Magnetization of a Diffusive Ring: Beyond the Perturbation Theory

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Abstract

Average persistent current over a set of diffusive metallic rings with fixed number of electrons is considered. We study the case where the phase breaking time is much greater than an inverse average inter-level distance. In such a situation, many return events for an electron have to be taken into account. As a result, one arrives at a non-perturbative problem for a fixed by an external magnetic field Cooperon mode. This multi-Cooperon problem has been considered previously by Altland et al., Europhys. Lett., 2, 155 (1992) and in several following papers within the framework of supersymmetric approach. Such an approach involves very tedious calculations which were performed using computer algebra package. Here we solve the problem in question with the help of replica trick. It is demonstrated that the replica trick in combination with a proper analytical continuation in the replica space allows one to obtain the result in much more explicit way.

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I. INTRODUCTION

Magnetic properties of small conductors were extensively studied during the last several years (see 1 and references therein). It has been understood that the magnetic moment (and the associated persistent current) induced by an external magnetic flux is a very specific manifestation of mesoscopic behavior. While originally predicted to appear in clean one-dimensional metallic rings 3, most of the recent discussion about persistent currents has focused on metallic rings containing impurities 4. Static magnetic properties of small rings and dots were studied by several authors 5–11. An important step in the understanding of magnetization of mesoscopic quantum rings was taking into account the difference between canonical and grand canonical ensembles 7,10,12–14. It has been shown that the magnetization of isolated rings with fixed number of particles is much larger than of the ensemble of rings kept under fixed chemical potential. As a result, the main contribution to the magnetic moment has been expressed in terms of the fluctuation of the number of particles at fixed chemical potential, \( \langle (\delta N)^2 \rangle \). The latter quantity has been analyzed in 10,14 under the condition \( \hbar / \Delta \tau_{\phi} \gg 1 \). Here \( \Delta \) is an average inter-level distance at the Fermi level, \( \Delta^{-1} = \nu V \) (\( \nu \) is the density of states at the Fermi level, while \( V \) is the volume). \( \tau_{\phi} \) is the phase-breaking time.

Let us discuss the physical meaning of the parameter \( \hbar / \Delta \tau_{\phi} \). As well known 15, in the absence of external magnetic field the quantum correction to the conductivity is proportional to the classical probability \( W \) for an electron with a velocity \( v \) and momentum \( p \) to return to the vicinity of the starting point (more exactly, into the volume of the order \( v dt (\hbar / p)^2 \) important for quantum interference). The probability \( W \) is given by the expression 15

\[
W \propto \frac{v \hbar^2}{p^2} \int_{t=0}^{\tau_{\phi}} dt P(r, t) \big|_{r=0},
\]

where \( P(r, t) \) is the probability density. Here we employ the fact that in a diffusive regime it is a smooth function of co-ordinates at the scale of mean free path \( \ell \). To estimate \( P(0, t) \) we take into account that the electron diffusion is restricted by finite volume of the sample. In such a case, we have
\[ P(0, t) \propto \frac{1}{V} \sum_{n, n_\perp} \exp \left( -D \left( \frac{n^2}{R^2} + \frac{n_\perp^2}{d_\perp^2} \right) t \right). \] (2)

Here \( D \) is the diffusion constant, \( R \) is the radius of the ring, while \( d_\perp \) is its transverse dimension. The numbers \( n, n_\perp \) have the meaning of quantum numbers for longitudinal and transverse diffusive modes, respectively. For a thin ring, \( d_\perp \ll R \), only \( n_\perp = 0 \) is important.

One can see that at \( D \tau_\phi / R^2 \gg 1 \) one cannot replace the sum over discrete \( n, n_\perp \) in (2) by an integral. On the contrary, only \( n = 0 \) is important, and \( W \sim \tau_\phi \Delta / \hbar \). If this quantity is small, one can restrict himself with a single return event.

Now let us concentrate on the case of external magnetic field. In a magnetic field, the number \( n \) in the expression (2) has to be replaced by \( n - \Phi / \Phi_0 \), where \( \Phi \) is the magnetic flux embedded in the ring, while \( \Phi_0 = \pi \hbar c/e \). It is clear that the quantum contribution is maximal if \( \Phi / \Phi_0 \) is close to some integer number \( n_0 \). If the difference \( \tilde{n} \equiv n_0 - \Phi / \Phi_0 = 0 \) we arrive at the same situation as in the absence of magnetic field – only the mode with \( n = n_0 \) is important. One can expect that this property is also the case at finite \( |\tilde{n}| \ll 1 \).

Indeed, for \( n \neq n_0 \)
\[ \delta W \propto \sum_{n \neq n_0} \frac{\Delta}{D \tilde{n}^2/R^2 + 1/\tau_\phi} \sim \frac{\Delta R^2}{D} \] (for the last estimate we have assumed \( D \tau_\phi / R^2 \gg 1 \)). Consequently, if \( \delta W \ll 1 \) one can neglect the contributions of all the modes with \( n \neq n_0 \) to the probability to return. However, the corresponding contribution of the mode with \( n = n_0 \) is not small at \( \Delta \tau_\phi / \hbar \gtrsim 1 \). Hence, we arrive at the problem of calculation of the localization contribution in the case
\[ D / R^2 \gg \Delta / \hbar \gtrsim 1 / \tau_\phi. \]

In this region we can still use a single-mode approximation, but the perturbation theory involving a single return event fails.

The problem in question was addressed by Altland et al.\(^\text{15,18}\) (see also Refs.\(^\text{19–21}\)). The authors used the so-called \( Q \)-Hamiltonian approach within the framework of the supersymmetric method. An intrinsic feature of this method is that one has to cancel out specific non-physical contributions. Therefore the supersymmetric approach involves tedious al-
gebraic calculations. Consequently, the authors of\textsuperscript{17,18} extensively used computer-algebra package. As a result, intermediate equations remain unpublished because, as it was stated, the computer printout covers many pages.

On the other hand, another approach - the so-called replica method – exists\textsuperscript{22}. According to this approach, one has to replace the system under consideration by $N$ systems identical to the original one and at the end tend $N \to 0$. Usually, after such a procedure one arrives at relatively simple expressions. The limiting transition $N \to 0$ (if done properly) automatically cancels out the non-physical contributions that has to be done explicitly within the supersymmetric approach.

To take the full advantage of this property one needs a regular procedure to calculate the limit $N \to 0$. The aim of the present paper is to suggest a procedure of analytical continuation of a non-perturbative two-particle Green’s function from integer $N$ to the whole complex plane which includes the point $N = 0$. Such a procedure allows one to calculate the limit rather automatically excluding necessity of direct cancelation of non-physical contributions. We obtain an analytical non-perturbative expression for persistent current in a mesoscopic diffusive ring and compare it with the results of\textsuperscript{17,18}.

The paper is organized as follows. In the Section \textbf{II} basic equations for the fluctuation of the number of particles, as well as for the persistent current are analyzed. The effective action in the single-mode approximation is considered in Section \textbf{III}. In Section \textbf{IV} the particle number auto-correlation function and persistent current are calculated in the non-perturbative region. The results are summarized in Discussion. In the following calculations we will put $\hbar = 1$. Then $\hbar$ will be restored in estimates and final results.

\section*{II. BASIC EQUATIONS}

According to\textsuperscript{19}, the main contribution to the persistent current $I$ can be expressed through the magnetic flux $\Phi$ embedded in the ring, as

$$ I = \frac{c\Delta}{2} \frac{\partial}{\partial \Phi} \langle (\delta N)^2 \rangle_{\mu = \langle \mu \rangle} , $$

(3)
where \( \langle (\delta N)^2 \rangle_{\mu=\mu_i} \) is the particle number auto-correlation function, calculated at a given value of chemical potential. The latter can be expressed in terms of single-electron Green’s functions as \(^{23}\)

\[
\langle (\delta N)^2 \rangle = \int_{-\mu}^{0} d\epsilon_1 d\epsilon_2 K(\epsilon_1, \epsilon_2),
\]

where

\[
K(\epsilon_1, \epsilon_2) = \frac{1}{\pi^2} \int d\mathbf{r}_1 d\mathbf{r}_2 \left\{ \langle G^R_{\epsilon_1}(\mathbf{r}_1, \mathbf{r}_1)G^R_{\epsilon_2}(\mathbf{r}_2, \mathbf{r}_2) \rangle - \langle G^R_{\epsilon_1}(\mathbf{r}_1, \mathbf{r}_1) \rangle \langle G^R_{\epsilon_2}(\mathbf{r}_2, \mathbf{r}_2) \rangle \right\}. \tag{5}
\]

Here \( \langle \cdot \cdot \cdot \rangle \) means the usual impurity average. The quantity \(^{[3]}\) has been calculated in Ref. \(^{10}\) in the limiting case \( \Delta \tau \phi \ll \hbar \). Our aim is to go beyond this limiting case, namely to calculate the correlation function for arbitrary \( \Delta \tau \phi/\hbar \), keeping \( p_F \ell \gg \hbar \). For this purpose we employ the approach by Efetov, Larkin, and Khmelnitskii\(^ {22}\) with minor modifications. Namely, we will use the so-called \( Q \)-Hamiltonian approach within the framework of the replica trick. The confined expression for the correlation function \( K(\omega) \) \( (\omega = \epsilon_1 - \epsilon_2) \) can be written in the form (cf. with Ref. \(^ {22}\))

\[
K(\omega) = \left[ \nu^2 \int DQ \exp(-F) \int d\mathbf{r}_1 d\mathbf{r}_2 \int DQ e^{-F} \text{Tr} [\Lambda Q(\mathbf{r}_1)] \text{Tr} [\Lambda Q(\mathbf{r}_2)] \right] \biggr|_{N \to 0}
\]

where \( F = \int d\mathbf{z} \text{Tr} \left[ D \left( \nabla Q + \frac{ie}{c} \mathbf{A}[Q, \Lambda]_\leftarrow \right)^2 + 2 \left( i\omega - \frac{1}{\tau \phi} \right) \Lambda Q \right] \).

Here \( \mathbf{A} \) is the vector-potential, \( [A, B]_\leftarrow \equiv AB - BA \). Having in mind to take into account only an elastic scattering by short-range non-magnetic impurities, we can specify \( Q \) as \( 2N \times 2N \) Hermitian matrices, \( Q^2 = 1, \text{Tr} Q = 0 \), \( N \) is the number of replicas, while

\[
\Lambda = \begin{pmatrix}
\hat{1} & 0 \\
0 & -\hat{1}
\end{pmatrix},
\]

where \( \hat{1} \) is the \( N \times N \) unit matrix. The parameter \( \tau \phi^{-1} \) is introduced phenomenologically. We assume that the phase breaking is due to inelastic processes. Following\(^ {22}\), we use the
parameterization
\[ Q = \Lambda \exp(W), \quad W = \begin{pmatrix} 0 & B \\ -B^+ & 0 \end{pmatrix}, \]
where \( B \) is an arbitrary \( N \times N \) matrix.

III. EFFECTIVE ACTION IN SINGLE-MODE APPROXIMATION

Consider a ring with the radius \( R \) and the width \( d_{\perp} \ll R \). Consequently, one can take into account only the dependence of the matrices \( B \) on the angular co-ordinate \( \varphi \). Expanding this dependence into the discrete Fourier series, \( B = \sum_n B_n \exp(in\varphi) \), we introduce the mode number \( n \). As it has been explained in Section I, only one mode with \( n = n_0 \) corresponding to \( \min(n - \Phi/\Phi_0) \) is important (this assumption will be justified at the end of Section IV A). Retaining only this mode and assuming
\[ \nabla \varphi Q = \frac{1}{R} \Lambda \frac{\partial}{\partial \varphi} e^W = (\nabla \varphi W)W^{-1} \sinh W = \frac{i n_0}{R} \sinh W. \]

Then, \( \sinh^2 W = (1/2)(\cosh 2W - 1) \) can be expanded as a series in
\[ W^{2k} = (-1)^k \left( \begin{array}{cc} \sqrt{B_{n_0}^+ B_{n_0}} & 0 \\ 0 & \sqrt{B_{n_0}^+ B_{n_0}} \end{array} \right)^{2k}. \]

The item \( \text{Tr} (\Lambda Q) \) can be treated in a similar way. As a result, we arrive at the following expression for \( F \)
\[ F = \frac{\pi}{2\Delta} \left[ \frac{D}{R^2} \left( n_0 - \frac{\Phi}{\Phi_0} \right)^2 \text{Tr} \sin^2 \left( \sqrt{B_{n_0}^+ B_{n_0}} \right) + 2 \left( i\omega - \frac{1}{\tau_{\varphi}} \right) \text{Tr} \cos \left( \sqrt{B_{n_0}^+ B_{n_0}} \right) \right]. \quad (8) \]

Now we arrive at an important point. An arbitrary complex \( N \times N \) matrix \( B \) can be described by 2 Hermitian \( N \times N \) matrices. We defined them as
\[ B = \rho \exp(i\varphi), \quad B^+ = \exp(-i\varphi) \rho. \quad (9) \]

The quantity \( F \) is dependent only on the matrix \( \rho \). On the other hand, an arbitrary Hermitian matrix could be diagonalized, the eigenvalues being \textit{real}. One can immediately observe...
that the integral over $\rho$ in the expression (6) with $F$ taken from Eq. (8) for the correlation function diverges. This divergence in fact does not occur, because the eigenvalues of $\rho$ must be defined within a finite interval. Indeed, one has to define the variables $\rho$ in a way to obtain one-to-one correspondence between $\rho$ and $Q$. On the other hand, one can explicitly show that

$$Q = \Lambda e^W = \begin{pmatrix} \cos \sqrt{BB^+} & \sin \sqrt{BB^+} \frac{B}{\sqrt{BB^+}} \\ \frac{B + \sin \sqrt{BB^+}}{\sqrt{BB^+}} - \cos \sqrt{B^+ B} & e^{-i\varphi} \sin \rho - \cos \rho^T \end{pmatrix},$$

where $\rho^T$ denotes the transposed matrix $\rho$. Here we have employed the relationship

$$\rho^T = \sqrt{e^{-i\varphi} \rho^2 e^{i\varphi}},$$

which is a consequence of the symmetry properties of the initial replica Hamiltonian (see Appendix A). It is clear that the matrix $Q$ is a periodic function of $\rho$, and one has to specify a region at least not larger than 1 period to get one-to-one correspondence. Moreover, to get proper analytical properties (damping is the lower semi-plane of the $\omega$-variable) of the action $F$ (8) we have to define the integration limits as $(-\pi/2, \pi/2)$. Finally, the action $F$ reads as

$$F = \frac{\pi}{2\Delta} \left[ \frac{D}{R^2} \left( n_0 - \frac{\Phi}{\Phi_0} \right)^2 \operatorname{Tr} \sin^2 \rho + 2 \left( i\omega - \frac{1}{\tau_0} \right) \operatorname{Tr} \cos \rho \right].$$

Now let us transform the variables from $B, B^+$ to $\rho, u \equiv \exp(i\varphi)$, the Jacobian being (see Appendix B)

$$\frac{D(B, B^+)}{D(\rho, u)} = 2(\operatorname{det} u^{-1} \rho)^N.$$

We observe that the variables $u$ can be integrated out and canceled with the denominator in Eq. (9).

**IV. PARTICLE NUMBER AUTO-CORRELATION FUNCTION**
A. Eigenvalue representation

Now we come back to Eq. (6). Taking into account that \( \text{Tr} \Lambda Q = 2 \text{Tr} \cos \rho \) we see that the integrand is dependent only on eigenvalues of \( \rho \). Hence, we have to transform the variables to the eigenvalues and some other ones which could be integrated out both in numerator and denominator. This transform is outlined in Appendix C. As a result,

\[
K_N = \frac{(V
u)^2 \int_0^1 \{d\lambda\} \theta(\lambda^{(i)}) \left[ \sum_{j=1}^N \cos(\pi\lambda^{(j)}/2) \right]^2}{(N^2) \int_0^1 \{d\lambda\} \theta(\lambda^{(i)})},
\]

where \( \{d\lambda\} \equiv \prod_{i=0}^{N-1} d\lambda^{(i)}|\lambda^{(i)}|^{2i+N} \), while

\[
\theta(x) = \exp \left[ -\frac{\pi D}{2\Delta R^2} \left( n_0 - \frac{\Phi}{\Phi_0} \right)^2 \sin^2 \left( \frac{\pi x}{2} \right) - \frac{\pi}{\Delta} \left( i(\epsilon_1 - \epsilon_2) - \frac{1}{\tau_{\phi}} \right) \cos \left( \frac{\pi x}{2} \right) \right].
\]

The expression (15) contains 3 dimensionless parameters,

\[
\gamma \equiv \frac{\pi \hbar}{\Delta \tau_{\phi}}, \quad \Omega \equiv \frac{\pi (\epsilon_1 - \epsilon_2)}{\Delta}, \quad \mathcal{E} \equiv \frac{\hbar \pi D \tilde{n}^2}{2R^2\Delta},
\]

where \( \tilde{n} \equiv (n_0 - \Phi/\Phi_0) \). It is important to keep in mind the following. If \( \max(\gamma, \Omega) \gg 1 \), only small \( \lambda \) are important. Hence, one arrives at the result, obtained in the framework of perturbation theory\(^{10,14}\). However, if both \( \gamma \) and \( \Omega \) are small one has to sum multi-Cooperon contributions that cannot be done in the framework of perturbation theory. There is a substantial simplification in the case

\[
\tilde{n} \ll 1, \quad \text{but} \quad \mathcal{E}_c \equiv \frac{\hbar \pi D}{2R^2\Delta} \gg 1.
\]

In this case, only one mode with \( |n_0 - \Phi/\Phi_0| \ll 1 \) is important, and it is the case where Eq. (15) is valid. Consequently, we consider the situation where the inequalities (17) hold, but the quantities \( \gamma \) and \( \Omega \) can be arbitrary. In fact, the mode \( n_0 \) must be considered by a non-perturbative way, while the other modes can be treated within the framework of the perturbation theory.
**B. Analytical continuation**

We are not able to calculate the expression (14) for an arbitrary \( N \) analytically. Instead, we will perform analytical continuation of this expression to arbitrary \( N \), and then calculate its limit at \( N \to 0 \).

Let us introduce the quantity

\[
Z_N = \prod_{k=0}^{N-1} \int_0^1 dx_k x_k^{2k+N} \theta(x_k),
\]  

(18)

where

\[
\theta(x) = \exp[-E \sin^2(\pi x/2) - (i\Omega - \gamma) \cos(\pi x/2)].
\]  

(19)

It is convenient to define

\[
\zeta_N \equiv \ln Z_N \equiv \zeta^R + \zeta^A,
\]

where

\[
\zeta^R \equiv \frac{1}{2} \sum_{k=\frac{N+1}{2}}^{\frac{N-1}{2}} \ln \int_0^1 dx \, x^{2k-1+\delta} \theta(x),
\]

\[
\zeta^A \equiv \frac{1}{2} \sum_{k=-\frac{N+1}{2}}^{-\frac{N-1}{2}} \ln \int_0^1 dx \, x^{-2k-1+\delta} \theta(x),
\]

(20)

\( \delta \) is a small positive number which later will be put zero. We introduce this parameter to keep the important integrals convergent at the limit \( N \to 0 \). The first step is to express the sum over \( k \) in terms of contour integral over complex \( k \). For this purpose let us take into account that the derivative \( \partial f[2\pi i(k + 1/2)]/\partial k \) [where \( f(z) = (e^z + 1)^{-1} \)] has second-order poles at integer numbers. Consequently, one can express \( \zeta_{R(A)} \) as

\[
\zeta_{R(A)} = \int_{C^\pm} dk \left( \frac{\partial f(2\pi ik)}{\partial k} \right) F^\pm(k),
\]

(21)

\[
F^\pm = \frac{1}{2} \int d k' \ln \left[ \int_0^1 dx \, x^{2k'-1+\delta} \theta(x) \right].
\]

(22)

The contours \( C^\pm \) are shown in Fig. 1. These expressions are correct only if other singularities except of the poles of \( f \) are unimportant. One can show that the function \( F^+(k) \) has
singularities only in the left-hand semi-plane of complex variable $k$, while the function $F^-$ has singularities only in the right-hand semi-plane of $k$. To prove this statement one has to expand the function $\theta(x)$ into a Taylor series. For the following, it is convenient to rotate the $k$-plane by $\pi/2$ by introducing a new variable $k_1 \equiv 2\pi i k$. The transformed contours $\tilde{C}^\pm$ are shown in Fig. 2.

Making use of exponential convergence of the integral due to the properties of $\partial f/\partial k$, we transform the contour integrals to the integrals along the real axis. For simplicity let us assume $N$ to be even. As a result,

$$
\zeta_{R(A)} = \mp \int_{-\infty}^{\infty} dk \left( \frac{\partial f(k)}{\partial k} \right) \left[ F^\pm \left( \frac{k \pm i\pi N}{2\pi i} \right) - F^\pm \left( \frac{k \pm 3i\pi N}{2\pi i} \right) \right].
$$

(23)

Now we are prepared to perform an analytical continuation over $N$. We have to do it in a different way for the functions $F^\pm$ for the reason to be discussed later. For this purpose we replace $iN$ by $\pm N_0$ in the functions $F^\pm$, respectively. Here $N_0$ is a real positive quantity which we are going to tend to zero later. Finally, we have

$$
\zeta_{N_0} = \int_{-\infty}^{\infty} dk \left[ \left( \frac{\partial f(k - \pi N_0)}{\partial k} \right) - \left( \frac{\partial f(k - 3\pi N_0)}{\partial k} \right) \right] \left[ F^- \left( \frac{k}{2\pi i} \right) - F^+ \left( \frac{k}{2\pi i} \right) \right].
$$

(24)

As a result, the lowest-order term in the $N_0$-expansion of the function $\zeta_{N_0}$ is $\propto N_0^2$. Finally, we get

$$
\zeta = 2\pi i N_0^2 \int_{-\infty}^{\infty} dk \left( \frac{\partial^2 f(k)}{\partial k^2} \right) \ln \left[ \int_0^1 dx \, x^{-ik/\pi - 1 + \delta} \theta(x) \right].
$$

(25)

The reason of splitting the function $\zeta_N$ into $\zeta^R$ and $\zeta^A$ with the replacements $N \to \pm iN_0$ is as follows. As $N$ tends to zero, the integration contour comes infinitely close to the cut of logarithm functions which enter the expressions for $F^\pm$. Such a situation is not the case for any finite $N$, and it leads to a non-physical pinch which has to be subtracted. Within the above mention procedure such a contribution is pure imaginary while the one we are interested in is real. The imaginary contributions to $F^+$ and $F^-$ have opposite signs. Thus the non-physical contribution is automatically canceled out in the sum $\zeta^R + \zeta^A$. We want to note that these terms are of the first order in $N_0$. They have to vanishes, otherwise the
two-particles Green’s function would be divergent. In fact, a similar trick has been used by Matsubara to formulate the thermal Green’s function technique (see e.g., Ref. [24]). Let us compare our analytical continuation of the function $\zeta$ to the analytical continuation of the two-particle Matsubara Green’s function $K(\Omega_m)$, where $\Omega_m$ is the external Matsubara frequency. In both cases one has to take two functions regular in the upper (retarded) and the lower (advanced) semi-plane, respectively. Then one combines the two above-mentioned functions into the one having a cut in its complex plane. The physical reason of such a splitting into $R$ and $A$ parts is to cancel out non-physical contributions. In the Matsubara case the non-physical contributions to $K(\Omega_m)$ arise in the point $\Omega_m \to 0$ and cancel out after the similar continuation $\Omega_m \to i\Omega$ of the sum over $\Omega$.

### C. Persistent current

Following [10], we express the current according to Eqs. (3),(4). On the other hand, $K(\epsilon_1, \epsilon_2) \propto e^{-\zeta} \frac{\partial^2}{\partial \epsilon_1 \partial \epsilon_2} e^{\zeta}$.

Finally, we get

$$I = -J_1 \tilde{n} G,$$  \hspace{1cm} (26)

where $J_0 = \hbar c D/R^2 \Phi_0 = eD/\pi R^2$, while

$$G = i \int_{-\infty}^{\infty} dk \frac{\partial^2 f(k)}{\partial k^2} \frac{\zeta_{N_0} (\epsilon_1 = \epsilon_2 = 0)}{N_0^2}.$$ \hspace{1cm} (27)

One can check directly that at

$$\Delta \ll \frac{\hbar}{\tau_0} \ll \frac{\hbar D}{R^2}, \quad |\tilde{n}| \ll 1$$

the above expressions lead to the expressions obtained in Refs. [10][11]. To show that one can calculate the integrals with the help of steepest-descent approach to get [10][14]

$$I = -J_1 \tilde{n}, \quad J_1 = \frac{e \Delta D \tau_0}{\pi^3 R^2}.$$ \hspace{1cm} (28)
In the region
\[ \gamma \ll 1, \quad \mathcal{E}_c \gg \mathcal{E} \gg 1 \]
one can also develop a perturbation theory. Indeed, only small \( x \) in the integrals in Eq. (27) are important. The physical reason of this fact is the magnetic-field-induced phase-breaking.

In this region, we arrive at the result
\[ G = \frac{1}{\pi^2 \tilde{n}^2} \frac{R^2 \Delta}{\hbar D}, \quad I = \frac{1}{\pi^3 \tilde{n}} \frac{e\Delta}{\hbar}. \] (29)

This result agrees with the asymptotic result of Ref. 14 for \( \gamma \gg 1, \sqrt{R^2/D\tau_\phi} \ll \tilde{n} \ll 1 \).

Note that the result (28) obtained for \( \gamma \ll 1 \) is valid in the region
\[ \sqrt{R^2\Delta/hD} \ll \tilde{n} \ll 1. \]

For the case \( \gamma \ll 1, \mathcal{E} \ll 1 \), where the perturbation theory is not applicable, one can put \( \theta(x, \Omega = 0) = 1 \). As a result, we get \( G = 0.21 \), the current being
\[ I = -0.21 \frac{eD}{\pi^3 R^2} \tilde{n}. \] (30)

We observe a maximum at \( \tilde{n} \sim \sqrt{R^2\Delta/hD} \), the maximal current being
\[ I_{\text{max}} \sim e\sqrt{\Delta D/hR^2}. \] (31)

Expressions (29) and (30) are fully consistent with the curve calculated in Ref. with a help of computer algebraic package. Let us discuss the dependence of the maximal current on \( \gamma \approx h/\Delta \tau_\phi \). At \( \gamma \gg 1 \) the perturbation theory predicts the maximum of the current at \( \tilde{n} \sim \sqrt{R^2/D\tau_\phi} \), the maximal value being
\[ I_{\text{max}} \sim e \frac{\Delta}{\hbar} \sqrt{\tau_\phi D/R^2}. \] (32)

Consequently \( I_{\text{max}} \propto \gamma^{-1/2} \) at \( \gamma \gg 1 \), and it \( \gamma \)-independent at \( \gamma \ll 1 \). In this region, the persistent current can be estimated also as
\[ J_{\text{max}} \sim \frac{ev_F}{R} \sqrt{\frac{\Delta \tau_\phi}{\hbar}}. \]
where $\tau_{el}$ is the elastic relaxation time. The quantity $\Delta \tau_{el}/\hbar$ for a typical metal can be estimated as $(\ell/R)(a^2/A)$ where $\ell$ is the mean free path, $a$ is a typical inter-atomic distance, while $A$ is the cross-section of the ring. Eq. (30) shows that at $\Delta \tau_{\phi}/\hbar \gg 1$ the phase-breaking time $\tau_{\phi}$ does not enter the expression for the persistent current. Coming back to Eq. (1) we have to conclude that at $t > \hbar/\Delta$ the electronic wave packet does not smear in space. That means that the $n_0$-mode of the Cooperon is localized in some sense, the localization length being of the order $\sqrt{\hbar D/\Delta}$. Of course, it does not mean complete localization because other modes are still under weak localization conditions.

The range of parameters where the theory above is applicable and leads to non-trivial results can be expressed as

$$1 \ll R/\ell \ll K, \ (1/K)(\tau_{\phi}/\tau_{el}),$$

where $K \sim (p_d/\hbar)^2$ is the number of transverse channels. The left inequality is the criterion for a diffusion motion, the first right inequality is just the Thouless criterion $E_c \gg 1$, while the last right one is the condition $\Delta \tau_{\phi}/\hbar \gg 1$. One can see that one needs low temperatures to meet the inequality $\tau_{\phi}/\tau_{el} \gg K \gg 1$, as well as samples of very small size. As far as we know, no previous experiments satisfy this set of conditions.

V. DISCUSSION

As one can see from the preceding sections, the results of the replica procedure being complicated for arbitrary integer $N$ are rather simple in the limit $N \to 0$. In this limit, the non-physical contributions are canceled out automatically, while within the supersymmetric method that has been done this explicitly. An important feature that leads to such a simplification is the employed procedure of analytical continuation which has been done before direct calculations. Namely, one has to $two$ functions analytical in the upper (lower) semi-plane of the complex plane of $N$, respectively. The proper analytical continuation is a combination of these functions. Consequently, in has a cut at $\Im N = 0$. The procedure above
allows one to cancel out automatically the non-physical pinch in the two-particles Green’s function, which otherwise would exist at \( N_0 = 0 \). We believe that such a construction is important in general for the calculations involving the replica trick. In such a way we reproduce analytically and rather simply the results obtained in \cite{17,18} by a computer algebraic package.

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**APPENDIX A: EFFECTIVE ACTION – DERIVATION**

Here we re-derive the expression \cite{22} following \cite{22} to make clear important symmetry properties. Following \cite{22} we use the replica trick and introduce a field operators as

\[
\Psi = \{\psi_1, \ldots, \psi_N, \psi_1^+, \ldots, \psi_N^+\}, \quad \Psi^+ = \begin{pmatrix}
\psi_1^+ \\
\vdots \\
\psi_N^+ \\
-\psi_1 \\
\vdots \\
-\psi_N
\end{pmatrix}.
\]

Here \( \psi_i \psi_i^+ + \psi_i^+ \psi_i = 0 \). The action can be written as

\[
F = i \int (d\mathbf{r}) \Psi^+(\mathbf{r})(\hat{E} - \hat{\mathcal{H}})\Psi(\mathbf{r}),
\]

\[
\hat{E} = E \hat{I},
\]

\[
\hat{\mathcal{H}} = [H_0 + U_{el}(\mathbf{r})] \hat{I} - \left( \frac{\omega}{2} + i\delta \right) \Lambda.
\]

Here \( H_0 \) is the free-electron Hamiltonian, \( \hat{I} \) is the \( 2N \times 2N \) unit matrix, while \( U_{el}(\mathbf{r}) = U_0 \sum_i^M \delta(\mathbf{r} - \mathbf{r}_i) \), \( M \) being the total number of impurities. The first \( N \) rows of \( \hat{\mathcal{H}} \) describe
an evolution of the retarded Green’s functions, while the last \( N \) rows describe an evolution of the advanced one. The following step is averaging over the positions of impurities. We have

\[
\Sigma \equiv \prod_{i=1}^{M} \int \frac{dr_i}{V} \exp \left[ iU_0 \sum_{i=1}^{M} \Psi^+(r_i) \Psi(r_i) \right] = \left[ \int \frac{dr}{V} \prod_{f=1}^{2N} \left( 1 + iU_0 \psi_f^+(r) \psi_f(r) \right) \right]^{M} \quad (A2)
\]

Here we have taken into account that only linear terms in \( \psi^+_f \) and \( \psi_f \) can enter the continual integral for the correlation function (Grassman algebra). Because of the same reason, one has to allow only for the terms with different \( f \) while calculating the product. For a weak scattering and in the thermodynamic limit \( M, V \to \infty \), \( M/V = \text{const} \),

\[
\Sigma \approx \exp(\delta \mu + i \Gamma),
\]

\[
\delta \mu = \frac{MU_0}{V} \int dr \, \Psi^+(r) \Psi(r),
\]

\[
\Gamma = \frac{g_0^2}{2} \int dr \sum_{f \neq g} \psi^+_f(r) \psi_f(r) \psi^+_g(r) \psi_g(r). \quad (A3)
\]

Here \( \delta \mu \) is a shift in chemical potential, while \( g_0^2 = 2MU_0^2/V \) (\( g_0 \) is the coupling constant). In a same way as it has been done in Ref. 22, we introduce an auxiliary scalar field represented by Hermitian matrices \( Q \). As a result, the effective \( \psi^4 \) interaction can be decoupled as

\[
\exp \left[ -\frac{g_0^2}{2} \int dr \sum_{f \neq g} \psi^+_f(r) \psi_f(r) \psi^+_g(r) \psi_g(r) \right] = \frac{\int \mathcal{D}Q \exp \left[ -\text{Tr} \int dr \left( \frac{\pi \nu}{\tau_{el}} Q(r) \Psi^+(r) \Psi(r) \right) \right]}{\int \mathcal{D}Q \exp \left[ -\text{Tr} \int dr \frac{\pi \nu}{\tau_{el}} Q^2(r) \right]}. \quad (A4)
\]

Here we have used the definition \( 2\pi \nu g_0^2 \tau_{el} = 1 \). This expression is just the same as Eq. (17) from Ref. 22. To analyze symmetry properties of the impurity-averaged Hamiltonian let us take into account that the initial Hamiltonian possesses the property \( H_{ij} |_{i,j \leq N} = H^*_{i+N,j+N} \). Such a property has to be kept after impurity averaging and introducing the field \( Q \). In terms of \( Q \) it reads as \( iQ_{ij} = -iQ^*_{i+N,j+N} \). Taking into account Eq. (11) we arrive at the relationship (11). The following steps are exactly the same as in Ref. 22.
APPENDIX B: CALCULATION OF THE JACOBIAN

Let us arrange the columns of the $2N^2 \times 2N^2$ matrix $\partial(B, B^+)/\partial(\rho, u)$ as

$$\{B_{11}, \ldots, B_{N1}; B_{12}, \ldots, B_{NN}; B_{11}^+, \ldots, B_{NN}^+\}$$

and the rows as

$$\{\rho_{11}, \ldots, \rho_{N1}; \rho_{12}, \ldots, \rho_{NN}; u_{11}, \ldots, u_{N1}; u_{12}, \ldots, u_{NN}\}.$$  

Taking into account the matrix identities

$$dB = d\rho u + \rho du,$$

$$dB^+ = u^{-1} d\rho - u^{-1} du u^{-1} \rho,$$  \hspace{1cm} (B1)

we express $\partial(B, B^+)/\partial(\rho, u)$ as

$$\frac{\partial(B, B^+)}{\partial(\rho, u)} = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}$$ \hspace{1cm} (B2)

where $\hat{A}_{ik}$ are $N^2 \times N^2$ matrices. One can show that

$$\hat{A}_{11} = u \times \hat{1}, \quad \hat{A}_{22} = -(u^{-1} \rho) \times u^{-1},$$

$$\hat{A}_{21} = \rho \otimes \rho \otimes \ldots \rho,$$

$$\hat{A}_{12} = u^{-1} \otimes u^{-1} \otimes \ldots u^{-1}.$$ \hspace{1cm} (B3)

Here $\times$ means the Kronecker product, while $\otimes$ means direct product. Making use of the identity

$$\det (A \times B) \equiv (\det A)^p (\det B)^q$$

(where $q, p$ are the ranges of the matrices $A$ and $B$, respectively), as well as the Laplace expansion of determinant we arrive at Eq. (13).
APPENDIX C: VARIABLE TRANSFORMATION

Consider the set of variables which includes eigenstates $\lambda_{(b)}$ and $N$ eigenvectors $X^{(b)}$. One can see from the definition $\rho_{ik}X^{(b)}_k = \lambda^{(b)}\delta_{ik}X^{(b)}_i$ that any vector of the type $e^{i\chi^{(b)}_i}X^{(b)}_i$ (where $\chi^{(b)}$ is an arbitrary phase) satisfies the equation with the same $\lambda^{(b)}$ and $\rho_{ik}$. Consequently, one must exclude $N$ extra variables $\chi^{(b)}$. For this purpose we require the diagonal elements $X^{(i)}_i$ to be real. Consequently, the matrix $X^{(i)}_k$ can be constructed according to the following procedure. The first column, $X^{(1)}_i$, contains $N - 1$ variables $X^{(1)}_i$, $i \neq 1$, while the last (real) one, $X^{(1)}_i$, is calculated from the requirement of normalization. In the next column, $X^{(2)}_i$, the last $N - 2$ variables are chosen are independent. The element $X^{(2)}_i$ is determined by the orthogonality of the vectors $X^{(2)}$ and $X^{(1)}$, while the last element, $X^{(2)}_2$ is determined by the normalization of $|X^{(2)}|$. The following elements are determined by continuation of this procedure. Note, that all the off-diagonal elements are complex ones, so one can consider real $(U^{(i)}_k)$ and imaginary $(V^{(i)}_k)$ parts. In this way we present $N^2$ independent elements of the matrix $\rho$ through $N$ eigenvalues $\lambda^{(i)}$, and $N^2 - N$ independent variables $U^{(i)}_k$ and $V^{(i)}_k$.

From the definition, $\rho_{ij} = \sum_k X^{(k)}_i \lambda^{(k)} X^{(k)*}_j$, one can express $\rho$ through $\{U, V, \lambda\}$ as

$$
\frac{\partial \rho_{ij}}{\partial U^{(k)}_i} = \lambda^{(k)}[U^{(k)}_j(1 + \delta_{ij}) - iV^{(k)}_j(1 - \delta_{ij})],
$$

$$
\frac{\partial \rho_{ij}}{\partial V^{(k)}_i} = \lambda^{(k)}[iU^{(k)}_j(1 - \delta_{ij}) + V^{(k)}_j(1 + \delta_{ij})],
$$

$$
\frac{\partial \rho_{ij}}{\partial \lambda^{(k)}} = U^{(k)}_i U^{(k)*}_j + V^{(k)}_i V^{(k)*}_j + i(V^{(k)}_i U^{(k)*}_j - U^{(k)}_i V^{(k)*}_j).
$$

(C1)

Note that the above formulas do not contain summation over repeated superscripts. To calculate the Jacobian we arrange the corresponding $N^2 \times N^2$ transformation matrix in the following way. The columns are labeled by $N^2$ “old” variables

$$\{\rho_{i1}, \rho_{i2}, \ldots, \rho_{iN}\}.$$

The rows are labeled by $N^2$ “new” variables

$$\{U^{(1)}_{k>1}, U^{(2)}_{k>2}, \ldots, U^{(N-1)}_N\}; \{V^{(1)}_{k>1}, V^{(2)}_{k>2}, \ldots, V^{(N-1)}_N\}; \{\lambda^{(1)}\}, \ldots, \{\lambda^{(N)}\}.$$
 Consequently, as follows from Eq. (C1), the first \((N - 1)\) rows contain the common factor \(\lambda^{(1)}\) times quantities which depend only on \(\{U, V\}\). The next \((N - 2)\) lines contain the factor \(\lambda^{(2)}\), and so on. The last \(N\) lines are \(\{\lambda\}\)-independent. As a result, the Jacobian can be expressed as \(\prod_{i=1}^{N} [\lambda^{(i)}]^{2(N-i)} \times \text{some function of } \{U, V\}\). This expression has to be multiplied by \((\det \rho)^N = (\prod_{i=1}^{N} \lambda^{(i)})^N\), and we arrive at Eq. (14).
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FIGURES

FIG. 1. The integration contours $C_\pm$ for $N = 4$

FIG. 2. The integration contours $\tilde{C}_\pm$ for $N = 4$