CHAOS AND ENTROPY FOR INTERVAL MAPS

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Abstract. In this paper, various chaotic properties and their relationships for interval maps are discussed. It is shown that the proximal relation is an equivalence relation for any zero entropy interval map. The structure of the set of f-nonseparable pairs is well demonstrated and so is its relationship to Li-Yorke chaos. For a zero entropy interval map, it is shown that a pair is a sequence entropy pair if and only if it is f-nonseparable.

Moreover, some equivalent conditions of positive entropy which relate to the number “3” are obtained. It is shown that for an interval map if it is topological null, then the pattern entropy of every open cover is of polynomial order, answering a question by Huang and Ye when the space is the closed unit interval.

1. Introduction

The study of the complexity or chaotic behavior is a central topic in topological dynamics. Starting from the work of Li and York [18] various authors introduce a lot of definitions of chaos according to their understanding of the phenomena. Among them, Li-Yorke chaos, Denavey chaos [7] and positive entropy [3] are popular ones. It is important to understand their relationships. Recently, it has been shown that for a general topological dynamical system, Devaney chaos implies Li-Yorke chaos [11] and positive entropy implies Li-Yorke chaos [5].

In the study of the so called “local entropy theory” (for a survey see [10]), a lots of notions are introduced to describe dynamical properties. It is not clear the relationship of those properties (related to entropy) with the chaotic behaviors for a given space. The purpose of the current paper is to study the relationship in the case when the given space is a closed interval. We believe that many results of the paper hold for a graph map even more general spaces.

To state our results, we introduce some notations first. Let $I$ be the closed unit interval $[0, 1]$ and $C(I, I)$ denote the class of continuous maps of $I$ to itself. For $f \in C(I, I)$, let $f^0$ be the identity, and for $n \in \mathbb{N}$, let $f^{n+1} = f^n \circ f$, where $\mathbb{N}$ stands for the set of positive integers.

A point $x \in I$ is called a periodic point of $f$ with period $n$ if $f^n(x) = x$, $f^k(x) \neq x$ for $1 \leq k < n$. A periodic point with period 1 is called a fixed point. The $\omega$-limit set of $x$, denoted by $\omega_f(x)$, is the set of limit points of $\{f^i(x)\}_{i=0}^{\infty}$. A set $W \subset I$ is called some $\omega$-limit set for $f$, if there exists an $x \in I$ such that $W = \omega_f(x)$. Denote the collection of all $\omega$-limit sets for $f$ by $\omega_f$.

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A point \( x \in X \) is called (1) a recurrent point, if for every neighborhood \( U \) of \( x \), there exists some \( n > 0 \), such that \( f^n(x) \in U \); (2) a strongly recurrent point, if for every neighborhood \( U \) of \( x \), there exists some \( N > 0 \), such that if \( f^m(x) \in U \) then \( f^{m+k}(x) \in U \) for some \( k \) with \( 0 < k \leq N \); (3) a regularly recurrent point, if for every neighborhood \( U \) of \( x \), there exists some \( N > 0 \), such that \( f^{kN}(x) \in U \) for all \( k > 0 \).

Denote \( \text{Per}(f) \), \( \text{Rec}(f) \), \( \text{SR}(f) \) and \( \text{RR}(f) \) by the set of periodic points, recurrent points, strongly recurrent points and regularly recurrent points, respectively. It is well known that

\[
\text{Per}(f) \subset \text{RR}(f) \subset \text{SR}(f) \subset \text{Rec}(f).
\]

The terminology “chaos” was first introduced by Li and Yorke \cite{18} to describe the complex behavior of trajectories. A pair \( \langle x, y \rangle \in I^2 \) is called proximal if \( \lim \inf_{n \to \infty} |f^n(x) - f^n(y)| = 0 \) and is called asymptotic if \( \lim_{n \to \infty} |f^n(x) - f^n(y)| = 0 \). A scrambled pair or Li-Yorke pair is one that is proximal but not asymptotic. A pair \( \langle x, y \rangle \) is called proper if \( x \neq y \).

For \( \delta > 0 \), a pair \( \langle x, y \rangle \) is said to be \( \delta \)-scrambled if

\[
\lim_{n \to \infty} |f^n(x) - f^n(y)| = 0 \text{ and } \lim_{n \to \infty} \sup |f^n(x) - f^n(y)| \geq \delta.
\]

A set \( C \subset I \) is called scrambled (resp. \( \delta \)-scrambled) if any proper pair \( \langle x, y \rangle \in C^2 \) is scrambled (resp. \( \delta \)-scrambled). The map \( f \) is called Li-Yorke chaotic (resp. \( \delta \)-Li-Yorke chaotic) if there exists an uncountable scrambled set (resp. \( \delta \)-scrambled set).

In \cite{18}, Li and Yorke proved that for an interval map period 3 implies Li-Yorke chaos. In \cite{14}, Jankova and Smital generalized this result as follows: if an interval map has positive entropy, then it is Li-Yorke chaotic.

The converse of this result is not true: Xiong \cite{26} and Smital \cite{24} constructed some interval maps with zero entropy which are Li-Yorke chaotic.

In \cite{24}, Smital also built some useful tools for zero entropy interval maps: the periodic portion of an \( \omega \)-limit set and \( f \)-nonseparable points. See \cite{3} for another approach to the periodic portion of an \( \omega \)-limit set.

In this paper, we discuss those various chaotic properties and their relationships for interval maps. In section 2, for preparation we recall some basic definitions and results for a general dynamical system. In section 3, we review the structure of the \( \omega \)-limit set and build a new approach to the periodic portion of an \( \omega \)-limit set.

In section 4, we deal with zero entropy interval maps. First, we show that the proximal relation is an equivalence relation for a zero entropy interval map. Second, some properties of \( f \)-nonseparable pairs are obtained. Third, we discuss the relationship between Li-Yorke chaos and \( f \)-nonseparable pair. Finally, after reviewing some recent results on the sequence entropy pair, we show that for a zero entropy interval map a pair is a sequence entropy pair if and only if it is \( f \)-nonseparable.

In section 5, we obtain some equivalent conditions of positive entropy which relate to the number “3” and show that strongly mixing is equivalent to topological K for interval maps. In section 6, we show that for an interval map if it is null, then the pattern entropy of every open cover is of polynomial order, which give an positive answer for a problem in \cite{13} for interval maps.
2. Preliminaries

In this section we briefly review some basic definitions and results for a general dynamical system. By a topological dynamical system (TDS for short), we mean a pair \((X, T)\), where \(X\) is a compact metric space with metric \(d\) and \(T : X \to X\) is a continuous map.

**Definition 2.1.** Let \((X, T)\) be a TDS. The system \((X, T)\) (or the map \(T\)) is called

1. **transitive** if for every two nonempty open subsets \(U, V\) of \(X\), there exists some \(n \geq 0\) such that \(T^n U \cap V \neq \emptyset\).
2. **weakly mixing** if \(T \times T\) is transitive on \(X \times X\).
3. **strongly mixing** if for every two nonempty open subsets \(U, V\) of \(X\), there exists some \(N \geq 0\) such that \(T^n U \cap V \neq \emptyset\) for all \(n \geq N\).
4. **minimal** if there is no non-trivial subsystem.

**Definition 2.2.** Let \((X, T)\) be a TDS. The system \((X, T)\) (or the map \(T\)) is called

1. **sensitive on initial conditions** (or just sensitive) if there exists \(\varepsilon > 0\) such that for every nonempty open subset \(U\) of \(X\), there are \(x, y \in U\) and \(n \geq 0\) such that \(d(T^n(x), T^n(y)) \geq \varepsilon\).
2. **Devaney chaotic**, if it is transitive and the set of periodic points is dense in \(X\).

In 1965, Adler, Konheim and McAndrew introduced topological entropy for a TDS. Let \(C_o X\) be the set of finite open covers of \(X\). Given two open covers \(U\) and \(V\), let \(U \lor V = \{U \cap V : U \in U, V \in V\}\).

We define \(N(U)\) as the minimum cardinality of subcovers of \(U\). The topological entropy of \(T\) with respect to \(U\) is

\[
h(T, U) = \lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-i} U \right).\]

The topological entropy of \(T\) is

\[
h(T) = \sup_{U \in C_o X} h(T, U).\]

In 1974, Goodman introduced sequence topological entropy. For an \(A = \{0 \leq t_1 < t_2 < \cdots\} \subset \mathbb{Z}_+\) and an open cover \(U\) of \(X\), the topological sequence entropy of \(T\) with respect to \(U\) and \(A\) is

\[
h_A(T, U) = \lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-t_i} U \right).\]

The topological sequence entropy of \((X, T)\) along \(A\) is

\[
h_A(T) = \sup_{U \in C_o X} h_A(T, U).\]

Recently, local entropy theory has aroused great interesting, see [10] for a survey on this topic. The notions of entropy tuple and sequence entropy tuple of length \(n\) were defined in paper [12]. Originally, entropy tuple and sequence entropy tuple are defined using open covers, now we state the equivalence definition using the notion
of independence set (see [12, 15]). Recall that a TDS \((X, T)\) is \emph{tame} if the cardinal number of its enveloping semigroup is not greater than the cardinal number of \(\mathbb{R}^9\), and a TDS \((X, T)\) is \emph{null} if \(\sup_A h_A(T) = 0\).

**Definition 2.3.** Let \((X, T)\) be a TDS. For a tuple \(\bar{A} = (A_1, \ldots, A_k)\) of subsets of \(X\), a subset \(J \subset \mathbb{Z}_+^\ast\) is called an independence set for \(\bar{A}\) if for every nonempty finite subset \(I \subset J\), we have
\[
\bigcap_{i \in I} T^{-i} A_{s(i)} \neq \emptyset
\]
for all \(s \in \{1, \ldots, k\}\).

In [15], Kerr and Li defined IE-tuple, IT-tuple and IN-tuple (standing for entropy, tame and null, respectively) as follows.

**Definition 2.4.** A tuple \(\bar{x} = \langle x_1, \ldots, x_k \rangle \in X^k\) is called
(1) an \emph{IE-tuple} if for every product neighborhood \(U_1 \times \cdots \times U_k\) of \(\bar{x}\) the tuple \((U_1, \ldots, U_k)\) has an independence set of positive density.
(2) an \emph{IT-tuple} if for every product neighborhood \(U_1 \times \cdots \times U_k\) of \(\bar{x}\) the tuple \((U_1, \ldots, U_k)\) has an infinite independence set.
(3) an \emph{IN-tuple} if for every product neighborhood \(U_1 \times \cdots \times U_k\) of \(\bar{x}\) the tuple \((U_1, \ldots, U_k)\) has arbitrarily long finite independence sets.

Recall that a tuple \(\langle x_1, \ldots, x_k \rangle \in X^k\) is said to be \emph{essential} if \(x_i \neq x_j\) for all \(1 \leq i < j \leq k\) and \emph{non-diagonal} if there are \(1 \leq i < j \leq k\) such that \(x_i \neq x_j\).

**Definition 2.5.** Let \((X, T)\) be a TDS. The system \((X, T)\) is called
(1) \emph{uniformly positive entropy} if every essential pair \(\langle x_1, x_2 \rangle \in X^2\) is an IE-tuple. In particular the system \((X, T)\) has zero entropy iff every IE-pair is diagonal.
(2) \emph{topological K} if every essential \(k\)-tuple \(\langle x_1, \ldots, x_k \rangle \in X^k\) is an IE-tuple for all \(k \geq 2\).

**Theorem 2.6.** [12, 15] Let \((X, T)\) be a TDS.
(1) A tuple is an entropy tuple iff it is a non-diagonal IE-tuple. In particular the system \((X, T)\) has zero entropy iff every IE-pair is diagonal.
(2) A tuple is a sequence entropy tuple iff it is a non-diagonal IN-tuple. In particular the system \((X, T)\) is null iff every IN-pair is diagonal.
(3) The system \((X, T)\) is tame iff every IT-pair is diagonal.

3. The structure of \(\omega\)-limit sets

We first recall some classical results on the structure of \(\omega\)-limit sets for interval maps. The following result is well known, see [3] for example.

**Theorem 3.1.** Let \(f \in C(I, I)\). The following conditions are equivalent:
(1) \(h(f) = 0\);
(2) the period of every periodic point is a power of 2;
(3) every \(\omega\)-limit set can not properly contain a periodic orbit.

The following result first appeared in [20], see also in [2].

**Theorem 3.2.** Let \(f \in C(I, I)\).
(1) If $\omega_1$ and $\omega_2$ are two $\omega$-limit sets and $a \in \omega_1 \cap \omega_2$ is an $\omega_3$-limit point from the left (resp., from the right) of both $\omega_1$ and $\omega_2$, then $\omega_1 \cup \omega_2$ is also an $\omega$-limit set of $f$.

(2) If $\omega_1 \subset \omega_2 \subset \cdots$ is a sequence of $\omega$-limit sets, then $\bigcup_{i=1}^{\infty} \omega_i$ is also an $\omega$-limit set of $f$.

On the basis of the above result and Zorn’s Lemma we have:

**Proposition 3.3.** Let $f \in C(I, I)$ and $\omega_f$ partially ordered by the inclusion relation. Then each maximal chain in $\omega_f$ has a maximal element.

**Lemma 3.4.** Let $f \in C(I, I)$ with $h(f) = 0$. If $\omega_f(x)$ and $\omega_f(y)$ are two maximal $\omega$-limit sets, then $\omega_f(x)$ and $\omega_f(y)$ either coincide or are disjoint.

**Proof.** Assume that $P = \omega_f(x) \cap \omega_f(y) \neq \emptyset$. If $P$ is finite, then $P$ contains a periodic point since $f(P) \subset P$. By Proposition 3.13, both $\omega_f(x)$ and $\omega_f(y)$ are periodic orbits, which implies $\omega_f(x) = \omega_f(y)$. If $P$ is infinite, then any limit point of $P$ is a limit point from the left or from the right of both $\omega_f(x)$ and $\omega_f(y)$. By Proposition 3.21, $\omega_f(x) \cup \omega_f(y)$ is also an $\omega$-limit set. Then the maximality of $\omega_f(x)$ and $\omega_f(y)$ imply that they coincide. 

**Proposition 3.5.** [3, Lemma VI.14] Let $f \in C(I, I)$ with $h(f) = 0$ and $x \in I$. Suppose $\omega_f(x)$ is infinite. For every $k \geq 1$ and $i = 0, 1, \ldots, 2^k - 1$, let

$$J_i^k = [\min_{f^{2^k}}(f^i(x)), \max_{f^{2^k}}(f^i(x))].$$

Then

1. $f(J_i^k) \supset J_{i+1}^{i+1 \mod 2^k}$,
2. the closed intervals $(J_i^k)_{0 \leq i < 2^k}$ are pairwise disjoint,
3. $J_{i}^{k+1} \cup J_{2^k+i}^{k+1} \subset J_i^k$ for $0 \leq i < 2^k$. Both $J_{i}^{k+1}$ and $J_{2^k+i}^{k+1}$ have an endpoint in common with $J_i^k$,
4. for every $0 \leq i < 2^k$, $J_i^k$ contains a periodic point of period $2^k$, but no periodic point of period less than $2^k$.

We call those intervals $(J_i^k)_{k \geq 1, 0 \leq i < 2^k}$ a periodic portion of the $\omega$-limit set $\omega_f(x)$. The periodic portion of an $\omega$-limit set does not depend on the choose of the base point, i.e. if $\omega_f(x) = \omega_f(y)$, then they have the same periodic portion.

**Theorem 3.6.** [3, Lemma VI.16, VI.18] Let $f \in C(I, I)$ with $h(f) = 0$ and $x \in I$. Suppose $\omega_f(x)$ is infinite. Let $(J_i^k)_{k \geq 1, 0 \leq i < 2^k}$ be the periodic portion of $\omega_f(x)$ and

$$C(x) = \bigcap_{k=1}^{\infty} \bigcup_{i=0}^{2^k-1} J_i^k.$$ 

Then $C(x)$ is closed, $f(C(x)) = C(x)$ and $C(x) \cap \text{Per}(f) = \emptyset$.

Moreover, for every nested sequence $J_1^{i_1} \supset J_2^{i_2} \supset \cdots$, put $K = \bigcap_{k=1}^{\infty} J_k^{i_k}$, then exactly one of the following alternatives holds:

1. $K = \{y\} \subset \omega_f(x)$ and $y$ is regularly recurrent,
2. $K = [y, z]$, $K \cap \omega_f(x) = \{y, z\}$ and both endpoints of $K$ are strongly recurrent but not regularly recurrent,
Moreover, if $y \in \omega_f(x)$, then $y$ is regularly recurrent iff $\lim_{n \to \infty} f^n(y) = y$.

**Theorem 3.7.** [3, Theorem VI.30] Let $f \in C(I, I)$ with $h(f) = 0$ and $x \in I$. If $Y = \omega_f(x)$ is infinite, then there exists a continuous map $\phi$ from $Y$ onto the adding machine $J$ such that except at most countable points in $J$ which have two preimages, other points have exactly one preimage and

$$\phi \circ f(y) = \tau \circ \phi(y), \forall y \in Y.$$ 

Moreover, $\phi$ maps $Y$ homeomorphically onto $J$ iff every point $y \in Y$ is regularly recurrent iff $(Y, f)$ is a minimal subsystem.

**Lemma 3.8.** Let $f \in C(I, I)$ with $h(f) = 0$ and $x \in I$. Suppose $\omega_f(x)$ is infinite. Let $(J_k^n)_{k \geq 1, 0 \leq i < 2^k}$ be the periodic portion of $\omega_f(x)$. For every $k \geq 1$, if $J^{r_1}_k$ and $J^{r_2}_k$ are two intervals in $(J_k^n)_{0 \leq i < 2^k}$, then there exists some periodic point between $J^{r_1}_k$ and $J^{r_2}_k$.

**Proof.** Recall that for every $k \geq 1$ there exists a periodic point of periodic $2^k$ in $J^1_k$ but no periodic point of periodic $2^j$ for any $j < k$ (see Proposition 3.3).

We use induction to show the result.

If $k = 1$, let $J_0 = [\min \omega_f(x), \max \omega_f(x)]$. Since $f(J_0) \supset J_0$, there is a fixed point $p$ in $J_0$, but neither $J^1_1$ or $J^1_1$ can contain a fixed point. Then $p$ lies between $J^1_1$ and $J^1_1$.

Assume for all $k \leq n$, the conclusion is made.

Let $k = n + 1$, we have two cases.

Case 1: $J^{r_1}_{n+1}$ and $J^{r_2}_{n+1}$ are contained in the same $J^{r_3}_n$. Since $f^{2^n}(J^{r_3}_n) \supset J^{r_3}_n$, there exists a periodic point $p$ in $J^{r_3}_n$ of periodic $2^n$, but neither $J^{r_1}_{n+1}$ nor $J^{r_2}_{n+1}$ can contain a periodic point of periodic $2^n$. Then $p$ lies between $J^{r_1}_{n+1}$ and $J^{r_2}_{n+1}$.

Case 2: $J^{r_1}_{n+1}$ and $J^{r_2}_{n+1}$ are contained in $J^{r_3}_n$ and $J^{r_4}_n$ respectively. By induction, there exists a periodic point $p$ between $J^{r_3}_n$ and $J^{r_4}_n$. However, $p$ also lies between $J^{r_1}_{n+1}$ and $J^{r_2}_{n+1}$. \qed

The following Lemma was proved in [21], for completeness we provide a proof.

**Lemma 3.9.** Let $f \in C(I, I)$ with $h(f) = 0$ and $x \in I$. Suppose $\omega_f(x)$ is infinite.

1. If $J$ is an interval containing three distinct points of $\omega_f(x)$, then $J$ contains a periodic point.
2. If $U$ is an open interval such that $U \cap \omega_f(x) \neq \emptyset$, then there exists $n \geq 0$ such that $f^n(U)$ contains a periodic point.

**Proof.** Let $(J^i_k)_{k \geq 1, 0 \leq i < 2^k}$ be the periodic portion of $\omega_f(x)$.

1. Let $J \cap \omega_f(x) \supset \{x_1, x_2, x_3\}$. Without loss of generality, assume $x_1 < x_2 < x_3$. Then there exists a nested sequence $J^{i_1}_1 \supset J^{i_2}_2 \supset \cdots$ such that $x_j \in \bigcap_{k=1}^{\infty} J^{i_k}_k$ for $j = 1, 2, 3$. Let $K_j = \bigcap_{k=1}^{\infty} J^{i_k}_k$, then $K_1 \cap K_3 = \emptyset$, since $\#(K_j \cap \omega_f(x)) \leq 2$ for $j = 1, 2, 3$. Then by Lemma 3.8 there exists a periodic point $p$ between $K_1$ and $K_3$. Hence, $p \in J$ since $J$ is connected.

2. Let $z \in U \cap \omega_f(x)$. If $U$ contains a periodic point, then the proof is complete. Otherwise, let $V \supset U$ be the maximal subinterval that contains no periodic point. By assumption, there are $0 < n_1 < n_2 < n_3$ such that $f^{n_3}(x) \in U$ for
The points \( \{z, f^{n_3-m_1}(z), f^{n_2-m_1}(z)\} \) are distinct and contained in \( \omega_f(x) \). If \( \{z, f^{n_3-m_1}(z), f^{n_2-m_1}(z)\} \subset V \), then the first part of the proof implies that \( V \) contains a periodic point, which contradicts the definition of \( V \). Thus, there exists \( j \in \{2, 3\} \) such that \( f^{n_j-m_1}(z) \notin V \). The interval \( f^{n_j-m_1}(U) \) contains both \( f^n(x) \in V \) and \( f^{n_j-m_1}(z) \notin V \). Therefore, \( f^{n_j-m_1}(U) \) must contains a periodic point by the maximality of the interval \( V \).

\[\text{Remark 3.10.} \text{ Let } f \in C(I, I) \text{ with } h(f) = 0 \text{ and } x \in I. \text{ Suppose } \omega_f(x) \text{ is infinite. Let } (J^i_k)_{k \geq 1, 0 \leq i < 2^k} \text{ be the periodic portion of } \omega_f(x). \text{ For every } k \geq 1, \text{ since } (J^i_k)_{0 \leq i < 2^k} \text{ are pairwise disjoint closed intervals, let } s_k = \min\{\frac{1}{k}, \frac{1}{4}d(J^i_k, J^{i+1}_k) : 0 \leq i < 2^k - 1\} > 0 \text{ and } J^k_{i, s_k} = B(J^i_k, s_k), \text{ where } d(J^i_k, J^{i+1}_k) = \inf\{|x - y| : x \in J^i_k, y \in J^{i+1}_k\} \text{ and } B(J^i_k, s_k) = \{x \in I : |x - y| < s_k \text{ for some } y \in J^i_k\}. \text{ Then } (J^k_{i, s_k})_{0 \leq i < 2^k} \text{ are pairwise disjoint open intervals and } d(J^k_{i, s_k}, J^{k+1}_{i, s_k}) > s_k \text{ for } 0 \leq i < 2^k - 1. \text{ For convenience, we also call } (J^k_{i, s_k})_{k \geq 1, 0 \leq i < 2^k} \text{ is the periodic portion of } \omega_f(x).\]

\[\text{Lemma 3.11.} \text{ Let } f \in C(I, I) \text{ with } h(f) = 0 \text{ and } x \in I. \text{ Suppose } \omega_f(x) \text{ is infinite. Let } (J^k_{j, s_k})_{k \geq 1, 0 \leq i < 2^k} \text{ be the periodic portion of } \omega_f(x). \text{ If } y \in I \text{ with } \omega_f(y) \subset \omega_f(x), \text{ then for every } k \geq 1 \text{ there exists some } n_k \geq 0 \text{ such that } f^n(y) \in J^n_{k, s_k} \text{ for all } n \geq n_k.\]

**Proof.** Since \( \omega_f(y) \subset \omega_f(x) \), then for every \( k \geq 1 \) there exists some \( n_{k_0} \geq 0 \) such that \( f^n(y) \in \bigcup_{i=0}^{2^k-1} J^i_{k, s_k} \) for all \( n \geq n_{k_0}. \)

**Claim:** For every \( k \geq 1 \) there exists \( n_{k_1} \geq n_{k_0} \), such that for every \( n \geq n_{k_1} \), if \( f^n(y) \in J^i_{k, s_k} \), then \( f^{n+1}(y) \in J^{i+1}_{k, s_k}. \)

**Proof of the Claim.** If not, then there exists a sequence \( \{n_q\} \), such that \( f^{n_q}(y) \in J^q_{k, s_k} \) and \( f^{n_q+1}(y) \notin J^{q+1}_{k, s_k} \). Without loss of generality, we assume \( \lim_{q \to \infty} f^{n_q}(y) = a \), \( f^{n_q}(y) \in J^0_{k, s_k} \) and \( f^{n_q+1}(y) \in J^1_{k, s_k} \) with \( i_1 \neq i_0 + 1 \text{ (mod } 2^k). \)

Then \( a \in J^0_k \) but \( f(a) \in J^1_k \). This contradicts the fact that \( f(\omega_f(x) \cap J^1_k) = \omega_f(x) \cap J^{1+1}_{k, s_k} \). Then the proof of the Claim is complete.

Now, choose \( n_k \geq n_{k_1} \) such that \( f^{n_k}(y) \in J^0_{k, s_k} \), then \( f^n(y) \in J^{n-n_k \text{ (mod } 2^k)}_{k, s_k} \) for all \( n \geq n_k. \)

4. Chaos for zero entropy maps

Throughout this section, if without any other statements, we assume that \( f \in C(I, I) \) with \( h(f) = 0. \)

4.1. Proximal relation. First, we consider the proximal relation of \( f \). If \( \langle x, y \rangle \in I^2 \) is proximal, then \( \omega_f(x) \cap \omega_f(y) \neq \emptyset \). Two maximal \( \omega \)-limit sets containing \( \omega_f(x) \) and \( \omega_f(y) \) respectively are not disjoint, then they coincide by Lemma [3.3].

We define a “kneading sequence” for one point according to the periodic portion of its maximal \( \omega \)-limit set, which can characterize the proximal pair following the idea in [1].

**Definition 4.1.** Let \( f \in C(I, I) \) with \( h(f) = 0 \) and \( x \in I \). If \( \omega_f(x) \) is infinite, then there exists a unique maximal \( \omega \)-limit set \( \omega_0 \) which contains \( \omega_f(x). \) Let
\((J_{i,k,s_k})_{k \geq 1, 0 \leq i < 2^k}\) be the periodic portion of \(\omega_0\). By Lemma 3.11, for every \(k \geq 1\) we can define

\[c_x(k) = \min \left\{ n_k \in \mathbb{N} : f^n(x) \in J_{k,s_k}^{n-n_k(\text{mod} \ 2^k)} \text{ for all } n \geq n_k \right\}\]

It is easy to see that \(c_x(k) = c_x(k + 1) \text{ (mod } 2^k)\).

**Proposition 4.2.** Let \(f \in C(I, I)\) with \(h(f) = 0\) and \(x, y \in I\). If \(\omega_f(x)\) and \(\omega_f(y)\) are contained in the same maximal limit set \(\omega_0\) which is infinite, then the following conditions are equivalent:

1. \(\langle x, y \rangle\) is proximal;
2. \(c_x(k) = c_y(k) \text{ (mod } 2^k)\) for all \(k \geq 1\).

**Proof.** Since \(\omega_0\) is infinite, \(\omega_f(x)\) and \(\omega_f(y)\) are also infinite. Let \((J_{i,k,s_k})_{k \geq 1, 0 \leq i < 2^k}\) be the periodic portion of \(\omega_0\).

(1) \(\Rightarrow\) (2). If not, there exists some \(k \geq 1\) such that \(c_k(x) \neq c_k(y) \text{ (mod } 2^k)\). Let 
\[\delta = \min_{0 \leq i < 2^k-1} d(J_{i,k,s_k}^i, J_{i,k,s_k}^{i+1}) > 0.\]
Then it is easy to verify that \(|f^n(x) - f^n(y)| \geq \delta\) for all \(n \geq \max\{c_k(x), c_k(y)\}\). This is a contradiction, since \(\langle x, y \rangle\) is proximal.

(2) \(\Rightarrow\) (1). For every \(\varepsilon > 0\), there exists an appropriate \(J_{i,k,s_k}\) such that \(\text{diam}(J_{i,k,s_k}) \leq \varepsilon\). Without loss of generality, we assume \(c_k(x) \geq c_k(y)\). Then \(f^{c_k(x)}(x), f^{c_k(x)}(y) \in J_{i,k,s_k}^0\) and \(f^{c_k(x)+i}(x), f^{c_k(x)+i}(y) \in J_{i,k,s_k}^i\). Thus, \(|f^{c_k(x)+i}(x) - f^{c_k(x)+i}(y)| \leq \varepsilon\), which implies \(\langle x, y \rangle\) is proximal since \(\varepsilon\) is arbitrary. \(\square\)

Let \(A\) be a subset of \(\mathbb{N}\), we call \(A\) has Banach density 1 if

\[\lim_{n \to \infty} \frac{\#(A \cap E_n)}{\#(E_n)} = 1\]

for any sequence \(\{E_n\}\) of intervals of positive integer, where \(E_n = \{a_n, a_n + 1, \ldots, b_n\}\) and \(\lim_{n \to \infty} \#(E_n) = \lim_{n \to \infty}(b_n - a_n + 1) = \infty\).

Let \(\mathcal{F}_{\text{bd1}} = \{A \subset \mathbb{N} : A\) has Banach density 1\}. A pair \(\langle x, y \rangle \in I^2\) is called \(\mathcal{F}_{\text{bd1}}\)-proximal if \(\{n \in \mathbb{N} : d(f^n(x), f^n(y)) < \frac{1}{p}\}\) is finite for all \(p \geq 1\). Clearly, if \(\langle x, y \rangle \in I^2\) is asymptotic, then \(\langle x, y \rangle\) is \(\mathcal{F}_{\text{bd1}}\)-proximal.

**Proposition 4.3.** Let \(f \in C(I, I)\) with \(h(f) = 0\) and \(x, y \in I\). If \(\langle x, y \rangle\) is proximal, then \(\langle x, y \rangle\) is \(\mathcal{F}_{\text{bd1}}\)-proximal.

**Proof.** If \(\omega_f(x)\) is finite, then it is easy to see that \(\langle x, y \rangle\) is asymptotic. Now assume that \(\omega_f(x)\) is infinite, then so is \(\omega_f(y)\). Let \(\omega_0\) be the maximal \(\omega\)-limit set which contains both \(\omega_f(x)\) and \(\omega_f(y)\), then \(\omega_0\) is also infinite. Let \((J_{i,k,s_k})_{k \geq 1, 0 \leq i < 2^k}\) be the periodic portion of \(\omega_0\).

Fix \(p \geq 1\) and \(k \geq 1\). Without loss of generality, assume \(c_k(x) \geq c_k(y)\). Since the intervals \((J_{i,k,s_k})_{0 \leq i < 2^k}\) are pairwise disjoint, there are at most \(p\) distinct sets \(J_{i,k,s_k}\) with \(\text{diam}(J_{i,k,s_k}) \geq 1/p\). Let \(\{E_n\}\) be a sequence of intervals of positive integer and
is not proximal, one has \( \liminf (\langle u \rangle) = \delta \), then there exists some \( k \) such that \( \lim_{n \to \infty} \#(E_n) \geq \#(E_n) - c_x(k) \)

\[
\geq 1 - \frac{\left(\frac{(E_n)}{2^k} + 2\right) p + c_x(k)}{\#(E_n)}
\]

\[
\to 1 - \frac{p}{2^k}, \quad \text{as } n \to \infty.
\]

Thus,

\[
\lim_{n \to \infty} \frac{\# \left\{ i \in E_n : |f^i(x) - f^i(y)| < \frac{1}{p} \right\}}{\#(E_n)} = 1
\]

since \( k \) is arbitrary. Therefore, \( \langle x, y \rangle \) is \( F_{\text{bdl}} \)-proximal since \( p \) and \( \{E_n\} \) are arbitrary.

\[\square\]

**Theorem 4.4.** Let \( f \in C(I, I) \) with \( h(f) = 0 \). Then the proximal relation \( \text{Prox}_f = \{ \langle x, y \rangle \in I^2 : (x, y) \text{ is proximal} \} \) is an equivalence relation.

**Proof.** The reflexivity and symmetry of \( \text{Prox}_f \) is obviously. The transitivity of \( \text{Prox}_f \) is followed by Proposition 4.3 and the property that the intersection of two Banach density 1 sets also has Banach density 1.

For every \( x \in I \), we define the *proximal cell* of \( x \) as \( \text{Prox}_f(x) = \{ y \in I : \langle x, y \rangle \text{ is proximal} \} \). It is easy to see that \( \{ \text{Prox}_f(x) : x \in I \} \) is a partition of \( I \).

**Proposition 4.5.** Let \( f \in C(I, I) \) with \( h(f) = 0 \). If \( \text{Prox}_f(x), \text{Prox}_f(y) \) are two different proximal cells, then there exists \( \delta > 0 \), such that

\[
\liminf_{n \to \infty} d(f^n(u), f^n(v)) \geq \delta
\]

for all \( u \in \text{Prox}_f(x), v \in \text{Prox}_f(y) \).

**Proof.** Let \( \omega_1, \omega_2 \) be two maximal \( \omega \)-limit sets which contain \( \omega_f(x) \) and \( \omega_f(y) \) respectively. If \( \omega_1 \neq \omega_2 \), then \( \omega_1 \cap \omega_2 = \emptyset \). Let \( \delta = d(\omega_1, \omega_2) > 0 \). Then \( \liminf_{n \to \infty} d(f^n(u), f^n(v)) \geq \delta \) for all \( u \in \text{Prox}_f(x), v \in \text{Prox}_f(y) \).

If \( \omega_1 = \omega_2 \), then there are two cases.

Case 1. \( \omega_1 \) is a periodic orbit. Then there exist two periodic points \( p_1 \) and \( p_2 \), such that \( \lim_{n \to \infty} |f^n(x) - f^n(p_1)| = 0 \) and \( \lim_{n \to \infty} |f^n(y) - f^n(p_2)| = 0 \). Since \( \langle x, y \rangle \) is not proximal, one has \( p_1 \neq p_2 \). Let \( \delta = \min_{n \geq 0} |f^n(p_1) - f^n(p_2)| > 0 \). Then \( \liminf_{n \to \infty} d(f^n(u), f^n(v)) \geq \delta \) for all \( u \in \text{Prox}_f(x), v \in \text{Prox}_f(y) \).

Case 2. \( \omega_1 \) is infinite. Let \( (J_{k,s}) \) be the periodic portion of \( \omega_1 \). Since \( \langle x, y \rangle \) is not proximal, there exists some \( k \geq 1 \) such that \( c_x(k) \neq c_y(k) \). Let \( \delta = \frac{1}{2} \min \{d(J_{k,s}, J_{k,s+1}) : 0 \leq i < 2^k - 1\} > 0 \). Then \( \liminf_{n \to \infty} d(f^n(u), f^n(v)) \geq \delta \) for all \( u \in \text{Prox}_f(x), v \in \text{Prox}_f(y) \).

\[\square\]

**Remark 4.6.** (1) It has been shown that the proximal relation is an equivalence relation for a zero entropy interval map, but it may not be closed in \( I \times I \). For example, let \( f(x) = x^2 \), then \( \langle 0, x \rangle \) is proximal for all \( x \in [0, 1) \), but \( \langle 0, 1 \rangle \) is not proximal.
(2). Recall that a pair \( (x, y) \in I^2 \) is called regionally proximal, if for every \( \varepsilon > 0 \), there exist \( u, v \) and \( n \geq 1 \) such that \( |x - u| < \varepsilon, |y - v| < \varepsilon \) and \( |f^n(u) - f^n(v)| < \varepsilon \). Obviously, every proximal pair is regionally proximal. Let \( Q_f \) denote the set of regionally proximal pairs. It is not hard to verify that

\[
Q_f = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} (f \times f)^{-n} V_k,
\]

where \( V_k = \{ (x, y) \in I^2 : |x - y| < 1/k \} \). Then \( Q_f \) is a closed subset of \( I^2 \), but it may not be an equivalence relation. For example, let \( f(x) = 2x^2 \) on \([0, 1/2]\) and \( 2(x - 1/2)^2 + 1/2 \) on \([1/2, 1]\), then \( 0, 1/2 \) and \( 1/2, 1 \) are regionally proximal, but \( 0, 1 \) is not regionally proximal.

4.2. \( f \)-nonseparable pair.

**Definition 4.7.** Let \( f \in C(I, I) \) with \( h(f) = 0 \). A proper pair \( (u, v) \in I^2 \) is called \( f \)-nonseparable, if there exists some \( x \in I \) with infinite \( \omega_f(x) \) and let \( (J^I_k)_{k \geq 1, 0 < i < 2^k} \) be the periodic portion of \( \omega_f(x) \), such that there exists a nested sequence \( J^I_1 \supset J^I_2 \supset \cdots \) satisfying \( \bigcap_{k=1}^{\infty} J^I_k = [u, v] \) or \([v, u]\).

The definition of \( f \)-nonseparability first appeared in [24]. Our definition is slightly different from the origin one, but it is easy to see that they are equivalent.

**Lemma 4.8.** Let \( f \in C(I, I) \) with \( h(f) = 0 \) and \( u < v \in I \). If \( (u, v) \) is \( f \)-nonseparable, then

1. \([u, v]\) is wandering, i.e. \( f^n([u, v]) \cap [u, v] = \emptyset \) for all \( n \geq 1 \),
2. \( u, v \) are not the endpoints of \( I \), i.e. \( u, v \in (0, 1) \).

**Proof.** (1) By the structure of \( \omega \)-limit set (see Theorem 3.6), it is easy to see that \([u, v]\) is wandering.

(2) Let \( u, v \in \omega_f(x) \). Then \( f^n([u, v]) \cap \text{Per}(f) = \emptyset \) for all \( n \geq 0 \) since \([u, v] \in C(x)\). If \( u = 0 \), then \([u, v]\) is open in \( I \). By Lemma 3.9 there exists some \( m \geq 0 \) such that \( f^m([u, v]) \cap \text{Per}(f) \neq \emptyset \). This is a contradiction. Thus, \( u > 0 \) and one can show \( v < 1 \) similarly. \( \square \)

A point \( x \in I \) is called an eventuall period point, if there exists some \( n \geq 0 \) such that \( f^n(x) \) is a periodic point. Denote \( EP(f) \) by the set of eventually periodic points. Then \( EP(f) \supset \bigcup_{x \in I} \omega_f(x) \). Moreover, if \( u < v \in I \) and \( (u, v) \) is \( f \)-nonseparable, then \([u, v] \cap EP(f) = \emptyset \). But \([u, v] \subset EP(f) \), i.e. \( u \) is a limit point of \( EP(f) \) from the left, \( v \) is a limit point of \( EP(f) \) from the right. Therefore, for every \( x \in I \) there is at most one point such that they can form a \( f \)-nonseparable pair.

**Proposition 4.9.** Let \( f \in C(I, I) \) with \( h(f) = 0 \) and \( x \in I \). Suppose that \( \omega_f(x) \) is infinite and \( u, v \in \omega_f(x) \).

1. If \( (u, v) \) is \( f \)-nonseparable, then so is \((f(u), f(v))\).
2. There exists a unique pair \((y, z) \in \omega_f(x) \times \omega_f(x)\) such that \( f(y) = u, f(z) = v \). Moreover, if \( (u, v) \) is \( f \)-nonseparable, then so is \((y, z)\).

**Proof.** By Theorem 3.6 Theorem 3.7 and the definition of \( f \)-nonseparability. \( \square \)
Proposition 4.10. Let \( f \in C(I,I) \) with \( h(f) = 0 \) and \( u < v \in I \). Then the following conditions are equivalent:

1. \( \langle u, v \rangle \) is \( f \)-nonseparable;
2. there exists an \( \omega \)-limit set \( \omega_0 \) which is infinite such that \( \{u, v\} \in \omega_0 \) and \( (u, v) \cap \text{Per}(f) = \emptyset \).

Proof. By the definition of \( f \)-nonseparability and Theorem 3.6 (1) \( \Rightarrow \) (2) is obvious. It remains to show (2) \( \Rightarrow \) (1). Let \( (J_k^i)_{k \geq 1, 0 \leq i < 2^k} \) be the periodic portion of \( \omega_0 \). If \( \langle u, v \rangle \) is not \( f \)-nonseparable, then there exist some \( k \geq 1 \) and two different intervals \( J_k^{i_1} \) and \( J_k^{i_2} \) such that \( u \in J_k^{i_1}, v \in J_k^{i_2} \). Then by Lemma 3.8 there exists a periodic point \( p \) between \( J_k^{i_1} \) and \( J_k^{i_2} \). Therefore, \( p \in (u, v) \). This is a contradiction. \( \square \)

Proposition 4.11. Let \( f \in C(I,I) \) with \( h(f) = 0 \). Then the set of \( f \)-nonseparable pairs is either empty or countable.

Proof. Suppose that the set of \( f \)-nonseparable pairs is not empty. Let \( A = \{x \in I : \omega_f(x) \) is a maximal \( \omega \)-limit set and there exist \( u, v \in \omega_f(x) \) such that \( \langle u, v \rangle \) is \( f \)-nonseparable\}. Then \( A \) is at most countable since for every \( x \in A \), \( C(x) \) contains a non-degenerate interval and any two different \( C(x) \) and \( C(y) \) are pairwise disjoint. Then by Theorem 3.6, Theorem 3.7 and Proposition 4.10 for every \( x \in A \), \( \omega_f(x) \times \omega_f(x) \) contains exactly countable \( f \)-nonseparable pairs. \( \square \)

4.3. Chaos in the sense of Li-Yorke.

Lemma 4.12. Let \( f \in C(I,I) \) with \( h(f) = 0 \) and \( u < v \in I \). Suppose \( \langle u, v \rangle \) is \( f \)-nonseparable with respect to \( \omega_f(x) \), \( (J_k^i)_{k \geq 1, 0 \leq i < 2^k} \) and \( \bigcap_{k=1}^{\infty} J_k^i = [u, v] \). If \( A_1 \) and \( A_2 \) are neighborhoods of \( u \) and \( v \) respectively, then there exists some \( n \geq 1 \) such that \( [u, v] \) is in the interior of \( f^{2^n}(A_1) \cap f^{2^n}(A_2) \).

Proof. By Lemma 3.9 there exists some \( n_j \geq 0 \) such that \( f^{n_j}(A_j) \) contains periodic points \( y_j \) for \( j = 1, 2 \). The periods of \( y_1, y_2 \) are some powers of \( 2 \). Let \( 2^q \) be a common multiple of their periods and let \( q > p \) such that \( 2^q > \max\{n_1, n_2\} \). Let \( z_j = f^{2^q-n_j}(y_j) \) for \( j = 1, 2 \). Then \( f^{2^q}(z_j) = z_j \in f^{2^q-n_j}(f^{n_j}(A_j)) = f^{2^q}(A_j) \) for \( j = 1, 2 \). Moreover, \( z_j \notin f^{k_j} \) since the period of \( z_j \) is less than \( 2^q \). Suppose for instance that \( z_1 \) is in the left of \( J_q^{i_1} \), the case in the right being symmetric.

Let \( g = f^{2^q} \). There exists \( i_{q_0} \geq 0 \) such that \( J_{q_1}^{i_{q_0}} \cup J_{q_1}^{i_{q_0}+1} \subset J_{q_1}^{i_{q_0}} \), \( g(J_{q_1}^{i_{q_0}+1}) \supset J_{q_1}^{i_{q_0}} \) and \( g(J_{q_1}^{i_{q_0}}) \supset J_{q_1}^{i_{q_0}+1} \). Take a fixed point \( c \) for \( g \) between \( J_{q_1}^{i_{q_0}} \) and \( J_{q_1}^{i_{q_0}+1} \).

Case 1. \( J_{q_1}^{i_{q_0}+1} \) is in the left of \( J_{q_1}^{i_{q_0}} \), then by connectedness \( g(A_1) \supset J_{q_1}^{i_{q_0}+1} \) since \( g(u) \in J_{q_2}^{i_{q_0}} \) and \( z_1 \in g(A_1) \). Hence, \( g^2(A_1) \supset J_{q_1}^{i_{q_0}} \cup \{z_1, c\} \) and by the connectedness \( g^2(A_1) \supset [z_1, c] \supset J_{q_1}^{i_{q_0}+1} \).

Case 2. \( J_{q_1}^{i_{q_0}+1} \) is in the right of \( J_{q_1}^{i_{q_0}}, \) then by the connectedness \( g^2(A_1) \supset J_{q_1}^{i_{q_0}} \cup \{z_1, c\} \) since \( z_1 \in g^2(A_1) \) and \( g^2(u) \in J_{q_2}^{i_{q_0}+1} \). Then \( g^3(A_1) \supset J_{q_1}^{i_{q_0}+1} \cup \{z_1, c\} \) and by the connectedness \( g^3(A_1) \supset [z_1, c] \). Therefore, \( [u, v] \) are in the interior of \( g^4(A_1) \), since there exists a point \( h \in J_{q_1}^{i_{q_0}} \cap \omega_f(x) \) such that \( g(h) = v \) and a neighborhood \( D \) of \( h \) with \( g(D) \subset A_2 \), then \( D \) contains an eventually periodic point \( e \) and \( g(e) \) must be in the right of \( v \).
As a conclusion, we get \([u,v]\) is in the interior of \(g^4(A_1)\). Similarly, we have \([u,v]\) is in the interior of \(g^4(A_2)\). Let \(n = q + 2\), then \([u,v]\) is in the interior of \(f^{2n}(A_1) \cap f^{2n}(A_2)\).

By Lemma 4.12 and induction, we have the following result.

**Proposition 4.13.** Let \(f \in C(I, I)\) with \(h(f) = 0\) and \(u,v \in I\). If \(\langle u,v \rangle\) is \(f\)-nonseparable, then there exist two sequences of closed intervals \(\{U_n\}\) and \(\{V_n\}\) which are neighborhoods of \(u\) and \(v\) respectively, and a sequence of positive number \(\{k_n\}\) such that

1. \(U_n \supset U_{n+1}, \lim_{n \to \infty} \text{diam}(U_n) = 0\),
2. \(V_n \supset V_{n+1}, \lim_{n \to \infty} \text{diam}(V_n) = 0\),
3. \(f^{2k_n}(U_n) \cap f^{2k_n}(V_n) \supset U_{n+1} \cup V_{n+1}\).

**Theorem 4.14.** Let \(f \in C(I, I)\) with \(h(f) = 0\), then the following conditions are equivalent:

1. \(f\) is Li-Yorke chaotic;
2. there exists a scrambled pair;
3. there exists a \(\delta\)-scrambled Cantor set for some \(\delta > 0\);
4. there exists an \(f\)-nonseparable pair.

**Proof.** (3)\(\Rightarrow\)(1)\(\Rightarrow\)(2) is trivial.

(2)\(\Rightarrow\)(4). Let \(\langle x, y \rangle \in I^2\) be a scrambled pair. Then \(\omega_f(x)\) and \(\omega_f(y)\) are contained in the same maximal \(\omega\)-limit set \(\omega_0\) which is infinite. Let \((J_{k,s})\) be the periodic portion of \(\omega_0\). Since \(\lim_{n \to \infty} |f^n(x) - f^n(y)| > 0\), there exists a sequence \(\{n_i\}\) such that \(\lim_{i \to \infty} f^{n_i}(x) = a\) and \(\lim_{i \to \infty} f^{n_i}(y) = b\) for \(a \neq b\). Then \(\langle a, b \rangle\) is \(f\)-nonseparable. In fact, if not, then there exist \(k \geq 1\) and \(i_0 \neq i_1\) such that \(a \in J_{k,i_0}\) and \(b \in J_{k,i_1}\). Thus, there are infinitely many \(n_i\) such that \(f^{n_i}(x) \in J_{k,i_0}\) and \(f^{n_i}(y) \in J_{k,i_1}\).

This is a contradiction since \(\langle x, y \rangle\) is proximal.

(4)\(\Rightarrow\)(3). Let \(\langle u,v \rangle \in I^2\) be an \(f\)-nonseparable pair and \(\delta = |v-u|\). Let \(\{U_n\}\), \(\{V_n\}\) as in Proposition 4.13. Let \(t_0 = 0\) and \(t_m = \sum_{n=1}^{m} 2k_n\) for \(m \geq 1\).

First we build a family of closed subintervals \(\{E_{a_0a_1,...,a_m} : m \geq 0, a_i \in \{0,1\}\}\) satisfying the following properties:

- (a) \(E_{a_0a_1,...,a_ma_{m+1}} \subset E_{a_0a_1,...,a_m}\),
- (b) \(E_{a_0a_1,...,a_m} \cap E_{b_0b_1,...,b_m} = \emptyset\) if \(a_0a_1, \ldots, a_m \neq b_0b_1, \ldots, b_m\),
- (c) for \(\alpha = a_0a_1, \ldots, a_m\), \(f^{t_m}(E_\alpha) \subset W_i\) for \(i = 0, 1, \ldots, m-1\) and \(f^{t_m}(E_\alpha) = W_m\) where \(W_i = U_i\) if \(a_i = 0\) and \(W_i = V_i\) if \(a_i = 1\).

Let \(E_0 = U_1, E_1 = V_1\). Suppose that \(E_{a_0a_1,...,a_m}, a_i \in \{0,1\}\) are already defined. For \(a_0a_2, \ldots, a_ma_{m+1}\), we have \(E_{a_0a_1,...,a_m} = \overrightarrow{f^{t_m}} W_m \overrightarrow{f^{2k_{m+1}}} W_{m+1}\), where notation \(A \overrightarrow{\rightarrow} B\) means \(g(A) \supset B\). Let \(F\) be a subinterval of \(W_m\) of minimal length such that \(f^{2k_{m+1}}(F) = W_{m+1}\) and \(E_{a_0a_1,...,a_ma_{m+1}}\) be a subinterval of \(E_{a_0a_1,...,a_m}\) of minimal length such that \(f^{t_m}(E_{a_0a_1,...,a_ma_{m+1}}) = F\). Then it is easy to verify that \(E_{a_0a_1,...,a_ma_{m+1}}\) satisfies the requirement.

With \(Z_+ = \{0,1,2,\ldots\}\) let \(\Sigma\) be the Cantor space \(\{0,1\}^{\mathbb{Z}_+}\) regard as the set of infinite words. For every \(\alpha = (a_0a_1, a_2, \ldots) \in \Sigma\), let \(E_\alpha = \bigcap_{m=0}^{\infty} E_{a_0a_1,...,a_m}\). Then \(E_\alpha\) is either a nonempty compact interval or a single point. Moreover \(E_\alpha \cap E_\beta = \emptyset\) if \(\alpha \neq \beta \in \Sigma\).
Let $\Lambda = \{ \alpha \in \Sigma : E_\alpha \text{ is not reduced to a single point}\}$. The set $\Lambda$ is at most countable because the sets $(E_\alpha)_{\alpha \in \Lambda}$ are pairwise disjoint and nondegenerate intervals. Let

$$X = \bigcup_{\alpha \in \Sigma} E_\alpha \setminus \bigcup_{\beta \in \Lambda} \text{int}(E_\beta).$$

It is easy to see that $X$ is a totally disconnected compact set. Define $\phi : X \to \Sigma$ by $\phi(x) = \alpha$ if $x \in E_\alpha$. Clearly, the map $\phi$ is well defined, continuous and onto.

Fix $\gamma = (c_0c_1 \ldots) = (010101 \ldots)$. Define $\psi : \Sigma \to \Sigma$ by

$$\psi((a_n)_{n \in \mathbb{Z}_+}) = (a_0c_0a_0c_1 \ldots a_0a_1 \ldots a_n c_0c_1 \ldots c_n \ldots).$$

The map $\psi$ is clearly continuous, thus $\psi(X)$ is compact.

For every $\alpha \in \Sigma$, choose $x_\alpha \in X$ such that $\phi(x_\alpha) = \psi(\alpha)$ and let $S = \{ x_\alpha \in X : \alpha \in \Sigma \}$. If $\psi(\alpha) \notin \Lambda$ then there is a unique choice for $x_\alpha$ and if $\psi(\alpha) \in \Lambda$ then there are two possible choices. Consequently, $S$ is equal to $\phi^{-1}(\psi(X))$ deprived of a countable set. Let $\alpha, \beta$ be two distinct elements of $\Sigma$. By the definition of $\psi$, for every $N \geq 0$ there exists some $m \geq N$ such that the $m$-th coordinates of $\psi(\alpha)$ and $\psi(\beta)$ are distinct. Then either $f^m(x_\alpha) \in U_{\psi(\alpha)_m}$ and $f^m(x_\beta) \in V_{\psi(\beta)_m}$, or the converse. Thus,

$$\limsup_{n \to \infty} |f^n(x_\alpha) - f^n(x_\beta)| \geq \delta.$$

According to the choice of $\gamma$ and the definition of $\psi$, for every $N \geq 0$ there exists some $m \geq N$ such that for every $\alpha \in \Sigma$ the $m$-th coordinate of $\psi(\alpha)$ is 0, which implies $f^m(x_\alpha) \in U_{\psi(\alpha)_m}$. Thus,

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$$

for all $x, y \in S$ since $\lim_{n \to \infty} \text{diam}(U_n) = \lim_{n \to \infty} \text{diam}(V_n) = 0$. By the choice of $\gamma$, it is also easy to see that $\{u, v\} \in \omega_f(x)$ for all $x \in S$.

Since $X$ is totally disconnected and $\phi^{-1}(\psi(X))$ is an uncountable closed subset of $X$, then $S$ contains some Cantor subset $K$. Therefore, $K$ is a $\delta$-scrambled Cantor set.

\begin{remark}
It should be noticed that in the above Theorem (3) $\iff$ (4) was proved in [24] and (2) $\iff$ (3) was proved in [16], but here we give a new proof.
\end{remark}

\begin{corollary}
Let $f \in C(I, I)$ with $h(f) = 0$. If $\langle u, v \rangle$ is $f$-nonseparable, then $\langle u, v \rangle$ is an IT-pair.
\end{corollary}

\begin{proof}
Let $\{U_n\}$, $\{V_n\}$ and $\{k_n\}$ as in Proposition 4.13. Let $t_n = \sum_{j=1}^n k_n$. If $U, V$ are neighborhood of $u, v$ respectively, then there exists some $N \geq 1$ such that $U_n \subset U$ and $V_n \subset V$ for all $n \geq N$.

For every $s \in \{0, 1\}^{\{0, 1, \ldots, k\}}$, by the proof of Theorem 4.14 there exists a $w_s$ such that for $0 \leq i \leq k$, $f^{iN+i}(w_s) \in U_{N+i+1}$ if $s(i) = 0$ and $f^{iN+i}(w_s) \in V_{N+i+1}$ if $s(i) = 1$, then

$$w_s \in \bigcap_{i=0}^k f^{-iN+i} A_{s(i)}$$

where $A_0 = U$, $A_1 = V$.

Thus, $\{t_N, t_{N+1}, \ldots\}$ is an infinite independence set of $(U, V)$. Therefore, $\langle u, v \rangle$ is an IT-pair and $(X, T)$ is not tame since $\langle u, v \rangle$ is not in the diagonal.
\end{proof}
Definition 4.17. Let \((X,T)\) be a TDS and \(S,R \subset X\).

(1) \(S\) and \(R\) are called equivalent if there exists a bijection \(\phi : S \rightarrow R\) such that
\[
\lim_{n \to \infty} d(f^n(\phi(x)), f^n(x)) = 0 \quad \text{for any } x \in S.
\]

(2) \(S\) and \(R\) are called separable if there exists \(\delta > 0\) such that
\[
\liminf_{n \to \infty} d(f^n(x), f^n(y)) \geq \delta \quad \text{for any } x \in S \text{ and } y \in R.
\]

Proposition 4.18. Let \(f \in C(I,I)\) with \(h(f) = 0\). Then every two maximal scrambled sets are either equivalent or separable.

Proof. Let \(S\) and \(R\) be two maximal scrambled sets. If \(S\) and \(R\) are contained in different proximal cells, then by Proposition 4.5 there exists \(\delta > 0\) such that
\[
\liminf_{n \to \infty} |f^n(x) - f^n(y)| \geq \delta \quad \text{for any } x \in S \text{ and } y \in R.
\]
Now assume that \(S\) and \(R\) are contained in the same proximal cell \(\text{Prox}_f(z)\). We define a relationship \(\sim\) in \(\text{Prox}_f(z)\). Let \(a, b \in \text{Prox}_f(z)\), \(a \sim b\) iff \((a,b)\) is asymptotic. Then \(\sim\) is an equivalence relation on \(\text{Prox}_f(z)\). It is easy to see that in \(\text{Prox}_f(z)\) every maximal scrambled set contains exactly one representative point for every one of those \(\sim\) equivalent classes. Thus, there exists a bijection \(\phi : S \rightarrow R\) such that
\[
\lim_{n \to \infty} |f^n(x) - f^n(\phi(x))| = 0 \quad \text{for any } x \in S.
\]

\(\square\)

Remark 4.19. The Proposition 4.18 was proved by Balibrea and Lopez in [1]. It seems that they used some lemmas which are only available for piecewise monotone maps.

Question: If \(h(f) = 0\), is every maximal scrambled set uncountable?

Definition 4.20. Let \((X,T)\) be a TDS and \(n \geq 2\). A tuple \(\langle x_1, x_2, \ldots, x_n \rangle \in X^n\) is called \(n\)-scrambled if
\[
\liminf_{k \to \infty} \max_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) = 0, \quad \limsup_{k \to \infty} \min_{1 \leq i < j \leq n} d(f^k(x_i), f^k(x_j)) > 0.
\]

The system \((X,T)\) (or the map \(T\)) is called \(n\)-chaotic in the sense of Li-Yorke if there exists an uncountable subset \(S \subset X\) such that every essential tuple \(\langle x_1, x_2, \ldots, x_n \rangle \in S^n\) is \(n\)-scrambled.

Theorem 4.21. Let \(f \in C(I,I)\) with \(h(f) = 0\). Then there is no 3-scrambled tuple.

Proof. If \(\langle x_1, x_2, x_3 \rangle \in f^3\) is a 3-scrambled tuple, then there exists a sequence \(\{n_q\}\) in \(\mathbb{N}\), such that \(\lim_{q \to \infty} f^{n_q}(x_i) = a_i\) for \(i = 1, 2, 3\) and \(a_1, a_2\) and \(a_3\) are pairwise distinct. Let \(\omega_0\) be the maximal \(\omega\)-limit set contains \(\omega_f(x_i)\) for \(i = 1, 2, 3\), then \(\omega_0\) is infinite. Let \((J_{k,q}^i)_{k \geq 0, 1 \leq i \leq 2^k}\) be the periodic portion of \(\omega_0\). Since \(\langle x_1, x_2 \rangle\) and \(\langle x_1, x_3 \rangle\) are proximal, by Proposition 4.2 for every \(k \geq 1, a_1, a_2\) and \(a_3\) must be in the same \(J_{k,a}^i\). Thus, \(\{a_1, a_2, a_3\} \subset \bigcap_{k=1}^{\infty} J_{k,1}^{r_k}\), which implies \(\#(\bigcap_{k=1}^{\infty} J_{k,1}^{r_k} \cap \omega_0) \geq 3\). This contradicts the structure of \(\omega\)-limit set (see Theorem 3.6).

\(\square\)

Corollary 4.22. There exists a TDS which is 2-chaotic in the sense of Li-Yorke but does not have any 3-scrambled tuple.

Proof. Any interval map, which is chaotic in the sense of Li-Yorke and has zero entropy, satisfies the requirement.

\(\square\)
4.4. **Sequence entropy pair.** In [8], Franzova and Smítal showed that positive sequence topological entropy can characterize Li-Yorke chaos:

**Theorem 4.23.** Let \( f \in C(I, I) \). Then \( f \) is Li-Yorke chaotic iff it is not null.

**Theorem 4.24.** [10] Let \( f \in C(I, I) \). Then \( f \) is null iff it is tame.

The structure of the set of IN-pairs (or sequence entropy pairs) was studied by Tan, Ye and Zhang in [25].

**Theorem 4.25.** [25] Let \( f \in C(I, I) \) with \( h(f) = 0 \). If \( f \) is not null, then there exist exactly countable IN-pairs, but no essential 3-IN-tuple.

**Theorem 4.26.** [25] Let \( f \in C(I, I) \) with \( h(f) = 0 \). If \( \langle x, y \rangle \) is an IN-pair and \( x < y \), then

1. both \( \omega_f(x) \) and \( \omega_f(y) \) are infinite,
2. \( [x, y] \) is wandering, i.e. \( f^n([x, y]) \cap [x, y] = \emptyset \) for all \( n \geq 1 \),
3. \( \langle x, y \rangle \) is asymptotic,
4. \( [x, y] \cap E\text{Per}(f) = \emptyset \), but \( \{x, y\} \cap \overline{\text{Per}(f)} \neq \emptyset \) and \( \{x, y\} \subset E\text{Per}(f) \).

**Theorem 4.27.** Let \( f \in C(I, I) \) with \( h(f) = 0 \) and \( x < y \in I \). Then the following conditions are equivalent:

1. \( \langle x, y \rangle \) is \( f \)-nonseparable;
2. \( \langle x, y \rangle \) is an IT-pair;
3. \( \langle x, y \rangle \) is an IN-pair.

**Proof.** (1)\(\Rightarrow\)(2) is proved in Corollary 4.16 and (2)\(\Rightarrow\)(3) is trivial.

It remains to show (3)\(\Rightarrow\)(1). Assume that \( \langle x, y \rangle \) is an IN-pair. Without loss of generality, assume that \( x \) is a limit point of \( \text{Per}(f) \) from the left. Let \( U_1 \) and \( U_2 \) be two disjoint connected neighborhoods of \( x \) and \( y \) respectively. Then there are periodic points \( p, q \) and \( n \geq 1 \) such that \( p \in U_1 \) and \( q \in f^n(U_2) \). Without loss of generality, assume that \( p \) and \( q \) are fixed points and \( n = 1 \), since the periods of \( p, q \) are the powers of 2 and \( \langle x, y \rangle \) is also an IN-pair for \( f^n \) for every \( n \geq 1 \).

**Claim:** There exists \( n \geq 1 \) such that the subinterval \( [x, y] \) is in the interior of \( f^n(U_1) \cap f^n(U_2) \).

**Proof of the Claim:** Clearly, we have \( p < x \), but there are two cases about the position of \( q \).

Case 1, \( q > y \). (a) \( f(x) > y, f(y) > y \). Since \( q \) is a fixed point and \( f(x) > y \), then by the connectedness \([x, y]\) is in the interior of \( f^k(U_1) \) for all \( k > 1 \). Since \( \langle x, y \rangle \) is an IN-pair, there exist \( z \in U_2 \) and \( 1 < n_1 < n_2 \) such that \( f^{n_1}(z) \in U_1 \) and \( f^{n_2}(z) \in U_1 \). Then one of \( f^{n_1}(z), f^{n_2}(z) \) must be on the left side of \( x \) since \([x, y]\) is wandering. Since \( q > y \), \( q \) is a fixed point and \( q \in f(U_2) \), by the connectedness we have \([x, y]\) is in the interior of \( f^{n_1}(U_2) \) or \( f^{n_2}(U_2) \). (b) \( f(x) < x, f(y) < x \). This is the symmetric case of (a).

Case 2, \( q < x \). (a) There exists some \( k \geq 1 \) such that \( f^k(x) > y, f^k(y) > y \). Without loss of generality, we can assume \( k = 1 \). Since \( q \) is a fixed point and \( f(x) > y \), then by the connectedness \([x, y]\) is in the interior of \( f(U_1) \). Since \( p \in f(U_2) \) and \( f(y) > y \), then by the connectedness \([x, y]\) is in the interior of \( f(U_2) \). (b) We have \( f^k(x) < x, f^k(y) < x \) for all \( k \geq 1 \). Then \( f^k([x, y]) \) is in the left of \([x, y]\) for all \( k \geq 1 \). Thus if \( z \in U_2 \) and \( f^n(z) \in U_2 \) for some \( n \geq 1 \), then \( z > y \). Since \( \langle x, y \rangle \) is an IN-pair,
there exist \( u \in U_1, v \in U_2 \) and \( 1 < n_1 < n_2 \) such that \( f^{n_1}(u) \in U_2, f^{n_2}(u) \in U_2 \) and \( f^{n_1}(v) \in U_2, f^{n_2}(v) \in U_2 \). Then \( f^{n_1}(u) \succ y \) and \( f^{n_1}(v) \succ y \). Thus \([x,y]\) is in the interior of \( f^{n_1}(U_1) \) and \( f^{n_1}(U_2) \). This completes the proof of the Claim.

Similarly to the usage of Lemma 4.12 to prove Theorem 4.14, we can get that there exists some \( z \in I \) such that \( x,y \in \omega_f(z) \). Thus, by Proposition 4.10 \( \langle x,y \rangle \) is \( f \)-nonseparable.

5. Positive entropy Maps

**Theorem 5.1.** Let \( f \in C(I,I) \). Then the following conditions are equivalent:

1. \( h(f) > 0 \);
2. there exists a subsystem which is Devaney chaotic [17];
3. there exists \( n \geq 1 \) such that \( f^n \) has a strongly mixing subsystem [28];
4. \( f \) is distributionally chaotic [19].

**Definition 5.2.** [27] Let \((X,T)\) be a TDS and \( n \geq 2 \). \((X,T)\) is said to be \( n \)-sensitive, if there exists some \( \delta > 0 \) such that for every nonempty open subset \( U \subset X \), there exist \( n \) distinct points \( x_1,x_2,\ldots,x_n \in U \) and \( k \geq 1 \) satisfying \( \min_{1 \leq i < j \leq n}\{d(f^k(x_i), f^k(x_j))\} \geq \delta \).

**Theorem 5.3.** Let \((X,T)\) be a TDS. If \((X,T)\) is Devaney chaotic, then

1. \((X,T)\) is infinite sensitive (i.e. \( n \)-sensitive for all \( n \geq 2 \)) [29];
2. \((X,T)\) is infinite chaotic in the sense of Li-Yorke (i.e. there is an uncountable subset \( S \) of \( X \) which is \( n \)-srambled for all \( n \geq 2 \)) [27].

**Lemma 5.4.** Let \((X,T)\) be a TDS. If \((X,T)\) is transitive and has two periodic points, then

1. \( \text{Prox}_T \) is not an equivalence relation,
2. there are two maximal scrambled sets which are neither equivalent nor separable.

**Proof.** (1) Without loss of generality, we assume that there are two fixed points \( p_1 \) and \( p_2 \), since for every \( n \geq 1 \) \( \langle x_1,x_2 \rangle \in X^2 \) is proximal for \( T \) iff so is for \( T^n \). Then it is easy to see that for every transitive point \( x \in X \), \( \langle x,p_1 \rangle \) and \( \langle x,p_2 \rangle \) are proximal, but \( \langle p_1,p_2 \rangle \) can not be proximal.

(2) Let \( S \) and \( R \) be the maximal scrambled sets which contain \( \{x,p_1\} \) and \( \{x,p_2\} \) respectively. Clearly, \( S \) and \( R \) are not separable. Next, we show that \( S \) and \( R \) also are not equivalent. If there exists a bijection \( \phi : S \rightarrow R \) such that \[ \lim_{n \to \infty} d(f^n(y) - f^n(\phi(y))) = 0 \text{ for any } y \in S, \]
then \( \langle p_1,\phi(p_1) \rangle \) is asymptotic. Thus, \( \langle p_1,p_2 \rangle \) is proximal since \( \langle p_2,\phi(p_1) \rangle \) is proximal. This is a contradiction. \( \Box \)

Now we state the main result of this paper: there are various equivalent conditions of positive entropy which may relate to the number “3”.

**Theorem 5.5.** Let \( f \in C(I,I) \). Then the following conditions are equivalent:

1. \( h(f) > 0 \);
2. \( \text{Prox}_f \) is not an equivalence relation;
(3) there exist two maximal scrambled sets which are neither equivalent nor separable;

(4) there exists some 3-scrambled tuple;

(5) there exists some 3-sensitive transitive subsystem;

(6) there exists some essential 3-IN-tuple.

Proof. (1)⇒(2) By Theorem 5.1(2) and Lemma 5.4(1).

(2)⇒(1) By Theorem 4.4.

(1)⇒(3) By Theorem 5.1(2) and Lemma 5.4(2).

(3)⇒(1) By Theorem 4.18.

(4)⇒(1) By Theorem 5.1(2) and Theorem 5.3(2).

(4)⇒(1) By Theorem 4.21.

(5)⇒(1) By Theorem 5.1(2) and Theorem 5.3(1).

(5)⇒(1) If $h(f) = 0$, by Theorem 3.7, any infinite transitive subsystem is at most 2 to 1 extension of the adding machine, then it is not 3-sensitive [22].

(1)⇒(6) In [12], it shows that if $h(f) > 0$, then there exists some essential 3-IN-tuple.

(6)⇒(1) By Theorem 4.25. □

Theorem 5.6. [3] Let $f \in C(I, I)$. If $f$ is transitive, then exactly one of the following alternatives holds:

(1) $f$ is strongly mixing,

(2) there exists a fixed point $c \in (0, 1)$ such that $f([0, c]) = [c, 1]$, $f([c, 1]) = [0, c]$ and both $f^2|[0, c]$ and $f^2|[c, 1]$ are strongly mixing.

Theorem 5.7. [3] Let $f \in C(I, I)$. Then the following conditions are equivalent:

(1) $f^2$ is transitive;

(2) $f$ is weakly mixing;

(3) $f$ is strongly mixing;

(4) for any $\varepsilon > 0$ and non degenerate subinterval $J \subset I$, there exists a $N > 0$ such that

$$f^n(J) \supset [\varepsilon, 1 - \varepsilon]$$

for all $n \geq N$.

Theorem 5.8. Let $f \in C(I, I)$. Then the following conditions are equivalent:

(1) $f$ is strongly mixing;

(2) $\mathcal{C}(\mathcal{U}) > 2$ for every open cover $\mathcal{U}$ which consists of two non dense open sets, where $\mathcal{C}(\mathcal{U}) = \lim_{n \to \infty} N(\bigvee_{i=1}^{n-1} f^{-i}(\mathcal{U}))$;

(3) $f$ is uniformly positive entropy;

(4) $f$ is topological K system.

Proof. (4)⇒(3) and (3)⇒(2) is trivial.

(2)⇒(1). We first prove a claim which implies that $f$ is transitive.

Claim: Let $(X,T)$ be a TDS. If $\mathcal{C}(\mathcal{U}) > 2$ for every open cover $\mathcal{U}$ of $X$ which consists of two non dense open sets, then $T$ is transitive.

Proof of the Claim: We follow the idea in [6]. If not, there exist two nonempty open subsets $U, V$ of $X$ such that $T^n U \cap V = \emptyset$ for all $n \geq 1$.

Case 1. If there is some $n \geq 1$ such that $T^n V \cap U \neq \emptyset$, let $m$ be the minimum of such values. Let $E = V \cap T^{-m} U$, then $E \cap T^{-n} E = \emptyset$ for all $n \geq 1$. Choose two
closed subsets $U_1$ and $V_1$ of $E$ and $T^{-1}E$ respectively which have non-empty interior. Let $\mathcal{U} = \{U_1^i, V_1^i\}$, it is easy to verify that $\mathcal{C}(\mathcal{U}) = 2$. This is a contraction.

Case 2. $T^nV \cap U = \emptyset$ for all $n \geq 1$. Choose two closed subsets $U_1$ and $V_1$ of $U$ and $V$ respectively which have non-empty interior. Let $\mathcal{U} = \{U_1^i, V_1^i\}$, it is easy to verify that $\mathcal{C}(\mathcal{U}) = 2$. This also is a contraction.

Thus, the proof of the Claim is complete.

So $f$ is transitive. Now assume that $f$ is not strongly mixing, then there exists a fixed point $c \in (0, 1)$ such that $f([0, c]) = [c, 1]$, $f([c, 1]) = [0, c]$. There exists $\varepsilon > 0$ such that $c + \varepsilon, c - \varepsilon \in (0, 1)$. Let $\mathcal{U} = \{[0, c + \varepsilon), (c - \varepsilon, 1]\}$. It is easy to verify that $\mathcal{C}(\mathcal{U}) = 2$. This also is a contradiction.

(1) $\Rightarrow$ (4) To show that $f$ is topological $K$, it is sufficient to show that every $k$-tuple of non-empty open subsets $(U_1, \ldots, U_k)$ has an independence set of positive density.

Since $f$ is strongly mixing, by Theorem 5.7(4), there exist some $n \geq 1$ and non-empty open subset $V_i \subset U_i$ for $1 \leq i \leq k$ such that $\bigcap_{i=1}^{n} f^nV_i \supset \bigcup_{i=1}^{k} V_i$. Then $n\mathbb{N} = \{n, 2n, 3n, \ldots\}$ is an independence set for $(U_1, \ldots, U_k)$ since

$$f^{-n}V_{s(1)} \cap f^{-2n}V_{s(2)} \cap \cdots \cap f^{-mn}V_{s(m)} \neq \emptyset$$

holds for all $m \geq 1$ and all $s \in \{1, 2, \ldots k\}^m$. $\square$

**Remark 5.9.** Recall a TDS $(X, T)$ is called to be of completely positive entropy if each of its non-trivial factors has positive entropy. In [4], it showed that there exists a TDS which is of completely positive entropy but not of uniformly positive entropy. There are also some examples for interval maps. For example, let $f(x) = 1/2 + 2x$ on $[0, 1/4]$, $3/2 - 2x$ on $[1/4, 1/2]$ and $1 - x$ on $[1/2, 1]$. It is easy to check that $f$ is of completely positive entropy but not of uniformly positive entropy. Moreover, $f^2$ is also of completely positive entropy but not transitive.

### 6. Topological null system

In [13], Huang and Ye introduced the notion of maximal pattern entropy. For a TDS $(X, T)$, $n \in \mathbb{N}$ and a finite open cover $\mathcal{U}$, let

$$p^*_{X, \mathcal{U}}(n) = \max_{(t_1 < t_2 < \cdots < t_n) \in \mathbb{Z}_+^n} N\left(\bigvee_{i=1}^{n} T^{-t_i} \mathcal{U}\right).$$

The maximal pattern entropy of $T$ with respect to $\mathcal{U}$ is defined by

$$h^*_{\text{top}}(T, \mathcal{U}) = \lim_{n \to \infty} \frac{1}{n} \log p^*_{X, \mathcal{U}}(n).$$

The maximal pattern entropy of $(X, T)$ is

$$h^*_{\text{top}}(T) = \sup_{\mathcal{U} \in \mathcal{C}^X} h^*_{\text{top}}(T, \mathcal{U}).$$

Then a TDS $(X, T)$ is null iff $h^*_{\text{top}}(T) = 0$, since $h^*_{\text{top}}(T) = \sup_{A} h_A(T)$.

In [13], Huang and Ye proved that for a null TDS defined on a zero dimensional space, $p^*_{X, \mathcal{U}}(n)$ is of polynomial order for each open cover $\mathcal{U}$ of $X$. They also

**Conjecture:** If a TDS $(X, T)$ is null, then it is true that $p^*_{X, \mathcal{U}}(n)$ is of polynomial order for each open cover $\mathcal{U}$ of $X$.

In the following section, we prove that the conjecture holds for interval maps. Before doing this, we need some lemmas. Let $\omega(f) = \bigcup \{\omega_f(x) : x \in I\}$. 
Lemma 6.1. [8] Let $f \in C(I, I)$. If $f$ is null, then for every $\varepsilon > 0$ there are points $x_1, x_2, \ldots, x_k \in \omega(f)$ and an open set $U \supset \omega(f)$ with the following property: if

$$f^j(x) \in U \quad \text{for} \quad 0 \leq j \leq r,$$

then there exists some $i \in \{1, 2, \ldots, k\}$ such that for any $j$ with $0 \leq j \leq r$,

$$|f^j(x) - f^j(x_i)| < \varepsilon.$$

Lemma 6.2. ([23] or [3] Corollary IV.13) Let $f \in C(I, I)$. Then for any neighborhood $U$ of $\omega(f)$ there is an integer $q > 0$ such that the number of points of an arbitrary trajectory lying outside $U$ is less than $q$.

Theorem 6.3. Let $f \in C(I, I)$. If $f$ is null, then $p_{i, U}^\star(n)$ is of polynomial order for each open cover $U$ of $I$.

Proof. We follow the idea in [8]. Let $U$ be an open cover of $I$ with Lebesgue number $\delta$ and $n \in \mathbb{N}$. For any $\bar{t} = (t_1 < t_2 < \cdots < t_n) \in \mathbb{Z}_+^n$, it is well known that

$$N\left(\bigvee_{i=1}^n T^{-t_i}U\right) \leq S\left(\bar{t}, f, \frac{\delta}{2}\right),$$

where $S\left(\bar{t}, f, \frac{\delta}{2}\right)$ is the minimal cardinality of $(\bar{t}, f, \frac{\delta}{2})$-spanning sets. Recall that a set $E \subset I$ is called a $(\bar{t}, f, \varepsilon)$-spanning set, if for any $x \in I$, there exists some $y \in E$ such that $|f^{t_i}(x) - f^{t_i}(y)| < \varepsilon$ for $1 \leq i \leq n$. Let $\varepsilon = \frac{\delta}{4}$ and $U$ and $x_1, \ldots, x_k$ be as in Lemma 6.1. Let $\{K_i\}_{i=1}^k$ be pairwise disjoint set with $\text{diam}(K_i) < \varepsilon$ for any $i$, and $K_1 \cup \cdots \cup K_k = I \setminus U$. Assign to any $x \in I$ an itinerary $\alpha_{\bar{t}}(x) = \{\alpha_{t_i}(x)\}_{i=1}^n$ such that $\alpha_{t_i}(x) = K_j$ if $f^{t_i}(x) \in K_j$. If $f^{t_i}(x) \in U$, let $M(t_i)$ be the maximal subinterval of the set of nonnegative integers such that $t_i \in M(t_i)$ and $f^{t_i}(x) \in U$, for all $k \in M(t_i)$. Then by Lemma 6.1 there exists some $r \in \{1, \ldots, k\}$ such that $|f^{t_i}(x) - f^{t_i}(x_r)| < \varepsilon < \frac{\delta}{2}$ for any $k \in M(t_i)$. Put $\alpha_{t_i}(x) = r$ for any $i \in M(t_i)$.

It is easy to see that for any $x, y \in I$, $\alpha_{\bar{t}}(x) = \alpha_{\bar{t}}(y)$ implies $|f^{t_i}(x) - f^{t_i}(y)| < 2\varepsilon = \frac{\delta}{2}$ for all $1 \leq i \leq n$. So we get

$$S\left(\bar{t}, f, \frac{\delta}{2}\right) \leq C\left(\bar{t}\right),$$

where $C\left(\bar{t}\right)$ is the number of all the possible codes $\alpha_{\bar{t}}(x)$. By Lemma 6.2, there exists an integer $q > 0$ such that the number of points of an arbitrary trajectory lying outside $U$ is less than $q$. Consequently, every code $\alpha_{\bar{t}}(x)$ consists of at most $2q + 1$ blocks, and each block is formed by only one of the symbols $1, \ldots, k, K_1, \ldots, K_s$ (with possible repetitions). Therefore,

$$N\left(\bigvee_{i=1}^n T^{-t_i}U\right) \leq C\left(\bar{t}\right) \leq (k + s)^{2q + 1}n^q.$$

We remark that the choice of $k, s$ and $q$ depend only on $U$ but not $\bar{t}$. Thus,

$$p_{i, U}^\star(n) \leq (k + s)^{2q + 1}n^q$$

for all $n \geq 1$. \qed
Remark 6.4. Recall that a space $X$ is called a tree if it is a connected space that is a union of finite number of intervals, but does not contain a subset homeomorphic to a circle. If one is acquainted with the dynamical properties of the tree maps, it is not hard to see that Theorem 5.8 and Theorem 6.3 hold for tree maps, but in Theorem 5.8(2), the number “2” should be replaced by a sufficient large integer which is associated with the number of endpoints of the tree.

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References

[1] F. Balibrea and V. J. Lopez, A characterization of chaotic functions with entropy zero via their maximal scrambled sets, Math. Bohemica, 120 (1995), no. 3, 293–298.
[2] A. Block, A. M. Bruckner, P. D. Humer and J. Smítal, The space of $\omega$-limit sets of a continuous map of the interval, Trans. Amer. Math. Soc., 384 (1996), no. 4, 1357–1372.
[3] L. S. Block and W. A. Copple, Dynamics in one dimension, Lecture Notes in Mathematics, 1513, Springer-Verlag, 1992.
[4] F. Blanchard, Fully positive topological entropy and topological mixing, Symbolic Dynamics and its Applications, 135 (ContemporaryMathematics). Amer. Math. Soc., Providence, RI, 1992, 95–105.
[5] F. Blanchard, E. Glasner, S. Kolyada and A. Maass, On Li-Yorke pairs, J. Reine Angew. Math., 547 (2002), 51–68.
[6] F. Blanchard, B. Host and A. Maass, Topological complexity, Ergod. Th. and Dynam. Sys., 20 (2000), 641–662.
[7] R. L. Devaney, An introduction to chaotic dynamical systems, Addison-Wesley Publishing Company Advanced Book Program, RedwoodCity, CA, second edition, 1989.
[8] N. Franzova and J. Smítal, Positive sequence topological entropy characterizes chaotic maps, Proc. Amer. Math., 112 (1991), no. 4, 1083–1086.
[9] E. Glasner, On tame dynamical systems, Colloq. Math., 105 (2006), 283–295.
[10] E. Glasner and X. Ye, Local entropy theory, Ergod. Th. and Dynam. Sys., 29 (2009), no. 2, 321–356.
[11] W. Huang and X. Ye, Devaney’s chaos or 2-scattering implies Li-Yorke’s chaos, Topology Appl., 117 (2002), no. 3, 259–272.
[12] W. Huang and X. Ye, A local variational relation and applications. Israel J. Math., 151 (2006), 237–280.
[13] W. Huang and X. Ye, Combinatorial lemmas and applications to dynamics, Adv. Math., 220 (2009), no. 6, 1689–1716.
[14] K. Jankova and J. Smítal, A characterization of chaos, Bull. Austral. Math. Soc., 34 (1986), no. 2, 283–292.
[15] D. Kerr and H. Li, Independence in topological and $C^*$-dynamics, Math. Ann., 338 (2007), 869–926.
[16] M. Kuchta and J. Smítal, Two-point scrambled set implies chaos, In European Conference on Iteration Theroy (Caldes De Malavella, 1987), World Sci. Publishing, Teaneck, NJ, 1989, 427–430.
[17] S. Li, $\omega$-chaos and topological entropy, Trans. Amer. Math. Soc., 339 (1993), no. 1, 243–249.
[18] T. Li and J. Yorke, Period three implies chaos, Amer. Math. Monthly, 82 (1975), no. 10, 985–992.
[19] B. Schweizer and J. Smítal, Measure of chaos and a spectral decomposition of dynamical systems on the interval, Trans. Amer. Math. Soc., 344 (1994), 737–754.
[20] A. N. Sharkovsky, The partially ordered system of attracting sets, Soviet Math. Dokl. 7 (1966), no. 5, 1384–1386.
[21] S. Ruette, *Chaos for continuous interval maps: a survey of relationship between the various sorts of chaos*, 2003. Available on [http://www.math.u-psud.fr/~ruette/](http://www.math.u-psud.fr/~ruette/).

[22] S. Shao, X. Ye and R. Zhang, *Sensitivity and regionally proximal relation in minimal systems*, Sci. China Ser. A., 51 (2008), 987–994.

[23] A. N. Sharkovskii, *On a theorem of G. Birkhoff*, Dopovidi Akad. Nauk Ukrain. RSR. Ser. A., 5 (1967), 429–433.

[24] J. Smital, *Chaotic functions with zero topological entropy*, Trans. Amer. Math. Soc., 297 (1986), no. 1, 269–282.

[25] F. Tan, X. Ye and R. Zhang, *The set of sequence entropy for a given space*, Nonlinearity, 23 (2010), 159–178.

[26] J. Xiong, *A chaotic map with topological entropy [zero]*, Acta Math. Sci. (English Ed.), 6 (1986), no. 4, 439–443.

[27] J. Xiong, *Chaos in topological transitive systems*, Sci. China Ser. A., 48 (2005), 929–939.

[28] J. Xiong and Z. Yang, *Chaos caused by a topological mixing map*, Dynamical systems and related topics (Nagoya, 1990), 550–572, Adv. Ser. Dynam. Systems, 9, World Sci. Publ., River Edge, NJ, 1991.

[29] X. Ye and R. Zhang, *On sensitive sets in topological dynamics*, Nonlinearity, 21 (2008), no. 7, 1601–1620.

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