SYMPELTIC PARABOLICITY AND $L^2$ SYMPLECTIC HARMONIC FORMS*

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Abstract

In this paper, we study the symplectic cohomologies and symplectic harmonic forms which introduced by Tseng and Yau. Based on this, we get if $(M^{2n}, \omega)$ is a closed symplectic parabolic manifold which satisfies the hard Lefschetz property, then its Euler number satisfies the inequality $(-1)^n \chi (M^{2n}) \geq 0$.

1. Main results

This paper is related to a special case of the Chern conjecture claiming that the topological Euler characteristic of a real $2n$-dimensional closed manifold $M$ of negative curvature satisfies $\text{sign} \chi (M) = (-1)^n$. This conjecture is true in dimensions 2 and 4 [8]. In dimension 2, the answer follows immediately from the Gauss–Bonnet formula, that is, a closed manifold of negative sectional curvature has negative Euler number. In dimension 4, it is proved by Milnor (see [8]) that negative sectional curvature implies that Gauss–Bonnet integrand is pointwise positive.

A differential form $\alpha$ on a Riemannian manifold $(M, g)$ is called $d$ (bounded) if $\alpha$ is the exterior differential of a bounded form $\gamma$, that is, $\alpha = d \gamma$, where $\|\gamma\|_{L^\infty} = \sup_{x \in M} \|\gamma(x)\|_g < \infty$. A form $\alpha$ on a Riemannian manifold $(M, g)$ is called $\overline{d}$ (bounded) if the lift $\bar{\alpha}$ of $\alpha$ to the universal covering $\bar{M} \to M$ is $d$ (bounded) on $\bar{M}$ with respect to the lift metric $\bar{g}$. Gromov gave the definition of Kähler hyperbolic in [12]. A closed complex manifold is called Kähler hyperbolic if it admits a Kähler metric whose Kähler form $\omega$ is $\overline{d}$ (bounded). Similarly, we can define symplectic hyperbolic manifold. Let $(M, \omega)$ be a closed symplectic manifold. Choose a $\omega$-compatible almost complex structure $J$ on $M$ [20]. Define an almost Kähler metric, $g(\cdot, \cdot) = \omega(\cdot, J \cdot)$, on $M$. Then the

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triple \((g, J, \omega)\) is called an almost Kähler structure on \(M\) and the quadruple \((M, g, J, \omega)\) is called a closed almost Kähler manifold.

**Definition 1.1** A closed almost Kähler manifold \((M, g, J, \omega)\) is called symplectic hyperbolic if the lift \(\tilde{\omega}\) of \(\omega\) to the universal covering \((\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})\) is \(d(\text{bounded})\) on \((\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})\).

Gromov [12] introduced the notion of Kähler hyperbolicity and proved the above conjecture in the Kähler case. After Gromov’s work, Jost and Zuo [16] obtained that if \(M\) is a \(2n\)-dimensional closed Riemannian manifold of non-positive sectional curvature and homotopy equivalent to a closed Kähler manifold, then \((-1)^n\chi(M) \geq 0\). It is well known that the geodesic flow on the unit tangent bundle of a negatively curved closed Riemannian manifold is an Anosov geodesic flow. Cheng [9] has proven that let \(M\) be a \(2n\)-dimensional closed Riemannian manifold with Anosov geodesic flow. If \(M\) is homotopy equivalent to a closed Kähler manifold, then \((-1)^n\chi(M) > 0\).

Gromov has proven that if \((M, g)\) is complete simply connected and has strictly negative sectional curvature, then every smooth bounded closed form of degree \(k \geq 2\) is \(d(\text{bounded})\) [12, 0.1 B].

We want to single out a condition which is weaker than \(d\)-boundedness and can be applied to Kähler manifolds of non-positive curvature. A differential form \(\alpha\) on a closed Riemannian manifold is called \(d(\text{sublinear})\) (cf. [14, 16]) if there exist a differential form \(\beta\) and a number \(c > 0\) such that \(\alpha = d\beta\) and \(\|\beta(x)\| \leq c(\rho(x_0, x) + 1)\), where \(\rho(x_0, x)\) stands for the Riemannian distance between \(x\) and a base point \(x_0\). A form \(\alpha\) on a Riemannian manifold \((M, g)\) is called \(d(\text{sublinear})\) if the lift \(\tilde{\alpha}\) of \(\alpha\) to the universal covering \(\tilde{M} \to M\) is \(d(\text{sublinear})\) on \(\tilde{M}\) with respect to the lift metric \(\tilde{g}\).

**Definition 2.1** A closed almost Kähler manifold \((M, g, J, \omega)\) is called symplectic parabolic if the lift \(\tilde{\omega}\) of \(\omega\) to the universal covering \((\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})\) is \(d(\text{sublinear})\) on \((\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})\).

If \(L^2\Omega^k\) denotes the Hilbert space of \(L^2\) \(k\)-forms, then the \(L^2\)-cohomology group \(L^2H^k_{\text{IR}}\) is defined as the quotient of the space of closed \(L^2\) \(k\)-forms by the closure of the space \(d\)\(L^2\Omega^{k-1} \cap L^2\Omega^k\). It is a theorem that on a complete manifold any harmonic \(L^2\) \(k\)-form is closed and coclosed and so represents a class in \(L^2H^k_{\text{IR}}\). Hitchin [14] has proven that the \(L^2\) harmonic forms on a complete non-compact Kähler parabolic manifold lie in the middle dimension, that is, if the Kähler form \(\omega\) on a complete non-compact Kähler manifold is \(d(\text{sublinear})\), then the only \(L^2\) harmonic forms lie in the middle dimension. In this paper, we want to prove some similar results for another two \(L^2\) harmonic forms (\(L^2\) symplectic harmonic).

Let \((M, g, J, \omega)\) be a closed almost Kähler \(2n\)-manifold, that is, \(M\) is a closed differential manifold with an almost Kähler structure on \(M\). Symplectic Hodge theory was introduced by Ehresmann and Libermann [10] and was rediscovered by Brylinski [4]. They defined the symplectic star operator \(*_s: \Omega^k(M) \to \Omega^{2n-k}(M)\) analogously to the Hodge star operator, but with respect to the symplectic form \(\omega\). As in Riemannian Hodge theory, define \(d_\lambda = (-1)^{k+1}*_sd*_s\) on \(\Omega^k(M)\) (cf. [18]). A form \(\alpha\) is called symplectic harmonic if it satisfies \(d\alpha = d_\lambda\alpha = 0\). Brylinski conjectured that on a closed symplectic manifold, every de Rham cohomology class contains a symplectic harmonic representative. Some evidence for his conjecture was presented in his paper [4] and he proved the conjecture for closed Kähler manifolds. Several years later, his conjecture for closed symplectic manifolds was disproved by Oliver Mathieu [19]. Brylinski’s conjecture is equivalent to the question of the existence of a Hodge decomposition in the sympletic sense. The uniqueness
of the decomposition in this case is evidently not true. Mathieu gave two ways to give counter-
examples to Brylinski’s conjecture. In fact, Mathieu proved that every de Rham cohomology
\( H^*_dR(M) \) class contains a symplectic harmonic form if and only if the symplectic manifold satisfies
the hard Lefschetz property, that is, the map

\[
H^k_dR(M) \rightarrow H_{dR}^{2n-k}(M), \quad A \mapsto [\omega]^{n-k} \wedge A,
\]

is an isomorphism for all \( k \leq n \).

Mathieu’s theorem is a generalization of the hard Lefschetz theorem for closed Kähler mani-
folds. His proof involves the representation theory of quivers and Lie superalgebras. Dong Yan
[25] provided a simpler, more direct proof of this fact. Yan’s proof follows the idea of the standard
proof of the hard Lefschetz theorem.

Both the existence and the uniqueness of symplectic harmonic forms may be not expected to
hold in de Rham cohomology for closed almost Kähler manifolds. So Tseng and Yau thought that
the de Rham cohomology may be not the appropriate cohomology to consider symplectic Hodge
theory. They [24] introduced some new cohomology groups \( H^k_{d+d^h}(M) \) and \( H^{d+d^h}_{dR}(M) \) for a sym-
plectic manifold \( (M, \omega) \). These two cohomologies are similarly paired and share many analogous
properties with the pair Bott–Chern cohomology and Aeppli cohomology defined on complex
manifolds. Indeed, both can be shown to be finite dimensional on closed complex manifolds by
constructing self-adjoint fourth-order differential operators (cf. [17]). Similar to the construction
in [17], Tseng and Yau found out the associated Laplacian operators \( \Delta_{d+d^h} \) and \( \Delta_{dR+d^h} \) such that \( \ker \Delta_{d+d^h} = \ker D_{d+d^h}, \ker \Delta_{dR+d^h} = \ker D_{dR+d^h} \). If \( (M, \omega) \) is closed, Tseng and Yau have proven that both
\( H^k_{d+d^h}(M) \) and \( H^{d+d^h}_{dR}(M) \) are finite dimensional (see [24, Theorem 3.5, 3.16]). Then we can
define the \( k \)th symplectic Betti numbers

\[
\beta^{s,1}_k \triangleq \dim H^k_{d+d^h}(M), \quad \beta^{s,2}_k \triangleq \dim H^{d+d^h}_{dR}(M)
\]

and the symplectic Euler numbers

\[
\chi^{s,1}(M) \triangleq \sum_{k=0}^{2n} (-1)^k \beta^{s,1}_k, \quad \chi^{s,2}(M) \triangleq \sum_{k=0}^{2n} (-1)^k \beta^{s,2}_k.
\]

For an almost Kähler manifold, we denote the spaces of \( d + d^h \) harmonic \( k \)-forms and \( dd^h \) har-
monic \( k \)-forms by \( \mathcal{H}^k_{d+d^h}(M) \) and \( \mathcal{H}^{d+d^h}(M) \) that are the kernel spaces of \( \Delta_{d+d^h} \) and \( \Delta_{dR+d^h} \), respectively. If \( (M, g, J, \omega) \) is closed, Tseng and Yau gave the Hodge decompositions for \( \mathcal{H}^k_{d+d^h}(M) \) and
\( \mathcal{H}^{d+d^h}(M) \). Then they got \( H^k_{d+d^h}(M) \cong \mathcal{H}^k_{d+d^h} \) and \( \mathcal{H}^{d+d^h}(M) \cong H^k_{dR+d^h} \) on a closed symplectic
manifold. Here, our first main result is considered on the complete non-compact almost Kähler mani-
fold \( (M, g, J, \omega) \). In the following section, the notation \( L^2 \) on \( (M, g, J, \omega) \) is meant with respect
to the almost Kähler metric \( g(\cdot, \cdot) = \omega(\cdot, J\cdot) \), where \( J \) is an almost complex structure on \( M \) com-
patible with \( \omega \).

Suppose that \( (M, g, J, \omega) \) is a complete non-compact almost Kähler manifold of \( 2n \)-dimension.
In general, \( J \) is not integrable, hence \( \nabla J = 0, \nabla \omega = 0 \), where \( \nabla \) is the Levi–Civita connection
induced from the metric \( g \) (cf. [7]). An almost Kähler manifold \( (M, g, J, \omega) \) is of bounded
geometry if $(\nabla)^k J, (\nabla)^k \omega$ have uniformly point-wise bounded on $M$. There are many complete
non-compact almost Kähler manifolds with bounded geometry, for example, the universal covering $(\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})$ of a closed almost Kähler manifold $(M, g, J, \omega)$ whose $\pi_1(M)$ is infinite is a non-compact almost Kähler manifold with bounded geometry.

Notice that if $(M, g, J, \omega)$ is a non-compact almost Kähler manifold with bounded geometry, Hilbert spaces $L^2_t \Omega^k$ are the closure of $\Omega^k_t(M), l \in \mathbb{Z}^+$, where $\Omega^k_t(M)$ is the space of $C^\infty k$-forms with compact support in $M$ (cf. [3, Remark 2.7]).

Let $H$ be a Hilbert space and $T: \text{dom}(T) \to H$ be a (not necessarily bounded) linear operator
defined on a dense linear subspace $\text{dom}(T)$ which is called (initial) domain. We call $T$ closed if its
graph $\text{gr}(T) \triangleq \{(u, T(u)): u \in \text{dom}(T)\} \subset H \times H$ is closed. We say that $S: \text{dom}(S) \to H$ is an extension of $T$ and write $T \subset S$ if $\text{dom}(T) \subset \text{dom}(S)$ and $S(u) = T(u)$ holds for all $u \in \text{dom}(T)$. We write $T = S$ if $\text{dom}(T) = \text{dom}(S)$ and $S(u) = T(u)$ holds for all $u \in \text{dom}(T)$. We call $T$ closable if and only if $T$ has a closed extension. Since the intersection of an arbitrary family of closed sets is closed again, a closable unbounded densely defined operator $T$ has a unique minimal closure, also called minimal closed extension, that is, a closed operator $T_{\text{min}}: \text{dom}(T_{\text{min}}) \to H$ which $T \subset T_{\text{min}}$ such that $T_{\text{min}} \subset S$ holds for any closed extension $S$ of $T$. Explicitly $\text{dom}(T_{\text{min}})$ consists of elements $u \in H$ for which there exists a sequence $(u_n)_{n \geq 0}$ in $\text{dom}(T)$ and an element $v$ in $H$ satisfying $\lim_{n \to \infty} u_n = u$ and $\lim_{n \to \infty} T(u_n) = v$. Then $v$ is uniquely determined by this property and we put $T_{\text{min}}(u) = v$. Equivalently, $\text{dom}(T_{\text{min}})$ is the Hilbert space completion of $\text{dom}(T)$ with respect to the inner product

$$<u, v>_{gr} = <u, v>_{H} + <T(u), T(v)>_{H}. \tag{1.1}$$

If not stated otherwise we always use the minimal closed extension as the closed extension of a closable unbounded densely defined linear operator.

The adjoint of $T$ is the operator $T^*: \text{dom}(T^*) \to H$ whose domain consists of elements $v \in H$
for which there is an element $u$ in $H$ such that $<u', u>_{H} = <T(u'), v>_{H}$ holds for all $u' \in \text{dom}(T)$. Then $u$ is uniquely determined by this property and we put $T^*(v) = u$. Notice that $T^*$ may not have a dense domain in general. If $T$ is closable, then $T_{\text{min}} = T^*$ and $T_{\text{min}} = (T^*)^\text{*}$. We call $T$ symmetric if $T \subset T^*$ and self-adjoint if $T = T^*$. Any self-adjoint operator is necessarily closed and symmetric. A bounded operator $T: H \to H$ is always closed and is self-adjoint if and only if it is symmetric. We call $T$ essentially self-adjoint if $T_{\text{min}}$ is self-adjoint. The maximal closure $T_{\text{max}}$ of $T$ is defined by the adjoint of $(T^*)_{\text{min}}$. In fact, $\text{dom}(T_{\text{max}})$ is the space of all $u \in H$
such that $Tu \in H$ (as a distribution). For any closure $\tilde{T}$ of $T: \text{dom}(T) \to H$, we have $T_{\text{min}} \subset \tilde{T} \subset T_{\text{max}}$. Hence, if $T$ is essentially self-adjoint, then $T_{\text{min}} = T_{\text{max}}$. For more details, see [2, 15, 23].

**Theorem 1.3** If $(M, g, J, \omega)$ is a $2n$-dimensional complete non-compact almost Kähler manifold
with bounded geometry whose symplectic form $\omega$ is $d$ (sublinear), and $D_{d+d^*}, D_{dd^*}$ are essentially
self-adjoint elliptic operators on $M$, then their $L^2$ symplectic harmonic forms satisfy $L^2\mathcal{H}^{k}_{d+d^*} = 0$
and $L^2\mathcal{H}^{k}_{dd^*} = 0$, unless $k = n$.

In general, the symplectic Betti numbers $\beta^k, 0 \leq k \leq 2n$, are not topological invariants. If the closed symplectic manifold $(M, \omega)$ satisfies the hard Lefschetz property, then
\[ H^k_{d+d^\Lambda}(M) \cong H^k_{dR}(M) \cong H^{2n-k}_{dD}(M) \]

and \( \chi(M) = \chi^{s,1}(M) = \chi^{s,2}(M) \). Suppose that \((M, \omega)\) is a closed symplectic manifold and \((\tilde{M}, \tilde{\omega})\) is the universal covering space. We consider \(L^2\)-symplectic harmonic forms on \((\tilde{M}, \tilde{\omega})\) and define \(L^2\)-symplectic Euler characteristics \(L^2\chi^{s,1}(\tilde{M}), L^2\chi^{s,2}(\tilde{M})\) on it. Then we get the following theorem.

**Theorem 1.4** Let \((M, \omega)\) be a \(2n\)-dimensional closed symplectic manifold, \(\Pi: (\tilde{M}, \tilde{\omega}) \to (M, \omega)\) the universal covering map. If \((M, \omega)\) satisfies the hard Lefschetz property, then

\[ L^2\chi^{s,1}(\tilde{M}) = L^2\chi^{s,2}(\tilde{M}) = L^2\chi(\tilde{M}) = \chi(M) = \chi^{s,1}(M) = \chi^{s,2}(M). \]

At last, we want to consider Chern conjecture on a closed symplectic parabolic manifold. One of the powerful tools for Gromov achieving Chern conjecture on a Kähler manifold is that the Lefschetz operator commutes with the Hodge Laplacian operator \(\Delta_d = dd^* + d^*d\). But in general, the Lefschetz operator does not commute with \(\Delta_d\) on symplectic manifold. This makes us think that de Rham cohomology and its harmonic forms are inappropriate to be seen as a tool to solve problems on symplectic manifolds. Fortunately, by considering Tseng and Yau’s new symplectic cohomologies on symplectic parabolic manifold, we get some interesting results. At last, with the hard Lefschetz property which ensures that de Rham cohomology consists with the new symplectic cohomology, we can obtain the third main result.

**Theorem 1.5** If \((M, \omega)\) is a \(2n\)-dimensional closed symplectic parabolic manifold which satisfies the hard Lefschetz property, then the Euler number satisfies \((-1)^n \chi(M) \geq 0\).

**Remark 1.6** (1) Jianguo Cao and Frederico Xavier in [5] (see also Jost and Zuo’s work [16]) have proven that a bounded closed \(k\)-form, \(k \geq 1\), is \(d\) (sublinear) on a complete simply connected manifold of non-positive sectional curvature. By their Lemma 3 in [5], we can also get: Let \(M\) be a closed \(2n\)-Riemannian manifold of non-positive sectional curvature. If \(M\) is homotopy equivalent with a closed symplectic manifold which satisfies the hard Lefschetz property, then the Euler number of \(M\) satisfies the inequality \((-1)^n \chi(M) \geq 0\).

(2) It is well known that a closed Kähler manifold \(M\) such that \(\eta_1(M)\) is word-hyperbolic in the sense of [11] and \(\eta_2(M) = 0\) is a Kähler hyperbolic manifold. Hence, we conjecture that: Let \((M, \omega)\) be a \(2n\)-dimensional closed symplectic manifold, if \(\eta_1(M)\) is infinite and \(\eta_2(M) = 0\), then \(\omega\) is \(d\) (bounded).

Since the hard Lefschetz property in this article is a technical condition, we have the following question:

**Question 1.7** If we drop the condition that \((M, \omega)\) satisfies the hard Lefschetz property in Theorem 1.4 and Theorem 1.5, could the same conclusion hold?
2. $L^2$ symplectic Hodge theory

Let us recall some definitions and some results of Hodge theory. Let $M$ be a closed oriented Riemannian manifold with metric $g$. The Hodge star operator $*_{g}: \Omega^k(M) \to \Omega^{m-k}(M)$ is a linear map which satisfies $\alpha \wedge *_g \beta = (\alpha, \beta)_g d\text{vol}_g$ for all $\alpha, \beta \in \Omega^k(M)$. Here $\Omega^k(M)$ is the space of the smooth $k$-forms on $M$. We denote the adjoint operator of the differential operator $d$ by $d^*$ associated to $g$. By a direct calculation, we will find that $d^* = (-1)^{m+k+1} *_g d*_{g}$ on $\Omega^k(M)$. A form $\alpha$ is called harmonic if it is both $d$-closed and $d^*$-closed. The Laplacian operator is given by $\Delta_g = dd^* + d^*d: \Omega^k(M) \to \Omega^k(M)$, then $\alpha$ is harmonic if and only if $\Delta_g \alpha = 0$. By the theory of elliptic operator we conclude that the kernel of $\Delta_g$ is finite dimensional. And the Hodge decomposition tells us every cohomology class has a unique harmonic representative.

Let $(M, \omega)$ be a closed symplectic $2n$-manifold. Symplectic Hodge theory was introduced by Ehresmann and Libermann [10] and was rediscovered by Brylinski [4]. They defined the symplectic star operator analogously to the Hodge star operator, but with respect to the symplectic form $\omega$. The symplectic star operator $*_\omega$ acts on a differential $k$-form $\alpha$ by

$$\alpha \wedge *_{\omega} \alpha' = (\omega^{-1})^k (\alpha, \alpha') d\text{vol}$$

$$= \frac{1}{k!} (\omega^{-1})^{i_1 j_1} \ldots (\omega^{-1})^{i_k j_k} \alpha_{i_1 \ldots i_k} \alpha'_{j_1 \ldots j_k} \frac{\omega^n}{n!}$$

with repeated indices summed over. Direct calculation shows

$$\alpha \wedge *_{\omega} \beta = (-1)^k \beta \wedge *_{\omega} \alpha,$$  

(2.1)

where $\alpha$ and $\beta$ are $k$-forms. The adjoint of the standard exterior derivative with respect to $\omega$ takes the form (cf. [18])

$$d^\omega = (-1)^{k+1} *_{\omega} d_{\omega}.$$  

Fix an almost Kähler structure $(g, J, \omega)$ on $(M, \omega)$. See some standard Hodge adjoint of the differential operators. Denote by

$$d^* = -*_g d*_{g},$$

$$d^\omega * = *_g d^\omega *_{g},$$

and

$$(dd^\omega)^* = (-1)^{k+1} *_{g} dd^\omega *_{g}$$

act on $k$-forms. By using the properties $d^2 = (d^\omega)^2 = 0$ and the anti-commutively $dd^\omega = -d^\omega d$, we will find that any form that is $dd^\omega$-exact is also $d$- and $d^\omega$-closed. This gives a differential complex.
Tseng and Yau [24] considered the symplectic cohomology group $H^k_{d+d^\Lambda}(M)$ which are just the symplectic version of well-known cohomologies in complex geometry already studied by Kodaira and Spencer [17], for example. With complex (2.2), they define

$$H^k_{d+d^\Lambda}(M) = \frac{\ker (d + d^\Lambda) \cap \Omega^k(M)}{\im d \cap \im d^\Lambda \cap \Omega^k(M)}.$$  \hspace{1cm} (2.3)

From the differential complex, the Laplacian operator associated with the cohomology is

$$\Delta_{d+d^\Lambda} = dd^\Lambda(dd^\Lambda)^* + \lambda(d^*d + d^\Lambda*d^\Lambda),$$

where we have inserted an undetermined real constant $\lambda > 0$ that gives the relative weight between the terms. The Laplacian is a fourth-order self-adjoint differential operator, but not elliptic. However, Tseng and Yau introduce a related fourth-order elliptic operator (cf. [17, 24])

$$D_{d+d^\Lambda} = dd^\Lambda(dd^\Lambda)^* + (dd^\Lambda)^*(dd^\Lambda) + d^*d^\Lambda d^\Lambda^*d + d^\Lambda^*dd^\Lambda d^\Lambda + \lambda(d^*d + d^\Lambda^*d^\Lambda).$$  \hspace{1cm} (2.4)

The solution space of $D_{d+d^\Lambda} \alpha = 0$ is identical to that of $\Delta_{d+d^\Lambda} \alpha = 0$. A differential form $\alpha$ is called $d + d^\Lambda$-harmonic if $\Delta_{d+d^\Lambda} \alpha = 0$, or equivalently,

$$d\alpha = d^\Lambda \alpha = 0 \text{ and } (dd^\Lambda)^* \alpha = 0.$$  \hspace{1cm} (2.5)

Denote the space of $d + d^\Lambda$-harmonic $k$-forms by $\mathcal{H}^k_{d+d^\Lambda}(M)$. Tseng and Yau [24] proved that the space of $d + d^\Lambda$-harmonic $k$-forms $\mathcal{H}^k_{d+d^\Lambda}(M)$ are finite dimensional and isomorphic to $H^k_{d+d^\Lambda}(M)$.

**Proposition 2.1 ([24, Theorem 3.5])** Let $(M, g, J, \omega)$ be a closed almost Kähler manifold. Then:

1. $\dim \mathcal{H}^k_{d+d^\Lambda}(M) < \infty$.
2. There is an orthogonal decomposition

$$\Omega^k = \mathcal{H}^k_{d+d^\Lambda} \oplus dd^\Lambda \Omega^k \oplus (d^*\Omega^{k+1} + d^\Lambda^*\Omega^{k-1}).$$

3. There is a canonical isomorphism: $\mathcal{H}^k_{d+d^\Lambda}(M) \cong H^k_{d+d^\Lambda}(M)$.

Interestingly, simply reversing the arrows of the complex (2.2) leads to another symplectic cohomology group $H^k_{dd^\Lambda}(M)$ (cf. [24]),

$$H^k_{dd^\Lambda}(M) = \frac{\ker (dd^\Lambda) \cap \Omega^k(M)}{(\im d + \im d^\Lambda) \cap \Omega^k(M)}.$$  \hspace{1cm} (2.6)

The Laplacian operator associated with the cohomology is

$$\Delta_{dd^\Lambda} = (dd^\Lambda)^*dd^\Lambda + \lambda(dd^\Lambda + d^\Lambda^*d^\Lambda).$$
The Laplacian is also not elliptic. A differential form $\alpha$ is called $dd^\Lambda$-harmonic if $\Delta_{dd^\Lambda}\alpha = 0$, or equivalently,

$$d^k \alpha = 0, \quad \delta \alpha = 0 \quad \text{and} \quad dd^\Lambda \alpha = 0.$$  

Denote the space of $dd^\Lambda$-harmonic $k$-forms by $\mathcal{H}^k_{dd^\Lambda}(M)$. Then Tseng and Yau introduce a fourth-order elliptic operator

$$D_{dd^\Lambda} = (dd^\Lambda)^k \delta + (dd^\Lambda)(dd^\Lambda)^* + dd^\Lambda d^\Lambda d^\Lambda + \lambda (dd^\Lambda + d^\Lambda d^\Lambda^*),$$

which satisfies $\ker D_{dd^\Lambda} = \ker \Delta_{dd^\Lambda} = \mathcal{H}^k_{dd^\Lambda}(M)$. Tseng and Yau [24] proved that the space of $dd^\Lambda$-harmonic $k$-forms $\mathcal{H}^k_{dd^\Lambda}(M)$ are finite dimensional and isomorphic to $H^k_{dd^\Lambda}(M)$.

**Proposition 2.2 ([24, Theorem 3.16])** Let $(M, g, J, \omega)$ be a closed almost Kähler manifold. Then:

1. $\dim \mathcal{H}^k_{dd^\Lambda}(M) < \infty$.
2. There is an orthogonal decomposition

$$\Omega^k = \mathcal{H}^k_{dd^\Lambda} \oplus (dd^\Lambda)^k \Omega^k \oplus (d\Omega^{k-1} + d^\Lambda \Omega^{k+1}).$$

3. There is a canonical isomorphism: $\mathcal{H}^k_{dd^\Lambda}(M) \cong H^k_{dd^\Lambda}(M)$.

Using the symplectic form $\omega = \sum_{ij} \omega_{ij} dx^i \wedge dx^j$, the Lefschetz operator $L: \Omega^k(M) \to \Omega^{k+2}(M)$ and the dual Lefschetz operator $\Lambda: \Omega^k(M) \to \Omega^{k-2}(M)$ are defined by

$$L: \quad L(\alpha) = \omega \wedge \alpha,$$

$$\Lambda: \quad \Lambda(\alpha) = \frac{1}{2} (\omega^{-1})^j i \partial_j i \partial_j \alpha,$$

where $i$ denotes the interior product.

**Proposition 2.3 ([24, Corollaries 3.8 and 3.19])** On a closed almost Kähler manifold $(M, g, J, \omega)$ of dimension $2n$, the Lefschetz operator defines isomorphisms

$$L^{n-k}: \mathcal{H}^k_{dd^\Lambda} \cong \mathcal{H}^{2n-k}_{dd^\Lambda}$$

and

$$L^{n-k}: \mathcal{H}^k_{dd^\Lambda} \cong \mathcal{H}^{2n-k}_{dd^\Lambda}$$

for $k \leq n$.

The compactness becomes important when one integrates by parts. For example, by applying the Stokes formula.
we can derive the desired relation \( \langle d\varphi, \psi \rangle_g = \langle \varphi, d^*\psi \rangle_g \). If \( M \) is non-compact, then (2.7) is not true generally. Fortunately, Gromov has proven that (2.7) remains true for all \( L^1 \)-forms on a complete manifold.

**Lemma 2.4** ([12, Lemma 1.1.A]) Suppose \( M \) is a complete \( m \)-manifold. Let \( \alpha \) be an \( L^1 \)-form on \( M \) of degree \( m-1 \) such that the differential \( d\alpha \) is also \( L^1 \). Then

\[
\int_M d\alpha = 0.
\]

**Remark 2.5** The above relation for \( C^\infty \) forms easily yields the statement for non-smooth \( \eta \) where \( d\eta \) is understood as a distribution (cf. [12, 15]).

Let \( \Delta_d = dd^* + d^*d \) be the de Rham Laplacian. Denote the space of \( \Delta_d \)-harmonic \( k \)-forms by \( \mathcal{H}^k_d(M) \). In [12], Gromov has obtained

\[
L^2\Omega^k = L^2\mathcal{H}^k_d \oplus [d(L^2\Omega^{k-1}) \oplus d^*(L^2\Omega^{k+1})].
\]  

Along the lines used by Gromov (see also [15]), we want to obtain another two decompositions.

**Lemma 2.6** Suppose that \((M, g, J, \omega)\) is a complete non-compact almost Kähler manifold with bounded geometry. Then we can get

1. \( \langle d\alpha, \beta \rangle_g = \langle \alpha, d^*\beta \rangle_g \), for \( \alpha, \beta, d\alpha, d^*\beta \in L^2\Omega^*(M) \),
2. \( \langle d^k\alpha, \beta \rangle_g = \langle \alpha, d^{k+1}\beta \rangle_g \), for \( \alpha, \beta, d^k\alpha, d^{k+1}\beta \in L^2\Omega^*(M) \),
3. \( \langle dd^k\alpha, \beta \rangle_g = \langle \alpha, (dd^k)\beta \rangle_g \), for \( \alpha, \beta, dd^k\alpha, dd^k\beta \in L^2\Omega^*(M) \).

**Proof.** (1) By observing the formula

\[
d\alpha \wedge *\beta - \alpha \wedge *d^*\beta = \pm d(\alpha \wedge *\beta),
\]

and note that both \( \alpha \wedge *\beta \) and \( d(\alpha \wedge *\beta) \) are \( L^1 \)-forms, we can easily get

\[
\langle d\alpha, \beta \rangle_g = \langle \alpha, d^*\beta \rangle_g
\]

by applying the Lemma 2.4.

(2) Suppose that \( \alpha \) is a \( k \)-form and \( \beta \) is a \( k-1 \)-form.

\[
(d^k\alpha, \beta)_g dvol_g = (-1)^{k+1} *_g d_* \alpha \wedge *_g \beta
= (-1)^{(k-1)^2} d_* \alpha \wedge *_g \beta
= (-1)^{(k-1)^2} [d (*_g \alpha \wedge *_g \beta) + (-1)^{k-1} *_g \alpha \wedge d_* *_g \beta].
\]

By the assumption of conditions, we find that both \( *_g \alpha \wedge *_g \beta \) and \( d (*_g \alpha \wedge *_g \beta) \) are \( L^1 \)-forms. Taking integral of both sides of the above equation, we obtain
\[<d^{k}\alpha, \beta>_g = \int_M (-1)^{(k-1)^2} d^*(\star_{s} \alpha \wedge \star_{s} \star_{g} \beta) + \int_M (-1)^{k^2-k} \star_{s} \alpha \wedge d^* \star_{s} \star_{g} \beta \]
\[= (-1)^{k^2-k} \int_M \star_{s} \alpha \wedge d^* \star_{s} \star_{g} \beta \]
\[= \int_M \alpha \wedge \star_{s} d^* \star_{s} \star_{g} \beta \]
\[= \int_M \alpha \wedge (-1)^{k} d^{\Lambda} \star_{g} \beta \]
\[= <\alpha, (-1)^{k^2+k} \star_{g} \star_{g} \beta>_g \]
\[= <\alpha, d^{\Lambda} \star_{g} \beta>_g. \]

(3) The third conclusion is an obvious result following (1) and (2). \[\square\]

With Lemma 2.6, we can obtain another very useful lemma. Before giving the useful lemma, we claim that there exists a family of cutoff functions \(a_\varepsilon\) such that

\[a_\varepsilon \geq 0, \quad \|\nabla^1_g a_\varepsilon\|_g \leq \varepsilon (a_\varepsilon)^{\frac{1}{2}}, \quad \| (\nabla^1_g)^2 a_\varepsilon\|_g \leq \varepsilon^2\]

and the subsets \(a_\varepsilon^{-1}(1) \subset M\) exhaust \(M\) as \(\varepsilon \to 0\) on complete non-compact manifold \(M\), where \(\nabla^1_g\) is the second canonical connection with respect to the metric \(g\) and almost complex structure \(J\) on \(M\) (cf. [7]), that is, \(\nabla^1_g g = 0 = \nabla^1_g J\), hence \(\nabla^1_g \omega = \nabla^1_g (J \cdot \cdot) = 0\). Here, we only give the case on \(\mathbb{R}\). Let

\[f(x) = \begin{cases} 
\exp\left(\frac{-1}{x}\right), & x > 0 \\
0, & x \leq 0.
\end{cases} \quad (2.9)
\]

Define

\[\psi(x) = \frac{f(x)}{f(x) + f(1-x)}.\]

Note that

- \(0 \leq \psi(x) \leq 1\) for \(0 < x < 1\),
- if \(x \leq 0\), \(\psi(x) = 0\) and if \(x \geq 1\), \(\psi(x) = 1\),
- \(\psi, \psi'\) and \(\psi''\) are all bounded.

Finally, for \(x \geq 0\), let

\[a_\varepsilon = \psi^2(2 - \varepsilon x).\]

Clearly, \(a_\varepsilon(x) = 1\) on \([0, \frac{1}{\varepsilon}]\) and \(a_\varepsilon(x) = 0\) on \([\frac{2}{\varepsilon}, \infty)\). For \(\frac{1}{\varepsilon} < x < \frac{2}{\varepsilon}\), we have

\[a'_\varepsilon(x) = -2\varepsilon \psi(2 - \varepsilon x) \psi'(2 - \varepsilon x).\]
Since $\psi'$ is bounded, we see that $-\varepsilon C_1 \sqrt{\alpha_\varepsilon} \leq \alpha_\varepsilon' (x) \leq 0$ for some constant $C_1$. Moreover,

$$a_\varepsilon''(x) = 2\varepsilon^2 \psi''(2 - \varepsilon x) + 2\varepsilon^2 \psi(2 - \varepsilon x) \psi''(2 - \varepsilon x).$$

Since $\psi, \psi'$ and $\psi''$ are bounded, we have $|a_\varepsilon''(x)| \leq C_2 \varepsilon^2$ for some constant $C_2$.

Let $(M, g, J, \omega)$ be a $2n$-dimensional complete, non-compact almost Kähler manifold with bounded geometry. Then $L^2_\omega \Omega^k(M), 0 \leq k \leq 2n$, is completion of $\Omega^k_c(M)$ for any non-negative integer $l$. $\Delta_{d+d^\wedge}$ and $D_{d+d^\wedge}$ are two formal self-adjoint fourth-order differential operators, that is,

$$<\Delta_{d+d^\wedge} u, v> = <u, \Delta_{d+d^\wedge} v>, \quad <D_{d+d^\wedge} u, v> = <u, D_{d+d^\wedge} v>, \quad (2.10)$$

for $u, v \in \Omega^k_c(M)$. If $D_{d+d^\wedge}$ is essentially self-adjoint elliptic operator, then for $u \in L^2_\omega \Omega^k(M)$, $D_{d+d^\wedge} u = 0$ (as a distribution) implies that $u \in L^2_\omega \Omega^k(M)$ (cf. [2, 15, 23]). In fact, $D_{d+d^\wedge} : \Omega^k_c(M) \rightarrow \Omega^k_c(M)$ is an elliptic operator of fourth-order, $0 \leq k \leq 2n$, that is,

$$<D_{d+d^\wedge} u, v> = <u, D_{d+d^\wedge} v>, \quad u, v \in \Omega^k_c(M). \quad (2.11)$$

When fourth-order elliptic operator $D_{d+d^\wedge}$ is essentially self-adjoint in $L^2 \Omega^k(M)$, then its closure is a self-adjoint operator in $L^2 \Omega^k(M)$ with the domain

$$\text{dom}(D_{d+d^\wedge,\text{min}}) = \text{dom}(D_{d+d^\wedge,\text{max}}) = L^2_\omega \Omega^k(M),$$

hence, for any $u \in L^2_\omega \Omega^k(M)$, $D_{d+d^\wedge} u = 0$ (in the sense of distribution) implies that $u \in L^2_\omega \Omega^k(M)$ (cf. M. Shubin [23, Theorem on page 18-6] or W. Lück [15, Lemma 1.75]). Recall that

$$\Delta_{d+d^\wedge} = dd^\Lambda(dd^\Lambda)^* + \lambda (d^*d + d^\Lambda^*d^\Lambda), \quad \lambda > 0$$

and

$$D_{d+d^\wedge} = dd^\Lambda(dd^\Lambda)^* + (dd^\Lambda)^*(dd^\Lambda) + d^*d^\Lambda d^\Lambda^*d^\Lambda + d^\Lambda^*dd^\Lambda + d^\Lambda d^\Lambda^*d^\Lambda, \quad \lambda > 0,$$

it is easy to get that $D_{d+d^\wedge}$ and $\Delta_{d+d^\wedge}$ have the same kernel (cf. [2, 23]). For $D_{dd^\wedge}$ and $\Delta_{dd^\wedge}$, we have the similar results.

With the Gaffney cutoff trick, we get the following lemma.

**Lemma 2.7** Suppose that $(M, g, J, \omega)$ is a $2n$-dimensional complete, non-compact almost Kähler manifold with bounded geometry. If $D_{d+d^\wedge}$ is of essential self-adjointness, and an $L^2$ $k$-form $\alpha$, $0 \leq k \leq 2n$, is $d + d^\Lambda$-harmonic, that is, $\Delta_{d+d^\wedge} \alpha = 0$, then $\alpha$ satisfies $d \alpha = d^\Lambda \alpha = 0$ and $(dd^\Lambda)^* \alpha = 0$.

To prove the above lemma, we need point-wise estimate for $d \alpha$, $\alpha \in \Omega^*(M)$. Suppose that $(M, g, J, \omega)$ is a $2n$-dimensional complete, non-compact almost Kähler manifold with bounded geometry. For $p \in M$, choose a local unitary frame $\{e_1, \ldots, e_n\}$ for $T^{1,0}M$ near $p$ with respect to the almost Hermitian inner product induced from $g$, and let $\{\theta^1, \ldots, \theta^n\}$ be a dual coframe. Let $\nabla^i_g$ be
the second canonical connection with respect to the metric $g$. Locally there exists a matrix of complex valued $1$-forms $\{\theta^i_j\}$, called the connection $1$-forms, such that

$$\nabla_g e_i = \theta^i_j e_j, \quad \theta^i_j(p) = 0.$$ 

Hence,

$$\nabla^1_g e_i|_p = 0, \quad \nabla^1_g \theta^i_j|_p = 0, \quad 1 \leq i \leq n. \tag{2.12}$$

It is easy to see that $\{\theta^i_j\}$ satisfies the skew-Hermitian property $\theta^i_j + \overline{\theta^i_j} = 0$. Now define the torsion $\Theta = (\Theta^1, \ldots, \Theta^n)$ of $\nabla_g^1$ by

$$d\theta^i = -\theta^i_j \wedge \theta^j + \Theta^i, \quad \tag{2.13}$$

for $i = 1, \ldots, n$. Notice that the $\Theta^i$ are $2$-forms. Equation (2.13) is known as the first structure equation. Define the curvature $\Psi = \{\Psi^i_j\}$ of $\nabla_g^1$ by

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Psi^i_j. \tag{2.14}$$

Note that $\{\Psi^i_j\}$ is a skew-Hermitian matrix of $2$-forms. Equation (2.14) is known as the second structure equation. Since $d\omega = 0$,

$$\Theta^i = N^i_{lm} \overline{\theta^l} \wedge \overline{\theta^m}, \tag{2.15}$$

where $N^i_{lm}$ is the Nijenhuis tensor which is defined as

$$N^i_j(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y], \quad X, Y \in TM.$$

Now, let us estimate $d\alpha$. For simplicity, $\alpha \in \Omega^{1,1}(M)$ is locally written as $\alpha = \alpha_{ij} \theta^i \wedge \theta^j$.

$$d\alpha = d(\alpha_{ij} \theta^i \wedge \theta^j)$$

$$= (\nabla^1_g \alpha_{ij}) \wedge \theta^i \wedge \theta^j + \alpha_{ij} (d\theta^i) \wedge \theta^j - \alpha_{ij} \theta^i \wedge (d\theta^j)$$

$$= (\nabla^1_g \alpha_{ij}) \wedge \theta^i \wedge \theta^j + \alpha_{ij} N^j_{lm} \overline{\theta^l} \wedge \overline{\theta^m} - \alpha_{ij} \overline{N^j_{lm}} \theta^l \wedge \theta^m.$$

Since $(M, g, \omega)$ is of bounded geometry, by the definition of $N^i_{lm}$ (cf. [20]) (notice that $N^i_j(X, Y) = 2J(\nabla_X J)Y - 2J(\nabla_Y J)X, \quad X, Y \in TM$, where $\nabla$ is the Levi–Civita connection induced from the metric $g$), $N^i_{lm}$ is uniformly bounded on $M$. Hence,

$$\|d\alpha\|_g \leq C(J, \omega) (\|\nabla^1_g \alpha\|_g + \|\alpha\|_g), \tag{2.16}$$

where $\|\alpha\|^2_g = (\alpha, \alpha)_g$.

**Proof of Lemma 2.7.** We want again to justify the integral identity
<\Delta_{d+d^\perp} \alpha, \alpha>_g = \langle (dd^\perp)^* \alpha, (dd^\perp)^* \alpha>_g + \lambda <d\alpha, d\alpha>_g + \lambda <d^\perp \alpha, d^\perp \alpha>_g.

If $d\alpha$, $d^* \alpha$, $d^\perp \alpha$, $d^\perp^* \alpha$, $dd^\perp \alpha$ and $(dd^\perp)^* \alpha$ are all $L^2$ forms, then the above equation follows by Lemma 2.6.

To handle the general case, we will use the Gaffney cutoff trick. Let $\alpha \in \ker \Delta_{d+d^\perp} = \ker D_{d+d^\perp}$. We cutoff $\alpha$ and obtain by a simple computation

$$I_1(\varepsilon) = \int_M a_\varepsilon (\lambda \|d\alpha\|_g^2 + \lambda \|d^\perp \alpha\|_g^2 + \|(dd^\perp)^* \alpha\|_g^2)$$

and

$$|I_2(\varepsilon)| \leq C_1 \int_M \|\nabla_g a_\varepsilon\|_g \cdot \|\alpha\|_g \cdot (\|(dd^\perp)^* \alpha\|_g + \lambda \|d\alpha\|_g + \lambda \|d^\perp \alpha\|_g)$$

$$+ C_2 \int_M \|(dd^\perp)^* \alpha\|_g \cdot \|\nabla_g a_\varepsilon\|_g \cdot \|\nabla^1_g a_\varepsilon\|_g$$

$$+ C_3 \int_M \|(dd^\perp)^* \alpha\|_g \cdot \|\nabla^1_g a_\varepsilon\|_g \cdot \|\alpha\|_g,$$

where $\|d\alpha\|_g^2 = (d\alpha, d\alpha)_g$ and $C_1, C_2, C_3$ are some positive constants. Indeed, since on $k$-forms, $d^\perp = (-1)^{k+1} * d * g$ and $(dd^\perp)^* = (-1)^{k+1} * g d^\perp * g$, we have

$$<\Delta_{d+d^\perp} \alpha, a_\varepsilon \alpha>_g = <(dd^\perp)^* \alpha, (dd^\perp)^* a_\varepsilon \alpha>_g$$

$$+ \lambda <d\alpha, d a_\varepsilon \alpha>_g + \lambda <d^\perp \alpha, d^\perp a_\varepsilon \alpha>_g$$

$$= <(dd^\perp)^* \alpha, (1)^{k+1} * g d^\perp * g a_\varepsilon \alpha>_g$$

$$+ \lambda <d^\perp \alpha, (1)^{k+1} * g d * g a_\varepsilon \alpha>_g$$

$$+ \lambda <d\alpha, d a_\varepsilon \wedge \alpha>_g + \lambda <d\alpha, a_\varepsilon d\alpha>_g$$

$$= <(dd^\perp)^* \alpha, * g d * g a_\varepsilon \alpha>_g$$

$$+ \lambda <d^\perp \alpha, (1)^{k+1} * g d * g a_\varepsilon \wedge * g \alpha>_g$$

$$+ \lambda <d\alpha, d a_\varepsilon \wedge \alpha>_g + \lambda <d\alpha, a_\varepsilon d\alpha>_g$$

$$= <(dd^\perp)^* \alpha, a_\varepsilon * g d * g a_\varepsilon \alpha>_g$$

$$+ \lambda <d^\perp \alpha, (1)^{k+1} * g d * g a_\varepsilon \wedge * g \alpha>_g$$

$$+ \lambda <d\alpha, d a_\varepsilon \wedge \alpha>_g + \lambda <d\alpha, a_\varepsilon d\alpha>_g.$$
\[ I_2(\varepsilon) = + (dd^\Lambda)^* \alpha, *_g d \varepsilon \wedge *_s d *_g \alpha \gg_g \\
+ (dd^\Lambda)^* \alpha, *_g d *_s d \varepsilon \wedge *_s *_g \alpha \gg_g \\
+ \lambda < d^\Lambda \alpha, (-1)^{k+1} *_s d \varepsilon \wedge *_s \alpha \gg_g \\
+ \lambda < d \alpha, d \varepsilon \wedge \alpha \gg_g. \]

Then

\[ I_1(\varepsilon) = (dd^\Lambda)^* \alpha, a_\varepsilon (dd^\Lambda)^* \alpha \gg_g \\
+ \lambda < d \alpha, a_\varepsilon d \alpha \gg_g + \lambda < d^\Lambda \alpha, a_\varepsilon d^\Lambda \alpha \gg_g. \]

Since \( \nabla^1_g J = 0, \nabla^1_g g = 0 \) and \( \nabla^1_g \omega = 0 \), then \( \nabla^1_g *_g = 0 \) and \( \nabla^1_g *_s = 0 \). By (2.16), we have

\[ \| *_g d \varepsilon \wedge *_s d *_g \alpha \|_g \leq C \{ \| a'_\varepsilon \nabla^1_g (*_s *_g \alpha) \|_g + \| a'_\varepsilon \alpha \|_g \} \]

\[ \leq C \{ \| a'_\varepsilon \nabla^1_g \alpha \|_g + \| a'_\varepsilon \alpha \|_g \}, \]

\[ \| *_g d *_s d \varepsilon \wedge *_s *_g \alpha \|_g \leq C \{ \| \nabla^1_g (*_s d \varepsilon \wedge *_s *_g \alpha) \|_g + \| *_s d \varepsilon \wedge *_s *_g \alpha \|_g \} \]

\[ \leq C \{ \| a'_\varepsilon \nabla^1_g \alpha \|_g + \| a''_\varepsilon \alpha \|_g + \| a'_\varepsilon \alpha \|_g \}. \]

Therefore,

\[ |I_2(\varepsilon)| \leq C_2 \int_M \| (dd^\Lambda)^* \alpha \|_g \| \nabla^1_g a_\varepsilon \|_g \| \nabla^1_g \alpha \|_g \\
+ C_3 \int_M \| (dd^\Lambda)^* \alpha \|_g \| (\nabla^1_g)^2 a_\varepsilon \|_g \| \alpha \|_g \\
+ C_4 \int_M \| (dd^\Lambda)^* \alpha \|_g \| \nabla^1_g a_\varepsilon \|_g \| \alpha \|_g \\
+ C_5 \int_M (\lambda \| d \alpha \|_g + \lambda \| d^\Lambda \alpha \|_g) \| \nabla^1_g a_\varepsilon \|_g \| \alpha \|_g, \]

where \( C_4, C_5 \) are some positive constants and \( C_1 = \max \{ C_4, C_5 \} \). Without loss of generality, we assume \( C_1 = C_2 = C_3 = 1 \). Choose cutoff functions \( a_\varepsilon \), such that

\[ a_\varepsilon \geq 0, \quad \| \nabla^1_g a_\varepsilon \|_g \leq \varepsilon (a_\varepsilon)^{\frac{1}{2}}, \quad \| (\nabla^1_g)^2 a_\varepsilon \|_g \leq \varepsilon^2 \]

and the subsets \( a_\varepsilon^{-1}(1) \subset M \) exhaust \( M \) as \( \varepsilon \to 0 \). Then
\[ |I_2(\varepsilon)| \leq \varepsilon \int_M (a_\varepsilon)^{\frac{1}{2}} \cdot \|\alpha\|_g \cdot \left( \|\langle (dd^\Lambda)^* \alpha \rangle\|_g + \lambda \|d\alpha\|_g + \lambda \|d^\Lambda \alpha\|_g \right) \\
+ \varepsilon \int_M (a_\varepsilon)^{\frac{1}{2}} \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g + \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g \\
\leq \varepsilon \|\alpha\|_{L^2} \left( \int_M a_\varepsilon \|\langle (dd^\Lambda)^* \alpha \rangle\|_g^2 \right)^{\frac{1}{2}} + \left( \int_M a_\varepsilon \lambda \|d\alpha\|_g^2 \right)^{\frac{1}{2}} + \left( \int_M a_\varepsilon \lambda \|d^\Lambda \alpha\|_g^2 \right)^{\frac{1}{2}} \\
+ \varepsilon \int_M (a_\varepsilon)^{\frac{1}{2}} \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g + \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g \\
\leq 2 \varepsilon \|\alpha\|_{L^2} \left( \int_M a_\varepsilon \|\langle (dd^\Lambda)^* \alpha \rangle\|_g^2 \right)^{\frac{1}{2}} + \int_M a_\varepsilon \lambda \|d\alpha\|_g^2 + \int_M a_\varepsilon \lambda \|d^\Lambda \alpha\|_g^2 \\
+ \varepsilon \int_M (a_\varepsilon)^{\frac{1}{2}} \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g + \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g \\
= 2 \varepsilon \|\alpha\|_{L^2} \cdot I_1(\varepsilon)^{\frac{1}{2}} + \varepsilon \int_M (a_\varepsilon)^{\frac{1}{2}} \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g \\
+ \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g \\
\leq 4 \varepsilon^2 \|\alpha\|_{L^2}^2 + \frac{1}{4} I_1(\varepsilon) + \varepsilon \int_M (a_\varepsilon)^{\frac{1}{2}} \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g \\
+ \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g \\
\leq 4 \varepsilon^2 \|\alpha\|_{L^2}^2 + \frac{1}{4} I_1(\varepsilon) + \varepsilon \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g \\
+ \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g. \\
\]

Since \( I_1(\varepsilon) = |I_2(\varepsilon)| \), note that \( \langle (dd^\Lambda)^* \alpha \rangle \) and \( \nabla^I_\Lambda \alpha \) are in \( L^2 \)-space since \( D_{d+d^\Lambda} \) is of essential self-adjointness (hence, \( \text{dom} (D_{d+d^\Lambda}, \text{min}) = \text{dom} (D_{d+d^\Lambda}, \text{max}) \), \( \ker \Delta_{d+d^\Lambda} = \ker D_{d+d^\Lambda} \), and \( D_{d+d^\Lambda} \alpha = 0 \) in the sense of distribution implies that \( \alpha \in L^2_4 \Omega^4 \)), we get

\[ I_1(\varepsilon) \leq \frac{16}{3} \varepsilon^2 \|\alpha\|_{L^2}^2 + \frac{4}{3} \varepsilon \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\nabla^I_\Lambda \alpha\|_g + \frac{4}{3} \varepsilon^2 \int_M \|\langle (dd^\Lambda)^* \alpha \rangle\|_g \cdot \|\alpha\|_g, \]

and hence \( I_1(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \). This completes the proof of Lemma 2.7. \( \square \)

Let \( (M, g, J, \omega) \) be a \( 2n \)-dimensional complete non-compact almost Kähler manifold with bounded geometry. It is same to the classical de Rham Laplacian operator \( \Delta_d \), we can define \( L^2 \)-symplectic cohomology groups as follows:

\[ L^2 H^k_{d+d^\Lambda}(M) = \frac{\ker (d + d^\Lambda) \cap L^2 \Omega^k}{\ker (dd^\Lambda) \cap L^2 \Omega^k}, \quad (2.17) \]

\[ L^2 H^k_{d^\Lambda}(M) = \frac{\ker (dd^\Lambda) \cap L^2 \Omega^k}{\ker (d + d^\Lambda) \cap L^2 \Omega^k}. \quad (2.18) \]
Since $D_{d+d^\perp}$ is fourth-order elliptic operator, if $D_{d+d^\perp}$ is of essential self-adjointness, then $\Delta_{d+d^\perp}$ and $D_{d+d^\perp}$ have the same kernel. For $\Delta_{d+d^\perp}$ and $D_{d+d^\perp}$, we have the similar results.

With Lemma 2.6 and Lemma 2.7 one concludes, as in the closed manifolds, that the $L^2(T^k(M))$ of exterior $k$-forms on a complete manifold $M$ with bounded geometry admits the Hodge decomposition (cf. [2, 15, 23]).

**Proposition 2.8** Suppose $(M, g, J, \omega)$ is a complete, non-compact almost Kähler manifold with bounded geometry, and $D_{d+d^\perp}$, $D_{d+d^\perp}$ are of essential self-adjointness. Then

$$L^2(T^k) = L^2(H^k_{d+d^\perp} \oplus d\bar{d}(L^2(T^k)) \oplus [d^*(L^2(T^{k+1})) + d\bar{d}^*(L^2(T^{k-1}))],$$

where $\bar{d}(\cdots)$ is the closure in $L^2(T^k)$ of the intersection of $L^2(T^k)$ with the image of $d$. Similarly, we can get

$$L^2(T^k) = L^2(H^k_{d+d^\perp} \oplus (\bar{d}d^\perp)(L^2(T^k)) \oplus [d(L^2(T^{k-1})) + \bar{d}(L^2(T^{k+1}))].$$

If $D_{d+d^\perp}$ is an essentially self-adjoint elliptic operator, then $\Delta_{d+d^\perp}$ and $D_{d+d^\perp}$ have the same kernel. It is easy to get an isometric isomorphism

$$L^2(H^k_{d+d^\perp}) \simeq L^2(H^k_{d+d^\perp}).$$

Similarly, if $D_{d+d^\perp}$ is an essentially self-adjoint elliptic operator, then

$$L^2(H^k_{d+d^\perp}) \simeq L^2(H^k_{d+d^\perp}).$$

More details, see [2, 15, 23].

Similar to Corollaries 3.8 and 3.19 in [24], we get the following isomorphism.

**Proposition 2.9** Suppose that $(M, g, J, \omega)$ is a $2n$-dimensional complete, non-compact almost Kähler manifold with bounded geometry. If $D_{d+d^\perp}$ and $D_{d+d^\perp}$ are of essential self-adjointness, then the Lefschetz operator defines isomorphisms

$$L^{n-k}: L^2(H^k_{d+d^\perp}) \simeq L^2(H^{2n-k}_{d+d^\perp}),$$

$$L^{n-k}: L^2(H^k_{d+d^\perp}) \simeq L^2(H^{2n-k}_{d+d^\perp}).$$

**Proof.** For an almost Kähler manifold $(M, g, J, \omega)$, there exists a connection called second canonical connection $\nabla^1_g$ whose torsion tensor has vanishing $(1, 1)$-part. Moreover, $\nabla^1_g$ satisfies $\nabla^1_g g = 0$, $\nabla^1_g J = 0$ and $\nabla^1_g \omega = 0$ (cf. [7, 20]). Since $\nabla^1_g \omega = 0$, it implies that $\omega$ is bounded. We only prove the first isomorphism, and the other is similar. Since $\omega$ is bounded, if $\alpha$ is $L^2$, the same $L^{n-k}\alpha$. Note that $\Delta_{d+d^\perp}$ preserves the degree of forms and $[\Delta_{d+d^\perp}, L] = 0$, $[\Delta_{d+d^\perp}, \Lambda] = 0$ (cf. [24, Lemma 3.7]), by Proposition 2.8, we get that $L^{n-k}: L^2(H^k_{d+d^\perp}) \rightarrow L^2(H^{2n-k}_{d+d^\perp})$ is an isomorphism. $\square$
Decomposition (2.8) and Proposition 2.9 lead the Lefschetz vanishing property which is similar with Hitchin’s result (see [14, Theorem 2]).

Proof of Theorem 1.3. It is clear that the symplectic form $\omega$ is bounded with respect to the given almost Kähler metric $g$. By hypothesis, we assume that $\omega = d\eta$, where $\eta$ satisfy

$$||\eta(x)||_g \leq c(\rho(x_0, x) + 1).$$

Then for every closed $L^2$ $k$-form $\alpha$, $k < n$, the form $L^{n-k}\alpha = \omega^{n-k} \wedge \alpha = d\beta$, where $\beta = \eta \wedge \omega^{n-k-1} \wedge \alpha$. By [14, Proof of Theorem 1], we obtain that $\omega^{n-k} = \eta \wedge \omega^{n-k-1}$ is $d$-sub-linear. Then applying [14, Theorem 1] again we get that $L^{n-k}\alpha = d(\eta \wedge \omega^{n-k-1} \wedge \alpha) = d\beta$ lies in the closure of $dL^2\Omega^{2n-k-1} \cap L^2\Omega^{2n-k}$. In particular, if $\alpha$ is $dd^g$-harmonic, by Proposition 2.9, $L^{n-k}\alpha$ is also $dd^g$-harmonic, that is, $d\beta$ is $dd^g$-harmonic. Then $d\beta$ is $\Delta_g$-harmonic. Hence, by decomposition (2.8), $L^{n-k}\alpha = d\beta = 0$. Proposition 2.9 has stated that $L^{n-k}$ is an isomorphism from $L^2H_{dd^g}^k$ to $L^2H_{dd^g}^{2n-k}$ for $k < n$. Therefore, $\alpha = 0$. At last, we can summarize that both $L^2H_{dd^g}^k = 0$ for $k < n$ and $L^2H_{dd^g}^k \neq 0$ for $k > n$.

By simple calculation, we find that the Laplacians $\Delta_{d+d^g}$ and $\Delta_{dd^g}$ satisfy

$$*g\Delta_{d+d^g} = \Delta_{dd^g} *g.$$

Hence, we have $*g$: $L^2H_{dd^g}^k \to L^2H_{d+d^g}^{2n-k}$ is an isomorphism. Therefore, we can also summarize that $L^2H_{d+d^g}^k = 0$, unless $k = n$. \hfill $\square$

3. Symplectic Euler characteristics

A Hilbert space $\mathcal{H}$ with a unitary action of a countable group $\Gamma$ is called a $\Gamma$-module if $\mathcal{H}$ is isomorphic to a $\Gamma$-invariant subspace in the space of $L^2$-functions on $\Gamma$ with values in some Hilbert space $H$. To each $\Gamma$-module, one assigns the Von Neumann dimension, also called $\Gamma$-dimension, $0 \leq \dim_{\Gamma}\mathcal{H} \leq \infty$, which is a non-negative real number or $+\infty$ (see [2, 15, 22, 23]). The precise definition is not important for the moment, but the following properties convey the idea of $\dim_{\Gamma}\mathcal{H}$ as some kind of size of the ‘quotient space’ $\mathcal{H}/\Gamma$:

(i) $\dim_{\Gamma}\mathcal{H} = 0$ $\iff$ $\mathcal{H} = 0$.

(ii) If $\Gamma$ is a finite group, then $\dim_{\Gamma}\mathcal{H} = \dim\mathcal{H}/\text{card}\,\Gamma$.

(iii) $\dim_{\Gamma}\mathcal{H}$ is additive. Given $0 \to \mathcal{H}_1 \to \mathcal{H}_2 \to \mathcal{H}_3 \to 0$, one has $\dim_{\Gamma}\mathcal{H}_2 = \dim_{\Gamma}\mathcal{H}_1 + \dim_{\Gamma}\mathcal{H}_3$.

(iv) If $\mathcal{H}$ equals the space of $L^2$-functions $\Gamma \to H$, then $\dim_{\Gamma}\mathcal{H} = \dim H$. In particular, if $H = \mathbb{R}^n$, then $\dim_{\Gamma}\mathcal{H} = n$.

Here we are interested in the situation where $\Gamma$ is a discrete faithful group of symplectomorphisms of a symplectic manifold $(M, \omega)$. Find an almost Kähler structure $(g, J, \omega)$ on $(M, \omega)$, then one has an almost Kähler manifold (cf. [20]). It is not hard to see that the given group $\Gamma$ acts on $(M, g, J, \omega)$ as a deck transformation group [7]. One can easily show that the spaces $L^2H_{d+d^g}^k$, $L^2H_{dd^g}^k$ of harmonic $L^2$-forms are $\Gamma$-module for all degrees $k$ (see [2, 15, 22, 23]), and then one defines the $L^2$-symplectic Betti numbers $L^2\beta_{k+1}^g \triangleq \dim_{\Gamma}L^2H_{d+d^g}^k$ and $L^2\beta_{k}^{dd^g} \triangleq \dim_{\Gamma}L^2H_{dd^g}^k$. The most interesting case is when $M/\Gamma$ is closed. Then, the $L^2$-symplectic Betti numbers are finite $L^2\beta_k^g < \infty$ and the $L^2$-symplectic Euler characteristics is defined by
First recall how Hodge theory works on a complete non-compact Riemannian manifold \([14]\). If \(L^2 \Omega^k\) denotes the Hilbert space of \(L^2\) \(k\)-forms, then the \(L^2\)-de Rham cohomology group \(L^2 H^k_{dR}\) is defined as the quotient of the space of closed \(L^2\) \(k\)-forms by the closure of the space \(dl^2 \Omega^{k-1} \cap L^2 \Omega^k\).

Similarly, we can define the \(L^2\)-symplectic cohomology group on a complete, non-compact almost Kähler manifold \((M, g, J, \omega)\) with bounded geometry by

\[
L^2 H^k_{d+d^\Lambda} = \frac{\ker(d + d^\Lambda) \cap L^2 \Omega^k}{dd^\Lambda L^2 \Omega^{k-1} \cap L^2 \Omega^k}.
\]

If \(D_{d+d^\Lambda}\) is of essential self-adjointness by decomposition (2.8) and Proposition 2.8, we can find that

\[
L^2 H^k_{dR} \cong L^2 H^k_d, \quad L^2 H^k_{d+d^\Lambda} \cong L^2 H^k_{d+d^\Lambda},
\]

and every \(L^2\) cohomology class has an \(L^2\) harmonic representative form which is the only one.

A closed symplectic manifold \((M, \omega)\) is said to satisfy the \(dd^\Lambda\)-Lemma if every \(d\)-exact, \(d^\Lambda\)-closed form is \(dd^\Lambda\)-exact. In fact, it turns out that the following conditions are equivalent on a closed symplectic manifold \((M, \omega)\) \([1, 6, 13, 19, 21, 24, 25]\):

- \((M, \omega)\) satisfies the \(dd^\Lambda\)-Lemma;
- the natural homomorphism \(H^*_d(M; \mathbb{R}) \rightarrow H^*_d(M; \mathbb{R})\) is actually an isomorphism;
- every de Rham cohomology class admits a representative being both \(d\)-closed and \(d^\Lambda\)-closed;
- the hard Lefschetz Condition holds on \((M, \omega)\).

More generally, Dong Yan has obtained the following result on a symplectic manifold which may be not compact.

**Proposition 3.1** \([25]\). Let \((M, \omega)\) be a symplectic manifold with dimension \(2n\). Then the following two assertions are equivalent:

1. every de Rham cohomology class admits a representative being both \(d\)-closed and \(d^\Lambda\)-closed;
2. For any \(k \leq n\), the cup product \(H^k_{dR}(M; \mathbb{R}) \rightarrow H^{n+k}_{dR}(M; \mathbb{R})\) is surjective.

**Remark 3.2** The above assertion 2 is just the definition of hard Lefschetz property on a symplectic manifold which may be not compact.
Let \((M, g, J, \omega)\) be a complete non-compact almost Kähler manifold with bounded geometry. Denote the Sobolev space

\[
L^2_i \Omega^k = \left\{ \alpha \in \Omega^k \left| \sum_{i=0}^{k} (\nabla^i g)^\alpha |_{\theta} \in L^2(M) \right. \right\},
\]

where \(\nabla^i g\) be the second canonical connection with respect to the given almost Kähler structure \((g, J, \omega)\) (cf. [7, 20]). Now we define the \(L^2-dd^A\)-Lemma on a complete non-compact almost Kähler manifold.

**Definition 3.3** Let \((M, g, J, \omega)\) be a complete non-compact almost Kähler manifold with bounded geometry. Let \(\alpha \in L^2 \Omega^k\) be a \(d\)- and \(dd^A\)-closed differential form. We say that the \(L^2-dd^A\)-Lemma holds if the following properties are equivalent:

(i) \(\alpha = d\beta, \beta \in L^2_i \Omega^{k-1}\);

(ii) \(\alpha = d^A \gamma, \gamma \in L^2_i \Omega^{k+1}\);

(iii) \(\alpha = dd^A \theta, \theta \in L^2_i \Omega^k\).

**Proposition 3.4** Let \((M, g, J, \omega)\) be a 2n-dimensional closed almost Kähler manifold, \(\Pi: (\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega}) \to (M, g, J, \omega)\) the universal covering map. If \((M, g, J, \omega)\) satisfies the hard Lefschetz property, then \(L^2-dd^A\)-Lemma holds on \((\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})\).

**Proof.** We may assume without loss of generality that \(M\) is a connected manifold. Denote by \(\pi_1(M)\) the fundamental group of \(M\). Let \(\Gamma\) be the deck transformation group of the covering. Then \(\Gamma\) is isomorphic to \(\pi_1(M)\). Notice that \((\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega})\) is a complete, non-compact almost Kähler manifold with bounded geometry. Since \(\tilde{D}_{d+d^A} = \pi^* D_{d+d^A}, \tilde{D}_{dd^A} = \pi^* D_{dd^A}\) which are \(\Gamma \cong \pi_1(M)\)-invariant fourth-order elliptic operators, \(\tilde{D}_{d+d^A}\) and \(\tilde{D}_{dd^A}\) are essential self-adjointness (cf. [2, 15, 23]). Suppose \(F \subseteq \tilde{M}\) is the fundamental domain of the universal covering. It is well known that \(\Pi(F)\) is an open set of \(M\) and \(\Pi(F) = M\) (cf. [7]), moreover both \(\partial F\) and \(M \setminus \Pi(F)\) satisfy the Hausdorff dimension less than or equal to \(2n - 1\) (cf. [7]). For any \(\phi \in \Gamma\), \(\phi: \tilde{M} \to \tilde{M}\) is a homeomorphism. Denote by \(\tilde{F}_\phi = \phi(\tilde{F})\), then \(\phi: F \to \tilde{F}_\phi\) is a diffeomorphism and \(F \cap \tilde{F}_\phi = \emptyset\) for any \(\phi \neq e\). Moreover, \(\tilde{M} = \cup_{\phi \in \Gamma} \tilde{F}_\phi\).

Since \((M, \omega)\) satisfies the hard Lefschetz property, then the \(dd^A\)-Lemma holds on \(M\), that is

\[
\text{Im} d \cap \ker d_{dd^A} = \ker d \cap \text{Im} d_{dd^A} = \text{Im} d_{dd^A}.
\]

Suppose that \(\alpha\) is a \(d\)-closed \(k\)-form on \(M\) and \(\alpha = d^A \beta\). Then there exists \(k - 1\)-form \(\gamma\) such that \(\alpha = dA \beta = d^A \gamma\). Note that \(\gamma \in \Omega^{k-1} = H^1_{d^A}(\Omega^{k-1}) \oplus d(\Omega^{k-2}) \oplus d^*(\Omega^k)\), without loss of generality, we can assume \(\gamma = d^A \eta\), where \(\eta\) is a \(k\)-form. Then \(\alpha = dd^A \eta\). Using the Hodge decomposition again, we can assume \(\eta = d\xi\), where \(\xi \in \Omega^{k-1}\). Since \(dd^A: \Omega^{k-1} \to \Omega^{k-1}\) is an elliptic linear operator and essentially self-adjoint (cf. [2, 15, 23]), we can obtain

\[
\|\eta\|_{L^2(M)} \leq c_M \|\alpha\|_{L^2(M)}.
\]

where \(c_M\) is constant which only depends on \(M\). Indeed, we have obtained the following property in distribution sense. If \(\alpha = d^A \beta\) is a \(d\)-closed \(L^2\) form on \(M\), then we can find an \(L^2_i\) form \(\gamma\) such that \(\alpha = d\gamma\). Since \(\Pi: F \leftrightarrow M\) is a diffeomorphism and \(M \setminus \Pi(F)\) satisfy the Hausdorff dimension less than or equal to \(2n - 1\) (cf. [7]), we obtain that if \(\alpha_F = d^A \beta_F\) is a \(d\)-closed \(L^2\) form on \(F\),
then we can find an $L^2$ form $\gamma_F$ such that $\alpha_F = d\gamma_F$. Suppose that $\bar{\alpha} = d^\Lambda \beta$ is a $d$-closed k-form on $\tilde{M}$, moreover, $\bar{\alpha}$ is $L^2$. Restricted to $F_\phi$, we can find $\tilde{\eta}_{F_\phi}$ such that $\|\tilde{\eta}_{F_\phi}\|_{L^2(F_\phi)} \leq c(F_\phi) \|\bar{\alpha}\|_{L^2(F_\phi)}$ and $\bar{\alpha}|_{F_\phi} = d^\Lambda \tilde{\eta}_{F_\phi} = dd^* \tilde{\eta}_{F_\phi}$, where $\tilde{\eta}_{F_\phi} \triangleq d^* \tilde{\eta}_{F_\phi}$. Define $\tilde{\eta} = \sum_{\phi \in \Gamma} \tilde{\eta}_{F_\phi}$ and $\tilde{\gamma} = d^* \tilde{\eta}$. It is easy to see that $\tilde{\eta} \in L^2_2(\cup_{\phi \in \Gamma} F_\phi) = L^2_2(\cup_{\phi \in \Gamma} F_\phi) = L^2_2(\tilde{M})$. Then we will get $\bar{\alpha} = d\delta \tilde{\gamma}$ and $\tilde{\gamma} \in L^2_k(\tilde{M})$.

Suppose that $\alpha$ is a $d^\Lambda$-closed k-form on $M$ and $\alpha = \delta \beta$. Then by $dd^\Lambda$-Lemma, there exists $k$-form $\gamma$ such that $\alpha = dd^\Lambda \gamma$. Note that $\Delta_{d^\Lambda} = d^\Lambda d^{\Lambda \ast} + d^{\Lambda \ast} d^\Lambda$ is an elliptic differential operator (see [24, Proposition 3.3]). Applying elliptic theory to the $\Delta_{d^\Lambda}$ then implies the Hodge decomposition

$$\Omega^k(M) = \mathcal{H}_{d^\Lambda}^k(M) \oplus d^\Lambda \Omega^{k+1}(M) \oplus d^{\Lambda \ast} \Omega^{k-1}(M).$$

(3.1)

Without loss of generality, we can assume $\gamma = d^\Lambda \delta \eta$ and $\alpha = dd^\Lambda d^{\Lambda \ast} \eta$. Using the Hodge decomposition (3.1) again, we can assume $\eta = d^\Lambda \xi$. Since $d^\Lambda d^{\Lambda \ast}$, $d^\Lambda \Omega^k \to d^\Lambda \Omega^{k+1}$ is an elliptic linear operator, we can obtain

$$\|\eta\|_{L^2_{2k}(M)} \leq c_{M,1} \|d^\Lambda d^{\Lambda \ast} \eta\|_{L^2_k(M)},$$

where $c_{M,1}$ is constant which only depends on $M$. Note that $\alpha = d(d^\Lambda d^{\Lambda \ast} \eta)$, we can assume $d^\Lambda d^{\Lambda \ast} \eta = d^\Lambda \theta$, since $d^\Lambda d^{\Lambda \ast} \eta \in \Omega^{k-1} = \mathcal{H}_{d^\Lambda}^{k-1} \oplus d(\Omega^k \ominus d^{\Lambda \ast} \Omega^k)$, where $\theta$ is a $k$-form. It is well known that $d + d^\ast$ is an elliptic linear operator (cf. [2, 7]), so $d^\ast : d^\Lambda \Omega^k \to d^\Lambda \Omega^{k-1}$ is an elliptic linear operator. Hence, $\|d^\Lambda d^{\Lambda \ast} \eta\|_{L^2_k(M)} \leq c_{M,2} \|\alpha\|_{L^2_k(M)}$. Therefore, we can obtain

$$\|\eta\|_{L^2_{2k}(M)} \leq c_M \|\alpha\|_{L^2_k(M)},$$

where $c_M$ is constant which only depends on $M$. We have obtained the following property in distribution sense. If $\alpha = \delta \beta$ is a $d^\Lambda$-closed $L^2$ form on $M$, then we can find an $L^2$ form $\gamma$ such that $\alpha = dd^\Lambda \gamma$. At last, we obtain that: If $\alpha_F = d^\Lambda_\phi \gamma_F$ is a $d^\Lambda$-closed $L^2$ form on $F_\phi$, then we can find an $L^2_2$ form $\gamma_F$ such that $\alpha_F = dd^\Lambda \gamma_F$. Suppose that $\bar{\alpha} = d^\Lambda \beta$ is a $d^\Lambda$-closed $k$-form on $\tilde{M}$, moreover $\bar{\alpha}$ is $L^2$. Restricted to $F_\phi$, we can find $\tilde{\eta}_{F_\phi}$ such that $\|\tilde{\eta}_{F_\phi}\|_{L^2_k(F_\phi)} \leq c(F_\phi) \|\bar{\alpha}\|_{L^2_k(F_\phi)}$ and $\bar{\alpha}|_{F_\phi} = d^\Lambda \tilde{\eta}_{F_\phi} = dd^\Lambda d^\Lambda \ast \tilde{\eta}_{F_\phi}$, where $\tilde{\eta}_{F_\phi} \triangleq d^\Lambda \ast \tilde{\eta}_{F_\phi}$. Define $\tilde{\eta} = \sum_{\phi \in \Gamma} \tilde{\eta}_{F_\phi}$ and $\tilde{\gamma} = d^\Lambda \ast \tilde{\eta}$. It is easy to see that $\tilde{\eta} \in L^2_2(\cup_{\phi \in \Gamma} F_\phi) = L^2_2(\cup_{\phi \in \Gamma} F_\phi) = L^2_2(\tilde{M})$, since $\tilde{M} \backslash \bigcup_{\phi \in \Gamma} F_\phi$ has Hausdorff dimension $\leq 2n - 1$. Then we will get $\bar{\alpha} = dd^\Lambda \tilde{\gamma}$ and $\tilde{\gamma} \in L^2_k(\tilde{M})$. \hfill \Box

**Remark 3.5** Indeed, in the above Proposition, we have proven that

- $\alpha = d^\Lambda \beta$, $\alpha \in L^2$ and $\alpha$ is $d$-closed $\Rightarrow \alpha = d^\Lambda \gamma$, $\gamma \in L^2_2$;
- $\alpha = d \beta$, $\alpha \in L^2$ and $\alpha$ is $d^\Lambda$-closed $\Rightarrow \alpha = dd^\Lambda \gamma$, $\gamma \in L^2_2$.

**Proposition 3.6** Let $(M, g, J, \omega)$ be a $2n$-dimensional closed almost Kähler manifold, $\Pi : (\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega}) \to (M, g, J, \omega)$ the universal covering map. If $(M, g, J, \omega)$ satisfies the hard Lefschetz property, then the canonical homomorphism

$$L^2 H^k_{d + d^\Lambda}(\tilde{M}; \mathbb{R}) \to L^2 H^k_{dR}(M; \mathbb{R})$$

is an isomorphism for all $k$. 
Proof. Notice that since $D_{d^A}$ is a $\Gamma = \pi_1(M)$-invariant elliptic operator on $\tilde{M}$, then $D_{d^A}$ is of essential self-adjointness (cf. [23, Lecture 18]). Hence, for any $\alpha \in L^2\Omega^k$, $D_{d^A}\alpha = 0$ in the sense of distribution implies that $\alpha \in L^2\Omega^k$. Suppose that $\alpha$ is a $d$-closed and $d^A$-closed $L^2$ $k$-form such that $[\alpha]_{dR} = 0$, that is $\alpha = d\beta$ for some $\beta \in L^2\Omega^{k-1}(\tilde{M})$. By the proof of Proposition 3.4, we can find $\gamma \in L^2(\tilde{M})$ such that $dA\gamma$. It follows that $[\alpha]_{d^A} = 0$. This proves that the homomorphism is injective.

For any $[\alpha]_{dR} \in L^2H^{d^A}_{k}(\tilde{M}; \mathbb{R})$, by the decomposition 2.8, we can assume $\alpha \in L^2H^k_{d^A}$ without loss of generality. If $d^A\alpha = 0$, then $[\alpha]_{d^A} \in L^2H^{k}_{d^A}(\tilde{M})$ whose image under this map is $[\alpha]_{dR}$. Suppose that $d^A\alpha \neq 0$. Note that $\Delta_d\alpha = 0$ and $\alpha \in L^2$, therefore, we can obtain that $\alpha$ is smooth and $\|\alpha\|_{L^2_{d^A}(\tilde{M})} \leq c(k)$ for any $k = 0, 1, 2, \ldots$. Since $dd^A\alpha = 0$, by the proof of Proposition 3.4, we can find $\gamma \in L^2(\tilde{M})$ such that $d^A\alpha = dd^A\gamma$. Hence, $d^A(\alpha + d\gamma) = 0$ and $d(\alpha + d\gamma) = 0$. It follows that $[\alpha + d\gamma]_{d^A} \in L^2H^{k}_{d^A}(\tilde{M})$ whose image under this map is $[\alpha]_{dR}$. This proves that the homomorphism is also surjective. So it is an isomorphism.

Let $(M, \omega)$ be a closed symplectic manifold of dimension $2n$. Then one can find an almost Kähler structure $(g, J, \omega)$ on $(M, \omega)$. $(M, g, J, \omega)$ is a closed almost Kähler manifold of dimension $2n$ (cf. [20]). Let $\Pi: (\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega}) \rightarrow (M, g, J, \omega)$ be the universal covering map. If $(M, g, J, \omega)$ satisfies the hard Lefschetz property, then from Proposition 3.6, by M. Atiyah’s result [2], it is easy to get Theorem 1.4.

Proof of Theorem 1.5. Let $(M, g, J, \omega)$ be a $2n$-dimensional closed almost Kähler parabolic manifold which satisfies the hard Lefschetz property,

$$
\Pi: (\tilde{M}, \tilde{g}, \tilde{J}, \tilde{\omega}) \rightarrow (M, g, J, \omega)
$$

the universal covering map. Therefore, by Proposition 3.6,

$$
L^2H^{k}_{d^A}(\tilde{M}; \mathbb{R}) \rightarrow L^2H^{k}_{d^A}(\tilde{M}; \mathbb{R})
$$

is an isomorphism for all $k$. Since $L^2H^{k}_{d^A}(\tilde{M}; \mathbb{R}) \cong L^2H^{k}_{d^A}(\tilde{M}; \mathbb{R})$, by Theorem 1.3, we know $L^2H^{k}_{d^A}(\tilde{M}; \mathbb{R}) = 0$ for $k \neq n$. Hence, $L^2H^{k}_{d^A}(\tilde{M}; \mathbb{R}) = 0$ for $k \neq n$. The Atiyah index theorem for covers [2] then gives $(-1)^n\chi(M) \geq 0$. Then, the conclusion follows. \(\square\)

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