Entanglement in theory space

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Abstract – We propose a new concept of entanglement for quantum systems: entanglement in theory space. This is defined by decomposing a theory into two by an un-gauging procedure. We provide two examples where this newly introduced entanglement is closely related to conventional geometric entropies: deconstruction and AGT-type correspondence.

Introduction. – Entanglement entropy is an indispensable measure for intrinsically quantum properties of quantum systems, and plays crucial roles in a number of different disciplines, such as quantum information and computation, many-body systems, quantum field theories and black hole physics (see, e.g., refs. [1–3] for reviews).

The goal of this letter is to introduce a new concept of entanglement. As we will review momentarily the conventional definition of entanglement entropy involves the division of a spatial region into two. By contrast our entanglement entropy is defined by decomposing a gauge theory into two by an un-gauging procedure. Since the decomposition here refers not to the geometric/spatial regions but to more abstract “space of quantum theories”, our entanglement entropy will be called entanglement in theory space, or theory-space entanglement; for definiteness the conventional concept of entanglement will be hereafter called geometric entanglement.

While the concept of theory-space entanglement is rather unexplored and deserves further study, we point out that there are some examples where theory-space entanglement is closely related to, or even equal to, the geometric entanglement.

Geometric entanglement. – Let us first briefly recall the more conventional version, i.e. the geometric entanglement entropy.

Suppose that we have a quantum mechanical system; this could either be a discrete lattice system or a continuous field theory. In the canonical quantization we obtain a Hilbert space $\mathcal{H}_{\text{tot}}$ on a time slice. Let us divide the spatial regions into a region $A$ and its complement $B$. The total Hilbert space then factorizes into a product of those associated with regions $A$ and $B$:

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B.$$  

Now consider the ground state of the total theory and the associated density matrix $\rho_{\text{tot}}$. We can then define the reduced density matrix by

$$\rho_A = \text{Tr}_{\mathcal{H}_B} \rho_{\text{tot}},$$

and the entanglement entropy $S_{\text{ent}}$ as the von Neumann entropy for $\rho_A$ (see footnote):\footnote{The entanglement entropy for $A$ and $B$ coincide when $\rho_{\text{tot}}$ is constructed from a pure state in the total Hilbert space $\mathcal{H}_{\text{tot}}$.}

$$S_{\text{ent}} = -\text{Tr} \rho_A \log \rho_A.$$  

(Un-)gauging. – In the definition above of the geometric entanglement entropy, the essential ingredients are that i) there is a Hilbert space $\mathcal{H}_{\text{tot}}$ and the ground-state density matrix $\rho_{\text{tot}}$; ii) the total Hilbert space factorizes as in (1). We can then define the entanglement entropy $S_{\text{ent}}$ by (2), (3).

While the spatial division gives rise to natural decomposition of the Hilbert space, it is not the only possibility. For example, ref. [4] proposes entanglement entropy in the momentum space. Our proposal in this letter is more drastic, and relies on the gauging/un-gaung procedure, which we now explain.

Suppose that we have two theories $\mathcal{T}_A$ and $\mathcal{T}_B$, with global symmetries $G_A$ and $G_B$, respectively. Concretely this means that we have two different Lagrangians $\mathcal{L}_A$ and $\mathcal{L}_B$ with global symmetries $G_A$ and $G_B$, respectively. We assume the two theories are weakly gauged, i.e., the current $j^\mu_A$ ($j^\mu_B$) for the global symmetry could be coupled with the background gauge field $A^\mu_A$ ($A^\mu_B$) by including a term $\int A^\mu_A j^\mu_A$ ($\int A^\mu_B j^\mu_B$) in the Lagrangian $\mathcal{L}_A$ ($\mathcal{L}_B$). Note that at this point the gauge fields $A^\mu$ do not have kinetic terms and hence are not yet dynamical.

Suppose now that $G_A$ and $G_B$ contain a common subgroup $G$. We can then define a new theory $\mathcal{T}_G$ by
1) first identifying the $G$-components of the corresponding background gauge fields $\mathcal{A}_A$ and $\mathcal{A}_B$ and 2) second, adding a kinetic term $\frac{1}{2} \text{Tr} F_{\mu \nu} F^{\mu \nu}$ for the $G$-gauge field identified in the first step, where $F_{\mu \nu}$ is the field strength for the gauge field $\mathcal{A}_A$.

After this gauging, the two theories $\mathcal{T}_A$ and $\mathcal{T}_B$ now interact with each other through the dynamical gauge field, and should be regarded as a single interacting theory $\mathcal{T}_{\text{tot}}$. The coupling constant $g$ determines how strong this interaction is. In the limit $g \to 0$, the theory $\mathcal{T}_{\text{tot}}$ decomposes into two decoupled theories $\mathcal{T}_A$ and $\mathcal{T}_B$, each of which are coupled only with the non-dynamical background gauge fields (i.e., weakly gauged). This is known as un-gauging, the opposite of the gauging procedure.

This definition of gauging/un-gauging does not really require the Lagrangian descriptions, and in fact some of the examples we discuss later are without Lagrangians. Suppose again that we have two theories $\mathcal{T}_A$ and $\mathcal{T}_B$ with global symmetries $G_A$ and $G_B$, respectively. We can then gauge the diagonal $G$-symmetry inside $G_A \times G_B$ (fig. 1) to define a new theory $\mathcal{T}_{\text{tot}}$ (see footnote\(^\text{2}\)). We schematically write this as

$$\mathcal{T}_{\text{tot}} = \mathcal{T}_A \cup_G \mathcal{T}_B. \quad (4)$$

The details of gauging might differ depending on the symmetries we wish to preserve. For example, when $\mathcal{T}_A$ and $\mathcal{T}_B$ have supersymmetry we can supersymmetrize the procedure (4) by adding superpartners to the coupling $\int A^\mu j_\mu$.

The gauging procedure described here is rather general, and can describe a wide variety of phenomena involving gauge fields. For example, we can take the subtheories $\mathcal{T}_A$ and $\mathcal{T}_B$ to be the states of atoms, each interacting with the background photons; the atoms interact with each other only through the long-range interaction mediated by photons. Another example coming from high-energy physics is the gauge-mediated supersymmetry breaking [5], where $\mathcal{T}_A$ is the supersymmetry-breaking sector (together with messengers), $\mathcal{T}_B$ is the supersymmetric generalizations of standard model, and the gauge group $G$ is the $SU(5)$-gauge group for the grand unified theory.

As these examples show the decomposition (1) is a physical decomposition — it is a choice of the duality frame. This should be contrasted with the case of the geometric entanglement entropy, whose decomposition of spatial regions is often non-physical; we can define the geometric entanglement entropy for a region $A$ with any shape, and we use the choice of $A$ to extract different quantities representing the entanglement of the theory.

More conceptually, we can regard the gauging as a procedure of constructing more complicated theories out of simple ingredients, and by repeating this procedure we obtain a zoo of quantum (field) theories with rich structures. This viewpoint has been recently explored extensively in the context of supersymmetric gauge theories, and we will discuss some of these examples later.

**Theory-space entanglement.** – We can now define the theory-space entanglement.

Suppose that $\mathcal{T}_{\text{tot}}$ is a $D$-dimensional theory obtained by gauging two $D$-dimensional theories $\mathcal{T}_A$ and $\mathcal{T}_B$, as in (4). Let us consider the theory $\mathcal{T}_{\text{tot}}$ on a $D$-dimensional manifold of the form $\mathbb{R}_t \times \mathcal{S}$, where $\mathcal{S}$ is a compact $(D-1)$-dimensional manifold and $\mathbb{R}_t$ is the time direction\(^3\). In the canonical quantization we obtain a Hilbert space $\mathcal{H}_{\text{tot}}$ for a fixed time $t$, and the ground-state density matrix $\rho_{\text{tot}}$.

By repeating this procedure for $\mathcal{T}_A$ ($\mathcal{T}_B$) we also obtain $\mathcal{H}_A$ ($\mathcal{H}_B$).

The basic idea is now clear: since we have $\mathcal{H}_{\text{tot}}$, $\mathcal{H}_A$, $\mathcal{H}_B$ and $\rho_{\text{tot}}$, we can define the entanglement entropy by using the same formulas (2), (3).

There is one important subtlety, however; the factorization of the Hilbert space (1) does not hold, and we only have an embedding

$$\iota : \mathcal{H}_{\text{tot}} \hookrightarrow \mathcal{H}_A \otimes \mathcal{H}_B. \quad (5)$$

The reason is that the states $|\psi_A\rangle$ in $\mathcal{H}_A$ ($|\psi_B\rangle$ in $\mathcal{H}_B$) in general is charged non-trivially under the global symmetry $G$ before un-gauging, but then the product state $|\psi_A\rangle \otimes |\psi_B\rangle$ does not make sense as a state of $\mathcal{H}_{\text{tot}}$ since it is charged under $G$, which is now promoted to a gauge symmetry in $\mathcal{T}_{\text{tot}}$ after gauging. Nevertheless we can define the embedding (5) by incorporating the degrees of freedom for the gauge group $G$ (and their superpartners) in the definition of $\mathcal{H}_A$, thus effectively doubling the degrees of freedom of $G$. To emphasize this some readers might prefer the notation $\mathcal{H}_{A+G}$, $\mathcal{H}_{B+G}$.

The non-factorization, however, is not really a problem, and a small modification saves the definition. The embedding $\iota$ induces the embedding of the ground-state density matrix $\rho_{\text{tot}} = |\psi_0\rangle \langle \psi_0|$:

$$\iota^* (\rho_{\text{tot}}) := \iota (|\psi_0\rangle) \iota (\langle \psi_0|). \quad (6)$$

Note by definition $\iota$ maps a pure state into a pure state. We modify eq. (2) by

$$\rho_A = \text{Tr}_{\mathcal{B}} \iota^* (\rho_{\text{tot}}), \quad (7)$$

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\(^2\)After gauging the commutant of $G$ inside $G_{A,B}$ remains as global symmetries of $\mathcal{T}_{A,B}$.

\(^3\)It is not necessary to consider the Lorentzian signature. For the Euclidean signature the “time” is just one of the directions inside $D$-dimensions.
and can define the theory-space entanglement by the same equation (3). By the Schmidt decomposition it follows immediately that the answer does not change when we exchange the roles of $A$ and $B$. This concludes our definition of theory-space entanglement\(^4\).

The theory-space entanglement defined here depends non-trivially on the gauge coupling constant $g$ for the dynamical gauge field, as well as on the choice of the ground-state wave function. The latter choice will be crucial for the supersymmetric examples discussed in the latter part of this letter.

The definition of the theory-space entanglement requires not only the theory $\mathcal{T}$ itself, but also the choice of the decomposition (4) and the compactification manifold $S$. Neither of these choices is unique.

The choice of the decomposition (4) comes in since (as discussed above) we need a physical decomposition of the Hilbert space. This is closely related to the issue of duality; the theory $\mathcal{T}_{\text{tot}}$, defined in (4), in general could have a different decomposition

$$\mathcal{T}_{\text{tot}} = \mathcal{T}_A \cup G' \mathcal{T}_B,$$

and $G'$ can be rather different from the gauge group $G$ in another frame; gauge symmetry is by definition a redundancy for describing physics, and there is no unique way to associate a unique gauge symmetry for a given physical system\(^5\). The fact that we need a physical choice is natural since entanglement itself is a physical property of the theory. It should be kept in mind that an analogous choice is present for conventional geometric entanglement entropies [7]; the notion of the geometric entanglement depends on the physical choice of operationally accessible interactions and measurements.

We can think of the choice of the compactification manifold $S$ as a IR regulator of the theory. The fact that the theory-space entanglement depends on $S$ is somewhat analogous to the fact that the conventional geometric entanglement entropy depends on the choice of the spatial region $A$.

For continuous systems (such as quantum field theories) there are also UV divergences. If we choose a small UV regulator $\epsilon$, the leading contribution diverges as powers of $1/\epsilon$. However, as in the case of geometric entropies, we expect that the subleading constant (i.e., order $\epsilon^0$) term or the coefficient of the $\log \epsilon$ term is universal, depending on whether the dimension is odd or even. Note that this does not follow from the corresponding statement for geometric entanglement entropies, since theory-space entanglement is different from geometric entanglement.

**Comparison with lattice gauge theories.** – Some readers might be alarmed by the non-factorization of the Hilbert space (5), since the standard treatment of entanglement entropy assumes factorization. However, let us point out that the factorization actually in general does not hold, even for conventional geometric entanglement entropies (see ref. [8]).

The issue arises for gauge theories. For concreteness let us consider lattice gauge theories. In the Hamiltonian formulation the gauge-invariant degrees of freedom are given by strings of non-Abelian electric fluxes, and are not localized in space [9]. Such fluxes in general spread both in regions $A$ and its complement $B$, and the spatial division violates the Gauss law on the boundary $\partial A$. This explains the non-factorization of the Hilbert space.

To put it another way, the problem is that in lattice gauge theories the basic degrees of freedom resides in the links connecting vertices, and not in the vertices. The boundary $\partial A$ pass through some of the links, which are charged under some of the global symmetries. Note that this is not just a conceptual problem, but is of practical importance for numerical simulations of entanglement entropy.

To define geometric entropy for lattice gauge theories [8], we associate a new vertex for each link on the boundary and divide the link into two smaller links, one associated with region $A$ and another region $B$. We then define the Hilbert space $\mathcal{H}_A$ ($\mathcal{H}_B$) to be the functionals of the connections of the links in region $A$ ($B$) which are gauge invariant with respect to the gauge transformations associated with the vertices in the interior of $A$ ($B$) but not necessarily with respect to the newly introduced vertices on the boundary (see fig. 2). We then have the natural embedding (5) and the geometric entanglement entropy is defined by (7), (2). This is very analogous to the definition of the theory-space entanglement above.

The analogy goes even further in the context of deconstruction [10,11]. Let us begin with a quiver diagram on the circle, where quiver is simply a graph consisting of vertices and links (fig. 3). Given a quiver we can construct a graph theory by the rule that 1) we associate a $U(N_v)$ gauge group to each vertex $v$ and 2) we associate a bifundamental matter with respect to $U(N_v) \times U(N_w)$ for a

\(^4\)We can generalize the definition to the case where we gauge the diagonal global symmetry for a set of theories $\mathcal{T}_A$, each with a global symmetry $G$. In the graphical representation of fig. 1 this will be a multi-valent vertex.

\(^5\)A prototypical example for this is the Seiberg duality [6] for 4d $\mathcal{N} = 1$ supersymmetric gauge theories.
link connecting vertices $v$ and $w$. The precise matter content can vary depending on the context, for example the amount of supersymmetry and the dimensionality of spacetime. For example (in the original example of ref. [10]) we consider the four-dimensional (4d) theory, the link of the quiver is oriented, and the associated matter is a Weyl fermion with chirality determined by the orientation. For simplicity we take $N_c$ to be independent of $v$, and denote the corresponding integer by $N$.

Now the claim of refs. [10,11] is that in the IR limit and in the limit of large number of quiver vertices, the 4d theory on the Higgs branch coincides with the 5d gauge theory, where only one of the directions is latticed the direction of the quiver becomes the extra dimension $S^1_{\text{extra}}$ in the IR.

Since we have a 5d lattice gauge theory, we can define the geometric entanglement entropy along the $S^1_{\text{extra}}$, by dividing $S^1_{\text{extra}}$ into two. More precisely let us take 5d lattice gauge theory dimensionally reduced on a compact 3-manifold $S$, and consider the geometric entanglement for the resulting 2d theory on $\mathbb{R}_t \times S^1_{\text{extra}}$ (see footnote 6).

As we have seen already, the definition of geometric entanglement in lattice gauge theories involves introducing new vertices on the boundaries of regions $A,B$. In the language of 4d quiver gauge theories, adding a node is translated into adding $U(N)$ symmetry. Since the Hilbert spaces $H_{A,B}$ are not necessarily invariant under the $U(N)$ symmetry (as we discussed above), we should regard the $U(N)$ as a global symmetry acting on $H_{A,B}$; to obtain $H$ we need to gauge this symmetry. Since these are the same ingredients as in the definition of theory-space entanglement above\(^7\), we learn that the geometric entanglement in the deconstructed 5d theory (dimensionally reduced on $S$) coincides with the theory-space entanglement for the 4d quiver gauge theory (defined on the same manifold $S$)! In other words we naturally arrive at the definition of the theory-space entanglement if we want to extend the notion of geometric entanglement of the deconstructed theory to the quiver gauge theory. This is one justification for our definition, and illustrates nicely the close relation between geometric and theory-space entanglement.

**Geometric/theory-space duality.** – Let us provide another (and more non-trivial) example of the relation between geometric and theory-space entanglement.

This examples deals with the case of 4d $\mathcal{N} = 2$ superconformal field theories arising from the compactification of 6d $(2,0)$ theories of type $A_N$ on a punctured Riemann surface $C$ [12]. From the viewpoint of 4d gauge theory, the geometry $C$ is the defining data of the 4d theory.

A punctured Riemann surface $C$ can be decomposed into a collections of three-punctured spheres (trinions) (fig. 4). This is known as a pants decomposition. In the theories defined in ref. [12], a trinion is associated with a theory called $T_N$, with global symmetries $SU(N)^3$ (see footnote 8); each of the $SU(N)$ symmetries are associated with one of the punctures. When we glue such trinions, we gauge the diagonal of the associated $SU(N)$ global symmetries; this is the gauging procedure of (4). In other words, gauging of (4) for 4d gauge theories is translated into the geometrical operation of gluing $C = C_A \cup C_B$. Different choices of pants decompositions are argued to be different descriptions of the same 4d $\mathcal{N} = 2$ superconformal IR fixed point, and thus are $S$-dual to each other.

Since the definition of the theory involves a gauging (4), we can define the theory-space entanglement by compactifying the 4d theory on $\mathbb{R}_t \times S$, where $S$ is a compact 3-manifold. The Hilbert space $H_{T[C]}$ for our 4d theory $T[C]$ depends on the choice of $S$. We here choose a 1-parameter family of the 3-sphere $S^3_0$ whose metric is given by $b^2(x_1^2 + x_2^2) + b^{-2}(x_1^2 + x_2^2) = 1$ [13].

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\(^6\)Alternatively we could choose to lattice all the four dimensions.

\(^7\)The symmetry gauged in this case is $\prod_v U(N_v)$, where $v$ runs over all the links on the boundary.

\(^8\)Here we only consider the so-called full punctures. We can generalize the discussion to more general punctures labeled by Young diagrams.
Now the surprise is that the Hilbert space $\mathcal{H}_{T[C]}$ contains a subspace ("BPS Hilbert space") $\mathcal{H}^{\text{BPS}}_{T[C]}$ which coincides with the Hilbert space of the 3d $SL(N)$ Chern-Simons theory on the spatial Riemann surface:

$$\mathcal{H}[S^3_{(b)};T[C]] = \mathcal{H}^{\text{3d CS}}[C],$$

where the deformation parameter $b$ of $S^3_b$ is translated into the level $t$ of 3d $SL(N)$ Chern-Simons theory [14]. This is part of the statement of the "3d/3d duality" [14] (see also refs. [15,16]). In fact, we could regard $S^3_t$ of refs. [17,18] as $S^3_b$ fibered over an interval, with boundary conditions at both ends.

Let us consider the 6d $(2,0)$ theory on $\mathbb{R} \times S^3 \times C$. We can regard this either as i) $(\mathbb{R}t \times S^3_b) \times C$, giving rise to 4d $N = 2$ theory on $\mathbb{R}t \times S^3_b$, or ii) $(\mathbb{R}t \times C) \times S^3_b$, giving rise to a 3d $SL(N)$ Chern-Simons theory on $\mathbb{R}t \times C$ [14]. Since (9) is the equivalence of the Hilbert space, we automatically have the equivalence of the density matrix and the corresponding entanglement spaces. The correspondence is rather non-trivial since gauging of 4d $N = 2$ theories (fig. 1) is translated into the geometrical gluing operation on the 2d surface (fig. 4); the theory-space entanglement in the BPS Hilbert space of the 4d theory is identified with the geometric entanglement in the 3d $SL(N)$ Chern-Simons theory on the geometric surface $C$.

We can also discuss a similar correspondence for a different compactification manifold $S$; we can for example take the 4d theory on $S^1 \times S^3 \times C$ but with a twist along the $S^1$ direction [19]. The corresponding 3d theory is the $SU(N)$ Chern-Simons theory on $S^3 \times C$, which in turn gives 2d $q$-deformed Yang-Mills theory [20] on $C$ (cf. ref. [21]).

Note in both of these cases the definition of entanglement entropy depends on the choice of the duality frame, and is not $S$-duality invariant.

**Strong Subadditivity.** — Geometric entropies satisfy one crucial relation, the strong subadditivity

$$S_{\text{ent}}(A_1 \cup A_2) + S_{\text{ent}}(A_2 \cup A_3) \geq S_{\text{ent}}(A_1 \cup A_2 \cup A_3) + S_{\text{ent}}(A_2),$$

for three spatial regions $A_1,2,3$.

We conjecture that there exists a counterpart for this statement in theory-space entanglement. To be concrete, suppose that the theory $T$ has a decomposition into four:

$$T = T_1 \cup T_2 \cup T_3 \cup T_4.$$  \hspace{1cm} (11)

It is then natural to define

$$T_{1,2} = T_1 \cup T_2, \hspace{1cm} T_{2,3} = T_2 \cup T_3, \hspace{1cm} T_{1,2,3} = T_1 \cup T_2 \cup T_3, \hspace{1cm} T_2 = T_2 \cup T_3.$$  \hspace{1cm} (12)

The counterpart of (10) is

$$S_{\text{th}}(T_{1,2}) + S_{\text{th}}(T_{2,3}) \geq S_{\text{th}}(T_{1,2,3}) + S_{\text{th}}(T_2).$$  \hspace{1cm} (13)

where $S_{\text{th}}$ is the theory-space entanglement defined with respect to the total theory $T$ in (11). The proof of (13) will be similar to that of (10) (see, e.g., ref. [23]), however we have to carefully take the non-factorization (5) into account.

**Summary and discussion.** — In this paper we discussed a new notion of entanglement, the *theory-space entanglement*, which quantifies the entanglement of two theories interacting through gauge interactions.

While the idea might sound unfamiliar at first, the definition follows that of the conventional geometric entropies, with the only difference being that the division into two regions refers not to division in the geometric regions, but in more abstract theory space. Moreover, we have shown that the theory-space entanglement entropies for a class of theories are equivalent with the geometric entanglement entropies for the dual theories.

The notion of the theory-entanglement entropies can further be generalized — it is not crucial for our definition that the interactions between theories $A$ and $B$ are mediated by gauge interactions. For example, the two theories can interact through Yukawa interactions between bosons in $A$ and fermions in $B$. This example is simpler than the case with gauge interactions since there is no counterpart of dynamical gauge bosons.

Our theory-space entanglement entropies, when generalized in this way, are a rather general quantitative tool to measure entanglement between two theories interacting through some (e.g., gauge) long-range interactions. There are many such examples in physics, indicating the utility of the entanglement in a wide range of physical phenomena. Let us here mention a few of them for illustration.

A good example for our entanglement is the discussion of vacuum entanglement in ref. [24], which analyzes the vacuum entanglement for a scalar field interacting with two atoms/detectors $A$ and $B$ — in this case the “theories” are simply atoms/detectors and their mutual interactions are mediated by the scalar field. We can also generalize the discussion there by replacing the scalar field by the gauge field.

Another example is the gauge-mediated supersymmetry-breaking scenario [5] discussed previously in this letter. In this case the theory-space entanglement quantifies the degree to which the standard model physics (or its grand unified versions) is sensitive to the physics in the hidden sector. This partly answers the question of whether we can distinguish between different models of hidden sectors, which is of great phenomenological interest.

Interestingly, in many cases the separation in the abstract theory space actually coincides with the physical separation in the spatial regions, since the theories $A, B$ could for example refer to materials placed/localized in...
some geometrical regions, and their interactions are given by long-range forces such as a photons. Such geometrical separation also occurs when we choose to take $A, B$ to be theories on the branes in the brane-world scenario, where the two theories are localized in extra dimensions and interact though gravity.

More ambitiously, we believe that theory-space entanglement will provide useful tools to explore the space of quantum field theories in various dimensions, and learn about their mutual relations, perhaps along the lines of the Zamolodchikov metric for CFT. A general inequality among theory-space entanglement entropies, such as the strong subadditivity discussed in this letter, could constrain the possible forms of interactions between the two theories $A$ and $B$.

Finally, it would be interesting to systematically compute the theory-space entanglement entropies for concrete examples. The replica trick [25], which works well for geometric entanglement entropies, in itself does not work here since the meaning of the $n$-fold cover in the theory space is not clear. We can instead choose to compute the theory-space entanglement order by order in the gauge coupling constant in the perturbative expansion. It would also be interesting to ask if the theory-space entanglement has the counterpart of the Ryu-Takayanagi formula in the holographic description.

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