Precoloring Extension
Involving Pairs of Vertices of Small Distance

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Abstract

In this paper, we consider coloring of graphs under the assumption that some vertices are already colored. Let $G$ be an $r$-colorable graph and let $P \subset V(G)$. Albertson [J. Combin. Theory Ser. B 73 (1998), 189–194] has proved that if every pair of vertices in $P$ have distance at least four, then every $(r+1)$-coloring of $G[P]$ can be extended to an $(r+1)$-coloring of $G$, where $G[P]$ is the subgraph of $G$ induced by $P$. In this paper, we allow $P$ to have pairs of vertices of distance at most three, and investigate how the number of such pairs affects the number of colors we need to extend the coloring of $G[P]$. We also study the effect of pairs of vertices of distance at most two, and extend the result by Albertson and Moore [J. Combin. Theory Ser. B 77 (1999) 83–95].

Keywords: coloring, precoloring, distance

1 Introduction

Graph coloring has a number of applications. One example is a job scheduling problem. In this problem, each job is represented by a vertex, and a pair of vertices are joined by an edge if the corresponding jobs cannot be processed concurrently. In this model, an independent set represents a set of jobs which can be preformed at the same time, and if we assume that each job is processed

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In a unit time, the chromatic number gives the minimum amount of time in which we can finish all the jobs in a concurrent environment.

In the real world, however, the job scheduling may not be tackled from scratch. In many cases, the schedule of some jobs are already fixed and cannot be changed. In graph coloring, it corresponds to a situation in which some vertices are already colored. A precoloring extension is a problem to handle this situation. In this problem, a graph \( G \), a set of vertices \( P \subset V(G) \) and a coloring \( d: P \to \mathbb{Z} \) of \( G[P] \) are given, where \( G[P] \) is the subgraph of \( G \) induced by \( P \). We call \( d \) a precoloring. Our task is to find a coloring \( V(G) \to \mathbb{Z} \) of \( G \) whose restriction into \( P \) coincides with \( d \). If \( P \) is sufficiently sparse, we may expect to extend \( d \) to a coloring of \( G \) with a few extra colors. For the measure of sparseness, Albertson [1] and Albertson and Moore [5] have considered the minimum distance between the vertices in \( P \).

**Theorem A** ( [1] ) Let \( G \) be a graph with chromatic number at most \( r \), and let \( P \subset V(G) \). Suppose every pair of distinct vertices in \( P \) have distance at least four. Then every \((r+1)\)-coloring of \( P \) can be extended to an \((r+1)\)-coloring of \( G \).

**Theorem B** ( [5] ) Let \( G \) be a graph with chromatic number at most \( r \) and let \( P \subset V(G) \). Suppose every pair of distinct vertices in \( P \) have distance at least three. Then every \((r+1)\)-coloring of \( P \) can be extended to a \( \lceil \frac{3r+1}{2} \rceil \)-coloring of \( G \).

These theorems give insight into the relationship between the distance of precolored vertices and the number of colors necessary to extend the precoloring to a coloring of the whole graph.

On the other hand, again in the real world, the assumptions of Theorems A and B may be idealistic. For example, while we have Theorem A we may have to deal with a set \( P \) of precolored vertices which contains pairs of vertices of distance three. In this case, we might be forced to use more than \( r+1 \) colors to extend the given precoloring. But if the number of the pairs of distance three is sufficiently small, we expect that the number of additional colors is also small. Theorem A does not answer this question.

Motivated by this observation, we investigate the situation in which the set of precolored vertices contains pairs of distance at most three and two, and investigate how these pairs affect the conclusion of Theorems A and B respectively.

For a graph \( G \), \( P \subset V(G) \) and a positive integer \( k \), we define \( D_G(P,k) \) by

\[ D_G(P,k) = \{ \{x,y\} \subset P : x \neq y \text{ and } d_G(x,y) \leq k \}, \]

where \( d_G(x,y) \) is the distance between \( x \) and \( y \) in \( G \).
In the next section, we give an upper bound to the number of additional colors to extend a given precoloring of $P$ to a coloring of $G$, which is described in terms of $|D_G(P, 3)|$. In Section 3, we give another bound, which is described in terms of $|D_G(P, 2)|$. In Section 4, we give some concluding remarks.

We remark that this paper is neither the only nor the first one to extend Theorems A and B. The problem of extending a precoloring to the entire graph has been studied in many papers. We refer the readers who are interested in this problem to \cite{2, 4, 6, 7, 10, 17}. In particular, Albertson and Hutchinson \cite{3} and Hutchinson and Moore \cite{12} have considered the situation in which the set of precolored vertices induces a graph with several components, and studied distance conditions among these components that guarantee the extension without using an additional color. They have given best-possible results in many cases.

For graph-theoretic notation and definitions not explained in this paper, we refer the reader to \cite{9}. Let $G$ be a graph. Then we denote by $\Delta(G)$ and $\chi(G)$ the maximum degree and the chromatic number of $G$, respectively. For $x \in V(G)$, we denote the neighborhood of $x$ in $G$ by $N_G(x)$. In this paper, we often deal with the closed neighborhood of $G$, which is denoted and defined by $N_G[x] = N_G(x) \cup \{x\}$. If $A, B \subset V(G)$ and $A \cap B = \emptyset$, we define $E_G(A, B)$ by $E_G(A, B) = \{ab \in E(G) : a \in A \text{ and } b \in B\}$.

Let $P \subset V(G)$. As we have already seen, a coloring of $G[P]$ is called a precoloring of $P$ in $G$. In this paper, we always perceive a coloring of $G$ as a mapping $f: V(G) \to \mathbb{Z}$. If $d: P \to \mathbb{Z}$ is a precoloring of $P$ in $G$ and $f: V(G) \to \mathbb{Z}$ is a coloring of $G$ with $f(v) = d(v)$ for every $v \in P$, we say that $f$ extends $d$. For a positive integer $r$, we denote the set $\{1, 2, \ldots, r\}$ by $[r]$. An $r$-coloring of $G$ is a coloring of $G$ which uses at most $r$ colors. In this paper, an $r$-coloring is often perceived as a function from $V(G)$ to $[r]$. For $t \in [r]$, $f^{-1}(t)$ is the set of vertices that receive the color $t$. We call it the color class of $V(G)$ with respect to the color $t$.

If $e = uv$ is an edge of a graph $G$, we denote $\{u, v\}$ by $V(e)$. Moreover, for $F \subset E(G)$, we write $V(F)$ for $\bigcup_{e \in F} V(e)$. A matching of $G$ is a set of independent edges in $G$. Hence if $M$ is a matching, then the order of $M$, denoted by $|M|$, is the number of edges in $M$, and $|V(M)| = 2|M|$. If $M$ is a maximum matching of $G$, $|V(G)| - |V(M)|$ is called the deficiency of $G$. Concerning the deficiency of a graph, Berge’s Formula is well-known. We denote by $o(G)$ the number of components of odd order in $G$.

**Theorem C (Berge’s Formula)** \cite{8} For a graph $G$, the deficiency of $G$ is given by $\max\{o(G - S) - |S| : S \subset V(G)\}$.

A matching $M$ in $G$ is called a perfect matching if $V(M) = V(G)$, and $M$ is called an almost
perfect matching if $|V(M)| = |V(G)| - 1$.

2 Pairs of Vertices of Distance Three

In this section, we investigate the effect of the number of vertices which are of distance at most three. Theorem A states that for a graph $G$ with $\chi(G) \leq r$ and $P \subset V(G)$ with $D_G(P, 3) = \emptyset$, every $(r + 1)$-coloring of $P$ extends to an $(r + 1)$-coloring of $G$. If $D_G(P, 3) \neq \emptyset$, we may need more than $r + 1$ colors. The purpose of this section is to prove that for $t = |D_G(P, 3)|$, $r + O(\sqrt{t})$ colors suffice.

Theorem 1 Let $k$ be a positive integer. Let $G$ be a graph with $\chi(G) \leq r$ and let $P \subset V(G)$.

Suppose $|D_G(P, 3)| \leq \frac{1}{2}k(k + 1)$. Then for each precoloring $d: P \to [r + 1]$ in $G$, there exists a coloring $f: V(G) \to [r + k]$ with $f(u) = d(u)$ for each $u \in P$.

We prove several lemmas to give a proof to Theorem 1. The following lemma has already been proved in [1]. But we tailor its statement so that it fits the subsequent arguments. For the completeness of the paper, we give a proof to it. For two colorings $f$, $g$ of a graph $G$, we define $X(f, g)$ by $X(f, g) = \{v \in V(G): f(v) \neq g(v)\}$.

Lemma 2 ([1]) Let $G$ be a graph with an $r$-coloring $c: V(G) \to [r]$. Let $P$ be a set of vertices of $G$ with $D_G(P, 3) = \emptyset$. Then for every precoloring $d: P \to [r + 1]$ in $G$, there exists an $(r + 1)$-coloring $f: V(G) \to [r + k]$ such that

(1) $f(x) = d(x)$ for every $x \in P$, and

(2) for each $v \in X(c, f)$, there exists a unique vertex $x \in P$ such that $v \in N_G(x)$. Moreover, if $v \neq x$, then $c(v) = d(x)$.

Proof. For each $x \in P$, if $d(x) \neq c(x)$, then give the color $r + 1$ to all the vertices $v$ in $N_G(x)$ with $c(v) = d(x)$ and then assign $d(x)$ to $x$. Since $D_G(P, 3) = \emptyset$, no two vertices receiving the color $r + 1$ are adjacent. Hence this gives a proper $(r + 1)$-coloring $f$ of $G$. By the construction, $f$ satisfies both (1) and (2). $\square$

Let $k$ be a positive integer. Then a coloring $c$ of a graph $G$ is said to be an almost $k$-coloring if

(1) $c$ is a $k$-coloring of $G$, or

(2) $c$ is a $(k + 1)$-coloring such that at least one color class is a singleton set.

Lemma 3 For a positive integer $k$, a graph $H$ with $|E(H)| \leq \frac{1}{2}k(k + 1)$ has an almost $k$-coloring.
Proof. We proceed by induction on \( k \). If \( k = 1 \), then \( |E(H)| \leq 1 \), and it is easy to see that \( H \) has an almost 1-coloring. Suppose \( k \geq 2 \). If \( \chi(H) \leq k \), then a \( k \)-coloring of \( H \) is an almost \( k \)-coloring of \( H \). Thus, we may assume \( \chi(H) \geq k + 1 \). Then by Brook’s Theorem, we have \( \Delta(H) \geq k \). Let \( x \) be a vertex of maximum degree, and let \( H' = H - x \). Then \( |E(H)| \leq \frac{1}{2}k(k+1) - k = \frac{1}{2}k(k-1) \).

By the induction hypothesis, \( H \) has an almost \((k-1)\)-coloring. Then by assigning a new color to \( x \), we obtain an almost \( k \)-coloring of \( H \). \( \square \)

Lemma 4 Let \( G \) be a graph and let \( P \subset V(G) \). Suppose \( P \) has a partition \( \{P_0, P_1, \ldots, P_k\} \) with \( D_G(P_i, 3) = \emptyset \) for each \( i \) with \( 0 \leq i \leq k \). Let \( d: P \to [r+1] \) be a precoloring of \( P \) in \( G \). Suppose \( G \) has an \( r \)-coloring \( f \) with \( f(v) = d(v) \) for each \( v \in P_0 \). Then for each \( t \) with \( 0 \leq t \leq k \), there exists an \((r+t)\)-coloring \( f_t \) of \( G \) with \( f_t(v) = d(v) \) for each \( v \in \bigcup_{i=0}^{t} P_i \). In particular, \( f_k \) is an \((r+k)\)-coloring of \( G \) with \( f_k(v) = d(v) \) for each \( v \in P \).

Proof. We proceed by induction on \( t \). If \( t = 0 \), the lemma trivially follows with \( f_0 = f \). Suppose \( t \geq 1 \). By the induction hypothesis, there exists an \((r+t-1)\)-coloring \( f_{t-1} \) of \( G \) with \( f_{t-1}(v) = d(v) \) for each \( v \in \bigcup_{i=0}^{t-1} P_i \). Without loss of generality, we may assume \( f_{t-1}: V(G) \to [r+t-1] \). By Lemma 2 there exists a coloring \( f_t: V(G) \to [r+t] \) of \( G \) such that

(1) \( f_t(x) = d(x) \) for each \( x \in P_t \), and

(2) for each \( v \in X(f_{t-1}, f_t) \), there exists a unique vertex \( x \) in \( P_t \) with \( v \in N_G[x] \). Moreover, if \( v \neq x \), then \( f_{t-1}(v) = d(x) \).

Suppose \( f_t(v) \neq d(v) \) for some \( v \in \bigcup_{i=0}^{t} P_i \). By (1), \( v \in \bigcup_{i=0}^{t-1} P_i \) and hence \( f_{t-1}(v) = d(v) \). This implies \( f_t(v) \neq f_{t-1}(v) \) and hence \( v \in X(f_{t-1}, f_t) \). Then there exists a unique vertex \( x \in P_t \) with \( v \in N_G[x] \). Since \( v \notin P_t \), we have \( v \neq x \) and \( f_{t-1}(v) = d(x) \) by (2). Now we have \( d(v) = f_{t-1}(v) = d(x) \), which contradicts the assumption that \( d \) is a precoloring of \( P \). Therefore, we have \( f_t(v) = d(v) \) for each \( v \in \bigcup_{i=0}^{t} P_i \). \( \square \)

Proof of Theorem 1 Define an auxiliary graph \( H \) by \( V(H) = P \) and \( E(H) = \{uv: \{u, v\} \in D_G(P, 3)\} \). Then \( |E(H)| \leq \frac{1}{2}k(k+1) \). By Lemma 3 \( H \) has an almost \( k \)-coloring. Let \( f_0: V(H) \to \{0\} \cup [k] \) be an almost \( r \)-coloring of \( H \) with \( |f_0^{-1}(0)| \leq 1 \). Let \( P_i = f_0^{-1}(i) \) \( (0 \leq i \leq k) \). Since \( \chi(G) \leq r \), there exists a coloring \( c: V(G) \to [r] \) of \( G \). If \( P_0 \neq \emptyset \), then let \( P_0 = \{u_0\} \) and by taking an appropriate permutation of colors, we may assume \( c(u_0) = d(u_0) \). Then by Lemma 4 \( G \) has an \((r+k)\)-coloring \( f \) with \( f(v) = d(v) \) for each \( v \in P \). \( \square \)
3 Pairs of Vertices of Distance Two

In this section, we consider the effect of pairs of vertices of distance two. As we have seen in the introduction, under the assumption of $D_G(P, 2) = \emptyset$, Albertson and Moore [5] have proved that an $(r+1)$-coloring of $P$ can be extended to a $\left\lceil \frac{3r+1}{2} \right\rceil$-coloring of $G$. They have also given infinitely many examples of $(G, P, d)$ for each positive integer $r$ such that (1) $G$ is a graph with $\chi(G) \leq r$, (2) $P \subset V(G)$ with $D_G(P, 2) = \emptyset$, (3) $d: P \rightarrow [r+1]$ is a precoloring of $P$ in $G$, and (4) every coloring of $G$ that extends $d$ uses at least $\left\lceil \frac{3r+1}{2} \right\rceil$-colors. Therefore, Theorem B is best possible in this sense. On the other hand, the assumption on $|D_G(P, 2)|$ has room for relaxation. We prove that even if $D_G(P, 2) \neq \emptyset$, we can still extend an $(r+1)$-coloring of $P$ to a $\left\lceil \frac{3r+1}{2} \right\rceil$-coloring of $G$ as long as $|D_G(P, 2)|$ is sufficiently small. We also consider the case in which $P$ is colored in more than $r+1$ colors.

If $k \leq r$, we prove the following theorem.

**Theorem 5** Let $k$ and $r$ be positive integers with $r \geq 2$ and $k \leq r$, and let $G$ be a graph with $\chi(G) \leq r$. Let $P \subset V(G)$ and let $d: P \rightarrow [r+k]$ be a precoloring of $P$ in $G$.

(1) If $r + k \equiv 0 \pmod{2}$ and $|D_G(P, 2)| < 2(r+k-1)$, then $d$ can be extended to a $\left\lceil \frac{3r+k}{2} \right\rceil$-coloring of $G$.

(2) If $r + k \equiv 1 \pmod{2}$, $r + k \geq 13$ and $|D_G(P, 2)| < 3(r+k-1)$, then $d$ can be extended to a $\left\lceil \frac{3r+k+1}{2} \right\rceil$-coloring of $G$.

We also prove that the bound of $|D_G(P, 2)|$ in the above theorem is best-possible in the following sense.

**Theorem 6** For every pair of integers with $r$ and $k$ with $r \geq 2$, $r + k \equiv 0 \pmod{2}$ and $k \leq r$, there exist infinitely many triples $(G, P, d)$ such that

(1) $G$ is a graph of $\chi(G) \leq r$,

(2) $P \subset V(G)$ and $|D_G(P, 2)| = 2(r+k-1)$,

(3) $d: P \rightarrow [r+k]$ is a precoloring of $P$ in $G$, and

(4) $d$ cannot be extended to a $\left\lceil \frac{3r+k}{2} \right\rceil$-coloring of $G$.

**Theorem 7** For every pair of integers $r$ and $k$ with $r \geq 2$, $r + k \equiv 1 \pmod{2}$ and $k \leq r$, there exist infinitely many triples $(G, P, d)$ such that
Let $G$ be a graph. Then

1. $G$ is a graph with $\chi(G) \leq r$,

2. $P \subset V(G)$ and $|D_G(P, 2)| = 3(r + k - 1)$,

3. $d: P \rightarrow [r + k]$ is a precoloring of $P$ in $G$, and

4. $d$ cannot be extended to a $3\frac{r + k + 1}{2}$-coloring of $G$.

In the range of $k > r$, the situation changes. We no longer need an additional color to extend a precoloring of $P$. Moreover, different bounds of $|D_G(P, 2)|$ from those in Theorem 6 appear.

**Theorem 8** Let $r$ and $k$ be positive integers with $k > r \geq 2$, and let $G$ be a graph with $\chi(G) \leq r$. Let $P \subset V(G)$ and let $d: P \rightarrow [r + k]$ be a precoloring of $P$ in $G$.

1. If $r < k \leq 3r^2/k - 7/k$ and $|D_G(P, 2)| < \min \{\frac{1}{2}(k + 3r - 4)(k - r + 3), (k - r + 2)(k + r - 1)\}$, then $d$ can be extended to an $(r + k)$-coloring of $G$.

2. If $k > \frac{3r^2}{k} - 7/k$ and $|D_G(P, 2)| < \min \{\frac{1}{2}(k + 1)(k + 2), (k - r + 2)(k + r - 1)\}$, then $d$ can be extended to an $(r + k)$-coloring of $G$.

Note that the bounds of $|D_G(P, 2)|$ in Theorem 6 are linear functions of $k$ and $r$ and they are sharp for $k \leq r$, while the bounds in the above theorem are quadratic functions of $k$.

In order to prove Theorems 6 and 8 we use matchings among colors. For this purpose, we introduce several definitions.

For a set $X$, a triple $E = (E_0, E_1, E_2)$ is said to be an ordered partition of $X$ if $E_0 \cap E_1 = E_0 \cap E_2 = E_1 \cap E_2 = \emptyset$ and $X = E_0 \cup E_1 \cup E_2$. Note that in this definition, possibly $E_i = \emptyset$ for some $i$, $1 \leq i \leq 3$. Let $G$ be a graph, and let $E = (E_0, E_1, E_2)$ be an ordered partition of $E(G)$. Let $M$ be a matching. Then $M$ is said to be a good matching for $E$ if $M \cap E_2 = \emptyset$ and $|M \cap E_1| \leq 1$.

Let $r$ and $k$ be positive integers. Let $G$ be a graph with $\chi(G) \leq r$ and let $P \subset V(G)$. Suppose a precoloring $d: P \rightarrow [r + k]$ in $G$ is given. Now for a pair of distinct colors $\{i, j\}$, we count how many times it appears as $\{d(x), d(y)\}$ with $\{x, y\} \in D_G(P, 2)$. Let $H$ be the complete graph with $V(H) = [r + k]$. For each edge $e = ij$ of $H$, we define $\varphi(e)$ by $\varphi(e) = |\{x, y\} \in D_G(P, 2): \{d(x), d(y)\} = \{i, j\}|$. Then define $E^d_0$, $E^d_1$, $E^d_2$ and $E^d$ by

\[
E^d_0 = \{e \in E(H): \varphi(e) = 0\}, \\
E^d_1 = \{e \in E(H): \varphi(e) = 1\}, \\
E^d_2 = \{e \in E(H): \varphi(e) \geq 2\}, \quad \text{and} \\
E^d = (E^d_0, E^d_1, E^d_2).
\]
Clearly, $E^d$ is an ordered partition of $E(H)$.

We make the following observation. Though it is easy, it is an important step in the proofs of Theorems 5 and 8.

Lemma 9 Let $G, P, d, H, \varphi$ and $E^d$ be as above. Then $|E^d_1| + 2|E^d_2| \leq |D_G(P, 2)|$.

Proof. By the definition of $\varphi$ and $E^d$,

$$|D_G(P, 2)| = \sum_{e \in E(H)} \varphi(e) = \sum_{k \geq 0} k|\{e \in E(H) : \varphi(e) = k\}| \geq |E^d_1| + 2|E^d_2|. \quad \square$$

We next prove a lemma which shows a relationship between color extensions and good matchings.

Lemma 10 Let $r$ and $k$ be positive integers, and let $G$ be a graph with $\chi(G) \leq r$. Let $P \subset V(G)$ and let $d: P \rightarrow [r+k]$ be a precoloring of $P$ in $G$. Let $H$ be the complete graph with $V(H) = [r+k]$.

1. If $k \leq r$ and $H$ has a good matching for $E^d$ of order $\lceil \frac{1}{2}(r + k) \rceil$, then $d$ can be extended to a $\left\lceil \frac{3r+k}{2} \right\rceil$-coloring of $G$.

2. If $k > r$ and $H$ has a good matching for $E^d$ of order $r$, then $d$ can be extended to an $(r+k)$-coloring of $G$.

Proof. Let $M = \{m_i n_i : 1 \leq i \leq \min \{r, \left\lceil \frac{1}{2}(r+k) \right\rceil\}\}$ be a good matching for $E^d$ in $H$. We may assume $M \cap E^d_1 \subset \{m_1 n_1\}$, and if $M \cap E^d_i = \{m_1 n_1\}$, then let $\{x_1, y_1\}$ be the unique pair in $D_G(P, 2)$ with $d(x_1) = m_1$ and $d(y_1) = n_1$.

Since $\chi(G) \leq r$, there exists a coloring $g: V(G) \rightarrow [r]$ of $G$. Let $D_i = g^{-1}(i)$ and $C_i = D_i - P$ ($1 \leq i \leq r$). Without loss of generality, we may assume $g(x_1) = 1$. Then $N_G(x_1) \cap C_1 = \emptyset$.

We first prove (1). We define a mapping $f: V(G) \rightarrow Z$ in the following way.

1. If $v \in C_i$ with $1 \leq i \leq \left\lceil \frac{1}{2}(r+k) \right\rceil$ and $N_G(v) \cap d^{-1}(m_i) = \emptyset$, let $f(v) = m_i$.

2. If $v \in C_i$ with $1 \leq i \leq \left\lceil \frac{1}{2}(r+k) \right\rceil$ and $N_G(v) \cap d^{-1}(m_i) \neq \emptyset$, let $f(v) = n_i$.

3. If $v \in C_i$ with $\left\lceil \frac{1}{2}(r+k) \right\rceil + 1 \leq i \leq r$, let $f(v) = \left\lceil \frac{1}{2}(r+k) \right\rceil + i$.

4. If $v \in P$, let $f(v) = d(v)$.

Note that the total number of colors used by $f$ is at most $\left\lceil \frac{1}{2}(r+k) \right\rceil + r = \left\lceil \frac{3r+k}{2} \right\rceil$.

We prove that $f$ is a proper coloring of $G$. Assume, to the contrary, that $f(u) = f(v)$ for some adjacent pair of vertices $u$ and $v$ in $G$. Since the restriction of $f$ into $P$ coincides with
Then the total number of colors used by $v \in \mathcal{V}$ is at most $r + k$.

Proof. Let $E = (E_0, E_1, E_2)$ be an ordered partition of $E(H)$. If $|E_1| + 2|E_2| < 2(r + k - 1)$, then $H$ contains a perfect matching which is also a good matching for $E$.

**Proof.** Since $H$ is a complete graph of even order, $E(H)$ can be decomposed into $r + k - 1$ perfect matchings $M_1, \ldots, M_{r+k-1}$. Assume none of them is a good matching. Then for each $i$, $1 \leq i \leq r + k - 1$, either $|M_i \cap E_1| \geq 2$ or $|M_i \cap E_2| \geq 1$ holds. Then in either case, we have $|M_i \cap E_1| + 2|M_i \cap E_2| \geq 2$, and

\[
|E_1| + 2|E_2| = \sum_{i=1}^{r+k-1} |E_1 \cap M_i| + 2 \sum_{i=1}^{r+k-1} |E_2 \cap M_i| \\
= \sum_{i=1}^{r+k-1} (|M_i \cap E_1| + 2|M_i \cap E_2|) \geq 2(r + k - 1).
\]
This contradicts the assumption, and hence at least one of $M_1, \ldots, M_{r+k-1}$ is a good matching for $E$. \hfill \Box

By combining Lemmas 9 and 10 we obtain a proof of Theorem 5(1).

Next, we prove Theorem 5(2). We further prove several lemmas.

**Lemma 12** Let $H$ be a complete graph, and let $E = (E_0, E_1, E_2)$ be an ordered partition of $E(H)$. Let $M$ be a matching in $H$ with $M \subseteq E_0$ and $|M| < \left\lfloor \frac{1}{2}|V(H)| \right\rfloor$, and let $X = V(H) - V(M)$. Then

1. if $E(H[X]) \cap (E_0 \cup E_1) \neq \emptyset$ then $H$ contains a good matching for $E$ of order $|M| + 1$, and
2. if $|E_H(V(M), X) \cap E_0| > |E_H(V(M), X) \cap E_2|$, then $H$ contains a good matching for $E$ of order $|M| + 1$.

**Proof.** (1) Let $e \in E(H[X]) \cap (E_0 \cup E_1)$. Then $M \cup \{e\}$ is a good matching for $E$.

(2) Assume, to the contrary, that $H$ does not contain a good matching for $E$ of order $|M| + 1$. Since

$$\sum_{e \in M} |E_H(V(e), X) \cap E_2| = |E_H(V(M), X) \cap E_2|$$

we have $|E_H(V(e), X) \cap E_2| < |E_H(V(e), X) \cap E_0|$ for some $e \in M$. Then $|E_H(V(e), X) \cap E_0| = \sum_{e \in M} |E_H(V(e), X) \cap E_0|$, we have $|E_H(V(e), X) \cap E_2| \leq |X| - 1$. Let $e = uv$. Since $E_H(V(e), X) \cap E_0 \neq \emptyset$, we may assume $ux_0 \in E_0$ some $x_0 \in X$. If $vx \in E_0 \cup E_1$ for some $x \in X - \{x_0\}$, then $(M - \{uv\}) \cup \{ux_0, vx\}$ is a good matching for $E$, a contradiction. Therefore, $vx \in E_2$ for every $x \in X - \{x_0\}$. Since $|E_H(V(e), X) \cap E_2| \leq |X| - 1$, This implies that $E_H(V(e), X) \cap E_2 = \{vx : x \in X - \{x_0\}\}$. Then $|E_H(V(e), X) \cap E_0| \geq |X|$. Since $|M| < \left\lfloor \frac{1}{2}|V(H)| \right\rfloor$, $|X| \geq 2$. Hence we can take a vertex $x_1 \in X - \{x_0\}$. Then $\{ux_1, vx_0\} \cap E_2 = \emptyset$. On the other hand, since $|E_H(V(e), X) \cap E_0| \geq |X|$, $\{ux_1, vx_0\} \cap E_0 \neq \emptyset$. Then $(M - \{uv\}) \cup \{ux_1, vx_0\}$ is a good matching for $E$. This contradicts the assumption, and the lemma follows. \hfill \Box

For a positive integer $n$, we define an integer-valued function $h_n$ on $\mathbb{Z}^+$ by

$$h_n(t) = \begin{cases} \frac{1}{2}(t - 1)(-t + 2n) & \text{if } t < \frac{2}{3}(n + 1) \\ (t - 1)(2t - 1) & \text{if } t \geq \frac{2}{3}(n + 1) \end{cases}$$

**Lemma 13** Let $n$ and $t$ be positive integers with $t \leq \left\lfloor \frac{1}{2}n \right\rfloor$. Let $G$ be a graph of order $n$. If $|E(G)| > h_n(t)$, then $G$ contains a matching of order $t$.
Proof. Choose a graph $G_0$ of order $n$ without a matching of order $t$ so that $|E(G_0)|$ is as large as possible. We prove $|E(G_0)| = h_n(t)$.

By the assumption, the deficiency of $G_0$ is at least $n - 2(t - 1)$. By Berge’s Formula, $o(G_0 - S) \ge |S| + n - 2t + 2$ for some $S \subset V(G)$. Let $|S| = s$, and let $C_1, \ldots, C_k$ and $D_1, \ldots, D_l$ be the odd and even components of $G - S$, respectively. Since $t \le \left\lceil \frac{n}{2} \right\rceil$, $k \ge n - 2t + 2 + s \ge 2$. We may assume $|C_1| \ge |C_2| \ge \cdots \ge |C_k|$.

If $l \ge 1$, then replace $C_1 \cup D_1$ with a complete graph of order $|C_1| + |D_1|$. Since $|C_1| + |D_1|$ is an odd number, we have a new graph of order $n$ which has the same deficiency as that of $G_0$ and contains no edges. This contradicts the maximality of $|E(G_0)|$. Therefore, $l = 0$, and $G_0 - S$ has no even components.

If $k \ge s + n - 2t + 4$, then $k \ge 3$. Replace $C_1 \cup C_2 \cup C_3$ with a complete graph of order $|C_1| + |C_2| + |C_3|$. Let $G_1$ be the resulting graph. Then $|V(G_1)| = n$, $o(G_1 - S) = k - 2 \ge s + n - 2t + 2$ and $|E(G_1)| > |E(G_0)|$. This again contradicts the maximality of $|E(G_0)|$. Thus, we have $k \le s + n - 2t + 3$. However, since $s + k \equiv n \pmod{2}$, the equality does not hold, and hence we have $k = s + n - 2t + 2$. By the maximality of $|E(G_0)|$, each of $C_1, C_2, \ldots, C_{s+n-2t+2}$ and $S$ induces a complete graph. Moreover, there exists an edge between every vertex in $S$ and every vertex in $\bigcup_{i=1}^{s+n-2t+2} C_i$.

Assume $|C_2| \ge 3$ and replace $C_1$ and $C_2$ with complete graphs of order $|C_1| + 2$ and $|C_2| - 2$. Let $G_2$ be the resulting graph. Then $G_2$ has the same deficiency as that of $G_0$, and

$$|E(G_2)| = |E(G_0)| - \frac{1}{2}|C_1||C_1| - 1 - \frac{1}{2}|C_2||C_2| - 1$$
$$+ \frac{1}{2}|C_1| + 2)(|C_1| + 1) + \frac{1}{2}|C_2| - 2)|C_2| - 3 = |E(G_0)| + 2(|C_1| - |C_2|) + 2.$$

Since $|C_1| \ge |C_2|$, we have $|E(G_2)| > |E(G_0)|$. This contradicts the maximality of $|E(G_0)|$. Therefore, $|C_2| = |C_3| = \cdots = |C_{s+n-2t+2}| = 1$. Then $|C_1| = n - (s + s + n - 2t + 2 - 1) = 2t - 2s - 1$, and hence

$$|E(G_0)| = \frac{1}{2}(2t - 2s - 1)(2t - 2s - 2) + \frac{1}{2}s(s - 1) = \frac{3}{2}s^2 - \left(4t - \frac{5}{2} - n\right)s + (t - 1)(2t - 1).$$

Let $f(s) = \frac{3}{2}s^2 - \left(4t - \frac{5}{2} - n\right)s + (t - 1)(2t - 1)$. Since $|C_1| \ge 1$, $2t - 2s - 1 \ge 1$, and $s \le t - 1$. In the range of $0 \le s \le t - 1$, $f(s)$ takes the maximum value at either $s = 0$ or $s = t - 1$. Since $f(0) = (t - 1)(2t - 1)$ and $f(t - 1) = \frac{1}{2}(t - 1)(-t + 2n)$, we have $|E(G_0)| = \max \{ (t - 1)(2t - 1), \frac{1}{2}(t - 1)(-t + 2n) \} = h_n(t)$. □
Lemma 14 Let \( r \) and \( k \) be positive integers with \( r + k \equiv 1 \pmod{2} \) and \( k \leq r \), and let \( H \) be the complete graph with \( V(H) = [r+k] \). Let \( E = (E_0, E_1, E_2) \) be an ordered partition of \( E(H) \). If \( |E_1| + 2|E_2| < \min \{ 3(r+k-1), \frac{1}{3}r(r+k+3)(r+k+5) \} \), then \( H \) has an almost perfect matching which is also a good matching for \( E \).

Proof. First, we claim that there exists a matching \( M \) with \( M \subset E_0 \) and \( |M| \geq \frac{1}{2}(r+k-3) \).

Assume the contrary. Then by Lemma 13 we have

\[
|E_0| \leq h_{r+k} \left( \frac{r+k-3}{2} \right) = \begin{cases} \\
\frac{1}{2}(r+k-4)(r+k-5) & \text{if } r+k \geq 19 \\
\frac{3}{8}(r+k-5)(r+k+1) & \text{if } r+k \leq 17.
\end{cases}
\]

Then

\[
|E_1| + 2|E_2| \geq |E_1| + |E_2| \geq \frac{1}{2}(r+k)(r+k-1) - h_{r+k} \left( \frac{r+k-3}{2} \right) = \begin{cases} \\
4(r+k) - 10 & \text{if } r+k \geq 19 \\
\frac{1}{8}(r+k+3)(r+k+5) & \text{if } r+k \leq 17.
\end{cases}
\]

This immediately yields a contradiction if \( r+k \leq 17 \). If \( r+k \geq 19 \), we have \( 4(r+k) - 10 \leq |E_1| + 2|E_2| < 3(r+k+1) \), which yields \( r+k < 13 \), again a contradiction. Therefore, the claim follows.

If \( |M| = \frac{1}{2}(r+k-1) \), then \( M \) is a required matching. Therefore, we may assume that \( |M| = \frac{1}{2}(r+k-3) \) and \( H \) does not have a good matching of order \( |M| + 1 \). Let \( X = V(G) - V(M) \). Then \( |X| = 3 \). By Lemma 12 we have \( E(H[X]) \subset E_2 \) and \( |E_H(V(M), X) \cap E_2| \geq |E_H(V(M), X) \cap E_0| \).

Then

\[
|E_1| + 2|E_2| \geq |E_1 \cap E_H(V(M), X)| + 2(|E_2 \cap E_H(V(M), X)| + E(H[X])) = |E_1 \cap E_H(V(M), X)| + 2|E_2 \cap E_H(V(M), X)| + 2 \cdot 3 \geq |E_1 \cap E_H(V(M), X)| + |E_2 \cap E_H(V(M), X)| + |E_0 \cap E_H(V(M), X)| + 6 = |E_H(V(M), X)| + 6 = 3(r+k-3) + 6 = 3(r+k-1).
\]

This contradicts the assumption, and the lemma follows. \( \square \)

By combining Lemmas 9, 10 and 14 we obtain the following theorem, which is slightly stronger than Theorem 5 (2).

Theorem 15 Let \( r \) and \( k \) be positive integers with \( r \geq 2 \), \( r + k \equiv 1 \pmod{2} \) and \( k < r \), and let \( G \) be a graph with \( \chi(G) \leq r \). Let \( P \subset V(G) \) and let \( d: P \to [r+k] \) be a precoloring of \( P \) in \( G \). If \( |D_G(P, 2)| < \min \{ 3(r+k-1), \frac{1}{8}(r+k+3)(r+k+5) \} \), then \( d \) can be extended to a \( \frac{3r+k+1}{2} \)-coloring of \( G \).
Since
\[
\min \{3(r + k - 1), \frac{1}{8}(r + k + 3)(r + k + 5)\}
\]
\[
= \begin{cases} 
3(r + k - 1) & \text{if } r + k \leq 3 \text{ or } r + k \geq 13 \\
\frac{1}{8}(r + k + 3)(r + k + 5) & \text{if } 3 < r + k < 13,
\end{cases}
\]
\[
\frac{1}{8}(r + k + 3)(r + k + 5)
\]
appears as a bound only if \(r + k \in \{5, 7, 9, 11\}\). The difference between \(\frac{1}{8}(r + k + 3)(r + k + 5)\) and \(3(r + k - 1)\) in the range of \(3 \leq r + k \leq 13\) is shown in the following table.

| \(3(r + k - 1)\) | 3 | 5 | 7 | 9 | 11 | 13 |
|------------------|---|---|---|---|----|----|
| \(\frac{1}{8}(r + k + 3)(r + k + 5)\) | 6 | 10 | 15 | 21 | 28 | 36 |

Next, we prove Theorems 6 and 7.

**Proof of Theorem 6** Let \(q\) be an integer with \(q \geq r + k\) and consider the following construction.

1. Construct a balanced complete \(r\)-partite graph \(H\) with partite sets \(C_0, C_1, \ldots, C_{r-1}\), where \(|C_0| = |C_1| = \cdots = |C_{r-1}| = q\).
2. Take a set of \(2(r + k)\) vertices \(P = \{x_1, \ldots, x_{r+k}, y_1, \ldots, y_{r+k}\}\), which is disjoint from \(V(H)\).
3. For each \(i\), \(0 \leq i \leq r - 1\), choose \(r + k - 1\) distinct vertices \(z_{i1}^j, z_{i2}^j, \ldots, z_{ir+k}^j\) in \(C_i\).
4. Add edges \(\{x_1z_i^0, x_jz_i^0: 2 \leq j \leq r + k\} \cup \{y_1z_i^j, y_jz_i^j: 1 \leq i \leq r - 1, 2 \leq j \leq r + k\}\).

Let \(G^q\) be the resulting graph. Since \(G^q\) is an \(r\)-partite graph with \(C_0 \cup \{y_1, \ldots, y_{r+k}\}, C_1 \cup \{x_1, \ldots, x_{r+k}\}, C_2, \ldots, C_{r-1}\), we have \(\chi(G^q) = r\). By the construction, we have \(\mathcal{D}_{G^q}(P, 2) = \{\{x_1, x_j\}, \{y_1, y_j\}: 2 \leq j \leq r + k\}\) and hence \(\mathcal{D}_{G^q}(P, 2) = 2(r + k - 1)\). Define \(d: P \to [r + k]\) by \(d(x_j) = d(y_j) = j, 1 \leq j \leq r + k\).

We prove that \(d\) cannot be extended to a \(\frac{3r+k}{2}\)-coloring of \(G^q\). Assume, to the contrary, that there exists a \(\frac{3r+k}{2}\)-coloring \(f: V(G^q) \to \left[\frac{3r+k}{2}\right]\) of \(G^q\) such that the restriction of \(f\) into \(P\) coincides with \(d\). Choose \(f\) so that \(\left|\bigcup_{i=0}^{r-1} f(C_i)\right|\) is as small as possible. Since \(H\) is a complete \(r\)-partite graph, \(f(C_i) \cap f(C_j) = \emptyset\) if \(i \neq j\). If \(\{c_1, c_2\} \subset f(C_i)\) for some \(c_1\) and \(c_2\) with \(c_1 > r + k\) and \(c_1 \neq c_2\), define \(f': V(G^q) \to \left[\frac{3r+k}{2}\right]\) by
\[
f'(v) = \begin{cases} 
c_1 & \text{if } v \in C_i \\
f(v) & \text{if } v \in V(G^q) - C_i.
\end{cases}
\]
Since \(c_1 > r + k\), \(c_1 \notin f(P)\). Thus \(f'\) is also a proper coloring of \(G^q\) extending \(d\). Moreover, \(\bigcup_{i=0}^{r-1} f'(C_i) \subset \bigcup_{i=0}^{r-1} f(C_i) - \{c_2\}\). This contradicts the choice of \(f\). Therefore, if \(c \in f(C_i)\) for
some \( c > r + k \), then \( f(C_i) = \{ c \} \). Since \( \frac{3r + k}{2} - (r + k) = \frac{r - k}{2} \), at most \( \frac{r - k}{2} \) partite sets of \( H \) receive a color beyond \( r + k \), and hence at least \( \frac{r - k}{2} \) partite sets of \( H \) are colored only in colors in \([r + k]\). Let \( s = \frac{r + k}{2} \) and let \( C_{i_1}, C_{i_2}, \ldots, C_{i_s} \) be partite sets of \( H \) with \( f(C_{i_j}) \subset [r + k] \) (\( 1 \leq j \leq s \)). Since \( (N_G(x_l) \cup N_G(y_l)) \cap C_{i_j} \neq \emptyset \) for each \( l \), \( 1 \leq l \leq r + k \), we have \( |f(C_{i_j})| \geq 2 \) for each \( j \), \( 1 \leq j \leq s \). Therefore, we have \( \bigcup_{j=1}^s f(C_{i_j}) = \sum_{j=1}^s |f(C_{i_j})| \geq 2s = r + k \). This implies that \( \bigcup_{j=1}^s f(C_{i_j}) = [r + k] \) and \( |f(C_{i_j})| = 2 \) for each \( j \), \( 1 \leq j \leq s \). Then \( 1 \in f(C_{i_j}) \) for some \( j \). Let \( f(C_{i_j}) = \{ 1, c \} \) (\( 2 \leq c \leq r + k \)). If \( i_j \neq 0 \), then \( z^{i_j}_{C} \in N_G(y_1) \cap N_G(y_c) \) and hence \( f(z^{i_j}_{C}) \cap \{ 1, c \} = \emptyset \). On the other hand, if \( i_j = 0 \), then \( z^0_{C} \in N_G(x_1) \cap N_G(x_c) \) and hence \( f(z^0_{C}) \cap \{ 1, c \} = \emptyset \). Therefore, we have a contradiction in either case, and hence \( d \) cannot be extended to a \( \frac{3r + k}{2} \)-coloring of \( G^n \).

\( \square \)

**Proof of Theorem** Let \( q \) be an integer with \( r \geq 2 \) and \( q \geq r + k \). We consider the following construction.

1. Construct a complete \( r \)-partite graph \( H \) with partite sets \( C_0, C_1, \ldots, C_{r-1} \), where \( |C_0| = |C_1| = \cdots = |C_{r-1}| = q \).

2. Take a set of \( 2(r + k) \) vertices \( P = \{ x_1, x_2, \ldots, x_{r+k}, y_1, y_2, \ldots, y_{r+k} \} \) which is disjoint from \( \bigcup_{i=0}^{r-1} C_i \).

3. Choose \( r + k - 2 \) distinct vertices \( z^0_{C}, z^0_{C}, z^0_{C}, \ldots, z^0_{r+k} \) from \( C_0 \).

4. For every \( i \) with \( 1 \leq i \leq r - 1 \), choose \( r + k - 3 \) disjoint vertices \( z^1_{C}, z^1_{C}, \ldots, z^1_{r+k} \).

5. Add edges

\[
\{ x_1 z^1_{C}, x_2 z^1_{C}, x_3 z^1_{C}, x_j z^1_{C} : 4 \leq j \leq r + k, 1 \leq i \leq r - 1 \} \\
\bigcup \{ y_1 z^0_{C}, y_2 z^0_{C}, y_3 z^0_{C} \} \bigcup \{ y_j z^0_{C} : 4 \leq j \leq r + k \}.
\]

Let \( G^n \) be the resulting graph. Define \( d : P \to [r + k] \) by \( d(x_i) = d(y_i) = i \) (\( 1 \leq i \leq r + k \)).

Since \( G^n \) is an \( r \)-partite graph with partite sets \( C_0 \cup \{ x_1, \ldots, x_{r+k} \}, C_1 \cup \{ y_1, \ldots, y_{r+k} \}, C_2, \ldots, C_{r-1} \), we have \( \chi(G^n) = r \). By the definition of \( P \),

\[
D_{G^n}(P, 2) = \{ \{ x_i, x_j \} : 1 \leq i \leq 3, 4 \leq j \leq r + k \} \\
= \bigcup \{ \{ x_1, x_2 \}, \{ x_1, x_3 \}, \{ x_2, x_3 \} \} \bigcup \{ \{ y_1, y_2 \}, \{ y_1, y_3 \}, \{ y_2, y_3 \} \}
\]

and hence \( |D_{G^n}(P, 2)| = 3(r + k - 3) + 6 = 3(r + k - 1) \).

We prove that \( G^n \) does not have a \( \frac{3r + k + 1}{2} \)-coloring which extends \( d \). Assume, to the contrary, that \( G^n \) has a coloring \( f : V(G^n) \to \left[ \frac{3r + k + 1}{2} \right] \) which extends \( d \). Choose \( f \) so that \( \bigcup_{i=0}^{r-1} f(C_i) \) is as
small as possible. By the same argument as in the proof of Theorem [8], we have that if \( c \in f(C_i) \) for some \( c > r + k \), then \( f(C_i) = \{c\} \).

Since the number of colors beyond \( r + k \) is \( \frac{1}{2}(3r + k + 1) - (r + k) = \frac{1}{2}(r - k + 1) \), these colors appear in at most \( \frac{1}{2}(r - k + 1) \) partite sets of \( H \). Let \( C_{i_1}, C_{i_2}, \ldots, C_{i_r} \) be the partite sets of \( H \) with \( f(C_{i_j}) \subset [r + k] \) \((1 \leq j \leq r)\). Then \( r' \geq r - \frac{1}{2}(r - k + 1) = \frac{1}{2}(r + k - 1) \).

By the construction, \( f(C_{i_t}) \cap f(C_{i_r}) = \emptyset \) for each \( s, t \) with \( 1 \leq s < t \leq r' \). Moreover, \( \{y_1, y_2, y_3\} \subset N_G(z_i^0), \ y_j \in N_G(z_i^0) \ and \ \{x_1, x_2, x_3, x_j\} \subset N_G(z_i^0) \) for every \( i, j \) with \( 1 \leq i \leq r - 1 \) and \( 4 \leq j \leq r + k \). These imply \( |f(C_{i_j})| \geq 2 \) for each \( t, 1 \leq t \leq r' \). Therefore,

\[
r + k \geq \sum_{j=1}^{r'} |f(C_{i_j})| = 2r' \geq r + k - 1
\]

Since \( r + k \equiv 1 \pmod{2} \), we have \( r' = \frac{1}{2}(r + k - 1) \).

Suppose \( |f(C_{i_1})| = |f(C_{i_2})| = \cdots = |f(C_{i_{r'}})| = 2 \). Then \( \bigcup_{j=1}^{r'} f(C_{i_j}) = 2r' = r + k - 1 \). Let \( c_0 \) be the unique color in \([r + k] - \bigcup_{j=1}^{r'} f(C_{i_j})\). By the symmetry of the colors 1, 2, and 3, we may assume \( c_0 \notin \{1, 2\} \) and \( 1 \in f(C_{i_1}) \). Let \( f(C_{i_1}) = \{1, c_1\} \). If \( i_1 \neq 0 \), consider \( z^{i_1}_{i_1} \in C_{i_1} \). Since \( \{x_1, x_{c_1}\} \subset N_G(z^{i_1}_{i_1}), f(x_1) = d(x_1) = 1 \) and \( f(x_{c_1}) = d(x_{c_1}) = c_1 \), neither 1 nor \( c_1 \) is allowed for \( f(z^{i_1}_{i_1}) \), a contradiction. Therefore, \( i_1 = 0 \) and \( 1 \in f(C_0) \). By the same argument, we also have \( 2 \notin f(C_0) \), and hence \( f(C_0) = \{1, 2\} \). However, since \( \{y_1, y_2\} \subset N_G(z_0^0), d(y_1) = 1 \) and \( d(y_2) = 2 \), neither 1 nor \( c_2 \) can be assigned to \( z_0^0 \). This is again a contradiction. Therefore, \( |f(C_{i_j})| \geq 3 \) for some \( j, 1 \leq j \leq r' \). Then we may assume \( |f(C_{i_j})| = 3 \) and \( |f(C_{i_j})| = 2 \) for each \( j \) with \( 2 \leq j \leq r' \). This implies \( \bigcup_{j=1}^{r'} f(C_{i_j}) = [r + k] \).

Assume \( |f(C_{i_1}) \cap \{1, 2, 3\}| \leq 1 \). Then we may assume \( \{1, 2\} \cap f(C_{i_1}) = \emptyset \). Let \( 1 \in f(C_{i_j}), 2 \leq j \leq r', \) and let \( f(C_{i_j}) = \{1, c_1\} \). By the same argument as in the previous paragraph, we have \( i_j = 0 \) and \( 1 \in f(C_0) \). Similarly, \( 2 \in f(C_0) \). Then \( f(C_0) = \{1, 2\} \), a contradiction. Therefore, \( |f(C_{i_1}) \cap \{1, 2, 3\}| \geq 2 \). Since \( \{x_1, x_2, x_3\} \subset \bigcap_{i=1}^{r-1} N_G(z_i^0) \) and \( \{y_1, y_2, y_3\} \subset N_G(z_0^0), f(C_{i_1}) \neq \{1, 2, 3\} \). Therefore, we have \( f(C_{i_1}) = \{1, 2, c\} \) for some \( c \) with \( 4 \leq c \leq r + k \). Since \( \{x_1, x_2, x_c\} \subset \bigcap_{i=1}^{r-1} N_G(z_i^0), i_1 \notin \{1, 2, \ldots, r - 1\} \) and hence \( i_1 = 0 \), or \( f(C_0) = \{1, 2, c\} \). This implies \( f(C_{i_j}) = \{3, c'\} \) for some \( j, 2 \leq j \leq r' \), and \( c' \in [r + k] \). However, since \( \{x_3, x_{c'}\} \subset \bigcap_{i=1}^{r-1} N_G(z_i^0), i_j \notin \{1, \ldots, r - 1\} \) and hence \( i_j = 0 \). This means \( f(C_0) = \{3, c'\} \). This is a final contradiction, and the theorem follows. \( \Box \)

Now we prove Theorem [8]. In view of Lemmas [9] and [10], it suffices to prove the following lemma.

**Lemma 16** Let \( r \) and \( k \) be positive integers with \( k > r \), and let \( H \) be the complete graph of order
$r + k$. Let $E = (E_0, E_1, E_2)$ be an ordered partition of $E(H)$. Then

(1) if $r < k \leq \frac{3r-7}{2}$ and $|E_1| + 2|E_2| < \min \left\{ \frac{1}{2}(k + 3r - 4)(k - r + 3), (k - r + 2)(k + r - 1) \right\}$, then $H$ has a good matching of order $r$, and

(2) if $k > \frac{3r-7}{2}$ and $|E_1| + 2|E_2| < \min \left\{ \frac{1}{2}(k + 1)(k + 2), (k - r + 2)(k + r - 1) \right\}$, then $H$ has a good matching of order $r$.

**Proof.** Note

$$h_{r+k}(r-1) = \begin{cases} (2r - 3)(r - 2) & \text{if } r < k \leq \frac{3r-7}{2} \\ \frac{1}{2}(r-2)(r+2k+1) & \text{if } k > \frac{3r-7}{2}. \end{cases}$$

Let $\bar{h}(r, k) = \frac{1}{2}(r+k)(r+k-1) - h_{r+k}(r-1)$. Then

$$\bar{h}(r, k) = \begin{cases} \frac{1}{2}(k+3r-4)(k-r+3) & \text{if } r < k \leq \frac{3r-7}{2} \\ \frac{1}{2}(k+1)(k+2) & \text{if } k > \frac{3r-7}{2}. \end{cases}$$

Hence we have $|E_1| + 2|E_2| < \bar{h}(r, k)$ both in (1) and (2).

Let $H_0 = (V(H), E_0)$. Since $|E_0| + |E_1| + |E_2| = \frac{1}{2}(r+k)(r+k-1)$, we have

$$|E(H_0)| = \frac{1}{2}(r+k)(r+k-1) - (|E_1| + |E_2|)$$

$$\geq \frac{1}{2}(r+k)(r+k-1) - (|E_1| + 2|E_2|)$$

$$> \frac{1}{2}(r+k)(r+k-1) - \bar{h}(r, k) = h_{r+k}(r-1),$$

$H_0$ has a matching $M_0$ with $|M_0| \geq r - 1$.

Assume $H$ does not have a good matching for $E$ of order $r$, then $|M_0| = r - 1$. Let $X = V(H) - V(M_0)$. Then $|X| = (r+k) - 2(r-1) = k-r+2 \geq 3$ since $k > r$. By Lemma 12 $E(H[X]) \subset E_2$ and $|E_H(V(M), X) \cap E_2| \geq |E_H(V(M), X) \cap E_0|$. Therefore,

$$|E_1| + 2|E_2| \geq |E_H(V(M), X) \cap E_1| + 2(|E_H(V(M), X) \cap E_2| + |E(H[X])|)$$

$$= |E_H(V(M), X) \cap E_1| + 2|E_H(V(M), X) \cap E_2| + 2|E(H[X])|$$

$$\geq |E_H(V(M), X) \cap E_1| + |E_H(V(M), X) \cap E_2|$$

$$+ |E_H(V(M), X) \cap E_0| + 2|E(H[X])|$$

$$= |E_H(V(M), X)| + 2 \cdot \frac{1}{2}(k-r+2)(k-r+1)$$

$$= 2(r-1)(k-r+2) + (k-r+2)(k-r+1)$$

$$= (k-r+2)(k+r-1).$$

This contradicts the assumption, and the lemma follows. \qed
4 Conclusion

In this paper, we have studied extension of precoloring. For a set of precolored vertices $P$ in a graph $G$, we define $D_G(P, k)$ to be the set of pairs of vertices in $P$ whose distance is at most $k$. We have investigated how $|D_G(P, 3)|$ and $|D_G(P, 2)|$ affect the bounds on the number of additional colors required to extend the precoloring of $P$, which have been given by Albertson [1] and Albertson and Moore [5]. We have proved that the sharpness of Theorem [6]. However, the sharpness of Theorems [1] and [8] is unknown.

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