On a conjecture of Ashbaugh and Benguria about lower eigenvalues of the Neumann laplacian

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Abstract
In this paper, we prove an isoperimetric inequality for lower order eigenvalues of the free membrane problem on bounded domains in a Euclidean space or a hyperbolic space which strengthens the well-known Szegö–Weinberger inequality and supports a celebrated conjecture of Ashbaugh–Benguria.

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1 Introduction

Let $(M, g)$ be complete Riemannian manifold of dimension $n, n \geq 2$. We denote by $\Delta$ the Laplace operator on $M$. For bounded domain $\Omega$ with smooth boundary in $M$ we consider the free membrane problem

\[
\begin{align*}
\Delta f &= -\mu f \quad \text{in } \Omega, \\
\frac{\partial f}{\partial \nu} &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.1)

Here $\frac{\partial}{\partial \nu}$ denotes the outward unit normal derivative on $\partial \Omega$. It is well known that the problem (1.1) has discrete spectrum consisting in a sequence

$\mu_0 = 0 < \mu_1 \leq \mu_2 \leq \cdots \rightarrow +\infty.$
In the two dimensional case, G. Szego [11] proved via conformal mapping techniques that if \( \Omega \subset \mathbb{R}^2 \) is simply connected, then
\[
\mu_1(\Omega) A(\Omega) \leq (\mu_1 A)|_{\text{disk}} = \pi p_{1,1}^2
\]
(1.2)
where \( A \) denotes the area. Later, using more general methods, Weinberger [12] showed that (1.2) and its \( n \)-dimensional analogue,
\[
\mu_1(\Omega) \leq \left( \frac{\omega_n}{|\Omega|} \right)^{2/n} p_{n/2,1}^2,
\]
(1.3)
hold for arbitrary domains in \( \mathbb{R}^2 \) and \( \mathbb{R}^n \), respectively. Here \( J_v \) is the Bessel function of the first kind of order \( v \), \( p_{v,k} \) is the \( k \)th positive zero of the derivative of \( x^{1-v}J_v(x) \) and \( |\Omega| \) denotes the volume of \( \Omega \). Szegö and Weinberger also noticed that Szegö’s proof of (1.2) for simply connected domains in \( \mathbb{R}^2 \) extends to prove the bound
\[
\frac{1}{\mu_1} + \frac{1}{\mu_2} \geq \frac{2A}{\pi p_{1,1}^2},
\]
(1.4)
for such domains. The bounds of Szegö and Weinberger are isoperimetric with equality if and only if \( \Omega \) is a disk (\( n \)-dimensional ball in the case of Weinberger’s result (1.3)). A quantitative improvement of (1.2) was made by Brasco and Pratelli in [5] who showed that for any bounded domain with smooth boundary \( \Omega \subset \mathbb{R}^n \) we have
\[
\omega_n^{2/n} p_{n/2,1}^2 - \mu_1(\Omega)|\Omega|^{2/n} \geq c(n)A(\Omega)^2.
\]
(1.5)
Here, \( c(n) \) is positive constant depending only on \( n \) and \( A(\Omega) \) is the so called Fraenkel asymmetry, defined by
\[
\mathcal{A}(\Omega) = \inf \left\{ \frac{|\Omega \Delta B|}{|\Omega|} : B \text{ ball in } \mathbb{R}^n \text{ such that } |B| = |\Omega| \right\}.
\]
Nadirashvilli obtained in [9] a quantitative improvement of (1.4) which states that there exists a constant \( C > 0 \) such that for every \( \Omega \subset \mathbb{R}^2 \) smooth simply connected bounded open set it holds
\[
\frac{1}{|\Omega|} \left( \frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} \right) - \frac{1}{|B|} \left( \frac{1}{\mu_1(B)} + \frac{1}{\mu_2(B)} \right) \geq \frac{1}{C} \mathcal{A}(\Omega)^2,
\]
(1.6)
where \( B \) is any disk in \( \mathbb{R}^2 \). On the other hand, Ashbaugh and Benguria [2] showed that
\[
\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_n(\Omega)} \geq \frac{n}{n+2} \left( \frac{|\Omega|}{\omega_n} \right)^{2/n}
\]
(1.7)
holds for any $\Omega \subset \mathbb{R}^n$. Some generalizations to (1.7) have been obtained e.g., in [8,13].

In [2], Ashbaugh and Benguria also proposed the following important

**Conjecture I** ([2]). For any bounded domain $\Omega$ with smooth boundary in $\mathbb{R}^n$, we have

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_n(\Omega)} \geq \frac{n (|\Omega|/\omega_n)^{2/n}}{p_{n/2,1}^2}$$

(1.8)

with equality holding if and only if $\Omega$ is a ball in $\mathbb{R}^n$.

Ashbaugh [1] and Henrot [7] mentioned this conjecture again.

The Szegö-Weinberger inequality (1.3) has been generalized to bounded domains in a hyperbolic space by Ashbaugh-Benguria [3] and Xu [14] independently. In his book, Chavel [6] mentioned that one can use Weinberger’s method to prove this result. In [3], Ashbaugh-Benguria also proved the Szegö-Weinberger inequality for bounded domains in a hemisphere. One can also consider similar estimates for lower order eigenvalues of the Neumann Laplacian for bounded domains in a hyperbolic space or a hemisphere.

**Conjecture II.** Let $M$ be an $n$-dimensional complete simply connected Riemannian manifold of constant sectional curvature $\kappa \in \{-1, 1\}$ and $\Omega$ be a bounded domain in $M$ which is contained in a hemisphere in the case that $\kappa = 1$. Let $B_\Omega$ be a geodesic ball in $M$ such that $|\Omega| = |B_\Omega|$ and denote by $\mu_1(B_\Omega)$ the first nonzero eigenvalue of the Neumann Laplacian of $B_\Omega$. Then the first $n$ non-zero eigenvalues of the Neumann Laplacian of $\Omega$ satisfy

$$\frac{1}{\mu_1(\Omega)} + \frac{1}{\mu_2(\Omega)} + \cdots + \frac{1}{\mu_n(\Omega)} \geq \frac{n}{\mu_1(B_\Omega)}$$

(1.9)

with equality holding if and only if $\Omega$ is isometric to $B_\Omega$.

In this paper, we prove an isoperimetric inequality for the sums of the reciprocals of the first $(n-1)$ non-zero eigenvalues of the Neumann Laplacian on bounded domains in $\mathbb{R}^n$ or a hyperbolic space which supports the above conjectures.

**Theorem 1.1** Let $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^n$. Then

$$\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_{n-1}(\Omega)} \geq \frac{(n-1)(|\Omega|/\omega_n)^{2/n}}{p_{n/2,1}^2}$$

(1.10)

with equality holding if and only if $\Omega$ is a ball in $\mathbb{R}^n$.

**Theorem 1.2** Let $\mathbb{H}^n$ be an $n$-dimensional hyperbolic space of curvature $-1$ and $\Omega$ be a bounded domain in $\mathbb{H}^n$. Let $B_\Omega$ be a geodesic ball in $\mathbb{H}^n$ such that $|\Omega| = |B_\Omega|$. Then we have

$$\frac{1}{\mu_1(\Omega)} + \cdots + \frac{1}{\mu_{n-1}(\Omega)} \geq \frac{n-1}{\mu_1(B_\Omega)}$$

(1.11)

with equality holding if and only if $\Omega$ is isometric to $B_\Omega$. 

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2 A proof of Theorem 1.1.

In this section, we shall prove the following result which implies Theorem 1.1.

**Theorem 2.1** Let $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^n$. There exists a positive constant $d(n)$ depending only on $n$ such that the first $(n-1)$ nonzero Neumann eigenvalues of the Laplacian of $\Omega$ satisfy the inequality

$$\omega_n^{2/n} p_{n/2,1}^2 - \frac{(n-1)|\Omega|^{2/n}}{\mu_1 + \ldots + \frac{1}{\mu_{n-1}}} \geq d(n) A(\Omega)^2,$$

with equality holding if and only if $\Omega$ is an $n$-ball.

**Remark** One can easily see that (2.1) strengthens (1.5).

Before proving Theorem 2.1, we recall some known facts we need (Cf. [6,7,10]). Let $\{u_j\}_{j=0}^{\infty}$ be an orthonormal set of eigenfunctions of the problem (1.1), that is,

$$\begin{cases}
\Delta u_i = -\mu_i u_i & \text{in } \Omega, \\
\frac{\partial u_i}{\partial \nu} |_{\partial \Omega} = 0, \\
\int_{\Omega} u_i u_j dv_g = \delta_{ij}.
\end{cases}$$

(2.2)

where $dv_g$ denotes the volume element of the metric $g$. For each $i = 1, 2, \ldots$, the variational characterization of $\mu_i(\Omega)$ is given by

$$\mu_i(\Omega) = \inf_{u \in H^1(\Omega) \setminus \{0\}} \left\{ \frac{\int_{\Omega} |\nabla u|^2 dv_g}{\int_{\Omega} u^2 dv_g} : \int_{\Omega} uu_j dv_g = 0, j = 0, \ldots, i-1 \right\}.$$  

(2.3)

Let $B_r$ be a ball of radius $r$ centered at the origin in $\mathbb{R}^n$. It is known that $\mu_1(B_r)$ has multiplicity $n$, that is, $\mu_1(B_r) = \cdots = \mu_n(B_r)$. This value can be explicitly computed together with its corresponding eigenfunctions. A basis for the eigenspace corresponding to $\mu_1(B_r)$ consists of

$$\xi_i(x) = |x|^{1-\frac{n}{2}} J_{n/2} \left( \frac{p_{n/2,1}|x|}{r} \right) \frac{x_i}{|x|}, \quad i = 1, \ldots, n.$$  

(2.4)

The radial part of $\xi_i$

$$g(|x|) = |x|^{1-\frac{n}{2}} J_{n/2} \left( \frac{p_{n/2,1}|x|}{r} \right),$$

(2.5)

satisfies the differential equation of Bessel type

$$\begin{cases}
g''(t) + \frac{n-1}{t} g'(t) + \left( \mu_1(B_r) - \frac{n-1}{t^2} \right) g(t) = 0, \\
g(0) = 0, \quad g'(r) = 0.
\end{cases}$$

(2.6)
We can compute

$$\mu_1(B_r) = \frac{\int_{B_r} \left( g'(|x|)^2 + (n - 1) \frac{g(|x|)^2}{|x|^2} \right) dx}{\int_{B_r} g(|x|)^2 dx}$$

$$= \left( \frac{p_n/2,1}{r} \right)^2. \tag{2.7}$$

**Proof of Theorem 2.1.** Let

$$r = \left( \frac{|\Omega|}{\omega_n} \right)^{1/n} \tag{2.8}$$

and define $G : [0, +\infty) \to \mathbb{R}$ by

$$G(t) = \begin{cases} 
  g(t), & t \leq r, \\
  g(r), & t > r.
\end{cases} \tag{2.9}$$

We need to choose suitable trial functions $\phi_i$ for each of the eigenfunctions $u_i$ and insure that these are orthogonal to the preceding eigenfunctions $u_0, \ldots, u_{i-1}$. For the $n$ trial functions $\phi_1, \phi_2, \ldots, \phi_n$, we choose:

$$\phi_i = G(|x|) \frac{x_i}{|x|}, \quad \text{for } i = 1, \ldots, n, \tag{2.10}$$

but before we can use these we need to make adjustments so that

$$\phi_i \perp \text{span}\{u_0, \ldots, u_{i-1}\} \tag{2.11}$$

in $L^2(\Omega)$. In order to do this, let us fix an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathbb{R}^n$. From the well-know arguments of Weinberger in [12] by using the Brouwer fixed point theorem, we know that it is always possible to choose the origin of $\mathbb{R}^n$ so that

$$\int_{\Omega} \langle x, e_i \rangle G(|x|) \frac{G(|x|)}{|x|} dx = 0, \quad i = 1, \ldots, n, \tag{2.12}$$

that is, $\langle x, e_i \rangle G(|x|) \frac{G(|x|)}{|x|} \perp u_0$ (which is actually just the constant function $1/\sqrt{|\Omega|})$. Here $dx$ and $\langle , \rangle$ denote the standard Lebesgue measure and the inner product of $\mathbb{R}^n$, respectively. Now we show that there exists a new orthonormal basis $\{e'_i\}_{i=1}^n$ of $\mathbb{R}^n$ such that

$$\langle x, e'_i \rangle G(|x|) \frac{G(|x|)}{|x|} \perp u_j, \tag{2.13}$$
for \( j = 1, \ldots, i - 1 \) and \( i = 2, \ldots, n \). To see this, we define an \( n \times n \) matrix \( Q = (q_{ij}) \) by

\[
q_{ij} = \int_\Omega (x, e_i) \frac{G(|x|)}{|x|} u_j(x) \, dx, \quad i, j = 1, 2, \ldots, n.
\] (2.14)

Using the orthogonalization of Gram and Schmidt (QR-factorization theorem), we know that there exist an upper triangle matrix \( T = (T_{ij}) \) and an orthogonal matrix \( U = (a_{ij}) \) such that \( T = UQ \), i.e.,

\[
T_{ij} = \sum_{k=1}^{n} a_{ik} q_{kj} = \int_\Omega \sum_{k=1}^{n} a_{ik} (x, e_k) \frac{G(|x|)}{|x|} u_j(x) \, dx = 0, \quad 1 \leq j < i \leq n.
\]

Letting \( e'_i = \sum_{k=1}^{n} a_{ik} e_k, \ i = 1, \ldots, n \); we arrive at (2.13). Let us denote by \( x_1, x_2, \ldots, x_n \) the coordinate functions with respect to the base \( \{e'_i\}_{i=1}^{n} \), that is, \( x_i = (x, e'_i), \ x \in \mathbb{R}^n \). From (2.12) and (2.13), we have

\[
\int_\Omega \phi_i u_j \, dx = \int_\Omega G(|x|) \frac{x_i}{|x|} u_j(x) \, dx = 0, \quad i = 1, \ldots, n, \ j = 0, \ldots, i - 1(2.15)
\]

It then follows from the variational characterization (2.3) that

\[
\mu_i \int_\Omega \phi_i^2 \, dx \leq \int_\Omega |\nabla \phi_i|^2 \, dx, \quad i = 1, \ldots, n.
\] (2.16)

Substituting

\[
|\nabla \phi_i|^2 = G'(|x|)^2 \frac{x_i^2}{|x|^2} + \frac{G(|x|)^2}{|x|^2} \left( 1 - \frac{x_i^2}{|x|^2} \right)
\]

\[
= \frac{G(|x|)^2}{|x|^2} + \left( G'(|x|)^2 - \frac{G(|x|)^2}{|x|^2} \right) \frac{x_i^2}{|x|^2}
\] (2.17)

into (2.16) and dividing by \( \mu_i \), one gets for \( i = 1, \ldots, n \) that

\[
\int_\Omega \phi_i^2 \, dx \leq \frac{1}{\mu_i} \int_\Omega \frac{G(|x|)^2}{|x|^2} \, dx + \frac{1}{\mu_i} \int_\Omega \left( G'(|x|)^2 - \frac{G(|x|)^2}{|x|^2} \right) \frac{x_i^2}{|x|^2} \, dx. \quad (2.18)
\]

Summing over \( i \), we get

\[
\int_\Omega G(|x|)^2 \, dx \leq \sum_{i=1}^{n} \frac{1}{\mu_i} \int_\Omega \frac{G(|x|)^2}{|x|^2} \, dx
\]

\[
+ \sum_{i=1}^{n} \frac{1}{\mu_i} \int_\Omega \left( G'(|x|)^2 - \frac{G(|x|)^2}{|x|^2} \right) \frac{x_i^2}{|x|^2} \, dx. \quad (2.19)
\]
Lemma 2.2 We have $g'|_{(0,r)} > 0, g|_{(0,r)} > 0$ and $g'(t) - \frac{g(t)}{t} \leq 0, \forall t \in (0, r]$.

**Proof of Lemma 2.2.** The Bessel function of the first kind $J_v(t)$ is given by

$$J_v(t) = \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{t}{2}\right)^{2k+v}}{k!\Gamma(k + v + 1)}. \quad (2.22)$$

which, combining with (2.5), gives

$$g(t) = \left(\frac{p_{n/2,1}}{2r}\right)^{\frac{3}{2}} \frac{1}{2} \sum_{k=0}^{+\infty} \frac{(-1)^k \left(\frac{p_{n/2,1}}{2r}\right)^{2k}}{k!\Gamma(k + \frac{3}{2} + 1)}. \quad (2.23)$$

Thus, $g(0) = 0, g'(0) > 0$. Since $r$ is the first positive zero of $g'$, we have $g|_{(0,r)} > 0$ and $g'|_{(0,r)} > 0$. Observe that

$$\lim_{t \to 0} \left( g'(t) - \frac{g(t)}{t} \right) = 0, \quad g'(r) - \frac{g(r)}{r} < 0. \quad (2.24)$$
Let us assume by contradiction that there exists a \( t_0 \in (0, r) \) such that
\[
g'(t_0) - \frac{g(t_0)}{t_0} > 0. \tag{2.25}
\]

In this case, we know from (2.24) that the function \( g'(t) - \frac{g(t)}{t} \) attains its maximum at some \( t_1 \in (0, r) \) and so we have
\[
g''(t_1) = \frac{t_1g'(t_1) - g(t_1)}{t_1^2} = 0. \tag{2.26}
\]

From (2.6), we have
\[
g''(t_1) + \frac{n-1}{t_1} g'(t_1) + \left( \mu_1(B_r) - \frac{n-1}{t_1^2} \right) g(t_1) = 0. \tag{2.27}
\]

Eliminating \( g''(t_1) \) from (2.26) and (2.27), we get
\[
\frac{n}{t_1} \left( g'(t_1) - \frac{g(t_1)}{t_1} \right) = -\mu_1(B_r) g(t_1) < 0. \tag{2.28}
\]

This is a contradiction and completes the proof of Lemma 2.1. \( \square \)

From Lemma 2.1 and the definition of \( G \), we know that
\[
G'(|x|)^2 - \frac{G(|x|^2)}{|x|^2} \leq 0 \quad \text{on} \quad \Omega. \tag{2.29}
\]

Hence
\[
\sum_{i=1}^{n-1} \int_{\Omega} \left( \frac{1}{\mu_i} - \frac{1}{\mu_n} \right) \left( G'(|x|)^2 - \frac{G(|x|^2)}{|x|^2} \right) \frac{x_i^2}{|x|^2} dx \leq 0. \tag{2.30}
\]

Combining (2.19), (2.21) and (2.30), one gets
\[
\int_{\Omega} G(|x|)^2 dx \leq \frac{1}{\mu_n} \int_{\Omega} \left( G'(|x|)^2 - \frac{G(|x|^2)}{|x|^2} \right) dx \\
+ \sum_{i=1}^{n} \frac{1}{\mu_i} \int_{\Omega} \frac{G(|x|^2)}{|x|^2} dx \\
= \frac{1}{\mu_n} \int_{\Omega} G'(|x|)^2 + \sum_{i=1}^{n-1} \frac{1}{\mu_i} \int_{\Omega} \frac{G(|x|)^2}{|x|^2} dx \\
\leq \frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\mu_i} \int_{\Omega} \left( G'(|x|)^2 + (n-1) \frac{G(|x|^2)}{|x|^2} \right) dx, \tag{2.31}
\]
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that is,

\[
\frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \int_{\Omega} G(|x|) dx \leq \int_{\Omega} \left( G(|x|)^2 + (n-1) \frac{G(|x|)^2}{|x|^2} \right) dx. \quad (2.32)
\]

Using the fact that \( G(t) \) is increasing, one gets

\[
\int_{\Omega} G(|x|)^2 dx = \int_{\Omega \cap B_r} G(|x|)^2 dx + \int_{\Omega \setminus B_r} G(|x|)^2 dx \geq \int_{\Omega \cap B_r} G(|x|)^2 dx + g(r)^2 |\Omega \setminus B_r| = \int_{\Omega \cap B_r} g(|x|)^2 dx + \int_{B_r \setminus \Omega} g(|x|)^2 dx \geq \int_{\Omega \cap B_r} g(|x|)^2 dx + \int_{B_r \setminus \Omega} g(|x|)^2 dx = \int_{B_r} g(|x|)^2 dx, \quad (2.33)
\]

which, combining with (2.32), gives

\[
\frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \int_{B_r} g(|x|)^2 dx \leq \int_{\Omega} \left( G(|x|)^2 + (n-1) \frac{G(|x|)^2}{|x|^2} \right) dx. \quad (2.34)
\]

We know from (2.7) that

\[
\left( \frac{p_{n/2,1}}{r} \right)^2 \int_{B_r} g(|x|)^2 dx = \int_{B_r} \left( g'(|x|)^2 + (n-1) \frac{g(|x|)^2}{|x|^2} \right) dx = \int_{B_r} \left( G'(|x|)^2 + (n-1) \frac{G(|x|)^2}{|x|^2} \right) dx. \quad (2.35)
\]

Consequently, we have

\[
\left( \left( \frac{p_{n/2,1}}{r} \right)^2 - \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \right) \int_{B_r} g(|x|)^2 dx \geq \int_{B_r} \left( G'(|x|)^2 + (n-1) \frac{G(|x|)^2}{|x|^2} \right) dx - \int_{\Omega} \left( G'(|x|)^2 + (n-1) \frac{G(|x|)^2}{|x|^2} \right) dx. \quad (2.36)
\]

We have

\[
\frac{d}{dt} \left[ G'(t)^2 + (n-1) \frac{G(t)^2}{t^2} \right] = 2G'(t)G''(t) + 2(n-1)(tG(t)G'(t) - G(t)^2)/t^3.
\]
For \( t > r \) this is negative since \( G \) is constant there. For \( t \leq r \) we use the differential equation (2.6) to obtain

\[
\frac{d}{dt} \left[ G'(t)^2 + (n - 1) \frac{G(t)^2}{t^2} \right] = -2\mu_1(B_r)GG' - (n - 1)(tG' - G)^2/t^3 < 0.
\]

Thus the function \( G'(t)^2 + (n - 1) \frac{G(t)^2}{t^2} \) is decreasing for \( t > 0 \).

**Lemma 2.3** ([5]) Let \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) be a decreasing function. Then we have

\[
\int_{B_r} f(|x|)dx - \int_{\Omega} f(|x|)dx \geq n\omega_n \int_{\rho_1}^{\rho_2} |f(t) - f(r)|t^{n-1}dt. \tag{2.37}
\]

Here

\[
\rho_1 = \left( \frac{|\Omega \cap B_r|}{\omega_n} \right)^{\frac{1}{n}} \quad \text{and} \quad \rho_2 = \left( \frac{|\Omega| + |\Omega \setminus B_r|}{\omega_n} \right)^{\frac{1}{n}}. \tag{2.38}
\]

Taking \( f(t) = G'(t)^2 + (n - 1) \frac{G(t)^2}{t^2} \) in Lemma 2.3, we obtain

\[
\int_{B_r} \left( G'(|x|)^2 + (n - 1) \frac{G(|x|)^2}{|x|^2} \right)dx - \int_{\Omega} \left( G'(|x|)^2 + (n - 1) \frac{G(|x|^2)}{|x|^2} \right)dx
\]

\[
\geq n\omega_n \int_{r}^{\rho_2} |f(t) - f(r)|t^{n-1}dt
\]

\[
= n\omega_n \int_{r}^{\rho_2} (f(r) - f(t))t^{n-1}dt. \tag{2.39}
\]

Observe that

\[
(f(r) - f(t))t^{n-1} = (n - 1)g(r)^2 \left( \frac{1}{r^2} - \frac{1}{t^2} \right) t^{n-1}, \quad \text{for } \rho_2 \geq t \geq r. \tag{2.40}
\]

Therefore,

\[
\int_{r}^{\rho_2} (f(r) - f(t))t^{n-1}dt = g(r)^2 \cdot \begin{cases} 
\frac{n-1}{n^2} (\rho_2^n - r^n) - \frac{n-1}{n^2} (\rho_2^{n-2} - r^{n-2}) , & \text{if } n > 2, \\
\frac{1}{2\pi^2} (\rho_2^2 - r^2) - \ln \frac{\rho_2}{r} , & \text{if } n = 2.
\end{cases} \tag{2.41}
\]
By using the definition of $\rho_2$ we have when $n > 2$,

$$
\rho_2^{n-2} - r^{n-2} = r^{n-2} \left[ \left( 1 + \frac{|\Omega \setminus B_r|}{|\Omega|} \right)^{\frac{n-2}{n}} - 1 \right]
$$

(2.42)

$$
\leq r^{n-2} \left( \frac{n-2}{n} \frac{|\Omega \setminus B_r|}{|\Omega|} - \frac{(n-2)2^{-\frac{2}{n}} - 1}{n^2} \left( \frac{|\Omega \setminus B_r|}{|\Omega|} \right)^2 \right),
$$

thanks to the elementary inequality

$$(1 + t)^\delta \leq 1 + \delta t + \frac{\delta(\delta - 1)}{2} \cdot 2^{\delta - 2} t^2, \forall \delta \in (0, 1), \forall t \in [0, 1],$$

and when $n = 2$,

$$
\ln \frac{\rho_2}{r} = \frac{1}{2} \ln \left( 1 + \frac{|\Omega \setminus B_r|}{|\Omega|} \right)
$$

$$
\leq \frac{1}{2} \left( \frac{|\Omega \setminus B_r|}{|\Omega|} - \frac{1}{4} \left( \frac{|\Omega \setminus B_r|}{|\Omega|} \right)^2 \right),
$$

(2.43)

thanks to the elementary inequality

$$
\ln(1 + t) \leq t - \frac{t^2}{4}, \forall t \in [0, 1].
$$

Since $|B_r| = |\Omega|$, we have $|\Omega \Delta B_r| = 2|\Omega \setminus B_r|$ and so

$$
\frac{|\Omega \setminus B_r|}{|\Omega|} \geq \frac{1}{2} A(\Omega).
$$

It then follows by substituting (2.42) and (2.43) into (2.41) that

$$
\int_r^{\rho_2} (f(r) - f(t)) t^{n-1} dt
$$

$$
\geq g(r)^2 \left( \frac{|\Omega \setminus B_r|}{|\Omega|} \right)^2 \cdot \begin{cases} 
    r^{n-2} \cdot \frac{(n-1)2^{-\frac{2}{n}} - 1}{n^2}, & \text{if } n > 2, \\
    \frac{1}{8}, & \text{if } n = 2.
\end{cases}
$$

$$
\geq \frac{1}{4} g(r)^2 A(\Omega)^2 \cdot \begin{cases} 
    r^{n-2} \cdot \frac{(n-1)2^{-\frac{2}{n}} - 1}{n^2}, & \text{if } n > 2, \\
    \frac{1}{8}, & \text{if } n = 2.
\end{cases}
$$

(2.44)
Thus, concerning the right hand side of (2.36), one gets from (2.39) and (2.44) that
\[
\int_{B_r} \left( G'(\|x\|)^2 + (n - 1) \frac{G(\|x\|)^2}{\|x\|^2} \right) dx - \int_{\Omega} \left( G'(\|x\|)^2 + (n - 1) \frac{G(\|x\|)^2}{\|x\|^2} \right) dx \\
\geq \frac{\omega_n}{4} g(r)^2 A(\Omega)^2 \cdot \left\{ \begin{array}{ll}
r^{n-2} \cdot \left( \frac{n-1}{n} \right)^{-\frac{3}{n}} & \text{if } n > 2, \\
\frac{1}{4} & \text{if } n = 2,
\end{array} \right.
\]
\[
= \omega_n \frac{J_{n/2}(p_{n/2,1})^2 A(\Omega)^2}{4} \cdot \left\{ \begin{array}{ll}
\frac{(n-1)^2}{n} & \text{if } n > 2, \\
\frac{1}{4} & \text{if } n = 2,
\end{array} \right.
\]
\[
\equiv \alpha(n) A(\Omega)^2. \tag{2.45}
\]

Concerning the left hand side of (2.36), we have
\[
\left( \frac{p_{n/2,1}}{r} \right)^2 - \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \int_{B_r} g(\|x\|)^2 dx \\
= \left( \frac{p_{n/2,1}}{r} \right)^2 - \frac{n-1}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} r^2 \int_{\{|y| \leq 1\}} |y|^{2-n} J_{n/2}(p_{n/2,1}|y|)^2 dy \\
= \left( \frac{p_{n/2,1}^2}{r^2} \right)^{2/n} - \frac{(n-1)|\Omega|^{2/n}}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \beta(n), \tag{2.46}
\]
where
\[
\beta(n) = \omega_n^{-2/n} \int_{\{|y| \leq 1\}} |y|^{2-n} J_{n/2}(p_{n/2,1}|y|)^2 dy.
\]

Combining (2.36), (2.45) and (2.46), we obtain
\[
p_{n/2,1}^2 \omega_n^{2/n} - \frac{(n-1)|\Omega|^{2/n}}{\sum_{i=1}^{n-1} \frac{1}{\mu_i}} \geq \alpha(n) \beta(n)^{-1} A(\Omega)^2 \equiv d(n) A(\Omega)^2. \tag{2.47}
\]

Moreover, we can see that equality holds in (2.47) only when $\Omega$ is a ball. This completes
the proof of Theorem 2.1.

3 A proof of Theorem 1.2

In this section, we shall prove Theorem 1.2. Firstly, we list some important facts we
need. About each point $p \in \mathbb{H}^n$ there exists a coordinate system $(t, \xi) \in [0, +\infty) \times \mathbb{S}^{n-1}$
relative to which the Riemannian metric reads as
\[
d s^2 = dt^2 + \sinh^2 t d\sigma^2, \tag{3.1}
\]
where $d\sigma^2$ is the canonical metric on the $(n - 1)$-dimensional unit sphere $\mathbb{S}^{n-1}$.
Lemma 3.1 (Cf. [6,14]). Let $B(p, r)$ be a geodesic ball of radius $r$ with center $p$ in $\mathbb{H}^n$. Then the eigenfunction corresponding to the first nonzero eigenvalue $\mu_1(B(p, r))$ of the Neumann problem on $B(p, r)$ must be

$$h(t, \xi) = f(t)\omega(\xi), \quad \xi \in S^{n-1},$$

where $\omega(\xi)$ is an eigenfunction corresponding to the first nonzero eigenvalue of $S^{n-1}$, $f$ satisfies

$$\begin{cases} f'' + (n - 1) \coth t + \left( \mu_1(B(p, r)) - \frac{n-1}{\sinh^2 t} \right) f = 0, \\ f(0) = f'(r) = 0, \quad f'|_{[0,r)} \neq 0, \end{cases}$$

and

$$\mu_1(B(p, r)) = \frac{\int_{B(p,r)} (f'(t)^2 + (n - 1)\frac{(f(t))^2}{\sinh^2 t}) dv}{\int_{B(p,r)} f(t)^2 dv}.$$  

**Proof of Theorem 1.2.** Assume that the radius of $B_\Omega$ is $r$. Let $f$ be as in Lemma 3.1. Noticing $f(t) \neq 0$ when $0 < t \leq r$, we may assume that $f(t) > 0$ for $0 < t \leq r$ and so $f$ is nondecreasing on $[0, r]$. Let $\{e_i\}_{i=1}^n$ be an orthonormal basis of $\mathbb{R}^n$ and set $\omega_i(\xi) = \langle e_i, \xi \rangle, \quad \xi \in S^{n-1} \subset \mathbb{R}^n$. Define

$$F(t) = \begin{cases} f(t), & t \leq r, \\ f(r), & t > r. \end{cases}$$

Let us take a point $p \in \mathbb{H}^n$ such that in the above coordinate system at $p$ we have

$$\int_{\Omega} F(t)\omega_i(\xi)d\nu = 0, \quad i = 1, \ldots, n.$$  

Here, $d\nu$ is the volume element of $\mathbb{H}^n$. By using the same arguments as in the proof of Theorem 2.1, we can assume further that

$$\int_{\Omega} F(t)\omega_i(\xi)u_j d\nu = 0,$$

for $i = 2, 3, \ldots, n$ and $j = 1, \ldots, i - 1$. Here $\{u_i\}_{i=0}^{+\infty}$ is an orthonormal set of eigenfunctions corresponding to the eigenvalues $\{\mu_i(\Omega)\}_{i=0}^{+\infty}$. Hence, we conclude from the Rayleigh-Ritz variational characterization (2.3) that

$$\mu_i(\Omega) \int_{\Omega} F(t)^2 \omega_i^2(\xi)d\nu \leq \int_{\Omega} |\nabla(F(t)\omega_i(\xi))|^2 d\nu$$

$$= \int_{\Omega} \left( |F'(t)|^2 \omega_i^2(\xi) + F^2(t)|\tilde{\nabla}\omega_i(\xi)|^2 \sinh^{-2} t \right) d\nu, \quad i = 1, \ldots, n.$$
where $\nabla \omega$ denotes the gradient operator of $S^{n-1}$. Thus

$$\int_\Omega F(t)^2 \omega_i^2(\xi) d\nu \leq \frac{1}{\mu_i(\Omega)} \int_\Omega |F'(t)|^2 \omega_i^2(\xi) d\nu + \frac{1}{\mu_i(\Omega)} \int_\Omega F^2(t) |\nabla \omega_i(\xi)|^2 \sinh^{-2} t d\nu. \quad (3.9)$$

Observing $F'(t) = 0, t \geq r$, one gets

$$\int_\Omega |F'(t)|^2 \omega_i^2(\xi) d\nu = \int_{\Omega \cap B(p,r)} |F'(t)|^2 \omega_i^2(\xi) d\nu$$

$$\leq \int_{B(p,r)} |F'(t)|^2 \omega_i^2(\xi) d\nu$$

$$= \int_0^r \int_{S^{n-1}} |F'(t)|^2 \omega_i^2(\xi) \sinh^{n-1} t dA dt$$

$$= \frac{1}{n} \int_0^r \int_{S^{n-1}} |F'(t)|^2 \sinh^{n-1} t dA dt$$

$$= \frac{1}{n} \int_{B(p,r)} |F'(t)|^2 d\nu, \quad (3.10)$$

where $dA$ denotes the area element of $S^{n-1}$. Since

$$|\nabla \omega_i(\xi)| \leq 1, \quad \sum_{i=1}^n |\nabla \omega_i(\xi)|^2 = n - 1, \quad (3.11)$$

we have

$$\sum_{i=1}^n \frac{1}{\mu_i(\Omega)} |\nabla \omega_i(\xi)|^2$$

$$= \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} |\nabla \omega_i(\xi)|^2 + \frac{1}{\mu_n(\Omega)} \sum_{i=1}^{n-1} \left(1 - |\nabla \omega_i(\xi)|^2\right)$$

$$\leq \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} |\nabla \omega_i(\xi)|^2 + \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \left(1 - |\nabla \omega_i(\xi)|^2\right)$$

$$= \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)}. \quad (3.12)$$
Summing on \( i \) from 1 to \( n \) in (3.9) and using (3.10) and (3.12), we get
\[
\int_{\Omega} F(t)^2 \, dv 
\leq \sum_{i=1}^{n} \frac{1}{n \mu_i(\Omega)} \int_{B(p,r)} |F(t)|^2 \, dv + \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \int_{\Omega} F^2(t) \sinh^{-2} t \, dv.
\] (3.13)

We need the following lemma.

**Lemma 3.2** The function \( h(t) = \frac{F(t)}{\sinh t} \) is decreasing.

**Proof of Lemma 3.2.** Observe that
\[
\lim_{t \to 0} h(t) = f'(0).
\]

Let us show that
\[
\gamma(t) \equiv f'(t) - \coth t f(t) \leq 0, \quad t \in (0, r].
\] (3.14)

Since
\[
\lim_{t \to 0} \gamma(t) = 0, \quad \gamma(r) = - \coth rf(r) < 0,
\] (3.15)

if \( \gamma(t_0) > 0 \) for some \( t_0 \in (0, r) \), then \( \gamma \) attains its maximum at some \( t_1 \in (0, r) \) and so
\[
0 = \gamma'(t_1) = f''(t_1) + \frac{f(t_1)}{\sinh^2 t_1} - \coth t_1 f'(t_1).
\] (3.16)

We have from (3.3) that
\[
f''(t_1) + (n - 1) \coth t_1 f'(t_1) + \mu_1(B(r)) f(t_1) - \frac{n - 1}{\sinh^2 t_1} f(t_1) = 0.
\] (3.17)

Hence
\[
f'(t_1) - \frac{f(t_1)}{\cosh t_1 \sinh t_1} = - \frac{\mu_1(B(r)) f(t_1) \sinh t_1}{n \cosh t_1} < 0,
\] (3.18)

which contradicts to
\[
f'(t_1) - \coth t_1 f(t_1) > 0.
\] (3.19)

Thus (3.14) holds. Consequently \( h'(t) \leq 0, \quad \forall t \in (0, r] \) and \( h \) is decreasing. The proof of Lemma 3.2 is completed. \( \square \)
Now we go on the proof of Theorem 1.2. Since $F$ is increasing and $F(t)/\sinh t$ is decreasing, we can use the same arguments as in the proof of (2.33) to conclude that

$$\int_{\Omega} F(t)^2 dv \geq \int_{B(p,r)} f(t)^2 dv$$  \hspace{1cm} (3.20)

and

$$\int_{\Omega} \frac{F(t)^2}{\sinh^2 t} dv \leq \int_{B(p,r)} \frac{f(t)^2}{\sinh^2 t} dv.$$  \hspace{1cm} (3.21)

Substituting (3.20) and (3.21) into (3.13), one gets

$$\frac{1}{n-1} \sum_{i=1}^{n-1} \frac{1}{\mu_i(\Omega)} \geq \frac{\int_{B(p,r)} f(t)^2 dv}{\int_{B(p,r)} \left( f'(t)^2 + (n-1) \frac{f(t)^2}{\sinh^2 t} \right) dv}$$

$$= \frac{1}{\mu_1(B(p,r))}$$  \hspace{1cm} (3.22)

and equality holds if and only if $\Omega = B(p,r)$. This completes the proof of Theorem 1.2.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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