On the Michor–Mumford phenomenon in the infinite dimensional Heisenberg group

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Abstract
In the infinite dimensional Heisenberg group, we construct a left invariant weak Riemannian metric that gives a degenerate geodesic distance. The same construction yields a degenerate sub-Riemannian distance. We show how the standard notion of sectional curvature adapts to our framework, but it cannot be defined everywhere and it is unbounded on suitable sequences of planes. The vanishing of the distance precisely occurs along this sequence of planes, so that the degenerate Riemannian distance appears in connection with an unbounded sectional curvature. In the 2005 paper by Michor and Mumford, this phenomenon was first observed in some specific Fréchet manifolds.

Keywords Hilbert manifold · Infinite dimensional Heisenberg group · Weak Riemannian metric · Geodesic distance · Sectional curvature

Mathematics Subject Classification Primary 58B20. Secondary 53C22

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1 Introduction

Geodesic distances naturally appear in the geometry of infinite dimensional manifolds. A new aspect is that they may also vanish on distinct points. In general, the vanishing of the geodesic distance may occur for certain Riemannian metrics, where no special conditions are assumed, namely for weak Riemannian metrics, [1, Definition 5.2.12]. These metrics are important, since they are the only possible metrics when the manifold is not modelled on a Hilbert space.

Vanishing geodesic (or Riemannian) distances in Fréchet manifolds were first found in [10, 19, 20] and further examples were studied in [3–5, 8, 14, 15]. A simple example of vanishing geodesic distance can be also constructed in a Hilbert manifold, [17]. However, one may still wonder whether replacing a weak Riemannian metric with a left invariant weak Riemannian metric with respect to a Hilbert Lie group structure might give a condition to have positive geodesic distance on distinct points.

The answer to this question does not seem intuitively clear. For instance, we observe that connected, simply connected and commutative Banach Lie groups, equipped with a bi-invariant weak Riemannian metric have positive geodesic distance on distinct points. In short, their geodesic distance is actually a distance. The proof of this fact essentially follows from [22, Proposition IV.2.7], observing that the exponential mapping is a local Riemannian isometry.

Thus, the question is whether considering a left invariant weak Riemannian metric on a noncommutative, connected and simply connected Banach Lie group may prevent the vanishing of the geodesic distance. Our first result answers this question in the negative.

Theorem 1.1 There exists a left invariant weak Riemannian metric on the infinite dimensional Heisenberg group $H$, whose Riemannian distance is not positive on all couples of distinct points.

In analogy with the finite dimensional case, the infinite dimensional Heisenberg group $H$ can be defined starting from the Heisenberg Lie algebra, due to the Baker–Campbell–Hausdorff formula. We use $\ell^2 \times \ell^2 \times \mathbb{R}$ to model $H$ as a Hilbert manifold, where $\ell^2$ denotes the standard linear space of real square-summable sequences. The Lie product associated to $H$ is defined in (1). More information on the Lie group $H$ is provided in Sect. 2.1.

Sub-Riemannian distances naturally appears also in infinite dimensional manifolds. We mention for instance [9, Theorem C.2], where strong sub-Riemannian metrics were considered in connection with Wiener spaces, lower bounds on the Ricci curvature and logarithmic Sobolev inequalities. In this case, the space of admissible velocities is strictly contained in the tangent space and we have an a priori smaller family of connecting curves. The possible vanishing of a sub-Riemannian distance between some distinct points of an in infinite dimensional manifold was explicitly mentioned in [12, Remark 2]. The following theorem seemingly provides a first example of such vanishing phenomenon for an infinite dimensional sub-Riemannian manifold.
Theorem 1.2  There exists a left invariant weak sub-Riemannian metric on the infinite dimensional Heisenberg group $\mathbb{H}$ such that its associated sub-Riemannian distance is not positive on all couples of distinct points.

The sub-Riemannian metric and the sub-Riemannian distance that we consider on $\mathbb{H}$ are described in Sect. 2.3. Both Theorem 1.1 and Theorem 1.2 are contained in Theorem 3.3 and their proof rather surprisingly relies on the same sequence of length-minimizing curves. The proof of these results shows that both the Riemannian and sub-Riemannian distance are vanishing between points that have the same projection on the subspace $\ell^2 \times \ell^2 \times \{0\}$. Remark 3.2 completes the picture, showing that when the projections of two points on $\ell^2 \times \ell^2 \times \{0\}$ are different, then both their Riemannian and sub-Riemannian distance are positive. In sum, all distinct points with vanishing geodesic distance are characterized.

From another perspective, dealing with a left invariant weak Riemannian metric has the advantage to find the sectional curvature by more manageable formulas. In [19], Michor and Mumford proved that in different Fréchet manifolds with a vanishing geodesic distance the sectional curvature is unbounded. Theorem 1.3 below presents the same phenomenon for the left invariant weak Riemannian metric $\sigma$ defined in (10), in the infinite dimensional Heisenberg group $\mathbb{H}$.

From the standard formula for the sectional curvature of Lie groups, see for instance [2] and [7], the sectional curvature of $\mathbb{H}$ with respect to $\sigma$ can be defined on “many planes” of the Lie algebra $\text{Lie}(\mathbb{H})$. We wish to point out that for general weak Riemannian metrics the existence of the Levi–Civita (and then of the sectional curvature) is not guaranteed a priori. An example of this fact can be found in [6], where more information on the problem is available. We also observe that the “finite dimensional formula” for the sectional curvature through the structure coefficients of $\text{Lie}(\mathbb{H})$, [21, Lemma 1.1], converges on the previous planes to the same sectional curvature obtained by [2, Theorem 5]. Broadly speaking, we may think of the convergence of the sectional curvature in Milnor’s paper [21] as a computation of the sectional curvature of $\mathbb{H}$ through a finite dimensional approximation by an orthonormal basis. On the other side, we also observe that this convergence does not hold on all 2-dimensional subspaces of $\text{Lie}(\mathbb{H})$, as shown in Remark 4.1.

The next theorem shows that the sectional curvature with respect to the weak Riemannian metric $\sigma$ is unbounded on a certain sequence of planes.

Theorem 1.3  Let $\mathbb{H}$ be the infinite dimensional Heisenberg group equipped with the left invariant weak Riemannian metric $\sigma$. Then there exists two sequences of orthonormal vectors $a_{1j}, a_{2j} \in \text{Lie}(\mathbb{H})$ and $b \in \text{Lie}(\mathbb{H})$ with $j \geq 1$ such that $K_\sigma(a_{1j}, b) = K_\sigma(a_{2j}, b)$,

$$\lim_{j \to \infty} K_\sigma(a_{1j}, a_{2j}) = -\infty \quad \text{and} \quad \lim_{j \to \infty} K_\sigma(a_{1j}, b) = +\infty.$$  

The numbers $K_\sigma(a_{1j}, a_{2j})$ and $K_\sigma(a_{1j}, b)$ are the sectional curvatures of the planes of $\text{Lie}(\mathbb{H})$ spanned by the orthonormal bases $(a_{1j}, a_{2j})$ and $(a_{1j}, b)$.

The proof of this theorem is provided in Sect. 4, where also more information on the vectors $a_{1j}, a_{2j}$ and $b$ can be found. Inspecting the proofs of Theorem 3.3 and
Theorem 1.3 another interesting phenomenon appears. The curves whose lengths converge to zero and that connect two distinct points are precisely contained in the span of the planes where the sectional curvature blows-up. We finally mention Proposition 4.2, where we prove that the sectional curvature is unbounded also on some sequences of converging planes.

2 Preliminary notions

In this section, we present an infinite dimensional version of the classical Heisenberg group equipped with either a Riemannian or a sub-Riemannian structure. The construction is well known and it has connections with different areas of Mathematics.

2.1 A short introduction to the infinite dimensional Heisenberg group

We denote by $\ell^2$ the linear space of all real and square summable sequences. Its scalar product $\langle \cdot, \cdot \rangle$ has the associated norm $\|x\| = \sqrt{\sum_{j=1}^{\infty} x_j^2}$ for any element $x = \sum_{j=1}^{\infty} x_j e_j$. The set of unit vectors $\{e_j : j \geq 1\}$ denotes the canonical orthonormal basis of $\ell^2$. For each integer $n \geq 1$, the element $e_n$ of $\ell^2$ has $n$-th entry equal to 1 and all the others are zero.

We consider $\ell^2 \times \ell^2 \times \mathbb{R}$ endowed with its standard structure of product of Hilbert spaces and we introduce the continuous and skew-symmetric function

$$\beta((h_1, h_2), (h'_1, h'_2)) = \langle h_1, h'_2 \rangle - \langle h_2, h'_1 \rangle.$$ 

defined on $(\ell^2 \times \ell^2) \times (\ell^2 \times \ell^2)$. Then we introduce a continuous Lie product on $\ell^2 \times \ell^2 \times \mathbb{R}$:

$$[(h_1, h_2, \tau), (h'_1, h'_2, \tau')] = 2 \beta((h_1, h_2), (h'_1, h'_2)) (0, 0, 1) \in \ell^2 \times \ell^2 \times \mathbb{R},$$

(1)

that makes $\ell^2 \times \ell^2 \times \mathbb{R}$ an infinite dimensional Lie algebra. Due to the Baker–Campbell–Hausdorff formula (in short BCH), we equip $\ell^2 \times \ell^2 \times \mathbb{R}$ with a noncommutative and analytic Lie group operation:

$$(h_1, h_2, \tau)(h'_1, h'_2, \tau') = (h_1 + h'_1, h_2 + h'_2, \tau + \tau' + \beta((h_1, h_2), (h'_1, h'_2)))$$

(2)

for all elements $(h_1, h_2, \tau), (h'_1, h'_2, \tau') \in \ell^2 \times \ell^2 \times \mathbb{R}$. We denote by $\mathbb{H}$ the Hilbert Lie group arising from the previous group operation, that is the infinite dimensional Heisenberg group modelled on the Hilbert space $\ell^2 \times \ell^2 \times \mathbb{R}$.

From the viewpoint of Mathematical Physics, the group $\mathbb{H}$ naturally appears in the theory of representations of infinite dimensional Lie algebras, see [13] and the
references therein. In the theory of infinite dimensional Lie groups, $\mathbb{H}$ can be seen as a special instance of more general BCH-Lie groups. They are infinite dimensional Lie groups with a local exponential mapping that is also a bianalytic diffeomorphism around the origin, [11], [22]. For infinite dimensional nilpotent Lie algebras, the BCH formula defines a global group operation, [23]. This viewpoint was followed in [16] to define infinite dimensional Banach homogeneous groups as suitable direct sums of Banach spaces, equipped with an analytic structure and an everywhere converging BCH formula. In the same work, several examples of Banach homogeneous groups were provided. We mention infinite producs of Engel groups using either $\ell^p$ or $L^p$ spaces, an infinite product of Heisenberg groups modelled on $\ell^2 \times \ell^1$ and other analogous analytic constructions.

In relation to the understanding of hypoellipticity in infinite dimensions [18], infinite dimensional Heisenberg-like groups based on a Wiener space, along with their Brownian motion were introduced and studied in [9]. In this connection, also some Ricci curvature lower bounds are obtained, using a left invariant Riemannian metric.

2.2 Weak Riemannian metrics on $\mathbb{H}$

For each $p \in \mathbb{H}$, we denote by $L_p : \mathbb{H} \to \mathbb{H}$ the left multiplication by $p$, defined as $L_p(r) = p \cdot r$ for all $r \in \mathbb{H}$. The group operation (2) gives a simple formula for the differential of $L_p$ at a point $q$, namely

$$(dL_p)_q(v) = \lim_{t \to 0} \frac{L_p(q + tv) - L_p(q)}{t} = (v_1, v_2, v_3 + \langle p_1, v_2 \rangle - \langle p_2, v_1 \rangle)$$

for every $v = (v_1, v_2, v_3) \in T_q \mathbb{H}$, with $p = (p_1, p_2, \tau)$. Notice that we have identified $T_q \mathbb{H}$ with $\mathbb{H}$, using the Hilbert space structure $\mathbb{H}$. We also notice that our formula for the differential $(dL_p)_q$ does not depend on the point $q$. We consider a scalar product

$$\sigma_0 : T_0 \mathbb{H} \times T_0 \mathbb{H} \to \mathbb{R}$$

on the tangent space $T_0 \mathbb{H}$ of $\mathbb{H}$ at the origin, which is continuous with respect to the product topology of $T_0 \mathbb{H} \times T_0 \mathbb{H}$.

Then for every $p \in \mathbb{H}$ and $v, w \in T_p \mathbb{H}$ the following scalar product

$$\sigma_p(v, w) = \sigma_0((dL_{p^{-1}})_p v, (dL_{p^{-1}})_p w) = \sigma_0((dL_{-p})_p v, (dL_{-p})_p w)$$

(3)

defines a left invariant weak Riemannian metric $\sigma$ on $\mathbb{H}$. The associated Riemannian norm is denoted by $\| \cdot \|_\sigma$.

If for any piecewise smooth curve $\gamma : [0, 1] \to \mathbb{H}$ we define its Riemannian length as

$$\ell_\sigma(\gamma) = \int_0^1 \| \dot{\gamma}(t) \|_\sigma \, dt.$$
then the associated geodesic distance \( d : \mathbb{H} \times \mathbb{H} \to [0, +\infty) \) between \( p, q \in \mathbb{H} \) is
\[
d(p, q) = \inf \{ \ell_\sigma(\gamma) : \gamma \text{ is a piecewise smooth curve with } \gamma(0) = p, \gamma(1) = q \}.
\]

(4)

Clearly \( d \) is left invariant, symmetric and it satisfies the triangle inequality.

### 2.3 Weak sub-Riemannian metrics on \( \mathbb{H} \)

Identifying \( \mathbb{H} \) with \( T_0 \mathbb{H} \), the set \( \ell^2 \times \ell^2 \times \{0\} \) can be seen as a closed subspace of \( T_0 \mathbb{H} \), that we denote by \( H_0 \mathbb{H} \). We may obtain a left invariant \emph{horizontal subbundle}, denoted by \( H \mathbb{H} \), introducing the fibers
\[
Hp\mathbb{H} = (dL_p)_0(H_0\mathbb{H}) \subset T_p\mathbb{H}
\]
for every \( p = (p_1, p_2, \tau) \in \mathbb{H} \). We note that \( v = (v_1, v_2, v_3) \in Hp\mathbb{H} \) if and only if
\[
(dL_p)_p(v) = (v_1, v_2, v_3 - \langle p_1, v_2 \rangle + \langle p_2, v_1 \rangle) \in H_0\mathbb{H}
\]
and the previous condition corresponds to the equality
\[
v_3 - \langle p_1, v_2 \rangle + \langle p_2, v_1 \rangle = 0.
\]

(5)

We have a precise formula to define the \emph{horizontal curves} associated to \( H \mathbb{H} \). They are continuous and piecewise smooth curves \( \gamma : [0, 1] \to \mathbb{H} \) of the form \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{H} \), such that for almost every \( t \in [0, 1] \) we have
\[
\dot{\gamma}_3(t) - \langle \gamma_1(t), \dot{\gamma}_2(t) \rangle + \langle \gamma_2(t), \dot{\gamma}_1(t) \rangle = 0.
\]

The previous differential constraint means that \( \dot{\gamma}(t) \in H_{\gamma(t)} \mathbb{H} \).

On the horizontal fibers \( Hp\mathbb{H} \) of \( H \mathbb{H} \) we can fix a scalar product. A \emph{left invariant weak sub-Riemannian metric} \( g \) on \( H \mathbb{H} \) is defined by a continuous inner product
\[
g_0 : H_0\mathbb{H} \times H_0\mathbb{H} \to \mathbb{R},
\]
so that for all \( p \in \mathbb{H} \) and \( v, w \in Hp\mathbb{H} \) we have
\[
g_p(v, w) = g_0((dL_p^{-1})_p v, (dL_p^{-1})_p w) = g_0((dL_{-p})_p v, (dL_{-p})_p w).
\]

(6)

The associated \emph{sub-Riemannian norm} is denoted by \( \| \cdot \|_g \) and the length of a horizontal curve \( \gamma : [0, 1] \to \mathbb{H} \) is defined by
\[
\ell_g(\gamma) = \int_0^1 \| \dot{\gamma}(t) \|_g \, dt.
\]
For any couple of points in $\mathbb{H}$, it is easy to construct a piecewise smooth horizontal curve that connects them, hence the following sub-Riemannian distance

$$\rho(p, q) = \inf \{ \ell_g(\gamma) : \gamma \text{ is a horizontal curve with } \gamma(0) = p, \gamma(1) = q \} \quad (7)$$

is finite for every couple of points $p, q \in \mathbb{H}$, hence we have $\rho : \mathbb{H} \times \mathbb{H} \to [0, +\infty)$. One immediately notices that $\rho$ is left invariant, symmetric and it satisfies the triangle inequality.

### 3 Degenerate geodesic distances in the infinite dimensional Heisenberg group

This section is devoted to the construction of special left invariant weak Riemannian and sub-Riemannian metrics that yield degenerate geodesic distances.

We introduce the linear and continuous operator $A : \ell^2 \to \ell^2$, which associates to each $x \in \ell^2$ of components $(x_k)_{k \geq 1}$ the element $Ax \in \ell^2$, whose $k$-th component is $(Ax)_k = x_k/k$. Then we define the scalar product $\eta : \ell^2 \times \ell^2 \to \mathbb{R}$ as

$$\eta(v, w) = \langle Av, w \rangle$$

for all $v, w \in \ell^2$ and its associated norm

$$\|v\|_\eta = \sqrt{\eta(v, v)} = \sqrt{\langle Av, v \rangle}. \quad (8)$$

We use $\eta$ to define the new scalar product

$$g_0((v_1, v_2), (w_1, w_2)) = \eta(v_1, w_1) + \eta(v_2, w_2) \quad (9)$$

for every $(v_1, v_2), (w_1, w_2) \in \ell^2 \times \ell^2$. By our identification, $g_0$ can be seen as a scalar product on $H_0\mathbb{H}$, so that using (6) we obtain a left invariant weak sub-Riemannian metric $g$ on $\mathbb{H}$. We follow the notation of the previous section, denoting by $\rho$ the special sub-Riemannian distance associated to this choice of $g$ through formula (7).

To obtain a left invariant weak Riemannian metric $\sigma$ on $\mathbb{H}$, we extend $g_0$ as follows

$$\sigma_0((v_1, v_2, v_3), (w_1, w_2, w_3)) = \sigma_0((v_1, v_2), (w_1, w_2)) + v_3 w_3 \quad (10)$$

for every $(v_1, v_2, v_3), (w_1, w_2, w_3) \in T_0\mathbb{H}$, where $\sigma_0 : T_0\mathbb{H} \times T_0\mathbb{H} \to \mathbb{R}$. From (3), the scalar product in (10) immediately defines a left invariant weak Riemannian metric $\sigma$ on $\mathbb{H}$. The Riemannian distance associated to $\sigma$ through (4) will be denoted by $d$.

**Remark 3.1** Let us consider the sub-Riemannian metric $g$, the Riemannian metric $\sigma$ and their associated geodesic distances $d$ and $\rho$ defined above (the Riemannian distance and the sub-Riemannian distance, respectively). The family of piecewise smooth curves connecting two points also contains the horizontal curves connecting the same points. Since the restriction of $\sigma$ to the horizontal subbundle $H\mathbb{H}$ coincides with $g$, the infimum defining $d$ is taken over a larger family, hence $d \leq \rho$.  

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Remark 3.2 It is easy to notice that both $d$ and $\rho$ are not everywhere vanishing on $H$. We consider $(p_1, p_2, \tau), (q_1, q_2, s) \in H$ with $(p_1, p_2) \neq (q_1, q_2)$ and we choose any piecewise smooth curve $\gamma = (\gamma_1, \gamma_2, \gamma_3) : [0, 1] \to H$ with $\gamma(0) = (p_1, p_2, \tau)$ and $\gamma(1) = (q_1, q_2, s)$. Let $i_0 \in \{1, 2\}$ be such that $p_{i_0} \neq q_{i_0}$ and let $k_0 \geq 1$ such that $p_{i_0k_0} \neq q_{i_0k_0}$, where

$$p_{i_0} = \sum_{j=1}^{\infty} p_{i_0j} e_j \quad \text{and} \quad q_{i_0} = \sum_{j=1}^{\infty} q_{i_0j} e_j.$$  

We consider the component $\gamma_{i_0} = \sum_{j=1}^{\infty} \gamma_{i_0j} e_j$ and the following inequalities

$$\ell_\sigma(\gamma) \geq \int_0^1 \sqrt{\|\dot{\gamma}_1\|^2_\eta + \|\dot{\gamma}_2\|^2_\eta} \, dt \geq \int_0^1 \|\dot{\gamma}_{i_0}\|_\eta \, dt \geq \int_0^1 \frac{|\dot{\gamma}_{i_0k_0}|}{\sqrt{k_0}} \, dt \geq \frac{|p_{i_0k_0} - q_{i_0k_0}|}{\sqrt{k_0}} > 0.$$  

In particular, we have shown that

$$0 < \frac{|p_{i_0k_0} - q_{i_0k_0}|}{\sqrt{k_0}} \leq d((p_1, p_2, \tau), (q_1, q_2, s)) \leq \rho((p_1, p_2, \tau), (q_1, q_2, s)).$$

The previous computation also shows that both $d$ and $\rho$ are actually distances, if restricted to any hyperplane $\ell^2 \times \ell^2 \times \{\kappa\}$ with $\kappa \in \mathbb{R}$.

We are now in a position to prove the following theorem.

Theorem 3.3 There exist a left invariant weak sub-Riemannian metric and a left invariant weak Riemannian metric on $H$ such that their associated geodesic distances are not positive on all couples of distinct points.

Proof For each $p \in H$, we denote the norm of a horizontal vector

$$v = (v_1, v_2, v_3) \in H_p H$$

with respect to $g$ as follows

$$\|v\|_g = \|(dL_\rho)_p v\|_g = \|(v_1, v_2, 0)\|_g,$$  

where the last equality is due to (5) and $(v_1, v_2, 0)$ is identified with a vector of $H_0 H$.

Since the subspaces $\ell^2 \times \{0\} \times \{0\}$ and $\{0\} \times \ell^2 \times \{0\}$ of $H_0 H$ are orthogonal with respect to $g_0$, the previous equalities give

$$\|v\|^2_g = \|v_1\|^2_\eta + \|v_2\|^2_\eta,$$
where \( \| \cdot \|_\eta \) is defined in (8). Thus, the length of a horizontal curve \( \gamma : [0, 1] \to \mathbb{H} \) with respect to \( g \) satisfies the formula

\[
\ell_g(\gamma) = \int_0^1 \sqrt{\| \dot{\gamma}_1 \|^2_\eta + \| \dot{\gamma}_2 \|^2_\eta} \, dt,
\]

(12)

where \( \gamma(t) = (\gamma_1(t), \gamma_2(t), \gamma_3(t)) \).

Next, we wish to show that whenever \((p_1, p_2, s_1), (p_1, p_2, s_2) \in \mathbb{H}\), then

\[
\rho((p_1, p_2, s_1), (p_1, p_2, s_2)) = 0.
\]

(13)

To do this, the main point is to prove that for all \( s > 0 \), we have \( \rho((0, 0, 0), (0, 0, s)) = 0 \). We will construct a sequence of horizontal curves connecting \((0, 0, 0)\) to \((0, 0, s)\), whose length converges to zero. Such sequence is obtained by gluing different sequences of horizontal curves. We fix \( c > 0 \) and consider \( \gamma^n : [0, 1] \to \mathbb{H} \) defined by

\[
\gamma^n(t) = (\gamma^n_1(t), \gamma^n_2(t), \gamma^n_3(t)) = \left( \frac{t^2}{2} e_n, -te_n, \frac{t^3}{6}c^2 \right),
\]

where the unit vector \( e_n \) is the \( n \)-th vector of the fixed orthonormal basis \( \{ e_j : j \geq 1 \} \) of \( \ell^2 \). By definition (8), we get

\[
\| \dot{\gamma}^n_1(t) \|^2_\eta = \frac{t^2 c^2}{n} \quad \text{and} \quad \| \dot{\gamma}^n_2(t) \|^2_\eta = \frac{c^2}{n}.
\]

(14)

From the form of \( \gamma^n \), it is immediate to check that the differential constraint

\[
\dot{\gamma}^n_3 - \langle \dot{\gamma}^n_1, \dot{\gamma}^n_2 \rangle + \langle \dot{\gamma}^n_2, \dot{\gamma}^n_1 \rangle = 0
\]

is satisfied for all \( t \in [0, 1] \), hence \( \gamma^n \) is horizontal. Thus, formula (12) holds and the expressions of (14) immediately prove that \( \ell_g(\gamma^n) \to 0 \) as \( n \to +\infty \).

Now we define the sequence of curves \( \alpha^n : [0, 1] \to \mathbb{H} \) as

\[
\alpha^n(t) = (\alpha^n_1(t), \alpha^n_2(t), \alpha^n_3(t)) = \left( c \left( \frac{1}{2} - \frac{t^2}{2} \right) e_n, c(t - 1)e_n, c^2 \left( \frac{1}{6} + \frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2} \right) \right).
\]

We immediately obtain

\[
\| \dot{\alpha}^n_1(t) \|^2_\eta = \frac{t^2 c^2}{n} \quad \text{and} \quad \| \dot{\alpha}^n_2(t) \|^2_\eta = \frac{c^2}{n}
\]

(15)

and the differential constraint

\[
\dot{\alpha}^n_3 - \langle \dot{\alpha}^n_1, \dot{\alpha}^n_2 \rangle + \langle \dot{\alpha}^n_2, \dot{\alpha}^n_1 \rangle = 0
\]
is satisfied for all \( t \in [0, 1] \). All curves \( \alpha^n \) are horizontal, hence combining (12) and (15), we conclude that \( \ell_g(\alpha^n) \to 0 \) as \( n \to +\infty \). We note that

\[
\alpha^n(0) = \left( \frac{c}{2} e_n, -ce_n, \frac{c^2}{6} \right) = \gamma^n(1)
\]

for all \( n \in \mathbb{N} \), hence we can consider the gluing \( \alpha^n * \gamma^n : [0, 1] \to \mathbb{H} \) of \( \alpha^n \) and \( \gamma^n \), that is a piecewise smooth curve. Clearly \( \alpha^n * \gamma^n \) is a horizontal curve and for all \( n \in \mathbb{N} \) we have

\[
\alpha^n * \gamma^n(0) = \gamma^n(0) = (0, 0, 0)
\]

and

\[
\ell_g(\alpha^n * \gamma^n) = \ell_g(\alpha^n) + \ell_g(\gamma^n) \to 0 \quad \text{as} \quad n \to \infty.
\]

We have proved that

\[
\rho \left( (0, 0, 0), \left( 0, 0, \frac{c^2}{3} \right) \right) = 0,
\]

hence \( \rho((0, 0, 0), (0, 0, s)) = 0 \) for all \( s > 0 \). By the left invariance of \( \rho \), we have

\[
\rho((0, 0, 0), (0, 0, -s)) = \rho((0, 0, s), (0, 0, 0)) = 0,
\]

therefore \( \rho((0, 0, 0), (0, 0, t)) = 0 \) for every \( t \in \mathbb{R} \). We conclude that

\[
\rho((p_1, p_2, s_1), (p_1, p_2, s_2)) = \rho((p_1, p_2, 0)(0, 0, s_1), (p_1, p_2, 0)(0, 0, s_2))
\]

\[
= \rho((0, 0, s_1), (0, 0, s_2))
\]

\[
= \rho((0, 0, 0), (0, 0, s_2 - s_1)) = 0,
\]

that proves (13). According to Remark 3.1, the inequality \( d \leq \rho \) implies that for all \( (p_1, p_2, s_1), (p_1, p_2, s_2) \in \mathbb{H} \), we have

\[
d((p_1, p_2, s_1), (p_1, p_2, s_2)) = 0.
\]

(16)

This concludes the proof.

\[\square\]

4 On the sectional curvature of a weak Riemannian Heisenberg group

In this section, we study the sectional curvature of \( \mathbb{H} \) equipped with a specific left invariant weak Riemannian metric. Following Sect. 3, we define the unique left invariant weak Riemannian metric \( \sigma \), such that

\[
\sigma_0((v_1, v_2, v_3), (w_1, w_2, w_3)) = g_0((v_1, v_2), (w_1, w_2)) + v_3w_3
\]
for \((v_1, v_2, v_3), (w_1, w_2, w_3) \in T_0 \mathbb{H}\), according to (10). We recall the formula
\[ g_0((v_1, v_2), (w_1, w_2)) = \eta(v_1, w_1) + \eta(v_2, w_2) = \langle Av_1, w_1 \rangle + \langle Av_2, w_2 \rangle \]
and \(Ax = \sum_{k=1}^{\infty} \frac{x_k}{k}, x = \sum_{k=1}^{\infty} x_k e_k \in \ell^2\). For every positive integer \(j\), we use the notation
\[ e_1^j = (e_j, 0, 0), \quad e_2^j = (0, e_j, 0) \quad \text{and} \quad e_3 = (0, 0, 1), \]
to indicate the standard orthonormal basis of \(\mathbb{H}\) seen as the Hilbert space \(\ell^2 \times \ell^2 \times \mathbb{R}\).

One easily realizes that the natural linear isomorphism between \(\text{Lie}(\mathbb{H})\) and \(\mathbb{H}\) is also an isomorphism of Lie algebras, where we equip \(\mathbb{H}\) with the Lie product (1). Thus, by slight abuse of notation, the left invariant vector fields of \(\text{Lie}(\mathbb{H})\) isomorphically associated with the basis \(e_1^j, e_2^j, e_3\) are denoted by the same symbols. It follows that for all \(i, j \geq 1\) and \(l = 1, 2\) we have
\[ [e_1^i, e_2^j] = 2\delta_{ij}e_3 \quad \text{and} \quad [e_1^i, e_1^j] = 0, \tag{17} \]
where \(e_1^i, e_1^j, e_3\) are now understood as left invariant vector fields of \(\text{Lie}(\mathbb{H})\).

Now we consider a left invariant weak Riemannian metric \(\nu\) on \(\mathbb{H}\). The associated scalar product on \(\text{Lie}(\mathbb{H})\) is denoted by \(\langle \cdot, \cdot \rangle_{\nu}\). We consider two orthonormal vectors \(X, Y \in \text{Lie}(\mathbb{H})\) with respect to \(\langle \cdot, \cdot \rangle_{\nu}\). By virtue of [2, Theorem 5], the sectional curvature \(K_{\nu}(X, Y)\) of the plane in \(\text{Lie}(\mathbb{H})\) spanned by \(X\) and \(Y\) can be obtained by the adjoint operator \(\text{ad}(Y)^\top(X)\), that we now introduce. We define \(\text{ad}(Y)(Z) = [Y, Z]\) and consider (in case it exists) the unique vector \(\text{ad}(Y)^\top(X) \in \text{Lie}(\mathbb{H})\) that satisfies the equalities
\[ \langle [Y, Z], X \rangle_{\nu} = \langle \text{ad}(Y)(Z), X \rangle_{\nu} = \left\langle Z, \text{ad}(Y)^\top(X) \right\rangle_{\nu}, \tag{18} \]
for every \(Z \in \text{Lie}(\mathbb{H})\). Then we define
\[ B_{\nu}(X, Y) = \text{ad}(Y)^\top(X) \in \text{Lie}(\mathbb{H}). \tag{19} \]

In the case \(\nu\) is a strong Riemannian metric, [1, Definition 5.2.12], the existence of \(B_{\nu}(X, Y)\) is always ensured, but not for any weak Riemannian metric. For instance, in Remark 4.1 below, we show the nonexistence of \(B_{\sigma}(X, Y)\) for a specific choice of \(X\) and \(Y\), where \(\sigma\) is the weak Riemannian metric introduced at the beginning of this section.

From formula (53) of [2], we have
\[ K_{\nu}(X, Y) = \langle \delta, \delta \rangle_{\nu} + 2 \langle \alpha, \beta \rangle_{\nu} - 3 \langle \alpha, \alpha \rangle_{\nu} - 4 \langle B_X, B_Y \rangle_{\nu}, \tag{20} \]
where we define
\[
\delta = \frac{1}{2} (B_v(X, Y) + B_v(Y, X)), \quad \beta = \frac{1}{2} (B_v(X, Y) - B_v(Y, X)),
\]
\[
\alpha = \frac{1}{2} [X, Y] \tag{21}
\]
\[
B_X = \frac{1}{2} B_v(X, X) \quad \text{and} \quad B_Y = \frac{1}{2} B_v(Y, Y). \tag{22}
\]

The proof of Theorem 1.3 follows from the application of (20) with respect to \(\sigma\) on suitable choices of planes. We denote by \(\langle \cdot, \cdot \rangle_\sigma\) the scalar product in \(\text{Lie}(\mathbb{H})\) induced by the left invariant weak Riemannian metric \(\sigma\). The associated norm on \(\text{Lie}(\mathbb{H})\) is denoted by \(\| \cdot \|_\sigma\). We assume that for \(X, Y \in \text{Lie}(\mathbb{H})\) the adjoint \(B_\sigma(X, Y) = \text{ad}(Y) \top(X)\) with respect to \(\sigma\) exists. In this case, its scalar product with a vector \(Z \in \text{Lie}(\mathbb{H})\) is assigned by the following formula
\[
\langle \text{ad}(Y) \top(X), Z \rangle_\sigma = \langle [Y, Z], X \rangle_\sigma = 2\beta(\pi(Y), \pi(Z))x^3, \tag{23}
\]
as a consequence of (1), where \(\pi : \mathbb{H} \to \ell^2 \times \ell^2\) is the canonical projection defined by
\[
X = (\pi(X), x^3) = (\pi(X), 0) + x^3e^3.
\]
We use the fixed orthonormal basis \(e^1_j, e^2_j, e^3\) of \(\mathbb{H}\) with respect to the standard Hilbert product of \(\ell^2 \times \ell^2 \times \mathbb{R}\), getting
\[
\text{ad}(Y) \top(X) = \sum_{j=1}^\infty [\text{ad}(Y) \top(X)]_j^1 e^1_j + \sum_{j=1}^\infty [\text{ad}(Y) \top(X)]_j^2 e^2_j + [\text{ad}(Y) \top(X)]^3 e^3.
\]
Formula (23) yields
\[
\sum_{j=1}^\infty \frac{1}{j} [\text{ad}(Y) \top(X)]_j^1 Z^1_j + \sum_{j=1}^\infty \frac{1}{j} [\text{ad}(Y) \top(X)]_j^2 Z^2_j + [\text{ad}(Y) \top(X)]^3 Z^3
\]
\[
= 2\beta(\pi(Y), \pi(Z))x^3 \tag{24}
\]
for arbitrary \(Z = Z^3 e^3 + \sum_{j=1}^\infty Z^1_j e^1_j + Z^2_j e^2_j\). In the case \(X = \pi(X)\), formula (24) shows the existence of \(\text{ad}(Y) \top(\pi(X))\) and yields
\[
B_\sigma(\pi(X), Y) = \text{ad}(Y) \top(\pi(X)) = 0. \tag{25}
\]
In the case $X = e^3$, again (24) for $Z = e^1_j$ and $Z = e^2_j$ respectively, gives

$$\text{ad}(Y) (e^3) ]_j^1 = 2 j \beta(\pi(Y), e^1_j) \quad \text{and} \quad \text{ad}(Y) (e^3) ]_j^2 = 2 j \beta(\pi(Y), e^2_j).$$

(26)

For $Z = e^3$, applying (24) we get

$$\text{ad}(Y) (e^3) ]^3 = 0.$$  

(27)

Assuming the existence of $\text{ad}(Y) (e^3)$, we have shown that

$$B_\sigma(e^3, Y) = \text{ad}(Y) (e^3) = 2 \sum_{j=1}^{\infty} j \beta(\pi(Y), e^1_j)e^1_j + 2 \sum_{j=1}^{\infty} j \beta(\pi(Y), e^2_j)e^2_j.$$

Writing $Y = Y^3 e^3 + \sum_{j=1}^{\infty} (Y^1_j e^1_j + Y^2_j e^2_j)$, we finally get

$$B_\sigma(e^3, Y) = 2 \sum_{j=1}^{\infty} j (Y^1_j e^2_j - Y^2_j e^1_j).$$

(28)

Then the assumption about the existence of $B_\sigma(e^3, Y)$ corresponds to the convergence of its series. The next remark shows a choice of $Y$ for which the series (28) does not converge.

**Remark 4.1** If we consider the vector

$$W = \sum_{j=1}^{\infty} \frac{e^1_j}{j} \in \text{Lie}(\mathbb{H}),$$

(29)

then it is easy to check that the series (28) representing $B_\sigma(e^3, W)$ does not converge. As a consequence, the adjoint $\text{ad}(W) (e^3)$ cannot be defined. In addition, Arnold’s formula (20) for the sectional curvature of the plane span $\{W, e^3\}$ does not apply.

**Proposition 4.2** We consider the orthonormal elements $W_k, e^3 \in \text{Lie}(\mathbb{H})$ with $k \geq 1$ and

$$W_k = \left( \sum_{j=1}^{k} j^{-3} \right)^{-1/2} \sum_{j=1}^{k} \frac{e^1_j}{j} \in \text{Lie}(\mathbb{H}).$$

As the subspace span $\{W_k, e^3\}$ converges to span$\{W_\infty, e^3\}$ for $k \to \infty$, with

$$W_\infty = \left( \sum_{j=1}^{\infty} j^{-3} \right)^{-1/2} \sum_{j=1}^{\infty} \frac{e^1_j}{j} \in \text{Lie}(\mathbb{H}),$$

(30)
it follows that

\[ K_\sigma(W_k, e^3) \to +\infty. \] (31)

The convergence of \( \text{span}\ \{W_k, e^3\} \) to \( \text{span}\{W_\infty, e^3\} \) is considered in the Grassmannian of the 2-dimensional planes contained in \( \text{Lie}(\mathbb{H}) \).

**Proof** First of all, the pointwise convergence of \( W_k \) to \( W_\infty \) implies the convergence of \( \text{span}\ \{W_k, e^3\} \) to \( \text{span}\{W_\infty, e^3\} \). To compute \( K_\sigma(W_k, e^3) \), we first apply (25), getting

\[ B_\sigma(W_k, e^3) = \text{ad}(e^3)^\top(W_k) = 0 \] (32)

for all \( k \geq 1 \). From (28), it follows that \( B_\sigma(e^3, e^1_j) = 2 je_j^2 \), hence

\[ B_\sigma(e^3, e^1_j/j) = 2e_j^2. \]

The bilinearity of \( B_\sigma(\cdot, \cdot) \) yields

\[ B_\sigma(e^3, W_k) = 2 \left( \sum_{j=1}^{k} j^{-3} \right)^{-1/2} \sum_{j=1}^{k} e_j^2 \] (33)

From (21), taking \( \delta = \left( B_\sigma(W_k, e^3) + B_\sigma(e^3, W_k) \right) / 2 \), we obtain

\[ \langle\delta, \delta\rangle_\sigma = \frac{1}{4} \|B_\sigma(e^3, W_k)\|_\sigma^2 = \left( \sum_{j=1}^{\infty} j^{-3} \right)^{-1} \left( \sum_{j=1}^{k} j^{-3} \right)^{-1} \sum_{j=1}^{\infty} j^{-1} = \left( \sum_{j=1}^{\infty} j^{-3} \right)^{-1} \sum_{j=1}^{k} j^{-1} \] (34)

From (21), (22), (25) and (28), we find

\[ \alpha = \frac{1}{2} B_\sigma(W_k, W_k) = \frac{1}{2} B_\sigma(e^3, e^3) = 0. \] (35)

Finally, by formula (20), we have proved that

\[ K_\sigma(W_k, e^3) = \langle\delta, \delta\rangle_\sigma = \left( \sum_{j=1}^{\infty} j^{-3} \right)^{-1} \sum_{j=1}^{k} j^{-1} \to +\infty \] (36)

as \( k \to \infty \). This concludes the proof. \( \square \)

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Proof of Theorem 1.3 Following the notation of the present section, we define
\[ a_1^j = \sqrt{j} e_1^j \quad \text{and} \quad a_2^j = \sqrt{j} e_2^j \]
of Lie(\mathbb{H}), that are orthonormal with respect to \( \langle \cdot, \cdot \rangle_\sigma \) and do not commute. To apply (20) for finding \( K_\sigma(a_1^j, a_2^j) \), we use (21) and (22). Due to (25), we get
\[ B_\sigma(a_1^j, a_2^j) = 0. \]
As a result, we have
\[ K_\sigma(a_1^j, a_2^j) = -\frac{3}{4} \left\| [a_1^j, a_2^j] \right\|_\sigma^2 = -\frac{3}{4} j^2. \] (37)

Now we wish to compute \( K_\sigma(a_1^j, e_3^j) \) and \( K_\sigma(a_2^j, e_3^j) \). We first apply (25) and (28), getting
\[ B_\sigma(e_l^j, e_3^j) = \text{ad}(e_3^j)^\top(e_l^j) = 0, \quad B_\sigma(e_3^j, e_1^j) = 2j e_2^j \quad \text{and} \quad B_\sigma(e_3^j, e_2^j) = -2j e_1^j \] (38)
for all \( l = 1, 2 \) and \( k \geq 1 \). From (21), taking \( \delta = \left( B_\sigma(a_1^j, e_3^j) + B_\sigma(e_3^j, a_1^j) \right) / 2 \), we obtain
\[ \langle \delta, \delta \rangle_\sigma = \frac{1}{4} \left\| B_\sigma(a_1^j, e_3^j) \right\|_\sigma^2 = \frac{1}{4} \left\| \sqrt{j} B_\sigma(e_3^j, e_1^j) \right\|_\sigma^2 = \frac{j}{4} \left\| 2j e_2^j \right\|_\sigma^2 \] (39)
\[ = j^3 \left\langle e_1^2, e_2^2 \right\rangle_\sigma = j^3 \left\langle Ae_1^2, e_2^2 \right\rangle_\sigma = j^2. \] (40)
From (21), (22), (25) and (28), we find
\[ \alpha = \frac{1}{2} B_\sigma(e_1^j, e_1^j) = \frac{1}{2} B_\sigma(e_3^j, e_3^j) = 0. \] (41)
Due to the formula for the sectional curvature (20), we have established that
\[ K_\sigma(a_1^j, e_3^j) = \langle \delta, \delta \rangle_\sigma = j^2. \] (42)
In analogous setting \( \delta = \left( B_\sigma(a_2^j, e_3^j) + B_\sigma(e_3^j, a_2^j) \right) / 2 \), we obtain
\[ \langle \delta, \delta \rangle_\sigma = \frac{1}{4} \left\| B_\sigma(e_3^j, a_2^j) \right\|_\sigma^2 = \frac{j}{4} \left\| B_\sigma(e_3^j, e_1^j) \right\|_\sigma^2 = \frac{j}{4} \left\| 2j e_1^j \right\|_\sigma^2 = \frac{j^3}{4} \left\| e_1^j \right\|_\sigma^2 = j^2. \] (43)
Again (21), (22), (25) and (28) imply that
\[ \alpha = \frac{1}{2} B_\sigma(e_2^j, e_2^j) = \frac{1}{2} B_\sigma(e_3^j, e_3^j) = 0. \] (44)
Due to (20), we have also proved that

\[ K_\sigma(a_2 j, e^3) = \langle \delta, \delta \rangle_\sigma = j^2. \]  

(45)

Taking into account (42) and (45), setting \( b = e^3 \), we have completed the proof. \( \square \)

**Remark 4.3** A direct verification shows that the computations of sectional curvature, to prove Theorem 1.3, could be also carried out extending the finite dimensional formula of [21, Lemma 1.1] for the countable structure coefficients of \( \text{Lie}(\mathbb{H}) \). These coefficients are obtained from the orthonormal vectors \( \sqrt{j} e^1_j, \sqrt{j} e^2_j, e^3 \) of \( \text{Lie}(\mathbb{H}) \) with respect to \( \langle \cdot, \cdot \rangle_\sigma \).

Following the notation of this section, the sequence of curves whose length converges to zero in the proof of Theorem 3.3 can be written as

\[
\gamma_j(t) = \frac{ct}{2} e^1_j - cte^2_j + \frac{c^2 t^3}{6} e^3 \in \mathbb{H} \quad \text{and} \\
\alpha_j(t) = c \left( \frac{1}{2} - \frac{t^2}{2} \right) e^1_j + c(t - 1) e^2_j + c^2 \left( \frac{1}{6} + \frac{t^3}{6} - \frac{t^2}{2} + \frac{t}{2} \right) e^3 \in \mathbb{H}.
\]

It is interesting to notice that all such curves are contained in the span of the planes

\[ \text{span}\{e^1_j, e^2_j\}, \quad \text{span}\{e^1_j, e^3\} \quad \text{and} \quad \text{span}\{e^2_j, e^3\}. \]

When these planes are seen in the Lie algebra, Theorem 1.3 shows that their sectional curvature blows-up, as the length of the curves converges to zero.

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