Canonical spectral representation for exchangeable max-stable sequences

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Abstract

The set $L$ of infinite-dimensional, symmetric stable tail dependence functions associated with exchangeable max-stable sequences of random variables with unit Fréchet margins is shown to be a simplex. Except for a single element, the extremal boundary of $L$ is in one-to-one correspondence with the set $\mathcal{F}_1$ of distribution functions of non-negative random variables with unit mean. Consequently, each $\ell \in L$ is uniquely represented by a pair $(b, \mu)$ of a constant $b$ and a probability measure $\mu$ on $\mathcal{F}_1$. A canonical stochastic construction for arbitrary exchangeable max-stable sequences and a stochastic representation for the Pickands dependence measure of finite-dimensional margins of $\ell$ are immediate corollaries.

1 Motivation and mathematical background

We denote by $[0, \infty)^N_{00}$ the set of sequences $\vec{t} = (t_1, t_2, \ldots)$ with non-negative members that are eventually zero, i.e. $t_k = 0$ for almost all $k \in \mathbb{N}$. Let $\vec{Y} = (Y_1, Y_2, \ldots)$ be an exchangeable sequence of random variables with $\mathbb{E}[Y_1] = 1$ and such that $\min_{k \geq 1} \{ Y_k/t_k \}$ has an exponential distribution with rate $\ell(\vec{t})$ for arbitrary $\vec{t} \in [0, \infty)^N_{00}$ except the zero sequence (or this one included when interpreting the exponential distribution with rate zero as an atom at $\infty$). The sequence $\vec{Y}$ is said to be min-stable multivariate exponential and its law is uniquely described by the function $\ell$, which is called the stable tail dependence function of the sequence $\vec{Y}$. Indeed,

$$\mathbb{P}(\vec{Y} > \vec{t}) = \mathbb{P}\left( \min_{k \geq 1} \{ Y_k/t_k \} > 1 \right) = \exp\left\{ -\ell(\vec{t}) \right\}, \quad \vec{t} \in [0, \infty)^N_{00},$$

with the first “$>$”-sign understood componentwise. The function $\ell$ is symmetric in its arguments by exchangeability of $\vec{Y}$. Our goal is to determine the shape of the set of
all symmetric stable tail dependence functions \( \ell : [0, \infty) \to [0, \infty) \), which we denote by \( \mathcal{L} \) in the sequel. The sequence \( 1/Y = (1/Y_1, 1/Y_2, \ldots) \) is called \emph{max-stable with unit Fréchet margins}, which makes the probability law of \( Y \) interesting for the field of multivariate extreme-value theory, for background the interested reader is referred to \cite{Segers2012} and \cite{Joe1997} Chapter 6. Concretely, the set of all \( d \)-variate functions \( C_\ell : [0, 1]^d \to [0, 1] \), defined by

\[
C_\ell(u_1, \ldots, u_d) := \exp \left\{ -\ell \left( -\log(u_1), \ldots, -\log(u_d), 0, 0, \ldots \right) \right\}, \quad \ell \in \mathcal{L},
\]
equals the family of those \( d \)-variate \emph{extreme-value copulas} associated with random vectors on \( [0, 1]^d \) that have a stochastic representation which is conditionally iid in the sense of De Finetti’s Theorem.

It is well-known at least since \cite{DeHaan1984} that \( \ell \) can be represented as

\[
\ell(t) = -\log \{ \mathbb{P}(Y > t) \} = \mathbb{E} \left[ \max_{k \geq 1} \{ t_k X_k \} \right],
\]

for some sequence \( \vec{X} = (X_1, X_2, \ldots) \) of random variables with finite means\(^2\) As an example\(^3\) for the representation \((1)\), if \( Y \) has independent components which all have the unit exponential distribution, that is \( \ell(t) = \sum_{k \geq 1} t_k \), the probability law of \( \vec{X} \) can be defined via a vector of discrete probabilities \( \vec{p} = (p_1, p_2, \ldots) \) as

\[
\mathbb{P} \left( \vec{X} = \frac{\vec{e}_k}{p_k} \right) = p_k > 0, \quad k \geq 1, \quad \sum_{k \geq 1} p_k = 1,
\]

where \( \vec{e}_k \) denotes the sequence with all members equal to zero except for the \( k \)-th. The representation \((1)\) of a stable tail dependence function is called a \emph{spectral representation}. As \((2)\) shows, it is not unique in general (i.e. different \( \vec{X} \) can imply the same \( \ell \), hence \( Y \)). Furthermore, even though \( Y \) is assumed to be exchangeable in the present work, \( \vec{X} \) needs not be exchangeable and, in fact, the proof in \cite{DeHaan1984} constructs \( \vec{X} \) from \( Y \) in such a way that \( \vec{X} \) is not exchangeable (the particular example \((2)\) demonstrates this). Conversely, however, the spectral representation \((1)\) can be used to construct models for \( Y \) by choosing convenient models for \( \vec{X} \) that allow the expected value in \((1)\) to be computed in closed form, as highlighted in \cite{Segers2012}. If one pursues this strategy and starts with an exchangeable \( \vec{X} \), one obtains an exchangeable sequence \( Y \), but to the best of our knowledge it is an open question (solved by the present article) whether all exchangeable min-stable multivariate exponential \( Y \) can be obtained in such way.

\(^1\)For background, the interested reader is referred to \cite{GudendorfSegers2009}.
\(^2\)Even though we are only interested in distributional statements throughout, for the sake of a more intuitive exposition we find it sometimes convenient to express formulas like \((1)\) in probabilistic notation with probability measure \( \mathbb{P} \) and expectations \( \mathbb{E} \) (as compared to writing integrals), with the generic random objects \( \vec{X}, Y \) being viewed as defined on some generic probability space \((\Omega, \mathcal{G}, \mathbb{P})\), on which we do not necessarily work.
\(^3\)This is the example on page 1198 in \cite{DeHaan1984}.
1 Motivation and mathematical background

We denote by $\ell^{(d)}$ the restriction of $\ell$ to the first $d \in \mathbb{N}$ components, i.e. $\ell^{(d)}$ determines the law of $(Y_1, \ldots, Y_d)$. For stable tail dependence functions in finite dimensions, such as $\ell^{(d)}$, there exist different methods to obtain uniqueness of the spectral representation [1] by imposing certain restrictions on the law of $\tilde{X}$. The most prominent one is the Pickands representation, named after [Pickands (1981)], see also [De Haan, Resnick (1977), Ressell (2013)], which states that if $\ell^{(d)}$ is the stable tail dependence function associated with some min-stable multivariate exponential random vector $\bar{Y}^{(d)} = (Y_1, \ldots, Y_d)$, then there is a random vector $\bar{X}^{(d)} = (X_{1}^{(d)}, \ldots, X_{d}^{(d)})$, uniquely determined in law, which takes values on the $d$-dimensional unit simplex $S_d := \{ \tilde{q} \in [0, 1]^d : q_1 + \ldots + q_d = 1 \}$ and satisfying $E[X_k^{(d)}] = 1/d$ for all $k = 1, \ldots, d$, such that

$$\ell^{(d)}(\bar{t}) = d E[\max\{ t_1 X_1^{(d)}, \ldots, t_d X_d^{(d)} \}], \quad \bar{t} \in [0, \infty)^N, \quad (3)$$

In our infinite-dimensional setting, even though we assume exchangeability of $\bar{Y} = (Y_1, Y_2, \ldots)$, an unfortunate aspect of the Pickands representation is that the relation between the laws of $\bar{X}^{(d)}$ and $\bar{X}^{(d+1)}$ is not easy to understand, in particular $\bar{X}^{(d)}$ is not a re-scaled $d$-margin of $\bar{X}^{(d+1)}$, like one might naively hope on first glimpse. Consequently, describing the infinite-dimensional, symmetric stable tail dependence function $\ell$ in terms of the collection of its finite-dimensional Pickands measures is neither easily accomplished nor convenient or algebraically natural.

In the main body of this article, we derive a natural and convenient spectral representation for symmetric stable tail dependence functions. To wit, each $\ell \in \mathcal{E}$ can be represented as

$$\ell(\bar{t}) = b \sum_{k \geq 1} t_k + (1 - b) E[\max_{k \geq 1} \{ t_k X_k \}], \quad \bar{t} \in [0, \infty)^N, \quad (4)$$

with a constant $b \in [0, 1]$ and an exchangeable sequence $\bar{X}$ satisfying $E[X_1] = 1$ (hence $E[X_k] = 1$ for each $k \geq 1$). The law of $\bar{X}$ is still not unique in general, but it becomes unique if we postulate in addition that the conditional mean of $\bar{X}$, that is $\bar{X} := \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} X_k$, is identically equal to one. In particular, the stable tail dependence function $\ell(\bar{t}) = \sum_{k \geq 1} t_k$, corresponding to independent members in $\bar{Y}$, cannot be represented via an exchangeable $\bar{X}$. However, the canonical representation [1] shows that the independence case occupies an isolated role in this regard.

By virtue of De Finetti’s Theorem, see [De Finetti (1931), De Finetti (1937)] and [Aldous (1985)] p. 19 ff., studying the law of the exchangeable sequence $\bar{Y}$ is tantamount to a study of the law of a random distribution function $F = \{ F_t \}_{t \geq 0}$ that is defined by $F_t := \mathbb{P}(Y_1 \leq t \mid \mathcal{T})$, with $\mathcal{T}$ denoting the tail-$\sigma$-field of $\bar{Y}$. A result of [Mai, Scherer (2014)] shows that the stochastic process $H := - \log(1 - F)$ is strongly infinitely divisible with respect to time [Kopp, Molchanov (2018)] call the “strong IDT” processes “time-stable” processes, but we prefer to stick with the original nomenclature.
(strong IDT), meaning that
\[ \{ H_t \}_{t \geq 0} \overset{d}{=} \{ \frac{H_t^{(1)}}{n} + \ldots + \frac{H_t^{(n)}}{n} \}_{t \geq 0}, \quad \forall n \in \mathbb{N}, \]
where \( \overset{d}{=} \) denotes equality in law and \( H_t^{(i)} \) are independent copies of \( H \). Conversely, given a non-decreasing, right-continuous strong IDT process \( H \) and an independent sequence \( \{ \epsilon_k \}_{k \geq 1} \) of iid unit exponential variables, the exchangeable, min-stable multivariate exponential sequence \( \tilde{Y} \) can be represented as
\[ Y_k := \inf \{ t > 0 : H_t > \epsilon_k \}, \quad k \in \mathbb{N}, \]
establishing a canonical stochastic representation, which is conditionally iid in the sense of De Finetti’s Theorem. If \( H \) is normalized to satisfy \( \mathbb{E}[\exp(-H_1)] = \exp(-1) \), it follows that \( \mathbb{E}[Y_1] = 1 \), so that the function
\[ \ell(\vec{t}) := -\log \{ \mathbb{P}(\vec{Y} > \vec{t}) \} = -\log \left\{ \mathbb{E} \left[ e^{-\sum_{k \geq 1} H_{tk}} \right] \right\}, \quad \vec{t} \in [0, \infty)_{\mathbb{R}^0}, \]
lies in \( \mathcal{L} \).

In fact, in our proof of (4) we rely heavily on the concept of strong IDT processes, for which \([\text{Kopp, Molchanov (2018)}]\) recently have derived a convenient series representation, which we make use of. Translating the analytical result (4) on symmetric stable tail dependence functions into the language of these processes then implies that each non-decreasing strong IDT process is uniquely determined by a triplet \((b,c,\mu)\) of constants \( b \geq 0, \quad c > 0 \) and a probability measure \( \mu \) on the set of distribution functions of non-negative random variables with unit mean, as we will see.

Regarding the organization of the remaining article, we prove and discuss the main result (4) in Section 2 and we conclude in Section 3.

2 The structure of \( \mathcal{L} \)

We denote by \( \ell_{\Pi}(\vec{t}) := \sum_{k \geq 1} t_k \) the stable tail dependence function associated with an iid sequence of unit exponentials. We denote by \( \mathfrak{F} \) (resp. \( \mathfrak{F}_1 \)) the set of all distribution functions of non-negative random variables with finite (resp. unit) mean. Then with \( F \in \mathfrak{F}_1 \) the function
\[ \ell_F(\vec{t}) := \int_0^\infty 1 - \prod_{k \geq 1} F\left( \frac{s}{t_k} \right) \, ds, \quad \vec{t} \in [0, \infty)_{\mathbb{R}^0}, \]
defines a symmetric stable tail dependence function, which is investigated thoroughly in \([\text{Mai (2018)}]\). In this definition, implicitly we mean \( 1/0 = \infty \) and \( F(\infty) = 1 \) for those \( t_k \) that are zero. We seek to show that \( \mathcal{L} \) (equipped with the topology of pointwise convergence) is a simplex with extremal boundary \( \partial \mathcal{L} = \{ \ell_{\Pi} \} \cup \{ \ell_F : F \in \mathfrak{F}_1 \} \), which is the main contribution of the present work, see Theorem 2.2 and Corollary 2.3 below.
The key idea to prove this relies on the aforementioned link to strong IDT processes, found in [Mai, Scherer (2014), Theorem 5.3]. In a recent article, [Kopp, Molchanov (2018), Theorem 4.2] show that a non-negative càdlàg, stochastically continuous, strong IDT process without Gaussian component admits a LePage series representation

\[
\{H_t\}_{t \geq 0} = \left\{ b t + \sum_{k \geq 1} f^{(k)} t^{1+\gamma_k} \right\}_{t \geq 0}, \quad t \geq 0, \tag{5}
\]

where \(\{f^{(k)}\}_{k \geq 1}\) is a sequence of independent copies of a non-vanishing càdlàg stochastic process \(f = \{f_t\}_{t \geq 0}\) with \(f_0 = 0\) (denote the space of all such functions by \(\mathfrak{D}\) in the sequel) and, independently, \(\{\epsilon_k\}_{k \geq 1}\) is a list of iid unit exponentials. Furthermore, the process \(f\) satisfies

\[
\int_{\mathfrak{D}}\int_0^\infty \min\{1, |f(u)|\} \frac{du}{u^2} \gamma(df) < \infty, \quad (6)
\]

with \(\gamma\) denoting the probability law of \(f\) on \(\mathfrak{D}\). In general, the law of \(f\) in this representation of a non-negative strong IDT process is non-unique. However, the following auxiliary lemma shows that if \(H\) is non-decreasing, we can at least learn that \(f\) is non-decreasing as well. This proof is the most technical step towards Theorem 2.2 below.

**Lemma 2.1 (Non-decreasing strong IDT processes)**

If \(H\) is a non-decreasing, right-continuous strong IDT process, the stochastic process \(f\) of any LePage series representation of a non-negative strong IDT process is non-decreasing. However, the following auxiliary lemma shows that if \(H\) is non-decreasing, we can at least learn that \(f\) is non-decreasing as well. This proof is the most technical step towards Theorem 2.2 below.

**Proof**

First notice that non-decreasingness, right-continuity, and the strong IDT property imply stochastic continuity of \(H\). This follows from the fact that for each fixed \(t > 0\) the random variable \(H_{t-} := \lim_{x \uparrow t} H_x\) exists in \([0, \infty]\) by non-decreasingness and has the same infinitely divisible law as the random variable \(H_t\) (since \(E[\exp(-x H_u)] = \exp\{-t \Psi_H(u)\}\) for some Bernstein function \(\Psi_H\), see [Mai, Scherer (2014), Lemma 3.7]). Thus, the non-negative random variable \(H_t - H_{t-}\) has zero expectation and is thus identically equal to zero. On the other hand, right-continuity implies that \(H_{t+} := \lim_{x \downarrow 0} H_x = H_t\) almost surely. Consequently, the stochastic continuity assumption of [Kopp, Molchanov (2018), Theorem 4.2] is satisfied, hence there is a LePage series representation.

Consider a probability space \((\Omega, \mathcal{G}, \mathbb{P})\), on which \(H\) is defined by the right-hand side of (5). We first prove that \(f\) is non-decreasing. The heuristic idea is to consider

\[
H_{\epsilon_1 t} = f^{(1)}_{\epsilon_1 t} + b \epsilon_1 t + \sum_{k \geq 2} f^{(k)}_{\epsilon_1 t^{1+\gamma_k}},
\]

where \(\{f^{(k)}_{\epsilon_1 t^{1+\gamma_k}}\}_{k \geq 1}\) is the sequence of independent copies of a non-vanishing càdlàg stochastic process \(f = \{f_t\}_{t \geq 0}\) with \(f_0 = 0\) (denote the space of all such functions by \(\mathfrak{D}\) in the sequel) and, independently, \(\{\epsilon_k\}_{k \geq 1}\) is a list of iid unit exponentials. Furthermore, the process \(f\) satisfies

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with \(\gamma\) denoting the probability law of \(f\) on \(\mathfrak{D}\). In general, the law of \(f\) in this representation of a non-negative strong IDT process is non-unique. However, the following auxiliary lemma shows that if \(H\) is non-decreasing, we can at least learn that \(f\) is non-decreasing as well. This proof is the most technical step towards Theorem 2.2 below.

**Lemma 2.1 (Non-decreasing strong IDT processes)**

If \(H\) is a non-decreasing, right-continuous strong IDT process, the stochastic process \(f\) of any LePage series representation is necessarily non-decreasing and \(b \geq 0\).

**Proof**

First notice that non-decreasingness, right-continuity, and the strong IDT property imply stochastic continuity of \(H\). This follows from the fact that for each fixed \(t > 0\) the random variable \(H_{t-} := \lim_{x \uparrow t} H_x\) exists in \([0, \infty]\) by non-decreasingness and has the same infinitely divisible law as the random variable \(H_t\) (since \(E[\exp(-x H_u)] = \exp\{-t \Psi_H(u)\}\) for some Bernstein function \(\Psi_H\), see [Mai, Scherer (2014), Lemma 3.7]). Thus, the non-negative random variable \(H_t - H_{t-}\) has zero expectation and is thus identically equal to zero. On the other hand, right-continuity implies that \(H_{t+} := \lim_{x \downarrow 0} H_x = H_t\) almost surely. Consequently, the stochastic continuity assumption of [Kopp, Molchanov (2018), Theorem 4.2] is satisfied, hence there is a LePage series representation.

Consider a probability space \((\Omega, \mathcal{G}, \mathbb{P})\), on which \(H\) is defined by the right-hand side of (5). We first prove that \(f\) is non-decreasing. The heuristic idea is to consider

\[
H_{\epsilon_1 t} = f^{(1)}_{\epsilon_1 t} + b \epsilon_1 t + \sum_{k \geq 2} f^{(k)}_{\epsilon_1 t^{1+\gamma_k}},
\]

More generally, [Kopp, Molchanov (2018)] consider strong IDT processes without Gaussian component, of which the non-negative ones form a subset, which is of interest for us.

See [Schilling et al. (2010)] for a textbook treatment on Bernstein functions.
Furthermore, it is obvious from the definition that $\tilde{c}$ ad $\tilde{a}$, and satisfies

By [Kopp, Molchanov (2018), Theorem 4.2], the process $\tilde{d}$ for Poisson random measure to observe for

To fill this intuitive idea with some mathematical rigor is what’s done in the sequel. We assume a violation of non-decreasingness of $f^{(1)}$, which means that there exists $\epsilon > 0$ and $0 \leq x_1 < x_2 < \infty$ such that $\mathbb{P}(A_f) > 0$ for the event

Our goal is to show that this implies a violation of the non-decreasingness of $H$. For later use we define the $\sigma$-algebra $\mathcal{H} := \sigma(\epsilon_k, f^{(k)} : k \geq 2)$ generated by all involved stochastic objects except for $\epsilon_1, f^{(1)}$.

For a moment consider independent copies $\{g^{(k)}\}_{k \geq 1}$ of $g := |f|$ and for each $x \in (0, 1]$, $t \geq 0$, let

By [Kopp, Molchanov (2018), Theorem 4.2], the process $\tilde{H}$ is non-negative, strong IDT, càdlàg, and satisfies $\tilde{H}_0 = 0$. Right-continuity in zero implies for each $x \in (0, 1]$ that the first exit time of $\{\tilde{H}_t\}_{t \geq 0}$ from the interval $[0, \epsilon/4]$ is almost surely positive, i.e.

Furthermore, it is obvious from the definition that $\tilde{T}_x = \tilde{T}_1/x$, which implies that $\lim_{x \uparrow 0} \tilde{T}_x = \infty$. We use the Laplace functional formula [Resnick (1987), Proposition 3.6] for Poisson random measure to observe for $d \in \mathbb{N}$ and $t_1, y_1, \ldots, t_d, y_d \geq 0$ arbitrary that

having applied the substitution $u = x/(x+s)$. An analogous computation with the substitution $u = x/s$ shows that

$$
\mathbb{E}\left[e^{-\sum_{i=1}^{d} y_i \tilde{H}^{(x)}_{i}}\right] = \exp \left(- \int_{0}^{\infty} \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(x t_i/(x+s))} \, ds \, \gamma(\,df\,)\right)
$$

$$
= \exp \left(- x \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(u t_i)} \, du \, \gamma(\,df\,)\right),
$$

$$
\mathbb{E}\left[e^{-\sum_{i=1}^{d} y_i \tilde{H}^{(s)}_{i}}\right] = \exp \left(- \int_{0}^{\infty} \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(x t_i/s) 1_{s>|x|}} \, ds \, \gamma(\,df\,)\right)
$$

$$
= \exp \left(- x \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(u t_i)} \, du \, \gamma(\,df\,)\right),
$$

$$
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$$

$$
= \exp \left(- x \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(u t_i)} \, du \, \gamma(\,df\,)\right),
$$

$$
\mathbb{E}\left[e^{-\sum_{i=1}^{d} y_i \tilde{H}^{(s)}_{i}}\right] = \exp \left(- \int_{0}^{\infty} \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(x t_i/s) 1_{s>|x|}} \, ds \, \gamma(\,df\,)\right)
$$

$$
= \exp \left(- x \int_{0}^{1} 1 - e^{\sum_{i=1}^{d} y_i |f|(u t_i)} \, du \, \gamma(\,df\,)\right).
$$
which implies that \( \{ \tilde{H}_t^{(x)} \}_{t \geq 0} \) has the same law as \( \{ H_t^{(x)} \}_{t \geq 0} \). This implies that the first exit time of \( H^{(x)} \) from \([0, \epsilon/4]\) is equal in law to that of \( H^{(x)} \). The process \( H^{(x)} \) evidently satisfies

\[
\tilde{H}_t^{(x)} = \sum_{k \geq 2} g^{(k)}_{x} x \frac{1}{x^2 + \ldots + \epsilon k} \leq \sum_{k \geq 2} g^{(k)}_{x} x = H_t^{(x)}.
\]

For its first exit time from \([0, \epsilon/4]\) this gives the lower bound

\[
\inf \{ t > 0 : \tilde{H}_t^{(x)} > \epsilon/4 \} \geq \inf \{ t > 0 : H_{xt} > \epsilon/4 \} = \hat{T}_x,
\]

which was shown to converge to infinity almost surely as \( x \searrow 0 \).

Now the process of interest for us is

\[
H_t^{(x)} := \sum_{k \geq 2} f^{(k)}_{x} x \frac{1}{x^2 + \ldots + \epsilon k}, \quad t \geq 0,
\]

which satisfies \( |H_t^{(x)}| \leq \tilde{H}_t^{(x)} \) for all \( t \). Consequently, the first exit time \( T_x \) of \( H^{(x)} \) from the interval \([0, \epsilon/4]\) is almost surely larger than that of \( \tilde{H}^{(x)} \), which was shown above to converge to infinity almost surely as \( x \searrow \infty \). Consequently, we find an \( H \)-measurable (notice that \( H^{(x)} \) is \( H \)-measurable) random variable \( Z > 0 \) such that \( T_x \geq x_2 \) for all \( x \leq Z \). In particular, on the event \( \{ \epsilon_1 \leq Z \} \) we have that \( T_{\epsilon_1} \geq x_2 \) and hence \( \sup_{t \in [0, x_2]} |H_t^{(\epsilon_1)}| \leq \epsilon/4 \). Finally, on the event \( A_\epsilon := \{ \epsilon_1 < \epsilon/(4 x_2 |b|) \} \) we have \( |b \epsilon_1 t| \leq \epsilon/4 \) for all \( t \leq x_2 \). Notice that \( A_\epsilon \) has positive probability (possibly equal to one if \( b = 0 \)). Summing up all terms, we show that the event

\[
A_H := \left\{ \{ H_{\epsilon_1 t} \}_{t \geq 0} \text{ not non-decreasing on } [x_1, x_2] \text{ and } \inf_{x \in [x_1, x_2]} \{ H_{\epsilon_1 x} - H_{\epsilon_1 x_1} \} \leq -\frac{\epsilon}{4} \right\}
\]

has positive probability. To this end, by construction \( (A_f \cap A_\epsilon \cap \{ \epsilon_1 \leq Z \}) \subset A_H \), since on this set we have for \( x \in [x_1, x_2] \) that

\[
H_{\epsilon_1 x} - H_{\epsilon_1 x_1} = f^{(1)}_x (x - x_1) + b \epsilon_1 (x - x_1) + H^{(\epsilon_1)}_x - H^{(\epsilon_1)}_{x_1} \leq -\frac{\epsilon}{4}.
\]

Hence,

\[
P(A_H) \geq E[1_{A_f} 1_{A_\epsilon} 1_{\{ \epsilon_1 \leq Z \}}] = P(A_f) E[1_{A_\epsilon} 1_{\{ \epsilon_1 \leq Z \}} | H] = P(A_f) E \left[ 1 - e^{-\min\{Z, \epsilon/(4 |b| x_2)\}} \right] > 0.
\]

That the last expression is strictly positive follows from the fact that the random variable \( \min\{Z, \epsilon/(4 |b| x_2)\} \) is not almost surely zero (it is even almost surely positive). Since \( A_H \) has positive probability, \( H \) cannot be non-decreasing almost surely, hence the assumption was wrong and \( f \) needs to be non-decreasing.
Next, we prove that $b \geq 0$. To this end, we know that $H_1$ is non-negative and infinitely divisible, consequently it has a non-negative drift $b_H \geq 0$ in its Lévy-Khinchin representation, see \cite{Bertoin1999, Schilling2010} for background. Also, the stochastic process
\[
\tilde{H}_t := \sum_{k \geq 1} f^{(k)} \frac{t}{1 + \ldots + k}, \quad t \geq 0,
\]
is strong IDT by \cite{KoppMolchanov2018} Theorem 4.2], hence $\tilde{H}_1$ is infinitely divisible. But by what we have shown, each $f^{(k)}$ is non-negative almost surely, so $\tilde{H}_1 \geq 0$. Since $f \geq 0$ by what we have just shown, the Bernstein function $\Psi_{\tilde{H}}$ associated with $\tilde{H}_1$ via $\Psi_{\tilde{H}}(x) = -\log(E[\exp(-x \tilde{H}_1)])$ can be computed using the Laplace functional formula for Poisson random measure, cf. \cite{Resnick1987} Proposition 3.6], which yields
\[
\Psi_{\tilde{H}}(x) = \int_{\mathcal{D}} \int_0^{\infty} 1 - e^{-x \tilde{f}(u)} \frac{du}{u^2} \gamma(df).
\]
By non-negativity of $f$, we get the estimate $(1 - \exp(-x \tilde{f}(u)))/x \leq \min\{1/x, f(u)\} \leq \min\{1, f(u)\}$, which holds for all $x \geq 1$ and all $u > 0$. By the dominated convergence theorem, using (6), we may thus conclude that $\Psi_{\tilde{H}}(x)/x$ converges to zero as $x \to \infty$.
Consequently, the random variable $\tilde{H}_1$ has no drift in its Lévy-Khinchin representation. This implies $b = b_H \geq 0$. \hfill $\Box$

We are now in the position to derive the main contribution of the present article. We denote by $\mathcal{M}_1^+(\mathcal{E})$ the set of probability measures on some set $\mathcal{E}$.

**Theorem 2.2 (The structure of $\mathcal{L}$)**

Let $\ell \in \mathcal{L}$, not equal to $\ell_H$. There exists a pair $(b, \mu) \in [0, 1) \times \mathcal{M}_1^+(\mathcal{F}_1)$ such that $\ell = b\ell_H + (1 - b) \int_{\mathcal{F}_1} \ell_F \mu(dF)$.

**Proof**

Given $\ell \in \mathcal{L}$, there exists an exchangeable sequence $\bar{Y} = (Y_1, Y_2, \ldots)$ of random variables on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$ such that $\mathbb{P}(\bar{Y} > \bar{t}) = \exp(-\ell(\bar{t}))$ for $\bar{t} \in [0, \infty)^N$. By \cite{MaiScherer2014} Theorem 5.3], the stochastic process $H_t := -\log\left(\mathbb{P}(Y_1 > t | T)\right)$, $t \geq 0$, with $T$ the tail-$\sigma$-field of $\bar{Y}$, is strong IDT, non-decreasing and not identically equal to $H_t = t$ (since $\ell \neq \ell_H$). Lemma 2.1 proves existence of $b \geq 0$ and a non-vanishing, right-continuous, non-decreasing stochastic process $f = \{f_t\}_{t \geq 0}$ with $f_0 = 0$ (denote the space of all such functions by $\mathcal{D}_+$ in the sequel) such that (5) holds. Denoting the probability law of $f$ by $\gamma$, (3) now reads
\[
\int_{\mathcal{D}_+} \int_0^{\infty} \min\{1, f(u)\} \frac{du}{u^2} \gamma(df) < \infty. \tag{7}
\]
From the properties of $f$ (non-decreasingness, right-continuity and $f(0) = 0$) we conclude that the function
\[
\tilde{F}_t := e^{-\lim_{s \downarrow t} f_{1/s}}, \quad t \geq 0,
\]
2 The structure of Λ

is almost surely the distribution function of some non-negative random variable, which is not identically zero (since \( f \) is non-vanishing). In the sequel, we denote the probability measure of \( \tilde{F} \) by \( \tilde{\mu} \). We get from (7) with the estimate \( 1 - x \leq \min\{1, -\log(x)\} \) for \( x \in [0, 1] \) and the substitution \( t = 1/u \) that

\[
\int_\mathbb{R} \int_0^\infty 1 - F(t) \, dt \, \tilde{\mu}(dF) \leq \int_\mathbb{R} \int_0^\infty \min\{1, -\log(F(t))\} \, dt \, \tilde{\mu}(dF) = \int_{\mathbb{R}_+} \int_0^\infty \frac{\min\{1, f(u)\}}{u^2} \, du \, \gamma(df) < \infty,
\]

i.e. \( \tilde{\mu} \) is an element of \( M_1^+(\mathbb{R}) \). For arbitrary \( F \in \mathfrak{F} \) we denote by \( M_F := \int_0^\infty 1 - F(t) \, dt \) its mean. By (8), the positive random variable \( M_{\tilde{F}} \) has finite mean \( c > 0 \) (note that \( M_{\tilde{F}} \) is positive almost surely and \( c = 0 \) is ruled out since \( f \) is non-vanishing, hence \( \tilde{F} \) not almost surely identically equal to one), i.e. \( \int_{\mathbb{R}} M_{\tilde{F}} \tilde{\mu}(dF) = c \). Consequently,

\[
\hat{\mu}(dF) := \frac{M_F}{c} \tilde{\mu}(dF)
\]

defines an equivalent probability measure on \( \mathfrak{F} \). Finally, we denote by \( \mu \) the probability measure that describes the law of the process \( \{\tilde{F}_{M_{\tilde{F}}} t\}_{t \geq 0} \) under the measure \( \hat{\mu} \), and observe that

\[
M_{F_{M_{\tilde{F}}}} = \int_0^\infty 1 - F(t) \, ds = \frac{1}{M_{\tilde{F}}} \int_0^\infty 1 - \tilde{F}_s \, ds = 1
\]

almost surely. Consequently, \( \mu \in M_1^+(\mathfrak{F}) \). Putting together the pieces, we may re-write (5) as

\[
\{H_t\}_{t \geq 0} \overset{d}{=} \left\{ b t + \sum_{k \geq 1} -\log \left[ \frac{\tilde{F}_t^{(k)} \epsilon_1 + \ldots + \epsilon_k}{\tilde{F}_t^{-k}} \right] \right\}_{t \geq 0}, \quad t \geq 0,
\]

and we observe that \( \sum_{k \geq 1} \delta_{(\epsilon_1 + \ldots + \epsilon_k, F^{(k)})} \) is a Poisson random measure on \([0, \infty) \times \mathfrak{F}\) with mean measure \( dx \times \tilde{\mu}(dF) \). Hence, the Laplace functional formula [Resnick (1987)]

---

[7] Here, \( \delta_x \) denotes the Dirac measure at a point \( x \) in a measure space \( X \).
Proposition 3.6], applied in the third equality below, gives
\[
P(\vec{Y} > \vec{t}) = \mathbb{E} \left[ e^{-\sum_{k \geq 1} H_{t_k}} \right] = e^{-b \ell(\vec{t})} \mathbb{E} \left[ \exp \left\{ -\sum_{k \geq 1} -\log \left( \prod_{i \geq 1} F_{i+\ldots+k}^{(k)} \right) \right\} \right]
\]
\[
= e^{-b \ell(\vec{t})} \exp \left\{ -\int_{\mathbb{R}} \int_{0}^{\infty} 1 - \prod_{i \geq 1} F \left( \frac{x}{t_i} \right) \, dx \, \tilde{\mu}(dF) \right\}
\]
\[
= e^{-b \ell(\vec{t})} \exp \left\{ -\int_{\bar{\mathbb{R}}} \int_{0}^{\infty} 1 - \prod_{i \geq 1} F \left( \frac{M_F x}{t_i} \right) \, dx \, M_F \tilde{\mu}(dF) \right\}
\]
\[
= e^{-b \ell(\vec{t})} \exp \left\{ -c \int_{\bar{\mathbb{R}}} \int_{0}^{\infty} 1 - \prod_{i \geq 1} F \left( \frac{x}{t_i} \right) \, dx \, \mu(dF) \right\}
\]
\[
= \exp \left\{ -b \ell(\vec{t}) - c \int_{\bar{\mathbb{R}}} \ell_F(\vec{t}) \, \mu(dF) \right\}
\]

Notice that in the pen-ultimate equation we make use of the fact that $M_F = 1$ almost surely under the probability measure $\mu$. Plugging $\vec{t} = (1, 0, 0, \ldots)$ into the last equation, we observe that
\[
1 = \ell(\vec{t}) = b + c.
\]

From this we can easily conclude that $c = 1 - b$, hence
\[
\ell = b \ell + (1 - b) \int_{\bar{\mathbb{R}}} \ell_F \, \mu(dF),
\]
as claimed. \hfill \Box

The following corollary is of particular relevance when thinking about potential further research concerning the parameter estimation of non-decreasing strong IDT processes or exchangeable max-stable sequences, resp. extreme-value copulas.

**Corollary 2.3 (Uniqueness)**
The pair $(b, \mu)$ in Theorem 2.2 is unique.

**Proof**
It is convenient to study uniqueness in terms of the probability law of the uniquely associated strong IDT process $H$, determined by
\[
\mathbb{E} \left[ e^{-\sum_{k \geq 1} H_{t_k}} \right] = e^{-b \ell(\vec{t}) - (1 - b) \int_{\bar{\mathbb{R}}} \ell_F(\vec{t}) \, \mu(dF)}, \quad \vec{t} \in [0, \infty)^{\mathbb{N}}.
\]

The constant $b$ is clearly unique, because it is the unique drift constant in the Lévy-Khinchin representation of the infinitely divisible random variable $H_1$. Regarding $\mu$, it
follows from Lemma 2.1 and the proof of Theorem 2.2 that the probability law of \( f \) in any Le Page series representation for \( H \) is necessarily supported by the set
\[
\mathcal{G} := \left\{ g : [0, \infty) \to [0, \infty] : g(0) = 0, \text{g right-continuous, non-decreasing,} \right. \\
\lim_{t \to \infty} g(t) = \infty, \left. \int_0^\infty 1 - e^{-g(t)} \, dt < \infty \right\}.
\]

For each \( g \in \mathcal{G} \) there is a unique \( c > 0 \), namely
\[
c := \int_0^\infty 1 - e^{-g(t)} \, dt,
\]
such that \( g = c \circ g^{(1)} \), where \( g^{(1)} \) lies in the smaller set
\[
\mathcal{G}_1 := \left\{ g \in \mathcal{G} : \int_0^\infty 1 - e^{-g(t)} \, dt = 1 \right\}.
\]

By [Kopp, Molchanov (2018), Remark 4.1] the law of \( f^{(1)} \) is unique. This implies that the measure \( \mu \) is unique, since by the proof of Theorem 2.2 it equals the law of
\[
F_t := e^{-\lim_{s \to t} f^{(1)}(s)} \quad t \geq 0,
\]
and this transformation maps \( \mathcal{G}_1 \) to \( \mathcal{F}_1 \) in a bijective manner.

We end this section with a few explanatory remarks ((a)-(c), (f)) and immediate consequences ((d),(e)):

(a) \( \mathcal{L} \) is compact (in the topology of pointwise convergence).

Proof

Let \( \{\ell_n\}_{n \in \mathbb{N}} \subset \mathcal{L} \). Then we find strong IDT processes \( \{H^{(n)}\}_{n \in \mathbb{N}} \) associated with these \( \ell_n \). The processes \( F^{(n)} := 1 - \exp(-H^{(n)}) \) define random variables on the space of distribution functions of non-negative random variables. The set of distribution functions of random variables taking values in \([0, \infty] \) (equipped with the topology of pointwise convergence) is compact (by Helly’s Selection Theorem) and Hausdorff (since it is metrizable by the Lévy metric, see [Sibley (1971)]). Thus, the probability measures on this set (equipped with the weak topology) form a Bauer simplex by [Alfsen (1971), Corollary II.4.2, p. 104], in particular form a compact set. Since the probability measures of the given sequence \( \{F^{(n)}\} \) lies in this set, we find a convergent subsequence and a limiting law, hence a limiting stochastic process \( H \). Then \( F := 1 - \exp(-H) \in M^1_+(\mathcal{F}) \) almost surely, since
\[
\mathbb{E} \left[ \int_0^\infty 1 - F_s \, ds \right] = \mathbb{E} \left[ \int_0^\infty e^{-H_s} \, ds \right] = \int_0^\infty \mathbb{E} \left[ e^{-H_s} \right] \, ds \quad = \int_0^\infty \lim_{\kappa \to \infty} \mathbb{E} \left[ e^{-H^{(\kappa)}_s} \right] \, ds \quad = \int_0^\infty e^{-s} \, ds = 1,
\]
\footnote{We use the notation of [Kopp, Molchanov (2018)], denoting \((c \circ g^{(1)})(t) := g^{(1)}(c t)\), for \( t \geq 0 \) and \( c > 0 \).}
where the third equality follows from the bounded convergence theorem and the fourth from the fact that $\mathbb{E}[\exp(-H^{(n)}_{k})] = \exp(-s \ell_{n}(1,0,0,\ldots)) = \exp(-s)$ for each $n \in \mathbb{N}$. Furthermore, to see that $H$ is strong IDT, it suffices to verify that the homogeneity of order one of the $\ell_{n}$ carries over to the limit (obviously). Finally, 
\[
\ell(t) := -\log \left\{ \mathbb{E} \left[ e^{-\sum_{k \geq 1} H_{k}} \right] \right\}, \quad t \in [0, \infty)^{[n]},
\]
is the limit of $\ell_{n}$ by bounded convergence and lies in $\mathcal{L}$. □

$\mathcal{L}$ is obviously convex and by Theorem 2.2 the extremal boundary of $\mathcal{L}$ is $\partial \mathcal{L} = \{ \ell_{\Pi} \} \cup \{ \ell_{F} : F \in \mathfrak{I}_{1} \}$. Thus, $\mathcal{L}$ is a simplex, since the boundary integral representation is unique by Corollary 2.3.

(b) There is another way to think about the identification of $\mathcal{L}$ with pairs $(b, \mu)$. We denote by $\mathfrak{I}_{1}$ the set of all distribution functions of non-negative random variables with mean less than or equal to one, equipped with the topology of pointwise convergence. For an arbitrary sequence in $\mathfrak{I}_{1}$, Helly’s Selection Theorem gives a convergent subsequence and a limit distribution function $F$, which a priori belongs to some random variable taking values in $[0, \infty]$. However, Fatou’s Lemma guarantees that $F \in \mathfrak{I}_{1}$, which in particular shows that $F$ belongs to a random variable that is almost surely finite. Thus, $\mathfrak{I}_{1}$ is compact. We define an equivalence relation $\sim$ on $\mathfrak{I}_{1}$ by
\[
F \sim G :\iff \exists c > 0 \text{ with } F(ct) = G(t) \text{ for all } t \geq 0.
\]
The distribution function $F_{0}$ of a random variable that is identically zero, i.e. $F_{0}(t) := 1$ for all $t \geq 0$, is its own equivalence class. For each $G \in \mathfrak{I}_{1} \setminus \{ F_{0} \}$ there is a unique element $F \in \mathfrak{I}_{1}$ such that $G \sim F$, namely $F(t) = G(M_{G}t)$ with $M_{G}$ as in the proof of Theorem 2.2. Consequently, the compact quotient space $\mathfrak{I}_{1}/\sim$ can be identified with $\{ F_{0} \} \cup \mathfrak{I}_{1}$. The derived identification of elements in $\mathcal{L}$ with pairs $(b, \mu)$ can be viewed as a homeomorphism between the set $M_{1}^{1}(\mathfrak{I}_{1}/\sim)$ of probability measures on $\mathfrak{I}_{1}/\sim$ and $\mathcal{L}$, given by
\[
\delta_{F_{0}} \mapsto \ell_{\Pi}, \quad b \delta_{F_{0}} + (1 - b) \mu \mapsto b \ell_{\Pi} + (1 - b) \int_{\mathfrak{I}_{1}} \ell_{F} \mu(dF),
\]
where $(b, \mu) \in [0,1) \times M_{1}^{1}(\mathfrak{I}_{1})$, with $b$ denoting the probability mass on $F_{0}$. See the Appendix for details.

(c) One noticeable aspect about the topology on $\mathcal{L}$ is that the seemingly isolated point $\ell_{\Pi}$ is in fact not isolated. The sequence $\{ \ell_{F_{n}} \}_{n \in \mathbb{N}} \subset \partial \mathcal{L} \cap \{ \ell_{\Pi} \}$ converges (pointwise) to $\ell_{\Pi}$ if and only if $\{ F_{n} \}_{n \in \mathbb{N}} \subset \mathfrak{I}_{1}$ converges (pointwise) to $F_{0}$. To see this, we point out for $F \in \mathfrak{I}_{1}$ that
\[
\ell_{F}^{(2)}(1,1) = 2 - \int_{0}^{\infty} (1 - F_{n}(s))^{2} \, ds = 2 - ||F_{0} - F||^{2}_{L_{2}}. \tag{9}\]

9$\mathfrak{I}_{1}/\sim$ is the (compact) image of the compact set $\mathfrak{I}_{1}$ under the continuous quotient map.
Thus, $\ell_{F_n}^{(2)}(1, 1)$ converges to 2 as $n \to \infty$ if and only if $F_n$ converges pointwise to $F_0$. But the only element $\ell \in \mathcal{L}$ satisfying $\ell^{(2)}(1, 1) = 2$ is $\ell_{\Pi}$, since $\ell_F^{(2)}(1, 1) < 2$ for each $F \in \mathfrak{F}_1$.

(d) The Pickands representation of $\ell^{(d)}$ has been derived in [Mai (2018), Lemma 4]. Furthermore, the Pickands representation of $\ell^{(d)}_{\Pi}$ is well-known to correspond to a uniform distribution on $\{1, \ldots, d\}$. Combining these facts with Theorem 2.2 immediately implies for the random vector $\mathbf{X}^{(d)}$ in (3) associated with $\ell^{(d)}$ for $\ell \in \mathcal{L}$, represented by $(b, \mu)$, that

$$\mathbf{X}^{(d)} \overset{d}{=} \left( \frac{W_1^{(d)}}{||W^{(d)}||_1}, \ldots, \frac{W_d^{(d)}}{||W^{(d)}||_1} \right),$$

where the random vector $\mathbf{W}^{(d)}$ can be simulated as follows:

- Draw a random variable $D$ which is uniformly distributed on $\{1, \ldots, d\}$.
- Draw a Bernoulli random variable with success probability $b$. If success, define $W_k^{(d)} := 1_{\{k = D\}}$ for $k = 1, \ldots, d$ and return. Otherwise, proceed with the following steps.
- Simulate the random distribution function $F = \{F_t\}_{t \geq 0}$ from the probability measure $\mu \in M_1^+(\mathfrak{F}_1)$ and draw a random variable $Z$ with distribution function $t \mapsto \int_0^t s dF_s$, $t \geq 0$.
- Draw iid random variables $Z_1, \ldots, Z_d$ with distribution function $F$.
- Define $W_k^{(d)} := 1_{\{k = D\}} Z + 1_{\{k \neq D\}} Z_k$ for $k = 1, \ldots, d$ and return.

This algorithm to simulate the random vector $\mathbf{X}^{(d)}$ can be used to derive an exact simulation algorithm for the random vector $(Y_1, \ldots, Y_d)$, see [Dombry et al. (2016), Algorithm 1].

(e) The probability law of a non-decreasing, right-continuous, strong IDT process $H = \{H_t\}_{t \geq 0}$, which is not deterministic (i.e. not identically $H_t = bt$ for some $b \geq 0$), is uniquely described by a triplet $(b, c, \mu) \in [0, \infty) \times (0, \infty) \times M_1^+(\mathfrak{F}_1)$, and has the LePage series representation

$$H_t = bt + c \sum_{k \geq 1} \delta_{(\epsilon_1 + \ldots + \epsilon_k, F^{(k)})} - \log \left\{ \frac{F^{(k)}_1}{\epsilon_1 + \ldots + \epsilon_k} \right\},$$

where $\sum_{k \geq 1} \delta_{(\epsilon_1 + \ldots + \epsilon_k, F^{(k)})}$ is a Poisson random measure on $[0, \infty) \times \mathfrak{F}_1$ with mean measure $dt \times \mu$. Lemma [2.1] and the proof of Theorem 2.2 show that other LePage series representations of the form (5) can only differ from this canonical one by changing from the unique measure $\mu \in M_1^+(\mathfrak{F}_1)$ to some $\tilde{\mu} \in M_1^+(\mathfrak{F})$ (and potentially adjusting the constant $c$ accordingly). The unboundedness of $b$ as well as the additional constant $c > 0$ in the triplet $(b, c, \mu)$, when compared to $(b, \mu)$ in
Theorem 2.2 is due to the fact that a stable tail dependence function $\ell$ is normalized to satisfy $\ell(1, 0, 0, \ldots) = 1$ (corresponding to $E[Y_1] = 1$), while the triplet $(b, c, \mu)$ describes the law of an arbitrary non-decreasing, right-continuous strong IDT process $H$ via the relation

$$E \left[ e^{-\sum_{i \geq 1} H_{t_i}} \right] = e^{-b \ell_1(t) - c \int_{F_1} \ell_F(t) \mu(dF)}.$$

(f) Recall that a Lévy subordinator $L = \{L_t\}_{t \geq 0}$, see Bertoin (1999) for a textbook treatment, is a non-decreasing, right-continuous strong IDT process with independent and stationary increments, which implies that its law is fully determined by the law of $L_1$. By the well-known Lévy-Khinchin formula for infinitely divisible distributions, the probability law of $L_1$ is canonically described in terms of a pair $(b_L, \nu_L)$ of a drift constant $b_L \geq 0$ and a Radon measure $\nu_L$ on $(0, \infty]$ subject to the condition $\int_0^1 x \nu_L(dx) < \infty$, the so-called Lévy measure. The (non-deterministic) Lévy subordinator $L$ associated with the pair $(b_L, \nu_L)$ is obtained when specifying $b := b_L$, $c := \int_{(0,\infty]} 1 - \exp(-x) \nu_L(dx)$, and $\mu$ as the law of the random distribution function

$$F_t := e^{-\Theta} + \left(1 - e^{-\Theta}\right) 1_{\{1-e^{-\Theta} \geq 1/t\}}, \quad t \geq 0, \quad \Theta \sim \left(1 - e^{-x}\right) \nu_L(dx)/c.$$

Conditioned on the randomized parameter $\Theta$, this $F$ corresponds to a random variable taking the value $1/(1 - \exp(-\Theta))$ with probability $1 - \exp(-\Theta)$, and the value zero with complementary probability $\exp(-\Theta)$. The random parameter $\Theta$ itself is drawn from the probability measure $(1 - e^{-x}) \nu_L(dx)/c$. Notice that every probability measure on $(0, \infty]$ is possible for $\Theta$, but the law of $\Theta$ is invariant with respect to changes of $c$, that is when changing from $\nu_L$ to $\beta \nu_L$ for some $\beta > 0$.

3 Conclusion

It has been shown that the set $\mathcal{L}$ of infinite-dimensional, symmetric stable tail dependence functions is a simplex. The boundary of the simplex and a respective boundary integral representation for $\ell \in \mathcal{L}$ has been derived in terms of a pair $(b, \mu)$ of a constant $b \in [0, 1]$ and a probability measure $\mu$ on the set of distribution functions of non-negative random variables with unit mean. Equivalently, the pair $(b, \mu)$ was shown to conveniently describe the probability law of a non-decreasing, right-continuous stochastic process which is strongly infinitely divisible with respect to time, subject to a normalizing condition.

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Appendix

We provide a proof for the claim that $M_1^+(\mathfrak{F} \leq 1/\sim)$ is homeomorphic to $\mathfrak{L}$. To this end, we denote the mapping from $M_1^+(\mathfrak{F} \leq 1/\sim)$ to $\mathfrak{L}$ by $\Phi$, and for the sake of a more compact notation denote the element $\delta_{F_0} \in M_1^+(\mathfrak{F} \leq 1/\sim)$ artificially as a pair $(1, \delta_{F_0})$. That $\Phi$ defines a bijection is clear. We have to prove that $(b_n, \mu_n)$ converges to $(b, \mu)$ in $M_1^+(\mathfrak{F} \leq 1/\sim)$ if and only if $\Phi((b_n, \mu_n))$ converges to $\Phi((b, \mu))$ in $\mathfrak{L}$.

$\Rightarrow$ Assume $(b_n, \mu_n)$ converges to $(b, \mu)$ in $M_1^+(\mathfrak{F} \leq 1/\sim)$. We distinguish two cases. First, assume that $(b, \mu) = (1, \delta_{F_0})$. Let $(b_n, \mu_n)$ be the subsequence of elements that are not equal to the limit. If this sequence is finite, we are done. If not, we denote $\ell_n := \Phi((b_n, \mu_n))$ and see that

$$\int_{\mathfrak{F}} \ell_n^{(2)}(1, 1) \mu_{n_k}(dF) \geq 2 - \int_{\mathfrak{F}} (1 - F(0)) \mu_{n_k}(dF),$$

which converges to 2 for $k \to \infty$, since $\mu_{n_k}$ converges to $\delta_{F_0}$. Consequently,

$$\lim_{k \to \infty} \ell_{n_k}^{(2)}(1, 1) \geq \lim_{k \to \infty} b_{n_k}^2 + (1 - b_{n_k})^2 = 2,$$

which shows that $\lim_{k \to \infty} \ell_{n_k} = \ell_{\Pi}$, the only element in $\mathfrak{L}$ with this property. Second, assume that $(b, \mu) \in [0, 1) \times M_1^+(\mathfrak{F}_1)$. This implies that almost all $(b_n, \mu_n) \in [0, 1) \times M_1^+(\mathfrak{F}_1)$ (here we implicitly use that $F_0$ can be separated from any $F \in \mathfrak{F}_1$ by two disjoint open sets in the quotient topology on $\mathfrak{F} \leq 1/\sim$, which is shown separately at the end of the proof). For arbitrary $\vec{t} \in [0, \infty)^N_0$ and $F \in \mathfrak{F}_1$ we have $\ell_{\vec{t}}(\vec{F}) \leq \ell_{|\vec{d}|} d$, with

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the maximal index such that \( t_d > 0 \) and \( t_{[d]} := \max\{t_1, \ldots, t_d\} \). Thus, the bounded convergence theorem gives

\[
\lim_{n \to \infty} b_n \ell_n(t) + (1 - b_n) \int_{\delta_1} \ell_F(t) \mu_n(dF) = b \ell_{0H}(t) + (1 - b) \int_{\delta_1} \ell_F(t) \mu(dF),
\]

hence the claim.

\[ \ll ] Conversely, assume that \( \{\ell_n\}_{n \in \mathbb{N}} \subset \mathcal{L} \) converges to \( \ell \in \mathcal{L} \), and denote \( \Phi^{-1}(\ell) \) by \((\bar{b}, \mu)\). First, we assume again that \( \ell = \ell_{0H} \), and let \( (\ell_n) \) be the subsequence of elements different from the limit. If this subsequence is finite, we are done. If not, we observe that

\[
2 = \lim_{k \to \infty} \ell_n(1, 1) = \lim_{k \to \infty} b_n 2 + (1 - b_n) \left(2 - \int_{\delta_1} \|F_0 - F\|_2^2 \mu_n(dF)\right).
\]

This means that we must have either \( \lim_{k \to \infty} b_n = 1 \) or \( \mu_n \) converges weakly to \( \delta_{F_0} \). In both cases, it follows that \( (b_n, \mu_n) \) converges to \((1, \delta_{F_0})\) in \( M^1_{\mathfrak{F}}(\mathfrak{F} \leq \gamma) \). Second, assume \( \ell \neq \ell_{0H} \). Then \( \ell_n \neq \ell_{0H} \) for almost all \( n \), and we denote the pair \( \Phi^{-1}(\ell_n) \) by \((b_n, \mu_n)\) in \([0,1) \times M^1_{\mathfrak{F}}(\mathfrak{F}_1)\). By Helly's Selection Theorem we find a subsequence \( \mu_n \)
converging to some probability law \( \hat{\mu} \) on \( \mathfrak{F} \leq 1 \). Denoting for a Borel set \( B \subset \mathfrak{F}_1 \) by \([B] \subset \mathfrak{F} \leq 1 \) the set of all elements \( G \in \mathfrak{F} \leq 1 \) such that \( G \sim F \) for some \( F \in B \), we define an element in \( M^1_{\mathfrak{F}}(\mathfrak{F} \leq \gamma) \), written as pair \((\hat{b}, \hat{\mu})\), via

\[
\hat{b} := \hat{\mu}(\{F_0\}), \quad \hat{\mu}(B) := \hat{\mu}([B]).
\]

Without loss of generality, let the subsequence \( \{n_k\}_{k \geq 1} \) be such that the bounded sequence \( b_{n_k} \) converges in \([0,1]\) to some \( \bar{b} \) (otherwise introduce further subsequence). It follows that the measure \( b_{n_k} \delta_{F_0} + (1 - b_{n_k}) \mu_{n_k} \) converges in \( M^1_{\mathfrak{F}}(\mathfrak{F} \leq \gamma) \) to the probability measure

\[
(\bar{b} + (1 - \bar{b}) \hat{b}) \delta_{F_0} + (1 - \bar{b}) (1 - \hat{b}) \hat{\mu},
\]

i.e. \((b_{n_k}, \mu_{n_k})\) converges to \((\bar{b} + (1 - \bar{b}) \hat{b}, \hat{\mu})\) in \( M^1_{\mathfrak{F}}(\mathfrak{F} \leq \gamma) \). By the same bounded convergence argument as before, and with the knowledge that \( \ell_{F_n} \to \ell_{0H} \) if and only if \( F_n \to F_0 \), we obtain for \( t \in [0,\infty)^N \) that

\[
\lim_{k \to \infty} \int_{\delta_1} \ell_{F_n}(t) \mu_{n_k}(dF) = \hat{b} \ell_{0H}(t) + (1 - \hat{b}) \int_{\delta_1} \ell_{F}(t) \hat{\mu}(dF).
\]

Since \( \ell_{n_k}(t) \) converges to \( \ell(t) \) by assumption, we conclude by limit rules that

\[
\hat{b} \ell_{0H}(t) = \lim_{k \to \infty} \left( \ell_{n_k}(t) - (1 - b_{n_k}) \int_{\delta_1} \ell_{F}(t) \mu_{n_k}(dF) \right) \\
= (\bar{b} - (1 - \bar{b}) \hat{b}) \ell_{0H}(t) + \int_{\delta_1} \ell_{F}(t) \left( (1 - b) \mu - (1 - \hat{b}) (1 - \hat{b}) \hat{\mu} \right)(dF)
\]

References
Since the last equation holds for arbitrary $\tilde{t}$, we conclude that
\[ b = \tilde{b} + (1 - \tilde{b}) \hat{b}, \quad \mu = \hat{\mu}. \]

Thus, we have shown that $(b_{nk}, \mu_{nk})$ converges to $(b, \mu)$ in $M^+_1(\tilde{F} \leq 1 \sim)$, as $k \to \infty$. Let $(b_{mk}, \mu_{mk})$ be any (possibly different) convergent subsequence of $(b_n, \mu_n)$, using that $M^+_1(\tilde{F} \leq 1 \sim)$ is compact. By the same proof, we find a subsequence $(b_{mk_i}, \mu_{mk_i})$ converging to $(b, \mu)$. So the limit of $(b_{mk}, \mu_{mk})$ equals $(b, \mu)$. Thus, every convergent subsequence converges to the same limit $(b, \mu)$, hence $(b_n, \mu_n)$ itself converges to $(b, \mu)$.

Finally, it remains to be shown that $F_0$ can be separated from any given $F \in \mathfrak{F}_1$ by two disjoint open sets in the quotient topology on $\tilde{F} \leq 1 \sim$, which has been used implicitly in “$\Rightarrow$” above. For the sake of clarity, when showing this we denote the equivalence class of $F \in \mathfrak{F}_1$ in $\tilde{F} \leq 1 \sim$ by $[F]$ in order to distinguish it from $F$ itself. Fix $\epsilon > 0$ and consider the following subsets of $\tilde{F} \leq 1$:
\[
U_{\epsilon,>} := \{ F \in \tilde{F} \leq 1 : F(0) > 1 - \epsilon \},
\]
\[
U_{\epsilon,<} := \{ F \in \tilde{F} \leq 1 : F(0) < 1 - \epsilon \}.
\]

These two obviously disjoint sets are open, since their complements are easily seen to be closed with respect to the topology of pointwise convergence. Similarly, consider the following disjoint subsets of $\tilde{F} \leq 1$:
\[
[U]_{\epsilon,>} := \{ [F] \in \tilde{F} \leq 1 / \sim : F \in \mathfrak{F}_1, F(0) > 1 - \epsilon \},
\]
\[
[U]_{\epsilon,<} := \{ [F] \in \tilde{F} \leq 1 / \sim : F \in \mathfrak{F}_1, F(0) < 1 - \epsilon \}.
\]

Denoting by $q : \tilde{F} \leq 1 \to \tilde{F} \leq 1 / \sim$ the quotient map, we observe that
\[
q^{-1}([U]_{\epsilon,>} ) = \{ F \in \tilde{F} \leq 1 : F(c \cdot 0) > 1 - \epsilon \text{ for some } c > 0 \} = U_{\epsilon,},
\]
\[
q^{-1}([U]_{\epsilon,<} ) = \{ F \in \tilde{F} \leq 1 : F(c \cdot 0) < 1 - \epsilon \text{ for some } c > 0 \} = U_{\epsilon,<}.
\]

Consequently, $[U]_{\epsilon,>}$ and $[U]_{\epsilon,<}$ are both open in $\tilde{F} \leq 1 / \sim$. It is now clear that for $F \in \mathfrak{F}_1$ the elements $\{ F_0 \}$ and $[F]$ can be separated in $\tilde{F} \leq 1 / \sim$ by the two open sets $[U]_{1-F(0)},<$ and $[U]_{1-F(0)}>,$. 

**References**

