Generalized Parameter Estimation-based Observers: Application to Power Systems and Chemical-Biological Reactors

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Abstract

In this paper we propose a new state observer design technique for nonlinear systems. It consists of an extension of the recently introduced parameter estimation-based observer, which is applicable for systems verifying a particular algebraic constraint. In contrast to the previous observer, the new one avoids the need of implementing an open loop integration that may stymie its practical application. We give two versions of this observer, one that ensures asymptotic convergence and the second one that achieves convergence in finite time. In both cases, the required excitation conditions are strictly weaker than the classical persistent of excitation assumption. It is shown that the proposed technique is applicable to the practically important examples of multimachine power systems and chemical-biological reactors.

Key words: Estimation parameters, nonlinear systems, observers, time-invariant systems, power systems.

1 Problem Formulation

In this paper we are interested in the design of state observers for nonlinear control systems whose dynamics is described by

\[ \begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u),
\end{align*} \]

where \( x \in \mathbb{R}^n \) is the system state, \( u \in \mathbb{R}^m \) is the control signal, and \( y \in \mathbb{R}^p \) are the measurable output signals.

The problem is to design a dynamical system

\[ \begin{align*}
\dot{\xi} &= F(y, \xi, u) \\
\dot{x} &= H(y, \xi, u)
\end{align*} \]

with \( \xi \in \mathbb{R}^{n_c} \), such that

\[ \lim_{{t \to \infty}} |\hat{x}(t) - x(t)| = 0, \]

where \( |\cdot| \) is the Euclidean norm. We are also interested in the case when the observer ensures finite-time convergence (FTC), that is, when there exists \( t_c \in [0, \infty) \) such that

\[ \hat{x}(t) = x(t), \quad \forall t \geq t_c. \]

Following standard practice in observer theory we assume that \( u \) is such that the state trajectories of (1) are bounded.

Since the publication of the seminal paper \[14\], which dealt with linear time-invariant (LTI) systems, this prob-
lem has been extensively studied in the control literature. We refer the reader to [3,5,7] for a review of the literature. In this paper we propose an extension of the parameter estimation-based observer (PEBO) design technique reported in [18]. The main novelty of PEBO is that it translates the task of state observation into an on-line parameter estimation problem.

The main features of the new observer design technique proposed in the paper, called generalized PEBO (GPEBO), are the following.

(F1) The key “linearizability” condition of the original PEBO [18, Assumption 1] is significantly relaxed.

(F2) We identify a class of systems for which the second key condition of PEBO [18, Assumption 2]—which relates with the, far from obvious, solution of the parameter estimation problem—is obviated. The class is identified via a particular algebraic constraint.

(F3) It avoids the need of open-loop integration which stymies the practical application of this observer for systems subject to high noise environments—see [18, Remark R5].

(F4) Using the dynamic regressor extension and mixing (DREM) procedure [2], which is a novel, powerful, parameter estimation technique, we propose a variation of GPEBO achieving FTC, that is, for which (4) holds, under the weakest sufficient excitation assumption [11].

(F5) It is proven that both conditions are satisfied by the practically important case of multimachine power systems, while the first one is verified by chemical-biological reactors.

For the multimachine power systems we consider the classical three-dimensional “flux-decay” model of a large-scale power system [13,25], consisting of N generators interconnected through a transmission network, which we assume to be lossy, that is, we explicitly take into account the presence of transfer conductances. We prove that, using the measurements of active and reactive power and rotor angle at each generator—a reasonable assumption given the current technology [12,25]—the application of GPEBO allows us to recover the full state of the system, even in the presence of lossy lines. To the best of the authors’ knowledge, this is the first globally convergent solution to the problem.

For the reaction problem we consider the classical dynamical model of the concentration components, e.g., equation (1.43) in [4, Section 1.5], which describes the behavior of a large class of chemical and bio-chemical reaction systems. We propose a state observer whose convergence rate is faster than the standard asymptotic observers [4,8]. Similarly to the case of power systems, using DREM, we can ensure FTC for the particular case when the reaction rates are linear in the unmeasurable states.

The remainder of the paper is organized as follows. In Section 2 we give the main results. Section 3 is devoted to some discussion. Section 4 presents the application of the observer to an academic example and two practical problems. The paper is wrapped-up with concluding remarks in Section 5. The proofs of the main propositions are given in appendices at the end of the paper.

2 Main Results

The GPEBO designs are based on the following two propositions. For ease of presentation we consider the case where we are interested in observing all state variables. In many applications it is only necessary to reconstruct some of these state variables, a case that can be treated with slight modifications to these propositions. Also, we present first the version of GPEBO that ensures asymptotic convergence and then, in Proposition 3, the one ensuring FTC. The proofs of both propositions are given in the appendix.

2.1 An asymptotically convergent GPEBO

Proposition 1 Consider the system (1). Assume there exist mappings

\[
\phi : \mathbb{R}^n \to \mathbb{R}^n, \quad \phi^L : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n, \quad B : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n
\]

\[
\Lambda : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{n \times n}, \quad L : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^{n \times n},
\]

\[
C : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}^n
\]

satisfying the following.

(i) The GPEBO partial differential equation (PDE)

\[
\nabla \phi^T(x)f(x,u) = \Lambda(u,h(x))\phi(x) + B(u,h(x)),
\]

(ii) \(\phi^L\) is a “left inverse” of \(\phi\), in the sense that it satisfies

\[
\phi^L(\phi(x),h(x)) = x.
\]

(iii) The algebraic constraint

\[
L(u,h(x))\phi(x) = C(u,h(x))
\]

is satisfied.

(iv) For the given \(u\), all solutions of the LTV system

\[
\dot{z} = \Lambda(u(t),y(t))z,
\]

with \(y\) generated by (1), are bounded.

\[\text{See [20] for an FTC version of DREM, [21] for an interpretation as a Luenberger observer and [17,22] for two recent applications of DREM+PEBO techniques.}\]
The GPEBO dynamics
\[
\dot{\xi} = \Lambda(u, y)\xi + B(u, y) \quad (8)
\]
\[
\dot{\Phi} = \Lambda(u, y)\Phi, \; \Phi(0) = I_n \quad (9)
\]
\[
\dot{Y} = -\lambda Y + \lambda\Psi^T[C(u, y) - L(u, y)\xi] \quad (10)
\]
\[
\dot{\Omega} = -\lambda\Omega + \lambda\Phi\Phi^T \quad (11)
\]
\[
\dot{\theta} = -\gamma (\Delta \dot{\theta} - \gamma), \quad (12)
\]
with \(\lambda > 0\) and \(\gamma > 0\), with the definitions
\[
\Psi := L(u, y)\Phi \quad (13)
\]
\[
\mathcal{Y} := \text{adj}\{\Omega\}Y \quad (14)
\]
\[
\Delta := \text{det}\{\Omega\}, \quad (15)
\]
the state estimate
\[
\dot{x} = \phi^h(\xi + \Phi \dot{\theta}, y), \quad (16)
\]
ensures (3) with all signals bounded provided
\[
\Delta \notin \mathcal{L}_2. \quad (17)
\]

2.2 An GPEBO with FTC

A variation of GPEBO that ensures FTC is given in Proposition 3. To streamline its presentation we need the following sufficient excitation condition [11].

Assumption 2 Fix a small constant \(\mu \in (0, 1)\). There exists a time \(t_c > 0\) such that
\[
\int_0^{t_c} \Delta^2(\tau)d\tau \geq -\frac{1}{\gamma} \ln(1 - \mu). \quad (18)
\]

Proposition 3 Consider the system (1), verifying the conditions (i)-(iii) of Proposition 1. Fix \(\gamma > 0\) and \(\mu \in (0, 1)\). The state observer defined by (8)-(12) and the state estimate
\[
\dot{x} = \phi^h\left(\xi + \Phi \frac{1}{1 - w_c}[\dot{\theta} - w_c\hat{\theta}(0)], y\right), \quad (19)
\]
with
\[
\dot{w} = -\gamma \Delta^2w, \; w(0) = 1, \quad (20)
\]
and \(w_c\) defined via the clipping function
\[
w_c = \begin{cases} 
w & \text{if } w < 1 - \mu \\ 1 - \mu & \text{if } w \geq 1 - \mu, \end{cases}
\]
ensures (4) with all signals bounded provided \(\Delta\) verifies Assumption 2.

3 Discussion

D1 The GPEBO PDE (5) is a generalization of the PDEs that are imposed in the Kazantzis-Kravaris-Luenberger observer (KKLO), first presented in [9] as an extension to nonlinear systems of Luenberger’s observer, and further developed in [1]. Indeed, in KKLO the mapping \(\Lambda(u, y)\) is a constant, Hurwitz matrix—see [6] for a recent extension to the non-autonomous case where the mapping \(\phi\) depends on time (or the systems input). It also generalizes the PDE required in PEBO where \(\Lambda\) is equal to zero.

D2 As discussed in [18] a drawback of the original PEBO is that it involves an open-loop integration, namely
\[
\dot{\xi} = B(u, y),
\]
which stymies the practical application of PEBO in the presence of noise—see [18, Remark R5]. Due to the presence of \(\Lambda\) in the dynamics of \(\xi\) given in (8), this difficulty is conspicuous by its absence in GPEBO. It should be pointed out that, using an alternative technique that relies on the Swapping Lemma [24, Lemma 3.6.5], this shortcoming of PEBO has been overcome in [23] for a class of electromechanical systems.

D3 It is interesting to compare the KKLO with PEBO from the geometric viewpoint. The former generates an attractive and invariant manifold
\[
\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi = \phi(x)\},
\]
and the state is reconstructed, via \(\phi^h\), with \(\xi\). On the other hand, PEBO generates an invariant foliation
\[
\{(x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \mid \xi = \phi(x) + \theta, \theta \in \mathbb{R}^n\}.
\]
To reconstruct the state—again via \(\phi^h\)—it is necessary to identify the leaf via the estimation of \(\theta\). See Fig. 1. See also [26] where it is proposed to combine PEBO and KKLO to extend the realm of application of these observers.

![Fig. 1. Geometric interpretation of KKLO and PEBO](attachment:image.png)

D4 Imposing the algebraic constraint (ii) of Proposition 1 is, clearly, a strong assumption. It is interesting that—as shown in Section 4—it is satisfied for the, practically relevant, power systems example. See also [23] where similar constraints are shown to be satisfied by a class of electromechanical systems.
Indeed, it is easy to check that the mappings $\phi = x_2$, $\Lambda = 0$ and $B = -y$ verify conditions (i) and (ii) of Proposition 1 for the system (21). Hence, following the procedure of Proposition 1, we define

$$\dot{\xi} = -y.$$  (22)

**Proposition 4** Consider the system (21). Define the state estimate

$$\dot{x}_2 = \xi + \hat{\theta},$$

with (22) and the scalar parameter estimator

$$\hat{\theta} = -\gamma \Delta (\Delta \hat{\theta} - \gamma)$$

where we define

$$\gamma = \frac{\lambda}{p + \lambda},$$

$$\Delta = \det \left[ \phi \right],$$

$$\phi = \Lambda - \gamma L^\top \left[ B^\top C^\top(u, y) - L(u, y) \right].$$

where $p := \frac{d}{dt}$ and $u_{-1}(t)$ is a step signal. Then, (3) holds provided (17) is verified.

**PROOF.** Clearly $x_2 = \xi + \theta$, where $\theta := x_2(0) - \xi(0)$. Replacing in (21), and developing the cubic power yields

$$\dot{x}_1 = \xi^3 + 3\xi^2 \theta + 3\xi \theta^2 + \theta^3.$$  

Applying the filter $\frac{\lambda}{p + \lambda}$, and using the definitions of $Y$ and $\phi$ above yields

$$Y = \phi^\top \Theta, \Theta := \begin{bmatrix} \theta \\ \theta^2 \\ \theta^3 \end{bmatrix}.$$ 

Multiplying the equation above by $\phi$, applying again the filter $\frac{\lambda}{p + \lambda}$, and multiplying by $\text{adj} \left[ \Omega \right]$ yields $Y = \Delta \Theta$. The proof is completed replacing $Y_1 = \Delta \theta$ in the parameter estimator to get the error equation (A.3).

**4 Applications**

In this section we illustrate with an academic example and two physical systems the applicability of the proposed GPEBO. Towards this end, we identify all the mappings required to verify some (or all) of the conditions of Proposition 1.

**4.1 An academic example**

In [6] the problem of state observation of the following system is considered

$$\begin{aligned}
\dot{x}_1 &= x_2^3 \\
\dot{x}_2 &= -x_1 \\
y &= x_1.
\end{aligned}$$  \hspace{1cm} (21)

**4.1.1 Solution via PEBO+DREM**

The proposition below shows that this problem can be trivially—and robustly—solved using PEBO+DREM.
choice of the observer ICs, the matrix Ω is rank deficient. Also, as expected, the rate of convergence is improved increasing the adaptation gain γ. These transients should be compared with the ones shown in Fig. 1 of [6], which are generated with a far more complicated KKLO.

4.2 Multimachine power systems

The dynamical model of the i-th generator of n interconnected machines can be described using the classical third order model \(^3\) [13,25]

\[
\begin{align*}
\dot{\delta}_i &= \omega_i \\
M_i \dot{\omega}_i &= -D_{mi} \omega_i + \omega_0 (P_{mi} - P_{ei}) \\
\tau_i \dot{E}_i &= -E_i - (x_{di} - x'_{di}) I_{di} + E_{fi} + \nu_i,
\end{align*}
\]

where the state variables are the rotor angle \(\delta_i \in \mathbb{R}\), rad, the speed deviation \(\omega_i \in \mathbb{R}\) in rad/sec and the generator quadrature internal voltage \(E_i \in \mathbb{R}_+\). \(I_{di}\) is the d axis current, \(P_{ei}\) is the electromagnetic power, the voltages \(E_{fi}\) and \(\nu_i\) are the constant voltage component applied to the field winding, and the control voltage input, respectively. \(D_{mi}, M_i, P_{mi}, \tau_i, \omega_0, x_{di}\) and \(x'_{di}\) are positive parameters.

The active power \(P_{ei}\) and reactive power \(Q_{ei}\) are defined as

\[
\begin{align*}
P_{ei} &= E_i I_{qi} \\
Q_{ei} &= E_i I_{di},
\end{align*}
\]

where \(I_{qi}\) is the q axis current.

These currents establish the connections between the machines and are given by

\[
\begin{align*}
I_{qi} &= G_{mi} E_i + \sum_{j=1, j \neq i}^{n} E_j Y_{ij} \sin(\delta_{ij} + \alpha_{ij}) \\
I_{di} &= -B_{mi} E_i - \sum_{j=1, j \neq i}^{n} E_j Y_{ij} \cos(\delta_{ij} + \alpha_{ij}),
\end{align*}
\]

where we defined \(\delta_{ij} := \delta_i - \delta_j\) and the constants \(Y_{ij} = Y_{ji}\) and \(\alpha_{ij} = \alpha_{ji}\) are the admittance magnitude and admittance angle of the power line connecting nodes \(i\) and \(j\), respectively. Furthermore, \(G_{mi}\) is the shunt conductance and \(B_{mi}\) the shunt susceptance at node \(i\). Finally, combining (23), (24) and (25) results in the well-known compact form

\[
\begin{align*}
\dot{\delta}_i &= \omega_i \\
\dot{\omega}_i &= -D \omega_i + P_i - d_i \left[ G_{mi} E_i^2 \\
&\quad - E_i \sum_{j=1, j \neq i}^{n} E_j Y_{ij} \sin(\delta_{ij} + \alpha_{ij}) \right] \\
\dot{E}_i &= -a_i E_i + b_i \sum_{j=1, j \neq i}^{n} E_j Y_{ij} \cos(\delta_{ij} + \alpha_{ij}) + u_i,
\end{align*}
\]

where we have defined the signal

\[
u_i := \frac{1}{\tau_i} (E_{fi} + \nu_i)
\]

and the positive constants

\[
\begin{align*}
D_i := \frac{D_{mi}}{M_i}, \quad P_i := d_i P_{mi}, \quad d_i := \frac{\omega_0}{M_i} \\
a_i := \frac{1}{\tau_i} [1 - (x_{di} - x'_{di}) B_{mi}], \quad b_i := \frac{1}{\tau_i} (x_{di} - x'_{di}).
\end{align*}
\]

To formulate the observer problem we consider that all parameters are known, and make the following assumption on the available measurements.

**Assumption 5** The signals \(u_i, \delta_i, P_{ei}\) and \(Q_{ei}\) of all generating units are measurable.

4.2.1 Verifying the conditions of Proposition 1

We make the following observation. Using (24) and (25), the rotor speed dynamics (26) may be written as

\[
\dot{\omega}_i = -D_\omega \omega_i + P_i - d_i P_{ei}.
\]

Considering that \(P_{ei}\) is measurable, while \(P_i, D_i\) and \(d_i\) are known positive constants, the design of an observer
for this system is trivial. For instance,
\[
\begin{align*}
\dot{\xi}_i &= -D_i \dot{\omega}_i + P_i - d_i P_{ei} - k_{wi} \dot{\omega}_i \\
\dot{\omega}_i &= \xi_{wi} + k_{wi} \delta_i, \quad k_{wi} > 0,
\end{align*}
\]
yields the LTI, asymptotically stable error dynamics
\[
\dot{\hat{\omega}}_i = -(D_i + k_{wi}) \hat{\omega}_i.
\]
Therefore, we concentrate in the estimation of the voltages \(E_i\). Its dynamics may be written as
\[
\dot{E} = \Lambda(\delta) E + u, \quad (27)
\]
where \(E := \text{col}(E_1, \ldots, E_n)\), \(\delta := \text{col}(\delta_1, \ldots, \delta_n)\), and we defined matrix
\[
\Lambda(\delta) := (\Lambda_1(\delta) \Lambda_2(\delta) \ldots \Lambda_n(\delta)),
\]
where
\[
\Lambda_1(\delta) := \begin{bmatrix}
-a_1 \\ b_2 Y_{21} \cos(\delta_{12} + \alpha_{12}) \\ b_n Y_{n1} \cos(\delta_{n1} + \alpha_{n1})
\end{bmatrix},
\]
\[
\Lambda_2(\delta) := \begin{bmatrix}
-b_1 Y_{12} \cos(\delta_{12} + \alpha_{12}) \\ -a_2 \\ b_n Y_{n2} \cos(\delta_{n2} + \alpha_{n2})
\end{bmatrix},
\]
\[
\Lambda_n(\delta) := \begin{bmatrix}
-b_1 Y_{1n} \cos(\delta_{1n} + \alpha_{1n}) \\ b_2 Y_{2n} \cos(\delta_{2n} + \alpha_{2n}) \\ -a_n
\end{bmatrix}
\]
and we recall that \(\delta\) is measurable. The remaining mappings of (i) and (ii) of Proposition 1 are given as \(\phi = E\) and \(B = u\). To define the mappings \(L\) and \(C\) of (7) we state the following simple lemma.

**Lemma 6** There exists a measurable matrix \(L(P_c, Q_c, \delta) \in \mathbb{R}^{n \times n}\) such that
\[
LE = 0
\]

**PROOF.** From (24) we have that
\[
P_c I_d - Q_c I_q = 0.
\]
Clearly, the equations (25)—which are linearly dependent on \(E\)—may be written in the compact form
\[
I_q = S(\delta) E, \quad I_d = T(\delta) E,
\]
for some suitably defined \(n \times n\) matrices \(S(\delta), T(\delta)\). The proof is completed by replacing (30) in the identity above and defining
\[
L(P_c, Q_c, \delta) := \begin{bmatrix}
P_{c1} T_1^T(\delta) - Q_{c1} S_1^T(\delta) \\ \vdots \\ P_{cn} T_n^T(\delta) - Q_{cn} S_n^T(\delta)
\end{bmatrix},
\]
where \(T_1^T(\delta), S_1^T(\delta)\) are the rows of the matrices \(T(\delta)\) and \(S(\delta)\), respectively.

Equipped with this lemma and selecting the mapping \(C = 0\) completes the verification of all the conditions of Proposition 1.

### 4.2.2 Simulations

For simulation we use the two-machine system considered in [19]. The dynamics of the system result in the sixth-order model
\[
\begin{align*}
\dot{\delta}_1 &= \omega_1, \\
\dot{\omega}_1 &= -D_1 \omega_1 + P_1 - G_11 E_1^2 - Y_{12} E_1 E_2 \sin(\delta_{12} + \alpha_{12}) \\
\dot{E}_1 &= -a_1 E_1 + b_1 E_2 \cos(\delta_{12} + \alpha_{12}) + E_{f1} + \nu_1; \\
\dot{\delta}_2 &= \omega_2, \\
\dot{\omega}_2 &= -D_2 \omega_2 + P_2 - G_{22} E_2^2 + Y_{21} E_1 E_2 \sin(\delta_{12} + \alpha_{12}) \\
\dot{E}_2 &= -a_2 E_2 + b_2 E_1 \cos(\delta_{21} + \alpha_{21}) + E_{f2} + \nu_2.
\end{align*}
\]
with the current equations defined as
\[
\begin{align*}
I_{q1} &= G_{11} E_1 + E_2 Y_{12} \sin(\delta_{12} + \alpha_{12}) \\
I_{d1} &= -B_{11} E_1 - E_2 Y_{12} \cos(\delta_{12} + \alpha_{12}) \\
I_{q2} &= G_{22} E_2 + E_1 Y_{21} \sin(\delta_{21} + \alpha_{21}) \\
I_{d2} &= -B_{22} E_2 - E_1 Y_{21} \cos(\delta_{21} + \alpha_{21}).
\end{align*}
\]
In this case we have that
\[
A(t) = \begin{bmatrix}
-a_1 & b_1 \cos(\delta_{12}(t) + \alpha_{12}) \\ b_2 \cos(\delta_{21}(t) + \alpha_{21}) & -a_2
\end{bmatrix}
\]
\[
S(\delta) = \begin{bmatrix}
G_{11} & Y_{12} \sin(\delta_{12} + \alpha_{12}) \\ Y_{21} \sin(\delta_{21} + \alpha_{21}) & G_{22}
\end{bmatrix}
\]
\[
T(\delta) = \begin{bmatrix}
-B_{11} & -Y_{12} \cos(\delta_{12} + \alpha_{12}) \\ -Y_{21} \cos(\delta_{21} + \alpha_{21}) & -B_{22}
\end{bmatrix}
\]
For the observer design we selected the simplest filter
\[
F(p) = \begin{bmatrix}
1 & 0 \\ \frac{k}{p + \gamma} & 0
\end{bmatrix},
\]
with \(k > 0\). The parameters of the model (31) are taken from [19] and are given in Table 1.
Table 1
System parameters

| Parameter | Initial values | After load change |
|-----------|----------------|-------------------|
| Y_{12}   | 0.1032         | 0.1032            |
| Y_{21}   | 0.1032         | 0.1032            |
| b_1      | 0.0223         | 0.02236           |
| b_2      | 0.0265         | 0.0265            |
| D_1      | 1              | 1                 |
| D_2      | 0.2            | 0.2               |
| ν_1      | 1              | 1                 |
| ν_2      | 1              | 1                 |
| B_{11}   | -0.4373        | -0.5685           |
| B_{22}   | -0.4294        | -0.5582           |
| G_{11}   | 0.0966         | 0.1256            |
| G_{22}   | 0.0926         | 0.1204            |
| a_1      | 0.2614         | 0.2898            |
| a_2      | 0.2532         | 0.2864            |
| P_1      | 28.22          | 28.22             |
| P_2      | 28.22          | 28.22             |
| E_{f1}   | 0.2405         | 0.2405            |
| E_{f2}   | 0.2405         | 0.2405            |

Fig. 3. Transients of the first voltage observation error $E_1 - \hat{E}_1$ for DREM and FTC observers with a 30% load change $t = 10$ sec

Fig. 4. Transients of the second voltage observation error $E_2 - \hat{E}_2$ for DREM and FTC observers with a 30% load change at $t = 10$ sec

Fig. 5. Transients of the first speed observation error $\omega_1 - \hat{\omega}_1$

Fig. 6. Transients of the second speed observation error $\omega_2 - \hat{\omega}_2$

To simplify the notation we partition the vector $c$ as $c = \text{col}(y, x)$, and rewrite (36) as

$$\dot{y} = -uy + K_y r(y, x) + \chi_y$$
$$\dot{x} = -ux + K_x r(y, x) + \chi_x.$$  
(37)

To simplify the presentation we assume that there are more measurements than reaction rates, that is, $p \geq q$ and rank $\{K_y\} = q$. \(^4\)

\(^4\) See [17] for a relaxation of this assumption.

4.3 Chemical-biological reactors

We consider reaction systems whose dynamical model is given by [4, Section 1.5]

$$\dot{c} = -uc + K r(c) + \chi$$
$$y = \left[ I_p : 0_{p \times d} \right] c,$$  
(36)

with $c \in \mathbb{R}^n, \chi \in \mathbb{R}^n, u \in \mathbb{R}^p, y \in \mathbb{R}^p, r : \mathbb{R}^n \rightarrow \mathbb{R}^p, d := n - p, q < n$. It is assumed that $y, u, \chi$ and $K$ are known.
4.3.1 Solution via GPEBO

The following lemma identifies the mappings $\phi$, $\Lambda$ and $B$ required to satisfy conditions (i) and (ii) of Proposition 1.

**Lemma 7** Consider the system (37). The mappings

\[
\phi := x - K_x K_y^\dagger y \\
\Lambda := -u \\
B := -K_x K_y^\dagger \chi_y
\]

where

\[
K_y^\dagger := (K_y^\top K_y)^{-1} K_y^\top
\]

satisfy the PDE (5). More precisely,

\[
\dot{\phi} = \Lambda \phi + B.
\]

**PROOF.** From (37) and (38) we get

\[
\phi = -u \phi + \lambda \phi = \phi + \lambda \phi = \phi = \xi + \Phi \theta,
\]

for some $\theta \in \mathbb{R}^d$. To obtain a bona fide regressor equation, that is a linear relation between measurable signals and $\theta$ we would assume condition (iii) of Proposition 1. That is, assume the existence of measurable mappings $C$ and $L$ such that (7) holds, that is $L \phi = C$. Unfortunately, in this example it is not possible to satisfy this condition. However, we can still obtain the required linear regression, needed for the parameter estimation using DREM, as shown in the lemma below.

**Lemma 8** Assume that the rate vector $r(y, x)$ depends *linearly* on the unmeasurable components of the state $x$, that is, it is of the form

\[
r(y, x) = R(y) x
\]

where $R : \mathbb{R}^p \to \mathbb{R}^{q \times d}$ is a known matrix.\(^5\) There exists measurable signals $\mathcal{Y} \in \mathbb{R}^d$ and $\Delta \in \mathbb{R}$ such that

\[
\mathcal{Y} = \Delta \theta.
\]

**PROOF.** Defining the partial coordinate $y^\dagger = K_y^\dagger y$, we see from (37) that its dynamics takes the form

\[
\dot{y}^\dagger = -u y^\dagger + R(y) x + K_y^\dagger \lambda y
\]

where we used (38) to get the second identity, (40) in the third identity and we defined the measurable signals

\[
\lambda := -u y^\dagger + K_y^\dagger \lambda y + R(y) (\xi + \Phi \theta + K_x y^\dagger)
\]

This completes the proof.

4.3.2 Simulations

To illustrate the performance of the PEBO+DREM observer proposed in the previous section we consider the model of the anaerobic digestion reactor reported in [15]. The dynamics, given in equations (55)-(59) of [15], maybe written in the form (37), (41) with the choices $n = 4, q = 2, p = 2$

\[
K_y = \begin{bmatrix} -k_3 & 0 \\ k_4 & -k_1 \end{bmatrix}, \quad K_x = I_2
\]

\[
R(y) = \begin{bmatrix} \mu_1(y_1) & 0 \\ 0 & \mu_2(y_2) \end{bmatrix}, \quad \xi_y = \begin{bmatrix} u s_{1,0} \\ u s_{2,0} \end{bmatrix}, \quad \xi_x = 0,
\]

\(^5\) As usual in adaptive control, we neglect an additive exponentially decaying term in (44) that is due to the filters initial conditions.

\(^6\) See [17] for the case of nonlinear dependence on $x$.\]
where \( y_1, x_1, y_2 \) and \( x_2 \) represent the organic matter concentration (g/l), the acidogenic bacteria concentration (g/l), the volatile fatty acid concentration (mmol), the methanogenic bacteria concentration (g/l) and \( u \) is the dilution rate. The positive constants \( s_{1.0} \) and \( s_{2.0} \) denote the concentration of the substrate in the feed, and \( k_1, k_3 \) and \( k_4 \) are yield positive coefficients.

The two specific growth rates \( \mu_1 \) and \( \mu_2 \) are given by

\[
\begin{bmatrix}
\mu_1(y_1) \\
\mu_2(y_2)
\end{bmatrix} =
\begin{bmatrix}
\frac{\mu_{m,1} + y_1}{K_{S,1} + y_1 + y_2 + K_I y_2^2} \\
\frac{\mu_{m,2} + y_2}{K_{S,2} + y_2 + K_I y_2^2}
\end{bmatrix},
\]

where \( \mu_{m,1}, \mu_{m,2}, K_{S,1}, K_{S,2} \) and \( K_I \) are yield positive coefficients.

Notice that \( K_y \) is square and full rank, consequently

\[
y^\dagger = K_y^{-1} y = \begin{bmatrix} \frac{y_1}{k_1} \\
\frac{y_2}{k_2} + \frac{k_4 y_1}{k_1 k_3}
\end{bmatrix}.
\]

To design the observer we first identify the signals (38) of Lemma 7 as

\[
\Lambda = -u \\
B = -K_y^{-1} \chi_y = -u \begin{bmatrix} -s_{1.0} & \frac{s_{1.0}}{k_3} \\
-k_{2.0} & \frac{k_4 s_{1.0}}{k_1 k_3}
\end{bmatrix}.
\]

Consequently, (8) and (9) become

\[
\dot{\xi} = -u \xi + u \begin{bmatrix} s_{1.0} & \frac{s_{1.0}}{k_3} \\
-k_{2.0} & \frac{k_4 s_{1.0}}{k_1 k_3}
\end{bmatrix} \\
\Phi = -u \Phi, \Phi(0) = I_n.
\]

Then, we follow the proof of Lemma 8 to construct the signals

\[
\chi = u \begin{bmatrix} \frac{y_1}{k_1} \\
\frac{y_2}{k_2} + \frac{k_4 y_1}{k_1 k_3}
\end{bmatrix} - u \begin{bmatrix} s_{1.0} & \frac{s_{1.0}}{k_3} \\
-k_{2.0} & \frac{k_4 s_{1.0}}{k_1 k_3}
\end{bmatrix} \\
+ \begin{bmatrix} \mu_1(y_1) & 0 \\
\mu_2(y_2) & 0
\end{bmatrix} \Phi_x.
\]

For the simulations we used the parameters of [15], that is, \( k_1 = 268 \text{ mmol/g, } k_3 = 42.14, k_4 = 116.5 \text{ mmol/g, } \alpha = 1, \mu_{m,1} = 1.2 \text{ d}^{-1}, K_{S,1} = 8.85 \text{ g/l, } \mu_{m,2} = 0.74 \text{ d}^{-1}, K_{S,2} = 23.2 \text{ mmol, } K_I = 0.0039 \text{ mmol}^{-1}, S_{1.0} = 1, S_{2.0} = 1 \text{ and } u = 0.1.

The initial conditions for the anaerobic digester were set to \( x_1(0) = 0.1 \text{ g/l, } y_1(0) = 0.05 \text{ g/l, } x_2(0) = 0.5 \text{ g and } y_2(0) = 4 \text{ mmol/l. We used the parameter } \lambda = 100 \text{ in the filters of (45). Fig. 7 and Fig. 8 show the transient behavior of the state estimation errors for different values of the adaptation gain. Notice that, although the convergence rate is increased with larger } \gamma, \text{ an undesirable peak appears in the first estimation error.}

\[
\begin{array}{c}
\text{Fig. 7. Transients of the error } x_1 - \hat{x}_1 \\
\text{Fig. 8. Transients of the error } x_2 - \hat{x}_2
\end{array}
\]

5 Concluding Remarks

An extension to the PEBO technique reported [18] has been proposed in the paper. It allows us to simplify the task of solving the key PDE and avoid a, sometimes problematic, open-loop integration required in PEBO. Also, we have identified a condition—verification of the algebraic equation (7)—that trivializes the task of estimating the unknown parameters. In the original version of PEBO this was left as an open problem to be solved.

7 In [15] there is a constant \( \alpha = 0.5 \) entering into the dynamics of \( x \) as \( \dot{x} = -\alpha u x + K_{r} x \). To avoid clattering the notation, and without loss of generality, we assume this constant is equal to one.
It is shown that this condition is satisfied for the practically important problem of power systems.

It has been shown that combining PEBO with DREM it is possible, on one hand, to relax the excitation conditions to ensure parameter convergence. On the other hand, it allows us to design an observer with FTC under extremely weak excitation assumptions.

As an additional example we show the application of PEBO+DREM to reaction systems. Notice that the use of DREM is necessary to solve the parameter estimation problem in this example. Although there are many ways to design an estimator from the linear regression (44), there exists a fundamental obstacle to ensure its convergence. Indeed, from the definition of $\Phi$, that is $\Phi = -u\Phi$ with $u(t) > 0$, we have that $\Psi(t) \to 0$, hence $\Psi(t) \to 0$—loosing identifiability of the parameter $\theta$. In particular the matrix $\Psi$ cannot satisfy the well-known persistency of excitation condition

$$\int_t^{t+\kappa} \Psi^T(s)\Psi(s)ds \geq \kappa \text{I}_d,$$

which is the necessary and sufficient condition for exponential convergence of the classical gradient and least-squares estimators [24, Theorem 2.5.1].

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A Proof of Proposition 1

From (5) we have that
\[ \dot{\phi} = \Lambda \phi + B. \]

Hence, defining the error signal
\[ e := \phi - \xi \]  \hspace{1cm} (A.1)

and taking into account the \( \xi \) dynamics of the observer, we obtain an LTV system \( \dot{e} = A(t)e \) where we defined \( A(t) := \Lambda(u(t), y(t)) \). Now, from the second equation in (12) we see that \( \Phi \) is the state transition matrix of the \( e \) system, which is bounded in view of condition (iv). Consequently, there exists a constant vector \( \theta \in \mathbb{R}^n \) such that
\[ e = \Phi \theta, \]

namely \( \theta = e(0) \). We now have the following chain of implications
\[ e = \Phi \theta \Leftrightarrow \phi = \xi + \Phi \theta \hspace{1cm} (\Leftrightarrow (A.1)) \]
\[ \Rightarrow L \phi = L \xi + L \Phi \theta \hspace{1cm} (\Leftrightarrow L \times) \]
\[ \Rightarrow C - L \xi = L \Phi \theta \hspace{1cm} (\Leftrightarrow (7)) \]
\[ \Leftrightarrow C - L \xi = \Psi \theta \hspace{1cm} (\Leftrightarrow (13)) \]
\[ \Rightarrow \Psi^\top (C - L \xi) = \Psi^\top \Psi \theta \hspace{1cm} (\Leftrightarrow \Psi^\top \times) \]
\[ \Rightarrow Y = \Omega \theta \hspace{1cm} \left( \Leftrightarrow \frac{\lambda}{P + \lambda} \right) \text{ and } (10), (11) \]
\[ \Rightarrow \Delta \theta = \gamma, \hspace{1cm} (\Leftrightarrow \text{adj}\{\Omega\} \times \text{ and } (14), (15)) \]

where we have used the fact that for any, possibly singular, \( n \times n \) matrix \( K \) we have \( \text{adj}\{K\}K = \det\{K\}I_n \) in the last line.

From \( \phi = \xi + \Phi \theta \) and (6) it is clear that, if \( \theta \) is known, we have that
\[ x = \phi^+(\xi + \Phi \theta, y). \]  \hspace{1cm} (A.2)

Hence, the remaining task is to generate an estimate for \( \theta \), denoted \( \hat{\theta} \), to obtain the observed state via \( \tilde{x} = \phi^+(\xi + \Phi \hat{\theta}, y) \). This is, precisely, generated with (12), whose error equation is of the form
\[ \dot{\hat{\theta}} = -\gamma \Delta^2 \hat{\theta}, \]  \hspace{1cm} (A.3)

where \( \tilde{\theta} := \hat{\theta} - \theta \). The solution of this equation is given by
\[ \tilde{\theta}(t) = e^{-\gamma \int_0^t \Delta^2(s)ds} \tilde{\theta}(0). \]  \hspace{1cm} (A.4)

Given the standing assumption on \( \Delta \) we have that \( \tilde{\theta}(t) \to 0 \). Hence, invoking (16) and (A.2) we conclude that \( \tilde{x}(t) \to 0 \), where \( \tilde{x} := \hat{x} - x \).

B Proof of Proposition 3

First, notice that the definition of \( w_c \) ensures that \( \hat{x} \), given in (19), is well-defined. Now, from (A.4) and the definition of \( w \) we have that
\[ \hat{\theta} = w \tilde{\theta}(0). \]

Clearly, this is equivalent to
\[ (1 - w) \theta = \hat{\theta} - w \tilde{\theta}(0). \]

On the other hand, under Assumption 2, we have that \( w_c(t) = w(t), \forall t \geq t_c \). Consequently, we conclude that
\[ \frac{1}{1 - w_c} [\hat{\theta} - w_c \tilde{\theta}(0)] = \theta, \forall t \geq t_c. \]

Replacing this identity in (19) completes the proof.