PRODUCTS OF IDEALS OF LINEAR FORMS IN QUADRIC HYPERSURFACES

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Abstract. Conca and Herzog proved in [10] that any product of ideals of linear forms in a polynomial ring has a linear resolution. The goal of this paper is to establish the same result for any quadric hypersurface. The main tool we develop and use is a flexible version of Derksen and Sidman’s approximation systems [13].

1. Introduction

Let $R$ be a standard graded algebra over a field $k$, i.e., $R = S/J$ where $S = k[x_1, \ldots, x_n]$ is a graded polynomial ring with variables in degree 1 and $J$ a homogeneous ideal. For a finitely generated graded $R$-module $M$, denote by $\text{reg}_R M$ the Castelnuovo-Mumford regularity of $M$ over $R$. The Castelnuovo-Mumford regularity computed over $S$ is denoted by $\text{reg}_S M$. By definition $R$ is Koszul if $\text{reg}_R k = 0$. Koszul algebras are known to share with polynomial rings several important features. For example, a beautiful and elegant result of Avramov, Eisenbud and Peeva [3, 4] asserts that Koszul algebras are exactly the algebras over which every finitely generated graded module has a finite Castelnuovo-Mumford regularity. Another very special feature of Koszul algebras is that they tend to have syzygies of relatively low degrees [2]. But not all Koszul algebras are born equal and some of them are better than others. For example, Roos says in [20] that a Koszul algebra $R$ is ‘good’ if all the Poincaré series of finitely generated graded $R$-modules are rational functions sharing a common denominator. He provides in the same paper examples of good and bad Koszul algebras.

Other important notions are that of absolutely Koszul algebras, see [11, 16, 17], and of universally Koszul algebras, see [7, 8]. The latter play an important role in our discussion so let us recall the definition. Let $\mathcal{L}(R)$ be the collection of ideals generated by linear forms of $R$, and $\mathcal{L}^\infty(R)$ be the collection of products of ideals generated by linear forms of $R$. The algebra $R$ is said to be universally Koszul if every ideal of $\mathcal{L}(R)$ has a linear resolution over $R$. We refer the reader to [7, 8] for basic facts about universally Koszul algebras and for the characterization of universally Koszul Cohen-Macaulay domains and universally Koszul algebras defined by monomials. In [10] the following result was proved.

Theorem 1.1 (Conca–Herzog, [10, Theorem 3.1]). Let $S$ be a polynomial ring over $k$. Then any ideal $I$ of $\mathcal{L}^\infty(S)$ has a linear resolution.

This result suggests the following definition.

Definition 1.2. We say that $R$ is universally* Koszul if every ideal in $\mathcal{L}^\infty(R)$ has a linear resolution over $R$.

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In [9, Section 5, Question 9] the authors ask:

**Question 1.3.** Are universally Koszul algebras necessarily universally* Koszul?

A positive answer is given in [9, Section 5, Theorem 11] for algebras of small dimension. However, Question 1.3 remains open in general. The main result of this paper answers in the positive Question 1.3 for quadric hypersurfaces (which are known to be universally Koszul).

**Theorem 1.4** (Theorem 5.1). *Any quadric hypersurface is universally* Koszul.*

To prove the theorem we introduce an invariant, the Chardin mixed regularity $\text{creg}_S(M, N)$ of a pair of graded $S$-modules $M$ and $N$, as follows:

$$\text{creg}_S(M, N) = \sup \{ \text{reg} \text{Tor}_i^S(M, N) - i : i \geq 0 \}.$$ 

We say that $R$ is Tor-linear with respect to a collection $F$ of ideals of $S$ if

$$\text{creg}_S(R, S/I) \leq \text{reg}(S/I) + 1$$ 

for every $I \in F$.

Then we observe in Proposition 3.5 that:

1. If $R$ is Tor-linear with respect to $L(S)$ then $R$ is universally Koszul,
2. If $R$ is Tor-linear with respect to $L^\infty(S)$ then $R$ is universally* Koszul.

Finally we prove the following:

**Theorem 1.5.** If $R = S/(f)$ is a quadric hypersurface (namely deg $f = 2$), then $R$ is Tor-linear with respect to $L^\infty(S)$.

We deduce Theorem 1.5 from Theorem 5.2, which asserts that

$$\text{creg}_S(S/I, S/(f)) = \text{reg} S/I + \text{reg} S/(f)$$

where $I \in L^\infty(S)$ and $f$ is a homogeneous element of $S$. Our proof of this equality uses a generalization of an inductive method due to Derksen and Sidman [13]. Roughly speaking, Derksen and Sidman’s idea is to keep the regularity of a module $N$ under control by means of an approximation system, that is a family of surjective maps $\phi_i : N \to N_i$ such that $\sum_i \text{Ann}(\text{Ker} \phi_i)$ is (primary to) the graded maximal ideal of $S$. For the applications we have in mind, we need a more flexible mechanism where the maps $\phi_i$ need not be surjective. This gives rise to the notion of generalized approximation system that we introduce and discuss in Section 4.

This paper is the continuation of an effort started by the last two authors in [19] with the goal of understanding resolutions of products of ideals generated by linear forms in quotient rings.

## 2. Regularities

Let $k$ be a field and let $R$ be a standard graded $k$-algebra. In other words, $R$ is a commutative, $\mathbb{N}$-graded algebra with $R_0 = k$ and $R$ is generated over $k$ by finitely many elements of degree 1. As such, $R$ can be represented as $R = S/J$ where $S$ is a polynomial ring (i.e. the symmetric algebra of $R_1$) and $J$ a homogeneous ideal of $S$. We also denote by $k$ the $R$-module $R/ \oplus_{i>0} R_i$. Let $M$ be a finitely generated graded $R$-module. The Castelnuovo-Mumford regularity of $M$ over $R$ is

$$\text{reg}_R M = \sup \{ j - i : \text{Tor}^R_i(k, M)_j \neq 0 \},$$
if \( M \neq 0 \) and, by convention, \( \text{reg}_R 0 = -\infty \). We say that \( M \) has a linear resolution over \( R \) if for some integer \( d \), the equality \( \text{Tor}_i^R(k, M)_j = 0 \) holds for all \( j \neq i + d \). In that case, we also say that \( M \) has a \( d \)-linear resolution over \( R \).

Let \( \mathfrak{m} \) be the graded maximal ideal of \( S \). The Castelnuovo-Mumford regularity of \( M \) computed over the polynomial ring \( S \) has also a well-known cohomological interpretation:

\[
\text{reg}_S M = \sup \{ i + j : H^j_{\mathfrak{m}}(M) \neq 0 \}.
\]

Here \( H^j_{\mathfrak{m}}(M) \) denotes the \( i \)-th local cohomology of \( M \) with support in \( \mathfrak{m} \). Because of this, the regularity computed over \( S \) is sometimes called the absolute Castelnuovo-Mumford regularity and it is denoted simply by \( \text{reg} M \) while that over \( R \) is called the relative Castelnuovo-Mumford regularity. Throughout the paper we will use without reference the fact that (absolute/relative) Castelnuovo-Mumford regularity behaves well with respect to short exact sequences (see [9, Section 2, Lemma 9] for details). We will also use the following well-known fact (see [10, Lemma 1.1]).

**Lemma 2.1.** Let \( M \) be a graded \( R \)-module of finite length. Then \( \text{reg} M = \sup \{ i : M_i \neq 0 \} \). If furthermore \( 0 \to M \to P \to N \to 0 \) is an exact sequence where \( P, N \) are finitely generated graded \( R \)-modules, then \( \text{reg} P = \max \{ \text{reg} M, \text{reg} N \} \).

The following result is important in our discussion of Tor-linear algebras.

**Proposition 2.2.** Let \( \varphi : Q \to R \) be a homomorphism of standard graded Koszul \( k \)-algebras such that via scalar restriction, \( R \) is a finitely generated \( Q \)-module (e.g. \( \varphi \) is surjective). Let \( M \) and \( N \) be finitely generated graded modules over \( Q \) and \( R \), respectively. Then

\[
\text{reg}_R(M \otimes_Q N) \leq \max \left\{ \text{reg}_Q M + \text{reg}_R N, \sup_{i \geq 1} \left\{ \text{reg}_Q \text{Tor}_i^Q(M, N) - (i + 1) \right\} \right\}.
\]

**Proof.** Let \( F \) be the minimal graded free resolution of \( M \) over \( Q \). Denote \( C = F \otimes_Q N \). Then we have \( H_i(C) = \text{Tor}_i^Q(M, N) \) for all \( i \geq 0 \). Observe that \( \text{reg}_R C_i = \text{reg}_Q F_i + \text{reg}_R N \) for all \( i \). Now using [9, Section 2, Lemma 9(2)], we obtain

\[
\text{reg}_R(M \otimes_Q N) = \text{reg} H_0(C) \leq \max \left\{ \sup_{i \geq 0} \{ \text{reg}_R C_i - i \}, \sup_{j \geq 1} \{ \text{reg}_R H_j(C) - (j + 1) \} \right\}
\]

\[
= \max \left\{ \sup_{i \geq 0} \{ \text{reg}_Q F_i - i \} + \text{reg}_R N, \sup_{j \geq 1} \{ \text{reg}_R H_j(C) - (j + 1) \} \right\}
\]

\[
= \max \left\{ \text{reg}_Q M + \text{reg}_R N, \sup_{j \geq 1} \{ \text{reg}_R \text{Tor}_j^Q(M, N) - (j + 1) \} \right\}
\]

\[
\leq \max \left\{ \text{reg}_Q M + \text{reg}_R N, \sup_{j \geq 1} \{ \text{reg}_Q \text{Tor}_j^Q(M, N) - (j + 1) \} \right\}.
\]

The last inequality follows from [18, Corollary 6.3]. \( \square \)

**3. Strong versions of Koszulness**

We keep the following notation for this section: \( R \) is a standard graded \( k \)-algebra with the presentation \( R = S/J \) where \( S = \text{Sym}(R_1) \) is the symmetric algebra on the \( k \)-vector space \( R_1 \).
By definition, $R$ is Koszul if $\text{reg}_R k = 0$. For generalities about Koszul algebras and related properties we refer the reader to [9]. Here we simply recall the notion of universally Koszul algebras and introduce universally* Koszul and Tor-linear algebras.

Let $\mathcal{L}(R)$ be the collection of ideals generated by linear forms of $R$, and $\mathcal{L}^\infty(R)$ be the collection of the products of ideals generated by linear forms of $R$.

**Definition 3.1.** We say that $R$ is universally Koszul if every ideal in $\mathcal{L}(R)$ has a linear resolution over $R$. We say $R$ is universally* Koszul if every ideal in $\mathcal{L}^\infty(R)$ has a linear resolution over $R$.

See [7, 8] for generalities on universally Koszul rings. Conca and Herzog’s Theorem 1.1 asserts that polynomial rings are universally* Koszul.

We say that $R$ is a quadric hypersurface if its defining ideal is generated by a quadratic form. By [7, Proposition 3.1], any quadric hypersurface is universally Koszul. In [9, Section 5, Question 9] it is asked whether every universally Koszul algebra is indeed universally* Koszul. To study this question, we now introduce Tor-linear algebras. For finitely generated graded $R$-modules $M$ and $N$ define the Chardin mixed regularity $\text{creg}_R(M, N)$ of $M$ and $N$ as:

$$\text{creg}_R(M, N) = \sup_{i \geq 0} \{\text{reg}_R \text{Tor}_i^R(M, N) - i\}.$$ 

Chardin [6] proved the following

**Theorem 3.2.** Let $M$ and $N$ be finitely generated graded $S$-modules. Then

$$\text{creg}_S(M, N) \geq \text{reg} M + \text{reg} N,$$

and equality holds if $\dim \text{Tor}_1^S(M, N) \leq 1$.

Indeed, this is a special case of much more general results in ibid.. The first assertion follows from [6, Lemma 5.1(i)–(ii)] by setting $S = R$, $F =$ the minimal graded free resolution of $N$ and observing that condition (B) in loc. cit. is trivially satisfied since $F$ is minimal. The second assertion follows from [6, Corollary 5.8]. Theorem 3.2 was inspired by previous results on the regularity of Tor and product of ideals by Sidman [21], Conca and Herzog [10], Caviglia [5] and Eisenbud, Huneke and Ulrich [14].

**Definition 3.3.** Let $\mathcal{F}$ be a collection of ideals of $S$. We say that $R$ is Tor-linear with respect to $\mathcal{F}$ if $\text{creg}_S(S/I, R) \leq \text{reg}(S/I) + 1$ for every $I \in \mathcal{F}$.

**Remark 3.4.** Our choice of terminology is justified as follows. Assume that $R$ is Tor-linear with respect to $\mathcal{F}$ and $\mathcal{F}$ consists of ideals with linear resolutions (which is the case if $\mathcal{F} = \mathcal{L}(S)$ or $\mathcal{L}^\infty(S)$). Then for all $I \in \mathcal{F}$ and $i \geq 2$, the $S$-module $\text{Tor}_i^S(R, S/I)$ has a linear resolution. Indeed, let $I \in \mathcal{F}$ be an ideal with a $d$-linear resolution. By assumption

$$\text{reg} \text{Tor}_i^S(R, S/I) \leq i + \text{reg}(S/I) + 1 = i + d.$$ 

On the other hand, let $F$ be the minimal graded free resolution of $R$ over $S$, then for $i \geq 2$, $F_{i-1}$ is generated in degree at least $i$. Thus $\text{Tor}_i^S(R, S/I) = \text{Tor}_i^S(R, I)$, being a subquotient of $F_{i-1} \otimes_S I$, is generated in degree at least $i + d$. In particular, it has an $(i + d)$-linear resolution.

The importance of Tor-linearity is that this property translates questions about resolutions over singular Koszul rings, usually infinite, to questions about the resolutions of Tor over polynomial rings, necessarily finite.
Proposition 3.5. One has:

(i) If $R$ is Tor-linear with respect to $\mathcal{L}(S)$ then $R$ is universally Koszul and $\text{reg} R \leq 1$.

(ii) If moreover $R$ is Tor-linear with respect to $\mathcal{L}^\infty(S)$ then $R$ is universally $^\ast$ Koszul.

Proof. First we prove (ii). Let $I \in \mathcal{L}^\infty(S)$. Using Proposition 2.2 for the map $S \to R$, $M = S/I$ and $N = R$, we get

$$\text{reg}_R(R/IR) \leq \max \left\{ \text{reg}(S/I), \sup_{i \geq 1} \{ \text{reg} \text{Tor}^S_i(S/I, R) - (i + 1) \} \right\} = \text{reg } S/I.$$ 

Hence by Theorem 1.1, $IR$ has a linear resolution over $R$. In other words, $R$ is universally $^\ast$ Koszul. The proof of the first assertion of (i) is similar. The second assertion of (i) follows from the inequality of Theorem 3.2 and the hypothesis. \qed

Polynomial ring $\xrightarrow{\text{Tor-linear w.r.t. } L^\infty(S)}$ Universally $^\ast$ Koszul $\xrightarrow{(2)}$ Universally Koszul $\xrightarrow{(3)}$ Koszul $\xrightarrow{\text{reg } R \leq 1}$

Figure 1. Properties of a standard graded $k$-algebra $R$

Figure 1 summarizes what we known about the relationships between the properties we have discussed. Examples are known showing that all the implications, with the exception of (1), (2), (3), are not reversible. Note that (3) is related to the open Question 1.3. If the latter turns out to have a positive answer then (3) is reversible and, in view of Example 3.6, (2) is not reversible. We do not know whether (1) can be reversed.

Example 3.6. Let $A = k[x_1, x_2, x_3, y_1, y_2, y_3]$ and $R = S/J$ where $J$ is the ideal of 2-minors of the matrix

$$\begin{pmatrix} x_1 & x_2 & y_1 & y_2 \\ x_2 & x_3 & y_2 & y_3 \end{pmatrix}.$$ 

Then $R$ is universally Koszul and $\text{reg } R = 1$ by [7, Theorem 3.3]. Nevertheless $R$ is not Tor-linear with respect to $\mathcal{L}(S)$. Indeed, for $I = (x_1, x_3) \in \mathcal{L}(S)$ one has

$$\text{reg} \text{Tor}^S_i(S/(x_1, x_3), R) = 3 > 2$$

as one can check with CoCoA [1], Macaulay2 [15], or by direct computation.

4. Generalized approximation

Inspired by [10] and [12], Derksen and Sidman introduce in [13] the approximation systems, a powerful tool to bound inductively the regularity. In the current section, we propose a refinement of Derksen and Sidman’s construction, namely generalized approximation.

In the sequel, $R$ will be a ring and $I \subset R$ an ideal. We start with an observation concerning the transfer of kernel-cokernel annihilation in tensor products.
Remark 4.1. Let $\phi : N \to N'$ be an $R$-linear map and $I$ an ideal such that $I\ker(\phi) = 0$ and $I\coker(\phi) = 0$. Let $B$ be an $R$-module and $\phi \otimes_R B : N \otimes_R B \to N' \otimes_R B$ the induced map. Then simple homological considerations imply that $I^2\ker(\phi \otimes_R B) = 0$ and $I\coker(\phi \otimes_R B) = 0$ but $I\ker(\phi \otimes_R B) \neq 0$ in general, see 4.2. So one might ask under which additional assumptions on $\phi$ does it follow that $I\ker(\phi \otimes_R B)$ vanishes for all $B$. We will see in 4.9 that if $\phi$ is an $I$-approximation (to be defined below) then one can actually deduce that $I\ker(\phi \otimes_R B) = 0$ for all $B$.

Example 4.2. Let $R = k[a,b,c]/(a^2, ab)$, $I = (\overline{a})$, $\phi : R^2 = Re_1 \oplus Re_2 \to R$ the linear map defined by $\phi(e_1) = \overline{a}$ and $\phi(e_2) = \overline{b}$. Then $\ker \phi = \langle (\overline{a}, 0), (\overline{b}, 0), (0, \overline{a}) \rangle$ so that $I\ker \phi = 0$ and $I\coker \phi = 0$. Now for $B = R/(ac + bc)$ one has $(\overline{a}, \overline{c}) \in \ker(\phi \otimes_R B)$ and $I(\overline{c}, \overline{c}) \neq 0$ in $B^2$.

Next we define the main ingredient to construct generalized approximations.

Definition 4.3. Let $\phi : N \to N'$ be a homomorphism of $R$-modules. We say that $\phi$ is an $I$-approximation, if there exist an $R$-module $Z$ and a commutative diagram

$$
\begin{array}{ccc}
N & \xrightarrow{\phi} & N' \\
\downarrow{\alpha} & & \downarrow{\beta} \\
Z & \xrightarrow{\beta} & N'
\end{array}
$$

in which $\alpha, \beta$ are $R$-module homomorphisms such that:

(i) $\alpha$ is injective and $\beta$ is surjective,
(ii) $I\ker \alpha = 0$ and $I\ker \beta = 0$.

If $R$ is a graded ring, $I$ is a homogeneous ideal and $\phi$ is a degree 0 homomorphism of graded $R$-modules, then we require additionally that $Z$ is a graded $R$-module, and $\alpha$ and $\beta$ are graded of degree 0.

In the proof of our main result in Section 5, we will use the following equivalent formulation of the notion of $I$-approximations.

Proposition 4.4. Let $\phi : N \to N'$ be a homomorphism of $R$-modules. Then $\phi$ is an $I$-approximation if and only if there exist an $R$-module $Q$ and submodules $M, P, M', P'$ of $Q$ such that the following conditions hold:

(i) $M \subseteq P$ and $M' \subseteq P'$,
(ii) $IM' \subseteq M \subseteq M'$ and $IP' \subseteq P \subseteq P'$,
(iii) There exist isomorphisms $N \cong P/M, P'/M' \cong N'$ such that we have a commutative diagram

$$
\begin{array}{ccc}
P/M & \xrightarrow{\iota} & P'/M' \\
\cong \downarrow & & \cong \downarrow \\
N & \xrightarrow{\phi} & N'
\end{array}
$$

in which $\iota$ is induced by the inclusions $P \subseteq P'$ and $M \subseteq M'$.

If $R$ is a graded ring, $I$ is a homogeneous ideal, $\phi$ is a homomorphism of graded $R$-modules, then we require additionally that $Q$ is a graded $R$-module, $M, P, M', P'$ are graded submodules of $Q$, and the isomorphisms $N \cong P/M$ and $P'/M' \cong N'$ respect the gradings.
Proof. Assume that $\phi$ is an $I$-approximation, so that there exist a module $Z$ and a commutative diagram as in Definition 4.3. Denote $P = \text{Im}\alpha$, $M = 0$, $M' = \text{Ker}\beta$, which are submodules of $P' = Z$. Set $Q = Z$. We have a commutative diagram
\[
\begin{array}{ccc}
P/0 & \longrightarrow & Z/M' \\
\alpha & \downarrow & \downarrow \phi \\
N & \longrightarrow & N'
\end{array}
\]
where $\phi$ is induced by $\beta$. Both vertical maps are isomorphisms.

Since $\text{Coker}\alpha = Z/P$ and $\text{Ker}\beta = M'$ are annihilated by $I$, we have
\[IZ \subseteq P \subseteq Z,
IM' \subseteq M'.\]
Therefore the modules $Q, M, P, M', P'$ satisfy the required conditions.

Conversely, assume that there exist modules $Q, M, P, M', P'$ as in the statement of Proposition 4.4. We can identify $N$ with $P/M$, $N'$ with $P'/M'$ and $\phi$ with the map $\iota$. Then let $Z = P'/M$, $\alpha$ and $\beta$ be the natural maps $P/M \to P'/M$ and $P'/M \to P'/M'$. Clearly $\alpha$ is injective, $\beta$ is surjective and $\phi = \beta \circ \alpha$. Moreover $\text{Coker}\alpha = P'/P$ and $\text{Ker}\beta = M'/M$ are both annihilated by $I$. Hence $\phi$ is an $I$-approximation. Straightforward variations of the arguments given work in the graded setting as well. \qed

Some elementary but useful properties of $I$-approximations are the following:

**Lemma 4.5.** Let $\phi : N \to N'$ be an $I$-approximation. Then $I\text{Ker}\phi = 0$ and $I\text{Coker}\phi = 0$.

**Proof.** Using the notation of Definition 4.3, there is an injective map $\text{Ker}\phi \to \text{Ker}\beta$ and a surjective map $\text{Coker}\alpha \to \text{Coker}\phi$. \qed

**Lemma 4.6.** Let $\phi : N \to N'$ be a homomorphism of $R$-modules.

(i) If $\phi$ is injective, then $\phi$ is an $I$-approximation if and only if $I\text{Coker}\phi = 0$.

(ii) If $\phi$ is surjective, then $\phi$ is an $I$-approximation if and only if $I\text{Ker}\phi = 0$.

The proof is immediate from Definition 4.3 and Lemma 4.5. We introduce now some notation that is useful in dealing with tensor products.

**Remark 4.7.** Given an ideal $J$ and an $R$-module $M$ then the product $JM$ can be seen as the image of the natural map $J \otimes_R M \to R \otimes_R M \cong M$. Furthermore $(R/J) \otimes_R M \cong M/JM$. Similarly if $G$ is a free $R$-module and $W \subseteq G$ a submodule we will denote by $WM$ the image of the map
\[W \otimes_R M \to G \otimes_R M.
\]
In particular $GM$ gets identified with $G \otimes_R M$ and
\[G/W \otimes_R M \cong GM/WM.
\]
In practice, let $\{e_i : i \in \Lambda\}$ be a basis for $G$, where $\Lambda$ is an index set, then $GM \cong M^\Lambda$ and $WM$ gets identified with the submodule of $M^\Lambda$ generated by elements of the form $(a_i, m : i \in \Lambda) \in M^\Lambda$ with $m \in M$ and $(a_i : i \in \Lambda) \in W$. Note that, given another free module $G_1$ with a submodule $W_1$ we can identify $W_1(WM)$ with $(W_1W)M$ and with $W(W_1M)$. 7
The following lemma will be employed in the proofs of Theorem 4.12.

**Lemma 4.8.** Let \( \phi : N \to N' \) be an I-approximation. Let \( B = G/W \) be an \( R \)-module where \( G \) is a free \( R \)-module and \( W \subseteq G \) a submodule.

(i) With the notation of 4.7, the natural map \( WN \to WN' \) induced by \( \phi \) is an I-approximation.

(ii) The map \( N \otimes R B \xrightarrow{\phi \otimes R B} N' \otimes R B \) is an I-approximation.

(iii) Assume further that \( \phi \) is surjective. Then for all \( i \geq 1 \), the natural map
\[
\Tor_i^R(N, B) \xrightarrow{\Tor_i^R(\phi, B)} \Tor_i^R(N', B)
\]
is an I-approximation.

**Proof.** By definition, we can assume that there exist \( R \)-modules \( U \subseteq Z \) such that \( N \subseteq Z, N' = Z/U \), and a commutative diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{\alpha} & N' \\
\downarrow{\beta} & & \downarrow{\phi} \\
N & \xrightarrow{\phi} & N'
\end{array}
\]

where \( \alpha, \beta \) are respectively natural inclusion and projection maps, such that \( \Coker \alpha = Z/N \) and \( \Ker \beta = U \) are annihilated by \( I \).

(i) Denote by \( \phi' \) the natural map \( WN \to WN' \). Note that \( WN' = (WZ + GU)/GU \), so we have the following commutative diagram

\[
\begin{array}{ccc}
WZ & & \\
\downarrow{\alpha'} & \xrightarrow{\phi'} & WN' \\
WN & \xrightarrow{\phi'} & WN'
\end{array}
\]

where \( \alpha' \) is injective and \( \beta' \) is surjective. Clearly \( \Coker \alpha' = WZ/WN \) and \( \Ker \beta' = WZ \cap GU \) are annihilated by \( I \). Hence \( \phi' \) is an I-approximation.

(ii) By Remark 4.7, we may identify \( N \otimes R B \) with \( GN/WN \) and \( N' \otimes R B \) with \( GN'/WN' \). Denote \( \pi = \phi \otimes R B \). Observe that \( GN'/WN' = GZ/(WZ + GU) \), so we have an induced commutative diagram

\[
\begin{array}{ccc}
GZ/WN & & \\
\downarrow{\pi} & \xrightarrow{\overline{\pi}} & GN'/WN' \\
GN/WN & \xrightarrow{\overline{\pi}} & GN'/WN'
\end{array}
\]

where \( \overline{\pi} \) is injective and \( \overline{\beta} \) is surjective. Clearly \( \Coker \overline{\pi} = GZ/GN \) and \( \Ker \overline{\beta} = (WZ + GU)/WN \) are annihilated by \( I \). Therefore \( \pi \) is an I-approximation.

(iii) If \( i \geq 1 \), replacing \( B \) by a suitable syzygy in a free resolution of it, we can assume that \( i = 1 \). Since \( \phi \) is surjective, we can write \( N = F/U, N' = F/V \) where \( F \) is a free \( R \)-module, \( U \subseteq V \subseteq F \). Then
\[
\Tor_1^R(N, B) = \Ker(U \otimes_R B \to F \otimes_R B) = (GU \cap WF)/WU.
\]

Similarly
\[
\Tor_1^R(N', B) = (GV \cap WF)/WV.
\]
Since $I$ kills $\text{Ker} \phi = V/U$, it holds that $IV \subseteq U \subseteq V$. It remains to apply Proposition 4.4.

So combining the results obtained we can give an answer to Question 4.1:

**Corollary 4.9.** Let $\phi : N \to N'$ be an $I$-approximation, then $I\text{Ker}(\phi \otimes_R B) = 0$ and $I\text{Coker}(\phi \otimes_R B) = 0$ for every $R$-module $B$.

In the remaining of this section $S$ will be a polynomial ring over $k$ with the graded maximal ideal $m$. We will define generalized approximations and present regularity bounds obtained by means of them.

Let $N$ be a finitely generated graded $S$-module and $t \geq 1$ an integer. A **generalized approximation system of degree $t$** for $N$ is a collection of finitely generated graded $S$-modules $N_1, \ldots, N_d$ and surjections $\phi_i : N \to N_i$ for $i = 1, \ldots, d$ which fulfill the following conditions:

1. $\phi_i$ is an $I_i$-approximation for $i = 1, \ldots, d$,
2. $m^t \subseteq I_1 + \cdots + I_d$.

We will say that $N_1, \ldots, N_d$ is a generalized approximation system of degree $t$ for $N$ if there are maps $\phi_i$ with the properties above.

**Remark 4.10.** Derksen and Sidman [13, Section 3] defined approximation systems as follows. If $N$ is any finitely generated graded $S$-module, then an approximation system of degree $t$ for $N$ is a finite collection of graded $S$-modules $N_1, \ldots, N_d$ and surjections $\phi_i : N \to N_i$ for $i = 1, \ldots, d$ which fulfill the following conditions:

1. $I_i \text{Ker} \phi_i = 0$,
2. $m^t \subseteq I_1 + \cdots + I_d$.

From Lemma 4.6 we see that any approximation system (in the sense of Derksen and Sidman) is a generalized approximation system (in our sense).

Our next goal is to show that the results in [13] on regularity bounds obtained using approximation systems can be extended to generalized approximation systems. Following [22], a homogeneous element $y \in S$ is called $N$-filter-regular if the map $N_m \to N_{m + \deg y}$ is injective for all $m \gg 0$. Denote $a_0(N) = \max\{i : H^i_m(N) \neq 0\}$. A useful feature of filter-regular elements is (see [10, Proposition 1.2]): If $y$ is a linear form of $S$ which is $N$-filter-regular, then

$$\text{reg } N = \max\{\text{reg}(N/yN), a_0(N)\}.$$

**Theorem 4.11.** Let $N$ be a finitely generated graded $S$-module. Let $N_1, \ldots, N_d$ be a generalized approximation system of degree $t$ for $N$ (where $d, t \geq 1$). Suppose that $y \in S_1$ is a linear form which is filter-regular with respect to $N$ and $N_i$ for all $i = 1, \ldots, d$. Then

$$\text{reg } N \leq \max\left\{ \max_{i=1,\ldots,d} \{\text{reg } N_i\} + 1, \text{reg}(N/yN) \right\} + t - 1.$$

**Proof.** The proof of [13, Theorem 3.3] carries over verbatim, taking Lemma 4.5 into consideration. Note that the condition $M \to M_i$ is surjective for $i = 1, \ldots, d$ in that result is not necessary. The only crucial conditions are that $I_i$ annihilates the kernel of the map $M \to M_i$ for every $i$, and $m^t \subseteq I_1 + \cdots + I_d$.

The following result is the most important feature of generalized approximation systems. We will use it only in the case $t = 1$. 

9
Theorem 4.12. Let $n = \dim S$. Let $N$ be a finitely generated graded $S$-module and $N_1, \ldots, N_d$ be a generalized approximation system of degree $t$ for $N$. Then:

$$\reg N \leq \max \left\{ \max_{i=1,\ldots,d} \{ \reg N_i \} + 1, b \right\} + (t - 1)n$$

where $b$ is the largest degree of a minimal generator of $N$.

Proof. The argument of the proof [13, Theorem 3.5] carries over verbatim using Theorem 4.11 and that, thanks to Lemma 4.8(ii), the module $N/KN$ admits $N_1/KN_1, \ldots, N_d/KN_d$ as a generalized approximation system of degree $t$ for any homogeneous ideal $K$ of $S$. □

5. Quadric hypersurfaces

In this section $S$ will denote a standard graded polynomial ring over a field $k$ with the graded maximal ideal $m$. Throughout the section we fix $I_1, \ldots, I_d$ ideals generated by linear forms, i.e. $I_i \in \mathcal{L}(S)$, where $d \geq 1$. Furthermore we set $I = I_1 \cdots I_d$.

Denote $[d] = \{1, 2, \ldots, d\}$. For each non-empty index set $A \subseteq [d]$, we let $I_A = \sum_{i \in A} I_i$. For any $i = 1, \ldots, d$ we set $J_i = \prod_{j \neq i} I_j$ and more generally for $1 \leq i_1 < \cdots < i_p \leq d$ we set

$$J_{i_1 \ldots i_p} = \prod_{j \notin \{i_1, \ldots, i_p\}} I_j.$$ 

Our main result is the following:

Theorem 5.1. Let $f \in S$ be a quadric and $R = S/(f)$. Then $R$ is Tor-linear with respect to $\mathcal{L}^\infty(S)$. In particular, $R$ is universally Koszul.

Indeed, we prove the following more general statement.

Theorem 5.2. Let $f \neq 0$ be a homogeneous form of positive degree of $S$. Then there is an equality

$$\creg S (S/I, S/(f)) = \reg S/I + \reg S/(f).$$

The proof of Theorem 5.2 employs generalized approximation as well as the next lemma.

Lemma 5.3. For $i = 1, \ldots, d$ let $V_i \subseteq S_1$ be the vector space of linear forms that generates the ideal $I_i$ and $V = \sum_{i=1}^d V_i$. Then the ideal $I : f$ is generated by elements that belong to the polynomial subring $k[V]$ of $S$.

To prove Lemma 5.3, the following result will be useful.

Lemma 5.4 (Conca–Herzog, [10, Lemma 3.2 and its proof]). One has

$$I = J_1 \cap \cdots \cap J_d \cap I_{[d]}$$

and, furthermore, one has the following (possibly redundant) primary decomposition:

$$I = \bigcap_{A \subseteq [d], A \neq \emptyset} I_A^{[A]}.$$

Proof of Lemma 5.3. By 5.4 we have that $I : f$ is the intersection of $I_A^{[A]} : f$ where $A \subseteq [d], A \neq \emptyset$. Since $I_A$ is generated by linear forms it is a prime complete intersection. Hence the powers of $I_A$ are primary and $I_A^{[A]} : f$ is a power of $I_A$. Therefore $I : f$ is an intersection of powers of $I_A$ as $A$ varies in the set of the non-empty subsets of $[d]$. Since each $I_A$ is generated by elements of $k[V]$, the claim follows. □
Next we come to the proof of Theorem 5.2.

Proof of Theorem 5.2. Denote \( p = \deg(f) \). Since \( \text{pd}_S S/(f) = 1 \), we have \( \text{Tor}_i^S(S/I, S/(f)) = 0 \) for \( i \geq 2 \). Together with Theorems 1.1 and 3.2, the desired equality is equivalent to

\[
\max \left\{ \reg \frac{S}{I + (f)}, \reg \text{Tor}_1^S(S/I, S/(f)) - 1 \right\} \leq d + p - 2.
\]

By Proposition 2.2 for \( M = S/I \) and \( N = S/(f) \),

\[
\reg \frac{S}{I + (f)} \leq \max \left\{ d + p - 2, \reg \text{Tor}_1^S(S/I, S/(f)) - 2 \right\}.
\]

Hence it suffices to show that \( \reg \text{Tor}_1^S(S/I, S/(f)) \leq d + p - 1 \), equivalently \( \reg(I : f)/I \leq d - 1 \), noting that \( \text{Tor}_1^S(S/I, S/(f)) \cong (I : f)/(I(-p)) \).

For every \( i = 0, \ldots, d \) and \( 0 \leq t \leq d - 1 \) we set

\[
M_i = \frac{(I : f) \cap J_1 \cap \cdots \cap J_i}{I},
\]

\[
N_i = \frac{(I : f) \cap J_1 \cap \cdots \cap J_t + J_{t+1}}{J_{t+1}}.
\]

Note that, by definition \( M_d \subseteq M_{d-1} \subseteq \cdots \subseteq M_0 \) and \( M_i/M_{i+1} \cong N_i \) for all \( i = 0, \ldots, d - 1 \).

We prove by induction on \( d \geq 0 \) the following statements:

(S1) For \( i = 0, \ldots, d \) one has \( \reg M_i \leq d - 1 \),

(S2) For every \( i = 0, \ldots, d - 1 \) one has \( \reg N_i \leq d - 1 \).

In particular, for \( i = 0 \) in (S1), since \( M_0 = (I : f)/I \) we obtain the desired inequality.

The case \( d = 0 \) is immediate. For \( d = 1 \) both assertions are obviously true: (S1) holds since for \( i = 0 \), \( M_0 \) is either \( S/I_1 \) or 0 and \( M_1 = 0 \) while (S2) holds since \( N_0 = 0 \).

Now assume that \( d \geq 2 \) and (S1) and (S2) hold up to \( d - 1 \), we will establish both (S1) and (S2) for \( d \).

First we prove that (S2) holds for \( d \). We treat in full detail the case \( 1 \leq i \leq d - 1 \); a similar argument works for the case \( i = 0 \).

Denote

\[
P_1 = \frac{(J_{i+1} : f) \cap J_{1,i+1} \cap \cdots \cap J_{i,i+1}}{(I : f) \cap J_1 \cap \cdots \cap J_i + J_{i+1}}
\]

and

\[
P_0 = \frac{(J_{i+1} : f) \cap J_{1,i+1} \cap \cdots \cap J_{i,i+1}}{J_{i+1}}
\]

so that we have an exact sequence

\[
0 \to N_i \to P_0 \to P_1 \to 0.
\]

Now \( P_0 \) corresponds to the module \( M_i \) associated to the family of ideals \( I_j \) with \( j = 1, \ldots, d \) and \( j \neq i + 1 \). Since (S1) holds for \( d - 1 \), we have that

\[
\reg P_0 \leq d - 2.
\]

Hence, to prove that \( \reg N_i \leq d - 1 \) it suffices to show that \( \reg P_1 \leq d - 2 \).
We will define $I_j$-approximation maps $P_1 \to P_{1,j}$ for $j = 1, \ldots, d$. For $j = 1, \ldots, i + 1$ one easily checks that $I_j$ annihilates $P_1$ so it suffices to take $P_{1,j} = 0$. For $i + 2 \leq j \leq d$ set
\[
U_j = (J_{i+1,j} : f) \cap J_{i,i+1,j} \cap \cdots \cap J_{i,i+1,j},
\]
so that $Y_j \subseteq U_j$. We check that
\[
\text{(a) } I_j U_j \subseteq (J_{i+1} : f) \cap J_{i,i+1} \cap \cdots \cap J_{i,i+1} \subseteq U_j,
\]
\[
\text{(b) } I_j Y_j \subseteq (I : f) \cap J \cap \cdots \cap J_{i,i+1} \subseteq Y_j,
\]
\[
\text{(c) setting } P_{1,j} = U_j/Y_j \text{ one has } \text{reg } P_{1,j} = d - 3.
\]

Conditions (a)–(b) are easy to check. To show that $\text{reg } P_{1,j} \leq d - 3$ we consider the exact sequence
\[
0 \to \frac{Y_j}{J_{i+1,j}} \to \frac{U_j}{J_{i+1,j}} \to P_{1,j} \to 0.
\]

Now we observe that $U_j/J_{i+1,j}$ is a module of type $M_u$ associated to the family $\{I_u : 1 \leq u \leq d, u \neq i + 1, j\}$. So by induction hypothesis we have by (S1) that $\text{reg } (U_j/J_{i+1,j}) \leq d - 3$. Similarly, $Y_j/J_{i+1,j}$ is a module of type $N_d$ associated to the family $\{I_u : 1 \leq u \leq d, u \neq j\}$. Hence by induction hypothesis for (S2) we have $\text{reg } (Y_j/J_{i+1,j}) \leq d - 2$. Therefore
\[
\text{reg } P_{1,j} \leq \max \left\{ \text{reg } \frac{U_j}{J_{i+1,j}}, \text{reg } \frac{Y_j}{J_{i+1,j}} - 1 \right\} \leq d - 3,
\]
as claimed.

By Lemma 5.3, the modules $P_1$ and also $P_{1,1}, \ldots, P_{1,d}$ are quotients of ideals defined in $k[V]$ (notation as in 5.3). Hence their regularities can be computed over $k[V]$. But the graded maximal ideal of $k[V]$ is $I_1 + \cdots + I_d$. Hence the family of maps $P_1 \to P_{1,j}$ defined above is a generalized approximation system of degree 1. Finally note that $P_1$ is generated in degree at most $d - 2$, being a quotient of $P_0$ that has regularity $\leq d - 2$. Summing up, by Theorem 4.12, we conclude that $\text{reg } P_1 \leq d - 2$. This concludes the proof of (S2) for $d$.

Now we proceed to establish (S1) by showing that $\text{reg } M_i \leq d - 1$ by reverse induction on $i$. For $i = d$, by Lemma 5.4, $I : f = (J_1 : f) \cap \cdots \cap (J_d : f) \cap (I_{[d]}^d : f)$. Hence if $f \not\in I_{[d]}$ one has $M_d = 0$. Else if $f \in I_{[d]}$, it follows that $J_1 \cap \cdots \cap J_d \subseteq I : f$ so that
\[
M_d = \frac{J_1 \cap \cdots \cap J_d}{I} = \frac{J_1 \cap \cdots \cap J_d + I_{[d]}^d}{I_{[d]}^d}
\]

All the ideals involved in the expression are generated by elements that belong to $k[V]$ (notation as in 5.3), thus the regularity of $M_d$ can be computed over $k[V]$. Since $I_{[d]}$ is the graded maximal ideal of $k[V]$ we get that $\text{reg } M_d \leq d - 1$ by Lemma 2.1.

Finally for $i < d$, since $M_i/M_{i+1} \cong N_i$ one has
\[
\text{reg } M_i \leq \max \{ \text{reg } M_{i+1}, \text{reg } N_i \}
\]
and $\text{reg } N_i \leq d - 1$ because (S2) has been already proved for $d$. Hence we conclude by reverse induction on $i$ that $\text{reg } M_i \leq d - 1$. The proof is completed. \hfill \square

We are now ready to prove Theorem 5.1.
Proof of Theorem 5.1. Let $I \in \mathcal{L}_\infty(S)$ be an ideal. Since $\deg f = 2$, by Theorem 5.2,
$$\text{creg}_S(S/I, R) = \text{reg}(S/I) + \text{reg} S/(f) = \text{reg}(S/I) + 1.$$Hence $R$ is Tor-linear. The last assertion follows from Proposition 3.5(ii). This finishes the proof. □

As the following example shows if $J \subset S$ has a linear resolution over $S$ and $R$ is a quadric hypersurface then, in general, $JR$ need not have a linear resolution over $R$. This show that Theorem 5.1 cannot be extended to the family of ideals of $S$ with a linear resolution.

Example 5.5. Let $S = k[x_1, x_2, y, z]$, $f = x_1^2$ and $R = S/(f)$. Let $J$ be the ideal of 2-minors of the following matrix
$$\begin{pmatrix} x_1 & x_2 & y \\ x_2 & 0 & z \end{pmatrix}$$By computations with CoCoA [1], $\text{reg} J = 2$ but $\text{reg}_R JR \geq 3$.

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13
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