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Tensor products and sums of $p$-harmonic functions, quasiminimizers and $p$-admissible weights

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Abstract. The tensor product of two $p$-harmonic functions is in general not $p$-harmonic, but we show that it is a quasiminimizer. More generally, we show that the tensor product of two quasiminimizers is a quasiminimizer. Similar results are also obtained for quasisuperminimizers and for tensor sums. This is done in weighted $\mathbb{R}^n$ with $p$-admissible weights. It is also shown that the tensor product of two $p$-admissible measures is $p$-admissible. This last result is generalized to metric spaces.

Key words and phrases: doubling measure, metric space, $p$-admissible weight, $p$-harmonic function, Poincaré inequality, quasiminimizer, quasisuperminimizer, tensor product, tensor sum.

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1. Introduction

It is well known (and easy to prove) that the tensor product and tensor sum of two harmonic functions are harmonic, i.e. if $u_j$ is harmonic in $\Omega_j \subset \mathbb{R}^{n_j}$, $j = 1, 2$, then $u_1 \otimes u_2$ and $u_1 \oplus u_2$ are harmonic in $\Omega_1 \times \Omega_2 \subset \mathbb{R}^{n_1+n_2}$. Here

$$(u_1 \otimes u_2)(x,y) := u_1(x)u_2(y) \quad \text{and} \quad (u_1 \oplus u_2)(x,y) := u_1(x) + u_2(y).$$

It is also well known that the corresponding property for $p$-harmonic functions fails. However, as we show in this note, the tensor product of two $p$-harmonic functions is a quasiminimizer.

Here $u \in W_{\text{loc}}^{1,p}(\Omega)$ is $p$-harmonic in the open set $\Omega \subset \mathbb{R}^n$ if it is a continuous weak solution of the $p$-Laplace equation

$$\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad 1 < p < \infty.$$ 

Moreover, $u \in W_{\text{loc}}^{1,p}(\Omega)$ is a $Q$-quasiminimizer if

$$\int_{\varphi \neq 0} |\nabla u|^p \, dx \leq Q \int_{\varphi \neq 0} |\nabla (u + \varphi)|^p \, dx$$

for all boundedly supported Lipschitz functions $\varphi$ vanishing outside $\Omega$. A quasiminimizer always has a continuous representative, and if $Q = 1$ this representative is a $p$-harmonic function.

In this note we show the following result.
Theorem 1. Let $1 < p < \infty$, and let $u_j$ be a $Q_j$-quasiminimizer in $\Omega_j \subset \mathbb{R}^{n_j}$ with respect to a $p$-admissible weight $w_j$, $j = 1, 2$. Then $u = u_1 \otimes u_2$ and $v = u_1 \otimes v_2$ are $Q$-quasiminimizers in $\Omega_1 \times \Omega_2$ with respect to the $p$-admissible weight $w = w_1 \otimes w_2$, where

$$Q = \begin{cases} \left( Q_1^{2/(p-2)} + Q_2^{2/(p-2)} \right)^{p-2/2}, & \text{if } p \neq 2, \\ \max\{Q_1, Q_2\}, & \text{if } p = 2. \end{cases}$$

(1)

In particular, if $u_1$ and $u_2$ are $p$-harmonic, then $u$ and $v$ are $Q$-quasiminimizers with $Q = 2^{(p-2)/2}$.

We also obtain a corresponding result for quasisuperminimizers. We pursue our studies on weighted $\mathbb{R}^n$ with respect to so-called $p$-admissible weights. To do so, we first show that the product of two $p$-admissible measures is $p$-admissible, which we do in Section 2. This generalizes some earlier special cases from Lu–Wheeden [14, Lemma 2], Kilpeläinen–Koskela–Masaoka [12, Lemma 2.2] and Björn [4, Lemma 11], but we have not seen it proved in this form in the literature. In fact, our result holds in the generality of metric spaces, see Remark 4.

Usually, $Q \geq 1$ in the definition of $Q$-quasiminimizers but here it is convenient to also allow for $Q = 0$ (which happens exactly when $u$ is a.e. constant in every component of $\Omega$). For example, if $Q_2 = 0$ then $Q = Q_1$ in Theorem 1. Even this special case of Theorem 1 seems to have gone unnoticed in the literature.

Quasiminimizers were introduced by Giaquinta and Giusti [7], [8] in the early 1980s as a tool for a unified treatment of variational integrals, elliptic equations and quasiregular mappings on $\mathbb{R}^n$. In those papers, De Giorgi’s method was extended to quasiminimizers, yielding in particular their local Hölder continuity. Quasiminimizers have since then been studied in a large number of papers, first on unweighted $\mathbb{R}^n$ and later on metric spaces, see Appendix C in Björn–Björn [3] and the introduction in Björn [5] for further discussion and references.

Quasiminimizers form a much more flexible class than $p$-harmonic functions. For example, Martio–Sbordone [15] showed that quasiminimizers have an interesting and nontrivial theory also in one dimension, and Kinnunen–Martio [13] developed an interesting nonlinear potential theory for quasiminimizers, including quasisuperharmonic functions. Unlike $p$-harmonic functions and solutions of elliptic PDEs, quasiminimizers can have singularities of any order, as shown in Björn–Björn [2].

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2. Tensor products of $p$-admissible measures

Let $w$ be a weight function on $\mathbb{R}^n$, i.e. a nonnegative locally integrable function, and let $d\mu = w \, dx$. In this section we also let $1 \leq p < \infty$ be fixed. For a ball $B = B(x_0, r) := \{ x : |x - x_0| < r \}$ in $\mathbb{R}^n$ we use the notation $\lambda B = B(x_0, \lambda r)$.

Definition 2. The measure $\mu$ (or the weight $w$) is $p$-admissible if the following two conditions hold:

- It is doubling, i.e. there exists a doubling constant $C > 0$ such that for all balls $B$,
  $$0 < \mu(2B) \leq C \mu(B) < \infty.$$

- It supports a $p$-Poincaré inequality, i.e. there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B$ and all bounded locally Lipschitz functions $u$ on $\lambda B$,
  $$\int_B |u - u_B| \, d\mu \leq C \operatorname{diam}(B) \left( \int_{\lambda B} |\nabla u|^p \, d\mu \right)^{1/p},$$

where $u_B$ denotes the average of $u$ over $B$.
where \( \nabla u \) is the a.e. defined gradient of \( u \) and \( u_B := \int_B u \, d\mu := \mu(B)^{-1} \int_B u \, d\mu \).

This is one of many equivalent definitions of \( p \)-admissible weights in the literature, see e.g. Corollary 20.9 in Heinonen–Kilpeläinen–Martio [10] (which is not in the first edition) and Proposition A.17 in Björn–Björn [3]. It can be shown that on \( \mathbb{R}^n \), the dilation \( \lambda \) in the Poincaré inequality can be taken equal to 1, see Jerison [11], Hajłasz–Koskela [9] and the discussion in [10, Chapter 20].

It is not known whether there exist any admissible measures on \( \mathbb{R}^n, \ n \geq 2 \), which are not absolutely continuous with respect to the Lebesgue measure (and thus given by admissible weights). (On \( \mathbb{R} \) all \( p \)-admissible measures are absolutely continuous, and even \( A_p \) weights, see Björn–Buckley–Keith [6].) We therefore formulate our next result in terms of \( p \)-admissible measures.

**Theorem 3.** Let \( \mu_1 \) and \( \mu_2 \) be \( p \)-admissible measures on \( \mathbb{R}^{n_1} \) and \( \mathbb{R}^{n_2} \), respectively. Then the product measure \( \mu = \mu_1 \times \mu_2 \) is \( p \)-admissible on \( \mathbb{R}^{n_1+n_2} \).

For a function \( u \) on an open subset \( \Omega \subset \mathbb{R}^{n_1+n_2} \) we will denote the gradient by \( \nabla u \). The gradients with respect to the first \( n_1 \) resp. the last \( n_2 \) variables will be denoted by \( \nabla_x u \) and \( \nabla_y u \). In this section we will only consider gradients of locally Lipschitz functions, which are thus defined a.e. and coincide with the Sobolev gradients determined by the admissible measures, see Heinonen–Kilpeläinen–Martio [10, Lemma 1.11].

**Proof.** Let \( z = (z_1, z_2) \in \mathbb{R}^{n_1+n_2} \) and \( r > 0 \). We denote balls in \( \mathbb{R}^{n_1}, \mathbb{R}^{n_2} \) and \( \mathbb{R}^{n_1+n_2} \), by \( B', B'' \) and \( B \), respectively. Let

\[
Q(z, r) = B'(z_1, r) \times B''(z_2, r)
\]

and note that

\[
B(z, r) \subset Q(z, r) \subset B(z, \sqrt{2}r).
\]

It follows that for \( B = B(z, r) \) we have

\[
\mu(2B) \leq \mu(Q(z, 2r)) = \mu_1(B'(z_1, 2r))\mu_2(B''(z_2, 2r)) \leq C\mu_1(B'(z_1, \frac{1}{2}r))\mu_2(B''(z_2, \frac{1}{2}r)) = C\mu(Q(z, \frac{1}{2}r)) \leq C\mu(B),
\]

and hence \( \mu \) is doubling. Here and below, the letter \( C \) denotes various positive constants whose values may vary even within a line.

We now turn to the Poincaré inequality. As mentioned above we can assume that the \( p \)-Poincaré inequalities for \( \mu_1 \) and \( \mu_2 \) hold with dilation \( \lambda = 1 \). Let \( B = B(z, r) \) and \( Q = Q(z, r) = B' \times B'' \). Also let \( u \) be an arbitrary bounded locally Lipschitz function on \( 2B \) and set

\[
c = \int_{Q} u \, d\mu = \int_{B''} \int_{B'} u(s, t) \, d\mu_1(s) \, d\mu_2(t).
\]

Then by the Fubini theorem,

\[
\int_{Q} |u - c| \, d\mu \leq \int_{B''} \left( \int_{B'} |u(x, y) - \int_{B'} u(s, y) \, d\mu_1(s)| \, d\mu_1(x) \right) \, d\mu_2(y) \tag{3}
\]

\[
+ \int_{B'} \left( \int_{B''} u(s, y) \, d\mu_1(s) - \int_{B'} \int_{B''} u(s, t) \, d\mu_2(t) \, d\mu_1(s) \right) \, d\mu_2(y)
\]

\[
=: I_1 + I_2.
\]
The first integral $I_1$ can be estimated using the $p$-Poincaré inequality for $\mu_1$ and $u(\cdot,y)$ on $B'$, and then the Hölder inequality with respect to $\mu_2$, as follows
\[
I_1 \leq \int_{B'} C r \left( \int_{B'} |\nabla_x u(x,y)|^p \, d\mu_1(x) \right)^{1/p} \, d\mu_2(y) \leq C r \left( \int_{B'} |\nabla_x u(x,y)|^p \, d\mu_1(x) \right)^{1/p} \leq C r \left( \int_{Q} |\nabla u|^p \, d\mu \right)^{1/p}.
\]
As for the second integral $I_2$ in (3) we have by the Fubini theorem,
\[
I_2 \leq \int_{B'} \int_{B'} \left| u(s,y) - \int_{B'} u(s,t) \, d\mu_2(t) \right| \, d\mu_1(s) \, d\mu_2(y) = \int_{B'} \int_{B'} \left| u(s,y) - \int_{B'} u(s,t) \, d\mu_2(t) \right| \, d\mu_1(s) \, d\mu_2(y)
\]
which can be estimated in the same way as $I_1$, by switching the roles of the variables. Thus
\[
I_2 \leq C r \left( \int_{Q} |\nabla u|^p \, d\mu \right)^{1/p}.
\]
Summing the estimates for $I_1$ and $I_2$ and using the doubling property for $\mu$ we see that
\[
\int_B |u - c| \, d\mu \leq C \int_Q |u - c| \, d\mu \leq C r \left( \int_{Q} |\nabla u|^p \, d\mu \right)^{1/p} \leq C r \left( \int_{2B} |\nabla u|^p \, d\mu \right)^{1/p}.
\]
Finally, a standard argument allows us to replace $c$ on the left-hand side by $u_B$ at the cost of an extra factor 2 on the right-hand side, cf. [3, Lemma 4.17]. We conclude that $\mu$ supports a $p$-Poincaré inequality on $\mathbb{R}^{n_1+n_2}$, and thus that $\mu$ is $p$-admissible.

**Remark.** The proof of Theorem 3 easily generalizes to metric spaces. More precisely, if $(X_j, d_j)$, $j = 1, 2$, are (not necessarily complete) metric spaces equipped with doubling measures $\mu_j$, supporting $p$-Poincaré inequalities with dilation constant $\lambda$ then $X = X_1 \times X_2$, equipped with the product measure $\mu = \mu_1 \times \mu_2$, supports a $p$-Poincaré inequality with dilation constant $2\lambda$ and $\mu$ is a doubling measure. See e.g. Björn–Björn [3] for the precise definitions of these notions in metric spaces.

Poincaré inequalities in metric spaces are defined using so-called upper gradients, and the main property needed for the proof of Theorem 3 in the metric setting is that whenever $g(\cdot, \cdot)$ is an upper gradient of $u(\cdot, \cdot)$ in $X$ and $y \in X_2$, then $g(\cdot, y)$ is an upper gradient of $u(\cdot,y)$ with respect to $X_1$, and similarly for $g(x, \cdot)$ and $u(x, \cdot)$ with $x \in X_1$. For this to hold, the metric on $X_1 \times X_2$ can actually be defined using
\[
d((x_1, y_1), (x_2, y_2)) = \| (d_1(x_1, x_2), d_2(y_1, y_2)) \|
\]
with an arbitrary norm $\| \cdot \|$ on $\mathbb{R}^2$. In this generality we cannot assume that $\lambda = 1$, and therefore $\lambda$ also needs to be inserted at suitable places in the proof. (If the norm does not satisfy $\| (x,0) \| \leq \| (x,y) \|$ and $\| (0,y) \| \leq \| (x,y) \|$, then the inclusions (2) need to be modified, necessitating similar changes also later in the proof.) We refrain from this generalization in this note. Also Theorem 5 below can be similarly generalized to metric spaces.

We conclude this section by showing that Theorem 3 admits a converse.

**Theorem 5.** Assume that $\mu = \mu_1 \times \mu_2$ is a $p$-admissible measure on $\mathbb{R}^{n_1+n_2}$. Then $\mu_1$ and $\mu_2$ are $p$-admissible measures on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively.

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Definition 6. A function $u : \Omega \to [-\infty, \infty]$ is a $Q$-quasi $(\text{sub/super})$ minimizer with respect to a $p$-admissible weight $w$ in a nonempty open set $\Omega \subset \mathbb{R}^n$ if $u \in W^{1,p}_{\text{loc}}(\Omega; \mu)$ and

$$\int_{\varphi \neq 0} |\nabla u|^p d\mu \leq Q \int_{\varphi \neq 0} |\nabla (u + \varphi)|^p d\mu$$

for all (nonpositive/nonnegative) $\varphi \in \text{Lip}_0(\Omega)$.

By splitting $\varphi$ into its positive and negative parts, it is easily seen that a function is a $Q$-quasiminimizer if and only if it is both a $Q$-subquasiminimizer and a $Q$-superquasiminimizer.

The Sobolev space $W^{1,p}_{\text{loc}}(\Omega; \mu)$ is defined as in Heinonen–Kilpeläinen–Martio [10] (although they use the letter $H$ instead of $W$). See [10, Section 1.9] and [3, Proposition A.17] for the definition of the gradient $\nabla u$ for $u \in W^{1,p}_{\text{loc}}(\Omega; \mu)$, which need not be the distributional gradient of $u$. 

Proof. It suffices to show the $p$-admissibility of $\mu_1$. Let $B' = (z', r) \subset \mathbb{R}^{n_1}$ be a ball and let $B'' := B(0, r) \subset \mathbb{R}^{n_2}$. Let $u$ be an arbitrary bounded locally Lipschitz function on $B'$ and for $(x, y) \in B' \times B''$ define $v(x, y) = u(x)$. Then

$$v_{B' \times B''} = \int_{B'} \int_{B''} v(x, y) d\mu_1(x) d\mu_2(y) = u_{B'}.$$ 

Note that for $z = (z', 0) \in \mathbb{R}^{n_1+n_2}$,

$$B(z, r) \subset B' \times B'' \subset B(z, \sqrt{2}r) =: \tilde{B} \subset 2B' \times 2B'' \subset B(z, 2\sqrt{2}r).$$

It then follows from the doubling property of $\mu$ that $\mu_1(2B')\mu_2(2B'') \leq \mu(B(z, \sqrt{2}r)) \leq C \mu_1(B')\mu_2(B'')$ and division by $\mu_2(2B'') \geq \mu_2(B'')$ yields $\mu_1(2B') \leq C\mu_1(B')$, i.e. $\mu_1$ is doubling.

As for the Poincaré inequality, we have by (4), the doubling property of $\mu$ and [3, Lemma 4.17] that

$$\int_{B'} |u - u_{B'}| d\mu_1 = \int_{B' \times B''} |v - v_{B' \times B''}| d\mu \leq 2 \int_{B' \times B''} |v - v_{B'}| d\mu \leq C \int_{B} |v - v_{B}| d\mu.$$ 

The last integral is estimated using the $p$-Poincaré inequality for $\mu$ and the fact that $\nabla v(x, y) = \nabla u(x)$ as follows

$$\int_{B} |v - v_{B}| d\mu \leq C r \int_{B' \times B''} |\nabla v|^p d\mu \leq C r \int_{2B' \times 2B''} |\nabla v|^p d\mu \leq C r \int_{4B'} |\nabla u|^p d\mu.$$
Definition 6 is one of several equivalent definitions of quasi(sub/super)minimizers, see Björn [1, Proposition 3.2], where this was shown on metric spaces. It follows from Propositions A.11 and A.17 in [3] that the metric space definitions coincide with the usual ones on weighted $\mathbb{R}^n$ (with a $p$-admissible weight).

For quasisuperminimizers, an analogue of Theorem 1 takes the following form.

**Theorem 7.** Let $u_j$ be a $Q_j$-quasisuperminimizer in $\Omega_j \subset \mathbb{R}^{n_j}$ with respect to $p$-admissible weights $w_j$, $j = 1, 2$, and $Q$ be given by (1). Then $u_1 \otimes u_2$ is a $Q$-quasisuperminimizer in $\Omega = \Omega_1 \times \Omega_2$ with respect to $w = w_1 \otimes w_2$.

In addition, if both $u_1$ and $u_2$ are nonnegative/nonpositive, then $u_1 \otimes u_2$ is a $Q$-quasisuper/subminimizer in $\Omega$ with respect to $w$.

By considering $-u_1$ and $-u_2$, we easily obtain a corresponding result for quasisubminimizers. Usually, $Q_j \geq 1$ but we also allow for $Q_j = 0$. This can only happen when $u_j$ is constant (a.e. in each component of $\Omega_j$), but when this is fulfilled in Theorem 1 or 7 it immediately implies the following conclusion.

**Corollary 8.** If $u$ is a $Q$-quasi(super)minimizer in $\Omega \subset \mathbb{R}^{n_1}$ with respect to a $p$-admissible weight $w_1$, and we let $v(x, y) = u(x)$ for $(x, y) \in \Omega \times \mathbb{R}^{n_2}$, then $v$ is a $Q$-quasi(super)minimizer in $\Omega \times \mathbb{R}^{n_2}$ with respect to $w = w_1 \otimes w_2$, whenever $w_2$ is a $p$-admissible weight on $\mathbb{R}^{n_2}$.

**Proof.** As $v = u \otimes 0$, where $0$ is the zero function, this follows directly from Theorems 1 and 7. $\square$

**Proof of Theorem 1.** Since $u_1$ and $u_2$ are finite a.e., and the quasiminimizing property is the same for all representatives of an equivalence class in the local Sobolev space, we may assume that $u_1$ and $u_2$ are finite everywhere.

First, we show that $u := u_1 \otimes u_2$ is a $Q$-quasiminimizer. Note that

$$|\nabla u(x, y)|^p = (|\nabla_x u(x, y)|^2 + |\nabla_y u(x, y)|^2)^{p/2},$$

where $\nabla_x u(x, y) = u_2(y) \nabla u_1(x)$ and $\nabla_y u(x, y) = u_1(x) \nabla u_2(y)$.

Let $\varphi \in \text{Lip}_0(\Omega)$ be arbitrary. For a fixed $y \in \Omega_2$, let

$$\Omega_y^0 = \{x \in \Omega_1 : \varphi(x, y) \neq 0\}.$$

As $u_1$ is a $Q_1$-quasiminimizer in $\Omega_1$, so is $u(\cdot, y) = u_2(y) u_1(\cdot)$. Since $\varphi(\cdot, y) \in \text{Lip}_0(\Omega_1^y)$, we get

$$\int_{\Omega_y^0} |\nabla_x u(x, y)|^p \, d\mu_1(x) \leq Q_1 \int_{\Omega_y^0} |\nabla_x (u(x, y) + \varphi(x, y))|^p \, d\mu_1(x).$$

Integrating over all $y \in \Omega_2$ with nonempty $\Omega_y^0$ yields

$$\int_{\varphi \neq 0} |\nabla_x u|^p \, d\mu \leq Q_1 \int_{\varphi \neq 0} |\nabla_x (u + \varphi)|^p \, d\mu. \tag{5}$$

Similarly,

$$\int_{\varphi \neq 0} |\nabla_y u|^p \, d\mu \leq Q_2 \int_{\varphi \neq 0} |\nabla_y (u + \varphi)|^p \, d\mu. \tag{6}$$

Now we consider four cases.

**Case 1.** $Q_1 = 0$. In this case, $\nabla u_1 \equiv 0$ a.e., and so $\nabla_x u \equiv 0$ a.e. Hence, by (6),

$$\int_{\varphi \neq 0} |\nabla u|^p \, d\mu = \int_{\varphi \neq 0} |\nabla_y u|^p \, d\mu \leq Q_2 \int_{\varphi \neq 0} |\nabla_y (u + \varphi)|^p \, d\mu \leq Q_2 \int_{\varphi \neq 0} |\nabla(u + \varphi)|^p \, d\mu,$$
and thus $u$ is a $Q_2$-quasiminimizer.

Case 2. $Q_2 = 0$. This is similar to Case 1.

Case 3. $p \leq 2$. In this case, summing (5) and (6) gives
\[
\int_{\varphi \neq 0} |\nabla u|^p \, d\mu \leq \int_{\varphi \neq 0} (|\nabla_x u|^p + |\nabla_y u|^p) \, d\mu \\
\leq \int_{\varphi \neq 0} (Q_1 |\nabla_x (u + \varphi)|^p + Q_2 |\nabla_y (u + \varphi)|^p) \, d\mu.
\]
This proves the result for $p = 2$. For $p < 2$, the Hölder inequality applied to the sum $Q_1 a^p + Q_2 b^p$ in the last integrand shows that
\[
\int_{\varphi \neq 0} |\nabla u|^p \, d\mu \leq \left( Q_1^{2/(2-p)} + Q_2^{2/(2-p)} \right)^{1-p/2} \\
\times \int_{\varphi \neq 0} (|\nabla_x (u + \varphi)|^2 + |\nabla_y (u + \varphi)|^2)^{p/2} \, d\mu \\
= \left( Q_1^{2/(2-p)} + Q_2^{2/(2-p)} \right)^{1-p/2} \int_{\varphi \neq 0} |\nabla (u + \varphi)|^p \, d\mu.
\]

Case 4. $p \geq 2$ and $Q_1, Q_2 > 0$. Rewrite $|\nabla u|^p$ as
\[
|\nabla u|^p = (|\nabla_x u|^2 + |\nabla_y u|^2)^{p/2} = \left( Q_1^{2/p} \left( \frac{1}{Q_1} \right)^{2/p} |\nabla_x u|^2 + Q_2^{2/p} \left( \frac{1}{Q_2} \right)^{2/p} |\nabla_y u|^2 \right)^{p/2}.
\]
The Hölder inequality applied to the sum $Q_1^{2/p} a^2 + Q_2^{2/p} b^2$ implies
\[
|\nabla u|^p \leq \left( Q_1^{2/(p-2)} + Q_2^{2/(p-2)} \right)^{(p-2)/2} \left( \frac{1}{Q_1} |\nabla_x u|^p + \frac{1}{Q_2} |\nabla_y u|^p \right).
\]
Integrating over the set $\{(x, y) \in \Omega : \varphi(x, y) \neq 0\}$ and using (5) and (6) we obtain
\[
\int_{\varphi \neq 0} |\nabla u|^p \, d\mu \leq \left( Q_1^{2/(p-2)} + Q_2^{2/(p-2)} \right)^{(p-2)/2} \\
\times \int_{\varphi \neq 0} (|\nabla_x (u + \varphi)|^p + |\nabla_y (u + \varphi)|^p) \, d\mu.
\]
As $p/2 \geq 1$, the elementary inequality $a^p + b^p \leq (a^2 + b^2)^{p/2}$ concludes the proof for $u$.

We now turn to $v := u_1 \oplus u_2$. Let $\varphi \in \text{Lip}_b(\Omega)$ be arbitrary. Note that
\[
|\nabla v(x, y)|^p = (|\nabla_x v(x, y)|^2 + |\nabla_y v(x, y)|^2)^{p/2} = (|\nabla u_1(x)|^2 + |\nabla u_2(x)|^2)^{p/2}
\]
and
\[
|\nabla (v + \varphi)|^p = (|\nabla_x (v + \varphi)|^2 + |\nabla_y (v + \varphi)|^2)^{p/2}.
\]
For a fixed $y \in \Omega_2$, let
\[
\Omega_1^y = \{ x \in \Omega_1 : \varphi(x, y) \neq 0 \}.
\]
As $u_1$ is a $Q_1$-quasiminimizer in $\Omega_1$ and $\varphi(\cdot, y) \in \text{Lip}_b(\Omega_1^y)$, we get
\[
\int_{\Omega_1^y} |\nabla u_1(x)|^p \, d\mu_1(x) \leq Q_1 \int_{\Omega_1^y} |\nabla (u_1(x, y) + \varphi(x, y))|^p \, d\mu_1(x).
\]
Integrating over all $y \in \Omega_2$ with nonempty $\Omega_1^y$ yields
\[
\int_{\varphi \neq 0} |\nabla u_1|^p \, d\mu_2(x) \, d\mu_2(y) \leq Q_1 \int_{\varphi \neq 0} |\nabla (v + \varphi)|^p \, d\mu_2(x) \, d\mu_2(y),
\]
i.e. (5) holds. Similarly, (6) holds and the rest of the proof is as for $u$. $\square$
Proof of Theorem 7. This proof is very similar to the proof above. In this case we of course assume that \( \varphi \in \text{Lip}_0(\Omega) \) is nonnegative/nonpositive.

The only other difference in the proof is that since \( u_1 \) is a \( Q_1 \)-quasisuperminimizer in \( \Omega_1 \) and \( u_2(y) \) is nonnegative/nonpositive, we can conclude that

\[
u(\cdot, y) = u_2(y)u_1(\cdot)
\]

is a \( Q_1 \)-quasisuper/subminimizer in \( \Omega_1 \). The rest of the proof is the same; in particular the proof for \( v \) needs no nontrivial changes, and is thus valid also when \( u_1 \) and \( u_2 \) change sign.

For tensor sums one can use Theorem 7 to deduce (the corresponding part of) Theorem 1. For tensor products this is not possible as in this case the quasisuperminimizers in Theorem 7 need to be nonnegative. This nonnegativity is an essential assumption for quasisuperminimizers, which is not required for quasiminimizers. (To see this consider what happens when \( u_2 \equiv -1 \).) We can however obtain the following result.

Theorem 9. Let \( u_1 \) be a \( Q_1 \)-quasisub/superminimizer in \( \Omega_1 \) and \( u_2 \geq 0 \) be a \( Q_2 \)-quasiminimizer in \( \Omega_2 \), with respect to \( p \)-admissible weights \( w_1 \) and \( w_2 \), respectively.

Then \( u_1 \otimes u_2 \) is a \( Q \)-quasisub/superminimizer in \( \Omega = \Omega_1 \times \Omega_2 \) with respect to \( w = w_1 \otimes w_2 \), where \( Q \) is given by (1).

Proof. This is proved using a similar modification of the proof of Theorem 1 as we did when proving Theorem 7. The key fact is that quasiminimizers are preserved under multiplication by real numbers, while the corresponding fact for quasisuperminimizers is only true under multiplication by nonnegative real numbers.

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