AN STRATIFICATION OF $B^4(2, K_C)$ OVER A GENERAL CURVE.

ABEL CASTORENA AND GRACIELA REYES-AHUMADA

ABSTRACT. For a general curve $C$ of genus $g \geq 10$, we show that the Brill-Noether locus $B^4(2, K_C)$ contains irreducible sub-varieties $B_3 \supset B_4 \supset \cdots \supset B_n$, where $B_n$ is of dimension $3g - 10 - n$ and $B_3$ is an irreducible component of the expected dimension $3g - 13$.

1. Introduction

We study the Brill-Noether locus $B^4(2, K_C)$ that parametrizes classes of stable rank two vector bundles with canonical determinant and four sections over a general irreducible complex projective curve $C$.

In [2], the authors construct an irreducible component of $B^4(2, K_C)$ of the expected dimension by studying the space of extensions of line bundles following the spirit in [3]. In this work we generalize such construction over the general curve for the case of rank two bundles with four sections in the following sense: we exhibit a chain of irreducible closed subloci $B_3 \supset \cdots \supset B_n$ inside $B^4(2, K_C)$ where $B_3$ is an irreducible component of expected dimension $3g - 13$ ($B_3$ is the component constructed in [2]) and the dimension of any stratum $B_n$ is $3g - 10 - n$. In order to give such stratification we analyze the space $Ext^1 := Ext^1(K_C(-D_n), O_C(D_n))$ which parametrizes extensions of the form

$$u : 0 \to O_C(D_n) \to E_u \to K_C(-D_n) \to 0;$$

where $D_n$ is a general effective divisor of degree $n$ over $C$. We construct the locus $B_n$ as the image of an irreducible determinantal locus $\Delta_n \subset \mathbb{P}Ext^1$ through the rational map $\pi : \Delta_n \to U(2, K_C), [u] \to [E_u]$, where $U(2, K_C)$ denote the moduli space of rank two stable bundles with canonical determinant. By looking at the deformations inside the global space of extensions constructed in [3] and let the divisor $D_n$ to vary, we conclude that such $B_n$ stratify the component $B_3$ of $B^4(2, K_C)$. Our main result is Theorem 3.1 and we prove it by applying from Lemma 3.2 to Lemma 3.5.

With these Lemmas we construct an stratification of $B^4(2, K_C)$ by the loci $B_n$.

We would like to emphasize that in this work we do not find a new component of the Brill-Noether locus $B^4(2, K_C)$, but we do give a partial answer to the problem to describe deformations of a regular component by studying deformations of extensions of line bundles. An interesting question (which remains open) coming from

2000 Mathematics Subject Classification. 14C20(primary), 14J26(secondary).

The first author was supported by PAPIIT IN100716, Universidad Nacional Autónoma de México).

Second author was supported by a FORDECYT(CONACyT, México) fellowship.
Given a line bundle over a smooth curve defines $B_{n+1}$ inside $B_n$ at least for curves of small genus.

2. Extensions of Line Bundles and Geometrically Ruled Surfaces.

2.1. Global space of extensions. We restrict our attention to special line bundles $L, N$ over a smooth curve $C$, and we assume that for any integer $t \geq 1$, $h^1(N) \geq t$ and $h^0(L) \geq \max\{1, h^1(N) - t\}$. For a more general context, see [3].

Given a line bundle $M$ over a smooth curve, we denote by $\rho_M$ (or simply $\rho$, when $M$ is understood) the Brill-Noether number of $M$.

For any $u \in H^1(N \otimes L^\vee) \simeq \text{Ext}^1(L, N)$, consider the coboundary map $\partial_u : H^0(L) \to H^1(N)$ and denote by $\text{coker}(\partial_u) := \dim(\text{Cokernel}(\partial_u))$. Consider the cup product map $\cup : H^0(L) \otimes H^1(N \otimes L^\vee) \to H^1(N)$. We have that $\partial_u(s) = s \cup u$. By Serre duality, $\cup$ is equivalent to consider for any subspace $W \subseteq H^0(K_C \otimes N^\vee)$, the multiplication map $\mu_W : H^0(L) \otimes W \to H^0(K_C \otimes L \otimes N^\vee)$.

For $u \in H^1(N \otimes L^\vee) \simeq \text{Ext}^1(L, N) \simeq H^0(C, K_C \otimes L \otimes N^\vee)$, we denote by $H_u := \{u = 0\} \subset H^0(C, K_C \otimes L \otimes N^\vee)$ the hyperplane defined by $u$, so if $u \in \text{Ext}^1(L, N)$ is such that $\text{coker}(\partial_u) \geq t$, then $W := \text{Im}(\partial_u) = H^0(K_C - N)$ satisfies that $\text{Im}(\mu_W) \subset H_u$, thus we define:

$$W_t = \{u \in \text{Ext}^1(L, N) \mid \exists W \subset H^0(K_C - N) \text{ with } \dim(W) \geq t \text{ and } \text{Im}(\mu_W) \subset H_u\}.$$ Note that $W_t$ has a natural structure of determinantal variety.

Let $C$ be a general curve of genus $g \geq 3$. Let $0 \to N \to E \to L \to 0$ be an exact sequence such that $\deg(N) = d - \delta > 0$, $\deg(L) = \delta > 0$ and $h^0(L) \cdot h^1(L) > 0$, $h^0(N) \cdot h^1(N) > 0$. Let $\ell := h^0(L), j := h^1(L), n := h^0(N), r := h^1(N)$. Set $Y := \text{Pic}^{\ell-r}(C) \times W_{\delta-1}(C), Z := W_{n-1}(C) \times W_{\delta-1}(C) \subset Y$. One has $\dim(Y) = g + \rho_L$ and $\dim(Z) = \rho_L + \rho_N$. Since $C$ is general, we have that $Y$ and $Z$ are irreducible when $\rho > 0$; otherwise one replace the (reducible) zero-dimensional Brill-Noether locus with one of its irreducible components to construct $Y$ and $Z$ as above.

When $2\delta - d \geq 1$ one can construct a vector bundle $E \to Y$ of rank $m = 2\delta - d + g - 1$ (see [1], p. 176-180), together with a projective bundle morphism $\gamma : \mathbb{P}(E) \to Y$ where, for $y = (N, L) \in Y$, the fiber $\gamma^{-1}(y) = \mathbb{P}(\text{Ext}^1(N, L)) = \mathbb{P}$. We have that $\dim(\mathbb{P}(E)) = \dim(Y) + m - 1$ and $\dim(\mathbb{P}(E)|_Z) = \dim(Z) + m - 1$. Since (semi)stability is an open condition, for $2\delta - d \geq 2$ there is an open, dense subset $\mathbb{P}(E)^0 \subseteq \mathbb{P}(E)$ and a morphism $\pi_{\delta, \delta} : \mathbb{P}(E)^0 \to U_C(d)$. In [2] the authors showed that on a general curve $C$ of genus $g \geq 8$, there exists an irreducible component of $W_3$, $\Delta_3 \subset \mathbb{P}(E)$ of dimension $3g - 13$. Moreover, this component fills up an irreducible component $B_3 \subseteq B^4(2, K_C)$ of dimension $3g - 13$.

Following the above construction, We will show in the next section that for $n \geq 4$, there exist closed and irreducible subschemes $B_n$ all of them contained in $B_3$ such that $B_{n+1} \subset B_n$ and $\dim(B_n) = 3g - 10 - n$.

2.2. Unisecant curves in geometrically ruled surfaces. Let $E$ be a rank-two vector bundle over a smooth irreducible complex projective curve $C$ of genus $g$. Let $S := \mathbb{P}(E)$ be the (geometrically) ruled surface associated to it, with structure
map $p : S \to C$. For any $x \in C$ we denote by $f_x = p^{-1}(x) \simeq \mathbb{P}^1$ and we write $f_D = p^*(D)$ for $D \in \text{Div}(C)$. We denote by $f$ a general fiber of $p$ and by $\mathcal{O}_S(1)$ the tautological line bundle on $S$. We recall that there is a one-to-one correspondence between sections $\Gamma$ of $S$ and surjective maps $E \to L$ with $L$ a line bundle on $C$ (cf. [6], chapter V). One has an exact sequence $0 \to N \to E \to L \to 0$, where $N \in \text{Pic}(C)$. The surjection $E \to L$ induces an inclusion $\Gamma = \mathbb{P}(L) \subset S = \mathbb{P}(E)$. If $L = \mathcal{O}_C(B)$, $B \in \text{Div}(C)$ with $b = \deg(B)$, then $b = H \cdot \Gamma$ and $\Gamma \sim H + f_N$ where $N = L \otimes \det(E)^\vee \in \text{Pic}(C)$.

An element $\Gamma \in |H + f_D|$ is called an unisecant curve, and irreducible unisecants are called sections of $S$. For any $m \in \mathbb{N}$, denote by $\text{Div}^{1,m}(S)$ the Hilbert scheme of unisecant curves of $S$ of degree $m$ with respect to $\mathcal{O}_S(1)$. Since elements of $\text{Div}^{1,m}(S)$ correspond to quotients of $E$, therefore $\text{Div}^{1,m}(S)$ can be endowed with a natural structure of Quot-Scheme (cf. [7], section 4.4), and one has an isomorphism (see e.g. [3], Section 2.4.)

$$\Phi_{1,m} : \text{Div}^{1,m}(S) \xrightarrow{\simeq} \text{Quot}_{E,m-g+1}, \quad \Gamma \to (E \to L)$$

Let $\Gamma \in \text{Div}^{1,m}(S)$ be. We say that:

(i). $\Gamma$ is linearly isolated if $\dim(\mathcal{O}_S(\Gamma)) = 0$;

(ii). $\Gamma$ is called special unisecant if $h^1(\Gamma, \mathcal{O}_S(1) \otimes \mathcal{O}_{\Gamma}) > 0$.

Inside $\text{Div}^{1,m}(S)$, we consider the scheme $S^{1,n}$ that parametrizes degree $n$, special unisecant curves of $S$.

Let $\mathcal{F} \subset \text{Div}^{1,n}(S)$ be a subscheme, and let $\Gamma$ be a special unisecant of $S$. Assume that $\Gamma \in \mathcal{F}$, where $\mathcal{F} \subset \text{Div}^{1,n}(S)$, $\Gamma$ is called specially isolated in $\mathcal{F}$ if $\dim_{\Gamma}(\mathcal{F} \cap S^{1,n}) = 0$.

3. Construction of components in the space of extensions.

The following construction is a generalization of the one given in [2], and this is an adapted argument due to Robert Lazarsfeld.

Lemma 3.1. Fix an integer $n \geq 4$. Let $C$ be a non-hyperelliptic curve of genus $g \geq n + 5$. Let $D_n$ be a general effective divisor of degree $n$ over $C$. There exists a vector bundle $E \to C$ satisfying the following properties:

i: rank($E$) = 2;

ii: det($E$) = $K_C(-D_n)$;

iii: $h^0(E) = 3$;

iv: $E$ is globally generated;

v: $h^0(E^\vee) = 0$.

Proof. Set $k = g - 2 - n$ and let $D_k$ be a general effective divisor of degree $k$. Let $\text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k))$ be the space parametrizing extensions of the form

$$u : 0 \to \mathcal{O}_C(D_k) \to E \to K_C(-D_n - D_k) \to 0.$$
Since $\text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k)) = H^1(D_k - K_C + D_n + D_k) = H^0(2K_C - D_n - 2D_k)^\vee$, then $\dim(\text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k))) = 3g - 3 - n - 2k = 3g - 3 - n - 2g + 4 + 2n = g + n + 1$. Note that for such extensions $u$, the corresponding bundles $E$ satisfy $\text{rank}(E) = 2$ and $\text{det}(E) = K_C(-D_n)$. Consider the coboundary map
\[ \partial : \text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k)) \to H^0(K_C(-D_n - D_k))^\vee \otimes H^1(\mathcal{O}_C(D_k)) \]
which associates to an extension $u$ its coboundary map in cohomology
\[ \partial_u : H^0(K_C(-D_n - D_k)) \to H^1(\mathcal{O}_C(D_k)) \]
For these extensions $u \in \text{Ker}(\partial)$ we have that $h^0(E) = 3$. Since the map $\partial$ is dual to the map $\mu : H^0(K_C(-D_n - D_k)) \otimes H^0(K_C(-D_k)) \to \text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k))^\vee$, then $\dim(\text{Ker}(\partial)) = \text{coker}(\mu)$. By the base-point-free pencil trick to the pencil defined by $K_C(-D_n - D_k)$, we have that $\text{Ker}(\mu) \cong H^0(K_C(-D_k) - (K_C(-D_n - D_k))) = H^0(\mathcal{O}_C(D_n))$, since $h^0(\mathcal{O}_C(D_n)) = 1$, then $\dim(\text{Ker}(\mu)) = 1$ and $\dim(\text{Ker}(\partial)) = \text{coker}(\mu) = (g + n + 1) - (2g - k) - 1 = g - n - 2 = k$.
Consider now $u \in \text{Ker}(\partial)$. For every $x \in C$, $h^0(E(-x)) \geq 1$, if $x \notin \text{Supp}(D_k)$, then tensoring $u$ by $\mathcal{O}(-x)$ we get
\[ 0 \to \mathcal{O}_C(D_k - x) \to E(-x) \to K_C(-D_n - D_k - x) \to 0. \]
Then $h^0(E(-x)) \leq h^0(\mathcal{O}_C(D_k - x)) + h^0(K_C(-D_n - D_k - x)) = 1$ and the equality holds. Then $E$ could fail to be generated at most at the points in $\text{Supp}(D_k)$.
Suppose $E$ is not generated at $x \in \text{Supp}(D_k)$, then the extension $u$ induces an extension
\[ u' : 0 \to \mathcal{O}_C(D_k - x) \to E' \to K_C(-D_n - D_k) \to 0, \]
such that $u' \in \text{Ker}(\partial') : \text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k - x)) \to H^0(K_C(-D_n - D_k))^\vee \otimes H^1(\mathcal{O}_C(D_k - x)).$
Note that $\text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k - x)) = H^1(D_k - x - K_C + D_n + D_k) = H^0(2K_C(-D_n - 2D_k + x))^\vee$, then $\dim(\text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k - x))) = h^0(2K_C-D_n-2D_k+x) = 3g-3-n-2k+1 = 3g-3-n-2g+4+2n+1 = g+n+2$.
The coboundary map $\partial'$ is dual to $\mu' : H^0(K_C(-D_n - D_k)) \otimes H^0(K_C(-D_k + x)) \to \text{Ext}^1(K_C(-D_n - D_k), \mathcal{O}_C(D_k - x))^\vee$, then $\dim(\text{Ker}(\partial')) = \text{coker}(\mu')$ and by the base-point-free pencil trick, $\text{Ker}(\mu') \cong H^0(K_C(-D_k + x) - K_C(-D_n - D_k)) = H^0(\mathcal{O}_C(D_n + x))$, then $\dim(\text{Ker}(\mu')) = 1$ and $\dim(\text{Ker}(\partial')) = \text{coker}(\mu') = (g + n + 2) - (2g - k + 1) - 1 = g - n - 3$.
thus, the locus of bundles which are not generated has codimension at least one inside $\text{Ker}(\partial)$, so there exists a bundle $E$ which is globally generated.
Suppose there is a non-zero section $s \in H^0(E^\vee)$. Then $s$ induces an injection $\mathcal{O}_C \hookrightarrow E^\vee$ by tensoring by $K_C(-D_n)$ and using the isomorphism $E \cong E^\vee \otimes K_C(-D_n)$ we
Let $B$ be a non-hyperelliptic curve of genus $g \geq n + 5$, and let $D_n$ be a general effective divisor of degree $n$ over $C$. For the general $V \in G(3, H^0(K_C - D_n))$, the multiplication map
\[ \mu_V : V \otimes H^0(K_C - D_n) \to H^0(2K_C - 2D_n) \]
has kernel of dimension three.

**Proof.** By uppersemicontinuity it is enough to show there exists an element $V \in G(3, H^0(K_C - D_n))$ such that $dim(Ker(\mu_V)) = 3$. Consider the bundle $E$ from the previous Lemma. Since $E$ is globally generated we have the exact sequence
\[ 0 \to (K_C(-D_n))^\vee \to H^0(E) \otimes O_C \to E \to 0, \]
by dualizing this sequence and computing the cohomology, we get an injective map $H^0(E)^\vee \to H^0(K_C - D_n)$. Denote by $V$ the image of this injective map. By twisting the dual of the above sequence by $K_C - D_n$ and identifying $E^\vee \otimes K_C(-D_n) \cong E$, we have a cohomology sequence
\[ 0 \to H^0(E) \to V \otimes H^0(K_C - D_n) \to H^0(2K_C - 2D_n) \]
and the kernel of $\mu_V$ is identified with $H^0(E)$, which is of dimension three. This completes the proof. \qed

Let $X$ be a smooth curve contained in a projective space $\mathbb{P}$. For any positive integer $h$, denote by $Sec_h(X)$, the $h^{th}$-secant variety of $X$, defined as the closure of the union of all linear subspaces $< \phi(D) > \subset \mathbb{P}$, for all effective general divisors of degree $h$. One has that $dim(Sec_h(X)) = min\{dim(\mathbb{P}), 2h - 1\}$ (see [4]).

**Theorem 3.1.** Let $C$ be a non-hyperelliptic curve of genus $g \geq 10$ with general moduli. Let $3 \leq n \leq g - 4$ be an integer. There exist closed irreducible loci $B_n \subset B^4(2, K_C)$ of dimension $3g - 10 - n$ whose general point corresponds to a stable bundle $E$ fitting in an exact sequence of the form
\[ 0 \to O_C(D_n) \to E \to K_C(-D_n) \to 0, \]
where $D_n$ is a general effective divisor of degree $n$. In particular for $n = 3$, $B_3$ is a regular component of $B^4(2, K_C)$ and the loci $B_3 \supset B_4 \supset \cdots \supset B_n$ stratify $B_3$.

We recall that the component $B_3$ was constructed in [2], thus we need only to construct the components $B_n$ for $n \geq 4$. To do this we start by showing the following lemma

**Lemma 3.2.** For every integer $n \geq 3$, inside the locus $W_3$ there exist loci $\Delta_n \subset W_3$, such that $\Delta_n$ is an irreducible component of dimension $3g - 10 - 2n$.

**Proof.** Consider the space of extensions $Ext^1(K_C(-D_n), O_C(D_n))$ which parametrizes exact sequences of the form
\[ u : 0 \to O_C(D_n) \to E \to K_C(-D_n) \to 0. \]
Set \( \mathbb{P} := \mathbb{P} \text{Ext}^1(K_C(-D_n), \mathcal{O}_C(D_n)) \) and consider the incidence variety
\[
J = \{ V \times u | \text{Im}((\mu \nu) \subset H_u \} \subset G(3, H^0(K_C - D_n)) \times \mathbb{P}.
\]

Let \( \pi_1 : J \to G(3, H^0(K_C - D_n)) \) and \( \pi_2 : J \to \mathbb{P} \) be the projections. By Proposition 3.1, for the general element \( V \in G(3, H^0(K_C - D_n)) \), the codimension of \( \text{Im}((\mu \nu) \subset H^0(2K_C - 2D_n) \) is equal to \( n \), then \( \text{Im}((\mu \nu) \subset H^0(2K_C - 2D_n) \) is contained in a hyperplane, and \( J \) is not empty. Then there exists an irreducible component \( J_n \subset J \) dominating the Grassmannian through \( \pi_1 \). The general fiber of \( \pi_1|_{J_n} \) is isomorphic to the linear system of hyperplanes containing \( \text{Im}((\mu \nu) \subset H^0(2K_C - 2D_n) \), so the general fiber of \( \pi_1|_{J_n} \) is of dimension \( n - 1 \) and \( \text{dim}(J_n) = 3g - 2n - 10 \). Set \( \Lambda_n := \pi_2(J_n) \subset \mathbb{P} \), and consider the image \( X \) of the map \( C \hookrightarrow \mathbb{P} \) induced by the linear system \( |(K_C(-D_n))^{\otimes 2}| \). Note that for \( \epsilon \in \{0, 1\} \) and for \( \sigma = g - 8 - \epsilon > 0 \) with \( \sigma \) even, \( \text{dim}(\Lambda_n) = 3g - 2n - 10 > 3g - 2n - 11 - \epsilon = \text{dim}(\text{Sec}_{\mathbb{P}}^{2g-2n-5-\sigma}(X)) \).

By Proposition 1.1 of [5], we have that for the general extension \( u \in \Lambda_n \), the corresponding vector bundle \( E = E_u \) is stable, so we have a rational map
\[
\pi : \Lambda_n \to B^4(2, K_C), \pi(u) := [E_u],
\]
and in particular \( \pi(\Lambda_n) \) is non-empty.

Let \( u \in \text{Ext}^1(K_C(-D_n), \mathcal{O}_C(D_n)) \) be and let \( \partial_u : H^0(C, K_C(-D_n)) \to H^1(\mathcal{O}_C(D_n)) \) be the corresponding coboundary map. By Serre duality we have that \( \partial_u \) is a symmetric map, that is \( \partial_u = \partial_u^\vee \), then \( \ker(\partial_u) = \ker(\partial_u^\vee) = (\text{Im}(\partial_u))^\perp \), in particular \( \ker(\partial_u) \) is uniquely determined, so the general fiber of the map \( \pi_2|_{J_n} : J_n \to \mathbb{P} \) is irreducible and zero-dimensional, then \( \Delta_n := \overline{\Lambda_n} \subset \mathbb{P} \) is an irreducible component of \( \mathcal{W}_3 \) of dimension \( 3g - 10 - 2n \).

Until here we construct the locus \( \Delta_n \) consisting in extensions whose general element defines (through the map \( \pi \)) an stable rank two vector bundle with canonical determinant with four sections. To conclude the proof of the Theorem 3.1 it remains to study the image \( \pi(\Delta_n) \subset B^4(2, K_C) \). By let the divisor \( D_n \) to vary in all general effective divisors of degree \( n \), in notation as in Section 2.1 for \( d = 2g - 2, \delta = 2g - 2 - n \) we construct the locus \( \Delta_n \) in the global space of extensions and we study the moduli map \( \pi_{d, \delta}|_{\Delta_n} : \Delta_n \to B^4(2, K_C) \).

Let \( u : 0 \to \mathcal{O}_C(D_n) \to E_u \to K_C(-D_n) \to 0 \) be a general extension in \( \Delta_n \). Consider the quotient \( E = E_u \to K_C(-D_n) \) and let \( \Gamma_{u_n} = \Gamma \) the corresponding section of such quotient. Let \( S = \mathbb{P}(E) \) and \( p : S \to C \) be the structure map. Since \( \mathcal{N}_{\Gamma/S} \simeq K_C(-2D_n) \), then \( h^0(\Gamma, \mathcal{N}_{\Gamma/S}) = g - 2n \) and \( h^1(\Gamma, \mathcal{N}_{\Gamma/S}) = 1 \). With this notation we have the following theorem

**Lemma 3.3.** (i). The couple \((S, \Gamma)\) is not obstructed, that is, the first-order infinitesimal deformations of the closed embedding \( \Gamma \hookrightarrow S \) are unobstructed with \( S \) not fixed, in particular \( \Gamma \) is unobstructed in \( S \) fixed and \( \Gamma \) varies in a family \((g - 2n)\)-dimensional.

(ii). The section \( \Gamma \) is linearly isolated.

(iii). The section \( \Gamma \) is specially isolated.
Proof: The proof of this result is similar to the proof in \[\text{Lemma 2, Step 2.}\] In order to make the paper self-contained we show only (iii).

**Proof of (iii).** Let \( N = \mathcal{O}_C(D_n) \) be; tensoring by \( N^\vee = \mathcal{O}_C(-D_n) \) the exact sequence

\[
u : 0 \to N \to E_u \to K_C \otimes N^\vee \to 0,
\]

we have in cohomology the exact sequence

\[
0 \to H^0(\mathcal{O}_C) \to H^0(E \otimes N^\vee) \to H^0(K_C \otimes (N^\vee)^2) \xrightarrow{\delta} H^1(\mathcal{O}_C) \to \cdots
\]

Since \(|\mathcal{O}_S(\Gamma)| \simeq \mathbb{P}H^0(E(-D_n))\), the fact that \( \Gamma \) is linearly isolated implies that \( h^0(E(-D_n)) = 1 \), then \( \delta \) is an isomorphism onto its image. Denote by \(<, >\) the Serre duality pairing, and let \( V_1 := H^0(K_C \otimes (N^\vee)^2) \) be. By (i) and (ii) and Section 2.2, we can identify \( \delta(V_1) \) with the tangent space at \( N \) to \( \text{Quot}_E^\vee \), the Quot-Scheme that parametrizes quotient line bundles of \( E \). Thus, to prove that \( \Gamma \) is specially isolated we need to prove that \( \delta(V_1) \cap T|_N(W_3^0(C)) = (0) = \delta(V_1) \cap (\text{Im}(\mu_N))^\perp \), where for \( \ell = 1, 2, \mu_N : H^0(\mathcal{O}_C(\ell D_n)) \otimes H^0(K_C(-\ell D_n)) \to H^0(C, K_C) \) is the Petri map. Since \( \text{Im}(\mu_N) = V_1 \subset \text{Im}(\mu_N) = H^0(K_C \otimes N^\vee) \), then \( \text{Im}(\mu_N)^\perp \subset V_1^\perp \).

Let \( \delta(\omega) \in \delta(V_1) \cap (\text{Im}(\mu_N))^\perp \subset \delta(V_1) \cap V_1^\perp \). Then \( < \delta(\omega), v >= 0 \) for all \( v \in V_1 \), that is, \( \delta(\omega) \in \text{Ker}(V_1 \to V_1^\vee) \), where \( \beta : x \to \beta_x, \beta_x(v) = < x, v > \). By duality, \( \beta \) is an isomorphism, then \( \delta(\omega) = 0 \); since \( \delta \) is injective, \( \omega = 0 \). This proves that \( \Gamma \) is specially isolated.

\[\square\]

**Lemma 3.4.** The map \( \pi : \Delta_n \to B^4(2, K_C) \) is generically injective.

**Proof.** For a general element \([E_u] \in \pi(\Delta_n) \subset B^4(2, K_C)\), the fiber \( \pi^{-1}([E_u]) \) corresponds to the extensions \( u' \in \Delta_n \) such that there exists a diagram

\[
\begin{array}{ccc}
u : 0 & \to & \mathcal{O}_C(D_n) \xrightarrow{\iota_1} E_u \xrightarrow{\iota_2} K_C(-D_n) \xrightarrow{\phi} 0 \\
\end{array}
\]

where \( \phi : E_u \to E_u' \) is an isomorphism of stable bundles. The maps \( \phi \circ \iota_1 \) and \( \iota_2 \) determine two non-zero sections \( \sigma_1 \neq \sigma_2 \) in \( H^0(E_u(-D_n)) \). For the general extension \( u \in \Delta_n \), we have by Lemma 3.3 that \( h^0(E_u(-D_n)) = 1 \), so \( \phi \circ \iota_1 = \lambda \iota_2 \), for some scalar \( \lambda \neq 0 \). Since \( E_u \) is stable, then \( E_u = E_u' \), i.e \( u, u' \) are proportional in \( \Delta_n \), then \( \pi \) is generically injective.

For a general effective divisor of degree \( n \), \( D_n, L = K_C(-D_n) \) depends on \( \rho(L) = n \) parameters, then with notation as in Section 2.1 we have an irreducible component \( \Delta_n \subset \mathbb{P}(\mathcal{E}) \) of dimension \( 3g - 10 - 2n + \rho(L) = 3g - 10 - n \), where a point in \( \Delta_n \) corresponds to the datum of a pair \((y, u)\) with \( y = (\mathcal{O}_C(D_n), K_C(-D_n)) \), \( D_n \) a general effective divisor of degree \( n \) and \( u \in \Delta_n \) an extension as in Lemma 3.1. For \( d = 2g - 2, \delta = 2g - 2 - n \), we have that the moduli map \( \pi_{d, \delta} : \Delta_n \to B^4(2, K_C) \) is generically injective and its image fills up an irreducible closed sublocus \( \mathcal{E}_n := \pi_{d, \delta}^{-1}(\Delta_n) \subset B^4(2, K_C) \) of dimension \( 3g - 10 - n \). \[\square\]

With the following Lemma we complete the proof of Theorem 3.1.
Lemma 3.5. For $n \geq 3$, $B_{n+1} \subset B_n$

Proof. Note that by an inductive step, it is enough to show that $B_4 \subset B_3$.
Consider a general element $E \in \pi_{d,\delta}[[\widetilde{\Delta}_4]]$, then there exists a general effective divisor $D_4$ of degree four such that $E$ fits into an exact sequence

$$u : 0 \to \mathcal{O}_C(D_4) \to E \to K_C(-D_4) \to 0,$$

where $V := \text{Coker}(\partial_u)$ is general in the Grassmannian $G(3, H^0(K_C(-D_4)))$. Fix three points in $\text{Supp}(D_4)$ and consider the degree three effective divisor $D_3$ given by these three points, then we can write $D_4 = D_3 + q$, $q \in C$ a general point. The natural inclusion $H^0(K_C(-D_4)) \hookrightarrow H^0(K_C(-D_3))$ inside $H^0(K_C)$ induces an inclusion in Grassmannians

$$G(3, H^0(K_C(-D_4))) \hookrightarrow G(3, H^0(K_C(-D_3))).$$

From Lemma 3.2 we have in particular that for $n = 3, 4$ the loci $J_n := \{W \times u | \text{Im}(\mu_W) \subset \{u = 0\}\} \subset G(3, H^0(K_C(-D_n))) \times \mathbb{P} \text{Ext}^1(K_C(-D_n), \mathcal{O}_C(D_n))$, are such that $\pi_1|_{J_n} : J_n \to G(3, H^0(K_C(-D_n)))$ is dominant. Note also that for every $n \geq 3$, elements $u \in \Delta_n$ are such that $V := \text{Coker}(\partial_u)$ satisfy the conditions of Proposition 3.1. Moreover, we have that $\Delta_4 = \overline{\pi_2(J_4)} \subset \Delta_3 = \overline{\pi_2(J_3)}$. By the construction from Section 2.1 we have that $E \in B_3$, then $B_4 \subset B_3$. \hfill \Box

Note that the proof of Theorem 3.1 is obtained by applying from Lemma 3.2 to Lemma 3.5.

References

[1] E. Arbarello, M. Cornalba, P. Griffiths, J. Harris, Geometry of Algebraic Curves. Volume I, Series: Grundlehren der mathematischen Wissenschaften Vol. 267 (Springer Verlag 1985).
[2] A. Castorena, G. Reyes-Ahumada, Rank two bundles with canonical determinant and four sections, Rendiconti del Circolo Matematico di Palermo (1952 -) (2015) Volume 64 Issue 2 261-272.
[3] C. Ciliberto, F. Flamini, Extensions of line bundles and Brill-Noether loci of rank-two vector bundles on a general curve, Rev. Rumaine Math. Pures Appl. 60 (2015) no. 3 201-255
[4] H. Lange, Higher secant varieties of curves and the theorem of Nagata on ruled surfaces, Manuscripta Math. (1984) 47 263 doi:10.1007/BF01174597.
[5] H. Lange, M.S. Narasimhan, Maximal Subbundles Of Rank Two Vector Bundles on Curves, Math. Ann. 266 (1983) no. 1 55-72.
[6] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics Volume 52 (Springer-Verlag 1977).
[7] E. Sernesi, Deformations of algebraic schemes, Volume 334 Grundlehen der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics (Springer Verlag).

Centro de Ciencias Matemáticas (Universidad Nacional Autónoma de México), Campus Morelia, Apdo. Postal 61-3(Xangari). C.P. 58089, Morelia, Michoacán. México.
Instituto de Matemáticas (Universidad Nacional Autónoma de México), Sede Oaxaca, León 2 Centro, C.P. 68000, Oaxaca, Oaxaca. México.

E-mail address: abel@matmor.unam.mx, grace@matmor.unam.mx