On the covering radius of small codes versus dual distance

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Abstract

Tietävänäinen’s upper and lower bounds assert that for block-length-\(n\) linear codes with dual distance \(d\), the covering radius \(R\) is at most \(\frac{n}{2} - (\frac{1}{2} - o(1)) \sqrt{dn} \) and typically at least \(\frac{n}{2} - \Theta(\sqrt{dn \log \frac{n}{d}})\). The gap between those bounds on \(R - \frac{n}{2}\) is an \(\Theta(\sqrt{\log n})\) factor related to the gap between the worst covering radius given \(d\) and the sphere-covering bound. Our focus in this paper is on the case when \(d = o(n)\), i.e., when the code size is subexponential and the gap is \(w(1)\). We show that up to a constant, the gap can be eliminated by relaxing the covering requirement to allow for missing \(o(1)\) fraction of points. Namely, if the dual distance is at least \(d \geq 7\), where \(d = o(n)\) is odd, then for sufficiently large \(n\), almost all points can be covered with radius \(R \leq \frac{n}{2} - \frac{1}{13} (d - 5) \sqrt{n \log d} - \frac{1}{2}\). Compared to random linear codes, our bound on \(R - \frac{n}{2}\) is asymptotically tight up to a factor less than 3. We give applications to dual BCH codes. The proof builds on the author’s previous work on the weight distribution of cosets of linear codes, which we simplify in this paper and extend from codes to probability distributions on \(\{0, 1\}^n\), thus enabling the extension of the above result to \((d - 1)\)-wise independent distributions.

1 Introduction

The covering radius of a subset \(C\) of the Hamming cube \(\{0, 1\}^n\) is the minimum \(r\) such that any vector in \(\{0, 1\}^n\) is within Hamming distance at most \(r\) from \(C\). That is, the covering radius of \(C\) is the minimum \(r\) such that \(H_n(C; r) = \{0, 1\}^n\), where \(H_n(C; r) = \cup_{x \in C} H_n(x; r)\) is the \(r\)-neighborhood of \(C\) with respect to the Hamming distance. See [1] for a comprehensive reference on covering codes.

In 1990, Tietävänäinen derived an upper bound on the covering radius \(R\) of a block-length-\(n\) linear code \(C\) in terms of only its minimum dual distance \(d\):

**Theorem 1.1 (Tietävänäinen [2, 3]) (Upper bound on covering radius of codes in terms of dual distance)** Let \(C \subset \mathbb{F}_2^n\) be an \(\mathbb{F}_2\)-linear code whose dual has minimum distance \(d \geq 2\). Then the covering radius \(R\) of \(C\) is at most

\[
\begin{cases}
\frac{n}{2} - \sqrt{s(n - s)} + s^{1/6} \sqrt{n - t} & \text{if } d = 2s \text{ is even} \\
\frac{n}{2} - \sqrt{s(n - 1 - s)} + s^{1/6} \sqrt{n - 1 - t - \frac{1}{2}} & \text{if } d = 2s + 1 \text{ is odd}.
\end{cases}
\]

Prior to Tietävänäinen’s work, the relation between the covering radius and dual distance was investigated in [4]-[6]. In the \(d = \Theta(n)\) regime, Tietävänäinen’s bound was later improved in a sequence of works [7] - [16] (see also [17]). For small values of \(d\) including the \(d = o(n)\) regime, it is still the best known upper bound. In a recent paper [18], we showed that for \(d \leq \frac{2^{1/3}}{\log^2 n}\),

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Tietäväinen’s bound on $R - \frac{n}{2}$ is asymptotically tight up to a factor of 2 for $(d - 1)$-wise independent probability distributions on $\{0,1\}^n$, of which linear codes with dual distance $d$ are special cases.

By combining the sphere-covering bound and Gilbert-Varshamov bound, Tietäväinen established also a simple lower bound on the covering radius as function of the dual distance. For comparison purposes, we need the following version of the lower bound tailored to the small $d$ regime.

**Lemma 1.2 (Small codes version of Tietäväinen’s lower bound on the covering radius in terms of dual distance)** Let $n \geq 1$ be an integer and $n \leq K \leq 2^{n-1}$ be an integer power of 2. Then, all but at most $\frac{1}{n}$ fraction of $\mathbb{F}_2$-linear codes $C \subset \mathbb{F}_2^n$ of size $K$ have covering radius

$$R \geq \frac{n}{2} - \sqrt{\frac{dn}{2} \log \frac{en}{d}} + n \log (n + 1),$$

where $d$ is the minimum distance of $C^\perp$.

For a proof of Lemma 1.2 see Lemma 1.3 with $\varepsilon = 0$.

The difference between Tietäväinen’s upper and lower bounds on $R - \frac{n}{2}$ is a $\Theta(\sqrt{\log \frac{n}{d}})$ factor. The focus of this brief paper is on linear codes with dual distance $d = o(n)$, which corresponds to the case when the code size is subexponential and the factor $\Theta(\sqrt{\log \frac{n}{d}})$ grows with $n$. Our study is motivated by this gap which is related to the gap between the typical and the worst possible covering radius given $d$. In what follows, we highlight the gap by comparing the covering radius of dual BCH codes with the typical covering radius of linear codes of the same size.

It follows from the work of Cohen and Blinovskii that the typical covering radius of linear codes achieve the sphere-covering bound. Cohen’s showed that there are linear codes up to the sphere-covering bound:

**Theorem 1.3 (Cohen [19]; see also [1, Chapter 12])** (Linear codes up to the sphere-covering bound) For any $n \geq 1$ and $0 < R \leq n$, there exists an $\mathbb{F}_2$-linear code $C \subset \mathbb{F}_2^n$ of covering radius $R$ and dimension

$$k \leq \left\lceil \log_2 \frac{n(\log 2)}{\nu_n(R)} \right\rceil,$$

where $\nu_n(R)$ is the probability with respect to the uniform distribution of the radius-$R$ hamming ball $\mathcal{H}_n(0; R)$.

Later, Blinovskii [20, 21] showed that almost all linear codes achieve the sphere-covering bound. See also [1, Chapter 12] and the references therein.

To illustrate the gap in the case of dual BCH code, we need the following immediate corollary to Cohen’s theorem customized to small codes. We include a proof in Appendix A for completeness.

**Corollary 1.4 (Explicit version for small codes)** For each $\varepsilon > 0$, there exists $\delta > 0$ such that the following holds. Let $n \geq 1$ be an integer and $s > 1$ be such that $s \log_2 n \leq \delta n$. Then for $n$ large enough, there exists an $\mathbb{F}_2$-linear code $C \subset \mathbb{F}_2^n$ of dimension at most $\lceil s \log_2 n \rceil$ and covering radius

$$R \leq \frac{n}{2} - \sqrt{\frac{(s - 1)n \log n}{2 + \varepsilon}} + \sqrt{2n} + 2.$$

Before moving to the next section, we note that a related result is the explicit construction of polynomial size codes of covering radius of $\frac{n}{2} - c\sqrt{n \log n}$, for any constant $c$, by Alon, Rabani, and Shpilka [22].
1.1 The gap in the case of dual BCH codes

Consider the block-length-$n$ dual BCH codes $C = BCH(s, m)^{\perp}$, where $m \geq 2$ and $s \geq 1$ are integers such that $2s-2 < 2^{m/2}$, i.e., $s < \frac{1}{2}\sqrt{2^{m}+1}$, and $n = 2^{m}-1$. The dimension of $C$ is $k = sm = s \log_2(n+1) > s \log_2 n$, the minimum distance of $C^{\perp}$ is at least $d = 2s+1$, and the covering radius $R$ of $C$ satisfies:

$$R \leq \frac{n}{2} - \sqrt{s(n-1-s)} + s^{1/6}\sqrt{n-s} - \frac{1}{2} \quad (1)$$

$$R \geq \frac{n}{2} - (s-1)\sqrt{n} + 1 - \frac{1}{2} \quad (2)$$

The upper bound (1) is Tietäväinen’s bound and the lower bound (2) is Weil-Carlitz-Uchiyama bound (see Section 2.5). Applying Corollary 1.4 to linear codes of dimension comparable to the minimum distance $\sqrt{n}$, we get that there exists an $\mathbb{F}_2$-linear code $C' \subset \mathbb{F}_2^n$ of dimension $k' \leq [s \log_2 n] \leq k$ and covering radius

$$R' \leq \frac{n}{2} - \sqrt{(s-1)n \log n} + \sqrt{2n} + 2 \quad (3)$$

Comparing (1) and (3), we see that the upper bound on $R - \frac{n}{2}$ in (1) is worse than that in (3) by a factor of $\Theta(\sqrt{\log n})$. The same factor appears if we compare the lower bound (2) with the upper bound (3) when $s = \Theta(1)$. That is, in the $s = \Theta(1)$ regime, while the actual covering radius of $BCH(s, m)^{\perp}$ is $R = \frac{n}{2} - \Theta(\sqrt{n})$, linear codes of smaller dimension have covering radius $R = \frac{n}{2} - \Theta(\sqrt{n \log n})$.

1.2 Summary of results

For dual BCH codes $BCH(s, m)^{\perp}$, where $s \geq 3$ and $2s-2 < 2^{m/2}$, we show that the $\Theta(\sqrt{\log n})$ gap can be eliminated by relaxing the covering requirement: instead of covering all the vectors in $\{0, 1\}^n$, we can guarantee covering all but $o(1)$ fraction of them with radius $\frac{n}{2} - \Theta(\sqrt{sn \log n})$. More generally, we show that if a linear code has dual minimum distance at least $d$, where $d \geq 7$ is odd and sublinear in $n$, then for $n$ large enough, almost all the points in $\{0, 1\}^n$ can be covered with radius $R \leq \frac{n}{2} - \sqrt{\frac{1}{13}(d-5)n \log \frac{n}{d-1}}$. This bound on $R - \frac{n}{2}$ asymptotically matches Tietäväinen lower bound up to factor less than 3. It also asymptotically matches up to the same factor an adaptation of Tietäväinen lower bound to almost-all-coverings, i.e., compared to random linear codes, it is tight up to a constant factor less than 3. The proof builds on the author’s previous work on the weight distribution of cosets of linear codes with given bilateral minimum distance [23].

We also simplify in this paper a part of the proof in [23] which makes it possible to extend the results in [23] as well as the above results from codes to probability distributions. In particular, we extend the above results on the almost-all covering radius from codes with dual distance $d$ to $(d-1)$-wise independent distributions, of which linear codes with dual distance $d$ are special cases. A probability distribution $\mu$ on $\{0, 1\}^n$ is called $k$-wise independent if for $(x_1, \ldots, x_n) \sim \mu$, each $x_i$ is equally likely to be 0 or 1 and any $k$ of the $x_i$’s are statistically independent. Note that the covering radius of a probability distribution on $\{0, 1\}^n$ is interpreted as the covering radius of its support.

We elaborate below on the results in the case of linear codes. Their extensions to distributions are presented in Section 5.

Definition 1.5 (Almost-all covering) Let $0 \leq \varepsilon \leq 1$. The \(\varepsilon\)-covering radius of a subset $C$ of the Hamming cube $\{0, 1\}^n$ is the minimum $r$ such that the fraction of points of the Hamming cube not contained the $r$-Hamming-neighborhood $H_n(C; r)$ of $C$ is at most $\varepsilon$. 


Thus the covering radius corresponds to \( \varepsilon = 0 \). The notion of almost-all-covering goes back to the argument of Blinovskii [20 21] in his proof that almost all linear codes achieve the sphere-covering bound.

First we establish the following nonasymptotic bound.

**Theorem 1.6 (Upper bound on the almost-all-covering radius of small codes in terms of dual distance)** Let \( C \subseteq \mathbb{F}_2^n \) be an \( \mathbb{F}_2 \)-linear code whose dual \( C^\perp \) has minimum distance at least \( d \), where \( d \geq 7 \) is an odd integer. Let \( R > 0 \) be a real number. Then the fraction of points in hamming cube not covered by \( H_n(C; R) \) is at most

\[
\varepsilon = \frac{d}{v_{n+d}(R)} \left( e \log \frac{n+d}{d-1} \right)^{d-1} \left( \frac{d-1}{n+d} \right)^{\frac{d-5}{d}},
\]

where \( v_{n+d}(R) \) is the probability with respect to the uniform distribution of the radius-\( R \) hamming ball \( H_{n+d}(0; R) \) in \( \{0,1\}^{n+d} \). That is, if \( \varepsilon \leq 1 \), then the \( \varepsilon \)-covering radius of \( C \) is at most \( R \).

The proof of Theorem 1.6 builds on [23], where it is shown that for an \( \mathbb{F}_2 \)-linear code \( Q \) with dual bilateral minimum distance at least \( b \), the average \( L_1 \)-distance between the weight distribution of a random cosets of \( Q \) and the binomial distribution decays quickly in \( b \), and namely, it is bounded by \( b \left( e \ln \frac{n}{n-1} \right)^{b-1} \left( \frac{b-1}{n} \right)^{b-5} \), if \( b \geq 7 \) is odd. The bilateral minimum distance of a non-zero linear code is the maximum \( b \) such that all nonzero codewords have weights between \( b \) and \( n - b \). The proof of Theorem 1.6 boils down to using Markov Inequality and applying the above result to the block-length \( n + d \) code \( Q \) constructed from \( C \) by appending \( d \) independent bits to \( C \). This simple construction turns the lower bound \( d \) on the minimum distance of \( C^\perp \) into a lower bound on the bilateral minimum distance of \( Q^\perp \).

Then, based on the entropy estimate of the binomial coefficients, we conclude the following bound in the \( d = o(n) \) regime.

**Corollary 1.7 (Explicit asymptotic version)** Let \( C \subseteq \mathbb{F}_2^n \) be an \( \mathbb{F}_2 \)-linear code whose dual \( C^\perp \) has minimum distance at least \( d \), where \( d \geq 7 \) is an odd integer such that \( d = o(n) \). Then, for sufficiently large \( n \), the \( \left( \frac{1}{n} \right)^\Delta \)-covering radius of \( C \) is at most \( R = \frac{n}{d} - \Delta \), where

\[
\Delta = \sqrt{\frac{1}{13} (d-5) n \log \frac{n}{d-1}}.
\]

Comparing the bounds on \( R - \frac{n}{d} \) in Corollary 1.7 and Lemma 1.2 we see that the guarantee given by Corollary 1.7 on \( R - \frac{n}{d} \) is asymptotically not far from Tietäväinen’s lower bound on the covering radius of random linear codes by more than a factor of \( \sqrt{\frac{d}{2}} \leq 3 \). Actually, for almost-all-coverings, the upper bound of Corollary 1.7 is asymptotically tight up to the same factor in comparison to random linear codes. This follows from the following simple variation of Lemma 1.2.

**Lemma 1.8 (Variation of Tietäväinen’s lower bound: lower bound on the almost-all-covering radius of small codes in terms of dual distance)** Consider any \( 0 \leq \varepsilon < 1 \) and let \( n \geq 1 \) be an integer and \( n \leq K \leq 2^{n-1} \) be an integer power of 2. Then, all but at most \( \frac{1}{n} \) fraction of \( \mathbb{F}_2 \)-linear codes \( C \subseteq \mathbb{F}_2^n \) of size \( K \) have \( \varepsilon \)-covering radius

\[
R \geq \frac{n}{d} - \sqrt{\frac{dn}{2} \log \frac{en}{d} + n \log \frac{n+1}{1 - \varepsilon}},
\]

where \( d \) is the minimum distance of \( C^\perp \).
Applying Corollary 1.7 to the dual BCH codes $BCH(s, m)\perp$ with $d = 2s + 1$, where $s \geq 3$ so that $d \geq 7$, we get the following:

**Corollary 1.9 (Application to dual BCH codes)** Let $m \geq 2$ be an integer and $n = 2^m - 1$. Let $s \geq 3$ be an integer such that $2s - 2 < 2^{m/2}$, i.e., $s < \frac{1}{2}\sqrt{n+1} + 1$ and consider the dual BCH code $C = BCH(s, m)\perp$. Then, for sufficiently large $n$, the $(\frac{2s}{m})^{2s-2}$-covering radius of $C$ is at most

$$R = \frac{n}{2} - \sqrt[3]{\frac{1}{13}(2s - 4)n \log \frac{n}{2s}}.$$

For instance, for $s = 3$, we have $R = \frac{n}{2} - \sqrt[3]{\frac{2}{11}n \log \frac{n}{2}}$. Thus, for $BCH(3, m)\perp$, even though we need an $\frac{n}{2} - \Theta(\sqrt{n})$ radius to cover all points in $\{0, 1\}^n$, we can cover almost all of them using an $\frac{n}{2} - \sqrt[3]{\frac{2}{11}n \log \frac{n}{2}}$ radius.

Using Cohen's iterative argument for showing the existence of linear coverings up the sphere-covering bound [19], we conclude from Corollary 1.7 that there exists a small $2^{m/n}$-dimensional linear code which can be added to $C$ to turn the almost cover into a total cover.

**Corollary 1.10 (Adding a small code)** Let $C \subseteq \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code whose dual $C\perp$ has minimum distance at least $d$, where $d \geq 7$ is an odd integer such that $d = o(n)$. Then, there exists an $\mathbb{F}_2$-linear code $D$ of dimension at most $[\log_2 n]$ such that, for sufficiently large $n$, the covering radius of $C + D$ is at most $R = \frac{n}{2} - \sqrt[3]{\frac{1}{13}(d - 5)n \log \frac{n}{d-1}}$.

See Section [3] for a related open problem on dual BCH codes.

Before outlining the rest of the paper in the next section, we highlight additional links with the existing literature. Turning an almost-all linear cover into a total cover goes back to the work of Blinovskii [20, 21]. The notion of bilateral minimum distance $b$ of a linear code is equivalent to its width $\sigma$ which is given by $\sigma = n - 2b$. For small values of $b$, it is more convenient to work with $b$ rather than $\sigma$. In the high rate regime, the relation between the covering radius and the dual width was studied by Sole and Stokes [7].

### 1.3 Paper outline

After introducing some preliminaries in Section 2, we prove Theorem 1.9 in Section 3. In Section 4, we prove Corollaries 1.7 and 1.10. In Section 5, we extend the results from codes to distributions.

### 2 Preliminaries

In this section, we compile some notations and definitions used throughout the paper.

#### 2.1 Notations

Throughout the paper, $n \geq 1$ is an integer and log means $\log_e$. We will use the following notations as in [18]. The finite field structure on $\{0, 1\}$ is denoted by $\mathbb{F}_2$. The set $\{0, \ldots, n\}$ is denoted by $[n]$. The binomial distribution on $[0 : n]$ is denoted by $B_n$, i.e., $B_n(w) = \binom{n}{w}$. The uniform distribution on $\{0, 1\}^n$ is denoted by $U_n$, i.e., $U_n(x) = \frac{1}{2^n}$, for all $x \in \{0, 1\}^n$.

The *Hamming weight* of a vector $x \in \{0, 1\}^n$, which is denoted by $|x|$, is the number of nonzero coordinates of $x$. If $r \geq 0$ is a real number and $x \in \{0, 1\}^n$, $H_n(x; r)$ denotes the *Hamming ball* of radius $r$ centered at $x$, i.e., $H_n(x; r) = \{x \in \{0, 1\}^n : |x+y| \leq r\}$. If $C$ is a subset of $\{0, 1\}^n$, $H_n(C; r)$ denotes the $r$-neighborhood of $C$, i.e., $H_n(C; r) = \bigcup_{x \in C} H_n(x; r)$. 

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See Appendix [3] for a proof of Lemma [8].
Thus, in terms of the above notations, the \( \varepsilon \)-covering radius of a subset \( C \subset \{0, 1\}^n \) is the minimum \( r \) such that \( U_n(H_n(C; r)) \geq 1 - \varepsilon \).

If \( \mu \) is a probability distribution, \( \mathbb{E}_\mu \) denotes the expectation with respect to \( \mu \) and “\( x \sim \mu \)” denotes the process of sampling a random vector \( x \) according to \( \mu \).

**Weight distributions** We will also use the following notations as in [23]. If \( \mu \) is a probability distribution on \( \{0, 1\}^n \), \( \tilde{\mu} \) denotes the corresponding weight distribution on \( [0 : n] \), i.e., for all \( w \in [0 : n] \), \( \tilde{\mu}(w) \define \mu(x \in \{0, 1\}^n : |x| = w) \).

If \( A \subset \{0, 1\}^n \), \( \mu_A \) denotes the probability distribution on \( \{0, 1\}^n \) uniformly distributed on \( A \). Thus \( \mu_A(w) \) is the fraction of points in \( A \) of weight \( w \).

### 2.2 Hamming Ball Volume

Let \( v_n(R) \) denote the probability with respect to the uniform distribution of the radius-\( R \) hamming ball, i.e.,

\[
v_n(R) = \mathbb{U}_n(H_n(0; R)) = \sum_{w \leq R} \mathbb{B}_n(w).
\]

The proofs of Corollaries 1.4 and 1.7 use the lower bound on \( v_n(R) \) in Lemma 2.1 below. The lower bound is based on the following estimate of the binomial coefficients: if \( n \geq 1 \) is an integer and \( 0 < \lambda < 1 \) is such that \( \lambda n \) is an integer, then

\[
\binom{n}{\lambda n} \geq \frac{2^n H(\lambda)}{\sqrt{8 n \lambda (1 - \lambda)}},
\]

where \( H(x) = -x \log_2 x - (1 - x) \log_2 (1 - x) \) is the binary entropy function (see, e.g., [1, Lemma 2.4.2]).

**Lemma 2.1** For each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that the following holds. Let \( R = \frac{n}{\delta} - \Delta \), where \( \Delta > 0 \) is a real number such that \( \Delta \leq \delta n \). Then, for \( n \) large enough,

\[
v_n(R) \geq e^{-2+\epsilon} \left( \frac{\Delta + \sqrt{n + 2}}{\delta} \right)^2.
\]

**Proof:** With \( \lambda = \frac{1}{n} \left( \frac{n}{\delta} - \Delta - \sqrt{2n + 1} \right) \), we have

\[
v_n(R) = \sum_{w \leq \frac{n}{\delta} - \Delta} \mathbb{B}_n(w) \geq \sqrt{2n} \mathbb{B}_n(\lambda n) \geq \sqrt{\frac{2n}{8 n \lambda (1 - \lambda)}} 2^{-n(1 - H(\lambda))} \geq 2^{-n(1 - H(\lambda))}.
\]

Let \( x = \frac{\Delta + \sqrt{n + 2}}{n} \), hence \( \lambda \geq \frac{1}{2} - x \). Since \( H(\frac{1}{2} - x) = 1 - \frac{2x^2}{\log 2} - O(x^4) \), let \( \delta > 0 \) so that \( H(\frac{1}{2} - x) \geq 1 - \frac{(2+\epsilon)x^2}{\log 2} \) for each \( 0 \leq x \leq 2\delta \). Thus

\[
v_n(R) \geq 2^{-n(1 - H(\lambda))} \geq 2^{-n(1 - H(1 - x))} \geq e^{-(2+\epsilon)n} = e^{-2+\epsilon} \left( \frac{\Delta + \sqrt{n + 2}}{\delta} \right)^2.
\]

The claim then holds for \( n \) sufficiently large so that \( \frac{\sqrt{n + 2}}{\delta} \leq \delta \). \( \blacksquare \)

### 2.3 Fourier transform preliminaries

We compile in this section harmonic analysis preliminaries as in [23] [18]. See [23] Section IV for a more detailed treatment. The notions in this section are used in Sections 2.4 and 5.
Consider the abelian group structure \( \mathbb{Z}_2^n = (\mathbb{Z}/2\mathbb{Z})^n \) on the hypercube \( \{0,1\}^n \) and the \( \mathbb{C} \)-vector space \( L(\mathbb{Z}_2^n) = \{ f : \mathbb{Z}_2^n \to \mathbb{C} \} \) endowed with the inner product:

\[
    \langle f, g \rangle = \mathbb{E}_{u \in \mathbb{Z}_2^n} f(u) \overline{g(u)} = \frac{1}{2^n} \sum_x f(x) \overline{g(x)}.
\]

The characters of \( \mathbb{Z}_2^n \) are \( \{ \chi_z \}_{z \in \mathbb{Z}_2^n} \), where \( \chi_z : \mathbb{Z}_2^n \to \{-1,1\} \) is given by \( \chi_z(x) = (-1)^{\langle x,z \rangle} \) and \( \langle x,z \rangle = \sum_{i=1}^n x_i z_i \). They form an orthonormal basis of \( L(\mathbb{Z}_2^n) \), i.e., \( \langle \chi_z, \chi_{z'} \rangle = \delta_{z,z'} \), for each \( z,z' \in \{0,1\}^n \), where \( \delta \) is the Kronecker delta function.

The Fourier transform of a function \( f \in L(\mathbb{Z}_2^n) \) is the function \( \hat{f} \in L(\mathbb{Z}_2^n) \) given by the coefficients of the unique expansion of \( f \) in terms of the characters:

\[
    f(x) = \sum_z \hat{f}(z) \chi_z(x) \quad \text{and} \quad \hat{f}(z) = \langle f, \chi_z \rangle = \mathbb{E}_{u \in \mathbb{Z}_2^n} f(x) \chi_z(x).
\]

### 2.4 Codes, limited independence, and bilateral limited independence

The minimum distance of non-zero \( \mathbb{F}_2 \)-linear code is the minimum weight of a nonzero codeword. If \( D \) is a nonzero \( \mathbb{F}_2 \)-linear code, the bilateral minimum distance of \( D \) is the maximum \( b \) such that \( b \leq |z| \leq n - b \), for each nonzero \( z \in D \).

The following notions on probability distributions are used in Section 5. A probability distribution \( \mu \) on \( \{0,1\}^n \) is called \( k \)-wise independent if for \( (x_1, \ldots, x_n) \sim \mu \), each \( x_i \) is equally likely to be 0 or 1 and any \( k \) of the \( x_i \)'s are statistically independent [24, 25]. Equivalently, in terms of the characters \( \{ \chi_z \}_{z \in \mathbb{Z}_2^n} \), \( \mu \) is \( k \)-wise independent if \( \mathbb{E}_{u \in \mathbb{Z}_2^n} \chi_z = 0 \) for each nonzero \( z \in \{0,1\}^n \) such that \( |z| \leq k \). Linear codes with dual distance at least \( k + 1 \) are special cases of \( k \)-wise independent distributions. Namely, if \( \mu \) is uniformly distributed on an \( \mathbb{F}_2 \)-linear code \( C \subset \mathbb{F}_2^n \), i.e., \( \mu = \mu_C \), then \( \mu \) being \( k \)-wise independent is equivalent to \( C \) having dual minimum distance at least \( k + 1 \).

Accordingly, we define the notion of bilateral \( k \)-wise independence to match the dual bilateral minimum distance in the case of linear codes. We call a probability distribution \( \mu \) on \( \{0,1\}^n \) bilaterally \( k \)-wise independent if \( \mathbb{E}_{u \in \mathbb{Z}_2^n} \chi_z = 0 \) for each nonzero \( z \in \{0,1\}^n \) such that \( |z| \leq k \) or \( |z| \geq n - k \). Thus, if \( \mu \) is uniformly distributed on an \( \mathbb{F}_2 \)-linear code \( C \subset \mathbb{F}_2^n \), then \( \mu \) being bilaterally \( k \)-wise independent is equivalent to \( C \) having bilateral dual minimum distance at least \( k + 1 \).

### 2.5 Dual BCH codes

For a general reference on dual BCH codes, see [20]. We compile in this section some their basic properties used in the introduction. Let \( m \geq 2 \) be an integer and \( n = 2^m - 1 \). Consider the finite field \( \mathbb{F}_{2^m} \) on \( 2^m \) elements and let \( \mathbb{F}_{2^m}^\times \) be \( \mathbb{F}_{2^m} \) excluding zero. Let \( s \geq 1 \) be an integer such that \( 2s - 2 < 2^{m/2} \), i.e., \( s < \frac{1}{2} \sqrt{n+1} + 1 \). Consider the BCH code \( BCH(s,m) \subset \mathbb{F}_2^n \):

\[
    BCH(s,m) = \{(f(a))_{a \in \mathbb{F}_{2^m}^\times} : f \in \mathbb{F}_{2^m}[x] \text{ such that } deg(f) < 2^m - 2s - 1 \} \cap \mathbb{F}_{2^m}^\times.
\]

We have:

a) \( \text{dim}(BCH(s,m)^+) = ms \)

b) The minimum distance of \( BCH(s,m) \) is at least \( 2s + 1 \)

c) (Weil-Carlinz-Uchiyama Bound) For each non-zero codeword \( x \in BCH(s,m)^+ \), we have \( ||x| - 2^{m-1}| \leq (s-1)2^{m/2} \), hence \( ||x| - \frac{n+1}{2}| \leq (s-1)\sqrt{n+1} \).
Let $R$ be the covering radius of dual BCH code $BCH(s, m)\perp$. It follows from (c) that
\[ R \geq \frac{n}{2} - (s - 1)\sqrt{n + 1} - \frac{1}{2}. \]

This holds because with $\bar{1}$ denoting the all-ones vector, we have for each $x \in BCH(s, m)\perp$,
\[ |\bar{1} + x| = n - |x| \geq n - \left(\frac{n+1}{2} - (s - 1)\sqrt{n + 1}\right) = \frac{n}{2} - (s - 1)\sqrt{n + 1} - \frac{1}{2}. \]

## 3 Proof of Theorem 1.6

The statement of Theorem 1.6 is restated below for convenience.

**Theorem 1.6 (Upper bound on the almost-all-covering radius of small codes in terms of dual distance)** Let $C \subseteq F_2^n$ be an $F_2$-linear code whose dual $C\perp$ has minimum distance at least $d$, where $d \geq 7$ is an odd integer. Let $R > 0$ be a real number. Then the fraction of points in hamming cube not covered by $\mathcal{H}_n(C; R)$ is at most
\[ \frac{d}{v_n(R)} \left( e \log \frac{n + d}{d - 1} \right) \frac{d}{n} \frac{d - 1}{n}. \]

The proof builds on [23].

**Theorem 3.1 [23] Corollary 3** (Dual bilateral minimum distance versus weight distribution of cosets of small codes; $L_1$-bound) Let $Q \subseteq F_2^n$ be an $F_2$-linear code whose dual $Q\perp$ has bilateral minimum distance at least $b$, where $b \geq 7$ is an odd integer. Then
\[ E_{u \sim U_n} \| \mu_{Q + u} - B_n \|_1 \leq b \left( e \log \frac{n}{b - 1} \right) \frac{b - 1}{n} \frac{b - 1}{n}. \]

See Section 3.1 for weight distribution notations.

Using Markov Inequality, we obtain the following corollary to Theorem 3.1.

**Corollary 3.2 (Upper bound on the almost-all-covering radius of small codes in terms of dual bilateral distance)** Let $Q \subseteq F_2^n$ be an $F_2$-linear code whose dual $Q\perp$ has bilateral minimum distance at least $b$, where $b \geq 7$ is an odd integer. Let $R > 0$ be a real number. Then the fraction $p$ of points in hamming cube not covered by $\mathcal{H}_n(Q; R)$ is at most
\[ \frac{b}{v_n(R)} \left( e \log \frac{n}{b - 1} \right) \frac{b - 1}{n} \frac{b - 1}{n}. \]

**Proof:** Choose a uniformly random $u \in \{0, 1\}^n$, thus $p$ is the probability that $Q \cap \mathcal{H}_n(u; R) = \emptyset$. Let $f : \{0, 1\}^n \to \{0, 1\}$ be the indicator function of $\mathcal{H}_n(0; R)$, i.e., $f(x) = 1$ if $|x| \leq R$ and $f(x) = 0$ otherwise. If $Q \cap \mathcal{H}_n(0; R) = \emptyset$, i.e., $E_{Q + u} f = 0$, then $|E_{Q + u} f - E_{U_n} f| \geq E_{U_n} f$. Therefore, by Markov Inequality,
\[ p \leq \frac{1}{E_{U_n} f} E_{u \sim U_n} |E_{Q + u} f - E_{U_n} f|. \]

Note that $f$ is a symmetric function in the sense that its value on $x$ depends only on the weight $|x|$ of $x$. Thus, for any $u \in \{0, 1\}^n$, $E_{Q + u} f = E_{\bar{f}} E_{Q + u} \bar{f}$ and $E_{U_n} f = E_{\bar{f}} E_{B_n} \bar{f}$, where $\bar{f} : [0 : n] \to \{0, 1\}$ is 1 if $w \leq R$ and zero otherwise. Therefore,
\[ |E_{Q + u} f - E_{U_n} f| = |E_{\bar{f}} E_{Q + u} \bar{f} - E_{B_n} \bar{f}| \leq \| \mu_{Q + u} - B_n \|_1. \]

Noting that $E_{U_n} f = v_n(R)$, we get
\[ p \leq \frac{1}{v_n(R)} E_{u \sim U_n} \| \mu_{Q + u} - B_n \|_1. \]

The lemma then follows from Theorem 3.1. □
Proof of Theorem 1.6. Let $m = n + d$ and extend $C$ to the $\mathbb{F}_2$-linear code $Q = C \times \{0, 1\}^d \subset \{0, 1\}^m$. Thus $Q^\perp = C^\perp \times 0_J$, where $J = \{n+1, \ldots, n+d\}$ and $0_J \in \{0, 1\}^J$ is the all-zeros vector.

By construction, the bilateral minimum distance of $Q^\perp$ is at least $d$ since $d \leq |z| \leq n = m - d$ for each nonzero $z \in Q^\perp$. Applying Corollary 3.2 to $Q$, we get

$$U_m(\mathcal{H}_m(Q; R)) \geq 1 - \frac{d}{v_{n+d}(R)} \left( e \log \frac{n+d}{d-1} \right)^{\frac{d+1}{2}} \left( \frac{d-1}{n+d} \right)^{\frac{d+3}{2}}. \quad (4)$$

On the other hand,

$$\mathcal{H}_m(Q; R)|_I \subset \mathcal{H}_n(C; R), \quad (5)$$

where $I = \{1, \ldots, n\}$ and $\mathcal{H}_m(Q; R)|_I$ is the restriction of $\mathcal{H}_m(Q; R)$ to $I$. To see why (5) holds, note that for any $x \in \mathcal{H}_m(Q; R)$, we have $|x| \leq R$ for some $y' \in C$ and $y'' \in \{0, 1\}^d$.

Thus $|x|_I + y'\leq R$.

The claim then follows from (5) and (4) via the bounds:

$$U_n(\mathcal{H}_n(C; R)) \geq U_n(\mathcal{H}_m(Q; R)|_I) = U_m(\mathcal{H}_m(Q; R)|_I \times \{0, 1\}^J) \geq U_m(\mathcal{H}_m(Q; R)). \quad \blacksquare$$

4 Proofs of Corollary 1.7 and 1.10

The statements of the both corollaries are restated below for convenience.

Corollary 1.7 (Explicit asymptotic version) Let $C \subseteq \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code whose dual $C^\perp$ has minimum distance at least $d$, where $d \geq 7$ is an odd integer such that $d = o(n)$. Let $R = \frac{d}{2} - \Delta$, where

$$\Delta = \sqrt{\frac{1}{13} (d - 5)n \log \frac{n}{d-1}}.$$

Then, for sufficiently large $n$, the fraction of points in hamming cube not covered by $\mathcal{H}_n(C; R)$ is at most $(\frac{d-1}{n})^{\frac{d+3}{2}}$.

Proof: Write $d = 2t + 5$, where $t \geq 1$ is an integer. Let $m = n + d$, $R' = \frac{d}{2} - \Delta'$, where

$$\Delta' = \sqrt{\frac{(d - 5)m}{12} \log \frac{m}{d-1} - \sqrt{2m} - 2},$$

and

$$p' = \frac{d}{v_m(R')} \left( e \log \frac{m}{d-1} \right)^{t+2} \left( \frac{d-1}{m} \right)^{\frac{t}{2}}.$$

By Theorem 1.6, it is enough to show that for $n$ large enough,

$$R' \leq R \quad (6)$$

and

$$p' \leq \left( \frac{d-1}{n} \right)^{t/6.5}. \quad (7)$$

Note that since $d = o(n)$, we have $m = n(1 + o(1))$ and $d = o(m)$.

Proof of (7): We have $\Delta' = o(m)$ since $d = o(m)$. To see why this holds, note that

$$\Delta' \leq \sqrt{\frac{(d-1)m}{12} \log \frac{m}{d-1}} = \frac{m}{\sqrt{12}} \sqrt{\frac{d-1}{m} \log \frac{m}{d-1}}$$

and that the function $x \log \frac{1}{x}$ is zero at $x = 0$.

Hence, by Corollary 2.1, for any $\epsilon > 0$ and for $m$ large enough

$$v_m(R') \geq e^{-(2+\epsilon)\frac{\Delta'^2 (d-1)}{m+2}} = \left( \frac{d-1}{m} \right)^{\frac{(2+\epsilon)m}{6+2\epsilon}}.$$
Therefore
\[ p' \leq d \left( e \log \frac{m}{d-1} \right)^{t+2} \left( \frac{d-1}{m} \right)^{\frac{(t-1)}{a}}. \]

Let \( a = \frac{m}{d-1} \) and note that \( a = o(n) \) since \( d = o(m) \). We have \( d = 2t + 5 = 2(t + 2) + 1 \leq 2^{t+2} \), hence \( d \left( e \log \frac{m}{d-1} \right)^{t+2} \leq (2e \log a)^{t+2} \leq (2e \log a)^{3t} \). Therefore,
\[ p' \leq \left( \frac{(2e \log a)^{t+2}}{a^{\frac{(t-1)}{a}}} \right)^{\frac{(t-1)}{a}} \leq \left( \frac{1}{a} \right)^{\frac{t}{a}} = \left( \frac{d-1}{n+d} \right)^{\frac{t}{a}} \leq \left( \frac{d-1}{n} \right)^{\frac{t}{a}}, \]
where the second inequality holds for \( \epsilon \) sufficiently small and for \( a \) sufficiently large, i.e., for \( n \) sufficiently large.

**Proof of (9):** We have
\[ R' = \frac{n}{2} - \sqrt{\frac{(d-5)(n+d)}{12}} \log \frac{n+d}{d-1} + \frac{2(n+d)}{d} + 2. \]

Since \( d = o(n) \), we have \( \frac{d}{2} + 2 = o \left( \sqrt{(d-5)(n+d)} \right) \) and \( \sqrt{2(n+d)} = o \left( \sqrt{(n+d) \log \frac{n+d}{d-5}} \right) \), hence, for \( n \) large enough,
\[ R' \leq \frac{n}{2} - \sqrt{\frac{(d-5)(n+d)}{13}} \log \frac{n+d}{d-1} \leq \frac{n}{2} - \sqrt{\frac{(d-5)n}{13}} \log \frac{n}{d-1}. \]

**Corollary 1.10 (Adding a small code)** Let \( C \subseteq \mathbb{F}_2^n \) be an \( \mathbb{F}_2 \)-linear code whose dual \( C^\perp \) has minimum distance at least \( d \), where \( d \geq 7 \) is an odd integer such that \( d = o(n) \). Then there exists an \( \mathbb{F}_2 \)-linear code \( D \) of dimension at most \( \lceil \log_2 n \rceil \) such that, for sufficiently large \( n \), the covering radius of \( C + D \) is at most \( R = \frac{n}{2} - \sqrt{\frac{13}{13}(d-5)n} \log \frac{n}{d-4} \).

**Proof:** Cohen’s argument is based on iteratively augmenting the code by adding points in \( \mathbb{F}_2^n \) to minimize the number of uncovered points ([19]; see also [1, Section 12.3]). Consider the set of points not \( R \)-covered by \( C \), i.e., \( \mathcal{H}_n(C; R) \). Choose \( x^{(1)} \in \mathbb{F}_2^n \) to minimize the number of points not \( R \)-covered by \( C^{(1)} = C \cup (C + x^{(1)}) \). Thus \( U_n(\mathcal{H}_n(C^{(1)}; R)^c) = U_n(\mathcal{H}_n(C; R)^c \cap (\mathcal{H}_n(C; R) + x^{(1)})^c) \). By [1, Lemma 12.3.1], for each \( A \subseteq \mathbb{F}_2^n \), there exists \( x \in \mathbb{F}_2^n \) such that \( U_n(A \cap (A + x)) \leq U_n(A)^2 \). Thus \( U_n(\mathcal{H}_n(C^{(1)}; R)^c) \leq U_n(\mathcal{H}_n(C; R)^c)^2 \).

By repeating this process \( i \) steps, we get that there exists an \( \mathbb{F}_2 \)-linear code \( D \) of dimension \( i \) such that \( U_n(\mathcal{H}_n(C + D; R)^c) \leq U_n(\mathcal{H}_n(C; R)^c)^{2^i} < 2^{-2^i} \) assuming that \( n \) is large enough so that \( \left( \frac{d-1}{n} \right)^{\frac{2^i}{a}} < \frac{1}{2} \). Thus, for \( i = \lceil \log_2 n \rceil \), \( U_n(\mathcal{H}_n(C + D; R)^c) < 2^{-n} \), i.e., \( \mathcal{H}_n(C + D; R) = \{0, 1\}^n \).

**5 Extension from codes to distributions**

In this section, we simplify a part of the proof in [23], which makes it possible to extend the results in [23] and accordingly the results reported in this paper from codes to distributions. Namely, we extend the results in [23] on the weight distribution of cosets of codes with bilateral dual distance at least \( b \) to translations of bilaterally \( k \)-wise independent probability distributions, where \( k = b - 1 \). In particular, we show that if \( \mu \) is a bilaterally \( k \)-wise independent probability distribution on \( \{0, 1\}^n \), then the average \( L_1 \)-distance between the weight distribution of a random translation of \( \mu \) and the binomial distribution decays quickly in \( b \). The decay is exactly the same.
as in [23] with $b - 1$ replaced with $k$. This immediately extends the results reported in this paper on the $\epsilon$-covering radius from codes with dual distance $d$ to $k$-wise independent distributions on $\{0, 1\}^n$, where $k = d - 1$ and the $\epsilon$-covering radius of a distribution is interpreted as that of its support.

In this section, we use the weight distributions, Fourier transform, and limited independence notations and definitions given in Sections 2.1, 2.3, 2.4, respectively. We also need the following definitions.

If $\mu$ is a probability distribution on $\{0, 1\}^n$ and $u \in \{0, 1\}^n$, define $\sigma_u \mu$ to be the translation over $\mathbb{F}_2$ of $\mu$ by $u$, i.e., $(\sigma_u \mu)(x) = \mu(x + u)$. If $f, g : \{0, 1\}^n \rightarrow \mathbb{C}$, define their convolution $f \star g : \{0, 1\}^n \rightarrow \mathbb{C}$ with respect to addition in $\mathbb{Z}_2^n$ by $(f \star g)(x) = \sum_y f(y) g(x + y)$. If $\mu_1, \mu_2$ are probability distributions on $\{0, 1\}^n$, then their convolution $\mu_1 \ast \mu_2$ is a probability distributions on $\{0, 1\}^n$; to sample from $\mu_1 \ast \mu_2$, sample $a \sim \mu_1$, $b \sim \mu_2$, and return $a + b$.

In the proofs of the main results in [23], the only part which relies on the linearity of the code is the following lemma.

**Lemma 5.1** [23] Lemma 14] If $0 \leq \theta < 2\pi$, define $e_{\theta} : \{0, 1\}^n \rightarrow \mathbb{C}$ by $e_{\theta}(x) = e^{i\theta |x|}$. Let $Q \subset \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code and $0 \leq \theta < 2\pi$. Then

$$E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}} - E_{U_n} e_{\theta}|^2 = E_{y \sim \mu_Q} (\cos \theta)^{|y|} - \left(\frac{\cos \theta + 1}{2}\right)^n.$$

**Lemma 5.1** is used in the proof of [23] Theorem 5:

**Theorem 5.2** [23] Theorem 5 (Mean-square-error bound) Let $Q \subset \mathbb{F}_2^n$ be an $\mathbb{F}_2$-linear code whose dual $Q^\perp$ has bilateral minimum distance at least $b = 2t + 1$, where $t \geq 1$ is an integer. Then, for each $0 \leq \theta < 2\pi$, we have the bounds:

a) (Small dual distance bound)

$$E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}} - E_{U_n} e_{\theta}|^2 \leq \left(e \ln \frac{n}{2t}\right)^{2t} \left(\frac{2t}{n}\right)^t.$$

b) (Large dual distance bound)

$$E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}} - E_{U_n} e_{\theta}|^2 \leq 2e^{-\frac{t}{4}}.$$

**Lemma 5.3** below extends Lemma 5.1 from codes to probability distributions and it has a simpler proof.

**Lemma 5.3** Let $\mu$ be a probability distribution on $\{0, 1\}^n$ and $0 \leq \theta < 2\pi$. Then

$$E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}} - E_{U_n} e_{\theta}|^2 = E_{y \sim \mu \ast \mu} (\cos \theta)^{|y|} - E_{y \sim U_n} (\cos \theta)^{|y|}.$$

Note that $E_{y \sim U_n} (\cos \theta)^{|y|} = \left(\frac{\cos \theta + 1}{2}\right)^n$. Moreover, if $\mu = \mu_Q$, where $Q \subset \mathbb{F}_2^n$ is an $\mathbb{F}_2$-linear code, then $\mu_Q \ast \mu_Q = \mu_Q$.

**Proof** We have

$$E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}} - E_{U_n} e_{\theta}|^2 = E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}}|^2 - |E_{U_n} e_{\theta}|^2$$

and

$$E_{u \sim U_n} |E_{\sigma_u \epsilon_{\theta}}|^2 = E_{u \sim U_n} \left|\sum_x \mu(x) e_{\theta}(x + u)\right|^2$$

$$= E_{u \in \{0, 1\}^n} \left|\sum_{x,y} \mu(x) \mu(y) e_{\theta}(u + x) \overline{e_{\theta}(u + y)}\right|$$

$$= \sum_{x,y} \mu(x) \mu(y) E_{u \in \{0, 1\}^n} |e_{\theta}(u + x + y)|$$

$$= E_{\mu \ast \mu} e_{\theta} \otimes \overline{e_{\theta}}.$$
where ⊗ is the weighted convolution operator: if \( f, g : \{0, 1\}^n \to \mathbb{C} \), their weighted convolution \( f \otimes g = \frac{1}{2} f \ast g \), i.e., \( (f \otimes g)(x) = \mathbb{E}_y f(y)g(x+y) \), hence \( \hat{f} \otimes g = \hat{f} \hat{g} \).

Then the Lemma follows from the fact that
\[
(e^\theta \otimes e^\theta)(x) = (\cos \theta)^{|z|}. \tag{8}
\]

Note that \( \mathbb{E}_{U_n} e^\theta \otimes e^\theta = |\mathbb{E}_{U_n} e^\theta|^2 \). Thus, by (8), \( |\mathbb{E}_{U_n} e^\theta|^2 = \mathbb{E}_{W^i \sim U_n} (\cos \theta)^{|y|} \).

To verify (8), we go to the Fourier domain. In the Fourier domain, (8) is equivalent \( e^\theta \hat{\otimes} e^\theta = \hat{g} \cos \theta \), where \( \hat{g}_r(x) = r^{|x|} \). Since \( \hat{f} \otimes g = \hat{f} \hat{g} \), we have to show that \( |\hat{e}|^2 = \hat{g} \cos \theta \). We need the following basic lemma about the Fourier transform of exponential function, e.g., [23, Lemma 11]:

**Lemma 5.4** Let \( r \) be complex number and \( g_r : \{0, 1\}^n \to \mathbb{C} \) be given by \( g_r(x) = r^{|x|} \). Then \( \hat{g}_r(z) = (1+\frac{r}{2})^n \left(1 - \frac{z}{1+r}\right)^{|z|} \). Moreover, if \( r = e^{i\theta} \), then \( \hat{g}_r(z) = e^{in\theta/2} (\cos \theta)^n (-i \tan \theta)^{|z|} \).

Therefore
\[
|\hat{e}_\theta(z)|^2 = \left( \cos \frac{\theta}{2} \right)^{2n} \left( \tan \frac{\theta}{2} \right)^{2|z|} = \left( \frac{1+\cos \theta}{2} \right)^n \left( \frac{1-\cos \theta}{1+\cos \theta} \right)^{|z|} = g_{\cos \theta}(z),
\]

where the second equality follows from the trigonometric identities \( (\cos \frac{\theta}{2})^2 = \frac{1+\cos \theta}{2} \) and \( (\tan \frac{\theta}{2})^2 = \frac{1-\cos \theta}{1+\cos \theta} \).

As explicitly noted in [23, Section V p. 6499], the proof of Theorem 5.2 bounds the term \( \mathbb{E}_{y \in \mu_Q}(\cos \theta)^{|y|} - (\cos \frac{\theta}{2})^2 \) by ignoring the linearity of \( Q \) and using only the bilateral k-wise independence property of \( \mu_Q \), where \( k = b-1 \). This property holds also for \( \mu \ast \mu \); if \( \mu \) is bilaterally \( k \)-wise independent, then so is \( \mu \ast \mu \). This follows from the definition of bilateral \( k \)-wise independence since \( \mathbb{E}_{\mu \ast \mu \chi_z} = (\mathbb{E}_\mu \chi_z)^2 \), for all \( z \in \{0, 1\}^n \).

Accordingly, using Lemma 5.4 as well as [23, Theorem 2] (\( L_\infty \)-bound) and [23, Corollary 3] (\( L_1 \)-bound, i.e., Theorem 5.41 in this paper) extend as follows from codes with bilateral minimum distance at least \( b \) to bilaterally \( k \)-wise independent probability distributions, where \( k = b-1 \).

**Theorem 5.5** (Bilateral limited independence versus weight distribution of translates; mean-square-error bound) Let \( \mu \) be a bilaterally \( k \)-wise independent probability distribution on \( \{0, 1\}^n \), where \( k \geq 2 \) is an even integer. Then, for each \( 0 \leq \theta < 2\pi \), we have the bounds:

a) (Small \( k \))
\[
\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e^\theta - \mathbb{E}_{U_n} e^\theta|^2 \leq \left( e \ln \frac{n}{k} \right)^\frac{k}{n}
\]

b) (Large \( k \))
\[
\mathbb{E}_{u \sim U_n} |\mathbb{E}_{\sigma_u \mu} e^\theta - \mathbb{E}_{U_n} e^\theta|^2 \leq 2e^{-\frac{n}{4k}}.
\]

**Theorem 5.6** (Bilateral limited independence versus weight distribution of translates; \( L_\infty \)-bound) Let \( \mu \) be a bilaterally \( k \)-wise independent probability distribution on \( \{0, 1\}^n \), where \( k \geq 2 \) is an even integer. Then, we have the bounds:

a) (Small \( k \))
\[
\mathbb{E}_{u \sim U_n} \|\sigma_u \mu - B_n\|_\infty \leq \left( e \ln \frac{n}{k} \right)^\frac{1}{2} \left( \frac{k}{n} \right)^\frac{k}{4}.
\]

b) (Large \( k \))
\[
\mathbb{E}_{u \sim U_n} \|\sigma_u \mu - B_n\|_\infty \leq \sqrt{2}e^{-\frac{n}{8k}}.
\]
Theorem 5.7 (Bilateral limited independence versus weight distribution of translates; $L_1$-bound) Let $\mu$ be a bilaterally $k$-wise independent probability distribution on $\{0, 1\}^n$, where $k \geq 6$ is an even integer. Then, we have the bounds:

a) (Small $k$ bound)
\[
\mathbb{E}_{u \sim U_n} \|\sigma_{u\mu} - B_n\|_1 \leq k + 1 \left( e \log \frac{n}{k} \right)^{\frac{k}{2}} \left( \frac{k}{n} \right)^{\frac{k}{2} - 1}.
\]

b) (Large $k$ bound)
\[
\mathbb{E}_{u \sim U_n} \|\sigma_{u\mu} - B_n\|_1 \leq \sqrt{2(n + 1)} e^{-\frac{n}{k}}.
\]

As in [23], Theorem 5.7 follows from Theorem 5.6, which in turns follows from Theorem 5.5.

Accordingly, Theorem 1.6 (Nonasymptotic bound) and Corollaries 1.7 (Explicit asymptotic bound) and 1.10 (Adding a small code) extend as follows from codes with minimum distance at least $d$ to $k$-wise independent probability distributions, where $k = d - 1$.

Definition 5.8 ($\varepsilon$-covering radius of a probability distribution) Let $\mu$ be a probability distribution on $\{0, 1\}^n$. The covering radius of $\mu$ is the covering radius of its support. Equivalently, the covering radius of $\mu$ is the minimum $r$ such that $\mu(H_n(x; r)) \neq 0$ for each $x \in \{0, 1\}^n$.

More generally, if $0 \leq \varepsilon \leq 1$, the $\varepsilon$-covering radius of $\mu$ is the $\varepsilon$-covering radius of its support. Equivalently, the $\varepsilon$-covering radius of $\mu$ is the minimum $r$ such that the fraction of points $x \in \{0, 1\}^n$ such that $\mu(H_n(x; r)) = 0$ is at most $\varepsilon$.

Theorem 5.9 (Limited independence versus almost-all-covering radius) Let $\mu$ be a $k$-wise independent probability distribution on $\{0, 1\}^n$, where $k \geq 6$ is an even integer. Let $R > 0$ be a real number. Let
\[
\varepsilon = \frac{d}{v_{n+k+1}(R)} \left( e \log \frac{n + k + 1}{k} \right)^{\frac{k}{2}} \left( \frac{k}{n + k + 1} \right)^{\frac{k}{2} - 1}
\]
and assume that $\varepsilon \leq 1$. Then the $\varepsilon$-covering radius of $\mu$ is most $R$.

To adapt the proof of Theorem 1.6 into a the setup of Theorem 5.9, given a $k$-wise independent probability distribution $\mu$ on $\{0, 1\}^n$, consider the probability distribution $\gamma = \mu \times U_d$ on $\{0, 1\}^m$, where $d = k + 1$ and $m = n + d$. Then $\gamma$ is bilaterally $k$-wise independent. The reason is that if $z \in \{0, 1\}^m$ is such that $|z| > m - d = n$, then with $I = \{1, \ldots, n\}$ and $J = \{n + 1, \ldots, n + d\}$, we have $z|_I \neq 0$, hence $\mathbb{E}_{\mu|_I} \chi_z = 0$, and accordingly $\mathbb{E}_{\gamma|_I} \chi_z = 0$ since $\mathbb{E}_{\gamma|_I} \chi_z = \mathbb{E}_{\mu|_I} \mathbb{E}_{\mu|_I} \chi_z$.

Corollary 5.10 (Explicit asymptotic version) Let $\mu$ be a $k$-wise independent probability distribution on $\{0, 1\}^n$, where $k \geq 6$ is an even integer such that $k = o(n)$. Then, for sufficiently large $n$, the $(\frac{k}{n})^{k+1}$-covering radius of $\mu$ is at most $R = \frac{2}{\tau} - \Delta$, where
\[
\Delta = \sqrt{\frac{1}{13} (k - 4)n \log \frac{n}{k}}.
\]

Corollary 5.11 (Convolution with a small code) Let $\mu$ be a $k$-wise independent probability distribution on $\{0, 1\}^n$, where $k \geq 6$ is an even integer such that $k = o(n)$. Then there exists an $F_2$-linear code $D$ of dimension at most $\lceil \log_2 n \rceil$ such that, for sufficiently large $n$, the covering radius of $\mu \ast \mu_D$ is at most $R = \frac{2}{\tau} - \sqrt{\frac{1}{13} (k - 4)n \log \frac{n}{k}}$. 
6 Open problems

We conclude with the following open questions:

- As noted in the introduction, the upper bound of Corollary 1.7 on \(R - \frac{n}{2}\), where \(R\) is the almost-all covering radius, is asymptotically tight up to a factor of \(\sqrt{\frac{13}{2}}\) in comparison to random linear codes (see Lemma 1.8). The proofs of Theorem 1.6 and Corollary 1.7 can be easily tuned to bring the \(\sqrt{\frac{13}{2}}\) factor down to \(2 + \epsilon\), for any \(\epsilon > 0\). The gain is at the cost is increasing the fraction of uncovered points while keeping it \(o(1)\). Is it possible to go below \(2\)?

- Corollary 1.7 assumes that the dual distance \(d\) is at least 7. Is it possible to extend it to smaller values of \(d\)?

- Consider the block-length-\(n\) dual BCH code \(C = BCH(s, m)\perp\), where \(m \geq 2\) is an integer, \(n = 2^m - 1\), and \(s\) is an integer such that \(2s - 2 < 2^m/2\). If \(s \geq 3\), we know from Corollaries 1.9 and 1.10 that there exists a small \(\mathbb{F}_2\)-linear code \(D\) of dimension at most \(\left\lceil \frac{s \log n}{\log 2} \right\rceil\) such that, for sufficiently large \(n\), the covering radius of \(C + D\) is at most \(n^2 - (s-1)n\log n \geq n^2 - \Theta(\sqrt{n^3\log n})\). It would be interesting to explicitly construct such a code \(D\) using algebraic tools.

Appendix

A Proof of Corollary 1.4

The corollary is restated below for convenience.

**Corollary 1.4** For each \(\epsilon > 0\), there exists \(\delta > 0\) such that the following holds. Let \(n \geq 1\) be an integer and \(s > 1\) be such that \(s \log n \leq \delta n\). Then for \(n\) large enough, there exists an \(\mathbb{F}_2\)-linear code \(C \subset \mathbb{F}_2^n\) of dimension at most \(\left\lceil \frac{s \log n}{\log 2} \right\rceil\) and covering radius

\[
R \leq n^2 - \sqrt{(s-1)n\log n} + \sqrt{2n} + 2.
\]

Let \(\Delta = \sqrt{(s-1)n\log n} - \sqrt{2n} - 2\). By Theorem 1.3 it is enough to show that \(\log_2 \frac{n\log 2}{vin(\frac{n}{2} - \Delta)} \leq s \log_2 n\), i.e., \(vin(\frac{n}{2} - \Delta) \geq \frac{n\log 2}{\log 2 + s}\). Since \(s \log_2 n \leq \delta n\), we have \(\Delta \leq \sqrt{\frac{2n^3\log n}{2^s}}\). Applying Lemma 2.1 we get that for sufficiently small \(\delta\) and sufficiently large \(n\),

\[
vin(n/2 - \Delta) \geq e^{-(2+\epsilon)(\Delta + \frac{n\log 2}{\log 2})^2} = \frac{1}{n^s-1} > \log_2 2^{n^{s-1}}.
\]

B Proof of Lemma 1.8

The lemma is restated below for convenience.

**Lemma 1.8** Consider any \(0 \leq \epsilon < 1\) and let \(n \geq 1\) be an integer and \(n \leq K \leq 2^{n-1}\) be an integer power of 2. Then, all but at most \(\frac{1}{n}\) fraction of \(\mathbb{F}_2\)-linear codes \(C \subset \mathbb{F}_2^n\) of size \(K\) have \(\epsilon\)-covering radius

\[
R \geq n^2 - \sqrt{\frac{dn}{2} \log \frac{en}{d} + n \log \frac{n+1}{1-\epsilon}},
\]

where \(d\) is the minimum distance of \(C\perp\).
Lemma B.1 (Sphere-covering bound adaptation to almost-all covers) Let $0 \leq \varepsilon < 1$ and $n \geq 1$. Then for any code $C \subset \{0, 1\}^n$ of size $K$, where $K \geq 1$, the $\varepsilon$-covering radius of $C$ at least

$$R \geq \frac{n}{2} - \sqrt{\frac{1}{2} n \log \frac{K}{1 - \varepsilon}}.$$  

The proof of Lemma B.1 follows from exactly the same counting argument used to establish the sphere-covering bound [1, Theorem 12.5.1].

The upper bound on $K$ in terms of $d$ comes from Gilbert-Varshamov bound. Choose the generator matrix $G_{k \perp \times n}$ of the dual code $C^\perp$ uniformly at random, where $k_{\perp} = n - \log_2 K$. Let $d$ be the minimum distance of $C^\perp$ and let $1 \leq d \leq \frac{n}{2} + 1$ be an integer. The probability that $d < d$ is at most

$$\left( |C^\perp| - 1 \right) \left( \frac{n}{d} \right) \left( \frac{en}{d - 1} \right)^{d - 1} \leq \frac{1}{n}$$

if $K \geq f(d)$, where $f(x) = nx \left( \frac{en}{x - 1} \right)^{x - 1}$. Since $f(x)$ is increasing in $x$ for all $1 \leq x \leq n + 1$, we conclude that, with probability at least $1 - \frac{1}{n}$, $d \geq \lceil f^{-1}(K) \rceil$ if $1 \leq \lfloor f^{-1}(K) \rfloor \leq \frac{n}{2} + 1$, hence $K \leq f(d + 1)$ if $f(1) \leq K \leq f(\frac{n}{2} + 1)$. We have $f(1) = n \leq K$ and $f(\frac{n}{2} + 1) > 2^n$ for all $n \geq 1$. Therefore,

$$R \geq \frac{n}{2} - \sqrt{\frac{1}{2} n \log \left( \frac{n(d + 1)}{1 - \varepsilon} \left( \frac{en}{d} \right)^d \right)} \geq \frac{n}{2} - \sqrt{\frac{dn}{2} \log \frac{en}{d} + n \log \frac{n + 1}{1 - \varepsilon}}.$$

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