Multiple solutions
for $p$-Laplacian type problems
with asymptotically $p$-linear terms
via a cohomological index theory

A.M. Candela∗,1, G. Palmieri∗,2, K. Perera†
∗Dipartimento di Matematica
Università degli Studi di Bari “Aldo Moro”
Via E. Orabona 4, 70125 Bari, Italy
1annamaria.candela@uniba.it, 2giuliana.palmieri@uniba.it
†Department of Mathematical Sciences
Florida Institute of Technology
150 W. University Blvd, Melbourne, FL 32901, USA
kperera@fit.edu

Abstract
The aim of this paper is investigating the existence of weak solutions
of the quasilinear elliptic model problem

$$
\begin{align*}
-\text{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p} A_t(x,u)|\nabla u|^p &= f(x,u) \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \partial \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $p > 1$, $A$ is a given function
which admits partial derivative $A_t(x,t) = \frac{\partial A}{\partial t}(x,t)$ and $f$ is asymptotically
$p$-linear at infinity.

Under suitable hypotheses both at the origin and at infinity, and if
$A(x,\cdot)$ is even while $f(x,\cdot)$ is odd, by using variational tools, a cohomological index theory and a related pseudo–index argument, we prove a
multiplicity result if $p > N$ in the non–resonant case.

2000 Mathematics Subject Classification. 35J35, 35J60, 35J92, 47J30, 58E05.
Key words. $p$-Laplacian type equation, asymptotically $p$-linear problem, Palais–Smale condition, cohomological index theory, pseudo–index theory.

∗The authors acknowledge the support of Research Funds PRIN2009 and Fondi d’Ateneo 2010.
†This work was done while the third–named author was visiting the Dipartimento di Matematica at the Università degli Studi di Bari, and he is grateful for the kind hospitality of the department.
1 Introduction

Let us consider the $p$-Laplacian type equation

\[
(P) \begin{cases} 
-\operatorname{div}(A(x,u)|\nabla u|^{p-2}\nabla u) + \frac{1}{p} A_t(x,u)|\nabla u|^p = f(x,u) & \text{in } \Omega, \\
  u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $N \geq 2$, $p > 1$, $A, f : \Omega \times \mathbb{R} \to \mathbb{R}$ are given functions such that the partial derivative $A_t(x,t) = \frac{\partial A}{\partial t}(x,t)$ exists for a.e. $x \in \Omega$, all $t \in \mathbb{R}$.

If we set $F(x,t) = \int_0^t f(x,s) ds$, we can associate with problem $(P)$ the functional $\mathcal{J} : D \subset W^{1,p}_0(\Omega) \to \mathbb{R}$ defined by

\[
\mathcal{J}(u) = \frac{1}{p} \int_{\Omega} A(x,u) |\nabla u|^p \, dx - \int_{\Omega} F(x,u) \, dx. \tag{1.1}
\]

In general, if no growth assumption is made on $A$ with respect to $t$, the natural domain $D$ of $\mathcal{J}$ is contained in, but is not equal to, the Sobolev space $W^{1,p}_0(\Omega)$. Anyway, under the assumptions

$(H_0)$ $A, A_t$ are Carathéodory functions on $\Omega \times \mathbb{R}$ such that

\[
\sup_{|t| \leq r} |A(\cdot,t)| \in L^\infty(\Omega), \quad \sup_{|t| \leq r} |A_t(\cdot,t)| \in L^\infty(\Omega) \quad \text{for any } r > 0;
\]

$(h_0)$ $f$ is a Carathéodory function on $\Omega \times \mathbb{R}$ such that

\[
\sup_{|t| \leq r} |f(\cdot,t)| \in L^\infty(\Omega) \quad \text{for any } r > 0,
\]

the functional $\mathcal{J}$ is surely well-defined on the Banach space

\[
X := W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \quad \|u\|_X = \|u\| + |u|_\infty, \tag{1.2}
\]

with

\[
\|u\|^p = \int_{\Omega} |\nabla u|^p \, dx, \quad |u|_\infty = \operatorname{ess~sup}_{x \in \Omega} |u(x)|,
\]

and, for any $u, v \in X$, its Gâteaux derivative with respect to $u$ in the direction $v$ is given by

\[
(\langle d\mathcal{J}(u), v \rangle) = \int_{\Omega} A(x,u) |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx
\]

\[
+ \frac{1}{p} \int_{\Omega} A_t(x,u) |\nabla u|^p v \, dx - \int_{\Omega} f(x,u) \, v \, dx.
\]

As our aim is investigating the existence of weak solutions of $(P)$ when it is an asymptotically $p$-linear elliptic problem, we assume that $A$ and $f$ satisfy the following hypotheses:
there exists $\alpha_0 > 0$ such that

$$A(x,t) \geq \alpha_0 \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R};$$

(H2) there exists $A^\infty \in L^\infty(\Omega)$ such that

$$\lim_{|t| \to +\infty} A(x,t) = A^\infty(x) \quad \text{uniformly a.e. in } \Omega;$$

(h1) there exist $\lambda^\infty \in \mathbb{R}$ and a (Carathéodory) function $g^\infty : \Omega \times \mathbb{R} \to \mathbb{R}$ such that

$$f(x,t) = \lambda^\infty |t|^{p-2} t + g^\infty(x,t),$$

where

$$\lim_{|t| \to +\infty} \frac{g^\infty(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly a.e. in } \Omega. \quad (1.3)$$

As $J$ is a $C^1$-functional on $X$ under these hypotheses (see Proposition 3.1), we can seek weak solutions of $(P)$ by means of variational tools.

In the asymptotically linear case, i.e. under the hypotheses $(h_0)$ and $(h_1)$, a variational approach was first used for $p = 2$ and $A(x,t) \equiv 1$ (see the seminal papers [1, 5]). On the contrary, only a few results have been obtained when $p \neq 2$, but always for $A(x,t) \equiv 1$ or, at worst, for $A(x,t) = A(x)$ independent of $t$ (see [2, 4, 6, 10, 13, 21, 22, 23, 24, 25]). In fact, when $p > 1$ is arbitrary, the main difficulty is that, while the structure of the spectrum of $-\Delta$ in $H_0^1(\Omega)$ is known, the full spectrum of $-\Delta_p$ is still unknown, even though various authors have introduced different characterizations of eigenvalues and definitions of quasi–eigenvalues.

Clearly, the same problem arises in our setting when $A(x,t)$ depends on $t$. Furthermore, we have difficulties with the Palais–Smale condition as well, and have to consider the asymptotic behavior, both at the origin and at infinity, not only of the term $f(x,t)$, but also of the coefficient $A(x,t)$.

When $(h_1)$ is replaced with different conditions at infinity, weaker versions of the Palais–Smale condition hold for arbitrary $p > 1$, and the existence of critical points of $J$ in $X$ have been proved (see [10, 13]). However, these approaches do not distinguish between different critical points at the same critical level (see [11, 12]), and therefore, up to now, multiplicity results via a cohomological index theory have been obtained only for $p > N$ (see [9, 13]). In fact, in this case the Sobolev Imbedding Theorem implies $X = W_0^{1,p}(\Omega)$ and the classical Cerami’s variant of the Palais–Smale condition can be verified.

In this paper, we will prove a multiplicity result for problem $(P)$ when $p > N$ and $f(x,t)$ is asymptotically $p$-linear at infinity. To this aim, by considering some sequences of eigenvalues defined by means of the cohomological index, we will prove the classical Palais–Smale condition and, by means of a cohomological index theory and a related pseudo–index argument, we will extend the result
in \[25\] to our setting (see \[14\] for a result obtained by using the approach in \[5\]). In particular, let us point out that, if the coefficient \(A\) depends on \(t\), the boundedness of each Palais–Smale sequence of \(J\) requires a careful proof also in the non–resonant assumption, unlike the \(t\)–independent case (see Proposition \[3.5\]).

2 Abstract tools

The aim of this section is to recall the abstract tools we need for the proof of our main result. Hence, let \((B, \| \cdot \|_B)\) be a Banach space with dual space \((B', \| \cdot \|_{B'})\) and let \(J \in C^1(B, \mathbb{R})\).

Furthermore, fixing a level \(\beta \in \mathbb{R}\), a point \(u_0 \in B\), a set \(C \subset B\) and a radius \(r > 0\), let us denote

- \(K^J = \{ u \in B : dJ(u) = 0 \}\) the set of critical points of \(J\) in \(B\);
- \(K^J_\beta = \{ u \in B : J(u) = \beta, \ dJ(u) = 0 \}\) the set of critical points of \(J\) in \(B\) at the level \(\beta\) (clearly, \(K^J_\beta = \emptyset\) if \(\beta\) is a regular value);
- \(J^\beta = \{ u \in B : J(u) \leq \beta \}\) the sublevel set of \(J\) associated with \(\beta\);
- \(B^B_r(u_0) = \{ u \in B : \| u - u_0 \|_B \leq r \}\) the closed ball in \(B\) centered at \(u_0\) of radius \(r\), with boundary \(\partial B^B_r(u_0)\);
- \(\text{dist}_B(u, C) = \inf_{v \in C} \| v - u \|_B\) the distance from \(C\) to \(u \in B\).

We say that a sequence \((u_n)_n \subset B\) is a Palais–Smale sequence at the level \(\beta\), briefly a \((PS)_\beta\)–sequence, if

\[
J(u_n) \to \beta \quad \text{and} \quad \| dJ(u_n) \|_{B'} \to 0 \quad \text{as} \quad n \to +\infty.
\]

The functional \(J\) satisfies the Palais–Smale condition at the level \(\beta\) in \(B\), \((PS)_\beta\) condition for short, if every \((PS)_\beta\)–sequence admits a subsequence that converges in \(B\).

Now, we assume that \(J\) is even and \(J(0) = 0\), and use the \(\mathbb{Z}_2\)-cohomological index of Fadell and Rabinowitz in \[20\] and the associated pseudo-index of Benci in \[7\] to obtain multiple critical points.

Let us first recall the definition and some basic properties of the cohomological index.

Let \(A\) be the class of symmetric subsets of \(B \setminus \{0\}\). For \(A \in A\), we denote by

- \(\overline{A} = A/\mathbb{Z}_2\) the quotient space of \(A\) with each \(u\) and \(-u\) identified,
- \(f : \overline{A} \to \mathbb{R}P^\infty\) the classifying map of \(\overline{A}\),
- \(f^* : H^*(\mathbb{R}P^\infty) \to H^*(\overline{A})\) the induced homomorphism of the Alexander–Spanier cohomology rings.
Then the cohomological index of $A$ is defined by

$$i(A) = \begin{cases} 
\sup \{ m \geq 1 : f^*(\omega^{m-1}) \neq 0 \} & \text{if } A \neq \emptyset, \\
0 & \text{if } A = \emptyset,
\end{cases}$$

where $\omega \in H^1(\mathbb{R}P^\infty)$ is the generator of the polynomial ring $H^*(\mathbb{R}P^\infty) = \mathbb{Z}_2[\omega]$.

For example, if $S^{n-1}$ is the unit sphere in $\mathbb{R}^n$, $1 \leq n < +\infty$, then $i(S^{n-1}) = n$ as the classifying map of $S^{n-1}$ is the inclusion $\mathbb{R}P^{n-1} \subset \mathbb{R}P^\infty$, which induces isomorphisms on $H^q$ for $q \leq n - 1$.

**Proposition 2.1** (Fadell–Rabinowitz [20]). The index $i : \mathcal{A} \to \mathbb{N} \cup \{0, +\infty\}$ has the following properties:

(i$_1$) **Definiteness**: $i(A) = 0$ if and only if $A = \emptyset$;

(i$_2$) **Monotonicity**: If there is an odd continuous map from $A$ to $B$ (in particular, if $A \subset B$), then $i(A) \leq i(B)$. Thus, equality holds when the map is an odd homeomorphism;

(i$_3$) **Dimension**: $i(A) \leq \dim \mathcal{B}$;

(i$_4$) **Continuity**: If $A \in \mathcal{A}$ is closed, then there is a closed neighborhood $N \in \mathcal{A}$ of $A$ such that $i(N) = i(A)$. When $A$ is compact, $N$ may be chosen to be a $\delta$-neighborhood $N_\delta(A) = \{ u \in \mathcal{B} : \text{dist}(u, A) \leq \delta \}$;

(i$_5$) **Subadditivity**: If $A, B \in \mathcal{A}$ are closed, then $i(A \cup B) \leq i(A) + i(B)$;

(i$_6$) **Stability**: If $SA$ is the suspension of $A \neq \emptyset$, obtained as the quotient space of $A \times [−1, 1]$ with $A \times \{1\}$ and $A \times \{−1\}$ collapsed to different points, then $i(SA) = i(A) + 1$;

(i$_7$) **Piercing property**: If $A, A_0, A_1$ are closed and $\varphi : A \times [0, 1] \to A_0 \cup A_1$ is a continuous mapping such that $\varphi(−u, t) = −\varphi(u, t)$ for all $(u, t) \in A \times [0, 1]$, $\varphi(A \times [0, 1])$ is closed, $\varphi(A \times \{0\}) \subset A_0$, and $\varphi(A \times \{1\}) \subset A_1$, then $i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A)$;

(i$_8$) **Neighborhood of zero**: If $U$ is a bounded closed symmetric neighborhood of $0$, then $i(\partial U) = \dim \mathcal{B}$.

For any integer $k \geq 1$, let

$$\mathcal{A}_k = \{ A \in \mathcal{A} : A \text{ is compact and } i(A) \geq k \}$$

and set

$$c_k := \inf_{A \in \mathcal{A}_k} \max_{u \in A} J(u).$$

Since $\mathcal{A}_{k+1} \subset \mathcal{A}_k$, then $c_k \leq c_{k+1}$. Furthermore, for any $k$-dimensional subspace $V$ of $\mathcal{B}$ and $\delta > 0$, by (i$_8$) we have $\partial B^\delta_\delta(0) \cap V \in \mathcal{A}_k$, while by continuity it results

$$\sup_{u \in \partial B^\delta_\delta(0)} J(u) \to J(0) \text{ as } \delta \to 0,$$
so $c_k \leq J(0)$.

The following theorem is standard (see, e.g., [24, Proposition 3.36]).

**Theorem 2.2.** Assume that $J \in C^1(B, \mathbb{R})$ is even and $J(0) = 0$. If

$$-\infty < c_k \leq \cdots \leq c_{k+m-1} < 0$$

and $J$ satisfies $(PS)_{c_i}$ for $i = k, \ldots, k+m-1$, then $J$ has $m$ distinct pairs of nontrivial critical points.

Now, let us recall the definition and some basic properties of a pseudo–index related to the cohomological index $i$.

Let $A^*$ denote the class of symmetric subsets of $B$, let $\mathcal{M} \in A$ be closed, and let $\Gamma$ denote the group of odd homeomorphisms $\gamma$ of $B$ such that $\gamma|_{\mathcal{M}}$ is the identity. Then the pseudo-index of $A \in A^*$ related to $i$, $\mathcal{M}$, and $\Gamma$ is defined by

$$i^*(A) = \min_{\gamma \in \Gamma} i(\gamma(A) \cap \mathcal{M}).$$

**Proposition 2.3** (Benci [7]). The pseudo-index $i^* : A^* \to \mathbb{N} \cup \{0, +\infty\}$ has the following properties:

1. (i$_1^*$) If $A \subset B$, then $i^*(A) \leq i^*(B)$;
2. (i$_2^*$) If $\eta \in \Gamma$, then $i^*(\eta(A)) = i^*(A)$;
3. (i$_3^*$) If $A \in A^*$ and $B \in A$ are closed, then $i^*(A \cup B) \leq i^*(A) + i(B)$.

For any integer $k \geq 1$ such that $k \leq i(M)$, let

$$A_k^* = \{ A \in A^* : A \text{ is compact and } i^*(A) \geq k \}$$

and set

$$c_k^* := \inf_{A \in A_k^*} \max_{u \in A} J(u).$$

From $A_{k+1}^* \subset A_k^*$, it follows $c_k^* \leq c_{k+1}^*$. The following theorem is standard (see, e.g., [24, Proposition 3.42]).

**Theorem 2.4.** Assume that $J \in C^1(B, \mathbb{R})$ is even and $J(0) = 0$. If

$$0 < c_k^* \leq \cdots \leq c_{k+m-1}^* \leq +\infty$$

and $J$ satisfies $(PS)_{c_i^*}$ for $i = k, \ldots, k+m-1$, then $J$ has $m$ distinct pairs of nontrivial critical points.
3 The Palais–Smale condition

From here on, let $X$ be the Banach space in (1.2) and let $\mathcal{J} : X \to \mathbb{R}$ be the functional in (1.1). Furthermore, we denote by

- $(X', \| \cdot \|_{X'})$ the dual space of $(X, \| \cdot \|_X)$,
- $(W^{-1,p'}(\Omega), \| \cdot \|_{W^{-1}})$ the dual space of $(W_0^{1,p}(\Omega), \| \cdot \|)$,
- $L^q(\Omega)$ the Lebesgue space equipped with the canonical norm $| \cdot |_q$ for any $q \geq 1$,
- $\text{meas}(\cdot)$ the usual Lebesgue measure in $\mathbb{R}^N$.

By definition, $X \hookrightarrow W_0^{1,p}(\Omega)$ and $X \hookrightarrow L^\infty(\Omega)$ with continuous imbeddings; moreover, if $p^*$ is the critical exponent, i.e. $p^* = \frac{pN}{N-p}$ if $p \in [1, N[$, $p^* = +\infty$ otherwise, by the Sobolev Imbedding Theorem, for any $1 \leq q < p^*$, a constant $\gamma_q > 0$ exists such that

$$|u|_q \leq \gamma_q \|u\| \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(3.1)

In particular, for $1 \leq p < p^*$, we have

$$|u|_p \leq \gamma_p \|u\|, \quad |u|_1 \leq \gamma_1 \|u\| \quad \text{for all } u \in W_0^{1,p}(\Omega),$$

(3.2)

while, under the stronger assumption $p > N$, we have

$$|u|_\infty \leq \gamma_\infty \|u\| \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

(3.3)

Letting $g^\infty$ be as in $(h_1)$ and setting $G^\infty(x, t) = \int_0^t g^\infty(x, s)ds$, if $(h_0)$ and $(h_1)$ hold, then $g^\infty$ is a Carathéodory function on $\Omega \times \mathbb{R}$ such that

$$\sup_{|t| \leq r} |g^\infty(\cdot, t)| \in L^\infty(\Omega) \quad \text{for any } r > 0;$$

(3.4)

furthermore, (3.4), respectively (1.3), implies that

$$\sup_{|t| \leq r} |G^\infty(\cdot, t)| \in L^\infty(\Omega) \quad \text{for any } r > 0,$$

(3.5)

$$\lim_{|t| \to +\infty} \frac{G^\infty(x, t)}{|t|^p} = 0 \quad \text{uniformly a.e. in } \Omega.$$ 

(3.6)

Hence (1.3) and (3.4), respectively (3.5) and (3.6), imply that for any $\varepsilon > 0$ a constant $L_c > 0$ exists such that

$$|g^\infty(x, t)| \leq L_c + \varepsilon |t|^{p-1} \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R},$$

(3.7)

$$|G^\infty(x, t)| \leq L_c + \varepsilon |t|^p \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}. $$

(3.8)

Throughout this section, we consider the parametrized family of functionals $\mathcal{J}_\lambda : X \to \mathbb{R}$ defined by

$$\mathcal{J}_\lambda(u) = \frac{1}{p} \int_\Omega (A(x, u)|\nabla u|^p - \lambda |u|^p) \, dx - \int_\Omega G^\infty(x, u) \, dx.$$ 

(3.9)
Proposition 3.1. Let $p \geq 1$ and assume that the conditions $(H_0)$, $(h_0)$ and $(h_1)$ hold. If $(u_n)_n \subset X$, $u \in X$ are such that
\[
\|u_n - u\| \to 0 \quad \text{as } n \to +\infty
\] (3.10)
and $k > 0$ exists so that
\[
|u_n|_\infty \leq k \quad \text{for all } n \in \mathbb{N},
\] (3.11)
then for any $\lambda \in \mathbb{R}$, we have
\[
\mathcal{J}_\lambda(u_n) \to \mathcal{J}_\lambda(u) \quad \text{and} \quad \|d\mathcal{J}_\lambda(u_n) - d\mathcal{J}_\lambda(u)\|_{X'} \to 0 \quad \text{as } n \to +\infty.
\]
In particular, $\mathcal{J}_\lambda \in C^1(X, \mathbb{R})$ with derivative $d\mathcal{J}_\lambda : u \in X \mapsto d\mathcal{J}_\lambda(u) \in X'$ defined by
\[
\langle d\mathcal{J}_\lambda(u), \varphi \rangle = \int_\Omega A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx + \frac{1}{p} \int_\Omega A_t(x,u)\varphi|\nabla u|^p \, dx
\]
\[
- \lambda \int_\Omega |u|^{p-2}u\varphi \, dx - \int_\Omega g^\infty(x,u)\varphi \, dx,
\]
for any $u, \varphi \in X$.

Proof. The proof is essentially a simpler version of [10, Proposition 3.1], but, for completeness, here we point out its main tools.

First of all, consider the functional $\bar{\mathcal{J}} : X \to \mathbb{R}$ which is defined as
\[
\bar{\mathcal{J}}(w) = \frac{1}{p} \int_\Omega A(x,w)|\nabla w|^p \, dx, \quad w \in X,
\]
whose Gâteaux differential in $w$ along direction $\varphi (w, \varphi \in X)$ is
\[
\langle d\bar{\mathcal{J}}(w), \varphi \rangle = \int_\Omega A(x,w)|\nabla w|^{p-2}\nabla w \cdot \nabla \varphi \, dx + \frac{1}{p} \int_\Omega A_t(x,w)\varphi|\nabla w|^p \, dx.
\]
Now, let $(u_n)_n \subset X$, $u \in X$ be such that (3.10) and (3.11) hold. A direct consequence of (3.11) and $(H_0)$ is the existence of a constant $b > 0$, $b$ depending only on $k$ and $|u|_\infty$, such that for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$ we have
\[
|A(x,u)| \leq b, \quad |A(x,u_n)| \leq b, \quad |A_t(x,u)| \leq b, \quad |A_t(x,u_n)| \leq b.
\] (3.12)

On the other hand, by (3.10) it follows that
\[
(u_n, \nabla u_n) \to (u, \nabla u) \quad \text{in measure on } \Omega.
\]
Thus, being $A$ and $A_t$ Carathéodory functions, there results
\[
A(x,u_n)|\nabla u_n|^p \to A(x,u)|\nabla u|^p,
\]
\[
A_t(x,u_n)|\nabla u_n|^p \to A_t(x,u)|\nabla u|^p,
\]
\[
A(x,u_n)|\nabla u_n|^{p-2}\nabla u_n \to A(x,u)|\nabla u|^{p-2}\nabla u
\]
in measure on $\Omega$, too, i.e., for all $\varepsilon > 0$ it is
\[ \text{meas}(\Omega_{n,\varepsilon}) \to 0, \quad \text{meas}(\Omega_{n,\varepsilon}^d) \to 0, \quad \text{meas}(\Omega_{n,\varepsilon}^{p-1}) \to 0, \quad (3.13) \]
where
\[ \Omega_{n,\varepsilon} = \{ x \in \Omega : |A(x, u_n)\nabla u_n|^p - A(x, u)\nabla u|^p | \geq \varepsilon \}, \]
\[ \Omega_{n,\varepsilon}^d = \{ x \in \Omega : |A_t(x, u_n)\nabla u_n|^p - A_t(x, u)\nabla u|^p | \geq \varepsilon \}, \]
\[ \Omega_{n,\varepsilon}^{p-1} = \{ x \in \Omega : |A(x, u_n)|\nabla u_n|^{p-2}\nabla u_n - A(x, u)\nabla u|^{p-2}\nabla u| \geq \varepsilon \}. \]

So, fixing $\varepsilon > 0$, by applying Vitali–Hahn–Saks Theorem and taking into account the absolutely continuity of the Lebesgue integral, there exists $\delta_\varepsilon > 0$ (eventually, $\delta_\varepsilon \leq \varepsilon$), such that if $E \subset \Omega$, meas$(E) < \delta_\varepsilon$, then
\[ \int_E |\nabla u|^p dx < \varepsilon, \quad \int_E |\nabla u|^p dx < \varepsilon \quad \text{for all } n \in \mathbb{N}; \quad (3.14) \]
moreover, by (3.13) an integer $n_\varepsilon$ exists such that
\[ \text{meas}(\Omega_{n,\varepsilon}) < \delta_\varepsilon, \quad \text{meas}(\Omega_{n,\varepsilon}^d) < \delta_\varepsilon, \quad \text{meas}(\Omega_{n,\varepsilon}^{p-1}) < \delta_\varepsilon \quad \text{for all } n \geq n_\varepsilon. \quad (3.15) \]

Then, from (3.12), (3.14), (3.15) and direct computations it follows that
\[ |\mathcal{J}(u_n) - \mathcal{J}(u)| \leq \frac{1}{p} \int_{\Omega_{n,\varepsilon}} (|A(x, u_n)|\nabla u_n|^p + |A(x, u)|\nabla u|^p) dx + \frac{1}{p} \int_{\Omega \setminus \Omega_{n,\varepsilon}} |A(x, u_n)|\nabla u_n|^{p-2}\nabla u_n - A(x, u)\nabla u|^{p-2}\nabla u| \nabla \varphi| dx < b_1 \varepsilon \]
for all $n \geq n_\varepsilon$, where $b_1 > 0$ is a suitable constant independent of $\varepsilon$. Whence, $\mathcal{J}(u_n) \to \mathcal{J}(u)$.

Now, fixing any $\varepsilon > 0$ and taking any $\varphi \in X$, we have
\[ |\langle d\mathcal{J}(u_n) - d\mathcal{J}(u), \varphi \rangle | \leq \int_{\Omega_{n,\varepsilon}^{p-1}} |A(x, u_n)|\nabla u_n|^{p-1}\nabla \varphi| dx + \int_{\Omega_{n,\varepsilon}^d} |A(x, u)|\nabla u|^p |d\varphi| dx + \int_{\Omega \setminus \Omega_{n,\varepsilon}} |A(x, u_n)|\nabla u_n|^{p-2}\nabla u_n - A(x, u)\nabla u|^{p-2}\nabla u| | \nabla \varphi| dx + \frac{1}{p} \int_{\Omega_{n,\varepsilon}} |A_t(x, u_n)|\nabla u_n|^p + |A_t(x, u)|\nabla u|^p | \varphi| dx + \frac{1}{p} \int_{\Omega \setminus \Omega_{n,\varepsilon}} |A_t(x, u_n)|\nabla u_n|^p - A_t(x, u)\nabla u|^p | \varphi| dx. \]

Thus, reasoning as above, from (1.2), (3.12), (3.14), (3.15) and direct computations, a constant $b_2 > 0$, $b_2$ independent of $\varepsilon$ and $\varphi$, exists such that
\[ |\langle d\mathcal{J}(u_n) - d\mathcal{J}(u), \varphi \rangle | \leq (2b \varepsilon^{1-\frac{1}{p}} + \varepsilon \text{meas}(\Omega))^{1-\frac{1}{p}} ||\varphi|| + \frac{\varepsilon}{p} (2b + \text{meas}(\Omega)) ||\varphi||_{\infty} \leq b_2 \max\{\varepsilon, \varepsilon^{1-\frac{1}{p}}\} ||\varphi||_X \]
for all $n$ large enough. Hence, by the arbitrariness of $\varepsilon$ and $\varphi \in X$, we have
\[ \|d\overline{J}(u_n) - d\overline{J}(u)\|_{X'} \to 0. \]

On the other hand, from (3.7) and standard arguments (see, e.g., [17, Subsection 2.1]), it follows that the functional
\[ u \in W_0^{1,p}(\Omega) \mapsto \lambda \frac{1}{p} \int_{\Omega} |u|^p dx + \int_{\Omega} G(x,u) dx \in \mathbb{R} \]
is $C^1$ in $(W_0^{1,p}(\Omega), \| \cdot \|)$, and so in $(X, \| \cdot \|_X)$; hence, the thesis follows. 

Thus, if conditions $(H_0)$, $(h_0)$ and $(h_1)$ hold, for each $p \geq 1$, problem $(P)$ has a variational structure and its bounded weak solutions are critical points of $J = J_{\lambda\infty}$ in the Banach space $X$.

As our aim is applying variational methods to the study of critical points of $J$ in the asymptotically $p$-linear case, we introduce the following further conditions:

(H3) we have
\[ \lim_{|t| \to +\infty} A_t(x,t) = 0 \quad \text{uniformly a.e. in } \Omega; \]

(H4) there exists $\alpha_1 > 0$ (without loss of generality, $\alpha_1 \leq 1$) such that
\[ A(x,t) + \frac{1}{p} A_t(x,t) t \geq \alpha_1 A(x,t) \quad \text{a.e. in } \Omega, \text{ for all } t \in \mathbb{R}. \]

Remark 3.2. Hypothesis $(H_2)$ implies that
\[ \lim \inf_{|t| \to +\infty} A_t(x,t) \to 0 \quad \text{a.e. in } \Omega; \]

hence, condition $(H_3)$ is quite natural.

Remark 3.3. By $(H_2)$ and $(H_3)$, for each $\varepsilon > 0$, a radius $R_\varepsilon > 0$ exists such that

\[ |A(x,t) - A^\infty(x)| < \varepsilon \quad \text{for a.e. } x \in \Omega, \text{ if } |t| \geq R_\varepsilon, \]  
\[ |A_t(x,t)| < \varepsilon \quad \text{for a.e. } x \in \Omega, \text{ if } |t| \geq R_\varepsilon. \]

Since (3.16) implies
\[ |A(x,t)| \leq |A^\infty|_{\infty} + \varepsilon \quad \text{for a.e. } x \in \Omega, \text{ if } |t| \geq R_\varepsilon, \]
it follows from $(H_0)$ and (3.17) that
\[ |A(x,t)| \leq b, \quad |A_t(x,t)| \leq b \quad \text{for a.e. } x \in \Omega, \text{ for all } t \in \mathbb{R}, \]
for a suitable $b > 0$. 

10
Remark 3.4. In the proof of Proposition 3.1, assumption (3.11) is required only for the boundedness conditions (3.12), which are necessary for investigating the smoothness of $\bar{J}$. However, this uniform bound can be avoided in the hypotheses $(H_2)$ and $(H_3)$ as (3.18) holds. Whence, in this set of hypotheses, for any $p \geq 1$, the functional $\bar{J}$ is continuous in $W^{1,p}_0(\Omega)$, and so is $J_\lambda$ for any $\lambda \in \mathbb{R}$. However, in general, $J_\lambda$ is not $C^1$ in $W^{1,p}_0(\Omega)$ as it is Gâteaux differentiable in $u \in W^{1,p}_0(\Omega)$ only along bounded directions.

Here and in the following, by $\sigma(A^{\infty}_p)$ we denote the spectrum of the operator

$$A^{\infty}_p : u \in W^{1,p}_0(\Omega) \mapsto - \text{div}(A^{\infty}_p(x) |\nabla u|^{p-2} \nabla u) \in W^{-1,p'}(\Omega),$$

which is the set of $\lambda \in \mathbb{R}$ such that the nonlinear eigenvalue problem

$$\begin{cases} - \text{div}(A^{\infty}_p(x) |\nabla u|^{p-2} \nabla u) = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial \Omega \end{cases}$$

has a nontrivial (weak) solution in $W^{1,p}_0(\Omega)$, i.e. some $u \in W^{1,p}_0(\Omega)$, $u \not\equiv 0$, exists such that

$$\int_{\Omega} A^{\infty}_p(x)|\nabla u|^{p-2}\nabla u \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} |u|^{p-2} u \varphi \, dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega).$$

**Proposition 3.5.** If $p > 1$, the hypotheses $(H_0)$–$(H_4)$, $(h_0)$ and $(h_1)$ hold, and $\lambda \notin \sigma(A^{\infty}_p)$, then for all $\beta \in \mathbb{R}$, each $(PS)_\beta$–sequence of $J_\lambda$ in $X$ is bounded in the $W^{1,p}_0$–norm.

**Proof.** Taking $\beta > 0$, let $(u_n)_n \subset X$ be a $(PS)_\beta$–sequence, i.e.

$$J_\lambda(u_n) \to \beta \quad \text{and} \quad \|dJ_\lambda(u_n)\|_{X'} \to 0 \quad \text{if } n \to +\infty. \quad (3.20)$$

Arguing by contradiction, we suppose that

$$\|u_n\| \to +\infty. \quad (3.21)$$

Hence, without loss of generality, for any $n \in \mathbb{N}$ we assume $\|u_n\| > 0$, and define

$$v_n = \frac{u_n}{\|u_n\|} \quad \text{hence} \quad \|v_n\| = 1. \quad (3.22)$$

Then, there exists $v \in W^{1,p}_0(\Omega)$ such that, up to subsequences, we have

$$v_n \to v \text{ weakly in } W^{1,p}_0(\Omega), \quad (3.23)$$

$$v_n \to v \text{ strongly in } L^q(\Omega) \text{ for each } 1 \leq q < p^*, \quad (3.24)$$

$$v_n \to v \text{ a.e. in } \Omega. \quad (3.25)$$

In order to yield a contradiction, we organize the proof in some steps:

1. $v \neq 0$;
2. a constant $b_0 > 0$ exists such that for any $\mu > 0$ there exists $n_{\mu} \in \mathbb{N}$ such that

$$
\int_{\Omega \setminus \Omega_{n_{\mu}}^\mu} |\nabla v_n|^p dx \leq b_0 \max\{\mu, \mu^p\} \quad \text{for all } n \geq n_{\mu},
$$

where

$$
\Omega_{n_{\mu}}^\mu = \{ x \in \Omega : |v_n(x)| \geq \mu \};
$$

3. taking $\Omega_0 = \{ x \in \Omega : v(x) = 0 \}$, if $\text{meas}(\Omega_0) > 0$ then

$$
\int_{\Omega_0} |\nabla v|^p dx = 0,
$$

which implies $\nabla v = 0$ a.e. in $\Omega_0$ and, clearly,

$$
\int_{\Omega_0} A_{p}^\infty(x) |\nabla v|^{p-2} \nabla v \cdot \nabla \varphi dx = \int_{\Omega_0} \lambda |v|^{p-2} v \varphi dx \quad \text{for all } \varphi \in W^{1,p}_0(\Omega);
$$

4. taking any $\varphi \in X$ we have

$$
\int_{\Omega} A_{p}^\infty(x)|\nabla v_n|^{p-2} \nabla v_n \cdot \nabla \varphi dx - \lambda \int_{\Omega} |v_n|^{p-2} v_n \varphi dx \to 0; \quad (3.29)
$$

5. $\lambda \in \sigma(A_{p}^\infty)$, in contradiction with the hypotheses.

For simplicity, here and in the following $b_i$ denotes any strictly positive constant independent of $n$.

**Step 1.** Firstly, let us point out that for any $\varepsilon > 0$ from (3.2), (3.8) and (3.22) it follows

$$
| \int_{\Omega} \frac{G_{p}^\infty(x, u_n)}{\|u_n\|^p} dx | \leq \frac{L \varepsilon \text{meas}(\Omega)}{\|u_n\|^p} + \varepsilon \gamma_p^p;
$$

hence, (3.21) implies

$$
| \int_{\Omega} \frac{G_{p}^\infty(x, u_n)}{\|u_n\|^p} dx | \leq \varepsilon (1 + \gamma_p^p) \quad \text{for all } n \geq n_{\varepsilon}
$$

for $n_{\varepsilon}$ large enough. Thus, we have

$$
\int_{\Omega} \frac{G_{p}^\infty(x, u_n)}{\|u_n\|^p} dx \to 0. \quad (3.30)
$$

Furthermore, (3.20) and (3.21) give

$$
\frac{\mathcal{J}_\lambda(u_n)}{\|u_n\|^p} \to 0. \quad (3.31)
$$
Now, arguing by contradiction, assume \( v \equiv 0 \). Then, from (3.24) it follows

\[
\int_{\Omega} |v_n|^p dx \to 0, \quad (3.32)
\]

but for any \( n \in \mathbb{N} \), condition \((H_1), (3.35)\) and \( (3.22)\) imply

\[
0 < \frac{\alpha_0}{p} = \frac{\alpha_0}{p} \|v_n\|^p \leq \frac{1}{p} \int_{\Omega} A(x, u_n)|\nabla v_n|^p dx
\]
\[
= \frac{\mathcal{J}_\lambda(u_n)}{\|u_n\|^p} + \frac{\lambda}{p} \int_{\Omega} |v_n|^p dx + \int_{\Omega} \frac{G_\infty(x,u_n)}{\|u_n\|^p} dx
\]

in contradiction with \( (3.30) - (3.32) \).

**Step 2.** Taking any \( \phi \in X \), we have

\[
\langle d \mathcal{J}_\lambda(u_n), \frac{\phi}{\|u_n\|^p} \rangle = \int_{\Omega} A(x, u_n) |\nabla v_n|^{p-2} \nabla v_n \cdot \nabla \phi dx
\]
\[
+ \frac{1}{p} \int_{\Omega} A_t(x, u_n) |u_n| |\nabla v_n|^{p-1} \phi dx - \lambda \int_{\Omega} |v_n|^{p-2} v_n \phi dx
\]
\[
- \int_{\Omega} \frac{g_{\infty}(x, u_n)}{\|u_n\|^p} \phi dx. \quad (3.33)
\]

Fix any \( \mu > 0 \) and \( \varepsilon > 0 \). From one hand,

\[
|\langle d \mathcal{J}_\lambda(u_n), \frac{\phi}{\|u_n\|^p} \rangle| \leq \frac{|d \mathcal{J}_\lambda(u_n)|_{\lambda' \|u_n\|^{p-1}}}{{\|\phi\|}} \quad \text{for all } n \in \mathbb{N}; \quad (3.34)
\]

while from (3.22), (3.21), (3.22) and the Hölder inequality it follows

\[
|\int_{\Omega} \frac{g_{\infty}(x, u_n)}{\|u_n\|^p} \phi dx| \leq \left( \varepsilon \gamma_p^p + \frac{L \nu_1}{\|u_n\|^p} \right) \|\phi\| \quad \text{for all } n \in \mathbb{N}. \quad (3.35)
\]

On the other hand, by (3.22) and (3.27) we have

\[
|u_n(x)| > \mu \|u_n\| \quad \text{for all } x \in \Omega_n^\mu, \ n \in \mathbb{N}. \quad (3.36)
\]

Then, from (3.21) an integer \( n_{\mu, \varepsilon} \), independent of \( \phi \), exists such that (3.20) and (3.35) imply

\[
|\langle d \mathcal{J}_\lambda(u_n), \frac{\phi}{\|u_n\|^p} \rangle| \leq \varepsilon \|\phi\| \quad \text{for all } n \geq n_{\mu, \varepsilon}, \quad (3.37)
\]

while inequality (3.35) becomes

\[
|\int_{\Omega} \frac{g_{\infty}(x, u_n)}{\|u_n\|^p} \phi dx| \leq \varepsilon \left( \gamma_p^p + 1 \right) \|\phi\| \quad \text{for all } n \geq n_{\mu, \varepsilon}, \quad (3.38)
\]

and from (3.17) and (3.30) it follows

\[
|A_t(x, u_n(x))u_n(x)| < \varepsilon \quad \text{for a.e. } x \in \Omega_n^\mu, \text{ if } n \geq n_{\mu, \varepsilon}. \quad (3.39)
\]
Hence, for all \( n \geq n_{\mu, \varepsilon} \) by (3.22), (3.27) and (3.39), direct computations imply

\[
\int_{\Omega_{\mu}^n} \frac{|A_t(x, u_n)u_n|}{|v_n|} |\nabla v_n|^p dx \leq \frac{\varepsilon}{\mu}, \tag{3.40}
\]

and then

\[
|\int_{\Omega_{\mu}^n} A_t(x, u_n)\|u_n\| |\nabla v_n|^p \phi \, dx| \leq \int_{\Omega_{\mu}^n} \frac{|A_t(x, u_n)u_n|}{|v_n|} |\nabla v_n|^p |\phi| \, dx \leq \frac{\varepsilon}{\mu} |\phi|_{\infty}. \tag{3.41}
\]

Now, for any \( n \in \mathbb{N} \), let us consider the cut–off function \( T_{\mu} : \mathbb{R} \to \mathbb{R} \) such that

\[
T_{\mu}(t) = \begin{cases} 
\frac{t}{\mu} & \text{if } |t| < \mu, \\
1 & \text{if } |t| \geq \mu.
\end{cases}
\]

As

\[
T_{\mu}(v_n(x)) = \begin{cases} 
v_n(x) & \text{for a.e. } x \in \Omega \setminus \Omega_{\mu}^n, \\
\frac{v_n(x)}{\mu/|v_n(x)|} & \text{if } x \in \Omega_{\mu}^n,
\end{cases} \tag{3.42}
\]

\[
|\nabla T_{\mu}(v_n(x))| = \begin{cases} 
\nabla v_n(x) & \text{for a.e. } x \in \Omega \setminus \Omega_{\mu}^n, \\
0 & \text{for a.e. } x \in \Omega_{\mu}^n, \tag{3.43}
\end{cases}
\]

then \( T_{\mu}(v_n) \in X \) with

\[
\|T_{\mu}(v_n)\| \leq 1, \quad |T_{\mu}(v_n)|_{\infty} \leq \mu. \tag{3.44}
\]

Thus, applying (3.33) on the test function \( \phi = T_{\mu}(v_n) \), we have

\[
\int_{\Omega} A(x, u_n)|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla T_{\mu}(v_n) dx + \frac{1}{p} \int_{\Omega} A_t(x, u_n)\|u_n\||\nabla v_n|^p T_{\mu}(v_n) dx \\
= \langle dJ_{\lambda}(u_n), \frac{T_{\mu}(v_n)}{\|u_n\|_{p-1}} \rangle + \lambda \int_{\Omega} |v_n|^{p-2}v_n T_{\mu}(v_n) dx + \int_{\Omega} \frac{g_\infty(x, u_n)}{|u_n|_{p-1}} T_{\mu}(v_n) dx,
\]

where (3.22), (3.34) and (3.43) imply

\[
\int_{\Omega} A(x, u_n)|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla T_{\mu}(v_n) dx + \frac{1}{p} \int_{\Omega} A_t(x, u_n)\|u_n\||\nabla v_n|^p T_{\mu}(v_n) dx \\
= \int_{\Omega \setminus \Omega_{\mu}^n} A(x, u_n)|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla T_{\mu}(v_n) dx + \frac{1}{p} \int_{\Omega \setminus \Omega_{\mu}^n} A_t(x, u_n)\|u_n\||\nabla v_n|^p T_{\mu}(v_n) dx \\
+ \frac{\mu}{p} \int_{\Omega_{\mu}^n} A_t(x, u_n)\|u_n\| \frac{v_n}{|v_n|} |\nabla v_n|^p dx \\
= \int_{\Omega \setminus \Omega_{\mu}^n} \langle A(x, u_n) + \frac{1}{p} A_t(x, u_n)u_n \rangle |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla T_{\mu}(v_n) dx + \frac{\mu}{p} \int_{\Omega_{\mu}^n} \frac{A_t(x, u_n)u_n}{|v_n|} |\nabla v_n|^p dx,
\]

while (3.1) with \( q = p - 1 \), (3.22), (3.27) and (3.42) give

\[
|\int_{\Omega} |v_n|^{p-2}v_n T_{\mu}(v_n) dx| \leq \mu^p \text{meas}(\Omega) + \mu^{p-1}_{p-1}.
\]
Whence, from (3.37), (3.38), (3.40) and (3.44) it follows

$$\int_{\Omega_{n}^\mu} (A(x,u_n) + \frac{1}{p} A_t(x,u_n) u_n) |\nabla v_n|^p dx \leq \varepsilon (\gamma_p^p + 2 + \frac{1}{p}) + |\lambda| (\mu^p \text{meas}(\Omega) + \mu^{p-1})$$

for all $n \geq n_{\mu,\varepsilon}$.

As $\mu$ and $\varepsilon$ are any and independent one from the other, we can fix $\varepsilon = \mu$; hence, $n_{\mu} = n_{\mu,\mu}$ and (3.45) becomes

$$\int_{\Omega \setminus \Omega_{n}^\mu} (A(x,u_n) + \frac{1}{p} A_t(x,u_n) u_n) |\nabla v_n|^p dx \leq b_1 \max\{\mu, \mu^p\}$$

(3.46)

for all $n \geq n_{\mu}$, where $b_1 = \gamma_p^p + 2 + \frac{1}{p} + |\lambda| \text{meas}(\Omega) + |\lambda|^{p-1} > 0$.

Vice versa, by assumptions $(H_1)$ and $(H_4)$ we have

$$\alpha_0 \alpha_1 \int_{\Omega \setminus \Omega_n^\mu} |\nabla v_n|^p dx \leq \alpha_1 \int_{\Omega \setminus \Omega_n^\mu} A(x,u_n) |\nabla v_n|^p dx$$

$$\leq \int_{\Omega \setminus \Omega_n^\mu} (A(x,u_n) + \frac{1}{p} A_t(x,u_n) u_n) |\nabla v_n|^p dx;$$

whence, summing up, (3.46) implies (3.26) with $b_0 = \frac{b_1}{\alpha_0 \alpha_1}$.

Step 3. Firstly, we claim that if $\text{meas}(\Omega_0) > 0$ then for any $\mu > 0$ there exists $n^\mu \in \mathbb{N}$ such that

$$\text{meas}(\Omega_0 \cap \Omega_n^\mu) = 0 \quad \text{for all } n \geq n^\mu.$$  

(3.47)

In fact, arguing by contradiction, we assume that $\bar{\mu} > 0$ exists such that, up to subsequences,

$$\text{meas}(\Omega_0 \cap \Omega_n^\mu) > 0 \quad \text{for all } n \in \mathbb{N}.$$  

From (3.26) a set $\Omega \subset \Omega$ exists such that $\text{meas}(\Omega) = 0$ and $v_n(x) \rightarrow v(x)$ for all $x \notin \Omega$; hence, for all $n \in \mathbb{N}$ it results $\text{meas}((\Omega_0 \cap \Omega_n^\mu) \setminus \Omega) > 0$ and for all $x \in (\Omega_0 \cap \Omega_n^\mu) \setminus \Omega$ we have both $|v_n(x)| \geq \bar{\mu}$ for all $n \in \mathbb{N}$ and $v_n(x) \rightarrow 0$ as $n \rightarrow +\infty$: a contradiction.

Now, from Step 2, (3.26) and (3.47) imply that

$$\int_{\Omega_0} |\nabla v_n|^p dx = \int_{\Omega_0 \setminus \Omega_n^\mu} |\nabla v_n|^p dx \leq \int_{\Omega \setminus \Omega_n^\mu} |\nabla v_n|^p dx \leq b_0 \max\{\mu, \mu^p\}$$

for all $n$ large enough, where from the weak lower semi–continuity of norms we have

$$\int_{\Omega_0} |\nabla v|^p dx \leq \liminf_{n \rightarrow +\infty} \int_{\Omega_0} |\nabla v_n|^p dx \leq b_0 \max\{\mu, \mu^p\}.$$  

Hence, for the arbitrariness of $\mu > 0$, (3.28) holds.
Step 4. Fixing any $\rho > 0$, we introduce another cut-off function $\chi_\rho \in C^1(\mathbb{R}, \mathbb{R})$ which has to be even, nondecreasing in $[0, +\infty)$ and such that

$$
\chi_\rho(t) = \begin{cases}
0 & \text{if } |t| < \rho, \\
1 & \text{if } |t| \geq 2\rho,
\end{cases}
$$

with $|\chi'_\rho(t)| \leq 2$ for all $t \in \mathbb{R}$.

Taking any $\varphi \in X$, $n \in \mathbb{N}$, we denote $\omega_{\rho, n} = \chi_\rho(v_n)\varphi$, hence, by definition,

$$
\omega_{\rho, n}(x) = \begin{cases}
0 & \text{if } |v_n(x)| < \rho, \\
\varphi(x) & \text{if } |v_n(x)| > 2\rho,
\end{cases}
$$

(3.48)

and

$$
\nabla \omega_{\rho, n}(x) = \begin{cases}
0 & \text{if } |v_n(x)| < \rho, \\
\nabla \varphi(x) & \text{if } |v_n(x)| > 2\rho,
\end{cases}
$$

(3.49)

so direct computations imply $\omega_{\rho, n} \in X$ with

$$
\|\omega_{\rho, n}\| \leq \|\varphi\| + 2\|\varphi\|_\infty \leq 3\|\varphi\|_X, \quad |\omega_{\rho, n}|_\infty \leq |\varphi|_\infty \leq \|\varphi\|_X.
$$

(3.50)

Thus, we consider the test function $\phi = \omega_{\rho, n}$ in (3.33), and, from (3.27) with $\mu = \rho$, we have

$$
\int_{\Omega_n} A(x, u_n)|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \omega_{\rho, n} dx - \lambda \int_{\Omega_n} |v_n|^{p-2}v_n \omega_{\rho, n} dx
\leq \langle d\mathcal{J}_\lambda(u_n), \frac{\omega_{\rho, n}}{|u_n|^{p-1}} \rangle - \frac{1}{p} \int_{\Omega_n} A_t(x, u_n)\|u_n\||\nabla v_n|^{p-1}\omega_{\rho, n} dx
$$

(3.51)

$$
+ \int_{\Omega} \frac{\gamma_p \omega_{\rho, n}}{|u_n|^{p-1}}\omega_{\rho, n} dx.
$$

Whence, by using (3.37) with $\phi = \omega_{\rho, n}$ and $\varepsilon = \rho$, (3.38) with $\phi = \omega_{\rho, n}$ and $\varepsilon = \rho$, (3.41) with $\phi = \omega_{\rho, n}$, $\varepsilon = \rho^2$ and $\mu = \rho$, equation (3.51) with estimates (3.50) implies

$$
\left| \int_{\Omega_n} A(x, u_n)|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \omega_{\rho, n} dx - \lambda \int_{\Omega_n} |v_n|^{p-2}v_n \omega_{\rho, n} dx \right|
\leq \rho \|\omega_{\rho, n}\| + \left( \frac{\rho}{p} |\omega_{\rho, n}|_\infty + \rho (\gamma_p^p + 1)\|\omega_{\rho, n}\| \right) \leq \rho b_2 \|\varphi\|_X
$$

(3.52)

for all $n \geq n^1_\rho$, with $n^1_\rho$ large enough and $b_2 = 3\gamma_p^p + 6 + \frac{1}{p}$.

On the other hand, by (3.16) with any $\varepsilon > 0$, (3.21) and (3.36) with $\mu = \varepsilon$ (and with $\Omega_n^\varepsilon$ as in (3.27)), an integer $n^\varepsilon$ exists such that

$$
|A(x, u_n(x)) - A^\infty(x)| < \varepsilon \quad \text{for a.e. } x \in \Omega_n^\varepsilon, \text{ if } n \geq n^\varepsilon,
$$

(3.53)

then, taking any $\phi \in X$, for all $n \geq n^\varepsilon$ by (3.22) and (3.53), the Hölder inequality and direct computations imply

$$
\left| \int_{\Omega_n} (A(x, u_n) - A^\infty(x))|\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \phi dx \right| \leq \varepsilon \|\phi\|.
$$

(3.54)
In particular, if we take $\varepsilon = \rho$ and $\phi = \omega_{\rho,n}$ in (3.54), an integer $n_{\rho}^{2} \geq n_{\rho}^{1}$ is such that from (3.50) and (3.52), it follows

$$\int_{\Omega_{n}} A^{\infty}(x)|\nabla v_{n}|^{p-2}\nabla v_{n} \cdot \nabla \omega_{\rho,n} dx - \lambda \int_{\Omega_{n}} |v_{n}|^{p-2} v_{n} \omega_{\rho,n} dx \leq \rho b_{3} \|\varphi\|_{X}$$

(3.55)

for all $n \geq n_{\rho}^{2}$, with $b_{3} = 3 + b_{2}$.

Now, from definitions (3.27) with $\mu = 2\rho$, direct computations and (3.48), (3.49) imply

$$\int_{\Omega} A^{\infty}(x)|\nabla v_{n}|^{p-2}\nabla v_{n} \cdot \nabla \varphi dx - \lambda \int_{\Omega} |v_{n}|^{p-2} v_{n} \varphi dx$$

$$\leq \int_{\Omega \setminus \Omega_{n}^{2\rho}} A^{\infty}(x)|\nabla v_{n}|^{p-2}\nabla v_{n} \cdot \nabla ((1 - \chi_{\rho}(v_{n})) \varphi) dx$$

$$+ |\lambda| \int_{\Omega \setminus \Omega_{n}^{2\rho}} |v_{n}|^{p-2} v_{n} (1 - \chi_{\rho}(v_{n})) \varphi dx$$

$$+ \lambda \int_{\Omega_{n}} |v_{n}|^{p-2} v_{n} \varphi_{\rho,n} dx - \lambda \int_{\Omega} |v_{n}|^{p-2} v_{n} \omega_{\rho,n} dx,$$

where (3.20) with $\mu = 2\rho$ (in Step 2), (3.50), (H2) and the Hölder inequality give

$$\int_{\Omega \setminus \Omega_{n}^{2\rho}} A^{\infty}(x)|\nabla v_{n}|^{p-2}\nabla v_{n} \cdot \nabla ((1 - \chi_{\rho}(v_{n})) \varphi) dx$$

$$\leq |A^{\infty}|_{\infty} \left( \int_{\Omega \setminus \Omega_{n}^{2\rho}} |\nabla v_{n}|^{p} dx \right)^{1-\frac{1}{p}} \|\varphi - \omega_{\rho,n}\|$$

$$\leq b_{4} \max\{\rho^{p-1}, \rho^{1-\frac{1}{p}}\} \|\varphi\|_{X}$$

for all $n \geq n_{\rho}^{3}$, with $n_{\rho}^{3}$ large enough and $b_{4} > 0$ independent of both $\rho$ and $\varphi$, while (3.22) implies

$$\int_{\Omega \setminus \Omega_{n}^{2\rho}} |v_{n}|^{p-2} v_{n} (1 - \chi_{\rho}(v_{n})) \varphi dx \leq (2\rho)^{p-1} \int_{\Omega \setminus \Omega_{n}^{2\rho}} |\varphi| dx$$

$$\leq \rho^{p-1} 2^{p-1} \gamma_{1} \|\varphi\| \leq \rho^{p-1} 2^{p-1} \gamma_{1} \|\varphi\|_{X}.$$

Whence, taking $n_{\rho} \in \mathbb{N}$ large enough, from (3.55) it follows

$$\int_{\Omega} A^{\infty}(x)|\nabla v_{n}|^{p-2}\nabla v_{n} \cdot \nabla \varphi dx - \lambda \int_{\Omega} |v_{n}|^{p-2} v_{n} \varphi dx$$

$$\leq \max\{\rho^{1-\frac{1}{p}}, \rho, \rho^{p-1}\} b_{5} \|\varphi\|_{X} \quad \text{for all } n \geq n_{\rho},$$

(3.56)

with $b_{5} > 0$ independent of both $\rho$ and $\varphi$. Thus, from the arbitrariness of $\rho$, (3.56) implies (3.20).
Step 5. Firstly, we apply (3.29) to \( \varphi = v_n - v \) by taking into account (3.24), then by considering (3.23) we have

\[
\int_{\Omega} A_{\infty}(x) \left( (|\nabla v_n|^{p-2}\nabla v_n - |\nabla v|^{p-2}\nabla v) \cdot (\nabla v_n - \nabla v) \right) \, dx \to 0.
\]

Whence, from the properties of \( A_{\infty} \) and the uniform convexity of \((W_0^{1,p}(\Omega), ||\cdot||)\) (as \( p > 1 \)) it follows that \( ||v_n - v|| \to 0 \). Thus,

\[
\int_{\Omega} A_{\infty}(x) |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla \varphi \, dx \to \int_{\Omega} A_{\infty}(x) |\nabla v|^{p-2}\nabla v \cdot \nabla \varphi \, dx
\]

for any \( \varphi \in W_0^{1,p}(\Omega) \), and by (3.24) and (3.29) it results

\[
\int_{\Omega} A_{\infty}(x) |\nabla v|^{p-2}\nabla v \cdot \nabla \varphi \, dx = \lambda \int_{\Omega} |v|^{p-2} v \varphi \, dx,
\]

for any \( \varphi \in X \), or better any \( \varphi \in W_0^{1,p}(\Omega) \). \( \square \)

As pointed out in Remark 3.4, even if \( A \) and \( A_t \) are bounded, we cannot simply replace \( X \) with \( W_0^{1,p}(\Omega) \), so the classical Palais–Smale condition for \( J \) in \( X \) requires the convergence not only in the \( W_0^{1,p} \)-norm, but also in the \( L^{\infty} \)-norm. This problem can be overcome if \( p > N \) since then \( X = W_0^{1,p}(\Omega) \) and the two norms \( ||\cdot|| \) and \( ||\cdot||_X \) are equivalent.

Proposition 3.6. If \( p > N \) and the hypotheses \((H_0)-(H_4)\), \((h_0)\) and \((h_1)\) hold, then for any \( \lambda \notin \sigma(A_{\infty}^p) \), the functional \( J_\lambda \) satisfies the \((PS)_\beta \) condition in \( W_0^{1,p}(\Omega) \) at each level \( \beta \in \mathbb{R} \).

Proof. Taking \( \beta > 0 \), let \((u_n)_n \subset W_0^{1,p}(\Omega) \) be a \((PS)_\beta \)-sequence, i.e. (3.20) holds. From Proposition 3.5 and (3.3) a constant \( L > 0 \) exists such that

\[
||u_n|| \leq L \quad \text{and} \quad |u_n|_{\infty} \leq \gamma_\infty L \quad \text{for all} \ n \in \mathbb{N}.
\]

(3.57)

Hence, up to subsequences, there exists \( u \in W_0^{1,p}(\Omega) \) such that

\[
u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),
\]

\[
u_n \to u \text{ strongly in } L^q(\Omega) \text{ for each } q \geq 1,
\]

\[
u_n \to u \text{ a.e. in } \Omega,
\]

(3.58) \hspace{1cm} (3.59) \hspace{1cm} (3.60)

and \( h \in L^p(\Omega) \) exists such that

\[
|u_n(x)| \leq h(x) \quad \text{a.e. in } \Omega, \text{ for all } n \in \mathbb{N}.
\]

(3.61)

We claim that \( u_n \to u \) strongly in \( W_0^{1,p}(\Omega) \). This proof is essentially as in Step 4. of the proof of [10] Proposition 4.6 and follows some arguments in [9] according to an idea introduced in [8]. Anyway, for completeness, here we prove it.
Let us consider the real map \( \psi(t) = t e^{\eta t^2} \), where \( \eta > \left( \frac{\beta_1}{2 \beta_2} \right)^2 \) will be fixed once \( \beta_1, \beta_2 > 0 \) are chosen, later on, in a suitable way. By definition,

\[
\beta_1 \psi'(t) - \beta_2 |\psi(t)| > \frac{\beta_1}{2} \text{ for all } t \in \mathbb{R}.
\]  

(3.62)

Taking \( w_n = u_n - u \), from (3.57) it follows

\[
|w_n|_\infty \leq \gamma_\infty L + |u|_\infty;
\]

moreover, (3.58) – (3.61) imply

\[
\left\{ \begin{array}{l}
w_n \rightharpoonup 0 \text{ weakly in } W^{1,p}_0(\Omega), \\
w_n \rightarrow 0 \text{ strongly in } L^q(\Omega) \text{ for all } q \geq 1, \\
w_n \rightarrow 0 \text{ a.e. in } \Omega,
\end{array} \right.
\]

and \( |w_n(x)| \leq h(x) + |u(x)| \text{ a.e. in } \Omega \), for all \( n \in \mathbb{N} \), with \( h + |u| \in L^p(\Omega) \).

Hence, \( \sigma_0 > 0 \) exists such that

\[
\left\{ \begin{array}{l}
\psi(w_n) \leq \sigma_0, \\
0 < \psi'(w_n) \leq \sigma_0 \text{ a.e. in } \Omega, \text{ for all } n \in \mathbb{N}, \\
\psi(w_n) \rightarrow 0, \psi'(w_n) \rightarrow 1 \text{ a.e. in } \Omega \text{ if } n \rightarrow +\infty.
\end{array} \right.
\]

(3.64)

(3.65)

Thus, \( (\psi(w_n))_n \) is bounded in \( W^{1,p}_0(\Omega) \), and (3.66) implies

\[
\langle dJ_\lambda(u_n), \psi(w_n) \rangle \rightarrow 0 \text{ as } n \rightarrow +\infty,
\]

(3.66)

where it is

\[
\langle dJ_\lambda(u_n), \psi(w_n) \rangle = \int_\Omega \psi'(w_n) A(x, u_n) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w_n \, dx
\]

\[
+ \frac{1}{p} \int_\Omega A_t(x, u_n) \psi(w_n) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w_n \, dx - \int_\Omega |u_n|^{p-2} u_n \psi(w_n) \, dx - \int_\Omega g_\infty(x, u_n) \psi(v_{k,n}) \, dx.
\]

(3.67)

By (3.7), (3.60), (3.61), (3.64) and (3.65), the Lebesgue Dominated Convergence Theorem implies

\[
\int_\Omega |u_n|^{p-2} u_n \psi(w_n) \, dx \rightarrow 0, \quad \int_\Omega g_\infty(x, u_n) \psi(w_n) \, dx \rightarrow 0;
\]

whence, by (3.66) and (3.67) we have

\[
\int_\Omega \psi'(w_n) A(x, u_n) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w_n \, dx
\]

\[
+ \frac{1}{p} \int_\Omega A_t(x, u_n) \psi(w_n) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w_n \, dx = \varepsilon_{1,n},
\]

(3.68)

19
with $\varepsilon_{1,n} \to 0$. On the other hand, from $(H_1)$ and (3.18) it follows

$$
\left| \int_{\Omega} A_t(x,u_n)\psi(w_n)|\nabla u_n|^pdx \right| \leq \frac{b}{\alpha_0} \int_{\Omega} A(x,u_n)|\psi(w_n)|\ |\nabla u_n|^pdx
$$

$$
= \frac{b}{\alpha_0} \int_{\Omega} A(x,u_n)|\psi(w_n)|\ |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla w_n \ dx
+ \frac{b}{\alpha_0} \int_{\Omega} A(x,u_n)|\psi(w_n)|\ |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u \ dx,
$$

where (3.18), Hölder inequality, (3.57), (3.64), (3.65), and the Lebesgue Dominated Convergence Theorem give

$$
\int_{\Omega} A(x,u_n)|\psi(w_n)|\ |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla u \ dx \to 0.
$$

Whence, a sequence $\varepsilon_{2,n} \to 0$ exists such that from (3.68) and the above estimates it follows

$$
\varepsilon_{2,n} \geq \int_{\Omega} \left( \psi'(w_n) - \frac{b}{\rho \alpha_0} |\psi(w_n)| \right) A(x,u_n)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla w_n \ dx \ (3.69)
$$

for all $n \in \mathbb{N}$. Now, taking $\beta_1 = 1$ and $\beta_2 = \frac{b}{\rho \alpha_0}$ in the definition of $\psi$, and denoting $h_n = \beta_1 \psi'(w_n) - \beta_2 |\psi(w_n)|$, from (3.62) and (3.64) it follows

$$
\frac{1}{2} \leq h_n(x) \leq \sigma_0(1 + \beta_2) \ \text{a.e. in } \Omega, \text{ for all } n \in \mathbb{N}; \ (3.70)
$$

while from (3.65) it is

$$
h_n(x) \to 1 \ \text{a.e. in } \Omega, \text{ as } n \to +\infty. \ (3.71)
$$

Moreover, it is

$$
\int_{\Omega} h_n A(x,u_n)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla w_n dx = \int_{\Omega} A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla w_n dx
+ \int_{\Omega} \left( h_n A(x,u_n) - A(x,u) \right) |\nabla u|^{p-2}\nabla u \cdot \nabla w_n \ dx
+ \int_{\Omega} h_n A(x,u_n)(|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla w_n \ dx,
$$

where (3.63) implies

$$
\int_{\Omega} A(x,u)|\nabla u|^{p-2}\nabla u \cdot \nabla w_n \ dx \to 0,
$$

while Hölder inequality, (3.57), and also (3.18), (3.60), (3.70), (3.71) and the Lebesgue Dominated Convergence Theorem, imply

$$
\int_{\Omega} (h_n A(x,u_n) - A(x,u)) |\nabla u|^{p-2}\nabla u \cdot \nabla w_n \ dx \to 0.
$$
Thus, the convexity condition

\( (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla w_n \geq 0 \) a.e. in \( \Omega \),

\((H_1)\), \((3.69)\) and \((3.70)\) give

\[
\varepsilon_{3,n} \geq \int_{\Omega} h_n A(x,u_n) (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla w_n \, dx
\]

\[
\geq \frac{\alpha_0}{2} \int_{\Omega} (|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u) \cdot \nabla w_n \, dx \geq 0.
\]

for a suitable \( \varepsilon_{3,n} \to 0 \). Whence, \( \|u_n - u\| \to 0 \).

\[ \square \]

4 Main result

In addition to the hypotheses \((H_0)-(H_1)\), \((h_0)\) and \((h_1)\), we assume

\((h_2)\) there exist \( \lambda^0 \in \mathbb{R} \) and a (Carathéodory) function \( g^0 : \Omega \times \mathbb{R} \to \mathbb{R} \) such that \( f(x,t) = \lambda^0 |t|^{p-2} t + g^0(x,t) \)

and

\[
\lim_{t \to 0} \frac{g^0(x,t)}{|t|^{p-1}} = 0 \quad \text{uniformly a.e. in } \Omega.
\]

From \((h_2)\) it follows

\[
\lim_{t \to 0} \frac{G^0(x,t)}{|t|^p} = 0 \quad \text{uniformly a.e. in } \Omega, \tag{4.1}
\]

where \( G^0(x,t) = \int_{\Omega} g^0(x,s) \, ds \).

Moreover, if we write \( A^0(x) = A(x,0) \), then \((H_0)\) implies \( A^0 \in L^\infty(\Omega) \), while from \((H_1)\) it follows \( A^0(x) \geq \alpha_0 > 0 \) a.e. in \( \Omega \). Furthermore, \((H_0)\) and \((H_3)\) imply \((3.18)\); whence,

\[
\lim_{t \to 0} A(x,t) = A^0(x) \quad \text{uniformly a.e. in } \Omega. \tag{4.2}
\]

For simplicity, as in \((3.19)\), we introduce the operator \( A_p^0 : u \in W_0^{1,p}(\Omega) \mapsto -\text{div}(A^0(x)|\nabla u|^{p-2}\nabla u) \in W^{-1,p'}(\Omega) \)

and denote its spectrum by \( \sigma(A_p^0) \).

For \( p = 0, \infty \), let

\[
P^p(u) = \int_{\Omega} A^p(x) |\nabla u|^p \, dx, \quad u \in W_0^{1,p}(\Omega),
\]
and let
\[ M^\sharp = \{ u \in W^{1,p}_0(\Omega) : I^\sharp(u) = 1 \}. \]

Since the hypotheses imply that
\[ A^\sharp \in L^\infty(\Omega) \quad \text{and} \quad A^\sharp(x) \geq \alpha_0 > 0 \text{ for a.e. } x \in \Omega, \tag{4.3} \]
then \( M^\sharp \subset W^{1,p}_0(\Omega) \setminus \{0\} \) is a bounded symmetric complete \( C^1 \)-Finsler manifold radially homeomorphic to the unit sphere in \( W^{1,p}_0(\Omega) \). Let
\[ \Psi(u) = \frac{1}{\int_\Omega |u|^p dx}, \quad u \in W^{1,p}_0(\Omega) \setminus \{0\}. \]

Then \( \lambda \in \sigma(A^\sharp) \) if and only if \( \lambda \) is a critical value of \( \Psi|_{M^\sharp} \) by the Lagrange multiplier rule.

Now, let \( F^\sharp \) denote the class of compact symmetric subsets of \( M^\sharp \) and set
\[ \lambda_k^\sharp := \inf_{M \in F_k^\sharp} \max_{u \in M} \Psi(u), \quad k \geq 1, \]
where \( F_k^\sharp = \{ M \in F^\sharp : i(M) \geq k \} \), and \( i \) is the cohomological index. Then \( \lambda_k^\sharp \in \sigma(A_k^\sharp) \) and \( 0 < \lambda_k^\sharp \nearrow +\infty \) (see [24, Proposition 3.52]). In particular,
\[ \lambda_1^\sharp \int_\Omega |u|^p dx \leq \int_\Omega A^\sharp(x)|\nabla u|^p dx \quad \text{for all } u \in W^{1,p}_0(\Omega). \tag{4.4} \]

Our main result is the following.

**Theorem 4.1.** Assume that \( p > N, (H_0)-(H_4) \) and \((h_0)-(h_2)\) hold, and

- \( A(x,\cdot) \) is an even function for a.a. \( x \in \Omega \) and \( f(x,\cdot) \) is an odd function for a.a. \( x \in \Omega \),
- \( \lambda^\infty \not\in \sigma(A^\infty_\infty) \).

If \( m, l \in \mathbb{N} \), \( l \neq m \), exist such that one of the two following conditions hold:

(i) \( l > m \) and \( \lambda_l^0 < \lambda^0 \) \quad \lambda^\infty < \lambda^\infty_{m+1};

(ii) \( l < m \) and \( \lambda^0 < \lambda^0_{l+1} \), \quad \lambda^\infty_m < \lambda^\infty;

then problem \((P)\) has at least \( |l - m| \) distinct pairs of nontrivial solutions.

From here on, let \( p > N \) and assume that the hypotheses of Theorem 4.1 hold. Thus, \( X = W^{1,p}_0(\Omega) \) and, from [14] and condition \((h_1)\),
\[ J(u) = \frac{1}{p} \int_\Omega (A(x,u)|\nabla u|^p - \lambda^\infty |u|^p) dx - \int_\Omega G^\infty(x,u) dx, \quad u \in W^{1,p}_0(\Omega), \]
Lemma 4.2. For any $u$ for $\exists 0, \infty$, we write

$$J(u) = \frac{1}{p} \int_\Omega (A(x,u) |\nabla u|^p - \lambda^0 |u|^p) \, dx - \int_\Omega G^0(x,u) \, dx, \quad u \in W_0^{1,p}(\Omega).$$

Furthermore, for $\exists 0, \infty$, we write

$$J^\varepsilon(u) = \frac{1}{p} \int_\Omega (A^\varepsilon(x) |\nabla u|^p - \lambda^\varepsilon |u|^p) \, dx = \frac{1}{p} (I^\varepsilon(u) - \lambda^\varepsilon |u|_p^p); \quad (4.5)$$

whence,

$$J(u) - J^\varepsilon(u) = \frac{1}{p} \int_\Omega (A(x,u) - A^\varepsilon(x)) |\nabla u|^p \, dx - \int_\Omega G^\varepsilon(x,u) \, dx, \quad (4.6)$$

for $u \in W_0^{1,p}(\Omega)$.

In order to prove our main result, we need the following lemmas.

**Lemma 4.2.** For any $\varepsilon > 0$, a suitable $r_\varepsilon > 0$ exists such that

$$u \in W_0^{1,p}(\Omega), \ |u|_\infty \leq r_\varepsilon \implies |J(u) - J^0(u)| \leq \frac{\varepsilon}{p} I^0(u). \quad (4.7)$$

**Proof.** Fixing any $\varepsilon > 0$, by (4.2), respectively (4.1), there is a $r_\varepsilon > 0$ such that

$$|t| \leq r_\varepsilon \implies |A(x,t) - A^0(x)| \leq \frac{\varepsilon \alpha_0}{2}, \ |G^0(x,t)| \leq \frac{\varepsilon \lambda^0}{2p} |t|^p \ \text{a.e. in } \Omega,$$

where $\alpha_0$ is as in $(H_1)$. Then, if $|u|_\infty \leq r_\varepsilon$ from (4.3), the estimates (4.1) and (4.4) imply

$$|J(u) - J^0(u)| \leq \frac{\varepsilon}{2p} \alpha_0 \int_\Omega |\nabla u|^p \, dx + \frac{\varepsilon}{2p} \lambda^0 \int_\Omega |u|^p \, dx \leq \frac{\varepsilon}{p} I^0(u). \quad \Box$$

**Lemma 4.3.** Let $\mathcal{K}_\infty$ be a compact subset of $\mathcal{M}_\infty$. Then, for any $\varepsilon > 0$ there exists a constant $C_\varepsilon = C(\mathcal{K}_\infty, \varepsilon) > 0$ such that

$$|J(Ru) - J^\infty(Ru)| < \frac{\varepsilon}{p} I^\infty(Ru) + C_\varepsilon \text{ for all } R \geq 0, \ u \in \mathcal{K}_\infty. \quad (4.8)$$

**Proof.** We organize the proof in different steps:

(a) if $\mathcal{K}$ is a compact subset of $W_0^{1,p}(\Omega)$, taking any $\varepsilon > 0$ there exists $\rho_\varepsilon = \rho(\mathcal{K}, \varepsilon) > 0$ such that

$$\int_{\Omega_{\rho_\varepsilon}} |\nabla u|^p \, dx < \varepsilon \text{ for all } u \in \mathcal{K}, \ \text{with } \Omega_{\rho_\varepsilon} = \{x \in \Omega : |u(x)| < \rho_\varepsilon\};$$

(b) if $\mathcal{K}$ is a compact subset of $W_0^{1,p}(\Omega)$, taking any $\varepsilon > 0$ there exists $R_\varepsilon^* = R^*(\mathcal{K}, \varepsilon) > 0$ such that

$$\left| \int_\Omega (A(x,Ru) - A^\infty(x)) |\nabla u|^p \, dx \right| < \varepsilon \text{ for all } R \geq R_\varepsilon^*, \ u \in \mathcal{K};$$
(c) if \( K_\infty \) is a compact subset of \( \mathcal{M}_\infty \), taking any \( \varepsilon > 0 \) a constant \( C_\varepsilon = C(K_\infty, \varepsilon) > 0 \) exists such that the estimate (4.8) holds.

**Step (a)** Firstly, we claim that for any \( u \in W^{1,p}_0(\Omega) \) and \( \varepsilon > 0 \) there exists \( r_\varepsilon > 0 \) such that

\[
\int_{\Omega_{r_\varepsilon}} |\nabla u|^p \, dx < \varepsilon. \tag{4.9}
\]

In fact, the monotonicity property of the Lebesgue integral implies

\[
\lim_{r \to 0} \int_{\Omega_r} |\nabla u|^p \, dx = \int_{\Omega_0^u} |\nabla u|^p \, dx, \quad \text{with } \Omega_0^u = \{ x \in \Omega : u(x) = 0 \}, \tag{4.10}
\]

where

\[
\int_{\Omega_0^u} |\nabla u|^p \, dx = 0 \tag{4.11}
\]

not only if \( \text{meas}(\Omega_0^u) = 0 \) but also if \( \text{meas}(\Omega_0^u) > 0 \) as it is known that

\[
\text{meas}(\{ x \in \Omega : u(x) = 0, \nabla u(x) \neq 0 \}) = 0
\]

(see, e.g., [19, Ex. 17, pp. 292]). Whence, (4.9) follows from (4.10) and (4.11).

Now, arguing by contradiction, assume that for the compact \( K \) the thesis in **Step (a)** does not hold; hence, there exist a constant \( \bar{\varepsilon} > 0 \) and a sequence \((u_n)_n \subset K\) such that

\[
\int_{\Omega_n} |\nabla u_n|^p \, dx \geq \bar{\varepsilon} \quad \text{for all } n \geq 1, \quad \text{with } \Omega_n = \{ x \in \Omega : |u_n(x)| < \frac{1}{n} \}. \tag{4.12}
\]

As \( K \) is compact, then \( \bar{u} \in K \) exists such that, up to subsequences,

\[
\|u_n - \bar{u}\| \to 0, \quad \text{and so } u_n \to \bar{u} \text{ a.e. in } \Omega. \tag{4.13}
\]

Now, taking \( \varepsilon < \bar{\varepsilon} \), from (4.9) applied to \( \bar{u} \), there exists \( \bar{r} > 0 \) such that

\[
\int_{\Omega_\bar{r}} |\nabla \bar{u}|^p \, dx < \frac{\varepsilon}{2}.
\]

Then, taking a \( \rho < \bar{r} \), if \( n \) is large enough, not only we have \( \Omega_n \subset \Omega_{u_n} \) but also from (4.13) it follows that

\[
\int_{\Omega_{u_n}} |\nabla u_n|^p \, dx < \varepsilon
\]

in contradiction with (4.12).

**Step (b)** For the compactness of \( K \), a constant \( \gamma_K > 0 \) exists such that

\[
\|u\|^p \leq \gamma_K \quad \text{for all } u \in K. \tag{4.14}
\]
Furthermore, taking $\varepsilon > 0$, let $\rho_{\varepsilon} > 0$ be as in Step (a) so that
\[ \int_{\Omega_{\rho_{\varepsilon}}} \left| \nabla u \right|^p \, dx < \frac{\varepsilon}{2(b + |A^\infty|_{\infty})} \quad \text{for all } u \in \mathcal{K}, \quad (4.15) \]
where $b > 0$ is as in (3.18). On the other hand, from (H2), a constant $\sigma_{\varepsilon} > 0$ exists such that
\[ |A(x, t) - A^\infty(x)| < \frac{\varepsilon}{2\gamma} \quad \text{for a.e. } x \in \Omega, \text{ if } |t| \geq \sigma_{\varepsilon}, \quad (4.16) \]
then, taking $R_{\ast}^\varepsilon = \frac{\sigma_{\varepsilon}}{\rho_{\varepsilon}}$, for all $u \in \mathcal{K}$, $R \geq R_{\ast}^\varepsilon$, from (3.18), (4.14) – (4.16) we have
\[
\left| \int_{\Omega} (A(x, Ru) - A^\infty(x)) \left| \nabla u \right|^p \, dx \right| \leq \int_{\Omega_{\rho_{\varepsilon}}} (|A(x, Ru)| + |A^\infty(x)|) \left| \nabla u \right|^p \, dx \\
+ \int_{\Omega \setminus \Omega_{\rho_{\varepsilon}}} |A(x, Ru) - A^\infty(x)| \left| \nabla u \right|^p \, dx < \varepsilon.
\]
Step (c) Consider $\mathcal{K}_{\infty}$, compact subset of $\mathcal{M}_{\infty}$, and take any $\varepsilon > 0$. Firstly, let us remark that, by (3.8), there is a $L_{\varepsilon} > 0$ such that
\[ |G^\infty(x, t)| \leq \frac{\varepsilon \lambda_{\infty}}{2p} |t|^p + L_{\varepsilon} \quad \text{for a.a. } x \in \Omega \text{ and all } t \in \mathbb{R}. \]
Hence, (4.4) implies that
\[ \left| \int_{\Omega} G^\infty(x, u) \, dx \right| \leq \frac{\varepsilon \lambda_{\infty}}{2p} I^\infty(u) + L_{\varepsilon} \text{meas}(\Omega) \quad \text{for all } u \in W^{1,p}_0(\Omega). \quad (4.17) \]
Now, taking $u \in \mathcal{K}_{\infty}$, from Step (b) applied to $\mathcal{K}_{\infty}$ and $\frac{\rho_{\varepsilon}}{\rho}$, a constant $R_{\ast}^\varepsilon > 0$ exists such that two cases may occur.
If $R \leq R_{\ast}^\varepsilon$, then (3.18), (4.3), (4.6) and (4.17) imply that
\[ |J(Ru) - J^\infty(Ru)| \leq \frac{(R_{\ast}^\varepsilon)^p}{p} \left( b + |A^\infty|_{\infty} \right) \gamma_{\mathcal{K}_{\infty}} + \frac{\varepsilon}{2p} I^\infty(Ru) + L_{\varepsilon} \text{meas}(\Omega). \quad (4.18) \]
On the contrary, if $R > R_{\ast}^\varepsilon$, then by (4.6), (4.17) and Step (b), as $\mathcal{K}_{\infty} \subset \mathcal{M}_{\infty}$, it follows
\[ |J(Ru) - J^\infty(Ru)| \leq R^p \frac{\varepsilon}{2p} I^\infty(u) + \frac{\varepsilon}{2p} I^\infty(Ru) + L_{\varepsilon} \text{meas}(\Omega) \]
\[ = R^p \frac{\varepsilon}{2p} I^\infty(u) + \frac{\varepsilon}{2p} I^\infty(Ru) + L_{\varepsilon} \text{meas}(\Omega) \]
\[ = \frac{\varepsilon}{p} I^\infty(Ru) + L_{\varepsilon} \text{meas}(\Omega). \quad (4.19) \]
Thus, (4.8) follows from (4.18) and (4.19) if we choose $C_{\varepsilon} > 0$ large enough. \qed

25
Now, we are ready to prove Theorem 4.1.

**Proof of Theorem 4.1.** Firstly, we note that by Proposition 3.6, \( J \) satisfies the \((PS)_\beta\) condition for all \( \beta \in \mathbb{R} \).

Then, we split the proof in two steps.

(i) Case \( l > m \). Let \( \mathcal{A} \) denote the class of symmetric subsets of \( W^{1,p}_0(\Omega) \setminus \{0\} \) and

\[
\mathcal{A}_k = \{ M \in \mathcal{A} : M \text{ is compact and } i(M) \geq k \}.
\]

Set

\[
c_k := \inf_{M \in \mathcal{A}_k} \max_{u \in M} J(u), \quad m + 1 \leq k \leq l.
\]

We will show that \(-\infty < c_{m+1} \leq \cdots \leq c_l < 0\), so we can apply Theorem 2.2.

In order to see that \( c_l < 0 \), let \( \epsilon > 0 \) be so small that \((1 + \epsilon) \lambda_0 \leq \lambda_0 \). Then, there is \( M_0 \in \mathcal{F}_l^0 \) such that \( \Psi(u) \leq \lambda_0 + \epsilon \) for all \( u \in M_0 \). Let \( r = \frac{r}{\gamma_\infty} \) with \( r_\epsilon \) as in Lemma 4.2 and \( \gamma_\infty \) as in (3.3), and let

\[
\tilde{M}_0 = \{ v = r \frac{u}{\|u\|} : u \in M_0 \}.
\]

As the map \( v \in M_0 \mapsto u \in \tilde{M}_0 \) is an odd homeomorphism, then \( \tilde{M}_0 \) is compact and \( i(\tilde{M}_0) = i(M_0) \geq l \) by (4.2), so \( \tilde{M}_0 \in \mathcal{A}_l \). By (4.5) and (4.7), for any \( v \in \tilde{M}_0 \) we have

\[
J(v) \leq J^0(v) + \frac{\epsilon}{p} I^0(v) = \frac{1}{p} \left( (1 + \epsilon) I^0(v) - \lambda_0 |v|^p \right)
\]

\[
= \frac{1}{p} \left( (1 + \epsilon) I^0(u) - \lambda_0 |u|^p \right) \leq \frac{1}{p} \left( 1 + \epsilon - \frac{\lambda_0}{\lambda_0^p + \epsilon} \right) < 0;
\]

so \( c_l < 0 \).

For seeing that \( c_{m+1} > -\infty \), take any \( M_\infty \in \mathcal{A}_{m+1} \) and let \( \epsilon > 0 \) be so small that \((1 - \epsilon) \lambda_0 \). Then, consider

\[
\tilde{M}_\infty = \{ u = v/|I^\infty(v)|^{1/p} : v \in M_\infty \} \subset \mathcal{M}\infty.
\]

As the map \( v \in M_\infty \mapsto u \in \tilde{M}_\infty \) is an odd homeomorphism, then \( \tilde{M}_\infty \) is compact and \( i(\tilde{M}_\infty) = i(M_\infty) \geq m + 1 \) by (4.2). So, \( \tilde{M}_\infty \in \mathcal{F}_{m+1} \); hence,

\[
\max_{u \in M_\infty} \Psi(u) \geq \lambda_0 \in M_{m+1}.
\]

Now, let \( C_\epsilon \) be as in Lemma 4.3 with \( K_\infty = M_\infty \). By (4.5) and (4.8), for any \( v \in M_\infty \), it results

\[
J(v) \geq J^\infty(v) - \frac{\epsilon}{p} I^\infty(v) - C_\epsilon \geq \frac{1}{p} \left( (1 - \epsilon) I^\infty(v) - \lambda_\infty |v|^p \right) - C_\epsilon
\]

\[
= \frac{I^\infty(v)}{p} \left( (1 - \epsilon - \lambda_\infty |u|^p \right) - C_\epsilon,
\]

26
with $I^∞(v) ≥ 0$. Whence,

$$\max_{v ∈ M_∞} J(v) ≥ -C_ε;$$

thus $c_{m+1} ≥ -C_ε$.

(ii) Case $l < m$. Let $A^*$ denote the class of symmetric subsets of $W_0^{1,p}(Ω)$, $Γ$ the group of odd homeomorphisms $γ$ of $W_0^{1,p}(Ω)$ such that $\gamma|_{\{J ≤ 0\}}$ is the identity, and $i^*$ the pseudo-index related to $i$, $∂B^W_B(0)$, and $Γ$, where $W = W_0^{1,p}(Ω)$. Then, let

$$A_k^* = \{ M ∈ A^* : M \text{ is compact and } i^*(M) ≥ k \}$$

and set

$$c^*_k := \inf_{M ∈ A_k^*} \max_{u ∈ M} J(u), \quad l + 1 ≤ k ≤ m.$$ 

We will show that $0 < c^*_{l+1} ≤ \cdots ≤ c^*_m < +∞$ if $r > 0$ is sufficiently small, and then we can apply Theorem 2.4.

In order to see that $c^*_{l+1} > 0$, fix $ε > 0$ so small that $(1 - ε) λ^0_{l+1} > λ^0$, define $r = r_ε$ with $r_ε$ as in Lemma 4.2 and $γ_∞$ as in (3.3), take any $M_0^* ∈ A_{l+1}^*$, and consider

$$\tilde{M}_0^* = \{ u = v[1_{0}^∞(v)]^{1/p} : v ∈ M_0^* ∩ ∂B^W_B(0) \} \subset M_0^*.$$

The map $v ∈ M_0^* ∩ ∂B^W_B(0) ↦ u ∈ M_0^*$ is an odd homeomorphism; hence, $\tilde{M}_0^*$ is compact and

$$i(\tilde{M}_0^*) = i(M_0^* ∩ ∂B^W_B(0)) ≥ i^*(M_0^*) ≥ l + 1$$

by (12). So $\tilde{M}_0^* ∈ F_{l+1}^0$ and hence

$$\max_{u ∈ \tilde{M}_0^*} Ψ(u) ≥ λ^0_{l+1}.$$ 

By 4.3 and 4.7, for any $v ∈ M_0^* ∩ ∂B^W_B(0)$ we have

$$J(v) ≥ J^0(v) - \frac{ε}{p} I^0(v) = \frac{1}{p} ((1 - ε) I^0(v) - λ^0 | v|^p_p) = \frac{I^0(v)}{p} (1 - ε - \frac{λ^0 | u|^p_p}{λ^0_{l+1}}).$$

Since $I^0(v) ≥ α_0 \| v \|_p^p$, it results

$$δ := \inf_{v ∈ ∂B^W_B(0)} I^0(v) ≥ α_0 r^p > 0.$$ 

Whence, it follows that

$$\max_{v ∈ M_0^*} J(v) ≥ \frac{δ}{p} \left( 1 - ε - \frac{λ^0}{λ^0_{l+1}} \right) > 0;$$

27
so $c^*_m < +\infty$, let $\varepsilon > 0$ be so small that $(1 + \varepsilon)\lambda^\infty_m + \varepsilon < \lambda^\infty$. There is a $M^*_\infty \in \mathcal{F}^m_\infty$ such that $\Psi(u) \leq \lambda^\infty_m + \varepsilon$ for all $u \in M^*_\infty$. Let $C_\varepsilon$ be as in Lemma 4.3 with $K_\infty = M^*_\infty$ and consider

$$
\tilde{M}^*_R = \{v = Ru : u \in M^*_\infty\}, \quad R > 0.
$$

The map $u \in M^*_\infty \mapsto v \in \tilde{M}^*_R$ is an odd homeomorphism; hence, $\tilde{M}^*_R$ is compact and $i(\tilde{M}^*_R) = i(M^*_\infty) \geq m$ by (i.2). By (4.5) and (4.8), for any $v \in \tilde{M}^*_R$ we have

$$
\mathcal{J}(v) \leq \frac{R_p}{p} \left( (1 + \varepsilon) I^\infty(u) - \lambda^\infty |u|^p_p \right) + C_\varepsilon.
$$

Fixing $R$ so large that the last term of the previous estimates is $\leq 0$, consider

$$
\tilde{M}^*_\infty = \{tv : v \in \tilde{M}^*_R, t \in [0, 1]\} \in \mathcal{A}^*.
$$

Since $\tilde{M}^*_R$ is compact, so is $\tilde{M}^*_\infty$. Since $\mathcal{J}(v) \leq 0$ on $\tilde{M}^*_R$, for any $\gamma \in \Gamma$ it results $\gamma|_{\tilde{M}^*_R}$ is the identity. Thus, by applying the piercing property (i.2) to

$$
A = \tilde{M}^*_R, \quad A_0 = B^W_r(0), \quad A_1 = W^{1,p}_0(\Omega) \setminus B^W_r(0),
$$

$$
\varphi : (v, t) \in A \times [0, 1] \mapsto \gamma(tv) \in A_0 \cup A_1
$$

(r as in the first part of the proof of this case), we have

$$
i(\gamma(M^*_\infty) \cap \partial B^W_r(0)) = i(\varphi(A \times [0, 1]) \cap A_0 \cap A_1) \geq i(A) = i(\tilde{M}^*_R) \geq m.
$$

So $i^*(\tilde{M}^*_\infty) \geq m$, and hence $\tilde{M}^*_\infty \in \mathcal{A}^*_m$. Then,

$$
c^*_m \leq \max_{u \in \tilde{M}^*_\infty} \mathcal{J}(u) < +\infty.
$$

References

[1] H. Amann, E. Zehnder, *Nontrivial solutions for a class of nonresonance problems and applications to nonlinear differential equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 7 (1980), 539-603.

[2] A. Anane, J.P. Gossez, *Strongly nonlinear elliptic problems near resonance: a variational approach*, Comm. Partial Differential Equations 15 (1990), 1141-1159.

[3] D. Arcoya, L. Boccardo, *Critical points for multiple integrals of the calculus of variations*, Arch. Rational Mech. Anal. 134 (1996), 249-274.

[4] D. Arcoya, L. Orsina, *Landesman–Lazer conditions and quasilinear elliptic equations*, Nonlinear Anal. 28 (1997), 1623-1632.
Abstract critical point theorems and applications to some nonlinear problems with “strong” resonance at infinity, Nonlinear Anal. 7 (1983), 981–1012.

R. Bartolo, A.M. Candela, A. Salvatore, p-Laplacian problems with nonlinearities interacting with the spectrum, NoDEA Nonlinear Differential Equations Appl. 20 (2013), 1701-1721. DOI:10.1007/s00030-013-0226-1.

V. Benci, On critical point theory for indefinite functionals in the presence of symmetries, Trans. Amer. Math. Soc. 274 (1982), 533–572.

L. Boccardo, F. Murat, J.P. Puel, Existence of bounded solutions for nonlinear elliptic unilateral problems, Ann. Mat. Pura Appl. IV Ser. 152 (1988), 183-196.

A.M. Candela, E. Medeiros, G. Palmieri, K. Perera, Weak solutions of quasilinear elliptic systems via a cohomological index, Topol. Methods Nonlinear Anal. 36 (2010), 1–18.

A.M. Candela, G. Palmieri, Infinitely many solutions of some nonlinear variational equations, Calc. Var. Partial Differential Equations 34 (2009), 495–530.

A.M. Candela, G. Palmieri, Some abstract critical point theorems and applications, in: “Dynamical Systems, Differential Equations and Applications” (X. Hou, X. Lu, A. Miranville, J. Su & J. Zhu Eds), Discrete Contin. Dyn. Syst., AIMS Press (2009), 133-142.

A.M. Candela, G. Palmieri, An abstract three critical points theorem and applications, Proc. Dynamic Systems and Applications 6 (2012), 70-77.

A.M. Candela, G. Palmieri, Multiplicity results for some quasilinear equations in lack of symmetry, Adv. Nonlinear Anal. 1 (2012), 121-157.

A.M. Candela, G. Palmieri, Multiple solutions for p-Laplacian type problems with an asymptotically p-linear term, In: Proc. Workshop in “Nonlinear Differential Equations” (to appear).

A.M. Candela, G. Palmieri, K. Perera, Nontrivial solutions of some quasilinear problems via a cohomological local splitting, Nonlinear Anal. 73 (2010), 2001–2009.

D.G. Costa, C.A. Magalhães, Existence results for perturbations of the p-Laplacian, Nonlinear Anal. 24 (1995), 409-418.

G. Dinca, P. Jebelean, J. Mawhin, Variational and topological methods for Dirichlet problems with p-Laplacian, Port. Math. (N.S.) 58 (2001), 339–378.

P. Drábek, S. Robinson, Resonance problems for the p-Laplacian, J. Funct. Anal. 169 (1999), 189-200.
[19] L.C. Evans, *Partial Differential Equations*, Grad. Stud. Math. *19*, Amer. Math. Soc., Providence RI, 1998 (Reprint 2002).

[20] E.R. Fadell, P.H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. *45* (1978), 139–174.

[21] S. Liu, S. Li, *Existence of solutions for asymptotically ‘linear’ p-Laplacian equations*, Bull. London Math. Soc. *36* (2004), 81-87.

[22] G. Li, H.S. Zhou, *Asymptotically linear Dirichlet problem for the p-Laplacian*, Nonlinear Anal. *43* (2001), 1043-1055.

[23] G. Li, H.S. Zhou, *Multiple solutions to p-Laplacian problems with asymptotic nonlinearity as \( u^{p-1} \) at infinity*, J. London Math. Soc. *65* (2002), 123-138.

[24] K. Perera, R.P. Agarwal, D. O’Regan, *Morse Theoretic Aspects of p-Laplacian Type Operators*, Math. Surveys Monogr. *161*, Amer. Math. Soc., Providence RI, 2010.

[25] K. Perera, A. Szulkin, *p-Laplacian problems where the nonlinearity crosses an eigenvalue*, Discrete Contin. Dyn. Syst. *13* (2005), 743-753.