An alternative way to observe percolation phase transitions

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Abstract

We present an alternative way to observe phase transitions of percolating systems using information theory. As expected, our approach shows the difference in character of classical and explosive percolation transitions: smooth transition for the classical case and abrupt transition for the explosive case. It enables to assign at which point the birth of the giant (percolating) cluster occurs and, as an unexpected result, we show that this birth does not match the transition threshold for the explosive percolation.

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From a more general context, percolation[1] is the study of formation and distribution of clusters in networks[2] and these networks are used to model since porous media to social interactions. There is a phase transition associated to percolation and it is determinate by the existence of a giant cluster with size comparable to the size of the entire network. Focusing attention on the network model developed by Erdös and Rényi[3] (random network), one of its main features is the smoothly continuous phase transition, as shown in Fig. 1. However, with the advent of explosive percolation paradigm[4], the continuous nature of the percolation phase transition came to be questioned.

The main characteristic of the mechanisms for the placement of edges in random networks exhibiting an abrupt percolation transition is that a choice is done before the placement of each edge. This choice process has the aim of delaying the onset of the giant cluster and, as a consequence, it grows more abruptly after its birth, allowing a abrupt phase transition be observed. The first proposed mechanism was called product rule (PR) and it works as follows. Two node pairs are selected and an edge is placed between the pair whose product of the number of nodes in the connecting clusters is the smallest. The explosive percolation caused by the PR is shown in Fig. 1.

Disregarding the discussion about if the nature of such transition is continuous[5–9], we are proposing a new alternative way to observe percolation phase transitions able to determine the transition threshold even without the specific study of the order parameter (relative size of the network’s largest cluster for classical percolation on random networks[1, 8]). Although others mechanisms towards abrupt percolation transition have been proposed[10, 11], in this article we focus only on: (i) the product rule and (ii) the rule developed by da Costa et al.[5, 8] (dCR). The explosive percolations resulting from these two mechanisms and the classical percolation were simulated on random networks. So, we analyzed them through the alternative approach described below.

Our alternative way to observe the percolation phase transition is based on information theory. We use the Shannon entropy[12], that is defined as

$$H = -K \sum_i (p_i \ln p_i),$$  \hspace{1cm} (1)

in which the set \( \{p_i\} \) is a probability distribution. It is a function whose output, that we call uncertainty, is a type of measure of information contained in the probability distribution. The Shannon entropy is such that the more flat is the probability distribution, that is, the
more close to the equiprobability, the more high is the uncertainty. Thus, in the percolation context, we expect the following behaviour for the uncertainty. Initially, when there are only isolated nodes without any edges (clusters with size one), the uncertainty is zero. So, when edges are added to the network, it starts to increase because of the emergence of new clusters with different sizes. But, there is a moment when the giant cluster will be born from the agglutination of the clusters in the network. At this moment, the network is populated by several clusters with several sizes and the cluster size distribution has its flatter form. Thereafter, it’s expected that the uncertainty decreases continuously because the absorption of small clusters by the giant cluster. This is basically the dynamics observed for the classical percolation, while that resulting from the explosive rules has more ingredients.

Initially, we analyze only the classical percolation and the explosive percolation via PR. Plot A in Fig. 2 shows graphically what we report above. To do it, we use Eq. 1 with $K = 1$ and the uncertainty was repeatedly calculated after a fixed number of edges was added to the networks. The result of the uncertainty analysis for classical percolation follows the expected behaviour and the maximum uncertainty matches the transition threshold. In turn, the result for explosive percolation shows that the maximum point does not correspond to the transition threshold, so that the transition happens only after the addition of more edges. Thus, in the explosive case the onset of the giant cluster and the start of percolation transition do not happen at the same time. Between these two points the uncertainty declines fast, but the size of the giant cluster don’t change significantly, as we saw in Fig. 1. After that, a sequence of straight plateaus with slow decline and small regions with fast decline can be observed. This pattern is a manifestation of the finite size of the system. Nevertheless, we can justify it. The straight plateaus corresponds to periods in which the giant cluster stay almost without growth, with small clusters slowly joining together to form clusters slightly bigger, and so these clusters are swallowed by the giant cluster at the fast declines. It matches the step-shaped pattern observed after the transition in Fig. 1. The explanation for the existence of these plateaus is that, after the onset of the giant cluster, more and more edges are added internally to the clusters, mainly internally to the giant, instead of connecting pairs of different clusters, which means that the uncertainty relative to clusters size tends to stay unchanged. It’s noteworthy that others rules generating explosive transitions don’t exhibit peaks after the transition threshold (see Fig. 3).

From the analysis of the Fig. 2 it’s possible realize that the PR has three consequences
over the giant cluster: (i) delaying its onset in comparison to the classical case, (ii) suppressing its growth temporarily after its onset, and (iii) causing its size increases abruptly (explodes) when it guls others clusters in the network. It’s worth reinforcing that only the examination of the order parameter is insufficient to observe the separation between the moments when the giant cluster is born and the transition threshold for the explosive case.

The data exposed on plot A alone is not enough to explain the positioning of explosive percolation transition threshold based on the uncertainty analysis. To solve this, it’s necessary to calculate the uncertainty first derivative, shown in plot B (Fig. 2). Such as expected, the curve relative to the classical percolation is smooth. Yet, the striking feature of this figure is the sharpened curve for the explosive case. It’s possible to see the explosive behaviour being reproduced. A divergence point occurs around the transition threshold, so that this discontinuity can be used to determine the threshold value. The pattern of straight plateaus followed by fast decline regions observed in the uncertainty data can be noticed in its derivative as a sequence of small peaks after the transition point. All these small peaks are finite and, indeed, they tend to disappear when we increase the network size towards the thermodynamic limit, because they are consequences of the finite size of the system. However, the peak which marks the transition threshold diverges in this limit.

The application of our method to the explosive percolation caused by the dCR is shown in Fig. 3. The dCR imposes that edges are added as follows: (i) choose two sets of \( m \) nodes; (ii) for each set, select the node in the smallest cluster; and (iii) put the edge connecting these two nodes. While the case \( m = 1 \) recovers the classical percolation, if \( m > 1 \) we obtain explosive percolations. As in Fig. 2 the giant cluster onset occurs before the transition threshold pointed by the divergence on the uncertainty derivative for the explosive cases. We can see that the higher is \( m \), the more retarded are these points in comparison to the classical percolation. Furthermore, these points become closer when \( m \) is increased, such that they should coincide when the value of \( m \) is too high. However, if \( m \) is comparable to the size of the network, the edges will be added deterministically. They will always connect the two smallest clusters in the network at every moment. In this case, the cluster size distribution is fully known and calculate its uncertainty will not give more insights about the network structure. Therefore, our approach only applies when there is an amount of randomness involved.

According to what was discussed, although the analysis of the order parameter is the
canonical way to observe phase transitions of percolating systems, some aspects of these transitions are not apparent from this approach. In this article we presented an alternative way to observe these transitions using the information theory framework to analyze the cluster size probability distribution. In this new way, we propose a method that enables to identify exactly the giant cluster onset. An immediate and unexpected result of our approach was to note that the start of the explosive percolation transition does not occur simultaneously to the giant cluster onset. This will have consequences for future researches and may even add arguments to the discussion about the continuous nature of the explosive percolation. In further works, we expect use our approach to analyze others particularities of systems in which the cluster size distribution is important and reveal new features of the dynamics of these systems.

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FIG. 1. (Color online) Conventional graphic form of observing percolation phase transitions: a smooth growing curve for the classical percolation and a step-shaped curve for the explosive percolation via product rule (PR). The data show the fraction of nodes in the networks’ largest cluster against the density of edges in networks, i.e. the ratio between the number of edges and the number of nodes. The position of the dashed and dotted lines are: rightmost dashed line (black, at 0.5) matches the onset of the giant cluster and the start of the transition on networks undergoing the classical percolation; the central dotted line (red, around 0.81) marks the onset of the giant cluster on networks undergoing the explosive percolation; the leftmost dashed line (red, around 0.89), in turn, points to the start of the explosive transition. The reason for the existence of the dotted line marking the onset of the giant cluster before the transition threshold for explosive percolation will be clarified in Fig. 2. These results, and all others hereafter, are based on simulation of 500 percolation transitions of each type over networks with $2 \times 10^5$ nodes.
FIG. 2. (Color online) Proposed alternative way to observe the percolation phase transitions using the information theory framework. These curves show how the uncertainty (Eq. \[ \text{I} \] with $K = 1$) relative to the cluster size probability distribution (plot A) and its numerical first derivative (plot B) behave when the density of edges in the networks increases. The red data are the result for networks whose edges are added through product rule (PR), while the black data are the result for networks undergoing the classical percolation. The dashed lines mark where the percolation transition occurs: accurately at 0.5 for classical percolation and around 0.89 for explosive percolation. We observe a maximum value for the uncertainty and this point define the moment at which the number of cluster with different sizes reaches its maximum and begins to decrease. We interpret this point as the moment in which the giant cluster emerges. Exactly at this point, the classical percolation transition begins. Looking at explosive percolation data, the maximum point occurs after that for classical percolation and, more important, the transition don’t happen at the maximum point but afterward. So, we realize that the giant cluster is born at around 0.81 (red dotted line), but indeed it arises just at 0.89, what is a visible manifestation of the fact that the PR delays the the onset of the giant cluster and, furthermore, suppresses its growth temporarily after that. In plot B, we note that the explosive percolation’s abrupt nature is present through the divergence around the transition threshold, while the classical percolation transition retains its continuous character. The small peaks that appear after the transition are due to the finite size of the system.
FIG. 3. (Color online) Using the alternative way to observe the explosive percolation through dCR \[5, 8\]. Like Fig. 2, these curves show the behaviour of the uncertainty relative to the cluster size probability distribution (plot A) and its numerical first derivative (plot B) when we increase the density of edges. Ordering from the curve whose maximum is more to the left: \(m = 1\) (black), \(m = 2\) (red) and \(m = 4\) (blue). The higher is \(m\), the more restrictive is the edges addition. As we can see, the giant cluster onset and the percolation threshold are retarded by higher \(m\), so the abrupt character becomes more strong. The separation between these two moments is apparent, however they become closer for higher \(m\). While the giant cluster onset occurs around 0.86 and the percolation threshold around 0.92 for \(m = 2\), these same points are around 0.98 and 0.99 for \(m = 4\), respectively. In plot B, we restricted the data showed for \(m = 4\), otherwise the data for \(m = 2\) would not be visible properly. In these plots, we hide the lines marking the birth of the giant clusters and the transitions threshold.