On distance $r$-dominating and $2r$-independent sets in sparse graphs

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Abstract

Dvořák [5] gave a bound on the minimum size of a distance $r$ dominating set in the terms of the maximum size of a distance $2r$ independent set and generalized coloring numbers, thus obtaining a constant factor approximation algorithm for the parameters in any class of graphs with bounded expansion. We improve and clarify this dependence using an LP-based argument inspired by the work of Bansal and Umboh [3].

A set $X$ of vertices of a graph $G$ is dominating if each vertex of $G$ either belongs to or has a neighbor in $X$, and it is independent if no two vertices of $X$ are adjacent. The domination number $\gamma(G)$ of $G$ is the minimum size of a dominating set in $G$, and the independence number $\alpha(G)$ of $G$ is the maximum size of an independent set in $G$. Determining either of these parameters in a general graph is NP-complete [9]. Even approximating them is hard. No polynomial-time algorithm approximating the domination number of an $n$-vertex graph within a factor better than $O(\log n)$ exists [16], unless $P = \text{NP}$. Even worse, for every $\varepsilon > 0$, no polynomial-time algorithm approximating the independence number of an $n$-vertex graph within a factor better than $O(n^{1-\varepsilon})$ exists [3], unless $\text{ZPP} = \text{NP}$.

Both parameters become more tractable in sparse graphs—they have a PTAS in planar graphs [2] and other related graph classes, most generally in all graph classes with strongly sublinear separators [7]. To obtain constant-factor approximation, much weaker constraints suffice. Lenzen and Wattenhofer [12] proved that the domination number can be approximated within factor $a^2 + 3a + 1$ on
graphs with *arboricity* at most $a$ (i.e., for graphs whose edge sets can be partitioned into at most $a$ forests). This was improved to $3a$ by Bansal and Umboh [3] using a simple LP-based argument, even under a weaker assumption.

**Theorem 1** (Bansal and Umboh [3]). *For any positive integer $a$, if a graph $G$ has an orientation with maximum indegree at most $a$, then a dominating set in $G$ of size at most $3a \gamma(G)$ can be found in polynomial time.*

Regarding independent sets, let us remark that if $G$ has an orientation with maximum indegree at most $a$, then its maximum average degree is at most $2a$. Consequently, $G$ has a proper coloring using at most $2a + 1$ colors, and one of the color classes gives an independent set of size at least $|V(G)|/(2a + 1)$, which approximates the independence number within the factor $2a + 1$.

We consider distance generalizations of domination and independence number and the relationship between them. A set $X \subseteq V(G)$ of vertices of a graph $G$ is *$r$-dominating* if each vertex of $G$ is at distance at most $r$ from $X$. For a vertex $v \in V(G)$, let $N_r[v]$ denote the set of vertices of $G$ at distance at most $r$ from $v$. A set $Y \subseteq V(G)$ is *$2r$-independent* if the distance between any two vertices of $Y$ is greater than $2r$, or equivalently, if $|N_r[v] \cap Y| \leq 1$ for each $v \in V(G)$.

Since each vertex $r$-dominates at most one vertex of a $2r$-independent set, we have $|Y| \leq |X|$ for every $r$-dominating set $X$ and $2r$-independent set $Y$ in the graph $G$. Hence, defining $\gamma_r(G)$ as the minimum size of an $r$-dominating set in $G$ and $\alpha_{2r}(G)$ as the maximum size of a $2r$-independent set in $G$, we have the following inequality.

**Observation 2.** *For any positive integer $r$, every graph $G$ satisfies*

$$\alpha_{2r}(G) \leq \gamma_r(G).$$

The relationship between $\gamma_r$ and $\alpha_{2r}$ becomes clearer when we consider their LP relaxations. Let

$$\gamma^*_r(G) = \min \sum_{v \in V(G)} x_v$$

subject to

$$\sum_{v \in N_r[u]} x_v \geq 1$$

for all $u \in V(G)$

$$x_v \geq 0$$

for all $v \in V(G)$,
and

$$\alpha^*_2(G) = \max \sum_{u \in V(G)} y_u$$

subject to

$$\sum_{u \in N_r[v]} y_u \leq 1 \quad \text{for all } v \in V(G)$$

$$y_u \geq 0 \quad \text{for all } u \in V(G).$$

Since the programs defining the two parameters are dual, we obtain the following chain of inequalities.

**Observation 3.** For any positive integer \( r \), every graph \( G \) satisfies

$$\alpha_2(G) \leq \alpha^*_2(G) = \gamma^*_r(G) \leq \gamma_r(G).$$

Note that \( \alpha^*_2(G) = \gamma^*_r(G) \) can be determined exactly in polynomial time by solving the linear programs that define them.

For any integer \( r \geq 1 \), the ratio \( \gamma_r(G)/\alpha_2(G) \) can be arbitrarily large even for graphs of arboricity at most 3, the class of graphs studied by Lenzen and Wattenhofer [12], as we will see below. Dvořák [5] found a bound on this ratio in terms of a stronger sparsity parameter. Let \( v_1, v_2, \ldots, v_n \) be an ordering of the vertices of a graph \( G \). A vertex \( v_a \) is weakly \( k \)-accessible from \( v_b \) if \( a \leq b \) and there exists a path \( v_a = v_{i_0}, v_{i_1}, \ldots, v_{i_\ell} = v_b \) of length \( \ell \leq k \) in \( G \) such that \( a \leq i_j \) for \( 0 \leq j \leq \ell \). For a fixed ordering of \( V(G) \), let \( Q_k(v) \) denote the set of vertices that are weakly \( k \)-accessible from \( v \) and let \( q_k(v) = |Q_k(v)| \). The weak \( k \)-coloring number of the ordering is the maximum of \( q_k(v) \) over \( v \in V(G) \). The weak \( k \)-coloring number \( \text{wcol}^k(G) \) of \( G \) is the minimum of the weak \( k \)-coloring numbers over all orderings of \( V(G) \).

Weak coloring numbers were first defined by Kierstead and Yang [11] as a distance generalization of degeneracy or ordinary coloring number—for any integer \( d \), a graph has weak 1-coloring number at most \( d + 1 \) if and only if \( G \) is \( d \)-degenerate, i.e., each subgraph of \( G \) has a vertex of degree at most \( d \). Weak 2-coloring number is similarly related to another well studied graph parameter, arrangeability [4,10,17]. They also play an important role in the theory of graph classes with bounded expansion [14]. The starting point of our discourse is the following bound on the ratio \( \gamma_r(G)/\alpha_2(G) \).

**Theorem 4 (Dvořák [5]).** For any positive integer \( r \), every graph \( G \) satisfies

$$\gamma_r(G) \leq \text{wcol}^2_{2r}(G) \alpha_2(G).$$
On the other hand, the ratio $\gamma_r(G)/\alpha_{2r}(G)$ cannot be bounded by a function of $\text{wcol}_{2r-1}(G)$. Let us now give a construction showing this fact. For a hypergraph $H$ and a positive integer $r$, let $H^{(r)}$ be the graph obtained as follows. Let $H_1$ be the incidence graph of $H$, i.e., the bipartite graph with parts $V(H)$ and $E(H)$ such that $v \in V(H)$ is adjacent to $e \in E(H)$ iff $v \in e$. Let $H_2$ be the graph obtained from $H_1$ by subdividing each edge by $r-1$ vertices. Finally, $H^{(r)}$ is obtained from $H_2$ by adding a new vertex $u$ joined by new paths of length $r$ to all vertices of $V(H_2) \setminus V(H)$. The relevant properties of $H^{(r)}$ are given by the following lemma, which we prove in Section 1.

**Lemma 5.** Let $H$ be a hypergraph of minimum degree $\delta \geq 1$ and let $r$ be a positive integer.

- $\text{wcol}_{2r-1}(H^{(r)}) \leq r^2 - r + 3$.
- If each edge of $H$ has size most $t$, then $\text{wcol}_{2r-1}(H^{(r)}) \leq r^2 - r + t + 2$.
- If each two vertices of $H$ are incident with a common edge, then $\alpha_{2r}(H^{(r)}) \leq 2$.
- If no $p$ edges of $H$ cover all vertices of $H$, then $\gamma_r(H^{(r)}) > p$.
- $\gamma^*_r(G) = \alpha^*_r(G) \leq |E(H)|/\delta + 1$.

In particular, taking $H$ as the complete graph $K_n$, we have $\alpha_{2r}(K_n^{(r)}) \leq 2$, $\gamma_r(K_n^{(r)}) \geq n/2$, and $\text{wcol}_{2r-1}(K_n^{(r)}) \leq r^2 - r + 4$, showing that $\gamma_r/\alpha_{2r}$ cannot be bounded by a function of $\text{wcol}_{2r-1}$. Furthermore, observe that the graph $K_n^{(r)}$ has arboricity at most 3.

By Observation 3, $\alpha^*_r(G) = \gamma^*_r(G)$ approximates $\gamma_r(G)$ and $\alpha_{2r}(G)$ within factor of $\text{wcol}_{2r}(G)$. Let us remark that Amiri et al. 11 gave an improved approximation algorithm for $\gamma_r$, within factor of $\text{wcol}_{2r}(G)$.

In this note, we aim to clarify the relationship between distance domination number, distance independence number, and weak coloring numbers. Firstly, generalizing the LP-based approach of Bansal and Umboh 3, we show in Theorem 9 that $\gamma_r(G)$ can be approximated within a factor expressed in terms of $\text{wcol}(G)$, by bounding the ratio $\gamma_r(G)/\gamma^*_r(G)$; we also show that the ratio cannot be bounded in terms of $\text{wcol}_{2r-1}(G)$.

Note that $\alpha^*_r(K_n^{(r)}) \geq n/2$, as shown by setting $y_v = 1/2$ for $v \in V(K_n)$ and $y_v = 0$ for all other vertices $v$ of $K_n^{(r)}$; and thus the ratio $\alpha^*_r(G)/\alpha_{2r}(G)$ cannot be bounded even in terms of $\text{wcol}_{2r-1}(G)$. To work around this issue, we consider
a relaxed version of 2r-independent set. We say that a set \( Y \subseteq V(G) \) is \( (2r, b)\)-independent if \( |N_r[v] \cap Y| \leq b \) for all \( v \in V(G) \). Let \( \alpha_{2r,b}(G) \) be the maximum size of a \( (2r, b)\)-independent set in \( G \). Note that setting \( y_v = 1/b \) for \( v \in Y \) and \( y_v = 0 \) for \( v \in V(G) \setminus Y \) gives a feasible solution to the program defining \( \alpha^*_2 \), and thus we have the following.

**Observation 6.** For any graph \( G \) and positive integers \( r \) and \( b \),

\[
\alpha_{2r}(G) \leq \alpha_{2r,b}(G) \leq ba^*_2(G).
\]

We show in Theorem 15 that for \( b = \text{wcol}^2_r(G) \), the ratio \( \alpha^*_2(G)/\alpha_{2r,b}(G) \) is at most 2.

Finally, we link the results with Theorem 4. In Lemma 16, we show that the ratio \( \alpha_{2r,b}(G)/\alpha_{2r}(G) \) is bounded in terms of \( \text{wcol}_{2r}(G) \). Composing all the results, we obtain the following chain of inequalities.

**Theorem 7.** For any graph \( G \) and a positive integer \( r \),

\[
\frac{1}{\text{wcol}^2_r(G)} \gamma_r(G) \leq \gamma^*_r(G) = \frac{\alpha^*_2(G) - \alpha_{r, \text{wcol}^2_r(G)}(G) \leq 2\alpha_{2r, \text{wcol}^2_r(G)}(G) \leq 4\text{wcol}^2_r(G)\text{wcol}_{2r}(G)\alpha_{2r}(G)}.
\]

In particular, \( \gamma_r(G) \leq 4\text{wcol}^2_r(G)\text{wcol}_{2r}(G)\alpha_{2r}(G) \), improving Theorem 4 when \( \text{wcol}_{2r}(G) \) is large enough compared to \( \text{wcol}_r(G) \). By Theorem 7, \( \gamma^*_r(G) = \alpha^*_2(G) \) approximates \( \gamma_r(G) \) and \( \alpha_{2r}(G) \) up to factors \( \text{wcol}^2_r(G) \) and \( 4\text{wcol}^2_r(G)\text{wcol}_{2r}(G) \), respectively. Inspection of our arguments shows that they are constructive and give a polynomial-time algorithm to return an \( r\)-dominating set of size at most \( w^2_r \cdot \gamma^*_r(G) \) and a \( 2r\)-independent set of size at least \( \frac{1}{4w^2_r} \cdot \alpha^*_2(G) \), assuming that orderings of vertices of \( G \) with weak \( r\)-coloring number \( w_r \) and weak \( 2r\)-coloring number \( w_{2r} \) are given. See [5, 6] for a discussion of the complexity and algorithms to obtain such orderings.

## 1 Weak coloring number and augmentations

Let us start by showing the properties of the graphs \( H^{(r)} \) declared in Lemma 5. Let us fix the notation as in the construction of the graph. For a vertex \( v \in V(H^{(r)}) \setminus (V(H) \cup E(H) \cup \{u\}) \), let \( R_v \) denote the set of vertices reachable from \( v \) by paths in \( H^{(r)} \) not containing vertices of \( V(H) \cup E(H) \cup \{u\} \), and for the unique \( e \in R_v \cap E(H) \), let \( R'_v = R_v \cup e \). We have \( |R_v| \leq r^2 - r + 3 \) and \( |R'_v| \leq r^2 - r + |e| + 2 \).
Proof of Lemma: Consider the ordering of the vertices of $H^{(r)}$ where $u$ is the smallest vertex, followed by vertices of $V(H)$ in any order, vertices of $E(H)$ in any order, and finally all other vertices of $H^{(r)}$ in any order. Clearly $Q_r(u) = \{u\}$ for every $r'$. If $v \in V(H)$, then $Q_r(v) \subseteq \{u, v\}$ when $r' \leq 2r - 1$. If $e \in E(H)$, then $Q_r(e) = \{e\}$ when $r' \leq r - 1$ and $Q_r(e) \subseteq e \cup \{e, u\}$ when $r' \leq 2r - 1$. For any other vertex $v \in V(H^{(r')})$, we have $Q_r(v) \subseteq R_v$ when $r' \leq r$ and $Q_r(v) \subseteq R_v'$ when $r' \leq 2r$.

We conclude that $\text{wcol}_{r-1}(H^{(r)}) \leq r^2 - r + 3$ and $\text{wcol}_{2r-1}(H^{(r)}) \leq r^2 - r + t + 2$.

Note that any two vertices of $V(H^{(r')}) \setminus V(H)$ are at distance at most $2r$ from one another, as shown by a path through $u$. If $u, v \in e$ for some $e \in E(H)$, then the distance between $u$ and $v$ in $H^{(r')}$ is $2r$, as shown by a path through $e$. Hence, if each two vertices of $H$ are contained in a common edge, then any $2r$-independent set in $H^{(r')}$ has at most two vertices (one in $V(H)$ and one in $V(H^{(r')}) \setminus V(H)$), and thus $\alpha_{2r}(H^{(r)}) \leq 2$.

No vertex of $V(H)$ is at distance at most $r$ from $u$ in $H^{(r)}$. If $v \in V(H)$, then no vertex of $V(H) \setminus \{v\}$ is at distance at most $r$ from $v$ in $H^{(r)}$. If $v = e \in E(H)$, or $v \in V(H^{(r)}) \setminus (V(H) \cup E(H) \cup \{u\})$ and $e$ is the unique element of $R_v \cap E(H)$, then no vertices of $V(H) \setminus e$ are at distance at most $r$ from $u$. Hence, for each $v \in V(H^{(r)})$, there exists $e_v \in E(H)$ such that all vertices of $V(H)$ at distance at most $r$ from $v$ in $H^{(r)}$ belong to $e_v$. If $D$ is a dominating set in $H^{(r)}$, it follows that $\bigcup_{v \in D} e_v = V(H)$. If no $p$ edges of $H$ cover all vertices of $H$, then we conclude that $\gamma^*_r(H^{(r)}) > p$.

Finally, setting $x_u = 1$, $x_r = 1/\delta$ for all $e \in E(H)$, and $x_v = 0$ for all $v \in V(H^{(r)})$ gives a feasible solution to the program defining $\gamma^*_r(G)$, which implies that $\gamma^*_r(G) \leq |E(H)|/\delta + 1$. \hfill \qed

In the rest of the paper, it is convenient to perform the arguments in terms of certain directed graphs rather than weak coloring numbers; this also makes the connection to the result of Bansal and Umboh more transparent.

For a positive integer $r$, an $r$-augmentation $\widehat{G}$ of a graph $G$ is an orientation of a supergraph of $G$ with $V(\widehat{G}) = V(G)$ such that each edge $e \in E(\widehat{G})$ is assigned $\text{length } \rho(e) \in \{0, 1, \ldots, r\}$ with the following properties:

(LOOP) Each vertex of $\widehat{G}$ is incident with a loop of length 0.

(DIST) For any non-negative integer $r' \leq r$ and vertices $u, v \in V(G)$, the distance between $u$ and $v$ in $G$ is at most $r'$ if and only if there exists a common inneighbour $x$ of $u$ and $v$ in $\widehat{G}$ and $\rho(xu) + \rho(xv) \leq r'$.

Because of (LOOP), if $uv \in E(\widehat{G})$, then (DIST) applied with $x = u$ shows that the distance between $u$ and $v$ in $G$ is at most $\rho(uv)$. Essentially, we augment $G$ by
adding (directed) edges $uv$ representing certain paths in $G$, with $\rho(uv)$ being the length of the corresponding path between $u$ and $v$. It follows that $\rho(uv) > 0$ for all $uv \in E(\hat{G})$ with $u \neq v$, and that if $x$ and $y$ are adjacent vertices of $G$, then at least one of the directed edges $xy$ or $yx$ appears in $\hat{G}$ with length 1.

For a non-negative integer $r' \leq r$ and $v \in V(G)$, let $\deg_{r',\hat{G}}(v)$ be the number of inneighbors $u$ of $v$ with $\rho(uv) \leq r'$, and let $\Delta_{r'}(\hat{G})$ be the maximum of $\deg_{r',\hat{G}}(v)$ over all vertices of $G$. Let us note a connection between weak coloring numbers and augmentations.

**Observation 8.** Consider any ordering of vertices of a graph $G$ and a non-negative integer $r$. Let $\hat{G}$ be the directed graph in which $uv \in E(\hat{G})$ iff $u \in Q_r(v)$, and let $\rho(uv)$ be the minimum $r'$ such that $u \in Q_r(v)$. Then $\hat{G}$ is an $r$-augmentation of $G$ and $\Delta_{r'}(\hat{G})$ is equal to the weak $r'$-coloring number of the ordering for any non-negative $r' \leq r$.

**Proof.** Note that $\Delta_{r'}(\hat{G})$ is equal to the weak $r'$-coloring number of the ordering for every $r' \leq r$ by the choice of $\rho$, and thus it suffices to argue that $\hat{G}$ is an $r$-augmentation.

Since $v \in Q_0(v)$ for all $v \in V(G)$, (LOOP) is satisfied by $\hat{G}$. If vertices $u, v \in V(G)$ have a common inneighbor $x$ in $\hat{G}$, then their distance in $G$ is at most the sum of distances from $x$ to $u$ and $v$, which is at most $\rho(xu) + \rho(xv)$. Conversely, suppose that the distance between $u$ and $v$ is $r' \leq r$. Let $P$ be a path of length $r'$ from $u$ to $v$, and let $x$ be the smallest vertex of $P$ in the considered ordering. Then $x \in Q_{r_1}(u) \cap Q_{r_2}(v)$, where $r_1$ and $r_2$ are the lengths of the subpaths of $P$ from $x$ to $u$ and $v$, and $r' = r_1 + r_2 \geq \rho(xu) + \rho(xv)$. We conclude that (DIST) holds. $\square$

Note that it is possible to obtain $r$-augmentations in other ways, e.g., using the transitive fraternal augmentation procedure of Nešetřil and Ossona de Mendez [13].

## 2 Domination

We are now ready to give the approximation argument for $\gamma_r$.

**Theorem 9.** Let $r$ be a positive integer. If $\hat{G}$ is an $r$-augmentation of a graph $G$, then

$$\gamma_r(G)/\gamma_r^*(G) \leq (\Delta_{r-1}(\hat{G}) + 1)\Delta_r(\hat{G}) - \Delta_{r-1}(\hat{G}).$$
Proof. Let $a = (\Delta_{r-1}(\hat{G}) + 1)\Delta_r(G) - \Delta_{r-1}(\hat{G})$. Consider any optimal solution to the linear program defining $\gamma^*_r(G)$, and let $X_0$ be the set of vertices $v \in V(G)$ such that $x_v \geq 1/a$ in this solution. Let $v_1, \ldots, v_n$ be any ordering of vertices of $G$. For $i = 1, \ldots, n$, if a vertex at distance at most $r$ from $v_i$ belongs to $X_{i-1}$, then let $X_i = X_{i-1}$; otherwise, $X_i$ is obtained from $X_{i-1}$ by adding all inneighbors $x$ of $v_i$ such that $\rho(xv_i) \leq r - 1$.

Clearly, $X_n$ is an $r$-dominating set of $G$; hence, it suffices to bound its size. We have

$$|X_0| \leq a \sum_{u \in X_0} x_u.$$ 

To bound $|X_n \setminus X_0|$, we perform a charge redistribution argument. Vertices start with zero charge. For $i = 1, \ldots, n$, if $X_i \neq X_{i-1}$, then we increase by $x_u$ the charge of each vertex $u \in N_i[v_i]$ such that $uv_i \notin E(\hat{G})$. Let $\delta_i$ denote the total amount of charge added in this step. Observe that since $X_i \neq X_{i-1}$, none of vertices in $N_i[v_i]$ belongs to $X_{i-1}$. In particular, no inneighbor of $v_i$ belongs to $X_0$, and thus $\sum_{uv \in E(\hat{G})} x_u \leq \Delta_r(\hat{G})/a$. Since we are considering a solution to the linear program defining $\gamma^*_r(G)$, we have the following bound on the charge increase.

$$\delta_i = \sum_{u \in N_i[v_i], uv \notin E(\hat{G})} x_u \geq 1 - \sum_{uv \in E(\hat{G})} x_u \geq 1 - \Delta_r(\hat{G})/a = \Delta_{r-1}(\hat{G})(\Delta_r(\hat{G}) - 1)/a.$$ 

Consequently,

$$|X_i \setminus X_{i-1}| \leq \Delta_{r-1}(\hat{G}) \leq \delta_i \frac{a}{\Delta_r(\hat{G}) - 1},$$

and letting $\delta = \sum_{i=1}^n \delta_i$ be the total amount of charge created, we have

$$|X_n \setminus X_0| \leq \delta \frac{a}{\Delta_r(\hat{G}) - 1}.$$ 

On the other hand, by (DIST), when $u \in N_i[v_i]$ and $uv_i \notin E(\hat{G})$, then $u$ and $v_i$ have a common inneighbor $x$ and $r \geq \rho(xu) + \rho(xv_i) \geq 1 + \rho(xv_i)$. Consequently, whenever the charge of $u$ is increased, some inneighbor of $u$ (distinct from $u$) is added to the $r$-dominating set, and thus the final charge of $u$ is at most $(\Delta_r(\hat{G}) - 1)x_u$. Furthermore, as we observed before, charge is only added to vertices not belonging to $X_0$. Summing over all vertices of $G$, we obtain

$$\delta \leq (\Delta_r(\hat{G}) - 1) \sum_{u \in X_0} x_u.$$ 

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Combining these bounds, we obtain

\[ |X_n \setminus X_0| \leq a \sum_{u \in X_0} x_u, \]

and thus

\[ \gamma_r(G) \leq |X_0| + |X_n \setminus X_0| \leq a \sum_{u \in V(G)} x_u = a \gamma_r^*(G), \]

as required. \( \square \)

Note that \((x + 1)y - x = y^2 - (y - x)(y - 1)\), and thus if \(1 \leq x \leq y\), then \((x + 1)y \leq y^2\). Hence, \((\Delta_{r-1}(\hat{G}) + 1)\Delta_r(\hat{G}) - \Delta_{r-1}(\hat{G}) \leq \Delta_2^2(\hat{G})\). By Observation 8, Theorem 9 has the following consequence.

**Corollary 10.** For any positive integer \(r\) and a graph \(G\),

\[ \gamma_r(G)/\gamma_r^*(G) \leq \text{wcol}_r^2(G). \]

On the other hand, the ratio cannot be bounded in terms of \(\text{wcol}_{r-1}\), as shown by the following example. Let \(n\) be an odd integer and let \(H\) be the hypergraph with vertex sets consisting of all subsets of \([1, \ldots, n]\) of size \((n + 1)/2\), with edges \(e_1, \ldots, e_n\) such that for \(1 \leq i \leq n\), the edge \(e_i\) consists of the sets in \(V(H)\) that contain \(i\). For any \(I \subseteq \{1, \ldots, n\}\) of size \((n - 1)/2\), the vertex \(\{1, \ldots, n\} \setminus I\) is not incident with any of the edges \(e_i\) for \(i \in I\); hence, Lemma 5 implies \(\gamma_r(H^{(r)}) \geq (n + 1)/2\). Each vertex of \(H\) is incident with \((n + 1)/2\) edges and \(|E(H)| = n\), and thus \(\gamma_r^*(H^{(r)}) \leq \frac{n}{(n+1)/2} + 1 \leq 3\). Also, Lemma 5 implies \(\text{wcol}_{r-1}(H^{(r)}) \leq r^2 - r + 3\).

Corollary 10 implies that \(\gamma_r(G)\) can be approximated in polynomial time within factor of \(\text{wcol}_r^2(G)\). This bound can be improved in the special case \(r = 1\). If \(G\) has an orientation with maximum indegree at most \(d\), then giving each edge length 1 and adding loops of length 0 on all vertices results in a 1-augmentation \(\hat{G}\) with \(\Delta_1(\hat{G}) \leq d + 1\) and \(\Delta_0(\hat{G}) = 1\). Hence, we have the following.

**Corollary 11.** If a graph \(G\) has an orientation with maximum indegree at most \(d\), then \(\gamma(G)\) can be approximated in polynomial time within factor of \(2d + 1\).

Note that Bansal and Umboh give the approximation factor as \(3d\), however this is just because we are slightly more careful in the analysis—their algorithm is exactly the same as the one of Theorem 9 in the case \(r = 1\).
3 Independence

To prove a bound on the ratio $\alpha^*_2(G)/\alpha_{2r,b}(G)$, we use a result of Parekh and Pritchard [15] on generalized hypergraph matching. Let $H$ be a hypergraph and let $b$ be a positive integer. A $b$-matching in $H$ is a set $M$ of edges of $H$ such that each vertex of $H$ is incident with at most $b$ edges of $M$. Let $\mu_b(H)$ denote the maximum size of a $b$-matching in $H$. Let $\mu^*_b(H)$ be the fractional relaxation of this parameter, defined as

$$\mu^*_b(H) = \max \sum_{e \in E(H)} m_e$$

subject to

- $\sum_{e \ni v} m_e \leq b$ for all $v \in V(H)$
- $0 \leq m_e \leq 1$ for all $e \in E(H)$

Clearly, $\mu^*_b(H) \geq \mu_b(H)$. Conversely, we have the following.

**Theorem 12** (Parekh and Pritchard [15]). If $H$ is a hypergraph with all edges of size at most $k$ and $b$ is a positive integer, then

$$\mu_b(H) \geq \frac{k}{k^2 - k + 1} \mu^*_b(H).$$

Furthermore, a $b$-matching of size at least $\frac{k}{k^2 - k + 1} \mu^*_b(H)$ can be found in polynomial time.

**Corollary 13.** If $H$ is a hypergraph with all edges of size at most $k \geq 1$, then then $\mu_k(H) \geq \mu^*_1(H)/2$.

**Proof.** Consider an optimal solution to the linear program defining $\mu^*_1(H)$. Let $M_1 = \{ e \in E(H) : m_e \geq 1/k \}$ and $s_1 = \sum_{e \in M_1} m_e$. Clearly, $M_1$ is a $k$-matching in $H$, and thus $\mu_k(H) \geq |M_1| \geq s_1$. If $s_1 \geq \mu^*_1(H)/2$, the desired bound on $\mu_k(H)$ follows, and thus assume that $s_1 < \mu^*_1(H)/2$.

Let $m'_e = km_e$ for all $e \in E(H)$ such that $m_e \leq 1/k$ and $m'_e = 0$ for all other $e \in E(H)$. This gives a feasible solution to the program defining $\mu^*_k(H)$, and thus

$$\mu^*_k(H) \geq k(\mu^*_1(H) - s_1) > k\mu^*_1(H)/2.$$ 

By Theorem 12 we have

$$\mu_k(H) > \frac{k}{k^2 - k + 1} k\mu^*_1(H)/2 \geq \mu^*_1(H)/2,$$

as required. □
We use this result to find sets intersecting outneighborhoods in $r$-augmentations only in a bounded number of vertices.

**Lemma 14.** Let $G$ be a graph, let $\widehat{G}$ be an $r$-augmentation of $G$, and let $k = \Delta_r(\widehat{G})$. There exists a set $Y \subseteq V(G)$ such that each vertex of $\widehat{G}$ has at most $k$ outneighbors in $Y$ and

$$|Y| \geq \alpha^*_{2r}(G)/2.$$

**Proof.** For a vertex $u \in V(G)$, let $e_u$ be the set of inneighbors of $u$ in $\widehat{G}$. Let $H$ be the hypergraph with vertex set $V(G)$ and edge set $\{e_u : u \in V(G)\}$; each edge of $H$ has size at most $k$. Note that $M$ is a $k$-matching in $H$ if and only if each vertex of $\widehat{G}$ has at most $k$ outneighbors in $Y = \{u : e_u \in M\}$. Hence, it suffices to prove that $\mu_k(H) \geq \alpha^*_{2r}(G)/2$.

Consider an optimal solution to the linear program defining $\alpha^*_{2r}(G)$, and for every $u \in V(G)$, let $m_{e_u} = y_u$. For each $v \in V(H)$, we have

$$\sum_{e_u \ni v} m_{e_u} = \sum_{u \in E(\widehat{G})} y_u \leq \sum_{u \in N[v]} y_u \leq 1,$$

and thus this gives a feasible solution to the program defining $\mu^*_k(H)$. We conclude that $\mu^*_k(H) \geq \alpha^*_{2r}(G)$, and thus the claim follows from Corollary 13. $\square$

We are now ready to show existence of large $(2r, b)$-independent sets.

**Theorem 15.** Let $G$ be a graph, let $\widehat{G}$ be an $r$-augmentation of $G$, and let $b = (\Delta_{r-1}(\widehat{G}) + 1)\Delta_r(\widehat{G}) - \Delta_{r-1}(\widehat{G})$. We have

$$\alpha_{2r,b} \geq \alpha^*_{2r}(G)/2.$$

**Proof.** Let $k = \Delta_r(\widehat{G})$, and let $Y$ be the set obtained by applying Lemma 14 such that every vertex of $\widehat{G}$ has at most $k$ outneighbors belonging to $Y$. For any $v \in V(G)$ and $y \in N_r[v] \cap Y$, (DIST) implies that either $y$ is an inneighbor of $v$ in $\widehat{G}$ and $\rho(yv) = r$, or $y$ and $v$ have a common inneighbor $x$ with $\rho(xv) \leq r - 1$. Hence, we have

$$|N_r[v] \cap Y| \leq (\deg_{r,\widehat{G}}(v) - \deg_{r-1,\widehat{G}}(v)) + \sum_{x \in E(\widehat{G}), \rho(xy) \leq r - 1} ||y \in Y : xy \in E(\widehat{G})||$$

$$\leq (\deg_{r,\widehat{G}}(v) - \deg_{r-1,\widehat{G}}(v)) + \deg_{r-1,\widehat{G}}(v) \cdot k$$

$$= \deg_{r,\widehat{G}}(v) + (k - 1) \deg_{r-1,\widehat{G}}(v) \leq \Delta_r(\widehat{G}) + (k - 1)\Delta_{r-1}(\widehat{G}) = b,$$

$$\sum_{e_u \ni v} m_{e_u} = \sum_{u \in E(\widehat{G})} y_u \leq \sum_{u \in N[v]} y_u \leq 1.$$
and thus \( Y \) is a \((2r, b)\)-independent set in \( G \). Consequently,

\[
\alpha_{2r,b}(G) \geq |Y| \geq \alpha^*_r(G)/2,
\]
as required. \( \square \)

The following lemma clarifies the relationship with Theorem 4.

**Lemma 16.** If \( \hat{G} \) is a \( 2r \)-augmentation of \( G \), then for every positive integer \( b \),

\[
\alpha_{2r}(G) \geq \frac{1}{2b\Delta_2(G)}\alpha_{2r,b}(G).
\]

**Proof.** Let \( Y \) be a \((2r, b)\)-independent set in \( G \) of size \( \alpha_{2r,b}(G) \). Let \( G_1 \) be the graph with vertex set \( Y \) and distinct vertices \( y_1, y_2 \in Y \) adjacent iff their distance in \( G \) is at most \( 2r \). Orient the edges of \( G_1 \) as follows: if \( v \) is an inneighbor of \( y_1 \) in \( \hat{G} \) and \( y_2 \in N_{\hat{G}}[v] \), then direct the edge from \( y_2 \) to \( y_1 \). Since \( Y \) is \((2r, b)\)-independent, we have \( |N_\hat{G}[v] \cap (Y \setminus \{y_1\})| \leq b \) for each inneighbor \( v \) of \( y_1 \), and the inequality is strict when \( v = y_1 \). Hence, the maximum indegree of \( G_1 \) is less than \( d = b\Delta_2(\hat{G}) \). Furthermore, all edges of \( G_1 \) are directed in at least one direction by (DIST). Consequently, each subgraph \( F \) of \( G_1 \) has less than \( d|V(F)| \) edges, and thus \( G_1 \) is \((2d - 1)\)-degenerate. Consequently, \( \chi(G_1) \leq 2d \), and thus \( G_1 \) contains an independent set \( Y_1 \) of size at least \( \frac{|Y|}{2d} = \frac{\alpha_{2r,b}(G)}{2b\Delta_2(G)} \). Observe that \( Y_1 \) is a \( 2r \)-independent set in \( G \), which gives the required lower bound on \( \alpha_{2r}(G) \). \( \square \)

Composing Theorems 9 and 15 with Lemma 16 and using Observation 8, we obtain the following inequalities, implying Theorem 7.

**Corollary 17.** Let \( G \) be a graph, let \( \hat{G}_1 \) be an \( r \)-augmentation of \( G \) and let \( \hat{G}_2 \) be a \( 2r \)-augmentation of \( G \). Let \( b = (\Delta_{r-1}(G_1) + 1)\Delta_2(\hat{G}_1) - \Delta_{r-1}(G_1) \). Then

\[
\frac{1}{b}\gamma_r(G) \leq \gamma^*_r(G) = \alpha^*_r(G) \leq 2\alpha_{2r,b}(G) \leq 4b\Delta_2(\hat{G}_2)\alpha_{2r}(G).
\]

In particular,

\[
\frac{1}{\text{wcol}_r^2(G)}\gamma_r(G) \leq \gamma^*_r(G) \leq 2\alpha^*_r(G) \leq 4\text{wcol}_r^2(G)\text{wcol}_{2r}(G)\alpha_{2r}(G),
\]

and if \( G \) has an orientation with maximum indegree at most \( d \), then

\[
\frac{1}{2d+1}\gamma(G) \leq \gamma^*_1(G) = \alpha^*_2(G) \leq 2\alpha_{2,2d+1}(G) \leq 4(2d+1)\text{wcol}_2(G)\alpha_2(G).
\]
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