Quantum symmetries in discrete gauge theories

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Abstract

We analyse the fusion, braiding and scattering properties of discrete non-abelian anyons. These occur in (2+1)-dimensional theories where a gauge group $G$ is spontaneously broken down to some discrete subgroup $H$. We identify the quantumnumbers of the electrically and magnetically charged sectors of the remaining discrete gauge theory, and show that on the quantum level the symmetry group $\bar{H}$ is extended to the (quasi-triangular) Hopf algebra $D(\bar{H})$. A conjugacy class paired with a centralizer representation forms an irreducible representation. The fusion rules for arbitrary discrete non-abelian anyons are completely determined by the nontrivial comultiplication (i.e. tensorproduct) of this algebra. It allows for a clear interpretation of Cheshire charge. Also the braid-matrix $R$, which determines the statistical properties and the two particle Aharonov-Bohm scattering, is fixed and satisfies the Yang-Baxter equation. Most of our considerations are relevant for discrete gauge theories in (3+1)-dimensional space time as well.

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Introduction. In two spatial dimensions particles or excitations with (fractional) charge and (fractional) flux may have exceptional spin-statistics properties. These so-called anyons are of particular interest because of their possible relevance in the description of the fractional quantum Hall effect, and their appearance in models for two dimensional superconductivity. So far, most attention has been given to the abelian anyons [1]. In this paper we focus on the non-abelian variety, which appear as excitations in gauge theories, where some continuous non-abelian group $G$ is spontaneously broken down to a finite non-abelian group $H$ by a nonvanishing expectation value of some Higgs field $\Phi$. In $(2+1)$ dimensions this leads to particle-like topological excitations which carry a non-abelian magnetic flux, labelled by elements of the homotopy group $\pi_1(G/H) \simeq \pi_1(\tilde{G}/\tilde{H}) \simeq \bar{\tilde{H}}$, where $\tilde{G}$ is the covering group of $G$ and $\tilde{H}$ the lift of $H$ in $\tilde{G}$. There has been a recent increase of interest in these peculiar objects and many interesting though partial results on their fundamental properties have been obtained [4, 6, 7]. In this paper we propose a general quantum description of these magnetic- and all the other (anyonic) excitations in this discrete $\tilde{H}$ gauge theory. To be explicit, we give a treatment of their fusion, braiding and scattering properties, based on an extension of the residual group $\tilde{H}$ to a new symmetry algebra $D(\tilde{H})$. The results reported here have been obtained by exploiting a striking analogy with certain aspects of string- and conformal field theory (notably orbifold models), and consequently, also with topological field theory. For convenience, we have restricted ourselves to (2+1)-dimensional space time, but it should be stressed that most of our considerations are relevant for discrete gauge theories in (3+1) dimensions as well. In the latter case the magnetic excitations, labelled by $\pi_1(G/H)$, are string-like.

The model. For the sake of concreteness we will restrict our considerations to the case where $G = SO(3)$, but our analysis applies in general. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + (D_\mu \Phi)^* \cdot (D^\mu \Phi) - V(\Phi),$$

where $A^a_\mu (\mu = 0, 1, 2; a = 1, 2, 3)$ is the Yang-Mills potential and $F^a_{\mu\nu}$ the corresponding fieldstrength. We take the covariant derivative as $D_\mu \Phi = (\partial_\mu + ieA^a_\mu T_a) \Phi$, with $T_a$ the (hermitean) generators of $SO(3)$ in the representation of the Higgs field $\Phi$.

Topological flux sectors. With a suitable choice for the representation of $\Phi$ and its invariant potential $V(\Phi)$ one may break $SO(3)$ down to any discrete subgroup $H$, i.e. a cyclic- $\mathbb{Z}_N$, a dihedral- $D_N$ or a pointgroup like $T, O$ or $Y$. The broken phase supports topological excitations with magnetic fluxes labelled by the elements $h$ of the double group $\tilde{H}$. Although regular time independent finite energy solutions can be constructed, we are here only interested in their long range behaviour, which is fixed by minimising the three terms in the energy density seperatedly. From $|F|^2 = 0$ we conclude that $F = 0$, so $A$ becomes locally a pure gauge. Minimising $V(\Phi)$ tells us that $\Phi$ should take values in the
vacuum manifold $\Phi = \Phi_0 \in G/H$. Finally, the condition $D\Phi = 0$ can be integrated along a closed path $\gamma$ in space to yield the condition

$$P \exp \int_{\gamma} A^a_i(s) T_a ds^i = \Gamma(h) \quad h \in H,$$

where $P$ denotes path ordering and $\Gamma$ an $SO(3)$ representation. This condition states that the holonomy is restricted to lie in $H$ \[2\]. Indeed, after taking the limit where the symmetry breaking scale (and hence all masses) go to infinity, one is basically left with (dually charged) pointlike excitations which interact purely topologically through locally flat gauge connections with holonomy restricted to the discrete group $H$. All the statements we will make pertain to the theory in the aforementioned limit, thus effectively describing the long range interactions between topologically charged particles.

Under the residual gauge group $\bar{H}$ the magnetic charges $h$ transform by conjugation

$$h \mapsto bhb^{-1},$$

with $b \in \bar{H}$. This suggests that we should think of the conjugacy classes $C$ as degenerate multiplets. This will indeed turn out to be the case.

**Fusion and braiding of fluxes.** Let us consider the fusion and braiding properties of the magnetic excitations. The classical fusion- or composition rule for two fluxes $h$ and $k$ corresponds to the group multiplication $h \cdot k$, which indicates the importance of the ordering if $\bar{H}$ is non-abelian. Indeed, if we simply braid $h$ and $k$ by interchanging them such that $k$ passes behind $h$ then $k$ suffers from a metamorphosis:

$$|h> |k > \mapsto |hkh^{-1}> |h>,$$

so the ordered product remains invariant \[2, 3\]. This nontrivial braiding property has important consequences on the quantum level. Clearly, if we want to discuss the quantum statistical properties of particles carrying these fluxes, we have to diagonalise the braid-matrix $R$ on the two-particle hilbertspace, thus we have to consider the vectorspace spanned by the group elements, the so-called group algebra $C(\bar{H})$. One expects that already at this level, $R$ may have eigenvalues different from $\pm 1$, meaning that such particles have anyonic properties and exhibit Aharonov-Bohm scattering \[4\].

The group $\bar{H}$ acts by conjugation on $C(\bar{H})$, which is reducible under this action. In fact, even the conjugacy classes generally do not form irreducible representations. It is important to note that there are now two notions of fusion:

- the tensor product of representations in $C(\bar{H})$ (i.e. the ordinary representation ring of $\bar{H}$),
- the class algebra obtained by multiplication of all the elements in the two classes.
Surprisingly enough, these two notions are not compatible (see for instance the discussion after equation (19)), which suggests that either the classical notion of flux composition, or the implementation of the $\hat{H}$ symmetry needs adjustment. This problem is nicely resolved in the generalized framework that we are to propose.

**Electric charges.** First of all, we know that in the magnetic vacuum sector, the electric charges will correspond to representations of the residual symmetry group $H$. If we add fields in representations of the original gauge group $SO(3)$ (or $SU(2)$), then we know how they decompose into a direct sum of irreducible $H$ (or $\hat{H}$) representations. However, in a sector with nontrivial flux one can only implement global transformations belonging to the centralizer $A_N \subset H$ (or $\hat{H}$) of the group element $h$, which labels the magnetic charge $[5, 6, 7]$. The centralizer is obviously a subgroup of $H$ (or $\hat{H}$), and depends only on the class $A_C \subset \hat{H}$ to which $h$ belongs. So in general, we should think of flux/charge sectors labelled as follows:

$$|A_C, \alpha_{\Gamma}>,$$

where $\alpha_{\Gamma}$ labels the $\alpha$-th irreducible representation of the centralizer $A_N$. These sectors are spanned by the states $\{|A g_j, \alpha_{\nu_i}>\}_{j=1,...,\dim A}$, with the basis elements of the representations $\alpha_{\Gamma}$ of $A_N$ denoted by $\alpha_{\nu_i}$. Clearly, the number of sectors is finite.  

We can assign a spin $s$ to the dyonic sectors of the theory (by, for example calculating the conserved angular momentum of the classical solution). The spin factor $w(A, \alpha)$ for

$$w(A, \alpha) = \exp 2\pi is = \alpha(A g_1)/\alpha(e), (6)$$

with $\alpha(A g_1) = \text{Tr} T(A g_1)$. In models in which the conventional spin-statistics connection is realized (such as the abelian case), we have the identity $w(A, \alpha) = \exp i\theta(A, \alpha)$, where $\theta$ denotes the statistical parameter typically equal to the product of the charge $q$ and the flux $\phi$. As we will see later, the relation between spin and statistics is more involved in the non-abelian case.

**The lattice of charge/flux sectors.** To illustrate the situation and facilitate forthcoming discussions, we start with the diagram of admissible electric charges and magnetic fluxes in a compact $U(1)$ theory (Figure 1). Both of them are quantised (say $q = ne$ and $\phi = 2\pi m/e$, obeying the Dirac condition), and the lattice extends indefinitely in both directions. If we consider this $U(1)$ as a subgroup of $SU(2)$, then the charges can be half integral ($q = ne/2$) and consequently the flux should double ($\phi = 4\pi m/e$). However, in two spatial dimensions there are no topological flux sectors in the unbroken $SU(2)$, because its fundamental group is trivial (i.e. $m = m \mod 1$). In the group

\[\begin{align*}
|A_C, \alpha_{\Gamma}> & , \\

Note that the sum of the squares of the dimensions of the representations equals the squared order of $\hat{H}$: $\sum_{A,\alpha} |A_C|^2 |\alpha_{\Gamma}|^2 = \sum_{A} |A_C|^2 |A_N| = |\hat{H}|^2$. As we will see later, this is the order of the extended symmetry algebra.
Figure 1: The charge/flux lattice for $\bar{H} = Z_6$. In $SO(3)$ only the charges denoted by the filled circles occur.

$SO(3) \simeq SU(2)/Z_2$, one has again that $q = ne$ and $\phi = 2\pi m/e$ where now $m = m \mod 2$ so that only one nontrivial magnetic sector remains. If we now break $SO(3)$ down to some discrete group $H$, we can determine the magnitude of the flux $h$ by looking at $\bar{H}$ or better at its centralizer $hN \subset \bar{H}$. In most cases this is some group $Z_{2M}$. It is clear that the charges will be $q = ne/2$ with $n = n \mod 2M$ and $\phi = 4\pi m/2Me$ with $m = m \mod 2M$ for $SU(2)$, or alternatively $q = ne$ with $n = n \mod M$ and $\phi = 2\pi m/Me$ with $m = m \mod 2M$ in $SO(3)$. We see that the charge/flux lattice becomes periodic, but the periodicity in the electric direction depends on the magnitude of the flux. One should note that for a non-abelian group $H$, the reduction of the allowed charges/fluxes to a two-dimensional lattice is of course an oversimplification, if for example more then one particle is in question (as in the case of braiding and fusion). This is one of the reasons to develop some more algebraic apparatus in the following sections.

The symmetry algebra $D(\bar{H})$. Once we have identified the superselection sectors $\bar{H}$ in our discrete gauge theory, we should answer the following questions: How do we implement the $\bar{H}$ symmetry? And consequently, what are the fusion braiding and scattering properties of the spectrum? Specifically, we have to define the action of an arbitrary group element $\bar{h} \in \bar{H}$ in any centralizer representation $\Gamma$, and we need to develop a tensor calculus for excitations $\Gamma$, which involves somehow ‘tensoring’ representations of different centralizers. A similar problem is encountered with holomorphic orbifolds in conformal field theory $\bar{H}$. As was shown by Dijkgraaf, Pasquier and Roche, the solution amounts to an extension of the symmetry group $\bar{H}$ to the quasi-triangular Hopf algebra $D(\bar{H})$. We summarize some of its characteristics $\bar{H}$. For a more general discussion of Hopf algebras in connection with conformal field theory, we refer to $\bar{H}$.
\(D(\bar{H})\) is a finitely generated algebra of order \(|\bar{H}|^2\). A basis is given by \(\{gL_x\}_{g,x \in \bar{H}}\), with multiplication
\[
gL_x \cdot hL_y = \delta_{g,xh^{-1}}gL_{xy}. \tag{7}
\]
Here, \(\delta_{g,h} = 1\) if \(g = h\) and 0 otherwise. As a sidenote, we mention that the subgroup \(\bar{H}\) is generated by the elements
\[
1L_x \equiv \sum_g gL_x. \tag{8}
\]
Note that for fixed \(g\), and \(x, y \in \bar{g}N\), the product (7) reduces to the product in the subgroup \(\bar{g}N\). The full representation theory of \(D(\bar{H})\) is obtained by inducing the representations of these subgroups \([10]\). Let \(\{\bar{A}C\}\) be the set of conjugacy classes of \(\bar{H}\) and introduce a fixed but arbitrary ordering \(\bar{A}C = \{\bar{A}g_1, \bar{A}g_2, \ldots, \bar{A}g_k\}\). Let \(\bar{A}N\) be the centralizer of \(\bar{A}g_1\) and \(\{\bar{A}x_1, \bar{A}x_2, \ldots, \bar{A}x_k\}\) be a set of representatives of the equivalence classes of \(G/\bar{A}N\), such that \(\bar{A}g_i = \bar{A}x_i \bar{A}g_1^{-1}\) and choose \(\bar{A}g_1 = e\). Consider the complex vectorspace \(V_\bar{A}g\) spanned by the basis \(\{|\bar{A}g_j, \bar{\alpha}v_i\rangle\}_{i=1}^{\dim \bar{A}g}\). We denote the basis elements of the unitary irreducible representation \(\bar{\alpha}\Gamma\) of \(\bar{A}N\) by \(\bar{\alpha}v_i\). This vectorspace carries a representation \(\Pi_\bar{A}\) of \(D(\bar{H})\), given by
\[
\Pi_\bar{A}(gL_x)|\bar{A}g_i, \bar{\alpha}v_j\rangle = \delta_{g,x\bar{A}g_i^{-1}}|x\bar{A}g_i x^{-1}, \bar{\alpha}(\bar{A}x_i^{-1} x \bar{A}x_i) \bar{\alpha}v_j\rangle, \tag{9}
\]
with \(\bar{A}x_k\) defined by \(\bar{A}g_k = x\bar{A}g_i x^{-1}\).

The set \(\{\Pi_\bar{A}\}\) provides the complete set of representations \([12]\). Note that this set coincides with the spectrum \([\mathbb{F}]\) of the \(\bar{H}\) discrete gauge theory, as discussed in the previous paragraph. What’s more; \(D(\bar{H})\) gives rise to consistent (and partly anticipated) fusion rules and braid properties of the particles in this spectrum.

**Fusion.** The fusion rules contain information about all possible invariant couplings between the different particles in the spectrum, they are as such relevant in the computation of non-elastic scattering cross-sections. In order to incorporate the action of \(D(\bar{H})\) on these couplings, one formally needs to extend the action of \(D(\bar{H})\) to \(D(\bar{H}) \otimes D(\bar{H})\). This is done by means of the comultiplication \(\Delta\). For the algebra at hand, the comultiplication reads as
\[
\Delta(gL_x) = \sum_{h,k \in G \atop hk = g} hL_x \otimes kL_x. \tag{10}
\]
Since \(\Delta\) is an algebra-morphism, the tensor product of two representations \(\Pi_\alpha^A\) and \(\Pi_{\beta}^B\) is again a representation. This representation is in general not irreducible, but rather gives rise to the decomposition
\[
\Pi_\alpha^A \otimes \Pi_{\beta}^B = N_{\alpha\beta\gamma}^C \Pi_{\gamma}^C, \tag{11}
\]
where $N_{\alpha\beta\gamma}^{AB\eta}$ is the multiplicity of the irreducible representations $\Pi^C_{\eta}$. Relation (11) is called a fusion rule of the algebra $D(\bar{H})$. The fusion algebra is a commutative associative algebra and can therefore be diagonalized by a single matrix, $S$. For $D(\bar{H})$, this matrix takes the following form

$$S_{\alpha\beta}^{AB} = \sum_{A_{gi} \in AC, B_{gj} \in BC} [A_{gi}, B_{gj}] = e \alpha^*(A_{x_i}^{-1} B_{gj} A_{x_i}) \beta^*(B_{x_j}^{-1} A_{gi} B_{x_j}).$$

(12)

This so-called ‘modular’ $S$ matrix plays a profound role in conformal field theory. It contains all the information about the fusion algebra [13]:

$$N_{\alpha\beta\gamma}^{AB\eta} = \sum_{D,\delta} S_{AD}^{\alpha\delta} S_{BD}^{\beta\delta} (S^*)^{CD}_{\gamma\delta}.$$  

(13)

It is related to charge conjugation through $S^2 = C$. In the present context, the $S$ matrix realises a higher-order electric-magnetic duality on the space of states.

**Braiding.** The universal $R$-matrix on $D(\bar{H})$ reads

$$R = \sum_{g \in G} g \mathcal{L}_e \otimes \mathcal{L}_g.$$  

(14)

To obtain its action, we consider the two-particle state $|A_{gi}, \alpha v_j > |B_{gk}, \beta v_l > \in |^AC, \alpha T > \otimes |^BC, \beta T >$. Using (14), we define

$$\mathcal{R}_{\alpha\beta}^{AB} = \sigma \circ (\Pi_{\alpha}^{A} \otimes \Pi_{\beta}^{B})(R),$$  

(15)

where $\sigma$ is the permutation operator. The operator $\mathcal{R}$ implements a (positively oriented) interchange of two particles. Explicitly, on the state $|A_{gi}, \alpha v_j > |B_{gk}, \beta v_l >$ the braid operation reads

$$\mathcal{R}_{\alpha\beta}^{AB} |A_{gi}, \alpha v_j > |B_{gk}, \beta v_l > = |A_{gi} B_{gk} A_{g_i}^{-1}, \beta^*(x_m^{-1} A_{gi} B_{x_k}) \beta v_j > |A_{gi}, \alpha v_j >,$$

(16)

where $B_{xm}$ is defined through $B_{gm} = A_{gi} B_{gk} A_{g_i}^{-1}$. A few remarks are in order at this stage:

- the braiding of two pure magnetic fluxes gives the flux metamorphosis (4),
- encircling a pure electric charge around a pure magnetic flux $A_{gj}$ (a process effectuated by $\mathcal{R}_{\alpha\beta}^{00} \mathcal{R}_{\alpha\beta}^{00}$), boils down to a transformation of the electric charge with $T(A_{gj})$. A result that also coincides with earlier observations [7],
- consistency of the braid operation $\mathcal{R}$ with the extended symmetry $D(\bar{H})$ is assured since $\mathcal{R}$ commutes with the $D(\bar{H})$ that is defined on the tensor product through the comultiplication (10),
• $\mathcal{R}$ satisfies the Yang-Baxter equation $\mathcal{R}_1 \mathcal{R}_2 \mathcal{R}_1 = \mathcal{R}_2 \mathcal{R}_1 \mathcal{R}_2$. Here $\mathcal{R}_1$ acts on the three-particle states as $\mathcal{R} \otimes 1$ and $\mathcal{R}_2$ as $1 \otimes \mathcal{R}$.

• The braid-matrix is related to the modular $S$ matrix: $S = \frac{1}{H} \text{Tr}(\mathcal{R}^2)$.

**Braid statistics.** The $\mathcal{R}$-matrix commutes with the comultiplication $\Delta$, so states which have the same eigenvalues under $\mathcal{R}$ also form representations of $D(H)$, and vice versa. Since in the fusion product of a representation with itself different representations can occur, the braid matrix generically has distinct eigenvalues. According to our earlier observation \[6\] we can associate to each representation of $D(H)$ a spinfactor $w(A, \alpha)$. The phases $m^{AB\gamma}_{\alpha\beta C}$ that appear in the monodromy matrix $\mathcal{R}^2$, depend for two representations ($\Pi^A_{\alpha}$ and $\Pi^B_{\beta}$) on the channel ($\Pi^C_{\gamma}$) allowed for by the fusion rule of these representations. One obtains the following formula \[10\]:

$$m^{AB\gamma}_{\alpha\beta C} = \frac{w(C, \gamma)}{w(A, \alpha)w(B, \beta)}.$$  \hspace{1cm} (17)

The conventional spin-statistics connection \[14\] is retrieved from $m^{A\bar{A}0}_{\alpha\bar{\alpha}e}$, where $(\bar{A}, \bar{\alpha})$ denotes the charge conjugated sector of $(A, \alpha)$.

**Aharonov-Bohm scattering.** The cross sections of elastic two-particle Aharonov-Bohm scattering are completely determined by the monodromy matrix $\mathcal{R}^2$. The explicit relation can be cast in the following form \[15\]

$$\frac{d\sigma}{d\phi} = \frac{1}{2\pi k}\frac{1}{\sin^2(\phi/2)} \left[ 1 - \text{Re} <\psi_{in}|\mathcal{R}^2|\psi_{in}> \right].$$  \hspace{1cm} (18)

Here $|\psi_{in}>$ denotes the incoming two-particle state, $k$ the relative wave vector. Besides the conventional abelian Aharonov-Bohm scattering \[13\] and the non-abelian pure flux scattering \[14\], it can also be verified that this formula governs scattering of pure electric charge $\Gamma$ off pure magnetic flux $\mathcal{A}$. In the latter case, the $H$ multiplet $\Gamma$ indeed decomposes into multiplets of the centralizer $^A N$ of $^A C$, such that different particles in the same centralizer multiplet acquire the same Aharonov-Bohm phase \[7\].

**An example: the quaternion group $\mathcal{H} = \mathcal{D}_2$.** Consider the example of the double dihedral group $\mathcal{D}_2$. This is a group of order 8, with five conjugacy classes, denoted $e, \bar{e}, X_1, X_2$ and $X_3$. Table 1a exhibits the elements of these conjugacy classes together with their centralizers. Table 1b and 1c are the character tables of the occurring centralizers.

| Conjugacy class | Centr. | $D_2$ | $\bar{e}$ | $e$ | $X_1$ | $X_2$ | $X_3$ | $Z_4$ | $\hat{g}$ | $\hat{g}^2$ | $\hat{g}^3$ |
|-----------------|--------|--------|----------|------|--------|--------|--------|--------|---------|---------|---------|
| $e = 1$         | $D_2$  | $T$    | 1        | 1    | 1      | 1      | 1      | 1      | 1       | 1       | 1       |
| $\bar{e} = -1$ | $D_2$  | $T$    | 1        | 1    | 1      | -1     | -1     | -1     | -1      | -1      | -1      |
| $X_1 = \{i\sigma_1, -i\sigma_1\}$ | $Z_4$ | $T$    | 1        | 1    | -1     | 1      | 1      | 1      | 1       | 1       | 1       |
| $X_2 = \{i\sigma_2, -i\sigma_2\}$ | $Z_4$ | $T$    | 1        | 1    | -1     | -1     | 1      | 1      | 1       | 1       | 1       |
| $X_3 = \{i\sigma_3, -i\sigma_3\}$ | $Z_4$ | $T$    | 2        | -2   | 0      | 0      | 0      | 0      | 0       | 0       | 0       |
The representations of $D(\bar{D}_2)$ are labelled as follows

\[
\begin{align*}
1 & \simeq |e, \bar{T}>, \\
\bar{1} & \simeq |\bar{e}, \bar{T}>, \\
\sigma_a^+ & \simeq |X_a, \bar{T}>, \\
\tau_a^+ & \simeq |X_a, \bar{T}>, \\
J_a & \simeq |e, T>, \\
\bar{J}_a & \simeq |\bar{e}, T>, \\
\sigma_a^- & \simeq |X_a, T>, \\
\tau_a^- & \simeq |X_a, T>, \\
\phi & \simeq |e, T>, \\
\bar{\phi} & \simeq |\bar{e}, T>,
\end{align*}
\]

where $a$ runs from 1 to 3. These representations constitute the complete set of inequivalent irreducible representations of $D(\bar{D}_2)$. In particular, we have 22 representations, divided among 8 1-dimensional- and 14 2-dimensional representations. Indeed, this adds up to the dimension of the algebra: $8^1 + 14^2 = 8^2$.

The sets $\{J_a, \phi\}$ and $\{\bar{1}, \sigma_a^+\}$ respectively, form the pure electric- and pure magnetic excitations in our discrete $\bar{D}_2$ gauge theory. The other representations are dyonic excitations.

Using the algebraic machinery of $D(\bar{D}_2)$ we can compute the $S$-matrix (Table 2) and subsequently the fusion rules. We discuss some of these.

|  | 1 | 1 | $J_a$ | $\bar{J}_a$ | $\phi$ | $\bar{\phi}$ | $\sigma_a^+$ | $\sigma_a^-$ | $\tau_a^+$ | $\tau_a^-$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| 1 | 1 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 |
| $J_b$ | 1 | 1 | 1 | 1 | 2 | 2 | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ |
| $J_b$ | 1 | 1 | 1 | 1 | 2 | 2 | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ | $-2\epsilon_{ab}$ | $-2\epsilon_{ab}$ |
| $\phi$ | 2 | -2 | 2 | -2 | 4 | -4 | 0 | 0 | 0 | 0 |
| $\phi$ | 2 | -2 | 2 | -2 | 4 | -4 | 0 | 0 | 0 | 0 |
| $\sigma_b^+$ | 2 | 2 | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ | 0 | 0 | $4\delta_{ab}$ | $-4\delta_{ab}$ | 0 | 0 |
| $\sigma_b^-$ | 2 | 2 | $2\epsilon_{ab}$ | $2\epsilon_{ab}$ | 0 | 0 | $-4\delta_{ab}$ | $4\delta_{ab}$ | 0 | 0 |
| $\tau_b^+$ | 2 | -2 | $2\epsilon_{ab}$ | $-2\epsilon_{ab}$ | 0 | 0 | 0 | 0 | $-4\delta_{ab}$ | $4\delta_{ab}$ |
| $\tau_b^-$ | 2 | -2 | $2\epsilon_{ab}$ | $-2\epsilon_{ab}$ | 0 | 0 | 0 | 0 | $4\delta_{ab}$ | $-4\delta_{ab}$ |

Table 2. The modular $S$-matrix of $\bar{D}_2$, $\epsilon_{ab} = 1$ if $a = b$ and $-1$ otherwise.

The fusion of the pure electric charges is dictated by the $\bar{D}_2$ representation ring

\[
J_a \times J_a = 1, \quad J_a \times J_b = J_c, \quad \phi \times \phi = 1 + \sum J_a,
\]

while amalgamation of electric charges with magnetic fluxes may yield dyonic excitations

\[
J_a \times \sigma_a^+ = \sigma_a^+, \quad J_a \times \sigma_b^+ = \sigma_b^+, \quad \phi \times \sigma_a^+ = \tau_a^+ + \tau_a^-.
\]

Remarkably, the fusion of two pure magnetic fluxes seems to give rise to electric charge creation, as for example in the fusion product of $\sigma_a^+$ with itself

\[
\sigma_a^+ \times \sigma_a^+ = 1 + J_a + \bar{1} + \bar{J}_a.
\]
Before fusion this electric charge was present in the form of so-called non-localizable Cheshire charge [3] (i.e. the non-trivial representation of the gauge group $D_2$) carried by the magnetic flux pair. In terms of these Cheshire charges we have
\[
(\hat{\mathcal{T}} + \hat{T}) \times (\hat{\mathcal{T}} + \hat{T}) = \hat{\mathcal{T}} + \hat{T} + \hat{\mathcal{T}} + \hat{T}.
\] (20)
Of course, charge creation is not the proper expression in this context. A similar argument holds for the common centralizer $Z_4 = \{e, X_a, \bar{e}, \bar{X}_a\}$. In terms of the representations of this group, the fusion (19) reads
\[
(\hat{\mathcal{T}} + \hat{T}) \times (\hat{\mathcal{T}} + \hat{T}) = \hat{\mathcal{T}} + \hat{T} + \hat{\mathcal{T}} + \hat{T}.
\]
So, there is no electric charge creation at any level of symmetry $D(\bar{D}_2) \supset \bar{D}_2 \supset Z_4$.

We also note that the class algebra is still respected as an overall selection rule. In (19), the associated class multiplication yields
\[
X_a \ast X_a = 2e + 2\bar{e}.
\]
Clearly, the interpretation of the magnetic fluxes in terms of $\bar{H}$ representations due to (3) yields a result that is incompatible with (20). (See the discussion in the paragraph ‘Fusion and braiding of fluxes’.)

Finally, there is also the peculiar occurrence of the vacuum in the fusion of for instance $\tau_a^+$ with itself
\[
\tau_a^+ \times \tau_a^+ = 1 + J_a + \sum_{b \neq a} \bar{J}_b.
\] (21)
This is different from the usual situation where charge conjugation relates complex conjugated (electric) representations. This naively relates $\tau_a^+$ to $\tau_a^-$. However, due to the interplay between magnetic and electric quantum numbers, this relation is altered. Indeed, in this example we find $S^2 = 1$.

It is also instructive to determine the Aharonov-Bohm cross-sections [18] for the elastic scattering processes in this model. We shall do this for a simple, yet nontrivial, example: scattering a $\bar{\phi}$ particle off a $\tau_1^+$ particle. Assume this two particle system to be in the following four component quantumstate
\[
|\psi_{in} > = |\bar{e}, \cos \theta v_1 + \sin \theta v_2 > |\cos \theta' X_1 + \sin \theta' \bar{X}_1, \bar{\theta} v >.
\]
Under encircling the $\tau_1^+$ particle with the $\bar{\phi}$ particle this quantumstate transforms as
\[
R^2|\psi_{in} > = R^2(|\bar{e}, \cos \theta v_1 + \sin \theta v_2 > |\cos \theta' X_1, \bar{\theta} v >
+|\bar{e}, \cos \theta v_1 + \sin \theta v_2 > |\sin \theta' \bar{X}_1, \bar{\theta} v >)
= |\bar{e}, \cos \theta \mathcal{T}(X_1) v_1 + \sin \theta \mathcal{T}(X_1) v_2 > |\cos \theta' X_1, \mathcal{T}(\bar{e}) \bar{\theta} v >
+|\bar{e}, \cos \theta \mathcal{T}(\bar{X}_1) v_1 + \sin \theta \mathcal{T}(\bar{X}_1) v_2 > |\sin \theta' \bar{X}_1, \mathcal{T}(\bar{e}) \bar{\theta} v >
= |\bar{e}, \bar{\theta} \cos \theta v_2 + \bar{\theta} \sin \theta v_1 > |\cos \theta' X_1 - \sin \theta' \bar{X}_1, -\bar{\theta} v >.
\]
In the last equality we used Table 1, in particular the fact that the representation $\mathcal{T}$ involves the pauli matrices: $\mathcal{T}(X_1) = -\mathcal{T}(\bar{X}_1) = i\sigma_1$. The real part of the inproduct
\[ <\psi_{in}|\mathcal{R}^2|\psi_{in}> \text{ vanishes. Plugging this result in (18) yields the following cross-section}
\[
\frac{d\sigma}{d\varphi}(\bar{\phi}, \tau_1^+)=\frac{1}{2\pi k \sin^2(\varphi/2)} \frac{1}{2}.
\] (22)

The calculations of the other cross-sections, although in some cases more involved, are completely analogous.

**Concluding remarks.** In this paper we proposed a general framework to describe quantum properties of discrete \( \hat{H} \) gauge theories, based on the Hopf algebra \( D(\hat{H}) \). It incorporates among others an unified description of the Aharonov-Bohm scatterings, fusion, and a clear interpretation of Cheshire charges in this model. In a forthcoming paper \[17\], we discuss the inclusion of a Chern-Simons term in the action. We show that this corresponds to the introduction of a non-trivial 3-cocycle on the algebra \( D(\hat{H}) \). The representation theory, and consequently the physical properties like fusion, braiding and scattering are modified. In so doing we will argue that the C-S parameter is not just quantized, but also periodic. Furthermore, a correspondence with the discrete topological field theories introduced by Dijkgraaf and Witten emerges.

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