Nonextensive Generalizations of the Jensen-Shannon Divergence

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Abstract—Convexity is a key concept in information theory, namely via the many implications of Jensen’s inequality, such as the non-negativity of the Kullback-Leibler divergence (KLD). Jensen’s inequality also underlies the concept of Jensen-Shannon divergence (JSD), which is a symmetrized and smoothed version of the KLD. This paper introduces new JSD-type divergences, by extending its two building blocks: convexity and Shannon’s entropy. In particular, a new concept of $q$-convexity is introduced and shown to satisfy a Jensen’s $q$-inequality. Based on this Jensen’s $q$-inequality, the Jensen-Tsallis $q$-difference is built, which is a nonextensive generalization of the JSD, based on Tsallis entropies. Finally, the Jensen-Tsallis $q$-difference is characterized in terms of convexity and extrema.

Index Terms—Convexity, Tsallis entropy, nonextensive entropies, Jensen-Shannon divergence, mutual information.

I. INTRODUCTION

The central role played by the Shannon entropy in information theory has stimulated the proposal of several generalizations and extensions during the last decades (see, e.g., [1], [2], [3], [4], [5], [6], [7]). One of the best known of these generalizations is the family of Rényi entropies, which has the Shannon entropy as a limit case [1], and has been used in several applications (e.g., [8], [9]). The Rényi and Shannon entropies share the well-known additivity property, under which the joint entropy of a pair of independent random variables is simply the sum of the individual entropies. In other generalizations, namely those introduced by Havrda-Charvát [2], Daróczy [3], and Tsallis [7], the additivity property is abandoned, yielding the so-called nonextensive entropies. These nonextensive entropies have raised great interest among physicists in modeling certain physical phenomena (such as those exhibiting long-range interactions and multifractal behavior) and as a framework for nonextensive generalizations of the classical Boltzmann-Gibbs statistical mechanics [10], [11]. Nonextensive entropies have also been recently used in signal/image processing (e.g., [12], [13], [14]) and many other areas [15].

Convexity is a key concept in information theory, namely via the many important corollaries of Jensen’s inequality [16], such as the non-negativity of the relative Shannon entropy, or Kullback-Leibler divergence (KLD) [17]. The Jensen inequality is also at the basis of the concept of Jensen-Shannon divergence (JSD), which is a symmetrized and smoothed version of the KLD [18], [19]. The JSD is widely used in areas such as statistics, machine learning, image and signal processing, and physics.

The goal of this paper is to introduce new extensions of JSD-type divergences, by extending its two building blocks: convexity and the Shannon entropy. In previous work [?], we investigate how these extensions may be applied in kernel-based machine learning. More specifically, the main contributions of this paper are:

- The concept of $q$-convexity, as a generalization of convexity, for which we prove a Jensen $q$-inequality. The related concept of Jensen $q$-differences, which generalize Jensen differences, is also proposed. Based on these concepts, we introduce the Jensen-Tsallis $q$-difference, a nonextensive generalization of the JSD, which is also a “mutual information” in the sense of Furuichi [20].
- Characterization of the Jensen-Tsallis $q$-difference, with respect to convexity and its extrema, extending results obtained by Burbea and Rao [21] and by Lin [19] for the JSD.

The rest of the paper is organized as follows. Section II-A reviews the concepts of nonextensive entropies, with emphasis on the Tsallis case. Section II discusses Jensen differences and divergences. The concepts of $q$-differences and $q$-convexity are introduced in Section IV where they are used to define and characterize some new divergence-type quantities. Section V defines the Jensen-Tsallis $q$-difference and derives some properties. Finally, Section VI contains concluding remarks and mentions directions for future research.

II. NONEXTENSIVE ENTROPIES

A. Suyari’s Axiomatization

Inspired by the Shannon-Khinchin axiomatic formulation of the Shannon entropy [22], [23], Suyari proposed an axiomatic framework for nonextensive entropies and a uniqueness theorem [24]. Let

$$\Delta^{n-1} := \left\{ (p_1, \ldots, p_n) \in \mathbb{R}^n : p_i \geq 0, \sum_{j=1}^n p_i = 1 \right\}$$  \hspace{1cm} (1)
denote the \((n - 1)\)-dimensional simplex. The Suyari axioms (see Appendix) determine the function \(S_{q,\phi} : \Delta^{n-1} \to \mathbb{R}\) given by
\[
S_{q,\phi}(p_1, \ldots, p_n) = \frac{k}{\phi(q)} \left( 1 - \sum_{i=1}^{n} p_i^q \right),
\]
where \(q, k \in \mathbb{R}^+_+\), \(S_{1,\phi} := \lim_{q \to 1} S_{q,\phi}\), and \(\phi : \mathbb{R}^+_+ \to \mathbb{R}\) is a continuous function satisfying the following three conditions:

(i) \(\phi(q)\) has the same sign as \(q - 1\);
(ii) \(\phi(q)\) vanishes if and only if \(q = 1\);
(iii) \(\phi\) is differentiable in a neighborhood of 1 and \(\phi'(1) = 1\).

For any \(\phi\) satisfying these conditions, \(S_{q,\phi}\) has the pseudoadditivity property: for any two independent random variables \(A\) and \(B\), with probability mass functions \(p_A \in \Delta^{n_A}\) and \(p_B \in \Delta^{n_B}\), respectively,
\[
S_{q,\phi}(A \times B) = S_{q,\phi}(A) + S_{q,\phi}(B) - \frac{\phi(q)}{k} S_{q,\phi}(A) S_{q,\phi}(B),
\]
where we denote (as usual) \(S_{q,\phi}(A) := S_{q,\phi}(p_A)\).

For \(q = 1\), we recover the Shannon entropy,
\[
S_{1,\phi}(p_1, \ldots, p_n) = H(p_1, \ldots, p_n) = -k \sum_{i=1}^{n} p_i \ln p_i,
\]
thus pseudoadditivity turns into additivity.

B. Tsallis Entropies

Several proposals for \(\phi\) have appeared [2], [3], [7]. In the rest of the paper, we set \(\phi(q) = q - 1\), which yields the Tsallis entropy:
\[
S_q(p_1, \ldots, p_n) = \frac{k}{q - 1} \left( 1 - \sum_{i=1}^{n} p_i^q \right).
\]
To simplify, we let \(k = 1\) and write the Tsallis entropy as
\[
S_q(X) := S_q(p_1, \ldots, p_n) = -\sum_{x \in X} p(x)^q \ln q p(x),
\]
where \(\ln_q(x) := (x^{1-q} - 1)/(1-q)\) is the \(q\)-logarithm function, which satisfies \(\ln_q(xy) = \ln_q(x) + x^{1-q} \ln_q(y)\) and \(\ln_q(1/x) = -x^{1-1} \ln_q(x)\).

Furuichi derived some information theoretic properties of Tsallis entropies [20]. Tsallis joint and conditional entropies are defined, respectively, as
\[
S_q(X, Y) := -\sum_{x, y} p(x, y)^q \ln_q p(x, y)
\]
and
\[
S_q(X|Y) := -\sum_{x, y} p(x, y)^q \ln_q p(x|y)
= \sum_{y} p(y)^q S_q(X|y),
\]
and the chain rule \(S_q(X, Y) = S_q(X) + S_q(Y|X)\) holds.

For two probability mass functions \(p_X, p_Y \in \Delta^n\), the Tsallis relative entropy, generalizing the KLD, is defined as
\[
D_q(p_X||p_Y) := -\sum_{x} p_X(x) \ln \left( \frac{p_Y(x)}{p_X(x)} \right).
\]
Finally, the Tsallis mutual entropy is defined as
\[
I_q(X;Y) := S_q(X) - S_q(X|Y) = S_q(Y) - S_q(Y|X),
\]
generalizing (for \(q > 1\)) Shannon’s mutual information [20]. In Section \[\text{V}\], we establish a relationship between Tsallis mutual entropy and a quantity called Tsallis q-difference, generalizing the one between mutual information and the JSD [25].

Furuichi considers an alternative generalization of Shannon’s mutual information,
\[
\tilde{I}_q(X;Y) := D_q(p_{X,Y}||p_X \otimes p_Y),
\]
where \(p_{X,Y}||p_X \otimes p_Y\) denotes their joint probability if they were independent [20]. This alternative definition has also been used as a “Tsallis mutual entropy” [26]; notice that \(I_q(X;Y) \neq \tilde{I}_q(X;Y)\) in general, the case \(q = 1\) being a notable exception. In Section \[\text{V}\] we show that this alternative definition also leads to a nonextensive analogue of the JSD.

C. Denormalization of Tsallis Entropies

In the sequel, we extend the domain of Tsallis entropies from \(\Delta_{n-1}\) to the set of unnormalized measures, \(\mathbb{R}^+_+ := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \forall i \ x_i \geq 0\}\). The Tsallis entropy of a measure is defined as
\[
S_q(x_1, \ldots, x_n) := -\sum_{i=1}^{n} x_i^q \ln_q x_i = \sum_{i=1}^{n} \varphi_q(x_i),
\]
where \(\varphi_q : \mathbb{R}^+_+ \to \mathbb{R}\) is given by
\[
\varphi_q(y) = -y \ln_q y = \begin{cases} -y \ln y, & \text{if } q = 1, \\ (y - y^q)/(q - 1), & \text{if } q \neq 1. \end{cases}
\]

III. JENSEN DIFFERENCES AND DIVERGENCES

A. The Jensen Difference

Jensen’s inequality states that, if \(f\) is a concave function and \(X\) is an integrable real-valued random variable,
\[
f(E[X]) - E(f(X)) \geq 0.
\]
Burbea and Rao studied the difference in the left hand side of \[\text{[13]}\] with \(f := H_{\varphi}\), where \(H_{\varphi} : [a, b] \to \mathbb{R}\) is a concave function, called a \(\varphi\)-entropy, defined as
\[
H_{\varphi}(x) := -\sum_{i=1}^{n} \varphi(x_i),
\]
where \(\varphi : [a, b] \to \mathbb{R}\) is convex [21]. The result is called the Jensen difference, as formalized in the following definition.

**Definition 1**: The Jensen difference \(J_{\varphi}^{\Psi} : \mathbb{R}^{nm} \to \mathbb{R}\) induced by a (concave) generalized entropy \(\Psi : \mathbb{R}^+_+ \to \mathbb{R}\) and weighted by \((\pi_1, \ldots, \pi_m) \in \Delta^{m-1}\) is
\[
J_{\varphi}^{\Psi}(x_1, \ldots, x_m) := \Psi \left( \sum_{j=1}^{m} \pi_j x_j \right) - \sum_{j=1}^{m} \pi_j \Psi(x_j),
\]
where both expectations are with respect to \((\pi_1, \ldots, \pi_m)\).

In the following subsections, we consider several instances of Definition \[\text{[1]}\] leading to several Jensen-type divergences.
B. The Jensen-Shannon Divergence

Let $P$ be a random probability distribution taking values in \{\(p_i\)\}_{i=1,\ldots,m} \subseteq \Delta^{n-1}$ according to a distribution \(\pi = (\pi_1, \ldots, \pi_m) \in \Delta^{m-1}\). In classification/estimation theory parlance, \(\pi\) is called the prior distribution and \(p_y := p(.|y)\) the likelihood function. Then, (15) becomes
\[
J^\pi_q(p_1, \ldots, p_m) = \Psi(E[P]) - E[\Psi(P)],
\]
where the expectations are with respect to \(\pi\).

Let now \(\Psi = H\), the Shannon entropy. Consider the random variables \(Y\) and \(X\), taking values respectively in \(Y = \{1, \ldots, m\}\) and \(X = \{1, \ldots, n\}\), with probability mass functions \(\pi(y) := \pi_y\) and \(p(x) := \sum_{y=1}^{m} p(x|y)\pi(y)\). Using standard notation of information theory [17],
\[
J^\pi_q(P) := J^\pi_q(p_1, \ldots, p_m) = H(X) - H(X|Y) = I(X;Y),
\]
where \(I(X;Y)\) is the mutual information between \(X\) and \(Y\). Since \(I(X;Y)\) is also equal to the KLD between the joint distribution and the product of the marginals [17], we have
\[
J^\pi_q(P) = H(E[P]) - E[H(P)] = E[D(P||E[P])]. \tag{18}
\]

The quantity \(J^\pi_q(P)\) is called the Jensen-Shannon divergence (JSD) of \(p_1, \ldots, p_m\), with weights \(\pi_1, \ldots, \pi_m\) [21], [19]. Equality (18) allows two interpretations of the JSD: (i) the Jensen difference of the Shannon entropy of \(P\); or (ii) the expected KLD between \(P\) and the expectation of \(P\).

A remarkable fact is that \(J^\pi_q(P) = \min_{Q} E[D(P||Q)]\), i.e., \(Q^* = E[P]\) is a minimizer of \(E[D(P||Q)]\) with respect to \(Q\). It has been shown that this property together with equality (18) characterize the so-called Bregman divergences: they hold not only for \(\Psi = H\), but for any concave \(\Psi\) and the corresponding Bregman divergence, in which case \(J^\pi_q\) is the Bregman information (see [27] for details).

When \(m = 2\) and \(\pi = (1/2, 1/2)\), \(P\) may be seen as a random distribution whose value on \([p_1, p_2]\) is chosen by tossing a fair coin. In this case, \(J^{(1/2,1/2)}(P) = JS(p_1, p_2)\), where
\[
JS(p_1, p_2) = H\left(\frac{p_1 + p_2}{2}\right) - \frac{H(p_1) + H(p_2)}{2} = \frac{1}{2}D\left(\frac{p_1 + p_2}{2}\right) + \frac{1}{2}D\left(\frac{p_1 + p_2}{2}\right), \tag{19}
\]
as introduced in [19]. It has been shown that \(\sqrt{JS}\) satisfies the triangle inequality (hence being a metric) and that, moreover, it is an Hilbertian metric [28], [29].

C. The Jensen-Rényi Divergence

Consider again the scenario above (Subsection III-B), now with the Rényi q-entropy
\[
R_q(p) = \frac{1}{1-q} \ln \sum_{i=1}^{n} p_i^q \tag{20}
\]
replacing the Shannon entropy. The Rényi q-entropy is concave for \(q \in [0,1)\) and has the Shannon entropy as the limit when \(q \to 1\) [1]. Letting \(\Psi = R_q\), (16) becomes
\[
J^\pi_{R_q}(p_1, \ldots, p_m) = R_q(E[P]) - E[R_q(P)]. \tag{21}
\]

Unlike in the JSD case, there is no counterpart of equality (18) based on the Rényi q-divergence
\[
D_{R_q}(p_1||p_2) = \frac{1}{q-1} \ln \sum_{i=1}^{n} p_i^q p_2^{1-q}. \tag{22}
\]

The quantity \(J^\pi_{R_q}\), in (21) is called the Jensen-Rényi divergence (JRD). Furthermore, when \(m = 2\) and \(\pi = (1/2, 1/2)\), we write \(J^\pi_{R_q}(P) = JR_q(p_1, p_2)\), where
\[
JR_q(p_1, p_2) = R_q\left(\frac{p_1 + p_2}{2}\right) - R_q(p_1) + R_q(p_2). \tag{23}
\]

The JRD has been used in several signal/image processing applications, such as registration, segmentation, denoising, and classification [30], [31], [32].

D. The Jensen-Tsallis Divergence

Burbea and Rao have defined divergences of the form (16) based on the Tsallis q-entropy \(S_q\), defined in (11) [21]. Like the Shannon entropy, but unlike the Rényi entropies, the Tsallis q-entropy is an instance of a \(\varphi\)-entropy (see (14)). Letting \(\Psi = S_q\), (16) becomes
\[
J^\pi_{S_q}(p_1, \ldots, p_m) = S_q(E[P]) - E[S_q(P)]. \tag{24}
\]

Again, like in Subsection III-C if we consider the Tsallis q-divergence,
\[
D_{q}(p_1||p_2) = \frac{1}{1-q} \left(1 - \sum_{i=1}^{n} p_i^q p_2^{1-q}\right), \tag{25}
\]
there is no counterpart of the equality (18).

The quantity \(J^\pi_{S_q}\) in (24) is called the Jensen-Tsallis divergence (JTD) and it has also been applied in image processing [33]. Unlike the JSD, the JTD lacks an interpretation as a mutual information. In spite of this, for \(q \in [1,2]\), the JTD exhibits joint convexity [21]. In the next section, we propose an alternative to the JTD which, amongst other features, is interpretable as a nonextensive mutual information (in the sense of Furuichi [20]) and is jointly convex, for \(q \in [0,1]\).

IV. \(q\)-Convexity and \(q\)-Differences

A. Introduction

This section introduces a novel class of functions, termed Jensen \(q\)-differences (JqD), that generalizes Jensen differences. We will later (Section VII) use the JqD to define the Jensen-Tsallis \(q\)-difference (JTqD), which we will propose as an alternative nonextensive generalization of the JSD, instead of the JTD discussed in Subsection III-D.

We begin by recalling the concept of \(q\)-expectation, which is used in nonextensive thermodynamics [7].

**Definition 2:** The unnormalized \(q\)-expectation of a finite random variable \(X \in \mathcal{X}\), with probability mass function \(P_X(x)\), is
\[
E_q[X] := \sum_{x \in \mathcal{X}} x P_X(x)^q. \tag{26}
\]

Of course, \(q = 1\) corresponds to the standard notion of expectation. For \(q \neq 1\), the \(q\)-expectation does not correspond
to the intuitive meaning of average/expectation (e.g., $E_q[1] \neq 1$ in general). Nonetheless, it has been used in the construction of nonextensive information theoretic concepts such as the Tsallis entropy, which can be written compactly as $S_q(X) = -E_q[\ln_q p(X)]$.

B. $q$-Convexity

We now introduce the novel concept of $q$-convexity and use it to derive a set of results, among which we emphasize a $q$-Jensen inequality.

**Definition 3**: Let $q \in \mathbb{R}$ and $\mathcal{X}$ be a convex set. A function $f : \mathcal{X} \to \mathbb{R}$ is $q$-convex if for any $x,y \in \mathcal{X}$ and $\lambda \in [0,1]$, 
\[
f(\lambda x + (1-\lambda)y) \leq \lambda^q f(x) + (1-\lambda)^q f(y).
\]

Naturally, $f$ is $q$-concave if $-f$ is $q$-convex. Of course, $1$-convexity is the usual notion of convexity. The next proposition states the $q$-Jensen inequality.

**Proposition 4**: If $f : \mathcal{X} \to \mathbb{R}$ is $q$-convex, then for any $n \in \mathbb{N}$, $x_1, \ldots, x_n \in \mathcal{X}$ and $\pi = (\pi_1, \ldots, \pi_n) \in \Delta^{n-1}$, 
\[
f \left( \sum \pi_i x_i \right) \leq \sum \pi_i^q f(x_i).
\]

**Proof**: Use induction, exactly as in the proof of the standard Jensen inequality (e.g., [17]).

**Proposition 5**: Let $f \geq 0$ and $q \geq q' \geq 0$; then, 
\[
f \text{is } q\text{-convex} \implies f \text{is } q'\text{-convex} \tag{29}\]
\[
f \text{is } q\text{-convex} \implies -f \text{ is } q\text{-convex}. \tag{30}\]

**Proof**: Implication (29) results from 
\[
f(\lambda x + (1-\lambda)y) \leq \lambda^q f(x) + (1-\lambda)^q f(y) \leq \lambda^q f(x) + (1-\lambda)^q f(y),
\]
where the first inequality states the $q$-convexity of $f$ and the second one is valid because $f(x), f(y) \geq 0$ and $\lambda^q \geq \lambda^{q'} \geq 0$, for any $t \in [0,1]$ and $q \geq q'$. The proof of (30) is analogous.

C. Jensen $q$-Differences

We now generalize Jensen differences, formalized in Definition 4 by introducing the concept of $q$-Jensen differences.

**Definition 6**: For $q \geq 0$, the Jensen $q$-difference induced by a (concave) generalized entropy $\Psi : \mathbb{R}_+^n \to \mathbb{R}$ and weighted by $(\pi_1, \ldots, \pi_m) \in \Delta^{m-1}$ is 
\[
T_{q,\Psi}(x_1, \ldots, x_m) \triangleq \Psi \left( \sum_{j=1}^m \pi_j x_j \right) - \sum_{j=1}^m \pi_j^q \Psi(x_j) = \Psi \left( E[X] \right) - E_q[\Psi(X)], \tag{31}
\]
where the expectation and the $q$-expectation are with respect to $(\pi_1, \ldots, \pi_m)$.

Burbea and Rao established necessary and sufficient conditions for the Jensen difference of a $\varphi$-entropy to be convex [21]. The following proposition generalizes that result, extending it to Jensen $q$-differences.

**Proposition 7**: Let $\varphi : [0,1] \to \mathbb{R}$ be a function of class $C^2$ and consider the ($\varphi$-entropy [21]) function $\Psi : [0,1]^m \to \mathbb{R}$ defined by $\Psi(z) := -\sum_{t=1}^m \varphi(z)$. Then, the $q$-difference $T_{q,\Psi} : [0,1]^m \to \mathbb{R}$ is convex if and only if $\varphi$ is convex and $-1/\varphi''$ is $\text{(2 - q)-convex}$. 

**Proof**: The case $q = 1$ corresponds to the Jensen difference and was proved by Burbea and Rao (Theorem 1 in [21]). Our proof extends that of Burbea and Rao to $q \neq 1$.

In general, $y = \{y_1, \ldots, y_m\}$, where $y_t = \{y_{1t}, \ldots, y_{nt}\}$, thus 
\[
T_{q,\Psi}(y) = \Psi \left( \sum_{t=1}^m \pi_t y_t \right) - \sum_{t=1}^m \pi_t^q \Psi(y_t) = \sum_{t=1}^n \left[ \sum_{i=1}^m \pi_t^q \varphi(y_{it}) - \varphi \left( \sum_{i=1}^m \pi_t y_{it} \right) \right],
\]
showing that it suffices to consider $n = 1$, i.e.,
\[
T_{q,\Psi}(y_1, \ldots, y_m) = \sum_{t=1}^m \pi_t^q \varphi(y_t) - \varphi \left( \sum_{t=1}^m \pi_t y_t \right) \tag{32}
\]
this function is convex on $[0,1]^m$ if and only if, for every fixed $a_1, \ldots, a_m \in [0,1]$, and $b_1, \ldots, b_m \in \mathbb{R}$, the function 
\[
f(x) = T_{q,\Psi}(a_1 + b_1 x, \ldots, a_m + b_m x) \tag{33}
\]
is convex in $\{x \in \mathbb{R} : a_t + b_t x \in [0,1], t = 1, \ldots, m\}$. Since $f$ is $C^2$, it is convex if and only if $f''(t) \geq 0$

We first show that convexity of $f$ (equivalently of $T_{q,\Psi}$) implies convexity of $\varphi$. Letting $c_t = a_t + b_t x$,
\[
f''(x) = \sum_{t=1}^m \pi_t b_t^2 \varphi''(c_t) - \left( \sum_{t=1}^m \pi_t b_t \right)^2 \varphi'' \left( \sum_{t=1}^m \pi_t c_t \right). \tag{34}
\]
By choosing $x = 0$, $a_t = a_1$ and $b_t = c_t$ for $t = 1, \ldots, m$, and $b_1, \ldots, b_m$ satisfying $\sum t \pi_t b_t = 0$ in (34), we get
\[
f''(0) = \varphi''(a) \sum_{t=1}^m \pi_t b_t^2,
\]
hence, if $f$ is convex, $\varphi''(a) \geq 0$ thus $\varphi$ is convex.

Next, we show that convexity of $f$ also implies $(2 - q)$-convexity of $-1/\varphi''$. By choosing $x = 0$ (thus $c_t = a_t$) and $b_t = \pi_{1-t}^{-q}(\varphi''(a_t))^{-1}$, we get 
\[
f''(0) = \sum_{t=1}^m \pi_t b_t^2 \varphi''(a_t) - \left( \sum_{t=1}^m \pi_t b_t \right)^2 \varphi'' \left( \sum_{t=1}^m \pi_t a_t \right) = \left( \sum_{t=1}^m \pi_t b_t \right)^2 \varphi'' \left( \sum_{t=1}^m \pi_t a_t \right) \times \left[ \frac{1}{\varphi''(\sum_{t=1}^m \pi_t a_t)} - \sum_{t=1}^m \pi_t b_t^2 \right],
\]
where the expression inside the square brackets is the Jensen $(2 - q)$-difference of $1/\varphi''$ (see Definition 4). Since $\varphi''(x) \geq 0$, the factor outside the square brackets is non-negative, thus the Jensen $(2 - q)$-difference of $1/\varphi''$ is also non-negative and $-1/\varphi''$ is $(2 - q)$-convex.
Finally, we show that if \( \varphi \) is convex and \(-1/\varphi''\) is \((2- q)\)-convex, then \( f'' \) is \((q-1)\)-convex. Let \( r_t = (q_t^{2-q}/\varphi''(c_t))^1/2 \) and \( s_t = b_t/p_t^{1-q}\varphi''(c_t)/q_t^{1/2} \); then, the non-negativity of \( f'' \) results from the following chain of inequalities/equalities:

\[
0 \leq \left( \sum_{t=1}^{m} r_t^2 \right) \left( \sum_{t=1}^{m} s_t^2 \right) - \left( \sum_{t=1}^{m} r_t s_t \right)^2
\]

\[
= \sum_{t=1}^{m} \frac{\pi_t^{2-q}}{\varphi''(c_t)} \sum_{t=1}^{m} b_t^2 \pi_t^{q} \varphi''(c_t) - \left( \sum_{t=1}^{m} b_t \pi_t \right)^2
\]

\[
\leq \frac{1}{\varphi'' \left( \sum_{t=1}^{m} \pi_t c_t \right)} \sum_{t=1}^{m} b_t^2 \pi_t^{q} \varphi''(c_t) - \left( \sum_{t=1}^{m} b_t \pi_t \right)^2
\]

\[
= \frac{1}{\varphi'' \left( \sum_{t=1}^{m} \pi_t c_t \right)} \cdot f''(t),
\]

where: (35) is the Cauchy-Schwarz inequality; equality (36) results from the definitions of \( r_t \) and \( s_t \) and from the fact that \( r_t s_t = b_t \pi_t \); inequality (37) states the \((2-q)\)-convexity of \(-1/\varphi''\); equality (38) results from (34).

\section{V. The Jensen-Tsallis q-Difference}

\subsection{A. Definition}

As in Subsection [11-13] let \( P \) be a random probability distribution taking values in \( \{ p_y \}_{y=1,...,m} \). According to a distribution \( \pi = (\pi_1, ..., \pi_m) \in \Delta^{m-1} \), then, we may write

\[
T_{\pi, q}(p_1, ..., p_m) = \Psi(E[P]) - E_q[\Psi(P)],
\]

where the expectations are with respect to \( \pi \). Hence Jensen q-differences may be seen as deformations of the standard Jensen differences [16], in which the second expectation is replaced by a \( q \)-expectation.

Let now \( \Psi = S_q \), the nonextensive Tsallis q-entropy. Introducing the random variables \( Y \) and \( X \), with values respectively in \( Y = \{ 1, ..., m \} \) and \( X = \{ 1, ..., n \} \), with probability mass functions \( \pi(y) := \pi_y \) and \( p(x) := \sum_{y=1}^{m} p(x|y)\pi(y) \), we have (writing \( T_{\pi, q} \) simply as \( T_q \))

\[
T_q(p_1, ..., p_m) = S_q(X) - S_q(X|Y) = I_q(X;Y),
\]

where \( S_q(X|Y) \) is the Tsallis conditional q-entropy, and \( I_q(X;Y) \) is the Tsallis mutual q-entropy, as defined by Fujiuchi [20]. Observe that (40) is a nonextensive analogue of (17). Since, in general, \( I_q \neq I_q \) (see (10)), unless \( q = 1 \) \((I_1 = I_1 = I)\), there is no counterpart of (13) in terms of q-differences. Nevertheless, Lamberti and Majtey have proposed a non-logarithmic version of the JSD, which corresponds to using \( I_q \) for the Tsallis mutual q-entropy (although this interpretation is not explicitly mentioned by those authors) [26].

We call the quantity \( T_{\pi} \) the Jensen-Tsallis q-difference (JTqD) of \( p_1, ..., p_m \) with weights \( \pi_1, ..., \pi_m \). Although the JTqD is a generalization of the Jensen-Shannon divergence, for \( q \neq 1 \), the term “divergence” would be misleading in this case, since \( T_{\pi} \) may take negative values (if \( q < 1 \)) and does not vanish in general if \( P \) is deterministic.

When \( m = 2 \) and \( \pi = (1/2, 1/2) \), define \( T_q := T_q^{1/2,1/2} \),

\[
T_q(p_1, p_2) = S_q \left( \frac{p_1 + p_2}{2} \right) - \frac{S_q(p_1) + S_q(p_2)}{2q}. \quad (41)
\]

Notable cases arise for particular values of \( q \):

\begin{itemize}
  \item For \( q = 0 \), \( S_0(p) = -1 + \| x \|_0 \), where \( \| x \|_0 \) denotes the so-called 0-norm (although it’s not a norm) of vector \( x \), i.e., its number of nonzero components. The Jensen-Tsallis 0-difference is thus
    \[
    T_0(p_1, p_2) = 1 - \| p_1 \odot p_2 \|_0, \quad (42)
    \]
    where \( \odot \) denotes the Hadamard-Schur (i.e., elementwise) product. We call \( T_0 \) the Boolean difference.
  \item For \( q = 1 \), since \( S_1(p) = H(p) \), \( T_1 \) is the JSD,
    \[
    T_1(p_1, p_2) = JS(p_1, p_2). \quad (43)
    \]
  \item For \( q = 2 \), \( S_2(p) = 1 - (p, p) \), where \( (x, y) = \sum_i x_i y_i \) is the usual inner product between \( x \) and \( y \). Consequently, the Tsallis 2-difference is
    \[
    T_2(p_1, p_2) = \frac{1}{2} - \frac{1}{2} (p_1, p_2), \quad (44)
    \]
    which we call the linear difference.
\end{itemize}

\subsection{B. Properties of the JTqD}

This subsection presents results regarding convexity and extrema of the JTqD, for several values of \( q \), extending known properties of the JSD (\( q = 1 \)).

Some properties of the JSD are lost in the transition to nonextensivity. For example, while the former is nonnegative and vanishes if and only if all the distributions are identical, this is not true in general with the JTqD. Nonnegativity of the JTqD is only guaranteed if \( q \geq 1 \), which explains why some authors (e.g., [20]) only consider values of \( q \geq 1 \), when looking for nonextensive analogues of Shannon’s information theory. Moreover, unless \( q = 1 \), it is not generally true that \( T_q(p_1, ..., p_m) = 0 \) or even that \( T_q(p_1, ..., p_m, p') \geq T_q(p_1, ..., p, p') \). For example, the solution to the optimization problem

\[
\min_{p_1 \in \Delta^m} T_q(p_1, p_2),
\]

is, in general, different from \( p_2 \), unless \( q = 1 \). Instead, this minimizer is closer to the uniform distribution, if \( q \in [0,1) \), and closer to a degenerate distribution, for \( q \in (1,2] \). This is not so surprising: recall that \( T_2(p_1, p_2) = \frac{1}{2} - \frac{1}{2} (p_1, p_2) \); in this case, (43) becomes a linear program, and the solution is not \( p_2 \), but \( p^*_1 = \delta_j \), where \( j = \arg \max_{i} s_i \).

We start by recalling a basic result, which essentially confirms that Tsallis entropies satisfy one of the Suyari axioms (see Axiom A2 in the Appendix), which states that entropies should be maximized by uniform distributions.

\begin{itemize}
  \item \textbf{Proposition 8:} The uniform distribution maximizes the Tsallis entropy for any \( q \geq 0 \).
\end{itemize}

\begin{itemize}
  \item \textbf{Proof:} Consider the problem
    \[
    \max_p S_q(p), \quad \text{subject to } \sum_i p_i = 1 \text{ and } p_i \geq 0.
    \]
\end{itemize}
Equating the gradient of the Lagrangian to zero, yields
\[
\frac{\partial}{\partial p_i} (S_q(p) + \lambda(\sum_i p_i - 1)) = -q(q - 1)^{-1} p_i^{q-1} + \lambda = 0,
\]
for all \(i\). Since all these equations are identical, the solution is the uniform distribution, which is a maximum, due to the concavity of \(S_q\).

The following corollary of Proposition 7 establishes the joint convexity of the JTqD, for \(q \in [0, 1]\). This complements the joint convexity of the JTD, for \(q \in [1, 2]\), which was proved by Burbea and Rao [21].

**Corollary 9:** For \(q \in [0, 1]\), the JTqD is a jointly convex function on \(\Delta^{n-1}\). Formally, let \(\{p_i^{(l)}\}_{l=1}^m\) be a collection of \(l\) sets of probability distributions on \(X = \{1, \ldots, n\}\); then, for any \((\lambda_1, \ldots, \lambda_l) \in \Delta^{l-1},
\[
T_q^n \left( \sum_{i=1}^l \lambda_i p_i^{(l)} \right) \leq \sum_{i=1}^l \lambda_i T_q^n (p_i^{(l)})
\]
Proof: Observe that the Tsallis entropy \(\varepsilon\) of a probability distribution \(p_t = \{p_1, \ldots, p_m\}\) can be written as
\[
S_q(p_t) = -\sum_{i=1}^m \varphi(p_{ti}), \quad \text{where} \quad \varphi(x) = \frac{x - x^q}{1 - q};
\]
thus, from Proposition 7, \(T_q^n\) is convex if and only if \(\varphi_q\) is a convex and the reciprocal \(1/\varphi_q^n\) is a convex function. Since \(\varphi_q^n(x) = x^{-q}\), \(\varphi_q(\pi)\) is convex for \(x \geq 0\) and \(q \geq 0\). To show the \((2-q)\)-diagonality of \(1/\varphi_q^n\), we use a version of the mean power inequality [34],
\[
-\left( \sum_{i=1}^l \lambda_i x_i \right)^{2-q} \leq -\sum_{i=1}^l \left( \lambda_i x_i \right)^{2-q} \leq -\sum_{i=1}^l \lambda_i^{2-q} x_i^{2-q},
\]
thus concluding that \(-1/\varphi_q^n\) is in fact \((2-q)\)-convex.

The next corollary, which results from the previous one, provides an upper bound for the JTqD, for \(q \in [0, 1]\). Although this result is weaker than that of Proposition 11 below, we include it since it provides insight about the upper extremum of the JTqD.

**Corollary 10:** Let \(q \in [0, 1]\). Then, \(T_q^n(p_1, \ldots, p_m) \leq S_q(\pi)\).

Proof: From Corollary 9 for \(q \in [0, 1]\), \(T_q^n(p_1, \ldots, p_m)\) is convex. Since its domain is a convex polytope (the cartesian product of \(m\) simplices), its maximum occurs on a vertex, i.e., when each argument \(p_t\) is a degenerate distribution at \(x_t\), denoted \(\delta_{x_t}\). In particular, if \(n \geq m\), this maximum occurs at the vertex corresponding to disjoint degenerate distributions, i.e., such that \(x_i \neq x_j\) if \(i \neq j\). At this maximum,
\[
T_q^n(\delta_{x_1}, \ldots, \delta_{x_m}) = S_q \left( \sum_{t=1}^m \pi_t \delta_{x_t} \right) = S_q \left( \sum_{t=1}^m \pi_t \delta_{x_t} \right) = S_q(\pi),
\]
where the equality in (46) results from \(S_q(\delta_{x_t}) = 0\). Notice that this maximum may not be achieved if \(n < m\).

The next proposition establishes (upper and lower) bounds for the JTqD, extending Corollary 10 to any non-negative \(q\).

**Proposition 11:** For \(q \geq 0\),
\[
T_q^n(p_1, \ldots, p_m) \leq S_q(\pi),
\]
and, if \(n \geq m\), the maximum is reached for a set of disjoint degenerate distributions. As in Corollary 10, this maximum may not be attained if \(n < m\).

For \(q \geq 1\),
\[
T_q^n(p_1, \ldots, p_m) \geq 0,
\]
and the minimum is attained in the pure deterministic case, i.e., when all distributions are equal to the same degenerate distribution. Results (48) and (49) still hold when \(X\) and \(Y\) are countable sets.

For \(q \in [0, 1]\),
\[
T_q^n(p_1, \ldots, p_m) \geq S_q(\pi)[1 - n^{1-q}],
\]
This lower bound (which is zero or negative) is attained when all distributions are uniform.

Proof: The proof of (48), for \(q \geq 0\), results from
\[
T_q^n (p_1, \ldots, p_m) = \frac{1}{q - 1} \left[ 1 - \sum_j \left( \sum_i \pi_i p_t^q \right)^q - \sum_i \pi_i^q \left( 1 - \sum_j (p_t^q)^q \right) \right] = S_q(\pi) + \frac{1}{q - 1} \sum_j \left[ \sum_i \left( \pi_i p_t^q \right)^q - \sum_i \pi_i^q p_t^q \right],
\]
where the inequality holds since, for \(y_t \geq 0\): if \(q \geq 1\), then \(\sum y_t^q \leq (\sum y_t)^q\); if \(q \in [0, 1]\), then \(\sum y_t^q \geq (\sum y_t)^q\).

The proof that \(T_q^n \geq 0\) for \(q \geq 1\), uses the notion of q-convexity. For countable \(X\), the Tsallis entropy \(\varepsilon\) is nonnegative. Since \(-S_q\) is 1-convex, then, by Proposition 5, it is also \(q\)-convex for \(q \geq 1\). Consequently, from the \(q\)-Jensen inequality (Proposition 4), for finite \(Y\), with \(|Y| = m\),
\[
T_q^n (p_1, \ldots, p_m) = S_q \left( \sum_{t=1}^m \pi_t p_t \right) - \sum_{t=1}^m \pi_t^q S_q(p_t) \geq 0.
\]
Since \(S_q\) is continuous, so is \(T_q^n\), thus the inequality is valid in the limit as \(m \to \infty\), which proves the assertion for \(Y\) countable. Finally, \(T_q^n(\delta_1, \ldots, \delta_1) = 0\), where \(\delta_1\) is some degenerate distribution.
Finally, to prove (50), for $q \in [0, 1]$ and $\mathcal{X}$ finite,

$$
T_q^\pi(p_1, \ldots, p_m) = S_q\left(\sum_{i=1}^m \pi_i p_i\right) - \sum_{i=1}^m \pi_i^q S_q(p_i)
$$

$$
\geq \sum_{i=1}^m \pi_i S_q(p_i) - \sum_{i=1}^m \pi_i^q S_q(p_i)
$$

(52)

$$
= \sum_{i=1}^m (\pi_i - \pi_i^q) S_q(p_i)
$$

$$
\geq S_q(U) \sum_{i=1}^m (\pi_i - \pi_i^q)
$$

(53)

$$
= S_q(\pi)[1 - n^{1-q}],
$$

(54)

where the inequality (52) results from $S_q$ being concave, and the inequality (53) holds since $\pi_i - \pi_i^q \leq 0$, for $q \in [0, 1]$, and the uniform distribution $U$ maximizes $S_q$ (Proposition 8), with $S_q(U) = (1 - n^{-q})/(q - 1)$. 

Finally, the next proposition characterizes the convexity/concavity of the JTqD. As before, it holds more generally when $\mathcal{X}$ and $\mathcal{Y}$ are countable sets.

**Proposition 12:** The JTqD is convex in each argument, for $q \in [0, 2]$, and concave in each argument, for $q \geq 2$.

**Proof:** Notice that the JTqD can be written as

$$
T_q^\pi(p_1, \ldots, p_m) = \sum_j \psi(p_{1j}, \ldots, p_{mj}),
$$

with

$$
\psi(y_1, \ldots, y_m) = \frac{1}{q - 1} \left[ \sum_i (\pi_i - \pi_i^q) y_i + \sum_i \pi_i^q y_i^q - \left( \sum_i \pi_i y_i \right)^q \right].
$$

(55)

It suffices to consider the second derivative of $\psi$ with respect to $y_1$. Introducing $z = \sum_{i=2}^m \pi_i y_i$,

$$
\frac{\partial^2 \psi}{\partial y_1^2} = q \left[ \pi_1^q y_1^{q-2} - \pi_1^2 (\pi_1 y_1 + z)^{q-2} \right] = q \pi_1^2 \left[ (\pi_1 y_1)^{q-2} - (\pi_1 y_1 + z)^{q-2} \right].
$$

(56)

Since $\pi_1 y_1 \leq (\pi_1 y_1 + z) \leq 1$, the quantity in (56) is nonnegative for $q \in [0, 2]$ and non-positive for $q \geq 2$. 

**VI. CONCLUSION**

In this paper we have introduced new Jensen-Shannon-type divergences, by extending its two building blocks: convexity and entropy. We have introduced the concept of $q$-convexity, for which we have stated and proved a Jensen $q$-inequality. Based on this concept, we have introduced the Jensen-Tsallis $q$-divergence, a nonextensive generalization of the Jensen-Shannon divergence. We have characterized the Jensen-Tsallis $q$-difference with respect to convexity and extrema, extending previous results obtained in [21], [19] for the Jensen-Shannon divergence.

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**APPENDIX**

In [24], Suyari proposed the following set of axioms (above referred as Suyari’s axioms) that determine nonextensive entropies $S_{q,\phi} : \Delta^{n-1} \to \mathbb{R}$ of the form stated in (2). In what follows, $q$ is fixed and $f_q$ is a function defined on $\Delta^{n-1}$.

(A1) **Continuity:** $f_q$ is continuous in $\Delta^{n-1}$ and $q \geq 0$;

(A2) **Maximality:** For any $q \geq 0, n \in \mathbb{N}$, and $(p_1, \ldots, p_n) \in \Delta^{n-1}$, $f_q(p_1, \ldots, p_n) \leq f_q(1/n, \ldots, 1/n)$;

(A3) **Generalized additivity:** For $i = 1, \ldots, n, j = 1, \ldots, m_i, p_{ij} \geq 0$, and $p_i = \sum_{j=1}^{m_i} p_{ij}$,

$$
f_q(p_1, \ldots, p_{mn}) = f_q(p_1, \ldots, p_n) + \sum_{i=1}^m p_i^q f_q\left(\frac{p_1}{p_i}, \ldots, \frac{p_{m_i}}{p_i}\right);
$$

(A4) **Expandability:** $f_q(p_1, \ldots, p_n, 0) = f_q(p_1, \ldots, p_n)$.

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