AUTODUALITY OF COMPACTIFIED JACOBIANS FOR CURVES WITH PLANE SINGULARITIES

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Abstract. Let $C$ be an integral projective curve with planar singularities. Consider its Jacobian $J$ and the compactified Jacobian $\overline{J}$. We construct a flat family $\overline{P}$ of Cohen-Macaulay sheaves on $\overline{J}$ parametrized by $\overline{J}$, its restriction to $J \times \overline{J}$ is the Poincaré line bundle. We prove that the Fourier-Mukai transform given by $\overline{P}$ is an auto-equivalence of the derived category of $\overline{J}$.

Introduction

Let $C$ be a smooth irreducible projective curve over a field $k$, and let $J$ be the Jacobian of $C$. As an abelian variety, $J$ is self-dual. More precisely, $J \times J$ carries a natural line bundle (the Poincaré bundle) $P$ that is universal as a family of topologically trivial line bundles on $J$.

The Poincaré bundle defines the Fourier-Mukai functor

$$\Psi : D^b(J) \to D^b(J) : G \mapsto Rp_{2,*}(p_1^*(G) \otimes P).$$

Here $D^b(J)$ is the bounded derived category of quasi-coherent sheaves on $J$ and $p_{1,2} : J \times J \to J$ are the projections. Mukai proved that $\Psi$ is an equivalence of categories ([25]).

Now suppose that $C$ is a singular curve, assumed to be projective and integral. The Jacobian $J$ is no longer projective, but it admits a natural compactification $\overline{J} \supset J$ ([2], [11]). By definition, $\overline{J}$ is the moduli space of torsion-free sheaves $F$ on $C$ such that $F$ has generic rank one and $\chi(F) = \chi(O_C)$; $J$ is identified with the open subset of locally free sheaves. It is natural to ask whether $\overline{J}$ is self-dual.

In this paper, we prove such self-duality assuming that $C$ is an integral projective curve with planar singularities over a field $k$ of characteristic zero. We construct a Poincaré sheaf $\overline{P}$ on $\overline{J} \times \overline{J}$. The sheaf is flat over each copy of $\overline{J}$; we can therefore view it as a $\overline{J}$-family of sheaves on $\overline{J}$. We prove that this family is universal in the sense that it identifies $\overline{J}$ with a connected component of the moduli space of torsion-free sheaves of generic rank one on $\overline{J}$. This generalizes autoduality results of [13], [14], [5] and answers the question posed in [14]. We also prove that the corresponding Fourier-Mukai functor

$$\Psi : D^b(\overline{J}) \to D^b(\overline{J}) : G \mapsto Rp_{2,*}(p_1^*(G) \otimes \overline{P})$$

is an equivalence.

Remarks. (1) If $C$ is a plane cubic (nodal or cuspidal), these results are known: see the remark after Theorem C.

(2) It is easy to write a formula for the Poincaré line bundle $P$ on $J \times \overline{J}$; see [11]. Our result is thus a construction of an extension of $P$ to a sheaf on $\overline{J} \times \overline{J}$ satisfying certain natural properties.
We studied $P$ in [5]. There, we prove weaker versions of the results of the present paper: that $P$ is a universal family of topologically trivial line bundles on $\mathcal{J}$, and that the corresponding Fourier-Mukai functor
$$\mathfrak{F}_J : D^b(J) \to D^b(\mathcal{J})$$
is fully faithful. Note that $\mathfrak{F}_J = \mathfrak{F} \circ R_j^*$, where $j : J \hookrightarrow \mathcal{J}$ is the open embedding.

(3) Compactified Jacobians appear as (singular) fibers of the Hitchin fibration for group $GL(n)$; therefore, our results imply a kind of autoduality of the Hitchin fibration for $GL(n)$. Such autoduality can be viewed as a ‘classical limit’ of the (conjectural) Langlands transform. For arbitrary reductive group $G$, one expects a duality between the Hitchin fibrations of $G$ and its Langlands dual $L^\vee G$. For smooth fibers of the Hitchin fibration, such duality is proved by R. Donagi and T. Pantev in [11] (assuming some non-degeneracy conditions).

(4) Our construction of $P$ can be obtained as a ‘classical limit’ of V. Drinfeld’s construction of automorphic sheaves for group $GL(2)$ ([12]), see Section 4.5 for more details.

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1. Main results

1.1. Summary of main results. Fix a ground field $k$ of characteristic zero. To simplify notation, we also assume that $k$ is algebraically closed; this assumption is not necessary for the argument. Let $C$ be an integral projective curve over $k$. Denote by $g$ the arithmetic genus of $C$, and let $J$ be the Jacobian of $C$, that is, $J$ is the moduli space of line bundles on $C$ of degree zero. Denote by $\mathcal{J}$ the compactified Jacobian; in other words, $\mathcal{J}$ is the moduli space of torsion-free sheaves on $C$ of generic rank one and degree zero. (For a sheaf $F$ of generic rank one, the degree is $\deg(F) = \chi(F) - \chi(O_C)$.) Then $\mathcal{J}$ is an irreducible projective variety; it is locally a complete intersection of dimension $g$ ([11]). Clearly, $J \subset \mathcal{J}$ is an open smooth subvariety.

Let $P$ be the Poincaré bundle; it is a line bundle on $J \times \mathcal{J}$. Its fiber over $(L, F) \in J \times \mathcal{J}$ equals

$$P_{(L,F)} = \det R\Gamma(L \otimes F) \otimes \det R\Gamma(O_C) \otimes \det R\Gamma(L)^{-1} \otimes \det R\Gamma(F)^{-1}.$$  

More explicitly, we can write $L \simeq O(\sum a_ix_i)$ for a divisor $\sum a_ix_i$ supported by the smooth locus of $C$, and then

$$P_{(L,F)} = \bigotimes (F_{x_i})^{\otimes a_i}.$$  

The same formula ([11] defines $P_{(F_1, F_2)}$ for any pair $(F_1, F_2) \in \mathcal{J} \times \mathcal{J}$ with either $F_1 \in J$ or $F_2 \in J$. Equivalently, $P_{|J \times J}$ is symmetric under the permutation of factors in $J \times J$; and therefore $P$ naturally extends to a line bundle on $J \times \mathcal{J} \cup \mathcal{J} \times J \subset \mathcal{J} \times \mathcal{J}$. We denote this extension by the same letter $P$.
For the rest of the paper, we assume that $C$ has planar singularities; in other words, the tangent space to $C$ at any point is at most two-dimensional.

**Theorem A.** There exists a coherent sheaf $\mathcal{P}$ on $\mathcal{J} \times \mathcal{J}$ with the following properties:

1. $\mathcal{P}|_{\mathcal{J} \times \mathcal{J} \cup \mathcal{J} \times \mathcal{J}} \simeq \mathcal{P}$;
2. $\mathcal{P}$ is flat for the projection $p_2 : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$, and the restriction $\mathcal{P}|_{\mathcal{J} \times \{F\}}$ is a Cohen-Macaulay sheaf for every $F \in \mathcal{J}$.

**Remark.** As explained in Section 6, Theorem A uniquely determines $\mathcal{P}$ as a ‘Cohen-Macaulay extension’ of $\mathcal{P}$ under the embedding $j : \mathcal{J} \cup \mathcal{J} \hookrightarrow \mathcal{J}$ in $\mathcal{J}$. In fact, $\mathcal{P} = j^* \mathcal{P}$.

By Theorem A(2), $\mathcal{P}$ is a family of coherent (Cohen-Macaulay) sheaves on $\mathcal{J}$ parametrized by $\mathcal{J}$. For fixed $F \in \mathcal{J}$, denote the corresponding coherent sheaf on $\mathcal{J}$ by $\mathcal{P}_F$. In other words, $\mathcal{P}_F$ is the restriction $\mathcal{P}|_{\mathcal{J} \times \{F\}}$.

Let $\text{Pic}(\mathcal{J})^= = \text{Pic}(\mathcal{J})$ be the moduli space of torsion-free sheaves of generic rank one on $\mathcal{J}$. A. Altman and S. Kleiman proved in [4, 3] that connected components of $\text{Pic}(\mathcal{J})^=$ are proper schemes ([3, Theorem 3.1]). The correspondence $F \mapsto \mathcal{P}_F$ can be viewed as a morphism $\rho : \mathcal{J} \to \text{Pic}(\mathcal{J})^=$.

Denote by $\text{Pic}^0(\mathcal{J}) \subset \text{Pic}(\mathcal{J})^=$ the irreducible component of the trivial bundle $O_{\mathcal{J}} \in \text{Pic}(\mathcal{J}) \subset \text{Pic}(\mathcal{J})^=$ (since $O_{\mathcal{T}} \in \text{Pic}(\mathcal{T})^=$ is a smooth point, it is contained in a single component). We prove that $\mathcal{J}$ is self-dual in the following sense.

**Theorem B.** $\rho$ is an isomorphism $\mathcal{J} \sim \text{Pic}^0(\mathcal{J})$. Moreover, $\text{Pic}^0(\mathcal{J}) \subset \text{Pic}(\mathcal{J})^=$ is a connected component.

**Remark.** The first statement of the theorem follows immediately from Theorem A using [14, Theorem 2.6]. (Although [14, Theorem 2.6] is formulated for curves with double singularities, the same argument works in the case of planar singularities if we use [5, Theorem C].) The second statement relies on Theorem [C].

Finally, we show that $\mathcal{P}$ also provides a ‘categorical autoduality’ of $\mathcal{J}$ in the sense that the corresponding Fourier-Mukai functor is an equivalence of categories.

**Theorem C.** Let $D^b(\mathcal{J})$ be the bounded derived category of quasicoherent sheaves on $\mathcal{J}$. The Fourier-Mukai functor

$$\mathfrak{F} : D^b(\mathcal{J}) \to D^b(\mathcal{J}) : \mathcal{G} \mapsto Rp_2_*(p_2^*(\mathcal{G}) \otimes \mathcal{P})$$

is an equivalence of categories. Its quasi-inverse is given by

$$D^b(\mathcal{J}) \to D^b(\mathcal{J}) : \mathcal{G} \mapsto Rp_2_*(p_2^*(\mathcal{G}) \otimes \mathcal{P}^\vee) \otimes \det(H^1(\mathcal{C}, O_C))^{-1}[g].$$

Here $\mathcal{P}^\vee = \mathcal{H}om(\mathcal{P}, O_{\mathcal{J} \times \mathcal{J}})$.

**Remarks.** (1) These results are known in the case of singular plane cubics (nodal or cuspidal). The sheaf $\mathcal{P}$ is constructed by E. Esteves and S. Kleiman in [14] for curves $C$ with any number of nodes and cusps; they also prove that $\mathcal{P}$ is universal (the first statement of Theorem B). If $C$ is a singular plane cubic, Theorem C is proved by I. Burban and B. Kreussler ([9]). J. Sawon proves it for nodal or cuspidal curves of genus two in [28].

(2) For simplicity, we consider a single curve $C$ in this section, but all our results hold for families of curves. Actually, the universal family of curves is used in the proof of Theorem C.
1.2. Organization. The rest of the paper is organized as follows.

Sections 2 and 3 contain preliminary results on Cohen-Macaulay sheaves and punctual Hilbert schemes of surfaces. These are used in the construction of the Poincaré sheaf $P$ in Section 4. The proof of the key step in the construction is contained in Section 5. This completes the proof of Theorem A.

Theorem A easily implies certain simple properties of $P$ that are given in Section 6. In Section 7, we derive Theorem C from these properties, and show that Theorem C implies Theorem B.

2. Cohen-Macaulay sheaves

Our argument is based on certain properties of Cohen-Macaulay sheaves. Let us summarize these properties.

Let $X$ be a scheme (all schemes are assumed to be of finite type over $k$). Denote by $\mathcal{D}_{coh}(X) \subset \mathcal{D}(X)$ the coherent derived category of $X$. Suppose that $X$ has pure dimension. Let us normalize the dualizing complex $\mathcal{D}X \in \mathcal{D}_{coh}(X)$ by the condition that its stalk at generic points of $X$ is concentrated in cohomological degree 0. If $X$ is Gorenstein, $\mathcal{D}X$ is an invertible sheaf, which we denote $\omega_X$.

Consider the duality functor $\mathcal{D}: \mathcal{D}_{coh}(X) \to \mathcal{D}_{coh}(X): G \mapsto R\mathcal{H}om(G, \mathcal{D}X)$.

Let $M$ be a coherent sheaf on $X$. Set $d := \text{codim}(\text{supp}(M))$. Then $H^i(\mathcal{D}(M)) = 0$ for $i < d$. Recall that $M$ is Cohen-Macaulay (of codimension $d$) if and only if $H^i(\mathcal{D}(M)) = 0$ for all $i \neq d$, so that $\mathcal{D}(M)[d]$ is a coherent sheaf.

2.1. Families of Cohen-Macaulay sheaves.

Lemma 2.1. Let $X$ and $Y$ be schemes of pure dimension, and suppose that $Y$ is Cohen-Macaulay. Suppose that a coherent sheaf $M$ on $X \times Y$ is flat over $Y$, and that for every point $y \in Y$, the restriction $M|_{X \times \{y\}}$ is Cohen-Macaulay of some fixed codimension $d$.

1. $M$ is Cohen-Macaulay of codimension $d$.
2. If in addition $Y$ is Gorenstein, then $\mathcal{D}(M)[d]$ is also flat over $Y$, and

$$\mathcal{D}(M)[d]|_{X \times \{y\}} \cong \mathcal{D}(M|_{X \times \{y\}})[d]$$

for all points $y \in Y$.

Proof. For (1), we need to show that $M_z$ is a Cohen-Macaulay $O_z$-module for all $z \in X \times Y$. Set $y = p_2(z) \in Y$. The claim then follows from [20] Corollary 6.3.3 applied to morphism $O_y \to O_z$, the $O_y$-module $O_y$ and the $O_z$-module $M_z$.

(2) follows from the identity

$$L_\iota^*(R\mathcal{H}om(M, \mathcal{D}X \times Y)) = R\mathcal{H}om(L_\iota^*M, L_\iota^*\mathcal{D}X \times Y),$$

where $\iota: X \times \{y\} \hookrightarrow X \times Y$ is the embedding. \hfill $\square$

Remark. In principle, the lemma can be stated for all (not necessarily closed) points $y \in Y$; then the restriction $M|_{X \times \{y\}}$ should be understood as an appropriate inverse image. This form of Lemma 2.1 is more natural, and it suffices for our purposes. On the other hand, it is easy to see that it suffices to check the Cohen-Macaulay property of $M_z$ for closed points $z \in X \times Y$. 

2.2. **Extension of Cohen-Macaulay sheaves.** Recall that a Cohen-Macaulay sheaf is **maximal** if it has codimension zero. Maximal Cohen-Macaulay sheaves are normal in the sense that their sections extend across subsets of codimension two.

**Lemma 2.2.** As before, let $X$ be a scheme of pure dimension. Let $M$ be a maximal Cohen-Macaulay sheaf on $X$. Then for any closed subset $Z \subset X$ of codimension at least two, we have $M = j_*(M|_{X-Z})$ for the embedding $j : X - Z \hookrightarrow X$.

**Proof.** This is a special case of [20, Theorem 5.10.5]. (Actually, it suffices to require that $M$ has property $(S_2)$.) □

2.3. **Acyclicity.**

**Lemma 2.3.** Let $f : Y \to X$ be a morphism of schemes. Suppose that $X$ is a Gorenstein scheme of pure dimension, and that $f$ has finite Tor-dimension. Let $M$ be a maximal Cohen-Macaulay sheaf on $X$.

1. $L \cdot f^* M = f^* M$.
2. In addition, suppose $Y$ is Cohen-Macaulay. Then $f^* M$ is maximal Cohen-Macaulay.

**Proof.** The statement is local on $X$, so we may assume that it is affine without losing generality. Essentially, the statement follows because $M$ is an $\infty$-syzygy sheaf. Indeed, we can include $M$ into a short exact sequence

$$0 \to M \to E \to M' \to 0,$$

for some vector bundle $E$ and a maximal Cohen-Macaulay sheaf $M'$. Then

$$L_i f^* M \simeq L_{i+1} f^* M' \quad \text{for all } i > 0,$$

and (1) follows by induction.

If $Y$ is Cohen-Macaulay, we can assume that it is of pure dimension (since this is true locally). Then a similar argument shows that

$$\mathcal{E}xt^i(f^* M, \mathbb{D}C_Y) \simeq \mathcal{E}xt^{i+1}(f^* M', \mathbb{D}C_Y) \quad \text{for all } i > 0,$$

which implies (2). □

3. **Punctual Hilbert schemes of surfaces**

Let $S$ be a smooth surface. Let us review some properties of the Hilbert scheme of $S$. Fix an integer $n > 0$.

3.1. **Hilbert scheme of points.** Let $\operatorname{Hilb}_S = \operatorname{Hilb}_S^n$ be the Hilbert scheme of finite subschemes $D \subset S$ of length $n$. It is well known that $\operatorname{Hilb}_S$ is smooth of dimension $2n$, and that it is connected if $S$ is connected. This statement is due to J. Fogarty ([15]).

Fix a point $0 \in S$, and let $\operatorname{Hilb}_{S,0} \subset \operatorname{Hilb}_S$ be the closed subset of $D \in \operatorname{Hilb}_S$ such that $D$ is (set-theoretically) supported at $0$. If we fix local coordinates $(x, y)$ at $0$, we can identify $\operatorname{Hilb}_{S,0}$ with the scheme of codimension $n$ ideals in $k[[x, y]]$.

**Lemma 3.1** (J. Briançon). $\dim(\operatorname{Hilb}_{S,0}) = n - 1$; $\operatorname{Hilb}_{S,0}$ has a unique component of maximal dimension.

**Proof.** See [8], [22], [27], or [7]. □

**Remark.** In fact, $\operatorname{Hilb}_{S,0}$ is irreducible. However, we do not need this claim; without it, Lemma 3.1 is much easier, see [7, Theorem 2].
Consider the symmetric power $\text{Sym}^n S$. We write its elements as 0-cycles
\[
\zeta = \sum_{x \in S} \zeta_x \cdot x \quad (\zeta_x \geq 0, \sum_x \zeta_x = n).
\]
Set
\[
\text{supp}(\zeta) := \{x \in S | \zeta_x \neq 0\} \quad (\zeta = \sum_x \zeta_x \cdot x \in \text{Sym}^n S).
\]

**Corollary 3.2.** Consider the Hilbert-Chow morphism
\[
\text{HC} : \text{Hilb}_S \to \text{Sym}^n S : D \mapsto \sum_{x \in S} (\text{length}_x D) \cdot x.
\]
For any $\zeta \in \text{Sym}^n S$, the preimage $\text{HC}^{-1}(\zeta)$ has a unique component of maximal dimension; its dimension equals $n - |\text{supp}(\zeta)|$.

**Proof.** The preimage equals $\prod_x \text{Hilb}_{S,x}^{\zeta_x}$. \qed

Denote by $\text{Hilb}'_S \subset \text{Hilb}_S$ the open subscheme parametrizing $D \in \text{Hilb}_S$ such that $D$ can be embedded into a smooth curve (which can be assumed to be $\mathbb{A}^1$ without loss of generality). Equivalently, $D \in \text{Hilb}'_S$ if and only if the tangent space to $D$ at every point is at most one-dimensional.

**Lemma 3.3.** $\text{codim}(\text{Hilb}_S - \text{Hilb}'_S) \geq 2$.

**Proof.** If $D \in \text{Hilb}_S - \text{Hilb}'_S$, then $|\text{supp}(\text{HC}(D))| \leq n - 2$, and the lemma follows from Corollary 3.2. \qed

### 3.2. Flags of finite schemes.

Let $\text{Flag}_S'$ be the moduli space of flags
\[
\emptyset = D_0 \subset D_1 \subset \cdots \subset D_n \subset S,
\]
where each $D_i$ is a finite scheme of length $i$ and $D_n \in \text{Hilb}'_S$. It is equipped with maps
\[
\psi : \text{Flag}_S' \to \text{Hilb}'_S : (\emptyset = D_0 \subset D_1 \subset \cdots \subset D_n) \mapsto D_n
\]
and
\[
\sigma : \text{Flag}_S' \to S^n : (\emptyset = D_0 \subset D_1 \subset \cdots \subset D_n) \mapsto (\text{supp}(\text{ker}(O_{D_i} \to O_{D_{i-1}})))_{i=1}^n.
\]
Moreover, $\text{Flag}_S'$ carries an action of $S_n$.

**Example 3.4.** For $D \in \text{Hilb}'_S$, choose an embedding $t : D \hookrightarrow \mathbb{A}^1$. Then $t(D) = Z(f)$ for a monic degree $n$ polynomial $f \in k[t]$. The fiber $\psi^{-1}(D)$ is identified with the scheme
\[
\text{Flag}_f := \{(t_1, \ldots, t_n) \in \mathbb{A}^n | f(t) = (t-t_1) \cdots (t-t_n)\}
\]
the identification sends $(t_1, \ldots, t_n) \in \text{Flag}_f$ to the flag
\[
\emptyset \subset Z(t-t_1) \subset Z((t-t_1)(t-t_2)) \subset \cdots \subset Z(f).
\]
The group $S_n$ acts on $\text{Flag}_f$ by permuting $t_i$’s.

The following claim is well known.

**Proposition 3.5.**

1. $\psi$ is a degree $n!$ finite flat morphism.
2. There exists a unique action of $S_n$ on $\text{Flag}_S'$ such that $\psi$ and $\sigma$ are equivariant. Here $S_n$ acts on $\text{Hilb}'_S$ trivially and on $S^n$ by permutation of factors.
3. The fiber of $\psi_* (O_{\text{Flag}_S'})$ over every point of $\text{Hilb}'_S$ is isomorphic to the regular representation of $S_n$. 

Lemma 3.6. There is a natural $S_n$-equivariant identification

$$\psi_*(O_{\text{Flag}}_S) = \left((A|_{\text{Hilb}_S})^\otimes n\right)_N,$$

where the lower index $N$ denotes the maximal quotient on which $A^\times$ acts via the character $N$.

Remarks. (1) Let us describe the map $(N, \text{character})$. Consider the $n$-fold fiber product

$$\mathcal{D}_n := \mathcal{D} \times_{\text{Hilb}_S} \cdots \times_{\text{Hilb}_S} \mathcal{D} = \{(D, s_1, \ldots, s_n) \in \text{Hilb}_S \times S^n | s_i \in D \text{ for all } i\}.$$

The projection $h_n : \mathcal{D}_n \to \text{Hilb}_S$ is finite and flat of degree $n^n$ over Hilb$_S$, and $h_n : (O_{\mathcal{D}_n}) = A^\otimes n$. Since the image of the map

$$(\psi, \sigma) : \text{Flag}_S \to \text{Hilb}_S \times S^n$$

is contained in $\mathcal{D}_n$, we obtain a morphism of sheaves of algebras

$$h_n : (O_{\mathcal{D}_n})|_{\text{Hilb}_S} \to \psi_*(O_{\text{Flag}}_S).$$

(2) Lemma 3.6 is similar to the description of $\psi_*(O_{\text{Flag}}_S)$ given in [21]. Namely, $\psi_*(O_{\text{Flag}}_S)$ is the quotient of $(A|_{\text{Hilb}_S})^\otimes n$ by the kernel of the symmetric form

$$A^\otimes n \times A^\otimes n \to \det A : (f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n) \mapsto \prod_{i=1}^n (f_ig_i).$$

This identification extends to Hilb$_S$ and provides a description of the isospectral Hilbert scheme (see Section 3.3). I do not know whether the identification of Lemma 3.6 also extends to Hilb$_S$. Such an extension would provide another formula for the Poincaré sheaf.

Proof of Lemma 3.6. Take $D \in \text{Hilb}_S$, and choose an embedding $t : D \hookrightarrow \mathbb{A}^1$. According to Example 3.4, we have to identify $k[X_f]$ and $(k[D]^\otimes n)_N$. Explicitly, for

$$f = t^n - a_1 t^{n-1} + \cdots + (-1)^n a_n,$$

we have

$$k[X_f] = k[t_1, \ldots, t_n]/(a_1 - (t_1 + \cdots + t_n), \ldots, a_n - (t_1 \cdots t_n))$$

and

$$k[D]^\otimes n = k[t_1, \ldots, t_n]/(f(t_1), \ldots, f(t_n)).$$

The multiplicative group $k[D]^\times$ is generated by linear polynomials. The polynomial $a(b-t)$ acts $k[D]^\otimes n$ as $a^n(b-t_1) \cdots (b-t_n)$, while $N(a(b-t)) = a^n f(b)$.
Therefore, $(k[D]^{(n)})^X$ is the quotient of $k[t_1, \ldots, t_n]$ modulo the ideal generated by $f(t_1), \ldots, f(t_n)$ and $(b - t_1) \ldots (b - t_n) - f(b)$ for all $b \in k$ such that $f(b) \neq 0$. The statement follows. □

3.3. **Isospectral Hilbert scheme.** Let us keep the notation of Section 3.2. The following result is a part of M. Haiman’s $n!$ Conjecture.

**Proposition 3.7.** There exists a scheme $\tilde{\text{Hilb}}_S$ and a commutative diagram

$$
\begin{array}{ccc}
S^n & \xrightarrow{\sigma} & \text{Flag}'_S \\
\downarrow & & \downarrow \psi \\
S^n & \xrightarrow{\psi} & \text{Hilb}'_S \\
\end{array}
$$

in which the right square is Cartesian and $\tilde{\text{Hilb}}_S \to \text{Hilb}_S$ is a degree $n!$ finite flat morphism. The scheme and the diagram are unique up to a unique isomorphism.

**Proof.** The scheme $\tilde{\text{Hilb}}_S$ can be constructed by extending $\psi_*O_{\text{Flag}'_S}$ to a rank $n!$ sheaf of algebras on $\text{Hilb}_S$. Since $\text{codim}(\text{Hilb}_S - \text{Hilb}'_S) \geq 2$, such extension is unique if it exists. This implies the uniqueness claim.

The existence claim is local in the étale topology on $\text{Hilb}_S$; it therefore reduces to the case $S = \mathbb{A}^1$ proved by M. Haiman in [21]. □

Following [21], we call $\text{Hilb}_S$ the **isospectral Hilbert scheme** of $S$. We keep the notation $\psi$ and $\sigma$ for the extended morphisms $\text{Hilb}_S \to \text{Hilb}_S$ and $\text{Hilb}_S \to S^n$. Finally, note that the action of $S_n$ on $\text{Flag}'_S$ extends to its action on $\text{Hilb}_S$ (because $\text{Hilb}_S$ is unique), and that $\psi$ and $\sigma$ are equivariant.

**Remark.** In [21], it is shown that the map $(\psi, \sigma) : \tilde{\text{Hilb}}_S \to \text{Hilb}_S \times S^n$ is an embedding, so that $\text{Hilb}_S$ can be defined as the closure of the image of $\text{Flag}'_S$ in $\text{Hilb}_S \times S^n$. We do not use this property. This gives us a choice of two possible references for Proposition 3.7; the original argument of [21] and V. Ginzburg’s paper [18], which provides a construction of $\text{Hilb}_S$ based on Hodge $D$-modules.

3.4. **Remark: stack of finite schemes.** Let us define universal versions of $\text{Flag}'_S$ and $\text{Hilb}_S$.

Let $\mathcal{S}\text{ch}$ be the algebraic stack parametrizing finite schemes of length $n$. Denote by $\mathcal{S}\text{ch}_1 \subset \mathcal{S}\text{ch}$ the open substack of schemes $D \in \mathcal{S}\text{ch}$ that are isomorphic to a closed subscheme of $\mathbb{A}^1$. Denote by $\text{Flag}'_{univ}$ the stack of flags

$$(\emptyset = D_0 \subset D_1 \subset \cdots \subset D_n),$$

where $D_i$ is a finite scheme of length $i$ and $D_n \in \mathcal{S}\text{ch}_1$. The natural morphism $\text{Flag}'_{univ} \to \mathcal{S}\text{ch}_1$ has an action of $S_n$, and the map $\psi$ is obtained from it by base change via $\text{Hilb}'_S \to \mathcal{S}\text{ch}_1$.

The morphism $\text{Flag}'_S \to \text{Hilb}'_S$ is a cameral cover for the group $\text{GL}(n)$ in the sense of [10]. Moreover, $\mathcal{S}\text{ch}_1$ is identified with the stack of cameral covers for $\text{GL}(n)$, and $\text{Flag}'_{univ} \to \mathcal{S}\text{ch}_1$ is the universal cameral cover.

Consider $\text{Hilb}_{\mathbb{A}^1}$ (the Hilbert scheme of finite subschemes $D \subset \mathbb{A}^1$ of length $n$). The natural map $\text{Hilb}_{\mathbb{A}^1} \to \mathcal{S}\text{ch}_1$ is a presentation, so $\mathcal{S}\text{ch}_1$ is a quotient of $\text{Hilb}_{\mathbb{A}^1}$ by an action of a groupoid. We can identify $\text{Hilb}_{\mathbb{A}^1}$ with the affine space
of monic degree $n$ polynomials in $k[t]$ (as in Example 3.4). The elements of the groupoid acting on $\text{Hilb}_{k^1}$ are then interpreted as Tschirnhaus transformations of polynomials. In this way, the stack $\mathfrak{Sch}_1$ goes back to the seventeenth century [29]. This relation was pointed out to me by V. Drinfeld.

Now let $\mathfrak{Sch}_2 \subset \mathfrak{Sch}$ be the open substack of schemes $D$ that admit an embedding into a smooth surface, which may be assumed to be $k^2$ without loss of generality. The natural morphism $\text{Hilb}_{k^2} \to \mathfrak{Sch}_2$ is a presentation, and the scheme $\text{Hilb}_{k^2}$ defined by M. Haiman descend to a flat finite stack $\text{Hilb}_{\text{univ}}$ over $\mathfrak{Sch}_2$. We can view $\text{Hilb}_{\text{univ}}$ as the universal isospectral Hilbert scheme: for every smooth surface $S$, we have

$$\text{Hilb}_S = \text{Hilb}_{\text{univ}} \times_{\mathfrak{Sch}_2} \text{Hilb}_S.$$  

4. Poincaré sheaf

We prove Theorem A by constructing $\mathcal{P}$. Actually, we construct a sheaf not on $\mathcal{J} \times \mathcal{J}$ but on its smooth cover, and then show that the sheaf descends to $\mathcal{J} \times \mathcal{J}$. This is similar to the construction of automorphic sheaves (12).

4.1. Construction of the Poincaré sheaf. Fix an integer $n > 0$, and let $\text{Hilb}_C$ be the Hilbert scheme of finite subschemes $D \subset C$ of degree $n$. Recall that $\text{Hilb}_C$ is an irreducible locally complete intersection of dimension $n$ (11). For $n \gg 0$, $\text{Hilb}_C$ is a smooth cover of $\mathcal{J}$. More precisely, fix a smooth point $p_0 \in C$. It defines an Abel-Jacobi map

$$\alpha : \text{Hilb}_C \to \mathcal{J} : D \mapsto \mathcal{J}^\vee (-np_0) = \mathcal{H}om(\mathcal{I}_D, \mathcal{O}_C(-np_0)).$$

Here $\mathcal{I}_D$ is the ideal sheaf of $D \subset C$. For $n \gg 0$, the map $\alpha : \text{Hilb}_C \to \mathcal{J}$ is smooth and surjective.

Our goal is to construct a sheaf $Q$ on $\text{Hilb}_C \times \mathcal{J}$, and then show that it descends to a sheaf $\mathcal{P}$ on $\mathcal{J} \times \mathcal{J}$ when $n \gg 0$. The construction of $Q$ makes sense even if $n$ is not assumed to be large.

Let $\mathcal{F}$ be the universal sheaf on $C \times \mathcal{J}$. Thus, for every $F \in \mathcal{J}$, the restriction $\mathcal{F}|_{C \times \{F\}}$ is identified with $F$. We normalize $\mathcal{F}$ by framing it over $p_0$, so we have

$$\mathcal{F}|_{\{p_0\} \times \mathcal{J}} = O_{\{p_0\} \times \mathcal{J}}.$$

Now consider the sheaf

$$\mathcal{F}_n := p_{1,n+1}^* \mathcal{F} \otimes \cdots \otimes p_{n,n+1}^* \mathcal{F}$$

on $C^n \times \mathcal{J}$. In other words, $\mathcal{F}_n$ is the family of sheaves $F^{\otimes n}$ on $C^n$ parametrized by $F \in \mathcal{J}$. The sheaf $\mathcal{F}_n$ is $S_n$-equivariant in the obvious way.

Choose a closed embedding $i : C \hookrightarrow S$ into a smooth surface $S$, and consider the diagram

$$\text{Hilb}_S \times \mathcal{J} \xrightarrow{(\psi \otimes i)} \text{Hilb}_S \times \mathcal{J} \xrightarrow{(\sigma \otimes i)} S^n \times \mathcal{J} \xrightarrow{(i^n \otimes i)} C^n \times \mathcal{J} \xrightarrow{p_1} \text{Hilb}_S.$$ 

Set

$$(4.1)\quad Q := ((\psi \otimes i_\mathcal{J})_* (\sigma \otimes i_\mathcal{J})^* (i^n \otimes i_\mathcal{J})_* \mathcal{F}_n)^{\text{sign}} \otimes p_1^* \det(\mathcal{A})^{-1}.$$
Here the upper index ‘sign’ stands for the space of anti-invariants with respect to the action of $S_n$. Recall that $\det \mathcal{A}$ is the line bundle on $\text{Hilb}_S$ whose fiber over $D \in \text{Hilb}_S$ is $\det(\mathcal{k}[D])$.

Note that (4.1) defines a sheaf on $\text{Hilb}_S \times \mathcal{J}$. Let us identify $\text{Hilb}_C$ with a closed subscheme of $\text{Hilb}_S$ using $\iota$. The following claim shows that

$$\text{supp}(Q) \subseteq \text{Hilb}_C \times \mathcal{J}.$$  

This is not immediate, because $\psi(\sigma^{-1}(\iota(C)^n)) \not\subseteq \text{Hilb}_C$.

**Proposition 4.2.** As above, $\iota : C \rightarrow S$ is a closed embedding of a reduced curve $C$ into a smooth surface $S$ (it is not necessary to assume that $C$ is projective or irreducible). Let $F$ be a torsion-free sheaf of generic rank one on $C$, and consider the sheaf $(\iota_* F)^\boxtimes_n$ on $S^n$. Then

1. $L\sigma^*(\iota_* F)^\boxtimes_n = \sigma^*(\iota_* F)^\boxtimes_n$;
2. $\sigma^*(\iota_* F)^\boxtimes_n$ is Cohen-Macaulay of codimension $n$;
3. $\psi_*(\sigma^*(\iota_* F)^\boxtimes_n)^\text{sign}$ is supported by the subscheme $\text{Hilb}_C \subseteq \text{Hilb}_S$.

It is also not hard to check that $Q$ given by formula (4.1) agrees with $P$ in the following sense. Let $\text{Hilb}'_C \subseteq \text{Hilb}_C$ (resp. $\text{Hilb}''_C \subseteq \text{Hilb}_C$) be the open subscheme parametrizing $D \in \text{Hilb}_C$ such that $D$ is isomorphic to a subscheme of $A^1$ (resp. $D$ is reduced and contained in the smooth locus $C^{\text{sm}} \subseteq C$). Note that $\text{Hilb}'_C = \text{Hilb}_C \cap \text{Hilb}_C$ and $\text{Hilb}''_C \subseteq \text{Hilb}_C$. Also, note that $\text{Hilb}'_C$ is dense in $\text{Hilb}_C$ (because $\text{Hilb}_C$ is irreducible) and $\text{Hilb}''_C \subseteq \text{Hilb}_C$ is a complement of subset of codimension at least two (this is easy to see from Corollary 3.2).

**Lemma 4.3.** The restrictions of the sheaves $Q$ and $(\alpha \times \text{id}_{\mathcal{J}})^* P$ to $(\text{Hilb}'_C \times \mathcal{J})$ or $(\text{Hilb}''_C \times \mathcal{J})$ are naturally isomorphic.

We postpone the proof of Proposition 4.2 until Section 5. Lemma 4.3 follows from Proposition 4.5 below. Let us show that Proposition 4.2 and Lemma 4.3 imply Theorem A.

**Proof of Theorem A** Proposition 4.2 implies that $Q$ is flat over $\mathcal{J}$. By Proposition 4.2, $Q$ is a flat $\mathcal{J}$-family of Cohen-Macaulay sheaves of codimension $n$ on $\text{Hilb}_S$. Therefore, $Q$ is a Cohen-Macaulay sheaf of codimension $n$ on $\text{Hilb}_S \times \mathcal{J}$ by Lemma 2.4.

By Proposition 4.2, the restriction $Q|_{\text{Hilb}_S \times \{F\}}$ is supported by $\text{Hilb}_C \times \{F\}$ for every $F \in \mathcal{J}$. Therefore, $Q$ is a maximal Cohen-Macaulay sheaf on $\text{Hilb}_C \times \mathcal{J}$.

Finally, consider the Abel-Jacobi map $\alpha : \text{Hilb}_C \rightarrow \mathcal{J}$ for $n \gg 0$. By Lemma 4.3 $Q$ coincides with the pullback $(\alpha \times \text{id}_{\mathcal{J}})^* P$ on the complement of a subset of codimension at least two. It follows that $Q$ descends to $\mathcal{J} \times \mathcal{J}$: we can extend the descent data across codimension two using Lemma 2.2. We thus obtain a sheaf $\mathcal{P}$ on $\mathcal{J} \times \mathcal{J}$. It is clear that $\mathcal{P}$ has the properties required by Theorem A.

4.2. **Restriction to** $\text{Hilb}'_S \subseteq \text{Hilb}_S$. The rest of this section contains comments on the formula (4.1).

Recall that $\text{Hilb}'_S \subseteq \text{Hilb}_S$ is the open subscheme of $D \in \text{Hilb}_S$ such that $D$ is isomorphic to a subscheme of $A^1$. Over $\text{Hilb}'_S$, we can identify $\text{Hilb}_S$ with the space of flags $\text{Flag}'_S$. In this way, (4.1) is more explicit if we are only interested in the restriction $Q|_{\text{Hilb}'_S \times \mathcal{J}}$. To make the formula more concrete, let us fix $F \in \mathcal{J}$.
**Lemma 4.4.** Consider the diagram

\[
\text{Hilb}'_S \xleftarrow{\psi} \text{Flag}'_S \xrightarrow{\alpha} S^n \xrightarrow{\iota^n} C^n.
\]

For every \( F \in \mathcal{J} \), we have

\[
Q|_{\text{Hilb}'_S \times \{ F \}} = \left( \psi_* \sigma^* (\iota^n)_* F^\boxtimes n \right)^{\text{sign}} \otimes \det(A)^{-1}.
\]

**Proof.** Clear. \( \square \)

We can also rewrite the formula for \( Q|_{\text{Hilb}'_S \times \mathcal{J}} \) using Lemma 3.6. Use the diagram

\[
\begin{array}{c}
\text{Hilb}'_S \\
\bigg\downarrow p_1 \\
\text{Hilb}_S
\end{array}
\begin{array}{c}
\xleftarrow{\psi} \\
\xrightarrow{\alpha} \\
\xrightarrow{S \times \mathcal{J}} C \times \mathcal{J}
\end{array}
\]

\[
\xrightarrow{g \times \id_{\mathcal{J}}} D \times \mathcal{J} \xrightarrow{\id_{\mathcal{J}}} S \xrightarrow{\iota \times \id_{\mathcal{J}}} C \times \mathcal{J}
\]

\[
\xrightarrow{\h \times \id_{\mathcal{J}}}
\]

to define the sheaf

\[
Q' := \left( \left( \bigwedge^n (h \times \id_{\mathcal{J}})_*(g \times \id_{\mathcal{J}})^* (\iota \times \id_{\mathcal{J}})_* F) \right) \otimes p_1^* \det(A)^{-1} \right)_{p_1^{-1}(A^*)}.
\]

Recall that \( \mathcal{F} \) is the universal sheaf on \( C \times \mathcal{J} \); see Section 3.2 for definitions of the remaining objects. Explicitly, the fiber of \( Q' \) over \( (D, F) \in \text{Hilb}'_S \times \mathcal{J} \) equals

\[
Q'_{(D, F)} = \left( \left( \bigwedge^n H^0(D, \iota_* F) \right) \otimes (\det k[D])^{-1} \right)_{k[D]^\times}.
\]

Up to the twist by \( \det k[D]^{-1} \), the fiber is the largest quotient of \( \bigwedge^n H^0(D, \iota_* F) \) on which \( k[D]^\times \) acts by the norm character. (Notice the similarity with [12, Lemma 3].)

**Proposition 4.5.** The restrictions of \( Q \) and \( Q' \) to \( \text{Hilb}'_S \times \mathcal{J} \) are naturally isomorphic.

**Proof.** Follows from Lemma 3.6 \( \square \)

Lemma 3.5 follows immediately from Proposition 4.5.

4.3. Curves with double singular points. As explained in Section 4.2, the restriction \( Q|_{\text{Hilb}'_S \times \mathcal{J}} \) is more explicit than \( Q \) itself.

Suppose that all singularities of \( C \) are at most double points. (So that at every point \( c \in C \), there exists \( f \in O_c \) such that \( \text{length}(O_c/fO_c) = 2 \)).

**Proposition 4.6.** For \( n \gg 0 \), the morphism \( \alpha : \text{Hilb}'_C \to \mathcal{J} \) is surjective.

**Proof.** Let \( C^{\text{sing}} \) be the singular locus of \( C \). For every \( p \in C^{\text{sing}} \), choose an invertible subsheaf \( \mathcal{I}_p^{(2)} \subset O_C \) of degree \(-2\) such that \( \mathcal{I}_p^{(2)} \supset O_p^{(2)} \). Such \( \mathcal{I}_p^{(2)} \) exists because \( C \) has only double singular points.

Consider \( D \in \text{Hilb}_C \). Then \( D \in \text{Hilb}'_C \) if and only if \( \mathcal{I}_D \not\subset \mathcal{I}_p^{(2)} \) for every \( p \in C^{\text{sing}} \).

In particular, \( D \in \text{Hilb}'_C \) provided \( \mathcal{I}_D \not\subset \mathcal{I}_p^{(2)} \) for all \( p \in C^{\text{sing}} \).

Now take \( F \in \mathcal{J} \) and \( n \geq 2g + 1 \). Recall that \( p_0 \in C \) is the smooth point used to define the Abel-Jacobi map \( \alpha : \text{Hilb}_C \to \mathcal{J} \). Choose a non-zero morphism \( \phi : O(-np_0) \to F \). By the Riemann-Roch Theorem, the space of such morphisms \( \phi \) has dimension \( n - g + 1 \). Then \( F = \alpha(D) \), where \( \mathcal{I}_D \subset O_C \) is the image of

\[
\phi^\vee : F^\vee(-np_0) \to O_C.
\]
For fixed \( p \in C^{sing} \), the space of morphisms \( \phi \) such that \( \phi(O(-np_0)) \subset F \otimes \mathcal{I}_p^{(2)} \) has dimension \( n - g - 1 \) by the Riemann-Roch Theorem. Thus, if \( \phi \) is generic, we have \( \phi(O(-np_0)) \not\subset F \otimes \mathcal{I}_p^{(2)} \) for all \( p \in C^{sing} \), and \( D \in \text{Hilb}'_C \).

By Proposition 4.4, we see that the sheaf \( \mathcal{P} \) on \( \mathcal{J} \times \mathcal{J} \) can be constructed as the descent of \( Q|_{\text{Hilb}_{\mathcal{C}} \times \mathcal{J}} \), assuming \( C \) has at most double singular points. Thus, in this case it is possible to describe \( \mathcal{P} \) without using isospectral Hilbert schemes.

**Remark.** Suppose the singularities of \( C \) are arbitrary planar, and denote by \( \mathcal{J} \subset \mathcal{J} \) the image \( \alpha(\text{Hilb}'_S) \) for \( n \gg 0 \). It is easy to see that for \( n \gg 0 \), the image does not depend on \( n \) or the choice of \( p_0 \). The restriction \( \mathcal{F}_{\mathcal{J} \times \mathcal{J}} \) can be constructed without using isospectral Hilbert schemes.

### 4.4. Independence of embedding.

The definition of \( Q \) involves the embedding \( \iota : C \to S \), and the argument of Section 3 relies on the properties of the Hilbert scheme \( \text{Hilb}_S \). On the other hand, it is not hard to see that the restriction \( Q|_{\text{Hilb}_{\mathcal{C}} \times \mathcal{J}} \) is independent of this embedding. By Proposition 4.4 this restriction coincides with \( Q \).

Let us provide a formula for \( Q|_{\text{Hilb}_{\mathcal{C}} \times \mathcal{J}} \). Set \( \text{Hilb}_C := \psi^{-1}_1 \text{Hilb}_C \subset \text{Hilb}_S \). The morphisms \( \psi : \text{Hilb}_C \to \text{Hilb}_C \) and \( \sigma : \text{Hilb}_C \to C^n \) are \( S_n \)-equivariant for the natural actions, and \( \psi \) is a degree \( n! \) finite flat morphism. It is easy to see that \( \text{Hilb}_C \) does not depend on \( \iota : C \to S \). Indeed, \( \text{Hilb}_C \) is obtained from \( \text{Hilb}_{\text{univ}} \) by base change \( \text{Hilb}_C \to \mathfrak{F}\text{Sch}_2 \) (see Section 3.2). For another explanation, note that the preimage \( \psi^{-1}_1(\text{Hilb}_{\mathcal{C}}) \) is identified with the moduli space \( \text{Flag}_{\mathcal{C}} \) of flags of finite subschemes in \( C \), and \( \text{Hilb}_{\text{univ}} \) can be viewed as its extension to a finite flat scheme over \( \text{Hilb}_C \). Such an extension is unique because \( \text{codim}(\text{Hilb}_C - \text{Hilb}_{\mathcal{C}}) \geq 2 \) and \( \text{Hilb}_C \) is Gorenstein.

Now consider the diagram

\[
\begin{array}{ccc}
\text{Hilb}_C \times \mathcal{J} & \xrightarrow{\psi \times \text{id}} & \text{Hilb}_C \times \mathcal{J} \\
\downarrow{p_1} & & \downarrow{p_1} \\
\text{Hilb}_C & \xrightarrow{\sigma \times \text{id}} & C^n \times \mathcal{J}
\end{array}
\]

Clearly,

\[
Q|_{\text{Hilb}_{\mathcal{C}} \times \mathcal{J}} := ((\psi \times \text{id})^*(\sigma \times \text{id})^* \mathcal{F}_n)^{sign} \otimes p_1^* \det(\mathcal{A})^{-1}.
\]

Similarly, one can describe the restriction \( Q|_{\text{Hilb}_{\mathcal{C}} \times \mathcal{J}} \) without choosing \( \iota : C \to S \) by rewriting the formulas of Section 4.2. We leave this description to the reader.

### 4.5. Poincaré sheaf and automorphic sheaves.

As we already mentioned, the formula for \( \mathcal{P} \) can be interpreted as a classical limit of V. Drinfeld’s formula for automorphic sheaves for \( GL(2) \) ([12]). Let us sketch the relation.

Let \( X \) be a smooth projective absolutely irreducible curve over a finite field, and let \( \mathcal{E} \) be a geometrically irreducible \( \ell \)-adic local system on \( X \). In [12], V. Drinfeld constructs an automorphic perverse sheaf \( \text{Aut}_{\mathcal{E}} \) on the moduli stack \( \text{Bun}_2 \) of rank two vector bundles on \( X \).

We now apply two ‘transformations’ to this construction. Firstly, let us assume that \( X \) is a (smooth projective connected) curve over a field \( k \) of characteristic zero, which we assume to be algebraically closed for simplicity. Also, let us replace...
perverse sheaves with $D$-modules. Now the input of the construction is a rank two bundle with connection $\mathcal{E}$ on $X$, and its output is a $D$-module $\text{Aut}_\mathcal{E}$ on $\text{Bun}_2$.

The second transformation is a ‘classical limit’ (‘classical’ here refers to the relation between classical and quantum mechanics). This involves replacing $D$-modules on a smooth variety (or stack) $Z$ with $O$-modules on the cotangent bundle $T^*Z$. Now the input $\mathcal{E}$ of the construction is a rank two Higgs bundle on $X$; equivalently, $\mathcal{E}$ is an $O$-module on $T^*X$ whose direct image to $X$ is locally free of rank two. The output is an $O$-module $\text{Aut}_\mathcal{E}$ on $T^*\text{Bun}_2$.

In particular, let $\iota : C \hookrightarrow T^*X$ be an irreducible reduced spectral curve for $\text{GL}(2)$: that is, the map $C \to X$ is finite of degree two. Note that $C$ has at most double singular points. We can then take $\mathcal{E} = \iota_* F$ for a torsion-free sheaf $F$ on $C$ of generic rank one. By interpreting Drinfeld’s construction in these settings, we obtain a formula for a sheaf $\text{Aut}_\mathcal{E}$ on $T^*\text{Bun}_2$ (more precisely, on the cotangent space to a smooth cover of $\text{Bun}_2$). From the point of view of geometric Langlands program, it is natural to expect that the sheaf is supported on the compactified Jacobian $\widetilde{\text{Jac}}$ of $C$, which is embedded in $T^*\text{Bun}_2$ as a fiber of the Hitchin fibration.

Thus, given $F \in \widetilde{\text{Jac}}$, we have a conjectural construction of a sheaf on $\widetilde{\text{Jac}}$. This is the construction of $\mathcal{T}_F$ provided by Lemma 4.4 with surface $S$ being $T^*X$.

From this point of view, the general formula (4.1) is obtained by extending Lemma 4.4 from $\text{Hilb}_S$ to $\text{Hilb}_\mathbb{C}$. Presumably, it is the classical limit of the formula for automorphic sheaves for $\text{GL}(n)$ suggested by G. Laumon ([24, 23]) and proved by E. Frenkel, D. Gaitsgory, and K. Vilonen ([16, 17]).

Although (4.1) is inspired by results in the area of geometric Langlands conjecture, it is not clear whether the proofs from [12, 16, 17] can be adapted to our settings. The argument of this paper is based on different ideas.

5. Proof of Theorem A

It remains to prove Proposition 4.2.

Proof of Proposition 4.2. The morphism $\sigma : \text{Hilb}_S \to S^n$ has finite Tor-dimension, because $S^n$ is smooth. The subvariety $C^n \subset S^n$ is locally a complete intersection of codimension $n$. By Corollary 3.2, its preimage $\sigma^{-1}(C^n) \subset \text{Hilb}_S$ also has codimension $n$. In other words, $C^n \subset S^n$ is locally cut out by a length $n$ regular sequence of functions on $S^n$, and the pull-back of this regular sequence to $\text{Hilb}_S$ remains regular. This implies that the restriction $\sigma : \sigma^{-1}(C^n) \to C^n$ has finite Tor-dimension. Now (1) follows from Lemma 2.3.1.

Recall that $\text{Hilb}_S$ is Cohen-Macaulay (it is finite and flat over $\text{Hilb}_S$, which is smooth). As we saw, $\sigma^{-1}(C^n) \subset \text{Flag}'_S$ is locally a complete intersection. Therefore, $\sigma^{-1}(C^n)$ is also Cohen-Macaulay. Lemma 2.3.2 implies (2).

Let us prove (3). Since $\psi$ is finite and flat, it follows that $\psi_*(\sigma^* (\iota_* F)^{\otimes n})$ is a Cohen-Macaulay sheaf of codimension $n$. The same is true for its direct summand

$$M := \psi_*(\sigma^* (\iota_* F)^{\otimes n})^{\text{sign}}.$$ 

Clearly, $\text{supp}(M) \subset \psi(\sigma^{-1}(C^n))$, which is a reducible scheme of dimension $n$. One of its irreducible components is $\text{Hilb}_C$, and we need to show that $M$ is supported by this component.

Let us first verify this on the level of sets. Set

$$Z := \text{Hilb}_S \cap \psi(\sigma^{-1}((C^{\text{sm}})^n)).$$
Corollary 3.2 implies that \( \dim(\psi(\sigma^{-1}(C^n)) - Z) < n \). Since \( M \) is Cohen-Macaulay, it suffices to check that
\[
\text{supp}(M|_Z) \subset \text{Hilb}C \cap Z.
\]
But this follows from Proposition 4.5. Indeed, for \( D \in Z \), we have
\[
H^0(D, i_* F) \simeq H^0(D, i_* O_C),
\]
because \( F \) and \( O_C \) are isomorphic in a neighborhood of \( D \). Therefore, if \( D \not\subset C \), we have \( \dim H^0(D, i_* F) < n \), and \( M_D = 0 \) by Proposition 4.5.

Thus, \( M \) is supported by \( \text{Hilb}C \) in the set-theoretic sense. Note that \( \psi(\sigma^{-1}(C^n)) \) is reduced at the generic point of \( \text{Hilb}C \) (in fact, it contains \( \text{Hilb}''C \) as an open set).

Since \( M \) is Cohen-Macaulay, we see that its support is reduced, as required. \( \square \)

Remarks. (1) Suppose that \( S \) admits a symplectic form (in fact, Proposition 4.2 is local on \( S \), so we can make this assumption without losing generality). Fix a symplectic form on \( S \); it is well known that it induces a symplectic form on \( \text{Hilb}S \).

One can check that the image \( \psi(\sigma^{-1}(C^n)) \subset \text{Hilb}S \) is Lagrangian. (Here one can assume that \( C \) is smooth, in which case the observation is due to I. Grojnowski \[19, Proposition 3\].) This provides a conceptual explanation why its dimension equals \( n \).

(2) The irreducible components of \( \psi(\sigma^{-1}(C^n)) \) have the following description (also contained in \[19\]). Consider the ‘diagonal stratification’ of \( C^n \). The strata are indexed by the set \( \Sigma \) of all equivalence relations \( \sim \) on the set \( \{1, \ldots, n\} \) (in other words, \( \Sigma \) is the set of partitions of the set \( \{1, \ldots, n\} \) into disjoint subsets). Given \( \sim \in \Sigma \), the corresponding stratum is
\[
C^n_\sim := \{(x_1, \ldots, x_n) \in C^n | x_i = x_j \text{ if and only if } i \sim j\}.
\]
In particular, the open stratum \( C^n_\sim \) corresponds to the usual equality relation \( = \in \Sigma \).

For every \( \sim \in \Sigma \), the preimage \( \sigma^{-1}(C^n_\sim) \) is irreducible of dimension \( n \) (see Corollary 3.2). The irreducible components of \( \psi(\sigma^{-1}(C^n)) \) are of the form \( \psi(\sigma^{-1}(C^n_{\sim})) \); they are indexed by \( \Sigma/S_n \) (which is the set of partitions of \( n \)).

(3) Let us keep the notation of the previous remark. It is not hard to check that at the generic point of the component corresponding to \( \sim \in \Sigma \), the fiber of \( \psi_* \sigma^* (i_* (F)^{\otimes n}) \) is isomorphic to the space of functions on \( S_n/S_\sim \) as a \( S_n \)-module. Here
\[
S_\sim := \{\tau \in S_n | \tau(i) \sim i \text{ for all } i \} \subset S_n
\]
is the subgroup given by the partition \( \sim \). In particular, if \( \sim \) is not discrete, the generic fiber has no anti-invariants under the action of \( S_n \). This provides another explanation of Proposition 4.2(3).

6. Properties of the Poincaré sheaf

Consider the Poincaré sheaf \( \mathcal{P} \) on \( J \times J \) provided by Theorem A.

6.1. \( \mathcal{P} \) as an extension.

Lemma 6.1. Let \( j : J \times J \cup J \times J \hookrightarrow J \times J \) be the open embedding.

(1) \( \mathcal{P} \) is a maximal Cohen-Macaulay sheaf on \( J \times J \);
(2) \( \mathcal{P} = j_* P \);
(3) \( \mathcal{P} \) is equivariant with respect to the permutation of the factors \( p_{21} : J \times J \to J \times J \).
Proof. (1) follows from Lemma 2.1. Now Lemma 2.2 implies (2). Finally, (3) is also clear, because $P$ is equivariant under $p_{21}$. □

In particular, Theorem A(2) and Lemma 6.1(3) imply that $\mathcal{P}$ is flat with respect to both projections $\mathcal{J} \times \mathcal{J} \to \mathcal{J}$, and that for every $F \in \mathcal{J}$, the restrictions $\mathcal{P}|_{\{F\} \times \mathcal{J}}$ and $\mathcal{P}|_{\mathcal{J} \times \{F\}}$ give the same sheaf on $\mathcal{J}$, which we denoted by $\mathcal{P}_F$.

6.2. $\mathcal{P}$ and duality. Consider now the duality involution
$$\nu: \mathcal{J} \to \mathcal{J}: F \mapsto F^\vee := \text{Hom}(F, O_{\mathcal{C}}).$$

Note that $\nu$ is an algebraic map by Lemma 2.1.

Lemma 6.2. (1) $(\nu \times \text{id}_{\mathcal{J}})^* \mathcal{P} = (\text{id}_{\mathcal{J} \times \mathcal{J}})^* \mathcal{P} = \mathcal{P}^\vee$;
(2) $(\nu \times \nu)^* \mathcal{P} = \mathcal{P}$.

Proof. By Lemma 6.1, all of the sheaves in the statement are maximal Cohen-Macaulay. It remains to notice that over $(J \times \mathcal{J} \cup \mathcal{J} \times J) \subset \mathcal{J} \times \mathcal{J}$ the required identifications are clear from the definition of $\mathcal{P}$. □

Corollary 6.3. Let $\mathcal{F}: D^b(\mathcal{J}) \to D^b(\mathcal{J})$ be the Fourier-Mukai functor of Theorem C. Its restriction to $D^b_{coh}(\mathcal{J})$ satisfies
$$\mathcal{F} \circ \mathcal{D} \simeq (\nu^* \circ \mathcal{D} \circ \mathcal{G})[-g].$$

Proof. By Serre’s duality, the functor $\mathcal{D} \circ \mathcal{G} \circ \mathcal{D}$ is given by
$$D^b_{coh}(\mathcal{J}) \to D^b_{coh}(\mathcal{J}): G \mapsto Rp_{1*}(p_2^*(\mathcal{G} \otimes \mathcal{P}^\vee) \otimes \omega_{\mathcal{J}}[-g]).$$

Now Lemma 6.2 implies the statement. □

6.3. Theorem of the Square. Consider the universal Abel-Jacobi map
$$A: J \times C \to \mathcal{J}: (L, c) \mapsto L(c - p_0) := \text{Hom}(\mathcal{I}_c, L(-p_0)).$$

(Recall that $p_0 \in C$ is a fixed smooth point.) It is easy to see that $\mathcal{P}$ agrees with it in the following sense:

Lemma 6.4. Consider the diagram
$$\begin{array}{ccc}
J \times \mathcal{J} & \xrightarrow{p_{13}} & J \times C \times \mathcal{J} & \xrightarrow{p_{23}} & C \times \mathcal{J} \\
& & \downarrow A \times \text{id}_{\mathcal{J}} & \\
& & \mathcal{J} \times \mathcal{J}.
\end{array}$$

We have $(A \times \text{id}_{\mathcal{J}})^* \mathcal{P} = p_{23}^*(\mathcal{F}) \otimes p_{13}^! P$. Recall that $\mathcal{F}$ is the universal sheaf on $C \times \mathcal{J}$.

Proof. Both sides are maximal Cohen-Macaulay, and their restrictions to $J \times (C_{sm} \times \mathcal{J} \cup C \times J)$ are identified. Here $C_{sm} \subset C$ is the smooth locus of $C$. □

Remark. Set $Y := (\mathcal{J} \times C_{sm} \cap J \times C) \subset \mathcal{J} \times C$. The map $A: J \times C \to \mathcal{J}$ extends to a regular map $Y \to \mathcal{J}$. Lemma 6.4 remains true for this extension: it provides an isomorphism of sheaves on $Y \times \mathcal{J}$. 

Similarly, we check that $\overline{\mathcal{P}}$ satisfies the Theorem of the Square. Namely, consider the action

$$\mu : J \times J \to J : (L, F) \to L \otimes F.$$  

Clearly, $\mu$ is a smooth algebraic morphism.

**Lemma 6.5.** Consider the diagram

$$
\begin{array}{ccc}
J \times J & \xrightarrow{P_{13}} & J \times J \\
\downarrow & & \downarrow \\
J \times J & \xrightarrow{P_{23}} & J \times J \\
\mu \times \text{id}_J & & \text{id}_J \\
\end{array}
$$

We have $$(\mu \times \text{id}_J)^* \overline{\mathcal{P}} = p_{13}^*(P) \otimes p_{23}^*\overline{\mathcal{P}}.$$  

**Proof.** Both sides are maximal Cohen-Macaulay, and their restrictions to $J \times (J \times J \cup J \times J)$ are identified. □

**Remark.** Lemmas 6.4 and 6.5 are contained, in a form, in [14]. Namely, Lemma 6.4 is contained in the proof of [14, Theorem 2.6], while Lemma 6.5 is equivalent to [14, Proposition 2.5]. Both statements are formulated for curves with double singularities, but this assumption can be removed using [5, Theorem C].

6.4. **Hecke eigenproperty.** Let us also state a less obvious property of $\overline{\mathcal{P}}$, which is motivated by the Langlands program. Essentially, we claim that $\overline{\mathcal{P}}$ is an ‘eigenobject’ with respect to natural ‘Hecke endofunctors’. The proof of the property will be given elsewhere; it is not used in this paper.

Let Hecke be the moduli space of collections $(F_1, F_2, f)$, where $F_1, F_2 \in J$ and $f$ is a non-zero map $F_1 \hookrightarrow F_2(p_0)$, defined up to scaling. Informally, $F_2(p_0)$ is an elementary upper modification of $F_1$ at the point $\text{supp}(\text{coker}(f)) \in C$. The space $\text{Hecke}$ is equipped with maps $\phi_1, \phi_2 : \text{Hecke} \to J$ and $\gamma : \text{Hecke} \to C$ that send $(F_1, F_2, f)$ to $F_1 \in J$, $F_2 \in J$, and $\text{supp}(\text{coker}(f))$ respectively.

The Hecke eigenproperty claims that for every $F \in J$, we have

$$R(\phi_1, \gamma)_* \phi_2^*(\overline{\mathcal{P}}_F) \simeq \overline{\mathcal{P}}_F \boxtimes F.$$  

We also have a ‘universal’ Hecke property for the sheaf $\overline{\mathcal{P}}$; its precise statement is left to the reader. The Hecke eigenproperty generalizes Lemma 6.4.

7. **Fourier-Mukai transform**

Set

$$\Psi := Rp_{13*}(p_{12}^*\overline{\mathcal{P}}' \otimes p_{23}^*\overline{\mathcal{P}}) \in D^b(J \times J).$$  

Our goal is to prove

**Proposition 7.1.** $\Psi \simeq O_\Delta[-g] \otimes_k \det H^1(C, O_C).$

Proposition 7.1 implies Theorem C by the argument completely analogous to the proof of [25, Theorem 2.2]. The proof of Proposition 7.1 follows the same pattern as the proof of [5, Theorem A].
Let \( C \), if \( \mathrm{Corollary} \ 7.3 \). by the Abel-Jacobi map \( C \). Same proof as \([5, \mathrm{Corollary} \ 2]\): pull back the isomorphism of Proposition \( 7.2 \). Proposition \( 7.4 \) implies that the fibers of the projection \( \mu \). \( \mathrm{Lemma} \ 7.7 \). \( \mathrm{Corollary} \ 7.6 \). Suppose \( \mathrm{Corollary} \ 7.3 \). if \( \mu \). \( \mu \) is a countable union of subvarieties of dimension \( g = \tilde{g} \). (In particular, it does not contain generic points of subschemes of higher dimension.) □

Corollary \( 7.6 \). Suppose \( C \) is singular, so \( \tilde{g} < g \). Then \( \dim(\mathrm{supp}(\Psi)) < 2g - \tilde{g} \).

Proof. Since \( \mu : J \to \ov{J} \to J \) is smooth of relative dimension \( g \), it suffices to show that
\[
\dim(\mu \times \mathrm{id}_J)^{-1}(\mathrm{supp}(\Psi)) < 3g - \tilde{g}.
\]
Proposition \( 7.4 \) implies that the fibers of the projection \( (\mu \times \mathrm{id}_J)^{-1}(\mathrm{supp}(\Psi)) \to \ov{J} \times J \) have dimension at most \( g - \tilde{g} \), while \([5, \mathrm{Theorem} \ A]\) shows that over \( J \times J \), the fibers are zero-dimensional. □

7.2. Proof of Theorem \( \mathbb{C} \)

Lemma \( 7.7 \). Let \( X \) be a scheme of pure dimension. Suppose \( \mathcal{G} \in \mathcal{D}^b_{\mathrm{coh}}(X) \) satisfies the following conditions:

1. \( \text{codim}(\mathrm{supp}(\mathcal{G})) \geq d \);
2. \( H^i(\mathcal{G}) = 0 \) for \( i > 0 \);
3. \( H^i(\mathcal{D}\mathcal{G}) = 0 \) for \( i > d \).

Then \( \mathcal{G} \) is a Cohen-Macaulay sheaf of codimension \( d \).
Proof. The proof is naturally given in the language of perverse coherent sheaves (6). Indeed, (3) claims that $D^G \in D_{\leq 0}(X)$ for perversity $p : |X| \to \mathbb{Z}$ given by $p(x) = d$. Here $|X|$ is the set of (not necessarily closed) points of $X$. Therefore, $G \in D_{\geq 0}(X)$ for the dual perversity $p : |X| \to \mathbb{Z}$ given by $p(x) = \text{codim}(\{x\}) - d$. Here $|X|$ is the set of (not necessarily closed) points of $X$. Therefore, $G \in D_{\geq 0}(X)$ for the dual perversity $p : |X| \to \mathbb{Z}$ given by $p(x) = \text{codim}(\{x\}) - d$. Now (1) implies that $G \in D_{\geq 0}(X)$, and (2) implies that $G$ is a sheaf. Since $D^G[d]$ also satisfies conditions (1)–(3), it is also a sheaf, and therefore $G$ is Cohen-Macaulay of codimension $d$. □

Note that the statement of Lemma 7.7 is local in smooth topology on $X$; therefore, the lemma still holds if $X$ is an algebraic stacks (locally of finite type over $\mathbb{Z}$).

Proof of Proposition 7.1. Both $P$ and $\Psi$ are defined for families of curves with plane singularities. Let us consider the universal family. Let $M$ be the moduli stack of (reduced irreducible projective) curves of fixed arithmetic genus $g$ with plane singularities. Let $C \to M$ be the universal curve, and let $J$ (resp. $J \subset C$) be the relative compactified Jacobian (resp. the relative Jacobian) of $C$ over $M$. The properties of these objects are summarized in [5]. As $C \in M$ varies, the family of Poincaré sheaves gives a Cohen-Macaulay sheaf $P_{\text{univ}}$ on $J \times_M J$; similarly, $\Psi$ is naturally defined as an object of the derived category $\Psi_{\text{univ}} \in D^b(J \times_M J)$. The restriction of $\Psi_{\text{univ}}$ to the fiber over a particular curve $C \in M$ is $\Psi$.

Denote by $j$ the rank $g$ vector bundle on $M$ whose fiber over $C \in M$ is $H^1(C, O_C)$. Alternatively, $j$ can be viewed as the bundle of (commutative) Lie algebras corresponding to the group scheme $J \to M$. Denote the projection $J \times_M J \to M$ by $\pi$ and diagonal in $J \times_M J$ by $\Delta$. Our goal is to prove that

$$(7.8) \quad \Psi_{\text{univ}}[g] \simeq O_{\Delta} \otimes \pi^* \det(j).$$

Consider on $M$ the stratification $M(\tilde{g})$, where $M(\tilde{g})$ parametrizes curves of geometric genus $\tilde{g}$. By [5, Proposition 6], $\text{codim}(M(\tilde{g})) \geq g - \tilde{g}$. Now Corollary 7.6 implies that $\dim(\text{supp}(\Psi)) = \dim(M) + g$, and moreover, every maximal-dimensional component of $\text{supp}(\Psi)$ meets $\pi^{-1}(M(\tilde{g}))$. Since $M(\tilde{g})$ parametrizes smooth curves, $\supp(\Psi) \cap \pi^{-1}(M(\tilde{g})) = \Delta \cap \pi^{-1}(M(\tilde{g}))$.

This is a reformulation of Mumford’s result [26, Section II.8.(vii)].

Finally, $H^i(\Psi_{\text{univ}}) = 0$ for $i > g$ and Serre’s duality implies that $H^i(D\Psi_{\text{univ}}) = 0$ for $i > 0$. (For instance, we have

$$D\Psi \simeq p_2^* \Psi[g]$$

over a fixed curve $C \in M$.) Thus, Lemma 7.7 shows that $\Psi_{\text{univ}}[g]$ is a Cohen-Macaulay sheaf of codimension $g$. Outside of a set of codimension $g + 1$, we see that $\text{supp}(\Psi_{\text{univ}})$ coincides with $\Delta$, therefore,

$$\text{supp}(\Psi_{\text{univ}}) \subset \Delta.$$ 

Also, [5, Theorem 10] provides the required isomorphism (7.8) over $J \times_M J$. Thus both sides of (7.8) are Cohen-Macaulay sheaves on $\Delta$ that are isomorphic outside of a subset of codimension two. By Lemma 2.2, they are isomorphic. □
As we have seen, Proposition 7.1 implies Theorem C.

7.3. Autoduality of the compactified Jacobian. It remains to prove Theorem B. As we already mentioned, the first statement follows from Theorem A and the results of [14]. On the other hand, both statements easily follow from Theorem C.

Note that Pic(J) is not claimed to be a fine moduli space; that is, there may be no universal family of torsion-free sheaves on J of generic rank one parametrized by Pic(J). However, locally in the étale topology of Pic(J), such a family exists and is unique ([3, Theorem 3.1]). In particular, points of Pic(J) are in bijection with isomorphism classes of torsion-free sheaves on J of generic rank one. We will make no distinction between these two objects, so that M ∈ Pic(J) means “M is a torsion-free sheaf on J of generic rank one, defined up to non-canonical isomorphism”.

Proof of Theorem B. Given M ∈ Pic(J), consider its Fourier-Mukai transform F(M). Its cohomology sheaves are concentrated in degrees between 0 and g. Fix an ample line bundle on J, and denote by hi(F(M)) ∈ Q[t] the Hilbert polynomial of the cohomology sheaf Hi(G(M)) for 0 ≤ i ≤ g.

Proposition 7.1 implies that if M = ρ(F) = P for F ∈ J, then G(M) ∼= O_{F'}[-g]. Here O_{F'} is the structure sheaf of the point F' ∈ J. In particular,

$$h^i(G(M)) = \begin{cases} 1, & i = g \\ 0, & i \neq g \end{cases}$$

On the other hand, consider h^i(G(M)) as functions of M ∈ Pic(J). They are semicontinuous with respect to the order on Q[t] given by

$$f > g \quad \text{if} \quad f(t) > g(t) \quad \text{for} \quad t \gg 0 \quad (f, g \in Q[t]).$$

Therefore, also holds for M in a neighborhood of ρ(J) ⊂ Pic(J).

However, if M satisfies (7.2), then G(M) ∼= O_{F'}[-g] for some point F ∈ J. Therefore, G(M) ∼= G(F'), and Theorem C implies M ∼= P. Hence ρ(J) is a connected component of Pic(J).

We also see that the inverse of the map ρ : J → ρ(J) is given by

$$M \mapsto \nu(\text{supp}(G(M))) : \rho(J) \to J.$$ 

Clearly, this is an algebraic map. This completes the proof.

□

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