Asymptotically periodic behavior of solutions to fractional non-instantaneous
impulsive semilinear differential inclusions with sectorial operators

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Abstract
In this paper, we prove two results concerning the existence of S-asymptotically
ω-periodic solutions for non-instantaneous impulsive semilinear differential
inclusions of order $1 < \alpha < 2$ and generated by sectorial operators. In the first result,
we apply a fixed point theorem for contraction multivalued functions. In the second
result, we use a compactness criterion in the space of bounded piecewise continuous
functions defined on the unbounded interval $J = [0, \infty)$. We adopt the fractional
derivative in the sense of the Caputo derivative. We provide three examples
illustrating how the results can be applied.

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Asymptotically periodic solutions; Caputo fractional derivative; Non-instantaneous
impulses

1 Introduction
Fractional calculus has become a well-established branch of mathematical analysis. It has
many applications in engineering and science. Much work has appeared studying vari-
ous models involving fractional differential boundary value problems and providing so-
lutions to those models using analytical methods or numerical methods. We highlight
some recent work involving fractional differential equations. Agarwal et al. [1]i
vestigated existence and uniqueness results on time scales for fractional nonlocal thermistor
problems in the conformable sense. Sunarto et al. [2] developed a numerical method us-
ing a quarter-sweep and PAOR to solve a one-dimensional time-fractional mathemati-
cal physics model. Rezapour et al. [3] showed the existence and uniqueness of solutions
for a general multi-term fractional BVP involving the generalized $\psi$-RL operators. Then
they suggested two numerical algorithms, namely, the Daftardar-Gejji and Jafari method
(DGJIM) and the Adomian decomposition method (ADM) in which a series of approxi-
mate solutions converge to the exact ones. Agarwal et al. [4] discussed the existence and
uniqueness of solutions for a nonlocal problem with integral transmitting condition for mixed parabolic–hyperbolic type equations with Caputo fractional derivative. Agarwal et al. [5] provided a detailed description of impulsive fractional differential equations using Lyapunov functions and overviewed results for the stability in Caputo’s sense. Khan et al. [6] focused on the existence and uniqueness of solutions and Hyers–Ulam stability for ABC-fractional DEs with p-Laplacian operator. Khan et al. [7] studied the stability and numerical simulation of a fractional order plant-nectar-pollinator model. Khan et al. [8] proved the existence and Hyers–Ulam stability of solutions to a class of hybrid fractional differential equations with p-Laplacian operator.

The problem of existence of non-constant periodic solutions for fractional order models has become one of the most interesting topics to conduct research on. This is particularly due to the differences between systems of integers order and systems of fractional orders in terms of the existence of non-constant periodic solutions. In much work, such as [9–14], the authors have shown that non-constant periodic solutions of fractional order systems do not exist contrary to the case where the order of the system is an integer. Therefore, the concept of an asymptotically periodic solution for fractional differential equations or inclusions is introduced and discussed in much work. For example, in [12, 15–17], the authors considered semilinear differential equations of order \( \alpha \in (0, 1) \) generated by a \( C_0 \)-semigroup, while the papers [18–20] addressed semilinear differential equations of order \( \alpha \in (0, 1) \) generated by sectorial operators. Moreover, the asymptotically periodic solutions for delayed fractional differential equations with almost sectorial operator of order \( \alpha \in (0, 1) \) are examined in [21]. Rogovchenko et al. [22] studied the asymptotic properties of solutions for a certain classes of second order nonlinear differential equations. Very recently, Wang et al. [23] discussed the asymptotic behavior of solutions to time-fractional neutral functional differential equations of order \( \alpha \in (0, 1) \).

For more information regarding this subject, we refer the reader to [24, 25].

It is worth noting that the problems discussed in all cited work above do not contain impulse effects, whether it is instantaneous or non-instantaneous, and the nonlinear term is a single-valued function.

Let \( \alpha \in (1, 2) \), \( E \) be a Banach space, \( m \) be a natural number, \( \omega > 0 \), \( J = [0, \infty) \),

\[
0 = s_0 < \theta_1 \leq s_1 < \theta_2 < \cdots < \theta_m \leq s_m = \omega < \theta_{m+1} = \omega + \theta_1 \leq s_{m+1} = \omega + s_1 < \cdots,
\]

with \( \lim_{i \to \infty} \theta_i = \infty \), \( s_{m+i} = s_i + \omega \), and \( \theta_{m+i} = \theta_i + \omega \); \( i \in \mathbb{N} = \{1, 2, 3, \ldots\} \) and \( A : D(A) \subseteq E \to E \) be a sectorial operator of type \( \{M, \varphi, \alpha, \mu\} \), where \( M > 0, \varphi \in (0, \frac{\pi}{2}) \) and \( \mu \in \mathbb{R} \). Moreover, let \( F : J \times E \to 2^E - \{\phi\} \) be a multivalued function, \( g_i : [\theta_i, s_i] \times E \to E; i \in \mathbb{N}, x_0 \in D(A) \) and \( x_1 \in E \).

Motivated by the above cited work, we prove two results concerning the existence of \( S \)-asymptotically \( \omega \)-periodic mild solutions to the following non-instantaneous impulsive semilinear differential inclusion:

\[
\begin{align*}
\alpha_{\rho, \theta}^p x(\theta) &\in Ax(\theta) + F(\theta, x(\theta)), \quad \text{a.e. } \theta \in (\theta_i, \theta_{i+1}], i \in \mathbb{N} \cup \{0\}, \\
x(\theta_i^+) &\in g_i(\theta_i, x(\theta_i^-)), \quad i \in \mathbb{N}, \\
x(\theta) &\in g_i(\theta, x(\theta_i^-)), \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, \\
x(0) &\in x_0, \quad x'(0) = x_1,
\end{align*}
\]  

(1)
where $\mathcal{D}^\alpha_{s_i,\theta}x(\theta)$ is the Caputo derivative of the function $x$ at the point $\theta$ and with lower limit at $s_i$ [26]. In the first result, we apply a fixed point theorem for contraction multivalued functions, and, in the second result, we use a compactness criterion in the space of bounded piecewise continuous functions defined on the unbounded interval $J = [0, \infty)$. Our work generalizes much recent work such as [18–20] to the case where there are impulse effects, and the right-hand side is a multivalued function.

To the best of our knowledge, there is no work on $S$-asymptotic $\omega$-periodic behavior of solutions to fractional non-instantaneous impulsive differential inclusions with order $\alpha \in (1, 2)$ and generated by sectorial operators, and this fact is the main goal in the present paper.

To clarify the advantage of this study, we mention that two methods have been provided to demonstrate the existence of $S$-asymptotic $\omega$-periodic solutions for semilinear fractional differential inclusions in the presence of non-instantaneous impulse effects, and in which the nonlinear part is a multivalued function, and the linear part is a sectorial operator. Moreover, the technique presented in this paper can be used to generalize the work in [12, 15–21, 23–25] to the case where the linear part is a sectorial operator, the nonlinear part is a multivalued function, and there is impulse effects. In addition, Problem (1) can be investigated on time scales using the arguments in [1], and using the arguments in [3, 6, 8], one can examine the asymptotic periodic solutions for Problem (1) when the Caputo derivative is replaced by the $\psi$-Caputo derivative, $\psi$-RL derivative, Atangana–Baleanu derivative or $p$-Laplacian operator. Also, the technique used in this paper can be applied to study the asymptotic periodic solutions for many fractional differential equations or inclusions generated by sectorial operators or almost sectorial operators.

For more information related to fractional differential equations and inclusions with non-instantaneous impulse effects, we refer the reader to [27–31]. See [32–35] for more information about semilinear differential equations and inclusions with sectorial operators.

It is worth noting that Refs. [36–44] contain very important and interesting topics in mathematics as well as their applications such as differential equations, fractional calculus and ABC-fuzzy-Volterra integro-differential equation.

The paper is organized as follows. Section 2 includes definitions and basic information that we need to prove our results. In Sect. 3, we provide two existence results of $S$-asymptotic $\omega$-periodic solutions for Problem (1). In Sect. 4, we give three examples to illustrate our theoretical results.

2 Preliminaries and notations

Let $J_0 = [0, t_1]$ and $J_i = (\theta_i, \theta_{i+1}]$, $i \in \mathbb{N}$. It is known that the vector spaces

$$PC(J, E) := \{ x : J \to E, x \text{ is bounded}, x|_{J_i} \in C(J_i, E) \}$$

and

$$SAP_\omega PC(J, E) := \{ x : J \to E, x \text{ is bounded}, x|_{J_i} \in C(J_i, E), \lim_{\theta \to \infty} \| x(\theta + \omega) - x(\theta) \| = 0 \}$$

are Banach spaces endowed with the norm

$$\| x \| := \max_{\theta \in J} \| x(\theta) \|.$$
Definition 1 ([45]) Let $M > 0$, $\varphi \in (0, \frac{\pi}{2})$, and $\mu \in \mathbb{R}$. A closed linear operator $A : D(A) \subseteq E \to E$ with dense domain is called sectorial of type $\{M, \varphi, \alpha, \mu\}$ if:

(i) $\delta^{\alpha} \notin S_{\varphi} + \mu \Rightarrow \|R(\delta^{\alpha}, A)\|$ exists, where $R(\delta^{\alpha}, A)$ is the $\alpha$-resolvent operator of $A$ defined by

$$R(\delta^{\alpha}, A) := (\delta^{\alpha} I - A)^{-1}$$

and

$$S_{\varphi} + \mu = \{\mu + \delta^{\alpha} : \delta \in \mathbb{C}, |\text{Arg}(-\delta^{\alpha})| < \varphi\}.$$  

(ii)

$$\|R(\delta^{\alpha}, A)\| \leq \frac{M}{|\delta^{\alpha} - \mu|}, \quad \delta^{\alpha} \notin S_{\varphi} + \mu.$$  

Remark 1 ([45], Remark 2.1) If $A$ is a sectorial operator of type $\{M, \varphi, \alpha, \mu\}$, then it is the infinitesimal generator of a $\alpha$-resolvent family of operators $\{T_\alpha(\theta) : \theta \geq 0\}$ in $E$ defined by

$$T_\alpha(\theta) = \frac{1}{2\pi i} \int_{\gamma} e^{\delta^{\alpha} \theta} R(\delta^{\alpha}, A) d\delta,$$  

where $\gamma$ is a suitable path and $\delta^{\alpha} \notin S_{\varphi} + \mu$ for $\delta \in \gamma$.

Definition 2 ([45], Definition 3.1) Let $A$ be a sectorial operator of type $\{M, \varphi, \alpha, \mu\}$ and $f \in L^1([0, b], E)$. A continuous function $u : [0, b] \to E$ is called a mild solution to the Cauchy problem:

$$\begin{align*}
\frac{d}{d\theta} x(\theta) &= Ax(\theta) + f(\theta), \quad \theta \in [0, b], \\
x(0) &= x_0, \quad x'(0) = x,
\end{align*}$$

if

$$x(\theta) = S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 + \int_{0}^{\theta} T_\alpha(\theta - \tau) f(\tau) d\tau, \quad \theta \in [0, b],$$

where

$$S_\alpha(\theta) = \frac{1}{2\pi i} \int_{\gamma} e^{\delta^{\alpha} \theta} R(\delta^{\alpha}, A) d\delta,$$

$$K_\alpha(\theta) = \frac{1}{2\pi i} \int_{\gamma} e^{\delta^{\alpha} \theta} R(\delta^{\alpha}, A) R(\delta^{\alpha}, A) d\delta,$$

and $T_\alpha(\theta)$ is given by (2).

The following lemma provides estimates on $\|S_\alpha(\theta)\|$, $\|K_\alpha(\theta)\|$ and $\|T_\alpha(\theta)\|$.
Lemma 1 ([45], Theorems 3.3, 3.4) Let $A$ be a sectorial operator of type $\{M, \varphi, \alpha, \mu\}$. Suppose $\mu < 0$. Then $L = L(M, \varphi, \alpha) > 0$ such that

$$
\|S_\alpha(\theta)\| \leq \frac{L}{1 + |\mu|\theta^\alpha}, \quad \|K_\alpha(\theta)\| \leq \frac{L(\theta + 1)}{1 + |\mu|\theta^\alpha}, \quad \text{and}
$$

$$
\|T_\alpha(\theta)\| \leq \frac{L\theta^{\alpha-1}}{1 + |\mu|\theta^\alpha}, \quad \forall \theta > 0.
$$

(3)

Remark 2 ([19], Remark 3) In view of (6), we get:

(i)

$$
\|S_\alpha(\theta)\| \leq L, \quad \forall \theta > 0.
$$

(4)

(ii)

$$
\|K_\alpha(\theta)\| \leq L + \frac{L\theta}{1 + |\mu|\theta^\alpha}
$$

\[=egin{cases} 
L + L\theta, & \text{if } 0 < \theta < 1, \\
L + \frac{L}{|\mu|\theta^{\alpha-1}}, & \text{if } \theta \geq 1
\end{cases}
$$

\[\leq L \left(1 + \max \left\{1, \frac{1}{|\mu|}\right\}\right), \quad \forall \theta > 0.
$$

(5)

(iii) As in (ii), we derive

$$
\|T_\alpha(\theta)\| \leq \frac{L\theta^{\alpha-1}}{1 + |\mu|\theta^\alpha}
$$

\[\leq L \max \left\{1, \frac{1}{|\mu|}\right\}, \quad \forall \theta > 0.
$$

(6)

Based on Definition 2, we can give the definition of an $S$-asymptotically $\omega$-periodic mild solution for Problem (1).

Definition 3 A function $x \in SAP_\omega PC(J, E)$ is called an $S$-asymptotically $\omega$-periodic mild solution for Problem (1) if it has the form

$$
x(\theta) =
\begin{cases}
S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\
+ \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, & \theta \in [0, \theta_1], \\
g_i(\theta, x(\theta^-)), & \theta \in (\theta_1, s_1], i \in \mathbb{N}, \\
S_\alpha(\theta-s_i)g_i(s_i, x(s_i^-)) + K_\alpha(\theta-s_i)g_i(s_i, x(s_i^-)) \\
- \int_0^{\theta-s_i} T_\alpha(s_i-\tau)f(\tau)\,d\tau \\
+ \int_0^\theta T_\alpha(\theta-\tau)f(\tau)\,d\tau, & \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N},
\end{cases}
$$

where $f(\tau) \in F(\tau, x(\tau))$, a.e. for $\tau > 0$.

3 Existence of $S$-asymptotically $\omega$-periodic mild solutions for Problem (1)

In order to give the first result, we need the following lemma which is due to [46].
Lemma 2 Let \((X,d)\) be a metric space and \(G\) be a contraction multivalued function from \(X\) to the family of non-empty closed subsets of \(X\). Then \(G\) has a fixed point.

For notations about multivalued functions we refer the reader to [47].

Theorem 1 Suppose the following assumptions are satisfied.

\((HA)\) \(A : D(A) \rightarrow E\) is a sectorial operator of type \([M,\varphi,\alpha,\mu]\), where \(M > 0, \varphi \in (0, \frac{\pi}{2})\), and \(\mu \in \mathbb{R}\).

\((HF)\) \(F : J \times E \rightarrow P_{ck}(E)\) is a multivalued function such that

(i) For any \(z \in E\), the multivalued function \(\theta \rightarrow F(\theta, z)\) is strongly measurable.

(ii) For any \(x \in PC(J,E)\), the set

\(S_{F(x,J)}^1 := \{ \psi : J \rightarrow E, \psi\ \text{is locally integrable and } \psi(\tau) \in F(\tau, x(\tau)), \ \text{a.e. } \tau \in J \}\)

is not empty.

(iii) There is a continuous function \(L_1 : J \rightarrow (0,\infty)\) such that

\[ h(F(\theta, z_1), F(\theta, z_2)) \leq L_1(\theta) \|z_1 - z_2\|, \ \forall \theta \in J, z_1, z_2 \in E, \]  

where \(h\) is the Hausdorff distance.

(iv) There is a continuous function \(L_2 : J \rightarrow (0,\infty)\) such that

\[ h(F(\theta + \omega, z), F(\theta, z)) \leq L_2(\theta) \|1 + x\|, \ \forall \theta \in J, z \in E. \]  

(v) The function \(\sigma(\tau) := \|F(\tau, 0)\| = \sup_{x \in F(\tau, 0)} \|x\|\) is continuous, bounded on \(J\) and satisfies the relation

\[ \lim_{\theta \rightarrow \infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(\theta - \tau)\alpha} \sigma(\tau) d\tau = 0. \]  

\((Hg)\) For any \(i \in \mathbb{N}, g_i : [\theta_i, \theta_i] \times E \rightarrow E\) such that, for any \(x \in E\), the function \(\theta \mapsto g_i(\theta, x)\) is differentiable at \(s_i\) and

(i) \[ \lim_{\theta \rightarrow \infty} \|g_i(\theta + \omega, z) - g_i(\theta, z)\| = 0, \ \forall z \in E. \]  

(ii) There is \(N > 0\) such that for any \(i \in \mathbb{N}\)

\[ \|g_i(\theta, z_1) - g_i(\theta, z_2)\| \leq N \|z_1 - z_2\|, \ \forall \theta \in J, \forall z_1, z_2 \in E. \]  

(iii) There is \(N' > 0\) such that for any \(i \in \mathbb{N}\)

\[ \|g_i'(s_i, z_1) - g_i'(s_i, z_2)\| \leq N' \|z_1 - z_2\|, \ \forall z_1, z_2 \in E. \]
(iv) There is $\kappa_1 > 0$ such that
\[ \sup_{i \in \mathbb{N}} \sup_{\theta \in I} \| g_i(\theta, z) \| \leq \kappa_1 (\| z \| + 1), \quad \forall z \in E. \tag{13} \]

(v) There is $\kappa_2 > 0$ with
\[ \sup_{i \in \mathbb{N}} \| g'_i(s_i, z) \| \leq \kappa_2 (\| z \| + 1), \quad \forall z \in E. \tag{14} \]

Then Problem (1) has an $S$-asymptotically $\omega$-periodic mild solution provided that the following conditions are verified:
\[ \lim_{\theta \to \infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu| (\theta - \tau)^\mu} L_1(\tau) d\tau = 0, \tag{15} \]
\[ \lim_{\theta \to \infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu| (\theta - \tau)^\mu} L_2(\tau) d\tau = 0, \tag{16} \]
and
\[ L \left( 2 \xi + N + \left( 1 + \max \left\{ 1, \frac{1}{|\mu|} \right\} \right) N \right) < 1, \tag{17} \]
where $\xi = \sup_{\theta \in I} \int_0^\theta \frac{(\theta - \tau)^{\mu-1}}{1 + |\mu| (\theta - \tau)^\mu} L_1(\tau) d\tau$.

**Proof** Due to (HF)(ii), one can define a multivalued function $\Theta$ on $SAP_\omega PC(J, E)$ in the following manner: an element $y \in \Theta(x)$ if and only if
\[
y(\theta) = \begin{cases} 
S_\omega(\theta)x_0 + K_\omega(\theta)x_1 \\
+ \int_0^\theta T_\omega(\theta - \tau)f(\tau) d\tau, & \theta \in [0, \theta_1], \\
+ \int_0^\theta T_\omega(\theta - \tau)f(\tau) d\tau, & \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}, \\
\end{cases}
\tag{18}
\]
where $f \in S^1_{f(x)}$.

Obviously, $y \in PC(J, E)$. We clarify that, if $x \in SAP_\omega PC(J, E)$, then $\Theta(x)$ is a closed subset of $SAP_\omega PC(J, E)$. We do this in the following steps.

Step 1. We demonstrate that, if $y \in \Theta(x)$, we have
\[ \lim_{\theta \to \infty} \| y(\theta + \omega) - y(\theta) \| = 0. \tag{19} \]

Since $x \in SAP_\omega PC(J, E)$,
\[ \lim_{\theta \to \infty} \| x(\theta + \omega) - x(\theta) \| = 0. \tag{20} \]

Now, we consider two cases.
(i) If $\theta \in (\theta_0, s_i]$ for some $i \in \mathbb{N}$, then $\theta + \omega \in [\theta_0 + s_i + \omega_0, \theta] = [s_i, \theta_i]$, and hence, using (21) and (18), it yields

$$
\|g_{i,m}(\theta + \omega, x(\theta^-_{i,m})) - g_i(\theta, x(\theta^-_i))\|
\leq \|g_{i,m}(\theta + \omega, x(\theta^-_{i,m})) - g_{i,m}(\theta, x(\theta^-_{i,m}))\|
+ \|g_{i,m}(\theta, x(\theta^-_{i,m})) - g_i(\theta, x(\theta^-_i))\|
\leq \|g_{i,m}(\theta + \omega, x(\theta^-_{i,m})) - g_{i,m}(\theta, x(\theta^-_{i,m}))\|
+ N\|x(\theta^-_{i,m}) - x(\theta^-_i)\|.
$$ (21)

(ii) If $\theta \in [s_i, \theta_i]$ for some $i \in \mathbb{N}$, then $\theta + \omega \in [s_i + \omega, \theta_{i+1} + \omega] = [s_i, \theta_i]$. Using (4) and arguing as in (21), it follows that

$$
\|S_\omega(\theta + \omega - (s_i + \omega))g_{i,m}(s_i + \omega, x(\theta^-_{i,m})) - S_\omega(\theta - s_i)g_i(s_i, x(\theta^-_i))\|
= \|S_\omega(\theta - s_i)\|\|g_{i,m}(s_i + \omega, x(\theta^-_{i,m})) - g_i(s_i, x(\theta^-_i))\|
\leq L\|g_{i,m}(s_i + \omega, x(\theta^-_{i,m})) - g_i(s_i, x(\theta^-_i))\|.
$$ (22)

Moreover, in view of (3) and (14), we arrive at

$$
\|K_\omega(\theta + \omega - (s_i + \omega))g'_{i,m}(s_i + \omega, x(\theta^-_{i,m})) - K_\omega(\theta - s_i)g'_i(s_i, x(\theta^-_i))\|
= \|K_\omega(\theta - s_i)\|\|g'_{i,m}(s_i + \omega, x(\theta^-_{i,m})) - g'_i(s_i, x(\theta^-_i))\|
\leq \|K_\omega(\theta - s_i)\|\|g'_{i,m}(s_i + \omega, x(\theta^-_{i,m})) - g'_i(s_i, x(\theta^-_i))\|
\leq 2\kappa_2(\|x\| + 1)\frac{L(\theta - s_i + 1)}{1 + |\mu|(|\lambda s_i|^\tau)}.
$$ (23)

Next,

$$
\left\|\int_0^{\theta + \omega} T_\omega(\theta + \omega - \tau)\mathbf{f}(\tau)\,d\tau - \int_0^\theta T_\omega(\theta - \tau)\mathbf{f}(\tau)\,d\tau\right\|
= \left\|\int_0^{\theta + \omega} T_\omega(\theta + \omega - \tau\mathbf{f}(\tau)\,d\tau - \int_0^\theta T_\omega(\theta - \tau)\mathbf{f}(\tau)\,d\tau\right\|
\leq \int_0^{\theta + \omega} T_\omega(\theta - \tau)\|\mathbf{f}(\tau + \omega)\,d\tau
+ \int_0^\theta T_\omega(\theta - \tau)\|\mathbf{f}(\tau + \omega) - \mathbf{f}(\tau)\,d\tau
= I_1 + I_2.
$$ (24)

Let $\tau \in [-\omega, 0]$ be fixed. Since $F(\tau + \omega, 0)$ is compact, there is $\nu_{\tau+\omega} \in F(\tau + \omega, 0)$ such that

$$
\|\mathbf{f}(\tau + \omega) - \nu_{\tau+\omega}\| = d\mathbf{f}(\tau + \omega), F(\tau + \omega, 0)
\leq h(F(\tau + \omega, x(\tau + \omega)), F(\tau + \omega, 0)).
$$ (25)
From (7), (HF)(v) and (25), we get

\[
\|f(\tau + \omega)\| \leq h(F(\tau + \omega, x(\tau + \omega)), F(\tau + \omega, 0)) + \| v_{\tau + \omega}\|
\]

\[
\leq L_1(\tau + \omega) \| x(\tau + \omega) \| + \sigma(\tau + \omega)
\]

\[
\leq \| x \| L_1(\tau + \omega) + \sigma(\tau + \omega).
\]

(26)

Since \( L_1 \) and \( \sigma \) are continuous on \( f \), there are two positive numbers \( \omega_1, \omega_2 \) such that

\[
\sup_{\tau \in [0,\omega]} \left| L_1(\tau) \right| \leq \omega_1, \quad \text{and} \quad \sup_{\tau \in [0,\omega]} |\sigma(\theta)| \leq \omega_2.
\]

(27)

Then, from (3), (26) and (27), we have

\[
I_1 = \left\| \int_{-\omega}^{\theta} T_\alpha(\theta - \tau)f(\tau + \omega) \, d\tau \right\|
\]

\[
\leq (\omega_1 \| x \| + \omega_2)L \int_{-\omega}^{\theta} \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(|\theta - \tau|^\alpha)} \, d\tau
\]

\[
\leq (\omega_1 \| x \| + \omega_2)L \int_{-\omega}^{\theta} \frac{|\mu|(|\theta - \tau|^\alpha)}{1 + |\mu|(|\theta - \tau|^\alpha)} \, d\tau
\]

\[
= \frac{(\omega_1 \| x \| + \omega_2)L}{\alpha |\mu|} \ln \frac{1 + |\mu|(|\theta + \omega|^\alpha)}{1 + |\mu|(|\theta|^\alpha)}.
\]

(28)

Next, let \( \tau \in [0,\theta] \) be fixed. From the fact that \( F(\tau + \omega, x(\tau)) \) is compact, there are \( z_{\tau+\omega}, z_\tau \in F(\tau, x(\tau + \omega)) \) such that \( d(f(\tau + \omega), z_{\tau+\omega}) = d(f(\tau + \omega), F(\tau, x(\tau + \omega))) \) and \( d(f(\tau), z_\tau) = d(f(\tau, F(\tau, x(\tau + \omega)))) \). Then, by (12), (13) and (25), we arrive at

\[
\|f(\tau + \omega) - f(\tau)\|
\]

\[
\leq \|f(\tau + \omega) - z_{\tau+\omega}\| + \| z_{\tau+\omega} - z_\tau\| + \| z_\tau - f(\tau)\|
\]

\[
\leq d(f(\tau + \omega), F(\tau + \omega, x(\tau))) + \| z_{\tau+\omega} - z_\tau\|
\]

\[
+ d(f(\tau, F(\tau + \omega, x(\tau)))
\]

\[
\leq h(F((\tau + \omega), x((\tau + \omega)), F(\tau, x(\tau + \omega))
\]

\[
+ 2\|F(\tau, x((\tau + \omega))\| + h(F(\tau, x((\tau + \omega)), F(\tau, x(\tau)))
\]

\[
\leq L_2(\tau)\|1 + x(\tau + \omega)\| + 2\|F(\tau, 0)\| + 2h(F(\tau, x((\tau + \omega)), F(\tau, 0))
\]

\[
+ L_1(\tau)\|x(\tau + \omega) - x(\tau)\|
\]

\[
\leq L_2(\tau)(1 + \|x\|) + 2\sigma(\tau) + 3L_1(\tau)\|x\|.
\]

(29)

By (3) and (29), we arrive at

\[
I_2 = \int_0^\theta \| T_\alpha(\theta - \tau)\| \| f(\tau + \omega) - f(\tau) \| \, d\tau
\]

\[
\leq 3L\|x\| \int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|(|\theta - \tau|^\alpha)}L_1(\tau) \, d\tau
\]
\[ + 2L \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|((\theta - \tau)^{\alpha})} \sigma(\tau) \, d\tau \\
+ L \left(1 + \|x\|\right) \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|((\theta - \tau)^{\alpha})} L_2(\tau) \, d\tau. \] (30)

By arguing as in (24), (28) and (30), one can show

\[
\left\| \int_{s_i}^{s_i+\omega} T_\alpha(s_i + \omega - \tau)f(\tau) \, d\tau - \int_{s_i}^{s_i} T_\alpha(s_i - \tau)f(\tau) \, d\tau \right\| \\
\leq \frac{(\omega_1\|x\| + \omega_2)L}{\alpha|\mu|} \ln \frac{1 + |\mu|(s_i + \omega)^{\alpha}}{1 + |\mu|s_i^{\alpha}} \\
+ 3L\|x\| \int_{s_i}^{s_i} \frac{(s_i - \tau)^{\alpha - 1}}{1 + |\mu|(s_i - \tau)^{\alpha}} L_1(\tau) \, d\tau \\
+ 2L \int_{s_i}^{s_i} \frac{(s_i - \tau)^{\alpha - 1}}{1 + |\mu|(s_i - \tau)^{\alpha}} \sigma(\tau) \, d\tau \\
+ L \left(1 + \|x\|\right) \int_{s_i}^{s_i} \frac{(s_i - \tau)^{\alpha - 1}}{1 + |\mu|(s_i - \tau)^{\alpha}} L_2(\tau) \, d\tau. \] (31)

Since \(s_i \to \infty\) when \(\theta \to \infty\), we can derive (19) from (9), (10), (15), (16), (20), (21)–(23), (28), (30) and (31).

Then, due to (21)–(24), (28), (30) and (31), we arrive at (19).

Step 2. In this step, we show that, if \(x \in SAP_\omega PC(J, E)\) and \(y \in \Theta(x)\), then \(y\) is bounded.

(i) Let \(\theta \in [0, \theta_1]\). Then, using (4)–(6), we get

\[
\|y(\theta)\| \leq L\|x_0\| + L\|x_1\| \left(1 + \max \left\{1, \frac{1}{|\mu|}\right\}\right) \\
+ L \max \left\{1, \frac{1}{|\mu|}\right\} \int_0^\theta \|f(\tau)\| \, d\tau. \] (32)

On the hand, from (7), (9) and (27), we derive

\[
\|f(\tau)\| \leq \|F(\tau, x(\tau))\| \leq \|F(\tau, 0)\| + h(F(\tau, 0), F(\tau, x(\tau))) \\
\leq \sigma(\tau) + L_1(\tau)\|x\| \leq \omega_1 + \omega_2\|x\|. \] (33)

Then, by (32) and (33), we conclude that

\[
\sup_{\theta \in [0, \theta_1]} \|y(\theta)\| \leq L\|x_0\| + L\|x_1\| \left(1 + \max \left\{1, \frac{1}{|\mu|}\right\}\right) \\
+ L \max \left\{1, \frac{1}{|\mu|}\right\} \left[\omega_1 + \omega_2\|x\|\right]. \] (34)

(ii) Let \(\theta \in (\theta_i, s_i]; i \in \mathbb{N}\). In view of (18) and (19), it follows that

\[
\|g_i(\theta, x(\theta_i^-))\| \leq \kappa_1(\|x\| + 1) \] (35)

and

\[
\|g_i'(s_i, x(\theta_i^-))\| \leq \kappa_2(\|x\| + 1). \] (36)
(iii) Let $\theta \in [s_i, \theta_{i+1}]$; $i \in \mathbb{N}$. In view of (4), (5), (35) and (36), we have

$$\sup_{\theta \in [s_i, \theta_{i+1}]} \left\| S_\alpha (\theta - s_i)x_i + K_\alpha (\theta - s_i)x_i \right\| + K_\alpha (\theta - s_i)g_i (s_i, x(\theta_i^-))$$

$$\leq L \kappa_1 (\|x\| + 1) + L \max \left\{ 1, \frac{1}{|\mu_1|} \right\} \kappa_2 (\|x\| + 1).$$

(37)

Furthermore, using (3) and (33), we arrive at

$$\left\| \int_0^\xi T_\alpha (\theta - \tau)f(\tau) d\tau \right\| \leq L \int_0^\xi \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu| \|\theta - \tau\|^\alpha} \|f(\tau)\| d\tau$$

$$\leq L \int_0^\xi \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu| \|\theta - \tau\|^\alpha} \sigma(\tau) d\tau$$

$$+ \|x\| L \int_0^\xi \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu| \|\theta - \tau\|^\alpha} L_1(\tau) d\tau.$$

(38)

From (9), (15) and (38), there is $\xi > 0$ with

$$\sup_{\theta \in J} \left\| \int_0^\xi T_\alpha (\theta - \tau)f(\tau) d\tau \right\| \leq \zeta$$

(39)

and

$$\sup_{i \in \mathbb{N}} \left\| \int_0^{s_i} T_\alpha (s_i - \tau)f(\tau) d\tau \right\| \leq \zeta.$$

(40)

As a result of (34), (35), (37), (39) and (40), we conclude that $y$ is bounded on $J$.

Hence, $\Theta$ is a multivalued function from $SAP_\omega PC(J, E)$ to the non-empty subsets of $SAP_\omega PC(J, E)$.

Step 3. The values of $\Theta$ are closed.

To show this, let $x \in SAP_\omega PC(J, E)$ and $y_n \in \Theta(x)$, $\forall n \geq 1$, with $y_n \to y$ in $SAP_\omega PC(J, E)$. Then we have $f_n \in S_{\theta, \xi}(x(\tau))$ such that

$$y_n(\theta) = \begin{cases} 
S_\alpha (\theta)x_0 + K_\alpha (\theta)x_1 \\
+ \int_0^\theta T_\alpha (\theta - \tau)f_n(\tau) d\tau, \quad \theta \in [0, \theta_1], \\
g_i(\theta, x(\theta_i^-)), \quad \theta \in (s_i, s_{i+1}), i \in \mathbb{N}, \\
S_\alpha (\theta - s_i)x_i + K_\alpha (\theta - s_i)x_i' + K_\alpha (\theta - s_i)g_i (s_i, x(\theta_i^-)) \\
- \int_0^\theta T_\alpha (s_i - \tau)f_n(\tau) d\tau \\
+ \int_0^\theta T_\alpha (\theta - \tau)f_n(\tau) d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}.
\end{cases}$$

(41)

Let $\theta$ be a fixed point in $[0, \theta_1]$ and put $J_0 = [0, \theta]$. Then

$$\left\| f_n(\tau) \right\| \leq \left\| F(\tau, x(\tau)) \right\| \leq \left\| F(\tau, 0) \right\| + L_1(\tau) \left\| x(\tau) \right\|$$

$$\leq \sigma(\tau) + L_1(\tau) \|x\|, \quad \forall \tau \in J_0.$$

(42)
This relation with the fact that $\sigma$ and $L_1$ are continuous guarantee that the family $\{f_n : n \geq 1\}$ is bounded in $L^2(J_0, E)$, and hence, by Mazur’s lemma, there is a sequence $(z_n)_{n \geq 1}$ of convex combinations of $f_n$ with $z_n \to f$ strongly in $L^2(J_0, E)$. Hence, we can suppose, without loss of generality, that $z_n(\theta) \to f(\theta)$, a.e. $\theta \in J_0$. Moreover, from (6) and (42), we get

$$
\| T_\alpha(\theta - \tau) f_n(\tau) \| \\
\leq L \max \left\{ 1, \frac{1}{|\mu|} \right\} (\sigma(\tau) + L_1(\tau))\|x\|, \quad \forall \tau \in J_0.
$$

Therefore, by the Lebesgue dominated convergence theorem and the continuity of $T_\alpha(\theta - \tau)$, we obtain

$$
\lim_{n \to \infty} \int_0^\theta T_\alpha(\theta - \tau) z_n(\tau) \, d\tau \\
= \int_0^\theta T_\alpha(\theta - \tau) f(\tau) \, d\tau.
$$

Consequently,

$$
\lim_{n \to \infty} y_n(\theta) = S_\alpha(\theta) x_0 + K_\alpha(\theta) x_1 + \int_0^\theta T_\alpha(\theta - \tau) f(\tau) \, d\tau, \quad \theta \in [0, \theta_1]. \quad (43)
$$

Similarly, one can show that, for any $\theta \in [s_i, \theta_{i+1}]$, $i \in \mathbb{N}$, we have

$$
\lim_{n \to \infty} y_n(\theta) = S_\alpha(\theta - s_i) g_i(s_i, x(\theta_i^-)) + K_\alpha(\theta - s_i) g_i'(s_i, x(\theta_i^-)) \\
- \int_0^{s_i} T_\alpha(s_i - \tau) f(\tau) \, d\tau + \int_0^\theta T_\alpha(\theta - \tau) f(\tau) \, d\tau. \quad (44)
$$

Remark that $(HF)(iv)$ implies that $f(\theta) \in F(\theta, x(\theta))$, a.e. $\theta \in J$. So, from (43) and (44), we arrive at

$$
\gamma(\theta) = \begin{cases} 
S_\alpha(\theta) x_0 + K_\alpha(\theta) x_1 \\
+ \int_0^\theta T_\alpha(\theta - \tau) f(\tau) \, d\tau, \quad \theta \in [0, \theta_1], \\
g_i(\theta, x(\theta_i^-)), \quad \theta \in [\theta_i, s_i], i \in \mathbb{N}, \\
S_\alpha(\theta - s_i) g_i(s_i, x(\theta_i^-)) + K_\alpha(\theta - s_i) g_i'(s_i, x(\theta_i^-)) \\
- \int_0^{s_i} T_\alpha(s_i - \tau) f(\tau) \, d\tau \\
+ \int_0^\theta T_\alpha(\theta - \tau) f(\tau) \, d\tau, \quad \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}.
\end{cases}
$$

Then $\gamma \in \Theta(x)$.

Step 4. $\Theta$ is a contraction.
Let $u_1, u_2 \in SAP_r\text{PC}(J, E)$ and $y_1 \in \Theta(u_1)$. Then we have $f_1 \in S^1_{F, (\alpha \in J)}$ such that

$$y_1(\theta) = \begin{cases} 
S_{\alpha}(\theta)x_0 + K_{\alpha}(\theta)x_1 \\
+ \int_0^\alpha T_{\alpha}(\theta - \tau)f_1(\tau) \, d\tau, & \theta \in [0, \theta_1], \\
+ g_{\alpha}(\theta, u_1(\theta^-)), & \theta \in (\theta_i, \theta_{i+1}], i \in \mathbb{N}, \\
C_{\alpha}(\theta - s_i)g_{\alpha}(s_i, u_1(\theta_i^-)) + K_{\alpha}(\theta - s_i)g_{\alpha}(s_i, u_1(\theta_i^+)) \\
- \int_0^\alpha T_{\alpha}(\theta - \tau)f_2(\tau) \, d\tau, & \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}.
\end{cases} \quad (45)$$

Consider the multivalued function $\Psi : J \to 2^E$ defined by

$$\Psi(\theta) = \{z \in E : \|z - f_1(\theta)\| \leq L_1(\theta)\|u_1(\theta) - u_2(\theta)\|, \text{a.e. } \theta \in J\}.$$

We clarify that the values of $\Psi$ are non-empty. Let $\theta \in J$. From (5), we get

$$h(F(\theta, u_1(\theta))), F(\theta, u_2(\theta))) \leq L_1(\theta)\|u_1(\theta) - u_2(\theta)\|.$$

So, there is $z_0 \in F(\theta, u_2(\theta))$ such that

$$\|f_1(\theta) - z_0\| \leq h(F(\theta, u_1(\theta))), F(\theta, u_2(\theta))) \leq L_1(\theta)\|u_1(\theta) - u_2(\theta)\|,$$

which leads to $\Psi(\theta) \neq \emptyset; \theta \in J$. Moreover, the set $\Lambda(\theta) = \Psi(\theta) \cap F(\theta, u_2(\theta)); \theta \in J$ is not empty. Because the functions $f_1, L_1, u_1, u_2$ are measurable, Proposition 3.4 in [47] or (Corollary 1.3.1(a) in [48]) guarantees that the multivalued map $\theta \to \Lambda(\theta)$ is measurable. Notice that the set $\Psi(\theta); \theta \in J$ is closed. Consequently, the values of $\Lambda$ are non-empty and compact, and hence, there exists a measurable selection $f_2$ for $\Lambda$ with

$$\|f_1(\theta) - f_2(\theta)\| \leq L_1(\theta)\|u_1(\theta) - u_2(\theta)\|, \text{ a.e. } \theta \in J. \quad (46)$$

We define $y_2 : J \to E$ as follows:

$$y_2(\theta) = \begin{cases} 
S_{\alpha}(\theta)x_0 + K_{\alpha}(\theta)x_1 \\
+ \int_0^\alpha T_{\alpha}(\theta - \tau)f_2(\tau) \, d\tau, & \theta \in [0, \theta_1], \\
+ g_{\alpha}(\theta, u_2(\theta^-)), & \theta \in (\theta_i, \theta_{i+1}], i \in \mathbb{N}, \\
C_{\alpha}(\theta - s_i)g_{\alpha}(s_i, u_2(\theta_i^-)) + K_{\alpha}(\theta - s_i)g_{\alpha}(s_i, u_2(\theta_i^+)) \\
- \int_0^\alpha T_{\alpha}(\theta - \tau)f_2(\tau) \, d\tau, & \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N}.
\end{cases} \quad (47)$$

Obviously, $y_2 \in \Theta(u_1)$. Now, we estimate the quantity $\|y_1 - y_2\|$. To do this, we consider three cases.

Case 1. $\theta \in [0, \theta_1]$. In view of (3), (15), (47), (46) and (47), we have

$$\|y_1(\theta) - y_2(\theta)\| \leq \| \int_0^\theta T_{\alpha}(\theta - \tau)\|f_1(\tau) - f_2(\tau)\| \, d\tau.$$
≤ L\|u_1 - u_2\| \int_0^\vartheta \frac{(\vartheta - \tau)^{\alpha - 1}}{1 + |\mu|(|\vartheta - \tau)^\alpha} L_1(\tau) \, d\tau \\
≤ \xi L\|u_1 - u_2\|. \quad (48)

Case 2. \( \theta \in (\theta_i, s_i] \). Using (11), we get
\[
\|y_1(\theta) - y_2(\theta)\| \\
= \|g(\theta, u_1(\theta^-)) - g(\theta, u_2(\theta^-))\| \\
≤ N\|u_1(\theta^-) - u_2(\theta^-)\| \leq N\|u_1 - u_2\|. \quad (49)
\]

Case 3. \( \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N} \). From (4) and (49), we derive
\[
\|S_{\alpha}(\theta - s_i)g_i(s_i, u_1(\theta^-)) - S_{\alpha}(\theta - s_i)g_i(s_i, u_2(\theta^-))\| \\
≤ L\|g_i(s_i, u_1(\theta^-)) - g_i(s_i, u_2(\theta^-))\| \leq LN\|u_1 - u_2\|. \quad (50)
\]

By (4) and (12), we arrive at
\[
\|K_{\alpha}(\theta - s_i)g'_i(s_i, u_1(\theta^-)) - K_{\alpha}(\theta - s_i)g'_i(s_i, u_2(\theta^-))\| \\
≤ \|K_{\alpha}(\theta - s_i)\| \|g'_i(s_i, u_1(\theta^-)) - g'_i(s_i, u_2(\theta^-))\| \\
≤ L\left(1 + \max\left\{1, \frac{1}{|\mu|}\right\}\right)N\|u_1 - u_2\|. \quad (51)
\]

Moreover, as in (48), one can show that
\[
\left\|\int_0^\vartheta T_{\alpha}(\tau) f_1(\tau) d\tau - \int_0^\vartheta T_{\alpha}(\tau) f_2(\tau) d\tau\right\| \\
≤ L\xi \|u_1 - u_2\| \quad (52)
\]
and
\[
\left\|\int_0^{s_i} T_{\alpha}(s_i - \tau) f_1(\tau) d\tau - \int_0^{s_i} T_{\alpha}(s_i - \tau) f_2(\tau) d\tau\right\| \\
≤ L\xi \|u_1 - u_2\|. \quad (53)
\]

Now, by (48)–(53), we conclude that
\[
\|y_1 - y_2\| \leq (u - v) L\left(2\xi + N + \left(1 + \max\left\{1, \frac{1}{|\mu|}\right\}\right)N\right). \quad (54)
\]

As a consequence of (17), Eq. (54) becomes
\[
\|y_1(\theta) - y_2(\theta)\| < \vartheta \|u - v\|, \quad (55)
\]
where \( \vartheta = L(2\xi + N + (1 + \max\{1, \frac{1}{|\mu|}\})N) < 1 \). By interchange of the role of \( y_1 \) and \( y_2 \) in the above discussion, we conclude that \( \Theta \) is a contraction. So, applying Lemma 2, shows
Lemma 3 ([49], Corollary 3.3.1) Let \( W \) be a closed convex subset of a Banach space \( X \) and \( N : E \to P_{ck}(W) \) be a closed multifunction which is \( \vartheta \)-condensing on every bounded subset of \( W \), where \( \vartheta \) is a non-singular measure of noncompactness defined on subsets of \( W \), then the set of fixed points for \( N \) is non-empty.

Lemma 4 ([50], Lemma 1.2) Let \( D \subseteq PC(J,E) \). Assume that

(i) \( \lim_{\theta \to \infty} \| u(\theta) \| = 0 \), uniformly for \( u \in D \).

(ii) The set \( D_J \) is equicontinuous for every \( i \in \mathbb{N} \), where

\[
D_J = \{ y^* \in C(J,E) : y^*(t) = y(t), t \in J_i = (t_i, t_{i+1}], y^*(t_i) = y(t_i^*), y \in D \}.
\]

(iii) For any \( i \in \mathbb{N} \), and any \( \theta \in J \), the set \( \{ y^*(t) : y^* \in D_J \} \) is relatively compact in \( E \).

Then \( D \) is relatively compact in \( PC(J,E) \).

Theorem 2 Assume that (HA) and the following conditions are verified:

\((HF)^*\) \( F : J \times E \to P_{ck}(E) \) such that:

(i) For every \( x \in E \), the multivalued function \( t \mapsto F(t,x) \) is measurable.

(ii) For almost \( t \in J \), the multivalued function \( x \mapsto F(t,x) \) is upper semicontinuous.

(iii) For any \( x \in PC(J,E) \), the set

\[
S^J_{F(x(t))} := \{ \psi : J \to E, \psi \text{ is locally integrable and } \psi(\tau) \in F(\tau,x(\tau)), \quad a.e. \tau \in J \}
\]

is not empty.

(iv) There exists a continuous function \( \varphi : J \to (0,\infty) \) with

\[
\| F(t,z) \| \leq \varphi(t)(1 + \| z \|) \quad \forall (t,z) \in J \times E.
\]

\((Hg)^*\) For any \( i \in \mathbb{N} \), \( g_i : [\theta_i, s_i] \times E \to E \) \((i \in \mathbb{N})\) is uniformly continuous on bounded sets and for any \( z \in E \), the function \( \theta \mapsto g_i(\theta, z) \) is continuously differentiable at \( s_i \) such that (10), (11), (14) and the following conditions are satisfied:

(i) There is a bounded continuous function \( h^* : J \to J \) with \( \lim_{\theta \to \infty} h^*(\theta) = 0 \) and

\[
\| g_i(\theta, z) \| \leq h^*(\theta)(\| z \| + 1), \quad \forall (i,\theta, z) \in \mathbb{N} \times [t_i, s_i] \times E.
\]
(ii) For any \( t \in J \), the function \( z \to g_i(t, z) \) is compact.

Then Problem (1) has an S-asymptotically \( \omega \)-periodic mild solution provided that the families \( \{S_\alpha(t) : t > 0\} \), \( \{K_\alpha(t) : t > 0\} \) and \( \{T_\alpha(t) : t > 0\} \) are compact,

\[
\lim_{\theta \to \infty} \left( \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|((\theta - \tau)\alpha)} \right) (\varphi(\tau + \omega) + \varphi(\tau)) = 0, \tag{57}
\]

\[
\sup_{\theta \in J} \int_0^\theta \varphi(t) d\tau < \frac{1}{4L \max\{1, \frac{1}{|\mu|}\}}, \tag{58}
\]

and

\[
\eta + L\eta + L\kappa_2 \max \left\{ 1, \frac{1}{|\mu|} \right\} < \frac{1}{2}, \tag{59}
\]

where \( \eta = \sup_{t \in J} \|h^*(t)\| \), and \( \kappa_2 \) is as defined in (14).

**Proof** Due to \((HF)^\ast(iii)\), we can consider a multioperator \( \Theta \) on \( SAP_\omega PC(J, E) \) defined as in (18).

In the following steps we show that \( \Theta \) satisfies the assumptions of Lemma 3.

Step 1. In this step, we demonstrate that, if \( x \in SAP_\omega PC(J, E) \), \( \gamma \in \Theta(x) \), then

\[
\lim_{\theta \to \infty} \|\gamma(\theta + \omega) - \gamma(\theta)\| = 0.
\]

Notice that condition \((Hg_j)^\ast(i)\) implies \((Hg_j)(ii)\), so Eqs. (21)–(23) are satisfied. Now, let \( f \in S_{1, \alpha,i}^1 \) and \( \theta \in [s_1, \theta_{i+1}]; i > 1 \). As in (24), we have

\[
\left\| \int_0^{\theta + \omega} T_\alpha(\theta + \omega - \tau)f(\tau) d\tau - \int_0^\theta T_\alpha(\theta - \tau)f(\tau) d\tau \right\|
\leq \int_0^{\theta + \omega} T_\alpha(\theta - \tau)\|f(\tau + \omega)\| d\tau
+ \int_0^\theta T_\alpha(\theta - \tau)\|f(\tau + \omega) - f(\tau)\| d\tau
= I_1 + I_2. \tag{60}
\]

From the continuity of \( \varphi \), there is \( \rho \) such that \( \sup_{t \in \{0,\omega\}} \varphi(t) \leq \rho \), and hence

\[
\sup_{\tau \in [-\omega,0]} \|f(\tau + \omega)\| \leq \rho (1 + \|x\|). \tag{61}
\]

Then, using (3), \((HF)^\ast(iv)\) and (61), we get

\[
I_1 \leq (1 + \|x\|)\rho L \int_{-\omega}^0 \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|((\theta - \tau)\alpha)} d\tau
= \frac{(1 + \|x\|)\rho L}{\alpha|\mu|} \ln \left( 1 + \frac{|\mu|((\theta + \omega)\alpha)}{1 + |\mu|((\theta - \tau)\alpha)} \right). \tag{62}
\]
Also, in view of (3) and \((HF)^*\text{(iv)}, we arrive at

\[
I_2 \leq \int_0^\theta \left\| T_\theta (\theta - \tau) \right\| f(\tau + \omega) - f(\tau) \, d\tau
\]

\[
\leq 2L(1 + ||x||) \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + ||\mu||(\theta - \tau)^\alpha} (\psi(\tau + \omega) + \psi(t)) \, d\tau.
\]

Then, by (60), (62) and (63), it follows that

\[
\left\| \int_0^{\theta+\omega} T_\theta (\theta + \omega - \tau) f(\tau) \, d\tau - \int_0^\theta T_\theta (\theta - \tau) f(\tau) \, d\tau \right\|
\]

\[
\leq \frac{(1 + ||x||)\rho L}{\alpha ||\mu||} \ln \frac{1 + ||\mu||(\theta + \omega)^\alpha}{1 + ||\mu||\theta^\alpha}
\]

\[
+ 2L(1 + ||x||) \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + ||\mu||(\theta - \tau)^\alpha} (\psi(\tau + \omega) + \psi(t)) \, d\tau.
\]

Similarly,

\[
\left\| \int_0^{s_i+\omega} T_\theta (s_i + \omega - \tau) f(\tau) \, d\tau - \int_0^{s_i} T_\theta (s_i - \tau) f(\tau) \, d\tau \right\|
\]

\[
\leq \frac{(1 + ||x||)\rho L}{\alpha ||\mu||} \ln \frac{1 + ||\mu||(s_i + \omega)^\alpha}{1 + ||\mu||s_i^\alpha}
\]

\[
+ 2L(1 + ||x||) \int_0^{s_i} \frac{(s_i - \tau)^{\alpha - 1}}{1 + ||\mu||(s_i - \tau)^\alpha} (\psi(\tau + \omega) + \psi(t)) \, d\tau.
\]

Then, from (21)–(23), (57), (64) and (65), we derive \(\lim_{\alpha \to \infty} ||y(\theta + \omega) - y(\theta)|| = 0\).

Step 2. Put \(D_\lambda = \{u \in SAP_\alpha PC J, E : ||u|| \leq \lambda\},\) where

\[
\lambda = \frac{L\|x_0\| + L(||x_1|| + \kappa_2)\{1 + \max\{1, \frac{1}{|\mu|}\}\} + \eta + L\eta + \frac{1}{2}}{1 - [\eta + L\eta + L\kappa_2 \max\{1, \frac{1}{|\mu|}\} + \frac{1}{2}]}.
\]

Due to Eq. (59), \(\lambda\) is well defined. In this step, we show that, if \(x \in D_\lambda\) and \(y \in \Theta(x),\) then \(||y|| \leq \lambda\).

(i) Let \(\theta \in [0, \theta_1].\) Then from (4)–(6) and \((HF)^*\text{(iv)}, we get

\[
\left\| y(\theta) \right\| \leq L\|x_0\| + L\|x_1\| \left(1 + \max\{1, \frac{1}{|\mu|}\}\right)
\]

\[
+ (1 + ||x||)\rho L \max\{1, \frac{1}{|\mu|}\} \int_0^{\theta_1} \psi(t) \, d\tau
\]

\[
\leq L\|x_0\| + L\|x_1\| \left(1 + \max\{1, \frac{1}{|\mu|}\}\right) + \frac{1 + ||x||}{4}
\]

\[
\leq L\|x_0\| + L\|x_1\| \left(1 + \max\{1, \frac{1}{|\mu|}\}\right) + \frac{1 + \lambda}{4}.
\]

(ii) Let \(\theta \in (\theta_i, s_i],\) for some \(i \in \mathbb{N}.\) In view of (56), we have

\[
\left\| g(\theta, x(\theta^-)) \right\| \leq \eta(\lambda + 1),
\]

(68)
where \( \eta = \sup_{t \in J} \| h^*(t) \| \). Moreover, from (15), it follows that

\[
\sup_{i \in \mathbb{N}} \| g_i(s_i, x(\theta_i^-)) \| \leq \kappa_2 ( \lambda + 1 ). \tag{69}
\]

Next, let \( \theta \in [s_i, \theta_{i+1}] \). In view of (4), (5) and (69), we arrive at

\[
\| S_\alpha (\theta - s_i) g_i(s_i, x(\theta_{i+1}^-)) + K_\alpha (\theta - s_i) g_i(s_i, x(\theta_i^-)) \|
\leq \left( L \eta + L \kappa_2 \max \left\{ 1, \frac{1}{|\mu|} \right\} \right) (\lambda + 1). \tag{70}
\]

Furthermore, by (6) and \((HF)^*\)(iv), and (58), we obtain

\[
\left\| \int_0^\theta T_\alpha (\theta - t) f(t) \, dt \right\|
\leq L \max \left\{ 1, \frac{1}{|\mu|} \right\} (1 + \| x \|) \int_0^\theta \varphi(t) \, dt
\leq (1 + \| x \|) L \max \left\{ 1, \frac{1}{|\mu|} \right\} \sup_{\theta \in \Theta} \int_0^\theta \varphi(t) \, dt < \frac{\lambda + 1}{4}. \tag{71}
\]

Similarly, we obtain

\[
\sup_{i \in \mathbb{N}} \left\| \int_0^{s_i} T_\alpha (s_i - t) f(t) \, dt \right\| \leq \frac{\lambda + 1}{4}. \tag{72}
\]

As a result of (67), (68), (70)- (72), we arrive at

\[
\sup_{\theta \in J} \| y(\theta) \|
\leq L \| x_0 \| + L \| x_1 \| \left( 1 + \max \left\{ 1, \frac{1}{|\mu|} \right\} \right) + \frac{1 + \lambda}{4}
+ \eta (\lambda + 1) + \left( L \eta + L \kappa_2 \max \left\{ 1, \frac{1}{|\mu|} \right\} \right) (\lambda + 1)
+ \frac{\lambda + 1}{4}
\leq L \| x_0 \| + L \| x_1 \| \left( 1 + \max \left\{ 1, \frac{1}{|\mu|} \right\} \right) + \eta
+ L \eta + L \kappa_2 \max \left\{ 1, \frac{1}{|\mu|} \right\} + \frac{1}{2}
+ \lambda \left[ \eta + L \eta + L \kappa_2 \max \left\{ 1, \frac{1}{|\mu|} \right\} + \frac{1}{2} \right] \leq \lambda.
\]

Therefore, our aim in this step is achieved.

Now, as a result of Steps 1 and 2, \( \Theta \) is a multivalued function from \( D_\lambda \subseteq SAP_{\omega} PC(J, E) \) to the non-empty subsets of \( D_\lambda \).

Step 3. \( \Theta \) is closed (its graph is closed) on \( D_\lambda \).
Let \((x_n)_{n \geq 1}\), \((y_n)_{n \geq 1}\) be two sequences in \(D\) with \(x_n \to x\), \(y_n \to y\) and \(y_n \in \Theta(x_n), \forall n \geq 1\). Then we have \(f_n \in S_{F,(x_n)}\) such that

\[
y_n(\theta) = \begin{cases} 
S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\
+ \int_0^\theta T_\alpha(\theta - \tau)f_\alpha(\tau)\,d\tau, & \theta \in [0,\theta_1], \\
g_\alpha(\theta,x_\alpha(\theta_1^-)), & \theta \in (\theta_1,s_1], i \in \mathbb{N}, \\
S_\alpha(\theta - s_1)g_\alpha(s_1,x_\alpha(\theta_1^-)) + K_\alpha(\theta - s_1)g_\alpha'(s_1,x_\alpha(\theta_1^-)) \\
- \int_0^\theta T_\alpha(s_1 - \tau)f_\alpha(\tau)\,d\tau \\
+ \int_0^\theta T_\alpha(\theta - \tau)f_\alpha(\tau)\,d\tau, & \theta \in [s_1,\theta_{i+1}], i \in \mathbb{N}. 
\end{cases}
\]

Let \(\theta\) be a fixed point in \([0,\theta_1]\) and \(J_0 = [0,\theta]\). In view of \((HF)^*(iv)\), we have

\[
\|f_n(t)\| \leq \varphi(t)(1 + \lambda), \quad \text{a.e. } t \in [0,\theta]. 
\] (73)

Using similar arguments to Step 3 in the proof of Theorem 1, one can show, by (73), that \(f_n \rightharpoonup f\) weakly in \(L^2(J_0,E)\) and there is a sequence of convex combinations \((z_n)\) of \((f_n)\) such that \(z_n \to f\), a.e. \(t \in J_0\), and

\[
\lim_{n \to \infty} \int_0^\theta T_\alpha(\theta - \tau)z_n(\tau)\,d\tau = \int_0^\theta T_\alpha(\theta - \tau)f(\tau)\,d\tau; \quad \theta \in [0,\theta_1]. 
\] (74)

Now, due to the continuity of \(g_\alpha(\theta,\cdot), g_\alpha'(s_1,\cdot), S_\alpha(\theta - s_1)\) and \(K_\alpha(\theta - s_1)\), we arrive at

\[
\lim_{n \to \infty} g_\alpha(\theta,x_n(\theta_1^-)) = g_\alpha(\theta,x(\theta_1^-)) 
\] (75)

and

\[
\lim_{n \to \infty} S_\alpha(\theta - s_1)g_\alpha(s_1,x_n(\theta_1^-)) + K_\alpha(\theta - s_1)g_\alpha'(s_1,x_n(\theta_1^-)) \\
= S_\alpha(\theta - s_1)g_\alpha(s_1,x(\theta_1^-)) + K_\alpha(\theta - s_1)g_\alpha'(s_1,x(\theta_1^-)). 
\] (76)

Noting that \((z_n)\) is a subsequence of \((f_n)\), and hence by (74)–(76), there is a subsequence of \((y_n)\) that converge to

\[
y^*(t) = \begin{cases} 
S_\alpha(\theta)x_0 + K_\alpha(\theta)x_1 \\
+ \int_0^\theta T_\alpha(\theta - \tau)f(\tau)\,d\tau, & \theta \in [0,\theta_1], \\
g_\alpha(\theta,x(\theta_1^-)), & \theta \in (\theta_1,s_1], i \in \mathbb{N}, \\
S_\alpha(\theta - s_1)g_\alpha(s_1,x(\theta_1^-)) + K_\alpha(\theta - s_1)g_\alpha'(s_1,x(\theta_1^-)) \\
- \int_0^\theta T_\alpha(s_1 - \tau)f(\tau)\,d\tau \\
+ \int_0^\theta T_\alpha(\theta - \tau)f(\tau)\,d\tau, & \theta \in [s_1,\theta_{i+1}], i \in \mathbb{N}. 
\end{cases}
\]

Because \(y_n \to y\), we arrive at \(y = y^*\). Moreover, \((HF)^*(ii)\) ensures that \(f(\tau) \in F(\tau,x(\tau))\), a.e. \(\tau \in J\). So, \(y \in \Theta(x)\).

Step 4. \(\lim_{\theta \to \infty} \|y(\theta)\| = 0\) uniformly on \(D_x\).
Let \( x \in D_\alpha \) and \( y \in \Theta(x) \). According to the definition of \( \Theta \), there is \( f \in S_{F(x)}^1 \) such that \( y \) is given by (18). We consider two cases:

Case 1. \( \theta \in (\theta_i, \theta_{i+1}], i \in \mathbb{N} \). By (56), we get

\[
\| y(\theta) \| = \| g_i(\theta, x(\theta_i^-)) \| \leq (\lambda + 1) \| h^*(\theta) \|.
\]  

(77)

Case 2. \( \theta \in [\theta_i, \theta_{i+1}], i \in \mathbb{N} \). In view of (3), (15), (56) and \((HF)^*(iv)\), we find

\[
\| y(\theta) \| = \| S_{\nu}(\theta - s_i)g_i(s_i, x(\theta_i^-)) \| \\
+ \| K_{\nu}(\theta - s_i)g_i^*(s_i, x(\theta_i^-)) \| \\
+ \int_{0}^{\theta_i} f(u - \tau) d\tau \\
+ \int_{0}^{\theta} f(u - \tau) d\tau \\
\leq (\lambda + 1) \eta \frac{L}{1 + |\mu| (\theta - s_i)^\mu} + (\lambda + 1) \kappa_2 \frac{L(\theta - s_i + 1)}{1 + |\mu| (\theta - s_i)^\mu} \\
+ L(1 + \lambda) \int_{0}^{\theta_i} \frac{L(s_i - \tau)^{\alpha - 1}}{1 + |\mu| (s_i - \tau)^\mu} \psi(\tau) d\tau \\
+ L(1 + \lambda) \int_{0}^{\theta} \frac{L(\theta_i - \tau)^{\alpha - 1}}{1 + |\mu| (\theta_i - \tau)^\mu} \psi(\tau) d\tau.
\]  

(78)

It follows from (77) (78) and (57) that \( \lim_{\nu \to \infty} \| y(\theta) \| = 0 \) uniformly on \( D_\alpha \).

Step 5. Let \( D = \Theta(D_\alpha) \). In this step, we claim that the set \( D_{\mathcal{I}} \) is equicontinuous for every \( i \in \mathbb{N} \), where

\[
D_{\mathcal{I}} = \{ y^* \in C(\mathcal{I}, E) : y^*(\theta) = y(\theta), t \in \mathcal{I} = [\theta_i, \theta_{i+1}], y^*(\theta_i) = y(\theta_i^+), y \in D \}.
\]

Let \( y^* \in D_{\mathcal{I}} \). Then we have \( x \in D \) and \( f \in S_{F(x)}^1 \) such that

\[
y(\theta) = \begin{cases} 
S_{\nu}(\theta)x_0 + K_{\nu}(\theta)x_1 \\
+ \int_{0}^{\theta} f(u - \tau) d\tau, & \theta \in [0, \theta_1], \\
S_{\nu}(\theta - s_i)g_i(s_i, x(\theta_i^-)) + K_{\nu}(\theta - s_i)g_i^*(s_i, x(\theta_i^-)) \\
- \int_{0}^{\theta_i} f(u - \tau) d\tau + \int_{0}^{\theta} f(u - \tau) d\tau, & \theta \in [s_i, \theta_{i+1}], i \in \mathbb{N},
\end{cases}
\]

and \( y^*(\theta) = y(\theta^*) \).

Case 1. Let \( \theta_1, \theta_2 \in \mathcal{I}_0 = [0, \theta_1] \) with \( \theta_1 < \theta_2 \). We have

\[
\| y^*(\theta_2) - y^*(\theta_1) \| = \| y(\theta_2) - y(\theta_1) \| \\
\leq \| S_{\nu}(\theta_2) - S_{\nu}(\theta_1) \| ||x_0|| + \| K_{\nu}(\theta_2) - K_{\nu}(\theta_1) \| ||x_1|| \\
+ \int_{0}^{\theta_2} f(u - \tau) d\tau - \int_{0}^{\theta_1} f(u - \tau) d\tau
\]
From the compactness of the families \( \{S_{\theta}(\theta) : \theta > 0\} \) and \( \{K_{\theta}(\theta) : \theta > 0\} \), we get

\[
\lim_{\theta_2 \to \theta_1} Q_1 = \lim_{\theta_2 \to \theta_1} \left\| S_{\theta}(\theta) - S_{\theta}(\theta_1) \right\| \| x_0 \| + \left\| K_{\theta}(\theta_2) - K_{\theta}(\theta_1) \right\| \| x_1 \| = 0, \tag{80}
\]

and from the continuity of \( \psi \), we arrive at

\[
\lim_{\theta_2 \to \theta_1} Q_2 = \lim_{\theta_2 \to \theta_1} \int_{\theta_1}^{\theta_2} \left\| T_{\theta}(\theta_2 - \tau) \right\| \| f(\tau) \| d\tau 
\leq (1 + \lambda)L \max \left\{ 1, \frac{1}{|\mu|} \right\} \lim_{\theta_2 \to \theta_1} \int_{\theta_1}^{\theta_2} \psi(\tau) d\tau = 0. \tag{81}
\]

Moreover, since the family \( \{T_{\theta}(\theta) : \theta > 0\} \) is compact, we get

\[
\lim_{\theta_2 \to \theta_1} Q_3 = \lim_{\theta_2 \to \theta_1} \int_{0}^{\theta_1} \left\| T_{\theta}(\theta_2 - \tau) - T_{\theta}(\theta_1 - \tau) \right\| \| f(\tau) \| d\tau 
\leq (1 + \lambda) \lim_{\theta_2 \to \theta_1} \int_{0}^{\theta_1} \left\| T_{\theta}(\theta_2 - \tau) - T_{\theta}(\theta_1 - \tau) \right\| \psi(\tau) d\tau 
= 0. \tag{82}
\]

Equations (79)–(82) lead to \( \lim_{\theta_2 \to \theta_1} \left\| y^{*}(\theta_2) - y^{*}(\theta_1) \right\| = 0 \).

Case 2. Let \( \theta_1, \theta_2 \in (\theta_i, s_i) \) (\( i \in \mathbb{N} \)) with \( \theta_1 < \theta_2 \). From the fact that \( g_i \) is uniformly continuous on bounded sets,

\[
\lim_{\theta_2 \to \theta_1} \left\| y^{*}(\theta_2) - y^{*}(\theta_1) \right\| = \lim_{\theta_2 \to \theta_1} \left\| y(\theta_2) - y(\theta_1) \right\| 
\leq \lim_{\theta_2 \to \theta_1} \left\| g_i(\theta_2, x(\theta_2)) - g_i(\theta_1, x(\theta_1)) \right\| 
\leq \lim_{\theta_2 \to \theta_1} \sup_{\| z \| \leq \lambda} \left\| g_i(\theta_2, z) - g_i(\theta_1, z) \right\| = 0. \tag{83}
\]

Case 3. Let \( \theta_1, \theta_2 \in (s_i, \theta_{i+1}) \) (\( i \in \mathbb{N} \)) with \( \theta_1 < \theta_2 \).

\[
\lim_{\theta_2 \to \theta_1} \left\| y^{*}(\theta_2) - y^{*}(\theta_1) \right\| = \lim_{\theta_2 \to \theta_1} \left\| y(\theta_2) - y(\theta_1) \right\| 
\leq \lim_{\theta_2 \to \theta_1} \left\| S_{\theta}(\theta_2 - s_i) - S_{\theta}(\theta_1 - s_i) \right\| \| g_i(s_i, x(\theta_1)) \| 
+ \lim_{\theta_2 \to \theta_1} \left\| K_{\theta}(\theta_2 - s_i) - K_{\theta}(\theta_1 - s_i) \right\| \| g_i(s_i, x(\theta_1)) \| 
+ \lim_{\theta_2 \to \theta_1} \int_{0}^{\theta_1} T_{\theta}(\theta_2 - \tau)f(s) \, ds - \int_{0}^{\theta_1} T_{\theta}(\theta_1 - \tau)f(s) \, ds. \tag{84}
\]
Again, from the compactness of the families \( \{S_a(\theta) : \theta > 0\} \) and \( \{K_a(\theta) : \theta > 0\} \), we have
\[
\lim_{\theta_2 \to \theta_1} \left\| S_a(\theta_2 - s_i) - S_a(\theta_1 - s_i) \right\| g_i(s_i, x(\theta_i^*)) = \lim_{\theta_2 \to \theta_1} \left\| K_a(\theta_2 - s_i) - K_a(\theta_1 - s_i) \right\| g_i(s_i, x(\theta_i^*)) = 0. \tag{85}
\]

Furthermore, by repeating the arguments employed in Case 1, one can show that
\[
\lim_{\theta_2 \to \theta_1} \left\| \int_0^{\theta_2} T_a(\theta_2 - \tau) f(s) d\tau - \int_0^{\theta_1} T_a(\theta_1 - \tau) f(s) d\tau \right\| = 0. \tag{86}
\]

Equations (84) and (86) ensure that \( \lim_{\theta_2 \to \theta_1} \| y^*(\theta_2) - y^*(\theta_1) \| \). Therefore, \( T_{\mathcal{F}} \) is equicontinuous for any \( i \in \mathbb{N} \).

Step 6. Our goal in this step is showing that, for any \( i \in \mathbb{N} \) and any \( \theta \in J_i \), the set \( Z_i^\delta := \{y(\theta) : y \in D = \Theta(D_\lambda)\} \) is relatively compact in \( E \).

Case 1. Let \( \theta \in J_0 = [0, \theta_1] \). If \( \theta = 0 \), then \( Z_i^\delta = \{x_0\} \) is compact. Let \( \theta \in (0, \theta_1] \) be a fixed point. We have
\[
Z_i^\delta = \{ y(\theta) : y \in \Theta(D_\lambda) \} = \left\{ S_a(\theta)x_0 + K_a(\theta)x_1 + \int_0^\theta T_a(\theta - \tau) f(\tau) d\tau : f \in S_{F(\alpha)}, x \in D_\lambda \right\}.
\]

Now, for any \( \delta \in (0, \theta) \), let
\[
Z_{i,\delta}^1 := \left\{ S_a(\theta)x_0 + K_a(\theta)x_1 + \int_0^{\theta-\delta} T_a(\theta - \tau) f(\tau) d\tau : f \in S_{F(\alpha)}, x \in D_\lambda \right\}.
\]

Notice that \( \| f(\tau) \| \leq (1 + \lambda)\phi(\tau) \), for any \( \tau \in [0, \theta - \delta] \), any \( n \geq 1 \). Because \( \phi \) is continuous on \( [0, \theta] \), the set \( \{f_n(\tau) : n \geq 1\} \) is bounded, and hence, by the compactness of \( T_a(\theta - \tau) \); \( \tau \in [0, \theta - \delta] \), the set \( Z_{i,\delta}^1 \) is relatively compact in \( E \). Moreover, for any \( x \in D_\lambda \) and any \( f \in S_{F(\alpha)} \), we get from (6), \((HF)^*(iv)\) and the continuity of \( \phi \),
\[
\lim_{\delta \to 0} \left\| \int_0^\theta T_a(\theta - \tau) f(\tau) d\tau - \int_0^{\theta-\delta} T_a(\theta - \tau) f(\tau) d\tau \right\| \leq L \max \left\{ 1, \frac{1}{|\mu|} \right\} (1 + \lambda) \lim_{\delta \to 0} \int_0^\theta \phi(\tau) d\tau = 0.
\]

Then there exist relatively compact sets that can be arbitrarily approximated to the set \( Z_i^\delta \). Then it is relatively compact in \( E \).

Case 2. Let \( \theta \in [s_i, \theta_1] ; i \in \mathbb{N} \) be a fixed point. Since the set \( \{x(\theta_i^+) : x \in D_{\lambda i}\} \) is bounded, then by \((Hg)^*(ii)\), the set \( \{y(\theta) : y \in \Theta(x), x \in D_\lambda \} = \{g_i(\theta, x(\theta_i^+)) : x \in D_{\lambda i}\} \) is relatively compact in \( E \).

Case 3. Let \( \theta \in [\theta_i, s_{i+1}] ; i \in \mathbb{N} \) be a fixed point. As in Case 2, the set \( \{g_i(\theta, x(\theta_i^+)) : x \in D_{\lambda i}\} \) is relatively compact in \( E \), and hence, by the compactness of \( S_a(\theta - s_i) \), we conclude that
\[ \{ S_{\alpha}(\theta - s_i)g(s_i, x(\theta^{-1})) : s \in D_{\lambda} \} \text{ is relatively compact in } E. \] Also, due to the compactness of \( K_\alpha(t) ; t > 0 \), the set \( \{ K_\alpha(\theta - s_i)g(s_i, x(\theta^{-1})) : s \in D_{\lambda} \} \text{ is relatively compact in } E. \] Furthermore, using the same arguments as in Case 1, one can show that the sets \( \{ \int_0^t T_\alpha(\theta - \tau)f(\tau) \, d\tau : f \in S_{F_0, x(\theta)}^1(x \in D_{\lambda}) \} \) and \( \{ \int_0^t T_\alpha(s_i - \tau)f(\tau) \, d\tau : f \in S_{F_0, x(\theta)}^1(x \in D_{\lambda}) \} \) are relatively compact.

As a result of this discussion, we conclude that, for any \( i \in \mathbb{N} \) and any \( t \in J_i \), the set \( Z_0^i = \{ y(\theta) : y \in \Theta(D_{\lambda}) \} \) is relatively compact in \( E \).

Now, according to Lemma 4, Steps 4, 5 and 6 imply the set \( T = \Theta(D_{\lambda}) \) is relatively compact in \( SAP_{\omega} PC(J, E) \).

Now, as a result of Steps 1–6, we conclude that \( \Theta \) is closed and completely continuous from \( D_{\lambda} \) to the family on non-empty convex compact of \( D_{\lambda} \). Applying Lemma 3, \( \Theta \) has a fixed point which is an \( S \)-asymptotically \( \omega \)-periodic mild solution to Problem (1).

\[ \] \[ 4 \text{ Examples} \]

**Example 1** Let \( \alpha = \frac{3}{2}, m = 4, \omega = 2\pi, f = [0, \infty), s_i = i \pi ; i \in \{0\} \cup \mathbb{N} \), and \( \theta_i = (2i - 1) \frac{2\pi}{4} ; i \in \mathbb{N} \).

Observe that, for \( i \in \mathbb{N} \), \( s_{i+m} = s_{i+4} = (i + 4) \pi = s_i + 2\pi \), and \( \theta_{i+m} = (2(i + 4) - 1) \frac{2\pi}{4} = \theta_i + 2\pi \).

Let \( \Omega = \{ s = (s_1, s_2) : s_1^2 + s_2^2 \leq 1 \} \), and \( E = L^2(\Omega) \). Define an operator \( A : D(A) \subseteq E \rightarrow E \) by

\[ A(u) := \Delta u - u, \quad (88) \]
with \( D(A) = H^2(\Omega) \cap H^1_0(\Omega) \). It is known that (see [19]) \( A \) is a sectorial operator of type \([M, \varphi, \alpha, \mu] \) with \( \mu = -1 \) and \( L = 3 \). Let \( Z \) be a non-empty, compact and convex subset of \( E, v = \sup_{z \in Z} \|z\| \). Consider the multivalued function \( F : J \times E \rightarrow 2^E \) defined by

\[
F(\theta, \upsilon) := \{v_{\theta, \upsilon, z} : \upsilon \in Z\},
\]

where for any \( \upsilon \in Z \) any \( (\theta, \upsilon) \in J \times E, v_{\theta, \upsilon, z} : \Omega \rightarrow \mathbb{R} \); \( v_{\theta, \upsilon, z}(s) = \frac{1 + \rho \sin(\alpha)}{(\theta + 1)\upsilon} \|z\| \); \( \rho > 0 \). Since \( Z \) is a non-empty, convex and compact subset, the values of \( F \) also are. Moreover,

\[
\|F(\theta, 0)\| = \sup_{\upsilon \in Z} \|v_{\theta, \upsilon, z}\| \leq \frac{1}{\theta + 1} = \sigma(\theta), \quad \forall \theta \in J,
\]

\[
h(F(\theta, \upsilon), F(\theta, \nu)) \leq \left( \int_{\Omega} \left| v_{\theta, \upsilon, z}(s) - v_{\theta, \upsilon, z}(s) \right|^2 ds \right)^{\frac{1}{2}}, \quad \forall \upsilon \in Z
\]

\[
= \frac{\rho}{\theta + 1} \int_{\Omega} \left| u(s) - \sin(\nu(s)) \right|^2 ds \frac{1}{\theta + 1}
\]

\[
\leq \frac{\rho}{\theta + 1} \int_{\Omega} \left| u(s) - v(s) \right|^2 ds \frac{1}{\theta + 1}
\]

\[
\leq \frac{\rho}{\theta + 1} \|u - v\|,
\]

and

\[
h(F(\theta + 2\pi, \upsilon), F(\theta, \upsilon)) \leq \left( \int_{\Omega} \left| v_{\theta + 2\pi, \upsilon, z}(s) - v_{\theta, \upsilon, z}(s) \right|^2 ds \right)^{\frac{1}{2}}, \quad \forall \upsilon \in Z
\]

\[
= \int_{\Omega} \left| \rho \sin(u(s) + 1) \right| \left| u(s) - \frac{\rho \sin(u(s) + 1)}{\theta + 1} \right| ds \frac{1}{\theta + 1}
\]

\[
\leq \frac{1}{(\theta + 1 + 2\pi)(\theta + 1)} \left( \int_{\Omega} \left| \rho \sin(u(s) + 1) \right|^2 ds \right)^{\frac{1}{2}}
\]

\[
\leq \frac{2\pi (\rho + 1)}{(\theta + 1 + 2\pi)(\theta + 1)} (\|u\| + 1) = L_2(\theta)(\|u\| + 1).
\]

Due to (91) and (92), Eqs. (7) and (8) are verified with \( L_1(\theta) = \frac{\rho}{\theta + 1} \) and \( L_2(\theta) = \frac{2\pi (\rho + 1)}{(\theta + 1 + 2\pi)(\theta + 1)} \); \( \theta \in [0, \infty) \).

Next, in view of (87), and since \( |\mu| = 1 \), it follows that

\[
\frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|^2(\theta - \tau)^2} \leq \frac{2}{1 + \theta - \tau}, \quad \forall \theta \geq 0, \tau \in [0, \theta].
\]

Therefore,

\[
\int_0^\theta \frac{(\theta - \tau)^{\alpha-1}}{1 + |\mu|^2(\theta - \tau)^2} L_1(\tau) \, d\tau
\]

\[
\leq 2\rho \int_0^\theta \frac{1}{1 + \theta - \tau} \frac{1}{\tau + 1} \, d\tau
\]

\[
= 2\rho \int_0^\theta \frac{1}{\theta + 2} \left[ \frac{1}{1 + \theta - \tau} + \frac{1}{\tau + 1} \right] \, d\tau
\]

\[
= 4\rho \frac{\ln|1 + \theta|}{(\theta + 2)},
\]
which yields \( \lim_{\theta \to \infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|(\theta - \tau)^2} L_1(\tau) \, d\tau = 0. \) Similarly, \( \lim_{\theta \to \infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|(\theta - \tau)^2} \sigma(\tau) \, d\tau = 0. \) Hence, (9) and (15) are verified. Moreover,

\[
\xi = \sup_{t \in J} \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|(\theta - \tau)^2} L_1(\tau) \, d\tau \leq 4\rho. \tag{93}
\]

Now, again from (87), one has

\[
\lim_{\theta \to \infty} \int_0^\theta \frac{(\theta - \tau)^{\alpha - 1}}{1 + |\mu|(\theta - \tau)^2} L_2(\tau) \, d\tau \\
\leq (\rho + 1) \lim_{\theta \to \infty} \int_0^\theta \frac{1}{1 + \theta - \tau} \left( \frac{2\pi}{\tau + 1 + 2\pi}(\tau + 1 + 2\pi) \right) \, d\tau \\
\leq (\rho + 1) \lim_{\theta \to \infty} \int_0^\theta \left[ \frac{1}{1 + \theta - \tau} + \frac{1}{\tau + 1 + 2\pi} \right] \, d\tau \\
= (\rho + 1) \lim_{\theta \to \infty} \frac{2\ln(1 + \theta)}{\theta + 2} \\
- (\rho + 1) \lim_{\theta \to \infty} \frac{1}{\theta + 2 + 2\pi} \left[ \ln(1 + \theta) + \ln(1 + 2\pi + \theta) - \ln(2\pi + 1) \right] \\
= 0, \tag{94}
\]

which means that (16) holds.

Furthermore, for any \( i \in \mathbb{N}, \) define \( g_i : [t_i, s_i] \times E \to E \) by

\[
g_i(\theta, u)(s) := \frac{\varrho u(s) \sin i\theta}{i}; \quad (\theta, u) \in [\theta_i, s_i] \times E, s \in \Omega, \tag{95}
\]

where \( \varrho \) is a positive real number. Obviously for any \( u \in E, \) \( \frac{d}{dt}(g_i(\theta, u))(s) = \varrho(\cos i\theta)u(s); \) \( t \in J. \) Notice that, for any \( u \in E, \)

\[
\lim_{i \to \infty} \left\| g_{i+m}(\theta + 2\pi, u) - g_i(\theta, u) \right\| \\
= \lim_{i \to \infty} \left( \int_{\Omega} \left| \frac{\varrho u(s) \sin (i + m)(\theta + 2\pi)}{i + m} - \frac{\varrho u(s) \sin (i\theta)}{i} \right|^2 \, ds \right)^{\frac{1}{2}} \\
\leq \lim_{i \to \infty} \frac{2\varrho}{i} \| u \| = 0.
\]

Then (10) is satisfied. Moreover, for any \( u_1, u_2 \in E, \)

\[
\left\| g_i(\theta, u_1) - g_i(\theta, u_2) \right\| \\
= \frac{\varrho}{i} \left( \int_{\Omega} \left| u_1(s) \sin i\theta - u_2(s) \sin i\theta \right|^2 \, ds \right)^{\frac{1}{2}} \\
\leq \varrho \| u_1 - u_2 \| \tag{96}
\]
and

\[
\|g'(s,t) - g'(s,u)\| = \left( \int_{\Omega} \|u_1(s)\cos i\theta - u_2(s)\cos i\theta\|^2 \, ds \right)^{\frac{1}{2}} 
\leq \frac{\theta}{I} \|u_1 - u_2\|. \tag{97}
\]

From (96) and (97), it follows that (11) and (12) hold with \(N = \mathcal{N} = \varrho\). In addition,

\[
\|g(\theta, u)\| = \left( \int_{\Omega} \|u(s)\sin i\theta\|^2 \, ds \right)^{\frac{1}{2}} \leq \frac{\theta}{I} \|u\| \leq \varrho(\|u\| + 1) \tag{98}
\]

and

\[
\|g'(s,t)\| = \left( \int_{\Omega} \|u(s)\cos is\|^2 \, ds \right)^{\frac{1}{2}} \leq \varrho \|u\| \leq \varrho(\|u\| + 1). \tag{99}
\]

By (98) and (99), we arrive at (13) and (14) are verified where \(\kappa_1 = \kappa_1 = \varrho\). Now, all the assumptions of Theorem 1 are satisfied, so the problem

\[
\begin{align*}
\partial \mathfrak{D}_{\omega}\mathfrak{a} x(\theta, s) \in & A x(\theta, s) + F(\theta, x(\theta)), \quad \text{a.e. } \theta \in (s, \theta_{i+1}], i \in \mathbb{N} \cup \{0\}, s \in \Omega, \\
x(\theta^*, s) = & g_i(\theta, x(\theta^*, s)), \quad i \in \mathbb{N}, s \in \Omega, \\
x(\theta, s) = & g_i(\theta, x(\theta^*, s)), \quad \theta \in (\theta_i, \theta_{i+1}], i \in \mathbb{N}, s \in \Omega, \\
x(0, s) = & x_0(s), \quad x'(0) = x_1(s); \quad s \in \Omega, \\
x(\theta, s) = & 0, \quad \theta \in J, s \in \partial \Omega,
\end{align*}
\]

has an \(S\)-asymptotically \(2\pi\)-periodic mild solution, \(x : J \rightarrow L^2(\Omega)\), provided that

\[
8\varrho + 3\varrho < \frac{1}{3}, \tag{100}
\]

where \(A, F, g_i\) are given by (88), (90) and (94). By choosing \(\varrho\) and \(\varrho\) sufficiently small, we can arrive at (100).

**Example 2** Let \(A, \alpha, E, \omega = 2\pi, J, s_i, \theta_i; i \in \mathbb{N}\) be as in Example 1, \(Z\) be a non-empty convex compact subset of \(E\), the families \([S_a(t) : t > 0], [K_a(t) : t > 0]\) and \([T_a(t) : t > 0]\) are compact [19]. Define a multivalued function \(F : J \times E \rightarrow P_{\mathfrak{c}}(E)\) by

\[
F(\theta, u) = \frac{\rho(\|u\| + 1)}{\nu(1 + \theta)^{\frac{1}{2}}} Z, \tag{101}
\]

where \(\rho > 0\) and \(\nu = \text{Sup}(\|z\| : z \in Z)\). Clearly for every \(x \in E, \theta \rightarrow F(\theta, x)\) is strongly measurable, \(F(\theta, \cdot)\) is upper semicontinuous and, for any \(u \in PC(J, E)\), the function \(f(\theta) = \frac{\rho(\|u\| + 1)}{\nu(1 + \theta)^{\frac{1}{2}}} z_0; z_0 \in Z\) is locally integrable and \(f(\theta) \in F(\theta, u(\theta)); \theta \in J\). Moreover, in view of (101), for any \(u \in E\) and any \(\theta \in J\),

\[
\|F(\theta, u)\| \leq \frac{\rho(\|u\| + 1)}{(1 + \theta)^{\frac{1}{2}}} = \varphi(\theta)(1 + \|u\|), \tag{102}
\]
where
\[
\varphi(\theta) = \frac{\rho}{(1 + \theta)^{\frac{3}{2}}}; \quad \theta \in \mathcal{J}.
\] (103)

We show that (57) and (58) are verified. In view of (87), and by arguing as in (94), one has
\[
\lim_{\theta \to \infty} \int_0^\theta \frac{(\theta - \tau)^{\frac{3}{2}}}{1 + |\mu|((\theta - \tau)^2 (\varphi(\tau + 2\pi) + \varphi(\tau))) d\tau
\]
\[
= \rho \lim_{\theta \to \infty} \int_0^\theta \left[ \frac{1}{1 + \theta - \tau} \left( \frac{1}{(\tau + 1 + 2\pi)^2} + \frac{1}{(\tau + 1)^2} \right) \right] d\tau
\]
\[
\leq \rho \lim_{\theta \to \infty} \int_0^\theta \left[ \frac{1}{1 + \theta - \tau} \left( \frac{1}{(\tau + 1 + 2\pi)} + \frac{1}{\tau + 1} \right) \right] d\tau
\]
\[
\leq 2 \rho \lim_{\theta \to \infty} \int_0^\theta \frac{1}{1 + \theta - \tau} \frac{1}{\tau + 1} d\tau = 0.
\]

Then (57) holds. Furthermore,
\[
\sup_{\theta \in \mathcal{J}} \int_0^\theta \varphi(\tau) d\tau = \sup_{\theta \in \mathcal{J}} \frac{2\rho}{\sqrt{1 + \theta}} = 2\rho.
\] (104)

Next, let \( K : D(K) = E \to E \) be a linear bounded compact operator and for any \( i \in \mathbb{N} \), define \( g_i : [\theta_i, s_i] \times E \to E \) by
\[
g_i(\theta, u)(s) = \frac{(Ku)(s)}{i(1 + \theta)}, \quad \forall s \in \Omega.
\] (105)

Notice that
\[
g_i'(s_i, u)(s) = \frac{-i(1 + s_i)Ku(s)}{i(1 + s_i)^2}, \quad \forall s \in \Omega.
\] (106)

In view of (105) and (106), we get
\[
\lim_{i \to \infty} \left\| g_{i+m}(\theta + 2\pi, u) - g_i(\theta, u) \right\|
\]
\[
= \|K\| \lim_{i \to \infty} \left( \int_\Omega \left| \frac{u(s)}{(i + 2\pi)(1 + \theta + 2\pi)} - \frac{u(s)}{i(1 + \theta)} \right|^2 ds \right) \frac{1}{2}
\]
\[
= \|K\| \|u\| \lim_{i \to \infty} \left| \frac{1}{(i + 2\pi)(1 + \theta + 2\pi)} - \frac{1}{i(1 + \theta)} \right| = 0,
\]

\[
\left\| g_i(\theta, u_1) - g_i(\theta, u_2) \right\|
\]
\[
= \left( \int_\Omega \left| \frac{(Ku_1)(s)}{i(1 + \theta)} - \frac{(Ku_2)(s)}{i(1 + \theta)} \right|^2 ds \right)^{\frac{1}{2}}
\]
\[
\leq \frac{\|K\|}{i(1 + \theta)} \| u_1 - u_2 \| \leq \|K\| \|u_1 - u_2\|,
\]
\[\|g_i(\theta, u)\| = \left( \int_{\Omega} \left\| \frac{(Ku)(s)}{i(1 + \theta)} \right\|^2 ds \right)^{1/2} \leq \|K\| \|u\| + h^* (\theta) \|u\|,\]

and

\[\|g_i'(s, u)\| = \left( \int_{\Omega} \left\| \frac{(Ku)(s)}{i(1 + s)^2} \right\|^2 ds \right)^{1/2} \leq \frac{\|K\| \|u\|}{i(1 + s)^2} \leq \|K\| (\|u\| + 1).\]

Then (10), (11), (14) and (56) are verified with \(N = N = \|K\| \text{ and } h^* (\theta) = \frac{\|K\|}{1 + \theta}; \theta \in J \) and \(\eta = \kappa_2 = \|K\|\). Notice that, by the compactness of \(K\), the function \(g_i(\theta, \cdot); i \in \mathbb{N}\) is compact, and hence all assumptions of Theorem 2 are satisfied. So, by applying Theorem 2, Problem (1), where \(A\) as be in example (1) and \(F, g_i\) are given by (102) and (105), has an \(S\)-asymptotically \(2\pi\)-periodic mild solution provided that \(2 \rho < \frac{1}{14} = \frac{1}{14}\) and \(\|K\| < \frac{1}{14}\).

**Example 3** Let \(A, \alpha, E, m, \omega = 2\pi, J, \Omega, s_i, \theta_i; i \in \mathbb{N}\) be as in Example 1, \(Z\) a non-empty convex compact subset of \(E\) and \(x_0, x_1\) two fixed elements of \(E\). For any \((\theta, s) \in (J, \Omega)\) and any \(x : J \rightarrow E\), we denote \(x(\theta)(s)\) by \(x(\theta, s)\). Consider the impulsive semilinear differential inclusion:

\[
\begin{cases}
\frac{d^\alpha}{ds^\mu} x(\theta, s) & \in \Delta x(\theta, s) - x(\theta, s) + \frac{\cos x(\theta, s)}{40(1 + \theta)} Z, \\
\quad \text{a.e. } \theta \in (s_i, \theta_{i+1}], i \in \mathbb{N} \cup \{0\}, s \in \Omega, \\
x(\theta^+, s) = \frac{x(\theta^-, s)}{40^\alpha} \sin \theta^-, & \quad i \in \mathbb{N}, s \in \Omega, \\
x(\theta, s) = \frac{x(\theta, s)}{40^\alpha} \sin \theta, & \quad \theta \in (\theta_i, s_i], i \in \mathbb{N}, s \in \Omega, \\
x(0, s) = x_0(s), & \quad x'(0) = x_1(s); \quad s \in \Omega, \\
x(\theta, s) = 0, & \quad \theta \in J, s \in \partial \Omega.
\end{cases}
\]

Let \(F : J \times E \rightarrow 2^E\) be defined by

\[F(\theta, u) = \frac{\cos u(s)}{40(1 + \theta)} Z.\]

Obviously, \(F\) verifies (i) and (ii) of (HF). Moreover,

\[\|F(\theta, 0)\| \leq \frac{1}{40(1 + \theta)} = \sigma(\theta), \quad \forall \theta \in J,\]

\[h(F(\theta, u), F(\theta, v)) \leq \frac{1}{40(1 + \theta)} \|u - v\|,\]

and

\[h(F(\theta + 2\pi, u), F(\theta, u)) \leq \frac{1}{40(\theta + 1 + 2\pi)(\theta + 1)} \|u\| \leq \frac{1}{40(\theta + 1 + 2\pi)(\theta + 1)} (\|u\| + 1).\]
It shows that (7) and (8) are satisfied with \( L_1(\theta) = \frac{1}{40(1+\theta)} \) and \( L_2(\theta) = \frac{1}{40(1+2\pi)(1+\theta)} \); \( \theta \in [0, \infty) \). Define

\[
g_i(\theta, u)(s) := \frac{u(s)}{40i} \sin \frac{\theta s}{40}; \quad (\theta, u) \in [\theta_i, s_i] \times E, s \in \Omega.
\]  

(112)

Using the same arguments as in Example 1, one can demonstrate that (9) and (10)–(16) are obtained with \( N = N = \kappa_1 = \kappa_1 = \frac{1}{40} \) and \( \xi \leq \frac{1}{10} \). Notice that \( L = 3 \). So, \( L(2\xi + N + (1 + \max\{1, \frac{1}{|\mu|}\})N) < 1 \) By applying Theorem 1, Problem (107) has an \( S \)-asymptotically \( 2\pi \)-periodic mild solution.

**Conclusion** Two existence results of \( S \)-asymptotically \( \omega \)-periodic of mild solutions to non-instantaneous impulsive semilinear differential inclusions of order \( 1 < \alpha < 2 \) and generated by sectorial operators are given. This work generalizes much recent work such as [18–20] to the case when there are impulse effects and the right-hand side is a multivalued function. Moreover, our technique can be used to develop the work in [12, 15–17, 21, 23–25] to the case when the linear part is a sectorial operator, the nonlinear part is a multivalued function and we have impulse effects. There are many directions for future work, for example:

1. With the help of technique in [1], we study the existence of solutions for Problem (1) on a time scales.
2. Investigation an existence theorem for a nonlinear singular-delay-fractional differential equation considered in [42, 43], when it contains a sectorial operator as a linear term and the nonlinear term becomes a multivalued function instead of single-valued function.
3. With the help of technique in [3], discuss the numerical solutions for Problem (1) on a closed bounded interval.
4. Study the \( S \)-asymptotically periodic solutions to Problem (1) when the sectorial operator is replaced by almost sectorial.
5. Study the \( S \)-asymptotically periodic solutions to Problem (1) when it involves \( p \)-Laplacian operator \( \varphi_p \) as well as when the Caputo derivative is replaced by the \( \psi \)-Caputo or \( \psi \)-Riemann–Liouville derivative. For contributions on BVP involving the \( \psi \)-Riemann–Liouville derivative, see [3] and for references on BVP containing the \( p \)-Laplacian operator \( \varphi_p \), see [6, 8].

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