Rational Conchoids of Algebraic Curves.*

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Abstract

We study the rationality of the components of the conchoid to an irreducible algebraic affine plane curve, excluding the trivial cases of the isotropic lines, of the lines through the focus and the circle centered at the focus and radius the distance involved in the conchoid. We prove that conchoids having all their components rational can only be generated by rational curves. Moreover, we show that reducible conchoids to rational curves have always their two components rational. In addition, we prove that the rationality of the conchoid component, to a rational curve, does depend on the base curve and on the focus but not on the distance. Also, we provide an algorithm that analyzes the rationality of all the components of the conchoid and, in the affirmative case, parametrizes them. The algorithm only uses a proper parametrization of the base curve and the focus and, hence, does not require the previous computation of the conchoid. As a corollary, we show that the conchoid to the irreducible conics, with conchoid-focus on the conic, are rational and we give parametrizations. In particular we parametrize the Limaçons of Pascal. We also parametrize the conchoids of Nicomedes. Finally, we show to find the focuses from where the conchoid is rational or with two rational components.

1 Introducción

The conchoid is a classical geometric construction. Intuitively speaking, if \( C \) is a plane curve (the base curve), \( A \) a fixed point in the plane (the focus), and \( d \) a non-zero fixed field element (the distance), the conchoid of \( C \) from the focus \( A \) at distance \( d \) is the (closure of) set of points \( Q \) in the line \( AP \) at distance \( d \) of a point \( P \) varying in the curve \( C \). The two classical and most famous conchoids are the Conchoid of Nicomedes (\( C \) is a line and \( A \notin C \)) and the Limaçons of Pascal (\( C \) is a circle and \( A \in C \)). Conchoids are useful in many applications as construction of buildings, astronomy, electromagnetic research, physics, optics, engineering

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in medicine and biology, mechanical in fluid processing, etc (see the introduction of [3] for references).

In this paper, we deal with the problem of analyzing the rationality of the components of a conchoid and, in the affirmative case, the actual computation of parametrizations. Clearly, the problem can be approached by computing the implicit equation of the conchoid to afterwards factor it to finally apply to each factor any parametrization algorithm. Nevertheless, we want to avoid all these computations solving the problem directly from the input base curve and the focus. For this purpose, similarly as in [3], we work over an algebraically closed field $\mathbb{K}$ of characteristic zero, and curves are considered reduced; that is, they are the zero set in $\mathbb{K}^2$ of non-constant square-free polynomials of $\mathbb{K}[y_1, y_2]$. Furthermore, if a curve is defined by the square-free polynomial $f$, when we speak about its components, we mean the curves defined by the non-constant irreducible factors (over $\mathbb{K}$) of $f$ (see [4] for further details).

In [3] we presented a theoretical analysis of the concept and main properties of conchoids to irreducible curves (see Section 2 for a brief summary). In this analysis, three different types of curves have an exceptional behavior: the isotropic lines $y_1 \pm \sqrt{-1} y_2 = 0$ (its conchoid is empty), the circle centered at the focus and radius the distance involved in the conchoid (its conchoid has a zero-dimensional component) and the lines through the focus (all conchoid components are special; see Section 2 for this concept). For all the other cases, the most remarkable property in [3] is that the conchoid is a plane algebraic curve with at most two component, being at least one of them simple (see Section 2 for the notion of simple component).

In this paper, we exclude w.l.o.g. the above three exceptional types of curves. In this situation, we prove that conchoids having all their components rational can only be generated by rational curves. Moreover, we show that reducible conchoids to rational curves have always their two components rational; we call this case double rationality. Furthermore, we characterize rational conchoids and double rational conchoids. From these results, one deduces that the rationality of the conchoid component, to a rational curve, does depend on the base curve and on the focus but not on the distance. To approach the problem we use similar ideas to those in [1] introducing the notion of reparametrization curve (see Def. 3.3) as well as the notion of RDF parametrization (see Def. 3.1). The RDF concept allows us to detect the double rationality while the reparametrization curve is a much simpler curve than the conchoid, directly computed from the input rational curve and the focus, and that behaves equivalently as the conchoid in terms of rationality. As a consequence of these theoretical results we provide an algorithm to solve the problem. Given a proper parametrization of the base curve and the focus, the algorithm analyzes the rationality of all the components of the conchoid and, in the affirmative case, parametrizes them. We note that the algorithm does not require the computation of the conchoid. In addition, we show that the conchoid to the irreducible conics, with conchoid-focus on the conic, are rational and we give parametrizations. In particular we parametrize the Limacons of Pascal. We also parametrize the conchoids of Nicomedes. Finally, we show to find the focuses from where the conchoid is rational or with two rational components.
2 Preliminaries on Conchoids and General Assumptions.

In this section we recall the notion of conchoid as well as its main properties. For further details, we refer to [3]. Let $\mathbb{K}$ be an algebraically closed field of characteristic zero. In $\mathbb{K}^2$ we consider the symmetric bilinear form

$$b((x_1, x_2), (y_1, y_2)) = x_1y_1 + x_2y_2,$$

which induces a metric vector space with light cone of isotropy $L = \{P \in \mathbb{K}^2 \mid b(P, P) = 0\}$ (see [5]). That is, $L$ is the union of the two lines defined by $x_1 \pm \sqrt{-1}x_2 = 0$. In this context, the circle of center $P \in \mathbb{K}^2$ and radius $d \in \mathbb{K}$ is the plane curve defined by $b(\bar{x} - P, \bar{x} - P) = d^2$, with $\bar{x} = (x_1, x_2)$. We say that the distance between $P, Q \in \mathbb{K}^2$ is $d \in \mathbb{K}$ if $P$ is on the circle of center $Q$ and radius $d$. The notion of “distance” is hence defined up to multiplication by $\pm 1$. On the other hand, if $P \in \mathbb{K}^2$ is not isotropic (i.e. $P \notin L$) we denote by $\|P\|$ any of the elements in $\mathbb{K}$ such that $\|P\|^2 = b(P, P)$, and if $P \in \mathbb{K}^2$ is isotropic, then $\|P\| = 0$. In this paper we usually work with both solutions of $\|P\|^2 = b(P, P)$. For this reason we use the notation $\pm\|P\|$.

In this situation, let $\mathcal{C}$ be the affine irreducible plane curve defined by the irreducible polynomial $f(\bar{y}) \in \mathbb{K}[\bar{y}]$, $\bar{y} = (y_1, y_2)$, let $d \in \mathbb{K}^*$ be a non-zero field element, and let $A = (a, b) \in \mathbb{K}^2$. We consider the (conchoid) incidence variety

$$\mathcal{B}(\mathcal{C}) = \left\{(\bar{x}, \bar{y}, \lambda) \in \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \mid \begin{array}{l} f(\bar{y}) = 0 \\ \|\bar{x} - \bar{y}\|^2 = d^2 \\ \bar{x} = A + \lambda(\bar{y} - A) \end{array} \right\}$$

and the incidence diagram

$$\mathcal{B}(\mathcal{C}) \subset \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K}$$

$$\pi_1 \downarrow \quad \pi_2$$

$$\pi_1(\mathcal{B}(\mathcal{C})) \subset \mathbb{K}^2 \quad \mathcal{C} \subset \mathbb{K}^2$$

where

$$\pi_1 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2, \quad \pi_2 : \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K} \longrightarrow \mathbb{K}^2$$

$$(\bar{x}, \bar{y}, \lambda) \longmapsto \bar{x} \quad (\bar{x}, \bar{y}, \lambda) \longmapsto \bar{y}.$$ 

Then, we define the conchoid of $\mathcal{C}$ from the focus $A$ and distance $d$ as the algebraic Zariski closure in $\mathbb{K}^2$ of $\pi_1(\mathcal{B}(\mathcal{C}))$, and we denote it by $\mathcal{C}(\mathcal{C})$; i.e.

$$\mathcal{C}(\mathcal{C}) = \overline{\pi_1(\mathcal{B}(\mathcal{C}))}.$$

For details on how to compute the conchoid see [3]. In general, $A$ and $d$ are just precise elements in $\mathbb{K}^2$ and $\mathbb{K}^*$, respectively. When this will not be the case (for instance in Section 5) the conchoid will be denoted by $\mathcal{C}(\mathcal{C}, A, d)$ instead of $\mathcal{C}(\mathcal{C})$ to emphasize this fact.

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Throughout this paper, we assume w.l.o.g. that:
1. $C$ is none of the two lines defining the light cone of isotropy $L$. This ensures that $C(C) \neq \emptyset$.

2. $C$ is not a circle centered at $A$ and radius $d$. If $C$ is such a circle, then $C(C)$ decomposes as the focus union the circle centered at $A$ and radius $2d$. This assumption avoids that the conchoid has zero-dimensional components (compare to Theorem 2.1).

3. $C$ is not a line through the focus. If $C$ is such a line, then $C(C) = C$. This assumption avoids that the conchoid has all components special (compare to Theorem 2.2).

The following theorem (see Theorem 1 in [3]) states the main property on conchoids

**Theorem 2.1.** $C(C)$ has at most two components and all of them have dimension 1.

Now, we recall the notion of simple and special components of a conchoid that, as shown in [3], play an important role when studying the rationality. More precisely, an irreducible component $\mathcal{M}$ of $C(C)$ is called simple if there exists a non-empty Zariski dense subset $\Omega \subset \mathcal{M}$ such that, for $Q \in \Omega$, $\text{Card}(\pi_2(\pi_1^{-1}(Q))) = 1$. Otherwise $\mathcal{M}$ is called special. The next theorem states the main property on the existence of simple components (see Theorem 3 in [3]).

**Theorem 2.2.** $C(C)$ has at least one simple component.

The next lemma (see Lemma 5 in [3]) connects the birationality of the maps in incidence diagram and the simple components of the conchoid.

**Lemma 2.3.** Let $\pi_1, \pi_2$ be the projections in the incidence diagram of $\mathcal{C}$, and $\mathcal{M}$ an irreducible component of $C(C)$.

1. If $C(C)$ is reducible, the restricted map $\pi_2|_{\pi_1^{-1}(\mathcal{M})} : \pi_1^{-1}(\mathcal{M}) \to \mathcal{C}$ is birational.

2. The restricted map $\pi_1|_{\pi_1^{-1}(\mathcal{M})} : \pi_1^{-1}(\mathcal{M}) \to \mathcal{M}$ is birational iff $\mathcal{M}$ is simple.

3 Rational Conchoids

We know that conchoids are either irreducible or with two components (see Theorem 2.1). In this section, we characterize the conchoids having all their components rational. We see that these conchoids can only be generated by rational curves. Moreover, we characterize the cases where the conchoid is rational or it is reducible with the two components rational. As a consequence, we prove that the conchoid of the irreducible conics, from a focus on the conic, are rational; in particular all Limaçons of Pascal are rational. We also see that all conchoids of Nicomedes are rational.

Let $\mathcal{P}(t)$ be a rational parametrization of $\mathcal{C}$. Taking into account the definition of the incidence variety $\mathfrak{B}(\mathcal{C})$, one has that

$$\left( \mathcal{P}(t) + \frac{d}{\pm \| \mathcal{P}(t) - A \|} (\mathcal{P}(t) - A), \mathcal{P}(t), 1 + \frac{d}{\pm \| \mathcal{P}(t) - A \|} \right) \in \mathfrak{B}(\mathcal{C}).$$
So, \( T^\pm(t) = P(t) + \frac{d}{\pm\|P(t) - A\|} (P(t) - A) \in \mathcal{C}(C) \). Therefore, if \( \pm\|P(t) - A\| \in \mathbb{K}(t) \), \( T^\pm(t) \) parametrizes the components of \( \mathcal{C}(C) \). This motivates the next definition.

**Definition 3.1.** We say that a parametrization \( P(t) \in \mathbb{K}(t)^2 \) is at rational distance to the focus if \( ||P(t) - A||^2 = m(t)^2 \), with \( m(t) \in \mathbb{K}(t) \). For short, we express this fact saying that \( P(t) \) is RDF or \( A\text{-RDF} \) if we need to specify the focus.

**Remark 3.2.** Note that:

1. The notion of RDF depends on the focus. For instance, if \( P(t) = (P_1, P_2) = (t, t^2) \) then
   \[
   P_1^2 + P_2^2 = t^2(t^2 + 1), \quad \text{and} \quad P_1^2 + (P_2 - \frac{1}{4})^2 = \frac{1}{16}(1 + 4t^2)^2.
   \]
   So \( P(t) \) is not \((0, 0)\)-RDF but is \((0, 1/4)\)-RDF.

2. If \( P(t) \) is \( A\text{-RDF} \), every re-parametrization of \( P(t) \) is also \( A\text{-RDF} \). However, if can happen that a re-parametrization of a non RDF parametrization is RDF. For instance, as we have seen above \((t, t^2)\) is not \((0, 0)\)-RDF but \( (\frac{2t}{t^2 - 1}, \frac{4t^2}{(t^2 - 1)^2}) \) is \((0, 0)\)-RDF since
   \[
   \left(\frac{2t}{t^2 - 1}\right)^2 + \left(\frac{4t^2}{(t^2 - 1)^2}\right) = 4t^2(t^2 + 1)^2.
   \]
   So, we have that if \( C \) has a proper RDF parametrization then all the parametrizations of \( C \) are RDF with respect to the same focus. Nevertheless, it might happen that \( C \) does not have proper RDF parametrizations but has non-proper RDF parametrizations.

Checking whether a given parametrization is RDF is easy. However deciding, and actually computing, the existence of RDF reparametrizations of non RDF parametrizations is not so direct. For dealing with this, we introduce the next notion.

**Definition 3.3.** Let \( P(t) \in \mathbb{K}(t)^2 \) be a rational parametrization of \( C \). We define the reparametrizing curve of \( P(t) \), and we denote it by \( \Psi(P) \), as the curve generated by the primitive part with respect to \( x_2 \) of the numerator of \( b((-2x_2, x_2^2 - 1), P(x_1) - A) \).

**Remark 3.4.** We observe that:

1. \( \Psi(P) \) does not depend on the representatives of the rational functions in \( P(t) \).

2. The defining polynomial of \( \Psi(P) \) has degree 2 w.r.t. \( x_2 \) and it is primitive w.r.t. \( x_2 \). So, if \( \Psi(P) \) is reducible then it has two factors, both depending linearly on \( x_2 \).

3. Let \( P(t), Q(t) \) be parametrizations of \( C \), and \( \varphi(t) \in \mathbb{K}(t) \) such that \( Q(t) = P(\varphi(t)) \). Let
   \[
   M_1 = b((-2x_2, x_2^2 - 1), P(x_1) - A), \quad M_2 = b((-2x_2, x_2^2 - 1), Q(x_1) - A).
   \]
   Then, \( M_1(\varphi(x_1), x_2) = M_2(x_1, x_2) \). \( \blacksquare \)
The following theorem characterizes the conchoids, having all the components rational, by means of the notions of RDF and reparametrizing curve. In fact, we show that conchoids having all their components rational can only be generated by rational curves; indeed iff the base curve is rational and has RDF parametrizations.

**Theorem 3.5.** The following statements are equivalent:

1. $\mathcal{C}$ is rational and has an RDF parametrization.
2. $\mathcal{C}(\mathcal{C})$ has at least one rational simple component.
3. There exists a proper parametrization of $\mathcal{C}$ which reparametrizing curve has at least one rational component.
4. The reparametrizing curve of every proper parametrization of $\mathcal{C}$ has at least one rational component.
5. All the components of $\mathcal{C}(\mathcal{C})$ are rational.

**Proof.** We prove that all the statements are equivalent to (1). To prove that (2) implies (1), let $\mathcal{M}$ be a rational simple component of $\mathcal{C}(\mathcal{C})$ parametrized by $\mathcal{R}(t) = (R_1(t), R_2(t))$. We consider the diagram:

$$
\Gamma = \pi^{-1}(\mathcal{M}) \subset \mathcal{B}(\mathcal{C}) \subset \mathbb{K}^2 \times \mathbb{K}^2 \times \mathbb{K}
$$

$$
\tilde{\pi}_1 = \pi_1|_{\pi^{-1}(\mathcal{M})} \quad \tilde{\pi}_2 = \pi_2|_{\pi^{-1}(\mathcal{M})}
$$

$$
\mathcal{M} \subset \mathcal{C}(\mathcal{C}) \subset \mathbb{K}^2 \quad \mathcal{C} \subset \mathbb{K}^2 \quad \mathcal{R} \quad \mathbb{K}
$$

Since $\mathcal{M}$ is simple, by Lemma 2.3, $\tilde{\pi}_1$ is birational. So, $\mathcal{Q}(t) = \tilde{\pi}_2(\tilde{\pi}_1^{-1}(\mathcal{R}(t)))$ parametrizes $\mathcal{C}$. Let us see that $\mathcal{Q}(t):=(Q_1(t), Q_2(t))$ is RDF. By construction, $\tilde{\pi}_1^{-1}(\mathcal{R}(t)) = (\mathcal{R}(t), \mathcal{Q}(t), \lambda) \in \mathcal{B}(\mathcal{C})$, where $\lambda = (R_1 - a)/(Q_1 - a) = (R_2 - b)/(Q_2 - b)$ and $\|\mathcal{R}(t) - \mathcal{Q}(t)\|^2 = d^2$. Note that $\mathcal{C}$ is not a line passing through $A$, and hence $Q_1 \neq a, Q_2 \neq b$. Moreover, $\lambda \neq 1$, since otherwise $\mathcal{R}(t) = \mathcal{Q}(t)$ that yields to $d = 0$. So $\mathcal{Q}(t) - A = (\mathcal{Q}(t) - \mathcal{R}(t)) + (\mathcal{R}(t) - A) = (\mathcal{Q}(t) - \mathcal{R}(t)) + \lambda(\mathcal{Q}(t) - A)$, and hence $\|\mathcal{Q}(t) - A\|^2 = d^2/(\lambda - 1)^2$.

In order to prove that (1) implies (2), let $\mathcal{P}(t)$ be an RDF parametrization of $\mathcal{C}$. Let $\|\mathcal{P}(t) - A\|^2 = m(t)^2$. Then

$$
\left(\mathcal{P}(t) \pm \frac{d}{m(t)}(\mathcal{P}(t) - A), \mathcal{P}(t), 1 \pm \frac{d}{m(t)}\right) \in \mathcal{B}(\mathcal{C}).
$$

Moreover, since $\mathcal{P}(t)$ generates a dense subset of $\mathcal{C}$, by Lemma 3 in [3], $\mathcal{P}(t) \pm \frac{d}{m(t)}(\mathcal{P}(t) - A)$ generates a dense in $\mathcal{C}(\mathcal{C})$. So, all components of $\mathcal{C}(\mathcal{C})$ are rational. Now, the result follows from Theorem 2.2.
To see that (1) implies (3), let \( P(t) = (P_1(t), P_2(t)) \) be an RDF parametrization of \( C \), and \( \|P(t) - A\|^2 = m(t)^2 \). Then, \( 1/m(t)(P(t) - A) \) parametrizes the circle \( x_1^2 + x_2^2 = 1 \). Since \( R(t) = (\frac{2}{m(t)}, \frac{2 \phi(t)}{m(t)}) \) is a proper parametrization of the circle, it holds that there exists \( \phi(t) \in \mathbb{K}(t) \) such that \( R(\phi(t)) = 1/m(t)(P(t) - A) \). This implies that \( b((-2\phi(t), \phi(t)^2 - 1), P(t) - A) = 0 \). Therefore (1) implies (3).

To prove that (3) implies (1), let \((\phi_1, \phi_2)\) be a parametrization of one component of \( \mathbb{G}(\mathcal{P}) \), where \( P(t) = (P_1(t), P_2(t)) \) is a proper parametrization of \( C \). Then, \( b((-2\phi_2(t), \phi_2(t)^2 - 1), P(\phi_1(t)) - A) = 0 \). Note that \( \phi_2 \) is not identically zero since otherwise it would imply that \( P_2(\phi_1) = b \) and \( C \) is not a line passing through the focus. Then, it follows that \( P(\phi_1(t)) \) is RDF; indeed

\[
\|P(\phi_1(t)) - A\| = \frac{(\phi_2(t)^2 + 1)^2}{(2\phi_2(t))^2 - (P_2(\phi_1(t)) - b)^2}.
\]

Trivially (4) implies (3). In order to prove that (3) implies (4), let \( P(t) \) and \( Q(t) \) be two proper parametrizations of \( C \), such that \( \mathbb{G}(\mathcal{P}) \) has at least one rational component. Let \((\phi_1, \phi_2)\) be a parametrization of one component of \( \mathbb{G}(\mathcal{P}) \). Note that, because of Remark 3.4 (2), \( \phi_1(t) \) is not constant. Since both parametrizations are proper, there exists an invertible \( \varphi \in \mathbb{K}(t) \) such that \( Q(t) = P(\varphi(t)) \). Let \( M_1(x_1, x_2) = b((-2x_2, x_2^2 - 1), P(x_1) - A) \) and \( M_2(x_1, x_2) = b((-2x_2, x_2^2 - 1), Q(x_1) - A) \). Let \( D_i \) be the denominator of \( M_i \) and let \( C_i, H_i \) be, respectively, the content and primitive part w.r.t. \( x_2 \) of the numerator of \( M_i \). Then, by Remark 3.4 (3),

\[
C_1(x_1)H_1(x_1, x_2)D_2(\varphi^{-1}(x_1)) = D_1(x_1)C_2(\varphi^{-1}(x_1))H_2(\varphi^{-1}(x_1), x_2).
\]

So, \( D_1(\phi_1)C_2(\varphi^{-1}(\phi_1))H_2(\varphi^{-1}(\phi_1), \phi_2) = 0 \). Since \( \phi_1 \notin \mathbb{K} \), then \( \varphi^{-1}(\phi_1(t)) \notin \mathbb{K} \). Since \( D_1, H_2 \) are non-zero univariate polynomials, \( D_1(\phi_1)C_2(\varphi^{-1}(\phi_1)) \neq 0 \). Therefore, \( H_2(\varphi^{-1}(\phi_1), \phi_2) = 0 \). Hence, \((\varphi^{-1}(\phi_1), \phi_2)\) parametrizes a component of \( \mathbb{G}(\mathcal{Q}) \). Therefore one concludes (4).

For the implication of (1) implies (5) see the proof of (1) implies (2). Furthermore, if (5) holds, then \( \mathbb{C}(C) \) has at least one rational simple component, and by (2) one concludes (1).

\[\square\]

Remark 3.6. Theorem 3.5 implies that:

1. Conoids with all their components rational can only be generated by rational curves.

2. The rationality of all the components of the conoid does depend on the base curve and the focus, but not on the distance.

Corollary 3.7. The conoid of a rational curve is either rational, or it is reducible with two rational components, or it is irreducible but non-rational.

\[\text{Proof.} \ \text{Let} \ C \ \text{be the base curve, and let} \ \mathbb{C}(C) \ \text{be reducible. Then, by Corollary 3 in} \ [3] \ \text{and Theorem 2.2, at least one conoid simple component is rational. Now the corollary follows from Theorem 3.5.} \]
Remark 3.8. If $C$ is non-rational, it might happen that its conchoid is reducible with two non-rational components or with one component non-rational and the other rational. For instance, if $C$ is the curve defined by the polynomial
\[
f(y_1,y_2) = 81 + 162y_2^2y_1^6 + 1521y_1^4y_2^2 + 972y_1y_2^3 + 162y_1^2y_2^4 - 1944y_1^3y_2^3 - 1944y_1^4y_2 + 864y_1^5y_2^2 - 2898y_1^6y_2^3 - 8730y_1^7y_2 - 1404y_2^2 - 1458y_2 + 108y_1 - 324y_1^2y_2 + 81y_1^3y_2 - 17388y_1^4y_2 + 972y_1^5y_2 - 162y_2^2y_1^2 - 1024y_2^3y_2 - 13122y_2^4y_2 + 6480y_2^5 - 162y_2^6 - 972y_2y_1 + 8694y_2^2y_1 + 4203y_2^3y_2^2 + 1449y_2^4 - 1932y_1^6 + 5590y_1^5 + 972y_1^4 + 81y_1^3 + 8532y_1^2 + 4356y_1 + 81y_2^6,
\]
then the conchoid of $C$ from the focus $A = (-3,0)$ and distance $d = 1/3$, has two components defined by the factors
\[
(-x_2 + x_1^2)(1296 + 1728x_1 - 5832x_2 + 6237x_2^2 - 3888x_2x_1 + 3690x_1^2x_2 - 4968x_1^3 - 13122x_1^2x_2 - 7584x_1^3 + 2689x_1^4 - 162x_1^5x_2 + 1422x_1^6 + 972x_1^7 - 162x_1^5x_2^2 + 972x_1^6x_2^2 - 17064x_1^7x_2 - 648x_2^3 + 81x_2^4 - 8676x_1x_2^3 + 8532x_1x_2^4 + 81x_1 - 8244x_1^2x_2^3 + 7884x_2^5 + 540x_1^2x_2^4 - 1944x_1^2x_2 + 4302x_2^5 - 1944x_1^3x_2^3 + 162x_1^3x_2^2 + 972x_1x_2^3 + 1467x_1^2x_2^2 + 162x_2^6x_1^6 + 81x_2^6 - 324x_1^3x_2^3),
\]
one of them is the parabola (rational) and the other is a non-rational curve of genus 1.

Note that the reason is that simple components of reducible conchoids are birationally equivalent to $C$ (see Corollary 3 in [3]). Also, take into account that for almost all values of $d$ the conchoid has all the components simple (see Theorem 4 in [3]).

Corollary 3.9. Let $P(t)$ be a parametrization of $C$ such that $\mathcal{C}(P)$ has at least one rational component $M$, and let $(\phi_1(t),\phi_2(t))$ be a parametrization of $M$. Then $P(\phi_1(t))$ is RFD.

Proof. It follows from the proof of (3) implies (1) in Theorem 3.5.

Corollary 3.10. Let $P(t)$ be a proper parametrization of $C$. Then, the following statements are equivalent:

(1) All the components of $\mathcal{C}(C)$ are rational.

(2) There exists $\varphi \in \mathbb{K}(t)$ of degree at most two such that $P(\varphi(t))$ is RFD.

Proof. (1) implies (2) follows from Theorem 3.5. (2) implies (1) follows from Theorem 3.5 from Corollary 3.9 and using that the partial degree of $\mathcal{C}(P)$ w.r.t. $x_2$ is 2 (see Theorem 4.21 in [3]).

In the sequel, we analyze the case of conchoids of rational curves, characterizing the rational conchoid and conchoids with two rational components; we refer to this case as double rationality.

Lemma 3.11. $\mathcal{C}(P)$ is reducible if and only if $P(t)$ is RFD.

Proof. Let $H$ be the primitive part w.r.t. $x_2$, and $M(x_1)$ the content w.r.t. $x_2$ of the numerator of $b((-2x_2,x_2^2-1),P(x_1)-A)$. All factors of $H$ depend on $x_2$. Thus, $\mathcal{C}(P)$ is reducible if and only if $H$ has two factors depending on $x_2$, or equivalently, the discriminant $\Delta_H$ w.r.t. $x_2$ is the square of a polynomial. Therefore, since $M(x_1)^2\Delta_H = 4\|P(t) - A\|^2$, one has that $\mathcal{C}(P)$ is reducible if and only if $P(t)$ is RFD.
Theorem 3.12. (Characterization of double rational conchoids) Let \( C \) be rational. The following statement are equivalent:

1. \( \mathcal{C}(C) \) is reducible.
2. \( \mathcal{C}(C) \) has exactly two components and they are rational.
3. There exists an RDF proper parametrization of \( C \).
4. Every proper parametrization of \( C \) is RDF.
5. There exists a proper parametrization of \( C \) which reparametrizing curve is reducible.
6. The reparametrizing curve of every proper parametrization of \( C \) is reducible.

Proof. By Corollary 3.7, (1) implies (2). (2) implies (1) trivially. In order to prove that (2) implies (3), let \( \mathcal{R}(t) \) be a proper parametrization of a simple component \( \mathcal{M} \) of \( \mathcal{C}(C) \). We consider the diagram used in the proof of Theorem 3.5. By Lemma 2.3, \( \tilde{\pi}_2 \circ \tilde{\pi}_1^{-1} \circ \mathcal{R} : \mathbb{K} \to \mathcal{C} \) is birational. Therefore, \( Q(t) = \pi_2(\pi_1^{-1}(\mathcal{R}(t))) \) is a proper parametrization of \( C \). Furthermore, reasoning as in the proof of “(2) implies (1)”, in Theorem 3.5, one has that \( Q(t) \) is RDF. (3) implies (4) follows from the Remark 3.2.

In order to see that (4) implies (2), let \( \mathcal{P}(t) = (P_1(t), P_2(t)) \) be an RDF proper parametrization of \( C \). Reasoning as in the proof of Theorem 3.5, we get that

\[
\mathcal{M}^\pm(t) = \left( \mathcal{R}^\pm(t), \mathcal{P}(t), \frac{R_1^\pm(t) - a}{P_1(t) - a} \right)
\]

are two rational parametrizations of \( \mathcal{B}(C) \). Moreover, since \( \mathcal{P}(t) \) is proper then \( \mathbb{K}(t) = \mathbb{K}(\mathcal{P}(t)) \subseteq \mathbb{K}(\mathcal{M}^\pm(t)) \subseteq \mathbb{K}(t) \). So, each \( \mathcal{M}^\pm(t) \) is proper. So, there exists a linear rational function \( \varphi(t) \) such that \( \mathcal{M}^+(\varphi(t)) = \mathcal{M}^-(t) \). Thus, \( \varphi(t) = t \) and, since \( d \neq 0 \), \( \mathcal{P}(t) = A \) which is a contradiction.

Applying Lemma 3.11 one has that (4) implies (5). The implication “(5) implies (6)” follows from Lemma 3.11 and Remark 3.2. Finally, “(6) implies (4)” follows directly from Lemma 3.11. \( \square \)

Theorem 3.13. (Characterization of rational conchoids) Let \( C \) be rational. The following statement are equivalent:

1. \( \mathcal{C}(C) \) is rational.
2. There exists a proper parametrization of \( C \) which reparametrizing curve is rational.
Corollary 3.18. Conchoids of Nicomedes are rational.

Proof. Let $\mathcal{C}(C)$ be rational. By Theorem 3.15 there exists a proper parametrization $P(t)$ of $C$ such that $\mathcal{G}(P)$ has at least one rational component; say $M$. Furthermore, by Theorem 3.12 $P(t)$ is not RDF. Thus, by Lemma 3.11 $\mathcal{G}(P)$ is irreducible, and hence rational. So, (1) implies (2).

We prove that (2) implies (3). Let $P(t)$ proper such that $\mathcal{G}(P)$ is rational and let $Q(t)$ be another proper parametrization of $C$. Since $\mathcal{G}(P)$ is irreducible, by Lemma 3.11 $P(t)$ is not RDF. By Remark 3.2 $Q(t)$ is not RDF. So, by Lemma 3.11 $\mathcal{G}(Q)$ is irreducible. Now, the result follows from Theorem 3.5.

Finally, we prove that (3) implies (1). Let $\mathcal{G}(P)$ be rational, with $P(t)$ proper. By Lemma 3.11 $P(t)$ is not RDF. Thus, by Theorem 3.12 $\mathcal{C}(C)$ is irreducible. Now, the result follows from Theorem 3.5.

We apply these results to the case of conchoids of conics with the focus on the conic (in particular to Limaçons of Pascal), and to the case of conchoids of Nicomedes.

Lemma 3.14. Let $P(t) = (p_1(t)/p(t), p_2(t)/p(t))$ be a proper parametrization of $C$ with $\gcd(p_1, p_2, p) = 1$ and $\deg_p(p_i/p) \leq 2$ for $i = 1, 2$. If $A \in C$, then $\mathcal{C}(C)$ is rational.

Proof. The defining polynomial $g(x_1, x_2)$ of $\mathcal{G}(P)$ is the primitive part w.r.t. $x_2$ of $K(x_1, x_2) = -2x_2(p_1(x_1) - ap(x_1)) + (x_2^2 - 1)(p_2(x_1) - bp(x_1))$. Moreover, the content $C(x_1)$ of $K$ w.r.t. $x_2$ is $\gcd(p_1(x_1) - ap(x_1), p_2(x_1) - bp(x_1))$. First, we observe that $\deg_{x_1}(g) > 0$. Indeed, if it is zero, it implies that there exist $\lambda, \mu \in K$ such that $P(t) = (a + \lambda C(t)/p(t), b + \mu C(t)/p(t))$ and, hence, $C$ would be a line passing through the focus.

Let us assume that $A = (a, b)$ is reachable by $P(t)$; say $P(t_0) = A$. Then $x_1 - t_0$ divides $C(x_1)$, and hence $\deg_{x_1}(g) = 1$. So $\mathcal{G}(P)$ is rational and, by Theorem 3.14 $\mathcal{C}(C)$ is rational. Now, if $A$ is not reachable by $P(t)$, then, for $i = 1, 2$, $\deg(p_i) \leq \deg(p)$ (see Section 6.3. in [3]). Say

\[ p_i(x_1) = a_{i,n}x_1^n + \cdots + a_{i,0}, \quad p(x_1) = b_nx_1^n + \cdots + b_0, \]

where $a_{i,n}$ might vanishes. Then, $A = (a_{1,n}/b_n, a_{2,n}/b_n)$ (see Section 6.3. in [3]). So, $\deg_{x_1}(g) = 1$. Thus, reasoning as above we get the result.

Now, taking into account the parametrizations of the irreducible conics, by Lemma 3.14 one deduces the following result (see also Examples 4.1, 4.2, 4.3, 4.4).

Corollary 3.15. Let $C$ be an irreducible conic, and let $A \in C$, then $\mathcal{C}(C)$ is rational.

Corollary 3.16. Limaçons of Pascal are rational.

Remark 3.17. In general it is not true that if the focus is on the curve, the conchoid is rational. For instance, let $C$ be the curve defined by $y_1^3y_2 = 1$, $P(t) = (1/t, t^3)$, and $A = (1, 1) \in C$. Then, $\mathcal{G}(P)$ is defined by $x_2^2x_1^3 - x_1^3 + x_1^2x_2^3 - x_1^2 + x_1x_2^2 - x_1 + 2x_2$ and its genus is 2.

Finally we analyze the conchoids of Nicomedes (see also Example 4.5).

Corollary 3.18. Conchoids of Nicomedes are rational.
Proof. Conchoids of Nicomedes appear when $C$ is a line and $A \not\in C$. Let $P(t) = (p_1(t), p_2(t)) = (a_1 + \lambda_1 t, a_2 + \lambda_2 t)$. The defining polynomial of $G(P)$ is $g(x_1, x_2) = -2x_2(p_1(x_1) - a) + (x_2^2 - 1)(p_2(x_1) - b)$. Note that, since $A \not\in C$, $g$ is primitive w.r.t. $x_2$. Now the result follows from Theorem 3.13 and noting that $\deg_{x_1}(g) = 1$.

4 Parametrization of Conchoids.

In this section we apply the results in Section 3 to derive an algorithm to check the rationality of the components of a conchoid and, in the affirmative case, to parametrize them. For this purpose, in the sequel, let $C$ be rational and $P(t)$ be a proper parametrization of $C$. We also assume that the focus $A$ is fixed. However, we consider $d$ generic. Recall that we have assumed that $C$ is not a line through the focus neither a circle centered at the focus and radius $d$; nevertheless, observe that the problem for these two excluded cases is trivial.

First, we check whether $P(t)$ is RDF; equivalently one can check whether $G(P)$ is reducible. If so, by Theorem 3.12 $C(C)$ is not rational. If it is not rational, by Theorem 3.13 $C(C)$ is not rational. If $G(P)$ is rational, by Theorem 3.13 $C(C)$ is rational. In order to parametrize $C(C)$, we get a proper parametrization $(\phi_1(t), \phi_2(t))$ of $G(P)$ (see [4] for this). Then, by Corollary 3.9 $Q(t) = P(\phi_1(t))$ is RDF. Therefore, any of the parametrizations

$$Q(t) + \frac{d}{\pm||Q(t) - A||}(Q(t) - A)$$

parametrizes $C(C)$. Summarizing we get the following procedure:

1. Compute the primitive part $g(x_1, x_2)$ w.r.t. $x_2$ of the numerator of $b((-2x_2, x_2^2 - 1), P(x_1) - A)$.

2. If $g$ is reducible return that $C(C)$ is double rational and that $P(t) + \frac{d}{\pm||P(t) - A||}(P(t) - A)$ parametrize the two components.

3. Check whether the genus of $G(P)$ is zero. If not return that $C(C)$ is not rational.

4. Compute a proper parametrization $(\phi_1(t), \phi_2(t))$ of $G(P)$ and return that $C(C)$ is rational and that $P(\phi_1(t)) + \frac{d}{\pm||P(\phi_1(t)) - A||}(P(\phi_1(t)) - A)$ parametrizes $C(C)$.

We illustrate the algorithm by means of some examples.

Example 4.1. (Conchoid of Parabolas) Let $C$ be the parabola defined by $f(y_1, y_2) = y_2 - \mu_1 y_1^2 + \mu_2 y_1 + \mu_3$, with $\mu_1 \neq 0$. We consider the proper parametrization $P(t) = (t, \mu_1 t^2 + \mu_2 t + \mu_3)$, and the focus $A = (\lambda, \mu_1 \lambda^2 + \mu_2 \lambda + \mu_3)$ being any point on $C$. By Corollary 3.15
we know that \( \mathcal{E}(\mathcal{C}) \) is rational. Here, we indeed compute a parametrization. The polynomial \( g \) defining \( \mathcal{G}(\mathcal{P}) \) is irreducible:

\[
g(x_1, x_2) = \mu_1 x_1 x_2^2 - \mu_1 x_1 + \mu_2 x_2^2 - \mu_2 - 2x_2 + \lambda \mu_1 x_2^2 - \lambda \mu_1.
\]

Moreover \( \mathcal{G}(\mathcal{P}) \) is rational and can be parametrized as (recall that \( \mu_1 \neq 0 \))

\[
\phi(t) = (\phi_1(t), \phi_2(t)) = \left( -\frac{t^2 \mu_2 - \mu_2 - 2t + \lambda t^2 \mu_1 - \lambda \mu_1}{\mu_1 (t^2 - 1)}, t \right).
\]

Therefore, \( Q(t) = (\phi_1(t), \mu_1 \phi_1(t)^2 + \mu_2 \phi_1(t) + \mu_3) \) is RDF and \( Q(t) + \frac{d}{\|Q(t) - A\|}(Q(t) - A) \) parametrizes \( \mathcal{E}(\mathcal{C}) \).

**Example 4.2. (Conchoid of Ellipses)** Let \( \mathcal{C} \) be the ellipse defined by

\[
f(y_1, y_2) = \frac{y_1^2}{r_1^2} + \frac{y_2^2}{r_2^2} - 1,
\]

with \( r_1 r_2 \neq 0 \). We consider the proper parametrization

\[
\mathcal{P}(t) = \left( \frac{2r_1 t}{t^2 + 1}, \frac{r_2 (t^2 - 1)}{t^2 + 1} \right),
\]

and the focus \( A = \mathcal{P}(\lambda) \) being a point on \( \mathcal{C} \). By Corollary 3.15, we know that \( \mathcal{E}(\mathcal{C}) \) is rational. Here, we indeed compute a parametrization. The polynomial \( g \), defining \( \mathcal{G}(\mathcal{P}) \), is irreducible:

\[
g(x_1, x_2) = 2 x_1 x_2 r_1 \lambda + x_1 r_2 x_2^2 - r_2 x_1 - 2x_2 r_1 + \lambda r_2 x_2^2 - r_2 \lambda.
\]

Moreover \( \mathcal{G}(\mathcal{P}) \) is rational and can be parametrized as

\[
\phi(t) = (\phi_1(t), \phi_2(t)) = \left( -\frac{2r_1 t + \lambda r_2 t^2 - r_2 \lambda}{2 r_1 \lambda + r_2 t^2 - r_2}, t \right).
\]

Therefore, \( Q(t) = \mathcal{P}(\phi_1(t)) \) is RDF and \( Q(t) + \frac{d}{\|Q(t) - A\|}(Q(t) - A) \) parametrizes \( \mathcal{E}(\mathcal{C}) \).

**Example 4.3. (Limaçon of Pascal)** Taking \( r_1 = r_2 \neq 0 \) in Example 4.2, we get a parametrization of the Limaçons of Pascal.

**Example 4.4. (Conchoid of Hyperbolas)** Let \( \mathcal{C} \) be the hyperbola defined by

\[
f(y_1, y_2) = \frac{y_1^2}{r_1^2} - \frac{y_2^2}{r_2^2} - 1,
\]

with \( r_1 r_2 \neq 0 \). We consider the proper parametrization

\[
\mathcal{P}(t) = \left( \frac{-r_1 \left( r_1^2 + r_2^2 t^2 \right)}{-r_1^2 + r_2^2 t^2}, \frac{2r_2 r_1 t}{-r_1^2 + r_2^2 t^2} \right),
\]

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and the focus $A = \mathcal{P}(\lambda)$ being a point on $\mathcal{C}$. By Corollary 3.15, we know that $\mathcal{E}(\mathcal{C})$ is rational. Here, we indeed compute a parametrization. The polynomial $g$, defining $\mathcal{E}(\mathcal{P})$, is irreducible:

$$g(x_1, x_2) = -x_1 x_2^2 \lambda r_2^2 - 2 x_1 x_2 r_1^2 + \lambda r_2^2 x_1 - x_2^2 r_1^2 + r_1^2 - 2 \lambda x_2 r_1^2.$$  

Moreover $\mathcal{E}(\mathcal{P})$ is rational and can be parametrized as

$$\phi(t) = (\phi_1(t), \phi_2(t)) = \left(\frac{-r_1^2 (t^2 - 1 + 2 \lambda t)}{\lambda r_2^2 t^2 + 2 r_1^2 t - \lambda r_2^2}, t\right).$$

Therefore, $\mathcal{Q}(t) = \mathcal{P}(\phi_1(t))$ is RDF and $\mathcal{Q}(t) + \frac{d}{\pm \|\mathcal{Q}(t) - A\|}(\mathcal{Q}(t) - A)$ parametrizes $\mathcal{E}(\mathcal{C})$.

**Example 4.5. (Conchoid of Nicomedes)** Let $\mathcal{C}$ be the line parametrized by

$$\mathcal{P}(t) = (a_1 + t\lambda_1, a_2 + t\lambda_2),$$

and the focus $A = (a, b) \notin \mathcal{C}$. Then, $\mathcal{E}(\mathcal{C})$ is the conchoid of *Nicomedes*. By Corollary 3.15, we know that $\mathcal{E}(\mathcal{C})$ is rational. Here, we indeed compute a parametrization. The polynomial $g$, defining $\mathcal{E}(\mathcal{P})$, is irreducible because $A \notin \mathcal{C}$:

$$g(x_1, x_2) = -2x_2a_1 - 2x_2x_1\lambda_1 + 2x_2a + x_2^2a_2 + x_2^2x_1\lambda_2 - x_2^2b - a_2 - x_1\lambda_2 + b.$$  

Moreover $\mathcal{E}(\mathcal{P})$ is rational and can be parametrized as

$$\phi(t) = (\phi_1(t), \phi_2(t)) = \left(\frac{2ta_1 + t^2b - 2ta - t^2a_2 - b + a_2}{-2t\lambda_1 - \lambda_2 + t^2\lambda_2}, t\right).$$

Therefore, $\mathcal{Q}(t) = \mathcal{P}(\phi_1(t))$ is RDF and $\mathcal{Q}(t) + \frac{d}{\pm \|\mathcal{Q}(t) - A\|}(\mathcal{Q}(t) - A)$ parametrizes $\mathcal{E}(\mathcal{C})$.

5 **Detecting focuses to Parametrize Conchoids.**

In the previous section we have seen how to decide whether the conchoid to a rational curve is rational or double rational and, in the affirmative case, how to parametrize the components of the conchoid. Nevertheless, in that reasoning the focus is fixed. In this section, we analyze a slightly different problem. We assume that we are given a proper parametrization

$$\mathcal{P}(t) = \left(\frac{p_1(t)}{p(t)}, \frac{p_2(t)}{p(t)}\right),$$

where $\gcd(p_1, p_2, p) = 1$, of a rational curve $\mathcal{C}$, and we look for $A_0 \in \mathbb{K}^2$ such that the conchoid $\mathcal{E}(\mathcal{C}, A_0, d)$ has all the components rational. We know that this implies that either $\mathcal{E}(\mathcal{C}, A_0, d)$ has two rational components or it is rational. In the first case we say that the $A_0$ is a *double rational focus* and, in the second, that $A_0$ is a *rational focus*. For this purpose, in the sequel, $A = (a, b)$ is treated generically, and hence $a, b$ are unknowns.
getting double rational focuses. The strategy is as follows. First we determine a set $F$ in $\mathbb{K}^2$ containing the possible double rational focuses. Afterwards, we prove that $F$ is the union of $\mathcal{C}$ and finitely many lines. So all components of $F$ are rational, and using a parametrization of each component we determine conditions on the parameter to get double rational focuses.

More precisely, let

$$\Delta_1(a, t) = p_1(t) - ap(t), \quad \Delta_2(b, t) = p_2(t) - bp(t),$$

$$\Sigma_1(a, b, t) = \Delta_1 + \sqrt{1} Delta_2, \quad \Sigma_2(a, b, t) = \Delta_1 - \sqrt{1} \Delta_2.$$

So,

$$\|\mathcal{P}(t) - A\|^2 = \frac{1}{p(t)^2}(\Delta_1^2 + \Delta_2^2) = \frac{1}{p(t)^2} \Sigma_1 \Sigma_2.$$ 

Thus, a necessary condition for $A_0 = (a_0, b_0) \in \mathbb{K}^2$ to be a double rational focus of $\mathcal{P}(t)$ is that $\Sigma_1(a_0, b_0, t)\Sigma_2(a_0, b_0, t)$ is either constant or it has multiple roots. Let us see that the first condition cannot happen; recall that we have excluded the case where $\mathcal{C}$ is a line and $A_0 \in \mathcal{C}$.

**Lemma 5.1.** For every focus $A_0 = (a_0, b_0) \in \mathbb{K}^2$, $\Sigma_1(a_0, b_0, t)\Sigma_2(a_0, b_0, t)$ is not constant.

**Proof.** Let $a_0, b_0 \in \mathbb{K}$ be such that $\Sigma_1(a_0, b_0, t)\Sigma_2(a_0, b_0, t) \in \mathbb{K}$. We prove that then $\mathcal{C}$ is a line passing through $A_0 = (a_0, b_0)$ and we have excluded this case. It holds that either there exists $i \in \{1, 2\}$ such that $\Sigma_i(a_0, b_0, t) = 0$ (say, $i = 1$) or both $\Sigma_i(a_0, b_0, t), i = 1, 2$, are constant. In the first case, $p_1(t)/p(t) = a$, and this implies that $\mathcal{C}$ is the line $y_1 = a$. So $\Sigma_i(a_0, b_0, t) \in \mathbb{K}$ for $i = 1, 2$. This implies that $\Delta_1(a_0, t), \Delta_2(b_0, t)$ are constants. Say $\Delta_1(a_0, t) = \mu_1, \Delta_2(b_0, t) = \mu_2$, with $\mu_1, \mu_2 \in \mathbb{K}$. However, this implies that $\mathcal{C}$ is the line $\mu_2(y_1 - a) = \mu_1(y_1 - b)$ that passes through $A_0$.

Therefore, if $A_0$ is a double rational focus then at least one the following holds: (i) $\Sigma_1(a_0, b_0, t), \Sigma_2(a_0, b_0, t)$ have a common root or, equivalently, $\Delta_1(a_0, t), \Delta_2(b_0, t)$ have a common root; (ii) $\Sigma_1(a_0, b_0, t)$ has a multiple root; (iii) $\Sigma_2(a_0, b_0, t)$ has a multiple root.

Now, let $R(a, b)$ be the square-free part of the resultant of $\Delta_1(a, t), \Delta_2(b, t)$ w.r.t. $t$, and $D_i(a, b)$ be the square-free part of the resultant w.r.t. $t$ of $\Sigma_i(a, b, t), \partial \Sigma_i(a, b, t)/\partial t$, respectively; note that $D_i$ is the square-free part of the discriminant of $\Sigma_i$ w.r.t. $t$ multiplied by the leading coefficient of $\Sigma_i$ w.r.t. $t$. Then, the double rational focuses belong to the algebraic set $F$ in $\mathbb{K}^2$ defined by $R(a, b)D_1(a, b)D_2(a, b) = 0$.

By Theorem 4.41 in [4], $R$ is the defining polynomial of $\mathcal{C}$. Moreover, since

$$\Sigma_i(a, b, t) = (p_1(t) \pm \sqrt{1} p_2(t)) - (a \pm \sqrt{1} b)p(t),$$

$D_i(a, b)$ can be expressed as a polynomial in $(a \pm \sqrt{1} b)$ and hence it is a product of linear factors in $a, b$. Thus, we have the following proposition.

**Proposition 5.2.** $F$ decomposes as $\mathcal{C}$ union finitely many lines.
Now, we take a parametrization \(Q(h) = (Q_1(h), Q_2(h))\) of each component of \(\mathcal{F}\), and we consider the rational function \(\Delta(h, t) := \Delta_1(Q_1(h), t)^2 + \Delta_2(Q_2(h), t)^2\). Repeating a similar argument as above we determine necessary conditions on \(h\) such that \(Q(h)\) is a double rational focus, and a final checking detect the double rational focuses, when they exist.

**Example 5.3. (Double rational focuses for the parabola)** Let \(\mathcal{C}\) be the parabola over \(\mathbb{C}\) parametrized by \(\mathcal{P}(t) = (t, t^2)\). Using the above notation, \(R(a, b) = b^2 - a, D_1(a, b) = 4a + 4bi - i,\) and \(D_2(a, b) = 4a - 4bi + i\). By Corollary 3.15 for \(A \in \mathcal{C}\) the conchoid is rational. We analyze the lines given by \(D_1\) and \(D_2\). We take the parametrization \(Q(h) = (\frac{1}{2}i - h, h)\) of \(D_1\). The rational function \(\Delta(h, t)\) is

\[
\Delta(h, t) = \frac{1}{16}(4t^2 + 4it + 1 - 8h)(2t - i)^2.
\]

The discriminant of \(4t^2 + 4it + 1 - 8h\) w.r.t. \(t\) is \(128(1 - 4h)\). So, the only candidate generated by \(D_1\) is \(Q(1/4) = (0, \frac{1}{4})\) that, indeed, is a double rational focus. Analyzing \(D_2\) one reaches the same point. So, the only double rational focus for the parabola \(\mathcal{C}\) is \((0, \frac{1}{4})\) (compare to Remark 3.2 and Example 5.7); note that we have got the focus of the parabola.

**Example 5.4. (Double rational focuses for the circle)** Let \(\mathcal{C}\) be the circle over \(\mathbb{C}\) parametrized by

\[
\mathcal{P}(t) = \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right).
\]

One has that \(R(a, b) = a^2 + b^2 - 1, D_1(a, b) = -4(a + bi - i)(a + bi), D_2 = -4(a - bi + i)(a - bi)\). Therefore, \(\mathcal{F}\) is \(\mathcal{C}\) union the four lines defined \(D_{1,1} := a + bi, D_{1,2} := a + bi - i, D_{2,1} := a - bi, D_{2,2} := a - bi + i\). By Corollary 3.15 we only need to analyze the lines given by \(D_{i,j}\). We take the parametrization \(Q(h) = (-ih, h)\) of \(D_{1,1}\). Then

\[
\Delta(h, t) = -(t - i + 2th - 2ih)(t + i)(t - i)^2.
\]

The resultant of \((-t - i + 2th - 2ih)(t + i)\) and its derivative w.r.t. \(t\) is \(16h^2(-1 + 2h)\). So, the candidates generated by \(D_{i,1}\) are \(Q(0) = (0, 0)\) and \(Q(1/2) = (-\frac{i}{2}, \frac{1}{2})\). One checks that \((0, 0)\) is double rational but \((-\frac{i}{2}, \frac{1}{2})\) is not. Reasoning with the other three lines no new focuses are found. So, the only double rational focus is the center of the circle.

**Example 5.5. (Double rational focuses for the ellipse)** Let \(\mathcal{C}\) be the ellipse over \(\mathbb{C}\) parametrized by

\[
\mathcal{P}(t) = \left(\frac{4t}{t^2 + 1}, \frac{3(t^2 - 1)}{t^2 + 1}\right).
\]

One has that \(R(a, b) = a^2/4 + b^2/9 - 1, D_1(a, b) = -4D_{1,1}D_{1,2}D_{1,3}, D_2 = -4D_{2,1}D_{2,2}D_{2,3}\), where \(D_{1,1} = a + bi + \sqrt{5}i, D_{1,2} = a + bi - \sqrt{5}i, D_{1,3} = a + bi - 3i,\) and \(D_{2,j}\) is the conjugate polynomial of \(D_{1,j}\). By Corollary 3.15 we only need to analyze the lines given by \(D_{i,j}\). We take the parametrization \(Q(h) = (-ih - i\sqrt{5}, h)\) of \(D_{1,1}\). Then

\[
\Delta(h, t) = \frac{-(\sqrt{5} + 3)}{4}(-3t^2 - 4it + 3 + \sqrt{5} + 2ht^2 + 2h + t^2\sqrt{5})(2t - 3i + i\sqrt{5})^2.
\]
The resultant of $3t^2 - 4it + 3 + \sqrt{5} + 2ht^2 + 2h + t^2\sqrt{5}$ and its derivative w.r.t. $t$ is 

$$32(5/2 - 3/2h - 3/2\sqrt{5} + h^2 + 3/2h\sqrt{5})h.$$ 

So, the candidates generated by $D_{1,1}$ are (let $\alpha = 3/2 - 1/2\sqrt{5}$)

$$Q(\alpha) = (-i\alpha - i\sqrt{5}, \alpha), \; Q(-\sqrt{5}) = (0, -\sqrt{5}), \; Q(0) = (-i\sqrt{5}, 0).$$

One checks that $Q(-\sqrt{5})$ and $Q(0)$ are double rational but $Q(\alpha)$ is not. Using $D_{1,2}$ one deduces that $(0, \sqrt{5})$ and $(i\sqrt{5}, 0)$ are also double rational focuses. Reasoning with the other lines no new focuses are found. So, the only double rational focus are $\{(0, \pm\sqrt{5}), (\pm i\sqrt{5})\}$. Note that the focuses of the ellipse have appeared.

**Example 5.6. (Double rational focuses for the hyperbola)** Let $C$ be the hyperbola over $\mathbb{C}$ parametrized by

$$\mathcal{P}(t) = \left(\frac{-1 - t^2}{-1 + t^2}, \frac{2t}{-1 + t^2}\right).$$

One has that $R(a, b) = a^2 - b^2 - 1, D_1(a, b) = -4D_{1,1}D_{1,3}, D_2 = -4D_{2,1}D_{2,2}D_{2,3},$ where $D_{1,1} = a + bi - \sqrt{2}, D_{1,2} = a + bi + \sqrt{2}, D_{1,3} = a + bi + 1,$ and $D_{2,j}$ is the conjugate polynomial of $D_{1,j}$. Reasoning as before, one deduces that the double rational focuses are $(\pm\sqrt{2}, 0), (0, \pm i\sqrt{2})$. Note that Note that the focuses of the hyperbola have appeared.

**Detecting rational focuses.** For analyzing the existence, and actual computation, of rational focuses we apply Theorem 3.13. Therefore, we consider a proper parametrization $\mathcal{P}(t)$ of $\mathcal{C}$ and we analyze the rationality of $\mathfrak{S}(\mathcal{P})$. For this purpose, we analyze the genus of $\mathfrak{S}(\mathcal{P})$ in terms of the parameters $a, b$ that define the focus. In the following example, we illustrate these ideas in the case of the parabola.

**Example 5.7. (Rational focuses for the parabola)** Let $C$ be the parabola over $\mathbb{C}$ parametrized by $\mathcal{P}(t) = (t, t^2)$. Let 

$$g(x_1, x_2, a, b) = -2x_2x_1 + 2x_2a + x_2^3 - x_1^2b - x_1^2 + b,$$

where $A = (a, b)$ is generic and let $G(x_1, x_2, x_3, a, b)$ be the homogenization of $g$ in the variables $x_1, x_2.$ We first observe that $g$ is primitive w.r.t. $x_2$ iff $A \notin C.$ On the other hand, by Corollary 3.15 every focus on $C$ is rational. So, we can assume w.l.o.g. that $A \notin C,$ and hence that $g$ is primitive w.r.t. $x_2.$ We also know, by Lemma 3.11 and Example 5.3 that the primitive part w.r.t. $x_2$ of $g$ is irreducible iff $A \neq (0, 1/4)$.

We start analyzing the points at infinity. $G(x_1, x_2, 0, a, b) = x_1^3x_2^2$. Thus, the points at infinity of $\mathfrak{S}(\mathcal{P})$ are $P_1 := (1 : 0 : 0)$ and $P_2 := (0 : 1 : 0)$ independently on $A.$ Moreover $P_1, P_2$ are, independently on $A$, double points. Moreover, $P_1$ is always ordinary (the tangents are given by $x_2^2 - x_3^2$) and, if $b \neq 0,$ $P_2$ is ordinary too (the tangents are given by $x_1^2 - bx_3^2$). Now, we analyze the affine singular locus. For this purpose, we compute a reduced Gröbner
basis $G$ of \( g, \partial g/\partial x_1, \partial g/\partial x_2 \}, as polynomials in \( \mathbb{C}(a, b)[x_1, x_2] \), w.r.t. the graded reverse lexicographic order with \( x_1 < x_2 \). The basis is \( G = \{ 1 \} \) and

\[
1 = \frac{a_1(x_1, x_2, a, b)}{c(a, b)} g + \frac{a_2(x_1, x_2, a, b)}{c(a, b)} \frac{\partial g}{\partial x_1} + \frac{a_3(x_1, x_2, a, b)}{c(a, b)} \frac{\partial g}{\partial x_2},
\]

where

\[
a_1(x_1, x_2, a, b) = 12 x_2 x_1 a^2 + 8 x_2 x_1 b^2 - 6 x_2 x_1 b + x_2 x_1 - 8 x_2^2 b a^2 + 4 x_2^2 b^2 - x_2^2 b - 4 a x_1 b - a x_1 + 8 x_2 a^3 + 8 x_2 a b^2 - 2 x_2 ab + x_2 a + 8 a^2 b - 16 a^2 - 16 b^2 + 8 b - 1,
\]

\[
a_2(x_1, x_2, a, b) = -6 x_1^2 x_2 a^2 - 4 x_1^2 x_2 b^2 + 3 x_1^2 x_2 b - 1/2 x_1^2 x_2 + 2 a x_1^2 b + 1/2 a x_1^2 + 4 a^3 x_1 x_2 - 2 a x_1 x_2 b + 1/2 a x_1 x_2 - 4 a x_1 b^2 + 7 x_1 a^2 + 8 a x_1 b^2 - 4 a x_1 b + 1/2 x_1 + 2 b x_2 a^2 + 4 b^2 x_2 - b^2 x_2 - 8 a^3 - 6 a b^2 - 9/2 a b - 1/2 a,
\]

\[
a_3(x_1, x_2, a, b) = 4 b x_2^3 a^2 - 2 b^2 x_2^3 + 1/2 b x_2^3 - 8 a^3 x_2^2 - 4 a x_2^3 b^2 + 3 a x_2^3 b - a x_2^3 - 4 b x_2 a^2 + 6 x_2 a^2 + 6 b^2 x_2 - 7/2 b x_2 + 1/2 x_2 + 8 a^3 + 4 a b^2 - 5 a b + 1/2 a,
\]

\[
c(a, b) = (-b + a^2)(16 b^2 - 8 b + 1 + 16 a^2).
\]

Moreover, \( g, \partial g/\partial x_1, \partial g/\partial x_2 \) are monic w.r.t. the above order. So, by exercise 7 page 284 in \cite{2}, if \( c(a, b) \neq 0 \) the Gröbner basis specializes properly. Then, let \( \mathcal{W} \) be the curve in \( \mathbb{C}^2 \) defined by \( b(-b + a^2)(16 b^2 - 8 b + 1 + 16 a^2) \). So \( \mathcal{W} \) is the parabola \( \mathcal{C} \ union the lines \( b = 0 \) (call it \( \mathcal{L} \)), and \( a \pm (1/4 - b)i = 0 \) (call them \( \mathcal{L}^\pm \)). We distinguish several cases in our analysis:

1. If \( A \notin \mathcal{W} \), the genus of \( \mathcal{G}(\mathcal{P}) \) is 1; note that \( (0, 1/4) \in \mathcal{W} \), and hence \( \mathcal{G}(\mathcal{P}) \) is irreducible. Therefore, \( A \) is not rational.

2. If \( A \in \mathcal{C} \), we already know that \( A \) is rational.

3. Let \( A \in \mathcal{L}^+ \). If \( b = 0 \), then \( A = (-1/4i, 0) \) and \( \mathcal{G}(\mathcal{P}) \) has genus 0. So \( (-1/4i, 0) \) is rational. Let \( b \neq 0 \). Then, \( P_1 \) and \( P_2 \) are ordinary double points. So, we only need to analyze the affine singular locus. It holds that, independently on \( b, (-i/2, i) \) is a double point of \( \mathcal{G}(\mathcal{P}) \). So, if \( a \neq 0 \) (i.e. \( A \neq (0, 1/4) \)), \( \mathcal{G}(\mathcal{P}) \) has genus 0. Therefore, every \( A \in \mathcal{L}^+ \ \\{ (0, 1/4) \} \) is rational.

4. Let \( A \in \mathcal{L}^- \). Reasoning as above, one gets that every \( A \in \mathcal{L}^- \ \\{ (0, 1/4) \} \) is rational.

5. Let \( A \in \mathcal{C} \). If \( A \) is also on another component of \( \mathcal{W} \), we already know the classification.

So, we assume w.l.o.g. that \( A \neq (0, 0), A \neq (\pm i/4, 0) \). Therefore, in this case, \( \mathcal{G}(\mathcal{P}) \) has no affine singularity. Thus, we only need to analyze \( P_2 \) that now is a non-ordinary singularity. For this purpose, we blow up the curve at \( P_2 \) (see e.g. Chapter 3 in \cite{4}). Applying a suitable projective linear change of coordinates, for instance \( \{ x_1 = x_1^\ast - x_3^\ast, x_2 = x_2^\ast - x_3^\ast, x_3 = x_1^\ast + x_3^\ast \} \), and the Cremona transformation, one deduces that if \( a \neq 0 \) then \( P_2 \) has none neighboring singularities. So, in this case the genus of \( \mathcal{G}(\mathcal{P}) \) is 1, and thus no new rational focuses appear.

Summarizing, jointly with Example 5.3, one has the following table
Double rational focuses & Rational focuses
\begin{tabular}{|c|c|c|}
\hline
Parabola & \((0, \frac{1}{4})\) & \((a, a^2), a \in \mathbb{C}\) \\
\hline\end{tabular}

\((\pm (\frac{1}{4} - b)i, b)\), \(b \in \mathbb{C} \setminus \{\frac{1}{4}\}\)

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