Strong Gravitational Lensing by Kiselev Black Hole

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Abstract: We investigate the gravitational lensing scenario due to Schwarzschild-like black hole surrounded by quintessence (Kiselev black hole). We discuss here three special cases of Kiselev black hole: non-extreme, extreme and naked singularity. We present the detailed derivation for the bending angles of light as its traverses in the equatorial plane of the black hole. We also calculate the approximate bending angle and compare it with exact bending angle. In the weak field approximation, we compute the position and total magnification of relativistic images as well.

Keywords: Black hole; gravitational lensing; null-geodesics; quintessence; relativistic images.

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I. INTRODUCTION

A very useful tool to test General Relativity (GR) is the “gravitational lensing” (GL). It is well-known that GR predicts that when the light passes through a massive object it will be bent due to its strong gravitational field. Several works have used GL to test the theory and also to study, for example highly redshifted galaxies, quasars, supermassive black holes, exoplanets, dark matter candidates, primordial gravitational wave signatures, etc. [1]. Einstein was the first who predicted a gravitational lensing due to a compact object in 1911 [2]. In 1915, Schwarzschild provided the solution of an uncharged, static and spherically symmetric source by solving the vacuum Einstein field equations [3]. In 1919, Eddington confirming the prediction of Einstein (bending of light from the distant star) during the solar eclipse. He also found that the secondary images are formed when light is deflected around a relativistic compact object [4]. This result was very important since it was the first verification of GR. All the classical computations were in the frame of a weak gravitational field since all the astronomical observations are in this regime, where photons are deflected about only some arcseconds. For weak gravitational lensing, (WGL) small bending angle up to first order was calculated in [5]. Nevertheless, for a compact object such as a black hole, WGL is not valid and it is needed to proceed for the necessary corrections in the strong (or extreme) deflection limit. The main interest in this subject is to test GR up to a higher-order using, for example, the supermassive black hole in the center of our galaxy. Zwicky’s prediction (1937) about the gravitational lensing gave motivation for searching the source of macrolensing for seeing the cosmologically distant objects [6]. In 1979, Zwicky’s vision true when Walsh, Weymann and Carswell discovered the first example of GL through which they obtained first multiple images of a binary quasar (QSO0957 + 561). These images of QSO were separated by six arcsecond. Further, these images of quasar allowed the measurement of Hubble constant, which certify the Universe’s expansion rate [7].

In 1959, Charles Darwin calculated for the first time the light deflection due to a strong gravitational field using the Schwarzschild metric [8]. Deflection angle and intensities of the images are studied for the Schwarzschild Black Hole (SBH) in detail in [9]. Considering the SBH for the strong gravitational lensing, Virbhadra and Ellis obtained the lens equation and introduced a method to calculate bending angles. They also studied the lensing problem for the galactic supermassive black hole numerically [10]. For the study of the SBH lensing in the strong field limit, bending angle was also evaluated analogous to the weak field limit. Besides the weak field limit of relativistic images, magnifications and critical curves formulae were also formulated [11]. Bozza treated the strong
lensing phenomenon by a spherically symmetric black hole where an infinite sequence of higher order images are formed and also he interpreted the results with the iron K-lines\cite{12} and also for spinning black hole\cite{13}. However, so far, there is no direct observation of gravitational lensing by black holes or other compact objects. For small sized black holes, the detection of images and lensing phenomenon is hard. However large supermassive black holes such as Sgr A*, are excellent astrophysical laboratories to test the lensing and the behavior of the light in the strong gravity regime\cite{14}. The gravitational lensing has also been analyzed for more esoteric objects including static and non-static naked singularities\cite{15}, wormholes\cite{16}, boson and fermion stars\cite{17}, magnetized black hole\cite{18} and modified gravity black holes\cite{19}.

About two decades ago, a very important astronomical observation (using Supernovas type Ia) suggested that the Universe is in a state of an accelerating expansion\cite{20}. This study was a revolution in physics and the dark energy was named to be responsible for this accelerating scenario. Cosmologists proposed different models in order to explain this strange behavior of the Universe such as the dark energy that can be interpreted as a cosmological constant (with a state parameter of $w = -1$) or dynamic scalar fields\cite{21, 22}. The first has some observational problems like the so-called “the cosmological constant problem” where the value of the cosmological constant differs about $10^{120}$ orders of magnitude of the experimental value\cite{23}. The second approach is a model in which a candidate for dark energy is a dynamic scalar field such as quintessence, phantoms, k-essence, etc\cite{24–26}. Generally, a quintessence model has a state parameter $w(t) = p(t)/\rho(t)$ that varies with time depending on the energy potential $V(\Phi)$ and scalar field $\Phi$. In addition, it is important to mention that a Quintessence field is minimally coupled to gravity and the energy potential decreases as the field increases. This model is the simplest case without having theoretical problems like Laplacian instabilities or ghosts. For a more detailed review of the quintessence, see\cite{27–29}

One important solution related to the quintessence model was discovered by Kiselev in 2003\cite{30}. He derived static and spherically symmetric solutions of quintessence matter surrounding the black hole at the range of state parameter $-1 < w_q < -\frac{1}{3}$, where $w_q$ is the ratio of pressure and energy density of quintessence. The author obtained the Schwarzschild-like and Reissner-Nordström-de Sitter black holes surrounded by the quintessence. This solution physically describes a spherically symmetric and static exterior spacetime filled with a quintessence energy field, hence a non-vacuum solution.

In this paper, we study the gravitational lensing due to a Kiselev black hole (KBH)\cite{30} and consider three possibilities: two distinct horizons (non-extreme), unique horizon (extreme black
hole) and no horizon (naked singularity). From the astrophysical point of view, it is a hard task to distinguish between the signatures and properties of black holes and naked singularities

The paper is structured as follows: In Sec. II, we study the geodesics and effective potential for non-extreme and naked singularity. In Sec. III, we discuss critical variables and equation of path for photons and calculate the relations between closest approach \( r_o \) and impact parameter \( b \). In Sec. IV, we derive bending angle in terms of elliptical integrals for both non-extreme KBH and naked singularity at different values of quintessence parameter \( \sigma \) and than make comparison with bending angle for Schwarzschild BH. In Sec. V, we study the geodesics and effective potential for extreme KBH. In Sec. VI, we discuss critical variables and equation of path for photons and calculate the relationship between the closest approach and impact parameter for extreme lensing scenario. In Sec. VII, we calculate bending angle expression in terms of elliptical integrals for extreme Kiselev KBH at fixed value of \( \sigma \) and compare it with Schwarzschild bending angle as a reference. In Secs. VIII, IX, X, we use an alternative method for finding bending angle to study the relativistic images. We adopt the units \( c = G = 1 \).

II. BASIC EQUATIONS FOR NULL GEODESICS IN KISELEV SPACETIME

We consider a quintessence scalar field surrounded by a Schwarzschild black hole whose equation of state parameter \( w_q \) is given by

\[
w_q = \frac{p_q}{\rho_q} = \frac{\frac{1}{2}\dot{\Phi}^2 - V(\Phi)}{\frac{1}{2}\dot{\Phi}^2 + V(\Phi)},
\]

where \( p_q \) and \( \rho_q \) are defined in terms of the kinetic energy term (i.e. \( \frac{1}{2}\dot{\Phi}^2 \)) and potential energy \( V(\Phi) \) respectively. Here, the dot represents the differentiation with respect to cosmic time.

Based on the above point of view, the geometry of a static spherically symmetric black hole surrounded by the quintessence is given by

\[
ds^2 = f(r)dt^2 - \frac{1}{f(r)}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2,
\]

\[
f(r) = 1 - \frac{2M}{r} - \frac{\sigma}{r^{3w_q+1}},
\]

where \( M \) is the mass of the black hole and \( \sigma \) is the quintessence parameter (normalization factor) that is related to the energy density as follows:

\[
\rho_q = -\frac{\sigma}{2} \frac{3w_q}{r^{3(1+w_q)}},
\]
For \((w_q = -1)\), the function \(f(r)\) for the metric (2) reduces to
\[
f(r) = 1 - \frac{2M}{r} - cr^2, \tag{4}
\]
that is the Schwarzschild-de-Sitter black hole spacetime. For this case, the lensing phenomenon has been studied by Pavel Bakala [32, 33]. In this paper, our focus is on the special case \(w_q = -\frac{2}{3}\), which it the geometry of spacetime (2) reduces to the Schwarzschild-like black hole surrounded by quintessence matter. So, in this case the function \(f(r)\) becomes
\[
f(r) = 1 - \frac{2M}{r} - \sigma r, \quad (0 < \sigma < \frac{1}{8M}). \tag{5}
\]
and also it can be written as
\[
f(r) = \frac{\sigma}{r}(r - r_-)(r - r_+). \tag{6}
\]
The metric (2) becomes ill-defined at \(r = 0\), i.e. \((g_{00} \to \infty)\) which gives a curvature singularity. For \(f(r) = 0\), we get two values of \(r\), namely
\[
r_+ = \frac{1 + \sqrt{1 - 8M\sigma}}{2\sigma}, \quad r_- = \frac{1 - \sqrt{1 - 8M\sigma}}{2\sigma}. \tag{7}
\]
The region \(r = r_-\) corresponds to black hole event horizon while \(r = r_+\) represents the cosmological event horizon. Note that \(r_-\) and \(r_+\) are the two coordinate singularities in the metric (2). The coordinate singularities arise when \(0 < \sigma < \frac{1}{8M}\). However when \(\sigma > \frac{1}{8M}\), both \(r_+\) and \(r_-\) become imaginary, giving a naked singularity. When \(\sigma = 0\) we have that \(r = r_-\) and then it becomes the SBH’s event horizon.

The Lagrangian for a photon traveling in Kiselev spacetime is given by
\[
\mathcal{L} = (1 - \frac{2M}{r} - \sigma r)\dot{t}^2 - \frac{1}{1 - \frac{2M}{r} - \sigma r}r^2 - r^2\dot{\theta}^2 - r^2 \sin^2 \theta \dot{\phi}^2. \tag{8}
\]
We will work in a isotropic gravitational field, thus we can permit us to restrict the orbits of photon in the equatorial plane \((\theta = \frac{\pi}{2})\). Therefore, Eq. (8) becomes
\[
\mathcal{L} = (1 - \frac{2M}{r} - \sigma r)\dot{t}^2 - \frac{1}{1 - \frac{2M}{r} - \sigma r}r^2 - r^2 \dot{\phi}^2. \tag{9}
\]
By using the Euler-Lagrange equations for null geodesics and using \(\lambda\) as an affine parameter, we get
\[
\dot{t} \equiv \frac{dt}{d\lambda} = \frac{E}{1 - \frac{2M}{r} - \sigma r}. \quad (E = \text{Energy per unit mass.}) \tag{10}
\]
\[
\dot{\phi} \equiv \frac{d\phi}{d\lambda} = \frac{L}{r^2}. \quad (L = \text{Angular momentum per unit mass.}) \tag{11}
\]
Using the null condition of the 4-velocity $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$, where $\mu, \nu = t, r, \theta, \phi$. We get the equation of motion for photons, that is:

$$\dot{r} = L \sqrt{\frac{1}{b^2} - \frac{1}{r^2} \left(1 - \frac{2M}{r} - \sigma r\right)}, \text{ where } b = \frac{|L|}{E}. \quad (12)$$

Here $b$ is impact parameter for a photon of finite rest mass [34]. Geodesics motion is a force free unaccelerated motion. In the presence of gravitational field, photons experience gravitational force and this force comes due to the effective potential. Here, the effective potential for photons traveling in spacetime (2) is given by

$$V_{\text{eff}} = \frac{L}{r^2} \left(1 - \frac{2M}{r} - \sigma r\right). \quad (13)$$

Effective potential also has different limits of quintessence parameter $\sigma$ for non-extreme and naked singularity. i.e. non-extreme $0 < \sigma < \frac{1}{8M}$ while for naked singularity $\sigma > \frac{1}{8M}$. When $\sigma = 0$ then Eq. (13) reduces to Schwarzschild’s effective potential i.e.

$$V_{\text{eff}}^s = \frac{L}{r^2} \left(1 - \frac{2M}{r}\right). \quad (14)$$

![Effective potential of photons as a function of distance from black hole](image)

FIG. 1: Effective potential $V_{\text{eff}}$ of photons as a function of distance $r$ from black hole, setting $M = 1$. Upper curve for Schwarzschild black hole, middle two curves for non-extreme while bottom two curves for naked singularity.
In Fig. 1, the effective potential $V_{\text{eff}}$ is plotted to study the behavior of photons near the considered spacetime \(^{2}\) for different values of quintessence parameter $\sigma$.\(^{1}\) Hence $\sigma = 0$ corresponds to SBH and $\sigma = 0.06$ and 0.1 corresponds to non-extreme BH. For these cases photons do not cross the horizon while at $\sigma = 0.14$ and $\sigma = 0.15$ photons cross the horizon. In each curve, there is no minima. Therefore there is no stable orbit for the photons, only an unstable orbit exists in each case which gives a maximum value that corresponds to $V_{\text{max}}$.

### III. CRITICAL VARIABLES AND EQUATION OF PATH FOR PHOTONS

To find the radius of circular orbit of photons, we use the condition $\frac{dV_{\text{eff}}}{dr} = 0$ to obtain

$$r_{c\pm} = \frac{1 \pm \sqrt{1 - 6M\sigma}}{\sigma}. \quad (15)$$

Here $r_{c+}$ is greater than the outer horizon $r_+$ while $r_{c-}$ lies between inner and outer horizon ($r_- < r_{c-} < r_+$). The region of interest is between the horizons. Therefore, the radius of unstable circular orbit for photon is $r_{c-}$ which is also called the photon sphere

$$r_{ps} = \frac{1 - \sqrt{1 - 6M\sigma}}{\sigma}. \quad (16)$$

For the critical value of photon sphere, the conditions imposed on $\sigma$ are: for non-extreme $0 < \sigma < \frac{1}{8M}$ and for naked singularity $\sigma > \frac{1}{8M}$. In the limit, $\sigma \rightarrow 0$, we get the radius of photon sphere for SBH as

$$r_{ps}^s = 3M. \quad (17)$$

Now, for the equation of path, we convert the equation of motion \(^{12}\) in terms of $u = \frac{1}{r}$. Therefore, applying the chain rule successively we obtain $\frac{dr}{d\lambda} = \left(-\frac{1}{u^2 \frac{d\phi}{d\lambda}}\right)(\frac{d\phi}{d\lambda})$. From here we find

$$\frac{du}{d\phi} = -\frac{u^2 \dot{r}}{\dot{\phi}}. \quad (18)$$

Taking square on both sides

$$\left(\frac{du}{d\phi}\right)^2 = \frac{u^4 r^2}{\phi^2}. \quad (19)$$

Using Eqs. \(^{11}\) and \(^{12}\) in Eq. \(^{19}\), we obtain the equation of path for photon

$$\left(\frac{du}{d\phi}\right)^2 - B(u) = 0, \quad (20)$$

\(^{1}\) We take $M = 1$ for plotting, $\sigma = 1/8 = 0.125$ and the limits on $\sigma$ becomes as: for non extreme $0 < \sigma < 0.125$, for extreme case $\sigma = 0.125$ and for naked singularity $\sigma > 0.125$. 
where

\[ B(u) = \frac{1}{b^2} - u^2 \left( 1 - 2Mu - \frac{\sigma}{u} \right). \]  (21)

For critical value of the closest approach \( \frac{du}{d\phi} = 0 \). Identifying this point as \( u = u_2 \), we determine from Eq. (20):

\[ \frac{1}{b^2} = u_2^2 - 2Mu_2^3 - \sigma u_2. \]  (22)

Substituting the value of \( u_2 = \frac{1}{r_{ps}} \) from Eq. (16) in Eq. (22), we obtain the critical value of impact parameter for circular orbits

\[ b_{sc} = \sqrt{\frac{r_{ps}^3}{r_{ps} - 2M - \sigma r_{ps}^2}}. \]  (23)

The value of impact parameter also imposes the same limits on the quintessence parameter \( \sigma \), for both non-extreme and naked singularity as mentioned above. When \( \sigma = 0 \), Eq. (23) gives the value of impact parameter for SBH as

\[ b_{sc}^g = 3\sqrt{3}M. \]  (24)

We can see that the right-hand side of Eq. (21) is a rational functional in \( u \). According to the circular orbit condition (setting \( B(u) = 0 \)) and after solving that equation, we get one real root \( u_1 \) and two other roots \( u_2 \) and \( u_3 \), \( (u_3 > u_2 > u_1) \) which they are

\[ u_1 = \frac{r_o - 2M - \sqrt{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}}{4Mr_o}, \]  (25)

\[ u_2 = \frac{1}{r_o}, \]  (26)

\[ u_3 = \frac{r_o - 2M + \sqrt{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}}{4Mr_o}. \]  (27)

Thus Eq. (21) becomes

\[ B(u) = 2M(u - u_1)(u - u_2)(u - u_3). \]  (28)

Substituting Eq. (27) in (20), yields

\[ \frac{du}{d\phi} = \pm \sqrt{2M(u - u_1)(u - u_2)(u - u_3)}. \]  (29)
For a ray of light, both closest approach \( r_o \) and impact parameter \( b \) are obviously different from each other. Here \( b \) is the distance perpendicular from the center of black hole to the normal line on the ray of light intersecting at the observer at infinity \[35\]. Now relation between \( b \) and \( r_o \) can be obtained by solving the cubic equation:

\[
r_o^3 + \sigma b^2 r_o^2 - b^2 r_o + 2Mb^2 = 0. \tag{30}
\]

Using Cardano's method \(^2\) for solving a cubic equation, it is possible to convert Eq. \[30\] into a depressed cubic equation \((x^3 + px + q = 0)\). Thus if we substitute \((r_o = x - \frac{\sigma b^2}{3})\) into Eq. \[30\], where \(x\) is a new variable, we get

\[
x^3 - \left(\frac{\sigma b^4}{3} + b^2\right)x + \left(\frac{2\sigma^3 b^6}{27} + \frac{\sigma b^4}{3} + 2Mb^2\right) = 0. \tag{31}
\]

In this case \(p = -\left(\frac{\sigma b^4}{3} + b^2\right)\) and \((q = \frac{2\sigma^3 b^6}{27} + \frac{\sigma b^4}{3} + 2Mb^2)\). For one real negative root we used trigonometric formula \(^3\) which has the condition \(p < 0\), which it yields to

\[
r_o = 2\sqrt{\frac{\sigma^2 b^4 + 3b^2}{9}} \cos\left[\frac{1}{3} \cos^{-1}\left(-\frac{2\sigma^3 b^6 + 9\sigma b^4 + 54Mb^2}{6\sigma^2 b^4 + 18b^2}\right)\right] - \frac{\sigma b^2}{3}. \tag{32}
\]

At \(\sigma = 0\), it consistently reduces to the SBH lensing case \[35\],

\[
r_o = \frac{2b}{\sqrt{3}} \cos\left[\frac{1}{3} \cos^{-1}\left(-\frac{3\sqrt{3}M}{b}\right)\right]. \tag{33}
\]

From Fig. 2, we observe that by increasing the value of \(b\), \(r_o\) increases. In the region of the photon sphere \((\sigma = 0 \sim 0.1)\), \(r_o\) is related to \(b\) from the quintessence parameter \(\sigma\). Additionally, in that region, as value the of \(\sigma\) increases, light moves closer to BH and the value of \(r_o\) from BH decreases. Therefore, \(\sigma = 0\) corresponds to SBH (taken as a reference) while \(\sigma = 0.02\) to \(\sigma = 0.1\) correspond to the non-extreme BH. Beyond the photon sphere (region where no horizon exist) \(\sigma = 0.150\), the light goes into the BH, \(r_o \to b\), where naked singularity occurs.

\section*{IV. BENDING ANGLE}

Suppose that a light ray coming from infinity (say \(-\infty\)) reaches the black hole at \(r_o\) and finally it approaches again to infinity (say \(+\infty\)) (observer). Due to this a change, the angular coordinate

\(^2\) convert pure cubic equation \((ax^3 + bx^2 + cx + d = 0)\) into depressed cubic equation \((ax^3 + bx + c = 0)\)

\(^3\) \(x_k = 2\sqrt{-\frac{a}{d}} \cos\left(\frac{1}{3} \cos^{-1}\left(\frac{\Delta}{\sqrt{-\frac{2}{p} - k\frac{2a}{3}}}\right)\right)\) where \(k = 0, 1, 2\)
FIG. 2: The figure shows closest approach $r_o$ as a function of impact parameter $b$ $(M = 1)$. We discuss here the relation between the closest approach $r_o$ and impact parameter $b$ for Kiselev black hole lensing cases: non-extreme and naked-singularity and compared it with Schwarzschild black hole lensing case for different values of $\sigma$.

$\phi$ is twice from infinity to $r_o$. The light ray deflects from a straight line path at the difference of $\pi$ which results in bending angle $\hat{\alpha}$

$$\hat{\alpha} = 2 \int_{\frac{1}{b}}^{r_o} \frac{d\phi}{du} du - \pi. \quad (34)$$

If we substitute Eq. (29) into Eq. (34), we obtain

$$\hat{\alpha} = 2 \int_{0}^{\frac{1}{b}} \frac{1}{\sqrt{2M(u-u_1)(u-u_2)(u-u_3)}} du - \pi. \quad (35)$$

If we write Eq. (35) in terms of complete elliptic integral $^4$ and an incomplete elliptic integral $^5$ we need to separate the integral limits into two parts:

$$\hat{\alpha} = \sqrt{\frac{2}{M}} \left[ \int_{u_1}^{\frac{1}{b}} \frac{1}{\sqrt{(u_1-u)(u-u_2)(u_3-u)}} du - \int_{u_1}^{0} \frac{1}{\sqrt{(u_1-u)(u-u_2)(u_3-u)}} du \right] - \pi. \quad (36)$$

$^4$ The integral involving a rational function which contains square roots of cubic or quartic polynomials. Generally, here a definite cubic integral that has a build in command in Mathematica as:

$$K(m) = F\left(\frac{\pi}{2} \mid m\right) = \int_{0}^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta$$

$^5$ $F(\phi, m)$ having range $-\frac{\pi}{2} < \phi < \frac{\pi}{2}$ by $F(\phi \mid m) = \int_{0}^{\phi} \frac{1}{\sqrt{1 - m \sin^2 \theta}} d\theta$
Here the integrals can be recognized in term of first kind of elliptical integral, where \( u_3 > u_2 > u_1 \) \(^\text{[37]}\). Hence
\[
\hat{\alpha} = 2 \sqrt{\frac{2}{M}} \left[ \frac{F(\Psi_1, k)}{\sqrt{u_3 - u_1}} - \frac{F(\Psi_2, k)}{\sqrt{u_3 - u_1}} \right] - \pi. \tag{37}
\]
The integral variables can be defined as
\[
\Psi_1 = \frac{\pi}{2}, \tag{38}
\]
\[
\Psi_2 = \sin^{-1} \sqrt{\frac{r_o - 2M - \sqrt{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}}{r_o - 6M - \sqrt{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}}}. \tag{39}
\]
In the elliptical integral modulus \( k \) has range \( 0 \leq k^2 \leq 1 \), where
\[
k = \sqrt{\frac{6M - r_o + \sqrt{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}}{2\sqrt{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}}}. \tag{40}
\]
Now \( F(\frac{\pi}{2}, k) \equiv K(k) \) defines a complete elliptical integral and \( F(\Psi, k) \) is an incomplete elliptic integral. By simplifying Eq. \((37)\), an exact bending angle can be obtained:
\[
\hat{\alpha} = 4 \sqrt{\frac{r_o}{(1 - 8M\sigma)r_o^2 + 4Mr_o - 12M^2}} \left( K(k) - F(\Psi, k) \right) - \pi. \tag{41}
\]
From the last expression, \( \hat{\alpha} \) can be deduced for non-extreme KBH under \( 0 < \sigma < \frac{1}{8M} \) and for naked singularity KBH under \( \sigma > \frac{1}{8M} \). For \( \sigma = 0 \), this it reduces to the Schwarzschild bending angle \( \hat{\alpha}_S \) \(^\text{[35]}\).
FIG. 3: Bending angle is a function of impact parameter $b$. This is the case of non-extreme Kiselev black hole lensing and its maximum deflection value depends on the quintessence parameter $0 < \sigma < 1/8M$. Here Schwarzschild case occurs at $\sigma = 0$ while $\sigma = 0.02M$ to $0.08M$ for non-extreme case. In all cases we take $M = 1$.

Fig. 3 shows that the maximum deflection of light will occur at the critical value of the impact parameter $b_{sc}$ in Eq. (21). Before this value, there will be no deflection and after it, we will get a continuous deflection (light circulates around black hole). Each single curve shows that by increasing the value of $b$, the bending angle decreases at different value of $\sigma$. Nevertheless, originally when we increases the value of $\sigma$, the critical value of closest approach decreases since the light goes closer to the black hole. Similarly, the value of the critical impact parameter (near the photon sphere, where we get the maximum deflection) decreases and the bending angle increases.
FIG. 4: Bending angle $\hat{\alpha}$ is a function of $b$ for the naked singularity. Here $\sigma > 1/8M$, where we take $M = 1$. It can be observed that this is the case where no horizon exists and the value of the closest approach $r_o$ approaches to the impact parameter $b$.

FIG. 5: Graph for the naked singularity where $\hat{\alpha}$ is a function of $b$. In this case, as $b$ increases $\hat{\alpha}$ remains constant.

Figures (4) and (5) display the behavior of naked singularity. In Fig. 4 for a single curve at short distances, as $b$ increases the bending angle increases. In Fig. 5 for a long distance, as $b$ increases the bending angle remains constant. But when we observe the whole phenomena,
we see that the bending angle also depends on $\sigma$. As we increase the value of $\sigma$, the bending angle decreases for both short and long ranges distances. Furthermore, when we compare the non-extreme bending angle $\hat{\alpha}$ with the extreme bending angle $\hat{\alpha}^e$, we observe that for there cases (and for naked singularity) the bending angle are reciprocal to each other.

V. GRAVITATIONAL LENSING BY EXTREME KISELEV BLACK HOLE

Extreme gravitational lensing is very amazing for some important phenomenon but it demands a great effort to be observed. In extreme gravitational lensing, where a Kiselev black hole is used as a lens, we need to discuss about the bending of photons which pass very close to the lens and then they suffer a very large deflection.

The extreme case of Kiselev black hole condition is $\sigma = 1/8M$, thus the function $f(r)$ becomes

$$f(r) = 1 - \frac{2M}{r} - \frac{r}{8M}.$$  

This is an extreme black hole case for which $f(r) = 0$ gives degenerate solution (one horizon),

$$r^e_H = 4M.$$  

This value is twice the SBH horizon, so it can be written as

$$r^e_H = 2r^s_H.$$  

Choosing the equatorial plane ($\theta = \frac{\pi}{2}$) for the photon trajectory, the Lagrangian becomes

$$\mathcal{L} = (1 - \frac{2M}{r} - \frac{r}{8M})\dot{t}^2 - \frac{1}{1 - \frac{2M}{r} - \frac{r}{8M}}\dot{r}^2 - r^2\dot{\phi}^2.$$  

By using the Euler-Lagrange equation, we get

$$\dot{t} \equiv \frac{dt}{d\lambda} = \frac{E}{1 - \frac{2M}{r} - \frac{r}{8M}}.$$  

For EKBH, $\dot{\phi}$ is similar to Eq. (11). Again using the null condition of the 4-velocity, we get

$$\dot{r} = L\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r} - \frac{r}{8M}\right)}.$$  

The effective potential is

$$V^e_{\text{eff}} = \frac{L}{r^2}\left(1 - \frac{2M}{r} - \frac{r}{8M}\right).$$
FIG. 6: Effective potential $V_{\text{eff}}$ is shown as a function of distance $r$ taking for extreme Kiselev lensing phenomenon. Observe that there is no minima and only an unstable orbit exists which correspond to $V_{\text{max}}$. Schwarzschild’s effective potential is taken as a reference ($\sigma = 0$).

Taking the square of Eq. (47), to get

$$r^2 = L^2 \left[ \frac{1}{b^2} - \frac{1}{r^2} \left( 1 - \frac{2M}{r} - \frac{r}{8M} \right) \right],$$

(49)

Converting Eq. (11) from variable $r$ to $u$ as:

$$\dot{\phi}^2 = L^2 u^4.$$

(50)

VI. EQUATION OF PATH AND CRITICAL VALUES FOR EXTREME KISELEV BLACK HOLE

If we substitute Eqs. (49) and (50) into Eq. (19), we can obtain the first order non-linear differential equation of path

$$\left( \frac{du}{d\phi} \right)^2 - B^e(u) = 0,$$

(51)

where

$$B^e(u) = \frac{1}{b^2} - u^2 \left( 1 - 2Mu - \frac{1}{8Mu} \right).$$

(52)
In Eq. [52], the circular orbit condition is needed to apply. This condition gives a cubic equation that has one real expression $u_e^1 < 0$ and two distinct positive roots $u_e^3 > u_e^2 > 0$. Thus the roots are

$$u_e^1 = \frac{r_o^e - 2M - 2\sqrt{(r_o^e - 3M)M}}{4Mr_o^e}, \quad (53)$$

$$u_e^2 = \frac{1}{r_o^e}, \quad (54)$$

$$u_e^3 = \frac{r_o^e - 2M + 2\sqrt{(r_o^e - 3M)M}}{4Mr_o^e}. \quad (55)$$

Therefore, Eq. [52] can be rewritten as

$$B^e(u) = 2M(u - u_e^1)(u - u_e^2)(u - u_e^3). \quad (56)$$

If we replace again this equation into the equation of path, Eq. [51], we obtain

$$\frac{du}{d\phi} = \pm \frac{1}{\sqrt{2M(u - u_e^1)(u - u_e^2)(u - u_e^3)}}. \quad (57)$$

In the limit $u = 0 \ (r \to \infty)$, Eq. [51] gives

$$u = \frac{\phi}{b} + \text{constant}. \quad (58)$$

For the critical value for the EKBH of the closest approach (radius of photon sphere $r_o$), applying the second circular orbit condition $\frac{du}{d\phi}|_{u=\frac{1}{r_o}} = 0$, and then the condition $\frac{dB^e(u)}{d\phi}|_{u=\frac{1}{r_o}} = 0$ in Eq. [51], we get the quadratic equation such as

$$6Mu^2 - 2u + \frac{1}{8M} = 0. \quad (59)$$

It gives us two values

$$r_e^{c+} = 4M, \quad r_e^{c-} = 12M. \quad (60)$$

Here $r_e^{c+} = r_H^e$ gives a degenerate solution (here $b = 0$) whereas $r_e^{c-}$ gives the photon sphere. Thus

$$r_{ps}^e = 12M. \quad (61)$$

Now, by putting the Eq. [61] into Eq. [51] and using the condition of circular orbit $B^e(u) = 0$, we get the critical value of impact parameter, which is

$$b_{sc}^e = 6\sqrt{6}M. \quad (62)$$
In order to find the value of \( r_o \), we need to use the cubic equation

\[
(r_o^e)^3 + \frac{b^2}{8M} (r_o^e)^2 - b^2 r_o^e + 2Mb^2 = 0. \tag{63}
\]

Converting Eq. (63) into a depressed cubic equation of the form \( (x^3 + px + q = 0) \) and substituting \( r_o^e = x - \frac{b^2}{24M} \), where \( x \) is a new variable, we get

\[
x^3 - \left(\frac{b^4 + 192M^2b^2}{192M^2}\right)x + \frac{b^6 + 288M^2b^4 + 13824M^4b^2}{6912M^3} = 0. \tag{64}
\]

Here \( p = -\frac{b^4 + 192M^2b^2}{192M^2} \) and \( q = \frac{b^6 + 288M^2b^4 + 13824M^4b^2}{6912M^3} \). In order to get one real root for depressed cubic equation (64), we can use the same trigonometry formula as it was mention in the section before 3 with the condition \( p < 0 \). Thus, we obtain

\[
r_o^e = \frac{b\sqrt{b^2 + 192M^2}}{12M} \cos\left(\frac{1}{3} \cos^{-1}\left(-\frac{(b^4 + 288b^2 + 13824)}{b^2(b^2 + 192M^2)^{\frac{3}{2}}}\right)\right) - \frac{b^2}{24M}. \tag{65}
\]

Equation (65) provides the relationship between \( r_o^e \) and \( b \). We also see their behavior in Fig. (7).

**FIG. 7:** Figure shows relation between closest approach \( r_o \) as a function of impact parameter \( b \). We see that by increasing the value of impact parameter \( b \) the closest approach \( r_o \) increases. At \( \sigma = 0 \), a Schwarzschild curve is taken as a reference while for extreme case we take \( \sigma = 0.125M \) with \( M = 1 \).
VII. BENDING ANGLE FOR EXTREME KISELEV BLACK HOLE

The bending angle for a Extreme Kiselev Black Hole (EKBH) can be obtained by putting Eq. (57) into (34) where $r_o \rightarrow r_o^e$. Therefore we obtain

$$\hat{\alpha}_e = 2 \int_0^{\frac{1}{r_o^e}} \frac{1}{\sqrt{2M(u-u_1^c)(u-u_2^c)(u-u_3^c)}} du - \pi. \quad (66)$$

We can decompose the limits and convert the integral into complete and incomplete elliptical integral form, i.e.

$$\hat{\alpha}_e = \sqrt{\frac{2}{M}} \int_0^{\frac{1}{r_o^e}} \frac{1}{\sqrt{(u_1^c - u)(u-u_2^c)(u_3^c - u)}} du - \int_{u_{c1}}^{0} \frac{1}{\sqrt{(u_1^c - u)(u-u_2^c)(u_3^c - u)}} du - \pi. \quad (67)$$

Both integrals can be recognized in terms of first kind of elliptical integral [37], where integrand has condition $u_3^c > u_2^c > u_1^c$. Thus we will have

$$\hat{\alpha}_e = \sqrt{\frac{2}{M}} \left[ \frac{2F(\Psi_1^e, k_e)}{\sqrt{u_3^c - u_1^c}} - \frac{2F(\Psi_2^e, k_e)}{\sqrt{u_3^c - u_1^c}} \right] - \pi. \quad (68)$$

Simplifying Eq. (68) becomes

$$\hat{\alpha}_e = 4 \sqrt{\frac{2r_o^e}{(r_o^e - 3M)M}} \left[ \frac{F(\Psi_1^e, k_e)}{\sqrt{u_3^e - u_1^e}} - \frac{F(\Psi_2^e, k_e)}{\sqrt{u_3^e - u_1^e}} \right] - \pi. \quad (69)$$

For EKBH, elliptic integral parameters can be defined as:

$$\Psi_1^e = \frac{\pi}{2}, \quad (70)$$

$$\Psi_2^e = \arcsin \sqrt{\frac{r_o^e - 2M - 2\sqrt{(r_o^e - 3M)M}}{r_o^e - 6M - 2\sqrt{(r_o^e - 3M)M}}}. \quad (71)$$

Modulus $k_e$ has range $0 \leq k_e^2 \leq 1$, where

$$k_e = \sqrt{\frac{6M - r_o^e + 2\sqrt{(r_o^e - 3M)M}}{4\sqrt{(r_o^e - 3M)M}}}. \quad (72)$$

Thus, the exact bending angle for EKBH lensing is given by

$$\hat{\alpha}_e = 2 \sqrt{\frac{2r_o}{(r_o^e - 3M)M}} \left[ K(k_e) - F(\Psi^e, k_e) \right] - \pi, \quad (73)$$

where $F(\frac{\pi}{2}, k_e) \equiv K(k_e)$ defines the complete elliptical integral and $F(\Psi^e, k_e)$ is an incomplete elliptical integral.
FIG. 8: For extreme Kiselev black hole lensing, the bending angle $\hat{\alpha}$ is a function of the impact parameter $b$ (setting $M = 1$). In this case, the bending angle also depends on the value of the quintessence parameter $\sigma$. In this figure, $\sigma = 0.125$ is the value for the extreme case while $\sigma = 0$ is the value for the Schwarzschild bending angle that it has been taken as a reference.

Fig. 8 shows that by increasing the value of $b$, the bending angle decreases. The dashed curve shows the extreme KBH lensing bending angle, while the solid curve it shows the SBH lensing bending angle. Both curves display the same behavior since they have one horizon. In the extreme Kiselev black hole lensing, the event horizon is twice the Schwarzschild’s horizon see Eq. (14). However, the difference between these two bending angles is that in the extreme case, the bending angle is larger than the SBH bending angle because if we increase the value of the quintessence parameter $\sigma$, the bending angle also it will increase.

VIII. ALTERNATIVE APPROACH FOR FINDING BENDING ANGLE

Gravitational lensing phenomena involve the study of the null geodesic equations. When the solution of the space-time geometry (2) extends, an event horizons exist at $r_+$ and $r_-$ see (Eq. 7). Our main interest is in the region that lies between the horizons, which it is called the photon sphere $r_{ps}$ (Eq. 15). Therefore, the deflection will occur when the ray of light passes through that region with closest approach $r_o$. In order to compute the bending angle $\hat{\alpha}$ we need to compute the
value of impact $b$. If we divide Eq. (11) with (12) we obtain

$$\frac{d\phi}{dr} = \frac{1}{r^2 \sqrt{1 - \frac{1}{r^2} (1 - \frac{2M}{r} - \sigma r)}}. \quad (74)$$

Now, for the closest approach $r = r_o$ and $\frac{dr}{d\phi}|_{r=r_o} = 0$, we will have

$$b(r_o) = \frac{r_o}{\sqrt{1 - \frac{2M}{r_o} - \sigma r_o}}. \quad (75)$$

By substituting Eq. (75) into Eq. (74) we obtain

$$\frac{d\phi}{dr} = \frac{1}{r \sqrt{\left(\frac{r}{r_o}\right)^2 (1 - \frac{2M}{r_o} - \sigma r_o) - (1 - \frac{2M}{r} - \sigma r)}}. \quad (76)$$

We adopt the procedure of [36], thus we will use the following bending angle formula:

$$\alpha = 2 \int_{r_o}^{\infty} \frac{d\phi}{dr} dr - \pi. \quad (77)$$

By using Eq. (76), the deflection angle for a light ray becomes

$$\alpha(r_o) = 2 \int_{r_o}^{\infty} \frac{dr}{r \sqrt{\left(\frac{r}{r_o}\right)^2 (1 - \frac{2M}{r_o} - \sigma r_o) - (1 - \frac{2M}{r} - \sigma r)}} - \pi. \quad (78)$$

Our gravitational lensing phenomenon can be represented as it is showed in Fig. (9). The lens equation can be expressed as [10]

$$\tan \beta = \tan \theta - \frac{D_{LS}}{D_{OS}} [\tan(\alpha - \theta) + \tan \theta], \quad (79)$$

$$b(r_o) = D_{OL} \sin \theta. \quad (80)$$

Angular positions of source and images are represented by $\beta$ and $\theta$ respectively while the deflection angle due to black hole is denoted by $\alpha$ as it is showed in the Fig. (9).

Now, if we convert the distance and the impact parameter in terms of the Schwarzschild BH’s radius i.e.

$$X = \frac{r}{2M}, \quad X_o = \frac{r_o}{2M}, \quad b(r_o) = 2Mb(X_0).$$

$$d_{ol} = \frac{D_{OL}}{2M}, \quad d_{os} = \frac{D_{OS}}{2M}, \quad d_{ls} = \frac{D_{LS}}{2M}. \quad (81)$$
FIG. 9: The lens diagram. The positions of observer (O), source (S), lens (L) and image (I) are shown in the figure. The observer-lens, observer-source and lens-source distances are represented by $D_{ol}$, $D_{os}$ and $D_{ls}$, respectively.

From here, we will introduce a new quintessence parameter in terms of the Schwarzschild radius:

$$\sigma_\ell = 2M\sigma.$$  \hspace{1cm} (82)

Using Eqs. (80) and (81) in Eqs. (78), (75), (7) and (16) respectively, we get

$$\alpha(X_o) = 2\int_{X_o}^{\infty} \frac{dX}{X\sqrt{\left(\frac{X}{X_o}\right)^2 - \left(1 - \frac{1}{X_o} - \sigma_\ell X_o\right) - \left(1 - \frac{1}{X} - \sigma_\ell X\right)}} - \pi,$$  \hspace{1cm} (83)

$$b(X_o) = \frac{X_o}{\sqrt{\left(1 - \frac{1}{X_o} - \sigma_\ell X_o\right)}} = d_{ol}\sin\theta,$$  \hspace{1cm} (84)

$$X_H = \frac{1}{2\sigma_\ell} \pm \frac{1}{\sigma_\ell \sqrt{\frac{1}{4} - \sigma_\ell}},$$  \hspace{1cm} (85)
\[ X_{ps} = \frac{1 - \sqrt{1 - 3\sigma_{\ell}}}{\sigma_{\ell}}, \]  
(86)

where \(X_{H}\) denotes the distance to the horizons and \(X_{ps}\) the distance from the photon sphere.

In order to find the position of the images, we need to solve Eq. (79) for the source position \(\beta\) along with the Eqs. (83) and (84).

Generally, for a circular symmetric lens, the magnification is given by \[ \mu = \left| \frac{\sin \beta \, d\beta}{\sin \theta \, d\theta} \right|^{-1}. \]  
(87)

Here, the tangential magnifications and the radial magnifications are respectively defined as \[ \mu_t = \left( \frac{\sin \beta}{\sin \theta} \right)^{-1}, \quad \mu_r = \left( \frac{d\beta}{d\theta} \right)^{-1}. \]  
(88)

By differentiating both sides of Eq. (79) we get \[ \frac{d\beta}{d\theta} = \left( \frac{\cos \beta}{\cos \theta} \right)^2 \left[ 1 - \frac{d_{ls}}{d_{os}} \left( 1 + \left( \frac{\cos \theta}{\cos(\alpha - \theta)} \right)^2 \left( \frac{d\alpha}{d\theta} - 1 \right) \right) \right], \]  
(89)

where \(\frac{d\alpha}{d\theta} = \frac{d\alpha}{dX_o} \frac{dX_o}{d\theta}\). By taking derivative with respect to \(X_o\) in Eq. (83), we obtain \[ \frac{d\alpha}{dX_o} = \int_{X_o}^{\infty} \frac{X(2X_o - 3 - \sigma_{\ell}X_o^2)}{2X_o^2 \left[ (\frac{X}{X_o})^2(1 - \frac{1}{X_o} - \sigma_{\ell}X_o) - (1 - \frac{1}{X} - \sigma_{\ell}X) \right]^{3/2}} \]  
(90)

Finally, by differentiating Eq. (76) with respect to \(\theta\) on both sides and doing some simplifications we get \[ \frac{dX_o}{d\theta} = \frac{X_o(1 - \frac{1}{X_o} - \sigma_{\ell}X_o)^{3/2} \sqrt{1 - (\frac{X}{X_o})^2(1 - \frac{1}{X_o} - \sigma_{\ell}X_o)^{-1}}}{2\sqrt{d_{ls}(2X_o - 3 - \sigma_{\ell}X_o^2)}}. \]  
(91)

**IX. STRONG FIELD LIMIT**

We are going to take some approximations in this section. If the source and the lens are aligned, then we can replace \(\tan \beta\) by \(\beta\) and \(\tan \theta\) by \(\theta\). For the relativistic images we can write \(\Delta \alpha = 2n\pi + \Delta \alpha_n\), (where \(n\) is an integer) and \(0 < \Delta \alpha_n \leq 1\). Hence, we can replace \(\tan(\alpha - \theta)\) by \(\Delta \alpha_n - \theta\). If the ray of light reaches the observer after it turns around the black hole, the deflection angle \(\alpha\) must be very close to \(2\pi\). Therefore, Eq. (79) becomes \[ \beta = \theta - \frac{D_{ls}}{D_{os}} \Delta \alpha_n = \theta - \frac{d_{ls}}{d_{os}} \Delta \alpha_n, \]  
(92)

and the impact parameter is \[ b = d_{ol} \theta. \]  
(93)
Relativistic images are formed only if the ray of light passes very close to the photon sphere. For the closest approach \( X_o \), it is convenient to write

\[
X_o = X_{ps} + \varepsilon, \quad 0 \leq \varepsilon \ll 1
\]  

(94)

For SBH the approximated deflection angle will be \([11]\)

\[
\alpha \sim -2 \ln\left(\frac{2 + \sqrt{3}}{18} \varepsilon\right) - \pi.
\]  

(95)

Therefore, we shall also look for a similar approximation \([39]\)

\[
\alpha = -A \ln(B\varepsilon) - \pi,
\]  

(96)

where \( A \) and \( B \) are positive numbers that we take from \([39]\). However, in our case these numbers will depend only on \( \sigma_\ell \). Therefore, we will have

\[
A = \lim_{X_o \to X_{ps}} \left[-(X_o - X_{ps}) \frac{d\alpha_{exact}}{dX_o}\right],
\]  

(97)

\[
B = \lim_{X_o \to X_{ps}} \exp\left[-\frac{(\alpha_{exact} + \pi)}{A}\right] \frac{1}{(X_o - X_{ps})}.
\]  

(98)

Now, by taking the value of \( X_o \) from Eq. (94) and by putting that expression into the Eq. [84], we can get the impact parameter in terms of \( \varepsilon \)

\[
b(X_{ps} + \varepsilon) = \frac{X_{ps} + \varepsilon}{\sqrt{1 - \frac{1}{X_{ps} + \varepsilon} - \sigma_\ell(X_{ps} + \varepsilon)}},
\]  

(99)

If we use a Taylor expansion in \( \varepsilon \) up to second order in Eq. [99], we get the impact parameter as

\[
b = C - D\varepsilon^2,
\]  

(100)

where

\[
C = \sqrt{\frac{1 - \sqrt{1 - 3\sigma_\ell}}{\sigma_\ell}} \left[\frac{\sqrt{1 - \sqrt{1 - 3\sigma_\ell}}}{\sigma_\ell}\right]^\frac{3}{2},
\]  

(101)

and

\[
D = \left[\frac{-2 + 2\sqrt{1 - 3\sigma_\ell} + \sigma_\ell(8 - 5\sqrt{1 - 3\sigma_\ell} - 6\sigma_\ell)}{2(-1 + \sqrt{1 - 3\sigma_\ell} + 2\sigma_\ell)^2} \left(\frac{-2 + 2\sqrt{1 - 3\sigma_\ell} + 5\sigma_\ell - 2\sigma_\ell\sqrt{1 - 3\sigma_\ell}}{\sigma_\ell}\right)\right].
\]  

(102)

Now we can find the value of \( \varepsilon \) from Eq. [100], that it will be

\[
\varepsilon = \sqrt{\frac{C - b}{D}}.
\]  

(103)
Finally, if we substitute the value of $\varepsilon$ (from Eq. (103)) into (96) we obtain our approximated bending angle expression
\[ \alpha = -A \ln \left( B \sqrt{\frac{C - d_{ol} \theta}{D}} \right) - \pi. \] (104)

**X. RELATIVISTIC IMAGES**

Virbhadra and Ellis defined “relativistic images” of a gravitational lens as those images which occur due to light deflections by angles $\hat{\alpha} > 3\pi/2$ [10]. Similarly, when $\beta = 0$ and $\hat{\alpha} > 2\pi$, the location of relativistic Einstein rings are specified [38]. For any fixed value of $\beta$, we can get $\theta$ related to the positions of the corresponding images. Thus, we can do an approximation using a first order Taylor expansion around $\alpha = 2n\pi$ for the position of the $n$th relativistic image [39]
\[ \theta \approx \theta_n^\circ - \rho_n \Delta \alpha_n, \] (105)
where $\theta = \theta_n^\circ$ at $\alpha = 2n\pi$ and
\[ \rho_n = -\frac{d\theta}{d\alpha} \bigg|_{\alpha=2n\pi}. \] (106)

For the value of $\theta$ we take Eq. (104), and get
\[ \theta = \frac{1}{d_{ol}} \left[ C - \frac{D}{B^2} \exp\left( \frac{-2(\alpha + \pi)}{A} \right) \right], \] (107)
and
\[ \theta_n^\circ = \frac{1}{d_{ol}} \left[ C - \frac{D}{B^2} \exp\left( \frac{-2}{A}(2n + 1)\pi \right) \right]. \] (108)

If we differentiate Eq. (108) on both sides and then we substitute it into the Eq. (106) we obtain
\[ \rho_n = -\frac{1}{d_{ol}} \left[ \frac{2D}{AB^2} \exp\left( \frac{-2}{A}(2n + 1)\pi \right) \right]. \] (109)

Now, from Eq. (105), we have
\[ \Delta \alpha_n \approx \frac{\theta_n - \theta_n^\circ}{-\rho_n}. \] (110)

Using Eqs. (108) and (109) in (110), we get
\[ \Delta \alpha_n \approx \frac{A}{2} \left[ \left( \frac{d_{ol} B^2}{D} \exp\left( \frac{2}{A}(2n + 1)\pi \right) \right) \theta_n - \left( \frac{B^2 C}{D} \exp\left( \frac{2}{A}(2n + 1)\pi \right) - 1 \right) \right]. \] (111)

Substituting Eq. (110) into (82) yields
\[ \beta = \theta_n - \frac{d_{ls}}{d_{os}} \Delta \alpha_n. \] (112)
Putting Eq. (111) into (112), we get
\[ \beta = \left[ 1 + \frac{d_{ls}d_{ol}}{d_{os}} \left( \frac{AB^2}{2D} \exp\left( \frac{2}{A}(2n+1)\pi \right) \right) \right] \theta_n - \frac{d_{ls}}{d_{os}} \left[ \frac{A}{2} \left( \frac{B^2C}{D} \exp\left( \frac{2}{A}(2n+1)\pi \right) - 1 \right) \right]. \] (113)

In order to obtain the approximate position for the relativistic images, we neglect 1 because \( \frac{d_{ls}d_{ol}}{d_{os}} \gg 1 \) in this approximation. Therefore, we have
\[ \theta_n = \frac{d_{os}}{d_{ls}d_{ol}} \left[ \frac{2D}{AB^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right] \beta + \frac{1}{\beta d_{ol}} \left[ C - \frac{D}{B^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right]. \] (114)

Here in Eq. (114), if the source, lens and image are perfectly aligned, \( \beta = 0 \) and then we obtain the Einstein ring with angular radius
\[ \theta_n^E = \frac{1}{d_{ol}} \left[ C - \frac{D}{B^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right] = \theta_n^o. \] (115)

The amplification of the \( n \)th relativistic image is given by
\[ \mu_n \approx \left| \frac{\beta}{\theta_n d\theta_n} \right|^{-1}. \] (116)

Tangential magnification for relativistic images is
\[ \mu_t = \frac{\theta_n}{\beta} = \frac{d_{os}}{d_{ls}d_{ol}} \left[ \frac{2D}{AB^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right] + \frac{1}{\beta d_{ol}} \left[ C - \frac{D}{B^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right]. \] (117)

Radial magnification for relativistic images is
\[ \mu_r = \frac{d\theta_n}{d\beta} = \frac{d_{os}}{d_{ls}d_{ol}} \left[ \frac{2D}{AB^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right]. \] (118)

Thus, the total amplification of the \( n \)th relativistic images can be calculated by combining both tangential magnification Eq. (117) and radial magnification Eq. (118) in (116), which it yields
\[ \mu_n = \frac{1}{|\beta|} \frac{d_{os}}{d_{ls}d_{ol}} \left[ \frac{2D}{AB^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right] \left[ \frac{1}{d_{ol}} \left[ C - \frac{D}{B^2} \exp\left( \frac{-2}{A}(2n+1)\pi \right) \right] \right]. \] (119)

Here, if the observer, lens and source are aligned \((\beta = 0)\), the amplification will diverge. Therefore, the size of the relativistic images becomes very small and the brightness will be low. For the total magnification of relativistic images, the relativistic images and its sums are taken into account.

\[ \mu_R = 2\sum_{n=1}^{\infty} \mu_n = \frac{2}{|\beta|} \frac{d_{os}}{d_{ls}} \sum_{n=1}^{\infty} \theta_n^o \rho_n. \] (120)

Now, by using the geometrical series \( \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} \) for \(|a| < 1\), thus the total magnification of the relativistic images will be
\[ \mu_R \approx \frac{2}{|\beta|} \frac{d_{os}}{d_{ls}d_{os}} \frac{2D}{AB^2} \left[ \frac{D}{B^2} \frac{\exp(-12\pi/A)}{1 - \exp(-8\pi/A)} - C \frac{\exp(-6\pi/A)}{1 - \exp(-4\pi/A)} \right]. \] (121)
XI. DISCUSSION

We have studied the gravitational lensing scenario for non-extreme, naked singularity and extreme cases for the Kiselev black hole. For these cases, we discussed the null-geodesics of a Schwarzschild black hole surrounded by quintessence in order to study the behavior of the scalar field, where we take the state parameter as \( w_q = -\frac{2}{3} \). Therefore, we observed that effective potential and the null-geodesics trajectories depend on the quintessence parameter. We take the negative value of \( w_q \), thus we observed that in the three cases mentioned, the potential does not have a minimum value and also we noticed that for photons there are no stable circular orbits. Moreover, there are only unstable orbits with a single maximum value \((V_{\text{max}})\).

We also studied the behavior of the light in the lensing process. For this, we calculated the equation of the path and the bending angle \( \alpha \). After that, we converted this expression in terms of elliptic integrals. Bending angle depends on the value of \( \sigma \). For each case, it has different limits. Using numerical techniques, we found the bending angle expression and then we observed graphically how it behaves the light around the black hole.

When we have a GL phenomenon for non-extreme KBH we have that \( 0 < \sigma < \frac{1}{8M} \). In this case, if we observed each single curve for each \( \sigma \), we see that as the value of the impact parameter increases, the bending angle decreases. Nevertheless, for the whole process, as the value of \( \sigma \) is increased the light goes closer to the black hole and bending angle will increase. Furthermore, when we compared it with the Schwarzschild bending angle \( \alpha_S \), we observed that \( \alpha_S \) is smaller than the non-extreme bending angle.

For a GL phenomenon for EKBH, we have \( \sigma = \frac{1}{8M} \). In this case, we noticed that as the impact parameter \( b \) increases, the bending angle for EKBH \( \alpha^e \) decreases. When we compared it with SBH, we observed that its behavior is similar to the SBH bending angle \( \alpha_S \) since EKBH has only one horizon which is the double in size of the Schwarzschild’s horizon. However, \( \alpha^e \) is smaller than the \( \alpha_S \).

When we want to study a GL phenomena for naked singularity, we need to take \( \sigma > \frac{1}{8M} \). In this case, the behavior of the light is totally different since light goes into the black hole. No horizon exists and the value of the closest approach was \( r_o \rightarrow b \). In this case for each single curve with different value of sigma, we can see that as we increase the value of \( b \), the bending angle increases. However, if we observed the whole curves, then the bending angle decreases as \( \sigma \) increases.

For the extreme bending angle \( \alpha^e \), we found that it was the largest bending from non-extreme, naked singularity and Schwarzschild bending angles. The non-extreme bending angle was greater
than the Schwarzschild and naked singularity bending angles but smaller than $\hat{\alpha}^e$. Additionally, naked singularity bending angle is reciprocal to both non-extreme and extreme bending angles.

Finally, we used another approach to calculate the bending angle for KBHL and then we used the weak field approximation in order to calculate the approximated bending angle. After that, we found the expression for the magnification of relativistic images.

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