A REMARK ON EINSTEIN WARPED PRODUCTS

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Abstract. We prove triviality results for Einstein warped products with non-compact bases. These extend previous work by D.-S. Kim and Y.-H. Kim. The proofs, from the viewpoint of “quasi-Einstein manifolds” introduced by J. Case, Y.-S. Shu and G. Wei, rely on maximum principles at infinity and Liouville-type theorems.

1. Introduction

The main purpose of this note is to prove the following triviality result for Einstein warped products which extends, to the case of non-compact bases, a recent theorem by D.-S. Kim and Y.-H. Kim, [7].

Theorem 1. Let \( N^{n+m} = M^n \times_u F^m, m > 1 \), be a complete Einstein warped product with non-positive scalar curvature \( \overline{N}S \leq 0 \), warping function \( u(x) = e^{-\frac{f(x)}{m}} \) satisfying \( \inf_M f = f_* > -\infty \) and complete Einstein fibre \( F \). Then \( N \) is simply a Riemannian product if either one of the following further conditions is satisfied:

(a) \( f \) has a local minimum.

(b) the base manifold \( M \) is complete and non-compact, the warping function satisfies \( \int_M |f|^p e^{-\frac{f}{m}} \, dv_{\text{vol}} < +\infty \), for some \( 1 < p < +\infty \), and \( f(x_0) \leq 0 \) for some point \( x_0 \in M \).

Note that, in case \( M \) is compact, from the point (a) we recover the main result in [7].

Our proof of Theorem 1 will rely on the link between Einstein warped product metrics and the so called “quasi-Einstein metrics” recently introduced by J. Case, Y.-S. Shu and G. Wei, [3]. In the spirit of [13], i.e. using methods from stochastic analysis and \( L^p \)-Liouville type theorems, we shall prove scalar curvature estimates and triviality results for a complete quasi-Einstein manifold that largely extend previous theorems in [3]. Whence, the main theorem will follow immediately.

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In a final section, using similar techniques, we extend another triviality result for Einstein warped products obtained in the very recent [2]. A non-existence result is also discussed.

2. QUASI-EINSTEIN MANIFOLDS

Consider the weighted manifold \((M^n, g_M, e^{-f} d\text{vol})\), where \(M\) is a complete \(n\)-dimensional Riemannian manifold, \(f\) is a smooth real valued function on \(M\) and \(d\text{vol}\) is the Riemannian volume density on \(M\). A natural extension of the Ricci tensor to weighted manifolds is the \(m\)-Bakry-Emery Ricci tensor

\[ \text{Ric}_f^m = \text{Ric} + \text{Hess} f - \frac{1}{m} df \otimes df, \quad \text{for} \quad 0 < m \leq \infty. \]

When \(f\) is constant, this is the usual Ricci tensor and when \(m = \infty\) this is the Ricci Bakry-Emery tensor \(\text{Ric}_f\). We call a metric \(m\)-quasi-Einstein if the \(m\)-Bakry-Emery Ricci tensor satisfies the equation

\[ \text{Ric}_f^m = \lambda g_M, \]

for some \(\lambda \in \mathbb{R}\). This equation is especially interesting in that when \(m = \infty\) it is exactly the gradient Ricci soliton equation. When \(f\) is constant, it gives the Einstein equation and we call the quasi-Einstein metric trivial. When \(m\) is a positive integer, it corresponds to warped product Einstein metrics.

Indeed, in [3], elaborating on [7], it is observed the following characterization of quasi-Einstein metrics.

**Theorem 2.** Let \(M^n \times_u F^m\) be an Einstein warped product with Einstein constant \(\lambda\), warping function \(u = e^{-f} \) and Einstein fibre \(F^m\). Then the weighted manifold \((M^n, g_M, e^{-f} d\text{vol})\) satisfies the quasi-Einstein equation (1). Furthermore the Einstein constant \(\mu\) of the fibre satisfies

\[ \Delta f - |\nabla f|^2 = m\lambda - m\mu e^{\frac{m}{m}f}. \]

Conversely if the weighted manifold \((M^n, g_M, e^{-f} d\text{vol})\) satisfies (1), then \(f\) satisfies (2) for some constant \(\mu \in \mathbb{R}\). Consider the warped product \(N^{n+m} = M^n \times_u F^m\), with \(u = e^{-f}\) and Einstein fibre \(F\) with \(F\text{Ric} = \mu g_F\). Then \(N\) is Einstein with \(N\text{Ric} = \lambda g_N\).

3. SCALAR CURVATURE ESTIMATES

In this section, in the same spirit of Theorem 3 of [13], we generalize the scalar curvature estimates in Proposition 3.6 of [3] to quasi-Einstein manifolds with non-constant scalar curvature. Possible rigidity at the endpoints is also discussed.
Theorem 3. Let \((M^n, g_M, e^{-f}dvol)\) be a geodesically complete \(m\)-quasi-Einstein manifold, \(1 < m < +\infty\), with scalar curvature \(S\) and let \(S_* = \inf_M S\).

(a) If \(\lambda > 0\), then \(M\) is compact and

\[
\frac{n(n-1)}{m+n-1}\lambda < S_* \leq n\lambda.
\]

Moreover \(S_* \neq n\lambda\) unless \(M\) is Einstein.

(b) If \(\lambda = 0\) and \(\inf_M f = f_* > -\infty\) then \(S_* = 0\). Moreover, either \(S > 0\) or \(S(x) \equiv 0\). In this latter case, either \(f\) is constant (and \(M\) is trivial) or \(M\) is isometric to the Riemannian product \(\mathbb{R} \times \Sigma\) where \(\Sigma\) is a Ricci-flat, totally geodesic hypersurface.

(c) If \(\lambda < 0\) and \(\inf_M f = f_* > -\infty\), then

\[
n\lambda \leq S_* \leq \frac{n(n-1)}{m+n-1}\lambda
\]

and \(S(x) > n\lambda\) unless \(M\) is Einstein.

The proof of Theorem 3 will require the following formula obtained in [3], which generalizes to the case \(m < +\infty\) similar formulas for Ricci solitons \((m = +\infty)\) obtained previously by P. Petersen and W. Wylie, [12]. Following the terminology introduced in [11], the \(f\)-Laplacian on the weighted manifold \((M, g_M, e^{-f}dvol)\) is the diffusion type operator defined by \(\Delta f u = e^f \text{div}(e^{-f}\nabla u)\). It is clearly a symmetric operator on \(L^2(M, e^{-f}dvol)\).

Lemma 4. Let \(\text{Ric}_f^m = \lambda g_M\), for some \(\lambda \in \mathbb{R}\) and \(m < +\infty\). Set \(\bar{f} = \frac{m+2}{m} f\). Then

\[
\frac{1}{2} \Delta \bar{f} S = -\frac{m-1}{m} |\text{Ric} - \frac{1}{n} S g_M|^2 - \frac{m+n-1}{mn} (S - n\lambda)(S - \frac{n(n-1)}{m+n-1}\lambda).
\]

Proof (of Theorem 3). First of all, we show that \(\inf_M S > -\infty\). According to Qian version of Myers’ theorem this is obvious if \(\lambda > 0\) because \(M\) is compact, see also the Appendix. In the general case \(\lambda \in \mathbb{R}\) we proceed as follows. Since

\[
-|\text{Ric} - \frac{1}{n} S g_M|^2 = -|\text{Ric}|^2 + \frac{S^2}{n},
\]

from (5) we obtain

\[
\frac{1}{2} \Delta \bar{f} S = -\frac{m-1}{m} |\text{Ric}|^2 - \frac{1}{m} S^2 + \frac{m+2n-2}{m} S\lambda - \frac{n(n-1)}{m} \lambda^2.
\]

\[
\leq -\frac{1}{m} S^2 + \frac{m+2n-2}{m} \lambda S.
\]
Let \( S_- (x) = \max \{-S(x), 0\} \). Then

\[
\Delta \tilde{f} S_- \geq 2 \frac{S_-^2}{m} + \frac{2(m + 2n - 2)}{m} \lambda S_-.
\]

Observe now that from Qian’s estimates of weighted volumes ([14], see also section 2 in [8] and references therein), since \( \text{vol}_f (B_r) \leq e^{-\frac{1}{2} \tilde{f}} \text{vol}_f (B_r) \), we can apply the “a-priori” estimate in Theorem 12 of [13] to inequality (7) on the complete weighted manifold \((M, g_M, e^{-\tilde{f}} d\text{vol})\) and we obtain that \( S_- \) is bounded from above, or equivalently, \( S_\ast = \inf_M S > -\infty \). Again from the volume estimates in [14] and by Theorem 9 in [13] applied to \((M, g_M, e^{-\tilde{f}} d\text{vol})\), the weak maximum principle at infinity for the \( \tilde{f} \)-laplacian holds on \( M \). This produces a sequence \( \{x_k\} \) such that \( \Delta \tilde{f} S (x_k) \geq -\frac{1}{k} \) and \( S(x_k) \to S\ast \). Taking the \( \lim \inf \) in (5) along \( \{x_k\} \) shows that, for \( m > 1 \),

\[
0 \leq -\frac{m + n - 1}{mn} (S\ast - n\lambda)(S - \frac{n(n-1)}{m + n - 1}) \lambda
\]

We now distinguish three cases.

(a) Assume \( \lambda > 0 \), so that \( M \) is compact. Equation (5) yields \( \frac{n(n-1)}{m + n - 1} \lambda \leq S\ast \leq n \lambda \). Assume now that \( S\ast = n \lambda > 0 \). Then \( S \geq n \lambda \geq \frac{n(n-1)}{m + n - 1} \lambda \) and from (5) we get

\[
\frac{1}{2} \Delta \tilde{f} S \leq -\frac{m + n - 1}{mn} (S - n\lambda)(S - \frac{n(n-1)}{m + n - 1}) \leq 0.
\]

Since \( M \) is compact, \( S \) must be constant. Hence \( S = S\ast = n \lambda \). Substituting in (5) we obtain that \( \text{Ric} = \frac{1}{n} S g_M \) and thus that \( M \) is Einstein.

Now we show that \( S\ast > \frac{n(n-1)}{m + n - 1} \lambda \). Indeed, suppose that \( S \) attains its minimum \( \frac{n(n-1)}{m + n - 1} \lambda \). Since the non-negative function \( v(x) = S(x) - \frac{n(n-1)}{m + n - 1} \lambda \) satisfies

\[
\frac{1}{2} \Delta \tilde{f} v \leq -\frac{m + n - 1}{mn} v^2 + \lambda v \leq +\lambda v,
\]

and \( v \) attains its minimum \( v(x_0) = 0 \), it follows from the minimum principle, (see p. 35 in [6]), that \( v \) vanishes identically. Hence \( S \equiv \frac{n(n-1)}{m + n - 1} \lambda \) is constant and, substituting in (5), we get that \( M \) is Einstein with

\[
\text{Ric} = \frac{n-1}{m + n - 1} \lambda g_M.
\]

Using this information into (1) we obtain that

\[
\text{Hess}(f) = \frac{1}{m} |\nabla f|^2 + \frac{m}{m + n - 1} \lambda g_M > 0.
\]

But this is clearly impossible because \( M \) is compact.
(b) Assume $\lambda = 0$. From (8) we conclude that $S = 0$. Note that, according to (5), $\Delta \tilde{f} f S \leq 0$. Therefore, either $f$ is constant and $M$ is Einstein, or the non constant function $u = e^{-\frac{f}{m}}$ satisfies $\text{Hess}(u) = 0$. A Cheeger-Gromoll type argument now shows that $M$ is isometric to the Riemannian product $\mathbb{R} \times \Sigma$ along the Ricci flat, totally geodesic hypersurface $\Sigma$ of $M$.

(c) Assume $\lambda < 0$. From (8) we deduce that $n\lambda \leq \Delta \tilde{f} f S \leq n(n-1) + n - 1 \lambda$. Suppose that $S(x_0) = n\lambda < 0$ for some $x_0 \in M$. Since the non-negative function $w(x) = S(x) - n\lambda$ satisfies

$$\frac{1}{2} \Delta \tilde{f} w \leq -\frac{m+n-1}{mn} w^2 - \lambda w \leq -\lambda w,$$

and $w$ attains its minimum $w(x_0) = 0$, it follows from the minimum principle that $w$ vanishes identically. Hence $S \equiv n\lambda$ is constant and substituting in (5) we get that $M$ is Einstein. □

4. Triviality results under $L^p$ conditions

It is well known that steady or expanding compact Ricci solitons are necessarily trivial. The same result is proven in [7] for quasi-Einstein metrics on compact manifolds with finite $m$. For Ricci solitons a generalization to the complete non-compact setting is obtained in [13].

In this section using the scalar curvature estimates of Theorem 3, we get triviality for (not necessarily compact) quasi-Einstein metrics with $m < +\infty$, $\lambda \leq 0$.

**Theorem 5.** Let $(M^n, g_M, e^{-f}d\text{vol})$ be a geodesically complete non-compact $m$-quasi-Einstein manifold, $1 \leq m < +\infty$. If the quasi-Einstein constant $\lambda$ is non-positive and $f$ satisfies, for some $1 < p < +\infty$,

$$f \in L^p(M, e^{-\frac{f}{m}}d\text{vol}),$$

and $\inf_M f = f_* > -\infty$, then either $f \equiv \text{const} \leq 0$ and $M$ is Einstein or $f > 0$.

**Proof.** (of Theorem 5) Tracing (1) and letting $\hat{f} = \frac{1}{m} f$ we have that

$$\Delta \hat{f} = n\lambda - S.$$

Since $\lambda \leq 0$ and $f_* > -\infty$, from (4) of Theorem 3 we obtain that $\Delta \hat{f} \leq 0$.

Applying Theorem 14 in [13] to $f_- = \max\{-f, 0\} \in L^p(M, e^{-\hat{f}}d\text{vol})$, gives that $f_-$ is constant. Hence, if there exists a point $x_0 \in M$ such that $f(x_0) \leq 0$ then $f \equiv f(x_0) \leq 0$. □
Remark 6. From the proof it follows that if either $M$ is compact or $f$ attains its absolute minimum then $f \equiv \text{const}$. Actually, it was pointed out to us by Dezhong Chen that the same conclusion holds if we merely assume that $f$ attains a local minimum at some point $x_0 \in M$. Indeed the following proposition holds.

Proposition 7. Let $(M, g_M, e^{-f}d\text{vol})$ be a geodesically complete non-compact $m$-quasi-Einstein manifold, $1 < m < +\infty$. If the quasi-Einstein constant $\lambda$ is non positive and $f$ satisfies $f_* > -\infty$, then any local minimum of $f$ is actually an absolute minimum.

Proof. Assume that $f$ attains a local minimum $x_0 \in M$. Evaluating (10) at $x_0$, we get

$$S(x_0) \leq n\lambda.$$ 

Hence, since $\lambda \leq 0$, by Theorem 3, $M$ is Einstein and $S$ is identically $n\lambda$. Thus the quasi-Einstein equation (11) reads

$$\text{Hess}(f) = \frac{1}{m}df \otimes df.$$ 

In particular $\text{Hess}(f)$ is positive semi-definite on $M$ and this implies the thesis.

5. Proof of the main theorem

Putting together the results of the previous sections we easily obtain a proof of Theorem 1.

Indeed, according to Theorem 2, $M$ is quasi-Einstein. Statement (a) follows immediately from Remark 6 and Proposition 7. In case (b), since $(n + m)\lambda = NS \leq 0$, we get by Theorem 5 that $f$, and so $u$, is a constant function.

6. Other triviality results

Another triviality result for Einstein warped products has been obtained by J. Case in [2].

Theorem 8. (Case) Let $N^{n+m} = M^n \times_u F^m$ be a complete warped product with warping function $u(x) = e^{-\frac{f(x)}{m}}$, scalar curvature $NS \geq 0$ and complete Einstein fibre $F$. Then $N$ is simply a Riemannian product provided the base manifold $M$ is complete and the scalar curvature of $F$ satisfies $FS \leq 0$.

In the following theorem we obtain the same conclusion in case the fibers have non-negative scalar curvature, up to assume an integrability condition on the warping function $u$. We observe that non-trivial examples with $NS \leq 0$ and $FS \geq 0$ are constructed in ([1], 9.118). Thus the integrability assumption is necessary.
Theorem 9. Let $N^{n+m} = M^n \times_u F^m$ be a complete Einstein warped product with warping function $u(x) = e^{-\frac{f(x)}{m}}$, scalar curvature $N^S \leq 0$, and complete Einstein fibre $F$. Then $N$ is simply a Riemannian product provided the base manifold $M$ is complete, the warping function satisfies $\int_M e^{-\frac{f(x)}{m}}d\text{vol} < +\infty$ for some $1 < p < +\infty$, and the scalar curvature of $F$ satisfies $F^S \geq 0$. In this case $M$ and $F$ are Ricci flat and $M$ is compact.

Combining Theorem 8 and Theorem 9 immediately gives the following

Corollary 10. Let $N$ be a complete Ricci flat warped product with complete Einstein fibre $F$ and warping function $u(x) = e^{-\frac{f(x)}{m}}$ satisfying $u \in L^p(M, e^{-f}d\text{vol})$, for some $1 < p < +\infty$. Then $N$ is simply a Riemannian product.

Proof. (of Theorem 9) Just observe that computing the $f$-laplacian of $u$ and using (2) one obtains the following equation

$$\Delta_f u = \mu u^{-1} - \lambda u + \frac{u}{m^2} |\nabla f|^2.$$  

Thus, in our assumptions, we obtain that $\Delta_f u \geq 0$. Since $0 < u \in L^p(M, e^{-f}d\text{vol})$, by Theorem 14 in [13], we obtain the constancy of $u$. Up to a rescaling of the metric of $F$ we can suppose $u = 1$.

Now, since the Riemannian product $M \times F$ is Einstein, both $M$ and $F$ are Einstein manifolds with the same Einstein constant. In particular, $M^S$ and $F^S$ have the same sign. By our assumption on the signs of $N^S$ and $F^S$ we thus obtain that both $M$ and $F$ are Ricci flat. Finally, since $u$ (and thus $f$) is constant, from the integrability condition we obtain that $\text{vol}(M) < +\infty$. Thus, by a result of Calabi-Yau, [14], we obtain that $M$ must be compact. □

We end this section with a non-existence result. Recall that by the volume estimates in [14] and Theorem 9 in [13] the weak maximum principle for the $f$-laplacian holds on $(M, g_M, e^{-f}d\text{vol})$ provided $\text{Ric}_f^M = \lambda g_M$ for some $\lambda \in \mathbb{R}$, $m < +\infty$.

Theorem 11. There is no complete Einstein warped product $N = M^n \times_u F^m$ with warping function $u = e^{-\frac{f}{m}} \in L^\infty(M)$, scalar curvature $N^S < 0$ and Einstein fibre $F$ with $F^S \geq 0$.

Proof. Since $m\mu = F^S \geq 0$, from (12), we have that

$$\Delta_f u \geq -u\lambda.$$  

Since, by assumption, $u$ satisfies $\sup_M u = u^* < +\infty$, by the weak maximum principle at infinity for the $f$-laplacian, there exists a sequence $\{x_k\} \subset M$ along which $u(x_k) \geq u^* - \frac{1}{k}$ and $\Delta_f u(x_k) \leq \frac{1}{F}$. Thus evaluating (13) along $\{x_k\}$ and taking the limit as $k \to +\infty$ we obtain that $\lambda u^* \geq 0$ and since $u^* > 0$ we cannot have $\lambda < 0$. □
Appendix

An extension of Myers’ theorem to weighted manifolds with a positive lower bound on the $m$-Bakry-Emery Ricci tensor ($m$ finite) is obtained by Qian in [14]. For generalizations of Myers’ theorem in a different direction see [10].

In this section we extend Qian theorem by allowing some negativity of the $m$-Bakry-Emery Ricci tensor. The starting point of our considerations is the following Bochner formula for the $m$-Bakry-Emery Ricci tensor; see e.g. [15].

Let $u : M^n \to \mathbb{R}$ be a smooth function on a complete weighted manifold $(M^n, g_M, e^{-f}d\text{vol})$ then

$$\frac{1}{2} \Delta_f |\nabla u|^2 = |\text{Hess}(u)|^2 + g_M(\nabla u, \nabla \Delta_f u) + \text{Ric}^m_f(\nabla u, \nabla u) + \frac{1}{m} \left| g_M(\nabla f, \nabla u) \right|^2. \quad (14)$$

Using this formula one obtains the following generalization of a well-known lemma which estimate the integral of Ricci along geodesics. The proof is modelled on [14].

**Lemma 12.** Let $(M^n, g_M, e^{-f}d\text{vol})$ be a complete weighted manifold, and consider the $m$-Bakry-Emery Ricci tensor $\text{Ric}^m_f$ for $m$ finite. Fix $o \in M$ and let $r(x) = \text{dist}(x, o)$. For any point $q \in M$, let $\gamma_q : [0, r(q)] \to M$ be a minimizing geodesic from $o$ to $q$ such that $|\dot{\gamma}_q| = 1$. If $h \in \text{Lip}_{\text{loc}}(\mathbb{R})$ is such that $h(0) = h(r(q)) = 0$, then for every $q \in M$, it holds

$$0 \leq \int_0^{r(q)} (m + n - 1) (h')^2 \, ds - \int_0^{r(q)} h^2 \text{Ric}^m_f(\dot{\gamma}_q, \dot{\gamma}_q) \, ds. \quad (15)$$

**Proof.** Fix a point $q \notin \text{cut}(o)$. Straightforward computations show that

$$\frac{(\Delta_f r)^2}{m + n - 1} \leq \left( \frac{(\Delta r)^2}{n-1} + \frac{|g_M(\nabla f, \nabla r)|^2}{m} \right), \quad (16)$$

$$|\text{Hess}(r)|^2 \geq \frac{(\Delta r)^2}{n-1}. \quad (17)$$

Using (16) and (17), from the Bochner formula (14) applied to the distance function $r(x)$ we obtain that

$$0 \geq \frac{(\Delta_f r)^2}{m + n - 1} + g_M(\nabla r, \nabla \Delta_f r) + \text{Ric}^m_f(\nabla r, \nabla r).$$

Evaluating this along a minimizing geodesic $\gamma_q$ such that $|\dot{\gamma}_q| = 1$, we get

$$0 \geq \frac{(\Delta_f r \circ \gamma_q)^2}{m + n - 1} + \frac{d}{ds}(\Delta_f(r \circ \gamma_q)) + \text{Ric}^m_f(\dot{\gamma}_q, \dot{\gamma}_q). \quad (18)$$
If $h \in \text{Lip}_{\text{loc}}(\mathbb{R})$, $h \geq 0$, $h(0) = 0$, multiplying (18) by $h^2$ and integrating on $[0, t]$, we obtain

$$0 \geq \int_0^t h^2 \frac{(\Delta f \circ \gamma_q)^2}{m+n-1} \, ds + \int_0^t \frac{d}{ds} (\Delta f \circ \gamma_q) h^2 \, ds + \int_0^t h^2 \text{Ric}^m_f (\dot{\gamma}_q, \dot{\gamma}_q).$$

Since $(\Delta f \circ \gamma_q) h^2 \to 0$ as $r \to 0$, integrating by parts we have that

$$0 \geq \int_0^t h^2 \frac{(\Delta f \circ \gamma_q)^2}{m+n-1} \, ds + \int_0^t h^2 (\Delta f \circ \gamma_q)(t) \, ds - 2 \int_0^t h h'(\Delta f \circ \gamma_q) \, ds + \int_0^t h^2 \text{Ric}^m_f (\dot{\gamma}_q, \dot{\gamma}_q) \, ds.$$

Since

$$-2 h h'(\Delta f \circ \gamma_q) \geq -\frac{h^2 (\Delta f \circ \gamma_q)^2}{m+n-1} - (m + n - 1)(h')^2,$$

we deduce that

$$0 \geq h^2 (\Delta f \circ \gamma_q) - \int_0^t (m + n - 1)(h')^2 \, ds + \int_0^t \text{Ric}^m_f (\dot{\gamma}_q, \dot{\gamma}_q) h^2 \, ds.$$

Thus, taking $t = r(q)$ and choosing $h$ such that $h^2(r(q)) = 0$, we get (15) for $q \notin \text{cut}(o)$. To treat the general case one can use the Calabi trick. Namely suppose that $q \in \text{cut}(o)$. Translating the origin $o$ to $\alpha_e = \gamma_q (\epsilon)$ so that $q \notin \text{cut}(\alpha_e)$, using the triangle inequality and, finally, taking the limit as $\epsilon \to 0$, one checks that (15) holds also in this case. \hfill \Box

From Lemma 12 some Myers’ type results can be proven. Here we state the following which generalizes a theorem of G. J. Galloway, [4].

**Theorem 13.** Let $(M^n, g_M, e^{-f} \text{dvol})$ be a complete weighted manifold. Given two different points $p, q \in M$, let $\gamma_{p,q}$ be a minimizing geodesic from $p$ to $q$ parameterized by arc length. Suppose that there exist constants $c$ and $G \geq 0$ such that for each pair of points $p, q$ it holds

$$\text{Ric}^m_f (\dot{\gamma}_{p,q}, \dot{\gamma}_{p,q}) \vert_{\gamma_{p,q}(t)} \geq (m + n - 1) \left[ c^2 + \frac{d}{dt} (g \circ \gamma_{p,q}) \right],$$

for some $C^1(M)$ function $g$ satisfying $\sup_M |g| \leq G$, $m < +\infty$. Then $M$ is compact and

$$\text{diam}(M) \leq \frac{1}{c} \left[ \frac{2G}{c} + \sqrt{\frac{4G^2}{c^2} + \pi^2} \right].$$

(19)
Proof. Define $L$ to be the length of $\gamma_{p,q}$ between $p$ and $q$ and set $h(t) := \sin(\frac{\pi}{L} t)$. Compute

$$
\int_0^L h^2(t) dt = \int_0^L \sin^2(\frac{\pi}{L} t) dt = \frac{L}{2}; \quad \int_0^L h''(t) dt = \frac{\pi^2}{L^2} \int_0^L \cos^2(\frac{\pi}{L} t) dt = \frac{\pi^2}{2L}.
$$

Then, applying Lemma 12, we have

$$
\frac{\pi^2 (m+n-1)}{2L} = \int_0^L (m+n-1) h'^2 \geq \int_0^L h^2 \text{Ric}_f^m(\dot{\gamma}_{p,q}, \dot{\gamma}_{p,q}) |_{\gamma_{p,q}} ds
$$

$$
\geq c^2(m+n-1) \int_0^L h^2 + (m+n-1) \int_0^L h^2 \frac{d}{dt}(g \circ \gamma_{p,q})
$$

$$
= \frac{c^2(m+n-1)L}{2} + (m+n-1)h^2 g(\gamma_{p,q}) \bigg|_0^L
$$

$$
- (m+n-1) \left[ \int_0^{\frac{L}{2}} \left( \frac{d}{dt} h^2 \right)(g \circ \gamma_{p,q}) + \int_{\frac{L}{2}}^L \left( \frac{d}{dt} h^2 \right)(g \circ \gamma_{p,q}) \right]
$$

$$
\geq \frac{c^2(m+n-1)L}{2} - (m+n-1)G \left[ \int_0^{\frac{L}{2}} \left( \frac{d}{dt} h^2 \right) + \int_{\frac{L}{2}}^L \left( \frac{d}{dt} h^2 \right) \right]
$$

$$
\geq \frac{c^2(m+n-1)L}{2} - 2(m+n-1)G
$$

Finally, this latter can be written as

$$
c^2 L^2 - 4GL - \pi^2 \leq 0,
$$

which in turn implies (19), because $p$ and $q$ are arbitrary. \(\square\)

Reasoning as in the classical case, ([5], [9]) the validity of (15) and an integration by parts shows that the compactness of $M$ depends on the behavior, and on the position of the zeros, of the solution of the differential equation along minimizing geodesics

$$
- h''(t) - \frac{\text{Ric}_f^m(\dot{\gamma}, \dot{\gamma})}{m+n-1} h(t) = 0 \tag{21}
$$

We are thus reduced to find sufficient condition on $\text{Ric}_f^m$ for which solutions of the differential equation (21) have a first zero at finite time. Minor changes to the proofs of the results contained in [9] lead to similar compactness results in the weighted setting. In particular we state the following theorem in which a Myers’ type conclusion is obtained assuming a nonpositive lower bound on $\text{Ric}_f^m$. 


Theorem 14. Let $\text{Ric}_f^m \geq -(m + n - 1)B^2$, for some constant $B \geq 0$, $m < +\infty$. Suppose there is a point $q \in M$ such that along each geodesic $\gamma : [0, +\infty) \to M$ parameterized by arc length, with $\gamma(0) = q$, it holds either

$$\int_a^b \frac{\text{Ric}_f^m(\dot{\gamma}, \dot{\gamma})}{m + n - 1} dt > B \left\{ b + \frac{a^2B - 1}{e^{2Ba} - 1} \right\} + \frac{1}{4} \log \left( \frac{b}{a} \right),$$

or

$$\int_a^b \frac{\text{Ric}_f^m(\dot{\gamma}, \dot{\gamma})}{m + n - 1}(t) dt > B \left\{ b^\alpha + \frac{a^2B - 1}{e^{2Ba} - 1} \right\} + \frac{\alpha^2}{4(1 - \alpha)} \left\{ a^{\alpha - 1} - b^{\alpha - 1} \right\}$$

for some $0 < a < b$ and $\alpha \neq 1$. Then $M$ is compact.

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