Light-cone Gauge NSR Strings in Noncritical Dimensions

Yutaka Baba, Nobuyuki Ishibashi, Koichi Murakami

\textit{a}Theoretical Physics Laboratory, RIKEN, Wako, Saitama 351-0198, Japan

\textit{b}Institute of Physics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

Abstract

Light-cone gauge NSR string theory in noncritical dimensions should correspond to a string theory with a nonstandard longitudinal part. Supersymmetrizing the bosonic case, we formulate a superconformal worldsheet theory for the longitudinal variables $X^\pm, \psi^\pm$. We show that with the transverse variables and the ghosts combined, it is possible to construct a nilpotent BRST charge.


1 Introduction

In the light-cone gauge formulation of string theory, it is possible to consider the theory in noncritical space-time dimensions. Since the Lorentz invariance is broken in such dimensions, it should correspond to a string theory in a Lorentz noninvariant background. In the previous paper [1], we consider bosonic string theory and identify the worldsheet CFT for the longitudinal variables $X^\pm$ which corresponds to such a background.

What we would like to do in this paper is to supersymmetrize the results of Ref. [1]. We propose a superconformal field theory for the longitudinal variables $X^\pm, \psi^\pm$ which should correspond to the longitudinal part of the light-cone gauge NSR string in $d$ $(d \neq 10)$ dimensions. We show that the superconformal field theory has the right properties so that we can construct a nilpotent BRST charge with the transverse variables and ghosts combined.

This paper is organized as follows. In section 2, we propose a superconformal field theory for $X^\pm, \psi^\pm$. We present the energy momentum tensor and the action, and define the correlation functions. In section 3 we evaluate the partition function $e^{-\Gamma_{\text{super}}}$ of light-cone gauge NSR strings on the tree super light-cone diagram for $N$ strings, which is necessary to calculate the correlation functions for $X^\pm, \psi^\pm$. In section 4 we calculate the correlation functions and check that the superconformal field theory possesses the desired properties. Section 5 is devoted to conclusions and discussions. In appendix A we present properties of the interaction points on the super light-cone diagram, which are useful for the computation of $\Gamma_{\text{super}}$. In appendix B we show that $\Gamma_{\text{super}}$ obtained in section 3 is consistent with the results in Refs. [2, 3].

2 $X^\pm$ CFT

2.1 Energy momentum tensor

In the bosonic case [1], the energy momentum tensor of the worldsheet CFT for the longitudinal variables $X^\pm$ is given as

$$T_{X^\pm}(z) = \partial X^+ \partial X^- - \frac{d-26}{12} \{X^+, z\} , \quad (2.1)$$

where $\{X^+, z\}$ is the Schwarzian derivative.

We would like to consider the supersymmetrized version of this theory. In order to deal with the operators, it is convenient to introduce the supercoordinate $z \equiv (z, \theta)$ and the
superfields $X^\pm (z, \bar{z})$ which can be expanded as

$$X^\pm (z, \bar{z}) = x^\pm + i \theta \psi^\pm + i \bar{\theta} \bar{\psi}^\pm + i \theta \bar{\theta} F^\pm . \quad (2.2)$$

In this paper we follow the notations of Refs. [4, 5, 6] and define

$$D = \partial_\theta + \theta \partial_z , \quad z - z' = z - z' - \theta \theta'. \quad (2.3)$$

The supersymmetric generalization of eq.(2.1) will be

$$T_{X^\pm} (z) = \frac{1}{2} DX^+ \partial X^- + \frac{1}{2} DX^- \partial X^+ - \frac{d-10}{4} S(z, X^+_L) . \quad (2.4)$$

Here $X^+_L = (X^+_L (z), \Theta^+(z))$; $X^+_L (z)$ denotes the holomorphic part of the superfield $X^+(z, \bar{z})$ and $\Theta^+(z)$ is defined as

$$\Theta^+(z) = \frac{DX^+}{(\partial X^+)^\frac{1}{2}} (z) , \quad (2.5)$$

so that the map $z = (z, \theta) \mapsto X^+_L (z) = (X^+_L (z), \Theta^+(z))$ is a superconformal mapping. $S(z, X^+_L)$ is the super Schwarzian derivative defined as

$$S(z, X^+_L) = \frac{D^4 \Theta^+}{D \Theta^+} - 2 \frac{D^3 \Theta^+ D^2 \Theta^+}{(D \Theta^+)^2}$$

$$= - \frac{1}{4} D \Phi \partial \Phi + \frac{1}{2} \partial D \Phi , \quad (2.6)$$

where

$$\Phi (z, \bar{z}) = \ln \left( -4 (D \Theta^+)^2 (z) (\bar{D} \Theta^+)^2 (\bar{z}) \right) . \quad (2.7)$$

In the following, we will study the superconformal field theory with the energy momentum tensor $T_{X^\pm} (z)$ in eq.(2.4), on the complex plane.

### 2.2 Action and correlation functions

From the energy momentum tensor (2.4), we obtain the action for the theory:

$$S_{X^\pm} = - \frac{1}{2\pi} \int d^2 z \left( \bar{D} X^+ D X^- + \bar{D} X^- D X^+ \right) + \frac{d-10}{8} \Gamma_{\text{super}} [\Phi]$$

$$= - \frac{1}{2\pi} \int d^2 z \left( \partial x^+ \bar{\partial} x^- + \partial x^- \bar{\partial} x^+ \right.$$

$$+ \bar{\psi}^+ \bar{\partial} \bar{\psi}^- + \psi^- \partial \bar{\psi}^+ + \bar{\psi}^+ \partial \bar{\psi}^- + \bar{\psi}^- \partial \bar{\psi}^+ \right) + \frac{d-10}{8} \Gamma_{\text{super}} [\Phi] . \quad (2.8)$$
Here we have used the measure $d^2z = d(\text{Re} z) d(\text{Im} z) d\theta d\bar{\theta}$. $\Gamma_{\text{super}}[\Phi]$ coincides with the super Liouville action

$$\frac{d - 10}{8} \Gamma_{\text{super}}[\Phi] = -\frac{d - 10}{16\pi} \int d^2z \bar{D} \Phi D \Phi,$$

(2.9)

and is responsible for the super Schwarzian derivative part of the energy momentum tensor.

In order for the super Liouville action to be well-defined, $e^{\Phi}$ should have a nonzero expectation value. As in the bosonic case, we always consider the theory in the presence of the vertex operators

$$e^{-ip^+ X^-} (Z_r, \bar{Z}_r)$$

for $r = 1, \ldots, N$, with $\sum_{r=1}^{N} p_r^+ = 0$ and $Z_r = (Z_r, \Theta_r)$. Thus the quantities in which we are interested are the expectation values of functionals $F [X^+, X^-]$ defined as

$$\left\langle F [X^+, X^-] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r) \right\rangle$$

$$\equiv \int [dX^+ dX^-] e^{-S_{X^+} X^-} F [X^+, X^-] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r).$$

(2.10)

For the functionals $F [X^+]$ which do not depend on $X^-$, it is easy to calculate them up to an overall factor as in the bosonic case [1] and obtain

$$\left\langle F [X^+] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r) \right\rangle$$

$$\sim F \left[ -\frac{i}{2} (\rho + \bar{\rho}) \right] \exp \left( -\frac{d - 10}{8} \Gamma_{\text{super}} \left[ \ln \left( (D\xi)^2 (\bar{D}\bar{\xi})^2 \right) \right] \right),$$

(2.11)

where

$$\rho (z) = \sum_{r=1}^{N} \alpha_r \ln(z - Z_r), \quad \xi (z) = \frac{D\rho}{(\partial\rho)^{\frac{1}{2}}} (z), \quad \alpha_r = 2p_r^+.$$

(2.12)

$\rho (z)$ coincides with the super Mandelstam mapping and $\xi (z)$ is defined so that the mapping $z = (z, \theta) \mapsto \rho = (\rho, \xi)$ is superconformal. From eq.(2.11), one can see that $X^+ (z, \bar{z})$ has the expectation value

$$-\frac{i}{2} (\rho(z) + \bar{\rho}(\bar{z})).$$

It is convenient to define

$$\left\langle F [X^+, X^-] \right\rangle_{\rho} \equiv \frac{\left\langle F [X^+, X^-] \prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r) \right\rangle}{\prod_{r=1}^{N} e^{-ip^+ X^-} (Z_r, \bar{Z}_r)}.$$
Various correlation functions can be obtained by differentiating eq. (2.11) with respect to $\alpha_r$'s. For example, introducing

$$\rho'(z) \equiv \sum_{r=0}^{N+1} \alpha_r \ln (z - Z_r), \quad \alpha_{N+1} = -\alpha_0,$$  \hspace{1cm} (2.14)

and $\xi'(z)$ accordingly, we get the one-point function $\langle DX^- (Z_0) \rangle_\rho$ as

$$\langle DX^- (Z_0) \rangle_\rho = 2i \partial_{\alpha_0} D_{Z_0} \left( -\frac{d-10}{8} \Gamma_{\text{super}} \left[ \ln \left( (D\xi')^2 (\bar{D}\xi')^2 \right) \right] \right)|_{\alpha_0=0}.$$

The manipulation is essentially the same as that in the bosonic case [1] and it is straightforward to obtain

$$\langle DX^- (Z_0) DX^- (z) \rangle_\rho = 2i \partial_{\alpha_0} D_{Z_0} \langle DX^- (z) \rangle_\rho' \bigg|_{\alpha_0=0} + \langle DX^- (Z_0) \rangle_\rho \langle DX^- (z) \rangle_\rho,$$

$$\langle DX^- (z) F [X^+] \rangle_\rho = \langle DX^- (z) \rangle_\rho F \left[ -\frac{i}{2} (\rho + \bar{\rho}) \right] + \int d^2z' \frac{\theta - \theta'}{z - z'} \frac{\delta F [X^+]}{\delta X+(z')} \bigg|_{X^+ = -\frac{1}{2}(\rho + \bar{\rho})}.$$

$$\langle DX^- (z) DX^- (z') F [X^+] \rangle_\rho = \langle DX^- (z) DX^- (z') \rangle F \left[ -\frac{i}{2} (\rho + \bar{\rho}) \right]$$

$$+ \langle DX^- (z) \rangle_\rho \int d^2z'' \frac{\theta' - \theta''}{z'''} \frac{\delta F [X^+]}{\delta X+(z''')} \bigg|_{X^+ = -\frac{1}{2}(\rho + \bar{\rho})},$$

$$+ \int d^2z'' d^2z''' \frac{\partial - \theta''}{z - z''} \frac{\delta^2 F [X^+]}{\delta X+(z'' \delta X+(z''')} \bigg|_{X^+ = -\frac{1}{2}(\rho + \bar{\rho})},$$

and so on. In this way we can, in principle, derive all the correlation functions from the one-point function in eq. (2.15). From eq. (2.10), one can read off the OPE’s

$$X^+ (z, \bar{z}) X^+ (z', \bar{z}') \sim \text{regular},$$

$$DX^- (z) X^+ (z', \bar{z}') \sim \frac{\theta - \theta'}{z - z'}.$$  \hspace{1cm} (2.17)

In order to obtain the OPE $X^- X^-$, we need the explicit form of $\Gamma_{\text{super}} \left[ \ln \left( (D\xi')^2 (\bar{D}\xi')^2 \right) \right]$, which will be denoted as $\Gamma_{\text{super}}$ in the following.
3 Evaluation of $\Gamma_{\text{super}}$

As in the bosonic case, even with the expectation value $-\frac{i}{2} (\rho + \bar{\rho})$ for $X^+$, the super Liouville action (2.9) is not well-defined because of the singularities of $\langle \Phi (\mathbf{z}, \bar{\mathbf{z}}) \rangle_{\rho}$. In order to define $\Gamma_{\text{super}}$, we should regularize the singularities and carefully take account of various effects [7, 2, 3]. Here we would like to calculate $\Gamma_{\text{super}}$ in the same way as was done in Ref. [1]. As in the bosonic case, the form obtained by such a method is more convenient for the calculations of the correlation functions. In appendix B, we show that $\Gamma_{\text{super}}$ we get is consistent with the results of Ref. [2, 3].

3.1 Procedure

$\rho$ can be considered as the supercoordinate on the super light-cone diagram. We would like to obtain $\Gamma_{\text{super}}$ by integrating the variation $\delta \Gamma_{\text{super}}$ under the variations of moduli. The moduli space of the super light-cone diagram corresponds to the space of parameters $Z_r$ and $\Theta_r$ ($r = 1, \cdots, N$) modded out by the superprojective transformation

$$z \mapsto \frac{az + b + \alpha \theta}{cz + d + \beta \theta}, \quad \theta \mapsto -\alpha + \beta \frac{az + b}{cz + d} + \frac{\theta}{cz + d}, \quad (3.1)$$

where $a, b, c, d$ are Grassmann even, $\alpha, \beta$ are Grassmann odd and they satisfy

$$ad - bc = 1 + \alpha \beta. \quad (3.2)$$

Therefore there are $N - 3$ Grassmann even and $N - 2$ Grassmann odd moduli parameters.

In order to define the moduli parameters explicitly, we need to define the interaction points on the super light-cone diagram. The definition of the interaction points is a little bit complicated compared with the bosonic case [2, 3, 6, 8]. One can define $\tilde{z}_I$ ($I = 1, \cdots, N - 2$) which satisfy

$$\partial \rho (\tilde{z}_I) = 0, \quad \partial D \rho (\tilde{z}_I) = 0. \quad (3.3)$$

The interaction points $\tilde{z}_I$’s may be considered as a straightforward generalization of the bosonic version, but they do not transform covariantly under the superprojective transformations. In order to remedy the covariance, in place of $\tilde{z}_I$ we introduce $z_I$ [3, 6] which is defined so that with an appropriately chosen Grassmann odd parameter $\xi_I$,

$$\hat{\rho} (\mathbf{z}) \equiv \rho (\mathbf{z}) - \rho (\mathbf{z}_I) - [2 (\rho (\mathbf{z}) - \rho (\mathbf{z}_I))]^{-\frac{1}{2}} \xi (\mathbf{z}) \xi_I \quad (3.4)$$

can be expanded as

$$\hat{\rho} (\mathbf{z}) = \frac{1}{2} \partial^2 \hat{\rho} (\mathbf{z}_I) (\mathbf{z} - \mathbf{z}_I)^2 + \cdots \quad (3.5)$$
for \( z \sim z_I \). We summarize properties of \( z_I \) and \( \xi_I \) in appendix A. \( \xi_I \) (\( I = 1, \cdots, N - 2 \)) correspond to the odd moduli of the super Riemann surface and are determined in eq. (A.4).

As we see in appendix A, the above definition of \( z_I \) leads to eq. (A.5). It follows that the interaction points \( z_I \) can be considered to be defined as the solutions to the equations

\[
\partial \rho(z) - \frac{1}{2} \frac{\partial^2 D \rho \partial \rho}{\partial^2 \rho}(z) = 0, \quad \partial D \rho(z) - \frac{1}{6} \frac{\partial^3 \rho \partial \rho}{\partial^2 \rho}(z) = 0.
\]

(3.6)

Since these equations transform covariantly under the superprojective transformations, the interaction points \( z_I \) transform covariantly as well and thus have a physical meaning. As shown in appendix A, \( D \rho(\tilde{z}_I) (= D \rho(z_I)) \) and the difference between \( z_I \) and \( \tilde{z}_I \) are proportional to \( \xi_I \). This yields

\[
\rho(z_I) = \rho(\tilde{z}_I) + \left( \theta_I - \tilde{\theta}_I \right) D \rho(\tilde{z}_I) + (z_I - \tilde{z}_I) \partial \rho(\tilde{z}_I)
\]

\[
= \rho(\tilde{z}_I).
\]

(3.7)

Therefore the even moduli parameters \( T_I \) (\( I = 1, \cdots, N - 3 \)) can be defined as

\[
T_I \equiv \rho(z_{I+1}) - \rho(z_I) = \rho(\tilde{z}_{I+1}) - \rho(\tilde{z}_I).
\]

(3.8)

Now we would like to calculate the variation of \( -\Gamma_{\text{super}} \) when we modify the moduli parameters as \( T_I \mapsto T_I + \delta T_I, \xi_I \mapsto \xi_I \). For this variation, we can calculate \( \delta (-\Gamma_{\text{super}}) \) in a way similar to the bosonic case [1] and we obtain

\[
\delta (-\Gamma_{\text{super}}) = \sum_I \delta T_I \oint_{C_I} dz \frac{\delta S(z, \rho)}{2\pi i D\xi(z)}
\]

\[
= \sum_r \oint_{Z_r} dz \frac{\delta \rho(z_r^I)}{2\pi i (D\xi(z))^2} (-2S(z, \rho)) + \sum_I \oint_{z_I} dz \frac{\delta \rho(z_I)}{2\pi i (D\xi(z))^2} (-2S(z, \rho))
\]

\[
+ \oint_{z_{I+1}} dz \frac{\delta \rho(z_{I+1})}{2\pi i (D\xi(z))^2} (-2S(z, \rho))
\]

\[+ \text{c.c.}, \]

(3.9)

where the integration contour \( C_I \) lies between the consecutive interaction points \( \rho(z_{I+1}) \) and \( \rho(z_I) \) as in the case of bosonic case [1], \( z_r^I \) denotes the interaction point where the \( r \)-th external string interacts and \( z_{I+1}^\infty \) denotes the interaction point closest to \( \infty \). Using the fact that the possible singularities of \( \frac{\delta \rho(z) - \delta \xi(z)}{(D\xi(z))} S(z, \rho) \) are at \( z = Z_r, z_I, \infty \), we can express the
variation of $-\Gamma_{\text{super}}$ as
\[
\delta (-\Gamma_{\text{super}}) = -\sum_r \oint_{z_r} \frac{dz}{2\pi i} \frac{\delta \rho(z) - \delta \rho(z_r) - \delta \xi \xi(z)}{(D \xi)^2(z)} (-2S(z, \rho))
- \sum_I \oint_{z_I} \frac{dz}{2\pi i} \frac{\delta \rho(z) - \delta \rho(z_I) - \delta \xi \xi(z)}{(D \xi)^2(z)} (-2S(z, \rho))
- \oint_\infty \frac{dz}{2\pi i} \frac{\delta \rho(z) - \delta \rho(z_I(\infty)) - \delta \xi \xi(z)}{(D \xi)^2(z)} (-2S(z, \rho))
+ \text{c.c.}
\] (3.10)

We will evaluate $\delta (-\Gamma_{\text{super}})$ by performing the contour integrations, and integrate it to obtain $-\Gamma_{\text{super}}$. We cannot fix the dependence on $\xi_I$ and $\alpha_r$ by this method. However, since the dependence on $T_I$ is known, we can take the limit $T_I \rightarrow \infty$. Imposing the factorization conditions on $-\Gamma_{\text{super}}$, the entire $-\Gamma_{\text{super}}$ can be obtained.

In order to calculate the right hand side of eq.(3.10), we need to know the behavior of the integrand for $z \sim z_r, z_I, \infty$. The super Schwarzian derivative $S(z, \rho)$ can be written as
\[
S(z, \rho) = -\frac{1}{4} D \ln \omega(z) \partial \ln \omega(z) + \frac{1}{2} \partial D \ln \omega(z),
\] (3.11)
where
\[
\omega(z) \equiv (D \xi)^2(z) = \partial \rho(z) - \frac{\partial D \rho \partial \rho}{\partial \rho}(z). \] (3.12)
\omega plays a role similar to $\partial \rho$ in the bosonic case. In the bosonic case, the fact that $\partial \rho$ can be written as
\[
\partial \rho(z) = \frac{(\sum_r \alpha_r Z_r) \prod_I (z - z_I)}{\prod_s (z - Z_s)}
\] (3.13)
makes the calculations easier. Here we need a similar identity for $\omega$.

3.2 $\omega(z)$

In order to obtain an expression like eq.(3.13) for $\omega$, we need some facts about the polynomials of the supercoordinate $z = (z, \theta)$.

Polynomials in superspace

We restrict ourselves to the situation in which there are a finite number of Grassmann odd parameters. In general, a Grassmann even number $a$ can be decomposed as
\[
a = a^{(0)} + a^{(2)} + a^{(4)} + \cdots,
\] (3.14)
where $a^{(n)}$ involves $n$ Grassmann odd parameters. In the present situation, such an expansion terminates at a finite order. It is easy to prove that if $a^{(0)} \neq 0$, there exists the inverse $a^{-1}$ of $a$.

Let us consider a Grassmann even superanalytic function $f(z)$ in superspace. This can be expressed as

$$f(z) = f_0(z) + \theta f_1(z) .$$  \hspace{1cm} (3.15)

In the following, let us assume that $f_0(z)$ and $f_1(z)$ are $N$-th order polynomials of $z$. The polynomial $f_0(z)$ can also be decomposed as

$$f_0(z) = f_0^{(0)}(z) + f_0^{(2)}(z) + \cdots ,$$  \hspace{1cm} (3.16)

where the coefficients of $f_0^{(n)}$ involves $n$ Grassmann odd parameters. $f_0^{(0)}(z)$ is referred to as the body part of $f(z)$.

If there exists $z_i = (z_i, \theta_i)$ such that

$$f(z_i) = D f(z_i) = 0 ,$$  \hspace{1cm} (3.17)

by Taylor expanding $f(z)$ around $z = z_i$, one can show that

$$f(z) = (z - z_i) \tilde{f}(z) , \quad \tilde{f}(z) = \tilde{f}_0(z) + \theta \tilde{f}_1(z) ,$$  \hspace{1cm} (3.18)

where $\tilde{f}_0(z)$ and $\tilde{f}_1(z)$ are polynomials of $z$ whose orders are at most $N - 1$. Suppose that there exists $z_j = (z_j, \theta_j)$ for this $f(z)$ such that $z_j^{(0)} \neq z_i^{(0)}$ and

$$f(z_j) = D f(z_j) = 0 .$$  \hspace{1cm} (3.19)

Substituting eq.(3.18) into eq.(3.19), one obtains

$$(z_j - z_i) \tilde{f}(z_j) = 0 ,$$

$$(\theta_j - \theta_i) \tilde{f}(z_j) + (z_j - z_i) D \tilde{f}(z_j) = 0 .$$  \hspace{1cm} (3.20)

Since $z_j - z_i$ has the inverse, one can show that $\tilde{f}(z_j) = D \tilde{f}(z_j) = 0$ and thus $f(z)$ can be further factorized as

$$f(z) = (z - z_i)(z - z_j) \tilde{\tilde{f}}(z) .$$  \hspace{1cm} (3.21)

If one can go on like this, one can show that the function $f(z)$ can be factorized as the usual polynomials.

However, this is not always the case. For example, for

$$f(z) = z^2 + a ,$$  \hspace{1cm} (3.22)
with $a^{(0)} = 0$, one cannot find a solution to eq. (3.17) and $f(z)$ cannot be factorized. The problem about this kind of polynomials is that the body part has a double root. Indeed one can show that if $f_0^{(0)}(z)$ can be factorized as

$$f_0^{(0)}(z) = a \prod_i \left(z - z_i^{(0)}\right),$$

with $z_i^{(0)} \neq z_j^{(0)}$ for $i \neq j$, $f(z)$ can be factorized. One can find $z_i$ satisfying $f_0(z_i) = 0$ by expanding

$$z_i = z_i^{(0)} + z_i^{(2)} + z_i^{(4)} + \cdots.$$  

(3.24)

$z_i^{(2n)}$ can be obtained order by order by solving equations of the form

$$z_i^{(2n)} \prod_{j \neq i} \left(z_i^{(0)} - z_j^{(0)}\right) + \text{(contributions from lower order)} = 0.$$  

(3.25)

Then one can show that $z_i = \left(z_i, -\frac{f_1(z_i)}{\partial f_0(z_i)}\right)$ satisfies $f(z_i) = Df(z_i) = 0$.

**Factorization of $\omega(z)$**

Using above facts, it is easy to show

$$\partial \rho(z) = \sum_r \frac{\alpha_r}{z - Z_r} = \frac{\left(\sum_r \alpha_r Z_r + \theta \sum_r \alpha_r \Theta_r\right) \prod_f (z - \tilde{z}_I)}{\prod_r (z - Z_r)},$$

(3.26)

assuming that the body part $z_i^{(0)}$ satisfies $z_i^{(0)} \neq z_j^{(0)}$ for $I \neq J$. One can also show

$$-\frac{\partial D \rho \partial \rho}{\partial \rho}(z) = \frac{\sum_r \left(\alpha_r (\theta - \Theta_r) \prod_{s \neq r} (z - Z_s)^2\right) \sum_r \left(\alpha_r (\theta - \Theta_r) \prod_{s \neq r} (z - Z_s)\right)}{\left(\sum_r \alpha_r Z_r + \theta \sum_r \alpha_r \Theta_r\right) \prod_r (z - Z_r)^2 \prod_f (z - \tilde{z}_I)}.$$  

(3.27)

It is easy to see that the numerator is factorized as

$$F(z) \prod_r (z - Z_r),$$

where $F(z)$ is a polynomial of order $2N - 4$ whose coefficients involve at least two Grassmann odd parameters. Therefore $\omega$ can be written as

$$\omega = \frac{\left(\sum_r \alpha_r Z_r + \theta \sum_r \alpha_r \Theta_r\right)^2 \prod_f (z - \tilde{z}_I)^2 + F(z)}{\left(\sum_r \alpha_r Z_r + \theta \sum_r \alpha_r \Theta_r\right) \prod_r (z - Z_r) \prod_f (z - \tilde{z}_I)}.$$  

(3.28)

It is obvious that the body part of the numerator has double roots.
In order to get a factorized form of $\omega$, let us “regularize” it [8, 6] by introducing

$$\omega_\varepsilon \equiv \frac{\left( \sum_\alpha \alpha_r Z_r + \theta \sum_\alpha \alpha_r \Theta_r \right)^2 \prod_I (z - \tilde{z}_I)^2 + F(z) - \varepsilon^2}{\left( \sum_\alpha \alpha_r Z_r + \theta \sum_\alpha \alpha_r \Theta_r \right) \prod_r (z - Z_r) \prod_{I \neq I} (z - \tilde{z}_I)}, \quad (3.29)$$

where $\varepsilon (|\varepsilon| \ll 1)$ is a complex number. $\omega_\varepsilon$ can be factorized as

$$\omega_\varepsilon = A \prod_I \left[ (z - z_{I+}) (z - z_{I-}) \right] \prod_r (z - Z_r) \prod_{I \neq I} (z - \tilde{z}_I), \quad (3.30)$$

where

$$A \equiv \sum_\alpha \alpha_r Z_r - \frac{\sum_\alpha \alpha_r \Theta_r \sum_\alpha \alpha_r \Theta_r Z_r}{\sum_\alpha \alpha_r Z_r}. \quad (3.31)$$

Let us define $B, \gamma$ as

$$B^2 = -g(\tilde{z}_I), \quad \gamma = Dg(\tilde{z}_I),$$

$$g(z) = -\frac{1}{(\sum_\alpha \alpha_r Z_r + \theta \sum_\alpha \alpha_r \Theta_r) \prod_r (z - Z_r) \prod_{I \neq I} (z - \tilde{z}_I)}, \quad (3.32)$$

so that

$$\lim_{z \to \tilde{z}_I} [(z - \tilde{z}_I) \omega_\varepsilon(z)] = -\varepsilon^2 B^2, \quad \lim_{z \to \tilde{z}_I} D[(z - \tilde{z}_I) \omega_\varepsilon(z)] = -D\rho(\tilde{z}_I) + \varepsilon^2 \gamma. \quad (3.33)$$

Using these variables, we obtain for $|\varepsilon| \ll 1$

$$z_{I\pm} - \tilde{z}_I \sim \pm \frac{\varepsilon B}{(\varepsilon^2 \rho)^{1/2}} (\tilde{z}_I) + \frac{\partial^2 D \rho \partial \rho}{2 (\varepsilon^2 \rho)^2}(\tilde{z}_I),$$

$$\theta_{I\pm} - \tilde{\theta}_I \sim \pm \frac{1}{2 \varepsilon B (\varepsilon^2 \rho)^{1/2}} (\tilde{z}_I) + \frac{3}{4} \varepsilon^2 B \frac{\partial^2 D \rho}{(\varepsilon^2 \rho)^2}(\tilde{z}_I) \pm \frac{\varepsilon \gamma}{2 B (\varepsilon^2 \rho)^{1/2}} (\tilde{z}_I). \quad (3.34)$$

Hence, taking the limit $\varepsilon \to 0$ in eq.(3.30), we obtain

$$\omega = \lim_{\varepsilon \to 0} \omega_\varepsilon = A \prod_I \left[ (z - \tilde{z}_I) - \frac{\partial^2 D \rho \partial \rho}{(\varepsilon^2 \rho)^2}(\tilde{z}_I) - \frac{\theta - \tilde{\theta}_I}{z - \tilde{z}_I} \frac{\partial \rho}{\partial \rho}(\tilde{z}_I) \right] \frac{1}{\prod_r (z - Z_r)}. \quad (3.35)$$

### 3.3 Calculation of the contour integrals

With the form of $\omega$ in eq.(3.35), one can calculate the contour integrals on the right hand side of eq.(3.10).

For $z \sim Z_r$, the super Schwarzian derivative behaves as

$$S(z, \rho) \sim \frac{1}{4 (z - Z_r)^2} + \frac{1}{z - Z_r} \frac{1}{4} D \ln f_r(Z_r) + \frac{\theta - \Theta_r}{z - Z_r} \frac{1}{2} \partial \ln f_r(Z_r), \quad (3.36)$$
Thus it is straightforward to obtain

\[
\oint \ln f_r (z) = \ln A + \sum_I \ln \left[ (z - \tilde{z}_I) - \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) - \frac{\theta - \tilde{\theta}_I D\rho}{z - \tilde{z}_I \partial^2 \rho} (\tilde{z}_I) \right] - \sum_{s \neq r} \ln (z - Z_s) .
\]

Thus it is straightforward to obtain

\[
\oint \frac{dz}{2\pi i} \frac{\delta \rho(z) - \delta \rho(z_I^{(r)}) - \delta \xi(z)}{(D\xi)^2(z)} 2S(z, \rho)
\]

\[
= - (\delta Z_r - \delta \Theta_r, \Theta_r) \partial \ln f_r (Z_r) - \delta \Theta_r, D \ln f_r (Z_r) - \frac{1}{2} \delta \tilde{N}_{00}.
\]

where

\[
\tilde{N}_{00} = \frac{\rho(\tilde{z}_I^{(r)})}{\alpha_r} - \sum_{s \neq r} \frac{\alpha_s}{\alpha_r} \ln (Z_r - Z_s) .
\]

For \( z \sim \infty \), it is easy to see that

\[
S(z, \rho) \sim 0 \cdot \frac{1}{z^3} + \mathcal{O} (z^{-3}) , \quad \frac{\delta \rho(z) - \delta \rho(z_I) - \delta \xi(z)}{(D\xi)^2(z)} \sim \mathcal{O} (z^2) ,
\]

and we obtain

\[
\oint \frac{dz}{2\pi i} \frac{\delta \rho(z) - \delta \rho(z_I^{(\infty)}) - \delta \xi(z)}{(D\xi)^2(z)} S(z, \rho) = 0 .
\]

The most difficult is the integral around \( z_I \). From eq. (3.35) one can derive

\[
S(z, \rho) \sim \frac{1}{(z - \tilde{z}_I)^3} \frac{5 D\rho}{4 \partial^2 \rho} (\tilde{z}_I)
\]

\[
+ \frac{\theta - \tilde{\theta}_I}{(z - \tilde{z}_I)^3} \left[ -3 \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) - \frac{1}{2} \frac{D\rho}{\partial^2 \rho} (\tilde{z}_I) D \ln f_I (\tilde{z}_I) \right]
\]

\[
+ \frac{1}{(z - \tilde{z}_I)^2} \left[ \frac{1}{4 \partial^2 \rho} (\tilde{z}_I) \partial D \ln f_I (\tilde{z}_I) - \frac{1}{4} \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) D \ln f_I (\tilde{z}_I) \right]
\]

\[
+ \frac{\theta - \tilde{\theta}_I}{(z - \tilde{z}_I)^2} \left[ \frac{-3}{4 \partial^2 \rho} (\tilde{z}_I) \partial D \ln f_I (\tilde{z}_I)
\]

\[
- \frac{1}{2} \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) \partial \ln f_I (\tilde{z}_I) - \frac{3}{4} \right]
\]

\[
+ \frac{1}{z - \tilde{z}_I} \left[ \frac{-1}{4} D \ln f_I (\tilde{z}_I) - \frac{1}{4} \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) \partial D \ln f_I (\tilde{z}_I)
\]

\[
+ \frac{1}{4 \partial^2 \rho} (\tilde{z}_I) \partial^2 \ln f_I (\tilde{z}_I) \right]
\]

\[
+ \frac{\theta - \tilde{\theta}_I}{z - \tilde{z}_I} \left[ \frac{-1}{2} D \ln f_I (\tilde{z}_I) - \frac{1}{2} \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) \partial^2 \ln f_I (\tilde{z}_I) \right]
\]
− \frac{1}{2} \frac{D \rho}{\partial^2 \rho} (\tilde{z}_I) \partial^2 D \ln f_I (\tilde{z}_I) \right), \quad (3.42)

where \( \ln f_I (z) \) is defined by

\[
\ln f_I (z) = \ln A + \sum_{j \neq I} \ln \left[ (z - \tilde{z}_I) - \frac{\partial^2 D \rho D \rho}{(\partial^2 \rho)^2} (\tilde{z}_I) - \frac{\theta - \tilde{\theta}_I D \rho}{z - \tilde{z}_I} \frac{\partial^2 D \rho}{\partial^2 \rho} (\tilde{z}_I) \right] - \sum_r \ln (z - Z_r), \quad (3.43)
\]

and thus

\[
\omega(z) = \left[ (z - \tilde{z}_I) - \frac{\partial^2 D \rho D \rho}{(\partial^2 \rho)^2} (\tilde{z}_I) - \frac{\theta - \tilde{\theta}_I D \rho}{z - \tilde{z}_I} \frac{\partial^2 D \rho}{\partial^2 \rho} (\tilde{z}_I) \right] f_I (z). \quad (3.44)
\]

In order to see the behavior of \( \delta \rho (z) - \delta \rho (z_I) - \delta \xi (z) (D \xi)^2 (z) \) for \( z \sim z_I \), it is convenient to introduce \( w(z) \) and \( \eta(z) \) such that

\[
\rho (z) - \rho (z_I) = \frac{1}{2} (w(z))^2 + \eta(z) \xi_I ,
\]

\[
\xi (z) = (w(z))^\frac{1}{2} \eta(z) + (w(z))^{-\frac{1}{2}} \xi_I , \quad (3.45)
\]

where \( \xi_I \) is the Grassmann odd parameter introduced in eq. (3.4). The map \( \rho = (\rho, \xi) \mapsto w \equiv (w, \eta) \) is superconformal, and

\[
\frac{\delta \rho (z) - \delta \rho (z_I) - \delta \xi (z)}{(D \xi)^2 (z)} = \frac{\delta w - \delta \eta}{(D \eta)^2 (z)} \quad (3.46)
\]

for a variation satisfying \( \delta \xi_I = 0 \). By using eqs. (3.42) and (3.46) and relations given in appendix A it is now straightforward to evaluate the contour integral,

\[
\oint_{z_I} \frac{dz}{2\pi i} \frac{D \rho (z) - D \rho (z_I) - D \xi (z)}{(D \xi)^2 (z)} 2S (z, \rho)
\]

\[
= - \left( \delta \tilde{z}_I - \delta \tilde{\theta}_I \tilde{\theta}_I \right) \partial F_I (\tilde{z}_I) - \delta \tilde{\theta}_I D F_I (\tilde{z}_I)
\]

\[
+ \delta \left( \frac{\partial^2 D \rho D \rho}{(\partial^2 \rho)^2} (\tilde{z}_I) \right) \partial \ln f_I (\tilde{z}_I) + \delta \left( \frac{D \rho}{\partial^2 \rho} (\tilde{z}_I) \right) \partial D \ln f_I (\tilde{z}_I)
\]

\[
- \frac{3}{4} \delta \left[ \ln \left( \partial^2 \rho \frac{13 \partial^2 D \rho D \rho}{9 (\partial^2 \rho)^2} + \frac{8 \partial^2 D \rho D \rho}{3 (\partial^2 \rho)^2} \right) (\tilde{z}_I) \right], \quad (3.47)
\]

where

\[
F_I (z) \equiv - \ln f_I (z) - \frac{\partial^2 D \rho D \rho}{(\partial^2 \rho)^2} (\tilde{z}_I) \partial \ln f_I (z) - \frac{D \rho}{\partial^2 \rho} (\tilde{z}_I) \partial D \ln f_I (z). \quad (3.48)
\]
Now we introduce $-W_{\text{super}}$ defined as

$$-W_{\text{super}} \equiv \sum_{r>s} \ln (Z_r - Z_s) + \sum_{I>J} P_I P_J \ln (\tilde{Z}_I - \tilde{Z}_J) - \sum_r \sum_I P_I \ln (Z_r - \tilde{Z}_I),$$

$$P_I \equiv 1 + \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{Z}_I) \tilde{\theta}_I + \frac{D\rho}{\partial^2 \rho} (\tilde{Z}_I) \tilde{\theta}_I \tilde{D}_I,$$  \hspace{1cm} (3.49)

where $\tilde{\theta}_I = \partial \tilde{z}_I$ and $\tilde{D}_I = \partial \tilde{\theta}_I + \tilde{\theta}_I \partial \tilde{z}_I$. It satisfies

$$\delta (-W_{\text{super}}) = \sum_r \left[ - (\delta Z_r - \delta \Theta_r \Theta_r) \partial \ln f_r (Z_r) - \delta \Theta_r D \ln f_r (Z_r) \right]$$

$$+ \sum_I \left[ - (\delta \tilde{Z}_I - \tilde{\theta}_I \tilde{\theta}_I) \partial F_I (\tilde{Z}_I) - \delta \tilde{\theta}_I D F_I (\tilde{Z}_I) \right.$$  

$$+ \tilde{\delta} \left( \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{Z}_I) \right) \partial \ln f_I (\tilde{Z}_I) + \tilde{\delta} \left( \frac{D\rho}{\partial^2 \rho} (\tilde{Z}_I) \right) \partial D \ln f_I (\tilde{Z}_I) \right].$$  \hspace{1cm} (3.50)

Putting eqs. (3.38), (3.41), (3.47) and (3.50) together, we finally obtain

$$\delta (-\Gamma_{\text{super}}) = \delta \left[ -W_{\text{super}} - \frac{1}{2} \sum_r N_{00}^{rr} \right. \right.$$

$$- \frac{3}{4} \sum_I \ln \left( \partial^2 \rho - \frac{13}{9} \frac{\partial^3 D\rho D\rho}{\partial^2 \rho} + \frac{8}{3} \frac{\partial^3 \rho \partial^2 D\rho D\rho}{(\partial^2 \rho)^2} \right) (\tilde{Z}_I) \left. \right] + \text{c.c.}.$$  \hspace{1cm} (3.51)

Since we are dealing with the variation satisfying $\delta \xi_I = 0$, $\Gamma_{\text{super}}$ is given up to $\xi_I$ dependent terms. In the next subsection, we will fix the $\xi_I$ dependent terms by checking the factorization properties and obtain

$$-\Gamma_{\text{super}} = -W_{\text{super}} - \frac{1}{2} \sum_r N_{00}^{rr} \right.$$

$$- \frac{3}{4} \sum_I \ln \left( \partial^2 \rho - \frac{13}{9} \frac{\partial^3 D\rho D\rho}{\partial^2 \rho} + \frac{8}{3} \frac{\partial^3 \rho \partial^2 D\rho D\rho}{(\partial^2 \rho)^2} \right) (\tilde{Z}_I) \left. \right] + \text{c.c.}.$$  \hspace{1cm} (3.52)

### 3.4 Properties of $\Gamma_{\text{super}}$

We expect that $\Gamma_{\text{super}}$ is invariant under the superprojective transformation (3.1). It is easy to see that the expression on the right hand side of eq. (3.10) is superprojective invariant. After tedious but straightforward calculations, it is possible to check that the right hand side of eq. (3.52) is invariant under the superprojective transformation.
When all the $\Theta_r$’s vanish, $\Gamma_{\text{super}} \left[ \ln \left( \left( D\xi \right)^2 \left( \bar{D}\xi \right)^2 \right) \right]$ should be proportional to the bosonic counterpart $\Gamma$ in Ref. [1]. It is easy to see

$$\Gamma_{\text{super}} \left[ \ln \left( \left( D\xi \right)^2 \left( \bar{D}\xi \right)^2 \right) \right] \bigg|_{\Theta_r=0} = \frac{1}{2} \Gamma \left[ \ln \left( \partial \rho \bar{\partial} \bar{\rho} \right) \right] \bigg|_{\Theta_r=0}. \quad (3.53)$$

Finally we explore the factorization properties of $\Gamma_{\text{super}}$. Let us consider the super light-cone diagram depicted in figure 1 in the limit $\text{Re} \ T \to \infty$. Here we attach the superscript $[N]$ to the $\rho$ coordinate on the super light-cone diagram to indicate that the diagram is with $N$ external lines. We introduce a shorthand notation

$$\Gamma_{\text{super}}^{[N]} (1, 2, \ldots, N) \equiv \Gamma_{\text{super}} \left[ \ln \left( \left( D\xi^{[N]} \right)^2 \left( \bar{D}\xi^{[N]} \right)^2 \right) \right], \quad (3.54)$$

where $1, \ldots, N$ label the external lines. Since $\exp \left[ -\Gamma_{\text{super}}^{[N]} (1, 2, \ldots, N) \right]$ can be regarded as the partition function corresponding to the super light-cone diagram, it should satisfy

$$-\Gamma_{\text{super}}^{[N]} (1, 2, \ldots, N)
\sim -\Gamma_{\text{super}}^{[N-1]} (1, 2, \ldots, N-2, m') - \Gamma_{\text{super}}^{[3]} (m, N-1, N) + \frac{\text{Re} \ T}{\alpha_m}, \quad (3.55)$$

in the limit $\text{Re} \ T \to \infty$. Here $m, m'$ correspond to the intermediate string. It is straightforward to show that $\Gamma_{\text{super}}$ given in eq. (3.52) satisfies eq. (3.55). Eq. (3.55) no longer holds, if we add to $\Gamma_{\text{super}}$ the terms which depend on $\xi_I$ but are independent of $\mathcal{T}_I$. Therefore we conclude that the right hand side of eq. (3.52) gives $-\Gamma_{\text{super}}$.  

---

1Here we consider the factorization in the context of CFT. In the context of string field theory, we need to take care of the extra contributions coming from the integration measure for the moduli parameters.
4 Correlation functions

4.1 One-point function $\langle DX^-(z) \rangle_\rho$

Using $\Gamma_{\text{super}}$ obtained in eq.(3.52), we can calculate the correlation functions and show that the energy momentum tensor (2.4) satisfies the desired properties. Plugging eq.(3.52) into eq.(2.15), we can evaluate the one-point function,

$$
\langle DX^-(z) \rangle_\rho = 2i\frac{d-10}{8}
$$

$$
x_D \sum I \left\{ \frac{\theta - \tilde{\theta}_I}{(z-\tilde{z}_I)^4} \frac{13}{2} \frac{D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I) + \frac{1}{(z-\tilde{z}_I)^3} \frac{4\partial^2 D\rho D\rho}{(\partial^2 \rho)^3} (\tilde{z}_I) 
+ \frac{\theta - \tilde{\theta}_I}{(z-\tilde{z}_I)^3} \left( \frac{2D\rho}{(\partial^2 \rho)^2} \partial \ln f_I - \frac{4\partial^3 \rho D\rho}{(\partial^2 \rho)^3} \right) (\tilde{z}_I) 
+ \frac{1}{(z-\tilde{z}_I)^2} \left( \frac{D\rho}{(\partial^2 \rho)^2} \partial^2 \ln f_I + \frac{2\partial^2 D\rho D\rho}{(\partial^2 \rho)^3} \partial \ln f_I 
+ \frac{3}{4\partial^2 \rho} + \frac{13}{6} \frac{\partial^3 D\rho D\rho}{(\partial^2 \rho)^3} - \frac{6}{(\partial^2 \rho)^4} \right) (\tilde{z}_I) 
+ \frac{\theta - \tilde{\theta}_I}{(z-\tilde{z}_I)^2} \left( \frac{D\rho}{(\partial^2 \rho)^2} \partial^2 \ln f_I - \frac{\partial^3 \rho D\rho}{(\partial^2 \rho)^3} \partial \ln f_I - \frac{1}{\partial^2 \rho} D\ln f_I 
+ \frac{3}{4\partial^2 \rho} - \frac{13}{12} \frac{\partial^4 \rho D\rho}{(\partial^2 \rho)^3} - \frac{6}{(\partial^2 \rho)^4} + 2 \frac{(\partial^3 \rho)^2 D\rho}{(\partial^2 \rho)^4} \right) (\tilde{z}_I) 
+ \frac{1}{z-\tilde{z}_I} \left( \frac{D\rho}{(\partial^2 \rho)^2} \partial^2 \ln f_I + \frac{2\partial^2 D\rho D\rho}{(\partial^2 \rho)^3} \partial \ln f_I - \frac{\partial^3 \rho D\rho}{(\partial^2 \rho)^3} \partial D\ln f_I 
+ \frac{1}{\partial^2 \rho} + \frac{3\partial^3 D\rho D\rho}{(\partial^2 \rho)^3} - \frac{37}{12} \frac{\partial^4 \rho D\rho D\rho}{(\partial^2 \rho)^3} 
- \frac{25}{6} \frac{\partial^3 D\rho \partial^3 \rho D\rho}{(\partial^2 \rho)^5} + 8 \frac{(\partial^3 \rho)^2 \partial^2 D\rho D\rho}{(\partial^2 \rho)^5} \right) (\tilde{z}_I) 
+ \frac{\theta - \tilde{\theta}_I}{z-\tilde{z}_I} \left( \frac{1}{\partial^2 \rho} \partial D\ln f_I - \frac{\partial^2 D\rho}{(\partial^2 \rho)^2} \partial \ln f_I - \frac{13}{12} \frac{\partial^3 D\rho}{(\partial^2 \rho)^2} + \frac{2}{(\partial^2 \rho)^3} \right) (\tilde{z}_I) \right\} 
- \frac{1}{2} \sum_r \frac{1}{\alpha_r} \left\{ \ln \frac{z - \tilde{z}_I^{(r)}}{z - \tilde{z}_r} - \frac{\theta - \tilde{\theta}_I^{(r)}}{(z - \tilde{z}_I^{(r)})^2} \frac{D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I^{(r)}) - \frac{1}{z - \tilde{z}_I^{(r)}} \frac{\partial^2 D\rho D\rho}{(\partial^2 \rho)^2} (\tilde{z}_I^{(r)}) \right\}.
\right. \right.
\hfill (4.1)
4.2 Energy momentum tensor

Behavior of the energy momentum tensor can be deduced from eq. (4.1). Using eq. (2.16), one can find that

\[
\langle T_{X^\pm} (z) \rangle = \left\langle \left( \frac{1}{2} : DX^+ \partial X^- : (z) + : DX^+ DX^- : (z) \right) - \frac{d - 10}{4} S (z, \rho) \right\rangle F [X^+] \rho
\]

\[
= \langle T_{X^+} (z) \rangle F \left[ - \frac{i}{2} (\rho + \bar{\rho}) \right]
- \frac{i}{4} \partial \rho(z) \int d^2 z'' \frac{\delta F [X^+]}{z' - z''} \left|_{X^+ = - \frac{i}{2} (\rho + \bar{\rho})} \right.
- \frac{i}{4} D \rho(z) \int d^2 z'' \frac{1}{z' - z''} \frac{\delta F [X^+]}{\delta X^+(z'')} \left|_{X^+ = - \frac{i}{2} (\rho + \bar{\rho})} \right.
\]

where

\[
\langle T_{X^\pm} (z) \rangle = \frac{1}{2} \left( - \frac{i}{2} D \rho(z) \langle \partial X^- (z) \rangle - \frac{i}{2} \partial \rho(z) \langle DX^- (z) \rangle \right) - \frac{d - 10}{4} S (z, \rho).
\]

We can show that

\[
\langle T_{X^\pm} (z) \rangle \sim \begin{cases} 
\text{regular} & (z \sim z_I) \\
\text{regular} & (z \sim \infty) \\
\frac{1}{z - Z_r} D_r \left( - \frac{d - 10}{8} \Gamma_{\text{super}} \right) + \frac{\theta - \Theta_r}{z - Z_r} \partial_r \left( - \frac{d - 10}{8} \Gamma_{\text{super}} \right) & (z \sim Z_r)
\end{cases}
\]

where \( D_r = \partial_{\Theta_r} + \Theta_r \partial_{Z_r} \) and \( \partial_r = \partial_{Z_r} \). Thus we see that the energy momentum tensor is regular at \( z = z_I, \infty \), if there are no operator insertions at these points. Taking into account the definition (2.13) of the correlation function, from eq. (4.4) for \( z \sim Z_r \) we can read off the OPE

\[
T_{X\pm} (z) e^{-ip^+ x^-} (Z_r, Z_r) \sim \frac{1}{z - Z_r} D e^{-ip^+ x^-} (Z_r, Z_r) + \frac{\theta - \Theta_r}{z - Z_r} e^{-ip^+ x^-} (Z_r, Z_r).
\]

The non-local operator \( e^{-ip^+ x^-} \) therefore behaves as a primary field of weight 0, as in the bosonic case [1].

4.3 Super Virasoro algebra

Substituting eq. (4.1) into the first relation in eq. (2.16), one can obtain the two-point function \( \langle DX^- (z) DX^- (z') \rangle \). From its singular part for \( z \sim z' \), we can deduce the OPE of \( DX^- \):

\[
DX^- (z) DX^- (z')
\]
Using this OPE, one can show that the energy momentum tensor $T_{X^\pm}(z)$ satisfies the super Virasoro algebra with the central charge $\hat{c} = 12 - d$:

$$T_{X^\pm}(z) T_{X^\pm}(z') \sim -\frac{d - 10}{4 \left(\frac{\partial X^+}{\partial X^+}\right)^3} \left[ \theta - \theta' \right] \frac{3 DX_i^+}{(z - z')^3} (z')$$

$$+ \frac{1}{(z - z')^2} \left( \frac{1}{2} + \frac{4 \partial X^+ DX^+}{(\partial X^+)^4} \right) (z')$$

$$+ \frac{\theta - \theta'}{(z - z')^2} \left( - \frac{\partial^2 X^+}{(\partial X^+)^3} + \frac{5 \partial^2 X^+ DX^+}{2 (\partial X^+)^4} \right) (z')$$

$$+ \frac{1}{z - z'} \left( - \frac{\partial^2 X^+}{2 (\partial X^+)^3} + \frac{2 \partial^2 X^+ DX^+}{(\partial X^+)^4} - \frac{8 \partial^2 X^+ DX^+}{(\partial X^+)^5} \right) (z')$$

$$+ \theta - \theta' \left( - \frac{\partial^2 X^+}{2 (\partial X^+)^3} + \frac{3 \partial^2 X^+ DX^+}{2 (\partial X^+)^4} - \frac{\partial^3 X^+ DX^+}{2 (\partial X^+)^4} \right)$$

$$+ \left( \frac{\partial^2 X^+}{(\partial X^+)^5} - \frac{\partial^2 DX^+ \partial^2 X^+ DX^+}{(\partial X^+)^5} \right) (z') \right]. \quad (4.6)$$

It follows that combined with the transverse variables $X^i(z, \bar{z})$, the total central charge of the system becomes $\hat{c} = 10$. This implies that with the ghost superfields $B(z)$ and $C(z)$ defined as

$$B(z) = \beta(z) + \theta b(z), \quad C(z) = c(z) + \theta c(z), \quad (4.8)$$

it is possible to construct a nilpotent BRST charge

$$Q_B = \oint \frac{dz}{2\pi i} \left[ -C \left( T_{X^\pm} - \frac{1}{2} DX^i \partial X^i \right) + \left( C \partial C - \frac{1}{4} (DC)^2 \right) B \right]. \quad (4.9)$$

5 Conclusions

In this paper, we have constructed the supersymmetric generalization of the theory proposed in Ref. [1]. Although it is much more complicated because of the presence of the odd supermoduli, it is possible to generalize the results in Ref. [1]. We have evaluated $\Gamma_{\text{super}}$, which can be used to define and calculate the correlation functions. We have shown that the energy momentum tensor is regular at $z = z_I$ and $\infty$ if no operators are inserted there. With the transverse variables and ghosts, we have shown that it is possible to construct a nilpotent BRST charge.
With the BRST charge, now it is possible to construct BRST invariant worldsheet theory corresponding to string theory in noncritical dimensions. It can be used to dimensionally regularize the light-cone gauge superstring field theory \[9\]. In order to do so, a careful study on the Ramond sector is also necessary. We will deal with such an application elsewhere.

Acknowledgements

N.I. would like to thank the organizers of the workshop “Branes, Strings and Black Holes” at YITP Kyoto, for the hospitality, where part of this work was done. This work was supported in part by Grant-in-Aid for Scientific Research (C) (20540247) and Grant-in-Aid for Young Scientists (B) (19740164) from the Ministry of Education, Culture, Sports, Science and Technology (MEXT).

A Properties of $\tilde{z}_I$, $z_I$ and $\xi_I$

In order to fix the notation, in this appendix we present some properties concerning the interaction points $\tilde{z}_I, z_I$ and the odd moduli parameters $\xi_I$ introduced in eq.(3.4). These properties are given in Refs. \[3, 6\].

$w(z)$ introduced in eq.(3.45) is useful to derive them. By the definitions (3.4) and (3.45), we have

$$\hat{\rho}(z) = \frac{1}{2} (w(z))^2 .$$

(A.1)

Eq.(3.5) therefore means that

$$w(z_I) = 0 , \quad Dw(z_I) = 0 .$$

(A.2)

Using these equations and the relation $\eta(z) = \frac{Dw}{(\partial w)^{\frac{3}{2}}}(z)$, one can obtain from eq.(3.45)

$$D\rho(z_I) = (\partial w(z_I))^{\frac{3}{2}} \xi_I , \quad \partial \rho(z_I) = \frac{\partial Dw}{(\partial w)^{\frac{3}{2}}}(z_I) \xi_I ,$$

$$\partial D\rho(z_I) = \frac{1}{2} \frac{\partial^2 w}{(\partial w)^{\frac{3}{2}}}(z_I) \xi_I , \quad \partial^2 \rho(z_I) = (\partial w(z_I))^2 + \left( \frac{\partial^2 Dw}{(\partial w)^{\frac{3}{2}}} - \frac{\partial^2 w \partial Dw}{(\partial w)^{\frac{3}{2}}} \right)(z_I) \xi_I ,$$

$$\partial^2 D\rho(z_I) = 2 \partial w \partial Dw(z_I) + O(\xi_I) , \quad \partial^2 \rho(z_I) = 3 \partial^2 w \partial w(z_I) + O(\xi_I) ,$$

$$\partial^3 D\rho(z_I) = 3 \partial^2 Dw \partial w(z_I) + 3 \partial^2 w \partial Dw(z_I) + O(\xi_I) .$$

(A.3)

These relations yield

$$\xi_I = \frac{D\rho}{(\partial^2 \rho)^{\frac{3}{4}}}(z_I) ,$$

(A.4)
and
\[ \partial \rho(z_I) - \frac{1}{2} \frac{\partial^2 D \partial \rho(z_I)}{\partial^2 \rho} = 0, \quad \partial D \rho(z_I) - \frac{1}{6} \frac{\partial^3 \rho D \rho(z_I)}{\partial^2 \rho} = 0. \] (A.5)

Taylor expanding eq. (3.3) around \( \tilde{z}_I \sim z_I \), one obtains equations for \( \tilde{z}_I - z_I \) and \( \tilde{\theta}_I - \theta_I \). Solving them, we have
\[ \tilde{z}_I - z_I = - \frac{\partial \rho}{\partial^2 \rho}(z_I) = - \frac{\partial Dw}{(\partial w)^2}(z_I) \xi_I, \quad \tilde{\theta}_I - \theta_I = - \frac{\partial D \rho}{\partial^2 \rho}(z_I) = - \frac{1}{2} \frac{\partial^2 w}{(\partial w)^2}(z_I) \xi_I. \] (A.6)

We note that \( \tilde{z}_I - z_I \) and \( \tilde{\theta}_I - \theta_I \) are proportional to \( \xi_I \). This yields
\[ D \rho(z_I) = D \rho(\tilde{z}_I), \quad \xi_I = \frac{D \rho}{(\partial^2 \rho)}(z_I) = \frac{D \rho}{(\partial^2 \rho)}(\tilde{z}_I), \] (A.7)
as well as eq. (3.7).

B Another Expression for \( \Gamma_{\text{super}} \)

In this appendix, we recast \( \Gamma_{\text{super}} \) given in eq. (3.52) into a convenient expression to show that this is consistent with the results in Refs. [2, 3].

From eqs. (3.35) and (3.44), one can find that
\[ \ln \omega(z) = \ln A - \sum_r \ln(z - Z_r) + \sum_l P_l \ln(z - \tilde{z}_I), \]
\[ \ln f_I(z) = \ln A - \sum_r \ln(z - Z_r) + \sum_{j \neq I} P_j \ln(z - \tilde{z}_j), \] (B.1)

where \( P_I \) is defined in eq. (3.49). Using eq. (B.1) and the relation \( \lim_{z \to Z_r}(z - Z_r)\omega(z) = \alpha_r \), we can rewrite \( W_{\text{super}} \) given in eq. (3.49) into
\[ - W_{\text{super}} = - \frac{1}{2} \sum_I F_I(\tilde{z}_I) - \frac{1}{2} \sum_r \ln \alpha_r + \ln A, \] (B.2)

where \( F_I(z) \) is defined in eq. (3.48). Using eqs. (3.12) and (3.44), one can get
\[ \ln f_I(z) = \ln \frac{\partial \rho(z)}{z - \tilde{z}_I} - \frac{\partial D \partial \rho(z)}{(\partial \rho)^2} \frac{z - \tilde{z}_I}{(z - \tilde{z}_I)^2} \frac{\partial^2 D \partial \rho(z)}{(\partial^2 \rho)^2} + \frac{\theta - \tilde{\theta}_I}{(z - \tilde{z}_I)^2} \frac{\partial D \rho(z)}{(\partial^2 \rho)^2} \] (B.3)

This yields
\[ \ln f_I(\tilde{z}_I) = \ln \frac{\partial^2 \rho(\tilde{z}_I)}{2} - \frac{\partial^3 D \partial \rho(\tilde{z}_I)}{(\partial \rho)^2} + \frac{\partial^3 \rho \partial^2 D \partial \rho(\tilde{z}_I)}{(\partial^2 \rho)^3}, \]
\[ \partial \ln f_I(\tilde{z}_I) = \frac{1}{2} \frac{\partial^3 \rho}{\partial^2 \rho}(\tilde{z}_I) + (\text{terms proportional to } D \rho(\tilde{z}_I)), \]
\[ D \ln f_I(\tilde{z}_I) = \frac{5}{6} \frac{\partial^3 \rho}{\partial^2 \rho}(\tilde{z}_I) - \frac{\partial^3 \rho \partial^2 D \partial \rho(\tilde{z}_I)}{(\partial^2 \rho)^2} + (\text{terms proportional to } D \rho(\tilde{z}_I)). \] (B.4)
Substituting these equations into eq. (B.2), we obtain

\[
-W_{\text{super}} = \frac{1}{2} \sum_I \ln \left( \partial^2 \rho - \frac{4}{3} \frac{\partial^3 D \rho D \rho}{\partial^2 \rho} + \frac{5}{2} \frac{\partial^3 D \rho D \rho}{(\partial^2 \rho)^2} \right) (\bar{z}_I) - \frac{1}{2} \sum_r \ln \alpha_r + \ln A , \quad (B.5)
\]

and thus

\[
-\Gamma_{\text{super}} = -\frac{1}{4} \sum_I \ln \left( \partial^2 \rho - \frac{5}{3} \frac{\partial^3 D \rho D \rho}{\partial^2 \rho} + \frac{3}{2} \frac{\partial^3 D \rho D \rho}{(\partial^2 \rho)^3} \right) (\bar{z}_I)
\]

\[
- \frac{1}{2} \sum_r \tilde{N}^{rr}_{00} - \frac{1}{2} \sum_r \ln \alpha_r + \ln A 
\]

\[+ \text{c.c.} . \quad (B.6)\]

We set \(Z_N = \infty\) using the superprojective invariance of \(\Gamma_{\text{super}}\). Then \(e^{-\Gamma_{\text{super}}}\) can be compared with eq.(4.10) in Ref. [3]. Taking into account the fact that a factor \(e^{\tilde{N}^{rr}_{00}}\) comes from \(\hat{\epsilon}_r\), eq.(4.10) in Ref. [3] coincides with \(e^{-\Gamma_{\text{super}}}\).

References

[1] Y. Baba, N. Ishibashi, and K. Murakami, “Light-Cone Gauge String Field Theory in Noncritical Dimensions,” arXiv:0909.4675 [hep-th]

[2] N. Berkovits, “CALCULATION OF SCATTERING AMPLITUDES FOR THE NEVEU-SCHWARZ MODEL USING SUPERSHEET FUNCTIONAL INTEGRATION,” Nucl. Phys. B276 (1986) 650

[3] N. Berkovits, “SUPERSHEET FUNCTIONAL INTEGRATION AND THE INTERACTING NEVEU-SCHWARZ STRING,” Nucl. Phys. B304 (1988) 537

[4] D. Friedan, E. J. Martinec, and S. H. Shenker, “Conformal Invariance, Supersymmetry and String Theory,” Nucl. Phys. B271 (1986) 93.

[5] D. Friedan, “NOTES ON STRING THEORY AND TWO-DIMENSIONAL CONFORMAL FIELD THEORY,”. Proc. of Workshop on Unified String Theories, Santa Barbara, CA, Jul 29 - Aug 16, 1985.

[6] K. Aoki, E. D’Hoker, and D. H. Phong, “UNITARITY OF CLOSED SUPERSTRING PERTURBATION THEORY,” Nucl. Phys. B342 (1990) 149–230.
[7] S. Mandelstam, “THE INTERACTING STRING PICTURE AND FUNCTIONAL INTEGRATION,”. Lectures given at Workshop on Unified String Theories, Santa Barbara, CA, Jul 29 - Aug 16, 1985.

[8] S. Mandelstam, “The n loop string amplitude: Explicit formulas, finiteness and absence of ambiguities,” Phys. Lett. B277 (1992) 82–88.

[9] Y. Baba, N. Ishibashi, and K. Murakami, “Light-Cone Gauge Superstring Field Theory and Dimensional Regularization,” JHEP 10 (2009) 035, arXiv:0906.3577 [hep-th].