Miura Transformation between two Non-Linear Equations in $2 + 1$ dimensions

J.M. Cerveró and P.G. Estévez*

Area de Física Teórica
Facultad de Física
Universidad de Salamanca
37008 Salamanca. Spain

September 18, 2018

Abstract

A Dispersive Wave Equation in $2 + 1$ dimensions (2LDW) widely discussed by different authors is shown to be nothing but the modified version of the Generalized Dispersive Wave Equation (GLDW). Using Singularity Analysis and techniques based upon the Painlevé Property leading to the Double Singular Manifold Expansion we shall find the Miura Transformation which converts the 2LDW Equation into the GLDW Equation. Through this Miura Transformation we shall also present the Lax pair of the 2LDW Equation as well as some interesting reductions to several already known integrable systems in $1 + 1$ dimensions. As the 2LDW Equation arises from a Miura Transformation we propose that it should be treated conventionally as a Modified Equation. In this case, we propose its designation as The MGLDW Equation

PACS Numbers 02.30 and 03.40K

*e-mail: pilar@sonia.usal.es
I. Introduction

Recently the following system of non-linear partial differential equations in 2 + 1 dimensions has prompted a great deal of interest and several papers have appeared dealing with this system \[1\], \[2\]. It could also be considered as a Long Dispersive Wave Equation (2LDW) in the form:

\[
\begin{align*}
0 &= \lambda_0 q_t + q_{xx} - 2qV \\
0 &= \lambda_0 r_t - r_{xx} + 2rV \\
0 &= (qr)_x - V_y = \int_{-\infty}^{y} (qr)_x dy'
\end{align*}
\]  

(1.1)

- For \(\lambda = i\) and \(r = q^*\), it is identical to the equation proposed by Fokas \[3\] as the simplest integrable equation in 2 + 1 dimensions. In this reference, the author presents the Lax pair of this particular case of (1.1). It is also worth pointing out that when \(x = y\) the equation proposed by Fokas reduces to the Non-Linear Schrödinger Equation \[4\].

- The real version of (1.1) has been obtained by Chakravarty, Kent and Newman \[5\] as a particular reduction of the General Self-Dual Yang Mills Equations while the complex version of (1.1) has recently been discussed by Maccari \[6\] using the so-called Asymtotically Exact Reduction Method with the Kadomtsev-Petviashily Equation as a starting point.

- The Painlevé Property \[7\], \[8\] as well as the Bilinear Hirota Equations for the equation under consideration have been studied by Porsezian \[2\] for the complex version and by Radha and Lakhsmanan \[1\] for the real case. In this latter reference, a systematic use of the Bilinear Formalism is made in order to obtain soliton-like solutions of dromion type.

The aim of this paper is to show that equation (1.1) is nothing but the modified version of the Generalized Dispersive Wave Equation (GLDW) which has been thoroughly studied by Boiti, León and Pempinelli \[9\]. In order to show this, we shall be using singularity analysis of the GLDW that has already been succesfully applied to this equation by one of the present authors \[10\]. Also using techniques based upon Painlevé Analysis,
we shall find the Miura transformation which converts (1.1) into the GLDW. In view of these results, it appears reasonable to refer to equation (1.1) as the Modified Generalized Dispersive Wave Equation (MGDLW). Since the Lax pair of the GLDW equation is known [1], [9], the Miura transformation that we shall present in this paper can easily be used to find the Lax pair for equation (1.1). As a final point, we shall offer several interesting reductions of our equation to some integrable equations in $1 + 1$ dimensions.

II. The Miura Transformation between the GDLW and the MGDLW

In 1987 Boiti et al. [9] proposed a dispersive wave equation in $2 + 1$ dimensions

\begin{align*}
0 &= u_{ty} + (\eta_{xy} + 2uu_y)_x \\
0 &= \eta_{ty} + (u_{xy} + 2u\eta_y)_x,
\end{align*}

(2.1)

which we shall hereafter call the Generalized Dispersive Long Wave Equation (GDLW) since its reduction when $x = y$ is the dispersive wave equation also called the Kaup System [12] or Classical Boussinesq System [13]. Furthermore, the stationary version of (2.1) corresponds to the Sinh-Gordon Equation [9], [14].

- The Painlevé Property One of the present authors [10] has studied and discussed the system given by (2.1) from the point of view of Painlevé Analysis. As is well known [11] the Painlevé Property (PP), for practical purposes can be summarized in the statement that all solutions of equation (2.1) could be written in the form:

\begin{align*}
&u = \sum_{j=0}^{\infty} u_j \chi^{j-a} \\
&\eta = \sum_{j=0}^{\infty} \eta_j \chi^{j-b}
\end{align*}

(2.2)
where $\chi = 0$ is an arbitrary function called the Movable Singularities Manifold while $u_j$ y $\eta_j$ are analytic functions in an arbitrary neighbourhood of this manifold. Inserting (2.2) in (2.1), the dominant terms balance easily yields

$$a = b = 1$$

(2.3)

$$u_o = \pm \chi_x \quad \eta_o = \chi_x$$

(2.4)

where $\pm$ means that two Painlevé expansions are in principle possible. In reference [10] a detailed discussion is presented on how to deal with both expansions simultaneously.

• The Double Singular Manifold Method  In 1983 Weiss [8] proposed the Singular Manifold Method (SMM) which has been proved to be extremely successful in dealing with a large class of non-linear equations. The method is based on the restriction of the solutions of (2.1) for which expansion (2.2) becomes truncated at the constant level. In this case, one can prove that $\chi$ is no longer an arbitrary function since it must verify the truncation condition. Accordingly, henceforth it is called Singular Manifold. As system (2.1) presents two different Painlevé branches, the suggestion made in [10] is to use two singular manifolds $\phi$ and $\hat{\phi}$ (one for each expansion branch) in such a way that the truncation of (2.2) adopts the form:

$$u' = u + \frac{\phi x}{\phi} - \frac{\hat{\phi} x}{\hat{\phi}}$$

$$\eta' = \eta + \frac{\phi x}{\phi} + \frac{\hat{\phi} x}{\hat{\phi}}$$

(2.5)

• Superposition of Solutions:  Expansion (2.5) suggests the following change of dependent variables:

$$u = m - \hat{m}$$

$$\eta = m + \hat{m}$$

(2.6)

such that $m$ and $\hat{m}$ satisfy individual equations, each with just one branch of the Painlevé expansion. In order to find which equations $m$ y $\hat{m}$ should
Miura trans. for two equations in $2 + 1$

satisfy, we add and substract (2.1), and using (2.6) we finally obtain:

\[ 0 = m_{ty} + [m_{xy} + 2(m - \hat{m})m_y]_x \]
\[ 0 = \hat{m}_{ty} + [-\hat{m}_{xy} + 2(m - \hat{m})\hat{m}_y]_x \]  

(2.7)

Let us now define

\[ m_t = n_x \]
\[ \hat{m}_t = \hat{n}_x \]  

(2.8)

After an integration over the variable $x$ in (2.7) we obtain

\[ 0 = n_y + m_{xy} + 2(m - \hat{m})m_y \]  

(2.9)
\[ 0 = \hat{n}_y - \hat{m}_{xy} + 2(m - \hat{m})\hat{m}_y \]  

(2.10)

These equations allow us to obtain $m$ and $\hat{m}$ from (2.9) and (2.10). The result is:

\[ m = \hat{m} + \frac{m_{xy} - \hat{n}_y}{2\hat{m}_y} \]  

(2.11)
\[ \hat{m} = m + \frac{m_{xy} + n_y}{2m_y} \]  

(2.12)

By substituting (2.11) in (2.9) and (2.12) in (2.10), one can easily decouple these equations, thus obtaining one that is satisfied by $m$ and another one that is satisfied just by $\hat{m}$. Is is trivial to check that both equations are one and the same but written with different variables. In fact $(m, n)$ satisfies:

\[ 0 = m_t - n_x \]
\[ 0 = m_y^2 (n_{yt} - m_{xyy}) + m_{xy} \left( n_y^2 - m_{xy}^2 \right) 
  + 2m_y (m_{xy} m_{xxy} - n_y n_{xy}) - 4m_y^3 m_{xx} \]  

(2.13)

while $(\hat{m}, \hat{n})$ satisfies:

\[ 0 = \hat{m}_t - \hat{n}_x \]
\[ 0 = \hat{m}_y^2 (\hat{n}_{yt} - \hat{m}_{xyy}) + \hat{m}_{xy} \left( \hat{n}_y^2 - \hat{m}_{xy}^2 \right) 
  + 2\hat{m}_y (\hat{m}_{xy} \hat{m}_{xxy} - \hat{n}_y \hat{n}_{xy}) - 4\hat{m}_y^3 \hat{m}_{xx} \]  

(2.14)

which is obviously the same equation. Therefore, $m$ and $\hat{m}$ are both solutions of equation (2.13) but constrained by the transformation (2.11-12).

It is not difficult to recognize the constraint as an **Auto-Bäcklund Transformation** between two solutions of (2.13).
• **The Miura Transformation:** With regard to \( u \) and \( \eta \), these can be expressed through (2.6) as a linear superposition of \( m \) and \( \hat{m} \). Taking into account (2.6) and (2.11), one can write:

\[
\begin{align*}
\nu &= m - \hat{m} = \frac{\hat{m}_{xy} - \hat{n}_y}{2\hat{m}_y} = -\frac{m_{xy} + n_y}{2m_y} \\
\eta &= m + \hat{m} = 2\hat{m} + \frac{\hat{m}_{xy} - \hat{n}_y}{2\hat{m}_y} = 2m + \frac{m_{xy} + n_y}{2m_y} \\
\end{align*}
\]

(2.15)

In order to express \( m \) and \( \hat{m} \) as a function of \( u \) and \( \eta \) with the help of (2.15), we must previously realize that the first of the equations (2.1) can be integrated in the form:

\[
\eta_x = -\partial_x^{-1}u_t - u_x^2
\]

(2.16)

in such a way that \( m \) and \( \hat{m} \) can be finally expressed as:

\[
\begin{align*}
2m_x &= u_x + \eta_x = u_x - u^2 - \partial_x^{-1}u_t \\
2\hat{m}_x &= \eta_x - u_x = -u_x - u^2 - \partial_x^{-1}u_t
\end{align*}
\]

(2.17)

Expressions (2.17) are Miura transformations between solutions of the non-linear systems (2.1) and (2.14). In the next Section we shall show that system (2.14) is nothing but the initial system given by (1.1). Obviously, (2.13) must be considered as the **Modified GLDW Equation** since it has been transformed through the **Miura transformation** (2.17) from the GLDW equation. Hereafter, we shall refer to (2.13) as the MGLDW equation.

### III. The Modified GDLW Equation

We now attempt to show that (1.1) must truly be considered the modified GDLW equation (2.13)

#### III.1 Equation (1.1) as the Modified GDLW Equation

We first try to write down (1.1) as a equation for just one field. In order to do so, we make the change

\[
V = -m_x
\]

(3.1)
Miura trans. for two equations in $2 + 1$

in such a way that (1.1) becomes:

\[ 0 = q_t + q_{xx} + 2qm_x \quad (3.2) \]
\[ 0 = r_t - r_{xx} - 2rm_x \quad (3.3) \]
\[ 0 = qr + m_y \quad (3.4) \]

where time has been rescaled in the form:

\[ t \rightarrow \lambda t \]

Taking $q$ out of (3.4) and substituting it in (3.2), we find:

\[ 0 = \frac{m_{yt} + m_{xxy}}{r} - 2m_{xy} \frac{r_x}{r^2} + m_y \left( \frac{2m_x}{r} - \frac{r_t}{r^2} - \frac{r_{xx}}{r^2} - 2 \frac{r_x}{r^3} \right) \quad (3.5) \]

Using (3.3) in (3.5), we also obtain

\[ 0 = m_{xxy} + m_{yt} - \left( \frac{m_y r_x}{r} \right)_x \quad (3.6) \]

which can easily be integrated by setting $m_t = n_x$, which yields

\[ \frac{r_x}{r} = \frac{m_{xy} + n_y}{2m_y} \quad (3.7) \]

Substituting (3.7) in (3.3), we obtain

\[ \frac{r_t}{r} = 2m_x + \frac{m_{xxy} + n_x}{2m_y} - \frac{m_{xy}^2 - n_y^2}{4m_y} \quad (3.8) \]

Next, we calculate the obvious identity $\left( \frac{r_t}{r} \right)_x = \left( \frac{r_x}{r} \right)_t$ using (3.7) and (3.8), and finally we obtain for $m$ the equation:

\[ 0 = m_t - n_x \]
\[ 0 = m_y^2 (n_{yt} - m_{xxy}) + m_{xy} \left( n_y^2 - m_{xy}^2 \right) + 2m_y (m_{xy} m_{xxy} - n_y n_{xy}) - 4m_y^3 m_{xx} \quad (3.9) \]

which is precisely equation (2.13). Taking into account (3.1) and (2.17), we also find

\[ V = \frac{1}{2} \left( u_x + u^2 + \partial_x^{-1} u_t \right) \quad (3.10) \]
that establishes the relationship between the $V$-field of (1.1) and the $u$-field of (2.1).

Also, one can compare (3.7-8) and (2.15) yielding

$$\frac{r_x}{r} = -u$$
$$\frac{r_t}{r} = \eta_x + u^2$$  

(3.11)

III..2 The Lax pair for MGDLW

The Lax pair of the GLDW equation (2.1) is well known and can easily be written as [10]:

$$0 = \varphi_t - \varphi_{xx} + 2u\varphi_x$$
$$0 = \varphi_{xy} - \partial_x^{-1} u_y \varphi_x + \frac{\eta_y - u_y}{2} \varphi$$  

(3.12)

Alternatively, one can also write the Lax pair by using the gauge transformation:

$$\varphi = e^{\partial_x^{-1} u} \psi$$

which leads equation (3.12) to the expression:

$$0 = \psi_t - \psi_{xx} + \left[ \partial_x^{-1} u_t - u_x + u^2 \right] \psi$$
$$0 = \psi_{xy} + u\psi_y + \frac{\eta_y + u_y}{2} \psi$$  

(3.13)

This is the form of the Lax pair obtained by other authors [11], [9]. Using (2.15-17), expression (3.13) becomes:

$$0 = \psi_t - \psi_{xx} - 2m_x \psi$$
$$0 = \psi_{xy} - \frac{m_{xy} + \eta_y}{2m_y} \psi_y + m_y \psi$$,  

(3.14)
which can be considered as the Lax pair for the MGLDW equation (2.13). Taking into account (3.1), (3.4) and (3.8), this latter expression can also be written as

\[
0 = \psi_t - \psi_{xx} + 2V \psi \\
0 = \psi_{xy} - \frac{r_x}{r} \psi_y - qr \psi,
\]

(3.15)

which would be the same Lax pair but for form (3.2-4) of the MGLDW equation. Since (3.1-4) is invariant under the transformation

\[
t \rightarrow -t \\
r \rightarrow q,
\]

the following pair of equations

\[
0 = \dot{\psi}_t + \dot{\psi}_{xx} - 2V \dot{\psi} \\
0 = \dot{\psi}_{xy} - \frac{q_x}{q} \dot{\psi}_y - qr \dot{\psi},
\]

(3.16)

would also be a Lax pair for the MGLDW equation.

IV. Particular Cases

Let us now examine the Miura transformations induced in some interesting reductions of the MGLDW equation:

IV.1 The stationary case: Sinh-Gordon versus AKNS

- Let us assume that the fields of the MGLDW equation (3.9) do not to depend on \(t\). The equation takes the form:

\[
0 = -m_y^2 m_{xxy} - m_y^3 + 2m_y m_{xy} m_{xxy} - 4m_y^3 m_{xx}
\]

(4.1)

which can be integrated with respect to \(x\) to give:

\[
2m_{xxy} m_y + 8m_x m_y^2 - m_{xy}^2 = 0
\]

(4.2)

Eliminating \(m_{xxy}\) between (4.1) and (4.2), we finally obtain:

\[
m_{xxy} + 8m_x m_{xy} + 4m_y m_{xx} = 0,
\]

(4.3)

which is the AKNS equation in 1 + 1 dimensions [15].
• With the same reduction of time-independent fields, The GLDW equation (2.1) can also be written as:

\begin{align}
0 &= \eta_{xy} + 2uu_y \\
0 &= u_{xy} + 2u\eta_y
\end{align} \tag{4.4} \tag{4.5}

Let us now multiply (4.4) by \(\eta_y\), (4.5) by \(u_y\), and subtract. The result is:

\[
\frac{d(\eta_y^2 - u_y^2)}{dx} = 0
\]

Next, we introduce the change of variables

\[
\eta_y = a_0 \cosh 2q \\
u_y = a_0 \sinh 2q
\] \tag{4.6}

and so equations (4.4-5) take the form

\[
u = -q_x
\] \tag{4.7}

Alternatively, using (4.6) we finally obtain:

\[
q_{xy} = a_0 \sinh 2q
\] \tag{4.8}

which is the Sinh-Gordon equation \cite{I1}, \cite{I4}

• Therefore, the particularization of the Miura transformation (2.17) to the stationary case:

\[
2m_x = u_x - u^2 = -q_{xx} - q_x^2 \\
2\dot{m}_x = -u_x - u^2 = q_{xx} - q_x^2
\] \tag{4.9}

yields the corresponding Miura transformation between solutions of the AKNS equation (4.3) and the Sinh-Gordon equation (4.8).

**IV..2 Non-Local Boussinesq Equation versus the Kaup System**

In the next subsection we shall deal with the reduction which arises from setting \(x = y\)
In the present case, the MGLDW equation (3.10) is reduced to:

\[
0 = m_t - n_x \\
0 = m_x^2(n_{xt} - m_{xxxx}) + m_{xx}(n_x^2 - m_{xx}^2) \\
= 2m_x(m_{xx}m_{xxx} - n_xn_{xx}) - 4m_x^2m_{xx}
\] (4.10)

The set of equations (4.10) can be integrated with respect to \(x\). One easily obtains:

\[
0 = m_t - n_x \\
0 = m_x(n_t - m_{xxx}) - (n_x^2 - m_{xx}^2) - 2m_x^3
\] (4.11)

Alternatively, we can eliminate the field \(n\). Thus, for \(m\) we find the following equation:

\[
m_{tt} = \left[ m_{xxx} + 2m_x^2 + \frac{n_x^2 - m_{xx}^2}{m_x} \right]_x
\] (4.12)

Lambert et al. \[16\], \[17\] have called (4.12) the non-local Boussinesq equation (NLBq).

We now apply the reduction \(x = y\) to the GLDW equation (2.1):

\[
0 = u_t + \eta_{xx} + 2uu_x \\
0 = \eta_t + u_{xx} + 2u\eta_x
\] (4.13)

This system has been known as the classical Boussinesq system \[13\] or the Kaup system \[18\], \[12\], \[19\], \[16\].

The Miura transformation (2.17) establishes a relationship between the solutions of The non-local Boussinesq equation (4.12) and the Kaup system (4.13):
An important subcase of this reduction is the **non-linear Schrödinger equation**. To see this, we must notice that the reduction $x = y$ applied to the MGLDW in its form given by (3.2-4) appears as:

\[
0 = q_t + q_{xx} - 2q^2 r \\
0 = r_t - r_{xx} + 2r^2 q
\]

which for imaginary time, $t \rightarrow it$ and $q = r^*$, is clearly the non-linear Schrödinger equation [3], [20].

**V. Conclusions**

- Firs, starting from Painlevé analysis of the Dispersive Long Wave Equation in $2 + 1$ dimensions (GDLW), we have been able to decompose the solutions of this equation as a **linear superposition** of two solutions of another equation in $2 + 1$ that has been called Modified Dispersive Long Wave Equation (MGDLW).

- The expansion of the solutions of the GDLW as a linear superposition of its modified version (MGDLW) is shown to be invertible and gives rise to the Miura transformation between both equations. This Miura transformation has been obtained explicitly.

- In Section III we have shown that the MGLDW equation is nothing but the equation proposed by several authors [3], [1], [6], [2] as one of the simplest possible integrable equation in $2 + 1$ dimensions.

- Since the Lax pair of the GDLW equation is already known [9], the Miura transformation allows one to pass from the GDLW equation to the MGDLW equation. In particular, we can easily find the Lax pair of the latter equation.

- In Section IV some interesting reductions of the MGDLW equation are presented using the Miura transformation between GDLW and MGDLW.
In this way, one can obtain the Miura transformations between AKNS and Sinh-Gordon in $1+1$ and between non-local Boussinesq equation and the Kaup system, also in $1+1$ dimensions.

- In view of the above results one could conjecture that when two or more expansion branches in the Painlevé expansion appear in a given non-linear partial differential equation, a Miura transformation exists that converts the initial equation into one with just one branch in this expansion.

- Systematic application of the Singular Manifold Method and use of it to find the Darboux transformation for the MGDLW equation pave the way for finding extremely simple algorithms for the construction of solutions that generalize the ones already obtained by other authors using the bilinear formalism. We shall shortly be presenting these results in a forthcoming publication.

**ACKNOWLEDGEMENTS** This research has been supported in part by the DGICYT under contract PB95-0947.

**References**

[1] Radha R. and Lakshmanan, J. Math. Phys. **24**, 292 (1983).

[2] Porsezian K., J. Math. Physics **38**, 4675 (1997).

[3] Fokas A., Inverse Problems **10**, L19 (1994).

[4] Zakharov V. and Shbat A., Sov. Phys.-JETP **34**, 62 (1972)

[5] Chakravarty S., Kent S.L and Newman E.T., J. Math. Phys. **36**, 763 (1995).

[6] Maccari A., J. Math. Phys. **37**, 6207 (1996).

[7] Weiss J., Tabor M. and Carnevale G., J. Math. Phys. **24**, 522 (1983).

[8] Weiss J., J. Math. Phys. **24**, 1405-14013 (1983).
[9] Boiti M., Leon J. and Pempinelli F, Inv. Problems 3, 37 (1987).

[10] Estévez P.G. and Gordoa P.R., Inverse Problems 13, 939 (1997).

[11] Ablowitz M. J and Clarkson P., “Solitons, Nonlinear Evolution Equations and Inverse Scattering”. London Mathematical Society. Lecture Note Series 149, Cambridge University Press. (1991).

[12] Kaup D., Prog. Theor. Phys. 54, 396 (1975).

[13] Estévez P.G., Gordoa P.R., Martinez-Alonso L. and Medina-Reus E., J. Phys. A 26, 1915 (1993).

[14] Estévez P.G., Conde-Calvo E. and Gordoa P.R., Jour. Nonlinear. Math. Phys. In press (1998).

[15] Ablowitz M.J., Kaup D.J., Newell A.C. and Segur H., Stud. Appl. Math. 53, 249 (1974).

[16] Lambert F, Loris I, Springael J and Willox R., J. Phys A: Math Gen. 27, 5325 (1994).

[17] Willox R., Loris I. and Springael J., J. Phys A 28, 5963 (1995).

[18] Conte R. Musette M. and Pickering A., J. Phys A: Math. Gen. 27, 2831 (1994).

[19] Kawamoto S., J. Phys. Soc. Jap. 53, 2922 (1984)

[20] Tabor M., Chaos and Integrability in Nonlinear Dynamics, Wiley and sons, (1989).