ACTION INTEGRALS AND DISCRETE SERIES

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Abstract. Let $G$ be a complex semisimple Lie group and $G_{\mathbb{R}}$ a real form that contains a compact Cartan subgroup $T_{\mathbb{R}}$. Let $\pi$ be a discrete series representation of $G_{\mathbb{R}}$. We present geometric interpretations in terms of concepts associated with the manifold $M := G_{\mathbb{R}}/T_{\mathbb{R}}$ of the constant $\pi(g)$, for $g \in Z(G_{\mathbb{R}})$. For some relevant particular cases, we prove that this constant is the action integral around a loop of Hamiltonian diffeomorphisms of $M$. As a consequence of these interpretations, we deduce lower bounds for the cardinal of the fundamental group of some subgroups of $\text{Diff}(M)$. We also geometrically interpret the values of the infinitesimal character of the differential representation of $\pi$.

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1. Introduction

An irreducible unitary representation of a Lie group is a discrete series representation if it can be realized as a direct summand of the left regular representation $\mathbb{L}$. If a group possesses discrete series representations, then it contains a compact Cartan subgroup. Kostant and Langlands conjectured the realization of the discrete series by $L^2$-cohomology of holomorphic line bundles over the quotient of the group by the compact Cartan subgroup. This conjecture has been proved for “most” discrete series representations by Schmid in [13] and fully in [14].

On the other hand, the Orbit Method [4, 21] suggests the existence of a correspondence between the space of coadjoint orbits of a Lie group and its unitary dual (i.e. the set of equivalence classes of unitary irreducible representations), and the existence of some relations between geometric properties of the orbit and properties of the representation. In the spirit of the Orbit Method and using the geometric construction of Schmid, we describe here interpretations of some invariants of discrete series representations in terms of geometric concepts relative to the corresponding orbits.

The correspondence between coadjoint orbits and the unitary dual is bijective for connected simply connected nilpotent groups; but, in general, there is not such a bijection. However, it is possible to associate an irreducible unitary representation to each hyperbolic orbit of a reductive group (see [20, 21]). In [19], we started with a hyperbolic orbit and then we analyzed the geometric meaning of some invariants of the corresponding representation. Here, we start with a discrete series representation of a semisimple Lie group and using its realization by $L^2$-cohomology we will give the geometric interpretations above mentioned, which allow us to obtain results about the homotopy of some groups of diffeomorphisms.

Key words and phrases. Orbit method, geometric quantization, coadjoint orbits, discrete series.

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To formulate our results, we review briefly the geometric construction of Schmid and introduce notations that will be used in the sequel.

The Schmid construction

Let $G$ be a complex connected semisimple Lie group and $\mathfrak{g} := \text{Lie}(G)$. For $A \in \mathfrak{g}$ and $g \in G$, we put $g \cdot A := \text{Ad}_g(A)$. For $\xi \in \mathfrak{g}^*$, let $g \cdot \xi := \text{Ad}^*_g(\xi)$. Let $\mathfrak{g}_R \subset \mathfrak{g}$ be a real form (non-necessarily compact). By $G_R$ we denote the connected subgroup of $G$ with Lie algebra $\mathfrak{g}_R$. We will assume that $G_R$ contains a compact Cartan subgroup $T_R$ with Lie algebra $t_R$. We denote by $K_R$ a maximal compact subgroup of $G_R$ such that $T_R \subset K_R$.

Let $\Delta$ be a positive root system of $\mathfrak{t} := \mathfrak{t}_R \otimes \mathbb{R} \subset \mathfrak{g}$. From now on, $\rho$ stands for the half the sum of the positive roots, $\rho = (1/2) \sum_{\nu \in \Delta} \nu$. We set

$$\mathfrak{u} = \bigoplus_{\nu \in \Delta} \mathfrak{g}^{-\nu}, \quad \mathfrak{u} = \bigoplus_{\nu \in \Delta} \mathfrak{g}^{\nu},$$

$\mathfrak{g}^{\nu}$ being the root space of $\nu$. A root $\nu$ is said to be compact if $\mathfrak{g}^{\nu} \subset \mathfrak{t}$, and noncompact otherwise. We denote by $\mathfrak{b}$ the Borel subalgebra $\mathfrak{t} \oplus \mathfrak{u}$. The flag variety of $\mathfrak{g}$ will be denoted by $B(\mathfrak{g})$, or simply by $B$. It can be identified with the complex smooth variety $G/B$, where $B = N_G(\mathfrak{b})$, the normalizer of $\mathfrak{b}$ in $G$.

Let $\phi$ be an element of the weight lattice of $\mathfrak{t}_R$. The corresponding character of $T_R$ is denoted by $\Phi$. By complex linearity, $\phi$ can be extended to a map from $\mathfrak{t}$ to $\mathbb{C}$ and finally to an element of $\mathfrak{g}^*$ by putting $\phi|_{\mathfrak{u} \otimes \mathbb{R}} = 0$. The character $\Phi$ extends to a group homomorphism on the Borel subgroup $B$ as well.

We define the following holomorphic line bundle over $B \simeq G/B$.

$$\mathcal{V} = G \times_B \mathbb{C} = \{(g, z) | g \in G, z \in \mathbb{C}\}/\sim,$$

where $(g, z) \sim (gb, \Phi(b^{-1})z)$, with $b \in B$. The natural $G$-action on $\mathcal{V}$ covers the action of $G$ on $B$; that is, $\mathcal{V}$ is a $G$-equivariant holomorphic bundle.

The set of orbits of the $G_R$-action on $B$ is finite. Since the Cartan subgroup $T_R$ is compact the $G_R$-orbit of $\mathfrak{b}$ is an open orbit, which can be identify with the quotient $M := G_R/T_R$. The complex structure of $B$ gives rise to a $G_R$-invariant complex structure on $M$, with $T_{cT_R} = \mathfrak{u}$. Thus, $M$ is a homogeneous complex manifold acted transitively by the group $G_R$.

The restriction of the holomorphic line $\mathcal{V}$ to $M$ will be also denoted by $\mathcal{V}$. Denoting by $L_A$ the left invariant vector field on $G_R$ defined by $A \in \mathfrak{g}_R$, the space of the holomorphic sections of $\mathcal{V}$ can be identified with the space of the functions $f \in C^\infty(G_R)$ such that, $f(gt) = \Phi(t^{-1})f(g)$ for all $t \in T_R$ (i.e., they are $\Phi$-equivariant) and satisfy $L_C(f) = 0$ for all $C \in \mathfrak{u}$. Obviously, the $G_R$-action on $M$ lifts to the sheaf $\mathcal{O}(\mathcal{V})$ of germs of holomorphic sections of $\mathcal{V}$.

On the space of compactly supported $\mathcal{V}$-valued $(0, *)$-forms on $M$ we have the operator $\bar{\partial}$,

$$\bar{\partial} : \mathcal{A}^{0,*}(\mathcal{V}) \to \mathcal{A}^{0,*+1}(\mathcal{V}).$$

Furthermore, the group $G_R$ acts on the space $\mathcal{A}^{0,i}(\mathcal{V})$ by translation and the action commutes with the operator $\bar{\partial}$. By means of $G_R$-invariant Hermitian metrics on $M$ and on $\mathcal{V}$, we define the operator $\partial^*$, the formal adjoint of $\bar{\partial}$. The space of square integrable, $C^\infty$, $\mathcal{V}$-valued $(0, i)$-forms on $M$ which belong to $\ker(\bar{\partial}) \cap \ker(\partial^*)$ is denoted by $H^{i}_{(2)}(M, \mathcal{O}(\mathcal{V}))$. 


Denoting by \((\ldots)\) the bilinear form on \(t^*\) induced by the Killing form, we put \(q\) for the integer \(q\).

\[
\mathcal{O}(\Delta) := \{ \nu \in \Delta \mid \nu \text{ compact, } (\phi \oplus \rho, \nu) < 0 \} + \mathcal{O}(\Delta) := \{ \nu \in \Delta \mid \nu \text{ noncompact, } (\phi \oplus \rho, \nu) > 0 \}.
\]

According to the Langlands conjecture, if \(\phi \oplus \rho\) is regular, the action of \(G_{\mathbb{R}}\) on \(\mathcal{H} := H^q(\mathcal{M}, \mathcal{O}(\mathcal{V}))\) is an irreducible unitary representation \(\pi\), equivalent to a discrete series representation (see [1] for the omitted details).

Note that every \(\sigma \in \mathcal{H}\) is, in fact, a \(\partial\)-closed smooth Dolbeault form, but \(\mathcal{H}\) cannot be identified with \(H^q(M, \mathcal{O}(\mathcal{V}))\), since \(M\) is not compact, in general.

**Statement of main results**

Throughout, we will assume that the element \(\phi \oplus \rho \in i t^*_{\mathbb{R}}\) is regular, and we will denote by \(\mathcal{H}_{K_\mathbb{R}}\) the space of \(K_{\mathbb{R}}\)-finite vectors of \(\mathcal{H}\). The differential representation, on \(\mathcal{H}_{K_\mathbb{R}}\), of the above irreducible unitary representation \(\pi\) will be denoted by \(\pi'\).

The purpose of this paper is to give geometric interpretations of some invariants of \(\pi\) and \(\pi'\). Using these results we will find lower bounds for the cardinal of the homotopy groups of some subgroups of \(\text{Diff}(M)\), the group of diffeomorphisms of \(M\).

On \(W := \mathbb{C} \otimes (\wedge^q \mathbb{u})^*\) we consider the representation \(\Psi\) of \(T_{\mathbb{R}}\), tensor product of \(\Phi\) by the \(q\)-exterior product of \(\text{Ad}^*\). We denote by \(\mathcal{P}\) the \(\text{GL}(W)\)-principal bundle over \(M\) determined by \(\Psi\), and by \(\mathcal{W}\) the associated vector bundle with fiber \(W\) (defined by the standard representation of \(\text{GL}(W)\)). Thus, the vector space \(\mathcal{H}\) is contained in the space \(\Gamma(\mathcal{W})\) of sections of \(\mathcal{W}\). These bundles will be the geometric framework for our developments.

1. **The representation \(\pi'\) and the “quantization operators”**

   Using the root structure of \(\mathfrak{g}\) and \(\Psi'\), the differential representation of \(\Psi\), we will define a \(G_{\mathbb{R}}\)-invariant connection \(\Omega\) on \(\mathcal{P}\). The corresponding covariant derivative on sections of \(\mathcal{W}\) is denoted \(\nabla\). By means of \(\nabla\) and \(\Psi\), for each \(A \in \mathfrak{g}_{\mathbb{R}}\), we define a first order differential operator \(\mathcal{Q}_A\) which acts on the sections of \(\mathcal{W}\). The first order term of \(\mathcal{Q}_A\) is \(-\nabla_{X_A}\), with \(X_A\) the vector field on \(M\) defined by \(A\), and the zeroth order term is related with the value of \(\Psi'\) at \(A\). Abusing of language, the operators \(\mathcal{Q}_A\) will be called “quantization operators” (see paragraph below Theorem 5). We will prove in Theorem 5 that the operators \(\mathcal{Q}_A\) restricted to \(\mathcal{H}_{K_\mathbb{R}}\) are a representation of \(\mathfrak{g}_{\mathbb{R}}\) equivalent to \(\pi'\).

   The universal enveloping algebra of \(\mathfrak{g}\) is denoted by \(U(\mathfrak{g})\) and its center by \(\mathcal{Z}(\mathfrak{g})\). Let \(\chi\) denote the infinitesimal character \([\mathfrak{g}]\) of the \(U(\mathfrak{g})\)-module \(\mathcal{H}_{K_\mathbb{R}}\). Let \(\{C_1, \ldots, C_r\}\) be a basis of \(t_{\mathbb{R}}\) and \(E_\nu\) a basis of \(\mathfrak{g}^*\). According to Proposition 5.34 of [9], each element \(J \in \mathcal{Z}(\mathfrak{g})\) is a linear combination, \(p(E_{-\nu}, C_i, E_\nu)\), of terms of the following two types:

   \(\text{(a) } C_1^{m_1} \cdots C_r^{m_r}, \text{ with } m_j \in \mathbb{Z}_{\geq 0}\).

   \(\text{(b) } E_1^{n_1} \cdots E_1^{n_1} C_1^{m_1} \cdots C_r^{m_r} E_{-\nu_i} \cdots E_{-\nu_i} E_\nu, \text{ with } \nu_i, \nu \in \Delta \text{ and } q_j, n_j, q_j \in \mathbb{Z}_{\geq 0}\).

   We will prove that the corresponding differential operator \(p(\mathcal{Q}_{E_{-\nu}}, \mathcal{Q}_{C_i}, \mathcal{Q}_{E_\nu})\) on the space \(\mathcal{H}\) is the scalar one defined by the constant \(\chi(J)\) (Theorem 7).
2. Invariants defined by Schur’s Lemma. Given an element $g_1$ in the center of $G_R$, by Schur’s Lemma \[5\]
\[\pi(g_1) = \kappa \mathrm{Id}_\mathcal{H},\]
$\kappa$ being a complex number of modulus 1. In this paper, we will give geometric interpretations of the invariant $\kappa$ associated with $\pi$:

(i) in terms of “evolution equations” for elements in $\mathcal{H}$, equations generated by families of “quantization operators”;

(ii) as a gauge transformation of the fibre bundle $\mathcal{P}$ that is the time-1 map of a flow in $\mathcal{P}$ that preserves the connection.

For the explanation of (i) and (ii) we need to introduce some notations. Henceforth, $\{g_t \mid t \in [0, 1]\}$ stands for a smooth curve in $G_R$ with the initial point at $e$. For brevity, such a curve will be called a path in $G_R$. We denote by $\{A_t\} \subset \mathfrak{g}_R$ the corresponding velocity curve, that is,
\[A_t = \dot{g}_tg_t^{-1}.\]

By means of the $G_R$-action on $M$, the path $g_t$ determines an isotopy of $M$, which will be denoted by $\{\varphi_t\}_{t \in [0, 1]}$ in the sequel; that is,
\[\varphi_t(gT) = g_tgT.\]

Given a path $\{g_t\}$ in $G_R$, we can consider the set of sections $\sigma_t$ of $\mathcal{W}$ defined by the following “evolution equations”:
\[\frac{d\sigma_t}{dt} = Q_{A_t}(\sigma_t), \quad \sigma_0 = \sigma.\]

If $g_1$ belongs to $Z(G_R)$, the center of $G_R$, then we will prove in Theorem \[12\] that $\sigma_1 = \kappa \sigma$, for any $\sigma \in \mathcal{H}_K$.

For each $A \in \mathfrak{g}_R$, the vector field $X_A$ on $M$ determines a vector field $U(X_A)$ on $\mathcal{P}$ such that the Lie derivative of $\Omega$ with respect this vector field vanishes. So, a path $\{g_t\}$ in $G_R$ defines the time-dependent vector field $U(X_{A_t})$, which in turn determines a flow $\mathbf{H}_t$ on $\mathcal{P}$. If $g_1 \in Z(G_R)$, we will prove that $\mathbf{H}_1$ is the gauge transformation $\mathbf{H}_1(p) = \kappa p$ (Theorem \[22\]).

When $G_R$ is compact and $\phi$ is a regular dominant weight, the representation $\pi$ is the one provided by the Borel-Weil Theorem. In this case $M$ is the flag variety $\mathcal{B}(g)$ of $g$, a compact manifold diffeomorphic to the coadjoint orbit of $\phi \in \mathfrak{g}^\ast$. On $M$ we will consider the symplectic Kirillov structure $\varpi$ \[3\]. When $g_1 \in Z(G_R)$, $\{\varphi_t\}$ is a loop in $\text{Ham}(M, \varpi)$, the Hamiltonian group of $(M, \varpi)$ \[11\] \[12\]. By studying equation \[15\], we will find that $\kappa$ is the exponential of the symplectic action \[22\] \[17\] around the loop $\{\varphi_t\}$ (see Proposition \[19\]).

3. Lower bounds for the cardinal of the homotopy group of some subgroups of $\text{Diff}(M)$. Let $\mathfrak{X}(M)$ denote the Lie algebra of all vector fields on $M$. We will consider subalgebras of $\mathfrak{X}(M)$, denoted by $\mathfrak{X}'$, such that each $Z \in \mathfrak{X}'$ determines a vector field $U$ on $\mathcal{P}$ satisfying $\mathfrak{L}_U\Omega = 0$ ($\mathfrak{L}$ = Lie derivative). The precise conditions imposed to the algebras $\mathfrak{X}'$ are detailed in the third paragraph of Section \[8\]. Let $\mathcal{G}$ be a connected Lie subgroup of $\text{Diff}(M)$, which contains the isotopies associated with paths in $G_R$ and such that $\text{Lie}(\mathcal{G})$ is subalgebra of some algebra $\mathfrak{X}'$. Using the above interpretation of $\kappa$ as a gauge transformation which is the final point of a curve consisting of automorphisms of $\mathcal{P}$, we will prove that:
\[\sharp(\pi_1(\mathcal{G})) \geq \sharp(\Psi(g) \mid g \in Z(G_R)),\]
When $G_\mathbb{R}$ is compact and $\phi$ is a regular dominant weight, the above result adopts the following form: Let $G$ be a connected subgroup of the symplectomorphism group of $(M, \varpi)$ which contains $G_\mathbb{R}$, then $\sharp\{\Phi(g) \mid g \in Z(G_\mathbb{R})\}$ is a lower bound of the cardinal of $\pi_1(G)$ (Theorem 24).

Theorem 6 of [18] gives a lower bound for $\pi_1(\text{Ham}(O))$, where $O$ is a coadjoint orbit of $\text{SU}(n+1)$ diffeomorphic to a (partial) flag manifold of $G = \text{SL}(n + 1, \mathbb{C})$ (i.e. diffeomorphic to the the quotient of $\text{SL}(n + 1, \mathbb{C})$ by a parabolic subgroup). This lower bound is $\leq 2N$, where $N$ is the minimal Chern number of the manifold on spheres. For the flag variety $B(\text{sl}(n + 1, \mathbb{C}))$ the minimal Chern number is 2 (see [10, page 117]). Hence, the lower bound given by the mentioned theorem applied to this flag manifold is $\leq 4$, for any $n \geq 1$. In contrast with the result of [18], the lower bound given for $\sharp\pi_1(\text{Ham}(M, \varpi))$ in Theorem 24 (when it is applied to $G_\mathbb{R} = \text{SU}(n + 1)$) depends on $n$ and also on $\phi$. For example, this lower bound is equal to $n + 1$, when $\Phi$ is injective on $Z(\text{SU}(n + 1))$.

This article is organized as follows. In Section 2, we define the principal bundle $P$, the connection $\Omega$ on it and the vector bundle $W$. We prove Theorem 8, which gives the representation $\pi'$ in terms of the “quantization operators”. We also prove Proposition 13 that relates $\kappa$ with the action integral around a Hamiltonian loop in $M$. The final part of this section concerns the interpretation of the infinitesimal character in terms of “quantization operators”.

In Section 3, we prove Theorem 20 about $\kappa$ as a gauge transformation of $P$. We also prove Theorems 23 and 24 which give lower bounds for the cardinal of the homotopy groups of subgroups of $\text{Diff}(M)$.

2. The “Quantization Operators”

Let $E_\nu$ be a basis of $g^\nu$, then the decomposition $g = t \oplus u \oplus \bar{u}$ gives rise to the following direct sum decomposition

$$g_\mathbb{R} = t_\mathbb{R} \oplus l,$$

where

$$l := \bigoplus_{\nu \in \Delta} \left( \mathbb{R}(E_\nu + \bar{E}_\nu) \right) \oplus \bigoplus_{\nu \in \Delta} \left( i\mathbb{R}(E_\nu - \bar{E}_\nu) \right).$$

Here the bar designates conjugation with respect to the real form $g_\mathbb{R}$. The component of $A \in g_\mathbb{R}$ in $t_\mathbb{R}$ will be denoted by $A_0$. The decomposition (2.1) is preserved by $\text{ad}(C)$, when $C \in t_\mathbb{R}$. Thus,

$$t \cdot A_0 = t \cdot A_0, \quad \text{for all } t \in T_\mathbb{R}.$$  

Since $\phi$ belongs to the lattice weight of $t_\mathbb{R}$, it gives rise to a group homomorphism $\Phi : T_\mathbb{R} \to \mathbb{U}(1)$. On the other hand, we have the adjoint representation $\text{Ad} : T_\mathbb{R} \to \text{Aut}(u)$ and the corresponding exterior product of its dual on $(\wedge^q u)^*$, where $q$ is the nonnegative integer defined by (1.1). On the vector space $W = \mathbb{C} \otimes (\wedge^q u)^*$ we will consider the representation of $T_\mathbb{R}$ tensor product $\Phi \otimes (\wedge^q \text{Ad})^*$. As we said, this representation will be denoted by $\Psi$.

For $A \in g_\mathbb{R}$, we define $h_A : G_\mathbb{R} \to \mathfrak{gl}(W)$ by the formula

$$h_A(g) = \Psi((g^{-1} \cdot A)_0),$$

(see Theorem 24).
where $\Psi'$ is the derivative of $\Psi$. By (2.3) $h_A$ induces a map on $M$, which will also be denoted by $h_A$.

It is straightforward to prove the following lemma:

**Lemma 1.** For any $g \in G_R$ and every $t \in T_R$,

$$\Psi(t) h_A(g) = h_A(g) \Psi(t).$$

On the other hand,

$$[h_A(g), h_C(g)]_{gl} = [\Psi'((g^{-1} \cdot A)g), \Psi'((g^{-1} \cdot C)g)]_{gl} = \Psi'((g^{-1} \cdot A)g, (g^{-1} \cdot C)g],$$

where $[,]_{gl}$ is the bracket in the Lie algebra $gl(W)$. As $t_R$ is an abelian Lie algebra, we have

(2.5) $[h_A, h_C]_{gl} = 0$.

Given $A \in g_R$, we set $R_A$ for right invariant vector field on $G_R$ defined by $A$. The vector field on $M$ defined by $A$ will be denoted by $X_A$, as we said. From the definitions, it follows that

(2.6) $R_A(h_C) = -h_{[A, C]}$

(2.7) $X_A(h_C) = -h_{[A, C]}$ (with $h_C$ considered as function on $M$).

We denote by $P$ the following $GL(W)$-principal bundle over $M = G_R/T_R$:

$$P := G_R \times g \frac{\text{pr}}{\text{pr}} M.$$

That is, $P = \{(g, \alpha) \mid g \in G_R, \alpha \in GL(W)\}$ with $[g, \alpha] = [gt, \Psi(t^{-1})\alpha]$, for $t \in T_R$. The right $GL(W)$-action on $P$ will be denoted by $R$ and by $V_y$ the vertical vector field associated with $y \in gl(W)$.

We denote by $L$ the left natural $G_R$-action on $P$; this action gives to $P$ the structure of $G_R$-equivariant bundle. Given $A \in g_R$, the vector field on $P$ determined by $A$ through $L$ will be denoted by $Y_A$. So $(L_y)_*(Y_A) = Y_A$.

For $C \in t_R$, the trivial curve $\{(ge^{tC}, \Psi(e^{-tC})\alpha)\}_{t}$ in $P$ defines the vector

$$Y_{g \cdot C}[g, \alpha] - V_y[g, \alpha],$$

with $y = \text{Ad}_{a^{-1}}(\Psi(C))$; here $\text{Ad}$ is adjoint action of $GL(W)$ on its Lie algebra. Hence, the tangent space to $P$ at $[g, \alpha]$ is

(2.8) $T_{[g, \alpha]}P = \left\{ Y_A[g, \alpha] \mid A \in g_R \right\} \oplus \left\{ V_y[g, \alpha] \mid y \in gl(W) \right\} / \left\{ Y_{g \cdot C}[g, \alpha] - \text{Ad}_{a^{-1}}(\Psi(C))[g, \alpha] \mid C \in t_R \right\}$.

We define the following $gl(W)$-valued 1-form on $P$:

(2.9) $\Omega(Y_A[g, \alpha] + V_y[g, \alpha]) = \text{Ad}_{a^{-1}}(\Psi((g^{-1} \cdot A)g)) + y$.  

Obviously, $\Omega$ is well-defined on the quotient (2.8). It is straightforward to check its invariance under $L$ and that $R^*_\alpha \Omega = \text{Ad}_{a^{-1}} \circ \Omega$, for all $\alpha \in GL(W)$. That is, we have the following proposition:

**Proposition 2.** The 1-form $\Omega$ is a $G_R$-invariant connection on the $GL(W)$-principal bundle $P$. 
We set \( h_A \langle g, \alpha \rangle := \text{Ad}_{\alpha^{-1}} h_A(g) \). By Lemma 1 together with (2.3), it follows that \( h_A \) is a \( \mathfrak{g}(W) \)-valued well defined map on \( \mathcal{P} \). In this notation

\[
(2.10) \quad \Omega(Y_A + V_y) = h_A + y.
\]

The horizontal lift of \( X_A \) at the point \( [g, \alpha] \) is denoted by \( X^h_A[g, \alpha] \). From (2.10), it follows that

\[
(2.11) \quad X^h_A[g, \alpha] = Y_A[g, \alpha] + V_y[g, \alpha], \text{ with } y = -h_A[g, \alpha].
\]

The following lemma is immediate:

**Lemma 3.** Given \( A, C \in \mathfrak{g}_R, y \in \mathfrak{g}(W) \) and \( p \) a point of \( \mathcal{P} \),

\[
Y_A(h_C) = -h_{[A,C]} \quad \text{and} \quad V_y(p)(h_A) = -[y, h_A(p)]_{\mathfrak{g}_1}.
\]

The curvature of the connection \( \Omega \) will be denoted by \( K \). As the curvature is a tensorial 2-form \( \mathcal{P} \), the following proposition determines \( K \):

**Proposition 4.** For all \( A, C \in \mathfrak{g}_R \),

\[
(2.12) \quad K(Y_A, Y_C) = -h_{[A,C]}
\]

**Proof.** By the structure equation

\[
K(Y_A, Y_C) = d\Omega(Y_A, Y_C) + [\Omega(Y_A), \Omega(Y_C)]_{\mathfrak{g}_1}.
\]

By (2.3), \( [\Omega(Y_A), \Omega(Y_C)]_{\mathfrak{g}_1} = 0 \). From Lemma 3 it follows that

\[
Y_A(\Omega(Y_C)) = h_{[C,A]} = -Y_C(\Omega(Y_A)).
\]

On the other hand, \( \Omega([Y_A, Y_C]) = -\Omega(Y_{[A,C]}) = -h_{[C,A]} \). Thus, (2.12) follows.

We denote by \( D \) the covariant derivative determined by \( \Omega \). From (2.11) together with Lemma 3 and Proposition 4 it follows

\[
(2.13) \quad Dh_C(Y_A) = -K(Y_C, Y_A), \quad \text{for all } A, C \in \mathfrak{g}_R.
\]

The vector bundle on \( M \) associated with \( \mathcal{P} \),

\[
G_R \times_{\Psi} (\mathbb{C} \otimes (\Lambda^d u)^*)
\]

will be denoted \( W \). The elements of \( W \) are classes \((g, w)\), of pairs \((g, w) \in G_R \times W\), with \( (g, w) = (gt, \Psi(t^{-1})w) \), for all \( t \in T_R \). \( W \) is a \( G_R \)-equivariant vector bundle with the \( G_R \)-action \( g' \cdot (g, w) = (g'g, w) \).

The \( C^\infty \) sections \( \sigma \) of \( \mathcal{W} \) can be identified with the \( C^\infty \) \( \Psi \)-equivariant functions \( s : G_R \to W \), i.e. \( C^\infty \) functions that satisfy

\[
(2.14) \quad s(gt) = \Psi(t^{-1})s(g), \quad \text{for all } g \in G_R \text{ and } t \in T_R.
\]

The section \( \sigma \) is related with the \( \Psi \)-equivariant function \( s \) by the formula

\[
(2.15) \quad \sigma(gT_R) = (g, s(g)).
\]

The \( G_R \)-action on the section \( \sigma \) is given by \( (g \cdot \sigma)(x) = g \cdot \sigma(g^{-1}x) \), for all \( x \in M \). From (2.13), it follows the following proposition, which gives the action of the representation \( \pi \) on \( \Psi \)-equivariant functions:

**Proposition 5.** The \( \Psi \)-equivariant function associated with \( g \cdot \sigma \) is \( s \circ L_{g^{-1}} \), where \( L_{g^{-1}} \) is the left multiplication by \( g^{-1} \) in \( G_R \). In particular, if \( \sigma \in \mathcal{H} \), then \( \pi(g)(s) = s \circ L_{g^{-1}} \).
From Proposition 6 and taking into account that the section associated with (2.16), it follows being a section of $\sigma$ equivariant as a consequence of Lemma 1.

By Proposition 6, the representation differential $\pi'$ of the discrete series representation $\pi$, considered as an action on the space $\mathcal{E}$ of the $\Psi$-equivariant functions which correspond to the elements of $\mathcal{H}_{K_{\mathbb{R}}}$, is given by

\[ \pi'(A)(s) = -R_A(s). \]

From Proposition 6 and taking into account that the section associated with $h_A(s)$ is $F_A(\sigma)$, we obtain the following theorem:

\[ (2.16) \quad s^\flat[g, \alpha] = \alpha^{-1}s(g). \]

From (2.16), it follows

\[ (2.17) \quad Y_A(s^\flat) = (R_A(s))^\flat, \quad V_{h_A}(s^\flat) = -(h_A(s))^\flat, \]

where $V_{h_A}(s^\flat)$ is the map $p \in \mathcal{P} \mapsto V_{h_A}(p)(s^\flat) \in W$ and $h_A(s)$ is the mapping on $G_{\mathbb{R}}$ which at $g$ takes the value $h_A(g)(s(g))$. The function $h_A(s)$ is in fact $\Psi$-equivariant as a consequence of Lemma 1.

On the other hand, a $\Psi$-equivariant function

\[ \text{determines a pseudotensorial \cite{7}} \]

map $s^\flat: \mathcal{P} \to W$ of type standard; that is, $s^\flat(p\beta) = \beta^{-1}s^\flat(p)$, for all $\beta \in \text{GL}(W)$. The maps $s$ and $s^\flat$ are related by

\[ (2.18) \quad [\nabla_{X_{\mathbb{A}}}, \nabla_{X_{\mathbb{C}}} - \nabla\{X_{\mathbb{A}}, X_{\mathbb{C}}\}](\sigma) = -F_{[\mathbb{A}, \mathbb{C}]}(\sigma), \]

$\sigma$ being a section of $\mathcal{W}$.

**Proposition 6.** Given $A \in \mathfrak{g}_{\mathbb{R}}$ and a section $\sigma$ of $\mathcal{W}$, the $\Psi$-equivariant map associated with $\nabla_{X_{\mathbb{A}}} \sigma$ is $R_A(s) + h_A(s)$.

**Proof.** The pseudotensorial map associated with $\nabla_{X_{\mathbb{A}}} \sigma$ is $X^\flat_A(s^\flat)$. From (2.17) together with (2.11), it follows the proposition.

\[ \nabla_{X_{\mathbb{A}}} \tau \text{ over the identity determined by } h_A \text{ is denoted by } F_A; \]

that is, $F_A((g, w)) = (g, h_A(g)(w))$.

**Proposition 7.** For $A, C \in \mathfrak{g}_{\mathbb{R}}$ one has

\[ (2.19) \quad R_A(R_C(s) + h_C(s)) = h_{[\mathbb{A}, \mathbb{C}]}(s) + h_A(R_C(s) + h_C(s)). \]

By (2.16) $R_A(h_C(s)) = h_{[\mathbb{A}, \mathbb{C}]}(s) + h_C(R_A(s))$. So, $\nabla_{X_{\mathbb{A}}} (\nabla_{X_{\mathbb{C}}} \sigma)$ is associated with

\[ (2.20) \quad - (R_{[\mathbb{A}, \mathbb{C}]}(s) + h_{[\mathbb{A}, \mathbb{C}]}(s)). \]

From (2.19), (2.20) and (2.16), it follows that the equivariant function corresponding to the left hand side of (2.18) is $-h_{[\mathbb{A}, \mathbb{C}]}(s)$, and the proposition is proved.

By Proposition 5 the representation differential $\pi'$ of the discrete series representation $\pi$, considered as an action on the space $\mathcal{E}$ of the $\Psi$-equivariant functions which correspond to the elements of $\mathcal{H}_{K_{\mathbb{R}}}$, is given by

\[ (2.21) \quad \pi'(A)(s) = -R_A(s). \]
Theorem 8. For any \( A \in \mathfrak{g}_R \), \( \pi'(A) \) acting on elements of \( \mathcal{H}_{\mathcal{K}_\pi} \) is the operator
\[
Q_A := -\nabla_{X_A} + F_A.
\]

Given a closed symplectic quantizable manifold \( N \), each Hamiltonian vector field on \( N \) has associated a differential operator called quantization operator \( Q \), which acts on the sections of a line bundle over \( N \) called a prequantum bundle. On the other hand, if \( G_R \) is compact and \( \phi \) is a regular dominant weight, then the integer \( q \) is zero; so, \( \mathcal{W} \) is a line bundle on \( M \). In this case \( M \) equipped with the Kirillov symplectic structure \( \varpi \) is quantizable and \( \mathcal{W} \) is a prequantum bundle. Furthermore, the above operator \( Q \) is the quantization operator associated with the vector field \( X_A \). Thus, we will also call “quantization operators” the differential operators defined in (2.22) for the general case.

Let \( \{g_t\}_{t \in [0,1]} \) be a path in \( G_R \). It determines the velocity curve \( \{A_t\} \), defined in (1.3). This path also gives rise to the isotopy \( \{\varphi_t\} \) on \( M \) defined by (1.4). Moreover, the time-dependent vector field on \( M \) determined by \( \varphi_t \) is \( X_{A_t} \); that is,
\[
\frac{d\varphi_t}{dt} = X_{A_t} \circ \varphi_t.
\]

We consider the evolution equation (1.5) for sections of \( \mathcal{W} \). By Proposition 6, the \( \Psi \)-equivariant function \( s_t \) associated with the solution of (1.5) satisfies
\[
(2.23) \quad \frac{ds_t}{dt} = -R_{A_t}(s_t), \quad s_0 = s.
\]

It is easy to prove the following proposition:

Proposition 9. The solution of (2.23) is \( s_t = s \circ L_{g_t^{-1}} \).

By Proposition 5 when \( s \in \mathcal{E} \) Proposition 9 can be rephrased as saying that the solution of (2.23) is
\[
(2.24) \quad s_t = \pi(g_t)(s).
\]

Proposition 10. If \( g_1 \in Z(G_R) \), and \( s \) is the equivariant function associated with an element \( \sigma \in \mathcal{H} \), then \( s_1 = \kappa s \).

Proof. The proposition follows from (2.24) together with (1.2).

Corollary 11. If \( g_1 \in Z(G_R) \), then \( \Psi(g_1) = \kappa \text{Id}_W \).

Proof. By Proposition 9 we have for all \( g \in G_R \)
\[
s_1(g) = s(g_1^{-1} g) = s(g g_1^{-1}) = \Psi(g_1) s(g),
\]
and the corollary follows from Proposition 10.

By the equivalence between (1.5) and (2.23), Proposition 10 gives rise to the following theorem:

Theorem 12. If \( \{g_t\}_{t \in [0,1]} \) is a path in \( G_R \) with \( g_1 \in Z(G_R) \) and \( \sigma \in \mathcal{H} \), then the solution of the evolution equation (1.5) satisfies \( \sigma_t = \kappa \sigma \).

Let \( x \) be a point of \( M \), we set \( x_t := \varphi_t(x) \). Evaluating (1.5) at the point \( x_t \), and we obtain
\[
(2.25) \quad \frac{d\sigma_t}{dt}(x_t) = -\iota(X_{A_t}(x_t))\nabla \sigma_t + F_{A_t}(\sigma_t(x_t)).
\]
Let \( \{\mu_a\}_a \) be a local frame for \( \mathcal{W} \) on an open set \( U \), the solution \( \sigma_t \) of (1.5) can be written as
\[
(2.26) \quad \sigma_t = \sum_a m^a_t \mu_a,
\]
with \( m^a_t \) a complex function defined on \( U \). We denote by \( \vartheta \) the connection form in this frame; that is, \( \nabla \mu_a = \sum_c \vartheta^c_a \mu_c \). Thus,
\[
(2.27) \quad \nabla \sigma_t = \sum_a \left( dm^a_t + \sum_c m^c_t \vartheta^a_c \right) \mu_a.
\]
On the other hand, \( F_{\mathcal{A}}(\mu_a) = \sum_c (F_t)^c_a \mu_c \), with \( (F_t)^c_a \) a complex function defined on \( U \).

If \( \{x_t\} \) is contained in \( U \), let \( m^a(t) := m^a_t(x_t) \). It follows from (2.25), (2.27) and (2.26)
\[
(2.28) \quad \frac{dm^a(t)}{dt} = \sum_c \left( - \vartheta^a_c(X_{\mathcal{A}}(x_t)) + (F_t)^c_a(x_t) \right) m^c(t).
\]
If \( \dim \mathcal{W} = 1 \), then the above expression reduces to
\[
(2.29) \quad \frac{dm(t)}{dt} = \left( - \vartheta(X_{\mathcal{A}}(x_t)) + h_{\mathcal{A}}(x_t) \right) m(t).
\]
So,
\[
m(t) = m(0) \exp \left( \int_0^t \left( - \vartheta(X_{\mathcal{A}}(x_t)) + h_{\mathcal{A}}(x_t) \right) dt \right).
\]

Next, we consider the particular case when \( G_\mathbb{R} \) is compact and \( \phi \) is a regular dominant weight. In this case \( M \) is the flag manifold of \( \mathfrak{g} \) and it supports the symplectic structure \( \varpi \), given by the 2-form
\[
(2.30) \quad \varpi(X_{\mathcal{A}}(gT_\mathbb{R}), X_{\mathcal{C}}(gT_\mathbb{R})) = \phi((g^{-1} \cdot [A,C])_0).
\]

Now, as \( q \) equals zero, \( \mathcal{W} \) is a complex line bundle and
\[
(2.31) \quad h_{\mathcal{A}}([g, A]) = h_{\mathcal{A}}(g) = \phi(g^{-1} \cdot A)_0).
\]
The curvature \( \mathbf{K} \) projects a 2-form \( \omega \) on \( M \). By Proposition 11 (or by (2.18))
\[
\omega(X_{\mathcal{A}}, X_{\mathcal{C}}) = -h_{[A,C]}; \text{ thus, } \omega = -\varpi. \quad \text{From (2.13), it follows that}
\]
\[
(2.31) \quad dh_{\mathcal{C}} = \varpi(X_{\mathcal{C}}, \cdot).
\]
That is, \( h_{\mathcal{C}} \) is a Hamiltonian function associated with the vector field \( X_{\mathcal{C}} \). If \( g_1 \in Z(G_\mathbb{R}) \), then the evaluation curve \( x_t \) is closed and nullhomologous in \( M \). From Stokes theorem
\[
(2.32) \quad \frac{m(1)}{m(0)} = \exp \left( \int_S \varpi + \int_0^1 h_{\mathcal{A}}(x_t) dt \right),
\]
\( S \) being a 2-chain whose boundary is \( x_t \).

The right hand side of (2.32) is the exponential of the action integral around the loop \( \varphi_t \). From Theorem 12, we deduce the following proposition:

**Proposition 13.** If \( G_\mathbb{R} \) is compact, \( \phi \) is a regular dominant weight and \( g_1 \in Z(G_\mathbb{R}) \), then \( \kappa \) is the exponential of the action integral around the Hamiltonian loop \( \varphi_t \).
From now on until the end of this section, we will concern with the infinitesimal character of the \(U(g)\)-module \(H_{K_a}\).

The “representation” of the associative algebra \(U(g)\) induced by \(\pi'\) on the space \(H_{K_a}\) will also be denoted by \(\pi'\).

We denote the extension of \(A \in g_R \mapsto \Psi'(A_0) \in \mathfrak{gl}(W)\) to a map on \(g\) by \(\psi\); that is,

\[
(2.33) \quad \psi(A + iB) = \Psi'(A_0) + i\Psi'(B_0).
\]

**Proposition 14.** If \(C \in t\) and \(s\) is a \(\Psi\)-equivariant function of \(E\) (i.e. associated with an element of \(H_{K_a}\)), then \(\pi'(C) s = \psi(C) s\).

**Proof.** By \(2.33\), we can assume that \(C\) is an element of \(t_R\). The proposition follows from \(2.21\) together with the fact that \(s\) is \(\Psi\)-equivariant.

Since \(\pi\) is unitary and irreducible, the Harish-Chandra module \(H_{K_a}\) has an infinitesimal character \(\chi : Z(g) \to \mathbb{C}\) (see \([5, Corollary 8.14]\)).

Let \(\{C_1, \ldots, C_T\}\) be a basis of \(t\), and

\[
J' = \sum z_{m_1 \ldots m_r}^{i_1 \ldots i_r} C_{i_1}^{m_1} \ldots C_{i_r}^{m_r} \in Z(g),
\]

where \(z_{m_1 \ldots m_r}^{i_1 \ldots i_r}\) is a complex number and \(m_j\) a positive integer. By Proposition 14, \(\pi'(J')\) acting on \(E\) is the operator

\[
(2.34) \quad \sum z_{m_1 \ldots m_r}^{i_1 \ldots i_r} (\psi(C_{i_1}))^{m_1} \ldots (\psi(C_{i_r}))^{m_r},
\]

which is the multiplication by the constant \(\chi(J')\), since \(J'\) belongs to \(Z(g)\). This result expressed in terms of the corresponding “quantization operators” gives rise to the following proposition:

**Proposition 15.** If \(J' = \sum z_{m_1 \ldots m_r}^{i_1 \ldots i_r} C_{i_1}^{m_1} \ldots C_{i_r}^{m_r} \in Z(g)\), then the differential operator

\[
\sum z_{m_1 \ldots m_r}^{i_1 \ldots i_r} (Q_{C_{i_1}})^{m_1} \ldots (Q_{C_{i_r}})^{m_r},
\]

acting on \(H_{K_a}\) is the scalar operator defined by the constant \(\chi(J')\).

As \(t\) is abelian, the algebra \(U(t)\) can be identified with the algebra of polynomial functions on \(t^*\). In this way, the constant \(2.34\) is \(J'(\psi)\), the evaluation at \(\psi\) of the corresponding polynomial.

Next, we will extend the result stated in Proposition 15 to general elements of \(Z(g)\).

\(\rho\), the half the sum of the positive roots, induces the linear map \(C \in t \mapsto C - \rho(C) \in U(t)\). By the universal property of \(U(t)\), it extends to an algebra automorphism \(\tau\) of \(U(t)\).

By Proposition 5.34 in \([6]\), if \(J \in Z(g)\), then

\[
(2.35) \quad J \in U(t) \oplus \mathcal{P}, \quad \text{where } \mathcal{P} = \sum_{\nu \in \Delta} U(g)E_{\nu}.
\]

The projection of \(J\) into \(U(t)\) will be denoted \(\tilde{J}\).

The algebra of the elements in \(U(t)\) invariant under the Weyl group \(N_G(T)/T\)
(with \(T =\) complexification of \(T_R\)) will be denoted by \(U(t)'\). The Harish-Chandra isomorphism \(\gamma : Z(g) \to U(t)'\) is defined by \(\gamma(J) = \tau(\tilde{J})\) \([6]\).
As the Harish-Chandra module $\mathcal{H}_{K_\mathfrak{s}}$ has an infinitesimal character, there exists $\lambda \in \mathfrak{t}^*$ such that
\begin{equation}
\chi(J) = \gamma(J)(\lambda),
\end{equation}
for all $J \in \mathcal{Z}(\mathfrak{g})$. In particular, for the element $J'$ considered above, $J'(\psi)$ must be equal to $\gamma(J')(\lambda)$; that is,
\begin{equation}
J'(\psi) = (J' - J'(\rho))(\lambda).
\end{equation}
Since $J'(\rho)$, the evaluation of the polynomial $J'$ on $\rho$, is a constant number (i.e. a constant polynomial), $(J' - J'(\rho))(\lambda) = J'(\lambda) - J'(\rho)$. Hence, $\lambda = \psi + \rho$ is the element of $\mathfrak{t}^*$ which determines the infinitesimal character $\chi$ by means of (2.36).

By (2.36), any element $J \in \mathcal{Z}(\mathfrak{g})$ is of the form
\begin{equation}
J = \sum (\text{const}) C_i^{n_i} \ldots C_r^{n_r} + \sum (\text{const}) E_{\nu_1}^{p_1} \ldots E_{-\nu_1}^{p_1} E_{\nu_1}^{q_1} \ldots E_{\nu_1}^{q_1}.
\end{equation}
For abbreviation, we write $J = p(E_{-\nu_1}, C_1, E_\nu)$.

As the space $\mathcal{H}_{K_\mathfrak{s}}$ is dense in $\mathcal{H}$ [15], we can state the following theorem that relates $\chi(J)$ with the corresponding composition of “quantization operators” acting on $\mathcal{H}$:

**Theorem 16.** If $J \in \mathcal{Z}(\mathfrak{g})$ is of the form (2.37), then differential operator
\begin{equation}
p(Q_{E_{-\nu_1}}, Q_{C_1}, Q_{E_\nu})
\end{equation}
acting on the space $\mathcal{H}$ is the scalar operator defined by the constant $\chi(J) = \gamma(J)(\psi + \rho)$.

### 3. Homotopy groups of some subgroups of $\operatorname{Diff}(M)$

In this section, we assume that the spaces of smooth functions between two manifolds are endowed with the Whitney $C^1$-topology; in particular, the Lie algebras of vector fields on a manifold.

We will consider subalgebras $\mathfrak{X}'$ of $\mathfrak{X}(M)$, the Lie algebra consisting of the vector fields on $M$, such that each of its elements admits a lift to a $\operatorname{GL}(W)$-invariant vector field on $\mathcal{P}$ which is an infinitesimal symmetry of the connection $\Omega$. We will deal with the following points related with such an algebra $\mathfrak{X}'$: (i) Let $\{\psi_t \mid t \in [0, 1]\}$ be a loop in the group $\operatorname{Diff}(M)$ generated by a vector field of $\mathfrak{X}'$; we prove that there is a lift of $\psi_t$ to a flow $H_t$ in $\mathcal{P}$, such that $H_1$ is a gauge transformation of $\mathcal{P}$ which preserves $\Omega$. (ii) We check that $H_1$ is the multiplication by a constant on each orbit $\{H_t(p)\}_{t}$. (iii) We will construct loops in $\operatorname{Diff}(M)$ such that the corresponding transformations $H_1$ are the multiplication by a constant on the total space $\mathcal{P}$. (iv) If $\varphi, \varphi'$ are loops of those considered in (iii), such that the corresponding constant are different, then $\varphi$ and $\varphi'$ won’t be homotopic in any Lie subgroup of $\operatorname{Diff}(M)$ whose Lie algebra is contained in $\mathfrak{X}'$.

We state the precise conditions imposed to the algebras $\mathfrak{X}'$. Let $\mathfrak{X}'$ be a Lie subalgebra of $\mathfrak{X}(M)$ satisfying the following conditions:

1. There is a continuous $\mathbb{R}$-linear map $Z \in \mathfrak{X}' \mapsto U(Z) \in \mathfrak{X}(\mathcal{P})$ such that $Z^h$ is the horizontal part of $U(Z)$.
2. For each $Z \in \mathfrak{X}'$, there is a $C^\infty$ map $a(Z) : \mathcal{P} \to \mathfrak{gl}(W)$ such that:
2a) \( a(Z)(p\beta) = \beta^{-1}a(Z)(p)\beta \), for all \( \beta \in \text{GL}(W) \), \( p \in P \). That is, \( a(Z) \) is a pseudotensorial function on \( P \) of type \( \text{Ad} \).

2b) \( U(Z) = Z^2 + V_a(Z) \).

2c) \( Da(Z) = -K(Z^2, \cdot, \cdot) \).

2d) The map \( Z \mapsto a(Z) \) is \( \mathbb{R} \)-linear and continuous.

**Example.** For \( X_C \), with \( C \in \mathfrak{g}_R \) we define \( U(X_C) := Y_C \) and \( a(X_C) := h_C \). By (3.11) and (2.13), the Lie algebra \( X' := \{ X_C \mid C \in \mathfrak{g}_R \} \), with the above choices for \( U(X_C) \) and \( a(X_C) \), satisfies the conditions 1)-2d).

**Theorem 17.** For any \( Z \in X' \), the Lie derivative \( \mathfrak{L}_{U(Z)}\Omega \) vanishes.

**Proof.** We will prove that the 1-form \( d(i_U\Omega) + i_U(d\Omega) \) is zero, where \( U := U(Z) \).

Let \( p \) be an arbitrary point of \( P \), we will check that

\[
(d(i_U\Omega) + i_U(d\Omega))(E) = 0,
\]

when \( E \) is a vertical vector at \( p \) and when \( E \) is horizontal. In any case, by 2b)

\[
d(i_U\Omega) = da,
\]

where \( a := a(Z) \).

If \( E \) is a vertical vector \( E = V_y(p) \), with \( y \in \mathfrak{gl}(W) \), by (3.2) and 2a)

\[
d(i_U\Omega)(E) = da(V_y) = -[y, a(p)].
\]

We extend \( E \) to a vertical vector field. By the structure equation and 2b), we have

\[
(i_U(d\Omega))_p(E) = K_p(U, E) - [\Omega_p(U), \Omega_p(E)] = -[a(p), y].
\]

Thus, (3.1) holds when \( E \) is a vertical vector.

If \( E \) is horizontal, then by (3.2) and 2c)

\[
d(i_U\Omega)(E) = -K(Z^2, E).
\]

Now, we extend \( E \) to an horizontal field; by the structure equation

\[
(i_U(d\Omega))(E) = K(U, E) = K(Z^2, E).
\]

So, (3.1) also holds when \( E \) is a horizontal vector.

Since \( a(Z) \) satisfies 2a) and \( U(Z) = Z^2 + V_a(Z) \), we have the following proposition:

**Proposition 18.** For any \( \beta \in \text{GL}(W) \) and any \( Z \in X' \),

\[
(\mathcal{R}_\beta)_*U(Z) = U(Z).
\]

Let \( \{ Z_t \}_{t \in [0, 1]} \) be a time-dependent vector field on \( M \), with \( Z_t \in X' \). Let \( \{ \psi_t \} \) be the isotopy defined by

\[
\frac{d\psi_t}{dt} = Z_t \circ \psi_t, \quad \psi_0 = \text{Id}_M.
\]

As \( M \) is a homogeneous space that admits a Hermitian metric, the flow \( \psi \) is defined on \([0, 1] \times M \) [9, page 85].

We have the time-dependent vector field \( U_t := U(Z_t) \) on \( P \) and the corresponding flow \( H_t \) which is also defined on \([0, 1] \times P \):

\[
\frac{dH_t(p)}{dt} = U_t(H_t(p)), \quad H_0 = \text{Id}_P.
\]

From Theorem 17, it follows that the diffeomorphism \( H_t \) preserves the connection; that is, \( H_t^*\Omega = \Omega \).
We will prove that \( H \) morphisms of \( f \) (3.9) gauge transformation \( H \) (3.5) \( \Omega = H \). In particular, if \{\psi_t|t \in [0, 1]\} is a closed curve, then \( H_1(p) \in \text{pr}^{-1}(x) \).

Let us assume that \( \psi_t = \psi_1 = \text{Id}_M \), that is \{\psi_t|t \in [0, 1]\} is a loop of diffeomorphisms of \( M \); by Proposition \ref{prop:1} \( H_1 \) is a gauge transformation of \( \mathcal{P} \). Thus, there exists a map \( f: \mathcal{P} \rightarrow \text{GL}(W) \) satisfying \( f(p\beta) = \beta^{-1}f(p)\beta \) and such that \( H_1(p) = pf(p) \), for all \( p \in \mathcal{P} \). As
\[
(3.5) \quad \Omega = H_1^*(\Omega) = f^{-1}df + \text{Ad}(f^{-1}) \circ \Omega,
\]
we deduce
\[
(3.6) \quad df(E) = 0, \quad \text{if } E \text{ is horizontal.}
\]
By Proposition \ref{prop:1} \( \Omega(U_t) = \text{Ad}_{\beta^{-1}}(\Omega(U_t)) \), for all \( \beta \in \text{GL}(W) \). From \ref{eq:3.5}, it follows
\[
\text{Ad}_{\beta^{-1}}(\Omega(U_t)) = f^{-1}df(U_t) + \text{Ad}(f^{-1})(\Omega(U_t)).
\]
Given \( p \in \mathcal{P} \), taking \( \beta = f(p) \), we conclude that
\[
(3.7) \quad df(U_t) = 0.
\]
Thus, \( f \) is constant on the orbit \{\( H_t(p) \mid t \in [0, 1] \}\).

Let \{\( g_t \)\} be a path in \( G_R \) and \{\( A_t \)\} the corresponding velocity curve. Let us assume that \( X_{A_t} \in \mathfrak{X}' \). This family gives rise to the time-dependent vector field on \( \mathcal{P} \), \( U_t := U(X_{A_t}) \), which in turn defines a flow \( H_t \) on \( \mathcal{P} \). By \ref{eq:2.11}, \( U(X_{A_t}) = Y_{A_t} \).

So,
\[
(3.8) \quad \frac{d}{dt} H_t = Y_{A_t} \circ H_t, \quad \mathbf{H}_0 = \text{Id}_P.
\]

**Lemma 19.** The bundle diffeomorphism \( H_t \) defined in \ref{eq:3.8} is the left multiplication by \( g_t \) in \( \mathcal{P} \); i.e. \( H_t = L_{g_t} \).

**Proof.** Given \( p \in \mathcal{P} \)
\[
\frac{d}{dt} \bigg|_{t=0} g_u p = \frac{d}{dt} \bigg|_{t=0} g_u g_t^{-1} g_t p = Y_{A_t}(g_t p).
\]

If \( g_1 \) is an element of the center of \( G_R \), then \( \phi_0 = \phi_1 \) and \( H_1 \) is a gauge transformation of \( \mathcal{P} \).

**Theorem 20.** If \( g_1 \in Z(G_R) \), then for all \( p \in \mathcal{P} \), \( H_1(p) = pk \), where \( k \) is given by \ref{eq:1.3}.

**Proof.** By Lemma \ref{lem:1} together with Corollary \ref{cor:11}
\[
H_1([g, \alpha]) = [g_1, g, \alpha] = [gg_1, \alpha] = [g, \Psi(g_1)\alpha] = [g, \alpha]k.
\]

**Remark.** Note that paths with the same final point in \( Z(G_R) \) determine the same gauge transformation \( H_1 \).

Let \( G \) be a connected Lie subgroup of \( \text{Diff}(M) \) such that:

(i) \( \text{Lie}(G) \subset \mathfrak{X}' \).

(ii) If \{\( g_t \mid t \in [0, 1] \}\} is a path in \( G_R \), then the isotopy \{\( \phi_t \)\} is contained in \( G \).

We will prove that
\[
\# \pi_1(G) \geq \# \{\Psi(g) \mid g \in Z(G_R)\},
\]
the loop $\varphi$ is a lift to $C$. As before, $f$ but we need some previous results.

By hypothesis, $\{\zeta^s\}$ is as close to $\varphi$ in $G$. Then the gauge transformation is a horizontal curve with respect the invariant connection in $M$ (see Figure 1). If $\{\zeta^s\}$ is a deformation of $\varphi$ in $G$, then the gauge transformation $H^1_\gamma$ of $P$ defined by $\zeta^s$ satisfies

$$H^1_\gamma(p) = pk \text{ for all } p \in (pr)^{-1}(eT_R),$$

with $\kappa Id_W = \Psi(g_1)$.

**Proof.** Let $x_0$ denote the point $eT_R \in M$. We have the following closed curves in $M$:

$$\{\varphi(x_0) \mid t \in [0, 1]\}, \{\zeta^s(x_0) \mid t \in [0, 1]\}.$$

By hypothesis, $\{\zeta^s\}$ is the horizontal lift of $\{\varphi(x_0)\}$ at the point $e \in G_R$.

Fixed $s$, for $a \leq s$, let $\gamma^a$ be the curve in $G_R$ horizontal lift of $\{\zeta^s(x_0) \mid t \in [0, 1]\}$ at the point $e$. We can construct a path $\{\hat{g}_t \mid t \in [0, 1]\}$ in

$$\{\gamma^a \mid a \in [0, s], t \in [0, 1]\} \subset G_R,$$

with $\hat{g}_1 = g_1$ and such that the loop $\{\hat{g}_t x_0 \mid t \in [0, 1]\}$ is as close to $\{\zeta^s(x_0) \mid t \in [0, 1]\}$ as we wish; hence, $\{\hat{g}_t x_0\}_t$ is contained in an arbitrarily small tubular neighborhood of $\{\zeta^s(x_0)\}_t$ in $M$ (see Figure 1).

By (3.9) (ii), $\{\hat{g}_t\}$ defines a loop in $G$. With $\{\hat{g}_t\}$ we construct the corresponding flow $\tilde{H}_\gamma$ in $P$ (see (3.8)). Fix an arbitrary point $p$ in $P$ such that $pr(p) = x_0$. The curves in $P$ $\{H_\gamma(p)\}_t$ and $\{H^*_\gamma(p)\}_t$, which are lifts at $p$ of $\{\hat{g}_t x_0 \mid t \in [0, 1]\}$ and $\{\zeta^s(x_0) \mid t \in [0, 1]\}$, will be as close to each other as we will wish.

Furthermore, $\tilde{H}_\gamma$ is defined by a $GL(W)$-valued map $\hat{f}$ on $P$. According to the Remark below Theorem 20, $\hat{f} = \kappa Id_{GL(W)}$, with $\kappa Id_W = \Psi(g_1)$ (Corollary 11).

Thus, for each tubular neighborhood of $\{H^*_\gamma(p) \mid t \in [0, 1]\}$ in $P$, there is a loop in $G$, defined by a path $\{\hat{g}_t\}$, which in turn determines a flow $\tilde{H}_\gamma$ in $P$ such that the corresponding evaluation curve $\{\tilde{H}_\gamma(p)\}$ is in that neighborhood and the gauge transformation $H^1_\gamma$ is the multiplication by $\kappa$. On the other hand, $H^1_\gamma$ on
Figure 1. The path \( \hat{g}_t \)

\[ \{ H_t(p) \mid t \in [0, 1] \} \] is defined by the constant function \( f^* \). By continuity, \( f^*(p') = p' \kappa \), for any point \( p' \) in \( \{ H_t(p) \}_t \).

\[ \square \]

**Corollary 22.** Let \( \{ g_t \} \) and \( \{ g'_t \} \) be paths in \( G_\mathbb{R} \) satisfying the hypotheses of Theorem 21. If \( \Psi(g_1) \neq \Psi(g'_1) \), then the corresponding loops \( \varphi = \{ \varphi_t \} \) and \( \varphi' = \{ \varphi'_t \} \) are not homotopic in \( G \).

The Lie algebra of the holonomy group at \( e \) of the invariant connection on \( G_\mathbb{R} \) is generated by the vectors of the form \( [A, C]_0 \), with \( A, C \in \mathfrak{g} \) (see [7, Theorem 11.1]). On the other hand, the vectors \( [E, E_\nu] \) with \( \nu \in \Delta \) span \( \mathfrak{t}_\mathbb{R} \). So, from (2.2), it follows that the Lie algebra of the mentioned holonomy group is \( \mathfrak{t}_\mathbb{R} \). Since \( Z(G_\mathbb{R}) \subset \mathfrak{t}_\mathbb{R} \), each element of \( Z(G_\mathbb{R}) \) can be joined to \( e \) by a horizontal curve in \( G_\mathbb{R} \) (horizontal with respect to the invariant connection). Thus, from Corollary 22, it follows the following theorem:

**Theorem 23.** If \( \mathcal{G} \) is a connected Lie subgroup of \( \text{Diff}(M) \), for which the conditions (3.9) hold, then

\[
\sharp(\pi_1(\mathcal{G})) \geq \sharp(\{ \Psi(g) \mid g \in Z(G_\mathbb{R}) \}).
\]

As we said in Section 2 when \( G_\mathbb{R} \) is compact and \( \phi \) is a regular dominant weight, \( (M, \varpi) \) is the coadjoint orbit associated with \( \phi \); thus, it is a compact symplectic manifold. In this case, as we will see, a possible subalgebra \( \mathfrak{X}' \) is the one consisting of all locally Hamiltonian vector fields; that is, the set of all vector fields \( Z \) on \( M \) such that \( \mathfrak{L}_Z \varpi = 0 \).

In fact, \( M \) is simply connected [2, page 33]; so, such a vector field \( Z \) is Hamiltonian. We denote by \( a(Z) \) the corresponding normalized Hamiltonian function; that is, \( a(Z) \) is the function on \( M \) such that

\[
da(Z) = \iota_Z \varpi, \quad \text{and} \quad \int_M a(Z) \varpi^n = 0,
\]
2n being the dimension of $M$. We put $U(Z)$ for the vector field on $P$ defined by $U(Z) = Z^2 + V_a(Z)$. As $K$ projects on $M$ the form $-\omega$ (see paragraph before (2.31)), it is easy to check that the algebra $\mathcal{X}$ of Hamiltonian vector fields on $M$, the Hamiltonian functions $a(Z)$ and the vector fields $U(Z)$ satisfy the conditions $1), 2\alpha) - 2d)$ introduced at the beginning of this Section.

Since any Lie subgroup $G$ of $\text{Symp}(M, \omega)$, the group of all symplectomorphisms of $(M, \omega)$, satisfies the condition (3.9)(i), it follows from Theorem 23 the following result:

**Theorem 24.** Let us assume that $G_\mathbb{R}$ is compact and that $\phi$ is a regular dominant weight. If $G$ is any connected Lie subgroup of $\text{Symp}(M, \omega)$ which contains $G_\mathbb{R}$, then

$$\sharp(\pi_1(G)) \geq \sharp\{\Phi(g) \mid g \in Z(G_\mathbb{R})\}.$$

**Remark.** For $G_\mathbb{R} = SU(2)$, the corresponding flag manifold is $\mathbb{C}P^1$. Given $[z_0 : z_1] \in \mathbb{C}P^1$ with $z_0 \neq 0$, we put $x + iy = z_1/z_0$. It is easy to check that the vector fields $X_C$ and $X_D$ defined by the matrices of $su(2)$

$$C = \text{skew diagonal } (-c, c), \quad D = \text{skew diagonal } (d, di),$$

take at the point $(x = 0, y = 0)$ the values $X_C = -c \partial_x, \quad X_D = d \partial_y$. We denote by $\omega$ the Fubini-Study form on $\mathbb{C}P^1$, then $\omega[1:0](X_C, X_D) = -cd/\pi$.

On the other hand, let $\phi$ be the weight defined by $\phi(\text{diagonal}(ai, -ai)) = a$. Then $\omega[1:0](X_C, X_D) = \phi([C, D]) = 2cd$. By the invariance of $\omega$ and $\omega$ under the action of $SU(2)$ we conclude that $\omega = -2\pi \omega$. So, the Hamiltonian groups of $(\mathbb{C}P^1, \omega)$ and $(\mathbb{C}P^1, \omega)$ are isomorphic. By Theorem 23 $\sharp\pi_1(\text{Ham}(\mathbb{C}P^1, \omega)) \geq 2$. This result is consistent with the fact that $\pi_1(\text{Ham}(\mathbb{C}P^1, \omega))$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ [12, page 52].

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