From Lagrangian to Hamiltonian formulations of the Palatini action

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Abstract

We work on the Lagrangian and the Hamiltonian formulations of the Palatini action. In the Lagrangian formulation, we find that we need to assume the metric compatibility and the torsion zero or to assume the tetrad compatibility to describe General Relativity. In the Hamiltonian formulation, we obtain the Einstein’s equations only with assuming the tetrad compatibility. The Hamiltonian from assuming the metric compatibility and the torsion zero should be used to quantize General Relativity.

1 Introduction

The tetrad and the internal connection formulation of General Relativity has been studied more than 30 years, yet it is still obscure what should be assumed beforehand and what are derived afterward from the Euler-Lagrange equations in the beginning Lagrangian formulation of this program. In this paper, we clear this up once and for all. This makes the Hamiltonian formulation more interesting than previously known.

We derive the Palatini action from the Einstein-Hilbert action. From the variational principle, we find that varying the connection, we have the compatibility condition of the connection with the tetrad when we assume the metric compatibility and the torsion zero conditions. Varying the tetrad, we have the Einstein equations. When the torsion is not zero, varying the connection gives us the torsion zero condition if the connection is compatible with the tetrad. In the Lagrangian formulation, we find these two approaches to describe General Relativity, which we apply to the Hamiltonian formulation.

We perform the Legendre transformation and obtain the Hamiltonian. There are 2nd class constraints. From the lesson above, we solve these and obtain the scalar, vector and Gauss constraints. In the first approach of the metric compatibility and the torsion zero conditions, the Hamiltonian equations of motion are different from the Einstein’s equations. In the second approach of the tetrad compatibility condition, the Hamiltonian equations of motion become the Einstein’s equations after solving the Gauss constraint.

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In section 2, we introduce Riemannian geometry \[1\]. Spacetime and spatial tensor indices are denoted by the alphabet \(a, b, \cdots\), while internal indices are denoted by the alphabet \(i, j, \cdots\) for 3-dimension and \(I, J, \cdots\) for 4-dimension. The signature of the spacetime metric \(g_{ab}\) is taken to be \((-+++\)).

\section{Connection and Torsion}

Consider a 4-dimensional manifold \(M\), and let \(V\) be a 4-dimensional vector space with Minkowski metric \(\eta_{IJ}\) having signature \((-+++\)). A tetrad at \(p \in M\) is an isomorphism \(e^a_I(p) : V \to T_pM\) and can act on tensors. For example

\[\eta_{IJ} = g_{ab} e^a_I e^b_J.\] (1)

The inverse of \(e^a_I\) will be denoted by \(e_I^a\). It satisfies

\[\eta_{IJ} e^I_a e^J_b = g_{ab}.\] (2)

Spacetime tensor fields with additional internal indices \(I, J, \cdots\) will be called generalized tensor fields on \(M\). Spacetime indices are raised and lowered with the spacetime metric \(g_{ab}\); internal indices are raised and lowered with the Minkowski metric \(\eta_{IJ}\).

A generalized derivative operator obey the linearity, Leibnitz rule, and commutativity with contraction with respect to both the spacetime and the internal indices. We require that all generalized derivative operators be compatible with \(\eta_{IJ}\). If \(\partial_a\) is a derivative operator, then any other generalized derivative operator \(D_a\) is defined by a pair of generalized tensor fields \(A_{ab}^c\) and \(w_{aI}^J\):

\[D_a H_{bI} \equiv \partial_a H_{bI} + A_{ab}^c H_{cI} + w_{aI}^J \delta_{b}^{J}.\] (3)

From \(D_a \eta_{IJ} = 0\), we obtain

\[w_{aIJ} = w_{a[IJ]}.\] (4)

If \(D_a g_{bc} = 0\),

\[A_{ab}^c = \Gamma_{ab}^c + \frac{1}{2} \{ - T_{ab}^c b + T_{bc}^c a + T_{ab}^e \},\] (5)

where \(\Gamma_{ab}^c\) is the Christoffel symbols,

\[\Gamma_{ab}^c = - \frac{1}{2} g^{cd} (\partial_b g_{ad} + \partial_a g_{bd} - \partial_d g_{ab})\] (6)

and \(T_{ab}^c\) is the torsion,

\[T_{ab}^c = A_{ab}^c - A_{ba}^c,\] (7)

which measures the failure of the closure of the parallelogram made up of small displacement vectors and their parallel transports \[2\] and the non-commutativity of the derivative operator on a scalar field \(f\) such that

\[D_a D_b f - D_b D_a f = T_{ab}^c D_c f.\] (8)
If $T_{ab}^c = 0$, just as a compatibility with a spacetime metric $g_{ab}$ defines a unique, torsion-free spacetime derivative operator, compatibility with $e_I^a$ defines a unique torsion-free generalized derivative operator $\nabla_a$ defined by

$$\nabla_a e_b I \equiv \partial_a e_b I + \Gamma_{ab}^c e_c I + w_{ak}^K e_b K = 0.$$  

(9)

Whether the torsion is zero or not, the compatibility condition gives

$$w_{aI}^J = -e^b_I(J(\partial_a e_b I + A_{ab}^c e_c I)).$$  

(10)

In this case, $w_{aI}^J$ is the spin connection. It is related to the spacetime geometry and has informations about the torsion and the curvature.

In the notation of differential form, the torsion is defined as

$$T^I \equiv d e^I + w^I_J \wedge e^J$$  

(11)

which means

$$T_{ab}^I = 2\partial_{[a} e_{b]}^I + w_{aI}^J e_b^J - w_{bI}^J e_a^J.$$  

(12)

In Riemannian geometry, (10) is always satisfied. In this case $T_{ab}^I$ and $T_{ab}^c$ are equivalent:

$$T_{ab}^c = T_{ab}^I e_I^c.$$  

(13)

For the zero torsion, we can write $w_{aI}^J$ in terms of $e_I^a$ using (12) and it turns out to be equivalent to (10) with $A_{ab}^c = \Gamma_{ab}^c$. For the non-zero torsion, if we plug (10) into (12), we obtain (7). In the connection formulation of the Palatini action, $e_I^a$ and $w_{aI}^J$ are the basic independent variables. Therefore (10) and (13) are not satisfied in general and (12) does not have the geometrical meanings of Riemannian geometry.

Given a generalized derivative operator $D_a$, we can construct curvature tensors by commuting derivatives. For the torsion zero, the internal curvature tensor $F_{abI}^J$ and the spacetime curvature tensor $\tilde{F}_{abc}^d$ are defined by

$$2D_{[a} D_{b]} H_I \equiv F_{abI}^J H_J \quad \text{and}$$

$$2D_{[a} D_{b]} H_c \equiv \tilde{F}_{abc}^d H_d.$$  

(14)

(15)

From these

$$F_{abI}^J = 2\partial_{[a} w_{b]}^I + [w_a, w_b]^I_J \quad \text{and}$$

$$\tilde{F}_{abc}^d = 2\partial_{[a} A_{bc]}^d + [A_a, A_b]^d_{c}.$$  

(16)

(17)

Here $[w_a, w_b]^I_J = (w_{aI}^K w_{bK}^J - w_{bI}^K w_{aK}^J)$ and $[A_a, A_b]^d_{c} = (A_{ac}^e A_{be}^d - A_{bc}^e A_{ae}^d)$. For the non-zero torsion, we have an additional term from the torsion to keep the linearity of the curvature tensor [3]

$$(2D_{[a} D_{b]} - T_{ab}^c D_c) H_I \equiv F_{abI}^J H_J \quad \text{and}$$

(18)
\( (2D_{[a}D_{b]} - T_{ab}^d D_d) H_c \equiv \tilde{F}_{abc}^d H_d. \)  

(19)

We denote internal and spacetime curvature tensors of the unique torsion-free generalized derivative operator \( \nabla_a \) by \( R_{ab}^J \) and \( R_{abc}^d. \) From (14) and (15), we can see that they are related by

\[
R_{ab}^J = R_{abc}^d e_\ell^c e_\ell_d. \tag{20}
\]

3 Palatini theory: Lagrangian formulation

The Einstein-Hilbert action is

\[
S_{EH}(g^{ab}) = \int_M \sqrt{-g} R \tag{21}
\]

and

\[
\sqrt{-g} R = \frac{1}{4} \hat{\eta}^{abcd} \epsilon_{abcd} R_{cd} \tag{22}
\]

where \( \hat{\eta}^{abcd} \) is the Levi-Civita tensor density of weight 1 and

\[
\epsilon_{abcd} = \epsilon_{IJKL} e_a^I e_b^J e_c^K e_d^L, \tag{23}
\]

which relates the volume element \( \epsilon_{abcd} \) of \( g_{ab} \) to the volume element \( \epsilon_{IJKL} \) of \( \eta_{IJ}. \) The Einstein-Hilbert action in terms of a co-tetrad \( e_I^a \) is

\[
S_{EH}(e) = -\frac{1}{4} \int_M \hat{\eta}^{abcd} \epsilon_{IJKL} e_a^I e_b^J e_c^K e_d^L R_{cd}^{KL}. \tag{24}
\]

In the Palatini action, \( e_I^a \) and \( w_{a}^{IJ} \) are the basic independent variables. By replacing \( R_{ab}^J \) in (24) with the internal curvature tensor \( F_{ab}^J \) of an arbitrary generalized derivative operator \( D_a \) defined by (3), we obtain the 3+1 Palatini action based on \( SO(3, 1): \)

\[
S_p(e, w) \equiv -\frac{1}{8} \int_M \hat{\eta}^{abcd} \epsilon_{IJKL} e_a^I e_b^J e_c^K e_d^L F_{cd}^{KL}. \tag{25}
\]

An additional factor \( 1/2 \) which will not affect the Euler-Lagrange equations of motion is included for the Hamiltonian formulation. With

\[
\hat{\eta}^{abcd} \epsilon_{IJKL} e_a^K e_d^L = -4 \sqrt{-g} e_I^{[a} e_j^{b]}, \tag{26}
\]

the Palatini action is

\[
S_p(e, w) \equiv \frac{1}{2} \int_M \sqrt{-g} e_I^{a} e_j^{b} F_{ab}^{IJ}. \tag{27}
\]

\( \sqrt{-g} \) is the determinant of a metric \( g_{ab}, \) which is the determinant of \( e_I^a \) from \( g_{ab} = \eta_{IJ} e_I^a e_j^b. \) Because we are interested in the role of the metric compatibility condition, this expression \( \sqrt{-g} \) here is useful.
It is also important to write the exact statement of a relation between the metric compatibility condition, the torsion zero condition and the tetrad compatibility condition: If $D_ag_{bc} = 0$ and $T_{ab}^c = 0$, then $T_{ab}^I = 0$ if and only if $D_a e_I^b = 0$ \[4\]. Stokes’s theorem holds for a torsion-free derivative operator on a orientable manifold and Gauss’s theorem holds when the metric compatibility condition is satisfied once a volume element is chosen by a metric. Because great care must be taken to apply the variational principle without $D_ag_{bc} = 0$ or the torsion zero condition, let’s work on a simple model first:

$$S \equiv \int_M \sqrt{-g} P^a Q^b D_a R_b. \quad (28)$$

If $D_ag_{bc} = 0$ and $T_{ab}^c = 0$,

$$\partial_a (\sqrt{-g} P^a) = \sqrt{-g} D_a P^a \quad (29)$$

where we used the formula:

$$\partial_a \sqrt{-g} = \frac{1}{2} \sqrt{-g} g^{bc} \partial_a g_{bc} = -\sqrt{-g} (A_{ba}^b - T_{ba}^b). \quad (30)$$

Note that the first equality holds also for $D_a$ and we have

$$D_a \sqrt{-g} = \partial_a \sqrt{-g} + A_{ab}^b \sqrt{-g}. \quad (31)$$

Generally without assuming $D_ag_{bc} = 0$,

$$\partial_a (\sqrt{-g} P^a) = \sqrt{-g} D_a P^a + (\partial_a \sqrt{-g} + \sqrt{-g} A_{ba}^b) P^a = D_a (\sqrt{-g} P^a) + \sqrt{-g} T_{ba}^b P^a. \quad (32)$$

Let’s see what we have when we vary $R_a$. From $\delta S = 0$, we have

$$\sqrt{-g} D_a (P^a Q^b) + (\partial_a \sqrt{-g} + \sqrt{-g} A_{ba}^b) P^a = 0, \quad (33)$$

where we used $\delta R_a = 0$ on the boundary. Note that the second term does not disappear as in (32). If $T_{ab}^c = 0$, we have

$$D_a (\sqrt{-g} P^a Q^b) = 0. \quad (34)$$

We can see that integration by parts works for $D_a$ when $T_{ab}^c = 0$. A solution $D_a (P^a Q^b) = 0$ is obtained only when $D_ag_{bc} = 0$ and $T_{ab}^c = 0$.

Let’s work on the Palatini action with the variational method. To see what we have when we vary $e_I^a$, note that $\tilde{\eta}^{abcd}$ and $\epsilon_{IJ,KL}$ are -1 or 0 or 1 depending on their indices, so they are independent of $e_I^a$. With this, varying $e_I^a$ in (25) gives

$$\tilde{\eta}^{abcd} \epsilon_{IJ,KL} e^I_b F_{cd}^{KL} = 0. \quad (35)$$

For $w_a^{IJ}$, we need the following formula:

$$\delta F_{ab}^{IJ} = 2 D_a \delta w_{[b]}^{IJ} - T_{ab}^c \delta w_{c}^{IJ}. \quad (36)$$

We can see immediately that the variational calculations of the Palatini action (27) with respect to $w_a^{IJ}$ are very similar to those of our simple action (28).
If we assume $D_ag_{bc} = 0$ and $T_{ab}^c = 0$, varying $w_a^{IJ}$ gives us
\[ D_a(e_I^{[a} e_J^{b]}) = 0. \] (37)

To determine what (37) gives, let us express $D_a$ in terms of the unique, torsion-free generalized derivative operator $\nabla_a$ compatible with $\epsilon_a^I$, and $C_{aI}^J$ defined by
\[ D_a H_a^b = \nabla_a H_a^b + C_{aI}^J H_J^b. \] (38)

Note that this expression is possible only when $D_ag_{bc} = 0$. Multiplying $\epsilon_a^J$ to (37), we have
\[ D_a \epsilon_a^I = 0. \]

Since $\epsilon_a^I$ is invertible, combining these we get
\[ \epsilon_a^I C_{aJK} - \epsilon_b^J C_{aIK} = 0. \] (39)

Multiplying $\epsilon_a^J e_b^I$, we have
\[ C_{bJK} = \epsilon_a^I C_{aIK}. \] (40)

With $C_{bJK} + C_{bKJ} = 0$, we have
\[ \epsilon_a^I C_{aIK} + \epsilon_b^J C_{aIJ} = 0. \] (41)

With index substitutions $I \rightarrow K, J \rightarrow I, K \rightarrow J,$
\[ \epsilon_a^I C_{aJK} + \epsilon_b^J C_{aIK} = 0. \] (42)

With (39) and (42), we obtain $C_{aIJ} = 0$. Algebraically there are 24 homogeneous linear equations of 24 variables $C_{aIJ}$, so $C_{aIJ} = 0$. Since $C_{aI}^J = 0$, we find that one equation of motion implies that $D_a = \nabla_a$ and $F_{ab}^{IJ} = R_{ab}^{IJ}$. The remaining Euler-Lagrange equation of motion becomes
\[ \tilde{\eta}^{abcd} e_{IJ}^{KL} e_b^J R_{cd}^{KL} = 0. \] (43)

When (43) is contracted with $e^{eI}$, we get the 3+1 vacuum Einstein’s equation, $G^{ac} = 0$.

If we do not assume $D_ag_{bc} = 0$ but only assume $T_{ab}^c = 0$, we have
\[ \sqrt{-g} D_a(e_I^{[a} e_J^{b]}) + (\partial_a \sqrt{-g} + \sqrt{-g} A_{ca}^c)(e_I^{[a} e_J^{b]}) = 0. \] (44)

In this case, we need to add $B_{ac}^c = A_{ac}^c - \Gamma_{ac}^c$ in (38) to determine what (44) gives such that
\[ D_a H_a^b = \nabla_a H_a^b - B_{ac}^b H_a^c + C_{aI}^J H_J^b. \] (45)

If we express (44) with (45), there are 24 inhomogeneous linear equations of 24 variables $C_{aI}^J$, so $C_{aI}^J \neq 0$. In this case, $D_a e_I^b$ is not zero. The Palatini action does not become the Einstein-Hilbert action and we do not have the Einstein’s equations. If we assume 40 components of $D_a e_I^b$ are zero, which are linear relations between $B_{ac}^c$ and $C_{aI}^J$, we obtain other 24 components of $D_a e_I^b$ are zero from (44). However, this assumption is not covariant. Therefore we must assume $D_ag_{bc} = 0$.

Finally if we do not assume $T_{ab}^c = 0$, varying $w_a^{IJ}$ gives us
\[ 2D_a(\sqrt{-g} e_I^{[a} e_J^{b]}) + \sqrt{-g}(2e_I^{[a} e_J^{b]} T_{ca}^c + e_I^{[c} e_J^{b]} T_{ac}^c) = 0. \] (46)
If we assume $D_a e^b_I = 0$, we multiply $e^b_I$ to both sides and obtain $T_{ac}^e = 0$. Thus we have $T_{ab}^c = 0$ and the Palatini action describe General Relativity.

Since $D_a e^b_I = 0$ means $D_a g_{bc} = 0$, we can see that we must assume $D_a g_{bc} = 0$ to have the Einstein's equations from the Palatini action. Because this condition is assumed from the beginning, it must be preserved in quantization. We also need to assume either $T_{ab}^c = 0$ or $D_a e^b_I = 0$ to have the Einstein's equations, which should also be preserved in quantization. The conditions $D_a g_{bc} = 0$ and $T_{ab}^c = 0$ are what Einstein assumed when he constructed General Relativity [6]. With these two conditions, geodesic is an extremal length between two spacetime points, which is related to the Principle of Equivalence. On the other hand, assuming $D_a e^b_I = 0$ is based on Riemannian geometry. It is straightforward to check that our results also hold for the Holst action [7].

4  Palatini theory: Hamiltonian formulation

Before working on the Hamiltonian formulation of the Palatini action, let’s discuss the equivalence of the Lagrangian and the Hamiltonian formulation. To construct the Hamiltonian, we define the momentum variable $p_i$ from the Lagrangian $L(q, \dot{q})$:

$$p_i = \frac{\partial L(q, \dot{q})}{\partial \dot{q}_i}.$$  (47)

We obtain the Hamiltonian with the Legendre transformation:

$$H(q, p) = \dot{q}_i p_i - L(q, \dot{q}).$$  (48)

With this, we obtain the Hamiltonian equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i},$$  (49)

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}.$$  (50)

The Euler-Lagrange equations are equivalent to the Hamiltonian equations when (47) is equivalent to (49). In the Palatini action, the independent variables are $e^a_I$ and $w_a^{IJ}$. $\dot{q}_i$ of $w_a^{IJ}$ comes from $F_{ab}^{IJ}$, but where is $\dot{q}_i$ of $e^a_I$? Because the metric compatibility condition and the torsion zero condition deal only with $A_{ab}^c$, $\dot{q}_i$ of $e^a_I$ comes from the tetrad compatibility condition. Therefore we will see that only in the second approach, the Lagrangian and the Hamiltonian formulation are equivalent. The Hamiltonian equations of motion from the first approach should be treated as one of modifications of General Relativity for quantization [8].

Let’s work on the Hamiltonian formulation of the first approach. To perform the Legendre transformation, we introduce a foliation $\{\Sigma\}$ in space-time and a time-like vector field $t^a$ whose integral curves intersect each $\Sigma$ of the foliation precisely once. Let $n^a$ denote the unit normal to the foliation. We can then decompose the time-evolution vector field $t^a$ normal and tangential to the foliation:

$$t^a = N n^a + N^a,$$

$$n^a N_a = 0.$$  (51)

The function $N$ is called the lapse function and the vector field $N^a$ is called the shift vector [9]. Given $n^a$, it follows that $q_0^b = \delta^b_0 + n^a n_b$ is a projection operator into the foliation. Let $E_i^a = q_0^b e^b_i$. Let $^4D$
denote an derivative operator on $M$ and $D_a = q_a^b D_b$ on $\Sigma$. We can now decompose the action (27):

\[
e^a_1 e^b_J F^I_{ab} = e^a_1 e^b_J (q_a^c - n^c n_a) (q_b^d - n^d n_b) F^I_{cd}.
\] (52)

The first term becomes $E^a_I E^b_J F^I_{ab}$, the last term becomes zero and the cross terms are:

\[
-2e^a_1 e^b_J q_a^c n^d n_b F^I_{cd} = -2E^a_i n_J (\frac{1}{N} t^d - \frac{N^d}{N}) F^I_{cd}.
\] (53)

where $t^d F^I_{cd} = -\mathcal{L}_t w^I + 4 D_e (t^d w^I_d)$. The action becomes:

\[
S_p = \frac{1}{2} \int dt \int d^3 x \sqrt{q} \left( N E^a_I E^b_J F^I_{ab} + 2 n^a E^a_i n_J D_a (w \cdot t)^I J - 2 n^a E^a_i w^I + 2 N^a n_J E^a_{i J} F^I_{ab} \right),
\] (54)

where $\sqrt{q} = N \sqrt{q}$. Note that all $a, b, \cdots$ are spatial and now $F_{ab}^I J$ in (54) is the curvature tensor of $D_a$. To further simplify the action, we define

\[
\hat{E}^a_I = \sqrt{q} E^a_I,
\] (55)

\[
\hat{E}^a_{IJ} = E^a_{[IJ]}.
\] (56)

Because there is not much confusion, we keep using $\hat{E}$ for $\hat{E}^a_{IJ}$. With this

\[
\text{tr}(\hat{E}^a \hat{E}^b F_{ab}) = \hat{E}^a_{[IJ]} \hat{E}^{K[I} F^J_{ab K} = \frac{1}{4} \hat{E}^a_I \hat{E}^b_{IJ} F^I_{ab},
\] (57)

where we have used that $E^a_i n^J = 0$. In this way, the action becomes:

\[
S_p = \int dt \int d^3 x \frac{\sqrt{q}}{2} \left( -N \hat{E}^a \hat{E}^b F_{ab} + N^a \hat{E}^b F_{ab} - (w \cdot t) D_a \hat{E}^a - \hat{E}^a \dot{w}^a \right),
\] (58)

where we used the fact that the torsion zero condition in 4-dimension makes the torsion in 3-dimension vanish.

We can see that $w^a_{IJ}, -\hat{E}^a_{IJ}$ are canonical variables and $N, N^a, (w \cdot t)^I J$ are non-dynamical. They serve as Lagrange multipliers. Variation of the action with respect to these fields yields the constraints:

\[
H_s = \frac{2}{\sqrt{q}} \text{tr}(\hat{E}^a \hat{E}^b F_{ab}) \approx 0,
\] (59)

\[
V_a = -\text{tr}(\hat{E}^b F_{ab}) \approx 0,
\] (60)

\[
G_{IJ} = -D_a \hat{E}^a_{IJ} \approx 0.
\] (61)

The Hamiltonian up to surface terms is

\[
H = \int \Sigma d^3 x \left( N H_s + N^a V_a + (t \cdot w)^I J G_{IJ} \right).
\] (62)
There are second class constraints in this formulation. Not all $\tilde{E}_{IJ}^a$ are independent and we have a primary constraint

$$\phi^{ab} \equiv \epsilon^{IJKL} \tilde{E}_{IJ}^a \tilde{E}_{KL}^b \approx 0,$$

which is obvious from (56). All Poisson brackets between constraints vanish weakly except one between $H_a$ and $\phi^{ab}$ \textsuperscript{10}. The secondary constraints from this is

$$\chi^{ab} \equiv \epsilon^{IJKL} D_a [\tilde{E}_b^c, \tilde{E}_c^d]_{KL} + (a \leftrightarrow b) \approx 0,$$

where $[\tilde{E}_b^c, \tilde{E}_c^d]_{KL} = \tilde{E}_b^c \tilde{E}_c^d_{KL} - \tilde{E}_b^c \tilde{E}_c^d_{KL}$. The Poisson bracket between $\chi^{ab}$ and the total Hamiltonian vanishes weakly, and

$$\{ \phi^{ab}(x), \chi^{cd}(y) \} \approx 8q(2q^{ab}q^{cd} - q^{ad}q^{bc} - q^{ac}q^{bd}) \neq 0.$$

Thus we do not have any more constraints and $\phi^{ab}, \chi^{ab}$ are the second class constraints.

Now how to solve the second class constraints? We have learned from the Lagrangian formulation of the Palatini theory that we need the tetrad compatibility condition to have the Einstein Equation. For the Hamiltonian formulation, we break 4-dimensional diffeomorphic covariance to 1+3, but we still have 3-dimensional covariance. Therefore we might guess that the 3-dimensional triad compatibility condition can solve the 2nd class constraints. We will see that this turns out to be the case.

To solve (64), we fix $n_I$ by $\partial_a n_I = 0$. This makes an internal vector field $n_I$ become an internal vector, which means we break 4-dimensional internal covariance to 3+1. With this, $\tilde{E}_{IJ}^a$ has 9 degrees of freedom from $\tilde{E}_{I}^a$. To make $w_{IJ}^a$ also have 9 degrees of freedom, we also request

$$n^I G_{IJ} = 0,$$

because (64) has only 6 components, which are equations of $w_{IJ}^a$ with only spatial $I$ and $J$. To solve (64) and (66), we express $D_a$ in terms of the unique, torsion-free generalized derivative operator $\nabla_a$ compatible with $E_I^a$, and $c_{aI}^J$ defined by

$$D_a H_I^b = \nabla_a H_I^b + c_{aI}^J H_J^b.$$

This is possible from the metric compatibility assumption. (64) and (66) are 9 independent homogeneous equations of $c_{aI}^J$ with spatial $I$ and $J$, so it is zero. Thus $w_{IJ}^a$ with spatial $I, J$ is the spin connection which is completely determined by $\tilde{E}_I^a$. Because the boost part of $w_{IJ}^a$ is free, we can write as

$$w_{IJ}^a = \Gamma_{IJ}^a + 2K_{a}^{[I} n_{J]},$$

where $\Gamma_{IJ}^a$ is the spin connection on $\Sigma$ and $K_{IJ}^a n_I = 0$. Because $\tilde{E}_I^a n^I = 0$ also, we will use 3-dimensional internal index $i$ and write these variables as $(K_i^a, \tilde{E}_I^a)$. Thus, after eliminating the 2nd class constraints and fixing $n_I$, the phase space of the Palatini theory is the pair $(K_i^a, \tilde{E}_I^a)$ and the only non-vanishing Poisson bracket is

$$\{ K_i^a(x), \tilde{E}_j^b(y) \} = \delta_i^b \delta_j^a \delta^3(x, y).$$
Starting from 16 components of $e^a_i$, 40 of $A_{a\ell}^c$ and 24 $w_a^{IJ}$, we are left with 18 degrees of freedom by 40 of $D_ag_{bc} = 0$, 3 of $\partial_a n_I = 0$, 9 of $\Gamma_{aIJ}^I$ and 10 non-dynamical $N, N^a, (w \cdot t)^{IJ}$. With the 7 first class constraints, we have 2 degrees of freedom [9].

Finally let’s write down the 7 first class constraints with this pair. It is straightforward if we write down $F_{ab}^{IJ}$ using (68):

$$H_a = -\frac{1}{2}\sqrt{q}\mathcal{R} - \frac{1}{\sqrt{q}}\tilde{E}_a^i\tilde{E}_b^jK_a^iK_b^j \approx 0,$$

$$V_a = -2\tilde{E}_a^bD_aK_b^i \approx 0,$$

$$G_{ij} = \tilde{E}_a^iK_{a[j]} \approx 0,$$

where $\mathcal{R}$ denotes the scalar curvature of $D_a$ which is the unique torsion-free derivative operator compatible with $E^a_i$. We will call (70), (71), and (72) the scalar, vector, and Gauss constraints. If $4D_ae^b_I = 0$, $-K_{ab}$ is an extrinsic curvature:

$$E^a_bK_{ai} = e^b_IE_a^I = -e^b_\ell q^c_\ell D_c n_I = -q^c_\ell D_c n_b \text{ if } 4D_ae^b_I = 0.$$

Because $K_{ab} = K_{ba}$, $G_{ij} = 0$ is automatically satisfied. In this case, (70) and (71) become the scalar and vector constraints of the standard Einstein-Hilbert action. However $K_{aI}^i$ is not the extrinsic curvature because we do not assume $4D_ae^b_I = 0$. We will see that the Hamiltonian formulation of the 3+1 Palatini theory in this approach is not the metric description of General Relativity.

Suppose we start with the metric compatibility, the torsion zero and the 3-dimensional triad compatibility conditions with fixing $n_I$. Then there is no 2nd class constraint. This method can be applied to the Holst action and we obtain the phase space variables and the constraints of Loop Quantum Gravity, which are originally derived by the canonical transformation from $(K_{aI}^i, \tilde{E}_a^I)$ [11].

So far we have solved the second class constraints assuming the metric compatibility and the torsion zero with fixing $n_I$. The other approach is to assume the tetrad compatibility condition. Here more second class constraints come from (36), which are solved by the torsion zero on $\Sigma$. We can solve (64) with a more covariant way directly from our assumption $4D_ae^b_I = 0$ with some care because

$$D_aE^b_I = q^c_\ell q^b_d D_cE^d_I = \begin{cases} q^c_\ell q^b_d e^c_d D_cq^d_e \text{ with } 4D_ae^b_I = 0 \\ q^c_\ell q^b_d D_c n^d n_I \text{ with } 4D_a g_{bc} = 0 \end{cases} = K_{aI}^b n_I,$$

which is not zero, and $\tilde{K}_{ab}$ is the extrinsic curvature. Therefore we need to use

$$q^b_J D_aE^b_I = 0,$$

where $q^b_J = q^b_aE^a_jE^b_J$. It is straightforward to check that (75) solves (64):

$$\chi^{ab} = 4\epsilon^{IJKL}\tilde{E}_I^a\tilde{E}_K^b\tilde{E}_L^cD_cn_J + (a \leftrightarrow b) = 0.$$
Furthermore only \( q^I_K q^J_L G_{I,J} \) survives:

\[
n^I q^J_K G_{I,J} = 0. \tag{77}
\]

As we mentioned, \( \phi^{ab} = 0 \) automatically by our construction.

Now we have 7 first class constraints. In the same way as the first approach, we fix \( n_I \) by \( \partial_a n_I = 0 \). With this, \( \tilde{E}^I_{ij} \) has 9 degrees of freedom from \( \tilde{E}^a_i \). \( w^a_{ij} \) also has 9 degrees of freedom with spatial \( I, J \) becoming the spin connection because (75) becomes the triad compatibility condition on \( \Sigma \). Therefore the phase space and the constraints are the same with those of the first approach. Because \( K_{ab} \) is the extrinsic curvature from (73), this approach is the metric description of General Relativity.

To see what is going on more clearly, we parameterize the foliation \( \{ \Sigma_t \} \) by a global time function \( t \) which is possible if \( M \) is globally hyperbolic. We also pick up a coordinate \( \{ x^\mu \} \) on \( \Sigma \). Let \( t^a \) in (51) satisfy \( t^a \nabla_a t = 1 \) and \( t^a \nabla_a x^\mu = 0 \). Let \( N^a \) satisfy \( N^a \nabla_a t = 0 \) and \( N^a \nabla_a x^\mu = N^\mu \). In this coordinate, \( t^a = (1, 0)^a, N^a = (0, N^\mu) \) and \( n^a = \frac{1}{N}, -N^\mu/N \). From (1), \( e^a_0 = (e^t_0, e^\mu_0) \) and \( e^a_i = (e^t_i, e^\mu_i) \) are orthonormal vectors. If we choose \( e^a_0 = n^a \), then \( e^a_i = (0, E^a_i) \) by \( \partial_a n_I = 0 \). To make \( q^a_5 D_{c} n_b \) symmetric with \( (a,b) \), we need \( q^a_5 q^b_5 T_{cd} c = 0 \). We can easily see that this comes from the Gauss constraint. Thus by solving the Gauss constraint, this approach becomes the metric description of General Relativity.

Finally let’s come back to the first approach and write down the Hamiltonian equations of motion:

\[
\dot{E}^a_i = \{ \tilde{E}^a_i, H \} = N(\tilde{E}^a_i K - \tilde{E}^b_i K^a_0) - \tilde{E}^a_i D_b N^b + \tilde{E}^b_i D_b N^a + (t \cdot w)_i^j \tilde{E}^a_j, \tag{78}
\]

\[
\dot{K}^a_i = \{ K^a_i, H \} = -N(R^a_i + K K^a_i - K^b_i K^a_b) + N^b_i D_a K^a_b - N^b_i D_b K^a_i + (t \cdot w)_i^j K_{aj}, \tag{79}
\]

where \( R_{ab} \) is the Ricci tensor on \( \Sigma \) and we impose the triad compatibility condition after functional derivatives.

5 Conclusion

In the Lagrangian formulation of the Palatini action, we found that there are two approaches to describe General Relativity. One is to assume the metric compatibility and the torsion zero conditions and the other is to assume the tetrad compatibility condition. In the Hamiltonian formulation, we found that only the second approach describes General Relativity. This is the metric description which is very hard to quantize.

In the first approach of the metric compatibility and the torsion zero assumptions, the time evolution of the tetrad is different from that of General Relativity. This is a very unexpected result. We do not know whether this has any meaning classical mechanically because General Relativity is an established theory with experiments. We will see what it means to quantized General Relativity with this modification.
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