ON PROBLEMS DUAL TO UNIFICATION:
THE STRING-REWRITING CASE

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Abstract. In this paper, we investigate problems which are dual to the unification problem, namely the Fixed Point (FP) problem, Common Term (CT) problem and the Common Equation (CE) problem for string rewriting systems (SRS). Our main motivation is computing fixed points in systems, such as loop invariants in programming languages. We show that the fixed point (FP) problem is reducible to the common term problem. Our new results are: (i) the fixed point problem is undecidable for finite convergent string rewriting systems (SRS) whereas it is decidable in polynomial time for finite, convergent and dwindling string rewriting systems, (ii) the common term problem is undecidable for the class of dwindling string rewriting systems, and (iii) for the class of finite, monadic and convergent systems, the common equation problem is decidable in polynomial time but for the class of dwindling string rewriting systems, common equation problem is undecidable.

Key words and phrases: unification, convergent string-rewriting systems, fixed point problem, common term problem, common equation problem, conjugacy problem, common multiplier problem.

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Introduction

Unification, with or without background theories such as associativity and commutativity, is an area of great theoretical and practical interest. The former problem, called *equational* or *semantic* unification, has been studied from several different angles. Here we investigate some problems that can clearly be viewed as dual to the unification problem. Our main motivation for this work is theoretical, but, as explained below, we begin with a practical application that is shared by many fields.

In every major research field, there are variables or other parameters that change over time. These variables are modified — increased or decreased — as a result of a change in the environment. Computing *invariants*, or expressions whose values do not change under a transformation, is very important in many areas such as Physics, e.g., invariance under the *Lorentz* transformation.

In Computer Science, the issue of obtaining invariants arises in *axiomatic semantics* or *Floyd-Hoare semantics*, in the context of formally proving a loop to be correct. A *loop invariant* is a condition, over the program variables, that holds before and after each iteration. Our research is partly motivated by the related question of finding expressions, called *fixed points*, whose values will be the same before and after each iteration, i.e., will remain unchanged as long as the iteration goes on. For instance, for a loop whose body is

\[
X = X + 2; \quad Y = Y - 1;
\]

the value of the expression \(X + 2Y\) is a fixed point.

In this paper, we explore the Fixed Point, Common Term and Common Equation problems in *convergent* string rewriting systems (SRS), more specifically in the subclass of *dwindling* string rewriting systems.

We prove that the Fixed Point problem is undecidable for convergent SRS, yet polynomial for dwindling SRS. In the case of the Common Term and Common Equation problems, we will demonstrate their undecidability within the context of dwindling SRS. Additionally, we will showcase that the Common Equation problem attains polynomial complexity when considering monadic SRS. Our results and the current literature for these three problems are summarized in Table 1, the *rectangle* boxes indicating our own results:

|        | Convergent | Length-reducing | Dwindling | Monadic |
|--------|------------|-----------------|-----------|---------|
| FP     | undecidable | NP-complete     | P         | P       |
| CT     | undecidable | undecidable     | undecidable | P       |
| CE     | undecidable | undecidable     | P         | P       |

Table 1. Complexity results of the problems in String Rewriting Systems.

An important contribution of this paper is completing the complexity results for these three problems for convergent, length-reducing, dwindling and monadic SRS. Length-reducing
and monadic string rewriting systems are thoroughly investigated by [Ott86], [NOW84], [OND98], [NO97]. The explanation of these different subclasses of string rewriting systems can be found under the section “Notation and Preliminaries.”

Dwindling convergent systems are especially important because they are a special case of subterm-convergent theories which are widely used in the field of protocol analysis [AC06, Bau05, CDK09, CD09]. Tools such as TAMARIN subterm-convergent theories since these theories have nice properties (e.g., finite basis property [CR10]) and decidability results [AC06].

### Notation and Preliminaries

We start by presenting some notation and definitions on term rewriting systems and particularly string rewriting systems. Only some definitions are given in here, but for more details, refer to the books [BN99] for term rewriting systems and [BO93] for string rewriting systems.

A signature $\Sigma$ consists of finitely many ranked function symbols. Let $X$ be a (possibly infinite) set of variables. The set of all terms over $\Sigma$ and $X$ is denoted as $T(\Sigma, X)$. $\text{Var}(t)$ is the set of variables for term $t$ and a term is a ground term iff $\text{Var}(t) = \emptyset$. The set of ground terms, or terms with no variables is denoted $T(\Sigma)$. A term rewriting system (TRS) is a set of rewrite rules that are defined on the signature $\Sigma$, in the form of $l \rightarrow r$, where $l$ and $r$ are called the left-hand-side and right-hand-side ($\text{lhs}$ and $\text{rhs}$) of the rule, respectively. The rewrite relation induced by a term rewriting system $R$ is denoted by $\rightarrow_R$. The reflexive and transitive closure of $\rightarrow_R$ is denoted $\rightarrow^*_R$. A TRS $R$ is called terminating iff there is no infinite chain of terms. A TRS $R$ is confluent iff, for all terms $t, s_1, s_2$, if $s_1 \rightarrow^*_R t \rightarrow^*_R s_2$, then there exists a term $t'$ such that $s_1 \rightarrow^*_R t' \leftarrow^*_R s_2$. A TRS $R$ is convergent iff it is both terminating and confluent.

A term is irreducible iff no rule of TRS $R$ can be applied to that term. The set of terms that are irreducible modulo $R$ is denoted by $\text{IRR}(R)$ and also called as terms in their normal form $s$. A term $t'$ is said to be an $R$-normal form of a term $t$ when $R$ is understood from the context.

String rewriting systems (SRS) are a restricted class of term rewriting systems where all functions are unary. These unary operators, that are defined by the symbols of a string, applied in the order in which these symbols appear in the string, i.e., if $g, h \in \Sigma$, the string $gh$ will be seen as the term $h(g(x))$. The set of all strings over the alphabet $\Sigma$ is denoted by $\Sigma^*$ and the empty string is denoted by the symbol $\lambda$. Thus the term rewriting system

$$\{p(s(x)) \rightarrow x, s(p(x)) \rightarrow x\}$$

is equivalent to the string-rewriting system

$$\{sp \rightarrow \lambda, ps \rightarrow \lambda\}$$

If $R$ is a string rewriting system over alphabet $\Sigma$, then the single-step reduction on $\Sigma^*$ can be written as:

For any $u, v \in \Sigma^*$, $u \rightarrow_R v$ iff there exists a rule $l \rightarrow r \in R$ such that $u = xly$ and $v = xry$ for some $x, y \in \Sigma^*$; i.e.,

$$\rightarrow_R = \{(xly, xry) \mid (l \rightarrow r) \in R, x, y \in \Sigma^*\}$$
For any string rewrite system $R$ over $\Sigma$, the set of all irreducible strings, $\text{IRR}(R)$, is a regular language: in fact, $\text{IRR}(R) = \Sigma^* \setminus \{ \Sigma^*l_1 \Sigma^* \cup \ldots \cup \Sigma^*l_n \Sigma^* \}$, where $l_1, \ldots, l_n$ are the left-hand sides of the rules in $R$.

Throughout the rest of the paper, $a, b, c, \ldots, h$ will denote elements of the alphabet $\Sigma$, and $l, r, u, v, w, x, y, z$ will denote strings over $\Sigma$. Concepts such as normal form, terminating, confluent, and convergent have the same definitions for string rewriting systems as they have for term rewriting systems. An SRS $T$ is called canonical if and only if it is convergent and inter-reduced, i.e., no lhs is a substring of another lhs and each rhs is an irreducible string.

For a string $x \in \Sigma^*$, the element at position $i$ is denoted $x_{[i]}$, and the substring from position $i$ to position $j$ (inclusive) is denoted as $x_{[i:j]}$ where $i \leq j$ and this shall denote the empty string when $i > j$. We will write $x_{[i]}$ to denote $x_{[i:|x|]}$ when it is cumbersome to use $|x|$. Additionally, for an index sequence $\beta = (\beta_1, \beta_2, \ldots, \beta_c)$, we use $x_{[\beta]} := x_{[\beta_1]}x_{[\beta_2]}\ldots x_{[\beta_c]}$. Parenthesized superscripts shall be a general way to index elements in a sequence of strings e.g., $(x^{(1)}, x^{(2)}, x^{(3)}, \ldots)$.

A string rewrite system $T$ is said to be:
- **monadic** iff the rhs of each rule in $T$ is either a single symbol or the empty string, e.g., $abc \rightarrow b$.
- **dwindling** iff, for every rule $l \rightarrow r$ in $T$, the rhs $r$ is a proper prefix of its lhs $l$, e.g., $abc \rightarrow ab$.
- **length-reducing** iff $|l| > |r|$ for all rules $l \rightarrow r$ in $T$, e.g., $abc \rightarrow ba$.

**Figure 1.** Some of the classes of String Rewriting Systems

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw (0,0) circle (1.5cm);
\node at (0,0) {Length-Reducing};
\node at (-1,0) {Monadic};
\node at (1,0) {Dwindling};
\end{tikzpicture}
\end{figure}

**Motivation and Problem Statements**

The Fixed Point, Common Term and Common Equation problems can be formulated in terms of properties of substitutions modulo a term rewriting system. We plan to assist the reader by providing these formulations and examples to clarify the problems for readers with no particular expertise in String Rewriting Systems.

\[1\] The trivial forms of monadic rules such as $a \rightarrow b$ can be ignored. We can get rid of such rules by changing every occurrence of $a$ to $b$. 
Fixed Point Problem (FP).

**Input:** A substitution $\theta$ and an equational theory $E$.

**Question:** Does there exist a non-ground term $t \in T(Sig(E), Dom(\theta))$ such that $\theta(t) \approx_E t$?

Example 1: Suppose $E$ is a theory of integers which contains linear arithmetic. As an example, a similar equational theory can be found in the Example 3 below. Let $\theta = \{x \mapsto x - 2, y \mapsto y + 1\}$ and we would like to find a term $t$ such that $\theta(t) \approx_E t$. Note that $x + 2y$ is such a term, since
\[
\theta(x + 2y) = (x - 2) + 2(y + 1) \approx_E x + 2y
\]

Example 2: What is the fixed point/invariant of the given loop?

**Algorithm 1 Fixed Point Loop**

1:  $\{x = X_0, y = Y_0\}$
2:  while do $x > 0$
3:     $x = x - 1$ 
4:     $y = y + x$
5:  end while

Note that the value of the expression $y + \frac{x * (x - 1)}{2}$ is unchanged, since

- Before Iteration: $y + \frac{x * (x - 1)}{2}$
- After Iteration:
\[
y + x - 1 + \frac{(x - 1) * (x - 2)}{2} = y + \frac{x^2 - 3x + 2 + 2(x - 1)}{2} = y + \frac{x * (x - 1)}{2}
\]

Thus $y + \frac{x * (x - 1)}{2}$ is a fixed point of $\theta$ as defined in Algorithm 1.

Note that fixed points may not be unique. Consider the term rewriting system
\[
\{a(b(y)) \rightarrow a(y), \ c(b(z)) \rightarrow c(z)\}
\]
and let $\theta = \{x \mapsto b(x)\}$. We can see that both $a(x)$ and $c(x)$ are fixed points of $\theta$.

We plan to explore two related formulations, both of which can be viewed as dual to the well-known unification problem. Unification deals with solving symbolic equations: thus a typical input would be either two terms, say $s$ and $t$, or an equation $s \approx_E t$. The task is to find a substitution such that $\theta(s) \approx_E \theta(t)$. For example, given two terms $s_1 = f(a,y)$ and $s_2 = f(x,b)$, where $f$ is a binary function symbol, $a$ and $b$ are constants, and $x$ and $y$ are variables, the substitution $\sigma = \{x \mapsto a, \ y \mapsto b\}$ unifies $s_1$ and $s_2$, or equivalently, $\sigma$ is a unifier for the equation $s_1 \approx_E s_2$.

There are two ways to “dualize” the unification problem:
Common Term Problem (CT):

**Input:** Two ground substitutions \( \theta_1 \) and \( \theta_2 \), and an equational theory \( E \). (i.e., \( \forall \text{Ran}(\theta_1) = \emptyset \) and \( \forall \text{Ran}(\theta_2) = \emptyset \ ))

**Question:** Does there exist a non-ground term \( t \in T(\text{Sig}(E), \text{Dom}(\theta_1) \cup \text{Dom}(\theta_2)) \) such that \( \theta_1(t) \approx_E \theta_2(t) \)?

**Example 3:** Consider the two substitutions
\[
\theta_1 = \{ x \mapsto p(a), y \mapsto p(b) \} \text{ and } \theta_2 = \{ x \mapsto a, y \mapsto b \}.
\]

The following term rewriting system \( R_{lin}^1 \) specifies a fragment of linear arithmetic using successor and predecessor operators:

\[
\begin{align*}
x - 0 & \rightarrow x \\
x - x & \rightarrow 0 \\
s(x) - y & \rightarrow s(x - y) \\
p(x) - y & \rightarrow p(x - y) \\
x - p(y) & \rightarrow s(x - y) \\
x - s(y) & \rightarrow p(x - y) \\
p(s(x)) & \rightarrow x \\
s(p(x)) & \rightarrow x
\end{align*}
\]

If we take the term rewriting system \( R_{lin}^1 \) as our background equational theory \( E \), then the common term \( t = x - y \) satisfies \( \theta_1(t) \approx_E \theta_2(t) \).

\[
\theta_1(x - y) \approx_E p(a) - p(b) \approx_E a - b
\]

and

\[
\theta_2(x - y) \approx_E a - b
\]

We can easily show that the fixed point problem can be reduced to the CT problem, as seen below in Lemma 0.1.

**Lemma 0.1.** The fixed point problem is reducible to the common term problem.

**Proof.** Let \( \theta_2 \) be the identity substitution. Assume that the fixed point problem has a solution, i.e., there exists a term \( t \) such that \( \theta(t) \approx_E t \). Then the CT problem for \( \theta \) and \( \theta_2 \) has a solution since \( \theta_2(t) \approx_E t \) (because \( \theta_2(s) = s \) for all \( s \)). The “only if” part is trivial, again because \( \theta_2(s) = s \) for all \( s \).

Alternatively, suppose that \( \text{Dom}(\theta) \) consists of \( n \) variables, where \( n \geq 1 \). If we map all the variables in \( \forall \text{Ran}(\theta) \) to new constants, this will create a ground substitution

\[
\theta_1 = \{ x_1 \mapsto a_1, x_2 \mapsto a_2, ..., x_n \mapsto a_n \}.
\]

\( \theta_1 \) will be the one of the substitutions for the CT problem. The other substitution, \( \theta_2 \), is the composition of the substitutions \( \theta \) and \( \theta_1 \). The substitution \( \theta_1 \) will replace all of the variables in \( \forall \text{Ran}(\theta) \) with the new constants, thus making \( \theta_2 \) a ground substitution. Now if \( \theta(t) \approx_E t \), then \( \theta_2(t) = \theta_1(\theta(t)) \approx_E \theta_1(t) \); in other words, \( t \) is a solution to the common term problem.
The “only if” part can also be explained in terms of the composition above. Suppose that \( \theta_1(s) \) and \( \theta_2(s) \) are equivalent, i.e., \( \theta_1(s) \approx_E \theta_2(s) \) for some \( s \). Since \( \theta_2 = \theta_1 \circ \theta \), the equation can be rewritten as \( \theta_1(\theta(s)) \approx_E \theta_1(s) \). Since \( a_1, \ldots, a_n \) are new constants and are not included in the signature of the theory, for all \( t_1 \) and \( t_2 \), \( \theta_1(t_1) \approx_E \theta_1(t_2) \) holds if and only if \( t_1 \approx_E t_2 \) (See [BN99], Section 4.1, page 60) Thus \( \theta_1(\theta(s)) \approx_E \theta_1(s) \) implies that \( \theta(s) \approx_E s \), making \( s \) a fixed point.

**Common Equation Problem (CE):**

**Input:** Two substitutions \( \theta_1 \) and \( \theta_2 \) with the same domain, and an equational theory \( E \).

**Question:** Does there exist a non-ground, non-trivial \((t_1 \not\approx_E t_2)\) equation \( t_1 \approx_E t_2 \), where \( t_1, t_2 \in T(\text{Sig}(E), \text{Dom}(\theta_1)) \) such that both \( \theta_1 \) and \( \theta_2 \) are \( E \)-unifiers of \( t_1 \approx_E t_2 \)?

By trivial equations, we mean equations which are identities in the equational theory \( E \), i.e., an equation \( s \approx_E t \) is trivial if and only if \( s \approx_E t \). We exclude this type of trivial equations in the formulation of this question.

**Example 4:** Let \( E = \{ p(s(x)) \approx x, s(p(x)) \approx x \} \). Given two substitutions

\[
\theta_1 = \{ x_1 \mapsto s(s(a)), x_2 \mapsto s(a) \} \quad \text{and} \quad \theta_2 = \{ x_1 \mapsto s(a), x_2 \mapsto a \},
\]

we can see that \( \theta_1(t_1) \approx_E \theta_1(t_2) \) and \( \theta_2(t_1) \approx_E \theta_2(t_2) \), with the equation \( p(x_1) \approx_E x_2 \). However, there is no term \( t \) on which the substitutions agree, i.e., there aren’t any solutions for the common term problem in this example. Thus, CT and CE problems are not equivalent as we observe in the example above.

In the subsequent sections, this paper delves into an in-depth examination of these three problems within the realm of string rewriting systems: the Fixed Point problem, the Common Term problem, and the Common Equation problem.

### 1. Fixed Point Problem

For a string rewriting system \( R \), the fixed point problem can be stated as follows.

**Input:** A string-rewriting system \( R \) on an alphabet \( \Sigma \), and a string \( \alpha \in \Sigma^+ \).

**Question:** Does there exist a string \( W \) such that \( \alpha W \xleftarrow{*} R W \)?

Since the fixed point problem is a particular case of the common term problem, it is decidable in polynomial time for finite, monadic and convergent string rewriting systems. The fixed point problem is also a subcase of the conjugacy problem. The conjugacy problem seeks to determine whether two given words, \( w \) and \( w' \), in the group \( G \) are conjugate. In other words, the question is whether there exists a word \( z \) in \( G \) such that the conjugation of \( w \) by \( z \), i.e., \( zwz' \), is equal to \( w' \). The conjugacy problem is both decidable in \( \text{NP} \) and \( \text{NP} \)-hard for finite, length-reducing and convergent systems. [NOW84]
1.1. Fixed Point Problem for Finite and Convergent Systems: Theorem 1.4 shows that the fixed point problem is undecidable for finite and convergent string rewriting systems.

**Theorem 1.1.** The following problem is undecidable:

\[ \Sigma \]

Suitably modifying the construction given in [Car91] in a straightforward way, we can prove the undecidability of this problem with a reduction from the well-known Post Correspondence Problem (PCP). Recall that an instance of PCP is a collection of pairs of strings over an alphabet \( \Sigma \) ("dominos" in [Sip96]) and the question is if there exists a sequence of indices \( i_1, i_2, \ldots, i_n \) such that \( x_{i_1}x_{i_2} \cdots x_{i_n} = y_{i_1}y_{i_2} \cdots y_{i_n} \). Alternatively, we can define the PCP in terms of two homomorphisms \( \psi \) and \( \phi \) from \( C^* = \{c_1, \ldots, c_n\}^* \) to \( \Sigma^* = \{a, b\}^* \). The question now is whether there exists a non-empty string \( m_C \) such that \( \psi(m_C) = \phi(m_C) \).

The reduction proof is done by validating the solution string \( w \in \Sigma^* \), for the two homomorphisms using the Deterministic Linear Bounded Automata (DLBA) \( M \) [Kur64] with the following configuration, \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_f) \). Assume \( w = w_1 w_2 \) where \( w_1 \in C^* \) and \( w_2 \in \Sigma^* \). \( M \) will check the correctness of \( w \) by going over one symbol at a time in the string \( w_1 \) and its corresponding mapping in \( w_2 \). \( M \) uses a marking technique, marking the symbols \( c_i \) and their correct mappings with overlined symbols. If every letter in the string is overlined, then clearly the string is "okay" according to the first homomorphism; \( M \) will then replace the overlined letters with the same non-overlined letters as before and repeat the same steps for the second homomorphism. Thus the DLBA will accept and re-create \( w_1 w_2 \) if and only if it is a solution string. \( L(M) \) is empty if and only if the instance of PCP has no solution.

We construct a string rewriting system \( R \) from the above-mentioned DLBA \( M \). The construction of \( R \) is similar to the one in [BO84]. Let \( M = (Q, \Sigma, \Gamma, \delta, q_0, q_f) \). Here \( \Sigma \) denotes the input alphabet, \( \Gamma \) is the tape alphabet, \( Q \) is the set of states, \( q_0 \in Q \) is the initial state, \( q_a \in Q \) the accepting state and \( q_r \in Q \) the rejecting state. We assume that the tape has two end-markers: \( \ell \in \Gamma \) denotes the left end-marker and \( \ell \in \Gamma \) is the right end-marker. We also assume that on acceptance the DLBA comes to a halt at the left-end of the tape. Finally, \( \delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\} \) is the transition function of \( M \).

The alphabet of \( R \) is \( \Gamma = \Sigma \cup \Sigma' \cup Q \) where \( \Sigma' \) is a replica of the alphabet \( \Sigma \) such that \( \Sigma' \cap \Sigma = \emptyset \). \( R \) has the rules

\[
q_i a_k \to a'_k q_j \quad \text{if} \quad (q_i, a_k, q_j, a'_k, R) \in \delta
\]

\[
a'_k q_i a_k \to q_j a_j a_k \quad \text{for all} \quad a'_k \in \Sigma', \quad \text{if} \quad (q_i, a_k, q_j, a_l, L) \in \delta
\]

\[
q_a \ell \to \ell
\]

Since the linear bounded automaton is deterministic, \( R \) is locally confluent. Besides, \( M \) ultimately always halts, and that means there will be no infinite chain of rewrites for \( R \), and thus \( R \) is terminating.

**Lemma 1.2.** \( M \) accepts \( w \) iff \( q_0 \ell w \mathrel{\rightarrow^*} q_a \ell w \).

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Proof. By inspection of the rules we can see that \( M \) makes the transition \( u_1 q_1 v_1 \vdash_M u_2 q_2 v_2 \) if and only if \( u'_1 q_1 v_1 \to_R u'_2 q_2 v_2 \).

**Lemma 1.3.** \( \cdot w \cdot \) is a fixed point for \( q_0 \) in \( R \) iff \( M \) accepts \( w \).

**Proof.** For the “if” part, suppose \( M \) accepts \( w \). Observe that \( q_0 \cdot w \cdot \to^+_R q_0 \cdot w \cdot \) by Lemma 1.2, and \( R \) has the rule \( q_0 \cdot \to \cdot \). Thus, we get \( q_0 \cdot w \cdot \to^+_R q_0 \cdot w \cdot \to \cdot w \cdot \).

For the “only if” part, suppose \( \cdot w \cdot \) is a fixed point for \( q_0 \), i.e., \( q_0 \cdot w \cdot \to^+_R \cdot w \cdot \). Now note that the only rule that can remove a state-symbol from a string is the rule \( q_0 \cdot \to \cdot \). But once that rule is applied, no other rules are applicable. Therefore, there must be a reduction sequence such that \( q_0 \cdot w \cdot \to^+_R q_0 \cdot w \cdot \). This proves that \( w \) is accepted by the DLBA \( M \).

**Theorem 1.4.** The fixed point problem is undecidable for finite and convergent string rewriting systems.

### 1.2. Fixed Point Problem for Dwindling Convergent Systems:

Given a string \( \alpha \in \Sigma^* \) and a dwindling, convergent, and finite string rewriting system \( R \), the fixed point problem is equivalent to determining whether there exists \( W \in \Sigma^* \) such that \( \alpha W \to^+_R W \).

We shall define a shorthand notation for working with dwindling systems. Observe that applying a dwindling rule \( pq \to p \) to a string \( X^{(1)}pqX^{(2)} \) has the effect of ”deleting” the substring \( q \), and we are left with \( X^{(1)}pX^{(2)} \). We can say that for any \( X \), if \( X \to_R X' \) then there exists an index sequence \( j_1, \ldots, j_n \) such that \( X' = X[j_1, \ldots, j_n] \), where each \( j_i \) is the original position of its symbol. There may be several index sequences that satisfy \( X' = X[j_1, \ldots, j_n] \), but for any given sequence of reductions applied to \( X \) to obtain \( X[j_1, \ldots, j_n] \), each rule is applied at a specific position and consequently yields a particular index sequence \( j_1, \ldots, j_n \), derived as a subsequence of the previous. To denote this, we write \( X \sim^* \cdot X[j_1, \ldots, j_n] \) and extend this shorthand in the obvious manner e.g. \( AX \sim^* A[j_1, \ldots, j_n]X[k_1, \ldots, k_m] \), with the understanding that \( \sim_R \) is actually a relation on tuples e.g. \((X, (1, 2, \ldots, |X|)), (X, (j_1, \ldots, j_n)), (AX, (j_1, \ldots, j_n, k_1 + |A|, \ldots, k_m + |A|)).

**Lemma 1.5.** Let \( A \in \Sigma^+ \) irreducible, \( X \in \Sigma^+ \), and \( R \) dwindling, convergent and finite. If \( AX \to^+_R Y \), then \( Y = A_{[1:m]}X[j_1, \ldots, j_n] \) for some \( m, j_1, \ldots, j_n \) such that

\[
AX \sim^*_R A_{[1:m]}X[j_1, \ldots, j_n].
\]

Additionally, if \( AX \sim^*_R A_{[1:m]}X[j_1, \ldots, j_n] \) then \( n < |X| \).

**Proof.** This is easily shown by an inductive argument. Since \( A \) is irreducible, no rule \( pq \to p \in R \) can match a substring of only \( A \), so any sequence of reductions on \( AX \) will yield a string in the form \( A_{[1:m]}X[j_1, \ldots, j_n] \).

**Theorem 1.6.** If some instance \( R, \alpha \) of the fixed point problem has a solution and \( R \) is dwindling, convergent and finite then a minimal-length solution can be found in polynomial time.

**Proof.** The size of a minimal-length solution is bounded by \( 2\delta|\alpha| \), as the fixed point problem is also a special case of the conjugacy problem.[NO85] Since \( R \) is convergent, it may be assumed that \( \alpha \) is irreducible without loss of generality. Suppose \( W \) is a minimal-length solution.
By lemma 1.5, \( \alpha X \sim^1_R A_{[1:m]} W_{[j_1, \ldots, j_n]} \). Ignoring the trivial case where \( \alpha \) is the empty string, we have \( n < |W| \).

Since \( n < |W| \) and \( \alpha_{[1:m]} W_{[j_1, \ldots, j_n]} = W \), it must be that \( W_1 = \alpha_1 \). Suppose now that the first \( i < |W| \) characters of \( W \) are known and let \( \alpha^{(i)} := (\alpha_{[1:m]} W_{[i+1:|W|]})_1 \). Since we assume \( W \) is a minimal solution, \( \alpha^{(i)} W_{[i+1:|W|]} \) must be reducible and so \( \alpha^{(i)} W_{[i+1:|W|]} \sim^1_R \alpha^{(i)}_{[1:m(i+1)]} W_{[j_1, \ldots, j_n(i+1)]} \). Since \( n(i+1) < |W_{[i+1:|W|]}| \), we must have that \( |\alpha^{(i)}_{[1:m(i+1)]}| \geq i + 1 \) and so we have found \( W_{[i+1]} \).

The algorithm above is efficient and we can readily obtain an asymptotic bound:

**Theorem 1.7.** Any instance \( R, \alpha \) of a fixed point problem where \( R \) is dwindling, convergent and finite can be solved in \( O(\delta^2 |\alpha|) \) time.

**Proof.** Letting \( \alpha^{(0)} := \alpha \) and \( \alpha^{(k+1)} := (\alpha^{(k)} \alpha^{(k)})_1 \), either \( \alpha^{(0)} [1] \alpha^{(1)} [2] \ldots \alpha^{(k)} [k] \) is a solution for \( 1 \leq k \leq \delta |\alpha| \) or there is no solution. Since \( \alpha^{(k+1)} \) is by definition irreducible, the string \( \alpha^{(k)} \alpha^{(k+1)} \) is only reducible if the left-hand side of a rule in \( R \) can be matched to a substring including the last symbol. By the nature of dwindling rules, reducing \( \alpha^{(k)} \alpha^{(k+1)} \) to its normal form will require at most a single rewrite. The rules of \( R \) can be arranged into a tree structure so that any matching rules can be found and applied to \( \alpha^{(k)} \alpha^{(k+1)} \) in \( O(\delta) \) time.

**Corollary 1.8.** If the rhs of each rule in \( R \) is also of length \( \leq 1 \) (the cases where \( R \) is a dwindling convergent system that is also special or monadic) and solutions exist, then there must exist a minimal-length solution that’s a prefix of \( \alpha \).

**Proof.** Assume solutions exist and that \( W \) is a solution found using the procedure in Theorem 1.6. Let \( W_{[1:k]} \) be the smallest prefix of \( W \) such that \( \alpha W_{[1:k]} \) is reducible. Then \( \alpha^{(k)} := (\alpha W_{[1:k]})_1 \) is a prefix of \( \alpha \), since the reduction must match a substring including \( \alpha W_{[1:k]} \), remove a postfix including at least \( W_{[1:k]} \) and therefore leave only a prefix \( \alpha_{[1:i]} \). Inductively, if \( \alpha W \sim^1_R \alpha_{[1:i]} W_{[k+1:|W|]} \) and \( k \neq W \) then there must be some minimal \( \bar{k} \) such that \( \alpha_{[1:i]} W_{[k+1:k]} \) is reducible, and again some \( i \leq \bar{i} \) such that \( \alpha_{[1:i]} W_{[k+1:k]} \sim^1_R \alpha_{[1:i]} \).

## 2. Common Term Problem

Note that for string rewriting systems the common term problem is equivalent to the following problem:

**Input:** A string-rewriting system \( R \) on an alphabet \( \Sigma \), and two strings \( \alpha, \beta \in \Sigma^* \).

**Question:** Does there exist a string \( W \) such that \( \alpha W \leftrightarrow^* R \beta W \)?

This is also known as *Common Right Multiplier Problem* which has been shown to be decidable in polynomial time for monadic and convergent string-rewriting systems (see,
e.g., [OND98], Lemma 3.7). However, the CT problem is undecidable for convergent and length-reducing string rewriting systems in general [Ott86].

In this section, we focus on the decidability of the CT problem for convergent and dwindling string rewriting systems.

2.1. Common Term Problem for Convergent and Dwindling Systems. We show that the CT (Common Term) problem is undecidable for string rewriting systems that are dwindling and convergent. We define CT as the following decision problem:

Given: A finite, non-empty alphabet \( \Sigma \), strings \( \alpha, \beta \in \Sigma^* \) and a dwindling, convergent string rewriting system \( S \).

Question: Does there exist a string \( W \in \Sigma^* \) such that \( \alpha W \approx_S \beta W \)?

Note that interpreting concatenation the other way, i.e., \( ab \) as \( a(b(x)) \), will make this a unification problem.

We show that the Generalized Post Correspondence Problem (GPCP) reduces to the CT problem, where GPCP stands for a variant of the modified post correspondence problem such that we will provide the start and finish dominoes in the problem instance. This slight change does not effect the decidability of the problem in any way, i.e., GPCP is also undecidable [EKR82, Nic08].

Given: A finite set of tuples \( \{(x_i, y_i)\}_{i=0}^{n+1} \) such that each \( x_i, y_i \in \Sigma^* \), i.e., for all \( i \), \( |x_i| > 0 \), \( |y_i| > 0 \), and \((x_0, y_0), (x_{n+1}, y_{n+1})\) are the start and end dominoes, respectively.

Question: Does there exist a sequence of indices \( i_1, \ldots, i_k \) such that

\[
\begin{align*}
x_0 & \ x_{i_1} \ \ldots \ x_{i_k} \ x_{n+1} = y_0 & \ y_{i_1} \ \ldots \ y_{i_k} & \ y_{n+1}.
\end{align*}
\]

To understand the problem described above, consider the following example. Let \( C = c_0, c_1, c_2, c_3 \) be a set of dominoes, where each domino is represented as a tuple of two strings. Specifically, \( c_0 \) is \( (a, b) \), \( c_1 \) is \( (a, b) \), \( c_2 \) is \( (b, a) \) and \( c_3 \) is \( (b, abb) \). We designate \( c_0 \) as the start domino, and \( c_3 \) as the end domino. In the start domino, \( c_0 \), \( x_0 \) is equal to \( abb \) and \( y_0 \) is equal to \( a \). The problem at hand is to find a sequence of indices that generates equal strings for both elements of the tuples. Formally we seek a sequence of dominoes \( c_{i_0}, c_{i_1}, \ldots, c_{i_k} \) such that resulting strings satisfy: \( x_0 \ x_{i_1} \ \ldots \ x_{i_k} \ x_{n+1} = y_0 \ y_{i_1} \ \ldots \ y_{i_k} \ y_{n+1} \).

In our example, a closer inspection of the dominoes reveals that the sequence of \( c_0, c_1, c_2, c_3 \) provides us a combination that yields equal strings: \( abbaabb \) for both combinations.

We work towards showing that the CT problem defined above is undecidable by a many-one reduction from GPCP. First, we show how to construct a string-rewriting system that is dwindling and convergent from a given instance of GPCP.

Let \( \{(x_i, y_i)\}_{i=1}^{n} \) be the set of “intermediate” dominoes and \((x_0, y_0), (x_{n+1}, y_{n+1})\), the start and end dominoes respectively, be given. Suppose \( \Sigma \) is the alphabet given in the instance of GPCP. Without loss of generality, we may assume \( \Sigma = \{a, b\} \). Then the set \( \hat{\Sigma} := \{a, b\} \cup \{c_0, \ldots, c_{n+1}\} \cup \{\epsilon_1, \epsilon_2, B, a_1, a_2, a_3, b_1, b_2, b_3\} \) will be our alphabet for the instance of CT.

\(^2\)In fact, Otto et al. [OND98] showed that there is a fixed convergent length-reducing string rewriting system for which the CT problem is undecidable.
Next we define a set of string homomorphisms used to simplify the discussion of the reduction. Namely, we have the following:

\[ h_1(a) = a_1 a_2 a_3, \quad h_2(a) = a_1 a_2, \quad h_3(a) = a_1 \]
\[ h_1(b) = b_1 b_2 b_3, \quad h_2(b) = b_1 b_2, \quad h_3(b) = b_1 \]

such that each \( h_i : \Sigma \rightarrow \hat{\Sigma}^+ \) is a homomorphism.

**Reduction and form of solution:** For the convenience of the reader, we state the form of the \( CT \) problem instance and its solutions before explaining the details. In particular, \( \alpha = c_1, \beta = c_2 \). For instances such that a solution \( Z \), which is constructed over the alphabet of the SRS \( S \), exists, it shall take the form \( h_1(Z_1)Z_2 \), where \( Z_1 \in \{a, b\}^+ \) is the string of the GPCP solution and \( Z_2 = c_{n+1}Bc_{i_k} \ldots Bc_{i_1}Bc_0 \). The sequence \( c_{n+1}, c_{i_k}, \ldots, c_{i_1}, c_0 \) is the (reversed) sequence of tuple indices, that is, the GPCP solution itself, and they are separated by the symbol \( B \). Informally, the purpose of the homomorphisms and \( c_1, c_2 \) is to ensure that \( \alpha Z \) and \( \beta Z \) are processed with two separate sets of rules, corresponding to the sets of first and second tuple elements, respectively.

We are now in a position to construct the string rewriting system \( S \), with the following collections of rules, named as the Class D rules:

\[ c_1 h_1(a) \rightarrow c_1 h_3(a), \quad c_2 h_1(a) \rightarrow c_2 h_2(a) \]
\[ c_1 h_1(b) \rightarrow c_1 h_3(b), \quad c_2 h_1(b) \rightarrow c_2 h_2(b) \]

and,

\[ h_i(a) h_1(a) \rightarrow h_i(a) h_i(a), \quad h_i(a) h_1(b) \rightarrow h_i(a) h_i(b) \]
\[ h_i(b) h_1(a) \rightarrow h_i(b) h_i(a), \quad h_i(b) h_1(b) \rightarrow h_i(b) h_i(b) \]

for \( i \in \{2, 3\} \).

The erasing rules of our system consists of three classes. Class I rules are defined as:

\[ c_1 h_3(x_0) B c_0 \rightarrow \lambda \]
\[ c_2 h_2(y_0) c_0 \rightarrow \lambda \]

and Class II rules (for each \( i = 1, 2, \ldots, n \)),

\[ h_3(x_i) B c_i \rightarrow \lambda \]
\[ h_2(y_i) c_i B \rightarrow \lambda \]

and finally Class III rules,

\[ h_3(x_{n+1}) c_{n+1} \rightarrow \lambda \]
\[ h_2(y_{n+1}) c_{n+1} B \rightarrow \lambda \]

Clearly given an instance of GPCP the above set of rules can effectively be constructed from the instance data. Also, by inspection, we have that our system is confluent (there are no overlaps among the left-hand sides of rules), terminating, and dwindling.

We then set \( \alpha = c_1 \) and \( \beta = c_2 \) to complete the constructed instance of \( CT \) from GPCP.

It remains to show that this instance of \( CT \) is a “yes” instance if and only if the given instance of GPCP is a “yes” instance, i.e., the \( CT \) has a solution if and only if the GPCP does. In that direction, we prove some results relating to \( S \).
Lemma 2.1. Suppose \( c_1 h_3(w_1) B \gamma \rightarrow^1 \lambda \) and \( c_2 h_2(w_2) \gamma \rightarrow^1 \lambda \) for some \( w_1, w_2 \in \{a, b\}^* \), then \( \gamma \in \{c_1 B, c_2 B, \ldots, c_n B\}^*c_0 \).

Proof. Suppose \( \gamma \) is a minimal counterexample with respect to length and \( \gamma \in IRR(S) \). In order for the terms to be reducible, \( \gamma = c_i B \gamma' \) (this follows by inspection of \( S \)). After we replace the \( \gamma \) at the equation in the lemma, we get:

\[
\begin{align*}
\gamma_1 h_3(w_1) B c_i B \gamma' & \rightarrow \gamma_1 h_3(w_1) B \gamma' \rightarrow^1 \lambda \\
\gamma_2 h_2(w_2) c_i B \gamma' & \rightarrow \gamma_2 h_2(w_2) \gamma' \rightarrow^1 \lambda
\end{align*}
\]

by applying the Class II rules and finally Class I rule to erase the \( \gamma \) signs. Then, however, \( \gamma' \) is also a counterexample, and \( |\gamma'| < |\gamma| \), which is a contradiction.

We are now in a position to state and prove the main result of this section.

Theorem 2.2. The CT problem is undecidable for dwindling convergent string-rewriting systems by a reduction from GPCP.

Proof. We first complete the “only if” direction. Suppose CT has a solution such that \( Z_1 \) is a minimal solution that is constructed over the alphabet of the string-rewriting system. We show that \( Z \) corresponds to a solution for GPCP. Let \( Z_1 \) be the longest string such that \( h_1(Z_1) \) is a prefix of \( Z \), and denote the rest of \( Z \) by \( Z_2 \), so that \( Z \) can be written \( Z = h_1(Z_1)Z_2 \).

\( h_1(Z_1) \) can be rewritten to \( h_3(Z_1) \) and \( h_2(Z_1) \) by applying the Class D rules. Thus, we will get

\[
\begin{align*}
\gamma_1 h_3(Z_1) Z_2 & \rightarrow^* \gamma_1 h_3(Z_1) Z_2 \\
\gamma_2 h_2(Z_1) Z_2 & \rightarrow^* \gamma_2 h_2(Z_1) Z_2
\end{align*}
\]

In order for both terms to be reducible, \( Z_2 \) must be of the form \( Z_2 = c_{n+1} B Z_2' \). Thus

\[
\begin{align*}
\gamma_1 h_3(Z_1) Z_2 & = \gamma_1 h_3(Z_1) c_{n+1} B Z_2' \\
\gamma_2 h_2(Z_1) Z_2 & = \gamma_2 h_2(Z_1) c_{n+1} B Z_2'
\end{align*}
\]

i.e., \( Z_1 = Z_1' x_{n+1} \) and \( Z_1 = Z_1'' y_{n+1} \). By applying the Class III rules, these equations will reduce to:

\[
\begin{align*}
\gamma_1 h_3(Z_1) c_{n+1} B Z_2' & \rightarrow \gamma_1 h_3(Z_1') B Z_2' \\
\gamma_2 h_2(Z_1) c_{n+1} B Z_2' & \rightarrow \gamma_2 h_2(Z_1'') Z_2'
\end{align*}
\]

We now apply Lemma 2.1 to conclude that \( Z_2' \in \{c_1 B, c_2 B, \ldots, c_n B\}^*c_0 \).

At this point we have that:

\[
Z_2 = c_{n+1} B c_{i_k} B c_{i_{k-1}} \cdots B c_{i_2} B c_{i_1} B c_0 \quad \text{for some } i_1, \ldots, i_k
\]

Then the sequence of dominoes

\[
(x_0, y_0), (x_{i_1}, y_{i_1}), \ldots, (x_{i_k}, y_{i_k}), (x_{n+1}, y_{n+1})
\]

will be a solution to the given instance of GPCP with solution string \( Z_1 \) since the left-hand sides of the Class I, II, III rules consist of the images of domino strings under \( h_2 \) and \( h_3 \). More specifically, there is a finite number of \( B \)'s and \( c_i \)'s in \( Z_2 \), so there must be a decomposition of \( h_1(Z_1) \):

\[
\begin{align*}
h_1(Z_1) = h_1(x_0) h_1(x_{i_1}) \cdots h_1(x_{i_k}) h_1(x_{n+1})
\end{align*}
\]
We show that

\[ h_1(Z_1) = h_1(y_0)h_1(y_{i_1}) \cdots h_1(y_{i_k})h_1(y_{n+1}) \]

Thus, we have the following reductions with Class D rules:

\[ \varphi_1 h_1(Z_1) Z_2 \rightarrow^* \varphi_1 h_3(Z_1) Z_2 \]
\[ \varphi_2 h_1(Z_1) Z_2 \rightarrow^* \varphi_2 h_2(Z_1) Z_2 \]

Finally, by Class I, II, III rules:

\[ \varphi_1 h_3(Z_1) Z_2 \rightarrow^* \varphi_1 h_3(x_0) B c_0 \rightarrow \lambda \]
\[ \varphi_2 h_2(Z_1) Z_2 \rightarrow^* \varphi_2 h_2(y_0) c_0 \rightarrow \lambda \]

and \( Z_1 \) is a solution to the instance of the GPCP.

We next prove the "if" direction. Assume that the given instance of GPCP has a solution. Let \( Z_1 \) be the string corresponding to the matching dominoes, and let

\[(x_0, y_0), (x_{i_1}, y_{i_1}), \ldots, (x_{i_k}, y_{i_k}), (x_{n+1}, y_{n+1})\]

be the sequence of tiles that induces the match. Let \( Z_2 = c_{n+1}Bc_{i_k}Bc_{i_{k-1}} \cdots Bc_{i_2}Bc_1Bc_0 \). We show that \( \varphi_1 h_1(Z_1) Z_2 \downarrow \varphi_2 h_1(Z_1) Z_2 \). First apply the Class D rules to get:

\[ \varphi_1 h_1(Z_1) Z_2 \rightarrow^* \varphi_1 h_3(Z_1) Z_2 \]
\[ \varphi_2 h_1(Z_1) Z_2 \rightarrow^* \varphi_2 h_2(Z_1) Z_2 \]

but then we can apply Class I, II, III rules to reduce both of the above terms to \( \lambda \). \( \square \)

This result strengthens the earlier undecidability result of Otto [Ott86] for string-rewriting systems that are length-reducing and convergent.

To clarify the results of the theorem, let us revisit the previous example 2.1 introduced earlier in this paper. Given the string \( abbaabb \) in the example, we begin by applying homomorphism and Class D rules. This application yields the following sequence for the first string pair, \( x_0 x_1 x_1 x_2 x_3 \): \( \varphi_1 h_3(abb)h_3(a)h_3(b)h_3(b)c_3Bc_2Bc_1Bc_1Bc_0 \). In order to establish the reduction from GPCP to CT, we proceed to apply erasing rules to both strings. First, we employ Class III rule to erase the last domino \( h_3(b)c_3 \). Subsequently, Class II rule is applied three times to the inner dominoes: first for the homomorphism \( h_3(b)Bc_2 \), then for \( h_3(a)Bc_1 \) twice. Finally, the erasing rule Class I removes the \( \varphi \) symbol along with the remaining homomorphism, resulting in: \( \varphi_1 h_3(abb)Bc_0 \rightarrow \lambda \). This sequence of application of the rules demonstrates the reduction process from GPCP to CT, providing a concrete illustration of the theorem’s applicability.

3. Common Equation Problem

To clarify the CE problem for string rewriting systems, let us consider two substitutions \( \theta_1 \) and \( \theta_2 \) such that

\[ \theta_1 = \{ x_1 \mapsto \alpha_1, \ x_2 \mapsto \alpha_2 \} \]
\[ \theta_2 = \{ x_1 \mapsto \beta_1, \ x_2 \mapsto \beta_2 \} \]

Think of the letters of the alphabet as monadic function symbols as mentioned in the notation and preliminaries section. We have two cases for the equation \( e_1 = e_2 \): (i) both \( e_1 \) and \( e_2 \) have the same variable in them, or (ii) they have different variables, i.e., one has \( x_1 \)
and the other $x_2$. Thus in the former, which we call the “one-mapping” case, we are looking for different irreducible strings $W_1$ and $W_2$ such that either

1. $\alpha_1 W_1 \xrightarrow{R} \alpha_1 W_2$ and $\beta_1 W_1 \xrightarrow{R} \beta_1 W_2$, or
2. $\alpha_2 W_1 \xrightarrow{R} \alpha_2 W_2$ and $\beta_2 W_1 \xrightarrow{R} \beta_2 W_2$.

In the latter (“two-mappings case”) case, we want to find strings $W_1$ and $W_2$, not necessarily distinct, such that

$$\alpha_1 W_1 \xrightarrow{R} \alpha_2 W_2$$

$$\beta_1 W_1 \xrightarrow{R} \beta_2 W_2$$

The loss of explicit inclusion of the variables by the passage from unary term rewriting systems to string rewriting systems is handled by the definitions of the separate problems.

The one-mapping case can be illustrated by an example. Consider the term rewriting system

\[ \{ a(a(b(z))) \rightarrow a(b(z)), a(b(z)) \rightarrow b(c) \} \]

and two substitutions $\theta_1 = \{ x \rightarrow b(c) \}$ and $\theta_2 = \{ x \rightarrow b(b(c)) \}$. Now $a(a(x)) = a(x)$ is a common equation. Considering this in the string rewriting setting, we have $R = \{ ba a \rightarrow ba \}$, $\alpha = b$ and $\beta = bb$. Now $W_1 = aa$ and $W_2 = a$ is a solution.

Hence we define CE as the following decision problem:

**Input:** A string-rewriting system $R$ on an alphabet $\Sigma$, and strings $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Sigma^*$.

**Question:** Do there exist irreducible strings $W_1, W_2 \in \Sigma^*$ such that one of the following conditions is satisfied?

i. $\alpha_1 W_1 \xrightarrow{R} \alpha_2 W_2$, $\beta_1 W_1 \xrightarrow{R} \beta_2 W_2$, $\alpha_1 \neq \alpha_2 \lor \beta_1 \neq \beta_2$

ii. $\alpha_i W_1 \xrightarrow{R} \alpha_i W_2$, $\beta_i W_1 \xrightarrow{R} \beta_i W_2$, $i \in \{1, 2\} \land W_1 \neq W_2$

CE is also undecidable for dwindling systems, since in the string-rewriting case CT is a particular case of CE. To see this, consider the case where $\alpha_1 \neq \alpha_2$ and $\beta_1 = \beta_2 = \lambda$, i.e., consider the substitutions

\[ \theta_1 = \{ x_1 \rightarrow \alpha_1, x_2 \rightarrow \alpha_2 \} \]
\[ \theta_2 = \{ x_1 \rightarrow \lambda, x_2 \rightarrow \lambda \} \]

where $\alpha_1 \neq \alpha_2$. This has a solution if and only if there are irreducible strings $W_1, W_2 \in \Sigma^*$ such that either

i. $\alpha_1 W_1 \xrightarrow{R} \alpha_2 W_2$,

\[ W_1 \xrightarrow{R} W_2, \]

or

ii. $\alpha_i W_1 \xrightarrow{R} \alpha_i W_2$, $i \in \{1, 2\} \land W_1 \neq W_2$

Since $W_1$ and $W_2$ are irreducible strings, $W_1 \xrightarrow{R} W_2$ makes $W_1$ equal to $W_2$. Thus, we eliminate the second condition.
With the second condition being out of the picture, we only consider the first condition which shows a similarity with the definition of Common Term (CT) problem. The CE problem for $\theta_1$ and $\theta_2$ has a solution if and only if the CT problem for $\alpha_1$ and $\alpha_2$ has a solution. Therefore, CT is reducible to CE problem. We now show that CE is decidable for monadic string rewriting systems. We start with the two-mappings case first, since the solution for one-mapping case is similar to the two-mapping case with a slightly simpler approach.

In the following we will need the concept of a right-factor and minimal product (MP), which we define below.

**Definition 1.** Given a monadic, finite and convergent string rewriting system $R$ and irreducible strings $x$ and $y$, let $RF(x, y)$ define the set of right factors needed to derive $y$, i.e.,

$$RF(x, y) = \{ z \in IRR(R) \mid xz \rightarrow^*_R y \}.$$  

**Definition 2.** Let $R$ be a convergent monadic SRS. For an irreducible string $\alpha$, let

$$MP(\alpha) = \{ w \mid w \in PREF(\alpha) \circ (\Sigma \cup \{\lambda\}) \}.$$  

MP stands for the term Minimal Product and $PREF$ is the set of prefixes of given string.

### 3.1. Two-mapping CE Problem for Monadic Systems.

For monadic and convergent string rewriting systems, the two-mappings case of Common Equation (CE) problem is decidable. This can be shown using Lemma 3.6 in [OND98]. (See also Theorem 3.11 of [OND98].) In fact, the algorithm runs in polynomial time as explained below:

**Theorem 3.1.** Common Equation (CE) problem, given below, is decidable in polynomial time for monadic, finite and convergent string rewriting systems.

**Input:** A string rewriting system $R$ on an alphabet $\Sigma$, and strings $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Sigma^*$.

**Question:** Do there exist strings $X, Y \in \Sigma^*$ such that $\alpha_1 X \leftrightarrow_R \alpha_2 Y$ and $\beta_1 X \leftrightarrow_R \beta_2 Y$?

**Proof.** The CE problem is a particular case of the simultaneous $E$-unification problem of [OND98], but with a slight difference: CE consists of only two equations, while simultaneous $E$-unification problem is defined for an arbitrary number of equations. Besides, simultaneous $E$-unification problem is PSPACE-hard. We will use their construction, but we will modify it to obtain our polynomial time result.

Note: $RF(x, y)$ is a regular language for all $x, y$, where $|y| \leq 1$ [OND98] and a DFA for it can be constructed in polynomial time in $|R|$, $|x|$ and $|y|$. A DFA for $IRR(R)$ can also be constructed in polynomial time [Gil79], [Boo87]. We can characterize the solutions of the equation $\alpha_1 X \downarrow_R \alpha_2 Y$ by an analysis similar to that used in Lemma 3.6 of [OND98] and its proof. Since $R$ is monadic, there exist $a, b \in \Sigma \cup \{\lambda\}$ and partitions of the strings $\alpha_1 = \alpha_{11} \alpha_{12}$, $\alpha_2 = \alpha_{21} \alpha_{22}$, $X = X_1 X_2$ and $Y = Y_1 Y_2$, such that

$$\alpha_{12} X_1 \rightarrow^1 a,$$

$$\alpha_{22} Y_1 \rightarrow^1 b,$$

and

$$\alpha_{11} a X_2 = \alpha_{21} b Y_2.$$
Now there are two main cases:

(a) $X_2$ is a suffix of $Y_2$ i.e., $Y_2 = Z X_2$ and $\alpha_{11} a = \alpha_{21} b Z$, therefore

$$(X_1, Y_1 Z) \in RF(\alpha_{12}, a) \times RF(\alpha_{22}, b) \cdot Z.$$ 

(b) $Y_2$ is a proper suffix of $X_2$ i.e., $X_2 = Z'' Y_2, \alpha_{11} a Z'' = \alpha_{21} b$, and $\alpha_{11} a U = \alpha_{21}, Z'' = U b$, therefore $(X_1 Z'', Y_1) \in RF(\alpha_{12}, a) \cdot Z'' \times RF(\alpha_{22}, b)$.

Similar partitioning can be done for the second equation.

Let $Sol(\alpha_1, \alpha_2)$ stand for a set of ‘minimal’ solutions$^3$:

$$Sol(\alpha_1, \alpha_2) = \bigcup_{a, b \in \Sigma \cup \{\lambda\}} RF(\alpha_{12}, a) \times RF(\alpha_{22}, b) \cdot Z$$ 

$$\cup \bigcup_{a, b \in \Sigma \cup \{\lambda\}} RF(\alpha_{12}, a) \cdot Z'' \times RF(\alpha_{22}, b)$$

Note that this is a finite union of cartesian products of regular languages. More precisely, it is an expression of the form

$$(L_{11} \times L_{12}) \cup \ldots \cup (L_{m1} \times L_{m2})$$

where $m$ is a polynomial over $|\alpha_1|, |\alpha_2|$ and $|\Sigma|$ and each $L_{ij}$ has a DFA of size polynomial in $|R|$ and $max(|\alpha_1|, |\alpha_2|)$.

To find the complexity of a DFA concatenated with a letter or string, refer to the Lemma 3.2 for the former and Lemma 3.3 for the latter.

The set of all solutions for $\alpha_1 X \downarrow_R \alpha_2 Y$ is

$$\Delta(\alpha_1, \alpha_2) = \left\{ (w_1 x_1, z_1 x_1) \mid (w_1, z_1) \in Sol(\alpha_1, \alpha_2) \text{ and } x_1 \in IRR(R) \right\}$$

The minimal solutions for the second equation with $\beta_1$ and $\beta_2$, $Sol(\beta_1, \beta_2)$, can be found by following the same steps. Thus $Sol(\beta_1, \beta_2)$ can also be expressed as the union of cartesian products of regular languages:

$$(L_{11}' \times L_{12}') \cup \ldots \cup (L_{n1}' \times L_{n2}')$$

where $n$ is also a polynomial over $|\beta_1|, |\beta_2|$ and $|\Sigma|$. The set of all solutions for $\beta_1 X = \beta_2 Y$ equals to

$$\Delta(\beta_1, \beta_2) = \left\{ (w_2 x_2, z_2 x_2) \mid (w_2, z_2) \in Sol(\beta_1, \beta_2) \text{ and } x_2 \in IRR(R) \right\}$$

The solutions for both the equations are the tuples $(w, z) \in \Delta(\alpha_1, \alpha_2) \cap \Delta(\beta_1, \beta_2)$. That is, there must be $w_1, w_2, z_1, z_2, x_1, x_2$ such that

$$(w_1, z_1) \in Sol(\alpha_1, \alpha_2), (w_2, z_2) \in Sol(\beta_1, \beta_2) \text{ and }$$

$$w = w_1 x_1 = w_2 x_2 \text{ and }$$

$$z = z_1 x_1 = z_2 x_2$$

$^3$We can define minimality in terms of the partial order $(x, y) \subseteq (xz, yz)$ for all $z$. 

ON PROBLEMS DUAL TO UNIFICATION: THE STRING-REWRITING CASE
If $x_1$ is a suffix of $x_2$, i.e., $x_2 = x_2'x_1$, then

$$w_1 = w_2 x_2'$$

$$z_1 = z_2 x_2'$$

(Similarly we repeat the same steps when $x_2$ is a suffix of $x_1$.)

Recall that $(w_1, z_1) \in L_i \times L_j$ for some $i > 0$, and $(w_2, z_2) \in L'_i \times L'_j$ for some $j > 0$. Thus $x'_2 \in L_i \setminus L'_i$ and $x'_2 \in L_j \setminus L'_j$ where \ stands for the left quotient operation on languages, defined as $A \setminus B := \{v \in \Sigma^* | \exists u \in B : uv \in A\}$ (See Lemma 3.4 for more details, previous lemmas build up to Lemma 3.4). Thus there is a solution if the intersection of $L_i \setminus L'_i$ and $L_j \setminus L'_j$ is nonempty. This check has to be repeated for every $i, j$. The process of finding the intersection of two languages is explained in Lemma 3.5 and to be able to find the strings in the intersection and the quotient, you need to follow the steps in Lemma 3.6.

**Lemma 3.2.** Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA and $a \in \Sigma$. Then there exists a DFA $M' = (Q', \Sigma, \delta', q'_0, F')$ that recognizes $L(M) \circ \{a\}$ such that $|F'| \leq |F|$ and $|Q'| \leq |Q| + |F|$.

**Proof.** The concatenation of a letter with a DFA can be easily achieved by adding extra transitions from each final state $q_i$ to a new state $p_i$ for the symbol $a$. However, that turns the DFA $M$ into a non-deterministic finite automaton (NFA). We claim that there exists a DFA $M'$ with $|F|$ as the upper bound for accepting states.

Consider Figure 2: $q_{f1}$ through $q_{fn}$ are the $n$ final states of the given DFA $M$. Suppose $\delta(q_{fi}, a) = r_i$. Then the subset construction gives us the new transition $\delta'([q_{fi}], a) = \{r_i, p_i\}$, where $\{r_i, p_i\}$ is a label for a new state. The new accepting states for the new DFA $M'$ will be $\{r_i, p_i\}$, such that $1 \leq i \leq n$.

![Diagram](a.png) \hspace{1cm} ![Diagram](b.png)

**Figure 2.** DFA $M$ concatenation with a single letter $a$.

Besides, if the transitions for the letter $a$ from two earlier accepting states have the same destination state, we can combine the new accepting states that were created. Thus in Figure 3, the state $\{r_1, p_1\}$ can be assigned to $\delta'([q_{fn}], a)$, avoiding needless duplication.
Thus the number of final states, $|F'|$ for the DFA $M'$ is less than or equal to the original number of final states, $|F|$, in DFA $M$.

Total number of states $|Q'|$ for $M'$ is bounded by the number of the final states $|F|$ in $M$ as well as the number of total states, $|Q|$, in $M$. Therefore, the number of states for $M'$ can be less than or equal to the both of the factors, i.e., $|Q'| \leq |Q| + |F|$. □

**Lemma 3.3.** Concatenation of a deterministic finite automaton (DFA) with a single string has the time complexity $O(|F| \times |Z| \times |\Sigma|)$, where $|F|$ is the number of final states in the DFA, $|Z|$ is the length of the string and $|\Sigma|$ is the size of the alphabet.

**Proof.** Recall that the previous lemma proved that the number of states in the new DFA after the concatenation of one letter is at most $|Q| + |F|$ and that the number of (new) accepting states is at most $F$. Thus repeatedly applying this operation will result in a DFA with at most $|Q| + |Z| \times |F|$ states and at most $|F|$ accepting states. The number of new edges will be at most $|F| \times |Z| \times |\Sigma|$. Thus the overall complexity is polynomial in the size of the original DFA. The Figure 4 shows an example of DFA M’s concatenation with the symbols in string Z. □
Lemma 3.4. Let $M_1$ and $M_2$ be DFAs. Then a multiple-entry DFA (MEFA) for $\mathcal{L}(M_2) \setminus \mathcal{L}(M_1)$ can be computed in polynomial time.

Proof. Let $M_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $M_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$, where $\delta_i : Q_i \times \Sigma \rightarrow Q_i$ are the transition functions and $\delta_i^* \subseteq Q_i \times \Sigma^*$ denotes their extensions to $Q_i \times \Sigma^*$.

Let $L_{\text{quo}} = \mathcal{L}(M_2) \setminus \mathcal{L}(M_1)$. Then

$$y \in L_{\text{quo}} \iff \exists x \in \mathcal{L}(M_2) \text{ such that } xy \in \mathcal{L}(M_1) \text{ by definition of left quotient}$$

$$\iff \delta^*_2(q_{02}, x) = q \in F_2 \text{ and } \delta^*_1(xy, q_{01}) = p' \in F_1$$

$$\iff \delta^*_2(q_{02}, x) = q \in F_2 \text{ and } \delta^*_1(x, q_{01}) = p \in Q_1 \text{ and } \delta^*_1(y, p) = p'$$

Let $P$ be the product transition system of the two automata, i.e.,

$$P = (Q_1 \times Q_2, \Sigma, \delta, (q_{01}, q_{02}))$$

where $\delta$ is defined as

$$\delta((r, r'), c) = (\delta_1(r, c), \delta_2(r', c))$$

for all $c \in \Sigma$, $r \in Q_1$, and $r' \in Q_2$. We can assume that states which are not reachable from $(q_{01}, q_{02})$ have been removed. This can be done in time linear in the size of the transition graph.

Now in terms of the product transition system we can say that a string $y$ belongs to $L_{\text{quo}}$ if and only if there exists a string $x$ and states $(p, q) \in Q_1 \times F_2$ and $(p', q') \in F_1 \times Q_2$ such that

$$\delta^*((q_{01}, q_{02}), x) = (p, q) \text{ and } \delta^*((p, q), y) = (p', q')$$

We can now convert $P$ into a multiple-entry DFA (MEFA). In the above case, $(p, q)$ has to be one of the initial states of the new MEFA, and $(p', q')$ one of its final states. Therefore, the states that are reachable from $(q_{01}, q_{02})$ that are in $F_2 \times Q_1$ will be the initial states of the MEFA and $F_1 \times Q_2$ will be the final states of the MEFA. \qed

Lemma 3.5. Given two MEFAs, we can check whether their intersection is empty in polynomial time.

Proof. Consider two MEFAs $A_1 = (Q_1, \Sigma, \delta_1, Q_{s_1}, F_{A_1})$ and $A_2 = (Q_2, \Sigma, \delta_2, Q_{s_2}, F_{A_2})$, $Q_{s_1}$ and $Q_{s_2}$ may include more than one initial state. A string $w$ is accepted by both MEFAs if and only if there exist states $q_{\text{init}}^1 \in Q_{s_1}$ and $q_{\text{init}}^2 \in Q_{s_2}$ such that

$$\delta^*_1(q_{\text{init}}^1, w) \in F_{A_1} \text{ and } \delta^*_2(q_{\text{init}}^2, w) \in F_{A_2}.$$ 

To find such a string $w$, we take the product transition system of the two MEFAs, named as $T$, i.e., $T = (Q_1 \times Q_2, \Sigma, \delta, (Q_{s_1} \times Q_{s_2}))$ where

$$\delta((r, r'), c) = (\delta_1(r, c), \delta_2(r', c))$$

for all $c \in \Sigma$, $r \in Q_1$, and $r' \in Q_2$. A string $w$ is accepted by both of the MEFAs $A_1$ and $A_2$ if and only if there exist states $p, q, p', q'$ such that $p \in Q_{s_1}$, $q \in Q_{s_2}$, $p' \in F_{A_1}$, and $q' \in F_{A_2}$, and

$$\delta^*((p, q), w) = (p', q')$$

We can now apply depth-first search (DFS) to check, in time linear in the size of $T$, if there exists a path from some state in $Q_{s_1} \times Q_{s_2}$ to a state in $F_{A_1} \times F_{A_2}$. \qed
Lemma 3.6. The following problem is decidable in polynomial time:

**Input:** DFAs $M$ and $N$.

**Question:** Do there exist strings $x, y, z$ such that $x \neq y$, $x, y \in \mathcal{L}(M)$, and $xz, yz \in \mathcal{L}(N)$?

**Proof.** Suppose there exist strings $x, y, z$ such that $x \neq y$, $x, y \in \mathcal{L}(M)$, and $xz, yz \in \mathcal{L}(N)$. We call the triple $(x, y, z)$ a solution. Thus, we have two cases:

(i) Both $x$ and $y$ start from an initial state $q_0$ and reach the same state, $q$, in $N$, i.e.,

$$\exists q: \delta^*(q_0^N, x) = \delta^*(q_0^N, y) = q \text{ and } \delta^*(q, z) \in F_N.$$ 

(ii) $x$ and $y$ reach different states, say $q'$ and $q''$, in $N$, i.e.,

$$\exists q', q'': \delta^*(q_0^N, x) = q' \neq q'' = \delta^*(q_0^N, y) \text{ and } \delta^*(q', z) \in F_N \land \delta^*(q'', z) \in F_N.$$ 

Let $A = (Q, \Sigma, \delta, s, F)$ be a DFA and $p$ be a state in $A$. By $A^{F=p}$, we denote a replication of $A$, with the sole difference of $p$ being the only accepting state of $A$. Thus $N^{F \{q\}}$ denotes a replication of $N$, with $q$ being the accepting state of $N$. Then, we classify these states of $N$ which are not dead states into GREEN, ORANGE and BLUE states. Note that confirming the status of $q$ being a dead state can be done in linear time w.r.t. to the size of graph.

- **GREEN states:** \{ $q \mid |\mathcal{L}(N^{F=q}) \cap \mathcal{L}(M)| > 1$ \}.

The state $q$ mentioned in case (i) is a GREEN state. (See Figure 5a)

- **ORANGE states:** \{ $q \mid |\mathcal{L}(N^{F=q}) \cap \mathcal{L}(M)| = 1$ \}.

Suppose that case (i) does not apply, i.e., there are no GREEN states in $N$. Then case (ii) must apply and the states $q'$ and $q''$ must be ORANGE states; in other words, the intersection of $\mathcal{L}(M)$ individually with the two DFAs, $\mathcal{L}(N^{F=q'})$ and $\mathcal{L}(N^{F=q''})$ gives us exactly 1 string for each. Note also that $x$ and $y$ are two strings in $\mathcal{L}(M)$ which are not equal to each other since $q' \neq q''$. (See Figure 5b)

- **BLUE states:** \{ $q \mid |\mathcal{L}(N^{F=q}) \cap \mathcal{L}(M)| = 0$ \}.

The algorithm for finding the triple $(x, y, z)$ is constructed as follows. First, we identify the green and orange states. If there exists a green state, then we have a solution. Otherwise we explore whether there exists an $z$ such that $\delta^*(q', z) \in F_N \land \delta^*(q'', z) \in F_N$ for orange states $q'$ and $q''$, i.e., we check whether

$$\{ z \mid \exists(q', q''): q', q'' \text{ are orange states } \land \delta^*(q', z) \in F_N \land \delta^*(q'', z) \in F_N \}$$

is empty.

Given orange states $q'$ and $q''$, we use DFA intersection to check whether there is a string $z$ that takes both to an accepting state. Let $N_{s=q'}$ denote a replication of $N$, with the difference of $q'$ being the initial state of $N$. $N_{s=q''}$ is similar to the $N_{s=q'}$, but this time $q''$ is the initial state. After creating these two DFAs, we can find if there exists a string $z$ by intersecting the DFA $N_{s=q'}$ with $N_{s=q''}$. This process may have to be repeated for every tuple $(q', q'')$ of orange states. \qed
Lemma 3.7. Let $\mu, \omega, X, Y \in IRR(R)$. Then $\mu X \downarrow \omega Y$ if and only if there exist strings $X', Y', W, \gamma$ such that

1. $\gamma \in MP(\mu) \cup MP(\omega)$,
2. $X = X'W, Y = Y'W$, and
3. $\mu X' \xrightarrow{1_R} \gamma \xleftarrow{1_R} \omega Y'$.

Proof. This proof follows from [OND98] (see Lemma 3.6).

Lemma 3.8. Let $\alpha_1, \alpha_2, \beta_1, \beta_2, X, Y \in IRR(R)$. Then $\alpha_1 X \downarrow \alpha_2 Y$ and $\beta_1 X \downarrow \beta_2 Y$ if and only if there exist strings $X', Y', V, W, \gamma_1, \gamma_2$ such that

1. $\gamma_1 \in MP(\alpha_1) \cup MP(\alpha_2)$,
2. $\gamma_2 \in MP(\beta_1) \cup MP(\beta_2)$,
3. $X = X'VW, Y = Y'VW$, and
4. either
   (a) $\alpha_1 X'V \xrightarrow{1_R} \gamma_1 \xleftarrow{1_R} \alpha_2 Y'V$ and
   (b) $\beta_1 X' \xrightarrow{1_R} \gamma_2 \xleftarrow{1_R} \beta_2 Y'$.
   or
   (c) $\alpha_1 X' \xrightarrow{1_R} \gamma_1 \xleftarrow{1_R} \alpha_2 Y'$ and
   (d) $\beta_1 X'V \xrightarrow{1_R} \gamma_2 \xleftarrow{1_R} \beta_2 Y'V$.

Proof. This proof also follows from [OND98] (see Lemma 3.6).

Corollary 3.9. Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in IRR(R)$. Then there exist irreducible strings $X$ and $Y$ such that $\alpha_1 X \downarrow \alpha_2 Y$ and $\beta_1 X \downarrow \beta_2 Y$ if and only if there exist strings $X', Y', V, \gamma_1, \gamma_2$ such that
Theorem 3.11. Let \( \gamma_1 \in MP(\alpha_1) \cup MP(\alpha_2) \), \( \gamma_2 \in MP(\beta_1) \cup MP(\beta_2) \), \( X'V \) and \( Y'V \) are irreducible, and either
\[
(a) \quad \alpha_1 X'V \xrightarrow{1}_R \gamma_1 \xrightarrow{1}_R \alpha_2 Y'V \quad \text{and}
(b) \quad \beta_1 X' \xrightarrow{1}_R \gamma_2 \xrightarrow{1}_R \beta_2 Y'.
\]
or
\[
(c) \quad \alpha_1 X' \xrightarrow{1}_R \gamma_1 \xrightarrow{1}_R \alpha_2 Y' \quad \text{and}
(d) \quad \beta_1 X'V \xrightarrow{1}_R \gamma_2 \xrightarrow{1}_R \beta_2 Y'.
\]

Proof. The “if” part is obvious since \( \alpha_1 X'V \downarrow \alpha_2 Y'V \) and \( \beta_1 X' \downarrow \beta_2 Y'V \) in both cases (a) and (b). The “only if” part follows from Lemma 3.8. \( \square \)

Recall the definition \( RF(x, y) = \{ z \in IRR(R) \mid xz \rightarrow^*_R y \} \). Thus Corollary 3.9 can be restated as

Lemma 3.10. Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in IRR(R) \). Then there exist irreducible strings \( X \) and \( Y \) such that \( \alpha_1 X \downarrow \alpha_2 Y \) and \( \beta_1 X \downarrow \beta_2 Y \) if and only if there exist strings \( X', Y', V, \gamma_1, \gamma_2 \) such that
\[
(1) \quad \gamma_1 \in MP(\alpha_1) \cup MP(\alpha_2),
(2) \quad \gamma_2 \in MP(\beta_1) \cup MP(\beta_2),
(3) \quad \text{either}
\]
\[
(a) \quad X'V \in RF(\alpha_1, \gamma_1), \quad Y'V \in RF(\alpha_2, \gamma_1), \quad X' \in RF(\beta_1, \gamma_2) \quad \text{and} \quad Y' \in RF(\beta_2, \gamma_2)
\]
or
\[
(b) \quad X' \in RF(\alpha_1, \gamma_1), \quad Y' \in RF(\alpha_2, \gamma_1), \quad X'V \in RF(\beta_1, \gamma_2) \quad \text{and} \quad Y'V \in RF(\beta_2, \gamma_2)
\]

Note that the existence of \( X', Y' \) and \( V \) in statements (a) and (b) in the above lemma can be formulated in terms of the left quotient operation. Thus

Theorem 3.11. Let \( \alpha_1, \alpha_2, \beta_1, \beta_2 \in IRR(R) \). Then there exist irreducible strings \( X \) and \( Y \) such that \( \alpha_1 X \downarrow \alpha_2 Y \) and \( \beta_1 X \downarrow \beta_2 Y \) if and only if there exist strings \( \gamma_1, \gamma_2 \) such that
\[
(1) \quad \gamma_1 \in MP(\alpha_1) \cup MP(\alpha_2),
(2) \quad \gamma_2 \in MP(\beta_1) \cup MP(\beta_2),
(3) \quad \text{either}
\]
\[
(a) \quad RF(\beta_1, \gamma_2) \setminus RF(\alpha_1, \gamma_1) \cap RF(\beta_2, \gamma_2) \setminus RF(\alpha_2, \gamma_1) \neq \emptyset
\]
or
\[
(b) \quad RF(\alpha_1, \gamma_1) \setminus RF(\beta_1, \gamma_2) \cap RF(\alpha_2, \gamma_1) \setminus RF(\beta_2, \gamma_2) \neq \emptyset
\]

Proof. The proof follows from the above lemmas and corollaries. \( \square \)
3.2. One-mapping CE Problem for Monadic Systems. The one-mapping case of the CE problem is decidable for monadic and convergent string rewriting systems. We can show it using a construction similar to the two-mappings case. However it will be a slightly simpler approach since we only have two input strings $\alpha$ and $\beta$ as opposed to four. The algorithm for the one-mapping case also runs in polynomial time as explained below:

**Theorem 3.12.** The following CE problem is decidable in polynomial time for monadic, finite and convergent string rewriting systems.

**Input:** A string-rewriting system $R$ on an alphabet $\Sigma$, and irreducible strings $\alpha, \beta \in \Sigma^*$.  
**Question:** Do there exist distinct irreducible strings $X, Y \in \text{IRR}(R)$ such that $\alpha X \downarrow_R \alpha Y$ and $\beta X \downarrow_R \beta Y$?

**Proof.** This follows from Lemma 3.16 and 3.17, since we only need to check whether there are strings $X', Y', V, \gamma_1, \gamma_2$ such that $X'$ and $Y'$ are distinct, $\gamma_1$ and $\gamma_2$ belong to $MP(\alpha)$ and $MP(\beta)$ respectively, and one of the following symmetric cases hold:

(a) $X'V, Y'V \in RF(\alpha, \gamma_1)$ and  
(b) $X', Y' \in RF(\beta, \gamma_2)$

or

(c) $X', Y' \in RF(\alpha, \gamma_1)$ and  
(d) $X'V, Y'V \in RF(\beta, \gamma_2)$

First of all there are only polynomially many strings in $MP(\alpha)$ and $MP(\beta)$. The two cases can be checked in polynomial time by Lemma 3.6. 

Figure 6 illustrates how $\alpha X$ reduces to its normal form $\alpha_1 a X_2$, ($\alpha$ and $X$ are irreducible strings.)
Below are the lemmas for the one-mapping case:

**Lemma 3.13.** Let \( \mu, X, Y \in \text{IRR}(R) \) where \( X \neq Y \). Then \( \mu X \downarrow \mu Y \) if and only if there exist strings \( X', Y', W, \gamma \) such that

1. \( \gamma \in MP(\mu) \),
2. \( X = X'W, Y = Y'W, X' \neq Y' \) and
3. \( \mu X' \xrightarrow{1} R \gamma \xrightarrow{1} \mu Y' \).

**Proof.** Let \( Z \) be the normal form of \( \mu X \) and \( \mu Y \). Then there exists strings \( \mu_1, \mu_2, \mu_3, \mu_4, X_1, X_2, Y_1, Y_2 \) such that \( \mu = \mu_1 \mu_2 = \mu_3 \mu_4, X = X_1 X_2, Y = Y_1 Y_2 \) and

\[
\mu_2 X_1 \xrightarrow{1} a, \\
\mu_4 Y_1 \xrightarrow{1} b, \quad \text{and} \\
Z = \mu_1 a X_2 = \mu_3 b Y_2.
\]

where \( a, b \in \Sigma \cup \{\lambda\} \). If \( X_1 = Y_1 \), then the same reduction can be applied on both sides, i.e., \( \mu_2 = \mu_4 \) and \( a = b \). But the rest of the string \( X_2 = Y_2 \) since \( X \neq Y \). Therefore, we conclude that \( X_1 \neq Y_1 \). It can also be seen that \( \mu_1 a, \mu_3 b \in MP(\alpha) \).

We now consider two cases:

(a) \( X_2 \) is a suffix of \( Y_2 \): Let \( Y_2 = Y'_2 X_2 \). Then \( \mu_1 a = \mu_3 b Y'_2 \). We can take \( \gamma = \mu_1 a, X' = X_1, Y' = Y'_1 Y'_2 \) and \( W = X_2 \).

(b) \( Y_2 \) is a suffix of \( X_2 \): Let \( X_2 = X'_2 Y_2 \). Then \( \mu_1 a X'_2 = \mu_3 b \). In this case we can take \( \gamma = \mu_3 b, X' = X_1 X'_2, Y' = Y_1 \) and \( W = X_2 \).

\( \square \)

**Lemma 3.14.** Let \( \alpha, \beta \in \text{IRR}(R) \). Then there exist distinct irreducible strings \( X \) and \( Y \) such that \( \alpha X \downarrow \alpha Y \) and \( \beta X \downarrow \beta Y \) if and only if there exist irreducible strings \( X', Y', V, W, \gamma_1, \gamma_2 \) such that

1. \( X' \neq Y' \),
2. \( \gamma_1 \in MP(\alpha) \),
3. \( \gamma_2 \in MP(\beta) \),
4. \( X = X'VW, Y = Y'VW \), and
5. either
   
   (a) \( \alpha X'V \xrightarrow{1} R \gamma_1 \xrightarrow{1} \alpha Y'V \) and
   (b) \( \beta X' \xrightarrow{1} R \gamma_2 \xrightarrow{1} \beta Y' \),
   
   or

   (c) \( \alpha X' \xrightarrow{1} R \gamma_1 \xrightarrow{1} \alpha Y' \) and
   (d) \( \beta X'V \xrightarrow{1} R \gamma_2 \xrightarrow{1} \beta Y'V \).

**Proof.** Assume that there exist strings \( X', Y', V, \gamma_1, \gamma_2 \) that satisfy the properties above. Let us consider the fifth property. It shows that \( \alpha X'V \downarrow \alpha Y'V \) as well as \( \beta X'V \downarrow \beta Y'V \). Now suppose \( X = X'V \) and \( Y = Y'V \). Therefore, we can see that \( \alpha X \downarrow \alpha Y \) as well as \( \beta X \downarrow \beta Y \) and \( X \) and \( Y \) are distinct irreducible strings.
Conversely, assume that there exist distinct irreducible strings $X$ and $Y$ such that $\alpha X \downarrow \alpha Y$ and $\beta X \downarrow \beta Y$. We start by considering the case $\alpha X \downarrow \alpha Y$ such that $X \neq Y$. By Lemma 3.13, there must be strings $X'$, $Y'$, $Z$ and $\gamma_1$ such that $X = X'Z$, $Y = Y'Z$ and $\alpha X' \xrightarrow{1_r} \gamma_1 R \xrightarrow{1_r} \alpha Y'$ where $\gamma_1 \in MP(\alpha)$. Similarly, for the case $\beta X \downarrow \beta Y$, $X$ can be written as $X''Z'$ and $Y$ can be written as $Y''Z'$ such that $\beta X'' \xrightarrow{1_r} \gamma_2 R \xrightarrow{1_r} \beta Y''$ for some $\gamma_2 \in MP(\beta)$.

We have to consider two cases depending on whether $X'$ is a prefix of $X''$ or vice versa. It is not hard to see that they correspond to the two cases in condition 5.

**Corollary 3.15.** Let $\alpha, \beta \in IRR(R)$. Then there exist distinct irreducible strings $X$ and $Y$ such that $\alpha X \downarrow \alpha Y$ and $\beta X \downarrow \beta Y$ if and only if there exist irreducible strings $X', Y', V, \gamma_1, \gamma_2$ such that

1. $X' \neq Y'$,
2. $\gamma_1 \in MP(\alpha)$,
3. $\gamma_2 \in MP(\beta)$,
4. $X'V$ and $Y'V$ are irreducible, and
5. either
   
   (a) $\alpha X'V \xrightarrow{1_r} \gamma_1 R \xrightarrow{1_r} \alpha Y'V$ and
   
   (b) $\beta X' \xrightarrow{1_r} \gamma_2 R \xrightarrow{1_r} \beta Y'$.

   or

   (c) $\alpha X' \xrightarrow{1_r} \gamma_1 R \xrightarrow{1_r} \alpha Y'$ and
   
   (d) $\beta X'V \xrightarrow{1_r} \gamma_2 R \xrightarrow{1_r} \beta Y'V$.

**Proof.** The proof follows from the proof of Lemma 3.14. 

**Theorem 3.16.** Let $\alpha, \beta \in IRR(R)$. Then there exist distinct irreducible strings $X$ and $Y$ such that $\alpha X \downarrow \alpha Y$ and $\beta X \downarrow \beta Y$ if and only if there exist strings $X', Y', V, \gamma_1, \gamma_2$ such that

1. $X' \neq Y'$,
2. $\gamma_1 \in MP(\alpha)$,
3. $\gamma_2 \in MP(\beta)$,
4. either
   
   (a) $X'V, Y'V \in RF(\alpha, \gamma_1)$ and
   
   (b) $X', Y' \in RF(\beta, \gamma_2)$

   or

   (c) $X', Y' \in RF(\alpha, \gamma_1)$ and
   
   (d) $X'V, Y'V \in RF(\beta, \gamma_2)$

**Proof.** The proof follows from the above lemmas, corollaries and by the definition of right factor.

**Lemma 3.17.** Let $T$ be monadic, finite, and convergent string rewriting system. Let $x, y \in IRR(T)$. Then $RF(x, y)$ is a regular language and a DFA for it can be constructed in polynomial time.
Proof. A DFA accepting $RF(x, y)$ will be given by $M = (Q, \Sigma, \delta, q_0, F)$ where the set of states $Q \subseteq (MP(x) \times IRR(T)) \cup \{\bot\}$ and $\bot$ will be used to label the unique dead state of $M$. Let $q_0 = (x, \lambda) \in Q$ be the start state of $M$. The transition function is then

$$
\delta((\alpha, \beta), a) :=
\begin{cases}
\bot & \beta a \notin IRR(T) \\
(\alpha \beta a \downarrow, \lambda) & \beta a, \alpha \beta a \notin IRR(T) \\
(\alpha, \beta a) & \alpha \beta a \in IRR(T), \beta a \text{ proper substring of a lhs} \\
(\alpha, \beta) & \alpha \beta a \in IRR(T), \beta a \text{ not a proper substring of a lhs}, \alpha \beta a \text{ prefix of y} \\
\bot & \text{otherwise}
\end{cases}
$$

We then set $F = \{(u, v) \in MP(x) \times IRR(T) | y = uv\}$. Correctness of the construction follows from the fact that for all $(z_1, z_2) \in Q$ we have that $z_1 z_2 \in IRR(T)$ and that $z_2$ is a substring of either the left-hand sides of rules in $T$ or of $y$.

Additionally, $M$ can be constructed in polynomial time as $x, y$ are fixed and a DFA for $IRR(T)$ can be constructed in polynomial time.

Conclusion and Future Work

Inspired by past works done on string rewriting systems, we explored the Fixed Point, Common Term, and Common Equation problems. For these problems we looked at string rewriting systems that were convergent, length-reducing, dwindling, and monadic. We provided complexity results for the Fixed Point problem for convergent and dwindling SRSs, for the Common Term problem we provided complexity results for dwindling SRSs, and for the Common Equation problem we provided complexity results for dwindling and monadic SRSs.

For the sake of brevity and clarity, we only discussed string rewriting systems in this paper. Our future work will include the investigation of these problems for general term rewriting systems. Additionally, we will also explore forward closed SRSs.

References

[AC06] Martín Abadi and Véronique Cortier. Deciding knowledge in security protocols under equational theories. *Theoretical Computer Science*, 367(1-2):2–32, 2006.

[AIN17a] Zümruıt Akçam, Daniel S. Hono II, and Paliath Narendran. On problems dual to unification. *arXiv e-prints*, page arXiv:1706.05607, June 2017.

[AIN17b] Zümruıt Akçam, Daniel S. Hono II, and Paliath Narendran. On problems dual to unification. In Adrià Gascón and Christopher Lynch, editors, *Informal Proceedings of the 31st International Workshop on Unification (UNIF 2017)*, 2017. [https://unif-workshop.github.io/UNIF2017/](https://unif-workshop.github.io/UNIF2017/).

[Akc18] Zümruıt Akçam. *On Problems Dual to Unification: The String-Rewriting Case*. PhD thesis, State University of New York at Albany, ProQuest, 2018.

[Bau05] Mathieu Baudet. Deciding security of protocols against off-line guessing attacks. In *Proceedings of the 12th ACM Conference on Computer and Communications Security*, CCS ’05, pages 16–25, New York, NY, USA, 2005. ACM.

[BN99] Franz Baader and Tobias Nipkow. *Term Rewriting and All That*. Cambridge University Press, 1999.

[BO84] Günther Bauer and Friedrich Otto. Finite complete rewriting systems and the complexity of the word problem. *Acta Informatica*, 21(5):521–540, 1984.

[BO93] Ronald V Book and Friedrich Otto. *String-rewriting systems*. Springer, 1993.
[Boo87] Ronald V. Book. Thue systems as rewriting systems. Journal of Symbolic Computation, 3(1-2):39–68, 1987.

[Car91] Anne-Cécile Caron. Linear bounded automata and rewrite systems: Influence of initial configurations on decision properties. In TAPSOFT’91, pages 74–89. Springer, 1991.

[CD09] V. Cortier and S. Delaune. A method for proving observational equivalence. In 2009 22nd IEEE Computer Security Foundations Symposium, pages 266–276, July 2009.

[CDK09] Ştefan Ciobăcă, Stéphanie Delaune, and Steve Kremer. Computing Knowledge in Security Protocols under Convergent Equational Theories. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[CR10] Yannick Chevalier and Michaël Rusinowitch. Compiling and securing cryptographic protocols. Information Processing Letters, 110(3):116–122, 2010.

[EKR82] Andrzej Ehrenfeucht, Juhani Karhumäki, and Grzegorz Rozenberg. The (Generalized) Post Correspondence Problem with lists consisting of two words is decidable. Theoretical Computer Science, 21:119–144, 1982.

[Gil79] Robert H. Gilman. Presentations of groups and monoids. Journal of Algebra, 57(2):544–554, 1979.

[Kur64] S-Y Kuroda. Classes of languages and linear-bounded automata. Information and Control, 7(2):207–223, 1964.

[Nic08] François Nicolas. (Generalized) Post Correspondence Problem and semi-Thue systems. CoRR, abs/0802.0726, 2008.

[NO85] Paliath Narendran and Friedrich Otto. Complexity results on the conjugacy problem for monoids. Theoretical Computer Science, 35:227–243, 1985.

[NO97] Paliath Narendran and Friedrich Otto. Single versus simultaneous equational unification and equational unification for variable-permuting theories. J. Autom. Reasoning, 19(1):87–115, 1997.

[NOW84] Paliath Narendran, Friedrich Otto, and Karl Winklmann. The uniform conjugacy problem for finite Church-Rosser Thue systems is NP-complete. Information and Control, 63(1/2):58–66, 1984.

[OND98] Friedrich Otto, Paliath Narendran, and Daniel J. Dougherty. Equational unification, word unification, and 2nd-order equational unification. Theoretical Computer Science, 198(1-2):1–47, 1998.

[Ott86] Friedrich Otto. On two problems related to cancellativity. Semigroup Forum, 33(1):331–356, 1986.

[Sip96] Michael Sipser. Introduction to the Theory of Computation. International Thomson Publishing, 1st edition, 1996.