On small world Non-Sunada twins and cellular Voronoi diagrams

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Abstract. Special infinite families of regular graphs of unbounded degree of bounded diameter (small world graphs) are considered. Two families of small world graphs $G_i$ and $H_i$ form a family of Non-Sunada twins if $G_i$ and $H_i$ are isospectral of bounded diameter but groups $\text{Aut}(G_i)$ and $\text{Aut}(H_i)$ are nonisomorphic.

We say that a family of Non-Sunada twins are unbalanced if each $G_i$ is edge transitive but each $H_i$ is edge intransitive. If all $G_i$ and $H_i$ are edge transitive we have balanced family of small world Non-Sunada twins. We say that a family of Non-Sunada twins are strongly unbalanced if each $G_i$ is edge transitive but each $H_i$ is intransitive.

We use term edge disbalanced for the family of Non-Sunada twins such that all graphs $G_i$ and $H_i$ are edge intransitive.

We present explicite constructions of above defined families. Two new families of distance regular but not of distance transitive graphs will be introduced.

Keywords: Laplacians, isospectric graphs, small world graphs, distance regular graphs, Non-Sunada constructions, graph Voronoi diagram, thin Voronoi cells

1 Introduction

As everybody knows the answer for the famous question "Can you hear shape of graph?" is negative. In this paper we would like to stress that in fact we are so deaf that unable to recognise orchestras of large families of small world graphs. Two nonisomorphic families can produce the same symphony in absolutely identical manner.

In [1] Sunada presented a method of constructing pairs of nonisometric manifolds $\{M_1, M_2\}$ such that the eigenvalues of Laplace operator satisfy $\lambda_i(M_1) = \lambda_i(M_2)$ for all $i$. His method was based on interpreting the isospectrality condition in terms of finite group theory: if $G$ is a finite group acting freely on a manifold $M^G$, with $M_1$ and $M_2$ quotients of $M^G$ by subgroups $H_1$ and $H_2$ respectively, then $M_1$ and $M_2$ will be isospectral if the induced representations of the trivial representation $\text{ind}^G_{H_1}(I)$ and $\text{ind}^G_{H_2}(I)$ are equivalent as $G$-representations, where $I$ denotes the trivial representation. If $\Gamma_1$ and $\Gamma_2$ are
two graphs then one may define a Laplacian on them, and ask whether they are isospectral. As it is written in [2] "There are many constructions of isospectral graphs which appear to have little to do with the Sunada construction, see [3] for survey. However if we imposed the condition that \( \Gamma_1 \) and \( \Gamma_2 \) are \( k \)-regular, which is a natural condition to impose from the point of view of geometry, then most of these constructions do not apply. An exception to this is the construction of Seidel switching ([3]). In fact some other construction of isospectral distance regular graphs of Non Sunada nature were already known and presented in [4]. In particular Hemmeter’s [5] and Ustimenko’s [6] examples of distance regular but not distance transitive graphs via new construction of infinite families of isospectral pairs of graphs were reflected in [4]. In fact now we have other examples of isospectral graphs due to the recent progress in Distance Regular Graphs Theory (see [7]).

The purpose of this paper is to draw attention of specialists in applicled graph theory to the fact that theory of Tits geometries over Dynkin-Coxeter diagrams \( B_n \) and \( C_n \) provides a remarkable examples of infinite sequences of Non-Sunada isospectral pairs of graphs which form the families of small world graphs. Notice that similar situation had happened in Extremal Graph Theory, Benson [8] noticed only in 1966 that size of defined 7 years earlier by Tits finite regular generalised polygons [9] are on the upper bound from Erdos Even Circuite Theorem.

We conjecture that many other families of small world Non - Sunada twins can be defined via Borel subgroups and partitions of largest Schubert cell into special medium Schubert cells.

Two new families of distance regular but not distance transitive families of graphs are introduced.

Finally we introduce the concept of thin Voronoi cell connected with the idea of Voronoi graph diagram. Nice examples of Voronoi cells of geometrical nature are thin Schubert cells. It is interesting that isospectrality of small world graphs described in this paper can be easily explained via similarity of cellular Voronoi diagrams.

Two families of distance transitive graphs defined in terms of Lie geometry form pair of Non Sunada twins. Let \( \Gamma(B_n(q)) \) and \( \Gamma(C_n(q)) \) be totellations of elements of geometries of Chevalley groups \( B_n(q) \) and \( C_n(q) \), with type corresponding to node \( i \) of Coxeter Dynkin diagram where \( i = 1, 2, \ldots, n \). If \( q \) is odd, then binary relation \( \phi(B_n(q)) \) and \( \phi(C_n(q)) \) two elements of type \( n \) are incident to common element of type \( n - 1 \) defined on \( \Gamma_n(B_n(q)) \) and \( \Gamma_n(C_n(q)) \) are non isomorphic isospectral small world graphs.

Each family can be embedded in a larger family of small world graphs via the following scheme:

1. Take universal geometry of type \( B_n \) (or \( C_n \)) defined over commutative ring \( K \) defined in section 9, compute its specialization \( S \) which is incidence system with partition sets \( \Gamma^1(X_n), \Gamma^2(X_n), \ldots, \Gamma^n(X_n) \).

2. Consider graphs \( \phi_S(X_n, K) \) of the binary relation of incidency of two distinct elements of \( \Gamma^n(X_n) \) to some element of \( \Gamma^{n-1}(X_n) \).
Notice that new families of small world graphs $\phi_S(X_n, K)$ depends on the 
ring $K$ and specialisations $S$. Graphs $\phi_S(K)$ for the chosen ring $K$ and various 
$S$ have the same sets of vertices.

So the following problem appears naturally:

Find a ring $K$ and a pair of representations $S^1$ and $S^2$ such that nonisomorphic graphs $\phi_S(B_n, K)$ and $\phi_S(C_n, K)$ are isospectral. In the case of finite fields $F_q$ of odd order such a pair of $S^1$ and $S^2$ really exists.

We can consider families of distance regular graphs $\phi(D_{n+1}(q))$ (two vertices of $\Gamma_n(D_{n+1}(q))$ are incident to some element of type $n - 1$) and $\phi(X_n(q))$ (two vertices of $\Gamma_n(X_n(q))$ are incident to some element of type $n - 2$) where $X = B$ or $X = C$ and $n \geq 3$.

Notice that graph $\phi(D_{n+1}(q))$ and $\phi(C_n(q))$ are isomorphic and $\phi'(C_n(q))$ are Ustimenko graphs (see subject index of [4]). One can take universal geometries of $X_n$ over finite commutative ring $K$ for $X = B$, $X = C$ and $X = D$ and their specialisations.

The search for nonisomorphic isospectral pairs of graphs from the disjoint union of $\phi_S(D_{n+1}(q))$, $\phi_S'(B_n, K)$ and $\phi_S'(C_n, K)$ makes sense because of isospectrality of $\phi(D_{n+1}(q))$ and Ustimenko graphs for odd $q$ and $n \geq 3$.

We can consider the restrictions of $\phi(D_4(q))$, 

$(X_3(q))$ for $X = B$ and $X = C$ or $X = B$ on the largest Schubert cells of corresponding geometry which are also distance regular graphs (see section 6) and investigate restrictions of $\phi_S(D_4, K)$, $\phi'_S(B_3, K)$ and $\phi'_S'(B_3, K)$ on corresponding the largest VoronoiSchubert cells (see section 9).

The interpretation of geometry $\Gamma(n, q)$ of Chevalley group $X_n(q)$ as special interpretation of universal geometry $U(X_n, K)$ allows to generate this object in computer memory in time $O(|\Gamma(n, q)|)$ with fast algorithm $O(n^2)$ to check whether or not two elements of the geometry are incident. Notice that incident systems $U_S(X_n, K)$ where $K$ is a finite commutative ring are "computationally equivalent" to $\Gamma(n, q)$.

The interpretation of $\Gamma(n, q)$ as $U_S(X_n, K)$ for special $S$ allows effectively to generate small world graphs corresponding to double cosets of kind $P_1 \cdot P_2$ and more general graphs corresponding to $P g P g$ like Double Grassman graphs of geometry $A_{2k+1}(q)$ considered in section 4 (see [31]).

The concept of Voronoi orbitals allows us to define new small world graphs in terms of specialisations of $U(X_n, K)$ which are analogs of double coset graphs.

An important geometrical object is a partition of elements of geometry of Chevalley group into thin Schubert cells. In the case of Projective Geometry it was investigated by D. Hilbert. We conjecture that this is a partition of vertices of incidence graphs into Voronoi cells with respect of Weyl geometry naturally embedded into geometry of Chevalley group. In the case of projective geometry over the field of complex numbers this fact was proven by I. M. Gelfand and R. MacPherson (see [16] and further references).

For each specialization $S$ of universal geometry $U(X_n, K)$ one can find naturally embedded geometry of Weyl group $W(X_n)$. So Voronoi Schubert cells can be considered for each incidence graph of $U_S(X_n, K)$. We hope that properties
of this partition are important for spectrum investigation of small world graph defined in term of $U(X_n, K)$.

2 Some basic definitions of Graph Theory and Algebraic Combinatorics

2.1 On distance in graph and related constructions

Let $G$ be a finite simple graph. For vertices $x$ and $y$ of $G$, we denote by $d(x, y)$ the length of the shortest path from $x$ to $y$, and set $G_i(x) = \{y : d(x, y) = i\}$. We sometimes write $G(x)$ for $G_1(x)$. Also $x \leftrightarrow y$ is used to denote that $x \in G(y)$. Now fix $x$ and $y$ in $V(G)$ with $y \in G_i(x)$. Define $a_i = |G_i(x) \cap G_1(y)|$, $b_i = |G_{i+1}(x) \cap G_1(y)|$, $c_i = |G_{i-1}(x) \cap G_1(y)|$. If these parameters never depend on $x$ and $y$, but only on $i$, then $G$ is called a distance-regular graph. One example of a distance-regular graph is a distance-transitive graph: a graph $G$ is distance-transitive if, for all $x_1, y_1, x_2, y_2 \in V(G)$ with $d(x_1, y_1) = d(x_2, y_2)$, there exists an automorphism $\sigma$ of $G$ with $\sigma(x_1) = x_2$ and $\sigma(x_2) = y_2$.

Let $G$ be any graph. We define a bipartite graph $G^*$ (called the extended bipartite double of $G$, cf. [4], p. 261) as follows. Let $F_1$ and $F_2$ be disjoint sets of the same size as $V(G)$, with $f_i : V(G) \to F_i$ bijections, for $i = 1, 2$. The vertex set of $G^*$ is defined to be $F_1 \cup F_2$. For $u, v \in V(G)$, $f_1(u)$ will be adjacent to $f_2(v)$ in $G^*$ if $u = v$ or $u$ is adjacent to $v$ in $G$. For $x, y \in V(G^*)$ we will write $d^*(x, y)$ for the distance from $x$ to $y$. We also define $\phi : V(G^*) \to V(G)$ by $\phi|F_i = f_i^{-1}$.

**LEMMA 1.**

Let $x, y \in V(G^*)$ with $d^*(x, y) = i$. Suppose that $u = \phi(x)$ and $v = \phi(y)$ with $d(u, v) = j$. Then $j$ is either $i$ or $i - 1$. The diameter of $G^*$ is greater by one than that of $G$.

If a regular graph $G$ is bipartite, it is associated with another regular graph $G$, defined as follows. Suppose that $V(G) = X \cup Y$ is the bipartition of $G$. We let $V(G) = X$ and, for $x_1, x_2 \in X$, let $x_1 \in G_1(x_2)$ iff $x_1 \in G_2(x_2)$, $G$ is called a halved graph of $G$. More information on halved graphs can be found in [4].

2.2 On Coxeter groups, Tits systems and their geometries

The incidence system is the triple $(\Gamma, I, t)$ where $I$ is a symmetric antireflexive relation (simple graph) on the vertex set $\Gamma$ such that $\alpha I \beta$ and $t(\alpha) = t(\beta)$ implies $\alpha = \beta$. The flag $F$ is a nonempty subset in $\Gamma$ such that $\alpha, \beta \in F$ implies $\alpha I \beta$. We assume that $t(F) = \{t(x) | x \in F\}$ (see [16]).

An important example of the incidence system as above is the so-called group incidence system $\Gamma(G, G_s)_{s \in S}$. Here $G$ is the abstract group and $G_s \in S$ is the family of distinct subgroups of $G$. The objects of $\Gamma(G, G_s)$ are the left cosets of $G_s$ in $G$ for all possible $s \in S$. Cosets $\alpha$ and $\beta$ are incident precisely when $\alpha \cap \beta \neq \emptyset$. The type function is defined by $t(\alpha) = s$ where $\alpha = gG_s$ for some $s \in S$.  

4 On small world Non-Sunada twins
Let \((W,S)\) be a Coxeter system, i.e. \(W\) is a group with a set of distinguished generators given by \(S = \{s_1, s_2, \ldots, s_l\}\) and generic relation \((s_i \times s_j)^{m_{ij}} = e\). Here \(M = (m_{ij})\) is a symmetrical \(l \times l\) matrix with \(m_{ii} = 1\) and off-diagonal entries satisfying \(m_{ij} \geq 2\) (allowing \(m_{ij} = \infty\) as a possibility, in which case the relation \((s_i \times s_j)^{m_{ij}} = e\) is omitted). Letting \(W_i = S - \{s_i\}, 1 \leq i \leq l\) we obtain a group incidence system \(\Gamma(W) = \Gamma(W_i)_{1 \leq i \leq l}\) which is called the Coxeter geometry of \(W\).

The \(W_i\) are referred to as the maximal standard subgroups of \(W\) (see [32] or [16]).

Let \(T\) be the totality of all reflection, i.e. elements of kind \(gs_i g^{-1}\) where \(g \in W, i = 1, 2, \ldots, n\). We assume that elements of \(W\) and \(T\) are listed in the standard lexicographical order with respect to basis \(s_1, s_2, \ldots, s_n\).

Let \(l(g)\) be the length of minimal irreducible decomposition of \(g\) into elements of \(S\). For \(\alpha \in \Gamma(W)\) we define \(l(\alpha)\) as the minimal length of representative of this coset. We introduce \(\Delta(\alpha) = \{s \in T | l(\alpha s) < l(\alpha)\}\).

Let \(G\) be a group, \(B\) and \(N\) subgroups of \(G\), and \(S\) be a collection of cosets of \(B \cap N\) in \(N\). We call \((G, B, N, S)\) Tits system (or we say that \(G\) has a BN-pair) if

\((i) G = < B, N > \) and \(B \cap N\) is normal in \(N\),
\((ii) S\) is a set of involutions which generate \(W = N/(B \cap N)\),
\((iii) sBw\) is a subset in \(BwB \cup BswB\) for any \(s \in S\) and \(w \in W\),
\((iv) sBs \neq B\) for all \(s \in S\).

Properties (1)-(iv) imply that \((W,S)\) is a Coxeter system (see[32] or [16]). Whenever \((G, B, N, S)\) is Tits system, we call a group \(W\) Weyl group of the system, or more usually Weyl group of \(G\). The subgroups \(P_i\) of \(G\) defined by \(BW_iB\) are called the standard maximal parabolic subgroups of \(G\). The group incidence system \(\Gamma(G) = \Gamma(G, P_i)_{1 \leq i \leq l}\) is commonly referred to as Lie geometry of \(G\) (see [16], [32]). Note that Lie geometry of \(G\) and Coxeter geometry of the corresponding Weyl group have the same rank. In fact there is a type preserving morphism from \(\Gamma(G)\) onto \(\Gamma(W)\) given by \(gP_i \to wW_i\), where \(w\) is determined from the equality \(BwP_i = BwP_i\). This morphism \(\text{Ret}\) is called a retraction.

Let us consider the totality \(F\) of conjugates of Borel subgroup \(B\) of kind \(B^n = nBn^{-1}\), where \(n \in N\) and their orbits on \(\Gamma(G)\). They define natural equivalence \(xty\) if \(x\) and \(y\) are in the same orbit of each subgroup from \(F\). The classes of this equivalence are known as thin Schubert cells (see [16] and further references with descriptions of various applications of this equivalence relations).

### 2.3 On association schemes, permutation groups and hypergroups

Theory of association schemes is one of the important directions of Algebraic Combinatorics.

Association scheme \(\Omega\) consists of finite set \(X\) and a collection of binary relations \(R_0, R_1, \ldots, R_d\), such that

1. \(R_0 = \{(x,x) | x \in X\}\)
2. for each \( i, i = 1, 2, \ldots, d \) the inverse relation
\[
R_i^{-1} = \{(x, y) | (y, x) \in R_i\}
\]
coinsides with some \( R_j \) from the collection.

3. for \( i, j, k \in \{0, 1, 2, \ldots, d\} \) the number
\[
p^k_{i,j} = p^k_{i,j}(a, b) = \{x \in X | ((a, x) \in R_i, (x, b) \in R_j)\}
\]
does not dependent on the choice of the pair \((a, b) \in R^k\)

Let \((G, X)\) be the transitive permutation group acting on the set \(X\). Orbitals of \((G, X)\), i.e. orbits of natural action of \(G\) on the set \(X \times X\), are binary relations (subsets of Cartesian product of \(X\) with itself). Let \(a \in X\) and
\[
H = \{g \in G | g(a) = a\},
\]
then orbitals of \((G, X)\) are in one to one correspondence with the double cosets
\(HgH, \ g \in G\).

For each directed graph \(R_i\) of associative scheme we consider its adjacency matrix \(A_i\) of size \(|X| \times |X|\) with entries \(a(x, y) \in \{0, 1\}\) such that \(a(x, y) = 1\) if and only if \((x, y) \in R_i\). As it follows from definition matrix product of \(A_i\) and \(A_j\) is
\[
\sum p^k_{i,j} A_k.
\]
So the vector subspace \(A(\Omega) = \langle A_0, A_1, A_2, \ldots, A_d \rangle\) in \(M_n(C)\), \(n = |X|\) (matrix algebra over the field \(C\) of complex numbers) is closed under matrix multiplication. Such algebras appeared in different areas of mathematics under different names such as **Bose-Messner algebra**, **cellular algebras**, **V-Shur rings** (see [33], [34]).

In the case of association schemes of orbitals for permutation group \((G, X)\) we will use term **Hecke algebra** (see [32]) instead of Bose Messner algebra. Hecke algebra \(H = H(G, X)\) can be identified with the centralizator of the totality of all permutations from \((G, X)\):
\[
H = \{M \in M_n(C) | A_{\pi} M = M A_{\pi} \text{ for all } \pi G\}
\]
\(A_{\pi}\) here is a permutational \(n \times n\) matrix corresponding to permutation \(\pi\).

We can also consider the totality \(HG\) of formal linear combinations of formal symbols \(a_0, a_1, \ldots, a_d\) such that relations
\[
a_i \times a_j = \sum z^k_{i,j} a_k
\]
give us definition of associative algebra with unity \(a_0\). This object is known in Functional Analysis as **hypergroup** (see, [35], [36]). Notice that hypergroup is an abstract algebra with fixed basis \(a_0, a_1, \ldots, a_d\) and integer structure constants \(p^k_{i,j} \geq 0\).

We will refer to nonisomorphic matrix Bose-Messner algebras with the same hypergroups as **hyperequivalent BM algebras**.
3 Two Cartan matrices with the same Coxeter groups and large families of isospectral graphs

Let us take the Coxeter-Dynkin diagrams $X_n$ with $n$ nodes, where X stands for one of the letters B, C and D with $n \geq 3$. Assume that $W(X_n)$ are their Weyl groups, and $X_n(q)$ are Chevalley groups in cases of diagrams $X_n$ and of finite fields $F_q$, where $q$ stands for some prime power.

Let $s_1, s_2, \ldots, s_n$ be Coxeter generators of $W = W(X_n)$ with $X_n = B_n$ or $X_n = C_n$ and $s_n$ corresponds to extremal right node of the diagram. Let us consider an action of the group $W(X_n)$ on its left cosets by standard subgroup $W_n$ generated by elements $s_1, s_2, \ldots, s_{n-1}$. One can identify the variety $(W : W_n)$ with a vector space $F_q^n$. The permutation group $(W, (W : W_n))$ coincides with the affine transformations $T_{A,b}$ of kind $x \rightarrow xA + b$, where $A$ is a permutational matrix, $x$ and $b$ are row vectors from $F_q^n$. So $W$ is a semidirect product of symmetric group $S_n$ and additive group of vector space $F_q^n$.

Let $T$ be a linear map on $F_q^n$ which transforms $(x_1, x_2, \ldots, x_n)$ into $(x_1 + x_2 + \cdots + x_n, x_2, x_3, \ldots, x_n)$. Then linear group $< T, s_1, s_2, \ldots, s_{n-1} >$ is isomorphic to $S_{n+1}$. Notice that group $W' = < W, T >$ is isomorphic to Weyl group $W(D_{n+1})$. It is known that $W(X_n)$ preserves Hamming distance $d$ on $F_q^n$ defined by the rule $d((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = |i : x_i \neq y_i|$.

So orbitals of transitive permutation group $W(X_n)$ are binary relations $\phi_i = \{(x, y) : d(x, y) = I, i = 0, 1, \ldots, n\}$.

One can see that orbitals of $W'$ are $\phi_0$ (equality relation) and $\phi_1 \cup \phi_2, \phi_3 \cup \phi_4, \ldots, \phi_{n-1} \cup \phi_n$ in the case of even $n$, Or $\phi_1 \cup \phi_2, \phi_3 \cup \phi_4, \ldots, \phi_{n-2} \cup \phi_{n-1}, \phi_n$ in the case of odd $n$.

Let us consider Chevalley groups $X_n(q)$ with its Borel subgroup $B$ acting on the totality of its left cosets $(X_n(q) : P_n)$ by standard parabolic subgroup $P_n = BW_nB$ corresponding to extremal right node of the Coxeter-Dynkin diagram. There is natural one to one correspondence between orbitals of $(X_n(q), (X_n(q) : P_n))$ and orbitals of $(W(X_n), (W : W_n))$ (see [14]). Let $\phi'_i$ be the corresponding orbital of $X_n(q)$ to orbital $\phi_i$ of Weyl group $W$. Proper subgroups $S$ of symmetric group $S(\Omega)$ for $\Omega = (X_n(q) : P_n)$ containing $X_n(q)$ are described in ([10], [11]). The proof does not use classification of simple finite group theorem.

If $X = C$ then $X_n(q) < G < N(X_n(q))$, where normaliser in $S(\Omega)$ is simply an extension of $X_n(q)$ via automorphisms of the ground field $F_q$. So orbitals of $N(C_n(q))$ and $C_n(q)$ are the same.

If $X = B$, then $B_n(q) < G < N(B_n(q))$ or $D_{n+1}(q) < G < N(D_{n+1}(q))$. The orbitals of groups $B_n(q)$ and $D_{n+1}(q)$ coincide with orbitals of their normalisers which are extentions of corresponding Chevalley groups by automorphisms of ground field $F_q$ (see [12] for the details).

We can describe orbitals of $D_{n+1}(q)$ via special subsets of $N = \{1, 2, \ldots, n\}$.

In the case of even $n$ we refer to $\{i, i + 1\}, i = 1, 2, \ldots, n/2$ as generic subsets.

In the case of odd $n$ we take the following list of generic subsets $\{i, i + 1\}$, $i = 1, 2, \ldots, n - 1)/2, \{n\}$. Orbitals of $D_{n+1}(q)$ are unions of $\phi'_i$, where $i$ belongs to single generic subset.
Let $J$ be a proper subset of $N = \{1, 2, \ldots, n\}$ and $C_J = C_J(n, q)$, $B_J = B_J(n, q)$ are graphs of binary invariant relations of permutation groups $(C_n(q), (C_n(q) : P_n))$ and $(B_n(q), (B_n(q) : P_n))$ which are unions of $\phi_i'$ for $i \in J$, Hecke algebras of $(B_n(q), (B_n(q) : P_n))$ and $(C_n(q), (C_n(q) : P_n))$ have the same intersectional indices (see [13] for closed formulae). This fact also follows from a distance regularity of $\phi'$. In fact structure constants of Hecke algebras of groups $B_n(q)$ and $C_n(q)$ acting on left cosets by maximal parabolic subgroup corresponding to any node of Coxeter diagram are also the same. This fact can be easily deduced from the relation in the definition of Tits generic algebra of Coxeter group $B_n$.

THEOREM 1.

Let $J$ be a proper subset of $N$. For each $n$ graphs $C_J(n, q)$ and $B_J(n, q)$, $n \geq 3$ where $q$ running through various odd prime powers, form a family of Non - Sunada twins for which $\operatorname{Aut}(C_J(n, q)) = N(C_n(q)) \times Z_2$ $\operatorname{Aut}(B_J(n, q)) = N(D_{n+1}(q))$ if $J$ is union of generic subsets and $\operatorname{Aut}(B_J(n, q)) = N(B_J(n + 1(q))$ in other cases.

If $J$ is generic set of cardinality 2, then for each $n, n \geq 3$ graphs $C_J(n, q)$ and $B_J(n, q)$ form a family of unbalanced Non-Sunada twins.

COROLLARY.

If $J$ is a singleton and differs from $\{n\}$, where $n$ is odd integer, then graphs $C_J(n, q)$ and $B_J(n, q)$ form balanced family of Non-Sunada small world twins with automorphism groups $N(C_n(q))$ and $N(B_n(q))$. If $J = \{n\}$, and $n$ is odd then graphs $C_J(n, q)$ and $B_J(n, q)$ form a balanced family of small world graphs Non-Sunada twins with automorphism groups $N(C_n(q))$ and $N(D_{n+1}(q))$.

J. Hemmeter observed that $D'(n + 1, q)$ is isomorphic to $B_{1,2}(n, q)$ and introduce graphs $H(n, q) = C_{1,2}(n, q)^*$ [5]. These graphs are known as Hemmeter graphs (see [4]).

THEOREM 2.

Let $J$ be a proper subset of $N$. For each $n$ graphs $C_J(n, q)^*$ and $B_J(n, q)^*$, where $q$ runs through various prime powers, form a family of Non - Sunada twins for which $\operatorname{Aut}(C_J(n, q))^* = N(C_n(q)) \times Z_2$ $\operatorname{Aut}(B_J(n, q))^* = N(D_{n+1}(q)) \times Z_2$ if $J$ is union of generic subsets and $\operatorname{Aut}(B_J(n, q)) = N(D_{n+1}(q)) \times Z_2$ in other cases. If $J$ is generic set of cardinality 2, then for each $n, n \geq 3$ graphs $C_J(n, q)^*$ and $B_J(n, q)^*$ form a family of unbalanced Non-Sunada twins.

4 The case of other nodes

We introduce orbitals of Coxeter group $W$ of type $B_n$ (or $C_n$) acting on left cosets by maximal standard subgroup $W_i = \langle s_1, s_2, \ldots, s_{i-1}, s_{i+1}, s_{i+2}, \ldots, s_n \rangle$, $i = 1, 2, \ldots, n$. The geometry of group $W$ is a disjoint union $\Gamma$ of $\Gamma_i = (W : W_i)$ of left cosets of $W$ by $W_i$ with incidence relation $I$ such that $\alpha \beta$ for $\alpha \in \Gamma_i, \beta \in \Gamma_j$ where $i \neq j$ if and only if $|\alpha \cap \beta| \neq 0$. It can be interpreted as follows. Let us take set $N = \{1, 2, \ldots, n\}$ and consider $H_i = \{(A, f) | A \in 2^N, |A| = i, f \rightarrow \{0, 1\}\}$, $i = 1, 2, \ldots, n$. A disjoint union $H$ of $H_i$ with incidence relation $I: (A, f)I(B, g)$ for $(A, f) \in H_i$ and $(B, g) \in H_j$ with $i \neq j$ iff $A$ is a subset of $B$ or $B$ is a subset
of \(A\) and \(f(x) = g(x)\) for \(x \in A \cap B\). The incidence system \(H\) is isomorphic to \(\Gamma\). As always \(2^N\) stands for totality of all subsets of \(N\) (see for instance [6]).

The orbitals of \(W\) acting transitively on \(H_t\), \(1 \leq t \leq n\) can be described as
\[
\phi_{r,s} = \{(A,f), (B,g)| |A \cap B| = \text{sand} |\{x : f(x) = g(x)\}| = r\text{ where }0 \leq r \leq s \leq t.
\]

Let us consider action of \(G = X_n(q)\), where \(X = B\) or \(X = C\), with the standard Borel subgroup \(B\) on left cosets by maximal standard parabolic subgroup \(P_t = Bw_tB\), \(t = 1,2,\ldots, n\). The orbitals \(X_{r,s}\) of the permutation groups \((G : P_t)\) are in natural one to one correspondence with \(\phi_{r,s}\) of \((W : W_t)\).

**THEOREM 3.**

Let \(M\) be a proper subset of \(M(t)\), \(t < n\), \(G = X_n(q)\), where \(X = B\) or \(X = C\) acts on the set \(X(t,n,q) = (G : P_t)\) of left cosets of \(G\) by maximal standard parabolic subgroup \(P_t\). Assume that \(X_M(n,t,q)\) are disjoint unions of \(X_{r,s}\) via \((r,s) \in M(t)\). Then for chosen parameters \(n\), \(t\) and subset \(M\) pairs \((B_M(n,t,q), C_M(n,t,q))\) with odd \(q\), \(q > 3\) form a family of Non-Sunada graphs with automorphism groups \(N(B_n(q))\) and \(N(C_n(q))\) respectively.

5 Remarks on Schubert geometry, some conjectures

Let us consider actions of groups \(X_n(q)\) on the geometry \(\Gamma(X_n(q))\) which is disjoint union of \(X(n,t,q)\) via \(t = 1,2,\ldots, n\) and natural action of these permutation groups on \(X^2 = X^2(n,t,q) = X(n,t,q) \times X(n,t,q)\). Brurat’s decomposition gives a following connection of these actions with permutational representation of Weyl group \(W\) of \(X_n(q)\) on the elements of its geometry \(\Gamma(W_n(q))\), which is disjoint union of \(W(n,t) = (W : W_t)\).

Orbits of \((X_n(q), X^2(n,t,q))\) and \((W,W(n,t))\) are in natural one to one correspondence \(D\) induced by correspondence between double cosets \(P_tW_P_j\) and \(W_iW_j\) for \(1 \leq i,j \leq t\) (see [14], [15] and further references).

Introduced above interpretation allows us to identify \(W(n,t)\) with \(\{(A,f)| |A| = t, f \in 2^A\}\) and orbits on \((W(n,i) \times W(n,j)\) with bipartite graphs \(W(n,i,j,t,s) = (\{(A,f), (B,g)| (A,f) \in W(n,i), (B,g) \in W(n,j)\}, |A \cap B| = t, |\{x : f(x) = g(x)\}| = s\) where \(\max(i + j - n, 0) \leq t \leq \min(i,j)\).

The orbits of \(G = X_n(q)\) on \(\Gamma(G)\) are in natural one to one correspondence with \(W(n,i,j,t,s)\). We use symbol \(X(n,i,j,t,s)\) for orbits corresponding to \(W(n,i,j,t,s)\). We define type \(t(X(n,i,j,t,s))\) as tuple \((n,i,j,t,s)\).

Let \(T\) be a maximal torus of \(X_n(q)\). The totality of elements of \(\Gamma(X_n(q))\) containing \(T\) with the restriction of its incidence relation \(I\) on this subset is isomorphic to Weyl geometry \(\Gamma(W_n(q))\). There is a well defined homomorphic retraction map \(\text{Retr}\) from \(\Gamma(X_n(q))\) onto \(\Gamma(W_n(q))\) (see [16] and further references).

Without loss of generality we assume that \(\text{Retr}(P_i)\) coincides with \(\alpha_i = (1,2,\ldots,dots,i,f)\), where \(f(j) = 0\) for \(j = 1,2,\ldots, i\) and elements \(\text{Retr}(wP_i)\), where \(w\) is a Coxeter element of Weyl group \(W\), coincides with \(\beta_i = (\{n,n-\)
On small world Non-Sunada twins

Let $H_i = H^{X_i} = H^{X_i}(n,q)$ be the subset of elements $hP_i$ from $\Gamma_i(X_n(q))$ such that $\text{Retr}(hP_i) = \beta_i$. We refer to the incidence system $H = H_1 \cup H_2 \cdots \cup H_t$ with the restriction $I'$ on this set as Schubert geometry $H = H(X_n(q))$. It is easy to check that standard Borel subgroup $B$ acts regularly on maximal flags of $H$.

Notice that the restriction of $I'$ onto $H_i \cup H_j$ with $i \neq j$ is biregular bipartite graph. We can assume that $\Gamma(W_n)$ is a subset of elements of $\Gamma(X_n(q))$ containing maximal torus. For each element $\gamma$ from $H_i(X_n(q))$ and each $\beta_j$ we consider list $L^j_i(\gamma)$ of types of orbits $R$ of $(X_n(q), \Gamma_i \times \Gamma_j)$ which contain $(\gamma, \beta_j)$. We say that $\gamma_1$ and $\gamma_2$ from $H_i$ are Weyl equivalent if $L^j_i(\gamma_1) = L^j_i(\gamma_2)$ for all $j$. As it follows from the definition classes of Weyl equivalences are unions of thin Schubert cells (see [17] and further references). In fact each class is medium Schubert cell with respect to set of Weyl elements which contains $\alpha_i$ and $\beta_i$ for $i = 1, 2, \ldots, n$ (see [18]).

It means that number of classes of Weyl equivalence is bounded by constant $c(n)$, which does not depend on parameter $q$.

We say that two orbitals $\Phi_1$ and $\Phi_2$ of transitive permutation group $(B, H_i)$ are Weyl equivalent if $(\beta_i, \gamma_1) \in \Phi_1$, $(\beta_i, \gamma_2) \in \Phi_2$ and $\gamma_1$ and $\gamma_2$ are Weyl equivalent.

We refer to unions of Weyl equivalent orbitals as Weyl pseudoorbitals of $(B, H_i)$. We define type of pseudoorbital as list $L^j_i(\gamma)$, $j = 1, 2, \ldots, n$ where $(\beta_i, \gamma) \in R$. The partition into pseudoorbitals is defined in terms of Coxeter matroid of group $W(B_n) = W(C_n)$. So cardinalities of pseudoorbitals of $B(B_n(q))$ and $B(C_n(q))$ of given type are of the same cardinality.

**CONJECTURE 1.**

Let $\Omega$ be the set of all types of pseudoorbitals of $(B, H_i)$ and $J$, $J \neq \emptyset$ be its proper subset. For each triple $(n, i, J)$ we consider disjoint union $X(n, i, J, q)$ of pseudoorbitals of type belonging to $J$. Let $i \neq n$ then graphs $B(n, i, J, q)$ and $C(n, i, J, q)$ form a family of Non-Sunada twins with automorphism groups $N(B(B_n(q)))$, $N(B(C_n(q)))$. If $n$ odd then graphs $B(n, n, J, q)$ and $C(n, n, J, q)$ form families of Non-Sunada twins with automorphism groups $N(B(D_{n+1}(q)))$ and $N(B(C_n(q)))$.

### 6 New distance regular graphs and Non-Sunada twins

For prime powers $q$ Pasechnik [18] constructed a distance-regular graph with intersection array \{\$q^3 - 1, q^3 - q, q^3 - q^2 + 1, 1, q, q^2 - 1, q^3\$\} as a subgraph of the dual polar graph $D_4(q)$, in fact it is the induced subgraph on the set of vertices at maximal distance from an edge. Brouwer [18] constructed related distance-regular graphs $Z = Z_q$ with intersection array \{\$q^3 - 1, q^3 - q, q^3 - q^2 + 1, 1, q, q^2 - 1\$\} as follows. Consider the vector space $F_q^3$ equipped with a cross product $\times$. The vertex set is $(F_q)^3$, where a pair $(u, v)$ is adjacent to a distinct pair $(u_0, v_0)$ if and only if $u_0 = u + v \times v_0$. The extended bipartite doubles of these graphs are the
forms graphs for distance regular graphs with the same intersection array as (bipartite, diameter 4). Pasechnik and Yoshiara [19] constructed distance regular graphs with the same intersection array as (bipartite, diameter 3) Kasami graphs, (see [4], p. 285-286, and [20]). This shows that $Z$ is a distance-regular with the claimed parameters. The spectrum follows. The fact that the extended bipartite double is distance-regular, and has the stated above intersection array, follows from [4], Theorem 1.11.2(vi). The fact that $Z_3$ is strongly regular follows from [4], proposition 4.2.17(ii) (which says that this happens when $Z$ has eigenvalue $-1$). For $q = 2$, the graphs here are (i) the folded 7-cube, (ii) the folded 8-cube, (iii) the halved folded 8-cube. All are distance-transitive. For $q > 2$ these graphs are not distance-transitive. When $q$ is a power of two, the graphs $Z$ have the same parameters as certain Kasami graphs, but for $q > 2$ these are nonisomorphic. Notice that set of the vertices $Z(q)$ of graph $Z_q$ is the largest Schubert cell of $B_3(q)$ acting on dual polar spaces, i.e., the largest orbit of Borel subgroup $B^3 = B_3(q)$ of $B_2(q)$. Group $D_4(q)$ is an overgroup of permutation group $(B_3(q), B(3, q))$, the stabilizer of $Z(q)$ in $D_4(q)$ is isomorphic of Borel subgroup $B^4 = B_4(q)$ of this group. Group $D_4(q)$ has orbitals which form a fusion scheme of association scheme for of orbitals $(B_3(q), Z(q))$. The distance in the graph $Z_q$ defines association scheme $R$ which is a fusion of orbital schemes of $D_4(q)$ and $B_3(q)$. For two types $T_1$ and $T_2$ of pseudoorbitals $\delta_1$ and $\delta_2$ of $B_3(q)$ we write $T_1 \leftrightarrow T_2$ mod $(D_4)$ and say that they are $D_4$ equivalent if there is an orbital $\phi$ of $B(D_4(q))$ such that $\phi \cap \delta_i \neq \emptyset$ for $i = 1, 2$. We refer to minimal sets of $D_4$-equivalent types as generic sets. Each graph of association scheme $R$ coincides with $B(3, 3, J, q)$, where $J$ is a union of generic sets. Notice that we have 4 generators of this associative scheme $\tau^i = z_q, z^i, i = 2, 3, 4$. Let $J_1, J_2, J_3$ and $J_4$ such that $z^i = B(3, 3, J_i, q)$. It turns out that relations $s_i = B(3, 3, J_i, q)$ for $i = 1, 2, 3, 4$ form an association scheme with the same intersection indices with the scheme $R$. Let $s_q = s_1$. It is easy to see that $S_q = Z_q$ in the case of even $q$. In case of odd $q$ groups $\text{Aut}(Z_q)$ and $\text{Aut}(S_q)$ are normalisers $N_1 = N(B^3)$ and $N_2 = N(B(\text{PS}P_3(q)))$ of Borel subgroups of $D_4(q)$ and $C_3(q)$ in corresponding symmetric groups.

**THEOREM 4.**

Distance regular graphs $Z_q$ and $S_q$ with the same parameters in case of odd $q$ form a family of edge intransitive Non-Sunada small world twins with automorphism groups $N(D_4(q))$ and $N(C_3(q))$.

**PROPOSITION 1.**

Let $M$ be a nonempty proper subset of $\{1, 2, 3, 4\}$ then unions $(Z^M_q$ of $z^i_q, i \in M$ and $(S^M_q$ of $s^i_q, i \in M$ form a family of Non-Sunada small world twins when $q$ runs through all odd prime powers.

**THEOREM 5.**
Extended bipartite doubles $Z'_{q}$ and $S'_{q}$ of $Z_{q}$ and $S_{q}$ with odd $q$ form a family of distance regular Non-Sunada twins with automorphism group $N_{1} \times Z_{2}$ and $N_{2} \times Z_{2}$.

7 Case of diagram $A_{n}$, the twisted Grassmann graphs

Van Dam and Koolen [21] constructed the first family of non-vertex-transitive distance regular graphs with unbounded diameter. These graphs have the same intersection array as certain Grassmann graphs and they are constructed as follows. Let $q$ be a prime power and let $D \geq 2$ be an integer. Let $V$ be a $(2D + 1)$-dimensional vector space over $F_{q}$, and let $H$ be a hyperplane in $V$. Vertices are the $(D + 1)$-dimensional subspaces of $V$ that are not contained in $H$, and the $(D + 1)$-dimensional subspaces of $H$. Two vertices of the first kind are adjacent if they intersect in a $D$-dimensional subspace; a vertex of the first kind is adjacent to a vertex of the second kind if the first contains the second; and two vertices of the second kind are adjacent if they intersect in a $(D - 2)$-dimensional subspace. This graph is distance-regular with the same intersection array as the Grassmann graph $J_{q}(2D + 1, D)$. In fact, this Grassmann graph and the twisted Grassmann graph $\tilde{J}_{q}(2D + 1, D)$ are the point graph and line graph, respectively, of a partial linear space where points are the $D$-dimensional subspaces of $V$, and where a $(D + 1)$-dimensional subspace of $V$ that is not contained in $H$ is incident to the $D$-dimensional subspaces that it contains, and a $(D + 1)$-dimensional subspace of $H$ is incident to the $D$-dimensional subspaces of $H$ containing it. The twisted Grassmann graph is not vertex-transitive (it has two orbits of vertices), and hence it is not isomorphic to the Grassmann graph. Fujisaki, Koolen, and Tagami [22] showed that the automorphism group of the twisted Grassmann graphs is the subgroup of $P(2D + 1, q)$ that axes $H$. Bang, Fujisaki, and Koolen [23] determined the spectra of the local graphs, and studied in some detail its Terwilliger algebras. Remarkably, these algebras with respect to vertices in distinct orbits are not the same. The twisted Grassmann graphs are also counterexamples to two conjectures by Terwilliger ([24], p. 207–210), see [23]. Jungnickel and Tonchev [25] constructed designs that are counterexamples for Hamada’s conjecture. Munemasa and Tonchev [26] showed that the twisted Grassmann graphs are isomorphic to the block graphs of these designs. Munemasa [27] showed that the twisted Grassmann graphs can also be obtained from the Grassmann graphs by Godsil-McKay switching (cf. [28], X1.8.3).

PROPOSITION 2.

For each $D > 2$ graphs $\tilde{J}_{q}(2D + 1, D)$ and $J_{q}(2D + 1, D)$ and $q$ running through all prime powers form a family of strongly unbalanced Non-Sunada graphs with automorphism groups $P(2D + 1, q)$ and the subgroup of $P(2D + 1, q)$ that axes $H$. 

8 On Voronoi diagrams and cells, thin Schubert cells and isospectral graphs

The classical Voronoi diagram is a distance-based decomposition of a metric space relative to a discrete set, Voronoi sites. Given a set of points (Voronoi sites), the Voronoi decomposition leads to regions (Voronoi regions) consisting of all points that are the closest to a specific site. Mehlhorn [29] and Erwig [30] proposed an analogous decomposition, Graph Voronoi Diagram, for undirected and directed graphs respectively.

Definition 1. (Graph Voronoi Diagram [29], [30]). In a graph \( \Gamma = (V, E, w) \), where \( w \) is weight function, Voronoi diagram for a set of nodes \( S = \{v_1, \ldots, v_s\} \), \( v_i \in V \) is a disjoint partition \( \text{Vor}(\Gamma, S) = V_1, V_2, \ldots, V_s \) of \( V \) such that for each node \( u \in V_i \), \( d(u, v_i) \leq d(u, v_j) \) for all \( j \in \{1, 2, \ldots, s\} \).

The \( V_i \) are called Voronoi regions. The graph Voronoi diagram is not necessarily unique, as a node \( u \) may have the same distance to more than one Voronoi node.

We introduce Voronoi cells as classes of equivalence relation \( \tau \) on \( V \): \( x \tau y \) if and only if \( d(x, v_i) = d(y, v_i) \) for all \( i = 1, 2, \ldots, n \). The connection with graph Voronoi diagram is clear: Voronoi regions are disjoint unions of Voronoi cells.

Let \( C(\Gamma, S) \) be a totality of Voronoi cells. We assume that the weight of each edge is 1 and Voronoi Diagram is unique and consider Voronoi-Schubert diagram \( \text{VorSch}(\Gamma, S) \) which is a partition of \( C(\Gamma, S) \) corresponding to equivalence relation: two Voronoi cells are in the same Voronoi regions.

For each \( c \in C(\Gamma, S) \) we define its trace \( t(c) = (d_1, d_2, \ldots, d_n) \) where \( d_i = d(v_i, c) = d(v_i, x) \) for \( x \in c \). If \( \Gamma \) is a finite graph we consider the order \( |c| \) of Voronoi cell \( c \) which is simply a cardinality of this subsets of nodes.

We refer to \( (\Gamma, S) \) as graph with selected nodes. We say that \( (\Gamma_1, S_1) \) and \( (\Gamma_2, S_2) \) are Voronoi equivalent if there is a bijections \( \eta : S_1 \to S_2 \) and \( \mu : C(\Gamma_1, S_1) \to C(\Gamma_2, S_2) \) such that \( d(v_i, c) = d(\eta(v_i), \mu(c)) \). We refer to pair \((\eta, \mu)\) as diagram isomorphism.

In the case of finite graphs we say that \( (\Gamma_1, S_1) \) and \( (\Gamma_2, S_2) \) are strongly diagram equivalent if they are Voronoi equivalent and \( |c| = |\mu(c)| \) for each Voronoi cell \( c \).

Let \( F \) be a field. Chevalley group \( X_l(F) \) is a \( BN \)-pair with finite irreducible Weyl group \( W \). Fixed points of maximal torus \( T \) for its action on \( (X_l(F)) \) form subset \( W \), such that the restriction of incidence relation \( I \) on this set is isomorphic to Weyl geometry of \( W(X_l) \). So we have natural embedding of \( W(X_l) \) into \( I = \Gamma(X_l(F)) \). Let \( I(X_l(F)) \) stand for the incidence relation of \( \Gamma \). We are interested in studies \( \text{Vor}(\Gamma(G), S) \).

Conjecture 2.

Let \( \Gamma(G) \) be a geometry of simple group \( G \) with Borel subgroup \( B \), Weyl group \( W \) and maximal torus \( T \). Then Voronoi cells for \( \Gamma(G), \Gamma(W) \) and thin Schubert cells form the same partition.

In the case of projective geometry the conjecture is supported by known theorem by I. Gelfand and R. MacPherson (see [16] and further references).
CONJECTURE 3.
Let $F$ be a field. Pairs $(I(B_n(F)), W(B_n))$ and $(I(C_n(F)), W(C_n))$ are Voronoi equivalent. In case of finite field they are strongly Voronoi equivalent.

Let us consider a pair $\Gamma, (a, b)$, where $(a, b) \in V^2$ and take $S = \{a, b\}$ of cardinality at most two. We say that $(a_1, b_1)$ and $(a_2, b_2)$ are $\Gamma$-equivalent if there is diagram isomorphism $(\eta, \mu)$ of $(\Gamma, \{a_1, b_1\})$ and $(\Gamma, \{a_2, b_2\})$ with $\eta(a_1) = a_2$ and $\eta(b_1) = b_2$.

We refer to classes of $\Gamma$-equivalence as $\Gamma$-orbitals. We say that partition into $\Gamma$-orbitals is transitive if there is a permutation group $(G, V)$ such that its orbits on $V^2$ coincide with $\Gamma$-orbitals.

We refer to Voronoi - Schubert diagram of kind $\Gamma, \{a, b\}$ with the trace function $tr$ on Voronoi cells as binary $VS$-diagram. If for each cell $c$ the value $|c|$ is given we use term scaled $VS$-diagram.

Two graphs $\Gamma_1$ and $\Gamma_2$ are Voronoi equivalent (strongly diagram equivalent) if their orbital partitions are transitive and sets of binary $VS$ diagrams (scaled $VS$-diagrams) are coincides.

We can use technique of [31] and prove that

THEOREM 6.
Let $F$ be arbitrary field. Incidence graphs $\Gamma(B_n(F))$ and $\Gamma((C_n(F))$ are Voronoi equivalent.

THEOREM 7.
Incidence graphs of $\Gamma(B_n(q))$ and $\Gamma((C_n(F))$ are strongly diagram equivalent.

REMARK. The isospectrality of graphs of corresponding orbitals $(B_n(q), \Gamma_1(B_n(q))$ and $(C_n(q), \Gamma_1(C_n(q))$ can be deduced from this statement.

Let us consider a partition $P = \{R_0, R_1, \ldots, R_k\}$ of Cartesian product of $V$ with itself such that all graphs $R_i$ are regular, $R_0$ is equality relation and partition of $V^2$ into $R_i$ orbitals coincides with some fusion of $P$. We refer to such $P$ as Voronoi graph configuration.

EXAMPLE. Let $(G, V)$ be a transitive transformation group of set $V$. Then partition $V^2$ into orbitals $R_i$ will be Voronoi graph configuration.

THEOREM 8.
Let us consider groups $B_n(F)$ and $C_n(F)$ defined over arbitrary field $F$ acting on totally isotropic subspaces of same dimension. Then their Voronoi graph configurations are isomorphic.

9 Linguistic graphs and some conjectures

We refer to a triple consisting of set $V$, its partition $V = P \cup L$ and symmetric and antireflexive binary relation $I$ (incidence) on the set $V$, such that $xIy$ implies $x \in P, y \in L$ or $x \in L$ and $y \in P$ as incidence structure. The pair $\{x, y\}, x \in P, y \in L$ such that $xIy$ is called a flag of incidence structure $I$. Clearly incidence structure is an incidence system of rank 2.

Let $K$ be a finite commutative ring. We refer to an incidence structure with a point set $P = P_{r,m} = K^{r+m}$ and a line set $L = L_{r,m} = K^{r+m}$ as linguistic
incidence structure \( I_m \) if point

\[
(x) = (x_1, x_2, \ldots, x_s, x_{s+1}, x_{s+2}, \ldots, x_{s+m})
\]
is incident to line

\[
[y] = [y_1, y_2, \ldots, y_r, y_{r+1}, y_{r+2}, \ldots, y_{r+m}]
\]
if and only if the following relations hold

\[
\begin{align*}
\xi_1 x_{s+1} + \xi_1 y_{r+1} &= f_1(x_1, x_2, \ldots, x_s, y_1, y_2, \ldots, y_r) \\
\xi_2 x_{s+2} + \xi_2 y_{r+2} &= f_2(x_1, x_2, \ldots, x_s, x_{s+1}, y_1, y_2, \ldots, y_r, y_{r+1}) \\
&\quad \ldots \\
\xi_m x_{s+m} + \xi_m y_{r+m} &= f_m(x_1, x_2, \ldots, x_{s+m-1}, y_1, y_2, \ldots, y_{r+m-1})
\end{align*}
\]

where \( \xi_j, j = 1, 2, \ldots, m \) are not zero divisors, and \( f_j \) are multivariate polynomials with coefficients from \( K \). Brackets and parenthesis allow us to distinguish points from lines (see [22]). The colour \( \rho(x) = \rho([x]) \) \( \rho(y) = \rho([y]) \) of point \( (x) \) (line \([y]\)) is defined as projection of an element \((x)\) \( ([y])\) from a free module on its initial \( s \) (relatively \( r \)) coordinates. As it follows from the definition of linguistic incidence structure for each vertex of incidence graph there exists unique neighbour of a chosen colour. We also consider a linguistic incidence structures defined by infinite number of equations. We refer to relation \( I \) incidence structure for each vertex of incidence graph.

Let us consider the union \( L \) of two large Schubert cells. Let us consider the action of Borel Subgroup \( B \) of \( BN \)-pair \( G \) with finite Weyl group on the geometry \( \Gamma(G) \). Orbits of this action are known as large Schubert cells. Let us consider the action of Borel Subgroup \( B \) of \( BN \)-pair \( G \) with finite Weyl group on the geometry \( \Gamma(G) \). Orbits of this action are known as large Schubert cells.

CONJECTURE 4.

Binary relations two points are neighbours and two lines are neighbours of Schubert graphs \( S_{i,j}(B_n(q)) \) and \( S_{i,j}(B_n(q)) \) where \( q \) stands for arbitrary odd power form two families of edge transitive non Sunada graphs.

CONJECTURE 5.

Let \( F \) be an arbitrary field. Then graphs \( S_{i,j}(B_n(F)) \) and \( S_{i,j}(C_n(F)) \) are Voronoi equivalent.

REMARK. In the case of the field of odd characteristic or characteristic zero graphs \( S_{i,j}(B_n(F)) \) and \( S_{i,j}(C_n(F)) \) are non isomorphic.
10 Towards $K$-theory of linguistic configurations

Linguistic graph with parameters $m, s, r$ can be interpreted in a following way. Let $\Omega$ be a set with distinguished subsets $A_1$ and $A_2$, such that $|A_1 \cap A_2| = m$. Below we define below a linguistic graph $L(A_1, A_2; \Omega)$ with partition sets $K^{A_1}$ and $K^{A_2}$ and with colour spaces $K^{A_12}$ and $K^{A_22}$, where $A_{1,2} = A_1 - (A_1 \cap A_2)$, $A_{2,1} = A_2 - (A_1 \cap A_2)$ such that $\rho(f)$, $f \in A_1$ is the restriction of $f$ onto $A_1 - A_1 \cup A_2$.

To do this without loss of generality we can take $A_1 \cap A_2 = \{1, 2, \ldots, m\}$, $A_1 - A_2 = \{m+1, m+2, \ldots, m+s\}$, $A_2 - A_1 = \{m+s+1, m+s+2, \ldots, m+s+r\}$ and identify $(A_1, g)$, $(A_2, h)$ with the tuples $(g(m + 1), g(m + 2), \ldots, g(m + s), g(1), g(2), g(m)) = (x_1, x_2, \ldots, x_{s+m})$ and $h(m + s + 1), h(m + s + 2), \ldots, h(m + s + r), h(1), h(2), \ldots, h(m)] = [y_1, y_2, \ldots, y_{r+m}] = [y]$ respectively. The set $\Omega$ can be chosen as any set which contains $\{1, 2, \ldots, m + s + r\}$.

Let $\Gamma$ be a simple graph with vertex set $\{1, 2, \ldots, n\}$. Let us assume that there is a set $\Omega$ and function $\eta$ from $V(\Gamma)$ onto $2^\Omega$ and for each edge $i, j \in \Gamma$ the linguistic graph $L_{i,j}$ is given with its $(\eta_i, \eta_j)$ interpretation. We refer to such data as linguistic configuration with a governing triple $(\Gamma, \eta, \Omega)$ and linguistic relations $(\eta_i, i, j) \in E(\Gamma)$ over commutative ring $K$. For fixed linguistic configuration we define a linguistic blow up $\tilde{\Gamma}$ of the governing graph $\Gamma$ which is a totality of new vertices of kind $(i, x)$, $x \in K^n(i)$ such that $(i, x)$ and $(j, x)$ are neighbours in $\tilde{\Gamma}$ if and only if $i, j$ are neighbours in $\Gamma$ and $x$ an $y$ are adjacent in the linguistic graph $L_{i,j}$. For each pair $(i, j)$ of ordered neighbouring vertices in a governing graph $\Gamma$ we consider $\Delta_{i,j} = \eta(i) - \eta(i) \cap \eta(j)$. We refer to $\eta$ as governing map and use notation $A_\eta$ for set $\eta(i)$. For each set $A$ of $\Omega$ we consider the list of all its elements $a_1, a_2, \ldots, a_m$ and the ring $K[A] = K[x_{a_1}, x_{a_2}, \ldots, x_{a_m}] = K[A]$ where $x_{a_i}$ are formal variables.

11 On Coxeter and Lie geometries and corresponding linguistic configurations

Let $\alpha, \beta$ be a flag of cardinality 2 ($\alpha I \beta$). Defined in the case of general linguistic configuration $\Delta(\alpha, \beta)$ will be written as $\Delta(\alpha) = \Delta(\alpha) \cap \Delta(\beta)$.

Let us consider a class of Coxeter linguistic configurations with the collecting incidence graph $I$ of Coxeter geometry $\Gamma(W)$ and boolean embedding $\Delta : \alpha \to 2^I$ which sends $\alpha$ to defined above $\Delta(\alpha)$. To define a representative from this class we have to choose a commutative ring $K$ and for each edge of Coxeter geometry a linguistic relation $L_{i, \alpha, \beta}$ together with its $\Delta(\alpha), \Delta(\beta), I$ interpretation. Each blow up of $\Gamma$ corresponding to such linguistic configuration is an incidence relation of the incidence system such that type of vertex $(\alpha, f)$, where $f : K^{\Delta(\alpha)}$ is defined simply as $I(\alpha)$ for $\alpha \in \Gamma(W)$. We say that linguistic configuration $L(\Gamma, \eta, \Omega)(K)$ defined via the choice of linguistic graphs $L_{i, \alpha, \beta}$ is bilinear if there is a bilinear product $\times$ on $L = K^{\Omega}$ and family of functions $l_i : \Omega \to K$, $i \in V(\Gamma)$ such that incidence relation of each linguistic graph can be written for $x \in L_i$ and $y \in L_j$ as $x(s)l_j(s) - y(s)l_i(s) = (x \times y)(s)$, $s \in \Delta(i) \cup \Delta(j)$. 


REMARK. We have to choose linear order for each pair \((i, j)\). In the case of skew symmetric bilinear order \((x \times y = -y \times x)\) the order on pairs is immaterial. In alternative case we will talk on alternative bilinear linguistic configuration.

Throughout this section \((G, B, N, S)\) is Tits system which arises in connection with Chevalley group \(G\) or their generalisations.. We write \(G = X_1(K)\) to signify that \(G\) is Chevalley group over the field \(K\), with associated Dynkin diagram \(X_1\). We are mostly interested in the case when \(K\) is finite, and we shall write \(X_1(q)\) instead of \(X_1(F_q)\) in that case. So, fix Chevalley group \(G = X_1(K)\) with corresponding Weyl group \(W\). As in the previous section \(\Gamma(W)\) and \(\Gamma(G)\) are associated Coxeter and Lie geometries.

PROPOSITION 3.

Let \(\Gamma = \Gamma(X_1(F))\) be the geometry of simple Chevalley group over the field \(F\) of characteristic \(p > 3\) or 0. There is a Coxeter linguistic skew symmetric bilinear configuration of normal type \(L(\Gamma(W), \Delta, T)(F)\) such that its blow up is isomorphic to \(\Gamma\).

This statement shows that many flag transitive algebraic incidence systems can be constructed in terms of linguistic configurations defined over the field. The proof of the statement is constructive. In fact it is a byproduct of the theorem on the embedding of Lie Geometries into Borel subalgebra of corresponding Lie Algebra (see [17] and further references).

12 BN-pairs, their geometries as linguistic configurations

On universal geometries of Chevalley groups \(X_n(K)\) over commutative rings

Let us consider \(\Gamma_W, \Delta, T\) and Cartan function \(f(x, y)\). For each incident vertices \(\alpha\) and \(\beta\) of \(\Gamma_W\) we take functions \(d_{\alpha, \beta}\) and \(d_{\beta, \alpha}\) from \(\Delta(\alpha) \cap \Delta(\beta)\) to \(K^*\) and parameters \(\lambda(r, s, \alpha, \beta) \in K\) for pairs \((r, s)\) such that \(r \in \Delta(\alpha), s \in \Delta(\beta)\) and \(f(r, s)\) is well defined. We consider the bilinear map \([x, y]\) from \(L_\alpha \times L_\beta\) into \(L_\alpha \alpha \Delta(\beta) \cap L_\beta\) such that \([\alpha, \beta] = \lambda_{\alpha, \beta} e_f(r, s)\) for the pair \(r, s\) from the domain of \(f\) and \([\alpha, \beta] = 0\) for \((r, s)\) which is outside of the domain.

We say that \(x \in L_\alpha\) and \(y \in L_\beta\) are incident \(x(s) d_{\alpha, \beta}(s) - y(s) d_{\beta, \alpha}(s) = [x, y](s)\), \(s \in \Delta(\alpha) \cup \Delta(\beta)\). We use termi \(Generalised Coxeter linguistic configuration\) (GCLC) for the defined above object. We talk about universal GCLC. This object UGLCN for each edge \(\alpha, \beta\) of the Coxeter geometry has functions \(d_{\alpha}, d_{\beta}\) defined on \(\Delta(\alpha) \cap \Delta(\beta)\) with symbolic values \(d_{\alpha}(r) = x(\alpha, r)\) and
B \in X$

Chevalley group

$to identify set

irreducible Weyl group with Cartan matrix corresponding to root systems

A

case of GCLC is Cartan-Coxeter linguistic configuration for which

$W$

is a finite

irreducible Weyl group with Cartan matrix corresponding to root systems.

To define such a configuration we have to identify set $T$ with positive roots $R$ of corresponding root system and use

to define such a configuration we have to identify set $T$ with positive roots $R$ of corresponding root system and use

$f(x, y)$ such that

$f(r, s) = r + s$ if $r + s \in R$ and 0 in opposite case. We assume

that $\lambda_{\alpha, \beta} \neq 0$ if one of the roots $r, s$ is simple. Notice that Universal Cartan-Coxeter linguistic configuration is uniquely determined by pair Cartan matrix $A$ and commutative ring $K$. Let $X_n$ stand for one of the listed above root systems. We use symbol $UCC(X_n, K)$ for the universal Coxeter-Cartan linguistic configuration with Cartan matrix $K$ defined over commutative ring $K$. For the investigation of such algebraic objects we use universal geometry $UG(X_n(K))$ of Chevalley group $X_n(K)$ in which parameters $d(\alpha, \beta)(s) \in K^*$, $d(\beta, \alpha)(s) \in K^*$, $s \in \Delta(\alpha) \cap \Delta(\beta)$ and $\lambda_{\alpha, \beta} \in K$, $s \in \Delta(\alpha)$, $r \Delta(\beta)$ are variables. Specialisation $S$ of Universal Cartan Coxeter linguistic configuration $UG_S(X_n, K)$ over $K$, prescribes values from $K$ for all these parameters. The proposition of the previous section shows that in the case of fields special choice of $S$ provides Coxeter Cartan linguistic configuration $V_S$ with blow up isomorphic to geometry of Chevalley group $X_n(F)$. Specializations of universal geometry $UG(X_n, K)$ form an interesting class of incidence systems which contains flag transitive geometry of Chevalley group in the case of field $K$. Computer simulation demonstrates that change of the coefficients in the generic equations of linguistic graphs of Coxeter Cartan configurations usually changes the spectrum of the linguistic graph. It means that majority of pairs of distinct specialisations provide nonisomorphic incidence systems. Representatives of this class of incidence system have some interesting common properties. For instance from each vertex of blow up $G(X_n, K)$ of $UG_S(X_n, K)$ to elements of kind $(W_i, 0)$ there is a walk of length restricted by linear function $l(n)$ which does not depend on $K$ and $S$. So diameter of $G_S(X_n, K)$ is bounded by $2l(n)$. In the case of fixed $n$ and $S$ and family of finite rings $K = K_m$ of increasing order graphs $G_S(X_n, K_m)$, $m = 1, 2, \ldots$ form a family of small world graphs with unbounded degree (see [37] for related geometrically based construction and further references).

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20 On small world Non-Sunada twins

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