LOGARITHMICALLY COMPLETE MONOTONICITY
OF RECIPROCAL ARCTAN FUNCTION

VLADIMIR JOVANOVIĆ¹ AND MILANKA TREML¹

Abstract. We prove the conjecture stated in F. Qi and R. Agarwal, On complete monotonicity for several classes of functions related to ratios of gamma functions, J. Inequal. Appl. (2019), 1-42, that the function $1/\arctan$ is logarithmically completely monotonic on $(0, \infty)$, but not a Stieltjes transform.

1. Introduction

By a completely monotonic function (shortly CM) we mean here an infinitely differentiable function $f : (0, \infty) \rightarrow \mathbb{R}$, such that

$$(-1)^n f^{(n)}(x) \geq 0, \quad n = 0, 1, 2, \ldots.$$ 

If $f'$ is completely monotonic and $f \geq 0$, then we call $f$ a Bernstein function. Here we are mostly interested in logarithmically completely monotonic functions, that is, infinitely differentiable functions $f : (0, \infty) \rightarrow (0, \infty)$ with the property

$$(-1)^n (\log f)^{(n)}(x) \geq 0, \quad n = 1, 2, 3, \ldots.$$ 

A basic fact concerning CM - functions is the Bernstein theorem: a function $f$ is CM if and only if there exists a non-decreasing function $\alpha$ on $(0, \infty)$ satisfying

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t),$$

for all $x > 0$ (see [6], p. 161). In some occasions it has been proven a stronger property which leads to complete monotonicity of a function $f$, namely that there exist $a \geq 0$ and a non-negative Borel measure $\mu$ on $[0, \infty)$ for which the equality

$$f(x) = a + \int_0^\infty \frac{d\mu(t)}{x + t}$$

holds for $x > 0$, where the measure $\mu$ fulfills the condition

$$\int_0^\infty \frac{d\mu(t)}{1 + t} < \infty.$$ 

Key words and phrases. Complete monotonicity, Stieltjes transform

2020 Mathematics Subject Classification. Primary: 26A48. Secondary: 30E20.
Such functions are called Stieltjes transforms. We recall that all Stieltjes transforms are logarithmically completely monotonic (see [2]), and the latter are CM (see [5]). In [4] the authors set the conjecture that the function \( f(x) = \frac{1}{\arctan x} \) is logarithmically completely monotonic on \((0, \infty)\), but not a Stieltjes transform. The aim of this paper is to justify these assertions. We will do it in the next section.

2. Formulations and proofs

**Theorem 2.1.** The function \( f(x) = \frac{1}{\arctan x} \) is logarithmically completely monotonic on \((0, \infty)\).

The idea of the proof of Theorem 2.1 is based on the Remark (1) in [4], where the authors suggest employing the residue theorem in an attempt to obtain integral representations of functions under consideration.

**Proof.** It suffices to prove that

\[
g(x) = -(\log f(x))' = \frac{1}{(x^2 + 1) \arctan x}
\]

is CM on \((0, \infty)\). In what follows we always assume that log denotes the principle value of logarithm, i. e. \( \log z = \ln |z| + i \arg z \), with \( \arg z \in (-\pi, \pi] \).
LOGARITHMICALLY COMPLETE MONOTONICITY OF RECIPROCAL ARCTAN FUNCTION

Let us consider the integral \( \int_{\Gamma_{R,r}} G(z) \, dz \), over the "keyhole" contour \( \Gamma_{R,r} \) given in Figure \( 1 \), where

\[
G(z) = \frac{z + 1}{z(z - z_0) \log z}
\]

and \( z_0 = \frac{i-x}{ix} \), for \( x > 0 \).

We assume \( R > 1 \) and \( r < 1 \). Note that \( |z_0| = 1 \) and that 1, \( z_0 \) are the only singularities of \( G \) lying inside \( \Gamma_{R,r} \). From the residue theorem, we have

\[
\int_{\Gamma_{R,r}} G(z) \, dz = 2\pi i (\text{Res}(G(z); z_0) + \text{Res}(G(z); 1)).
\]

Since \( z_0 \) is a first-order pole, it follows

\[
\text{Res}(G(z); z_0) = \frac{1 + z_0}{z_0 \log z_0} = \frac{1 + \frac{i-x}{x}}{\frac{i-x}{x} \log \frac{i-x}{x}} = \frac{2i}{(i-x)2i \arctan x} = \frac{i+x}{(x^2+1) \arctan x},
\]

where we used the fact that \( \arctan x = \frac{1}{2i} \log \frac{1+ix}{1-ix} \), for \( x > 0 \). Similarly,

\[
\text{Res}(G(z); 1) = \lim_{z \to 1} (z-1) \frac{1+z}{z \log (z-z_0)} = \frac{2}{1-z_0} = \frac{2}{1-i-x} = \frac{i+x}{x},
\]

whence,

(2.1) \[ g(x) = \frac{1}{x} - \frac{1}{2\pi i (x+i)} \int_{\Gamma_{R,r}} G(z) \, dz. \]

Now, it remains to calculate the integral \( \int_{\Gamma_{R,r}} G(z) \, dz \). In order to accomplish it, we start from the relation

(2.2) \[ \int_{\Gamma_{R,r}} G(z) \, dz = \int_{\Gamma_R} G(z) \, dz + \int_{\Gamma_r} G(z) \, dz + \int_{\Gamma_{R,r}^+} G(z) \, dz + \int_{\Gamma_{R,r}^-} G(z) \, dz. \]

The first two integrals vanish as \( R \to \infty \) and \( r \to 0^+ \). It follows from the estimates

\[
\left| \int_{\Gamma_R} G(z) \, dz \right| \leq 2R\pi \max_{|z|=R} \frac{|z + 1|}{|z| \log |z - z_0|} \leq 2\pi \frac{R + 1}{(\ln R - 2\pi)(R - 1)}
\]

and

\[
\left| \int_{\Gamma_r} G(z) \, dz \right| \leq 2r\pi \max_{|z|=r} \frac{|z + 1|}{|z| \log |z - z_0|} \leq 2\pi \frac{1 + r}{(-\ln r - 2\pi)(1 - r)}.
\]

We also have for \( t < 0 \),

\[
\lim_{x \to 0^+} G(z) = \frac{t + 1}{t(\ln(-t) + \pi i)(t - x_0)} = G^+(t)
\]
and

\[
\lim_{z \to t+} G(z) = \frac{t + 1}{t(\ln(-t) - \pi i)(t - x_0)} = G^-(t).
\]

Consequently,

\[
\int_{\Gamma_{R,r}^+} G(z)\,dz + \int_{\Gamma_{R,r}^-} G(z)\,dz = \int_{-R}^{-r} [G^-(t) - G^+(t)]\,dt
\]

Let us denote

\[
I = \lim_{R \to \infty} \lim_{r \to 0^+} \int_{\Gamma_{R,r}} G(z)\,dz.
\]

From (2.2) and (2.3) we obtain

\[
I = \int_0^\infty \frac{-2\pi i(t + 1)}{t(\log^2(-t) + \pi^2)(t - z_0)}\,dt
\]

Using \(z_0 = \frac{1}{1+i}x\), we have

\[
I = \int_0^\infty \frac{-2\pi i(1 - t)}{t(\log^2 t + \pi^2)(t + \frac{1}{1+i}x)}\,dt
\]

Note that (2.1) implies

\[
g(x) = \frac{1}{x} - \frac{1}{2\pi i(x + i)I}
\]

and since \(\frac{1}{2\pi i(x + i)I}\) is real, we conclude that

\[
\int_0^\infty \frac{(1 - t^2)}{t(x^2(1 - t)^2 + (1 + t)^2)(\log^2 t + \pi^2)}\,dt = 0.
\]

Therefore, from (2.4), it follows

\[
g(x) = \frac{1}{x} - \int_0^\infty \frac{(1 - t^2)x}{t(x^2(1 - t)^2 + (1 + t)^2)(\log^2 t + \pi^2)}\,dt
\]

Employing

\[
\frac{1}{x} = \int_0^\infty \frac{dt}{xt(\log^2 t + \pi^2)},
\]

we get

\[
g(x) = \int_0^\infty \frac{(1 + t)^2}{xt(x^2(1 - t)^2 + (1 + t)^2)(\log^2 t + \pi^2)}\,dt.
\]
The substitution $t \mapsto \frac{1}{t}$ implies
\[
\int_0^1 \frac{(1 + t)^2}{xt(x^2(1-t)^2 + (1+t)^2)(\log^2 t + \pi^2)} \, dt = \int_1^\infty \frac{(1 + t)^2}{xt(x^2(1-t)^2 + (1+t)^2)(\log^2 t + \pi^2)} \, dt.
\]
Hence,
\[
(2.6) \quad g(x) = 2 \int_0^1 \frac{(1 + t)^2}{xt(x^2(1-t)^2 + (1+t)^2)(\log^2 t + \pi^2)} \, dt,
\]
For $a, b, x > 0$, it is
\[
\frac{1}{x(a^2x^2 + b^2)} = \frac{1}{b^2} \left( \frac{1}{x} - \frac{1}{2} \left( \frac{1}{x + \frac{b}{a}} + \frac{1}{x - \frac{b}{a}} \right) \right),
\]
and using
\[
\frac{1}{x + \frac{b}{a}} = \int_0^\infty e^{-xs} e^{-\frac{bs}{a}} \, ds, \quad \frac{1}{x - \frac{b}{a}} = \int_0^\infty e^{-xs} e^{\frac{bs}{a}} \, ds
\]
and
\[
\frac{1}{x} = \int_0^\infty e^{-xs} \, ds,
\]
one obtains
\[
\frac{1}{x(a^2x^2 + b^2)} = \int_0^\infty e^{-xs} \left( 1 - \cos \left( \frac{bs}{a} \right) \right) \, ds.
\]
Setting $a = 1 - t$ and $b = 1 + t$ yields
\[
\frac{1}{x(x^2(1-t)^2 + (1+t)^2)} = \frac{1}{(1+t)^2} \int_0^\infty e^{-xs} \left( 1 - \cos \frac{1 + t}{1 - t} s \right) \, ds.
\]
From (2.6), we have
\[
g(x) = 2 \int_0^1 \left( \int_0^\infty e^{-xs} \left( 1 - \cos \frac{1 + t}{1 - t} s \right) \, ds \right) \, dt,
\]
and, finally, after interchanging integration order, we obtain
\[
(2.7) \quad g(x) = \int_0^\infty \left( \int_0^1 2(1 - \cos \frac{1 + t}{1 - t} s) \, dt \right) e^{-xs} \, ds.
\]
Now, it is evident that (2.7) implies complete monotonicity of $g$. \hfill \square

**Theorem 2.2.** The function $f(x) = \frac{1}{\arctan x}$ is not a Stieltjes transform on $(0, \infty)$.

For the proof of this theorem, we use the following result on Stieltjes transforms from [3].

**Proposition 2.1.** If $f \neq 0$ is a Stieltjes transform, then $1/f$ is a Bernstein function.
Proof of Theorem 2.2

The function \( h(x) = \frac{1}{f(x)} = \arctan x \) is not a Bernstein function, since

\[
h^{(3)}(x) = 2 \frac{1 - 3x^2}{(1 + x^2)^3}
\]

changes its sign on \((0, \infty)\). Therefore, according to Proposition 2.1, \( f \) if not a Stieltjes transform. \( \square \)

References

[1] H. Alzer and C. Berg, Some classes of completely monotonic functions, Annales Academiae Scientiarum Fennicae. Mathematica 27 (2002), 445–460.

[2] C. Berg, Integral representation of some functions related to the gamma function, Mediterr. J. Math. 1 (2004), 433–439.

[3] C. Berg and H. L. Pedersen, A completely monotone function related to the gamma function, Journal of Computational and Applied Mathematics 133 (2001), 219–230.

[4] F. Qi and R. Agarwal, On complete monotonicity for several classes of functions related to ratios of gamma functions, J Inequal Appl (2019), 1–42.

[5] F. Qi and C.-P. Chen, A complete monotonicity property of the gamma function, J. Math. Anal. Appl. 296 (2004), 603–607.

[6] D. V. Widder, The Laplace transform, Princeton University Press, 1946.

1Faculty of Sciences and Mathematics, University of Banja Luka, Mladena Stojanovića 2, Banja Luka, Republic of Srpska, Bosnia and Herzegovina

Email address: vladimir.jovanovic@pmf.unibl.org

Email address: milanka.treml@pmf.unibl.org