Extending tensors on polar manifolds

Ricardo A. E. Mendes

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1 Introduction

Let \((M, g)\) be a Riemannian manifold and \(G\) a Lie group acting on \(M\) properly by isometries. Recall that, by definition (see [21], [10]), this action is called polar if there exists an immersed sub-manifold \(\Sigma \to M\) meeting all \(G\)-orbits orthogonally. Such a submanifold \(\Sigma\) is called a section, and comes with a natural action by a discrete group of isometries \(W = W(\Sigma)\), called its generalized Weyl group. Sections are always totally geodesic, and the immersion \(\Sigma \to M\) induces an isometry \(\Sigma / W \to M / G\), so in particular \(M / G\) is a Riemannian orbifold.

Denote by \(C^\infty(T^k,l M)^G\), respectively \(C^\infty(T^k,l \Sigma)^W(\Sigma)\), the sets of smooth \((k,l)\)-tensors on \(M\), respectively \(\Sigma\), which are invariant under \(G\), respectively \(W\). Our main result states that the natural restriction map \(C^\infty(T^k,l M)^G \to C^\infty(T^k,l \Sigma)^W(\Sigma)\) is surjective:

**Theorem 1.1.** Let \(M\) be a polar \(G\)-manifold with immersed section \(i : \Sigma \to M\), and \(W(\Sigma)\) the generalized Weyl group associated to \(\Sigma\). Define the pull-back (restriction) map

\[
\psi = i^* : C^\infty(T^k,l M)^G \to C^\infty(T^k,l \Sigma)^W(\Sigma)
\]

by

\[
[\psi(\beta)](x)(v_1, \ldots, v_l) = P^{\otimes k}[\beta(i(x)((di)_x v_1, \ldots, (di)_x v_l)]
\]

where \(P : T_i(x)M \to T_\Sigma \Sigma\) is orthogonal projection. Then \(\psi\) is surjective.

In the case of functions, that is, \((k,l) = (0,0)\), the map \(\psi\) above is an isomorphism. This is known as the Chevalley Restriction Theorem — see [21].

Note that Theorem 1.1 applies to \((0,l)\)-tensors with symmetry properties, such as symmetric \(l\)-tensors, exterior \(l\)-forms, etc. This can be phrased naturally in terms of Weyl’s construction (see [9] Lecture 6). Recall that Weyl’s construction associates to each partition \(\lambda = (\lambda_1, \ldots, \lambda_k)\) of \(l \in \mathbb{N}\) a functor \(S_\lambda\) of vector spaces called its Schur functor. One recovers \(\wedge^l\) and \(\text{Sym}^l\) as the Schur functors associated to \(\lambda = (l)\) and \(\lambda = (1,1, \ldots, 1)\), respectively.

**Corollary 1.1.** Let \(M\) be a Riemannian manifold with an isometric polar action by \(G\). Let \(\lambda = (\lambda_1, \ldots, \lambda_k)\) be a partition of \(l \in \mathbb{N}\), and consider the associated
Schur functor $\mathbb{S}_\lambda$. Then the (surjective) restriction map $\psi: C^\infty(T^0\lambda M)^G \to C^\infty(T^0\lambda \Sigma)^W$ induces a surjective map

$$\psi_\lambda: C^\infty(\mathbb{S}_\lambda(T^\ast M))^G \to C^\infty(\mathbb{S}_\lambda(T^\ast \Sigma))^W$$

For context, consider two special cases of Corollary 1.1: exterior $l$-forms and symmetric 2-tensors. In the case of exterior forms the conclusion of Corollary 1.1 is implied by P. Michor’s Basic Forms Theorem — see [18] and [19]. In fact, Michor’s Theorem gives more precise information: it states that for a polar $G$-manifold $M$ with section $\Sigma$, every smooth $W(\Sigma)$-invariant $l$-form on $\Sigma$ can be extended uniquely to a smooth $G$-invariant $l$-form on $M$ which is basic, that is, vanishes when contracted to vectors tangent to the $G$-orbits.

The case of symmetric 2-tensors follows from [17], which is again a sharper statement in the sense that a set of basic tensors is identified. This is used in the following extension result for Riemannian metrics:

**Theorem 1.2.** Let $G$ act polarly on the Riemannian manifold $M$ with section $\Sigma$ and generalized Weyl group $W$. Consider the restriction map (which is surjective by Corollary 1.1):

$$\psi|_\Sigma: C^\infty(\text{Sym}^2 M)^G \to C^\infty(\text{Sym}^2 \Sigma)^W$$

For any Riemannian metric $\sigma \in C^\infty(\text{Sym}^2 \Sigma)^W$, there is a Riemannian metric $\tilde{\sigma} \in C^\infty(\text{Sym}^2 M)^G$ such that $\psi(\tilde{\sigma}) = \sigma$, and with respect to which the $G$-action is polar with the same section $\Sigma$.

For both Theorem 1.2 and Michor’s Basic Forms Theorem, the proof relies on polarization results in the Invariant Theory of finite reflection groups — see section 4. On the other hand, the main ingredient in the proof of Theorem 1.1 is a multi-variable version of the Chevalley Restriction Theorem due to Tevelev — see section 2.

An application of Theorem 1.2 is to give a partial answer to a natural question by K. Grove: Given a proper isometric action of $G$ on a Riemannian manifold $(M, g)$, describe the set of all metrics on $M/G$ which are induced by smooth $G$-invariant metrics $g_0$ on $M$. Theorem 1.2 answers this question under the additional hypothesis that $M$ is a polar $G$-manifold. Namely, that set of metrics on $M/G = \Sigma/W$ coincides with the set of smooth orbifold metrics.

Another application of Theorem 1.2 is an important step in the main reconstruction result in [10]. This was in fact our main motivation for Theorem 1.2.

The present paper is organized as follows.

In section 2 we state Tevelev’s multi-variable version of the Chevalley Restriction Theorem for isotropy representations of symmetric spaces (Theorem 2.1), and generalize it to the class of polar representations (Corollary 2.1).

Section 3 is concerned with the proofs of Theorem 1.1 and Corollary 1.1.

In section 4 we show how the algebraic results behind Michor’s Basic Forms Theorem [18] [19] and Theorem 1.2 (namely Solomon’s Theorem [24] and Theorem 4.1) are in fact results about polarizations in the Invariant Theory of
finite reflection groups. We then show in detail how Theorem 1.2 follows from
Theorem 4.1.

The Appendix provides a proof of Theorem 4.1. It is computer-assisted, and
mostly reproduced from the author’s PhD dissertation [17].

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2 Multi-variable Chevalley Restriction Theorem

Let \((G, K)\) be a symmetric pair, and consider the isotropy representation of \(K\)
on \(V = T_K G/K\), also called an \(s\)-representation. This is polar, and any maximal
abelian sub-algebra \(\Sigma \subset V\) is a section. Its generalized Weyl group \(W\) is also
called the “baby Weyl group”. The classic Chevalley Restriction Theorem says that
\[
|\Sigma : \mathbb{R}[V]^K \to \mathbb{R}[\Sigma]^W
\]
is an isomorphism (see [28], page 143).

Now consider the diagonal action of \(K\) on \(V^m\) (respectively \(W\) on \(\Sigma^m\)),
and the corresponding algebras of invariant \((m\text{-variable})\) polynomials \(\mathbb{R}[V^m]^K\)
(respectively \(\mathbb{R}[\Sigma^m]^W\)). In contrast with the single-variable case, the restriction
map \(|\Sigma\) is not injective. On the other hand, surjectivity is due to Tevelev:

**Theorem 2.1** ([26]). *In the notation above, the restriction map \(|\Sigma : \mathbb{R}[V^m]^K \to \mathbb{R}[\Sigma^m]^W\) is surjective.*

Remarks: The proof of Theorem 2.1 relies on the Kumar-Mathieu Theorem,
previously known as the PRV conjecture, see [14] and [16]. Joseph [13] proved
the Theorem above in the special case of the adjoint action, using similar tech-
niques. In [26] the Theorem above is stated only for \(m = 2\) factors. But on page
324 it is remarked that “Actually, this (and Joseph’s) Theorem also holds for
any number of summands [...] ”.

We observe that Theorem 2.1 generalizes to the class of polar representations.
(See [4] for a treatment of polar representations)

**Corollary 2.1.** Let \(K \subset O(V)\) be a polar representation, with section \(\Sigma\) and
generalized Weyl group \(W \subset O(\Sigma)\). Then the \(m\text{-variable restriction is surjec-
tive:}
\[
|\Sigma : \mathbb{R}[V^m]^K \to \mathbb{R}[\Sigma^m]^W
\]

Proof. Let \(K_0\) be the connected component of \(K\) which contains the identity.
It is polar with the same section \(\Sigma\). Let \(W_0\) be its generalized Weyl group, so
that \(W_0 \subset W\). From the classification of irreducible polar representations in
[4], it follows that the maximal subgroup \(\hat{K} \subset O(V)\), containing \(K_0\), that is
orbit-equivalent to \(K_0\), defines an \(s\)-representation. (This fact has been given a
classification-free proof in [5, .] Note that \(K_0\) and \(\hat{K}\) have the same sections and
generalized Weyl groups.
Theorem 2.1 states that
\[ |\Sigma : R[V^m]^K \to R[\Sigma^m] |_{W_0} \]
is surjective. But since \( K \supset K_0 \), we have \( R[V^m]^K \subset R[V^m]_{K_0} \), and so
\[ |\Sigma : R[V^m]_{K_0} \to R[\Sigma^m] |_{W_0} \]
is again surjective.

Finally, to show \( |\Sigma : R[V^m]^k \to R[\Sigma^m] |_{W_0} \) is surjective, let \( \beta \in R[\Sigma^m]_{W_0} \). Then there is \( \tilde{\beta}_0 \in R[V^m]_{K_0} \) which restricts to \( \beta \). Define
\[ \tilde{\beta} = \frac{1}{|K/K_0|} \sum_{h \in K/K_0} h \tilde{\beta}_0 \]
Since \( \tilde{\beta} \) equals the average of \( \tilde{\beta}_0 \) over \( K \), it is \( K \)-invariant. To show that \( \tilde{\beta}|_\Sigma = \beta \), we note that each coset \( hK_0 \in K/K_0 \) can be represented by some \( h \in N(\Sigma) \). Indeed, for an arbitrary \( h \in K \), \( h \Sigma \) is a section for \( K \), hence also for \( K_0 \). Since \( K_0 \) acts transitively on the sections, there is \( h_0 \in K_0 \) such that \( hh_0^{-1} \in N(\Sigma) \).

Therefore
\[ \tilde{\beta}|_\Sigma = \frac{1}{|K/K_0|} \sum_{h \in K/K_0} (h \tilde{\beta}_0)|_\Sigma = \frac{1}{|K/K_0|} \sum \beta = \beta \]
because \( \beta \) is \( W \)-invariant.

Note that the algebra of multi-variable polynomials \( R[V^m] \) is graded by \( m \)-tuples of natural numbers \((d_1, \ldots, d_m)\), and similarly for \( R[\Sigma^m] \). Consider the subspace generated by the polynomials of degree \((*,1,\ldots,1)\). These can be identified with those tensor fields of type \((0, m-1)\) which have polynomial coefficients, that is, members of \( R[V, (V^*)^{m-1}] \), respectively \( R[\Sigma, (\Sigma^*)^{m-1}] \).

Since this grading is preserved by the restriction map \( |\Sigma \), Corollary 2.1 implies:

**Corollary 2.2.** Let \( K \subset O(V) \) be a polar representation, with section \( \Sigma \) and generalized Weyl group \( W \subset O(\Sigma) \). Then the restriction map for polynomial-coefficient invariant \((0, l-1)\)-tensors
\[ |\Sigma : R[V, (V^*)^{l-1}]^K \to R[\Sigma, (\Sigma^*)^{l-1}] |_{W_0} \]
is surjective.

### 3 Extending tensors

The goal of this section is to provide proofs of Theorem 1.1 and Corollary 1.1. We start with two Lemmas that will be used in proving Theorem 1.1.

**Lemma 3.1.** Let \( V \) be a polar \( K \)-representation with section \( \Sigma \) and generalized Weyl group \( W \subset O(\Sigma) \). Then restriction to \( \Sigma \) is a surjective map
\[ |\Sigma : C^\infty(T^{0,l}V)^K \to C^\infty(T^{0,l}\Sigma)^W \]
Proof. \ The space of polynomial-coefficient \((0,l)-\)tensors \(\mathbb{R}[V, (V^*)^l]^K \subset C^\infty(T^{0,l}V)^K\) is generated, as an \(\mathbb{R}[V]^K\)-module, by finitely many (homogeneous) \(\sigma_1, \ldots, \sigma_r\).

(See \([23]\) Proposition 2.4.14)

Since \(\mathbb{R}[V]^K = \mathbb{R}[\Sigma]^W\), Corollary \([23]\) implies that the restrictions \(\sigma_1|_{\Sigma}, \ldots, \sigma_r|_{\Sigma}\) generate \(\mathbb{R}[\Sigma, (\Sigma^*)^l]^W\) as an \(\mathbb{R}[\Sigma]^W\)-module.

Then, by an argument involving the Malgrange Division Theorem and the fact that \(\mathbb{R}[\Sigma, (\Sigma^*)^l]^W\) is dense in \(C^\infty(T^{0,l}\Sigma)^W\) (see \([6]\) Lemma 3.1), we conclude that \(\sigma_1|_{\Sigma}, \ldots, \sigma_r|_{\Sigma}\) generate \(C^\infty(\Sigma, (\Sigma^*)^l)^W = C^\infty(T^{0,l}\Sigma)^W\) as a \(C^\infty(\Sigma)^W\)-module. This implies that \(|_\Sigma : C^\infty(T^{0,l}V)^K \to C^\infty(T^{0,l}\Sigma)^W\) is surjective. \(\square\)

The next Lemma describes the smooth \(G\)-invariant tensors on a tube \(U = G \times_K V\) in terms of smooth \(K\)-invariant tensors on the slice \(V\).

**Lemma 3.2.** Let \(K \subset G\) be Lie groups with \(K\) compact, and \(V\) be a \(K\)-representation. Define \(U = G \times_K V\) to be the quotient of \(G \times V\) by the free action of \(K\) given by \(k \cdot (g, v) = (gk^{-1}, kv)\), and identify \(V\) with the subset of \(U\) which is the image of \(\{1\} \times V \subset G \times V\) under the natural quotient projection \(G \times V \to U\).

Then there is a \(K\)-representation \(H\) and an isomorphism

\[
C^\infty(T^{0,l}V)^K \times C^\infty(V, H)^K \to C^\infty(T^{0,l}U)^G
\]

Under this identification the restriction map

\[
|_V : C^\infty(T^{0,l}U)^G \to C^\infty(T^{0,l}V)^K
\]

corresponds to projection onto the first factor. In particular \(|_V|\) is onto.

**Proof.** To describe \(H\), let \(p \in U\) be the image of \((1, 0) \in G \times V\) in \(U\). Then \((V^*)^\otimes l\) is a \(K\)-invariant subspace of \((T_p^*U)^\otimes l\), and we define \(H\) to be its \(K\)-invariant complement, so that

\[
(T_p^*U)^\otimes l = (V^*)^\otimes l \oplus H
\]
as \(K\)-representations.

We define \(\Psi : C^\infty(T^{0,l}V)^K \times C^\infty(V, H)^K \to C^\infty(T^{0,l}U)^G\) in the following way: Given \((\beta_1, \beta_2) \in C^\infty(T^{0,l}V)^K \times C^\infty(V, H)^K\), let \(\tilde{\beta} : G \times V \to T^{0,l}U\) be given by

\[
\tilde{\beta}(g, v) = g \cdot (\beta_1(v) + \beta_2(v))
\]

Since \(\tilde{\beta}\) is \(K\)-invariant, it descends to \(\beta = \Psi(\beta_1, \beta_2) : U \to T^{0,l}U\).

The map \(\beta\) is smooth because \(\tilde{\beta}\) is smooth and the action of \(K\) on \(G \times V\) is free. Moreover \(\beta\) is clearly a \(G\)-invariant cross-section of the bundle \(T^{0,l}U \to U\), and \(\beta|_V = \beta_1\). \(\square\)

Now the proof of Theorem \([1]\) essentially follows from Lemmas \([3, 3.2]\) together with the Slice Theorem (see \([2]\)) and partitions of unity:
Proof of Theorem 1.1. First note that it is enough to consider \((0,l)\) tensors. Indeed, \(\psi\) for \((k,l)\) tensors equals the composition of \(\psi\) for \((0,k+l)\)-tensors with raising and lowering indices (using the Riemannian metric on \(M\)) to transform between \((k,l)\)-tensors and \((0,k+l)\)-tensors.

It is enough to prove surjectivity of \(\psi\) locally around each orbit in \(M\), because of the existence of \(G\)-invariant partitions of unity subject to any cover by \(G\)-invariant open sets in \(M\).

So let \(p \in M\) be an arbitrary point, with orbit \(Gp\), isotropy \(K = Gp\), and slice \(V = (T_p Gp)^\perp\). The Slice Theorem (see [2]) then says that for an open \(G\)-invariant tubular neighborhood \(U\) of the orbit \(Gp\) there is a \(G\)-equivariant diffeomorphism

\[
E : G \times_K V \to U
\]

From now on we will identify \(U\) with \(G \times_K V\) through \(E\).

The slice representation of \(K\) on \(V\) is polar (see [21]). If \(\Sigma \subset V\) is a section with generalized Weyl group \(W(\Sigma)\), the quotients \(U/G, V/K\) and \(\Sigma/W\) are isometric.

Since the inclusion \(\Sigma \to U\) factors as \(\Sigma \to V \to U\), the restriction map \(\psi\) factors as

\[
|_V : C^\infty (T^{0,1} V)^K \to C^\infty (T^{0,1} \Sigma)^W \quad |_U : C^\infty (T^{0,1} U)^G \to C^\infty (T^{0,1} V)^K
\]

Both these maps are surjective, by Lemmas 3.1 and 3.2. Therefore \(\psi\) is surjective. \(\square\)

Now we turn to Corollary 1.1 about \((0,l)\)-tensors with symmetry properties, such as exterior forms and symmetric tensors.

Proof of Corollary 1.1. The Schur functor \(S_\lambda\) is defined in terms of a certain element \(c_\lambda \in \mathbb{Z}S_t\) in the group ring \(\mathbb{Z}S_t\), called the Young symmetrizer associated to \(\lambda\) — see [9] Lecture 6. Indeed, given a vector space \(V\), the group \(S_t\) acts on \(V^\otimes t\), and so \(c_\lambda\) determines a linear map \(V^\otimes t \to V^\otimes t\). The image of this map is defined to be \(S_\lambda(V)\).

Thus \(C^\infty(S_\lambda(T^* M))\) is simply the image of the natural map

\[
c_\lambda : C^\infty (T^{0,1} M) \to C^\infty (T^{0,1} M)
\]

and similarly for \(C^\infty(S_\lambda(T^* M))^G\) (because the actions of \(G\) and \(S_t\) commute), and \(C^\infty(S_\lambda(T^* \Sigma))^W\).

Since the restriction map \(\psi\) is \(S_t\)-equivariant and surjective, it takes the image of

\[
c_\lambda : C^\infty (T^{0,1} M)^G \to C^\infty (T^{0,1} M)^G
\]

onto the image of

\[
c_\lambda : C^\infty (T^{0,1} \Sigma)^W \to C^\infty (T^{0,1} \Sigma)^W
\]

completing the proof. \(\square\)
4 Polarizations and finite reflection groups

An alternative way of proving special cases of Theorem 2.1 is given by the polarization technique. This has the advantage of providing explicit lifts, which we exploit to give a proof of Theorem 1.2.

We start by recalling the definition of polarizations (see [23] for a reference). Let $U$ be an Euclidean vector space, and $H \to O(U)$ be a representation of the group $H$. Consider the diagonal action of $H$ on $m$ copies of $U$, and the corresponding algebra of invariant $(m$-variable) polynomials $\mathbb{R}[U^m]^H$. Identify $\mathbb{R}[U]^H$ with the elements of $\mathbb{R}[U^m]^H$ which depend only on the first variable.

The method of polarizations consists of generating multi-variable invariants from single-variable invariants. Indeed, assuming $f \in \mathbb{R}[U]^H$ is homogeneous of degree $d$, let $t_1, \ldots, t_m$ be formal variables, and formally expand

$$f(t_1v_1 + \ldots + t_mv_m) = \sum_{r_1+\ldots+r_m=d} t_1^{r_1} \cdots t_m^{r_m} f_{r_1,\ldots,r_m}(v_1,\ldots,v_m)$$

Then each $f_{r_1,\ldots,r_m}$ belongs to $\mathbb{R}[U^m]^H$, and is called a polarization of $f$.

An alternative but equivalent definition of polarizations is given in terms of polarization operators — see [27]. These are differential operators $D_{ij}$ (for $1 \leq i, j \leq m$) on $\mathbb{R}[U^m]^H$ defined by

$$(D_{ij} f)(u_1, \ldots, u_m) = \left. \frac{d}{dt} \right|_{t=0} f(u_1, \ldots, u_j + tu_i, \ldots, u_m)$$

Then one defines the subalgebra $\mathcal{P}^m \subset \mathbb{R}[U^m]^H$ of polarizations to be the smallest subalgebra of $\mathbb{R}[U^m]^H$ containing $\mathbb{R}[U]^H$ and stable under the operators $D_{ij}$.

For example, if $f \in \mathbb{R}[U]^H$, then the tensors $df = D_{2,1} f \in \mathbb{R}[U^2]^H$ and $\text{Hess} f = D_{2,1}(D_{3,1} f) \in \mathbb{R}[U^3]^H$ are polarizations. Similarly, if $f_1, \ldots, f_p \in \mathbb{R}[U]^H$, then $df_1 \otimes df_2 \otimes \cdots \otimes df_p = (D_{2,1} f_1) \cdots (D_{p+1,1} f_p)$ is a polarization, and so is $df_1 \wedge \cdots \wedge df_p$. (Here we are identifying tensor fields with multi-variable functions as in section 2.)

Now consider the special case where $H = W_0$ is a finite group generated by reflections on $U = \Sigma$. If $W_0$ is irreducible of type $A$, $B$, or dihedral, then $\mathcal{P}^m = \mathbb{R}[\Sigma^m]^W$ by [29], [12].

It was noted by Wallach [27] that $\mathbb{R}[\Sigma^m]^W$ is not generated by polarizations for $W_0$ of type $D_n$ for $n > 3$ and $m > 1$. He proposed a definition of generalized polarizations, and showed that these do generate all multi-variable invariants for type $D$. Unfortunately Wallach’s generalized polarizations fail to generate all multi-variable invariants for $W_0$ of type $F_4$ (see [12]).

For $W_0$ of general type, even though $\mathcal{P}^m \neq \mathbb{R}[\Sigma^m]^W$, one can still identify geometrically interesting subspaces of $\mathbb{R}[\Sigma^m]^W$ which are contained in $\mathcal{P}^m$. For example, Solomon’s Theorem [24] states that the subspace $\mathbb{R}[\Sigma, \Lambda^{m-1}\Sigma^*]^W \subset \mathbb{R}[\Sigma^m]^W$ of exterior $(m - 1)$-forms is contained in $\mathcal{P}^m$. Another example is the main ingredient in the proof of Theorem 1.2.
Theorem 4.1 (Hessian Theorem — [17]). Let $W_0 \subset O(\Sigma)$ be a finite group generated by reflections. Then every $W_0$-invariant symmetric 2-tensor field on $\Sigma$ is a sum of terms of the form $a\text{Hess}(b)$, for $a, b \in \mathbb{R}[\Sigma]^{W_0}$.

For the convenience of the reader, we provide a proof of Theorem 4.1 above in the Appendix.

Now assume $K \subset O(V)$ is a polar representation of the compact group $K$ with section $\Sigma$, and generalized Weyl group $W$. Recall that the connected component of the identity $K_0$ is polar with the same section $\Sigma$, and denote by $W_0$ its generalized Weyl group. By [4], $W_0$ is a finite group generated by reflections. Since the operators $D_{ij}$ commute with the restriction map $|\Sigma^m : \mathbb{R}[V^m]^{K_0} \rightarrow \mathbb{R}[\Sigma^m]^{W_0}$, and the single-variable invariants coincide by the Chevalley Restriction Theorem, the image of $|\Sigma^m$ must contain $\mathbb{P}^m$. In particular, this gives an alternative proof of Theorem 2.1 in the special case that $W_0$ is of classical type — see [12].

Similarly, Theorem 4.1 implies surjectivity of the restriction map for symmetric 2-tensors. In fact, we have the sharper statement:

**Lemma 4.1.** Let $K \subset O(V)$ be a polar representation of the compact group $K$, with section $\Sigma \subset V$ and generalized Weyl group $W$. Consider the restriction map for symmetric 2-tensor fields $|\Sigma : C^\infty(\text{Sym}^2V)^K \rightarrow C^\infty(\text{Sym}^2\Sigma)^W$.

This map is surjective. Moreover, given $\beta \in C^\infty(\text{Sym}^2\Sigma)^W$ there is $\tilde{\beta} \in C^\infty(\text{Sym}^2V)^K$ such that $|\Sigma^\beta = \beta$ and satisfying the following property:

For all $q \in V$, and $X,Y \in T_qV$ such that $X$ is vertical (that is, tangent to the $K$-orbit through $q$) and $Y$ is horizontal (that is, normal to the $K$-orbit through $q$), we have $\beta(X,Y) = 0$.

**Proof.** Let $K_0$ be the connected component of the identity. It is polar with the same section $\Sigma$, and generalized Weyl group $W_0$. By [4], $W_0$ is generated by reflections.

Let $\beta \in C^\infty(\text{Sym}^2\Sigma)^W$. By Theorem 4.1 together with [6], Lemma 3.1, $\beta$ is of the form $\beta = \sum_i a_i\text{Hess}(b_i)$, where $a_i, b_i \in C^\infty(\Sigma)^{W_0}$. By the Chevalley Restriction Theorem, $a_i, b_i$ extend uniquely to $\tilde{a}_i, \tilde{b}_i \in C^\infty(V)^{K_0}$.

Define $\tilde{\beta}_0 = \sum_i \tilde{a}_i\text{Hess}(\tilde{b}_i)$ and

$$\tilde{\beta} = \frac{1}{|K/K_0|} \sum_{h \in K/K_0} h\tilde{\beta}_0$$

Then $|\Sigma^\beta = \beta$ by the same argument as in Corollary 2.1.

To show that $\tilde{\beta}$ satisfies the additional property in the statement of the Lemma, it is enough to do so for each $\text{Hess}(\tilde{\beta}_i)$. Changing the section $\Sigma$ if necessary, we may assume that $q,Y \in \Sigma$. Extend the given $X,Y \in T_qV$ to parallel vector fields (in the Euclidean metric), also denoted by $X,Y$. Let $f = d\tilde{\beta}_i(X)$.

We claim that $f|\Sigma$ is identically zero. Indeed, since $X(q)$ is vertical, it is orthogonal to $\Sigma$, and so $X(p)$ is orthogonal to $\Sigma$ for every $p \in \Sigma$. Thus, for
regular \( p \in \Sigma \), \( X(p) \) is vertical. Since \( \tilde{\beta}_i \) is constant on orbits, \( f(p) = 0 \) for every regular \( p \in \Sigma \), and hence on all of \( \Sigma \) by continuity.

Therefore \( \text{Hess}(\tilde{\beta}_i)(X,Y) = df(Y) = 0 \), because \( Y \in \Sigma \).

The following Lemma is needed in the proof of Theorem 1.2.

**Lemma 4.2.** Let \( V \) be a polar \( K \)-representation with section \( \Sigma \subset V \) and generalized Weyl group \( W \). Let \( \tilde{\sigma} \in C^\infty(\text{Sym}^2V)^K \), and \( \sigma = \tilde{\sigma}|_{\Sigma} \). Then \( \sigma(0) \) is positive definite if and only if \( \tilde{\sigma}(0) \) is positive definite.

**Proof.** Denote by \( K_0 \) the connected subgroup of \( K \) containing the identity. Recall that the action of \( K_0 \) is polar with the same section \( \Sigma \). Denote by \( W_0 \) its generalized Weyl group. Consider a decomposition of \( V \) into \( K_0 \)-invariant subspaces 

\[
V = \mathbb{R}^m \oplus V_1 \oplus \cdots \oplus V_k
\]

where \( K_0 \) acts trivially on \( \mathbb{R}^m \), and each \( V_i \) is irreducible and non-trivial.

By Theorem 4 in [4], each \( V_i \) is a polar \( K_0 \)-representation, with section \( \Sigma_i = \Sigma \cap V_i \), and we have the decomposition into \( W_0 \)-invariant subspaces 

\[
\Sigma = \mathbb{R}^m \oplus \Sigma_1 \oplus \cdots \oplus \Sigma_k
\]

Moreover \( W_0 \) splits as a product \( W_1 \times \cdots \times W_k \) (see section 2.2 in [11]), where \( W_i \) is the generalized Weyl group associated to the section \( \Sigma_i \subset V_i \), so that \( \Sigma_i \) are pairwise inequivalent as \( W_0 \)-representations. This implies that \( V_i \) are pairwise inequivalent as \( K_0 \)-representations.

Since the quotients \( V_i/K_0 \) and \( \Sigma_i/W_0 \) are isometric, irreducibility of \( V_i \) as a \( K_0 \)-representation implies irreducibility of \( \Sigma_i \) as a \( W_0 \)-representation. (Indeed, a general representation of a compact group \( H \) on Euclidean space \( \mathbb{R}^n \) is irreducible if and only if the quotient \( S^{n-1}/H \) has diameter less than \( \pi/2 \))

By Schur’s Lemma together with the assumption \( \tilde{\sigma}|_{\Sigma} = \sigma \),

\[
\sigma(0) = A \oplus \lambda_1 \text{Id}_{\Sigma_1} \oplus \cdots \oplus \lambda_k \text{Id}_{\Sigma_k}
\]

\[
\tilde{\sigma}(0) = A \oplus \lambda_1 \text{Id}_{V_1} \oplus \cdots \oplus \lambda_k \text{Id}_{V_k}
\]

where \( A \) is a symmetric \( m \times m \) matrix, and \( \lambda_i \in \mathbb{R} \).

Therefore \( \sigma(0) > 0 \) if and only if \( \tilde{\sigma}(0) > 0 \).

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** As in the proof of Theorem 1.1, we use partitions of unity and the Slice Theorem to reduce to the case where \( M \) is a tube \( U = G \times_K V \), and \( V \) is a polar representation. Let \( \Sigma \subset V \) be a section, with generalized Weyl group \( W \), so that \( M/G = V/K = \Sigma/W \).

Note that it suffices to extend the given Riemannian metric \( \sigma \in C^\infty(\text{Sym}^2\Sigma)^W \) to a \( G \)-invariant Riemannian metric on a possibly smaller tube \( G \times_K V^\epsilon \) around the orbit \( G/K \), for some \( \epsilon > 0 \).

By Corollary 1.1, \( \sigma \) extends to \( \beta_1 \in C^\infty(\text{Sym}^2V)^K \). By Lemma 4.2, \( \beta_1(0) \) is positive-definite, and so by continuity, \( \beta_1 > 0 \) on \( V^\epsilon \) for some small \( \epsilon > 0 \).
Choose any smooth, $K$-invariant and positive-definite $\beta_2: V \to \text{Sym}^2(T_KG/K)$. Then, by Lemma 3.2, the pair $(\beta_1, \beta_2)$ defines $\tilde{\sigma} \in C^\infty(\text{Sym}^2M)^G$, which is positive-definite on $G \times_K V^\vee$ and extends the given $\sigma$. By construction, $\Sigma$ is $\tilde{\sigma}$-orthogonal to $G$-orbits.

A Appendix — Hessian Theorem for finite reflection groups

In this section we provide a proof of Theorem 4.1 for all finite reflection groups $W \subset O(\Sigma)$. Note that as far as the proof of Theorem 1.2 is concerned, the only case of Theorem 4.1 needed is that of crystallographic reflection groups (see [11] for a definition). Our proof includes the non-crystallographic case for the sake of completeness.

The structure of the proof is as follows. First we reduce to the case where $W$ is irreducible — see Lemma A.1. Then we point out that for $W$ irreducible of classical type, Theorem 4.1 follows from more general polarization results due to Weyl [29] and Hunziker [12]. Finally we tackle the case of the exceptional groups with the help of a computer.

Recall some facts about finite reflection groups: First, the algebra of invariants, $R[\Sigma]^W$, is a free polynomial algebra with as many generators as the dimension of $\Sigma$. This is known as Chevalley’s Theorem — see [1] Chapter V. Such a set of homogeneous generators is called a set of basic invariants. Second, $\Sigma$ is reducible as a $W$-representation if and only if $\Sigma = \Sigma_1 \times \Sigma_2$ and $W = W_1 \times W_2$ for two reflection groups $W_k \subset O(\Sigma_k)$ — see section 2.2 in [11]. Because of the latter, the following proposition reduces the proof of the Hessian Theorem to the irreducible case.

Lemma A.1. Let $W_k \subseteq O(\Sigma_k)$, $k = 1, 2$ be two finite reflection groups in the Euclidean vector spaces $\Sigma_k$, and let $W = W_1 \times W_2 \subset O(\Sigma) = O(\Sigma_1 \times \Sigma_2)$. Then the conclusion of the Hessian Theorem holds for $W \subset O(\Sigma)$ if and only if it holds for both $W_k \subseteq O(\Sigma_k)$, $k = 1, 2$.

Proof. Let $i_k: \Sigma_k \to \Sigma_1 \times \Sigma_2$ and $p_k: \Sigma_1 \times \Sigma_2 \to \Sigma_k$ be the natural inclusions and projections. As a $W$-representation, $\text{Sym}^2(\Sigma^*)$ decomposes as

$$\text{Sym}^2(\Sigma^*) = \text{Sym}^2(\Sigma_1^*) \oplus \text{Sym}^2(\Sigma_2^*) \oplus (\Sigma_1^* \otimes \Sigma_2^*)$$

Denote by $i_{11}$ and $i_{22}$ the natural inclusions of the first two summands. All these maps are $W$-equivariant.

Assume the conclusion of the Hessian Theorem holds for $W \subset O(\Sigma)$. Thus there are $Q_j \in R[\Sigma]^W$ whose Hessians form a basis for $R[\Sigma, \text{Sym}^2(\Sigma^*)]^W$. Then the restrictions $Q_j|_{\Sigma_k} = i_{kk}^*Q_j$ generate $R[\Sigma_k, \text{Sym}^2(\Sigma_k^*)]^{W_k}$ as an $R[\Sigma_k]^{W_k}$-module.

Indeed, given $\sigma \in R[\Sigma_k, \text{Sym}^2(\Sigma_k^*)]^{W_k}$, define

$$\tilde{\sigma} = i_{kk} \circ \sigma \circ p_k$$
Since $\hat{\sigma}$ is $W$-equivariant, there are $a_j \in \mathbb{R}[\Sigma]^W$ such that $\hat{\sigma} = \sum_j a_j \text{Hess}(Q_j)$. Therefore

$$\sigma = i_k^*(\hat{\sigma}) = \sum_j (a_j |_{\Sigma_k}) \text{Hess}(Q_j |_{\Sigma_k})$$

For the converse, assume the conclusion of the Hessian Theorem holds for $W_k \subset O(\Sigma_k)$. Let $\rho_j \in \mathbb{R}[\Sigma_1]^W$, $j = 1, \ldots, n_1$ and $\psi_j \in \mathbb{R}[\Sigma_2]^W$, $j = 1, \ldots, n_2$ be basic invariants on $\Sigma_1$ and $\Sigma_2$ respectively, and $Q_j \in \mathbb{R}[\Sigma_1]^W$, for $j = 1, \ldots, (n_1^2 + n_1)/2$, $R_j \in \mathbb{R}[\Sigma_2]^W$, for $j = 1, \ldots, (n_2^2 + n_2)/2$ be homogeneous invariants whose Hessians form a basis for the corresponding spaces of equivariant symmetric 2-tensors.

Claim: The Hessians of the following set of $W = W_1 \times W_2$-invariant polynomials on $\Sigma = \Sigma_1 \times \Sigma_2$ form a basis for the space of equivariant symmetric 2-tensors on $\Sigma$:

$$\{Q_j \} \cup \{R_j \} \cup \{\rho_i \psi_j, \ i = 1 \ldots n_1, \ j = 1 \ldots n_2\}$$

Indeed, $\mathbb{R}[\Sigma, \text{Sym}^2(\Sigma^*)]^W$ decomposes as

$$\mathbb{R}[\Sigma, \text{Sym}^2(\Sigma_1^*)]^W \oplus \mathbb{R}[\Sigma, \text{Sym}^2(\Sigma_2^*)]^W \oplus \mathbb{R}[\Sigma, \Sigma_1^* \otimes \Sigma_2^*]^W$$

The first two pieces are freely generated over $\mathbb{R}[\Sigma]^W$ by $\text{Hess}Q_j$ and $\text{Hess}R_j$. The third piece can be rewritten as $\mathbb{R}[\Sigma, \Sigma_1^* \otimes \Sigma_2^*]^W = \mathbb{R}[\Sigma_1, \Sigma_1^*]^W \oplus \mathbb{R}[\Sigma_2, \Sigma_2^*]^W$. By Solomon’s Theorem [24], $\mathbb{R}[\Sigma_k, \Sigma_k]^W$ are freely generated by $d\rho_i$ and $d\psi_j$, so that $\mathbb{R}[\Sigma, \Sigma_1^* \otimes \Sigma_2^*]^W$ is freely generated by $d\rho_i \otimes d\psi_j$. To finish the proof of the Claim one uses the product rule

$$\text{Hess}(\rho_i \psi_j) = d\rho_i \otimes d\psi_j + \rho_i \text{Hess}(\psi_j) + \psi_j \text{Hess}(\rho_i)$$

Irreducible finite reflection groups are classified by type — see [11]. For $W$ irreducible of type $A$, $B$ and dihedral, the statement of Theorem 4.1 follows from [29], [12], while for type $D$, it follows from [12], Theorem 3.1.

Finally we prove the Hessian Theorem for the six exceptional finite reflection groups $W \subset O(\Sigma)$ usually called by the names of their Dynkin diagrams: $H_3$, $H_4$, $F_4$, $E_6$, $E_7$ and $E_8$. Note that the subscript denotes the rank $n = \dim(\Sigma)$. In all cases our proof relies on calculations performed by a computer running GAP 3 (see [22]) using the package CHEVIE, which ultimately rely only on integer arithmetic. For the actual code that was used, see

[http://www.nd.edu/~rmendes/sym2.txt](http://www.nd.edu/~rmendes/sym2.txt)

Recall a way of describing $W \subset O(\Sigma)$ from its Cartan matrix $C = (C_{ij})$. $\Sigma$ has a basis $r_1, \ldots, r_n$ of simple roots with corresponding co-roots $r_1^\vee, \ldots, r_n^\vee$. This means that $W$ is generated by the reflections in the hyperplanes $\ker(r_i^\vee)$ given by:

$$R_i : v \mapsto v - r_i^\vee(v) r_i \quad i = 1, \ldots, n$$
Expressing $v \in \Sigma$ in the basis of simple roots $v = a_1 r_1 + \ldots + a_n r_n$, we get

$$R_i(v) = v - \left( \sum_j a_j r_i^\vee (r_j) \right) r_i$$

The coefficients $r_i^\vee (r_j) = C_{ij}$ form the Cartan matrix.

Here are the Cartan matrices for $H_3, H_4$ and $F_4$: (where $\zeta = \exp(2\pi i/5)$)

\[
H_3 : \begin{pmatrix} 2 & \zeta^2 + \zeta^3 & 0 \\ \zeta^2 + \zeta^3 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \quad H_4 : \begin{pmatrix} 2 & \zeta^2 + \zeta^3 & 0 & 0 \\ \zeta^2 + \zeta^3 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\]

\[
F_4 : \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}
\]

For the Cartan matrices in type E, refer to the tables at the end of [1].

We start the proof of the Hessian Theorem by describing how the program computes the polynomial

$$P_t(\mathbb{R}[\Sigma, \text{Sym}^2 \Sigma^*] W)$$

where $P_t(U) = \sum_{l=0}^{\infty} (\dim U_l) t^l$ denotes the Poincaré series of a graded vector space $U = \oplus_{l=0}^{\infty} U_l$.

We need to recall a few facts. Let $I$ be the ideal in $\mathbb{R}[\Sigma]$ generated by the homogeneous invariants of positive degree. The quotient $\mathbb{R}[\Sigma]/I$ is known to be isomorphic, as a $W$-representation, to the regular representation (see Theorem B in [3]), but it is also a graded vector space. Fixing an irreducible representation/character $\xi$, the Poincaré polynomial $\text{FD}_\xi(t)$ of the subspace of $\mathbb{R}[\Sigma]/I$ with components isomorphic to $\xi$ is called the fake degree of $\xi$. Moreover $\mathbb{R}[\Sigma]$ is isomorphic to $(\mathbb{R}[\Sigma]/I) \otimes \mathbb{R}[\Sigma]^W$. Thus the Poincaré series of the vector subspace in $\mathbb{R}[\Sigma]$ given by the direct sum of all irreducible subspaces isomorphic to $\xi$ equals $\text{FD}_\xi(t) P_t(\mathbb{R}[\Sigma]^W)$.

The way the program computes $P_t(\mathbb{R}[\Sigma, \text{Sym}^2 \Sigma^*] W)$ is as follows:

It first computes the character $\chi$ of $\text{Sym}^2 \Sigma^*$, and decomposes it into a sum of irreducible characters, using character tables that come with CHEVIE.

$$\chi = \sum_{\xi \text{ irreducible}} c_\xi \xi$$

It then uses a command in CHEVIE that returns the fake degrees of the irreducible characters $\xi$, and computes

$$\sum_{\xi} c_\xi \text{FD}_\xi(t)$$

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Using Schur’s Lemma one sees that this equals

\[ \frac{P_t(\mathbb{R}[\Sigma, \text{Sym}^2 \Sigma^*])}{P_t(\mathbb{R}[\Sigma])} \]

Here are the outputs:

\[
\begin{array}{c|c}
H_3 & t^{10} + t^8 + t^6 + t^4 + t^2 + 1 \\
H_4 & t^{38} + t^{30} + t^{28} + t^{22} + t^{20} + t^{18} + t^{12} + t^{10} + t^2 + 1 \\
F_4 & t^{14} + t^{12} + 2t^{10} + t^8 + 2t^6 + t^4 + t^2 + 1 \\
E_6 & t^{16} + t^{15} + t^{14} + t^{13} + 2t^{12} + t^{11} + 2t^{10} + 2t^9 + 2t^8 + t^7 + 2t^6 + t^5 + t^4 + t^3 + t^2 + 1 \\
E_7 & t^{26} + t^{24} + 2t^{22} + 2t^{20} + 3t^{18} + 3t^{16} + 3t^{14} + 3t^{12} + 3t^{10} + 2t^8 + 2t^6 + t^4 + t^2 + 1 \\
E_8 & t^{46} + t^{42} + t^{40} + t^{38} + 2t^{36} + 2t^{34} + t^{32} + 3t^{30} + 2t^{28} + 2t^{26} + 3t^{24} + 2t^{22} + 2t^{20} + 3t^{18} + t^{16} + 2t^{14} + 2t^{12} + t^{10} + t^8 + t^6 + t^2 + 1 \\
\end{array}
\]

Now we turn to the task of defining an explicit set of basic invariants \( \rho_1, \ldots, \rho_n \in \mathbb{R}[\Sigma]^W \). The degrees \( d_i = \deg(\rho_i) \) are well known: (see tables at the end of [1])

\[
\begin{array}{c|c}
\text{degrees} & d_1, \ldots, d_n \\
H_3 & 2, 6, 10 \\
H_4 & 2, 12, 20, 30 \\
F_4 & 2, 6, 8, 12 \\
E_6 & 2, 5, 6, 8, 9, 12 \\
E_7 & 2, 6, 8, 10, 12, 14, 18 \\
E_8 & 2, 8, 12, 14, 18, 20, 24, 30 \\
\end{array}
\]

We choose for each group a regular vector \( v \in \Sigma \) and identify it with the row vector of its coefficients in the basis of the simple roots \( r_i \). We also take one non-zero \( \lambda \in \Sigma^* \) with minimal \( W \)-orbit size, namely the one which in the basis \( \{v_i^\vee\} \) of simple co-roots is identified with the row vector

\[ \lambda = (0, \ldots, 0, 1) \cdot C^{-1} \]

Then the program computes the \( W \)-orbit \( \mathcal{O} \) of \( \lambda \). Here are our choices of \( v \) and the number of elements in the orbit \( \mathcal{O} \):

\[
\begin{array}{c|c|c}
v \text{ (in the basis } \{r_i\}) & \mathcal{O} \\
H_3 & (1, 2, 3) & 12 \\
H_4 & (1, 2, 3, 5) & 20 \\
F_4 & (2, -3, 5, 7) & 24 \\
E_6 & (2, -5, 41, 7, -9, 110) & 27 \\
E_7 & (2, -5, 41, 7, -9, 110, -87) & 56 \\
E_8 & (2, -5, 41, 7, -9, 110, -87, 11) & 240 \\
\end{array}
\]
Since $W$ permutes the linear polynomials in $\mathcal{O}$, for each natural number $m$ we get a $W$-invariant polynomial of degree $m$

$$\psi_m = \sum_{\lambda \in \mathcal{O}} \lambda^m$$

The invariants constructed this way are called the Chern classes associated to the orbit $\mathcal{O}$. See [20] chapter 4.

**Lemma A.2.** The polynomials $\rho_i = \psi_{d_i}$, $i = 1, \ldots, n$, form a set of basic invariants, and $v$ is indeed a regular vector.

**Proof.** Let $J$ be the Jacobian matrix

$$J = \left( \frac{\partial \rho_i}{\partial r_j} \right)_{i,j} = \left( \sum_{\lambda \in \mathcal{O}} d_i \lambda^{d_i-1} \frac{\partial \lambda}{\partial r_j} \right)_{i,j}$$

The program computes its determinant, evaluates it at the vector $v$, and checks that the value is non-zero. This proves both that $\rho_i$ are algebraically independent (see Proposition 3.10 in [11]) and hence a set of basic invariants because they have the right degrees; and that $v$ is indeed a regular vector, that is, does not belong to any of the reflecting hyperplanes (see section 3.13 in [11]). \qed

We point out that L. Flatto and M. Weiner studied the set of all $\lambda \in \Sigma^*$ that make the $\rho_i = \psi_{d_i}$ constructed above a set of basic invariants. They produce a distinguished set of basic invariants $J_1, \ldots, J_n$, determined up to non-zero constants, such that $\lambda \in \Sigma^*$ gives rise to a set of basic invariants if and only if $J_i(\lambda) \neq 0$ for all $i$ — see [8, 7] for more details.

**Theorem A.1.** Let $W \subset O(\Sigma)$ be one of the six exceptional finite reflection groups, and $\rho_1, \ldots, \rho_n$ the set of basic invariants described above. Let $T \subset \{\rho_i\} \cup \{\rho_i \rho_j\}$ be a subset with $n(n+1)/2$ elements such that $T$ contains $\{\rho_i\}$ and

$$\sum_{Q \in T} \ell^{\deg(Q)-2} = \frac{P_t(\mathbb{R}[\Sigma, \text{Sym}^2\Sigma^*]^W)}{P_t(\mathbb{R}[\Sigma]^W)}$$

There is at least one such $T$, and for each one, $\{\text{Hess}(Q) \mid Q \in T\}$ is a basis for $\mathbb{R}[\Sigma, \text{Sym}^2\Sigma^*]^W$ as a free module over $\mathbb{R}[\Sigma]^W$.

**Proof.** First the program finds a list of all subsets $T$ satisfying the condition in the statement of the Theorem. The number of elements in this list (choices for $T$) are:

| choices | $H_3$ | $H_4$ | $F_4$ | $E_6$ | $E_7$ | $E_8$ |
|---------|-------|-------|-------|-------|-------|-------|
|         | 2     | 2     | 12    | 48    | 96    |       |

For each $T$, the program constructs a square matrix $M$ of size $n(n+1)/2$. The rows are in correspondence with the set $\mathcal{H} = \{\text{Hess}(Q) \mid Q \in T\}$, and
the columns with the set $\mathcal{P}$ of upper triangular positions of an $n \times n$ matrix. The entry of $M$ associated with $\text{Hess}(Q) \in \mathcal{H}$ and a position $(a, b) \in \mathcal{P}$ is the $(a, b)$-entry of $\text{Hess}(Q)(v)$, that is,

$$\frac{\partial^2 Q}{\partial r^a \partial r^b}(v)$$

Then it proceeds to compute the determinant of $M$ and checks that it is non-zero. This implies that $\mathcal{H}$ is linearly independent at $v$, hence over $\mathbb{R}[\Sigma]$, and in particular over $\mathbb{R}[\Sigma]^W$.

Therefore $\text{span}_{\mathbb{R}[\Sigma]^W} \mathcal{H}$ is a submodule of $\mathbb{R}[\Sigma, \text{Sym}^{2*}\Sigma^*]^W$ with the same Poincaré series, and so they must coincide. □

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