ROKHLIN CONJECTURE AND TOPOLOGY
OF QUOTIENTS OF COMPLEX
SURFACES BY COMPLEX CONJUGATION

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Abstract. Quotients $Y = X/\text{conj}$ of complex surfaces by anti-holomorphic
involutions $\text{conj}: X \to X$ tend to be completely decomposable when they are
simply connected, i.e., split into connected sums, $n\mathbb{C}P^2 \# m\overline{\mathbb{C}P}^2$, if $w_2(Y) \neq 0$,
or into $n(S^2 \times S^2)$ if $w_2(Y) = 0$. If $X$ is a double branched covering over
$\mathbb{C}P^2$, this phenomenon is related to unknottedness of Arnold surfaces in $S^4 = \mathbb{C}P^2/\text{conj}$, which was conjectured by V.Rokhlin. The paper contains proof of
Rokhlin Conjecture and of decomposability of quotients for plenty of double
planes and in certain other cases.

This results give, in particular, an elementary proof of Donaldson’s result
on decomposability of $Y$ for $K3$ surfaces.

§1. Introduction

By a Real variety (e.g., Real surface, Real curve) we mean a pair $(X, \text{conj})$,
where $X$ is a complex variety and $\text{conj}: X \to X$ an anti-holomorphic involution called
the real structure or the complex conjugation. Given an algebraic
variety over $\mathbb{R}$ we consider its complexification with the natural complex con-
jugation (the Galois transformation) as the corresponding Real variety.

This paper is devoted to studying the topology of quotients $Y = X/\text{conj}$
for nonsingular Real surfaces $(X, \text{conj})$. The quotient $Y$ inherits from $X$
an orientation and a smooth structure, which makes the projection $q: X \to Y$
an orientation preserving and smooth 2-fold covering branched along the real
part $X_\mathbb{R}$ of $X$, the fixed point set of $\text{conj}$, which is identified, here and in
what follows, with its image $q(X_\mathbb{R})$.

It turns out that $Y$ tends to split into a connected sum of elementary pieces,
$\mathbb{C}P^2$, $\overline{\mathbb{C}P}^2$ and $S^2 \times S^2$, when it is simply connected; in particular, when $X$
is simply connected and $X_\mathbb{R} \neq \emptyset$. We call this property of complete decom-
POSABILITY for quotients CDQ-property and call $(X, \text{conj})$ a CDQ-surface if it is
satisfied. CDQ-property provides a link between the differential topology
of complex surfaces and differential topology of knotted surfaces in the de-
composable 4-manifolds, $X_\mathbb{R} \subset Y$, which was used in [FKV] in construction
of exotic knottings. A plenty of examples may suggest that CDQ-property is universal for Real surfaces, i.e. holds whenever $Y$ is simply connected; we will refer to it as to CDQ-conjecture.

The first and rather famous example of CDQ-surface is $\mathbb{CP}^2$ with $\mathbb{CP}^2/\text{conj} \cong S^4$ ($\cong$ is read “diffeomorphic”), which is known as Massey–Kuiper theorem. This fact has a long history as a folklore (see [A1, A2]), but, its proof, perhaps, was not published before [Ma, K]. The next examples are rational surfaces, $X \cong \mathbb{CP}^2 \# n\overline{\mathbb{CP}^2}$, obtained by blow-ups at real points of $\mathbb{CP}^2$. The Massey–Kuiper theorem prompts that the quotient will be the same, $Y \cong S^4 \# nS^4 \cong S^4$. CDQ-property for one more class of Real rational surfaces was set up in [Le].

Logarithmic transforms of an elliptic surface can also be made in the real category (real fibers on a Real surface) and does not change the quotient as well [FKV]. For example, by logarithmic transforms on $E(1) \cong \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$ we can get CDQ-Dolgachev surfaces, $D_{p,q}$, and, moreover, Real elliptic surfaces with arbitrary number of multiple fibers and arbitrary multiplicities with the quotient $Y$ being still diffeomorphic to $S^4$. Further, S.Donaldson noticed in [D] that CDQ-property holds for K3 surfaces and S. Akbulut [Ak] gave examples of CDQ-surfaces of general type by proving CDQ-property for a series of Real double planes.

We prove CDQ-property for several new families of Real surfaces, which include a plenty of double planes and, more generally, certain doubles of CDQ-surfaces. Some related problems of topology of algebraic curves and the topology of $X_R$ in $Y$ are also studied.

In §2 we introduce Arnold surfaces for Real algebraic curves and discuss the Rokhlin Conjecture, which is in fact a refined version of the CDQ-conjecture in the case of double planes. In §3 we prove the Rokhlin Conjecture for curves which can be obtained by perturbation of lines in generic position. In §4 we discuss some phenomena related to deformations of Real curves. The effect of such deformations for $Y$ is well-known [Le, W]. We give a more subtle description, which takes into account the topology of $X_R$ in $Y$ and the topology of Arnold surfaces in $S^4$. We show that vanishing of a torus component $T \subset X_R$ leads to a logarithmic transform of $Y$ along $T$. In §5 we prove a generalized version of the Rokhlin Conjecture for a certain class of “double” curves in CDQ-surfaces. As a corollary we get unknottedness for the images $q(A) \subset Y$ of imaginary curves $A \subset X$. In §6 a special case of Generalized Rokhlin Conjecture and Decomposability Problem is considered for fibered Real surfaces. In the case of elliptic surfaces, $P = E(n)_{m_1,\ldots,m_k}$, our results imply that the exotic phenomena which were found in [FKV] for the knottings $F_R \subset Q = (P/\text{conj})$, cannot straightforwardly produce exotic differential structures in the quotients of the double coverings over $P$. 
In §7 we discuss some applications for Real plane curves of degree $\leq 6$ and give in particular an elementary proof of CDQ-property for Real K3 surfaces, alternative to the Donaldson’s.

It may be worth to mention some other recent related results. The author proved CDQ-property for all Real rational and Enriques surfaces [F3] and for certain complete intersections of arbitrary multi-degree, namely, for the ones which can be constructed by method of a small perturbation [F2]. Note also that vanishing of Donaldson (Seiberg–Witten) invariants for $Y$, is a weaker version of CDQ-property, which is often easier to prove. For example, vanishing in the case when $X \cap Y$ contains an orientable component of genus $\geq 2$ is a trivial corollary of the adjunction formula (see §4). Another example: if $X = \emptyset$ then $Y$ is undecomposable, however, its Seiberg–Witten invariants vanish by the recent result of S.Wang.

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### §2. Rokhlin Conjecture

Given a nonsingular curve $A \subset \mathbb{C}P^2$, the zero set of a degree $2k$ real form $f$, consider the surfaces $W^\pm = W^\pm(A) = \{x \in \mathbb{R}P^2 \mid \pm f(x) \geq 0\}$. We have obviously $W^+ \cup W^- = \mathbb{R}P^2$, $W^+ \cap W^- = A \cap \mathbb{R}P^2$. One of these surfaces is, obviously, orientable, the other is not. Being free to change the sign of $f$ we can assume that the orientable surface is $W^+$. By taking unions, $\mathfrak{A}^\pm(A) = (A/\text{conj}) \cup W^\pm$, we get 2 closed surfaces in $S^4 = \mathbb{C}P^2/\text{conj}$ known as Arnold surfaces in Topology of Real Algebraic Curves. They are not smooth along $A \cap \mathbb{R}P^2$ but can be smoothed, since $A$ intersects $\mathbb{R}P^2$ normally. In what follows we will leave the notation $\mathfrak{A}^\pm$ for the smoothed Arnold surfaces.

#### 2.1. Rokhlin Conjecture. Arnold surfaces are standard for nonsingular Real curves $A$ of even degree with $A \cap \mathbb{R}P^2 \neq \emptyset$.

By a standard surface in $S^4$ we mean a connected sum of several copies of standard tori, or standard $\mathbb{R}P^2$. A standard torus, which can be characterized as a boundary of a solid torus embedded in $S^4$, is unique up to isotopy. $\mathbb{R}P^2$ can be embedded in $S^2$ in two (up to isotopy) standard ways distinguished by the normal numbers, which can be $\pm 2$. We will denote a standard torus by $(S^2, T^2)$ and standard $\mathbb{R}P^2$ with the normal numbers $-2, +2$ by $(S^4, \mathbb{R}P^2)$ and $(S^4, \mathbb{R}P^2)$ respectively.

Note that surfaces which bound solid handlebodies embedded in $S^4$ are standard. (By a solid handlebody we mean a 3-manifold homeomorphic to $F \times [0, 1]$, where $F$ is a connected compact surface with $\partial F \neq \emptyset$.)

Note also that $(\mathbb{C}P^2, \mathbb{R}P^2)/\text{conj} = (S^4, \mathbb{R}P^2)$. 


Remarks.

(1) When $A_R = \emptyset$ the Rokhlin Conjecture still may concern $A^+ = A/\text{conj}$. On the other hand, $A^-$ has two components, $\mathbb{RP}^2$ and $A^+$, which are linked, since the double covering over $S^4$ branched along $A^-$ has fundamental group $\mathbb{Z}/2$ (it would be $\mathbb{Z}$ for an unlink).

(2) There is a weaker version of the Rokhlin Conjecture studied in [F1]:

Complements of the Arnold surfaces for nonsingular Real curves $A$ of even degree, with $A_R \neq \emptyset$, have abelian fundamental groups.

We define in addition Arnold surfaces in $\mathbb{CP}^2$ when $A - A_R$ splits in two connected components, which happens if the fundamental class $[A_R] \in H_1(A; \mathbb{Z}/2)$ vanishes. If this is the case $A$ is said to be of type 1, otherwise it is of type 2. If $A$ is of type 1 then we denote by $A_k$, $k = 1, 2$ the components of $A - A_R$, put $A_k^\pm = W^\pm \cup A_k$ and call $A_k^\pm$ the Arnold surfaces in $\mathbb{CP}^2$. The Rokhlin conjecture may treat them if we define a standard surface in $\mathbb{CP}^2$ as a surface which is isotopic to a connected sum of a nonsingular algebraic curve in $\mathbb{CP}^2$ with a standard surface in $S^4$.

The Rokhlin conjecture for $A_k^\pm(A)$ is stronger then CDQ-conjecture for double planes $X$ branched along curves $A \subset \mathbb{CP}^2$, as it follows from the following construction well known after [A1].

Define $X$ by the equation $f(z_0 : z_1 : z_2) = w^2$ in the weighted quasi-projective 3-dimensional space with coordinates $z_0, z_1, z_2$ of weight 1 and $w$ of weight $k$. It has 2 real structures defined by the covering conjugations: $\text{conj}^\pm(z_0 : z_1 : z_2 : w) = (\overline{z_0} : \overline{z_1} : \overline{z_2} : \pm w)$.

The real part $X_R^\pm$ of $(X, \text{conj}^\pm)$ is mapped onto $W^\pm$ by the projection $p: X \to \mathbb{CP}^2$ as a double covering with sheets glued together along $\partial W^\pm = A_R$. For even $k$ this covering is trivial, hence, $X_R^\pm$ is the usual double of $W^\pm$. For odd $k$ it is the orientation covering, so, $X_R^\pm$ is orientable.

Further, $\text{conj}^+$ and $\text{conj}^-$ commute, give in product the covering transformation and thus define an action of $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ on $X$. Factorization by $\mathbb{Z}/2$ in different order yields the following commuting diagram:

$$
\begin{array}{ccc}
X & \xrightarrow{q^\pm} & Y^\pm = X/\text{conj}^\pm \\
p & & \downarrow{p^\pm} \\
\mathbb{CP}^2 & \xrightarrow{q} & S^4 = \mathbb{CP}^2/\text{conj},
\end{array}
$$

where $q, q^\pm$ are the quotient maps and $p^\pm$ is the double covering branched along $A^\pm$ (cf. [A1]).

This description implies
2.2. Theorem. If \( \mathfrak{A}^\pm \) for a curve \( A \) is standard, then \( (X, \text{conj}^\pm) \) is CDQ-surface.

Proof. Use the above diagram and note that the double branched coverings over \((S^4, T^2), (S^4, \mathbb{R}P^2)\) and \((S^4, \mathbb{C}P^2)\) are \( S^2 \times S^2, \mathbb{C}P^2 \) and \( \mathbb{C}P^2 \) respectively, and that the double covering over \((Y_1 \# Y_2, F_1 \# F_2)\) is the connected sum of the double coverings over \((Y_1, F_1)\) and \((Y_2, F_2)\).

\[ \square \]

§3. Rokhlin Conjecture for L-curves

Recall that the parameter space \( C_m \) of Real plane algebraic curves of degree \( m \) is a real projective space of dimension \( \binom{m+2}{2} - 1 \) and singular curves constitute the discriminant hypersurface, \( \Delta_m \subset C_m \). A deformation \( A_t \) of \( A_\alpha \in C_m \) is a path \([\alpha, \beta] \to C_m, t \mapsto A_t\). A perturbation of \( A_\alpha \in \Delta_m \) is a deformation \( A_t \) such that \( A_t \in C_m - \Delta_m, \forall t \in (\alpha, \beta) \).

We call \( A_\beta \) an L-curve if it can be obtained by a perturbation of a curve \( A_\alpha \) which splits into a union of \( m \) Real lines in general position (three lines should not have a common point).

3.1. Theorem. The Rokhlin Conjecture holds for \( \mathfrak{A}^\pm(A) \subset S^4 \) for L-curves \( A \).

3.2. Corollary. Double planes \( X \to \mathbb{C}P^2 \) branched along L-curves are CDQ-surfaces.

Proof of Theorem 3.1 is based on the following criterion:

3.3. Theorem [Li]. Let \( F \subset D^4 \subset S^4 \) be a closed surface which lies in \( S^3 = \partial D^4 \), except several 2-discs standardly embedded inside 4-disc \( D^4 \). Then \( F \) is a standard surface in \( S^4 \).

Let us describe first a suitable handle decompositions of \( \mathbb{C}P^2 \) and \( \mathbb{C}P^2/\text{conj} \). Choose a point \( b_0 \in \mathbb{R}P^2 \) with a small regular conj-symmetric neighborhood \( B_0 \) around it. Consider the central projection \( r: (\mathbb{C}P^2 - B_0) \to C \), from \( b_0 \) to some Real line \( C, b_0 \notin C \). Let \( C_1, C_2 \), be the closures of the connected components of \( C - C_\mathbb{R} \) and \( B_i = r^{-1}(C_i), i = 1, 2 \). Then \( \tilde{H} = B_1 \cap B_2 \) is a solid torus, a disc bundle over \( C_\mathbb{R} \), which contains a Möbius band \( M = \mathbb{R}P^2 - B_0 \). 4-discs \( B_1, B_2 \), \( B_0 \) can be considered as 0-, 2- and 4-handles giving a decomposition of \( \mathbb{C}P^2 \), like in [Ak]. The quotient \( q(B_0) \) is again a 4-disc. \( B_1 \) and \( B_2 \) are permuted by conj, hence, \( B = q(B_1 \cup B_2) = B_1/\sim, \) where \( \sim \) identifies conjugated points in \( \tilde{H} \), is a 4-disc as well. \( H = \tilde{H}/\text{conj} \) can be thought of as the trace of an isotopy of \( M \), fixed on \( \partial M \), which pushes \( M \) outside of the interior, \( \text{Int}(B) \), to \( \partial B \).

Consider now an L-curve \( A \) obtained from \( A_0 = L_1 \cup \cdots \cup L_m \) by a perturbation. It is well known that a perturbation of a curve does not change its
isotopy type outside a neighborhood of singularities. So, $A$ can be assumed to coincide with $A_0$ in the complement of small regular conj-symmetric open discs $\tilde{P}_{ij}$ around points $p_{ij} = L_i \cap L_j$, $1 \leq i < j \leq m$. Put $P_{ij} = q(\tilde{P}_{ij})$ and $P = \bigcup_{1 \leq i < j \leq m} P_{ij}$. We can assume also that $b_0$ is chosen inside $W^\pm$ and that $B_0$ does not intersect $A_0$ and $P_{ij}$.

The idea behind the proof is to apply Theorem 3.3 to $A_{\pm}$ using the 4-disc obtained from $B$ by cutting it along $H$. The result can be naturally identified with $B_1$ (or with $B_2$). It contains $W^\pm$ in its boundary and $(A/\text{conj}) - P$ consists of $m$ standard discs $(L_i/\text{conj}) - P$. More formally, we get a 4-disc $B'$ after deleting from $B$ a small regular neighborhood $N$ of $H - M$. Then we push by an isotopy the part of $A_{\pm}$ contained inside $N$ to $\partial B'$.

It suffices to construct such an isotopy inside $P$, since $(\mathfrak{A}^\pm - P) \cap \text{Int}(N) = \emptyset$. So, we only need to study the local question, specifically, possible positions of $\mathfrak{A}^\pm$ with respect to $H$ inside $P_{ij}$. It is determined, obviously, by the position of the lines $L_i$, $L_j$, with respect to the basepoint $b_0$ and with respect to $W^\pm$. The 4 possible cases are shown on Figure 1.

**Figure 1.** Positions of $L_i, L_j$ with respect to $W^\pm$ (the shaded region) and to $b_0$

Let $W_{ij} = W^\pm \cap P_{ij}$, $A_{ij} = (A/\text{conj}) \cap P_{ij}$, $H_{ij} = H \cap P_{ij}$, and $\mathfrak{A}_{ij} = \mathfrak{A}^\pm \cap P_{ij}$. $\partial A_{ij}$ splits into 2 pairs of arcs $l_1 = A_{ij} \cap \partial P_{ij}$ and $l_2 = A_{ij} \cap W_{ij}$. By standard arguments of [M], $A_{ij}$ can be pushed out to $\partial P_{ij}$ by an isotopy which is transversal to $H$, keeping fixed on $l_1$ and moves $l_2$ in $\mathbb{R}P^2$ so that $A_{ij}$ is preserved not to intersect normally with the latter. It gives an isotopy of $\mathfrak{A}^\pm$ which is fixed on $(A/\text{conj}) - P$, preserves $W^\pm - P$ invariant and pushes $A_{ij}$ to $\partial P_{ij}$. In the cases (a) and (b) this isotopy pushes $\mathfrak{A}^\pm$ outside of the interior of $P_{ij}$. In the case (d), the part $W_{ij}$ is still inside of $P_{ij}$ but it can be pushed out to $\partial P_{ij}$ along $H_{ij}$ by an isotopy without obstructions (see Figure 2).

In the case (c) we cannot do the same, because $A_{ij}$ is intersecting $H$ not only at the part of its boundary $l_2$ but also along the “middle line” $l_3$ (see...
Figure 2. Mutual position of $H$ (the disc) and $A_{ij}$ (the band) after pushing $A_{ij}$ off $\text{Int}(P_{ij})$. $A_{ij}$ is shaded

Figure 3. $l_3 \subset A_{ij} \cap H$ formed by intersection of $A$ with the pencil of Real lines passing through $b_0$. The isotopy leaves $W'_{ij}$ (which is shaded on the right) inside $P_{ij}$

So, after being pushed off $\text{Int}(P_{ij})$ along $H$, $A_{ij}$ intersects $W_{ij}$ as it is shown on Figure 2. But, we can avoid this and push to $\partial P_{ij}$ only a neighborhood of $l_1$ leaving the rest, the neighborhood $W'_{ij}$ of $l_3$, inside $P_{ij}$.

Finally, we add to $B'$ the discs $P_{ij}$ for which $A_{ij}$ lie in the positions (a) and (c) and excise from $B'$ the other ones. We get, clearly, a new 4-disc, $B''$ and can apply now Theorem 3.3 to $A^\pm$. To see how $A^\pm$ lies in $B''$ after this isotopy note that in the cases (a) and (c) $\partial B'' \cap \partial P_{ij} = \text{Cl}(N \cap \partial P_{ij})$ is a proper neighborhood of the 2-disc $H \cap \partial P_{ij}$. It is clear from Figure 2 that we can vary size of this neighborhood by varying size of $N$ so that it will contain $A_{ij}$ in the cases (a), and both $A_{ij}$ and $W_{ij} - W'_{ij}$ in the case (c). In the cases (b) and (d) $\partial B'' \cap \partial P_{ij}$ is the complement of the neighborhood $N \cap \partial P_{ij}$ in $\partial P_{ij}$. It will contain $A_{ij}$ in the case (b) if the neighborhood $N$ is small enough. It will also contain $A_{ij}$ in the case (d) after we push $W_{ij}$ to $\partial N \cap \partial P_{ij}$ by an isotopy.

After the above isotopy $A^\pm$ lies in the boundary of $B''$, except $m$ discs
\((A/\text{conj}) \cap \text{Int}(B'')\) left from \(L_i\) and discs \(W'_{ij}\), which lie all inside \(B''\) and are obviously unknotted. □

**Remarks.**

(1) \(L\)-curves form a pretty large collection of Real curves. They include, in particular, “maximal nest” curves, that is degree \(2k\) curves which real parts consist of \(k\) ovals linearly ordered by inclusion (see the example on Figure 8). Thus Theorem 3.1 generalizes results of [Ak], where maximal nests were under consideration.

(2) One can easily notice the upper bound, \(\frac{1}{3}m(m - 1)\), for the number of real components of \(L\)-curves of degree \(m \geq 3\). For \(m \geq 5\) it is less than the Harnak bound, \(\frac{1}{2}(m - 1)(m - 2) + 1\), effective for arbitrary Real curves of degree \(m\). This shows one of the restrictions to real schemes of \(L\)-curves. More examples and discussions of \(L\)-curves can be found in [F1], [FKV] (see also §7 below).

§4. **Deformations of real structures**

In this section we describe how the topology of \(A^{\pm}\), \(Y\) and \(X_R\) is changing along with the deformation of the branched locus \(A\).

Any pair of nonsingular plane Real curves, \(A_\alpha, A_\beta \in C_m\), can be connected by a deformation, say \(A_t, t \in [\alpha, \beta]\). If \(A_t \notin \Delta_m, \forall t \in [\alpha, \beta]\), the deformation is called rigid isotopy. In this case the conj-equivariant topological type of \((\mathbb{CP}^2, A_t)\) and, hence, the topological type of \(A^{\pm}\), \(Y\), \(X_R\) remains unchanged. If \(A_t\) is a generic deformation then it crosses \(\Delta_m\) transversally at several nonsingular points. Such a nonsingular point, say, \(A_0 \in \Delta_m\), represents a curve with one real node, \(P \in A_0\), which can be either isolated in \(\mathbb{RP}^2\) or cross-like. The topological effect of transversal crossing of \(\Delta_m\) for \(A^{\pm}\), \(Y\), \(X_R\), is local and depends, obviously, only on the type of the node at \(P\) and on the local position of \(P\) with respect to \(W^{\pm}\). Figure 4 shows 6 possible topologically different elementary modifications of \(W^{\pm}\), which will be called moves of \(W^{\pm}\) and denoted by \(M_i\) or \(M_i^{-1}, i = 0, 1, 2\).

Consider a variation \(A_t, t \in [-1, 1]\) which crosses \(\Delta_m\) transversally at \(t = 0\). Denote by \(X_t\) the double plane branched along \(A_t\) and by \(W_t\) one of the domains \(W^\pm(A_t)\) continuously following the curves \(A_t\) (we omit the superscript sign over \(W_t\), since the move \(M_i\) may change it). Denote by \(A_t, \text{conj}_t, X_{R,t}, Y_t\) the objects which follow continuously \(W_t\), so that if \(W_t = W^\pm(A_t)\), then \(A_t = A^{\pm}(A_t)\), but \(\text{conj}_t = \text{conj}^\mp, X_{R,t} = X^{\mp}_{R,t}\) and \(Y_t = Y^\mp\). Assume that \(M_i\) is the move which describes the modification of \(W_t\) as \(t\) increases.

4.1. **Theorem.** If \(M = M_1\) or \(M_0^{-1}\) then

(1) \(A_0\) is smoothable at \(P\), hence, \(Y_0\) is smoothable as well;
Figure 4. Elementary moves. $W$ is the shaded region

(2) $(S^4, \mathcal{A}_0) \cong (S^4, \mathcal{A}_{-1})$, hence, $Y_0 \cong Y_{-1}$;
(3) $(S^4, \mathcal{A}_1) \cong (S^4, \mathcal{A}_0) \# (S^4, \mathbb{RP}^2)$, hence, $Y_1 \cong Y_0 \# \mathbb{RP}^2$;
(4) After smoothing of $Y_0$, the real part, $X_{\mathbb{R},0} \subset Y_0$, is locally diffeomorphic at $P$ to a node, i.e., there exist an orientation preserving diffeomorphism $U \rightarrow \mathbb{C}^2$ of a sufficiently small regular neighborhood $U \subset S^4$ of $P$ which sends $X_{\mathbb{R},0} \cap U$ to the union of coordinate lines;
(5) $(Y_{-1}, X_{\mathbb{R},-1})$ is obtained from $(Y_0, X_{\mathbb{R},0})$ by an “algebraic” perturbation of the above singularity (i.e., the local effect of this deformation in the chart $U \cong \mathbb{C}^2$ is the perturbation of the node);
(6) $(Y_1, X_{\mathbb{R},1})$ is obtained from $(Y_0, X_{\mathbb{R},0})$ by blowing-up at $P$.

4.2. Theorem. Suppose that $M = M_2$. Then

(1) $\mathcal{A}_0$ is locally diffeomorphic to an algebraic curve near a simple tangent point singularity; more precisely, there exist an orientation preserving diffeomorphism $U \rightarrow \mathbb{C}^2$ of a sufficiently small regular neighborhood $U \subset S^4$ of $P$ which sends $\mathcal{A}_0 \cap U$ to the curve $z_2^2 - z_1^4 = 0$; in particular, this singularity has $(2, 4)$ torus link; $Y_0$ has a singularity at $P$ which link is the lens space $L(4, 1)$;
(2) $(S^4, \mathcal{A}_{-1})$ is obtained from $(S^4, \mathcal{A}_0)$ by removing a neighborhood of $P$ (a pair of tangent discs) and replacing it by a Seifert surface of the $(2, 4)$ torus link (see Figure 5); it implies that $Y_{-1}$ is obtained from $Y_0$ by exchanging a neighborhood of $P$ (the cone over $L(4, 1)$) for the total space of a $D^2$-bundle over $S^2$ with the normal number $-4$;
(3) $(S^4, \mathcal{A}_1)$ is obtained from $(S^4, \mathcal{A}_0)$ by replacing a neighborhood of $P$ by a disc and a Möbius band, as it is shown on Figure 5; $Y_1$ is obtained from $Y_0$ accordingly, by substituting of the cone over $L(4, 1)$ for the total space of a $D^2$-bundle over $\mathbb{RP}^2$ with the normal number $-1$;
(4) $X_{\mathbb{R},0}$ contains only an isolated point at $P$ in a neighborhood of $P$;
after the deformation this point disappears in $X_{R,1}$ and gives rise to a sphere component in $X_{R,-1}$; this sphere can be identified with the zero section of the disc bundle described in (2).

Figure 5. Bifurcation of $\mathfrak{A}_t$ after the move $M_2$. The disc component is not seen on the right figure, as it is pushed inside $D^4$.

4.3. Remarks. (1) The modification of $Y$ after move $M_2$ can be thought of as the “connected sum” with $\mathbb{C}P^2$ along a tubular neighborhood of a conic, say, the Klein conic defined by the equation $z_0^2 + z_1^2 + z_2^2 = 0$. Its complement is a tubular neighborhood of $\mathbb{RP}^2$. This sort of surgery is known as rational blow-down of degree 2 [FS2]. We can also think of the transform $\mathfrak{A}_{-1} \to \mathfrak{A}_1$ as of a “connected sum” with $\mathbb{RP}^2$ along a neighborhood of the circle (real conic) $x_0^2 + x_1^2 = x_2^2$ and consider it as a real rational blow-down.

(2) If instead of perturbation of $A_0$ we consider the resolution, i.e., blow up $\mathbb{CP}^2$ at $P \in A_0$ and take the double covering $\tilde{X}_0 \to \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ branched along the proper image $\tilde{A}_0$ of $A_0$, further, denote by $\tilde{\mathfrak{A}}_0$ the Arnold surface of $\tilde{A}_0$ in $S^4 = \mathbb{CP}^2 \# \overline{\mathbb{CP}^2}/\text{conj}$ (cf. §5), by $\tilde{X}_{R,0}$ the real part of $\tilde{X}_0$ and put $\tilde{Y}_0 = \tilde{X}_0/\text{conj}$, then we have $\tilde{\mathfrak{A}}_0 \cong \mathfrak{A}_0$, $\tilde{Y}_0 \cong Y_0$, $\tilde{X}_{R,0} \cong X_{R,-1}$ if the node at $P$ is hyperbolic (i.e., $M = M_1$ or $M_0^{-1}$), and $\tilde{\mathfrak{A}}_0 \cong \mathfrak{A}_1$, $\tilde{Y}_0 \cong Y_1$, $\tilde{X}_{R,0} \cong X_{R,1}$ for an elliptic node ($M = M_2$).

4.4. Corollary. Moves $M_0^{-1}$ and $M_1$ have both the effect of a real blow-up on the Arnold surface (i.e., $\# \mathbb{RP}^2$), a usual blow-up on $Y$, and a Morse modification of index 2 on $X_R$. $M_2$ makes a real rational blow-down on the Arnold surface, a rational blow-down of degree 2 on $Y$, and a Morse modification of index 3 on $X_R$. The moves opposite to the above ones have the obvious opposite effect. This shows, in particular, how the modifications on $Y$ is determined by the Morse modifications of $X_R$.

Given a pair of nonsingular curves, $A_0$, $A_1$, of the same even degree with $W_i = W^\pm(A_i)$, $\mathfrak{A}_i = \mathfrak{A}_i^\pm(A_i)$, $i = 0, 1$, let us write $W_0 \succ W_1$ if $W_1$ can be obtained from $W_0$ by a deformation connecting $A_0$ and $A_1$ which involves only the moves $M_0^{-1}$ and $M_1$. Clearly, $W_0 \succ W_1$ implies $\chi(W_0) > \chi(W_1)$. 

4.5. **Corollary.** If $\mathfrak{A}_0$ is standard and $W_0 \succ W_1$ then $\mathfrak{A}_1$ is standard as well.

**Proof of Theorems 4.1, 4.2.** Consider first the case of move $M_0$. In a suitable coordinate system near $P$, the projection $q: \mathbb{C}P^2 \to \mathbb{C}P^2/\text{conj}$ is modeled by the map $\mathbb{C}^2 \to \mathbb{C}^2$, with $(z_1, z_2) \mapsto (z_1^2, z_2)$, where $\mathbb{R}P^2$ is represented by the line $z_1 = 0$, $A_0$ is defined by $z_1^2 - z_2^2 = 0$. Then locally $\mathfrak{A}_0 = q(A_0)$ is defined by $z_1 - z_2^2 = 0$ and, hence, is smooth and tangent to $\mathbb{R}P^2$ at $P$. This proves 4.1(1) and 4.2(1).

4.1(2) Near $P$ (i.e., inside a small regular neighborhood), for $t > 0$, $\mathfrak{A}_t$ is a union of $q(A_t)$ and of a disc component $D_t$ of $W_t$, bounding an oval $\partial D_t \subset A_{t, \mathbb{R}}$, which can be considered as a vanishing cycle for the deformation $t \to 0$. $\mathfrak{A}_0$ is obtained from $\mathfrak{A}_t$ by contracting $D_t$. Hence, $\mathfrak{A}_0 \cong \mathfrak{A}_1$. 4.1(3) For $t < 0$, $\mathfrak{A}_t = q(A_t)$ is, near $P$, a Möbius band, which replaces the disc $D_t$. The direction of its “twisting” is, obviously, standard and can be easily determined on a model, for example, conic curve, by taking double covering branched over it. 4.1(4) Follows from the above local algebraic model describing the mutual position of $\mathbb{R}P^2$ and $A_0$ near $P$. 4.1(5) Follows from the arguments of (1) and (2). $X_{t, \mathbb{R}, 0}$ is obtained from $X_{t, \mathbb{R}}$ by the contraction of $D_t$, topologically equivalent to the contraction of a vanishing cycle which gives a double point singularity at $P$.

4.1(6) When $t \to -0$, the imaginary vanishing cycle in $\mathfrak{A}_t$ is pinched along the disc, which is covered by the exceptional curve of the blow-up, $E \subset Y_t \cong Y_0 \# \mathbb{C}P^2$. $X_{t, \mathbb{R}} \cap E$ consists of a pair of points projected to $P$. This gives the same topological description as for the resolution of a double point singularity.

Let us pass now to Theorem 4.2. 4.2(2) A Seifert surface on the $(2, 4)$-toric link, which replaces a pair of tangent discs, can be easily seen in the above local description of $\mathfrak{A}_0$. 4.2(3) The components are a disc part of $\mathbb{R}P^2$ and a Möbius band part of $q(A_t)$ (see above 4.1(3)), the mutual position of which is, obviously, the one described on Figure 5. 4.2(4) The sphere component of $X_{t, \mathbb{R}}$, $t < 0$, covers $D_t$; hence, it is the zero section. The rest is straightforward.

Now consider the case $M_1$.

4.1(1) The link of $A_0 \cup \mathbb{R}P^2$ at $P$ and its quotient in $\mathbb{C}P^2/\text{conj}$ are shown on Figure 6. The link of $\mathfrak{A}_0$ turns out to be an unknot.

![Figure 6](image-url) Links for a cross-like node in $\mathbb{C}P^2$ (on the left) and in $S^4$ (on the right)
4.1(2)-(3) Can be easily seen from the local description of $\mathcal{A}_t$ shown on Figure 2. 4.1(4) The link of singularity of $X_{R,0}$ is obtained from the right picture on Figure 6 as the double covering along the link of $\mathcal{A}_0$. The result is equivalent to the left picture. 4.1(5)-(6) Analogous to the case of move $M_0$, by making use of the local description of $\mathcal{A}^\pm$ on Figure 2. □

Let us write $W_0 \triangleright W_1$ if $W_1$ can be obtained from $W_0$ by making use of any moves except $M_2^{-1}$. It may look plausible that the relation $W_0 \triangleright W_1$ is always true whenever $W_0 \neq \mathbb{RP}^2$, however, it is not known for curves of degree $\geq 8$.

The following theorems yield vanishing of Donaldson (Seiberg–Witten) invariants for $Y$ in more general setting, when the above methods do not allow to prove decomposability of $Y$. Assume that $b_2^+(Y) > 1$ and odd, which is needed for the invariants to be well defined. Note that $b_2^+(Y) = p_g(X)$ (see Supplement 7.4). It follows from [FS1] and [FS2] that the vanishing property for Donaldson (Seiberg–Witten) invariants is preserved by usual blow-ups and -downs and by rational blow-downs. This immediately prompts the following

4.6. Corollary. If Donaldson (Seiberg–Witten) invariants vanish for $Y_0$ and $W_0 \triangleright W_1$, then they vanish for $Y_1$ as well.

We can also formulate this result in more general setting, since modifications of $Y$ are determined by the Morse modifications of $X_R$.

4.7. Corollary. Let $(X_t, \text{conj}_t)$, $t \in [0, 1]$ be a generic deformation of Real surfaces, i.e., $X_t$ nonsingular for all $t$ with finitely many exceptions for which $X_t$ has one double point singularity. Assume that sphere components of $X_{t,R}$ do not collapse while passing double points as $t$ increases. Then vanishing of Donaldson (Seiberg–Witten) invariants for $X_0/\text{conj}_0$ implies their vanishing for $X_1/\text{conj}_1$.

One can also easily deduce from [KM] and [FS2] the following

4.8. Corollary.

(1) If $X_R$ contains an orientable component of genus $\geq 2$, then the Donaldson polynomials of $Y$ vanish.

Furthermore, if $X$ is of simple type then

(2) fundamental classes of orientable components of $X_R$ of genus $\geq 1$ are orthogonal in $H_2(X)$ to the Kronheimer–Mrowka (Seiberg–Witten) basic classes $K_s$ of $X$;

(3) if $\Sigma \subset X_R$ is a sphere component, then $|\Sigma \cdot K_s| \leq 2$ for any basic class $K_s$. 12
Proof. By the adjunction formula (see [KM], FS2), if $X$ is of simple type, then
\[ \chi(\Sigma) + \Sigma \circ \Sigma + \max_s(K_s \circ \Sigma) \leq 0 \]
for a smooth essential connected surface $\Sigma \subset X$ with $\Sigma \circ \Sigma \geq 0$. If $\Sigma$ is a component of $X_\mathbb{R}$, then $\chi(\Sigma) + \Sigma \circ \Sigma = 0$, since the normal bundle of $X_\mathbb{R}$ is obtained from the tangent bundle by multiplication by $i$. Thus to prove (2), it suffices to recall that $J = \max_s(K_s \circ \Sigma) \geq 0$, and $J = 0$ if and only if $\Sigma$ is orthogonal to all basic classes $K_s$. (3) follows similarly from the adjunction formula for immersed spheres [FS2], which works as well when $\Sigma \circ \Sigma < 0$.

To obtain (1) note that if $\Sigma \subset X_\mathbb{R}$ has genus $g(\Sigma) \geq 2$ then $\Sigma$ is essential since it has nonzero self-intersection number. Furthermore, we have $\chi(\Sigma) + 2\Sigma \circ \Sigma = \Sigma \circ \Sigma > 0$, which contradicts to the adjunction formula applied to $\Sigma$ in $Y$. On the other hand, this inequality guarantee that $Y$ is of simple type. Hence, the basic classes do not exist. □

By [FS2] a logarithmic transform of multiplicity $m$ can be decomposed into a product of $m - 1$ usual blow-ups and a rational blow-down of degree $m$. For $m = 2$ it is one blow-up at a nodal point of a singular elliptic fiber, which yields a nonsingular rational curve with self-intersection $-4$, then a rational blow-down of degree 2 replacing this curve by $\mathbb{R}P^2$. Theorems 4.1—4.2 imply that the described combination is produced on $Y^\pm$ by the combination of moves $M_1$ and $M_2$ on $W^\pm$ which eliminate an annulus component of the complementary domain $W^\pm$.

4.9. Corollary. The combination of moves $M_1$ and $M_2$ of $W^\pm$ which makes vanish an annulus component of $W^\pm$ as it is shown on Figure 7 produces a logarithmic transform of multiplicity 2 on $Y^\pm$ along the torus component of $X_{\mathbb{R}}$ corresponding to the annulus. □

Figure 7. Logarithmic transform of multiplicity 2

Since the modifications involved are local, we can reformulate the above Corollary as follows.

4.10. Corollary. Let $(X_1, \text{conj}_1)$ be obtained from $(X_0, \text{conj}_0)$ by a deformation $(X_t, \text{conj}_t)$, which consists of nonsingular Real surfaces for all $t \in [0, 1]$
with 2 exceptions when \( X_t \) has a node. Suppose that this deformation make a torus component of \( X_{0,\mathbb{R}} \) vanish. Then \( Y_1 \) is obtained from \( Y_0 \) by a logarithmic transform of multiplicity 2 along this component.

§5. Generalized Rokhlin Conjecture

In this section we study a generalization of the Rokhlin Conjecture, when \( \mathbb{C}P^2 \) is replaced by an arbitrary nonsingular Real surface, \((P, \text{conj})\).

Call a curve \( A \subset P \) even if it is the zero set of a holomorphic section \( f: P \to L^2 \), where \( L^2 = L \otimes L \) is the square of some holomorphic linear bundle \( p: L \to P \). The latter is called a real bundle if it is supplied with an anti-linear involution \( \text{conj}_L: L \to L \), such that \( p \circ \text{conj}_L = \text{conj} \circ p \):

\[
\begin{array}{ccc}
L & \xrightarrow{\text{conj}_L} & L \\
p \downarrow & & \downarrow p \\
p \circ \text{conj}_L & \xrightarrow{\text{conj}} & p \\
P & \xrightarrow{\text{conj}} & P \\
\end{array}
\]

Denote by \( L_\mathbb{R} \) its real part, that is the fixed point set of \( \text{conj}_L \). The restriction of \( p \) gives a real line bundle \( L_\mathbb{R} \to P_\mathbb{R} \). \( L^2 \) has the real structure \( \text{conj}_{L^2} = (\text{conj}_L)^2 \) induced from \( L \), with the real part \( (L^2)_\mathbb{R} \) trivialized by the choice of the natural positive ray \( (L_\mathbb{R})^2 \). Hence, the sign of \( f(x) \) is well defined at real points \( x \in P_\mathbb{R} \) and we can put \( W^\pm = W^\pm(f) = \{ x \in P_\mathbb{R} : \pm f \geq 0 \} \) as in the case \( P = \mathbb{C}P^2 \). Again, we have \( W^+ \cup W^- = P_\mathbb{R} \), \( W^+ \cap W^- = A_\mathbb{R} \), and define generalized Arnold surfaces \( A^\pm = A^\pm(f) = W^\pm \cup q(A) \subset Q \), where \( Q = P/\text{conj} \) and \( q: P \to Q \) is the quotient map. If \( A \) is connected and is of type 1 then the Arnold surfaces \( A^\pm_\mathbb{R} \subset P \) can be defined like in the case \( P = \mathbb{C}P^2 \).

The collection of examples known to the author suggests that the Rokhlin Conjecture might be generalized as follows.

5.1. Generalized Rokhlin Conjecture. If \((P, \text{conj})\) is a CDQ-surface and \( A \subset P \) is a Real, nonsingular, even curve with \( A_\mathbb{R} \neq \emptyset \) then the Arnold surfaces in \( P \) and in \( Q \) are standard when connected.

By a standard surface in \( Q \) we mean a surface obtained as a connected sum of standard surfaces in \( S^4 \) and possibly surfaces \( \pm(\mathbb{C}P^2, \text{conic}) \), \( \pm(\mathbb{C}P^2, \emptyset) \), \((S^2 \times S^2, \emptyset)\). We will consider multi-component standard surfaces and admit connected sums in the both forms: \((Y_1 \# Y_2, F_1 \# F_2)\) and \((Y_1 \# Y_2, F_1 \sqcup F_2)\).

The above conjecture looks plausible for disconnected Arnold surfaces as well, unless they include some components of \( X_\mathbb{R} \) entirely. By a standard surface in \( P \) we mean a connected sum of \((P, C)\) and \((S^4, F)\), where \( C \) is a nonsingular complex curve in \( P \) and \( F \) a standard surface in \( S^4 \).
Take \( X = \{ v \in L \mid v^2 = f(p(v)) \} \), the double covering over \( P \) with the projection \( p|_X : X \to P \), and consider two conjugations \( \text{conj}^\pm(v) = \text{conj}_L(\pm v) \) which cover \( \text{conj} \). Put \( Y^\pm = X/\text{conj}^\pm \).

5.2. Theorem. If \((P, \text{conj})\) is a CDQ-surface and the Arnold surface \( \mathfrak{A}^+ \) (or \( \mathfrak{A}^- \)) for a Real curve \( A \) is connected and standard, then \((X, \text{conj}^-)\) (respectively \((X, \text{conj}^+)\)) is also CDQ-surface. If \( \mathfrak{A}^\pm \) is standard and has \( k \) connected components, then \( Y^\mp \cong R \# (k-1)(S^1 \times S^3) \), where \( R \) is a completely decomposable 4-manifold.

Proof. If \( \mathfrak{A}^\pm \) is connected then the proof is the same as for Theorem 2.2 with only one additional remark that the double branched covering over \( \pm(\mathbb{CP}^2, \text{conic}) \) is a quadric and, hence, standard. If \( \mathfrak{A}^\pm \) is not connected we can use that the double covering over \((Y_1 \# Y_2, F_1 \sqcup F_2)\) is obviously obtained from the covering over \((Y_1 \# Y_2, F_1 \# F_2)\) by adding a 1-handle. \( \square \)

Theorem 5.3 below provides patterns of curves \( A_i \) with standard Arnold surfaces. Their deformations can produce further examples via Corollary 4.5. Let \((P, \text{conj})\) be a nonsingular Real surface and \((L, \text{conj}_L)\) a real linear bundle of degree \( d \). Consider a real pencil, \( f_t : P \to L \), \( f_t(x) = tf_1(x) + (1-t)f_0(x) \), \( t \in \mathbb{C} \), where \( f_0, f_1 \) are real holomorphic sections. Denote by \( D(f_t) \) the zero divisor of a section \( f_t \) and assume that the curve \( B = D(f_0) \) is nonsingular, connected and intersects \( D(f_1) \) transversely at real distinct points, \( b_1, \ldots, b_d \in \mathbb{R} \). Assume further that there is a real section \( h : P \to L \), with \( b_i \notin D(h) \), \( 1 \leq i \leq d \). Define sections \( v_{\varepsilon, t}, u_{\varepsilon, t} : P \to L^2 \), parametrized by \( t, \varepsilon \in \mathbb{C} \), as

\[
v_{\varepsilon, t} = f_0^2 + f_t^2 - \varepsilon h^2, \quad u_{\varepsilon, t} = -(f_0 \cdot f_t + \varepsilon h^2).
\]

It can be easily seen that the curves \( D(f_t), D(v_{\varepsilon, t}), D(u_{\varepsilon, t}) \) are Real, nonsingular and connected for real and sufficiently small \( \varepsilon, t > 0 \). The real scheme of \( D(v_{\varepsilon, t}) \) of the curve \( D(v_{\varepsilon, t}) \) consists of \( d \) small ovals around the basepoints \( b_i \). \( D(u_{\varepsilon, t}) \) lies between \( D(f_0) \) and \( D(f_t) \), inside \( W^+(\pm h(f_t)) \), and is obtained from \( B \) by doubling the ovals of \( B \) which does not contain basepoints and replacing the ovals which contains \( r \) basepoints by \( r \) ovals.

The following theorem asserts that these curves have standard Arnold surfaces in \( P \) and \( Q \).

5.3. Theorem. For small enough \( \varepsilon, t > 0 \)

1. \( D(v_{\varepsilon, t}) \) is of type 1 and has standard Arnold surfaces, \( \mathfrak{A}^+_k(v_{\varepsilon, t}) \subset P \), isotopic to \( B \); \( D(u_{\varepsilon, t}) \) is of type 1 if \( B \) is. In the latter case the Arnold surfaces \( \mathfrak{A}^+_k(u_{\varepsilon, t}) \subset P \) are standard and isotopic to \( B \).

2. Arnold surfaces \( \mathfrak{A}^+ \subset Q \) for the both curves \( u_{\varepsilon, t} \) and \( v_{\varepsilon, t} \) bound solid handlebodies embedded in \( Q \). These handlebodies are orientable in the case of \( u_{\varepsilon, t} \) independently of the type of \( B \). In the case of \( u_{\varepsilon, t} \) they are orientable if and only if \( B \) is of type 1.
Proof. The degenerated curve $D(v_{0,t})$ splits into a pair of conjugated imaginary curves, $D(f_0 \pm itf_1)$, isotopic to $B$. After a perturbation they are deformed into a pair of halves of $D(v_{\varepsilon,t}) - D(v_{\varepsilon,t})_\mathbb{R}$. $D(u_{0,t})$ splits into a pair of Real curves, $B = D(f_0)$ and $D(f_t)$, which are both of the same type for a small enough $t > 0$. If it is type 1, then the opposite halves of $D(f_0) - D(f_0)_\mathbb{R}$ and $D(f_t) - D(f_t)_\mathbb{R}$ are fused after a perturbation and form the halves of $D(u_{\varepsilon,t}) - D(u_{\varepsilon,t})_\mathbb{R}$ (this way of gluing of “halves” is well known and can be understood easily). Consecutive degenerations as $\varepsilon \to 0$ and $t \to 0$ provide isotopies of the Arnold surfaces into $B$. To prove (2) we use first the degeneration $\varepsilon \to 0$ and then consider the trace of the variation $t \to 0$ as the solid handlebody. In the case of $v_{\varepsilon,t}$ it gives an isotopy between $A^+(v_{\varepsilon,t})$ and $A^+(v_{0,t}) = q(D(f_0 + itf_1))$. The latter bounds the handlebody $H' = q(H)$, where $H = \bigcup_{-\tau \leq \varepsilon \leq \tau} D(f_0) + itf_1$. In the case of $u_{\varepsilon,t}$ we get an isotopy between $A^+(u_{\varepsilon,t})$ and $A^+(u_{0,t})$. The latter bounds $H' = q(H)$, where $H = \bigcup_{0 \leq \tau \leq \varepsilon} D(f_{\tau})$.

In the both cases $H'$ is smoothable at basepoints since its links at $b_i$ are homeomorphic to discs (see Figure 6). Smoothability at the other points is obvious. $H'$ is a solid handlebody, since it is homeomorphic to the quotient of $A \times [-1,1]$ by the conjugation $\tau_1(x,t) = (\text{conj}|_A, t)$ in the case of $u_{\varepsilon,t}$, and $\tau_2(x,t) = (\text{conj}|_A, -t)$ in the case of $v_{\varepsilon,t}$. \[\square\]

Theorems 5.2 and 5.3 imply the following result.

5.4. Corollary. Let $Q$ be simply connected and $Y^\pm$ defined as above via $v_{\varepsilon,t}$ or $u_{\varepsilon,t}$. Then $Y^- \cong 2Q \# R$, where $R$ is a completely decomposable 4-manifold.

In particular, if $(P, \text{conj})$ is a CDQ-surface then $(X, \text{conj}^-)$ is CDQ as well.

Remarks.

(1) If $A^+$ is orientable and $g = g(A^+) = g(B)$ is the genus, then $R \cong g(S^2 \times S^2)$. The above condition is not satisfied if and only if $B$ is of type 2 and we consider the case of $u_{\varepsilon,t}$. Then $R \cong g(\mathbb{CP}^2 \# \mathbb{CP}^2)$.

(2) The proof of Theorem 5.3 works also if $B$ is not connected provided all components of $B$ have nonempty real part. Then $Y^-$ gets several additional 1-handles, the number of which is less by 1 than the number of connected components of $A^+$.

(3) In the case of $P = \mathbb{CP}^2$, the real part of curves $D(v_{\varepsilon,t})$ consists of $k^2$ disjoint ovals if $\deg(B) = k$. Curves $D(u_{\varepsilon,t})$ have real schemes consisting of $k^2$ disjoint ovals added to a doubled scheme of a degree $k$ curve, which belongs to type 2 (and can be empty). $L$-curves of degree $k$ can provide real schemes of type 1 with $k^2$ empty ovals as well. It seems to be interesting to study if these three types of curves with $k^2$ ovals are rigidly isotopic.
The arguments in the proof of Theorem 5.3 showing that $\mathfrak{A}^+(v_0, t) = q(D(f_0 + itf_1))$ bounds a handlebody, give the following

5.5. Corollary. If $Q$ is simply connected, $z \in \mathbb{C} - \mathbb{R}$ and $|z| > 0$ is small enough, then $q(D(f_z))$ is a standard surface in $Q$.

5.6. Corollary. Let $C \subset P$ be a nonsingular curve of degree $d$ which intersects $P_\mathbb{R}$ at $d$ points. Then $q(C)$ bounds a handlebody in $Q$ and, hence, is a standard surface if $Q$ is simply connected.

The latter Corollary follows from the previous one, since the pencil of curves containing $C$ and conj($C$) is real and, therefore contains Real curves.

The condition for the number of intersection points with $P_\mathbb{R}$ is important; otherwise, $q(C)$ is not an embedded surface.

5.7. Corollary. Nonsingular imaginary curves $A \subset \mathbb{CP}^2$ of degree $k$ which intersect $\mathbb{RP}^2$ at $k^2$ points are projected by the quotient map into standard surfaces in $S^4 = \mathbb{CP}^2/$conj. For example, the images of imaginary lines and imaginary conics intersecting $\mathbb{RP}^2$ in 4 points are unknots in $S^4$.

§6 Fibered surfaces

In this section we consider complex surfaces fibered over complex curves, $\pi: P \to C$, and study Arnold surfaces $\mathfrak{A}^\pm$ and quotients $Y^\pm$ associated with the curves $A$ consisting of several nonsingular fibers. A real structure on a fibered surface is given, by definition, by a pair of commuting complex conjugations $\text{conj}_C: C \to C$ and $\text{conj}: P \to P$:

\[
P \xrightarrow{\text{conj}} P \\
\pi \downarrow \quad \pi \downarrow \\
C \xrightarrow{\text{conj}_C} C
\]

Fibers over real points $c \in C$, $F = \pi^{-1}(c)$, inherit real structures, $\text{conj} |_F$, from $P$.

We assume for the rest of the section that

(1) $P$, $C$ and a generic fiber $F$ are nonsingular and connected.
(2) $P_\mathbb{R} \neq \varnothing$ and, in particular, $B_\mathbb{R} \neq \varnothing$.
(3) $A$ is nonsingular, real and even. Hence, it consists of even number of real fibers and several pairs of conjugated imaginary fibers $B_j = \pi^{-1}(b_j)$, $B'_j = \pi^{-1}(b'_j)$, $b'_j = \text{conj}(b_j)$, $j = 1, \ldots, s$.
(4) Real fibers of $A$ are double, i.e., split into pairs $A_i = \pi^{-1}(a_i)$, $A'_i = \pi^{-1}(a'_i)$, where $a_i, a'_i \in C_\mathbb{R}, i = 1, \ldots, r$, are close enough to each other.
The latter means, in particular, that there are no singular real fibers between \( a_i \) and \( a'_i \).

(5) Real parts \( A_{i,R} \) of real fibers \( A_i \), \( i = 1, \ldots, r \) are not empty.

Let \( W^+_i \cong A_{i,R} \times I \) denote the part of \( X_R \) between \( A_{i,R} \) and \( A'_{i,R} \), \( i = 1, \ldots, r \). Let \( q: P \to Q \) be the quotient map and \( \mathfrak{A}^\pm \), \( X \), \( \text{conj}^\pm \) and \( Y^\pm \) are defined by the curve \( A \) as in §5, with the sign superscripts determined so that \( \mathfrak{A}^+ = \bigcup_{i=1}^r \mathfrak{A}^+_i \cup \bigcup_{j=1}^s \mathfrak{B}_j \), where \( \mathfrak{A}^+_i = W^+_i \cup q(A_i) \cup q(A'_i) \) and \( \mathfrak{B}_j = q(B_j) = q(B'_j) \).

6.1. Theorem. \( \mathfrak{A}^+ \) bounds a disjoint union of \( r+s \) solid handlebodies embedded in \( Q \).

Similar to Corollary 5.4 this implies the following

6.2. Corollary. If \( Q \) is simply connected then

1. \( \mathfrak{A}^+ \) is standard,
2. \( Y^- \cong 2Q \# R \# (r+s-1)(S^3 \times S^1) \), where \( R \) is a completely decomposable 4-manifold.

6.3. Corollary. If \((P, \text{conj})\) is a CDQ-surface and \( Y^- \) is simply connected, then \((X, \text{conj}^-)\) is CDQ.

Proof of Theorem 6.1. Let \([a_i, a'_i]\) denote the small segment of \( B_{R} \) between \( a_i \) and \( a'_i \), and \( H_i = \pi^{-1}([a_i, a'_i]) \). Then \( H'_i = q(H_i) \cong q(A_i) \times [a_i, a'_i] \) is a solid handlebody bounded by \( \mathfrak{A}^+_i \).

Let \( q_C: C \to C/\text{conj}_C, \pi': Q \to C/\text{conj}_C \) denote the natural projections. Choose points \( c_j \in (C_R - \bigcup_{i=1}^r [a_i, a'_i]) \), \( j = 1, \ldots, s \), with nonempty real parts of the fibers \( \pi^{-1}(c_j) \), for example choose them near \( a_1 \) outside \([a_1, a'_1]\). Connect \( q_C(b_j) \) with \( c_j \) by disjoint set of smoothly embedded arcs \( l_j \subset C/\text{conj}_C \), which don’t intersect with \( G_{R} \), except endpoints \( c_j \), and don’t pass under singular fibers. Then \( G'_j = (\pi')^{-1}(l_j) \) are solid handlebodies bounded by \( \mathfrak{B}_j \).

\( H'_i \) and \( G'_j \) are disjoint by construction. \( \Box \)

Remarks.

1. Note that Theorem 6.1 implies Theorem 5.3(2), since blow-ups at the basepoints of the pencil \( f_i \) give a fibered surface and the curve \( A \) is becoming a double fiber. A blow-ups at a real basepoint does not change \( Q \), as it is pointed out in the introduction. A blow-up does not change also the Arnold surfaces \( \mathfrak{A}^+_0 \) of a curve \( A_0 \) when it is made at a cross-type point, and it does not change \( \mathfrak{A}^+_0 \) in the case of an isolated point of \( A_0 \). Therefore it does not change \( Y^- \) in both cases as well.

2. Orientability of \( H' \) and \( G' \) is, clearly, determined by the same criterion as in Theorem 5.3.
The topology of $\mathfrak{A}^- \subset Q$ is more complicated. However, we can describe the double covering $Y^+ \rightarrow Q$, branched over $\mathfrak{A}^-$, in terms of some surgery on $P \rightarrow Q$.

**Definition.** Given a pair of spaces $X_1, X_2$, with involutions $\theta_i: X_i \rightarrow X_i$, $i = 1, 2$, we define $\mathbb{Z}/2$-product of $(X_1, \theta_1)$ and $(X_2, \theta_2)$ as the quotient space $X_1 \times_{\mathbb{Z}/2} X_2 = (X_1 \times X_2)/\theta$, by the product involution $\theta = \theta_1 \times \theta_2$.

Let $N_i = F \times_{\mathbb{Z}/2} Z_i$, $i = 1, 2$, be $\mathbb{Z}/2$-products of a real fiber $F = \pi^{-1}(a)$, $a \in C_R$, supplied with the involution $\text{conj} | F$, and the cylinder $Z_i = S^1 \times [-1,1]$, supplied with the involution $\tau_i(z,t) = (z,-t)$, for $i = 1$, and with $\tau_2(z,t) = (-z,-t)$, for $i = 2$. Note, that $\partial N_i = F \times_{\mathbb{Z}/2} (\partial Z_i) \cong F \times S^1$.

**Definition.** By $\mathbb{Z}/2$-surgery along a real nonsingular fiber $F = \pi^{-1}(a)$ we mean replacing of its tubular neighborhood $\pi^{-1}(D) \cong D^2 \times F$, where $D \subset C$ is a small, regular, conj-symmetric neighborhood of $a$, by $N_1$ ($\mathbb{Z}/2$-surgery of type 1), or by $N_2$ ($\mathbb{Z}/2$-surgery of type 2).

The diffeomorphism type of the result is well defined since the trivializations of $\pi^{-1}(D)$ and $\partial N_i$ are, clearly, canonical up to diffeotopy.

The following examples are easy exercises.

**Examples:**

1. If $F \cong S^2$, a rational fiber with $F_{\mathbb{R}} \neq \emptyset$, then $\mathbb{Z}/2$-surgery of type 1 is the Morse modification of index 2 along $F$.

2. If $F$ is a rational fiber and $F_{\mathbb{R}} = \emptyset$, then $\mathbb{Z}/2$-surgery of type 1 replaces $F \times D^2$ by $(\mathbb{R}P^3 - D^3) \times S^1$.

3. If $F$ is an elliptic fiber, $F_{\mathbb{R}} \neq \emptyset$, then $\mathbb{Z}/2$-surgery of type 1 makes a logarithmic transform of multiplicity 0 along $F$, which is, by definition, the modification $D^2 \times S^1 \times S^1 \rightarrow S^1 \times D^2 \times S^1$, that is the Morse modification of index 1 in dimension 3, multiplied by $S^1$.  

6.4. **Theorem.** $Y^+$ can be obtained from $P$ by $r \mathbb{Z}/2$-surgeries of type 1 along $A_i$, $1 \leq i \leq r$, and $s \mathbb{Z}/2$-surgeries of type 2 along arbitrary real fibers $F_j \subset P$, $1 \leq j \leq s$, with $F_{j,\mathbb{R}} \neq \emptyset$.

**Proof.** Consider small regular conj-symmetric neighborhoods $H_i \supset H_i$, $G_j \supset G_j$ of the solid handlebodies of Theorem 6.1. The restriction of the covering $p: X \rightarrow P$ over $K = P - (\bigcup_i H_i \cup \bigcup_j G_j)$ is trivial and $\text{conj}^+ \text{ flips 2 copies of } K$. Thus $p^{-1}(K)/\text{conj}^+$ can be identified with $K$.

The restrictions of $\text{conj}^+$ to the pull backs of neighborhoods, $p^{-1}(H_i)$, $p^{-1}(G_j)$ are equivalent to $(\text{conj} | F) \times \tau_1$ on $F \times Z_1$ and $(\text{conj} | F) \times \tau_2$ on $F \times Z_2$. Thus quotients are $N_1, N_2$. □

Theorems 6.1—6.4 can be applied to elliptic surfaces; by making use of Example 3 we may summarise the above results as follows.
6.5. **Theorem.** Let $P$ be an elliptic surface and $A = A_1 \perp A'_1$ a double real fiber. Then

1. $Y^+$ is diffeomorphic to the manifold, obtained by a logarithmic transform of multiplicity 0 on $P$, along the fiber $A_1$.
2. If $Q$ is simply connected, then $Y^- \cong 2Q \# R$, where $R$ is $S^2 \times S^2$ if $A_{1,R}$ belongs to type 1, and $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ if it belongs to type 2.
   In particular, if $(P, \text{conj})$ is CDQ-surface then $(X, \text{conj}^-)$ is CDQ as well.

6.6. **Corollary.** Let $P = E(n)_{m_1, \ldots, m_k}$. Then $Y^+$ is completely decomposable if it is simply connected.

The last corollary follows from Theorem 6.5, since $Y^+ = E(n)_{m_1, \ldots, m_k, 0}$ is not simply connected unless $k = 0$; however $E(n)_{0}$ is decomposable (see [G]).

§7. **Some applications**

In this section we consider applications of the results of §3–5 to Real curves $A \subset \mathbb{CP}^2$ of degree $\leq 6$ and to Real $K3$ surfaces.

All nonsingular curves $A$ of degree $\leq 4$ with $A_R \neq \emptyset$ are known to be $L$-curves [F1]. Hence, Theorem 3.1 implies

7.1. **Corollary.** Arnold surfaces $\mathfrak{A}^\pm(A)$ for nonsingular curves $A$ of degree 2 and 4 with $A_R \neq \emptyset$ are standard.

The case $\deg(A) = 6$ is a somewhat more delicate. According to the classification of sextics (see, e.g., [V]) $A$ is determined up to rigid isotopy by the arrangement of the components (ovals) of $A_R$ in $\mathbb{RP}^2$ and the type of $A$. Similar to [V], we will use the notation $\langle \alpha \perp 1 \langle \beta \rangle \rangle_{\kappa}$ to code curves of type $\kappa = 1, 2$ which have an oval containing $\beta$ separate ovals inside and $\alpha$ separate ovals outside. In the case of sextics the total number of ovals is $\alpha + \beta + 1 \leq 11$. In addition to the schemes $\langle \alpha \perp 1 \langle \beta \rangle \rangle_{\kappa}$ sextics may have schemes $\langle \alpha \rangle_{\kappa}$, $\alpha \leq 10$, corresponding to $\alpha$ separate ovals, and the scheme $\langle 1\langle 1 \langle 1 \rangle \rangle \rangle$, a nest of 3 ovals. We may omit the type $\kappa$ in codes of the schemes when it is determined by the arrangement of ovals of a sextic, like we did it for the above nest, which can be only of type 1 (for possible schemes and their types see [V]).

We will also use notation $\langle \alpha \perp 1 \langle \beta \rangle \rangle_{\kappa}^\pm$ and likewise for the domain $W^\pm(A)$ of a curve $A$ with the real scheme $\langle \alpha \perp 1 \langle \beta \rangle \rangle_{\kappa}$.

7.2. **Theorem.** All nonsingular Real sextics $A$ with $A_R \neq \emptyset$ have standard Arnold surfaces $\mathfrak{A}^+(A)$. If the scheme of $A_R$ differs from $\langle 1 \perp 1 \langle 9 \rangle \rangle_1$, $\langle 1 \perp 1 \langle 8 \rangle \rangle_2$, $\langle 1 \langle 9 \rangle_2$ and $\langle 1 \langle 8 \rangle_2$, then it has also a standard Arnold surface $\mathfrak{A}^-(A)$.
Proof. It is well known and not difficult to see directly from the Hilbert and Gudkov constructions of nonsingular Real sextics (cf. [V]), that the ones with schemes \((\alpha \perp 1\langle \beta \rangle)_{\kappa}, \kappa = 1, 2,\) can be deformed to the both schemes

\[
(1) \ ((\alpha - 1) \perp 1\langle \beta \rangle)_{2} \text{ if } \alpha \geq 1, \text{ and}
\]
\[
(2) \ (\alpha \perp 1(\beta - 1))_{2} \text{ if } \beta \geq 2, \text{ or to } (\alpha)_{2} \text{ if } \beta = 1
\]

by passing through a cross-like real node which connects the ambient oval with an exterior oval in the first deformation and with an interior one in the second. The only exception is the scheme \((9 \perp 1\langle 1 \rangle)\), which can be reduced to \((10)_{2}\) only by contracting the interior oval.

Figure 8 shows that sextics with the nest scheme \((1\langle 1\langle 1 \rangle)\) and with the scheme \((10)_{2}\) are \(L\)-curves, thus have standard Arnold surfaces.

**Figure 8. Construction of \(L\)-curves \((1\langle 1\langle 1 \rangle)\) and \((10)_{2}\)**

The latter scheme gives the maximal possible value of \(\chi(W^{\pm})\) realizable for sextics, \(\chi(W^{+}) = 10\). We will show that the other schemes \(W^{\pm}\) can be obtained from it by moves \(M_{1}\), with 4 exceptions mentioned in Theorem 7.2. Hence, Corollary 4.5 can be applied to provide the result of the theorem.

It follows from the above remark on reduction of schemes that \((10)_{2}^{+} \succ (\alpha)_{2}^{+}\) and \((\alpha + 1)_{2}^{+} \succ (\alpha \perp 1\langle \beta \rangle)_{\kappa}^{+}\) for all schemes with \(\alpha \geq 0, \beta \geq 1, \kappa = 1, 2,\) whenever they are realizable by sextics, except \((9 \perp 1\langle 1 \rangle)^{+}\). Further, by Theorem 5.3 sextics with schemes \((9)_{1}, (9 \perp 1\langle 1 \rangle)\) have standard \(A^{+}\) as well. Altogether this shows that \(A^{+}\) are standard in all the cases. Figure 9 shows that \((9)_{2}^{+} \succ (1\langle 8 \rangle)_{1}^{-}\). By the same arguments as above we get \((1\langle 8 \rangle)_{1}^{-} \succ (1\langle \beta \rangle)_{2}^{-} \succ (\alpha \perp 1\langle \beta \rangle)_{\kappa}^{-}\) for all existing schemes of sextics with \(0 \leq \alpha, 0 \leq \beta \leq 7, \kappa = 1, 2.\)

This includes all cases with the 4 exceptions mentioned in Theorem 7.2. \(\square\)
7.3. Theorem. Any Real nonsingular K3-surface $(X, \text{conj})$ with $X_\mathbb{R} \neq \emptyset$ is CDQ-surface.

Proof. The components of the moduli space for nonsingular Real K3 surfaces $(X, \text{conj})$ can be described in terms of arithmetics in $H_2(X)$ and distinguished by topological types of $X_\mathbb{R}$ and vanishing of $[X_\mathbb{R}] \in H_2(X; \mathbb{Z}/2)$. This fact is well known and follows from the global Torelli theorem, as it was noticed by V. Kharlamov and V. Nikulin [N]. (In fact, [N] considers only algebraic K3 surfaces, but after Torelli theorem was extended to abstract K3 the arguments of [N] concerns the latter case as well.) It is known also that $X_\mathbb{R}$ may be homeomorphic to $S_1 \sqcup S_1$, or $S_g \sqcup kS_0$, $g+k \leq 11$, with certain restrictions to $g, k$ when $g+k = 11$ or 10 (cf. [N] or [V]); here $S_g$ denotes a genus $g$ orientable surface and $kS_0$ a disjoint union of $k$ spheres.

After comparing the classification of Real K3-surfaces with the classification of Real sextics, cf. [N, V], we see that all but one deformation types of Real K3-surfaces are represented by double coverings over $\mathbb{CP}^2$ branched along sextics, as $(X, \text{conj}^\pm)$ in the notation of §2. More precisely, all the types of K3 surfaces for which $X_\mathbb{R}$ has a non sphere component can be represented as $(X, \text{conj}^-)$. The ones for which $X_\mathbb{R} \cong kS_0$ and $[X_\mathbb{R}] \neq 0 \in H_2(X; \mathbb{Z}/2)$ can be represented as $(X, \text{conj}^+)$, when $A$ is a sextic having real scheme $\langle k \rangle$.

By Theorem 7.2, Arnold surfaces $\mathfrak{A}^+$ are standard for sextics with any
scheme, and \( \mathfrak{A}^- \) for sextics with schemes \( \langle k \rangle \). Hence, by Theorem 2.2 \((X, \text{conj})\) are CDQ-surfaces in all the cases (recall that \( \mathfrak{A}^\pm \) corresponds to \( \text{conj}^\mp \)).

The only real deformation type of \( K3 \) surfaces, \((X, \text{conj})\), which does not contain a double plane has \( X_\mathbb{R} \cong 8S_0 \) and \([X_\mathbb{R}] = 0\). We will see that there is a deformation of \((X, \text{conj})\) fusing a pair of components of \( X_\mathbb{R} \), and giving as the result \((X', \text{conj}')\) with \( X'_\mathbb{R} = 7S_0 \). \((X', \text{conj}')\) is already shown to be CDQ-surface, hence, by Corollary 4.4 \((X, \text{conj})\) is CDQ-surface as well. To see the fusing deformation connecting \((X, \text{conj})\) with \((X', \text{conj}')\) consider a family of double quadrics branched along Real curves of degree \((4, 4)\). It consists of the Real \( K3 \) surfaces and includes \((X, \text{conj})\) not realizable as a double plane. Explicitly, take the quadric \( P \subset \mathbb{CP}^3 \) defined by the equation \( x^2 + y^2 - z^2 = t^2 \), the projection \( f: P \to \mathbb{CP}^2, (x: y: z: t) \mapsto (x: y: z) \), and the pull back \( \tilde{A} = f^{-1}(A) \subset P \) of a plane curve \( A \) of degree 4, which real part \( A_\mathbb{R} \) has 4 ovals outside the oval of the branched locus of \( f \), i.e. the conic \( x^2 + y^2 - z^2 = 0 \). \( \tilde{A}_\mathbb{R} \) is of type 1 and has 8 separate ovals lying in the hyperboloid \( P_{\mathbb{R}} \). \((X, \text{conj})\) is the double covering over \( P \), branched along \( \tilde{A} \). A deformation fusing a pair of components of \( X_\mathbb{R} \) is obtained then from a deformation of \( \tilde{A} \), which fuses a pair of ovals. \( \square \)

Standard calculations (see Supplement below) give values \( b_2^+(Y) = 1, b_2^-(Y) = 9 + \frac{1}{2} \chi(X_\mathbb{R}) \) for Real \( K3 \) surfaces \((X, \text{conj})\). Hence, by Theorem 7.3 either \( Y \cong \mathbb{CP}^2 \# k\overline{\mathbb{CP}}^2 \), where \( k = 9 + \frac{1}{2} \chi(X_\mathbb{R}) \), or \( Y = S^2 \times S^2 \). The latter holds in 2 cases:

1. \( X_\mathbb{R} \cong S_{10} \perp S_0 \),
2. \( X_\mathbb{R} = S_0 \) and the class \([X_\mathbb{R}] \in H_2(X; \mathbb{Z}/2)\) vanishes,

as it follows from Theorem 5.3, since these two types of \( K3 \) surfaces are represented by double planes \((X, \text{conj}^-)\) branched along sextics with one of the schemes \( \langle 9 \perp 1 \rangle \), \( \langle 9 \rangle \), and because \( Y^\mp = S^2 \times S^2 \) if and only if the Arnold surface \( \mathfrak{A}^\pm \) is orientable.

**7.4. Supplement.** We reproduce here the calculation of \( b_2^\pm(Y) \) in more general setting, which completes the description of \( Y \) if it is decomposable. By the Hirzebruch signature formula and the Riemann–Hurwitz formula, we have

\[
\sigma(X) = 2\sigma(Y) - X_\mathbb{R} \circ X_\mathbb{R},
\]
\[
\chi(X) = 2\chi(Y) - \chi(X_\mathbb{R}),
\]

where \( \sigma \) denotes signature and the self-intersection number \( X_\mathbb{R} \circ X_\mathbb{R} \) is taken in \( X \). We have also \( X_\mathbb{R} \circ X_\mathbb{R} = -\chi(X_\mathbb{R}) \), as it is pointed out in the proof of Corollary 4.8. This gives if \( b_1(X) = 0 \) and, hence, \( b_1(Y) = 0 \)

\[
b_2^+(Y) = \frac{1}{2}(b_2^+(X) - 1),
\]
\[
b_2^-(Y) = \frac{1}{2}(b_2^-(X)) + \chi(X_\mathbb{R}) - 1,
\]
One can also eliminate terms involving $X$ when $(X, \text{conj}^\pm)$ is obtained as the double covering over $(P, \text{conj})$, branched along curve $A$, by making use of the relations

$$\sigma(X) = 2\sigma(P) - A \circ A,$$
$$\chi(X) = 2\chi(P) - \chi(A).$$

We can express $b_2^\pm(Y^\pm)$ in terms of the zero divisor $B$ of $L$ (for notation see §5), by making use of adjunction formula, applied to $A$ and $B$ in $P$.

$$b_2^+(Y^\pm) = b_2^+(P) - \frac{1}{2}\chi(B) = b_2^+(P) + \frac{1}{2}(B + K) \cdot B,$$
$$b_2^-(Y^\pm) = b_2^-(P) + \frac{1}{2}(-\chi(B) + \chi(X_\mathbb{R}^\pm)) + d = b_2^-(P) + \frac{1}{2}(3B + K) \cdot B + \frac{1}{2}\chi(X_\mathbb{R}^\pm),$$

where $d = B \cdot B = \text{deg}(L)$ and $K$ is the canonical divisor on $P$.

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