STOCHASTIC COMPARISONS OF SERIES-PARALLEL AND PARALLEL-SERIES SYSTEMS WITH DEPENDENCE BETWEEN COMPONENTS AND ALSO OF SUBSYSTEMS

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Abstract. In this paper, we consider series-parallel and parallel-series systems comprising dependent components that are drawn from a heterogeneous population consisting of \( m \) different subpopulations. The components within each subpopulation are assumed to be dependent, and the subsystems themselves are also dependent, with their joint distribution being modeled by an Archimedean copula. We consider a very general setting in which we assume that the subpopulations have different Archimedean copulas for their dependence. Under such a general setup, we discuss the usual stochastic, hazard rate and reversed hazard rate orders between these systems and present a number of numerical examples to illustrate all the results established here. Finally, some concluding remarks are made. The results established here extend the recent results of Fang et al. (2020) in which they have assumed all the subsystems to be independent.

1. Introduction. Global competition has made the manufacturing process of technical systems very demanding and challenging. A manufacturing process typically relies on parts or components coming from many different suppliers all over the world. Inevitably, this would result in differing quality and reliability measures for these components. However, the supply and demand would warrant the producers to still order their parts from different suppliers and then use them suitably in their production process. Statistically, this poses great difficulties since the modelling of lifetimes of systems becomes quite complex due to the dependence between the components within each subsystem and also the added dependence between the

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subsystems themselves. This, in turn, makes the comparative evaluation of different systems with regard to their reliability characteristics even more complex. One successful attempt in this direction is the recent work of [4] in which they considered a restricted set-up with the components within each subsystem being dependent (modelled by a copula), but the subsystems themselves function independently. Even though this work gave some key ideas and directions with regard to the modelling of systems as well as their reliability comparisons, for the reasons stated above, it becomes necessary to consider the general situation in which the subsystems are also dependent, possibly due to components coming from the same supplier being used for building different subsystems or possibly due to common environmental conditions impacting all subsystems simultaneously. In this work, we consider such a very general set-up and then focus on series-parallel and parallel-series systems, and establish various reliability comparative results between different systems.

It should be mentioned that though there are many coherent systems that could be formed and used, series, parallel, series-parallel, parallel-series and fail-safe systems are some of the most commonly used systems in practice. Series-parallel and parallel-series systems are commonly used in circuit systems. For this reason, we focus here on series-parallel and parallel-series systems under the very general setting described above, and then establish some reliability comparative results between different systems in terms of the usual stochastic, hazard rate and reversed hazard rate orders. These compressions are done for different cases: first when the number of subsystems is fixed and the number of components allocated within subsystems vary, second when the number of systems varies, and third when the selection probabilities or the distributions of subpopulations vary.

The optimal allocation of components on different series and parallel systems has been discussed extensively by many authors, including [3], [5], [17], [18], and the references therein. Several authors have subsequently discussed this issue for parallel-series and series-parallel systems, including [2], [13], [14], [1], [7], [16], [9] and [4].

We shall now formulate the problem of interest statistically and then describe as to how the reliability systems are formed under this general set-up. Suppose a series-parallel system comprising \( n \) components is to be formed with \( k \) subsystems connected in series, with each subsystem being in parallel comprising \( a_1, \ldots, a_k \) components, such that all \( a_j > 0 \) and \( \sum_{j=1}^{k} a_j = n \). As described earlier, we now assume that all the components are acquired from \( m \) different suppliers, and that the components for building each subsystem are procured from these \( m \) suppliers in proportions (or probabilities) \( p_1, \ldots, p_m \), with \( \sum_{i=1}^{m} p_i = 1 \). It is now evident as to why the components within each subsystem are dependent and at the same time as to why the \( k \) subsystems themselves are dependent. For capturing this double dependence, we assume different copulas for the dependence between components within the \( k \) subsystems, and yet another copula to model the dependence between the \( k \) subsystems.

In the statistical literature, there are many ways to model dependence between random variables [see [6]], and the theory of copulas is one popular tool for this purpose; see, for example, [12] for an elaborate discussion on the theory and applications of copulas. Though many copulas have been studied, Archimedean copulas have attracted considerable attention due to their flexibility and the fact that they include many well-known copulas, such as Clayton, Ali-Mikhail-Haq, Gumbel-Hougaard
and Frank copulas, as special cases. In addition, they also include the independence copula as a special case and consequently comparison results established under Archimedean copula for the joint distribution of lifetimes of components in the system and also of the subsystems are quite general in form and structure, and so would naturally include the corresponding results for the case of independence developed in the recent work of [4].

The rest of this paper proceeds as follows. In Section 2, we present briefly some basic definitions and lemmas that will be used in the subsequent sections. In Section 3, we establish some comparative results for series-parallel systems by using the concepts of weak majorization and majorization orders. The usual stochastic and reversed hazard rate orders between parallel-series systems are then established in Section 4. Several illustrative examples are presented by using considering dependence, Clayton and Ali-Mikhail-Haq copulas. Finally, some concluding remarks are made in Section 5 wherein a problem of further interest is also outlined.

2. Background. In this section, we briefly describe some well-known concepts about stochastic orders, majorization and associated orders, and copulas. Throughout, we assume all random variables under consideration to be nonnegative, we use “increasing” to mean “nondecreasing” and similarly “decreasing” to mean “nonincreasing”, and further assume all involved expectations to exist. For convenience in notation, use \( a \equiv b \) to denote that both sides of an equality have the same sign. We also set

\[
A = \left\{ M^{m}_{n}(k|a, p, X) : n, m, a \in \mathbb{N}_{+}^{k} \text{ with } \sum_{j=1}^{k} a_{j} = n, \ p \in [0, 1]^{m} \text{ such that } \sum_{i=1}^{m} p_{i} = 1 \right\},
\]

where \( M^{m}_{n}(k|a, p, X) \) denotes the lifetime of a system comprising \( k \) subsystems with a total of \( n \) components coming from \( m \) suppliers in proportions \( p = (p_{1}, \cdots, p_{m}) \), and with \( a = (a_{1}, \cdots, a_{k}) \) being the number of components in the \( k \) subsystems and \( X \) denoting the component lifetimes.

2.1. Stochastic orders. Let \( X \) and \( Y \) be two random variables with density functions \( f_{X} \) and \( f_{Y} \), distribution functions \( F_{X} \) and \( F_{Y} \), survival functions \( \bar{F}_{X} = 1 - F_{X} \) and \( \bar{F}_{Y} = 1 - F_{Y} \), hazard rate functions \( h_{X} = f_{X}/\bar{F}_{X} \) and \( h_{Y} = f_{Y}/\bar{F}_{Y} \), and reversed hazard rate functions \( r_{X} = f_{X}/\bar{F}_{X} \) and \( r_{Y} = f_{Y}/\bar{F}_{Y} \), respectively.

**Definition 2.1.** \( X \) is said to be larger than \( Y \) in the

(i) usual stochastic order (denoted by \( X \geq_{st} Y \)) if \( \bar{F}_{X}(t) \geq \bar{F}_{Y}(t) \), for all \( t \in \mathbb{R} \), or equivalently, \( E[\phi(X)] \geq E[\phi(Y)] \) for all increasing functions \( \phi : \mathbb{R} \rightarrow \mathbb{R} \);

(ii) hazard rate order (denoted by \( X \geq_{hr} Y \)) if and only if \( h_{Y}(t) \geq h_{X}(t) \), for all \( t \in \mathbb{R} \), or equivalently, \( \bar{F}_{X}(t)/\bar{F}_{X}(t) \) is increasing in \( t \in \mathbb{R} \);

(iii) reversed hazard rate order (denoted by \( X \geq_{rh} Y \)) if and only if \( r_{X}(t) \geq r_{Y}(t) \), for all \( t \in \mathbb{R} \), or equivalently, \( \bar{F}_{X}(t)/\bar{F}_{Y}(t) \) is increasing in \( t \in \mathbb{R} \).

The following implications are well-known between these orders:

\[
X \leq_{hr} Y \implies X \leq_{st} Y.
\]

We refer to the books by [11] and [15] for elaborate discussions on various stochastic orderings, and their inter-relationships and applications.
2.2. Majorization orders.

**Definition 2.2.** Let \( a = (a_1, \ldots, a_n) \) and \( b = (b_1, \ldots, b_n) \) be vectors with increasing arrangements \( a_{(1)} \leq \cdots \leq a_{(n)} \) and \( b_{(1)} \leq \cdots \leq b_{(n)} \), respectively. Then:

(i) Vector \( a \) is said to be majorized by vector \( b \) (denoted by \( a \preceq b \)) if \( \sum_{j=1}^{i} a_{(j)} \geq \sum_{j=1}^{i} b_{(j)} \) for \( i = 1, \ldots, n-1 \), and \( \sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)} \);

(ii) Vector \( a \) is said to be weakly supermajorized by vector \( b \) (denoted by \( a \preceq_w b \)) if \( \sum_{j=1}^{i} a_{(j)} \geq \sum_{j=1}^{i} b_{(j)} \), for \( i = 1, \ldots, n \).

**Definition 2.3.** A real-valued function \( \phi \), defined on a set \( A \subseteq \mathbb{R}^n \), is said to be Schur-convex (Schur-concave) on \( A \) if \( a \preceq b \) implies \( \phi(a) \leq (\geq) \phi(b) \) for any \( a, b \in A \).

For a detailed discussion on majorization and Schur functions, we refer the readers to [10]. One of the results presenting necessary and sufficient conditions for the characterization of Schur-convex and Schur-concave functions is as follows.

**Lemma 2.4.** ([10], p. 84) Suppose \( J \subset \mathbb{R} \) is an open interval and \( \phi : J^n \to \mathbb{R} \) is a continuously differentiable function. Then, necessary and sufficient conditions for \( \phi \) to be Schur-convex (Schur-concave) on \( J^n \) are

(i) \( \phi \) is symmetric on \( J^n \);

(ii) for all \( i \neq j \) and all \( z \in J^n \),

\[
(z_i - z_j) \left( \frac{\partial \phi(z)}{\partial z_i} - \frac{\partial \phi(z)}{\partial z_j} \right) \geq 0 \quad (\leq 0),
\]

where \( \frac{\partial \phi(z)}{\partial z_i} \) denotes the partial derivative of \( \phi \) with respect to its \( i \)-th argument.

**Lemma 2.5.** ([10], p. 87) Consider the real-valued function \( \varphi \), defined on a set \( A \subseteq \mathbb{R}^n \). Then, \( u \succeq v \) implies \( \varphi(u) \succeq \varphi(v) \) if and only if \( \varphi \) is decreasing and Schur-convex on \( A \).

3. Copulas. Many different stochastic comparisons of univariate random variables have been defined and discussed in the literature; see [11] and [15] for pertinent details. These involve comparisons of marginal distributions of the underlying variables, without taking dependence between the variables into account. Here, we discuss stochastic comparisons of series-parallel and parallel-series systems with dependent heterogeneous components, when the joint distribution is modelled by the flexible family of Archimedean copulas.

Archimedean copulas have been applied extensively due to their mathematical tractability as well as their ability to represent a wide range of dependence. For a decreasing and continuous function \( \phi : [0, \infty) \to [0, 1] \) such that \( \phi(0) = 1 \), \( \phi(+\infty) = 0 \), and \( \psi = \phi^{-1} \) being the pseudo-inverse,

\[ C_{\phi}(u_1, \ldots, u_n) = \phi(\psi(u_1)) + \cdots + \psi(u_n)), \quad \text{for all } u_i \in [0, 1], \ i = 1, \ldots, n, \]

is said to be an Archimedean copula with generator \( \phi \) if \( (-1)^k \phi^{(k)}(x) \geq 0 \), for \( k = 0, \ldots, n-2 \), and \( (-1)^{n-2} \phi^{(n-2)}(x) \) is decreasing and convex. In the above, \( \phi^{(k)}(x) \) denotes the \( k \)-th derivative of \( x \). The Archimedean copula family includes many well-known copulas, such as independence (product), Clayton, Ali-Mikhail-Haq (AMH) and Gumbel-Hougaard copulas, as special cases.
A function $f$ is said to be superadditive if $f(x + y) \geq f(x) + f(y)$ for all $x$ and $y$ in the domain of $f$. Then, based on Lemma A.1 of [8], it is known that for two dimensional Archimedean copulas $C_{\phi_1}(u)$ and $C_{\phi_2}(u)$ with respective generators $\phi_1$ and $\phi_2$ and pseudo-inverses $\psi_1$ and $\psi_2$, if $\psi_2 \circ \phi_1$ is superadditive, $C_{\phi_1}(u) \leq C_{\phi_2}(u)$ for all $u \in [0,1]^n$. In this case, for many sub-families of Archimedean copulas, the superadditivity of $\psi_2 \circ \phi_1$ can be roughly interpreted as follows: Kendall’s $\tau$ of the copula with generator $\phi_2$ is larger than that of the copula with generator $\phi_1$ and is therefore more positively dependent. For detailed discussions on copulas and their properties, one may refer to [12].

4. Results for series-parallel systems. In this section, we focus on a series-parallel system that comprises $k$ dependent subsystems connected in series, with the $j$-th subsystem having $a_j$ components connected in parallel, for $j = 1, 2, \cdots, k$. The components within each subsystem are assumed to come from the $i$-th subpopulation with probability $p_i$, $i = 1, 2, \cdots, m$. Now, in a very general set-up, we assume that the components within each subsystem and also the subsystems themselves are dependent. We also assume that all the subpopulations have different Archimedean copulas with generators $\phi_i$, $i = 1, 2, \cdots, m$. Then, with $M_n^m(k|\mathbf{a}, \mathbf{p}, \mathbf{X})$ denoting the lifetime of such a series-parallel system, its survival function is given by

$$
\bar{F}_{S_{M_n^m(k|\mathbf{a}, \mathbf{p}, \mathbf{X})}}(t) = \Phi \left( \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right), \quad t \geq 0.
$$

It is then of interest to determine now the $n$ components from $m$ different subpopulations should be distributed so that the resulting series-parallel system, with dependent subsystems, will be optimal in some stochastic sense. First, we fix the number of dependent subsystems within the series-parallel system to be $k$. We then show specifically that if the allocation vector for any model with dependent components is weakly majorized by another allocation vector, then the series-parallel system corresponding to the first vector is more reliable than the system corresponding to the second vector in the sense of usual stochastic order. In the following theorem, we relax the majorization assumption of Theorem 3.1 of [4] by the weak majorization order.

**Theorem 4.1.** Let $S_{M_n^m(k, \Phi_1|\mathbf{a}, \mathbf{p}, \mathbf{X})}$ denote the random lifetime of a series-parallel system corresponding to $M_n^m(k, \Phi_1|\mathbf{a}, \mathbf{p}, \mathbf{X}) \in A$ with different generator Archimedean copulas $\Phi_1$ and $\phi_i$, for $i = 1, 2, \cdots, m$. Also, let $S_{M_n^m(k, \Phi_2|\mathbf{b}, \mathbf{p}, \mathbf{X})}$ be the random lifetime of a series-parallel system corresponding to $M_n^m(k, \Phi_2|\mathbf{b}, \mathbf{p}, \mathbf{X}) \in A$ with different generator Archimedean copulas $\Phi_2$ and $\phi_i$, for $i = 1, 2, \cdots, m$. Let us assume further that $\Psi_2 \circ \Phi_1$ is superadditive. Then, we have

$\mathbf{a} \preceq \mathbf{b} \implies S_{M_n^m(k, \Phi_1|\mathbf{a}, \mathbf{p}, \mathbf{X})} \preceq_{st} S_{M_n^m(k, \Phi_2|\mathbf{b}, \mathbf{p}, \mathbf{X})}$

**Proof.** The survival functions of series-parallel systems $S_{M_n^m(k, \Phi_1|\mathbf{a}, \mathbf{p}, \mathbf{X})}$ and $S_{M_n^m(k, \Phi_2|\mathbf{b}, \mathbf{p}, \mathbf{X})}$ are given by

$$
\bar{F}_{S_{M_n^m(k, \Phi_1|\mathbf{a}, \mathbf{p}, \mathbf{X})}}(t) = \Phi \left( \sum_{j=1}^{k} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right), \quad t \geq 0,
$$
\[ F_{M_{m_1}^m(k, \phi_2) b p X}(t) = \Phi_2 \left( \sum_{j=1}^{k} \Psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (b_j \psi_i (F_i(t))) \right] \right), \quad t \geq 0, \]

respectively, where \( a_1 + \cdots + a_k = n_1, \ b_1 + \cdots + b_k = n_2 \) and \( \sum_{i=1}^{m} p_i = 1 \). The superadditivity of \( \Psi_2 \circ \Phi_1 \) implies that

\[ \Phi_1 \left( \sum_{i=1}^{k} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_j \psi_i (F_i(t))) \right] \right) \leq \Phi_2 \left( \sum_{i=1}^{k} \Psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (b_j \psi_i (F_i(t))) \right] \right). \]

Next, for \( i \neq j \), we obtain

\[ S(a) = F_{S_{M_{m_1}^m(k, \phi_1) a p X}}(t) = \Phi_1 \left( \sum_{i=1}^{k} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_j \psi_i (F_i(t))) \right] \right), \quad t \geq 0. \]

Then, according to Lemma 2.5, we only need to show that \( S(a) \) is increasing and Schur-concave in \( (a_1, \cdots, a_k) \), for any fixed \( t \geq 0 \). Taking the derivative of \( S(a) \) with respect to \( a_\alpha, \alpha \in \{1, 2, \cdots, k\} \), we obtain

\[ \frac{\partial S(a)}{\partial a_\alpha} = - \sum_{i=1}^{m} p_i \phi_i (F_i(t)) \phi'_s (a_\alpha \psi_i (F_i(t))) \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_\alpha \psi_i (F_i(t))) \right] \]

\[ \times \Phi_1 \left( \sum_{j=1}^{k} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_j \psi_i (F_i(t))) \right] \right) \]

\[ \geq 0. \]

Next, for \( 1 \leq \alpha < \beta \leq k \), we obtain

\[ J_1 := (a_\alpha - a_\beta) \left( \frac{\partial S(a)}{\partial a_\alpha} - \frac{\partial S(a)}{\partial a_\beta} \right) \]

\[ = (a_\alpha - a_\beta) \left( \sum_{i=1}^{m} p_i \psi_i (F_i(t)) \phi'_s (a_\beta \psi_i (F_i(t))) \left[ \Phi_1 \left( \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_\alpha \psi_i (F_i(t))) \right] \right) \right] \phi'_s (a_\alpha \psi_i (F_i(t))) \right) \]

\[ \times \Phi_1 \left( \sum_{j=1}^{k} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_j \psi_i (F_i(t))) \right] \right). \]

For \( i = 1, 2, \cdots, m \), we have

\[ \phi'_s (x) \leq 0, \quad \phi''_s (x) \geq 0, \quad \Phi'_s (x) \leq 0, \quad \Phi''_s (x) \geq 0. \]

So, for \( a_\alpha \geq a_\beta \), we obtain

\[ \sum_{i=1}^{m} p_i \psi_i (F_i(t)) \phi'_s (a_\beta \psi_i (F_i(t))) \leq \sum_{i=1}^{m} p_i \psi_i (F_i(t)) \phi'_s (a_\alpha \psi_i (F_i(t))) \]

and

\[ \Phi'_s \left( \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_\alpha \psi_i (F_i(t))) \right] \right) \leq \Phi'_s \left( \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i (a_\beta \psi_i (F_i(t))) \right] \right). \]
From (3) and (4), we get
\[
\Phi_1 \left( \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \left( a_i \psi_i \left( F_i(t) \right) \right) \right] \right) \leq \Phi_1 \left( \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \left( a_i \psi_i \left( F_i(t) \right) \right) \right] \right),
\]
(5)

From (1) and (5), we obtain
\[
(a_{\alpha} - a_{\beta}) \left( \frac{\partial S(a)}{\partial a_\alpha} - \frac{\partial S(a)}{\partial a_\beta} \right) \leq 0,
\]
and similarly, if \( a_\alpha \leq a_\beta \), we obtain
\[
(a_{\alpha} - a_{\beta}) \left( \frac{\partial S(a)}{\partial a_\alpha} - \frac{\partial S(a)}{\partial a_\beta} \right) \leq 0.
\]
Thus, for all \( 1 \leq \alpha, \beta \leq k \), we get
\[
(a_{\alpha} - a_{\beta}) \left( \frac{\partial S(a)}{\partial a_\alpha} - \frac{\partial S(a)}{\partial a_\beta} \right) \leq 0,
\]
which means that \( S(a) \) is Schur-concave in \((a_1, \cdots, a_k)\), for any fixed \( t \geq 0 \), as required.

**Example 4.2.** Let us consider \( F_i(t) = 1 - e^{-\lambda_i t}, k = 3, m = 4, \phi_i(t) = (\theta_i t + 1)^{-1/\theta_i}, \theta_i \geq 0, i = 1, \cdots, 4 \). Let us further choose \( a = (4, 6, 8), b = (8, 6, 15), \)
\( p = (0.4, 0.3, 0.2, 0.1) \), \( \theta = (1, 2, 3, 4) \), \( \lambda = (1, 2, 3, 4) \), \( \alpha = 0.5, \beta = 2 \) and \( \mu = 2 \). Now, let us consider the following situations:

(i) Suppose \( \Phi_1 = \Phi_2 = (1 - \alpha)/(e^t - \alpha) \), for \( \alpha \in [0, 1) \). Then, explicit expressions of \( \bar{F}_{S_{M_{n}^{(k, \Phi_1)}(a \cdot p \cdot X)} (t)} \) and \( \bar{F}_{S_{M_{n}^{(k, \Phi_2)}(b \cdot p \cdot X)} (t)} \) are as follows:

\[
\bar{F}_{S_{M_{n}^{(k, \Phi_1)}(a \cdot p \cdot X)} (t)} = \frac{1 - \alpha}{\exp \left[ \sum_{j=1}^{3} \ln \left( \frac{1 - \alpha}{1 - \sum_{i=1}^{4} p_i \left( a_j \left[ \frac{1 - \alpha}{\left( 1 - e^{-\lambda_i t} \right)^{\theta_j} - 1 + 1 \right]^{\frac{\mu}{\theta_i}} + \alpha \right] \right)} \right] - \alpha),
\]

\[
\bar{F}_{S_{M_{n}^{(k, \Phi_2)}(b \cdot p \cdot X)} (t)} = \frac{1 - \alpha}{\exp \left[ \sum_{j=1}^{3} \ln \left( \frac{1 - \alpha}{1 - \sum_{i=1}^{4} p_i \left( b_j \left[ \frac{1 - \alpha}{\left( 1 - e^{-\lambda_i t} \right)^{\theta_j} - 1 + 1 \right]^{\frac{\mu}{\theta_i}} + \alpha \right] \right)} \right] - \alpha),
\]

(ii) Suppose \( \Phi_1 = \Phi_2 = e^{1-(1+ t)^{\beta}}, \) for \( \beta \in [1, \infty) \). Then, explicit expressions of \( \bar{F}_{S_{M_{n}^{(k, \Phi_1)}(a \cdot p \cdot X)} (t)} \) and \( \bar{F}_{S_{M_{n}^{(k, \Phi_2)}(b \cdot p \cdot X)} (t)} \) are as follows:

\[
\bar{F}_{S_{M_{n}^{(k, \Phi_1)}(a \cdot p \cdot X)} (t)} = \exp \left[ 1 - \left( -2 + \sum_{j=1}^{3} 1 - \sum_{i=1}^{4} p_i \left( a_j \left[ \left( 1 - e^{-\lambda_i t} \right)^{\theta_j} - 1 \right]^{\frac{\mu}{\theta_i}} \right) \right) \right];
\]

\[
\bar{F}_{S_{M_{n}^{(k, \Phi_2)}(b \cdot p \cdot X)} (t)} = \exp \left[ 1 - \left( -2 + \sum_{j=1}^{3} 1 - \sum_{i=1}^{4} p_i \left( b_j \left[ \left( 1 - e^{-\lambda_i t} \right)^{\theta_j} - 1 \right]^{\frac{\mu}{\theta_i}} \right) \right) \right];
\]
(iii) Suppose \( \Phi_1 = \Phi_2 = (\mu t + 1)^{-1/\mu} \), for \( \mu \geq 0 \). Then, explicit expressions of 
\[ F_{S_{M^{(k,a,p,X)}}(t)}(t) \] and 
\[ F_{S_{M^{(k,b,p,X)}}(t)}(t) \] are as follows:

\[
F_{S_{M^{(k,a,p,X)}}(t)} = -2 + \sum_{j=1}^{3} \left( 1 - \sum_{i=1}^{4} p_i \left( a_j \left( 1 - e^{-\lambda_t j} \right)^{\theta_i} - 1 \right) + 1 \right)^{\frac{-\alpha}{\beta}} \right)^{\frac{-1}{\beta}},
\]

\[
F_{S_{M^{(k,b,p,X)}}(t)} = -2 + \sum_{j=1}^{3} \left( 1 - \sum_{i=1}^{4} p_i \left( b_j \left( 1 - e^{-\lambda_t j} \right)^{\theta_i} - 1 \right) + 1 \right)^{\frac{-\alpha}{\beta}} \right)^{\frac{-1}{\beta}}.
\]

Plots of the reliability functions of these series-parallel systems under Clayton copula for the components and the copulas considered in Parts (i)-(iii) above for the subsystems are presented in Figure 1. Because \( a \succ b \), based on Theorem 4.1, we have 
\[ S_{M^{(k,a,p,X)}} \preceq \mu S_{M^{(k,b,p,X)}} \] for all three parts above. But, Theorem 3.1 of [4] cannot be used in this case to obtain this stochastic ordering result here, since neither \( a \not\succ b \) nor \( b \not\succ a \), as the sums of the two vectors are unequal.

![Figure 1: Plots of the reliability functions under Clayton copula for the components and copulas considered in Parts (i)-(iii) for the subsystems, from left to right.](image)

**Remark 1.** Observe that Theorem 4.1 presents the usual stochastic order between series-parallel systems without any restriction on the generator functions of the copulas for the subsystems. Now, if we take \( \Phi_1(t) = \Phi_2(t) = e^{-t} \), corresponding to independence of the subsystems, then the results of Fang et al. (2020) are readily deduced.

**Remark 2.** It should be noted that the condition “\( \Psi_2 \circ \Phi_1 \) is superadditive” in Theorem 4.1 is quite general and is easy to verify for many Archimedean copulas. For example, consider the Gumbel-Hougaard copula with generator \( \Phi(t) = e^{1-(1+t)^{\theta}} \) for \( \theta \in [1, \infty) \), and then take \( \Phi_1(t) = e^{1-(1+t)^{\alpha}} \) and \( \Phi_2(t) = e^{1-(1+t)^{\beta}} \). Then, we observe that \( \Psi_2 \circ \Phi_1(t) = (1+t)^{\alpha/\beta-1} \), which when differentiated twice with respect to \( t \), yields \( \Psi_2 \circ \Phi_1(t)'' = (\frac{\alpha}{\beta})^{(\alpha/\beta-1)}(1+t)^{(\alpha/\beta)-2} \geq 0 \), for \( \alpha > \beta > 1 \), which implies the superadditivity of \( \Psi_2 \circ \Phi_1(t) \).

**Theorem 4.3.** Let \( S_{M^{(k,a,p,X)}} \) denote the random lifetime of a series-parallel system corresponding to \( M^{(k,a,p,X)} \) with different generator Archimedean copulas \( \Phi_1 \) and \( \Phi_i \), for \( i = 1,2,\cdots,m \). Also, let \( S_{M^{(k-1,b,p,X)}} \) denote the random lifetime of a series-parallel system corresponding to \( M^{(k-1,b,p,X)} \) with different generator Archimedean copulas \( \Phi_2 \) and \( \Phi_i \), for \( i = 1,2,\cdots,m \). Let \( a = (a_1,\cdots,a_k) \) and \( b = (b_1,\cdots,b_{k-1}) \), and assume
further that \( \Psi_2 \circ \Phi_1 \) is superadditive. Then, we have 
\[ a_j \leq b_j \text{ for all } j = 1, \cdots, k - 1 \implies S_{M_{n_1}^m (k, \Phi_1 | a.p.X)} \leq_{st} S_{M_{n_2}^m (k-1, \Psi_2 | b.p.X)}. \]

Proof. From (2), for all \( t \geq 0 \), we have 
\[
F_{S_{M_{n_1}^m (k, \Phi_1 | a.p.X)} (t)} = \Phi_1 \left( \sum_{i=1}^{k} \Psi_1 \left( 1 - \sum_{i=1}^{m} p_i \{ a_j \psi_i (F_i) \} \right) \right) 
\leq \Phi_1 \left( \sum_{i=1}^{k} \Psi_1 \left( 1 - \sum_{i=1}^{m} p_i \{ a_j \psi_i (F_i) \} \right) \right) 
\leq \Phi_2 \left( \sum_{i=1}^{k} \Psi_2 \left( 1 - \sum_{i=1}^{m} p_i \{ b_j \psi_i (F_i) \} \right) \right) \quad \text{(as } \Psi_2 \circ \Phi_1 \text{ is superadditive)} 
\]
which means that \( S_{M_{n_1}^m (k, \Phi_1 | a.p.X)} \leq_{st} S_{M_{n_2}^m (k-1, \Psi_2 | b.p.X)} \), as required. \( \square \)

**Theorem 4.4.** Let \( S_{M_{n}^m (k, \Phi | a.p.X)} \) denote the random lifetime of a series-parallel system corresponding to \( M_{n}^m (k, \Phi | a.p.X) \in A \) with different generator Archimedean copulas \( \Phi \) and \( \phi_i \), for \( i = 1, 2, \cdots, m \). Then, we have 
\[
S_{M_{n}^m (1| a.p.X)} \geq_{st} S_{M_{n}^m (k| a.p.X)} \geq_{st} S_{M_{n}^m (1| b.p.X)}. 
\]

Proof. Let \( a_l \in \{ a_1, a_2, \cdots, a_k \} \). Because \( a_l \leq n \) for \( l = 1, \cdots, n \), we get 
\[
\Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ n \psi_i (F_i) \} \right) \leq \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_i \psi_i (F_i) \} \right) 
\leq \sum_{j=1}^{k} \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i) \} \right). \quad \text{(6)}
\]
We also observe that 
\[
\Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_i \psi_i (F_i) \} \right) \leq \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (F_i) \} \right),
\]
and so 
\[
\sum_{j=1}^{k} \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i) \} \right) \leq k \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (F_i) \} \right) 
\leq n \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (F_i) \} \right). \quad \text{(7)}
\]
As \( \Phi(x) \) is a decreasing function, from (6) and (7), we obtain 
\[
1 - \sum_{i=1}^{m} p_i \{ n \psi_i (F_i) \} = \Phi \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \{ n \psi_i (F_i) \} \right) \right) 
\geq \Phi \left( \sum_{j=1}^{k} \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i) \} \right) \right) 
\geq \Phi \left( n \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (F_i) \} \right) \right), \quad \text{(8)}
\]
which completes the proof of the theorem.

**Theorem 4.5.** Let $S_{M_n(k|a,p,X)}$ denote the random lifetime of a series-parallel system corresponding to $M_n(k|a,p,X) \in A$ with different generator Archimedean copulas $\Phi$ and $\phi_i$, for $i = 1, 2, \ldots, m$. Let us further assume that $\Phi$ is log-concave. Then, we have

$$S_{M_n(1|n,p,X)} \geq h_r S_{M_n(k|a,p,X)} \geq h_r S_{M_n(n|1,p,X)}.$$  

**Proof.** Note that the hazard rate function of $S_{M_n(k|a,p,X)}$ is given by

$$h_{S_{M_n(k|a,p,X)}}(t) = \sum_{i=1}^{k} \frac{\prod_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (\alpha_i \psi_i (F_i(t)))}{\prod_{i=1}^{m} p_i \phi_i (\alpha_i \psi_i (F_i(t)))} \times \frac{\Phi' \left( \sum_{i=1}^{k} \left[ p_i \phi_i (\alpha_i \psi_i (F_i(t))) \right] \right)}{\sum_{i=1}^{k} \left[ p_i \phi_i (\alpha_i \psi_i (F_i(t))) \right]}.$$  

Because $a_1 + a_2 + \cdots + a_k = n$, we have the following two special cases:

$$h_{S_{M_n(1|n,p,X)}}(t) = \sum_{i=1}^{k} \frac{\prod_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (n \psi_i (F_i(t)))}{\prod_{i=1}^{m} p_i \phi_i (n \psi_i (F_i(t)))} \times \frac{\Phi' \left( \sum_{i=1}^{k} \left[ n \psi_i (F_i(t)) \right] \right)}{\sum_{i=1}^{k} \left[ n \psi_i (F_i(t)) \right]}.$$  

$$h_{S_{M_n(n|1,p,X)}}(t) = \sum_{i=1}^{k} \frac{\prod_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t)))}{\prod_{i=1}^{m} p_i \phi_i (F_i(t)))} \times \frac{\Phi' \left( \sum_{i=1}^{k} \left[ F_i (t) \right] \right)}{\sum_{i=1}^{k} \left[ F_i (t) \right]}.$$  

Now, in order to show that $S_{M_n(1|n,p,X)} \geq h_r S_{M_n(k|a,p,X)} \geq h_r S_{M_n(n|1,p,X)}$, we need to establish, for all $i = 1, 2, \ldots, k$, that

$$0 \geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (n \psi_i (F_i(t))) \geq h_r \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (\alpha_i \psi_i (F_i(t)))$$  

$$\geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t))) \geq h_r \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t)))$$  

$$\sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t))) \geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t)))$$  

$$\sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t))) \geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i' (F_i(t)))$$  

$$(9)$$
and

\[
0 \geq \frac{\Phi' (\Psi [1 - \sum_{i=1}^{m} p_i \{ n \psi_i (F_i(t)) \}])}{\Phi (\Psi [1 - \sum_{i=1}^{m} p_i \{ n \psi_i (F_i(t)) \}])} \\
\geq \frac{\Phi' \left( \sum_{i=1}^{k} \Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ a_i \psi_i (F_i(t)) \}] \right)}{\Phi \left( \sum_{i=1}^{k} \Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ a_i \psi_i (F_i(t)) \}] \right)} \\
\geq \frac{\Phi' (n \Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (F_i(t)) \}])}{\Phi (n \Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (F_i(t)) \}])}.
\]

For \( x \geq 0 \), let us consider

\[
g(x) = \frac{\sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i'(x \psi_i(F_i(t)))}{\Phi' \left( \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ x \psi_i(F_i(t)) \} \right] \right)}.
\]

Then, the derivative of \( g(x) \) with respect to \( x \) can be obtained as

\[
g'(x) = \left[ \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i'(x \psi_i(F_i(t))) \right] \Phi' \left( \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ x \psi_i(F_i(t)) \} \right] \right)
\]

\[
\times \left[ \Phi' \left( \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ x \psi_i(F_i(t)) \} \right] \right) \right]^2
\]

\[
+ \left[ \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(F_i(t)) \phi_i'(x \psi_i(F_i(t))) \right] \left[ \sum_{i=1}^{m} \psi_i(F_i(t)) \right] \Phi' \left( \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ x \psi_i(F_i(t)) \} \right] \right)
\]

\[
\times \left[ \Phi' \left( \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ x \psi_i(F_i(t)) \} \right] \right) \right]^2
\]

\[
\geq 0;
\]
so, \( g'(x) \) is increasing in \( x \geq 0 \), which means that Eq. (9) is true. Also, if \( \Phi(t) \) is a log-concave function, due to (6), Eq. (10) is true. Hence, the theorem is proven. \( \square \)

**Remark 3.** The condition “\( \Phi \) is log-concave” in Theorem 4.5 is quite general and can be easily verified for many Archimedean copulas. For example, let us consider the Gumbel-Hougaard copula with generator \( \Phi(t) = e^{1-(1+t)^\theta} \), for \( \theta \in [1, \infty) \). It is then easy to verify that \( \log \Phi(t) = 1 - (1 + t)^\theta \) is concave in \( t \in [0, 1] \). \( \square \)

In the following theorem, we study the influence of selection probabilities and the distributions of subpopulations on the reliability of a series-parallel system.

**Theorem 4.6.** Let \( M_{\mu_1}^m(k, \Phi_1(a,p,X)) \) denote the random lifetime of a series-parallel system corresponding to \( M_{\mu_1}^m(k, \Phi_1(a,p,X)) \in A \) with different generator Archimedean copulas \( \Phi_1 \) and \( \phi_i \), for \( i = 1, 2, \ldots, m \). Also, let \( M_{\mu_2}^m(k, \Phi_2(a,q,X)) \) denote the random lifetime of a series-parallel system corresponding to \( M_{\mu_2}^m(k, \Phi_2(a,q,X)) \in A \) with different generator Archimedean copulas \( \Phi_2 \) and \( \phi_i \), for \( i = 1, 2, \ldots, m \). Further, suppose \( F_1 \leq \cdots \leq F_m \), \( p_1 \geq p_2 \geq \cdots \geq p_n \), and \( q_1 \geq q_2 \geq \cdots \geq q_n \), and \( \Psi_1 \circ \Phi_2 (\Psi_2 \circ \Phi_1) \) is superadditive. Then, we have

\[
\mathbf{p} \geq \mathbf{q} \implies M_{\mu_1}^m(k, \Phi_1(a,p,X)) \geq_{st} (\leq_{st}) M_{\mu_2}^m(k, \Phi_2(a,q,X)).
\]

**Proof.** For all \( t \geq 0 \), we have

\[
\bar{F}_{M_{\mu_1}^m(k, \Phi_1(a,p,X))}(t) = \Phi_1 \left( \sum_{j=1}^{k} \psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right),
\]

\[
\bar{F}_{M_{\mu_2}^m(k, \Phi_2(a,q,X))}(t) = \Phi_2 \left( \sum_{j=1}^{k} \psi_2 \left[ 1 - \sum_{i=1}^{m} q_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right).
\]

The superadditivity of \( \Psi_1 \circ \Phi_2 \) implies that

\[
\Phi_2 \left( \sum_{i=1}^{m} \psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right) \leq \Phi_1 \left( \sum_{i=1}^{m} \psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right).
\]

Then, to prove the desired result, it is sufficient to show that

\[
\Phi_2 \left( \sum_{i=1}^{m} \psi_2 \left[ 1 - \sum_{i=1}^{m} q_i \phi_i \{ b_j \psi_i (F_i(t)) \} \right] \right) \leq \Phi_2 \left( \sum_{i=1}^{m} \psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right).
\]

Now, let us define

\[
S(p) = \bar{F}_{M_{\mu_2}^m(k, \Phi_2(a,p,X))}(t) = \Phi_2 \left( \sum_{j=1}^{k} \psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right).
\]

According to Lemma 2.4, we then need to show that \( S(p) \) is decreasing and Schur-convex (Schur-concave) in \( (p_1, \cdots, p_m) \), for any fixed \( t \geq 0 \). Taking the derivative
of \( S(p) \) with respect to \( p_\alpha, \alpha \in \{1, 2, \cdots, k\} \), we get
\[
\frac{\partial S(p)}{\partial p_\alpha} = - \sum_{j=1}^{k} \left[ \varphi_\alpha \{ a_j \psi_\alpha (F_\alpha(t)) \} \Psi_2 \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right) \right] \\
\times \Phi' \left( \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right) \\
\leq 0.
\]
So,
\[
J_2 = (p_\alpha - p_\beta) \left( \frac{\partial S(p)}{\partial p_\alpha} - \frac{\partial S(p)}{\partial p_\beta} \right) \\
= \sum_{j=1}^{k} \left( \varphi_\beta \{ a_j \psi_\beta (F_\beta(t)) \} - \varphi_\alpha \{ a_j \psi_\alpha (F_\alpha(t)) \} \right) \Psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \\
\times \Phi' \left( \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right).
\]
For all \( 1 \leq \alpha < \beta \leq m \), if \( F_\alpha(t) \leq F_\beta(t) \), then \( \psi_\beta(t) \leq \psi_\alpha(t) \) and \( \phi_\alpha(t) \leq \phi_\beta(t) \) \( \left( F_\alpha(t) \geq F_\beta(t), \psi_\alpha(t) \geq \psi_\beta(t) \right) \). Now, upon using (2), we obtain
\[
\phi_\alpha \{ a_j \psi_\alpha (F_\alpha(t)) \} \leq (\geq) \phi_\alpha \{ a_j \psi_\beta (F_\beta(t)) \} \leq (\geq) \phi_\beta \{ a_j \psi_\beta (F_\beta(t)) \},
\]
which implies that
\[
\left( \frac{\partial S(p)}{\partial p_\alpha} - \frac{\partial S(p)}{\partial p_\beta} \right) \geq (\leq) 0.
\]
Hence, the theorem is proven.

\[ \square \]

Example 4.7. Let us consider \( F_i(t) = 1 - e^{-\lambda_i t}, \ k = 4, \ m = 5, \ \phi_i(t) = (\theta_i t + 1)^{-1/\theta_i}, \ \theta_i \geq 0, \ i = 1, \cdots, 5 \). Let us further choose \( \alpha = (5, 10, 20, 8), \ \lambda = (1, 2, 3, 4, 5), \ p = (0.4, 0.3, 0.1, 0.1, 0.1), \ q = (0.3, 0.2, 0.2, 0.2, 0.1), \ \theta = (1, 2, 3, 4, 5), \ \alpha = 0.5, \ \beta = 2 \) and \( \mu = 2 \). Let us now consider the following situations:

(i) Suppose \( \Phi_1 = \Phi_2 = (1 - \alpha)/(e^t - \alpha) \), for \( \alpha \in [0, 1] \). Then, explicit expressions of \( F_{\text{M}_m^{(k, \Phi_1 \{ a.p.X \})}}(t) \) and \( F_{\text{M}_m^{(k, \Phi_2 \{ a.q.X \})}}(t) \) are given by
\[
F_{\text{M}_m^{(k, \Phi_1 \{ a.p.X \})}}(t) = \exp \left[ \frac{1 - \alpha}{\ln \left( \frac{1 - a}{1 - \sum_{i=1}^{m} p_i \left( 1 - e^{-\lambda_i t} \right)^{\theta_i - 1} + 1} \right) - \alpha \right],
\]
\[
F_{\text{M}_m^{(k, \Phi_2 \{ a.q.X \})}}(t) = \exp \left[ \frac{1 - \alpha}{\ln \left( \frac{1 - a}{1 - \sum_{i=1}^{m} q_i \left( 1 - e^{-\lambda_i t} \right)^{\theta_i - 1} + 1} \right) - \alpha \right].
\]
(ii) Suppose $\Phi_1 = \Phi_2 = e^{-(1+t)^\beta}$, for $\beta \in [1, \infty)$. Then, explicit expressions of $\tilde{F}_{S_M^{(k, \Phi_1)}(a, p, X)}(t)$ and $\tilde{F}_{S_M^{(k, \Phi_2)}(a, q, X)}(t)$ are given by

$$
\tilde{F}_{S_M^{(k, \Phi_1)}(a, p, X)}(t) = \exp \left[ 1 - \left( -3 + \sum_{j=1}^{4} \left( 1 - \sum_{i=1}^{5} p_i \left( 1 - e^{-\lambda_i t} \right)^{\theta_j} - 1 \right) \right) \right],
$$

$$
\tilde{F}_{S_M^{(k, \Phi_2)}(a, q, X)}(t) = \exp \left[ 1 - \left( -3 + \sum_{j=1}^{4} \left( 1 - \sum_{i=1}^{5} q_i \left( 1 - e^{-\lambda_i t} \right)^{\theta_j} - 1 \right) \right) \right].
$$

(iii) Suppose $\Phi_1 = \Phi_2 = (\mu t + 1)^{-1/\mu}$, for $\mu \geq 0$. Then, explicit expressions of $\tilde{F}_{S_M^{(k, \Phi_1)}(a, p, X)}(t)$ and $\tilde{F}_{S_M^{(k, \Phi_2)}(a, q, X)}(t)$ are given by

$$
\tilde{F}_{S_M^{(k, \Phi_1)}(a, p, X)}(t) = \left[ -3 + \sum_{j=1}^{4} \left( 1 - \sum_{i=1}^{5} p_i \left( 1 - e^{-\lambda_i t} \right)^{\theta_j} + 1 \right) \right]^{-\mu/\mu},
$$

$$
\tilde{F}_{S_M^{(k, \Phi_2)}(a, q, X)}(t) = \left[ -3 + \sum_{j=1}^{4} \left( 1 - \sum_{i=1}^{5} q_i \left( 1 - e^{-\lambda_i t} \right)^{\theta_j} + 1 \right) \right]^{-\mu/\mu}.
$$

Plots of the reliability functions of these series-parallel systems under Clayton copula for the components and the copulas considered in Parts (i)-(iii) above for the subsystems are presented in Figure 2. Because $p \geq q$, based on Theorem 4.6, we have $S_M^{(k, \Phi_1)}(a, p, X) \geq S_M^{(k, \Phi_2)}(a, q, X)$ for all three parts above.

Figure 2: Plots of the reliability functions under Clayton copula for the components and copulas considered in Parts (i)-(iii) for the subsystems, from left to right.

5. **Results for parallel-series systems.** In this section, we focus on a parallel-series system that comprises $k$ dependent subsystems connected in parallel, with the $j$-th subsystem having $a_j$ components connected in series, for $j = 1, 2, \ldots, k$. The components within each subsystem are assumed to come from the $i$-th subpopulation with probability $p_i$, $i = 1, 2, \ldots, m$. As in the preceding section, we assume that the components within each subsystem and also the subsystems themselves are dependent. We also assume that all the subpopulations have different Archimedean copulas with generators $\phi_i$, $i = 1, 2, \ldots, m$. Then, with $M_m^{(k)}(a, p, X)$ denoting
the lifetime of such a parallel-series system, its cumulative distribution function is given by

\[ F_{H_{M_{n}^{m}(k, \Phi_{1})a.p.X}}(t) = \Phi \left( \sum_{j=1}^{k} \sum_{i=1}^{m} a_j \psi_i \left( \bar{F}_i(t) \right) \right), \quad t \geq 0. \quad (11) \]

It is then of interest to determine how the \( n \) components from \( m \) different sub-populations should be distributed so that the resulting parallel-series system, with dependent subsystems, will be optimal in some stochastic sense. We first take the number of dependent subsystems within the parallel-series system to be fixed as \( k \).

We then show specifically that if the allocation vector for any model with dependent components is majorized by another allocation vector, then the parallel-series system corresponding to the second vector will be more reliable than the system corresponding to the first vector in the sense of usual stochastic order. In the following theorem, we strengthen Theorem 4.1 of [4] by relaxing the majorization order in their result to the weak majorization order.

**Theorem 5.1.** Let \( H_{M_{n_{1}}^{m}(k, \Phi_{1})a.p.X} \) denote the lifetime of a parallel-series system corresponding to \( M_{n_{1}}^{m}(k, \Phi_{1})a.p.X \) \( \in A \) with different generator Archimedean copulas \( \Phi_{1} \) and \( \phi_{i} \), for \( i = 1, 2, \ldots, m \). Also, let \( H_{M_{n_{2}}^{m}(k, \Phi_{2})b.p.X} \) denote the lifetime of a parallel-series system corresponding to \( M_{n_{2}}^{m}(k, \Phi_{2})b.p.X \) \( \in A \) with different generator Archimedean copulas \( \Phi_{2} \) and \( \phi_{i} \), for \( i = 1, 2, \ldots, m \). Further, let \( \Psi_{2} \circ \Phi_{1} \) is superadditive. Then,

\[ a \overset{w}{\succ} b \Rightarrow H_{n_{1}}^{m}(k, \Phi_{1})a.p.X \succeq_{st} H_{n_{2}}^{m}(k, \Phi_{2})b.p.X. \]

**Proof.** The distributions functions of parallel-series systems \( H_{M_{n_{1}}^{m}(k, \Phi_{1})a.p.X} \) and \( H_{M_{n_{2}}^{m}(k, \Phi_{2})b.p.X} \) are given by

\[ F_{H_{M_{n_{1}}^{m}(k, \Phi_{1})a.p.X}}(t) = \Phi_{1} \left( \sum_{j=1}^{k} \sum_{i=1}^{m} a_j \psi_i \left( \bar{F}_i(t) \right) \right), \quad t \geq 0, \]

\[ F_{H_{M_{n_{2}}^{m}(k, \Phi_{2})b.p.X}}(t) = \Phi_{2} \left( \sum_{j=1}^{k} \sum_{i=1}^{m} b_j \psi_i \left( \bar{F}_i(t) \right) \right), \quad t \geq 0, \]

respectively. The superadditivity of \( \Psi_{2} \circ \Phi_{1} \) implies that

\[ \Phi_{1} \left( \sum_{i=1}^{n} \Psi_{1} \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \left( \bar{F}_i(t) \right) \right] \right) \leq \Phi_{2} \left( \sum_{i=1}^{n} \Psi_{2} \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \left( \bar{F}_i(t) \right) \right] \right). \]

Then, to prove the desired result, it is sufficient to show that

\[ \Phi_{1} \left( \sum_{i=1}^{n} \Psi_{1} \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \left( \bar{F}_i(t) \right) \right] \right) \leq \Phi_{1} \left( \sum_{i=1}^{n} \Psi_{1} \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \left( \bar{F}_i(t) \right) \right] \right). \]

Let us set \( H(a) = F_{H_{M_{n_{1}}^{m}(k, \Phi_{1})a.p.X}}(t) = \Phi_{1} \left( \sum_{j=1}^{k} \sum_{i=1}^{m} a_j \psi_i \left( \bar{F}_i(t) \right) \right). \)

Then, by using arguments similar to those used for proving Theorem 4.1, we can show that \( H(a) \) is increasing in \( a_{i} \), and for all \( 1 \leq \alpha, \beta \leq k \), that

\[ (a_{\alpha} - a_{\beta}) \left( \frac{\partial H(a)}{\partial a_{\alpha}} - \frac{\partial H(a)}{\partial a_{\beta}} \right) \leq 0, \]
which means that \( H_{n1}^m(k|a, p, X) \geq_{st} H_{n1}^m(k|b, p, X) \), as required.

\[\text{Example 5.2.} \text{ Let us consider } \bar{F}_i(t) = e^{-\lambda_i t}, \ k = 4, \ m = 6, \ \phi_i(t) = \frac{1}{e^{1-\theta_i}}, \ \theta_i \in [0, 1], \ i = 1, \ldots, 6. \text{ Let us further choose } a = (7, 5, 3, 2), \ b = (6.9, 16, 3), \ p = (0.1, 0.2, 0.1, 0.2, 0.1, 0.3), \ \Theta = (1/2, 1/3, 1/5, 1/7, 1/9, 1/4), \ \lambda = (1, 2, 3, 4, 5, 6), \ \alpha = 0.8, \ \beta = 2 \text{ and } \mu = 2. \text{ Now, let us consider the following situations:}

(i) Suppose \( \Phi_1 = \Phi_2 = (1 - \alpha)/(e^{t} - \alpha) \), for \( \alpha \in [0, 1) \). Then, explicit expressions of \( F_{H_{Mn}^m(k, \Phi_1|a, p, X)}(t) \) and \( F_{H_{Mn}^m(k, \Phi_2|b, p, X)}(t) \) are given by

\[ F_{H_{Mn}^m(k, \Phi_1|a, p, X)}(t) = \exp \left\{ -3 + 4 \sum_{j=1}^{4} \left[ 1 - \sum_{i=1}^{6} \frac{p_i(1-\theta_i)}{(1-\theta_i)e^{\lambda_i t} + \theta_i} \right] \right\}^{\frac{1}{\mu}}. \]

(ii) Suppose \( \Phi_1 = \Phi_2 = e^{1-(1+t)\beta} \), for \( \beta \in [1, \infty) \). Then, explicit expressions of \( F_{H_{Mn}^m(k, \Phi_1|a, p, X)}(t) \) and \( F_{H_{Mn}^m(k, \Phi_2|b, p, X)}(t) \) are given by

\[ F_{H_{Mn}^m(k, \Phi_1|a, p, X)}(t) = \exp \left\{ -3 + 4 \sum_{j=1}^{4} \left[ 1 - \sum_{i=1}^{6} \frac{p_i(1-\theta_i)}{(1-\theta_i)e^{\lambda_i t} + \theta_i} \right] \right\}^{\frac{1}{\beta}}. \]

(iii) Suppose \( \Phi_1 = \Phi_2 = (\mu t + 1)^{-1/\mu} \), for \( \mu \geq 0 \). Then, explicit expressions of \( F_{H_{Mn}^m(k, \Phi_1|a, p, X)}(t) \) and \( F_{H_{Mn}^m(k, \Phi_2|b, p, X)}(t) \) are given by

\[ F_{H_{Mn}^m(k, \Phi_1|a, p, X)}(t) = \left[ -3 + 4 \sum_{j=1}^{4} \left[ 1 - \sum_{i=1}^{6} \frac{p_i(1-\theta_i)}{(1-\theta_i)e^{\lambda_i t} + \theta_i} \right] \right]^{-\frac{1}{\mu}}. \]

Plots of the distribution functions of these parallel-series systems under Ali-Mikhail-Haq copula for the components and the copulas considered in Parts (i)-(iii) above for the subsystems are presented in Figure 3. Because \( a \geq w \ b \), based on Theorem 5.1, we have \( H_{Mn}^m(k, \Phi_1|a, p, X) \geq_{st} H_{Mn}^m(k, \Phi_2|b, p, X) \) for all three parts. But, Theorem 4.1 of Fang et al. (2020) can not be used in this case to provide this result, since neither \( a \geq_{w} b \) nor \( b \not\geq_{w} a \), as the sums of the two vectors are unequal.
The following theorem shows that for a parallel-series system, under certain conditions, if the number of subsystems becomes larger, then the system becomes more reliable.

**Theorem 5.3.** Let $H_{M_{mn}^n(k,\Phi_1|\mathbf{a},\mathbf{p},\mathbf{X})}$ be the random lifetime of a parallel-series system corresponding to $M_{mn}^n(k,\Phi_1|\mathbf{a},\mathbf{p},\mathbf{X}) \in A$ with different generator Archimedean copulas $\Phi_1$ and $\phi_i$, for $i = 1, 2, \ldots, m$. Let $H_{M_{mn}^n(k-1,\Phi_1|\mathbf{a},\mathbf{p},\mathbf{X})}$ be the random lifetime of a parallel-series system corresponding to $M_{mn}^n(k,\Phi_1|\mathbf{a},\mathbf{p},\mathbf{X}) \in A$ with different generator Archimedean copulas $\Phi_1$ and $\phi_i$, for $i = 1, 2, \ldots, m$. Let $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_{k-1})$, and that $\Psi_2 \circ \Phi_1$ be superadditive. Then, we have

$$a_j \leq b_j \text{ for all } j = 1, \ldots, k-1 \implies H_{M_{mn}^n(k,\Phi_1|\mathbf{a},\mathbf{p},\mathbf{X})} \geq_{st} H_{M_{mn}^n(k-1,\Phi_2|\mathbf{b},\mathbf{p},\mathbf{X})}.$$  

**Proof.** From (11), for all $t \geq 0$, we have

$$F_{H_{M_{mn}^n(k,\Phi_1|\mathbf{a},\mathbf{p},\mathbf{X})}}(t) = \Phi_1 \left( \sum_{i=1}^{k} \phi_i \left( a_j \psi_i (\bar{F}_i(t)) \right) \right) \leq \Phi_1 \left( \sum_{i=1}^{k-1} \phi_i \left( a_j \psi_i (\bar{F}_i(t)) \right) \right) \leq \Phi_2 \left( \sum_{i=1}^{k-1} \phi_i \left( b_j \psi_i (\bar{F}_i(t)) \right) \right) \text{ (as } \Psi_2 \circ \Phi_1 \text{ is superadditive) }$$

$$= F_{H_{M_{mn}^n(k-1,\Phi_2|\mathbf{b},\mathbf{p},\mathbf{X})}}(t),$$

as required. \qed

Let $M_{n}^m(\mathbf{1},\mathbf{p},\mathbf{X})$ denote a parallel-series system having only one subsystem with $n$ components connected in series and $M_{n}^m(\mathbf{n}|\mathbf{1},\mathbf{p},\mathbf{X})$ denote a parallel-series system having $n$ subsystems connected in parallel with each subsystem having only one component. Then, the following theorem shows that $M_{n}^m(\mathbf{n}|\mathbf{1},\mathbf{p},\mathbf{X})$ is an optimal system and $M_{n}^m(\mathbf{1},\mathbf{p},\mathbf{X})$ is the worst among all possible such parallel-series systems, in the sense of usual stochastic order.
Theorem 5.4. Let $H_{M^n_m(k, \Phi | a, p, X)}$ denote the random lifetime of a parallel-series system corresponding to $M^n_m(k, \Phi_1 | a, p, X) \in A$ with different generator Archimedean copulas $\Phi$ and $\phi_i$, for $i=1, 2, \cdots, m$. Then, we have

\[ H_{M^n_m(1 | n, p, X)} \geq_{st} H_{M^n_m(k | a, p, X)} \geq_{st} H_{M^n_m(n | 1, p, X)}. \]

Proof. Let $a_l \in \{a_1, a_2, \cdots, a_k\}$. Then, from (11), we see that

\[ \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ n \psi_i (\bar{F}_i(t)) \} \right] \leq \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_l \psi_i (\bar{F}_i(t)) \} \right] \leq \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (\bar{F}_i(t)) \} \right]. \] (12)

Also, for $a_l \in \{a_1, a_2, \cdots, a_k\}$, we have

\[ \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_l \psi_i (\bar{F}_i(t)) \} \right) \leq \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (\bar{F}_i(t)) \} \right), \]

and so

\[ \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (\bar{F}_i(t)) \} \right] \leq k \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (\bar{F}_i(t)) \} \right] \leq n \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (\bar{F}_i(t)) \} \right]. \] (13)

As $\Phi(x)$ is a decreasing function, from (12) and (13), we obtain

\[ 1 - \sum_{i=1}^{m} p_i \{ n \psi_i (\bar{F}_i(t)) \} = \Phi \left( \Psi \left[ 1 - \sum_{i=1}^{m} p_i \{ n \psi_i (\bar{F}_i(t)) \} \right] \right) \geq \Phi \left( \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (\bar{F}_i(t)) \} \right] \right) \geq \Phi \left( n \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_i (\bar{F}_i(t)) \} \right] \right), \] (14)

from which the desired result follows readily. \qed

In the same set-up as in Theorem 8, we prove in the following theorem the result in Theorem 8, but in the sense of reversed hazard rate order.
Theorem 5.5. Let \( H_{M_n^m(k,\Phi|a,p,X)} \) denote the random lifetime of a parallel-series system corresponding to \( M_n^m(k,\Phi|a,p,X) \in A \) with different generator Archimedean copulas \( \Phi \) and \( \phi_i \), for \( i = 1,2,\cdots,m \). Further, suppose \( \Phi \) is log-concave. Then, we have

\[
H_{M_n^m(1|n,p,X)} \geq_r H_{M_n^m(k|a,p,X)} \geq_r H_{M_n^m(n|1,p,X)}.
\]

Proof. The reversed hazard rate function of \( H_{M_n^m(k|a,p,X)} \) is given by

\[
r_{H_{M_n^m(k|a,p,X)}}(t) = \sum_{i=1}^{k} \frac{\sum_{i=1}^{m} p_i f_i(t) a_i \psi_i' (F_i(t)) \phi_i' (a_i \psi_i (F_i(t)))}{\Phi' \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i (a_i \psi_i (F_i(t))) \right) \right)} \times \Phi' \left( \Phi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \left( a_i \psi_i \left( F_i(t) \right) \right) \right) \right).\]

As \( a_1 + a_2 + \cdots + a_k = n \), we have the following two special cases:

\[
r_{H_{M_n^m(1|n,p,X)}}(t) = \sum_{i=1}^{k} \frac{\sum_{i=1}^{m} p_i f_i(t) a_i \psi_i' (F_i(t)) \phi_i' (n \psi_i (F_i(t)))}{\Phi' \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i (n \psi_i (F_i(t))) \right) \right)} \times \Phi' \left( \Phi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \left( n \psi_i \left( F_i(t) \right) \right) \right) \right),
\]

\[
r_{H_{M_n^m(n|1,p,X)}}(t) = \sum_{i=1}^{k} \frac{\sum_{i=1}^{m} p_i f_i(t) a_i \psi_i' (F_i(t)) \phi_i' (\psi_i (F_i(t)))}{\Phi' \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i (\psi_i (F_i(t))) \right) \right)} \times \Phi' \left( \Phi \left( 1 - \sum_{i=1}^{m} p_i \phi_i \left( \psi_i \left( F_i(t) \right) \right) \right) \right).\]

Now, in order to prove \( H_{M_n^m(n|1,p,X)} \geq_r H_{M_n^m(k|a,p,X)} \geq_r H_{M_n^m(1|n,p,X)} \), we need to prove, for all \( l = 1,2,\cdots, k \), that

\[
0 \geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i' (F_i(t)) \phi_i' (n \psi_i (F_i(t))) \geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i' (F_i(t)) \phi_i' (a_i \psi_i (F_i(t))) \geq \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i' (F_i(t)) \phi_i' (\psi_i (F_i(t)))
\]

\[
\Phi' \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i (n \psi_i (F_i(t))) \right) \right) \geq \Phi' \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i (a_i \psi_i (F_i(t))) \right) \right) \geq \Phi' \left( \Psi \left( 1 - \sum_{i=1}^{m} p_i \phi_i (\psi_i (F_i(t))) \right) \right)
\]

(15)
and

\[
0 \geq \frac{\Phi'(\Psi [1 - \sum_{i=1}^{m} p_i \{ n \psi_i (\bar{F}_i(t)) \}])}{\Phi(\Psi [1 - \sum_{i=1}^{m} p_i \{ n \psi_i (\bar{F}_i(t)) \}])}
\]

\[
\geq \frac{\Phi'(\sum_{i=1}^{k} \Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ a_i \psi_i (\bar{F}_i(t)) \}])}{\Phi(\sum_{i=1}^{k} \Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ a_i \psi_i (\bar{F}_i(t)) \}])}
\]

\[
\geq \frac{\Phi'(n\Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_1 (\bar{F}_i(t)) \}])}{\Phi(n\Psi [1 - \sum_{i=1}^{m} p_i \phi_i \{ \psi_1 (F_i(t)) \}])}.
\]

For \( x \geq 0 \), let us set

\[
g(x) = \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(\bar{F}_i(t)) \phi_i' (x \psi_i (\bar{F}_i(t)))
\]

\[
\Phi'(\Psi (1 - \sum_{i=1}^{m} p_i \phi_i (x \psi_i (\bar{F}_i(t))))\right)^2.
\]

Then, we find

\[
g'(x) = \frac{\left[ \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(\bar{F}_i(t)) \psi_i (\bar{F}_i(t)) \phi_i' (x \psi_i (\bar{F}_i(t))) \phi_i'' (x \psi_i (\bar{F}_i(t))) \right] \Phi'(\Psi (1 - \sum_{i=1}^{m} p_i \phi_i (x \psi_i (\bar{F}_i(t)))))}{\left[ \Phi'(\Psi (1 - \sum_{i=1}^{m} p_i \phi_i (x \psi_i (\bar{F}_i(t))))\right]^2}
\]

\[
+ \frac{\left[ \sum_{i=1}^{m} p_i f_i(t) a_i \psi_i'(\bar{F}_i(t)) \phi_i' (x \psi_i (\bar{F}_i(t))) \right] \left[ \sum_{i=1}^{m} \psi_i (f_i(t)) p_i \phi_i (x \psi (\bar{F}_i(t))) \right]}{\left[ \Phi'(\Psi (1 - \sum_{i=1}^{m} p_i \phi_i (x \psi_i (\bar{F}_i(t))))\right]^2}
\]

\[
\times \psi' \left( 1 - \sum_{i=1}^{m} p_i \phi_i (x \psi_i (\bar{F}_i(t))) \right) \Phi'' \left( \Psi (1 - \sum_{i=1}^{m} p_i \phi_i (x \psi_i (\bar{F}_i(t)))) \right)
\]

\[
\geq 0.
\]
and so \( g'(x) \) is increasing in \( x \geq 0 \), which means that Eq. (15) is true. Also, by the assumption that \( \Phi(t) \) is a log-concave function, and so according to (13), Eq. (16) is true. Hence, the theorem is proven.

In the following theorem, we examine the influence of selection probabilities and the distributions of subpopulations on the system reliability of parallel-series systems in terms of majorization order.

**Theorem 5.6.** Let \( H_{M_{mn}^m(k, \Phi_1, a, p, X)} \) denote the random lifetime of a parallel-series system corresponding to \( M_{mn}^m(k, \Phi_1, a, p, X) \in A \) with different generator Archimedean copulas \( \Phi_1 \) and \( \phi_i \), for \( i = 1, 2, \ldots, m \). Also, let \( H_{M_{mn}^m(k, \Phi_2, a, q, X)} \) denote the random lifetime of a parallel-series system corresponding to \( M_{mn}^m(k, \Phi_2, a, q, X) \in A \) with different generator Archimedean copulas \( \Phi_2 \) and \( \phi_i \), for \( i = 1, 2, \ldots, m \). Further, suppose \( F_1 \leq \cdots \leq F_m \leq F_{mn} \), \( p_1 \geq p_2 \geq \cdots \geq p_n \) and \( q_1 \geq q_2 \geq \cdots \geq q_n \), and that \( \Psi_1 \circ \Phi_2 \) (\( \Psi_2 \circ \Phi_1 \)) is superadditive. Then, we have

\[
p_m^m \implies H_{M_{mn}^m(k, \Phi_1, a, p, X)} \leq \text{st} (\geq \text{st}) H_{M_{mn}^m(k, \Phi_2, a, q, X)}.
\]

**Proof.** The distribution functions of parallel-series systems \( H_{M_{mn}^m(k, \Phi_1, a, p, X)} \) and \( H_{M_{mn}^m(k, \Phi_2, a, q, X)} \) are given by

\[
F_{H_{M_{mn}^m(k, \Phi_1, a, p, X)}}(t) = \Phi_1 \left( \sum_{j=1}^{k} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right), \quad t \geq 0,
\]

\[
F_{H_{M_{mn}^m(k, \Phi_2, a, q, X)}}(t) = \Phi_2 \left( \sum_{j=1}^{k} \Psi_2 \left[ 1 - \sum_{i=1}^{m} q_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right), \quad t \geq 0,
\]

respectively. The superadditivity of \( \Psi_1 \circ \Phi_2 \) implies that

\[
\Phi_2 \left( \sum_{i=1}^{n} \Psi_2 \left[ 1 - \sum_{i=1}^{m} q_i \phi_j \{ a_j \psi_j (F_i(t)) \} \right] \right) \leq \Phi_1 \left( \sum_{i=1}^{n} \Psi_1 \left[ 1 - \sum_{i=1}^{m} p_i \phi_j \{ a_j \psi_j (F_i(t)) \} \right] \right).
\]

Then, to prove the desired result, it is sufficient to show that

\[
\Phi_2 \left( \sum_{i=1}^{n} \Psi_2 \left[ 1 - \sum_{i=1}^{m} q_i \phi_j \{ a_j \psi_j (F_i(t)) \} \right] \right) \leq \Phi_2 \left( \sum_{i=1}^{n} \Psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_j \{ a_j \psi_j (F_i(t)) \} \right] \right).
\]

For all \( t \geq 0 \), let us set

\[
H(p) = F_{H_{M_{mn}^m(k, \Phi_2, a, p, X)}}(t) = \Phi_2 \left( \sum_{j=1}^{k} \Psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right).
\]

Now, according to Lemma 2.4, we need to show that \( H(p) \) is Schur-convex (Schur-concave) in \( (p_1, \ldots, p_m) \), for any fixed \( t \geq 0 \). Taking the derivative of \( H(p) \) with respect to \( p_\alpha \), \( \alpha \in \{1, 2, \ldots, k\} \), we have

\[
\frac{\partial H(p)}{\partial p_\alpha} = - \sum_{j=1}^{k} \phi_\alpha \{ a_j \psi_\alpha (F_\alpha(t)) \} \Psi_2' \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right)
\]

\[
\times \Phi_2' \left( \sum_{j=1}^{k} \Psi_2 \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (F_i(t)) \} \right] \right),
\]
so that

\[ J_3 := (p_\alpha - p_\beta) \left( \frac{\partial H(p)}{\partial p_\alpha} - \frac{\partial H(p)}{\partial p_\beta} \right) \]

\[
= \sum_{j=1}^{k} \left[ \phi_\beta \{ a_j \psi_\beta (\tilde{F}_\beta(t)) \} - \phi_\alpha \{ a_j \psi_\alpha (\tilde{F}_\alpha(t)) \} \right] \Psi_j \left( 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (\tilde{F}_i(t)) \} \right) 
\times \Phi_j \left( \sum_{j=1}^{k} \Psi \left[ 1 - \sum_{i=1}^{m} p_i \phi_i \{ a_j \psi_i (\tilde{F}_i(t)) \} \right] \right).
\]

For all \( 1 \leq \alpha < \beta \leq m \), if \( \tilde{F}_\alpha(t) \leq \tilde{F}_\beta(t) \), then \( \psi_\beta(t) \leq \psi_\alpha(t) \) and \( \phi_\alpha(t) \leq \phi_\beta(t) \) \( (\tilde{F}_\alpha(t) \geq \tilde{F}_\beta(t), \psi_\beta(t) \geq \psi_\alpha(t) \) and \( \phi_\alpha(t) \geq \phi_\beta(t) \). Now, upon using (2), we get

\[
\phi_\alpha \{ a_j \psi_\alpha (\tilde{F}_\alpha(t)) \} \leq (\geq) \phi_\alpha \{ a_j \psi_\beta (\tilde{F}_\beta(t)) \} \leq (\geq) \phi_\beta \{ a_j \psi_\beta (\tilde{F}_\beta(t)) \},
\]

and then

\[
\left( \frac{\partial H(p)}{\partial p_\alpha} - \frac{\partial H(p)}{\partial p_\beta} \right) \geq (\leq) 0.
\]

This completes the proof of the theorem. \( \square \)

**Example 5.7.** Let us consider \( \tilde{F}_i(t) = e^{-\lambda_i t}, k = 4, m = 5, \phi_i(t) = \frac{1-\theta_i}{\alpha_i-\theta_i}, \theta_i \in [0, 1], i = 1, \cdots, 5. \) Let us further choose \( \alpha = (7, 10, 4, 3), \beta = (0.4, 0.3, 0.1, 0.1, 0.1), \psi = (0.3, 0.2, 0.2, 0.2, 0.1), \theta = (1/3, 1/2, 1/7, 1/9, 1/4), \lambda = (10, 8, 6, 4, 2), \alpha = 0.2, \beta = 2 \) and \( \mu = 2. \) It is then easy to observe that \( p \succ q. \) Now, let us consider the following situations:

(i) Suppose \( \Phi_1 = \Phi_2 = (1-\alpha)/(e^t-\alpha) \), for \( \alpha \in [0, 1) \). Then, explicit expressions of \( F_{HM^{(k, \cdot, \cdot)}_M a, p, X}(t) \) and \( F_{HM^{(k, \cdot, \cdot)}_M a, q, X}(t) \) are given by

\[
F_{HM^{(k, \cdot, \cdot)}_M a, p, X}(t) = \frac{1-\alpha}{\exp \left[ \sum_{j=1}^{4} \ln \left( \frac{1-\alpha}{1 - \sum_{i=1}^{5} \frac{p_i (\theta_i - \theta_i)}{(1-\theta_i) e^{\lambda_i t} + \theta_i}} + \alpha \right) \right] - \alpha};
\]

\[
F_{HM^{(k, \cdot, \cdot)}_M a, q, X}(t) = \frac{1-\alpha}{\exp \left[ \sum_{j=1}^{4} \ln \left( \frac{1-\alpha}{1 - \sum_{i=1}^{5} \frac{q_i (\theta_i - \theta_i)}{(1-\theta_i) e^{\lambda_i t} + \theta_i}} + \alpha \right) \right] - \alpha};
\]

(ii) Suppose \( \Phi_1 = \Phi_2 = e^{1-(1+t)^\beta} \), for \( \beta \in [1, \infty) \). Then, explicit expressions of \( F_{HM^{(k, \cdot, \cdot)}_M a, p, X}(t) \) and \( F_{HM^{(k, \cdot, \cdot)}_M a, q, X}(t) \) are given by

\[
F_{HM^{(k, \cdot, \cdot)}_M a, p, X}(t) = \exp \left[ 1 - \left( -3 + \sum_{j=1}^{4} \left( 1 - \ln \left[ 1 - \sum_{i=1}^{5} \frac{p_i (\theta_i - \theta_i)}{(1-\theta_i) e^{\lambda_i t} + \theta_i} \right]^\beta \right)^\beta \right);\]

\[
F_{HM^{(k, \cdot, \cdot)}_M a, q, X}(t) = \exp \left[ 1 - \left( -3 + \sum_{j=1}^{4} \left( 1 - \ln \left[ 1 - \sum_{i=1}^{5} \frac{q_i (\theta_i - \theta_i)}{(1-\theta_i) e^{\lambda_i t} + \theta_i} \right]^\beta \right)^\beta \right);\]
(iii) Suppose \( \Phi_1 = \Phi_2 = (\mu t + 1)^{-1/\mu} \), for \( \mu \geq 0 \). Then, explicit expressions of 
\[ F_{H_{M_1}^{m}(k, \Phi_1 \mid a \cdot p X)}(t) \] 
and 
\[ F_{H_{M_2}^{m}(k, \Phi_2 \mid a \cdot q X)}(t) \]
are given by
\[
F_{H_{M_1}^{m}(k, \Phi_1 \mid a \cdot p X)}(t) = -3 + 4 \sum_{j=1}^{4} \left[ 1 - \sum_{i=1}^{5} \frac{p_i (1 - \theta_i)}{[(1 - \theta_i) e^{\lambda_i t} + \theta_i]^{m_j} - \theta_i} \right]^{\frac{1}{\mu}},
\]
\[
F_{H_{M_2}^{m}(k, \Phi_2 \mid a \cdot q X)}(t) = -3 + 4 \sum_{j=1}^{4} \left[ 1 - \sum_{i=1}^{5} \frac{q_i (1 - \theta_i)}{[(1 - \theta_i) e^{\lambda_i t} + \theta_i]^{m_j} - \theta_i} \right]^{\frac{1}{\mu}}.
\]

Plots of the distribution functions of these parallel-series systems under Ali-Mikhail-Haq copula for the components and the copulas considered in Parts (i)-(iii) above for the subsystems are presented in Figure 4. Because \( p \geq q \), based on Theorem 5.6, we have 
\[ H_{M_1}^{m}(k, \Phi_1 \mid a \cdot p X) \geq H_{M_2}^{m}(k, \Phi_2 \mid a \cdot q X) \]
for all three parts.

![Figure 4: Plots of the distribution functions of parallel-series systems under Ali-Mikhail-Haq copula for the components and copulas considered in Parts (i)-(iii) for the subsystems, from left to right.](image)

6. Concluding remarks. The manufacturing process of many reliability systems has truly become global in the sense that different parts or components of the system are acquired from many different suppliers from all over the world. Clearly, the quality and reliability of components received from different suppliers will vary; but, still acquiring these parts or components from different suppliers, with varying quality and reliability characteristics, becomes inevitable due to supply and demand situations. From a statistical perspective, this poses a great difficulty in the modelling and also in examining characteristics of different reliability systems since components in each subsystem are not only dependent, but also the subsystems themselves become dependent as they are often formed with some of the components coming from the same supplier.

So, under such a general dependence setting, we have discussed here the system reliability of series-parallel and parallel-series systems for three different cases: first by fixing the number of subsystems and then presenting relationships between allocation vectors, second by varying the number of subsystems, and finally by varying the selection probabilities or the distributions of subpopulations. We have then used the theory of stochastic orders and majorization to examine comparative reliability characteristics of different possible systems that could be formed in this case. Even
though the results established are true for the general family of Archimedean copulas, we have used the special cases of independence, Clayton and Ali-Mikhail-Haq copulas as specific examples for illustrating all the results established here. Noting that we have focused here on the well-known series-parallel and parallel-series systems, it will be of great interest to extend the results to series-$k$-out-of-$n$ and parallel-$k$-out-of-$n$ systems as such systems are also in common use. We are currently looking into this problem and hope to report the findings in a future paper.

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