WHEN ARE STARS THE LARGEST CROSS INTERSECTING FAMILIES?

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Abstract. We provide a necessary and sufficient condition for stars to be the largest cross intersecting families.

1. Introduction

Let $2^{[n]}$ denote the power set of $[n] := \{1, 2, \ldots, n\}$, and let $[n]_k$ denote the set of all $k$-element subsets of $[n]$. A family of subsets $A \subseteq 2^{[n]}$ is intersecting if $A \cap A' \neq \emptyset$ for all $A, A' \in A$. If $n < 2k$ then $[n]_k$ itself is intersecting. But if $n \geq 2k$ then $[n]_k$ is no longer intersecting, and we can ask the maximum size of intersecting families in $[n]_k$. The Erdős–Ko–Rado Theorem answers this question, which tells us that if $n \geq 2k$ and $A \subseteq [n]_k$ is intersecting, then $|A| \leq \binom{n-1}{k-1}$.

Problem 1. Let $n, k, l$ be positive integers. What is the maximum of the product $|A||B|$ among cross intersecting families $A \subseteq [n]_k$ and $B \subseteq [n]_l$?

Let $M(n, k, l)$ denote the maximum. If $k + l > n$ then $M(n, k, l)$ is clearly $\binom{n}{k}\binom{n}{l}$. It is not difficult to see that if $k + l = n$ then $M(n, k, l) = \frac{1}{2} \binom{n}{k} \left( \frac{1}{2} \binom{n}{l} \right)$. The first non-trivial result was obtained by Pyber [6], and later extended by Matsumoto and the author [5], which states that if $n \geq \max\{2k, 2l\}$ then

$$M(n, k, l) = \binom{n-1}{k-1} \binom{n-1}{l-1}. \tag{1}$$

Moreover the stars are the only optimal configurations if $n > \max\{2k, 2l\}$. Bey [1] gave an alternative combinatorial proof, and Suda and Tanaka [7] established a semidefinite programming approach to obtain related results including (1). These are perhaps all the results known about $M(n, k, l)$. It is open for the remaining cases $k + l < n$ and $n < \max\{2k, 2l\}$. These cases look more interesting and more difficult than the known cases, for most likely infinitely many different structures appear as optimal families. This paper addresses the problem of finding the necessary and sufficient condition for stars to be the largest cross intersecting families.

Key words and phrases. Cross intersecting families; Erdős-Ko-Rado theorem; Shadow; Kruskal–Katona Theorem.

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1 Note that $A \in A$ implies $|A| = |[n] \setminus A|$, and $|B| \leq \binom{n}{l} - |A| = \binom{n}{l} - |A|$. 

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sufficient conditions for \(n, k, l\) such that (1) holds. To begin with let us introduce the following cross intersecting families \(A_j^{(k)}\) and \(B_j^{(l)}\) for \(j \geq 0\) defined by

\[
A_j^{(k)} := \{A \in \binom{[n]}{k} : 1 \in A\} \cup \{A \in \binom{[n]}{k} : A \cap [j + 2] = [j + 2] \setminus \{1\}\},
B_j^{(l)} := \{B \in \binom{[n]}{l} : 1 \in B\} \setminus \{B \in \binom{[n]}{l} : B \cap [j + 2] = \{1\}\}.
\]

To ensure (1) it is necessary that \(|A_j^{(k)}|\|B_j^{(l)}|\) does not exceed the RHS of (1) for every \(j \geq 0\). Indeed we conjecture that this is sufficient as well.

**Conjecture 1.** Let \(n, k, l\) be positive integers with \(k + l < n < \max\{2k, 2l\}\). Suppose that \(|A_j^{(k)}|\|B_j^{(l)}| < \binom{n - 1}{k-1}\binom{n-1}{l-1}\) for all \(j \geq 0\). Then \(M(n, k, l) = \binom{n - 1}{k-1}\binom{n-1}{l-1}\).

Moreover if \(A \subset \binom{[n]}{k}\) and \(B \subset \binom{[n]}{l}\) are cross intersecting families satisfying \(|A||B| = M(n, k, l)\) then \(A = \{A \in \binom{[n]}{k} : i \in A\} \) and \(B = \{B \in \binom{[n]}{l} : i \in B\}\) for some \(i \in [n]\).

We prove the conjecture under some additional conditions. But before stating our results let us introduce a measure counterpart of the problem, which is closely related to the original problem and easier to understand the corresponding conditions. For a real number \(p \in (0, 1)\) and a family of subsets \(\mathcal{F} \subset 2^n\) we define the measure of the family \(\mu_p(\mathcal{F})\) by

\[
\mu_p(\mathcal{F}) := \sum_{F \in \mathcal{F}} p^{|F|}(1 - p)^{n - |F|}.
\]

Then we can ask the following.

**Problem 2.** Let \(n \in \mathbb{N}\) and \(\alpha, \beta \in (0, 1)\). What is the maximum of the product \(\mu_\alpha(A)\mu_\beta(B)\) among cross intersecting families \(A, B \subset 2^n\)?

Let \(m(n, \alpha, \beta)\) denote the maximum. Since \(m(n, \alpha, \beta) \leq 1\) and \(m(n, \alpha, \beta)\) is increasing\(^2\) in \(n\), we can also define

\[m(\alpha, \beta) := \lim_{n \to \infty} m(n, \alpha, \beta).\]

Note that \(m(n_0, \alpha, \beta) = c\) and \(m(\alpha, \beta) = c\) imply \(m(n, \alpha, \beta) = c\) for all \(n \geq n_0\). A simple computation shows that \(m(\alpha, \beta) = 1\) if \(\alpha + \beta > 1\), and \(m(\alpha, \beta) = \frac{1}{2}\) if \(\alpha + \beta = 1\). In [8] it is shown that if \(\max\{\alpha, \beta\} \leq \frac{1}{2}\) then

\[m(n, \alpha, \beta) = \alpha\beta\]

for all \(n \geq 1\). Moreover it is shown in [8] that if \(\max\{\alpha, \beta\} < \frac{1}{2}\) and \(n \geq 4\), then the only optimal families are the stars, that is, \(A = B = \{F \in 2^n : i \in F\}\) for some \(i\). We illustrate the known values of \(m(\alpha, \beta)\) in Figure 1. The two red triangles indicate the regions where \(m(\alpha, \beta)\) is unknown. By symmetry we focus on the upper left triangle

\[\Omega := \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha > 0, \beta > \frac{1}{2}, \alpha + \beta < 1\}.
\]

\(^2\) Indeed if \(m(n, \alpha, \beta) = \mu_\alpha(A)\mu_\beta(B)\) for some \(A, B \subset 2^n\), then \(\mathcal{A} := A \cup \{A \cup \{n + 1\} : A \in A\}\) and \(\mathcal{B} := B \cup \{B \cup \{n + 1\} : B \in B\}\) satisfy \(\mu_\alpha(A)\mu_\beta(B) = \mu_\alpha(\mathcal{A})\mu_\beta(\mathcal{B})\), implying \(m(n + 1, \alpha, \beta) \geq m(n, \alpha, \beta)\).
In this paper, we provide the necessary and sufficient conditions for \((\alpha, \beta) \in \Omega\) satisfying (2). A trivial necessary condition comes from the following cross intersecting families \(A_j\) and \(B_j\) for \(j \geq 0\):

\[
A_j := \{ A \in 2^{[n]} : 1 \in A \} \cup \{ A \in 2^{[n]} : A \cap [j + 2] = [j + 2] \setminus \{1\} \}
\]

\[
B_j := \{ B \in 2^{[n]} : 1 \in B \} \setminus \{ B \in 2^{[n]} : B \cap [j + 2] = \{1\} \}.
\]

If (2) holds, then \(\mu_\alpha(A_j)\mu_\beta(B_j)\) does not exceed \(\alpha \beta\). Our main result states that the converse is also true.

**Theorem 1.** Let \((\alpha, \beta) \in \Omega\). Suppose that \(\mu_\alpha(A_j)\mu_\beta(B_j) < \alpha \beta\) for all \(j \geq 0\). Then \(m(n, \alpha, \beta) = \alpha \beta\) for all \(n \geq 1\).

**Conjecture 2.** Let \(\alpha\) and \(\beta\) satisfy the premises in Theorem 1. If \(F\) and \(G\) are cross intersecting families in \(2^{[n]}\) satisfying \(\mu_\alpha(F)\mu_\beta(G) = \alpha \beta\), then \(F = G = \{ F \in 2^{[n]} : i \in F \}\) for some \(i \in [n]\).

To visualize the conditions for \(\alpha\) and \(\beta\) in Theorem 1, let us compute the measures of \(A_j, B_j\). We have

\[
\mu_\alpha(A_j) = \alpha + (1 - \alpha) \alpha^{j+1},
\]

\[
\mu_\beta(B_j) = \beta - \beta(1 - \beta)^{j+1}.
\]

Thus \(\mu_\alpha(A_j)\mu_\beta(B_j) < \alpha \beta\) if and only if

\[
(1 + (1 - \alpha)\alpha^j)(1 - (1 - \beta)^{j+1}) < 1.
\]

By solving (3) for \(\beta\) in terms of \(\alpha\) we get \(\beta < e_j\), where

\[
e_j = e_j(\alpha) := 1 - \left( \frac{\alpha^j - \alpha^{j+1}}{1 + \alpha^j - \alpha^{j+1}} \right)^{\frac{1}{j+1}}.
\]

see Figure 2. Then the condition \(\mu_\alpha(A_j)\mu_\beta(B_j) < \alpha \beta\) for all \(j \geq 0\) is equivalent to

\[
\beta < \min\{ e_j : j \geq 0 \}.
\]
Let $\Delta \subset \Omega$ be the set of $(\alpha, \beta)$ satisfying (5), see Figure 3. Theorem 1 tells us that if $(\alpha, \beta) \in \Delta$ then $m(\alpha, \beta) = \alpha\beta$, and conversely if $(\alpha, \beta) \in \Omega$ and $m(\alpha, \beta) = \alpha\beta$, then $(\alpha, \beta)$ is in $\Delta$ or its boundary.

We will derive Theorem 1 from the following result which is a partial solution to Conjecture 1.

**Theorem 2.** Let $(\alpha, \beta) \in \Delta$ be fixed. Then there exists $n_0 = n_0(\alpha, \beta)$ such that if $n > n_0$ then $M(n, k, l) = \binom{n-1}{k-1}\binom{n-1}{l-1}$, where $k = \lfloor \alpha n \rfloor$ and $l = \lfloor \beta n \rfloor$. Moreover if $\mathcal{A} \subset \binom{[n]}{k}$ and $\mathcal{B} \subset \binom{[n]}{l}$ are cross intersecting families satisfying
To state the result, let stronger condition than (3) then we get another partial solution to Conjecture 1.

\[ |A| |B| = \binom{n-1}{k-1} \binom{n-1}{l-1} \] then \( A = \{ A \in \binom{[n]}{i} : i \in A \} \) and \( B = \{ B \in \binom{[n]}{l} : i \in B \} \) for some \( i \in [n] \).

To relate Theorem 1 and Theorem 2 let us compare the sizes of \( A \) and \( B \) with their measures. We have

\[
|A| |B| = \binom{n-1}{k-1} + \binom{n-j-2}{k-j-1},
|B| |A| = \binom{n-1}{l-1} - \binom{n-j-2}{l-1}.
\]

If \( \alpha, \beta, \) and \( j \) are fixed, and \( n \to \infty \) with \( k = \alpha n \) and \( l = \beta n \), then it follows that

\[
\frac{|A| |B|}{|B|} \to \mu_{\alpha}(A) \text{ and } \frac{|B| |A|}{|A|} \to \mu_{\beta}(B).
\]

This means that the two conditions

\[
|A| |B| < \binom{n-1}{k-1} \binom{n-1}{l-1} \text{ and } \mu_{\alpha}(A) \mu_{\beta}(B) < \alpha \beta
\]

are corresponding to each other provided

\[
\alpha = \frac{k}{n} \text{ and } \beta = \frac{l}{n}.
\]

In Theorem 2 we assume that \( n, k, \) and \( l \) are large. Instead, if we assume a stronger condition than (5) then we get another partial solution to Conjecture 1. To state the result, let

\[
\Omega' := \{ (k, l) \in \mathbb{Z}^2 : k > 0, l > \frac{n}{2}, k + l < n \}.
\]

**Theorem 3.** Let \((k, l) \in \Omega'. \) Suppose that

\[
1 + \frac{n-k}{n-1} \frac{l-1}{n-1} < 1,
\]

\[
(n-k) \sum_{i=n-l}^{n-2} \frac{1}{i} - (n-l) \sum_{i=k}^{n-2} \frac{1}{i} < 0.
\]

Then \( M(n, k, l) = \binom{n-1}{k-1} \binom{n-1}{l-1} \). Moreover if \( A \subset \binom{[n]}{k} \) and \( B \subset \binom{[n]}{l} \) are cross intersecting families satisfying \( |A| |B| = M(n, k, l) \) then \( A = \{ A \in \binom{[n]}{k} : i \in A \} \) and \( B = \{ B \in \binom{[n]}{l} : i \in B \} \) for some \( i \in [n] \).

To compare Theorems 2 and 3 we note that if \((\alpha, \beta) \in \Omega \) then \((k, l) \in \Omega' \), where \( k = \lfloor \alpha n \rfloor \) and \( l = \lfloor \beta n \rfloor \), provided \( n > n_0(\alpha, \beta) \). The condition (7) is equivalent to

\[
|A| |B| < \binom{n-1}{k-1} \binom{n-1}{l-1}.
\]

If we assume (6), then (7) corresponds to

\[
(2 - \alpha) \beta < 1,
\]

which comes from (3) at \( i = 0 \). Also the condition (8) corresponds to

\[
(1 - \alpha) \log \frac{1}{1 - \beta} - (1 - \beta) \log \frac{1}{\alpha} < 0,
\]

which is a stronger requirement than (3) for \( i \geq 1 \). Let \( \Delta' \subset \Delta \) be the set of \((\alpha, \beta) \) satisfying (9) and (10). Then \( \Delta' \subset \Delta \) and \( \Delta' \) is illustrated in Figure 4 filled with light blue.
The rest of the paper is organized as follows. In Section 2 we introduce our main tool based on the Kruskal–Katona Theorem. In Section 3 we prove the first part of Theorem 2 (the determination of $M(n, k, l)$), and in Section 3 we prove the second part of Theorem 2 (the uniqueness of the extremal structure). Then, in Section 4, we prove Theorem 1 using Theorem 2. In the last section we outline the proof of Theorem 3.

2. Preliminaries

In this section we state a consequence of the Kruskal–Katona Theorem, which is one of the main tools for the proof of our results. For a real number $x$ and an integer $t$ with $x \geq t > 0$ define $\binom{x}{t} := \frac{x(x-1)\cdots(x-t+1)}{t!}$. We also define $\binom{x}{0} = 1$ and $\binom{x}{t} = 0$ if $x < t$. For given positive integers $m$ and $u$ we can represent $m$ in the form as

$$m = \binom{a_u}{u} + \binom{a_{u-1}}{u-1} + \cdots + \binom{a_{u-t}}{u-t} \quad (11)$$

with $a_u > a_{u-1} > \cdots > a_{u-t} \geq u-t \geq 1$, where $t, a_u, \ldots, a_{u-t}$ are all integers. The representation is unique, and it is called the $u$-cascade form of $m$. For a family $\mathcal{F} \subset \binom{[n]}{u}$ and an integer $v$ with $0 < v < u$, let us define the $v$-shadow of $\mathcal{F}$ by

$$\sigma_v(\mathcal{F}) := \{G \in \binom{[n]}{v} : G \subseteq F \text{ for some } F \in \mathcal{F}\}.$$

If $|\mathcal{F}| = m$ and its $u$-cascade form is given by (11), then it follows from the Kruskal–Katona Theorem [3, 2] that

$$|\sigma_v(\mathcal{F})| \geq \binom{a_u}{v} + \binom{a_{u-1}}{v-1} + \cdots + \binom{a_{u-t}}{v-t}.$$
If, moreover, we choose an integer \( s \) and a real number \( x \) with \( 0 < s < t \) and \( a_{u-s-1} \leq x < a_{u-s} \) such that

\[
m = \left( \frac{a_u}{u} \right) + \left( \frac{a_{u-1}}{u-1} \right) + \cdots + \left( \frac{a_{u-s}}{u-s} \right) + \left( \frac{x}{u-s-1} \right),
\]

in other words, if we choose \( s \) and \( x \) so that

\[
\left( \frac{a_{u-s-1}}{u-s-1} \right) + \cdots + \left( \frac{a_{u-t}}{u-t} \right) = \left( \frac{x}{u-s-1} \right),
\]

then it follows from the Lovász version of the Kruskal–Katona Theorem \(^4\) that

\[
|\sigma_v(F)| \geq \left( \frac{a_u}{v} \right) + \left( \frac{a_{u-1}}{v-1} \right) + \cdots + \left( \frac{a_{u-s}}{v-s} \right) + \left( \frac{x}{v-s-1} \right).
\]

If \( \mathcal{F} \subset \binom{[n]}{u} \) and \( \mathcal{G} \subset \binom{[n]}{v} \) are cross-intersecting, then \( \sigma_v(\mathcal{F}^c) \cap \mathcal{G} = \emptyset \), where \( \mathcal{F}^c := \{ [n] \setminus F : F \in \mathcal{F} \} \). Thus \( |\mathcal{G}| \leq \binom{n}{v} - |\sigma_v(\mathcal{F}^c)| \). Noting that \( |\mathcal{F}| = |\mathcal{F}^c| \), we have the following lemma, which we will use repeatedly in the proof of Theorem \(^2\).

**Lemma 1.** Let \( \mathcal{F} \subset \binom{[n]}{u} \) and \( \mathcal{G} \subset \binom{[n]}{v} \) be cross-intersecting. If \( m = |\mathcal{F}| \) is represented by \(^{12}\), then \( |\mathcal{G}| \leq \binom{n}{v} - ( \text{the RHS of } ^{13}) \).

Let \( (k, l) \in \Omega \), and let \( \mathcal{A} \subset \binom{[n]}{k} \) and \( \mathcal{B} \subset \binom{[n]}{l} \) be cross intersecting. To prove \( |\mathcal{A}||\mathcal{B}| \leq \binom{n-1}{k-1}\binom{n-1}{l-1} \) we may assume that \( |\mathcal{A}| > \binom{n-1}{k-1} \) or \( |\mathcal{B}| > \binom{n-1}{l-1} \). We can settle the latter case easily.

**Claim 2.** If \( |\mathcal{B}| > \binom{n-1}{l-1} \) then \( |\mathcal{A}||\mathcal{B}| < \binom{n-1}{k-1}\binom{n-1}{l-1} \).

**Proof.** Let \( |\mathcal{B}| = \binom{n-1}{l-1} + \binom{x}{n-l-1} \) for \( n - l - 1 \leq x \leq n - 1 \). Then, by Lemma \(^1\)

\[
|\mathcal{A}| \leq \binom{n-1}{k} - \binom{n-1}{k-1} - \binom{x}{n-l-1} = \binom{n-1}{k-1} - \binom{x}{n-l-1}.
\]

To prove \( |\mathcal{A}||\mathcal{B}| < \binom{n-1}{k-1}\binom{n-1}{l-1} \) it suffices to show that \( \binom{n-1}{k-1} - \binom{x}{n-l-1} > \binom{n-1}{k-1}\binom{n-1}{l-1} \). This is equivalent to

\[
(n - k) \cdot l > (x - k + 1) \cdot (x - n + l + 2)(n - l).
\]

Both sides consist of \( n - k - l + 1 \) products, and by comparing the corresponding terms one by one we see that \( n - k - i + 1 \geq x - k - i + 2 \) for \( 1 \leq i \leq n - l - k \), and \( l > n - l \).

Note that we did not use none of \(^3\), \(^9\), or \(^10\) to prove Claim \(^2\). Thus Claim \(^2\) is valid for both Theorems \(^2\) and \(^3\).

### 3. Proof of Theorem \(^2\) The inequality

In this section we prove \( M(n, k, l) = \binom{n-1}{k-1}\binom{n-1}{l-1} \) under the premises of Theorem \(^2\). In short, we just apply Lemma \(^1\) several times according to the size of \( \mathcal{A} \), and compute the corresponding upper bounds. However this task is rather involved because we need to choose the interval for the size of \( \mathcal{A} \) carefully to obtain the right bound.

Fix \( (\alpha, \beta) \in \Delta \). Recall that \( \alpha \) and \( \beta \) satisfy \(^3\) for all \( j \geq 0 \), or equivalently, \( \beta < e_j \) from \(^3\). For every fixed \( 0 < \alpha < 1/2 \) we have \( e_j \to 1 - \alpha \) as \( j \to \infty \). Solving \( e_0 = e_1 \) for \( \alpha \) we get \( \alpha = 1 - \frac{1}{\sqrt{2}} =: \tilde{\alpha} \). In this case \( e_0 = e_1 = 2\tilde{\alpha} =: \tilde{\beta} \). Thus, by \(^3\), we have

\[
\beta \leq \min\{e_0, e_1\} \leq \tilde{\beta}.
\]

Indeed there is a cusp at \( (\alpha, \beta) = (\tilde{\alpha}, \tilde{\beta}) \) in Figure \(^3\). Let \( \tilde{\alpha} := 1 - \alpha \) and \( \tilde{\beta} := 1 - \beta \).
Proof. By differentiating the LHS of (15) with respect to 
\(i\) as 
\[\frac{(n-s)}{(n-k-t)} / \binom{n}{k} \to a^{s-t} \alpha^i, \quad \frac{(n-s)}{(l-t)} / \binom{n}{l} \to \beta^t \beta^{s-t}\]
as \(n \to \infty\).

Claim 3. Let \(k, l, n \in \mathbb{N}\) with \(k = |\alpha n|, \ l = |\beta n|\). For fixed \(0 < t \leq s\) with \(t \in \mathbb{N}, s \in \mathbb{R}\) it follows that
\[\left(\frac{n-s}{n-k-t}\right) / \binom{n}{k} \to a^{s-t} \alpha^i, \quad \left(\frac{n-s}{l-t}\right) / \binom{n}{l} \to \beta^t \beta^{s-t}\]
for \(n \to \infty\).

Proof. Recall the following basic properties of the Gamma function:
\[\lim_{n \to \infty} \frac{\Gamma(n+1)}{n^y \Gamma(n)} = 1\]
for \(x \in \mathbb{R}\) with \(x \geq n\), and
\[\lim_{n \to \infty} \frac{\Gamma(y+n)}{n^y \Gamma(n)} = 1\]
for \(y \in \mathbb{R}\). Thus we have
\[\lim_{n \to \infty} \left(\frac{n-s}{n-k-t}\right) / \binom{n}{k} = \lim_{n \to \infty} \frac{\Gamma(n-s+1)}{\Gamma(n+1)} \frac{\Gamma(k+1)}{\Gamma(k-s+t+1)} \frac{\Gamma(n-k+1)}{\Gamma(n-k-t+1)} = \lim_{n \to \infty} n^{-s-k^2} (n-k)^t = a^{s-t} \alpha^i.\]

One can prove the second item similarly. \(\square\)

The goal of this section is to show the following lemma.

Lemma 4. Let \(k, l, n\) satisfy the premises of Theorem 2. Suppose that \(A \subset \binom{n}{k}\) and \(B \subset \binom{n}{l}\) are cross intersecting. Then \(|A||B| < \binom{n-1}{k-1}\binom{n-1}{l-1}\) unless \(|A| = \binom{n-1}{k-1}\) and \(|B| = \binom{n-1}{l-1}\).

The case \(|B| > \binom{n-1}{l-1}\) is shown in Claim 2. So from now on we assume that \(|A| > \binom{n-1}{k-1}\). We will apply Lemma 1 with \(\mathcal{F} = A\) and \(\mathcal{G} = B\) to show that the upper bound for \(|A||B|\) from the lemma is less than \(\binom{n-1}{k-1}\binom{n-1}{l-1}\). We break the proof into several cases according to the size of \(A\). To this end we introduce a constant \(i_0\) based on the following fact.

Claim 5. There exists a constant \(i_0 = i_0(\alpha)\) such that
\[\frac{1}{1 - e_i - 2} < \log \frac{1}{\alpha}\]holds for all \(i \geq i_0\).

Proof. By differentiating the LHS of (10) with respect to \(i\) we get
\[\frac{1}{\alpha^i (i-1)^2} h(i),\]
where
\[\lim_{i \to \infty} h(i) = \alpha^2 \log(\alpha^{-1} - 1) > 0\]
for \(0 < \alpha < \frac{1}{2}\). Thus the LHS of (15) is increasing in \(i\) provided \(i\) sufficiently large. Moreover, using \(e_i \to 1 - \alpha\) as \(i \to \infty\), it follows that the LHS of (15) goes to \(\log \frac{1}{\alpha}\) as \(i \to \infty\). This means that (15) holds if \(i\) is large enough. \(\square\)

We fix an integer \(i_0\) from Claim 5. Note that the \(i_0\) is independent of \(n\). Then we consider the following four cases separately:
- \(\binom{n-1}{n-k} < |A| \leq \binom{n-i_0}{n-k} + \binom{n-i_0}{n-k-1}\)
- \(\binom{n-1}{n-k} + \binom{n-i_0}{n-k-1} < |A| \leq \binom{n-3}{n-k-1} + \binom{n-3}{n-k-1}\).
We actually prove a slightly stronger inequality

\[
|A| \leq (n-1) + (n-2) + (n-3).
\]

We will often use the following easy observation.

**Lemma 6.** Let \(0 < \alpha < 1\) be a fixed real number, and let \(n, k\) be positive integers with \(k = \lfloor \alpha n \rfloor\). Let \(\mathcal{F}_n\) be a sequence of families with \(\mathcal{F}_n \subset \binom{[n]}{k}\), and let \(\{c_n\}\) be a sequence of real numbers with \(0 < c_n < 1\). Suppose that \(\lim_{n \to \infty} |\mathcal{F}_n|/\binom{n}{k} < \lim_{n \to \infty} c_n\). Then there exists an \(n_0\) such that \(|\mathcal{F}_n|/\binom{n}{k} < c_n\) for all \(n > n_0\).

Now we prove several inequalities (Claims 7–12) which we need for the proof of Lemma 6. We mention that we could prove Claims 7, 9, and 11 in a unified way, but to make the description simpler we prove them separately. We also mention that the main idea of the proof of these claims are taken from [3]. After proving these claims Lemma 6 will follow easily.

Here we recall our assumption throughout this section: \((\alpha, \beta) \in \Delta, k = \lfloor \alpha n \rfloor\), and \(l = \lfloor \beta n \rfloor\).

**Claim 7.** Let \(i \geq 2\) and \(\epsilon \geq 0\) be fixed integers. Let \(X = \binom{n-1}{n-k} + \binom{n-1}{n-k-1}\), \(Y = \binom{i-1}{i-1} - \binom{i-1}{i-1}\), and define a polynomial \(F(x)\) by

\[
F(x) := \left( X + \binom{x}{n-k-2} \right) \left( Y - \binom{x}{l-2} \right).
\]

Let \(M := \max\{XY, F(n-i-\epsilon)\}\). Then there exists \(n_0 = n_0(i, \epsilon)\) such that for all \(n > n_0\) the following holds: if

\[
\frac{1 - \beta^{i-1}}{\beta^{i-2} + \epsilon} \log \frac{1}{\beta} \leq \frac{1 + \alpha^{i-2} \bar{\alpha}}{\alpha^{i-3} + \epsilon \bar{\alpha}^2} \log \frac{1}{\alpha},
\]

then

\[
F(x) < M
\]

for all \(x\) with \(n-k-3 < x < n-i-\epsilon\).

**Proof.** The inequality (17) is rewritten as

\[
\frac{F(x) - XY}{\binom{x}{l-2}} = -X + Y\binom{x}{n-k-2} - \binom{x}{n-k-2} < M - XY\binom{x}{l-2}.
\]

This is equivalent to \(f(x) < (M - XY)\frac{(n-k-2)!}{\binom{x}{l-2}}\), where \(f(x)\) is defined by

\[
f(x) := (F(x) - XY)\frac{(n-k-2)!}{\binom{x}{l-2}} = -(n-k-2)!X + (l-2)!Y \prod_{j=1}^{n-k-3} (x-j) - \prod_{j=0}^{n-k-3} (x-j).
\]

We actually prove a slightly stronger inequality

\[
f(x) < (M - XY)\frac{(n-k-2)!}{\binom{n-i-\epsilon}{l-2}} =: m
\]

for \(n-k-3 < x < n-i-\epsilon\). Clearly it is true at the two ends, indeed we have \(f(n-k-3) = -(n-k-2)!X < 0 \leq m\), and \(f(n-i-\epsilon) \leq m\) follows from the definition of \(f(x)\) with \(F(n-i-\epsilon) \leq M\).
We prove \((19)\) by contradiction. Suppose that there is some \(y\) with \(n - k - 3 < y < n - i - \epsilon\) such that \(f(y) > m\). Since \(f(x) \leq m\) at the two ends we may assume that \(\frac{df}{dy}(y) = 0\). This yields
\[
\prod_{j=0}^{n-k-3} (y-j) = (l-2)! Y \prod_{j=l-2}^{n-k-3} (y-j) \frac{1}{\sum_{j=l-2}^{n-k-3} \frac{1}{y-j}} \left( \sum_{j=0}^{n-k-3} \frac{1}{y-j} \right).
\]
Substituting the RHS into the last term in \((18)\) we get a new inequality
\[
m < g(y) := -(n-k-2)! X + (l-2)! Y \prod_{j=l-2}^{n-k-3} (y-j) \sum_{j=0}^{l-3} \frac{1}{y-j} \left( \sum_{j=0}^{n-k-3} \frac{1}{y-j} \right).
\]
Since
\[
(y-n+k+3) \sum_{j=0}^{l-3} \frac{1}{y-j} = y - n + k + 3 \frac{1}{y} + \ldots + y - n + k + 3 \frac{1}{y-l+3}
\]
is increasing in \(y\), \(g(y)\) is also increasing in \(y\). So we must have \(g(n-i-\epsilon) > m\).

We will show that this cannot happen. Using
\[
\lim_{n \to \infty} (n-k-2)! Y \frac{k! n^2}{n!} = \frac{\alpha + \alpha^{-1} \bar{\alpha}}{\alpha^2},
\]
\[
\lim_{n \to \infty} (l-2)! Y \prod_{j=l-2}^{n-k-3} (y-j) \frac{k! n^2}{n!} = \frac{\beta - \beta \bar{\beta}^{-1}}{\beta^2} \left( \frac{\alpha}{\beta} \right)^{i+\epsilon-2},
\]
\[
\lim_{n \to \infty} \sum_{j=0}^{l-3} \frac{1}{(n-i-\epsilon) - j} = \lim_{n \to \infty} \int_{n-i-3-\epsilon}^{n-i+1-\epsilon} \frac{1}{y} dy = \log \frac{1}{\beta},
\]
and
\[
\lim_{n \to \infty} \sum_{j=0}^{n-k-3} \frac{1}{(n-i-\epsilon) - j} = \lim_{n \to \infty} \int_{n-i-3-\epsilon}^{n-i+1-\epsilon} \frac{1}{y} dy = \log \frac{1}{\alpha},
\]
it follows from \(g(n-i-\epsilon) > m > 0\) that
\[
-\frac{\alpha + \alpha^{-1} \bar{\alpha}}{\alpha^2} + \frac{\beta - \beta \bar{\beta}^{-1}}{\beta^2} \left( \frac{\alpha}{\beta} \right)^{i+\epsilon-2} \log \frac{1}{\beta} \log \frac{1}{\alpha} > 0.
\]
However this reduces to the opposite inequality to \((16)\), a contradiction. \(\square\)

**Claim 8.** Let \(i, \epsilon, X, Y, F(x)\) and \(M\) be as in Claim 7. If one of the following holds,
\begin{enumerate}
  \item \(i = 2, 3\) and \(\epsilon = 1\),
  \item \(i = 3, \epsilon = 0\), and \(\alpha < 0.27\),
  \item \(i \geq 4\) and \(\epsilon = 0\),
\end{enumerate}
then for sufficiently large \(n\) it follows \(F(x) < \binom{n-1}{k-1} \binom{n-1}{i-1}\) for \(n-k-3 < x < n-i-\epsilon\).

**Proof.** We use Claim 7. To this end we show \((16)\) and \(M < \binom{n-1}{k-1} \binom{n-1}{i-1}\) in each case. Then the result will follow from \((17)\). In all cases \(XY < \binom{n-1}{k-1} \binom{n-1}{i-1}\) for \(n\)
sufficiently large. Indeed we have \(\lim_{n \to \infty} \frac{XY}{\binom{k}{l} \binom{i}{l}} = (\alpha + \alpha^{-1} \bar{\alpha}) (\beta - \beta \bar{\beta}^{-1}) < \alpha \beta\) for \((3)\) at \(j = i-2\), and this together with Lemma 9 gives us the upper bound for \(XY\). Thus, to prove \(M < \binom{n-1}{k-1} \binom{n-1}{i-1}\), we only need to show \(F(n-i-\epsilon) < \binom{n-1}{k-1} \binom{n-1}{i-1}\).
This is true if \(\epsilon = 0\). Indeed, in this case, we have \(\lim_{n \to \infty} F(n-i) / \binom{n}{l} \binom{i}{l} = (\alpha + \alpha^{-1} \bar{\alpha}) (\beta - \beta \bar{\beta}^{-1})\) and this is less than \(\alpha \beta\) by \((3)\) at \(j = i-1\).
Consequently we have

\[ M < \frac{(n-i-1)}{(n-1)} \]

and in view of Lemma 8 we need to show that

\[ (1 + \alpha^i - \alpha^i) < 1. \]  

The LHS is increasing in \( \alpha \). We note that

\[ \frac{(n-1)}{(k-1)(n-1)} = \frac{\alpha + \alpha^i - \alpha^i}{\alpha^2} \log 1 \alpha. \]

The LHS is increasing in \( \beta \). By (5) it suffices to show the inequality at \( \beta = \epsilon_0 \) if \( i = 2 \), and at \( \beta = \min\{e_0, e_1\} \) if \( i = 3 \). In these cases we can verify (20) by direct computation.

Next we show (16), that is,

\[ \frac{1 - \beta^i}{\beta} \leq \frac{1 + \alpha^i}{\alpha}. \]

The LHS is increasing in \( \beta \), and in view of Lemma 8 we need to show that

\[ (1 + \alpha^i - \alpha^i) < 1. \]  

The LHS is increasing in \( \beta \). By (5) it suffices to show the inequality at \( \beta = \epsilon_0 \) if \( i = 2 \), and at \( \beta = \min\{e_0, e_1\} \) if \( i = 3 \). In these cases we can verify (20) by direct computation.

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\[ \frac{1 - \beta^i}{\beta} \leq \frac{1 + \alpha^i}{\alpha}. \]

The LHS is increasing in \( \beta \), and in view of Lemma 8 we need to show that

\[ (1 + \alpha^i - \alpha^i) < 1. \]  

The LHS is increasing in \( \beta \). By (5) it suffices to show the inequality at \( \beta = \epsilon_0 \) if \( i = 2 \), and at \( \beta = \min\{e_0, e_1\} \) if \( i = 3 \). In these cases we can verify (20) by direct computation.

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Next we show (16), that is,

\[ \frac{1 - \beta^i}{\beta} \leq \frac{1 + \alpha^i}{\alpha}. \]

The LHS is increasing in \( \beta \), and in view of Lemma 8 we need to show that

\[ (1 + \alpha^i - \alpha^i) < 1. \]  

The LHS is increasing in \( \beta \). By (5) it suffices to show the inequality at \( \beta = \epsilon_0 \) if \( i = 2 \), and at \( \beta = \min\{e_0, e_1\} \) if \( i = 3 \). In these cases we can verify (20) by direct computation.
We also note that both $e_{i-3}$ and $e_{i-2}$ are decreasing in $\alpha$. (Here we need $i \geq 4$, because $e_0$ is increasing in $\alpha$.)

If $\beta = e_{i-3}$ then, using $\tilde{\beta}^{i-2} = \frac{\alpha^{i-3-\bar{\alpha}}}{1 + \alpha^{i-3-\bar{\alpha}}}$, it follows that

$$z(\beta = e_{i-3}) = \frac{(1 + \beta \alpha^{i-3}) \bar{\alpha} \log \frac{1}{\beta}}{\beta (1 + \alpha^{i-2}) \log \frac{1}{\alpha}}. \quad (23)$$

Then, using $\alpha^{i-3} \bar{\alpha} = \frac{\tilde{\beta}^{i-2}}{1 - \tilde{\beta}^{i-2}}$, $z(\beta = e_{i-3}) \leq 1$ reduces to

$$\frac{1 - \tilde{\beta}^{i-1}}{\beta (1 - \tilde{\beta}^{i-2})} \log \frac{1}{\beta} \leq \frac{1 + \alpha^{i-2} \bar{\alpha}}{\alpha \bar{\alpha} (1 + \alpha^{i-2}) \log \frac{1}{\alpha}}.$$

We need to check this inequality for $0 < \alpha \leq \alpha_*(i)$. Since the LHS is increasing in $\beta$ and the RHS is decreasing in $\alpha$, it suffices to check the inequality at $(\alpha, \beta) = (\alpha_*(i), \beta_*(i))$.

If $\beta = e_{i-2}$ then, using $\tilde{\beta}^{i-1} = \frac{\alpha^{i-2} \bar{\alpha}}{1 + \alpha^{i-2} \bar{\alpha}}$, it follows that

$$z(\beta = e_{i-2}) = \frac{\bar{\alpha} \beta \log \frac{1}{\beta}}{\alpha \beta (1 + \alpha^{i-2} \bar{\alpha}) \log \frac{1}{\alpha}}, \quad (24)$$

and $z(\beta = e_{i-2}) \leq 1$ is equivalent to

$$\frac{\tilde{\beta}}{\beta} \log \frac{1}{\beta} \leq \frac{\alpha (1 + \alpha^{i-2} \bar{\alpha})}{\alpha \bar{\alpha}} \log \frac{1}{\alpha}.$$

We need to check this inequality for $\alpha_*(i) \leq \alpha < \frac{1}{2}$. In this case the LHS is decreasing in $\beta$ and the RHS is increasing in $\alpha$, so again it suffices to check the inequality at $(\alpha, \beta) = (\alpha_*(i), \beta_*(i))$.

Now we consider the case $(\alpha, \beta) = (\alpha_*(i), \beta_*(i))$. For simplicity let us just write $\alpha_*$ and $\beta_*$ omitting $i$, and let $\bar{\alpha}_* = 1 - \alpha_*$ and $\bar{\beta}_* = 1 - \beta_*$. Then, by (23) and (24), $z(\beta = e_{i-3}) = z(\beta = e_{i-2})$ implies that $\beta_* = \frac{\alpha_*}{1 + \alpha_* \bar{\alpha}_*}$, and substituting this into (24) we have

$$z_* := z(\beta = \beta_*) = \frac{\bar{\beta}_* \log \frac{1}{\beta_*}}{\alpha_* \log \frac{1}{\alpha_*}}.$$

Recall that we always assume $\alpha + \beta < 1$, and in particular, $\frac{1}{\epsilon} < \alpha_* < \beta_*$. (Here we need $i \geq 4$ because $\alpha_*(3) < \frac{1}{\epsilon}$.) Since $x \log \frac{1}{x}$ is decreasing for $\frac{1}{\epsilon} < x$ we get $z_* < 1$, as required.

**Claim 9.** Let $i \geq 2$ and $\epsilon \geq 1$ be fixed integers. Let $X = \binom{n-1}{n-k} + \binom{n-i}{n-k-1} + \binom{n-i-1}{n-k-2}$ and $Y = \binom{n-i-1}{n-i-1} - \binom{n-i-1}{n-i-2}$. Define a polynomial $F(x)$ by

$$F(x) := \left( X + \binom{x}{n-k-3} \right) \left( Y - \binom{x}{n-i-3} \right).$$

Let $M := \max\{XY, F(n - i - \epsilon)\}$. Then there exists $n_0 = n_0(i, \epsilon)$ such that for all $n > n_0$ the following holds: if

$$\frac{1 - \beta_1^{i-1} - \beta_2 \beta_1^{i-1}}{\beta^2 \beta_1^{i-3+\epsilon}} \log \frac{1}{\beta} \leq \frac{1 + \bar{\alpha} \alpha_2^{i-2} + \bar{\alpha}^2 \alpha_1^{i-2}}{\beta^3 \alpha^{i-4+\epsilon}} \log \frac{1}{\alpha}, \quad (25)$$

then

$$F(x) < M$$

for all $x$ with $n - k - 4 < x < n - i - \epsilon$. 

\end{document}
Proof. The proof is almost identical to the proof of Claim 9, and we only include a sketch. Let
\[ f(x) := (F(x) - XY) \frac{(n - k - 3)!}{(l - 3)} \]
\[ = -(n - k - 3)! X + (l - 3)! Y \prod_{j=0}^{n-k-4} (x - j) - \prod_{j=0}^{n-k-4} (x - j). \]

Then \( F(x) < M \) follows from
\[ f(x) < (M - XY) \frac{(n - k - 3)!}{(n-i-\epsilon)} =: m. \]

Suppose, to the contrary, that there is some \( y \) such that \( f(y) \geq m \). Then we may assume that
\[ m < g(y) := -(n - k - 3)! X + (l - 3)! Y \prod_{j=0}^{n-k-4} \frac{1}{y-j} \sum_{j=0}^{n-k-4} \frac{1}{y-j}. \]

Since \( g(y) \) is increasing in \( y \), \( g(n - i - \epsilon) > m > 0 \) must hold. However, considering \( n \to \infty \), \( g(n - i - \epsilon) > 0 \) yields the opposite inequality to (25), a contradiction. \( \square \)

Claim 10. Let \( i, \epsilon, X, Y, F(x) \) and \( M \) be as in Claim 9 If either

(i) \( i = 2 \) and \( \epsilon = 2 \), or
(ii) \( i = 3, \epsilon = 1 \), and \( \alpha > 0.23 \),

then for sufficiently large \( n \) it follows \( F(x) < \binom{n-1}{k-1} \binom{n-1}{l-1} \) for \( n-k-4 < x < n-i-\epsilon \).

Proof. (i) First we show that \( M < \binom{n-1}{k-1} \binom{n-1}{l-1} \). To bound \( XY \) we note that
\[ \lim_{n \to \infty} \frac{XY}{\binom{n}{k} \binom{n}{l}} = (\alpha + \alpha \bar{\alpha} + \alpha \bar{\alpha}^2)(\beta - \beta \bar{\beta} - \beta^2 \bar{\beta}). \]

In view of Lemma 6 we need to show that \( (1 + \bar{\alpha} + \bar{\alpha}^2)(1 - \beta - \beta \bar{\beta}) < 1 \). The LHS is increasing in \( \beta \), and it suffices to check the inequality at \( \beta = e_0 \), that is, \( \frac{3 - 3\alpha + \alpha^2}{2(2 - \alpha)^2} < 1 \). This is indeed true for \( 0 < \alpha < 1/2 \). Similarly as for \( F(n - 4) \) it suffices to check \( (1 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3)(1 - \beta - \beta \bar{\beta} - \beta^2 \bar{\beta}) < 1 \) at \( \beta = e_0 \), and it is also true.

Next we check (25), that is,
\[ \frac{1 - \beta - \beta \bar{\beta}}{\beta^2 \bar{\beta}} \log \frac{1}{\beta} \leq \frac{1 + \bar{\alpha} + \bar{\alpha}^2}{\bar{\alpha}^3} \log \frac{1}{\alpha}. \]

Since the LHS is increasing in \( \beta \) it suffices to check the inequality at \( \beta = e_0 \), which can be verified by direct computation.

(ii) In this case \( M < \binom{n-1}{k-1} \binom{n-1}{l-1} \) follows by showing \( (1 + \alpha \bar{\alpha} + \alpha \bar{\alpha}^2)(1 - \beta - \beta \bar{\beta} - \beta^2 \bar{\beta}) < 1 \) and \( (1 + \bar{\alpha})(1 - \beta) < 1 \). The latter is equivalent to (3) for \( j = 0 \). The LHS of the former is increasing in \( \beta \) and the inequality is indeed true at \( \beta = \min\{e_0, e_1\} \). As for (25), we need to show
\[ \frac{1 - \beta^2 - \beta \bar{\beta}^2}{\beta^2 \bar{\beta}} \log \frac{1}{\beta} \leq \frac{1 + \alpha \bar{\alpha} + \alpha \bar{\alpha}^2}{\bar{\alpha}^3} \log \frac{1}{\alpha}. \]
The LHS is increasing in \( \beta \), and one can verify the inequality at \( \beta = e_0 \) for \( \alpha > 0.23 \) in this case. \( \square \)
Claim 11. Let \( i \geq 2 \) be a fixed integer. Let \( X = \binom{n-1}{n-k} \) and \( Y = \binom{n-1}{l-1} \). Define a polynomial \( F(x) \) by

\[
F(x) := \left( X + \binom{x}{n-k-1} \right) \left( Y - \binom{x}{l-1} \right).
\]

Then there exists \( n_0 = n_0(i) \) such that for all \( n > n_0 \) the following holds: if

\[
\frac{1}{\beta^{i-1}} \log \frac{1}{\beta} < \frac{1}{\alpha^{i-2} \hat{\alpha}} \log \frac{1}{\alpha},
\]

then

\[
F(x) < XY
\]

for all \( x \) with \( n - k - 2 < x < n - i \).

Proof. The proof is similar to and easier than that of Claims 7 and 9. Note that \( F(n-i) < XY \) follows from (3) at \( j = i - 2 \). Let

\[
f(x) := (F(x) - XY) \frac{(n-k-1)!}{(x-1)!} \]

\[
= -(n-k-1)! X + (l-1)! Y \prod_{j=l-1}^{n-k-2} (x-j) - \prod_{j=0}^{n-k-2} (x-j).
\]

Then \( F(x) < XY \) follows from \( f(x) < 0 \). Suppose, to the contrary, that there is some \( y \) such that \( f(y) \geq 0 \). Then we may assume that

\[
0 \leq g(y) := -(n-k-1)! X + (l-1)! Y \prod_{j=l-1}^{n-k-2} (y-j) \sum_{j=0}^{l-2} \frac{1}{y-j} \left( \sum_{j=0}^{n-k-2} \frac{1}{y-j} \right).
\]

Since \( g(y) \) is increasing in \( y \), \( g(n-i) \geq 0 \) must hold, that is,

\[
\frac{(k-1)!(n-k)}{(n-l)!} \prod_{j=l-1}^{n-k-2} (n-i-j) \sum_{j=0}^{l-2} \frac{1}{n-i-j} \geq \sum_{j=0}^{n-k-2} \frac{1}{n-i-j}.
\]

But this reduces to the opposite inequality to (26) by considering \( n \to \infty \). \( \square \)

Claim 12. Let \( i, X, Y \), and \( F(x) \) be as in Claim 11. If \( i = i_0 \), that is,

\[
(1 + \alpha^{i-2} \hat{\alpha}) \log \frac{1}{1 - e_{i-2}} < \log \frac{1}{\alpha},
\]

then for sufficiently large \( n \) it follows \( F(x) < \binom{n-1}{k-1} \binom{n-1}{l-1} \) for \( n - k - 2 < x < n - i \).

Proof. Since \( XY = \binom{n-1}{k-1} \binom{n-1}{l-1} \) we only need to show (26). The LHS of (26) is increasing in \( \beta \), it suffices to check the inequality at \( \beta = e_{i-2} \). In this case, using \( \beta^{i-1} = \frac{\alpha^{i-2} \hat{\alpha}}{1 + \alpha^{i-2} \hat{\alpha}} \), the inequality (26) reduces to (28). \( \square \)

Using the above claims we are going to prove

\[
|A||B| < \binom{n-1}{k-1} \binom{n-1}{l-1}.
\]

3.1. The case \( \binom{n-1}{n-k} < |A| \leq \binom{n-1}{n-k} + \binom{n-1}{n-k-1} \) for \( i = i_0 \). Let \( |A| = \binom{n-1}{n-k} + \binom{x}{n-k-1} \) for \( n - k - 1 \leq x \leq n - i \). Then, by Lemma \( \ref{lem:1} \), \( |B| \leq \binom{n-1}{l-1} - \binom{x}{l-1} \). By Claim \( \ref{claim:12} \) we have (29).
3.2. The case \((n^{-1-\gamma}) + (n^{-1-\gamma}) < |A| \leq (n^{-1-\gamma}) + (n^{-1-\gamma})\) for some \(i\) with \(4 \leq i \leq \iota_0\). Let \(|A| = X + (\frac{x}{n-k-2})\) where \(X = (n^{-1-\gamma}) + (n^{-1-\gamma})\) and \(n - k - 2 \leq x \leq n - i\).

Then, by Lemma 1, \(|B| \leq Y - (\frac{y}{l-1})\), where \(Y = (n^{-1-\gamma}) - (n^{-1-\gamma})\). By Claim 3, we have (29).

3.3. The case \((n^{-1-\gamma}) + (n^{-1-\gamma}) < |A| \leq (n^{-1-\gamma}) + (n^{-1-\gamma})\). Let \(|A| = X + (\frac{x}{n-k-2})\), where \(X = (n^{-1-\gamma}) + (n^{-1-\gamma})\) and \(x \geq n - k - 2\). Then, by Lemma 1, \(|B| \leq Y - (\frac{y}{l-1})\), where \(Y = (n^{-1-\gamma}) - (n^{-1-\gamma})\). By Claim 3, we have (29) for \(x \leq n - 4\), and for \(x \leq n - 3\) and \(\alpha < 0.27\). Thus, for the remaining, we may assume that \((n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma}) < |A| \leq (n^{-1-\gamma}) + (n^{-1-\gamma})\) and \(\alpha \geq 0.27\). Let \(|A| = X + (\frac{x}{n-k-2})\), where \(X = (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma})\) and \(x \geq n - k - 3\). Then, by Lemma 1, \(|B| \leq Y - (\frac{y}{l-1})\), where \(Y = (n^{-1-\gamma}) - (n^{-1-\gamma}) - (n^{-1-\gamma})\). In this case, by Claim 10, we have (29) for \(x \leq n - 3\) and \(\alpha > 0.23\).

3.4. The case \((n^{-1-\gamma}) + (n^{-1-\gamma}) < |A|\). First suppose that \((n^{-1-\gamma}) + (n^{-1-\gamma}) < |A| \leq (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma})\). Let \(|A| = (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma})\) for \(n - k - 2 \leq x \leq n - 3\). Then, by Lemma 1, \(|B| \leq (n^{-1-\gamma}) - (\frac{y}{l-1})\), and (29) follows from Claim 3(i).

Next suppose that \((n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma}) < |A| \leq (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma})\). Let \(|A| = (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma}) + (n^{-1-\gamma})\) for \(n - k - 3 \leq x \leq n - 4\). Then, by Lemma 1, \(|B| \leq (n^{-1-\gamma}) - (\frac{y}{l-1})\), and (29) follows from Claim 10(i).

Finally suppose that \((n^{-1-\gamma}) + (n^{-1-\gamma}) + \cdots + (n^{-1-\gamma}) < |A| \leq (n^{-1-\gamma}) + (n^{-1-\gamma}) + \cdots + (n^{-1-\gamma})\) for some \(t \geq 4\). Then, by Lemma 1, \(|B| \leq (n^{-1-\gamma}) - (\frac{y}{l-1})\). In this case we generically estimate

\[
\lim_{n \to \infty} \frac{|A|}{n^\gamma} \leq \alpha(1 + \bar{\alpha} + \bar{\alpha}^2 + \cdots + \bar{\alpha}^t), \quad \text{and} \quad \lim_{n \to \infty} \frac{|B|}{n^\gamma} \leq \beta^t.
\]

Let \(\gamma := 1 + \bar{\alpha} + \bar{\alpha}^2 + \cdots + \bar{\alpha}^t\). In view of Lemma 6, it suffices to show that \(\gamma \beta^t - 1 < 1\). Recall from (14) that \(\beta \leq \bar{\beta} = 2 - \sqrt{2}\). If \(t = 4\), then \(\gamma = \frac{1 - \alpha^4}{\alpha} \leq 4\) and \(\beta^t - 1 \leq \bar{\beta}^t < 0.21\), so \(\gamma \beta^t - 1 < 1\). Let \(t \geq 5\). Since \(\gamma \beta^t - 1 \leq (t + 1)\bar{\beta}^t - 1\) and the RHS is decreasing in \(t\), it is maximized at \(t = 5\). Thus \(\gamma \beta^t - 1 \leq 6\bar{\beta}^4 < 1\).

This completes the proof of Lemma 4.

4. Proof of Theorem 2. Uniqueness of the optimal families.

In the previous section, we have proved Lemma 4, that is, if \(|A| > \binom{n-1}{k-1} - \binom{n-1}{l-1}\), then \(|A||B| < \binom{n-1}{k-1} - \binom{n-1}{l-1}\) (under the assumptions of Theorem 2). Thus we have

\[
|A||B| \leq \binom{n-1}{k-1} - \binom{n-1}{l-1},
\]

and equality holds if and only if \(|A| = \binom{n-1}{k-1}\) and \(|B| = \binom{n-1}{l-1}\). In this case we show that \(A\) and \(B\) are stars.

Suppose that equality holds in (30). Then \(|A| = |A| = \binom{n-1}{k-1}\) and, by Lemma 1,

\[
\binom{n-1}{l-1} = |B| \leq \binom{n-1}{l-1} - |\sigma_t(A')| \leq \binom{n-1}{l-1} - \binom{n-1}{l-1} = \binom{n-1}{l-1}.
\]
Thus \(|\sigma_l(A^c)| = \binom{n-1}{n-1}\) must hold, and it follows from the Kruskal–Katona Theorem that this is possible only when \(A^c = \binom{[n]}{n-k}\) for some \(i \in [n]\). In this case we have \(A = \{A \in \binom{[n]}{n-k} : i \in A\}\) and \(B = \{B \in \binom{[n]}{n-k} : i \in B\}\), that is, both families are stars.

5. Proof of Theorem 3: The measure version

The proof is essentially written in [9], but for convenience we include it here.

Recall that \(\Delta\) is the set of \((\alpha, \beta) \in \Omega\) satisfying \(\mu_\alpha(A_j) \mu_\beta(B_j) < \alpha \beta\) for all \(j \geq 0\). Let \((\alpha, \beta) \in \Delta\) be fixed. Then we can choose \(\epsilon > 0\) sufficiently small so that \(\alpha - \epsilon < \alpha' < \alpha + \epsilon\) and \(\beta - \epsilon < \beta' < \beta + \epsilon\) imply \((\alpha', \beta') \in \Delta\). Let

\[ I := ((\alpha - \epsilon)n, (\alpha + \epsilon)n) \cap N \text{ and } J := ((\beta - \epsilon)n, (\beta + \epsilon)n) \cap N. \]

Let \(F\) and \(G\) be cross intersecting families in \([2^n]\). Let \(F^{(k)} := F \cap \binom{[n]}{k}\) and \(G^{(k)} := G \cap \binom{[n]}{k}\). If \(k \in I\) and \(l \in J\) then we can apply Theorem 2 to \(F^{(k)}\) and \(G^{(k)}\), and we get \(|F^{(k)}||G^{(k)}| \leq \binom{n}{k}^2\) provided \(n > n_0(\alpha, \beta)\). We also use the fact that the binomial distribution \(B(n, \alpha)\) is concentrated around \(\alpha n\), which yields

\[
\lim_{n \to \infty} \sum_{k \in I} \binom{n-1}{k} \alpha^k (1 - \alpha)^{n-k} = \alpha, \quad \text{and} \quad \lim_{n \to \infty} \sum_{k \in J} \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} = 0,
\]

and the corresponding formula for \(\beta\) and \(J\). Thus, as \(n \to \infty\), we have

\[
\mu_\alpha(F) \mu_\beta(G) \leq \left( \sum_{k \in I} |F^{(k)}| \alpha^k (1 - \alpha)^{n-k} + \sum_{k \in J} \binom{n}{k} \alpha^k (1 - \alpha)^{n-k} \right) \times \left( \sum_{l \in J} |G^{(l)}| \beta^l (1 - \beta)^{n-l} + \sum_{l \in I} \binom{n}{l} \beta^l (1 - \beta)^{n-l} \right)
\]

\[
= \left( \sum_{k \in I} |F^{(k)}| \alpha^k (1 - \alpha)^{n-k} \right) \left( \sum_{l \in J} |G^{(l)}| \beta^l (1 - \beta)^{n-l} \right) + o(1)
\]

\[
= \sum_{k \in I} \sum_{l \in J} |F^{(k)}||G^{(l)}| \alpha^k (1 - \alpha)^{n-k} \beta^l (1 - \beta)^{n-l} + o(1)
\]

\[
\leq \sum_{k \in I} \sum_{l \in J} \binom{n-1}{k} \binom{n-1}{l} \alpha^k (1 - \alpha)^{n-k} \beta^l (1 - \beta)^{n-l} + o(1)
\]

\[
= \left( \sum_{k \in I} \binom{n-1}{k} \alpha^k (1 - \alpha)^{n-k} \right) \left( \sum_{l \in J} \binom{n-1}{l} \beta^l (1 - \beta)^{n-l} \right) + o(1)
\]

\[
= \alpha \beta + o(1),
\]

and \(m(\alpha, \beta) \leq \alpha \beta\).

Now, to prove \(m(n, \alpha, \beta) = \alpha \beta\) for all \(n \geq 1\), suppose on the contrary that for some \(n\) and \(\delta > 0\) there exist cross intersecting families \(F, G \subset [2^n]\) with \(\mu_\alpha(F) \mu_\beta(G) \geq \alpha \beta + \delta\). Set \(F' = F \cup \{F \cup \{n+1\} : F \in F\}\) and \(G' = G \cup \{G \cup \{n+1\} : G \in G\}\), then \(F'\) and \(G'\) are cross intersecting. Since \(\mu_\alpha(F') = \mu_\alpha(F)(\alpha + (1 - \alpha)) = \mu_\alpha(F)\) and also \(\mu_\beta(G') = \mu_\beta(G)\), we have \(\mu_\alpha(F') \mu_\beta(G') \geq \alpha \beta + \delta\). This means \(m(n', \alpha, \beta) \geq \alpha \beta + \delta\) for all \(n' \geq n\), and \(m(\alpha, \beta) \geq \alpha \beta + \delta\), a contradiction. \(\square\)

6. Proof of Theorem 3: A sketch

The proof is similar to the proof of the main result in [5], and much easier than that of Theorem 2. So we only include a sketch here.
Recall that the case $|\mathcal{B}| > \binom{n-1}{l-1}$ is already settled by Claim 2, and we may assume that $|\mathcal{A}| > \binom{n-1}{k-1}$. Let

$$|\mathcal{A}| = \sum_{i=1}^{t} \binom{n-i}{n-k-i+1} + \binom{x}{n-k-t}$$

for some $t$ with $1 \leq t \leq n-k$ and $x$ with $n-k-t \leq x \leq n-t-1$. Then, by Lemma 1, we have

$$|\mathcal{B}| \leq \binom{n-t}{l-t} - \binom{x}{l-t}.$$

We need to show the following.

**Lemma 13.** For $1 \leq t \leq n-k$ and $n-k-t \leq x \leq n-t-1$,

$$\left( \sum_{i=1}^{t} \binom{n-i}{n-k-i+1} + \binom{x}{n-k-t} \right) \left( \binom{n-t}{l-t} - \binom{x}{l-t} \right) < \binom{n-1}{k-1} \binom{n-1}{l-1}.$$

The proof of the lemma breaks down into four cases: $t = 1$, $t = 2$, $t = 3$, and $t \geq 4$. The case $t \geq 4$ is easy, and one can argue as in Subsection 3.4. The proof of the other three cases are similar, and we only present the case $t = 1$ here. In this case Lemma 13 reads as follows.

**Claim 14.** For $n \geq 2$ and $n-k-1 \leq x \leq n-2$ we have

$$\left( \binom{n-1}{k-1} + \binom{x}{n-k-1} \right) \left( \binom{n-1}{l-1} - \binom{x}{l-1} \right) < \binom{n-1}{k-1} \binom{n-1}{l-1}.$$

**Proof.** We argue exactly in the same way as in the proof of Claim 11. Then we get (27) for $i = 2$, which reduces to

$$\frac{1}{n-k} \sum_{j=0}^{l-2} \frac{1}{n-2-j} \geq \frac{1}{n-l} \sum_{j=0}^{n-k-2} \frac{1}{n-2-j}.$$

This is the opposite inequality to (8), and this contradiction proves the claim. □

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