I. INTRODUCTION

The last couple of decades have observed a steadily growing interest in the field of systems at mesoscopic scales, thanks to the growing understanding of machines and engines with smaller dimensions. This has led to the area of stochastic thermodynamics which provides a framework for extending notions of classical thermodynamics to small systems wherein concepts of work, heat, and entropy are extended to the level of individual trajectories during nonequilibrium processes (ensembles). Research in this area has given birth to a group of exact and powerful theorems that dictate the behavior of such systems. They are commonly referred to as the fluctuation theorems (FTs) (9-17), and these theorems are valid even far from equilibrium, a feat that is beyond the scope of the well-established linear response theory. The theorems provide stringent restrictions on the probabilities of phase space trajectories in which second law is transiently “violated”. They show that at the level where fluctuations are comparable to the relevant energy exchanges of the system, one needs to replace the associated quantities in the statement of the second law by their averages: $\langle W \rangle \geq \Delta F$ or $\langle \Delta s_{\text{stat}} \rangle \geq 0$ (9-13, 14). Here the angular brackets represent the ensemble average. Thus, they in essence uphold the second law, even at the mesoscopic level, however, for the average properties.

The Crooks Fluctuation theorem (CFT) for heat states that the ratio of the probabilities of forward trajectory and the corresponding reverse trajectory for given initial states is given by (15, 16)

$$P[X|x_0]/P[X|x_+]=e^{\beta Q}. \quad (1)$$

Here, $X$ is the short form of the phase space trajectory along the forward process $x_0, x_1, ..., x_\tau$ generated by the protocol $\lambda(t)$. $x_+$ represents the phase space point at time $t_\tau$. $\tilde{X}$ is the corresponding reverse trajectory generated by the time reversed protocol $\lambda(\tau-t)$, where $\tau$ is the time of observation. $x_0$ is a given initial state of the forward process. The reverse process begins from the state $\tilde{x}_\tau$, which is the time-reversal of the final state $x_\tau$ of the forward process.

Using CFT, several other theorems like the Jarzynski equality and entropy production FT, can be easily derived (13, 16).

In this paper, we study the validity of these FTs in the presence of coarse-graining, when we transform the underdamped Langevin equation to the overdamped one, in the limit of high friction. We find that a prominent difference in the analysis is observed between the overdamped (coarse-grained) and the underdamped systems, when the friction coefficient is space-dependent (18-21). It should be noted that space-dependent friction does not alter the equilibrium state. However, Langevin dynamics of the system gets modified especially for the overdamped case. There are several physical systems wherein friction is space-dependent (see [21] and the references therein).

II. CROOKS THEOREM IN PRESENCE OF SPACE-DEPENDENT FRICTION

In the presence of space-dependent friction $\gamma(x)$, the equation of motion of the underdamped system of mass $m$ moving in a time-dependent potential $U(x,t)$ is given by

$$mv = -\gamma(x)v - U'(x,t) + \sqrt{2\gamma(x)T} \xi(t). \quad (2)$$

Note that the above equation contains multiplicative noise term. Here, $T$ is the temperature of the bath, while $\xi(t)$ is the delta-correlated Gaussian noise with zero mean: $\langle \xi(t) \rangle = 0; \langle \xi(t)\xi(t') \rangle = \delta(t-t')$. The overhead dot denotes time-derivative, whereas prime represents space derivative. Eq. (2) has been derived microscopically by invoking system and bath coupling (19, 20). It is shown that the high damping limit of eq. (2) is not equivalent to ignoring only inertial term (18, 21). The detailed treatment leads to an extra term that is crucial for system to reach equilibrium state in absence of time-dependent perturbations (see eq. (19) below).

Roughly speaking, this happens in the overdamped case because the random forces $\xi(t)$ appear as delta-function pulses that cause jumps in $x$. It then becomes unclear what value of $x$ must be provided in the argument

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of the function \( g \), because the value of the position at
the time the delta-peak appears becomes undefined \cite{22}. It
does not converge to a unique value even in the limit of
small time step \( \Delta t \). In fact, we can plug in any value
of position in-between \( x(t) \) (position before the jump) and
\( x(t + \Delta t) \) (position after the jump). These different val-
ues of position lead to different discretization schemes.
The case is simpler in case of underdamped Langevin
equation. There, the jumps are caused in the velocities,
while the position is a much smoother variable (being
an integral over the velocities). In other words, it does
not feel the noise as delta peaks, but instead as a more
well-behaved function. In that case, in the limit of small
\( \Delta t \), the argument of \( g \) is given by the unambiguous value
\( x(t) \). Thus, in this case, an update in the values of \( x \) and
\( v \) will be unique in each time step.

Let us now check the validity of CFT in both the un-
derdamped and overdamped cases.

A. Underdamped case

At first we want to calculate the ratio of path prob-
abilities between forward and reverse process. In a
given process, let the evolution of the system in phase
space be denoted by the phase space trajectory \( X(t) \equiv
\{x_0, x_1, \cdots, x_T\} \). Here, \( x_k \) represents the phase point
at time \( t = t_k \). In general, the phase point includes both
the position and the velocity coordinates of the system.
In the overdamped case, however, it would consist of
the position coordinate only. Now, a given path \( X(t) \), for
a given initial point \( x_0 \), would be fully determined if the se-
quence of noise terms for the entire time of observation is
available (this happens because there is no unambigu-
ity in either the positions or the velocities, while updating
their values by using the underdamped Langevin equa-
tion, as discussed above): \( \xi \equiv \{\xi_0, \xi_1, \cdots, \xi_{T-1}\} \).
The probability distribution of \( \xi_k \) is given by

\[
P(\xi_k) \propto e^{-\xi_k^2/2}.
\]

Therefore, the probability of obtaining the sequence \( \xi \)
will be \cite{12,23}

\[
P(\xi(t)) \propto \exp \left[ -\frac{1}{2} \int_0^T \xi^2(t) dt \right].
\]

Now, from the probability \( P(\xi(t)) \) of the path \( \xi(t) \) in
noise space, we can obtain the probability \( P[X(t)|x_0] \).
These two probability functionals are related by the Ja-
cobian \cite{22,27}. Thus, we can as well write \cite{12}

\[
P[X(t)|x_0] \propto \exp \left[ -\frac{1}{2} \int_0^T \xi^2(t) dt \right],
\]

where the proportionality constant is different from that
in eq. (3). In eq. (5), we then substitute the expression
for \( \xi(t) \) from the Langevin equation (Eq. (2)):

\[
P[X(t)|x_0] \propto \exp \left[ -\frac{1}{4} \int_0^T dt \frac{(m \dot{v} + U'(x,t) + \gamma(x)v)^2}{\gamma(x) T} \right].
\]

For the reverse process, \( v \to -v \), but the Jacobian is
same. The ratio of probability of the forward to the
reverse path can be readily shown to be \cite{12,24}

\[
\frac{P[X(t)|x_0]}{P[X(t)|x_{\tau}]} = \frac{\exp \left[ -\int_0^T dt(m \dot{v} + U'(x,t) + \gamma(x)v)^2/4\gamma(x) T \right]}{\exp \left[ -\int_0^T dt(4m \gamma(x)v v + 4U'(x,t) \gamma(x)v)^2/4\gamma(x) T \right]}
\]

\[
= \exp \left[ -\beta \int_0^T dt (m \dot{v} + U'(x,t)v) \right] = e^{\beta Q},
\]

where \( Q \) is the heat dissipated by the system into the
bath, defined as

\[
Q = \int_0^T \{\gamma(x)v - \sqrt{2\gamma(x)T} \xi(t)\} v dt = -\int_0^T \{m \dot{v} + U'(x,t)v\} v dt.
\]

This definition follow from the stochastic energetics de-
veloped by Sekimoto \cite{25,26} from the definition of First
Law using Langevin dynamics. Eq. (7) is the celebrated
CFT, from which several FT follow.

B. Integral and detailed fluctuation theorems

We have,

\[
P[X(t)|x_0] = e^{\beta Q},
\]

where \( Q \) is the heat dissipated, as obtained from the first
law. Multiplying by the ratio of the initial equilibrium
distributions, for forward and reverse processes, namely
by \( p_0(x_0)/p_1(x_\tau) \), we get \cite{13}

\[
\frac{P[X(t)|x_0]}{P[X(t)|x_{\tau}]} = \frac{P[X]}{P[X]} = e^{-\beta E_0} \cdot Z(\lambda_\tau) e^{-\beta E_\tau}
\]

\[
= e^{\beta (Q + \Delta E - \Delta F)} = e^{\beta (W - \Delta F)}.
\]

We have used the expression for equilibrium initial dis-
tribution \( p_0(x_0) = e^{-\beta E_0}/Z(\lambda_0) \) and \( p_1(x_\tau) = e^{-\beta E_\tau}/Z(\lambda_\tau) \). Here,
\( \Delta E = E_\tau - E_0 \), and we have made use of the relation
\( Z = e^{-\beta F} \), between the partition function and the free
energy. \( Z(\lambda_0) \) and \( Z(\lambda_\tau) \) are the partition functions cor-
responding to the protocol values at the initial time and
the final time, respectively. In the final step, the first law
for the work done on the system, \( W = Q + \Delta E \), has been
invoked. The above relation can be readily converted to the Crooks work theorem \[16\], given by

$$\frac{P(W)}{P(-W)} = e^{\beta(W - \Delta F)}.$$  \hspace{1cm} (11)

Here, $P(W)$ is the probability of work done $W$ on the system in the forward process. $P(-W)$ is the probability of $W$ amount of work extracted from the system in the reverse process. By cross-multiplication and integration over $W$, we get the Jarzynski equality \[9\]:

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}. \hspace{1cm} (12)$$

If the initial distributions for the forward and reverse processes are not equilibrium ones and $p_1(x_\tau)$ is the solution of the Fokker-Planck Equation at the final time $\tau$ of the forward process, we get, instead of eq. \[10\], the relation \[13, 14\]

$$\frac{P[X]}{P[X]} = e^{\beta Q + \ln(p_0(x_0)/p_1(x_\tau))} = e^{\Delta_{s_{tot}}}. \hspace{1cm} (13)$$

We then arrive at the relations for change of total entropy $\Delta_{s_{tot}}$ which is nothing but sum of change of system entropy $\Delta_{s_{sys}} = \ln(p_0(x_0)/p_1(x_\tau))$ (in the units of Boltzmann constant $k_B$) and entropy production in the bath $s_B = \beta Q$.

$$\Delta_{s_{tot}} = \ln(p_0(x_0)/p_1(x_\tau)) + \beta Q. \hspace{1cm} (14)$$

From Eq. \[13\], integral fluctuation theorem follows, which hold for all times, namely,

$$\langle e^{-\Delta_{s_{tot}}} \rangle = 1. \hspace{1cm} (15)$$

From the integral forms of the fluctuation theorems, given by eqs. \[12\] and \[15\], using Jensen’s inequality we easily obtain the second law inequalities \[9, 14\]

$$\langle W \rangle \geq \Delta F; \hspace{1cm} (16)$$

$$\langle \Delta_{s_{tot}} \rangle \geq 0. \hspace{1cm} (17)$$

Thus, in the underdamped limit second law retains same form for a system in presence of space-dependent friction. This completes our treatment for some FTs in the underdamped case for a particle moving in space dependent friction.

Above exact FTs do not give any information about probability distribution of work ($P(W)$), entropy $P(\Delta_{s_{tot}})$ etc. These distributions depend crucially on the specific problem being investigated.

Here, we study these distributions for the case of driven particle in harmonic trap. Apart from verifying FTs we also see how the space dependent friction modifies the distribution of ($P(W)$), and $P(\Delta_{s_{tot}})$ as compared to the particle moving in a space independent frictional coefficient $\gamma$ (which is the space average of $\gamma(x)$). The underdamped Langevin equation is given by

$$m\ddot{v} = -\gamma(x)v - kx + A\sin(\omega t) + \sqrt{2\gamma(x)T} \xi(t). \hspace{1cm} (18)$$

$A\sin(\omega t)$ is driven sinusoidal force of frequency $\omega$ and amplitude $A$. For this model analytical solution can be obtained for space independent case only for both overdamped and underdamped case \[33, 34\].

For simplicity in our study, we restrict ourselves to two cases of space dependent friction (i) $\gamma(x) = \gamma = \text{constant}$ (ii) $\gamma(x) = \gamma + c\tanh(\alpha x)$

In fig. (1) we have plotted the transient work distribution obtained after driving a system for one-fourth of a cycle for forward ($P(W)$) and corresponding reverse ($\tilde{P}(-W)$) protocol. Initially the system is equilibrated at appropriate initial values of protocol for forward and reverse process. In all our simulations, we have used the Heun’s method of numerical integration \[27\], and have generated $\sim 10^8$ realizations. Implementing the Heun’s method tantamounts to using the Stratonovitch discretization scheme \[28\]. Henceforth we have used all the quantities in dimensionless form and taken $k=1, m=1$
and $\gamma = 1$. For case (i), both distributions are Gaussian nature, and they cross each other at $\Delta F = -0.044$, which is the free energy difference over one-fourth cycle. This is obtained numerically from $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$, while theoretically we have $\Delta F = -0.045$. This is well within our numerical accuracy.

In fig. (2) we have plotted the same for space dependent friction $\gamma(x) = 1.0+0.9 \tanh(20x)$. Here the distributions are non-Gaussian but the crossing point is same as in space independent case. This is because the equilibrium distribution remain same in both cases.

In fig. (3) we have plotted distribution of total entropy production of a driven Brownian particle confined in a harmonic trap for one-fourth cycle. Here the underlying dynamics is underdamped and we find that the distribution is Gaussian for space independent case, while it is non-Gaussian for space dependent case. If we take particle to be initially equilibrated at different temperature $T = 0.1$ and then connected instantaneously to the given bath of temperature $T = 0.3$, and driven by same external force (i.e., for athermal case), we find that the distribution of total entropy production is non-Gaussian even for space independent case. This is consistent with the results in [29]. Numerically we find $\langle e^{-\Delta s_{tot}} \rangle = 1.002$ which is well within our numerical accuracy. In all these distributions, we find that there is a finite weight for realizations having $W < \Delta F$ and $\Delta s_{tot} < 0$, although the mean values follow the second law inequalities. These realizations are called transient Second Law violating trajectories. This finite weight is necessary to satisfy the fluctuation theorems [30].

After establishing the well known FTs in the underdamped case, we turn our attention to the overdamped dynamics of the particle, in a space-dependent frictional medium. Going to the overdamped regime implies coarse-graining. Instead of evolution in full phase space (coordinates and momenta), we restrict the evolution of the system to the position space only. This is equivalent to ignoring the information contained in the velocity variables.

### C. Overdamped case

The treatment of overdamped case is more subtle and a proper methodology must be followed. In order to obtain a unique Fokker-Planck equation (which is needed for a unique equilibrium distribution), the overdamped Langevin equation must be modified, depending on the discretization process that is being used. It can be written as

$$
\dot{x} = f(x, t) + g(x)\xi(t)
$$

$$
= -\Gamma(x)U'(x, t) + (1 - \alpha)g(x)g'(x) + g(x)\xi(t).
$$

For detail we refer to [21]. Such ambiguity of discretization process does not arises in the underdamped case as discussed in detail in [22, 31]. Here, $g(x) = \sqrt{2T\Gamma(x)} = \sqrt{2T/\gamma(x)}$. $\alpha \in [0, 1]$. $\alpha = 0$ for Ito convention, while $\alpha = 1/2$ and $\alpha = 1$ for Stratonovich and and isothermal conventions, respectively. In earlier literature [18, the overdamped Langevin equation in a Stratonovich prescription is derived. In [21], it has been shown that for all values of $\alpha$, the same equilibrium distribution is obtained for a given value of the protocol. Now we closely follow the treatment given in [21]. From eq. (19), the path probability for a single trajectory in position space can be shown to be given by

$$
P[X(t)|x_0] \sim e^{-S[X]},
$$

(20)

where

$$
S[X] = \int_0^T dt \left( \frac{1}{2g^2}[\dot{x} - f(x, t) + \alpha gg']^2 + \alpha f'(x, t) \right).
$$

Using $f(x, t) = -U'(x, t)\Gamma(x) + (1 - \alpha)g(x)g'(x)$, we get

$$
S[X] = \int_0^T dt \left( \frac{1}{2g^2}[\dot{x} + U'\Gamma + (2\alpha - 1)gg']^2 + \alpha[-U'\Gamma' + (1 - \alpha)(gg'' + g^2)] \right).
$$

(22)

For reverse path, (see eq. (22) of [32, 33]) one has to replace $\dot{x} \rightarrow -\dot{x}$, and $\alpha \rightarrow 1 - \alpha$. Thus the action for reverse path is given by,

$$
\tilde{S}[X] = \int_0^T dt \left( \frac{1}{2g^2}[-\dot{x} - f(x, t) + (1 - \alpha)gg']^2 + (1 - \alpha)f'(x, t) \right).
$$

(23)

Once again, substituting $f(x, t) = -U'(x, t)\Gamma(x) + \alpha g(x)g'(x)$, we get

$$
\tilde{S}[\tilde{X}] = \int_0^T dt \left( \frac{1}{2g^2}[-\dot{\tilde{x}} + U''\Gamma' + (2\alpha - 1)gg'' - \alpha(gg'' + g^2)] \right).
$$

(24)
However we restrict our analysis to $\alpha = 1/2$, i.e., Stratonovich discretization scheme. For this we have

$$S[X] = \int_0^\tau dt \left( \frac{1}{2} g^2 [\dot{x} + U']^2 + \frac{1}{2} \left[ -U'' \Gamma - U' \Gamma' + \frac{1}{2} (gg'' + g'^2) \right] \right).$$

(25)

Similarly,

$$\tilde{S}[\tilde{X}] = \int_0^\tau dt \left( \frac{1}{2} g^2 [-\dot{x} + U']^2 + \frac{1}{2} \left[ U'' \Gamma - U' \Gamma' + \frac{1}{2} (gg'' + g'^2) \right] \right).$$

(26)

Thus, the path ratio become simply

$$\frac{P[X|x_0]}{P[\tilde{X}|x_\tau]} = e^{\tilde{S}[\tilde{X}] - S[X]} = \exp \left[ -\int_0^\tau dt \dot{x} U' \right] = e^{\beta Q},$$

(27)

where $Q \equiv -\int_0^\tau dt \dot{x} U'(x, t)$. Thus, under Stratonovich scheme, the Crooks fluctuation theorem for trajectories remains unaffected in the overdamped regime, even in the presence of multiplicative noise. Since the Stratonovich scheme is considered to be the physically correct one for a Brownian particle in a heat bath [22], we may conclude that all the fluctuation theorems retain their forms as in the underdamped case [36].

As in the underdamped case we study the nature of probability distribution for work and entropy for simple model of driven harmonic oscillator for both space independent and space dependent case and verifying FTs numerically using Heun’s method (which is equivalent to following Stratonovich description as discussed earlier).

The corresponding Langevin equation is given by

$$\gamma(x) \ddot{x} = -k x + A \sin(\omega t) - \frac{\gamma' (x)}{2 \gamma(x)} T + \sqrt{2 \gamma(x) T} \xi(t).$$

(28)

In fig. (4) and fig. (5), we have plotted the transient work distributions for forward and reverse processes, for space independent and space dependent friction, respectively. The functional form of $\gamma(x)$ are same as studied in underdamped case. All the units are in dimensionless form and we take $k = 1, \gamma = 1$. We find that the distributions are Gaussian for space independent case while for space dependent it is non-Gaussian. But, the crossing point is same. From $\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$, the numerically obtained free energy difference $\Delta F = -0.045$, which is equal to the theoretical value, thus reassuring that space dependent friction does not alter the equilibrium distribution.

In fig. (6) we have plotted the distribution of total entropy production for the overdamped particle. We found that for space independent case the distribution is Gaussian. This is true only if initial distribution is the ther-
mal one. We have verified separately that for initial nonequilibrium distribution, \( P(\Delta s_{\text{tot}}) \) is non Gaussian. But for space dependent case \( P(\Delta s_{\text{tot}}) \) is non-Gaussian even initial equilibrium distribution. Numerically we find \( \langle e^{-\Delta s_{\text{tot}}} \rangle = 1.002 \) which is well within our numerical accuracy.

III. DEFINITION OF HEAT IN OVERDAMPED CASE

We can, following Sekimoto [26], derive the expression for dissipated heat using the overdamped Langevin dynamics (substituting \( \alpha = 1/2 \) in eq. (19)):

\[
\dot{x} = -\Gamma(x)U'(x, t) + \frac{1}{2} \gamma g(x)g'(x) + T \xi(t),
\]

(29)

We found that microscopic reversibility gives (see eq. (27))

\[
Q = -\int_0^T dt \dot{x}U'(x, t).
\]

(30)

The above two equations then give

\[
Q = \int_0^T dt \frac{\dot{x}}{\Gamma(x)} \left[ \dot{x} - \frac{1}{2} \gamma g(x)g'(x) - g(x)\xi(t) \right]
\]

\[
= \int_0^T dt \dot{x} \left[ \gamma(x) \dot{x} + \frac{\gamma'(x)T}{2\gamma(x)} - \sqrt{2\gamma(x)T} \xi(t) \right]
\]

\[
= \int_0^T dt \dot{x} \left[ \gamma(x) \dot{x} - \sqrt{2\gamma(x)T} \xi(t) \right] + \frac{T}{2} \ln \frac{\gamma(x_T)}{\gamma(x_0)}
\]

\[
= Q_{\text{conv}} + \frac{T}{2} \ln \frac{\gamma(x_T)}{\gamma(x_0)},
\]

(31)

where \( Q_{\text{conv}} \) is the conventional definition of heat. We thus get an extra boundary term in the definition, which assigns the logarithm of \( \sqrt{\gamma(x)} \) with the physical meaning of an entropy term. The presence of this term implies that if the particle begins from a given position \( x_0 \) with a small friction coefficient, then it dissipates more heat into the bath if it travels to a position \( x_T \) with a greater friction coefficient.

IV. CONCLUSION

In this work, we have considered the validity of FTs in presence of space-dependent friction, for both under-

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However, note that in our case, the equilibrium distribution is independent of discretization scheme, unlike in [32].

The above reasoning is correct for systems where the noise is not exactly delta-correlated, but has a very short correlation time. Now if we take the limit of correlation time going to zero, we get the Fokker-Planck equation that corresponds to the Stratonovich discretization scheme. As is evident, this is the case with most stochastic systems in nature.