RETURNING FUNCTIONS WITH CLOSED GRAPH ARE CONTINUOUS

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ABSTRACT. A function $f : X \to \mathbb{R}$ defined on a topological space $X$ is called returning if for any point $x \in X$ there exists a positive real number $M_x$ such that for every path-connected subset $C_x \subset X$ containing the point $x$ and any $y \in C_x \setminus \{x\}$ there exists a point $z \in C_x \setminus \{x, y\}$ such that $|f(z)| \leq \max\{M_x, |f(y)|\}$. A topological space $X$ is called path-inductive if a subset $U \subset X$ is open if and only if for any path $\gamma : [0, 1] \to X$ the preimage $\gamma^{-1}(U)$ is open in $[0, 1]$. The class of path-inductive spaces includes all first-countable locally path-connected spaces and all sequential locally contractible spaces. We prove that a function $f : X \to \mathbb{R}$ defined on a path-inductive space $X$ is continuous if and only if it is returning and has closed graph. This implies that a (weakly) Świątkowski function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it has closed graph, which answers a problem of Maliszewski, inscribed to Lviv Scottish Book.

Let $X$ and $Y$ be topological spaces. We say that a function $f : X \to Y$ has closed graph if its graph
\[
\Gamma_f := \{(x, f(x)) : x \in X\}
\]
is closed in the product $X \times Y$. It is well-known that each continuous function $f : X \to Y$ to a Hausdorff topological space $Y$ has closed graph. Trivial examples show that the converse statement is not true in general.

There exist many (algebraic or topological) properties of functions, which being combined with the closedness of the graph yield the continuity, see Tao’s blog [21]. For example, by the classical Closed Graph Theorem, a linear operator between Banach spaces is continuous if and only if it has closed graph.

Known topological properties implying the continuity of functions with closed graph include the Darboux property, the subcontinuity, the almost continuity, the near continuity, the $B$-quasicontinuity, the weak Gibson property, etc. (see [1], [3], [4], [5], [6], [7], [8], [13], [16], [17], [18], [19], [22]).

The inspiration for our present investigation was a problem, written by the third author to the Lviv Scottish Book on 01.05.2018. Actually, the problem was originally asked several years ago by Aleksander Maliszewski: Is any Świątkowski function with closed graph continuous? Our main theorem gives an affirmative answer to this question and a posteriori to many its modifications (for peripherally continuous functions, peripherally bounded functions, Darboux functions, etc.).

One of the most general properties, responsible for the continuity of real-valued functions with closed graph is introduced in the following definition.
**Definition 1.** A real-valued function \( f : X \to \mathbb{R} \) on a topological space \( X \) is defined to be

- **returning at a point** \( x \in X \) if there exists a positive real number \( M_x \) such that for every path-connected subset \( C_x \subset X \) containing the point \( x \) and any \( y \in C_x \setminus \{x\} \) there exists a point \( z \in C_x \setminus \{x,y\} \) such that \( |f(z)| \leq \max\{M_x, |f(y)|\};
- **returning** if \( f \) is returning at each point.

Another property that will be used in our characterization of the continuity of functions with the closed graphs is called the path-continuity.

**Definition 2.** A function \( f : X \to Y \) between topological spaces is defined to be **path-continuous** if for any continuous function \( \gamma : [0,1] \to X \) the continuity of the restriction \( f \circ \gamma | [0,1) \) implies the continuity of the composition \( f \circ \gamma \).

The following theorem is the main result of this paper.

**Theorem 1.** For a function \( f : \mathbb{R} \to \mathbb{R} \) the following conditions are equivalent:

1. \( f \) is continuous;
2. \( f \) has closed graph and is returning;
3. \( f \) has closed graph and is path-continuous.

For the proof of this theorem we shall need some auxiliary notions and results.

For two points \( a < b \) on the real line let \([a,b]\) and \((a,b)\) be the closed and open intervals with end-points \( a, b \), respectively. To avoid confusion between notations for open intervals and ordered pairs, an ordered pair of real numbers \( a, b \) will be denoted by \((a, b)\).

A function \( f : X \to \mathbb{R} \) defined on a topological space \( X \) is called

- **weakly discontinuous** if for any non-empty closed subset \( A \subset \mathbb{R} \) the set \( C(f|A) \) of continuity points of the restriction \( f|A \) has non-empty interior in \( A \);
- **locally bounded** at a point \( x \in X \) if \( x \) has a neighborhood \( U_x \subset X \) such that the set \( f(U_x) \) is bounded in \( \mathbb{R} \).

The following lemma is known (see [2], [6]) and its proof is included for the convenience of the reader.

**Lemma 1.** Assume that a function \( f : X \to \mathbb{R} \), defined on a closed subset \( X \subset \mathbb{R} \) has closed graph. Then

1. \( f \) is weakly discontinuous;
2. \( f \) is continuous at a point \( x \in X \) if and only if \( f \) is locally bounded at \( x \);
3. the set \( C(f) \) of continuity points is open and dense in \( X \).

**Proof.** For every \( n \in \mathbb{N} \) consider the set \( K_n = \{ x \in X : \max\{|x|, |f(x)| \} \leq n \} \) and observe that it is compact, being the projection of the compact subset \( \Gamma_f \cap [-n, n]^2 \) onto the real line. Next, observe that the restriction \( f|K_n \) is continuous since it has compact graph \( \Gamma_f \cap [-n, n]^2 \). For every closed subset \( A \subset X \), the Baire Theorem yields a number \( n \in \mathbb{N} \) such that \( A \cap K_n \) contains a non-empty relatively open subset \( U \) of \( A \). Taking into account that \( U = C(f|U) \subset C(f|A) \), we see that \( f \) is weakly discontinuous.

Assuming that \( f \) is locally bounded at some point \( x \in X \), we can find a bounded neighborhood \( U_x \subset X \) such that \( f(U_x) \) is bounded and hence \( U_x \cup f(U_x) \subset (\neg n, n) \). Then \( U_x \subset K_n \) and the continuity of \( f|U_x \) follows from the continuity of \( f|K_n \).

The weak discontinuity of \( f \) implies the density of the set \( C(f) \) in \( X \). To see that \( C(f) \) is open in \( X \), take any continuity point \( x \in X \) of \( f \) and find a neighborhood \( U_x \subset X \) whose image \( f(U_x) \) is bounded in the real line. By Lemma [2], \( f|U_x \) is continuous and hence \( U_x \subset C(f) \). \( \square \)
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Proof of Theorem 1. The implication (1) ⇒ (2) is trivial. To prove that (2) ⇒ (3), assume that \( f: \mathbb{R} \to \mathbb{R} \) is a returning function with closed graph. To prove that \( f \) is path-continuous, take any continuous map \( \gamma: [0, 1] \to \mathbb{R} \) such that \( f \circ \gamma | [0, 1] \) is continuous. The closedness of the graph of the function \( f \) implies the closedness of the graph of the function \( f \circ \gamma \). By Lemma 2, the continuity of \( f \circ \gamma \) will follow as soon as we check that the function \( f \circ \gamma \) is locally bounded at 1.

To derive a contradiction, assume that \( f \circ \gamma \) is not locally bounded at 1. We claim that

\[
\liminf_{t \to 1} |f \circ \gamma(t)| = \infty.
\]

Assuming that \( \liminf_{t \to 1} |f \circ \gamma(t)| < \infty \), we can fix any \( M > \max\{|f \circ \gamma(1)|, \liminf_{t \to 1} |f \circ \gamma(t)|\} \) and conclude that for every \( \varepsilon > 0 \) the interval \((1-\varepsilon, 1) \subset [0, 1]\) contains a point \( t \) such that \( |f \circ \gamma(t)| < M \).

Since the graph \( \Gamma_{f \circ \gamma} \) of the function \( f \circ \gamma \) is closed and the points \((1, M)\) and \((1, -M)\) do not belong to \( \Gamma_{f \circ \gamma} \), there exists \( \varepsilon > 0 \) such that the set \([1-\varepsilon, 1] \times \{-M, M\}\) is disjoint with \( \Gamma_{f \circ \gamma} \).

Since \( f \circ \gamma \) is not locally bounded at 1, there exists a point \( y \in (1-\varepsilon, 1) \) such that \( |f(y)| > M > \liminf_{t \to 1} |f \circ \gamma(t)| \).

Now the definition of \( \liminf_{t \to 1} |f \circ \gamma(t)| < M \) yields a point \( x \in (y, b) \) such that \( |f \circ \gamma(x)| < M < |f \circ \gamma(y)| \). By the Mean Value Theorem, for some point \( z \in [x, y] \subset [0, 1] \subset C(f \circ \gamma) \) we have \( |f \circ \gamma(z)| = M \) and hence \( (z, f \circ \gamma(z)) \in \Gamma_{f \circ \gamma} \cap ([1-\varepsilon, 1] \times \{-M, M\}) \), which contradicts the choice of \( \varepsilon \). This contradiction completes the proof of the equality \( \liminf_{t \to 1} |f \circ \gamma(t)| = \infty \).

Since the function \( f \) is returning at the point \( \gamma(1) \), there exists a positive real constant \( M \) such that for every \( \lambda < 1 \) there exists a point \( x \in \gamma([\lambda, 1]) \setminus \{\gamma(1), \gamma(1)\} \) such that \( |f(x)| \leq \max\{|M, |f \circ \gamma(\lambda)|\} \). Since \( \liminf_{t \to 1} |f \circ \gamma(t)| = \infty \), there exists a point \( c \in [0, 1] \) such that \( |f \circ \gamma(c)| > M \).

Taking into account that the function \( f \circ \gamma | [0, 1] \) is continuous and tends to infinity at 1, we conclude that the set \( L := \{x \in [c, 1] : |f \circ \gamma(x)| \leq |f \circ \gamma(c)|\} \) is compact and hence has the largest element \( \lambda \in L \). By the choice of the constant \( M \), there exists a point \( x \in (\lambda, 1) \) such that \( |f \circ \gamma(x)| \leq \max\{|M, |f \circ \gamma(\lambda)|\} \) \( |f \circ \gamma(\lambda)| \). Then \( x \in L \) and hence \( x \leq \lambda \), which contradicts the inclusion \( x \in (\lambda, 1) \). This contradiction completes the proof of the continuity of \( f \circ \gamma \), which implies the path-continuity of \( f \).

(3) ⇒ (1) Now assume that \( f \) has closed graph and \( f \) is path-continuous. By Lemma 3, the set \( C(f) \) of continuity points of \( f \) is open and dense in \( \mathbb{R} \). If \( C(f) = \mathbb{R} \), then \( f \) is continuous and we are done. So, assume that \( C(f) \neq \mathbb{R} \).

Let \( C \) be the family of connected components of the set \( C(f) \). It follows that each set \( C \in C \) is an open connected subset of the real line. By the path-continuity of \( f \), for each \( C \in C \) and its closure \( \overline{C} \) in \( \mathbb{R} \) the restricted function \( f|\overline{C} \) is continuous.

Claim 1. Any distinct sets \( C_1, C_2 \in C \) have disjoint closures.

Proof. Assuming that \( C_1 \cap \overline{C}_2 \neq \emptyset \), we conclude that the set \( \overline{C}_1 \cup \overline{C}_2 \) is convex and so is its interior \( U \). The continuity of the restrictions \( f|\overline{C}_1 \) and \( f|\overline{C}_2 \) imply that \( f|U \) is continuous and hence \( U \subset C(f) \) and \( C_1 \cup C_2 \subset U \subset C \) for some connected component \( C \subset C \), which is not possible as distinct connected components of \( C(f) \) are disjoint.

Claim 1 implies that the complement \( F = \mathbb{R} \setminus \mathcal{C}(f) \) has no isolated points. By Lemma 4, the function \( f \) is weakly discontinuous, so there exist points \( a < b \) such that \( \emptyset \neq [a, b] \cap F \subset C(f) \).

By the continuity of \( f \) on the compact set \( K := [a, b] \cap F \), there exists a number \( M \in \mathbb{N} \) such that \( |f(x)| < M \) for all \( x \in K \). The restriction \( f| [a, b] \) is discontinuous and hence unbounded. Then we can choose a sequence \( (x_n)_{n \in \omega} \) of points of \( [a, b] \) such that \( |f(x_n)| > M + n \) for all \( n \in \omega \). For every \( n \in \omega \) let \( C_n \in C \) be the unique connected component of \( C(f) \), containing the point \( x_n \). Replacing \( x_n \) by a suitable subsequence, we can assume that \( |f(x_n)| > \max\{|f(x) : x \in [a, b] \cup \bigcup_{k<n} \overline{C}_k\}| \) for all \( n \in \omega \). This ensures that the components \( C_n \) are pairwise disjoint and hence \( \operatorname{diam}(C_n) \to 0 \).
Replacing \((x_n)_{n \in \omega}\) by a suitable subsequence, we can assume that \(C_n \subset [a, b]\) for all \(n \in \omega\) and \((x_n)_{n \in \omega}\) converges to some point \(c \in [a, b]\), which is a discontinuity point of \(f\) as \(\lim_{n \to \infty} |f(x_n)| = \infty \neq |f(c)|\). Then \(|f(c)| < M\) by the choice of \(M\). The graph \(\Gamma_f\) of \(f\) is closed and hence is disjoint with the set \([c - \varepsilon, c + \varepsilon] \times \{-M, M\}\) for some \(\varepsilon > 0\). For every \(n \in \omega\) write the component \(C_n\) as \(C_n = (a_n, b_n)\) for some points \(a_n < b_n\) in the set \([a, b]\). Since \(x_n \to c\) and \(|b_n - a_n| \to 0\), there exists \(n \in \omega\) such that \((a_n, b_n) \subset [c - \varepsilon, c + \varepsilon]\).

Since \(|f(a_n)| < M < |f(x_n)|\), the Mean Value Theorem applied to the continuous function \(f[a_n, x_n]\) yields a point \(z_n \in [a_n, z_n]\) with \(|f(z_n)| = M\). Then \((z_n, f(z_n)) \in \Gamma_f \cap \((c - \varepsilon, c + \varepsilon) \times \{-M, M\}\) = \(\emptyset\), which is a desired contradiction completing the proof of Theorem 1.

Theorem 1 admits a generalization to real-valued functions defined on path-inductive topological spaces.

By a path in a topological space \(X\) we understand any continuous function \(\gamma: [0, 1] \to X\).

We define a topological space \(X\) to be \(\text{path-inductive}\) if a subset \(U \subset X\) is open if and only if for any path \(\gamma: [0, 1] \to X\) the preimage \(\gamma^{-1}(U)\) is open in \([0, 1]\).

**Proposition 1.** A topological space \(X\) is path-inductive if \(X\) is either sequential and locally contractible or \(X\) is first-countable and locally path-connected.

**Proof.** Assume that \(X\) is either sequential and locally contractible or \(X\) is first-countable and locally path-connected. Given a non-open set \(A \subset X\) we should find a path \(\gamma: [0, 1] \to X\) such that \(\gamma^{-1}(A)\) is not open in \([0, 1]\).

By the sequentiality or the first-countability of \(X\), there exists a sequence \(\{x_n\}_{n \in \omega} \subset X \setminus A\) that converges to a point \(x_\omega \in A\).

If \(X\) is locally contractible, then there exists a neighborhood \(V \subset X\) of the point \(x_\omega\) and a continuous map \(h: V \times [0, 1] \to X\) such that \(h(x, 0) = x_\omega\) and \(h(x, 1) = x_\omega\) for all \(x \in V\). Replacing \((x_n)_{n \in \omega}\) by a suitable subsequence, we can assume that \(\{x_n\}_{n \in \omega} \subset V\). Consider the homotopy \(H: V \times [0, 2] \to X\) defined by

\[
H(x, t) = \begin{cases} h(x, 1 - t) & \text{if } t \in [0, 1] \\ h(x, t - 1) & \text{if } t \in [1, 2] \end{cases}
\]

for \((x, t) \in V \times [0, 2]\), and observe that \(H(x, 0) = H(x, 2) = x_\omega\) and \(H(x, 1) = x_\omega\) for every \(x \in V\).

Define a path \(\gamma: [0, 1] \to X\) by \(\gamma(0) = x_\omega\) and \(\gamma(t) = H(x_\omega, 2^n + 2t - 2)\) where \(n \in \omega\) is the unique number such that \(\frac{1}{2^n} < t \leq \frac{1}{2^{n+1}}\). It is clear that the function \(\gamma\) is continuous at each point \(t \in (0, 2)\). To see that \(\gamma\) is continuous at 0, fix any neighborhood \(W \subset X\) of \(\gamma(0) = x_\omega\) and using the continuity of the function \(H\) at points of the compact set \(\{x_n\}_{n \in [0, 2]} \subset H^{-1}(x_\omega) \subset H^{-1}(W)\), find a neighborhood \(U \subset V\) of the point \(x_\omega\) such that \(H(U \times [0, 2]) \subset W\). Since the sequence \((x_n)_{n \in \omega}\) converges to \(x_\omega\), there exists a number \(m \in \omega\) such that \(\{x_n\}_{n \geq m} \subset U\). Then

\[
\gamma([0, \frac{1}{2^m}]) \subset \{x_\omega\} \cup \bigcup_{n \geq m} H(\{x_n\} \times [0, 2]) \subset H(U \times [0, 2]) \subset W,
\]

witnessing that \(\gamma\) is continuous at 0.

Next, assume that the space \(X\) is first-countable and locally path-connected. In this case we can find a countable neighborhood base \(\{U_n\}_{n \in \omega}\) at \(x_\omega\) such that for every \(n \in \omega\) and point \(x \in U_{n+1}\) there exists a path \(\gamma_x: [0, 1] \to U_n\) such that \(\gamma_x(0) = x\) and \(\gamma_x(1) = x_\omega\). Since the sequence \((x_n)_{n \in \omega}\) converges to \(x_\omega\), for every \(n \in \omega\) there exists a number \(k_n \in \omega\) such that \(x_{k_n} \in U_{n+1}\). By our assumption, there exists a number \(\frac{1}{2^n} < t \leq \frac{1}{2^{n+1}}\) such that \(\gamma_n(\frac{1}{2^n}) = \gamma_n(\frac{1}{2^{n+1}}) = x_\omega\) and \(\gamma_n(\frac{3}{2^n}) = x_{k_n}\). The paths \(\gamma_n, n \in \omega\), compose a continuous path \(\gamma: [0, 1] \to X\) such that \(\gamma(0) = x_\omega\) and \(\gamma(\frac{1}{2^n}) = \gamma(\frac{3}{2^n}) = \gamma_n\) for all \(n \in \omega\).
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Taking into account that $0 \in \gamma^{-1}(x_\omega) \subset \gamma^{-1}(A)$ and $\frac{3}{k_n} \in \gamma^{-1}(x_{k_n}) \subset \gamma^{-1}(X \setminus A) = X \setminus \gamma^{-1}(A)$ for all $n \in \omega$, we see that the set $\gamma^{-1}(A)$ is not open in $[0, 1]$. □

**Remark 1.** Proposition 1 implies that each sequential linear topological space over the field $\mathbb{R}$ is path-inductive (being locally contractible). In particular, the inductive limit $\mathbb{R}^\infty = \varinjlim \mathbb{R}^n$ of an increasing sequence of finite-dimensional Euclidean spaces is a sequential path-inductive space, which is not first-countable. On the other hand, the Sierpiński triangle is first-countable locally path-connected but not locally contractible.

Theorem 1 implies the following its self-generalization.

**Theorem 2.** For a real-valued function $f : X \to \mathbb{R}$ on a path-inductive topological space $X$ the following conditions are equivalent:

1. $f$ is continuous;
2. $f$ is a returning function with closed graph;
3. $f$ is a path-continuous function with closed graph.

**Proof.** The implication $(1) \Rightarrow (2)$ is trivial and $(2) \Rightarrow (3)$ follows from Theorem 1. To prove that $(3) \Rightarrow (1)$, assume that $f$ is path-continuous and has closed graph. If $f$ is not continuous, then by the path-inductivity of $X$, there exists a path $\gamma : [0, 1] \to X$ such that the function $f \circ \gamma$ is not continuous. Let $g : \mathbb{R} \to \mathbb{R}$ be the extension of $f \circ \gamma$ such that $g((-\infty, 0]) = \{g(0)\}$ and $g([1, +\infty)) = \{g(1)\}$. The closedness of the graph of the function $f$ implies the closedness of the graph of the functions $f \circ \gamma$ and $g$. The path-continuity of the function $f$ implies the path-continuity of $g$. By Theorem 1 the function $g$ is continuous, which contradicts the choice of $\gamma$. This contradiction completes the proof. □

The following examples show that the closedness of the graph is essential in Theorems 1 and 2.

**Example 1.** The function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0; \end{cases}$$

is returning but not path-continuous.

**Example 2.** Let $\xi : \mathbb{Q} \to \mathbb{N}$ be any bijective function. The function

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{|x|} - \sum_{Q \ni q < |x|} 2^{-\xi(q)} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0; \end{cases}$$

is path-continuous but not returning.

Theorems 1 and 2 motivate studying returning functions in more details. The following proposition was suggested by the referee.

**Proposition 2.** For any function $f : \mathbb{R} \to \mathbb{R}$ the set

$$X := \{x \in \mathbb{R} : f \text{ is not returning at } x\}$$

is countable and nowhere dense in $\mathbb{R}$.

**Proof.** Assuming that $X$ is uncountable, we can find $n \in \mathbb{N}$ such that the set $X_n = \{x \in X : |f(x)| \leq n\}$ is uncountable. Being uncountable, the set $X_n$ contains a point $x$ such that for any $\varepsilon > 0$ the sets $X_n \cap (x, x + \varepsilon)$ and $X_n \cap (x - \varepsilon, x)$ are not empty. Now Definition 1 ensures that $f$ is returning at $x$, which contradicts the choice of $x \in X$. 301
Next, we show that the set $X$ is nowhere dense in $\mathbb{R}$. To derive a contradiction, assume that the closure of $X$ contains some interval $I = (a, b)$. For every $n \in \mathbb{N}$ consider the set $I_n := \{ x \in I : |f(x)| \leq n \}$. Since $I = \bigcup_{n \in \mathbb{N}} I_n$, we can apply the Baire Theorem, and find $n \in \mathbb{N}$ such that the closure of the set $I_n$ contains some open interval $(c, d) \subset (a, b)$. Now Definition 1 ensures that $f$ is returning at each point $x \in (c, d)$, which implies that $X \cap (c, d) = \emptyset$ and hence $X \cap (a, b)$ is not dense in $(a, b)$. \hfill \square

**Remark 2.** It is easy to construct a function $f : \mathbb{R} \to \mathbb{R}$ which is not returning at a countable dense set of points of the standard Cantor set in $\mathbb{R}$.

Next, we show that the class of returning functions on the real line is quite wide and contains many known classes of real-valued functions possessing some generalized continuity properties.

Let us recall that a function $f : X \to \mathbb{R}$ on a topological space $X$ is called

- Świątkowski if for any connected subset $C \subset X$ and points $a, b \in C$ with $f(a) < f(b)$ there exists a continuity point $x \in C$ of $f$ such that $f(a) < f(x) < f(b)$;
- weakly Świątkowski if for any connected subset $C \subset X$ and points $a, b \in C$ with $f(a) < f(b)$ there exists a point $x \in C$ such that $f(a) < f(x) < f(b)$;
- Darboux if for any connected subset $C \subset X$ the image $f(C)$ is connected;
- almost continuous (or else nearly continuous) if for any open set $V \subset X$ the preimage $f^{-1}(V)$ is contained in the interior of $\overline{f^{-1}(V)}$;
- quasicontinuous if for each point $x \in X$, neighborhood $V \subset Y$ of $f(x)$ and neighborhood $O_x \subset X$ of $x$, there exists a non-empty open set $G \subset O_x$ with $f(G) \subset V$;
- $B$-quasicontinuous if for each point $x \in X$, neighborhood $V \subset Y$ of $f(x)$ and an open connected set $O \subset X$ with $x \in \overline{O}$, there exists a non-empty open set $G \subset O$ with $f(G) \subset V$;
- (weakly) Gibson if $f(O) \subset \overline{f(O)}$ for any open (connected) subset $O \subset X$;
- peripherally continuous if for any point $x \in X$ and neighborhoods $O_x \subset X$ and $O_{f(x)} \subset \mathbb{R}$ of $x$ and $f(x)$ there exists a neighborhood $V_x \subset O_x$ of $x$ such that $f(\partial V_x) \subset O_{f(x)}$.
- peripherally bounded if for any point $x \in X$ there exists a bounded set $B \subset \mathbb{R}$ such that for any neighborhood $O_x \subset X$ of $x$ there exists a neighborhood $V_x \subset O_x$ of $x$ such that $f(\partial V_x) \subset B$.

Here by $\partial V_x$ we denote the boundary of $V_x$ in the topological space $X$. For more information on these classes of functions, see the survey of Gibson and Natkaniec [11].

For any function $f : X \to \mathbb{R}$ these notions relate as follows (by a simple arrow we denote the implications holding under the additional assumption $X = \mathbb{R}$):

$$
\begin{array}{ccc}
\text{Świątkowski} & \text{bounded} & \text{almost continuous} \\
\downarrow & \downarrow & \downarrow \\
\text{weakly Świątkowski} \xrightarrow{\text{ returning}} & \text{peripherally bounded} & \text{peripherally continuous} \\
\uparrow & \uparrow & \uparrow \\
\text{Darboux} & B\text{-quasicontinuous} & \text{weakly Gibson} \\
\end{array}
$$

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Example 3. The Dirichlet function

\[ f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \notin \mathbb{Q}; \end{cases} \]

is returning but not weakly Świątkowski.

Example 4. The function

\[ f: \mathbb{R} \to \mathbb{R}, \quad f(x) = \begin{cases} \frac{1}{2^n} |x| & \text{if } 2^n < |x| \leq 2^{n+1} \text{ for some } n \in \mathbb{Z}; \\ 0 & \text{if } x = 0; \end{cases} \]

is returning but not peripherally bounded.

Since each (weakly) Świątkowski function is returning, Theorem 1 implies the following corollary that answers the original problem of Maliszewski.

Corollary 1. A (weakly) Świątkowski function \( f: \mathbb{R} \to \mathbb{R} \) is continuous if and only if it has closed graph.

It is interesting to compare Theorems 1 and 2 to the following known results on the continuity of functions with closed graphs.

Theorem 3. A function \( f: X \to \mathbb{R} \) defined on a topological space \( X \) is continuous if it has closed graph and one of the following conditions is satisfied:

1. \( f \) is almost continuous (Long and McGehee [16]);
2. \( f \) is nearly continuous and \( X \) is Baire (Moors [18]);
3. \( f \) is Darboux and \( X \) is locally connected (Wójcik [23: Corollary 18]);
4. \( f \) is peripherally continuous and \( X = \mathbb{R} \) (Hagan [12]);
5. \( X = \mathbb{R} \) and \( f \) is bilaterally quasicontinuous (Doboš [9]);
6. \( f \) is \( B \)-quasicontinuous and \( X \) is locally connected (Borsík [5]);
7. \( f \) is weakly Gibson and \( X \) is locally connected (Das and Nesterenko [8]).

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