OBSTRUCTIONS TO DEFORMING CURVES ON AN ENRIQUES-FANO 3-FOLD

HIROKAZU NASU

Dedicated to Professor Shigeru Mukai on the occasion of his 65-th birthday

Abstract. We study the deformations of a curve $C$ on an Enriques-Fano 3-fold $X \subset \mathbb{P}^n$, assuming that $C$ is contained in a smooth hyperplane section $S \subset X$, that is a smooth Enriques surface in $X$. We give a sufficient condition for $C$ to be (un)obstructed in $X$, in terms of half pencils and $(-2)$-curves on $S$. Let $\text{Hilb}^{sc} X$ denote the Hilbert scheme of smooth connected curves in $X$. By using the Hilbert-flag scheme of $X$, we also compute the dimension of $\text{Hilb}^{sc} X$ at $[C]$ and give a sufficient condition for $\text{Hilb}^{sc} X$ to contain a generically non-reduced irreducible component of Mumford type.

1. Introduction

We work over an algebraically closed field $k$ of characteristic 0. Given a projective scheme $X$ over $k$, we denote by $\text{Hilb}^{sc} X$ the Hilbert scheme of smooth connected curves in $X$. Mumford [16] first proved that $\text{Hilb}^{sc} \mathbb{P}^3$ contains a generically non-reduced (irreducible) component. In [15, 17, 18] for smooth Fano 3-folds $X$, $\text{Hilb}^{sc} X$ has been studied from the viewpoint of generalizations of Mumford’s example and more recently it has been proved in [19] that if $X$ is a prime Fano 3-fold then $\text{Hilb}^{sc} X$ contains a generically non-reduced component whose general member is contained in a smooth hyperplane section $S \sim -K_X$ in $\text{Pic} X (= \mathbb{Z}[-K_X])$, i.e. a smooth $K3$ surface $S$ in $X$.

In this paper, we study the Hilbert scheme $\text{Hilb}^{sc} X$ for Enriques-Fano 3-folds $X$ (see Definition 2.3) and discuss the existence of its generically non-reduced components. The 3-folds $X$ in this class contain a (smooth) Enriques surface $S$ as a hyperplane section, and were originally studied by Fano in his famous paper [8] and the study was followed in many papers, e.g. [4, 5, 2, 21, 20, 12]. It is known that every Enriques-Fano 3-fold $X$ has isolated singularities and $-K_X$ is not a Cartier divisor but numerically equivalent to the hyperplane section $S \in |O_X(1)|$. The number $g := (-K_X)^3/2 + 1$ is called the genus of $X$, and it is known that for every $X$ we have $g \leq 17$ (cf. [20, 12]). It follows from a general theory that on every Enriques surface $S$ there exists an effective divisor $E$ on $S$ such that $|2E|$ is base-point-free and defines an elliptic fibration on $S$. (Such a divisor $E$ is called a half pencil on $S$.) Let $N_{E/X}$ denote the normal bundle of $E$ in $X$ and let $N_{E/X}(E)$ be defined by $N_{E/X}(E) := N_{E/X} \otimes E N_{E/S}$. The following is our main theorem.

2010 Mathematics Subject Classification. Primary 14C05; Secondary 14H10, 14D15.

Key words and phrases. Hilbert scheme, obstruction, Enriques surface, Enriques-Fano threefold.
Theorem 1.1. Let $X$ be an Enriques-Fano 3-fold of genus $g$, and $S \subset X$ a smooth hyperplane section, i.e. an Enriques surface $S$ in $X$. If there exists a half pencil $E$ on $S$ of degree $e := (-K_X.E) \geq 2$ such that $H^1(E, N_{E/X}(E)) = 0$, then Hilb$^{sc}$ $X$ contains a generically non-reduced component $W$ of dimension $2g + 2e - 1$ whose general member $C$ satisfies:

1. $C$ is contained in an Enriques surface $S'$ in $X$, and
2. $C$ is linearly equivalent to $-K_X|_{S'} + 2E'$ in $S'$ for some half pencil $E'$ on $S'$.

In Example 4.2 we give a few examples of Enriques-Fano 3-folds $X$ (of genus $g = 6, 9, 13$) satisfying the assumption of Theorem 1.1. For these $X$, there exist a smooth Fano 3-fold $Y$ and a $K3$ surface $M \subset Y$ which double cover $X$ and $S$, respectively. We use the geometry of elliptic fibrations on $M$ to show the existence of the desired half pencil $E$ on $S$. It might be notable that for these $X$ we have

1. every general member $C$ of the component $W$ is contained in a general hyperplane section $S$ of $X$ (cf. Remark 2.7), and
2. for the smooth Fano cover $Y$ of $X$, there exists a generically non-reduced component $V$ of Hilb$^{sc}$ $Y$ of double dimension as Hilb$^{sc}$ $X$ (i.e. dim $V = 2$ dim $W$ for the component $W \subset$ Hilb$^{sc}$ $X$), whose general member is contained in a $K3$ surface $M$ in $Y$, but $M$ is not general in $| - K_Y|$ (cf. Remark 4.5).

One can compare Theorem 1.1 with Proposition 4.4, which gives a sufficient condition for the Hilbert scheme of a smooth Fano 3-fold to have a generically non-reduced component. Theorem 1.1 is obtained as an application of Theorem 1.2, which enables us to compute the dimension of Hilb$^{sc}$ $X$ at $[C]$ and determines the (un)obstructedness of $C$ in $X$ for curves $C$ contained in $S$.

Theorem 1.2. Let $X$ be an Enriques-Fano 3-fold of genus $g$, $S$ a smooth hyperplane section of $X$ and $C$ a smooth connected curve on $S$ satisfying $H^1(S, O_S(C)) = 0$. We define a divisor $D$ on $S$ by $D := C + K_X|_S$.

1. If $H^1(S, D) = 0$, then $C$ is unobstructed in $X$.
2. If there exists a half-pencil $E$ on $S$ such that $D \sim 2mE$ or $D \sim K_S + (2m+1)E$ for an integer $m \geq 1$, then we have $h^1(S, D) = m$. If moreover $H^1(E, N_{E/X}(E)) = 0$, then $C$ is obstructed in $X$.
3. If $D \geq 0$, $D^2 \geq 0$ and there exists a $(-2)$-curve $E$ on $S$ such that $E.D = -2$ and $H^1(S, D - 3E) = 0$, then we have $h^1(S, D) = 1$. If moreover the $\pi$-map $\pi_{E/S}(E)$ (cf. (2.7)) for $(E, S)$ is not surjective, then $C$ is obstructed in $X$.

In (1), (2) and (3), if we assume furthermore that $h^0(S, K_S - D) = 0$ and $m = 1$ (for (2)), then Hilb$^{sc}$ $X$ is of dimension $g + g(C) - 1$ at $[C]$, where $g(C)$ denote the (arithmetic) genus of $C$. 
If $H^1(S, \mathcal{O}_S(C)) = 0$, then the Hilbert-flag scheme $HF^{sc}_X$ of $X$ is nonsingular at $(C, S)$ of expected dimension $g + g(C) - 1$ (cf. Lemma 2.9). If moreover $H^1(S, D) = 0$, then the first projection $HF^{sc}_X \to \text{Hilb}^{sc}_X$, $(C, S) \mapsto [C]$ is smooth at $(C, S)$, and thus Theorem 1.2 (1) follows from a property of smooth morphisms. We partially prove that $C$ is obstructed in $X$ if $H^1(S, D) \neq 0$ by using half pencils and $(-2)$-curves on $S$ together with a result in [18]. See [18] for a result on the (un)obstructedness of curves lying on a $K3$ surface in a smooth Fano 3-fold.

The organization of this paper is as follows. In §2.1 and §2.2 we recall some properties of Enriques surfaces and Enriques-Fano 3-folds, respectively. In §2.3 we recall some known results on Hilbert-flag schemes and obstructions to lifting first order deformations of curves on a 3-fold to second order deformations (i.e. primary obstructions). These results will be used in §3 and §4 to prove Theorems 1.2 and 1.1, respectively.

Acknowledgments. I would like to thank Prof. Hiromichi Takagi for his comment, which motivated me to research the topic of this paper. I would like to thank Prof. Shigeru Mukai for letting me know examples of Enriques-Fano 3-folds. This paper was written during my stay as a visiting researcher at the department of mathematics at the University of Oslo (UiO), Norway. I thank UiO for providing the facilities. I thank Prof. Kristian Ranestad, Prof. John Christian Ottem and Prof. Jan Oddvar Kleppe for helpful and inspiring discussions during the stay. Last but not least, I thank the referee for giving helpful comments improving the readability and quality of this paper. This work was supported in part by JSPS KAKENHI Grant Numbers JP17K05210 and JP20K03541.

2. Preliminaries

2.1. Enriques surfaces. In this section, we recall some properties of Enriques surfaces. We refer to [7] and [1] for proofs and more general theories on Enriques surfaces. A smooth projective surface $S$ is called an Enriques surface if $H^i(S, \mathcal{O}_S) = 0$ for $i = 1, 2$ and $2K_S \sim 0$. Every Enriques surface $S$ is isomorphic to the quotient $M/\theta$ of a smooth $K3$ surface $M$ by a fixed-point-free involution $\theta$ of $M$. Here $M$ is the canonical cover of $S$ and called the $K3$ cover of $S$. It is well known that $S$ admits an elliptic fibration $\psi : S \to \mathbb{P}^1$, whose general fiber is a smooth curve of arithmetic genus $1$ (cf. e.g. [1, VIII, §17]). For each $\psi$ there exist exactly two multiple fibers $2E$ and $2E'$ of $\psi$, i.e. the double fibers of $\psi$, and we have $K_S \simeq \mathcal{O}_S(E - E') \simeq \mathcal{O}_S(E' - E)$. Such divisors $E$ and $E'$ are called half pencils on $S$. Every nef and primitive divisor $E \geq 0$ on $S$ with self-intersection number $E^2 = 0$ is a half pencil on $S$.

Let $S$ be an Enriques surface and $\pi : M \to S$ its $K3$ cover, i.e. there exists a fixed-point-free involution $\theta$ of $M$ such that $S \simeq M/\theta$. Let $\varphi : M \to \mathbb{P}^1$ be an elliptic fibration on $M$. Then there exists a non-constant rational function $u \in k(M)$, which is defined by $u = \varphi^* t$ and $k(\mathbb{P}^1) = k(t)$. Here $u$ is called the elliptic parameter of $\varphi$ and unique only up to the linear fractional transformations (of $t$) (cf. [13]).
Lemma 2.1. Let \( u \) be the elliptic parameter of \( \varphi : M \to \mathbb{P}^1 \) and \( F \) the fiber of \( \varphi \) defined by \( u \), i.e. \( F = \text{div}_0(u) \). If \( u \) is \( \theta \)-anti-invariant, i.e. \( \theta^*u = -u \), then the image \( E = \pi(F) \) is a half pencil on \( S \) and the pull-back \( \pi^*E \) coincides with \( F \).

Proof. Let \( \pi' \) be the map on \( \mathbb{P}^1 \) defined by \( t \mapsto t^2 \). Since \( u^2 \) is invariant under \( \theta \), the composition \( \pi' \circ \varphi \) factors through \( \pi \) and there exists a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\varphi} & \mathbb{P}^1 \\
\pi \downarrow & & \pi' \downarrow \\
S & \xrightarrow{\psi} & \mathbb{P}^1,
\end{array}
\]

where \( \psi \) defines an elliptic fibration on \( S \). Since \( \pi' \) is ramified at \( (t = 0) \) (and \( (t = \infty) \)) on \( \mathbb{P}^1 \), \( \psi \) has the double fibers \( 2E \) at \( (t = 0) \). By commutativity, we have proved the lemma. \( \square \)

Let \( E \) be a half pencil on \( S \). Note that the normal bundle \( N_{E/S} \simeq \mathcal{O}_E(2E) \) of \( E \) in \( S \) is a 2-torsion in \( \text{Pic} E \). Here \( \mathcal{O}_E(2E) \) has no sections, but \( \mathcal{O}_E(E) \) has a unique nonzero section up to constants. We also note that for every integer \( k \geq 0 \), the restriction map \( H^0(S,kE) \to H^0(E,kE) \) is surjective. It follows from an exact sequence \( 0 \to \mathcal{O}_S((k-1)E) \to \mathcal{O}_S(kE) \to \mathcal{O}_E(kE) \to 0 \) that the surjectivity is equivalent to the injectivity of the natural induced map \( H^1(S, (k-1)E) \to H^1(S, kE) \). We need the following lemma for our proof of Theorem 1.2.

Lemma 2.2. Let \( D \) be an effective divisor on \( S \) with \( D^2 \geq 0 \).

(1) \( H^1(S, D) \neq 0 \) if and only if

[i] \( D \sim mE \) or \( D \sim K_S + (m + 1)E \) for a half pencil \( E \) on \( S \) and an integer \( m \geq 2 \) (then \( h^1(S, D) = \lfloor m/2 \rfloor \)), or

[ii] \( D, \Delta \leq -2 \) for some divisor \( \Delta \geq 0 \) on \( S \) with \( \Delta^2 = -2 \).

(2) If there exists \((-2)\)-curve \( E \) on \( S \) such that \( E.D = -2 \) and \( H^1(S, D - 3E) = 0 \) then we have \( h^1(S, D) = 1 \) and \( H^1(S, D - E) = 0 \).

Proof. (1) is a special case of [11]. (2) follows from [18, Claim 4.1], whose proof works also for divisors on Enriques surfaces. \( \square \)

2.2. Enriques-Fano 3-folds. In this section, we collect some known results on Enriques-Fano 3-folds and prepare a lemma on the elliptic fibrations on their hyperplane sections (cf. Lemma 2.6). This lemma will be used in §4 to show the existence of a generically non-reduced component of the Hilbert scheme of some Enriques-Fano 3-folds.

Definition 2.3. A normal projective 3-dimensional variety \( X \subset \mathbb{P}^N \) is called Enriques-Fano if it contains an Enriques surface \( S \) as a hyperplane section and \( X \) is not a cone over \( S \).
An equivalent definition is to assume that a general hyperplane section is a smooth Enriques surface. In this section, we mainly consider the Enriques-Fano 3-folds $X$ with only terminal cyclic quotient singularities. There is a classification of such $X$ due to Bayle [2] and Sano [21]. We summarize the properties of $X$:

1. $-2K_X \sim 2H$ in $\text{Pic} \, X$ for $H \in |O_X(1)|$.
2. The canonical cover
   
   $$Y = \text{Spec}_X(O_X \oplus O_X(K_X + H)) \xrightarrow{\pi} X$$

   of $X$ is a smooth Fano 3-fold, and $Y$ is isomorphic to one of the 3-folds in Table 1.
3. The covering transformation $\theta$ of $\pi$ is an involution of $Y$ and $\theta$ fixes just 8 points on $Y$.
4. The fixed points on $Y$ give rise to the singularity on $X \simeq Y/\theta$ of type $\frac{1}{2}(1, 1, 1)$.

Table 1. Enriques-Fano 3-folds $X$

| No. | canonical cover $Y \xrightarrow{(2:1)} X$ | $g$ |
|-----|----------------------------------------|----|
| 1   | a complete intersection $2 \cap (4) \subset \mathbb{P}(1, 1, 1, 1, 1, 2)$ | 2 |
| 2   | a complete intersection $2 \cap (2) \subset \mathbb{P}^6$ | 3 |
| 3   | the blow-up $\text{Bl}_{V_2}$ of $V_2$ with a center an elliptic curve $\gamma$ | 3 |
| 4   | $\mathbb{P}^1 \times S_2^c$ | 4 |
| 5   | a double cover of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ branched along a divisor $R \sim (2, 2, 2)$ | 4 |
| 6   | a double cover of $(1, 1) \subset \mathbb{P}^2 \times \mathbb{P}^2$ branched along $R \in |-K|$. | 4 |
| 7   | the blow-up $\text{Bl}_{V_4}$ of $V_4$ with a center an elliptic curve $\gamma$ | 5 |
| 8   | a weighted hypersurface $(4) \subset \mathbb{P}(1, 1, 1, 1, 2)$ (i.e. $V_2$) | 5 |
| 9   | a complete intersection $(1, 1) \cap (1, 1) \cap (1, 1) \subset \mathbb{P}^3 \times \mathbb{P}^3$ | 6 |
| 10  | $\mathbb{P}^1 \times S_4$ | 6 |
| 11  | a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of multidegree $(1, 1, 1, 1)$ | 7 |
| 12  | a complete intersection $2 \cap (2) \subset \mathbb{P}^5$ (i.e. $V_4$) | 9 |
| 13  | $\mathbb{P}^1 \times S_6$ | 10 |
| 14  | $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ | 13 |

In this table, $V_n$ denotes a del Pezzo 3-fold of degree $n$.

$\gamma$ is a complete intersection $H_1 \cap H_2 \subset V_2$ where $H_i \in |(-1/2)K_{V_2}|$ for $i = 1, 2$.

$S_n$ denotes a del Pezzo surface of degree $n$.

A hypersurface in $\mathbb{P}^2 \times \mathbb{P}^2$ of multidegree $(1, 1)$ is isomorphic to $\mathbb{P}_2(T_{\mathbb{P}^2})$.

$\gamma$ is a complete intersection $(1) \cap (1) \cap (2) \subset \mathbb{P}^5$.

Minagawa [14] showed that every Enriques-Fano 3-fold $X_0$ with at most terminal singularities admits a $\mathbb{Q}$-smoothing, which assures us that $X_0$ is obtained as a flat specialization.
of Enriques-Fano 3-folds $X_t$ ($t \neq 0$) with only terminal cyclic quotient singularities. This fact implies that we have $g \leq 13$ for any $X_0$.

**Remark 2.4.** Prokhorov [20] and Knutsen-Lopez-Munoz [12] proved that $g \leq 17$ for any Enriques-Fano 3-fold $X$. As far as we know, the problem of classifying all Enriques-Fano 3-folds $X$ with non-terminal singularities is still open.

In what follows, we recall some well known examples of Enriques-Fano 3-folds $X_g$ of genus $g = 6, 9$ and 13. Here $X_g$ has the Picard rank 1, 2 and 3 for $g = 9, 6$ and 13, respectively.

**Example 2.5.** In the following example, $Y$ is a smooth Fano 3-fold and there exists an involution $\theta$ of $Y$ fixing just 8 points on $Y$. There exists a smooth K3 surface $M$ in $Y$ on which $\theta_M := \theta|_M$ acts without fixed points. Thereby the quotient $S := M/\theta_M$ is an Enriques surface and $X := Y/\theta$ is an Enriques-Fano 3-fold of genus $g = (-K_X)^3/2 + 1 = (-K_Y)^3/4 + 1$.

(1) (No.14 in Table 1) Let $Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We define an involution $\theta$ of $Y$ by

\[ (x_0 : x_1) \times (y_0 : y_1) \times (z_0 : z_1) \mapsto (x_0 : -x_1) \times (y_0 : -y_1) \times (z_0 : -z_1). \]

Here we say that $\theta$ is of type $(-1, -1, -1)$. Then $\theta$ fixes just 8 coordinate points on $Y$. There exist exactly 14 $\theta$-invariant monomials, which correspond to the 14 ($= \bullet \times 8 + \circ \times 6$) vertices in Figure 1(a). Then these monomials span the linear subsystem $\Lambda$ of $|-K_Y| = |\mathcal{O}_Y(2, 2, 2)| \simeq \mathbb{P}^{26}$ of dimension 13. There exists a smooth member $M \in \Lambda$ not passing through the 8 fixed points. Thus $X_{13} := Y/\theta$ is an Enriques-Fano 3-fold of genus 13, and $X_{13}$ is embedded into $\mathbb{P}^{13}$ by the linear system $\Lambda \simeq |\mathcal{O}_X(2, 2, 2)| \simeq \mathbb{P}^{13}$.

(2) (No.12 in Table 1) Let $\mathbb{P}^5$ be the projective 5-space and $x_0, \ldots, x_5$ its homogeneous coordinates. Let $Y \subset \mathbb{P}^5$ be a smooth complete intersection of two hyperquadrics, whose defining equations are of the forms

\[ q_i(x_0, x_1, x_2) + q'_i(x_3, x_4, x_5) \quad (i = 0, 1). \]
We define an involution \( \theta' \) of \( \mathbb{P}^5 \) by

\[
(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_0, x_1, x_2, -x_3, -x_4, -x_5).
\]

Then \( \theta' \) defines an involution \( \theta \) of \( Y \) by restriction. The the fixed locus \( \text{Fix}(\theta') \) of \( \theta' \) is equal to \( (x_0 = x_1 = x_2 = 0) \cup (x_3 = x_4 = x_5 = 0) \). Thereby \( \theta \) fixes just 8 points on \( Y \). We consider the third quadratic form \( q_2(x_0, x_1, x_2) + q'_2(x_3, x_4, x_5) \) of the same type and the hyperquadric \( Q_2 \) in \( \mathbb{P}^5 \) defined by it. Then the intersection \( M := Y \cap Q_2 \) is a \( K3 \) surface, and moreover we can take \( q_2 \) and \( q'_2 \) so that \( M \cap \text{Fix}(\theta) = \emptyset \). Thus \( X_9 := Y/\theta \) is an Enriques-Fano 3-fold of genus 9. Then the linear system \( \Lambda (\cong \mathbb{P}^9) \) of \( \theta \)-invariant quadratic forms on \( Y \) of type (2.2) defines an embedding of \( X_9 \) into \( \mathbb{P}^9 \).

(3) (No.9 in Table 1) Let \( Y \) be a smooth complete intersection of three hypersurfaces \( Q_i \) \( (i = 0, 1, 2) \) in \( \mathbb{P}^3 \times \mathbb{P}^3 \) of bidegree \( (1, 1) \). Suppose that for each \( i \), \( Q_i \) is defined by a symmetric bilinear form \( q_i(x, y) \) on \( \mathbb{P}^3 \times \mathbb{P}^3 \). We consider the diagonal action \( (x, y) \mapsto (y, x) \) on \( \mathbb{P}^3 \times \mathbb{P}^3 \) and define an involution \( \theta \) on \( Y \) by its restriction. Then \( \theta \) fixes just 8 diagonal points on \( Y \). We consider the fourth symmetric bilinear form \( q_3(x, y) \) and the hypersurface \( Q_3 \subset \mathbb{P}^3 \times \mathbb{P}^3 \) defined by it. If \( q_3 \) is general, then the intersection \( M := Y \cap Q_3 \) is a \( K3 \) surface in \( Y \), on which \( \theta |_M \) acts without fixed points. Thus \( X_6 := Y/\theta \) is an Enriques-Fano 3-fold of genus 6. The linear system \( \Lambda (\cong \mathbb{P}^6) \) of symmetric bilinear forms on \( Y \) defines an embedding of \( X_6 \) into \( \mathbb{P}^6 \).

Let \( X \) be an Enriques-Fano 3-fold with at most terminal cyclic quotient singularities. Then the canonical (double) cover \( \pi : Y \to X \) of \( X \) is a smooth Fano 3-fold and there exists an involution \( \theta \) of \( Y \) such that \( X \cong Y/\theta \). (Here and later we call \( Y \) the smooth Fano cover of \( X \).) If a \( K3 \) surface \( M \subset Y \) is invariant under \( \theta \) and an elliptic fibration \( \varphi : M \to \mathbb{P}^1 \) has a \( \theta \)-anti-invariant elliptic parameter \( u \in k(M) \), then by Lemma 2.1, \( \varphi \) induces an elliptic fibration on the Enriques quotient \( S \subset X \) of \( M \), and the image \( E = \pi(F) \) of \( F = (u = 0) \) becomes a half-pencil on \( S \).

**Lemma 2.6.** If \( X \) is either \( X_{13}, X_9 \) or \( X_6 \) in Example 2.5, then there exist a \( \theta \)-invariant \( K3 \) surface \( M \) in the smooth Fano cover \( Y \) of \( X \) and an elliptic fibration \( \varphi : M \to \mathbb{P}^1 \) with a \( \theta \)-anti-invariant elliptic parameter \( u \) such that

\[
H^1(F, N_{F/Y}) = 0
\]

for the invariant elliptic curve \( F = (u = 0) \) on \( M \). More explicitly, \( F \) is described as follows:

1. \( F \) is a complete intersection of two hypersurfaces of tridegree \( (1, 1, 1) \) in \( Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \) if \( X = X_{13} \),
2. \( F \) is a linear section \( (1) \cap (1) \cap Y \) of \( Y = (2) \cap (2) \subset \mathbb{P}^5 \) if \( X = X_9 \), and
(3) $F$ is a complete intersection of two hypersurfaces of bidegree $(1, 0)$ and $(0, 1)$ with $Y = (1, 1)^3 \subset \mathbb{P}^3 \times \mathbb{P}^3$ if $X = X_6$.

Proof. The lemma is proved in case by case.

(1) Let $X = X_{13}$. We denote by $R^+$ (resp. $R^-$) the space of invariant (resp. anti-invariant) $(1,1,1)$-forms on $Y$. Then $R^+$ and $R^-$ are spanned by the four monomials $x_iy_jz_k$ corresponding to the vertices $\bullet$ and $\circ$ in Figure 1(b), respectively. We take four general $(1,1,1)$-forms $l^+, m^+ \in R^+$ and $l^-, m^- \in R^-$ and define a smooth K3 surface $M$ in $Y$ by a $(2,2,2)$-form $f = l^+m^+ - l^-m^-$ on $Y$. Then since $f$ is $\theta$-invariant, so is $M$. We consider a rational function $u = l^+/m^- = (l^-/m^+)$ on $M$ and the elliptic fibration $\varphi : M \to \mathbb{P}^1$ with the elliptic parameter $u$. Then $u$ is clearly $\theta$-anti-invariant. The fiber $F$ at $(t = 0) \in \mathbb{P}^1$ is defined by $l^+ = l^- = 0$ in $Y$, and hence $F$ is a complete intersection in $Y$. Since $N_{F/Y} \simeq \mathcal{O}_F(1,1,1)^{\oplus 2}$ and $\mathcal{O}_F(1,1,1)$ is ample, we have $H^1(F, N_{F/Y}) = 0$.

(2) Let $X = X_9$ and let $M$ be the K3 surface in Example 2.5(2). Then $M$ is defined in $\mathbb{P}^5$ by three quadratic polynomials $q_i(x) + q'_i(x')$ $(i = 0, 1, 2)$, where $x = (x_0, x_1, x_2)$ and $x' = (x_3, x_4, x_5)$. Then for each $i$, there exist two $3 \times 3$ symmetric matrices $A_i$ and $A'_i$ corresponding to $q_i$ and $q'_i$, respectively. We see that there is a one-to-one correspondence between the set of elliptic fibrations on $M$ (on $S = M/\theta$) and the 9 points $\lambda = (\lambda_0, \lambda_1, \lambda_2)$ in $\mathbb{P}^2$ defined by

$$\det(\lambda_0A_0 + \lambda_1A_1 + \lambda_2A_2) = \det(\lambda_0A'_0 + \lambda_1A'_1 + \lambda_2A'_2) = 0.$$ 

In fact, if $\lambda$ satisfies this equation, then there exist four linear forms $l, m, l', m'$ on $\mathbb{P}^2$ such that $\sum_{i=0}^2 \lambda_i(q_i(x) + q'_i(x')) = l(x)m(x) - l'(x')m'(x')$. Then as in (1), there exists an elliptic fibration $\varphi : M \to \mathbb{P}^1$ on $M$ defined by the elliptic parameter $u = l(x)/m'(x')$ ($= l'(x')/m(x)$), which is $\theta$-anti-invariant. Since the fiber $F = (u = 0)$ of $\varphi$ is defined by $l(x) = l'(x') = 0$ in $Y$, $F$ is a linear section of $Y$. Thus $H^1(F, N_{F/Y}) = H^1(F, \mathcal{O}_F(1)^{\oplus 2}) = 0$.

(3) Let $X = X_6$ and let $M$ be the K3 surface in Example 2.5(3). We recall that $M$ is a complete intersection of four hypersurfaces $Q_i \subset \mathbb{P}^3 \times \mathbb{P}^3$ $(i = 0, 1, 2, 3)$ of bidegree $(1, 1)$, each of which is defined by a symmetric bilinear form $q_i$ on $\mathbb{P}^3 \times \mathbb{P}^3$. For each $i$, let $A_i$ be the $4 \times 4$ symmetric matrix corresponding to $q_i$. Then the elliptic fibrations on $M$ (on $S = M/\theta$) are in one-to-one correspondence with the points $\lambda = (\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ in $\mathbb{P}^3$ satisfying

$$\text{rank}(A(\lambda)) \leq 2, \quad \text{where } A(\lambda) := \lambda_0A_0 + \lambda_1A_1 + \lambda_2A_2 + \lambda_3A_3.$$ 

These points correspond to the 10 nodes of a quartic surface in $\mathbb{P}^3$ defined by $\det(A(\lambda))$, which is well known as a Cayley’s quartic symmetroid (cf. [3]). For each such $\lambda$, there exist two linear forms $l$ and $m$ on $\mathbb{P}^3$ such that $^t\!xA(\lambda)y = l(x)l(y) + m(x)m(y)$. By changing the coordinates of $\mathbb{P}^3$, we may assume that $^t\!xA(\lambda)y = x_0y_0 + x_1y_1$. We define a rational function $u \in k(M)$ by $u = (x_1 - x_0\sqrt{-1})/(x_1 + x_0\sqrt{-1})$ and take it as an elliptic
Remark 2.7. We note that in Lemma 2.6, the $K3$ surface $M$ is required to contain a pencil of elliptic curves $F$ in $Y$ and hence not a general member of $|-K_Y|$. However its image $S = \pi(M)$ is a general hyperplane section of $X_g \subset \mathbb{P}^g$. It is rather easy to see this for $g = 6, 9$. For $g = 13$ we consider a linear map

$$\Phi : \text{Sym}^2(R^+) \oplus \text{Sym}^2(R^-) \rightarrow H^0(X, \mathcal{O}_X(2, 2, 2)) \simeq k^{14},$$

where $R^+$ (resp. $R^-$) is the 4-dimensional vector space of (resp. anti-)invariant $(1, 1, 1)$-forms on $Y$ (cf. Figure 1(b)), and $\text{Sym}^2(V)$ denotes the symmetric square of a $k$-vector space $V$. Then we see that the kernel of $\Phi$ is of dimension 6 and hence $\Phi$ is surjective.

Remark 2.8. In Lemma 2.6, $F$ is a complete intersection in $Y$. This implies that $H^1(Y, \mathcal{I}_{F/Y} \otimes_Y \mathcal{O}_Y(-K_Y)) = 0$. Then there exists a short exact sequence

$$0 \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{I}_{F/Y} \otimes_Y \mathcal{O}_Y(-K_Y) \longrightarrow N_{M/Y}(-F) \longrightarrow 0$$

on $Y$ and it follows from this sequence that $H^1(M, N_{M/Y}(-F)) = 0$.

2.3. Hilbert-flag scheme and Primary obstruction. In this section, we recall some properties of Hilbert-flag schemes (cf. [9, 23]) and a result in [18] on primary obstructions to deforming curves on a 3-fold. In this paper, Hilbert-flag schemes play an important role. Given a projective scheme $X$, we denote by $HF_X$ the Hilbert-flag scheme, that is the projective scheme parametrizing all pairs $(C, S)$ of closed subschemes $C$ and $S$ of $X$ satisfying $C \subset S$. For a pair $(C, S)$, its normal sheaf $N_{(C,S)/X}$ in $X$ is defined by the Cartesian diagram

$$\begin{array}{ccc}
N_{(C,S)/X} & \xrightarrow{\pi_2} & N_{S/X} \\
\downarrow{\pi_1} & & \downarrow{\mid_C}
\end{array}$$

where $\mid_C$ and $\pi_{C/S}$ are the restriction and the projection, respectively. Suppose that the two embeddings $C \hookrightarrow S$ and $S \hookrightarrow X$ are regular (cf. [23]). Then $H^0(X, N_{(C,S)/X})$ and $H^1(X, N_{(C,S)/X})$ respectively represent the tangent space and the obstruction space of $HF_X$ at $(C, S)$. It follows from a general theory on Hilbert schemes that

$$h^0(X, N_{(C,S)/X}) - h^1(X, N_{(C,S)/X}) \leq \dim_{(C,S)} HF_X \leq h^0(X, N_{(C,S)/X}).$$

Moreover, there exist two fundamental exact sequences

$$0 \longrightarrow \mathcal{I}_{C/S} \otimes_S \mathcal{N}_{S/X} \longrightarrow N_{(C,S)/X} \longrightarrow \mathcal{N}_{C/X} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{I}_{C/S} \otimes \mathcal{N}_{S/X} \longrightarrow \mathcal{N}_{(C,S)/X} \longrightarrow \mathcal{N}_{C/X} \longrightarrow 0$$
Lemma 2.9. Suppose that $-K_X$ is ample and $p_g(S) = 0$.

(1) If $H^i(S, \mathcal{O}_S(C)) = 0$ then $H^i(X, N_{(C,S)/X}) = 0$ for all $i > 0$, which implies that $H^{sec}X$ is nonsingular at $(C, S)$ of expected dimension $\chi(X, N_{(C,S)/X})$.

(2) If $H^1(S, \mathcal{O}_S(C)) = H^1(S, \mathcal{O}_S(C + K_X|_S)) = 0$, then $C$ is unobstructed in $X$.

Proof. (1) There exists an exact sequence $0 \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_S(C) \rightarrow N_{C/S} \rightarrow 0$. Since $H^2(S, \mathcal{O}_S) = 0$, we have $H^i(C, N_{C/S}) = 0$ for all $i > 0$. By adjunction, we have $N_{S/X} = -K_X|_S + K_S$ and hence $H^i(S, N_{S/X}) = 0$ for all $i > 0$ by assumption. Thus we obtain the first conclusion of the lemma by (2.5) and (2.3).

(2) By Serre duality, we see that

$$H^1(S, N_{S/X}(-C)) \simeq H^1(S, -K_X|_S + K_S - C) \simeq H^1(S, C + K_X|_S)^\vee = 0.$$ 

Then the first projection $pr_1$ is smooth at $(C, S)$ by [10, Lemma A10] (cf. [19, §2.2]). Thus as a consequence of (1) we have proved the lemma.

Remark 2.10. If $X$ is an Enriques-Fano 3-fold and $S$ is its hyperplane section, then by using (2.5) the number $\chi(X, N_{(C,S)/X})$ is computed as

$$\chi(S, N_{S/X}) + \chi(C, N_{C/S}) = (-K_X)^3/2 + 1 + C^2/2 = g + g(C) - 1,$$

where $g$ is the genus $g = (-K_X)^3/2 + 1$ of $X$.

Next we recall primary obstructions to deforming curves on a 3-fold. Let $\alpha$ be a global section of $N_{C/X}$, i.e. a first order (infinitesimal) deformation $\tilde{C}$ of $C$ in $X$. We note that $C$ is a locally complete intersection in $X$. Then the primary obstruction $ob(\alpha)$, i.e. the obstruction to extend $\tilde{C}$ to a deformation $\tilde{C}$ of $C$ over $k[t]/(t^3)$, is contained in $H^1(C, N_{C/X})$ and expressed as a cup product of cohomology classes on $C$ (cf. e.g. [18, Theorem 2.1]). If $ob(\alpha) \neq 0$ for some $\alpha \in H^0(C, N_{C/X})$, then $C$ is obstructed in $X$. In [15, 18] a sufficient condition for $ob(\alpha) \neq 0$ was given under the presence of an intermediate smooth surface $S$ satisfying $C \subset S \subset X$. Let $\pi_{C/S} : N_{C/X} \rightarrow N_{S/X}|_C$ be the natural projection of

\[
0 \longrightarrow N_{C/S} \longrightarrow N_{(C,S)/X} \xrightarrow{\pi_2} N_{S/X} \longrightarrow 0
\]
normal bundles, and $\pi_{C/S}(\alpha) \in H^0(S, N_{S/X}|_C)$ the image of $\alpha$ by the projection (i.e. the exterior component of $\alpha$). We suppose furthermore that $\pi_{C/S}(\alpha)$ lifts to a global section $\beta \in H^0(S, N_{S/X}(E))$ for some effective divisor $E \geq 0$ on $S$, i.e. we have
\begin{equation}
(2.6) \quad r(\pi_{C/S}(\alpha), E) = \beta|_E \quad \text{in} \quad H^0(C, N_{S/X}(E)|_C).
\end{equation}
Here and later, for a sheaf $\mathcal{F}$ on $S$ and a cohomology class $* \in H^i(S, \mathcal{F})$, we denote by $r(*, E)$ the image of $*$ by the natural map $H^i(S, \mathcal{F}) \to H^i(S, \mathcal{F}(E))$ (and we use similar notation for $C$). The rational section $\beta$ of $N_{S/X}$ admitting a pole along $E$ is called an infinitesimal deformation of $S$ in $X$ with pole (along $E$). Tensoring the projection $\pi_{E/S}$ with $N_{E/S} \simeq \mathcal{O}_E(E)$ and taking the map induced on the space of global sections, we define the $\pi$-map
\begin{equation}
(2.7) \quad \pi_{E/S}(E) : H^0(E, N_{E/X}(E)) \longrightarrow H^0(E, N_{S/X}(E)|_E)
\end{equation}
for $(E, S)$. The following theorem is crucial to our proof of Theorems 1.1 and 1.2.

**Theorem 2.11** (cf. [18, Theorem 1.1 and Corollary 3.2]). *The primary obstruction $\text{ob}(\alpha)$ is nonzero if*

(i) the natural map $H^1(S, kE) \longrightarrow H^1(S, (k+1)E)$ is injective for all integer $k \geq 0$,
(ii) the restriction map $H^0(S, \Delta) \longrightarrow H^0(E, \Delta|_E)$ is surjective for $\Delta := C + K_X|_S - 2E \in \text{Pic} S,$
(iii) $\beta$ is not contained in $H^0(S, N_{S/X})$, equivalently, the principal part $\beta|_E$ of $\beta$ is nonzero in $H^0(E, N_{S/X}(E)|_E),$
(iv) $E$ is an irreducible curve of arithmetic genus of $g(E)$ and $(\Delta, E) = 2g(E) - 2 - 2E^2$,
(v) the $\pi$-map $\pi_{E/S}(E)$ is not surjective, and
(vi) $H^1(S, C - E) = 0$.

Here the natural map in the item (i) is induced by an inclusion $\iota : \mathcal{O}_S(kE) \hookrightarrow \mathcal{O}_S((k + 1)E)$ of sheaves on $S$.

**Remark 2.12.** In [15, 18] the authors reduced the computation of $\text{ob}(\alpha)$ to the image $\text{ob}_S(\alpha) (= \pi_{C/S}(\text{ob}(\alpha)))$ in $H^1(S, N_{S/X}|_C)$. They deduced the nonzero of $\text{ob}_S(\alpha)$ from the nonzero of a cup product on the (polar) curve $E$. In fact, let $\partial_E$ denote the coboundary map of the short exact sequence
\begin{equation}
(2.8) \quad [0 \longrightarrow N_{E/S} \longrightarrow N_{E/X} \xrightarrow{\pi_{E/S}} N_{S/X}|_E \longrightarrow 0] \otimes_E \mathcal{O}_E(E)
\end{equation}
on $E$. Then the nonzero of $\text{ob}_S(\alpha)$ is deduced from that of $\partial_E(\beta|_E) \cup \beta|_E$, where the cup product is taken by the map
$$
H^1(E, \mathcal{O}_E(2E)) \times H^0(E, N_{S/X}(E - C)|_E) \xrightarrow{\cup} H^1(E, N_{S/X}(3E - C)|_E).
$$

The condition (iv) assures us that the invertible sheaf $N_{S/X}(E - C)|_E$ on $E$ is trivial, while (v) and (vi) imply that the coboundary image $\partial_E(\beta|_E)$ is nonzero. Thus they obtained
the nonzero of the cup product on $E$. Theorem 2.11 looks technical, however it has many application in e.g. [17, 18, 19]. We refer to [18] for the proof.

Now we assume that $X$ is an Enriques-Fano 3-fold and $S$ is its hyperplane section. Let $E$ be a half pencil on $S$.

**Lemma 2.13.** (1) The $\pi$-map $\pi_{E/S}(E)$ for $(E, S)$ is not surjective if and only if $H^1(E, N_{E/X}(E)) = 0$.

(2) Suppose that there exists a commutative diagram

\[
\begin{array}{ccc}
F & \rightarrow & M \\
\downarrow^{2:1} & & \downarrow^{2:1} \\
E & \rightarrow & S \\
\downarrow^{2:1} & & \downarrow^{2:1} \\
& & \pi \\
E & \rightarrow & S & \rightarrow & X,
\end{array}
\]

where $\pi$ is the canonical cover of $X$, $M$ is the $K3$-cover of $S$ in $Y$ and $F$ is the pullback of $E$ in $M$. Then we have $H^1(E, N_{E/X}(E)) = 0$ if $H^1(F, N_{F/Y}(F)) = 0$.

**Proof.** (1) By adjunction, we have $\left[H^1(E, N_{S/X}(E))|^E\right] \simeq H^1(E, -K_X|_E + K_E) = 0$. Thus it follows from (2.8) that $\pi_{E/S}(E)$ is surjective if and only if the map

$\Phi : H^1(E, N_{E/S}(E)) \rightarrow H^1(E, N_{E/X}(E))$

induced by the sheaf homomorphism $[N_{E/S} \hookrightarrow N_{E/X}] \otimes E \mathcal{O}_E(E)$ is an isomorphism. Since $N_{E/S}(E) \simeq \mathcal{O}_E(2E)$ is trivial, we see that $H^1(E, N_{E/S}(E)) \simeq k$. Thus we obtain the first assertion.

(2) Note that $X$ is a quotient scheme $Y/G$ of $Y$ by a finite group scheme $G$ (of order 2). Then $N_{E/X}(E)$ is the $G$-invariant part of $\pi_* N_{F/Y}(F)$. Therefore $N_{E/X}(E)$ is a direct summand of $\pi_* N_{F/Y}(F)$, due to the existence of the Reynolds operator (cf. [22, §2]). Then there exists a natural injection

$H^1(E, N_{E/X}(E)) \hookrightarrow H^1(F, N_{F/Y}(F))$.

Since $\mathcal{O}_F(F) \simeq N_{F/M}$ is trivial, we conclude that $H^1(E, N_{E/X}(E)) = 0$ by assumption. \qed

### 3. Deformations of curves on Enriques-Fano 3-folds

In this section, we prove Theorem 1.2. Let $X$ be an Enriques-Fano 3-fold of genus $g$, $S$ an Enriques surface in $X$ and $C$ a smooth connected curve on $S$ of genus $g(C)$.

**Proof of Theorem 1.2.** We first show a strategy of the proof, which is very similar to that of [18, Theorem 1.2]. By Lemma 2.9, we have $H^1(X, N_{(C,S)/X}) = 0$, which implies that the Hilbert-flag scheme $HF^{sc} X$ of $X$ is nonsingular at $(C, S)$. Moreover, it follows from (2.4) that there exists an exact sequence

\[
\begin{array}{c}
H^0(X, N_{(C,S)/X}) \xrightarrow{p_1} H^0(C, N_{C/X}) \xrightarrow{\delta} H^1(S, N_{S/X}(-C)) \longrightarrow 0,
\end{array}
\]
where \( p_1 \) is the tangent map of the first projection \( p_{r_1} : \text{HF}^{sc} X \to \text{Hilb}^{sc} X, (C', S') \mapsto [C'] \). We define a divisor \( D \) on \( S \) as in the statement. Then since \( N_{S/X}(-C) \simeq \mathcal{O}_S(K_S - D) \), the cokernel of \( p_1 \) is isomorphic to \( H^1(S, D)^\vee \) by Serre duality. Therefore, \( p_1 \) is surjective if \( H^1(S, D) = 0 \) and we have proved Theorem 1.2(1) by virtue of Lemma 2.9. On the other hand, under the settings of (2) and (3) of the theorem, by Lemma 2.2, \( H^1(S, D) \) is of dimension \( m \), and \( 1 \), respectively. Therefore, there exists a global section \( \alpha \) of \( N_{C/X} \) not contained in the image of \( p_1 \). For the proofs of (2) and (3), it suffices to prove that the primary obstruction \( \text{ob}(\alpha) \) is nonzero for such an \( \alpha \). We prove this only for (2) and skip the proof of (3), because in the latter case, \( S \) is a nodal Enriques surface and the proof of \( \text{ob}(\alpha) \neq 0 \) is more similar to that of [18, Theorem 1.2 (2)]. We refer to [18, §4] for more details of the proof of (3).

We prove (2). Suppose that there exist a half pencil \( E \) on \( S \) and an integer \( m \geq 1 \) such that \( D \sim 2mE \) or \( D \sim K_S + (2m + 1)E \). Then by Lemma 2.2, we see that \( h^1(S, D) = m \) and \( h^1(S, D - E) = m - 1 \). Since \( E \) is effective, there exists a natural map \( H^1(S, N_{S/X}(-C)) \to H^1(S, N_{S/X}(E - C)) \), where \( N_{S/X}(-C) \simeq \mathcal{O}_S(K_S - D) \) in \( \text{Pic} \, S \). Then by dimension, there exists a nonzero element \( \gamma \) in the kernel of this map, i.e. we have \( r(\gamma, E) = 0 \), using the same notation in (2.6). Let \( \delta \) be the coboundary map of (2.4). Then it follows from (3.1) that there exists a global section \( \alpha \) of \( N_{C/X} \) such that \( \delta(\alpha) = \gamma \). Let \( \pi_{C/S}(\alpha) \) be the exterior component of \( \alpha \) (cf. §2.3). We see that \( \delta \) factors through the coboundary map of the short exact sequence

\[
\begin{align*}
[0 & \longrightarrow \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \xrightarrow{|c|} \mathcal{O}_C \longrightarrow 0] \otimes N_{S/X} \\
\end{align*}
\]

on \( S \) (cf. [19, §2.2]). Thus we obtain that \( \gamma = \pi_{C/S}(\alpha) \cup k_C \) for the extension class \( k_C \in \text{Ext}^1(\mathcal{O}_C, \mathcal{O}_S(-C)) \) of (3.2). Since the reduction \( r(\ast, E) \) and the cup product map \( \cup k_C \) are compatible, we have

\[
\begin{align*}
\gamma = \pi_{C/S}(\alpha) \cup k_C = r(\pi_{C/S}(\alpha) \cup k_C, E) = r(\gamma, E) = 0. \\
\end{align*}
\]

Then it follows from the exact sequence (3.2) tensored with \( \mathcal{O}_S(E) \) that there exists an element \( \beta \) in \( H^0(S, N_{S/X}(E)) \) (i.e. an infinitesimal deformation of \( S \) with a pole along \( E \)) such that \( \beta\big|_C = r(\pi_{C/S}(\alpha), E) \) in \( H^0(C, N_{S/X}(E)) \). It is easy to check that all the conditions (from (i) to (vi)) of Theorem 2.11 are satisfied. In fact, (i) is clear (cf. §2.1). Since \( \Delta = C + K_X \big|_S - 2E \), we have \( \Delta \sim (2m - 2)E \) or \( \Delta \sim K_S + (2m - 1)E \) and hence we obtain (ii). Since we have \( \pi_{C/S}(\alpha) \cup k_C = \gamma \neq 0 \) in \( H^1(S, N_{S/X}(-C)) \), (iii) is a consequence of [18, Lemma 3.1]. (iv) follows from \( E^2 = 0 \) and \( g(E) = 1 \). Since \( H^1(E, N_{E/X}(E)) = 0 \), (v) follows from Lemma 2.13. Since \( -K_X \) is ample, so is \( C - E = -K_X \big|_S + E + \Delta \) and hence we obtain (vi). Thus we have proved (2) of Theorem 1.2.

Finally we prove the last statement, which is concerned with the dimension of \( \text{Hilb}^{sc} X \). Let \( \mathcal{O}_{X,x} \) denote the local ring of a scheme \( X \) at a point \( x \). We note that \( H^0(S, N_{S/X}(-C)) \simeq \)
Then it follows from [19, Theorem 2.4] that there exist inequalities
\[ \dim \mathcal{O}_{\text{HF}^{sc} X,(C,S)} \leq \dim \mathcal{O}_{\text{Hilb}^{sc} X,(C)} \leq \dim \mathcal{O}_{\text{HF}^{sc} X,(C,S)} + h^1(S, D), \]
and the inequality to the right is strict if and only if $C$ is obstructed in $X$. By assumption, we have $h^1(S, D) \leq 1$ and $C$ is obstructed in $X$ if $h^1(S, D) = 1$. Thus we have $\dim \mathcal{O}_{\text{Hilb}^{sc} X,(C)} = \dim \mathcal{O}_{\text{HF}^{sc} X,(C,S)} = g + g(C) - 1$ by Remark 2.10.

**Remark 3.1.** Let $D := C + K_X|_S$. If $D \sim (2m + 1)E$ or $D \sim K_S + (2m + 2)E$ for $m \geq 1$, then we still have $H^1(S, D) \neq 0$. However, Theorem 2.11 does not apply to $C$ in this case. In fact, in the proof of Theorem 2.11, the nonzero of $\text{ob}(\alpha)$ is reduced to that of the cup product $\partial_E(\beta|_E) \cup \beta|_E$ in $H^1(E, N_S/X(3E - C)|_E)$ (cf. Remark 2.12). We see that this cohomology group is zero in the above case, because $N_S/X(3E - C) \simeq \mathcal{O}_S(K_S - D - 3E)$ and we have $H^1(E, \mathcal{O}_E(nE)) = 0$ for odd $n$ and $H^1(E, \mathcal{O}_E(K_S + nE)) = 0$ for even $n$.

## 4. Non-reduced components of the Hilbert scheme

In this section, we prove Theorem 1.1. We also give some examples of Enriques-Fano 3-folds satisfying the assumption of the theorem (cf. Example 4.2). In our examples, every Enriques-Fano 3-fold $X$ has only terminal cyclic quotient singularities and there exist a smooth Fano 3-fold $Y$ that double covers $X$. We also prove that $\text{Hilb}^{sc} Y$ also contains a generically non-reduced component and compare its properties (e.g. the dimension of the component) with that of $\text{Hilb}^{sc} X$ (cf. Remark 4.5). In what follows, we fix an Enriques-Fano 3-fold $X$ of genus $g$ and a smooth hyperplane section $S$ of $X$ and consider (a family of) curves on $X$ (or $S$).

**Proof of Theorem 1.1.** We consider a complete linear system
\[ \Lambda := |-K_X|_S + 2E| \]
of divisors on $S$ for the half pencil $E$ in the theorem. Let $\Phi$ denote the Cossec-Dolgachev function (cf. [7, 6]). Then we have $\Phi(-K_X|_S + 2E) = (-K_X.E) = e \geq 2$, which indicates that $\Lambda$ is base-point-free (cf. [6, Chap. IV. §4]). By Bertini’s theorem, $\Lambda$ contains a smooth connected member $C$, which is a curve on $S$ of genus $g(C) = C^2/2 + 1 = g + 2e$. Since $\mathcal{O}_S(C)$ is ample, we see that $H^1(S, \mathcal{O}_S(C)) = 0$. Then by Lemma 2.9, the Hilbert-flag scheme $\text{HF}^{sc} X$ is nonsingular at $(C, S)$ of expected dimension $\chi(X, N_{(C,S)/X}) = g + g(C) - 1$ (cf. Remark 2.10). Therefore there exists a unique irreducible component $\mathcal{W}$ of $\text{HF}^{sc} X$ passing through $(C, S)$. Let $W$ be its image by the first projection $pr_1 : \text{HF}^{sc} X \to \text{Hilb}^{sc} X$, $(C, S) \mapsto [C]$. Then $W$ is an irreducible closed subset of $\text{Hilb}^{sc} X$. Since $N_{S/X}(-C) \simeq K_S - 2E$, we see that $H^0(S, N_{S/X}(-C)) = 0$. This implies that there exists a (Zariski) open neighborhood $\mathcal{U} \subset \mathcal{W}$ of $(C, S)$, the restriction of $pr_1$ to which is an embedding. Thus we see that $\dim W = \dim \mathcal{W} = g + g(C) - 1$. Moreover, since
\( h^1(S, N_{S/X}(-C)) = 1 \), it follows from (3.1) that
\[
h^0(C, N_{C/X}) = h^0(X, N_{(C,S)/X}) + 1 = g + g(C).
\]

Applying Theorem 1.2 to \( C \), we see that \( C \) is obstructed in \( X \). Moreover, since \( \dim W \) attains the dimension of \( \text{Hilb}^{sc} X \) at \([C]\), \( W \) is an irreducible component of \( \text{(Hilb}^{sc} X)_{\text{red}} \).

Let \( (C', S') \) be a general member of \( W \). Then \( S' \) is a (smooth) Enriques surface. Since \( H^1(S, \mathcal{O}_S) = 0 \), the Picard group of \( S \) does not change under the smooth deformation of \( S \) and hence \( \text{Pic} S' \cong \text{Pic} S \). Since \( H^1(S, \mathcal{O}_S(E)) = 0 \), the half pencil \( E \) is deformed to a half pencil \( E' \) on \( S' \) (of the same degree). Thereby \( C' \) is linearly equivalent to \( -K_X|_{S'} + 2E' \) for some \( E' \). By upper semicontinuity, we have \( H^1(E', N_{E'/X} \otimes_{E'} N_{E'/S'}) = 0 \). Again by Theorem 1.2, \( C' \) is obstructed in \( X \). Thus \( \text{Hilb}^{sc} X \) is generically singular along \( W \). Since \( g + g(C) - 1 = 2g + 2e - 1 \), we have completed the proof. \( \square \)

**Corollary 4.1.** Let \( X \) be an Enriques-Fano 3-fold with only terminal cyclic quotient singularities\(^*\), \( Y \) the smooth Fano cover of \( X \) and \( \theta \) an involution of \( Y \) such that \( X \cong Y/\theta \). If there exist

(1) a \( \theta \)-invariant K3 surface \( M \subset Y \) not passing through the fixed points of \( \theta \), and

(2) an elliptic fibration on \( M \) with an \( \theta \)-anti-invariant elliptic parameter \( u \) such that

\[ H^1(F, N_{F/Y}) = 0 \]

for \( F = (u = 0) \),

and if moreover \( f := (-K_Y.F) \geq 4 \), then \( \text{Hilb}^{sc} X \) contains a generically non-reduced component of dimension \( 2g + f - 1 \).

**Proof.** Let \( S \subset X \) be the Enriques quotient of \( M \) (by \( \theta_M := \theta |_M \)), that is a smooth hyperplane section of \( X \). Then by Lemma 2.1, the image \( E \) of \( F \) in \( S \) is a half-pencil. We see that \( e = (-K_X.E) = f/2 \geq 2 \) and \( H^1(E, N_{E/X}(E)) = 0 \) by Lemma 2.13. Thus the corollary follows from Theorem 1.1. \( \square \)

**Example 4.2.** Suppose that \( X \) is one of the Enriques-Fano 3-folds \( X_g \) of genus \( g = 13, 9, 6 \) in Example 2.5. Then by Lemma 2.6, there exist a K3 surface \( M \subset Y \) and an elliptic fibration on \( M \) with \( \theta \)-invariant fiber \( F \subset M \) satisfying the assumption of Corollary 4.1. Moreover \( F \) is a complete intersection in \( Y \) of degree \( f = (-K_Y.F) = 12, 8, 6 \) for \( X = X_{13}, X_9, X_6 \), respectively. Therefore \( \text{Hilb}^{sc} X \) contains a generically non-reduced component \( W \) with the following properties:

(1) every general member \( C \) of \( W \) is contained in an Enriques surface \( S \cong Q -K_X \),

(2) \( C \sim -K_X|_S + 2E \) in \( \text{Pic} S \), where \( E \) is a half pencil on \( S \), and

(3) \( h^0(C, N_{C/X}) = \dim W + 1 \) and \( \dim W = 37 \) if \( X = X_{13} \), \( \dim W = 25 \) if \( X = X_9 \) and \( \dim W = 17 \) if \( X = X_6 \).

**Remark 4.3.** In Example 4.2 every general member \( C \) of \( W \) is contained in a general hyperplane section \( S \) of \( X \subset \mathbb{P}^g \) by Remark 2.7.

\( \ast \)i.e. \( X \) is one of the 3-folds listed in Table 1
It may be worthwhile to note that if a smooth Fano 3-fold $Y$ contains a $K3$ surface and an elliptic curve on the surface, then under some extra assumptions, $\text{Hilb}^{sc} Y$ also contains a generically non-reduced component.

**Proposition 4.4** (cf. [18, 19]). Let $Y$ be a smooth Fano 3-fold anti-canonically embedded and $M$ a smooth $K3$ surface in $Y$. If there exists an elliptic curve $F$ on $M$ such that

$$H^1(F, N_{F/Y}) = H^1(M, \mathcal{O}_M(F + K_Y|_M)) = 0,$$

then $\text{Hilb}^{sc} Y$ contains a generically non-reduced component $V$ such that for its general member $C$, we have

1. $C$ is contained in a smooth $K3$ surface $M' \subset | - K_Y|$, 
2. $C \sim -K_Y|_M + 2F'$ for some elliptic curve $F'$ on $M$, and 
3. $\dim V = (-K_Y)^3 + 2f + 2$ and $h^0(C, N_{C/Y}) = \dim V + 1$, where $f$ denotes the degree $(-K_Y \cdot F)$ of $F$.

**Proof.** By adjunction and Serre duality, we see that

$$H^1(M, N_{M/Y}(-F)) \simeq H^1(M, \mathcal{O}_M(-K_Y|_M - F)) = 0.$$

It follows from (2.4) that $H^1(Y, N_{(F,M)/Y}) = 0$. We consider a complete linear system $\Lambda := |- K_Y|_M + 2F|$ on $M$. Since $\Lambda$ is base-point-free, there exists a smooth connected curve $C$ on $S$, whose genus is computed as $g(C) = C^2/2 + 1 = (-K_Y)^3/2 + 2f + 1$. Since $H^1(M, F) = 0$, by virtue of [19, Lemma 2.12] \footnote{There is a typo in this lemma. The assumption $\text{char} k \neq 0$ is wrong and $\text{char} k = 0$ is correct.}, we deduce from $H^1(Y, N_{(F,M)/Y}) = 0$ that $H^1(Y, N_{(C,M)/Y}) = 0$. This implies that there exists a first order deformation $\tilde{M}$ of $M$ in $Y$, to which $C$ does not lift by [19, Lemma 2.8]. Then [18, Theorem 1.2 and Corollary 1.3] show that $C$ is obstructed in $Y$ and moreover, there exists a generically non-reduced component $V$ of $\text{Hilb}^{sc} Y$ passing through $[C]$. Since $N_{M/Y}(-C) \simeq -2F$, we have $h^0(M, N_{M/Y}(-C)) = 0$ and then $\dim V$ is equal to

$$h^0(C, N_{(C,M)/Y}) = (-K_Y)^3/2 + g(C) + 1 = (-K_Y)^3 + 2f + 2,$$

that is the expected dimension of the Hilbert-flag scheme $\text{HF}^{sc} Y$ at $(C, M)$. It follows from $h^1(M, N_{M/Y}(-C)) = 1$ and (2.4) that $h^0(C, N_{C/Y}) = h^0(Y, N_{(C,M)/Y}) + 1$. Thus the proposition has been proved. \hfill $\Box$

**Remark 4.5.** One can compare Theorem 1.1 with Proposition 4.4. It might be also interesting to note that in Example 4.2 the $K3$ surface $M$ and the elliptic curve $F$ satisfy the assumption of Proposition 4.4 (cf. Remark 2.8). Therefore the Hilbert scheme $\text{Hilb}^{sc} Y$ of the smooth Fano cover $Y$ of $X$ contains a generically non-reduced component $V$. Moreover, we have

$$\dim V = (-K_Y)^3 + 2f + 2 = 2((-K_X)^3 + 2e + 1) = 2 \dim W,$$

where $W$ is the non-reduced component of $\text{Hilb}^{sc} X$ in Example 4.2.
For every example $X$ of the Enriques-Fano 3-folds in Example 4.2, the smooth Fano cover $\pi : Y \to X$ is étale in a neighborhood of $C$. Thus this example gives us some affirmative evidence to the following question.

**Question 4.6.** Let $\pi : Y \to X$ be a finite covering of a projective scheme $X$, $C$ a smooth curve on $X$. Suppose that $\pi$ is étale in a neighborhood of $C$. Then is $\pi^{-1}(C)$ obstructed in $Y$ if so is $C$ in $X$?

We remark that there is no morphism $\text{Hilb} Y \to \text{Hilb} X$ in general.

**References**

[1] W. P. Barth, K. Hulek, C. A. M. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, second edition, 2004.

[2] L. Bayle. Classification des variétés complexes projectives de dimension trois dont une section hyperplane générale est une surface d’Enriques. *J. Reine Angew. Math.*, 449:9–63, 1994.

[3] P. Cayley. A Memoir on Quartic Surfaces. *Proc. Lond. Math. Soc.*, 3:19–69, 1869/71.

[4] A. Conte. Two examples of algebraic threefolds whose hyperplane sections are Enriques surfaces. In *Algebraic geometry—open problems (Ravello, 1982)*, volume 997 of Lecture Notes in Math., pages 124–130. Springer, Berlin, 1983.

[5] A. Conte and J. P. Murre. Algebraic varieties of dimension three whose hyperplane sections are Enriques surfaces. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 12(1):43–80, 1985.

[6] F. R. Cossec and I. V. Dolgachev. *Enriques surfaces. I*, volume 76 of Progress in Mathematics. Birkhäuser Boston, Inc., Boston, MA, 1989.

[7] I. V. Dolgachev. A brief introduction to Enriques surfaces. In *Development of moduli theory—Kyoto 2013*, volume 69 of Adv. Stud. Pure Math., pages 1–32. Math. Soc. Japan, [Tokyo], 2016.

[8] G. Fano. Sulle varietà algebriche a tre dimensioni le cui sezioni iperplane sono superficie di genere zero e bigenere uno. Mem. Mat. Sci. Fis. Natur. Soc. Ital. Sci., III. Ser. 24, 41-66 (1938)., 1938.

[9] J. O. Kleppe. The Hilbert-flag scheme, its properties and its connection with the Hilbert scheme. Applications to curves in 3-space. Preprint (part of thesis), Univ. of Oslo, 1981.

[10] J. O. Kleppe. Nonreduced components of the Hilbert scheme of smooth space curves. In *Space curves (Rocca di Papa, 1985)*, volume 1266 of Lecture Notes in Math., pages 181–207. Springer, Berlin, 1987.

[11] A. L. Knutsen and A. F. Lopez. A sharp vanishing theorem for line bundles on $K3$ or Enriques surfaces. *Proc. Amer. Math. Soc.*, 135(11):3495–3498, 2007.

[12] A. L. Knutsen, A. F. Lopez, and R. Muñoz. On the extendability of projective surfaces and a genus bound for Enriques-Fano threefolds. *J. Differential Geom.*, 88(3):485–518, 2011.

[13] M. Kuwata and T. Shioda. Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface. In *Algebraic geometry in East Asia—Hanoi 2005*, volume 50 of Adv. Stud. Pure Math., pages 177–215. Math. Soc. Japan, Tokyo, 2008.

[14] T. Miñagawa. Deformations of $\mathbb{Q}$-Calabi-Yau 3-folds and $\mathbb{Q}$-Fano 3-folds of Fano index 1. *J. Math. Sci. Univ. Tokyo*, 6(2):397–414, 1999.

[15] S. Mukai and H. Nasu. Obstructions to deforming curves on a 3-fold. I. A generalization of Mumford’s example and an application to Hom schemes. *J. Algebraic Geom.*, 18(4):691–709, 2009.

[16] D. Mumford. Further pathologies in algebraic geometry. *Amer. J. Math.*, 84:642–648, 1962.
[17] H. Nasu. Obstructions to deforming curves on a 3-fold, II: Deformations of degenerate curves on a del Pezzo 3-fold. *Ann. Inst. Fourier (Grenoble)*, 60(4):1289–1316, 2010.

[18] H. Nasu. Obstructions to deforming curves on a 3-fold, III: Deformations of curves lying on a $K3$ surface. *Internat. J. Math.*, 28(13):1750099, 30, 2017.

[19] H. Nasu. Obstructions to deforming curves on a prime Fano 3-fold. *Math. Nachr.*, 292(8):1777–1790, 2019.

[20] Y. G. Prokhorov. On Fano-Enriques varieties. *Mat. Sb.*, 198(4):117–134, 2007.

[21] T. Sano. On classifications of non-Gorenstein $Q$-Fano 3-folds of Fano index 1. *J. Math. Soc. Japan*, 47(2):369–380, 1995.

[22] M. Schlessinger. Rigidity of quotient singularities. *Invent. Math.*, 14:17–26, 1971.

[23] E. Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

**DEPARTMENT OF MATHEMATICAL SCIENCES, TOKAI UNIVERSITY, 4-1-1 KITAKANAME, HIRATSUKA, KANAGAWA 259-1292, JAPAN**

*Email address: nasu@tokai-u.jp*