Competing Optimally Against An Imperfect Prophet

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Abstract

Consider a gambler who observes the realizations of $n$ independent, non-negative, distribution-labeled random variables arriving in a uniform random order and can stop the sequence at any time to obtain a reward of the most recent observation. In 2017, Correa et al. showed that when all distributions are identical, it is possible to design a stopping time that achieves a $\beta \approx 0.745$ fraction of the maximum value (the “prophet” benchmark), matching an upper bound of Hill and Kertz. In 2019, Correa et al. showed that when the distributions differ, it is no longer possible to achieve this bound: they prove an upper bound of $\sqrt{3} - 1 < 0.74$ on the best approximation ratio.

We show that it is possible to asymptotically achieve the $\beta \approx 0.745$ bound even in the case of non-identical distributions, as long as we are allowed to remove a small constant number of distributions. Formally, we show that for any $\varepsilon$, there exists a constant $C_\varepsilon = \text{poly}(1/\varepsilon)$ (independent of $n$) such that after removing $C_\varepsilon$ distributions of our choice, we can achieve a $(\beta - \varepsilon)$-approximation to the resulting maximum. We additionally show it is possible to asymptotically achieve an exact $\beta$ approximation ratio for several natural classes of problem instances, including small prophets (where each distribution is concentrated near zero) and frequent prophets (where each distribution occurs at least some number of times).

1 Introduction

The problem of prophet inequalities is a classic problem in optimal stopping theory consisting of designing stopping times for sequences of independent random variables that always achieve (in expectation) a certain fraction of the maximum of the sequence. Formally, it asks for the maximum constant $\alpha$ such that for any independent random variables $X_1, \ldots, X_n$ with known distributions, it is possible to design a stopping time $\tau$ such that $E[X_\tau] \geq \alpha \cdot E[\max_i X_i]$.

Another way to phrase this problem is to imagine $n$ boxes: each is labelled with a distribution $F_i$ and inside the box there is a random variable $X_i$ distributed independently according to $F_i$. A gambler is then presented with the boxes in some sequence. After opening
each box, the algorithm inspects the value of \( X_i \) and either takes it and ends the game or discards it (discarded items are forever lost) and continues opening boxes. The goal is to design algorithms for the gambler that are competitive against a prophet who sees the content of the boxes without having to open them (and therefore just picks the box containing the maximum variable). Throughout this paper, we will assume (unless otherwise specified) that the algorithm is presented with the boxes in a uniform random order; this is sometimes referred to as the “prophet secretary” model (see e.g. Esfandiari et al. [2017]; Ehsani et al. [2018]; Correa et al. [2019]).

One special case is the case where the distributions in all boxes are identical. An upper bound for this case was given by Kertz [1986] who showed that the best possible approximation is at most the solution \( \beta \approx 0.745 \) to the equation:

\[
\int_0^1 \left[y(\log y - 1) - (\beta^{-1} - 1)\right]^{-1}dy = 1 \tag{1}
\]

This case was recently closed by Correa et al. [2017], who demonstrated a policy matching this lower bound.

A major open problem is to determine the optimal factor for non-identical distributions. Given that the iid case is a special case, the bound \( \beta \approx 0.745 \) was established as the golden standard for prophet inequalities and a major question is whether there is a gap between the iid and non-iid cases. For the random order case, this question was resolved by Correa et al. [2019], who show an upper bound on the approximation ratio in this setting of \( \sqrt{3} - 1 \approx 0.732 \), strictly less than the Kertz bound of 0.745. The same paper provides a policy achieving an approximation ratio of \( \approx 0.669 \).

Our main result is that if we eliminate a constant number of variables, the gap between iid and non-iid disappears. This implies optimal prophet inequalities (with factor \( \beta \approx 0.745 \)) for a variety of settings of interest which we describe below.

1.1 Our Results

In this paper we demonstrate a policy that achieves the approximation ratio of \( \beta - \epsilon \) when compared to the expected max of all but a constant (depending on \( \epsilon \) but not on \( n \)) number of boxes. In fact, we prove something slightly stronger: that by removing \( \text{poly}(\epsilon^{-1}) \) random variables from an instance, we can find a stopping time for the resulting instance that is \( (\beta - \epsilon) \)-competitive with the prophet benchmark.

**Theorem A (Restatement of Theorem 16).** Consider a collection of \( n \) random variables \( X = \{X_1, X_2, \ldots, X_n\} \). Then, for any \( \epsilon > 0 \), there exists a subset \( X' \) of \( X \) containing \( n - \tilde{\Theta}(\epsilon^{-3}) \) of these variables and a stopping rule \( \tau \) for \( X' \) such that \( \mathbb{E}[X'_\tau] \geq (\beta - \epsilon) \mathbb{E}[\max_i X'_i] \). Moreover, it is possible to efficiently construct such a subset \( X' \) and a stopping rule \( \tau \) achieving this guarantee in polynomial time.

One way to interpret this result is that all instances of this problem are “close” to instances where it is possible to achieve the Kertz bound: the only obstructions to achieving
the Kertz bound for non-identical distributions are some small (constant-sized) sets of distributions.

Theorem A has a number of interesting applications. For example, Theorem A immediately implies an asymptotically tight prophet inequality for the $\Theta(\varepsilon^{-3})$th order statistic (Corollary 18). This is what we call the imperfect prophet case, where the benchmark picks the $k$-largest variable instead of the largest. This is equivalent to the $k$-optimal price benchmark in [Goldberg et al. 2006].

Theorem A also implies improved results for frequent prophets. Frequent prophets, introduced in [Abolhassani et al. 2017], are a subclass of prophet inequality instances where each distribution must be repeated at least some number of times; an instance is $m$-frequent if each distribution appears at least $m$ times. In [Abolhassani et al. 2017], the authors show how to design a 0.738-competitive policy for $\Theta(\log n)$-frequent instances. We improve on this by showing how to obtain a $(\beta - \varepsilon)$-competitive policy for $O(1)$-frequent instances.

**Theorem B** (Restatement of Theorem 19). Consider an $m$-frequent collection of independent random variables. If $m = \Omega(\varepsilon^{-2})$ for some $\varepsilon > 0$, it is possible to construct a stopping time $\tau$ such that $\mathbb{E}[X_\tau] \geq (\beta - \varepsilon) \mathbb{E}[\max_i X_i]$, where $\beta$ is the Kertz bound. Furthermore, it is possible to construct a policy achieving this guarantee in polynomial time.

Both the bounds in Theorems A and B are tight. It is impossible to obtain better-than-$\beta$ approximation removing any constant (independent of $n$) number of variables. The frequent instances in Theorem B have the iid case as a special case, for which it is also known that we can’t obtain better-than-$\beta$ bounds.

Our proof of these theorems proceeds by first proving analogous results about a class of prophet instances we call small prophets. In small prophets, the distribution of each random variable is concentrated near zero; formally, we say a random variable $X$ is $(\varepsilon, \delta)$ small if $\Pr[X \geq \delta] \leq \varepsilon$. We show that for prophet instances composed of small variables, the Kertz bound is asymptotically achievable.

**Theorem C** (Restatement of Corollary 10). Given a set of $(\varepsilon, \delta)$-small variables $X = \{X_1, X_2, \ldots, X_n\}$, then there exists a stopping rule $\tau$ such that

$$\mathbb{E}[X_\tau] \geq (\beta - O(\varepsilon)) \cdot \mathbb{E}[\max_i X_i] - \delta.$$

To prove Theorem C, we design what we call time based policies for small prophets instances. In such a policy, we interpret each random variable arriving at a uniform random time in $[0, 1]$. Then, in order of arrivals, we choose whether to accept each variable based on whether it is above a time-based threshold (e.g. if variable $X$ arrives at time $t$, we select it if $X \geq r(t)$ for some function $r$). For small prophets, we can show that the performance of the optimal such time-based policy is exactly the Kertz bound.

With our result for small prophets in hand, we can prove Theorem A. Here we proceed via a thresholding argument. Given any instance $X = \{X_1, X_2, \ldots, X_n\}$ of random variables and a threshold $t > 0$ we consider a new set of random variables formed by $X_i - \max(X_i, t)$. As we increase $t$, more and more of these random variables are small. Our algorithm is
then as follows: if we ever see an item larger than \( t \), we accept it; otherwise, we run the small prophets algorithm on these small residues. When we choose \( t \) so that \( \text{poly}(\varepsilon^{-1}) \) of the random variables are small, we can show that the above algorithm leads to a \( \beta \)-competitive policy against imperfect prophets (establishing the full strength of Theorem A requires some additional ideas; see Section 4 for more details).

1.2 Related Work

The literature on prophet inequalities [Krengel and Sucheston, 1977] is vast; we provide here a high-level overview of the prophet inequalities landscape, primarily focusing on the case where a single random variable out of \( n \) needs to get picked.

**Adversarial order.** When the random variables are independent but not identical and they arrive in an adversarially chosen order, Krengel and Sucheston [1978] showed that the gambler can obtain at least \( \frac{1}{2} \) of the prophet benchmark. Later Samuel-Cahn [1984] showed that the same \( \frac{1}{2} \)-approximation can be obtained by a simple threshold policy, that posts a single threshold and accepts the first random variable to exceed the threshold.

In the special case where the random variables are i.i.d. (and hence the adversarial, random and free orders coincide), Hill and Kertz [1982] show that the gambler can obtain at least \( 1 - \frac{1}{e} \) of the prophet benchmark and also show examples that prove that one cannot obtain a factor beyond \( \frac{1.342}{1.342} \sim 0.745 \). Kertz [1986] later conjectured that \( \frac{1}{1.342} \sim 0.745 \) is the best possible approximation. The first formal proof that one can go beyond \( 1 - \frac{1}{e} \) was given by Abolhassani et al. [2017] and Correa et al. [2017]. In the former, Abolhassani et al. give a 0.738 approximation when \( n \) is larger than a large constant \( n_0 \). Simultaneously and independently, Correa et al. [2017] show a 0.745 approximation for this problem, thereby completely closing the gap between upper and lower bounds for the i.i.d. case.

**Random order.** The random order prophet inequality problem, or the prophet secretary problem, was first studied by Esfandiari et al. [2017], where they show a \( 1 - \frac{1}{e} \) approximation for large \( n \). They also show that with a single threshold it is impossible to get better than a \( \frac{1}{2} \)-approximation. Correa et al. [2017] were the first to show that one can obtain a \( 1 - \frac{1}{e} \) approximation for any \( n \), and they do this via non-adaptive thresholds. Azar et al. [2018] were the first to beat the \( 1 - 1/e \) barrier for the random order prophet problem and show an approximation factor of \( 1 - \frac{1}{e} + 0.0025 \). The notion of time based policies that we crucially use in our paper (see Section 3.1) was introduced in Ehsani et al. [2018] to obtain a \( 1 - 1/e \) approximation for random order prophets with matroid feasibility constraints (i.e., the set of random variables the gambler can feasibly pick has to be an independent set of an underlying matroid). These time based policies were used later by Singla [2018] for obtaining an alternative proof of 0.745 approximation for the i.i.d. case (obtained originally by Correa et al. [2017]), although this proof works only for very large \( n \). The time based

\[^1\text{This constant } \frac{1}{e} \text{ cannot be improved, even if the algorithm was allowed to use adaptive strategies and even if we were in the } m \text{-frequent prophet setting for arbitrarily large } m.\]
policies were subsequently used by [Correa et al. 2019] to show a 0.669 approximation factor which remains the best known factor to date for the random order prophet problem. In the same paper [Correa et al. 2019] show that it is impossible to get larger than $\sqrt{3} - 1$ approximation factor. When each distribution occurs at least $\Theta(\log n)$ times, [Abolhassani et al. 2017] show that one can obtain a 0.738 approximation for random order prophets.

Free order. The most intriguing open problem in the prophet inequalities landscape is whether one can obtain a 0.745 approximation factor in the free order prophets problem, namely, when the gambler gets to choose the order of inspection of random variables. While the upper bound of $\sqrt{3} - 1$ from [Correa et al. 2019] is not known to hold in the free order case, the 0.669 bound established by [Correa et al. 2019] in the same paper for the random order case continues to be the best known approximation factor for the free order case as well, for general $n$. There are a few special cases where better approximation factors are known. For small $n$, [Beyhaghi et al. 2018] design better approximation factors through factor revealing LPs. When each distribution occurs at least a constant $m_0$ times, [Abolhassani et al. 2017] show that one can obtain a 0.738 approximation for free order prophets.

2 Prophet inequalities setting

In the prophet inequalities setting we have $n$ random variables $X_i$ with known distributions $F_i$ arriving in random order. Upon the arrival, an algorithm learns its realization and decides to either stop and obtain that value as reward or reject that variable and continue.

It is useful to think of the random variables as $n$ boxes each labeled with the distribution $F_i$. Inside each box is a sample $X_i \sim F_i$. The distributions are known in advance, but the boxes are given to the algorithm in random order. The identity of the $i$-th box is only revealed when that box arrives. Upon arrival, the algorithm can inspect the content of that box.

Given a collection of random variables $X = \{X_1, \ldots, X_n\}$ we will denote by $\text{OPT}(X)$ the reward of the optimal policy. This will be compared with a prophet who can see the value inside all the boxes in advance. The reward of the prophet is given by $\text{MAX}(X) = \mathbb{E}[\max_{i=1..n} X_i]$. The prophet inequality problem asks how good is the optimal online policy when compared with the prophet. In other words, what is the largest factor $\alpha$ for which the following inequality is true:

$$\text{OPT}(X) \geq \alpha \cdot \text{MAX}(X)$$

We will often omit $X$ and just refer to $\text{OPT}$ and $\text{MAX}$ when clear from context. For bounding $\text{OPT}$ we will often define a certain feasible policy $\text{ALG}$ and then bound it with respect to $\text{MAX}$. This will immediately imply a prophet inequality since $\text{OPT} \geq \text{ALG}$ for any feasible online policy.

2.1 Useful definitions

We present certain definitions that will be useful throughout the paper:
Definition 1 ((ε, δ)-Small Variables). We say that a random variable $X_i \geq 0$ with cdf $F_i$ is $(\varepsilon, \delta)$-small if

$$1 - F_i(r) \leq \varepsilon, \quad \forall r > \delta$$

or in other words, $1 - \varepsilon$ of its mass is in $[0, \delta]$. We say that a variable is $\varepsilon$-small if it is $(\varepsilon, 0)$-small.

Definition 2 (Imperfect Prophet). We say that for a set of random variables $X$, the Imperfect Prophet benchmark $\text{MAX}_k(X)$ corresponds to the expectation of the $k$-th largest value of $X_i$.

Definition 3 (Free-order policy). The free-order optimum $\text{OPT}_{\text{free}}(X)$ is the reward of the optimal online policy that can choose the order in which the boxes are inspected.

Definition 4 ($m$-frequent variables). We say that a set of random variables $X = \{X_1, \ldots, X_n\}$ is $m$-frequent if for any variable $X_i$ there are at least other $m - 1$ variables with the same distribution.

2.2 Kertz upper bound

Kertz shows that even if the random variables are iid (in which case the order in which boxes are inspected is irrelevant) the maximum possible factor in prophet inequalities is $\beta \approx 0.745$:

Theorem 5 (Kertz [1986]). Let $\beta$ be the solution to Kertz’s equation (1). Then for every $\epsilon$ there is a set of iid random variables $X$ such that $\text{OPT}(X) \leq (\beta + \epsilon) \cdot \text{MAX}(X)$.

3 Small prophets

Our first step is to prove a prophet inequality for small variables (Definition 1). We first argue that the Kertz’ upper bound (Theorem 5) is still valid when restricted to $\varepsilon$-small distributions. Then we give a policy for small prophets achieving $\beta - O(\varepsilon)$.

Lemma 6. For any $\varepsilon, \delta > 0$ there is a set $X = \{X_1, \ldots, X_n\}$ of $\varepsilon$-small variables such that $\text{OPT}(X) \leq (\beta + \delta) \cdot \text{MAX}(X)$ where $\beta$ is the Kertz bound.

The proof is based on the following observation. This and other omitted proofs can be found in the appendix.

Lemma 7. Let $X$ be a set of $n$ iid variables with cdf $F$ and $Y$ be a set of $nk$ variables with cdf $F^{1/k}$. Then $\text{MAX}(X) = \text{MAX}(Y)$ and $\text{OPT}(X) \geq \text{OPT}(Y)$.

This lemma allows us to convert an upper bound of $\text{OPT}(X) \leq (\beta + \delta) \cdot \text{MAX}(X)$ for cdf $F$ to an upper bound of type $\text{OPT}(Y) \leq (\beta + \delta) \cdot \text{MAX}(Y)$ for cdf $F^{1/k}$. If we add a tiny probability mass at zero and take $k$ to be large enough, the distribution $F^{1/k}$ becomes $\varepsilon$-small. A formal proof is given in the appendix.

Our main result in this section is an algorithm achieving the optimal bound.
Theorem 8. Given a (non-iid) set of $\varepsilon$-small variables $X$, then

$$\text{OPT}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X),$$

where $\beta$ is the Kertz bound.

The proof of this theorem will be given in the following subsections. Before presenting the proof we would like to point out two consequences. Firstly, we can obtain the optimal algorithm for iid (but not necessarily small) variables as a corollary, providing an alternate proof of the result by Correa et al [Correa et al. 2017]:

Corollary 9. If $X$ is a set of iid variables (not necessarily small), then

$$\text{OPT}(X) \geq \beta \cdot \text{MAX}(X)$$

Proof. Consider an iid prophet instance with $n$ random variables $X_1, X_2, \ldots, X_n$ each with cdf $F$. As in the proof of Lemma 6, perturb $F$ slightly so that $F(0) > 0$ and choose $k$ such that $F^{1/k}(0) \geq 1 - \varepsilon$. Consider the $nk$ variables $Y_1, Y_2, \ldots, Y_{kn}$. They are $\varepsilon$-small, so by Theorem 8, $\text{OPT}(Y) \geq \beta \cdot (1 - O(\varepsilon)) \cdot \text{MAX}(Y)$. Lemma 7 then implies that $\text{OPT}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X)$. This is true for any $\varepsilon$ and hence $\text{OPT}(X) \geq \beta \cdot \text{MAX}(X)$. 

Finally, we show that Theorem 8 extends to $(\varepsilon, \delta)$-small prophets at the cost of an additive $\delta$. This fact will be useful in Section 4.

Corollary 10. Given a (non-iid) set of $(\varepsilon, \delta)$-small variables $X$, then

$$\text{OPT}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X) - \delta$$

Proof. Apply Theorem 8 to the variables $\tilde{X}_i = X_i \cdot 1\{X_i > \delta\}$. 

3.1 Time Based Policies

The policy we will construct in the proof of Theorem 8 has the form of a time based policy, which we describe below. Let $\pi$ be the random permutation of the boxes, i.e., the $i$-th box inspected by the algorithm in $F_{\pi(i)}$. It is useful to think of the random arrivals of boxes in terms of timestamps, i.e., that each box arrives uniformly at random in a time $t \in [0, 1]$. We will assign timestamps to boxes that are iid uniform and consistent with $\pi$. This can be done in the following manner:

1. Sample timestamps $t'_1, \ldots, t'_n$ iid from the uniform distribution over $[0, 1]$. Sort the numbers and let $t'_{(i)}$ be the $i$-th smallest sampled timestamp.

2. Given a permutation $\pi$, assign timestamp $t'_{\pi(i)}$ to box $\pi(i)$ by setting $t_{\pi(i)} = t'_{(i)}$.

Lemma 11. The variables $t_1, \ldots, t_n$ are iid uniform.
**Proof.** The variables are obtained by sampling iid uniform random variable and then applying a random permutation, hence the result must be also iid uniform.

The timestamps encode the permutation in which the boxes arrive, so from this point on we will reason solely in terms of timestamps. Note that in the original problem there is no notion of time, only arrival order, but the construction above (sample timestamps, sort them and assign timestamps in the order of arrival) allows us to think in terms of time arrivals.

**Time based threshold** The policy we will consider is parametrized by a decreasing threshold function $r : [0, 1] \to \mathbb{R}_+$. If a variable arrives at time $t$, we will pick that variable with probability $(1 - \varepsilon)^2$ if $X_i \geq r(t_i)$ and not pick it otherwise.

### 3.2 Notation and Useful Inequalities

We start by establishing some notation and some useful inequalities. We will use the notation $\bar{F}_i(r)$ to denote $1 - F_i(r)$. The cdf of $\max_i X_i$ is $F(r) := \prod_i F_i(r)$. For any threshold $r$ note that:

$$\tilde{R}_i(r) := \mathbb{E}[X_i \cdot 1\{X_i \geq r\}] = r\bar{F}_i(r) + \int_r^\infty \bar{F}_i(s)ds \quad (2)$$

In various places of the proof, it will be useful to analyze the sum of the above quantity over all variables and the following notation will come in handy:

$$\bar{w}(r) = \sum_i \bar{F}_i(r) \quad (3)$$

$$\tilde{R}(r) := \sum_i \mathbb{E}[X_i \cdot 1\{X_i \geq r\}] = r\bar{w}(r) + \int_r^\infty \bar{w}(s)ds \quad (4)$$

Whenever $\bar{F}_i(r)$ is small it will be convenient to approximate $\bar{w}(r)$ by:

$$w(r) := -\sum_i \log(1 - \bar{F}_i(r)) = -\log F(r) \quad (5)$$

Since the variables are $\varepsilon$-small and $x \leq -\log(1 - x) \leq (1 + \varepsilon)x$ for $x \in [0, \varepsilon]$ we have that

$$w(r) \leq \bar{w}(r) \leq (1 + \varepsilon) \cdot w(r) \quad (6)$$

It will be equally convenient to define an approximation of $\bar{R}$ using $w$ as follows:

$$\tilde{R}(r) = rw(r) + \int_r^\infty w(s)ds = -\int_r^\infty sw'(s)ds \quad (7)$$

And by equation $\tilde{R}$ we have:

$$R(r) \leq \tilde{R}(w) \leq (1 + \varepsilon) \cdot \tilde{R}(r) \quad (8)$$

We will use $\text{ALG}$ to denote the performance of the policy that that picks a variable with probability $(1 - \varepsilon)^2$ whenever $X_i \geq r(t_i)$. In the next subsection we will establish bounds on $\text{ALG}$ and $\text{MAX}$. We will use $\text{ALG}$ as a lower bound for $\text{OPT}$. 

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3.3 Bounding the Prophet and the Algorithm

Lemma 12 (Prophet bound). The prophet benchmark can be written as:

\[
\text{MAX} = \int_0^\infty R(r)e^{-w(r)}w'(r)dr
\]

Proof. If \( F \) is the cdf of \( \max_i X_i \) we have:

\[
\text{MAX} = \int_0^\infty 1 - F(r)dr = \int_0^\infty 1 - e^{-w(r)}dr = -\int_0^\infty rw'(r)e^{-w(r)}dr
\]

where the last equality follows by integration by parts. Integrating by parts again we obtain the result in the statement.

We now lower bound the performance of the time-based threshold policy:

Lemma 13 (Policy bound). Given a non-increasing threshold function \( r : [0, 1] \rightarrow \mathbb{R}_+ \) the performance of the policy that picks a variable with probability \((1 - \varepsilon)^2\) whenever \( X_i \geq r(t_i) \) is lower bounded as follows:

\[
\text{ALG} \geq (1 - \varepsilon)^2 \int_0^1 R(r(t)) \exp \left( -\int_0^t w(r(t))dt \right) dt
\]

Proof. We prove the bound in three steps:

**Step 1:** Let’s look at a single variable \( i \) and compute the probability it is not picked before time \( t \). This is

\[
1 - (1 - \varepsilon)^2 \cdot \int_0^t \tilde{F}_i(r(t))dt \geq \exp \left( -(1 - \varepsilon) \cdot \int_0^t \tilde{F}_i(r(t))dt \right)
\]

using the fact that for \( x \in [0, \varepsilon] \) we have \( 1 - x(1 - \varepsilon) \geq e^{-x} \).

**Step 2:** The probability that no variable is picked before time \( t \) is:

\[
\prod_i \exp \left( -(1 - \varepsilon) \cdot \int_0^t \tilde{F}_i(r(t))dt \right) = \exp \left( -(1 - \varepsilon) \cdot \int_0^t \tilde{w}(r(t))dt \right) \geq \exp \left( -\int_0^t w(r(t))dt \right)
\]

**Step 3:** Now if variable \( i \) arrives in interval \([t, t + dt]\) the reward is \((1 - \varepsilon)^2 \tilde{R}_i(r(t))\) (see equation 2) times the probability that none of the other variables have already been previously picked. This probability is lower bounded in the previous step. Integrating, we obtain that the expected reward from variable \( i \) is at least:

\[
(1 - \varepsilon)^2 \int_0^1 \tilde{R}_i(r(t)) \exp \left( -\int_0^t w(r(t))dt \right) dt
\]

Summing over all \( i \) and using that \( \sum_i \tilde{R}_i(r(t)) = \tilde{R}(r(t)) \geq R(r(t)) \) (see equation 8) we get the bound in the lemma.
3.4 Relating the bounds to Kertz’s equation

Our next step is to reduce the problem to solving a differential equation:

Lemma 14. If $y : [0, 1] \rightarrow [0, 1]$ is a function satisfying

$$\exp \left( \int_0^t \log y(t) \right) = -\beta \cdot y'(t), \quad y(0) = 1 \text{ and } y(1) = 0 \tag{10}$$

then the threshold policy that sets $r(t) = F^{-1}(y(t))$ is a $(1-\varepsilon)^2 \beta$-approximation to the prophet benchmark, i.e. $\text{OPT} \geq \text{ALG} \geq (1-\varepsilon)^2 \beta \cdot \text{MAX}$.

Proof. Since $r(t)$ goes from the top to the bottom of the support of $F$ when we vary $t$ from 0 to 1, we can apply the change of variables $r = r(t)$ in the prophet bound in Lemma 12 obtaining:

$$\text{MAX} = \int_0^1 R(r(t)) e^{-w(r(t))} w'(r(t)) r'(t) dt \tag{11}$$

Now, substituting $y(t) = e^{-w(r(t))}$ and $\hat{R}(t) = R(r(t))$, we get:

$$\text{MAX} = \int_0^1 \hat{R}(t)[-y'(t)] dt \tag{12}$$

$$\text{ALG} \geq (1-\varepsilon)^2 \int_0^1 \hat{R}(t) \exp \left( \int_0^t \log y(t) \right) dt$$

The conditions in equation (10) together with the expression above directly imply that the policy is a $\beta$-approximation.

The final step is to find a function satisfying the conditions in equation (10) with $\beta$ equal to the the Kertz bound. It turns out that that equation (10) can be transformed in the equation defining the Kertz bound in equation (1). The proof of the following lemma can be found in the appendix.

Lemma 15. There is a solution to equation (10) with $\beta$ equal to the Kertz bound.

Taken together, the previous lemmas imply a proof to Theorem 8.

4 Imperfect Prophets

The Kertz bound $\beta \approx 0.745$ is the golden standard for prophet inequalities since the iid case establishes a natural upper bound on what can be achieved in any other setting. Our main result in this paper is that all prophet instances are near optimal: by removing $\text{poly}(\varepsilon^{-1})$ variables from an instance, we can find a stopping time for the resulting instance that achieves a $(\beta - \varepsilon)$ fraction of its maximum.

It is useful to contrast this with the result in Correa et al. [2019], which shows an instance $X$ where the gap between $\text{OPT}(X)$ and $\text{MAX}(X)$ is strictly less than $\beta$. Their result shows
an inherent gap between the iid and non-iid cases. We show that by removing a constant number of variables, this gap disappears. In fact, if we compete with a slightly imperfect prophet - who can obtain the k-th largest value as reward (\(\text{MAX}_k\) in Definition 2) then we are able to achieve the Kertz bound and this is tight.

**Theorem 16.** Consider a collection of \(n\) random variables \(X = \{X_1, X_2, \ldots, X_n\}\). Then, for any \(\varepsilon > 0\), there exists a subset \(X'\) of \(X\) containing \(n - \Theta(\varepsilon^{-3} \log \varepsilon^{-1})\) of these variables so that

\[
\text{OPT}(X') \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X')
\]

Moreover, it is possible to efficiently construct such a subset \(X'\) and a policy achieving this guarantee in polynomial time.

If our stopping rule is allowed to use variables that the prophet is not, then we can get a slightly better dependence on \(\varepsilon\).

**Theorem 17.** Consider a collection of \(n\) random variables \(X = \{X_1, X_2, \ldots, X_n\}\). Then, for any \(\varepsilon > 0\), there exists a subset \(X'\) of \(X\) containing \(n - \Theta(\varepsilon^{-2} \log \varepsilon^{-1})\) of these variables so that \(\text{OPT}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X')\). Moreover, it is possible to efficiently construct such a subset \(X'\) and a policy achieving this guarantee in polynomial time.

Theorems 16 and 17 have a number of almost immediate interesting applications. For example, Theorem 17 immediately implies a nearly optimal prophet inequality for the \(\Theta(\varepsilon^{-2})\)th-order statistic of \(X\).

**Corollary 18.** If \(X\) is any collection of \(n\) random variables, then for any \(\varepsilon > 0\) there is a \(k = \Theta(\varepsilon^{-2} \log \varepsilon^{-1})\) such that

\[
\text{OPT}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}_k(X).
\]

Proof. Let \(X^{(k)}\) be a random variable denoting the value of the \(k\)th largest element of \(X\). Note that for any subset \(X'\) of \(X\) with \(|X'| = n - k\), \(\max(X') \geq X^{(k)}\). The result then immediately follows from Theorem 17. \(\square\)

We can also use Theorem 17 to design optimal stopping rules for \(m\)-frequent instances (see Definition 4) approaching the Kertz bound as \(m\) grows large.

**Theorem 19.** Consider an \(m\)-frequent collection \(X\) of independent random variables with \(m = \Omega(\varepsilon^{-2} \log \varepsilon^{-1})\) for some \(\varepsilon > 0\). Then:

\[
\text{OPT}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X)
\]

We will defer the proof of this theorem to Section 4.3.

All of the above theorems are tight in the sense that it is impossible to replace the Kertz bound \(\beta\) by any larger constant. For frequent prophets (Theorem 19) this follows since iid prophets are a special case of frequent prophets and the Kertz upper bound holds there. For imperfect prophets, a similar reduction to the iid case holds.
Lemma 20. Choose any $\alpha > \beta$ (where $\beta$ is the Kertz bound) and positive integer $r$. Then (for a sufficiently large $n$) there exists a collection of $n$ random variables $X = \{X_1, X_2, \ldots, X_n\}$ such that for any subset $X'$ of $X$ containing $n - r$ of these variables,

$$\text{OPT}(X) < \alpha \cdot \text{MAX}(X').$$

A proof of Lemma 20 can be found in Appendix B.

The remainder of this section is structured as follows. In Section 4.1, we begin by proving Theorems 16 and 17 in the weaker model of free-order prophets, where the algorithm can choose the order in which it looks at the items (in addition to being a slightly simpler proof, we also get a slightly better dependence on $\varepsilon$). In Section 4.2 we extend this to random-order prophets and prove Theorems 16 and 17. Finally, in Section 4.3 we apply these theorems to prove Theorem 19 for frequent prophets.

4.1 Warm-up: free order

For simplicity, we will begin by proving Theorem 17 in the free-order setting, where the algorithm has the power to choose the order in which they encounter the random variables.

Recall that this is the weaker analogue of Theorem 16, where the algorithm is allowed to use all the random variables (but the prophet is only allowed to choose the max of a specific subset of all but poly($\varepsilon$) of the variables).

Theorem 21. Consider a collection of $n$ random variables $X = \{X_1, X_2, \ldots, X_n\}$. Then, for any $\varepsilon > 0$, there exists a subset $X'$ of $X$ of size $n - \Theta(\varepsilon^{-1} \log \varepsilon^{-1})$ so that

$$\text{OPT}_{\text{free}}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X').$$

Moreover, it is possible to efficiently construct such a subset $X'$ and a policy achieving this guarantee in polynomial time.

Proof. For any real $t \geq 0$, define the random variables

$$Y_i(t) = \max(X_i, t) \quad \text{and} \quad Z_i(t) = Y_i(t) - t$$

Note that as $t$ increases, the number of variables $Z_i(t)$ that are $(\varepsilon, \varepsilon t)$-small increases. In particular, if $Z_i(t)$ is $(\varepsilon, \varepsilon t)$-small for a given $t$, $Z_i(t')$ is $(\varepsilon, \varepsilon t')$ small for all $t' \geq t$. Also, note that every variable $Z_i(t)$ is $(\varepsilon, \varepsilon t)$-small for a sufficiently large $t$ (e.g. $t = \max(1, F_i^{-1}(1 - \varepsilon))$).

Let $k = \Theta\left(\frac{1}{\varepsilon} \log \frac{1}{\varepsilon}\right)$, and let $t^*$ be the supremum over all $t$ such that exactly $k$ of the variables $Z_i(t^*)$ are not $(\varepsilon, \varepsilon t^*)$-small. This means that for $k$ values of $i$, $\Pr[Z_i(t^*) \geq \varepsilon t^*] \geq \varepsilon$; in particular, this implies that $\Pr[X_i \geq t^*(1 + \varepsilon)] \geq \varepsilon$.

Let $X_{\text{big}}$ be the set of $X_i$ such that $Z_i(t^*)$ is not $(\varepsilon, \varepsilon t^*)$-small and $X' = X \setminus X_{\text{big}}$. Our policy will proceed as follows. We will order the items so that all the elements of $X'$ come before the elements of $X_{\text{big}}$. We will start by running the small prophets policy of Section 3 on the elements in $X'$ (specifically, mapping each $X_i$ to $Z_i = \max(X_i - t^*, 0)$ and accepting each $X_i$ with the probability $Z_i$ would have been accepted under the corresponding small
prophets instance). If we reach the end of $X'$ without accepting any items, we accept the first item we see in $X_{big}$ that is larger than $t^*$.

Let us analyze the performance of this policy. First, we claim that with high probability, $X_{big}$ contains an $X_i$ greater than or equal to $t^*$. Recall that each $X_i$ in $X_{big}$ satisfies $\Pr[X_i \geq t^*(1 + \varepsilon)] \geq \varepsilon$. Therefore,

$$\Pr[\max(X_{big}) \leq t^*] \leq (1 - \varepsilon)^{|X_{big}|} = (1 - \varepsilon)^k = (1 - \varepsilon)^{\Theta(\varepsilon^{-1} \log \varepsilon^{-1})} \leq \varepsilon.$$

It follows that with probability at least $(1 - \varepsilon)$, $X_{big}$ contains an $X_i$ satisfying $X_i \geq t^*$. In particular, with probability at least $(1 - \varepsilon)$, our policy is guaranteed to receive reward at least $t^*$ (since we also only accept an $X_i \in X'$ if $X_i > t^*$).

By Corollary [10] running the small prophets policy on $Z'$ guarantees us an additional $(\beta - O(\varepsilon)) \cdot \max(Z') - \varepsilon t^*$ reward over our guaranteed $t^*$. The total expected reward of this policy is therefore at least

$$\text{OPT}(X) \geq (1 - \varepsilon)((1 - O(\varepsilon))\beta \cdot \max(Z') - \varepsilon t^* + t^*)$$

$$\geq (\beta - O(\varepsilon)) \cdot (\max(Z') + t^*)$$

$$= (\beta - O(\varepsilon)) \cdot \max(X').$$

We now extend this to the setting where the algorithm is also restricted to the same subinstance as the prophet. To do this, we roughly proceed as follows. If the expected maximum of the subinstance is close to the expected maximum of the original instance (i.e. $\max(X') \geq (1 - \varepsilon)\max(X)$), then we are done – we can just take our final subinstance to be $X$. If it is not, then we can recurse and apply Theorem 21 to $X'$: if it has a subinstance $X''$ with $\max(X'') \geq (1 - \varepsilon)\max(X')$ then we are also done (we can take our final subinstance to be $X'$). Otherwise, we continue recursing.

We repeat until we have done this poly$(1/\varepsilon)$ times. At this point we are looking at some subinstance $X^*$ of $X$ (with $|X^*| = |X| - \text{poly}(1/\varepsilon)$). Since we have never stopped and reported a valid subinstance, we know that $\max(X^*) \leq (1 - \varepsilon)\text{poly}(1/\varepsilon)\max(X) \leq \text{poly}(\varepsilon)\max(X)$. But in this case we can show that one of the variables we discarded has expectation much larger than $\max(X^*)$; by including it in our subinstance and just picking it, we can achieve a competitive ratio very close to 1. We formalize this in the following theorem.

**Theorem 22.** Consider a collection of $n$ random variables $X = \{X_1, X_2, \ldots, X_n\}$. Then, for any $\varepsilon > 0$, there exists a subset $X'$ of $X$ of size $n - \Theta((\varepsilon^{-1} \log \varepsilon^{-1})^2)$ so that

$$\text{OPT}_{\text{free}}(X') \geq (\beta - O(\varepsilon)) \cdot \max(X')$$

Moreover, it is possible to efficiently construct such a subset $X'$ and a policy achieving this guarantee in polynomial time.
Proof. We will define a sequence of instances in the following way. Let \( X_0 = X \) be the original instance, and for each \( i \), let \( X_{i+1} \) be the subset of \( X_i \) constructed by Theorem 21 when applied to \( X_i \). From the conditions of Theorem 21 for each \( k \geq 0 \) we know that \( |X_k| \geq n - O(k \varepsilon^{-1} \log \varepsilon^{-1}) \) and that

\[
\text{OPT}_{\text{free}}(X_k) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X_{k+1}).
\]

Now, if there exists a \( k < k_{\text{max}} = 10 \varepsilon^{-1} \log \varepsilon^{-1} \) such that \( \text{MAX}(X_{k+1}) \geq (1 - \varepsilon)\text{MAX}(X_k) \), this immediately implies the desired result (for \( X' = X_k \)). Thus, assume that for all such \( k \), \( \text{MAX}(X_{k+1}) < (1 - \varepsilon)\text{MAX}(X_k) \).

This implies that, for all \( k > 0 \), \( \text{MAX}(X_k) < (1 - \varepsilon)^k \text{MAX}(X_0) \). In particular, for \( k = 1 \), we know that \( \text{MAX}(X_1) < (1 - \varepsilon)\text{MAX}(X) \). We’ll now argue that there exists a random variable \( X^* \in X \setminus X_1 \) such that \( \mathbb{E}[X^*] \geq \text{poly}(\varepsilon)\text{MAX}(X) \). We will then argue that we can construct a good policy for \( X' = X_{k_{\text{max}}} \cup \{X^*\} \) (in particular, \( X^* \) will have such large expectation compared to \( \text{MAX}(X_{k_{\text{max}}} \) that it will suffice to just accept \( X^* \)).

Let \( X_{\text{big}} = X \setminus X_1 \). Recall that for any non-negative random variables \( U \) and \( V \) that \( \mathbb{E}[U] \geq \mathbb{E}[\max(U,V)] - \mathbb{E}[V] \) (since \( U + V \geq \max(U,V) \)). Applying this to \( U = X_{\text{big}} \) and \( V = X_1 \), we have that

\[
\mathbb{E}[\max(X_{\text{big}})] \geq \text{MAX}(X) - \text{MAX}(X_1) > \varepsilon \text{MAX}(X).
\]

Since \( |X_{\text{big}}| \leq \Theta(\varepsilon^{-1} \log \varepsilon^{-1}) \), this implies that there exists a variable \( X^* \in X_{\text{big}} \) such that \( \mathbb{E}[X^*] > \Theta(\varepsilon^2 / \log \varepsilon^{-1})\text{MAX}(X) \).

Now, consider the subset of variables \( X' = X_{k_{\text{max}}} \cup \{X^*\} \). Note that

\[
\frac{\text{MAX}(X')}{\text{OPT}_{\text{free}}(X')} \leq \frac{(1 - \varepsilon)^{k_{\text{max}}} \text{MAX}(X) + \mathbb{E}[X^*]}{\mathbb{E}[X^*]}
\]

\[
= 1 + \frac{(1 - \varepsilon)^{k_{\text{max}}} \text{MAX}(X)}{\mathbb{E}[X^*]}
\]

\[
\leq 1 + \frac{(1 - \varepsilon)^{k_{\text{max}}}}{\Theta(\varepsilon^2 / \log \varepsilon^{-1})}
\]

\[
\leq 1 + \frac{\exp(-\varepsilon \cdot (10 \varepsilon^{-1} \log \varepsilon^{-1}))}{\Theta(\varepsilon^2 / \log \varepsilon^{-1})}
\]

\[
\leq 1 + \frac{\exp(-10 \varepsilon \log \varepsilon^{-1}))}{\Theta(\varepsilon^2 / \log \varepsilon^{-1})}
\]

\[
\leq 1 + O(\varepsilon^8 \log \varepsilon^{-1}).
\]

It follows that \( \text{OPT}_{\text{free}}(X') \geq (1 - O(\varepsilon)) \cdot \text{MAX}(X) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X') \), as desired. \( \square \)

4.2 Random order

We now prove Theorems 16 and 17 by extending the proofs of Theorems 21 and 22 to the random-order model. The only place in Section 4.1 where we use our ability to control the
order of the variables is in Theorem 21 where we place all the items in \( X_{\text{big}} \) after those in \( X' \); other than this, everything works given the variables in a random order. Instead of ordering the items in \( X_{\text{big}} \) after those in \( X' \), we’ll instead show that enough of the variables in \( X_{\text{big}} \) will occur late enough (e.g. after we have seen \( 1 - \text{poly}(\varepsilon) \)) of the items) that we can replicate the proof with these items.

To do this, we will first need to slightly strengthen our policy for small prophets so that it can work even when restricted to select an item from the first \((1 - \varepsilon)\) fraction of random variables.

**Lemma 23.** Let \( X \) be a collection of \( n \) \((\varepsilon, \delta)\)-small variables. Then, if \( n > \varepsilon^{-2} \log(1/\varepsilon) \), there exists a stopping time \( \tau \) such that \( \tau \leq n(1 - \varepsilon) \) (the policy always stops before element \((1 - \varepsilon)n\)) and such that

\[
E[X_{\tau}] \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X) - \delta.
\]

**Proof.** Consider a slightly different policy, which begins by sampling a positive integer \( R \) from \( \text{Binomial}(n, (1 - \varepsilon)) \) and guarantees that its stopping time \( \tau \) is at most \( R \). We will show that we can construct such a policy which is \((\beta - O(\varepsilon))\) competitive with \( \text{MAX}(X) \).

To do this, consider the collection \( X' \) of variables defined via \( X'_i = X_i \) with probability \( 1 - \varepsilon \), and \( X'_i = 0 \) with probability \( 0 \). Note that \( X' \) is also a collection of \((\varepsilon, \delta)\)-small variables. We will show how to transform the small prophets policy for \( X' \) to a small prophets policy for \( X \) that stops within the first \( R \) elements.

Recall that the small prophets for \( X' \) can be implemented as follows. We start by sampling \( n \) random times uniformly from the interval \([0, 1]\); in sorted order, let these times be \( t'_1 \leq t'_2 \leq \cdots \leq t'_n \). The small prophets policy \( \tau' \) provides a threshold function \( r(t) \) such that we should select the \( i \)th item \( X'_{\pi(i)} \) we encounter if \( X'_{\pi(i)} \geq r(t'_i) \) (and we have not selected any earlier item). Importantly, note that the \( i \)th threshold does not depend on the individual identity of the \( i \)th item, just on the randomly sampled \( t_i \) and this global threshold function \( r \). By Corollary 10, this guarantees an expected reward at least \( E[X'_{\tau'}] \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X') - \delta \).

Now, consider the policy \( \tau \) for \( X \) where we sample \( R \) random times uniformly from \([0, 1]\) (call them \( t_1 \leq t_2 \leq \cdots \leq t_R \)), and select the \( i \)th item we encounter if \( X_{\pi(i)} \geq r(t_i) \) (if we pick no item by \( X_{\pi(r)} \), we end the protocol without picking anything). We claim that this policy gets exactly the same reward in expectation as \( \tau' \); i.e. that \( E[X_{\tau}] = E[X'_{\tau'}] \). To see this, note that we can couple executions of \( \tau' \) on \( X' \) with executions of \( \tau \) on \( X \). Specifically, \( R \) should equal the number of indices \( i \) where \( X'_i \) is chosen to equal \( X_i \) (instead of 0), and \( t_i \) should equal the \( i \)th value of \( t'_i \) that corresponds to a “non-zero” \( X_i \) (it is straightforward to verify that the distributions of \( R \) and \( t_i \) match the distributions generated by this coupling process).

It follows that \( E[X_{\tau}] = E[X'_{\tau'}] \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X') - \delta \). Now, note that \( \text{MAX}(X') \geq (1 - \varepsilon)\text{MAX}(X) \) (one way to see this is to note that with probability \((1 - \varepsilon)\), the maximum item in \( X \) does not get erased and remains the maximum in \( X' \)). It follows that \( E[X_{\tau}] \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X') - \delta \).
Finally, all the previous discussion was for a modified stopping time which observes the first \( \text{Binomial}(n, (1 - \varepsilon)) \) items in \( X \). But to prove the theorem statement, we wish to show that there exists a stopping time which never observes more than \((1 - \varepsilon)n\) items. To do this, we will apply the first result for \( \varepsilon' = 2\varepsilon \) so that we only observe the first \( R' = \text{Binomial}(n, (1 - \varepsilon')) \) items (since \( \varepsilon' = O(\varepsilon) \), our guarantee is still the same). Then note that by Hoeffding’s inequality, the probability \( R' \) is larger than \((1 - \varepsilon)n\) is at most
\[
\exp(-2\varepsilon^2 n) \leq \varepsilon^2 \quad \text{(since } n > \varepsilon^{-2} \log 1/\varepsilon).\]
On the other hand, if \( R' > (1 - \varepsilon)n \), we lose reward at most \( \varepsilon^2 \text{MAX}(X) \) (since the choice of \( R' \) is independent from the realizations of \( X \)). This means that if we follow the stopping time for \( R' \) and quit if \( R' > (1 - \varepsilon)n \), our stopping time satisfies
\[
\mathbb{E}[X_r] \geq (\beta - O(\varepsilon))\text{MAX}(X) - \varepsilon^2 \text{MAX}(X) \geq (\beta - O(\varepsilon))\text{MAX}(X) - \delta.
\]

We can now proceed to prove Theorem 17.

**Proof of Theorem 17.** Similarly to the proof of Theorem 21, set \( k = \Theta(\varepsilon^{-2} \log \varepsilon^{-1}) \), and let \( t^* \) be the supremum over all \( t \) such that exactly \( k \) of the variables \( Z_i(t^*) \) are not \((\varepsilon, \varepsilon t^*)\)-small. Let \( X_{\text{big}} \) be this subset of \( k \) non-small variables, and let \( X' = X \setminus X_{\text{big}} \).

Our policy will operate in two parts. We will first use the small prophets policy on the variables \( Z_i \) to process elements of \( X' \) (ignoring elements of \( X_{\text{big}} \)) until we have seen \((1 - \varepsilon)|X'| \) elements of \( X' \). If we have not chosen an item by this time, we will pick the next item we see with value at least \((1 - O(\varepsilon)) \cdot \text{MAX}(X') - \varepsilon t^* \). It therefore follows that, conditioned on our policy picking an item, our policy achieves expected reward at least
\[
(\beta - O(\varepsilon)) \cdot \text{MAX}(Z_{\text{rand}}) - \varepsilon t^* + t^* \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X_{\text{rand}}) \geq (\beta - O(\varepsilon)) \cdot \text{MAX}(X').
\]

We now claim that our policy selects an item with probability at least \((1 - O(\varepsilon)) \), thus implying the theorem. First, note that each \( X_i \in X_{\text{big}} \) has an independent \( \varepsilon \) probability of occurring after the \((1 - \varepsilon)|X'| \)th item in \( X' \) (to see this, imagine constructing the uniform random order by first choosing the order of elements in \( X' \) and then randomly inserting the elements in \( X_{\text{big}} \)). It follows from Hoeffding’s inequality that the probability we see at least \( k\varepsilon/2 \) variables in \( X_{\text{big}} \) in the second part of the policy is at least
\[
1 - 2\exp(-\varepsilon^2 k/2) = 1 - 2\exp(-\Theta(\varepsilon^{-1} \log \varepsilon^{-1})) \geq 1 - O(\varepsilon).
\]
Since each of the variables \( X_i \in X_{\text{big}} \) satisfies \( \Pr[X_i \geq t^*(1 + \varepsilon)] \geq \varepsilon \), the probability that maximum of these \( k\varepsilon/2 \) variables is less than \( t^* \) is at most \((1 - \varepsilon)^{k\varepsilon/2} = (1 - \varepsilon)^{\Theta(\varepsilon^{-1} \log \varepsilon^{-1})} \leq \varepsilon \). It follows that with probability at least \( 1 - O(\varepsilon) \) that in the second part of our policy we see a variable with value at least \( t^* \), as desired.
Proof of Theorem 16. Follows from the proof of Theorem 22 using Theorem 17 in place of Theorem 21. \qed

4.3 Frequent Prophets

Finally, we show the proof of Theorem 19 as an application of Theorem 17. The following lemma relates the expected max of a subinstance of a frequent instance to the expected max of the instance itself.

Lemma 24. Let \( X \) be an \( m \)-frequent collection of random variables. Let \( X' \subseteq X \) have size \( |X'| \geq |X| - k \). Then
\[
\text{MAX}(X') \geq \left(1 - \frac{k}{m}\right) \text{MAX}(X).
\]

Proof. We will show that with probability \( 1 - \frac{k}{m} \), \( \text{max}(X') = \text{max}(X) \), hence implying the result.

Recall that in an \( m \)-frequent collection of random variables, each random variable is distributed according to some distribution \( D_i \), and each distribution \( D_i \) has at least \( m \) random variables distributed according to it. Let us condition on the event that the maximum variable in \( X \) is distributed according to \( D_i \). By symmetry, any of the (at least) \( m \) random variables distributed according to \( D_i \) has an equal chance of being the maximum. Moreover, at least \( m - k \) of these variables also belong to \( X' \). It follows that the probability that the maximum variable belongs to \( X' \) (conditioned on the variable being distributed according to \( D_i \)) is at least \( 1 - \frac{k}{m} \). Since this is true for each \( D_i \), it follows (by the law of total probability) that it is true in general, as desired. \qed

The result for frequent prophets follows as a direct corollary.

Proof of Theorem 19. Follows from Lemma 24 and Theorem 17. \qed

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A Missing proofs from Section 3

Proof of Lemma 7 To see that $\max(X) = \max(Y)$ observe that the distribution of

$$\Pr[\max_i X_i \leq t] = F(t)^n = (F(t)^{1/k})^{nk} = \Pr[\max_i Y_i \leq t].$$
Since max_i Y_i and max_i X_i have the same cdf, they should have the same expectation.

To show that OPT(X) ≥ OPT(Y) consider a stopping time τ such that E[Y_τ] = OPT(Y). Now we will construct a stopping time τ' such that E[X_τ'] = E[Y_τ].

Let Y_k be the distribution over k-element sequences (Y_1, Y_2, . . . , Y_k) where each element Y_i is iid with cdf F^{1/k}. Let Y_k(x) be the resulting distribution of Y_k conditioned on max Y_i = x. Consider the following procedure for generating a stopping time τ. So the solution can be obtained by simple integration:

Proof of Lemma 15. Let F be the cdf of the iid variables X in Theorem 5. Assume without loss of generality that F(0) > 0 (if not, we can modify X_i so that it equals 0 with probability δ' ≪ δ while preserving the inequality). If these random variables X_i are ε-small, then we are done. If not, note that for some sufficiently large k, a random variable with cdf F^{1/k} is ε-small. In particular, it suffices to take k ≥ log F(0)/log(1 − ε) to ensure that F(0)^{1/k} ≥ (1 − ε). Now consider a set Y of nk random variables distributed cdf F^{1/k}. By Lemma 7 if we had OPT(Y) > (β + δ) MAX(Y) then we would have also OPT(X) > (β + δ) · MAX(X) violating Kertz’s upper bound.

Proof of Lemma 15. Consider the following ordinary differential equation problem:

\[ \frac{dy}{dt} = (y \log y - 1) - (\beta^{-1} - 1) \quad \text{and} \quad y(0) = 1 \]  

(13)

It is simpler to solve the inverse function of y(t) which we denote by t(y). We know that:

\[ \frac{dt}{dy} = \left[(y \log y - 1) - (\beta^{-1} - 1)\right]^{-1} \]

So the solution can be obtained by simple integration:

\[ t(y) = t(1) - \int_y^1 \left[(y \log y - 1) - (\beta^{-1} - 1)\right]^{-1} dy \]

Hence by the Kertz equation, if we take β to be the Kertz bound, t(0) = 1. Therefore the inverse y(t) satisfied y(0) = 1, y(1) = 0 and equation (13). Finally, we show that it also satisfies condition (10). To see that, derive equation (13), obtaining:

\[ y'' = y'(\log y - 1) + \frac{y'}{y} = y' \cdot \log y \]

which can be re-written as:

\[ \log(y) = [\log(-y')]' \]
Notice that since $y' < 0$ we need to write $\log(-y')$ instead of $\log(y')$. Integrating from 0 to $t$ each expression, we get:

$$
\int_0^t \log y(s) ds = \log(-y'(t)) - \log(-y'(0))
$$

Since $y'(0) = \beta^{-1}$ by replacing $y(0) = 1$ in equation (13), we have that:

$$
\int_0^t \log y(s) ds = \log(-y'(t)) - \log(-\beta^{-1}) = \log(\beta \cdot y'(t))
$$

which is exactly condition in equation (10).

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**B  Tightness of Kertz bound for imperfect prophets**

In this section we prove Lemma 20 showing that the Kertz bound in Theorem 16 cannot be improved.

**Proof of Lemma 20.** By the Kertz upper bound (see Kertz [1986]), for any $\varepsilon$ and sufficiently large $n$ we can construct an iid prophets instance $X$ where $\text{OPT}(X) < (\beta + \varepsilon)\text{MAX}(X)$. Since this is an iid prophets instance, we additionally have that $\text{MAX}(X') \geq (1 - \frac{\varepsilon}{n})\text{MAX}(X)$ (one way to see this is that since all the rvs are identical, with probability $(n - r)/n$, the maximum value in $X$ will be among the variables in $X'$). Let $\alpha - \beta = \delta$. If we pick $\varepsilon < \delta/2$ and $n > r(1 + \delta/2)$ it follows that $\text{OPT}(X) < \alpha\text{MAX}(X')$, as desired.  

\hfill \Box