Research Article

On Submersion of CR-Submanifolds of l.c.q.K. Manifold

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We study submersion of CR-submanifolds of an l.c.q.K. manifold. We have shown that if an almost Hermitian manifold $B$ admits a Riemannian submersion $\pi : M \to B$ of a CR-submanifold $M$ of a locally conformal quaternion Kaehler manifold $\tilde{M}$, then $B$ is a locally conformal quaternion Kaehler manifold.

1. Introduction

The concept of locally conformal Kaehler manifolds was introduced by Vaisman in [1]. Since then many papers appeared on these manifolds and their submanifolds (see [2] for details). However, the geometry of locally conformal quaternion Kaehler manifolds has been studied in [2–4] and their QR-submanifolds have been studied in [5].

A locally conformal quaternion Kaehler manifold (shortly, l.c.q.K. manifold) is a quaternion Hermitian manifold whose metric is conformal to a quaternion Kaehler metric in some neighborhood of each point. The main difference between locally conformal Kaehler manifolds and l.c.q.K. manifolds is that the Lee form of a compact l.c.q.K. manifold can be chosen as parallel form without any restrictions [2].

The study of the Riemannian submersion $\pi : M \to B$ of a Riemannian manifold $M$ onto a Riemannian manifold $B$ was initiated by O'Neill [6]. A submersion naturally gives rise to two distributions on $M$ called the horizontal and vertical distributions, respectively, of which the vertical distribution is always integrable giving rise to the fibers of the submersion which are closed submanifold of $M$. The notion of Cauchy-Riemann (CR) submanifold was introduced by Bejancu [7] as a natural generalization of complex submanifolds and totally real submanifolds. A CR-submanifolds $M$ of a l.c.q.K. manifold $\tilde{M}$ requires a differentiable...
holomorphic distribution $D$, that is, $J_xD_x = D_x$ for all $x \in M$, whose orthogonal complement $D^\perp$ is totally real distribution on $M$, that is, $J_xD^\perp_x \subset TM^\perp$ for all $x \in M$. A CR-submanifold is called holomorphic submanifold if $\dim D^\perp_x = 0$, totally real if $\dim D_x = 0$ and proper if it is neither holomorphic nor totally real.

A CR-submanifold of a l.c.q.K. manifold $\overline{M}$ is called a CR-product if it is Riemannian product of a holomorphic submanifold $N^\perp$ and a totally real submanifold $N^\perp$ of $\overline{M}$. Kobayashi [8] has proved that if an almost Hermitian manifold $B$ admits a Riemannian submersion $\pi : M \to B$ of a CR-submanifold $M$ of a Kaehler Manifold $\overline{M}$, then $B$ is a Kaehler manifold. However, Deshmukh et al. [9] studied similar type of results for CR-submanifolds of manifolds in different classes of almost Hermitian manifolds, namely, Hermitian manifolds, quasi-Kaehler manifolds, and nearly Kaehler manifolds.

In the present paper, we investigate submersion of CR-submanifold of a l.c.q.K. manifold $\overline{M}$ and prove that if an almost Hermitian manifold $B$ admits a Riemannian submersion $\pi : M \to B$ of a CR-submanifold $M$ of a l.c.q.K. manifold $\overline{M}$, then $B$ is a l.c.q.K. manifold.

2. Preliminaries

Let $(\overline{M}, g, H)$ be a quaternion Hermitian manifold, where $H$ is a subbundle of end $(T\overline{M})$ of rank 3 which is spanned by almost complex structures $J_1$, $J_2$, and $J_3$. The quaternion Hermitian metric $g$ is said to be a quaternion Kaehler metric if its Levi-Civita connection $\nabla$ satisfies $\nabla H \subset H$.

A quaternion Hermitian manifold with metric $g$ is called a locally conformal quaternion Kaehler (l.c.q.K.) manifold if over neighborhoods $\{U_i\}$ covering $\overline{M}$, $g|_{U_i} = e^{f_i}g_i$, where $g_i$ is a quaternion Kaehler metric on $U_i$. In this case, the Lee form $\omega$ is locally defined by $\omega|_{U_i} = df_i$ and satisfies [3]

$$d\theta = \omega \wedge \theta, \quad dw = 0. \quad (2.1)$$

Let $\overline{M}$ be l.c.q.K. manifold and $\nabla$ denotes the Levi-Civita connection of $\overline{M}$. Let $B$ be the Lee vector field given by

$$g(X, B) = w(X). \quad (2.2)$$

Then for l.c.q.K. manifold, we have [3]

$$(\nabla_X J_a)Y = \frac{1}{2} \left[ \theta(Y)X - w(Y)J_aX - g(X, Y)A - \Omega(X, Y)B \right]$$

$$+ Q_{ab}(X)J_bY + Q_{ac}(X)J_cY \quad (2.3)$$

for any $X, Y \in T\overline{M}$, where $Q_{ab}$ is skew-symmetric matrix of local forms $\theta = w \circ J_a$ and $A = -J_aB$. 

We also have
\[ \theta(X) = g(J_a X, B), \quad \Omega(X, Y) = g(X, J_a Y). \tag{2.4} \]

Let \( M \) be a Riemannian manifold isometrically immersed in \( \overline{M} \). Let \( T(M) \) be the Lie algebra of vector fields in \( M \) and \( TM^\perp \), the set of all vector fields normal to \( M \). Denote by \( \nabla \) the Levi connection of \( M \). Then the Gauss and Weingarten formulas are given by
\[
\nabla_X Y = \nabla_X Y + h(X, Y), \quad \nabla_X N = -\overline{A}_N X + \nabla^\perp_X N \tag{2.5, 2.6}
\]
for any \( X, Y \in T(M) \), and \( N \in TM^\perp \), where \( \nabla^\perp \) is the connection in the normal bundle \( TM^\perp \), \( h \) is the second fundamental form, and \( \overline{A}_N \) is the Weingarten endomorphism associated with \( N \). The second fundamental form and shape operator are related by
\[ g(\overline{A}_N X, Y) = g(h(X, Y), N). \tag{2.7} \]

The curvature tensor \( R \) of the submanifold \( M \) is related to the curvature tensor \( \overline{R} \) of \( \overline{M} \) by the following Gauss formula:
\[
\overline{R}(X, Y; Z, W) = R(X, Y; Z, W) + g(h(X, Z), h(Y, W)) - g(h(X, W), h(Y, Z)), \tag{2.8}
\]
for any \( X, Y, Z, W \in T(M) \).

For submersion of a l.c.q.K. manifold onto an almost Hermitian manifold, we have the following.

**Definition 2.1.** Let \( M \) be a CR-submanifold of a locally conformal quaternion Kaehler manifold \( \overline{M} \). By a submersion \( \pi : M \to B \) of \( M \) onto an almost Hermitian manifold \( B \), we mean a Riemannian submersion \( \pi : M \to B \) together with the following conditions:

(i) \( D^\perp \) is the kernel of \( \pi_* \), that is, \( \pi_* D^\perp = \{0\} \),

(ii) \( \pi_* : D_p \to D^*_{\pi(p)} \) is a complex isometry of the subspace \( D_p \) onto \( D^*_{\pi(p)} \) for every \( p \in M \), where \( D^*_{\pi(p)} \) denotes the tangent space of \( B \) at \( \pi(p) \),

(iii) \( J \) interchanges \( D^\perp \) and \( \nu \), that is, \( JD^\perp = TM^\perp \).

For a vector field \( X \) on \( M \), we set \[ X = HX + VX, \tag{2.9} \]
where \( H \) and \( V \) denoted the horizontal and vertical part of \( X \).
We recall that a vector field $X$ on $M$ for submersion $\pi: M \to B$ is said to be a basic vector field if $X \in D$ and $X$ is $\pi$ related to a vector field on $B$, that is, there is a vector field $X_*$ on $B$ such that

$$\left(\pi_*X\right)_p = (X_*)_p$$ for each $p \in M. \quad (2.10)$$

If $J$ and $J'$ are the almost complex structures on $\overline{M}$ and $B$, respectively, then from Definition 2.1(ii) we have $\pi_* \circ J = J' \circ \pi_*$ on $D$.

We have the following lemma for basic vector fields [6].

**Lemma 2.2.** Let $X$ and $Y$ be basic vector fields on $M$. Then

(i) $g(X,Y) = g_*(X_*,Y_*) \circ \pi$, $g$ is the metric on $M$, and $g_*$ is the Riemannian metric on $B$;

(ii) the horizontal part $H[X,Y]$ of $[X,Y]$ is a basic vector field and corresponds to $[X_*,Y_*]$, that is,

$$\pi_*H[X,Y] = [X_*,Y_*] \circ \pi; \quad (2.11)$$

(iii) $H(\nabla_X Y)$ is a basic vector field corresponding to $\nabla^*_X Y_*$, where $\nabla^*$ is a Riemannian connection on $B$;

(iv) $[X,W] \in D^\perp$ for $W \in D^\perp$.

For a covariant differentiation operator $\nabla^*$, we define a corresponding operator $\tilde{\nabla}^*$ for basic vector fields of $M$ by

$$\tilde{\nabla}^*_X Y = H(\nabla_X Y), \quad X,Y \in D, \quad (2.12)$$

then $\tilde{\nabla}^*_X Y$ is a basic vector field, and from the above lemma we have

$$\pi_* \left(\tilde{\nabla}^*_X Y\right) = \nabla^*_X Y_* \circ \pi. \quad (2.13)$$

Now, we define a tensor field $C$ on $M$ by setting

$$\nabla_X Y = H(\nabla_X Y) + C(X,Y), \quad X,Y \in D, \quad (2.14)$$

that is, $C(X,Y)$ is the vertical component of $\nabla_X Y$.

In particular, if $X$ and $Y$ are basic vector fields, then we have

$$\nabla_X Y = \tilde{\nabla}^*_X Y + C(X,Y). \quad (2.15)$$

The tensor field $C$ is skew-symmetric and it satisfies

$$C(X,Y) = \frac{1}{2} V[X,Y], \quad X,Y \in D. \quad (2.16)$$
For $X \in D$ and $V \in D^\perp$ define an operator $A$ on $M$ by setting $\nabla_X V = \nu(\nabla_X V) + A_X V$, that is, $A_X V$ is the horizontal component of $\nabla_X V$. Using (iv) of Lemma 2.2 we have

$$H(\nabla_X V) = H(\nabla_V X) = A_X V.$$  \hspace{1cm} (2.17)

The operator $C$ and $A$ are related by

$$g(A_X V, Y) = -g(V, C(X, Y)), \quad X, Y \in D, \quad V \in D^\perp. \hspace{1cm} (2.18)$$

For a CR-submanifold $M$ in a locally conformal quaternion Kaehler manifold $\overline{M}$, we denote by $\nu$ the orthogonal complement of $JD^\perp$ in $TM^\perp$. Hence, we have the following orthogonal decomposition of the normal bundle:

$$TM^\perp = JD^\perp \oplus \nu, \quad JD^\perp \perp \nu. \hspace{1cm} (2.19)$$

Set

$$PX = \tan(JX), \quad FX = \text{nor}(JX), \quad \text{for } X \in TM,$$

$$tN = \tan(JN), \quad fN = \text{nor}(JN), \quad \text{for } N \in TM^\perp. \hspace{1cm} (2.20)$$

Here, $\tan_x$ and $\text{nor}_x$ are the natural projections associated with the orthogonal direct sum decomposition

$$T_x\overline{M} = T_x M \oplus TM^\perp_x \quad \text{for any } x \in M. \hspace{1cm} (2.21)$$

Then the following identities hold:

$$P^2 = -I - tF, \quad FP + tF = 0, \quad Pt + tf = 0, \quad f^2 = -I - Ft, \hspace{1cm} (2.22)$$

where $I$ is the identity transformation.

We have following results.

**Lemma 2.3.** Let $M$ be a CR-submanifold in a l.c.q.K. manifold $\overline{M}$. Then

(i) holomorphic distribution $D$ is integrable iff

$$h(X, J_a Y) - h(J_a X, Y) + \Omega(X, Y) \text{ nor } (B) = 0, \quad \forall X, Y \in D \hspace{1cm} (2.23)$$

or equivalently,

$$\tilde{g}(h(X, J_a Y) - h(J_a X, Y) + \Omega(X, Y)B, J_a Z) = 0, \quad \forall X, Y \in D, \quad Z \in D^\perp; \hspace{1cm} (2.24)$$
(ii) anti-invariant distribution $D^\perp$ of $M$ is integrable iff

$$A_{J_aW} T = A_{J_aT} W, \quad \forall W, T \in D^\perp. \quad (2.25)$$

Proof. (i) Using (2.3), we have

$$\nabla_X J_a Y = P_a \nabla_X Y + t_a h(X, Y) + \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - g(X, Y) \tan(A) - \Omega(X, Y) \tan(B) \}$$

$$+ Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y,$$

$$h(X, J_a Y) = F_a \nabla_X Y + f_a h(X, Y) - \frac{1}{2} \{ g(X, Y) \text{nor}(A) + \Omega(X, Y) \text{nor}(B) \}. \quad (2.26)$$

From the second of these equations, we have

$$h(X, J_a Y) - h(Y, J_a X) + \Omega(X, Y) \text{nor}(B) = F_a [X, Y], \quad \forall X, Y \in D. \quad (2.27)$$

If we need $D$ to be integrable, we have

$$h(X, J_a Y) - h(Y, J_a X) + \Omega(X, Y) \text{nor}(B) = 0 \quad (2.28)$$

or

$$g(h(X, J_a Y) - h(Y, J_a X) + \Omega(X, Y) (B), J_a Z) = 0, \quad \forall Z \in D^\perp. \quad (2.29)$$

$v$-part of $h(X, J_a Y) - h(J_a X, Y) + \Omega(X, Y) B$ vanishes for all $X, Y \in D$.

(ii) We have

$$\nabla_X J_a Y = J_a \nabla_X Y + \frac{1}{2} \{ \theta(Y) X - \omega(Y) J_a X - \Omega(X, Y) B + g(X, Y) J_a B \}$$

$$+ Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y. \quad (2.30)$$

Then for any $T, W \in D^\perp$, and $X \in D$, we have

$$g(\nabla_T J_a W, X) = g(J_a \nabla_T W, X) + \frac{1}{2} \theta(W) g(T, X) - \frac{1}{2} \omega(W) g(J_a T, X)$$

$$- \frac{1}{2} \Omega(T, W) g(B, X) + \frac{1}{2} g(T, W) g(J_a B, X)$$

$$+ Q_{ab}(T) g(J_b W, X) + Q_{ac}(T) g(J_c W, X)$$

$$\implies -A_{J_a W} T, X = - \langle \nabla_T, J_a X \rangle - \frac{1}{2} g(T, W) g(B, J_a X). \quad (2.32)$$
So, we have

\[ \langle A_{j,T}W, X \rangle = \langle \bar{\nabla}_T W, J_a X \rangle + \frac{1}{2} \bar{g}(W, J_a X), \]

(2.33)

\[ \langle A_{j,T}W, X \rangle = \langle \bar{\nabla}_W T, J_a X \rangle + \frac{1}{2} g(W, T) g(B, J_a X). \]

From these two equations, we have

\[ \langle A_{j,W}T - A_{j,T}W, X \rangle = \langle \bar{\nabla}_T W - \bar{\nabla}_W T, J_a X \rangle \]

(2.34)

\[ \implies \langle A_{j,W}T - A_{j,T}W, X \rangle = \langle [W, T], J_a X \rangle. \]

So, we conclude that if \( A_{j,W}T = A_{j,T}W \) then \( D^\perp \) is integrable. Converse is obvious. \( \square \)

**Lemma 2.4.** Let \( M \) be a CR-submanifold of l.c.q.K. manifold. Then

\[ \nabla_X J_a Y = \nabla_Y J_a X \]  

(2.35)

iff Lee vector field \( B \) is orthogonal to anti-invariant distribution \( D^\perp \).

**Proof.** Since \( \bar{\nabla} \) is metric connection, for \( X, Y \in D \), and \( Z \in D^\perp \), using (2.3), we have

\[
\begin{aligned}
\langle \bar{\nabla}_X J_a Y, Z \rangle &= \langle J_a \bar{\nabla}_X Y, Z \rangle + \frac{1}{2} \theta(Y)(X, Z) - \frac{1}{2} \Omega(X, Y) g(B, Z) \\
&\quad - \frac{1}{2} \omega(Y) g(J_a X, Z) + \frac{1}{2} g(X, Y) g(J_a B, Z) \\
&\quad + Q_{ab}(X) g(J_b Y, Z) + Q_{ac}(X) g(J_c Y, Z) \\
&= -\langle \bar{\nabla}_X Y, J_a Z \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) - \frac{1}{2} g(X, Y) g(B, J_a Z) \\
&= \langle Y, \bar{\nabla}_X J_a Z \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) - \frac{1}{2} g(X, Y) g(B, J_a Z)
\end{aligned}
\]

or

\[
\begin{aligned}
\langle \nabla_X J_a Y + h(X, J_a Y) Z \rangle &= \langle Y, -A_{J_a Z} X + \bar{\nabla}_X J_a Z \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) \\
&\quad - \frac{1}{2} g(X, Y) g(B, J_a Z).
\end{aligned}
\]

(2.37)
This gives
\begin{align}
\langle \nabla_{X}J_{a}Y, Z \rangle &= -\langle Y, A_{l,z}X \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) - \frac{1}{2} g(X, Y) g(B, J_{a}Z), \\
\langle \nabla_{Y}J_{a}X, Z \rangle &= -\langle X, A_{l,z}Y \rangle - \frac{1}{2} \Omega(Y, X) \omega(Z) - \frac{1}{2} g(Y, X) g(B, J_{a}Z).
\end{align}
(2.38)

The above two equations give
\begin{align}
\langle \nabla_{X}J_{a}Y - \nabla_{Y}J_{a}X, Z \rangle \\
&= -\langle A_{l,z}X, Y \rangle + \langle X, A_{l,z}Y \rangle - \frac{1}{2} \Omega(X, Y) \omega(Z) + \frac{1}{2} \Omega(Y, X) \omega(Z) \\
&= \Omega(Y, X) \omega(Z)
\end{align}
(2.39)

or
\begin{align}
\langle \nabla_{X}J_{a}Y - \nabla_{Y}J_{a}X, Z \rangle &= \Omega(Y, X) g(B, Z).
\end{align}
(2.40)

This gives $\nabla_{X}J_{a}Y = \nabla_{Y}J_{a}X$ iff $\omega(Z) = 0$.\qed

## 3. Submersions of CR-Submanifolds

On a Riemannian manifold $M$, a distribution $S$ is said to be parallel if $\nabla_{X}Y \in S$, $X, Y \in S$, where $\nabla$ is a Riemannian connection on $M$. It is proved earlier that horizontal distribution $D$ is integrable. If, in addition, $D^{\perp}$ is parallel, then we prove the following.

**Proposition 3.1.** Let $\pi : M \rightarrow B$ be a submersion of a CR-submanifold $M$ of a locally conformal quaternion Kaehler manifold $\bar{M}$ onto an almost Hermitian manifold $B$. If (horizontal distribution) $D$ is integrable and (vertical distribution) $D^{\perp}$ is parallel, then $M$ is a CR-product (Riemannian product $M_{1} \times M_{2}$, where $M_{1}$ is an invariant submanifold and $M_{2}$ is a totally real submanifold of $\bar{M}$).

**Proof.** Since the horizontal distribution $D$ is integrable for $X, Y \in D$, we have $[X, Y] \in D$. Therefore, $V[X, Y] = 0$. Then from (2.16), we have
\begin{align}
C(X, Y) = 0, \quad \forall X, Y \in D.
\end{align}
(3.1)

Thus, from the definition of $C$, we have
\begin{align}
\nabla_{X}Y = \tilde{\nabla}_{X}^{\ast}Y \in D, \quad \text{that is, } D \text{ is parallel.}
\end{align}
(3.2)

Since $D$ and $D^{\perp}$ are both parallel, using de Rham’s theorem, it follows that $M$ is the product $M_{1} \times M_{2}$, where $M_{1}$ is invariant submanifold of $\bar{M}$ and $M_{2}$ is totally real submanifold of $\bar{M}$. Hence, $M$ is a CR-product.\qed
In [10], Simons defined a connection and an invariant inner product on $H(T, V) = \text{Hom}(T(M), V(M))$, where $V(M)$ is vector bundle over $M$ and $T(M)$ be tangent bundle of $M$. In fact, if $r, s \in H(T, V)_m$, we set

$$\langle r, s \rangle = \sum_{i=1}^{p} \langle r(e_i), s(e_i) \rangle, \quad \text{where } \{e_i\} \text{ is a frame in } T(M)_m.$$  

(3.3)

Define $Q_{ab}(X) = \langle \overline{D} J_a, J_b \rangle$, which implies $Q_{ab}(X) J_a Y = \langle \overline{D} J_a, J_b \rangle J_a Y$.

Let $D$ be $4n$ dimensional distribution whose basis is given by $\{e_1, \ldots, e_n, e_a, \ldots, e_{da}, e_{b_1}, \ldots, e_{b_2}, e_{c_1}, \ldots, e_{ea}\}$ where $e_{a_i} = J_a(e_i)$, $e_{b_i} = J_b(e_i)$, $e_{c_i} = J_c(e_i)$ and $J_a \circ J_b = J_c$, $J_b \circ J_c = J_a$, $J_c \circ J_a = J_b$.

Now, component of $Q_{ab}(X)$ is defined as follows:

$$Q_{ab}(X) = \langle \overline{D} X J_a, J_b \rangle = \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D} X J_a(e_{a_i}), J_b(e_{b_i}) \rangle + \sum_{i=1}^{n} \langle \overline{D} X J_a(e_{c_i}), J_b(e_{c_i}) \rangle$$

(3.4)

$$+ \sum_{j=1}^{q} \langle \overline{D} X J_a(e_j), J_b(e_j) \rangle.$$

So, we have

$$Q_{ab}(X) = \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D} X J_a(e_{a_i}), J_b(e_{b_i}) \rangle + \sum_{i=1}^{n} \langle \overline{D} X J_a(e_{c_i}), J_b(e_{c_i}) \rangle + \sum_{j=1}^{q} \langle \overline{D} X J_a(e_j), J_b(e_j) \rangle$$

(3.5)

$$= \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^{n} \langle \overline{D} X J_a(e_{a_i}), J_b(e_{b_i}) \rangle + \sum_{i=1}^{n} \langle \overline{D} X J_a(e_{c_i}), J_b(e_{c_i}) \rangle - \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), e_i \rangle - \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), J_a(e_i) \rangle - \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), J_a(e_i) \rangle$$

$$+ \sum_{i=1}^{n} \langle \overline{D} X J_a(e_i), J_b(e_i) \rangle + \sum_{i=1}^{q} \langle \overline{D} X J_a(e_j), J_b(e_j) \rangle.$$
or

\[
Q_{ab}(X)J_bY = \sum_{i=1}^{n} \left< D_X J_a(e_i), J_b(e_i) \right> J_bY + \sum_{i=1}^{n} \left< D_X e_i, J_c(e_i) \right> J_bY
+ \sum_{i=1}^{n} \left< D_X J_a(e_i), J_b(e_i) \right> J_bY
- \sum_{i=1}^{n} \left< D_X J_b(e_i), J_a(e_i) \right> J_bY
\]

(3.6)

\[
\sum_{i=1}^{n} \left< D_X J_a(e_i), J_b(e_i) \right> J_bY + \sum_{i=1}^{n} \left< D_X J_a(e_i), J_b(e_i) \right> J_bY
- \sum_{i=1}^{n} \left< D_X J_b(e_i), J_a(e_i) \right> J_bY
- \sum_{i=1}^{n} \left< D_X J_a(e_i), J_b(e_i) \right> J_bY
\]

Applying \( \pi^* \) and using Lemma 2.2, we get

\[
\pi^* Q_{ab}(X)J_bY = \sum_{i=1}^{n} \left< D_X J'_a e_i^*, J'_b e_i^* \right> J'_b Y_* + \sum_{i=1}^{n} \left< D_X e_i^*, J'_c e_i^* \right> J'_b Y_*
+ \sum_{i=1}^{n} \left< D_X J'_a e_i^*, J'_b e_i^* \right> J'_b Y_*
- \sum_{i=1}^{n} \left< D_X J'_b e_i^*, e_i^* \right> J'_b Y_*
- \sum_{i=1}^{n} \left< D_X J'_a e_i^*, J'_b e_i^* \right> J'_b Y_*
+ \sum_{i=1}^{n} \left< D_X J'_b e_i^*, J'_a e_i^* \right> J'_b Y_*
= \sum_{i=1}^{n} \left< D_X J'_a J'_b e_i^*, J'_b Y_* \right> J'_b Y_*
\]

(3.7)

or

\[
\pi^* Q_{ab}(X)J_bY = Q^*_{ab}(X)J'_b Y_*.
\]

(3.8)

Now, we prove the main result of this paper.

**Theorem 3.2.** Let \( \overline{M} \) be an l.c.q.K. manifold and \( M \) be a CR-submanifold of \( \overline{M} \). Let \( B \) be an almost Hermitian manifold and \( \pi : M \rightarrow B \) be a submersion. Then \( B \) is an l.c.q.K. manifold.

**Proof.** Let \( X, Y \in D \) be basic vector fields. Then from (2.5) and (2.15), we have

\[
\nabla_X Y = \tilde{\nabla}_X Y + C(X, Y) + h(X, Y).
\]

(3.9)

Replacing \( Y \) by \( J_a Y \) in (3.9), we have

\[
\nabla_X J_a Y = \tilde{\nabla}_X J_a Y + C(X, J_a Y) + h(X, J_a Y).
\]

(3.10)
Using (2.3), we get

\[ \nabla^*_X j_a Y + C(X, j_a Y) + h(X, j_a Y) \]
\[ = j_a \nabla^*_X Y + \frac{1}{2} \left\{ \theta(Y) X - \omega(Y) j_a X - \Omega(X, Y) B + g(X, Y) j_a B \right\} \]
\[ + Q_{ab}(X) j_b Y + Q_{ac}(X) j_c Y \]

or

\[ \nabla^*_X j_a Y + C(X, j_a Y) + h(X, j_a Y) = j_a \nabla^*_X Y + j_a C(X, Y) + j_a h(X, Y) \]
\[ + \frac{1}{2} \left\{ \theta(Y) X - \omega(Y) j_a X - \Omega(X, Y) B + g(X, Y) j_a B \right\} \]
\[ + Q_{ab}(X) j_b Y + Q_{ac}(X) j_c Y. \]

Thus, we have

\[ \left( \nabla^*_X j_a \right) Y + C(X, j_a Y) + h(X, j_a Y) - j_a C(X, Y) - j_a h(X, Y) \]
\[ - \frac{1}{2} \left\{ \theta(Y) X - \omega(Y) j_a X - \Omega(X, Y) B + g(X, Y) j_a B \right\} \]
\[ - Q_{ab}(X) j_b Y - Q_{ac}(X) j_c Y = 0. \]

Comparing horizontal, vertical, and normal components in the above equation to get

\[ \left( \nabla^*_X j_a \right) Y - \frac{1}{2} \left\{ \theta(Y) X - \omega(Y) j_a X - \Omega(X, Y) B + g(X, Y) j_a B \right\} \]
\[ - Q_{ab}(X) j_b Y - Q_{ac}(X) j_c Y = 0, \]
\[ C(X, j_a Y) = j_a h(X, Y), \]
\[ h(X, j_a Y) = j_a C(X, Y). \]

from (3.14), we have

\[ \nabla^*_X j_a Y - j_a \nabla^*_X Y - \frac{1}{2} \left[ g(j_a Y, B) X - g(B, Y) j_a X - g(X, j_a Y) B + g(X, Y) j_a B \right] \]
\[ - Q_{ab}(X) j_a Y - Q_{ac}(X) j_c Y = 0. \]
Then for any \( X_*, Y_* \in \chi(B) \), and \( J' \) being almost complex structure on \( B \), we have after operating \( \pi^* \) on the above equation

\[
\nabla^*_X J'_a Y_* - J'_a \nabla^*_X Y_* - \frac{1}{2} \left\{ g_*(J'_a Y_*, B_*) X_* - g_*(B_*, Y_*) J'_a Y_* - g_*(X_*, J'_a Y_*) B_* + g_*(X_*, Y_*) J'_a B_* \right\} \\
- Q^*_{ab}(X_*) J'_a(Y_*) - Q^*_{ac}(X_*) J'_c(Y_*) = 0.
\]

(3.18)

This gives

\[
(\nabla^*_X J'_a) Y_* - \frac{1}{2} \left\{ \theta'(Y_*) X_* - \omega'(Y_*) J'_a X_* - \Omega(X_*, Y_*) B_* + g_*(X_*, Y_*) J'_a B_* \right\} \\
- Q^*_{ab}(X_*) J'_a Y_* - Q^*_{ac}(X_*) J'_c Y_* = 0.
\]

(3.19)

This shows that \( B \) is l.c.q.K. manifold.

Now, using (2.17) and (2.18), we obtain a relation between curvature tensor \( R \) on \( M \) and curvature tensor \( R^* \) of \( B \) as follows:

\[
R(X, Y, Z, W) = R^*(X_*, Y_*, Z_*, W_*) - g(C(Y, Z), C(X, W)) \\
+ g(C(X, Z), C(Y, W)) + 2g(C(X, Y), C(Z, W)),
\]

(3.20)

where \( \pi_* X = X_* \), \( \pi_* Y = Y_* \), \( \pi_* Z = Z_* \), and \( \pi_* W = W_* \in B \).

Now, using the above equation together with (2.8) and using the fact that \( C \) is skew-symmetric, we obtain

\[
\nabla^*_X H(X) = \nabla^*_X \tilde{R}(X, J_a X, J_a X, X) \\
= H^*(X_*) + \|h(X, J_a X)\|^2 - g(h(J_a X, J_a X), h(X, X)) \\
- 3\|C(X, J_a X)\|^2,
\]

(3.21)

where \( \nabla^*_X H(X) \) and \( H^*(X_*) \) are the holomorphic sectional curvature tensors of \( \tilde{M} \) and \( B \), respectively.

If we assume that \( D \) is integrable then using Lemma 2.3(i), we have

\[
h(J_a X, J_a X) = -h(X, X).
\]

(3.22)

Also from (3.15), we have \( C(X, J_a X) = 0 \). Then, (3.21) reduces to

\[
\nabla^*_X H(X) = H^*(X_*) + \|h(X, J_a X)\|^2 + \|h(X, X)\|^2, \quad \forall X \in D.
\]

(3.23)

This gives \( \nabla^*_X H(X) \geq H^*(X_*) \).

Thus, we have the following result.
Theorem 3.3. Let $M$ be a CR-submanifold of a l.c.q.K. manifold $\overline{M}$ with integrable $D$. Let $B$ be an almost Hermitian manifold and $\pi : \overline{M} \to B$ be a submersion. Then holomorphic sectional curvatures $\overline{H}$ and $H^*$ of $\overline{M}$ and $B$, respectively, satisfy

$$\overline{H}(X) \geq H^*(X), \quad \text{for all unit vectors } X \in D. \quad (3.24)$$

Note. The above result was obtained in [9] by taking $\overline{M}$ to be quasi-Kaehler manifold. Later, similar type of relation was derived in [11], considering $\overline{M}$ to be l.c.K manifold.

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