THE DEGENERATE RESIDUAL SPECTRUM OF QUASI-SPLIT FORMS OF Spin$_8$ ASSOCIATED TO THE HEISENBERG PARABOLIC SUBGROUP

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ABSTRACT. In [GS15] and [Seg17], the twisted standard $L$-function $L(s, \pi, \chi, st)$ of a cuspidal representation $\pi$ of the exceptional group of type $G_2$ was shown to be represented by a family of new-way Rankin-Selberg integrals. These integrals connect the analytic behaviour of $L(s, \pi, \chi, st)$ with that of a family of degenerate Eisenstein series $E_E(\chi, f_s, s, g)$ on quasi-split forms $H_E$ of Spin$_8$, induced from Heisenberg parabolic subgroups. The analytic behaviour of the series $E_E(\chi, f_s, s, g)$ in the right half-plane $\Re(s) > 0$ was studied in [Seg]. In this paper we study the residual representations associated with $E_E(\chi, f_s, s, g)$.

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1. Introduction

In [GS15] and [Seg17], the twisted standard $L$-function $L(s, \pi, \chi, st)$ of a cuspidal representation $\pi$ of the exceptional group of type $G_2$ was shown to be represented by a family of new-way Rankin-Selberg integrals. These integrals links the analytic behaviour of $L(s, \pi, \chi, st)$ with that of a family of degenerate Eisenstein series $E_E(\chi, f_s, s, g)$ on quasi-split forms $H_E$ of Spin$_8$ induced from Heisenberg parabolic subgroups.

The analytic behaviour of $E_E(\chi, f_s, s, g)$ in the right half-plane $\Re(s) > 0$ was studied in [Seg]. As a consequence, it was shown that $L(s, \pi, \chi, st)$ is holomorphic at any $s \neq 1, 2$ such
that $\Re (s) > 0$. It was further shown that the orders of the poles of $L(s, \pi, \chi, st)$ at $s = 1$ and $s = 2$ are bounded as follows:

- At $s = 1$, $L(s, \pi, \chi, st)$ may admit at most a simple pole when $\chi$ is a quadratic character. For any other $\chi$, $L(s, \pi, \chi, st)$ is holomorphic there.
- At $s = 2$, $L(s, \pi, \chi, st)$ may admit at most a double pole when $\chi$ is trivial and at most a simple pole when $\chi$ is of order 2 or 3. For any other $\chi$, $L(s, \pi, \chi, st)$ is holomorphic there.

This information was applied, in [Seg, Part 2] and [GSb], to classify all cuspidal representations $\pi$ of $G_2 (\mathbb{A})$ such that $L(s, \pi, \chi, st)$ admits a pole at $s = 2$ in terms of functorial lifts. More precisely, it is shown that

- (Seg) If $L(s, \pi, 1, st)$ admits a pole of order two at $s = 2$ or $L(s, \pi, \chi, st)$ admits a simple pole at $s = 2$ for $\chi \neq 1$, then $\pi$ is a lift from a group of finite type.
- (GSb) If $L(s, \pi, 1, st)$ admits a simple at $s = 2$ then $\pi$ is a Rallis-Schiffman lift from $\tilde{SL}_2$.

These calculations make use of the residual representation of $E_E (\chi, f_s, s, g)$ at $s = \frac{3}{2}$.

The classification of cuspidal representations $\pi$ of $G_2$ such that $L(s, \pi, \chi, st)$ admits a pole at $s = 1$ in terms of functorial lifts is an open problem. This paper is a study of the residual representations of $E_E (\chi, f_s, s, g)$ for $\Re (s) > 0$, and in particular at $s = \frac{1}{2}$. Applying the results of this paper to the classification of cuspidal representations of $G_2$ in terms of the analytic behaviour of $L(s, \pi, \chi, st)$ at $s = 1$ is a work in progress.

The square-integrable residual representations associated with $E_E (\chi, f_s, s, g)$ are listed in Theorem 5.2. The statement of this theorem will not be quoted here since it requires a detailed description of the irreducible quotients of the local degenerate principal series $I_\nu (\chi_\nu, s)$ associated to $E_E (\chi, f_s, s, g)$. However, we note here the main interesting feature of this theorem. The square-integrable residues of $E_E (\chi, f_s, s, g)$ at $s = \frac{1}{2}$, when $\chi$ is of order 2, are given by parity conditions on the cardinality of certain subsets of the set of places $S$ where the local representation is ramified. Most of these parity conditions are similar to those found in similar computations (for example, see [Han18, Han15, Kim01, Mg94]). However, to the best of the author’s knowledge, there is no previous example of a parity condition similar to the one in Equation (5.23).

It is worth noting that the residual spectrum of $H_E$, when $E$ is a field, was computed in [Lao16]. In particular, decomposing the square-integrable spectrum of $H_E$ with respect to cuspidal components, we get

$$L^2 (H_E (F) \backslash H_E (\mathbb{A})) = \bigoplus_{[M, \sigma]} L^2_{[M, \sigma]}, \tag{1.1}$$

where the sum goes over all pairs of a standard Levi subgroup $M$ and a cuspidal representation $\sigma$ of $M$, up to $W$-conjugation. The space $L^2_{[M, \sigma]}$ is the subspace of $L^2 (H_E (F) \backslash H_E (\mathbb{A}))$ spanned by automorphic forms with cuspidal data $[M, \sigma]$. 
We also write
\begin{equation}
L^2 (H_E (F) \backslash H_E (\mathbb{A})) = L^2_{\text{cusp}} \oplus L^2_{\text{res}} \oplus L^2_{\text{cont}},
\end{equation}
where $L^2_{\text{cusp}}$ denotes the cuspidal spectrum of $H_E (\mathbb{A})$, $L^2_{\text{res}}$ denotes its residual spectrum and $L^2_{\text{cont}}$ denotes the continuous spectrum. Let $L^2_{[M, \sigma], \text{res}}$ denote $L^2_{[M, \sigma]} \cap L^2_{\text{res}}$.

By the general theory of Eisenstein series, the degenerate residual spectrum, computed in Theorem 5.2, is contained in
\begin{equation}
\bigoplus_{\chi : F^\times \backslash \mathbb{A}^\times \to S^1} L^2_{[T_E, \mu_\chi], \text{res}},
\end{equation}
where $T_E$ is a maximal torus in $H_E$ and $\mu_\chi$ is a character of $T_E$ obtained by restriction of a character of the Levi subgroup of the Heisenberg parabolic subgroup of $H_E$; see Section 3 for more details. Note that $L^2_{[T_E, \mu_\chi], \text{res}}$ might be (0). A comparison of Theorem 5.2, when $E$ is a field, and the results of [Lao16, Theorem 5.15], shows that the square-integrable degenerate residual spectrum spans all of the space 1.3.

The non-square-integrable residual representations associated with $\mathcal{E}_E (\chi, f, s, g)$ are listed in Theorem 6.6. The non-square-integrable residual representation at $s = \frac{1}{2}$ was essentially computed in [GSb]. When $s = \frac{1}{2}$, a pole occurs when $E = F \times K$ and $\chi \circ \text{Nm}_{K/F} \equiv 1$. For any such $\chi$, the residual representation is shown to be irreducible, while the technique used for the proof varies for different $\chi$.

For $\chi = 1$, we prove a Siegel-Weyl-type identity between the residue of $\mathcal{E}_E (1, f, s, g)$ at $s = \frac{1}{2}$ and the special value of another Eisenstein series. This identity is especially interesting as the other Eisenstein series is evaluated on the unitary axis, which makes the associated degenerate principal series semi-simple.

When $\chi \neq 1$, the computation involves a detailed study of the images of certain intertwining operators, for which the results of the previous case are surprisingly useful.

This paper is structured as follows:

- Section 2 describes general notation and results used throughout this paper.
- In Section 3, the groups $H_E$ and the Eisenstein series $\mathcal{E}_E (\chi, f, s, g)$ are introduced. Also, the main results of [Seg] are summarized.
- Section 4 studies the irreducible quotients of the local degenerate principal series $I_\nu (\chi_\nu, s)$, associated with $\mathcal{E}_E (\chi, f, s, g)$. Some parts of the computation, for Archimedean places, are performed in Appendix A.

These irreducible quotients are then identified as eigenspaces of various intertwining operators acting on the maximal semi-simple quotient of $I_\nu (\chi_\nu, s)$.

- In Section 5, the square-integrable degenerate residual spectrum is computed. The results of this section are summarized, at its end, in Theorem 5.2.
- In Section 6, the non-square-integrable degenerate residual spectrum is computed. The results of this section are summarized, at its end, in Theorem 6.6.
- In Appendix A, we demonstrate the application of the software "atlas of lie groups" (ATLAS) for certain calculations in $I_\nu (\chi_\nu, s)$ at Archimedean places.
• In Appendix B a few complementary calculations for Section 6 are carried out.

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2. Background Theory

In this section we consider notations and preliminary results we use in this paper. For a comprehensive account on the theory of Eisenstein series consider [MW95]. Some of the facts discussed in this section are described in more detail in the survey in [Seg, Section 2].

2.1. Notation

Let $F$ denote a number field with a set of places $\mathcal{P}$ and a ring of adeles $\mathbb{A} = \mathbb{A}_F$.

Let $G$ be a quasi-split, simple group of relative rank $n$, defined over $F$ and let $B = T \cdot N$ denote a Borel subgroup of $G$ with maximal torus $T$ and unipotent radical $N$, all defined over $F$. Also denote by $S \subset T$ a maximal split torus defined over $F$. We let $\Phi = \Phi(G,S)$ denote the relative root system of $G$ with respect to $B$ with simple roots $\Delta = \Delta(G,S)$. For $\alpha \in \Phi$, we denote by $F_\alpha$ its field of definition. Let $\Phi^+$ denote the set of positive roots in $\Phi$ with respect to $B$.

Let $W = W(G,B)$ denote the Weyl group of $G$. The Weyl group $W$ is generated by the simple reflections $w_\alpha$ along the simple roots $\alpha \in \Delta$.

We recall the correspondence

$$\begin{align*}
\left\{ \text{Subsets of } \Delta \right\} & \leftrightarrow \left\{ \text{Standard parabolic subgroups of } G \right\} \\
\Psi \subset \Delta & \leftrightarrow \Psi \subset \Delta
\end{align*}$$

Furthermore, for $\Psi \subset \Delta$, we write $P_\Psi = M_\Psi \cdot U_\Psi$, where $U_\Psi$ denote the unipotent radical of $P_\Psi$ and $M_\Psi$ denotes its Levi subgroup. We also write $\Delta_M = \Psi$ for the set of simple roots of $M_\Psi$, $\Phi_M$ for the set of roots of $M_\Psi$ and $\Phi^+_M$ for the set of positive roots.

Fix a standard parabolic subgroup $P = P_\Psi$ of $G$, with $M = M_\Psi$ and $U = U_\Psi$, all defined over $F$. Let $a^{*}_{M,C} = X^*(M)_F \otimes \mathbb{C}$, where $X^*(M)_F$ denote the $F$-rational characters of $M$. Also, let $W_M = W(M,M \cap B)$ denote the relative Weyl group of $M$ and note that the quotient $W_M \backslash W$ is well defined; let $W(M,G)$ denote the set of shortest representatives of the cosets $W_M \backslash W$. 
For any $\alpha \in \Delta$, let $\omega_\alpha$ denote the fundamental weight associated to $\alpha$. The fundamental weights give rise to the isomorphism

$$\mathbb{C}^n \xrightarrow{\phi} a^*_{T,\mathbb{C}}$$

$$\bar{s} = (s_\alpha)_{\alpha \in \Delta} \mapsto \lambda_{\bar{s}} = \sum_{\alpha \in \Delta} s_\alpha \omega_\alpha.$$

Throughout, we denote the contragredient of a representation $\pi$ by $\pi^*$. Let $1_G$ denote the trivial representation of $G$ and if there is no source of confusion, it will simply be denoted by 1. Also, in any vector space $V$ over $\mathbb{C}$, we denote the zero vector by $\bar{0}$.

### 2.2. Characters on Levi Subgroups and their Restriction to the Torus

For a Levi subgroup $M$ of $G$, let $X_M$ denote the complex manifold of characters of $M(\mathbb{A})$ trivial on $M(\mathbb{F})$. There is a natural embedding of $a^*_{M,\mathbb{C}}$ in $X_M$. One can choose a direct sum complement $X_M = a^*_{M,\mathbb{C}} \oplus X_{M,0}$, where the characters in $X_{M,0}$ are of finite order.

We note that the restriction from $M$ to $T$ gives rise to natural embeddings

$$\iota_M : X_M \hookrightarrow X_T, \quad X^* (M) \hookrightarrow X^* (T), \quad a^*_{M,\mathbb{C}} \hookrightarrow a^*_{T,\mathbb{C}}.$$  

The image of these embeddings can be identified by restriction to $M^{der}$. Namely, for $\chi \in X_T$ it holds that $\chi \in \iota_M (X_M)$ if and only if $\langle \chi, \alpha^\vee \rangle = 0$ for all $\alpha \in \Delta_M$. Similarly for $X^* (M)$ and $a^*_{M,\mathbb{C}}$.

In particular, any element of $\lambda \in a^*_{M,\mathbb{C}}$ is of the form

$$\lambda = \sum_{\alpha \notin \Delta_M} s_\alpha \omega_\alpha,$$

where $s_\alpha \in \mathbb{C}$ and any $\chi \in X_M$ is of the form

$$\chi = \sum_{\alpha \notin \Delta_M} \chi_\alpha \circ \omega_\alpha,$$

where $\chi_\alpha \in X_{GL_1}$.

**Remark 2.1.** Given the split component $A_M$ of the center of $M$, we have $A_M \subset T$. In particular, the restriction yields surjective maps

$$r_M : X_T \twoheadrightarrow X_{A_M}, \quad X^* (T) \twoheadrightarrow X^* (A_M), \quad a^*_{T,\mathbb{C}} \twoheadrightarrow a^*_{A_M,\mathbb{C}}.$$  

In particular, since $a^*_{A_M,\mathbb{C}} \cong a^*_{M,\mathbb{C}}$, the composition of the maps

$$a^*_{M,\mathbb{C}} \xrightarrow{\iota_M} a^*_{T,\mathbb{C}} \xrightarrow{r_M} a^*_{M,\mathbb{C}}$$

is the identity map on $a^*_{M,\mathbb{C}}$. For more details, see [BJ08, pg. 3].
2.3. Degenerate Eisenstein Series

We fix a Hecke character \( \mu : M(F) \setminus M(\mathbb{A}) \to \mathbb{C}^\times \). We will usually assume that it is of finite order, i.e. a Dirichlet character. For \( \lambda \in \mathfrak{a}_{\mathbb{M}, \mathbb{C}}^* \), we consider the normalized parabolic induction

\[
I_P(\mu, \lambda) = \text{Ind}_{P(A)}(\mu \otimes \lambda).
\]

For a standard section \( f_\lambda \in I_P(\mu, \lambda) \) we form the associated Eisenstein series

\[
E_P(\mu, f, \lambda, g) = \sum_{\gamma \in \mathbb{P}(F) \setminus \mathbb{G}(F)} f_\lambda(\gamma g).
\]

This series converges for \( \Re(\lambda) \gg 0 \) and admits a meromorphic continuation to \( \mathfrak{a}_{\mathbb{M}, \mathbb{C}}^* \). More precisely, let

\[
\mathfrak{S}_M^+ = \{ \lambda \mid \Re\left(\langle \lambda - \rho_B, \alpha^\vee \rangle \right) > 0 \quad \forall \alpha \in \Phi^+ \setminus \Phi_M^+ \}.
\]

In particular, we write \( \mathfrak{S}_M^+ = \mathfrak{S}_T^+ \). The series on the right hand-side of Equation (2.2) converges if \( \iota_M(\lambda) \in \mathfrak{S}_M^+ \).

We note that the leading terms of this series are intertwining operators into the space of automorphic forms on \( \mathbb{G}(\mathbb{A}) \). Namely, they give an automorphic realization to a quotient of \( I(\mu, \lambda) \).

Denote the half-sum of the roots in \( \mathfrak{u} = \text{Lie}(U) \) by \( \rho_P \). It is known that

\[
I_P(\mu, \lambda) \hookrightarrow \text{Ind}_{B(A)}^{\mathbb{G}(A)}(\mu \otimes \lambda \otimes |\rho_P - \rho_B|)
\]

\[
\text{Ind}_{B(A)}^{\mathbb{G}(A)}(\mu \otimes \lambda \otimes |\rho_B - \rho_P|) \twoheadrightarrow I_P(\lambda, \mu),
\]

where we implicitly used the inclusions in Equation (2.1). Note that, under this inclusion, \( \rho_B - \rho_P = \rho_B \cap M \). We note that for any \( f \in I_P(\mu, \lambda) \) it holds that

\[
E_P(\mu, f, \lambda, g) = E_B(\mu, f, \lambda \otimes |\rho_P - \rho_B|, g).
\]

This is proven in Proposition 2.4 below.

Throughout this paper, we use the conventions for Dedekind \( \zeta \)-functions, Hecke \( L \)-functions and their \( \epsilon \)-factors, specified in [Seg][Sec. 3.2]. In particular, given a number field \( L \), \( \xi_L(s) \) denote the \( \zeta \)-function of \( L \) normalized so that it satisfies the functional equation \( \xi_L(1-s) = \xi_L(s) \) and \( L(s, \chi) \) denotes the Hecke \( L \)-function of \( \chi : L^\times \setminus \mathbb{A}_L^\times \to \mathbb{C}^\times \).

We also note that, if \( \mu = 1 \), we may drop it from our notation.

2.4. Intertwining Operators and the Constant Term Formula

For \( \lambda \in \mathfrak{a}_{\mathbb{M}, \mathbb{C}}^* \), a Hecke character \( \mu : T(F) \setminus T(\mathbb{A}) \to \mathbb{C}^\times \) and \( w \in W \) we consider the standard intertwining operator given by the integral

\[
M(w, \mu, \lambda) f_\lambda(g) = \int_{\mathbb{N}(\mathbb{A}) \cap w^{-1} \mathbb{N}(\mathbb{A}) w \setminus \mathbb{N}(\mathbb{A})} f_\lambda(wg) 
\]
This integral converges on the shifted positive Weyl chamber \( \mathfrak{F}^+ \) to a holomorphic family of operators and admits a meromorphic continuation to \( \mathfrak{a}^*_{T,C} \). At points of holomorphy, \( M(w, \mu, \lambda) \) defines an intertwining operator

\[
M(w, \mu, \lambda) : I_B(\mu, \lambda) \to I_B\left( w^{-1} \cdot \mu, w^{-1} \cdot \lambda \right).
\]

We note the following cocycle relation on the standard intertwining operators

**Lemma 2.2.** For any \( w, w' \in W \) we have

\[
M\left( w w', \mu, \lambda \right) = M\left( w', w^{-1} \cdot \mu, w^{-1} \cdot \lambda \right) \circ M\left( w, \mu, \lambda \right).
\]

**Remark 2.3.** Note that is an alternative definition of \( M(w, \mu, \lambda) \) which yields a different cocycle relation. Namely, for

\[
\tilde{M}(w, \mu, \lambda) f_\lambda(g) = \int_{N(\mathcal{A}) \cap wN(\mathcal{A})w^{-1}\backslash N(\mathcal{A})} f_\lambda(w^{-1}ug) \, du
\]

the following cocycle relation holds

\[
M\left( w w', \mu, \lambda \right) = M\left( w, w' \cdot \mu, w^{-1} \cdot \lambda \right) \circ M\left( w', \mu, \lambda \right).
\]

The constant term of \( \mathcal{E}_P(\mu, f, \lambda, g) \) along \( B \) is given by

\[
\mathcal{E}_P(\mu, f, \lambda, g)_{CT} = \int_{N(F) \backslash N(\mathcal{A})} \mathcal{E}_P(\mu, f, \lambda, ug) \, du.
\]

When restricted to \( T(\mathcal{A}) \), this is an automorphic form on \( T(\mathcal{A}) \); however, it is beneficial to consider this also as a function of \( G(\mathcal{A}) \); in particular

\[
f_\lambda \mapsto \mathcal{E}_P(\mu, f, \lambda, \cdot)_{CT}
\]

is a \( G(\mathcal{A}) \)-equivariant map.

The constant term formula, computed as in [GRS97], is given as follows:

\[
(2.6) \quad \mathcal{E}_P(\mu, f, \lambda, g)_{CT} = \sum_{w \in W(M, G)} M(w, \mu, \lambda) f_\lambda(g).
\]

A simple application of the constant term formula is the following useful well-known result.

**Proposition 2.4.** For any \( f \in I_P(\mu, \lambda) \) it holds that

\[
(2.7) \quad \mathcal{E}_P(\mu, f, \lambda, g) = \mathcal{E}_B(\mu, f, \lambda \otimes |\rho_P - \rho_B|, g).
\]

**Proof.** We recall the inclusion

\[
I_P(\mu, \lambda) \hookrightarrow \text{Ind}^{G(\mathcal{A})}_{B(\mathcal{A})} (\mu \otimes \lambda \otimes |\rho_P - \rho_B|),
\]

which follows from induction in parts. Namely,

\[
\text{Ind}^{G(\mathcal{A})}_{B(\mathcal{A})} (\mu \otimes \lambda \otimes |\rho_P - \rho_B|) = \text{Ind}^{G(\mathcal{A})}_{P(\mathcal{A})} \left( \text{Ind}^{M(\mathcal{A})}_{B(\mathcal{A}) \cap M(\mathcal{A})} |\rho_P - \rho_B| \right) \otimes \mu \otimes \lambda.
\]
Since the trivial representation of $M(\mathbb{A})$ is the unique irreducible subrepresentation of $\text{Ind}_{M(\mathbb{A})}^{M(\mathbb{A})} |\rho_P - \rho_B|$, it is the kernel of $M(w, \mu, \lambda)$ for any non-trivial $w \in W_M$.

This can be rephrased as follows. For any $w \not\in W(M, G)$ it holds that $M(w, \mu, \lambda) f = 0$. It follows that we have an equality of the constant terms of the two sides of Equation (2.7) since both are equal to

$$\sum_{w \in W(M, G)} M(w, \mu, \lambda) f_{\lambda}(g).$$

We consider the difference between the left and right-hand sides in Equation (2.7),

$$\mathcal{E}_P(\mu, f, \lambda, g) - \mathcal{E}_B(\mu, f, \lambda \otimes |\rho_P - \rho_B|, g).$$

By construction, the cuspidal support (see [MW95, pg. 38]) of both $I_P(\mu, \lambda)$ and $I_B(\mu, \lambda \otimes |\rho_P - \rho_B|)$ lies along $T$ and hence, the above difference of Eisenstein series is cuspidal. However, the cuspidal spectrum is orthogonal to Eisenstein series and hence

$$\mathcal{E}_P(\mu, f, \lambda, g) - \mathcal{E}_B(\mu, f, \lambda \otimes |\rho_P - \rho_B|, g) = 0.$$

□

Remark 2.5. One can similarly prove the following, closely related, fact. For any $f \in I_P(\mu, \lambda)$ there exists a section $\tilde{f} \in I_B(\mu, \lambda \otimes |\rho_P - \rho_B|)$ such that $M(w_{M,t}, \mu, \lambda) f = \tilde{f}$ and

$$\mathcal{E}_P(\mu, f, \lambda, g) = \left( \prod_{\alpha \in \Phi_M^+} (\langle \lambda, \alpha^\vee \rangle - 1) \right) \mathcal{E}_B(\mu, \tilde{f}, \lambda \otimes |\rho_B - \rho_P|, g).$$

2.5. Local and Global Intertwining Operators

For a Hecke character $\mu : M(F) \backslash M(\mathbb{A}) \to \mathbb{C}^\times$, we write

$$\mu = \bigotimes_{\nu \in \mathcal{P}}' \mu_\nu,$$

where $\mu_\nu : M(F_\nu) \to \mathbb{C}^\times$ is unramified for almost all $\nu \in \mathcal{P}$.

For a place $\nu \in \mathcal{P}$, we consider the degenerate principal series

$$I_{P,\nu}(\mu, \lambda) = \text{Ind}_{P(F_\nu)}^G(\mu_\nu \otimes \lambda)$$

of $G(F_\nu)$. The global degenerate principal series is a restricted tensor product of local ones. Namely,

$$I_P(\mu, \lambda) = \bigotimes_{\nu \in \mathcal{P}}' I_{P,\nu}(\mu, \lambda).$$

For a place $\nu \in \mathcal{P}$, $\lambda \in a_{T,C}^*$, a unitary character $\mu_\nu : T(F_\nu) \to \mathbb{C}^\times$, a section $f_{\lambda,\nu} \in I_B(\mu, \lambda)$ and $w \in W$ we consider the standard local intertwining operator given by the
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integral

\begin{align}
M_\nu (w, \mu_\nu, \lambda) f_{\lambda, \nu} (g) &= \int_{N(F_\nu) \cap w^{-1} N(F_\nu) \setminus N(F_\nu)} f_{\lambda, \nu} (wng) \, dn.
\end{align}

This integral converges absolutely to an analytic function in $\mathfrak{F}^+$ and admits a meromorphic continuation to $\mathfrak{a}_{T,C}^\ast$. Furthermore, it holds that the global intertwining operator decomposes into a restricted tensor product of local operators

\[
M (w, \mu, \lambda) = \bigotimes_{\nu \in \mathcal{P}} 'M_{\nu} (w, \mu_\nu, \lambda).
\]

Namely, given a pure tensor $f_\lambda = \bigotimes f_{\lambda, \nu}$ it holds that

\begin{align}
M (w, \mu, \lambda) f_\lambda &= \bigotimes_{\nu \in \mathcal{P}} 'M_{\nu} (w, \mu_\nu, \lambda) f_{\lambda, \nu}
\end{align}

2.6. Decomposition into Rank-1 Operators

The set of intertwining operators satisfies a cocycle condition. Namely, for any $w, w' \in W$ it holds that

\begin{align}
M (ww', \mu, \lambda) &= M (w', w^{-1} \cdot \mu, w^{-1} \cdot \lambda) \circ M (w, \mu, \lambda).
\end{align}

In particular, writing $w = w_{i_1} w_{i_2} \cdots w_{i_k}$, where the $w_{i_j}$ are simple reflections, it holds that

\begin{align}
M (w, \mu, \lambda) &= M \left( w_{i_k}, \left( w_{i_1} \cdots w_{i_{k-1}} \right)^{-1} \cdot \mu, \left( w_{i_1} \cdots w_{i_{k-1}} \right)^{-1} \cdot \lambda \right) \\
&\circ \cdots \circ M \left( w_{i_2}, w_{i_1}^{-1} \cdot \mu, w_{i_1}^{-1} \cdot \lambda \right) \circ M (w_{i_1}, \mu, \lambda)
\end{align}

This allows us to reduce many calculations to a sequence of calculations in rank-1.

Let $\mathcal{B} = T \cdot N$ be the Borel subgroup of $SL_2$ with torus $T$ and unipotent radical $N$. Also let $\dot{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ be a representative in $SL_2$ of the non-trivial element in the Weyl group of $SL_2$.

For any simple root $\alpha$, let $\iota_\alpha : SL_2 \to G$ denote the associated structure map.

**Lemma 2.6.** For a place $\nu \in \mathcal{P}$, the following diagram is commutative

\[
\begin{array}{ccc}
I_{\mathcal{B},\nu} (\mu, \lambda) & \xrightarrow{M_\nu (w, \mu_\nu, \lambda)} & I_{\mathcal{P},\nu} (w_{\alpha}^{-1} \cdot \mu, w_{\alpha}^{-1} \cdot \lambda) \\
\downarrow \iota_{\alpha} & & \downarrow \iota_{\alpha} \\
\text{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)} (\mu_\nu \otimes \lambda) \circ \iota_\alpha & \xrightarrow{M_\nu \circ \text{Ind}_{\mathcal{B}(F_\nu)}^{SL_2(F_\nu)} (w_{\alpha}^{-1} \cdot \mu_\nu \otimes \lambda) \circ \iota_\alpha} & \\
\end{array}
\]

where the vertical maps should be understood as the pull-back map.
2.7. Normalized Intertwining Operators

For \( \nu \in \mathcal{P} \), \( s \in \mathbb{C} \) and a unitary character \( \sigma_{\nu} : \mathcal{T} (F_{\nu}) \to \mathbb{C}^\times \), let \( f_{s,\nu}^0 \in \text{Ind}_{B(F_{\nu})}^{SL_2(F_{\nu})} (\sigma_{\nu} \otimes (s \cdot \rho_B)) \) denote the spherical vector normalized so that \( f_{s,\nu}^0 (1) = 1 \). The rank-one Gindikin-Karpelevich formula states that

\[
M (\tilde{w}, \sigma_{\nu}, s) f_{s,\nu}^0 = \frac{\mathcal{L}_{F_{\nu}} (s, \sigma_{\nu})}{\mathcal{L}_{F_{\nu}} (s + 1, \sigma_{\nu})} f_{-s,\nu}^0.
\]

This formula suggests a normalization of intertwining operators so that normalized spherical vectors are sent to normalized spherical vectors.

For a unitary character \( \mu_{\nu} : \mathcal{T} (F_{\nu}) \to \mathbb{C}^\times \) and \( \lambda \in \mathfrak{a}_{T,\mathbb{C}}^* \), define the normalized intertwining operator to be

\[
N_{\nu} (w, \mu_{\nu}, \lambda) = \prod_{\alpha > 0, \ w^{-1} \alpha < 0} \frac{\mathcal{L}_{F_{\alpha,\nu}} ((\lambda, \alpha^\vee) + 1, \mu_{\nu} \circ \alpha^\vee)}{\mathcal{L}_{F_{\alpha,\nu}} ((\lambda, \alpha^\vee), \mu_{\nu} \circ \alpha^\vee) \epsilon_{F_{\alpha,\nu}} ((\lambda, \alpha^\vee), \mu_{\nu} \circ \alpha^\vee, \psi_{\nu})} M (w, \mu_{\nu}, \lambda).
\]

The normalized intertwining operator satisfy a cocycle condition similar to Equation (2.11). Namely, for any \( w, w' \in W \), it holds that

\[
N_{\nu} (ww', \mu_{\nu}, \lambda) = N_{\nu} (w', w^{-1} \cdot \mu_{\nu}, w^{-1} \cdot \lambda) \circ N_{\nu} (w, \mu_{\nu}, \lambda).
\]

Recall that for \( \sigma_{\nu} \) unramified, it holds that \( \epsilon_{F_{\nu}} (s, \sigma_{\nu}, \psi_{\nu}) = 1 \). If \( \mu_{\nu} \) is unramified, let \( f_{\lambda,\nu}^0 \) be the spherical vector in \( I_{B,\nu} (\mu_{\nu}, \lambda) \) so that \( f_{\lambda,\nu}^0 (1) = 1 \). It follows from Equation (2.14), Equation (2.15) and Equation (2.13), that

\[
N_{\nu} (w, \mu, \lambda) f_{\lambda,\nu}^0 = f_{w^{-1} \cdot \lambda,\nu}^0.
\]

2.8. The Gindikin-Karpelevich Formula

For \( \lambda \in \mathfrak{a}_{T,\mathbb{C}}^* \) and a Hecke character \( \mu : T(F) \setminus T(\mathbb{A}) \to \mathbb{C}^\times \) we define the local Gindikin-Karpelevich factor to be

\[
J_{\nu} (w, \mu_{\nu}, \lambda) = \prod_{\alpha > 0, \ w^{-1} \alpha < 0} \frac{\mathcal{L}_{F_{\alpha,\nu}} ((\lambda, \alpha^\vee), \mu_{\nu} \circ \alpha^\vee)}{\mathcal{L}_{F_{\alpha,\nu}} ((\lambda, \alpha^\vee) + 1, \mu_{\nu} \circ \alpha^\vee)}
\]

and the global Gindikin-Karpelevich factor is defined by

\[
J (w, \mu, \lambda) = \prod_{\nu \in \mathcal{P}} J_{\nu} (w, \mu_{\nu}, \lambda) \prod_{\alpha > 0, \ w^{-1} \alpha < 0} \frac{\mathcal{L}_{F_{\alpha}} ((\lambda, \alpha^\vee), \mu \circ \alpha^\vee)}{\mathcal{L}_{F_{\alpha}} ((\lambda, \alpha^\vee) + 1, \mu \circ \alpha^\vee)}.
\]

Let \( f_\lambda = \otimes f_{\lambda,\nu} \) be a pure tensor in \( I_B (\mu, \lambda) \) and let \( S \subset \mathcal{P} \) be a finite set so that \( f_{\lambda,\nu} = f_{\lambda,\nu}^0 \) for all \( \nu \notin S \).
By Equation (2.10) and Equation (2.16), for any $w \in W$ and $\lambda \in a^*_\mathfrak{T}$, it holds that

$$
(2.19) M(w, \mu, \lambda) f_{\lambda} = \left( \bigotimes_{\nu \in S} M_{\nu}(w, \mu_{\nu}, \lambda) \right) \left( \bigotimes_{\nu \notin S} J_{\nu}(w, \mu_{\nu}, \lambda) f^0_{\lambda, \nu} \right)
$$

This implies that the analytic behaviour of $M(w, \mu, \lambda) f_{\lambda}$ depends on the analytic behaviour of $J(w, \mu, \lambda)$ and the $N_{\nu}(w, \mu_{\nu}, \lambda) f_{\lambda, \nu}$ for $\nu \notin S$.

Since the partially normalized intertwining operators

$$
\prod_{\alpha > 0, \ w^{-1} \alpha < 0} L_{F_{\alpha, \nu}}(\langle \lambda, \alpha^\vee \rangle, \mu_{\nu} \circ \alpha^\vee) M_{\nu}(w, \mu_{\nu}, \lambda)
$$

are entire for all $\nu \in \mathcal{P}$ (see [KS88] when $\nu \mid \infty$ and [Win78] when $\nu \nmid \infty$), it follows that $N_{\nu}(w_{\alpha}, \mu_{\nu}, \lambda)$ is holomorphic whenever $\Re(\langle \lambda, \alpha^\vee \rangle) > -1$ and $\alpha \in \Delta$.

### 2.9. Degenerate Eisenstein Series Attached to Maximal Parabolic Subgroups

Let $P = M \cdot U$ be the maximal parabolic subgroup of $G$ associated to the set $\Delta \setminus \{\alpha\}$. The space of characters $a^*_M$ is one-dimensional and we fix an isomorphism

$$
\mathbb{C} \overset{\sim}{\rightarrow} a^*_M,
$$

$$
s \mapsto \Omega_{P,s} = s \cdot \omega_{\alpha}.
$$

Also, for a Hecke character $\chi : F^\times \backslash A^\times \rightarrow \mathbb{C}^\times$, we denote

$$
\mu_{\chi} = \chi \circ \omega_{\alpha}.
$$

In what follows, we replace $\Omega_{P,s}$ by $s$ and $\mu_{\chi}$ by $\chi$ in all notations, e.g. $I_P(\chi, s) = I_P(\mu_{\chi}, \Omega_{P,s})$.

Revisiting Equation (2.2), for

$$
(2.20) \lambda_s = \Omega_{P,s} \otimes |\rho_P - \rho_B| \quad \eta_s = \Omega_{P,s} \otimes |\rho_B - \rho_P|,
$$

it holds that

$$
(2.21) I_P(\chi, s) \leftrightarrow I_B(\mu_{\chi} \otimes \lambda_s) \quad I_B(\mu_{\chi} \otimes \eta_s) \rightarrow I_P(\chi, s).
$$

In what follows, we write

$$
(2.22) \chi_s = \lambda_s \otimes \mu_{\chi}
$$
and $M(w, \chi_s)$ for the restriction of $M(w, \mu, \lambda_s)$ to $I_P(\chi, s)$.

**Lemma 2.7.** Let $f_s \in I_P(\chi, s)$ be a holomorphic section. Assume that $E_P(\chi, f, s)$ admits a pole of order $m$ at $s_0 \in \mathbb{C}$ and let

$$\varphi(g) = \lim_{s \to s_0} [(s - s_0)^m E_P(\chi, f, s, g)].$$

Let $\varphi_{CT}$ denote the constant term of $\varphi$ along $N$. Then, $\varphi \equiv 0$ if and only if $\varphi_{CT} \equiv 0$.

**Proof.** The proof here is similar to that of Proposition 2.4. It is enough to prove that if $\varphi_{CT} \equiv 0$ then $\varphi \equiv 0$. By the construction, the cuspidal support of $I_P(\chi, s)$ lies along $B$ and hence, $\varphi_{CT} \equiv 0$ implies that $\varphi$ is cuspidal. However, the cuspidal spectrum is orthogonal to Eisenstein series and hence $\varphi \equiv 0$.

\[\square\]

**Corollary 2.8.** Under the assumptions of the previous lemma,

$$\text{Span}_C \left\{ \lim_{s \to s_0} (s - s_0)^m E_P(\chi, f_s) \mid f_s \in I_P(\chi, s) \right\}$$

\[\text{(2.23)}\]

$$\cong \text{Span}_C \left\{ \lim_{s \to s_0} (s - s_0)^m \sum_{w \in W_M \setminus W} M(w, \chi_s) f_s \mid f_s \in I_P(\chi, s) \right\}$$

**Proof.** We consider two maps $R$ and $R_{CT}$ given by

$$I_P(\chi, s) \xrightarrow{R} \text{Span}_C \left\{ \lim_{s \to s_0} (s - s_0)^m E_P(\chi, f_s) \mid f_s \in I_P(\chi, s) \right\}$$

$$f_s \mapsto \varphi = \lim_{s \to s_0} (s - s_0)^m E_P(\chi, f_s)$$

$$I_P(\chi, s) \xrightarrow{R_{CT}} \text{Span}_C \left\{ \lim_{s \to s_0} (s - s_0)^m \sum_{w \in W_M \setminus W} M(w, \chi_s) f_s \mid f_s \in I_P(\chi, s) \right\}$$

$$f_s \mapsto \varphi_{CT} = \lim_{s \to s_0} (s - s_0)^m \sum_{w \in W_M \setminus W} M(w, \chi_s) f_s.$$  

Obviously, $R_{CT} = CT \circ R$ and by the previous lemma $\ker(R) = \ker(R_{CT})$. It follows that both sides of Equation (2.23) are isomorphic to

$$I_P(\chi, s)/\ker(R) \cong I_P(\chi, s)/\ker(R_{CT}).$$

\[\square\]

2.10. **The Constant Term Formula Revisited**

Let $P = M \cdot U$ be the maximal parabolic subgroup of $G$ associated to the set $\Delta \setminus \{\alpha\}$. We consider Equation (2.6) in light of Equation (2.19).

For $s_0 \in \mathbb{C}$ and a Hecke character $\chi : F^\times / \mathbb{A}^\times \to \mathbb{C}^\times$ let

$$n = \sup \{ \operatorname{ord}_{s=s_0} M(w, \chi, \lambda_s) f_s(g) \mid w \in W(M, G), \ f_s \in I_P(\chi, s), \ g \in G(A) \},$$
where the order \( \text{ord}_{s=s_0} h(s) \) of a pole of a complex function \( h(s) \) at \( s_0 \) is the unique integer \( n \) such that
\[
\lim_{s \to s_0} (s - s_0)^n h(s) \in \mathbb{C}^\times.
\]

By Equation (2.19), \( n \) is finite for \( s_0 > 0 \). We assume that \( n > 0 \).

For \( w \in W(M, G) \), we denote
\[
\text{ord}_{s=s_0} M(w, \chi_s) = \sup \{ \text{ord}_{s=s_0} M(w, \chi_s) f_s(g) \mid f_s \in I_P(\chi, s), g \in G(\mathbb{A}) \}.
\]

For \( 0 < m \leq n \) let
\[
\Sigma^P_{(\chi,s_0,m)} = \{ w \in W(M, G) \mid \text{ord}_{s=s_0} M(w, \chi_s) \geq m \}.
\]

We say that the pole of order \( m \) cancels if
\[
\lim_{s \to s_0} (s - s_0)^m \sum_{w \in W(M, G)} M(w, \chi_s) = \lim_{s \to s_0} (s - s_0)^m \sum_{w \in \Sigma^P_{(\chi,s_0,m)}} M(w, \chi_s) \equiv 0.
\]

After, maybe, cancellation of higher orders of a pole, we wish to determine its actual order. Namely, for \( 0 < m \leq n \) we say that \( E_P(\chi, \lambda_{s_0})_{CT} \) admits a pole of order \( m \) at \( s_0 \) if
\[
\lim_{s \to s_0} (s - s_0)^{m+1} \sum_{w \in W(M, G)} M(w, \chi_s) = \lim_{s \to s_0} (s - s_0)^{m+1} \sum_{w \in \Sigma^P_{(\chi,s_0,m+1)}} M(w, \chi_s) \equiv 0
\]

and
\[
\lim_{s \to s_0} (s - s_0)^m \sum_{w \in W(M, G)} M(w, \chi_s) = \lim_{s \to s_0} (s - s_0)^m \sum_{w \in \Sigma^P_{(\chi,s_0,m)}} M(w, \chi_s) \not\equiv 0.
\]

In particular, for any holomorphic section \( f_s \in I_P(\chi, s) \) and any \( t \in T_E(\mathbb{A}) \), it holds that
\[
\lim_{s \to s_0} (s - s_0)^m \sum_{w \in W(M, G)} M(w, \chi_s) f_s(t) \in \mathbb{C}
\]
and the limit is non-zero for some \( f_s \in I_P(\chi, s) \) and \( t \in T_E(\mathbb{A}) \).

We define an equivalence relation on \( \Sigma^P_{(\chi,s_0,m)} \) by:
\[
(2.24) \quad w \sim_{(\chi,s_0)} w' \iff w^{-1} \cdot (\chi_{s_0}) = w'^{-1} \cdot (\chi_{s_0}).
\]

Clearly, cancellations of poles of intertwining operators can occur only within the same equivalence class. It is thus useful to write the above sums as follows
\[
\lim_{s \to s_0} (s - s_0)^m \sum_{w \in W(M, G)} M(w, \chi_s)
\]
In view of Corollary 2.8, in order to calculate the kernel of \( \lim_{s \to s_0} [(s - s_0)^m E_{\mathbf{P}}(\chi, f, s, g)] \), it is enough to calculate the kernel of the above sum. Since cancellations happen only within equivalency classes, it follows that

\[
\ker \left( \lim_{s \to s_0} (s - s_0)^m E_{\mathbf{P}}(\chi, f, s, g) \right) = \bigcap_{[w^r] \in \Sigma_{(\chi, s_0, m)^/}} \left[ \lim_{s \to s_0} (s - s_0)^m \sum_{w \in [w^r]} M(w, \chi_s) \right].
\]

2.11. The Langlands Quotient Theorem

We recall the Langlands classification, which will be used in Section 4. For a more detailed discussion, the reader may consult [BW00, Chapter IV, Sec. XI.2]. We fix a place \( \nu \in \mathcal{P} \).

**Theorem 2.9** (Langlands’ Unique Irreducible Quotient Theorem). Let \( \mathbf{P} = \mathbf{M} \cdot \mathbf{U} \) be a Levi subgroup of \( \mathbf{G} \) and let \( \sigma_\nu \) be a tempered representation of \( \mathbf{M}(F_\nu) \). Also let \( \lambda \in a_{M, \mathbb{C}}^* \) satisfy

\[
\langle \lambda, \alpha^\vee \rangle > 0, \quad \forall \alpha \in \Delta_M.
\]

Then, \( \text{Ind}_{\mathbf{M}(F_\nu)}^{\mathbf{G}(F_\nu)}(\sigma_\nu \otimes \lambda) \) admits a unique irreducible quotient which is the image of \( M_\nu(w, \sigma_\nu, \lambda) \), where \( w \) is the representative in \( W(\mathbf{M}, \mathbf{G}) \) of the coset of the longest Weyl element \( w_l \in W \).

In this theorem, \( M_\nu(w, \sigma_\nu, \lambda) \) is constructed in a similar way to Equation (2.9). In particular, if \( \sigma \) is a subrepresentation of some \( \text{Ind}_{\mathbf{B}(F_\nu) \cap \mathbf{M}(F_\nu)}^{\mathbf{M}(F_\nu)} \chi \), then \( M_\nu(w, \sigma_\nu, \lambda) \) equals the restriction of \( M_\nu(w, \chi, \lambda) \).

In this case, we say that \( \text{Ind}_{\mathbf{M}(F_\nu)}^{\mathbf{G}(F_\nu)}(\sigma_\nu \otimes \lambda) \) is a standard module. In fact, Langlands have shown that every irreducible admissible representation of \( \mathbf{G}(F_\nu) \) can be attained as a Langlands quotients of some standard module and that the inducing data is unique up to conjugation.

**Remark 2.10.** We note that there is an equivalent form of this theorem in terms of subrepresentations (see [BJ08]). Assuming that \( \lambda \) satisfy

\[
\langle \lambda, \alpha^\vee \rangle < 0, \quad \forall \alpha \in \Delta_M,
\]

the induction \( \text{Ind}_{\mathbf{M}(F_\nu)}^{\mathbf{G}(F_\nu)}(\sigma_\nu \otimes \lambda) \) admits a unique irreducible subrepresentation. This subrepresentation is the kernel of \( M_\nu(w, \sigma, \lambda) \).

2.12. Harish-Chandra’s Commuting Algebra Theorem and the \( R \)-group

We recall Harish-Chandra’s commuting algebra theorem, the definition of the \( R \)-groups and a few properties of it. For further information, the reader may consult [KS80] when \( \nu | \infty \) and [Keys82] when \( \nu \not{|}_{\infty} \).
Let $\nu$ be a place of $F$.

**Theorem 2.11** (Harish-Chandra’s Commuting Algebra Theorem). For a unitary character $\mu : T(F) \to \mathbb{C}^\times$, the representation $I_{B,\nu}(\mu, 0)$ is semi-simple and, by Harish-Chandra’s commuting algebra theorem, its endomorphism ring $\text{End}(I_{B,\nu}(\mu, 0))$ is spanned by the intertwining operators $N_{\nu}(w, \mu, 0)$ such that $w \in \text{Stab}_W(\mu)$.

Let $\mathcal{K}_W(\mu) = \{w \in \text{Stab}_W(\mu) \mid N_{\nu}(w, \mu, 0) = c \cdot I, c \in \mathbb{C}\}$

and let $\mathcal{R}_W(\mu)$ denote the quotient group $\text{Stab}_W(\mu)/\mathcal{K}_W(\mu)$.

The following theorem summarizes some of the properties of $\mathcal{R}_W(\mu)$ and its connection to the structure of $I_{B,\nu}(\mu, 0)$. For more details, consider [KS80] when $\nu|\infty$ or [GK81, Tad92, Win78, Key82] when $\nu \not|\infty$. Essentially, it is an analogue of the Artin-Wedderburn theorem.

**Theorem 2.12.** The following hold:

1. The exact sequence
   \[ \{1\} \to \mathcal{K}_W(\mu) \to \text{Stab}_W(\mu) \to \mathcal{R}_W(\mu) \to \{1\} \]
   splits and $\text{Stab}_W(\mu) = \mathcal{R}_W(\mu) \rtimes \mathcal{K}_W(\mu)$.

2. $\text{End}(I_{B,\nu}(\mu, 0)) \cong \mathbb{C}[\mathcal{R}_W(\mu)]$.

3. $\dim_{\mathbb{C}}(\text{End}(I_{B,\nu}(\mu, 0))) = |\mathcal{R}_W(\mu)|$.

4. The number of inequivalent irreducible components in $I_{B,\nu}(\mu, 0)$ equals the number of conjugacy classes in $\mathcal{R}_W(\mu)$.

5. $I_{P,\nu}(\mu, 0)$ decomposes with multiplicities equal to 1 if and only if $\mathcal{R}_W(\mu)$ is Abelian.

6. If $\mathbb{C}[\mathcal{R}_W(\mu)] = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$, then $I_{B,\nu}(\mu, 0)$ admits $k$ inequivalent irreducible subrepresentations with multiplicities $n_1, \ldots, n_k$.

### 3. The Degenerate Eisenstein Series on A Quasi-Split Form of $\text{Spin}_8$

Associated to the Heisenberg Parabolic Subgroup

This section is a reminder of the groups and series relevant to this paper. For further details please consult [GH06] or [Seg, Section 2].

#### 3.1. Quasi-Split Forms of $\text{Spin}_8$

We recall the bijection

$$
\left\{\text{Quasi-split forms of } \text{Spin}_8 \text{ over } F\right\} \longleftrightarrow \left\{ \varphi : \text{Gal}(\overline{F}/F) \to S_3 \right\} \longleftrightarrow \left\{ \text{Isomorphism classes of } \text{étale cubic algebras over } F \right\}.
$$

For any étale cubic algebra $E$ over $F$ we denote by $H_E = \text{Spin}_8^E$ the corresponding simply-connected quasi-split form of $H = \text{Spin}_8$. We denote by $\varphi_E : \text{Gal}(\overline{F}/F) \to S_3$ the corresponding action of $\text{Gal}(\overline{F}/F)$ on the Dynkin diagram of type $D_4$. 

We note that $\varphi_E$ also fix a twisted form $S_E = \text{Aut}_F(E)$ of $S_3$.
Also recall that an étale cubic algebra over $F$ is one of the following:

1. $F \times F \times F$ (the split cubic algebra).
2. $F \times K$, where $K$ is a quadratic field extension of $F$.
3. $E$, where $E$ is a cubic Galois field extension of $F$.
4. $E$, where $E$ is a cubic non-Galois field extension of $F$.

We call the first three Galois étale cubic algebras.

Given an étale cubic algebra $E$ over $F$ we fix a Chevalley-Steinberg system of épinglage $[\text{BT84}, \text{Sections 4.1.3-4.1.4}]$

$\{T_E, B_E, x_\gamma : \mathbb{G}_a \to (H_E)_\gamma, \gamma \in \Phi_D \}$

where $T_E \subset B_E$ is a maximal torus contained in a Borel subgroup (both defined over $F$) and $\Phi_D$ are the roots of $H_E \otimes \overline{F} \simeq \text{Spin}_8(F)$. For any $\gamma$ in the reduced root system of $H_E$ we denote by $F_\gamma$ the field of definition of $\gamma$. Let $\Phi_E$ denote the root system of $H_E$ with respect to $T_E$. Also, let $\Phi_E^+$ denote the positive roots of $H_E$ with respect to $B_E$ and let $\Delta_E$ denote the set of simple roots. For a root $\gamma$, we denote by $F_\gamma$ the field of definition of $x_\gamma$.

Let $W = W_{H_E}$ denote the relative Weyl group of $H_E$ with respect to $T_E$ and denote by $w_l$ the longest element of $W$. For elements of the Weyl group and of $\mathfrak{a}_{T_E,\mathbb{C}}$, we use the notations introduced at the end of Section 2.1 of $[\text{Seg}]$. In particular, we denote by $\tilde{\alpha}_i$ the simple roots of the absolute root system while $\alpha_i$ denote the roots of relative root system. We denote by $\tilde{w}_i$ the simple reflection in the Weyl group of the absolute root system associated to $\tilde{\alpha}_i$ and by $w_i$ the simple reflection in the Weyl group of the relative root system associated to $\alpha_i$. Similarly, we denote by $\tilde{\omega}_{\tilde{\alpha}_i}$ the fundamental weights of the absolute root system of $H_E$ and by $\omega_{\alpha_i}$ the fundamental weights of the relative root system. Also, we write $\tilde{w}_{i_1\ldots i_k}$ or $w_{[i_1,\ldots,i_k]}$ for $\tilde{w}_{i_1}\cdots\tilde{w}_{i_k}$ and write $w_{i_1,\ldots,i_k}$ or $w [i_1\ldots i_k]$ for $w_{i_1}\cdots w_{i_k}$.

For $E$ non-split we have:

- For a cubic extension $E/F$ it holds that

$$w_1 = \tilde{w}_{134} = \tilde{w}_1 \tilde{w}_3 \tilde{w}_4, \quad w_2 = \tilde{w}_2,$$

$$\alpha_1 = \tilde{\alpha}_1 + \tilde{\alpha}_3 + \tilde{\alpha}_4, \quad \alpha_2 = \tilde{\alpha}_2$$

$$\omega_\alpha = \tilde{\omega}_{\tilde{\alpha}_1} + \tilde{\omega}_{\tilde{\alpha}_3} + \tilde{\omega}_{\tilde{\alpha}_4}, \quad \omega_\alpha = \tilde{\omega}_{\tilde{\alpha}_2}.$$
• In the case where $E = F \times K$, we always use the following convention

\[
\begin{align*}
  w_1 &= \tilde{w}_1, \quad w_2 = \tilde{w}_2, \quad w_3 = \tilde{w}_{34} = \tilde{w}_3 \tilde{w}_4 \\
  \alpha_1 &= \tilde{\alpha}_1, \quad \alpha_2 = \tilde{\alpha}_2, \quad \alpha_3 = \tilde{\alpha}_3 + \tilde{\alpha}_4 \\
  \omega_{\alpha_1} &= \tilde{\omega}_{\alpha_1}, \quad \omega_{\alpha_2} = \tilde{\omega}_{\alpha_2}, \quad \omega_{\alpha_3} = \tilde{\omega}_{\alpha_3} + \tilde{\omega}_{\alpha_4}.
\end{align*}
\]

Note that here we make a choice of a distinct simple root in the absolute root system, $\tilde{\alpha}_1$, in the construction of the relative root system. Any other choice of simple "external" root in $\Delta$ would induce a different action of $SE$ on the Dynkin diagram; this, in turn, corresponds to a different quasi-split form of $\text{Spin}_8$. However, all such groups are isomorphic (as algebraic groups); this is phenomenon is called \textit{triality}.

Denote by $w_0 \in W(M_E, T_E)$ the representative of the class of the longest element $w_l$; it holds that $w_l = \tilde{w}_{134} w_0$.

We denote by $P_E = M_E \cdot U_E$ the Heisenberg parabolic subgroup of $H_E$ which is the maximal parabolic subgroup given by $P_\Psi$, where $\Psi = \{\alpha_1, \alpha_3, \alpha_4\}_{SE}$. The Levi subgroup $M_E$ of $P_E$ is isomorphic to $\left(\text{Res}_{E/F} GL_2\right)^0 = \{g \in \text{Res}_{E/F} GL_2 \mid \det(g) \in \mathbb{G}_m\}$.

Associated to $M_E$ is a determinant map, $\det_{M_E} : M_E \to \mathbb{G}_m$. The unipotent radical $U_E$ of $P_E$ is a 9-dimensional Heisenberg group over $F$.

3.2. The Degenerate Eisenstein Series Associated to the Heisenberg Parabolic Subgroup

Given $s \in \mathbb{C}$ and a finite order character $\chi$ of $F^x \backslash \mathbb{A}_x$, we form the normalized parabolic induction

\[
I(\chi, s) = \text{Ind}_{P_E}^{H_E} (\chi \circ \det_{M_E}) \otimes |\det_{M_E}|^s.
\]

Let

\[
\begin{align*}
  \lambda_s &= \left( -1, s + \frac{3}{2}, -1, -1 \right) \\
  \eta_s &= \left( 1, s - \frac{3}{2}, 1, 1 \right) \\
  \mu_\chi &= \chi \circ \det_{M_E} \\
  \chi_s &= \mu_\chi \otimes \lambda_s,
\end{align*}
\]

where $\lambda_s, \eta_s \in \mathfrak{a}^*_E$ are written in the coordinates coming from the absolute root system. As in Equation (2.3), it holds that

\[
\begin{align*}
  \text{Ind}_{B_E}^{H_E} \mu_\chi \otimes \eta_s &\to I(\chi, s) \\
  I(\chi, s) \leftrightarrow \text{Ind}_{B_E}^{H_E} \mu_\chi \otimes \lambda_s &= I_{B_E}(\chi, s).
\end{align*}
\]
In particular, note that, $I(\chi, s)$ is the image (the leading term in the Laurent series) of $N_\nu(w^{134}, \mu_\chi, \eta_\chi)$ and the kernel of $N_\nu(w^{134}, \mu_\chi, \lambda_\chi)$.

We also note that we also write $N(w, \chi_\lambda)$ for $N(w, \lambda)$.

For a holomorphic section $f_s \in I(\chi, s)$ we define the degenerate Eisenstein series

$$\mathcal{E}_E(\chi, f, s, g) = \mathcal{E}_{P_E}(\mu_\chi, f, \lambda_\chi, g) = \sum_{\gamma \in P_E(F) \backslash H_E(F)} f_s(\gamma g).$$

To any Galois étale cubic algebra $E$, we associate a finite order Hecke character $\chi_E$ as follows:

- If $E$ is a Galois field extension of $F$, let $\chi_E$ denote the cubic character associated to it by global class field theory.
- If $E = F \times K$, where $K$ is a field, let $\chi_E = \chi_K$ be the quadratic character associated to $K/F$ by global class field theory.
- IF $E = F \times F \times F$ let $\chi = 1$.

We recall the order of the poles of $\mathcal{E}(\chi, f, s, g)$ in the half-plane $\mathfrak{R}(s) > 0$ given in Seg, Theorem 4.1.

**Theorem 3.1.** The degenerate Eisenstein series $\mathcal{E}(\chi, f, s, g)$ is holomorphic at $s = s_0$ except for the following poles:

- $s_0 = \frac{1}{2}$ and $\chi^2 = 1$, a simple pole.
- $s_0 = \frac{3}{2}$ and $\chi = \chi_E$ a simple pole for $E$ non-split and a double pole when $E = F \times F \times F$.
- $s_0 = \frac{1}{2}$ and $\chi = 1$ when $E = F \times K$ and $K$ is a field, a simple pole.
- $s_0 = \frac{5}{2}$ and $\chi = 1$, a simple pole.

Further more, the residual representations at these points are square-integrable except for the following cases:

- $s_0 = \frac{1}{2}$, $E = F \times K$ and $\chi \circ \text{Nm}_{K/F} = 1$ (Here $K$ is either a field or split).
- $s_0 = \frac{3}{2}$, $\chi = 1$ and $E = F \times K$ is non-split.

Let $E$ be an étale cubic algebra over $F$, $\chi : F^\times \setminus \mathbb{A}^\times \to \mathbb{C}^\times$ a quadratic character and $s_0 \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\}$ such that $\mathcal{E}_E(\chi, f_s, s, g)$ admits a pole of order $n$ at $s_0$. We define the residue of $\mathcal{E}_E(\chi, f_s, s, g)$ at $s_0$ to be

$$\text{Res}(s_0, \chi, E) = \text{Span}_C \left\{ \lim_{s \to s_0} (s - s_0)^n \mathcal{E}_E(\chi, f_s, s, \cdot) \mid f_s \in I_{P_E}(\chi, s) \right\}.$$

In this paper we study $\text{Res}(s_0, \chi, E)$ for the various triples $(s_0, \chi, E)$. The results are summarized in Theorem 5.2 and Theorem 6.0.

**4. Local Degenerate Principal Series**

Write $I(\chi, s) = \otimes' I_\nu(\chi_\nu, s)$, where

$$I_\nu(\chi_\nu, s) = \text{Ind}_{P_{\E(F_\nu)}}^{H_{\E(F_\nu)}} (\chi \circ \text{det}_{M_E}) \otimes |\text{det}_{M_E}|_{F_\nu}^{s_0}.$$
In this section, we discuss the structure of \( I_\nu (\chi_\nu, s) \) in cases where \( \mathcal{E} (\chi, f, s, g) \) admits a pole. We also discuss the behaviour of these representations under various intertwining operators of importance to the calculations performed in Section 5 and Section 6. We record the structure of those local representations in the following theorem.

**Theorem 4.1.**

1. For any place \( \nu \), the representation \( I_\nu (\chi_\nu, \frac{5}{2}) \) admits a unique irreducible quotient which is trivial.
2. For any place \( \nu \) such that \( E_\nu \) is a Galois étale cubic algebra, the representation \( I_\nu (\chi_\nu, \frac{3}{2}) \) admits a unique irreducible quotient which is the minimal representation of \( H_E (F_\nu) \).
3. For any place \( \nu \) such that \( E_\nu \) is a field, the representation \( I_\nu (\chi_\nu, \frac{1}{2}) \) admits a unique irreducible quotient which is spherical.
4. For any place \( \nu \) such that \( E_\nu \) is not a field, the representation \( I_\nu (\chi_\nu, \frac{1}{2}) \) admits a maximal semi-simple quotient of the form \( \pi_1 \oplus \pi_{-2} \), where \( \pi_1 \) is spherical and both irreducible quotients are eigenspaces of certain intertwining operators (of eigenvalues 1 and \(-2\)).

The proof of this theorem will occupy the rest of this section, while some calculations performed at Archimedean places are performed in Appendix A.

**Remark 4.2.** In all cases, we shall identify the irreducible quotients of the various \( I_\nu (\chi_\nu, s) \) as eigenspaces of certain normalized intertwining operators. We will denote the irreducible quotients of \( I_\nu (\chi_\nu, s_0) \) by \( \pi_{\epsilon, \nu}^{(\chi_\nu, E_\nu, s_0)} \), where \( \epsilon \) is this eigenvalue, or tuple of eigenvalues. Usually, when there is no source of confusion, we will denote \( \pi_{\epsilon, \nu}^{(\chi_\nu, E_\nu, s_0)} \) by \( \pi_{\epsilon, \nu} \).

**Proof.** We consider the various possible \( s \), \( \chi \) and \( E \).

#### 4.1. \( s = \frac{5}{2} \) and \( \chi_\nu = 1_\nu \)

In this case, \( I_\nu (1, \frac{5}{2}) \) is a standard module with Langlands operator \( N_\nu \left( w_0, 1, \lambda_{\frac{5}{2}} \right) \). Since \( I_\nu (1, -\frac{5}{2}) \cong I_\nu (1, \frac{5}{2})^* \) admits \( 1_{H_E} \) as a subrepresentation, the unique irreducible quotient of \( I_\nu (1, \frac{5}{2}) \) is isomorphic to the \( 1_{H_E} \).
4.2. $s = \frac{3}{2}$ and $\chi_{\nu} = \chi_{E,\nu}$

We recall, from [GJ102, Proposition 4.3], that, for any $\nu \in \mathcal{P}$, the unique irreducible quotient of $I_{\nu}(\chi_{E,\nu}, \frac{3}{2})$ is the minimal representation $\Pi_{E,\nu}$ of $H_{E}(F_{\nu})$.

4.3. $s = \frac{3}{2}$ and $\chi_{\nu} = 1_{\nu}$

We assume that $E_{\nu} = F_{\nu} \times K_{\nu}$, where $K_{\nu}$ is a quadratic field extension of $F_{\nu}$. It follows from [GS5, Theorem 3.3(1)] that $I_{\nu}(1, \frac{3}{2})$ admits a unique irreducible quotient (which is spherical).

4.4. $s = \frac{1}{2}$ and $\chi_{\nu} = 1_{\nu}$

Fix a place $\nu \in \mathcal{P}$. We start by recalling from [NSS] that

$$\text{Ind}_{B_{E}(F_{\nu})}^{H_{E}(F_{\nu})} \lambda(-1,1,-1,-1) = \Pi_{-2,\nu} \oplus \Pi_{1,\nu},$$

where:

- $\Pi_{1,\nu} = \text{Ind}_{P_{(\alpha_{2})}(F_{\nu})}^{H_{E}(F_{\nu})} \lambda_{0}$, where $\lambda_{0} \in a_{M(\alpha_{2})}^{*}$ satisfy that, $\iota_{M}(\lambda_{0}) = \lambda(-\frac{1}{2},0,-\frac{1}{2},-\frac{1}{2})$, in terms of Equation 2.1.
- $\Pi_{-2,\nu} = \text{Ind}_{P_{(\alpha_{2})}(F_{\nu})}^{H_{E}(F_{\nu})} \text{St}_{M(\alpha_{2})} \otimes \lambda_{0}$, where $\text{St}_{M(\alpha_{2})}$ is the Steinberg representation of $M(\alpha_{2})$.
- Each of the representations $\Pi_{\nu}$ is the $e$-eigenspace of $\lim_{n \to \frac{1}{2}} N_{\nu}(w_{21342134}, 1, \lambda(-1,1,-1,-1))$ and each of them admits a unique irreducible subrepresentation.

It follows that

$$\text{Ind}_{B_{E}(F_{\nu})}^{H_{E}(F_{\nu})} \lambda(1,1,1,1) = \left[\text{Ind}_{B_{E}(F_{\nu})}^{H_{E}(F_{\nu})} \lambda(-1,1,-1,-1)\right]^{*} = \Pi_{-2,\nu}^{*} \oplus \Pi_{1,\nu}^{*},$$

and that both $\Pi_{1,\nu}^{*}$ and $\Pi_{-2,\nu}^{*}$ admit unique irreducible quotients, denoted by $\pi_{1}$ and $\pi_{-2}$ respectively. Note that $\pi_{1}$ is a spherical representation.

We also note that:

- $\lambda(-1,1,-1,1) = \lambda_{-\frac{1}{2}} = w_{2134}^{-1} \cdot \lambda_{\frac{1}{2}} = w_{21342134}^{-1} \cdot \lambda_{\frac{1}{2}}$.
- $\lambda(1,-1,1,1) = \eta_{\frac{1}{2}}$.

Hence, the maximal semi-simple quotient of $I_{\nu}(1, \frac{1}{2})$ is a quotient of $I_{B_{E,\nu}}(1, \eta_{\frac{1}{2}})$. Since $I_{\nu}(1, \frac{1}{2})$ is spherical, its maximal semi-simple quotient is either $\pi_{1,\nu}$ or $\pi_{1,\nu} \oplus \pi_{-2,\nu}$.

From the calculations in [NSS], it also follows that the maximal semi-simple quotient of $I_{\nu}(1, \frac{1}{2})$ is given by $N_{\nu}(w_{2134}, 1, \lambda_{\frac{1}{2}})$ ($I_{\nu}(1, \frac{1}{2})$).

Finally, it holds that:

**Lemma 4.3.** The maximal semi-simple quotient of $I_{\nu}(1, \frac{1}{2})$ is given as follows:

- If $E_{\nu}$ is not a field then $\pi_{1,\nu}$ is the unique irreducible quotient of $I_{\nu}(1, \frac{1}{2})$.
- If $E_{\nu}$ is a field then the maximal semi-simple quotient of $I_{\nu}(1, \frac{1}{2})$ is $\pi_{1,\nu} \oplus \pi_{-2,\nu}$. 

Proof. For $\nu|\infty$ it follows from Appendix $\textit{A}$ for $\nu /\infty$ it follows from $\textit{GSa}$; however, we supply a different proof.

We show that $I_\nu (1, -\frac{1}{2})$ has a semi-simple subrepresentation of length 2 if $E_\nu$ is a field and a unique irreducible subrepresentation if $E_\nu = F_\nu \times K_\nu$.

- Assume that $E_\nu$ is a field. In this case, a straight forward computation of Jacquet modules along $B_E$ yields that the multiplicity of $(-1,1,-1,-1)$:
  - The multiplicity of $(-1,1,-1,-1)$ in $J^{H_E}_{TE}(\text{Ind}^{H_E}_{B_E} \chi_\nu)$ is 2.
  - The multiplicity of $(-1,1,-1,-1)$ in $J^{H_E}_{TE}(I_\nu (1, -\frac{1}{2}))$ is 2.
  - The multiplicity of $(-1,1,-1,-1)$ in $J^{H_E}_{TE} (\pi_1,\nu)$ and $J^{H_E}_{TE} (\pi_2,\nu)$ is 1.

Since both $\pi_1,\nu$ and $\pi_2,\nu$ are constituents of $\text{Ind}^{H_E}_{B_E} \chi_\nu$, the claim follows.

- Assume that $E_\nu = F_\nu \times K_\nu$. In particular, we assume that $\alpha_1$ is defined over $F_\nu$.

Let $\Omega$ denote the 1-dimensional representation of $E_\nu$ given by the Jacquet functor $J^{H_E}_{M(\alpha_1)} \left( \frac{1}{2} \right)$.

Note that, by taking Jacquet module in stages, $\mathcal{J}^{M(\alpha_1)}_{TE}(\Omega) = (-1,1,-1,-1)$.

The Levi subgroup $M_{\alpha_1,\alpha_2}$ is isomorphic to $GL_3 \times \text{Res}_{K/F} (GL_1)$ and hence the (normalized) induction $\text{Ind}^{M(\alpha_1,\alpha_2)}_{M(\alpha_1)} \Omega$ is irreducible. It follows that

$$I_\nu (1, -\frac{1}{2}) \subseteq \text{Ind}^{G_E}_{P_E} \left( \text{Ind}^{P_E}_{P(\alpha_1)} \Omega \right)$$

$$\cong \text{Ind}^{H_E}_{P(\alpha_1)} \Omega$$

$$\cong \text{Ind}^{H_E}_{P(\alpha_1,\alpha_2)} \left( \text{Ind}^{M(\alpha_1,\alpha_2)}_{M(\alpha_1)} \Omega \right)$$

$$\subseteq \text{Ind}^{H_E}_{P(\alpha_1,\alpha_2)} \left( \text{Ind}^{M(\alpha_1,\alpha_2)}_{TE} \chi_{\lambda(0,-1,0,0)} \right)$$

Since $\text{Ind}^{H_E}_{TE} \chi_{\lambda(0,-1,0,0)}$ admits $\pi_1,\nu$ as its unique irreducible subrepresentation, the claim follows.

$\square$

4.5. $s = \frac{1}{2}$ and $\chi_{\nu}^2 = 1_\nu \neq \chi_{\nu}$

We first note that

$$I_{B_{E,\nu}} (\mu_{\lambda_{\nu}}, \eta_{\frac{1}{2}}) \cong I_{B_{E,\nu}} (\chi_{\nu} \circ (\tilde{\omega}_{\alpha_1} + \tilde{w}_{\alpha_2} + \tilde{w}_{\alpha_3} + \tilde{\omega}_{\alpha_4}) + \lambda_{(0,1,0,0)}, \lambda_{(0,1,0,0)})$$

where the isomorphism is given by $N_{\nu} (w_2, \mu_{\lambda_{\nu}}, \eta_{\frac{1}{2}})$.

By induction in stages, we have

$$I_{B_{E,\nu}} (\chi_{\nu} \circ (\tilde{w}_{\alpha_1} + \tilde{w}_{\alpha_2} + \tilde{w}_{\alpha_3} + \tilde{\omega}_{\alpha_4}) + \lambda_{(0,1,0,0)}, \lambda_{(0,1,0,0)}) =$$
We first consider the inner induction. Write $\tilde{\chi} = \chi_\nu \circ (\tilde{\omega}_{\alpha_1} + \tilde{\omega}_{\alpha_2} + \tilde{\omega}_{\alpha_3} + \tilde{\omega}_{\alpha_4})$.

As $\tilde{\chi}$ is unitary, Ind$_{E(F_\nu)}^{M_E(F_\nu) \cap M_E(F_\nu)} \tilde{\chi}$ is semi-simple and its structure could be understood via its $R$-group (see Subsection 2.12).

We first note that $Stab_{W}(\tilde{\chi}) \subset W_{M_E}$. Also, since any element in $W_{M_E}$ is an involution, for any $w \in Stab_{W}(\tilde{\chi})$, the space Ind$_{E(F_\nu)}^{M_E(F_\nu) \cap M_E(F_\nu)} \tilde{\chi}$ decompose into a direct sum of eigenspaces of $N(w, \tilde{\chi}, 0)$ with eigenvalues 1 and $-1$. While these eigenspaces are subrepresentations, they are not necessarily irreducible.

We calculate $Stab_{W}(\tilde{\chi}) \subset W_{M_E}$ and $R_{W_{M_E}}(\tilde{\chi})$ for the various $E_\nu$ and $\chi_\nu$.

- If $E_\nu$ is a field, then $Stab_{W_{M_E}}(\tilde{\chi}) = \{1\}$. It follows that $R_{W_{M_E}}(\tilde{\chi}) = \{1\}$. In particular, Ind$_{E(F_\nu)}^{M_E(F_\nu) \cap M_E(F_\nu)} (\chi_\nu)$ is an isomorphism, but is not an endomorphism.
- If $E_\nu = F_\nu \times K_\nu$, where $K_\nu$ is a field, then $Stab_{W_{M_E}}(\tilde{\chi}) = \{1, w_3\}$.
  * If $\chi_\nu \circ \text{Nm}_{E_\nu/F_\nu} = 1$, then $N_\nu(w_3, \tilde{\chi}, 0) = 1$. It follows that $R_{W_{M_E}}(\tilde{\chi}) = \{1\}$.
  * If $\chi_\nu \circ \text{Nm}_{E_\nu/F_\nu} \neq 1$, then $N_\nu(w_3, \tilde{\chi}, 0)$ is not a constant multiple. In fact, in this case, we have a decomposition Ind$_{E(F_\nu)}^{M_E(F_\nu) \cap M_E(F_\nu)} (\tilde{\chi}) = \sigma^{(-1)}_\nu \oplus \sigma^{(1)}_\nu$, where the irreducible subrepresentation $\sigma^{(c)}$ is the $c$-eigenspace of $N_\nu(w_3, \tilde{\chi}, 0)$. It follows that $R_{W_{M_E}}(\tilde{\chi}) = \{1, w_3\}$.

We note that $N_\nu(w_1, \tilde{\chi}, 0)$ acts as an isomorphism, but is not an endomorphism.
- If $E_\nu = F_\nu \times F_\nu \times F_\nu$, Stab$_{W_{M_E}}(\tilde{\chi}) = \{w_{13}, w_{14}, w_{34}\}$.
  
  We note that each of the intertwining operators $N_\nu(w_{13}, \tilde{\chi}, 0)$, $N_\nu(w_{14}, \tilde{\chi}, 0)$ and $N_\nu(w_{34}, \tilde{\chi}, 0)$ is not a constant multiple; namely, its $-1$-eigenspace is non-zero.

Indeed, by Lemma 2.5, any one of these intertwining operators factors through a subgroup $SL_2(F_\nu) \times SL_2(F_\nu) \times GL_2(F_\nu)$ of $M_E(F_\nu)$ associated to it. The decomposition in this case is a well known fact.

It follows that $R_{W_{M_E}}(\tilde{\chi}) = \{1, w_{13}, w_{14}, w_{34}\}$.

Since $R_{W_{M_E}}(\tilde{\chi})$ has 4 conjugacy classes, it follows that $I_{\nu}^{(\mu, \tilde{0})}$ admits 4 inequivalent components. We write Ind$_{E(F_\nu)}^{M_E(F_\nu) \cap M_E(F_\nu)} (\tilde{\chi}) = \sigma^{(-1)}_\nu \oplus \sigma^{(1)}_\nu \oplus \sigma^{(-1)}_\nu \oplus \sigma^{(1)}_\nu$, where:

- $\sigma^{(c)}_\nu$ is a $\nu$-eigenspace for $N_\nu(w_{13}, \tilde{\chi}, 0)$.
- $\sigma^{(c)}_\nu$ is a $\nu$-eigenspace for $N_\nu(w_{14}, \tilde{\chi}, 0)$.
- $\sigma^{(c)}_\nu$ is a $(\epsilon, \delta)$-eigenspace for $N_\nu(w_{34}, \tilde{\chi}, 0)$.

We note that all of $N_\nu(w_{13}, \tilde{\chi}, 0)$, $N_\nu(w_{14}, \tilde{\chi}, 0)$ and $N_\nu(w_{34}, \tilde{\chi}, 0)$ are isomorphisms, but are not endomorphisms.

Another way to view this decomposition, in the spirit of [GK81], is by restriction to $SL\big(2(F_\nu)\big) \times SL\big(2(F_\nu)\big) \times SL\big(2(F_\nu)\big) \subset M_E(F_\nu)$. Write Ind$_{E(F_\nu)}^{SL\big(2(F_\nu)\big)} (\tilde{\chi}) = \sigma_1 \oplus \sigma_{-1}$. Then
we have
\[
\text{Ind}_{[B \times B \times B]}^{[SL_2 \times SL_2 \times SL_2]}(F_\nu)(\chi_\nu \otimes \chi_\nu \otimes \chi_\nu) = \bigoplus_{\epsilon_1, \epsilon_3, \epsilon_4 \in \{1, -1\}} \sigma_{(\epsilon_1, \epsilon_3, \epsilon_4)},
\]
where
\[
\sigma_{(\epsilon_1, \epsilon_3, \epsilon_4)} = \sigma_{\epsilon_1} \boxtimes \sigma_{\epsilon_3} \boxtimes \sigma_{\epsilon_4}.
\]
By applying \(h_{\alpha_2}(\varpi) \in M_E\), we get
\[
\begin{align*}
\sigma_{\nu}^{(1,1)} &= \sigma_{(1,1,1)} \oplus \sigma_{(-1,1,1)} \\
\sigma_{\nu}^{(1,-1)} &= \sigma_{(1,1,-1)} \oplus \sigma_{(-1,1,-1)} \\
\sigma_{\nu}^{(-1,1)} &= \sigma_{(1,-1,1)} \oplus \sigma_{(-1,-1,1)} \\
\sigma_{\nu}^{(-1,-1)} &= \sigma_{(-1,1,1)} \oplus \sigma_{(1,-1,1)}.
\end{align*}
\]

Note that, for any \(E_\nu\) and \(\chi_\nu\), any component \(\sigma\) of \(\text{Ind}_{B_\nu}^{M_\nu} \tilde{\chi}\) is tempered. By the Langlands Quotient Theorem, \(\text{Ind}_{B_\nu}^{H_\nu} |_{\sigma \otimes \omega_{\alpha_2}}\) admits a unique irreducible quotient; this quotient is the image of \(N_\nu (w_0, w_2^{-1} \cdot \mu_\chi, \lambda_{(0,1,0,0)})\).

It follows that the maximal semi-simple quotient of \(IB_{E, \nu} (w_2^{-1} \cdot \mu_\chi, \lambda_{(0,1,0,0)})\) is given by
\[
\begin{cases}
\pi_{1,\nu}, & \text{if } E_\nu \text{ is a field or } \chi_\nu \circ \text{Nm}_{E_\nu / F_\nu} = 1, \\
\pi_{1,\nu} \oplus \pi_{-1,\nu}, & \text{if } E_\nu = F_\nu \times K_\nu, \text{ where } K_\nu \text{ is a field and } \chi_\nu \circ \text{Nm}_{E_\nu / F_\nu} \neq 1,
\end{cases}
\]
(4.1)
where \(\pi_{\epsilon,\nu}\) is the unique irreducible quotient of \(\text{Ind}_{F_\nu}^{H_\nu} |_{\sigma_{\nu}^\epsilon \otimes \omega_{\alpha_2}}\). Furthermore, this maximal semi-simple quotient is the image of \(N_\nu (w_0, \chi, \omega_{\alpha_2})\).

By Equation (2.13), we have
\[
N_\nu (w_{213421342} \cdot \mu_\chi, \omega_{\alpha_2}) = N_\nu (w_{213421342} \cdot \mu_\chi, \lambda_{\frac{1}{2}}) \circ N_\nu (w_{134} \cdot \mu_\chi, \eta_{\frac{1}{2}}) \circ N_\nu (w_{2} \cdot \mu_\chi, \omega_{\alpha_2}).
\]
Since \(N_\nu (w_2 \cdot \mu_\chi, \omega_{\alpha_2})\) is an isomorphism and the image of \(N_\nu (w_{134} \cdot \mu_\chi, \eta_{\frac{1}{2}})\) is \(I_\nu (\chi_\mu, \frac{1}{2})\), it follows that the image of \(N_\nu (w_{213421342} \cdot \chi, \omega_{\alpha_2})\) is a quotient of \(I_\nu (\chi_\mu, \frac{1}{2})\). In conclusion, Equation (4.1) is the maximal semi-simple quotient of \(I_\nu (\chi_\mu, \frac{1}{2})\).

We note that, by Lemma 2.6 and by the discussion above, the \(\pi_{\epsilon,\nu}\) are subrepresentations of \(IB_{E, \nu} (w_2 \cdot \mu_\chi, -\omega_2)\) and as such, they are eigenspaces of the intertwining operators mentioned above, in the following sense:

- If \(E_\nu = F_\nu \times K_\nu\), where \(K_\nu\) is a field, then \(\pi_{\epsilon,\nu}\) is an \(\epsilon\)-eigenspace of \(N_\nu (w_3, w_2 \cdot \mu_\chi, -\omega_2)\).
- If \(E_\nu = F_\nu \times F_\nu \times F_\nu\), then
  - \(\pi_{(\epsilon,\delta),\nu}\) is a \(\epsilon\)-eigenspace for \(N_\nu (w_{13}, w_2 \cdot \mu_\chi, -\omega_2)\).
  - \(\pi_{(\epsilon,\delta),\nu}\) is a \(\delta\)-eigenspace for \(N_\nu (w_{14}, w_2 \cdot \mu_\chi, -\omega_2)\).
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$-\pi_{(\varepsilon,\delta),\nu}$ is a $(\varepsilon \cdot \delta)$-eigenspace for $N_\nu(w_{34}, w_2 \cdot \mu_{\chi_\nu}, -\omega_2)$.

**Remark 4.4.** In the case $E_\nu = F_\nu \times F_\nu \times F_\nu$, we denote $\pi_{1,\nu} = \pi_{(1,1),\nu}$. For all cases, if $\chi_\nu$ is unramified, then $\pi_{1,\nu}$ is unramified.

□

5. Square Integrable Residues

We compute the square-integrable residues $\text{Res}(s_0, \chi, E)$ for the various values of $s_0$ and $\chi$ separately. The crux of the computations of the square integrable residues is that they decompose as a direct sum of irreducible representations. Namely, one can write

$$\text{Res}(s_0, \chi, E) = \hat{\oplus} \sigma_i,$$

where $\sigma_i$ are irreducible quotients of $I_{P_E}(\chi, s_0)$. In particular, if, using Flath’s theorem, we write $\sigma_i = \otimes_{\nu} \sigma_\nu$ then $\sigma_\nu$ is an irreducible quotient of $I_\nu(\chi_\nu, s_0)$, unramified for almost all $\nu \in \mathcal{P}$.

For each place $\nu$ denote by $\Sigma_\nu(s_0, \chi_E)$ the (finite) set of irreducible quotients of $I_\nu(\chi_\nu, s_0)$ described in Theorem 4.1.

**Remark 5.1.** Note that in all cases, the maximal semi-simple quotient of $I_\nu(\chi_\nu, s_0)$ is a direct sum of inequivalent representations.

It follows that

$$\text{Res}(s_0, \chi, E) = \hat{\oplus} \sigma_i \subset \bigotimes' \left( \bigoplus_{\pi_\nu \in \Sigma_\nu} \pi_\nu \right).$$

Hence, in order to compute $\text{Res}(s_0, \chi, E)$, it is enough to check which of the direct summands in the right hand side is realized by the residue of $\mathcal{E}_E(\chi, f_s, s, \cdot)$. Namely, assuming the pole of $\mathcal{E}_E(\chi, f_s, s, \cdot)$ at $s_0$ is of order $n$, $\text{Res}(s_0, \chi, E)$ is the direct sum of all $\pi = \otimes' \pi_\nu$, where $\pi_\nu \in \Sigma_\nu$ is spherical for almost all $\nu$, and appears in the image of

$$\lim_{s \to s_0} (s - s_0)^n \mathcal{E}_E(\chi, f_s, s, \cdot).$$

In order to determine which $\pi = \otimes' \pi_\nu$ appears in the image it is enough to check that

$$\lim_{s \to s_0} (s - s_0)^n \mathcal{E}_E(\chi, f_s, s, \cdot) \neq 0$$

for some $f_s \in I_{P_E} (\chi, s_0)$ whose image in $\pi$, under the quotient map, is non-zero. In order to do this, we apply Corollary 2.8 and show that

$$\lim_{s \to s_0} (s - s_0)^m \left( \sum_{w \in \Sigma_\nu} M(w, \chi, \lambda_s) f_s \right) \neq 0.$$
5.1. $s_0 = \frac{5}{2}$ and $\chi = 1$

In this case, $\mathcal{E}_E(1, f, s, g)$ admits a simple pole. We recall, from [Seg], that

$$\Sigma^{PE}_{(E, 1, \frac{1}{2}, 1)} = \{w_0\}$$

On the other hand, for any $\nu \in \mathcal{P}$ it holds that $N_{\nu} \left( w_0, 1, \lambda_{\frac{3}{2}} \left( I_{\nu} \left( 1, \frac{5}{2} \right) \right) \right) = 1_{H_E, \nu}$. Hence, $Res \left( \frac{5}{2}, 1, E \right) \equiv 1_{H_E}$, where $1_{H_E}$ is the trivial representation of $H_E$.

5.2. $s_0 = \frac{3}{2}$ and $\chi = \chi_E$

In [GGJ02, Section 5] it is proven that

$$Res \left( \frac{3}{2}, \chi_E, E \right) \equiv \Pi_E,$$

where $\Pi_E$ is the minimal representation of $H_E(\mathbb{A})$. Indeed, this follows from theorem 4.1 and Equation (5.1).

5.3. $s_0 = \frac{1}{2}, \chi = 1$ and $E$ is a Field

In this case, $\mathcal{E}_E(1, f, s, g)$ admits a simple pole. We recall, from [Seg], that

$$\Sigma^{PE}_{(E, 1, \frac{1}{2}, 2)} = \{w_{212}, w_{2121}\}$$

and that

$$\Sigma^{PE}_{(E, 1, \frac{1}{2}, 1)} = \{w_{21}, w_{212}, w_{2121}, w_{21212}\}$$

and that

$$\Sigma^{PE}_{(E, 1, \frac{1}{2}, 1)}/\sim_{(E, s_0)} = \{\{w_{21}, w_{21212}\}, \{w_{212}, w_{2121}\}\}.$$}

Let

$$\mathcal{P}_{\text{inert}} = \{\nu \in \mathcal{P} \mid E_{\nu} \text{ is a field}\}.$$

From Equation (5.1) it follows that

$$Res \left( \frac{1}{2}, 1, E \right) = \hat{\oplus} \Sigma_i \subseteq \bigoplus_{S \subseteq \mathcal{P}_{\text{inert}}} \pi_S,$$

where

$$\pi_S = \bigotimes_{\nu \in S} \pi_{-2, \nu} \otimes \bigotimes_{\nu \notin S} \pi_{1, \nu}.$$}

In particular, in order to determine $Res \left( \frac{1}{2}, 1, E \right)$ it is enough to determine for which finite subsets $S \subseteq \mathcal{P}_{\text{inert}}$ will $\pi_S$ appear in $Res \left( \frac{1}{2}, 1, E \right)$.

We note that, given a finite subsets $S \subseteq \mathcal{P}_{\text{inert}}$ and an element $\varphi \in \pi_S$, it holds that

$$M \left( w_{212}, w_{2121}, \chi_s \right) \varphi = (-2)^{|S|-1} \varphi.$$
So
\[ \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) \left[ M(w_{21}, \chi_s) + M(w_{21212}, \chi_s) \right] \]
vanish on \( I(1, \frac{1}{2}) \) if and only if \(|S| = 1\). Hence,
\begin{equation}
\bigoplus_{S \in \text{CP}_{\text{inert}}} \pi_S \subseteq \text{Res} \left( \frac{1}{2}, 1, E \right).
\end{equation}

We wish to show that this is an equality.

We consider the residue of the constant term
\[ \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) [E_E(1, f, s, g)_{\text{CT}}] \]
\[ = \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) [(M(w_{21}, \chi_s) + M(w_{21212}, \chi_s)) f_s(g)] \]
\[ + \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) [(M(w_{212}, \chi_s) + M(w_{2121}, \chi_s)) f_s(g)]. \]

We note that
\[ \ker \left[ \text{Res}_{s=\frac{1}{2}} E_E(1, \cdot, s, \cdot)_{\text{CT}} \right] \]
\[ = \ker \left[ \text{Res}_{s=\frac{1}{2}} (M(w_{21}, \chi_s) + M(w_{21212}, \chi_s)) \right] \bigcap \ker \left[ \text{Res}_{s=\frac{1}{2}} (M(w_{212}, \chi_s) + M(w_{2121}, \chi_s)) \right] \]

It is thus enough to show that
\begin{equation}
\ker \left[ \text{Res}_{s=\frac{1}{2}} (M(w_{21}, \chi_s) + M(w_{21212}, \chi_s)) \right] \subseteq \ker \left[ \text{Res}_{s=\frac{1}{2}} (M(w_{212}, \chi_s) + M(w_{2121}, \chi_s)) \right]
\end{equation}

Noting that
\[ M(w_{212}, \chi_s) + M(w_{2121}, \chi_s) = (I + M(w_1, w_{212}^{-1} \cdot \chi_s)) M(w_{212}, \chi_s) \]

We write the Laurent series of \( I + M(w_1, w_{212}^{-1} \cdot \chi_s) \) and \( M(w_{212}, \chi_s) \) in a neighbourhood of \( s = \frac{1}{2} \):
\[ I + M(w_1, w_{212}^{-1} \cdot \chi_s) = A_1 \left(s - \frac{1}{2}\right) + A_2 \left(s - \frac{1}{2}\right)^2 + ... \]
\[ M(w_{212}, \chi_s) = \frac{B_{-2}}{\left(s - \frac{1}{2}\right)^2} + \frac{B_{-1}}{\left(s - \frac{1}{2}\right)} + B_0 + ... \]

Composing the two series we get
\[ M(w_{212}, \chi_s) + M(w_{2121}, \chi_s) = \frac{A_1 \circ B_{-2}}{\left(s - \frac{1}{2}\right)^2} + (A_2 \circ B_{-2} + A_1 \circ B_{-1}) + ... \]

Hence
\[ \lim_{s \to \frac{1}{2}} \left(s - \frac{1}{2}\right) [(M(w_{212}, \chi_s) + M(w_{2121}, \chi_s)) f_s(g)] = A_1 \circ B_{-2}. \]
Since $\text{Im} (A_{-2}) = \pi_0$ Equation (5.7) follows.

Alternatively, the equality in Equation (5.6) follows from the results of [Lao16]. We note that

\begin{equation}
\text{Res} (\frac{1}{2}, 1, E) \subseteq L^2_{[B, \mu_\chi]},
\end{equation}

where $L^2_{[B, \mu_\chi]}$ is the subspace of $L^2 (H_E (F) \backslash H_E (\mathbb{A}))$ spanned by automorphic forms with cuspidal data $[B, \mu_\chi]$. We recall from [Lao16] that

\begin{equation}
L^2_{[B, \mu_\chi]} = \bigoplus_{\text{SCP \_ inert}} \pi_S
\end{equation}

and hence also

\begin{equation}
\text{Res} (\frac{1}{2}, 1, E) = \bigoplus_{\text{SCP \_ inert}} \pi_S.
\end{equation}

5.4. $s_0 = \frac{1}{2}$, $\chi^2 = 1 \neq \chi \circ \text{Nm}_{E/F}$

We separate the discussion into the various kinds of étale cubic algebras. We partition $\mathcal{P}$ as follows,

\begin{equation}
\mathcal{P} = \mathcal{P}^{(E, \chi)}_{\text{sph}} \bigcup \mathcal{P}^{(E, \chi)}_{1} \bigcup \mathcal{P}^{(E, \chi)}_{2} \bigcup \mathcal{P}^{(E, \chi)}_{3},
\end{equation}

where

- If $\nu \in \mathcal{P}^{(E, \chi)}_{\text{sph}}$ then $I_\nu (\chi_\nu, \frac{1}{2})$ admits a unique irreducible (spherical) quotient. Namely, $E_\nu$ is not a field and $\chi_\nu \circ \text{Nm}_{E_\nu/F_\nu} = 1_\nu$ or, $E_\nu$ is a field and $\chi_\nu \circ \text{Nm}_{E_\nu/F_\nu} \neq 1_\nu$.
- If $\nu \in \mathcal{P}^{(E, \chi)}_{1}$ then $E_\nu$ is a field and $\chi_\nu = \text{Id}_{E_\nu}$.
- If $\nu \in \mathcal{P}^{(E, \chi)}_{2}$ then $E_\nu = F_{\nu} \times K_{\nu}$ where $K_{\nu}$ is a field and $\chi_\nu \circ \text{Nm}_{K_{\nu}/F_{\nu}} \neq 1_\nu$.
- If $\nu \in \mathcal{P}^{(E, \chi)}_{3}$ then $E_\nu = F_{\nu} \times F_{\nu} \times F_{\nu}$ and $\chi_\nu \neq 1_\nu$.

Also, let

\begin{equation}
\mathcal{P}^{(E, \chi)}_{\text{non\_sph}} = \mathcal{P}^{(E, \chi)}_{1} \bigcup \mathcal{P}^{(E, \chi)}_{2} \bigcup \mathcal{P}^{(E, \chi)}_{3}.
\end{equation}

We now turn to compute $\text{Res} (\frac{1}{2}, \chi, E)$ for the various possible cases. This is done by comparing the action of

$$
\lim_{s \to s_0} (s - s_0) \sum_{w \in \Sigma^{E}_{\chi, \lambda_s}} M (w, \chi, \lambda_s)
$$

on quotients of $I (\chi, \frac{1}{2})$. This is done by identifying these quotients as images or eigenspaces of various intertwining operators.
5.4.1. \( E \) is a Field

In this case
\[
\Sigma P E \langle E,\chi,\frac{1}{2},1 \rangle / \sim (\chi,\frac{1}{2}) = \{ \{ w_{212} \}, \{ w_{2121} \}, \{ w_{21212} \} \},
\]
where all three elements are mutually inequivalent and their associated intertwining operators admit simple poles at \( s = \frac{1}{2} \). Let \( S \subset P_1^{(E,\chi)} \cup P_2^{(E,\chi)} \cup P_3^{(E,\chi)} \) be a finite subset such that
\[
\hat{S} = S_1 \cup S_2 \cup \left( S_{3,(1,-1)} \cup S_{3,(-1,1)} \cup S_{3,(-1,-1)} \right),
\]
with

- \( S_1 = S \cap P_1^{(E,\chi)} \subset P_1^{(E,\chi)} \),
- \( S_2 = S \cap P_2^{(E,\chi)} \subset P_2^{(E,\chi)} \),
- \( S_{3,(\epsilon_1,\epsilon_2)} \subset P_3^{(E,\chi)} \).

Note that \( \hat{S} \) is a set with a choice of the distinct subsets \( S_{3,(\epsilon_1,\epsilon_2)} \) and \( S \) denotes its underlying set. We let
\[
(5.12) \quad \pi_{\hat{S}} = \left( \bigotimes_{\nu \in S_1} \pi_{-2,\nu} \right) \otimes \left( \bigotimes_{\nu \in S_2} \pi_{-1,\nu} \right) \otimes \left( \bigotimes_{\nu \in S_{3,(\epsilon_1,\epsilon_2)}} \pi_{\epsilon_1,\epsilon_2,\nu} \right) \otimes \left( \bigotimes_{\nu \in S} \pi_{1,\nu} \right).
\]

It follows from the discussion above that \( \pi_{\hat{S},\nu} \subset N_{\nu} \left( w_{21212}, \chi_{\frac{1}{2},\nu} \right) \left( I_{\nu} \left( \chi_{\nu}, \frac{1}{2} \right) \right) \). It follows that \( \pi_{\hat{S}} \) is a direct summand of \( \text{Res} \left( \frac{1}{2}, \chi, E \right) \). On the other hand,
\[
\text{Res} \left( \frac{1}{2}, \chi, E \right) \subseteq \bigoplus_{S \subset P_{\text{non-sph}}^{(E,\chi)}} \pi_{\hat{S}}
\]
and hence
\[
(5.13) \quad \text{Res} \left( \frac{1}{2}, \chi, E \right) = \bigoplus_{S \subset P_{\text{non-sph}}^{(E,\chi)}} \pi_{\hat{S}}.
\]

5.4.2. \( E = F \times K \) and \( K \) is a Field

In this case
\[
\Sigma P E \langle E,\chi,\frac{1}{2},1 \rangle / \sim (\chi,\frac{1}{2}) = \{ \{ w_{2321}, w_{213213} \}, \{ w_{2132}, w_{21323} \}, \{ w_{21321}, w_{213213} \} \}.
\]

We recall from [Seg] that the poles of \( M \left( w_{2132}, \chi_s \right) \) and \( M \left( w_{21323}, \chi_s \right) \) cancel each other. The other poles do not cancel and we wish to analyze the image of the intertwining operator
\[
\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) \left[ M \left( w_{232123}, \chi_s \right) + M \left( w_{23212}, \chi_s \right) + M \left( w_{232132}, \chi_s \right) + M \left( w_{2321}, \chi_s \right) \right].
\]
Note that
\[
\begin{align*}
w_{2321232} &= w_{2321} w_{232}, \\
w_{213213} &= w_{21321} w_3
\end{align*}
\]
and
\[
\begin{align*}
J \left( w_{3}, w_{21321}^{-1} \cdot \chi_s \right) &= \frac{\mathcal{L}_K \left( s - \frac{1}{2}, \chi \circ \text{Nm}_{K/F} \right)}{\mathcal{L}_K \left( s + \frac{1}{2}, \chi \circ \text{Nm}_{K/F} \right)}, \\
J \left( w_{232}, w_{232}^{-1} \cdot \chi_s \right) &= \frac{\mathcal{L}_F \left( s - \frac{3}{2}, \chi \right) \mathcal{L}_F \left( s + \frac{1}{2}, \chi \right) \mathcal{L}_K \left( s - \frac{1}{2}, \chi \circ \text{Nm}_{K/F} \right)}{\mathcal{L}_F \left( s - \frac{1}{2}, \chi \right) \mathcal{L}_F \left( s + \frac{3}{2}, \chi \right) \mathcal{L}_K \left( s + \frac{1}{2}, \chi \circ \text{Nm}_{K/F} \right)}.
\end{align*}
\]
According to [Roh11, Theorem 2.2] it follows that \( \epsilon_F (s, \chi) \equiv 1 \) and \( \epsilon_K (s, \chi \circ \text{Nm}_{K/F}) \equiv 1 \) and hence
\[
\begin{align*}
\mathcal{L}_F \left( s, \chi \right) &= \mathcal{L}_F \left( 1 - s, \chi \right), \\
\mathcal{L}_K \left( s, \chi \circ \text{Nm}_{K/F} \right) &= \mathcal{L}_K \left( 1 - s, \chi \circ \text{Nm}_{K/F} \right).
\end{align*}
\]
It follows that
\[
\lim_{s \to \frac{1}{2}} J \left( w_{3}, w_{21321}^{-1} \cdot \chi_s \right) = 1, \quad \lim_{s \to \frac{1}{2}} J \left( w_{232}, w_{232}^{-1} \cdot \chi_s \right) = 1.
\]
Let \( S \subset P_2^{(E, \chi)} \cup P_3^{(E, \chi)} \) be a finite subset such that
\[
\hat{S} = S_2 \bigcup \left( S_{3,-1,-1} \cup S_{3,1,-1} \cup S_{3,-1,1} \right),
\]
with
\[
\begin{align*}
&\bullet \quad S_2 \subset P_2^{(E, \chi)}, \\
&\bullet \quad S_{3,(\epsilon_1, \epsilon_2)} \subset P_3^{(E, \chi)}.
\end{align*}
\]
We let
\[
\pi_{\hat{S}} = \left( \bigotimes_{\nu \in S_2} \pi_{-1,\nu} \right) \otimes \left( \bigotimes_{\nu \in S_{3,(\epsilon_1, \epsilon_2)}} \pi_{(\epsilon_1, \epsilon_2),\nu} \right) \otimes \left( \bigotimes_{\nu \not\in S} \pi_{1,\nu} \right).
\]
The irreducible representation \( \pi_{\hat{S}} \) appears in the images of all
\[
\begin{align*}
\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M \left( w_{2321} \cdot \chi_s \right) \\
\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M \left( w_{2321} \cdot \chi_s \right) \\
\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M \left( w_{2321} \cdot \chi_s \right) \\
\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M \left( w_{2321} \cdot \chi_s \right).
\end{align*}
\]
On the other hand, since \( w_3 = \tilde{w}_{34} \) and \( w_{232} = \tilde{w}_{2342} \), we note that both \( M \left( w_3, w_{21321}^{-1} \cdot \chi_{\frac{1}{2}} \right) \) and \( M \left( w_{232}, w_{2321}^{-1} \cdot \chi_{\frac{1}{2}} \right) \) acts on \( \pi_S \) by

\[
(-1)^{|S_2| + |S_{3,(-1,1)}| + |S_{3,(-1,1)}|} \text{Id}.
\]

It follows that

\[
(5.15) \quad \text{Res} \left( \frac{1}{2}, \chi, F \times K \right) = \bigoplus_{\delta \in \mathcal{P}(F, \chi)_{\text{non-sph}}} \pi_{\delta},
\]

where

\[
(5.16) \quad S^* = S_2 \cup S_{3,(-1,1)} \cup S_{3,(-1,1)}
\]

5.4.3. \( E = F \times F \times F \)

In this case

\[
\Sigma_{(E, \chi, \frac{1}{2}, 1)}^{F^e} / \sim (\chi, \frac{1}{2}) = \{ \{ w_{21324}, w_{21423}, w_{23421}, w_{213421342} \} \}
\]

\[
\{ w_{21342}, w_{2134213}, w_{2134214}, w_{2134234} \}
\]

\[
\{ w_{213421}, w_{213423}, w_{213424}, w_{21342134} \}
\]

Note that

\[
w_{213421342} = w_{21324} w_{2132} = w_{21423} w_{2142} = w_{23421} w_{2342}
\]

\[
w_{21342134} = w_{213424} w_{13} = w_{213423} w_{14} = w_{213421} w_{34}
\]

\[
w_{2134213} = w_{21342} w_{13}
\]

\[
w_{2134214} = w_{21342} w_{14}
\]

\[
w_{2134234} = w_{21342} w_{34}
\]

Furthermore,

\[
J \left( w_{2132}, w_{21324}^{-1} \cdot \chi_s \right) = J \left( w_{2142}, w_{21423}^{-1} \cdot \chi_s \right) = J \left( w_{2342}, w_{23421}^{-1} \cdot \chi_s \right) = \frac{\mathcal{L} \left( s - \frac{1}{2}, \chi \right) \mathcal{L} \left( s - \frac{3}{2}, \chi \right)}{\mathcal{L} \left( s + \frac{1}{2}, \chi \right) \mathcal{L} \left( s + \frac{3}{2}, \chi \right)}
\]

\[
J \left( w_{13}, w_{21342}^{-1} \cdot \chi_s \right) = J \left( w_{14}, w_{21342}^{-1} \cdot \chi_s \right) = J \left( w_{34}, w_{23421}^{-1} \cdot \chi_s \right) = \left( \frac{\mathcal{L} \left( s - \frac{1}{2}, \chi \right)}{\mathcal{L} \left( s + \frac{1}{2}, \chi \right)} \right)^2
\]

\[
J \left( w_{13}, w_{21342}^{-1} \cdot \chi_s \right) = J \left( w_{14}, w_{21342}^{-1} \cdot \chi_s \right) = J \left( w_{34}, w_{213421}^{-1} \cdot \chi_s \right) = \left( \frac{\mathcal{L} \left( s - \frac{1}{2}, \chi \right)}{\mathcal{L} \left( s + \frac{1}{2}, \chi \right)} \right)^2.
\]

According to \cite{Roh11} Theorem 2.2\] it follows that \( \epsilon_F (s, \chi) \equiv 1 \) and hence

\[
\mathcal{L}_F (s, \chi) = \mathcal{L}_F (1 - s, \chi)
\]
It follows that

$$\lim_{s \to \frac{1}{2}} \frac{\mathcal{L}(s - \frac{1}{2}, \chi) \mathcal{L}(s - \frac{3}{2}, \chi)}{\mathcal{L}(s + \frac{1}{2}, \chi) \mathcal{L}(s + \frac{3}{2}, \chi)} = \lim_{s \to \frac{1}{2}} \left( \mathcal{L}(s - \frac{1}{2}, \chi) \right)^2 = 1.$$ 

Let $S \subset \mathcal{P}_3^{(E, \chi)}$ be a finite subset such that

$$\dot{S} = S_{3,(-1,1)} \cup S_{3,(1,-1)} \cup S_{3,(-1,-1)},$$

with $S_{3,(\epsilon, \epsilon_2)} \subset \mathcal{P}_3^{(E, \chi)}$. We let

$$\pi_{\dot{S}} = \bigotimes_{\nu \in S_2} \pi_{-1, \nu} \otimes \bigotimes_{\nu \in S_{3,(\epsilon, \epsilon_2)}} \pi_{(\epsilon, \epsilon_2), \nu} \otimes \bigotimes_{\nu \not\in S_2} \pi_{1, \nu}. \tag{5.17}$$

The irreducible representation $\pi_{\dot{S}}$ appears in the images of all

$$\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M(w, \chi_s),$$

where $w \in \Sigma_{\mathcal{P}_3^{(E, \chi)}}^{(F \times F \times F, \chi, \frac{1}{2}, 1)}$.

Fix a standard section $f_s \in I(\chi, s)$ so that

$$\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M\left(w_{21342}, \chi_s\right) f_s = \varphi \in \pi_{\dot{S}}$$

A similar analysis to the one performed in the previous cases shows that

$$\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) \sum_{w \in \Sigma_{\mathcal{P}_3^{(E, \chi)}}^{(F \times F \times F, \chi, \frac{1}{2}, 1)}} M(w, \chi_s) = 4 \left( 1 + (-1)^{|S_{3,(-1,1)}|} + (-1)^{|S_{3,(1,-1)}|} + (-1)^{|S_{3,(1,-1)}|} \right) \varphi.$$ 

It follows that

$$\text{Res} \left( \frac{1}{2}, \chi, F \times F \times F \right) = \bigoplus_{\dot{S} \in \mathcal{P}_{\mathcal{V}}^{(E, \chi)}_{\text{non-sph}}} \pi_{\dot{S}}, \tag{5.18}$$

where

$$\mathcal{V}^{(E, \chi)} = \left\{ \dot{S} \subset \mathcal{P}_3^{(E, \chi)}_{\text{non-sph}} \mid \begin{array}{c} |S_{3,(-1,1)}| \cdot |S_{3,(1,-1)}| \equiv |S_{3,(1,-1)}| \cdot |S_{3,(-1,-1)}| \pmod{2} \\
|S_{3,(-1,1)}| \cdot |S_{3,(-1,-1)}| \equiv |S_{3,(1,-1)}| \cdot |S_{3,(-1,-1)}| \end{array} \right\} \tag{5.19}$$
5.5. Summary

We record the results of this section as a theorem.

**Theorem 5.2.** The square integrable residues $\text{Res}(s_0, \chi, E)$ with $\Re(s_0) > 0$ are given as follows:

1. If $s_0 = \frac{5}{2}$ and $\chi = 1$ then $\text{Res}(s_0, \chi, E) = 1$, the trivial representation of $H_E(\mathbb{A})$, for any $E$.
2. If $s_0 = \frac{3}{2}$ and $\chi = \chi_E$, where $E$ is a Galois étale cubic algebra over $F$, then $\text{Res}(s_0, \chi, E) = \Pi_E$, the minimal representation $\Pi_E$ of $H_E(\mathbb{A})$.
3. If $s_0 = \frac{1}{2}$, $\chi = 1$ and $E$ is a field extension then

$$\text{Res} \left( \frac{1}{2}, 1, E \right) = \bigoplus_{S \in \mathcal{P}_{\text{inert}} \setminus \{S \mid \text{inert} \}} \pi_S,$$

where $\mathcal{P}_{\text{inert}}$ is defined in Equation (5.23) and $\pi_S$ is given by Equation (5.5).

4. Assume that $s_0 = \frac{1}{2}$ and $\chi^2 = 1 \neq \chi \circ \text{Nm}_{E/F}$ and recall the definition of $\mathfrak{p}^{(E,\chi)}_{\text{non-sph}}$ from Equation (5.11).
   - If $E$ is field then

$$\text{Res} \left( \frac{1}{2}, \chi, E \right) = \bigoplus_{S \in \mathfrak{p}^{(E,\chi)}_{\text{non-sph}} \setminus \{S \mid |S| < \infty \}} \pi_S,$$

where $\pi_S$ is given by Equation (5.12).
   - If $E = F \times K$, where $K$ is field, then

$$\text{Res} \left( \frac{1}{2}, \chi, F \times K \right) = \bigoplus_{S \in \mathfrak{p}^{(E,\chi)}_{\text{non-sph}} \setminus \{S \mid |S^*| \text{ is even} \}} \pi_S,$$

where $\pi_S$ is given by Equation (5.14) and $S^*$ is given in Equation (5.16).
   - If $E = F \times F \times F$ then

$$\text{Res} \left( \frac{1}{2}, \chi, F \times F \times F \right) = \bigoplus_{S \in \mathfrak{p}^{(E,\chi)}_{\text{non-sph}} \setminus \hat{S} \in \mathcal{V}^{(E,\chi)}} \pi_S,$$

where $\pi_S$ is given by Equation (5.17) and $\mathcal{V}^{(E,\chi)}$ is given by Equation (5.19).

We note here again that, in item (4), $\hat{S}$ is a finite set $S \subset \mathcal{P}$ together with a choice $S_3 = S_{3,(-1,1)} \cup S_{3,(1,-1)} \cup S_{3,(-1,-1)}$, where $S_3 = S \cap \mathcal{P}^{(E,\chi)}_3$. 
6. Non-Square-integrable Residues

We compute the non-square-integrable residues $\text{Res}(s_0, \chi, E)$ for the various values of $s_0$ and $\chi$.

For $\chi = 1$, we compute $\text{Res}(s, \chi, E)$ using a Siegel-Weil type identity. Namely, we prove an identity between $\text{Res}(s, \chi, E)$ and a special value, or residue, of an Eisenstein series associated to an induction from a different parabolic subgroup of $H_E$. The case of $s = \frac{3}{2}$ was essentially computed in [GSb], we recall the relevant results and compute $\text{Res}(\frac{3}{2}, 1, F \times K)$. For $s = \frac{1}{2}$, we prove an identity between $\text{Res}(\frac{1}{2}, 1, F \times K)$ and the special value of the Eisenstein series associated to the degenerate principal series induced from $M_{\{\alpha_1, \alpha_2\}}$. We then use the fact that the relevant local degenerate principal series is semi-simple in order to compute $\text{Res}(\frac{1}{2}, 1, F \times K)$.

6.1. $\chi = 1$

6.1.1. $s_0 = \frac{3}{2}$

In this case, $E = F \times K$ Where $K$ is a Field. It was essentially dealt with in [GSb]. We recall the results from there and use them in order to compute $\text{Res}(\frac{3}{2}, 1, F \times K)$. We start with the Siegel-Weil-type identity [GSb Corollary 3.17]:

**Proposition 6.1.** There exists a non-zero constant $C$ such that for every $f \in I_{P_F \times K}(s)$ it holds that

$$
\left[ \left( s - \frac{3}{2} \right) E_{F \times K}(f, s, g) \right]_{s=\frac{3}{2}} = C \cdot E_{P(2,3,4)}(A(w_{232}) f, 1, g),
$$

where

- $E_{P(2,3,4)}$ is the degenerate Eisenstein series associated to the normalized parabolic induction $I_{P_2(2,3,4)}(s)$ of $\omega_{\alpha_1}^s$ from the maximal parabolic $P_{(2,3,4)}$ and
- $A(w_{232})$ is the leading term of $M(w_{232}, \lambda)$ at $\lambda_{\frac{3}{2}}$, given by

$$
A(w_{232}) = \lim_{s \to \frac{3}{2}} \left[ (s - \frac{3}{2}) M(w_{232}, \lambda_s) \right].
$$

We also recall that $A(w_{232}) = \otimes_{\nu \in P} A_{\nu}$ and $A_{\nu}$ is onto for any place $\nu \in P$.

**Proposition 6.2.**

$$
\text{Res}\left(\frac{3}{2}, 1, F \times K\right) \cong I_{P(2,3,4)}(1).
$$
We conclude that for any $K$ identity and apply this identity to calculate this using a similar approach to the previous case. We first establish a Siegel-Weil type identity, In this case, we have $E$ and the right-hand side is given by $\delta_{1,1}$. It follows from Corollary 6.2 that

$$\ker\left(\mathcal{E}_{P_{2,3,4}} (\cdot, 1, g)\right) = (0)$$

from which the claim follows.

6.1.2. $s_0 = \frac{1}{2}$

In this case, we have $E = F \times K$, where $K$ is a quadratic étale algebra over $F$. We approach this using a similar approach to the previous case. We first establish a Siegel-Weil type identity and apply this identity to calculate $\text{Res}\left(\frac{1}{2}, 1, F \times K\right)$.

We recall [GSb] Prop. 3.9. Let us define the following normalized spherical Eisenstein series

$$\mathcal{E}_{BE}^s (\lambda, g) = \prod_{\alpha \in \Phi^+} \xi_{F_{\alpha}} \left(\left\langle \lambda, \alpha^\vee \right\rangle + 1\right) I^+_{\alpha} (\lambda) I^-_{\alpha} (\lambda) \mathcal{E}_{BE} (f^0, \lambda, g),$$

where

$$I^\pm_{\alpha} (\lambda) = \left\langle \lambda, \alpha^\vee \right\rangle \pm 1.$$

Proposition 6.3. The normalized Eisenstein series $\mathcal{E}_{BE}^s (\lambda, g)$ is entire and $W_H$-invariant in the sense that for any $w \in W_H$ it holds that $\mathcal{E}_{BE}^s (w \cdot \lambda, g) = \mathcal{E}_{BE}^s (\lambda, g)$.

In particular, it holds that

$$\mathcal{E}_{BE}^s \left(\lambda_{(-1,2,-1,-1)}, g\right) = \mathcal{E}_{BE}^s \left(\lambda_{(-1,-1,1,1)}, g\right).$$

We evaluate both sides of the equation at Appendix 3. In particular, evaluating the left-hand side yields

$$\begin{cases} 2^7 \cdot 3 \cdot \xi_F (2) \cdot \xi_F (3) \cdot \xi_K (2) \cdot R_F^3 \cdot R_K^2 \cdot \text{Res}_{s=\frac{1}{2}} \left[ \mathcal{E}_E \left( f^0, s, g \right) \right], & \text{if } K \text{ is a field,} \\ -2^9 \cdot 3 \cdot \xi_F (2) \cdot \xi_F (3) \cdot R_F^7 \cdot \text{Res}_{s=\frac{1}{2}} \left[ \mathcal{E}_E \left( f^0, s, g \right) \right], & \text{if } K = F \times F, \end{cases}$$

while the right-hand side is given by

$$\begin{cases} 2^6 \cdot 3 \cdot \xi_F (2) \cdot \xi_K (2) \cdot R_F^4 \cdot R_K^2 \cdot \mathcal{E}_{P_{1,2}} (f^0, 0, g), & \text{if } K \text{ is a field,} \\ -2^8 \cdot 3 \cdot \xi_F (2) \cdot \xi_K (2) \cdot R_F^8 \cdot \mathcal{E}_{P_{1,2}} (f^0, 0, g), & \text{if } K = F \times F. \end{cases}$$

We conclude that for any $K$ it holds that

$$\text{Res}_{s=\frac{1}{2}} \left[ \mathcal{E}_E \left( f^0, s, g \right) \right] = \frac{R_F}{2 \cdot \xi_F (3)} \cdot \mathcal{E}_{P_{1,2}} (f^0, 0, g).$$
We let
\[ A(w_{21}) = \lim_{s \to \frac{1}{2}} \left[ \left( s - \frac{1}{2} \right) M(w_{21}, \lambda_s) \right] \]
and note that
\[ A(w_{21}) f_{\lambda \frac{1}{2}}^0 = \frac{R_F}{\xi_F(3)} f_{\lambda(-1,-1,1,1)}^0, \]
It follows that
\[ \text{Res}_{s = \frac{1}{2}} \left[ \mathcal{E}_E(f^0, s, g) \right] = \frac{1}{2} \cdot \mathcal{E}_{P(1,2)} (A(w_{21}) f_{\bar{0}}, g) . \]
On the other hand, by \[ \text{Lap11}, \text{Proposition 6} \], \( \mathcal{E}_{P(1,2)} (f, \lambda(s_1, s_2), g) \) is holomorphic at \( \lambda(0,0) \) for any section and hence the map \( \mathcal{E}_{P(1,2)} (\bar{f}, 0, g) \) is \( H_E(\mathbb{A}) \)-equivariant.

Since \( f_{\lambda \frac{1}{2}}^0 \) generates \( I(1, \frac{1}{2}) \) we have the following result.

**Proposition 6.4.** For any standard section \( f_s \in I(1, s) \) it holds that
\[ \text{Res}_{s = \frac{1}{2}} \left[ \mathcal{E}_E(f, s, g) \right] = \frac{1}{2} \cdot \mathcal{E}_{P(1,2)} (A(w_{21}) f_0, 0, g) . \]

We note that \( I_{P(1,2)} (0) \) is unitary and hence semi-simple. On the other hand, the maxim semi-simple quotient of \( I(1, \frac{1}{2}) \) is \( \bigotimes_{\nu \in \mathcal{P}} \pi_{1,\nu} \).

It follows that
\[ \text{Res} \left( \frac{1}{2} \cdot 1, F \times K \right) = \bigotimes_{\nu \in \mathcal{P}} \pi_{1,\nu} . \]

### 6.2. \( \chi \neq 1 \)

Here we assume that \( s_0 = \frac{1}{2}, \chi = \chi_K \) and \( E = F \times K \), where \( K \) is a quadratic field extension of \( F \).

We recall, from \[ \text{Seg} \], the intertwining operators involved in the simple pole of \( \mathcal{E}_E(\chi_K, f_s, s, g) \) \( B_E \) at \( s = \frac{1}{2} \).

\[ \Sigma_{(F \times K, \chi_K, \frac{1}{2}, 2)}^{P_E} = \{ w_{2321}, w_{2132}, w_{21321}, w_{21323}, w_{213213}, w_{2132132} \} \]
and
\[ w_{2321}^{-1} \cdot \chi_{\frac{1}{2}} (t) = w_{2132132}^{-1} \cdot \chi_{\frac{1}{2}} (t) = \chi_K (t_2) \frac{|t_2|_F}{|t_1|_F |t_3|_K} \]
\[ w_{2132}^{-1} \cdot \chi_{\frac{1}{2}} (t) = w_{21323}^{-1} \cdot \chi_{\frac{1}{2}} (t) = \chi_K (t_1) \frac{1}{|t_2|_F} \]
\[ w_{21321}^{-1} \cdot \chi_{\frac{1}{2}} (t) = w_{213213}^{-1} \cdot \chi_{\frac{1}{2}} (t) = \chi_K (t_1 t_2) \frac{1}{|t_2|_F} \]

\[ \Sigma_{(F \times K, \chi_K, \frac{1}{2}, 1)}^{P_E} \setminus \Sigma_{(F \times K, \chi_K, \frac{1}{2}, 2)}^{P_E} = \{ w_{23}, w_{232}, w_{23213} \} \]
w_{23}^{-1} \cdot \chi_{\frac{1}{2}} (t) = \chi_K (t_1 t_2) \frac{|t_1|_F}{|t_3|_K}, \quad w_{232}^{-1} \cdot \chi_{\frac{1}{2}} (t) = \chi_K (t_2) \frac{|t_1|_F}{|t_3|_K}, \quad w_{231}^{-1} \cdot \chi_{\frac{1}{2}} (t) = \chi_K (t_1) \frac{|t_2|_F}{|t_3|_K}.

Given a standard section \( f_s \in I_{P_E} (s) \) it generates a finite dimensional \( K \)-representation, where \( K \) is a fixed maximal compact subgroup of \( H_E (\mathbb{A}) \) as in [MW95, Section I.1.1]. We let \( \mathcal{F} \) denote the finite set of \( K \)-types determining the finite-dimensional subspace of \( \text{Ind}^K_{B (E (\mathbb{A}) \cap K)} (\chi_s |_{B (E (\mathbb{A}) \cap K)} \cdot f_s |_K) \). Note that both \( \text{Ind}^K_{B (E (\mathbb{A}) \cap K)} (\chi_s |_{B (E (\mathbb{A}) \cap K)} \cdot f_s |_K) \) and \( f = f_s |_K \) are independent of \( s \). For \( w \in W (P_E, H_E) \), we let \( M_{\mathcal{F}} (w, \chi_s) \) be the associated intertwining operator on \( \text{Ind}^K_{B (E (\mathbb{A}) \cap K)} (\chi_s |_{B (E (\mathbb{A}) \cap K)}) \).

We then may write

\[
\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) \mathcal{E}_E (\chi_K, f_s, s, g)_{CT} = \left( w_{23}^{-1} \cdot \chi_s \right) M_{\mathcal{F}} (w_{23}, \chi_s) f + \left( w_{23}^{-1} \cdot \chi_s \right) M_{\mathcal{F}} (w_{23}, \chi_s) f
\]

In particular, \( \text{Res} \left( \frac{1}{2}, 1, F \times K \right) \) is a quotient of

\[
\left[ \lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M (w_{23}, \chi_s) \left( I_{P_E} \left( \chi_K, \frac{1}{2} \right) \right) \right].
\]

**Lemma 6.5.** The local representation \( I_{P_E} \left( \chi_K, \frac{1}{2} \right) \) admits a unique irreducible quotient \( \pi_{1, \nu} \) for any \( \nu \in \mathcal{P} \) and

\[
N_{\nu} (w_{23}, \chi_s) \left( I_{P_E} \left( \chi_K, \frac{1}{2} \right) \right) = \pi_{1, \nu}.
\]

It follows that

\[
\left[ \lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) M (w_{23}, \chi_s) \left( I_{P_E} \left( \chi_K, \frac{1}{2} \right) \right) \right] = \bigotimes_{\nu \in \mathcal{P}} \pi_{1, \nu}
\]

and hence

\[
\text{Res} \left( \frac{1}{2}, \chi_K, F \times K \right) \cong \bigotimes_{\nu \in \mathcal{P}} \pi_{1, \nu}\tag{6.3}
\]
Proof of Lemma. First, assume that $K_\nu = F_\nu \times F_\nu$. In this case, the claim follows from the computation of $\text{Res} \left( \frac{1}{2}, 1, F \times F \times F \right)$.

The fact that $\text{Res} \left( \frac{1}{2}, 1, F \times F \times F \right) = \otimes_{\nu \in \mathcal{P}} \pi_{1, \nu}^{(1,F_\nu \times F_\nu \times F_\nu, 1/2)}$ is irreducible and that the pole of $\mathcal{E}_{F \times F \times F} (1, f_s, s, g)$ at $s = \frac{1}{2}$ implies that for any $[w] \in \Sigma_{E \times F \times F, 1, 1/2}$ it holds that

$$\lim_{s \to \frac{1}{2}} \left( s - \frac{1}{2} \right) \sum_{w \in [w]} M (w, \chi_s) \left( I \left( \frac{1}{2} \right) \right) \subseteq \otimes_{\nu \in \mathcal{P}} \pi_{1, \nu}^{(1,F_\nu \times F_\nu \times F_\nu, 1/2)}.$$

Since \{w_{23}, w_{24}\} $\in \Sigma_{E \times F \times F, 1, 1/2}$, it follows that

$$N_\nu \left( \text{Res} \left( F \times F \times F, 1, 1/2 \right) \right) \left( I_{PE} \left( \chi_K, \frac{1}{2} \right) \right) \subseteq \pi_{1, \nu}^{(1,F_\nu \times F_\nu \times F_\nu, 1/2)},$$

$$N_\nu \left( \text{Res} \left( F \times F \times F, 1, 1/2 \right) \right) \left( I_{PE} \left( \chi_K, \frac{1}{2} \right) \right) \subseteq \pi_{1, \nu}^{(1,F_\nu \times F_\nu \times F_\nu, 1/2)}.$$ 

Equalities follow since both intertwining operators sends a non-zero spherical vector to a non-zero spherical vector at $s = \frac{1}{2}$.

Now, assume that $K_\nu$ is a field. If $F_\nu = \mathbb{R}$, the claim is proved in Appendix A we assume that $\nu \neq \mathbb{R}$.

The idea, in this case, is similar to that of [Seg, Lem. 4.5]. As $N_\nu (w_3, \chi_s)$ is an isomorphism at $s = \frac{1}{2}$, it is enough to show that the unique irreducible subrepresentation of $I_\nu (\chi_{K, \nu}, \frac{1}{2})$ is a subquotient of the kernel of $N_\nu (w_3, w_2^{-1} \cdot \chi_s)$ at $s = \frac{1}{2}$.

A straight-forward computation of the Jacquet functor $J^{HE}_{T_E} \left( \text{Ind}^{HE}_{B_E} \chi_{\frac{1}{2}} \right)$ shows that any character appearing in it will appear with multiplicity 2; indeed, it follows immediately from the fact that $| \text{Stab}_{W} \left( \chi_{\frac{1}{2}, \nu} \right) | = 2$.

On the other hand, by Frobenius reciprocity, $\chi_{\frac{1}{2}}$ will appear in the Jacquet functor of the unique irreducible subrepresentation of $I_\nu (\chi_{K, \nu}, \frac{1}{2})$. One then checks that the multiplicity of $\chi_{\frac{1}{2}}$ in the kernel of $N_\nu (w_3, w_2^{-1} \cdot \chi_s)$ is 2 and hence the unique irreducible subrepresentation of $I_\nu (\chi_{K, \nu}, \frac{1}{2})$ lies in the kernel of $N_\nu (w_3, w_2^{-1} \cdot \chi_s)$. The claim then follows.

\[\square\]

6.3. Summary

We record the results of this section as a theorem.

Theorem 6.6. The non-square integrable residues $\text{Res} (s_0, \chi, E)$ with $\Re (s_0) > 0$ are given as follows:

1. If $s_0 = \frac{3}{2}$, $\chi = 1$ and $E = F \times K$, where $K$ is a field, the residue is given by

$$\text{Res} \left( \frac{3}{2}, 1, F \times K \right) \cong I_{P(2,3,4)} (1).$$
(2) If $s_0 = \frac{1}{2}$, $\chi = 1$ and $E = F \times K$ then residue is given by
\[ \text{Res} \left( \frac{1}{2}, 1, F \times K \right) \cong \bigotimes_{\nu \in \mathcal{P}} \pi_{1, \nu}. \]

(3) If $s_0 = \frac{1}{2}$, $\chi = \chi_K$ and $E = F \times K$, where $K$ is a quadratic field extension of $F$, then residue is given by
\[ \text{Res} \left( \frac{1}{2}, 1, F \times K \right) \cong \bigotimes_{\nu \in \mathcal{P}} \pi_{1, \nu}. \]

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Appendix A. Structure of Local Degenerate Principal Series at Archimedean Places

All the results in this section were obtained using the atlas of lie groups \[dCvL\]. We consider the cases where \(s_0 = \frac{1}{2}\) since the cases where \(s_0 = \frac{3}{2}\) and \(s_0 = \frac{5}{2}\) were dealt with in Section 4. We also wish to remind the reader that \(\mathbb{R}\) has no cubic field extensions and \(\mathbb{C}\) has no proper field extension and that the only finite-order characters of \(\mathbb{R}^\times\) are \(1\) and \(\text{sgn}\) while the only finite-order character of \(\mathbb{C}^\times\) is \(1\).

Proposition A.1. 

1. Assume \(F_\nu = \mathbb{R}\) and \(E_\nu = \mathbb{R} \times \mathbb{R} \times \mathbb{R}\).
   - \(I_\nu (1_\nu, \frac{1}{2})\) has length 5 with a unique irreducible quotient, which is spherical.
   - \(I_\nu (\text{sgn}_\nu, \frac{1}{2})\) has length 5 with a unique irreducible subrepresentation; the maximal semisimple quotient is a direct sum of four irreducible representations.

2. Assume \(F_\nu = \mathbb{R}\) and \(E_\nu = \mathbb{R} \times \mathbb{C}\). Both \(I_\nu (1_\nu, \frac{1}{2})\) and \(I_\nu (\text{sgn}_\nu, \frac{1}{2})\) has length 2 with a unique irreducible subrepresentation and a unique irreducible quotient.

3. Assume \(F_\nu = \mathbb{C}\) and \(E_\nu = \mathbb{C} \times \mathbb{C} \times \mathbb{C}\). \(I_\nu (1_\nu, \frac{1}{2})\) is irreducible.

In what follows, we supply the output from the ATLAS software regarding the above mentioned representations. It should be noted before hand that ATLAS thinks of irreducible representations of a real reductive Lie group \(G\) in terms of the Langlands parametrization, i.e. triples \((x, \lambda, \nu)\), called parameters, where:

- \(x\) is an element in \(K \setminus G/B\), where \(K\) is a fixed maximal compact subgroup and \(B\) is a fixed Borel subgroup. The element \(x\) gives rise to a Borel subgroup \(B_x\), a Cartan subgroup \(H_x\) and an involution \(\theta_x\) of \(G\).
- \(\lambda \in X^*/(1-\theta_x)X^*\), where \(X^*\) is the lattice of algebraic characters of \(G_x\). This is a character on the maximal compact subgroup of \(H_x\). When \(H_x\) is split, it can be identified with the group of connected components of \(H_x\).
- \(\nu \in (X^*)^{-\theta_x}\).
- The parameter \(p = (x, \lambda, \nu)\) corresponds to the Langlands quotient of \(J(p) = \text{Ind}^{H_x}_{B_x} (\lambda \otimes \nu)\).
- The infinitesimal character of \(J(p)\) is \(\gamma = \frac{1+\theta_x}{2}\lambda + \frac{1-\theta_x}{2}\nu = \frac{1+\theta_x}{2}\lambda + \nu\).

For more details the reader may consult [AvLEJ] (in particular Proposition 4.3 there), [AdC09] and [Ada08]. The reader is also advised to consider the documentation in [http://www.liegroups.org/](http://www.liegroups.org/).

The calculations bellow were performed on version 1.0.1 of ATLAS.

A.1. \(F_\nu = \mathbb{R}, E_\nu = \mathbb{R} \times \mathbb{R} \times \mathbb{R}\) \(s = \frac{1}{2}\) and \(\chi_\nu = 1_\nu\)

We start by defining the group \(H = H_E = \text{Spin}(4,4)\) and the subgroups \(B = B_E, T = T_E\), \(P = P_E\) and \(M = M_E\):

atlas> set H=Spin(4,4)
Variable H: RealForm
atlas> #KGB(H)
Value: 109
The degenerate residual spectrum of $\text{Spin}_E^F$ along the Heisenberg parabolic

Variable $x$: KGBelt
Value: disconnected split real group with Lie algebra
$\text{sl}(2, \mathbb{R}).\text{sl}(2, \mathbb{R}).\text{sl}(2, \mathbb{R}).\text{gl}(1, \mathbb{R})$'

Variable $B$: ([int], KGBelt)
Value: disconnected split real group with Lie algebra
$\text{gl}(1, \mathbb{R}).\text{gl}(1, \mathbb{R})$'

Then, we consider $I_\nu (1_{\nu}, \frac{1}{2})$ as a quotient of $\text{Ind}_{BE}^{HE} \eta_{\frac{1}{2}}$. First, we define $\eta_{\frac{1}{2}} = (1, -1, 1, 1)$ and consider the induction $\text{Ind}_{BE}^{ME} \eta_{\frac{1}{2}}$ and then we pick up the unique irreducible quotient of $\text{Ind}_{BE}^{ME} \eta_{\frac{1}{2}}$; this is the one-dimensional representation $|\text{det}_{ME}|^{\frac{1}{2}}$ of $M_E$.

Indeed we see that it has length 5 and we turn to show that the last parameter represents the unique irreducible quotient of $I_\nu (1_{\nu}, \frac{1}{2})$. One can further show that the direct sum of the other representations is the maximal semi-simple subrepresentation of $I_\nu (1_{\nu}, \frac{1}{2})$. 
The degenerate residual spectrum of Spin$^E$ along the Heisenberg parabolic

By Theorem 4.1, $\text{Ind}_{H}^{E} \lambda_{(1, -1, 1, 1)}$ admits two irreducible quotients. Since $I_{\nu} (1_{\nu}, \frac{1}{2})$ is a quotient of $\text{Ind}_{H}^{E} \lambda_{(1, -1, 1, 1)}$, any irreducible quotient of $I_{\nu} (1_{\nu}, \frac{1}{2})$ is an irreducible quotient of $\text{Ind}_{H}^{E} \lambda_{(1, -1, 1, 1)}$. We now compute the parameters of the two irreducible quotients of $\text{Ind}_{H}^{E} \lambda_{(1, -1, 1, 1)}$ and see that only one appears in the Jordan-Hölder series of $I_{\nu} (1_{\nu}, \frac{1}{2})$.

```
atlas> set Q=Parabolic :([1],x)
Variable Q: ([int],KGBElt)
atlas> set L=Levi(Q)
Variable L: RealForm
atlas> void: for q in monomials(real_induce_irreducible(p,L))
    do prints(q,",is_finite_dimensional (q)) od
final parameter (x=0,lambda=[-1,2,-1,-1]/2,nu=[1,0,1,1]/2) false
final parameter (x=1,lambda=[-1,2,-1,-1]/2,nu=[0,1,0,0]/1) true
atlas> set par0=monomials(real_induce_irreducible(p,L))
Variable par0: [Param]
atlas> real_induce_standard (par0[0],H)
Value: non-normal parameter (x=106,lambda=[1,1,1,1]/1,nu=[1,0,1,1]/2)
atlas> finalize($)
Value: [final parameter (x=65,lambda=[-1,4,-1,-1]/1,nu=[-1,3,-1,-1]/2)]
atlas> real_induce_standard (par0[1],H)
Value: final parameter (x=108,lambda=[1,1,1,1]/1,nu=[0,1,0,0]/1)
```

Namely, $\pi_{-2}$ is not a constituent of $I_{\nu} (1_{\nu}, \frac{1}{2})$. It follows that $\pi_{1}$ is the unique irreducible quotient of $I_{\nu} (1_{\nu}, \frac{1}{2})$.

A.2. $F_{\nu} = \mathbb{R}$, $E_{\nu} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, $s = \frac{1}{2}$ and $\chi_{\nu} = \text{sgn}_{\nu}$

This case was proven in Section 4. For the benefit of the reader we include the relevant calculation in ATLAS.

We set $H$, $B$, $T$, $P$ and $M$ as in the previous case. Then, we consider $I_{\nu} (\text{sgn}, \frac{1}{2})$ as a quotient of $\text{Ind}_{BE}^{E} \eta_{\text{sgn}} \otimes \eta_{\frac{1}{2}}$. First, we define $\eta_{\frac{1}{2}} = (1, -1, 1, 1)$ and consider the induction $\text{Ind}_{BE}^{E} \eta_{\frac{1}{2}}$ and then we pick up the unique irreducible quotient of $\text{Ind}_{BE}^{E} \eta_{\frac{1}{2}}$; this is the one-dimensional representation $|\text{det}_{ME}|^{\frac{1}{2}}$ of $M_{E}$.

```
atlas> set z=KGB(T,0)
Variable z: KGBElt
atlas> set u=vec:[1,-1,1,1]
Variable u: vec
atlas> set ud=dominant(H,u)
Variable ud: vec
atlas> ud
Value: [ 0, 1, 0, 0 ]```
THE DEGENERATE RESIDUAL SPECTRUM OF $\text{Spin}^c_E$ ALONG THE HEISENBERG PARABOLIC 43

atlas> set psgn=parameter(z,ud,u)
Variable psgn: Param

atlas> set par0=monomials(real_induce_irreducible(psgn,M))
Variable par0: [Param]

atlas>void:for q in par0 do prints(q," ",is_finite_dimensional (q)) od
final parameter (x=0,lambda=[2,-3,2,2]/2,nu=[0,1,0,0]/2) false
final parameter (x=2,lambda=[2,-3,2,2]/2,nu=[0,1,0,0]/2) false
final parameter (x=3,lambda=[2,-3,2,2]/2,nu=[0,1,0,0]/2) false
final parameter (x=4,lambda=[2,-3,2,2]/2,nu=[0,0,0,1]/2) false
final parameter (x=5,lambda=[2,-3,2,2]/2,nu=[0,0,0,1]/2) false
final parameter (x=6,lambda=[2,-3,2,2]/2,nu=[0,0,1,0]/2) false
final parameter (x=7,lambda=[2,-3,2,2]/2,nu=[0,0,1,0]/2) false
final parameter (x=8,lambda=[2,-3,2,2]/2,nu=[1,0,0,0]/2) false
final parameter (x=9,lambda=[2,-3,2,2]/2,nu=[1,0,0,0]/2) false
final parameter (x=10,lambda=[2,-3,2,2]/2,nu=[1,-1,2,2]/2) false
final parameter (x=11,lambda=[2,-3,2,2]/2,nu=[2,-1,0,2]/2) false
final parameter (x=12,lambda=[2,-3,2,2]/2,nu=[2,-1,2,0]/2) false
final parameter (x=13,lambda=[2,1,0,0]/2,nu=[1,-1,1,1]/1) true
atlas> set q=par0[13]
Variable q: Param

Another way to create the parameter $q$ is as follows:

atlas> set q=real_induce_standard(psgn,M)
Variable q: Param

We then consider the composition series of the induced representation $I_\nu (\text{sgn}, \frac{1}{2})$:

atlas> real_induce_irreducible(q,H)
Value:
1*final parameter (x=65,lambda=[1,4,-1,-1]/1,nu=[1,3,-1,-1]/2)
1*final parameter (x=91,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
1*final parameter (x=92,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
1*final parameter (x=93,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
1*final parameter (x=94,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)

Indeed, we see that it has length 5 and we turn to show that the last four parameters represent quotients of $I_\nu (\text{sgn}, \frac{1}{2})$ while the first one does not.

We also recall, from Section 4, that $\text{Ind}_{BE}^{ME} \mu_{\text{sgn}} \otimes \eta_{\frac{1}{2}}$ admits a maximal semi-simple quotient of length 4. We compute this quotient and show that it is the quotient of $I_\nu (\text{sgn}, \frac{1}{2})$.

We first decompose $\text{Ind}_{BE(\mathbb{R}) \cap ME(\mathbb{R})}^{ME(\mathbb{R})} \chi$

atlas> set psgnd=parameter(z,u,ud)
Variable psgnd: Param
THE DEGENERATE RESIDUAL SPECTRUM OF $\text{Spin}^E_8$ ALONG THE HEISENBERG PARABOLIC 44

\[ \text{atlas> real induce irreducible}(\text{psgn}, M) \]

Value:
1*final parameter \( (x=0, \lambda=[0,1,0,0]/2, \nu=[0,1,0,0]/1) \)
1*final parameter \( (x=1, \lambda=[0,1,0,0]/2, \nu=[0,1,0,0]/1) \)
1*final parameter \( (x=2, \lambda=[0,1,0,0]/2, \nu=[0,1,0,0]/1) \)
1*final parameter \( (x=3, \lambda=[0,1,0,0]/2, \nu=[0,1,0,0]/1) \)

For each of the irreducible constituents of \( \text{Ind}^M_{BE(R)}(R) \cap M_{BE(R)} \tilde{\chi} \) we compute its unique irreducible quotient.

\[ \text{atlas> void:for } \sigma_i \text{ in monomials(real induce irreducible(psgn,M))} \]
\[ \text{ do prints(real induce standard}(\sigma_i, H)\text{)) od } \]

final parameter \( (x=91, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
final parameter \( (x=94, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
final parameter \( (x=93, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
final parameter \( (x=92, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)

It remains to show that the multiplicity of each one of this quotients in \( \text{Ind}^{H_E}_{BE} \mu_{\text{psgn}} \otimes \eta^2_1 \) is 1.

\[ \text{atlas> real induce irreducible}(\text{psgn}, H) \]

Value:
1*final parameter \( (x=0, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
3*final parameter \( (x=1, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
3*final parameter \( (x=3, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
3*final parameter \( (x=4, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
1*final parameter \( (x=5, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
1*final parameter \( (x=6, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
1*final parameter \( (x=7, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
3*final parameter \( (x=9, \lambda=[0,1,0,0]/1, \nu=[0,0,0,0]/1) \)
1*final parameter \( (x=16, \lambda=[0,1,0,0]/1, \nu=[-1,2,-1,-1]/2) \)
1*final parameter \( (x=17, \lambda=[0,1,0,0]/1, \nu=[-1,2,-1,-1]/2) \)
1*final parameter \( (x=18, \lambda=[0,1,0,0]/1, \nu=[-1,2,-1,-1]/2) \)
1*final parameter \( (x=19, \lambda=[0,1,0,0]/1, \nu=[-1,2,-1,-1]/2) \)
2*final parameter \( (x=47, \lambda=[0,3,-2,0]/1, \nu=[0,1,-1,0]/1) \)
2*final parameter \( (x=48, \lambda=[0,3,-2,0]/1, \nu=[0,1,-1,0]/1) \)
2*final parameter \( (x=49, \lambda=[-1,2,-1,1]/1, \nu=[-1,1,0,0]/1) \)
2*final parameter \( (x=50, \lambda=[-1,2,-1,1]/1, \nu=[-1,1,0,0]/1) \)
2*final parameter \( (x=51, \lambda=[0,3,0,-2]/1, \nu=[0,1,0,-1]/1) \)
2*final parameter \( (x=52, \lambda=[0,3,0,-2]/1, \nu=[0,1,0,-1]/1) \)
4*final parameter \( (x=65, \lambda=[-1,4,-1,-1]/1, \nu=[-1,3,-1,-1]/2) \)
1*final parameter \( (x=91, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
1*final parameter \( (x=92, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
1*final parameter \( (x=93, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
1*final parameter \( (x=94, \lambda=[0,3,0,0]/1, \nu=[0,1,0,0]/1) \)
The degenerate residual spectrum of Spin\(^{E}_8\) along the Heisenberg parabolic

**Remark A.2.** We wish to remark on the output of ATLAS in case that induction from the parameter \(p\), entered to `real_induce_standard`, is not a standard module.

- If there exists a parameter \(p'\) conjugate to \(p\) such that induction from \(p'\) is a standard module, then ATLAS would compute the unique irreducible quotient of that induction.
- If there is no such parameter conjugate to \(p\), one can still find a "weakly-dominant" parameter \(p'\) conjugate to \(p\). In which case, ATLAS would return a parameter for the maximal semi-simple quotient of the induction from \(p'\). It is then possible to compute the parameters for the irreducible quotients of the induction from \(p'\).

This is the case of the parameter \((x = 0, \lambda = [0, 1, 0, 0]/1, \nu = [1, -1, 1, 1]/1)\).

```
  atlas> real_induce_standard(psgn,H)
  Value: non-dominant parameter (x=108,lambda=[1,2,1,1]/1,nu=[1,-1,1,1]/1)
```

As we already know, this induction has a maximal semi-simple quotient of length 4. In order to find the parameters of the irreducible quotients in this induction we need to finalize the parameter as follows.

```
  atlas> set standard_quotient=finalize(real_induce_standard (psgn,H))
  Variable standard_quotient: [Param]
  atlas> void:for t in standard_quotient do prints(t) od
  final parameter (x=91,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
  final parameter (x=92,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
  final parameter (x=93,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
  final parameter (x=94,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
```

A.3. \(F_\nu = \mathbb{R}, E_\nu = \mathbb{R} \times \mathbb{C}, s = \frac{1}{2}\) and \(\chi_\nu = 1_\nu\)

We start by defining the group \(H = H_E = Spin(5,3)\) and the subgroups \(B = B_E, T = T_E, P = P_E\) and \(M = M_E\).

```
  atlas> set H=Spin(5,3)
  Variable H: RealForm
  atlas> #KGB(H)
  Value: 40
  atlas> set x=KGB(H,39)
  Variable x: KGBElt
  atlas> set P=Parabolic :([0,2,3],x)
  Variable P: ([int],KGBElt)
  atlas> set M=Levi(P)
  Variable M: RealForm
  atlas> set B=Parabolic :([],x)
  Variable B: ([int],KGBElt)
  atlas> set T=Levi(B)
```
Variable $T$: RealForm

Then, we consider $I_{\nu} \left(1_{\nu}, \frac{1}{2}\right)$ as a quotient of $\text{Ind}_{HE}^{M} \eta_{\frac{1}{2}}$. First, we define $\eta_{\frac{1}{2}} = (1, -1, 1)$ and consider the induction $\text{Ind}_{HE}^{M} \eta_{\frac{1}{2}}$ and then we pick up the unique irreducible quotient of $\text{Ind}_{HE}^{M} \eta_{\frac{1}{2}}$; this is the one-dimensional representation $|det_{M}|^{\frac{1}{2}}$ of $M_{E}$.

atlas> set $z$=KGB($T$,0)
Variable $z$: KGBElt
atlas> set $u$=vec: [1, -1, 1, 1]
Variable $u$: vec
atlas> set $p$=parameter($z$,null(rank($H$)), $u$)
Variable $p$: Param
atlas> set par0=monomials(real_induce_irreducible($p$,$M$))
Variable par0: [Param]
atlas> void: for $q$ in par0 do prints($q$, " ",is_finite_dimensional ($q$)) od
final parameter (x=0,lambda=[2, -3, 2, 2]/2,nu=[0, 1, 0, 0]/2) false
final parameter (x=1,lambda=[2, -3, 2, 2]/2,nu=[1, 0, 0, 0]/1) false
final parameter (x=2,lambda=[2, -3, 2, 2]/2,nu=[0, -1, 2, 2]/2) false
final parameter (x=3,lambda=[2, -3, 2, 2]/2,nu=[1, -1, 1, 1]/1) true
atlas> set $q$=par0[3]
Variable $q$: Param

We then consider the induced representation $I_{\nu} \left(1_{\nu}, \frac{1}{2}\right)$:

atlas> real_induce_irreducible($q$,$H$)
Value:
1*final parameter (x=16,lambda=[0, 3, -1, -1]/1,nu=[-1, 3, -1, -1]/2)
1*final parameter (x=36,lambda=[1, 2, 1, -1]/1,nu=[0, 1, 0, 0]/1)

As explained in Section[4] $\text{Ind}_{H}^{B} \lambda_{(1,-1,1)}$ admits two irreducible quotients, $\pi_{1,\nu}$ and $\pi_{-2,\nu}$. Since $I_{\nu} \left(1_{\nu}, \frac{1}{2}\right)$ is a quotient of $\text{Ind}_{H}^{B} \lambda_{(1,-1,1)}$, any irreducible quotient of $I_{\nu} \left(1_{\nu}, \frac{1}{2}\right)$ is an irreducible quotient of $\text{Ind}_{H}^{B} \lambda_{(1,-1,1)}$.

We now compute the parameters of $\pi_{1,\nu}$ and $\pi_{-2,\nu}$. It turns out, as opposed to the split case $E_{\nu} = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, that both are constituents of $I_{\nu} \left(1_{\nu}, \frac{1}{2}\right)$. We then show that, in fact, the multiplicity of $\pi_{-2,\nu}$ in $\text{Ind}_{H}^{B} \lambda_{(1,-1,1)}$ is 2. It turns out that the $\pi_{-2,\nu}$ constituent of $I_{\nu} \left(1_{\nu}, \frac{1}{2}\right)$ is not a quotient. Namely, the exact sequence

$$(A.1) \quad 0 \rightarrow \pi_{-2,\nu} \rightarrow I_{\nu} \left(1_{\nu}, \frac{1}{2}\right) \rightarrow \pi_{1,\nu} \rightarrow 0$$

does not split.

First, we construct the parameters of $\pi_{1,\nu}$ and $\pi_{-2,\nu}$:

atlas> set $Q$=Parabolic:([1],x)
Variable $Q$: ([int],KGBElt)
atlas> set $L$=Levi($Q$)
Variable L: RealForm

\texttt{atlas> real\_induce\_standard (par0[0],H)}
\texttt{Value: non-normal parameter (x=38,lambda=[1,1,1,1]/1,nu=[1,0,1,1]/2)}
\texttt{atlas> finalize($)}
\texttt{Value: \{final parameter (x=16,lambda=[0,3,-1,-1]/1,nu=[-1,3,-1,-1]/2)\}}

\texttt{atlas> real\_induce\_standard (par0[1],H)}
\texttt{Value: non-normal parameter (x=39,lambda=[1,1,1,1]/1,nu=[0,1,0,0]/1)}
\texttt{atlas> finalize($)}
\texttt{Value: \{final parameter (x=36,lambda=[1,2,1,-1]/1,nu=[0,1,0,0]/1)\}}

And now, we show that the multiplicity of $\pi_{-2,\nu}$ in $\text{Ind}_{H}^{G} \lambda_{(1,1,1)}$ is 2, and hence it does not follow that $i_{\nu} (1,\frac{1}{2})$ is semi simple; in fact we will show that it is not.

\texttt{atlas> real\_induce\_irreducible(p,H)}
\texttt{Value:}
\texttt{1*final parameter (x=0,lambda=[0,1,0,0]/1,nu=[0,0,0,0]/1)}
\texttt{1*final parameter (x=1,lambda=[0,1,0,0]/1,nu=[0,0,0,0]/1)}
\texttt{2*final parameter (x=3,lambda=[0,1,0,0]/1,nu=[-1,2,-1,-1]/2)}
\texttt{2*final parameter (x=12,lambda=[-1,2,0,0]/1,nu=[-1,1,0,0]/1)}
\texttt{1*final parameter (x=14,lambda=[-1,2,0,0]/1,nu=[-1,1,0,0]/1)}
\texttt{2*final parameter (x=16,lambda=[0,3,-1,-1]/1,nu=[-1,3,-1,-1]/2)}
\texttt{1*final parameter (x=36,lambda=[1,2,1,-1]/1,nu=[0,1,0,0]/1)}

The idea of showing that the sequence in Equation (A.1) does not split is similar to the ideas in [Sah95]. The sequence splits if the subrepresentation $\pi_{-2,\nu}$ of $I_{\nu} (1,\frac{1}{2})$ is also a quotient. In order to show that it is not a quotient, it is enough to show that, given a $K$-types $\rho_1$ and $\rho_{-2}$ of $\pi_{1,\nu}$ and $\pi_{-2,\nu}$, there is a non-zero element in the universal enveloping algebra ($H_{C}$) of the complexified Lie algebra $H_{C} = \text{Lie} (H) \otimes \mathbb{C}$, sending vectors from $\rho_1$ to $\rho_{-2}$. We will show the existence of such an operator.

The maximal compact subgroup of $H_{E} (F_{\nu})$ is $K = \text{Spin} (5) \times SU (2) / \mu_2 \equiv Sp (2) \times SU (2) / \mu_2$. Any finite dimensional irreducible representation of $K$ is of the form $V_{(x,y)} \boxtimes V_{z}$, where:

- $V_{(x,y)}$ is an irreducible representation of $Sp (2)$ with highest weight $(x,y)$; in particular $x, y \in \mathbb{N}$ and $x \geq y \geq 0$.
- $V_{z}$ is an irreducible representation of $SU (2)$ with highest weight $z$; in particular $z \in \mathbb{N}$ and the dimension of $V_{z}$ is $z + 1$.
- $x + y + z$ is even.

We recall [AGPS] Lemma 6.6.

\textbf{Lemma A.3.} The type $V_{(0,0)} \boxtimes V_{n}$ with $n > 0$ appears in $\pi_{1}$ for $n$ odd and in $\pi_{-2}$ for $n$ even, with multiplicity 1.
We also note that the trivial representation of $K$, $V_{(0,0)} \boxtimes V_0$, appears with multiplicity 1 in $\pi_1$ as $\pi_1$ is spherical and $\pi_{-2}$ is not.

Fix a highest weight vector $v_n$ in $V_{(0,0)} \boxtimes V_n$ and let $X_+$ denote a raising operator in $\mathfrak{su}(2)$. For $s \in \mathbb{C}$ we consider the $(\mathfrak{sl}_2, K)$-module associated to it. The operator $X_+^2$ will send $v_0$ to $f(s)v_4$, where $f(s)$ is a quadratic polynomial in $s$.

Since $\pi_1$ is a subrepresentation of $I_{\nu}(1, -\frac{1}{2})$, it follows that $f\left(-\frac{1}{2}\right) = 0$. Also, since the unique irreducible representation of $I_{\nu}(1, -\frac{5}{2})$ is trivial, it also follows that $f\left(-\frac{5}{2}\right) = 0$. We conclude that $f\left(\frac{1}{2}\right) \neq 0$ from which the claim follows.

A.4. $F_\nu = \mathbb{R}$, $E_\nu = \mathbb{R} \times \mathbb{C}$, $s = \frac{1}{2}$ and $\chi_\nu = \text{sgn}_\nu$

This case is proved in Section 4. However, for the benefit of the reader, we demonstrate how to find the irreducible constituents of $I_{\nu}(\text{sgn}, \frac{1}{2})$ and the parameter of its unique irreducible quotient using ATLAS. We further perform a simple calculation which is used in Section 6.

We set $H$, $B$, $T$, $P$ and $M$ as in the previous case. We then compute $I_{\nu}(\text{sgn}, \frac{1}{2})$ as a quotient of $\text{Ind}_{B_E}^{H_E} \mu_{\text{sgn}} \otimes \eta_{\frac{1}{2}}$. We define $\eta_{\frac{1}{2}} = (1, -1, 1)$ and consider the induction $\text{Ind}_{M_E}^{H_E} \mu_{\text{sgn}} \otimes \eta_{\frac{1}{2}}$ and then we pick up the unique irreducible quotient of $\text{Ind}_{M_E}^{H_E} \mu_{\text{sgn}} \otimes \eta_{\frac{1}{2}}$; this is the one-dimensional representation $(\chi \circ \text{det}_{M_E}) \otimes |\text{det}_{M_E}|^{\frac{1}{2}}$ of $M_E$.

We then consider the induced representation $I_{\nu}(\text{sgn}, \frac{1}{2})$:

Value:

1*final parameter (x=14, lambda=[-2, 3, 2, 2]/2, nu=[0, 1, 0, 0]/1) 1*final parameter (x=32, lambda=[-2, 3, 2, 2]/2, nu=[0, 1, 0, 0]/1)
We check that the last parameter is the unique irreducible quotient of \( \text{Ind}_{BE}^{HE} \mu_{\text{sgn}} \otimes \eta_{\frac{1}{2}} \) (and hence of \( I_{\nu} (\text{sgn}, \frac{1}{2}) \)):

```
atlas> real_induce_standard(psgn ,H)
Value: non-dominant parameter (x=39,lambda=[1,2,0,0]/1,nu=[1,-1,1,1]/1)
atlas> finalize($)
Value: [final parameter (x=32,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)]
```

Finally, we compute the irreducible constituents of \( \text{Ind}_{BE}^{HE} \mu_{\text{sgn}} \otimes \eta_{\frac{1}{2}} \).

```
atlas> real_induce_irreducible(psgn ,H)
Value:
1*parameter(x=0,lambda=[0,1,0,0]/1,nu=[0,0,0,0]/1)
1*parameter(x=1,lambda=[0,1,0,0]/1,nu=[0,0,0,0]/1)
2*parameter(x=3,lambda=[0,1,0,0]/1,nu=[-1,2,-1,-1]/2)
1*parameter(x=12,lambda=[-1,2,0,0]/1,nu=[-1,1,0,0]/1)
1*parameter(x=14,lambda=[-1,2,0,0]/1,nu=[-1,1,0,0]/1)
1*parameter(x=24,lambda=[-1,1,1,1]/1,nu=[-1,1,0,0]/1)
1*parameter(x=16,lambda=[0,3,-1,-1]/1,nu=[-1,3,-1,-1]/2)
1*parameter(x=32,lambda=[0,3,0,0]/1,nu=[0,1,0,0]/1)
```

In Section [4] we use the fact that the unique irreducible quotient of \( I_{\nu} (\text{sgn}, \frac{1}{2}) \) is the image of \( N_{\nu} (w_{23}, \chi_s) \). In order to prove this, it is enough to show that \( N_{\nu} (w_{2}, \chi_s) \) acts as an isomorphism on \( I_{\nu} (\text{sgn}, \frac{1}{2}) \) and that the irreducible subrepresentation of \( I_{\nu} (\text{sgn}, \frac{1}{2}) \) lies in the kernel of \( N_{\nu} (w_{3}, w_{2}^{-1} \cdot \chi_s) \). Indeed, \( N_{\nu} (w_{2}, \chi_s) \) is an isomorphism:

```
atlas> set P2=Parabolic :([1],x)
Variable P2: ([int],KGBElt)
atlas> set M2=Levi(P2)
Variable M2: RealForm
atlas> set u=[-1,2,-1,-1]
Variable u: [int]
atlas> set sgn=vec:[0,1,0,0]
Variable sgn: vec
atlas> set p=parameter(z,sgn,u)
Variable p: Param
atlas> set q=real_induce_standard (p,M2)
Variable q: Param
atlas> real_induce_irreducible(q,H)
Value:
2*final parameter (x=3,lambda=[0,1,0,0]/1,nu=[-1,2,-1,-1]/2)
1*final parameter (x=12,lambda=[-1,2,0,0]/1,nu=[-1,1,0,0]/1)
1*final parameter (x=14,lambda=[-1,2,0,0]/1,nu=[-1,1,0,0]/1)
1*final parameter (x=16,lambda=[0,3,-1,-1]/1,nu=[-1,3,-1,-1]/2)
1*final parameter (x=24,lambda=[-1,1,1,1]/1,nu=[-1,1,0,0]/1)
```
THE DEGENERATE RESIDUAL SPECTRUM OF $Spin^E_8$ ALONG THE HEISENBERG PARABOLIC

1*final parameter (x=32, lambda=[0,3,0,0]/1, nu=[0,1,0,0]/1)
and the irreducible subrepresentation of $I_\nu (\text{sgn}, \frac{1}{2})$ lies in the kernel of $N_\nu (w_3, w_2^{-1} \cdot \chi_s)$:

atlas> set P3=Parabolic :([2,3],x)
Variable P3: ([int],KGBElt)
atlas> set M3=Levi(P3)
Variable M3: RealForm
atlas> set u=[1,-2,1,1]
Variable u: [int]
atlas> set sgn=vec:[1,1,0,0]
Variable sgn: vec
atlas> set p=parameter(z,sgn,u)
Variable p: Param
atlas> set q=real_induce_standard(p,M3)
Variable q: Param
atlas> real_induce_irreducible(q,H)
Value:
1*final parameter (x=1, lambda=[0,1,0,0]/1, nu=[0,0,0,0]/1)
1*final parameter (x=16, lambda=[0,3,-1,-1]/1, nu=[-1,3,-1,-1]/2)
1*final parameter (x=24, lambda=[-1,1,1,1]/1, nu=[-1,1,0,0]/1)
1*final parameter (x=32, lambda=[0,3,0,0]/1, nu=[0,1,0,0]/1)
atlas> set q=monomials(real_induce_irreducible(p,M3))[0]
Variable q: Param
atlas> real_induce_irreducible(q,H)
Value:
1*final parameter (x=0, lambda=[0,1,0,0]/1, nu=[0,0,0,0]/1)
2*final parameter (x=3, lambda=[0,1,0,0]/1, nu=[-1,2,-1,-1]/2)
1*final parameter (x=12, lambda=[-1,2,0,0]/1, nu=[-1,1,0,0]/1)
1*final parameter (x=14, lambda=[-1,2,0,0]/1, nu=[-1,1,0,0]/1)

A.5. $F_\nu = \mathbb{C}$, $E_\nu = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$ s = $\frac{1}{2}$ and $\chi_\nu = 1_\nu$

We start this case by reminding the reader that a complex reductive Lie group $G(\mathbb{C})$ is realized in ATLAS as the product $G(\mathbb{R}) \times G(\mathbb{R})$, where $G(\mathbb{R})$ is the real split form of $G(\mathbb{C})$.

We start by defining the group $H = Spin(8, \mathbb{C})$ and the subgroups $B_E, T_E, P_E$ and $M_E$:

atlas> set G=Spin(4,4)
Variable G: RealForm
atlas> set H=complex(G)
Variable H: RealForm
atlas> #KGB(H)
Value: 192
THE DEGENERATE RESIDUAL SPECTRUM OF $\text{Spin}_E^8$ ALONG THE HEISENBERG PARABOLIC

atlas> set x=KGB(H,191)
Variable x: KGBElt
atlas> set P=Parabolic :([0,2,3,4,6,7],x)
Variable P: ([int],KGBElt)
atlas> set M=Levi(P)
Variable M: RealForm
atlas> M
Value: connected quasisplit real group with Lie algebra
'sl(2,C).sl(2,C).sl(2,C).gl(1,C)'
atlas> set B=Parabolic :([],x)
Variable B: ([int],KGBElt)
atlas> set T=Levi(B)
Variable T: RealForm

Then, we consider $I_\nu \left(1_\nu, \frac{1}{2}\right)$ as a quotient of $\text{Ind}_{BE}^{H_E} \eta_\frac{1}{2}$. First, we define $\eta_\frac{1}{2} = (1,-1,1,1)$ and consider the induction $\text{Ind}_{BE}^{ME} \eta_1$ and then we pick up the unique irreducible quotient of $\text{Ind}_{BE}^{ME} \eta_\frac{1}{2}$; this is the one-dimensional representation $|\text{det}_{ME}|^\frac{1}{2}$ of $M_E$.

atlas> set z=KGB(T,0)
Variable z: KGBElt
atlas> set u=vec:[1,-1,1,1,-1,1,1]
Variable u: vec
atlas> set p=parameter(z,null(rank(H)),u)
Variable p: Param
atlas> set par0=monomials(real_induce_irreducible(p,M))
Variable par0: [Param]
atlas> void:for q in par0 do prints(q," ",is_finite_dimensional (q)) od
final parameter (x=0,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=1,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=2,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=3,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=4,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=5,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=6,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) false
final parameter (x=7,lambda=[2,-3,2,2,-3,2,2]/2,nu=[0,0,1,0,0,1,0,0]/1) true
atlas> set q=par0[7]
Variable q: Param
Inducing to $H_E(C)$, we see that $I_\nu \left(1_\nu, \frac{1}{2}\right)$ is irreducible.

atlas> real_induce_irreducible(q,H)
Value: 1*final parameter (x=163,lambda=[2,0,1,1,-2,4,-1,-1]/1,nu=[0,1,0,0,1,0,0,0]/1)
Appendix B. Calculation for Non-Square Integrable Residues

In this section we evaluate the normalized Eisenstein series of Equation (6.1) at certain points, useful to the proof of the Siegel-Weil identities of Section 6.

For any number field \( L \) we write

\[
\xi_L(s) = \frac{R_L}{s-1} + a_0 + a_1(s-1) + ...
\]

From the functional equation \( \xi_L(s) = \xi_L(1-s) \) we deduce that

\[
\xi_L(s) = \xi_L(1-s) = \frac{R_L}{(1-s)-1} + a_0 + a_1((1-s)-1) + ... = \frac{-R_L}{s} + a_0 - a_1s + ...
\]

From both, one can deduce the following identities

\[
\begin{align*}
\lim_{s \to -1} (s+1) \xi_L(s+1) &= -R_L & \lim_{s \to -1} (s+1) \xi_L(s+1) &= -R_L \\
\lim_{s \to 1} (s-1) \xi_L(s-1) &= -R_L & \lim_{s \to 1} (s-1) \xi_L(s) &= R_L \\
\lim_{s \to 2} (s-2) \xi_L(s-1) &= R_L & \lim_{s \to 2} (s-2) \xi_L(s-2) &= -R_L \\
\lim_{s \to 1} (2s-2) \xi_L(2s-1) &= R_L & \lim_{s \to 1} (2s-2) \xi_L(2s-2) &= -R_L \\
\lim_{s \to s'} (s+s'-1) \xi_L(s+s'-1) &= R_L & \lim_{s \to s'} (s+s'-2) \xi_L(s+s'-2) &= R_L
\end{align*}
\]

B.1. \( K = F \times F \)

First, we write the normalized Eisenstein series in this case:

\[
E^F_{BE}(\lambda,g) = (s_1-1)(s_1+1)\xi_F(s_1+1) \\
(2s_2-1)(s_2+1)\xi_F(s_2+1) \\
(s_3-1)(s_3+1)\xi_F(s_3+1) \\
(s_4-1)(s_4+1)\xi_F(s_4+1) \\
(s_1+s_2-1)(s_1+s_2+1)\xi_F(s_1+s_2+1) \\
(s_2+s_3-1)(s_2+s_3+1)\xi_F(s_2+s_3+1) \\
(s_2+s_4-1)(s_2+s_4+1)\xi_F(s_2+s_4+1) \\
(s_1+s_2+s_3-1)(s_1+s_2+s_3+1)\xi_F(s_1+s_2+s_3+1) \\
(s_1+s_2+s_4-1)(s_1+s_2+s_4+1)\xi_F(s_1+s_2+s_4+1) \\
(s_2+s_3+s_4-1)(s_2+s_3+s_4+1)\xi_F(s_2+s_3+s_4+1) \\
(s_1+s_2+s_3+s_4-1)(s_1+s_2+s_3+s_4+1)\xi_F(s_1+s_2+s_3+s_4+1) \\
(s_1+2s_2+s_3+s_4-1)(s_1+2s_2+s_3+s_4+1)\xi_F(s_1+2s_2+s_3+s_4+1)E^F_{BE}(f^0_\lambda,\lambda,g).
\]
First, we write the normalized Eisenstein series in this case:

\[ E^B_{s} = 2^5 \frac{3}{s} \cdot \xi_F (s + 1) (s_f^2 - 3)^3 \xi_F (s_f) (s_f - 2) \xi_F (2s_f - 2) \mathcal{E}_E \left( f^0, s_f - \frac{3}{2}, g \right). \]

Taking the limit as \( s_f \to 2 \) yields

\[ E^B_{s} = 2^9 \cdot \xi_F (3) \cdot \xi_F (2) \mathcal{E}_E \left( f^0, \frac{1}{2}, g \right). \]

Plugging in \( \lambda_{(-1,-1,-1,-1)} \), we get:

\[ E^B_{s} = 2^8 \cdot \xi_F (2) \cdot R_F^2 \]

\[ (s_f - 3) (s_f + 1) (s_f - 1)^2 s_f (s_f + 1) \xi_F (s_f - 1) \xi_F (s_f) \xi_F (s_f + 1) \]

\[ (s_f - 3) (s_f + 1) (s_f + 1)^2 s_f (s_f + 1) \xi_F (s_f - 1) \xi_F (s_f) \xi_F (s_f + 1) \]

\[ (s_f + s_f - 3) (s_f + s_f - 2) (s_f + s_f - 2) (s_f + s_f - 1) (s_f + s_f) \]

\[ \xi_F (s_f + s_f - 1) \xi_F (s_f + s_f) \xi_F (s_f + s_f - 2) \mathcal{E}_{P_{(1,2)}} \left( f^0, \lambda_{s_f-1,s_f-1}, g \right). \]

Taking the limit as \( s_f, s_f \to 1 \) yields

\[ E^B_{s} = 2^8 \cdot \xi_F (2) \mathcal{E}_E \left( f^0, 0, g \right). \]

B.2. \( K \) is a Field

First, we write the normalized Eisenstein series in this case:

\[ E^B_{s} = (s_f - 1) (s_f + 1) \xi_F (s_f + 1) \]
\[(s_2 - 1) (s_2 + 1) \xi_F (s_2 + 1)\]
\[(s_3 - 1) (s_3 + 1) \xi_K (s_3 + 1)\]
\[(s_1 + s_2 - 1) (s_1 + s_2 + 1) \xi_F (s_1 + s_2 + 1)\]
\[(s_2 + s_3 - 1) (s_2 + s_3 + 1) \xi_K (s_2 + s_3 + 1)\]
\[(s_1 + s_2 + s_3 - 1) (s_1 + s_2 + s_3 + 1) \xi_K (s_1 + s_2 + s_3 + 1)\]
\[(s_2 + 2s_3 - 1) (s_2 + 2s_3 + 1) \xi_F (s_2 + 2s_3 + 1)\]
\[(s_1 + s_2 + 2s_3 - 1) (s_1 + s_2 + 2s_3 + 1) \xi_F (s_1 + s_2 + 2s_3 + 1)\]
\[(s_1 + 2s_2 + 2s_3 - 1) (s_1 + 2s_2 + 2s_3 + 1) \xi_F (s_1 + 2s_2 + 2s_3 + 1) \mathcal{E}_{BE} (f^0, \lambda, g) .\]

Plugging in \(\lambda_{(-1,s_2,-1)}\), we get:
\[
\mathcal{E}_{BE}^2 (\lambda_{(-1,s_2,-1)}, g) = 2^2 \cdot R_F \cdot R_K
\]
\[
(2s_2 - 4) (2s_2 - 2) (s_2 - 4) (s_2 - 3)^2 (s_2 - 2)^3 (s_2 - 1)^3 s_2^2 (s_2 + 1)
\]
\[
\xi_F (2s_2 - 2) \xi_F (s_2 - 2) \xi_F (s_2 - 1) \xi_F (s_2) \xi_F (s_2 + 1)
\]
\[
\xi_K (s_2 - 1) \xi_K (s_2) \mathcal{E}_E \left( f^0, s_2 - \frac{3}{2}, g \right) .
\]

Taking the limit as \(s_2 \to 2\) yields
\[
\lim_{s_2 \to 2} \left[ (s_2 - 2)^4 \xi_F (s_2 - 2) \xi_F (s_2 - 1) \xi_K (s_2 - 1) \mathcal{E}_E \left( f^0, s_2 - \frac{3}{2}, g \right) \right]
\]
\[
= 2^7 \cdot 3 \cdot \xi_F (2)^2 \cdot \xi_F (3) \cdot \xi_K (2) \cdot R_F^2 \cdot R_K^2 \lim_{s \to \frac{3}{2}} \left[ \left( s - \frac{1}{2} \right) \mathcal{E}_E (f^0, s, g) \right] .
\]

Plugging in \(\lambda_{(-1,-1,s_3)}\), we get:
\[
\mathcal{E}_{BE}^2 (\lambda_{(-1,-1,s_3)}, g) = 2^4 \cdot 3 \cdot \xi_F (2) \cdot R_F^2
\]
\[
(2s_3 - 3) (2s_3 - 2)^2 (2s_3 - 1) (s_3 - 3) (s_3 - 2)^2 (s_3 - 1)^2 s_3^2 (s_3 + 1)
\]
\[
\xi_F (2s_3 - 2) \xi_F (s_3 - 1) \xi_F (2s_3) \xi_K (s_3 - 1) \xi_K (s_3) \xi_K (s_3 + 1)
\]
\[
\mathcal{E}_{L(1,2)} \left( f^0, \lambda_{s_3-1}, g \right) .
\]

Taking the limit as \(s_3 \to 1\) yields
\[
\mathcal{E}_{BE}^2 (\lambda_{(-1,-1,1)}, g) = 2^6 \cdot 3 \cdot \xi_F (2)^2 \cdot \xi_K (2) \cdot R_F^2
\]
\[
\lim_{s_3 \to 1} \left[ (2s_3 - 2)^2 (s_3 - 1)^2 \xi_F (2s_3 - 2) \xi_F (2s_3 - 1) \xi_K (s_3 - 1) \xi_K (s_3) \right]
\]

\[
\mathcal{E}_{P_{1,2}} \left( f^0, 0, g \right)
\]

\[
= 2^6 \cdot 3 \cdot \xi_F (2)^2 \cdot \xi_K (2) \cdot R_F^4 \cdot R_K^2 \mathcal{E}_{P_{1,2}} \left( f^0, 0, g \right).
\]