GEVREY AND Formal NILSSON SOLUTIONS OF A-HYPERGEOMETRIC SYSTEMS

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ABSTRACT. We prove that the space of Gevrey solutions of an $A$–hypergeometric system along a coordinate subspace is contained in a space of formal Nilsson solutions. Moreover, under some additional conditions, both spaces are equal. In the process we prove some other results about formal Nilsson solutions.

INTRODUCTION

We study formal solutions of $A$–hypergeometric systems, also known as GKZ–systems, as they were introduced by Gel’fand, Graev, Kapranov and Zelevinsky (see [GGZ87] and [GKZ89]). They are systems of linear partial differential equations associated with a pair $(A, \beta)$ where $A$ is a full rank $d \times n$ matrix $A = (a_{ij}) = (a_1 \cdots a_n)$ with $a_j \in \mathbb{Z}^d$ for all $j = 1, \ldots, n$ and $\beta \in \mathbb{C}^d$ is a vector of complex parameters. Recall that the toric ideal of $A$ is defined as

$$I_A := \langle \partial^{a_{+}} - \partial^{a_{-}} | u \in \mathbb{Z}^n, Au = 0 \rangle \subseteq \mathbb{C}[\partial_1, \ldots, \partial_n]$$

where $(u_{+})_j = \max \{ u_j, 0 \}$ and $(u_{-})_j = \max \{ -u_j, 0 \}$ for $j = 1, \ldots, n$. The $A$–hypergeometric system $H_A(\beta)$ is the left ideal of the Weyl algebra $D = \mathbb{C}[x_1, \ldots, x_n](\partial_1, \ldots, \partial_n)$ generated by $I_A$ and by the Euler operators $E_i - \beta_i := \sum_{j=1}^{n} a_{ij} x_j \partial_j - \beta_i$ for $i = 1, \ldots, d$. The $A$–hypergeometric $D$-module is nothing but the quotient $M_A(\beta) := D/H_A(\beta)$. It is well known that $M_A(\beta)$ is a holonomic $D$-module, see [Ado94] and [GKZ89]. In loc. cit., it was also proved that the holonomic rank of $M_A(\beta)$, i.e. the dimension of the space of its holomorphic solutions at a nonsingular point, equals the normalized volume of $A$, denoted by $\text{vol}(A)$ (see 1.1), when $\beta$ is generic. Moreover, $M_A(\beta)$ is regular holonomic if and only if $I_A$ is homogeneous (equivalently, $(1, \ldots, 1)$ lies in the rowspan of the matrix $A$), see [Hot98, Ch. II, 6.2, Thm.], [SST00, Thm. 2.4.11] and [SW08, Corollary 3.16].

Gevrey series solutions of a holonomic $D$–module $M$ along a variety $Y$ are closely related with the irregularity sheaf of $M$ along $Y$ defined by Mebkhout [Meb90] and with the so called slopes of $M$ along $Y$, see [LM99]. The slopes of $M_A(\beta)$ along a coordinate subspace $Y$ were computed in [SW08]. The spaces of Gevrey series solutions of $M_A(\beta)$ along $Y$ were described in [Fer10] (see also [FC11b], [FC11a]) for generic enough parameters $\beta \in \mathbb{C}^d$.

On the other hand, there is an algorithm that computes, for any regular holonomic left $D$–ideal $I$ and a generic vector $w \in \mathbb{R}^n$, a set of canonical series solutions of $I$ that belong to certain Nilsson ring. These series converge in a certain open set that depends on $w$ and form a basis of holomorphic solutions of $I$, see [SST00] chapters 2.5 and 2.6. In [DMM12] the authors introduced a notion of formal Nilsson solutions of $H_A(\beta)$ in the direction of $w$, denoted by $\mathcal{N}_w(H_A(\beta))$, and they used it to generalize various results in [SST00] to the case when $H_A(\beta)$ is not necessarily regular.

In the papers [Fer10] and [DMM12], some of the results assume $\beta$ to be (very) generic, meaning that it lies outside a certain infinite (but locally finite) collection of affine hyperplanes. In particular,
this condition is stronger than \( \beta \) being not rank jumping, a condition that only requires to avoid a concrete finite affine subspace arrangement of codimension at least two \([\text{MMW05}]\). The set of rank jumping parameters is \( \varepsilon(A) := \{ \beta \in \mathbb{C}^d \mid \text{rank}(M_A(\beta)) > \text{vol}(A) \} \) and it was computed in \([\text{MMW05}]\) in terms of the local cohomology modules of the toric ring \( S_A = \mathbb{C}[\mathcal{I}] / I_A \). In particular they proved that \( \varepsilon(A) = \emptyset \) if and only if \( S_A \) is Cohen–Macaulay.

In this note we prove, for all \( \beta \in \mathbb{C}^d \), that the space of Gevrey series solutions of \( M_A(\beta) \) along a coordinate subspace is contained in the space of formal Nilsson solutions of \( H_A(\beta) \) in a certain direction, see Theorem \([3.3] \). We also prove that under one additional condition both spaces coincide and that for \( \beta \notin \varepsilon(A) \) the dimension of this space is the normalized volume of certain submatrix of \( A \), see Theorem \([3.4] \). Moreover, in Section \([2] \) we provide some additional results about formal Nilsson solutions of \( H_A(\beta) \).

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1. Preliminaries.

1.1. Notations. Let \( A = (a_1 \cdots a_n) \) be a \( d \times n \) matrix with columns \( a_j \in \mathbb{Z}^d \) such that \( \mathbb{Z} A := \sum_{j=1}^n \mathbb{Z} a_j = \mathbb{Z}^d \).

For a given subset \( \tau \subseteq \{1, \ldots, n\} \), set \( \tau^c := \{1, \ldots, n\} \setminus \tau \). We shall identify \( \tau \) with the set of columns of \( A \) indexed by \( \tau \) and write \( A_{\tau} \) for the submatrix of \( A \) with column set \( \tau \). We denote by \( \Delta_{\tau} \) the convex hull in \( \mathbb{R}^d \) of all the columns of \( A_{\tau} \) and the origin. We also denote \( \text{pos}(\tau) := \sum_{j \in \tau} \mathbb{R}_{\geq 0} a_j \).

We say that a subset \( \sigma \subseteq \{1, \ldots, n\} \) is a maximal simplex if \( A_{\sigma} \) is an invertible matrix. We associate to a maximal simplex \( \sigma \) an \( n \times (n - d) \) matrix \( B_{\sigma} \), whose columns are indexed by \( \overline{\sigma} \), the \( j \)-th column of \( B_{\sigma} \) has \( \sigma \)-coordinates equal to \(- A_{\sigma}^{-1} a_j \), \( j \)-coordinate equal to one and the rest of coordinates equal to zero. In particular, the columns of \( B_{\sigma} \) form a basis of the kernel of \( A \).

For example, if \( \sigma = \{1, \ldots, d\} \) then

\[
B_{\sigma} = \begin{pmatrix}
-A_{\sigma}^{-1} a_{d+1} & -A_{\sigma}^{-1} a_{d+2} & \cdots & -A_{\sigma}^{-1} a_n \\
1 & 0 & 0 \\
0 & 1 & 0 \\
\vdots & \ddots & \vdots \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

Recall that for any subset \( \tau \subseteq \{1, \ldots, n\} \), the normalized volume of \( A_{\tau} \) (with respect to the lattice \( \mathbb{Z}^d \)) is given by:

\[
\text{vol}(A_{\tau}) := d! \text{vol}_{\mathbb{R}^d}(\Delta_{\tau}) \tag{1.1}
\]

where \( \text{vol}_{\mathbb{R}^d}(\Delta_{\tau}) \) denotes the Euclidean volume of \( \Delta_{\tau} \subseteq \mathbb{R}^d \). We also denote \( \text{vol}(\tau) := \text{vol}(A_{\tau}) \).

If \( \sigma \subseteq \{1, \ldots, n\} \) is a maximal simplex, then \( \text{vol}(\sigma) = [\mathbb{Z}^d : \mathbb{Z} A_{\sigma}] = | \det(A_{\sigma}) | \).

1.2. Regular triangulations. A vector \( w \in \mathbb{R}^n \) defines an abstract polyhedral complex \( T_w \) with vertex set contained in \( \{1, \ldots, n\} \) as follows: \( \tau \in T_w \) if and only if there exists a vector \( c \in \mathbb{R}^d \) such that

\[
\langle c, a_j \rangle = w_j \quad \text{for all} \quad j \in \tau, \tag{1.2}
\]
The Weyl Algebra says that a regular subdivision of $A$ if it satisfies $\text{pos}(A) = \cup_{\tau \in T_w} \text{pos}(\tau)$. This happens for example if $w \in \mathbb{R}_{>0}^n$ or if $A$ is pointed, i.e. the intersection of $\mathbb{R}_{>0}^n$ with the rowspan of $A$ is nonempty.

An element $\tau \in T_w$ is called a facet of $T_w$ if the rank of $A_\tau$ is $d$. Any regular subdivision is determined by its facets and from now on we will write only $\tau \in T_w$ when $\tau$ is a facet of $T_w$. We say that a regular subdivision $T_w$ of $A$ is a regular triangulation of $A$ if all its facets are simplices.

An important case of regular subdivision of $A$ is the following. If $w_j = 1$ for all $j = 1, \ldots, n$, a facet of $T_w$ is the same as a facet of $\Delta_A$ not containing the origin. This particular regular subdivision of $A$ is denoted by $\Gamma_A$.

Notice that for a maximal simplex $\sigma$, it is straightforward from (1.2) and (1.3) that
\[
\sigma \in T_w \iff wB_\sigma > 0.
\] (1.4)

If $T$ is any regular triangulation of $A$ then the set
\[
C(T) := \{w \in \mathbb{R}^n | T = T_w\} = \{w \in \mathbb{R}^n | wB_\sigma > 0, \forall \sigma \in T\}
\] (1.5)
is an open nonempty convex rational polyhedral cone. The closures of these cones and their faces form the so called secondary fan of $A$, introduced and studied by Gel’fand, Kapranov and Zelevinsky [GKZ94, Chapter 7]. When $A$ is pointed, it is easy to see that the secondary fan is a complete fan, i.e. its support is $\mathbb{R}^n$. In general, it is not necessarily complete but its support contains the orthant $\mathbb{R}^n_{\geq 0}$.

1.3. The $A$--hypergeometric fan. A vector $w \in \mathbb{R}^n$ defines a partial order on the monomials of the Weyl Algebra $D$ (and also on the monomials in $\mathbb{C}[\partial_1, \ldots, \partial_n]$) by defining the $(-w, w)$--weight of $x^\alpha \partial^\gamma \in D$ as the real value $\langle w, \gamma - \alpha \rangle$.

The initial form of an element $P = \sum_{\alpha, \gamma \in \mathbb{N}^n} c_{\alpha, \gamma} x^\alpha \partial^\gamma \in D$ with respect to $(-w, w)$, denoted by $\text{in}_{(-w, w)}(P)$, is the sum of the terms $c_{\alpha, \gamma} x^\alpha \partial^\gamma$, with $c_{\alpha, \gamma} \neq 0$, whose $(-w, w)$--weight is maximum. If $I$ is a left $D$--ideal, its initial ideal with respect to $(-w, w)$ is defined as $\text{in}_{(-w, w)}(I) := \langle \text{in}_{(-w, w)}(P) | P \in I, P \neq 0 \rangle$.

**Remark 1.1.** When $A$ is pointed, there is a vector $w' \in \mathbb{R}_{>0}^n$ in the rowspan of $A$ and we have that $\text{in}_{w'}(I_A) = I_A$ and $\text{in}_{(-w', w')}(H_A(\beta)) = H_A(\beta)$. Then, for any $w \in \mathbb{R}^n$ we have that $w^n := w' + \epsilon w \in \mathbb{R}_{>0}^n$, $\text{in}_{w}(I_A) = \text{in}_{w}(\text{in}_{w'}(I_A)) = \text{in}_{w'}(I_A)$ and $\text{in}_{(-w, w)}(H_A(\beta)) = \text{in}_{(-w, w)}(\text{in}_{(-w', w')}(H_A(\beta)))$ for $\epsilon > 0$ small enough, see [SST00, Lemma 2.1.6].

The Gröbner fan of $I_A$, see [Stu95, p. 13] (resp. the small Gröbner fan of $H_A(\beta)$, see [SST00, p. 60]) is a rational polyhedral fan in $\mathbb{R}^n$ whose cones $\mathcal{C}$ satisfy that $\text{in}_{w}(I_A) = \text{in}_{w'}(I_A)$ (resp. $\text{in}_{(-w, w)}(H_A(\beta)) = \text{in}_{(-w, w)}(H_A(\beta))$) for all $w, w' \in \mathcal{C}$, where $\mathcal{C}$ denotes the relative interior of $\mathcal{C}$. By Remark 1.1 these two fans are also complete fans when $A$ is pointed.

**Definition 1.2.** The $A$--hypergeometric fan (at $\beta$) is the coarsest rational polyhedral fan in $\mathbb{R}^n$ that refines both the Gröbner fan of $I_A$ and the small Gröbner fan of $H_A(\beta)$.

**Remark 1.3.** We notice that this fan is a refinement of the hypergeometric fan defined in [SST00, Section 3.3] when $I_A$ is homogeneous and $\beta$ is generic. By [Stu95, Proposition 8.15] and [BF19, Corollary 4.4] the $A$--hypergeometric fan is a refinement of the secondary fan of $A$. 

\[\langle c, a_j \rangle < w_j \text{ for all } j \notin \tau.\] (1.3)
1.4. $\Gamma$–series. Let us denote $\ker(Z(A)) := \{u \in \mathbb{Z}^n \mid Au = 0\}$. Following [GKZ89], for any vector $v \in \mathbb{C}^n$ such that $Av = \beta$, we consider the $\Gamma$–series

$$\varphi_v := \sum_{u \in \ker(Z(A))} \frac{x^{v+u}}{\Gamma(v+u+1)},$$

where $\Gamma(v+u+1) = \prod_{j=1}^n \Gamma(v_j + u_j + 1)$ and $\Gamma$ is the Euler Gamma function. These series are formally annihilated by $H_A(\beta)$. Moreover, when $I_A$ is homogeneous and $\beta \in \mathbb{C}^d$ is generic, a basis of convergent $\Gamma$–series solutions of $M_A(\beta)$ can be constructed by using any regular triangulation of $A$, see loc. cit. These $\Gamma$–series are handled in [SST00, Section 3.4] in the following way:

$$\phi_v := \sum_{u \in \mathbb{N}_v} \frac{[v]_u}{[v+u]_+} x^{v+u},$$

where $N_v = \{u \in \ker(Z(A)) \mid \forall j = 1, \ldots, n, v_j + u_j \in \mathbb{Z}_{<0} \text{ iff } v_j \in \mathbb{Z}_{<0}\}$ and

$$[v]_u = \prod_{j=1}^n v_j(v_j - 1) \cdots (v_j - u_j + 1).$$

Set $\text{nsupp}(v) := \{j \in \{1, \ldots, n\} \mid v_j \in \mathbb{Z}_{<0}\}$ for any $v \in \mathbb{C}^n$.

The series $\phi_v$ is annihilated by $H_A(\beta)$ if and only if $v$ has minimal negative support, i.e. there is no $u \in \ker(Z(A))$ such that $\text{nsupp}(v+u) \subsetneq \text{nsupp}(v)$, see [SST00, Proposition 3.4.13] whose proof works as well when $I_A$ is not homogeneous.

**Remark 1.4.** It is easy to check that $\Gamma(v+1)\varphi_v = \phi_v$ when $v \in (\mathbb{C} \setminus \mathbb{Z}_{<0})^n$. Notice that $\varphi_v = \varphi_{v+u}$ for any $u \in \ker(Z(A))$. Thus, for $u \in N_v$ there is a nonzero scalar $c \in \mathbb{C}$ such that $\phi_v = c \cdot \phi_{v+u}$.

1.5. Gevrey series solutions of $A$-hypergeometric $D$-modules. In this section we introduce the notion of Gevrey series and we recall some notations and results from [Fer10]. Let us denote, for a subset $\tau \subseteq \{1, \ldots, n\}$, $Y_\tau := \{x_j = 0 \mid j \notin \tau\}$ and $x_\tau$ for the set of variables $x_j$ with $j \in \tau$. We denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X = \mathbb{C}^n$ and by $\mathcal{O}_{X|Y_\tau}$ the sheaf of formal series along $Y_\tau$. A germ of $\mathcal{O}_{X|Y_\tau}$ at $p \in Y_\tau$ can be written as

$$f = \sum_{\alpha \in \mathbb{N}^\tau} f_\alpha(x_\tau) x_\tau^\alpha \in \mathcal{O}_{X|Y_\tau,p} \subseteq \mathbb{C}\{x_\tau - p_\tau\}[\![x_\tau]\!]$$

where $f_\alpha(x_\tau) \in \mathcal{O}_{Y_\tau}(U)$ for certain nonempty relatively open subset $U \subseteq Y_\tau$, $p \in U$. A formal series $f = \sum_{\alpha \in \mathbb{N}^\tau} f_\alpha(x_\tau) x_\tau^\alpha \in \mathcal{O}_{X|Y_\tau,p}$ is said to be Gevrey of order $s \in \mathbb{R}$ along $Y_\tau$ at $p \in Y_\tau$ if the series

$$\sum_{\alpha \in \mathbb{N}^\tau} \frac{f_\alpha(x_\tau)}{(\prod_{j \in \tau} \alpha_j)! s^{-1} \tau^\alpha}$$

is convergent at $p$.

Since $M_A(\beta)$ is a holonomic $D$–module, any of its formal solutions along $Y_\tau$ is Gevrey of some order. We denote by $\text{Hom}_D(M_A(\beta), \mathcal{O}_{X|Y_\tau,p})$ the space of all Gevrey solutions of $M_A(\beta)$ along $Y_\tau$ at $p \in Y_\tau$.

Given a maximal simplex $\sigma$ and a vector $k = (k_i)_{i \in \sigma} \in \mathbb{N}^\sigma$ we denote by $v_\sigma^k \in \mathbb{C}^n$ the vector with $\sigma$–coordinates equal to $A^{-1}_\sigma(\beta - A_\sigma k)$ and $\tau$–coordinates equal to $k$. Let $\Omega_\sigma \subseteq \mathbb{N}^\sigma$ be a set of representatives for the different classes with respect to the following equivalence relation

$\{v \in \mathbb{Z}^n \mid (v + k) \in \Lambda^\sigma, \forall k \in \mathbb{Z}^\sigma \text{ such that } k \not\in \Omega_\sigma\}$

It is easy to see that $\text{nsupp}(v_\sigma^k) = \Omega_\sigma$.
in $\mathbb{N}^\sigma$: we say that $k \sim k'$ if and only if $A_{\sigma}k - A_{\sigma}k' \in \mathbb{Z}A_{\sigma}$. Thus, $\Omega_{\sigma}$ is a set of cardinality $\text{vol}(\sigma) = [Z^d : \mathbb{Z}A_{\sigma}]$.

When $\beta$ is generic, the space of Gevrey solutions of $M_A(\beta)$ along $Y_\tau$ is explicitly described in [Fer10].

**Theorem 1.5.** [Fer10, Theorem 6.7 and Remark 6.8] If $T(\tau)$ is a regular triangulation of $A_x$ that refines $\Gamma_{A_x}$ and $\beta \in \mathbb{C}^d$ is generic enough, the set $\{\phi_{\sigma,k} : \sigma \in T(\tau), k \in \Omega_{\sigma}\}$ is a basis of the space of Gevrey series solutions of $M_A(\beta)$ along $Y_\tau$ at any point $p$ of a certain nonempty relatively open set $W_T(\tau) \subseteq Y_\tau$. In particular, $\dim_{\mathbb{C}}(\text{Hom}_D(M_A(\beta), \mathcal{O}_{X/Y_\tau,p})) = \text{vol}(\tau)$.

For more precise statements, including the Gevrey order of these series and its relation with the so-called slopes of $M_A(\beta)$ along $Y_\tau$, see [Fer10]. The slopes of $M_A(\beta)$ along $Y_\tau$ were described in [SW08].

Notice that if $\tau = A$ then $Y_\tau = \mathbb{C}^n$, $\mathcal{O}_{X/Y_\tau} = \mathcal{O}_X$, and Theorem 1.5 gives a basis of holomorphic functions of $M_A(\beta)$ at any point of $W_T(\tau)$ when $\beta \in \mathbb{C}^d$ is generic. Such a basis was first described in [OT09] (see also [GKZ89] when $I_A$ is homogeneous).

The following result is the first part of [Fer10, Theorem 6.2].

**Theorem 1.6.** If $p$ is a generic point of $Y_\tau$ then, for all $\beta \in \mathbb{C}^d$, $\dim_{\mathbb{C}}(\text{Hom}_D(M_A(\beta), \mathcal{O}_{X/Y_\tau,p})) \geq \text{vol}(\tau)$.

### 1.6. Formal Nilsson solutions of $A$-hypergeometric $D$-modules.

We recall here some definitions and results from [DMM12], see also [SST00] when $I_A$ is homogeneous. In the former paper the authors write the following results in terms of a regular triangulation of the matrix $\rho(A) := (\tilde{a}_0 \ a_1 \cdots \ a_n)$, that is constructed from $A$ by adding a first column of zeroes and then a first row of ones. It follows from the definition of regular subdivision (see (1.2) and (1.3)) that for a subset $\sigma \subseteq \{1, \ldots, n\}$, we have that $\sigma \in T_w$ if and only if $\{0\} \cup \sigma \in T_{(0,w)}$.

Given a cone $C \subseteq \mathbb{R}^n$, the dual cone of $C$, denoted by $C^*$, is a closed cone consisting of vectors $u \in \mathbb{R}^n$ such that $\langle w, u \rangle \geq 0$ for all $w \in C$ and all $u \in C^*$. If $C$ is full dimensional, then the cone $C^*$ is strongly convex (i.e., it doesn’t contain non trivial linear subspaces) and $\langle w, u \rangle > 0$ for all $w \in C$ and all nonzero $u \in C^*$.

We say that $w \in \mathbb{R}^n_{\geq 0}$ is a weight vector (for $H_A(\beta)$) if it belongs to the interior of a full dimensional cone of the $A$–hypergeometric fan.

For a weight vector $w \in \mathbb{R}^n$ we denote by $C_w$ the interior of the (full dimensional) cone in the $A$–hypergeometric fan such that $w \in C_w$. Notice that $C_w \subseteq C(T_w)$.

**Definition 1.7.** [DMM12, Definition 2.6] Let $w$ be a weight vector for $H_A(\beta)$. Write $\log(x) = (\log x_1, \ldots, \log x_n)$. A basic Nilsson solution of $H_A(\beta)$ in the direction of $w$ is a series of the form

$$\phi = x^v \sum_{w \in C} x^w p_w(\log(x)),$$

where $v \in \mathbb{C}^n$, that satisfies

i) $\phi$ is annihilated by the partial differential operators of $H_A(\beta)$;

ii) $C$ is contained in $\ker_{\mathbb{Z}}(A) \cap C^*$ for some strongly convex open cone $C \subseteq C_w$ such that $w \in C$;
iii) the $p_u$ are nonzero polynomials and there exists $K \in \mathbb{Z}$ such that $\deg(p_u) \leq K$ for all $u \in C$;
iv) $0 \in C$.

The set $\text{supp}(\phi) = \{v + u \mid u \in C\}$ is called the support of $\phi$.

The $\mathbb{C}$–linear span of all basic Nilsson solutions of $H_A(\beta)$ in the direction of $w$ is called the space of formal Nilsson solutions of $H_A(\beta)$ in the direction of $w$ and it is denoted by $\mathcal{N}_w(H_A(\beta))$.

We define the $w$–weight of a term $x^v p(\log(x))$, where $v \in \mathbb{C}^n$ and $p$ is a polynomial, as the real value $\text{Re}(\langle w, v \rangle)$. For a series $\phi$ consisting in a (possibly infinite) sum of terms, we say that it has an initial form if there exists the minimum for the set of $w$–weights of all its nonzero terms. In this case, its initial form in the direction of $w$, denoted by $\text{in}_w(\phi)$, consists in the sum of all the terms of $\phi$ with minimum $w$–weight.

**Remark 1.8.** Notice that if a series $\phi$ as in (1.7) satisfies all conditions in Definition 1.7 we have $\text{in}_w(\phi) = x^v p_0(\log(x))$.

**Proposition 1.9.** [DMM12, Proposition 2.11] Let $w \in \mathbb{R}^n$ be a weight vector for $H_A(\beta)$, then $\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) \leq \text{rank}(\text{in}_{(-w,w)}(H_A(\beta)))$.

We recall that a vector $v \in \mathbb{C}^n$ is called an exponent of $H_A(\beta)$ with respect to $w$ if $x^v$ is a solution of $\text{in}_{(-w,w)}(H_A(\beta))$.

**Theorem 1.10.** [DMM12, Theorem 4.8] If $\beta$ is generic and $w$ is a weight vector for $H_A(\beta)$ then the set

$$\{\phi_v \mid v \text{ is an exponent of } H_A(\beta) \text{ with respect to } w\}$$

is a basis of $\mathcal{N}_w(H_A(\beta))$, where $\phi_v$ is defined in (1.6).

**Theorem 1.11.** [DMM12, Corollaries 4.9 and 4.11] If $\beta$ is generic and $w$ is a weight vector for $H_A(\beta)$ then

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) = \text{rank}(\text{in}_{(-w,w)}(H_A(\beta))) = \deg(\text{in}_w(I_A)) = \sum_{\sigma \in T_w} \text{vol}(\sigma). \quad (1.8)$$

Let $w \in \mathbb{R}_{>0}^n$ be a weight vector for $H_A(\beta)$. We say that $w$ is a perturbation of a vector $w_0 \in \mathbb{R}^n$ if there exists a full dimensional cone $C$ of the $A$–hypergeometric fan such that $w_0 \in C$ and $w \in \hat{C}$.

We notice that a weight vector $w$ is a perturbation of $(1, \ldots, 1)$ if and only if the regular triangulation $T_w$ is a refinement of $\Gamma_A$.

**Theorem 1.12.** [DMM12, Theorem 6.4] Assume that $A$ is pointed. If $w$ is a perturbation of $(1, \ldots, 1)$ then, for all $\beta \in \mathbb{C}^d$,

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) = \text{rank}(\text{in}_{(-w,w)}(H_A(\beta))) = \text{rank}(M_A(\beta)). \quad (1.9)$$

More precisely, $\mathcal{N}_w(H_A(\beta))$ is the space of convergent series solutions of $M_A(\beta)$ at any point in a certain nonempty open set $\mathcal{U}_w \subseteq \mathbb{C}^n$.

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2. SOME REMARKS ON FORMAL NILSSON SOLUTIONS OF $H_A(\beta)$.

In this section we provide some additional results about $\mathcal{N}_w(H_A(\beta))$. 

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1. Theorem 1.11. [DMM12, Corollaries 4.9 and 4.11] If $\beta$ is generic and $w$ is a weight vector for $H_A(\beta)$ then

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) = \text{rank}(\text{in}_{(-w,w)}(H_A(\beta))) = \deg(\text{in}_w(I_A)) = \sum_{\sigma \in T_w} \text{vol}(\sigma). \quad (1.8)$$

Let $w \in \mathbb{R}_{>0}^n$ be a weight vector for $H_A(\beta)$. We say that $w$ is a perturbation of a vector $w_0 \in \mathbb{R}^n$ if there exists a full dimensional cone $C$ of the $A$–hypergeometric fan such that $w_0 \in C$ and $w \in \hat{C}$.

We notice that a weight vector $w$ is a perturbation of $(1, \ldots, 1)$ if and only if the regular triangulation $T_w$ is a refinement of $\Gamma_A$.

**Theorem 1.12.** [DMM12, Theorem 6.4] Assume that $A$ is pointed. If $w$ is a perturbation of $(1, \ldots, 1)$ then, for all $\beta \in \mathbb{C}^d$,

$$\dim_{\mathbb{C}}(\mathcal{N}_w(H_A(\beta))) = \text{rank}(\text{in}_{(-w,w)}(H_A(\beta))) = \text{rank}(M_A(\beta)). \quad (1.9)$$

More precisely, $\mathcal{N}_w(H_A(\beta))$ is the space of convergent series solutions of $M_A(\beta)$ at any point in a certain nonempty open set $\mathcal{U}_w \subseteq \mathbb{C}^n$. 

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2. SOME REMARKS ON FORMAL NILSSON SOLUTIONS OF $H_A(\beta)$.
Lemma 2.1. For any $\beta \in \mathbb{C}^d$ and any weight vector $w$,

$$\dim_{\mathbb{C}}(N_w(H_A(\beta))) \geq \sum_{\sigma \in T_w} \operatorname{vol}(\sigma).$$

(2.1)

Proof. By Theorem 1.11 equality holds in (2.1) when $\beta$ is generic and, by Theorem 1.10 there is a basis of $N_w(H_A(\beta))$ that consists of the set of series $\phi_\sigma$, see (1.6), for $v$ varying in the set of exponents of $H_A(\beta)$ with respect to $w$. In this situation, when $\beta$ is not generic, we can apply the same procedure as in the proof of [SST00, Theorem 3.5.1] and obtain a set of linearly independent formal Nilsson solutions of $H_A(\beta)$ in the direction of $w$. The cardinality of this set is the rightmost quantity in (2.1).

$$\square$$

Corollary 2.2. If $w$ is a weight vector, then (1.8) holds for any $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$.

Proof. By [SW08, Theorem 4.28] and [BF19, Lemma 3.1], $\operatorname{rank}(\text{in}(\neg w, w))(H_A(\beta)))$ is constant for $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$ and hence the second equality in (1.8) holds in this case too. The first equality in (1.8) now follows from Lemma 2.1 and Proposition 1.9.

The following result states that the basis given in Theorem 1.10 only depends, up to multiplication of their elements by nonzero scalars, on the regular triangulation $T_w$ and not on the cone $\mathcal{C}_w \subseteq C(T_w)$.

Proposition 2.3. If $\beta \in \mathbb{C}^d$ is generic and $w$ is a weight vector, then the set

$$\mathcal{B}_w(\beta) := \{\phi_\sigma^k : \sigma \in T_w, \ k \in \Omega_\sigma\}$$

is a basis of $N_w(H_A(\beta))$.

Proof. By the assumption on $\beta$, the difference between two vectors in the set $\{v_\sigma^k : \sigma \in T_w, \ k \in \Omega_\sigma\}$ is not an integer vector. Thus, the series in $\mathcal{B}_w(\beta)$ have pairwise disjoint supports, hence they are linearly independent. Moreover, $\text{nsupp}(v_\sigma^k) = \emptyset$, which implies that $\phi_\sigma^k$ is annihilated by $H_A(\beta)$, see [SST00, Proposition 3.4.13].

On the other hand, the support of $\phi_\sigma^k$ is the set

$$\text{supp}(\phi_\sigma^k) = \{v_\sigma^k + (B_\sigma m)^t : m \in \mathbb{Z}^\sigma, \ B_\sigma m \in \mathbb{Z}^n \text{ and } k + m \in \mathbb{N}^\sigma\}.$$ 

Notice that $v_\sigma^k + (B_\sigma m)^t = v_\sigma^0 + (B_\sigma(k + m))^t$ where $k + m \in \mathbb{N}^\sigma$. Thus, since $wB_\sigma > 0$ for any $\sigma \in T_w$, we have that $\Re(\langle w, v'\rangle) \geq \Re(\langle w, v_\sigma^0 \rangle)$ for all $v' \in \text{supp}(\phi_\sigma^k)$. It follows that there exists the initial form $\text{in}_w(\phi_\sigma^k)$ and that it consists in a finite sum of terms. By the proof of [SST00, Theorem 2.5.5] we also have that $\text{in}_w(\phi_\sigma^k)$ is a solution of $\text{in}(\neg w, w)(H_A(\beta))$, but a basis of its solutions is given by the set of monomials $v^\sigma v$ for $v$ varying in the set of exponents of $H_A(\beta)$ (see Theorems 1.10 and 1.11). It follows that there exists an exponent $\tilde{v_\sigma^k}$ of $H_A(\beta)$ with respect to $w$ such that $\tilde{v_\sigma^k} \in \text{supp}(\phi_\sigma^k)$, hence $\phi_\sigma^k = c_{\sigma, k} \cdot \phi_\sigma^k$ for some nonzero scalar $c_{\sigma, k} \in \mathbb{C}$, see Remark 1.4. This implies that $\mathcal{B}_w(\beta)$ is a basis of $N_w(H_A(\beta))$ in this case, by Theorem 1.10.

$$\square$$

Corollary 2.4. If $w, w'$ are weight vectors for $H_A(\beta)$, then we have the following:

i) If $\beta$ is generic, $N_w(H_A(\beta)) = N_{w'}(H_A(\beta))$ if and only if $T_w = T_{w'}$.

ii) For all $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, $N_w(H_A(\beta)) = N_{w'}(H_A(\beta))$ if $\mathcal{C}_w = \mathcal{C}_{w'}$.

iii) For all $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, the cone $\mathcal{C}$ in Definition 1.7 can be chosen to be $\mathcal{C}_w$ for any basic Nilsson solution of $H_A(\beta)$ in the direction of $w$. 

Proof. Proposition 2.3 directly implies i). Let us prove ii). We can take a basis $B = \{\phi_1, \ldots, \phi_r\}$ of $N_w(H_A(\beta))$ so that each $\phi_i = x^{w(i)} \sum_{u \in C_i} p_u^{(i)}(\log(x))$ is a basic Nilsson solution of $H_A(\beta)$ in the direction of $w$. Thus, for $i = 1, \ldots, r$, there exists a strongly convex open cone $C_i$ as in condition ii) of Definition 1.7, i.e. $w \in C_i \subseteq C_w$ and $C_i \subseteq C_i^* \cap \ker(A)$. Then the strongly convex open cone $C := \cap_{k=1}^r C_k \subseteq C_w$ satisfies that condition for all $i = 1, \ldots, r$, since $C \subseteq C_i$ implies $C_i^* \subseteq C_i^*$. It follows that for all $w' \in C$, the series $\phi_i$ are also basic Nilsson solutions in the direction of $w'$ and, by Remark 1.8, $\in_w(\phi_i) = x^{w(i)} p_0^{(i)}(\log(x))$. This implies that $N_w(H_A(\beta)) \subseteq N_w(H_A(\beta))$, for all $w' \in C$. But this last inclusion must be an equality since both spaces have the same dimension, see Corollary 2.2. It follows that the space $N_w(H_A(\beta))$, its subset of basic Nilsson solutions $\phi_i$ and their initial forms in $w_0(\phi_i)$ are locally constant with respect to the weight vector $w$, hence they are constant in the whole open cone $C_u$ of the $A$-hypergeometric fan at $\beta$. This proves ii). For iii), notice that, the fact that $\phi \not\in \in_w(\phi)$ for all some series $\phi \in B_w(\beta) = B_w(\beta)$ (in which case case $\in_{(-w,w)}(H_A(\beta)) \neq in_{(-w',w')}(H_A(\beta))$). For example, let us consider the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

and a generic $\beta \in \mathbb{C}^3$. The maximal simplex $\sigma = \{1, 4\}$ defines a regular triangulation of $A$ induced by any of the weight vectors $w^{(1)} = (1, 2, 5, 1)$ and $w^{(2)} = (1, 5, 2, 1)$. We can choose $\Omega_\sigma$ so that the series $\phi_v$ with $v = (\beta_1 - (\beta_2 + 2)/3, 1, 0, (\beta_2 - 1)/3)$, see (1.6), belongs to $B_{w^{(1)}}(\beta) = B_{w^{(2)}}(\beta)$. Notice that

$$N_v = \{x = -(2m_2 + m_3)/3, m_2, m_3, -(m_2 + 2m_3)/3 \in \mathbb{Z}^3 \} \geq -1, \ m_3 \geq 0 \}.$$ 

Thus, $\in_{w^{(1)}}(\phi_v) \neq in_{w^{(2)}}(\phi_v)$ since $v + (0, -1, 2, -1) \in supp(\phi_v) = v + N_v$ has smaller $w^{(2)}$-weight than $v$.

The following result improves [DMM12, Corollary 6.9].

**Corollary 2.6.** If $w$ is a weight vector, we have, for all $\beta \in \mathbb{C}^d$, that

$$\dim_{\mathbb{C}}(\{\phi \in N_w(H_A(\beta)) | \phi \text{ is convergent}\}) \geq \sum_{\sigma \in T_w^0} \text{vol}(\sigma) \tag{2.2}$$

where $T_w^0 = \{\sigma \in T_w | \sigma \subseteq \eta \text{ for some } \eta \in \Gamma_A\}$. Moreover, equality holds in (2.2) if $\beta$ is generic.

**Proof.** Assume first that $\beta \in \mathbb{C}^d$ is generic. Thus, $B_w(\beta)$ is a basis of $N_w(H_A(\beta))$ by Proposition 2.3. On the other hand, it follows from [Fer10, Theorem 3.11] and the definition of $B_w$, that $\phi_{\sigma,k}$ is convergent if and only if $(1, \ldots, 1) B_\sigma \geq 0$, that holds if and only if $\sigma$ is contained in a facet of $\Gamma_A$. Thus, $B_w^0(\beta) := \{\phi_{\sigma,k} : \sigma \in T_w^0, k \in \Omega_\sigma\}$ is a linearly independent set of convergent formal Nilsson solutions of $H_A(\beta)$ in the direction of $w$. The equality in (2.2) for generic $\beta$ follows from the fact that the series in $B_w(\beta) \setminus B_w^0(\beta)$ are all divergent and have pairwise disjoint supports, so no linear combination of them can be convergent. If $\beta \in \mathbb{C}^d$ is not assumed to be generic, we can consider a generic parameter $\beta'$ and apply to the set $B_w(\beta + \epsilon \beta')$, with $\epsilon \in \mathbb{C}$ such that $|\epsilon|$ is small enough, the same method as in [SST07, Theorem 3.5.1] to get the desired lower bound in (2.1). \qed
Lemma 2.7. [Sai02 Lemma 5.3] Let \( p(y) \in \mathbb{C}[y] \), \( v \in \mathbb{C}^n \) and \( \nu \in \mathbb{N}^n \), then
\[
\partial^\nu [x^\nu p(\log(x))] = x^{\nu-\nu} \sum_{0 \leq \nu' \leq \nu} \lambda_{\nu'} \partial^{\nu'-\nu}[p](\log(x))
\]
where the sum is over \( \nu' \in \mathbb{N}^n \) such that \( \nu'_j \leq \nu_j \) for all \( j \), and \( \lambda_{\nu'} \in \mathbb{C} \). In particular,
\[
\lambda_\nu = [v]_\nu = \prod_{j=1}^n v_j (v_j - 1) \cdots (v_j - \nu_j + 1).
\]

The following lemma guarantees that in the third condition of Definition 1.7 we can assume that the constant \( K \) is independent of \( \beta \).

Lemma 2.8. Let \( w \) be a weight vector. Then for any basic Nilsson solution \( \phi \) of \( H_A(\beta) \) in the direction of \( w \) as in (1.7) we have that
\[
\deg(p_u) \leq (n + 1)(2^{2(d+1)}\text{vol}(A) - 1)
\]
for all \( u \in C \).

Proof. Assume first that \( I_A \) is homogeneous. In this case \( M_{A_\beta - A_\alpha} \) is regular holonomic [Hor98] and \( \deg(p_u) \leq n(2^d \text{vol}(A) - 1) \) by [SST00 Theorem 2.5.14 and Corollary 4.1.2].

If \( I_A \) is not homogeneous, we can consider the \((d + 1) \times (n + 1)\) matrix \( \rho(A) \) as defined at the beginning of Subsection 1.6. Notice that \( \phi \) is also a basic Nilsson solution of \( H_A(\beta) \) in the direction of \( w' \) for all \( w' \in C \), where \( C \) is an open cone as in condition ii) in Definition 1.7. In particular, we can assume without loss of genericity that \( w \) is generic. This implies that \((0, w) + \lambda(1, \ldots, 1) \subset \mathbb{R}^{n+1} \) is generic if \( \lambda > 0 \) is generic. Since \((1, \ldots, 1) \in \text{row span of } \rho(A) \), we have that \((0, w) \) is a weight vector for \( H_{\rho(A)}(\beta_0, \beta) \), where \( \beta_0 \subset \mathbb{C} \) (see also [DMM12 Remark 2.5]).

Assume that \( \beta_0 \subset C \) is sufficiently generic and consider the following series, see [DMM12 Definition 3.16],
\[
\rho(\phi) := \sum_{u \in C} \partial^{|u|}_0 \left[ x_0^{\beta_0 - |u|} x^{u+u} \widehat{p}_u(\log(x_0), \ldots, \log(x_n)) \right]
\]
where \(|u| := \sum_{j=1}^n u_j \), \( \partial_0^{-k} \) is defined in [DMM12 Definition 3.13] when \( k > 0 \), and \( \widehat{p}_u \in \mathbb{C}[y_0, \ldots, y_n] \) is defined from \( p_u \) as in [DMM12 (3.2)]. We remark here that \( \deg(p_u) \leq \deg(\widehat{p}_u) \) because \( \widehat{p}_u(0, y_1, \ldots, y_n) = p_u(y_1, \ldots, y_n) \).

By Lemma 2.7 and [DMM12 Lemma 3.12 and Definition 3.13], we can write
\[
\rho(\phi) = \sum_{u \in C} x_0^{\beta_0 - |u| - |u|} x^u h_u(\log(x_0), \ldots, \log(x_n))
\]
where \( h_u(y) \subset \mathbb{C}[y_0, \ldots, y_n] \). By Lemma 2.7, if \(|u| \geq 0\), then \( h_u = [\beta_0 - |u|] \cdot \widehat{p}_u \) plus other polynomial of degree smaller than \( \deg(\widehat{p}_u) \). This implies that \( \deg(h_u) = \deg(\widehat{p}_u) \) when \(|u| \geq 0 \) because \( \beta_0 \) is generic. For \(|u| < 0\) it is also true that \( \deg(h_u) = \deg(\widehat{p}_u) \) by [DMM12 Lemma 3.12 and Definition 3.13].

On the other hand, by [DMM12 Proposition 3.17] the series \( \rho(\phi) \) is a basic Nilsson solution in the direction of \((0, w) \) of the hypergeometric system \( H_{\rho(A)}(\beta_0, \beta) \) and since \( I_{\rho(A)} \) is homogeneous, we have that \( \deg(h_u) \leq (n + 1)(2^{2(d+1)}\text{vol}(A) - 1) \), where \( \text{vol}(\rho(A)) = \text{vol}(A) \). Thus,
\[
\deg(p_u) \leq \deg(\widehat{p}_u) = \deg(h_u) \leq (n + 1)(2^{2(d+1)}\text{vol}(A) - 1).
\]
\( \square \)
Let $\tau \subseteq \{1, \ldots, n\}$ be a subset such that $A_\tau$ is pointed and $\text{rank}(A_\tau) = d$. In this section we prove, for all $\beta \in \mathbb{C}^d$, that the space of Gevrey solutions of $M_A(\beta)$ along a coordinate subspace $Y_\tau$ is contained in the space of formal Nilsson solutions of $H_A(\beta)$ in a certain direction, see Theorem 3.3. If we further assume that $\text{pos}(A) = \text{pos}(A_\tau)$, we also prove that both spaces are the same and, for $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, its corresponding dimension is $\text{vol}(\tau)$, see Theorem 3.4.

The following result follows from [DMM12, Lemma 3.6] (see also [SST00, Lemma 4.1.3]), [SVW95, (3.2)] (see also [SST00, (3.13)]) and [Stu95, Corollary 8.4].

**Lemma 3.1.** If $w$ is a weight vector and $v$ is an exponent of $H_A(\beta)$ with respect to $w$, there exists $\sigma \in T_w$ such that $v_j \in \mathbb{N}$ for all $j \notin \sigma$.

**Lemma 3.2.** For any regular triangulation $T(\tau)$ of $A_\tau$ there exists a regular triangulation $T$ of $A$ such that $T(\tau) \subseteq T$. In particular, if $\text{pos}(A) = \text{pos}(A_\tau)$ then $T(\tau) = T$.

**Proof.** By definition of regular triangulation, there is a weight vector $w(\tau) \in \mathbb{R}^\tau$ such that $T(\tau) = T_{w(\tau)}$. Then choose another generic vector $v(\tau) \in \mathbb{R}_{\geq 0}^\tau$, and consider $w \in \mathbb{R}^n$ to be a vector with $\tau$–coordinates $\epsilon w(\tau)$, with $\epsilon > 0$ small enough, and $\sigma$–coordinates $w(\tau)$. Since $\epsilon w(\tau), w(\tau)$ and $\epsilon > 0$ can be chosen to be generic, it follows that $w$ induces a regular triangulation $T := T_w$ of $A$. By using the definition of regular triangulation, see conditions (1.2) and (1.3) (but substitute $\tau$ there by a maximal simplex $\sigma$), it is easy to check that $T(\tau) \subseteq T$ if $\epsilon > 0$ is small enough.

Let $T(\tau)$ be a regular triangulation of $A_\tau$ refining $\Gamma_{A_\tau}$ and $w$ a weight vector for $H_A(\beta)$ chosen as in the proof of Lemma 3.2. Thus, by the assumption on $T(\tau)$ we have that $w(\tau)$ is a perturbation of $(1, \ldots, 1) \in \mathbb{R}^\tau$. Recall that $w, w(\tau)$ and $w(\tau)$ are all chosen to be generic.

**Theorem 3.3.** If $A_\tau$ is pointed and $\text{rank}(A_\tau) = d$, any Gevrey solution of $M_A(\beta)$ along $Y_\tau$ (at $p \in U_{w(\tau)}$) can be written as a formal Nilsson solution of $H_A(\beta)$ in the direction of $w$.

**Proof.** Let $f = \sum_{\alpha \in \mathbb{N}^\tau} f_\alpha(x_\tau) x_\tau^\alpha \in \mathcal{O}_{X[Y,p]}$ be any Gevrey series solution of $M_A(\beta)$. By [Fer10, Lemma 6.11] we have that $f_\alpha(x_\tau)$ is a holomorphic solution of $M_A(\beta - A_\tau \alpha)$ at $p$. Thus, by Theorem 1.12 each $f_\alpha(x_\tau)$ can be written as an element of $\mathcal{N}_{w(\tau)}(H_A(\beta - A_\tau \alpha))$, i.e. a finite linear combination of series of the form (1.7) in the variables $x_\tau$ convergent at $p$. In particular, $f$ can be rewritten as a sum of terms $x_\tau^\gamma q_\gamma(\log x_\tau)$ where $\gamma \in \mathbb{C}^n$ and $q_\gamma \in \mathbb{C}[x_\tau]$ are polynomials with degree $\deg(q_\gamma) \leq (n + 1)2^{d+1}\text{vol}(A_\tau)$, see Lemma 2.8. Moreover, notice that the result of applying a monomial $x_\tau^\nu \partial^\mu$ in the Weyl Algebra $D$ to $x_\tau^\gamma q_\gamma(\log x_\tau)$ is of the form $x_\tau^{\gamma - \mu + \lambda} g(\log(x_\tau))$ for some polynomial $g$. This implies that any subseries of $f$ of the form $\sum_{\gamma \in \mathbb{C}^n} x_\tau^{\gamma} q_\gamma(\log x_\tau)$, for some $\nu \in \mathbb{C}^n$, is still annihilated by $H_A(\beta)$ and hence defines a Gevrey solution of $M_A(\beta)$. Since the dimension of the space of Gevrey series solutions is finite, $f$ is a finite sum of such subseries, say $F_1, \ldots, F_r$. Since $F_k$ is annihilated by the Euler operators of $A$ we have that $A\gamma = \beta$ for all $\gamma \in \mathbb{C}^n$ such that $q_\gamma \neq 0$. Thus, we can write each $F_k$ in the form

$$F_k = x^v \sum_{u \in \ker_2(A)} x_\tau^u p_u(\log(x_\tau))$$

where $v \in \mathbb{C}^n$ and $p_u = q_{v+u}$. It is enough to prove that a Gevrey solution $F_k$ of $M_A(\beta)$ is a formal Nilsson solution of $H_A(\beta)$ in the direction of $w$ and we can assume for simplicity that the original $f = \sum_{\alpha \in \mathbb{N}^\tau} f_\alpha(x_\tau) x_\tau^\alpha \in \mathcal{O}_{X[Y,p]}$ can also be written in this form. Notice that we have shown that $f$
can be written as in (1.7) satisfying conditions i) and iii) in Definition 1.7. We need to prove that the support of \( f \) is of the form \( v' + C \) for some \( v' \in v + \ker_{Z}(A) \) and \( C \) satisfying conditions ii) and iv) in Definition 1.7.

Notice that for all \( \alpha \in \mathbb{N}^{r} \), we have

\[
 f_{\alpha}(x_{\tau}) = \sum_{u} x^{(v+u)_{\tau}} p_{u}(\log(x_{\tau}))
\]

where the sum is over the vectors \( u \in \ker_{Z}(A) \) such that \( (v + u)_{\tau} = \alpha \) (in particular, \( u_{\tau} \) varies in a translate of \( \ker_{Z}(A_{\tau}) \)).

Let us see that \( f \) has an initial form with respect to \( w \). It is enough to find a lower bound for the \( w \)-weights of all terms \( x^{v+u} p_{u}(\log(x_{\tau})) \) of \( f \) with \( p_{u} \neq 0 \).

By [SST00, Theorem 2.5.5], the initial form of \( f_{\alpha} \in N_{w(\tau)}(H_{A_{\tau}}(\beta - A_{\tau} \alpha)) \) with respect to \( w(\tau) \) is a solution of \( \in_{(-w(\tau),w(\tau))}(H_{A_{\tau}}(\beta - A_{\tau} \alpha)) \). Since \( w(\tau) \) is generic, \( \in_{w(\tau)}(f_{\alpha}) = x^{\tilde{v}}_{\tau} p_{w}(\log(x_{\tau})) \) for some \( \tilde{v}_{\tau} \in \mathbb{C}^{r} \) and \( w' \in \ker_{Z}(A) \). By [SST00] Theorems 2.3.3(2), 2.3.9, 2.3.11, \( \tilde{v}_{\tau} \) is an exponent of \( H_{A_{\tau}}(\beta - A_{\tau} \alpha) \) with respect to \( w(\tau) \). Thus, by Lemma 3.1 there is a maximal simplex \( \sigma \in T_{w(\tau)} \) such that \( \tilde{v}_{j} \in \mathbb{N} \) for all \( j \in \tau \setminus \sigma \).

Let us denote by \( \tilde{v} \in \mathbb{C}^{n} \) the vector with \( \tau \)-coordinates \( \tilde{v}_{\tau} = \alpha \) and \( \tau \)-coordinates equal to \( \tilde{v}_{\tau} \). Notice that \( \tilde{v}_{\tau} \in \mathbb{N}^{r} \) and since \( A \tilde{v} = \beta \) we have that \( \tilde{v}_{\sigma} = A^{-1}_{\sigma}(\beta - A_{\tau} \tilde{v}_{\tau}) \). We can write

\[
 \tilde{v} = \beta_{\sigma} + B_{\sigma} \tilde{v}_{\tau}
\]

where \( B_{\sigma} \) was defined in Subsection 1.1 and \( \beta_{\sigma} \in \mathbb{C}^{n} \) denotes the vector whose \( \sigma \)-coordinates agree with \( A_{\sigma}^{-1} \beta \) and whose other coordinates are zero.

Moreover, by Lemma 3.2, \( \sigma \in T_{w} \). Thus, by (1.4) and the fact that \( \tilde{v}_{\tau} \in \mathbb{N}^{r} \), we have that

\[
 \text{Re}(\langle w, \tilde{v} \rangle) = \text{Re}(\langle w, \beta_{\sigma} \rangle) + \langle w, B_{\sigma} \tilde{v}_{\tau} \rangle = \text{Re}(\langle w, \beta_{\sigma} \rangle) + \langle w B_{\sigma}, \tilde{v}_{\tau} \rangle \geq \text{Re}(\langle w, \beta_{\sigma} \rangle)
\]

where \( \text{Re}(\langle w, \tilde{v} \rangle) \) is the \( w \)-weight of \( \in_{w}(f_{\alpha}(x_{\tau})x^{\frac{\alpha}{\tau}}) = \in_{w(\tau)}(f_{\alpha}(x_{\tau}))x^{\frac{\alpha}{\tau}} \).

Then the minimum of the finite set

\[
 \{ \text{Re}(\langle w, \beta_{\sigma} \rangle) | \sigma \in T_{w(\tau)} \}
\]

is a lower bound for the \( w \)-weights of all the terms of \( f \). Thus, \( f \) has an initial form with respect to \( w \) and \( \in_{w}(f) \) must be a term \( x^{v+u'} p_{u'}(\log(x_{\tau})) \) for some \( u' \in \ker_{Z}(A) \) because \( w \) is generic. We may assume for simplicity that \( \in_{w}(f) = x^{v} p_{0}(\log(x_{\tau})) \) since \( v + \ker_{Z}(A) = v + u' + \ker_{Z}(A) \). This implies that \( x^{v} p_{0}(\log(x_{\tau})) \) is a solution of \( \in_{(-v,w)}(H_{A}(\beta)) \) by the same argument as in the proof of [SST00, Theorem 2.5.5], hence \( v \) is an exponent of \( H_{A}(\beta) \) with respect to \( w \). The set \( C := \{ u \in \ker_{Z}(A) | p_{u} \neq 0 \} \) satisfies condition iv) in Definition 1.7.

Let \( w' \in C_{w} \) be generic and such that \( w'_{\tau} \) is a perturbation of \( w(\tau) \). Then the previous argument works as well for \( w' \) instead of \( w \) and we have that \( f \) has also an initial form with respect to \( w' \) of the form \( \in_{w'}(f) = x^{v+u'} p_{u'}(\log(x_{\tau})) \) and that \( v + u' \) is an exponent of \( H_{A}(\beta) \) with respect to \( w' \). Since the set of exponents of \( H_{A}(\beta) \) with respect to \( w' \) is finite and constant for all \( w' \in C_{w} \), we can find an open cone \( C' \subseteq C_{w} \) such that \( w \in C' \) and \( \in_{w}(\phi) = x^{v} p_{0}(\log(x_{\tau})) \) for all \( w' \in C' \).

Hence, we have \( \langle w', u \rangle > 0 \) for all \( w' \in C' \) and all \( u \in C' \setminus \{0\} \). Thus, \( f \) satisfies condition ii) in Definition 1.7.

The following result provides a partial converse to Theorem 3.3.
Theorem 3.4. If $A$ is pointed, $\text{pos}(A_\tau) = \text{pos}(A)$ and $p \in U_{w(\tau)} \subseteq Y_\tau$, we have
\[
\text{dim}_\mathbb{C}(\text{Hom}_D(M_A(\beta), \mathcal{O}_{X_\tau}|_{Y_\tau,p})) = \text{dim}_\mathbb{C}(\mathcal{N}_w(H_A(\beta)))
\]
for all $\beta \in \mathbb{C}^d$. More precisely, $\mathcal{N}_w(H_A(\beta))$ is the space of Gevrey series solutions of $M_A(\beta)$ along $Y_\tau$ at any point $p \in U_{w(\tau)}$. Moreover, for $\beta \in \mathbb{C}^d \setminus \varepsilon(A)$, the dimension of this space is $\text{vol}(\tau)$.

Proof. By Theorem 3.3, it is enough to show that any basic Nilsson series $\phi$ in the direction of $w$ as in (1.7) is a Gevrey series along $Y_\tau$ at any point $p \in U_{w(\tau)}$.

Since $\text{pos}(A_\tau) = \text{pos}(A)$ we have that $T(\tau) = T_w$ by the construction of $w$, see the proof of Lemma 3.2 and the subsequent paragraph.

By [DMM12, Lemma 2.10] we have that $v$ is an exponent of $H_A(\beta)$ with respect to $w$. Thus, there exists $\sigma \in T_w = T(\tau)$ such that $v_j \in \mathbb{N}$ for all $j \notin \sigma$, see Lemma 3.1. In particular, we have $v_\tau \in \mathbb{N}^\tau$. Moreover, by [Sai02, Proposition 5.4] (whose proof is also valid when $I_A$ is not necessarily homogeneous) we have that $p_u(y) \in \mathbb{C}[y_j : j \in \text{vert}(T_w)]$ for all $u \in C$, where $\text{vert}(T_w)$ is the set of vertices of $T_w$. In particular, $p_u \in \mathbb{C}[y_j : j \in \tau]$. Thus, we can define $p_u(y_\tau) := p_u(y)$ and write
\[
\phi = \sum_{u \in C} x^{v+u}p_u(\log(x_\tau)) = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha(x_\tau)x^\alpha
\]
where
\[
f_\alpha(x_\tau) = \sum_{u \in C, (v+u)_\tau = \alpha} x^{(v+u)_\tau}p_u(\log(x_\tau))
\]
is annihilated by $H_{A_\tau}(\beta - A_\tau \alpha)$, see the proof of [Fer10, Lemma 6.11].

We need to prove that $f_\alpha = 0$ if $\alpha \notin \mathbb{N}^n$. That is, we need to show that $v_j + u_j \geq 0$ for all $j \notin \tau$ and $u \in C$, where $v + C = \text{supp}(\phi)$. Since $\text{in}_w(\phi) = x^v p_0(\log(x_\tau))$, we know that $\langle w, u \rangle > 0$ for all $u \in C \setminus \{0\}$. Assume to the contrary that there exist $u \in C$ and $j \in \tau$ such that $v_j + u_j < 0$ and choose such a vector $u$ so that $\langle w, u \rangle$ is minimal. Since $j \in \tau$ and $\text{pos}(A) = \text{pos}(A_\tau) = \cup_{\sigma \in T(\tau)} \text{pos}(\sigma)$, there exist $m_\sigma \in \mathbb{N}^\sigma$ and $m_j \in \mathbb{N}$ with $m_j \geq 1$ such that $m_j a_j = \sum_{i \in \sigma} m_i a_i$. Then $u(m) := \lambda_j m_j + \sum_{i \in \sigma} m_i e_i \in \ker(A)$, where $e_\ell$ denotes the $\ell$-th vector of the standard basis of $\mathbb{R}^n$. Notice that $P = \partial_u(m)_\tau - \partial_u(m) = \partial_{m_\sigma} - \partial_{m_j} \in H_A(\beta)$, hence $P(\phi) = 0$.

By Lemma 2.7 and using that $\partial_j[p_u] = 0$ for all $j \notin \tau$, we have that
\[
\partial_j^{m_j}[x^{v+u}p_u(\log(x_\tau))] = x^{v+u-m_j e_j} \lambda_j m_j p_u(\log(x_\tau)) \neq 0
\]
where $\lambda_j m_j = [v_j + u_j]_m \neq 0$ because $v_j + u_j < 0$ by assumption. Since $P(\phi) = 0$ and $v+u-m_j e_j = v+u+u(m)_\tau$ belongs to the support of the series $\partial_{m_\sigma} - \partial_{m_j}(\phi)$, we have that $v+u+u(m)$ must be in the support of $\phi$, but then $u+u(m) \in C$ and $v_j + u_j + u(m)_\tau < 0$. On the other hand, $\text{in}_w(P) = -\partial_{m_j}^{m_j}$ because $\partial_{m_\sigma} \notin \text{in}_w(I_A)$ for $\sigma \in T(\tau) = T_w$, see [Stu95, Corollary 8.4]. Thus, $\langle w, u(m) \rangle < 0$ and we obtain that $\langle w, u+u(m) \rangle < \langle w, u \rangle$ which is in contradiction with our assumption.

Finally, let us see that each $f_\alpha$ is convergent at $p \in U_{w(\tau)}$. Since $\phi$ is basic, there is an open and strongly convex cone $C \subseteq C_w$ such that $w \in C$ and $\langle w', u \rangle > 0$ for all $w' \in C$ and all $u \in C \setminus \{0\}$. In particular, $\text{in}_{w'}(\phi) = \text{in}_{w'}(\phi)$ for all $w' \in C$. This implies that for fixed $\alpha$ and all $w' \in C$ with $w'_\tau = w_\tau$ there exists the initial form $\text{in}_{w'}(f_\alpha)$. Notice that the set of all $w'_\tau$ for $w'$ as above is a neighborhood of $w_\tau \in \mathbb{R}^\tau$ and we can find a smaller neighborhood $U$ of $w_\tau$ such that $\text{in}_{w'_\tau}(f_\alpha) =
in \(w(\tau)(f_\alpha)\) for all \(w'_\tau \in U\). Then, by [DMM12, Remark 2.7], \(f_\alpha \in \mathcal{N}_{w(\tau)}(H_{A_\tau}(\beta - A_\tau \alpha))\) and by Theorem [1.12] \(f_\alpha\) is convergent at any point \(p \in U_{w(\tau)}\).

Thus, \(\phi\) is a formal solution of \(M_A(\beta)\) along \(Y_\tau\), which implies it is Gevrey of some order because \(M_A(\beta)\) is holonomic.

For \(\beta \in \mathbb{C}^d \setminus \varepsilon(A), \dim(\mathcal{N}_{w}(H_A(\beta))) = \text{vol}(\tau)\) by Corollary [2.2] where \(\cup_{\sigma \in T_w} \Delta_{\sigma} = \Delta_\tau\) in our case, and Theorem [1.6].

**Remark 3.5.** The additional condition \(\text{pos}(A) = \text{pos}(A_\tau)\) in Theorem [3.4] is necessary, even if \(\beta \in \mathbb{C}^d\) is generic, as shown by the following example.

**Example 3.6.** Let us consider the system \(H_A(\beta)\) for

\[
A = \begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & -1
\end{pmatrix},
\]

and a generic parameter \(\beta \in \mathbb{C}^2\). For \(\tau = \{1, 2\}\), we have \(Y_\tau = \{ x_3 = 0 \}\). Let \(w = (0, 0, 1) + \varepsilon(w(\tau), 0)\) where \(\varepsilon > 0\) is small enough and \(w(\tau) \in \mathbb{R}^2\) is a perturbation of \((1, 1)\). We have that \(T_w = \{ \tau, \sigma \}\) for \(\sigma = \{1, 3\}\), see Figure 1. Moreover, \(T(\tau) = \{ \tau \} \subseteq T_w\).

Notice that both simplices \(\tau, \sigma\) have normalized volume one. According to Proposition [2.3] \(\{ \phi_1 := \phi_{v_1}, \phi_2 := \phi_{v_2} \}\) is a basis of \(\mathcal{N}_{w}(H_A(\beta))\), where \(v_1 = (1, \beta_1, 2, 0)\) and \(v_2 = (1, \beta_2, 0, -\beta_2)\). Moreover \(\ker_\mathbb{Z}(A) = \mathbb{Z}(-3, 1, 1)\). Thus, \(\text{supp}(\phi_1) = \{ (1, 3m, \beta_2 + m, m) | m \in \mathbb{N} \}\) and \(\text{supp}(\phi_2) = \{ (1 + 3\beta_2 - 3m, m, -\beta_2 + m) | m \in \mathbb{N} \}\). It follows from Theorem [1.5] that \(\phi_1\) generates the space of Gevrey solutions of \(H_A(\beta)\) along \(Y_\tau = \{ x_3 = 0 \}\) at any point \(p \in \{ x_3 = 0; x_1x_2 \neq 0 \}\).

In particular, the space \(\mathcal{N}_{w}(H_A(\beta))\) is not contained in the space of Gevrey solutions of \(M_A(\beta)\) along \(Y_\tau\), although all the assumptions in Theorem [3.4] except the condition \(\text{pos}(A) = \text{pos}(A_\tau)\) are satisfied.

![Figure 1](image-url)

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