Tilting modules of affine quasi-hereditary algebras

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June 25, 2018

Abstract

We discuss tilting modules of affine quasi-hereditary algebras. We present an existence theorem of indecomposable tilting modules when the algebra has a large center and use it to deduce a criterion for an exact functor between two affine highest weight categories to give an equivalence. As an application, we prove that the Arakawa-Suzuki functor (Arakawa-Suzuki, 1998) gives a fully faithful embedding of a block of the deformed BGG category of $\mathfrak{gl}_m$ into the module category of a suitable completion of degenerate affine Hecke algebra of $GL_n$.

1 Introduction

The notion of highest weight category and its ring theoretic counterpart, quasi-hereditary algebra introduced by Cline-Parshall-Scott [6] enables us to study representation theory of algebras of Lie theoretic origin in terms of theory of Artin algebras. Ringel’s influential work [19] on tilting modules of quasi-hereditary algebras is one of the examples. In a highest weight category, there are two kinds of distinguished indecomposable modules called standard modules and costandard modules. In [19], Ringel presented tilting modules which are characterized to be filtered by both standard modules and costandard modules.

Recently, Kleshchev [16] defined the notion of affine highest weight category and affine quasi-hereditary algebra as a graded version of the definition of Cline-Parshall-Scott [6]. Examples of affine highest weight categories include the graded module categories of Khovanov-Lauda-Rouquier (KLR) algebras of finite Lie type ([5], [13]), those of current Lie algebras etc. There are two kinds of counterparts to standard modules in an affine highest weight category called standard modules (which are of infinite length in general) and proper standard modules (which are of finite length) respectively. The counterparts to costandard modules are called proper costandard modules (which are of finite length).

In this paper, we discuss tilting modules of affine quasi-hereditary algebras, which are defined to be filtered by both standard modules and proper costandard modules. Under some conditions on the center, we prove an existence theorem

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of indecomposable tilting modules and deduce a simple criterion for an exact functor between two affine highest weight categories to give an equivalence.

For more detailed explanation, let $H$ be an affine quasi-hereditary algebra over a field $k$. By definition, $H$ has a $\mathbb{Z}$-grading $H = \bigoplus_{n \in \mathbb{Z}} H_n$ such that we have $\dim H_n < \infty$ for any $n \in \mathbb{Z}$ and $H_n = 0$ for $n \ll 0$. Let $\{\Delta(\pi) \mid \pi \in \Pi\}$ be the set of standard modules of $H$, where $\Pi$ is a finite set with a partial order $\leq$ parameterizing simple modules of $H$. For any finitely generated graded $H$-module $V$ with $\Delta$-filtration and $\pi \in \Pi$, we consider the number $(V : \Delta(\pi))$ of appearance of some grading shift of $\Delta(\pi)$ as a subquotient, which is known to be finite. We consider the following property ($\spadesuit$):

($\spadesuit$) There is a central subalgebra $Z \subset H_{\geq 0}$ with $Z_0 = k \cdot 1$ such that $H$ is finitely generated as a $Z$-module.

Note that this is equivalent to simply saying that the algebra $H$ is a finitely generated module over its center.

Our main results are the followings:

**Theorem 1.1** (Theorem 3.6). Let $H$ be an affine quasi-hereditary algebra with property ($\spadesuit$). Then for each $\pi \in \Pi$, there exists a unique indecomposable tilting module $T(\pi)$ such that $(T(\pi) : \Delta(\pi)) = 1$ and $(T(\pi) : \Delta(\sigma)) = 0$ for any $\sigma \not< \pi$. Moreover every indecomposable tilting module is isomorphic to a grading shift of $T(\pi)$ for some $\pi \in \Pi$.

**Theorem 1.2** (Theorem 3.9). Suppose that an exact functor $\Phi$ between the module categories over affine quasi-hereditary algebras with property ($\spadesuit$) induces order-preserving bijections on standard modules and on proper costandard modules. Then $\Phi$ gives an equivalence of graded categories.

For example, the property ($\spadesuit$) is satisfied by KLR algebras. Therefore Theorem 1.1 says that for each affine quasi-hereditary structure of KLR algebra of finite Lie type, there exists a complete collection of indecomposable tilting modules (in the sense of Ringel).

As an application of Theorem 1.2, we prove that the Arakawa-Suzuki functor [2] gives a fully faithful embedding of a block of the deformed BGG category of $\mathfrak{gl}_n$ into the module category of a suitable completion of degenerate affine Hecke algebra $H_n$ of $GL_n$.

The organization of this paper is as follows: Section 2 is preliminary and a brief review of affine highest weight categories following Kleshchev [16]. After a homological characterization of tilting modules (3.1), we prove the existence theorem of indecomposable tilting modules (= Theorem 1.1) in Subsection 3.2. Our argument is similar to Donkin’s exposition [7]. Then we prove Theorem 1.2 in Subsection 3.3. In Section 4, we consider affine highest weight categories and affine quasi-hereditary algebras in the setting of topologically complete algebras. We also discuss the completion of (graded) affine quasi-hereditary algebras with property ($\spadesuit$). In Section 5, we apply the results of Section 3 and 4 to the Arakawa-Suzuki functor [2], [22]. We discuss an affine quasi-hereditary
structure of a central completion of the degenerate affine Hecke algebra $H_n$ \((5.1)\) and the affine highest weight structure of the deformed BGG category of $\mathfrak{gl}_n$ \((5.2)\). Then we apply the complete version of Theorem 1.2 to prove that the Arakawa-Suzuki functor is a fully faithful functor between two affine highest weight categories \((5.3)\).

**Acknowledgements**

The author is very grateful to Syu Kato for many discussions and encouragement. He also thanks Ryosuke Kodera for helpful conversations and comments.

2 Preliminary

2.1 Overall notation

Fix a field $k$ throughout Section 2, 3 and 4. In Section 5, we consider the case $k = \mathbb{C}$. For a $k$-algebra $H$, we write $H$-$\text{Mod}$ to indicate the $k$-linear category of all left $H$-modules and we write $H$-$\text{mod}$ to indicate the $k$-linear category of finitely generated left $H$-modules. The category $H$-$\text{mod}$ is abelian if $H$ is left Noetherian. We always assume that a $k$-linear abelian category is Schurian i.e. endomorphism ring of a simple object is isomorphic to $k$. The following general lemma will be used in the sequel.

**Lemma 2.1** (cf. [3] Lemma 1.1). Let $C$ be an abelian category and $U, V \in C$. We fix a non-negative integer $i$. Suppose that for a filtration $V = V_0 \supset V_1 \supset \cdots$ in $C$, we have a natural isomorphism $\lim_{\leftarrow n} V/V_n \cong V$ and $\text{Ext}^i_C(U, V_n/V_{n+1}) = 0$ for all $n \geq 0$. Then we have $\text{Ext}^i_C(U, V) = 0$.

2.2 Notation for graded categories

We always refer gradings as $\mathbb{Z}$-gradings. A graded category is a $k$-linear category $\mathcal{C}$ equipped with a self-equivalence $q$ called the grading shift functor and its quasi-inverse $q^{-1}$. For objects $V, V'$ in the graded category $\mathcal{C}$, we write $V \simeq V'$ to indicate $V \cong q^nV'$ for some integer $n$. We set $\text{hom}_\mathcal{C}(V, V')$ to be the direct sum of spaces $\text{hom}_\mathcal{C}(q^nV, V')$ of graded homomorphisms of degree $n \in \mathbb{Z}$ and define $\text{ext}_\mathcal{C}(V, V')$ similarly. A functor between graded categories is said to be graded if it commutes with grading shift functors.

For a graded $k$-algebra $H$, we denote by $H$-$\text{gmod}$ the category of all finitely generated graded left $H$-modules $V = \bigoplus_{n \in \mathbb{Z}} V_n$ whose morphisms preserve the gradings. The category $H$-$\text{gmod}$ is a graded category whose grading shift functor is defined as $(qV)_n = V_{n-1}$. We put $\text{hom}_H(V, V') := \text{Hom}_{H$-$\text{gmod}}(q^nV, V')$. In Section 2 and 3 any modules (resp. ideals) are assumed to be graded (resp. homogeneous).
A graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ is said to be Laurentian if we have $V_{-n} = 0$ for $n \gg 0$ and $\dim_k V_n < \infty$ for all $n \in \mathbb{Z}$. For a Laurentian graded vector space $V$, its graded dimension is defined as $g\dim_q V := \sum_{n \in \mathbb{Z}} (\dim V_n) q^n \in \mathbb{Z}[[q]]$. A graded $k$-algebra is said to be Laurentian if its underlying graded vector space is Laurentian. For a Laurentian graded algebra $H$, we write $\text{hd} V$ (resp. $\text{rad} V$) for the head (resp. radical) of an object $V$ in $H\text{-gmod}$.

2.3 Affine highest weight category

In this subsection, we review the definition and some properties of a (graded) affine highest weight category following Kleshchev [16].

Let $C$ be a graded category with a set $\{L(\pi) \mid \pi \in \Pi\}$ of simple objects, which is complete and irredundant up to isomorphisms and grading shifts.

**Definition 2.2.** A graded abelian category $C$ is called a Noetherian Laurentian category if the following three conditions hold:

1. Every object $V$ in $C$ is Noetherian and has a filtration $V \supset V_1 \supset V_2 \supset \cdots$ which is separated (i.e. $\bigcap_{n=1}^{\infty} V_n = 0$) such that each quotient $V/V_n$ has finite length;
2. Each simple object $L(\pi)$ has a projective cover $P(\pi) \twoheadrightarrow L(\pi)$;
3. For all $\pi, \sigma \in \Pi$, the graded vector space $\text{hom}_C(P(\pi), P(\sigma))$ is Laurentian.

For example, the category $H\text{-gmod}$ for a left Noetherian Laurentian $k$-algebra $H$ is a Noetherian Laurentian category.

The following lemma easily follows from [16] Lemma 3.3 (iii).

**Lemma 2.3.** Let $U, V$ be objects of a Noetherian Laurentian category $C$ and $V \supset V_1 \supset V_2 \supset \cdots$ be a filtration as in Definition 2.2 (1). Then:

1. there exists an integer $N$ such that $\text{Hom}_C(U, V_n) = \text{Ext}_C^1(U, V_n) = 0$ for all $n \geq N$;
2. we have a natural isomorphism $\lim_{\leftarrow n} \text{Hom}_C(U, V/V_n) \cong \text{Hom}_C(U, V)$. \hfill \square

We assume the set $\Pi$ to be equipped with a partial order $\leq$. For each $\pi \in \Pi$, we define the standard object $\Delta(\pi)$ and the proper standard object $\bar{\Delta}(\pi)$ as:

$$
\Delta(\pi) := P(\pi)/\left( \sum_{\sigma \leq \pi, f \in \text{hom}_C(P(\sigma), P(\pi))} \text{Im} f \right),
$$

$$
\bar{\Delta}(\pi) := P(\pi)/\left( \sum_{\sigma \leq \pi, f \in \text{hom}_C(P(\sigma), \text{rad} P(\pi))} \text{Im} f \right).
$$

We say that an object $V \in C$ has a $\Delta$-filtration if $V$ has a separated filtration $V = V_0 \supset V_1 \supset \cdots$ whose subquotients are $\simeq \Delta(\pi)$ for some $\pi \in \Pi$.

**Definition 2.4.** A Noetherian Laurentian category $C$ is called an affine highest weight category if the following three conditions hold:

4
(1) For each \( \pi \in \Pi \), the kernel of the natural quotient map \( P(\pi) \to \Delta(\pi) \) has a \( \Delta \)-filtration whose subquotients are \( \simeq \Delta(\sigma) \) with \( \sigma > \pi \);

(2) For each \( \pi \in \Pi \), the algebra \( B_\pi := \text{end}_C(\Delta(\pi)) \) is isomorphic to a graded polynomial ring \( k[z_1, \ldots, z_{n_\pi}] \) for some \( n_\pi \in \mathbb{Z}_{\geq 0} \) with \( \deg z_i \in \mathbb{Z}_{> 0} \);

(3) For any \( \pi, \sigma \in \Pi \), the \( B_\sigma \)-module \( \hom_C(P(\pi), \Delta(\sigma)) \) is graded free of finite rank.

Remark 2.5. Kleshchev’s definition of affine highest weight category in [16] does not require that each \( B_\pi \) is a polynomial ring. However, we require it in order to guarantee that the global dimension of \( H \) is finite by [16] Corollary 5.25, which is needed in Subsection 3.3.

Any affine highest weight category \( \mathcal{C} \) with finite index poset \( \Pi \) is equivalent to a graded module category \( H^-\text{gmod} \) over a left Noetherian Laurentian algebra \( H \). Such an algebra \( H \) is characterized as an affine quasi-hereditary algebra, without referring to its module category. See [16] Section 6 for the definition. More precisely, we have the following.

Theorem 2.6 ([16] Theorem 6.7). For a Noetherian Laurentian algebra \( H \), the category \( H^-\text{gmod} \) is an affine highest weight category if and only if the algebra \( H \) is an affine quasi-hereditary algebra.

From now on, we assume that the index poset \( \Pi \) is finite. Thanks to Theorem 2.6, we can use the expressions “\( H \) is an affine quasi-hereditary algebra” and “\( H^-\text{gmod} \) is an affine highest weight category” interchangeably.

For each \( \pi \in \Pi \), let \( I(\pi) \) be an injective hull of \( L(\pi) \) in the category of all graded \( H \)-modules. Let \( A(\pi) \) be the largest submodule of \( I(\pi)/L(\pi) \) among all of whose composition factors are of the form \( \simeq L(\sigma) \) for \( \sigma < \pi \). We define the proper costandard module \( \bar{\nabla}(\pi) \subset I(\pi) \) to be the preimage of \( A(\pi) \) with respect to the quotient map \( I(\pi) \to I(\pi)/L(\pi) \).

Theorem 2.7 ([16] Theorem 7.6). For each \( \pi \in \Pi \), the module \( \bar{\nabla}(\pi) \) has finite length and characterized by the property:

\[
\text{ext}_H^i(\Delta(\sigma), \bar{\nabla}(\pi)) = \begin{cases}
  k & i = 0, \sigma = \pi; \\
  0 & \text{otherwise}.
\end{cases}
\]

Thanks to Lemma 2.3 ([11]) and Theorem 2.7, we know that the length of any \( \Delta \)-filtration of \( V \in H^-\text{gmod} \) is finite and the multiplicity of grading shifts of \( \Delta(\pi) \) is equal to \( (V : \Delta(\pi)) := \text{gdim}_q \text{hom}_H(V, \bar{\nabla}(\pi)) \) for \( q = 1 \).

We say that an object \( V \in \mathcal{C} \) has a \( \bar{\nabla} \)-filtration if \( V \) has a separated filtration \( V = V_0 \supset V_1 \supset \cdots \) whose subquotients are \( \simeq \bar{\nabla}(\pi) \) for some \( \pi \in \Pi \).

The following lemma is frequently used in the sequel.

Lemma 2.8 ([16] Lemma 5.10 and 7.7). Let \( \sigma, \pi \in \Pi \).

(1) If \( \sigma \not< \pi \), then \( \text{ext}_H^1(\Delta(\sigma), \Delta(\pi)) = 0 \);

(2) If \( \sigma < \pi \), then \( \text{ext}_H^1(\bar{\nabla}(\sigma), \bar{\nabla}(\pi)) = 0 \).
3 Tilting modules

We keep the notation in the previous section.

Let $H$ be an affine quasi-hereditary algebra with the index poset $\Pi$. We write $\mathcal{F}(\Delta)$ (resp. $\mathcal{F}(\nabla)$) to indicate the full subcategory of $\text{H-gmod}$ consisting of all $\Delta$-filtered (resp. $\nabla$-filtered) modules.

**Definition 3.1.** A module $V \in \text{H-gmod}$ is called a tilting module if it is both $\Delta$-filtered and $\nabla$-filtered, i.e. it belongs to $\mathcal{T} := \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

3.1 Homological characterization

**Theorem 3.2.** Let $H$ be an affine quasi-hereditary algebra and $V \in \text{H-gmod}$.

1. $V$ belongs to $\mathcal{F}(\Delta)$ if and only if $\text{ext}_H^1(V, \nabla(\pi)) = 0$ for all $\pi \in \Pi$;
2. $V$ belongs to $\mathcal{F}(\nabla)$ if and only if $\text{ext}_H^1(\Delta(\pi), V) = 0$ for all $\pi \in \Pi$.

**Proof.** [11] is [16] Lemma 7.8. A proof of (2) will be given after Lemma 3.4.

**Remark 3.3.** For the case where $V$ is of finite length, Theorem 3.2 (2) has been proved by Kleshchev [16] Lemma 7.9.

For a module $V \in \text{H-gmod}$, we define its support set $\text{supp}(V) := \{ \pi \in \Pi \mid \text{hom}_H(P(\pi), V) \neq 0 \}$. We say that a subset $A$ of $\Pi$ is saturated if it has the property that $\pi \in A$ whenever $\pi \leq \sigma$ and $\sigma \in A$.

**Lemma 3.4.** Let $V \in \text{H-gmod}$ and fix a separated filtration $V = V_0 \supset V_1 \supset \cdots$ whose subquotients $V_r/V_{r+1}$ are simple for all $r \geq 0$. Assume that $\text{ext}_H^1(\Delta(\sigma), V) = 0$ for all $\sigma \in \Pi$. Then for each saturated subset $A \subseteq \Pi$, there exists a filtration $V = W(A)_0 \supset W(A)_1 \supset \cdots$ which satisfies the following two conditions:

(a) For each $r \geq 0$, the subquotient $W(A)_r/W(A)_{r+1}$ has a $\nabla$-filtration of finite length whose subquotients are of the form $\nabla(\sigma)$ for some $\sigma \in \Pi \setminus A$;

(b) For each $n \geq 0$, we have $\text{supp}(W(A)_N/(V_n \cap W(A)_N)) \subseteq A$ for $N \gg 0$.

**Proof.** We proceed by induction on the size of $\Pi \setminus A$. The case where $A = \Pi$ is trivial by setting $W(A)_i := V$ for every $i \geq 0$.

Let $\pi \in \Pi \setminus A$ be a minimal element and set $A' := A \cup \{\pi\}$. Then $A'$ is a saturated subset and $|A'| = |A| + 1$. By induction hypothesis, there exists a filtration $V = W(A')_0 \supset W(A')_1 \supset \cdots$ satisfying the conditions (a) and (b) with respect to $A'$.

Since for each $n$ the quotient $V/V_n$ is finite length, we find a sequence of integers $0 < N_1 < N_2 < \cdots$ such that for each $n$ we have natural isomorphisms $W(A')_{N_n}/V_n \cap W(A')_{N_n} \cong W(A')_{r}/V_n \cap W(A')_{r}$ for all $r \geq N_n$. Note that we have a natural surjective map

$$
\frac{W(A')_{N_{n+1}}}{V_{n+1} \cap W(A')_{N_{n+1}}} \to \frac{W(A')_{N_{n+1}}}{V_{n} \cap W(A')_{N_{n+1}}} \cong \frac{W(A')_{N_n}}{V_{n} \cap W(A')_{N_n}},
$$
for each $n \geq 0$. The projective limit

$$J(A') := \lim_{\to n} \frac{W(A')_{N_n}}{V_n \cap W(A')_{N_n}}$$

is contained in $W(A'_m)$ for all $m$. By construction and the condition $(\beta)$, we have $\text{supp} J(A') \subset A'$. For each $\sigma \in A'$, we apply the functor $\text{hom}_H(\Delta(\sigma), -)$ to the short exact sequence $0 \to J(A') \to V \to V/J(A') \to 0$ to get:

$$\text{hom}_H(\Delta(\sigma), V/J(A')) \to \text{ext}^1_H(\Delta(\sigma), J(A')) \to \text{ext}^1_H(\Delta(\sigma), V) = 0.$$ 

We observe that $\text{hom}_H(\Delta(\sigma), V/J(A')) = 0$, which follows from the fact $V/J(A')$ has a $\nabla$-filtration whose subquotients are of the form $\nabla(\rho)$ for some $\rho \in H \setminus A'$ and Theorem 2.7. Thus we have $\text{ext}^1_H(\Delta(\sigma), J(A')) = 0$ for all $\sigma \in A'$.

If we have $\text{supp}(W(A')_{N_n}/(V_n \cap W(A')_{N_n})) \subset A$ for all $n \geq 0$, then put $W(A)_n := W(A')_n$ for all $n \geq 0$ and we are done. If not, we prove the following claim:

**Claim.** There is a strictly increasing sequence of integers $0 \leq n_1 < n_2 < \cdots$, and for each $i \in \mathbb{Z}_{\geq 0}$ a filtration $V = W_i(A)_0 \supset W_i(A)_1 \supset \cdots$ such that:

1. For each $r \geq 0$, the subquotient $W_i(A)_r/W_i(A)_{r+1}$ has a $\nabla$-filtration of finite length whose subquotients are of the form $\nabla(\sigma)$ for some $\sigma \in H \setminus A$;
2. For each $0 \leq r \leq n_i$, we have $\text{supp}(W_i(A)/W_i(A)_{N_n}) \subset A$ for sufficiently large $N$;
3. $W_i(A)_r = W_{i-1}(A)_r$ for $0 \leq r < n_i$.

**Proof.** Let $n_1$ be the smallest integer satisfying $\pi \in \text{supp}(W(A')_{N_{n_1}}/(V_{n_1} \cap W(A')_{N_{n_1}})).$ We put $W := W(A'_{N_{n_1}}).$ Since $\pi \in \text{supp}(W/(V_{n_1} \cap W))$ and $\pi \not\in \text{supp}(W/(V_{n_1-1} \cap W))$, we see that there is a unique embedding $q^mL(\pi) \to W/(V_{n_1} \cap W)$ for some $m \in \mathbb{Z}$. We lift the natural embedding $q^mL(\pi) \to q^mI(\pi)$ to a morphism $\phi : W/(V_{n_1} \cap W) \to q^mI(\pi)$ so that we have $X := \text{Im}(\phi) \subset q^m\nabla(\pi) \subset q^mI(\pi)$. We set $Y := \text{Ker}(J(A') \to W/(V_{n_1} \cap W) \to X)$ and get an exact sequence:

$$0 \to Y \to J(A') \to X \to 0. \quad (3.1)$$

For each $\sigma \in A'$, we apply the functor $\text{hom}_H(\Delta(\sigma), -)$ to (3.1) to obtain an exact sequence

$$\text{hom}_H(\Delta(\sigma), J(A')) \to \text{hom}_H(\Delta(\sigma), X) \to \text{ext}^1_H(\Delta(\sigma), Y) \to \text{ext}^1_H(\Delta(\sigma), J(A')) = 0.$$ 

Since $X \subset q^m\nabla(\pi)$, Theorem 2.7 implies that $\text{hom}_H(\Delta(\sigma), X) = 0$ for $\sigma \neq \pi$. Therefore we have $\text{ext}^1_H(\Delta(\sigma), Y) = 0$ for $\sigma \in A = A' \setminus \{\pi\}$. When $\sigma = \pi$, we have

$$\text{hom}_H(\Delta(\pi), J(A')) \xrightarrow{\delta} \text{hom}_H(\Delta(\pi), X) \to \text{ext}^1_H(\Delta(\pi), Y) \to 0.$$ 


Note that the full subcategory $H\text{-gmod}_{A'} \subset H\text{-gmod}$ consisting of modules $V$ with $\text{supp}(V) \subset A'$ is also an affine highest weight category with the set of standard objects $\{\Delta(\sigma) \mid \sigma \in A'\}$ (cf. [16] Proposition 5.16). The map $a$ is surjective because $\pi$ is maximal in $A'$ and so $\Delta(\pi)$ is projective in $H\text{-gmod}_{A'}$. Thus we have $\text{ext}^1_H(\Delta(\pi), Y) = 0$.

We have seen that $\text{ext}^1_H(\Delta(\sigma), J(A')) = \text{ext}^1_H(\Delta(\sigma), Y) = 0$ for all $\sigma \in A'$. We can apply a standard argument as in [10] Section 6.12 to find that $\text{ext}^1_H(\Delta(\sigma), J(A')) = \text{ext}^1_H(\Delta(\sigma), Y) = 0$ for all $\sigma \in A'$ and $i \geq 1$. By considering the long exact sequence induced from the short exact sequence (5.1), we conclude that $\text{ext}^1_H(\Delta(\sigma), X) = 0$ for all $\sigma \in A'$ and $i \geq 1$. Since $X$ is of finite length, we can apply Theorem 3.2 to $X$ in $H\text{-gmod}_{A'}$ (see Remark 3.3) to find that $X = q^n\nabla(\pi)$.

We set $W_0(A)_r := W(A')$, for all $r \geq 0$ and define

$$W_1(A)_r := \begin{cases} W_0(A)_r & 0 \leq r < n_1; \\ W_0(A)_r \cap Z & n_1 \leq r, \end{cases}$$

where $Z := \text{Ker}(W \to W/\cap V_{n_1} \to X = q^n\nabla(\pi))$. Then we have a filtration $V = W_1(A)_0 \supset W_1(A)_1 \supset \cdots$ satisfying the conditions $(\alpha)_1, (\beta)_1$ and $(\gamma)_1$. By iterating the same argument, we prove the claim inductively.

To complete the proof of Lemma 5.4 we set $W(A)_r := \lim_{i \to \infty} W_i(A)_r$, for each $r \in \mathbb{Z}_{\geq 0}$, which is well-defined by the condition $(\gamma)_i$. Then the filtration $V = W(A)_0 \supset W(A)_1 \supset \cdots$ satisfies the required condition $(\alpha)$ and $(\beta)$.

**Proof of Theorem 3.2.** Assume that $V$ has a $\nabla$-filtration, say $V = V_0 \supset V_1 \supset \cdots$. We have $\text{ext}^1_H(\Delta(\pi), V_n/V_{n+1}) = 0$ for all $\pi \in \Pi$ and $n \geq 0$ by Theorem 2.7. Then by Lemma 2.1, we have $\text{ext}^1_H(\Delta(\pi), V) = 0$ for all $\pi \in \Pi$. The converse implication is a special case of Lemma 5.4 where $A = \emptyset$.

### 3.2 Existence theorem

We consider the following property for a left Noetherian Laurentian algebra $H$.

\[(\heartsuit)\] There is a central $\mathbb{Z}_{\geq 0}$-graded subalgebra $Z \subset H_{\geq 0}$ with $Z_0 = k \cdot 1$ such that $H$ is finitely generated as a $Z$-module.

Note that $Z$ is a graded local ring with maximal ideal $m := Z_{>0}$. For a module $M \in Z\text{-gmod}$, we have $\text{hd } M = M/mM$, which is finite-dimensional over $k$. If $H$ is a Noetherian Laurentian algebra with property $(\heartsuit)$, then $\text{ext}^i_H(U, V) \in Z\text{-gmod}$ for any $U, V \in H\text{-gmod}$ and $i \geq 0$.

For a subset $A \subset \Pi$, we define its closure by $\overline{A} := \{\pi \in \Pi \mid \pi \leq \sigma, \exists \sigma \in A\}$.

**Lemma 3.5.** Let $H$ be an affine quasi-hereditary algebra with property $(\heartsuit)$. For each $\pi \in \Pi$, let $\mathcal{F}_\pi(\Delta)$ (resp. $\mathcal{H}_\pi$) be the full subcategory of $\mathcal{F}(\Delta)$ (resp. $\mathcal{H}$) consisting of modules $V$ satisfying that $(V : \Delta(\pi)) = 1$ and $(V : \Delta(\sigma)) = 0$ for any $\sigma \not\leq \pi$. Then for each $V \in \mathcal{F}_\pi(\Delta)$, there is an embedding $V \hookrightarrow T$ into some $T \in \mathcal{F}_\pi$.  


Proof. For a module $V \in H\text{-}\mathsf{mod}$, we define a subset $\mathcal{D}(V) \subset \Pi$ by $\mathcal{D}(V) := \{ \sigma \in \Pi \mid \text{ext}^1_H(\Delta(\sigma), V) \neq 0 \}$. Assume to be the contrary to derive a contradiction. Let $\mathcal{E}_\pi(\Delta) \subset \mathcal{F}_\pi(\Delta)$ be the full subcategory of modules $V \in \mathcal{F}_\pi(\Delta)$ which have no embeddings into any tilting modules belonging to $\mathcal{F}_\pi$. Let $A \subset \Pi$ be a subset with the smallest size $|A|$ among those of the form $A = \mathcal{D}(V)$ for some $V \in \mathcal{E}_\pi(\Delta)$. Note that we have $\sigma < \pi$ for any $\sigma \in A$ because of Lemma 2.8 (1).

If $A = \emptyset$, there is a module $V \in \mathcal{E}_\pi(\Delta)$ with the property $\text{ext}^1_H(\Delta(\sigma), V) = 0$ for all $\sigma \in \Pi$. Then $V \in \mathcal{F}(\nabla)$ by Theorem 2.8 (2) and thus $V$ already belongs to $\mathcal{F}_\pi$, which contradicts to our assumption. For the case $A \neq \emptyset$, we fix a maximal element $\rho \in A$ and pick a module $V \in \mathcal{E}_\pi(\Delta)$ so that $\mathcal{D}(V) = A$ and $\dim_k(\text{hd}\text{ext}^1_H(\Delta(\rho), V))$ is as small as possible. Let $\theta \in \text{Ext}^1_H(q^m\Delta(\rho), V)$ be an element such that $\theta \neq 0 \mod \text{ext}^1_H(\Delta(\rho), V)$. Consider the extension $0 \to V \to V' \to q^m\Delta(\rho) \to 0$ corresponding to $\theta$. Then we have $V' \in \mathcal{F}_\pi(\Delta)$ and for any $\sigma \in \Pi$ we have an exact sequence in $Z\text{-}\mathsf{mod}$:

$$\text{hom}_H(\Delta(\sigma), q^m\Delta(\rho)) \to \text{ext}^1_H(\Delta(\sigma), V) \to \text{ext}^1_H(\Delta(\sigma), V') \to \text{ext}^1_H(\Delta(\sigma), q^m\Delta(\rho)).$$

First, we claim that $\mathcal{D}(V') \subset A$. When $\sigma \not\in A$, we have $\text{ext}^1_H(\Delta(\sigma), V) = 0$ and we have $\text{ext}^1_H(\Delta(\sigma), \Delta(\rho)) = 0$ since $\sigma \not\in \rho$ and Lemma 2.8 (1). Therefore, we conclude that $\text{ext}^1_H(\Delta(\sigma), V') = 0$ for any $\sigma \not\in A$, from which the claim follows.

Next, we consider the case $\sigma = \rho$. We have $\text{ext}^1_H(\Delta(\rho), q^m\Delta(\rho)) = 0$ by Lemma 2.8 (1). By taking $m\text{-coinvariants}$, we get an exact sequence:

$$q^{-m}\text{hd}(B_\rho) \to \text{hd}\text{ext}^1_H(\Delta(\rho), V) \to \text{hd}\text{ext}^1_H(\Delta(\rho), V') \to 0,$$

where the most left arrow is non-zero because of our choice of the extension $\theta$. Therefore, we have $\dim_k(\text{hd}\text{ext}^1_H(\Delta(\rho), V')) < \dim_k(\text{hd}\text{ext}^1_H(\Delta(\rho), V))$. By the minimality assumption, we have $V' \not\in \mathcal{E}_\pi(\Delta)$. Thus there exists an embedding $V' \to T'$ into some $T' \in \mathcal{F}_\pi$. Since $V$ is a submodule of $V'$, this contradicts to our assumption. \hfill $\square$

**Theorem 3.6.** Let $H$ be an affine quasi-hereditary algebra with property $(\spadesuit)$.

1. For each $\pi \in \Pi$, there exists an indecomposable tilting module $T(\pi) \in H\text{-}\mathsf{mod}$ such that $(T(\pi) : \Delta(\pi)) = 1$ and $(T(\pi) : \Delta(\sigma)) = 0$ for any $\sigma \not\leq \pi$. Such $T(\pi)$ is unique up to isomorphisms and grading shifts.

2. For each $\pi \in \Pi$, the algebra $\text{End}_H(T(\pi))$ is a local ring with quotient $\cong k$;

3. Every tilting module is a finite direct sum of the modules $q^nT(\pi)$ for some $\pi \in \Pi$ and $n \in \mathbb{Z}$.

**Proof.** We apply Lemma 3.3 to $\Delta(\pi)$ to find an embedding $\Delta(\pi) \hookrightarrow T$ for some $T \in \mathcal{F}$ which satisfies $(T : \Delta(\pi)) = 1$ and $(T : \Delta(\sigma)) = 0$ for any $\sigma \not\leq \pi$. By Theorem 2.2, the indecomposable direct summand $T(\pi)$ of $T$ with $(T(\pi) : \Delta(\pi)) = 1$ is also a tilting module and satisfies the required property in (1). We prove the uniqueness of $T(\pi)$ later.

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Let us prove the assertion (2). By Lemma 2.8 (1), we have an embedding \( \Delta(\pi) \hookrightarrow q^nT(\pi) \) for some \( m \in \mathbb{Z} \) such that \( T(\pi)/\Delta(\pi) \) also has a \( \Delta \)-filtration. By grading shift, we may normalize so that \( m = 0 \). Thus we have a short exact sequence: \( 0 \to \Delta(\pi) \to T(\pi) \to Q(\pi) \to 0 \), where \( Q(\pi) \) has a \( \Delta \)-filtration whose subquotients are of the form \( \simeq \Delta(\sigma) \) for some \( \sigma < \pi \). We apply the functor \( \text{hom}_H(-, \nabla(\pi)) \) to get:

\[
\text{hom}_H(Q(\pi), \nabla(\pi)) \to \text{hom}_H(T(\pi), \nabla(\pi)) \to \text{hom}_H(\Delta(\pi), \nabla(\pi)) \to \text{ext}^1_H(Q(\pi), \nabla(\pi)).
\]

By Theorem 2.7, the first and the fourth terms are zero. Therefore we have \( \text{hom}_H(T(\pi), \nabla(\pi)) \cong \text{hom}_H(\Delta(\pi), \nabla(\pi)) \cong \mathbb{k} \). This and Lemma 2.8 (2) show that there exists a unique submodule \( U(\pi) \subset T(\pi) \) such that \( T(\pi)/U(\pi) \cong \nabla(\pi) \). Then, by the graded version of Fitting’s lemma, we can see that \( I_\pi := \{ f \in \text{end}_H(T(\pi)) \mid f(T(\pi)) \subset U(\pi) \} \) is the Jacobson radical of \( \text{end}_H(T(\pi)) \), with \( \text{end}_H(T(\pi))/I_\pi \cong \mathbb{k} \). In particular, \( \text{end}_H(T(\pi)) \) is local.

To prove the assertion (3) and the uniqueness of \( T(\pi) \)'s, it is enough to show that, for each tilting module \( T \), there is a direct sum decomposition \( T \cong T' \oplus q^nT(\pi) \) for some \( \pi \in \Pi \), \( m \in \mathbb{Z} \) and \( T' \in \mathcal{F} \). Let \( \pi \) be a maximal element in \( \text{supp}(T) \). Then Lemma 2.8 (2) shows that there is an exact sequence: \( 0 \to U \to T \to q^n\nabla(\pi) \to 0 \) with \( U \in \mathcal{F}(\nabla) \). We then lift the surjection \( q^n\nabla(\pi) \to \nabla(\pi) \) to a map \( \phi : q^nT(\pi) \to T \). By a similar argument, we find a map \( \psi : T \to q^nT(\pi) \) of the converse direction which satisfies that \( \psi \circ \phi \equiv \text{id} \mod I_\pi \). Then from the assertion (2) proved in the previous paragraph, we see that \( \psi \circ \phi \) is an isomorphism and thus \( q^nT(\pi) \) is a direct summand of \( T \). The complement \( T/q^nT(\pi) \) is also tilting by Theorem 3.2.

**Proposition 3.7.** Let \( H \) be an affine quasi-hereditary algebra with property (\( \star \)). Then a module \( V \in H\text{-gmod} \) belongs to \( \mathcal{F}(\Delta) \) if and only if \( V \) has a finite right resolution by tilting modules.

**Proof.** Suppose that \( V \in \mathcal{F}(\Delta) \). To prove that \( V \) has a right tilting resolution, we proceed by induction on the size of the set \( \text{supp}(V) \). The case where \( \text{supp}(V) = \emptyset \) is trivial. Let \( \pi \in \text{supp}(V) \) be a maximal element. By Lemma 2.8 (1), we have a submodule \( U \in V \) which is a direct sum of some grading shifts of \( \Delta(\pi) \) and the quotient \( V/U \) has a \( \Delta \)-filtration with \( (V/U : \Delta(\pi)) = 0 \). Then we have \( \text{supp}(V/U) \subset \text{supp}(V) \setminus \{ \pi \} \). By induction hypothesis, we have a right tilting resolution \( V/U \to T'_0 \to T'_1 \to \cdots \) and get a map \( \phi : V \to V/U \to T'_* \). On the other hand, Theorem 3.6 shows that there is an exact sequence \( 0 \to \Delta(\pi) \to T(\pi) \to Q(\pi) \to 0 \), where \( Q(\pi) \in \mathcal{F}(\Delta) \) and \( (Q(\pi) : \Delta(\sigma)) = 0 \) for any \( \sigma \neq \pi \). Because the module \( Q(\pi) \) has a tilting resolution by the induction hypothesis, the standard module \( \Delta(\pi) \) has a tilting resolution. Thus \( U \) has a tilting resolution \( U \to T'_0 \to T'_1 \to \cdots \). Since \( \text{ext}_H^1(V/U, T'_0) = 0 \) we can lift the map \( U \to T'_* \) to a map \( \psi : V \to T'_* \). Therefore we have a map \( \phi \oplus \psi : V \to T'_* \oplus T'_*, \) which gives a tilting resolution of \( V \).

The other implication is a direct consequence of Theorem 3.2 (I).
3.3 A criterion for categorical equivalence

For an additive (resp. abelian) category \( C \), we denote its bounded homotopy (resp. derived) category by \( K^b(C) \) (resp. \( D^b(C) \)).

**Lemma 3.8.** Let \( H \) be an affine quasi-hereditary algebra with property (\( \spadesuit \)) and \( C := H_{-\text{gmod}} \). Then the natural functor \( K^b(\mathcal{F}) \to D^b(C) \), which is the composition of the natural embedding \( K^b(\mathcal{F}) \to K^b(C) \) and the quotient \( K^b(C) \to D^b(C) \), gives an equivalence of graded triangulated categories.

**Proof.** Note that every affine quasi-hereditary algebra has a finite global dimension by Remark 2.5. Thanks to Theorem 3.2, 3.6 and Proposition 3.7, we can apply the graded version of [9] Lemma 1.1 and 1.5 to the module \( \bigoplus \pi \in \Pi T(\pi) \). □

For each \( i = 1, 2 \), let \( H_i \) be an affine quasi-hereditary algebra with property (\( \spadesuit \)) and \( C_i := H_i_{-\text{gmod}} \) be the associated affine highest weight category, whose index poset is denoted by \( \Pi_i \). For each \( \pi \in \Pi_i \), we denote by \( \Delta_i(\pi) \) (resp. \( \overline{\Delta}_i(\pi), T_i(\pi) \)) the corresponding standard (resp. proper costandard, indecomposable tilting) module in \( C_i \). We denote by \( \mathcal{F}_i := \mathcal{F}(\Delta_i) \cap \mathcal{F}(\overline{\Delta}_i) \) the additive full subcategory of tilting modules.

**Theorem 3.9.** Under the above notation, assume that there is a graded exact functor \( \Phi : C_1 \to C_2 \) and a bijection \( \phi : \Pi_1 \to \Pi_2 \) preserving the partial orderings such that we have \( \Phi(\Delta_1(\pi)) \cong \Delta_2(\phi(\pi)) \) and \( \Phi(\overline{\Delta}_1(\pi)) \cong \overline{\Delta}_2(\phi(\pi)) \) for all \( \pi \in \Pi_1 \). Then the functor \( \Phi \) gives an equivalence of graded categories \( \Phi : C_1 \cong C_2 \).

First we prove the following lemma:

**Lemma 3.10.** Under the same assumption as in Theorem 3.9, we have a natural isomorphism \( \text{hom}_{C_1}(U, V) \cong \text{hom}_{C_2}(\Phi(U), \Phi(V)) \) for any \( U \in \mathcal{F}(\Delta_1), V \in \mathcal{F}(\overline{\Delta}_1) \). In particular, we have a natural graded isomorphism \( \text{end}_{C_1}(T) \cong \text{end}_{C_2}(\Phi(T)) \) for all \( T \in \mathcal{F}_1 \).

**Proof.** Let \( V = V_0 \supseteq V_1 \supseteq \cdots \) be a \( \overline{\Delta}_1 \)-filtration. By Theorem 2.7 and the five lemma, we have that \( \text{hom}_{C_1}(U, V, V_n) \cong \text{hom}_{C_2}(\Phi(U), \Phi(V), \Phi(V_n)) \) for all \( n \geq 0 \). Then Lemma 2.3 (2) completes the proof. □

**Proof of Theorem 3.9** By assumption, the functor \( \Phi \) sends a tilting module to a tilting module. By Theorem 3.6, we have a direct sum decomposition: \( \Phi(T_1(\pi)) \cong \bigoplus_{\sigma \in \Pi_1} T_2(\phi(\sigma))^{\oplus a_\sigma^*(q)} \) for each \( \pi \in \Pi_1 \), where \( a_\pi^*(q) \in \mathbb{Z}_{\geq 0}[q, q^{-1}] \).

Since the bijection \( \phi : \Pi_1 \to \Pi_2 \) preserves the partial orderings, we have \( a_\pi^*(q) = 1 \) and \( a_\sigma^*(q) = 0 \) for any \( \sigma \preceq \pi \). By Lemma 3.10, we have an isomorphism

\[
\text{end}_{C_1}(T_1(\pi)) \cong \text{end}_{C_2}(\Phi(T_1(\pi))) \\
\cong \text{end}_{C_2}(T_2(\phi(\pi))) \oplus \bigoplus_{\sigma < \pi} \text{end}_{C_2}(T_2(\phi(\sigma))^{\oplus a_\sigma^*(q)}) \oplus N,
\]

where \( N \) is a part included in the radical. Using Theorem 3.6 (2), we observe that \( a_\pi^*(q) = 0 \) for all \( \sigma < \pi \).
This shows that \( \Phi(T_1(\pi)) \cong T_2(\phi(\pi)) \). By Theorem 3.6 and Lemma 3.10, the functor \( \Phi \) induces a graded equivalence \( \mathcal{F}_1 \cong \mathcal{F}_2 \) and therefore yields an equivalence \( \Phi : K^b(\mathcal{F}_1) \cong K^b(\mathcal{F}_2) \) of graded triangulated categories. By Lemma 3.8, we have \( \Phi : D^b(C_1) \cong D^b(C_2) \). Since our functor \( \Phi \) is exact, it yields an equivalence of the hearts of the standard \( t \)-structures, which are naturally identified with the original abelian categories \( C_i \).

4 Completion

4.1 Complete category

Let \( C \) be a \( k \)-linear abelian category with a subfunctor \( \varphi = \text{Id}_C \) which preserves epimorphisms. Note that each object \( V \in C \) has a filtration \( V \supset \varphi(V) \supset \varphi^2(V) \supset \cdots \). We say that the category \( C \) is \( \varphi \)-adically complete if for each object \( V \in C \), the projective limit \( \varprojlim V/\varphi^n(V) \) exists and naturally isomorphic to \( V \).

An exact functor \( \Phi : C \to C' \) from a \( \varphi \)-adically complete category \( C \) to a \( \varphi' \)-adically complete category \( C' \) is said to be continuous if there is a positive integer \( n \) such that \( \Phi \circ \varphi^n \) is a subfunctor of \( \varphi' \circ \Phi \).

Assume that \( C \) have a complete set \( \{ L(\pi) \mid \pi \in \Pi \} \) of representatives of simple isomorphism classes in \( C \). We consider the following property (♣) for a \( \varphi \)-adically complete category \( C \):

(♣)\(_1 \) Each object \( V \in C \) is Noetherian;

(♣)\(_2 \) Each simple object \( L(\pi) \) has a projective cover \( P(\pi) \twoheadrightarrow L(\pi) \);

(♣)\(_3 \) For each object \( V \in C \), the quotient \( V/\varphi(V) \) is of finite length.

We give an example of such a category. Let \( H \) be a left Noetherian \( k \)-algebra and \( I \subset H \) be a two-sided ideal of finite codimension. Assume that \( H \) is \( I \)-adically complete i.e. \( H \cong \varprojlim H/I^n \). We define an endofunctor \( \varphi \) on \( C := H \text{-mod} \) by \( \varphi(V) := IV \). Then \( \varphi \) is naturally a subfunctor of \( \text{Id}_C \) preserving epimorphisms and \( C \) is a \( \varphi \)-adically complete category with property (♣).

Proposition 4.1. Let \( C \) be a \( \varphi \)-adically complete category with property (♣). Assume that the set \( \Pi \) is finite. We put \( P := \bigoplus_{\pi \in \Pi} P(\pi) \) and \( H := \text{End}_C(P)^{op} \). Then the algebra \( H \) is left Noetherian, complete with respect to the finite-codimensional ideal \( I := \text{Hom}_C(P, \varphi(P)) \). Moreover the functor \( \Phi := \text{Hom}_C(P, -) \) gives a continuous equivalence \( \Phi : C \cong H \text{-mod} \).

Proof. By property (♣), the category \( C \) has enough projectives and every projective object is isomorphic to a finite direct sum of \( P(\pi) \)'s for some \( \pi \in \Pi \). Therefore \( P \) is a projective generator of the category \( C \) and the equivalence \( \Phi : C \cong H \text{-mod} \) follows from a general result of abelian categories (see [3] Theorem II.1.3). The algebra \( H \) is left Noetherian by (♣)\(_1 \). We have to show that the ideal \( I \) is finite-codimensional and the algebra \( H \) is \( I \)-adically complete. First we claim that for any object \( V \in C \), we have \( I \cdot \text{Hom}_C(P, V) = \)}
Because \( P \) is a projective generator of \( \mathcal{C} \), we have an epimorphism \( a : P^N \to V \) for some \( N > 0 \). Since the functor \( \varphi \) preserves epimorphisms, the morphism \( \varphi(a) : \varphi(P)^{\oplus N} \to \varphi(V) \) is still epic and induces a surjection \( \text{Hom}_\mathcal{C}(P, \varphi(P))^{\oplus N} = I^{\oplus N} \to \text{Hom}_\mathcal{C}(P, \varphi(V)) \), from which the claim follows. Then we have \( \text{Hom}_\mathcal{C}(P, P/\varphi^n(P)) \cong \text{Hom}_\mathcal{C}(P, P)/\text{Hom}_\mathcal{C}(P, \varphi^n(P)) = \text{End}_\mathcal{C}(P)/I^n \). Therefore \( I \) is finite-codimensional by (♣)3. Because the category \( \mathcal{C} \) is \( \varphi \)-adically complete, we have \( \text{End}_\mathcal{C}(P) = \lim \leftarrow \text{Hom}_\mathcal{C}(P, P/\varphi^n(P)) = \lim \leftarrow \text{End}_\mathcal{C}(P)/I^n \), which proves that the algebra \( H \) is \( \bar{I} \)-adically complete.

4.2 Complete affine highest weight category

Let \( \mathcal{C} \) be a \( \varphi \)-adically complete category with property (♣). We write \( \text{hd}V \) (resp. \( \text{rad}V \)) to indicate the head (resp. radical) of \( V \in \mathcal{C} \). We assume that the index set \( \Pi \) is equipped with a partial order \( \leq \). Then we define the standard object \( \Delta(\pi) \) and the proper standard object \( \bar{\Delta}(\pi) \) for each \( \pi \in \Pi \) as:

\[
\Delta(\pi) := P(\pi)/\left( \sum_{\sigma \leq \pi, f \in \text{Hom}_\mathcal{C}(P(\sigma), P(\pi))} \text{Im } f, \right),
\]

\[
\bar{\Delta}(\pi) := P(\pi)/\left( \sum_{\sigma \leq \pi, f \in \text{Hom}_\mathcal{C}(P(\sigma), \text{rad } P(\pi))} \text{Im } f \right).
\]

**Definition 4.2.** Let \( \mathcal{C} \) be a \( \varphi \)-adically complete category with property (♣). The category \( \mathcal{C} \) is called a complete affine highest weight category if the following three conditions hold:

1. For each \( \pi \in \Pi \), the kernel of the natural quotient morphism \( P(\pi) \to \Delta(\pi) \) has a \( \Delta \)-filtration whose subquotients are \( \cong \Delta(\sigma) \) for some \( \sigma > \pi \);

2. For each \( \pi \in \Pi \), the algebra \( B_\pi := \text{End}_\mathcal{C}(\Delta(\pi)) \) is isomorphic to a ring \( k[[z_1, \ldots, z_{n_\pi}]] \) of formal power series in \( n_\pi \)-variables for some \( n_\pi \in \mathbb{Z}_{\geq 0} \);

3. For any \( \pi, \sigma \in \Pi \), the \( B_\sigma \)-module \( \text{Hom}_\mathcal{C}(P(\pi), \Delta(\sigma)) \) is free of finite rank.

In the case of complete category, we also have the parallel theory of affine highest weight category to the case of graded category as in Section [2] and [3], although we should replace the property (♠) with the following (♣):

(♣) There is a central local subalgebra \( Z \subset H \) with maximal ideal \( m \) such that \( Z/m \cong k \) and \( H \) is finitely generated as a \( Z \)-module.

Note that this is equivalent to simply saying that the algebra \( H \) is a finitely generated module over its center.

We omit the other precise statements and their proofs since they are quite similar to those of graded cases.
4.3 Completion functor

For a graded $k$-vector space $V$, we write $V^f$ to indicate that we forget its grading structure. For a graded $k$-algebra $H$, we have the forgetful functor $(-)^f : \text{H-gmod} \to \text{H}^f\text{-mod}; V \mapsto V^f$.

For a graded $k$-vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$, we define its formal completion by $\hat{V} := \prod_{n \in \mathbb{Z}} V_n$. If $H$ is a Laurentian graded algebra, $\hat{H}$ naturally becomes a $k$-algebra and we have a natural isomorphism $\hat{H} \cong \varprojlim H^I/I^n$ for any finite-codimensional ideal $I \subset (H_{>0})^f$. In this case, the completion $\hat{V}$ of a module $V \in \text{H-gmod}$ naturally becomes a module over $\hat{H}$.

**Lemma 4.3.** Let $H$ be a Noetherian Laurentian algebra with property $(\blacklozenge)$.

1. The completion $\hat{H}$ is left Noetherian and complete with respect to the finite-codimensional ideal $\mathfrak{m} := \mathfrak{m}H = \hat{H}\mathfrak{m}$;
2. We have $\hat{V} \cong \varprojlim V^f/\mathfrak{m}^n V^f \cong \hat{H} \otimes_H V^f$ for any $V \in \text{H-gmod}$;
3. The completion functor $\text{H-gmod} \to \hat{H}\text{-mod}; V \mapsto \hat{V}$ is exact.

**Proof.** See [13] Section D.V and the above observations.

**Lemma 4.4.** Let $H$ be a Noetherian Laurentian algebra with property $(\blacklozenge)$. Then for any $U, V \in \text{H-gmod}$, we have a natural isomorphism $\text{Hom}_H(U, V) \cong \bigoplus_{n \in \mathbb{Z}} \text{hom}_H(U, V)_n$. In particular, the completion functor $\text{H-gmod} \to \hat{H}\text{-mod}$ is faithful.

**Proof.** If $V$ is of finite length, we have

$$\text{Hom}_H(\hat{U}, \hat{V}) \cong \text{Hom}_{H^f}(U^f, V^f) \cong \text{hom}_H(U, V)^f.$$ 

The third term is equal to $\bigoplus_{n \in \mathbb{Z}} \text{hom}_H(U, V)_n$ because only finitely many direct summands are non-zero. In general, note that we have $\hat{V}/\mathfrak{m}^n \hat{V} \cong V^f/\mathfrak{m}^n V^f$ for all $n > 0$ and we have:

$$\text{Hom}_H(\hat{U}, \hat{V}) \cong \varprojlim \text{Hom}_H(\hat{U}, \hat{V}/\mathfrak{m}^n \hat{V}) \cong \varprojlim \text{hom}_H(U, V/\mathfrak{m}^n V)^f.$$ 

By Lemma 2.3 (1), there is an increasing sequence of integers $N_1 < N_2 < \cdots$ such that $\text{hom}_H(U, \mathfrak{m}^n V)_m = 0$ for all $m < N_n$. Then we have a natural short exact sequence of projective systems: $0 \to \bigoplus_{m < N_n} \text{hom}_H(U, \mathfrak{m}^n V)_m \to \bigoplus_{m \geq N_n} \text{hom}_H(U, \mathfrak{m}^n V)_m \to 0$. By taking the projective limits, we get $\varprojlim \bigoplus_{m \geq N_n} \text{hom}_H(U, \mathfrak{m}^n V)_m = 0$ and therefore we have:

$$\prod_{m \in \mathbb{Z}} \text{hom}_H(U, V)_m \cong \varprojlim_n \bigoplus_{m < N_n} \text{hom}_H(U, V)_m \cong \varprojlim_n \bigoplus_{m \in \mathbb{Z}} \text{hom}_H(U, V/\mathfrak{m}^n V)_m.$$ 

□
Lemma 4.5. Let $H$ be a Noetherian Laurentian algebra with property (♠). Let \( \{L(\pi) \mid \pi \in \Pi\} \) be a complete and irredundant set of simple modules in $H$-$\text{gmod}$ up to isomorphism and grading shift. Then:

1. the set \( \{L(\pi)^\wedge \in \hat{H}$-$\text{mod} \mid \pi \in \Pi\} \) is a complete set of isomorphism classes of simple modules in $\hat{H}$-$\text{mod}$;

2. the $\hat{H}$-module $P(\pi)^\wedge$ is a projective cover of the simple module $L(\pi)^\wedge$ for each $\pi \in \Pi$.

Proof. 1 Let $L \in \hat{H}$-$\text{mod}$ be a simple module. By Nakayama’s lemma, we have $mL = 0$. Then $L$ is gradable by Lemma 5.1.6.

2 Let $P \in H$-$\text{gmod}$ be a projective module. Then we have $\text{Ext}_H^1(\hat{P}, L(\pi)^\wedge) = \text{ext}_H^1(P, L(\pi)) = 0$ for all $\pi \in \Pi$. Then for any $V \in \hat{H}$-$\text{mod}$, we have $\text{Ext}_H^1(\hat{P}, V) = 0$ by Lemma 2.1. Therefore $\hat{P}$ is projective in $\hat{H}$-$\text{mod}$. Moreover, by Lemma 4.4, we have:

$$\text{Hom}_H(P(\pi)^\wedge, L(\sigma)^\wedge) = \begin{cases} \mathbb{K} & \sigma = \pi; \\ 0 & \sigma \neq \pi. \end{cases}$$

This shows that $P(\pi)^\wedge$ is a projective cover of $L(\pi)^\wedge$ in $\hat{H}$-$\text{mod}$. □

Combining the above lemmas, we conclude as follows.

Theorem 4.6. Let $H$ be a graded affine quasi-hereditary algebra with property (♠). Then $\hat{H}$ is a complete affine quasi-hereditary algebra satisfying the property (♣) with respect to $\hat{Z}$. The standard (resp. proper standard, proper costandard, indecomposable tilting) module in $\hat{H}$-$\text{mod}$ associated to $\pi \in \Pi$ is the completion of the counterpart in $H$-$\text{gmod}$. Moreover the complete affine algebra $\text{End}_H(\Delta(\pi)^\wedge)$ is equal to the formal completion $\hat{B}_\pi$ of the graded affine algebra $B_\pi = \text{end}_H(\Delta(\pi))$.

5 An application: The Arakawa-Suzuki functor

5.1 The degenerate affine Hecke algebra of $GL_n$

We fix a positive integer $n > 0$.

Definition 5.1. The degenerate affine Hecke algebra of $GL_n$ is the $\mathbb{C}$-algebra $H_n$ defined by the generators

$$\{x_1, \ldots, x_n\} \cup \{s_1, \ldots, s_{n-1}\}$$

subject to the following relations

$$
\begin{align*}
x_i x_j &= x_j x_i, & s_i^2 &= 1, \\
s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & s_i s_j &= s_j s_i \text{ if } |i - j| > 1, \\
x_{i+1} s_i &= s_i x_i + 1, & x_j s_i &= s_i x_j \text{ if } j \neq i, i + 1.
\end{align*}
$$
Let $P_n := \mathbb{C}[x_1, \ldots, x_n]$ be the polynomial ring and let $\mathbb{C}S_n$ be the group algebra of the symmetric group of degree $n$ which is generated by the simple reflections $\{s_1, \ldots, s_{n-1}\}$. From Definition 5.1 we have the natural $\mathbb{C}$-algebra homomorphisms $P_n \to H_n$ and $\mathbb{C}S_n \to H_n$, which are injective by Proposition 5.2 below.

Note that there is an anti-involutive algebra homomorphism $\psi : H_n \to H_n$ which fixes all the generators $x_i, s_i$. For a left $H_n$-module $V$, we define a left $H_n$-module structure on the dual space $V^\oplus := \text{Hom}_\mathbb{C}(V, \mathbb{C})$ by twisting the natural right $H_n$-action by the anti-involution $\psi$.

**Proposition 5.2** (cf. [15] Theorem 3.2.2 and 3.3.1).

1. As a $(\mathbb{C}S_n, P_n)$-bimodule, we have $H_n \cong \mathbb{C}S_n \otimes_{\mathbb{C}} P_n$;
2. The center $Z(H_n)$ of $H_n$ is equal to the subalgebra of symmetric polynomials in $P_n$. I.e. $Z(H_n) = (P_n)^{\mathfrak{s}n}$.

Let $Q := \bigoplus_{\alpha \in \mathbb{Z}^2} \mathbb{Z} \alpha_i$ be the root lattice of type $A_{\infty}$ and define $Q^+ := \sum_{i,j} \mathbb{Z} \alpha_{i,j}$ for each $i, j \in \mathbb{Z}$ with $i < j$, we define the positive root $\alpha(i, j)$ by $\alpha(i, j) := \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1} \in Q^+$. For each element $\beta = \sum_{i,j} n_{i,j} \alpha_{i,j} \in Q^+$, we define its height by $ht(\beta) := \sum_{i,j} n_{i,j}$.

We fix an element $\beta \in Q^+$ with $ht(\beta) = n$. We regard $\beta$ as an unordered set of $n$ integers with multiplicity i.e. $\beta \in \mathbb{Z}^n / S_n \subset \mathbb{N}^n / S_n = \text{Specm}(Z(H_n))$, where the equality is due to Proposition 5.2 (2). We denote the formal completion of the affine algebra $Z(H_n)$ at $\beta$ by $Z(H_n)_\beta$ and define $\hat{H}_\beta := H_n \otimes_{Z(H_n)} Z(H_n)_\beta$.

An unordered multiset $\pi = \{\alpha(i_1, j_1), \ldots, \alpha(i_m, j_m)\}$ of positive roots such that $\alpha(i_1, j_1) + \cdots + \alpha(i_m, j_m) = \beta$ is called a Kostant partition of $\beta$. We denote by $\text{KP}(\beta)$ the set of all Kostant partitions of $\beta$. This is a finite set. Let $\pi, \pi' \in \text{KP}(\beta)$. We write $\pi \leq \pi'$ if $\pi = \pi'$ or one of the following two conditions hold:

1. There are integers $i, j, k$ with $i < j < k$ such that $\alpha(i, k) \in \pi$ and $\alpha(i, j), \alpha(j, k) \in \pi'$ with $\pi \setminus \{\alpha(i, k)\} = \pi' \setminus \{\alpha(i, j), \alpha(j, k)\}$;
2. There are integers $i, j, k, l$ with $i < j < k < l$ such that $\alpha(i, l), \alpha(j, k) \in \pi$ and $\alpha(i, k), \alpha(j, l) \in \pi'$ with $\pi \setminus \{\alpha(i, l), \alpha(j, k)\} = \pi' \setminus \{\alpha(i, k), \alpha(j, l)\}$.

The relation $\leq$ generates a partial order on $\text{KP}(\beta)$, which we denote by the same symbol $\leq$.

**Theorem 5.3.** The algebra $\hat{H}_\beta$ is a complete affine quasi-hereditary algebra with the index poset $(\text{KP}(\beta), \leq)$ satisfying the property $(\bullet)$.

**Proof.** Brundan-Kleshchev-McNamara [5] and Kato [13] proved that the KLR algebra $R_\beta$ associated to $\beta$ is a graded affine quasi-hereditary algebra with the index poset $(\text{KP}(\beta), \leq)$ (see Remark 5.4 below). By Khovanov-Lauda [14] Corollary 2.10, we can easily see that $R_\beta$ satisfies the property $(\bullet)$. Then the completion $\hat{R}_\beta$ is a complete affine quasi-hereditary algebra with property $(\bullet)$. 

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by Theorem 4.6. Now we complete the proof by the isomorphism $\hat{R}_\beta \cong \hat{H}_\beta$ of topological C-algebras proved by Brundan-Kleshchev [4].

**Remark 5.4.** By [12], a partial order for an affine quasi-hereditary structure of $R_\beta$ comes from the orbit closure relation on the space $\mathfrak{L}(\beta) := \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathbb{C}(\mathbb{C}^n, \mathbb{C}^{n+1})$ of representations of linear quiver of dimension vector $\beta$ with respect to the action of the group $G(\beta) := \prod_{i \in \mathbb{Z}} GL(\mathbb{C}^n)$. The closure relations of $G(\beta)$-orbits in $\mathfrak{L}(\beta)$ has been described in Abeasis-DelFra-Kraft [1] Proposition 9.1, which asserts that the set $\text{KP}(\beta)$ naturally parametrizes the $G(\beta)$-orbits in $\mathfrak{L}(\beta)$ and our ordering $\preceq$ on $\text{KP}(\beta)$ corresponds to the opposite closure ordering of orbits.

We denote the standard (resp. proper standard, proper costandard) module of $\hat{H}_\beta$-mod associated to $\pi \in \text{KP}(\beta)$ by $\Delta_H(\pi)$ (resp. $\Delta_H(\pi), \nabla_H(\pi)$).

**Remark 5.5.** We define a total order $\preceq$ on the set of positive roots so that $\alpha(i,j) \preceq \alpha(k,l)$ if $i < k$, or $i = k$ and $j \leq l$. For each $\pi = \{\pi_1, \ldots, \pi_m\} \in \text{KP}(\beta)$ such that $\pi_1 \succ \pi_2 \succ \cdots \succ \pi_m$, we give an explicit construction of the standard module $\Delta_H(\pi)$ following [4] (see also [14] Section 8.4). Let $\pi_k = \alpha(i_k, j_k)$ for $1 \leq k \leq m$. Then we define an $H_{n_1} \otimes \cdots \otimes H_{n_m}$-module structure on the space $R_m := \mathbb{C}[z_1, \ldots, z_m]$ so that $x_t$ acts on by the multiplication of the scalar $i_k + (l - a_k) + z_k$ if $a_k \leq l < a_k + 1$ and the action of $\mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_m}$ is trivial. Then we define $\Delta_H(\pi)$ as the induced module $H_n \otimes (H_{n_1} \otimes \cdots \otimes H_{n_m}) R_m$, which naturally extends to a module over the completion $\hat{H}_\beta$. By construction, $\Delta_H(\pi)$ has a natural action of $R_m$ which commutes with the action of $\hat{H}_\beta$. Then we have $\Delta_H(\pi) = \Delta_H(\pi)/m\Delta_H(\pi)$, where $m$ is the maximal ideal of $R_m$, and $\nabla_H(\pi) = \Delta_H(\pi)^\circ$.  

### 5.2 The deformed BGG category of $\mathfrak{gl}_m$

Let $m \geq 2$ be a positive integer and $\mathfrak{g} := \mathfrak{gl}_m(\mathbb{C}) = \text{Mat}_m(\mathbb{C})$ be the general linear Lie algebra. Let $e_{ij} \in \mathfrak{g}$ denote the $ij$-matrix unit $(1 \leq i, j \leq m)$. Let $t$ be the abelian Lie subalgebra of $\mathfrak{g}$ consisting of diagonal matrices, i.e. $t = \bigoplus_{i=1}^m \mathbb{C}e_{ii}$. We have the natural pairing $(\cdot, \cdot) : t^* \times t \to \mathbb{C}$ and identify the space $t^*$ with $\mathbb{C}^m$ by $t^* \hookrightarrow \mathcal{C}^m; \lambda \mapsto (\langle \lambda, e_{ii} \rangle)$.

Note that the symmetric group $\mathfrak{S}_m$ acts on the space $t^* = \mathbb{C}^m$ by permuting the coordinates. Let $\varepsilon_i \in t^*$ be the element defined by $\langle \varepsilon_i, e_{ij} \rangle = \delta_{ij}$. We define the weight lattice by $\Lambda := \bigoplus_{i=1}^m \mathbb{Z}\varepsilon_i$ and the set of dominant weights by $\Lambda^+ := \{\lambda \in \Lambda \mid \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m\}$. Let $\Lambda_r := \bigoplus_{i=1}^m \mathbb{Z}(\varepsilon_i - \varepsilon_{i+1}) \subset t^*$ be the root lattice and $\Lambda_r^+ := \bigoplus_{i=1}^m \mathbb{Z}_{\geq 0}(\varepsilon_i - \varepsilon_{i+1})$. We define a partial order called the dominance order $\preceq$ on $\Lambda$ so that $\lambda \preceq \mu$ if $\mu - \lambda \in \Lambda_r^+$. We set a special dominant weight $\rho$ by $\rho := (0, -1, -2, \ldots, -m + 1)$. Let $n_+$ (resp. $n_-$) be the nilpotent radical of upper (resp. lower) triangular matrices. We set the standard Borel subalgebra $\mathfrak{b} := t \oplus n_+$.

We identify the affine coordinate ring $\mathcal{C}[t^*] = \text{Sym}(t)$ with the polynomial ring $\mathcal{C}[z_1, \ldots, z_m]$ of $m$-variables by setting $z_i = e_{ii} \in t \subset \mathcal{C}[t^*]$. Let $R := \mathcal{C}[z_1, \ldots, z_m]$ be the completion of $\mathcal{C}[t^*]$, which is a local $\mathbb{C}$-algebra with its maximal ideal $m := (z_1, \ldots, z_m)$. 

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Definition 5.6. We define the deformed BGG category $\widetilde{O}$ of $\mathfrak{g}$ as a full subcategory of the category $(U(\mathfrak{g}) \otimes_{\mathbb{C}} R)\text{-Mod}$ of all $(\mathfrak{g}, R)$-bimodules. A $(\mathfrak{g}, R)$-bimodule $M$ belongs to $\widetilde{O}$ if the following three conditions hold:

1. $M$ is finitely generated as a $(\mathfrak{g}, R)$-bimodule;
2. $M$ is locally finite over $U(n_+)$;
3. As a $(U(\mathfrak{t}), R)$-bimodule, $M$ has a direct sum decomposition $M = \bigoplus_{\lambda \in \Lambda} M(\lambda)$, where $M(\lambda) := \{ v \in M \mid (e_{ii} - \lambda_i)v = vz_i, (1 \leq i \leq m) \}$.

We refer the non-zero $(U(\mathfrak{t}), R)$-submodule $M(\lambda)$ as a generalized weight space of weight $\lambda$. If $M(\lambda) \neq 0$, the element $\lambda$ is called a weight of the module $M$. Note that in Definition 5.6 (3), we restrict all the weights to be integral (i.e. belong to $\Lambda$) for simplicity. Since the algebra $U(\mathfrak{g}) \otimes_{\mathbb{C}} R$ is left Noetherian, the category $\widetilde{O}$ is a left Noetherian abelian category.

Proposition 5.7 (Soergel [21], Fiebig [8]). We have a decomposition of $\varphi$-adically complete category: $\widetilde{O} = \bigoplus_{\lambda \in \Lambda^+} \widetilde{O}_\lambda$. 


We call the category $\tilde{O}_\lambda$ the block of the category $\tilde{O}$ associated to the dominant weight $\lambda$.

**Theorem 5.8** (Soergel, Fiebig). Let $\lambda$ be a dominant weight.

1. For each weight $\mu \in \Pi_\lambda$, the deformed Verma module $\widetilde{M}(\mu)$ belongs to $\tilde{O}_\lambda$. The set $\{L(\mu) \mid \mu \in \Pi_\lambda\}$ is the complete set of representatives of irreducible isomorphism classes of $\tilde{O}_\lambda$;

2. The category $\tilde{O}$ has enough projectives. For each $\mu \in \Pi_\lambda$, there exists a projective cover $P(\mu)$ of the simple module $L(\mu)$ in the block $\tilde{O}_\lambda$. It fits into an exact sequence $0 \to K(\mu) \to P(\mu) \to \widetilde{M}(\mu) \to 0$, where $K(\mu)$ has a filtration whose subquotients are $\simeq \widetilde{M}(\nu)$ for some $\nu \in \Pi_\lambda$ with $\nu > \mu$;

3. The $R$-module $\text{Hom}_{\tilde{O}}(P(\mu), \widetilde{M}(\nu))$ is free of finite rank for any $\mu, \nu \in \Pi_\lambda$.

**Proof.** See Soergel [20] Theorem 6 or Fiebig [8] Theorem 2.7. \(\square\)

Let $\tilde{M}$ be the full subcategory of $(U(g) \boxtimes R)$-Mod consisting of modules $M$ having a generalized weight space decomposition $M := \bigoplus_{\lambda \in \Lambda} M(\lambda)$ with each generalized weight space not necessary being finitely generated over $R$. For each $M \in \tilde{M}$, we define $g$-module structures on its restricted dual $M^\vee := \bigoplus_{\lambda \in \Lambda} \text{Hom}_C(M(\lambda), C)$ and its restricted topological dual $D(M) := \bigoplus_{\lambda \in \Lambda} \text{Fun}(M(\lambda), C)$ by twisting the natural right $g$-action by the transpose map, where we define $\text{Fun}(N, C) := \lim \text{Hom}_C(N/m^nN, C)$ for an $R$-module $N$. Then we have two contravariant endofunctors $(-)^\vee$ and $D$ satisfying that $(M^\vee)_{(\lambda)} = \text{Hom}_C(M(\lambda), C)$ and $D(M)_{(\lambda)} = \text{Fun}(M(\lambda), C)$. Note that the functor $(-)^\vee$ is exact and the functor $D$ is left exact. If all generalized weight spaces of $M \in \tilde{M}$ are free $R$-modules of finite rank, then we have $R^1D(M) = 0$ and $(D(M))^\vee \cong M$. Although these functors do not preserve the category $\tilde{O}$, they preserve the category $O$, on which we have $D = (-)^\vee$ and they are involutive. We have $L^\vee \cong L$ for any irreducible module $L \in O$. For each $\lambda \in \Lambda$, we define the dual Verma module to be $M(\lambda)^\vee$.

**Proposition 5.9.** Let $\lambda, \mu \in \Lambda$. Then we have:

$$\text{Ext}^i_{\tilde{O}}(\widetilde{M}(\lambda), M(\mu)^\vee) = \begin{cases} \mathbb{C} & i = 0, \lambda = \mu; \\ 0 & \text{otherwise}. \end{cases}$$

**Proof.** For $i = 0$, we observe that $\text{Hom}_{\tilde{O}}(\widetilde{M}(\lambda), M(\mu)^\vee) \cong \text{Hom}_O(M(\lambda), M(\mu)^\vee)$ and then the assertion is well-known (see [10] Theorem 3.3).

For $i = 1$, we consider an extension in $O$:

$$0 \to M(\mu)^\vee \to E \to \widetilde{M}(\lambda) \to 0. \quad (5.1)$$

If $\lambda \not\leq \mu$, then the weight $\lambda$ is a maximal weight of $E$. By the universal property of the deformed Verma module $M(\lambda)$, we see that the sequence (5.1) must be
split. If $\lambda < \mu$, we apply the functor $D$ defined above to the sequence (5.1) to get:

$$0 \to D(M(\lambda)) \to D(E) \to M(\mu) \to R^1 D(M(\lambda)) = 0. \tag{5.2}$$

Since $\lambda < \mu$, the weight $\mu$ is maximal in $D(E)$. Moreover, $R$ acts on $D(E)(\mu)$ trivially because it is 1-dimensional. Therefore by the universal property of the Verma module $M(\mu)$, we see that the sequence (5.2) is split. Since $D(M(\lambda))^\vee \cong M(\lambda)$, by applying $(-)^\vee$ on the sequence (5.2), we observe that the sequence (5.1) is also split. Thus we have $\text{Ext}^1_{\mathcal{O}}(\tilde{M}(\lambda), M(\mu)^\vee) = 0$ for any $\lambda, \mu \in \Lambda$. The cases $i > 1$ follow from the case $i = 1$ by a standard argument (cf. [10] Section 6.12).

We summarize the above results as follows.

**Theorem 5.10.** For each $\lambda \in \Lambda^+$, the block $\tilde{O}_\lambda$ of the category $\tilde{O}$ is a complete affine highest weight category with index poset $\Pi_\lambda$ satisfying the property (♠). The standard (resp. proper standard, proper costandard) modules of $\tilde{O}_\lambda$ are the deformed Verma modules $\tilde{M}(\mu)$ (resp. Verma modules $M(\mu)$, dual Verma modules $M(\mu)^\vee$).

### 5.3 The Arakawa-Suzuki functor

Let $V := \mathbb{C}^m$ be the vector representation of $\mathfrak{g} = \mathfrak{gl}_m$. We denote by $\{v_i\}$ the standard basis of $V$. The $\mathfrak{g}$-action on $V$ is written explicitly as $e_{ij} v_k = \delta_{jk} v_i$. We also write the dual space of $V$ by $V^\vee := \text{Hom}(V, \mathbb{C})$.

Define the Casimir operator $\Omega \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ by $\Omega := \sum_{1 \leq i,j \leq m} e_{ij} \otimes e_{ji}$. If $M_1$ and $M_2$ are both $\mathfrak{g}$-modules, then $\Omega$ acts on the tensor module $M_1 \otimes M_2$. Let $C := \sum_{1 \leq i,j \leq m} e_{ij} e_{ji}$ be the Casimir element of $\mathfrak{g}$, which is a central element of $U(\mathfrak{g})$. We have $\Omega = \frac{1}{2} (\delta(C) - C \otimes 1 - 1 \otimes C) \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$, where $\delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ denotes the comultiplication of $U(\mathfrak{g})$ defined by $\delta(X) := X \otimes 1 + 1 \otimes X$ for $X \in \mathfrak{g}$. Thus we have $\Omega \in \text{End}_{\mathfrak{g}}(M_1 \otimes M_2)$. Note that the action of $\Omega$ on the tensor representation $V \otimes V$ is the permutation of the tensor factors.

Let $M_0, M_1, \ldots, M_n$ be $\mathfrak{g}$-modules. We define the linear operator $\Omega^{(a,b)}$ for $0 \leq a, b \leq n$ on the tensor product $M_0 \otimes M_1 \otimes \cdots \otimes M_n$ as $\Omega^{(a,b)} := \sum_{1 \leq i,j \leq m} e_{ij}^{(a)} e_{ji}^{(b)}$, where we define $X^{(a)} := 1^{\otimes a} \otimes X \otimes 1^{\otimes (n-a)} \in U(\mathfrak{g})^{\otimes (n+1)}$ for $X \in \mathfrak{g}$. Note that $\Omega^{(a,b)} = \Omega^{(b,a)}$ and $\Omega^{(a,b)}$ commutes with the $\mathfrak{g}$-action on $M_0 \otimes M_1 \otimes \cdots \otimes M_n$.

**Proposition 5.11** (Arakawa-Suzuki [2, 22]). Let $M$ be a $\mathfrak{g}$-module. Then the formula $s_i = \Omega^{(i, i+1)}$ (for $1 \leq i < n$), $x_i = \sum_{0 \leq j < i} \Omega^{(i,j)}$ (for $1 \leq i \leq n$) defines a right action of $H_n$ on the tensor $\mathfrak{g}$-module $M \otimes V^\otimes n$, and thus we have a $\mathbb{C}$-algebra homomorphism $H_n \to \text{End}_{\mathfrak{g}}(M \otimes V^\otimes n)^{\text{op}}$.

**Definition 5.12.** By taking $M = \tilde{M}(\rho)$ in Proposition 5.11 above, we define the Arakawa-Suzuki functor

$$\Phi = \Phi_n^m : \tilde{O}(\mathfrak{gl}_m) \to H_n\text{-Mod}; M \mapsto \text{Hom}_{\tilde{O}}(\tilde{M}(\rho) \otimes V^\otimes n, M).$$
Note that the module $\tilde{M}(\rho)$ and hence $\tilde{M}(\rho) \otimes V^{\otimes n}$ is projective in $\tilde{O}$, since $\rho$ is a dominant weight. Therefore the Arakawa-Suzuki functor $\Phi$ is an exact functor.

**Remark 5.13.** The original definition [2], [22] of Arakawa-Suzuki functor uses the non-deformed Verma modules of general dominant weights, although we use the deformed Verma module of a special dominant weight $\rho$ for simplicity. Our discussion below is valid for any regular dominant weights instead of $\rho$ by some obvious modifications.

**Lemma 5.14.** Let $M$ and $N$ be $g$-modules. The Casimir operator defines the following linear operators

$$
\begin{align*}
\Omega^*_{M,V} : \text{Hom}_g(M \otimes V, N) &\rightarrow \text{Hom}_g(M \otimes V, N); \quad f \mapsto f \circ \Omega, \\
\Omega_{V,N} : \text{Hom}_g(M, \tilde{V} \otimes N) &\rightarrow \text{Hom}_g(M, \tilde{V} \otimes N); \quad f \mapsto \Omega \circ f.
\end{align*}
$$

We also have the natural isomorphism $\text{Hom}_g(M \otimes V, N) \cong \text{Hom}_g(M, \tilde{V} \otimes N)$. Then the following diagram is commutative.

$$
\begin{array}{ccc}
\text{Hom}_g(M \otimes V, N) & \xrightarrow{\cong} & \text{Hom}_g(M, \tilde{V} \otimes N) \\
\downarrow{\Omega^*_{M,V}} & & \downarrow{-\Omega_{V,N} - m} \\
\text{Hom}_g(M \otimes V, N) & \xrightarrow{\cong} & \text{Hom}_g(M, \tilde{V} \otimes N)
\end{array}
$$

**Proof.** Recall $\Omega = \frac{1}{2}(\delta(C) - C \otimes 1 - 1 \otimes C) \in U(g) \otimes U(g)$. For $f \in \text{Hom}_g(M \otimes V, N)$ and $g \in \text{Hom}_g(M, \tilde{V} \otimes N)$, we have

$$
\begin{align*}
2\Omega^*_{M,V}(f) &= 2f \circ \Omega \\
&= f \circ (\delta(C) - C_M \otimes 1 - 1 \otimes C_V) \\
&= C_N \circ f - f \circ (C_M \otimes 1) - mf, \\
2\Omega_{V,N}(g) &= 2\Omega \circ g \\
&= (\delta(C) - C_V \otimes 1 - 1 \otimes C_N) \circ g \\
&= g \circ C_M - mg - (1 \otimes C_N) \circ g,
\end{align*}
$$

where $C_M$ denotes the action of the Casimir element $C$ on $M$ and we use $C_V = m$ and $C_V = m$. From this, we obtain the assertion. \qed

**Lemma 5.15.** Let $M, N$ be $g$-modules.

1. The formula $s_i = \Omega^{(i-1, i)}$ ($1 \leq i < n$), $x_i = -\sum_{i \leq j \leq n} \Omega^{(i-1, j)} - m$ ($1 \leq i \leq n$) defines a left action of $H_n$ on the tensor product $g$-module $\tilde{V}^{\otimes n} \otimes N$.

2. The natural isomorphism $\text{Hom}_g(M \otimes V^{\otimes n}, N) \cong \text{Hom}_g(M, \tilde{V}^{\otimes n} \otimes N)$ commutes with $H_n$-actions, where we regard $\text{Hom}_g(M \otimes V^{\otimes n}, N)$ as an $H_n$-module by Proposition 5.11 and $\text{Hom}_g(M, \tilde{V}^{\otimes n} \otimes N)$ as an $H_n$-module by (1).
Proof. (1) is similar to Proposition 5.11 (2) follows from Lemma 5.14.

We fix a dominant weight \( \lambda \in \Lambda^+ \) satisfying \( \lambda_m > 0 \) and denote its \( \mathfrak{g}_m \)-orbit by \( \Pi_\lambda \) as before. We define an element \( \beta \in Q^+ \) by \( \beta := \sum_{i=1}^m \alpha(-i+1, \lambda_i) \) and we set \( n := \text{ht}(\beta) \). For each \( \mu \in \Pi_\lambda \), we associate a Kostant partition \( \pi_\mu \in \text{KP}(\beta) \) by:

\[
\pi_\mu := \{\alpha(0, \mu_1), \alpha(-1, \mu_2), \ldots, \alpha(-m+1, \mu_m)\}.
\]

**Proposition 5.16.** In the above notation, for each \( \mu \in \Pi_\lambda \), we have:

\[
\Phi(\tilde{M}(\mu)) \cong \Delta_H(\pi_\mu), \quad \Phi(M(\mu)) \cong \bar{\Delta}_H(\pi_\mu).
\]

Proof. For each module \( M \in \tilde{O} \), we compute as:

\[
\text{Hom}_{\tilde{O}} \left( \tilde{M}(\rho) \otimes V^\otimes n, M \right) \cong \text{Hom}_{\tilde{O}} \left( \tilde{M}(\rho), \tilde{V}^\otimes n \otimes M \right) \ncong \text{Hom}_{\tilde{O}} \left( \tilde{M}(\rho), \text{pr}_\rho(\tilde{V}^\otimes n \otimes M) \right) \ncong \left( \text{pr}_\rho(\tilde{V}^\otimes n \otimes M) \right)^n_{(0)},
\]

where \( \text{pr}_\rho : \tilde{O} \to \tilde{O}_\rho \) is the projection with respect to the block decomposition (Proposition 5.7). Since the weight 0 is the largest weight in the block \( \tilde{O}_\rho \), there are natural isomorphisms:

\[
\left( \text{pr}_\rho(\tilde{V}^\otimes n \otimes M) \right)^n_{(0)} \cong \left( \text{pr}_\rho(\tilde{V}^\otimes n \otimes M) \right)_{(0)} \cong \left( \tilde{V}^\otimes n \otimes M / n_- \right)_{(0)}.
\]

Set \( M = \tilde{M}(\mu) \), then we have

\[
\Phi(\tilde{M}(\mu)) \cong \left( \tilde{V}^\otimes n \otimes \tilde{M}(\mu) / n_- \right)_{(0)} \ncong \left( (U(\mathfrak{g}) \otimes U(\mathfrak{b}) (\tilde{V}^\otimes n \otimes R_{\mu-\rho}) / n_- \right)_{(0)} \ncong \left( \tilde{V}^\otimes n \otimes R_{\mu-\rho} \right)_{(0)}
\]

where we use the tensor identity for the second equality. Let \( u \) be the vector belonging to \( (\tilde{V}^\otimes n \otimes R_{\mu-\rho})_{(0)} \) defined by \( u := \epsilon_1^{n_1} \otimes \epsilon_2^{n_2} \otimes \cdots \otimes \epsilon_m^{n_m} \otimes 1 \), where we put \( n_i := \mu_i + i - 1 = \mu_i - \rho_i \) and \( \{ \epsilon_i \} \) is the basis of \( \tilde{V} \) dual to the standard basis \( \{ v_i \} \) of \( \tilde{V} \). Note that we have \( \epsilon_i \epsilon_j = -\delta_{ij} \epsilon_j \) for \( 1 \leq i, j, k \leq m \).

For each \( 1 \leq k \leq m \), we define the number \( a_k \) by \( a_k := \sum_{i=1}^{k-1} n_i + 1 \) as in
Thus we have
\[ x_{a_k} \sim \mu \rightarrow \text{the vector} \]
\[ \text{Suzuki [2, 22].} \]
We remark that this latter assertion is a special case of the results of Arakawa-M assertion \( \Phi(\cdots \times S) \).

Proof. We remark that \( M \) Lemma 5.17. For each \( M \in \mathcal{O} \), we have \( \Phi(M^\vee) \cong \Phi(M)^\oplus \). In particular, we have \( \Phi(M(\mu)^\vee) \cong \nabla_H(\pi_\mu) \) for each \( \mu \in \Pi_\lambda \).

\begin{proof}
We remark that \( N^\vee \cong N \) holds for any finite dimensional \( g \)-module \( N \) and \( (N \otimes M)^\vee \cong N \otimes M^\vee \) for any \( M \in \mathcal{O} \). We compute as:
\[
\Phi(M^\vee) \cong \text{Hom}_\mathcal{O}(\tilde{M}(\rho) \otimes V^\otimes n, M^\vee)
\cong \text{Hom}_\mathcal{O}(M(\rho) \otimes V^\otimes n, M^\vee)
\cong \text{Hom}_\mathcal{O}(M(\rho), \tilde{V}^\otimes n \otimes M^\vee)
\cong \text{Hom}_\mathcal{O}(M(\rho), (\tilde{V}^\otimes n \otimes M)^\vee)
\cong ((\tilde{V}^\otimes n \otimes M)/n_-)^*(0)
\cong \Phi(M)^\oplus,
\]
which proves the assertion.
\end{proof}

Finally, we prove the main theorem of this section.
Theorem 5.18. The Arakawa-Suzuki functor induces a fully faithful functor $\Phi : \tilde{O}_\lambda \to \hat{H}_\beta\text{-mod}$ between complete affine highest weight categories.

Proof. Note that the category $\tilde{O}_\lambda$ is the Serre subcategory of $\tilde{O}$ generated by deformed Verma modules $\tilde{M}(\mu)$ for $\mu \in \Pi_\lambda$. By Proposition 5.16, we see for each $M \in \tilde{O}_\lambda$ the $H_n$-action on $\Phi(M)$ is naturally extended to the action of $\hat{H}_\beta$. Therefore we can regard $\Phi$ as the exact functor $\Phi : \tilde{O}_\lambda \to \hat{H}_\beta\text{-mod}$.

We have to show that this functor $\Phi$ is fully faithful. Let us consider the subset $\text{KP}(\beta)_m \subset \text{KP}(\beta)$ which consists of Kostant partitions of $\beta$ consisting of $m$ positive roots i.e. $\text{KP}(\beta)_m := \{\pi \in \text{KP}(\beta) \mid ||\pi|| = m\}$. It is easy to show that the correspondence $\Pi_\lambda \to \text{KP}(\beta)_m; \mu \mapsto \pi_\mu$ is an isomorphism of posets and $\text{KP}(\beta)_m = \{\pi \in \text{KP}(\beta) \mid \pi \preceq \pi_\lambda\}$, which is saturated. Then the full subcategory $\hat{H}_\beta\text{-mod}_m \subset \hat{H}_\beta\text{-mod}$ consisting of modules $M$ with $\text{supp}(M) \subset \text{KP}(\beta)_m$ is an affine highest weight category whose standard (resp. proper standard, proper costandard) modules are $\Delta_H(\pi_\mu)$ (resp. $\Delta_H(\pi_\mu), \nabla_H(\pi_\mu)$). Thanks to Proposition 5.16 and Lemma 5.17 we can apply the complete version of Theorem 3.9 to the functor $\Phi : \tilde{O}_\lambda \to \hat{H}_\beta\text{-mod}_m$ to complete the proof. \hfill $\square$

Remark 5.19. The subcategory $\hat{H}_\beta\text{-mod}_m \subset \hat{H}_\beta\text{-mod}$ is naturally equivalent to the module category over the maximal quotient algebra of $\hat{H}_\beta$ whose support as a left $\hat{H}_\beta$-module is contained in $\text{KP}(\beta)_m$ (cf. [16] Lemma 3.13). Therefore the above proof of Theorem 5.18 says that the block $\tilde{O}_\lambda$ is equivalent to the module category over a quotient algebra of $\hat{H}_\beta$.

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