Exactly Solvable Single Lane Highway Traffic Model With Tollbooths

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(March 21, 2022)

Tolls are collected on many highways as a means of traffic control and revenue generation. However, the presence of tollbooths on highway surely slows down traffic flow. Here, we investigate how the presence of tollbooths affect the average car speed using a simple-minded single lane deterministic discrete traffic model. More importantly, the model is exactly solvable.

PACS numbers: 45.70.Vn, 05.60.-k, 89.40.+k

Modern computing power allows us to simulate highway and city traffic by looking at the microscopic behaviors of all cars. Perhaps the most well-known model of this kind is the cellular automaton based Biham-Middleton-Levine (BML) model of two-dimensional city traffic [1]. Numerical simulations of this model strongly suggest that a first order phase transition from the full-speed phase to the completely jamming phase occurs as the car density of the system increases. Moreover, this phase transition is primarily the result of the exclusion volume effect [1,2]. In spite of its simplicity, very little rigorous result is known for the BML model [3].

Different generations and variations of the BML model have been investigated in the literature [4]. These generalizations focus on different aspects of the problem. They include the introduction of more realistic traffic rules [5], study of the effects of over-passes and faulty traffic lights [6], application of cellular automaton based traffic rules to single and multiple lane highway traffic [7] and investigation of higher dimensional traffic behaviors [8]. In particular, Nagel and Schreckenberg [9] proposed a model of (one-dimensional) highway traffic. In addition to the regular acceleration and deceleration, they model the realistic behavior of car drivers by allowing them to apply their breaks in a stochastic manner. Later on, Fukui and Ishibashi (FI) [10] investigated a simple-minded deterministic model analogous to that of Nagel and Schreckenberg. Recently, Chowdhury and Schadschneider (CS) incorporated the one-dimensional highway traffic model of Nagel and Schrecken berg as well as the two-dimensional BML model together to study microscopic dynamics of city traffic [11].

It is not uncommon for local governments to set up tollbooths on highways. In fact, tolls can be used to control traffic flow and to increase revenue for local governments. Nevertheless, the presence of tollbooths will definitely slow down highway traffic. This is particularly true when tollbooths are set up on the highway rather than behind the entrance and exit ramps. Unfortunately, because of geographical and administrative considerations, tollbooths in many highways have to be built right on highways themselves. In fact, a few cellular automaton models of one-dimensional highway traffic flow with different kinds of blockages have been proposed and investigated [12]. They consider the effects of overtaking sites, bottleneck and quenched noise. In contrast, this paper investigates how the presence of tollbooths affects the traffic flow in a single lane highway using a simple-minded deterministic discrete model based on the cellular automaton traffic model of Fukui and Ishibashi [10] as well as the so-called green wave model of Torok and Kertesz [13].

In their original model, Fukui and Ishibashi consider a one-dimensional array of N sites with periodic boundary conditions. Each site may either be empty or have a single rightward moving car. They fix an integer $V_{\text{max}}$ known as the maximum intrinsic car speed. Since their model does not consider the effect of car acceleration and deceleration, so at each timestep a car moves $k$ steps to the right where $k$ is the minimum of $V_{\text{max}}$ and the number of consecutive empty sites to the left of the car. In addition, the motion of cars are updated in parallel [10]. They define the average car speed by

$$\langle V \rangle = \left\langle \frac{1}{N\rho} \sum_{i=1}^{N\rho} V_i \right\rangle,$$

where $\rho$ is the car density in the system (and hence $N\rho$ is the total number of cars in the system) and $V_i$ is the speed of the $i$th car. Note that the right hand side of Eq. (1) is averaged both over time and initial system configurations.

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To coarse gain the highway system, we combine the FI model \[\text{(10)}\] with the so-called green wave traffic model \[\text{[13]}\]. That is to say, we call a collection of consecutive sites all containing rightward moving cars a car cluster. We demand that cars in the same cluster to move altogether as a group with speeds equal to that of the leading car in the group except possibly when the car passes through a tollbooth. Besides, all updates are taken in parallel. Finally, we introduce the effects of tollbooths by selecting \(N_{\text{booth}}\) special sites on the system. Whenever a car reaches these special tollbooth sites, it has to stop immediately and to wait for \(t_{\text{wait}}\) timesteps before it is allowed to move again. This waiting car, therefore, may block the motion of the cars queuing behind it for \(t_{\text{wait}}\) timesteps. For simplicity, tollbooths are located uniformly on the system.

For example, if \(t_{\text{wait}} = 0\), then a passing by car has to stop at the tollbooth site at once and then that car may move in the next timestep. Let us denote 0 as an empty site, 1 as a site occupied by a car, and an underline as a tollbooth. tollbooths are located uniformly on the system.

Since we are interested only in the average car speed over all possible initial car configurations in the thermodynamic limit, therefore only the recurrent behavior of the system will affect the average car speed \(\langle V \rangle\). More precisely, \(\langle V \rangle\) depends only on the car density \(\rho\), the maximum car speed \(V_{\text{max}}\), the tollbooth density \(d_{\text{booth}} \equiv N_{\text{booth}}/N\) and the tollbooth stopping time \(t_{\text{wait}}\) of the system. Thus, \(t_{\text{wait}}\) and \(d_{\text{booth}}\) are the two controlling parameters in studying the effect of tollbooths.

In this paper, we are going to show that

**Theorem 1** Case (a): If \(t_{\text{wait}} = 0\), then the average car speed \(\langle V \rangle\) is given by

\[
\langle V \rangle = \begin{cases} 
\frac{1}{d_{\text{booth}} + \frac{1}{d_{\text{booth}}V_{\text{max}}}} & \text{for } 0 < \rho \leq \rho_a, \\
1/2\rho & \text{for } \rho_a < \rho \leq 1/2, \\
1 & \text{for } 1/2 < \rho < 1, \\
0 & \text{for } \rho = 1,
\end{cases}
\]

where \(\rho_a = d_{\text{booth}} \left[1/d_{\text{booth}}V_{\text{max}}\right]/2\).

Case (b): If \(t_{\text{wait}} > 0\), and \(1/d_{\text{booth}} = 1 \mod V_{\text{max}}\) then the average car speed is given by

\[
\langle V \rangle = \begin{cases} 
\frac{1}{d_{\text{booth}}(t_{\text{wait}} + \frac{1}{d_{\text{booth}}V_{\text{max}}})} & \text{for } 0 < \rho \leq \rho_b, \\
\rho(t_{\text{wait}} + 1)^{-1} & \text{for } \rho_b < \rho < 1, \\
0 & \text{for } \rho = 1,
\end{cases}
\]

where \(\rho_b = d_{\text{booth}}(t_{\text{wait}} + \frac{1}{d_{\text{booth}}V_{\text{max}}})/(t_{\text{wait}} + 1)\).

Case (c): For the remaining possibility, namely that \(t_{\text{wait}} > 0\) and \(1/d_{\text{booth}} \neq 1 \mod V_{\text{max}}\), the average car speed is given by

\[
\langle V \rangle = \begin{cases} 
\frac{1}{d_{\text{booth}}(t_{\text{wait}} + \frac{1}{d_{\text{booth}}V_{\text{max}}})} & \text{for } 0 < \rho \leq \rho_{c1}, \\
\rho(t_{\text{wait}} + 2)^{-1} & \text{for } \rho_{c1} < \rho \leq \rho_{c2}, \\
\frac{1}{d_{\text{booth}}(t_{\text{wait}} + 1 + \frac{1}{d_{\text{booth}}V_{\text{max}}})} & \text{for } \rho_{c2} < \rho \leq \rho_{c3}, \\
\rho(t_{\text{wait}} + 1)^{-1} & \text{for } \rho_{c3} < \rho < 1, \\
0 & \text{for } \rho = 1,
\end{cases}
\]

where \(\rho_{c1} = d_{\text{booth}}(t_{\text{wait}} + \frac{1}{d_{\text{booth}}V_{\text{max}}})/(t_{\text{wait}} + 2); \rho_{c2} = d_{\text{booth}}(t_{\text{wait}} + 1 + \frac{1}{d_{\text{booth}}V_{\text{max}}})/(t_{\text{wait}} + 2); \rho_{c3} = d_{\text{booth}}(t_{\text{wait}} + 1 + \frac{1}{d_{\text{booth}}V_{\text{max}}})/(t_{\text{wait}} + 1)\).

Before going on to prove this theorem, we remark that in all the three cases above, first order phase transitions in \(\langle V \rangle\) occur only at \(\rho = 1\). All other transition points are second order in nature. (See Fig. [1]a–c for typical shapes of the \(\rho\) versus \(\langle V \rangle\) curves in these three cases.)
to any car cluster passing through a tollbooth. If we denote the action of applying our traffic rule once by \( t_{\text{wait}} \). Proof: Suppose the contrary, we can find car A blocking car B in a recurrent state such that car A does not belong to any car cluster passing through a tollbooth. If we denote the action of applying our traffic rule once by \( T \), then the inverse map \( T^{-1} \) is well-defined on the set of all recurrent states. Clearly, under the action of \( T^{-1} \), car A must move backward by at least one site. Moreover, car A must be blocked by another car at least once in the previous \( t_{\text{wait}} + 1 \) timesteps. Inductively, by considering the repeated application of \( T^{-1} \), we end up with a car C located at a tollbooth site whose motion in the next timestep is blocked by a car D located two sites in front. But this is impossible as the inverse image of this configuration under \( T^2 \) is an empty set, contradicting the assumption that the state is recurrent.

Proposition 2 In case (c), whenever the car density lies between \( \rho_{c1} \) and \( \rho_{c2} \), a car in a recurrent state can only be blocked by another car right at a tollbooth site when the car occupying that tollbooth site is moving away. And
whenever the car density lies between $\rho_{c2}$ and $\rho_{c3}$, a car in a recurrent state can only be blocked by another car right at a tollbooth site when the car occupying that tollbooth site is going to move away in the next timestep.

Proof: Proposition 1 implies that cars begin to block one another when $\rho > \rho_{c1}$ in case (c). From Lemma 1 and the fact that $1/d_{booth} \neq 1 \bmod V_{max}$, we know that it is possible for a car A in a recurrent state to be blocked by a car located at a tollbooth site. If this event happens, car A has to take $(t_{wait} + 1 + \lceil 1/d_{booth} V_{max} \rceil)$ timesteps to move through that two successive tollbooth sites. In other words, car A takes one timestep more than the minimum possible value in order to move through the two successive tollbooths. Note that if $\rho \leq \rho_{c3}$, no recurrent state can contain a car cluster C making up of more than two cars. The reason is simple: for otherwise, there exits, at any instance, at least one interval J between two tollbooths containing no more than $(t_{wait} + \lceil 1/d_{wait} V_{max} \rceil)$ cars. But then the car outflow rate from this interval J is strictly less than one car per $(t_{wait} + 1)$ timesteps while from Lemma 2 the car outflow rate from the cluster C equals one car every $(t_{wait} + 1)$ timesteps. Thus eventually there is not enough car supply to maintain the car cluster C and new car clusters with more than two cars cannot be found elsewhere in the system due to the restrictions of both the car inflow and outflow rates in a tollbooth site. This contradicts our assumption that the configuration is recurrent.

Using the same trick as in the above argument that no car cluster with more than two cars can be formed in a recurrent configuration for $\rho \leq \rho_{c3}$, it is easy to show that when $\rho \in (\rho_{c1}, \rho_{c2})$, the recurrent state consists of intervals of freely moving cars with density $\rho_{c1}$ as well as intervals of cars with density $\rho_{c2}$ that can be blocked by a ready-to-move car in a tollbooth site. (All car densities mentioned here are averaged over $(t_{wait} + 2)$ timesteps.)

Similarly, when $\rho_{c2} < \rho \leq \rho_{c3}$, a recurrent state is made up of intervals of cars with density $\rho_{c2}$ that can be blocked by a ready-to-move car in a tollbooth site as well as intervals of cars with density $\rho_{c3}$ that can be blocked by a car in a tollbooth site that will move in the next timestep.

Lemma 2 Let $t_{wait} = 0$ and $V_{max} > 1$. Then if $\rho \leq 1/2$, all car clusters in a recurrent state consist of only one car. And if $\rho > 1/2$, the recurrent state contains no consecutive empty site.

Proof: Since $t_{wait} = 0$, all cars can move at least one step to the right in every timestep. Besides, it is not possible for two car clusters to merge. Since we are using green wave traffic rule and $V_{max} > 1$, a car cluster may break up into two only at a tollbooth site. Furthermore, consecutive empty sites will “move” to the left while car clusters move to the right. Thus in O($N$) timesteps, the system will evolve to a state with maximum possible number of car clusters. Hence the lemma is proved.

Proposition 3 Let $t_{wait} = 0$, $V_{max} > 1$ and $\rho_a \leq \rho < 1/2$, then exactly one car passes through a tollbooth every two timesteps.

Proof: From Lemma 2 and our traffic rule, we know that at most one car can pass through a tollbooth every two successive timesteps. And from Proposition 1, we know that at least one car is blocked in a recurrent state. Using a similar argument as in the proof of Proposition 2, one can always find a car whose motion is blocked by another about-to-go car locating at a tollbooth site in front. Hence, every car takes exactly two timesteps to pass through such a tollbooth.

Proposition 4 The average car speed formulae in case (b) and (c) are valid for $\rho_b < \rho < 1$ and $\rho_c < \rho < 1$, respectively.

Proof: From Lemma 1 and the assumption that $\rho_b < \rho < 1$ and $\rho_c < \rho < 1$ in case (b) and (c) respectively, we can always find a car cluster in a recurrent state making up of at least three cars lining up in front of a tollbooth. Moreover, such a car cluster can never dissolve (that is, the cluster never disintegrates to clusters of single cars) under the repeated action of our traffic rule. Since the leading car in this car cluster passes through the tollbooth at a rate of once every $(t_{wait} + 1)$ timesteps, we conclude that the average car speed $\langle V \rangle$ equals $1/\rho(t_{wait} + 1)$.

Proof of Theorem 1. Since $\langle V \rangle$ is clearly equal to 0 when $\rho = 1$, so it remains for us to consider the case when $\rho < 1$.

Let us first consider case (a). If $V_{max} = 1$, then our traffic rules reduce to moving each car forward one site at a time provided that $\rho < 1$. Hence, Eq. (2) is trivially true in this case. So we only need to consider the case when $V_{max} > 1$. And from Proposition 1, we know that for case (a), when $\rho \leq \rho_a$, cars will eventually moves as if there were no other cars in the system. Thus a car in a recurrent state takes $(t_{wait} + \lceil 1/d_{booth} V_{max} \rceil) = \lceil 1/d_{booth} V_{max} \rceil$ timesteps to travel through $1/d_{booth}$ sites. Hence, $\langle V \rangle = (d_{booth} \lceil 1/d_{booth} V_{max} \rceil)^{-1}$. And when $\rho_a < \rho \leq 1/2$, $\langle V \rangle = 1/2\rho$ is an immediate consequence of Proposition 2. Finally, when $1/2 < \rho < 1$, Lemma 2 tells us that two adjacent car clusters are separated by exactly one empty site. Hence, $\langle V \rangle = 1$ and Eq. (3) holds for $t_{wait} = 0$. 

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Now we are going prove Eq. (2b) in case (b). Clearly the validity of Eq. (2b) for $\rho \leq \rho_b$ follows directly from Proposition 1. And finally $(V)$ for $\rho_b < \rho < 1$ has just been proven in Proposition 2.

Lastly, the validity of Eq. (2c) in case (c) when $\rho \leq \rho_{c1}$ or $\rho > \rho_{c3}$ follows directly from Propositions 1 and 2. And if $\rho_{c1} < \rho \leq \rho_{c2}$, Proposition 2 tells us that car density (averaged over $(t_{\text{wait}}+2)$ timesteps) between any two successive tollbooths in a recurrent state is either equal to $\rho_{c1}$ or $\rho_{c2}$. Thus, with a steady and continual supply of cars from behind, the car density averaged over $(t_{\text{wait}}+2)$ timesteps in every interval between two successive tollbooths is a constant. Since a car is released once every $(t_{\text{wait}}+2)$ timesteps from the tollbooth in an interval with car density $\rho_{c2}$, hence $(V) = 1/\rho(t_{\text{wait}}+2)$ in this car density range. Finally, when $\rho_{c2} < \rho < \rho_{c3}$, Proposition 2 implies that cars pass through every tollbooth once every $(t_{\text{wait}}+2)$ timesteps. Hence, Eq. (2c) holds in this density range as well. □

In summary, we have investigated the behavior of a single lane deterministic highway traffic model in the presence of tollbooths. Our models are exactly solvable and the average car speed consists of a high speed, a partially jamming and a trivial completely jamming phases. The transition from the partially jamming phase to the completely jamming phase is first order in nature while all other transitions are second order.

Most importantly, our model suggests that the average car speed at high car density depends only on the car density and is independent of the detail arrangement of tollbooths in single lane traffic. While the regular placement of tollbooths and the deterministic traffic rules give rise to the unrealistically flat $(V)$ when car density lies between $\rho_{c2}$ and $\rho_{c3}$, the general observation that for $t_{\text{wait}} > 0$, the average car speed $\langle V \rangle$ is approximately inversely proportional to $t_{\text{wait}}$ is robust. Let us compare our results with the two-dimensional CS city traffic model. In the CS model, the time duration of red or green lights plays an analogous role of the tollbooths. As shown in their density versus flux per street curve in Fig 4 of Ref. [1], a linear region corresponding to cars moving with almost full speed is observed when the car density $\rho$ is low. More interestingly, a plateau region corresponding to the $\langle V \rangle \sim 1/\rho$ is observed when the time duration of red or green lights is large and $\rho$ is around 0.1 to 0.4. And $(V)$ starts to decrease at higher values of $\rho$ probably due to the effects of two-dimensional car blocking. Comparing the observations in the CS and our models, we believe that the $1/\rho$ behavior in high car density highway traffic is robust.

ACKNOWLEDGMENTS

H.F.C. and H.X. are supported in part by the Hong Kong SAR Government RGC Grant HKU 7098/00P and H.F.C. is also supported in part by the Outstanding Young Researcher Award of the University of Hong Kong.

[1] O. Biham, A. A. Middleton, D. Levine, Phys. Rev. A 46, R6124 (1992).
[2] K. Y. Wan, “Biham-Middleton-Levine Traffic Model in Different Spatial Dimensions”, M.Phil. Thesis, University of Hong Kong (1998).
[3] H. F. Chau, P. M. Hui, Y. F. Woo, J. Phys. Soc. Jpn. 64, 3570 (1995); H. F. Chau, K. Y. Wan, K. K. Yan, Physica A 254, 117 (1998).
[4] J. A. Cuesta, F. C. Martínez, J. M. Molera, A. Sánchez, Phys. Rev. E 48, R4175 (1993); J. M. Molera, F. C. Martínez, J. A. Cuesta, Phys. Rev. E 51, 175 (1995); K. Nagel, M. Paczuski, Phys. Rev. E 51, 2009 (1995); M. Schreckenberg, A. Schadschneider, K. Nagel, N. Ito, Phys. Rev. E 51, 2939 (1995); H. Emmerich, E. Rank, Physica A 216, 435 (1995); T. Nagatani, J. Phys. Soc. Jpn. 64, 4504 (1995); S.-I. Tadaki, Phys. Rev. E 54, 2409 (1996); P. M. Simon, H. A. Gutwitz, Phys. Rev. E 57, 2441 (1998); D. Chowdhury, L. Santen, A. Schadschneider, Phys. Reports 329, 199 (2000).
[5] M. Rickert, K. Nagel, Int. J. Mod. Phys. C 8, 483 (1997); J. Esser, M. Schreckenberg, Int. J. Mod. Phys. C 8, 1025 (1997); D. Helbing, B. A. Huberman, Nature 396, 738 (1998).
[6] K. H. Cheng, P. M. Hui, G. Q. Gu, Phys. Rev. E 51, 772 (1995).
[7] M. Rickert, K. Nagel, M. Schreckenberg, A. Latour, Physica A 231, 534 (1996); D. Chowdhury, D. E. Wolf, M. Schreckenberg, Physica A 235, 417 (1997); H. Emmerich, E. Rank, Physica A 234, 676 (1997); P. Wagner, K. Nagel, D. E. Wolf, Physica A 234, 687 (1997); A. Awazu, J. Phys. Soc. Jpn. 67, L1071 (1998); K. Nagel, D. E. Wolf, P. Wagner, P. Simon, Phys. Rev. E 58, 1425 (1998).
[8] H. F. Chau, K. Y. Wan, Phys. Rev. E 60, 5301 (1999).
[9] K. Nagel, M. Schreckenberg, J. Phys. I France 2, 2221 (1992); A. Schadschneider, Eur. Phys. J. B 10, 573 (1999).
[10] M. Fukui, Y. Ishibashi, J. Phys. Soc. Jpn. 65, 1868 (1996).
[11] D. Chowdhury, A. Schadschneider, Phys. Rev. E 59, R1311 (1999).
[12] S. Yukawa, M. Kikuchi, S.-I. Tadaki, J. Phys. Soc. Jpn. 63, 3609 (1994); K. H. Chung, P. M. Hui, J. Phys. Soc. Jpn. 63, 4338 (1994); Z. Csahók, T. Vicsek, J. Phys. A (Lett) 27, L591 (1994).
[13] J. Torok, J. Kertesz, Physica A 231, 515 (1996).