SOME NEW EXAMPLES OF NONLINEAR QUOTIENTS WITH NONNEGATIVE SECTIONAL CURVATURE

RAFAEL TORRES

Abstract: In this note, new examples of manifolds with Riemannian metrics of nonnegative sectional curvature are constructed. These manifolds are orbit spaces of free nonlinear orientation-preserving and orientation-reversing involutions on products of spheres, their fundamental group is of order two, and they realize homeomorphism classes that were not previously known to contain nonnegatively curved smooth manifolds. We remark that our examples are obtained as a cut-and-paste construction along spheres and real projective spaces on sphere bundles.

1. Introduction and main results

The purpose of this note is to describe some new examples of nonlinear quotients of involutions that admit Riemannian metrics with nonnegative sectional curvature. Their novelty lies within the realization of several homeomorphism classes that were not previously known to contain nonnegatively curved smooth manifolds. The precise statement of our result is as follows.

Theorem A. Suppose \( k \geq 2 \).

1. There exists a smooth closed orientable \((2k + 1)\)-dimensional Riemannian manifold \((X^{2k+1}(2), g_{2,2k+1})\) such that
   - \(X^{2k+1}(2)\) is not homeomorphic to the total space of a 2-sphere bundle over the real projective \((2k - 1)\)-space \(X^{2k+1}(0)\) yet they share the same homotopy groups, and
   - the sectional curvature of the metric \(g_{2,2k+1}\) is nonnegative.

2. There exists a smooth closed nonorientable \(2k\)-dimensional Riemannian manifold \((P^{2k}(2), g'_{2,2k})\) such that
   - \(P^{2k}(2)\) is not homeomorphic to the nonorientable total space of a 2-sphere bundle over the real projective \((2k - 2)\)-space \(P^{2k}(0)\) yet they share the same homotopy groups, and
   - the sectional curvature of the metric \(g'_{2,2k}\) is nonnegative.

The sphere bundles \(X^{2k+1}(0)\) and \(P^{2k}(0)\) arise as orbit spaces of a free isometric \(\mathbb{Z}/2\) action on \((S^2 \times S^n, g_{S^2} + g_{S^n})\), i.e., a linear involution. They are locally isometric to the product of round spheres, and have a metric of nonnegative sectional curvature. Although our examples of Theorem A are also orbit spaces of involutions on the product of spheres \(S^2 \times S^n\) and have \(\mathbb{Z}/2\) as fundamental group, we show...
in Proposition 2.51 that these involutions are nonlinear in the sense that they are not topologically conjugate to linear ones. The manifolds \( P^4(0) \) and \( P^4(2) \) are not even homotopy equivalent \[13\] and the latter represents a homotopy equivalence class of manifolds that was not previously known to admit a nonnegatively curved member; see Remark 2.52. The following corollary is an immediate consequence of lifting the metrics to the universal covers.

Corollary B. The moduli space of Riemannian metrics of nonnegative sectional curvature on \( S^2 \times S^n \) is disconnected for \( n \geq 2 \).

Our manifolds \( X^{2k+1}(2) \) and \( P^{2k}(2) \) are obtained by gluing two copies of real projective spaces along loops that represent the generator of their fundamental group (Definitions 2.14 and 2.19), and identifying the common boundaries using a diffeomorphism that is not isotopic to the identity. In order to construct the nonnegatively curved metrics \( \{g_{2,2k+1}, g'_{2,2k}\} \), the manifolds are deconstructed as an union of disk bundles over real projective spaces that are equipped with an appropriate metric. A detailed construction of these manifolds is given in Section 2.4

while the metrics \( \{g_{2,2k+1}, g'_{2,2k}\} \) are built in Section 2.6. The manifolds of Theorem A can also be obtained by carving out a submanifold from a sphere bundle and gluing it back in using a given diffeomorphism of the boundary. This cut-and-paste construction is discussed in detail in Proposition 2.48. Example 2.63 will serve the interested reader as a hands on summarized description of the construction.

The technique used is reminiscent of Cheeger’s construction of Riemannian metrics of nonnegative sectional curvature on connected sums of symmetric spaces [3]. Grove-Ziller [11, Theorem E] extended Cheeger’s construction to cohomogeneity one manifolds with codimension two singular orbits. A fundamental aspect of their construction of nonnegatively curved \( G \)-invariant metrics is that it relies on the use of the identity map as the diffeomorphism used to identify two bundles along their totally geodesic common boundaries two bundles [11, Section 2]. Their extension yielded infinitely many nonsimmetric metrics with nonnegative sectional curvature on quotients of nonlinear involutions on \( S^5 \), i.e., on every diffeomorphism type of closed manifolds that are homotopy equivalent to \( \mathbb{R}P^5 \) [11, Theorem G]. We justify the claim about the novelty of the examples in Theorem A with the following result.

Theorem C. (i) The manifolds \( X^{2k+1}(2) \) and \( P^{2k}(2) \) are not diffeomorphic to a connected sum of two compact rank one symmetric spaces, a homogeneous space or a biquotient for every \( k \geq 2 \).

(ii) Neither of the metrics \( \{g_{2,2k+1}, g'_{2,2k}\} \) is a cohomogeneity one \( G \)-invariant metric for every \( k \geq 2 \).

(iii) The manifolds \( X^5(2) \) and \( P^4(2) \) do not admit a cohomogeneity one action of any Lie group \( G \).

Theorem C contains all the previously known constructions of nonnegatively curved manifolds (see Ziller’s survey [21]). The main ingredients of its proof are Totaro’s classification of biquotients [19] (cf. [12]), Su’s classification of orbit spaces of \( \mathbb{Z}/2 \)-actions on \( S^2 \times S^3 \) up to diffeomorphism [18], Parker’s classification of cohomogeneity one 4-manifolds [16] (along with an omission corrected in [9]), and Hoelscher classification of cohomogeneity one actions on manifolds of dimension five [10]. The number of manifolds in Item (iii) can be enlarged considerably; see [9, Chapters 4 and 5] and Section 5.
and Grove-Ziller’s inspiring work. We hope that our technique will find further applications on existence questions of nonnegatively curved metrics on inequivalent smoothings within a homeomorphism class and on more homeomorphism classes within homotopically equivalent manifolds as we mention in Remark 2.60.

We finish the note with the following observation regarding Gromov’s minimal volume, minimal entropy and collapse with bounds on sectional curvature [4, 17] of our examples.

**Theorem D.** The minimal volume of the orientable manifolds of Theorem A is

\[ \text{MinVol}(X^{2k+1}(j)) = 0, \]

and \( X^{2k+1}(j) \) collapses with bounded sectional curvature for \( j \in \{0, 2\} \).

The minimal entropy of the nonorientable manifolds of Theorem A is zero,

\[ \text{Vol}_{K_n} (P^{2k}(j)) = 0, \]

and \( P^{2k}(j) \) collapses with sectional curvature bounded from below for \( j \in \{0, 2\} \).

The definitions of these Riemannian invariants are provided in Section 3, where Theorem D is proven. The relation of minimal volume and minimal entropy is given by the inequality

\[ [h(M)]^n \leq (n - 1)^n \text{MinVol}(M) \]

found in [17 page 417]. The manifolds constructed in Theorem A have zero minimal entropy.

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2. **Tools and results**

The purpose of this section is two-fold. We present the concepts and technology we build upon, and we prove our basic technical results. We employ the notation in [8, 7].

2.1. **Involutions and characteristic submanifolds.** Let \( M \) be a smooth closed orientable \( n \)-manifold with fundamental group \( \pi_1(M) = \mathbb{Z}/2 \), and let \( \pi : \tilde{M} \to M \) be its universal cover. We point out at this moment that these symbols will be subject to these assumptions throughout the paper, unless otherwise stated. Let

\[ T : \tilde{M} \to \tilde{M} \]

be a fixed point free involution such that \( M = \tilde{M}/T \). The following special submanifolds are key tools to identify orbit spaces of involutions.

**Definition 2.2** (Characteristic submanifolds). Take the decomposition

\[ \tilde{M} = A \cup T(A), \]
where $A \subset \tilde{M}$ is a connected compact submanifold with $\partial A = \tilde{P} = \partial T(A)$. The codimension one submanifold
\begin{equation}
P := \tilde{P}/T
\end{equation}
is the characteristic submanifold of $M$ and $\tilde{P}$ is the characteristic submanifold of the pair $(\tilde{M}, T)$.

**Example 2.5.** Let $\Lambda : S^n \to S^n$ be the antipodal involution $x \mapsto -x$ for $x \in S^n$ whose orbit space is the real projective $n$-space $\mathbb{R}P^n$. The characteristic submanifolds are the equatorial codimension one sphere $\tilde{P} = S^{n-1} \subset S^n$, and its corresponding $\tilde{P}/\Lambda = P = \mathbb{R}P^{n-1} \subset \mathbb{R}P^n$.

A characteristic submanifold for any given $T$ is connected provided the dimension of $M$ is at least three. For any given involution $T$, one can take the classifying map $f : M \to \mathbb{R}P^n$ for large $N$ and perturb it to make it transverse to $\mathbb{R}P^{n-1}$ and define $\tilde{P} := \pi^{-1}(f^{-1}(\mathbb{R}P^{n-1}))$. Hence, characteristic manifolds exist for any $T$. Moreover, we can assume that $\pi_1(P) = \mathbb{Z}/2$ and that the inclusion $i : P \hookrightarrow M$ induces an isomorphism of fundamental groups (see [15, Chapter I.1.2]).

### 2.2. Pin$^\pm$-structures.

The groups Pin$^\pm(n)$ are central extensions of the orthogonal group
\begin{equation}
1 \to \mathbb{Z}/2 \to \text{Pin}^\pm(n) \to \text{O}(n) \to 1.
\end{equation}
A closed smooth manifold $P$ admits a Pin$^\pm$-structure if the structure group O$(n)$ of its tangent bundle $TP$ can be reduced to the corresponding group Pin$^\pm(n)$. Suppose for the remainder of this section that $TP$ does admit such a structure. There is a transitive and effective action of $H^1(P; \mathbb{Z}/2)$ on the set of Pin$^\pm$-structures on $P$. If $\pi_1(P) = \mathbb{Z}/2$, then there are two Pin$^\pm$-structures $\phi$ and $-\phi$ on $P$. They are related through the action of the first Stiefel-Whitney class $\omega_1(P)$. Moreover, $[(P, \phi)]$ is the inverse of $[(P, -\phi)]$ in the respective Pin$^\pm$ bordism group (see [14] for a definition).

We adhere to the following considerations and notation. Let $P \subset M$ be a characteristic submanifold as in Definition 2.2 where both manifolds have fundamental group of order two. Let $L$ be the nontrivial line bundle over $M$. Fixing a Spin-structure on $TM$ or $TM \oplus 2L$ induces a Pin$^\pm$-structure on $P$ (see [7, p. 153] for further details). We will abuse notation and write $P := (P, \phi)$ and $\overline{P} := (P, -\phi)$ whenever the Pin$^\pm$-structure is invoked. More precisely, the real projective $2k$-space equipped with a Pin$^\pm$-structure $(\mathbb{R}P^{2k}, \phi)$ will be denoted by the symbol $\overline{\mathbb{R}P^{2k}}$. Whenever equipped with the other Pin$^\pm$-structure $-\phi$, we will write
\begin{equation}
\overline{\overline{\mathbb{R}P^{2k}}} := (\mathbb{R}P^{2k}, -\phi).
\end{equation}
Regarding the orientable projective spaces we have the following explanation for our choice of notation in the sequel. As characteristic submanifold we have
\begin{equation}
(\mathbb{R}P^{2k}, \phi) = R P^{2k} \subset R P^{2k+1}.
\end{equation}
We will write
\begin{equation}
\overline{\mathbb{R}P^{2k+1}}
\end{equation}
when the choice of Pin$^\pm$-structure on the characteristic submanifold is
\begin{equation}
(\mathbb{R}P^{2k}, -\phi) = \overline{\mathbb{R}P^{2k}} \subset \mathbb{R}P^{2k+1}.
\end{equation}
Existence of Pin$^\pm$ structures on even-dimensional real projective spaces is established in the following proposition. Its proof consists of computations of the topological obstructions given in [7, Lemma 2.1] in terms of Stiefel-Whitney classes.

**Proposition 2.11.** Let $n \in \mathbb{N}$. Then the following hold:

- $\mathbb{R}P^{4n}$ admits two Pin$^+$-structures.
- $\mathbb{R}P^{4n+1}$ does not admit a Pin$^\mp$-structure.
- $\mathbb{R}P^{4n+2}$ admits two Pin$^-$-structures.
- $\mathbb{R}P^{4n+3}$ admits two Pin$^\pm$-structures.

There is a topological version of the central extension (2.6) that yields groups TopPin$^\pm$ as covers of O($n$) with corresponding bordism groups $\Omega_n^{\text{TopPin}}$. A vector bundle with a TopPin$^\pm$- and a O($n$)-structure is equivalent to a Pin$^\pm$-bundle. We again say that a manifold has a TopPin$^\pm$-structure if its tangent bundle can be equipped with such structure. The reader is directed towards [8, Section 2] and references there for further details.

We will use the following observation that is essentially contained in [15, Lemma p. 11] to distinguish the homeomorphism types of the manifolds of Theorem A.

**Proposition 2.12.** Let

\[ \tilde{M} \xrightarrow{T_i} \tilde{M} \]

be involutions whose orbit spaces $\tilde{M}/T_i = M_i$ are homeomorphic for $i \in \{1, 2\}$. Fix a Spin-structure on $TM_i$ or $TM_i \oplus 2L$. If $M_1$ is homeomorphic to $M_2$, then there is a characteristic topological bordism between the corresponding characteristic submanifolds $P_1$ and $P_2$. In particular, $P_1$ is TopPin$^\pm$-bordant to $P_2$.

**Proof.** A Spin-structure on $TM_i$ or $TM_i \oplus 2L$ induces a Pin$^\pm$-structure on $P_i$. The classifying space is $B\pi_1(M_i) = B\mathbb{Z}/2 = \mathbb{R}P^\infty$, and let $f_i$ be classifying maps of $\pi_1(M_i)$. Any two such maps are homotopic. The characteristic submanifolds $P_i$ are transversal preimages of $f_i$ for $i = 1, 2$ and a homotopy between the maps yields a characteristic Pin$^\pm$-bordism [15, Chapter I.1]. Due to topological transversality, the same holds in the TOP category, and there is a TopPin$^\pm$-bordism between the characteristic submanifolds (see [7, Section 2.2] and references given there). This concludes the proof of the proposition. \hfill $\square$

### 2.3. Gluing construction: circle sums

The cut-and-paste construction used to produce the manifolds of Theorem A is defined as follows (cf. [7, Section 3]).

**Definition 2.14** (Orientable circle sum). Let $M_1$ and $M_2$ be closed smooth oriented $n$-manifolds with fundamental group $\mathbb{Z}/2$ and let $\gamma_j \subset M_j$ be a loop that represents the generator of $\pi_1(M_j) = \mathbb{Z}/2$. A tubular neighborhood $\nu(\gamma_j)$ is diffeomorphic to $S^1 \times D^{n-1}$. Let

\[ i_j : S^1 \times D^{n-1} \hookrightarrow M_j \]

be embeddings of the tubular neighborhood of $\gamma_j$ such that $i_1$ preserves the orientation and $i_2$ reverses it. The smooth manifold

\[ (M_1 \#_{S^1} M_2)(\varphi) := (M_1 - i_1(S^1 \times D^{n-1})) \cup_{\varphi} (M_2 - i_2(S^1 \times D^{n-1})) \]
is the circle sum of $M_1$ and $M_2$, and it depends up to diffeomorphism on the isotopy class of the diffeomorphism
\[ \varphi : S^1 \times S^{n-2} \longrightarrow S^1 \times S^{n-2} \]
that is used to identify the boundaries. The latter is either the identity id or the map $\varphi_\alpha$ that identifies $(\theta, x) \mapsto (\theta, \alpha(\theta) \cdot x)$ where $\alpha : S^1 \to SO(n-1)$ is essential \[20\] Section 16] for $x \in S^{n-2}$ and $\theta \in S^1$.

The choice of diffeomorphism $\varphi$ is given by an element of $\pi_1(SO(n-1)) = \mathbb{Z}/2$, where $\varphi_\alpha$ represents the generator of the group, and there are two manifolds produced as $(M_1 \# S_1 M_2)(\varphi)$. The choice of $\varphi$ is the culprit for the topological novelty of the manifolds of Theorem \[A\] If the fundamental group of both $M_1$ and $M_2$ has order two, the Seifert-van Kampen theorem implies that $\pi_1((M_1 \# S_1 M_2)(\varphi)) = \mathbb{Z}/2$ (independent of choice of $\varphi$). If both $M_1$ and $M_2$ admit orientation-reversing automorphisms, then the construction does not depend on the choice of orientation.

There is the following ambiguity in the cut-and-paste construction of Definition \[2.14\] regarding framings: an embedding $S^1 \hookrightarrow M_j$ for a fixed $j$ has two possible extensions to an embedding of a tubular neighborhood of the circle $S^1 \times D^{n-1}$ up to isotopy, since normal framings are classified by $\pi_1(SO(n)) \cong \mathbb{Z}/2$.

As it was done in \[7\], we will get rid of the ambiguity on the choice of framings by making use of the Pin$\pm$-structures mentioned in Section \[2.2\] as follows. If $P_1 \subset M_1$ and $P_2 \subset M_2$ are characteristic submanifolds, then $P_1 \# S_1 P_2$ is the characteristic submanifold of $M_1 \# S_1 M_2$. In our orientable examples $X^{2k+1}(2)$, we will use
\[ M_1 = \mathbb{R}P^{2k+1} = M_2, \]
and set $P_1 = \mathbb{R}P^{2k} = P_2$. Hence, we recall the definition of circle sum for non-orientable smooth manifolds (cf. \[8\] p. 651).

**Definition 2.19** (Nonorientable circle sum). Let $P_1$ and $P_2$ be closed smooth Pin$\pm$- nonorientable 2k-manifolds, and let $\gamma_j \subset P_j$ be a loop that represents the generator of the fundamental group $\pi_1(P_j) = \mathbb{Z}/2$. A tubular neighborhood $\nu(\gamma_j)$ is diffeomorphic to the nontrivial total space of the $D^{n-1}$-bundle over the circle $D^{n-1} \times S^1$. Let
\[ i_j : D^{n-1} \times S^1 \hookrightarrow P_j \]
be embeddings of $\nu(\gamma_j)$ such that $i_1$ preserves the Pin$\pm$-structure $\phi_1$, while $i_2$ reverses the Pin$\pm$-structure $\phi_2$ for Pin$\pm$-structures $(P_j, \phi_j)$. The smooth manifold
\[ (P_1 \# S_1 P_2)(\varphi') := (P_1 - i_1(D^{n-1} \times S^1)) \cup_{\varphi'} (P_2 - i_2(D^{n-1} \times S^1)) \]
is the circle sum of $P_1$ and $P_2$, and it depends up to diffeomorphism on the isotopy class of the diffeomorphism
\[ \varphi' : S^{n-2} \times S^1 \longrightarrow S^{n-2} \times S^1 \]
of the nonorientable $S^{n-2}$-bundle over $S^1$. There are two choices for the diffeomorphism $\varphi'$: it is either the identity id or the map $\varphi'_\alpha$ that identifies $(\theta, x) \mapsto (\theta, \alpha(\theta) \cdot x)$ for every $\theta$ on the circle base and $x$ on the $(n-2)$-sphere fiber (cf. Definition \[2.14\]).

There is the following topological relation between the circle sum of two manifolds and their disjoint union.
Proposition 2.23. Let \( (P_1, \phi_1) \) and \( (P_2, \phi_2) \) be closed smooth \( \text{Top}\Pin \pm \)-manifolds. Their circle sum is \( \text{Top}\Pin \pm \)-bordant to the disjoint union \( (P_1, \phi_1) \sqcup (P_2, \phi_2) \).

Proof. A \( \text{Top}\Pin \pm \)-bordism is given by
\[
(2.24) \quad (P_1 \times [0, 1]) \cup_{\nu(\gamma_1)} (\nu(\gamma_1) \times [0, 1]) \cup_{\nu(\gamma_2)} (\nu(\gamma_2) \times [0, 1]) \cup_{\nu(\gamma_2)} (P_2 \times [0, 1]).
\]

\[
2.4. \text{Constructions of the manifolds of Theorem A}.
\]

The manifolds \( X_{j}^{2k+1} \) and \( P_{j}^{2k} \) for \( j \in \{0, 2\} \) in Theorem A are orbit spaces of involutions
\[
(2.25) \quad S^2 \times S^n \to S^2 \times S^n
\]
and they have fundamental group of order two. We now describe how to obtain them through a cut-and-paste construction along submanifolds. Consider the decomposition of the \( (2k + 1) \)-sphere given by
\[
S^{2k+1} = \partial D_{2+2k}^{2k}
\]
\[
= \partial(D^2 \times D^{2k})
\]
\[
= (D^2 \times S^{2k-1}) \cup_{\text{id}} (S^1 \times D^{2k})
\]
Let \( \mathbb{A} : S^{2k+1} \to S^{2k+1} \) be the antipodal map, so that
\[
S^{2k+1}/\mathbb{A} = \mathbb{R}P^{2k+1}
\]
\[
= (D^2 \times \mathbb{R}P^{2k-1}) \cup_{\text{id}} (S^1 \times D^{2k})
\]
and
\[
(2.28) \quad (D^2 \times S^{2k-1})/(r, \mathbb{A}) = D^2 \times \mathbb{R}P^{2k-1},
\]
with \( r \) reflection along a line, is the total space of a nontrivial 2-disk bundle over the real projective \((2k - 1)\)-space.

A similar decomposition of the even dimensional sphere
\[
(2.29) \quad S^{2k} = \partial(D^2 \times D^{2k-1}) = (D^2 \times S^{2k-2}) \cup_{\text{id}} (S^1 \times D^{2k-1})
\]
yields the decomposition
\[
(2.30) \quad \mathbb{R}P^{2k} = (D^2 \times \mathbb{R}P^{2k-2}) \cup_{\text{id}} (D^{2k-1} \times S^1).
\]
The nonorientable total space of the nontrivial \( D^2 \)-bundle over the real projective \((2k - 2)\)-space is denoted by \( D^2 \times \mathbb{R}P^{2k-2} \) and \( D^{2k-1} \times S^1 \) denotes the nonorientable \( D^{2k-1} \)-bundle over the circle.

In what follows, we denote by \( \gamma \) the trivial real Hopf bundle over \( \mathbb{R}P^{2k-2} \), and hence \( S(2\gamma \oplus \mathbb{R}) \) is the total space of a nonorientable and nontrivial 2-sphere bundle over the real projective plane. It is constructed as the sphere bundle of the Whitney sum of two copies of \( \gamma \) and the trivial line bundle, hence the symbol.

Example 2.31 (Total spaces of sphere bundles over real projective spaces). Consider the involution
\[
(2.32) \quad T : S^2 \times S^{2k-1} \to S^2 \times S^{2k-1}
\]
given by
\[
(2.33) \quad (x, y) \mapsto (r(x), \mathbb{A}(y)),
\]
Example 2.38 (The manifolds $X^{2k+1}(0)$). The bundle defined in (2.31) can be expressed as the circle sum
\[
X^{2k+1}(0) = (\mathbb{R}P^{2k+1} \# S^1 \mathbb{R}P^{2k+1})(\text{id}).
\]
Using decompositions (2.27) and (2.28), one concludes that
\[
\mathbb{R}P^{2k+1} - S^1 \times D^4 = D^2 \times \mathbb{R}P^{2k-1}.
\]
In particular, the circle sum (2.39) can be expressed as
\[
(\mathbb{R}P^{2k+1} \# S^1 \mathbb{R}P^{2k+1})(\text{id}) = (D^2 \times \mathbb{R}P^{2k-1}) \cup_{\text{id}} (D^2 \times \mathbb{R}P^{2k-1}).
\]
The manifold $X^{2k+1}(0) = S^2 \times \mathbb{R}P^{2k-1}$ is the double of the disk bundle $D^2 \times \mathbb{R}P^{2k-1}$. Its characteristic submanifold is
\[
P^{2k}(0) = (\mathbb{R}P^{2k} \# S^1 \mathbb{R}P^{2k})(\text{id}) = S(2\gamma \oplus \mathbb{R}).
\]
As it was mentioned in Example 2.31, this manifold is the double of a nontrivial nonorientable 2-disk bundle over the real projective 2k-space.

Example 2.43 (The manifolds $X^{2k+1}(2)$). We now define the manifold $X^{2k+1}(2)$ as the circle sum
\[
X^{2k+1}(2) = (\mathbb{R}P^{2k+1} \# S^1 \mathbb{R}P^{2k+1})(\varphi_\alpha).
\]
Using the decompositions (2.27) and (2.28), we again have
\[
\mathbb{R}P^{2k+1} - S^1 \times D^4 = D^2 \times \mathbb{R}P^{2k-1},
\]
and the circle sum (2.44) can be expressed as
\[
(\mathbb{R}P^{2k+1} \# S^1 \mathbb{R}P^{2k+1})(\varphi_\alpha) = (D^2 \times \mathbb{R}P^{2k-1}) \cup_{\varphi_\alpha} (D^2 \times \mathbb{R}P^{2k-1}).
\]
Its characteristic submanifold $P^{2k}(2)$ can be expressed as the circle sum
\[
(\mathbb{R}P^{2k} \# S^1 \mathbb{R}P^{2k})(\varphi_\alpha) = (D^2 \times \mathbb{R}P^{2k}) \cup_{\varphi_\alpha} (D^2 \times \mathbb{R}P^{2k})
\]
using (2.30).
Example 2.38 and Example 2.43 indicated that the manifolds of Theorem A are obtained by carving out a tubular neighborhood $\nu(\mathbb{R}P^{n-2})$ of a codimension two real projective space embedded in a total space of a 2-sphere bundle over $\mathbb{R}P^{n-2}$, and gluing it back in using a diffeomorphism of the boundary. The precise statement is stated as the following proposition; its proof follows from the aforementioned examples.

**Proposition 2.48.** The manifold $X^{2k+1}(2)$ is diffeomorphic to
\[(2.49) \quad (X^{2k+1}(0) - \nu(\mathbb{R}P^{2k-1}) \cup_\varphi (D^2 \times \mathbb{R}P^{2k-1}),\]
and the manifold $P^{2k}(2)$ is diffeomorphic to
\[(2.50) \quad (P^{2k}(0) - \nu(\mathbb{R}P^{2k}) \cup_{\varphi'} (D^2 \times \mathbb{R}P^{2k}),\]
where the diffeomorphism $\{\varphi, \varphi'\}$ are as in Definitions 2.14 and 2.19.

The reader is referred to Example 2.63, where a summarized description of the technique used to prove Theorem A is given by coupling the cut-and-paste construction of Proposition 2.48 with isometric identifications of Riemannian metrics on compact manifolds.

### 2.5. Homeomorphism classes obtained for Theorem A

We now discern the homeomorphism classes of the manifolds of Theorem A from the ones previously known to contain Riemannian manifolds of nonnegative sectional curvature, i.e., the 2-sphere bundles over the real projective spaces; see Section 5 as well. The precise statement is the following proposition.

**Proposition 2.51.**

1. The manifolds $X^{2k+1}(0)$ and $X^{2k+1}(2)$ are not homeomorphic.
2. The manifolds $P^{2k}(0)$ and $P^{2k}(2)$ are not homeomorphic.

The cases $k = 2$ have been proven in 
[8, Theorem 1 and Corollary 1] and 
[7, Theorems 3.1 and 3.12].

**Proof.** To prove Item (1), Proposition 2.12 indicates that we need to see that the corresponding characteristic submanifolds are not TopPin$^\pm$-bordant. We claim that the submanifolds represent different classes in the corresponding TopPin$^\pm$-bordism group. Indeed, the characteristic submanifolds
\[S(2\gamma \oplus \mathbb{R}) \subset X^{2k+1}(0) \text{ and } \mathbb{R}P^{2k} \# S^1 \mathbb{R}P^{2k} \subset X^{2k+1}(2),\]
admit a Pin$^+$-structure for even $k$ and a Pin$^-$-structure for odd $k$ (see Proposition 2.11); they admit the respective TopPin$^\pm$-structure as well. The circle sum operation of Definition 2.14 is addition in the corresponding Pin$^\pm$- and TopPin$^\pm$-bordism groups (cf. Proposition 2.23). On the one hand, we have that $[\mathbb{R}P^{2k}, \phi] \neq 0 \in \Omega_{2k}^{\text{Pin}^\pm}$ (see [14, p. 437] and [11]). On the other hand, the zero element in the bordism group is represented by the bundle $S(2\gamma \oplus \mathbb{R})$, since it bounds. Thus, the characteristic submanifolds are neither Pin$^\pm$- nor TopPin$^\pm$-bordant. They represent the zero element and a non-zero element in the respective bordism group. This implies that they are not homeomorphic, and Item (2) follows. This concludes the proof of Proposition 2.51.

**Remark 2.52.** Homotopy types of the manifolds of Theorem A

It is proven in 
[7] that $X^5(0)$ and $X^5(2)$ are homotopy equivalent.

It is proven in 
[13] that $P^4(0)$ and $P^4(2)$ are not homotopy equivalent.
2.6. Metrics of nonnegative sectional curvature of Theorem\textsuperscript{A}. The goal in this section is to construct Riemannian metrics $g_{2,2k+1}$ and $g_{2,2k}'$ on $X^{2k+1}(2)$ and $P^{2k}(2)$, respectively, whose sectional curvature is nonnegative. The metrics arise from the decomposition of the manifolds stated in Example \textsuperscript{2.43}. We use polar coordinates to define a metric on the 2-disk and keep the notation for the round metric on the sphere as in Example \textsuperscript{2.31}.

**Proposition 2.53.** There exists a complete Riemannian metric on the total space $D^2 \times \mathbb{R}P^{2k-1}$ with nonnegative sectional curvature and that restricts to the product metric $d\theta^2 + g_{S^{2k-1}}$ near the boundary $S^1 \times S^{2k-1} = \partial(D^2 \times \mathbb{R}P^{2k-1})$.

Similarly, there exists a complete Riemannian metric on the total space $D^2 \times \mathbb{R}P^{2k}$ with nonnegative sectional curvature and that restricts to the bundle metric on the boundary $S^{2k} \times S^1 = \partial(D^2 \times \mathbb{R}P^{2k})$.

By bundle metric in the statement of Proposition \textsuperscript{2.53}, we mean the quotient metric obtained from the product metric $d\theta^2 + g_{S^{2k-1}}$ obtained from the corresponding identification $(S^1 \times S^{2k}/\sim) = S^{2k} \times S^1$ (cf. involution described in Example \textsuperscript{2.31}).

**Proof.** The disk bundle can be written as the quotient

$$D^2 \times \mathbb{R}P^{2k-1} = (D^2 \times S^{2k-1})/(r(x), \hat{\mathbb{A}}(y))$$

for all $x \in D^2$ and $y \in S^{2k-1}$ as it was mentioned in decomposition \textsuperscript{2.28}. We pick the following product metric on $D^2 \times S^{2k-1}$. Equip the sphere factor with the round metric. On the disk we have two choices of metrics that satisfy the conclusion of the proposition. On one hand, we can take a round hemisphere as pointed out to us by Peter Petersen. We may also choose a rotationally symmetric metric

$$g_D := dt^2 + f^2(t)d\theta^2$$

where $f$ is an odd concave function satisfying $f'(0) = 1$ and whose odd derivatives vanish at the boundary [3 Example 3], [11 Remark 2.7]. The involution $(r, \hat{\mathbb{A}})$ is an isometry of the metric $g_D + g_{S^{2k-1}}$ and the orbit space $D^2 \times \mathbb{R}P^{2k-1}$ inherits a metric $g_B$, which restricts to the product metric $d\theta^2 + g_{S^{2k-1}}$ on the boundary. In particular, we have a Riemannian submersion

$$\pi : (D^2 \times S^{2k-1}, g_D + g_{S^{2k-1}}) \longrightarrow (D^2 \times \mathbb{R}P^{2k-1}, g_B).$$

As it was explained in Example \textsuperscript{2.38} since we have an isometric action by a finite group we conclude that the sectional curvature of $g_B$ is nonnegative (cf. [11 Theorem 2.6]).

The Riemannian metric on the nonorientable disk bundle is obtained in a similar manner (please see Example \textsuperscript{2.63}).

**Proposition 2.57.** The Riemannian manifolds

$$(X^{2k+1}(2), g_{2,2k+1}) \text{ and } (P^{2k}(2), g_{2,2k}')$$

are nonnegatively curved in the sense that the sectional curvature of the metrics $g_{2,2k+1}$ and $g_{2,2k}'$ is nonnegative.

**Proof.** We provide the details of the construction of the metric on the orientable manifolds; the nonorientable case is similar (please see Example \textsuperscript{2.63}). The metrics
$g_{2,2k+1}$ are constructed coupling Proposition 2.53 with the decomposition already mentioned in Example 2.43
\begin{equation}
X^{2k+1}(2) = (D^2 \times \mathbb{R} D^{2k-1}) \cup_{\varphi_\alpha} (D^2 \times \mathbb{R} P^{2k-1})
\end{equation}
with the diffeomorphism $S^1 \times S^{2k-1} \to S^1 \times S^{2k-1}$ used to identify the common boundaries given by $\varphi_\alpha(\theta, x) = (\theta, \alpha(\theta) \cdot x)$ for $\theta \in S^1$ and $x \in S^{2k-1}$. Homotopic loops are isotopic due to the codimension and the map $\alpha$ represents the generator of the fundamental group of $SO(2k)$, which its given by the inclusion
\begin{equation}
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \mapsto \begin{pmatrix}
\cos \theta & -\sin \theta & \cdots & 0 \\
\sin \theta & \cos \theta & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\end{equation}
In particular the diffeomorphism $\varphi_\alpha$ is an isometry of the product metric on the boundary that was constructed in Proposition 2.53. We equip each copy of the disk bundle with the metric $g_B$ as in Proposition 2.53 and its proof to obtain the metric $g_{2,2k+1}$ by identifying them with the isometry $\varphi_\alpha$ along the $S^1 \times S^{2n-1}$ boundary. The conditions on the vanishing of all odd derivatives of the function $f(t)$ at the boundary for the torpedo metric on $D^2$ guarantee that the metric obtained is smooth.

We finish this section with an pair of examples that summarizes the coupling of the metric construction of Proposition 2.57 and the topological constructions that were discussed in Section 2.4. A classical example of a nonnegatively curved manifold is the connected sum of two copies of the complex projective plane with opposite orientations (cf. [3]).

Example 2.60. Nonnegatively curved Riemannian metrics on $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ and $\mathbb{R}P^4 \# S^1 \mathbb{R}P^4 = P^4(2)$ via a cut-and-paste construction along 2-spheres and real projective 2-planes. Let
\begin{equation}
M := (D^2 \times S^2) \cup_{\varphi} (D^2 \times S^2),
\end{equation}
where, as in Definition 2.14, the diffeomorphism $\varphi : S^1 \times S^2 \to S^1 \times S^2$ is given by $(\theta, x) \mapsto (\theta, \alpha(\theta) \cdot x)$ for every $\theta \in S^1$ and $x \in S^2$ and where $\alpha(\theta)$ is rotation by $\theta$ of the 2-sphere about the axis that goes through its north and south poles. A handlebody argument quickly reveals that $M$ is diffeomorphic to $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, i.e., the total space of the nontrivial $S^2$-bundle over $S^2$. In particular, it is well-known that $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ is obtained from $S^2 \times S^2$ through an application of a Gluck twist along a homologically essential 2-sphere
\begin{equation}
M = (S^2 \times S^2 - \nu(\{pt\} \times S^2)) \cup_{\varphi} (D^2 \times S^2).
\end{equation}
As in the proof of Proposition 2.57 equip $D^2 \times S^2$ with the product metric $g_B := g_{D^2} + g_{S^2}$, where $g_{D^2}$ is a torpedo metric on the 2-disk factor satisfying the conditions indicated in the proof of Proposition 2.57 and $g_{S^2}$ is the round metric on the 2-sphere. We mention again that one can also take $g_{D^2}$ to be the round hemisphere metric on $D^2$. The metric $g_B$ restricts to the product metric $d\theta^2 + g_{S^2}$ along the $S^1 \times S^2$ boundary, and the diffeomorphism $\varphi$ is an isometry for this metric. Indeed, the map $x \mapsto \alpha(\theta) \cdot x$ is an isometry of $\mathbb{R}^3$ and the round metric is the induced metric from embedding $S^2 \hookrightarrow \mathbb{R}^3$. Notice that this argument holds in
higher dimensions. A Riemannian metric of nonnegative sectional curvature on $M$ is obtained by identifying two copies of $g_B$ on the $D^2 \times S^2$ blocks using the isometry $\varphi$.

The manifold $P^4(2) = \mathbb{R}P^4 \# S^1 \mathbb{R}P^4$ is obtained from the total space $S(2\gamma \oplus \mathbb{R})$ of the $S^2$-bundle over $\mathbb{R}P^2$ by carving out a tubular neighborhood of an $\mathbb{R}P^2$ and gluing it back using the diffeomorphism $\varphi'$ by Proposition (2.48) In Example (2.43) we have described this manifold as $(D^2 \times \mathbb{R}P^2) \cup_{\varphi'} (D^2 \times \mathbb{R}P^2)$. Proposition (2.57) equips each copy of the nontrivial disk bundle over the real projective plane with a nonnegatively curved metric that restricts to the quotient metric of $d\theta + g_{S^2}$ on the common $S^2 \times S^1$ boundary obtained as. The diffeomorphism $\varphi'$ is an isometry of this Riemannian metric.

**Example 2.63.** Nonnegatively curved Riemannian metrics on the total space of an $S^{n-2}$-bundle over $S^2$ for $n \geq 4$. The following manifolds were known to admit metrics of nonnegative curvature [21, Section 2]. Besides the product $S^{n-2} \times S^2$, there is a nontrivial bundle that we denote by $S^{n-2} \times S^2$. They are constructed as

$$(2.64) \quad M(\varphi) := (D^2 \times S^{n-2}) \cup_{\varphi} (D^2 \times S^{n-2}),$$

where the diffeomorphism to identify the common boundary

$$(2.65) \quad \varphi : S^1 \times S^{n-2} \to S^1 \times S^{n-2}$$

is either the identity id or the map $\varphi_\alpha$ defined in Definition (2.14). In particular, both sphere bundles are obtained by surgery on $S^n$ along an embedded loop using (2.20) with $M(\text{id}) = S^{n-2} \times S^2$ and $M(\varphi_\alpha) = S^{n-2} \times S^2$. The discussion of Example readily generalizes to a cut-and-paste construction of a metric of nonnegative sectional curvature on both sphere bundles.

**Remark 2.66.** Further constructions of homeomorphism classes and inequivalent smoothings.

- Different choices of diffeomorphism

$$(2.67) \quad \varphi : S^1 \times S^{n-1} \to S^1 \times S^{n-1}$$

that is used to identify the building blocks described in Section (2.4) need not yield the same manifold and in several cases yield other homeomorphism and diffeomorphism classes, as well as different homotopy types. New manifolds can be constructed by using as gluing diffeomorphism $\varphi = (f, \varphi_{S^n})$ where $f \in \text{Diff}^+(S^1) \cong O(2)$ and $\varphi_{S^n} \in \text{Diff}^+(S^n)$. For example, homotopy spheres are constructed as

$$(2.68) \quad \Sigma^n = (D^2 \times S^{n-2}) \cup_{\varphi} (S^1 \times D^{n-1}).$$

- The construction (2.68) need not produce new manifolds. For any diffeomorphism $\varphi_{S^3} : S^3 \to S^3$ there is a diffeomorphism

$$(2.69) \quad (D^2 \times \mathbb{R}P^3) \cup_{\varphi} (S^1 \times D^4) \to \mathbb{R}P^5.$$
range of choices for a diffeomorphism $\varphi$ (see discussion on Gromoll groups in [6, Section 3]).

Metrics on the $n$-sphere and on the real projective $n$-space of nonnegative sectional curvature that do vanish on certain given 2-planes are constructed in a similar fashion using the decompositions stated at the beginning of Section 2.4.

3. Minimal volume and collapse with bounds on sectional curvature:

Proof of Theorem D

We prove in this section that the orientable manifolds considered in Theorem A have zero minimal volume and collapse with bounded sectional curvature [4], while the nonorientable ones have zero minimal entropy and collapse with sectional curvature bounded from below [17]. We assume the normalization $\text{Vol}((M, g)) = 1$.

The minimal volume was introduced by Gromov as

$$\text{MinVol}(M) := \inf_g \{\text{Vol}(M, g) : |K_g| \leq 1\},$$

where $K_g$ is the sectional curvature of a Riemannian metric $g$ with $\text{Vol}(M, g) = 1$. A smooth manifold $M$ collapses with bounded sectional curvature if and only if there exists a sequence of Riemannian metrics $\{g_j\}$ for which the sectional curvature is uniformly bounded and their volumes $\{\text{Vol}(M, g_j)\}$ approach zero as $j \to \infty$.

Analogously, one defines

$$\text{Vol}_{K_g}(M) := \inf_g \{\text{Vol}(M, g) : K_g \geq 1\},$$

and say that $M$ collapses with sectional curvature bounded from below if and only if there exists a sequence of Riemannian metrics $\{g_j\}$ for which the sectional curvature is uniformly bounded from below but their volumes $\{\text{Vol}(M, g_j)\}$ approach zero as $j \to \infty$. Finally, recall that the minimal entropy $h(M)$ is the infimum of the topological entropy of the geodesic flow of a smooth metric $g$ on $M$ [17].

Definition 3.3. [4, 17]. An $\mathcal{F}$-structure on a smooth closed manifold $M$ consists of

1. a finite open cover $\{U_1, \ldots, U_N\}$ of $M$;
2. a finite Galois covering $\pi_i : \tilde{U}_i \to U_i$ with $\Gamma_i$ a group of deck transformations for $1 \leq i \leq N$;
3. a smooth effective torus action with finite kernel of a $k_i$-dimensional torus $\phi_i : T^{k_i} \to \text{Diff}(\tilde{U}_i)$ for $1 \leq i \leq N$;
4. a representation $\Phi_i : \Gamma_i \to \text{Aut}(T^{k_i})$ such that $\gamma(\phi_i(t)(x)) = \phi_i(\Phi_i(\gamma)(t))(\gamma x)$ for all $\gamma \in \Gamma_i$, $t \in T^{k_i}$, and $x \in \tilde{U}_i$;
5. for any subcollection $\{U_{i_1}, \ldots, U_{i_l}\}$ that satisfies $U_{i_1} \cap \cdots \cap U_{i_l} \neq \emptyset$, the following compatibility condition holds: let $\tilde{U}_{i_1, \ldots, i_l}$ be the set of all points $(x_{i_1}, \ldots, x_{i_l}) \in \tilde{U}_{i_1} \times \cdots \times \tilde{U}_{i_l}$ such that $\pi_{i_1}(x_{i_1}) = \cdots = \pi_{i_l}(x_{i_l})$. 


Theorem 3.4. Cheeger-Gromov [4]. The precise statements of these results are the following. Petean [17] to study vanishing of minimal entropy and collapse with a lower bound if manifolds with bounded sectional curvature. Their results were extended by Paternain-Petean to study collapse with sectional curvature.

Theorem 3.5. Paternain-Petean [17]. Suppose $M$ admits an $\mathcal{F}$-structure. Then,
\begin{equation}
(3.6) \quad h(M) = 0 = \text{Vol}_K(M),
\end{equation}
and $M$ collapses with sectional curvature bounded from below.

We now prove Theorem 3.3.

Theorem 3.7. For every $k \geq 2$ and $j \in \{0, 2\}$ the manifold $X^{2k+1}(j)$ admits a polarized $\mathcal{T}$-structure, while the manifold $P^{2k}(j)$ admits a non-polarized $\mathcal{F}$-structure. Consequently,
\begin{equation}
(3.8) \quad \text{MinVol}(X^{2k+1}(j)) = 0 = \text{Vol}_K(P^{2k}(j)),
\end{equation}
$X^{2k+1}(j)$ collapses with bounded sectional curvature, and $P^{2k}(j)$ collapses with sectional curvature bounded from below.

Proof. The latter claims follow respectively from Theorems 3.4 and Theorem 3.5 once we construct the corresponding $\mathcal{F}$-structures. We proceed to do so beginning with the $\mathcal{T}$-structure on $X^{2k+1}(j)$. In comply with Items (1) - (2) in Definition 3.3 take
\begin{equation}
(3.9) \quad \tilde{U}_i = U_i := (D^2 \times S^{2k-1})/(\mathbb{R}, A) = D^2 \times \mathbb{R}P^{2k-1}
\end{equation}
as suggested by decomposition (2.28), which for $i = 1, 2$ yields a covering of $X^{2k+1}(j)$ (for both choices $j = 0, 2$). Item (4) is not relevant in these cases given our choice of open sets. The manifold $U_i$ can be equipped with a free circle action $\phi_i : S^1 \to \text{Diff}(U_i)$ as required in Item (3) by taking the Hopf action on the $(2k - 1)$-sphere factor of $D^2 \times S^{2k-1}$. This action commutes with the involution $\Lambda$ and hence descends to a free circle action on the disk bundle over $\mathbb{R}P^{2k-1}$. Moreover, when $\psi_i$ is restricted to the boundary $S^1 \times S^{2k-1}$ we recover the Hopf action on the $(2k + 1)$-sphere factor $\Lambda$ as it was explained in Example 2.36 and Example 2.4.3 the manifold $X^{2k+1}(j)$ can be deconstructed as two copies of the $D^2$-bundle over $\mathbb{R}P^{2k-1}$ identified along by a diffeomorphism $\varphi$. In the case $\varphi = \psi_2$, the circle actions paste together to yield a global circle action and $X^{2k+1}(0)$ has a free circle action. In the case $\varphi = \psi_2$, the circle actions commute with each other satisfying Item (5), hence complying with Item (6). The choice of circle actions satisfy the condition of Item (7), and hence $X^{2k+1}(2)$ admits a polarized $\mathcal{T}$-structure.
Similarly, the manifolds $P^{2k}(j)$ can be equipped with an $F$-structure with choices of covering sets

\begin{equation}
\bar{U}_i = D^2 \times S^{2k-2}
\end{equation}

and

\begin{equation}
U_i := D^2 \times \mathbb{R}P^{2k-2}
\end{equation}

, and $\Gamma_i = \mathbb{Z}/2$ for $i = 1, 2$. The circle acts on the $(2k-2)$-sphere factor of $D^2 \times S^{2k-2}$ by rotations along an axis. Verification of each item in Definition 3.3 is similar to the orientable cases. The $F$-structure on these manifolds is not polarized. 

\section{Proof of Theorem A: Construction of New Examples of Nonnegatively Curved Riemannian Manifolds}

Example 2.43 describes a cut-and-paste construction of the manifolds $X^{2k+1}(2)$ and $P^{2k}(2)$ where disk bundles over real projective spaces are used as building blocks. Since they have fundamental group of order two and their universal cover is $S^2 \times S^n$, they share the same homotopy groups with the corresponding sphere bundle. Proposition 2.51 says that the manifolds $X^{2k+1}(0)$ and $X^{2k+1}(2)$ are not homeomorphic, and neither are the manifolds $P^{2k}(0)$ and $P^{2k}(2)$. The manifolds $P^4(0)$ and $P^4(2)$ are not homotopy equivalent as distinguished by a quadratic function $\pi_2 \otimes \mathbb{Z}/2 \to \mathbb{Z}/4$ [13]. The existence of the Riemannian metrics $\{g_{2k+1}, g_{2k+2k}\}$ on these manifolds was proven in Proposition 2.57.

\section{Proof of Theorem C: Novelty of the Examples of Theorem A}

To prove Item (i), we argue as follows. Since $\pi_1$ has order two and its universal cover is the product of a 2-sphere with an $n$-sphere, the manifolds $X^{2k+1}(2)$ and $P^{2k}(2)$ are not diffeomorphic to a connected sum of a compact rank one symmetric space. A comparison with Totaro’s list [19] (cf, [12]) implies that they are not diffeomorphic to biquotients. Since they are not homeomorphic to a sphere bundle over the real projective $n$-space by Proposition 2.51, the manifolds $X^{2k+1}(2)$ and $P^{2k}(2)$ are not homeomorphic to a homogeneous space.

As discussed in [11, Section 2], a cohomogeneity one $G$-invariant nonnegatively curved metric arises from the identification of the bundles

\begin{equation}
D(B_{\pm}) = G \times_{K_{\pm}} D^2
\end{equation}

using the identity map to identify the common boundaries

\begin{equation}
id : \partial(G \times_{K_{\pm}} D^2) \to G \times_{K_{\pm}} S^1
\end{equation}

with the nonnegatively curved metrics of [11, Theorem 2.6]. It was discussed in Example 2.43 that the manifolds $X^{2k+1}(2)$ and $P^{2k}(2)$ are built identifying a pair of 2-disk bundles over a real projective $n$-space as (2.46) and (2.47) using a diffeomorphism that is not isotopic to the identity. The isometry used in construction of the metrics in Proposition 2.57 is not isotopic to the identity map. This proves Item (ii).

The classification of 4-manifolds that admit a cohomogeneity one $G$-action given in [10] along with the omission corrected in [9] imply that the nonorientable 4-manifold $P^4(2) = \mathbb{R}P^4 \# S^1 \mathbb{R}P^4$ does not admit such an action. We claim that the manifold $X^5(2)$ of Theorem A does not admit such an action using the classification of cohomogeneity on $G$-actions on 5-manifolds given in [10, Theorem A]. There
are five instances to consider, all of them involving a cohomogeneity action on its universal cover $S^2 \times S^3$ that descends to $X^5(2)$ as in the case [11, Theorem G]. The first two instances correspond to an isometric action on a symmetric space or a product action. Both of them descend to the 2-sphere bundle over the real projective 3-space $X^5(0)$ but not to $X^5(2)$ by Proposition 2.51. The next instance of [10, Theorem A] is the $SO(2)SO(n)$ action on a Brieskorn variety $\Sigma_{q}^{2n-1}$, which is a real algebraic submanifold of $\mathbb{C}^{n+1}$ defined by the equations

\begin{equation}
q_0 + q_1^2 + \ldots + q_n^2 = 0
\end{equation}

and

\begin{equation}
|q_0|^2 + |q_1|^2 + \ldots + |q_n|^2 = 1.
\end{equation}

The 5-dimensional manifolds $\Sigma_{q}^5$ are diffeomorphic to $S^2 \times S^3$ for even values of $d$ [18, Section 2.3]. There is a free involution $T : \Sigma_{q}^5 \rightarrow \Sigma_{q}^5$ given by

\begin{equation}
(z_0, z_1, z_2, z_3) \mapsto (z_0, -z_1, -z_2, -z_3),
\end{equation}

which commutes with the cohomogeneity one $SO(2)SO(n)$ action. The orbit spaces $\{\Sigma_{q}^5/T : q = 0, 2, 4, 6, 8\}$ (using the notation in [18]) have a cohomogeneity one action with codimension two singular orbits and they have a Riemannian metric of nonnegative sectional curvature [11, Theorem E]. The manifold $X^5(2)$ is not diffeomorphic to any of these manifolds by [18, Theorem 1]. To conclude the proof of Item (iii), we consider the remaining two classes of cohomogeneity one actions on 5-manifolds [10]. The action of type $P^5$ of [10, Table I] is on the Wu manifold $SU(3)/SO(3)$, and hence not on $X^5(2)$. The action of type $N^5$ of [10, Table II] is on $S^2 \times S^3$ and descends to the $S^2$-bundle over $\mathbb{R}P^3$ that we have denoted by $X^5(0)$ as seen using arguments in [10].

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Scuola Internazionale Superiore di Studi Avanzati (SISSA), Via Bonomea 265, 34136, Trieste, Italy
E-mail address: rtorres@sissa.it