BETTI NUMBERS AND INJECTIVITY RADIi

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Dedicated to José María Montesinos on the occasion of his 65th birthday

ABSTRACT. We give lower bounds on the maximal injectivity radius for a closed hyperbolic 3-manifold with first Betti number 2 under some additional topological hypotheses.

The theme of this paper is the connection between topological properties of a closed orientable hyperbolic 3-manifold $M$ and the maximal injectivity radius of $M$. In [4] we showed that if the first Betti number of $M$ is at least 3, then the maximal injectivity radius of $M$ is at least $\log 3$. By contrast, the best known lower bound for the maximal injectivity radius of $M$ with no topological restriction on $M$ is the lower bound of $\text{arcsinh}(\frac{1}{4}) = 0.24746\ldots$ due to Przeworski [7]. One of the results of this paper, Corollary 4, gives a lower bound of $0.32798$ for the case where the first Betti number of $M$ is 2 and $M$ does not contain a “fibroid” (see below). Our main result, Theorem 3, is somewhat stronger than this.

The proofs of our results combine a result due to Andrew Przeworski [7] with results from [5] and [6].

The results of [5] and [6] were motivated by applications to the study of hyperbolic volume, and these applications were superseded by the results of [2]. The applications presented in the present paper do not seem to be accessible by other methods.

As in [5], we define a book of $I$-bundles to be a compact, connected, orientable topological 3-manifold (with boundary) $W$ which has the form $W = \mathcal{P} \cup \mathcal{B}$, where

- $\mathcal{P}$ is an $I$-bundle over a non-empty compact 2-manifold-with-boundary,
- each component of $\mathcal{B}$ is homeomorphic to $D^2 \times S^1$,
- the set $\mathcal{A} = \mathcal{P} \cap \mathcal{B}$ is the vertical boundary of the $I$-bundle $\mathcal{P}$, and
- each component of $\mathcal{A}$ is an annulus in $\partial \mathcal{B}$ which is homotopically non-trivial in $\mathcal{B}$.

(Note that this terminology differs slightly from that of [1], where it is the triple $(W, \mathcal{B}, \mathcal{P})$ that is called a book of $I$-bundles.)

As in [5], we define a fibroid in a closed, connected, orientable 3-manifold $M$ to be a connected incompressible surface $F$ with the property that each component of the compact manifold obtained by cutting $M$ along $F$ is a book of $I$-bundles.

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Note that in defining a fibroid to be connected, we are following the convention of [3] rather than that of [6].

We define a function \( R(x) \) for \( 0 < x < \log 3 \) by

\[
R(x) = \frac{1}{2} \arccosh \left( \frac{e^{2x} + 2e^x + 5}{(\cosh \frac{x}{2})(e^x - 1)(e^x + 3)} \right).
\]

As in [3] Section 10], we define a function \( V(x) \) for \( 0 < x < \log 3 \) by

\[
V(x) = \pi x \sinh^2 R(x) = \frac{\pi x}{e^x - 1} \left( \frac{e^{2x} + 2e^x + 5}{2(e^x + 3) \cosh(x/2)} \right) - \frac{\pi x^2}{2}.
\]

Thus, in a closed hyperbolic 3-manifold, if a geodesic of length \( \ell \) is the core of an embedded tube of radius \( R(\ell) \), then this tube has volume \( V(\ell) \).

The following result is implicit in [6], but we will supply a proof for the sake of comprehensibility.

**Proposition 1.** Let \( M \) be a closed hyperbolic 3-manifold. Suppose that there is an infinite subset \( N \) of \( H_2(M; \mathbb{Z}) \) such that every element of \( N \) is represented by some connected, incompressible surface which is not a fibroid. Let \( \lambda \) be a positive number less than \( \log 3 \). Then either the maximal injectivity radius of \( M \) is at least \( \lambda/2 \) or \( M \) contains a closed geodesic \( C \) of length at most \( \lambda \) such that the maximal tube about \( C \) has radius at least \( R(\lambda) \) and volume at least \( V(\lambda) \), where \( R(\lambda) \) and \( V(\lambda) \) are defined by (1) and (2).

**Proof.** The hypothesis implies in particular that \( H_2(M; \mathbb{Z}) \) has infinitely many primitive elements, and so the first Betti number \( \beta_1(M) \) is at least 2. If \( \beta_1(M) \geq 3 \), then according to [4] Corollary 10.4, the maximal injectivity radius is at least \( \frac{1}{2} \log 3 > \lambda/2 \). We may therefore assume that \( \beta_1(M) = 2 \). Hence the quotient of \( H_1(M, \mathbb{Z}) \) by its torsion subgroup is a free abelian group \( L \) of rank 2. We let \( h: \pi_1(M) \to L \) denote the natural homomorphism.

We distinguish two cases. First consider the case in which \( M \) contains a non-trivial closed geodesic \( C \) of some length \( \ell < \lambda \) such that the conjugacy class represented by \( C \) is contained in the kernel of \( h \). Let \( T \) denote the maximal embedded tube about \( C \). According to [3] Corollary 10.5] we have \( \text{Vol } T \geq V(\lambda) \). If \( \rho \) denotes the radius of \( T \), this gives

\[
\pi \ell \sinh^2 \rho = \text{Vol } T \geq V(\lambda) = \pi \lambda \sinh^2 R(\lambda) > \pi \ell \sinh^2 R(\lambda),
\]

and hence \( \rho > R(\lambda) \).

Thus the second alternative of the proposition holds in this case.

We now turn to the case in which no non-trivial closed geodesic of length \( < \lambda \) represents a conjugacy class contained in the kernel of \( h \). Since \( M \) is closed, there are only a finite number \( n \geq 0 \) of conjugacy classes in \( \pi_1(M) \) that are represented by closed geodesics of length \( < \lambda \). Let \( \gamma_1, \ldots, \gamma_n \) be elements belonging to these \( n \) conjugacy classes. Then \( \tilde{\gamma}_i = h(\gamma_i) \) is a non-trivial element of \( L \) for \( i = 1, \ldots, n \).

Since \( L \) is a free abelian group of rank 2, there exists, for each \( i \in \{1, \ldots, n\} \), a homomorphism \( \phi_i \) of \( L \) onto \( \mathbb{Z} \) such that \( \phi_i(\tilde{\gamma}_i) = 0 \). Because \( \tilde{\gamma}_i \neq 0 \), the epimorphism \( \phi_i \) is unique up to sign.

The epimorphism \( \phi_i \circ h: \pi_1(M) \to \mathbb{Z} \) corresponds to a primitive element of \( H^1(M; \mathbb{Z}) \), whose Poincaré dual in \( H_2(M; \mathbb{Z}) \) we shall denote by \( c_i \). Since the set \( \mathcal{N} \subset H_2(M; \mathbb{Z}) \) given by the hypothesis of the theorem is infinite, there is an element \( c \) of \( \mathcal{N} \) which is distinct from \( \pm c_i \) for \( i = 1, \ldots, n \). Since \( c \in \mathcal{N} \) it follows
from the hypothesis that there is a connected incompressible surface $S \subset M$ which represents the homology class $c$ and is not a fibroid.

We now apply Theorem A of [5], which asserts that if $S$ is a connected non-fibroid incompressible surface in a closed, orientable hyperbolic 3-manifold $M$, and if $\lambda$ is any positive number, then either (i) $M$ contains a non-trivial closed geodesic of length $< \lambda$ which is homotopic in $M$ to a closed curve in $M - S$ or (ii) $M$ contains a hyperbolic ball of radius $\lambda/2$. In the present situation, with $\lambda$ chosen as above, we claim that alternative (i) of the conclusion of Theorem A of [5] cannot hold.

Indeed, suppose that $C$ is a non-trivial closed geodesic of length $< \lambda$ with the properties stated in (i). Since $C$ has length $< \lambda$, the conjugacy class represented by $C$ contains $\gamma_i$ for some $i \in \{1, \ldots, n\}$. Since $C$ is homotopic to a closed curve in $M - S$, it follows that the image of $\gamma_i$ in $H_1(M;\mathbb{Z})$ has homological intersection number 0 with $c$. Thus if $\psi: \pi_1(M) \to \mathbb{Z}$ is the homomorphism corresponding to the Poincaré dual of $c$, we have $\psi(\gamma_i) = 0$. Now since $L$ is the quotient of $H_1(M)$ by its torsion subgroup, $\psi$ factors as $\phi \circ h$, where $\phi$ is some homomorphism from $L$ to $\mathbb{Z}$. Since $c$ is primitive, $\psi$ is surjective, and hence so is $\phi$. But we have $\phi(\gamma_i) = \psi(\gamma_i) = 0$. In view of the uniqueness that we observed above for $\phi_i$, it follows that $\phi = \pm \phi_i$, so that $\psi = \pm \phi_i \circ h$ and hence $c = \pm c_i$. This contradicts our choice of $c$.

Hence (ii) must hold. This means that the maximal injectivity radius of $M$ is at least $\lambda/2$. Thus the first alternative of the proposition holds in this case. □

**Proposition 2.** Let $M$ be a closed hyperbolic 3-manifold. Suppose that there is an infinite subset $N$ of $H_2(M;\mathbb{Z})$ such that every element of $N$ is represented by a connected, incompressible surface which is not a fibroid. Then the maximal injectivity radius of $M$ exceeds $0.32798$.

**Proof.** We set $\lambda = 2 \times 0.32798 = 0.65596$.

According to Proposition 1 either the maximal injectivity radius of $M$ is at least $\lambda/2$ — so that the conclusion of the theorem holds — or $M$ contains a closed geodesic $C$ of length at most $\lambda$ such that the maximal tube about $C$ has volume at least $V(\lambda)$, where $V(\lambda)$ is defined by [2]. In the latter case, if $R$ denotes the radius of $T$, we have

$$R \geq R(\lambda) = 0.806787 \ldots.$$  

Now according to [7] Proposition 4.1, the maximal injectivity radius of $M$ is bounded below by

$$\text{arcsinh} \left( \frac{\tanh R}{2} \right) > \text{arcsinh} \left( \frac{\tanh 0.806787}{2} \right) > 0.32799.$$  

This gives the conclusion of the theorem in this case. □

**Theorem 3.** Let $M$ be a closed hyperbolic 3-manifold. Suppose that there is an infinite set $N$ of primitive elements of $H_2(M;\mathbb{Z})$ such that no element of $N$ is represented by a (connected) fibroid. Then the maximal injectivity radius of $M$ exceeds $0.32798$.

**Proof.** If $\pi_1(M)$ has a non-abelian free quotient, then by [8] Theorem 1.3, the maximal injectivity radius of $M$ is at least $\frac{1}{3} \log 3 = 0.549 \ldots$. Now suppose that $\pi_1(M)$ has no non-abelian free quotient. If $N$ is the set given by the hypothesis of Theorem 2 it now follows from [8] Proposition 2.1 that every element of $N$ is represented by a connected incompressible surface, which by hypothesis cannot...
be a fibroid. Thus \( \mathcal{N} \) has the properties stated in the hypothesis of Proposition 2. The latter result therefore implies that the maximal injectivity radius of \( M \) exceeds 0.32798.

If \( M \) is a 3-manifold whose first Betti number is at least 2, then \( H_2(M; \mathbb{Z}) \) has infinitely many primitive elements. If a non-trivial element of \( H_2(M; \mathbb{Z}) \) is represented by a connected surface, it must be primitive, since it has intersection number 1 with a class in \( H_1(M; \mathbb{Z}) \). Hence Theorem 3 implies:

**Corollary 4.** Let \( M \) be a closed hyperbolic 3-manifold. Suppose that the first Betti number of \( M \) is at least 2 and that \( M \) contains no non-separating fibroid. Then the maximal injectivity radius of \( M \) exceeds 0.32798.

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