UPPER BOUNDS FOR VIRTUAL DIMENSIONS OF SEIBERG-WITTEN MODULI SPACES

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Abstract. Given a closed four-manifold with $b_1 = 0$ and a prime number $p$, we prove that for any mod $p^r$ basic class, the virtual dimension of the Seiberg-Witten moduli space is bounded above by $2r(p - 1) - 2$ under some conditions on $r$ and $b_2^+$. As an application, we obtain adjunction inequalities for embedded surfaces with negative self-intersection number.

1. Introduction

The Seiberg-Witten invariant of a smooth four-manifold has been playing a fundamental role in the study of four-manifolds, and has been a rich source of ideas and applications. The basic ingredient in the construction of the invariant is the moduli space of solutions of the Seiberg-Witten equation, which is defined for each spin$^c$ structure on a four-manifold. The virtual dimension of the moduli space is of particular importance, and there is a fundamental conjecture on it. Let us recall it here. Let $X$ be a closed, connected, oriented and smooth four-manifold, and let $s$ be a spin$^c$ structure on $X$. We will omit a four-manifold $X$ in the notation if it is clear from the context. We say that (the isomorphism class of) $s$ is a basic class if the Seiberg-Witten invariant $SW(s) \not\equiv 0$. Let $d(s)$ denote the virtual dimension of the moduli space corresponding to $s$. We define that $X$ is of simple type if $d(s) = 0$ whenever $s$ is a basic class. Now we state the so-called simple type conjecture (see [14, Conjecture 1.6.2]).

Conjecture 1.1. Every closed, connected, oriented and smooth four-manifold with $b_2^+ \geq 2$ is of simple type.

The simple type conjecture was originally posed in connection to Witten’s conjecture on the relationship between the Donaldson and the Seiberg-Witten invariants for four-manifolds of simple type. By a partial solution due to Feehan and Leness [7], the simple type conjecture implies that the relation of these invariants holds under mild topological conditions only.

The simple type conjecture trivially holds in the case where $b_2^+ - b_1$ is even, since the Seiberg-Witten invariant always vanishes. In the case that $b_2^+ - b_1$ is odd, the simple type conjecture has been verified for all symplectic four-manifolds [23] and also for other very large families of four-manifolds. However, so far, there is no result without demanding a condition on a smooth structure, except for the following work. We say that $s$ is a mod $q$ basic class if $SW(s) \not\equiv 0 \mod q$. So we can consider the mod $q$ analogue of the simple type conjecture, and recently, Kato, Nakamura and Yasui [12] solved the mod 2 analogue under a mild condition on the cohomology ring, which depends only on the underlying topological structure.

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In this paper, under a simple topological condition, we give an upper bound for the virtual dimension, giving a new approach to the simple type conjecture. Now we state our main theorem. For a prime $p$ and integers $k, t$ with $k \equiv t - 2 \mod p$, we define an integer $a(k, t)$ by

$$k + (t - 3)(p - 1) = a(k, t)p + 1. \tag{1.1}$$

For an integer $r$, let $1 \leq r_p \leq p - 1$ be the integer such that $r_p = 1$ for $r \equiv 0 \mod p$ and $r \equiv r_p \mod p$ for $r \not\equiv 0 \mod p$.

**Theorem 1.2.** Let $p$ be a prime, and suppose that $b^+_2$ is odd with $b^+_2 > 2$ and $b_1 = 0$. Let $s$ be a spin$^c$ structure on $X$, and set $k = (b^+_2 - 1)/2$. Take an integer $r$ satisfying $1 \leq r < p(p - 1)$. If $s$ is a mod $p'$ basic class, then

$$d(s) \leq 2r(p - 1) - 2$$

whenever $k, r$ satisfy the following conditions:

1. $k \not\equiv 0, 1, \ldots, r_p - 1 \mod p$;
2. under the above condition, if an integer $t$ satisfies $t - 2 \equiv k \mod p$ and $3 \leq t \leq r$, then $a(k, t)$ satisfies

$$3a(k, t) + 5 \not\equiv 0 \mod p \quad (t \equiv 0 \mod p \text{ and } t \geq p > 3)$$

$$a(k, t) + 2 \not\equiv 0 \mod p \quad (t \equiv 1 \mod p \text{ and } t > p)$$

$$3a(k, t) + 4 \not\equiv 0 \mod p \quad (t \equiv 3 \mod p)$$

$$(2t - 3)a(k, t) + 3t - 5 \not\equiv 0 \mod p \quad (t \equiv 4, 5, \ldots, p - 1 \mod p).$$

To prove Theorem 1.2, we will employ the Bauer-Furuta invariant, which is a lift of the Seiberg-Witten invariant to the stable cohomotopy group. This will enable us to deduce the divisibility of the Seiberg-Witten invariant from a property of the $p$-localized cohomotopy groups of complex projective spaces, that will be proved by computing Toda brackets based on the $p$-local cell structure of a complex projective space. Using techniques in hard homotopy theory such as Toda brackets is new in the study of Seiberg-Witten invariant, though it is standard in algebraic topology.

**Corollary 1.3.** Let $p$ be a prime, and suppose that $b^+_2$ is odd with $b^+_2 > 2$ and $b_1 = 0$. If $k = (b^+_2 - 1)/2$ and an integer $r$ satisfies $k(k - 1) \cdots (k - r + 1) \not\equiv 0 \mod p$ and $s$ is a mod $p'$ basic class, then

$$d(s) \leq 2r(p - 1) - 2.$$  

**Proof.** If $k \not\equiv 0, 1, \ldots, r_p - 1 \mod p$ exists, then there is no integer $t$ satisfying $t - 2 \equiv k \mod p$ and $3 \leq t \leq r$. Then the statement follows from Theorem 1.2. \hfill $\Box$

**Corollary 1.4.** Let $p$ be a prime, and suppose that $b^+_2$ is odd with $b^+_2 > 2$ and $b_1 = 0$. If $(b^+_2 - 1)/2 \equiv 0 \mod p$ and $s$ is a mod $p$ basic class, then

$$d(s) \leq 2p - 4.$$  

**Proof.** Apply Theorem 1.2 for $r = 1$. Note that the condition of $k^2 \not\equiv 0 \mod p$ is equivalent to the one of $k \not\equiv 0 \mod p$. \hfill $\Box$

We can deduce Corollaries 1.3 and 1.4 from the result of Bauer and Furuta [4, Theorem 3.7] by a purely algebraic argument, as in Section 5. The authors thank the referee of the earlier draft for pointing out this algebraic argument for Corollary 1.4. On the other hand, Theorem 1.2 gives an upper bound better than the one deduced from [4, Theorem 3.7]. For instance, if $k = 1$ and $r = p$, then by Theorem 1.2, we get

$$d(s) \leq 2r(p - 1) - 2.$$
for $SW(s) \not\equiv 0 \mod p^r$, whereas we only can deduce a rather weaker inequality
$$d(s) \leq 2p^r - 4$$
from Bauer-Furuta [4, Theorem 3.7]. See Section 5.
A straightforward corollary of Corollary 1.4 below gives an upper bound for any basic class, not a mod $p$ basic class.

**Corollary 1.5.** Suppose that $b_2^+ \equiv 1$ and $b_1 = 0$. For a basic class $s$, let $p$ be the least prime not dividing $SW(s)$ and satisfying $(b_2^+ - 1)/2 \not\equiv 0 \mod p$. Then
$$d(s) \leq 2p - 4.$$

From this corollary, we can derive a coarse but more concrete upper bound.

**Corollary 1.6.** Suppose that $b_2^+ \equiv 1$ and $b_1 = 0$. Then every basic class $s$ satisfies
$$d(s) \leq \max\{2|SW(s)| - 6, b_2^+ - 7, 10\}.$$

**Remark 1.7.** Theorem 2.2 below shows that we can get a sharper upper bound than Corollary 1.6 if we assume either $|SW(s)|$ or $b_2^+$ is large enough. For example (Example 2.6), if $\max\{|SW(s)|, (b_2^+ - 1)/2\} \geq 44$, then
$$d(s) \leq \max\{\frac{4}{3}|SW(s)| - 4, \frac{2}{3}b_2^+ - \frac{14}{3}\}.$$

Let us consider a small prime $p$. Clearly, if $p$ is small, then the upper bound in Corollary 1.4 imposes a strong constraint on the virtual dimension $d(s)$. Here, we consider the case $p = 2, 3$ cases. For $p = 2$, $(b_2^+ - 1)/2 \not\equiv 0 \mod 2$ is equivalent to $b_2^+ \equiv 3 \mod 4$ in Corollary 1.4 because $SW(s) = 0$ for $b_2^+$ even. Then we obtain that if $b_2^+ \geq 2, b_1 = 0$ and $b_2^+ \equiv 3 \mod 4$, then $d(s) = 0$ for every mod 2 basic class $s$. This is the special case $b_1 = 0$ of [12, Corollary 1.4], the above mentioned solution to the mod 2 analogue of the simple type conjecture. Our proof is very different from theirs. Indeed, their proof relies on connected sum formulae for the Bauer-Furuta invariant [5, 11] and thus does not give upper bounds for virtual dimensions in the $p \geq 3$ case. For $p$ odd, $(b_2^+ - 1)/2 \not\equiv 0 \mod p$ is equivalent to $b_2^+ \not\equiv 1 \mod p$. Then for $p = 3$, we have:

**Corollary 1.8.** Suppose that $b_2^+ \equiv 1$ and $b_1 = 0$. If $b_2^+ \not\equiv 1 \mod 3$ and $s$ is a mod 3 basic class, then
$$d(s) = 0$$

We turn to an application of Corollary 1.4. A typical application of an affirmative solution to the simple type conjecture is an adjunction inequality for embedded surfaces with negative self-intersection number in the case $b_1 = 0$ [19]. Here, we prove an adjunction inequality from Corollary 1.4, instead of assuming manifolds being of simple type. For a second homology class $\alpha$ of a four-manifold, an adjunction inequality gives a lower bound for the genus of a smoothly embedded closed oriented surface representing $\alpha$, and adjunction inequalities have various powerful applications to four-dimensional topology (e.g. [1, 2, 9, 26, 12]). When $\alpha \cdot \alpha \geq 0$, adjunction inequalities were previously obtained in [13, 16, 20]. When $\alpha \cdot \alpha < 0$, Ozsváth and Szabó [20] proved an adjunction inequality for four-manifolds satisfying a simple type condition on the extended Seiberg-Witten invariant, where in the $b_1 = 0$ case, their simple type condition is the same as ours. Applying Corollary 1.4 and results of Ozsváth and Szabó [19, 20], we obtain adjunction inequalities without assuming any simple type condition.
Theorem 1.9. Let $p$ be a prime, and suppose that $b_2^+$ is odd with $b_2^+ > 2$, $b_1 = 0$, $(b_2^+ - 1)/2 \not\equiv 0 \mod p$ and $s$ is a mod $p$ basic class. Let $\alpha$ be a second homology class of $X$ which satisfies $\alpha \cdot \alpha < 0$ and is represented by a smoothly embedded closed oriented surface of genus $g$.

1. If $g \geq 2p - 3$, then
   \[ |\langle c_1(s), \alpha \rangle \rangle | + \alpha \cdot \alpha + 2d(s) \leq 2g - 2. \]

2. If $g \geq p - 1$, then
   \[ |\langle c_1(s), \alpha \rangle \rangle | + \alpha \cdot \alpha + d(s) \leq 2g - 2. \]

Remark 1.10. It is straightforward to generalize Theorem 1.9 for mod $p^r$ basic classes by using Theorem 1.2.

We note that the $p = 2$ case of this theorem is a special case of a result of Kato, Nakamura and Yasui [12, Theorem 1.7]. On the other hand, we can also derive adjunction inequalities for a basic class, instead of a mod $p$ basic class. For a basic class $s$, as seen from the proof of Corollary 1.6, we can find a prime $p$ with $p \leq \max\{|SW(s)| - 1, (b_2^+ - 3)/2, 7\}$ satisfying the assumption of the above theorem. Hence, the adjunction inequalities in (1) and (2) hold for any ordinary basic class $s$ if $g \geq \max\{|2SW(s)| - 5, b_2^+ - 6, 11\}$ and $g \geq \max\{|SW(s)| - 2, (b_2^+ - 5)/2, 6\}$, respectively. As well as Corollary 1.6, these conditions get better as either $|SW(s)|$ or $b_2^+$ get larger (see Remark 1.7).

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2. Upper bound

This section proves Theorem 1.2 and Corollary 1.6. Hereafter, let $X$ be a closed, connected, oriented and smooth four-manifold with $b_1 = 0$ and $b_2 \geq 2$, and let $s$ be a spin$^c$ structure on $X$.

We use the Bauer-Furuta invariant. Since there is a formula

\begin{equation}
(2.1) \quad d(s) = \frac{c_1(s)^2 - \text{sign}(X)}{4} - (1 + b_2^+),
\end{equation}

it follows from [4, Proposition 3.4] that the Bauer-Furuta invariant $BF(s)$ belongs to the stable cohomotopy group $\pi^{b_2^+-1}(\mathbb{C}P^{d-1})$, where $2d = d(s) + 1 + b_2^+$. Moreover, there is an identity

\begin{equation}
(2.2) \quad \text{hur}(BF(s)) = SW(s),
\end{equation}

where $\text{hur} : \pi^{b_2^+-1}(\mathbb{C}P^{d-1}) \to H^{b_2^+-1}(\mathbb{C}P^{d-1})$ denotes the Hurewicz homomorphism. Note that $H^{b_2^+-1}(\mathbb{C}P^{d-1})$ is isomorphic to 0 and $\mathbb{Z}$ according to $b_2^+$ being even and odd.

Let $\mathbb{Z}_{(p)}$ denote the localization of $\mathbb{Z}$ at the prime $p$, that is, it is the subring of $\mathbb{Q}$ consisting of fractions whose denominators are not divisible by $p$. We will prove the following theorem in Section 4

Theorem 2.1. Under the assumption in Theorem 1.2, the natural map

$$(\pi^{2k}(\mathbb{C}P^{k+(p-1)}) \otimes Z_{(p)})/\text{Tor} \to (\pi^{2k}(\mathbb{C}P^{k}) \otimes Z_{(p)})/\text{Tor}$$

is identified with $p' : Z_{(p)} \to Z_{(p)}$. 

Proof of Theorem 1.2. Since \( SW(s) = 0 \) for \( b_2^+ \) even, we only need to consider the case \( b_2^+ \) odd. Let \( 2n = b_2^+ + 2r(p - 1) - 1 \) and \( 2\delta = d(s) - 2r(p - 1) \). Then the Bauer-Furuta invariant \( BF(s) \) belongs to \( \pi^{2n-2r(p-1)}/(CP^{n+\delta}) \) as mentioned above. Suppose \( d(s) \geq 2r(p - 1) \). Then \( \delta \geq 0 \), and so there is a commutative diagram

\[
\begin{array}{ccc}
\pi^{2n-2r(p-1)}(CP^{n+\delta}) & \xrightarrow{\text{hur}} & H^{2n-2r(p-1)}(CP^{n+\delta}) \\
\downarrow i^*_1 & & \downarrow i^*_1 \\
\pi^{2n-2r(p-1)}(CP^n) & \xrightarrow{\text{hur}} & H^{2n-2r(p-1)}(CP^n) \\
\downarrow i^*_1 & & \downarrow i^*_1 \\
\pi^{2n-2r(p-1)}(CP^{n-r(p-1)}) & \xrightarrow{\text{hur}} & H^{2n-2r(p-1)}(CP^{n-r(p-1)})
\end{array}
\]

where \( i_1 : CP^{n-r(p-1)} \to CP^n \) and \( i_2 : CP^n \to CP^{n+\delta} \) are inclusions. We apply Theorem 2.1 above. Then, the map

\[ i^*_1 \circ 1 : (\pi^{2n-2r(p-1)}(CP^n) \otimes Z(p))/\text{Tor} \to (\pi^{2n-2r(p-1)}(CP^{n-r(p-1)}) \otimes Z(p))/\text{Tor} \]

is identified with \( p' : Z(p) \to Z(p) \), the multiplication by \( p' \), where \( Z(p) \) denotes the ring of all rational numbers whose denominators are not divisible by \( p' \). Then the image of the Hurewicz homomorphism

\[ \text{hur} : \pi^{2n-2r(p-1)}(CP^{n+\delta}) \to H^{2n-2r(p-1)}(CP^{n+\delta}) \]

is included in \( p'Z \subset Z \cong H^{2n-2r(p-1)}(CP^{n+\delta}) \). Thus by the identity (2.2), we obtain that \( SW(s) \) is divisible by \( p' \), which contradicts to the assumption that \( s \) is a mod \( p' \) basic class. Thus we must have \( d(s) \leq 2r(p - 1) - 1 \). Since \( s \) is a basic class, \( d(s) \) is even. Therefore we obtain \( d(s) \leq 2r(p - 1) - 2 \), proving the statement.

We prove the crucial part of Corollary 1.6 in a more general form. To this end, we consider primes in intervals. For \( c \in (1/2, 1] \), let \( S_{c,n} \) be the infimum of real numbers such that for each \( x \geq S_{c,n} \), there are at least \( n \) primes in \((x/2, x)\) for \( c = 1 \) and \((x/2, cx)\) for \( c < 1 \). Lemma 2.4 guarantees that \( S_{c,n} \) certainly exists for each \( c, n \), where we can easily see that \( S_{c,n} \) is actually the least real number satisfying the above condition.

**Theorem 2.2.** Let \( n = \max\{|SW(s)|, (b_2^+ - 1)/2\} \) and \( c \in (1/2, 1] \). If \( s \) is a basic class and \( n \geq S_{c,2} \), then

\[
d(s) \leq \begin{cases} 
2n - 6 & (c = 1) \\
2cn - 4 & (c < 1) 
\end{cases}
\]

Proof. Since \( n \geq S_{c,2} \), there are two primes \( p, q \) in \((n/2, n)\) for \( c = 1 \) and \((n/2, cn)\) for \( c < 1 \). Then \( p < n < 2p \) and \( q < n < 2q \), implying that \( n \) is not divisible by \( p \) and \( q \). Moreover, for any \( 1 \leq m < n \), we have \( m < pq \), implying \( m \) is not divisible by at least one of \( p, q \). Then \( SW(s) \cdot (b_2^+ - 1)/2 \) is not divisible by either \( p \) or \( q \). Thus by Corollary 1.5, \( d(s) \leq 2\max\{p, q\} - 4 \). Clearly, \( \max\{p, q\} \leq n - 1 \) for \( c = 1 \) and \( \max\{p, q\} \leq cn \) for \( c < 1 \), completing the proof.

**Remark 2.3.** For \( n < S_{c,2} \), we can alternatively apply Corollary 1.6 to get an upper bound for \( d(s) \).

To make Theorem 2.2 applicable, we give an upper bound for \( S_{c,n} \) in terms of a generalized Ramanujan prime introduced in [3]. For \( c \in (0, 1) \), the \( n \)-th \( c \)-Ramanujan prime \( R_{c,n} \) is defined to be the least number such that for any \( x \geq R_{c,n} \), \((cx, x)\) includes at least \( n \) primes. Clearly, \( R_{c,n} \) is a prime, and \( R_{c,n}^+ \) coincides with
implies that if we assume either

Let

There are inequalities

\[
S_{1,n} \leq R_{\frac{3}{2},n+1} \quad \text{and} \quad S_{c,n} \leq \frac{1}{c} R_{\frac{3}{2},n},
\]

where \( c \in (1/2, 1) \).

Proof. If \( x \geq R_{\frac{3}{2},n+1} \), then \( (x/2, x) \) includes at least \( n + 1 \) primes, implying that \( (x/2, x) \) includes at least \( n \) primes. Hence the first inequality is proved. Let \( c \in (1/2, 1) \). If \( cx \geq R_{\frac{3}{2},n} \), then there are at least \( n \) primes in \( (x/2, cx) \), implying the second inequality.

We give a coarse upper bound for \( R_{c,n} \), which helps evaluate \( S_{c,n} \) by Lemma 2.4.

Lemma 2.5. For \( n \geq 1 \) and \( c \in (0, 1) \), there is an inequality

\[
R_{c,n} \leq \max \left\{ (2\sqrt{2n+1})!, \exp \left( \frac{-\log c + \frac{3}{2}}{1-c}, \frac{e^2}{c} \right), 59 \right\},
\]

where \([x]\) denotes the least integer \( \geq x \).

Proof. Let \( \pi(x) \) denote the prime counting function, that is, \( \pi(x) \) is the number of primes \( \leq x \). By [21, Theorem 2 and Corollary 1], for \( x > \max\{59, e^2/c\} \), we have

\[
\pi(x) - \pi(cx) > \frac{x}{\log x} \left( 1 + \frac{1}{2 \log x} \right) - \frac{cx}{\log(cx) - \frac{3}{4}}.
\]

If \( x \geq \exp \left( \frac{-\log c + \frac{3}{2}}{1-c} \right) \), then \( \frac{x}{\log x} - \frac{cx}{\log(cx) - \frac{3}{4}} \geq 0 \), and so

\[
\pi(x) - \pi(cx) \geq \frac{x}{2(\log x)^2}.
\]

Clearly, \( \frac{x}{\log x} \geq n \) if and only if \( e^{\sqrt{x}} - x^{\sqrt{2n}} \geq 0 \). For \( x \geq (2\sqrt{2n+1})! \), we have

\[
e^{\sqrt{x}} - x^{\sqrt{2n}} > \frac{x^{\sqrt{2n+1}}}{(2\sqrt{2n+1})!} - x^{\sqrt{2n}} \geq 0.
\]

Thus the proof is complete.

Now we are ready to prove Corollary 1.6.

Proof of Corollary 1.6. Let \( n = \max\{|SW(s)|, (b_2^+ - 1)/2|\} \). By (2.3) and Lemma 2.4, we get \( S_{1,2} \leq R_{1,3} = 17 \). Then by Theorem 2.2, we obtain the inequality in the statement for \( n \geq 17 \). We can easily check that if \( 12 \leq n \leq 16 \), then there are two primes in \( (n/2, n) \), so that the proof of Theorem 2.2 for \( c = 1 \) works verbatim to show the inequality holds for \( 12 \leq n \leq 16 \). Suppose \( 1 \leq m \leq n \leq 11 \). Then \( m, n \) are divisible by at most two of \( 2, 3, 5, 7, 11 \). Moreover, at most one of \( m, n \) is divisible by \( 11 \), and if this is the case, \( mn \) is not divisible by at least one of \( 2, 3, 5, 7 \). We also have that at most one of \( m, n \) is divisible by \( 7 \), and if this is the case, \( mn \) is not divisible by at least one of \( 2, 3, 5 \). Then we obtain that \( SW(s) \cdot (b_2^+ - 1)/2 \) is not divisible by at least one of \( 2, 3, 5, 7 \). Thus by Corollary 1.5, \( d(s) \leq 2 \cdot 7 - 4 = 10 \), completing the proof.

Note that Proposition 2.2 implies that if we assume either \(|SW(s)|\) or \( b_2^+ \) is large enough, then we could get a sharper upper bound than Corollary 1.6. Here, we give such an example.
Example 2.6. Let $n = \max\{|SW(\mathfrak{s})|, (b_2^+ - 1)/2\}$, and let $\mathfrak{s}$ be a basic class. By (2.3) and Lemma 2.4, $S_{4,2}^c \leq \frac{2}{3}(R_{4,2}^c = \frac{3}{2} \cdot 29 = 43.5$, and so by Theorem 2.2, for $n \geq 44$, there is an inequality
\[ d(\mathfrak{s}) \leq \frac{4}{3}n - 4 = \max \left\{ \frac{4}{3}|SW(\mathfrak{s})| - 4, \frac{2}{3}b_2^+ - \frac{14}{3} \right\} . \]

3. Adjunction inequality

This section proves Theorem 1.9. To this end, we use the Seiberg-Witten invariant of the form
\[ SW_2 : A(X) \to \mathbb{Z}, \]
where $A(X) = (\Lambda H_1(X; \mathbb{Z})) \otimes \mathbb{Z}[U]$ such that elements of $H_1(X; \mathbb{Z})$ are assumed to be of degree 1 and $U$ is of degree 2. See [19, 20] for details. The above Seiberg-Witten invariant is an extension of the usual Seiberg-Witten invariant because there is an identity
\[ SW_2(U^{d(\mathfrak{s})/2}) = SW(\mathfrak{s}) \]
whenever $d(\mathfrak{s})$ is even.

Let $\alpha$ be a second homology class of $X$ represented by a smoothly embedded closed oriented surface $\Sigma$ of genus $g$. Let $PD(\alpha)$ denote the Poincaré dual of $\alpha$. Since $c_1(\mathfrak{s} + PD(\alpha)) = c_1(\mathfrak{s}) + 2PD(\alpha)$, it follows from (2.1) that
\[ d(\mathfrak{s} + PD(\alpha)) = d(\mathfrak{s}) + c_1(\mathfrak{s})PD(\alpha) + PD(\alpha)^2 \]
\[ = d(\mathfrak{s}) + (c_1(\mathfrak{s}), \alpha) + \alpha \cdot \alpha. \]

Since $c_1(\mathfrak{s}) \equiv w_2(M) \mod 2$, the Wu formula implies that $(c_1(\mathfrak{s}), \alpha) + \alpha \cdot \alpha$ is an even integer. In particular, $d(\mathfrak{s} + PD(\alpha))$ is even whenever so is $d(\mathfrak{s})$. We will freely use these facts.

Lemma 3.1. Suppose that $-(c_1(\mathfrak{s}), \alpha) + \alpha \cdot \alpha \geq \max\{2g - 2d(\mathfrak{s}), 0\}$ and $g \geq 1$. If $\mathfrak{s}$ is a mod $p$ basic class, then so is $\mathfrak{s} - PD(\alpha)$ too.

Proof. By (3.1) and $-(c_1(\mathfrak{s}), \alpha) + \alpha \cdot \alpha \geq 0$, there is an inequality $d(\mathfrak{s} - PD(\alpha)) \geq d(\mathfrak{s})$. Then since $-(c_1(\mathfrak{s}), \alpha) + \alpha \cdot \alpha + 2d(\mathfrak{s}) \geq 2g$ and $g \geq 1$, we can apply [20, Theorem 1.7] to get
\[ SW_{\mathfrak{s} - PD(\alpha)}(U^{d(\mathfrak{s} - PD(\alpha))/2}) = SW_2(U^{d(\mathfrak{s})/2}). \]

(In [20, Theorem 1.7], it is stated that $SW_2(U^d) = SW_{\mathfrak{s} - PD[\Sigma]}(U^{d'})$, where $d$ and $d'$ denote the dimensions of $\mathfrak{s}$ and $\mathfrak{s} - PD[\Sigma]$, respectively. We must be aware this is a typo by dimensionality, and the above equality is the correct one.) Thus by (3), $SW(\mathfrak{s} - PD(\alpha)) = SW(\mathfrak{s})$, implying that $SW(\mathfrak{s} - PD(\alpha))$ is a mod $p$ basic class, as stated. \[ \square \]

When $d(\mathfrak{s}) = 2n \geq 0$, e.g. $\mathfrak{s}$ is a basic class, we define
\[ \hat{X} = X \# nCP^2. \]

Clearly, $b_1(\hat{X}) = b_1(X) = 0$ and $b_2^+(\hat{X}) = b_2^+(X) \geq 2$. We may regard that the (co)homology of $X$ is a subgroup of the (co)homology of $\hat{X}$. Let $L = c_1(\mathfrak{s}) + 3PD(e_1) + \cdots + 3PD(e_n) \in H^2(\hat{X})$, where each $e_i$ is the second homology class of the $i$-th $CP^2$ represented by the exceptional sphere. By the blow-up formula [8, 17], there is a spin$^c$ structure $\hat{\mathfrak{s}}$ on $\hat{X}$ such that
\[ c_1(\hat{\mathfrak{s}}) = L \quad \text{and} \quad SW(\hat{\mathfrak{s}}) = SW(\mathfrak{s}). \]
Since \( \text{sign}(\tilde{X}) = \text{sign}(X) - n \), it follows from (2.1) that
\[
d(\tilde{\alpha}) = \frac{c_1(\tilde{\alpha})^2 - \text{sign}(\tilde{X})}{4} - (1 + b_+^2(\tilde{X}))
\]
\[
= \frac{c_1(\tilde{\alpha})^2 - \text{sign}(X)}{4} - (1 + b_+^2(X)) - 2n
\]
\[
= d(\alpha) - 2n
\]
\[
= 0.
\]

Let \( \tilde{\alpha} = \alpha - e_1 - \cdots - e_n \in H_2(\tilde{X}) \).

**Lemma 3.2.** Suppose that \( \alpha - \alpha < 0 \), \( \langle c_1(\tilde{\alpha}), \alpha \rangle + \alpha \cdot \alpha \geq 2g \) and \( g \geq 1 \). If \( \alpha \) is a mod \( p \) basic class, then so is \( \tilde{\alpha} = \text{PD}(\tilde{\alpha}) \).

**Proof.** Since \( \tilde{\alpha} \) is a basic class, we can consider \( \tilde{X} \) and \( \tilde{\alpha} \). We may assume that \( \Sigma \) is also embedded into \( \tilde{X} \). Let
\[
\xi(\Sigma) = (U - x_1 y_1) \cdots (U - x_g y_g) \in A(\tilde{X}),
\]
where \( \{x_1, y_1, \ldots, x_g, y_g\} \) is the image of the standard symplectic basis of \( H_1(\Sigma) \). Then by assumption and \( d(\tilde{\alpha}) = 0 \), we can apply [19, Theorem 1.3] to obtain
\[
SW_{\tilde{\alpha} + \text{PD}(\tilde{\alpha})}(\xi(\Sigma) U^m) = SW_{\tilde{\alpha}}(1),
\]
where \( 2m = \langle c_1(\tilde{\alpha}), \alpha \rangle + \alpha \cdot \alpha - 2g \geq 0 \). Since \( b_1(\tilde{X}) = 0 \), \( i_* (\xi(\Sigma)) U^m \) coincides with \( U^{g+m} \) modulo torsion elements. Since \( SW_{\tilde{\alpha} + \text{PD}(\tilde{\alpha})}: A(\tilde{X}) \to \mathbb{Z} \) is linear, it annihilates torsion elements, so that \( SW_{\tilde{\alpha} + \text{PD}(\tilde{\alpha})}(\xi(\Sigma) U^m) = SW_{\tilde{\alpha} + \text{PD}(\tilde{\alpha})}(U^{g+m}) \). Thus by (3),
\[
SW(\tilde{\alpha} + \text{PD}(\tilde{\alpha})) = SW_{\tilde{\alpha} + \text{PD}(\tilde{\alpha})}(U^{m+g}) = SW_{\tilde{\alpha}}(1) = SW(\tilde{\alpha}) = SW(\alpha),
\]
implying \( \tilde{\alpha} + \text{PD}(\tilde{\alpha}) \) is a mod \( p \) basic class, as desired. \( \square \)

We are ready to prove Theorem 1.9.

**Proof of Theorem 1.9.** (1) By Corollary 1.4 and the assumption, we have \( 2d(\alpha) \leq 2(2p - 4) \leq 2g - 2 \). It thus suffices to prove the case \( \langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha \geq 0 \).

By reversing the orientation of the embedded surface if necessary, we may assume \( \langle c_1(\alpha), \alpha \rangle \leq 0 \), so that \( -\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha \geq 0 \). We also have \( g \geq 2p - 3 \geq 1 \). Assume that \( -\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + 2d(\alpha) \geq 2g - 1 \). Then since \( -\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + 2d(\alpha) \) is even, \( -\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + 2d(\alpha) \geq 2g \). So by Lemma 3.1, \( \alpha - \text{PD}(\alpha) \) is a mod \( p \) basic class. Moreover,
\[
2d(\alpha - \text{PD}(\alpha)) = 2(-\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + d(\alpha))
\]
\[
\geq -\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + 2d(\alpha)
\]
\[
\geq 2g
\]
\[
\geq 2(2p - 3).
\]

Then we obtain a contradiction to Corollary 1.4. Therefore we must have \( -\langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + 2d(\alpha) \leq 2g - 2 \).

(2) Assume that \( \langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + d(\alpha) \geq 2g \). Then by Lemma 3.2, \( \tilde{\alpha} + \text{PD}(\tilde{\alpha}) \) is a mod \( p \) basic class. Moreover, by (3.1),
\[
d(\tilde{\alpha} + \text{PD}(\tilde{\alpha})) = \langle c_1(\tilde{\alpha}), \tilde{\alpha} \rangle + \tilde{\alpha} \cdot \tilde{\alpha} + d(\tilde{\alpha})
\]
\[
= (\langle c_1(\alpha), \alpha \rangle + 3n) + (\alpha \cdot \alpha - n) + 0
\]
\[
= \langle c_1(\alpha), \alpha \rangle + \alpha \cdot \alpha + d(\alpha)
\]
\[
\geq 2g
\]
\[
\geq 2p - 2,
\]
where \( d(s) = 2n \). So we obtain a contradiction to Corollary 1.4, and hence \( \langle c_1(s), \alpha \rangle + \alpha \cdot \alpha + d(s) \leq 2g - 1 \). Since \( \langle c_1(s), \alpha \rangle + \alpha \cdot \alpha + d(s) \) is even, we must have \( \langle c_1(s), \alpha \rangle + \alpha \cdot \alpha + d(s) \leq 2g - 2 \). By reversing the orientation of the embedded surface if necessary, we may assume \( \langle c_1(s), \alpha \rangle \geq 0 \). Thus the proof is complete. \( \square \)

4. Cohomotopy computation

This section proves Theorem 2.1. Since we are concerned with stable cohomotopy groups, all spaces and maps will be stabilized. We will also localize all spaces and maps at an odd prime \( p \), unless otherwise is specified. We refer to [10] for \( p \)-localization.

4.1. Reduction. We reduce the computation of this map to simpler spaces. The following (\( p \)-locally stable) splitting was proved in [15, Theorem 9.3 and Corollary 9.5].

**Lemma 4.1.** There is a homotopy equivalence

\[ \mathbb{C}P^n \simeq X_1^r \setminus \cdots \setminus X_{p-1}^r \]

such that \( X_i^r = S^{2i} \cup e^{2i+2(p-1)} \cup \cdots \cup e^{2i+2r(p-1)} \), where \( r = \left\lfloor \frac{n-1}{p-1} \right\rfloor \).

We can easily see from the proof of [15, Theorem 9.3] that the splitting of Lemma 4.1 is natural with respect to \( n \) in the sense that \( X_i^r \) for \( \mathbb{C}P^n \) is a subcomplex of \( X_i^r \) for \( \mathbb{C}P^{n+1} \), where \( i = 1, \ldots, p-1 \). Then for \( s \leq r \), we can consider the quotient \( X_i^r / X_s^r = S^{2i+2(s+1)(p-1)} \cup e^{2i+2(s+2)(p-1)} \cup \cdots \cup e^{2i+2r(p-1)} \).

**Lemma 4.2.** If \( i \neq k \) mod \( p-1 \), then \( \pi^{2k}(X_i^r) \) is a finite abelian group.

**Proof.** The cofiber sequence \( X_i^{r-1} \to X_i^r \to X_i^r / X_i^{r-1} = S^{2i+2r(p-1)} \) induces a long exact sequence of cohomotopy groups

\[ \cdots \to \pi^{2k}(S^{2i+2r(p-1)}) \to \pi^{2k}(X_i^r) \to \pi^{2k}(X_i^{r-1}) \to \cdots. \]

Then since \( \pi^*(S^0) \) is a finite abelian group for \( * > 0 \), the statement is proved by induction on \( r \). \( \square \)

Hereafter, we set \( k = i + s(p-1) \) for given integers \( 1 \leq i \leq p-1 \) and \( s \geq 0 \). Let \( Y_k^r = X_i^{r+1} / X_i^{r-1} \). Then we have

\[ Y_k^r = S^{2k} \cup e^{2k+2(p-1)} \cup \cdots \cup e^{2k+2r(p-1)}. \]

**Lemma 4.3.** The natural map \( \pi^{2k}(Y_k^r) \to \pi^{2k}(X_i^{r+1}) \) is an isomorphism. Moreover, there is an isomorphism

\[ \pi^{2k}(Y_k^r) / \text{Tor} \cong \mathbb{Z}(p). \]

**Proof.** Since the dimension of \( X_i^{s-1} \) is smaller than \( 2k \), we have \( \pi^{2k}(X_i^{s-1}) = 0 \). Then by the long exact sequence

\[ \cdots \to \pi^*(Y_k^r) \to \pi^*(X_i^{r+1}) \to \pi^*(X_i^{s-1}) \to \cdots \]

we obtain the first isomorphism. Consider the long exact sequence

\[ \cdots \to \pi^*(Y_k^{r+(p-1)}) \to \pi^*(Y_k^r) \to \pi^*(S^{2k}) \to \cdots. \]

Then since \( Y_k^{r+(p-1)} = (2(k + p - 1) - 1) \)-connected and \( \pi^{2k}(S^{2k}) \cong \mathbb{Z}(p) \), we obtain the second isomorphism. \( \square \)

Now we are ready to prove:
Proposition 4.4. There is a commutative diagram
\[
\pi^{2k}(\mathbb{C}P^n)/\text{Tor} \longrightarrow \pi^{2k}(\mathbb{C}P^{n-p+1})/\text{Tor} \\
\cong \quad \cong \\
\pi^{2k}(Y^r_k)/\text{Tor} \longrightarrow \pi^{2k}(Y^{r-1}_k)/\text{Tor}
\]
where the horizontal maps are induced from inclusions and \( r = \left[ \frac{n+1}{p-1} \right] \).

Proof. By Lemma 4.1, there is a commutative diagram
\[
\pi^{2k}(\mathbb{C}P^n) \longrightarrow \pi^{2k}(\mathbb{C}P^{n-p+1}) \\
\cong \quad \cong \\
\bigoplus_{i=1}^{p-1} \pi^{2k}(X^r_i) \longrightarrow \bigoplus_{i=1}^{p-1} \pi^{2k}(X^{r-1}_i)
\]
where the bottom map is the sum of the maps induced from the inclusions \( X^{r-1}_i \to X^r_i \). Then the statement follows from Lemmas 4.2 and 4.3.

By Proposition 4.4, we need to compute the map
\[ j^r_* : \pi^{2k}(Y^r_k)/\text{Tor} \to \pi^{2k}(Y^{r-1}_k)/\text{Tor} \]
where \( j^r_r : Y_i^{r-1} \to Y_i^r \) is the inclusion. To this end, we describe the attaching maps of cells of \( Y^r_i \). To this end, we recall the \((p,\text{local stable})\) homotopy groups of \( S^0 \) in a range. Let \( C\{x\} \) denotes an abelian group having a generator \( x \) which is isomorphic with a cyclic group \( C \).

The following two lemmas describe the cell structure of \( Y^r_k \).

Lemma 4.6. If \( k \equiv i - s \not\equiv 0 \mod p \), then there is a homotopy equivalence
\[ Y^1_k = X_i^{s+1} / X_i^{s-1} \simeq S^{2k} \cup \alpha_1 \epsilon^{2k+2(p-1)}. \]

Proof. By definition, there is a homotopy equivalence
\[ Y^1_k = X_i^{s+1} / X_i^{s-1} \simeq S^{2k} \cup \epsilon^{2k+2(p-1)} \]
If the attaching map \( \phi \) is non-trivial, then by Theorem 4.5, we can take \( \phi = \alpha_1 \). If \( \phi \) is trivial, then the Steenrod operation \( P^1 \) acts trivially on the mod \( p \) cohomology of \( Y^1_k = X_i^{s+1} / X_i^{s-1} \) because it is a wedge of spheres. Then it is sufficient to show that \( P^1 \) acts non-trivially on the mod \( p \) cohomology of \( Y^1_k = X_i^{s+1} / X_i^{s-1} \). Let \( X = \mathbb{C}P^{k+p-1} / \mathbb{C}P^{k+1} \). Since \( k \not\equiv 0 \mod p \), we have
\[ P^1(H^{2k}(X;\mathbb{Z}/p)) = H^{2k+2(p-1)}(X;\mathbb{Z}/p). \]
By Lemma 4.1, the inclusion \( Y^2_k = X_i^{s+1} / X_i^{s-1} \) induces an isomorphism in the mod \( p \) cohomology of dimension \( 2k, 2k + 2(p-1) \), implying that \( P^1 \) acts non-trivially on the mod \( p \) cohomology of \( Y^2_k = X_i^{s+1} / X_i^{s-1} \). Thus the statement is proved.

Lemma 4.7. If \( k = i + s(p-1) = ap + 1 \), then \( Y^3_k = X_i^{s+3} / X_i^{s-1} \) is homotopy equivalent to
\[ S^{2k} \cup \alpha_1 \epsilon^{2k+2(p-1)} \cup \epsilon^{2k+4(p-1)} \cup \alpha_1 \alpha_2 \epsilon^{2k+6(p-1)}. \]

Proof. The lemma follows from [6, Proposition 2.3] and its proof.
4.2. Toda bracket. Since our basic computation tool is the Toda bracket, we briefly recall its definition, where we refer to [24] for details. Suppose we are given maps

$$
\gamma : W \to X, \quad \beta : X \to Y, \quad \alpha : Y \to Z
$$
satisfying $\beta \circ \gamma = 0$ and $\alpha \circ \beta = 0$. Let $h : CX \to Z$ be a null-homotopy for $\alpha \circ \beta$. Then we get a map

$$
\tilde{\alpha} = \alpha \cup h : Y \cup_\beta CX \to Z
$$

which is called an extension of $\alpha$ by $\beta$. Clearly, an extension of $\alpha$ by $\beta$ depends on the choice of a null-homotopy for $\alpha \circ \beta = 0$. Let $\text{Ext}(\alpha, \beta)$ denote the set of all extensions of $\alpha$ by $\beta$. Define a map $\tilde{\gamma} : \Sigma W \to Y \cup_\beta CX$ by

$$
\tilde{\gamma}(y, t) = \begin{cases} 
 g(y, 1 - 2t) & 0 \leq t \leq \frac{1}{2} \\
 (\gamma(y), 2t - 1) & \frac{1}{2} \leq t \leq 1
\end{cases}
$$

where $g : CW \to Y$ is a null-homotopy for $\beta \circ \gamma = 0$. We call a map $\tilde{\gamma}$ a coextension of $\gamma$ by $\beta$. A coextension of $\gamma$ by $\beta$ depends on the choice of a null-homotopy for $\gamma \circ \beta = 0$ as well as an extension above, and $\text{Coext}(\beta, \gamma)$ denote the set of all coextensions of $\gamma$ by $\beta$.

**Definition 4.8.** The Toda bracket of the above $\alpha, \beta, \gamma$ is defined as the set

$$
\text{Ext}(\alpha, \beta) \circ \text{Coext}(\beta, \gamma) \subset [\Sigma W, Z]
$$

which we denote by $\langle \alpha, \beta, \gamma \rangle$.

We write $\langle \alpha, \beta, \gamma \rangle = \delta$ if the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ consists of a single element $\delta$. As in [25, Lemma 1.1], the Toda bracket $\langle \alpha, \beta, \gamma \rangle$ is a coset of the subgroup

$$
(\Sigma \gamma)^*([\Sigma X, Z]) + \alpha_*([\Sigma W, Y]) \subset [\Sigma W, Z]
$$

which is called the indeterminacy of $\langle \alpha, \beta, \gamma \rangle$. We prove a key lemma in our computation.

**Lemma 4.9.** Suppose that maps $\beta : X \to Y$ and $\alpha : Y \to Z$ satisfy $k(\alpha \circ \beta) = 0$ for some integer $k$. Then for any map $\gamma : \Sigma W \to Y \cup_\beta CX$ and any extension $\tilde{k}\alpha : Y \cup_\beta CX \to Z$ of $k\alpha$ by $\beta$, we have

$$
\tilde{k}\alpha \circ \gamma \in \langle k, \alpha \circ \beta, \Sigma^{-1}(\rho \circ \gamma) \rangle
$$

where $\rho : Y \cup_\beta CX \to \Sigma X$ denotes the pinch map.

**Proof.** Since there is a homotopy cofibration $Y \cup_\beta CX \overset{p}{\to} \Sigma X \overset{\Sigma \beta}{\to} \Sigma Y$, we have $\beta \circ \Sigma^{-1}p = 0$, implying $\beta \circ \Sigma^{-1}(\rho \circ \gamma) = 0$. Since we are stabilizing, $\gamma$ is a coextension of $\Sigma^{-1}(\rho \circ \gamma)$ by [18]. Then since $k(\alpha \circ \beta) = 0$, the Toda bracket $\langle k\alpha, \beta, \Sigma^{-1}(\rho \circ \gamma) \rangle$ is defined, and by definition, $\tilde{k}\alpha \circ \gamma$ belongs to this Toda bracket. On the other hand, by [25, Proposition 1.2], we have

$$
\langle k\alpha, \beta, \Sigma^{-1}(\rho \circ \gamma) \rangle \subset \langle k, \alpha \circ \beta, \Sigma^{-1}(\rho \circ \gamma) \rangle.
$$

Thus the statement is proved. \qed

4.3. Computation. As in [24], if we choose $\alpha_1 \in \pi_{2p-3}(S^0)$, then $\alpha_i \in \pi_{2i(p-1)-1}(S^0)$ for $i > 1$ are inductively defined by

$$
\alpha_i = \langle \alpha_{i-1}, p, \alpha_1 \rangle.
$$

The element $\alpha_1$ is defined as a generator of $\pi_{2p-3}(S^0) = \mathbb{Z}/p$ with mod $p$ Hopf invariant 1 ([24, p. 309]). If $i \equiv 0 \text{ mod } p$, then $\pi_{2i(p-1)-1}(S^0) = \mathbb{Z}/p^2$ and $\alpha'_i = \alpha_i/p$ is a generator.

We will use the following alternative description of $\alpha_i$
Proposition 4.10 ([24, Proposition 4.17]). Let $\alpha_{tp} = p\alpha'_tp$. If $s + t < p(p-1)$, then
\[
\langle p, \alpha_s, \alpha_t \rangle = \begin{cases} 
\frac{1}{s+t} t^s + t & s + t \not\equiv 0 \mod p \\
\frac{1}{s+t} t^{s+t} & s + t \equiv 0 \mod p,
\end{cases}
\]

Remark 4.11. Since $\alpha_t \circ \alpha_s = 0$ if $s + t < p(p-1)$ ([24, Proposition 4.17]), the Toda bracket $\langle p, \alpha_s, \alpha_t \rangle$ in Proposition 4.10 is well-defined. The statement of the theorem means that $(p, \alpha_s, \alpha_t)$ has only one element given in the right hand side.

Hereafter, we assume $t < p(p-1)$. Let $\varphi_t: S^{2k+2t(p-1)-1} \to Y^{t-1}_k$ denote the attaching map of the top cell of $Y^{t-1}_k = Y_k^{t-1} \cup e^{2k+2t(p-1)}$. We say that a map $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ detects $\varphi_t$, if the restriction of $\theta_{t-1}$ to the bottom cell $S^{2k} \subset Y^{t-1}_k$ is non-trivial and $\theta_{t-1} \circ \varphi_t$ generates $\pi_{2k}^2(S^{2k+2t(p-1)-1}) = \pi_{2k+2t(p-1)-1}(S^{2k})$, where $\pi_{2k+2t(p-1)-1}(S^{2k})$ is given by Theorem 4.5. We define
\[
q_t = \begin{cases} 
p & t \not\equiv 0 \mod p \\
p^2 & t \equiv 0 \mod p.
\end{cases}
\]

Lemma 4.12. If a map $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ detects $\varphi_t$, then the map $j^*_t: \pi_{2k}(Y^{t-1}_k)/\text{Tor} \to \pi_{2k}(Y^{t-1}_k)/\text{Tor}$ is identified with the map
\[
g_t: \mathbb{Z}(p) \to \mathbb{Z}(p).
\]

Proof. Consider the exact sequence
\[
\cdots \to \pi_{2k}(Y^{t-1}_k) \to \pi_{2k}(Y^{t-1}_k) \overset{\varphi_t^*}{\to} \pi_{2k}(S^{2k+2t(p-1)-1}) \to \cdots
\]
induced from the cofibration sequence $S^{2k+2t(p-1)-1} \overset{\varphi_t}{\to} Y^{t-1}_k \to Y^{t-1}_k$. By Lemma 4.3, we have $\pi_{2k}(Y^{t-1}_k)/\text{Tor} \cong \pi_{2k}(Y^{t-1}_k)/\text{Tor} \cong \mathbb{Z}(p)$, and by Theorem 4.5, we also have $\pi_{2k}(S^{2k+2t(p-1)-1}) \cong \mathbb{Z}/q_t$. Then it is sufficient to show that there is an element $\phi \in \pi_{2k}(Y^{t-1}_k)$ of infinite order such that $\varphi_t^*(\phi)$ generates $\pi_{2k}(S^{2k+2t(p-1)-1})$. Since $\theta_{t-1} \mid_{S^{2k}} \neq 0$ and $\pi_{2k}(S^{2k}) \cong \mathbb{Z}(p)$, $\theta_{t-1}$ is of infinite order. Moreover, $\varphi_t^*(\theta_{t-1})$ generates $\pi_{2k}(S^{2k+2t(p-1)-1})$, because $\theta_{t-1}$ detects $\varphi_t$. This completes the proof. \hfill $\square$

Lemma 4.13. If $k + t + 1 \not\equiv 0 \mod p$ with $t \geq 2$ and there is a map $\theta_{t-2}: Y^{t-2}_k \to S^{2k}$ detecting $\varphi_{t-1}$, then there is a map $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ detecting $\varphi_t$.

Proof. By Theorem 4.5, we may assume $\theta_{t-2} \circ \varphi_{t-1} = \begin{cases} 
\alpha_{t-1} & t - 1 \not\equiv 0 \mod p \\
\alpha'_{t-1} & t - 1 \equiv 0 \mod p,
\end{cases}$

We define $\phi = \frac{\alpha_{t-1}}{p} \circ \theta_{t-2}$. Then we have $p \circ \phi \circ \varphi_{t-1} = q_t \circ \theta_{t-2} \circ \varphi_{t-1} = q_t - \theta_{t-2} \circ \varphi_{t-1} = 0$ because we are stabilizing. Hence we can set $\theta_{t-1} = Y^{t-1}_k \cup_{\varphi_{t-1}} \varphi_{t-1} \circ \phi_{t-1}$ to be an extension of $p \circ \phi: Y^{t-2}_k \to S^{2k}$ by $\varphi_{t-1}$. We apply Lemma 4.9 for $\alpha = \phi$, $\beta = \varphi_{t-1}$ and $\gamma = \varphi_t$. Then the composite $\theta_{t-1} \circ \varphi_t$ belongs to the Toda bracket $(p, \phi \circ \varphi_{t-1}, \Sigma^{-1}(p \circ \varphi_{t}))$, where $\rho: Y^{t-1}_k \to Y^{t-1}_k / Y^{t-2}_k = S^{2k+2t(p-1)-1}(p-1)$ is the pinch map onto the top cell. Note that $Y^{t-1}_k / Y^{t-2}_k = Y^{t-1}_k / Y^{t-2}_k = Y^{t-1}_k / (t(p-1)) = S^{2k+2t(p-1)-1}(p-1) \cup e^{2k+2t(p-1)}$. Since $k + (t-1)(p-1) \not\equiv 0 \mod p$ by the assumption, Lemma 4.6 implies that $\rho \circ \varphi_t = \alpha_1$. We may assume $\Sigma^{-1}(p \circ \varphi_{t}) = \alpha_1$ because we are stabilizing. We also have
\[
\phi \circ \varphi_{t-1} = \frac{q_{t-1}}{p} \circ \theta_{t-2} \circ \varphi_{t-1} = \frac{q_{t-1}}{p} (\theta_{t-2} \circ \varphi_{t-1}) = \alpha_{t-1}
\]

because we are stabilizing, where $p \alpha'_{t-1} = \alpha_{t-1}$ for $t - 1 \equiv 0 \mod p$. Then the composite $\theta_{t-1} \circ \varphi_t$ belongs to the Toda bracket $\langle p, \alpha_{t-1}, \alpha_1 \rangle$ which is a subset of
Let us abbreviate Lemma 4.14.

**Lemma 4.14.** Let \( t \geq 4 \) and \( k - t + 2 \equiv 0 \mod p \). If there is a map \( \theta_{t-4} : Y_k^{t-4} \to S^{2k} \) detecting \( \varphi_{t-3} \), then there is a map \( \theta_{t-3} : Y_k^{t-3} \to S^{2k} \) such that \( \theta_{t-1}|S^{2k} \neq 0 \). Thus the proof is finished.

**Proof.** By Theorem 4.5, we may assume

\[
\theta_{t-4} \circ \varphi_{t-3} = \begin{cases} 
\alpha'_{t-3} & t \equiv 0 \mod p \\
\alpha_{t-3} & t \not\equiv 3 \mod p.
\end{cases}
\]

Let us abbreviate \( a(k, t) \) by \( a \). We can apply Lemma 4.6 and Lemma 4.7 because \( k + (t - 3)(p - 1) = ap + 1 \) and therefore \( k + (t - 4)(p - 2) \not\equiv 0 \mod p \). Then there is a homotopy equivalence

\[
Y_k^t \simeq Y_k^{t-4} \cup_{\alpha_1} e^{2k+2(t-3)(p-1)} \cup_{-\alpha_1} e^{2k+2(t-2)(p-1)}
\]

\[
\cup \left( \frac{a}{\alpha_1} \right) \alpha_2 e^{2k+2(t-1)(p-1)} \cup (-\frac{a}{\alpha_2}+\alpha_1) e^{2k+2(t-4)(p-1)}.
\]

Let \( \theta_{t-3} : Y_k^{t-3} \to S^{2k} \) be an extension of \( p \circ (\varphi_{t-4}) \) by \( \varphi_{t-4} \).

(1) The \( t \equiv 0 \mod p \) case.

As in the proof of Lemma 4.13, we can see that \( \theta_{t-3}|S^{2k} \neq 0 \). We apply Lemma 4.9 to the following setting

\[
\alpha = \frac{q_t}{p} \theta_{t-4} : Y_k^{t-4} \to S^{2k};
\]

\[
\beta = \varphi_{t-3} = \alpha_1 : S^{2k+2(t-3)(p-1)-1} \to Y_k^{t-4},
\]

\[
\gamma = \varphi_{t-2} + \varphi_{t-1} = -\alpha_1 \vee \left( \frac{a}{2} + 1 \right) \alpha_2 : S^{2k+2(t-2)(p-1)-1} \vee S^{2k+2(t-1)(p-1)-1} \to Y_k^{k-4} \cup_{\alpha_1} e^{2k+2(t-3)(p-1)}.
\]

Then

\[
\theta_{t-3} \circ (\varphi_{t-2} + \varphi_{t-1}) = -\langle p, \alpha_{t-3}, \alpha_1 \rangle + \left( \frac{a}{2} + 1 \right) \langle p, \alpha_{t-3}, \alpha_2 \rangle
\]

\[
= -\frac{1}{t-2} \alpha_{t-2} + \frac{a + 2}{t-1} \alpha_{t-1}
\]
by Theorem 4.10, (4.1) and (4.2). Now we let $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ be an extension of $p \circ \theta_{t-3}$ by $\varphi_t$. Then as in the proof of Lemma 4.13, we obtain that $\theta_{t-1}|_{S^{2k}} \neq 0$. We apply Lemma 4.9 to the following setting 

$$\alpha = \theta_{t-3}: Y^{t-3}_k \to S^{2k},$$

$$\beta = \varphi_{t-2} + \varphi_{t-1} = -\alpha_1 \bigcup \left( \frac{a}{2} + 1 \right) \alpha_2: S^{2k+2(t-2)(p-1)-1} \cup S^{2k+2(t-1)(p-1)-1} \to Y^{k-3}.$$

Then 

$$\gamma = \varphi_1: S^{2k+2(p-1)-1} \to Y^{t-3} \cup \varphi_{t-2} e^{2k+2(t-2)(p-1)} \cup \varphi_{t-1} e^{2k+2(t-1)(p-1)}$$

Then 

$$\theta_{t-1} \circ \varphi_t = \frac{a + 1}{2(t-2)} (p, \alpha_{t-2}, \alpha_2) + \frac{a + 2}{t-1} (p, \alpha_{t-1}, \alpha_1)$$

$$= \frac{p}{t} \left( \frac{a + 1}{t-2} + \frac{a + 2}{t-1} \right) \alpha_t = - \frac{p(3a + 5)}{2t} \alpha_t$$

by Lemma 4.9 and Theorem 4.10 together with (4.2).

(2) The $t \equiv 1 \pmod{p}$ case.

By Lemma 4.9 and (4.2), we have 

$$\theta_{t-3} \circ (\varphi_{t-2} + \varphi_{t-1}) = \alpha_{t-2} + \frac{p(a + 2)}{t-1} \alpha_{t-1}.$$ 

Then as in the case (1), there is a map $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ such that $\theta_{t-1}|_{S^{2k}} \neq 0$ and 

$$\theta_{t-1} \circ \varphi_t = \frac{a + 1}{2(t-2)} (p, \alpha_{t-2}, \alpha_2) + \frac{p(a + 2)}{t-1} (p, \alpha_{t-1}, \alpha_1)$$

$$= \frac{p(a + 2)}{t-1} (p, \alpha_{t-1}, \alpha_1) = - \frac{p(a + 2)}{t-1} \alpha_t$$

by Lemma 4.9, Theorem 4.10 and (4.2).

(3) The $t \equiv 2 \pmod{p}$ case.

By Lemma 4.9 and (4.2), we have 

$$\theta_{t-3} \circ (\varphi_{t-2} + \varphi_{t-1}) = - \frac{p}{t-2} \alpha_{t-2} + (a + 2) \alpha_{t-1}.$$ 

Then as in the case (1), there is a map $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ such that $\theta_{t-1}|_{S^{2k}} \neq 0$ and 

$$\theta_{t-1} \circ \varphi_t = \frac{p(a + 1)}{2(t-2)} (p, \alpha_{t-2}, \alpha_2) + (a + 2) (p, \alpha_{t-1}, \alpha_1)$$

$$= \frac{p(a + 1)}{2(t-2)} (p, \alpha_{t-1}, \alpha_1) = - \frac{p(a + 1)}{2(t-2)} \alpha_t$$

by Lemma 4.9, Theorem 4.10 and (4.2).

(4) The $t \equiv 3 \pmod{p}$ with $p > 3$ case.

By Lemma 4.9 and (4.2), we have 

$$\theta_{t-3} \circ (\varphi_{t-2} + \varphi_{t-1}) = - \alpha_{t-2} + \frac{a + 2}{2} \alpha_{t-1}.$$ 

Then as in the case (1), there is a map $\theta_{t-1}: Y^{t-1}_k \to S^{2k}$ such that $\theta_{t-1}|_{S^{2k}} \neq 0$ and 

$$\theta_{t-1} \circ \varphi_t = \frac{a + 1}{2} (p, \alpha_{t-2}, \alpha_2) + \frac{a + 2}{t-1} (p, \alpha_{t-1}, \alpha_1) = - \frac{a + 2}{t-1} \alpha_t$$

by Lemma 4.9, Theorem 4.10 and (4.2).

(5) The $t \not\equiv 0, 1, 2, 3 \pmod{p}$ case.

By Lemma 4.9 and (4.2), we have 

$$\varphi_{t-3} \circ (\varphi_{t-2} + \varphi_{t-1}) = - \frac{1}{t-2} \alpha_{t-2} + \frac{a + 2}{t-1} \alpha_{t-1}.$$
Then as in the case (1), there is a map \( \theta_{t-1} : Y^{t-1}_k \to S^{2k} \) such that \( \theta_{t-1}|_{S^{2k}} \neq 0 \) and
\[
\theta_{t-1} \circ \varphi_t = \frac{a + 1}{2(t - 2)}(p, \alpha_{t-2}, \alpha_2) + \frac{a + 2}{t - 1}(p, \alpha_{t-1}, \alpha_1) = \frac{1}{l} \left( \frac{a + 1}{t - 2} + \frac{a + 2}{t - 1} \right) \alpha_t
\]
by Lemma 4.9, Theorem 4.10 and (4.2). Thus the proof is complete.

**Lemma 4.15.** If \( k \equiv 1 \mod p \) and \( 3a(k, 3) + 4 \equiv 0 \mod p \), then there is a map \( \theta_2 : Y^2_k \to S^{2k} \) detecting \( \varphi_3 \).

**Proof.** By Lemmas 4.6 and 4.7, we have
\[
Y^3_k \simeq S^{2k} \cup_{-\alpha_1} e^{2k+2(p-1)} \cup \left( \frac{4}{5} + 1 \right) \alpha_2 e^{2k+4(p-1)} \cup \frac{3a}{2} \alpha_3 e^{2k+6(p-1)}
\]
where \( a = a(k, t) \). Then we can define a map \( \theta_2 : Y^2_k \simeq S^{2k} \cup_{-\alpha_1} e^{2k+2(p-1)} \cup \left( \frac{4}{5} + 1 \right) \alpha_2 e^{2k+4(p-1)} \to S^{2k} \) as an extension of \( p : S^{2k} \to S^{2k} \) by \( -\alpha_1 + \left( \frac{4}{5} + 1 \right) \alpha_2 \). Then \( \theta_2|_{S^{2k}} \neq 0 \), and by Lemma 4.9 and Proposition 4.10, the composite \( \theta_2 \circ \varphi_3 \) belongs to the Toda bracket
\[
a + \frac{1}{2}(p, \alpha_1, \alpha_2) + \left( \frac{a}{2} + 1 \right)(p, \alpha_2, \alpha_1) = \begin{cases} \frac{3a + 1}{6} \alpha_3 & p = 3 \\ \frac{3a + 4}{6} \alpha_3 & p > 3 \end{cases}
\]
Thus \( \theta_2 \) detects \( \varphi_3 \), completing the proof.

**Lemma 4.16.** Given an integer \( k, r \) with \( 1 \leq r < p(p-1) \), suppose the following conditions:

1. \( k(r - 1) \equiv 0 \mod p \);
2. under the above condition, for any integer \( 3 \leq t \leq r \) satisfying \( k - t + 2 \equiv 0 \mod p \), then we further assume
   
   \[
   3a(k, t) + 5 \equiv 0 \mod p \quad (t \equiv 0 \mod p)
   \]
   \[
   a(k, t) + 2 \equiv 0 \mod p \quad (t \equiv 1 \mod p)
   \]
   \[
   3a(k, t) + 4 \equiv 0 \mod p \quad (t \equiv 3 \mod p)
   \]
   \[
   (2r - 3a(k, t)) + 3t - 5 \equiv 0 \mod p \quad (t \equiv 4, 5, \cdots, p-1 \mod p)
   \]

Then there is a map \( \theta_{r-1} : Y^{r-1}_k \to S^{2k} \) detecting \( \varphi_r \).

**Proof.** We proceed by induction on \( r \) satisfying \( k - r + 1 \equiv 0 \mod p \). Note that we are considering not all \( r \) but satisfying \( k - r + 1 \equiv 0 \mod p \), for which we can perform induction. For \( r = 1 \), we have \( k - r + 1 = k \equiv 0 \mod p \) by assumption. Let \( \theta_0 : Y_k^0 = S^{2k} \to S^{2k} \) be the identity map of \( S^{2k} \). Since \( k \not\equiv 0 \mod p \), we have \( \varphi_1 = \alpha_1 \) by Lemma 4.6. Then we have \( \varphi_1(\theta_0) = \alpha_1 \), and so \( \theta_0 \) detects \( \varphi_1 \) by Theorem 4.5. For \( r = 2 \), we only need to consider the case \( k \equiv 1 \mod p \) because we are assuming \( k - r + 1 \equiv 0 \mod p \). Then by Lemma 4.13, we get a map \( \theta_1 \) detecting \( \varphi_2 \). Suppose \( r = 3 \). If \( k \equiv 1 \mod p \), then we have \( \theta_1 \) as above, and so by Lemma 4.13, we get a map \( \theta_2 \) detecting \( \varphi_3 \), where we are assuming \( k - r + 1 \equiv 0 \mod p \). If \( k \equiv 1 \mod p \), then we can apply Lemma 4.15 to get a map \( \theta_2 \) detecting \( \varphi_3 \), where we are assuming \( 3a(k, 3) + 4 \equiv 0 \mod p \). Now we assume that for each \( 4 \leq t \leq r - 1 \) with \( k - t + 1 \equiv 0 \mod p \), there is a map \( \theta_{t-1} \) detecting \( \varphi_t \). If \( k - r + 2 \equiv 0 \mod p \), then by the induction hypothesis, we have \( \theta_{r-2} \), and so by Lemma 4.13 and the assumption \( k - r + 1 \equiv 0 \mod p \), we get a map \( \theta_{r-1} \) detecting \( \varphi_r \). If \( k - r + 2 \equiv 0 \mod p \), then \( k - r + 5 \equiv 0 \mod p \), and so by the induction hypothesis, we have \( \theta_{r-4} \) detecting \( \varphi_{r-3} \). Thus by Lemma 4.14, we also get a map \( \theta_{r-1} \) detecting \( \varphi_r \), completing the proof.

**Lemma 4.17.** If \( \theta_{r-2} \) detects \( \varphi_{r-1} \), then the extension of \( q_{r-1} \circ \theta_{r-2} \) by \( \varphi_{r-1} \) is a generator in \( \pi^{2k}(Y^{r-1}_k)/\text{Tor} \).
Proof. The map \( Y_k^{r-1} \to S^{2k} \) is the extension of the restriction of \( Y_k^{r-2} \) by \( \varphi_{r-1} \). Consider the exact sequence

\[
\pi^2k(Y_k^{r-1}) \to \pi^2k(Y_k^{r-2}) \xrightarrow{\varphi_{r-1}} \pi^2k(S^{2k+2(r-1)(p-1)-1})
\]

induced from a cofiber sequence \( S^{2k+2(r-1)(p-1)-1} \xrightarrow{\varphi_{r-1}} Y_k^{r-2} \to Y_k^{r-1} \). Because \( \theta_{r-2} \) detects \( \varphi_{r-1} \), it follows from the exact sequence that if we restrict any map \( Y_k^{r-1} \to S^{2k} \) on \( Y_k^{r-2} \), then the restriction is a multiple of \( q_{r-1} \circ \theta_{r-2} \). Hence, if we extend \( q_{r-1} \circ \theta_{r-2} \) by \( \varphi_{r-1} \), then the extension is a generator in \( \pi^2k(Y_k^{r-1})/\text{Tor} \). \( \square \)

**Lemma 4.18.** Suppose that there is a map \( \theta_{r-2} : Y_k^{r-2} \to S^{2k} \) detecting \( \varphi_{r-1} \). Then the map \( j^* : \pi^2k(Y_k^r)/\text{Tor} \to \pi^2k(Y_k^{r-1})/\text{Tor} \) is an isomorphism whenever \( k-r+1 \equiv 0 \mod p \).

**Proof.** By the assumption and Lemma 4.17, an extension \( \phi : Y_k^{r-1} \to S^{2k} \) of \( q_{r-1} \circ \theta_{r-2} \) by \( \varphi_{r-1} \) is a generator of \( \pi^2k(Y_k^{r-1})/\text{Tor} \). By Lemma 4.9, the composite \( \phi \circ \varphi_r \) belongs to the Toda bracket

\[
\langle q_{r-1}, \theta_{r-2} \circ \varphi_{r-1}, \Sigma^{-1} \rho \circ \varphi_r \rangle,
\]

where \( \rho : Y_k^{r-1} \to S^{2k+2(r-1)(p-1)-1} \) is the pinch map onto the top cell. The indeterminacy of this Toda bracket is

\[
q_{r-1}(\pi^2k+2(r-1)(p-1)(S^{2k})) + \pi^2k+2(r-1)(p-1)+1(S^{2k}) \circ \rho_\ast(\varphi_r) \quad (*)
\]

The first term \( q_{r-1}(\pi^2k+2(r-1)(p-1)(S^{2k})) \) vanishes by Theorem 4.5. We claim \( \rho_\ast(\varphi_r) = 0 \). Since \( k + (r-2)(p-1) \equiv 0 \mod p \) by the assumption, Lemma 4.7 implies that

\[
Y_{k+r-2}(p-1) = Y_k^r / Y_k^{r-3} = S^{2k+2(r-2)(p-1)-1} \cup_{\partial e^{2k+2r(p-1)}} S^{2k+2(r-2)(p-1)}.
\]

Note that \( \varphi_r = (\Sigma^1 \varphi_r) \) and \( \alpha_2 : \partial e^{2k+2r(p-1)} \to S^{2k+2(r-2)(p-1)} \) is the attaching map to \( S^{2k+2(r-2)(p-1)} \). On the other hand, \( \rho \) is a map that collapses \( S^{2k+2(r-2)(p-1)} \). These imply \( \rho_\ast(\varphi_r) = 0 \).

Hence, \( (*) \equiv 0 \) and the above Toda bracket consists of a single element. Since \( k-r+1 \equiv 0 \mod p \), we have \( \Sigma^{-1} \rho \circ \varphi_r = 0 \) by Lemma 4.7. Then the above Toda bracket includes \( 0 \), implying the Toda bracket is trivial. Thus we obtain \( \phi \circ \varphi_r = 0 \).

Now we consider the exact sequence

\[
\pi^2k(Y_k^r) \to \pi^2k(Y_k^{r-1}) \xrightarrow{\varphi_r} \pi^2k(S^{2k+2(r-1)(p-1)-1})
\]

induced from a cofiber sequence \( S^{2k+2r(p-1)-1} \xrightarrow{\varphi_r} Y_k^{r-1} \to Y_k^r \). By the above computation, the map \( \varphi^*_r : \pi^2k(Y_k^{r-1}) \to \pi^2k(S^{2k+2(r-1)(p-1)-1}) \) is trivial, implying that the map \( j^* : \pi^2k(Y_k^r)/\text{Tor} \to \pi^2k(Y_k^{r-1})/\text{Tor} \) is surjective. By Lemma 4.3, \( \pi^2k(Y_k^r)/\text{Tor} \cong \pi^2k(Y_k^{r-1})/\text{Tor} \cong Z(p) \). Thus since any surjection \( Z(p) \to Z(p) \) is an isomorphism, which contradicts to our assumption. Therefore the proof is finished. \( \square \)

Now we are ready to prove:

**Theorem 4.19.** Given an integer \( k, r \) with \( 1 \leq r < p(p-1) \), suppose the following conditions:

1. \( k \not\equiv 0, 1, \ldots, r_p - 1 \mod p; \)
(2) under the above condition, for any integer $3 \leq t \leq r$ satisfying $t - 2 \equiv k \mod p$, we further assume

\begin{align*}
3a(k, t) + 5 &\not\equiv 0 \mod p \quad (t \equiv 0 \mod p \text{ and } t \geq p > 3) \\
a(k, t) + 2 &\not\equiv 0 \mod p \quad (t \equiv 1 \mod p \text{ and } t > p) \\
3a(k, t) + 4 &\not\equiv 0 \mod p \quad (t \equiv 3 \mod p) \\
(2t - 3)a(k, t) + 3t - 5 &\not\equiv 0 \mod p \quad (t \equiv 4, \cdots, p - 1 \mod p).
\end{align*}

Then the natural map $\pi^{2k}(\mathbb{C}P^{k+r}(p^{-1}))/\text{Tor} \to \pi^{2k}(\mathbb{C}P^k)/\text{Tor}$ is identified with $p^r: \mathbb{Z}(p) \to \mathbb{Z}(p)$.

**Proof.** By Proposition 4.4, the map $\pi^{2k}(\mathbb{C}P^{k+r}(p^{-1}))/\text{Tor} \to \pi^{2k}(\mathbb{C}P^k)/\text{Tor}$ is identified with $j_k^*: \pi^{2k}(Y_k^p)/\text{Tor} \to \pi^{2k}(Y_k^p)/\text{Tor}$. Observe that $r = pq + rp$ for a non-negative integer $q$ by the definition of $r_p$. Let $0 \leq s < q$. Then:

- The map $j_k^*$ is identified with $p: \mathbb{Z}(p) \to \mathbb{Z}(p)$ by Theorem 4.5 and Lemmas 4.12 and 4.14 for $ps < t < p(s + 1)$ with $k - t + 1 \equiv 0 \mod p$.
- The map $j_k^*: \mathbb{Z}(p) \to \mathbb{Z}(p)$ is identified with $p^2: \mathbb{Z}(p) \to \mathbb{Z}(p)$.
- There is exactly one $t$ such that $ps < t < p(s + 1)$ with $k - t + 1 \equiv 0 \mod p$ for which the map $j_k^*$ is identified with $1: \mathbb{Z}(p) \to \mathbb{Z}(p)$ by Lemma 4.18.

Then, the composite $j_{p+1}^* \circ j_{p+2}^* \circ \cdots \circ j_{p+1}^*$ is identified with $p^q: \mathbb{Z}(p) \to \mathbb{Z}(p)$. If $pq < t \leq r_p$, then $k - t + 1 \equiv 0 \mod p$ follows, because $k \equiv 0, 1, \ldots, r_p - 1 \mod p$. Hence, the map $j_k^*$ is identified with $p: \mathbb{Z}(p) \to \mathbb{Z}(p)$ by Theorem 4.5 and Lemmas 4.12 and 4.14. Hence the composite $j_{pq}^* \circ j_{pq+1}^* \circ \cdots \circ j_{pq+r_p}^*$ is identified with $p^{2q}: \mathbb{Z}(p) \to \mathbb{Z}(p)$. Thus the composite $j_1^* \circ j_2^* \circ \cdots \circ j_q^*$ is identified with $p^{q+1} = p^r: \mathbb{Z}(p) \to \mathbb{Z}(p)$, completing the proof. \qed

Recall that we have assumed for $p$ to be odd prime at the first paragraph of Section 4. Below we consider the $p = 2$ case.

**Proposition 4.20.** If $p = 2$ and $k \not\equiv 0 \mod p$, then the map $\pi^{2k}(\mathbb{C}P^{k+1})/\text{Tor} \to \pi^{2k}(\mathbb{C}P^k)/\text{Tor}$ is identified with $p: \mathbb{Z}(p) \to \mathbb{Z}(p)$.

**Proof.** Consider the exact sequence

$$
\pi^{2k-1}(\mathbb{C}P^{k-1}) \to \pi^{2k}(\mathbb{C}P^n)/\mathbb{C}P^{k-1}) \to \pi^{2k}(\mathbb{C}P^n) \to \pi^{2k}(\mathbb{C}P^k-1)
$$

induced from the homotopy cofibration $\mathbb{C}P^k-1 \to \mathbb{C}P^n \to \mathbb{C}P^n/\mathbb{C}P^{k-1}$ for $n \geq k$. Since $\mathbb{C}P^k-1$ is of dimension $2k - 2$, we have $\pi^{2k-1}(\mathbb{C}P^{k-1}) = \pi^{2k}(\mathbb{C}P^{k-1}) = 0$, and so the natural map $\pi^{2k}(\mathbb{C}P^n)/\mathbb{C}P^{k-1}) \to \pi^{2k}(\mathbb{C}P^n)$ is an isomorphism. Note that the inclusion $\mathbb{C}P^k \to \mathbb{C}P^{k+1}$ induces a commutative diagram

$$
\begin{array}{ccc}
\pi^{2k}(\mathbb{C}P^{k+1}/\mathbb{C}P^{k-1}) & \xrightarrow{\cong} & \pi^{2k}(\mathbb{C}P^{k+1}) \\
\pi^{2k}(\mathbb{C}P^{k}/\mathbb{C}P^{k-1}) & \xrightarrow{\cong} & \pi^{2k}(\mathbb{C}P^{k}).
\end{array}
$$

Then the map $\pi^{2k}(\mathbb{C}P^{k+1})/\text{Tor} \to \pi^{2k}(\mathbb{C}P^k)/\text{Tor}$ is identified with the map

$$
\pi^{2k}(\mathbb{C}P^{k+1}/\mathbb{C}P^{k-1})/\text{Tor} \to \pi^{2k}(\mathbb{C}P^k)/\mathbb{C}P^{k-1})/\text{Tor}.
$$

Clearly, $\mathbb{C}P^{k+1}/\mathbb{C}P^{k-1} \cong S^{2k}$ holds. Because $k \not\equiv 0 \mod 2$, it is well known that $\mathbb{C}P^{k+1} \cong S^{2k} \cup^Y \Sigma^{2k+2} \mathbb{C}^{k+2}$ such that the inclusion $\mathbb{C}P^k/\mathbb{C}P^{k-1} \to \mathbb{C}P^{k+1}/\mathbb{C}P^{k-1}$ is
Identified with the bottom cell inclusion, where \( \eta \) is a generator of \( \pi_{2k+1}(S^{2k}) \cong \mathbb{Z}/2 \). Consider the exact sequence

\[
\pi^{2k}(C/P^{k+1}/C/P^{k-1}) \to \pi^{2k}(C/P^k/C/P^{k-1}) \to \pi^{2k}(S^{2k+1})
\]

induced from the cofiber sequence \( S^{2k+1} \to C/P^k/C/P^{k-1} \to C/P^{k+1}/C/P^{k-1} \). By the above observation, this exact sequence is identified with the exact sequence

\[
\pi^{2k}(S^{2k} \cup_{\eta} e^{2k+2}) \to \pi^{2k}(S^{2k}) \xrightarrow{\eta} \pi^{2k}(S^{2k+1}) = \pi_{2k+1}(S^{2k})
\]

induced from the cofiber sequence \( S^{2k+1} \to S^{2k} \to S^{2k} \cup_{\eta} e^{2k+2} \). Since \( \eta^*(1) = \eta \) and \( \pi_{2k+1}(S^{2k}) \) is generated by \( \eta \), the second map is surjective. Then the map \( \pi^{2k}(C/P^{k+1}/C/P^{k-1})/\text{Tor} \to \pi^{2k}(C/P^k/C/P^{k-1})/\text{Tor} \) is identified with \( p: \mathbb{Z}_p \to \mathbb{Z}_{(p)} \) with \( p = 2 \), completing the proof.

Finally, we prove Theorem 2.1.

**Proof of Theorem 2.1.** Combine Theorem 4.19 and Proposition 4.20 below. □

### 5. Inexplicit upper bound

This section explains an upper bound mentioned in Section 1. So it is completely independent from other sections, and does not contain any result. Let \( X \) be a closed, connected, oriented and smooth four-manifold, and let \( s \) be a spin\(^*\) structure on \( X \). We recall a result of Bauer and Furuta [4, Theorem 3.7].

**Theorem 5.1.** If \( b_1^+ \geq 2 \) and \( b_1 = 0 \), then \( SW(s) \) is divisible by the denominator of \( a_i^{(k)} \) for \( 1 \leq i \leq d(s)/2 \), where \( k = (b_2 - 1)/2 \) and \( a_i^{(k)} \) is defined by

\[
\left( \frac{\log(1-x)}{x} \right)^k = \left( 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^{n-1}}{n} + \cdots \right)^k = 1 + \sum_{i \geq 1} a_i^{(k)} x^i.
\]

Let \( d(q,k) \) denote the greatest integer \( 2d \) such that the denominator of \( a_i^{(k)} \) is not divisible by \( q \) for \( 1 \leq i \leq d \). By Theorem 5.1, we get that if \( s \) is a mod \( p \) basic class for a prime \( p \) and \( k = (b_2 - 1)/2 \), then there is an inequality

\[
d(s) \leq d(p,k).
\]

By putting \( x = e^y - 1 \), we can see that the numbers \( a_i^{(k)} \) are computed from the Bernoulli numbers, and vice versa. Then it is quite hard to determine or evaluate \( a_i^{(k)} \), in general. Thus the upper bound \( d(p,k) \) is rather inexplicit, in general. However, we can compute \( d(p,q) \) in the following two special cases. First, we clearly have \( d(q,1) = 2q - 4 \). Then as mentioned in Section 1, for a prime \( p \) and \( k = 1 \), our upper bound \( 2r(p-1)/2 \) in Theorem 1.2 is much sharper than the upper bound \( d(p^r,1) = 2p^r - 4 \), except for a few cases. Second, we let for an integer \( 1 \leq r < p \). Then since \( r(p-1) < p^2 + 1 \), in the expansion of

\[
\left( 1 + \frac{x}{2} + \frac{x^2}{3} + \cdots + \frac{x^{n-1}}{n} + \cdots \right)^k,
\]

the coefficient of \( x^{r(p-1)} \) is

\[
\frac{1}{\lambda_1 p^{r_1}} + \cdots + \frac{1}{\lambda_n p^{r_n}} + \left( k \atop r \right) \frac{1}{p^r} \frac{p(\lambda_1 p^{r_1} + \cdots + \lambda_n p^{r_n} + \lambda_1 \cdots \lambda_n l_s)}{\lambda_1 \cdots \lambda_n p^r},
\]

where \( \lambda_1, \ldots, \lambda_n \not\equiv 0 \mod p \) and \( r_1, \ldots, r_n < r \). If \( k \not\equiv 0 \mod p \), and so the numerator is not divisible by \( p \). Hence \( d(p^r,k) \geq 2r(p-1)/2 \). On the other hand, we can see that the denominator of the coefficient of \( x^r \) for \( i < r(p-1) \) is not divisible by \( p^r \) quite similarly. Then we get \( d(p^r,k) \geq 2r(p-1)/2 \), hence \( d(p^r,k) = 2r(p-1) - 2 \). This gives an alternative proof of Corollary 1.3.
where the existence of an integer $k$ satisfying $k \not\equiv 0, 1, \ldots, r - 1 \pmod{p}$ implies $r < p$.

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