remark 2).

As in [MP09] (3.1.1) we see
\[
\Omega_S^2 \subset \left( m_{S, P} \cdot y_2 + m_{S, P} \cdot y_3^{(m-1)/2} \right) dy_2 \wedge dy_3 = m_{S, P} \cdot \omega_S,
\]
because for arbitrary elements $\phi_1, \phi_2$ of the set of generators
\[
\left\{ y_2^m, \ y_3^m, \ y_2^2y_3, \ y_2y_3^{(m+1)/2} \right\}
\]
of the ring $\mathcal{O}_{S^2}^{\mu_m}$ we have
\[
d\phi_1 \wedge d\phi_2 \in \left( (y_2, y_3)y_2 + (y_2, y_3)y_3^{(m-1)/2} \right) dy_2 \wedge dy_3.
\]
Thus the image of the homomorphism
\[
\delta : H^0(\mathcal{O}_X(-K_X)) \longrightarrow \omega_S \otimes \mathcal{O}_P \longrightarrow \left( \text{gr}^1_{C^\ast} \omega^\ast \right)^{\delta} \otimes \mathcal{O}_P,
\]
is equal to $(\text{gr}^1_{C^\ast} \omega^\ast)^{\delta} \otimes \mathcal{O}_P$ if $m = 3$, and $\mathbb{C} \cdot y_2 \Omega_P^{-1}$ if $m \geq 5$.

If $m \geq 5$, this implies that the coefficient of $y_2\Omega_P^{-1}$ in the image $\bar{\sigma}$ of $\sigma$ in $\text{gr}^1_{C^\ast} \omega^\ast$ is nonzero and hence nowhere vanishing. If $m = 3$, then the coefficients of $y_2\Omega_P^{-1}$ and $y_3\Omega_P^{-1}$ are independent and hence $\bar{\sigma}$ is a general global section of $\text{gr}^1_{C^\ast} \omega^\ast \simeq \mathcal{O}_C \oplus \mathcal{O}_C$. Then the proof of [KM92] (2.12.5) shows that $D$ is Du Val and that its graph is as given as type (k3A) in 5.1.

**Lemma 5.3.** In the notation of [5.4] there exists a deformation $(X_\lambda, C_\lambda \simeq \mathbb{P}^1)$ of $(X, C)$ which is trivial outside $R$ such that for $\lambda \neq 0$ the germ $(X_\lambda, C_\lambda)$ has a cyclic quotient singularity at $Q$, and is of type (kAD) of case [5.22] (resp. (k2A$_2$)) if $m \geq 5$ (resp. $m = 3$).
Proof. Let \((X_\lambda, C_\lambda)\) be the twisted extension \([M88\textbf{, 1b.8.1}]\) of the germ
\[
(X_\lambda, R) = \{ \gamma - \lambda w_2 = 0 \} \supset (C_\lambda, R) = (w_1\text{-axis})
\]
by \(u = (w_2, w_3, w_4)\). Then in \(gr_{C_\lambda}^1 \theta\) we have \(w_1 w_3 = \lambda w_2\) for \(\lambda \neq 0\). Since \(gr_{C_\lambda}^1 \omega^* = \theta C \cdot w_2 \Omega_R^{-1} \oplus \theta C \cdot w_4 \Omega_R^{-1}\) at \(R\), we have at \(R\):
\[
gr_{C_\lambda}^1 \omega^* = \theta C_\lambda \cdot w_3 \Omega_R^{-1} \oplus \theta C_\lambda \cdot w_4 \Omega_R^{-1},
\]
where \(w_3 \Omega_R^{-1} = (\lambda w_1)^{-1} w_2 \Omega_R^{-1}\). Whence,
\[
gr_{C_\lambda}^1 \omega^* = (R) \oplus (-1 + \frac{m + 3}{2} P^2).
\]
(5.18)
For \(\lambda \neq 0\) the germ \((X_\lambda, C_\lambda)\) is either of type (kAD) or (k2A_2). Comparing \((5.18)\) with \((5.31)\) (resp. in view of \((5.7)\)) we see that \((X_\lambda, C_\lambda)\) is (kAD) (resp. (k2A_2)) if \(m \geq 5\) (resp. \(m = 3\)). □

5.5. Case (kAD). The configuration of singular points on \((X, C)\) is the following: a type (IA) point \(P\) of odd index \(m \geq 3\) and a type (IA) point \(Q\) of index 2. According to \([M88\textbf{, A.3}], \[KM92\textbf{, (2.13)}\], \[M07\textbf{, (2.13)}\]) we can write
\[
(X, C, P) = (C^3_{y_1, y_2, y_3}, (y_1\text{-axis}), 0) / \mu_m(1, (m + 1)/2, -1),
(X, C, Q) = \{ \beta = 0 \} \subset C^4_{z_1, z_2, z_3, z_4}, (z_1\text{-axis}), 0) / \mu_2(1, 1, 1, 0),
\]
where \(\beta = (z_1, \ldots, z_4)\) is a semi-invariant with \(w(\beta) \equiv 0 \mod 2\).

For general divisor \(S \in |-2K_X|\) we have \(S \cap C = \{ P \}\) and \(S\) is given by \(y_1 = \xi(y_2, y_3)\) with \(\xi \in (y_2, y_3)^2\) such that \(w(\xi) \equiv 1 \mod m\). Thus
\[
S \simeq C^2_{y_1, y_2, y_3} / \mu_m \left( \left( \frac{m + 1}{2}, -1 \right) \right), \quad \omega_S = \left( \theta_{S, x} \cdot dy_2 \wedge dy_3 \right)^{\mu_m},
\]
\[
\omega_S \otimes C_P = C \cdot y_2 dy_2 \wedge dy_3 \oplus C \cdot y_3^{(m - 1)/2} dy_2 \wedge dy_3.
\]
Then
\[
gr_0^C \omega^* = (-1 + \frac{m + 1}{2} P^2 + Q^2) \simeq \theta C(-1),
\]
where an \(\ell\)-free \(\ell\)-basis at \(P\) and \(Q\), respectively, can be written as follows:
\[
\Omega_P^{-1} = (dy_1 \wedge dy_2 \wedge dy_3)^{-1}, \quad \Omega_Q^{-1} = \left( \frac{dz_1 \wedge dz_2 \wedge dz_3}{\partial \beta / \partial z_4} \right)^{-1}
\]
Hence, \(H^0(C, gr_0^C \omega^*) = 0\) and
\[
H^0(X, \theta X(-K_X)) = H^0(X, I_C \otimes \theta X(-K_X)),
\]
(5.21)
where \(I_C\) is the defining ideal of \(C\) in \(X\).

As in \([KM92\textbf{, (2.13.3)}]\) we distinguish two subcases:
\[
\ell(Q) \leq 1, \quad i_Q(1) = 1, \quad gr_{C_\lambda}^1 \theta \simeq \theta \oplus \theta(-1),
\]
\[
\ell(Q) = 2, \quad i_Q(1) = 2, \quad gr_{C_\lambda}^1 \theta \simeq \theta(-1) \oplus \theta(-1).
\]
(5.22)
(5.23)

5.6. Subcase \((5.23)\). It is treated similarly to \((5.4)\). Since \(\ell(Q) = 2\), we have
\[
\beta \equiv z_1^2 z_4 \mod (z_2, z_3, z_4)^2.
\]

As in \([KM92\textbf{, (2.13.4)}]\) we can arrange that
\[
gr_{C_\lambda}^1 \omega^* = (0) \oplus (-1 + \frac{m + 3}{2} P^2),
\]
(5.24)
where
\[
(y_2 \cdot \Omega_P^{-1}, z_2 \cdot \Omega_Q^{-1}), \quad (y_3 \cdot \Omega_P^{-1}, z_3 \cdot \Omega_Q^{-1})
\]
form an \(\ell\)-free \(\ell\)-basis at \(P\) and \(Q\) for \((0)\) and \((-1 + (m + 3)/2 P^2)\), respectively, and
\[
\beta \equiv z_1^2 z_4 + c_1 z_3^3 + c_2 z_2 z_3 + c_3 z_2^3 \mod (z_4, z_3^2, z_2 z_3, z_2^2) \cdot (z_2, z_3, z_4)
\]
10.
for some $c_1, c_2, c_3 \in \mathbb{C}$ such that $(c_1, c_2, c_3) \neq 0$ if $m = 3$ by the classification of 3-fold terminal singularities [R87, (6.1)], and $c_1 \neq 0$ if $m \geq 5$ (see [KM92, (2.12.6)] and [M07, Remark 2]). The rest of the argument is the same as [5.3] (the type (k3A)), except that we use [KM92, (2.13.5)] instead of [KM92, (2.12.6)].

**Remark 5.4.** We note that the lowest power of $\mu_2$-invariant variable $z_4$ that appear in $\beta$ (i.e. the axial multiplicity for $(X, P)$) remains the same for the defining equation of $D^z$ under the elimination of variable of $\text{wt} \equiv 1 \mod 2$. Thus the graph $\Delta(D, C)$ is as given as type (k3A) in [5.1].

**Lemma 5.5.** In the situation of [5.5] with (5.23), let $(X_\lambda, C_\lambda)$ be the twisted extension [M88, 1b.8.1] of the germ

$$(X_\lambda, Q) = \{\beta - \lambda z_4 = 0\}/\mu_2 \supset (C_\lambda, Q) = (z_1\text{-axis})/\mu_2$$

by $u = (z_1z_2, z_1z_3)$. Then for $\lambda \neq 0$ the germ $(X_\lambda, C_\lambda)$ is of type (k3A).

**Proof.** When $0 < |\lambda| \ll 1$, a small neighborhood $X_\lambda \ni Q$ has two singular points on $C_\lambda$: a cyclic quotient at $Q$ and a Gorenstein point at $(\sqrt{\lambda}, 0, 0, 0)$. □

5.7. **Subcase (5.22).** Note that in this case $m \geq 5$ (see [KM92, (2.13.10)] and [M07]). Since $\ell(Q) \leq 1$, we have

$$\beta \equiv z_4 \mod (z_2, z_3, z_4).$$

By [KM92, (2.13.10)] we have

$$\text{gr}_C \mathcal{O} = (\frac{m-1}{2}P^\xi + Q^\xi) \oplus (-1 + P^\xi + Q^\xi) \quad \text{(resp. } (\frac{m-1}{2}P^\xi) \oplus (1 + P^\xi)) \text{).} \quad (5.26)$$

Tensoring it with (5.22) we obtain

$$\text{gr}_C \omega^* = (1) \oplus (-1 + \frac{m+3}{2}P^\tau) \quad \text{(resp. } (Q^\tau) \oplus (-1 + \frac{m+5}{2}P^\tau)) \text{),} \quad (5.27)$$

where $(y_2 \Omega_P^1, z_3 \Omega_Q^-)$ (resp. $(y_2 \Omega_P^-1, z_4 \Omega_Q^1)$) is the $\ell$-free $\ell$-basis for the first $\ell$-summand of $\text{gr}_C \omega^*$ and $(y_3 \Omega_P^1, z_2 \Omega_Q^-)$ for the second. Take the ideal $J \subset I$ as in [KM92, (2.13.10-11)]. Thus

$$(I/J) \otimes \omega^* = (-1 + \frac{m+3}{2}P^\tau), \quad H^0(X, \mathcal{O}_X(-K_X)) = H^0(F(\omega^*, J),)$$

$$J^\tau = (y_3^2, y_2) \text{ at } P, \quad J^\tau = (z_2^2, z_3, z_4) \text{ at } Q.$$\]

Then we have by [KM92, (2.13.11)]

$$\text{gr}_C^2(\omega^*, J) = (0) \oplus (-1 + \frac{m+5}{2}P^\tau + Q^\tau)$$

where

$$y_2 \cdot \Omega_P^-1, \quad y_3^2 \cdot \Omega_P^1, \quad z_3 \cdot \Omega_Q^-1, \quad z_2 \cdot \Omega_Q^1 \text{ (resp. } z_4 \cdot \Omega_Q^1) \text{)}$$

form an $\ell$-free $\ell$-basis at $P$, $Q$. Thus we investigate

$$H^0(\mathcal{O}_X(-K_X)) = H^0(F(\omega^*, J)) \longrightarrow \text{gr}_C^2(\omega^*, J)$$

via the induced homomorphism

$$H^0(\mathcal{O}_X(-K_X)) \longrightarrow \text{gr}_C^2(\omega^*, J) \longrightarrow (\text{gr}_C^2(\omega^*, J)^\tau \otimes \mathbb{C}_{pt}),$$

which, by $(y_1 - \xi) \cdot (\text{gr}_C^2(\omega^*, J))^\tau \otimes \mathbb{C}_{pt} = 0$, factors as

$$H^0(\mathcal{O}_X(-K_X)) \longrightarrow \omega_S \longrightarrow (\text{gr}_C^2(\omega^*, J))^\tau \otimes \mathbb{C}_{pt}$$

and further to

$$\delta: H^0(\mathcal{O}_X(-K_X)) \longrightarrow \omega_S \otimes \mathbb{C}_P \longrightarrow (\text{gr}_C^2(\omega^*, J))^\tau \otimes \mathbb{C}_{pt}$$

since $\Omega_X^2 \subset m_{S, P} \cdot \omega_S$ as in the (k3A) case.
The image of $\delta$ is generated by $y_2 \cdot \Omega_P^{-1}$ if $m > 5$, and by $y_2 \cdot \Omega_P^{-1}$, $y_3^2 \cdot \Omega_P^{-1}$ if $m = 5$ (see 5.19 and 5.22). Hence if $\sigma_S$ is chosen general, the image $\bar{\sigma}$ of $\sigma$ in $\text{gr}^2(\omega^*, J)$ globally generates the direct summand $\mathcal{O}_C$ if $m > 5$ and a general global section of $\text{gr}^2(\omega^*, J) \simeq \mathcal{O}_C \oplus \mathcal{O}_C$ if $m = 5$. Hence,

$$
\bar{\sigma} \equiv \begin{cases} 
(\lambda_P y_2 + \mu_P y_3^2) \Omega_P^{-1} & \text{at } P, \\
(\lambda Q z_3 + \mu Q z_2^2) \Omega_Q^{-1} \text{ (resp.} (\lambda_Q z_3 + \mu Q z_3) \Omega_Q^{-1}) & \text{at } Q,
\end{cases}
$$

where

$$
m \geq 7 \implies \lambda_P(P) \lambda_Q(Q) \neq 0, \\
m = 5 \implies \lambda_P(P) \text{ and } \mu_P(P) \text{ are independent, and} \\
\lambda_Q(Q) \text{ and } \mu_Q(Q) \text{ are independent.}
$$

These mean that the corresponding $D \in \{-K_X\}$ is smooth outside $P$ and $Q$ and $D$ is Du Val at $P$ and $Q$ by [KM92 (2.13.6)]. See Remark 5.4 for further details.

**Lemma 5.6.** In the situation of 5.5 with (5.22), let $(X_\lambda, C_\lambda)$ be the twisted extension [M88 1b.8.1] of the germ $(X_\lambda, P) = (X, P) \supset (C_\lambda, P) = (C, P)$ by $u = (y_{1}^{(m-1)/2} y_2 + \lambda y_1 y_3, y_1, y_3)$. Then for $\lambda \neq 0$ the germ $(X_\lambda, C_\lambda)$ is of type $(k2A_2)$.

**Proof.** It is clear that $(X_\lambda, C_\lambda)$ is of type $(k2A_2)$ or $(k2A_D)$ with (5.22) because $\ell(Q) \leq 1$. In either case, there is only one non-zero section $s$ (up to constant multiplication) of $\text{gr} C^1 \mathcal{O}$ (cf. 5.31). Since $s = y_1^{(m-1)/2} y_2 \in \text{gr}^1 C^1 \mathcal{O}$ at $P$ (up to constant), we have an extension

$$s_\lambda = y_1^{(m-1)/2} y_2 + \lambda y_1 y_3 = y_1 y_1^{(m-3)/2} y_3 \in \text{gr}^1 C^1 \mathcal{O}
$$

on $(C_\lambda, P)$ which generates $(P^s) \subset \text{gr}^1 C^1 \mathcal{O}$. In view of (5.26) the germ $(X_\lambda, C_\lambda)$ is of type $(k2A_2)$ by 5.7.

**Lemma 5.7.** In the situation of 5.5 with (5.22) and $\ell(Q) = 1$, let $(X_\lambda, C_\lambda)$ be the twisted extension [M88 1b.8.1] of the germ

$$(X_\lambda, Q) = \{ \beta - \lambda z_4 = 0 \}/\mu_2 \supset (C_\lambda, Q) = (z_1 \text{-axis})/\mu_2
$$

by $u = (z_1 z_2, z_4)$. Then for $\lambda \neq 0$ the germ $(X_\lambda, C_\lambda)$ is of type $(kAD)$ and $\ell(Q) = 0$.

Indeed, a global section $s$ of $\text{gr} C^1 \mathcal{O}$ extends to $s_\lambda = y_4$ of $\text{gr}^1 C^1 \mathcal{O}$ at $Q$, and $s_\lambda$ vanishes at $P^s$ to order $(m-1)/2 > 1$ by 5.7. Thus $(X_\lambda, C_\lambda)$ is of type $(kAD)$.

5.8. **Case** $(k2A_2)$. This case comes from [KM92 (2.13.1) and (2.13.9)]. The configuration of singular points on $(X, C)$ is the following: a type (IA) point $P$ of odd index $m \geq 3$ and a type (IA) point $Q$ of index 2. According to [M88 (A.3)], [KM92 (2.13)], and [M07] we can write

$$(X, C, P) = \left( \{ \alpha = 0 \} \subset \mathbb{C}^{4}_{y_1, \ldots, y_4}, (y_1 \text{-axis}), 0 \right)/\mu_2(1, a, -1, 0),
$$

$$(X, C, Q) = \left( \{ \beta = 0 \} \subset \mathbb{C}^{4}_{z_1, \ldots, z_4}, (z_1 \text{-axis}), 0 \right)/\mu_2(1, 1, 1, 0),
$$

where $a$ is an integer prime to $m$ such that $m/2 < a < m$, and $\alpha$ and $\beta$ are invariants with

$$
\alpha = y_1 y_3 - \alpha_1(y_2, y_3, y_4), \quad \alpha_1 \in (y_2, y_3)^2 + (y_4),
$$

$$
\beta = z_1 z_3 - \beta_1(z_2, z_3, z_4), \quad \beta_1 \in (z_2, z_3)^2 + (z_4).
$$

Then

$$\text{gr}^0 C^1 \omega^* = (-1 + aP^s + Q^s) \simeq \mathcal{O}_C(-1), \quad (5.29)$$
where an \(\ell\)-free \(\ell\)-basis at \(P\) and \(Q\), respectively, can be written as follows:

\[
\Omega^1_P = \left( \frac{dy_1 \wedge dy_2 \wedge dy_3}{\partial \alpha / \partial y_4} \right)^{-1} \quad \Omega^1_Q = \left( \frac{dz_1 \wedge dz_2 \wedge dz_3}{\partial \beta / \partial z_4} \right)^{-1}
\]

Hence, \(H^0(C, \text{gr}_C^0 \omega^*) = 0\) and

\[
H^0(X, \mathcal{O}_X(-K_X)) = H^0\left( X, \mathcal{I}_C \otimes \mathcal{O}_X(-K_X) \right),
\]

where \(\mathcal{I}_C\) is the defining ideal of \(C\) in \(X\).

As in the argument in [KM92] (2.13.8–9], we have

\[
\text{gr}_C^1 \mathcal{O} = \mathcal{L} \oplus \text{gr}_C^0 \omega, \quad \text{gr}_C^1 \omega^* = \mathcal{L} \otimes (\text{gr}_C^0 \omega^*) \oplus (0),
\]

where \(\mathcal{L}\) is an \(\ell\)-invertible sheaf such that \(\mathcal{L} = (P^2 + Q^2)\) (resp. \((P^2); (Q^2); (0)) if \(y_4 \in \alpha\) and \(z_4 \in \beta\) (resp. \(y_4 \notin \alpha\) and \(z_4 \notin \beta\)). We also see that \(y_3 \Omega^{-1}_P\) (resp. \(y_4 \Omega^{-1}_P\)) and \(y_2 \Omega^{-1}_P\) form an \(\ell\)-free \(\ell\)-basis for \(\text{gr}_C^1 \omega^*\) at \(P\) if \(y_4 \in \alpha\) (resp. \(y_4 \notin \alpha\)).

For a general divisor \(S \in | -2K_X|\) we have \(S \cap C = \{P\}\) and \(S\) in \(X\) is given by

\[
\gamma := y_1^{2a-m} + y_2^2 + y_3^{2m-2a} + \cdots = 0
\]

with \(\text{wt}(\gamma) \equiv 2a \mod m\). Let \(\Omega\) be the determinant of the dualizing sheaf \(\omega_{S^t}\) of \(S^t\) at \(P^t\). Then

\[
\omega_S = (\mathcal{O}_{S^t, P^t} \omega)^{\mu m}, \quad \text{wt}(\Omega) \equiv -a \mod m,
\]

\[
\omega_S \otimes \mathbb{C}_P = \mathbb{C} \cdot y_2 \omega \otimes \mathbb{C} \cdot y_3^{m-a} \omega.
\]

**Lemma 5.8.** The induced map

\[
\Omega^2_S \longrightarrow \omega_S \otimes \mathbb{C}_P
\]

is zero, where \(\Omega^2_S\) is the sheaf of holomorphic 2-forms on \(S\).

**Proof.** We have

\[
\Omega = \frac{dy_1 \wedge dy_2}{\Delta_{3,4}} = \ldots = \frac{dy_3 \wedge dy_4}{\Delta_{1,2}}, \quad \Delta_{i,j} := \begin{vmatrix}
\frac{\partial \alpha}{\partial y_i} & \frac{\partial \alpha}{\partial y_j} \\
\frac{\partial \gamma}{\partial y_i} & \frac{\partial \gamma}{\partial y_j}
\end{vmatrix}
\]

Note that \(\text{wt}(\Delta_{i,j}) \equiv 2a - \text{wt}(y_i) - \text{wt}(y_j)\) and \(\text{wt}(\Omega) \equiv -a \mod m\). Since \(\omega_S = (\mathcal{O}_{S^t, \Omega})^{\mu m}\), it is sufficient to show that for any \(\phi_1, \phi_2 \in \mathbb{C}\{y_1, \ldots, y_4\}^{\mu m}\) the inclusion

\[
d\phi_1 \wedge d\phi_2 \in m_{S^t, 0} \cdot (\mathcal{O}_{S^t, \Omega})^{\mu m}
\]

holds. By (5.36) the form \(d\phi_1 \wedge d\phi_2\) is a linear combination of the following

\[
\frac{\partial (\phi_1, \phi_2)}{\partial y_i \partial y_j} dy_i \wedge dy_j = \frac{\partial (\phi_1, \phi_2)}{\partial y_i \partial y_j} \Delta_{k,l} \Omega, \quad \{i, j, k, l\} = \{1, \ldots, 4\}.
\]

Denote

\[
\Xi[i, j, k, l] := \frac{\partial \phi_1}{\partial y_i} \cdot \frac{\partial \phi_2}{\partial y_j} \cdot \frac{\partial \alpha}{\partial y_k} \cdot \frac{\partial \gamma}{\partial y_l}, \quad \{i, j, k, l\} = \{1, \ldots, 4\}.
\]

Since \(\partial (\phi_1, \phi_2)/(\partial y_i \partial y_j) \Delta_{k,l}\) are linear combinations of \(\Xi[i, j, k, l]\), it is sufficient to show that the following holds:

\[
\Xi[i, j, k, l] \mid_{S^t} \in (m_{S^t})_{\text{wt}=0} \cdot (\mathcal{O}_{S^t})_{\text{wt}=a}.
\]

First we note that (5.38) holds if \(\{1, 3\} \subset \{i, j, k\}\). Indeed, let, for example, \(i = 1, j = 3\). Then

\[
\frac{\partial \phi_1}{\partial y_1}, \frac{\partial \phi_2}{\partial y_3} \in m_{S^t}, \quad \text{wt} \left( \frac{\partial \phi_1}{\partial y_1} \cdot \frac{\partial \phi_2}{\partial y_3} \right) \equiv 0, \quad \text{wt} \left( \frac{\partial \alpha}{\partial y_k} \cdot \frac{\partial \gamma}{\partial y_l} \right) \equiv a
\]

(because \(\{\text{wt}(y_k), \text{wt}(y_l)\} = \{0, a\}\)). Thus,

\[
(\partial \phi_1/\partial y_1) \cdot (\partial \phi_2/\partial y_3) \in (m_{S^t})_{\text{wt}=0}, \quad (\partial \alpha/\partial y_k) \cdot (\partial \gamma/\partial y_l) \in (\mathcal{O}_{S^t})_{\text{wt}=a}.
\]
Therefore, \( l = 3 \) or \( 1 \).

Let \( l = 3 \). Then \( \text{wt} (\partial \gamma / \partial y_2) \equiv 2a + 1 \) and
\[
\text{wt} (\partial \phi_1 / \partial y_i), \text{wt} (\partial \phi_2 / \partial y_j), \text{wt} (\partial \alpha / \partial y_k) \equiv -1, -a, 0
\]
up to permutation of \( i, j, k \). We claim that any product \( \Pi \) of three monomials of weight
\(-1, -a, 2a + 1 \) belongs to \((m_{st})_{\text{wt}=0} \cdot (\mathcal{O}_{st})_{\text{wt}=-a} \). This is obvious if \( \Pi \) is divisible by \( y_4, y_1y_3 \), or \( y_2 \). So it is enough to consider the case \( \Pi \) is a power of \( y_1 \) or \( y_3 \). In the former case \( \Pi \) is divisible by
\( y_1^{m-1} \cdot y_1^a \cdot y_1^{2a+1} = y_1^m \cdot y_1^a \), and in the latter \( \Pi \) is divisible by
\( y_3 \cdot y_3^{2m-2a-1} = y_3^m \cdot y_3^{m-a} \), which settles the claim.

Finally, let \( l = 1 \). Similarly to the previous case, we show that any product \( \Pi \) of monomials
of weights \(-a, 1, 2a - 1 \) belongs to \((m_{st})_{\text{wt}=0} \cdot (\mathcal{O}_{st})_{\text{wt}=-a} \). Again we can assume \( \Pi \) is a power of
\( y_1 \) or \( y_3 \). In the latter case, \( \Pi \) is divisible by \( y_3^a \cdot y_3^{m-1} \cdot y_3^{2m-2a+1} = y_3^m \cdot y_3^{m-a} \). In the former case, \( \Pi \) similarly is divisible by
\( y_1^a \). By the equation \([5,32]\), the monomial \( y_1^{a-m} \) belongs to
\( (y_2, y_3)_{m_{st}} \) and we have \( y_1^a \in (m_{st})_{\text{wt}=-a} \cdot (\mathcal{O}_{st})_{\text{wt}=a} \) by \( a > 2a - m \). This concludes the proof of
Lemma 5.8.

By Lemma 5.8 the homomorphism \( \delta \) is factored as other cases:
\[
\delta : H^0(\mathcal{O}_X(-K_X)) \longrightarrow \omega_X \otimes \mathbb{C}_P \longrightarrow (\text{gr}_C^1 \omega^*)^2 \otimes \mathbb{C}_P. \quad (5.39)
\]
Thus we see that \( \delta \) is surjective if and only if \( a = m - 1 \) and \( y_4 \in \alpha \). If \( \delta \) is not surjective, then \( y_2\Omega_P^{-1} \)
generates its image which is the second summand \((0)\) of \( \text{gr}_C^1 \omega^* \) and hence \( \tilde{\sigma} \) is a nowhere vanishing
section. The rest is the same as \([KM92] (2.139)\). This concludes the proof of Theorem 5.2.

**Proposition 5.9.** Let \((X, \bar{C})\) be an extremal curve germ whose central fiber \( \bar{C} \) is reducible. Suppose that \( \bar{C} \) contains a component \( C \) of type \((k2A_2)\) and another component \( C' \) of type \((k1A)\) meeting at a point \( P \) of index \( m > 2 \). Assume further that \((X, \bar{C})\) satisfies the condition \([3,3^\ast]\). Then a general member \( D \in (-K_X) \) is Du Val in a neighborhood of \( C \cup C' \).

**Proof.** The \((k2A_2)\) case 5.8 of Theorem 5.2 shows that a general member \( D(\supset C) \) of \((-K_X) \) is Du Val in a neighborhood of \( C \). If \( D \not\supset C' \), then \( D \cap C' = \{P\} \) because otherwise \( D \) contains a Gorenstein point \( P' \) of \( X \) and so \( D \cdot C' > 1 \) which contradicts \( D \cdot C' = -K_X \cdot C' < 1 \) by \([M88] (2.3.1)\), \([MP08a] (3.1.1)\). Thus we may assume that \( D \supset C' \). We use the notation of 5.8. In view of \((IA)\) and \((IA')\) in \([M88] A.3\), the fact that \( D \) defined by \( y_2 + \cdots = 0 \) contains \( C' \) means that
\((X, C')\) is of type \((IA)\), \( C'^{\ast} \) is smooth at \( P \), and either \( y_1 \) or \( y_3 \) is a coordinate of \( C'^{\ast} \).

**Lemma 5.10.** \( y_3 \) is a coordinate of \( C'^{\ast} \), and hence we may assume that \( C'^{\ast} \) is the \( y_3 \)-axis modulo a \( \mu_m \)-equivariant change of coordinates.

**Proof.** Assume that \( y_1 \) is a coordinate of \( C'^{\ast} \). Then \( I_{C'^{\ast}}^\sharp \), is generated by
\[
y^{\ast}_i \gamma_2 + \delta_2, \quad y^{m-1}_i \gamma_3 + \delta_3, \quad y^m_i \gamma_4 + \delta_4
\]
with \( \gamma_i \in \mathcal{O}_X \) and \( \delta_i \in (y_2, y_3, y_4)^2 \mathcal{O}_X \). Thus
\[
I_{C'^{\ast}} \subset (y_2, y_3, y_4, y^a_1)
\]
and \( \omega^\ast_X \otimes \mathcal{O}_X/(I_{C'^{\ast}}^\sharp + I_{C'^{\ast}}^\sharp) \) contains a non-zero \( \mu_m \)-invariant element \( y^{m-a}_1 \Omega_P \) by \( m - a < a \). In view of the exact sequence
\[
0 \rightarrow \text{gr}_C^0(\omega) \longrightarrow \text{gr}_C^0(\omega \oplus \text{gr}_C^0(\omega) \longrightarrow \left(\omega^\ast_X \otimes \mathcal{O}_X/(I_{C'^{\ast}}^\sharp + I_{C'^{\ast}}^\sharp)\right)^{\mu_m} \rightarrow 0,
\]
we see that \( H^1(\text{gr}_C^0(\omega) \neq 0 \). This implies that \((X, C \cup C')\) is a conic bundle germ and \( C \cup C' \)
is a whole fiber of the conic bundle \([MP08a, Corollary 4.4.1]\). However \((C + C' \cdot D) < 2 \), and this is impossible. \( \square \)
From now on we assume that $C''$ is the $y_3$-axis. Hence $y_2, y_1$ (or $y_4$) form an $\ell$-free $\ell$-basis of $\text{gr}_C^1, \mathcal{O}$, and $y_2\Omega_P^{-1}, y_1\Omega_P^{-1}$ (or $y_4\Omega_P^{-1}$) form an $\ell$-free $\ell$-basis of $\text{gr}_C^1, \omega^*$ at $P$. Furthermore, we see that $\text{gr}_C^1(\omega^*)$ has a global section $\sigma = (y_2 + \cdots)\Omega_P^{-1}$ induced by $\sigma$ defining $D$. We also note $\text{gr}_C^0, \omega^* = ((m - a)P^2)$ since the weight $wt'$ for $C'$ is $wt' \equiv -wt \mod m$. According to the $(k2A_2)$ case of Theorem 5.2 the divisor $D$ is defined at $P$ by

$$y_2\Psi_1 + y_3^{m - a}\Psi_2 = 0$$

where $\Psi_i$ are invariant functions by (5.33)-(5.34). We observe the surjections

$$H^0(\mathcal{O}_X(-K_X)) \longrightarrow \omega_S/\Omega_S^2 \longrightarrow \omega_S \otimes \mathbb{C}_P$$

by (4.1) and (4.2) for the former and by Lemma 5.8 for the latter. In particular, $\Psi_2(P) \neq 0$. Since $C' = (y_3$-axis)/$\mu_m$, we have $D \not\subset C'$. This contradicts our assumption. Thus a general elephant $D$ of $(X, $C'$) is proved to be Du Val in a neighborhood of $C \cup C'$. \hfill \box

6. Proof of the main theorem

**Notation 6.1.** Let $(\bar{X}, \bar{C})$ be an extremal curve germ with reducible central fiber $\bar{C}$ such that on each irreducible component $C_i$ of $\bar{C}$ there exist at most one point of index $> 2$. Let $\{P_a\}_{a \in A}$ be the collection of such points. For each $C_i$ without points of index $> 2$, choose one general point of $C_i$. Let $\{P_b\}_{b \in B}$ be the collection of such points. For each $i \in A \cup B$, let $S_i \in | - 2K_{(X, P_i)}|$ be a general element on the germ $(X, P_i)$, and set $S = \sum_{i \in A \cup B} S_i$. Then $S$ extends to an element $| - 2K_X|$ by [M88 Thm. (7.3)].

**Proof of Main Theorem 1.3** Take a general element $\sigma_S \in \mathcal{O}_S(-K_X)$, and

$$\sigma_S := \sum_i \sigma_i \in \mathcal{O}_S(-K_X).$$

By 4.2(i) or 4.2(ii) the section $\sigma_S \mod \Omega_S^2$ lifts to

$$s \in H^0(X, \mathcal{O}_X(-K_X)).$$

Let $C_{\text{main}} \subset \bar{C}$ be the union of the irreducible component of type (IC), (IIB), (kAD), (k3A), or $(k2A_2)$. By Theorem 5.2 the divisor $D := \{s = 0\} \supset C_{\text{main}}$ is Du Val in a neighborhood of $C_{\text{main}}$ and, for each irreducible component $C$ of $C_{\text{main}}$, the graph $\Delta(D, C)$ is as described in 5.1 by Theorem 5.2. If $C_{\text{main}} = \varnothing$, then we are done by Propositions 3.1 and 3.2. So we assume $C_{\text{main}} \neq \varnothing$ and each irreducible component $C \subset \bar{C}$ intersects $C_{\text{main}}$ (because singular points of $\bar{C}$ are non-Gorenstein on $\bar{X}$, see [M88 Cor. 1.15], [MP08a Lemma 4.4.2]). Then $D$ is normal and $C_{\text{main}}$ is connected. If $C \not\subset D$, then

$$C \cap D \subset C_{\text{main}} \cap D$$

and $D$ is Du Val in a neighborhood of $C$ as well because $C \cdot D < 1$ [M88 (0.4.11.1)] and $D$ is Cartier outside $C_{\text{main}}$ [M88 Corollary 1.15].

Suppose $C_{\text{main}}$ contains a component of one of the types (IC), (IIB), (kAD), or (k3A) and let $C \subset D$. Let $v: D \rightarrow D_0$ be the contraction of $C_{\text{main}}$ and $P_0 := v(C_{\text{main}})$. Then the point $P_0 \in D_0$ is Du Val of type $D$ or $E$ (because $v$ is crepant and by Theorem 5.1). Apply Lemma 2.3 to $D_0$. We conclude that the surface $D_0$ has only Du Val singularities and the same is true for $D$.

If $C$ meets $C_{\text{main}}$ at an index 2 point, then $D \not\subset C$ by Lemma 4.3. The case where $C_{\text{main}}$ consists of curves of type $(k2A_2)$ and $C$ intersects $C_{\text{main}}$ at $P$ of index $m > 2$ is treated in Proposition 5.9. Theorem 1.3 is proved. \hfill \box
Proof of Corollary 1.4. If \( C_i \subset C \) is of type (IIB), then it contains a type cAx/4-point \( P \) and all other components \( C_j \subset C \) passing through \( P \) are of types (IIA) or (II'). But according to [M88, Th. 6.4 & 9.4] \( P \) is the only non-Gorenstein point on \( C_j \). Since \( X \) is not Gorenstein at any intersection point \( C_i \cap C_j \) by [M88 Cor. 1.15] and [MP08a Lemma 4.4.2], all the components \( C_k \subset C \) must pass through \( P \). This proves (i).

From now on we may assume that \( C = C_i \cup C_j \). We also may assume that \( C_j \) is not of type (k2A\(_{m,n} \)), \( n, m \geq 3 \) (otherwise there is nothing to prove). Thus \((X, C)\) satisfies the condition 1.3(*) Let \( f : (X, C) \to (Z, o) \) be the corresponding contraction. Consider a general member \( D \in |-K_X| \) and the Stein factorization

\[
f_D : D \supset C \overset{f'}{\longrightarrow} D_Z \ni o_Z \longrightarrow f(D) \ni o.
\]

By Theorem 1.3 the surface \( D \) has only Du Val singularities. The contraction \( f' : D \to D_Z \) is crepant and point \( D_Z \ni o_Z \) is Du Val. Now we note that the germs \((D, C_i)\) and \((D, C_j)\) are as described by Theorem 5.1. Thus the whole configuration \( \Delta(D, C) \) is one of the Dynkin diagrams A, D or E. In particular, \( \Delta(D, C) \) has no vertices of valency \( \geq 4 \) and at most one vertex, say \( v \), of valency 3. On the other hand, by Theorem 5.2 the configuration \( \Delta(D, C) \) is obtained by "gluing" the configurations \( \Delta(D, C_i) \) described in Theorem 5.1 along one connected component of white subgraph. Since the whole configuration \( \Delta(D, C) \) is Du Val, at most one component of \( C \) is of type (IC), (IIB), (kAD) or (k3A).

For (ii) it is enough to note that all the singularities along \( C_i \) are of type cA and so extremal germs cD/m, cAx/2, cE/2, (IIA), and (II') do not occur as a component of \((X, C)\). Similar argument is applied in (iii) but in this case singularities cD/2 and cAx/2 are allowed [M07]. It remains to prove (iv).

6.1. Let \( T \) be a point of \( C_j \). It is easy to observe that a twisted extension \((X_{j,\lambda}, C_{j,\lambda})\) [M88 1b.8.1] of the germ \((X_{j,\lambda}, T) \supset (C_{j,\lambda}, T)\) by \( u \) in a neighborhood of \( C_{j,\lambda} \) can naturally contain a neighborhood of \( C_i \) if

6.1a \( T \notin C_i \) or
6.1b \((X_{j,\lambda}, T) \supset (C_{j,\lambda}, T)\) is a trivial deformation, i.e. \( X_{j,\lambda} = X_j \) and \( C_{j,\lambda} = C_j \).

We can make a successive deformation of \((X, C)\) in a neighborhood of \( C_j \) which is trivial on a neighborhood of \( C_i \) in such a way that \((X, C_j)\) deforms

(def.1) from (kAD) of the case (5.23) to (k3A), as treated in Lemma 5.5
(def.2) from (k3A) to (k2A\(_2\)) or (kAD) of the case (5.22), as treated in Lemma 5.3
(def.3) from (kAD) of the case (5.22) to (k2A\(_2\)), as treated in Lemma 5.6

ultimately to \((X, C_j)\) of type (k2A\(_2\)). Indeed, 6.1a works with \( T \) as the type (III) point \( \notin C_i \) for deformation (def.2) 6.1b with \( T \) as the index \( m \) point for deformation (def.3).

6.2. For deformation (def.1) we need to take as \( T \) the index 2 point \( Q \) of \( \ell(Q) = 2 \). Suppose \( Q \in C_i \), in which case the divisor \( D \) cannot be Du Val because \( \Delta(D, C_j) \) of type \( A_q \) with \( q \geq 4 \) are connected at the index 2 point \( Q \). So \( Q \notin C_i \) and 6.1a applies, and we are left with the case \( C_j \) is of type (k2A\(_2\)). It remains to disprove the case where both components of \( C \) are of type (k2A\(_2\)). This follows from Lemma 6.4 below.

Lemma 6.2. Let \((X, C)\) be an extremal curve germ such that any component \( C_i \subset C \) is of type \((k2A)\). Assume that a general member \( D \in |-K_X| \) is Du Val. Then a general hyperplane section \( H \in |\mathcal{O}_X| \) passing through \( C \) has only cyclic quotient singularities and the pair \((H, C)\) is log canonical and purely log terminal outside \( \text{Sing}(C) \).
Remark 6.3. If in the conditions of Lemma \[6.2\] the germ \((X, C)\) is a \(\mathbb{Q}\)-conic bundle, then its base surface is smooth. Indeed, if the base surface is singular, then by \([\text{MP08}]\) Theorem 1.3 each component \(C_i \subset C\) must be locally imprimitive.

**Proof.** Let \(f : (X, C) \rightarrow (Z, o)\) be the corresponding contraction. Note that \(D \supset C\). Consider the Stein factorization \([6.1]\). It is easy to see that the configuration \(\Delta(D, C)\) is a linear chain. Therefore, \(D_Z \ni o_Z\) is a (Du Val) singularity of type A. Then the arguments in the proof of \([\text{MPTII}]\) Proposition 2.6] work and show that the pair \((X, D + H)\) is log canonical. Since \(D \supset C\), the pair \((X, H)\) is purely log terminal by Bertini’s theorem. Hence \(H\) is normal and by the inversion of adjunction the pair \((H, C)\) is log canonical. Moreover, \(C = D \cap H\) and \(K_H + C\) is a Cartier divisor on \(H\). By the classification of surface log canonical pairs \([\text{KM98}]\) Theorem 4.15] the singularities of \(H\) are cyclic quotients and the pair \((H, C)\) is purely log terminal outside \(\text{Sing}(C)\). □

**Lemma 6.4.** Let \((X, C)\) be an extremal curve germ, where \(C\) is reducible and has exactly two components. Then both components cannot be of type \((k2A_2)\).

**Proof.** Assume the contrary. The computation below are very similar to that in \([\text{M02}]\) Prop. 2.6]. Let \(C = C_1 \cup C_2\), let \(C_1 \cap C_2 = \{P_0\}\), and let \(P_i \in C_i\) for \(i = 1, 2\) be the non-Gorenstein point other than \(P_0\). Let \(H \in |O_X|\) be a general hyperplane section passing through \(C\). According to Lemma \[6.2\] the surface \(H\) has only cyclic quotient singularities. Consider the minimal resolution \(\mu : \tilde{H} \rightarrow H\) and write

\[
K_{\tilde{H}} = \mu^*K_H - \Theta, \tag{6.2}
\]

where \(\Theta\) is an effective \(\mathbb{Q}\)-divisor with support in the exceptional locus (codiscrepancy divisor). Let \(\tilde{C}_i\) be the proper transform of \(C_i\). Then \(K_{\tilde{H}} \cdot \tilde{C}_i < K_H \cdot C_i < 0\). Hence \(\tilde{C}_i\) is a \((-1)\)-curve on \(\tilde{H}\). Moreover,

\[
\Theta \cdot \tilde{C}_i = 1 + K_H \cdot C_i < 1.
\]

Since \(H\) is a Cartier divisor on \(X\) such that \((X, H)\) is purely log terminal, the singularities of \(H\) are of type \(T\) \([\text{KS88}]\). Hence,

\[
(H \ni P_i) \simeq \left(\mathbb{C}/\mu_{m_i^2p_i}(1, m_i p_i a_i - 1) \ni 0\right)
\]

for some positive \(m_i, p_i, a_i\) such that \(a_i < m_i\) and \(\gcd(m_i, a_i) = 1\). Here \(m_i\) is the index of \(P_i\).

Write \(\Theta = \Theta_0 + \Theta_1 + \Theta_2\) so that \(\text{Supp}(\Theta_i) = \mu^{-1}(P_i)\). Computations with weighted blowups \([\text{KM92}]\) 10.1-10.3] show that the coefficients of \(\Theta_i\) in the ends of the chain \(\text{Supp}(\Theta_i)\) are equal to \((m_i - a_i)/m_i\) and \(a_i/m_i\). Since \(\tilde{C}_1\) and \(\tilde{C}_2\) meet different ends of the chain \(\text{Supp}(\Theta_0)\), up to permutations \(P_1\) and \(P_2\) and changing generators of \(\mu_{m_i^2p_i}\) we have

\[
\Theta_1 \cdot \tilde{C}_1 = \frac{a_1}{m_1}, \quad \Theta_0 \cdot \tilde{C}_1 = \frac{m_0 - a_0}{m_0}, \quad \Theta_0 \cdot \tilde{C}_2 = \frac{a_0}{m_0}, \quad \Theta_2 \cdot \tilde{C}_2 = \frac{m_2 - a_2}{m_2}.
\]

Denote

\[
\delta_1 := a_0 m_1 - a_1 m_0, \quad \delta_2 := a_2 m_0 - a_0 m_2. \tag{6.3}
\]

Then by \[6.2\]

\[
-K_H \cdot C_1 = \frac{a_0}{m_0} - \frac{a_1}{m_1} = \frac{\delta_1}{m_0 m_1} > 0, \quad -K_H \cdot C_2 = \frac{a_2}{m_2} - \frac{a_0}{m_0} = \frac{\delta_2}{m_0 m_2} > 0.
\]

Further, put

\[
\Delta_i := m_0^2 p_0 + m_i^2 p_i - m_0 p_0 m_i p_i \delta_i, \quad i = 1, 2. \tag{6.4}
\]

Then

\[
C_i^2 = \frac{-\Delta_i}{m_0^2 p_0 m_i^2 p_i}, \quad C_1 \cdot C_2 = \frac{1}{m_0^2 p_0}.
\]

Since the configuration is contractible, we have \(\Delta_i > 0\) and

\[
\Delta_1 \Delta_2 - m_1^2 p_1^2 m_2^2 p_2^2 \geq 0. \tag{6.5}
\]
Assume $m_0 > 2$. Since $C_i$ is of type $(k2A_2)$, $m_1 = m_2 = 2$. Then $a_1 = a_2 = 1$ and (6.3) implies
\[
\delta_1 = 2a_0 - m_0 > 0, \quad \delta_2 = m_0 - 2a_0 > 0,
\]
which is impossible. Hence, $m_0 = 2$. Then $a_0 = 1$ and (6.4) can be written as follows
\[
\Delta_i = m_i^2 p_i - 2p_0(m_i p_i - 2) > 0, \quad i = 1, 2,
\]
where $m_i p_i \delta_i - 2 > 0$. Then (6.5) reads
\[
2p_0(m_1 p_1 \delta_1 - 2)(m_2 p_2 \delta_2 - 2) \geq (m_1 p_1 \delta_1 - 2)m_2^2 p_2 + (m_2 p_2 \delta_2 - 2)m_1^2 p_1.
\]
Combining these inequalities we obtain
\[
\frac{m_1^2 p_1}{m_1 p_1 \delta_1 - 2} > 2p_0 \geq \frac{m_2^2 p_2}{m_2 p_2 \delta_2 - 2} + \frac{m_1^2 p_1}{m_1 p_1 \delta_1 - 2}.
\]
The contradiction proves the lemma.

\section*{Appendix A. Examples}

A.1. As was explained in [MP19, Sect. 6.6], to construct an example of an extremal curve germ whose general member of $|\mathcal{E}_X|$ is normal, it is sufficient to construct a normal surface germ $(H, C)$ along a proper connected curve $C$ satisfying the following conditions:

(a) there exists a contraction $f_H : (H, C) \to (T, o)$ such that $-K_H$ is relatively ample and $f_H^{-1}(o)_{\text{red}} = C$,

(b) for every each singular point $P_i \in H$ there exists a threefold terminal singularity $(X_i, P_i)$ and an embedding $(H, P_i) \subset (X_i, P_i)$ such that $H$ is a Cartier divisor on $X_i$ (at $P_i$).

Then there exists an extremal curve germ $(X, C)$ and an embedding $(H, C) \subset (X, C)$ such that $H$ is a member of $|\mathcal{E}_X|$ on $X$. Moreover, near each point $P_i \in H \subset X$ there is an isomorphism $(X_i, P_i) \simeq (X, P_i)$ inducing the embedding $(H, P_i) \subset (X_i, P_i)$:

\[
(X_i, P_i) \simeq (X, P_i) \subset (H, P_i)
\]

Then for any component $C_j \subset C$, types of points $P_i \in (X, C_j)$ can be seen locally near each $P_i$.

A.2. To construct examples of an extremal curve germs with reducible central curve one can start with one of the configurations described in the Appendix of [KM92] or [MP14] and add extra black vertices. We need to check only the amleness of the anti-canonical divisor. Below we give a few examples. Of course, they do not exhaust all the possibilities. Below in the dual graphs of minimal resolutions we use the notation [5.1]. A number attached to a white vertex denotes the negative of the self-intersection number. We may omit 2 if the self-intersection equals $-2$.

\textbf{Example A.1.} Starting with an exceptional flipping extremal curve germ $(X, C_1)$ of type (IC) as in [KM92, A.3.2.1] we can construct the following configurations:

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{example.png}
\end{figure}

Computing the intersection numbers on the minimal resolution of $H$ we obtain $-K_H \cdot C_1 = -K_H \cdot C_2 = 1/m$ and $-K_H \cdot C_3 = (m - 1)/2m$. Each $\bullet$ corresponds to a $(-1)$-curve. Hence, $-K_H$ is ample and our construction gives an example of a divisorial contraction of type (IC)+(k1A)+(k1A). (see Corollary [4(ii)].)
Example A.2. Starting with an exceptional divisorial extremal curve germ \((X, C_1)\) of type (IIB) as in [MP14, 1.2.2] we can construct the following configurations:

\[
\begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\end{array}
\]

Computing the intersection numbers on the minimal resolution of \(H\) we obtain
\[-K_H \cdot C_1 = -K_H \cdot C_2 = 1/4.\]
Hence, \(-K_H\) is ample and the germ \((X, C_2)\) is locally primitive. Therefore, our construction gives an example of a divisorial contraction of type (IIB)+(IIA) (see Corollary 1.4(i)).

Example A.3. Starting with an exceptional divisorial extremal curve germ \((X, C_1)\) of type \(cD/3\) as in [MP11, 4.5.2.1] we can construct the following configurations:

\[
\begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\end{array}
\]

So, this is an example of a \(Q\)-conic bundle contraction of type \(cD/3+cD/3+cD/3\).

Appendix B. \(Q\)-conic bundles with irreducible central fiber

For convenience of references, we provide below the classification of irreducible \(Q\)-conic bundle extremal curve germs and their general anticanonical members.

Notation. Let \((X, C)\) be a \(Q\)-conic bundle extremal curve germ with irreducible \(C\), let \(f: X \to Z\) be the corresponding contraction, and let \(D \in |-K_X|\) be the general member. If \(D \supset C\), then \(\Delta(D, C)\) denotes the dual graph of the minimal resolution \(\mu: \tilde{D} \to D\). For this graph, we use the standard notation: each vertex \(\circ\) corresponds to a prime \(\mu\)-exceptional divisor (which is a \((-2)\)-curve on \(\tilde{D}\)) and vertex \(\bullet\) corresponds to the proper transform of \(C\) (which is a \((-1)\)-curve).

The cases with singular \(Z\) [MP08a].

Case (T), see [MP08a, 1.2.1]. \(D\) does not contain \(C\) and it is a disjoint union \(D = D_1 + D_2\), where \(D_1 \simeq D_2\) is a singularity of type \(A_{m-1}\), \(m \geq 2\).

Case (k2A), see [MP08a, 1.2.2 and Theorem 11.1]. \(D \supset C\), it is given by \(y_1 = 0\), and \(\Delta(D, C)\) is as follows:

\[
\begin{array}{c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\bullet & \bullet & \bullet \\
\end{array}
\]

\(m\) is odd \(\geq 3\).

Case (IA\(\nu\)), see [MP08a, 1.2.3]. The intersection \(D \cap C\) is a single point \(P\) and \((D, P)\) is of type \(A_7\).

Case (ID\(\nu\)), see [MP08a, 1.2.4]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(A_3\) or \(D_k, k \geq 4\).

Case (IA\(\nu\)), see [MP08a, 1.2.5]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(A_3\).

Case (II\(\nu\)), see [MP08a, 1.2.6]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(D_{2k+1}, k \geq 2\).

The cases with smooth \(Z\) [MP08a, MP09, MP11, MP14].

Gorenstein case. \(|-K_X|\) is base point free and \((D, P)\) is smooth.

Case (cAx/2), see [MP08a § 12], [MP11 § 7]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(D_k, k \geq 4\).
Case (cD/2), see \[\text{MP08a}, \S \, 12\]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(D_{2k}, \ k \geq 3\).

Case (cE/2), see \[\text{MP08a}, \S \, 12\]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(E_7\).

Case (k1A), see \[\text{MP08a}, \text{Theorem} \ 8.6(\text{iii})\]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(A_k, \ k \geq 1\).

Case (cD/3), see \[\text{MP08a}, \text{Theorem} \ 8.6(\text{iii})\]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(E_6\).

Case (IC), see \[\text{MP09, Theorem} \ 1.3\] and \[\text{MP14, Theorem} \ 1.1\]. \(D \supset C\), the unique non-Gorenstein point has index 5, and \(\Delta(D, C)\) is as follows:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\bullet \\
\circ \\
\end{array}
\]

Case (IIA), see \[\text{MP16}\]. \(D \cap C = \{P\}\) and \((D, P)\) is of type \(D_{2k+1}, \ k \geq 2\).

Case (IIB), see \[\text{MP09, Theorem} \ 1.3\]. \(D \supset C\) and \(\Delta(D, C)\) is as follows:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\bullet \\
\circ \\
\end{array}
\]

Case (kAD), see \[\text{MP09, Theorem} \ 1.3\]. \(D \supset C\) and \(\Delta(D, C)\) is as follows:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\circ \\
\bullet \\
\circ \\
\end{array}
\]

\[m - 1\]

where \(m\) is odd \(\geq 3\).

Case (k3A), see \[\text{MP09, Theorem} \ 1.3\]. \(D \supset C\) and \(\Delta(D, C)\) is as follows:

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\bullet \\
\circ \\
\end{array}
\]

\[m - 1\]

\[m \text{ is odd } \geq 3 \text{ or } m = 3\]

References

[K88] Yu. Kawamata. Crepant blowing-up of 3-dimensional canonical singularities and its application to degenerations of surfaces. \textit{Ann. of Math. (2)}, 127(1):93–163, 1988.

[KM92] J. Kollár and S. Mori. Classification of three-dimensional flips. \textit{J. Amer. Math. Soc.}, 5(3):533–703, 1992.

[KM98] J. Kollár and S. Mori. \textit{Birational geometry of algebraic varieties}, Cambridge Tracts in Mathematics, v. 134. Cambridge University Press, Cambridge, 1998.

[Ko99] J. Kollár. Real algebraic threefolds. III. Conic bundles. \textit{J. Math. Sci. (New York)}, 94(1):996–1020, 1999. Algebraic geometry, 9.

[KS88] J. Kollár and N. I. Shepherd-Barron. Threefolds and deformations of surface singularities. \textit{Invent. Math.}, 91(2):299–338, 1988.

[M88] S. Mori. Flip theorem and the existence of minimal models for 3-folds. \textit{J. Amer. Math. Soc.}, 1(1):117–253, 1988.

[M02] S. Mori. On semistable extremal neighborhoods. In \textit{Higher dimensional birational geometry (Kyoto, 1997), Adv. Stud. Pure Math.}, v. 35, pp. 157–184. Math. Soc. Japan, Tokyo, 2002.

[M07] S. Mori. Errata to: \[\text{KM92}\]. \textit{J. Amer. Math. Soc.}, 20(1):269–271, 2007.

[MP08a] S. Mori and Yu. Prokhorov. On \(Q\)-conic bundles. \textit{Publ. Res. Inst. Math. Sci.}, 44(2):315–369, 2008.

[MP08b] S. Mori and Yu. Prokhorov. On \(Q\)-conic bundles. II. \textit{Publ. Res. Inst. Math. Sci.}, 44(3):955–971, 2008.
[MP09] S. Mori and Yu. Prokhorov. On $\mathbb{Q}$-conic bundles, III. *Publ. Res. Inst. Math. Sci.*, 45(3):787–810, 2009.

[MP11] S. Mori and Yu. Prokhorov. Threefold extremal contractions of type IA. *Kyoto J. Math.*, 51(2):393–438, 2011.

[MP14] S. Mori and Yu. Prokhorov. Threefold extremal contractions of types (IC) and (IIB). *Proc. Edinburgh Math. Soc.*, 57(1):231–252, 2014.

[MP16] S. Mori and Yu. Prokhorov. Threefold extremal contractions of type (IIA), I. *Izv. Math.*, 80(5):884–909, 2016.

[MP19] S. Mori and Yu. Prokhorov. Threefold extremal curve germs with one non-Gorenstein point. *Izv. Math.*, 83(3):565–612, 2019.

[P97] Yu. Prokhorov. On the existence of complements of the canonical divisor for Mori conic bundles. *Russian Acad. Sci. Sb. Math.*, 188(11):1665–1685, 1997.

[R87] M. Reid. Young person’s guide to canonical singularities. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, *Proc. Sympos. Pure Math.*, v. 46, pp. 345–414. Amer. Math. Soc., Providence, RI, 1987.

[S92] V. Shokurov 3-fold log flips *Russ. Acad. Sci., Izv., Math.*, 40(1): 95–202, 1993.

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