ON THE CENTER CONJECTURE FOR THE CYCLOTOMIC KLR ALGEBRAS

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Abstract. The center conjecture for the cyclotomic KLR algebras \( R^\Lambda_\beta \) asserts that the center of \( R^\Lambda_\beta \) consists of symmetric elements in its KLR \( x \) and \( e(\nu) \) generators. In this paper we show that this conjecture is equivalent to the injectivity of some natural map \( \iota_{\Lambda, i, \beta} \) from the cocenter of \( R^\Lambda_\beta \) to the cocenter of \( R^{\Lambda+\Lambda_i}_\beta \) for all \( i \in I \) and \( \Lambda \in P^+ \). We prove that the map \( \iota_{\Lambda, i, \beta} \) is given by multiplication with a center element \( z(i, \beta) \in R^{\Lambda+\Lambda_i}_\beta \) and we explicitly calculate the element \( z(i, \beta) \) in terms of the KLR \( x \) and \( e(\nu) \) generators. We present an explicit monomial basis for certain bi-weight spaces of the defining ideal of \( R^\Lambda_\beta \) and of \( R^{\Lambda}_\beta \). For \( \beta = \sum_{j=1}^n \alpha_{i_j} \) with \( \alpha_{i_1}, \ldots, \alpha_{i_n} \) pairwise distinct, we construct an explicit monomial basis of \( R^\Lambda_\beta \), prove the map \( \iota_{\Lambda, i, \beta} \) is injective and thus verify the center conjecture for these \( R^\Lambda_\beta \).

Introduction

Given a Catan datum or a symmetrizable Cartan matrix, Khovanov-Lauda ([17, 18]) and Rouquier ([22]) introduced a vast family of algebras \( R_\beta \), called KLR algebras or quiver Hecke algebras, and use them to provide a categorification of the negative part of the quantum groups associated to the same Cartan datum. These algebras play an important role in the categorical representation theory of 2-Kac-Moody algebras. Khovanov, Lauda and Rouquier also defined their graded cyclo-
tomic quotients \( R^\Lambda_\beta \) associated with an integral dominant weight \( \Lambda \) and conjectured that they categorify the corresponding irreducible integrable highest weight module of the quantum groups. This conjecture was later proved by Kang-Kashiwara ([16]) and Webster ([26]).

The KLR algebras generalize the affine Hecke algebras in many aspects of structure and representation theory. For example, both of them admit a faithful polynomial representation and have monomial bases. The center of a KLR algebra is proved to consist of all symmetric elements in its KLR generators \( x_1, \ldots, x_n \) and \( e(\nu) \), which is
similar to the well-known Bernstein theorem on the centers of affine Hecke algebras. In the case of type $A^{(1)}_e$ and $A_\infty$, Brundan and Kleshchev ([6]) proved these cyclotomic quotients $R^A_\beta$ are isomorphic to the block algebras of the cyclotomic Hecke algebras of type $A$. The graded cellular bases of the cyclotomic quotients $R^A_\beta$ have been constructed in [13], [19], [20], [21] in some finite and affine types. For more results on the structure and representation theory of the cyclotomic KLR algebras $R^A_\beta$, see [1, 2, 3, 4, 14, 25].

One of the major unsolved open problems on $R^A_\beta$ is the center conjecture, which asserts that the centers of $R^A_\beta$ consists of symmetric elements in its KLR $x$ and $e(\nu)$ generators. Using Brundan-Kleshchev’s isomorphism, one can reduce the center conjecture for $R^A_\beta$ to the claim that the center of the cyclotomic Hecke algebra consists of symmetric polynomials in its Jucys-Murphy operators. The latter was proved for the degenerated cyclotomic Hecke algebras by Brundan ([5]). In the level one case, the latter is just the Dipper-James conjecture ([8]) for the Iwahori-Hecke algebras of type $A$, which was proved by Francis and Graham ([10]).

In this paper we show that the center conjecture for general $R^A_\beta$ is equivalent to the claim that the natural homomorphism $\overline{\tau}^A_{i,\beta}$ from the center of $R^{A+\Lambda}_\beta$ to the center of $R^A_\beta$ is surjective for any $i \in I$ and any $\Lambda \in P^+$ (which was first observed in [11]). The latter is clearly equivalent to the injectivity of the naturally induced dual map $\iota^A_{i,\beta}$ from the cocenter $R^A_\beta/[R^A_\beta,R^A_\beta]$ of $R^A_\beta$ to the cocenter $R^{A+\Lambda}_\beta/[R^{A+\Lambda}_\beta,R^{A+\Lambda}_\beta]$ of $R^{A+\Lambda}_\beta$ for any $i \in I$ and any $\Lambda \in P^+$. We show that the map $\iota^A_{i,\beta}$ is given by multiplication with a center element $z(i, \beta)$ and we explicitly calculate the element $z(i, \beta)$ in terms of the KLR $x$ and $e(\nu)$ generators. In the case of type $A^{(1)}_e$ and $A_\infty$, the multiplication with the center element $z(i, \beta)$ actually sends each graded cellular basis element (resp., monomial basis element) of $R^A_\beta$ to a graded cellular basis element (resp., monomial basis element) of $R^{A+\Lambda}_\beta$. Thus the studying of the injectivity of $\overline{\tau}^A_{i,\beta}$ seems to be a rather promising approach (which we named it as “cocenter approach”) as long as one can construct a basis of the cocenter represented by some graded cellular basis or monomial basis elements. In this paper, we construct a monomial basis for certain bi-weight space of the defining ideal of $R^A_{\beta}$. For $\beta = \sum_{j=1}^{n} \alpha_{i_j}$ with $\alpha_{i_1}, \ldots, \alpha_{i_n}$ pairwise distinct, we construct an explicit monomial basis for the cocenter of $R^A_{\beta}$. We introduce a new combinatorial notion of “indecomposable
relative to $\nu$ in Definition 4.3 which seems of independent interest. This notion plays a key role in our construction of the monomial basis of the cocenter. We then apply the “cocenter approach” to show that in this case the map $\mathcal{I}^\Lambda_i$ is injective and thus verify the center conjecture for these $\mathcal{R}_\beta^\Lambda$.

This paper is organized as follows. In Section 1, we recall some basic definitions and properties of the KLR algebra $\mathcal{R}_\beta$ and its cyclotomic quotient $\mathcal{R}_\beta^\Lambda$. In Section 2, we present our cocenter approach to the center conjecture for general $\mathcal{R}_\beta^\Lambda$. We show that in the center conjecture is equivalent to the injectivity of the naturally induced dual map $\mathcal{I}^\Lambda_i$ from the cocenter $\mathcal{R}_\beta^\Lambda/[\mathcal{R}_\beta^\Lambda, \mathcal{R}_\beta^\Lambda]$ of $\mathcal{R}_\beta^\Lambda$ to the cocenter $\mathcal{R}_\beta^\Lambda+A_i/[\mathcal{R}_\beta^\Lambda+A_i, \mathcal{R}_\beta^\Lambda+A_i]$ of $\mathcal{R}_\beta^\Lambda+A_i$ for any $i \in I$ and any $\Lambda \in P^+$, and the map $\mathcal{I}^\Lambda_i$ is given by multiplication with a center element $z(i, \beta)$. The main result is Theorem 2.30, where we explicitly calculate the element $z(i, \beta)$ in terms of the KLR $x$ and $e(\nu)$. The proof makes essential use of the explicit formulae for the bubbles presented in [24, Definition A.1, Proposition B.3]. In Section 3, we present an explicit monomial basis (in Theorem 3.3 and Corollary 3.10) for certain bi-weight spaces of the defining ideal of $\mathcal{R}_\beta^\Lambda$ and of the whole $\mathcal{R}_\beta^\Lambda$. In Section 4, we study the cyclotomic KLR algebras when $\beta = \sum_{j=1}^n \alpha_{i_j}$ with $\alpha_{i_1}, \ldots, \alpha_{i_n}$ pairwise distinct. We introduce a combinatorial notion of “indecomposable relative to $\nu$” in Definition 4.3 and give a complete characterization of all indecomposable elements in $\mathfrak{S}_n$ which is relative to a given $\nu \in I^\beta$ in Lemma 4.5. We construct in Theorem 4.8 an explicit monomial basis for the cocenter of $\mathcal{R}_\beta^\Lambda$. Finally, we show in Lemma 4.14 that in this case the map $\mathcal{I}^\Lambda_i$ is injective, and thus obtain in Theorem 4.16 that the Center Conjecture 1.9 holds for these $\mathcal{R}_\beta^\Lambda$.

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1. Preliminary

In this section, we shall recall some basic definitions and properties of the KLR algebras and their cyclotomic quotients.
Let $I$ be an index set. An integral square matrix $A = (a_{i,j})_{i,j \in I}$ is called a symmetrizable generalized Cartan matrix if it satisfies

1. $a_{ii} = 2, \forall i \in I$;
2. $a_{ij} \leq 0 (i \neq j)$;
3. $a_{ij} = 0 \iff a_{ji} = 0 (i, j \in I)$;
4. there is a diagonal matrix $D = \text{diag}(d_i \in \mathbb{Z}_{>0} \mid i \in I)$ such that $DA$ is symmetric.

A Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ consists of

1. a symmetrizable generalized Cartan matrix $A$;
2. a free abelian group $P$ of finite rank, called the weight lattice;
3. $\Pi = \{\alpha_i \in P \mid i \in I\}$, called the set of simple roots;
4. $P^\vee := \text{Hom}(P, \mathbb{Z})$, called the dual weight lattice and $\langle \cdot, \cdot \rangle : P^\vee \times P \to \mathbb{Z}$ the natural pairing;
5. $\Pi^\vee = \{\alpha_i^\vee \mid i \in I\} \subset P^\vee$, called the set of simple coroots;

satisfying the following properties:

1. $\langle \alpha_i^\vee, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
2. $\Pi$ is linearly independent,
3. $\forall i \in I, \exists \Lambda_i \in P$ such that $\langle \alpha_i^\vee, \Lambda_j \rangle = \delta_{ij}$ for all $j \in I$.

Those $\Lambda_i$ are called the fundamental weights. We denote by

$$P^+ := \{\lambda \in P \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I\}$$

the set of dominant integral weights. The free abelian group $Q := \bigoplus_{i \in I} \mathbb{Z} \alpha_i$ is called the root lattice. Set $Q^+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For $\beta = \sum_{i \in I} k_i \alpha_i \in Q^+$, we define the height of $\beta$ to be $|\beta| = \sum_{i \in I} k_i$. For each $n \in \mathbb{N}$, we set $Q^+_n := \{\beta \in Q^+ \mid |\beta| = n\}$. Let $\mathfrak{h} = Q \otimes_{\mathbb{Z}} P^\vee$. Since $A$ is symmetrizable, there is a symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ satisfying

$$(\alpha_i, \alpha_j) = d_i a_{ij} \quad (i, j \in I) \quad \text{and}$$

$$\langle \alpha_i^\vee, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)} \text{ for any } \lambda \in \mathfrak{h}^* \text{ and } i \in I.$$
Let $k = \oplus_{n \in \mathbb{N}} k_n$ be a graded commutative noetherian unital ring such that $k_0 = \mathbb{k}$ is a field. We fix a matrix $(Q_{i,j})_{i,j \in I}$ in $k[u,v]$ such that

\[
Q_{i,j}(u,v) = Q_{j,i}(v,u),
\]
\[
Q_{i,i}(u,v) = 0,
\]
\[
Q_{i,j}(u,v) = \sum_{p,q \geq 0} c_{i,j,p,q} u^p v^q \text{ if } i \neq j.
\]

where $c_{i,j,p,q} \in k_{2(\alpha_i, \alpha_j) - (\alpha_i, \alpha_i) - (\alpha_j, \alpha_j)q}$ and $c_{i,j,-a_{ij},0} \in k_0^\times$.

We denote by $\mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle$ the symmetric group on $n$ letters, where $s_i = (i, i+1)$ is the transposition on $i, i+1$. Then $\mathfrak{S}_n$ acts naturally on $I_n$ by:

\[
w.\nu := (\nu_{w^{-1}(1)}, \ldots, \nu_{w^{-1}(n)}),
\]

where $\nu = (\nu_1, \ldots, \nu_n) \in I_n$. The orbits of this action is identified with element of $Q_n^+$ and we denote by $I^\beta$ the corresponding orbit. For any $k, m \in \mathbb{N}$ with $k \leq m$, we set $[k, m] := \{k, k+1, \ldots, m\}$. 

**Definition 1.1.** The Khovanov-Lauda-Rouquier (KLR) algebra $R_\beta$ associated with a Cartan datum $(A, P, \Pi, P^\vee, \Pi^\vee)$, $(Q_{i,j})_{i,j \in I}$ and $\beta \in Q_n^+$ is the associative algebra over $k$ generated by $e(\nu) (\nu \in I^\beta)$, $x_k (1 \leq k \leq n)$, $\tau_l (1 \leq l \leq n-1)$ satisfying the
following defining relations:
\[ e(\nu) e(\nu') = \delta_{\nu, \nu'} e(\nu), \quad \sum_{\nu \in I^3} e(\nu) = 1, \]
\[ x_k x_l = x_l x_k, \quad x_k e(\nu) = e(\nu) x_k, \]
\[ \tau e(\nu) = e(s_l(\nu)) \tau_l, \quad \tau_l \tau_l = \tau_l \tau_l \quad \text{if } |k - l| > 1, \]
\[ \tau_k^2 e(\nu) = Q_{\nu_k, \nu_k+1}(x_k, x_{k+1}) e(\nu), \]
\[ (\tau_k x_l - x_{s_l(l)} \tau_k) e(\nu) = \begin{cases} -e(\nu) & \text{if } l = k, \nu_k = \nu_k + 1, \\ e(\nu) & \text{if } l = k + 1, \nu_k = \nu_k + 1, \\ 0 & \text{otherwise}, \end{cases} \]
\[ (\tau_k \tau_{k+1} \tau_{k+1} - \tau_k \tau_{k+1} \tau_k) e(\nu) = \begin{cases} \frac{Q_{\nu_k, \nu_{k+1}}(x_k, x_{k+1}) - Q_{\nu_{k+2}, \nu_{k+1}}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) & \text{if } \nu_k = \nu_k + 2, \\ 0 & \text{otherwise}. \end{cases} \]

In particular, \( R_0 \cong k \), and \( R_{\alpha_i} \) is isomorphic to \( k[x_1] \). We will write \( e(\beta - \alpha_i, i) = \sum e(\nu) \) and more generally, for \( \beta \geq \beta' \in Q^e_{n-l+1} \), and \( \nu' \in I^{\beta'} \),

\[ e(\beta - \beta', \nu') := \sum_{e(\nu) \in I^{\beta} \atop (\nu_1, \ldots, \nu_n) = \nu'} e(\nu) \]

We also abbreviate \( e(i \cdots i) \) as \( e(i^k) \).

The algebra \( R_{\beta} \) is \( \mathbb{Z} \)-graded whose grading is given by

\[ \deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg \tau_l e(\nu) = -(\alpha_{\nu_l}, \alpha_{\nu_l+1}). \]

For any \( f \in k[x_1, \ldots, x_n] \), define \( \partial_k(f) = \frac{I_{-s_k(l)}}{x_{k+1} - x_k} \) and call \( \partial_k \) the \( k \)-th Demazure operator.

**Lemma 1.2.** For any \( \nu \in I^3 \) with \( \nu_k = \nu_{k+1} \) and any \( f \in k[x_1, \ldots, x_n] \), we have

\[ f \tau_k e(\nu) = (\partial_k(f) + \tau_k f + \tau_k f \tau_k(x_k - x_{k+1})) e(\nu), \]

\[ \tau_k f \tau_k e(\nu) = \tau_k \partial_k(f) e(\nu), \]
Proof. These follow from the defining relations in Definition 1.1 and some induction on the degree of $f$. \qed

Proposition 1.5 ([17, 22, 23]). $R_{\beta}$ is a free $k$-module with basis

$$S_{\beta} = \{ \tau_w x^a e(\nu) \mid \nu \in I^\beta, w \in S_n, a = (a_1, \ldots, a_n) \in \mathbb{N}^n, 1 \leq i \leq n \},$$

where $\tau_w := \tau_{i_1} \cdots \tau_{i_r}$ with $w = s_{i_1} \cdots s_{i_r}$ a preferred choice of reduced decomposition of $w$ and $x^a := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$.

This basis is called the homogeneous monomial basis of $R_{\beta}$.

The centers of KLR algebras are also well-known. Recall that $S_n$ acts naturally on the polynomial subalgebra $P_{\beta} := \bigoplus_{\nu \in I^\beta} k[x_1, x_2, \ldots, x_n] e(\nu)$ by

$$w f(x_1, \cdots, x_n) := f(x_{w1}, \cdots, x_{wn}), \quad w(e(\nu)) := e(w.\nu).$$

We have

Proposition 1.6 ([17, 22]). The center $Z(R_{\beta})$ of $R_{\beta}$ consists of $S_n$-fixed points (i.e. symmetric elements on $x_1, \cdots, x_n, e(\nu), \nu \in I^\beta$) of $P_{\beta}$:

$$Z(R_{\beta}) = \left( \bigoplus_{\nu \in I^\beta} k[x_1, \ldots, x_n] e(\nu) \right)^{S_n}.$$

Let $\Lambda \in P^+$ be a dominant integral weight. We now study the cyclotomic KLR algebra $R_{\Lambda,\beta}$. For $i \in I$, choose a monic polynomial of degree $\langle \alpha_i^\vee, \Lambda \rangle$:

$$a_i^\Lambda(u) = \sum_{k=0}^{\langle \alpha_i^\vee, \Lambda \rangle} c_{i;k} u^{\langle \alpha_i^\vee, \Lambda \rangle - k}$$

with $c_{i;k} \in k_{\langle \alpha_i, \alpha_i \rangle}$ and $c_{i;0} = 1$.

For $\beta \in \mathbb{Q}_n^+$ and $1 \leq k \leq n$, we define

$$a^\Lambda_{\beta}(x_k) = \sum_{\nu \in I^\beta} a^\Lambda_{\nu_k}(x_k) e(\nu) \in R_{\beta}.$$

Hence $a^\Lambda(x_k)e(\nu)$ is a homogeneous element of $R_{\beta}$ with degree $2(\alpha_{\nu_k}, \Lambda)$.

Definition 1.7. Set $I_{\Lambda,\beta} = R_{\beta}a^\Lambda_{\beta}(x_1)R_{\beta}$. The cyclotomic KLR algebra $R_{\Lambda,\beta}$ is defined to be the quotient algebra

$$R_{\Lambda,\beta} = R_{\beta}/I_{\Lambda,\beta}.$$
The following is proved in [16, Corollary 4.1, Theorem 4.5].

**Proposition 1.8.** The k-algebra $R^\Lambda_\beta$ is a free k-module of finite rank.

A natural question arose: how to find out a subset $S^\Lambda_\beta$ of $S_\beta$ in Proposition 1.5, so that the canonical image of $S^\Lambda_\beta$ in $R^\Lambda_\beta$ gives rise to a basis of $R^\Lambda_\beta$. We will answer this question in Section 3 when $\beta = \sum_{j=1}^n \alpha_{ij}$ with $\alpha_{i1}, \ldots, \alpha_{in}$ pairwise distinct. Another unsolved open problems on $R^\Lambda_\beta$ is the following center conjecture which reveals the relation between the centers of KLR algebras and their cyclotomic quotients.

**Center Conjecture 1.9.** Let $\beta \in Q_\mathbb{N}^+$ and $\Lambda \in P^+$. Let $p^\Lambda_\beta : R_\beta \rightarrow R^\Lambda_\beta$ be the canonical surjection. Then the induced map $p^\Lambda_\beta|_{Z(R_\beta)} : Z(R_\beta) \rightarrow Z(R^\Lambda_\beta)$ is always surjective. In particular, the center $Z(R^\Lambda_\beta)$ consists of symmetric elements in its KLR $x$ and $e(\nu)$ generators.

In Section 2 we will give an approach to this conjecture and justify our approach in Section 4 when $\beta = \sum_{j=1}^n \alpha_{ij}$ with $\alpha_{i1}, \ldots, \alpha_{in}$ pairwise distinct.

## 2. The center conjecture and distinguished central elements

In this section, we assume that $k = k_0$ is a field. Fix $\Lambda \in P^+$ and $i \in I$. For any $j \in I$, $a^\Lambda_j(u) = u^{(\alpha'_j, \Lambda)}$ and $a^{\Lambda + \Lambda_i}(u) = u^{(\alpha'_j, \Lambda + \Lambda_i)}$. In particular, $a^\Lambda_j(u)|a^{\Lambda + \Lambda_i}(u)$. Therefore, $I_{\Lambda + \Lambda_i, \beta} \subset I_{\Lambda, \beta}$, we have canonical surjection $p^\Lambda_{\beta, i} : R_{\Lambda + \Lambda_i} \rightarrow R^\Lambda_\beta$. Furthermore, for any $\Lambda, \Lambda' \in P^+$, we have a canonical surjection $p^\Lambda_{\beta} : R^{\Lambda + \Lambda'}_\beta \rightarrow R^\Lambda_\beta$. When $\beta, \Lambda, \Lambda'$ are clear in the context, we write $p_{\beta}$ instead of $p^\Lambda_{\beta, \Lambda'}$ for simplicity. These induce canonical homomorphisms between centers $\overline{p}^\Lambda_{\beta} : Z(R^\Lambda_\beta) \rightarrow Z(R^\Lambda_\beta)$.

**Theorem 2.1.** Let $\beta \in Q_\mathbb{N}^+$. Then the center conjecture 1.9 holds for any $\Lambda \in P^+$ if and only if for any $\Lambda \in P^+$ and any $i \in I$, $\overline{p}^\Lambda_{\beta, i}$ is surjective.
Proof. If the center conjecture 1.9 holds, then clearly for any \( \Lambda \in P^+ \) and any \( i \in I \), \( \overline{p}^\Lambda_i \) is surjective.

Conversely, assume that for any \( \Lambda \in P^+ \) and any \( i \in I \), \( \overline{p}^\Lambda_i \) is surjective. Then it immediately implies that for any \( \Lambda, \Lambda' \in P^+ \), \( \overline{p}^\Lambda_{\Lambda'} \) is surjective.

Now, we fix \( \Lambda \in P^+ \) and let \( z^\Lambda \in Z(\mathcal{R}_\beta^+) \) be a homogeneous element of degree \( d_z \). Note that by Proposition 1.5, every homogeneous element in \( \mathcal{R}_\beta \) has degree \( \geq -2m_0 \ell(w_0) = -m_0 n(n-1) \), where \( w_0 \) is the unique longest element in \( S_n \), \( m_0 := \max \{d_i | i \in I\} \).

We can take \( \Lambda' \in P^+ \) with \( \langle \alpha_i, \Lambda' \rangle \gg 0 \) for any \( i \in I \), such that \( I_{\Lambda+\Lambda', \beta} \subset (\mathcal{R}_\beta)^{d_z+d} \), where

\[
d := \max \{0, \deg \tau e(\nu), \deg x_k e(\nu) | \nu \in I^\beta, 1 \leq l < n, 1 \leq k \leq n\}.
\]

Since \( \overline{p}^\Lambda_{\Lambda'} \) is surjective, we can choose \( z^{\Lambda+\Lambda'} \in Z(\mathcal{R}_\beta^{\Lambda+\Lambda'}) \) such that \( p^\Lambda_{\Lambda'}(z^{\Lambda+\Lambda'}) = z^\Lambda \). Let \( z \in \mathcal{R}_\beta \) be a homogeneous element of degree \( d_z \) such that \( p^\Lambda_{\Lambda'}(z) = z^{\Lambda+\Lambda'} \).

Since \( p^\Lambda_{\beta} = p^\Lambda_{\beta} \circ p^\Lambda_{\Lambda'} \), it follows that \( p^\Lambda_{\beta}(z) = z^\Lambda \). The assumption \( z^{\Lambda+\Lambda'} \in Z(\mathcal{R}_\beta^{\Lambda+\Lambda'}) \) implies that for any \( y \in \{e(\nu), \tau e(\nu), x_k e(\nu) | \nu \in I^\beta, 1 \leq l < n, 1 \leq k \leq n\} \),

\[
zy - yz \in I_{\Lambda+\Lambda', \beta}.
\]

Combining this with the assumption \( I_{\Lambda+\Lambda', \beta} \subset (\mathcal{R}_\beta)^{d_z+d} \), we can deduce that

\[
zy - yz = 0, \quad \forall y \in \{e(\nu), \tau e(\nu), x_k e(\nu) | \nu \in I^\beta, 1 \leq l < n, 1 \leq k \leq n\}.
\]

In other words, \( z \in Z(\mathcal{R}_\beta) \). This completes the proof of the theorem. \( \square \)

This theorem can also be reinterpreted in terms of cocenters since \( \mathcal{R}_\beta^\Lambda \) is a \( k \)-symmetric algebra. Here we only give the definition of symmetric algebras, for more properties, refer to [22] for more details.

**Definition 2.2.** Let \( R \) be a commutative domain and \( A \) an \( R \)-algebra which is finitely generated projective as \( R \)-module. An \( R \)-linear morphism \( t : A \rightarrow R \) is called a symmetrizing form if the morphism

\[
\hat{t} : A \rightarrow \text{Hom}_R(A, R), \quad a \mapsto (a' \mapsto t(a'a))
\]

is an \((A, A)\)-bimodule isomorphism. In this case, we call \((A, t)\) (or simply \( A \)) a symmetric algebra over \( R \).
In particular, if \( t : A \to R \) is a symmetrizing form, then \( t(aa') = t(a'a), \forall a, a' \in A \). The map \( t \) induces a perfect pairing on \( A \):

\[
A \times A \to R, \quad (a, a') \mapsto t(aa'),
\]

and a perfect pairing

\[
Z(A) \times A/[A, A] \to R, \quad (a, a') \mapsto t(aa')
\]

whenever \( R \) is a field.

Let \( p : A \twoheadrightarrow B \) be a surjective homomorphism between two symmetric \( k \)-algebras \((A, t_A)\) and \((B, t_B)\). The dual of \( p \) is an injective \((A, A)\)-bimodule homomorphism \( \iota : B \hookrightarrow A \). By restriction, the map \( p \) induces a homomorphism \( \overline{p} : Z(A) \to Z(B) \) between the two centers, which may not be surjective in general, Via the above perfect pairing the map \( \iota \) induces a homomorphism \( \iota^* : B/[B, B] \to A/[A, A] \) of \((Z(A), Z(A))\)-bimodules.

**Lemma 2.3.** Let \( p : A \twoheadrightarrow B \) be a surjective algebraic homomorphism between two symmetric \( k \)-algebras \((A, t_A)\) and \((B, t_B)\). Let \( \iota := p^* : B \hookrightarrow A \) be the dual of \( p \). Then \( z := \iota(1_B) \in A \) is the unique element in the center \( Z(A) \) of \( A \) which satisfies that \( t_A(az) = t_B(p(a)), \forall a \in A \). Moreover, we have

\[
\text{Ann}_A(z) = \text{Ker}(p), \quad \iota(p(a)) = az, \quad \forall a \in A.
\]

**Proof.** By the definition of dual map, we see that \( z := \iota(1_B) \in A \) satisfies that \( t_A(az) = t_B(p(a)), \forall a \in A \).

For any \( a, b \in A \), we have

\[
t_A((az - za)b) = t_A(azb) - t_A zab = t_A(baz) - t_A(abz) = t_A((ba - ab)z) = t_B(p(ba - ab)) = t_B(p(a)p(b) - p(b)p(a)) = 0.
\]

It follows that \( za - az = 0 \) and hence \( z = \iota(1_B) \in A \) is in the center \( Z(A) \) of \( A \). If \( z' \in Z(A) \) also satisfies that \( t_A(az') = t_B(p(a)), \forall a \in A \). Then \( t_A(a(z - z')) = t_B(p(a)) - t_B(p(a)) = 0, \forall a \in A \), which implies that \( z = z' \).

Let \( a \in \text{Ann}_A(z) \). For any \( b \in B \), we can write \( b = p(a') \), where \( a' \in A \) as \( p \) is surjective. Thus

\[
t_B(p(a)b) = t_B(p(aa')) = t_A(zaa') = 0.
\]
Since $t_B$ is non-degenerate and $b$ is arbitrary, we can deduce that $p(a) = 0$, i.e. $a \in \text{Ker}(p)$.

Conversely, if $a \in \text{Ker}(p)$, for any $a' \in A$, we have

$$t_A(a'az) = t_B(p(a'a)) = 0.$$  

The non-degeneracy of $t_A$ yields that $az = 0$, i.e. $a \in \text{Ann}_A(z)$. We are done. □

We now turn to study the morphism $\bar{\iota}_\Lambda,i,\beta$.

**Definition 2.4.** Let $\Lambda \in P^+$ and $\beta \in Q_n^+$. We define

$$d_{\Lambda,\beta} := 2(\Lambda, \beta) - (\beta, \beta).$$

Let $\beta \in Q_n^+$. For each $\nu \in I^3$, we define

$$r_\nu := \prod_{k<l} r_{\nu_k,\nu_l}, \quad \text{where} \quad r_{i,j} = \begin{cases} c_{i,j,\nu_i,\nu_j,0}, & \text{if } i \neq j; \\ 0, & \text{if } i = j, \end{cases}$$

and set

$$(2.5) \quad r(\beta, \nu_n) := \prod_{k=1}^{n-1} r_{\nu_k,\nu_n}.$$  

Let $\Lambda \in P^+$ and $\nu = (\nu_1, \cdots, \nu_n) \in I^3$. For each $1 \leq k \leq n$, let

$$\hat{\varepsilon}_{\nu_k} : e(\nu_1, \cdots, \nu_k) \mathbb{A}_\Lambda^k \sum_{j=1}^{k-1} \alpha_{\nu_j} e(\nu_1, \cdots, \nu_k) \to e(\nu_1, \cdots, \nu_{k-1}) \mathbb{A}_\Lambda^k \sum_{j=1}^{k-1} \alpha_{\nu_j} e(\nu_1, \cdots, \nu_{k-1})$$

be the homomorphism $\hat{\varepsilon}_{\nu_k}$ as defined in [24, A.3], which is the same as the restriction of $\hat{\varepsilon}'_{\nu_k,\Lambda-\sum_{i=1}^{k-1} \alpha_{\nu_i}}$ (in the notation of [24, Theorem 3.8]) to the bi-weight subspace $e(\nu_1, \cdots, \nu_k) \mathbb{A}_\Lambda^k \sum_{j=1}^{k-1} \alpha_{\nu_j} e(\nu_1, \cdots, \nu_k)$. Define a map $t_{\Lambda,\beta}$ as follows: for any $\nu, \nu' \in I^3$ and $x \in \mathbb{A}_\beta^\Lambda$,

$$t_{\Lambda,\beta}(e(\nu)x(e(\nu'))) := \begin{cases} r_{\nu,\nu_1} \circ \cdots \circ \hat{\varepsilon}_{n,\nu_n}(e(\nu)x(e(\nu'))), & \text{if } \nu = \nu'; \\ 0, & \text{if } \nu \neq \nu'. \end{cases}$$

We extend the above map $k$-linearly to a $k$-linear map $t_{\Lambda,\beta} : \mathbb{A}_\beta^\Lambda \to k$.

**Proposition 2.6 ([24]).** Let $\Lambda \in P^+$ and $\beta \in Q_n^+$. The map $t_{\Lambda,\beta}$ is a homogeneous symmetrizing form with degree $-d_{\Lambda,\beta}$ on $\mathbb{A}_\beta^\Lambda$. In particular, $\mathbb{A}_\beta^\Lambda$ is a symmetric $k$-algebra.
For each \( \Lambda, \Lambda' \in P^+ \). We denote by \( \vartheta^{i,L}_{\beta} : \mathcal{R}^{\Lambda}_{\beta} \to \mathcal{R}^{\Lambda+\Lambda'}/[\mathcal{R}^{\Lambda+\Lambda'_{\beta}}, \mathcal{R}^{\Lambda+\Lambda'_{\beta}}] \) the dual of the map \( \varpi^{i,L}_{\beta} : Z(\mathcal{R}^{\Lambda+\Lambda'_{\beta}}) \to Z(\mathcal{R}^{\Lambda}_{\beta}) \). Similarly, we use

\[
\vartheta^{i,L}_{\beta} : \mathcal{R}^{\Lambda}_{\beta} \to \mathcal{R}^{\Lambda+\Lambda'_{\beta}}/\mathcal{R}^{\Lambda+\Lambda'_{\beta}} \to \mathcal{R}^{\Lambda+\Lambda'_{\beta}}, \mathcal{R}^{\Lambda+\Lambda'_{\beta}}
\]

to denote the dual of the map \( \varpi^{i,L}_{\beta} : Z(\mathcal{R}^{\Lambda+\Lambda'_{\beta}}) \to Z(\mathcal{R}^{\Lambda}_{\beta}) \).

**Theorem 2.7.** Let \( \beta \in Q^+_n \). Then the Center Conjecture 1.9 holds for all \( \Lambda \in P^+ \) if and only if \( \varpi^{i,L}_{\beta} \) is injective for any \( \Lambda \in P^+ \) and \( i \in I \).

**Proof.** This is a consequence of Theorem 2.1. \( \square \)

**Proposition 2.8 ([24]).** For each \( \beta \in Q^+_n \) and \( \nu \in I^3 \), we have

\[
B_{\pm i, \lambda}(z)e(\nu) = z^{\mp t_i}a_i^\Lambda(z^{-1})^{\mp 1} \prod_{k=1}^n q_{k\nu}(z, x_k) e(\nu).
\]

where \( B_{\pm i, \lambda}(z) = \sum_{k \in \mathbb{Z}} B_{\pm i, \lambda}^k \in \mathcal{R}^{\Lambda}_{\beta}, e(\nu) \) and \( t_i = \langle \alpha_i, \Lambda \rangle \).

The following lemma plays a crucial role in the proof of the main result in this section.

**Lemma 2.9.** Let \( \beta \in Q^+_n, \nu \in I^3, i, j \in I \). Set \( \Lambda := \Lambda + \Lambda_i, \lambda = \Lambda - \beta, \tilde{\Lambda} := \Lambda - \beta \).

Let \( a \in e(\nu, j)\mathcal{R}^{\tilde{\Lambda}}_{\beta+\alpha_j} e(\nu, j) \). Then

1. If \( i \neq j \), then \( p_{\beta}^{L,i}(\hat{\varepsilon}_{j, \lambda}(a)) = \hat{\varepsilon}_{j, \lambda}(p_{\beta+\alpha_j}^{L,i}(a)) \);
2. If \( i = j \), then \( p_{\beta}^{L,i}(\hat{\varepsilon}_{j, \lambda}(ax_{n+1})) = \hat{\varepsilon}_{j, \lambda}(p_{\beta+\alpha_j}^{L,i}(a)) \).

**Proof.** 1) Set \( \lambda_j := \langle h_j, \lambda \rangle, \tilde{\lambda}_j := \langle h_j, \tilde{\lambda} \rangle \). If \( \lambda_j > 0 \), then by [24, (6)] we have a decomposition:

\[
a = e(\nu, j)\mu_\tau(\pi(a))e(\nu, j) + \sum_{k=0}^{\tilde{\lambda}_j-1} e(\nu, j)p_k(\alpha)x_{n+1}^k e(\nu, j),
\]

where \( \pi(a) \in \mathcal{R}^{\tilde{\Lambda}}_{\beta} e(\beta - \alpha_j, j) \otimes \mathcal{R}^{\tilde{\Lambda}}_{\beta} e(\beta - \alpha_j, j) \mathcal{R}^{\tilde{\Lambda}}_{\beta}, p_k(a) \in \mathcal{R}^{\tilde{\Lambda}}_{\beta}, k = 0, 1, \ldots, \tilde{\lambda}_j - 1, \) are unique elements such that (2.10) holds.

Note that the assumption \( i \neq j \) implies that \( \lambda_j = \tilde{\lambda}_j \). We can apply \( p_{\beta+\alpha_j}^{L,i} \) to (2.10) to get the decomposition ([24, (6)]) of \( p_{\beta+\alpha_j}^{L,i}(a) \) in \( \mathcal{R}^{\tilde{\Lambda}}_{\beta} \). By the uniqueness related to the decomposition we can deduce that \( (p_{\beta}^{L,i} \otimes p_{\beta}^{L,i})(\pi(a)) = \pi(p_{\beta+\alpha_j}^{L,i}(a)) \) and \( p_{\beta}^{L,i}(p_k(a)) = p_k(p_{\beta+\alpha_j}^{L,i}(a)), k = 0, 1, \ldots, \tilde{\lambda}_j - 1 = \lambda_j - 1 \). In particular, we have

\[
p_{\beta}^{L,i}(\hat{\varepsilon}_{j, \lambda}(a)) = p_{\beta}^{L,i}(p_{\lambda_j-1}(a)) = p_{\lambda_j-1}(p_{\beta+\alpha_j}^{L,i}(a)) = \hat{\varepsilon}_{j, \lambda}(p_{\beta+\alpha_j}^{L,i}(a)).
\]
If \( \lambda_j \leq 0 \), then we have \( \hat{\epsilon}_{j, \tilde{\lambda}}(a) = \mu_{x_n^{-\epsilon}}(\tilde{a}) = \mu_{x_n^{-\lambda_j}}(\tilde{a}) \), where \( \tilde{a} \in \mathcal{R}_\beta^\Lambda e(\beta - \alpha_j, j) \mathcal{R}_\beta^\Lambda \) is the unique element such that
\[
e(\nu, j)\mu_{x_n}(\tilde{a})e(\nu, j) = a, \quad \mu_{x_n}(\tilde{a}) = 0, \quad k = 0, 1, \ldots, -\tilde{\lambda}_j - 1 = -\lambda_j - 1.
\]

By the uniqueness we can deduce that
\[
(p_{\beta}^{\Lambda, i} \otimes p_{\beta}^{\Lambda, i})(\tilde{a}) = \tilde{\epsilon}_{j, \tilde{\lambda}}(a).
\]

It follows that
\[
p_{\beta}^{\Lambda, i}(\hat{\epsilon}_{j, \tilde{\lambda}}(a)) = p_{\beta}^{\Lambda, i}(\mu_{x_n^{-\epsilon}}(\tilde{a})) = \mu_{x_n^{-\lambda_j}}(p_{\beta}^{\Lambda, i}(a)) = \hat{\epsilon}_{j, \tilde{\lambda}}(p_{\beta+\alpha_i}^{\Lambda, i}(a)).
\]

This proves 1).

2) Suppose \( i = j \). Then \( \tilde{\lambda}_i = (\alpha_i', \tilde{\Lambda}) = (\alpha_i', \Lambda) + 1 = \lambda_i + 1 \). Our discussion will be divided into four cases:

**Case 1.** \( \lambda_i > 0 \). By \([24, (6)]\), we have

\[
a = e(\nu, i)\mu_{x_n}(\pi(a))e(\nu, i) + \sum_{k=0}^{\lambda_i} e(\nu, i)p_k(a)x_{n+1}^k e(\nu, i),
\]

where \( \pi(a) \in \mathcal{R}_\beta^\Lambda e(\beta - \alpha_i, i) \mathcal{R}_\beta^\Lambda, p_k(a) \in \mathcal{R}_\beta^\Lambda, k = 0, 1, \ldots, \lambda_i \). By the relation \( \tau_n x_{n+1} e(\beta - \alpha_i, i^2) = x_n \tau_n e(\beta - \alpha_i, i^2) + e(\beta - \alpha_i, i^2) \), we can deduce
\[
\hat{\epsilon}_{j, \tilde{\lambda}}(ax_{n+1}) = \hat{\epsilon}_{i, \tilde{\lambda}}(ax_{n+1}) = p_{\lambda_i}(ax_{n+1}) = p_{\lambda_i-1}(a) + p_{\lambda_i}(a)p_{\lambda_i}(x_{n+1}^{\lambda_i+1} e(\nu, i)).
\]

Applying \( p_{\beta+\alpha_i}^{\Lambda, i} \) to (2.11) and using the \( \mathcal{R}_\beta^\Lambda \)-bilinearity of \( \hat{\epsilon}_{i, \tilde{\lambda}} \) \([24, (6)]\), we can deduce that
\[
p_{\beta+\alpha_i}^{\Lambda, i}(a) = \mu_{x_n}(p_{\beta}^{\Lambda, i} \otimes p_{\beta}^{\Lambda, i})\pi(a)) + \sum_{k=0}^{\lambda_i-1} p_{\beta}^{\Lambda, i}(p_k(a)x_{n+1}^k e(\nu, i) + p_{\beta}^{\Lambda, i}(p_{\lambda_i}(a))x_{n+1}^{\lambda_i} e(\nu, i),
\]

and therefore
\[
\hat{\epsilon}_{j, \tilde{\lambda}}(p_{\beta+\alpha_i}^{\Lambda, i}(a)) = p_{\beta}^{\Lambda, i}(p_{\lambda_i-1}(a)) + p_{\beta}^{\Lambda, i}(p_{\lambda_i}(a))p_{\lambda_i}(x_{n+1}^{\lambda_i+1} e(\nu, i)).
\]

So Part 2) of the lemma will follow from the following equality:

\[
p_{\beta}^{\Lambda, i}(\hat{\epsilon}_{i, \tilde{\lambda}}(x_{n+1}^{\lambda_i+1} e(\nu, i))) = \hat{\epsilon}_{i, \tilde{\lambda}}(x_{n+1}^{\lambda_i} e(\nu, i)),
\]

where the element \( x_{n+1}^{\lambda_i+1} e(\nu, i) \) on the left handside of the above equality is defined in \( \mathcal{R}_\beta^\Lambda \), while the element \( x_{n+1}^{\lambda_i} e(\nu, i) \) on the right handside of the above equality
is defined in $\mathcal{R}^A$. The above equality follows from Proposition 2.8, the fact that $p_{\beta}^{\Lambda,i}(B_{+i,\lambda}^1) = B_{+i,\lambda}^1$ and the equalities ([24, Definition A.1])

$$B_{+i,\lambda}^1 e(\nu) = \hat{\epsilon}_{i,\lambda}(x_{n+1}^i e(\nu, i)) = \hat{\epsilon}_{i,\lambda}(x_{n+1}^{\lambda+1} e(\nu, i)), \quad B_{+i,\lambda}^1 e(\nu) = \hat{\epsilon}_{i,\lambda}(x_{n+1}^{\lambda} e(\nu, i)),$$

in this case.

**Case 2.** $\lambda_i = 0$. By [24, (6)], we have

$$a = e(\nu, i)\mu_{\tau_n}(\pi(a))e(\nu, i) + e(\nu, i)p_0(a)e(\nu, i),$$

where $\pi(a) \in \mathcal{R}^A e(\beta - \alpha_i, i) \otimes \mathcal{R}^A e(\beta - \alpha_i, i), \mathcal{R}^A, p_0(a) \in \mathcal{R}^A$. Again, since

$$\tau_n x_{n+1} e(\beta - \alpha_i, i^2) = x_n \tau_n e(\beta - \alpha_i, i^2) + e(\beta - \alpha_i, i^2),$$

we can deduce that (because $\lambda_i = 0$)

$$\hat{\epsilon}_{i,\lambda}(ax_{n+1}) = \mu_1(\pi(a)) + p_0(a)p_0(x_{n+1} e(\nu, i)).$$

Applying $p_{\beta+\alpha_i}^{\Lambda,i}$ to (2.13) and using the $R^A(\beta)$-bilinearity of the map $z \mapsto \pi(z)$ in [24, (6)], we get in $\mathcal{R}^A_{\beta+a_i}$ that

$$p_{\beta+\alpha_i}^{\Lambda,i}(a) = \mu_{\tau_n}((p_{\beta}^{\Lambda,i} \otimes p_{\beta}^{\Lambda,i})(\pi(a))) + p_{\beta}^{\Lambda,i}(p_0(a))e(\nu, i)$$

$$= \mu_{\tau_n}((p_{\beta}^{\Lambda,i} \otimes p_{\beta}^{\Lambda,i})(\pi(a))) + p_{\beta}^{\Lambda,i}(p_0(a))\mu_{\tau_n}(\pi(e(\nu, i))),$$

and therefore by applying [24, Theorem 3.8] we get that

$$\hat{\epsilon}_{i,\lambda}(p_{\beta+\alpha_i}^{\Lambda,i}(a)) = p_{\beta}^{\Lambda,i}(\mu_1(\pi(a))) + p_{\beta}^{\Lambda,i}(p_0(a))\mu_1(\pi(e(\nu, i))).$$

So Part 2) of the lemma will follow from the following equality:

$$p_{\beta}^{\Lambda,i}(\hat{\epsilon}_{i,\lambda}(x_{n+1} e(\nu, i))) = \hat{\epsilon}_{i,\lambda}(e(\nu, i));$$

where the element $x_{n+1} e(\nu, i)$ on the left handside of the above equality is defined in $\mathcal{R}^A_{\beta+\alpha_i}$, while the element $x_n e(\nu, i)$ on the right handside of the above equality is defined in $\mathcal{R}^A_{\beta}$. The above equality follows from Proposition 2.8, the fact that $p_{\beta}^{\Lambda,i}(B_{+i,\lambda}^1) = B_{+i,\lambda}^1$ and the equalities ([24, Definition A.1])

$$B_{+i,\lambda}^1 e(\nu) = \hat{\epsilon}_{i,\lambda}(x_{n+1} e(\nu, i)), \quad B_{+i,\lambda}^1 e(\nu) = \hat{\epsilon}_{i,\lambda}(e(\nu, i))$$

in this case.

**Case 3.** $\lambda_i < -1$. By [24, (8)], we have

$$a = e(\nu, i)\mu_{\tau_n}(\bar{a})e(\nu, i),$$

where
where $\tilde{a} \in R_{\beta}^\Lambda e(\beta - \alpha_i, i) \otimes R_{\beta - \alpha_j}^\Lambda e(\beta - \alpha_j, i) R_{\beta}^\Lambda$ is the unique element such that (2.15) holds and

$$\mu_{x_n^k}(\tilde{a}) = 0, \quad k = 0, 1, \ldots, -\tilde{\lambda}_i - 1 = -\lambda_i - 2.$$ 

By the relation $\tau_n x_{n+1} e(\beta - \alpha_i, i^2) = x_n \tau_n e(\beta - \alpha_i, i^2) + e(\beta - \alpha_i, i^2)$, we can get that

$$ax_{n+1} = \mu_{x_n \tau_n}(\tilde{a}) = \mu_{\tau_n} \left( b - \hat{\varepsilon}_{i,\hat{\lambda}}(a) \hat{\pi}_{-\lambda_i - 2} \right),$$

where $\mu_1(\tilde{a}) = 0$ follows from [24, (8)] and the assumption $\lambda_i < -1$.

Recall the definition of the elements $\{\hat{\pi}_\ell | 0 \leq \ell \leq -\tilde{\lambda}_i - 1 = -\lambda_i - 2\}$ in [24, (8)]. In particular, $\mu_{\tau_n}(\hat{\pi}_{-\lambda_i - 2}) = 0$. Therefore,

$$ax_{n+1} = \mu_{x_n \tau_n}(\tilde{a}) = \mu_{\tau_n} \left( b - \hat{\varepsilon}_{i,\hat{\lambda}}(a) \hat{\pi}_{-\lambda_i - 2} \right),$$

where $b := \sum z_1 x_n \otimes z_2$ and $\tilde{a} = \sum z_1 \otimes z_2$. Note that the element $b - \hat{\varepsilon}_{i,\hat{\lambda}}(a) \hat{\pi}_{-\lambda_i - 2}$ satisfies that

$$\mu_{x_n^k}(b - \hat{\varepsilon}_{i,\hat{\lambda}}(a) \hat{\pi}_{-\lambda_i - 2}) = 0, \quad k = 0, 1, \ldots, -\tilde{\lambda}_i - 1 = -\lambda_i - 2.$$ 

So by the uniqueness in the definition of $ax_{n+1}$ ([24, (8)]) we can deduce that $ax_{n+1} = b - \hat{\varepsilon}_{i,\hat{\lambda}}(a) \hat{\pi}_{-\lambda_i - 2}$. Therefore,

$$\hat{\varepsilon}_{i,\hat{\lambda}}(ax_{n+1}) = \mu_{x_n^{-\lambda_i - 1}}(ax_{n+1}) = \mu_{x_n^{-\lambda_i}}(\tilde{a}) - \hat{\varepsilon}_{i,\hat{\lambda}}(a) \mu_{x_n^{-\lambda_i - 1}}(\hat{\pi}_{-\lambda_i - 2}).$$

By definition, we have that

$$\mu_{x_n^{-\lambda_i - 1}}((p^\Lambda_{\beta} \otimes p^\Lambda_{\beta})\tilde{a}) = p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)).$$

We claim that

$$p^\Lambda_{\beta + \alpha_i}(\tilde{a}) = (p^\Lambda_{\beta} \otimes p^\Lambda_{\beta})\tilde{a} - p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)) \hat{\pi}_{-\lambda_i - 1}. \tag{2.19}$$

In fact, since

$$\mu_{\tau_n} ((p^\Lambda_{\beta} \otimes p^\Lambda_{\beta})(\tilde{a})) - \mu_{\tau_n} (p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)) \hat{\pi}_{-\lambda_i - 1})$$

$$= p^\Lambda_{\beta + \alpha_i}(a) - p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)) \mu_{\tau_n}(\hat{\pi}_{-\lambda_i - 1})$$

$$= p^\Lambda_{\beta + \alpha_i}(a) - 0 = p^\Lambda_{\beta + \alpha_i}(a),$$

and

$$\mu_{x_n^{-\lambda_i - 1}}((p^\Lambda_{\beta} \otimes p^\Lambda_{\beta})(\tilde{a})) - \mu_{x_n^{-\lambda_i - 1}} (p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)) \hat{\pi}_{-\lambda_i - 1})$$

$$= p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)) - p^\Lambda_{\beta}(\hat{\varepsilon}_{i,\hat{\lambda}}(a)) = 0,$$
and for \( k = 0, 1, \ldots, -\lambda_i - 2, \)
\[
\mu_{x_n^k}(p^\Lambda_i \otimes p^\Lambda_i)(\tilde{a})) - \mu_{x_n^k}(p^\Lambda_i(\tilde{\varepsilon}_{i,\lambda}(a))\tilde{\pi}_{-\lambda_i-1}) = 0 - 0 = 0,
\]
our claim follows from the uniqueness ([24, (8)]) in the definition of \( p^\Lambda_i(a) \).

As a result of (2.19), we see that
\[
\tilde{\varepsilon}_{i,\lambda}(p^\Lambda_i(a)) = \mu_{x_n}(\tilde{a}) - p^\Lambda_i(\tilde{\varepsilon}_{n+1,i}(a))\mu_{x_n^{-\lambda_i}}(\tilde{\pi}_{-\lambda_i-1}).
\]

Now comparing (2.17) with the above equality, to prove the part 2) of the lemma, it suffices to show that
\[
(2.20) \quad p^\Lambda_i \left( \mu_{x_n^{-\lambda_i-1}}(\tilde{\pi}_{-\lambda_i-2}) \right) = \mu_{x_n^{-\lambda_i}}(\tilde{\pi}_{-\lambda_i-1}).
\]
where the element \( \tilde{\pi}_{-\lambda_i-2} \) on the left handside of the above equality is defined in \( R^{\lambda\pm}_{\alpha_i} \), while the element \( \tilde{\pi}_{-\lambda_i-1} \) on the right handside of the above equality is defined in \( R^\Lambda_\beta \). The above equality follows from Proposition 2.8, the fact that \( p^\Lambda_i(B^1_{+i,\lambda}) = B^1_{+i,\lambda} \) and the equalities ([24, Definition A.1])
\[
B^1_{+i,\lambda} = -\mu_{x_n^{-\lambda_i-1}}(\tilde{\pi}_{-\lambda_i-2}), \quad B^1_{+i,\lambda} = -\mu_{x_n^{-\lambda_i}}(\tilde{\pi}_{-\lambda_i-1})
\]
in this case.

Case 4. \( \lambda_i = -1 \). By [24, (8)], we have
\[
(2.21) \quad a = e(\nu, i)\mu_{\tau_n}(\tilde{a})e(\nu, i),
\]
where \( \tilde{a} \in R^\Lambda_\beta e(\beta - \alpha_i, i) \otimes R^\Lambda_{\beta - \alpha_i} e(\beta - \alpha_i, i)R^\Lambda_\beta \) is the unique element such that (2.15) holds. By the relation \( \tau_n x_{n+1} e(\beta - \alpha_i, i^2) = x_n \tau_n e(\beta - \alpha_i, i^2) + e(\beta - \alpha_i, i^2) \), we can get that
\[
ax_{n+1} = \mu_{x_n\tau_n}(\tilde{a}) + \mu_1(\tilde{a}).
\]
Since \( \lambda_i = -1 \) implies that \( \lambda_i = 0 \), it follows that \( \tilde{\varepsilon}_{i,\lambda}(a) = \mu_1(\tilde{a}) \) ([24, Theorem 3.8]). Therefore,
\[
(2.22) \quad ax_{n+1} = \mu_{x_n\tau_n}(\tilde{a}) = \mu_{\tau_n}(b + \tilde{\varepsilon}_{i,\lambda}(a)\pi(e(\beta, i))),
\]
where \( b := \sum_z z_1 x_n \otimes z_2 \) and \( \tilde{a} = \sum_z z_1 \otimes z_2 \), and we have used the equalities: for \( e(\nu, i) \in R^\Lambda_{\beta + \alpha_i}, \pi(e(\beta, i)) = e(\beta, i) \) and \( \mu_{\tau_n}(e(\beta, i)) = e(\beta, i) \). So by the uniqueness
in the definition of $\tilde{a}x_{n+1}$ ([24, (8)]) we can deduce that $\tilde{a}x_{n+1} = b + \hat{e}_{i,\lambda}(a)\pi(e(\beta, i))$. Therefore,

\begin{equation}
\hat{e}_{i,\lambda}(\tilde{a}x_{n+1}) = \mu_1(\tilde{a}x_{n+1}) = \mu_{x_n}(\tilde{a}) + \hat{e}_{i,\lambda}(a)\mu_1(\pi(e(\beta, i))).
\end{equation}

By definition, we have that

\begin{equation}
\hat{\varepsilon}_{i,\lambda}(\tilde{a})(\tilde{a}x_{n+1}) = \mu_1((p^\lambda_i \otimes p^\lambda_i)\tilde{a}) = p^\lambda_i(\hat{\varepsilon}_{i,\lambda}(a)).
\end{equation}

We claim that

\begin{equation}
p^\lambda_{\beta+\alpha_i}(a) = (p^\lambda_\beta \otimes p^\lambda_\beta)(\tilde{a}) - \mu_1(\tilde{a})\tilde{\pi}_0,
\end{equation}

where the element $\tilde{\pi}_0$ in the righthand side of above equality is defined in $R^\lambda_\beta$.

In fact, since $\mu_{x_n}(\tilde{\pi}_0) = 0$ and $\mu_1(\tilde{\pi}_0) = 1$, our claim follows from the uniqueness ([24, (8)]) in the definition of $p^\lambda_{\beta+\alpha_i}(a)$ and the following two equalities:

\begin{equation}
\mu_{x_n}((p^\lambda_\beta \otimes p^\lambda_\beta)(\tilde{a})) = p^\lambda_{\beta+\alpha_i}(a), \quad \mu_1((p^\lambda_\beta \otimes p^\lambda_\beta)(\tilde{a})) = 0.
\end{equation}

As a result of (2.25), we see that

\begin{equation}
\hat{\varepsilon}_{i,\lambda}(p^\lambda_{\beta+\alpha_i})(a) = p^\lambda_\beta\left(\mu_{x_n}(\tilde{a})\right) - \mu_1(\tilde{a})\mu_{x_n}(\tilde{\pi}_0).
\end{equation}

Now comparing (2.23) with the above equality, to prove the part 2) of the lemma, it suffices to show that

\begin{equation}
p^\lambda_\beta\left(\mu_1(\pi(e(\beta, i)))\right) = -\mu_{x_n}(\tilde{\pi}_0).
\end{equation}

where the element $\pi(e(\nu, i)$ on the left handside of the above equality is defined in $R^\lambda_{\beta+\alpha_i}$, while the element $\tilde{\pi}_0$ on the right handside of the above equality is defined in $R^\lambda_\beta$. The above equality follows from Proposition 2.8, the fact that $p^\lambda_{\beta+\alpha_i}(B^1_{+i,\lambda}) = B^1_{+i,\lambda}$ and the equalities ([24, Definition A.1]

\begin{equation}
B^1_{+i,\lambda} = \mu_1(e(\beta, i)) = \mu_1(e(\beta, i)),
\end{equation}

\begin{equation}
B^1_{+i,\lambda} = -\mu_{x_n}(\tilde{\pi}_0)
\end{equation}

in this case. This completes the proof of the lemma.

\[
\text{□}
\]

**Definition 2.27.** Let $\beta \in Q^+_n$. For any $i \in I$, we define

\begin{equation}
z(i, \beta) = \sum_{\nu \in I^s \atop \nu_k = i} \prod_{1 \leq k \leq n \atop \nu_k = i} x_k e(\nu) \in R_\beta.
\end{equation}
For any $\Lambda' = \sum_{j \in I} n_j \Lambda_j \in P^+$, we define

$$z(\Lambda', \beta) = \prod_{i \in I} z(i, \beta)^{n_i} \in R_\beta,$$

where $z(i, \beta)$ is a symmetric elements in the KLR generators $x_1, \ldots, x_n, e(\nu), \nu \in I^\beta$. It follows from Proposition 1.6 that $z(i, \beta)$ and hence $z(\Lambda', \beta)$ belong to $Z(R_\beta)$.

Henceforth, by some abuse of notations, we shall often use the same notations $z(i, \beta), z(\Lambda', \beta)$ to denote their images in the cyclotomic KLR algebra $Z(R_{\tilde{\Lambda}})$. The following theorem is the main result of this section.

**Theorem 2.30.** Let $\Lambda \in P^+, \beta \in Q_+^n$ and $i \in I$. Then

$$t_{\beta, i}^\Lambda (1_{R_\beta^\Lambda}) = z(i, \beta) \in R_{\beta + \Lambda_i}.$$

In particular, for any $a \in R_{\beta + \Lambda_i}$,

$$t_{\beta, i}^\Lambda (p_{\beta, i}^\Lambda(a) + [R_{\beta}^\Lambda, R_{\beta}^\Lambda]) = az + [R_{\beta + \Lambda_i}, R_{\beta + \Lambda_i}],$$

and $\text{Ann}_{R_{\beta + \Lambda_i}}(z(i, \beta)) = \text{Ker}(p_{\beta, i}^\Lambda)$.

**Proof.** Set $\tilde{\Lambda} := \Lambda + \Lambda_i$. Note that $t_{\Lambda, \beta}(e(\mu)xe(\nu)) = 0$ whenever $\mu \neq \nu \in I^\beta$. In view of Lemma 2.3, to prove the theorem, it suffices to show that for any $\nu \in I^\beta$ and $a \in e(\nu)R_{\beta}^\Lambda e(\nu)$,

$$t_{\tilde{\Lambda}, \beta}(az(i, \beta)) = t_{\Lambda, \beta}(p_{\beta + \alpha_i}^\Lambda(a)).$$

(2.31)

We show this by induction on $n = |\beta|$. Suppose $\beta = \alpha_j$ for some $j \in I$. Then we have

i) If $j \neq i$, then $I_{\Lambda, \beta} = I_{\tilde{\Lambda}, \beta}$, hence $R_{\beta}^\Lambda = R_{\tilde{\Lambda}}^\Lambda$. In this case, $p_{\beta, i}^\Lambda$ is a natural isomorphism and $z(i, \beta)$ is the identity, we are done;

ii) If $j = i$, then $p_{\beta, i}^\Lambda$ can be identified with the canonical projection:

$$k[x_1]/(x_1^{r+1}) \rightarrow k[x_1]/(x_1^r), \quad x_1^r + (x_1^r) \mapsto x_1^r + (x_1^r),$$

where $r = \langle \alpha_i^\nu, \Lambda \rangle$. In this case, $z(i, \beta) = x_1$, the desired equality (2.31) of traces follows immediate since in this case the trace function $t_{\tilde{\Lambda}, \beta}$ (resp. $t_{\Lambda, \beta}$) sends a polynomial $f(x_1)$ in $x_1$ to its coefficient of $x_1^{r+1}$ (resp. $x_1^r$).
Suppose that (2.31) holds for any $\beta \in Q^+_n$ with $n \geq 1$, we want to show that it holds for any $\beta \in Q^+_{n+1}$. Let $\beta \in Q^+_n$, $\nu \in I^\beta$, $j \in I$ and $a \in e(\nu,j)\mathcal{R}_\beta^{\lambda + \alpha_j} e(\nu,j)$.

a) If $j \neq i$, then $\tilde{\lambda}_j = \lambda_j$ and $az(i, \beta + \alpha_j) = e(\nu,j)az(i, \beta)e(\nu,j)$, we have

$$t_{\tilde{\lambda},\beta+\alpha_j}(az(i, \beta + \alpha_j)) = r(\beta, j)t_{\tilde{\lambda},\beta}(\tilde{\varepsilon}_{j,\bar{\lambda}}(az(i, \beta + \alpha_j)))$$

$$= r(\beta, j)t_{\tilde{\lambda},\beta}(\tilde{\varepsilon}_{j,\bar{\lambda}}(a)z(i, \beta))$$

$$= r(\beta, j)t_{\tilde{\lambda},\beta}(p_\beta^{\lambda,i}(\tilde{\varepsilon}_{j,\bar{\lambda}}(a)))$$

$$= r(\beta, j)t_{\tilde{\lambda},\beta}(p_\beta^{\lambda,i}(\tilde{\varepsilon}_{j,\bar{\lambda}})(a))$$

$$= t_{\tilde{\lambda},\beta+\alpha_j}(p_\beta^{\lambda,i}(a)),$$

where the first and the last equalities are [24, (64)], the second one follows from the $\mathcal{R}_\beta$-bilinearity of $\tilde{\varepsilon}_{j,\bar{\lambda}}$ (see its definition in [24, Theorem 3.8]), the third one follows from induction hypothesis, the fourth one follows from Lemma 2.9.

b) If $j = i$, then $\tilde{\lambda}_i = \lambda_i + 1$ and $az(\beta + \alpha_i) = ax_{n+1} \cdot z(\beta)e(\beta,i)$, we have similarly

$$t_{\tilde{\lambda},\beta+\alpha_i}(az(i, \beta + \alpha_i)) = r(\beta, i)t_{\tilde{\lambda},\beta}(p_\beta^{\lambda,i}(\tilde{\varepsilon}_{i,\bar{\lambda}}(ax_{n+1})))$$

and

$$t_{\tilde{\lambda},\beta+\alpha_i}(p_\beta^{\lambda,i}(a)) = r(\beta, i)t_{\tilde{\lambda},\beta}(\tilde{\varepsilon}_{i,\bar{\lambda}}(p_\beta^{\lambda,i}(a))).$$

By Lemma 2.9, we have $p_\beta^{\lambda,i}(\tilde{\varepsilon}_{i,\bar{\lambda}}(ax_{n+1})) = \tilde{\varepsilon}_{i,\bar{\lambda}}(p_\beta^{\lambda,i}(a))$. It follows that (2.31) holds in this case.

This completes the proof of the theorem. \qed

Remark 2.32. We remark that Theorem 2.30 still holds when replacing $z(i, \beta), \Lambda + \Lambda_i$ by $z(\Lambda', \beta), \Lambda + \Lambda'$ respectively for arbitrary $\Lambda' \in P^+$. One can prove this by applying Theorem 2.30 repeatedly. Alternatively, one can also prove it directly. The key part in the argument is to check in the case when $i = j$ the following equality of bubbles:

$$p_\beta^{\Lambda,\Lambda'}(B_{k+i,\bar{\lambda}}^k) = B_{k+i,\bar{\lambda}}^k \quad \forall 0 \leq k \leq n_i,$$

where $n_i = \langle \alpha_i^\vee, \Lambda' \rangle$. 
Combine Lemma 2.3 and Theorem 2.30, we see that the map \( \iota^\Lambda_i : \mathbb{R}_\beta^\Lambda / [\mathbb{R}_\beta^\Lambda, \mathbb{R}_\beta^\Lambda] \to \mathbb{R}_\beta^{\Lambda+\Lambda_i}/[\mathbb{R}_\beta^{\Lambda+\Lambda_i}, \mathbb{R}_\beta^{\Lambda+\Lambda_i}] \) is induced by the \((\mathbb{R}_\beta, \mathbb{R}_\beta)\)-bilinear morphism \( \iota^\Lambda_i : \mathbb{R}_\beta \hookrightarrow \mathbb{R}_\beta, \quad x \mapsto z(i, \beta)x. \)

It is obvious that \( \iota^\Lambda_i \) sends monomial basis to monomial basis. Therefore, to show the Center Conjecture 1.9, it suffices to find out a monomial basis \( T^\Lambda_\beta \) for the cocenters \( \mathbb{R}_\beta^\Lambda / [\mathbb{R}_\beta^\Lambda, \mathbb{R}_\beta^\Lambda] \) as well as a monomial basis \( \tilde{T}^\Lambda_\beta \) for the cocenters \( \mathbb{R}_\beta^{\tilde{\Lambda}} / [\mathbb{R}_\beta^{\tilde{\Lambda}}, \mathbb{R}_\beta^{\tilde{\Lambda}}] \), and to show that the image of \( T^\Lambda_\beta \) under \( \iota^\Lambda_i \) is a subset of \( \tilde{T}^\Lambda_\beta \) and hence are \( K \)-linearly independent. We name this way as “cocenter approach” to the Center Conjecture 1.9. In Section 4 we shall construct such monomial bases for some special \( \beta \) and use the above “cocenter approach” to verify the Center Conjecture 1.9 for these special \( \beta \).

3. Bases of cyclotomic KLR algebras and their defining ideals

In this section we study the defining ideal \( I_{\Lambda, \beta} \) for \( \mathbb{R}_\beta^\Lambda \). We construct a monomial bases for certain bi-weight space \( e(\tilde{\nu})I_{\Lambda, \beta}e(\nu) \) of \( I_{\Lambda, \beta} \). As a result, we recover a result [14, Theorem 1.5] on the monomial bases of the bi-weight space \( e(\tilde{\nu})\mathbb{R}_\beta^\Lambda e(\nu) \), in particular, a result on the monomial bases of the cyclotomic KLR algebras \( \mathbb{R}_\beta^\Lambda \) when \( \beta \) satisfying (3.12). Compared to [14] and [15], our approach to these monomial bases is elementary in the sense that it does not use the Cyclotomic Categorification Conjecture ([17],[16]). Moreover, throughout this section, we work over an arbitrary domain \( \mathbb{k} \), and do not need to assume that \( \mathbb{k} \) is a field or a Noetherian domain as required in [14] and [15].

Note that \( I_{\Lambda, \beta} \) is a (right) \( \mathbb{k}[x_1, \ldots, x_n] \) submodule of \( \mathbb{R}_\beta \), we need the following useful lemma to construct monomial basis of \( \mathbb{k}[x_1, \ldots, x_n] \)-modules:

**Lemma 3.1.** 1) Let \( R \) be a commutative ring, \( g(x) \in R[x] \) be a monic polynomial whose leading term has degree \( k \), then \( R[x]/(g(x)) \) has an \( R \)-basis \( \{1, x, \ldots, x^{k-1}\} \).

2) Assume for each \( 1 \leq t \leq n \), there is a polynomials \( g_t \in R[x_1, \ldots, x_t] \) satisfying that \( g_t \) is monic when regarding as a polynomial in \( x_t \) and its leading term is of degree \( a_t \). Then the canonical image of the set

\[
B_1 := \{x^\underline{b} \mid \underline{b} \in \mathbb{N}^n, b_t < a_t, \forall 1 \leq t \leq n\}
\]
form an $R$-basis of $R[x_1, \ldots, x_n]/(g_1, \ldots, g_n)$. Consequently, the ideal $(g_1, \ldots, g_n) \subset R[x_1, \ldots, x_n]$ has a basis

$$B_2 := \left\{ x^b g_k \mid b = (b_1, \ldots, b_n) \in \mathbb{N}^n, 1 \leq k \leq n, \text{ and } b_t < a_t \text{ whenever } k < t \leq n \right\},$$

where $x^b := x_1^{b_1} \cdots x_n^{b_n}$ for each $b$.

**Proof.** The part 1) of the lemma is trivial. We now consider the part 2) of the lemma.

Set $R' := R[x_1, \ldots, x_{n-1}]/(g_1, \ldots, g_{n-1})$. Note that

$$R[x_1, \ldots, x_{n}]/(g_1, \ldots, g_n) \xrightarrow{\sim} R'[x_n]/(g_n),$$

One could argue by induction on $n$ to prove that $B_1$ is an $R$-basis of $R[x_1, \ldots, x_{n}]/(g_1, \ldots, g_n)$.

It remains to prove the last statement of the part 2) of the lemma. Recall the anti-lexicographic order on monomials:

$$x^b < x^c \iff \exists 1 \leq k \leq n, \text{ such that } b_k < c_k \text{ and } b_t = c_t, \forall k < t \leq n.$$

Then we have

$$x_1^{b_1} \cdots x_n^{b_n} g_k = c \cdot x_1^{b_1} \cdots x_k^{b_k + a_k} \cdots x_n^{b_n} + \text{"lower terms"},$$

where $c \in R^\times$ and "lower terms" is a $R$-linear combination of monomials $x^c$ such that $x^c < x^b \cdot x_k^{a_k}$. This yields that the elements in $B_2$ are $R$-linearly independent. Denote by $I$ the $R$-submodule of $R[x_1, \ldots, x_n]$ generated by $B_2$, then $I \subset (g_1, \ldots, g_n)$. On the other hand, it is easy to see that for any $1 \leq k \leq n$ and $f \in R[x_1, \ldots, x_n]$ that $fg_k \in I$. In fact, one can easily see that subtracting off suitable $R[x_1, \ldots, x_n]$-linear combination of $g_n, \ldots, g_{t+1}$ from $fg_k$ will produce an element living inside

$$R \text{-Span}\left\{ x^b g_k \mid b = (b_1, \ldots, b_n) \in \mathbb{N}^n, \text{ and } b_t < a_t \text{ whenever } k < t \leq n \right\}.$$

This completes the proof of the lemma. \(\square\)

**Definition 3.2.** Suppose that $\beta = \sum_{j=1}^p k_j \alpha_{i_j} \in Q^+_n$, where $i_s \neq i_t \in I$ for any $1 \leq s \neq t \leq p$. Let $\tilde{\nu} \in I^\beta$ be of the form

$$\tilde{\nu} = \left( i_1, \ldots, i_1, i_2, \ldots, i_2, \ldots, i_p, \ldots, i_p \right),$$

where $k_1$ copies $k_2$ copies $k_p$ copies

We define $i_s < i_t \iff s < t$. 

\(\square\)
Let \( w \in \mathfrak{S}_n \). Note that \( e(\bar{\nu})\tau_w \) is independent of the choice of the reduced expressions of \( w \in \mathfrak{S}_n \) since there is no triple \((r, s, t)\) satisfying \( r < s < t \) and \( \bar{\nu}_r = \bar{\nu}_t \neq \bar{\nu}_s \). So we can choose any reduced expression of \( w \) to calculate \( e(\bar{\nu})\tau_w \). In particular, \( e(\bar{\nu})\tau_w = e(\bar{\nu})\tau_{u}\tau_t \) whenever \( w = us_t \) with \( \ell(w) = \ell(u) + 1 \).

For any \( \nu, \nu' \in I^\beta \), we define \( \mathfrak{S}(\nu, \nu') := \{ w \in \mathfrak{S}_n | w\nu = \nu' \} \). The following is the main result of this section.

**Theorem 3.3.** Assume that \( \Lambda \in P^+ \). Let \( \beta \in Q^+_n \) and \( \bar{\nu} \in I^\beta \) be given as in Definition 3.2. For any \( \nu \in I^\beta \), we set \( g^\Lambda_{\nu,1} = a^\Lambda_{\nu,1}(x_1) \), and for \( 1 \leq k < n \), define

\[
\begin{align*}
g^\Lambda_{\nu,k+1} &= \begin{cases} 
\partial_k(g^\Lambda_{\nu,k}) & \text{if } \nu_k = \nu_{k+1}; \\
\nu_s(g^\Lambda_{\nu,k,v}) & \text{if } \nu_k \succ \nu_{k+1}; \\
\nu_s(g^\Lambda_{\nu,k,v}) \cdot Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1}) & \text{if } \nu_k < \nu_{k+1}.
\end{cases}
\end{align*}
\]

Then \( g_{\nu,k} \in \mathbf{k}[x_1, \ldots, x_k] \) is a monic polynomial on \( x_k \), and as \( \mathbf{k}[x_1, \ldots, x_n] \)-submodule of \( e(\bar{\nu})\mathcal{R}_{\beta}e(\nu) \), \( e(\bar{\nu})I_{\Lambda,\beta}e(\nu) \) is generated by the set

\[
\{ \tau_w g^\Lambda_{\nu,k}e(\nu) \mid 1 \leq k \leq n, w \in \mathfrak{S}(\nu, \bar{\nu}) \}.
\]

**Proof.** Since \( \Lambda \) is fixed, we shall write \( g_{\nu,k} = g_{\nu,k}^\Lambda \) for simplicity. By an induction on \( k \) we can show that \( g_{\nu,k} \in \mathbf{k}[x_1, \ldots, x_k] \) and the leading term of its \( x_k \)-expansion is monic.

We have the following compatible decompositions:

\[
\begin{align*}
\mathcal{R}_{\beta} &= \bigoplus_{\mu, \nu \in I^\beta} e(\mu)\mathcal{R}_{\beta}e(\nu) = \bigoplus_{w \in \mathfrak{S}_n, \nu \in I^\beta} \tau_w \mathbf{k}[x_1, \ldots, x_n]e(\nu), \\
I_{\Lambda, \beta} &= \bigoplus_{\mu, \nu \in I^\beta} e(\mu)I_{\Lambda, \beta}e(\nu), \\
\mathcal{R}_{\beta}^\Lambda &= \bigoplus_{\mu, \nu \in I^\beta} e(\mu)\mathcal{R}_{\beta}^\Lambda e(\nu).
\end{align*}
\]

In particular, \( e(\mu)\mathcal{R}_{\beta}^\Lambda e(\nu) \cong e(\mu)\mathcal{R}_{\beta}e(\nu)/e(\mu)I_{\Lambda, \beta}e(\nu) \). Each direct summand in these decompositions is a right \( \mathbf{k}[x_1, \ldots, x_n] \)-module.

By Definition 1.7 and Proposition 1.5, \( I_{\Lambda, \beta} \) is generated as a (right) \( \mathbf{k}[x_1, \ldots, x_n] \)-module by elements of form \( \tau_n a^\Lambda_{\nu-1}(x_1)\tau_n e(\nu) \), with \( \nu \) running through \( I^\beta \). We can decompose \( v \) as \( v = w \cdot s_1 s_2 \cdots s_t \) with \( w \in \mathfrak{S}_{\{2, 3, \ldots, n\}} \) and \( 0 \leq t \leq n - 1 \). We may
assume that the preferred decomposition we choose for \( w \) is a product of that of \( w \) with \( s_1s_2 \cdots s_l \). Then

\[
\tau_w a^\Lambda_{\nu_{v=1}(1)}(x_1)\tau_v e(\nu) = \tau_w \tau_u a^\Lambda_{\nu_{v+1}(1)}(x_1)\tau_1 \cdots \tau_t e(\nu).
\]

By Proposition 1.5 and the defining relation of \( R_\beta \), this can written as a (right) \( k[x_1, \ldots, x_n] \)-linear combination of some elements of the form:

\[
\tau_z a^\Lambda_{\nu_z}(x_1)\tau_1 \cdots \tau_k e(\nu), \quad z \in \mathcal{S}_n, \quad 0 \leq k \leq n - 1.
\]

For each \( 1 \leq l \leq k \), we define \( \nu^{1,k} := (s_1s_{l+1} \cdots s_{k-1})\nu \). In particular, \( \nu^{1,1} = \nu \) by convention. Then, as a right \( k[x_1, \ldots, x_n] \)-module, \( e(\mu)I_{\Lambda,\beta}e(\nu) \) is generated by

\[
\{ \tau_w g^{\nu_1,k,1}_{\nu_{1,k}} \tau_1 \tau_2 \cdots \tau_k e(\nu) \mid 1 \leq k \leq n, \nu \in \mathcal{S}(\nu^{1,k}, \mu) \}.
\]

Now, we concentrate on the case where \( \mu = \widetilde{\nu} \). For \( k > 1 \), if \( \nu_k < \nu_1 \), equivalently, \( \nu_2^{1,k} < \nu_1^{1,k} \), then \( \ell(ws_1) = \ell(w) + 1 \), and

\[
(3.5) \quad \tau_w g^{\nu_1,k,1}_{\nu_{1,k}} \tau_1 \tau_2 \cdots \tau_k e(\nu) = \tau_w \tau_1 s_1(g^{\nu_1,k,1}_{\nu_{1,k}}) \tau_2 \cdots \tau_{k-1} e(\nu) = \tau_{ws_1} g^{\nu_2,k,2}_{\nu_1,k} \tau_2 \cdots \tau_{k-1} e(\nu);
\]

If \( \nu_k > \nu_1 \), equivalently, \( \nu_2^{1,k} > \nu_1^{1,k} \), then \( \ell(ws_1) = \ell(w) - 1 \), and

\[
(3.6) \quad \tau_w g^{\nu_1,k,1}_{\nu_{1,k}} \tau_1 \tau_2 \cdots \tau_{k-1} e(\nu) = \tau_{ws_1} \tau_1^{2} s_1(g^{\nu_1,k,1}_{\nu_{1,k}}) \tau_2 \cdots \tau_{k-1} e(\nu) = \tau_{ws_1} g^{\nu_2,k,2}_{\nu_1,k} \tau_2 \cdots \tau_{k-1} e(\nu);
\]

If \( \nu_k = \nu_1 \), equivalently, \( \nu_2^{1,k} = \nu_1^{1,k} \), then \( \nu^{2,k} = \nu^{1,k} \). In this case, for \( w \) satisfying \( \ell(ws_1) = \ell(w) - 1 \), we have

\[
(3.7) \quad \tau_w g^{\nu_1,k,1}_{\nu_{1,k}} \tau_1 \tau_2 \cdots \tau_{k-1} e(\nu) = \tau_w \partial_1(g^{\nu_1,k,1}_{\nu_{1,k}}) \tau_2 \cdots \tau_{k-1} e(\nu) = \tau_w g^{\nu_2,k,2}_{\nu_1,k} \tau_2 \cdots \tau_{k-1} e(\nu),
\]

where the first equality follows from (1.4); while for \( w \) satisfying \( \ell(ws_1) = \ell(w) + 1 \), we have

\[
(3.8) \quad \tau_w g^{\nu_1,k,1}_{\nu_{1,k}} \tau_1 \tau_2 \cdots \tau_{k-1} e(\nu) = \tau_w \partial_1(g^{\nu_1,k,1}_{\nu_{1,k}}) \tau_2 \cdots \tau_{k-1} e(\nu) + \tau_w \tau_1 g^{\nu_1,k,1}_{\nu_{1,k}} (x_1 - x_2) \tau_2 \cdots \tau_{k-1} e(\nu)
\]

\[
+ \tau_w \tau_1 \partial_1(g^{\nu_1,k,1}_{\nu_{1,k}})(x_1 - x_2) \tau_2 \cdots \tau_{k-1} e(\nu)
\]

\[
= \tau_w g^{\nu_2,k,2}_{\nu_1,k} \tau_2 \cdots \tau_{k-1} e(\nu) + \tau_{ws_1} \tau_2 \cdots \tau_{k-1} g^{\nu_1,k,1}_{\nu_{1,k}} e(\nu)
\]

\[
+ \tau_{ws_1} g^{\nu_2,k,2}_{\nu_1,k} (x_1 - x_2) \tau_2 \cdots \tau_{k-1} e(\nu),
\]
where the first equality follows from (1.3) and (1.4). By the defining relations of \( \mathfrak{R}_3 \), we have
\[
(x_1 - x_2)\tau_2\tau_3\cdots\tau_{k-1}e(\nu) = \tau_2\tau_3\cdots\tau_{k-1}e(\nu)(x_1 - x_k) + \sum_{2 \leq j < k} \frac{\tau_2\tau_3\cdots\tilde{\tau}_j\cdots\tau_{k-1}e(\nu)}{\nu_j = \nu_k},
\]
where \( \tilde{\tau}_j \) means deleting \( \tau_j \). Note that by definition and induction on \( l \), we have
\[
g_{\nu^l,k,l} = g_{\nu^l,j,l}
\]
since \( \nu^l_j = \nu_k \) for \( 1 \leq t \leq l \). So
\[
(3.9) \\
\tau_{w\nu^1}g_{\nu^2,k,2}(x_1 - x_2)\tau_2\cdots\tau_{k-1}e(\nu) = \tau_{w\nu^1}g_{\nu^2,k,2}\tau_2\cdots\tau_{k-1}e(\nu)(x_1 - x_k)
\]
\[
+ \sum_{2 \leq j < k} \tau_{w\nu^1}\tau_{j+1}\cdots\tau_{k-1}g_{\nu^2,j,2}\tau_2\tau_3\cdots\tau_j-1e(\nu)
\]
\[
= I + II.
\]

Writing \( \tau_{w\nu^1}\tau_{j+1}\cdots\tau_{k-1}e(\nu^{2,j}) \) in terms of monomial basis, and argue by induction on \( j \), we see that \( II \) is a right \( k[x_1, \ldots, x_n] \) combination of \( \tau_u g_{\nu^{2,t}} \tau_2\tau_3\cdots\tau_{l-1}e(\nu) \) with \( t < k, \nu_t = \nu_k \) and \( u \in \mathfrak{S}(\nu^{2,k}, \tilde{\nu}) \).

Consider the following set
\[
\{\tau_w g_{\nu,k}e(\nu) \mid 1 \leq k \leq 2, \ w \in \mathfrak{S}(\nu, \tilde{\nu})\}
\]
\[
\bigcup\{\tau_w g_{\nu^2,k,2}\tau_2\cdots\tau_{k-1}e(\nu) \mid 2 < k \leq n, \ w \in \mathfrak{S}(\nu^{2,k}, \tilde{\nu})\}.
\]
Equations (3.5), (3.6), (3.7), (3.8) and (3.9) show that the right \( k[x_1, \ldots, x_n] \)-module generated by the above subset is the same as the right \( k[x_1, \ldots, x_n] \)-module generated by the set (3.4), and the transition matrix between these two sets of \( k[x_1, \ldots, x_n] \)-generators is upper-unitriangular (under suitable ordering).

By induction on \( 1 \leq t \leq n \) and repeating this argument, one can show that as a right \( k[x_1, \ldots, x_n] \)-module, \( e(\tilde{\nu})I_{\Lambda,\beta}e(\nu) \) is generated by
\[
\{\tau_w g_{\nu,k}e(\nu) \mid 1 \leq k \leq t, \ w \in \mathfrak{S}(\nu, \tilde{\nu})\}
\]
\[
\bigcup\{\tau_w g_{\nu^{l,k},l}\tau_1\cdots\tau_{k-1}e(\nu) \mid t < k \leq n, \ w \in \mathfrak{S}(\nu^{l,k}, \tilde{\nu})\}.
\]
Taking \( t = n \), we prove the theorem. \( \square \)
As a result, we obtain the following result which should be equivalent to [14, Theorem 1.5].

**Corollary 3.10.** For each \(1 \leq k \leq n\), let \(N_k\) denote the degree of the leading term of the \(x_k\)-expansion of \(g_{w,k}^\Lambda\). The (canonical image of the) set

\[
\{ \tau_w x^a e(\nu) \mid w \in \mathcal{G}(\nu, \tilde{\nu}), a \in \mathbb{N}^n, a_k < N_k, \forall 1 \leq k \leq n \}
\]

form a \(k\)-basis of \(e(\tilde{\nu})\mathcal{R}_\beta^\Lambda e(\nu)\).

**Proof.** This follows from Theorem 3.3 and Lemma 3.1 by replacing \(k[x_1, \ldots, x_n]\) with \(\tau_w k[x_1, \ldots, x_n]e(\nu)\) and \(g_k\) with \(g_{\nu,k}\). \(\Box\)

We also obtain the following result, where its first part is equivalent to [15, Theorem 1.9] in the case of cyclotomic KLR algebras.

**Corollary 3.11.** Assume that

\[
(3.12) \quad \beta = \alpha_1 + \cdots + \alpha_n, \quad \alpha_i \neq \alpha_j, \forall 1 \leq i \neq j \leq n
\]

Then \(\mathcal{R}_\beta^\Lambda\) has a \(k\)-basis

\[
\{ \tau_w x^a e(\nu) \mid \nu \in I^\beta, w \in \mathcal{G}_n, a \in \mathbb{N}^n, a_t < \langle \alpha_{\nu_t}^\vee, \Lambda \rangle - \sum_{1 \leq k < t} a_{\nu_{w(k)} \nu_{k}} \forall 1 \leq t \leq n \}.
\]

The two-sided ideal \(I_{\Lambda, \beta}\) has a \(k\)-basis

\[
\{ \tau_w x^a g_{w,\nu,k}^\Lambda e(\nu) \mid \nu \in I^\beta, w \in \mathcal{G}_n, a \in \mathbb{N}^n, a_t < \langle \alpha_{\nu_t}^\vee, \Lambda \rangle - \sum_{1 \leq k < t} a_{\nu_{w(k)} \nu_{k}} \forall 1 \leq k \leq n \}.
\]

where \(g_{w,\nu,k}^\Lambda\) is the same as \(g_{\nu,k}^\Lambda\) in Theorem 3.3 by setting \(\tilde{\nu} = w.\nu\).

We also recover the following result of [12, Theorem 2.34] which gives a monomial basis for cyclotomic NilHecke algebras of type \(A\).

**Corollary 3.13.** (\([12, \text{Theorem 2.34}]\)) Let \(\beta = n\alpha_i\) and set \(l = \langle \alpha_i^\vee, \Lambda \rangle\). Then \(\mathcal{R}_\beta^\Lambda\) has a basis

\[
\{ \tau_w x^a e(i^n) \mid w \in \mathcal{G}_n, a \in \mathbb{N}^n, a_t \leq l - t \}.
\]
4. Monomial basis for concenters and the Center Conjecture

Throughout this section, we shall assume that $\beta$ satisfies (3.12) and $k$ is a field. Under this assumption we shall construct a monomial basis for the cocenter $R_\beta^A/[R_\beta^A : R_\beta^A]$ and use it to verify the Center Conjecture.

Fix $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in I^\beta$. By assumption (3.12) we can assume without loss of generality that $I = \{\gamma_1, \ldots, \gamma_n\}$. Then $\gamma$ induces a total order on $I$ such that the map $\iota : \{1, \ldots, n\} \to I$, $t \mapsto \gamma_t$ is an order-preserving bijection. We use $\iota$ to identify these two sets.

For any $u, v \in S_n$, we define $u < v$ if and only if

$$\tag{4.1} (u(1), u(2), \ldots, u(n)) < (v(1), v(2), \ldots, v(n)),$$

where "<" is the lexicographic order on $\mathbb{N}^n$. That says, (4.1) holds if and only if there exists $t \in [1, n]$, such that $u(t) < v(t)$ and $u(k) = v(k)$ whenever $1 \leq k < t$.

We use the bijection $S_n \to I^\beta$, $u \mapsto u.\gamma$ to identify $S_n$ with $I^\beta$. Thus the total order "<" on $S_n$ induces a total order "$\prec_{\gamma}$" on $I^\beta$ and $\gamma$ is minimal element under this order "$\prec_{\gamma}$".

Note that the induced total order "$\prec_{\gamma}$" on $I^\beta$ is in general not the lexicographical order on $I^\beta$. The induced total order "$\prec_{\gamma}$" on $I^\beta$ can be described as follows: $\mu < \nu$ if and only if there exists $t \in [1, n]$, such that if $\mu_p = \nu_q = \gamma_k$ for $1 \leq k < t$, then $p = q$ and if $\mu_p = \nu_q = \gamma_t$, then $p < q$. We call $\gamma$ the initial weight. Since $\gamma$ is fixed in the whole section, we simply denote it by "$<$" instead of by "$\prec_{\gamma}$". For any $\nu \in I^\beta$ and $e \neq w \in S_n$, $e(w.\nu)R_\beta e(\nu) \subset [R_\beta, R_\beta]$ since the idempotents $\{e(\nu) \mid \nu \in I^\beta\}$ are mutually orthogonal. Therefore, by Definition 1.1 and Proposition 1.5, $R_\beta/[R_\beta, R_\beta]$ is isomorphic to $P_\beta/C_\beta$, where $P_\beta = \bigoplus_{\nu \in I^\beta} e(\nu)R_\beta e(\nu) = \bigoplus_{\nu \in I^\beta} k[x_1, \ldots, x_n]e(\nu)$ is the polynomial subalgebra and $C_\beta = P_\beta \cap [R_\beta, R_\beta]$ is generated as a $k$-module by some elements of the form:

$$e(\nu)\tau_{u-1}x^{a+b}e(\nu) = e(u.\nu)\tau_u x^{a+b}e(u.\nu)$$

$$= \tau_{u-1}\tau_u x^{a+b}e(\nu) - \tau_u x^{a+b}e(u.\nu)$$

$$= Q_{u,\nu}x^{a+b}e(\nu) - Q_{u-1,\nu}x^{a+b}e(u.\nu)$$

$$= Q_{u,\nu}x^{a+b}e(\nu) - u.(Q_{u,\nu}x^{a+b}e(\nu)),$$
where

\[ Q_{u,\nu} := \prod_{\substack{k \in I^\beta \setminus \nu \setminus u \setminus u(t) \setminus \nu_k \setminus \nu_t(x_k, x_t).}} \]

By the symmetry, we can easily see that \( C_\beta \) has a set of \( k \)-linear generators

\[ G_1 := \{ Q_{u,\nu} x^{u^{-1}\beta + b}e(\nu) - u \cdot (Q_{u,\nu} x^{u^{-1}\beta + b}e(\nu)) \mid \nu \in I^\beta, \nu < u, u, a \in \mathbb{N}^n \}. \]

The same argument shows that \( R^\Lambda_\beta /[R^\Lambda_\beta : R^\Lambda_\beta] \cong P^\Lambda_\beta / C^\Lambda_\beta \), where \( C^\Lambda_\beta \) is the image of \( C_\beta \) in \( R^\Lambda_\beta \). As \( R^\Lambda_\beta /[R^\Lambda_\beta : R^\Lambda_\beta] \cong R_\beta / \left( [R_\beta, R_\beta] + I_{A, \beta} \right) \), \( R^\Lambda_\beta /[R^\Lambda_\beta : R^\Lambda_\beta] \) is also isomorphic to \( P_\beta / D_\beta \), where \( D^\Lambda_\beta = [R_\beta, R_\beta] + I_{A, \beta} \) is a \( k \)-submodule generated by

\[ G'_1 = \{ Q_{u,\nu} x^{b}e(\nu) - u \cdot (Q_{u,\nu} x^{b}e(\nu)) \mid \nu \in I^\beta, \nu < u, u, a \in \mathbb{N}^n \} \]

\[ \bigcup \left\{ g^{A}_{e, e, k} x^{b}e(\nu) \mid b \in \mathbb{N}^n, \forall k < t \leq n, a_{e, t, t} < a_{e, t, t} - \sum_{1 \leq k < t} a_{e, t, t} \right\}, \]

where \( g^A_{e, e, k} \) is defined in Corollary 3.11 and \( a^A_{e, t, t} = a^A_{e, t, t}(x_t) \) for \( v = s_1 s_2 \cdots s_{t-1} \).

**Definition 4.3.** Let \( \nu \in I^\beta \) and \( u \in \mathcal{G}_n \) such that \( \nu < u, u \). We say that \( u \) is decomposable relative to \( \nu \), if \( u = u_1 u_2 \) such that \( \ell(u) = \ell(u_1) + \ell(u_2) \), \( u_1 \not\equiv e \) and \( \nu < u_2, u \), and call \( u = u_1 u_2 \) a decomposition of \( u \) relative to \( \nu \). Conversely, if there is no such decomposition, then we say that \( u \) is indecomposable relative to \( \nu \).

In particular, if \( u \) is indecomposable relative to \( \nu \), then for any reduced decomposition \( u = s_{i_k} s_{i_{k-1}} \cdots s_{i_1} \), we must have \( \nu^{(t)} < \nu < \nu^{(k)} \) for any \( 1 \leq t \leq k - 1 \), where \( \nu^{(t)} := s_{i_t} s_{i_{t-1}} \cdots s_{i_1} \), \( t = 1, 2, \cdots, k \).

If \( u \) is decomposable relative to \( \nu \) and \( u = u_1 u_2 \) is a decomposition relative to \( \nu \), then

\[ Q_{u,\nu} x^{b}e(\nu) - u \cdot (Q_{u,\nu} x^{b}e(\nu)) = Q_{u_1,\nu} x^{u_1^{-1}u_2}e(\nu) - u_2 \cdot (Q_{u_2,\nu} x^{u_1^{-1}u_2}e(\nu)) + Q_{u_1,\nu} x^{u_2}e(u_2) - u_1 \cdot (Q_{u_1,\nu} x^{u_2}e(u_2)). \]

This yields that \( Q_{u,\nu} x^{b}e(\nu) - u \cdot (Q_{u,\nu} x^{b}e(\nu)) \) can be written down as a linear combination of \( Q_{u,\nu} x^{b}e(\nu) - v \cdot (Q_{v,\nu} x^{b}e(\nu)) \) with \( v \) indecomposable relative to \( \nu \). Therefore,
$C_\beta$ has a set of $k$-generators

$$G_2 := \left\{ Q_{u,\nu}^\beta e(\nu) - u \cdot (Q_{u,\nu}^\beta e(\nu)) \mid \nu \in I^\beta, u \in S_n \setminus \{e\}, a \in \mathbb{N}^n \right\},$$

where we have used the condition that $Q_{i,j}(u, v) = Q_{j,i}(v, u)$, and $D_\beta^A$ is generated as $k$-module by

$$G'_2 = \left\{ Q_{u,\nu}^\beta e(\nu) - u \cdot (Q_{u,\nu}^\beta e(\nu)) \mid \nu \in I^\beta, u \in S_n \setminus \{e\}, a \in \mathbb{N}^n \right\}$$

$$\cup \left\{ y_{e,u,k}^\beta e(\nu) \mid \nu \in I^\beta, b \in \mathbb{N}^n, and \forall k < t \leq n, b_t < \langle \alpha_{\nu}^\vee, \Lambda \rangle - \sum_{1 \leq p < t} a_{\nu_p, \nu_p} \right\}.$$  

The following lemma describes all $u$ which is indecomposable relative to $\nu$.

**Lemma 4.5.** Let $\mu, \nu \in I^\beta$ and $u \in S_n$ such that $\mu = u.\nu$. Then $u$ is indecomposable relative to $\nu$ if and only if $u = s_k s_{k+1} \cdots s_t$ such that $\nu_k < \nu_{t+1} < \nu_p$ for $k+1 \leq p \leq t$, that is, $k = \max\{1 \leq s \leq t \mid \nu_s < \nu_{t+1}\}$. In particular, such $u$ is determined uniquely by $\nu$ and $t$, and thus, for any $1 \leq t < n$, there exists at most one such $u$ that is indecomposable relative to $\nu$.

**Proof.** The sufficiency is obvious, we show the necessity by induction on $n$. For $n = 1, 2$, the necessity is obvious.

For $n \geq 3$, assume that $u$ is indecomposable relative to $\nu$ and the necessity holds for any positive integer less or equal than $n - 1$. Let $v, w \in S_n$, such that $\nu = v.\gamma$, $\mu = w.\gamma$, then $w = uv$. In particular, $\nu_t = v^{-1}(t) \in I$ and $\mu_t = w^{-1}(t) \in I$. Since $\nu < \mu$, there exists $1 \leq t \leq n$, such that $v(k) = w(k), \forall 1 \leq k < t$ and $v(t) < w(t)$.

Assume $t > 1$. Let $p = v(1) = w(1)$. If $u = u_1 s_q s_{q-1} \cdots s_p u_2$, where $q \geq p$, $u_1 \in S_{[p+1,n]}$, $u_1$ is the distinguished minimal length left coset representative in $uS_{[p,n]}$. Then $u' := s_p u_2$ satisfies that $v(1) = p < p + 1 = u'v(1)$, which implies that $u_1 s_q s_{q-1} \cdots s_{p+1} \neq e$ and $\nu < u'.\nu$, contradict to the assumption that $u$ is indecomposable relative to $\nu$. If $u = u_1 u_2$, where $u_2 \in S_{[p+1,n]}$, $u_1$ is the distinguished minimal length left coset representative in $uS_{[p,n]}$. Then we must have $u_1 \in S_{p-1}$ because $p = v(1) = u(p) = u_1(p) < u_1(p + 1) < \cdots < u_1(n)$. Hence $u = u_1 u_2 = u_2 u_1$. 

Furthermore, if \( v(t) < p \), then for any \( 1 \leq k < t \),
\[
u_1 v(k) = \begin{cases} 
  u_1 u_2 v(k) = uv(k) = w(k) = v(k), & \text{if } v(k) < p; \\
  v(k), & \text{if } v(k) > p,
\end{cases}
\]
while \( u_1 u_2 v(t) = u(v(t)) = w(t) > v(t) \). Hence \( \nu < u_1 \nu \). In a similar way we can prove that if \( v(t) > p \) then \( \nu \prec u_2 \nu \). Now \( u \) is indecomposable relative to \( \nu \) implies that in the former case we have \( u_2 = e \) and \( u = u_1 \), while in the latter case \( u_1 = e \) and \( u = u_2 \). In both case we can apply the induction hypothesis to deduce that \( u \) is of the desired form.

Assume \( t = 1 \). Without loss of generality we may assume that \( \mu_n \neq \nu_n \), since otherwise \( u \in \mathfrak{S}_{n-1} \), and \( u \) is of the desired form by induction hypothesis. Therefore, we can write the element \( u \) uniquely as \( u = s_k s_{k+1} \cdots s_{n-1} u_1 \), where \( u_1 \in \mathfrak{S}_{n-1} \) and \( k \leq n-1 \). It remains to show that \( u_1 = e \). Suppose this is not the case, i.e., \( u_1 \neq e \). Set \( u' := s_k u \). Then \( u' \neq e \). Since \( u \) is indecomposable relative to \( \nu \), we have \( u' \nu \prec \nu \prec u \nu \). We can deduce that \( u' \nu(1) \in \{k, k+1\} \), because otherwise \( u' \nu(1) = w(1) > v(1) \) which contradicts to \( u' \nu \prec \nu \). Now \( u' \nu \prec \nu \) implies that \( w(1) = k+1 \). That is, \( u' \nu(1) = k = u_1 \nu(1) \). Because \( u_1 \nu < \nu \), \( k = u_1 \nu(1) \leq v(1) < w(1) = k + 1 \), we have \( v(1) = u_1 v(1) = k \). Using a similar argument as in the case \( t > 1 \), we can easily obtain that \( u_1 = u'_1 u'_2 \), with \( u'_1 \in \mathfrak{S}_{[1,k-1]} \) and \( u'_2 \in \mathfrak{S}_{[k+1,n-1]} \). We claim that \( u'_1 = e \) since otherwise \( u = u'_1 \cdot (s_k s_{k+1} \cdots s_{n-1} u'_2) \) is a decomposition of \( u \) relative to \( \nu \), contradiction. On the other hand, for each \( k+1 \leq j \leq n-2 \), \( s_k s_{k+1} \cdots s_{n-1} s_j s_{n-1} s_{n-2} \cdots s_k = s_{j+1} \). It follows that \( t = s_k s_{k+1} \cdots s_{n-1} u'_2 s_{n-1} s_{n-2} \cdots s_k \) satisfies that \( \ell(t) = \ell(u'_2) \). Thus \( u'_2 \neq e \) implies that \( t \cdot (s_k s_{k+1} \cdots s_{n-1}) \) is a decomposition of \( u \) relative to \( \nu \), which is a contradiction. In summary, \( u'_1 = e = u'_2 \) and \( u = s_k s_{k+1} \cdots s_{n-1} \).

Finally, if \( u = s_k s_{k+1} \cdots s_t \) is indecomposable relative to \( \nu \), then it is easy to check that \( \nu_k < \nu_{t+1} < \nu_p \) for \( k+1 \leq p \leq t \). We are done. \( \square \)

Remark 4.6. The notion of “indecomposable relative \( \nu \)” can be easily understood using the diagrammatic presentation of the symmetric group \( \mathfrak{S}_n \) (see, for example \cite[Exercise 1.5]{9}). A reduced expression of \( u \) gives a strand diagram in \( \mathbb{R} \times (0,1) \) matching \( \nu \) and \( \nu' = u \nu \). For example, \( u = \begin{pmatrix} 1, 2, 3, 4 \\ 4, 1, 3, 2 \end{pmatrix} = s_2 s_3 s_2 s_1 \in \mathfrak{S}_4 \), \( \nu = \)
\((\nu_1, \nu_2, \nu_3, \nu_4) = (1, 4, 3, 2)\). Then \(\nu = (1, 4, 3, 2) \prec u.\nu = (4, 2, 3, 1)\):

\[
\begin{array}{cccc}
\nu_1 & \nu_2 & \nu_3 & \nu_4 \\
\end{array}
\]

The element \(u\) is decomposable relative to \(\nu\) is equivalent to that we can find a strand diagram representing \(u\) and a horizontal line \(\mathbb{R} \times \{c\}\) with \(0 < c < 1\), such that their intersection \(\mu\) satisfies \(\nu \prec \mu\). In particular, the above diagram shows that \(w\) is decomposable relative to \(\nu\).

Let \(\nu\) as above. Now we consider the element \(v = \begin{pmatrix} 1, 2, 3, 4 \\ 2, 3, 4, 1 \end{pmatrix} = s_1 s_2 s_3 \in S_4\) and \(\nu'' = v.\nu = (2, 1, 4, 3)\), then \(v = s_1 s_2 s_3\) is the unique reduced decomposition of \(v\) and

\[
\begin{array}{cccc}
\nu_1 & \nu_2 & \nu_3 & \nu_4 \\
\end{array}
\]

it is easy to check that \(v\) is indecomposable relative to \(\nu\). One can also check that \(v\) is decomposable relative to \(\mu = (1, 4, 2, 3)\).

**Corollary 4.7.** If \(u = s_k s_{k+1} \cdots s_t \in S_n\) is indecomposable relative to \(\nu\), then \(Q_{u.\nu} = \prod_{p=k}^{t} Q_{\nu_p, \nu_{t+1}}(x_p, x_{t+1})\) is monic as polynomial of \(x_{t+1}\).

Now, we can construct the monomial basis for cocenter \(\mathcal{B}_\beta^\Lambda / [\mathcal{B}_\beta^\Lambda, \mathcal{B}_\beta^\Lambda]\).

**Theorem 4.8.** Assume that \(\beta\) satisfies (3.12), and fix an initial weight \(\gamma \in I^\beta\). For each \(\nu \in I^\beta\) and \(1 \leq t \leq n\), we define \(k_{\nu,t} := \max \{1 \leq p < t | \nu_p < \nu_t\}\); or 0 if
\{1 \leq p < t | \nu_p < \nu_t \} = \emptyset. Set

\begin{align*}
T^\Lambda_\gamma := \left\{ x^\omega e(\nu) \mid \nu \in I^\beta, \omega \in \mathbb{N}^n, \text{ and for any } 1 \leq t \leq n, \right. \\
\left. a_t < - \sum_{t = \max\{k_{u,v},1\}}^{t-1} a_{v_t,v_t} + \delta_{k_{u,v},0}(\alpha^\nu_{v_t},\Lambda) \right\}.
\end{align*}

Then the canonical image of \( T^\Lambda_\gamma \) in \( R^\Lambda_\beta / [R^\Lambda_\beta, R^\Lambda_\beta] \) gives a \( k \)-basis of \( R^\Lambda_\beta / [R^\Lambda_\beta, R^\Lambda_\beta] \).

**Proof.** Recall the definition of \( G'_2 \) in the paragraph above Lemma 4.5, where we have decomposed \( G'_2 \) as a disjoint union \( G'_2 = D_2 \sqcup M_2 \), such that \( D_2 \) consists of those commutators and \( M_2 \) is a basis of \( P_\beta \cap I_{\lambda,\beta} \) by Corollary 3.11. We are going to pick out a subset \( D'_2 \) of \( D_2 \) such that \( T^\Lambda_\gamma \sqcup D'_2 \sqcup M_2 \) form a \( k \)-basis of \( R_\beta \) and \( G'_2 \) is generated by \( D'_2 \sqcup M_2 \). Once this is done, it is clear that the canonical image of \( T^\Lambda_\gamma \) in \( R^\Lambda_\beta / [R^\Lambda_\beta, R^\Lambda_\beta] \) gives a \( k \)-basis of \( R^\Lambda_\beta / [R^\Lambda_\beta, R^\Lambda_\beta] \) because \( D'_2 \sqcup M_2 \) is a \( k \)-basis of \([R_\beta, R_\beta] + I_{\lambda,\beta}\).

For any \( \nu \in I^\beta, u \in S_n \) which is indecomposable relative to \( \nu \) and \( \omega \in \mathbb{N}^n \), we set \( P(u, \nu, \omega) := Q_{u,\nu}x^\omega e(\nu) - \mu (Q_{u,\nu}x^\omega e(\nu)) \). Define a subset \( D'_2 \) of \( D_2 \) as follows:

\begin{align*}
D'_2 = \left\{ P(u, \nu, \omega) \in D_2 \mid \nu \in I^\beta, \omega \in \mathbb{N}^n, u = s_p s_{p+1} \cdots s_{q-1}, \right. \\
\left. a_t < d_{u,v,t} \forall 1 \leq t \leq n \right\}
\end{align*}

where \( d_{u,v,t} \) (for \( u = s_p s_{p+1} \cdots s_{q-1} \)) is defined by

\begin{align*}
d_{u,v,t} = \left\{ \begin{array}{ll}
- \sum_{k = \max\{k_{u,v},1\}}^{t-1} a_{v_t,v_t} + \delta_{k_{u,v},0}(\alpha^\nu_{v_t},\Lambda) & \text{if } t > q; \\
(\alpha^\nu_{v_t},\Lambda) - \sum_{k = t-1}^{p-1} a_{v_t,v_k} & \text{if } t = q; \\
(\alpha^\nu_{v_t},\Lambda) - \sum_{k = 1}^{t-1} a_{v_t,v_k} & \text{if } 1 \leq t < q.
\end{array} \right.
\end{align*}

**Step 1.** We claim that \( T^\Lambda_\gamma \sqcup D'_2 \sqcup M_2 \) is a basis of \( P_\beta \). Recall that by Proposition 1.5, \( B_\beta = \{ x^\omega e(\nu) \mid \nu \in I^\beta, \omega \in \mathbb{N}^n \} \) is a \( k \)-basis of \( P_\beta \).

We define an total order \( \prec_\gamma \) on \( B_\beta \) as follows: \( x^\omega e(\nu) \prec_\gamma x^\omega e(\mu) \) if either \( \nu \prec_\gamma \mu \), or \( \nu = \mu \) and there is \( k \in [1,n] \) s.t. \( a_t = b_t \) for \( t > k \) and \( a_k > b_k \). The assumption (3.12) implies that \( g^\Lambda_{\nu,v,k}(\nu) = a^\Lambda_{\nu,k}(x_k) \prod_{p=1}^{k-1} Q_{\nu,v_p}(x_p, x_k) e(\nu) \). Then, for \( P(u, \nu, \omega) \in D_2 \)
and \( g_{e,v,k}^x e(\nu) \in M_2 \), we have

\[
P(u, \nu, \underline{a}) = c \cdot x_{\underline{a}} \cdot x_t - \sum_{p=k}^{t-1} a_{v_p u_p} e(\nu) + \text{"higher terms"}, \quad \text{if } u = s_{k} s_{k+1} \cdots s_{t-1},
\]

\[
g_{e,v,k}^x e(\nu) = c' \cdot x_{\underline{b}} \cdot x^k - \sum_{p=1}^{k-1} a_{v_p u_p + (a_{v_p} \Lambda)} e(\nu) + \text{"higher terms"}.
\]

where \( c, c' \) are invertible and "higher terms" means a \( k \)-linear combination of monomials \( x^a e(\mu) \in B_\beta \) that is larger than the "leading term" \( x^a \cdot x_t - \sum_{p=k}^{t-1} a_{v_p u_p} e(\nu) \) and

\[
x_{\underline{b}} \cdot x^k e(\nu) \text{ under the order } \prec_\gamma \text{ respectively. For any } x^a e(\nu) \in B_\beta, \text{ one can find a unique element } f \in T^A_\gamma \sqcup D'_2 \sqcup M_2, \text{ s.t. } f = c \cdot x^a e(\nu) + \text{"higher terms". This implies the claim of this step.}
\]

**Step 2.** We claim that \( D^A_\beta \) is generated as a \( k \)-module by \( D'_2 \sqcup M_2 \). Suppose this is not the case. Then we can find \( P(u, \nu, \underline{a}) \in D_2 \setminus D'_2 \), which does not belong to the \( k \)-module generated by \( D'_2 \sqcup M_2 \). As \( B_\beta \) is upper bounded under \( \prec_\gamma \), we can find such a \( P(u, \nu, \underline{a}) \) whose leading term is maximal under \( \prec_\gamma \). By the definition of \( D'_2 \), there is some \( t \in [1, n] \), such that \( a_t \geq d_{v,u,t} \) and we can choose \( t \) such that it is as maximal as possible. Now, by Lemma 4.5, \( u \) is form of \( s_p s_{p+1} \cdots s_{q-1} \).

(1) If \( 1 \leq t < q \), we have

\[
P(u, \nu, \underline{a}) = c \cdot \left( Q_{u,v} g_{e,v,t}^A x^a \cdot x_t^{-d_{v,u,t}} e(\nu) - u \left( Q_{u,v} g_{e,v,t}^A x^a \cdot x_t^{-d_{v,u,t}} e(\nu) \right) \right) + \text{"higher terms"}
\]

where \( c \in k^\times \) and "higher term" is a \( k \)-linear combination of some \( P(u, \nu, \underline{b}) \) whose leading terms are larger than that of \( P(u, \nu, \underline{a}) \). By our assumption, "higher terms" belong to the \( k \)-submodule generated by \( D'_2 \sqcup M_2 \). Note that \( Q_{u,v} g_{e,v,t}^A x^a \cdot x_t^{-d_{v,u,t}} e(\nu) \) belong to the \( k \)-submodule generated by \( M_2 \) and so does

\[
u (Q_{u,v} g_{e,v,t}^A x^a \cdot x_t^{-d_{v,u,t}} e(\nu)) = \tau_u \tau_{t-1} \tau_{t-2} \cdots \tau_1 a_{u_1}^A(x_1) \tau_1 \tau_2 \cdots \tau_{t-1} \tau_{u-1} e(u, \nu).
\]

This implies that \( P(u, \nu, \underline{a}) \) belongs to the \( k \)-module generated by \( D'_2 \sqcup M_2 \), contradiction.
(2) If \( t = q \), then as \( g^\Lambda_{e,\nu,q} = Q_{u,\nu} \cdot a^\Lambda_{\nu,q}(x_q) \cdot \prod_{k=1}^{p-1} Q_{\nu_k,\nu_q}(x_k, x_q) \), we have
\[
P(u, \nu, a) = c \cdot \left( g^\Lambda_{e,\nu,q} x^a_q \cdot x^{-d_{\nu,u,q}} e(\nu) - u \cdot (g^\Lambda_{e,\nu,q} x^a_q \cdot x^{-d_{\nu,u,q}} e(\nu)) \right) + \text{"higher terms"},
\]
where \( c \in \mathbb{k} \) and "higher term" is a \( k \)-combination of \( P(u, \nu, b) \) whose leading terms are larger than that of \( P(u, \nu, \bar{a}) \). A similar argument as in case (1) shows that both \( g^\Lambda_{e,\nu,q} x^a_q \cdot x^{-d_{\nu,u,q}} e(\nu) \) and \( u \cdot (g^\Lambda_{e,\nu,q} x^a_q \cdot x^{-d_{\nu,u,q}} e(\nu)) \) belong to the \( k \)-module generated by \( M_2 \) and this implies that \( P(u, \nu, a) \) belongs to the \( k \)-module generated by \( D'_2 \sqcup M_2 \), contradiction.

(3) If \( t > q \) and the set \( A_{\nu,t} = \{ 1 \leq i < t \mid \nu_i < \nu_t \} \neq \emptyset \), then we set \( k = \max A_{\nu,t} \),
\[
P(u, \nu, a) = c \cdot \left( Q_{u,\nu} Q_{v,\nu} x^a \cdot x^{-d_{\nu,u,t}} e(\nu) - u \cdot (Q_{u,\nu} Q_{v,\nu} x^a \cdot x^{-d_{\nu,u,t}} e(\nu)) \right)
\]
\[
+ \text{"higher terms"},
\]
where \( v = s_{k} s_{k+1} \cdots s_{t-1} \) and "higher terms" means a \( k \)-linear combination of \( P(u, \nu, b) \) whose leading terms are larger than that of \( P(u, \nu, a) \). By assumption, these \( P(u, \nu, b) \) belong to the \( k \)-module generated by \( D'_2 \sqcup M_2 \). On the other hand, set \( x^e = x^a \cdot x^{-d_{\nu,u,t}} \), the assumption on \( A_{\nu,t} \) yields that \( v \) is indecomposable relative to \( \nu \) and we have
\[
Q_{u,\nu} Q_{v,\nu} x^e e(\nu) - u \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) = Q_{u,\nu} Q_{v,\nu} x^e e(\nu) - v \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu))
\]
\[
+ v \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) - u \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu))
\]
It is easy to check that \( \ell(uv^{-1}) = \ell(u) + \ell(v^{-1}) \), and by definition, we have
\[
v \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) = \tau_v \tau_{u-1} \tau_{v-1} x^e u e(v, \nu) = Q_{uv^{-1},u,v} x^e e(v, \nu)
\]
\[
u \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) = \tau_u \tau_{v-1} \tau_{u-1} x^e v e(u, \nu) = Q_{vu^{-1},u,v} x^e e(u, \nu)
\]
\[
v \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) - u \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu))
\]
\[
= Q_{uv^{-1},u,v} x^e e(v, \nu) - uv^{-1} \cdot (Q_{uv^{-1},u,v} x^e e(v, \nu))
\]
Now, as \( \nu \prec_{\gamma} v \cdot \nu \) and \( \nu \prec_{\gamma} u \cdot \nu \), (4.4) implies that \( v \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) - u \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) \) is a \( k \)-linear combination of \( P(w, \mu, b) \) with \( \nu \prec_{\gamma} \mu \) and hence belongs to the \( k \)-submodule generated by \( D'_2 \sqcup M_2 \) by our assumption.

On the other hand, \( Q_{u,\nu} Q_{v,\nu} x^e e(\nu) - v \cdot (Q_{u,\nu} Q_{v,\nu} x^e e(\nu)) \) is a \( k \)-linear combination of some \( P(v, \nu, b) \) and the one having minimal leading term among those appear with
non-zero coefficients is $P(v, \nu, d)$ satisfying that $x^d e(\nu) = x^c x_q^{-\sum_{i=p}^{q-1} a_{\nu_i}} e(\nu)$. Note that the leading term of $P(u, \nu, a)$ is equal to that of $P(v, \nu, d)$ and this yields that $P(v, \nu, d)$ does not belong to the $k$-module generated by $D'_2 \sqcup M_2$ by our assumption. But for $t' > t > q$, we have $d_{\nu,v,t'} = d_{\nu,u,t'} = a_{t'} = d_{t'}$. As by assumption $P(v, \nu, d) \not\in D'_2$, we can find $p \in [1, n]$ as large as possible such that $d_p \geq d_{\nu,v,p}$. The argument above shows that $p \leq t$. Therefore, we are in a position to apply the case (1) (if $1 \leq p < t$); or the case (2) (if $p = t$) to $P(v, \nu, d)$. In both case, we obtain a contradiction.

(4) If $t > q$ and $A_{\nu,t} = \{1 \leq i < t \mid \nu_i < \nu_t\} = \emptyset$, then we have

$$P(u, \nu, a) = c \cdot \left( Q_{u,\nu} g_{e,v,t}^{\Lambda} x^a \cdot x_t^{-d_{\nu,u,t}} e(\nu) - u \cdot (Q_{u,\nu} g_{e,v,t}^{\Lambda} x^a \cdot x_t^{-d_{\nu,u,t}} e(\nu)) \right)$$

$$+ \text{"higher terms"}$$

where $c \in k^\times$ and “higher term” is a $k$-linear combination of $P(u, \nu, b)$ whose leading terms are larger than that of $P(u, \nu, a)$. An argument similar to the case (1) yields a contradiction.

Step 3. $D'_2 \sqcup M_2$ is a $k$-basis of $D^\Lambda_\beta$. This is a immediate consequence of Step 1 and 2. Moreover, the image of $T^\Lambda_{\gamma}$ is a basis of $\mathcal{R}_\gamma^\Lambda / [\mathcal{R}_\beta^\Lambda, \mathcal{R}_\beta^\Lambda]$. We’re done. \qed

Lemma 4.14. Assume that $\beta \in Q^+_n$ satisfies (3.12). For $i \in I$ and $\Lambda \in P^+$, set $\tilde{\Lambda} = \Lambda + \Lambda_i$. Then the $k$-linear homomorphism $\tilde{\tau}_{\beta,i}^\Lambda$ is injective. Equivalently, the following natural algebra homomorphism

$$P_{\beta,i}^\Lambda : Z(\mathcal{R}_\beta^\Lambda) \to Z(\mathcal{R}_\beta^\Lambda)$$

is surjective.

Proof. Pick $\gamma \in I^\beta$ such that $\gamma_1 = i$. Let $x^\gamma e(\nu) \in T^\Lambda_{\gamma}$. Then

$$z(i, \beta) x^\gamma e(\nu) = x^\gamma e(\nu),$$
where $b_t = a_t + \delta_{\nu_t,i}$. For $\nu \in I^\beta$, there is at most one $t \in [1, n]$, such that $\nu_t = i$, and in this case,

$$a_t < - \sum_{l=1}^{t-1} a_{\nu_l, \nu_l} + \langle \alpha_{\nu_l}^\vee, \Lambda \rangle,$$

$$b_t < - \sum_{l=1}^{t-1} a_{\nu_l, \nu_l} + \langle \alpha_{\nu_l}^\vee, \Lambda \rangle + 1 = - \sum_{l=1}^{t-1} a_{\nu_l, \nu_l} + \langle \alpha_{\nu_l}^\vee, \Lambda \rangle + \langle \alpha_{\nu_l}^\vee, \bar{\Lambda} \rangle$$

and for any $1 \leq p \leq n$ with $p \neq t$, we have

$$a_p < - \sum_{l=\max\{k_{\nu,p},1\}}^{p-1} a_{\nu_p, \nu_l} + \delta_{k_{\nu,p},0} \langle \alpha_{\nu_p}^\vee, \Lambda \rangle,$$

$$b_p < - \sum_{l=\max\{k_{\nu,p},1\}}^{p-1} a_{\nu_p, \nu_l} + \delta_{k_{\nu,p},0} \langle \alpha_{\nu_p}^\vee, \Lambda \rangle = - \sum_{l=\max\{k_{\nu,p},1\}}^{p-1} a_{\nu_p, \nu_l} + \delta_{k_{\nu,p},0} \langle \alpha_{\nu_p}^\vee, \bar{\Lambda} \rangle$$

So, $x^b e(\nu) \in T^\Lambda_\gamma$, and this completes the proof. \hfill \Box

**Corollary 4.15.** Assume that $\beta \in Q^+_n$ satisfies (3.12). For $\Lambda, \Lambda' \in P^+$, set $\bar{\Lambda} = \Lambda + \Lambda'$. The canonical homomorphism of algebras

$$Z(\mathcal{R}_{\beta}^{\bar{\Lambda}}) \rightarrow Z(\mathcal{R}_{\beta}^{\Lambda})$$

is surjective.

Now, by Theorem 2.1 or 2.7, we have

**Theorem 4.16.** Assume that $\beta \in Q^+_n$ satisfies (3.12). For any $\Lambda \in P^+$, the canonical algebra homomorphism

$$Z(\mathcal{R}_{\beta}) \rightarrow Z(\mathcal{R}_{\beta}^{\Lambda})$$

is surjective. In other words, the Center Conjecture 1.9 holds for $\mathcal{R}_{\beta}^{\Lambda}$.

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