Second-order schemes for solving decoupled forward backward stochastic differential equations

ZHAO WeiDong¹,*; LI Yang² & FU Yu¹

¹School of Mathematics, Shandong University, Jinan 250100, China; ²College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
Email: wdzhao@sdu.edu.cn, liyang19820816@163.com, nielf0614@126.com
Received April 14, 2013; accepted October 6, 2013; published online January 8, 2014

Abstract  In this paper, by using trapezoidal rule and the integration-by-parts formula of Malliavin calculus, we propose three new numerical schemes for solving decoupled forward-backward stochastic differential equations. We theoretically prove that the schemes have second-order convergence rate. To demonstrate the effectiveness and the second-order convergence rate, numerical tests are given.

Keywords  forward backward stochastic differential equations, second-order scheme, error estimate, trapezoidal rule, Malliavin calculus

MSC(2010)  65C20, 60H35

1 Introduction

Let \((\Omega, \mathcal{F}, \mathbb{F}, P)\) be a filtered complete probability space, where \(\mathbb{F} = (\mathcal{F}_t)_{0 \leq t \leq T}\) is the natural filtration of the standard \(d\)-dimensional Brownian motion \(W = (W_t)_{0 \leq t \leq T}\) on \((\Omega, \mathcal{F}, \mathbb{F}, P)\), and \(T\) is a fixed finite horizon. Let \(L^2 = L^2_F(0, T)\) be the set of all \(\mathcal{F}_t\)-adapted and mean-square-integrable vector/matrix processes.

In this paper, on \((\Omega, \mathcal{F}, \mathbb{F}, P)\), we consider numerical solutions to the decoupled forward-backward stochastic differential equations (FBSDEs)

\[
\begin{align*}
X_t &= X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad \text{(SDE)} \\
Y_t &= \varphi(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad \text{(BSDE)}
\end{align*}
\]

where \(X_0\) is the initial condition of the forward stochastic differential equation (SDE), \(b\) is the drift coefficient valued in \(\mathbb{R}^d\), \(\sigma\) is the diffusion matrix valued in \(\mathbb{R}^{d \times d}\), \(\varphi\) and \(f\) valued in \(\mathbb{R}^m\) are the terminal and generator functions of the backward stochastic differential equation (BSDE), respectively, and \(W_t\) is the standard \(d\)-dimensional Brownian motion. Note that the two integrals with respect to \(W_s\) in (1.1) are the Itô-type integrals.

*Corresponding author
A triple \((X_t, Y_t, Z_t) : [0, T] \times \Omega \to \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d}\) is called an \(L^2\)-adapted solution to (1.1) if it is \(\{\mathcal{F}_t\}\)-adapted, \(L^2\)-integrable, and satisfies (1.1). In 1990, under some standard conditions on the coefficients of (1.1), Pardoux and Peng \([13]\) originally proved the existence and uniqueness of the solution to nonlinear backward stochastic differential equations (BSDEs). The solution \((Y_t, Z_t)\) to (1.1) can be represented as (see \([4,14]\))

\[
Y_t = u(t, X_t), \quad Z_t = u_x(t, X_t) \sigma(t, X_t), \quad \forall t \in [0, T),
\]

where \(u(t, x)\) is the smooth solution to the following parabolic partial differential equation (PDE):

\[
\partial_t u(t, x) + \frac{1}{2} \sum_{i,j=1}^d [\sigma \sigma^*]_{i,j}(t, x) \partial_{x_i x_j}^2 u(t, x) + \sum_{i=1}^d b_i(t, x) \partial_{x_i} u(t, x) + f(t, x, u(t, x), u_x(t, x) \sigma(t, x)) = 0 \quad (1.3)
\]

with terminal condition \(u(T, x) = \varphi(x)\), where

\[
\begin{align*}
\partial_t u(t, x) &= 
\begin{bmatrix}
\frac{\partial u}{\partial x_1} \\
\frac{\partial u}{\partial x_2} \\
\vdots \\
\frac{\partial u}{\partial x_d}
\end{bmatrix}, \quad \partial_{x_i} u(t, x) =
\begin{bmatrix}
\frac{\partial u}{\partial x_i} \\
\frac{\partial u}{\partial x_{i+1}} \\
\vdots \\
\frac{\partial u}{\partial x_d}
\end{bmatrix}, \quad \partial_{x_i x_j}^2 u(t, x) =
\begin{bmatrix}
\frac{\partial^2 u}{\partial x_i \partial x_j} \\
\frac{\partial^2 u}{\partial x_i \partial x_{j+1}} \\
\vdots \\
\frac{\partial^2 u}{\partial x_i \partial x_d}
\end{bmatrix}.
\end{align*}
\]

The representations in (1.2) are called the Feynman-Kac formula for the nonlinear PDE.

Since the work \([13]\), FBSDEs have been extensively studied and have been found important applications in many fields, such as finance, risk measure, stochastic control, and so on. Thus, methods for solving the solution \((X_t, Y_t, Z_t)\) to FBSDEs, theoretically or numerically, are needed. It is often difficult to get the solution in closed form. So it is more important to study numerical methods for solving FBSDEs. There are many numerical schemes for solving FBSDEs. Some of them are Euler-type schemes for solving BSDEs or FBSDEs (see \([1–3,5,10,11,16]\)); the accuracy of these Euler-type schemes are often low with half-order strong convergence rate. Some of them are second-order (or higher) schemes for solving BSDEs (see \([9,17,18,21]\)).

In this paper, we first extend the second-order scheme proposed for BSDEs in \([17]\) to the decoupled FBSDEs. Then by using the trapezoidal rule and the integration-by-parts formula of Malliavin calculus, we propose two other new numerical schemes for solving decoupled FBSDEs. By Malliavin calculus theory we rigorously prove that the two new schemes have second-order convergence rate. Results of our numerical tests given in this paper show that the proposed schemes are second-order schemes, which is consistent with our theoretical results.

The paper is organized as follows. In Section 2, after introducing some needed preliminaries, we will propose new numerical schemes for solving the decoupled FBSDEs (1.1). In Section 3, we will give general error estimates results, which imply the stability of the schemes. By using the error estimate results obtained in Section 3, we obtain second-order error estimate results of the new schemes for the decoupled FBSDEs (1.1) in Section 4. In Section 5, two numerical examples are given to verify the theoretical results. Some conclusions are given in Section 6.

Now we introduce the following notations:

- \((\cdot)^T\): The transpose operator for a matrix or vector; \((\cdot)^{-1}\): the inverse operator for a matrix.
- \(\| \cdot \|\): The norm for vector or matrix defined by \(\|c\|^2 = \text{tr}(c^Tc)\).
- \(C^{l,k,k}_b\): The set of continuously differentiable functions \(\psi : [0, T] \times \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d} \to \mathbb{R}\) with uniformly bounded partial derivatives \(\partial^l \psi\) and \(\partial^l_X \partial^k \psi \partial^k \psi\) for \(l \leq l\) and \(k_1 + k_2 + k_3 \leq k\). The notations \(C^{l,k}_b\) and \(C^{l,k,k}_b\) are similarly defined.
- \(C^{k,\alpha}_b\) (\(\alpha \in (0, 1)\)): \(\phi \in C^{k,\alpha}_b\) means that it has uniformly bounded partial derivatives up to order \(k\), and \(\partial^k \phi(x)\) is Hölder continuous with index \(\alpha\) w.r.t. \(x\).
- \(\mathcal{F}^{l,x}_s(t \leq s \leq T)\): \(\sigma\)-field generated by the diffusion process \(\{X_r, t \leq r \leq s, X_t = x\}\) starting from the time-space point \((t, x)\). When \(s = T\), we use \(\mathcal{F}^{l,x}_t\) to denote \(\mathcal{F}^{l,x}_T\).
- \(E^{l,x}_s[\eta]\): The conditional mathematical expectation of the random variable \(\eta\) under the \(\sigma\)-field \(\mathcal{F}^{l,x}_s\) \((t \leq s \leq T)\), i.e., \(E^{l,x}_s[\eta] = E[\eta | \mathcal{F}^{l,x}_s]\). Let \(E^x[\eta] = E[\eta | \mathcal{F}^x_t]\).
2 Second-order schemes

Suppose that $H$ is a real separable Hilbert space with scalar product denoted by $\langle \cdot, \cdot \rangle_H$. The norm of an element $h \in H$ will be denoted by $\|h\|_H$. Let $\mathcal{B} = \{B(h), h \in H\}$ denote an isonormal Gaussian process associated with the Hilbert space $H$ on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let the operator $D^k$ [12] be the Malliavin derivative of order $k$. For any integer $p \geq 1$, $D^k p$ is the domain of $D^k (k \in \mathbb{N})$ in $L^p(\Omega)$, i.e., $D^k p$ is the closure of the class of smooth random variables $F$ with respect to the norm $\|F\|_{k,p} = [\mathbb{E}[|F|^p]]^{1/p} + \sum_{j=1}^{k} \mathbb{E}[\|D^j F\|_p^p]^{1/p}$. For $p = 2$, the space $D^{1,2}$ is a Hilbert space with the scalar product

$$\langle F, G \rangle = \mathbb{E}[FG] + \mathbb{E}[\langle DF, DG \rangle_H].$$

For the Malliavin derivative operator $D$, we introduce the following two lemmas.

**Lemma 2.1** (See [12]). For $F \in D^{1,2}$ and $u \in L^2(\Omega; H)$, we have

$$D_t \int_0^T u_s ds = \int_0^T D_t u_s ds, \quad 0 \leq t \leq T, \quad D_t \int_0^T u_s dW_s = u_t + \int_t^T D_t u_s dW_s, \quad 0 < t \leq T,$$

and the following Integration-by-parts formula:

$$\mathbb{E}\left[ \int_0^T u_t D_t F dt \right] = \mathbb{E}\left[ F \int_0^T u_t dW_t \right].$$

**Lemma 2.2** (See [6,12]). If the solution $(X_t, Y_t, Z_t)$ to (1.1) is in $D^{1,2}$, we have

$$D_s X_t I_{s \leq t} = \nabla_x X_t (\nabla_x X_t)^{-1} \sigma(s, X_s) I_{s \leq t}, \quad Z_t = D_t Y_t = \nabla_x Y_t (\nabla_x X_t)^{-1} \sigma(t, X_t),$$

$$D_s Y_t I_{s \leq t} = \nabla_y Y_t (\nabla_x X_t)^{-1} \sigma(s, X_s) I_{s \leq t} = Z_t \sigma^{-1}(t, X_t) \Delta \sigma^{-1}(s, X_s) I_{s \leq t}.$$

To propose our numerical schemes, we introduce a regular time partition: $0 = t_0 < \cdots < t_N = T$ for the time interval $[0, T]$. Let $\Delta t_n = t_{n+1} - t_n$ and $\Delta t = \max_{i=0}^{N-1} \Delta t_n$. We denote the increment $W_{t_{n+1}} - W_{t_n}$ of the Brownian motion $W_t$ by $\Delta W_{t_{n+1}}$.

2.1 Numerical schemes for solving SDEs

Since the FBSDEs (1.1) are decoupled, the SDE and the BSDE in (1.1) can be solved separately. There have been many numerical schemes for solving SDEs in literature, such as the Euler scheme, the Milstein scheme and the Itô-Taylor type schemes [7,15].

We assume that the numerical scheme used for solving the SDE in (1.1) is in the following form:

$$X^{n+1} = X^n + \phi(t_n, X^n, \Delta t_n, \xi^{n+1}), \quad n = 0, 1, \ldots, N - 1,$$

where $X^0 = X_0$, $\phi$ is a given $\mathbb{R}^d$-valued function, and $\xi^{n+1}$ is a random vector related to $\Delta W_{t_{n+1}} = W_{t_{n+1}} - W_{t_n}$, and under some conditions on the coefficients $b$ and $\sigma$ the function $\phi$ has the estimates (2.1) and (2.2) in the following two Hypothesis.

**Hypothesis 2.1.** For $n = 0, 1, \ldots, N - 1$, the $\phi$ in (2.1) has the estimate

$$\mathbb{E}_{t_n}^{X^n} \left[ |\phi(t_n, X^n, \Delta t_n, \xi^{n+1})|^2 \right] \leq C \Delta t_n,$$

and the Malliavin derivative $D_t X^{n+1}$ of $X^{n+1}$ defined by (2.1) has the approximate property

$$\mathbb{E}_{t_n}^{X^n} \left[ |D_t X^{n+1} - \sigma^n|^2 \right] \leq C \sqrt{\Delta t_n}, \quad t_n < t \leq t_{n+1}.$$

**Hypothesis 2.2.** The approximation solution $X^{n+1}$ to (2.1), $n = 0, 1, 2, \ldots, N - 1$, have the properties

$$|\mathbb{E}_{t_n}^{X^n} g(X^{n+1}_{t_{n+1}}) - g(X^{n+1})| \leq C g(\Delta t)^{\beta + 1},$$

$$|\mathbb{E}_{t_n}^{X^n} (g(X^{n+1}_{t_{n+1}}) - g(X^{n+1})) \Delta W_{t_{n+1}}^\tau| \leq C g(\Delta t)^{\gamma + 1},$$

where $g \in C_b^{2\beta+2}$, $\beta$ and $\gamma$ are nonnegative numbers, $C_g$ is a positive number which does not depend on time partition.
Numerical schemes for one-dimensional and multi-dimensional SDEs have been extensively studied \([7]\), and many of them can be written in the form of (2.1), where the \(\phi\) satisfies the above two hypothesis, such as the Euler scheme, the Milstein scheme and the order-2 weak Itô-Taylor type schemes. In the following, we take the three schemes for one-dimensional SDEs as examples to illustrate Hypothesis 2.1.

1. The Euler scheme is in the form of (2.1) with the \(\phi\) defined by
\[
\phi = b^n \Delta t_n + \sigma^n \xi_n^{n+1}, \quad \xi_n^{n+1} = \Delta W_{t_{n+1}}.
\]
The Malliavin derivative
\[
D_t X^{n+1} = \sigma^n, \quad t_n < t \leq t_{n+1}.
\] (2.2)

2. The Milstein scheme is in the form of (2.1) with the \(\phi\) defined by
\[
\phi = b^n \Delta t_n + \sigma^n \xi_n^{n+1} + \frac{1}{2} \sigma^n \xi_x^n (\xi_n^{n+1})^2 - \Delta t_n), \quad \xi_n^{n+1} = \Delta W_{t_{n+1}}.
\]
The Malliavin derivative
\[
D_t X^{n+1} = \sigma^n + \sigma^n \Delta W_{t_{n+1}}, \quad t_n < t \leq t_{n+1}.
\] (2.3)

3. The following scheme is an order-2 weak Itô-Taylor type scheme:
\[
X^{n+1} = X^n + b^n \Delta t_n + \sigma^n \xi_n^{n+1} + \frac{1}{2} \sigma^n \xi_x^n (\xi_n^{n+1})^2 - \Delta t_n
\]
\[
+ \frac{1}{2} \sigma^p + \sigma^p b^n + b^n \sigma^p + \frac{1}{2} (\sigma^n)^2 \sigma_x^n \Delta t_n + \frac{1}{2} b_t^n + b_t^n b^n + (\sigma^n)^2 b^n (\Delta t_n)^2, \quad \xi_n^{n+1} = \Delta W_{t_{n+1}}.
\] (2.4)

Obviously, the scheme (2.4) is in the form of (2.1). And the Malliavin derivative
\[
D_t X^{n+1} = \sigma^n + \sigma^n \Delta W_{t_{n+1}} + \frac{\Delta t_n}{2} \left[ \sigma^p + \sigma^p b^n + b^n \sigma^p + \frac{1}{2} (\sigma^n)^2 \sigma_x^n \right], \quad t_n < t \leq t_{n+1}.
\] (2.5)

Here, for function \(\psi = \psi(t, x)\), we use \(\psi^n\) to denote \(\psi(t_n, X^n)\).

**Remark 2.1.** It is well known in \([7]\) that under proper conditions on \(b\) and \(\sigma\), the following conclusions hold:

1. The Euler scheme is a strong-order 0.5 and weak order 1.0 scheme, and Hypothesis 2.1 and 2.2 hold with \(\beta = \gamma = 1\);
2. The Milstein scheme is a strong-order 1.0 and weak order 1.0 scheme, and Hypothesis 2.1 and 2.2 hold with \(\beta = \gamma = 1\);
3. The order-2 weak Itô-Taylor scheme is a strong-order 1.0 and weak order 2.0 scheme, and Hypothesis 2.1 and 2.2 hold with \(\beta = \gamma = 2\).

### 2.2 Reference equations for solving BSDEs

Let the triple \((X_t^{t_n, X^n}, Y_t^{t_n, X^n}, Z_t^{t_n, X^n})_{t \in [t_n, T]}\) be the solution to the following FBSDEs:

\[
\begin{align*}
X_t^{t_n, X^n} &= X^n + \int_{t_n}^t b(s, X_s^{t_n, X^n})ds + \int_{t_n}^t \sigma(s, X_s^{t_n, X^n})dW_s, \\
Y_t^{t_n, X^n} &= \varphi(X_T^{t_n, X^n}) + \int_t^T f(s, X_s^{t_n, X^n}, Y_s^{t_n, X^n}, Z_s^{t_n, X^n})ds - \int_t^T Z_s^{t_n, X^n}dW_s,
\end{align*}
\] (2.6)

Then
\[
Y_t^{t_n, X^n} = Y_{t_n}^{t_{n+1}, X^n} + \int_{t_n}^{t_{n+1}} f_s^{t_n, X^n}ds - \int_{t_n}^{t_{n+1}} Z_{s}^{t_n, X^n}dW_s,
\] (2.7)
where \( f_{t_n}^n \cdot X^n := f(s, X^n_{t_n}, Y^n_{t_n}, Z^n_{t_n}). \) Taking \( \mathbb{E} X^n [\cdot] \) on both sides of (2.7) leads to
\[
Y_{t_n}^{t_n, X^n} = \mathbb{E} X^n [Y_{t_{n+1}}^{t_{n+1}, X^n}] + \int_{t_n}^{t_{n+1}} \mathbb{E} X^n [f_{s}^{t_n, X^n}] ds,
\]
and using trapezoidal rule to approximate \( \int_{t_n}^{t_{n+1}} \mathbb{E} X^n [f_{s}^{t_n, X^n}] ds \) gives
\[
Y_{t_n}^{t_n, X^n} = \mathbb{E} X^n [Y_{t_{n+1}}^{t_{n+1}, X^n}] + \frac{1}{2} \Delta t_n f_{t_n}^{t_n, X^n} + \frac{1}{2} \Delta t_n \mathbb{E} X^n [f_{t_{n+1}}^{t_n, X^n}] + R^n_Y,
\]
where
\[
R^n_Y = \int_{t_n}^{t_{n+1}} \left( \mathbb{E} X^n [f_{s}^{t_n, X^n}] - \frac{1}{2} \mathbb{E} X^n [f_{t_{n+1}}^{t_n, X^n}] - \frac{1}{2} f_{t_n}^{t_n, X^n} \right) ds,
\]
\[
R^n_{Y_1} = \mathbb{E} X^n [Y_{t_{n+1}}^{t_{n+1}, X^n} - Y_{t_{n+1}}^{t_{n+1}, X^n}], + \frac{1}{2} \Delta t_n \mathbb{E} X^n [f_{t_{n+1}}^{t_n, X^n} - f_{t_{n+1}}^{t_{n+1}, X^n}].
\]
Now, we multiply BSDE (2.7) by \( \Delta W_{t_{n+1}} \) and then take \( \mathbb{E} X^n [\cdot] \) on both sides of the derived equation to obtain
\[
-\mathbb{E} X^n [Y_{t_{n+1}}^{t_{n+1}, X^n} \Delta W_{t_{n+1}}^T] = \int_{t_n}^{t_{n+1}} \mathbb{E} X^n [f_{s}^{t_n, X^n} \Delta W_s^T] ds - \int_{t_n}^{t_{n+1}} \mathbb{E} X^n [Z_{s}^{t_n, X^n}] ds,
\]
which by trapezoidal rule again yields
\[
\Delta t_n Z_{t_n}^{t_n, X^n} = -\frac{1}{2} \Delta t_n \mathbb{E} X^n [Z_{t_{n+1}}^{t_{n+1}, X^n}] + \mathbb{E} X^n [Y_{t_{n+1}}^{t_{n+1}, X^n} \Delta W_{t_{n+1}}^T] + \frac{1}{2} \Delta t_n \mathbb{E} X^n [f_{t_{n+1}}^{t_n, X^n} \Delta W_{t_{n+1}}^T] + R^n_Z,
\]
where
\[
R^n_Z = \int_{t_n}^{t_{n+1}} \left( \mathbb{E} X^n [f_{s}^{t_n, X^n} \Delta W_s^T] - \frac{1}{2} \mathbb{E} X^n [f_{t_{n+1}}^{t_n, X^n} \Delta W_{t_{n+1}}^T] \right) ds
- \int_{t_n}^{t_{n+1}} \left( \mathbb{E} X^n [Z_{s}^{t_n, X^n}] - \frac{1}{2} Z_{t_n}^{t_n, X^n} - \frac{1}{2} \mathbb{E} X^n [Z_{t_{n+1}}^{t_n, X^n}] \right) ds.
\]

Based on the two reference equations (2.8) and (2.10), we will propose our second order schemes in the following subsections.

**Remark 2.2.** Note that except for the initial condition for the forward SDEs, (1.1) and (2.6) have the same coefficients \( b, \sigma, f \) and \( \varphi. \) Thus, according to the nonlinear Feynman-Kac formula, their solution can be represented as
\[
Y_t = u(t, X_t), \quad Z_t = u_x(t, X_t)\sigma(t, X_t), \quad 0 \leq t < T,
\]
and
\[
Y_{t_n}^{t_n, X^n} = u(t, X_{t_n}^{t_n, X^n}), \quad Z_{t_n}^{t_n, X^n} = u_x(t, X_{t_n}^{t_n, X^n})\sigma(t, X_{t_n}^{t_n, X^n}), \quad t_n \leq t < T,
\]
respectively, where \( u \) is the unique smooth solution to the PDE (1.3).

### 2.3 Scheme I: Crank-Nicolson-type scheme

Based on (2.8) and (2.10), Zhao et al. [17] proposed a kind of \( \theta \)-scheme for solving BSDEs. The \( \theta \)-scheme becomes a Crank-Nicolson-type Scheme by letting the parameters equal \( \frac{1}{2} \) in the \( \theta \)-scheme. Now by directly withdrawing the truncation terms in (2.8) and (2.10), we extend the Crank-Nicolson-type Scheme in [17] to the decoupled FBSDEs (1.1) as follows.
Scheme 2.1. Given $Y^N = \varphi(X^N)$, $Z^N = \varphi_x(X^N)\sigma(t_N, X^N)$ for $n = N - 2, \ldots, 1, 0$, solve random variables $X^{n+1}$, $Z^n$ and $Y^n$ by

$$X^{n+1} = X^n + \phi(t_n, X^n, \Delta t_n, \xi_{n+1}^n),$$

$$\frac{1}{2} \Delta t_n Z^n = -\frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} [Z^{n+1}] + \mathbb{E}_{t_n}^{X^n} [Y^{n+1} \Delta W_{t_{n+1}}^\top] + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} [f(t_{n+1}, X^{n+1}, Y^{n+1}, Z^{n+1}) \Delta W_{t_{n+1}}^\top],$$

$$Y^n = \mathbb{E}_{t_n}^{X^n} [Y^{n+1}] + \frac{1}{2} \Delta t_n f(t_n, X^n, Y^n, Z^n) + \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} [f(t_{n+1}, X^{n+1}, Y^{n+1}, Z^{n+1})].$$

Different from getting the above Crank-Nicolson-type scheme, in the following two subsections, we will propose two new second-order schemes for solving FBSDEs (1.1) by using the integration-by-parts formula in Lemma 2.1 to the second term on the right-hand side of (2.10).

2.4 Scheme II

By Lemma 2.1, we deduce

$$\mathbb{E}_{t_n}^{X^n} [Y_{t_{n+1}}^{t_n}, X^n] \Delta W_{t_{n+1}}^\top = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} [D_t Y_{t_{n+1}}^{t_n}, X^n] dt. \quad (2.12)$$

Then using trapezoidal rule to approximate the above integral gives

$$\mathbb{E}_{t_n}^{X^n} [Y_{t_{n+1}}^{t_n}, X^n] \Delta W_{t_{n+1}}^\top = \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} [D_{t_{n+1}} Y_{t_{n+1}}^{t_n}, X^n] + D_{t_n} Y_{t_{n+1}}^{t_n}, X^n] + R^n_{Z1},$$

where

$$R^n_{Z1} = \int_{t_n}^{t_{n+1}} \mathbb{E}_{t_n}^{X^n} \left[ D_t Y_{t_{n+1}}^{t_n}, X^n - \frac{1}{2} D_{t_{n+1}} Y_{t_{n+1}}^{t_n}, X^n - \frac{1}{2} D_{t_{n+1}} Y_{t_{n+1}}^{t_n}, X^n \right] dt \quad (2.14)$$

with

$$D_t Y_{t_{n+1}}^{t_n}, X^n = Z_{t_{n+1}}^{t_n}, X^n \sigma^{-1}(t_{n+1}, X_{t_{n+1}}^{t_n}, X^n) \nabla_x X_{t_{n+1}}^{t_n}, X^n (-\nabla_x X_{t_{n+1}}^{t_n}, X^n)^{-1} \sigma(t, X_{t_{n+1}}^{t_n}, X^n), \quad (2.15)$$

for $t \in [t_n, t_{n+1}]$. Here, $\nabla_x X_{t_{n+1}}^{t_n}, X^n$ satisfies the SDE

$$\nabla_x X_{t_{n+1}}^{t_n}, X^n = I_d + \int_{t_n}^{t} \partial_x b(r, X_{r}^{t_{n+1}}, X^n) \nabla_x X_{t_{n+1}}^{t_n}, X^n dr + \sum_{j=1}^{d} \int_{t_n}^{t} \partial_x \sigma_j(r, X_{r}^{t_{n+1}}, X^n) \nabla_x X_{t_{n+1}}^{t_n}, X^n dW^j, \quad (2.16)$$

where $I_d$ is the $d \times d$ identity matrix, $\sigma_j(\cdot)$ is the $j$-th column of the matrix $\sigma(\cdot)$, $\partial_x \sigma_j$ is the Jacobi-matrix of $\sigma_j(\cdot)$ w.r.t. $x$, and $\nabla_x X_{t_{n+1}}^{t_n}, X^n$ is the variation of $X_{t_{n+1}}^{t_n}, X^n$ w.r.t. $X^n$. From (2.13) and (2.15), we obtain

$$\mathbb{E}_{t_n}^{X^n} [Y_{t_{n+1}}^{t_n}, X^n] \Delta W_{t_{n+1}}^\top = \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ Z_{t_{n+1}}^{t_n}, X^n (I_d + \sigma^{-1}(t_{n+1}, X_{t_{n+1}}^{t_n}, X^n) \nabla_x X_{t_{n+1}}^{t_n}, X^n \sigma(t, X_{t_{n+1}}^{t_n}, X^n)) \right] + R^n_{Z1},$$

which by using the Euler method to approximate $\nabla_x X_{t_{n+1}}^{t_n}, X^n$ defined by (2.16) yields

$$\mathbb{E}_{t_n}^{X^n} [Y_{t_{n+1}}^{t_n}, X^n] \Delta W_{t_{n+1}}^\top = \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ Z_{t_{n+1}}^{t_n}, X^n (I_d + \sigma^{-1}(t_{n+1}, X_{t_{n+1}}^{t_n}, X^n) A^n \sigma(t, X_{t_{n+1}}^{t_n}, X^n)) \right] + R^n_{Z1} + R^n_{Z2}, \quad (2.17)$$

where $A^n = I_d + \partial_x b(t_n, X^n) \Delta t_n + \sum_{j=1}^{d} \partial_x \sigma_j(t_n, X^n) \Delta W^j_{t_{n+1}}$, and

$$R^n_{Z2} = \frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ Z_{t_{n+1}}^{t_n}, X^n \sigma^{-1}(t_{n+1}, X_{t_{n+1}}^{t_n}, X^n) \nabla_x X_{t_{n+1}}^{t_n}, X^n (A^n - A^n) \sigma(t, X_{t_{n+1}}^{t_n}, X^n) \right]. \quad (2.18)$$

Then by (2.10) and (2.17), we deduce

$$\Delta t_n Z_{t_{n+1}}^{t_n}, X^n = -\frac{1}{2} \Delta t_n \mathbb{E}_{t_n}^{X^n} \left[ Z_{t_{n+1}}^{t_n}, X^n - Z_{t_{n+1}}^{t_n}, X^n \sigma^{-1}(t_{n+1}, X_{t_{n+1}}^{t_n}, X^n) A^n \sigma(t, X_{t_{n+1}}^{t_n}, X^n) \right]$$

$$+ \mathbb{E}_{t_n}^{X^n} [Y_{t_{n+1}}^{t_n}, X^n] \Delta W_{t_{n+1}}^\top + \Delta t_n \mathbb{E}_{t_n}^{X^n} [f_{t_{n+1}}^{t_n}, X^n] \Delta W_{t_{n+1}}^\top + 2R^n_{Z1} + 2 \sum_{j=1}^{2} R^n_{Zj} \quad (2.19)$$
Based on (2.1) and the two reference equations (2.8) and (2.19), we propose our second scheme in

Then by (2.21) and (2.23), we deduce

Just as what we did to (2.12), by Lemmas 2.1 and 2.2, we get

In this subsection, without any discretization of \( X^{n+1}, Z^{n+1} \) and \( Y^{n+1} \), solve random variables \( X^{n+1}, Z^{n+1}, Y^{n+1} \) by

where \( R_Z^n, R^n_{Z1}, \) and \( R^n_{Z2} \) are defined by (2.11), (2.14) and (2.18), respectively, and

By equation (2.1) and the two reference equations (2.8) and (2.19), we propose our second scheme in the following:

**Scheme 2.2.** Given \( Y^N = \varphi(X^N) \) and \( Z^N = \varphi_x(X^N)\sigma(t_N, X^N) \), for \( n = N-1, \ldots, 1, 0 \), solve random variables \( X^{n+1}, Z^{n+1} \) and \( Y^{n+1} \) by

\[
X^{n+1} = X^n + \sigma(t_n, X^n) \Delta t_n + \epsilon^n_n + 1, \\
\Delta t_n Z^{n+1} = -\frac{1}{2} \Delta t_n E^X_n \left[ Z^{n+1} - Z^n_\sigma^{n+1} \right] + A^n \sigma(t_n, X^n) + E^X_n \left[ Y^{n+1} \Delta W^\top_{t_n+1} \right] \\
+ \Delta t_n E^X_n \left[ f(t_n+1, X^{n+1}, Y^{n+1}, Z^{n+1}), \Delta W^\top_{t_n+1} \right], \\
Y^{n+1} = E^X_n \left[ Y^{n+1} \right] + \frac{1}{2} \Delta t_n f(t_n, X^n, Y^n, Z^n) + \frac{1}{2} \Delta t_n E^X_n \left[ f(t_n+1, X^{n+1}, Y^{n+1}, Z^{n+1}) \right],
\]

where \( \sigma_n = \sigma(t_n, X^n) \), \( A^n = I_d + \partial_y b^n \Delta t_n + \sum_{j=1}^{d} \partial_y \sigma_j^n \Delta W^j_{t_n+1} \), and \( \sigma^{n+1} \) is the inverse matrix of \( \sigma^n \).

**2.5 Scheme III**

In this subsection, without any discretization of \( E^X_n \left[ Y^{n+1}_t X^{n+1}_t \Delta W^\top_{t_n+1} \right] \), we rewrite (2.10) in the following equivalent form:

\[
\frac{1}{2} \Delta t_n Z^{n+1}_t \left[ t_{n+1} \right] = -\frac{1}{2} \Delta t_n E^X_n \left[ Z^{n+1}_t - Z^{n+1}_n \right] + E^X_n \left[ Y^{n+1}_t \Delta W^\top_{t_n+1} \right] \\
+ \frac{1}{2} \Delta t_n E^X_n \left[ f(t_n, X^n, Y^{n+1}, \Delta W^\top_{t_n+1}) \right] + R^n_{Z4} + R^n_{Z1},
\]

where

\[
R^n_{Z4} = -\frac{1}{2} \Delta t_n E^X_n \left[ Z^{n+1}_n - Z^{n+1}_n \right] + E^X_n \left[ (Y^{n+1}_t - Y^{n+1}_{t_n+1}) \Delta W^\top_{t_n+1} \right] \\
+ \frac{1}{2} \Delta t_n E^X_n \left[ f(t_n, X^n, Y^{n+1}_n) \Delta W^\top_{t_n+1} \right].
\]

Just as what we did to (2.12), by Lemmas 2.1 and 2.2, we get

\[
E^X_n \left[ Y^{n+1}_t \Delta W^\top_{t_n+1} \right] = \int_t^{t_n+1} E^X_{t_n} \left[ D_t Y^{n+1}_t \right] dt \\
= E^X_{t_n} \left[ Z^{n+1}_t \Delta t_n + \int_t^{t_n+1} D_t X^{n+1} dt \right].
\]

Then by (2.21) and (2.23), we deduce

\[
\Delta t_n Z^{n+1}_t = E^X_{t_n} \left[ Z^{n+1}_t \left( \sigma \right)^n + \int_t^{t_n+1} D_t X^{n+1} dt + \Delta t_n \right] + 2R^n_{Z4} + 2R^n_{Z1}.
\]
and then use these schemes backwardly to get \( Y^n \), \( Z^n \) as follows: 

**Remark 2.3.** Given \( Y^N = \phi(X^N) \) and \( Z^N = \phi_x(X^N)\sigma(t_N, X^N), \) for \( n = N - 1, \ldots, 1, 0, \) solve random variables \( X^{n+1}, Z^n \) and \( Y^n \) by:

\[
X^{n+1} = X^n + \phi(t_n, X^n, \Delta t_n, \xi_{n+1}^{(n)}), \\
\Delta t_n Z^n = \mathbb{E}X^n \left[ Z^{n+1} \left( \sigma^{n+1} - 1 \int_{t_n}^{t_{n+1}} D_t X^{n+1} dt - \Delta t_n \right) \right] + \mathbb{E}X^n \left[ Y^{n+1} \Delta W_{t_{n+1}}^T \right], \\
+ \Delta t_n \mathbb{E}X^n \left[ f(t_{n+1}, X^{n+1}, Y^{n+1}, Z^{n+1}) \Delta W_{t_{n+1}}^T \right], \\
Y^n = \mathbb{E}X^n [Y^{n+1}] + \frac{1}{2} \Delta t_n f(t_n, X^n, Y^n, Z^n) + \frac{1}{2} \Delta t_n \mathbb{E}X^n \left[ f(t_{n+1}, X^{n+1}, Y^{n+1}, Z^{n+1}) \right],
\]

where \( \sigma^n = \sigma(t_n, X^n), D_t \) is the Malliavin derivative.

**Remark 2.5.** Schemes 2.1–2.3 are time discrete schemes. To use these schemes to solve decoupled FBSDEs, the conditional mathematical expectations should be approximated. In this paper, we will obtain the error estimate results for the time discrete Schemes 2.2 and 2.3 in Sections 3 and 4, and use the Guass-Hermite quadrature rule to approximate the conditional expectations in our numerical experiments. The details of the approximations will be introduced in Section 5.

### 3 General error estimates

In this section, we only give general error estimate results for Schemes 2.2 and 2.3. It should be noted that we do not know how to derive the error estimate for Scheme 2.1.
We assume that the time partition step $\Delta t_n$ has the regularity constraint:
\[
\frac{\max_{0 \leq n \leq N-1} \Delta t_n}{\min_{0 \leq n \leq N-1} \Delta t_n} \leq c_0,
\]
where $c_0$ is a positive constant. In the sequel, $C$ is a generic constant which does not depend on the time partition and may be different from line to line. In order to get the error estimates of Schemes 2.2 and 2.3, we also make the following Hypothesis.

**Hypothesis 3.1.** The functions $|\partial_x b|$, $\sum_{j=1}^d |\partial_x \sigma_j|$ and $|(\sigma)^{-1}|$ are bounded, and the generator $f = f(t, x, y, z)$ of the decoupled FBSDEs (1.1) is uniformly Lipschitz continuous with respect to $(x, y, z) \in \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{m \times d}$.

**Theorem 3.1.** Let $(X^n_t, X^n, Y^n_t, Y^n, Z^n_t, Z^n)$ $t_n \leq t \leq T$ and $(X^n, Y^n, Z^n)$ $(n = 0, 1, \ldots, N)$ be the solutions to the FBSDEs (2.6) and Scheme 2.2, respectively. Then under Hypothesis 2.1 and 3.1, for sufficiently small time step $\Delta t$, we have
\[
c_0 |Y^n_t - Y^n|^2 + \frac{1}{14} |\Delta t| |Z^n_t - Z^n|^2 \\
\leq C) \sum_{i=0}^{N-n-1} \left( \frac{1}{1-C(\Delta t)} \right) \frac{\mathbb{E}_t^n \left[ |R^n_{f+1}|^2 + |R^n_0|^2 + |R^n_{f+1}|^2 + \sum_{j=1}^3 |R^n_{j2}|^2 \right]}{\Delta t},
\]
where $R^n_{f+1}$, $R^n_{f+1}$, $R^n_{f+1}$ and $R^n_{f+1}$ are defined in (2.9), (2.11), (2.14) and (2.20), respectively.

**Proof.** Let $e^n_t = Y^n_{t_n} - Y^n_{t_n}$, $e^n_{t_n} = Z^n_{t_n} - Z^n_{t_n}$, $f^n_{t_n} = f(t_n, X^n_{t_n}, Y^n_{t_n}, Z^n_{t_n}), f^n = f(t_n, X^n, Y^n, Z^n)$ and $e^n = f(t_n, X^n_{t_n}, Y^n_{t_n}, Z^n_{t_n}) - f(t_n, X^n, Y^n, Z^n)$ for $n = 0, 1, \ldots, N$. Then by (2.8) and Scheme 2.2, we obtain
\[
|e^n_t| = |\mathbb{E}_t^n [e^n_{t_n}]| + \frac{1}{2} |\Delta t| e^n_f + \frac{1}{2} |\Delta t| e^n_{f+1} + |R^n_Y + R^n_{Y1}| \\
\leq |\mathbb{E}_t^n [e^n_{t_n} + e^n_f]| + |\Delta t| |e^n_f + e^n_{f+1}| + |e^n_Y + e^n_Z| + |R^n_Y + R^n_{Y1}|.
\]
where $L$ is the Lipschitz constant of $f$. It follows from taking square on both sides of (3.1) and applying the inequality $(a + b)^2 \leq (1 + \gamma_1 |\Delta t_n|)a^2 + (1 + \frac{1}{\gamma_1 |\Delta t_n|})b^2$ $(\gamma_1 > 0)$ that
\[
|e^n_t|^2 \leq (1 + \gamma_1 |\Delta t_n|) |\mathbb{E}_t^n [e^n_{t_n}]| + C \left( 1 + \frac{1}{\gamma_1 |\Delta t_n|} \right) ((\Delta t_n)^2 (|e^n_f|^2 + |e^n_{f+1}|^2) \\
+ (\Delta t_n)^2 |\mathbb{E}_t^n [e^n_f]| + |e^n_{f+1}|^2 + |e^n_Y|^2 + |e^n_Z|^2 + |R^n_Y|^2 + |R^n_{Y1}|^2).
\]
By (2.19) and Scheme 2.2, we deduce
\[
|\Delta t_n e^n_0| = \frac{1}{2} |\Delta t_n| |\mathbb{E}_t^n [e^n_{t_n}] (Id - \sigma^{n+1} A^n \sigma^n) + \mathbb{E}_t^n [e^n_{t_n} + \Delta W_{t_{n+1}^T}] \\
+ |\Delta t_n| |\mathbb{E}_t^n [e^n_{f+1} \Delta W_{t_{n+1}^T}] + 2R^n_Z + 2 \sum_{j=1}^3 |R^n_{Zj}|
\]
where $A^n = Id + \partial_x b \Delta t_n + \frac{d}{j=1} \partial_x \sigma_j A^n \Delta W_{t_{n+1}^T}$. By taking square on both sides of the equation (3.3), the inequality $(a_1 + a_2 + \cdots + a_7)^2 \leq 7(a_1^2 + a_2^2 + \cdots + a_7^2)$ and Hölder’s inequality, we get
\[
(\Delta t_n)^2 |e^n_t|^2 \leq \frac{7}{4} (\Delta t_n)^2 |\mathbb{E}_t^n [e^n_{t_n}]| + |Id - \sigma^{n+1} A^n \sigma^n| + 1 |\mathbb{E}_t^n [e^n_1 + \Delta W_{t_{n+1}^T}]| + 7 |\mathbb{E}_t^n [e^n_{f+1} \Delta W_{t_{n+1}^T}]|^2 \\
+ 7 (\Delta t_n)^2 |\mathbb{E}_t^n [e^n_{f+1}]| |\mathbb{E}_t^n [\Delta W_{t_{n+1}^T}]|^2 + 28 \left( |R^n_Z|^2 + \sum_{j=1}^3 |R^n_{Zj}|^2 \right).
\]
Thanks to Cauchy-Schwartz inequality, we obtain
\[|E_{t_n}^X [e_{n+1} W_{t_{n+1}}^T]|^2 = |E_{t_n}^X [(e_{n+1} - E_{t_n}^X e_{n+1}) W_{t_{n+1}}^T]|^2 \leq \Delta t_n (E_{t_n}^X |e_{n+1}|^2 - |E_{t_n}^X e_{n+1}|^2).\] (3.5)

Under the conditions of the theorem, we have
\[E_{t_n}^X [\sigma^2 - \sigma_{n+1}^2] = E_{t_n}^X [\sigma(t_{n+1}, X^{n+1}) - \sigma(t_n, X^n)]^2 \leq E_{t_n}^X [L^2 (\Delta t_n + |X^{n+1} - X^n|)^2] \leq 2L^2 (\Delta t_n)^2 + 2L^2 |E_{t_n}^X [\phi(t_n, X^n, \Delta t_n, \xi_{n+1})]|^2 \leq C \Delta t_n,\] (3.6)

where \(L\) is the Lipschitz constant of \(\sigma\), we deduce
\[E_{t_n}^X [|I_d - [\sigma^2 - \sigma_{n+1}^2]|^2] = E_{t_n}^X \left[|\sigma(t_{n+1}) - \sigma(t_n)|^2 - (\partial_x b \Delta t_n + \sum_{j=1}^d \partial_x \sigma_j \Delta W_j^T|^2 \right] \leq C E_{t_n}^X [|\sigma^2 - \sigma_{n+1}^2|^2] + C \Delta t_n \leq C \Delta t_n.\] (3.7)

By the inequalities (3.4), (3.6) and (3.7), we obtain
\[(\Delta t_n)^2 |e_Z^n|^2 \leq C(\Delta t_n)^3 E_{t_n}^X [|e_Z^n|^2 + |e_{n+1}^2|^2] + 7 \Delta t_n (E_{t_n}^X [|e_{n+1}^2|^2] - E_{t_n}^X [e_{n+1}^2])^2 + 28 \left(R_{n+1}^2 + \sum_{j=1}^3 |R_j|^2\right).\] (3.8)

Dividing both sides of the inequality (3.8) by \(\frac{7(\Delta t_n)^2}{\Delta t}\) leads to
\[\frac{1}{\Delta t} |e_Z^n|^2 \leq C(\Delta t)^2 E_{t_n}^X [|e_Z^n|^2 + |e_{n+1}^2|^2] + C_0 E_{t_n}^X [|e_{n+1}^2|^2] - c_0 |E_{t_n}^X [e_{n+1}^2]]^2 + 4 c_0^2 \left(R_{n+1}^2 + \sum_{j=1}^3 |R_j|^2\right).\] (3.9)

By choosing \(\gamma_1 = 14C_0\) in (3.2), multiplying (3.2) by \(c_0\) and plugging the derived inequality into (3.9), we deduce
\[c_0 |e_Y^n|^2 + \frac{1}{\Delta t} |e_Z^n|^2 \leq c_0 (1 + 14C \Delta t) E_{t_n}^X [e_{n+1}^2] + \frac{1 + 14C \Delta t}{14} |e_Y^n|^2 + E_{t_n}^X [e_{n+1}^2 + |e_{n+1}^2|^2] + c_0 E_{t_n}^X [e_{n+1}^2] - c_0 E_{t_n}^X [e_{n+1}^2]^2 + 4 c_0^2 \left(R_{n+1}^2 + \sum_{j=1}^3 |R_j|^2\right),\]

which by combining the following inequality:
\[c_0 (1 + 14C \Delta t) |E_{t_n}^X [e_{n+1}^2] + c_0 E_{t_n}^X [e_{n+1}^2] - c_0 E_{t_n}^X [e_{n+1}^2] \leq c_0 (1 + 14C \Delta t) E_{t_n}^X [e_{n+1}^2]\]
implies
\[c_0 |e_Y^n|^2 + \frac{1}{14} |e_Z^n|^2 \leq \frac{1 + 14C \Delta t}{1 - C \Delta t} E_{t_n}^X \left[c_0 |e_{n+1}^2|^2 + \frac{1}{14} \Delta t |e_{n+1}^2|^2\right]
+ \frac{C}{1 - C \Delta t} \left(R_{n+1}^2 + |R_{n+1}^2|^2 + |R_{n+2}^2|^2 + \sum_{j=1}^3 |R_j|^2\right)\Delta t.\]
Remark 4.1. Bounded. Then under Hypothesis 4.1–4.4.

Proof. To prove the above theorem, we introduce Lemmas 4.1–4.4.

\[ \sum_{i=0}^{N-n-1} \left( \frac{1+C\Delta t}{1-C\Delta t} \right)^{i} \mathbb{E}_{t_{n}}^{X_{n}} \left[ |R_{Y_{n+1}}^{n+1}|^{2} + |R_{Y_{1}}^{n+1}|^{2} + |R_{Z_{n+1}}^{n+1}|^{2} + \sum_{j=1}^{3} |R_{Z_{j}}^{n+1}|^{2} \right]. \]

The proof is completed. \(\square\)

Theorem 3.2. Let \( (X_{t}^{n}, Y_{t}^{n}, Z_{t}^{n})_{t_{n} \leq t \leq T} \) and \( (X^{n}, Y^{n}, Z^{n}) \) \((n = 0, 1, \ldots, N)\) be the solutions to the FBSDEs (2.6) and Scheme 2.3, respectively. Then under the conditions of Theorem 3.1, for sufficiently small time step \( \Delta t \), we have

\[ \mathbb{E}_{t_{n}}^{X_{n}} \left[ c_{0} |Y_{t_{n}}^{n}|^{2} - Y_{n}^{n} \right] + \mathbb{E}_{t_{n}}^{X_{n}} \left[ c_{0} |Y_{t_{n}}^{n}|^{2} - Y_{n}^{n} \right] \]

where \( R_{Y}^{n}, R_{Y_{1}}^{n}, R_{Z}^{n} \) and \( R_{Z_{2}}^{n} \) are defined in (2.9), (2.11) and (2.22), respectively.

Proof. Let \( e_{Y_{n}}^{n} = Y_{t_{n}}^{n} - Y_{n}^{n} \) and \( e_{Z_{n}}^{n} = Z_{t_{n}}^{n} - Z_{n}^{n} \). By (2.24) and Scheme 2.3, we deduce

\( \Delta t e_{Z_{n}}^{n} = \mathbb{E}_{t_{n}}^{X_{n}} \left[ e_{Z_{n}}^{n+1} \left( \sigma_{n+1}^{-1} \int_{t_{n}}^{t_{n+1}} D_{t} X_{t}^{n+1} dt - \Delta t_{n} \right) \right] \)

Then under the conditions of the theorem, by the techniques used in the proof of Theorem 3.1, we can get the estimate (3.10) similarly. The proof is completed. \(\square\)

4 Error estimates

In this section, we first give error estimates for Schemes 2.2 and 2.3.

Theorem 4.1. Let \((X_{t}^{n}, Y_{t}^{n}, Z_{t}^{n})_{t_{n} \leq t \leq T} \) \((n = 0, 1, 2, \ldots, N)\) be the solution to the FBSDEs (2.6), and \((X^{n}, Y^{n}, Z^{n}) \) \((n = 0, 1, 2, \ldots, N)\) be the solution to Schemes 2.2 or 2.3 with \( Y^{n} = \varphi(X^{n}) \) and \( Z^{n} = \varphi_{2}(X^{n}) \sigma(t_{n}, X^{n}) \). Suppose \( \varphi \in C_{b}^{4, \alpha} \) \((\alpha \in [0, 1])\), \( b, \sigma \in C_{b}^{2,4} \), \( f \in C_{b}^{2,4,4,4} \), and \( |(\sigma)^{-1}| \) is bounded. Then under Hypothesis 2.1 and 2.2, we have the estimate

\[ \max_{0 \leq \rho \leq N} \left[ (\bar{Y}_{t}^{n} - Y^{n})^{2} + \Delta t \bar{Z}_{t}^{n} - Z^{n})^{2} \right] \leq C(\Delta t)^{\min\{4, 3, \beta, 2, \gamma\}} \]

where \( \bar{Y}_{t}^{n} \) and \( \bar{Z}_{t}^{n} \) are the values of \( Y_{t} \) and \( Z_{t} \) at the time-space point \((t_{n}, X^{n})\).

Remark 4.1. Theorem 4.1 implies that the convergence rate is 2.0 for solving \( Y_{t} \) and 1.5 for solving \( Z_{t} \). But our numerical tests in Section 5 show that the convergence rate is 2.0 for solving both \( Y_{t} \) and \( Z_{t} \). So the error estimate in this theorem is not optimal for \( Z_{t} \). Up to now, by our analysis we can only get the estimate (4.1). The optimal error estimate for these schemes should be studied in our future work.

To prove the above theorem, we introduce Lemmas 4.1–4.4.
Lemma 4.1 (See [19]). Let $R_1^n$ and $R_2^n$ be the truncation errors defined in (2.9) and (2.22), respectively. If $b, \sigma \in C^{2,4}_b$, $f \in C^{2,4,4}_b$ and $\varphi \in C^{4,\alpha}_b (\alpha \in (0, 1))$, we have the estimates

$$
\max_{0 \leq n \leq N} |R_1^n| \leq C(\Delta t_n)^3, \quad \max_{0 \leq n \leq N} |R_2^n| \leq C(\Delta t_n)^3.
$$

For simplicity presentation, we set

$$
\n = (\nabla_x X_t^{t_n,X^n})^{-1}\sigma(t, X_t^{t_n,X^n}) (t_n \leq t \leq t_{n+1}).
$$

Then from Lemma 2.2, we get

$$
D_t X_t^{t_{n+1},X^n} = Z_t^{t_{n+1},X^n} - (t_{n+1}, X_t^{t_{n+1},X^n}) (t_n \leq t \leq t_{n+1}),
$$

and $R_{Z1}^n$ becomes

$$
R_{Z1}^n = \int_{t_n}^{t_{n+1}} \mathbb{E}_t^n \left[ F_t^{t_n,X^n} \left( H_t^{t_{n+1},X^n} - \frac{1}{2} H_t^{t_{n+1},X^n} - \frac{1}{2} H_t^{t_{n+1},X^n} \right) \right] dt.
$$

By Itô’s formula, we obtain

$$
\int_{t_n}^{t_{n+1}} \mathbb{E}_t^n [F_t^{t_n,X^n} H_t^{t_{n+1},X^n}] dt
= \mathbb{E}_t^n [F_t^{t_n,X^n} H_t^{t_{n+1},X^n}] \Delta t_n + \mathbb{E}_t^n \left[ \int_{t_n}^{t_{n+1}} F_t^{t_n,X^n} \left( \int_{t_n}^{t} L^0 H_t^{t_{n+1},X^n} ds + \int_{t_n}^{t} L^1 H_t^{t_{n+1},X^n} dW_s \right) \right] dt
= \mathbb{E}_t^n [F_t^{t_n,X^n} H_t^{t_{n+1},X^n}] \Delta t_n + \mathbb{E}_t^n \left[ \int_{t_n}^{t_{n+1}} F_t^{t_n,X^n} \left( \int_{t_n}^{t} L^0 H_t^{t_{n+1},X^n} ds + \int_{t_n}^{t} L^1 H_t^{t_{n+1},X^n} dW_s \right) \right] dt
+ \mathbb{E}_t^n \left[ \int_{t_n}^{t_{n+1}} F_t^{t_n,X^n} \left( \int_{t_n}^{t} L^1 H_t^{t_{n+1},X^n} dW_s \right) \right] dt
+ R_{Z1}^n,
$$

where

$$
R_{Z1}^n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^n \left[ F_t^{t_n,X^n} \left( \int_{t_n}^{t} L^0 H_t^{t_{n+1},X^n} d\tau + \int_{t_n}^{t} L^1 H_t^{t_{n+1},X^n} dW_\tau \right) \right] ds dt.
$$

By Lemma 2.1 and Itô’s formula, we get

$$
R_{Z1}^n = \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^n \left[ F_t^{t_n,X^n} L^0 H_t^{t_{n+1},X^n} \right] d\tau ds dt.
$$

By Lemma 2.1 and Itô’s formula, we deduce

$$
\mathbb{E}_t^n \left[ \int_{t_n}^{t_{n+1}} F_t^{t_n,X^n} \left( \int_{t_n}^{t} L^1 H_t^{t_{n+1},X^n} dW_s \right) \right]
= \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^n \left[ D_s (F_t^{t_n,X^n}) L^1 H_t^{t_{n+1},X^n} \right] ds dt
= \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^n \left[ D_s (F_t^{t_n,X^n}) + \int_{t_n}^{t} \left( L^0 F_t^{t_n,X^n} d\tau + L^1 F_t^{t_n,X^n} dW_\tau \right) \right] dW_s \right] \right]
= \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^n \left[ \int_{t_n}^{t} \left( D_s (L^0 F_t^{t_n,X^n}) \right) d\tau + D_s (L^1 F_t^{t_n,X^n}) dW_\tau \right] L^1 H_t^{t_{n+1},X^n} \right] ds dt
+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^n \left[ L^1 F_t^{t_n,X^n} \right] dW_s \right] \right]
$$
It follows from the equations (4.3), (4.6) and (4.7) that
\[
\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^{X_n} [D_s \{ L^0 \hat{F}^{t_n,X_n}_s \} L^1 H^{t_n,X_n}_s] \, dt \, ds dt
\]
due to
\[
\mathbb{E}_t^{X_n} \left[ \left( \int_{t_n}^{t} D_s \{ L^1 \hat{F}^{t_n,X_n}_s \} \, dW_s \right) L^1 H^{t_n,X}_s \right] = \mathbb{E}_t^{X_n} \left[ \int_{t_n}^{t} D_s \{ L^1 \hat{F}^{t_n,X_n}_s \} \, dW_s \right] L^1 H^{t_n,X}_s = 0.
\]

For the last term in (4.6), by using the Itô’s formula again, we have
\[
\int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^{X_n} [L^1 \hat{F}^{t_n,X_n}_s L^1 H^{t_n,X}_s] \, dt \, ds dt
\]
\[
= \frac{1}{2} L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s (\Delta t_n)^2 + \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \mathbb{E}_t^{X_n} \left[ L^0 (L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s) \right] \, dt \, ds dt.
\] (4.7)

It follows from the equations (4.3), (4.6) and (4.7) that
\[
\int_{t_n}^{t_{n+1}} \mathbb{E}_t^{X_n} [F^{t_n,X}_s H^{t_n,X}_s] \, dt
\]
\[
= \mathbb{E}_t^{X_n} [F^{t_n,X}_s H^{t_n,X}_s \Delta t_n] + \frac{1}{2} \mathbb{E}_t^{X_n} [F^{t_n,X}_s L^0 H^{t_n,X}_s] (\Delta t_n)^2 + \frac{1}{2} L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s (\Delta t_n)^2
\]
\[
+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} [D_s \{ L^0 F^{t_n,X}_s \}] L^1 H^{t_n,X}_s \, dt \, ds \, ds dt
\]
\[
+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} \left[ L^0 (L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s) \right] \, dt \, ds dt + R^{t_n+1}_{t_n}.
\] (4.8)

Similarly, we obtain
\[
-\frac{1}{2} \int_{t_n}^{t_{n+1}} \mathbb{E}_t^{X_n} [F^{t_n,X}_s H^{t_n,X}_s] \, dt = -\frac{1}{2} \mathbb{E}_t^{X_n} [F^{t_n,X}_s H^{t_n,X}_s \Delta t_n] - \frac{1}{2} \mathbb{E}_t^{X_n} [F^{t_n,X}_s L^0 H^{t_n,X}_s] (\Delta t_n)^2
\]
\[
- \frac{1}{2} L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s (\Delta t_n)^2 + R^{t_n+1}_{t_n},
\] (4.9)

where
\[
R^{t_n+1}_{t_n} = -\frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} [D_s \{ L^0 \hat{F}^{t_n,X}_s \} L^1 H^{t_n,X}_s] \, dt \, ds dt
\]
\[
- \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} \left[ L^0 (L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s) \right] \, dt \, ds dt
\]
\[
- \frac{1}{2} \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} [F^{t_n,X}_s L^0 H^{t_n,X}_s] \, dt \, ds dt + D_s \hat{F}^{t_n,X}_s L^1 L^0 H^{t_n,X}_s \, dt \, ds dt.
\]

Now by the equalities (4.2), (4.5), (4.8) and (4.9), we deduce
\[
| R_{Z1}^{2} | = \int_{t_n}^{t_{n+1}} \mathbb{E}_t^{X_n} \left[ F^{t_n,X}_s \left( H^{t_n,X}_s - \frac{1}{2} H^{t_{n+1},X}_s - \frac{1}{2} H^{t_{n+1},X}_s \right) \right] \, dt
\]
\[
= \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} [D_s \{ L^0 \hat{F}^{t_n,X}_s \} L^1 H^{t_n,X}_s] \, dt \, ds dt
\]
\[
+ \int_{t_n}^{t_{n+1}} \int_{t_n}^{t} \int_{t_n}^{t} \mathbb{E}_t^{X_n} \left[ L^0 (L^1 \hat{F}^{t_n,X}_s L^1 H^{t_n,X}_s) \right] \, dt \, ds dt + R^{t_n+1}_{t_n} + R^{t_n+1}_{t_n}.
\]
\begin{equation}
\leq A_{t_n}^X (\Delta t_n)^3, \tag{4.10}
\end{equation}

where

\begin{align*}
A_{t_n}^X &= \mathbb{E}_{t_n}^X [ \mathbb{E}_{t_{n+1}}^X ] \left[ \sup_{t_n \leq \tau \leq t_{n+1}} |L^0 L^0 H_{\tau}^X|^2 \right]^{\frac{1}{2}} \\
&\quad + \mathbb{E}_{t_n}^X \left[ \sup_{t_n \leq \tau \leq t_{n+1}} |D_x F_{t_n}^X|^2 \right]^{\frac{1}{2}} \mathbb{E}_{t_n}^X \left[ \sup_{t_n \leq \tau \leq t_{n+1}} |L^1 L^0 H_{\tau}^X|^2 \right]^{\frac{1}{2}} \\
&\quad + \mathbb{E}_{t_n}^X \left[ \sup_{t_n \leq \tau \leq t_{n+1}} |L^0 (L^0 F_{\tau}^X | L^1 H_{\tau}^X)|^2 \right]^{\frac{1}{2}} \\
&\quad + \mathbb{E}_{t_n}^X \left[ \sup_{t_n \leq \tau \leq t_{n+1}} |D_{x} (L^0 F_{t_n}^X)|^2 \right] \mathbb{E}_{t_n}^X \left[ \sup_{t_n \leq \tau \leq t_{n+1}} |L^1 H_{\tau}^X|^2 \right]^{\frac{1}{2}}.
\end{align*}

From (1.2), we have

\begin{equation}
F_{t_n}^X = Z_{t_n}^X - \sigma^{-1}(t, X_{t_n}^X) \nabla_{x} X_{t_n}^X = u_x(t, X_{t_n}^X) \nabla_{x} X_{t_n}^X, \quad t \in [t_n, t_{n+1}]. \tag{4.11}
\end{equation}

According to (2.16), by applying Itô’s formula to \( \nabla_{x} X_{t_n}^X \), we have

\begin{equation}
\nabla_{x} X_{t_n}^X = \exp \left( \int_{t_n}^{t} \left\{ \left( \partial_x b - \frac{1}{2} (\partial_x \sigma)^2 \right) (r, X_{t_n}^X) dr + \partial_x \sigma (r, X_{t_n}^X) dW_r \right\} \right),
\end{equation}

\begin{equation}
\nabla_{x} X_{t_n}^X = \exp \left( - \int_{t_n}^{t} \left\{ \left( \partial_x b - \frac{1}{2} (\partial_x \sigma)^2 \right) (r, X_{t_n}^X) dr + \partial_x \sigma (r, X_{t_n}^X) dW_r \right\} \right), \tag{4.12}
\end{equation}

for \( t_n \leq t \leq T \). If the bounded functions \( b, \sigma \in C_{b}^{2,4} \), \( f \in C_{b}^{2,4,4,4} \) and \( \varphi \in C_{b}^{4,\alpha} (\alpha \in (0, 1]) \), then \( u \in C_{b}^{2,4} \) (see [4,8]), and the following inequality hold (see [12] for details):

\begin{equation}
\sup_{t_n \leq t \leq T} \mathbb{E}_{t_n}^X [ |(\nabla_{x} X_{t_n}^X)^{-1}|^p ] + \sup_{t_n \leq t \leq T} \mathbb{E}_{t_n}^X [ |D_x \nabla_{x} X_{t_n}^X|^p ] + |D_x \nabla_{x} X_{t_n}^X|^p ] \leq C_p, \tag{4.13}
\end{equation}

for \( \forall p \geq 2 \). Thus, under the conditions of the lemma, by (4.11)–(4.13) and \( u \in C_{b}^{2,4} \), we have the estimate \( A_{t_n}^X \leq C \) (independent of time partition) for \( \forall n \in \{0, \ldots, N-1\} \), where \( C \) depends on the upper bound of the derivatives of \( b, \sigma, f \) and \( \varphi \), which with the inequality (4.10) yields \( \left| R_{21}^n \right| \leq C (\Delta t_n)^3 \) for \( n = 0, 1, 2, \ldots, N \). The proof is completed.

\textbf{Lemma 4.3.} Suppose \( f \in C_{b}^{2,4,4,4}, \varphi \in C_{b}^{4,\alpha} (\alpha \in (0, 1]) \), and the bounded functions \( b, \sigma \in C_{b}^{2,4} \). Then for \( n = 1, 2, \ldots, N \), we have \( \left| R_{22}^n \right| \leq C (\Delta t_n)^3 \), where \( R_{22}^n \) is defined in (2.18).

\textbf{Proof.} By (2.16) and using the Itô-Taylor expansion, we obtain

\begin{equation}
\nabla_{x} X_{t_n}^X = A \bar{R}_{t_n}^{t_{n+1}} \tag{4.14}
\end{equation}

with the remainder

\begin{align*}
\bar{R}_{t_n}^{t_{n+1}} &= \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} (L^0 \{ b_x (\tau, X_{t_n}^X) \} \nabla_{x} X_{t_n}^X ) d\tau + L^1 \{ b_x (\tau, X_{t_n}^X) \} \nabla_{x} X_{t_n}^X ) dW_\tau ) ds \\
&\quad + \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} (L^0 \{ \sigma_x (\tau, X_{t_n}^X) \} \nabla_{x} X_{t_n}^X ) d\tau + L^1 \{ \sigma_x (\tau, X_{t_n}^X) \} \nabla_{x} X_{t_n}^X ) dW_\tau ) dW_s,
\end{align*}

where \( L^0 \) and \( L^1 \) are defined in (4.3). Then by (2.18), (4.14), Lemma 2.1, Hölder’s inequality, and the bounded functions \( b, \sigma \in C_{b}^{2,4} \), we have

\begin{align*}
\left| R_{22}^n \right| &= \frac{1}{2} \Delta t_n \left| \mathbb{E}_{t_n}^X [ Z_{t_n}^X \sigma^{-1}(t_n, X_{t_n}^X) \bar{R}_{t_n}^{t_{n+1}} ] \sigma(t_n, X_{t_n}^X) \right| \\
&= \frac{1}{2} \Delta t_n \int_{t_n}^{t_{n+1}} \int_{t_n}^{s} \mathbb{E}_{t_n}^X [ Z_{t_n}^X \sigma^{-1}(t_n, X_{t_n}^X) L^0 \{ b_x (\tau, X_{t_n}^X) \} \nabla_{x} X_{t_n}^X ) ]
\end{align*}
Then by (1.2), we have
\[ \bar{W}(\tau) = 0 \]
and Theorems 3.1 and 3.2 hold true. Thus, we have the estimates
\[ Z_{t_{n+1}}^{t_n,X^n} = u_x(t_{n+1}, X_{t_{n+1}}^{t_n,X^n}) \]
(4.16)

The conditions of the lemma imply \( u \in C_b^{2,4} \), and
\[ \sup_{t_n \leq s \leq t} \mathbb{E}_t X^n [ |\nabla X_{t_n}^{t_n,X^n}|^p ] \leq C_p, \]
\[ \mathbb{E}_t X^n [ |D_{\tau}X_{t_n}^{t_n,X^n}|^p + \sup_{t_n \leq s \leq t} |D_{\tau}X_{t_n}^{t_n,X^n}|^p ] \leq C_p, \]
(4.17)
for \( \forall p \geq 2 \) and \( n = 0, 1, 2, \ldots, N \). Now under the conditions of the lemma, by \( u \in C_b^{2,4} \) and (4.15)–(4.17), we have \( |R^n_{Z_2}| \leq C(\Delta t)^{\beta+1} \) for \( n = 0, 1, 2, \ldots, N \). The proof is completed. \( \square \)

**Lemma 4.4.** Assume Hypothesis 2.2 holds. Let \( b, \sigma \in C_b^{2,2+\alpha} \), \( f \in C_b^{2,2+2,2,2,2,2+\alpha}, \varphi \in C_b^{2,2,2,2,2,2,2+\alpha} \) \( (\alpha \in (0, 1)) \). Let \( R^n_{Y_1}, R^n_{Z_3} \) and \( R^n_{Z_4} \) are defined in (2.9), (2.20) and (2.22), respectively. Then for \( n = 1, 2, \ldots, N \), we have
\[ |R^n_{Y_1}| \leq C(\Delta t)^{\beta+1}, \quad |R^n_{Z_3}| \leq C(\Delta t)^{\min(\beta+1, \gamma+1)}, \quad |R^n_{Z_4}| \leq C(\Delta t)^{\min(\beta+1, \gamma+1)}. \]

**Proof.** From (1.2), we know
\[ Y_{t_n}^{t_n,X^n} = u(t_n, X_{t_n}^{t_n,X^n}), \quad Z_{t_n}^{t_n,X^n} = u_x(t_n, X_{t_n}^{t_n,X^n})\sigma(t_n, X_{t_n}^{t_n,X^n}), \quad \forall t_n \in (0, T). \]
(4.18)
The conditions of the lemma imply \( u \in C_b^{2,2+2} \). Thus, by (4.18) and the Hypothesis 2.2 (weak approximation properties), we deduce
\[ |R^n_{Y_1}| = \left| \mathbb{E}_t X^n [Y_{t_n}^{t_n,X^n} - Y_{t_n+1}^{t_n,X^n+1}] + \frac{1}{2}\Delta t \mathbb{E}_t X^n [f_{t_n}^{t_n,X^n} - f_{t_n+1}^{t_n,X^{n+1}}] \right| \]
\[ \leq \left| \mathbb{E}_t X^n [u(t_n, X_{t_n}^{t_n,X^n}) - u(t_{n+1}, X_{t_n}^{t_n,X^{n+1}})] \right| + \frac{1}{2}\Delta t \mathbb{E}_t X^n [f(t_{n+1}, X_{t_n}^{t_n,X^n}, u(t_{n+1}, X_{t_n}^{t_n,X^{n+1}}), u_x(t_{n+1}, X_{t_n}^{t_n,X^{n+1}})\sigma(t_{n+1}, X_{t_n}^{t_n,X^{n+1}})] \]
\[ - f(t_{n+1}, X_{t_n}^{t_n,X^n+1}, u(t_{n+1}, X_{t_n}^{t_n,X^{n+1}}), u_x(t_{n+1}, X_{t_n}^{t_n,X^{n+1}})\sigma(t_{n+1}, X_{t_n}^{t_n,X^{n+1}}))] \leq C(\Delta t)^{\beta+1}. \]
The other two estimates can be proved similarly. The proof is completed. \( \square \)

Now, we are ready to prove our main Theorem 4.1 in this section.

**Proof of Theorem 4.1.** Let \( (X^n, Y^n, Z^n) (n = 0, 1, 2, \ldots, N) \) be the solution to Scheme 2.2. Under the conditions of the theorem, by (1.2), we have \( Y_{t_n} = Y_{t_n}^{t_n,X^n} \) and \( Z_{t_n} = Z_{t_n}^{t_n,X^n} \). And further Lemmas 4.1–4.4 and Theorems 3.1 and 3.2 hold true. Thus, we have the estimates
\[ |R^n_{Y_1}| \leq C(\Delta t)^3, \quad |R^n_{Y_1}| \leq C(\Delta t)^{\beta+1}, \quad |R^n_{Z_2}| \leq C(\Delta t)^3, \quad |R^n_{Z_1}| \leq C(\Delta t)^3, \quad |R^n_{Z_2}| \leq C(\Delta t)^3, \quad |R^n_{Z_3}| \leq C(\Delta t)^{\min(\beta+1, \gamma+1)}, \quad |R^n_{Z_4}| \leq C(\Delta t)^{\min(\beta+1, \gamma+1)}. \]
(4.19)
Then by \( (Y^n, Z^n) = (\varphi(X^n), \varphi_x(X^n)\sigma(t_n, X^n)) \), the inequalities in (4.19) and Theorem 3.1, we deduce
\[ \max_{0 \leq n \leq N} \left( c_0|Y_{t_n}^{t_n,X^n} - Y^n|^2 + \frac{1}{14}\Delta t|Z_{t_n}^{t_n,X^n} - Z^n|^2 \right) \]
to compute the conditional mathematical expectations $Y_\omega$.

Let $\mathcal{B}$ be represented as a function of $X(t)$ and identity distributed standard normal random variables. Thus, in the sense of distribution, Schmidt orthogonalization process, this random vector can be represented as

$$\sum_{i=0}^{N-n-1} \left( \frac{1 + C\Delta t}{1 - C\Delta t} \right)^i,$$

which implies (4.1). If $(X^n, Y^n, Z^n) (n = 0, 1, 2, \ldots, N)$ are the solution of Scheme 2.3, the proof of the estimate (4.1) is similar. The proof is completed. 

\[ \square \]

5 Numerical experiments

In this section, we will perform two numerical experiments to demonstrate the effectiveness and accuracy of the three schemes proposed in this paper.

In order to numerically solve the FBSDEs by Schemes 2.1–2.3, the space partition is also needed. We partition the $d$-dimensional Euclidean space $\mathbb{R}^d$ by

$$\mathcal{R}_h = \mathcal{R}_h^1 \times \mathcal{R}_h^2 \times \cdots \times \mathcal{R}_h^d,$$

where $\mathcal{R}_h^k \ (k = 1, \ldots, d)$ are the partition of $(-\infty, \infty)$,

$$\mathcal{R}_h^k = \left\{ x^k_i \mid x^k_i \in \mathbb{R}, i \in \mathbb{Z}, x^k_i < x^k_{i+1}, \lim_{i \to -\infty} x^k_i = +\infty, \lim_{i \to \infty} x^k_i = -\infty \right\}.$$

Let

$$h = \max_{i \in \mathbb{Z}} |x^k_{i+1} - x^k_i|$$

denote the maximum step size of the partition $\mathcal{R}_h$. Let $x_i = (x_{i1}, x_{i2}, \ldots, x_{id}) \in \mathcal{R}_h$ with the tuple index $i = (i_1, i_2, \ldots, i_d)$.

To solve the values of $Y^n$ and $Z^n$ at every space grid point $x_i$ for $n = N-1, \ldots, 1, 0$ backwardly, we need to compute the conditional mathematical expectations $\mathbb{E}^G_{t_n}[G]$, where $G$ is one of the random variables $Y^{n+1}, Y^{n+1} + \Delta W_{t_{n+1}}, f^{n+1}, f^{n+1} + \Delta W_{t_{n+1}}, Z^{n+1}, Z^{n+1} + \sigma A^n \sigma n$ and $Z^{n+1} ((\sigma n)^{-1} f^{n+1} D_t X^{n+1} dt - \Delta t_n)$. By the definitions of the schemes, these random variables are functions of $X^{n+1}$ and $\Delta W_{t_{n+1}}$, and then functions of $t_n, x_i$ and $\xi^{n+1}_n$, where $\xi^{n+1}_n$ is a Gaussian distributed random vector. By the Gram-Schmidt orthogonalization process, this random vector can be represented as $\xi^{n+1}_n = \xi^{n+1}_n (U) = BU$ in law, where $B$ is a deterministic matrix and $U = (U_1, U_2, \ldots, U_k)$ with $U_i (i = 1, 2, \ldots, k)$ independent and identity distributed standard normal random variables. Thus, in the sense of distribution, $G$ can be represented as a function of $t_n, x_i$ and $\xi^{n+1}_n$. We denote this function by $H(t_n, x_i, U)$, i.e., $G = H(t_n, x_i, U)$. Now, we turn the approximation of $\mathbb{E}^G_{t_n}[G]$ to the approximation of $\mathbb{E}[H(t_n, x_i, U)]$, the expectation of functions of standard normal random vector.

It is well known that the Gauss-Hermite Quadrature is a very efficient and accurate method to approximate the expectations of functions of Gaussian type random variable. In our numerical experiments, we choose this method to approximate the conditional expectations in our schemes.

The Gauss-Hermite quadrature rule is an approximation for the values of integrals in the following form by a weighted sum of function values at specified points,

$$\int_{-\infty}^{+\infty} g(x) e^{-x^2} dx = \sum_{i=1}^{L} \omega_i g(a_i) + R(g, L),$$

where $\omega_i \ (i = 1, 2, \ldots, L)$ are weights defined by $\omega_i = 2^{L+i} L! \sqrt{\pi} \frac{H_L(a_i)}{H_L'(a_i)}$, $a_i \ (i = 1, 2, \ldots, L)$ are roots of the Hermite polynomial of degree $L$, $H_L(x) = (-1)^L e^{x^2} \frac{d^L}{dx^L} (e^{-x^2})$, and $R(g, L)$ is the truncation error expressed as

$$R(g, L) = \frac{L! \sqrt{\pi}}{2^{2L}(2L)!} g^{(2L)}(\zeta),$$
where $\zeta$ is some real number. For a $k$-dimensional function $g(x_1, x_2, \ldots, x_k)$, the Gauss-Hermite quadrature formula is
\[
\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} g(x_1, \ldots, x_k) e^{-(x_1^2 + \cdots + x_k^2)} \, dx_1 \cdots dx_k \approx \sum_{i_1=1}^{L} \cdots \sum_{i_k=1}^{L} \omega_{i_1} \cdots \omega_{i_k} g(a_{i_1}, \ldots, a_{i_k}).
\]
According to this formula, the conditional mathematical expectations appearing in the proposed schemes can be approximated as
\[
\mathbb{E}_{t_n}^x[G] = \mathbb{E}[H(t_n, x_i, U)] = \left( \frac{1}{\sqrt{2\pi}} \right)^k \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H(t_n, x_i, U) e^{-\frac{|U|^2}{2}} \, dU_1 \cdots dU_k
\]
\[
= \frac{1}{\pi^\frac{k}{2}} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} H(t_n, x_i, \sqrt{2}U) e^{-\frac{|U|^2}{2}} \, dU_1 \cdots dU_k
\]
\[
\approx \frac{1}{\pi^\frac{k}{2}} \sum_{i_1=1}^{L} \cdots \sum_{i_k=1}^{L} \omega_{i_1} \cdots \omega_{i_k} H(t_n, x_i, \sqrt{2}a_{i_1}, \ldots, \sqrt{2}a_{i_k}).
\]
To compute $H(t_n, x_i, \sqrt{2}a_{i_1}, \ldots, \sqrt{2}a_{i_k})$, the values of $Y^{n+1}$, $Z^{n+1}$ and $f^{n+1}$ at the space points $X^{n+1} = x_i + \phi(t_n, x_i, \Delta t, \xi^{n+1}(\sqrt{2}a_{i_1}, \ldots, \sqrt{2}a_{i_k}))$ on time level $t_{n+1}$ are needed. Generally, $X^{n+1}$ does not belong to $\mathcal{R}_h$. In our numerical experiments, we use the Lagrange interpolation method to approximate them.

By the theory of Lagrange interpolation, when the Lagrange polynomials of order $p$ are used, the errors of the interpolation is $O(h^{p+1})$. Since the aim of our experiments is to check the convergence rate with respect to time, the discrete space step $h$ should be chosen in such a way that the errors resulted from the discretization of FBSDEs and from the space interpolation are balanced. If the theoretical convergence rate of the time discrete FBSDE scheme is $\alpha$, then $h$ should be selected such that the local errors caused by the FBSDE scheme and the errors caused by the Lagrange interpolation are of the same, i.e.,
\[
(\Delta t)^{\alpha+1} = h^{p+1}, \quad h = (\Delta t)^{\frac{\alpha}{p+1}}.
\]
We note that in our numerical experiments, the number of the Gauss-Hermite quadrature points $L$ is also set to be big enough (we take 8 points) such that the errors contributed by the use of Gauss-Hermite quadrature are small and affect the convergence rate very little.

The three schemes proposed in this paper have common structures. Besides the terminal conditions, each scheme consists of three parts. The first part is a numerical scheme for solving the forward SDE, the second and third parts in the three schemes, respectively.

**Algorithm.**

**Step 1.** Let $n = N$. Compute $Y^n(x_i) = \varphi(x_i)$, $i \in \mathbb{Z}^d$. If the coefficients of FBSDE satisfy certain conditions let $Z^n(x_i) = \varphi_x(x_i)\sigma(T, x_i)$, otherwise, let $n = n - 1$ and compute $Y^n(x_i)$ and $Z^n(x_i)$ by the Euler scheme (2.25).

**Step 2.** Let $n = n - 1$. For all the space grid points $x_i \in \mathcal{R}_h$.

(a) For $i = 1, \ldots, L$, $l = 1, \ldots, k$.

i. Compute the points need to be interpolated on the time level $t_{n+1}$ by the first part of each scheme $X^{n+1} = x_i + \phi(t_n, x_i, \Delta t, \xi^{n+1}(\sqrt{2}a_{i_1}, \ldots, \sqrt{2}a_{i_k}))$.

ii. Compute the value of $H(t_n, x_i, \sqrt{2}a_{i_1}, \ldots, \sqrt{2}a_{i_k})$ by the Lagrange interpolation, where $H$ stands for the necessary functions of $t_n$, $x_i$ and $\xi^{n+1}$ whose conditional expectations will be used in the second and third parts in the three schemes, respectively.

(b) Compute the conditional expectations $\mathbb{E}_{t_n}^x[H]$ by the Gauss-Hermite quadrature rule.

(c) Compute $Z^n(x_i)$ by the second part of each scheme for each $i \in \mathbb{Z}^d$, which is explicit for $Z^n(x_i)$.

(d) Compute $Y^n(x_i)$ according to the third part of each scheme by iteration methods, which is implicit for $Y^n(x_i)$.

**Step 3.** If $n = 0$, then stop, otherwise go to Step 2.
Remark 5.1. In many practical problems, such as option pricing problems, we are only interested in the values of $(Y^0(x_i), Z^0(x_i))$ for $x_i$ in compact domain. Thus, under some accuracy, we just compute $(Y^n(x_i), Z^n(x_i))$ for $x_i$ in a bounded domain instead of all the $x_i \in \mathbb{Z}^d$.

According to Theorem 4.1, the convergence rate of Schemes 2.2 and 2.3 are also restricted by the weak convergence rate of the SDE scheme. Therefore, in our numerical tests, three different schemes for SDEs, the Euler scheme, the Milstein scheme and the order-2.0 weak Itô-Taylor scheme given in Subsection 2.1, are used to solve SDEs.

In the following examples, we take uniform partition in the time interval $[0, T]$, and set the terminal time $T = 1$. The time step sizes ranges from $\frac{T}{2^5}$ to $\frac{T}{2^7}$. In the following tables, the notations $|Y_0 - Y^0|$, $|Z_0 - Z^0|$ represent the errors between the exact solution $(Y_t, Z_t)$ of (1.1) at time $t = 0$ and the solution $(Y^n_t, Z^n_t)$ to the three schemes at $n = 0$. CR stands for the convergence rate with respect to time step $\Delta t$, and is obtained by using linear least square fitting to the errors.

Example 5.1 (Linear case). The considered FBSDEs is

$$
\begin{align*}
    dX_t &= \cos(X_t)dt + (\sin(X_t) + 3)\cos(X_t)dW_t, \\
    -dY_t &= \left( -Y_t - \frac{Z_t}{\sin(X_t) + 3} + 0.5 Z_t (1 + 3 \sin(X_t)) \right) dt + Z_t dW_t
\end{align*}
$$

with $X_0 = 1.0$ and $Y_T = \exp(T) \ln(\sin(X_T) + 3.0)$. The analytical solutions $Y_t$ and $Z_t$ are

$$
Y_t = \exp(t) \ln(\sin(X_t) + 3) \quad \text{and} \quad Z_t = \exp(t) \cos^2(X_t).
$$

Then the exact solution $(Y_t, Z_t)$ at the initial time is $(Y_0, Z_0) = (1.044, 0.292)$. Choose the number of points used in the Lagrange interpolation to be 4. The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$, and the convergence rates for different time partitions and different schemes are listed in Tables 1–3.

The run time of the three schemes for Example 5.1 are listed in Table 4, from which we can see that when the Euler scheme is applied, the run time of the three FBSDE schemes are shorter than the other two SDE schemes. And for the same SDE scheme, Scheme 2.3 is the most efficient.

**Figure 1** Errors and convergence rates of Scheme 2.1 for Example 5.1

| SDE scheme | Euler | Milstein | Weak order-2.0 |
|------------|-------|----------|----------------|
| $N$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ |
| 32 | 9.980e-03 | 4.538e-02 | 3.153e-03 | 6.411e-02 | 1.384e-03 | 3.067e-02 |
| 64 | 4.389e-03 | 2.144e-02 | 1.938e-03 | 2.520e-02 | 3.413e-04 | 7.764e-03 |
| 128 | 2.100e-03 | 1.049e-02 | 1.033e-03 | 1.104e-02 | 8.080e-05 | 1.636e-03 |
| 256 | 1.041e-03 | 5.202e-03 | 5.273e-04 | 5.329e-03 | 1.958e-05 | 3.776e-04 |
| 512 | 5.197e-04 | 2.596e-03 | 2.657e-04 | 2.633e-03 | 4.826e-06 | 9.314e-05 |
| CR | 1.060 | 1.030 | 0.902 | 1.145 | 2.045 | 2.109 |

**Figure 2** Errors and convergence rates of Scheme 2.2 for Example 5.1

| SDE scheme | Euler | Milstein | Weak order-2.0 |
|------------|-------|----------|----------------|
| $N$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ |
| 32 | 7.875e-03 | 2.845e-02 | 3.835e-03 | 6.908e-02 | 1.221e-03 | 2.740e-02 |
| 64 | 3.872e-03 | 1.805e-02 | 1.957e-03 | 3.082e-02 | 4.353e-04 | 9.359e-03 |
| 128 | 1.969e-03 | 1.022e-02 | 1.017e-03 | 1.339e-02 | 2.164e-04 | 2.480e-03 |
| 256 | 1.008e-03 | 5.434e-03 | 5.212e-04 | 6.167e-03 | 3.296e-05 | 6.168e-04 |
| 512 | 5.113e-04 | 2.802e-03 | 2.640e-04 | 2.961e-03 | 8.455e-06 | 1.531e-04 |
| CR | 0.983 | 0.842 | 0.963 | 1.141 | 1.807 | 1.889 |
Example 5.2 (Nonlinear case). The considered FBSDEs is

\[
\begin{align*}
    dX_t &= \frac{1}{1+2\exp(t+X_t)}dt + \frac{\exp(t+X_t)}{1+\exp(t+X_t)}dW_t, \\
    -dY_t &= \left( -\frac{2Y_t}{1+2\exp(t+X_t)} - \frac{1}{2} \left( \frac{Y_t Z_t}{1 + \exp(t+X_t)} - Y_t^2 Z_t \right) \right) dt - Z_t dW_t.
\end{align*}
\]

The terminal condition of $Y_t$ is

\[ Y_T = \frac{\exp(T + X_T)}{1 + \exp(T + X_T)}. \]

Then the analytic solutions of (5.3) are

\[ Y_t = \frac{\exp(t + X_t)}{1 + \exp(t + X_t)} \quad \text{and} \quad Z_t = \frac{(\exp(t + X_t))^2}{(1 + \exp(t + X_t))^3}, \]

respectively. Assume $X_0 = 1.0$ and $T = 1.0$, then the exact solution at the initial time $t = 0$ is $(Y_0, Z_0) = (0.731, 0.144)$. The errors $|Y_0 - Y^0|$, $|Z_0 - Z^0|$ and the convergence rates for different time partitions and different schemes of this nonlinear example are listed in Tables 5–7.
Figure 6 Errors and convergence rates of Scheme 2.2 for Example 5.2

| $N$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 32  | 2.685e-04  | 2.881e-04  | 2.110e-04  | 5.530e-04  | 1.609e-05  | 1.933e-05  |
| 64  | 1.372e-04  | 1.409e-04  | 1.083e-04  | 2.730e-04  | 4.063e-06  | 4.728e-06  |
| 128 | 6.934e-05  | 6.968e-05  | 5.487e-05  | 1.357e-04  | 1.028e-06  | 1.171e-06  |
| 256 | 3.486e-05  | 3.465e-05  | 2.762e-05  | 6.763e-05  | 2.548e-07  | 2.941e-07  |
| 512 | 1.748e-05  | 1.728e-05  | 1.385e-05  | 3.376e-05  | 6.335e-08  | 7.355e-08  |
| CR  | 0.986       | 1.014       | 0.983       | 1.008       | 1.997        | 2.008        |

Figure 7 Errors and convergence rates of Scheme 2.3 for Example 5.2

| $N$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ | $|Y_0 - Y^0|$ | $|Z_0 - Z^0|$ |
|-----|----------------|----------------|----------------|----------------|----------------|----------------|
| 32  | 2.913e-04  | 2.861e-04  | 1.823e-04  | 8.527e-04  | 1.582e-05  | 1.462e-05  |
| 64  | 1.487e-04  | 1.409e-04  | 9.366e-05  | 4.258e-04  | 3.994e-06  | 3.530e-06  |
| 128 | 7.514e-05  | 6.995e-05  | 4.747e-05  | 2.128e-04  | 1.011e-06  | 8.687e-07  |
| 256 | 3.777e-05  | 3.485e-05  | 2.390e-05  | 1.064e-04  | 2.505e-07  | 2.181e-07  |
| 512 | 1.893e-05  | 1.740e-05  | 1.199e-05  | 5.320e-05  | 6.226e-08  | 5.451e-08  |
| CR  | 0.986       | 1.009       | 0.982       | 1.000       | 1.997        | 2.015        |

In this example, the BSDE is nonlinear, which is different from Example 5.1, in which BSDE is linear. Table 8 shows the run time of Schemes 2.1–2.3 when the three different SDE schemes are applied, respectively.

From the errors and convergence rates listed in Tables 1–7, we conclude that:

1. The convergence rates of the three schemes for solving the decoupled FBSDEs, linear or nonlinear, depend only on the weak order of schemes used for solving the forward SDE. The convergence rates of the three schemes are of order 1 if the Euler scheme or the Milstein scheme is used for solving the SDE, and are of order 2 if the order-2 weak Itô-Taylor scheme is used for solving the SDE.

2. The convergence rates of Schemes 2.2 and 2.3 listed in Tables 2, 3, 6 and 7 are consistent with the theoretical results in Theorem 4.1. Tables 1 and 5 tell us that Scheme 2.1 is also a second-order scheme if order-2 weak schemes for solving SDEs are used.

3. Tables 2, 3, 6 and 7 also show that among the three schemes Schemes 2.1 and 2.3 are more accurate than Scheme 2.2. For Scheme 2.1, we have done many numerical tests, and all the tests showed that it was a second-order scheme for solving decoupled FBSDEs. Unfortunately, for Scheme 2.1, up to now, we still do not know how to theoretically get the error estimate results as stated in Theorem 4.1, though in [20], we have theoretically gotten second-order error estimate results when Scheme 2.1 is used to solve BSDEs.

Figure 8 Run Time of the Three Schemes for Example 5.2

| $N$ | 32 | 64 | 128 | 256 | 512 |
|-----|----|----|-----|-----|-----|
| Euler | 0.611 | 1.869 | 4.701 | 10.805 | 53.673 |
| Scheme 2.1 | 0.918 | 2.133 | 5.449 | 31.369 | 169.958 |
| Milstein | 0.841 | 2.262 | 5.288 | 32.177 | 231.696 |

| Euler | 0.459 | 2.094 | 4.080 | 11.656 | 53.916 |
| Scheme 2.2 | 0.717 | 2.031 | 5.288 | 32.585 | 199.839 |
| Milstein | 0.934 | 2.414 | 5.590 | 33.507 | 240.410 |

| Euler | 0.338 | 1.080 | 1.997 | 7.778 | 43.643 |
| Scheme 2.3 | 0.604 | 1.175 | 4.099 | 29.416 | 194.579 |
| Milstein | 0.806 | 2.292 | 6.944 | 34.870 | 231.650 |
4. Tables 4 and 8 show that the computational cost increased dramatically when we use the Milstein scheme and the weak order-2 scheme to solve the SDEs instead of the Euler scheme. The increasing is mainly caused by the fast increasing of the computation times of space interpolations when $h$ decreases.

5. The three schemes are more efficient when second-order (weak) scheme is used to solve SDEs. For example, in Table 8 of Example 5.2, the errors for solving $Y_0$ by Scheme 2.3 are similar when the Euler scheme is used with $N = 512$ and the weak order-2.0 scheme is used with $N = 32$, but the run times are 43.643 seconds and 0.806 seconds, respectively. The ratio is approximately 54.

6 Conclusions

We proposed three new high-order numerical schemes for solving the decoupled FBSDEs (1.1) by trapezoidal rule and the integration-by-parts formula of Malliavin calculus. By using Malliavin calculus and stochastic analysis theory, under some reasonable regularity conditions, we rigorously obtained general error estimate results for Schemes 2.2 and 2.3 for solving decoupled FBSDEs. Then we theoretically proved that the accuracy of Schemes 2.2 and 2.3 depends on the accuracy of numerical methods for solving SDEs in weak sense. Numerical experiments showed that the three numerical schemes are first-order ones if the weak-order 1 Euler and Milstein schemes are used for solving SDEs, and are second-order ones if weak-order 2 Itô-Taylor-type schemes are used. These convergence rate results obtained in the experiments are consistent with our theoretical results. Our theoretical and numerical experiment results show that the proposed schemes in this paper are stable, effective and accurate for solving decoupled FBSDEs. These schemes may be used in finance, stochastic control and other related problems. We will do some study on our schemes for these problems in our future research.

Acknowledgements This work was supported by National Natural Science Foundation of China (Grant Nos. 91130003 and 11171189) and Natural Science Foundation of Shandong Province (Grant No. ZR2011AZ002). The authors would like to thank the referees for their valuable suggestions, which improved the paper a lot.

References

1 Bender C, Denk R. A forward scheme for backward SDEs. Stochastic Process Appl, 2007, 117: 1793–1812
2 Bouchard B, Touzi N. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. Stochastic Process Appl, 2004, 111: 175–206
3 Chevance D. Numerical methods for backward stochastic differential equations. In: Numerical methods in finance. Cambridge: Cambridge University Press, 1997, 232–244
4 Evans L C. Partial Differential Equations. Providence, RI: Amer Math Soc, 1998
5 Gobet E, Labart C. Error expansion for the discretization of backward stochastic differential equations. Stochastic Process Appl, 2007, 117: 803–829
6 Karoui N E, Peng S G, Quenez M C. Backward stochastic differential equations in finance. Math Finance, 1997, 7: 1–71
7 Kloeden P E, Platen E. Numerical Solution of Stochastic Differential Equations. Berlin: Springer-Verlag, 1992
8 Ladyzenskaja O, Solonnikov V, Uralceva N. Linear and Quasilinear Equations of Parabolic Type. Providence, RI: Amer Math Soc, 1968
9 Li Y, Zhao W D. $L^p$-error estimates for numerical schemes for solving certain kinds of backward stochastic differential equations. Statist Probab Lett, 2010, 21–22: 1612–1617
10 Ma J, Protter P, San Martin J, et al. Numerical methods for backward stochastic differential equations. Ann Appl Probab, 2002, 12: 302–316
11 Ma J, Protter P, Yong J M. Solving forward-backward stochastic differential equations explicitly-a four step scheme. Probab Theory Related Fields, 1994, 98: 339–359
12 Nualart D. The Malliavin Calculus and Related Topics. Berlin: Springer Verlag, 1995
13 Pardoux E, Peng S G. Adapted solution of a backward stochastic differential equation. Systems Control Lett, 1990, 14: 55–61
14 Peng S G. Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. Stoch Rep, 1991, 37: 61–74
15. Tocino A, Vigo-aguiar J. Weak second order conditions for stochastic Runge-Kutta methods. SIAM J Sci Comput, 2003, 2: 507–523
16. Zhang J F. A numerical scheme for BSDEs. Ann Appl Probab, 2004, 14: 459–488
17. Zhao W D, Chen L F, Peng S G. A new kind of accurate numerical method for backward stochastic differential equations. SIAM J Sci Comput, 2006, 28: 1563–1581
18. Zhao W D, Li Y, Zhang G N. A Generalized $\theta$-Scheme for solving backward stochastic differential equations. Discrete Contin Dyn Syst Ser B, 2012, 5: 1585–1603
19. Zhao W D, Wang J L, Peng S G. Error estimates of the $\theta$-scheme for backward stochastic differential equations. Discrete Contin Dyn Syst Ser B, 2009, 4: 905–924
20. Zhao W D, Li Y, Ju L L. Error estimates of the Crank-Nicolson scheme for solving backward stochastic differential equations. Int J Numer Anal Model, 2013, 4: 876–898
21. Zhao W D, Zhang G N, Ju L L. A stable multistep scheme for solving backward stochastic differential equations. SIAM J Numer Anal, 2010, 4: 1369–1394