Corrected Entropy of Friedmann-Robertson-Walker Universe in Tunneling Method

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ABSTRACT: In this paper, we study the thermodynamic quantities of Friedmann-Robertson-Walker (FRW) universe by using the tunneling formalism beyond semiclassical approximation developed by Banerjee and Majhi[25]. For this we first calculate the corrected Hawking-like temperature on apparent horizon by considering both scalar particle and fermion tunneling. With this corrected Hawking-like temperature, the explicit expressions of the corrected entropy of apparent horizon for various gravity theories including Einstein gravity, Gauss-Bonnet gravity, Lovelock gravity, \(f(R)\) gravity and scalar-tensor gravity, are computed. Our results show that the corrected entropy formula for different gravity theories can be written into a general expression (4.39) of a same form. It is also shown that this expression is also valid for black holes. This might imply that the expression for the corrected entropy derived from tunneling method is independent of gravity theory, spacetime and dimension of the spacetime. Moreover, it is concluded that the basic thermodynamical property that the corrected entropy on apparent horizon is a state function is satisfied by the FRW universe.

KEYWORDS: Hawking-Like Radiation, Tunneling, Thermodynamics of Friedmann-Robertson-Walker Universe, Entropy
1. Introduction

Hawking radiation phenomenon of black holes shows that black holes are not completely black, but emit thermal radiations like a black body, with a temperature proportional to its surface gravity at the horizon and with an entropy proportional to its horizon area\[^1,2\]. The Hawking temperature and the horizon entropy together with the mass of the black hole obey the first law of thermodynamics\[^3\]. Modeling the phenomenon of Hawking radiation one should incorporate quantum fields moving in a background of classical gravity. Therefore quantum theory, gravitational theory and thermodynamics meet at black holes together. The first law of thermodynamics of black hole together with the quantum nature of black hole lead people to consider the connection between thermodynamics and gravity theory.

Inspired by black hole thermodynamics, Jacobson first showed that Einstein gravity can be derived from the fundamental thermodynamics relation (Clausius relation) \(\delta Q = TdS\) together with the proportionality of entropy and the horizon area, presuming that the relation holds for all Rindler causal horizons through each spacetime point\[^4\]. With the viewpoint of thermodynamics, Einstein equation is nothing but an equation of the state of spacetime. Applying this idea to \(f(R)\) gravity and scalar-tensor gravity, it turns out that a non-equilibrium thermodynamic setup has to be employed\[^5\,6\]. For another viewpoint, see\[^7\].
Jacobson’s derivation provides a convincing evidence for the connection between thermodynamics and gravity theory. Recently, this connection has been investigated extensively in literatures for Friedmann-Robertson-Walker (FRW) universe. By assuming the apparent horizon of FRW spacetime has an associated semiclassical Bekenstein-Hawking entropy \( S_{\text{BH}} \) and temperature \( T_0 \)

\[
S_{\text{BH}} = \frac{A}{4\hbar}, \quad T_0 = \frac{\hbar}{2\pi \tilde{r}_A},
\]

(1.1)

Cai et al. showed that Friedmann equations can be derived from the first law of thermodynamics \( dE = T_0 dS_{\text{BH}} \). Here \( \hbar \) is the Planck constant, \( A \) is the area of the apparent horizon, and \( \tilde{r}_A \) is the radius of the apparent horizon. Further they using the same procedure, derived also Friedmann equations in the Gauss-Bonnet gravity and the more general Lovelock gravity. That study has also been generalized to \( f(R) \) gravity and scalar-tensor gravity\[8, 9, 10\]. In \[8, 10\], the Friedmann equations for \( f(R) \) gravity and scalar-tensor gravity were derived from the first law of thermodynamics by adding non-equilibrium corrections. In this case, in order to construct the equilibrium thermodynamics in \( f(R) \) gravity and scalar-tensor gravity, a mass-like function should be introduced to define the energy flux crossing the apparent horizon\[11\]. Beside gravity theories in four dimensions, the first law form of thermodynamics also holds on apparent horizon in various braneworld scenarios\[12\]. Some other viewpoints and further developments in this direction see \[13\] and references therein. The fact that the first law of thermodynamics holds extensively in various spacetime and gravity theories suggests a deep connection between thermodynamics and gravity theory.

The thermodynamics behavior of spacetime is only one of the features of gravity. This feature connects gravity and thermodynamics together. Another feature is the quantum effects of spacetime, which is related to the radiation of quantum fields from the horizon of the spacetime. Of black holes, this radiation is usually called Hawking radiation and was first found by Hawking\[1\]. Hawking’s original derivation of this radiation was completely based on quantum field theory. Since then, several other derivations of Hawking radiation were subsequently presented in literatures. Among these derivations, a simple and physically intuitive picture is provided by the tunneling mechanism\[14\]. It has two variants namely null geodesic method\[14\] and Hamilton-Jacobi method\[15\]. The tunneling method has attracted a lot of attention and has been applied to various black hole spacetimes\[16\]. Among the applications of the tunneling method, the fermion tunneling from black hole horizon has also been investigated\[17, 36\]. Recently, a problem in the tunneling approach has been discussed which corresponds to a factor two ambiguity in the original Hawking temperature\[18\]. Later, the connection between tunneling formulism and the anomaly approach is discussed\[19\]. Recently, the derivation of Hawking black body spectrum in the tunneling formulism is addressed\[20\] and this derivation fills the gap in the existing tunneling formulations.

Now, inspired by the Hawking radiation of black hole spacetimes, a question raises. That is, is there a Hawking-like radiation from the apparent horizon of a FRW universe? In a recent paper\[21\], the scalar particles’ Hawking-like radiation from the apparent horizon
of a FRW universe was investigated by using the tunneling method. Subsequently, further investigations of the Hawking-like radiation as tunneling in a FRW universe have been done by many authors \[22, 23, 24\]. The calculation of the Hawking-like radiation in tunneling method shows that a FRW universe indeed emits particles with a physical Hawking-like temperature \( T_0 = \frac{\hbar}{2\pi r_A} \), which is just the assumed temperature on apparent horizon to construct the first law of thermodynamics in FRW universe. Knowing the expression of this Hawking-like temperature, one can apply the first law of thermodynamics to identify explicitly the expression of the entropy of apparent horizon in various gravity theories. However, as we have known, when one constructs the first law of thermodynamics in FRW universe, the expressions of the entropy for various gravity theories are only assumptions. Thus, the tunneling method provides an approach to directly calculate both the Hawking-like temperature and the corresponding entropy of apparent horizon for FRW universe.

However, the tunneling method used for the Hawking-like radiation in FRW universe is based on the semiclassical approximation. This means that the Hawking-like temperature \( T_0 = \frac{\hbar}{2\pi r_A} \) and the corresponding entropy are both semiclassical results. When the completely quantum effect is taken into account, both the Hawking-like temperature and entropy of the apparent horizon should undergo corrections. But it is not obvious how to go beyond this semiclassical approximation in the tunneling method. Recently, the question that how to go beyond semiclassical approximation in the tunneling method, have been discussed in a series of papers by Banerjee and Majhi \[23, 24\]. And the general formalism of tunneling beyond semiclassical approximation has been developed in \[23\]. This formalism provides an approach to investigate the quantum corrections to the semiclassical thermodynamic variables of spacetime and has been studied extensively recently \[27, 28, 29, 30, 31\]. Therefore, it is of interest to investigate whether this formalism can be generalized to the FRW universe.

In our previous work \[32\], by using the formalism of tunneling beyond semiclassical approximation, we have considered the Hawking-like radiation in a (3 + 1)-dimensional FRW spacetime. The result yields the corrected expression of the Hawking-like temperature and entropy of apparent horizon. Note that there is a similar work that also treats the Hawking-like radiation to obtain the corrected Hawking-like temperature in FRW universe via tunneling beyond semiclassical approximation \[33\]. However, the computations in \[32, 33\] are confined to a (3 + 1)-dimensional FRW spacetime in Einstein gravity. The corrected expression of the entropy for generalized gravity theories is generally not discussed. As we all have known, the first law of thermodynamics holds not only for Einstein gravity, but also for other gravity theories like Gauss-Bonnet gravity, Lovelock gravity, \( f(R) \) gravity and scalar-tensor gravity. Therefore, we must ask if the tunneling formalism beyond semiclassical approximation is still valid in investigating the quantum corrections to the Hawking-like temperature and the corresponding entropy of apparent horizon in generalized theories of gravity. In the present work, we are going to investigate this problem.

In this paper, we would like to study the corrected thermodynamic quantities of FRW universe by using the tunneling formalism beyond semiclassical approximation. Via the tunneling calculation, we obtain the corrected form of the Hawking-like temperature of apparent horizon for a \((n + 1)\)-dimensional FRW universe. With this corrected Hawking-
like temperature, the explicit expressions of the corrected entropy of apparent horizon for various gravity theories including Einstein gravity, Gauss-Bonnet gravity, Lovelock gravity, $f(R)$ gravity and scalar-tensor gravity, are computed. Our results show that the corrected entropy formula for different gravity theories can be written into a general expression of a same form and this expression is also valid for black holes. This might imply that the expression for the corrected entropy derived from tunneling method is independent of the gravity theory, spacetime and the dimension of the spacetime.

Therefore, the paper is organized as follows. In section 2, the tunneling of scalar particle and fermion are both used to calculate the corrected Hawking-like temperature of apparent horizon for a $(n+1)$-dimensional FRW universe. The derivation of the corrected entropy for various gravity theories appear in sections 3 and 4. In section 5 we test the expression for corrected entropy in black hole background, and section 6 is left for our conclusions.

2. Corrections to the semiclassical Hawking-like temperature

In this section, in order to obtain the corrected form of the Hawking-like temperature of apparent horizon for FRW universe, we consider both the scalar particle and fermion’s Hawking-like radiation by using the tunneling method beyond semiclassical approximation.

For convenience of our analysis let us first begin with the standard form of an $(n+1)$-dimensional FRW metric

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 d\Omega_{n-1}^2 \right), \tag{2.1}$$

where $d\Omega_{n-1}^2$ denotes the line element of an $(n-1)$-dimensional unit sphere, $a(t)$ is the scale factor of the universe and $k$ is the spatial curvature constant which can take values $k = +1$ (positive curvature), $k = 0$ (flat), $k = -1$ (negative curvature). Introducing $\tilde{r} = a(t)r$, the metric (2.1) can be rewritten as

$$ds^2 = h_{ab} dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2, \tag{2.2}$$

where $x^a = (t, r)$ and $h_{ab} = \text{diag}(-1, a^2/(1 - kr^2))$. In FRW universe, there is a dynamical apparent horizon, which is the marginally trapped surface with vanishing expansion and is defined by the equation

$$h^{ab} \partial_a \tilde{r} \partial_b \tilde{r} = 0. \tag{2.3}$$

Using the metric (2.2), one can easily get the radius of the apparent horizon for the FRW universe

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}, \tag{2.4}$$

where $H$ is the Hubble parameter, $H \equiv \dot{a}/a$ (the dot represents derivative with respect to the cosmic time $t$).
In the tunneling approach of reference [14] the Painlevé-Gulstrand coordinates are used for the Schwarzschild spacetime. Applying the change of radial coordinate, $\tilde{r} = ar$, along with the above definitions of $H$ and $\tilde{r}_A$ to the metric in (2.1) one obtains the Painlevé-Gulstrand-like metric for FRW spacetime

$$ds^2 = -\frac{1 - \tilde{r}^2/\tilde{r}_A^2}{1 - k\tilde{r}^2/a^2}dt^2 - \frac{2H\tilde{r}}{1 - k\tilde{r}^2/a^2}dtd\tilde{r} + \frac{1}{1 - k\tilde{r}^2/a^2}d\tilde{r}^2 + \tilde{r}^2d\Omega_{n-1}^2.$$  (2.5)

These coordinates have been used in both null geodesic method and Hamilton-Jacobi method [21, 22, 23, 32, 33] to study the Hawking-like radiation from a (3 + 1)-dimensional FRW metric.

2.1 Scalar particle tunneling

In this subsection we discuss scalar particle tunneling from apparent horizon. Although there are literatures [22, 33] for the computation of the corrections to the Hawking-like temperature via scalar particle tunneling, they are only confined to the (3 + 1)-dimensional case. Now, we shall do the computation for arbitrary $(n + 1)$-dimensional FRW universe.

A massless scalar field $\phi$ in FRW universe obeys the Klein-Gordon equation

$$\frac{-\hbar^2}{\sqrt{-g}}\partial_{\mu}(g^{\mu\nu}\sqrt{-g}\partial_{\nu})\phi = 0.$$  (2.6)

In the tunneling approach we are concerned about the radial trajectory, so that only the $(t - \tilde{r})$ sector of the metric (2.2) is relevant, thus by making the standard ansatz for scalar wave function

$$\phi(\tilde{r}, t) = \exp\left[\frac{i}{\hbar}I(\tilde{r}, t)\right],$$  (2.7)

the Klein-Gordon equation (2.6) can be simplified to

$$\frac{\partial^2 I}{\partial t^2} + \left(\frac{i}{\hbar}\right)\left(\frac{\partial I}{\partial t}\right)^2 + \frac{H}{1 - k\tilde{r}^2/a^2}\frac{\partial I}{\partial t} + \frac{\tilde{r}(H^2\tilde{r}_A^2 + 1 - k\tilde{r}^2/a^2)}{\tilde{r}_A^2(1 - k\tilde{r}^2/a^2)}\frac{\partial I}{\partial \tilde{r}} - \frac{i}{\hbar}(1 - \tilde{r}^2/\tilde{r}_A^2)\left(\frac{\partial I}{\partial \tilde{r}}\right)^2 + \frac{2i}{\hbar}H\tilde{r}\frac{\partial I}{\partial \tilde{r}}\frac{\partial I}{\partial t} + 2H\tilde{r}\frac{\partial^2 I}{\partial t\partial \tilde{r}} - (1 - \tilde{r}^2/\tilde{r}_A^2)\frac{\partial^2 I}{\partial \tilde{r}^2} = 0.$$  (2.8)

An expansion of $I(\tilde{r}, t)$ in powers of $\hbar$ gives,

$$I(\tilde{r}, t) = I_0(\tilde{r}, t) + \sum_i \hbar^i I_i(\tilde{r}, t),$$  (2.9)

where $i = 1, 2, 3, \ldots$. Substituting (2.9) into (2.8) and equating different powers of $\hbar$ at
both sides, after a straightforward calculation we obtain the following set of equations:

\[
\begin{align*}
\hbar^0 : & \quad \frac{\partial I_0}{\partial t} = (-H\tilde{r} \pm \sqrt{1 - k\tilde{r}^2/a^2}) \frac{\partial I_0}{\partial \tilde{r}}, \\
\hbar^1 : & \quad \frac{\partial I_1}{\partial t} = (-H\tilde{r} \pm \sqrt{1 - k\tilde{r}^2/a^2}) \frac{\partial I_1}{\partial \tilde{r}}, \\
\hbar^2 : & \quad \frac{\partial I_2}{\partial t} = (-H\tilde{r} \pm \sqrt{1 - k\tilde{r}^2/a^2}) \frac{\partial I_2}{\partial \tilde{r}},
\end{align*}
\]

(2.10)

and so on. The above equations have a same functional form. So their solutions are not independent and \(I_i\) are proportional to \(I_0\). Then, we write the Eq. (2.9) by

\[
I(\tilde{r}, t) = (1 + \sum_i \gamma_i \hbar^i) I_0(\tilde{r}, t).
\]

(2.11)

Here \(I_0\) denotes the semiclassical contribution and the extra value \(\sum_i \gamma_i \hbar^i I_0\) can be regarded as the quantum correction terms to the semiclassical analysis.

For the metric (2.5), since the metric coefficients are both radius and time dependent, there is no time translation Killing vector field as in the case of static spacetime. However, following Kodama\[34\], for spherically symmetric dynamical spacetime whose metric is like (2.5), there is a natural analogue, the Kodama vector

\[
K = \sqrt{1 - k\tilde{r}^2/a^2} \frac{\partial}{\partial t}.
\]

(2.12)

Thus, using the Kodama vector, the general form of the semiclassical action \(I_0(\tilde{r}, t)\) in FRW universe is given by

\[
I_0(\tilde{r}, t) = - \int \frac{\omega}{\sqrt{1 - k\tilde{r}^2/a^2}} dt + \int \frac{\partial I_0(\tilde{r}, t)}{\partial \tilde{r}} d\tilde{r},
\]

(2.13)

where \(\omega\) is the conserved quantity with respect to the Kodama vector \(K\). The Kodama vector gives a preferred flow of time, coinciding with the static Killing vector of standard black holes. It should be noted that the Kodama vector is timelike, null and spacelike as \(\tilde{r} < \tilde{r}_A\), \(\tilde{r} = \tilde{r}_A\) and \(\tilde{r} > \tilde{r}_A\), respectively.

Put (2.13) into the first equation of (2.11), and combine (2.9), one can obtain the solutions for \(I(\tilde{r}, t)\):

\[
I(\tilde{r}, t) = \left[ - \int \frac{\omega}{\sqrt{1 - k\tilde{r}^2/a^2}} dt + \omega \int \frac{-H\tilde{r} \pm \sqrt{1 - k\tilde{r}^2/a^2}}{(1 - \tilde{r}^2/\tilde{r}_A^2)\sqrt{1 - k\tilde{r}^2/a^2}} d\tilde{r} \right] \times \left( 1 + \sum_i \gamma_i \hbar^i \right),
\]

(2.14)

where the \(+(-)\) sign indicates the particle is outgoing (ingoing).
In Schwarzschild black hole, with the tunneling of a particle across the event horizon the nature of the time coordinate \( t \) changes. This change indicates that \( t \) coordinate has an imaginary part for the crossing of the horizon of the black hole and consequentially there will be a temporal contribution to the imaginary part of the action for the ingoing and outgoing particles. For FRW universe, the radiation is observed by a Kodama observer and the Kodama vector is timelike, null and spacelike for the regions outside, on and inside the apparent horizon, respectively. Because the energy of the particle is defined by the conserved quantity with respect to the Kodama vector, a discrepancy of Kodama vector inside and outside the horizon will effect the temporal part of the action. This means that the temporal part integral in (2.14) should also have an imaginary part. Therefore, outgoing and ingoing probabilities are given by

\[
P_{\text{out}} = |\phi_{\text{out}}|^2 = \left| \exp \left[ \frac{i}{\hbar} I_{\text{out}}(\tilde{r}, t) \right] \right|^2
\]

\[
= \exp \left[ -\frac{2}{\hbar} (1 + \sum_i \gamma_i h^i) \left( -\text{Im} \int \frac{\omega}{\sqrt{1 - k\tilde{r}^2/a^2}} dt \right)
\right.
\]

\[
+ \omega \text{Im} \int \left[ \frac{-H\tilde{r} + \sqrt{1 - k\tilde{r}^2/a^2}}{(1 - \tilde{r}^2/\tilde{r}_A^2) \sqrt{1 - k\tilde{r}^2/a^2}} \right] \left( \frac{-H\tilde{r} - \sqrt{1 - k\tilde{r}^2/a^2}}{(1 - \tilde{r}^2/\tilde{r}_A^2) \sqrt{1 - k\tilde{r}^2/a^2}} \right) d\tilde{r} \right], \tag{2.15}
\]

\[
P_{\text{in}} = |\phi_{\text{in}}|^2 = \left| \exp \left[ \frac{i}{\hbar} I_{\text{in}}(\tilde{r}, t) \right] \right|^2
\]

\[
= \exp \left[ -\frac{2}{\hbar} (1 + \sum_i \gamma_i h^i) \left( -\text{Im} \int \frac{\omega}{\sqrt{1 - k\tilde{r}^2/a^2}} dt \right)
\right.
\]

\[
+ \omega \text{Im} \int \left[ \frac{-H\tilde{r} + \sqrt{1 - k\tilde{r}^2/a^2}}{(1 - \tilde{r}^2/\tilde{r}_A^2) \sqrt{1 - k\tilde{r}^2/a^2}} \right] \left( \frac{-H\tilde{r} - \sqrt{1 - k\tilde{r}^2/a^2}}{(1 - \tilde{r}^2/\tilde{r}_A^2) \sqrt{1 - k\tilde{r}^2/a^2}} \right) d\tilde{r} \right]. \tag{2.16}
\]

In \cite{23}, the temporal part contribution to the action has been calculated in Schwarzschild-like coordinates of a FRW spacetime. The contribution of the temporal part of the action to the tunneling rate is canceled out when dividing the outgoing probability by the ingoing probability because the temporal part is completely the same for both the outgoing and ingoing solutions. It is no need to work out the result of the temporal part of the action.

In the WKB approximation, the tunneling probability is related to the imaginary part of the action as

\[
\Gamma \propto \frac{P_{\text{in}}}{P_{\text{out}}} = \exp \left[ \frac{4\omega}{\hbar} \left( 1 + \sum_i \gamma_i h^i \right) \text{Im} \int \frac{1}{(1 - \tilde{r}^2/\tilde{r}_A^2)} d\tilde{r} \right]. \tag{2.17}
\]

It is obvious that the integral function has a pole at the apparent horizon. Through a contour integral, the tunneling probability of ingoing particle now reads

\[
\Gamma \propto \exp \left[ -\frac{2}{\hbar} (1 + \sum_i \gamma_i h^i) \pi \omega \tilde{r}_A \right]. \tag{2.18}
\]

Now using the principle of “detailed balance” \cite{13},

\[
\Gamma \propto \exp \left( -\frac{\omega}{T} \right), \tag{2.19}
\]
the corrected Hawking-like temperature associated with the apparent horizon can be determined as

\[ T = \frac{\hbar}{2\pi r_A} \left( 1 + \sum_i \gamma_i \hbar^i \right)^{-1} = T_0 \left( 1 + \sum_i \gamma_i \hbar^i \right)^{-1}, \tag{2.20} \]

where \( T_0 \) is the semiclassical Hawking-like temperature and other terms are corrections coming from the higher order quantum effects.

### 2.2 Fermion tunneling

Recently, the tunneling of fermions beyond semiclassical approximation has been also investigated for black holes\cite{26}. Due to the fermion tunneling beyond semiclassical approximation, all the quantum corrections to the thermodynamics quantities of a black hole can be determined. In this subsection we turn to consider the fermion tunneling beyond semiclassical approximation in FRW universe. For the fermion tunneling, there is a paper which discusses this issue but only with the semiclassical computation\cite{22}. Here we shall do the analysis for the tunneling of massless fermions from a FRW universe by considering all the quantum corrections.

Now we calculate the fermions' Hawking-like radiation from the apparent horizon of a FRW universe via the tunneling formalism beyond semiclassical approximation. A massless spinor field \( \psi \) obeys the Dirac equation without a mass term

\[-i\hbar \gamma^\mu D_\mu \psi = 0, \tag{2.21}\]

where the covariant derivative \( D_\mu \) is given by

\[ D_\mu = \partial_\mu + \frac{i}{2} \Gamma^\alpha_\mu \Sigma_{\alpha\beta}, \quad \Sigma_{\alpha\beta} = \frac{i}{4} \{ \gamma^\alpha, \gamma^\beta \}, \tag{2.22}\]

and the gamma matrices satisfy the condition that

\[ \{ \gamma^\alpha, \gamma^\beta \} = 2g^{\alpha \beta} I. \tag{2.23}\]

In \((n + 1)\)-dimensional FRW spacetime, as in higher dimensional black hole\cite{36}, we can
choose the \( \gamma \) matrices for the metric (2.5) as

\[
\gamma_{m \times m}^t = \begin{pmatrix}
iI_{\frac{m}{2} \times \frac{m}{2}} & 0 \\
0 & -iI_{\frac{m}{2} \times \frac{m}{2}} \end{pmatrix},
\]
(2.24)

\[
\gamma_{m \times m}^r = H^r \begin{pmatrix}
iI_{\frac{m}{2} \times \frac{m}{2}} & 0 \\
0 & -iI_{\frac{m}{2} \times \frac{m}{2}} \end{pmatrix} + \sqrt{1 - k\bar{r}^2/a^2} \begin{pmatrix}0 & \gamma_{\frac{m}{2} \times \frac{m}{2}}^3 \\
\gamma_{\frac{m}{2} \times \frac{m}{2}}^3 & 0 \end{pmatrix},
\]
(2.25)

\[
\gamma_{m \times m}^\theta = \frac{1}{\bar{r}} \begin{pmatrix}0 & \gamma_{\frac{m}{2} \times \frac{m}{2}}^1 \\
\gamma_{\frac{m}{2} \times \frac{m}{2}}^1 & 0 \end{pmatrix},
\]
(2.26)

\[
\gamma_{m \times m}^\varphi = \frac{1}{\bar{r} \sin \theta} \begin{pmatrix}0 & \gamma_{\frac{m}{2} \times \frac{m}{2}}^2 \\
\gamma_{\frac{m}{2} \times \frac{m}{2}}^2 & 0 \end{pmatrix},
\]
(2.27)

\[
\gamma_{m \times m}^n = \sqrt{g^n} \begin{pmatrix}0 & \gamma_{\frac{m}{2} \times \frac{m}{2}}^l \\
\gamma_{\frac{m}{2} \times \frac{m}{2}}^l & 0 \end{pmatrix}, \quad 4 \leq l \leq n
\]
(2.28)

\[
\gamma_{m \times m}^{x_{n+1}} = \sqrt{g^{x_{n+1}}} \begin{pmatrix}0 & -iI_{\frac{m}{2} \times \frac{m}{2}} \\
iI_{\frac{m}{2} \times \frac{m}{2}} & 0 \end{pmatrix},
\]
(2.29)

where \( t = x^0 \) and \( \bar{r} = x^3 \) are time coordinate and radial coordinate respectively; \( \theta = x^1 \) and \( \varphi = x^2 \) are angular coordinates, \( \eta, \ldots, \eta_{n+1} \) are extra-dimensional coordinates; \( m = 2^{n+1} \) is the order of the matrix in even-(odd-)dimensional spacetime; \( I_{\frac{m}{2} \times \frac{m}{2}} \) is a unit matrix with \( \frac{m}{2} \times \frac{m}{2} \) orders; \( \gamma_{\frac{m}{2} \times \frac{m}{2}}^\mu \) is the \( \mu \)th gamma matrix with \( \frac{m}{2} \times \frac{m}{2} \) orders in flat spacetime. Note that Eq.(2.29) is only necessary in odd-dimensional spacetime.

In the tunneling approach we are concerned about the radial trajectory, so that only the \((t - \bar{r})\) sector of the metric (2.2) is relevant, thus the Dirac equation (2.21) can be expressed as

\[
ig_{m \times m}^t \partial_t \psi + i\gamma_{m \times m}^r \partial_r \psi + i\chi_{m \times m} \psi = 0,
\]
(2.30)

where \( \chi_{m \times m} \) is a matrix with \( m \times m \) orders and

\[
\chi_{m \times m} = \frac{i}{2} \left[ \gamma_{m \times m}^t \left( g^{tt} \Gamma_{tt}^t + g^{t\bar{r}} \Gamma_{t\bar{r}}^t - g^{\bar{r}t} \Gamma_{\bar{r}t}^t - g^{\bar{r}\bar{r}} \Gamma_{\bar{r}\bar{r}}^t \right) \\
+ \gamma_{m \times m}^r \left( g^{tt} \Gamma_{t\bar{r}}^r + g^{t\bar{r}} \Gamma_{t\bar{r}}^r - g^{\bar{r}t} \Gamma_{\bar{r}t}^r - g^{\bar{r}\bar{r}} \Gamma_{\bar{r}\bar{r}}^r \right) \right] \Sigma_{t\bar{r}}.
\]
(2.31)

Without loss of generality, we employ the following ansatz for spinor field in \((n + 1)-\)dimensional spacetime

\[
\psi(t, \bar{r}) = \begin{pmatrix}A_{\frac{m}{2} \times 1}(t, \bar{r}) \\
B_{\frac{m}{2} \times 1}(t, \bar{r}) \end{pmatrix} e^{\frac{it}{2} t(t, \bar{r})},
\]
(2.32)

where \( A_{\frac{m}{2} \times 1}(t, \bar{r}) \) and \( B_{\frac{m}{2} \times 1}(t, \bar{r}) \) are \( \frac{m}{2} \times 1 \) function column matrices, \( I(t, \bar{r}) \) is the one particle action which will be expanded in powers of \( h \). Substituting the ansatz (2.32) into
(2.30), one obtain

\[
(\gamma^l_{m\times n}\partial_l I + \gamma^r_{m\times n}\partial_r I) \left( \frac{A_{\frac{m+m}{2}}}{B_{\frac{m}{2}}\times 1} \right) \left( \frac{A_{\frac{m}{2}}}{B_{\frac{m}{2}}\times 1} \right) = 0. \tag{2.33}
\]

Since the terms in the second line of the above equation do not involve the single particle action, they will not contribute to the thermodynamic entities of the black hole. Therefore, we will drop these terms. We can expand \( I, A_{\frac{m}{2}}\times 1 \), and \( B_{\frac{m}{2}}\times 1 \) in powers of \( \hbar \) as

\[
I(t, \tilde{r}) = I_0(t, \tilde{r}) + \sum_i \hbar^i I_i(t, \tilde{r}), \tag{2.34}
\]

\[
A_{\frac{m}{2}}\times 1(t, \tilde{r}) = A_0(t, \tilde{r}) + \sum_i \hbar^i A_i(t, \tilde{r}), \tag{2.35}
\]

\[
B_{\frac{m}{2}}\times 1(t, \tilde{r}) = B_0(t, \tilde{r}) + \sum_i \hbar^i B_i(t, \tilde{r}), \tag{2.36}
\]

where \( i = 1, 2, 3, \ldots \). In above expansions, \( I_0, A_0, \) and \( B_0 \) are semiclassical values, and the other higher order terms are treated as quantum corrections. Substituting (2.34), (2.35), and (2.36) into (2.33), and then equating the different powers of \( \hbar \) on both sides, one obtain the following two sets of equations:

**Set I:**

\[
h^0 : \quad \hbar I_0 + H\tilde{r}\partial_t I_0 + i(\partial_t I_0 + H\tilde{r}\partial_t I_0) A_0 + \sqrt{1 - k\tilde{r}^2/a^2}\partial_t I_0 \gamma_{\frac{m}{2}}^{\frac{3}{2}} = 0, \tag{2.37}
\]

\[
h^1 : \quad \hbar I_1 + H\tilde{r}\partial_t I_1 + i(\partial_t I_1 + H\tilde{r}\partial_t I_1) A_0 + \sqrt{1 - k\tilde{r}^2/a^2}\partial_t I_1 \gamma_{\frac{m}{2}}^{\frac{3}{2}} = 0, \tag{2.38}
\]

\[
h^2 : \quad \hbar I_2 + H\tilde{r}\partial_t I_2 + i(\partial_t I_2 + H\tilde{r}\partial_t I_2) A_0 + \sqrt{1 - k\tilde{r}^2/a^2}\partial_t I_2 \gamma_{\frac{m}{2}}^{\frac{3}{2}} = 0, \tag{2.39}
\]

\[
\ldots \ldots
given that
\]
where the \( + (\cdot) \) sign indicates the particle is ingoing (outgoing). Thus the solution for \( I_0(t, \varpi) \) is

\[
I_0(t, \varpi) = -\int \frac{\omega}{\sqrt{1 - k \varpi^2 / a^2}} dt + \omega \int \frac{H \varpi \pm \sqrt{1 - k \varpi^2 / a^2}}{(\varpi^2 / \varpi_A^2 - 1) \sqrt{1 - k \varpi^2 / a^2}} d\varpi.
\]

Note that in the above by only solving (2.37) and (2.40), we obtain the solutions of \( I_0(t, \varpi) \). Substituting this solution (2.45) and (2.44) into the equations of Set I and Set II and then solving them we obtain relations between different orders in the expansion of \( A_{\varpi \times 1} \) and \( B_{\varpi \times 1} \):

\[
i A_j = \pm \gamma_{\varpi \times \varpi} B_j,
\]

where \( j = 0, 1, 2, 3, \ldots \). Using above relations about the equations of Set I and Set II, we get the simplified form of the equations of \( I_j \):

\[
\partial_t I_j = (H \varpi \pm \sqrt{1 - k \varpi^2 / a^2}) \partial_t I_j.
\]
The above set of equations have the same functional form. So their solutions are not independent and \( I_i \) are proportional to \( I_0 \). Thus one can write the action \( I \) as

\[
I(t, \tilde{r}) = \left( 1 + \sum_i \gamma_i \hbar^i \right) I_0(\tilde{r}, t).
\]

(2.48)

Therefore from the above equation and (2.45) one can immediately reach the solutions of the action \( I \),

\[
I(\tilde{r}, t) = \left[ -\int \frac{\omega}{\sqrt{1 - k\tilde{r}^2/a^2}} dt + \omega \int \frac{-H\tilde{r} \pm \sqrt{1 - k\tilde{r}^2/a^2}}{(1 - \tilde{r}^2/\tilde{r}_A^2)\sqrt{1 - k\tilde{r}^2/a^2}} d\tilde{r} \right]
\times \left( 1 + \sum_i \gamma_i \hbar^i \right),
\]

(2.49)

which is identical with the expression (2.14). Following the same step in Sec.(2.1) for scalar particle tunneling, one can obtain the corrected Hawking-like temperature for fermion tunneling. The result thus obtained is identical to (2.20).

3. Corrected entropy in Einstein gravity

In Einstein gravity, it is known that a FRW universe can be considered as a thermodynamical system with temperature \( T_0 = \frac{\hbar}{2\pi a} \) and Bekenstein-Hawking entropy \( S_{BH} = \frac{A}{4\hbar} \) on the apparent horizon. Here the temperature \( T_0 \) and the entropy \( S_{BH} \) are both semiclassical results. When the quantum effects come into play, the temperature and the entropy should alter. In this section, with the corrected Hawking-like temperature (2.20) on apparent horizon of the FRW universe given in the above section, we will explicitly calculate the corrections to the semiclassical Bekenstein-Hawking entropy \( S_{BH} \) with the help of the first law of thermodynamics on the apparent horizon of the FRW universe.

In the Hawking-like temperature expression (2.20), there are un-determined coefficients \( \gamma_i \). Obviously, \( \gamma_i \) should have the dimension \( \hbar^{-i} \). Now, we will perform the following dimensional analysis to express these \( \gamma_i \) in terms of dimensionless constants by invoking some basic macroscopic parameters of the FRW universe. In the \((n+1)\)-dimensional FRW spacetime, one sets the units as \( G_{n+1} = c = k_B = 1 \), where \( G_{n+1} \) is the \((n+1)\)-dimensional gravitation constant. In this setting, the Planck constant \( \hbar \) is of the order of \( m_p \cdot l_p \), where \( m_p \) is the Planck mass and \( l_p \) is the Planck length. Therefore, according to the dimensional analysis, the proportionality constants \( \gamma_i \) have the dimension of \( (m_p l_p)^{-i} \). In Einstein’s gravity, there is an important quantity, i.e., the Misner-Sharp mass

\[
M = \frac{n - 1}{16\pi} \Omega_{n-1} r^{n-2} (1 - h_{ab} \partial_a \tilde{r} \partial_b \tilde{r}),
\]

(3.1)

which is the total energy inside the sphere with radius \( \tilde{r} \) and has the dimension of \( m_p \). For FRW universe, the Misner-Sharp mass on the apparent horizon is

\[
M_A = \frac{n - 1}{16\pi} \Omega_{n-1} \tilde{r}_A^{n-2}.
\]

(3.2)
Remember that on apparent horizon one should use $h^{ab} \partial_a \tilde{\partial}_b \tilde{r} \mid _{\tilde{r}_A} = 0$ in the derivation of the above expression. Now we can make a dimensional analysis to express the proportionality constants $\gamma_i$ in terms of macroscopic parameters of the FRW universe as

$$\gamma_i = \beta_i (M_A \tilde{r}_A)^i, \quad (3.3)$$

where $\beta_i$ is a dimensionless constant. This is possible since the Misner-Sharp mass $M_A$ is independent of the radius of the apparent horizon $\tilde{r}_A$ and can be completely determined by $\tilde{r}_A$. Note that things will be a bit different in the next section while in the gravity theories beyond Einstein the Misner-Sharp mass on apparent horizon contains more than one parameters. Using (3.3) the Hawking-like temperature (2.20) now can be written as a new form

$$T = T_0 \left(1 + \sum_i \frac{\alpha_i \hbar^i}{(M_A \tilde{r}_A)^i} \right)^{-1}, \quad (3.4)$$

where $\alpha_i = \frac{1}{\beta_i}$ is also a dimensionless constant.

With the new form of the Hawking-like temperature (3.4), one can apply the first law of thermodynamics $dE = TdS$ on apparent horizon of the FRW universe, thus one can obtain the corrected entropy by the formula:

$$S = \int \frac{dE}{T}. \quad (3.5)$$

substituting the temperature (3.4) into (3.3) we obtain

$$S = \int \frac{T_0}{T} dS_{\text{BH}} = \int \left(1 + \sum_i \frac{\alpha_i \hbar^i}{(M_A \tilde{r}_A)^i} \right) dS_{\text{BH}}. \quad (3.6)$$

Note that the first law of thermodynamics with the semiclassical temperature $T_0$ and the semiclassical Bekenstein-Hawking entropy $S_{\text{BH}}$ holds on apparent horizon, i.e., $dE = T_0 dS_{\text{BH}}$. In a $(n+1)$-dimensional FRW universe, the area of the apparent horizon is $A = \Omega_{n-1} \tilde{r}_A^{n-1}$. Thus one can obtain $M_A \tilde{r}_A = \frac{n-1}{16\pi} A$, substitute it into (3.6), then we get

$$S = \int \left(1 + \sum_i \alpha_i \left(\frac{4\pi}{n-1}\right)^i \frac{1}{S_{\text{BH}}^{i-1}} \right) dS_{\text{BH}}$$

$$= S_{\text{BH}} + \frac{4\pi \alpha_1}{n-1} \ln S_{\text{BH}} + \sum_{i=2} \frac{\alpha_i}{1-i} \left(\frac{4\pi}{n-1}\right)^i \frac{1}{S_{\text{BH}}^{i-1}} + \text{const.} \quad (3.7)$$

The first term is the semiclassical Bekenstein-Hawking entropy $S_{\text{BH}} = \frac{A}{4\hbar}$. The other terms are the correction terms due to quantum effects. Interestingly in the correction terms, the leading order correction is logarithmic in $S_{\text{BH}}$ which is very famous in black hole physics and can be obtained by other approaches [37].

The above discussions show that in the tunneling method, the semiclassical Bekenstein-Hawking entropy should receive corrections due to quantum effects. In the derivation of the
correlated entropy $S^{\text{BH}} = \frac{A}{4\pi}$ play an important role. However, the Misner-Sharp mass and the entropy for the FRW universe in Einstein gravity is very special. In general, the Misner-Sharp mass and the entropy on the horizon in other gravity theories are more complicated than in Einstein gravity. Therefore, a crucial problem with the previous investigations arises. That is, is the above procedure and the result (3.7) still valid for more complicated gravity theories? We will answer this question in the next section.

4. Corrected entropy in generalized gravity theory

In this section, we will generalize the above discussions to the generalized gravity theories, including the Gauss-Bonnet gravity, Lovelock gravity, $f(R)$ gravity, and scalar-tensor gravity. We will carry out explicitly the expression of the corrected entropy.

4.1 Gauss-Bonnet gravity

The Lagrangian of the Gauss-Bonnet gravity in $(n+1)$-dimensional spacetime is

$$L = \frac{1}{16\pi} (R + \alpha R_{\text{GB}}),$$

(4.1)

where $\alpha$ is a parameter with the dimension $[\text{length}]^2$ and $R_{\text{GB}} = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\gamma\delta}R^{\mu\nu\gamma\delta}$ is the Gauss-Bonnet term. Gauss-Bonnet gravity is the natural generalization of Einstein gravity by including higher derivative correction term, i.e., the Gauss-Bonnet term to the original Einstein-Hilbert action. In this gravity theory, the semiclassical Bekenstein-Hawking entropy-area relationship that the entropy of horizon is proportional to its area, does not hold anymore. The relationship is now

$$S_{\text{GB}} = \frac{A}{4\hbar} \left(1 + \frac{n-1}{n-3} \frac{2\alpha}{r_+^2}\right),$$

(4.2)

where $A$ is the horizon area of a Gauss-Bonnet black hole and $r_+$ is the radius of the horizon.

In ref. [8], Cai et al applied the entropy formula (4.2) to the apparent horizon, assuming that the apparent horizon has an entropy with the same expression as (4.2) but replacing the black hole horizon radius $r_+$ by the apparent horizon radius $\bar{r}_A$. That is, the apparent horizon is supposed to have an entropy

$$S_{\text{GB}} = \frac{A}{4\hbar} \left(1 + \frac{n-1}{n-3} \frac{2\alpha}{\bar{r}_A^2}\right).$$

(4.3)

Then with the entropy $S_{\text{GB}}$ and temperature $T_0 = \frac{\hbar}{2\pi \bar{r}_A}$ on apparent horizon, Cai et al have shown explicitly that the first law of thermodynamics

$$dE = T_0 dS_{\text{GB}},$$

(4.4)

holds on apparent horizon of the FRW universe for Gauss-Bonnet gravity, where $dE$ is the amount of energy crossing the apparent horizon in Gauss-Bonnet gravity.
Now let us began to consider the dimensional analysis on the Hawking-like temperature (3.4) in Gauss-Bonnet gravity. In Gauss-Bonnet gravity, the mass parameter is the generalized Misner-Sharp mass, which is proposed in [38]. For FRW universe, the generalized Misner-Sharp mass on apparent horizon has the following form

\[
M_A = \frac{n-1}{16\pi} \Omega_{n-1} r_A^{n-2} \left( 1 + \frac{n-2\alpha}{n-3 r_A^n} \right). \tag{4.5}
\]

Unlike (3.2) that has only one independent parameter \(\tilde{r}_A\), the generalized Misner-Sharp mass \(M_A\) here have two independent parameters \(\tilde{r}_A\) and \(\alpha\). Thus it is not clear whether the combination \(M_A\tilde{r}_A\) is still valid for expressing the proportionality constants \(\gamma_i\) in terms of dimensionless constants. To be safe, one can express \(\gamma_i\) in terms of \(\alpha\) and \(\tilde{r}_A\) as

\[
\gamma_i = \beta_i (a_1 \tilde{r}_A^{n-1} + a_2 \alpha \tilde{r}_A^{n-3})^i, \tag{4.6}
\]

where \(a_1\) and \(a_2\) are dimensionless constants. Note that \(\alpha\) has the dimension \([\text{length}]^2\).

Now the Hawking-like temperature (2.20) has the form

\[
T = T_0 \left( 1 + \sum \frac{\alpha_i \hbar^i}{(a_1 \tilde{r}_A^{n-1} + a_2 \alpha \tilde{r}_A^{n-3})^i} \right) \tag{4.7}
\]

To fix the constants \(a_1\) and \(a_2\) let us first write the first law of thermodynamics with the corrected Hawking-like temperature (4.7) in the form

\[
dS = \frac{dE}{T} = \frac{T_0 dS_{GB}}{T} = \frac{T_0 \Omega_{n-1} d(r_A^{n-1})}{4hT} + \frac{n-1 T_0 \Omega_{n-1} d(\alpha r_A^{n-3})}{2hT} = \frac{T_0 \Omega_{n-1} dX}{4hT} + \frac{n-1 T_0 \Omega_{n-1} dY}{n-3 \frac{2hT}{2hT}}. \tag{4.8}
\]

We treat \(X = \tilde{r}_A^{n-1}, Y = \alpha \tilde{r}_A^{n-3}\) as two independent variables in the above equation. From the principle of the ordinary first law of thermodynamics one interprets entropy as a state function. In refs. [29, 30], this property of entropy has been used to investigate the first law of thermodynamics and entropy for black holes. Also, this property must be satisfied for FRW universe as well. Hence we can assume that the entropy of the FRW universe is a state function and consequently \(dS\) has to be an exact differential. As a result the following relation must hold:

\[
\frac{\partial}{\partial Y} \left( \frac{T_0 \Omega_{n-1}}{4hT} \right) \bigg|_X = \frac{\partial}{\partial X} \left( \frac{n-1 T_0 \Omega_{n-1}}{n-3 \frac{2hT}{2hT}} \right) \bigg|_Y. \tag{4.9}
\]

This relation is just the integrability condition that ensures \(dS\) is an exact differential. Using the corrected Hawking-like temperature (4.7) it follows that the above integrability condition is satisfied only for

\[
2 \frac{n-1}{n-3} a_1 = a_2. \tag{4.10}
\]
For convenience we choose \(a_1 = \frac{n-1}{16\pi}\Omega_{n-1}\), thus the proportionality constants \(\gamma_i\) is now given by

\[
\gamma_i = \beta_i \left[ \frac{n-1}{16\pi}\Omega_{n-1} \tilde{r}_A^n - 2(1 + \frac{n-1}{3}\tilde{r}_A) \right]^i = \beta_i (M_A \tilde{r}_A)^i. \tag{4.11}
\]

This shows that the combination \(M_A \tilde{r}_A\) still works for expressing the proportionality constants \(\gamma_i\) in terms of dimensionless constants. Therefore the corrected form of the Hawking-like temperature is given by

\[
T = T_0 \left( 1 + \sum_i \frac{\alpha_i \hbar^i}{(M_A \tilde{r}_A)^i} \right)^{-1}. \tag{4.12}
\]

Applying the first law of thermodynamics on apparent horizon and using (4.4), one immediately obtains the corrected entropy of apparent horizon in Gauss-Bonnet gravity

\[
S = S_{GB} + \frac{4\pi \alpha_1}{n-1} \ln S_{GB} + \sum_{i=2} \frac{\alpha_i}{1-i} \left( \frac{4\pi}{n-1} \right)^i \frac{1}{S_{GB}^{i-1}} + \text{const}, \tag{4.13}
\]

which is the same in form as (3.7) obtained in Einstein gravity. We see that the first term is the usual semiclassical entropy of the horizon in Gauss-Bonnet gravity and the other terms are the corrections from the quantum effects. Also interestingly the leading order correction appears as the logarithmic in \(S_{GB}\).

Thus, starting with the Hawking-like temperature (3.4) and applying the first law of thermodynamics to apparent horizon, we obtain the corrected entropy of apparent horizon in Gauss-Bonnet gravity. The correct entropy satisfies the same formula as that in Einstein gravity.

### 4.2 Lovelock gravity

Now we extend the above discussions to a more general case, the Lovelock gravity, which is a generalization of the Gauss-Bonnet gravity. The Lagrangian of the Lovelock gravity consists of the dimensionally extended Euler densities

\[
\mathcal{L} = \sum_{i=0}^m c_i \mathcal{L}_i, \tag{4.14}
\]

where \(c_i\) are constants, \(m \leq [n/2]\), and \(\mathcal{L}_i\) is the Euler density of a \((2i)\)-dimensional manifold

\[
\mathcal{L}_i = 2^{-i} \delta_{c_i d_i}^{a_1 b_1 \ldots a_{2i} b_{2i}} R_{a_1 b_1} \ldots R_{a_{2i} b_{2i}}. \tag{4.15}
\]

Here \(\mathcal{L}_1\) is the Einstein-Hilbert term, and \(\mathcal{L}_2\) is just the Gauss-Bonnet term discussed in the previous subsection. For the FRW universe in Lovelock gravity, the first law of thermodynamics also holds on apparent horizon. That is

\[
dE = T_0 dS_L, \tag{4.16}
\]
where $T_0 = \frac{n}{2\pi r_A}$ is the temperature and $S_L$ is the entropy of the apparent horizon of the FRW universe in Lovelock gravity, which has the following form

$$S_L = \frac{A}{4\hbar} \sum_{i=1}^{m} \frac{i(n-1)!}{(n-2i+1)!} c_i r_A^{2-2i}. \quad (4.17)$$

For the FRW universe in Lovelock gravity, the generalized Misner-Sharp mass on apparent horizon is

$$M_A = \frac{n-1}{16\pi} \Omega_{n-1} r_A^{n-2} \sum_{i=1}^{m} \frac{i(n-1)!}{(n-2i+1)!} c_i r_A^{2-2i}. \quad (4.18)$$

Now follow the same procedure in the above, we can write the corrected Hawking-like temperature (3.4) as the following form

$$T = T_0 \left( 1 + \sum_i \frac{\alpha_i \hbar^i}{(M_A r_A)^i} \right)^{-1}. \quad (4.19)$$

With this Hawking-like temperature we apply the first law of thermodynamics to the apparent horizon, we can obtain the corrected entropy

$$S = S_L + \frac{4\pi\alpha_1}{n-1} \ln S_L + \sum_{i=2} \frac{\alpha_i}{1-i} \left( \frac{4\pi}{n-1} \right)^i \frac{1}{S_L^{1-i}} + \text{const.} \quad (4.20)$$

Also, like the corrected entropy for Gauss-Bonnet gravity, this entropy formula follows the same form as that in Einstein gravity.

### 4.3 $f(R)$ gravity

The Lagrangian of the $f(R)$ gravity in $(n+1)$-dimensional spacetime is

$$\mathcal{L} = \frac{1}{16\pi} f(R), \quad (4.21)$$

where $f(R)$ is a continuous function of curvature scalar $R$. In the $f(R)$ gravity, the entropy of a black hole has a relation to its horizon

$$S_f = \frac{A}{4\hbar} f'(R), \quad (4.22)$$

where $f'(R)$ denotes the derivative with respect to the curvature scalar $R$. Also, one can assume that the apparent horizon of the FRW universe has an entropy with the same expression as (4.22). By further assuming that the temperature $T_0 = \frac{n}{2\pi r_A}$ still holds on the apparent horizon, one can investigate the thermodynamics behavior of Friedmann equations in $f(R)$ gravity. For $f(R)$ gravity, however, things are a bit different with that in the case of Einstein gravity, Gauss-Bonnet gravity, and Lovelock gravity. In this case, applying the first law of thermodynamics $dE = T_0 dS_f$ to the apparent horizon, one can not obtain the correct Friedmann equations. In order to get the correct Friedmann equations, one has to turn from the equilibrium thermodynamics relation $dE = T_0 dS_f$ to
a non-equilibrium one; an entropy production term needs to be added to the equilibrium thermodynamics relation\cite{10}. This means that the $f(R)$ gravity corresponds to a non-equilibrium thermodynamics of spacetime.

Recently, in \cite{11}, the authors have shown that there is a mass-like function connecting the first law of thermodynamics and the Friedmann equations in some gravity theories. For $f(R)$ gravity, the mass-like function can be written as

$$M = \frac{n - 1}{16\pi} \Omega_{n-1} f'(R) \tilde{r}^{n-2} (1 + h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}).$$  \hfill (4.23)

Using the mass-like function, the energy amount crossing the apparent horizon in an infinitesimal time interval can be defined as $dE = k^a \partial_a M dt$; then the equilibrium thermodynamics relation, i.e., the first law of thermodynamics $dE = T_0 dS_f$ holds on the apparent horizon.

The mass-like function has the dimension of $m_p$, thus we can choose it as the mass parameter in the case of the $f(R)$ gravity. For the FRW universe, the mass-like function on apparent horizon is

$$\mathcal{M}_A = \frac{n - 1}{16\pi} \Omega_{n-1} f'(R) \tilde{r}_A^{n-2}. \hfill (4.24)$$

Thus we can write the Hawking-like temperature (3.4) as the form

$$T = T_0 \left(1 + \sum_i \frac{\alpha_i h^i}{(\mathcal{M}_A \tilde{r}_A)^i}\right)^{-1}. \hfill (4.25)$$

Now the procedure same as the above subsections yields the corrected entropy formula on apparent horizon

$$S = S_f + \frac{4\pi \alpha_1}{n - 1} \ln S_f + \sum_{i=2}^{\infty} \frac{\alpha_i}{1 - i} \left(\frac{4\pi}{n - 1}\right)^i \frac{1}{S_f^{i-1}} + \text{const.} \hfill (4.26)$$

4.4 Scalar-Tensor gravity

The general scalar-tensor theory of gravity is described by the Lagrangian

$$\mathcal{L} = \frac{1}{16\pi} f(\phi) R - \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi), \hfill (4.27)$$

where $f(\phi)$ is a continuous function of the scalar field $\phi$ and $V(\phi)$ is its potential. The black hole entropy in scalar-tensor gravity has the following form\cite{12}

$$S_{ST} = \frac{A}{4\hbar} f(\phi). \hfill (4.28)$$

In order to investigate the thermodynamics properties on the apparent horizon of the FRW universe, one should assume that the entropy of the apparent horizon has the same form as (4.28). The thermodynamics behavior in scalar-tensor gravity is very similar with that in $f(R)$ gravity. Namely, one usually needs to treat the thermodynamics in scalar-tensor gravity as the non-equilibrium thermodynamics. As pointed out in \cite{11}, after introducing
the mass-like function, the equilibrium thermodynamics \(dE = T_0 dS_{ST}\) also holds on the apparent horizon.

The mass-like function in scalar-tensor gravity is defined as

\[
\mathcal{M} = \frac{n-1}{16\pi} \Omega_{n-1} f(\phi) \tilde{r}^{n-2} (1 + h^{ab} \partial_a \tilde{r} \partial_b \tilde{r}).
\]  (4.29)

For the FRW universe, the mass-like function on the apparent horizon is

\[
M_A = \frac{n-1}{16\pi} \Omega_{n-1} f(\phi) \tilde{r}_A^{n-2}.
\]  (4.30)

Thus one can write the Hawking-like temperature as

\[
T = T_0 \left( 1 + \sum_i \frac{\alpha_i h^i}{(\mathcal{M}_{A} \tilde{r}_A)^i} \right)^{-1}.
\]  (4.31)

Now, with the same procedure in the above, it is easy to obtain the corrected entropy of the apparent horizon,

\[
S = S_{ST} + \frac{4\pi \alpha_1}{n-1} \ln S_{ST} + \sum_{i=2}^{n} \frac{\alpha_i}{1-i} \left( \frac{4\pi}{n-1} \right)^i \left( \frac{1}{S_{ST}^{n-1}} \right) + \text{const.}
\]  (4.32)

Obvious, this expression is consistent with the corrected entropy formula in Einstein gravity, Gauss-Bonnet gravity, Lovelock gravity, and \(f(R)\) gravity.

It is well known that \(f(R)\) gravity can be written as a special scalar-tensor theories of gravity by redefining the field variable\(^\text{[43]}\). To see this we write the action of the \(f(R)\) gravity as

\[
I_{f(R)} = \frac{1}{16\pi} \int d^4 x \sqrt{-g} f(R).
\]  (4.33)

One can introduce a new field \(\chi = R\) and write the dynamically equivalent action

\[
I_{f(R)} = \frac{1}{16\pi} \int d^4 x \sqrt{-g} [f(\chi) + f'(\chi)(\chi - R)].
\]  (4.34)

Variation with respect to \(\chi\) leads to the equation

\[
f''(\chi - R) = 0.
\]  (4.35)

Therefore, \(\chi = R\) if \(f''(R) \neq 0\), which reproduces the action \(^{[43]}\). Then redefining the field \(\chi\) by \(\phi = f'(R)\) and setting

\[
V(\phi) = \chi(\phi) \phi - f(\chi(\phi)),
\]  (4.36)

the action takes the form

\[
I_{f(R)} = \frac{1}{16\pi} \int d^4 x [\phi R - V(\phi)].
\]  (4.37)
This is the Brans-Dicke action with a potential \( V(\phi) \) and a Brans-Dicke parameter \( \omega_0 = 0 \). Therefore there is a dynamical equivalence between \( f(R) \) gravity and a special scalar-tensor gravity. This equivalence means that the corrected entropy of FRW universe in \( f(R) \) gravity can be directly obtained from eq. (4.32). In the gravity theory described by the action \( (4.37) \), the entropy of black hole horizon takes the form \( S_{BD} = \frac{1}{4\pi}\phi A \). Thus from \( (4.32) \) the corrected entropy of FRW universe can be written as

\[
S = S_{BD} + \frac{4\pi\alpha_1}{n-1} \ln S_{BD} + \sum_{i=2}^{\infty} \frac{\alpha_i}{1-i} \left( \frac{4\pi}{n-1} \right)^i \frac{1}{S_{BD}^{i-1}} + \text{const}, \tag{4.38}
\]

noticing that \( \phi = f'(R) \), which shows above expression is just the corrected entropy formula \( (4.26) \) of the \( f(R) \) gravity.

Thus, summarizing Eqs. (3.7), (4.13), (4.20), (4.26) and (4.32), one can conclude that all these corrected entropy formulae in different gravity theories can be written into a general expression

\[
S = S_0 + \tilde{\alpha}_1 \ln S_0 + \sum_{i=2}^{\infty} \tilde{\alpha}_i \frac{1}{S_0^{i-1}} + \text{const}, \tag{4.39}
\]

where \( S_0 \) is the entropy on apparent horizon without quantum correction. This might imply that this general expression is independent of the concrete gravity theory. Also, one can see that the leading order correction in \( (4.39) \) appears as the logarithmic in \( S_0 \) and the sub-leading term is the standard inverse power of \( S_0 \). This character holds for arbitrary \( (n+1) \)-dimensional FRW spacetime.

5. Test the expression for corrected entropy with black holes

In above sections, it is shown that there is a general expression for the corrected entropy on apparent horizon by the tunneling method. However, the derivation of this general expression is confined to the FRW universe. As we all know the tunneling method has been used extensively to obtain the corrected black hole entropy. Thus, a question arises that is the general expression \( (4.39) \) still valid for black holes? Now, in order to answer this question we are going to check the corrected entropy from the tunneling method for a \((2+1)\)-dimensional BTZ black hole and a \((3+1)\)-dimensional Kerr-Newman black hole.

In ref. [29], Modak has considered the tunneling method beyond semiclassical approximation for BTZ black hole and obtained the corresponding corrected entropy for the BTZ black hole, which is

\[
S = S_{BH} + 4\pi\beta_1 \ln S_{BH} - \frac{16\pi^2\beta_2}{l} \left( \frac{1}{S_{BH}} \right) + \ldots, \tag{5.1}
\]

where \( S_{BH} = \frac{4\pi}{4\pi} \) is the semiclassical Bekenstein-Hawking entropy of the BTZ black hole and \( l \) is related to a negative cosmological constant \( \Lambda = -\frac{1}{l^2} \). It is obvious that this entropy formula \( (5.1) \) fit into the general expression \( (4.39) \).
For the Kerr-Newman black hole, its corrected entropy is \[ S = S_{\text{BH}} + 2\pi\beta_1 \ln S_{\text{BH}} - \frac{4\pi^2\beta_2}{S_{\text{BH}}^2} + \ldots, \] (5.2)

which also fits into the general expression (4.39). Now it is trivial, as one can check, that all other black holes, for example Schwarzschild, Kerr or Reissner-Nordstrom black hole, also fit into the general expression (4.39). Thus the universality of the expression (4.39) for black holes is justified.

Now, one can say that the general expression (4.39) is also valid for black holes. Here are some comments. First, the BTZ black hole is a black hole solution for (2+1)-dimensional gravity with a negative cosmological constant, this implies that (4.39) is robust for black hole even in low dimensional gravity theories. Second, we have noticed that prefactors of both the logarithmic term and the third term in (4.39) are dependent on the black holes. Also, from above section, it is easy to know that these prefactors are dependent on the dimension of the spacetime. Although the general expression is independent of gravity theory, spacetime and the dimension of the spacetime in form, this means that the prefactors may contain more detailed information of the spacetime. Third, in the derivation of the corrected entropy (5.1) in ref. [29] and (5.2) in ref. [30], the condition that the entropy of a black hole must be a state function is enforced. The property that entropy is a state function is a basic character of the ordinary first law of thermodynamics.

6. Conclusions

In this paper, we have investigated the thermodynamic quantities of FRW universe by using the tunneling formalism beyond semiclassical approximation developed by Banerjee and Majhi [25]. Both the scalar particle and fermion tunneling from apparent horizon are considered to obtain the corrected Hawking-like temperature in FRW universe. With this corrected Hawking-like temperature, the corresponding corrected entropy on apparent horizon for Einstein gravity, Gauss-Bonnet gravity, Lovelock gravity, \( f(R) \) gravity and scalar-tensor gravity are given. We found that the corrected entropy formula for different gravity theories can be written into a general expression (4.39) in form. We also show that this general expression is valid for black holes. These characteristics may imply that this general expression for the corrected entropy derived from tunneling method is independent of gravity theory, spacetime and dimension of the spacetime.

An important part in the derivation of the corrected entropy for various gravity theories is that we have use the combination \( M_A \tilde{r}_A \) to express the proportionality constants \( \gamma_i \) in terms of dimensionless constants by dimensional analysis. This combination is always valid for Einstein gravity, but in generalized gravity theories its validity is not clear. We have shown in Gauss-Bonnet gravity that this combination is an essential condition to ensure that the corrected entropy \( S \) is a state function, which is a basic property of ordinary first law of thermodynamics. This means that the basic thermodynamical property that corrected entropy on apparent horizon is a state function is satisfied by the FRW universe.

There is another significant point in the general expression (4.39) for corrected entropy is that it involves logarithmic in \( S_0 \) term as the leading correction together with the
standard inverse power of $S_0$ as sub-leading correction. The prefactors of the correction terms are dependent on the dimension of the FRW spacetime. In black holes, it has been proved that the prefactor of logarithmic correction term is related with the trace anomaly of the stress tensor near the horizon. In this paper, we have not discussed this issue for the FRW universe, and thus it remains an open issue to consider the connection between the prefactor of logarithmic term and the trace anomaly in FRW universe for various gravity theories.

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