On RG potentials in Yang-Mills Theories

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Abstract

We construct an RG potential for $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory, and
extract a positive definite metric by comparing its gradient with the recently discovered beta-
function for this system, thus proving that the RG flow is gradient in this four-dimensional
field theory. We also discuss how this flow might change after supersymmetry breaking,
provided the quantum symmetry group does not, emphasizing the non-trivial problem of
asymptotic matching of automorphic functions to perturbation theory.
Our understanding of the non-perturbative aspects of non-abelian gauge theories is at this point in time very limited. Most of our knowledge of the strongly coupled domain comes from exploiting approximate global continuous symmetries, whose origin are not well understood and may be coincidental, to construct low-energy effective field theories encoding the dynamics of the effectively massless modes whose existence is guaranteed by Goldstone’s theorem. Recently it has been realized that certain theories may possess exact global discrete quantum symmetries which are so constraining that the theory can essentially be solved exactly, even in the non-perturbative domain. Such symmetries may underlie the remarkable phase- and fixed point structure of the subtle quantum Hall system \[1\], and their discovery \[2\] in \(N = 2\) supersymmetric YM theories has led to a deep understanding of these systems, recently culminating in the construction of an exact representation of the contravariant (physical) beta-function \[3\]. It is the purpose of this letter to further explore the consequences of such symmetries and their relationship to supersymmetry in non-abelian gauge theories.

The main idea to be discussed here, originally devised for the quantum Hall system \[4\], is that the combined constraints of a quantum symmetry group (contained in the modular group) and matching to perturbation theory in the asymptotically flat domain (weak coupling) may be sufficient to completely harness the beta-function. In particular, automorphy of the contravariant beta-function is incompatible with holomorphy on the entire fundamental domain, and the way in which the analytical structure must be relaxed is indicated by the asymptotic behaviour of the beta-function:

\[
\beta^z = \frac{dz}{dt} = b_0 + \frac{b_1}{y} + \frac{b_2}{y^2} + \ldots + (q\text{-expansion}),
\]

where \(t\) is a scale parameter and for YM theory \(z = x + iy = \theta/2\pi + 4\pi i/g^2\), and \(q(z) = \exp(2\pi iz)\).

If the theory is \(N = 2\) supersymmetric, a strong non-renormalization theorem implies that there exists a renormalization scheme in which only one loop contributes, so that all \(b_i\) except \(b_0\) vanish \[5\]. Hence the beta-function is asymptotically holomorphic, and analyticity is only violated by a simple pole at \(z = 0\), which is the unique point on the boundary of the fundamental domain where the effective field theory breaks down. This is as close to holomorphic as a (sub-)modular beta-function can get, and its form is then completely fixed by the discrete quantum symmetry and asymptotic matching at \(z \to i\infty\) (to perturbation theory and a single-instanton calculation).

This idea \[6\] was very recently \[7\] applied to the contravariant beta-function for \(SU(2)\) and \(N = 2\) invariant YM theory \[3\], thus simplifying its derivation enormously. We first extend this line of reasoning to the \(N = 2\) covariant beta-function, which by comparison with the contravariant one allows us to extract a positive definite metric with reasonable physical properties. Since the covariant beta-function is gradient, this proves that the RG flow in this system is gradient \[8\].

If the theory is not supersymmetric, \(b_1\) does not vanish in any scheme and we see that the analytic structure changes dramatically. It seems unlikely that such a function could still be

\[\text{\footnote{Similar ideas have been discussed in Ref. [6] in the context of non-linear sigma models.}}\]
automorphic, but a remarkable “quasi-holomorphic” exception exists, and this can be used to build a candidate beta-function with similar uniqueness properties to the supersymmetric case. Thus, it is not inconsistent to discuss the properties of non-supersymmetric beta-functions automorphic under quantum symmetries. If such symmetries do exist they may or may not be survivors of supersymmetry breaking, but it seems plausible that it is the interplay of supersymmetry, quantum duality and holomorphy which must be elucidated if the spectacular success of $N = 2$ theories is to be extended to non-supersymmetric models.

A simple yet precise picture of the RG properties of $N = 2$ $SU(2)$ YM has recently emerged from the explicit construction of an exact non-perturbative beta-function $[3]$: 

$$
\beta^z(z) = \frac{1 - 4f(z)}{f'(z)} = -\frac{i}{\pi} \left( \frac{1}{\theta_3(z)^4} + \frac{1}{\theta_4(z)^4} \right) 
$$

(2)

where $f = -(\theta_3\theta_4/\theta_2^2)^4$ is invariant under the quantum symmetry group $\Gamma_T(2)$ of this system, and $\theta_i$ ($i = 2, 3, 4$) are the conventional elliptic theta-functions, related by $\theta_2^4 = \theta_3^4 - \theta_4^4$ $[8]$. This infinite non-abelian discrete quantum symmetry group is the largest sub-group of the modular group which contains translations by one. This is why it is $\Gamma_T(2)$ which appears both in YM and in the quantum Hall system. Translations by one $T : z \rightarrow z + 1$ is a symmetry because the theta parameter is periodic under shifts by $2\pi$, and guarantees that all functions automorphic under this group have $q$-expansions with $q = \exp(2\pi i z)$. The other generator of the group can be conveniently chosen as $ST^2S$.

Very recently a simple and illuminating derivation of this beta-function was given by Ritz $[7]$, which relies on the observation that the contravariant beta-function transforms like a weight $w = -2$ automorphic function $\gamma$:

$$
\beta^z(\gamma(z)) = \frac{d\gamma}{dz} \beta^z(z) = (cz + d)^{-2} \beta^z(z)
$$

(3)

for any $\gamma \in \Gamma_T(2)$ acting holomorphically by fractional linear transformations on the modular parameter $z$: $\gamma(z) = (az + b)/(cz + d)$. As observed in $[4]$, together with asymptotic matching to the perturbatively evaluated beta-function, this is a powerful constraint on the possible forms of this function. In the $N = 2$ case, where the beta-function is meromorphic, it does in fact uniquely fix the beta-function to the one given above, up to a single constant which will be discussed shortly.

We now adapt the argument to the covariant beta-function, which transforms as a weight $w = +2$ function. The first observation (Rankin $[9]$, p.111) is that for groups of genus zero (which includes $\Gamma(1) = SL(2, Z)$ and $\Gamma_T(2)$) there exists an invariant ($w = 0$) meromorphic function with a single pole (and a single zero) in the fundamental domain, which we have

\[^5\text{We follow the conventions of Rankin} \ [9] \text{as closely as possible, but adopt the standard physics notation for the generators of the modular group } SL(2, Z). \text{Thus, translations are denoted by } T \text{ (rather than } U), \text{ and simple duality } z \rightarrow -1/z \text{ is denoted by } S \text{ (rather than } V). \]

\[^\text{Mathematical textbooks like to reserve the words “function” and “form” for automorphic objects with special analyticity properties, but their usage is inconsistent and confusing, so we do not qualify our language in this way. We shall therefore have to be explicit about the number of poles possessed by our functions.} \]
called $f$ above. Any other weight zero function with total order $q$ of zeros (or poles) is given by a rational function $P(f)/Q(f)$, where the degree in $f$ of both the polynomials $P$ and $Q$ do not exceed $q$. Applying this theorem to the invariant function $g = \beta z f'$, and using the asymptotic behaviour of $f'$ and the $N = 2$ beta-function, it follows that $\beta^2 = (c_1 + c_2 f)/f'$, where the conventional choice of scheme corresponds to $c_2 = -4$ (from perturbation theory) and $c_1 = 1$ (one-instanton computation).

Similarly, assuming that there exists an asymptotically flat metric, the same argument applied to the invariant function $h = \beta z/f'$ gives

$$\beta_z = f' \left(\frac{f'}{1 - 4f} = \partial_z \Phi \quad \text{with} \quad \Phi = -\frac{1}{4} \log |1 - 4f|^2. \right.$$(4)

A contour plot of the RG potential is given in Fig. 1, where the horizontal axis is $x = \text{Re}z = \theta/2\pi$ and the vertical axis is $y = \text{Im}z = 4\pi/g^2$.

Figure 1: The phase and RG flow diagram of $N = 2$ supersymmetric $SU(2)$ Yang-Mills theory, obtained from the RG potential $\Phi$ exhibited in Eq.(4). Flow lines are downward and displayed only in the fundamental domain, while equipotential lines are shown for $\theta$ between $-\pi$ and $\pi$. 
A similar flow was very recently displayed in Ref. [10]. It is remarkable that this RG potential for $N = 2$ YM is so simply related to the unique univalent holomorphic function $f$ in $\Gamma_T(2)$. From this construction we can further extract a metric 

$$G_{zz} = \beta_z \bar{\beta}_z = \left| \frac{f'}{1 - 4f} \right|^2 = \partial_z \partial_{\bar{z}} K \quad \text{with} \quad K = c\Phi + \frac{1}{2} \Phi^2$$

which behaves correctly under the action of the Lie derivative, and is flat, finite and positive definite except at the fixed points. The Kähler potential $K$ can have any linear term since $\partial_z \beta_{\bar{z}} = 0$. At $z = i\infty$ it is finite, as behooves a metric adapted to a basis of noninteracting operators in the perturbative domain. Furthermore, it only carries mixed components, as expected from the OPE of YM in the perturbative domain, since $\langle (F F)(F^* F) \rangle = 0$. At the IR fixed point, $z = (1 + i)/2$, it has a coordinate singularity, which is to be expected if it is adapted to the perturbative basis of weakly interacting operators. At strong coupling we expect a different (“dual”) basis of weakly interacting composite operators to be more convenient, so it is natural for the metric adapted to the perturbative domain to have a coordinate singularity in the infrared. The curvature is zero, except at $z = 0$, where the metric is ill defined (not positive). This is also to be expected since this is where the effective field theory breaks down due to the presence of new massless modes. The Seiberg-Witten metric is quite different. It also has an essential singularity at $z = 0$, but it has a coordinate singularity in the perturbative domain, and is well behaved at the IR fixed point, indicating that it is better adapted to describing strong coupling.

These issues are not of any great relevance for us, since we have now achieved our goal: by exhibiting a potential function and a positive definite metric, we have shown that the RG flow is gradient. More precisely, we have just specified the elements in the following chain of equations:

$$- \beta^i \partial_i \Phi = - \beta^i \beta_i = - \beta^i \beta^j G_{ij} \leq 0 \ ,$$

where the inequality is obtained because $G_{ij}$ is positive definite. It is worth noting that this is the first explicit construction of a non-perturbative function of this kind for a $d = 4$ field theory. The much harder question of whether or not these functions have physical interpretations as the C-function, which counts effectively massless degrees of freedom at fixed points, and the Zamolodchikov metric, which is given by the correlators of relevant and marginal operators, is logically independent. All efforts addressing the irreversibility of RG flows are ultimately motivated by such physical considerations. Different approaches have recently converged on a solid candidate for the c-number (central charge) of the conformal field theories appearing at non-trivial fixed points. It is the coefficient of the Euler density in the trace anomaly in curved space (labeled as $a$ in [12] and $\beta_b$ in [13]), and this candidate has been shown, empirically, to be positive and to decrease along RG flows in a large number of non-trivial examples [14, 12]. Further work has related the sign of $\beta_b$ to a new form of the weak energy positivity theorem [13].

Although we at present are unable to relate $\Phi$ to a distinct physical observable, this is not required in order to prove that the RG flow is gradient. We are also not able to relate our metric $G_{zz}$ to the OPE of the operators in this theory. So, while it has many good
properties, being non-singular and automorphic as well as having correct asymptotics and Lie transport, we do not know how to relate it to the basic correlators of the theory.

Let us now turn to the issue of scheme dependence [15] alluded to above. As we have seen, the $\Gamma_T(2)$-symmetry reduces the mathematical freedom in the construction of the $\beta^z$-function to finding the appropriate values for $c_1$ and $c_2$. It was argued in Ref. [16] that $c_2$ controls the asymptotic behaviour of $\beta^z$, and therefore is fixed by the (one-loop) perturbative coefficient $b_0 = -2i/\pi$ of the beta-function, while $c_1$ was fixed by a one-instanton computation. It is obvious that $c_2$ is scheme independent. Let us argue that $c_1$ does not bring any scheme dependence either. A change of scheme corresponds to a finite redefinition of coupling constants which, in turn, leads to a different value of the RG invariant:

$$\Lambda = \mu e^{-\int \phi^{(\mu)} \beta^{-1}}.$$  \hspace{1cm} (7)

In our case we find:

$$\left(\frac{\Lambda}{\mu}\right)^{-c_2} = c_1 + c_2 f = \frac{-c_2}{256 q^2} + \left( c_1 + \frac{3c_2}{32} \right) - \frac{69c_2}{64} q^2 + \ldots .$$ \hspace{1cm} (8)

Although $c_1$ appears as a constant on the rhs of the above equation and looks suspiciously similar to the lhs, it cannot be absorbed by a finite redefinition of the coupling $z$. Furthermore, by plotting $\Phi = c_2^{-1} \log |c_1 + c_2 f|^2$ for different values of $c_1$ it is immediately evident that this function yields flows with different topologies, moving the IR attractor all over the $z$-plane. Therefore, rather than parametrizing different schemes, $c_1$ labels different universality classes: $c_1 = 1$ corresponds to $N = 2$ YM (see Fig. 1), $c_1 = 0$ leads to flows of the type discussed below (see Fig. 2), while other values of $c_1$ give exotic flows with no obvious interpretation.

In short, the contravariant beta-function for $N = 2$ $SU(2)$ YM is completely fixed by asymptotic matching to a one-loop and a one-instanton computation. The scheme is selected in such a way that the beta-function takes its simplest form with respect to modular transformations. Coupling constant reparametrizations will produce a change of scheme where the entanglement of the perturbative and non-perturbative parts of the beta-function can be very complicated. A deeper understanding of scheme dependence and the way in which reparametrizations of the moduli space can be used to simplify the description of a $C$-theorem is needed.

Finally we consider the possibility that non-supersymmetric gauge theories are also constrained by a quantum duality. The simplest origin of such a quantum symmetry $\Gamma$ would be if the $N = 0$ theory is a broken version of $N = 2$, and $\Gamma = \Gamma_T(2)$ is sufficiently robust to survive supersymmetry breaking (see the relevant instance proposed in [16]), but we do not need to commit ourselves to this scenario. There are several curious properties of ordinary YM gauge theory that may be construed as hinting at such a structure. The natural parametrization of YM moduli space is obtained by writing the action in terms of self-dual ($F_+$) and anti-self-dual ($F_-$) field configurations: $L = z F^2 + \bar{z} F^2$, where $z$ is the same complexified coupling constant that appears in the $N = 2$ case. Instanton corrections to the
beta-function have been argued [17] to be periodic in theta and exponentially suppressed in
perturbation theory, in precisely such a way that they are holomorphic in this parameter:
$q = \exp(i\theta - 8\pi^2/g^2) = \exp(2\pi iz)$, as required by the $q$-expansion of automorphic functions.
Note also that the $\theta$-parameter is ill-defined at the fixed point $z = i\infty$ of the (sub-)modular
group, as required by perturbation theory. Furthermore, we recall that the original intro-
duction of the modular group into physics was precisely in an attempt to understand the
oblique confinement scenario advocated by t’Hooft [18]. Unless this remarkable confluence of
central results is mere coincidence, it is not unnatural to conjecture that they are orches-
trated by a hidden quantum symmetry contained in the modular group. Since the vacuum
is invariant under translations of the theta-parameter by $2\pi$, $T$ is one of the generators and
it follows that if $\Gamma$ is not too small then $\Gamma = \Gamma_T(2)$.

![Figure 2](image)

Figure 2: The phase and RG flow diagram of the potential $\Phi_E$. Flow lines are downward and displayed
only in the fundamental domain, while equipotential lines are shown for $\theta$ between $-\pi$ and $\pi$.

We now formulate a precise conjecture which implies a unique form for the RG poten-
tial, and therefore the phase- and RG flow diagram in the non-supersymmetric case. We
exploit the remarkable mathematical fact that for $\Gamma_T(2)$ there exist two basic functions that
transform properly with weight $w = +2$ [9, 4]:

\[ E_T^T(z) = \partial \Phi_E(z, \bar{z}) = 1 + (q-\text{expansion}) \]

\[ H_T^T(z, \bar{z}) = \partial \Phi_H(z, \bar{z}) = \frac{1}{\pi y} + (q-\text{expansion}) \]

(9)

where, curiously, the potentials are built only from the lowest weight cusp forms ("instanton forms") for $\Gamma_T(2)$ and $\Gamma(1) = SL(2, \mathbb{Z})$, and $y = \text{Im} z$. They can be conveniently written as:

\[ \Phi_E(z, \bar{z}) = \frac{i}{\pi} \log |f|, \quad \Phi_H(z, \bar{z}) = \frac{i}{2\pi} \log |y^4 \theta_3^8 \theta_4^8|. \]

(10)

Notice that while both these functions are fully automorphic, and $E_T^T(z)$ is holomorphic, $H_T^T(z, \bar{z})$ is not. It is the unlikely existence of the latter, which we call a “quasi-holomorphic” or “Hecke” function, which allows us to formulate our conjecture. A contourplot of $\Phi_E$ is shown in Fig. 2, and the plot for $\Phi_H$ is very similar.

Our conjecture for the non-supersymmetric case can now be stated succinctly. If the quantum symmetry group is $\Gamma_T(2)$, and the holomorphic structure is violated only in the benign way exhibited above in the beta-function and in the Hecke form, by admitting only non-analytical terms of the type $1/y$ in the asymptotic domain (‘t Hooft scheme, where only $b_0$ and $b_1$ are different from zero), then the candidate covariant beta-function is a linear combination of $E_T^T$ and $H_T^T$ where one of the constants can be fixed by comparison with perturbation theory, provided the contravariant beta-function is constructed using an asymptotically flat metric. Thus, the final form of the conjecture is that $\Phi = b_0 \Phi_E + b \Phi_H$, where $b$ conspires with the sub-leading term in the metric to give $b_1$ etc. The phase- and RG flow diagram implied by this potential is evident from Fig. 2, showing an IR fixed point at $z = 0$, except when $\theta = \pi$ in which case $\theta$ is not renormalized and the IR fixed point is a critical point at $\alpha_s = g^2/4\pi = 2$. So this potential has the desirable property of having only one phase in which the theta-parameter renormalizes to zero in the IR domain, unless theta is exactly $\pi$. This suggests that the strong CP problem may be automatically resolved within YM theory by quantum duality, since it forces theta to flow towards zero at strong coupling, thus eliminating the need for axions or other ad-hoc constructions. The running of the theta-parameter would stop at the mass gap, yielding a finite but very small value of the physical theta-parameter. The existence of the critical point is in our context inescapable, and it may be possible to test this if the region with theta near $\pi$ can be probed in lattice simulations of $SU(2)$ Yang-Mills.

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