ON A BINARY SYSTEM OF PRENDIVILLE: THE CUBIC CASE

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ABSTRACT. We prove sharp decoupling inequalities for a class of two dimensional non-degenerate surfaces in $\mathbb{R}^3$, introduced by Prendiville [Pre13]. As a consequence, we obtain sharp bounds on the number of integer solutions of the Diophantine systems associated with these surfaces.

1. Introduction

Let $\Phi(t, s)$ be a homogeneous polynomial of degree three. Consider the two dimensional surface

$$S = \{(t, s, \Phi_t(t, s), \Phi_s(t, s), \Phi(t, s)) : (t, s) \in [0, 1]^2\}. \quad (1.1)$$

We say that $\Phi$ is non-degenerate if it cannot be written as $(\mu t + \nu s)^3$ for any $\mu, \nu \in \mathbb{R}$. This is the same as saying that, if we write $\Phi(t, s) = at^3 + bt^2s + cts^2 + ds^3$, then the matrix

$$\begin{pmatrix}
3a & 2b & c \\
b & 2c & 3d
\end{pmatrix} \quad (1.2)
$$

has rank two. This will be our assumption throughout the present paper.

Consider the following system of Diophantine equations

$$\begin{cases}
x_1 + x_2 + \ldots + x_r &= x_{r+1} + x_{r+2} + \ldots + x_{2r}, \\
y_1 + y_2 + \ldots + y_r &= y_{r+1} + y_{r+2} + \ldots + y_{2r}, \\
\Phi_t(x_1, y_1) + \ldots + \Phi_t(x_r, y_r) &= \Phi_t(x_{r+1}, y_{r+1}) + \ldots + \Phi_t(x_{2r}, y_{2r}), \\
\Phi_s(x_1, y_1) + \ldots + \Phi_s(x_r, y_r) &= \Phi_s(x_{r+1}, y_{r+1}) + \ldots + \Phi_s(x_{2r}, y_{2r}), \\
\Phi(x_1, y_1) + \ldots + \Phi(x_r, y_r) &= \Phi(x_{r+1}, y_{r+1}) + \ldots + \Phi(x_{2r}, y_{2r}).
\end{cases} \quad (1.3)
$$

Here $r$ is a positive integer and $x_i, y_i \in \mathbb{N}$ for each $1 \leq i \leq 2r$. For a large integer $N$, we let $J_r(N)$ denote the number of integer solutions $(x_1, \ldots, x_{2r}, y_1, \ldots, y_{2r})$ of the system (1.3) with $0 \leq x_i, y_i \leq N$ for each $1 \leq i \leq 2r$. We prove

**Theorem 1.1.** For each $r \geq 1$ and each $\epsilon > 0$, we have

$$J_r(N) \lesssim_{r, \epsilon} N^{2r+\epsilon} + N^{4r-9+\epsilon}. \quad (1.4)$$

Here the implicit constant depends only on $r$ and $\epsilon$. Moreover, up to the arbitrarily small factor $\epsilon$, the exponents of $N$ are sharp.

The lower bounds have been calculated by Parsell, Prendiville and Wooley [PPW13]. Our focus is to obtain the upper bounds (1.4). This will be done via proving a sharp decoupling inequality.

For a measurable set $R \subset [0, 1]^2$ and a measurable function $g : R \rightarrow \mathbb{C}$, define the extension operator associated with $S$ by

$$E_Rg(x) = \int_R g(t, s)e^{i\Phi_t(t, s) x_3 + i\Phi_s(t, s) x_4 + i\Phi(t, s) x_5} \, dt \, ds. \quad (1.5)$$

AMS subject classification: Primary 11L07; Secondary 42A45.
Here $x = (x_1, \ldots, x_5)$. For a ball $B = B(c, R) \subset \mathbb{R}^5$ with center $c$ and radius $R$, we use the weight

$$w_B(x) = (1 + \frac{\|x - c\|}{R})^{-C},$$

(1.6)

where $C$ is a large enough constant whose value will not be specified. For each $2 \leq q \leq p$ and $0 < \delta < 1$, let $B_{p,q}(\delta)$ be the smallest constant such that

$$\|E_{[0,1]^2}g\|_{L^p(w_B)} \leq B_{p,q}(\delta)(\sum_{\Delta: \text{square in } [0,1]^2, \|\Delta\| = \delta} \|E_{\Delta}g\|_{L^p(w_B)}^{q})^{1/q},$$

(1.7)

holds for each ball $B \subset \mathbb{R}^5$ of radius $\delta^{-3}$. Inequalities of this type are referred to as $l^qL^p$ decouplings. Via a standard reduction (see for instance Section 2 [BD16-2]), Theorem 1.1 follows from

**Theorem 1.2.** We have

$$B_{9,9}(\delta) \lesssim (\frac{1}{\delta})^{2(\frac{1}{2} - \frac{1}{8}) + \epsilon},$$

(1.8)

for each $\epsilon > 0$ and $0 < \delta \leq 1$.

The system (1.3) is the cubic case of a system considered by Prendiville [Pre13]. The way that the surface (1.1) and the Diophantine system (1.3) are formulated is slightly different from those in Prendiville [Pre13]. There the surface (1.1) is replaced by

$$S' = \{(\Phi_{tt}(t,s), \Phi_{ts}(t,s), \Phi_{ss}(t,s), \Phi_{tt}(t,s), \Phi_{ts}(t,s), \Phi(t,s)) : (t, s) \in [0,1]^2\}. $$

(1.9)

That is, the surface $S'$ is obtained by taking successive partial derivatives of the seed polynomial $\Phi$. However, under the non-degeneracy condition that the matrix (1.2) has rank two, we observe that the vector space $[\Phi_{tt}, \Phi_{ts}, \Phi_{ss}]$ is always the same as $[t, s]$. Hence the system of Diophantine equations associated with the surface $S'$ is always equivalent to that associated with the surface $S$, in the sense that they admit the same number of integer solutions.

To obtain a system analogous to (1.3) of higher degrees, one takes a seed polynomial $\Phi(t, s)$ of degree $k \geq 3$, extracts all the partial derivatives

$$\frac{\partial^{i_1+i_2} \Phi(t,s)}{\partial t^{i_1} \partial s^{i_2}} (i_1 \geq 0, i_2 \geq 0),$$

(1.10)

and forms a Diophantine system by using all these partial derivatives. If we take $\Phi(t, s)$ to be the monomial $t^{k_1} s^{k_2}$ with $k_1 \geq k_2 \geq 1$, then we recover the so-called simple binary systems

$$x_1^i y_1^j + \cdots + x_r^i y_r^j = x_{r+1}^i y_{r+1}^j + \cdots + x_{2r}^i y_{2r}^j,$$

with $i_1 \leq k_1, i_2 \leq k_2$ and $(i_1, i_2) \neq (0, 0)$, (1.11)

which appeared in recent work in quantitative arithmetic geometry (Section 4.15 [Tsc09] and [Val11]). Notice that if we take $\Phi$ to be a polynomial of degree $k$ that depends only on one variable, then we recover the Vinogradov system

$$x_1^i + \cdots + x_r^i = x_{r+1}^i + \cdots + x_{2r}^i,$$

with $1 \leq i \leq k$. (1.12)

All the systems mentioned above fall into the framework of translation-dilation invariant systems, which are intensively studied in [PPW13]. In our setting, this is reflected in the validity of the parabolic rescaling lemma (Lemma 3.1).

Parsell, Prendiville and Wooley [PPW13] proved (1.4) for $r \geq 21$, using the method of efficient congruencing. In the current paper we prove it for all $r \geq 1$, using the decoupling theory developed in [BD15] and [BD16]. When intending to generalise our proof to the above binary systems (1.10) or (1.11) of degrees higher than three, one encounters enormous difficulties. In comparison, the efficient congruencing method still provides bounds that are almost optimal. We refer to [PPW13] for the precise statement of the corresponding results.

Let us mention a further application of the result in Theorem 1.2. This application has been worked out carefully in [Pre13], [PPW13] and [Hen13], hence we mention it briefly. Let $\Phi$ be as above, a homogeneous polynomial of degree three that is non-degenerate. Take $r \in \mathbb{N}$. 
Let $c_1, c_2, \ldots, c_r$ with $c_1 + c_2 + \cdots + c_r = 0$ be a “non-singular” (Definition 1.1 [Pre13]) choice of coefficients for $\Phi$. Consider the equation

$$c_1 \Phi(x_1, y_1) + c_2 \Phi(x_2, y_2) + \cdots + c_r \Phi(x_r, y_r) = 0. \quad (1.13)$$

The solution $\{(x_1, y_1), \ldots, (x_r, y_r)\}$ to the above equation is called diagonal if they all lie on a line in the plane. Take a large number $N \in \mathbb{N}$. Let $A \subset [0, N]^2$ be a set which contains only diagonal solutions to the equation (1.13). Then a result in [Pre13] (further improved in [PPW13]) states that

$$|A| \ll N^2 (\log \log N)^{-1/(s-1)}, \quad (1.14)$$

for $s$ bigger than certain threshold. The validity of the estimate (1.8) will further lower down this threshold. We refer the interested reader to [Pre13] and [Hen15] for the details.

In the end, we mention some novelties of our proof and explain briefly the potential difficulties that appear when trying to adapt our argument to binary systems of higher degrees.

In decoupling theory, various Brascamp-Lieb inequalities (see (2.2)) play fundamental roles. In order to apply these inequalities, one needs to check a transversality condition (see (2.3)). When the dimensions and co-dimensions of the surfaces under consideration get higher and higher, checking these transversality conditions will become more and more difficult. In the current paper, we are dealing with a two dimensional surface in $\mathbb{R}^5$. To check (2.3), we further develop the idea introduced in [BDG16-2], where a specific two dimensional surface in $\mathbb{R}^9$ is considered. As currently we are dealing with a class of surfaces, certain algebraic structures need to be explored. For instance, see Subsection 2.2, in particular Lemma 2.3.

Notation: Throughout the paper we will write $A \lesssim B$ to denote the fact that $A \leq CB$ for a certain implicit constant $C$ that depends on the parameter $v$. Typically, this parameter is either $\epsilon$ or $K$. The implicit constant will never depend on the scale $\delta$, on the balls we integrate over, or on the function $g$. It will however most of the times depend on the Lebesgue index $p$.

We will denote by $B_R$ an arbitrary ball of radius $R$. We use the following notation for averaged integrals

$$\|F\|_{L_p^p(w_B)} = \left( \frac{1}{|B|} \int |F|^p w_B \right)^{1/p}.$$ 

$|A|$ will refer to either the cardinality of $A$ if $A$ is finite, or to its Lebesgue measure if $A$ has positive measure.

Acknowledgements. The author thanks Ciprian Demeter for reading this paper and giving several very useful suggestions. The authors also thanks Sean Prendiville for discussions on the applications of our main result.

2. Brascamp-Lieb inequalities and ball-inflation lemmas

Let $m$ be a positive integer. For $1 \leq j \leq m$, let $V_j$ be a $d$-dimensional linear subspace of $\mathbb{R}^n$. Let also $\pi_j : \mathbb{R}^n \to V_j$ denote the orthogonal projection onto $V_j$. Define

$$\Lambda(f_1, f_2, \ldots, f_m) = \int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\pi_j(x)) dx, \quad (2.1)$$

for $f_j : V_j \to \mathbb{C}$. We recall the following theorem due to Bennett, Carbery, Christ and Tao [BCCT10].

Theorem 2.1 ([BCCT10]). Given $p \geq 1$, the estimate

$$|\Lambda(f_1, f_2, \ldots, f_m)| \lesssim \prod_{j=1}^m \|f_j\|_p \quad (2.2)$$
holds if and only if \( np = dm \) and the following Brascamp-Lieb transversality condition is satisfied

\[
dim(V) \leq \frac{1}{p} \sum_{j=1}^{m} \dim(\pi_j(V)), \text{ for each linear subspace } V \subset \mathbb{R}^n. \tag{2.3}
\]

An equivalent formulation of the estimate (2.2) is

\[
\left\| \prod_{j=1}^{m} g_j \circ \pi_j \right\|_{L^q} \lesssim \left( \prod_{j=1}^{m} \|g_j\|_2 \right)^{1/m}, \tag{2.4}
\]

with \( q = \frac{2n}{d} \). The restriction that \( p \geq 1 \) becomes \( dm \geq n \). In our proof, \( m \) will always be a large constant, hence this condition is always satisfied. The transversality condition (2.3) becomes

\[
dim(V) \leq \frac{n}{dm} \sum_{j=1}^{m} \dim(\pi_j(V)), \text{ for each subspace } V \subset \mathbb{R}^n. \tag{2.5}
\]

Let us be more precise about the parameters in (2.5). We will take \( n = 5 \) as our surface \( S \) lives in \( \mathbb{R}^5 \). The degree \( m \) of multi-linearity will be chosen to be a large number. Our proof will make use of two different values of the parameter \( d \): First of all, we will use \( d = 2 \), which corresponds to that the surface \( S \) is two-dimensional; secondly, we also need to use \( d = 4 \), as at certain stage of the proof, we will view \( S \) as a four-dimensional surface in \( \mathbb{R}^5 \). For instance, see Lemma 2.5 in Subsection 2.3.

Recall that the surface we are looking at is \((t, s, \Phi_t(t, s), \Phi_s(t, s), \Phi(t, s))\). Its tangent space is spanned by

\[
n_1 = (1, 0, \Phi_{tt}, \Phi_{st}, \Phi_t) \text{ and } n_2 = (0, 1, \Phi_{ts}, \Phi_{ss}, \Phi_s). \tag{2.6}
\]

Moreover, we denote

\[
n_3 = (0, 0, 1, 0, t) \text{ and } n_4 = (0, 0, 0, 1, s). \tag{2.7}
\]

We will see from the following Lemma 2.3 that these two vectors span the “second order tangent space”. At a point \( \xi \in [0, 1]^2 \), let \( V_{\xi}^{(1)} \) be the linear space spanned by \( n_1(\xi) \) and \( n_2(\xi) \) given in (2.6). Let \( V_{\xi}^{(2)} \) be the linear space spanned by \( n_1, n_2, n_3, n_4 \) at the point \( \xi \).

Let \( K \in \mathbb{N} \) be a large number. It will be sent to infinity at the end of our proof. A \( K \)-square is defined to be a closed square of length \( 1/K \) inside \([0,1]^2\). The collection of all dyadic \( K \)-squares will be denoted by \( \text{Col}_K \).

**Proposition 2.2.** Take \( \Lambda \in \mathbb{N} \). Denote \( m = \Lambda K \). Let \( R_1, R_2, \ldots, R_m \) be different \( K \)-squares from \( \text{Col}_K \). For each \( 1 \leq i \leq m \), choose one point \( \xi_i \in R_i \). If we choose \( \Lambda \) sufficiently large, independently on any parameter, then the transversality condition (2.5) with \((d, n) = (2, 5)\) (respectively \((4, 5)\)) is satisfied for the collection of spaces \( \{V_{\xi_i}^{(1)}\}_{j=1}^{m} \) (respectively \( \{V_{\xi_i}^{(2)}\}_{j=1}^{m} \)).

We will prove Proposition 2.2 in the following two subsections. How to check the Brascamp-Lieb transversality condition (2.5) seems to have become a big obstacle in obtaining new decoupling inequalities associated with surfaces of high co-dimensions. For instance see [BDG16-2] where a particular two dimensional surface in \( \mathbb{R}^9 \) is considered. The forthcoming argument that corresponds to the case \( d = 4 \) in Subsection 2.2 further develops the idea introduced in [BDG16-2]. From our argument, in particular Lemma 2.3 it will become clear that more algebraic structures need to understood in order to push our current results to homogeneous polynomials of degrees higher than three.
2.1. Proof of Proposition 2.2: The case $d = 2$. In this subsection we prove the first part of Proposition 2.2. Let $\pi_\xi^{(1)}(V)$ denote the projection of the space $V$ on $V_\xi^{(1)}$. We will show that

$$\dim(V) \leq \frac{5}{2} \dim(\pi_\xi^{(1)}(V))$$

almost surely in $\xi$. \hfill (2.8)

Let us assume (2.8) for a moment and see how it implies the transversality condition (2.5). First of all, if we define an exceptional set

$$E_V := \{ \xi \in [0,1]^2 : \dim(V) > \frac{5}{2} \dim(\pi_\xi^{(1)}(V)) \},$$

then (2.8) implies that $E_V$ lies inside the zero set of a polynomial of degree less than 10. However, Wongkew’s lemma \cite{Won93} says that the 5-neighbourhood of the zero set of such a polynomial will intersect at most $C\mathcal{K}$ squares in $Col\mathcal{K}$ for some large constant $C$. The desired transversality condition (2.5) follows immediately if we choose $\Lambda = 100\mathcal{K}$.

**Case** $\dim(V) = 1$ or 2. The desired estimate (2.8) is reduced to

$$\dim(\pi_\xi^{(1)}(V)) = 1$$

almost surely. \hfill (2.10)

Suppose $V = \text{span}\{u\}$ with $u = (u_1, u_2, u_3, u_4, u_5)$. Then (2.10) is equivalent to

$$(u \cdot n_1, u \cdot n_2) \neq (0,0).$$

We argue by contradiction. Suppose $(u \cdot n_1, u \cdot n_2) = (0,0)$ for every $\xi \in [0,1]^2$. By checking the constant terms in the polynomials $u \cdot n_1$ and $u \cdot n_2$, we obtain $u_1 = u_2 = 0$. By checking the highest order terms, we obtain $u_5 = 0$. These two facts further imply that the cross product

$$(\Phi_{tt}, \Phi_{st}) \times (\Phi_{ts}, \Phi_{ss})$$

is constantly zero. However, by a direct calculation, this contradicts to the assumption that the polynomial $\Phi$ is non-degenerate.

**Case** $\dim(V) = 3$ or 4. We need to show that $\dim(\pi_\xi^{(1)}(V)) \geq 2$ almost surely. This is done via a direct calculation. Clearly the case $\dim(V) = 3$ is more difficult. Suppose $V = \text{span}\{u, v, w\}$. Then the dimension of $\pi_\xi^{(1)}(V)$ is equal to the rank of the matrix

$$\begin{pmatrix}
    u \cdot n_1 & v \cdot n_1 & w \cdot n_1 \\
    u \cdot n_2 & v \cdot n_2 & w \cdot n_2
\end{pmatrix}$$

(2.13)

We argue by contradiction and suppose that the determinants of all the two by two minors vanish constantly. We look at the two by two minor formed by the first two columns. The determinant of the matrix

$$\begin{pmatrix}
    u_1 + u_3\Phi_{tt} + u_4\Phi_{st} + u_5\Phi_t & v_1 + v_3\Phi_{tt} + v_4\Phi_{st} + v_5\Phi_t \\
    u_2 + u_3\Phi_{ts} + u_4\Phi_{ss} + u_5\Phi_s & v_2 + v_3\Phi_{ts} + v_4\Phi_{ss} + v_5\Phi_s
\end{pmatrix}$$

(2.14)

vanishes constantly. Denote

$$d_{i,j} := \det \begin{pmatrix} u_i & u_j \\ v_i & v_j \end{pmatrix}$$

(2.15)

We first look at the third order term, that is

$$d_{5,4}\Phi_{tt}\Phi_{ss} + d_{5,3}\Phi_{tt}\Phi_{ts} + d_{3,5}\Phi_{tt}\Phi_{ss} + d_{4,5}\Phi_{tt}\Phi_{ss}$$

$$= \left( d_{3,5} \frac{\partial}{\partial \Phi_t} \Phi_t + d_{4,5} \frac{\partial}{\partial \Phi_s} \Phi_s \right) \Phi_s^2 = 0.$$ \hfill (2.16)

This further implies $d_{3,5} = d_{4,5} = 0$. Moreover, we know $d_{1,2} = 0$ by checking the constant term of the determinant of the matrix (2.14). This further implies that

$$(u_5, v_5, w_5) = (0,0,0),$$

(2.17)
as otherwise we would derive a contradiction that \((u, v, w)\), when viewed as a matrix of order \(3 \times 5\), has rank two or smaller.

Substitute the identity (2.17) into (2.14), and look at the second order term of the determinant of (2.14). We obtain

\[
d_{3,4} \Phi_t \Phi_{ss} - d_{3,4} \Phi_{st} \Phi_{st} \equiv 0. \tag{2.18}
\]

By the non-degeneracy assumption on \(\Phi\), we obtain that \(d_{3,4} = 0\). This, together with (2.17) and \(d_{1,2} = 0\), implies that the \(3 \times 5\) matrix \((u, v, w)\) has rank two or smaller. Contradiction.

### 2.2. Proof of Proposition 2.2

The case \(d = 4\). We let \(\pi^{(2)}_\xi(V)\) denote the projection of the space \(V\) on \(V^{(2)}_\xi\). We need to show that

\[
\dim(V) \leq \frac{5}{4} \dim(\pi^{(2)}_\xi(V)) \text{ almost surely.} \tag{2.19}
\]

This amounts to calculating the dimension of

\[
\{(u \cdot n_1, u \cdot n_2, u \cdot n_3, u \cdot n_4) : u \in V\}. \tag{2.20}
\]

Following [BDG16-2], we define linear spaces

\[
S_1 = [t, s]; S_2 = [\Phi_t(t, s), \Phi_s(t, s)] \text{ and } S_3 = [\Phi(t, s)]. \tag{2.21}
\]

We need the following version of Taylor’s formula.

**Lemma 2.3.** If \(f \in S_3\), then

\[
\Delta f(t, s) \approx f_t(t, s) \Delta t + f_s(t, s) \Delta s + t \cdot f_t(\Delta t, \Delta s) + s \cdot f_s(\Delta t, \Delta s). \tag{2.22}
\]

Here \(\Delta f(t, s) = f(t + \Delta t, s + \Delta s) - f(t, s)\). The error produced by the approximate identity is a third order homogeneous polynomial in \(\Delta t\) and \(\Delta s\).

**Proof.** By linearity, it suffices to consider \(f(t, s) = \Phi(t, s)\). We calculate \(\Phi(t + \Delta t, s + \Delta s) - \Phi(t, s)\) and view it as a homogeneous polynomial of four variables \(t, s, \Delta t\) and \(\Delta s\). First, we collect the linear terms with respect to \(\Delta t\) and \(\Delta s\). By the first order Taylor expansion, they are given by \(\Phi_t(t, s) \Delta t + \Phi_s(t, s) \Delta s\), which are the former two terms on the right hand side of (2.22). Next, we collect the quadratic terms with respect to \(\Delta t\) and \(\Delta s\). These terms must be linear in the variables \(t\) and \(s\). We apply the first order Taylor expansion again, with the roles of \((t, s)\) and \((\Delta t, \Delta s)\) exchanged, and obtain \(t \Phi_t(\Delta t, \Delta s) + s \Phi_s(\Delta t, \Delta s)\), which gives the latter two terms in (2.22). \(\square\)

This lemma can be written in the following equivalent way.

\[
f(t, s) - f(t_0, s_0) \approx f_t(t_0, s_0)(t - t_0) + f_s(t_0, s_0)(s - s_0) + t_0 \cdot f_{tt}(t - t_0, s - s_0) + s_0 \cdot f_{ss}(t - t_0, s - s_0). \tag{2.23}
\]

According to this formula, let us consider

\[
f(t, s) = u_1 t + u_2 s + u_3 \Phi_t(t, s) + u_4 \Phi_s(t, s) + u_5 \Phi(t, s). \tag{2.24}
\]

At each point \(\xi = (t_0, s_0)\), denote \(\Delta t = t - t_0\) and \(\Delta s = s - s_0\). We define

\[
(P_\xi f)(t, s) = f(\xi) + f_t(\xi) \cdot \Delta t + f_s(\xi) \cdot \Delta s + (u_3 + u_5 t_0) \Phi_t(\Delta t, \Delta s) + (u_4 + u_5 s_0) \Phi_s(\Delta t, \Delta s). \tag{2.25}
\]

Here we observe that

\[
P_\xi f = f \text{ for } f \in S_1 \oplus S_2. \tag{2.26}
\]
We further define the canonical projection $\pi_{S_1 \oplus S_2}$ onto the space $S_1 \oplus S_2$. Hence
\[(\pi_{S_1 \oplus S_2} P_f)(t, s) = (f_t(\xi) + (u_3 + u_5 t_0)\Phi_{tt}(-t_0, -s_0) + (u_4 + u_5 s_0)\Phi_{ts}(-t_0, -s_0))t
+ (f_s(\xi) + (u_3 + u_5 t_0)\Phi_{ts}(-t_0, -s_0) + (u_4 + u_5 s_0)\Phi_{ss}(-t_0, -s_0))s
+ (u_3 + u_5 t_0)\Phi_t(t, s) + (u_4 + u_5 s_0)\Phi_s(t, s).
\] (2.27)

We can write
\[\pi_{S_1 \oplus S_2} P_f = (u_1 + u_5 \Phi_t(\xi) - u_5 t_0 \Phi_{tt}(\xi) - u_5 s_0 \Phi_{st}(\xi),
\]
\[u_2 + u_5 \Phi_s(\xi) - u_5 t_0 \Phi_{ts}(\xi) - u_5 s_0 \Phi_{ss}(\xi), u_3 + u_5 t_0, u_4 + u_5 s_0).
\] (2.28)

Recall the choice of the function $f$ in (2.24). Let us compare the vector in (2.28) with the vector in (2.20), which is given by
\[(u_1 + u_3 \Phi_{tt} + u_4 \Phi_{st} + u_5 \Phi_t, u_2 + u_3 \Phi_{ts} + u_4 \Phi_{ss} + u_5 \Phi_s, u_3 + u_5 t_0, u_4 + u_5 s_0).
\] (2.29)

By some simple row and column transformations, we see that
\[\dim\left(\{(u, v, n_1, u, n_2, u, n_3, u, n_4) : u \in V\}\right)
\]
\[= \dim\left(\{\pi_{S_1 \oplus S_2} P_f : f \in V\}\right).
\] (2.30)

Hence what we need to show becomes
\[\dim(V) \leq \frac{5}{4} \dim(\{\pi_{S_1 \oplus S_2} P_f : f \in V\}) \text{ almost surely.}
\] (2.31)

**Case** $\dim(V) = 1$. The is the same as the case $\dim(V) = 1$ and $d = 2$.

**Case** $\dim(V) = 2$. We need to show that $\dim(\pi_{S_1 \oplus S_2} P_f(V)) = 2$ almost surely. Argue by contradiction. Suppose $\dim(\pi_{S_1 \oplus S_2} P_f(V)) \leq 1$ everywhere. Then
\[V = \pi_{S_1 \oplus S_2}(V) \oplus S_3.
\] (2.32)

Let us calculate the projection of $S_3$ on $S_1 \oplus S_2$. Take
\[f(t, s) = \Phi(t, s) = at^3 + bt^2 s + cts^2 + ds^3.
\] (2.33)

Hence
\[\pi_{S_1 \oplus S_2} P_f = (-3at^2 - 2b t_0 s_0 - cs^2, -bt^2 - 2ct_0 s_0 - 3ds^2, t_0, s_0).
\] (2.34)

As we know that
\[V = \pi_{S_1 \oplus S_2}(V) \oplus S_3,
\] (2.35)

if we write $\pi_{S_1 \oplus S_2}(V) = \text{span}\{u\}$ with $u = (u_1, u_2, u_3, u_4)$, then the dimension of $\pi_{S_1 \oplus S_2} P_f(V)$ is equal to the rank of the matrix
\[
\begin{pmatrix}
-3at^2 - 2bt_0 s_0 - cs^2 & -bt^2 - 2ct_0 s_0 - 3ds^2 & t_0 & s_0 \\
 u_1 & u_2 & u_3 & u_4
\end{pmatrix}
\] (2.36)

For every nonzero vector $u$, this matrix has rank two almost surely.

**Case** $\dim(V) = 3$. We need to show that $\dim(\pi_{S_1 \oplus S_2} P_f(V)) = 3$ almost surely. Suppose not. Then by taking $\xi = (0, 0)$, we obtain that $\dim(\pi_{S_1 \oplus S_2}(V)) = 2$. Moreover,
\[V = \pi_{S_1 \oplus S_2}(V) \oplus S_3.
\] (2.37)

Write $\pi_{S_1 \oplus S_2}(V) = \text{span}\{u, v\}$ with $u = (u_1, u_2, u_3, u_4)$ and $v = (v_1, v_2, v_3, v_4)$. We need to show that the matrix
\[
\begin{pmatrix}
-3at^2 - 2bt_0 s_0 - cs^2 & -bt^2 - 2ct_0 s_0 - 3ds^2 & t_0 & s_0 \\
 u_1 & u_2 & u_3 & u_4
\end{pmatrix}
\] (2.38)
has rank three almost surely. By calculating the determinants of all the $3 \times 3$ minors, it is not difficult to see that this is indeed the case.

**Case** $\dim(V) = 4$. We need to show that $\dim(\pi_{S_1 \otimes S_2} P_\xi(V)) = 4$ almost surely. Similar as above, we prove by contradiction. In the end, we need to show that the matrix

$$
\begin{pmatrix}
-3at_0^2 - 2bt_0s_0 - cs_0^2 & -bt_0^2 - 2ct_0s_0 - 3ds_0^2 & t_0 & s_0 \\
u_1 & u_2 & u_3 & u_4 \\
v_1 & v_2 & v_3 & v_4 \\
w_1 & w_2 & w_3 & w_4
\end{pmatrix}
$$

has rank four almost surely. Argue by contradiction. Suppose the determinant of the above matrix vanishes constantly. By checking the linear terms in $t_0$ and $s_0$, we obtain that

$$
\det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix} = \det \begin{pmatrix} u_1 & u_2 & u_4 \\ v_1 & v_2 & v_4 \\ w_1 & w_2 & w_4 \end{pmatrix} = 0.
$$

This implies that the two vectors $(u_1, v_1, w_1)$ and $(u_2, v_2, w_2)$ are linearly dependent. That the determinant of the matrix (2.39) vanishes constantly contradicts to the non-degeneracy of the polynomial $\Phi$.

2.3. Multi-linear Kakeya inequalities and ball-inflation lemmas. In Proposition 2.2, we verified a transversality condition. As a consequence, we have the following multi-linear Kakeya inequality.

**Lemma 2.4** (Multi-linear Kakeya). Let $d_1 = 2$ and $d_2 = 4$. For every $\iota \in \{1, 2\}$, we have the following estimate: Let $M = \Lambda K$ where $\Lambda$ is the same as the one in Proposition 2.2. Let $R_1, \ldots, R_M$ be different sets from $Col_K$. Consider $M$ families $P_{\iota}$ consisting of rectangular boxes $P$ in $\mathbb{R}^5$, that we refer to as plates, having the following properties
1) For each $P \in P_{\iota}$, there exists $\xi_{\iota} = (t_\iota, s_\iota) \in R_\iota$ such that $d_\iota$ sides of $P$ have lengths equal to $R^{1/2}$ and span $V_{\xi_{\iota}}^{(\iota)}$, while the remaining $(5 - d_\iota)$ sides have lengths $R$;
2) all plates are subsets of a ball $B_{4R}$ of radius $4R$.

Then we have the following inequality

$$
\frac{1}{B_{4R}} \int_{B_{4R}} \left| \prod_{j=1}^M F_j \right|^{\frac{2}{M+2}} \lesssim_{\epsilon, \nu} R^\epsilon \left( \prod_{j=1}^M \left( \frac{1}{B_{4R}} \int_{B_{4R}} |F_j| \right)^{\frac{1}{M}} \right)^{\frac{5}{7}}
$$

(2.41)

for each function $F_j$ of the form

$$
F_j = \sum_{P \in P_{\iota}} c_P 1_P
$$

(2.42)

The implicit constant does not depend on $R$ or $c_P$.

Lemma 2.4 is essentially due to Guth [Guth15], and Bennett, Bez, Flock and Lee [BBFL16]. It follows from the Brascamp-Lieb inequalities in Theorem 2.1 via an induction-on-scales argument, after verifying the corresponding transversality conditions. Here we leave out the details. These multi-liner Kakeya inequalities have the following consequences.

**Lemma 2.5** (Ball-inflation lemmas). Let $d_1 = 2$ and $d_2 = 4$. For every $\iota \in \{1, 2\}$, we have the following estimate: Let $R_1, \ldots, R_M$ be different squares from $Col_K$. Let $B$ be an arbitrary ball in $\mathbb{R}^5$ of radius $\rho^{\iota-1}$. Let $B$ be a finitely overlapping cover of $B$ with balls $\Delta$ of radius
ρ−t. For each g : [0, 1]2 → C, we have

\begin{equation}
\frac{1}{|B|} \sum_{\Delta \in B} \left[ \prod_{i=1}^{M} \left( \sum_{R_i' \text{ square in } R_i \atop l(R_i') = \rho} \|E_{R_i'} g\|_{L^{\frac{q}{p}}_g(w_{\Delta})} \right)^{\frac{1}{p}} \right] \end{equation}

(2.43)

\begin{equation}
\lesssim_{\epsilon, \nu} \rho^{-\epsilon} \left[ \prod_{i=1}^{M} \left( \sum_{R_i' \text{ square in } R_i \atop l(R_i') = \rho} \|E_{R_i'} g\|_{L^{\frac{q}{p}}_g(w_{\Delta})} \right)^{\frac{1}{p}} \right]. \end{equation}

Proof. We will follow the proof in Bourgain, Demeter and Guth [BDG16], that is, we will derive Lemma 2.3 from Lemma 2.4. In order to apply Lemma 2.4 we need to check that the function \(|E_{R_i'} g\)| is essentially a constant on a plate whose two short sides span the linear space \(V_{(1)}\xi_j\) for some \(\xi_j \in R_j'\). Moreover, we need to check that for each ball \(\Delta\) of radius \(\rho^{-2}\), the function \(\|E_{R_i'} g\|_{L^{\frac{q}{p}}_g(w_{\Delta})}\) is essentially a constant on a plate whose four short sides span the linear space \(V_{(2)}\xi_j\) for some \(\xi_j \in R_j'\). The former statement follows from the standard Taylor expansion, and the latter one follows from Lemma 2.3. We comment here that this is what we meant previously by viewing the surface \(S\) as a four dimensional surface in \(\mathbb{R}^5\). For the rest of the details, we refer to [BDG16].

The idea of ball-inflations originated from the work of Bourgain, Demeter and Guth [BDG16] (see Theorem 6.6 there). If we replace the \(l^{\frac{q}{p}}\) summations over \(R_i' \subset R_i\) on both sides of (2.43) by \(l^2\) sums, essentially we arrive at the ball-inflation estimates that are proven in [BDG16]. Moreover, as has been pointed out above, the proof of Theorem 6.6 in [BDG16] also works for (2.43). However there are some subtle differences when it comes to applying these ball-inflation estimates in the iteration argument in Section 5.

In [BDG16], the authors there used \(l^2\) sums over \(R_i' \subset R_i\), in order to prove certain sharp \(l^2 L^p\) decoupling inequalities associated to moment curves. In our case, sharp \(l^2 L^p\) decouplings are no longer able to imply good enough estimates as in Theorem 1.1. Indeed, this issue already appeared in earlier attempts of trying to push the argument of [BDG16] to higher dimensions, see [BDG16-2] and [GZ18] (see [BD16-2] for an even earlier work). In the present paper, we follow the way in which ball-inflation lemmas are formulated in the work [GZ18] by Zhang and the author.

3. Parabolic rescaling

In this section we state the following result which is referred to as parabolic rescaling.

Lemma 3.1. Let \(0 < \delta < \sigma \leq 1\). Then for each square \(R \subset [0, 1]^2\) with side length \(\sigma\) and each ball \(B \subset \mathbb{R}^5\) with radius \(\delta^{-3}\) we have

\begin{equation}
\|E_{Rg}\|_{L^p(w_B)} \leq B_{p,q}(\frac{\delta}{\sigma}) \left( \sum_{R' \subset R: l(R') = \delta} \|E_{R'g}\|_{L^p(w_B)} \right)^{1/q}.
\end{equation}

(3.1)

The proof of this lemma is standard, see for instance Proposition 7.1 from [BD16-2]. One just needs to observe that our surface \(S\) is translation and dilation invariant, as can be seen via Lemma 2.3.

The parabolic rescaling lemma plays a determinant role in decoupling theory. It is used in every iteration step. First of all, it is used to run the Bourgain-Guth scheme, in order to show the equivalence between the linear and multilinear decoupling inequalities (Theorem 4.4).
Secondly, it is used in the iteration scheme in Section 5 to conclude the desired decoupling inequality (1.8).

### 4. Linear versus multilinear decoupling

In this section we introduce a multi-linear version of the desired decoupling inequality. Recall that $K$ is a large number and $M = \Lambda K$. We denote by $B_{p,q}(\delta, K)$ the smallest constant such that

$$
\left\| \left( \prod_{i=1}^{M} E_{R_i} g \right)^{1/M} \right\|_{L^p}(w_B) \leq B_{p,q}(\delta, K) \prod_{i=1}^{M} \left( \sum_{R_i' \subset R_i, l(R_i') = \delta} \left\| E_{R_i'} g \right\|_{L^q}(w_B) \right)^{1/M}.
$$

(4.1)

holds true for all distinct squares $R_i \in Col_K$, each ball $B \subset \mathbb{R}^9$ of radius $\delta - 3$, and each $g : [0, 1]^2 \to \mathbb{C}$.

By Hölder’s inequality, we see that the multi-linear decoupling constant $B_{p,q}(\delta, K)$ can be controlled by the linear decoupling constant $B_{p,q}(\delta)$. It turns out that, in the case $p = q$, the reverse direction also essentially holds true. That is,

**Theorem 4.1.** For each $p \geq 2$ and $K \in \mathbb{N}$, there exists $\Omega_{K,p} > 0$ and $\beta(K, p) > 0$ with

$$
\lim_{K \to \infty} \beta(K, p) = 0,
$$

such that for each small enough $\delta$, we have

$$
B_{p,p}(\delta) \leq \delta^{-\beta(K,p)} 2^{(\frac{1}{2} - \frac{1}{p})} + \Omega_{K,p} \log_K \left( \frac{1}{\delta} \right) \max_{\delta \leq \delta' \leq 1} \left( \frac{\delta'}{\delta} \right)^{2(\frac{1}{2} - \frac{1}{p})} B_{p,p}(\delta', K).
$$

(4.3)

The proof of this theorem is standard, and is essentially the same as that of Theorem 8.1 from [BD16-2]. Hence we leave it out.

### 5. Iteration

In this section, we run the final iteration argument. The consequence of this iteration, combined with Theorem 4.1, will lead to the desired decoupling inequality (1.8).

There will be two terms that are involved in the iteration procedure. They are

$$
D_p(q, B^r) := \left( \prod_{i=1}^{M} \sum_{J_{i,q} \subset R_i} \left\| E_{J_{i,q}} g \right\|_{L^p}(w_B)}^{p} \right)^{1/p}.
$$

(5.1)

and

$$
A_p(q, B^r, s) = \left( \frac{1}{|B_s(B^r)|} \sum_{B_r \in B_s(B^r)} D_2(q, B^r)^p \right)^{1/p}.
$$

(5.2)

Here for a positive number $r$, we use $B^r$ to denote a ball of radius $\delta^{-r}$, and $B_s(B^r)$ denotes a finitely overlapping collection of balls $B^s$ that lie inside of a ball $B^r$. In the notation $J_{i,q}$, the index $i$ indicates that this square lies in $R_i$, and $q$ indicates that the square $J_{i,q}$ has side length $\delta^q$.

Define $\alpha_1, \alpha_2, \beta_2 \in (0, 1)$ as follows

$$
\frac{1}{2p} = \frac{\alpha_1}{2p} + \frac{1 - \alpha_1}{2},
$$

$$
\frac{1}{4p} = \frac{\alpha_2}{p} + \frac{1 - \alpha_2}{6},
$$

$$
\frac{1}{6} = \frac{1 - \beta_2}{2} + \frac{\beta_2}{2p}.
$$
We will start our iteration with the term
\[ A_p(1, B^3, 1) = \left( \frac{1}{|B_1(B^3)|} \sum_{B^1 \in B_1(B^3)} D_2(1, B^1)^p \right)^{1/p}. \]  
(5.3)

By Hölder’s inequality, it can be bounded by
\[ \delta^{-2(\frac{1}{2} - \frac{2}{p})} \left( \frac{1}{|B_1(B^3)|} \sum_{B^1 \in B_1(B^3)} D_2(1, B^1)^p \right)^{1/p}. \]  
(5.4)

We apply Lemma 2.5 with \( \iota = 1 \) to (5.4) and bound it by
\[ \delta^{-2(\frac{1}{2} - \frac{2}{p})-\iota} \left( \frac{1}{|B_2(B^3)|} \sum_{B^2 \in B_2(B^3)} D_2(1, B^2)^p \right)^{1/p}. \]  
(5.5)

By Hölder’s inequality, the right hand side of (5.5) can be dominated by
\[ \delta^{-2(\frac{1}{2} - \frac{2}{p})-\iota} \left( \frac{1}{|B_2(B^3)|} \sum_{B^2 \in B_2(B^3)} D_2(1, B^2)^p \right)^{\frac{1}{1+\alpha}} \frac{\alpha}{2} A_p(2, B^3, 2)^{1-\alpha}. \]  
(5.6)

In the next step, we apply Lemma 2.5 with \( \iota = 2 \) and obtain
\[ \delta^{-2(\frac{1}{2} - \frac{2}{p})-\iota} D_2(1, B^3)^{\alpha_1} A_p(2, B^3, 2)^{1-\alpha_1}. \]  
(5.7)

The last term \( A_p(2, B^3, 2)^{1-\alpha_1} \) is ready for iteration. We further process the \( D \)-term. By Hölder’s inequality
\[ D_2(1, B^3) \lesssim D_6(1, B^3)^{1-\alpha_2} D_p(1, B^3)^{\alpha_2}. \]  
(5.9)

The second term on the right hand side is already of the form of the term in the decoupling inequality \(|\mathcal{E}|\), and it will not be further processed. It is the former term on the right hand side that needs further process.

Notice that in the term \( D_6(1, B^3) \), we are dealing with terms \( \|E_{j, k} g\|_{L^6(\mathbb{R}^3)} \). By the uncertainty principle, such a ball of radius \( \delta^{-3} \) is not able to distinguish the surface \( S \) from
\[ \{(t, s, \Phi_4(t, s), \Phi_5(t, s), 0) : (t, s) \in J_{4, 1}\} \]  
(5.10)

under certain affine transformations. By the \( L^6 \) decoupling estimate for the surface \(|\mathcal{E}|\) obtained in [BD16-1], we obtain
\[ D_6(1, B^3) \lesssim \delta^{-\frac{1}{4} - \frac{1}{p}} D_6(\frac{3}{2}, B^3). \]  
(5.11)

By Hölder’s inequality, this can be further bounded by
\[ D_6(1, B^3) \lesssim \delta^{-\frac{1}{4} - \frac{1}{p}} D_6(\frac{3}{2}, B^3)^{1-\beta_2} D_2(\frac{3}{2}, B^3)^{\beta_2} \lesssim \delta^{-\frac{1}{4} - \frac{1}{p}} D_2(3, B^3)^{1-\beta_2} D_6(\frac{3}{2}, B^3)^{\beta_2}. \]  
(5.12)

In the last step, we applied \( L^2 \) orthogonality. In the end, we have obtained so far can be organised as
\[ A_p(1, B^3, 1) \lesssim \delta^{-2\left(\frac{1}{2} - \frac{2}{p}\right) - \iota} A_p(2, B^3, 2)^{1-\alpha_1} A_p(3, B^3, 3)^{\alpha_1(1-\alpha_2)(1-\beta_2)} D_2(\frac{3}{2}, B^3)^{\alpha_1(1-\alpha_2)\beta_2} D_p(1, B^3)^{\alpha_1\alpha_2}. \]  
(5.13)
Now we run this iteration procedure for \( r \) many times. For all balls \( B \) of radius \( \delta^{-2} \left( \frac{3}{2} \right)^r \), we have

\[
A_p(1, B, 1) \lesssim_{\varepsilon, \delta} \left( \frac{1}{\delta} \right)^{\epsilon + 2(\frac{1}{2} - \frac{5}{2p})} \times \prod_{i=0}^{r-1} \left( \frac{1}{\delta} \right)^{\left( \frac{1}{2} - \frac{1}{6} \right) \alpha_1 (1 - \alpha_2) [(1 - \alpha_2) \beta_2]^i} \times A_p(2, B, 2)^{(1 - \alpha_2) \beta_2} \left( \frac{3}{2} \right)^r, B \right)^{\alpha_1 (1 - \alpha_2) \beta_2} \tag{5.14}
\]

Define

\[
\gamma_0 = 1 - \alpha_1; \gamma_i = \alpha_1 (1 - \alpha_2) [(1 - \alpha_2) \beta_2]^i, \text{ for } 1 \leq i \leq r; \\
b_i = 2 \cdot \left( \frac{3}{2} \right)^i, \text{ for } 0 \leq i \leq r; \\
\tau_r = \alpha_1 [(1 - \alpha_2) \beta_2]^r; \tau_i = \alpha_1 \alpha_2 [(1 - \alpha_2) \beta_2]^i, \text{ for } 0 \leq i \leq r - 1; \\
w_i = \frac{1 - \alpha_2}{2 \alpha_2} \tau_i, \text{ for } 0 \leq i \leq r - 1.
\]

We can write using Hölder

\[
D_{\frac{3}{2}} \left( \frac{3}{2} \right)^r, B \right) \lesssim D_p \left( \frac{3}{2} \right)^r, B \right).
\]

With these, the estimate \( (5.13) \) becomes

\[
A_p(1, B, 1) \lesssim_{\varepsilon, \delta} \left( \frac{1}{\delta} \right)^{\epsilon + 2(\frac{1}{2} - \frac{5}{2p})} \left( \prod_{i=0}^{r-1} \left( \frac{1}{\delta} \right)^{\left( \frac{1}{2} - \frac{1}{6} \right) b_i w_i} \right) \times \left( \prod_{i=0}^{r} A_p(b_i, B, b_i)^{\gamma_i} \right) \left( \prod_{i=0}^{r} D_p(b_i, B)^{\tau_i} \right) \tag{5.16}
\]

Using this and a simple rescaling argument, we can rewrite \( (5.14) \) as follows

\[
A_p(u, B, u) \lesssim_{\varepsilon, \delta} \left( \frac{1}{\delta} \right)^{\epsilon + 2u(\frac{1}{2} - \frac{5}{2p})} \left( \prod_{i=0}^{r-1} \left( \frac{1}{\delta} \right)^{\left( \frac{1}{2} - \frac{1}{6} \right) u b_i w_i} \right) \times \left( \prod_{i=0}^{r} A_p(b_i u, B, b_i u)^{\gamma_i} \right) \left( \prod_{i=0}^{r} D_p(b_i u, B)^{\tau_i} \right) \tag{5.17}
\]

Here \( B \) stands for a ball of radius \( \delta^{-3} \), and \( u \) is a sufficiently small positive constant such that \( u \cdot \left( \frac{3}{2} \right)^r \leq 1 \).

In the end, we iterate \( (5.17) \). To iterate, we will dominate each \( A_p(u b_i, B, u b_i) \) again by using \( (5.17) \). To enable such an iteration, we need to choose \( u \) to be even smaller. Let \( M \) be a large integer. Choose \( u \) such that

\[
\left[ 2 \left( \frac{3}{2} \right)^r \right]^M u \leq 2. \tag{5.18}
\]

This allows us to iterate \( (5.17) \) \( M \) times. To simplify the iteration, we bound all the powers of \( \frac{1}{\delta} \) by

\[
2u \left( \frac{1}{2} - \frac{5}{2p} \right) + \left( \sum_{i=0}^{\infty} u \left( \frac{1}{2} - \frac{1}{6} \right) b_i w_i \right). \tag{5.19}
\]
By a direct calculation,
\[ \sum_{j=0}^{\infty} b_j w_j = \frac{3(-5 + p)}{2(15 - 10p + p^2)}. \tag{5.20} \]
Moreover,
\[ \sum_{j=0}^{\infty} b_j \tau_j = \frac{75 - 25p + 2p^2}{15 - 10p + p^2}. \tag{5.21} \]
If we define
\[ \lambda_0 := 2\left(1 - \frac{5}{2p}\right) + \left(1 - \frac{1}{6}\right) \frac{3(-5 + p)}{2(15 - 10p + p^2)}, \tag{5.22} \]
then (5.17) can be rewritten as follows
\[ A_p(u, B, u) \lesssim r, \epsilon \delta^{-\epsilon - u \lambda_0} \left( \prod_{i=0}^{r} A_p(ub_i, B, ub_i)^{\gamma_i} \right) \left( \prod_{i=0}^{r} D_p(\frac{ub_i}{2}, B)^{\tau_i} \right), \tag{5.23} \]
for every ball $B$ of radius $\delta^{-3}$. Now we have arrived precisely at the estimate (6.51) from [BDG16-2]. The calculation there, from page 27 to page 30, can be repeated line by line. In the end, we obtain that
\[ \log_2 B_{g_9}(\delta) \leq \lim_{p \to \infty} \frac{\lambda_0(p)}{2(\sum_{j=0}^{\infty} b_j \tau_j(p))}. \tag{5.24} \]
By plugging in the calculation (5.20)–(5.21), we will be able to conclude the desired decoupling inequality \[ \leq \].

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