NON-PRIME 3-MANIFOLDS WITH OPEN BOOK GENUS TWO

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ABSTRACT. An open book decomposition of a 3-manifold $M$ induces a Heegaard splitting for $M$, and the minimal genus among all Heegaard splittings induced by open book decompositions is called the *open book genus* of $M$. It is conjectured by Ozbagci [8] that the open book genus is additive under the connected sum of 3-manifolds. In this paper, we prove that a non-prime 3-manifold which has open book genus 2 is homeomorphic to $L(p, 1) \# L(q, 1)$ for some integers $p, q \neq \pm 1$, that is, it has non-trivial prime pieces of open book genus 1. In particular, there cannot be a counter-example to additivity of the open book genus such that the connected sum has open book genus 2.

INTRODUCTION

Throughout this paper, any 3-manifold is closed, connected and orientable. A pair $(B, \pi)$ is called an *(embedded)* open book decomposition of a 3-manifold $M$ if $B$ is a link in $M$ and $\pi : (M \setminus B) \to S^1 = [0, 1]/\sim$ is a fibration with fibers realizing $B$ as boundary. The link $B$ is called the *binding* of the open book, and the closure of each fiber of $\pi$ in $M$ is called a *page* of the open book. Note that the pages have the homeomorphism type of a compact surface $\Sigma$. If $(B, \pi)$ is an open book decomposition of $M$, then $H_1 = (\pi^{-1}([0, 1/2]) \cup B)$ and $H_2 = (\pi^{-1}([1/2, 1]) \cup B)$ are handlebodies homeomorphic to $\Sigma \times I$ embedded in $M$. Moreover, $(H_1, H_2)$ defines a Heegaard splitting of genus $g = g(\Sigma \times I) = 1 - \chi(\Sigma)$ for $M$. The *open book genus* of a 3-manifold $M$, denoted by $\text{obg}(M)$, is the minimum genus over all Heegaard splittings of $M$ induced by open book decompositions. It immediately follows from the definition that $g(M) \leq \text{obg}(M)$ for any 3-manifold $M$, where $g(M)$ is the Heegaard genus of $M$. A Heegaard splitting is not necessarily induced by an open book decomposition. Furthermore, $\text{obg}(M)$ does not necessarily equal $g(M)$. For example, “most” 3-manifolds of Heegaard genus 2 have greater open book genus [10]. We also have the following classification of 3-manifolds with open book genus 0 and 1 (see [8]).

**Proposition 1.** Let $M$ be a 3-manifold. Then $\text{obg}(M) = 0$ if and only if $M \cong S^3$, and $\text{obg}(M) = 1$ if and only if $M \cong L(p, 1)$ for some integer $p \neq \pm 1$.

A well-known corollary of Haken’s Lemma is that the Heegaard genus is additive under the connected sum of 3-manifolds. The following conjecture is stated by Ozbagci [8].

**Conjecture 2.** The open book genus is additive under the connected sum of 3-manifolds.
For $M = M_1 \# M_2$, proving that $\text{obg}(M) \leq \text{obg}(M_1) + \text{obg}(M_2)$ is straightforward. For $i = 1, 2$, take open book decompositions of $M_i$ with pages of homeomorphism type $\Sigma_i$ inducing Heegaard splittings of genus $1 - \chi(\Sigma_i) = \text{obg}(M_i)$. A plumbing of these open book decompositions is an open book decomposition for $M = M_1 \# M_2$ with pages homeomorphic to $\Sigma = \Sigma_1 + \Sigma_2$, which is obtained by gluing closed rectangles $R_i \subset \Sigma_i$ in a certain way (see [4] for details). It follows that $\chi(\Sigma) = \chi(\Sigma_1) + \chi(\Sigma_2) - 1$. Since a plumbing of the open books induces a Heegaard splitting of genus $1 - \chi(\Sigma)$ for $M$, we get
\[
\text{obg}(M) \leq 1 - \chi(\Sigma) = 1 - \chi(\Sigma_1) + 1 - \chi(\Sigma_2) = \text{obg}(M_1) + \text{obg}(M_2).
\]
However, it is still unknown whether the $\text{obg}$ is super-additive, equivalently additive, or not. In this paper, we prove that there cannot be a counter-example to the Conjecture 2 such that the connected sum has open book genus 2. More precisely, we prove the following.

**Theorem 3.** A non-prime 3-manifold $M$ has open book genus 2 if and only if each non-trivial connected summand of $M$ has open book genus 1, that is, $M \cong L(p, 1) \# L(q, 1)$ for some integers $p, q \neq \pm 1$.

An open book decomposition induces a genus 2 Heegaard splitting if and only if it has pages of Euler characteristic $-1$. Therefore, the pages are either once-punctured tori or pairs of pants. We will use different tools to deal with each case. In section 1, we recall the fact that a 3-manifold that has an open book decomposition with once-punctured torus pages is a double branched cover of $S^3$ along a 3-braid link, and we list some known facts about double branched covers and braid closures. In section 2, we analyze 3-manifolds which possess open book decompositions with pair of pants pages. We argue that if such a manifold is not prime, then the open book decomposition should be “simple”. Our tool in section 2 is the theory of Seifert fibered spaces. Finally, we prove Theorem 3 in section 3.

Note that the author of [8] refers to the open book genus as the *contact genus* to draw attention to the strong relation between open book decompositions and contact structures, namely the Giroux correspondence. However, we do not benefit from this relation and use classical tools from 3-manifolds topology.

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1. **Open Book Decompositions of Double Branched Covers**

In this section, we recollect useful facts from classical Knot Theory. Let us denote the double branched cover of $S^3$ along a link $L$ by $M(L)$. See [9] for definition and details. The topology of $M(L)$ is strongly related to properties of $L$. It is known that every link $L$ can be represented as a braid closure [1]. The minimum number of strands required to
represent $L$ as a braid closure is called the braid index of $L$, denoted by $b(L)$. Assume that $L$ is represented as a braid closure along $k$ strands with braid axis $A$. Since $A$ is the unknot, it is the binding of an open book decomposition $(A, p)$ of $S^3$, where $p$ is the projection map $(S^3 \setminus A) = \text{int}(D^2) \times S^1 \to S^1$. Let $D_t$ be the open disk $p^{-1}(t)$ and $f : M(L) \to S^3$ the branched covering map. Define $B = f^{-1}(A)$ and $\pi = p \circ f : (M(L) \setminus B) \to S^1$. It follows that $\pi$ is a fibration with fibers $\Sigma_t = f^{-1}(D_t)$ realizing $B = f^{-1}(A)$ as boundary, i.e., $(B, \pi)$ is an open book decomposition of $M(L)$. On the other hand, the braid representation of $L$ on $k$ strands intersects each page $D_t$ in $k$ points, and $\Sigma_t$ is a double branched cover of $D_t$ along these $k$ points. In other words, the pages of $(B, \pi)$ are homeomorphic to a surface $\Sigma$, which is a double branched cover of $D^2$ along $k$ points. It follows that $\chi(\Sigma) = 2 - k$, and hence the induced open book decomposition $(B, \pi)$ of $M(L)$ induces a Heegaard splitting of genus $1 - \chi(\Sigma) = k - 1$. This proves the following.

**Proposition 4.** If $L$ is a link in $S^3$ with $b(L) = k$, then $\text{obg}(M(L)) \leq k - 1$.

Since the braid index of a link $L$ suggests an upper bound for the open book genus of $M(L)$, it is worth stating the following result concerning the additivity of the braid index under connected sum.

**Theorem 5 ([3], The Braid Index Theorem).** The braid index is minus one additive under the connected sum operation, that is, $b(L_1 \# L_2) = b(L_1) + b(L_2) - 1$.

In the discussion above, since $\Sigma$ is a double branched cover of $D^2$ along $k$ points, it is a compact genus $g$ surface with $b$ boundary components, where

$$
(g, b) = \begin{cases} 
g = (k - 1)/2 \text{ and } b = 1, & \text{if } k \text{ is odd,} 
g = (k - 2)/2 \text{ and } b = 2, & \text{if } k \text{ is even.}
\end{cases}
$$

In particular, if $L$ is a 3-braid (respectively 2-braid) link, then $M(L)$ has an open book decomposition with once-punctured torus (respectively annulus) pages. On the other hand, any 3-manifold $M$ that has an open book with once-punctured torus pages is homeomorphic to $M(L)$ for some 3-braid link $L$ (see [2], section 2). Thus, we have the following.

**Theorem 6.** A 3-manifold $M$ has an open book decomposition with once-punctured torus pages if and only if it is homeomorphic to a double branched cover of $S^3$ along a 3-braid link $L$.

**Remark 7.** This statement does not generalize for $k$-braid links when $k \geq 4$. In particular, there are open book decompositions with connected bindings which are not obtained as double branched covers along braids. This can be argued more carefully, but we simply note that every 3-manifold has an open book with connected binding, whereas there are 3-manifolds, e.g. $S^1 \times S^1 \times S^1$, which are not double branched covers of $S^3$ [9].
We can say more about the topology of $M(L)$ looking at the link $L$. For example, it is known that $M(L_1 \# L_2) \cong M(L_1) \# M(L_2)$ because a sphere $S$ that realizes the connected sum $L_1 \# L_2$ in $S^3$ lifts to a sphere in $M(L_1 \# L_2)$ that realizes the connected sum $M(L_1) \# M(L_2)$. It is also known that $M(L) \cong S^3$ if and only if $L$ is the unknot \cite{7}. These two facts imply that $M(L)$ is a non-prime manifold whenever $L$ is a composite link since if $L = L_1 \# L_2$ for non-trivial links $L_i$, then $M(L) \cong M(L_1) \# M(L_2)$ for non-trivial 3-manifolds $M(L_i)$. The converse is also true.

**Theorem 8** (\cite{6}, Corollary 4). A (non-split) link $L$ is prime if and only if $M(L)$ is prime.

### 2. Open Book Decompositions with Pair of Pants Pages

We can define open book decompositions in an abstract way. Let $\Sigma$ be a compact surface with boundary and $\phi$ an orientation-preserving self-homeomorphism of $\Sigma$, called a monodromy, such that $\phi|_{\partial \Sigma}$ is the identity map. Take the mapping torus $M_\phi$ and perform vertical Dehn fillings on $\partial M_\phi$, i.e., glue a solid torus $S^1 \times D^2$ to each torus component of $\partial M_\phi = \partial \Sigma \times S^1$ via a homeomorphism identifying a meridian $\{\ast\} \times S^1$ to $\{p\} \times S^1$ for some $p \in \partial \Sigma$. The resulting space forms a 3-manifold $M$ with an embedded open book decomposition such that the binding is the cores of the glued solid tori, and the pages are homeomorphic to $\Sigma$. The pair $(\Sigma, \phi)$ is called an abstract open book decomposition of $M$. Every embedded open book decomposition of a 3-manifold $M$ can be viewed as an abstract open book decomposition as well \cite{4}. We use the notation $M = (\Sigma, \phi)$. Note that isotoping or conjugating $\phi$ does not change the homeomorphism type of $M$.

An abstract open book decomposition $(\Sigma, \phi)$ of a 3-manifold $M$ defines an extrinsic Heegaard splitting for $M$. Namely, we take two copies of the handlebody $\Sigma \times I$ and glue them along their boundaries $\partial(\Sigma \times I) = \Sigma_0 \cup (\partial \Sigma \times I) \cup \Sigma_1$ via the homeomorphism $f$ defined by $f|_{\Sigma_0} = \text{id}$, $f|_{\partial \Sigma \times I} = \text{id}$ and $f|_{\Sigma_1} = \phi$, where $\Sigma_t$ denotes $\Sigma \times \{t\}$. This can be seen as follows. When we glue $\Sigma_0$’s via the id map, we obtain the trivial interval bundle over $\Sigma$. Then, gluing $\Sigma_1$’s via $\phi$, we obtain the mapping torus $M_\phi$. Finally, vertical Dehn fillings on $\partial M_\phi$ can be recovered by identifying $\partial \Sigma \times I$’s via the identity map. Hence, the open book decomposition and the Heegaard splitting form homeomorphic 3-manifolds.

![Figure 1. Generators of Mod(P).](image.png)

Some 3-manifolds with open book genus 2 can be obtained as abstract open book decompositions with pages homeomorphic to a pair of pants $P$. The mapping class group
\( \text{Mod}(P) \) of \( P \) is homomorphic to \( \mathbb{Z}^3 \), where the generators are the Dehn twists \( t_i \) about the curves \( c_i \) in Figure 1. Therefore, any given monodromy \( \phi \) of \( P \) can be written as a product of powers of \( t_i \) up to isotopy, that is, \( \phi \) can be taken to be \( t_1^{r_1} t_2^{r_2} t_3^{r_3} \) for some integers \( r_i \). In this section, we will argue that possible values of \( r_i \) are pretty restricted if the 3-manifold \( M = (P, \phi) \) is not prime.

**Lemma 9.** If no \( r_i \) equals 0, then \( M = (P, \phi) \) is a Seifert fibered space.

Apparently, this lemma is known to experts (see [8]), however, the author could not find a proof in the literature. We present a proof here.

**Proof.** Let \( d_1, d_2, d_3 \) be the boundary components of \( P \) parallel to the curves \( c_1, c_2, c_3 \) in Figure 1. For \( i = 1, 2, 3 \), let \( a_i \) be an embedded curve in \( P \) so that \( a_i \) and \( d_i \) bound an annulus \( A_i \) with core \( c_i \). Finally, let \( b_i \) be a spanning arc for \( A_i \), and \( p_i \) the endpoint of \( b_i \) on \( d_i \) as in Figure 2.

When we remove the annuli \( A_i \) from \( P \), we obtain another pair of pants \( P' \subset P \). Notice that \( \phi|_{P'} \) is the identity map, and so \( P' \times S^1 \) is embedded in the mapping torus \( M_\phi \). We will analyze how the vertical Dehn fillings on the boundary components \( d_i \times S^1 \) of \( M_\phi \) are realized on the boundary components \( a_i \times S^1 \) of \( P' \times S^1 \). Let \( M_i \) be the component of \( M_\phi \setminus (P' \times S^1) \) with boundary \( (d_i \times S^1) \cup (a_i \times S^1) \). In other words, \( M_i \) is the mapping torus of the monodromy \( t_i^{r_i} \) on the annulus \( A_i \), where \( t_i \) is the Dehn twist about \( c_i \).

**Claim.** There is a properly embedded annulus \( B_i \) in \( M_i \) such that \( B_i \cap (d_i \times S^1) = \{p_i\} \times S^1 \) and \( B_i \cap (a_i \times S^1) \) is an \((r_i, 1)\)-curve on the torus \( a_i \times S^1 \).

**Proof of the claim.** Take the rectangle \( R_i = b_i \times I \) in \( A_i \times I \), and let \( q_i \) be the vertex \((a_i \cap b_i) \times \{1\}\) of \( R_i \). Rotate \( q_i \) on \( a_i \times \{1\}\) for \( r_i \) times in the positive (negative) direction if \( r_i \) is positive (negative), while keeping the other vertices of \( \partial R_i \) fixed. In Figure 3a, we show \( R_i \) before rotation, and in Figure 3b, we depict \( \partial R_i \) after rotation in the \( r_i = 2 \) case. In particular, \( R_i \cap (A_i \times \{1\}) \) is the image of \( b_i \) under the monodromy \( t_i^{r_i} \), and \( R_i \cap (A_i \times \{0\}) \)
is the $b_i$ itself. Since $M_i$ is obtained from $A_i \times I$ by identifying $R_i \cap (A_i \times \{0\}) = b_i \times \{0\}$ with $R_i \cap (A_i \times \{1\}) = t_i^0(b_i) \times \{1\}$, then $R_i$ turns into a properly embedded annulus $B_i$ in $M_i$. By construction, the boundary of $B_i$ on $d_i \times S^1$ is $\{p_i\} \times S^1$, and the boundary of $B_i$ on $a_i \times S^1$ is an $(r_i, 1)$-curve. This completes the proof of the claim.

![Figure 3. Detecting the annulus $B_i$.](image)

Now, when we perform vertical Dehn fillings on $\partial M_\phi$, we glue a solid torus $S^1 \times D^2$ to each boundary component $d_i \times S^1$ by attaching a meridional disk $D = \{\ast\} \times D^2$ of the solid torus to $\{p_i\} \times S^1$. On the boundary component $a_i \times S^1$ of $\partial (P' \times S^1)$, this is realized as attaching the disk $D \cup_{\{p_i\} \times S^1} B_i$ to the non-infinity slope $(r_i, 1)$. Since the result of Dehn fillings along non-infinity slopes $(r_i, 1)$ on the components $a_i \times S^1$ of $\partial (P' \times S^1)$ is the Seifert fibered space $M(+0; 1/r_1, 1/r_2, 1/r_3)$, the result follows.

The proof suggests that when no $r_i$ is zero, $M = (P, \phi)$ has a Seifert fibration over the base space $S^2$ with three cone points of multiplicities $|r_i|$, possibly 1. We can prove the following statement using the theory of Seifert fibered spaces.

**Lemma 10.** If $M = (P, \phi)$ is not prime for $\phi = t_1^{r_1} t_2^{r_2} t_3^{r_3}$, one of the $r_i$’s should be 0.

**Proof.** Assume that no $r_i$ equals 0, so $M$ is Seifert fibred as above. Let $\pi : M \to S^2$ be the projection map of this fibration. We will prove that $M$ is either irreducible or homeomorphic to $S^2 \times S^1$, hence it is prime. If $M$ is reducible, then there exists an essential sphere $S$ in $M$ which intersects each Seifert fiber transversely, and hence $\pi|_S : S \to S^2$ is a branched cover along three cone points with multiplicities $|r_i|$ \[5\]. Hence, we obtain $\chi(S^2) - \chi(S)/n = \sum_{i=1}^3 (1 - 1/|r_i|)$ for $n$ the degree of the cover. As $\chi(S) = \chi(S^2) = 2$, we get $|r_i| = 1$ for each $i$, and $n = 1$. This implies that $S$ intersects each fiber once, and $M \setminus N(S)$ is homeomorphic to $S \times I$. In other words, $M$ is an orientable $S^2$ bundle over $S^1$, and thus $M$ is homeomorphic to $S^2 \times S^1$. \[\square\]
3. PROOF OF THEOREM 3

The if direction of Theorem 3 is straightforward. If \(M_1, M_2\) are 3-manifolds with open book genus 1, then they have Heegaard genus 1. Hence, \(g(M) = 2\) for \(M = M_1 \# M_2\). It follows that \(2 = g(M) \leq \text{obg}(M) \leq 2\), by the subadditivity of the open book genus. Therefore, we obtain \(\text{obg}(M) = 2\).

For the only if direction of Theorem 3, let \(M\) be a non-prime 3-manifold with open book genus 2. Pick an open book decomposition of \(M\) inducing a genus 2 Heegaard splitting. The pages of such an open book have Euler characteristic \(-1\), hence they are either once-punctured tori or pairs of pants. We analyze each case separately.

**Case 1.** \(M\) has an open book decomposition with once-punctured torus pages.

By Theorem 6, \(M\) is homeomorphic to a double branched cover of \(S^3\) along a 3-braid link \(L\). Theorem 8 implies that \(L\) is a composite link, that is, \(L = L_1 \# L_2\) for some non-trivial links \(L_i\), since \(M\) is not prime. On the other hand, by Proposition 4, \(b(L)\) must be 3 since otherwise \(\text{obg}(M(L))\) would be less than 2. Finally, it follows from Theorem 5 that \(b(L) = 3 = b(L_1) + b(L_2) - 1\) for braid indices, so \(b(L_1) = b(L_2) = 2\). Therefore, we get \(M \cong M(L_1 \# L_2) \cong M(L_1) \# M(L_2)\), which implies that the prime pieces \(M(L_1), M(L_2)\) of \(M\) are double branched covers of \(S^3\) along 2-braid links. The coverings suggest open book decompositions with annulus pages inducing genus 1 Heegaard splittings for \(M(L_1)\) and \(M(L_2)\). Hence, the result follows.

**Case 2.** \(M\) has an open book decomposition with pair of pants pages.

We can see \(M\) as an abstract open book \((P, \phi)\) for \(P\) the pair of pants with monodromy \(\phi = t_1^{r_1} t_2^{r_2} t_3^{r_3}\), where \(t_i\) are the Dehn twists about the curves \(c_i\) in Figure 1. By Lemma 10, one of the \(r_i\)’s is zero because \(M\) is not prime. Assume that \(r_3 = 0\), so \(\phi = t_1^{r_1} t_2^{r_2}\). We now analyze the Heegaard splitting induced by \((P, \phi)\).

![Figure 4. The induced Heegaard splitting when \(r_1 = 2\) and \(r_2 = -2\) is homeomorphic to \(L(2, 1) \# L(-2, 1)\).](image)

Take two distinct copies \(H_1, H_2\) of \(P \times I\). Let \(\alpha \subset P\) be a properly embedded essential arc with endpoints in \(d_3\) such that \(\alpha\) intersects neither \(c_1\) nor \(c_2\), hence \(\phi\) fixes \(\alpha\) pointwise. Let \(\alpha_1, \alpha_2 \subset P\) be properly embedded non-separating arcs which do not intersect \(\alpha\) and cut...
into a disk. Now take the vertical disks $D_1 = \alpha \times I$ and $D_2 = \alpha_2 \times I$ in $H_1$ as in Figure 4. The disks $D_1, D_2$ cut $H_1$ into a 3-ball, and hence the resulting 3-manifold is uniquely determined by where $\partial D_1, \partial D_2$ are mapped in $\partial H_2$ under the Heegaard map $f : \partial H_1 \to \partial H_2$ defined by $f|_{P_0} = \text{id}, f|_{\partial P \times I} = \text{id}$ and $f|_{P_1} = \phi$, where $P_1 = P \times \{t\}$. In Figure 4 we depict $f(\partial D_1), f(\partial D_2) \subset \partial H_2$ in the case $r_1 = 2$ and $r_2 = -2$. Figure 4 is suggestive, and in general, two distinct copies of $D$ glue along their boundaries to form a sphere $S \subset M$ that realizes the connected sum $L(r_1, 1) \# L(r_2, 1)$. Finally, note that $r_1, r_2 \neq \pm 1$. Otherwise, one of $L(r_i, 1)$ would be $S^3$, and $M$ would be homeomorphic to either $S^3$, $S^1 \times S^2$, or a lens space. Thus, the connected summands of $M$ have open book genus 1.

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