ENERGETICS AND COARSENING ANALYSIS OF A SIMPLIFIED NON-LINEAR SURFACE GROWTH MODEL

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ABSTRACT. We study a simplified multidimensional version of the phenomenological surface growth continuum model derived in [6]. The considered model is a partial differential equation for the surface height profile $u$ which possesses the following free energy functional:

$$E(u) = \int_{\Omega} \left[ \frac{1}{2} \ln \left( 1 + |\nabla u|^2 \right) - |\nabla u| \arctan (|\nabla u|) + \frac{1}{2} |\Delta u|^2 \right] dx,$$

where $\Omega$ is the domain of a fixed support on which the growth is carried out. The term $|\Delta u|^2$ designates the standard surface diffusion in contrast to the second order term which phenomenologically describes the growth instability. Hence minimizing such energy corresponds to reducing surface defects during the growth process from a given initial surface configuration. Our analysis is concerned with the energetic and coarsening behaviours of the equilibrium solution. The existence of global energy minimizers and a scaling argument are used to construct a sequence of equilibrium solutions with different wavelength. We apply our minimum energy estimates to derive bounds in terms of the linear system size $|\Omega|$ for the characteristic interface width and average slope. We also derive a stable numerical scheme based on the convex-concave decomposition of the energy functional and study its properties while accommodating these results with 1d and 2d numerical simulations.

1. Introduction. Epitaxial surface growth models occupy a central position in the process of manufacturing thin film devices with applications ranging from microelectronics, optoelectronic to photonics and biomechanics. The technique consists in depositing in a small quantity on an initial surface, called substrate, a material of the same or a different nature from that of the substrate. The deposited atoms will spread to the places where they can stick to form a new layer on top of the substrate. It is crucial to control, understand and predict the morphology of surfaces during growth. In practice, the structures obtained are mostly unstable generating all sort of defects such as (pyramidal structures, delaminations, wrinkling, step bunching

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etc.) limiting their industrial potentialities. It goes without saying that solving this problem is an extremely difficult task, both conceptually and technically. For more information regarding the growth phenomenon see the review paper [18].

For this purpose, the surface deposition is modelled in a 3D fixed Cartesian coordinate system $Ox_1x_2z$. The macroscopic height of the film surface at time $t$ is denoted by the height function $z = H(x,t)$, where $x = (x_1, x_2)$ varies in a domain $\Omega$ see Figure 1. The geometric domain $\Omega$ is usually taken as a square but in practice it reflects the geometry of the substrate which depends on the target application of the grown material e.g. the domain could be a disk on top of which cylindrical nano-wires are grown. The surface height $H(x,t)$ above the substrate satisfies a continuity equation:

$$\frac{\partial H}{\partial t} + \nabla \cdot J_{\text{surface}}(\nabla H) = F,$$

where $F$ is the incident mass flux out of the molecular beam.

![Figure 1. Thin film surface height in a co-moving frame ($Ox_1x_2z$)](image)

In general, the systematic current $J_{\text{surface}}$ can be expressed in terms of a gradient expansion of the whole surface configuration. Keeping only the most important expansion terms, applying the change of variable $u = H(x,t) - F t$, subtracting the mean height and using appropriately rescaled units of height, distance and time [23], equation (1) attains the dimensionless form:

$$\frac{\partial u}{\partial t} = -\Delta^2 u - \nabla \cdot \left[ f(|\nabla u|) \nabla u \right] \quad \text{in} \quad \Omega \times (0,T),$$

$$u(\cdot, t) \text{ is spatially } \overline{\Omega} \text{-periodic} \quad \forall t \in [0,T],$$

$$u(x,0) = u_0(x) \quad \text{in} \quad \Omega,$$

where $\overline{\Omega}$ is the closure of $\Omega$. The periodicity assumption is justified since the material grown has a highly symmetric and self-similar crystalline structure. The function $f$ models the Ehrlich-Schwoebel barrier defined in [24, 20] which is believed to be the origin of the morphological instabilities that occur during growth and $u_0$ represents the initial shape on which the deposition occurs. The non-linear term on the right-hand side of equation (2) is an anti-diffusive responsible of the roughening effect of the evolving surface. This last destabilizing term is countered by the classical regularizing surface diffusion biharmonic term. The evolution of the thin film profile is therefore a competition between the aforementioned terms.
In this framework, many models have been receiving a great deal of attention: the with slope selection model \( (x) = 1 - x^2 \) and the without slope selection model \( (x) = \frac{1}{1 + x^2} \) and have been studied recently by many authors [7, 2, 16, 17, 22, 10, 15]. A growing number of studies are also directed toward finding the dynamical properties and the construction of linear high order numerical schemes approximating these models [8, 14, 4, 5]. In [19] and [6], authors derive some highly non-linear models in the vicinity of the instability of a vicinal surface:

\[
\partial_t h + \partial_x \left\{ a \frac{\partial_x h}{1 + (\partial_x h)^2} + \frac{b}{\sqrt{1 + (\partial_x h)^2}} \partial_x \left[ \frac{\partial_{xx} h}{(1 + (\partial_x h)^2)^{3/2}} \right] \right\} = 0, \tag{3}
\]

and

\[
\partial_t h + \partial_x \left\{ a \arctan (\partial_x h) + \frac{b}{\sqrt{1 + (\partial_x h)^2}} \partial_x \left[ \frac{\partial_{xx} h}{(1 + (\partial_x h)^2)^{3/2}} \right] \right\} = 0, \tag{4}
\]

where \( a, b \) are positive parameters.

What caught our attention is the presence of the term \( \partial_x (\arctan (\partial_x u)) \), which phenomenologically describes the Ehrlich-Schwoebel effect: adatoms must overcome a higher energy barrier to stick to a step from an upper than a lower terrace [24, 20]. Our motivation in this work is to understand the effect of this second order term \( \partial_x (\arctan (\partial_x u)) \) on the structures of solution when the highly non-linear fourth order term is replaced with a simplified linear biharmonic operator. We are also interested in the ability of the arctangent term to capture the rough-smooth-rough instability which characterizes the effect of the Ehrlich-Schwoebel barrier. Hence, the objective of this work is to study the simplified multi-dimensional PDE:

\[
\partial_t u = -\nabla \cdot \left( \frac{\arctan (|\nabla u|)}{|\nabla u|} \nabla u \right) - \Delta^2 u, \tag{5}
\]

which corresponds to (2) with \( f(x) = \frac{\arctan (|x|)}{|x|} \). If the surface slope is small enough one could heuristically argue using Taylor series expansion that we have the following equivalence \( \frac{\arctan (|\nabla u|)}{|\nabla u|} \approx \left( 1 - |\nabla u|^2 \right) \) which formally links the studied model to a modified up to a constant with slope selection model. Hence, the underlying dynamics are expected to be close to those found in [16][1][3]. It is also associated with the \( L^2 \) gradient flow of the free energy:

\[
E(u) = \int_{\Omega} \left[ \frac{1}{2} \ln \left( 1 + |\nabla u|^2 \right) - |\nabla u| \arctan (|\nabla u|) + \frac{1}{2} |\Delta u|^2 \right] dx. \tag{6}
\]

Taking the open cube \( \Omega = (0,L)^d \), where \( d = 2 \) and \( \varepsilon = \frac{1}{L^2} \). One could mathematically study the scaled energy:

\[
E_\varepsilon(u) = \int_{\Omega_1} \left[ \frac{1}{2} \ln \left( 1 + |\nabla u|^2 \right) - |\nabla u| \arctan (|\nabla u|) + \frac{\varepsilon}{2} |\Delta u|^2 \right] dx, \tag{7}
\]

where \( \Omega_1 = (0,1)^d \), which is obtained by setting

\[
E(\hat{u}) = E_\varepsilon(u) \quad \text{with} \quad u(x) = \hat{u}(\hat{x})/L \quad \text{and} \quad \hat{x} = Lx. \tag{8}
\]

This factors the effect of the large domain \( \Omega \) into the expression of the functional \( E_\varepsilon \). Our goal in this paper is to derive some interesting averaging bounds on the
energetics, coarsening, and interfacial width to justify the growth dynamics of the model (5). The new scaled model is then
\[ \partial_t u = -\nabla \cdot \left( \frac{\arctan(|\nabla u|)}{|\nabla u|} \nabla u \right) - \varepsilon \Delta^2 u. \] (9)

2. Steady state equilibrium profile. In this section, we assume the space dimension is \( d = 1 \). We will refer to the \( L^2 \) gradient flow associated with (6):
\[ \partial_t h + \partial_x \{ \arctan(\partial_x h) + \partial_{xxx} h \} = 0, \] (10)
as the simplified model in further discussions. In preparation for the font evolution properties, we recall a general criterion about coarsening for some classes of one-dimensional non-linear evolution equations by Politi and Misbah [21]. They argued that a coarsening process occurs if and only if the period of the steady state solution is an increasing function of its amplitude, or its typical slope. In order to use this argument, periodic stationary solution to (10) are of primordial interest here. Steady states of (10) are invariant under translation and by conservation under the transformation \( x \to -x \), they also satisfy:
\[ \arctan(m) + m'' = 0, \] (11)
where \( m = \partial_x h \). Multiplying by \( m' \) and integrating leads to the energy equation:
\[ \frac{(m')^2}{2} + \left\{ m \arctan(m) - \frac{1}{2} \log(1 + m^2) \right\} = \Theta, \] (12)
where the energy constant \( \Theta \) is a positive parameter.

It is clear that the problem (11) admits a one-parameter family of periodic solutions parametrized by the maximum slope \( S = \max_x m \). Studying the relationship between \( m \) and \( m' \) from the last energy equation enables us to write
\[ m' = \sqrt{2} \left( \Theta - m \arctan(m) + \frac{1}{2} \log(1 + m^2) \right)^{\frac{1}{2}}. \] (13)
The phase space contains simple closed periodic orbit which proves the existence of periodic solutions. By a parametrisation of the energy constant by \( S \) as follows \( \Theta = (S \arctan(S) - \frac{1}{2} \log(1 + S^2)) \). A solution of the problem is given by:
\[ \int_0^{m(x)} \frac{dz}{\sqrt{S \arctan(S) - m \arctan(m) + \frac{1}{2} \log(1 + z^2)}} = \sqrt{2} x, \] (14)
for every \( |x| \leq \frac{1}{4} \) where \( \lambda \) is the period of the steady state and it is given by:
\[ \lambda(S) = 2\sqrt{2} \int_0^S \frac{dz}{\sqrt{S \arctan(S) - m \arctan(m) + \frac{1}{2} \log(1 + z^2)}}, \] (15)
We have the amplitude \( A \) is also given by
\[ A(S) = 2\sqrt{2} \int_0^S \frac{z \, dz}{\sqrt{S \arctan(S) - m \arctan(m) + \frac{1}{2} \log(1 + z^2)}}. \] (16)
A study of the integral form of the period analytically shows that the period is an increasing function of the typical slope \( S \). The question of the existence of the coarsening is then positively answered in this section due to Politi and Misbah criterion. An other important question concerns the behaviour of solutions for long
times. This is usually done by computing the coarsening and roughness exponent through the similarity ansatz:

\[ h(t, x) = t^{\beta} f(\eta), \quad \eta = x t^{-\alpha}, \quad (17) \]

where \( \alpha \) is the coarsening exponent and \( \beta \) is the roughness exponent. It should be stressed that seeking a solution to (10) can be inferred from the scaling argument above only for long time since differentiating the second order term in the equation leads to:

\[ \partial_t h + \frac{\partial_{xx} h}{1 + (\partial_x h)^2} + \partial_{xxxx} h = 0 \quad (18) \]

Based on the similarity ansatz, we obtain the same exponents computed in the case of the without slope selection model. Namely \( \alpha = \frac{1}{4} \) and \( \beta = \frac{1}{2} \) while \( f \) satisfies:

\[ \frac{1}{2} f - \frac{1}{4} \eta f' + \frac{f''}{t^{\frac{1}{2}}} + (f')^2 + f^{(iv)} = 0. \quad (19) \]

This is a slightly modified version of the equation studied by Guedda et al in [9]. The conclusion is that we cannot directly infer the coarsening behaviour using the previously mentioned similarity argument.

3. Existence of global minimizers. In this section, we study the variational problem of minimizing the scaled energy (7) defined on \( \Omega_1 = (0,1)^d \), over the class of smooth periodic function:

\[ \mathcal{H}(\Omega_1) = \left\{ u \in H^2_{\text{per}}(\Omega_1) / \int_{\Omega_1} u = 0 \right\}. \]

It is worth mentioning that \( D \)-periodic functions in \( H^2(D) \) are rigorously defined using density arguments. In fact, \( H^2_{\text{per}}(\Omega) \) is the closure in the usual \( H^2(D) \)-norm of the set of all restrictions onto \( \Omega \) of \( C^\infty \) functions defined on \( \mathbb{R}^d \). The properties of the considered space are reported in many references (see [16] and the references therein) including the Poincaré inequalities:

\[ \| u \| \leq C_4 \| \nabla u \|, \quad (20) \]

\[ \| \nabla u \| \leq C_5 \| \Delta u \|, \quad (21) \]

which hold true in \( \mathcal{H}(\Omega_1) \) where \( \| \cdot \| \) denotes the usual \( L^2 \)-norm for an underlying domain. We first show a lower bound estimate:

**Lemma 3.1.** Let \( \varepsilon > 0 \), we get for all functions \( u \in \mathcal{H}(\Omega_1) \), the lower bound estimate below:

\[ E_\varepsilon(u) \geq -C_1(\varepsilon) + \frac{\varepsilon}{4} \int_{\Omega_1} |\Delta u|^2, \quad (22) \]

where \( C_1(\varepsilon) = -\frac{C_5^2 \varepsilon^2}{4\varepsilon} \).

**Proof.** We have

\[
E_\varepsilon(u) = \int_{\Omega_1} \left[ \frac{1}{2} \ln \left( 1 + |\nabla u|^2 \right) - |\nabla u| \arctan \left( |\nabla u| + \frac{\varepsilon}{2} |\Delta u|^2 \right) \right] dx, \\
\geq \int_{\Omega_1} -|\nabla u| \arctan (|\nabla u|) \, dx + \frac{\varepsilon}{2} \int_{\Omega_1} |\Delta u|^2 \, dx, \\
\geq \int_{\Omega_1} -\frac{\pi}{2} |\nabla u| \, dx + \frac{\varepsilon}{2} \int_{\Omega_1} |\Delta u|^2 \, dx.
\]
Using Poincaré and Young’s inequality $|\nabla u| \leq \nu + \frac{1}{4\nu} |\nabla u|^2$ we get

$$E_\varepsilon(u) \geq -\frac{\nu\pi}{2} - \frac{\pi}{8\nu} \int_{\Omega_1} |\nabla u|^2 + \frac{\varepsilon}{2} |\Delta u|^2 \, dx,$$

$$\geq -\frac{\nu\pi}{2} + \left(\frac{\varepsilon}{4} - \frac{\pi C_5^2}{8\nu}\right) \int_{\Omega_1} |\Delta u|^2 \, dx + \frac{\varepsilon}{4} \int_{\Omega_1} |\Delta u|^2 \, dx.$$

Choosing $\nu = \frac{\pi C_5^2}{2\varepsilon}$, we obtain the desired result e.g.

$$E_\varepsilon(u) \geq -\frac{\nu\pi}{2} + \frac{\varepsilon}{4} \int_{\Omega_1} |\Delta u|^2 \, dx.$$ 

\[\square\]

**Theorem 3.2** (energy minimization). 1. For any $\varepsilon > 0$, there exists $u_\varepsilon \in \mathcal{H}(\Omega_1)$ such that

$$E_\varepsilon(u_\varepsilon) = \min_{u \in \mathcal{H}(\Omega_1)} E_\varepsilon(u). \quad (23)$$

2. There exists a constant $C_2$ such that for any $\varepsilon > 0$ and any global minimizer $u_\varepsilon$ of $E_\varepsilon$, the following estimate holds:

$$\|\nabla^m u_\varepsilon\| \leq \frac{C_2}{\varepsilon}, \quad m = 0, 1, 2 \quad (24)$$

where $\nabla^0 u = u, \nabla^1 u = \nabla u, \nabla^2 u = \Delta u$.

**Proof.** 1. According to the previous lemma, the set $\{E_\varepsilon(u)/u \in \mathcal{H}(\Omega_1)\}$ is lower-bounded and non-empty, thus there exists $\alpha_\varepsilon = \inf_{u \in \mathcal{H}(\Omega_1)} E_\varepsilon(u) > -\infty$.

Let $(u_n)_n$ be a minimizing sequence of $E_\varepsilon : \mathcal{H}(\Omega_1) \to \mathbb{R}$. It follows from the Poincaré inequalities and the identity (22) that $(u_n)_n$ is bounded in $H^2_{per}(\Omega_1)$. Hence, up to a subsequence, $u_n \rightharpoonup u_\varepsilon$ in $H^2(\Omega_1)$ for some $u_\varepsilon \in H^2_{per}(\Omega_1)$. In particular, $\Delta u_n \to \Delta u_\varepsilon$ in $L^2(\Omega_1)$ and, up to another subsequence if necessary, $u_n \to u_\varepsilon$ in $H^1(\Omega_1)$ as $n \to \infty$. Therefore, $u_\varepsilon \in \mathcal{H}(\Omega_1)$ thanks to the non-linearity in the gradient $\left(\frac{1}{2}\ln(1+x^2) - x \arctan(x)\right)$ which is globally Lipschitz continuous coupled with the fact that the $\|\Delta u\|_{L^2}$ is a norm in $\mathcal{H}(\Omega_1)$. We therefore get that the energy is sequentially lower semi-continuous. Hence,

$$\alpha_\varepsilon = \liminf_{n \to \infty} E_\varepsilon(u_n) \geq E_\varepsilon(u_\varepsilon) \geq \alpha_\varepsilon.$$

In conclusion, $u_\varepsilon$ is a global minimizer of $E_\varepsilon$ over $\mathcal{H}(\Omega)$.

2. Since $u_\varepsilon$ is a global minimizer of $E_\varepsilon$, it satisfies the Euler-Lagrange equation:

$$\varepsilon \Delta^2 u_\varepsilon + \nabla \cdot \left( \frac{\arctan(|\nabla u_\varepsilon|)}{|\nabla u_\varepsilon|} \nabla u_\varepsilon \right) = 0.$$

Consequently, multiplying by $u_\varepsilon$ and integrating over $\Omega_1$:

$$\varepsilon \int_{\Omega_1} |\Delta u_\varepsilon|^2 = \int_{\Omega_1} \arctan(|\nabla u_\varepsilon|) |\nabla u_\varepsilon| \quad \text{(upper bound on arctan(x))}$$

$$\leq \frac{\pi}{2} \int_{\Omega_1} |\nabla u_\varepsilon| \quad \text{(using Jensen’s inequality and } |\Omega_1| = 1)$$

$$\leq \frac{\pi}{2} \|\nabla u_\varepsilon\| \quad \text{(using poincaré inequality (21))}$$

$$\leq \frac{\pi C_5}{2} ||\Delta u_\varepsilon||$$
Thus,

$$||\Delta u_\varepsilon|| \leq \frac{\pi C_5}{2\varepsilon}$$

As a result, we get the desired bounds by applying the Poincaré inequalities (20) and (21) the existence the constant $C_2$ such that:

$$\|\nabla^m u\| \leq \frac{C_2}{\varepsilon}$$

for $m = 0, 1, 2$.

\[\square\]

4. Upper bounds on the surface width, gradient and energy. Let $T > 0$ and let $u : [0, T] \rightarrow L^2(\Omega)$. We change back to the original domain $\Omega = (0, L)^d$ using the transformation (8):

$$\int_\Omega |\nabla^m u(x)|^2 dx = \frac{1}{|\Omega|} \int_\Omega |\nabla^m u(x)|^2 dx = L^{2-2m} \int_{\hat{\Omega}} |\nabla^m u_\varepsilon(x)|^2 dx, \quad m = 0, 1, 2.$$  \hfill(25)

Thus, we have these estimates that were drawn in the last section:

$$\int_\Omega |\nabla^m u(x)|^2 dx \leq \frac{C_2^2}{\varepsilon^2} L^{2-2m} = C_2 L^{6-2m}, \quad m = 0, 1, 2,$$  \hfill(26)

where $u$ is any global minimizer of $E(u)$ in (6) parametrised by the linear system size $L$. The interface width for the surface is defined for any $t \in [0, T]$ by

$$w_u(t) = \sqrt{\int_\Omega |u(x,t) - \bar{u}(t)|^2 dx}, \quad \bar{u}(t) = \int_\Omega u(x,t) dx,$$

and in particular

$$w_u(t) = \sqrt{\int_\Omega |u(x,t)|^2 dx} \quad \forall u \in H(\Omega).$$

**Theorem 4.1** (Upper bounds). Let $u$ be a weak solution of (5) and let $t_0 > 0$, then we have:

1. an upper bound on the interface width:

$$w_u(t) \leq \sqrt{\pi C_2 L^2 (t - t_0) + [w_u(t_0)]^2} \quad \forall t \geq t_0,$$  \hfill(27)

2. an upper bounds on the gradients:

$$\int_{t_0}^t \int_\Omega |\Delta u(x, \tau)|^2 dx d\tau \leq \frac{\pi}{2} C_2 L^2 + \frac{1}{2(t - t_0)} [w_u(t_0)]^2 \quad \forall t > t_0,$$  \hfill(28)

and

$$\int_{t_0}^t \int_\Omega |\nabla u(x, \tau)|^2 dx d\tau \leq \frac{\pi}{2} C_2 L^2 \left[ t + t_0 \right] + \frac{2}{\pi C_2 L^2} [w_u(t_0)]^2 \sqrt{1 + \frac{(C_2 L^2)^{-1}}{(t-t_0)} [w_u(t_0)]^2}$$  \hfill(29)

3. a lower bound on the energy:

$$\int_{t_0}^t E(h(\tau)) d\tau \geq -\pi C_2 L^2 \sqrt{t}$$  \hfill(30)

when $t > t_0 + \frac{2}{\pi C_2 L^2} [w_u(t_0)]^2$. 


Proof. 1. Multiplying the front evolution equation (5):

\[
\partial_t u = -\nabla \cdot \left( \frac{\arctan(|\nabla u|)}{|\nabla u|} \nabla u \right) - \Delta^2 u,
\]

by \( u \) and integrating over \( \Omega \), we get

\[
\frac{1}{2} \frac{\partial}{\partial t} w_u(t) + \int_\Omega |\Delta u|^2 = \int_\Omega \arctan(|\nabla u|) |\nabla u|.
\]

Then, using Jensen’s inequality, a similar reasoning used in Lemma 3.1 and (26) leads to

\[
\frac{1}{2} \frac{\partial}{\partial t} w_u(t)^2 + \int_\Omega |\Delta u|^2 \leq \frac{\pi}{2} \left( \Omega |\nabla u|^2 \right)^{\frac{1}{2}} \leq \frac{\pi}{2} C_2 L^2.
\]

Consequently,

\[
w_u(t)^2 - w_u(t_0)^2 \leq \pi C_2 L^2 (t - t_0),
\]

hence the result:

\[
w_u(t) \leq \sqrt{\pi C_2 L^2 (t - t_0) + [w_u(t_0)]^2} \quad \forall t \geq t_0.
\]

2. Starting from (31), it follows after integrating from \( t_0 \) to \( t \)

\[
\frac{1}{2} \frac{\partial}{\partial t} w_u(t)^2 - \frac{1}{2} [w_u(t_0)]^2 + \int_{t_0}^t \int_\Omega |\Delta u(x, \tau)|^2 dxd\tau \leq \frac{\pi}{2} C_2 L^2
\]

Thus,

\[
\int_{t_0}^t \int_\Omega |\Delta u(x, \tau)|^2 dxd\tau \leq \frac{\pi}{2} C_2 L^2 + \frac{1}{2(t - t_0)} [w_u(t_0)]^2
\]

Moreover, Cauchy-Schwarz inequality implies

\[
\int_{t_0}^t \int_\Omega |\nabla u(x, \tau)|^2 dxd\tau = \int_{t_0}^t \int_\Omega [-u(x, \tau)] \Delta u(x, \tau) dxd\tau \\
\leq \left( \int_{t_0}^t \int_\Omega |u(x, \tau)|^2 dxd\tau \right)^{1/2} \left( \int_{t_0}^t \int_\Omega |\Delta u(x, \tau)|^2 dxd\tau \right)^{1/2}
\]

Which leads to

\[
\int_{t_0}^t \int_\Omega |\nabla u(x, \tau)|^2 dxd\tau \leq \sqrt{\frac{\pi}{2} C_2 L^2 (t + t_0) + [w_u(t_0)]^2} \sqrt{\frac{\pi}{2} C_2 L^2 + \frac{1}{2(t - t_0)} [w_u(t_0)]^2}
\]

\[
\leq \frac{\pi}{2} C_2 L^2 \sqrt{(t + t_0) + \frac{2}{\pi C_2 L^2} [w_u(t_0)]^2} \sqrt{1 + \frac{(\pi C_2 L^2)^{-1}}{(t - t_0)} [w_u(t_0)]^2}
\]

3. If \( t > t_0 + \frac{2}{\pi C_2 L^2} [w_u(t_0)]^2 \), then

\[
\sqrt{(t + t_0) + \frac{2}{\pi C_2 L^2} [w_u(t_0)]^2} \leq \sqrt{2t}, \quad \frac{(\pi C_2 L^2)^{-1}}{(t - t_0)} [w_u(t_0)]^2 \leq 1
\]
In addition,
\[
\int_{t_0}^{t} E_t(u) d\tau \geq \int_{t_0}^{t} \int_{\Omega} -|\nabla u| \arctan(|\nabla u|) dxd\tau \\
\geq - \int_{t_0}^{t} \int_{\Omega} |\nabla u(x, \tau)|^2 dxd\tau \\
\geq -\pi C_2 L^2 \sqrt{t}
\]

5. Discrete in time DC approximation scheme and numerical results. A
study of the integral form of the period analytically shows that the period is an
increasing function of the typical slope $S$. The question of the existence of the
coarsening is then positively answered in this section due to Politi and Misbah
criterion [21]. In this section, simulations of the simplified model confirms the
analytical results. In the spirit of [16, 13] for the without slope selection model,
a convex-concave decomposition of the function enables to compute steady states
through a linear semi-implicit first order scheme. In fact,
\[
E(u) = E_1(u) - E_2(u) \tag{34}
\]
\[
E_1(u) = \int_{\Omega} \left( \frac{\varepsilon^2}{2} |\Delta u|^2 + \gamma |\nabla u|^2 \right) dx \tag{35}
\]
\[
E_1(u) = \int_{\Omega} \left[ \gamma |\nabla u|^2 + |\nabla u| \arctan(|\nabla u|) - \frac{1}{2} \ln \left( 1 + |\nabla u|^2 \right) \right] dx \tag{36}
\]
where the integrand in the energy (36) is $\gamma x^2 + x \arctan(x) - \frac{1}{2} \ln \left( 1 + x^2 \right)$, which is
strongly convex when $\gamma$ strictly positive. The convex-concave decomposition leads
to the semi-discrete scheme below: For $\lambda > 0$ :
\[
\frac{u_{n+1}}{\lambda} + \varepsilon^2 \Delta^2 u_{n+1} - \gamma \Delta u_{n+1} = \frac{u_n}{\lambda} - \gamma \Delta u_n + N(\nabla u_n) \tag{37}
\]
where $N(\nabla u_n) = -\div \left( \frac{\arctan(|\nabla u_n|)}{|\nabla u_n|} \nabla u_n \right)$. Next, we show that the semi-discrete
scheme is unconditionally stable which alleviates the time-step restrictions associ-\nated with fourth order stiff systems.

**Theorem 5.1** (Energy stability). The linear DC scheme (37) satisfies the following
stability inequality for all $n > 0$:
\[
E(u^{n+1}) + c ||u^{n+1} - u^n||^2_{H^2(\Omega)} \leq E(u^n) \tag{38}
\]
where $c$ is any strictly positive constant such that $c \leq \min \left( \frac{1}{\lambda}, \frac{\gamma}{\lambda}, \frac{\varepsilon^2}{2} \right)$.

**Proof.** The proof originates from the convexity of the functional $E_1$ (35) and $E_2$
(36). See [13] for a similar proof in the case of the without slope selection model. \qed

We perform Pseudo-Spectral Fourier method for the spatial discretisation. The
one-dimensional discrete version of the scheme is as follows:
\[
\left( 1 + \lambda x^2 |k|^4 - \lambda \gamma |k|^2 \right) \hat{u}_{n+1}^k = \left( 1 - \lambda \gamma |k|^2 \right) \hat{u}_n^k + \lambda \hat{N}_k \quad \forall |k| \leq \frac{M_k}{2} \tag{39}
\]
where $\hat{u}_{n+1}^k$ and $\hat{N}_k$ are the Fourier modes of the solution $u_{n+1}$ and the non-linear
term $N(|\nabla u_n|)$, respectively, and $M_k$ is a fixed integer representing the number of
modes. We approximate the non-linearity using a pseudo-spectral methods where
we compute the products in the physical space and the resolution in Fourier space.
In one dimension, the front is evolved from the initial condition
\[ u_0(x) = 0.1(\sin(\frac{2}{3} \pi x) + \sin(\pi x) + \sin(\frac{4}{3} \pi x) + \sin(\frac{5}{3} \pi x)) \quad \forall x \in [0, 12] \quad (40) \]

Figure 2 illustrates the properties of the profile during evolutions. The starting configuration contains many modes that are dumped at the first stage of evolution. However at a sufficiently small scale, coarsening happens as was predicted theoretically. The steady states is then recovered after the energy stagnates. For a plot of the energy during evolution see Figure 3.

For the two-dimensional simulation, we consider the domain \( \Omega = [0, 2\pi] \times [0, 2\pi] \) and use the initial condition:
\[ u_0(x, y) = 0.01(\sin 3x \sin 2x + \sin 5x \sin 4y) \quad (x, y) \in \Omega \quad (41) \]

which is similar to the one used in Li and Liu [16] and [22] to observe the dynamics of the different sinusoidal modes. We show plots of the numerical height profile \( u \) and its amplitudes at different stages of evolution in Figure 8 and 9. One can deduce from these plots that the amplitudes of the simulated front reach higher values than those found in the with and without slope selection model. Moreover, the structural transitions from one stage to another corresponding to flat regions in the energy plot 4 and 5 indicate that these stages are local minima of the energy. In Figure 7, it can be seen that the magnitude of L2-error \( \|u^{n+1} - u^n\|_{L^2(\Omega)} \) increases when sinusoidal
modes fuse to transition from one stage to another. This behaviours, also shared by the roughness or interfacial width see figure 6. This phenomenon is referred to as a rough-smooth-rough instability and we establish from these simulations that this simplified model captures the instability and can also be used to model the effect of the Ehrlich-Schwoebel barrier.

6. Conclusion. The conclusion of this analysis shows that coarsening happens in the simplified model, solutions are regular and bounded. The regular structure of solutions is kept during evolution due to the presence of the strong biharmonic regularisation unlike the case studied in [12, 11] where solutions are unbounded. Our analysis on the variational problem of minimizing the energy helps understand and determine a time scale for bounding the interface width, slope as well as an estimate of the corresponding energy lower bound. There are a variety of issues that are not resolved particularly what happens when the inverse linear size $\varepsilon$ tends to 0. This last point is not obvious and requires further analysis and might require the derivation of sharper bounds. Furthermore, the numerical scheme is only first order and the convergence near equilibrium is slow. We intend to address this issue in a future investigation.

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Figure 4. Evolution of the energy $E(u^g)$ in two dimensions starting from the initial configuration (41) where $\varepsilon = 0.1$, $\lambda = 10^{-2}$, $M_k = 2^8$ and $\gamma = 0.2$. 
Figure 5. Evolution of the energy (6) in two dimensions starting from the initial configuration (41) where $\varepsilon = 0.01$, $\lambda = 10^{-2}$, $M_k = 2^8$ and $\gamma = 0.2$.

Figure 6. Evolution of the interfacial width or roughness $W_{\alpha}(t)$ starting from the initial configuration (41) where $\varepsilon = 0.1$, $\lambda = 10^{-2}$, $M_k = 2^8$ and $\gamma = 0.5$. 
Figure 7. Evolution of the $L^2$-error $\|u^{n+1} - u^n\|_{L^2}$ (6) starting from the initial configuration (41) where $\varepsilon = 0.01$, $\lambda = 10^{-2}$, $M_k = 2^8$ and $\gamma = 0.2$. 
Figure 8. Evolution dynamics of the front equation (10) starting from the initial condition (41) where $\varepsilon = 0.1$, $\lambda = 10^{-2}$, $M_k = 2^8$ and $\gamma = 0.2$
Figure 9. Evolution dynamics of the front equation (10) starting from the initial condition (41) where $\varepsilon = 0.01$, $\lambda = 10^{-2}$, $M_k = 2^8$ and $\gamma = 0.2$