High-dimensional Time Series Prediction with Missing Values

Hsiang-Fu Yu
The University of Texas at Austin
rofuyu@cs.utexas.edu

Nikhil Rao
Technicolor Research
nikhil.rao@technicolor.com

Inderjit S. Dhillon
The University of Texas at Austin
inderjit@cs.utexas.edu

Abstract

High-dimensional time series prediction is needed in applications as diverse as demand forecasting and climatology. Often, such applications require methods that are both highly scalable, and deal with noisy data in terms of corruptions or missing values. Classical time series methods usually fall short of handling both these issues. In this paper, we propose to adapt matrix matrix completion approaches that have previously been successfully applied to large scale noisy data, but which fail to adequately model high-dimensional time series due to temporal dependencies. We present a novel temporal regularized matrix factorization (TRMF) framework which supports data-driven temporal dependency learning and enables forecasting ability to our new matrix factorization approach. TRMF is highly general, and subsumes many existing matrix factorization approaches for time series data. We make interesting connections to graph regularized matrix factorization methods in the context of learning the dependencies. Experiments on both real and synthetic data show that TRMF outperforms several existing approaches for common time series tasks.

1 Introduction

Time series data plays a central role in many applications, from demand forecasting to speech and video processing to climatology. Such applications involve data collected over a large time frame, and also a very large number of possibly inter-dependent time series. For example, climatology applications involve data collected from possibly thousands of sensors, every hour (or less) over several days. Similarly, a store tracking its inventory would track thousands of items every day for multiple years. Not only is the scale of such problems huge, but they also typically involve missing values, due to sensor malfunctions, occlusions or simple human errors. Thus, modern time series applications present two challenges to practitioners: scalability in the presence of high-dimensional time series and the flexibility to handle missing values.

Classical approaches to handle time-varying data include autoregressive (AR) models or dynamic linear models (DLM) [8, 23]. These approaches fall short of handling the aforementioned issues [1]. Specifically, for $T$ time points of $n$ dimensional time series, an AR model of order $L$ requires $O(TL^2n^4 + L^3n^6)$ time to estimate $O(Ln^2)$ parameters, which is prohibitive even for moderate values of $n$. Similarly, Kalman Filter based DLM approaches need $O(kn^2T + k^3T)$ operations to update parameters, where $k$ is the latent dimensionality which may be larger than $n$ [15]. As a specific example, the maximum likelihood estimator implementation in the widely used R-DLM package [14], which relies on a general optimization solver, cannot scale beyond $n \geq 32$. (See Appendix B for details).

Recently, low rank matrix completion or matrix factorization (MF) methods have found use in large scale collaborative filtering [10, 26], multi-label learning [27], and many other applications where data often scales into the millions. A natural way to model high-dimensional time series data is in the form of a matrix, with rows corresponding to time series and columns corresponding to time points. In light of this, it is prudent to ask

"Can we generalize matrix factorization to analyze large scale data with temporal dependencies?"
Let $Y \in \mathbb{R}^{n \times T}$ be the observed data matrix of $n$ time series of length $T$. Many classic time series models can be described in the following general form [15, 23]:

$$
y_t = F x_t + \eta_t, \quad (1)
$$

$$
x_t = M \Theta \{x_{t-l} : l \in \mathcal{L}\} + \epsilon_t, \quad (2)
$$

where $y_t \in \mathbb{R}^n$ is the snapshot of $n$ time series at the $t$-th time point, $F \in \mathbb{R}^{n \times k}$, $x_t \in \mathbb{R}^k$ is the latent embedding for the $t$-th time point, and $\eta_t, \epsilon_t$ are Gaussian noise vectors. $M \Theta$ in (2) is a time series model parameterized by $\mathcal{L}$ and $\Theta$: $\mathcal{L}$ is a set containing each lag index $l$ denoting a dependency between $t$-th and $(t-l)$-th time points, while $\Theta$ captures the weighting information of temporal dependencies (such as the transition matrix in AR models). When we stack all the $x_t$s into a matrix $X$ and let $f_i^T$ be the $i$-th row of $F$ as shown in Figure 1, we clearly see that $Y \approx FX$ and we can attempt to learn the factors in (1) by considering a standard matrix factorization formulation (3):

$$
\min_{F,X} \sum_{(i,t) \in \Omega} (y_{it} - f_i^T x_t)^2 + \lambda_f R_f(F) + \lambda_x R_x(X), \quad (3)
$$

where $\Omega$ is the set of the observed entries, and $R_f(F), R_x(X)$ are regularizers for $F$ and $X$ respectively.

Figure 1: Matrix Factorization model for multiple time series. $F$ captures features for each time series in the matrix $Y$, and $X$ captures the latent and time-varying variables.

This connection made, there now arise two major challenges to extend MF approaches to analyze data with temporal dependencies:

- How to describe and incorporate the structure of temporal dependencies into the MF formulation?
- How to efficiently forecast values of future time points?

It is clear that the common choice of the regularizer $R_x(X) = \|X\|_F$ is no longer appropriate, as it does not take into account the dependencies among the columns of $X$. Most existing MF adaptations to handle temporal dependencies [2, 16, 18, 24, 28] graph-based approaches, where the dependencies are described by a graph and incorporated through a Laplacian regularizer [20]. However, graph-based regularization fails in cases where there are negative correlations between two time points. Furthermore, unlike scenarios where explicit graph information is available with the data (such as social network or product co-purchasing graph for recommender systems), explicit temporal dependency structures are usually unavailable and has to be inferred or approximated. Hence, this approach requires practitioners to either perform a separate procedure to estimate the dependencies or consider very short-term dependencies with simple fixed weights. Moreover, existing MF approaches, while yielding good estimations for missing values in past time points, are poor in terms of forecasting future values, which is crucial in time series analysis.

1.1 Our Contributions:

In this paper, we propose a novel temporal regularized matrix factorization (TRMF) framework for data with temporal dependencies. TRMF generalizes several existing temporal matrix factorization approaches [2, 16, 18, 24, 28] and connects to some existing latent time-series models [8, 11, 12, 15, 23]. Our approach not only supports data-driven
temporal dependency learning but also brings the ability to forecast future values to matrix factorization methods. Also, unlike classic time-series models, TRMF easily handles high dimensional time series data even in the presence of many missing values.

Next, based on AR models, we design a novel temporal regularizer to encourage temporal dependencies among the latent embeddings \( \{x_t\} \). We also make interesting connections between the proposed regularizer and graph-based approaches [20]. This connection not only leads to better understanding about the dependency structure incorporated by our framework but also brings the benefit of using off-the-shelf efficient solvers such as GRALS [17] directly to solve TRMF.

The rest of this paper is organized as follows. In Section 2, we review the existing approaches and their limitations on data with temporal dependencies. We present the proposed TRMF framework in Section 3, and show that the method is highly general and can be used for a variety of time series applications. We introduce a novel AR temporal regularizer in Section 4, and make connections to graph-based regularization approaches. We demonstrate the superiority of the proposed approach via extensive experimental results in Section 5 and conclude the paper in Section 6.

2 Data with Temporal Dependency: Existing Approaches and Limitations

2.1 Time-Series Models

Models such as AR and DLM are not suitable for modern multiple high-dimensional time series data (i.e., both \( n \) and \( T \) are large) due to their inherent computational inefficiency (see Section 1). To reduce the number of parameters and avoid overfitting in AR models, there have been studies with various structured transition matrices such as low rank and sparse matrices [5, 13]. The focus of most of these works is on devising better statistical consistency guarantees, and the issue of scalability of AR models remains a challenge. On the other hand, it is also challenging for many classic time-series models to deal with data with missing values [1].

DLM unifies many other time series models [15, 23]. The original idea goes back to the Kalman filter [8] in the control theory community. In many situations where the model parameters such as \( F \) in (1) and \( \Theta, L \) in (2) are either given or designed by practitioners, the Kalman filter approach is used to preform forecasting, while the Kalman smoothing approach is used to impute missing entries. In situations where model parameters are unknown, EM algorithms are applied to estimate both the model parameters and latent embeddings for DLM [3, 11, 12, 19, 21]. As most EM approaches for DLM contain the Kalman filter as a building block, they cannot scale to very high dimensional time series data. Indeed, as shown in Section 5, the popular R package for DLM’s does not scale beyond data with a few hundred observations and dimensions.

2.2 Existing Matrix Factorization Approaches for Time Series

In standard matrix factorization (3), the squared Frobenius norm \( R_x(X) = \|X\|_F = \sum_{t=1}^{T} \|x_t\|^2 \) is the usual regularizer of choice for \( X \). Because squared Frobenius norm assumes no dependencies among \( \{x_t\} \), standard MF formulation is invariant to column permutation and not applicable to data with temporal dependencies. Hence most existing temporal MF approaches [2, 16, 18, 24, 28] turn to the framework of graph-based regularization [20] for temporally dependent \( \{x_t\} \), with a graph encoding the temporal dependencies.

Graph regularization for temporal dependency. The framework of graph-based regularization is an approach to describe and incorporate general dependencies among variables. Let \( G \) be a graph over \( \{x_t\} \) and \( G_{ts} \) be the weight between the \( t \)-th node and \( s \)-th node. A popular regularizer to include as part of an objective function is the following:

\[
R_x(X) = G(X \mid G, \eta) := \frac{1}{2} \sum_{t \sim s} G_{ts} \|x_t - x_s\|^2 + \frac{\eta}{2} \sum_t \|x_t\|^2,
\]

where \( t \sim s \) denotes the edge between \( t \)-th node and \( s \)-th node, and the second summation term is used to guarantee strong convexity.
A large $G_{ts}$ will ensure that $x_t$ and $x_s$ are close to each other in the Euclidean sense, when (4) is minimized. Note that to guarantee the convexity of $\mathcal{G}(X \mid G, \eta)$, the weights $G_{ts}$ need to be non-negative.

To apply graph-based regularizers to temporal dependencies, we need to specify the (repeating) dependency pattern by a lag set $\mathcal{L}$ and a weight vector $w$ such that all the edges $t \sim s$ of distance $l$ (i.e., $|s - t| = l$) share the same weight $G_{ts} = w_l$. See Figure 2 for an example with $\mathcal{L} = \{1, 4\}$. Given $\mathcal{L}$ and $w$, the corresponding graph regularizer becomes

$$\mathcal{G}(X \mid G, \eta) = \frac{1}{2} \sum_{t \in \mathcal{L}} \sum_{t > l} w_l (x_t - x_{t-l})^2 + \frac{\eta}{2} \sum_{t} \|x_t\|^2.$$  

(5)

This direct use of graph-based approach for temporal dependency, while intuitive, has two issues: a) there might be negatively correlated dependencies between two time points; however, the standard graph regularization requires nonnegative weights. b) unlike many applications where such regularizers are used, the explicit temporal dependency structure is usually not available and has to be inferred. As a result, most existing temporal MF approaches consider only very simple temporal dependencies such with a small size of $\mathcal{L}$ (e.g., $\mathcal{L} = \{1\}$) and/or a uniform weight (e.g., $w_l = 1, \forall l \in \mathcal{L}$). For example, a simple chain graph is considered to design the smooth regularizer in TCF [24]. This leads to poor forecasting abilities of existing MF methods for large-scale time series applications.

### 2.3 Challenges to Learn Temporal Dependencies

One could try to learn the weights $w_l$ automatically, by using the same regularizer as in (5) but with the weights unknown. This would lead to the following optimization problem:

$$\min_{F,X,w \geq 0} \sum_{(i,t) \in \Omega} (Y_{it} - f_i^T x_t)^2 + \frac{\lambda_x}{2} \sum_{l \in \mathcal{L}} \sum_{t > 0} w_l (x_t - x_{t-l})^2 + \frac{\lambda_x \eta}{2} \sum_{l} \|x_t\|^2,$$

(6)

where $0$ is the zero vector, and $w \geq 0$ is the constraint imposed by graph regularization.

It is not hard to see that the above optimization yields the trivial all-zero solution for $w^*$, meaning the objective function is minimized when no temporal dependencies exist! To avoid the all zero solution, one might want to impose a simplex constraint on $w$ (i.e., $\sum_{l \in \mathcal{L}} w_l = 1$). Again, it is not hard to see that this will result in $w^*$ being a 1-sparse vector, with the non zero component $w_{l^*}$ being 1, where

$$l^* = \arg \min_{l \in \mathcal{L}} \sum_{t > l} \|x_t - x_{t-l}\|^2.$$  

Thus, looking to learn the weights automatically by simply plugging in the regularizer in the MF formulation is not a viable solution.

### 3 Temporal Regularized Matrix Factorization

In order to resolve the limitations mentioned in Sections 2.2 and 2.3, we propose the Temporal Regularized Matrix Factorization (TRMF) framework, which is a novel approach to incorporate temporal dependencies into matrix factorization models.

We propose to use well-known time series models (e.g. AR) to describe temporal dependencies in $X$ explicitly. Unlike the aforementioned graph-based approaches, there are many well-studied models which are specifically designed for data with temporal dependency [15]. Let $M_{ti}$ denote the time-series model for $\{x_t\}$ as introduced in
(2). To incorporate the temporal dependency in TRMF, we propose to design a new regularizer \( T_M(X \mid \Theta) \) which encourages the structure induced by \( M_\Theta \).

Taking a standard approach to model time series, we set \( T_M(X \mid \Theta) \) be the negative log likelihood of observing a particular realization of the \( \{x_t\} \) for a given model \( M_\Theta \):

\[
T_M(X \mid \Theta) = -\log \mathbb{P}(x_1, \ldots, x_T \mid \Theta).
\]

When \( \Theta \) is given, we can use \( R_\theta(X) = T_M(X \mid \Theta) \) in the MF formulation (3) to encourage \( \{x_t\} \) to follow the temporal dependency induced by \( M_\Theta \). When the \( \Theta \) is unknown, we can treat \( \Theta \) as another set of variables and include another regularizer \( R_\theta(\Theta) \) into (3) as follows.

\[
\min_{F,X,\Theta} \sum_{(i,t)\in\Omega} (Y_{it} - f_i^T x_t)^2 + \lambda_f R_f(F) + \lambda_x T_M(X \mid \Theta) + \lambda_\theta R_\theta(\Theta),
\]

which be solved by an alternating minimization procedure over \( F, X, \) and \( \Theta \).

**Data-driven Temporal Dependency Learning in TRMF:** Recall that in Section 2.3, we showed that directly using graph based regularizers to incorporate temporal dependencies leads to trivial solutions for the weights. We now show in TRMF can be learned automatically from data.

\[
\min_{\Theta} \lambda_x T_M(X \mid \Theta) + \lambda_\theta R_\theta(\Theta),
\]

which is a maximum-a-posterior (MAP) estimation problem (in the Bayesian sense, assuming priors on the data) to estimate the best \( \Theta \) for a given \( \{x_t\} \) under the \( M_\Theta \) model. There are well-developed algorithms to solve the MAP problem (9) and obtain non-trivial dependency parameter \( \Theta \). Thus, unlike most existing temporal matrix factorization approaches where the strength of dependencies is fixed, \( \Theta \) in TRMF can be learned automatically from data.

### 3.1 Time-Series Analysis with TRMF

We can see that TRMF (8) lends itself seamlessly to handle a variety of commonly encountered tasks in analyzing data with temporal dependency:

- **Time-series Forecasting:** As TRMF comes with a full specification of \( M_\Theta \) for latent embeddings \( \{x_t : 1, \ldots, T\} \), we can use \( M_\Theta \) to predict future latent embeddings \( \{x_t : t > T\} \) and have the ability to obtain non-trivial forecasting results for \( y_t = F x_t \) for \( t > T \).

- **Missing-value Imputation:** In some time-series applications, certain entries in \( Y \) might be unobserved, for example, due to faulty sensors in electricity usage monitoring or occlusions in the case of motion recognition in video. We can use \( f_i^T x_t \) to impute these missing entries, much like standard matrix completion. This task is useful in many recommender systems [24] and sensor networks [28].

- **Time-series classification/clustering:** The obtained \( f_i \) can be used as the latent embedding for the \( i \)-th time series of \( Y \). These latent features can be used to perform classification/clustering of the time series. Note that this can be done even when there are missing entries in the observed data, as missing entries are not a bottleneck for learning \( F \).

### 3.2 Extensions to Incorporate Extra Information

Like matrix factorization approaches, TRMF (8) can be extended to incorporate additional information. Below we only briefly describe three approaches, and more details on these extensions can be found in Appendix ??.

- **Known features for time series:** In many applications, one is given additional features along with the observed time series. Specifically, given a set of feature vectors \( \{a_i \in \mathbb{R}^d\} \) for each row of \( Y \), we can look to solve

\[
\min_{F,X,\Theta} \sum_{(i,t)\in\Omega} (Y_{it} - a_i^T F x_t)^2 + \lambda_f R_f(F) + \lambda_x T_M(X \mid \Theta) + \lambda_\theta R_\theta(\Theta).
\]


That is, the observation $Y_{it}$ is posited to be a bilinear function of the feature vector $a_i$ and the latent vector $x_t$. Such an inductive framework has two advantages: we can generalize TRMF to a new time series without any observations up to time $T$ (i.e., a new row $i'$ of $Y$ without any observations). As long as the feature vector $a_i$ is available, the model learned by TRMF can be used to estimate $Y_{i't} = a_i F x_{t'}$, $\forall t'$. Furthermore, prediction can be significantly sped up when $d \ll n$, since the dimension of $F$ is reduced from $n \times k$ to $d \times k$. Such methods for standard multi-label learning and matrix completion have been previously considered in [7, 25, 27].

- **Graph information among time series**: Often, separate features for the time series are not known, but other relational information is available. When a graph that encodes pairwise interactions among multiple time series is available, one can incorporate this graph in our framework using the graph regularization approach (4). Such cases are common in inventory and sales tracking, where sales of one item is related to sales of other items. Given a graph $G^j$ describing the relationship among multiple time series, we can formulate a graph regularized problem:

\[
\min_{F,X,\Theta} \sum_{(i,j) \in \Omega} (Y_{it} - f_i^T x_t)^2 + \lambda_f \mathcal{G}(F | G^j, \eta) + \lambda_x T_M(X | \Theta) + \lambda_\theta R_\theta(\Theta),
\]

(11)

where $\mathcal{G}(F | G^j, \eta)$ is the graph regularizer defined in (4) capturing pairwise interactions between time series. Graph regularized matrix completion methods have been previously considered in [17, 29].

- **Temporal-regularized tensor factorization**: Naturally, TRMF can be easily extended to analyze temporal collaborative filtering applications [21, 24], where the targeted data is a tensor with certain modes evolving over time. For example, consider $Y \in \mathbb{R}^{m \times n \times T}$ be a 3-way tensor with $Y_{ijt}$ encoding the rating of the $i$-th user for the $j$-th item at time point $t$. We can consider the following temporal regularization tensor factorization (TRTF) with $T_M(X | \Theta)$ as follows:

\[
\min_{P,Q,X,\Theta} \sum_{(i,j,t) \in \Omega} (Y_{ijt} - \langle p_i, q_j, x_t \rangle)^2 + \lambda_P \mathcal{R}_p(P) + \mathcal{R}_q(Q) + \mathcal{T}_M(X | \Theta) + \mathcal{R}_\Theta(\Theta),
\]

(12)

where $P = [p_1, \cdots, p_m]^T \in \mathbb{R}^{m \times k}$ and $Q = [q_1, \cdots, q_n]^T \in \mathbb{R}^{n \times k}$ are the latent embeddings for the $m$ users and $n$ items, respectively, and with some abuse of notation, we define $\langle p_i, q_j, x_t \rangle = \sum_{r=1}^k p_{ir} q_{jr} x_{tr}$.

### 4 A Novel Autoregressive Temporal Regularizer

Up until now, we described the TRMF framework in a very general sense, with the regularizer $T_M(X | \Theta)$ incorporating dependencies specified by the time series model $M_\Theta$. In this section, we specialize this to the case of AR models.

An AR model, parameterized by a lag set $L$ and weights $\mathcal{W} = \{W^{(l)} \in \mathbb{R}^{k \times k} : l \in L\}$, assumes that $x_t$ is a noisy linear combination of some previous points; that is,

\[
x_t = \sum_{l \in L} W^{(l)} x_{t-l} + \epsilon_t,
\]

(13)

where $\epsilon_t$ is Gaussian noise vector. For simplicity, we assume that the $\epsilon_t \sim \mathcal{N}(0, \sigma^2 I_k)$.

As a result, the temporal regularizer $T_M(X | \Theta)$ corresponding to this AR model can be written as follows:

\[
T_{AR}(X | L, \mathcal{W}, \eta) := \sum_{t=m}^T \frac{1}{2} \left\| x_t - \sum_{l \in L} W^{(l)} x_{t-l} \right\|^2 + \frac{\eta}{2} \sum_{t} \|x_t\|^2,
\]

(14)

1If the (known) covariance matrix is not identity, we can suitably modify the regularizer.
Then (14) can be written as follows:
\[ W = \text{diag}(x) \]
where \( x \) is a zero vector. Let \( \bar{w} = [\cdots, x_{r-1}, \cdots] \) be the \( r \)-th row of \( X \) and \( \bar{w}_r^T = [\cdots, \bar{W}_{rl}, \cdots] \) be the \( r \)-th row of \( \bar{W} \). Then (14) can be written as follows:
\[
\mathcal{T}_{\text{AR}}(X \mid \mathcal{L}, \mathcal{W}, \eta) = \sum_{r=1}^{k} \mathcal{T}_{\text{AR}}(\bar{x}_r \mid \mathcal{L}, \bar{w}_r, \eta),
\]
\[
\mathcal{T}_{\text{AR}}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) = \frac{1}{2} \sum_{t=m}^{T} \left( x_t - \sum_{l \in \mathcal{L}} w_l x_{t-l} \right)^2 + \frac{\eta}{2} \| \bar{x} \|^2,
\]
where \( x_t \) is the \( t \)-th element of \( \bar{x} \), and \( w_l \) is the \( l \)-th element of \( \bar{w} \).

Correlations among Multiple Time Series: TRMF retains the power to capture the correlations among time series via the factors \( \{ f_r \} \), even after simplifying \( \{ W_l \} \) to be diagonal, since it has effects only on the structure of latent embeddings \( \{ x_t \} \). Indeed, as the \( i \)-th dimension of \( \{ y_t \} \) is modeled by \( f_i^T X \) in (8), one can see the low rank \( F \) as a \( k \)-dimensional latent embedding of multiple time series. This low rank embedding captures correlations among multiple time series. Furthermore, \( \{ f_i \} \) acts as time series features, which can be used to perform classification/clustering even in the presence of missing values.

Choice of Lag Index Set \( \mathcal{L} \): Unlike most MF approaches mentioned in Section 2.2, the choice of \( \mathcal{L} \) in TRMF is more flexible. Thus, TRMF can provide the following two important advantages for practitioners: First, because there is no need to specify the weight parameters \( \mathcal{W} \), \( \mathcal{L} \) can be chosen to be larger to account for long range dependencies, which also yields more accurate and robust forecasts. Second, the indices in \( \mathcal{L} \) can be discontinuous so that one can easily embed domain knowledge about periodicity or seasonality. For example, one might consider \( \mathcal{L} = \{1, 2, 3, 51, 52, 53\} \) for weekly data with an one-year seasonality.

Connections to Graph Regularization: We now establish connections between \( \mathcal{T}_{\text{AR}}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) \) and graph regularization (4) for matrix factorization. Let \( \mathcal{L} := \mathcal{L} \cup \{0\} \), \( \bar{w}_0 = -1 \) so that (16) can be written as
\[
\mathcal{T}_{\text{AR}}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) = \frac{1}{2} \sum_{t=m}^{T} \left( \sum_{l \in \mathcal{L}} w_l x_{t-l} \right)^2 + \frac{\eta}{2} \| \bar{x} \|^2,
\]
and let \( \delta(d) := \{ l \in \mathcal{L} : l - d \in \mathcal{L} \} \). We then have the following result:

**Theorem 1.** Given a lag index set \( \mathcal{L} \), weight vector \( \bar{w} \in \mathbb{R}^L \), and \( \bar{x} \in \mathbb{R}^T \), there is a weighted signed graph \( G^{\text{AR}} \) with \( T \) nodes and a diagonal matrix \( D \in \mathbb{R}^{T \times T} \) such that
\[
\mathcal{T}_{\text{AR}}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) = \mathcal{G}(\bar{x} \mid G^{\text{AR}}, \eta) + \frac{1}{2} \bar{x}^T D \bar{x},
\]
where \( \mathcal{G}(\bar{x} \mid G^{\text{AR}}, \eta) \) is the graph regularization (4) for \( G^{\text{AR}} \). The edge weight for \( G^{\text{AR}} \) and the diagonal element of \( D \) are described as follows:
\[
G_{t,t+d}^{\text{AR}} = \begin{cases} 
\sum_{l \in \delta(d)} \sum_{m \leq t+l \leq T} -w_l w_{l-d} & \text{if } \delta(d) \neq \phi, \\
0 & \text{otherwise},
\end{cases}
\]
\[
D_{tt} = \left( \sum_{l \in \mathcal{L}} w_l \right) \left( \sum_{l \in \mathcal{L}} w_l \sum_{m \leq t+l \leq T} \right) \quad \forall t.
\]
See Appendix A.1 for the detailed proof of Theorem 1. From Theorem 1, we know that $\delta(d)$ is not empty if and only if there are edges of distance $d$ in $G^{AR}$. Thus, we can construct the dependency graph for $T_{AR}(\bar{x} \mid L, \bar{w}, \eta)$ by checking whether $\delta(d)$ is empty. Figure 3 demonstrates an example with $L = \{1, 4\}$. We can see that in addition to the edges of distance $d = 1$ and $d = 4$, there are also edges of distance $d = 3$ (dotted edges in Figure 3) because $4 - 3 \in L$ and $\delta(3) = \{4\}$.

Although Theorem 1 shows that AR-based regularizers are similar to the graph-based regularization framework, we note the following key differences:

- The graph $G^{AR}$ defined in Theorem 1 is a signed weighted graph, which contains both positive and negative edges. This implies that the AR temporal regularizer is able to support negative correlations, which the standard graph-based regularizer does not. This can make $G(\bar{x} \mid G^{AR}, \eta)$ non-convex.
- While $G(\bar{x} \mid G^{AR}, \eta)$ might be non-convex, the addition of the second term in (17) still leads to a convex regularizer $T_{AR}(\bar{x} \mid L, \bar{w}, \eta)$.
- Unlike the approach (5) where there is freedom to specify a weight for each distance, in the graph defined in Theorem 1, the weight values for the edges are more structured (e.g., the weight for $d = 3$ in Figure 3 is $-w_{1}w_{4}$). Hence, minimization w.r.t. $w$ is not trivial, and neither are the obtained solutions.

### 4.1 Optimization for TRMF with AR Temporal Regularization

Plugging $T_{M}(X \mid \Theta) = T_{AR}(X \mid L, \bar{w}, \eta)$ into (8), we obtain the following optimization problem:

$$
\min_{F, X, \bar{W}} \sum_{(i,t) \in \Omega} (Y_{it} - f_{i}^\top x_{t})^2 + \lambda f R_{f}(F) + \sum_{r=1}^{k} \lambda_{r} T_{AR}(\bar{x}_{r} \mid L, \bar{w}_{r}, \eta) + \lambda_{w} R_{w}(\bar{W}),
$$

where $R_{w}(\bar{W})$ is a regularizer for $\bar{W}$. We will refer to (18) as TRMF-AR. We can apply alternating minimization to solve (18), and in fact, solving for each variable reduces to well known methods, for which highly efficient algorithms exist:

**Updates for $F$.** When $X$ and $W$ are fixed, the subproblem of updating $F$ is the same as updating $F$ while $X$ fixed in (3). Thus, fast algorithms such as alternating least squares [10] or coordinate descent [26] can be applied directly to find $F$.

**Updates for $X$.** To update $X$ when $F$ and $W$ fixed, we solve:

$$
\arg \min_{X} \sum_{(i,t) \in \Omega} (Y_{it} - f_{i}^\top x_{t})^2 + \lambda x \sum_{r=1}^{k} T_{AR}(\bar{x}_{r} \mid L, \bar{w}_{r}, \eta).
$$

From Theorem 1, we see that $T_{AR}(\bar{x} \mid L, \bar{w}, \eta)$ shares the same form as the graph regularizer, and we can apply GRALS [17] to solve (19).

**Updates for $W$.** How to update $W$ while $F$ and $X$ fixed depends on the choice of $R_{w}(\bar{W})$. There are many parameter estimation techniques developed for AR with various regularizers [13, 22]. For simplicity, we consider the squared Frobenius norm: $R_{w}(\bar{W}) = ||\bar{W}||_{f}^2$. As a result, each row of $\bar{w}_{r}$ of $W$ can be updated independently by
solving the following one-dimensional autoregressive problem.

\[
\arg \min_{\hat{w}} \lambda_T T_{\text{AR}}(\hat{x}_r | \mathcal{L}, \tilde{w}, \eta) + \lambda_w \| \tilde{w} \|^2 \\
= \arg \min_{\hat{w}} \sum_{t=m}^{s} \left( x_t - \sum_{l \in \mathcal{L}} w_l x_{t-l} \right)^2 + \frac{\lambda_w}{\lambda_T} \| \tilde{w} \|^2.
\]

(21) is a simple \(|\mathcal{L}|\) dimensional ridge regression problem with \(T - m + 1\) instances, which can be solved efficiently by Cholesky factorization such as the backslash operator in MATLAB.

Note that since out method is highly modular, one can resort to any method to solve the optimization subproblems that arise for each module. Moreover, as seen from Section 3.2, TRMF can also be used with different regularization structures, making it highly adaptable.

### 4.2 Connections to Existing MF Approaches

TRMF-AR, while being a special family of TRMF, still can be seen as a generalization for many existing MF approaches to handle data with temporal dependencies. We demonstrate several cases which arise from a specific choice of \(\mathcal{L}\) and \(\{W^{(l)}\}\):

- **Temporal Collaborative Filtering** [24]: AR(1) with \(W^{(1)} = I_k\) on \(\{x_t\}\).
- **NMF with Temporal Smoothness** [2]: AR(L) with \(W^{(l)} = \alpha^{l-1}(1 - \alpha)I_k\), \(\forall l = 1, \ldots, L\), where \(\alpha\) is a pre-defined parameter.
- **Sparsity Regularized Matrix Factorization** [18, 28]: AR(1) with \(W^{(1)} = I_n\) on \(\{Fx_t\}\).
- **Temporal stability in low-rank structure** [16]: AR(2) with \(W^{(1)} = 2I_n\) and \(W^{(2)} = -I_n\) on \(\{Fx_t\}\). This is also corresponds to the Hodrick-Prescott filter [6, 9].
- **Dynamic Linear Model** [8]: DLM is a time series model which can be directly applied to model \(Y\). It is essentially a latent AR(1) model with a general \(W^{(1)}\), which can be estimated by expectation-maximization algorithms [3, 11, 19].

### 4.3 Connections to Gaussian Markov Random Field

The Gaussian Markov Random Field (GMRF) is a general way to model multivariate data with dependencies. GMRF assumes that data are generated from a multivariate Gaussian distribution with a covariance matrix \(\Sigma\) which describes the dependencies among \(T\) dimensional variables i.e., \(\tilde{x} \sim \mathcal{N}(0, \Sigma)\).\(^2\) If the unknown \(\tilde{x}\) is assumed to be generated from this model, The negative log likelihood of the data can be written as

\[\tilde{x}^\top \Sigma^{-1} \tilde{x},\]

ignoring the constants and where \(\Sigma^{-1}\) is the inverse covariance for the Gaussian distribution. This prior can be incorporated into an empirical risk minimization framework as a regularizer. Furthermore, it is known that if \((\Sigma^{-1})_{kl} = 0\), then \(x_l\) and \(x_s\) are conditionally independent, given the other variables. In Theorem 1 we established connections to graph based regularizers, and that such methods can be seen as regularizing with the inverse covariance matrix for Gaussians [29]. We thus have the following result:

**Corollary 1.** For any lag set \(\mathcal{L}\), \(\tilde{w}\), and \(\eta > 0\), the inverse covariance matrix \(\Sigma^{-1}\) of the GMRF model corresponding to the quadratic regularizer \(R_\Sigma(\tilde{x}) := T_{\text{AR}}(\tilde{x} | \mathcal{L}, \tilde{w}, \eta)\) shares the same off-diagonal non-zero pattern as \(G^{\text{AR}}\) defined in Theorem 1 for \(T_{\text{AR}}(\tilde{x} | \mathcal{L}, \tilde{w}, \eta)\).

A detailed proof is in Appendix A.2. As a result, our proposed AR-based regularizer is equivalent to imposing a Gaussian prior on \(\tilde{x}\) with a structured inverse covariance described by the matrix \(G^{\text{AR}}\) defined in Theorem 1. Moreover, the step to learn \(W\) has a natural interpretation: the lag set \(\mathcal{L}\) imposes the non-zero pattern of the graphical model on the data, and then we solve a simple least squares problem to learn the weights corresponding to the edges.

\(^2\)a non zero mean can be similarly handled.
5 Experimental Results

In this section, we perform extensive experiments on time series forecasting and missing value imputations on both synthetic and real-world datasets.

Datasets Used:
- **synthetic**: a small synthetic dataset with \( n = 16, T = 128 \). We generate \( \{ x_t \in \mathbb{R}^4 : t = 1, \ldots, 128 \} \) from the autoregressive process (13) with a lag index set \( \mathcal{L} = \{ 1, 8 \} \), randomly generated \( \{ W^{(i)} \} \), and an additive white Gaussian noise of \( \sigma = 0.1 \). We then randomly generate a matrix \( F \in \mathbb{R}^{16 \times 4} \) and obtain \( y_t = F x_t + \epsilon \), where \( \epsilon \sim \mathcal{N}(0, 0.1) \).
- **electricity**: the electricity usage in kW recorded every 15 minutes, for \( n = 370 \) clients. We convert the data to reflect hourly consumption, by aggregating blocks of 4 columns, to obtain \( T = 26,304 \).
- **traffic**: A collection of 15 months of daily data from the California Department of Transportation. The data describes the occupancy rate, between 0 and 1, of different car lanes of San Francisco bay area freeways. The data was sampled every 10 minutes, and we again aggregate the columns to obtain hourly traffic data to finally get \( n = 963, T = 10,560 \).
- **walmart-1 & walmart-2**: two propriety datasets from Walmart E-commerce contain weekly sale information of 1,350 and 1,582 items for 187 weeks, respectively. The time-series of sales for each item start and end at different time points; for modeling purposes we assume one start and end timestamp by padding each series of 1,350 and 1,582 items for 187 weeks, respectively. The time-series of sales for each item start and end at

\[ \text{Normalized deviation (ND): } \left( \frac{1}{|\Omega_{\text{test}}|} \sum_{(i,t) \in \Omega_{\text{test}}} |\hat{Y}_{it} - Y_{it}| \right) / \left( \frac{1}{|\Omega_{\text{test}}|} \sum_{(i,t) \in \Omega_{\text{test}}} |Y_{ij}| \right) \]

\[ \text{Normalized RMSE (NRMSE): } \sqrt{ \frac{1}{|\Omega_{\text{test}}|} \sum_{(i,t) \in \Omega_{\text{test}}} (\hat{Y}_{it} - Y_{it})^2 } / \left( \frac{1}{|\Omega_{\text{test}}|} \sum_{(i,t) \in \Omega_{\text{test}}} |Y_{ij}| \right) \]

For each method and data set, we perform the grid search over various parameters (such as \( k, \lambda \) values) and report the best ND and NRMSE. We search \( k \in \{ 2, 4, 8 \} \) for synthetic and \( \in \{ 20, 40 \} \) for other datasets. For TRMF-AR, SVD-AR(1), TCF, and AR(1), we search \( \lambda \in \{ 50, 5, 0.5, 0.05 \} \)

5.1 Forecasting

We compare the forecasting performance of various approaches. The detailed results are shown in Table 1.

Remark on the implementations of DLM: We first note that the numbers for R-DLM on electricity and traffic are not available because the R package crashes when the dimension of the time series is large (See Appendix B for
Table 1: Forecasting results: ND/ NRMSE for each approach. Lower values are better. ‘-’ indicates an unavailability due to scalability or an inability to handle missing values. DLM denotes the implementation provided in [12], while R-DLM denotes the popular DLM implementation in R [14]. Lower values are better. ‘-’ denotes that the result is not available because the implementation cannot scale up to size the data or deal with the scenario of missing values. TRMF-AR, which is the proposed approach (18), outperforms all other considered approaches in most situations.

| Dataset     | TRMF-AR | SVD-AR(1) | TCF | ND     | NRMSE   | AR(1) | DLM | R-DLM | Mean |
|-------------|---------|-----------|-----|--------|---------|-------|-----|-------|------|
| synthetic   | 0.373/ 0.487 | 0.444/ 0.872 | 1.000/ 1.424 | 0.928/ 1.401 | 0.936/ 1.391 | 0.996/ 1.420 | 1.000/ 1.424 |
| electricity | 0.255/ 1.397 | 0.257/ 1.865 | 0.349/ 1.838 | **0.219/ 1.439** | 0.435/ 2.753 | -/ - | 1.410/ 4.528 |
| traffic     | **0.185/ 0.421** | 0.555/ 1.194 | 0.624/ 0.931 | 0.275/ 0.536 | 0.639/ 0.951 | -/ - | 0.560/ 0.826 |

Forecasting with Missing Values.

| Dataset     | TRMF-AR | SVD-AR(1) | TCF | ND     | NRMSE   | AR(1) | DLM | R-DLM | Mean |
|-------------|---------|-----------|-----|--------|---------|-------|-----|-------|------|
| walmart-1   | **0.533/ 1.958** | -/ - | 0.540/ 2.231 | -/ - | 0.602/ 2.293 | -/ - | 1.239/ 3.103 |
| walmart-2   | **0.432/ 1.065** | -/ - | 0.446/ 1.124 | -/ - | 0.453/ 1.110 | -/ - | 1.097/ 2.088 |

Table 2: Missing value imputation on time series data sets. Note that TRMF outperforms all competing methods in almost all cases. When sufficiently large amount of data is available in the case of the electricity dataset, DLM is slightly better.

| Dataset   | ND     | NRMSE   |
|-----------|--------|---------|
| synthetic |        |         |
| electricity |       |         |
| traffic   |        |         |

Forecasting with Full Observations. We first compare various methods on the task of forecasting values in the test set, given fully observed training data. Three data sets are considered: synthetic, electricity, and traffic. For synthetic, we consider one-point ahead forecasting task and use the last ten time points as the test periods. For electricity and traffic, we consider the 24-hour ahead forecasting task and use last seven days as the test periods. The results are shown in the first part of Table 1. We can clearly observe the superiority of TRMF-AR. Other than the ND for electricity, TRMF-AR outperforms other existing matrix factorization approaches and classic time-series approaches.

Forecasting with Missing Values. We next compare the methods on the task of forecasting in the presence of missing values in the training data. We use the Walmart datasets here, and consider 6-week ahead forecasting and use last 54 weeks as the test periods. Note that SVD-AR(1) and AR(1) cannot handle missing values. The second part of Table 1 shows that we again outperform other methods.

---

5It uses LBFGS-B with a finite-difference approximation to obtain gradients.
5.2 Missing Value Imputation

We next consider the case if imputing missing values in the data. As considered in [11], we assume that entire blocks of data are missing. This corresponds to sensor malfunctions for example, over a length of time.

To create data with missing entries, we first fixed the percentage of data that we were interested in observing, and then uniformly at random occluded blocks of a predetermined length (2 for synthetic data and 5 for the real datasets). The goal was to predict the occluded values. Table ?? shows that TRMF outperforms the methods we compared to on almost all cases.

6 Conclusions

In this paper, we have proposed introduced a novel temporal regularized matrix factorization framework (TRMF) for large-scale multiple time series problems with missing values. TRMF not only models temporal dependency among the data points, but also supports data-driven dependency learning. Our method generalizes several well known methods, and also yields superior performance when compared to other state-of-the-art methods on real-world datasets.

Acknowledgments

This research was supported by NSF grant CCF-1320746 and gifts from Walmart Labs and Adobe. We thank Abhay Jha for the help to the experiments on the real-world datasets from Walmart E-commerce.

References

[1] Oren Anava, Elad Hazan, and Assaf Zeevi. Online time series prediction with missing data. In Proceedings of the International Conference on Machine Learning, pages 2191–2199, 2015.

[2] Zhe Chen and Andrzej Cichocki. Nonnegative matrix factorization with temporal smoothness and/or spatial decorrelation constraints. Laboratory for Advanced Brain Signal Processing, RIKEN, Tech. Rep. 68, 2005.

[3] Zoubin Ghahramani and Geoffrey E. Hinton. Parameter estimation for linear dynamical systems. Technical report, Technical Report CRG-TR-96-2, University of Toronto, Dept. of Computer Science, 1996.

[4] Ronald L. Graham, Donald E. Knuth, and Oren Patashnik. Concrete Mathematics: A Foundation for Computer Science. Addison-Wesley Longman Publishing Co., Inc., 2nd edition, 1994.

[5] Fang Han and Han Liu. Transition matrix estimation in high dimensional time series. In Proceedings of the International Conference on Machine Learning, pages 172–180, 2013.

[6] Robert J. Hodrick and Edward C. Prescott. Postwar business cycles: an empirical investigation. Journal of Money, Credit, and Banking, pages 1–16, 1997.

[7] Prateek Jain and Inderjit S. Dhillon. Provable inductive matrix completion. arXiv preprint arXiv:1306.0626, 2013.

[8] Rudolph Emil Kalman. A new approach to linear filtering and prediction problems. Journal of Fluids Engineering, 82(1):35–45, 1960.

[9] Seung-jean Kim, Kwangmoo Koh, Stephen Boyd, and Dimitry Gorinevsky. ℓ1 trend filtering. SIAM review, 51(2):339–360, 2009.

[10] Yehuda Koren, Robert M. Bell, and Chris Volinsky. Matrix factorization techniques for recommender systems. IEEE Computer, 42:30–37, 2009.

[11] Lei Li, James McCann, Nancy S Pollard, and Christos Faloutsos. DynaMMo: Mining and summarization of coevolving sequences with missing values. In ACM SIGKDD International Conference on Knowledge discovery and data mining, pages 507–516. ACM, 2009.
[12] Lei Li and B Aditya Prakash. Time series clustering: Complex is simpler! In Proceedings of the International Conference on Machine Learning, pages 185–192, 2011.

[13] William B. Nicholson, David S. Matteson, and Jacob Bien. Structured regularization for large vector autoregressions. Technical report, Technical Report, University of Cornell, 2014.

[14] Giovanni Petris. An r package for dynamic linear models. Journal of Statistical Software, 36(12):1–16, 2010.

[15] Giovanni Petris, Sonia Petrone, and Patrizia Campagnoli. Dynamic Linear Models with R. Use R! Springer, 2009.

[16] Swati Rallapalli, Lili Qiu, Yin Zhang, and Yi-Chao Chen. Exploiting temporal stability and low-rank structure for localization in mobile networks. In International Conference on Mobile Computing and Networking, MobiCom '10, pages 161–172. ACM, 2010.

[17] Nikhil Rao, Hsiang-Fu Yu, Pradeep K. Ravikumar, and Inderjit S. Dhillon. Collaborative filtering with graph information: Consistency and scalable methods. In Advances in Neural Information Processing Systems 27, 2015.

[18] Matthew Roughan, Yin Zhang, Walter Willinger, and Lili Qiu. Spatio-temporal compressive sensing and internet traffic matrices (extended version). IEEE/ACM Transactions on Networking, 20(3):662–676, June 2012.

[19] Robert H. Shumway and David S. Stoffer. An approach to time series smoothing and forecasting using the EM algorithm. Journal of time series analysis, 3(4):253–264, 1982.

[20] Alexander J. Smola and Risi Kondor. Kernels and regularization on graphs. In Learning theory and kernel machines, pages 144–158. Springer, 2003.

[21] John Z. Sun, Kush R. Varshney, and Karthik Subbian. Dynamic matrix factorization: A state space approach. In Proceedings of International Conference on Acoustics, Speech and Signal Processing, pages 1897–1900. IEEE, 2012.

[22] Hansheng Wang, Guodong Li, and Chih-Ling Tsai. Regression coefficient and autoregressive order shrinkage and selection via the lasso. Journal of the Royal Statistical Society: Series B (Statistical Methodology), 69(1):63–78, 2007.

[23] Mike West and Jeff Harrison. Bayesian Forecasting and Dynamic Models. Springer Series in Statistics. Springer, 2013.

[24] Liang Xiong, Xi Chen, Tzu-Kuo Huang, Jeff G Schneider, and Jaime G. Carbonell. Temporal collaborative filtering with Bayesian probabilistic tensor factorization. In SIAM International Conference on Data Mining, pages 223–234, 2010.

[25] Miao Xu, Rong Jin, and Zhi-Hua Zhou. Speedup matrix completion with side information: Application to multi-label learning. In C.j.c. Burges, L. Bottou, M. Welling, Z. Ghahramani, and K.q. Weinberger, editors, Advances in Neural Information Processing Systems 26, pages 2301–2309, 2013.

[26] Hsiang-Fu Yu, Cho-Jui Hsieh, Si Si, and Inderjit S. Dhillon. Parallel matrix factorization for recommender systems. Knowledge and Information Systems, 41(3):793–819, 2014.

[27] Hsiang-Fu Yu, Prateek Jain, Purushottam Kar, and Inderjit S. Dhillon. Large-scale multi-label learning with missing labels. In Proceedings of the International Conference on Machine Learning, pages 593–601, 2014.

[28] Yin Zhang, Matthew Roughan, Walter Willinger, and Lili Qiu. Spatio-temporal compressive sensing and internet traffic matrices. SIGCOMM Comput. Commun. Rev., 39(4):267–278, August 2009.

[29] Tinghui Zhou, Hanhuai Shan, Arindam Banerjee, and Guillermo Sapiro. Kernelized probabilistic matrix factorization: Exploiting graphs and side information. In SDM, volume 12, pages 403–414. SIAM, 2012.
A Proofs

A.1 Proof of Theorem 1

Proof. In this proof, we use the notations and summation manipulation techniques introduced by Knuth [4]. To prove (17), it suffices to prove that

$$\sum_{m \leq t \leq T} \left( \sum_{l \in L} w_l x_{t-l} \right)^2 = \sum_{1 \leq t \leq T} \sum_{1 \leq d \leq L} C_{t,t+d} (x_t - x_{t+d})^2 + \bar{x}^T D \bar{x}. \tag{22}$$

The LHS of the (22) can be expanded and regrouped as follows.

$$\sum_{m \leq t \leq T} \left( \sum_{l \in L} w_l x_{t-l} \right)^2 = \sum_{m \leq t \leq T} \left( \sum_{l \in L} w_l^2 x_{t-l}^2 + \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} 2w_l w_{l-d} x_{t-l} x_{t-l+d} \right)$$

$$= \sum_{m \leq t \leq T} \left( \sum_{l \in L} w_l^2 x_{t-l}^2 + \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} \left( -w_l w_{l-d} (x_{t-l} - x_{t-l+d})^2 + w_l w_{l-d} (x_{t-l}^2 + x_{t-l+d}^2) \right) \right)$$

Let’s look at the first term $G(\bar{x})$:

$$G(\bar{x}) = \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} \sum_{m \leq t \leq T} -w_l w_{l-d} (x_{t-l} - x_{t-l+d})^2$$

$$= \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} \sum_{m=1}^{t} -w_l w_{l-d} (x_{t-l} - x_{t+l})^2$$

$$= \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} \left( \sum_{1 \leq t \leq T} \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} -w_l w_{l-d} [m - l \leq t \leq T - l] (x_t - x_{t+d})^2 \right)$$

$$= \sum_{1 \leq t \leq T} \sum_{1 \leq d \leq L} \left( \sum_{l \in \delta(d)} -w_l w_{l-d} \right) (x_t - x_{t+d})^2,$$

where we can see that $G(\bar{x})$ is equivalent to the first term of RHS of (22).

Now, we consider the second term $D(\bar{x})$:

$$D(\bar{x}) = \sum_{m \leq t \leq T} \left( \sum_{l \in L} w_l^2 x_{t-l}^2 + \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} w_l w_{l-d} (x_{t-l}^2 + x_{t-l+d}^2) \right)$$

$$= \sum_{m \leq t \leq T} \sum_{l \in L} w_l^2 x_{t-l}^2 + \sum_{m \leq t \leq T} \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} w_l w_{l-d} x_{t-l} x_{t-l+d} + \sum_{m \leq t \leq T} \sum_{1 \leq d \leq L} \sum_{l \in \delta(d)} w_l w_{l-d} x_{t-l+d}^2$$
Based on the above fact and the results from Theorem 1, we obtain the quadratic form for

\begin{align*}
D_1(\bar{x}) &= \sum_{i \in \mathcal{L}} \sum_{m \leq t \leq T} w_i^2 x_{t-i}^2 = \sum_{i \in \mathcal{L}} w_i^2 \sum_{m \leq t \leq T} x_{t-i}^2 = \sum_{i \in \mathcal{L}} \left( \sum_{m \leq t \leq l \leq T} w_i^2 \right) x_{t-i}^2 \\
&= \sum_{t \leq l \leq T} \left( \sum_{i \in \mathcal{L}} \sum_{t \leq l \leq T} w_i^2 x_{t-l} [m \leq t \leq T] [l' = l] \right) x_{t-i}^2 \\
D_2(\bar{x}) &= \sum_{m \leq t \leq T} \sum_{1 \leq d \leq L} \sum_{l \in \mathcal{L}(d)} w_i w_{l-d} x_{t-l}^2 = \sum_{t \leq l \leq T} \left( \sum_{1 \leq d \leq L} \sum_{l \in \mathcal{L}(d)} w_i w_{l-d} [m \leq t \leq T] \right) x_{t-l}^2 \\
&= \sum_{t \leq l \leq T} \left( \sum_{i \in \mathcal{L}} \sum_{l' < l} w_i w_{l-i} [m \leq t \leq T] [l' < l] \right) x_{t-i}^2 \\
D_3(\bar{x}) &= \sum_{m \leq t \leq T} \left( \sum_{1 \leq d \leq L} \sum_{l \in \mathcal{L}(d)} w_i w_{l-d} x_{t-l+d}^2 = \sum_{t \leq l \leq T} \left( \sum_{1 \leq d \leq L} \sum_{l \in \mathcal{L}(d)} w_i w_{l-d} [m \leq t \leq l - d \leq T] \right) x_{t-l+d}^2 \\
&= \sum_{t \leq l \leq T} \left( \sum_{i \in \mathcal{L}} \sum_{l' > l} w_i w_{l-i} [m \leq t \leq T] [l' > l] \right) x_{t-i}^2
\end{align*}

Let \( D \in R^{T \times T} \) be a diagonal matrix with \( D_{tt} \) be the coefficient associated with \( x_t^2 \) in \( D(\bar{x}) \). Combining the results of \( D_1(\bar{x}), D_2(\bar{x}), \) and \( D_3(\bar{x}), D_t \) can be written as follows.

\[ D_{tt} = \left( \sum_{l \in \mathcal{L}} w_l \right) \left( \sum_{l \in \mathcal{L}} w_{l-m \leq t \leq T} \right) \forall t. \]

It is clear that \( D(\bar{x}) = \bar{x}^T D \bar{x} \). Note that \( \forall t = m, \ldots, T - L, D_{tt} = (\sum_{l \in \mathcal{L}} w_l)^2 \).

**A.2 Proof of Corollary 1**

**Proof.** It is well known that graph regularization can be written in the quadratic form [20] as follows.

\[ \frac{1}{2} \sum_{t \sim s} G_{ts}(x_t - x_s)^2 = \bar{x}^T \text{Lap}(G) \bar{x}, \]

where \( \text{Lap}(G) \) is the \( T \times T \) graph Laplacian for \( G \) defined as:

\[ \text{Lap}(G)_{ts} = \begin{cases} 
\sum_{j \sim s} G_{tj}, & t = s \\
-G_{ts}, & t \neq s \text{ and there is an edge } t \sim s \\
0, & \text{otherwise}
\end{cases} \]

Based on the above fact and the results from Theorem 1, we obtain the quadratic form for \( T_{AR}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) \) as follows.

\[ T_{AR}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) = \frac{1}{2} \bar{x}^T \left( \text{Lap}(G_{AR}) + D + \eta I \right) \bar{x}. \]

Because \( D + \eta I \) is diagonal, the non-zero pattern of the off-diagonal entries of the inverse covariance \( \Sigma^{-1} \) for \( T_{AR}(\bar{x} \mid \mathcal{L}, \bar{w}, \eta) \) is determined by \( \text{Lap}(G_{AR}) \) which shares the same non-zero pattern as \( G_{AR} \).

**B Details: Scalability Issue of R-DLM package**

In this section, we show the source code demonstrating that R-DLM fails to handle high-dimensional time series even with \( n = 32 \). Interested readers can run the following R code to see that the \( \text{d1mMLE()} \) function in R-DLM is able to run on a 16-dimensional time series. However, when we increase the dimension to 32, \( \text{d1mMLE()} \) crashes the entire R program.
library(dlm)

builderFactory <- function(n, k) {
  n = n;
  k = k;
  init = c(rep(0, k), rep(0.1, 3), 0.1*rnorm(n*k), 0.1*rnorm(k*k))
  build = function(x) {
    m0 = x[1:k]
    C0 = (abs(x[k+1]))*diag(k)
    V = (abs(x[k+2]))*diag(n)
    W = (abs(x[k+3]))*diag(k)
    FF = matrix(nrow=n, ncol=k, data=x[(k+3+1):(k+3+n*k)])
    GG = matrix(nrow=k, ncol=k, data=x[(k+3+n*k+1):(k+3+n*k+k*k)])
    return (dlm( m0=m0, C0=C0, FF=FF, GG=GG, V=V, W=W))
  }
  return (list(n=n, k=k, init=init, build=build))
}

Rdlm_train <- function(Y, k, maxit)
  if(missing(maxit)) { maxit=10 }
  if(ncol(Y)==3) {
    Ymat = matrix(nrow=max(Y[,1]), ncol=max(Y[,2]))
    Ymat[cbind(Y[,1], Y[,2])] = Y[,3]
  } else {
    Ymat = Y;
  }
  n = nrow(Ymat)
  TT = ncol(Ymat)
  dlm_builder = builderFactory(n, k)
  mle = dlmMLE(Ymat, dlm_builder$init, build=dlm_builder$build, control=list(maxit=10))
  dlm = dlm_builder$build(mle$par)
  dlm_filt = dlmFilter(Ymat, dlm)
  return (dlm_filt)
}

Rdlm_train(tmp, 4)
print('works')
Rdlm_train(tmp, 4)