The Tsallis entropy and the Shannon entropy of a universal probability

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Abstract—We study the properties of Tsallis entropy and Shannon entropy from the point of view of algorithmic randomness. In algorithmic information theory, there are two equivalent ways to define the program-size complexity $K(s)$ of a given finite binary string $s$. In the standard way, $K(s)$ is defined as the length of the shortest binary program for the universal self-delimiting Turing machine to output $s$. In the other way, the so-called universal probability $m$ is introduced first, and then $K(s)$ is defined as $-\log_2 m(s)$ without reference to the concept of program-size. In this paper, we investigate the properties of the Shannon entropy, the power sum, and the Tsallis entropy of a universal probability by means of the notion of program-size complexity. We determine the convergence or divergence of each of these three quantities, and evaluate its degree of randomness if it converges.

I. INTRODUCTION

Algorithmic information theory is a framework to apply information-theoretic and probabilistic ideas to recursive function theory. One of the primary concepts of algorithmic information theory is the program-size complexity (or Kolmogorov complexity) $K(s)$ of a finite binary string $s$, which is defined as the length of the shortest binary program for the universal self-delimiting Turing machine $U$ to output $s$. By the definition, $K(s)$ can be thought of as the information content of the individual finite binary string $s$. In fact, algorithmic information theory has precisely the formal properties of classical information theory (see Chaitin [2]). The concept of program-size complexity plays a crucial role in characterizing the randomness of a finite or infinite binary string.

The program-size complexity $K(s)$ is originally defined using the concept of program-size, as stated above. However, it is possible to define $K(s)$ without referring to such a concept, i.e., we first introduce a universal probability $m$, and then define $K(s)$ as $-\log_2 m(s)$.

In this paper, we investigate the properties of the Shannon entropy, the power sum, and the Tsallis entropy of a universal probability, from the point of view of algorithmic randomness, by means of the notion of program-size complexity. In particular, we show the following: (i) The Shannon entropy of any universal probability diverges to infinity. (ii) If $q$ is a computable real number with $q > 1$, then the power sum $\sum_s m(s)^q$ of any universal probability $m$ has the degree of randomness at least $1/q$. Here the notion of degree of randomness is a stronger notion than compression rate, and is defined using the program-size complexity [9], [10]. (iii) If $0 < q < 1$, then the power sum $\sum_s m(s)^q$ diverges to infinity. (iv) In the case where $q$ is a computable real number with $q > 1$, the Tsallis entropy $S_q(m)$ of a universal probability $m$ can have any computable degree of randomness. (v) If $0 < q < 1$, then the Shannon entropy $S_q(m)$ diverges to infinity.

II. PRELIMINARIES

We start with some notation about numbers and strings which will be used in this paper.

$\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers, and $\mathbb{N}^+$ is the set of positive integers. $\mathbb{Q}$ is the set of rational numbers, and $\mathbb{R}$ is the set of real numbers. $\{0, 1\}^* = \{\lambda, 0, 1, 00, 01, 10, 11, 000, 001, 010, \ldots\}$ is the set of finite binary strings where $\lambda$ denotes the empty string, and $\{0, 1\}^*$ is ordered as indicated. We identify any string in $\{0, 1\}^*$ with a natural number in this order, i.e., we consider $\varphi : \{0, 1\}^* \rightarrow \mathbb{N}$ such that $\varphi(s) = 1s - 1$ where the concatenation $1s$ of strings $1$ and $s$ is regarded as a dyadic integer, and then we identify $s$ with $\varphi(s)$. For any $s \in \{0, 1\}^*$, $|s|$ is the length of $s$. A subset $S$ of $\{0, 1\}^*$ is called a prefix-free set if no string in $S$ is a prefix of another string in $S$. $\{0, 1\}^\infty$ is the set of infinite binary strings, where an infinite binary string is infinite to the right but finite to the left. For any $\alpha \in \{0, 1\}^\infty$ and any $n \in \mathbb{N}^+$, $\alpha_n$ is the prefix of $\alpha$ length $n$. For any partial function $f$, the domain of definition of $f$ is denoted by $\text{dom} f$. We write “r.e.” instead of “recursively enumerable.”

Normally, $o(n)$ denotes any function $f : \mathbb{N}^+ \rightarrow \mathbb{R}$ such that $\lim_{n \rightarrow \infty} f(n)/n = 0$. On the other hand, $O(1)$ denotes any function $g : \mathbb{N}^+ \rightarrow \mathbb{R}$ such that there is $C \in \mathbb{R}$ with the property that $|g(n)| \leq C$ for all $n \in \mathbb{N}^+$.

Let $T$ be an arbitrary real number. $T \bmod 1$ denotes $T - [T]$, where $[T]$ is the greatest integer less than or equal to $T$. Hence, $T \bmod 1 \in [0, 1)$. We identify a real number $T$ with the infinite binary string $\alpha$ such that $0.\alpha$ is the base-two expansion of $T \bmod 1$ with infinitely many zeros. Thus, $T_n$ denotes the first $n$ bits of the base-two expansion of the real number $T \bmod 1$ with infinitely many zeros.

We say that a real number $T$ is computable if there exists a total recursive function $f : \mathbb{N}^+ \rightarrow \mathbb{Q}$ such that $|T - f(n)| < 2^{-n}$ for all $n \in \mathbb{N}^+$. We say that $T$ is right-computable if there exists a total recursive function $g : \mathbb{N}^+ \rightarrow \mathbb{Q}$ such that $T \leq g(n)$ for all $n \in \mathbb{N}^+$ and $\lim_{n \rightarrow \infty} g(n) = T$. We say that $T$ is left-computable if $-T$ is right-computable. It is then easy to see that, for any $T \in \mathbb{R}$, $T$ is computable if and only if $T$ is both right-computable and left-computable. See
e.g. Pour-El and Richards [6] and Weihrauch [14] for the detail of the treatment of the computability of real numbers and real functions on a discrete set.

A. Algorithmic information theory

In the following we concisely review some definitions and results of algorithmic information theory [2], [3], [4]. A computer is a partial recursive function \( C : \{0,1\}^* \rightarrow \{0,1\}^* \) such that \( \text{dom} C \) is a prefix-free set. For each computer \( C \) and each \( s \in \{0,1\}^* \), \( K_C(s) \) is defined by \( K_C(s) = \min \{ |p| \mid p \in \{0,1\}^* \text{ and } C(p) = s \} \). A computer \( U \) is said to be optimal if for each computer \( C \) there exists a constant \( \text{sim}(C) \) with the following property; if \( C(p) \) is defined, then there is a \( p' \) for which \( U(p') = C(p) \) and \( |p'| \leq |p| + \text{sim}(C) \). It is easy to see that there exists an optimal computer. Note that the class of optimal computers equals to the class of functions which are computed by universal self-delimiting Turing machines (see [2] for the detail). We choose a particular optimal computer \( U \) as the standard one for use, and define \( K(s) \) as \( K_U(s) \), which is referred to as the program-size complexity of \( s \), the information content of \( s \), or the Kolmogorov complexity of \( s \). Thus, \( K(s) \leq K_C(s) + \text{sim}(C) \) for any computer \( C \).

The program-size complexity \( K(s) \) is originally defined using the concept of program-size, as stated above. However, it is possible to define \( K(s) \) without referring to such a concept, i.e., as in the following, we first introduce a universal probability \( m \), and then define \( K(s) = -\log_2 m(s) \). We say that \( r \) is a semi-measure on \( \{0,1\}^* \) if \( r : \{0,1\}^* \rightarrow [0,1] \) such that \( \sum_{s \in \{0,1\}^*} r(s) \leq 1 \). A universal probability is defined as follows [15].

**Definition 1 (universal probability)** We say that \( r \) is a lower-computable semi-measure if \( r \) is a semi-measure on \( \{0,1\}^* \) and there exists a total recursive function \( f : \mathbb{N}^+ \times \{0,1\}^* \rightarrow \mathbb{Q} \) such that, for each \( s \in \{0,1\}^* \), \( \lim_{n \rightarrow \infty} f(n,s) = r(s) \) and \( \forall n \in \mathbb{N}^+ \ 0 \leq f(n,s) \leq r(s) \). We say that a lower-computable semi-measure \( m \) is a universal probability if for any lower-computable semi-measure \( r \), there exists a real number \( c > 0 \) such that, for all \( s \in \{0,1\}^* \), \( c r(s) \leq m(s) \).

The following theorem can be then shown (see e.g. Theorem 3.4 of Chaitin [2] for its proof). Here, \( P(s) \) is defined as \( \sum_{U(p)=s} 2^{-|p|} \) for each \( s \in \{0,1\}^* \).

**Theorem 2 Both \( 2^{-K(s)} \) and \( P(s) \) are universal probabilities.**

By Theorem 2 we see that, for any universal probability \( m \),

\[
K(s) = -\log_2 m(s) + O(1). \tag{1}
\]

Thus it is possible to define \( K(s) \) as \( -\log_2 m(s) \) with a particular universal probability \( m \) instead of as \( K_U(s) \). Note that the difference up to an additive constant is nonessential to algorithmic information theory. Any universal probability is not computable, as corresponds to the uncomputability of \( K(s) \). As a result, we see that \( 0 < \sum_{s \in \{0,1\}^*} m(s) < 1 \) for any universal probability \( m \).

For any \( \alpha \in \{0,1\}^\infty \), we say that \( \alpha \) is weakly Chaitin random if there exists \( c \in \mathbb{N} \) such that, for all \( n \in \mathbb{N}^+ \), \( n - c \leq K(\alpha_n) \) [2], [4]. As the total sum of the universal probability \( 2^{-K(s)} \), Chaitin [3] introduced the real number \( \theta \) by

\[
\theta = \sum_{s \in \{0,1\}^*} 2^{-K(s)}. \tag{2}
\]

Then [3] showed that \( \theta \) is weakly Chaitin random.

In the works [9], [10], we generalized the notion of the randomness of an infinite binary string so that the degree of the randomness can be characterized by a real number \( D \) with \( 0 < D \leq 1 \) as follows.

**Definition 3 (weakly Chaitin \( D \)-random)** Let \( D \in \mathbb{R} \) with \( D \geq 0 \), and let \( \alpha \in \{0,1\}^\infty \). We say that \( \alpha \) is weakly Chaitin \( D \)-random if there exists \( c \in \mathbb{N} \) such that, for all \( n \in \mathbb{N}^+ \), \( Dn - c \leq K(\alpha_n) \).

**Definition 4 (D-compressible)** Let \( D \in \mathbb{R} \) with \( D \geq 0 \), and let \( \alpha \in \{0,1\}^\infty \). We say that \( \alpha \) is \( D \)-compressible if \( K(\alpha_n) \leq Dn + o(n) \), which is equivalent to \( \lim_{n \rightarrow \infty} K(\alpha_n)/n \leq D \).

In the case of \( D = 1 \), the weak Chaitin \( D \)-randomness results in the weak Chaitin randomness. For any \( D \in \{0,1\} \) and any \( \alpha \in \{0,1\}^\infty \), if \( \alpha \) is weakly Chaitin \( D \)-random and \( D \)-compressible, then

\[
\lim_{n \rightarrow \infty} \frac{K(\alpha_n)}{n} = D, \tag{3}
\]

and therefore the compression rate of \( \alpha \) by the program-size complexity \( K \) is equal to \( D \). Note, however, that \( \Theta \) does not necessarily implies that \( \alpha \) is weakly Chaitin \( D \)-random.

In the work [10], we generalized \( \theta \) to \( \theta^D \) by

\[
\theta^D = \sum_{s \in \{0,1\}^*} 2^{-K^D(s)} \quad (D > 0). \tag{4}
\]

Thus, \( \theta = \theta^1 \). If \( 0 < D \leq 1 \), then \( \theta^D \) converges and \( 0 < \theta^D < 1 \), since \( \theta^D \leq \theta < 1 \). Theorem 5 below was mentioned in Remark 3.2 of Tadaki [10].

**Theorem 5 (Tadaki [10])** Let \( D \in \mathbb{R} \).

(i) If \( 0 < D \leq 1 \) and \( D \) is computable, then \( \theta^D \) is weakly Chaitin \( D \)-random.

(ii) If \( 0 < D \leq 1 \) and \( D \) is computable, then \( \theta^D \) is \( D \)-compressible.

(iii) If \( 1 < D \), then \( \theta^D \) diverges to \( \infty \).

III. THE SHANNON ENTROPY OF A UNIVERSAL PROBABILITY

We say that \( p = (p_1, \ldots, p_n) \) is a probability distribution if \( p_i \in [0,1] \) for all \( i = 1, \ldots, n \) and \( p_1 + \cdots + p_n = 1 \). For any probability distribution \( p = (p_1, \ldots, p_n) \), the Shannon entropy \( H(p) \) of \( p \) is defined by

\[
H(p) = -\sum_{i=1}^{n} p_i \ln p_i, \tag{5}
\]
where the ln denotes the natural logarithm [7]. We say that $p = (p_1, \ldots, p_n)$ is a semi-probability distribution if $p_i \in [0, 1]$ for all $i = 1, \ldots, n$ and $p_1 + \cdots + p_n \leq 1$. We define the Shannon entropy $H(p)$ also for any semi-probability distribution $p = (p_1, \ldots, p_n)$ by (5). Moreover, for any semi-measure $r$ on $\{0, 1\}^*$, we define the Shannon entropy $H(r)$ of $r$ by
\[ H(r) = - \sum_{s \in \{0, 1\}^*} r(s) \ln r(s) \]
in a similar manner to (5).

In this section, we prove that the Shannon entropy $H(m)$ of an arbitrary universal probability $m$ diverges to $\infty$. For convenience, however, we first prove the following more general theorem, Theorem 6, from which the result follows.

**Theorem 6** Let $A$ be an infinite r.e. subset of $\{0, 1\}^*$ and let $f : \mathbb{N}^+ \to \mathbb{N}$ be a total recursive function such that $\lim_{n \to \infty} f(n) = \infty$. Then the following hold.

(i) $\sum_{U(p) \in A} f(|p|)2^{-|p|}$ diverges to $\infty$.

(ii) If there exists $l_0 \in \mathbb{N}^+$ such that $f(l)2^{-l}$ is a nonincreasing function of $l$ for all $l \geq l_0$, then $\sum_{s \in A} f(K(s))2^{-K(s)}$ diverges to $\infty$.

**Proof:** (i) Contrarily, assume that $\sum_{U(p) \in A} f(|p|)2^{-|p|}$ converges. Then, there exists $d \in \mathbb{N}^+$ such that $\sum_{U(p) \in A} f(|p|)2^{-|p|} \leq d$. We define the function $r : \{0, 1\}^* \to [0, \infty)$ by
\[ r(s) = \frac{1}{d} \sum_{U(p)=s} f(|p|)2^{-|p|} \]
if $s \in A$; $r(s) = 0$ otherwise. We then see that $\sum_{s \in \{0, 1\}^*} r(s) \leq 1$ and therefore $r$ is a lower-computable semi-measure. Since $P(s)$ is a universal probability by Theorem 2, there exists $c \in \mathbb{N}^+$ such that $r(s) \leq cP(s)$ for all $s \in \{0, 1\}^*$. Hence we have
\[ \sum_{U(p)=s} (cd - f(|p|))2^{-|p|} \geq 0 \tag{6} \]
for all $s \in A$. On the other hand, since $A$ is an infinite set and $\lim_{n \to \infty} f(n) = \infty$, there is $s_0 \in A$ such that $f(|p|) > cd$ for all $p$ with $U(p) = s_0$. Therefore we have $\sum_{U(p)=s_0} (cd - f(|p|))2^{-|p|} < 0$. However, this contradicts (6), and the proof of (i) is completed.

(ii) We first note that there is $n_0 \in \mathbb{N}$ such that $K(s) \geq l_0$ for all $s$ with $|s| \geq n_0$. Now, let us assume contrarily that $\sum_{s \in A} f(K(s))2^{-K(s)}$ converges. Then, there exists $d \in \mathbb{N}^+$ such that $\sum_{s \in A} f(K(s))2^{-K(s)} \leq d$. We define the function $r : \{0, 1\}^* \to [0, \infty)$ by
\[ r(s) = \frac{1}{d} f(K(s))2^{-K(s)} \]
if $s \in A$ and $|s| \geq n_0$; $r(s) = 0$ otherwise. We then see that $\sum_{s \in \{0, 1\}^*} r(s) \leq 1$ and therefore $r$ is a lower-computable semi-measure. Since $2^{-K(s)}$ is a universal probability by Theorem 2, there exists $c \in \mathbb{N}^+$ such that $r(s) \leq c2^{-K(s)}$ for all $s \in \{0, 1\}^*$. Hence, if $s \in A$ and $|s| \geq n_0$, then $cd \geq f(K(s))$. On the other hand, since $A$ is an infinite set and $\lim_{n \to \infty} f(n) = \infty$, there is $s_0 \in A$ such that $|s_0| \geq n_0$ and $f(K(s_0)) > cd$. Thus, we have a contradiction, and the proof of (ii) is completed.

From Theorem 6 (ii), we obtain the following result, as desired.

**Corollary 7** Let $m$ be a universal probability. Then the Shannon entropy $H(m)$ of $m$ diverges to $\infty$.

**Proof:** We first note that there is a real number $x_0 > 0$ such that the function $x2^{-x}$ of a real number $x$ is decreasing for $x \geq x_0$. For this $x_0$, there is $n_0 \in \mathbb{N}$ such that $-\log_2 m(s) \geq x_0$ for all $s$ with $|s| \geq n_0$. On the other hand, by (1), there is $c \in \mathbb{N}$ such that $-\log_2 m(s) \leq K(s) + c$ for all $s \in \{0, 1\}^*$. Thus, we see that
\[
\sum_{s \in \{0, 1\}^* \& \ |s| \geq n_0} m(s) \log_2 m(s) \\
\geq \sum_{s \in \{0, 1\}^* \& \ |s| \geq n_0} (K(s) + c)2^{-K(s) - c} \\
= 2^{-c} \sum_{s \in \{0, 1\}^* \& \ |s| \geq n_0} K(s)2^{-K(s)} + c2^{-c} \sum_{s \in \{0, 1\}^* \& \ |s| \geq n_0} 2^{-K(s)}.
\]
Using Theorem 6(ii) with $A = \{0, 1\}^*$ and $f(n) = n$, we see that $\sum_{s \in \{0, 1\}^*} K(s)2^{-K(s)}$ diverges to $\infty$. It follows from (7) that $-\sum_{s \in \{0, 1\}^*} m(s) \log_2 m(s)$ also diverges to $\infty$. This completes the proof.

**IV. THE POWER SUM OF A UNIVERSAL PROBABILITY**

In this section, we investigate the convergence or divergence of the power sum $\sum_{s \in \{0, 1\}^*} m(s)^q$ of a universal probability $m$, and evaluate its degree of randomness if it converges, by means of the notions of the weak Chaitin $D$-randomness and the $D$-compressibility. We first consider the notion of the weak Chaitin $D$-randomness of the power sum of a universal probability. We can generalize Theorem 5 (i) and (iii) on the specific universal probability $2^{-K(s)}$ over an arbitrary universal probability as follows.

**Theorem 8** Let $m$ be a universal probability, and let $q \in \mathbb{R}$.

(i) If $q \geq 1$ and $q$ is a right-computable real number, then $\sum_{s \in \{0, 1\}^*} m(s)^q$ converges to a left-computable real number which is weakly Chaitin $1/q$-random.

(ii) If $0 < q < 1$, then $\sum_{s \in \{0, 1\}^*} m(s)^q$ diverges to $\infty$.

Theorem 8 (i) shows that, for any $q \in \mathbb{R}$ with $q \geq 1$, the right-computability of $q$ results in the weak Chaitin $1/q$-randomness of the power sum $\sum_{s \in \{0, 1\}^*} m(s)^q$ of a universal probability $m$. On the other hand, Theorem 8 below shows that...
the converse in a certain sense holds. Theorem 9 can be proved based on the techniques used in the proof of the fixed point theorem on compression rate [12].

**Theorem 9** Let $m$ be a universal probability, and let $q \in \mathbb{R}$ with $q \geq 1$. If $\sum_{s \in \{0,1\}^*} m(s)^q$ is a right-computable real number, then $q$ is weakly Chaitin $1/q$-random. \qed

Next, we consider the notion of the $D$-compressibility of the power sum of a universal probability. Theorem 5 (ii) shows that, for the specific universal probability $m(s) = 2^{-K(s)}$, if $q$ is a computable real number with $q > 1$, then the power sum $\sum_{s \in \{0,1\}^*} m(s)^q$ is $1/q$-compressible. Thus, the following question naturally arises: is $\sum_{s \in \{0,1\}^*} m(s)^q$ a $1/q$-compressible real number for any universal probability $m$ and any computable real number $q > 1$? As shown in Theorem 10, however, we cannot answer this question negatively.

**Theorem 10** There exists a universal probability $m$ such that, for every computable real number $q > 1$, $\sum_{s \in \{0,1\}^*} m(s)^q$ is weakly Chaitin random and therefore not $1/q$-compressible.

**Proof:** We choose any one universal probability $r$, and then choose any one $c \in \mathbb{N}$ with $2^{-c} \theta \leq r(\lambda)$, where $\theta$ is defined by (2). We define the function $m: \{0,1\}^* \rightarrow [0, \infty)$ by $m(s) = 2^{-s} \theta$ if $s = \lambda$; $m(s) = r(s)$ otherwise. Since $\sum_{s \in \{0,1\}^*} r(s) \leq 1$, it follows that $\sum_{s \in \{0,1\}^*} m(s) \leq 1$. Therefore, since $\theta$ is left-computable and $r$ is a lower-computable semi-measure, we see that $m$ is a lower-computable semi-measure. Note that $dr(s) \leq m(s)$ for all $s \in \{0,1\}^*$, where $d = 2^{-c} \theta/r(\lambda) > 0$. Thus, since $r$ is a universal probability, $m$ is also a universal probability.

On the other hand, since $\theta$ is weakly Chaitin random, $m(\lambda)$ is also weakly Chaitin random. Let $q$ be an arbitrary computable real number with $q > 1$. Then, since $q > 1$ is a computable real number with $q \neq 0$, it follows that $K((\alpha^q)_n) = K(\alpha_n) + O(1)$ for any real number $a > 0$. Thus, $K(m(\lambda)(n)^q) = K((\lambda)(n)^q) + O(1)$ and therefore $m(\lambda)^q$ is weakly Chaitin random. Note that $K(\alpha_n) \leq K((\alpha + b)_n) + O(1)$ for any left-computable real numbers $a, b$. This can be proved using the condition 2 of Lemma 4.4 and Theorem 4.9 of [1]. Thus, since $m(\lambda)^q$ and $\sum_{s \neq \lambda} m(s)^q$ are left-computable, we see that $\sum_{s \in \{0,1\}^*} m(s)^q$ is weakly Chaitin random. It follows from $q > 1$ that $\sum_{s \in \{0,1\}^*} m(s)^q$ is not $1/q$-compressible. \qed

V. THE TSEALLIS ENTROPY OF A UNIVERSAL PROBABILITY

The notion of Tsallis entropy has been introduced by Tsallis [13]. Let $q$ be a positive real number with $q \neq 1$. For any probability distribution $p = (p_1, \ldots, p_n)$, the Tsallis entropy $S_q(p)$ of $p$ is defined by

$$S_q(p) = \frac{1 - \sum_{i=1}^{n} p_i^q}{q - 1}. \quad (8)$$

When $q \rightarrow 1$, the Tsallis entropy recovers the Shannon entropy for any probability distribution. See [13], [5] for the detail of the theory and applications of Tsallis entropy.

We generalize the definition (8) for any semi-probability distribution $p = (p_1, \ldots, p_n)$ by

$$S_q(p) = \frac{\sum_{i=1}^{n} |p_i - p_i^q|}{q - 1}. \quad (9)$$

In fact, we see that, for any semi-probability distribution $p$, $\lim_{q \rightarrow 1} S_q(p) = H(p)$, and therefore this generalization is consistent with the Shannon entropy for a semi-probability distribution, defined in Section III. Thus, we define the Tsallis entropy $S_q(r)$ of any semi-measure $r$ on $\{0,1\}^*$ by

$$S_q(r) = \frac{1}{q - 1} \sum_{s \in \{0,1\}^*} \{r(s) - r(s)^q\}$$

in a similar manner to (9).

In what follows, we investigate the convergence or divergence of the Tsallis entropy $S_q(m)$ of a universal probability $m$, and evaluate its degree of randomness if it converges, in the same manner as the previous section. We first investigate the convergence and divergence of $S_q(m)$ as follows.

**Theorem 11** Let $m$ be a universal probability, and let $q \in \mathbb{R}$.

(i) If $q > 1$, then $S_q(m)$ converges.

(ii) If $0 < q < 1$, then $S_q(m)$ diverges to $\infty$.

**Proof:** Theorem 11 follows immediately from Theorem 8. \qed

Theorem 12 below shows that, if the total sum of a universal probability $m$ is small, then the Tsallis entropy of $m$ has to be maximally random with respect to the degree of randomness.

**Theorem 12** Let $m$ be a universal probability, and let $s$ be a computable real number with $q > 1$. If $m(s) \leq q^{-1}$ for all $s \in \{0,1\}^*$, then $S_q(m)$ is left-computable and weakly Chaitin random.

**Proof:** By Theorem 11 (i), there is $d \in \mathbb{N}^+$ such that $S_q(m) \leq d$. We define $r: \{0,1\}^* \rightarrow (0, \infty)$ by $r(s) = F(m(s))/d$, where $F: \{0,1\}^* \rightarrow (0, \infty)$ with $F(x) = (x - x^q)/(q - 1)$, and we show that $r$ is a universal probability.

Obviously, $\sum_{s \in \{0,1\}^*} r(s) \leq 1$. Since $m$ is a lower-computable semi-measure, there exists a total recursive function $f: \mathbb{N}^+ \times \{0,1\}^* \rightarrow \mathbb{Q}$ such that, for each $s \in \{0,1\}^*$, $\lim_{n \rightarrow \infty} f(n, s) = m(s)$ and $\forall n \in \mathbb{N}^+, 0 < f(n, s) \leq m(s)$. Since $F(x)$ is continuous and increasing for all $x \in (0, q^{-1}]$, it follows that, for each $s \in \{0,1\}^*$, $\lim_{n \rightarrow \infty} F(f(n, s)) = F(m(s))$ and $\forall n \in \mathbb{N}^+, 0 \leq F(f(n, s)) \leq F(m(s))$. On the other hand, since $q$ is computable, there exists a total recursive function $g: \mathbb{N}^+ \times \{0,1\}^* \rightarrow \mathbb{Q} \cap [0, \infty)$ such that, for each $s \in \{0,1\}^*$ and each $n \in \mathbb{N}^+$,

$$F(f(n, s)) - 2^{-n} \leq g(n, s) \leq F(f(n, s)).$$

Hence, $r$ is a lower-computable semi-measure. Note that $x/q \leq F(x)$ for all $x \in (0, q^{-1}]$. It follows that $m(s)/(qd) \leq r(s)$ for all $s \in \{0,1\}^*$. Thus, since $m$ is a universal probability, $r$ is also a universal probability.
It follows from Theorem 8 (i) that \( \sum_{s \in \{0,1\}^*} r(s) = S_q(m)/d \) is weakly Chaitin random. Note that \( K(a_n) \leq K((ab)_n) + O(1) \) for any left-computable real numbers \( a, b > 0 \). This can be proved using the condition 4 of Lemma 4.4 and Theorem 4.9 of [1]. Thus, since \( \sum_{s \in \{0,1\}^*} r(s) \) and \( d \) are left-computable positive real numbers, we see that \( S_q(m) \) is weakly Chaitin random and, obviously, left-computable. □

Based on Theorem 12, we can show a stronger result than Theorem 12 with respect to the range of the degree of randomness of the Tsallis entropy \( S_q(m) \). Theorem 13 and Corollary 14 show below that the Tsallis entropy of a universal probability can have any computable degree of randomness \( D \). Note, however, that Theorem 13 is not a generalization of Theorem 12. The reason is as follows: The Tsallis entropy \( S_q(m) \) is right-computable in Theorem 13 whereas it is not right-computable in Theorem 12.

**Theorem 13** Let \( q \) be a computable real number with \( q > 1 \). Then, for any right-computable real number \( y \in (0, q^{1/q}) \), there exists a universal probability \( m \) such that \( S_q(m) = y \).

**Proof:** Let \( F : (0, 1) \to [0, \infty) \) with \( F(x) = (x - x^q)/(q - 1) \), and let \( x_0 \) be the unique real number such that \( q^{1/q} < x_0 < 1 \) and \( F(x_0) = y/2 \). We choose any one rational number \( c \) such that \( 0 < c \leq \min\{q^{-1}, 1 - x_0, (q - 1)/2y\} \). We also choose any one universal probability \( r \). We then define a universal probability \( r_1 : \{0, 1\}^* \to (0, 1) \) by \( r_1(s) = cr(s) \). Since \( r_1(s) \leq q^{-1/q} \) for all \( s \in \{0, 1\}^* \), it follows from Theorem 12 that \( S_q(r_1) \) is left-computable.

Let \( \Theta = S_q(r_1) \). From \( \sum_{s \in \{0,1\}^*} r(s) \leq 1 \) we have \( \sum_{s \in \{0,1\}^*} r_1(s) \leq c \). Therefore,

\[
\Theta = \sum_{s \in \{0,1\}^*} F(r_1(s)) < \frac{1}{q-1} \sum_{s \in \{0,1\}^*} r_1(s) \leq \frac{c}{q-1}.
\]

Since \( c/(q-1) \leq y/2 \), it follows that \( y/2 < y - \Theta < y \).

Note that \( F(x) \) is continuous and decreasing for all \( x \in [q^{-1/q}, 1] \). Thus, since \( F(q^{-1/q}) = q^{-1/q} \geq y \) and \( y/2 > F(1) = 0 \), there exists the unique real number \( a \) such that \( q^{-1/q} < a < x_0 \) and \( F(a) = y - \Theta \). We see that \( a \) is left-computable. This is because \( y - \Theta \) is right-computable, \( q \) is computable, and \( F(x) \) is decreasing for all \( x \in (q^{-1/q}, x_0) \).

We define the function \( m : \{0, 1\}^* \to (0, \infty) \) by \( m(s) = a \) if \( s = \lambda \); \( m(s) = r_1(s-1) \) otherwise. Note here that \( \{0,1\}^* \) is identified with \( \mathbb{N} \). Then, it follows from \( c \leq 1 - x_0 \) and \( a < x_0 \) that \( \sum_{s \in \{0,1\}^*} m(s) < 1 \). Thus, since \( r_1 \) is a lower-computable semi-measure and \( a \) is left-computable, we see that \( m \) is a lower-computable semi-measure. Since \( r_1 \) is a universal probability and \( a > 0 \), we further see that \( m \) is a universal probability. On the other hand, \( S_q(m) = F(a) + S_q(r_1) = F(a) + \Theta = y \). This completes the proof. □

**Corollary 14** Let \( q \) be a computable real number with \( q > 1 \). Then, for any computable real number \( D \in [0, 1] \), there exists a universal probability \( m \) such that \( S_q(m) \) is weakly Chaitin \( D \)-random and \( D \)-compressible.

**Proof:** In the case of \( D = 0 \), consider a rational number \( y \in (0, q^{1/q}) \) in Theorem 13. In the case of \( D > 0 \), consider \( y = a(1 - \theta^D) \) in Theorem 13 where \( a \) is any one rational number with \( a \in (0, q^{1/q}) \) and \( \theta^D \) is defined by (4). In this case, the result follows from Theorem 5(i) and (ii). □

**VI. Conclusion**

In this paper, we have investigated the properties of the Shannon entropy, the power sum, and the Tsallis entropy of a universal probability, from the point of view of algorithmic randomness. Future work may aim at generalizing Rényi entropy over a universal probability properly and investigating its randomness properties.

**Acknowledgments**

This work was supported both by SCOPE (Strategic Information and Communications R&D Promotion Programme) from the Ministry of Internal Affairs and Communications of Japan and by KAKENHI, Grant-in-Aid for Scientific Research (C) (20540134).

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