PATH INTEGRAL SOLUTION OF A CLASS OF EXPLICITLY TIME-DEPENDENT POTENTIALS

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Abstract. A specific class of explicitly time-dependent potentials is studied by means of path integrals. For this purpose a general formalism to treat explicitly time-dependent space-time transformations in path integrals is sketched. An explicit time-dependent model under consideration is of the form $V(q,t) = V[q/\zeta(t)]/\zeta^2(t)$, where $V$ is a usual potential, and $\zeta(t) = (at^2 + 2bt + c)^{1/2}$. A recent result of Dodonov et al. for calculating corresponding propagators is incorporated into the path integral formalism by performing a space-time transformation. Some examples illustrate the formalism.

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1. Introduction

Explicitly time-dependent problems are of great importance in quantum mechanics, in particular in scattering theory, in cosmology, in systems with a time varying force field, or for the investigation of excitation spectra. However, there are only a few available exact solutions and even less discussions in the context of path integrals; here e.g. the famous forced harmonic oscillator [1] must be mentioned, reference [2] for a specific class of explicitly time-dependent one-dimensional problems, and the general quadratic Lagrangian with time-dependent coefficients [3, 4], as well as the time-dependent radial harmonic oscillator [5]. As a matter of fact, general formulæ are difficult to achieve and usually only special cases seem to allow to state explicit solutions, e.g. the “moving potentials” of reference [2].

Recently, Dodonov et al. [6] (and shortly later on Rogers and Spector [7]) have discussed a further class of explicitly time-dependent potentials. Of particular interest is the model of a particle moving inside an infinite square well of moving width, a model important in cosmology, see e.g. Makowski and Dembinski [8], Devoto and Pomorišac [9], and Da Luz and Cheng [10] and references therein.

The general structure of all these problems is that the corresponding quantum Hamiltonian has the following form

\[ H = \frac{p^2}{2m} + \frac{1}{\zeta^2(t)} V \left( \frac{x}{\zeta(t)} \right), \]

with \( x \) the spatial variable, \( p \) its conjugate momentum, \( \zeta(t) = (at^2 + 2bt + c)^{1/2} \), and \( a, b, c \) some constants. In reference [6] this kind of time-dependent potentials was solved by means of a particular integral of motion [11, 12]

\[ I(x, t) = \zeta^2(t) H - \frac{d}{dt} \frac{\zeta^2(t)}{4} (xp + px) + mx^2 \frac{d^2}{dt^2} \frac{\zeta^2(t)}{4}, \]

by looking for eigenfunctions of the operator \( I \), i.e. \( I(x, t) \Psi(x, t) = E \Psi(x, t) \). Having found the eigenfunctions \( \Psi(x, t) \) it is an easy task to construct the corresponding propagator, required, the propagator is known for the time-independent version of \( H \) in Eq. (1), plus an additional harmonic potential.

From the point of view of path integrals, this is an indirect reasoning and by no means satisfactory. In this Letter I want to show that the corresponding kind of time-dependent problems according to Eq. (1) can be done by a space-time transformation.

2. Explicitly Time-Dependent Duru–Kleinert Transformation

In order to discuss explicitly time-dependent space-time transformations we start by considering the usual path integral formulation according to

\[ K(x''', x'; t'', t') = \int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x) \right] dt \right\} \]
This gives the coordinate transformation formula

\[
K(h(q'', t''), h(q', t'); t'', t') = \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \prod_{j=1}^{N-1} dq_j \cdot \prod_{j=1}^{N} \hat{F}''(j) \cdot \prod_{j=1}^{N} d\hat{F}'(j)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \frac{m}{2\epsilon} \hat{F}''(j)^2 (\Delta q_j)^2 + \frac{m}{2\epsilon} \left( \hat{F}'(j) + 2 \hat{F}'(j) \hat{F}'(j) \Delta q_j \right)
\right\} (5)
\]

This gives the coordinate transformation formula

\[
K(h(q'', t''), h(q', t'); t'', t') = \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \prod_{j=1}^{N-1} dq_j \cdot \prod_{j=1}^{N} \hat{F}''(j) \cdot \prod_{j=1}^{N} d\hat{F}'(j)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \frac{m}{2\epsilon} \hat{F}''(j)^2 (\Delta q_j)^2 + \frac{m}{2\epsilon} \left( \hat{F}'(j) + 2 \hat{F}'(j) \hat{F}'(j) \Delta q_j \right)
\right\} (6)
\]

Here \(\Delta x_j = x_j - x_{j-1}, x_j = x(t_j), t_j = t' + j \epsilon, \epsilon = T/N, T = t'' - t'\) fixed, and I have used standard notation for path integrals [1]. In order to avoid cumbersome notation I only consider the one-dimensional case. I consider an explicitly time-dependent coordinate transformation according to \(x = h(q, t)\). Implementing this transformation one has to keep all terms of \(O(\epsilon)\) in the lattice definition of the path integral (3), and expands about midpoints \(\bar{q}_j = \frac{1}{2}(q_j + q_{j-1}), \bar{t}_j = \frac{1}{2}(t_j + t_{j-1})\). The measure is transformed according to

\[
\prod_{j=1}^{N-1} dx_j = \prod_{j=1}^{N-1} h'(q_j, t_j) dq_j
\]

\[
= [h'(q', t') h'(q', t'')]^{-1/2} \prod_{j=1}^{N} [h'(q_j, t_j) h'(q_j, t_j)]^{1/2} \prod_{j=1}^{N-1} dq_j
\]

This gives the coordinate transformation formula

\[
K(h(q'', t''), h(q', t'); t'', t') = \lim_{N \to \infty} \left( \frac{m}{2\pi i \epsilon \hbar} \right)^{N/2} \prod_{j=1}^{N-1} dq_j \cdot \prod_{j=1}^{N} \hat{F}''(j) \cdot \prod_{j=1}^{N} d\hat{F}'(j)
\]

\[
\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \frac{m}{2\epsilon} \hat{F}''(j)^2 (\Delta q_j)^2 + \frac{m}{2\epsilon} \left( \hat{F}'(j) + 2 \hat{F}'(j) \hat{F}'(j) \Delta q_j \right)
\right\} (6)
\]

\[
\equiv [h'(q'', t'') h'(q', t')]^{-1/2} \int_{q(t')=q''}^{q(t')=q'} h'(q, t) D_M p(q(t))
\]
\[ \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \left( \dot{h}'^2(q, t)q'^2 + \dot{h}^2(q, t) + 2h'(q, t)h(q, t)\dot{q} \right) - V(h(q, t)) - \frac{\hbar^2}{8m} h'^2(q, t) \right] dt \right\} . \]  

(7)

The notation \( \int \mathcal{D}_{MP}q\) means that the path integral is defined on mid-points. It is obvious that the path integral (7) is short of being completely satisfactory. Whereas the transformed potential \( V(h(q, t)) \) may have a convenient form when expressed in the new coordinate \( q \), the kinetic term \( \frac{m}{2} h'^2 q^2 \) is in general nasty, and the term linear in \( \dot{q} \) also cannot be seen as of practical use. To achieve a more convenient form of the path integral (7) I proceed in two steps, first, the term proportional to \( \dot{q} \) is gauged away, second, a time-transformation is performed ([13-19], in particular Refs. [18, 19], and c.f. [19] for a comprehensive but slightly different discussion)

To deal with the \( \dot{q} \) term we introduce the identity and expand it about midpoints

\[ 1 = \frac{g(q'', t'')}{g(q', t')} \prod_{j=1}^N \frac{g(q_{j-1}, t_{j-1})}{g(q_j, t_j)} \]

\[ \approx \frac{g(q'', t'')}{g(q', t')} \prod_{j=1}^N \left( 1 - \frac{\dot{g}'(j)}{g(j)} \Delta q_j + \frac{\Delta^2 q_j \dot{g}'^2(j)}{2 g_j^2} - \epsilon \frac{\dot{g}(j)}{g(j)} \right) \]

due to \( e^{-z} \approx 1 - z + z^2/2 \), \(|z| \ll 1\), and hence no term \( \propto (\Delta q_j)^2 \) is present. Here \( g(q, t) \) denotes a to-be-determined function in \( q \) and \( t \); by the symbol "\( \approx \)" we have denoted that we are keeping terms up to \( O(\epsilon) \propto O((\Delta q_j)^2) \). Implementing the identity (8) into the path integral (7) yields

\[ K(h(q'', t''), h(q', t'); t'', t') \]

\[ = [h'(q'', t'')h'(q', t')]^{-1/2} \frac{g(q'', t'')}{g(q', t')} \int \mathcal{D}_{MP}q(t) \]

\[ \times \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \left( h'^2(q, t)q'^2 + \dot{h}^2(q, t) \right) + \left( m h'(q, t) \dot{h}(q, t) - \frac{\hbar}{i} \frac{g'(q, t)}{g(q, t)} \right) \dot{q} \right. \]

\[ \left. - \frac{\hbar}{i} \frac{g(q, t)}{g(q, t)} - V(h(q, t)) - \frac{\hbar^2}{8m} h'^2(q, t) \right] dt \right\} . \]

(9)

To gauge away the \( \dot{q} \)-term we choose the function \( g(q, t) \) in such a way that

\[ \frac{\hbar}{m} h'(q, t) \dot{h}(q, t) = \frac{g'(q, t)}{g(q, t)} , \]

(10)
which gives for \( g(q, t) \) the solution

\[
g(q, t) = \exp \left( \frac{im}{\hbar} \int_{t'}^{t} h'(z, t) \hat{h}(z, t) dz \right). \tag{11}\]

Insertion into the path integral (9) then gives

\[
K(h(q'', t''), h(q', t'); t'', t') = A(q'', q'; t'', t') \tilde{K}(h(q'', t''), h(q', t'); t'', t') , \tag{12}\]

with the prefactor \( A(t'', t') \)

\[
A(q'', q'; t'', t') = \exp \left[ \frac{im}{\hbar} \left( \int_{t'}^{t''} h'(z, t'') \hat{h}(z, t'') dz - \int_{t'}^{t'} h'(z, t') \hat{h}(z, t') dz \right) \right], \tag{13}\]

and the path integral \( \tilde{K}(t'', t') \) is given by

\[
\tilde{K}(h(q'', t''), h(q', t'); t'', t') = [h'(q'', t'') h'(q', t')]^{-1/2} \times \int_{q(t')=q'}^{q(t'')=q''} h'(q, t) D_{MP} q(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \left( h'^2(q, t) q^2 + \hat{h}^2(q, t) \right) - V(h(q, t)) \right] \right. \]

\[
- \frac{\hbar^2}{8m} h'^2(q, t) - m \int_{t'}^{t''} \left( h'(z, t) \hat{h}(z, t) + \hat{h}(z, t) h'(z, t) \right) dz \left. \right] dt \right\}. \tag{14}\]

For the case \( \hat{h}'(q, t) \neq 0 \) this can simplified into

\[
\tilde{K}(h(q'', t''), h(q', t'); t'', t') = [h'(q'', t'') h'(q', t')]^{-1/2} \times \int_{q(t')=q'}^{q(t'')=q''} h'(q, t) D_{MP} q(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} h'^2(q, t) q^2 - V(h(q, t)) \right] \right. \]

\[
- \frac{\hbar^2}{8m} h'^2(q, t) - m \int_{t'}^{t''} h'(z, t) \hat{h}(z, t) dz \left. \right] dt \right\}. \tag{15}\]

It is also obvious that Eqs. (14,15) can be generalized to the \( D \) dimensional case.

Proceeding, one first for the resolvent makes use of the identity

\[
\frac{1}{\hat{H} - \hat{E}} = \hat{f}_{t}(q, t) \frac{1}{\hat{f}_{t}(q, t)(\hat{H} - \hat{E})\hat{f}_{r}(q, t)} \hat{f}_{r}(q, t) , \tag{16}\]

where \( \hat{H} \) is the Hamiltonian corresponding to the path integral \( K(t'', t') \), \( \hat{f}_{t,r}(q, t) \) are multiplication operators in \( q \) and \( t \), multiplying from the left, respectively from the
right, onto the operator \((\hat{H} - \hat{E})\), and and \(\hat{E} = i \hbar \partial_t\) is the energy operator, and second introduces a new pseudo-time \(s''\) defined by \([13-19]\):

\[
s'' \equiv \tau(t'') = \int_0^{t''} \frac{dt}{f_l(q, t)f_r(q, t)} .
\]  

(17)

Introducing the matrix element \([17, 19]\) \(\langle t'' | E | t' > = e^{-i E t'' / \sqrt{2 \pi \hbar}}\) with the corresponding representation of \(\delta(t'' - t' - s) = < t'' | e^{i E s / \hbar} | t' >\), together with the completeness relation \(\int dp \int dE |p, E > < p, E| = 1\), we obtain for the path integral \(K(t'', t')\)

\[
K(h(q'', t''), h(q', t'); t'', t') = f_r(x'', t'')f_l(x', t') \left[ h'(q'', t'')h'(q', t') \right]^{-1/2} \int_0^{\infty} ds'' \\
\times \lim_{N \to \infty} \prod_{j=1}^{N-1} \int dq_j \int dt_j \left[ \prod_{j=1}^{N} \delta(\Delta t_j - \epsilon_s f_{l,j} f_{r,j-1}) \right] \sqrt{\frac{m}{2 \pi \epsilon_s \hbar}} \cdot \frac{h_j h_{j-1}}{f_{l,j} f_{r,j-1}} \\
\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[ \frac{m}{2 \epsilon_s} \left( h_j - h_{j-1} \right)^2 - \epsilon_s f_{l,j} f_{r,j-1} V(h_j) \right] \right\} .
\]  

(18)

Here denote \(h_j = h(q_j, t_j)\), etc., and \(\epsilon_s\) and \(\epsilon\) are related by \(\epsilon = \epsilon_s f_{l,j} f_{r,j-1}\). We make the choice \(f_{l,j} = h_j', f_{r,j-1} = h_{j-1}'\), to guarantee a symmetric transformation with respect to initial and final coordinates, and again we expand about the midpoints \(\bar{q}_j\) and \(\bar{t}_j\). We obtain similarly as before [identify \(h(q, t(s)) = h(q, s); \dot{h}'(q, t) \neq 0\), c.f. the remark following Eq. (14)].

\[
K(h(q'', t''), h(q', t'); t'', t') = \left[ h'(q'', t'')h'(q', t') \right]^{1/2} A(q'', q'; t'', t') \int_0^{\infty} ds'' \\
\times \lim_{N \to \infty} \prod_{j=1}^{N-1} \int dq_j \int dt_j \left( \frac{m}{2 \pi \epsilon_s \hbar} \right)^{N/2} \prod_{j=1}^{N} \delta(\Delta t_j - \epsilon_s f_{l,j} f_{r,j-1}) \\
\times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[ \frac{m}{2 \epsilon_s} (\Delta q_j)^2 - \epsilon_s F_{j}(V(\bar{F}_{(j)}) - \epsilon_s \Delta V(\bar{q}_j, \bar{s}_j) \right] \right\} .
\]  

(19)

Here \(\Delta V\) denotes the quantum potential

\[
\Delta V(q, s) = \frac{\hbar^2}{8m} \left( 3 h''''(q, s) - 2 h'''(q, s) \right) + m h'(q, s) \int_{\bar{h}}^{\bar{q}} h'(z, s)\bar{h}(z, s)dz .
\]  

(20)

The \(\delta\)-function integrations successively determine the values of \(t_j\) by means of an iteration process

\[
t_j - t_{j-1} = \epsilon_s h_j' h_{j-1}' = \epsilon_s h_j'^2 + O(\epsilon^2) ,
\]  

(21)
which allows the actual iterated integration and evaluation of the argument of the \( \delta \)-function, respectively, where the \( O(\epsilon_s^2) \) can be ignored, yielding [19]

\[
t - t_j = \sum_{k=j}^{N} \epsilon_s h^{t_k^2}.
\]  

(22)

Because there is one more \( \delta \)-function as integration, the last one is expanded into its Fourier representation and thus we arrive finally for the combined transformations

\[
x = h(q, T) \quad \text{and} \quad dt/f_r(q, t)f_t(q, t) = dt/h^{t^2}(q, t) = ds \quad \text{at the space-time (Durum-Kleinert) transformation formulæ}
\]

\[
K(x'', x'; t'', t') = \left[ h'(q'', t'')h'(q', t') \right]^{1/2} A(q'', q'; t'', t') \\
\times \int_{-\infty}^{\infty} \frac{dE}{2\pi i} e^{-iET/h} G(q'', q; E)
\]  

(23)

\[
G(q'', q'; E) = \frac{i}{\hbar} \int_0^\infty \tilde{K}(q'', q'; s'')ds''
\]  

(24)

with the path integral \( \tilde{K}(s'') \) given by

\[
\tilde{K}(q'', q', s'')_{q(s'')=q''} = \int_{q(0)=q'} D_x D_p q(s) \exp \left\{ \frac{i}{\hbar} \int_0^{s''} \left[ \frac{m}{2} \dot{q}^2 - h^2(q, s) \left( V(h(q, s)) - E \right) - \Delta V(q, s) \right] ds \right\}.
\]  

(25)

Note that in comparison to Refs. [18, 19] there is no potential term \( \propto -i \hbar h'(q, t)\dot{h}'(q, t) \) present, c.f. the model in the next Section and the discussion in the summary. It must be noted that the whole procedure as sketched here remains only on a formal level, however with well-defined rules. Attempts to put it on a more sound mathematical basis can be e.g. found Ref. [20], and more recently c.f. Refs. [21, 22].

3. The Model

Let us consider the path integral formulation corresponding to Eq. (1)

\[
K(x'', x'; t'', t') = \int_{x(t')=x'} D_x(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{1}{\zeta(t)} V \left( \frac{x}{\zeta(t)} \right) \right] dt \right\}.
\]  

(26)

In order to discuss the path integral (26), I consider in the first step the (time-dependent) coordinate transformation \( x(t) = \zeta(t)q(t) \) according to the rules in
Section 2. Inserting this transformation into $\Delta V$ and the prefactor $A(t'', t')$ yields (note that we also can perform a partial integration in the Lagrangian in the exponential without the delay of $\Delta V$, because the transformation is linear in the spatial coordinates, $D$ the spatial dimension, $\zeta' = \zeta(t'), \zeta'' = \zeta(t'')$, etc.)

$$K(x'', x'; t'', t') = \left( \zeta'' \zeta' \right)^{-D/2} \exp \left[ \frac{i m}{2\hbar} \left( x''^2 \frac{\dot{\zeta}''}{\zeta''} - x'^2 \frac{\dot{\zeta}'}{\zeta'} \right) \right] \tilde{K}(q'', q'; t'', t') ,$$

with the path integral $\tilde{K}(t'')$ given by

$$\tilde{K}(q'', q'; t'', t') = \lim_{N \to \infty} \left( \frac{m \zeta_j \zeta_j^{-1}}{2\pi i \hbar} \right)^{ND/2 \left( N - 1 \right)} \prod_{j=1}^{N} \int dq_j \times \exp \left\{ \frac{i}{\hbar} \sum_{j=1}^{N} \left[ \frac{m}{2\epsilon} \zeta_j \zeta_j^{-1} (\Delta q_j)^2 - \frac{\epsilon}{\zeta_j \zeta_j^{-1}} \left( \frac{m}{2} \omega_j^{-2} q_j^2 + V(q_j) \right) \right] \right\} .$$

with $\omega^2 = ac - b^2$. The path integral (27) together to $\tilde{K}(t'', t')$ corresponds exactly to the path integral (14).

Proceeding we perform a time-transformation $dt/\zeta^2(t) = ds$, i.e. we set $f^2(s) = \zeta^2[q(t)]$, with a new time $s''$ defined by

$$s'' \equiv \tau(t'') = \int_{t'}^{t''} \frac{dt}{\zeta^2(t)} \left\{ \begin{array}{ll}
= \frac{1}{\omega'} \arctan \left( \frac{at + b}{\omega'} \right) \Big|_{t'}^{t''} & (\omega'^2 > 0) , \\
= - \frac{1}{|\omega'|} \arctanh \left( \frac{at + b}{|\omega'|} \right) \Big|_{t'}^{t''} & (\omega'^2 < 0) , \\
= \frac{a}{b} \left( \frac{t}{at + b} \right) \Big|_{t'}^{t''} & (\omega'^2 = 0) . 
\end{array} \right.$$  

This gives now the transformation formulæ

$$\tilde{K}(q'', q'; t'', t') = \zeta(t') \zeta(t'') \int^\infty_{-\infty} \frac{dE}{2\pi i} G(q'', q'; E) e^{-iET/\hbar}$$

$$G(q'', q'; E) = \frac{i}{\hbar} \int_{t_0}^{\infty} \tilde{K}(q'', q'; s'') \exp \left( \frac{iE}{\hbar} \int_{t_0}^{s''} \zeta^2(t(s))ds \right) ds'' ,$$

and the path integral $\tilde{K}(s'')$ given by

$$\tilde{K}(q'', q'; s'') = \int_{q(0) = q'}^{q(s'') = q''} Dq(s) \exp \left\{ \frac{i}{\hbar} \int_{s'}^{s''} \left[ \frac{m}{2} q'^2 - \frac{m}{2} \omega'^2 q^2 - V(q) \right] ds \right\} .$$
This path integral has no explicit time-dependence, however an additional harmonic part with frequency \( \omega' \) is present. Let us assume that we can write down the solution of the path integral (32) and call it \( K_{\omega',V}(s'') \). We obtain [c.f. Eqs. (17,18)]

\[
\tilde{K}(x'',x';t'',t') = \zeta(t')\zeta(t'') \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \int_{0}^{s''} ds'' \exp \left[ \frac{iE}{\hbar} \left( \int_{0}^{s''} \zeta^2(t(s))ds - T \right) \right] K_{\omega',V}(q'',q';s'')
\]

\[= K_{\omega',V} \left( \frac{x''}{\zeta''}, \frac{x'}{\zeta'} ; \tau(t'') \right), \tag{33}\]

and we therefore find for the propagator (27)

\[
K(x'',x';t'',t') = (\zeta''\zeta')^{-D/2} \exp \left[ \frac{im}{2\hbar} \left( \frac{m}{2} x''^2 - V(x-f(t)) \right) \right] K_{\omega',V} \left( \frac{x''}{\zeta''}, \frac{x'}{\zeta'} ; \tau(t'') \right), \tag{34}\]

which is the result of reference [6].

Let us note that for time-dependent potential problems according to \( V(x) \rightarrow V(x-f(t)) \) one derives from Eq. (14) the identity (c.f. [2], \( q' = x' - f' \), \( f' = f(t') \), etc., note \( \dot{f}'(q,t) = 0)\)

\[
\int_{x(t')=x'}^{x(t'')=x''} Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - V(x-f(t)) \right] dt \right\} \left[ \dot{f}''(x''-f'') - \dot{f}'(x'-f') + \frac{1}{2} \int_{t'}^{t''} \dot{f}^2(t) dt \right] \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{q}^2 - V(q) - m\dot{f}(t)q \right] dt \right\}. \tag{35}\]

4. Examples

Let us discuss this result shortly for the two cases \( \omega'^2 > 0 \) and \( \omega'^2 < 0 \), respectively.

1) \( \omega'^2 \neq 0 \). For exactly solvable solutions of equation (34) only harmonic potentials are relevant. Let us consider the harmonic potential \( V(x) = (m/2)\omega^2x^2 \), and set \( \Omega^2 = \omega^2 + \omega'^2 \) which may be positive or negative (for \( \Omega^2 < 0 \) we have a harmonic repeller). Let \( \Omega^2 > 0 \). Then we have the path integral identity \( (D = 1)\)

\[
\int_{x(t')=x'}^{x(t'')=x''} Dx(t) \exp \left[ \frac{im}{2\hbar} \int_{t'}^{t''} \left( \dot{x}^2 - \frac{\omega^2}{\zeta(t)} x^2 \right) dt \right]\]
\begin{align*}
(\zeta' \zeta'')^{-1/2} \exp \left[ \frac{i m}{2\hbar} \left( x'' \frac{\dot{z}''}{\zeta''} - x' \frac{\dot{z}'}{\zeta'} \right) \right] & \left( \frac{m \Omega}{2 \pi i \hbar \sin \Omega \tau(t'')} \right)^{1/2} \\
\times \exp \left\{ -\frac{m \Omega}{2i \hbar} \left[ \left( \frac{x''^2}{\zeta''^2} + \frac{x'^2}{\zeta'^2} \right) \cot \Omega \tau(t'') - \frac{2x' x''}{\zeta' \zeta'' \sin \Omega \tau(t'')} \right] \right\}.
\end{align*}

Similarly we obtain for a radial harmonic potential

\begin{equation}
V(r) = \frac{m}{2} \omega^2 r^2 + \frac{\lambda^2 - \frac{1}{4}}{2mr^2}
\end{equation}

the path integral identity \((\Omega \text{ and } \tau \text{ as before})\)

\begin{equation}
\int_{r(t')=r'}^{r(t'')=r''} \mathcal{D}r(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \frac{\dot{z}^2}{\zeta} - \frac{m \omega^2}{2} \frac{r^2}{\zeta'} - \frac{\lambda^2 - \frac{1}{4}}{2mr^2} \right) dt \right] = \left( \frac{r' r''}{\zeta' \zeta''} \right)^{1/2} \exp \left[ \frac{i m}{2\hbar} \left( r'' \frac{\dot{z}''}{\zeta''} - r' \frac{\dot{z}'}{\zeta'} \right) \right] \left( \frac{m \Omega}{i \hbar \sin \Omega \tau(t'')} \right)^{1/2} \times \exp \left[ -\frac{m \Omega}{2i \hbar} \left( \frac{r''^2}{\zeta''^2} + \frac{r'^2}{\zeta'^2} \right) \cot \Omega \tau(t'') \right] I_\lambda \left( \frac{m \Omega r' r''}{i \hbar \zeta' \zeta'' \sin \Omega \tau(t'')} \right).
\end{equation}

Equations (36,38) can also be achieved by considering the explicit time-dependence \(\Omega^2(t) = (\omega^2 + \omega'^2) \zeta^{-4}(t)\) and using the known solutions for the (radial) time-dependent harmonic oscillator [1, 5–6, 23], respectively.

2) \(\omega' = 0\). Here we have reduced the number of parameters by one and \(\zeta(t) = (at + b)/\sqrt{a}\). For this special case we can always rescale \(\tau\) according to \(\tau(t'') = \alpha T/a(1+\alpha T)\) \((\alpha = b/a)\) [6, 7], and every exactly solvable path integral solution can be substituted into Eq. (31), provided \(K_{0, V} \equiv K_V\) is known (see reference [6] for examples, only this particular case was treated in reference [7]).

We consider the example of the infinite well (IW) with one boundary fixed at \(x = 0\), and the other moving uniformly in time according to \(L(t) = L_0 \zeta(t)\) [6, 8-11]. The result then has the form \((\Theta(z, \tau)\text{ denotes a Jacobi-theta function})\)

\begin{equation}
K^{(IW)}(x'', x'; t'', t') = \frac{(\zeta' \zeta'')^{-1/2}}{2L_0} \exp \left[ \frac{i m}{2\hbar} \left( x'' \frac{\dot{z}''}{\zeta''} - x' \frac{\dot{z}'}{\zeta'} \right) \right] \\
\times \Theta_3 \left( \frac{x'' / \zeta'' - x' / \zeta'}{2L_0}, -\frac{\pi \hbar \tau(t'')}{2mL_0^2} \right) - \Theta_3 \left( \frac{x'' / \zeta'' + x' / \zeta'}{2L_0}, \frac{\pi \hbar \tau(t'')}{2mL_0^2} \right).
\end{equation}

For the second example we consider a time-dependent \(\delta\)-function perturbation according to \(V(x) = -\gamma \delta(x)/\zeta(t)\). We obtain the path integral identity

\begin{equation}
\int_{x(t')=x'}^{x(t'')=x''} \mathcal{D}x(t) \exp \left[ \frac{i}{\hbar} \int_{t'}^{t''} \left( \frac{m}{2} \frac{x^2}{\zeta(t)} + \frac{\gamma}{\zeta(t)} \delta(x) \right) dt \right]
\end{equation}
\[
\begin{aligned}
&= (\zeta' \zeta'')^{-1/2} \exp \left[ \frac{im}{2\hbar} \left( \frac{x'' \zeta''}{\zeta''} - x' \zeta' \right) \right] \\
&\times \left\{ \left( \frac{m}{2\pi i \hbar \tau(t'')} \right)^{1/2} \exp \left[ \frac{im}{2\hbar \tau(t'')} \left( \frac{x''}{\zeta''} - \frac{x'}{\zeta'} \right)^2 \right] \\
&\quad + \frac{m\gamma}{2\hbar^2} \exp \left[ -\frac{m\gamma}{\hbar^2} \left( \frac{|x''|}{\zeta''} + \frac{|x'|}{\zeta'} \right) + \frac{i}{\hbar} \tau(t'') \frac{m\gamma^2}{2\hbar^2} \right] \\
&\quad \times \text{erfc} \left[ \sqrt{\frac{m}{2i \hbar \tau(t'')}} \left( \frac{|x''|}{\zeta''} + \frac{|x'|}{\zeta'} - \frac{i}{\hbar} \gamma \tau(t'') \right) \right] \right\} \quad .
\end{aligned}
\]

(40)

However, for numerous examples only \( G_V(E) \) instead of \( K_V(t'') \) can be explicitly stated, i.e. the energy-dependent Green function is available [24]; here the best known example is the Coulomb-Green function (see however [6, 25]). Then, instead of Eq. (34), only Eq. (31) can be stated in closed form.

5. Summary and Discussion

In this Letter I have studied a particular model of explicitly time-dependent quantum mechanical problems by path integrals. In order to do this I have sketched how to treat general explicitly time-dependent space-time (Duru-Kleinert) transformation (“process-cum-time substitution” [20]) in path integrals. In comparison to, say, Ref. [19] there is no potential term \( \propto -i \hbar h'(q, t) h'(q, t) \) in the effective Lagrangian present. An analysis of the result of Ref. [19] shows that this extra term is due to another gauge, i.e. another function \( g(q, t) \) is chosen. Actually one has an additional factor \( [h'(q, t)]^{1/2} \) in \( g(q, t) \), suggested by the there-used postpoint expansion. Changing the gauge by taking \( [h'(q, t)]^{-1/2} \) instead leads to a cancellation of terms \( \propto -i \hbar h'(q, t) h'(q, t) \). This difference can be interpreted in the following way: In Ref. [19] the gauge has been chosen in such a way that a term \( \propto -i \hbar h'(q, t) h'(q, t) \) appears in the effective Lagrangian, hence an imaginary potential is present. The imaginary potential can be on the one hand understood as a source, respectively a sink for particles, because the transformation of a time-independent Hamiltonian to a time-dependent one, say, has the consequence that the new Hamiltonian does not conserve the energy; this is now exactly balanced by the imaginary potential in order to guarantee energy conservation of the entire (time-independent, say) system. On the other, this term can be interpreted as a “path-dependent measure” (as in [19]), respectively another gauge is chosen, as in this Letter; the latter case has the advantage that from the beginning on the effects of the explicitly time-dependent Duru-Kleinert transformation are incorporated in the weight factors of the wave-functions. Actually, in our model, a time-dependent system is transformed into a time-independent one, and the transformed integration measure is transformed just in the correct way that it
“guarantees that the probability density remains normalised in $D$-dimensional space” [11].

Therefore, the midpoint expansion leads in a very natural way to the gauge as chosen in (11) by putting additional contributions into the integration measure (Jacobian). Whereas in [19] $g(q, t)$ is chosen in such a way that the corresponding Schrödinger equation in the new coordinate $q$ and the pseudotime $s''$ does not have a first order partial derivation, the present formalism allows this by the introduction of a non-trivial momentum operator $p_q = -i \hbar (\partial_q + \frac{1}{2} \Gamma_q)$ ($\Gamma_q = \partial_q \ln \sqrt{g}$, with $\sqrt{g} dq$ the integration measure) [16, 26], which is hermitean with respect to the inner product $\int \sqrt{g} dq f^*(q) g(q)$. Furthermore, the postpoint expansion used in Ref. [19] leads to an expansion into many terms of order $\epsilon$ which in the midpoint expansion are not present, respectively they are cancelling each other. Therefore, our technique shows that the midpoint prescription is far simpler in handling, in the conceptual understanding, and gives unambiguous results in comparison with already existing models [6-11], in particular [6, 11].

Therefore we have shown that the technique of explicitly time-dependent space-time transformation in path integrals provides the necessary tools to treat explicitly time-dependent problems in a rigorous and explicit way, leading to a general formula for the corresponding propagator. I have unified the various approaches, and have presented a simpler derivation of the time-dependent Duru–Kleinert transformation by means of the midpoint prescription. I obtained a general formula for the incorporation of this kind of explicit time dependence, provided the propagator for the time-independent case is known. Whereas closed expressions for $K_{\omega',V}$ are only possible for $V$ also harmonic, and for $K_{0,V}$ there are only few, a formal spectral expansion is always possible, i.e. yielding $(\omega' = 0, D = 1)$

$$
\int_{x(t') = x'}^{x(t'') = x''} Dx(t) \exp \left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[ \frac{m}{2} \dot{x}^2 - \frac{1}{2} \zeta^2(t) V \left( \frac{x}{\zeta(t)} \right) \right] dt \right\} 
\begin{align*}
&= (\zeta'' \zeta')^{-1/2} \exp \left[ \frac{i m}{2 \hbar} \left( x'' \zeta'' \zeta' - x' \zeta'' \zeta' \right) \right] \\
&\times \int dE_\lambda \Psi_\lambda \left( \frac{x''}{\zeta''} \right) \Psi_\lambda^* \left( \frac{x'}{\zeta'} \right) \exp \left( -\frac{i}{\hbar} \frac{E_\lambda}{\hbar} \int_{t'}^{t''} \frac{dt}{\zeta^2(t)} \right),
\end{align*}
$$

where $\int dE_\lambda$ denotes a Stieltjes integral to include bound and scattering states $\Psi_\lambda$ with energy $E_\lambda$ of the corresponding time-independent problem. This general result concludes the discussion.
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