On some free boundary problem of the Navier-Stokes equations in the maximal $L_p$-$L_q$ regularity class

Yoshihiro SHIBATA *

Department of Mathematics and Research Institute of Science and Engineering
Waseda University, Ohkubo 3-4-1, Shinjuku-ku, Tokyo 169-8555, Japan

e-mail address: yshibata@waseda.jp

Abstract

This paper is concerned with the free boundary problem for the Navier-Stokes equations without surface tension in the $L_p$ in time and $L_q$ in space setting with $2 < p < \infty$ and $N < q < \infty$. A local in time existence theorem is proved in a uniform $W^{2-1/q}_2$ domain in the $N$-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) under the assumption that weak Dirichlet-Neumann problem is uniquely solvable. Moreover, a global in time existence theorem is proved for small initial data under the assumption that $\Omega$ is bounded additionally. This was already proved by Solonnikov [28] by using the continuation argument of local in time solutions which are exponentially stable in the energy level under the assumption that the initial data is orthogonal to the rigid motion. We also use the continuation argument and the same orthogonality for the initial data. But, our argument about the continuation of local in time solutions is based on some decay theorem for the linearized problem, which is a different point than [28].

Mathematics Subject Classification (2012). 35Q30, 76D05

Keywords. Navier-Stokes equations, free boundary problem, uniform $W^{2-1/q}_2$ domain, local in time unique existence theorem, bounded domain, global in time unique existence theorem

1 Introduction

The present paper deals with some local and global in time unique existence theorems of solutions to the Navier-Stokes equations describing the motion of a viscous incompressible fluid flow with free surface without taking surface tension into account. Our problem is formulated in the following. Let $\Omega$ be a domain in the $N$-dimensional Euclidean space $\mathbb{R}^N$ ($N \geq 2$) occupied by a viscous incompressible fluid. We assume that the boundary of $\Omega$ consists of two parts $S$ and $\Gamma$ with $S \cap \Gamma = \emptyset$. We may assume that $\Gamma$ is an empty set. Let $\Omega_t$ and $S_t$ be evolutions of $\Omega$ and $S$ with time variable $t > 0$ and we assume that $S_t \cap \Gamma = \emptyset$ for $t \geq 0$. The velocity vector field $\mathbf{v} = \mathbf{v}(x, t) = (v_1(x, t), \ldots, v_N(x, t))$ and the pressure $\pi = \pi(x, t)$ for $x = (x_1, \ldots, x_N) \in \Omega_t$ satisfy the Navier-Stokes equations

$$
(\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v}) - \text{Div} \mathbf{T}(\mathbf{v}, \pi) = 0, \quad \text{div} \mathbf{v} = 0. \tag{1.1}
$$

The initial conditions, the boundary conditions on the free boundary $S_t$ and the non-slip conditions on the fixed boundary $\Gamma$ have the following forms:

$$
v_{|t=0} = v_0 \quad \text{in } \Omega,
\quad \mathbf{T}(\mathbf{v}, \pi) n_t |_{S_t} = 0, \quad \mathbf{v} |_{\Gamma} = 0. \tag{1.2}
$$

Here, $n_t$ is the unit outward normal to $S_t$. Moreover, $\mathbf{T} = \mathbf{T}(\mathbf{v}, \pi)$ denotes the stress tensor of the form:

$$
\mathbf{T}(\mathbf{v}, \pi) = -\pi \mathbf{I} + \mu \mathbf{D}(\mathbf{v}) \tag{1.3}
$$

*Partially supported by JST CREST and JSPS Grant-in-aid for Scientific Research (S) # 24224004
where $\mu$ denotes a positive constant describing the viscosity coefficient, $D(v)$ the deformation tensor whose $(j,k)$ components are $D_{jk}(v) = (\partial_j v_k + \partial_k v_j)$ with $\partial_j = \partial/\partial x_j$, and $I$ the $N \times N$ identity matrix. Finally, for any matrix field $K$ with components $K_{ij}$, $i,j = 1, \ldots, N$, the quantity $\text{Div} \, K$ is an $N$-vector with $i$-th component $\sum_{j=1}^{N} \partial_j K_{ij}$, and also for any vector of functions $u = (u_1, \ldots, u_N)$ we set $\text{div} \, u = \sum_{j=1}^{N} \partial_j u_j, \quad u \cdot \nabla = \sum_{j=1}^{N} u_j \partial_j$ and $\partial t u = (\partial u_1/\partial t, \ldots, \partial u_N/\partial t)$.

Aside from the dynamical system (1.1), we impose a further kinematic condition:

$$\partial_t F + (v \cdot \nabla) F = 0 \quad \text{on } S_t,$$

where $S_t$ is defined by $F = F(x,t) = 0$ locally. In other words, $S_t$ is given by

$$S_t = \{ x \in \mathbb{R}^N \mid x = x(\xi, t) \quad (\xi \in S) \},$$

where $x = x(\xi, t)$ is the solution to the Cauchy problem: $x = dx/dt = v(x,t) (t > 0)$ with $x|_{t=0} = \xi$. This expresses the fact that the free boundary $S_t$ consists of the same particles for all $t > 0$, which do not leave it and are not incident from $\Omega$.

The free boundary problem for the Navier-Stokes equations has been studied by many mathematicians in the following two cases:

(1) The motion of an isolated liquid mass;

(2) The motion of a viscous incompressible fluid contained in an ocean of infinite content.

In case (1) the initial domain $\Omega$ is bounded. A local in time unique existence theorem was proved by Solonnikov [19], [27] and Nishida [7], Sylvestre [35] and Hataya [14], under the assumption that the initial domain $\Omega$ is a uniform $W^{2,1/2}$ $(N < q < \infty)$ domain and weak Dirichlet-Neumann problem is uniquely solvable [5], which includes the cases (1) and (2) without surface tension. And also, we prove a global in time unique existence theorem for small initial velocity was proved by Solonnikov [27] in the $L_p$ Sobolev-Slobodetskii space without surface tension; and by Solonnikov [27] in the $L_2$ Sobolev-Slobodetskii space and by Padula and Solonnikov [19] in the H"older spaces under the additional assumption that the initial domain $\Omega$ is sufficiently close to a ball with surface tension.

In case (2), the initial domain $\Omega$ is a perturbed layer like: $\Omega = \{ x \in \mathbb{R}^N \mid -b < X_N < \eta(x'), x' = (x_1, \ldots, x_{N-1}) \in \mathbb{R}^{N-1} \}$. A local in time unique existence theorem was proved by Beale [5], Allain [2] and Tani [39] in the $L_2$ Sobolev-Slobodetskii space with surface tension and by Abels [1] in the $L_p$ Sobolev-Slobodetskii space without surface tension. A global in time unique existence theorem for small initial velocity was proved in the $L_2$ Sobolev-Slobodetskii space by Beale [5] and Tani and Tanaka [37] with surface tension, and by Sylvester [34] without surface tension. The decay rate was studied by Beale and Nishida [7], Sylvestre [35] and Hataya [14].

The purpose of this paper is to prove a local in time unique existence theorem for problem (1.1) and (1.2) under the assumption that the initial domain $\Omega$ is a uniform $W^{2,1/2} (N < q < \infty)$ domain and weak Dirichlet-Neumann problem is uniquely solvable [5], which includes the cases (1) and (2) without surface tension. And also, we prove a global in time unique existence theorem for problem (1.1) and (1.2) for a small initial data in the $L_p$ in time and $L_q$ in space setting assuming that $\Omega$ is bounded in addition. This was mentioned in Shibata and Shimizu [23], but there was a serious gap in the proof, so that we reprove it in a different approach than [23] in this paper.

To prove a local in time unique existence theorem, the key step is to prove the maximal regularity theorem for the linearized equations given in the following:

$$\partial_t u - \text{Div} \, T(u, \theta) = f, \quad \text{div} \, u = g = \text{div} \, g \quad \text{in } \Omega \times (0,T),$$

$$T(u, \theta) \tilde{n}|_S = h|_S, \quad u|_{\Gamma} = 0, \quad u|_{t=0} = u_0 \quad \text{in } \Omega$$

with $0 < T \leq \infty$. Here, $\tilde{n}$ denotes the extension of $n$ to the whole space $\mathbb{R}^N$. In fact, as was seen in [13] (5.12)] (cf. also [12 Appendix]), we can define $\tilde{n}$ on $\mathbb{R}^N$ such that $\tilde{n}|_S = n$ and

$$\|f \tilde{n}\|_{W^{2,1}(\Omega)} \leq C\|f\|_{W^{2,1}(\Omega)}$$

where $\| \cdot \|_{W^{2,1}(\Omega)}$ is the norm in the $W^{2,1}(\Omega)$ space.

*These assumptions are exactly stated in Definition 2.1 and Definition 2.2 in the following.
for any $f \in W^1_q(\Omega)$ with some constant $C$ depending on $\Omega$ if $\Omega$ is a uniform $W^{2-1/r}_r$ domain with $N < r < \infty$.

To prove the maximal regularity theorem, problem \textbf{(1.6)} is reduced locally to the model problems in a neighbourhood of either an interior point or a boundary point by using the localization technique and the partition of unity associated with the domain $\Omega$. The boundary neighbourhood problem \textbf{(1.9)} is transformed to a problem in the half-space $x_N > 0$. By applying the Fourier transform with respect to time and tangential directions, problem \textbf{(1.6)} becomes a system of ordinary differential equations. Solonnikov \cite{Solonnikov} calculates explicitly the inverse Fourier transform of solutions of such ordinary differential equations and expresses them in the form of potentials in the half-space. Then, he estimates them in suitable norms. Mucha and Zajączkowski \cite{Mucha_Zajaczkowski} directly estimate them using the multiplier theorem of Marinkiewicz and Mikhlin type \cite{Marinkiewicz_Mikhlin}.

On the other hand, Shibata \cite{Shibata} proved the maximal regularity theorem\footnote{The maximal regularity theorems are given in Theorem 1.1 and Theorem 1.2 in Sect. 2 in the following.} by using the $R$-bounded solution operators to the corresponding resolvent problem of the form:

$$\lambda v - \text{Div } T(v, \kappa) = f, \quad \text{div } v = g = \text{div } g \quad \text{in } \Omega,$$

\[
T(v, \kappa) n|\partial \Omega = h|\partial \Omega, \quad v|\partial \Omega = 0. 
\]

In fact, according to the theorem in \cite{Shibata}, for any $\epsilon \in (0, \pi/2)$ there exist a constant $\lambda_0 \geq 1$ and an operator family $R(\lambda) \in \text{Hol}(\mathbb{C}, L(\mathcal{X}_q, W^2_q(\Omega)^N))$ such that for any $f \in L^q_q(\Omega), g \in W^1_q(\Omega), h \in W^1_q(\Omega)^N$ and $\kappa \in (0, \pi/2)$, problem \textbf{(1.8)} admits a unique solution $v = R(\lambda)(f, \lambda^{1/2} g, \nabla g, \lambda^{1/2} h, \nabla h)$ with some term $\kappa$, and $(\lambda, \lambda^{1/2} \nabla, \lambda^{1/2} \nabla^2) R(\lambda)$ is $R$ bounded for $\lambda \in \Sigma_{\epsilon, \lambda_0}$ with value in $L(\mathcal{X}_q, L^q_q(\Omega)^N)$. Here, $N = N + N^2 + N^3, \Sigma_{\epsilon, \lambda_0} = \{ \lambda \in \mathbb{C} \mid |\lambda| \geq \lambda_0, \arg \lambda \leq \pi - \epsilon \}, \mathcal{X}_q = \{ F = (F_1, \ldots, F_6) \mid F_1, F_2, F_3, F_4, F_5 \in L^q_q, \partial F_6 \in L^q_q(\Omega)^N \},$ and $F_1, F_2, F_3, F_4, F_5$ and $F_6$ are independent variables corresponding to $f, \lambda^{1/2} g, \nabla g, \lambda^{1/2} h$ and $\nabla h$, respectively. Moreover, $\text{Hol}(\Sigma_{\epsilon, \lambda_0}, \mathcal{L}(X,Y))$ denotes the set of all $\mathcal{L}(X,Y)$ valued holomorphic functions defined on $\Sigma_{\epsilon, \lambda_0}$ and $\mathcal{L}(X,Y)$ the set of all bounded linear operators from a Banach space $X$ to another Banach space $Y$. Since the solution $u$ for \textbf{(1.6)} is given by the Laplace inverse transform of $R(\lambda)(f, \lambda^{1/2} g, \nabla g, \lambda^{1/2} h, \nabla h)$, the maximal regularity is obtained with help of Weis’ operator valued Fourier multiplier theorem \cite{Weis}. Finally, we introduce some symbols used throughout the paper. For any domain $D$ and $1 \leq q \leq \infty$, $L^q_q(D)$ and $W^1_q(D)$ denote the usual Lebesgue space and Sobolev space, while $\| \cdot \|_{L^q_q(D)}$ and $\| \cdot \|_{W^1_q(D)}$ denote their norms, respectively. We set $W^0_q(D) = L^q_q(D). \quad C^\infty_0(D)$ denotes the set of all $C^\infty(\mathbb{R}^N)$ functions whose supports are compact and contained in $D$. We set $(f, g)_D = \int_D f(x)g(x)dx$. For any Banach space $X$ and $1 \leq p \leq \infty$, $L_p_p((a, b), X)$ and $W^1_p_p((a, b), X)$ denote the usual Lebesgue space and Sobolev space of $X$-valued functions defined on an interval $(a, b)$, while $\| \cdot \|_{L_p_p((a, b), X)}$ and $\| \cdot \|_{W^1_p_p((a, b), X)}$ denote their norms, respectively. For $0 < \theta < 1$, $B^\theta_p_p(D)$ denotes the real interpolation space defined by $B^\theta_p_p(D) = (L^1_p_p(D), L^\infty_p_p(D))_{\theta, p}$ with real interpolation functor $(\cdot, \cdot)_\theta, p$, while $\| \cdot \|_{B^\theta_p_p(D)}$ denotes its norm. We set $W^0_p_p = B^0_p_p. \quad$ The $d$-product space of $X$ is defined by $X^d = \{ f = (f_1, \ldots, f_d) \mid f_i \in X(i = 1, \ldots, d) \}$, while its norm is denoted by $\| \cdot \|_X$ instead of $\| \cdot \|_{X^d}$ for the sake of simplicity. $N, \mathbb{R}$ and $\mathbb{C}$ denote the sets of all natural numbers, real numbers and complex numbers, respectively. We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}. \quad$ For any multi-index $\kappa = (\kappa_1, \ldots, \kappa_N) \in \mathbb{N}_0^N$, we write $|\kappa| = \kappa_1 + \cdots + \kappa_N$ and $\partial^{\kappa} = \partial^{\kappa_1} \cdots \partial^{\kappa_N}$ with $x = (x_1, \ldots, x_N)$ and $\partial_j = \partial/\partial x_j$. For any scalar function $f$ and $N$-vector of functions $g$, we set

$$\nabla f = (\partial_1 f, \ldots, \partial_N f), \quad \nabla g = (\partial_ig_j \mid i, j = 1, \ldots, N),$$

$$\nabla^2 f = (\partial^2 f \mid \alpha = 2, \quad \nabla^2 g = (\partial^2 g \mid \alpha = 2, i = 1, \ldots, N).$$

For $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N) \in \mathbb{R}^N$, we set $a \cdot b = \sum_{j=1}^N a_j b_j$. For scalar functions $f, g$ and $N$-vectors of functions $f, g$, we set $(f, g)_D = \int_D f(x)g(x)dx$ and $(f, g)_D = \int_D f(x)g(x)dx$. The letter $C$ denotes generic constants and the constant $C_{a, b, \ldots}$ depends on $a, b, \ldots$. The values of constants $C$ and $C_{a, b, \ldots}$ may change from line to line.
2 Main Results

In this section, we state our main results. Since $\Omega$ should be decided, we transfer $\Omega$ to $\Omega$ by the Lagrange transformation as follows: If the velocity field $u(\xi, t)$ is known as a function of the Lagrange coordinates $\xi \in \Omega$, then the Euler coordinates $x \in \Omega_0$ is written in the form:

$$x = \xi + \int_0^t u(\xi, s) \, ds \equiv X_0(\xi, t),$$

where $u(\xi, t) = (u_1(\xi, t), \ldots, u_N(\xi, t)) = v(X_0(\xi, t), t)$. Let $A$ be the Jacobi matrix of the transformation $x = X_0(\xi, t)$ with elements $a_{ij} = \delta_{ij} + \int_0^t (\partial u_i/\partial \xi_j)(\xi, s) \, ds$. Since det $A = 1$ as follows from $\operatorname{div} v = 0$ in $\Omega_0$, denoting the cofactor matrix of $A$ by $\mathcal{A}$, we have $\nabla_x = A \nabla_\xi$ with $\nabla_x = \frac{T}{\partial x_1, \ldots, \partial x_N}$ and $\nabla_\xi = \frac{T}{\partial / \partial x_1, \ldots, \partial / \partial x_N}$ [1]. We can represent $A$ by $A = I + V_0(\int_0^t \nabla u(\xi, s) \, ds)$ with some matrix $V_0(K)$ of polynomials with respect to $K = (k_{ij})$ satisfying the condition: $V_0(0) = 0$, where $k_{ij}$ is a corresponding variable to $\int_0^t (\partial u_i/\partial \xi_j)(\xi, s) \, ds$. Let $\mathbf{n}$ be the unit outward normal to $S$, and then by (2.1) we have

$$\mathbf{n}_n = \frac{\mathbf{A}_n}{|\mathbf{A}_n|}.$$  (2.1)

We also see that

$$\operatorname{div}_x w = \operatorname{div}_\xi (\mathcal{A} \mathbf{w}) = \operatorname{tr}(\mathcal{A} \nabla_\xi \mathbf{w})$$  (2.2)

with $\dot{w}(\xi, t) = w(X_0(\xi, t), t)$, where $\mathcal{A} M$ denotes the trace of any matrix $M$. Moreover, what $A = (A^{-1})^{-1} = A^{-1}$ yields that

$$A^{-1} = I + V_1(\int_0^t \nabla u(\xi, s) \, ds)$$  (2.3)

with some matrix $V_1(K)$ of polynomials with respect to $K = (k_{ij})$ satisfying the condition: $V_1(0) = 0$. Using (2.1), (2.2) and (2.3), and setting $\theta(\xi, t) = \pi(X_0(\xi, t))$, we have the following Lagrangian description of problem (1.1):  

$$\partial_t u - \operatorname{Div}(T(u, \theta)) = F(u), \quad \operatorname{div} u = G(u) = \operatorname{div} G(u) \quad \text{in} \Omega \times (0, T), \quad T(u, \theta) \mathbf{n}|_{\partial S} = H(u) \mathbf{n}|_{\partial S}, \quad u|_{\Gamma} = 0, \quad u|_{t=0} = v_0 \quad \text{in} \Omega.$$  (2.4)

Here, $F(u)$, $g(u)$, $g(u)$ and $H(u)$ are nonlinear functions of the forms:

$$F(u) = -V_1\left(\int_0^t \nabla u \, ds\right) \partial_t u + V_2\left(\int_0^t \nabla u \, ds\right) \nabla^2 u + V_3(\int_0^t \nabla u \, ds) \int_0^t \nabla^2 u \, ds \cdot \nabla u,$$

$$G(u) = V_4(\int_0^t \nabla u \, ds) \nabla u, \quad G(u) = V_5(\int_0^t \nabla u \, ds) u, \quad H(u) = V_6(\int_0^t \nabla u \, ds) \nabla u, \quad (2.5)$$

with some matrices $V_i(K)$ ($i = 1, \ldots, 6$) of polynomials with respect to $K$ satisfying the conditions:

$$V_1(0) = 0, \quad V_2(0) = 0, \quad V_4(0) = 0, \quad V_5(0) = 0, \quad V_6(0) = 0.$$  (2.6)

We introduce the definition of uniform $W^{2-1/r}_r$ domain.

**Definition 2.1.** Let $1 < r < \infty$ and let $\Omega$ be a domain in $\mathbb{R}^N$ with boundary $\partial \Omega$. We say that $\Omega$ is a uniform $W^{2-1/r}_r$ domain, if there exist positive constants $\alpha$, $\beta$ and $K$ such that for any $x_0 = (x_{01}, \ldots, x_{0N}) \in \partial \Omega$ there exist a coordinate number $j$ and a $W^{2-1/r}_r$ function $h(x') (x' = (x_1, \ldots, x_j, \ldots, x_N))$ defined on $B'_a(x'_0)$ with $x_0 = (x_{01}, \ldots, \hat{x}_j, \ldots, x_{0N})$ and $\|h\|_{W^{2-1/r}_r(B'_a(x'_0))} \leq K$ such that

$$\Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j > h(x') (x' \in B'_a(x'_0)) \} \cap B_\beta(x_0),$$

$$\partial \Omega \cap B_\beta(x_0) = \{x \in \mathbb{R}^N \mid x_j = h(x') (x' \in B'_a(x'_0)) \} \cap B_\beta(x_0).$$  (2.7)

Here, $(x_1, \ldots, \hat{x}_j, \ldots, x_N) = (x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_N)$, $B'_a(x'_0) = \{x' \in \mathbb{R}^{N-1} \mid |x' - x'_0| < \alpha\}$ and $B_\beta(x_0) = \{x \in \mathbb{R}^N \mid |x - x_0| < \beta\}$. The notation $^T M$ denotes the transposed $M$. 

---

1. [1] M denotes the transposed $M$. 

---

4
To prove our local in time unique existence theorem for (2.4) in a uniform $W^{2-1/q}_q$ domain, we need the unique solvability of weak Dirichlet-Neumann problem to treat the divergence condition. But, in general it is not known except for the $L_2$ framework, so that we have to assume it in this paper. For this purpose, we introduce spaces $W^{1}_q(\Omega)$ and $W^{1}_{q,0}(\Omega)$ defined by $W^{1}_{q,0}(\Omega) = \{ \theta \in L_{q,loc}(\Omega) | \nabla \theta \in L_q(\Omega)^N, \theta |_{\partial \Omega} = 0 \}$ and $W^{1}_q(\Omega) = \{ \theta \in W^{1}_{q,0}(\Omega) | \theta |_{\partial \Omega} = 0 \}$.

**Definition 2.2.** Let $1 < q < \infty$ and $\Gamma$ be a closed subspace of $\hat{W}^{1}_q(\Omega)$ that contains $W^{1}_{q,0}(\Omega)$. Then, weak Dirichlet-Neumann problem is called uniquely solvable for $W^{1}_q(\Omega)$, if the following assertion holds: For any $f \in L_{q}(\Omega)^N$ there exists a unique $\theta \in W^{1}_{q,0}(\Omega)$ which satisfies the variational equation:

$$\langle \nabla \theta, \nabla \varphi \rangle = (f, \nabla \varphi)_{\Omega} \quad \text{for all } \varphi \in W^{1}_q(\Omega),$$

and the estimate: $\| \nabla \theta \|_{L_q(\Omega)} \leq C_q \| f \|_{L_q(\Omega)}$ for some constant $C_q$ independent of $f, \theta$ and $\varphi$.

**Remark 2.3.** (1) $W^{1}_q(\Omega) + W^{1}_{q}(\Omega) = \{ p = p_1 + p_2 | p_1 \in W^{1}_q(\Omega), \ p_2 \in W^{1}_{q}(\Omega) \}$ is the space for pressures.

(2) When $\Omega$ is a bounded domain, a half-space, a perturbed half-space, or a layer domain, weak Dirichlet-Neumann problem is uniquely solvable with $W^{1}_q(\Omega) = W^{1}_{q,0}(\Omega)$, while when $\Omega$ is an exterior domain, it is uniquely solvable with $W^{1}_q(\Omega)$ being the closure of $W^{1}_{q,0}(\Omega)$ by semi-norm $\| \nabla \cdot \|_{L_q(\Omega)}$. More examples of domains where the unique solvability of weak Dirichlet-Neumann problem holds were given in [21],[22].

To state the compatibility condition for initial data $v_0$, we introduce the solenoidal space $J_q(\Omega)$ defined by $J_q(\Omega) = \{ f \in L_q(\Omega)^N | \ (f, \nabla \varphi)_{\Omega} = 0 \ \text{for any } \varphi \in W^{1}_q(\Omega) \}$. Since $C^{\infty}(\Omega) \subset W^{1}_q(\Omega)$, we see that $\text{div } f = 0$ in $\Omega$ provided that $f \in J_q(\Omega)$. But, the opposite direction does not hold in general. We define $D_{q,p}(\Omega)$ by $D_{q,p}(\Omega) = (J_q(\Omega), J_q(\Omega))_{1-1/p/p}$ with

$$D_{q,p}(\Omega) = \{ f \in W^{1}_q(\Omega)^N | f \text{ satisfies the compatibility condition:} \quad (D(f) n - < D(f) n, n >) |_{\partial X} = 0, \ \text{if } f |_{\Gamma} = 0 \}. \quad (2.9)$$

From Steiger [22], we know that

$$D_{q,p}(\Omega) = \begin{cases} 
\{ f \in B^{2-1/p}_q(\Omega) \cap J_q(\Omega) | f \text{ satisfies (2.9)} \} & \text{when } 2(1 - 1/p) > 1 + 1/q, \\
\{ f \in J_q(\Omega) \cap B^{2-1/p}_q(\Omega) | f |_{\Gamma} = 0 \} & \text{when } 1/q < 2(1 - 1/p) < 1 + 1/q, \\
B^{2-1/p}_q(\Omega) \cap J_q(\Omega) & \text{when } 2(1 - 1/p) < 1/q.
\end{cases}$$

The following theorem is concerned with local in time unique existence theorem for (2.4).

**Theorem 2.4.** Let $2 < p < \infty$, $N < q < \infty$ and $R > 0$. Assume that $\Omega$ is a uniform $W^{2-1/q}_q$ domain and that weak Dirichlet-Neumann problem is uniquely solvable for $W^{1}_q(\Omega)$ and $W^{1}_{q}(\Omega)$ ($\theta = q/\delta - 1$). Then, there exists a time $T > 0$ depending on $R$ such that for any initial data $v_0 \in D_{q,p}(\Omega)$ with $\| v_0 \|_{B^{2-1/p}_q(\Omega)} \leq R$ problem (2.4) admits a unique solution $u \in L_q((0,T), W^{1}_q(\Omega)) \cap W^{1}_{p}(((0,T), L_q(\Omega))$ with some pressure term $\theta \in L_p((0,T), W^{1}_q(\Omega))$ possessing the estimate:

$$\| u \|_{L_q((0,T), W^{1}_q(\Omega))} + \| \partial_t u \|_{L_p((0,T), L_q(\Omega))} \leq M_q R$$

with some positive constant $M_q$ independent of $R$ and $T$.

**Remark 2.5.** (1) Employing the similar argumentation to Strömer [33], we can prove that there exists a positive number $\sigma > 0$ such that the map: $x = X_u(\xi, t)$ is diffeomorphism from $\Omega$ onto $\Omega_t$, $S$ onto $S_t$ and $\Gamma$ onto $\Gamma$ for any $t \in (0, T)$ provided that

$$\int_0^T \| \nabla u(\cdot, t) \|_{L_{q,\Omega}(t)} \, dt \leq \sigma, \quad (2.10)$$

so that from Theorem 2.4 $v(x, t) = u(x^{-1}(x, t), t)$ solves the original free boundary problem (1.1) for small $T > 0$ with some pressure term $\tau$, where $X_u^{-1}(x, t)$ denotes the inverse map of the correspondence: $x = X_u(\xi, t)$. 

5
It is easy to extend Theorem 2.4 to the equation:

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} - \text{Div } T(\mathbf{v}, \theta) = \mathbf{f}, \quad \text{div } \mathbf{v} = 0$$

(2.11) instead of (1.1) under similar assumption on \( \mathbf{f} \) to Solonnikov [23] and Shibata and Shimizu [24]. But, we only consider the case \( \mathbf{f} = 0 \) in this paper for simplicity.

Our global in time unique existence theorem is obtained under the assumption that \( \Omega \) is a bounded domain and the key issue is the orthogonality of the rigid motion. We introduce the rigid space \( \mathcal{R}_d \) defined by

$$\mathcal{R}_d = \{ Ax + b \mid A : N \times N \text{ anti-symmetric matrix, } b \in \mathbb{R}^N \}. \tag{2.12}$$

We know that \( \mathbf{u} \) satisfies \( \mathbf{D}(\mathbf{u}) = 0 \) if and only if \( \mathbf{u} \in \mathcal{R}_d \) (cf. [11]). Let \( \{ p_\ell \}_{\ell=1}^M \) be the orthogonal bases of \( \mathcal{R}_d \), that is \( p_\ell \in \mathcal{R}_d \) (\( \ell = 1, \ldots, M \)), and

$$(p_\ell, p_m)_\Omega = \delta_{\ell m} \quad (\ell, m = 1, \ldots, M), \tag{2.13}$$

where \( \delta_{\ell m} \) is the Kronecker delta symbol such that \( \delta_{\ell \ell} = 1 \) and \( \delta_{\ell m} = 0 \) with \( \ell \neq m \), \( M \) the dimension of \( \mathcal{R}_d \) and \((\cdot, \cdot)_{\Omega} \) the \( L_2 \) inner-product on \( \Omega \). The following theorem is our global in time unique existence result.

**Theorem 2.6.** Let \( 2 < p < \infty \) and \( N < q < \infty \). Assume that \( \Omega \) is a bounded domain, \( S \) and \( \Gamma \) are \( W^{2-1/q}_q \) compact hypersurfaces and that \( S \neq 0 \). Then, there exist numbers \( \epsilon > 0 \) and \( \gamma > 0 \) such that for any initial data \( \psi_0 \in \mathcal{D}_{q,p}(\Omega) \) with \( \| \psi_0 \|_{B^{1-1/p}_p(\Omega)} \leq \epsilon \) that satisfies, in addition, the orthogonality condition:

$$(\psi_0, p_\ell)_{\Omega} = 0 \quad (\ell = 1, \ldots, M) \quad \text{when } \Gamma = \emptyset, \tag{2.14}$$

and problem (1.1) with \( T = \infty \) admits a unique solution \( \mathbf{u} \in L_p((0, \infty), W^2_q(\Omega)) \cap W^1_p((0, \infty), L_q(\Omega)) \) possessing the estimate:

$$\| e^{\gamma t} \mathbf{u} \|_{L_p((0, \infty), W^2_q(\Omega))} + \| e^{\gamma t} \partial_t \mathbf{u} \|_{L_p((0, \infty), L_q(\Omega))} \leq C \epsilon$$

for some positive constant \( C \) independent of \( \epsilon \).

## 3 A proof of a local in time unique existence theorem

In this section, we prove Theorem 2.4. For this purpose, first we state our maximal \( L_p - L_q \) regularity theorem obtained by Shibata [22] for the linearized system (1.6). To state our maximal regularity result for (1.6), we introduce some symbols. For any Banach space \( X \) with norm \( \| \cdot \|_X \), integer \( m \geq 0 \) and \( \gamma_0 > 0 \), we set

$$W^m_{p,\gamma_0}(I, X) = \{ f : I \to X \mid e^{-\gamma_0 t} f(t) \in W^m_p(\mathbb{R}^+, X) \} \quad (I = \mathbb{R}^+, \mathbb{R}),$$

$$W^m_{p,0,\gamma_0}(\mathbb{R}, X) = \{ f : \mathbb{R} \to X \mid e^{-\gamma_0 t} f(t) \in W^m_p(\mathbb{R}, X), \quad f(t) = 0 \text{ for } t < 0 \},$$

where \( \mathbb{R}^+ = (0, \infty) \). We set \( W^0_{p,\gamma_0} = L_{p,\gamma_0} \) and \( W^0_{p,0,\gamma_0} = L_{p,0,\gamma_0} \). Let \( \mathcal{L} \) and \( \mathcal{L}^{-1} \) be the Laplace transform and its inverse transform defined by

$$\mathcal{L}[f](\lambda) = \int_{-\infty}^{\infty} e^{-\lambda t} f(t) \, dt, \quad \mathcal{L}^{-1}[g](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\lambda t} g(\gamma + i\tau) \, d\tau$$

with \( \lambda = \gamma + i\tau \in \mathbb{C} \). For any real number \( s \geq 0 \), let \( H^s_{p,\gamma_0}(\mathbb{R}, X) \) be the Bessel potential space of order \( s \) defined by

$$H^s_{p,\gamma_0}(\mathbb{R}, X) = \{ f \in L_{p,\gamma_0}(\mathbb{R}, X) \mid e^{-\gamma_0 t} \Lambda^s f \in L_p(\mathbb{R}, X) \quad \text{for any } \gamma \geq \gamma_0 \}$$

with \( [\Lambda^s f](t) = \mathcal{L}^{-1}[\lambda^s \mathcal{L}[f](\lambda)](t) \). We set \( H^s_{p,0,\gamma_0}(\mathbb{R}, X) = \{ f \in H^s_{p,\gamma_0}(\mathbb{R}, X) \mid f(t) = 0 \text{ for } t < 0 \} \). By using the \( \mathcal{R} \) bounded solution operator \( \mathcal{R}(\lambda) \) introduced in Sect.1, Shibata [22] proved the following maximal \( L_p - L_q \) result for problem (1.6).
Theorem 3.1. Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r \ (q' = q/(q - 1))$. Assume that $\Omega$ is a uniform $W^{2-1/r}_p$ domain and that the weak Dirichlet-Neumann problem is uniquely solvable for $W^1_0(\Omega)$ and $W_0^1(\Omega) \ (q' = q/(q - 1))$. Then, there exists a positive number $\gamma_0$ such that for any initial data $u_0 \in D_{q, \gamma}(\Omega)$ and right members $f$, $g$, $h$ and $\lambda$ with

$$f \in L_{p,0,\gamma_0}(\mathbb{R}, L_0(\Omega)^N), \quad g \in L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\Omega)) \cap H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_0(\Omega)),$$
$$g \in W^1_{0,\gamma_0}(\mathbb{R}, L_0(\Omega)^N), \quad h \in L_{p,0,\gamma_0}(\mathbb{R}, W^1_q(\Omega)) \cap H^{1/2}_{p,0,\gamma_0}(\mathbb{R}, L_0(\Omega)^N),$$

the weak Dirichlet-Neumann problem (1.6) admits a unique solution $u \in L_{p,0,\gamma}(\mathbb{R} +, W^1_q(\Omega)) \cap W^1_{0,\gamma}(\mathbb{R} +, L_0(\Omega)^N)$ with some pressure term $\theta \in L_{p,0,\gamma}(\mathbb{R} +, W^1_q(\Omega) + W^1_0(\Omega))$ possessing the estimate:

$$\|e^{-\gamma t} \partial_t u\|_{L_p(\mathbb{R} +, L_q(\Omega))} + \|e^{-\gamma t} g\|_{L_p(\mathbb{R}, W^1_q(\Omega))} \leq C\{\|u_0\|_{B^{1/2}_{p,\gamma_0}(\Omega)} + \|f\|_{L_p(\mathbb{R}, W^1_q(\Omega))} + \|\partial_t g\|_{L_p(\mathbb{R}, L_0(\Omega)^N)} + \|\mathbf{h}\|_{L_p(\mathbb{R}, L_0(\Omega)\mathbb{R}^N)}\}$$

for any $\gamma \geq \gamma_0$ with some constant $C$ independent of $\gamma \geq \gamma_0$.

To prove Theorem 2.2 we use the maximal $L_p$-$L_q$ regularity theorem for problem (1.1) in a finite time interval, which is derived from Theorem 3.1. But, we have to replace the nonlocal operator $A_\gamma^{1/2}$ with value in $L_q(\Omega)$ by the local operator $\partial_t$ with value in $W_q^{-1}(\Omega)$. For this purpose, first of all, we introduce the extension map $\iota : L_{1,\text{loc}}(\Omega) \rightarrow L_{1,\text{loc}}(\mathbb{R}^N)$ having the following properties:

1. For any $1 < q < \infty$ and $f \in W_q^{-1}(\Omega)$, $f \in W_q^{-1}(\mathbb{R}^N)$, $f = \iota f$ in $\Omega$ and $\|f\|_{W_q^{-1}(\mathbb{R}^N)} \leq C_q \|f\|_{W_q^{-1}(\Omega)}$ for $t = 0$ with some constant $C_q$ depending on $q$, $r$ and $\Omega$.

2. For any $1 < q < \infty$ and $f \in W_q^{-1}(\Omega)$, $\|(1 - \Delta)^{-1/2}f\|_{W_q^{-1}(\mathbb{R}^N)} \leq C_q \|f\|_{W_q^{-1}(\Omega)}$ with some constant $C_q$ depending on $q$, $r$ and $\Omega$.

Here, $(1 - \Delta)^{-1/2}$ is the operator defined by $(1 - \Delta)^{-1/2}f = F^{-1}((1 + |\xi|^2)^{1/4}F[f])$ with the help of Fourier transform $F$ and Fourier inverse transform $F^{-1}$ which are defined by

$$F[f](\xi) = \int_{\mathbb{R}^N} e^{-ix \xi} f(x) \, dx, \quad F^{-1}[g](\xi) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{ix \xi} g(\xi) \, d\xi.$$
Proof. Let $t$ be any number with $0 < t \leq T$. Given $f(\cdot, s)$ defined for $s \geq 0$, $f_0(\cdot, s)$ denotes the zero extension of $f$ to $s < 0$, that is $f_0(\cdot, s) = f(\cdot, s)$ for $s \geq 0$ and $f_0(\cdot, s) = 0$ for $s < 0$. Let $E_t f$ be the extension of $f$ defined by

$$E_t f = \begin{cases} 
  f_0(\cdot, s) & \text{for } s \leq t, \\
  f_0(\cdot, 2t - s) & \text{for } s \geq t.
\end{cases} \tag{3.3}$$

Note that $E_t f$ vanishes for $s \not\in [0, 2t]$. Moreover, if $f|_{s=0} = 0$, then

$$\partial_s E_t f = \begin{cases} 
  \partial_s f(\cdot, s) & \text{for } s \leq t, \\
  -\partial_s f(\cdot, 2t - s) & \text{for } s \geq t, \\
  0 & \text{for } s \not\in [0, 2t].
\end{cases} \tag{3.4}$$

Let $u^i(\cdot, s) = v(\cdot, s)$ and $\theta^i = \kappa(\cdot, s)$ be solutions to the equations:

$$\partial_s v - \text{Div } T(v, \kappa) = E_t f, \quad \text{Div } v = E_t g \equiv \text{Div } (E_t g) \quad \text{in } \Omega \times (0, \infty),$$

$$T(v, \kappa)|_s = E_t h|_s, \quad v|_{t=0} = v_0 \in \Omega. \tag{3.5}$$

Since $E_{t_1} f = E_{t_2} f$ for $0 < t_1, t_2 \leq T$, by the uniqueness of solutions yields that $u^{i_1}(\cdot, s) = u^{i_2}(\cdot, s)$ for $s \in [0, t]$ with $0 < t_1 < t_2 \leq T$. By Theorem 3.1,

$$\|e^{-\gamma s} \partial_s u^i\|_{L^p(\Omega)} + \|e^{-\gamma s} u^i\|_{L^q(\Omega)} \leq C \{\|u_0\|_{L^p(\Omega)} + \|e^{-\gamma s} f\|_{L^p(\Omega)} \}.$$

Noting (3.4), we see easily that

$$\|e^{-\gamma s} (E_t f, \partial_s E_t g)\|_{L^p(\Omega)} \leq 2 \|e^{-\gamma s} f, \partial_s g\|_{L^p(\Omega)},$$

Moreover, by (3.2) and (3.4), we have

$$\|e^{-\gamma s} \Lambda^{1/2}_\gamma (E_t g, E_t h)\|_{L^p(\Omega)} \leq C \{\|e^{-\gamma s} \partial_s (u, v)\|_{L^p(\Omega)} \}.$$

Setting $u = u^f$ and $\theta = \theta^f$, noting that $u^{(i)}(\cdot, s) = u^f(\cdot, s)$ for $0 < s < t$ and combining (3.5), (3.7) and (3.8), we complete the proof of Theorem 3.2. \hfill \Box

**A Proof of Theorem 2.3** In the following, we assume that $2 < p < \infty$ and $N < q < \infty$, that $\Omega$ is a uniform $W^{2-1/q}_q(\mathbb{R}^N)$ domain in $\mathbb{R}^N$ ($N \geq 2$), and that weak Dirichlet-Neumann problem is uniquely solvable for $W^1_q(\Omega)$ and $W^1_q(\Omega)$ ($q' = q/(q-1)$). By Sobolev’s imbedding theorem we have

$$W^1_q(\Omega) \subset L^\infty(\Omega), \quad \|m\|_{W^1_q(\Omega)} \leq C \|m\|_{L^q(\Omega)} \tag{3.9}$$

Let $T$ and $L$ be any positive numbers and we define a space $\mathcal{I}_{L,T}$ by

$$\mathcal{I}_{L,T} = \{v \in L^2_p((0,T), W^1_q(\Omega)) \cap W^1_q((0,T), L^q(\Omega)) \mid v|_{t=0} = v_0 \in \Omega, \quad \|v\|_{L^2_p(0,T), \Omega} \leq L \}, \tag{3.10}$$

where we have set $\|v\|_{L^2_p(0,T), \Omega} = \|v\|_{L^2_p((0,T), W^1_q(\Omega))} \bigoplus \|\partial_t v\|_{L^2_p((0,T), L^q(\Omega))}$. Given $w \in \mathcal{I}_{L,T}$, let $v$ and $\omega$ be solutions to problem:

$$\partial_t v - \text{Div } T(v, \omega) = F(w), \quad \text{Div } v = G(w) = \text{Div } G(w) \quad \text{in } \Omega \times (0, T),$$

$$\mathcal{T}(v, \omega)|_s = H(w)|_s, \quad v|_{t=0} = v_0 \in \Omega. \tag{3.11}$$
First, we estimate the right-hand sides of (3.11). By (3.9) and Hölder’s inequality we have
\[
\sup_{t \in (0,T)} \left\| \int_0^t \nabla w(\cdot, s) \, ds \right\|_{L^\infty(\Omega)} \leq M_1 T^{1/p'} L, \quad \sup_{t \in (0,T)} \left\| \int_0^t \nabla w(\cdot, s) \, ds \right\|_{W^{1,q}_t(\Omega)} \leq CT^{1/p'} L.
\] (3.12)
with \( p' = p/(p-1) \). Here and in the following, \( C \) denotes a generic constant independent of \( T \) and \( R \) and we use the letters \( M_i \) to denote some special constants independent of \( T \) and \( L \). The value of \( C \) may change from line to line. To treat nonlinear functions with respect to \( \int_0^t \nabla w(\cdot, s) \, ds \), we choose \( T \) so small that \( M_1 T^{1/p'} L \leq 1 \) in (3.12), so that
\[
\sup_{t \in (0,T)} \left\| \int_0^t \nabla w(\cdot, s) \, ds \right\|_{L^\infty(\Omega)} \leq 1.
\] (3.13)
By (3.12), (3.13), (3.9) and (2.9), we have
\[
\sup_{t \in (0,T)} \left\| V_i \left( \int_0^t \nabla w(\cdot, t) \, ds \right) \right\|_{W^1_2(\Omega)} \leq CT^{1/p'} L, \quad \sup_{t \in (0,T)} \left\| \nabla W \left( \int_0^t \nabla w(\cdot, t) \, ds \right) \right\|_{L^q(\Omega)} \leq CT^{1/p'} L
\] (3.14)
where \( i = 1, 2, 4, 5 \) and 6, and \( W = W(K) \) is any matrix of polynomials with respect to \( K \). By (2.5), (3.9), (3.12), (3.13) and (3.14), we have
\[
\left\| F(w) \right\|_{L_p((0,T),L_q(\Omega))} \leq CL^2 T^{1/p'}, \quad \left\| \left( G(w), H(w) \right) \right\|_{L_p((0,T),W^1_4(\Omega))} \leq CL^2 T^{1/p'}.
\] (3.15)
To obtain
\[
\sup_{t \in (0,T)} \left\| w(\cdot, t) \right\|_{B^{2(1-1/p)}_p(\Omega)} \leq C(\|w(0, T)\| + e^{\gamma T} \|v_0\|_{B^{2(1-1/p)}_p(\Omega)}),
\] (3.16)
we use the embedding relation:
\[
L_p(0, \infty, X_1) \cap W^1_4((0, \infty), X_0) \subset BUC(J, [X_0, X_1]_{1-1/p, p})
\] (3.17)
for any two Banach spaces \( X_0 \) and \( X_1 \) such that \( X_1 \) is dense in \( X_0 \) and \( 1 < p < \infty \) (cf. 3). In fact, let \( E_t \) be the extension operator defined in the proof of Theorem 3.2 and let \( Z \) and \( \Pi \) be solutions to problem:
\[
\frac{\partial}{\partial t} Z - \text{Div} T(Z, \Pi) = 0, \quad \text{div} Z = 0 \quad \text{in} \quad \Omega \times (0, \infty),
\]
\[
T(Z, \Pi) |_{\Gamma} = 0, \quad Z |_{t=0} = v_0 \quad \text{in} \quad \Omega.
\] (3.18)
By Theorem 3.1 (1), we know the unique existence of \((Z, \Pi)\) possessing the estimate:
\[
\left\| e^{-\gamma t} \frac{\partial}{\partial t} Z \right\|_{L_p(\mathbb{R}_+, L_q(\Omega))} + \left\| e^{-\gamma t} Z \right\|_{L_p(\mathbb{R}_+, W^1_2(\Omega))} \leq C \left\| v_0 \right\|_{B^{2(1-1/p)}_p(\Omega)} \quad (\gamma \geq \gamma_0)
\] (3.19)
for some constants \( \gamma_0 \) and \( C \), where \( C \) is independent of \( \gamma \geq \gamma_0 \). We choose \( \gamma \) so large and fix it in the following. Set \( z = w - Z \). Since \( z |_{t=0} = 0 \), by (3.3) and (3.4) we have
\[
\| e^{\gamma T} z(0, \infty) \| \leq C \| z(0, T) \| + e^{\gamma T} \| z(0, \infty) \|
\]
Thus, noting that \( w = Z + E_T z \) for \( t \in (0, T) \) and using (3.17), we have
\[
\sup_{t \in (0,T)} \left\| w(\cdot, t) \right\|_{B^{2(1-1/p)}_p(\Omega)} \leq \sup_{t \in (0,\infty)} \left\| E_T z(\cdot, t) \right\|_{B^{2(1-1/p)}_p(\Omega)} + e^{\gamma T} \sup_{t \in (0,\infty)} \left\| e^{-\gamma t} Z(\cdot, t) \right\|_{B^{2(1-1/p)}_p(\Omega)}
\]
\[
\leq C \left( \| E_T z(0, \infty) \| + e^{\gamma T} \| z(0, \infty) \| \right) \leq C \left( \| w(0, T) \| + e^{\gamma T} \| z(0, \infty) \| \right),
\]
which combined with (3.19) furnishes (3.16).
Since \( B^{2(1-1/p)}_p(\Omega) \subset W^1_4(\Omega) \) as follows from the assumption: \( 2 < p < \infty \), by (3.10) and (3.13) we have
\[
\sup_{t \in (0,T)} \left\| w(\cdot, t) \right\|_{W^1_4(\Omega)} \leq C \left( L + e^{\gamma T} \| v_0 \|_{B^{2(1-1/p)}_p(\Omega)} \right),
\]
\[
\sup_{t \in (0,T)} \left\| \frac{\partial}{\partial t} W \left( \int_0^t \nabla w(\cdot, s) \, ds \right) \right\|_{L^q(\Omega)} \leq C \left( L + e^{\gamma T} \| v_0 \|_{B^{2(1-1/p)}_p(\Omega)} \right).
\] (3.20)

9
Writing \( \partial_t G(w) = \{ \partial_t V_5(\int_0^t \nabla w \, ds) \} w + V_5(\int_0^t \nabla w \, ds) \partial_t w \) and using (3.10), (3.9), (3.16), (3.20) and (3.14), we have

\[
\|\partial_t G(w)\|_{L^p(0,T),L^q(\Omega)} \leq C \{ L^2 T^{1/p'} + (L + e^T \|v_0\|_{B_{(1-p)/(p')}^2(\Omega)})^2 T^{1/p} \}. \tag{3.21}
\]

To continue our estimate, we prepare the following lemma.

**Lemma 3.3.** Let \( 1 < p < \infty, \, N < q, r < \infty \) and let \( \Omega \) be a uniform \( W_r^{-1/r} \) domain. Let \( \iota \) be the extension map satisfying the properties (e-1) and (e-2). Then,

\[
\|\partial_t [(1 - \Delta)^{-1/2}(\iota((\nabla f)))]\|_{L^p(0,T),L^q(\mathbb{R}^N))} \leq C \left\{ \left( \int_0^T (\|\partial_t f(\cdot, t)\|_{L^q(\Omega)} \|g(\cdot, t)\|_{W_2^1(\Omega)})^p \, dt \right)^{1/p} + \left( \int_0^T (\|\nabla f(\cdot, t)\|_{L^q(\Omega)} \|\partial_t g(\cdot, t)\|_{L^q(\Omega)})^p \, dt \right)^{1/p} \right\}.
\]

**Proof.** To prove the lemma, we use an inequality:

\[
\|((1 - \Delta)^{-1/2}(\iota(fg)))\|_{L^q(\mathbb{R}^N)} \leq C \|f\|_{L^q(\Omega)} \|g\|_{L^q(\Omega)} \tag{3.22}
\]

provided that \( N < q < \infty \), which follows from the following observation: For any \( \varphi \in C_{0}^\infty(\mathbb{R}^N) \) by Hölder’s inequality and (e-1) we have

\[
\|((1 - \Delta)^{-1/2}(\iota(\varphi)))\|_{L^q(\mathbb{R}^N)} = \|\iota(\varphi)\|_{L^q(\mathbb{R}^N)} \leq C \|\iota\|_{L^q(\Omega)} \|\varphi\|_{L^q(\Omega)} \|((1 - \Delta)^{-1/2}\varphi)\|_{L^q(\mathbb{R}^N)},
\]

where \( s \) is an index such that \( 2q + 1/s = 1 \). Since \( N(1/q' - 1/s) = N/q < 1 \), by Sobolev’s imbedding theorem we have \( \|((1 - \Delta)^{-1/2}\varphi)\|_{L^q(\mathbb{R}^N)} \leq C \|\varphi\|_{L^q(\mathbb{R}^N)} \), which furnishes (3.22). Since \( \partial_t [(1 - \Delta)^{-1/2}(\iota((\nabla f)))] = (1 - \Delta)^{-1/2}[\nabla((\partial_t f)(\nabla g))] - (1 - \Delta)^{-1/2}[\partial_t(\iota(\iota(\nabla f)))] \), by (3.22), (3.9) and (e-2) we have Lemma 3.3. \( \square \)

Applying Lemma 3.3 to \( G(w) \) and \( H(w)\tilde{n} \) with \( f = w, \, g = V_4(\int_0^t \nabla (w) \, ds) \) and \( f = w, \, g = V_6(\int_0^t \nabla (w) \, ds) \tilde{n} \), respectively, and using (3.14), (3.16) and (3.20), we have

\[
\|\partial_t [(1 - \Delta)^{-1/2}(\iota(G(w), t(\iota(H(w)\tilde{n}))))\|_{L^p(0,T),L^q(\mathbb{R}^N))} \leq C \{ L^2 T^{1/p'} + (L + e^T \|v_0\|_{B_{(1-p)/(p')}^2(\Omega)})^2 T^{1/p} \}. \tag{3.23}
\]

Thus, applying Theorem 3.2 to problem (3.11) and using (3.15), (3.21) and (3.22), we have

\[
\|v_0\|_{B_{(1-p)/(p')}^2(\Omega)} \leq M_2 \|v_0\|_{B_{(1-p)/(p')}^2(\Omega)} + M_3 (L^2 T^{1/p'} + (L + e^T \|v_0\|_{B_{(1-p)/(p')}^2(\Omega)})^2 T^{1/p}). \tag{3.24}
\]

Let \( R \) be a number such that \( \|v_0\|_{B_{(1-p)/(p')}^2(\Omega)} \leq R \) and set \( L = (M_2 + 1)R \). Choosing \( T > 0 \) so small that

\[
M_3 (L^2 T^{1/p'} + (L + e^T \|v_0\|_{B_{(1-p)/(p')}^2(\Omega)})^2 T^{1/p}) \leq 1,
\]

by (3.24) we have \( \|v_0\|_{(0,T)} \leq L \), so that \( v \in I_{L,T} \). If we define a map \( \Phi \) by \( \Phi(w) = v \), then \( \Phi \) is a map from \( I_{L,T} \) into itself.

Next, we show the contractivity of the map \( \Phi \) on \( I_{L,T} \). Let \( w_i \in I_{L,T} \) and set \( v_i = \Phi(w_i) \) \( (i = 1, 2) \). Setting \( v = v_1 - v_2 \), we have

\[
\partial_t v - \text{Div } T(v, \omega) = f(w_1, w_2), \quad \text{div } v = g(w_1, w_2) = \text{div } g(w_1, w_2) \quad \text{in } \Omega \times (0,T), \quad T(v, \omega)\tilde{n}|_{\partial \Omega} = h(w_1, w_2)|_{\partial \Omega}, \quad v|_{\Gamma} = 0, \quad v|_{t=0} = \omega \quad \text{in } \Omega
\]

(3.25)

with some pressure term \( \omega \), where we have set

\[
f(w_1, w_2) = F(w_1) - F(w_2), \quad g(w_1, w_2) = G(w_1) - G(w_2), \quad h(w_1, w_2) = (H(w_1) - H(w_2))\tilde{n}
\]
By Theorem 3.2 we have
\[ I_{v_1-v_2}(0,T) \leq M_4 J(w_1,w_2)(T) \tag{3.26} \]
for some constant \( M_4 \) independent of \( T \) and \( R \) with
\[
\begin{align*}
J(w_1,w_2) &= \|(f(w_1,w_2), \partial_2 g(w_1,w_2))\|_{L^p_0(T),L^q(\Omega)} + \|(g(w_1,w_2), h(w_1,w_2))\|_{L^p_0(0,T),W^1_q(\Omega)} \\
& \quad + \|\partial_t [(1 - \Delta)^{-1/2}\mu g(w_1,w_2), \phi h(w_1,w_2))]\|_{L^p_0(0,T),L^q(\mathbb{R}^N)}.
\end{align*}
\]
We estimate each terms in the right-hand side of (3.25). Recalling that \( \|v_0\|_{B_v^{2(1-1/p)}(\Omega)} \leq R \) and \( L = (M_2 + 1)R \), by (3.12), (3.13), (3.14), (3.16) and (3.20) we have
\[
\sup_{t \in (0,T)} \|\nabla i(w_i(\cdot,s))\|_{L^\infty(\Omega)} \leq 1, \quad \sup_{t \in (0,T)} \|\nabla i(w_i(\cdot,s))\|_{W^1_q(\Omega)} \leq CR T^{1/p'}, \quad \|W_i(\cdot,t)\|_{W^1_q(\Omega)} \leq CRe^{\gamma T}, \quad \sup_{t \in (0,T)} \|\partial_t W_i(\cdot,s)\|_{L^q(\Omega)} \leq CRe^{\gamma T},
\]
where \( i = 1, 2 \) and \( j = 1, 2, 4, 5, 6 \). Thus, we have
\[
\sup_{t \in (0,T)} \|\nabla i(w_i(\cdot,s)) - \nabla i(w_j(\cdot,s))\|_{W^1_q(\Omega)} \leq CT^{1/p'} \|w_i - w_2\|_{L^2(0, T)}.
\]
Since \( (w_i - w_2)_{i=0} = 0 \), employing the similar argumentation to that in the proof of (3.10), we have
\[
\sup_{t \in (0,T)} \|w_i(\cdot,t) - w_j(\cdot,t)\|_{W^1_q(\Omega)} \leq C I_{w_i - w_2}(0,T),
\]
\[
\sup_{t \in (0,T)} \|\partial_t \{W_i(\cdot,s) - W_j(\cdot,s)\}\|_{W^1_q(\Omega)} \leq C(1 + R e^{\gamma T} T^{1/p'}) I_{w_i - w_2}(0,T).
\]
Using above estimates and Lemma 3.3 we have
\[ J(w_1,w_2)(T) \leq M_5 C(R,T) I_{w_1 - w_2}(0,T) \tag{3.27} \]
for some constant \( M_5 \) independent of \( R \) and \( T \) with
\[
C(R,T) = R^{1/p'} + (RT^{1/p'})^2 + (RT^{1/p'})^3 + RT^{1/p} + e^{\gamma T} \{(RT^{1/p})(RT^{1/p'}) + RT^{1/p} + RT^{1/p'} + (RT^{1/p'})^2\} + e^{2\gamma T}(RT^{1/p})(RT^{1/p}).
\]
Combining (3.27) with (3.20) furnishes that
\[ I_{\Phi(w_1) - \Phi(w_2)}(0,T) \leq M_4 M_5 C(R,T) I_{w_1 - w_2}(0,T). \tag{3.28} \]
Choosing \( T \) smaller in such a way that \( M_4 M_5 C(R,T) \leq 1/2 \), we have \( \Phi \) is a contraction map on \( \mathcal{I}_{L,T} \).
Thus, the Banach fixed point theorem tells us that \( \Phi \) has a unique fixed point \( u \) in \( \mathcal{I}_{L,T} \) satisfying the equations (2.4).

Finally, we prove the uniqueness. Given two \( v_i \in \mathcal{I}_{L,T} \) \((i = 1, 2)\) both of which satisfy the equations (2.4) with the same initial data \( v_0 \in B_v^{2(1-1/p)}(\Omega) \), employing the same argument as in proving (3.28) and replacing \( w_i \) by \( v_i \), we have \( I_{v_1 - v_2}(0,T) \leq M_4 M_5 C(R,T) I_{v_1 - v_2}(0,T) \). Since \( T \) has been chosen in such a way that \( M_4 M_5 C(R,T) \leq 1/2 \), we have \( I_{v_1 - v_2}(0,T) \leq \frac{1}{2} I_{v_1 - v_2}(0,T) \), which implies that \( v_1 = v_2 \). This completes the proof of Theorem 2.4.
4 Some decay properties of solutions to problem (1.6)

In this section, we discuss exponential stability of solutions to problem (1.6) assuming that $\Omega$ is bounded in addition. Let $\mathcal{R}(\lambda)$ be the $\mathcal{R}$ bounded solution operator for problem (1.8) introduced in Sect. 1. If we consider the time shifted equation of (1.6):

$$
\partial_t \nu + \lambda_1 \nu - \text{Div} T(\nu, \hat{\theta}) = f, \quad \text{div} \nu = g = \text{div} g \quad \text{in } \Omega \times (0, \infty),
$$

$$
T(\nu, \hat{\theta}) \mathbf{n}|_{S} = h|_{S}, \quad \nu|_{\Gamma} = 0, \quad \nu|_{\Gamma_0} = \mathbf{u}_0 \quad \text{in } \Omega,
$$

(4.1)

a solution $\nu$ is represented by using $\mathcal{R}(\lambda + \lambda_1)$, so that we have the following theorem concerning the exponential stability of solutions to (1.6).

**Theorem 4.1.** Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q - 1)$). Assume that $\Omega$ is a uniform $W^{r-1/r}_2$ domain and that weak Dirichlet-Neumann problem is uniquely solvable for $W^1_q(\Omega)$ and $W^1_q(\Omega)$ ($q' = q/(q - 1)$). Then, there exists a $\lambda_1 > 0$ such that problem (1.6) admits a unique solution $(\nu, \hat{\theta})$ with

$$
\nu \in L_p(\mathbb{R}_+, W^1_q(\Omega)) \cap W^1_q(\mathbb{R}_+, \mathcal{L}_q(\Omega)), \quad \hat{\theta} \in L_p(\mathbb{R}_+, W^1_q(\Omega) + W^1_q(\Omega))
$$

possessing the estimate:

$$
||e^{\gamma t} \partial_t \nu||_{L_p(\mathbb{R}_+, L_q(\Omega))} + ||e^{\gamma t} \nu||_{L_p(\mathbb{R}_+, W^1_q(\Omega))}
\leq C(||\mathbf{u}_0||_{L^\infty(\mathbb{R}_+, L^q(\Omega))}) + ||e^{\gamma t}(f, \tilde{\Lambda}^{1/2}_1 g, \partial_t g, \tilde{\Lambda}^{1/2}_1 h)||_{L_p(\mathbb{R}_+, L_q(\Omega))} + ||e^{\gamma t}(g, h)||_{L_p(\mathbb{R}_+, L_q(\Omega))})
$$

for any $\gamma \leq \lambda_0$ with some constants $C$ independent of $\gamma \leq \lambda_0$, provided that $\mathbf{u}_0 \in D_{q, p}(\Omega), e^{\gamma t} f \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} g \in L_p(\mathbb{R}_+, W^1_q(\Omega)), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 g \in L_p(\mathbb{R}_+, L_q(\Omega)), e^{\gamma t} \partial_t g \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N), e^{\gamma t} \tilde{\Lambda}^{1/2}_1 h \in L_p(\mathbb{R}_+, L_q(\Omega)^N),$ \((4.2)\)

and $f, g, \nu$ and $h$ vanish for $t < 0$. Here, we have defined $\tilde{\Lambda}^{1/2}_1 f$ by

$$
\tilde{\Lambda}^{1/2}_1 f = \mathcal{L}^{-1}_\lambda [(\lambda + \lambda_1)^{1/2} \mathcal{L}(f)(\lambda)] \quad \text{with } \lambda = -\gamma + i\tau.
$$

(4.3)

Since the $\mathcal{R}$ boundedness implies the usual boundedness of operators, we also see that for any $\lambda \in \Sigma_{\epsilon, \lambda_0}, f \in L_q(\Omega)^N, g \in W^1_q(\Omega), g \in W^1_q(\Omega)^N$ and $h \in W^1_q(\Omega)^N$, a unique solution $\nu$ of problem (1.6) possesses the generalized resolvent estimate:

$$
||(|\lambda| \nu, |\lambda|^{1/2} \nabla \nu, \nabla \nu)||_{L_q(\Omega)} \leq C(||(f, |\lambda|^{1/2} g, \nabla g, |\lambda| g, \tilde{\Lambda}^{1/2}_1 h, \nabla h)||_{L_q(\Omega)}),
$$

(4.4)

with some constant $C$ depending on $\epsilon$ and $\lambda_0$. Especially, we see the existence of a continuous semigroup $\{\mathcal{T}(t)\}_{t \geq 0}$ associated with problem (1.6), which is analytic.

To prove a global in time unique existence theorem for (2.4), we need the exponential stability of solutions to (1.6), so that from now on, we assume that $\Omega$ is bounded in addition. In this case, weak Dirichlet-Neumann problem is uniquely solvable for any exponent $q \in (1, \infty)$ with $W^1_q(\Omega) = W^1_q(\Omega)$ and $J_q(\Omega) = \{f \in L_q(\Omega)^N \mid \text{div} f = 0, \mathbf{n}_G \cdot f|_{\Gamma} = 0\}$, where $\mathbf{n}_G$ is the unit normal to $\Gamma$. When $\Omega$ is bounded, the uniqueness of solutions to problem (1.6) holds when $\Gamma \neq \emptyset$ up to $\lambda = 0$. When $\Gamma = \emptyset$, if we restrict the space of solutions to the quotient space $W^2_q(\Omega)/\mathcal{R}_d$, then we also have the uniqueness of solutions to (1.6). Namely, if $\nu \in W^1_q(\Omega)$ satisfies the equations (1.6) with $f = 0, g = 0, g = 0$ and $h = 0$ if $\mathbf{u}$ satisfies the orthogonal condition: $(\mathbf{u}, \mathbf{p})_{\mathbb{R}_{\delta}} = 0$ for $\ell = 1, \ldots, M$, then $u = 0$ up to $\lambda = 0$. Moreover, if $f \in L_q(\Omega)^N$ and $g \in W^1_q(\Omega)^N$ satisfy the condition: $(f, \mathbf{p})_{\mathbb{R}} + \mathbf{h} \cdot \mathbf{p} = 0$, then a solution $\nu$ to problem (1.6) also satisfies $(\mathbf{u}, \mathbf{p})_{\mathbb{R}} = 0$ whenever $\lambda \neq 0$. Here, $< f, g > = \int_{\Gamma} f(x)g(x) \ d\sigma$. $\mathbf{d}$ being the surface element of $S$. Using these facts and applying a homotopic argument, we see that $\{\mathcal{T}(t)\}_{t \geq 0}$ is exponentially stable. Namely, we have the following theorem which was already proved in Shibata and Shimizu [23] in the case of $\Gamma = \emptyset$.

12
Theorem 4.2. Let $1 < q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q - 1)$). Assume that $\Omega$ is a uniform $W^{2-1/r}_r$ domain and that $\Omega$ is bounded in addition. Then, there exists a continuous semigroup $\{T(t)\}_{t \geq 0}$ on $J_q(\Omega)$ associated with problem (1.6) such that $u = T(t)u_0$ with some pressure term $\theta$ solves problem (1.6) with $f = 0$, $g = 0$, $h = 0$. Moreover, $\{T(t)\}_{t \geq 0}$ is analytic and exponentially stable, that is
\[
\|T(t)u_0\|_{W^q_q(\Omega)} \leq C(1 + t^{-\ell/2})e^{-\gamma t}\|u_0\|_{L^q(\Omega)} \quad \text{for any } t > 0 \text{ and } \ell = 0, 1, 2 \quad (4.5)
\]
with some positive constants $C$ and $\gamma$ provided that $u_0 \in J_q(\Omega)$ when $\Gamma \neq \emptyset$ and $u_0 \in J_q(\Omega)$ satisfying the orthogonal condition: $(u_0, p_\Gamma) = 0$ for $\ell = 1, \ldots, M$ when $\Gamma = \emptyset$. Here, $W^q_q(\Omega) = L^q_q(\Omega)$.

By Theorem 4.2, we have the following Corollary which was proved in Shibata and Shimizu [23] in the case of $\Gamma = \emptyset$ under the assumption that the boundary of $\Omega$ is a $C^{1,1}$ hypersurface.

Corollary 4.3. Let $1 < q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q - 1)$). Assume that $\Omega$ is a uniform $W^{2-1/r}_r$ domain and that $\Omega$ is bounded in addition. Then, there exists a positive constant $\gamma_0$ such that problem (1.6) with $f = 0$, $g = 0$, $h = 0$, and $\Gamma = \emptyset$ admits unique solutions $u$ and $\theta$ with
\[
u \in L^q_q(\mathbb{R}, W^2_q(\Omega))^n \cap W^1_p(\mathbb{R}, L^q_q(\Omega))^n, \quad \theta \in L^p_p(\mathbb{R}, W^1_{q,1}(\Omega) + \bar{W}^1_{q,0}(\Omega))
\]
possessing the estimate:
\[
\|e^{\gamma t}\partial u\|_{L^p_p(\mathbb{R}, L^q_q(\Omega))} + \|e^{\gamma t}\theta\|_{L^p_p(\mathbb{R}, W^2_q(\Omega))} \leq C\|u_0\|_{B^{1-1/r}_{q,1}(\Omega)} \quad \text{for any } \gamma \leq \gamma_0
\]
with some positive constants $C$ independent of $\gamma \leq \gamma_0$ provided that $u_0 \in D_{q,p}(\Omega)$ when $\Gamma \neq \emptyset$ and $u_0 \in D_{q,p}(\Omega)$ and $u_0$ satisfies the orthogonal condition: $(u_0, p_\Gamma) = 0$ for $\ell = 1, \ldots, M$ when $\Gamma = \emptyset$.

Under the preparations mentioned above, we show the following theorem about the exponential stability of solutions to (1.6).

Theorem 4.4. Let $1 < p, q < \infty$, $N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q - 1)$). Assume that $\Omega$ is a uniform $W^{2-1/r}_r$ domain and that $\Omega$ is bounded in addition. Then, there exists a positive constant $\gamma_0$ such that the following assertion holds: Let $u_0 \in D_{q,p}(\Omega)$, and let right members $f$, $g$, $h$ for (1.6) satisfy the decay condition (4.3) and vanish for $t < 0$, then problem (1.6) with $T = \infty$ admits a unique solution $u \in L^p_p(\mathbb{R}, W^2_q(\Omega))^n \cap W^1_p(\mathbb{R}, L^q_q(\Omega))^n$ with some pressure term $\theta \in L^p_p(\mathbb{R}, W^1_{q,1}(\Omega) + \bar{W}^1_{q,0}(\Omega))$ possessing the estimate:
\[
\|e^{\gamma t}\partial u\|_{L^p_p(\mathbb{R}, L^q_q(\Omega))} + \|e^{\gamma t}\theta\|_{L^p_p(\mathbb{R}, W^2_q(\Omega))} \leq C\{J_{p,q} + \delta(\Gamma)\sum_{\ell=1}^{M} \left( \int_0^T |e^{\gamma t}(u(-t), p_\ell)|^p \, dt \right)^{1/p} \} \quad (4.6)
\]
for any $T > 0$ and $\gamma \leq \gamma_0$ with some constant $C$ independent of $T$. Here, $\delta(\Gamma)$ is a constant defined by $\delta(0) = 1$ when $\Gamma = \emptyset$ and $\delta(\Gamma) = 0$ if $\Gamma \neq \emptyset$, and we have set
\[
J_{p,q} = \|u_0\|_{B^{1-1/r}_{q,1}(\Omega)} + \|e^{\gamma t}(f, \tilde{\Lambda}_{q,1}^1 g, \tilde{\partial}_v h, \tilde{\Lambda}_{q,1}^1 h)\|_{L^p_p(\mathbb{R}, L^q_q(\Omega))} + \|e^{\gamma t}(g, h)\|_{L^p_p(\mathbb{R}, W^1_q(\Omega))}.
\]

Proof. We look for a solution $u$ of the form $u = v + w$, where $v$ and $w$ are a solution to (4.1) and a solution to problem:
\[
\partial_t w - \text{Div} T(w, \tilde{\theta}) = \lambda_1 v, \quad \text{div} w = 0 \quad \text{in } \Omega \times (0, \infty),
\]
\[
T(w, \tilde{\theta})\vec{n}|_S = 0, \quad w|_| = 0, \quad w|_{t=0} = 0 \quad \text{in } \Omega\quad (4.7)
\]
with some pressure term $\tilde{\theta} \in L^p_p(\mathbb{R}, W^1_{q,1}(\Omega) + \bar{W}^1_{q,0}(\Omega))$, respectively. By Theorem 4.3,
\[
\|e^{\gamma t}\partial v\|_{L^p_p(\mathbb{R}, L^q_q(\Omega))} + \|e^{\gamma t}v\|_{L^p_p(\mathbb{R}, W^2_q(\Omega))} \leq C J_{p,q}. \quad (4.8)
\]
If $\Gamma \neq \emptyset$, setting
\[
w(\cdot, t) = \int_0^t T(t-s)\lambda_1 v(\cdot, s) \, ds,
\]

13
by Duhamel’s principle we see that $w$ satisfies (1.7). Moreover, setting
\[
L_{q,a}(t) = ||a(\cdot, t)||_{W^2_q(\Omega)} + ||\partial_t a(\cdot, t)||_{L_q(\Omega)},
\]
by Theorem 4.2 and Hölder’s inequality we have
\[
L_{q,w}(t) \leq C \int_0^t e^{-\gamma_0(t-s)} L_{q,v}(s) \, ds \leq C(\gamma_0 p')^{-1/p'} \left( \int_0^t e^{-\gamma_0 p(p-1)} L_{q,v}(s)^p \, ds \right)^{1/p}
\]
with some $\gamma_0 > 0$ for some positive constant $C$ independent of $t > 0$, where $p' = p/(p-1)$. Thus, for $\gamma < \gamma_0$ we have
\[
\int_0^T (e^{\gamma t} L_{q,w}(t))^p \, dt \leq C(\gamma_0 p')^{-p'/p'} \int_0^T (e^{\gamma t} L_{q,v}(s))^p \left( \int_s^T e^{-\gamma_0 p(p-1)} \, ds \right)^{1/p} \, ds.
\]
which combined with (4.8) furnishes (4.6) with $\delta(\Gamma) = 0$.

Next, we consider the case of $\Gamma = \emptyset$. Setting $z(x, t) = \lambda_1 v(x, t) - \sum_{\ell=1}^M (\lambda_1 v(\cdot, t), p_\ell)_{\Omega} p_\ell(x)$, we have $(z(t), p_\ell)_{\Omega} = 0$ for $\ell = 1, \ldots, M$ and $t > 0$. Writing $\tilde{w}(t) = \int_0^t T(t-s)z(s) \, ds$, by Duhamel’s principle, we see that $\tilde{w}$ satisfies (1.7) replacing $\lambda_1 v$ by $z$. Moreover, by Theorem 4.2 and Hölder’s inequality
\[
L_{q,\tilde{w}}(t) \leq C \int_0^t e^{-\gamma_0(t-s)} L_{q,a}(s) \, ds.
\]
Thus, by (4.8) we have
\[
\int_0^T (e^{\gamma t} L_{q,\tilde{w}}(t))^p \, dt \leq C_{\gamma, 0, p} \int_0^T (e^{\gamma t} L_{q,a}(t))^p \, dt \leq C_{\gamma, 0, p} (J_{p,q})^p.
\]
Setting $u = v + w$ with $w = \tilde{w} + \sum_{\ell=1}^M \int_0^t T(t-s)(\lambda_1 v(\cdot, s), p_\ell)_{\Omega} p_\ell(s) \, ds$, we see that $u$ satisfies (1.6) with some pressure term $\theta \in L_p(\mathbb{R}_+, W^1_1(\Omega) + \dot{W}^1_1(\Omega))$, because $D(p_\ell) = 0$ and $\text{div} p_\ell = 0$. Moreover, we have $D(u) = D(v) + D(\tilde{w})$, so that by (4.8) and (4.9) we have
\[
\int_0^T ||D(u(\cdot, t))||_{L^q_1(\Omega)} \, dt \leq C(J_{p,q})^p.
\]
for any $T > 0$ with some constant $C$ independent of $T$. Since
\[
||a||_{W^2_q(\Omega)} \leq C(||D(a)||_{L^q_1(\Omega)} + \sum_{\ell=1}^M ||(a, p_\ell)_{\Omega}||)
\]
for any $a \in W^1_1(\Omega)^N$ as follows from the usual contradiction argument (cf. Duvaut and Lions [11]), by (4.10)
\[
\int_0^T (e^{\gamma t} ||u(\cdot, t)||_{W^2_q(\Omega)})^p \, dt \leq C(J_{p,q})^p + \sum_{\ell=1}^M \int_0^T e^{\gamma t} ||(\partial_t u(\cdot, t), p_\ell)_{\Omega}||^p \, dt.
\]
In addition, by (4.4) with $\lambda = \lambda_0 + 1$, we have
\[
||u(t)||_{W^2_q(\Omega)} \leq C(||\partial_t u(t)||_{L^q(\Omega)} + ||u(t)||_{L^q(\Omega)} + ||(f(t), g(t))||_{L^q(\Omega)} + ||(g(t), h(t))||_{W^1_1(\Omega)})
\]
(4.12)
Since $\partial_t u = \partial_t v + \partial_t \tilde{w} + \lambda_1 \sum_{\ell=1}^M (v(\cdot, t), p_\ell)_{\Omega} p_\ell$, by (4.8), (4.9), (4.11) and (4.12) we have (4.6), which completes the proof of Theorem 4.3. \qed
Finally, we prove the following theorem with help of Theorem 4.3

**Theorem 4.5.** Let $1 < p, q < \infty, N < r < \infty$ and $\max(q, q') \leq r$ ($q' = q/(q-1)$). Let $T$ be any positive number. Assume that $\Omega$ is a uniform $W_r^{2-1/r}$ domain and that $\Omega$ is bounded in addition. Then, there exists a positive constant $\gamma_0$ such that for any $u_0 \in D_{p,q}(\Omega)$ and right members $f, g, h$ and $\theta$ with

$$
\begin{align*}
&f \in L_p((0, T), L_q(\Omega)^N), \
g \in L_p((0, T), W^1_p(\Omega) \cap W^1_p((0, T), W^{-1}_q(\Omega))), \quad h \in L_p((0, T), W^1_p(\Omega)^N) \cap W^1_p((0, T), W^{-1}_q(\Omega)^N)),
\end{align*}
$$

satisfying the condition: $\|g\|_{t=0} = 0$, $\|g\|_{t=0} = 0$ and $\|h\|_{t=0} = 0$, problem (1.10) admits unique solutions $u$ and $\theta$ with

$$
\begin{align*}
u \in L_p((0, T), W^1_p(\Omega)^N) \cap W^1_p((0, T), L_q(\Omega)^N), \quad \theta \in L_p((0, T), W^1_p(\Omega) + \tilde{W}^1_{p, q, 0}(\Omega))
\end{align*}
$$

possessing the estimate:

$$
\begin{align*}
&\|e^{\gamma t} \partial_t u\|_{L_p((0, t), L_q(\Omega))} + \|e^{\gamma t} u\|_{L_p((0, t), W^2_p(\Omega))} \\
&\leq C e^{\gamma t} \left( \|u_0\|_{B^{2-1/q'}_{p, p}(\Omega)} + \delta(\Gamma) \sum_{t=1}^{M} \left( \int_{t_0}^{t} |e^{\gamma s}(u(\cdot, s), p(\cdot, s)) N| \right) ds \right)^{1/p} + \|e^{\gamma t}(f, \partial_t g)\|_{L_p((0, t), L_q(\Omega))} \\
&+ \|e^{\gamma t}(g, h)\|_{L_p((0, t), W^1_p(\Omega))} + \|e^{\gamma t} \partial_t \left( (1 - \Delta)^{-1/2}(g, \partial_t h) \right)\|_{L_p((0, t), L_q(\Omega))}
\end{align*}
$$

for any $t \in [0, T]$ and $0 < \gamma \leq \gamma_0$ with some constant $C$ independent of $T$ and $\gamma$. Here, $\delta(\Gamma)$ is the same number as in Theorem 4.3.

**Proof.** Let $E_1$ be the same operator as in the proof of Theorem 3.2. Let $\phi(s)$ be a function in $C^\infty(\mathbb{R})$ such that $\phi(s) = 1$ for $s \leq 0$ and $\phi(s) = 0$ for $s \geq 1$ and set $\psi_t(s) = \phi(s - t)$. Obviously, $\psi_t \in C^\infty(\mathbb{R})$, $\psi_t(s) = 1$ for $s \leq t$ and $\psi_t(s) = 0$ for $s \geq t + 1$. Let $u^t = v$ and $\theta^t = \omega$ be solutions to the equations:

$$
\begin{align*}
\partial_s v - \text{Div} T(v, \omega) &= \psi_t E_t f, \quad \text{div} v = \psi_t E_t g = \text{div} (\psi_t E_t g) \quad \text{in } \Omega \times (0, \infty), \\
T(v, \omega) \cdot \partial_N s &= \psi_t E_t h, \quad \text{v} \mid_{t=0} = 0, \quad \text{v} \mid_{s=0} = u_0 \quad \text{in } \Omega.
\end{align*}
$$

(4.13)

Since $(\psi_t E_t f)(\cdot, s) = f(\cdot, s)$ for $s \in [0, T]$, $u^t$ and $\theta^t$ solve problem (1.10) for $s \in (0, t)$. And, by the uniqueness of solutions, $u^t(\cdot, s) = u^{t'}(\cdot, s)$ for $s \in [0, t_1]$ when $0 < t_1 < t_2 \leq T$. By Theorem 4.4

$$
\begin{align*}
&\|e^{\gamma s} u^t\|_{L_p((0, t), W^2_p(\Omega))} + \|e^{\gamma s} \partial_s u^t\|_{L_p((0, t), L_q(\Omega))} \leq C \left( \|u_0\|_{B^{2-1/q'}_{p, p}(\Omega)} \right) \\
&+ \delta(\Gamma) \sum_{t=1}^{M} \left( \int_{t_0}^{t} |e^{\gamma s}(u(\cdot, s), p(\cdot, s)) N| \right) ds \right)^{1/p} + \|e^{\gamma s}(\psi_t E_t f, \partial_s (\psi_t E_t g))\|_{L_p((0, t), L_q(\Omega))} \\
&+ \|e^{\gamma s}(\psi_t E_t g, \psi_t E_t h)\|_{L_p((0, t), W^1_p(\Omega))} + \|e^{\gamma s}(\tilde{\lambda}^{1/2}(\psi_t E_t g), \tilde{\lambda}^{1/2}(\psi_t E_t h))\|_{L_p((0, t), L_q(\Omega))}.
\end{align*}
$$

(4.14)

Let $X$ be $L_q(\Omega)$ or $W^1_p(\Omega)$. Using the change of variable: $2t - s = r$, we have

$$
\begin{align*}
&\int_{0}^{2t} e^{\gamma s} \|\psi_t E_t f(\cdot, s)\| \, ds = \int_{0}^{2t} e^{\gamma s} \psi_t(s) \|f(\cdot, 2t - s)\|_{X} \, ds \leq e^{2\gamma t} \int_{0}^{2t} \psi_t(s) \|f(\cdot, r)\|_{X} \, ds \leq e^{2\gamma t} \int_{0}^{2t} \|f(\cdot, r)\|_{X} \, dr.
\end{align*}
$$

Thus, noting that $\psi_t E_t f$ vanishes for $s \notin [0, 2t]$, we have

$$
\begin{align*}
\|e^{\gamma t} \psi_t E_t f\|_{L_p(\mathbb{R}, X)} \leq e^{2\gamma t} \|e^{\gamma s} f\|_{L_p((0, t), X)}.
\end{align*}
$$

(4.15)

Noting (3.4) and using (4.13), we have

$$
\begin{align*}
&\|e^{\gamma s}(\partial_t E_t f, \partial_s (\psi_t E_t g))\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C e^{\gamma t} \|e^{\gamma s}(f, \partial_t g)\|_{L_p((0, t), L_q(\Omega))}, \\
&\|e^{\gamma s}(\partial_t E_t g, \partial_s (\psi_t E_t h))\|_{L_p(\mathbb{R}, W^1_p(\Omega))} \leq C e^{\gamma t} \|e^{\gamma s}(g, h)\|_{L_p((0, t), W^1_p(\Omega))}.
\end{align*}
$$

(4.16)
for any $\gamma \in (0, 70]$ with some constant independent of $\gamma$, $t$ and $T$.
In addition, applying the same argumentation as in the proof of the inequality (3.2) in the appendix, we have
\[
\|e^{\gamma T}\tilde{\Lambda}^{1/2}f\|_{L_p(R, L_q(\Omega))} \leq C\{\|e^{\gamma T}\partial_\nu[(1 - \Delta)^{-1/2}(f, f)]\|_{L_p(R, L_q(\mathbb{R}^n))} + \|e^{\gamma T}f\|_{L_p(R, W^1_q(\Omega))}\},
\] (4.17)
so that using (4.15) and (3.4), we have
\[
\|e^{\gamma T}(\tilde{\Lambda}^{1/2}(\phi_1 L_t g), \tilde{\Lambda}^{1/2}(\phi_1 L_t h))\|_{L_p(R, L_q(\Omega))}
\leq C \exp\{\|e^{\gamma T}\partial_\nu[(1 - \Delta)^{-1/2}(g, h)]\|_{L_p(0, T), L_q(\mathbb{R}^n))} + \|e^{\gamma T}(g, h)\|_{L_p(0, T), W^1_q(\Omega))}\}. \tag{4.18}
\]

Setting $u = u^T$ and $\theta = \theta^T$ and combining (4.14), (4.16) and (4.18), we have Theorem 4.5.

\section{A proof of a global in time unique existence theorem}

In this section, we prove Theorem 2.6 so that we assume that $\Omega$ is bounded in addition. Let $T_0$ be a positive number such that for any initial data $v_0 \in D_{q,p}(\Omega)$ with $\|v_0\|_{B_{q, p}^{(1 - 1/p)}(\Omega)} \leq 1$, problem (2.4) admits a solution $u \in S_{T_0}^{1,2}(0, T_0)$ satisfying (2.10). Here and in the following, we set
\[
S_{T_0}^{1,2}(a, b) = W_p^2((a, b), L_q(\Omega))^N \cap L_p((a, b), W^2_q(\Omega))^N,
\]
\[
\Gamma_{a, b} = \|e^{\gamma T}\partial_\nu\|_{L_p((a, b), L_q(\Omega))} + \|e^{\gamma T}v\|_{L_p((a, b), W^2_q(\Omega))}
\]
for any $a, b$ satisfying $0 < a < b \leq \infty$ for the notational simplicity, where $\gamma$ is a fixed positive number for which Theorem 2.4 and Theorem 4.3 hold. By Theorem 2.3 such $T_0 > 0$ exists.

Let $\epsilon$ be a small positive number $\leq 1$ that is determined later and we assume that $v_0 \in D_{q,p}(\Omega)$ and $\|v_0\|_{B_{q, p}^{(1 - 1/p)}(\Omega)} \leq \epsilon$. Let $T$ be a positive number such that problem (2.4) admits a solution $u \in S_{T_0}^{1,2}(0, T)$ that satisfies (2.10). Since $\|v_0\|_{B_{q, p}^{(1 - 1/p)}(\Omega)} \leq \epsilon \leq 1$, we have $T \geq T_0$. The main step is to prove that there exist constants $\epsilon_0 > 0$ and $M_\epsilon$ independent of $\epsilon$ and $T$ such that
\[
I_\epsilon(0, t) \leq M_\epsilon(\epsilon + I_\epsilon(0, t)^2) \tag{5.1}
\]
for any $t \in (0, T)$ provided that $0 < \epsilon \leq \epsilon_0$.

In fact, let $r_\pm(\epsilon)$ be two roots of the quadratic equation: $M_\epsilon(\epsilon + x^2) - x = 0$, that is $r_\pm(\epsilon) = (2M_\epsilon)^{-1} \pm \sqrt{(2M_\epsilon)^{-2} - \epsilon}$. We find a small positive number $\epsilon_1 > 0$ such that $0 < r_-(\epsilon) < r_+(\epsilon)$ whenever $0 < \epsilon < \epsilon_1$. In this case, $r_-(\epsilon) = M_\epsilon \epsilon + O(\epsilon^2)$ as $\epsilon \to 0$. Since $\|u(0, t)\| \to 0$ as $t \to 0$ and $I_\epsilon(0, t)$ is a continuous function with respect to $t$, by (5.1) we have $I_\epsilon(0, t) \leq r_-(\epsilon)$ for any $t \in (0, T]$, especially $I_\epsilon(0, T) \leq r_-(\epsilon)$. To prove
\[
\sup_{t \in (0, T)} \|u(t, \cdot)\|_{B_{q, p}^{1 - 1/p}(\Omega)} \leq M_\epsilon(I_\epsilon(0, T) + e^{-\epsilon T}\|v_0\|_{B_{q, p}^{1 - 1/p}(\Omega)}), \tag{5.2}
\]
we take $Z \in S_{\gamma, T}^{1,2}(0, \infty)$ which solves (3.13) with some pressure term $\Pi$. By Corollary 4.3 we have
\[
\|e^{\gamma T}\partial_\nu Z\|_{L_p(R, L_q(\Omega))} + \|e^{\gamma T}Z\|_{L_p(R, W^2_q(\Omega))} \leq M_\epsilon\|v_0\|_{B_{q, p}^{1 - 1/p}(\Omega)}
\]
with some constant $M_\epsilon$ independent of $\epsilon$ and $T$, because $v_0$ satisfies (2.14) when $\gamma = 0$. Employing the same argument as in the proof of (3.10), we have (5.2).

Since we may assume that $r_-(\epsilon) \leq 2M_\epsilon$, by (5.2) we have $\|u(\cdot, T - \epsilon)\|_{B_{q, p}^{1 - 1/p}(\Omega)} \leq M_\epsilon(2M_\epsilon + 1)\epsilon$, because $e^{-\epsilon T} \leq 1$. Choose $\epsilon$ so small that $M_\epsilon(2M_\epsilon + 1)\epsilon \leq 1$. By Theorem 2.4 there exists a unique solution $u \in S_{T, T_0}^{1,2}(0, T + T_0)$ of the equations:

\[
\partial_t u' - \text{Div} T(u', \theta') = F(u'), \quad \text{div} u' = G(u') = \text{div} G(u') \quad \text{in} \quad \Omega \times (T, T + T_0),
\]
\[
T(u', \theta')|_{S} = H(u')|_{S}, \quad u'|_{r = 0} = 0, \quad u'|_{t = T + 0} = u'|_{t = T - 0}.
\]
with some pressure term \( \theta' \). Choosing \( T_0 \) smaller if necessary, we may assume that

\[
\int_T^{T+T_0} \| \nabla u'(\cdot, t) \|_{L_\infty(\Omega)} dt \leq \sigma/2.
\]

Since \( \int_T^T \| \nabla u'(\cdot, t) \|_{L_\infty(\Omega)} dt \leq M_0 \| u(0, T) \| \) with some constant \( M_0 \) independent of \( \varepsilon \) and \( T \) as follows from (3.9), we choose \( \varepsilon \) so small that \( M_0 \varepsilon < \sigma/2 \), so that \( \int_T^{T+T_0} \| \nabla u'(\cdot, t) \|_{L_\infty(\Omega)} dt \leq \sigma/2 \). If we define \( u'' \) by \( u''(\cdot, t) = u(\cdot, t) \) for \( 0 \leq t \leq T \) and \( u''(\cdot, t) = u'(\cdot, t) \) for \( T \leq t \leq T + T_0 \), then \( u'' \) satisfies the equations (5.3) for \( t \in (0, T + T_0) \) with some pressure term \( \theta'' \) and the condition: \( \int_0^{T+T_0} \| u''(\cdot, t) \|_{L_\infty(\Omega)} dt \leq \sigma \).

Thus, by (5.3) \( I_{u''}(0, T + T_0) \leq M_0(\varepsilon + I_{u''}(0, T + T_0)^2) \). Repeating this argument, we can prolong \( u \) to any time interval \( (0, T) \) with \( I_u(0, T) \leq r_-(\varepsilon) \), which completes the existence of solution \( u \) globally defined in time with \( I_u(0, \infty) \leq r_-(\varepsilon) \). The uniqueness follows from the same arguments as in the proof of Theorem 2.4 with small \( \varepsilon > 0 \) instead of small \( T > 0 \). Therefore, our task is to prove (5.1).

Applying Theorem 4.3 to problem (2.3), we have

\[
I_u(0, t) \leq C\{ \| v_0 \|_{L_{\infty, p}^{(1-\frac{1}{p})}(\Omega)} + \delta(\Gamma) \sum_{l=1}^M \left( \int_0^t \| e^{\gamma s}(u(\cdot, s), p_1) \|_p ds \right)^{1/p} + \varepsilon \} \tag{5.3}
\]

for any \( t \in [0, T] \) with

\[
\kappa_u(0, t) = \| e^{\gamma s}(F(u), \partial_s G(u)) \|_{L_p((0, t), L_q(\Omega))} + \| e^{\gamma s}(G(u), H(u), \tilde{n}) \|_{L_p((0, t), W_q'(\Omega))} + \| e^{\gamma s} \delta_e[(1 - \Delta)^{-1/2}(G(u), H(u), \tilde{n})] \|_{L_p((0, t), L_q(\mathbb{R}^N))}.
\]

Here and in the following, \( C \) denotes a generic constant independent of \( \varepsilon, t \in [0, T] \) and \( T \).

When \( \Gamma = \emptyset, \delta(\Gamma) = 1 \), so that we have to estimate \( \int_0^t \| e^{\gamma s}(u(\cdot, s), p_1) \|_p ds \). Recalling Remark 2.5 (1), \( \psi(x, t) = u(X^{-1}(x, t), t) \) satisfies the equation (1.1) with (1.2), where \( X^{-1}(x, t) \) denotes the inverse map of the correspondence: \( x = \chi + \int_0^t u(\chi, s) ds = X_u(\chi, t) \). Since \( \frac{d}{dt} \int_T \psi(x, t) \|_{\mathbb{R}^N} dx = 0 \), by (2.14) we have \( \int_T \psi(x, t) \|_{\mathbb{R}^N} dx = 0 \), so that

\[
\int_T \psi(\chi, t) \|_{\mathbb{R}^N} dx = 0,
\]

which combined with (3.9) and Hölder’s inequality furnishes that

\[
\| \psi(\chi, t) \|_{\mathbb{R}^N} \leq C \| \psi(\chi, t) \|_{\mathbb{R}^N} \int_0^t \| \psi(\chi, t) \|_{\mathbb{R}^N} dt \leq C \| \psi(\chi, t) \|_{\mathbb{R}^N} \left( \int_0^t e^{-p'\gamma \tau} d\tau \right)^{1/p'} \left( \int_0^t \| \psi(\chi, t) \|_{\mathbb{R}^N}^p d\tau \right)^{1/p}.
\]

Thus, we have

\[
\delta(\Gamma) \sum_{l=1}^M \left( \int_0^t \| e^{\gamma s}(u(\cdot, s), p_1) \|_p ds \right)^{1/p} \leq C(I_u(0, t))^2 \tag{5.4}
\]

From now on, we estimate \( \kappa_u(0, t) \). By Hölder’s inequality we have

\[
\sup_{s \in (0, t)} \int_0^s \| \nabla u(\cdot, r) \|_{L_\infty(\Omega)} dr \leq \left( \int_0^t \| e^{-\gamma \tau} \|_{L_\infty(\Omega)}^p dr \right)^{1/p} \left( \int_0^t \| e^{-\gamma \tau} \|_{L_\infty(\Omega)}^{p'} dr \right)^{1/p'} \leq C(I_u(0, t)) \tag{5.5}
\]

Since (2.10) holds and since we may assume that \( \sigma \leq 1 \), by (2.6), (5.5) and (3.9)

\[
\sup_{s \in (0, t)} \| \nabla \mathbf{W}(s) \|_{L_\infty(\Omega)} \leq C(I_u(0, t)), \tag{5.6}
\]

17
where \(i = 1, 2, 4, 5\) and \(6\) and \(W = W(K)\) is any matrix of polynomials with respect to \(K\). By (3.3), (2.5) and (5.6), we have
\[
\|F(u)\|_{L_p((0,t),L_q(w))} \leq C|u(0,t)|^2, \quad \|(G(u), H(u)\tilde{n})\|_{L_p((0,t),W_q^2(\Omega))} \leq C|u(0,t)|^2. \tag{5.7}
\]
Since \(B^{(1-1/p)}_q(\Omega) \subset W_q^2(\Omega)\) as follows from the assumption: \(2 < p, \infty, \) by (3.2) we have
\[
\sup_{s \in (0,t)} \|u(s, \cdot)\|_{W_q^2(\Omega)} \leq C(|u(0,t) + \epsilon), \quad \sup_{s \in (0,t)} \|\partial_t W(\int_0^s \nabla u(\cdot, r) \, dr)\|_{L_q(\Omega)} \leq C(|u(0,t) + \epsilon). \tag{5.8}
\]
Thus, by (3.3), (5.6) and (5.8),
\[
\|\partial_s G(u)\|_{L_p((0,t),L_q(\Omega))} \leq C(|u(0,t)|^2 + \|u(0,t) + \epsilon\|_{L_q(\Omega)}) \leq 2C(|u(0,t)|^2 + \epsilon), \tag{5.9}
\]
because \(0 < \epsilon \leq 1\).

To estimate \(\|e^{\gamma t}\tilde{A}^{1/2} g\|_{L_p((0,t),L_q(\Omega))}\), we use the following lemma which can be proved in the same manner as in the proof of Lemma 3.3.

**Lemma 5.1.** Let \(1 < p, q < \infty, N < r, q < \infty\) and let \(\Omega\) be a uniform \(W_r^{2-1/r}\). Let \(\nu\) be the extension map satisfying the properties (e-1) and (e-2). Then

\[
\|e^{\gamma t}\tilde{A}^{1/2} ((\nabla f)g)\|_{L_p((0,t),L_q(\Omega))}^p \leq C\left\{ \int_0^t (e^{\gamma s}|\partial_s f(\cdot, s)\|_{L_q(\Omega)} g(\cdot, s)\|_{W_q^2(\Omega)})^p ds + \int_0^t (e^{\gamma s}|\nabla f(\cdot, s)\|_{L_q(\Omega)} |\partial_t g(\cdot, s)\|_{L_q(\Omega)})^p ds \right\}. \tag{5.6}
\]

Applying Lemma 5.1 and using (1.7), (5.6) and (5.7), we have
\[
\|e^{\gamma t}\partial_s [(1 - \Delta)^{-1/2} iG(u), h(t)\tilde{n}]\|_{L_p((0,t),L_q(\Omega))} \leq C(|u(0,t)|^2 + (|u(0,t) + \epsilon)\|_{L_q(\Omega)}) \leq 2(|u(0,t)|^2 + \epsilon), \tag{5.10}
\]
which combined with (5.3), (5.4), (5.7) and (5.9) furnishes (5.1). This completes the proof of Theorem 2.6.

### A A proof of the inequality (3.2)

First, we prove the inequality (3.2) in case of \(\Omega = \mathbb{R}^N\).

**Lemma A.1.** Let \(1 < p, q < \infty\) and set \((1 - \Delta)^{-1/2} f - F_s^{-1}(1 + |\xi|^2)^{-1/2} f(\xi)\) for \(f \in S'(\mathbb{R}^N)\) and \(s \in \mathbb{R}\). Here, \(s\) denotes the Fourier transform of \(f\), \(F_s^{-1}\) the inverse Fourier transform and \(S'(\mathbb{R}^N)\) the space of tempered distributions on \(\mathbb{R}^N\) in the sense of L. Schwartz. Then, we have
\[
\|e^{-\gamma t} A^{1/2} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \leq C\left\{ \|e^{-\gamma t} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|e^{-\gamma t}(1 - \Delta)^{-1/2} \partial_t f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} + \|e^{-\gamma t}(1 - \Delta)^{1/2} f\|_{L_p(\mathbb{R},L_q(\mathbb{R}^N))} \right\}. \tag{2.9}
\]

**Proof.** The idea of our proof here is the same as in the proof of Proposition 2.8 in [23]. Let \(\varphi_0(t)\) be a function in \(C^\infty(\mathbb{R})\) such that \(\varphi_0(t) = 1\) for \(|t| \leq 1\) and \(\varphi_0(t) = 0\) for \(|t| \geq 2\), and set \(\varphi_\infty(t) = 1 - \varphi_0(t)\). We define functions \(A_j(\xi, \lambda) (j = 1, 2)\) by
\[
A_1(\xi, \lambda) = \varphi_\infty(\tau) \varphi_0 \left( \frac{(1 + |\xi|^2)^{1/2}}{|\lambda|} \right) \varphi_\infty \left( \frac{1 + |\xi|^2}{1 + |\xi|^2} \lambda^{1/2} \right),
\]
\[
A_2(\xi, \lambda) = \varphi_\infty(\tau) \varphi_\infty \left( \frac{(1 + |\xi|^2)^{1/2}}{|\lambda|} \right) \varphi_\infty \left( \frac{1 + |\xi|^2}{1 + |\xi|^2} \lambda^{1/2} \right).
\]
We have
\[
|\partial^\alpha_s \partial^\beta_t A_j(\xi, \lambda)| \leq C_{\ell, \alpha} |\tau|^{-\ell} |\xi|^{-\alpha} \quad (\lambda = i\tau + \gamma, \ j = 1, 2)
\]
for any \(\ell \in \mathbb{N}_0\) and \(\alpha \in \mathbb{N}_0^N\), \(\xi \in \mathbb{R}^N \setminus \{0\}\), \(\tau, \gamma \in \mathbb{R} \setminus \{0\}\) with some constant \(C_{\ell,\alpha}\) depending solely on \(\ell\) and \(\alpha\). Set \(A_j(\lambda, D_x)f = \mathcal{F}_x^{-1}[A_j(\xi, \lambda)\hat{f}(\xi)]\) for any \(f \in \mathcal{S}'(\mathbb{R}^N)\), and then by Theorem 3.3 in [12] we know that the sets \(\{\tau^N\partial_x^N A_j(\lambda, D_x) \mid \tau \in \mathbb{R} \setminus \{0\}\}\) are \(\mathcal{R}\)-bounded families in \(\mathcal{L}(L_q(\mathbb{R}^N))\) and their \(\mathcal{R}\)-bounded are less than \(C_{q,N} \max_{|\alpha| \leq N+2} C_{k,\alpha}\) for \(k = 0, 1\) and \(j = 1, 2\), where \(\mathcal{L}(L_q(\mathbb{R}^N))\) is the set of all bounded linear operators on \(L_q(\mathbb{R}^N)\). Therefore, by Weis’ operator valued Fourier multiplier theorem [38] we have
\[
\|e^{-\gamma t} A_j(\partial_t, D_x)F\|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \leq C\|e^{-\gamma t} F\|_{L_p(\mathbb{R}, L_q(\Omega))} \quad (\gamma \neq 0) . \tag{A.1}
\]
Here, the operators \(A_j(\partial_t, D_x)\) are defined by
\[
A_j(\partial_t, D_x)F = \mathcal{L}_x^{-1}[A_j(\lambda, D_x)\mathcal{L}[F](\lambda, \cdot)](t) .
\]
Dividing \(\lambda^{1/2}\) into the following three parts:
\[
\lambda^{1/2} = \varphi_0(\tau)\lambda^{1/2} + \varphi_\infty(\tau)A_1(\xi, \lambda)\frac{\lambda}{(1 + \|\xi\|^2)^{1/4}} + \varphi_\infty(\tau)A_2(\xi, \lambda)(1 + |\xi|^{2})^{1/4} ,
\]
and using (A.1) and Bourgain’s Fourier multiplier theorem [8], we have Lemma \(A.1\).}

**Proof of the inequality (3.2).** To prove the lemma, we use the extension map \(E\) having the following properties:

\begin{enumerate}[(Ex-1)]
    \item \(\|Ef\|_{L_q(\mathbb{R}^N)} \leq C_q,\ell \|f\|_{L_q(\Omega)}\)
    \item \(\|(-\Delta)^{1/2}Ef\|_{L_q(\mathbb{R}^N)} \leq C_q,\ell \|f\|_{W^1_q(\Omega)}\)
    \item \(\|(-\Delta)^{-1/2}E(\nabla f)\|_{L_q(\mathbb{R}^N)} \leq C_q,\ell \|f\|_{L_q(\Omega)}\).
\end{enumerate}

Such extension map can be constructed under the assumption that \(N < q, r < \infty\). By Lemma \(A.1\), we have
\[
\|e^{-\tau t}A_0^{1/2}(\nabla f)g\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq \|e^{-\tau t}A_0^{1/2}E(\nabla f)g\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq \|e^{-\tau t}E(\nabla f)g\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\tau t}(1 - \Delta)^{-1/2}\partial_t E(\nabla f)g\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\tau t}E(\nabla f)\partial_t g\|_{L_p(\mathbb{R}, L_q(\Omega))}.
\]

Using the identity: \(\partial_t((\nabla f)g) = \nabla(\partial_t f \cdot g) - \partial_t f(\nabla g) + (\nabla f)\partial_t g\) and (Ex-3), we have
\[
\|e^{-\tau t}(1 - \Delta)^{-1/2}\partial_t E(\nabla f)g\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq \|e^{-\tau t}(1 - \Delta)^{-1/2}\partial_t f(\nabla g)\|_{L_p(\mathbb{R}, L_q(\Omega))} + \|e^{-\tau t}(1 - \Delta)^{-1/2}E(\nabla f)\partial_t g\|_{L_p(\mathbb{R}, L_q(\Omega))}.
\]

By (3.5) and (Ex-1), we have
\[
\|e^{-\tau t}(\partial_t f \cdot g)\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C\int_{-\infty}^\infty (e^{-\tau t}\|\partial_t f(\cdot, t)\|_{L_q(\Omega)}\|g(\cdot, t)\|_{W^1_q(\Omega)})^p dt,
\]
\[
\|e^{-\tau t}E(\nabla f)g\|_{L_p(\mathbb{R}, L_q(\Omega))} \leq C\int_{-\infty}^\infty (e^{-\tau t}\|\nabla f(\cdot, t)\|_{L_q(\Omega)}\|g(\cdot, t)\|_{W^1_q(\Omega)})^p dt.
\]

To estimate other terms, we use the inequality:
\[
\|(1 - \Delta)^{-1/2}E(f, g)\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}\|g\|_{L_q(\Omega)} . \tag{A.2}
\]
In fact, for any \(\varphi \in C_c^\infty(\mathbb{R}^N)\), we observe that
\[
\|(1 - \Delta)^{-1/2}E(f, \varphi)\|_{L_q(\Omega)} \leq \|E(f, \varphi)\|_{L_q(\Omega)}\|(1 - \Delta)^{-1/2}\varphi\|_{L_q(\Omega)} \leq C\|f\|_{L_q(\Omega)}\|\varphi\|_{L_q(\Omega)}\|(1 - \Delta)^{-1/2}\varphi\|_{L_q(\Omega)} .
\]
where $s$ is an index such that $1/s + 2/q = 1$. Since $2 \leq N < q < \infty$, we can choose such $s$ with $1 < s < \infty$. Since $N(1/q' - 1/s) = N(1 - 1/q - 1/s) = N/q < 1$, we have $(1 - \Delta)^{-1/2} \varphi \in L_q(\mathbb{R}^N)$, so that we have (A.2).

By (A.2) we have

$$\| e^{-\gamma t}(1 - \Delta)^{-1/2}E(\partial_t f(\nabla g)) \|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \leq C \int_{-\infty}^{\infty} (e^{-\gamma t}\| \partial_t f(\cdot, t)\|_{L_q(\Omega)}\| \nabla g(\cdot, t)\|_{L_q(\Omega)})^p dt,$$

$$\| e^{-\gamma t}(1 - \Delta)^{-1/2}E(\nabla f) \partial_t g) \|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \leq C \int_{-\infty}^{\infty} (e^{-\gamma t}\| \nabla f(\cdot, t)\|_{L_q(\Omega)}\| \partial_t g(\cdot, t)\|_{L_q(\Omega)})^p dt.$$

By (8.3) and (Ex-2) we have

$$\| e^{-\gamma t}(1 - \Delta)^{1/2}E(\nabla f) \|_{L_p(\mathbb{R}, L_q(\mathbb{R}^N))} \leq C \| e^{-\gamma t}(\nabla f) \|_{L_p(\mathbb{R}, W^1_q(\mathbb{R}^N))}^p \leq C \int_{-\infty}^{\infty} (e^{-\gamma t}\| \nabla f(\cdot, t)\|_{W^1_q(\Omega)}\| \nabla g(\cdot, t)\|_{W^1_q(\Omega)})^p dt.$$

This completes the proof of the inequality (3.2).

References

[1] H. Abels, *The initial-value problem for the Navier-Stokes equations with a free surface in $L^2$-Sobolev spaces*, Adv. Differential Equations, 10 (2005), 45–64.

[2] G. Allain, *Small-time existence for Navier-Stokes equations with a free surface*, Appl. Math. Optim., 16 (1987), 37–50.

[3] H. Amann, *Linear and Quasilinear Parabolic Problems*, Vol. I. Birkhäuser, Basel, 1995.

[4] H. Amann, *Anisotropic function spaces and maximal regularity for parabolic problem. Part 1. Function Spaces*, Jindrich Nácaš Center Math. Modelling Lecture Notes, Prague, vol. 6, 2009

[5] J. T. Beale, *The initial value problem for the Navier-Stokes equations with a free surface*, Commun. Pure Appl. Math., 34 (1981), 359–392.

[6] T. Beale, *Large-time regularity of viscous surface waves*, Arch. Rational Mech. Anal., 84 (1983/84), 307–352.

[7] J. T. Beale and T. Nishida, *Large-time behaviour of viscous surface waves*, Recent topics in nonlinear PDE, II, North-Holland Math. Stud., 128, North-Holland, Amsterdam, 1985, pp.1–14.

[8] J. Bourgain, *Vector-valued singular integrals and the $H^1$-BMO duality*, In: Probability Theory and Harmonic Analysis, D. Borkholder (ed.) Marcel Dekker, New York, 1–19 (1986)

[9] A. P. Calderón, *Lebesgue spaces of differentiable functions and distributions*, Proc. Symp. in Pure Math. 4, 33–49 (1961)

[10] R. Denk, M. Hieber and J. Prüß, *R-boundedness, Fourier multipliers and problems of elliptic and parabolic type*, Memoirs of AMS, Vol 166. No. 788. 2003.

[11] G. Duvaut and J. L. Lions, *Inequalities in mechanics and physis*, Lecture Notes in Mathematics 393, 1976, Springer-Verlag

[12] Y. Enomoto and Y. Shibata, *On the R-sectoriality and its application to some mathematical viscous compressible fluids*, Funk. Ekvaj. 56 (3), 2013, 441–505.

[13] Y. Enomoto, L. von Below and Y. Shibata, *On some free boundary problem for a compressible barotropic viscous fluid flow*, to appear in Annali dell Universita di Ferrara, DOI: 10.1007/s11565-013-0194-8.

[14] Y. Hataya, *Decaying solution of a Navier-Stokes flow without surface tension*, J. Math. Kyoto Univ., 49 (2009), 691–717.
[15] S. G. Mikhlin, *Fourier integrals and multiple singular integrals* (Russian), Vest. Leningrad Univ. Ser. Mat. **12** (1957), 143–145.

[16] I. Sh. Moglievski˘ı and V. A. Solonnikov, *On the solvability of a free boundary problem for the Navier-Stokes equations in the H¨older space of functions*, Nonlinear Analysis, Sc. Norm. Super. di Pisa Quaderni, Scuola Norm. Sup., Pisa, 1991, pp. 257–271.

[17] P. B. Mucha and W. Zajączkowski, *On the existence for the Cauchy-Neumann problem for the Stokes system in the $L_p$-framework*, Studia Math. **143**(1) (2000), 75–101.

[18] P. B. Mucha and W. Zajączkowski, *On local existence of solutions of the free boundary problem for an incompressible viscous self-gravitating fluid motion*, Appl. Math., **27**(3) (2000), 319–333.

[19] M. Padula and V. A. Solonnikov, *On the global existence of nonsteady motions of a fluid drop and their exponential decay to a uniform rigid rotation*, Quad. mat., **10** (2002), 185–218.

[20] B. Schweizer, *Free boundary fluid systems in a semigroup approach and oscillatory behavior*, SIAM J. Math. Anal., **28** (1997), 1135–1157.

[21] Y. Shibata, *Generalized resolvent estimates of the Stokes equations with first order boundary condition in a general domain*, J. Math. Fluid Mech. **15**(1) (2013), 1–40.

[22] Y. Shibata, *On the $R$-boundedness of solution operators for the Stokes equations with free boundary condition*, Diff. Integ. Eqn. **27**(27) (2014), 313–368.

[23] Y. Shibata and S. Shimizu, *On some free boundary problem for the Navier-Stokes equations*, Diff. Integ. Eqn. **20**(2007), 241–276.

[24] Y. Shibata and S. Shimizu, *On the $L_p$-$L_q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain*, J. Reine Angew. Math. **615** (2008), 157–209.

[25] V. A. Solonnikov, *Estimates of solutions of an initial-boundary value problem for the linear nonstationary Navier-Stokes system*, Zap. Nauchn. Sem. Leningrad Otdel. mat. Inst. Steklov, LOMI **59** (1976), 178–254 (in Russian).

[26] V. A. Solonnikov, *Solvability of the problem of evolution of an isolated amount of a viscous incompressible capillary fluid*, Zap. Nauchn. Sem. Leningrad Otdel. mat. Inst. Steklov, LOMI **140** (1984), 179–186 (in Russian).

[27] V. A. Solonnikov, *Unsteady flow of a finite mass of a fluid bounded by a free surface*, Zap. Nauchn. Sem. Leningrad Otdel. mat. Inst. Steklov, LOMI **152** (1986), 137–157 (in Russian); English transl: J. Soviet Math., **40** (1988), 672–686.

[28] V. A. Solonnikov, *On the transient motion of an isolated volume of viscous incompressible fluid*, Math. USSR Izvestiya **31** (1988) (2), 381–405.

[29] V. A. Solonnikov, *On nonstationary motion of a finite isolated mass of self-gravitating fluid*, Algebra i Analiz, **1** (1989), 207–249 (in Russian); English transl: Leningrad Math. J., **1** (1990), 227–276.

[30] V. A. Solonnikov, *Solvability of the problem of evolution of a viscous incompressible fluid bounded by a free surface on a finite time interval*, Algebra i Analiz, **3** (1991), 191–239.

[31] V. A. Solonnikov, *Lecures on evolution free boundary problems: classical solutions*, L. Ambrosio et al ed., Mathematical aspects of evolving interfaces, Lecture Notes in Math., **1812**, Springer-Verlag, 2003, pp.123–175.

[32] O. Steiger, *On Navier-Stokes equations with first order boundary conditions*, Ph.D thesis, Universität Zürich, 2004.

[33] G. Strömer, *About a certain class of parabolic- hyperbolic systems of differential equations*, Analysis **9** (1989), 1–39.

[34] D. L. G. Sylvester, *Large time existence of small viscous surface waves without surface tension*, Commun. Partial Differential Equations, **15** (1990), 823–903.
[35] D. L. G. Sylvester, *Decay rate for a two-dimensional viscous ocean of finite depth*, J. Math. Anal. Appl., 202 (1996), 659–666.

[36] A. Tani, *Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface*, Arch. Rational Mech. Anal., 133 (1996), 299–331.

[37] A. Tani and N. Tanaka, *Large-time existence of surface waves in incompressible viscous fluids with or without surface tension*, Arch. Rational Mech. Anal., 130 (1995), 303–314.

[38] L. Weis, *Operator-valued Fourier multiplier theorems and maximal $L_p$-regularity*, Math. Ann. 319, 733–758 (2001)