Design of composite members with curvilinear fiber trajectories

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Abstract. The design possibility of shaped constant area composite tensile test specimen and leaf spring is shown and the last can provide by chosen geometry threefold mass reduction for the given level of stored elastic energy. The use of unidirectional GFRP (glass fiber reinforced plastic) makes it possible to reduce the mass by approximately 15 times in compare with the steel analog for the same strength and stored energy requirements. The curvilinear trajectories of fibers tracing around the hole or rivet/bolt connection make it possible to increase load carry capacity many times.

1. Introduction
The emergence of polymer fibrous composites gave a new impetus to the study of equal strength composite elements [1-4] with curvilinear trajectories of fibers, because for the manufacture of profiled composite plates, unlike steel, does not require large additional energy costs. Pultrusion and pulforming allow directly, without machining, to obtain plates with any changes of cross sections dimensions, but with their sectional area saving, i.e. conserving of the uncut fibers number in each section. However, in the equal-strength constant area beam the fibers are misorientated, which must be taken into account. Even in Leonardo notes, it was found that the total cross-section of branches is conserved across branching nodes. In composites application, this means the cut fibers absence, that is, the conservation of the fiber number in element with equal strength variation of its cross section. Such a “Leonardo rule” [5, 6] can be applied in the design of equal strength tensile specimens (section 2), or a const-area beam (section 3), or a ring spring (section 4) [7-10]. Especially effective is the equally stressed curvilinear fibers tracing “flowing” around a free hole (Section 5).

Calculations of stress distributions in near-connection zones with application of finite elements method (FEM) are carried out with the help of ANSYS software [11]. Values and directions of the main tensile stresses in each element are designed and the algorithm of smoothing of a series of straight lines, connecting the vectors of main tensile stresses in each element is offered. These lines are considered as a trajectory of the fibers and for the material with such reinforcing scheme again the problem on stress distribution and specifications of trajectories of the fibers must be solved numerically. The programs of consecutive optimization of the form and structure of reinforcing...
composite plates with holes will be developed. To solve this problem analytically — it is necessary to create new mathematics for the media with non-homogeneous and curvilinear anisotropic properties [12-14]. It is too difficult problem, so our results are mainly the attempts to find rather simple engineering approaches for understanding bio-mechanical principles of strong bio-materials and fibrous structures design.

2. Constant area unidirectional FRP specimen for tensile test

The strength along the fibers of unidirectional reinforced plastic is the most important characteristic, but it is the hardest of all to correctly identify. The main problem is how to fasten the specimen without stress concentration and the grips effect because it strength affects also to prevent the specimen bearing and “cutting” from cross forces in standard self-tightening grips. It is proposed 3D printed constant area specimens, with variable dimensions, but with a constant cross-sectional area. The specimen contour is similar to free flowing liquid jet form (jet-type), therefore, the stresses along the contour remain constant, them similar to surface tension. During the junction from wide gripping part to the narrow working zone transfer, the specimen width decreases, and the thickness increases while the cross-sectional area conservation, which means a constant continuous fibre count. FEM modeling was performed to select a rational circuit shape. A specimens manufacturing technology was developed, and tensile tests showed the possibility of obtaining higher strength values than when testing standard specimens.

![Figure 1. Forms of specimen with a contour corresponding to free flowing liquid jet form (a, b) (jet-type) and the trajectories of the fibers in similar specimens (c).](image)

To create jet-type specimen of composite material, it is necessary to apply curvilinear reinforcement. Figure 1 shows the approximate fibers trajectories in the specimen with a tapered junction from the narrow working part to the winded grip part. As can be seen, the outer fibers are arranged along the free fluid jet shape. At the first stages, a plate specimen is considered, without taking into account the thickening in the constriction area. An algorithm for iterative tracing of continuous rational fiber trajectories has been created, in which, at the first step, the material is modeled as homogeneous, orthotropic with elastic properties corresponding to unidirectional carbon fiber reinforced plastic. In this case, the fiber trajectories are traced along the directions of the greatest tensile (or compressive) principal stresses. The coincidence of the fiber directions and the main stress eliminates shear stresses at the fiber–matrix interface, these stresses are most dangerous for polymer composites. After the trajectories tracing, a repeated iterative analysis of the stress field in a plate reinforced with curvilinear fibers is necessary. In second iteration in contradistinction to the first, it is necessary to take into account not only a new field of fiber directions, but also a change in their local volume fraction.

3. “Equal strength” const-area leaf spring

Let's consider an “equal strength” sheet of the leaf spring, and assume it’s half as a console beams with length $l$. The sizes of this beam cross-sections are changing following the power laws (Figure 2):
The “equal strength” condition means

\[ \frac{6}{w(x)} \frac{P_x}{h^2(x)} = \frac{6}{w(1)} \frac{P_l}{h^2(1)} \Rightarrow x^{\alpha + 2\beta} = \alpha + 2\beta = 1 \]  

(2)

Figure 2. Scheme of the cantilever beam with end section of constant width and thickness.

Deflection-formula for console beam has the form

\[ \nu = \int_0^1 \frac{12 P x^2}{E w(x) h^3(x)} \, dx = \frac{4 P l^3}{E w(1) h(1)^3 \left( 1 - \frac{\alpha}{3} - \beta \right)} = \nu_0 k_v, \]

where \( k_v = (1 - \alpha/3 - \beta)^{-1} \) — shape-factor for deflection; \( E \) — longitudinal Young’s modulus.

Formula for the one sheet mass has the form

\[ m = \rho \int_0^1 w(x) h(x) \, dx = \rho \frac{w(1) h(1)}{1 + \alpha + \beta} = m_{\alpha} k_m, \]

where \( k_m = (1 + \alpha + \beta)^{-1} \) — shape-factor for mass.

Conservation of the cross-section area (const-area) means

\[ w(x) h(x) = w(1) h(1) \rightarrow k_m = 1; \quad \alpha + \beta = 0 \quad \text{— straight line 2 on Figure 3, i.e. from equation (2) } \alpha = -1, \beta = 1 \text{ and for this case } k_v = 3. \]

As a result, after some calculations we can note that optimum-shape GFRP \( (g) \) leaf spring has a mass \( (m) \) many times lower than steel \( (s) \) analogue \((E_s = 210 \text{ GPa}, E_g = 45 \text{ GPa}, \rho_s = 7.8 \cdot 10^3 \text{ kg/m}^3, \rho_g = 2.5 \cdot 10^3 \text{ kg/m}^3, \sigma_s \approx \sigma_g)\):

\[ m = \frac{9 \rho P^2 E}{c \sigma^2}; \quad c = \frac{P}{v}. \]

So, the optimum length-variation of cross-section sizes and shape along the composite leaf spring makes it possible to achieve the mass reduction of spring up to 15 times in comparison with steel analogue, also due to a low elastic modulus \( E \) and low density \( \rho \) of GFRP. For example, an optimum leaf spring with constant area of the cross-sections provides 3 times rigidity decreasing in comparison with similar rectangular beam, and so it results in a side benefit — a three times mass reduction.
3.1. Fiber trajectories in constant area beam
The change in the dimensions of the “ideal” const-area beam cross section is described by power dependences (1). The “ideal” const-area beam means with the cross section dimensions change from edge to edge and thickness tend to the infinity, and width tends to zero on the end of beam. To take into account the influence on the deflection not only of the shape, but also of the misorientation of the fibers, it is necessary to propose some model of fibers distribution.

For an analytical solution, it is convenient to use the distributing principle, when only the direction of the infinitely thin fiber at each point remains. The trajectories are consistent with the shape of the beam (1) and are determined by the initial coordinates in the root section \( y(1), z(1) \):

\[
y(\bar{x}) = y(1)\bar{x}^\alpha; \quad z(\bar{x}) = z(1)\bar{x}^\beta; \quad \bar{x} = x/l; \quad \alpha = -1; \quad \beta = 1.
\]  

Derivatives of functions (6) are equal to the fiber trajectories tangents of the angles \( \varphi_1 \) and \( \varphi_2 \) of the trajectories inclination in the \( xy \) and \( xz \) planes

\[
\frac{dy(\bar{x})}{d\bar{x}} = \frac{y(1)}{\bar{x}} = \varphi_1; \quad \frac{dz(\bar{x})}{d\bar{x}} = -\frac{z(1)}{\bar{x}} = \varphi_2; \quad \bar{y} = \frac{y}{l}; \quad \bar{z} = \frac{z}{l}.
\]

Figure 3. The “equal strength” (1) and “const-area” (2) dependencies of \( \alpha(\beta) \).

Figure 4 shows a scheme for calculating the misorientation angle. The fiber element along axis 1 has projections \( dx, dy, dz \) on the axis \( x, y, z \). The angle \( \varphi \) between the axis 1 and the axis \( x \) is determined from the obvious relationships

\[
\cos \varphi = \frac{1}{\sqrt{1 + A}}; \quad \tan \varphi = \sqrt{A}; \quad A = \left(\frac{dy}{dx}\right)^2 + \left(\frac{dz}{dx}\right)^2.
\]

Having determined the misorientation angle from (8), using the approximate formula (9), we can calculate the local value of the longitudinal modulus of elasticity \( E_x \), being depending on the coordinates \( x, y(x), z(x) \) which connected from (6) with the initial coordinates of each “fiber” \( y(1), z(1) \)

\[
E_x(x, y, z, \varphi) \approx E_1^0 \cos^4 \varphi = \frac{E_1^0}{(1 + A)^2}.
\]

3.2. Calculation of the effective elastic modulus
The effective elastic modulus for the layer, for the section or for the beam as a whole, is meant the value of the modulus for a uniform element with the same dimensions and with the same total stiffness.
For the rod tensile, it is reasonable to assume that the strains distribution over each section is uniform, i.e. the strains for each structural element (“fiber”) are the same. According to this Voigt hypothesis, the effective modulus can be found by averaging

$$E_{\text{eff}} = \frac{E_0}{N} \sum_{i=1}^{N} \cos^4 \phi_i.$$  \hspace{1cm} (10)

Here “\(i\)” is the number of the element in the section, “\(N\)” is the total number of elements (“fibers”). Of course, nothing fundamentally changes if instead of the simplified equation (9), we use the full expression to transform the tensor of elastic modules.

The more complex problem of bending in the engineering approximation can be solved only on the basis of some natural kinematic hypotheses, which are accepted as follows:

- Along the height (along the \(z\) axis) from layer to layer, the deformations vary linearly — the traditional “flat cross-section hypothesis”.
- In each layer element across the width (along the \(y\) axis), the deformations are considered the same, so that the effective modulus of the layer can be determined by the formula (10).
- If the section height is small compared to the width, then only “flat” misorientation can be taken into account and the effective modulus in each layer in the section can be considered the same, and then averaging (10) can be used for the whole section.
- These hypotheses extremely simplify visual analytical and graphical estimates, but they are not required for FEM.

When bending for each section with dimensions \(w(x), h(x)\), it is possible to calculate the effective modulus

$$E_{\text{eff}}(x)w(x)h^3(x) = 12 \int_{-w/2}^{w/2} \int_{-h/2}^{h/2} E_i(x,y)z^2dz. \hspace{1cm} (11)$$

If we neglect the “fibers” misorientation across the height, then for beams with small ratios of height to width & to length, one can estimate the effective modulus by averaging it only across the width, and considering this average value to be constant over the whole section

$$E_{\text{eff}}(x) \approx \frac{1}{w(x)} \int_{-w/2}^{w/2} E_i(x,y)dy. \hspace{1cm} (12)$$

4. The spring of equal strength corrugated leaves

An important conclusion in Section 3 is that equal-strength profiling when loading with end force leads to a 3-fold mass leaf spring reduction. To summarize this conclusion, it is possible, apart from the power laws of the bending moment variation, to consider the moment variation proportional to the polar angle sine (even “weaker” than linear) during longitudinal tensile and compression of the semiring which is a ribbon spring element (Figure 5)

$$M = PR \sin \phi. \hspace{1cm} (13)$$

From conditions of strength \(6PR = \sigma^*w(\pi/2)t^2(\pi/2)\), uniform strength \(6PR \sin \phi = \sigma^*w(\phi)t^2(\phi)\), and semiring rectangular cross-section conservation \(w(\phi)t(\phi) = w(\pi/2)t(\pi/2) \rightarrow t(\phi) = t(\pi/2)\sin \phi\), one can find the dimensions of the most loaded section and the laws of their change according to polar angle:

$$w(\phi) = \frac{w(\pi/2)}{\sin \phi} \rightarrow I(\phi) = w\left(\frac{\pi}{2}\right)t^2\left(\frac{\pi}{2}\right)\sin^2 \phi / 12 = I \sin^2 \phi. \hspace{1cm} (14)$$
The elastic energy accumulated in the semiring with constant cross-section dimensions is calculated by the equation

$$U_o = \frac{1}{2} \int_0^\pi M^2(\varphi) R \, d\varphi = \frac{P^2 R^3}{2EI} \int_0^\pi \sin^2 \varphi \, d\varphi = \frac{\pi P^2 R^3}{4EI}. \quad (15)$$

For a const-area semiring (15), the accumulated energy is twice as large

$$U_* = \frac{P^2 R^3}{2EI} \int_0^\pi d\varphi = \frac{\pi P^2 R^3}{2EI} \quad (16)$$

and, therefore, equal-strength semiring profiling provides (in the extreme case, while conserving the strength and level of accumulated elastic energy), the reducing weight by half. This is less than a threefold mass reduction with a linear growth of the moment, since the sine changes more slowly than the linear dependence. The real weight reduction will be lower because the inability to create an unlimited wide of const area leaf. High strength, low density and low Young's modulus, fiberglass in such type of corrugated ribbons springs (Figure 5 and 6) is able to provide a weight reduction of about 15 times compared with steel analog.

5. Fiber tracing around a hole

One of the most important problem is to design the shape and a system of curvilinear fiber reinforcement in joints of composite structure elements (may be, using the analogy with structure of wood in zones of knots and roots, Figure 7).

Some problems concerning stress concentration near the holes in isotropic and anisotropic plates were solved. The structure of wood near knot was investigated. Using some photo of wood thin
section the corresponding functions were fitted which describe the trajectory of wood fibers around knot. This step makes it possible to formulate the problem on optimum design of curvilinear anisotropic fibrous composite structure for composite plate with a hole or with bolt/pin joint. The algorithms using the finite element method for constructing of fibers trajectories along the directions of the principal tensile stresses around the holes are developed for the future analysis of structures with curvilinear anisotropy of the elastic and strength properties.

6. Conclusions
Curvilinear equal strength reinforcement allows to increase the efficiency of using fibrous composites in elastic elements and in fitting location. Composite leaf and wave ribbon springs can be used as effective elastic elements, and they can accumulate three times greater elastic energy for a fixed mass due to profiling. Low-modulus and high-strength fiberglass reinforced plastic can give a weight reduction of about 15 times compared with steel analog. The optimal reinforcement structure in the holes area for riveted or/and bolted joints can significantly increase the load bearing capacity of the joint.

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