Closest separable state when measured by a quasi-relative entropy

Anna Vershynina

Department of Mathematics, Philip Guthrie Hoffman Hall, University of Houston, 3551 Cullen Blvd., Houston, TX 77204-3008, United States of America

E-mail: anna@math.uh.edu

Received 7 October 2020, revised 19 January 2021
Accepted for publication 28 January 2021
Published 22 February 2021

Abstract
It is well known that for pure states the relative entropy of entanglement is equal to the reduced entropy, and the closest separable state is explicitly known as well. The same holds for Renyi relative entropy per recent results. We ask the same question for a quasi-relative entropy of entanglement, which is an entanglement measure defined as the minimum distance to the set of separable states, when the distance is measured by the quasi-relative entropy. First, we consider a maximally entangled state, and show that the closest separable state is the same for any quasi-relative entropy as for the relative entropy of entanglement. Then, we show that this also holds for a certain class of functions and any pure state. And at last, we consider any pure state on two qubit systems and a large class of operator convex function. For these, we find the closest separable state, which may not be the same one as for the relative entropy of entanglement.

Keywords: entanglement, relative entropy, separable, entropy

1. Introduction

Entanglement describes strong quantum ‘connections’ (stronger than any classical ones) that parts of a system can possess even when separated by long distance and having no connecting matter in between. It found applications in quantum optics [8, 27], nuclear magnetic resonance [36, 37], condensed mater physics [19, 29], quantum cryptography [16, 30, 43], and quantum algorithms and protocols [4, 5]. Understanding the behavior of entanglement in a system, its creation and loss (decoherence), is one of the fundamental goals of quantum science. For an overview on entanglement see [26, 40].

The amount of entanglement in a quantum state can be captured using entanglement measures. Any such measure, $E(\rho)$ of a quantum state $\rho$, should fulfill at least the following criteria [45]:

*Author to whom any correspondence should be addressed.
(a) $E(\rho) \geq 0$, and $E(\rho) = 0$ if and only if $\rho$ is separable.
(b) Invariance under local unitary operations: $E(\rho) = E(U_A \otimes U_B \rho U_A^* \otimes U_B^*)$.
(c) $E$ cannot increase under local operations and classical communication (LOCC), i.e. $E(\Lambda(\rho)) \leq E(\rho)$ for any LOCC map $\Lambda$.

It is interesting to note that these conditions are not set in stone for all entanglement measures. They can be relaxed, strengthened or new ones added depending on specific circumstances one would like to consider. For example, even the simplest condition (a) is not satisfied for a distillable entanglement, as it is zero on the bound entangled states [24], which are entangled. However, the measure is nice enough for us to overlook that lapse, explaining that there are various kinds of entanglement, all of which cannot be captured by one single measure, yet. New conditions on entanglement measures sometimes take the form of normalization on the maximally entangled states, convexity, additivity, continuity, and reduction to the marginal state on pure states [25, 46].

A large class of entanglement measures is defined using a ‘distance’ between two quantum states [45, 46]. For a distance measure $D(\rho \parallel \sigma)$ between two quantum states $\rho$ and $\sigma$, which is not required to be a metric, entanglement measure is defined as

$$E(\rho) := \min_{\sigma \in S} D(\rho \parallel \sigma),$$

where $S$ is the set of separable states.

This construction gives entanglement measures such as: relative entropy of entanglement [45], Bures measure of entanglement [45], several Rényi relative entropies of entanglement [44, 47, 54], Tsallis relative entropy of entanglement [47], Groverian measure of entanglement [7, 41, 42], and possibly others.

In [46] it was shown that a relative entropy was a good candidate for a distance, since it induces an entanglement measure that satisfies (a)–(c). These conditions are:

(a) $D(\rho \parallel \sigma) = 0$ if and only if $\rho = \sigma$.
(b) Invariance under unitary operations: $D(\rho \parallel \sigma) = D(U \rho U^* \parallel U \sigma U^*)$.
(c) Data processing inequality (or monotonicity under completely positive trace-preserving (CPTP) maps): $D(\mathcal{N}(\rho) \parallel \mathcal{N}(\sigma)) \leq D(\rho \parallel \sigma)$, for any CPTP map $\mathcal{N}$.

In [46] it was shown that a relative entropy was a good candidate for a distance, since it induces an entanglement measure that satisfies (a)–(c). The relative entropy is defined as

$$S(\rho \parallel \sigma) = \text{Tr}(\rho \log \rho) - \text{Tr}(\rho \log \sigma).$$

It was also shown in [46] that a relative entropy of entanglement reduces to the marginal entropy on a maximally entangled state, and later in [45] it was generalized to all pure states: for a pure state $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$, the closest separable state when measured by the relative entropy is $\sigma^* = \sum_j p_j |jj\rangle \langle jj|$ and the entropy of entanglement is $E_r(|\Psi\rangle \langle \Psi|) = -\sum_j p_j \log p_j$. Sometimes this property of the relative entropy is elevated to a condition on an entanglement measure:

(a) Entanglement of a pure state $\rho = |\Psi\rangle \langle \Psi|$ is the marginal entropy, i.e. $E(|\Psi\rangle) = S(\rho_A)$, where $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi|$.

Similar results were found for the Renyi entropy of entanglement, defined by the minimal distance to the space of separable states when the distance is taken to be the Renyi relative
entropy

\[ S^R_{\alpha}(\rho||\sigma) = \frac{1}{\alpha - 1} \log \text{Tr}(\rho^\alpha \sigma^{1-\alpha}). \]

An explicit expression was found for the Renyi entropy of entanglement in [12, 54]. In [12], it was shown that for a pure state \( |\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle \) and \( \alpha < 2 \), the closest separable state is

\[ \sigma = \frac{1}{P_\alpha} \sum_j p_j^{1/\alpha} |jj\rangle \langle jj|, \]

where \( P_\alpha = \sum_j p_j^{1/\alpha} \). The Renyi entropy of entanglement is then given by

\[ E^R_\alpha(|\Psi\rangle) = \frac{\alpha}{\alpha - 1} \log \sum_j p_j^{1/\alpha}. \]

We consider a large class of distances, called quasi-relative entropy, defined as

\[ S_f(\rho||\sigma) = \text{Tr}(f(\Delta_{\sigma,\rho})\rho), \]

where the relative modular operator, \( \Delta_{A,B}(X) = L_A R_B^{-1}(X) = AXB^{-1} \). This class includes the regular relative entropy, the exponential Renyi relative entropy, and scaled Tsallis relative entropy.

Since the quasi-relative entropy satisfies conditions (a)–(c), the entanglement measure it induces, satisfy (a)–(c). The question now is what about (a)? Or, what is the closest separable state to a pure entangled state when measured by the quasi-relative entropy?

This optimization question was asked in [21], where the authors developed a method of solving this problem for any convex function on the set of quantum states. While authors considered the case of quasi-relative entropy as an example, they focused more on developing a general method rather than giving an explicit solution, as we aim in this manuscript. The method we are using is similar, if not the same, to the method in [21], but it could be simplified and traced back to the original relative entropy of entanglement paper [45].

In theorem 3.1 we consider a maximally entangled (Bell) state \( |\Psi^+\rangle = \sum_j \frac{1}{\sqrt{d}} |jj\rangle \), and show that the closest state is \( \sigma^* = \sum_j \frac{1}{d} \sum_j |jj\rangle \langle jj| \), which means that the quasi-relative entropy of entanglement is equal to

\[ E_f(|\Psi^+\rangle) = f(1/d). \]

In theorem 3.2 we consider any pure state, and a class of functions \( f \) that define quasi-relative entropy. In these conditions, we show that the closest separable state to a pure state \( |\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle \) is the state \( \sigma^* = \sum_j p_j |jj\rangle \langle jj| \). The quasi-relative entropy of entanglement is then

\[ E_f(|\Psi\rangle) = \sum_j p_j f(p_j). \]

In theorem 3.3 we take any qubit state \( |\Psi\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle \) and a large class of functions. We show that the closest separable state is in a form \( \sigma^* = q |00\rangle \langle 00| + (1 - q) |11\rangle \langle 11| \). We explicitly define these \( q \)'s in the proof of the theorem. In this case, the quasi-relative entropy becomes

\[ E_f(|\Psi\rangle \langle \Psi|) = \sum_j p_j f(q_j). \]
2. Preliminaries (for more details see [6, 11, 48, 49])

2.1. Operator monotone functions

Denote the set of operator monotone decreasing functions $f$ as $Q_{(0, \infty)}$.

**Example 2.1.** From [6, exercise V.4.8]. The following functions belong to $Q_{(0, \infty)}$:

- $f(x) = -\log x$.
- $f(x) = -x^p$ for $p \in [0, 1]$.
- $f(x) = x^p$ for $p \in [-1, 0]$.

According to [15, chapter II, theorem I] every function $f \in Q_{(0, \infty)}$, has a canonical integral representation

\[
f(x) = -a_f x - b_f + \int_0^\infty \left( \frac{1}{t + x} - \frac{t}{t^2 + 1} \right) d\mu_f(t),
\]

where $a_f := -\lim_{y \to \infty} \frac{f(iy)}{iy} \geq 0$, $b_f := -\Re f(i) \in \mathbb{R}$ and $\mu_f$ is a positive measure on $(0, \infty)$ such that \( \int_0^\infty \frac{1}{t + x} d\mu_f(t) < \infty \), and

\[
\mu_f(x_1) - \mu_f(x_0) = -\lim_{y \to 0} \frac{1}{\pi} \int_{x_0}^{x_1} \Im f(-x + iy) dx.
\]

Conversely, every such function belongs in $Q_{(0, \infty)}$.

We consider functions $f \in Q_{(0, \infty)}$ such that $f(1) = 0$. The last condition is equivalent to

\[
0 = f(1) = -a_f - b_f + \int_0^\infty \left( \frac{1}{t + 1} - \frac{t}{t^2 + 1} \right) d\mu_f(t),
\]

in other words,

\[
a_f + b_f = \int_0^\infty \left( \frac{1}{t + 1} - \frac{t}{t^2 + 1} \right) d\mu_f(t).
\]

Therefore, the operator monotone decreasing function $f$ such that $f(1) = 0$ has the following integral representation

\[
f(x) = a_f (1 - x) + \int_0^\infty \left( \frac{1}{t + x} - \frac{1}{t + 1} \right) d\mu_f(t).
\]

**Example 2.2.** Consider the power function $f(x) = -x^p$ for $p \in (0, 1)$. It is operator monotone decreasing. Then

\[
a_f = -\lim_{y \to \infty} f(iy)/(iy) = 0, \quad \text{and} \quad b_f = \cos(p\pi/2).
\]

For $x > 0$, \( \lim_{y \to 0} \Im f(-x + iy) = -x^p \sin(p\pi) \) so that

\[
d\mu_f(x) = \pi^{-1} \sin(p\pi)x^p dx.
\]

This yields the representation

\[
-x^p = -\cos(p\pi/2) + \frac{\sin(p\pi)}{\pi} \int_0^\infty t^p \left( \frac{1}{t + x} - \frac{t}{t^2 + 1} \right) dt.
\]

(2.5)
**Example 2.3.** Let \( f(x) = -\log(x) \). It is operator monotone decreasing. Then

\[
b_f = \text{Re}(\log(i)) = 0,
\]

and

\[
a_f = \lim_{y \to \infty} \log(iy)/(iy) = \lim_{y \to \infty} (\log(y + i\pi/2))/(iy) = 0.
\]

It is clear from (2.2) that

\[
d_{\mu_f}(x) = \frac{1}{\pi} \lim_{y \to 0} \text{Im} \log(-x + iy) dx = dx.
\]

Then the integral representation (2.1) gives the following formula for the logarithmic function

\[
-\log x = \int_0^\infty \left( \frac{1}{t + x} - \frac{t}{t^2 + 1} \right) dt,
\]

which is also obvious from the direct computation of the integral.

### 2.2. Quasi-relative entropy

Quantum quasi-relative entropy was introduced by Petz [38, 39] as a quantum generalization of a classical Csiszár’s \( f \)-divergence [13]. It is defined in the context of von Neumann algebras, but we consider only the Hilbert space setup. Let \( \mathcal{H} \) be a finite-dimensional Hilbert space, and \( \rho \) and \( \sigma \) be two states (given by density operators).

**Definition 2.4.** For an operator monotone decreasing function \( f \), such that \( f(1) = 0 \), and strictly positive states \( \rho \) and \( \sigma \) acting on a finite-dimensional Hilbert space \( \mathcal{H} \), the quasi-relative entropy (or sometimes referred to as the \( f \)-divergence) is defined as

\[
S_f(\rho||\sigma) = \text{Tr}(f(\Delta_{\sigma,\rho})\rho),
\]

where the relative modular operator, introduced by Araki [1],

\[
\Delta_{A,B}(X) = L_A R_B^{-1}(X) = AXB^{-1}
\]

is a product of left and right multiplication operators, \( L_A(X) = AX \) and \( R_B(X) = XB \). Throughout this paper we consider finite-dimensional setup, so the operators are invertible. (In general, \( A^{-1} \) is stands for the generalized inverse of \( A \).)

There is a straightforward way to calculate the quasi-relative entropy from the spectral decomposition of states. Let \( \rho \) and \( \sigma \) have the following spectral decomposition

\[
\rho = \sum_j \lambda_j |\phi_j\rangle \langle \phi_j|, \quad \sigma = \sum_k \mu_k |\psi_k\rangle \langle \psi_k|,
\]

where the eigenvalues are ordered:

\[
\lambda_n \leq \cdots \leq \lambda_1, \quad \mu_n \leq \cdots \leq \mu_1.
\]
the set \( \{|\phi_k\rangle \langle \psi_j|\}_{j,k} \) forms an orthonormal basis of \( B(\mathcal{H}) \), the space of bounded linear operators, with respect to the Hilbert–Schmidt inner product defined as \( \langle A, B \rangle = \text{Tr}(A^*B) \). By [51], the modular operator can be written as

\[
\Delta_{\sigma, \rho} = \sum_{j,k} \frac{\mu_k}{\lambda_j} P_{jk},
\]

(2.8)

where \( P_{jk} : B(\mathcal{H}) \to B(\mathcal{H}) \) is defined by

\[
P_{jk}(X) = |\psi_k\rangle \langle \phi_j| X |\psi_k\rangle \langle \phi_j|.
\]

The quasi-relative entropy is calculated as follows

\[
S_f(\rho||\sigma) = \sum_{j,k} \lambda_j f \left( \frac{\mu_k}{\lambda_j} \right) |\langle \psi_k| \sigma |\psi_k\rangle|^2.
\]

(2.9)

**Example 2.5.** For \( f(x) = -\log x \), the quasi-relative entropy becomes the Umegaki relative entropy

\[
S_{-\log}(\rho||\sigma) = S(\rho||\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma).
\]

**Example 2.6.** For \( p \in (-1, 2) \) and \( p \neq 0, 1 \) let us take the function

\[
f_p(x) = \frac{1}{p(1-p)}(1-x^p),
\]

which is operator convex. The quasi-relative entropy for this function is calculated to be

\[
S_{f_p}(\rho||\sigma) = \frac{1}{p(1-p)} \left( 1 - \text{Tr}(\sigma^p \rho^{1-p}) \right).
\]

**Example 2.7.** For \( p \in (-1, 1) \) take \( q = 1-p \in (0, 2) \), the function

\[
f_q(x) = \frac{1}{1-q}(1-x^{1-q})
\]

is operator convex. The quasi-relative entropy for this function is known as Tsallis \( q \)-entropy

\[
S_q(\rho||\sigma) = \frac{1}{1-q} \left( 1 - \text{Tr}(\rho^q \sigma^{1-q}) \right).
\]

3. Quasi-relative entropy of entanglement

Quasi-relative entropy of entanglement is defined as follows

\[
E_f(\rho) = \min_{\sigma \in \mathcal{S}} S_f(\rho||\sigma),
\]

where \( \mathcal{S} \) is the set of separable states.

Note that the quasi-relative entropy satisfies all conditions (a)–(c). Condition (a) was explicitly proved in [48], or could easily be derived from Pinsker inequality [23]. Condition (b) is clear from the explicit form of the quasi-relative entropy (2.9).

Throughout the paper consider a bipartite system \( AB \) with finite-dimensional Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \). Let \( \{|\tilde{j}_A\rangle\}_j \) and \( \{|\tilde{j}_B\rangle\}_j \) denote some bases for each Hilbert space (we will drop
the subscript $A$ and $B$ since these are the only two systems we consider. For a probability distribution $\{p_j\}_j$ (i.e. $0 \leq p_j \leq 1$ and $\sum_j p_j = 1$) and a basis $\{|j\}_j$ of each finite-dimensional Hilbert space, denote a separable state

$$\sigma(\{p_j\}) = \sum_j p_j |j\rangle \langle j|_{AB}.$$  \hfill (3.1)

**Theorem 3.1.** Let $f$ be an operator monotone decreasing function, such that $f(1) = 0$. For a maximally entangled pure state $|\Psi^+\rangle = \sum_j \frac{1}{\sqrt{2}} |jj\rangle$, the quasi-relative entropy of entanglement is reached for a state $\sigma(\{\frac{1}{2}\}) = \sum_j \frac{1}{2} |jj\rangle \langle jj|$, and becomes

$$E_f(|\Psi^+\rangle \langle \Psi^+|) = f(1/d).$$

**Proof.** For any state $\sigma = \sum_k \mu_k |\psi_k\rangle \langle \psi_k|$, the quasi-relative entropy is

$$S_f(|\Psi^+\rangle \langle \Psi^+| ||\sigma) = \sum_k f(\mu_k)|\langle \psi_k| \Psi^+\rangle|^2 = \langle \Psi^+| f(\sigma) |\Psi^+\rangle.$$  

Since $f$ is convex and monotonically decreasing, we have

$$\min_{\sigma \in S} S_f(|\Psi^+\rangle \langle \Psi^+| ||\sigma) = \min_{\sigma \in S} f(\langle \Psi^+| \sigma |\Psi^+\rangle)$$  \hfill (3.2)

$$\geq \min_{\sigma \in S} f\left(\frac{\langle \Psi^+| \sigma |\Psi^+\rangle}{\sigma_{\min}}\right)$$  \hfill (3.3)

$$\geq f\left(\max_{\sigma \in S} \langle \Psi^+| \sigma |\Psi^+\rangle \right) = f(1/d).$$  \hfill (3.4)

The last equality is due to [26], for example. All inequalities are reached for $\sigma(\{\frac{1}{2}\})$, and $S_f(|\Psi^+\rangle \langle \Psi^+| ||\sigma(\{\frac{1}{2}\})) = f(1/d).$ \hfill $\Box$

The proof of the following theorem is inspired by [45].

**Theorem 3.2.** Let $f$ be an operator monotone decreasing function, such that $f(1) = 0$ and $a_f = 0$ in (2.1). Let $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$ be a pure state. If there exists a constant $H_f > 0$ such that for any $p \in [0,1]$

$$\int_0^\infty p(t + p)^{-2} \, d\mu_f(t) = H_f,$$  \hfill (3.5)

where $\mu_f(t)$ is the measure in the integral representation (2.1), then the quasi-relative entropy of entanglement is reached at a state $\sigma(\{p_j\})$ (3.1), and becomes

$$E_f(|\Psi\rangle \langle \Psi|) = \sum_j p_j f(p_j).$$

**Proof.** Let $\rho = |\Psi\rangle \langle \Psi|$ and $\sigma^* := \sigma(\{p_j\}) = \sum_j p_j |jj\rangle \langle jj|$. Consider a function

$$F(x, \sigma) := S_f(\rho)(1 - x)\sigma^* + x\sigma.$$  

For any state $\sigma$, adopting notation $\Delta(x) := \Delta(1-a_f^*+x\rho)$ and $\Delta^* := \Delta_{\sigma^*,\rho}$, we have
Therefore, we obtain

\[
\frac{\partial F}{\partial x}(0, \sigma) = \lim_{x \to 0} \frac{1}{x} \text{Tr} \left[ \{ f(\Delta(x)) - f(\Delta^*) \} \right] \\
= \lim_{x \to 0} \frac{1}{x} \int_0^\infty \mu_f(t) \text{Tr} \left[ \{ (t\mathbb{I} + \Delta(x))^{-1} - (t\mathbb{I} + \Delta^*)^{-1} \} \right] \\
= \lim_{x \to 0} \frac{1}{x} \int_0^\infty \mu_f(t) \text{Tr} \left[ \{ (t\mathbb{I} + \Delta(x))^{-1}(\Delta^* - \Delta(x))(t\mathbb{I} + \Delta^*)^{-1} \} \right] \\
= \int_0^\infty \mu_f(t) \text{Tr} \left[ \{ (t\mathbb{I} + \Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{I} + \Delta^*)^{-1} \} \right]
\]  

(3.6)  

(3.7)  

(3.8)  

(3.9)

The formula \( A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \) holds for any invertible operators \( A \) and \( B \).

For \( \rho \) and \( \sigma^* \) defined above we have

\[
\Delta^* = \sum_j p_j P_j, \quad P_j(X) = \langle jj | X | jj \rangle \langle jj | \Psi | jj \rangle.
\]

Then

\[
(t\mathbb{I} + \Delta^*)^{-1}(\rho) = \sum_j (t + p_j)^{-1} P_j(t) = \sum_j (t + p_j)^{-1} \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle.
\]

And therefore,

\[
\{ (t\mathbb{I} + \Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{I} + \Delta^*)^{-1} \} \rho = \sum_j (t + p_j)^{-1} \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle.
\]

Denote

\[
g_t(p) := \sqrt{pq}(t + p)^{-1}(t + q)^{-1}.
\]

Then taking the trace above, we obtain

\[
\text{Tr} \left[ \{ (t\mathbb{I} + \Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{I} + \Delta^*)^{-1} \} \right] \\
= \sum_j (t + p_j)^{-1} \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle (\sigma^* - \sigma) (jj) \\
= \sum_j g_t(p_j, p_k) \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \\
= \sum_j p_j^2 (t + p_j)^{-2} - \sum_j g_t(p_j, p_k) \langle jj | \Psi | jj \rangle.
\]

Therefore,

\[
\frac{\partial F}{\partial x}(0, \sigma) = \int_0^\infty \mu_f(t) \sum_j p_j^2 (t + p_j)^{-2} - \sum_j g_t(p_j, p_k) \langle jj | \Psi | jj \rangle \\
= \sum_j \int_0^\infty p_j^2 (t + p_j)^{-2} \mu_f(t) - \int_0^\infty \mu_f(t) \langle jj | \Psi | jj \rangle \\
\times \text{Tr} \left[ \sigma \sum_j g_t(p_j, p_k) \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \right] \\
= \int_0^\infty \mu_f(t) \langle jj | \Psi | jj \rangle \\
\times \text{Tr} \left[ \sigma \sum_j g_t(p_j, p_k) \langle jj | \Psi | jj \rangle \langle jj | \Psi | jj \rangle \right].
\]

(3.10)  

(3.11)  

(3.12)  

(3.13)  

(3.14)
Define

\[ G_f(p, q) := \int_0^\infty g_f(t) \, dt, \]

and

\[ H_f(p) := G_f(p, p) = \int_0^\infty p(t + p)^{-2} \, dt. \]

Since \((t + p)(t + q) \geq (\sqrt{pq} + t)^2\), we have

\[ 0 \leq G_f(p, q) = \int_0^\infty \sqrt{pq}(t + p)^{-1}(t + q)^{-1} \, dt \leq \int_0^\infty \sqrt{pq}(t + \sqrt{pq})^{-2} \, dt = H_f(\sqrt{pq}). \]

Take \(\sigma = |\alpha\beta\rangle\langle\alpha\beta|\), where \(|\alpha\rangle = \sum j a_j |j\rangle\) and \(|\beta\rangle = \sum j b_j |j\rangle\). Then

\[ \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle\langle\alpha\beta|) = \sum_j p_j H_f(p_j) - \sum_{jk} G_f(p_j, p_k) \langle kk | \sigma | jj \rangle \]

\[ = \sum_j p_j H_f(p_j) - \sum_{jk} G_f(p_j, p_k) \langle ab | \langle ab | jj \rangle \]

\[ = \sum_j p_j H_f(p_j) - \sum_{jk} G_f(p_j, p_k) a_j b_j. \]

(3.15)

(3.16)

(3.17)

We assumed that the function \(f\) is such that \(H_f(p) = H_f\) is independent of \(p\), then

\[ \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle\langle\alpha\beta|) = H_f - \sum_{jk} G_f(p_j, p_k) a_j b_j. \]

And therefore,

\[ \left| \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle\langle\alpha\beta|) - H_f \right| \leq \sum_{jk} G_f(p_j, p_k) |a_j| |b_j| |a_k| |b_k| \]

\[ \leq \sum_{jk} H_f(\sqrt{p_j p_k}) |a_j| |b_j| |a_k| |b_k| \leq H_f. \]

Since every \(\sigma \in S\) can be written as a convex combination \(\sigma = \sum q_j |\alpha_j\beta_j\rangle\langle\alpha_j\beta_j|\), we obtain a non-negative partial derivate \(\frac{\partial F}{\partial x}(0, \sigma) \geq 0\). Moreover, quasi-relative entropy of entanglement for a pure state is calculated to be

\[ E_f(|\Psi\rangle \langle\Psi|) = S_f(|\Psi\rangle \langle\Psi| \| \sigma^* ) = Tr( f(\Delta^*) |\Psi\rangle \langle\Psi| ) = \sum_j p_j f( p_j ). \]
Theorem 3.3. Let $|\Psi\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle$. Let $f$ be an operator monotone decreasing function $f$, such that $f(1) = 0$ and $a_f = 0$ in (2.1). Let $f$ be such that for any $p \in [0,1]$ the function

$$H_f(p) = \int_0^\infty p(t + p)^{-2} \, d\mu_f(t),$$

is either monotonically increasing or decreasing in $p$ [here $\mu_f(t)$ is the measure in the integral representation (2.1)]. Then, the closest separable state to $|\Psi\rangle$, when measured by the quasi-relative entropy $S_f$, is in a form $\sigma(q, 1-q)$ (3.1) for some $0 < q < 1$. In this case, the quasi-relative entropy becomes,

$$E_f(|\Psi\rangle \langle \Psi|) = \sum_j p_j f(q_j).$$

Proof. Let us denote $\sigma^* = \sigma(q, 1-q) = q |00\rangle \langle 00| + (1-q) |11\rangle \langle 11|$. From a proof of theorem 3.2, we have

$$\Delta^* = \sum_j q_j P_j, \quad P_j(X) = \langle jj| X |jj\rangle \langle jj| \langle jj|.$$

Then denoting $\rho = |\Psi\rangle \langle \Psi|$, we have

$$(t\mathbb{1} + \Delta^*)^{-1}(\rho) = \sum_j (t + q_j)^{-1} P_j(\mathbb{1}) = \sum_j (t + q_j)^{-1} \langle jj| X |jj\rangle \langle jj| \langle jj|.$$

And therefore, for any state $\sigma$, we have

$$\Delta^* = \sum_j q_j P_j, \quad P_j(X) = \langle jj| X |jj\rangle \langle jj| \langle jj|.$$

And taking the trace, we obtain

$$\text{Tr} [((t\mathbb{1} + \Delta^*)^{-1}(\sigma^* - \sigma)(t\mathbb{1} + \Delta^*)^{-1})(\rho)] = \sum_{jk} (t + q_k)^{-1} (t + q_j)^{-1} \langle jj| |kk\rangle \langle kk| (\sigma^* - \sigma) |jj\rangle \langle jj| \langle jj|.$$

Therefore,

$$\frac{\partial F}{\partial x}(0, \sigma) = \int_0^\infty d\mu_f(t) \sum_j p_j q_j (t + q_j)^{-2} - \sum_{jk} g_t(p_j, p_k) \langle kk| (\sigma^* - \sigma) |jj\rangle \langle jj|$$

$$= \sum_j \int_0^\infty p_j q_j (t + q_j)^{-2} d\mu_f(t).$$
In other words, choose
\[ \sigma = |\alpha\beta\rangle \langle \alpha\beta| \]
with \( |\alpha\rangle = \sum_j a_j |j\rangle \) and \( |\beta\rangle = \sum_j b_j |j\rangle \), we obtain
\[
\frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle \alpha\beta|) = \sum_j p_j H_f(q_j) - \sum_k \frac{p_k}{q_{jk}} G_f(q_j, q_k) \sigma |jj\rangle.
\]
For \(|\Psi\rangle = \sqrt{p} |00\rangle + \sqrt{1-p} |11\rangle\), take \( \sigma = |00\rangle \langle 00| \),
\[
\frac{\partial g}{\partial x}(0, |00\rangle \langle 00|) = (1-p)H_f(1-q) - p \frac{1-q}{q} H_f(q).
\] (3.24)
And for \( \sigma = |11\rangle \langle 11| \),
\[
\frac{\partial F}{\partial x}(0, |11\rangle \langle 11|) = p H_f(q) - (1-p) \frac{q}{1-q} H_f(1-q) = -\frac{q}{1-q} \frac{\partial F}{\partial x}(0, |00\rangle \langle 00|).
\] (3.25)
Since we need both (3.24) and (3.25) to be non-negative, choose \( q \) such that \( \frac{\partial g}{\partial x}(0, |00\rangle \langle 00|) = 0 \). In other words, choose \( q \) such that
\[
\frac{1-p}{1-q} H_f(1-q) - \frac{p}{q} H_f(q) = 0.
\]
This expression can be written as
\[
p = \frac{q H_f(1-q)}{q H_f(1-q) + (1-q) H_f(q)} = \frac{\int_0^\infty (t+1-q)^{-2} d\mu_f(t)}{\int_0^\infty [(t+1-q)^{-2} + (t+q)^{-2}] d\mu_f(t)}.
\] (3.26)

Let us note a few cases here:

- If \( H_f(p) \) is a constant, then \( p = q \) for all \( p \in [0, 1] \), which is in line with theorem 3.2.
- If \( p = 1/2 \), then since the function \((t+x)^{-2}\) is monotone decreasing in \( x \), and \( \mu_f \) is a positive measure, we have \( q = 1/2 \). And vice versa. This is in line with theorem 3.1.
- If \( p > 1/2 \), then it implies that
\[
\int_0^\infty (t+q)^{-2} d\mu_f(t) < \int_0^\infty (t+1-q)^{-2} d\mu_f(t).
\]
Since the function \((t+x)^{-2}\) is monotone decreasing in \( x \), and \( \mu_f \) is a positive measure, we have that \( q > 1-q \), or \( q > 1/2 \). And vice versa, i.e. if \( q > 1/2 \), then \( p > 1/2 \).
- If \( H_f(q) \) is an increasing function and \( p > 1/2 \), then we also have that \( p < q \). So,
\[
\frac{1}{2} < p < q \leq 1.
\]
And similarly,
\[
0 < q < p < \frac{1}{2}.
\]
• Similarly, if $H_f(q)$ is a decreasing function, we have

$$0 < p < q < \frac{1}{2}, \quad \frac{1}{2} < q < p \leq 1.$$  

Consider the difference

$$\frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - \sum_j p_j H_f(q_j) = \left| \sum_k \sqrt{\frac{p_k}{q_j q_k}} G_f(q_j, q_k) a_k b_k \right|$$

(3.27)

$$\leq \sum_k \sqrt{\frac{p_k}{q_j q_k}} G_f(q_j, q_k) |a_k b_k|$$

(3.28)

$$= 2 \sum_{k<j} \sqrt{\frac{p_k}{q_j q_k}} G_f(q_j, q_k) |a_k b_k|$$

$$+ \sum_j \frac{p_j}{q_j} H_f(q_j) |a_j|^2 |b_j|^2.$$  

(3.29)

The last equality is due to the fact that $G_f$ is symmetric in its arguments. Therefore, using (3.26), we obtain

$$\left| \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - pH_f(q) - (1 - p)H_f(1 - q) \right|$$

(3.30)

$$= \left| \frac{\partial g}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - \frac{p}{q} H_f(q) \right|$$

(3.31)

$$= 2 \sqrt{\frac{p(1 - p)}{q(1 - q)}} G_f(q, 1 - q)|a_0 a_1 b_0 b_1| + \frac{p}{q} H_f(q) |a_0|^2 |b_0|^2$$

$$+ \frac{1 - p}{1 - q} H_f(1 - q) |a_1|^2 |b_1|^2$$

(3.32)

$$= 2 \sqrt{\frac{p(1 - p)}{q(1 - q)}} G_f(q, 1 - q)|a_0 a_1 b_0 b_1| + \frac{p}{q} H_f(q) (|a_0|^2 |b_0|^2 + |a_1|^2 |b_1|^2).$$

(3.33)

Let us denote $A := |a_0||b_0|$ and $B := |a_1||b_1|$. Then $(A + B)^2 \leq \sum_j |a_j|^2 \sum_j |b_j|^2 \leq 1$. Therefore, $B \leq 1 - A$. And thus

$$\left| \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - \frac{p}{q} H_f(q) \right|$$

(3.34)

$$= 2 \sqrt{\frac{p(1 - p)}{q(1 - q)}} G_f(q, 1 - q)|a_0 a_1 b_0 b_1| + \frac{p}{q} H_f(q) (|a_0|^2 |b_0|^2 + |a_1|^2 |b_1|^2)$$

(3.35)

$$\leq \frac{p}{q} H_f(q) + 2A(1 - A) \left[ \sqrt{\frac{p(1 - p)}{q(1 - q)}} G_f(q, 1 - q) - \frac{p}{q} H_f(q) \right].$$

(3.36)
Note that from (3.26) the following holds

\[
\sqrt{\frac{p(1-p)}{q(1-q)}} \sqrt{G_f(q, 1-q) - \frac{p}{q} H_f(q)}
\]

\[
= \sqrt{\frac{p(1-p)}{q(1-q)}} \left[ G_f(q, 1-q) - \sqrt{\frac{p(1-q)}{q(1-p)} H_f(q)} \right]
\]

\[
= \sqrt{\frac{p(1-q)}{q(1-p)}} \left[ G_f(q, 1-q) - \sqrt{H_f(q) H_f(1-q)} \right]
\]

\[
\leq 0.
\] (3.39)

The last inequality is due to the Cauchy–Swartz inequality:

\[
G_f(p,q)^2 = pq \left( \int_0^\infty (t+p)^{-1}(t+q)^{-1} d\mu_f(t) \right)^2
\]

\[
\leq pq \int_0^\infty (t+p)^{-2} d\mu_f(t) \int_0^\infty (t+q)^{-2} d\mu_f(t) = H_f(p) H_f(q).
\] (3.41)

Therefore, going back we obtain

\[
\left| \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) - \frac{p}{q} H_f(q) \right|
\]

\[
\leq \frac{p}{q} H_f(q) + 2A(1-A) \left[ \sqrt{\frac{p(1-p)}{q(1-q)}} G_f(q, 1-q) - \frac{p}{q} H_f(q) \right]
\]

\[
\leq \frac{p}{q} H_f(q).
\] (3.44)

This means that \( \frac{\partial F}{\partial x}(0, |\alpha\beta\rangle \langle\alpha\beta|) \geq 0 \). Since every \( \sigma \in S \) can be written as a convex combination \( \sigma = \sum_j q_j |\alpha_j\beta_j\rangle \langle\alpha_j\beta_j| \), we obtain a non-negative partial derivative \( \frac{\partial F}{\partial x}(0, \sigma) \geq 0 \). Therefore, the chosen \( \sigma^* \) is the closest separable state to \( |\Psi\rangle \). \( \square \)

As a consequence we obtain partial cases of Theorems discussed above:

- If \( H_f(p) \) is a constant, then \( p = q \) for all \( p \in [0,1] \). And the closest separable state to a state \( |\Psi\rangle = \sqrt{p}|00\rangle + \sqrt{1-p}|11\rangle \) is the state \( \sigma^* = p |00\rangle \langle 00| + (1-p) |11\rangle \langle 11| \), which is a case of a two-qubit system in theorem 3.2.

- If \( p = 1/2 \), then \( q = 1/2 \). Therefore, the closest separable state to a maximally entangled state \( |\Psi\rangle = \frac{1}{\sqrt{2}}( |00\rangle + |11\rangle ) \) is the state \( \sigma^* = \frac{1}{2}( |00\rangle \langle 00| + |11\rangle \langle 11|) \). This is a case of a two-qubit system in theorem 3.1.

### 4. Examples

**Example 4.1.** For \( f(x) = -\log(x) \), the quasi-relative entropy becomes regular Umegaki relative entropy

\[
S_{-\log}(\rho||\sigma) = S(\rho||\sigma) = \text{Tr}(\rho \log \rho - \rho \log \sigma).
\]
For this function, $H_f(p) = 1$ for all $p \in [0, 1]$. As it was shown in [45] the relative entropy of entanglement reduces to the marginal entropy on pure states, coinciding with theorem 3.2.

$$E(|\Psi\rangle \langle \Psi|) = S(\rho_A) = -\sum_j p_j \log p_j,$$

where $\rho_A = \text{Tr}_B |\Psi\rangle \langle \Psi| = \sum_j p_j |j\rangle \langle j|$. This holds for any pure state $\rho = |\Psi\rangle \langle \Psi|$. 

**Example 4.2.** The power function $f(x) = 1 - x^{1-\alpha}$ for $\alpha \in (0, 1)$ defines the quasi-relative entropy as

$$S_{\alpha}(\rho || \sigma) := 1 - \text{Tr}(\rho^\alpha \sigma^{1-\alpha}).$$

By theorem 3.1, for a maximally entangled state $|\Psi^+\rangle = \frac{1}{\sqrt{d}} \sum_j |jj\rangle$, the minimum of the $\alpha$-relative entropy over separable states is

$$\min_{\sigma \in S} S_{\alpha}( |\Psi^+\rangle \langle \Psi^+| || \sigma) = 1 - d^{\alpha-1}.$$

**Example 4.3.** Renyi entropy of entanglement is defined as

$$E_{\alpha}^R(\rho) = \min_{\sigma \in S} S_{\alpha}^R(\rho || \sigma),$$

for $\alpha \geq 0, \alpha \neq 1$, where the Renyi relative entropy is defined as

$$S_{\alpha}^R(\rho || \sigma) = \frac{1}{\alpha - 1} \log \text{Tr}(\rho^\alpha \sigma^{1-\alpha}) = \frac{1}{\alpha - 1} \log \left(1 - S_{\alpha}(\rho || \sigma)\right).$$

Consider the case when $\alpha < 1$. Then

$$E_{\alpha}^R(\rho) = \frac{1}{\alpha - 1} \log [1 - \min_{\sigma \in S} S_{\alpha}(\rho || \sigma)].$$

And therefore, for a maximally entangled state, we have

$$E_{\alpha}^R(|\Psi^+\rangle \langle \Psi^+|) = \frac{1}{\alpha - 1} \log d^{\alpha-1} = \log d.$$

This means that for a maximally entangled state, the Renyi entropy of entanglement reduces to the marginal Renyi entropy, since

$$S_{\alpha}^R(\rho_A) = \frac{1}{1 - \alpha} \log \text{Tr} \rho_A^\alpha = \log d = E_{\alpha}^R(|\Psi^+\rangle \langle \Psi^+|),$$

where $\rho_A = \text{Tr}_B |\Psi^+\rangle \langle \Psi^+| = \frac{1}{d} \sum_j |jj\rangle \langle jj|$. This is in line with the explicit expression found for the Renyi entropy of entanglement in [12, 54]. Note that in [12] it was shown that for a pure state $|\Psi\rangle = \sum_j \sqrt{p_j} |jj\rangle$ and $\alpha < 2$, the closest separable state is

$$\sigma = \frac{1}{P_\alpha} \sum_j p_j^{1/\alpha} |jj\rangle \langle jj|,$$

where $P_\alpha = \sum_j p_j^{1/\alpha}$. The Renyi entropy of entanglement is then given by

$$E_{\alpha}^R(\Psi) = \frac{\alpha}{\alpha - 1} \log \sum_j p_j^{1/\alpha}.$$
This would be a generalization of theorem 3.3 in Renyi relative entropy case, although more work is needed to explicitly show that $H_f(p)$ is monotonic in this case, and to explicitly find coefficients in the separable state to apply the theorem for a general pure state.

**Example 4.4.** Tsallis entropy of entanglement is defined as

$$E_{\alpha}^{T}(\rho) = \min_{\sigma \in S} S_{\alpha}^{T}(\rho \parallel \sigma),$$

for $\alpha \geq 0$, $\alpha \neq 1$, where Tsallis relative entropy is defined as

$$S_{\alpha}^{T}(\rho \parallel \sigma) = \frac{1}{1-\alpha} (1 - \text{Tr}(\rho^{\alpha} \sigma^{1-\alpha})) = \frac{1}{1-\alpha} S_{\alpha}(\rho \parallel \sigma).$$

By theorem 3.1, for $\alpha < 1$, and a maximally entangled state, we have

$$E_{\alpha}^{T}(\ket{\Psi^+} \bra{\Psi^+}) = \frac{1}{1-\alpha} \min_{\sigma \in S} S_{\alpha}(\ket{\Psi^+} \bra{\Psi^+} \parallel \sigma) = \frac{1}{1-\alpha} [1 - d^{\alpha-1}].$$

Interestingly enough, Tsallis entropy of entanglement does not reduce to the marginal Tsallis entropy on a maximally entangled state, since

$$S_{\alpha}^{T}(\rho_A) = \frac{1}{\alpha-1} (1 - \text{Tr} \rho_A^{\alpha}) = \frac{1}{\alpha-1} [1 - d^{\alpha-1}],$$

where $\rho_A = \text{Tr}_B \ket{\Psi^+} \bra{\Psi^+} = \sum_{j} \ket{jj} \bra{jj}$.

**5. Conclusion**

We asked a question: what is the closest separable state to a pure state when the distance between two states is measured by the quasi-relative entropy? We answered this question in three cases: when a state is maximally entangled; when a state is on a bipartite qubit systems with a large class of functions defining quasi-relative entropy; and for any pure state and a certain class of functions. It would be interesting to generalize a qubit case (theorem 3.3) to any finite dimension and/or remove the assumptions on the functions $f$, which would result in a complete answer of the initial question.

There are multiple directions one could take to further develop this problem. Let us list just a few possibilities, besides the direct improvement of the presented results.

- It is important to answer the same question for a variety of other entanglement measures, not obtained from the quasi-relative entropy of entanglement. These measures can be induced, for example, by sandwiched relative entropy [35, 52], $\alpha-z$-Renyi relative entropy [2, 10, 53], geometric Renyi divergence [18, 31], to name a few.

- A more general question of explicitly finding a closest separable state for a mixed bipartite entangled stats is still open. The way to find an answer for two-qubit systems has been discussed in [33] for some restricted cases. For a review on the problem, see [28]. In [9, 14, 17, 22, 55] authors provide an algorithm for the calculation of the relative entropy of entanglement and for the separability problem. The existence of a similar algorithm could be explored for a quasi-relative entropy.

- An inverse problem has been considered for a while: given a separable state, which entangled states are the closest to it? For two qubit states a closed formula for entangled states has been found in [33], which was generalized to any dimensions and any number of parties in [20]. A natural question arises to generalize this result for quasi-relative entropy.
As mentioned in introduction, there are a number of entanglement measure for a mixed state. For a review see, for example, [25]. It is interesting to note that if an entanglement of one state is less than an entanglement of another state when measured by, say, relative entropy, it is not necessarily the case that this order will be preserved using other measures. In fact it was demonstrated in [50] that all asymptotic entanglement measures satisfying (a)–(c) are either identical or have a different ordering on the set of all quantum states. So the following problem arises: for a fixed entanglement value of one kind of entanglement measure, find a state with a maximum or minimum value of entanglement measured by another. Some numerical results have been done, for example, in [3, 32, 34]. Adding quasi-relative entropy into the mix, poses new frontiers in this problem.

Acknowledgments

The author would like to thank anonymous referees who worked diligently to make this manuscript better on multiple levels. AV is supported by NSF Grant DMS-1812734.

Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Anna Vershynina © https://orcid.org/0000-0001-5108-1531

References

[1] Araki H 1976 Relative entropy of states of von Neumann algebras vol 11 (Publications of the Research Institute for Mathematical Sciences) (Switzerland: European Mathematical Society) pp 809–33
[2] Audenaert K M and Datta N 2015 α–β–Rényi relative entropies J. Math. Phys. 56 022202
[3] Bartkiewicz K, Horst B, Lemr K and Miranowicz A 2013 Entanglement estimation from Bell inequality violation Phys. Rev. A 88 052105
[4] Bennett C H and Brassard G 2020 Quantum cryptography: public key distribution and coin tossing (arXiv Preprint 2003.06557) (accessed: March 2020)
[5] Bennett C H, Brassard G, Crépeau C, Jozsa R, Peres A and Wootters W K 1993 Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels Phys. Rev. Lett. 70 1895
[6] Bhatia R 1997 Matrix Analysis (Berlin: Springer)
[7] Biham O, Nielsen M A and Osborne T J 2002 Entanglement monotone derived from Grover’s algorithm Phys. Rev. A 65 062312
[8] Boto A N, Kok P, Abrams D S, Braunstein S L, Williams C P and Dowling J P 2000 Quantum interferometric optical lithography: exploiting entanglement to beat the diffraction limit Phys. Rev. Lett. 85 2733
[9] Brandao F G and Vianna R O 2004 Separable multipartite mixed states: operational asymptotically necessary and sufficient conditions Phys. Rev. Lett. 93 220503
[10] Carlen E A, Frank R L and Lieb E H 2018 Inequalities for quantum divergences and the Audenaert–Datta conjecture J. Phys. A: Math. Theor. 51 483001
[11] Carlen E A and Vershynina A 2018 Recovery and the Data processing inequality for quasi-entropies IEEE Trans. Inf. Theory 64 6929–38
[12] Chitambar E, de Vicente J I, Girard M W and Gour G 2020 Entanglement manipulation beyond local operations and classical communication *J. Math. Phys.* 51 042201

[13] Csiszar I 1967 Information-type measures of difference of probability distributions and indirect observation *Stud. Sci. Math. Hung.* 2 229–318

[14] Doherty A C, Parrilo P A and Spedalieri F M 2002 Distinguishing separable and entangled states *Phys. Rev. Lett.* 88 187904

[15] Donoghue W F 1974 *Monotone Matrix Functions and Analytic Continuation* (Die Grundlehren der mathematischen Wissenschaften vol 207) (Berlin: Springer)

[16] Durt T, Kaszlikowski D, Chen J L and Kwek L C 2004 Security of quantum key distributions with entangled qudits *Phys. Rev. A* 69 032313

[17] Eisert J, Hyllus P, Guhne O and Curty M 2004 Complete hierarchies of efficient approximations to problems in entanglement theory *Phys. Rev. A* 70 062317

[18] Fang K and Fawzi H 2019 Geometric Rényi divergence and its applications in quantum channel capacities *(arXiv Preprint 1909.05758)* (accessed: September 2019)

[19] Fan H, Korepin V and Roychowdhury V 2004 Entanglement in a valence-bond solid state *Phys. Rev. Lett.* 93 237203

[20] Friedland S and Gour G 2011 An explicit expression for the relative entropy of entanglement in all dimensions *J. Math. Phys.* 52 052201

[21] Girard M W, Gour G and Friedland S 2014 On convex optimization problems in quantum information theory *J. Phys. A: Math. Theor.* 47 505302

[22] Girard M W, Zinchenko Y, Friedland S and Gour G 2015 Erratum: numerical estimation of the relative entropy of entanglement *Phys. Rev. A* 91 029901

[23] Hiai F and Mosonyi M 2017 Different quantum f-divergences and the reversibility of quantum operations *Rev. Math. Phys.* 29 1750023

[24] Horodecki M, Horodecki P and Horodecki R 1998 Mixed-state entanglement and distillation: is there a ‘bound’ entanglement in nature? *Phys. Rev. Lett.* 80 5239

[25] Horodecki M, Horodecki P and Horodecki R 2000 Limits for entanglement measures *Phys. Rev. Lett.* 84 2014

[26] Horodecki R, Horodecki P, Horodecki M and Horodecki K 2009 Quantum entanglement *Rev. Mod. Phys.* 81 865

[27] Jing J, Zhang J, Yan Y, Zhao F, Xie C and Peng K 2003 Experimental demonstration of tripartite entanglement and controlled dense coding for continuous variables *Phys. Rev. Lett.* 90 167903

[28] Krueger O and Werner R F 2005 Some open problems in quantum information theory *(arXiv preprint quant-ph/0504166)* (accessed: April 2005)

[29] Lewenstein M, Sanpera A, Ahufinger V, Damski B, Sen A and Sen U 2007 Ultracold atomic gases in optical lattices: mimicking condensed matter physics and beyond *Adv. Phys.* 56 243–379

[30] Lin S, Wen Q Y, Gao F and Zhu F C 2008 Quantum secure direct communication with ξ-type entangled states *Phys. Rev. A* 78 064304

[31] Matsumoto K 2015 A new quantum version of f-divergence *Nagoya Winter Workshop: Reality and Measurement in Algebraic Quantum Theory* (Berlin: Springer) pp 229–73

[32] Miranowicz A and Grudka A 2004 A comparative study of relative entropy of entanglement, concurrence and negativity *J. Opt. B: Quantum Semiclass. Opt.* 6 542

[33] Miranowicz A and Ishizaka S 2008 Closed formula for the relative entropy of entanglement *Phys. Rev. A* 78 032310

[34] Miranowicz A, Ishizaka S, Horst B and Grudka A 2008 Comparison of the relative entropy of entanglement and negativity *Phys. Rev. A* 78 052308

[35] Müller-Lennert M, Dupuis F, Szehr O, Fehr S and Tomamichel M 2013 On quantum Rényi entropies: a new generalization and some properties *J. Math. Phys.* 54 122203

[36] Nielsen M A, Knill E and Laflamme R 1998 Complete quantum teleportation using nuclear magnetic resonance *Nature* 396 52–5

[37] Peng X, Zhi X, Fang X, Feng M, Liu M and Gao K 2003 Experimental implementation of remote state preparation by nuclear magnetic resonance *Phys. Lett. A* 306 271–6

[38] Petz D 1985 Quasi-entropies for states of a von Neumann algebra vol 21 (Publications of the Research Institute for Mathematical Sciences) (Switzerland: European Mathematical Society) pp 787–800

[39] Petz D 1986 Quasi-entropies for finite quantum systems *Rep. Math. Phys.* 23 57–65
[40] Plenio M B and Virmani S S 2014 An introduction to entanglement theory Quantum Information and Coherence (Berlin: Springer) pp 173–209
[41] Shapira D, Shimon Y and Biham O 2006 Groverian measure of entanglement for mixed states Phys. Rev. A 73 044301
[42] Streltsov A, Kampermann H and Bruss D 2010 Linking a distance measure of entanglement to its convex roof New J. Phys. 12 123004
[43] Sun Y, Wen Q Y, Gao F and Zhu F C 2009 Robust variations of the Bennett-Brassard 1984 protocol against collective noise Phys. Rev. A 80 032321
[44] Takayanagi T, Ugajin T and Umemoto K 2018 Towards an entanglement measure for mixed states in CFTs based on relative entropy J. High Energy Phys. JHEP10(2018)166
[45] Vedral V and Plenio M B 1998 Entanglement measures and purification procedures Phys. Rev. A 57 1619
[46] Vedral V, Plenio M B, Rippin M A and Knight P L 1997 Quantifying entanglement Phys. Rev. Lett. 78 2275
[47] Vershynina A 2019 Entanglement rates for Rényi, Tsallis, and other entropies J. Math. Phys. 60 022201
[48] Vershynina A 2019 On quantum quasi-relative entropy Rev. Math. Phys. 31 1950022
[49] Vershynina A 2019 Upper continuity bound on the quantum quasi-relative entropy J. Math. Phys. 60 102201
[50] Virmani S and Plenio M B 2000 Ordering states with entanglement measures Phys. Lett. A 268 31–4
[51] Virosztek D 2016 Ph.D. thesis Quantum entropies, relative entropies, and related pre-solver problems Ph.D. thesis Budapest University of technology and economics, Hungary https://repozitorium.omikk.bme.hu/
[52] Wilde M M, Winter A and Yang D 2014 Strong converse for the classical capacity of entanglement-breaking and Hadamard channels via a sandwiched Rényi relative entropy Commun. Math. Phys. 331 593–622
[53] Zhang H 2020 From Wigner–Yanase–Dyson conjecture to Carlen–Frank–Lieb conjecture Adv. Math. 365 107053
[54] Zhu H, Hayashi M and Chen L 2017 Coherence and entanglement measures based on Rényi relative entropies J. Phys. A: Math. Theor. 50 475303
[55] Zinchenko Y, Friedland S and Gour G 2010 Numerical estimation of the relative entropy of entanglement Phys. Rev. A 82 052336