Brachistochrone of a Spherical Uniform Mass Distribution

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We solve the brachistochrone problem for a particle traveling through a spherical mass distribution of uniform density. We examine the connection between this problem and the popular “gravity elevator” result. The solution is compared to the well known brachistochrone problem of a particle in a uniform gravitational field.

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I. INTRODUCTION

A well known result quoted in popular fiction is that of the “gravity elevator.” The result consists of the fact that an object released through a hole through the center of the Earth undergoes simple harmonic motion with a period slightly under 1.5 hours. It is straightforward to show that this result holds true for any chord through the Earth; thus, one may consider a “gravity elevator” which shuttles between any two points on the Earth’s surface in approximately 42 minutes. Such a path is not the path of minimum time, as can be seen by objects undergoing small oscillations at the surface of the Earth. We consider, then, the following problem: determine the path of minimum time (brachistochrone) between two points on the surface of a spherical, uniformly distributed mass; i.e. the “fastest gravity elevator.”

The outline of this paper is as follows. We initially show a simple derivation for the gravity elevator result. We then review the solution of the brachistochrone problem in a uniform gravitational field using the calculus of variations. Finally, we extend this approach to solve for the spherical problem. Finally, we show the two solutions agree in the limit of small path length to mass radius ratio.

II. PREVIOUS RESULTS

In this section we reproduce the gravity elevator result for an arbitrary chord. We set the zero of the gravitational potential at the surface of the Earth (r=R).

Referring to Fig. 1 we note the following relationship between the latitude, \( \lambda \), and polar angle, \( \theta \):

\[
r(t) \cos(\theta) = R \sin(\lambda)
\]  

(1)

Gauss’ Law, when applied to a spherical mass distribution yields the radial dependent force as

\[
F[r(t)] = -\frac{gmr}{R} \hat{r},
\]  

(2)

where \( g \) is the gravitational acceleration, \( g = \frac{GM}{R^2} \).

Letting \( x = r \sin(\theta) \), applying Newton’s second law along the chord, and using Eq. (1), we find the following equation of motion:

\[
\ddot{x} + \frac{g}{R} x = 0,
\]  

(3)

which has angular frequency \( \sqrt{\frac{g}{R}} \), independent of the latitude, \( \lambda \).

III. THE BRACHISTOCHRONE PROBLEM

In this section we outline the solution to the Brachistochrone problem, commonly presented in an advanced undergraduate course in classical mechanics. We choose to present this review because it serves as a limit of the more complicated Brachistochrone problem solved below.

The problem may be stated as follows: given two points in a uniform gravitational field, find the equation for the path of least time between the two points.

IV. THE SPHERICAL BRACHISTOCHRONE PROBLEM

The kinetic energy is given by

\[
T = \frac{1}{2} m v^2
\]  

(4)
The potential energy is given by

\[ V = -\frac{GMm}{2R^2} \left[ R^2 - r^2 \right], \quad (5) \]

where the zero of the potential is chosen at the surface, \( r = R \).

We assume the particle begins at the surface with zero velocity with initially zero total energy. In this case, we have the following expression for velocity:

\[ v = \sqrt{\frac{GM}{R}} \left[ 1 - \left( \frac{r^2}{R} \right) \right]. \quad (6) \]

In our subsequent analysis we define the following dimensionless quantities:

\[ \tau = \frac{t}{\sqrt{\frac{g}{R}}} \]
\[ \rho = \frac{r}{R} \]
\[ \nu = \frac{v}{\sqrt{gR}}. \quad (7) \]

where \( \sqrt{\frac{g}{R}} \) is the inverse frequency of the gravity elevator obtained in section I.

Using these definitions there results a simple expression for the dimensionless velocity, \( \nu \) of the particle at a given dimensionless radius, \( \rho \):

\[ \nu = \sqrt{1 - \rho^2} \quad (8) \]

A simplification results if the trajectory is parameterized by the independent variable \( r \); that is, \( \theta = \theta(r) \), where \( \theta \) is constrained to be \( \theta_i \leq \theta \leq 0 \) (see Fig. 1). This approach explicitly solves for the initial half of the trajectory with the final half being a symmetric extension of this solution.

Using the dimensionless element for arclength, \( ds = \sqrt{1 + \rho^2 \theta''(\rho)} \), and the dimensionless velocity (\[8\]), the transit time, \( T \), to be minimized is

\[ T = \int d\tau = \int \frac{ds}{v} = \int \frac{\sqrt{1 + \rho^2 \theta''(\rho)} d\rho}{\sqrt{1 - \rho^2}} \quad (9) \]

and the brachistochrone problem reduces to finding the path \( \theta(\rho) \) that minimizes the transit time \( T \) in \( \[9\] \). This path may be found through the calculus of variations.

V. CALCULUS OF VARIATIONS SOLUTION

In this section we determine the path that minimizes the transit time, \( T \) (\[9\]), by using the calculus of variations. The integrand in \( \[9\] \) is a functional of the form \( f(\theta', \theta; \rho) \), where \( \theta' \) and \( \theta \) are functions of the independent variable \( \rho \). Explicitly, \( f \) is

\[ f(\theta', \theta; \rho) = \frac{\sqrt{1 + \rho^2 \theta'(\rho)^2}}{\sqrt{1 - \rho^2}}. \quad (10) \]

Upon minimization, the Euler-Lagrange equation \( \[1\] \) becomes

\[ \frac{d}{d\rho} \frac{\partial f}{\partial \theta'} - \frac{\partial f}{\partial \theta} = 0. \quad (11) \]

Inspecting Eq. \( \[10\] \), \( f \) is independent of \( \theta \) and so \( \frac{\partial f}{\partial \theta} = 0 \). Eq. \( \[11\] \) reduces to \( \frac{\partial f}{\partial \theta'} = \text{const} \), or

\[ \frac{\rho^2 \theta'(\rho)}{\sqrt{1 - \rho^2} \sqrt{1 + \rho^2 \theta'(\rho)^2}} = k. \quad (12) \]

On simplification this results in

\[ \theta' = \frac{\sqrt{1 - \rho^2}}{\rho \sqrt{k^2 + 1 - \rho^2}} \quad (13) \]

The trajectories are parameterized by a given \( k \), \( rho \) ranges from \([\rho_m, 1]\), where \( \rho_m \) is seen to be

\[ \rho_m = \frac{k^2}{k^2 + 1} \quad (14) \]

Using the substitution \( \rho = \sin(\alpha) \), Eq. \( \[13\] \) reduces to

\[ \theta = \int \frac{\cos^2 \alpha d\alpha}{\sin \alpha \sqrt{\frac{k^2 + 1}{k^2} \sin^2 \alpha - 1}} \quad (15) \]

On integration, this expression becomes \( \[2\] \)

\[ \theta(\alpha) = -\tan^{-1} \left[ \frac{\cos \alpha}{\sqrt{\frac{k^2 + 1}{k^2} \sin^2 \alpha - 1}} \right] + k \sin^{-1} \left[ \sqrt{k^2 + 1} \cos \alpha \right]. \quad (16) \]

in terms of \( \rho \), this becomes

\[ \theta(\rho) = -\tan^{-1} \left[ \frac{\sqrt{1 - \rho^2}}{\sqrt{\frac{k^2 + 1}{k^2} \rho^2 - 1}} \right] + k \sin^{-1} \left[ \sqrt{k^2 + 1} \sqrt{1 - \rho^2} \right]. \quad (17) \]

VI. CONCLUSION

We presented a variational approach to the brachistochrone problem of a particle traveling through a spherical mass distribution of uniform density. Earlier it was
shown that such a problem could be solved using Gauss’ Law yielding a period of oscillation of approximately 1 ½ hours. We show how this result may be improved upon using a variational calculus approach and the Euler-Lagrange equations.

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