An integrable version of the supersymmetric $U$ model with open boundary conditions and an
impurity situated at one end of the chain is introduced. The model is solved through the nested
algebraic Bethe ansatz method so that the Bethe ansatz equations are obtained.

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I. INTRODUCTION

The study of one-dimensional models of correlated electrons in the context of Luttinger liquids attracts a wealth of
activity presently. Particularly, such models in the presence of defects or impurities are of particular interest in the
effort to unify the results of both theoretical and experimental investigations. As a result, substantial research has
been devoted to this field from a variety of approaches.

Lee and Toner first considered the problem of an impurity coupled to a Luttinger liquid \cite{1} and the Kondo (magnetic
impurity) problem has been studied in detail using conformal field theory \cite{2,3}, perturbation theory \cite{4}, renormal-
ization group \cite{5} and Bethe ansatz methods \cite{6,7}. Still the problem of strongly correlated electron models with
impurities remains not well understood and recent discrepancies between observed experimental results and theoretical
predictions \cite{8} have justified the search for a deeper understanding.

Some of the well studied one-dimensional integrable spin models with impurities include the (supersymmetric) t-J
model \cite{12,13} and Heisenberg chain \cite{14,15,16,17}. However one discovers that when impurities are introduced into a
periodic lattice the Hamiltonian, though integrable, may contain terms that are physically unrealistic as the system
suffers from an absence of backscattering \cite{14,15,16,17}.

There exists another class of one-dimensional models with impurities. These are systems constructed with open
boundary conditions where the impurity appears on the end of the chain. It is desirable to introduce impurities into the
system in such a way that integrability is maintained. The quantum inverse scattering method (QISM) allows for
such a construction and following Sklyanin’s approach \cite{20} one may obtain an integrable impurity model with open
boundary conditions. This has been demonstrated for a number of models with Kondo impurities \cite{21,22}.

This article presents a new boundary impurity for the supersymmetric $U$ model \cite{28} maintaining integrability. Since
its discovery, the supersymmetric $U$ model has received considerable attention due partly to the fact that the model
contains a free integrability preserving coupling parameter. The Bethe ansatz equations and thermodynamic aspects
of the model on the closed chain were discussed in \cite{29} and an anisotropic generalization was presented in \cite{30}.

The boundary impurity we propose for the supersymmetric $U$ model, is constructed from an $R$-matrix solution of the
Yang-Baxter relation. The solution is obtained from the tensor product of one parameter family of typical
four-dimensional irreducible representations of $gl(2|1)$ characterized by distinct parameters. This produces boundary

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terms with a different coupling of the Hubbard interaction, compared with that of the bulk Hamiltonian. In addition, the boundary term also contains a spin interaction which is absent in the bulk interactions.

The paper is organised as follows. The construction of the Hamiltonian is presented in the next section. In section III the nested algebraic Bethe ansatz solution is presented and concluding remarks follow in section IV.

II. THE SUPERSYMMETRIC $U$ MODEL WITH BOUNDARY IMPURITY

In this section, we introduce the basic ingredients to construct an integrable impurity model with open boundary conditions through the QISM. We begin by writing the $R$-matrix \[ R_{\alpha\beta}(u) = \frac{u - \alpha - \beta}{-u - \alpha - \beta} P_1 - P_2 + \frac{-u - \alpha - \beta - 2}{u - \alpha - \beta - 2} P_3, \] (II.1) where the $\alpha, \beta$ arise from the one-parameter family of four-dimensional representations of $gl(2|1)$ and the projectors $P_1, P_2$ and $P_3$ are given explicitly in the appendix. In particular, for $\beta = \alpha$ we recover the $R$-matrix used to derive the usual supersymmetric $U$ model \[28\].

This $R$-matrix acts in the tensor product of two four-dimensional modules $V_{\alpha} \otimes V_{\beta}$ and is a solution to the $\mathbb{Z}_2$-graded quantum Yang-Baxter equation

\[ R_{12}^{\alpha \alpha}(u_1 - u_2) R_{13}^{\alpha \beta}(u_1 - u_3) R_{23}^{\beta \beta}(u_2 - u_3) = R_{23}^{\alpha \beta}(u_2 - u_3) R_{13}^{\alpha \beta}(u_1 - u_3) R_{12}^{\alpha \alpha}(u_1 - u_2). \] (II.2)

The usual notation is adopted, that is, $R_{jk}(u)$ acts on $j$-th and $k$-th superspaces and as an identity on the remaining superspace. Throughout the article we shall use the graded-tensor product law defined by

\[ (a \otimes b)(c \otimes d) = (-1)^{|b||c|} (ac \otimes bd). \] (II.3)

This $R$-matrix also satisfies unitarity

\[ R_{12}(u) R_{21}(-u) = \rho(u), \] and crossing unitarity

\[ R^{ist}_{12}(-u+1) R^{ist}_{12}(u) = \tilde{\rho}(u), \] where $\rho(u), \ \tilde{\rho}(u)$ are some scalar functions.

In order to deal with open boundary conditions we follow the necessary supersymmetric generalization of Sklyanin’s approach \[20\] and define the boundary transfer matrix $\tau(u)$ as

\[ \tau(u) = \text{str}_0 \left( K_+(u) T(u) K_-(u) T^{-1}(-u) \right). \] (II.4)

Here $T(u)$ is the monodromy matrix for a $L$-site chain

\[ T(u) = R_{01}^{\alpha \alpha}(u) R_{02}^{\alpha \alpha}(u) \cdots R_{0(L-1)}^{\alpha \alpha}(u) R_{0L}^{\alpha \beta}(u), \] (II.5)

$K_-(u), K_+(u)$ both obey the reflection equations

\[ R_{12}(u_1 - u_2) K_-(u_1) R_{21}(u_1 + u_2) K_-(u_2) = K_-(u_2) R_{12}(u_1 + u_2) K_-(u_1) R_{21}(u_1 - u_2), \]
\[ R^{ist}_{21}(ist)(-u_1 + u_2) K_+(ist)(u_1) R^{ist}_{12}(ist)(u_1 - u_2 + \nu) K^{ist}_{+}(u_2) \]
\[ = K^{ist}_{+}(u_2) R^{ist}_{21}(ist)(-u_1 - u_2 + \nu) K^{ist}_{+}(u_1) R^{ist}_{12}(ist)(-u_1 + u_2). \] (II.6)

Above $\nu$ is the so-called crossing parameter and $st_i$ stands for the supertransposition taken in the $i$-th space, whereas $ist_i$ is the inverse operation of $st_i$. Here we will choose $K_-(u), K_+(u)$ to be both equal to the identity matrix. Note that in the expression for the monodromy matrix $T(u)$ (II.5) the presence of an $R$-matrix intertwining between two different four-dimensional representations of $gl(2|1)$ characterized by $\alpha$ and $\beta$. The corresponding transfer matrix will be associated to a supersymmetric $U$ model with an impurity in the boundary.

Using the Yang-Baxter algebra (II.2), together with the reflection equations (II.6), it can be shown that the model is integrable, that is

\[ [\tau(u_1), \tau(u_2)] = 0. \] (II.7)
From this transfer matrix one may write the Hamiltonian for an open supersymmetric $U$ model with a boundary impurity through the second derivative of $\tau(u)$ with respect to the spectral parameter $u$, at $u = 0$. This is the procedure adopted when the supertrace of $K_+(u)$ is null [32]. The Hamiltonian is given by

$$H = -\frac{q^{\alpha+1} - q^{-\alpha-1}}{\ln q} H^R,$$

$$H^R = \frac{\tau''(0)}{4(A_1 + 2B_1)}$$

$$= \sum_{i=1}^{L-1} H_{i,i+1} + \frac{1}{2} K_{i}(0)$$

$$+ \frac{1}{2(A_1 + 2B_1)} \left[ \text{str}_0 \left( K_{i}(0)G_{L,0} \right) + 2 \text{str}_0 \left( K_{i}(0)H^R_{L,0} \right) + \text{str}_0 \left( K_{i}(0)(H^R_{L,0})^2 \right) \right], \quad \text{(II.8)}$$

where

$$A_1 = \text{str}_0 K_{i}(0), \quad B_1 = \text{str}_0 \left( K_{i}(0)H^R_{L,0} \right), \quad H^R_{i,i+1} = P_{i,i+1}R_{i,i+1}^R(0), \quad G_{i,i+1} = P_{i,i+1}R_{i,i+1}^R(0).$$

In the above, $P_{i,i+1}$ denotes the graded permutation operator acting on quantum spaces $i$ and $(i+1)$. With our choice for the boundary K matrices equal to the identity, the Hamiltonian (II.8) simplifies to

$$H^R = \frac{\tau''(0)}{8 \text{str}_0(H^R_{L,0})} = \sum_{i=1}^{L-1} H_{i,i+1} + \frac{1}{4 \text{str}_0(H^R_{L,0})} \left[ \text{str}_0(G_{L,0}) + \text{str}_0((H^R_{L,0})^2) \right]. \quad \text{(II.9)}$$

In view of the grading, the basis vectors of the module $V$ can be identified with the electronic states as follows

$$|1\rangle \equiv |\uparrow\rangle = c_{\downarrow}c_{\uparrow}1 \rangle, \quad |2\rangle \equiv |\uparrow\rangle = c_{\uparrow}c_{\uparrow}1 \rangle, \quad |3\rangle \equiv |\downarrow\rangle = c_{\downarrow}c_{\downarrow}1 \rangle, \quad |4\rangle \equiv |0\rangle. \quad \text{(II.10)}$$

We adopt the notation that $c_{\downarrow}^\dagger$ and $c_{\downarrow}^\dagger$ are spin up annihilation (spin down creation) operators. Also let $n_{i,\sigma} = c_{i,\sigma}^\dagger c_{i,\sigma}$ denote the number operator for electrons with spin $\sigma$ on site $i$ and let the total number of electrons be $n_i = n_{i,\uparrow} + n_{i,\downarrow}$.

The spin operators $S_i$, $S_i^\dagger$, $S_i^\dagger$, $S_i^\dagger = \frac{1}{2}(n_{i,\downarrow} - n_{i,\uparrow})$,

form an $sl(2)$ algebra. These operators commute with the Hamiltonian of the supersymmetric $U$ model which is given by

$$H_{i,i+1} = -\sum_{\sigma=\uparrow,\downarrow} (c_{i,\sigma}^\dagger c_{i+1,\sigma} + h.c.) \left( \frac{1 + \alpha}{\alpha} \right)^{1/2(n_{i,-\sigma} + n_{i+1,-\sigma})} + \frac{1}{\alpha} [n_{i,\uparrow}n_{i,\downarrow} + n_{i+1,\uparrow}n_{i+1,\downarrow}]$$

$$+ \frac{1}{\alpha} (c_{i,\sigma}^\dagger c_{i,-\sigma} c_{i+1,\sigma} + h.c.) + (n_i + n_{i+1}).$$

The impurity Hamiltonian constructed from (II.3) is found to be

$$H = \sum_{i=1}^{L-1} H_{i,i+1} + H_{L,0}^\beta \quad \text{(II.11)}$$

where the boundary term $H_{L,0}^\beta$ is given

$$H_{L,0}^\beta = l(t + \sum_{\sigma}(c_{L-1,\sigma}^\dagger c_{L,\sigma} + h.c.)) [h + d(\alpha,\beta)n_{L-1,-\sigma}n_{L,-\sigma} - 1] + d(\beta,\alpha)(n_{L-1,-\sigma} - 1)n_{L,-\sigma}$$

$$+ mn_{L-1,-\sigma}n_{L,-\sigma} - \frac{e}{2}(c_{L-1,\sigma}^\dagger c_{L-\sigma}^\dagger c_{L,\sigma} + h.c.) - f(S_{L-1}^\dagger S_L + h.c.) + f \sum_{\sigma} n_{L-1,\sigma} n_{L,-\sigma}$$

$$- (\alpha + \beta)^2 [(1 + \beta)n_{L-1} + (1 + \alpha)n_L] - 4(\beta(1 + \beta)n_{L-1,\uparrow}n_{L-1,\downarrow} + \alpha(1 + \alpha)n_{L,\uparrow}n_{L,\downarrow}). \quad \text{(II.12)}$$
with
\[ m = \sqrt{\alpha \beta}(2 + \alpha + \beta)^2 - (\alpha + \beta)^2 \sqrt{(1 + \alpha)(1 + \beta)}, \quad d(\alpha, \beta) = -\sqrt{1 + \beta}[4\sqrt{\alpha} + (\alpha + \beta)^2(\sqrt{\alpha} - \sqrt{1 + \alpha})], \]
\[ f = (\alpha - \beta)^2, \quad e = 4\sqrt{\alpha \beta}(1 + \alpha)(1 + \beta), \quad t = 4(\alpha + \beta)^2(2 + \alpha + \beta), \]
\[ h = (\alpha + \beta)^2 \sqrt{(1 + \alpha)(1 + \beta)}, \quad l = -\frac{4(\beta + 1)}{(\alpha + \beta)^2(2 + \alpha + \beta)^2}. \]

At this point a few interesting characteristics of the model should be pointed out. In contrast to the usual supersymmetric \( U \) model, one may observe that the above Hamiltonian contains an extra spin interaction term from the boundary contribution. Also the free parameter \( \alpha \) appears on the boundary as well as in the bulk terms of the Hamiltonian, in contrast the Kondo impurity model presented in [27]. Furthermore, in the limit \( \beta \to \alpha \) we recover the usual supersymmetric \( U \) model [28].

### III. THE BETHE ANSATZ SOLUTIONS

In order to find the Bethe ansatz equations of the Hamiltonian [11], we use Sklyanin’s generalized Bethe ansatz method to treat open boundary conditions together with Babujian and Tsvelick’s approach for higher spin chains. Therefore, we introduce the following doubled auxiliary monodromy matrix
\[
\hat{U}(u) = \left( \hat{T}(u) \hat{K}_-(u) \hat{T}^{-1}(-u) \right),
\]
where we will chose \( \hat{K}_-(u) \) to be the identity matrix and \( \hat{T}(u) \) is the monodromy matrix for a \( L \)-site chain
\[
\hat{T}(u) = R_{01}^\alpha(u) R_{02}^\alpha(u) \cdots R_{0(L-1)}^\alpha(u) R_{0L}^\beta(u).
\]
Above the matrix \( R^\alpha(u) \) acts on \( W \otimes V_\alpha \), where \( W \) is a three-dimensional module and it is given by [30]
\[
R^\alpha(u) =
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & b^* & 0 & 0 & e^* & 0 & 0 & d^* & 0 \\
0 & 0 & 0 & b^* & 0 & 0 & 0 & 0 & e^* & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
with
\[
b^* = 1 + 1/u, \quad c^* = 1/u, \quad d^* = \sqrt{\alpha}/u, \quad e^* = \sqrt{\alpha + 1}/u,
\]
\[
f^* = 1 + (\alpha + 1)/u, \quad w^* = 1 + \alpha/u, \quad g^* = 1 + (\alpha + 2)/u.
\]

The doubled auxiliary monodromy matrix [11] acts on \( W \otimes V_\alpha^{L-1} \otimes V_\beta \) and can be represented by the matrix
\[
\hat{U}(u) =
\begin{pmatrix}
A_{11}(u) & A_{12}(u) & B_1(u) \\
A_{21}(u) & A_{22}(u) & B_2(u) \\
C_1(u) & C_2(u) & D(u)
\end{pmatrix}.
\]

It is possible to show that it fulfills a modified Yang-Baxter equation of the form
\[
R_{12}(u_1 - u_2) \hat{U}(u_1) R_{21}(u_1 + u_2) \hat{U}(u_2) = \hat{U}(u_2) R_{12}(u_1 + u_2) \hat{U}(u_1) R_{21}(u_1 - u_2),
\]
where $R(u)$ is the usual $t - J$ $R$-matrix acting on $W \otimes W$.

\[ R(u) = \begin{pmatrix}
  u - 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & u & 0 & 0 & 0 & 0 \\
  0 & 0 & u & 0 & 0 & 0 \\
  -1 & 0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 0 & u & 0 \\
  0 & 0 & 0 & 0 & 0 & u + 1
\end{pmatrix}. \quad (III.7)

From the doubled auxiliary monodromy matrix \((III.1)\) we define an auxiliary transfer matrix

\[ \hat{\tau}(u) = \text{str}_0 \left( \hat{K}_+(u) \hat{U}(u) \right), \quad (III.8) \]

where $\hat{K}_+(u)$ is chosen to be the diagonal matrix with elements $\{ -1, -1, 1 \}$.

It can be shown that this auxiliary transfer matrix $\hat{\tau}(u)$ \((III.8)\) commutes with the transfer matrix $\tau(u)$ \((II.4)\), which means that they have a common set of eigenvectors. These vectors will be determined by applying the algebraic Bethe ansatz to $\hat{\tau}(u)$. For this purpose we begin by writing the Bethe vector $\langle \Omega \rangle$ (according to the first-level Bethe ansatz) as

\[ |\Omega\rangle = C_n(u_1) \cdots C_n(u_N) |\Psi\rangle F^{1 \cdots N}. \quad (III.9) \]

$|\Psi\rangle$ is the pseudovacuum and the coefficients $F^{1 \cdots N}$ will be determined later by the second level Bethe ansatz. The action of the elements of $\hat{U}(u)$ \((III.4)\) on $|\Psi\rangle$ is given by

\[ D(u) |\Psi\rangle = \begin{pmatrix} u + \alpha + 2 \\ -u + \alpha + 2 \end{pmatrix}^{-1} \begin{pmatrix} u + \beta + 2 \\ -u + \beta + 2 \end{pmatrix} |\Psi\rangle, \]

\[ B_d(u) |\Psi\rangle = 0, \quad C_d(u) |\Psi\rangle \neq 0, \]

\[ A_{db}(u) |\Psi\rangle = \begin{pmatrix} u + 1 \\ -u \end{pmatrix}^L \begin{pmatrix} u + \alpha + 1 \\ -u + \alpha + 2 \end{pmatrix}^{-1} \begin{pmatrix} u + \beta + 1 \\ -u + \beta + 2 \end{pmatrix} |\Psi\rangle, \quad (b \neq d) \]

\[ \hat{A}_{dd}(u) |\Psi\rangle = \begin{pmatrix} u + 1 \\ -u \end{pmatrix}^L \begin{pmatrix} u + \alpha + 1 \\ -u + \alpha + 2 \end{pmatrix}^{-1} \begin{pmatrix} u + \beta + 1 \\ -u + \beta + 2 \end{pmatrix} |\Psi\rangle + \frac{1}{2u + 1} \begin{pmatrix} u + \alpha + 2 \\ -u + \alpha + 2 \end{pmatrix}^L \begin{pmatrix} u + \beta + 1 \\ -u + \beta + 2 \end{pmatrix} |\Psi\rangle. \quad (III.10) \]

Using the Yang-Baxter relation \((III.6)\), together with the expressions for the matrices $W_R(u)$ \((III.7)\) and $\hat{U}(u)$ \((III.5)\), we can obtain the commutation relations between $D(u)$, $A_{db}(u)$ and $C_d(u)$. These are complicated relations but they will simplify considerably if instead of using $A_{db}(u)$ we work with $\hat{A}_{db}(u)$ according to the transformation

\[ \hat{A}_{db}(u) = A_{db}(u) - \frac{1}{2u + 1} \delta_{bd} D(u). \quad (III.11) \]

This transformation will allow us to more easily recognize the “wanted” and “unwanted” terms. More explicitly we have

\[ \hat{A}_{bd}(u_1) C_c(u_2) = \frac{(u_1 - u_2 - 1)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 1)} r_{gb}^b (u_1 + u_2 + 1) r_{cd}^c (u_1 - u_2) C_c(u_2) \hat{A}_{p}(u_1) - \frac{4u_1 u_2}{(u_1 + u_2 + 1)(2u_1 + 1)(2u_2 + 1)} r_{gb}^b (2u_1 + 1) C_g(u_1) D(u_2) \]

\[ + \frac{2u_1}{(u_1 - u_2)(2u_1 + 1)} r_{cd}^c (2u_2 + 1) C_g(u_1) \hat{A}_{b}(u_2), \quad (III.12) \]

\[ D(u_1) C_b(u_2) = \frac{(u_1 - u_2 - 1)(u_1 + u_2)}{(u_1 - u_2)(u_1 + u_2 + 1)} C_b(u_2) D(u_1) + \frac{2u_2}{(u_1 - u_2)(2u_2 + 1)} C_b(u_1) D(u_2) - \frac{1}{u_1 + u_2 + 1} C_d(u_1) \hat{A}_{db}(u_2). \quad (III.13) \]
Here the matrix $r(u)$, which satisfies the Yang-Baxter equation, takes the form,

$$
r_{bb}^{bd}(u) = 1, \quad r_{bd}^{ad}(u) = -\frac{1}{u-1}, \quad r_{dd}^{bd}(u) = \frac{u}{u-1}, \quad (b \neq d, \ b, d = 1, 2). \quad (III.14)$$

By applying the transfer matrix $\hat{\tau}(u)$ to the Bethe vector $|\Omega\rangle$ we can solve the eigenvalue problem

$$\hat{\tau}(u)|\Omega\rangle = \hat{\Lambda}(u)|\Omega\rangle.$$

We find the eigenvalue to be

$$\hat{\Lambda}(u) = \frac{2u-1}{2u+1} \left( \begin{array}{c}
  \frac{u+\alpha+2}{-u+\alpha+2} \\
  \frac{u+\alpha+2}{-u+\alpha+2}
\end{array} \right)^{L-1} \left( \begin{array}{c}
  \frac{u+\beta+2}{-u+\beta+2} \\
  \frac{u+\beta+2}{-u+\beta+2}
\end{array} \right) \prod_{j=1}^{N} \frac{(u+u_j)(u-u_j-1)}{(u-u_j)(u+u_j+1)} \frac{(u+u_j)(u-u_j-1)}{(u-u_j)(u+u_j+1)} \hat{\Lambda}^{(1)}(u; \{u_j\}), \quad (III.15)$$

provided the parameters $\{u_j\}$ satisfy a first Bethe ansatz equation given by

$$\left( \frac{-u_j}{u_j+1} \right)^L \left( \begin{array}{c}
  \frac{u_j+\alpha+2}{-u_j+\alpha+1} \\
  \frac{u_j+\alpha+2}{-u_j+\alpha+1}
\end{array} \right)^{L-1} \left( \begin{array}{c}
  \frac{u_j+\beta+2}{-u_j+\beta+1} \\
  \frac{u_j+\beta+2}{-u_j+\beta+1}
\end{array} \right) = \prod_{m=1}^{M} \frac{(u_j-v_m+\frac{1}{2})(u_j+v_m+\frac{1}{2})}{(u_j-v_m+\frac{1}{2})(u_j+v_m+\frac{1}{2})} + \frac{2u_j+1}{2u_j-1} \prod_{m=1}^{M} \frac{(u_j-v_m+\frac{1}{2})(u_j+v_m+\frac{1}{2})}{(u_j-v_m+\frac{1}{2})(u_j+v_m+\frac{1}{2})}. \quad (III.16)$$

Above, $\hat{\Lambda}^{(1)}(u; \{u_j\})$ is the eigenvalue of the transfer matrix $\hat{\tau}^{(1)}(u)$ for the reduced problem which arises out of the $r(u)$ matrices from the first term in the right hand side of $\hat{\tau}(u)$. The reduced transfer matrix $\hat{\tau}^{(1)}(u)$ may be recognized as that of the $N$-site inhomogeneous XXX spin-$\frac{1}{2}$ open chain and may be diagonalized following Ref. [24] with the reduced boundary K matrices are equal to the identity. We find the eigenvalue of the transfer matrix $\hat{\tau}^{(1)}(u)$ to be given by

$$\hat{\Lambda}^{(1)}(u; \{u_j\}) = \frac{2u-1}{2u} \prod_{m=1}^{M} \frac{(u-v_m+\frac{1}{2})(u+v_m+\frac{1}{2})}{(u-v_m+\frac{1}{2})(u+v_m-\frac{1}{2})} + \frac{2u+1}{2u} \prod_{m=1}^{M} \frac{(u-v_m+\frac{1}{2})(u+v_m-\frac{1}{2})}{(u-v_m+\frac{1}{2})(u+v_m-\frac{1}{2})},$$

provided the parameters $\{v_m\}$ satisfy a second Bethe ansatz equation given by

$$\prod_{j=1}^{N} \frac{(v_m-u_j+\frac{1}{2})(v_m+u_j+\frac{1}{2})}{(v_m-u_j+\frac{1}{2})(v_m+u_j+\frac{1}{2})} = \prod_{k \neq m}^{M} \frac{(v_m-v_k-1)(v_m+v_k-2)}{(v_m-v_k+1)(v_m+v_k)} \quad (III.17)$$

IV. CONCLUSION

We have presented the supersymmetric $U$ model with a new type of boundary impurity. The model is constructed with open boundary conditions and formulated within a modified QISM so that integrability is not lost. The impurity is introduced through an $R$-matrix solution of the Yang-Baxter equation, built from the tensor product of two different four-dimensional representations of $gl(2|1)$ which are characterized by different parameters. The resulting Hamiltonian contains an extra spin interaction term that does not appear in the usual supersymmetric $U$ model. Another feature of this impurity model is that the free parameter of the bulk appears in the boundary terms, in contrast to the Kondo impurity model presented in [27]. In the limit $\beta \rightarrow \alpha$ we recover the supersymmetric $U$ model on an open chain. In this work, we have also derived the Bethe ansatz equations using the nested algebraic Bethe ansatz method, which was appropriately modified to deal with a graded underlying algebra and open boundary conditions.

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APPENDIX: PROJECTION OPERATORS FOR THE $R^{αβ}$-MATRIX

We present the projectors for the tensor product $V_α ⊗ V_β$. For details on the construction of $R$-matrices from these projectors, the reader is referred to [31].

The tensor product basis is constructed from (1.10) and the tensor product decomposition is $V_α ⊗ V_β = V(0, 0, |α + β) ⊕ V(-1, -1, |α + β + 2) ⊕ V(0, -1, |α + β + 1)$. It is not necessary to calculate the three projectors as one may use the relation $Ident = \sum_i P_i$. The simplest projectors to calculate are $P_1$ and $P_3$ which are given as follows

$$P_1 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0&
