ON THE DYNAMICS OF RANDOM NEURONAL NETWORKS

PHILIPPE ROBERT AND JONATHAN TOUBOUL

Abstract. We study the mean-field limit and stationary distributions of a pulse-coupled network modeling the dynamics of a large neuronal networks. In contrast with the classical integrate-and-fire neuron, we take into account explicitly the intrinsic randomness of firing times. We analyze the behavior of this system for finite networks, and show that this jump process is well-posed and that no explosion occurs, contrasting with the integrate-and-fire model. This well-posedness persists in the thermodynamic limit, and we derive the McKean-Vlasov jump process governing the dynamics of the limit. Stationary distributions are investigated: we show that the system undergoes transitions as a function of the averaged connectivity parameter, and can support trivial states (where the network activity dies out, which is also the unique stationary state of finite networks) and self-sustained activity when connectivity level is sufficiently large, both being possibly stable.

Contents

1. Introduction 1
2. Stochastic Model 5
3. Finite Networks 7
4. Analysis of the McKean Vlasov process 13
5. Mean-Field Asymptotics 16
6. Analysis of Invariant distributions 24
Appendix A. A Few Elementary Technical Results 34
References 36

1. Introduction

We investigate a model of neuronal network in which the state of a neuron $i$ at time $t$ is given by a jump process describing the membrane potential $X_i(t)$ of the cell. Transitions occur either when the cell receives an action potential from another cell in the network, or when the cell is reset to a resting potential after firing a spike. This event occurs randomly at a state-dependent rate. The inspiration for the development of this model is the celebrated leaky integrate-and-fire neuron (see biological motivation below in section 1.2). The original integrate-and-fire model was studied under the assumptions that the membrane potential of each cell is noisy, and that spikes are fired as soon as a fixed threshold is $V_F$ exceeded. It can be described as an hybrid ODE with jumps:

- if $X_i(t^-) < V_F$
  \[ dX_i(t) = -X_i(t)\, dt + \sum_{j \neq i} w_{ji} \, dD_j(t) + dI^c_i(t), \]

\[ Date: \text{October 16, 2014.} \]
— if $X_i(t-) = V_F$

$$dD_j(t) = 1, X_i(t) = V_R,$$

with $f(t-)$ the left-limit of $f(\cdot)$ at $t$ and $D_j(t)$ the number of spikes of cell $j$ up to time $t$ and $w_{ji}$ is the input of cell $j$ sent to cell $i$ when it fires. The quantity $I_i^j(t)$ is the external input to cell $i$.

Caceres, Carrillo and Perthame in [6] have demonstrated the interesting property that in the mean-field limit, the solutions blow-up in finite time. While this may attributed to the instantaneity of spikes firings and their immediate effects on the firing of other cells, it remains a non-trivial issue since this blow-up also occurs when considering propagation delays [7].

It should be noted that in most of the works of the literature, that we briefly summarize in Section 1.3, the stochastic component of this dynamical system is generally the external input ($I_i^j(t)$) which is a Gaussian process describing the effect of the other population of neurons. A more accurate description incorporating the intrinsic spike time variability consists in considering that neurons fire as Poisson processes, with a rate being an increasing function of their voltage. The stochastic model investigated here describes the occurrences of spikes as follows: a cell with membrane potential $x > 0$ fires at rate $b(x)$ where $x \mapsto b(x)$ is a non-decreasing function. Note that the threshold model corresponds to the case $b(x) = 0$ if $x < V_F$ and $b(x) = +\infty$ otherwise. In this context, the analogue of the above differential equations are given by

$$dX_i(t) = -X_i(t) dt + \sum_{j \neq i} W_{ji}(t) N_j^i(b(X_j(t)))(dt) - X_i(t) N_i^j(b(X_i(t)))(dt)$$

where

— $N_j^i(dt)$ denotes a Poisson point process with rate $y \geq 0$;

— $(W_{ij}(t))$ is an i.i.d. sequence of non-negative random variables;

in particular, $V_R$ is set to be 0, after a spike the state of a neuron is 0 (see detailed formulation in Section 2). Note that intrinsic randomness occurs in this model through the firing mechanism with the Poisson processes but also through the synaptic efficacy $W_{ij}(t)$ representing the effect of a spike of neuron $j$ on the membrane potential of neuron $i$.

1.1. **Summary of the results.** The main results of the present paper go into two directions: (i) analysis of finite-sized networks, and (ii) limit of the system as the network size tends to infinity and the analysis of the corresponding invariant distributions.

**Finite Networks**

Concerning finite networks it is shown that if $b(0) > 0$ the there exists a unique non-trivial invariant distribution while if $b(0) = 0$ then, almost surely no spike occurs after some time. The Dirac mass at 0 is the unique invariant distribution in this case. An explicit representation of the invairant distribution is given when $b(\cdot)$ is a constant function.

**Mean-Field Asymptotics**

For large size networks, the variables $(W_{j,i}(t))$ are taken as $(V_{j,i}(t)/N)$ so that
Equation (1) of the evolution of \(X^N\) becomes

\[
dX^N_i(t) = -X^N_i(t) \, dt + \frac{1}{N} \sum_{j \neq i} V_{ji}(t) \mathcal{N}_{b(X^N_j(t))}(dt) - X^N_i(t) \mathcal{N}_{b(X^N_i(t))}(dt)
\]

It is shown that if \(b(\cdot)\) satisfies a growth condition of the type \(b'(x) < \gamma b(x) + c\) together with other technical conditions, then a mean-field result holds. In particular, propagation of chaos occurs and the empirical distribution of the state of the network converges in distribution to the solution of the non-linear stochastic differential equation, the associated McKean-Vlasov process \((Z(t))\)

\[
dZ(t) = -Z(t) \, dt + E(b(Z(t))) \, dt - Z(t-) \mathcal{N}_{b(Z(t-))}(dt).
\]

## Invariant States

The last part of the paper investigates the properties of the invariant distributions of Equation (3) and, when \(b(0) = 0\), the stability properties of the invariant distribution given by \(\delta_0\) the Dirac mass at 0. In the linear case, if \(b(x) = \lambda x\), it is shown that if \(\lambda E(V) < 1\) then \(\delta_0\) is the unique invariant distribution but if \(\lambda E(V) > 1\) there exists a non-trivial invariant distribution for \((Z(t))\). It suggests a quasi-stationary phenomenon: despite the networks is dying with probability 1 for a finite \(N\), in the limit, there seems to be a non-trivial equilibrium before absorption at 0.

More striking, it is also shown that if \(b(x) = \lambda x^\alpha\) with \(\alpha > 1\), then there exists some \(\rho_c > 0\) such that \(\lambda E(V) > \rho_c\) there exists at least two other invariant distributions for \((Z(t))\). Simulations presented at the end seem to suggest that one of them is stable but not the other one and that \(\delta_0\) has a non-empty basin of attraction.

## Methods

The general strategy of the proofs use tools from stochastic calculus for jump processes and mean-field limits, but the singularity of the system make it necessary to adapt these techniques and develop estimates of the values of empirical distribution for powers of \(b(\cdot)\).

The paper is organized as follows. In section 2 we present our model and the main results of the paper, before addressing in section 3 the properties of finite networks. The following sections are devoted to the analysis of the limit as the system size diverges: we analyze the properties of the McKean-Vlasov limit in section 4 and show the convergence of the network towards this limit in section 5. We conclude by the analysis of invariant distributions of the McKean-Vlasov equation in section 5.

1.2. Biological background. Neurons are intrinsically noisy electrically excitable cells, that transmit information through stereotyped electrical impulses, called action potentials, or spikes. Spikes are transmitted through synapses to all connected neurons, which have the effect of either increasing (excitatory) or decreasing (inhibitory) their membrane potential. In the cortex, neurons tends to form large populations of statistically identical cells, receiving the same input and strongly interconnected. These cortical areas (or cortical columns) are of the order of a few millimeters and contain hundreds to hundred of thousands of neurons. In such structures, neurons fire spikes when the difference of electrical potential between the intra- and extra-cellular domains is sufficiently large (hyperpolarization), therefore occuring in response to a sufficient amount of excitatory spikes received by the neuron.
The spiking nature of the neuronal activity motivated the introduction of a simple heuristic model of single cell, the integrate-and-fire neuron, which makes the assumption that the membrane potential linearly integrates the inputs it receives and fires a spike as soon as a fixed voltage threshold is reached. This model, introduced in the beginning of last century by Louis Lapique [20], has given rise to an extensive literature (see e.g. [4, 5] for reviews). The introduction of this model was instrumental in the advances made in the mathematical and computational understanding neurons’ activity. Though overly simplified, this model has allowed to take advantage of the stereotyped, fast nature of spikes to focus on the important problem of spike timings. Moreover, this model, for its relative simplicity, allowed very interesting mathematical developments and yielded better understanding of isolated cells [2].

In biological conditions, it has been observed that neurons display a noisy activity. The first model integrating randomness is due to Gerstein and Mandelbrot [14] who incorporated to the model random spike arrivals as a random walk. This model did not considered the intrinsic nature of the nerve cells firing times, but rather considered that this randomness was due to incoming spikes from cells outside of the network considered, i.e. disregarded the fact that isolated cells actually display an intrinsically noisy activity, i.e. fire irregularly even when disconnected from their network. This seminal model, interesting in many regards, lead to many developments. In particular, diffusion limits of the incoming spike strain using stochastic differential equations [32, 19] lead to the introduction of the celebrated leaky integrate-and-fire model. This type of model has been paramount in the study of noisy integrate-and-fire since then (see e.g. [2, 3, 12, 6] to cite a few) and lead to develop new technique to analyze their singular behavior.

At the level of one cell, it is natural to assume a random arrival of spikes, which in a diffusion limit could be modeled as an additive Brownian noise. But when one is interested in large populations of cells, it no more seems relevant to reject all noise to neurons being outside of the network considered. Randomness of spike times due to intrinsic variability, in that context, is a prominent phenomenon (see e.g. [31]). Another drawback of the classical noisy integrate-and-fire neuron is the presence of a fixed threshold. This feature, abstraction of the actual dynamics of neurons, induces a number of artifacts, among which the so-called avalanche effects occurring in excitatory networks, corresponding to an explosion of the spike rate (all neurons fire instantaneously at the same time inducing again all neurons to fire again). And these phenomena do occur, generically, in integrate-and-fire networks, as shown in [6]. All this context points to the fact that the integrate-and-fire neuron, extremely interesting in order to understand single isolated cells in a non-noisy context, may not be the best model when considering the behavior of very large networks of noisy neurons as appearing in physiological conditions in the brain.

Accounting for the intrinsic randomness of firing times has been a longstanding issue in computational neurosciences. The fact that nerve cells integrate the input received and fire spikes at random times with an intensity depending on their membrane potential, a very natural model is to consider that the membrane potential is rather the solution of a jump process with inhomogeneous Poisson jumps whose intensity is a function of the voltage. This natural model was introduced one century later than the integrate-and-fire model in [8] and called linear-nonlinear
Poisson model. This model, in phase with the basic principle of the integrate and fire, postulates linear integration of the input received, and spike emission according to an inhomogeneous Poisson process whose rate nonlinearly depends on the neuron’s voltage. This model seems particularly well suited in order to represent the neuronal firing, and displayed a good fit with experimental data, allowing precise prediction of spike trains [27, 28]. For these reasons, we will choose in the present study this model as the building block of our networks.

The model we shall study in the present manuscript has the interest of conserving the discrete nature of spikes, allowing to characterize the statistics of spike trains in the limit of large networks. Moreover, taking into account intrinsic noise, beyond its biological interest, ensures well-posedness of the system, allowing to describe the limit without resorting to additional phenomena. Eventually, an interesting property is that the dynamics of the limit equation can be partially characterized, and stationary solutions can be described. We see that depending on the sharpness of the spiking intensity function as well as the average coupling strength, self-sustained spontaneous activity may arise in infinite networks. This phenomenon opens interesting open problems.

1.3. Brief overview of the literature in neural mean-field dynamics. Networks of integrate and fire neurons or jump processes have been chiefly studied mathematically using functional analysis, in a number of settings [6, 7, 24, 26, 25]. Recently, these were complemented by very interesting probabilistic approaches [12, 17]. At the other hand of the spectrum of mathematical models for neuroscience are the detailed conductance-based models. These systems represent accurately the different ionic exchanges at play during the firing of an action potential. Such systems are described by continuous Markov processes, and maybe be analyzed using classical methods for the analysis of particle systems, including the celebrated coupling method [34]. This was the subject of a number of recent works in the domain (see e.g. [37, 38] in the context of spatially extended networks). Although these techniques lead to extremely complex equations, some simple models, representing the firing rate of neurons, are tractable. These models show very interesting phenomenology as a function of noise levels [36]. In a broader context, characterizing the behavior of large interacting systems has a long history in applied mathematics since the works of Kac [18] in the middle of last century. To cite just those results closest from our developments, mean-field limits and well-posedness of the mean-field equations were investigated in a number of situations, and make use of several techniques including coupling methods [33, 34], tightness results [15] for multi-class networks, geometric methods [13] and, appeared during the redaction of the present manuscript with applications to neurosciences, hydrodynamics limits [11].

2. Stochastic Model

We consider a network composed of $N$ neurons, whose state is described by a scalar variable representing its membrane potential $X^N_i(t)$ (i.e., the difference of electric potential between the intra- and extra-cellular domains). This quantity decays exponentially fast towards zero in the absence of input due to the leak currents and ions flowing across the cellular membrane, leading the voltage to its equilibrium value, assumed here to be 0. The timescale of this process is our time unit, i.e., for $1 \leq i \leq N$, if the neuron does not spike and does not receive any
spike in the interval of time \([T, T']\), the membrane potential of neuron \(i\) satisfies the ordinary differential equation:

\[
\frac{dX_i^N(t)}{dt} = -X_i^N(t).
\]

Neurons fire at random times, according to a voltage-dependent Poisson process with rate \(b(x)\) where \(x\) denotes the voltage of the neuron. After spiking, the neuron’s voltage is instantaneously reset to its rest potential \(X_i(t) = 0\), and the voltage of neurons \(j \neq i\) are instantaneously updated: their voltage is added the synaptic coefficient \(W_{ij}\), which are considered to be i.i.d. random variables with law \(F\):

\[
X_j(t) = X_j(t^-) + W_{ij}.
\]

For simplicity, it is assumed that the random variables \((W_{ij}, j \neq i)\) are positive i.i.d. and integrable and that their distribution does not depend on \(i\). In this way, it is easily seen that the process \((X(t))\) is Markov process. If \(x = (x_i) \in \mathbb{R}_+^N\), \(\|x\|\) denotes the \(l^1\) norm:

\[
\|x\| = \sum_{j=1}^N |x^j|
\]

**Evolution Equations.** An equivalent description of \((X(t))\) can be provided in terms of the solution of the following Stochastic Differential Equation (SDE),

\[
dX_i(t) = -X_i(t) \, dt + \sum_{j \neq i} \int_{\mathbb{R}_+^2} z_i \mathbb{1}_{\{0 \leq u \leq b(X_j(t^-))\}} N_j^i(du, dz, dt)
\]

\[
- X_i^N(t^-) \int_{\mathbb{R}_+^2} \mathbb{1}_{\{0 \leq u \leq b(X_j(t^-))\}} N_j^i(du, dz, dt),
\]

where \((N_j^i)\) are independent Poisson processes, for \(1 \leq i \leq N\), \(N_i\) has the intensity measure given by \(du \otimes \mathbb{W}^i(dz) \otimes dt\) where

\[
\mathbb{W}^i(dz) = \otimes_{j=1}^{i-1} \mathbb{W}(dz_j) \otimes \delta_0(dz_i) \otimes \otimes_{j=i+1}^N \mathbb{W}(dz_j)
\]

is the measure corresponding to the result of the emission of a spike by neuron \(i\) on the voltage of all neurons. In the latter expression, \(\delta_0\) is the Dirac distribution at 0 and \(\mathbb{W}(dx)\) is the common distribution of the random variable \((W_{ij}, j \neq i)\) on \(\mathbb{R}_+\) associated with the amount of excitation received by a neuron after a spike of another neuron.

Equivalently, it can be written as

\[
X_i(t) = -\int_0^t X_i(s) \, ds + \mathbb{E}(W_1) \sum_{j \neq i} \int_0^t b(X_j(s)) \, ds
\]

\[
- \int_0^t X_i^N(s)b(X_j(s)) \, ds + M_i(t),
\]

where

\[
M_i(t) = \sum_{j \neq i} \int_{s=0}^t \int_{u=0}^{b(X_i(s^-))} \left[ \int_{z=0}^{+\infty} z_i N_j^i(du, dz, ds) - \mathbb{E}(W_1) \, du \, ds \right]
\]

\[
- \int_{s=0}^t X_i(s^-) \int_{u=0}^{b(X_i(s^-))} \left[ \int_{z=0}^{+\infty} N_i^i(du, dz, ds) - \, du \, ds \right],
\]

\[
\int_{s=0}^t X_i(s^-) \int_{u=0}^{b(X_i(s^-))} \left[ \int_{z=0}^{+\infty} N_i^i(du, dz, ds) - \, du \, ds \right],
\]
is the associated local martingale. See Rogers and Williams \[30\] for example.

**Extinction properties of the network.** From the biological viewpoint, it is natural to assume that \( x \mapsto b(x) \) is a non-decreasing positive function since the higher the potential, the more likely a spike will occur. If the initial state of a neuron is \( x \) and if no spike occurs in the network (no neuron fires) in the time interval \([0, t]\), its state at time \( t \) is equal to \( x \exp(-t) \), and in particular its instantaneous firing rate at this time, \( b(x \exp(-t)) \) decreases to \( b(0) \) if \( t \) diverges. If this later quantity is 0, it may happen that the neuron will not spike with positive probability. In this case, if the components \( x_i(0), 1 \leq i \leq N \) of the initial value of the state of the network are too small, there would be an event of positive probability for which no spike occurs at all. The following lemma provides a sufficient condition on the behavior of the map \( b(x) \) at 0 under which extinction of the network activity does not occur. Theorem 1 below completes this result.

**Lemma 1.** If the condition

\[
\int_{[0,1]} \frac{b(s)}{s} \, ds = + \infty
\]

holds, then a node with a non-zero initial value spikes with probability 1.

**Proof.** The function \( x \mapsto b(x) \) being non-decreasing, Relation (7) also holds when \([0, 1]\) is replaced by \([0, a]\) with \( a > 0 \). Let \( 1 \leq i \leq N \) and \( X_i(0) = x > 0 \) then, if \( \tau_i \) denote the instant (possibly infinite) of the first spike of neuron \( i \), one has

\[
P(\tau_i > t) = \mathbb{E} \left( \exp \left( - \int_0^t b(X_i(s)) \, ds \right) \right)
\leq \exp \left( - \int_0^t b(x e^{-s}) \, ds \right) = \exp \left( - \int_0^{x} \frac{b(s)}{s} \, ds \right)
\]

since the relation \( X_i(t) \geq x \exp(-t) \) holds for \( t \leq \tau \) (the other neurons may only increase the state of neuron \( i \)) and that the function \( x \mapsto b(x) \) is non-decreasing. By letting \( t \) go to infinity, one gets that \( P(\tau_i = +\infty) = 0 \) by Condition 7. \( \square \)

Condition 7 together with the monotonicity and a convenient regularity property imply in fact that \( b(0) \) is positive. The quantity \( b(0) \) is the firing rate of a neuron with a flat potential, it can be seen as a representation of the external noise.

We now investigate the stability of the Markov process \((X_i(t))\). In the absence of spikes, each of the components decreases exponentially to 0 and it is reset to 0 when the corresponding node fires. The ergodicity property seems to be quite likely provided that it is proved that the nodes do not fire too quickly as in the PDE description of Caceres et al \[6\]. The analysis of these properties for finite-size networks are now investigated in order to ensure that these properties hold.

3. **Finite Networks**

In this section the number of neurons will be kept fixed, so we drop the upper index \( N \) throughout the section for simplicity of the notations.
3.1. Recurrence, Ergodicity and Invariant Measures. We start with a technical result related to an estimation of the mean return time in a specific compact set.

**Proposition 1.** There exists $C_0$ such that if

$$T_0 = \inf \left\{ s > 0 : X(t) \in [0, C_0]^N \right\}$$

then, for $X(0) = x = (x_i) \notin [0, C_0]^N$,

$$E_x(T_0) \leq \|x\| = x_1 + \cdots + x_N.$$  

**Proof.** Let

$$F = \left\{ x \in \mathbb{R}_+^N : \sum_{i=1}^{N} x_i [1 + b(x_i)] \leq (N-1)\mathbb{E}(W_1) \sum_{i=1}^{N} b(x_i) + 1 \right\}$$

due to the monotonicity property of the function $x \mapsto b(x)$, $F$ is a compact subset of $[0, C_0]^N$, with $C_0 = N^2\mathbb{E}(W_1) + 1$. Denote

$$T_F = \inf \left\{ u > 0 : X(u) \in F \right\},$$

then clearly $T_0 \leq T_F$.

If $X(0) = x \notin F$ and $t \geq 0$, define $S(t) = \|X(t)\| = X_1(t) + X_2(t) + \cdots + X_N(t)$, then Relation (5) gives the identity

$$S(t) = S(0) + \int_0^t \left[ (N-1)\mathbb{E}(W_1) \sum_{i=1}^{N} b(X_j(u)) - \sum_{i=1}^{N} X_i(u)(1 + b(X_i(u))) \right] \, du + \sum_{i=1}^{N} M_i(t),$$

where $(M_i(t))$ are the local martingales defined by Equation (6).

Assume that $X(0) \notin F$, since $T_F$ is a stopping time, one has

$$E(S(T_F \wedge t) - S(0)) = \mathbb{E} \left( \int_0^{t \wedge T_F} \left[ (N-1)\mathbb{E}(W_1) \sum_{i=1}^{N} b(X_j(u)) - \sum_{i=1}^{N} X_i(u)(1 + b(X_i(u))) \right] \, du \right) \leq -\mathbb{E}(t \wedge T_F).$$

One gets $E(t \wedge T_F) \leq S(0)$ and consequently the desired relation. \(\square\)

To state the stability properties of the Markov process $(X(t))$, the framework of Harris Markov processes now is used. See Nummelin [23] and Asmussen [1] for a general introduction.

**Theorem 1** (Stability).

(1) *If the condition*

$$\int_0^1 \frac{b(s)}{s} \, ds < +\infty,$$

*holds then, almost surely, no spike occurs after some finite time, in particular*

$$\lim_{t \to +\infty} (X_i(t), 1 \leq i \leq N) = 0$$

*and the Dirac mass at 0 is the unique invariant distribution.*
(2) If $b(0) > 0$ and if there exists $K > 0$ such that $W_i \leq K$ a.s., then the
Markov process $(X_i(t), 1 \leq i \leq N)$ is Harris ergodic.

Proof. Assume that Condition (9) holds. The notations of Proposition 1 are used.
Let $X(0) = x \in \mathbb{R}_+^N$, $x \neq 0$. If $X(0) = x \in [0, C_0]^N$, in the proof of Proposition 1 it
has been seen that if $\tau_1$ is the first time neuron $i$ spikes, $1 \leq i \leq N$, then in absence
of spikes of the other nodes,

$$
\mathbb{P}(\tau_1 = +\infty) = \exp \left( - \int_0^{x_i} \frac{b(s)}{s} \, ds \right) \geq \eta \overset{\text{def.}}{=} \exp \left( - \int_0^{C_0} \frac{b(s)}{s} \, ds \right) > 0.
$$

Consequently if $X(0) \in [0, C_0]^N$ there is a positive probability lower-bounded by
$\eta^N$ that none of the nodes spike. From Proposition 1 one gets that if $(X(t))$ leaves $F$ then it returns with probability 1, hence almost surely the process $(X(t))$ will stop having upward jumps after some time, (1) is proved.

Now it is assumed that $b(0) > 0$. The strategy of the proof is as follows, in
absence of spikes of other nodes, the duration of time for the next spike of a node
with value $y > 0$ can be decomposed as the minimum of two random variables
with one of them not depending of $y$. This property provides a way of having a
regeneration mechanism (i.e. forgetting the value $y$). If all nodes proceed along
the same line, then the initial value of the Markov process is forgotten after some time
on some event of positive probability, which gives the key regenerative structure of
a Harris Markov process.

If $X(0) = x = (x_i)$, one denotes by $E_i^1$ an exponential random variable
with parameter $b(0)$ and $E_i^{x_i,i}$ a random distribution such that

$$
\mathbb{P}(E_i^{x_i,i} \geq t) = \exp \left( - \int_0^t \left[ b(x_i e^{-u}) - b(0) \right] \, du \right).
$$

Hence, starting from the initial state $x$, if no other spike occurs before, the first
instant when node $i$ spikes has the same distribution as $E_i^1 \wedge E_i^{x_i,i}$. The variable $E_i^1$
is independent of the variables $(E_i^{x_i,i}, x > 0)$ which can be chosen so that $E_i^{x_i,i} \leq E_i^{x_i,i}$
for $x \geq y$.

For $n \geq 1$, one denotes by $Y_n = (y_{i,n})$ the state of the Markov process $(X(t))$
just after the $n$-th jump/spike. The sequence $(Y_n)$ is the embedded Markov chain,
for $n \geq 1$ $Y_n$ is the state of $(X(t))$ at the instant of the $n$th jump. Let $f$ be some
non-negative Borelian function on $\mathbb{R}_+^N$. If $X(0) = x \in F$, then, since $x_i \leq C_0$,

$$
\mathbb{E}_x(f(Y_1)) \geq \mathbb{E} \left( f(Y_1) \mathbb{1}_{\{E_i^1 \leq \min_{1 \leq i \leq N} (E_i^1 \wedge E_i^{x_i,i}) \}} \right) \geq \mathbb{E} \left( f(Y_1) \mathbb{1}_{A_1} \right),
$$

where

$$
A_1 = \left\{ E_i^1 \leq \min_{1 \leq i \leq N} (E_i^1 \wedge E_i^{x_i,i}) \right\}
$$

and, on $A_1$,

$$
Y_1 = (y_{1,1}, y_{2,1}, \ldots, y_{N,1}) = (0, x_2 e^{-E_2^1} + W_{1,2}, \ldots, x_N e^{-E_N^1} + W_{1,N}).
$$

This inequality is associated to the event that node 1 spikes first on its “$b(0)$-
component” $E_i^1$. As a result the lower bound of the above relation does not depend
on $x_1$ anymore. We proceed in the same way for the second step, with node 2
spiking this time, and since $y_{i,1} \leq C_0 + K$ for $1 \leq i \leq N$,

$$
\mathbb{E}_x(f(Y_2)) \geq \mathbb{E} \left( f(Y_2) \mathbb{1}_{A_1 \cap A_2} \right),
$$

(10)
with
\[ A_2 = \left\{ E_2^3 \leq \min_{1 \leq i \leq N} E_2^1 \land E_2^{C_0+K,i} \right\} \]

and
\[ Y_2 = (W_{2,1}, 0, y_{3,1}e^{-E_2^3} + W_{2,3}, \ldots, y_{N,1}e^{-E_2^3} + W_{2,N}) \]
on the event \( A_1 \cap A_2 \). This time the lower bound \( \mathbb{P} \) does not depend on \( x_1 \) and \( x_2 \). We can proceed recursively, and finally get the relation
\[ \mathbb{E}_x (f(Y_N)) \geq \mathbb{E}(f(Z)\mathbb{1}_A), \quad \forall x \in F, \]
where the random variable \( Z \) and the set \( A \) do not depend on \( x \) in \( F \) and that \( P(A) > 0 \). Consequently \( F \) is a regeneration set of the Markov chain \( (Y_n) \), see Asmussen [1, page 198] for example. The Harris property of the Markov chain has been established. To prove the ergodicity, it is enough to prove that if
\[ T_0^+ = \inf \{ u > 0 : X(u) \in [0,C_0]^N \} \text{ and } \exists v \leq u, X(v) \not\in [0,C_0]^N \}, \]
is the first return time to \( F \) after an exit, then
\[ \sup_{x \in F} \mathbb{E}_x(T_0^+) < +\infty. \]
See, for example, Asmussen [1] Theorem 3.2, page 200 and Robert [21], Proposition 8.12, page 221. For \( x \in F \), the first time \( (X(t)) \) is in \( y = (y_i) \) outside \( F \), necessarily \( y_i \leq C_0 + K \), consequently, by the strong Markov property,
\[ \mathbb{E}_x(T_0^+) \leq \sup_{y=(y_i) \not\in F, \max_i y_i \leq C_0 + K} E_y(T_0) \leq C_0 + K, \]
by Proposition [1]. The theorem is proved. \( \square \)

3.2. The State-Independent Process. The case when the firing rate function \( b \) is constant is investigated. In this setting, the neurons spike independently of their state. This is one of the very rare cases where one can get some substantial information on the distribution of the equilibrium of the network.

**Proposition 2.** If the firing rate is constant and equal to \( \lambda > 0 \), then the invariant distribution of the Markov process \( (X(t)) \) is the law of the vector \( (X_1, \ldots, X_N) \) with
\[ X_i = \sum_{j \neq i} \sum_{k \geq 1} W_{ik}^j(s)e^{-s^\lambda} \mathbb{1}_{\{s \leq t_{ik} \}}, \quad 1 \leq i \leq N, \]
where, the non-decreasing sequences \( (t_{jk}, k \geq 1) \) are \( N \) i.i.d. Poisson point processes on \( \mathbb{R}_+ \) with rate \( \lambda \) and the random variables \( (W_{ik}^j, 1 \leq j \leq N) \) are i.i.d. with the same distribution as \( W_{11} \).

**Proof.** The proof relies on a backward coupling argument, see Levin et al. [21] for a general presentation of so called coupling from the past methods and Loynes [22] for one of its early uses. Let \( (N_j, 1 \leq j \leq N) \) be \( N \) i.i.d. Poisson Processes on \( \mathbb{R}_+ \) with rate \( \lambda \), for \( 1 \leq j \leq N, N_j \) is the sequence of instants when the \( j \)th node spikes. Note that the time interval considered is \( (-\infty, +\infty) \). Assume that for some fixed \( T > 0, X_i(-T) = 0 \) for all \( 1 \leq j \leq N \), then, by using the invariance properties of Poisson processes, it is not difficult to see that \( (X_i(0), 1 \leq i \leq N) \) has the same distribution of the state of the network at time \( T \) when it starts empty at time 0.

For \( 1 \leq i \leq N \), if \( t_{i,-1} \) is the last instant of spike of node \( i \) before time 0, then if \( -T \leq t_{i,-1} \) state of this node at time 0 is determined by the spikes of the other
nodes after time \( t_{i-1} \). If node \( j \neq i \) spikes at time \( s, t_{i-1} \leq s \leq 0 \), then the contribution at time 0 for node \( i \) is the value of the spike multiplied by \( \exp(-s) \). Consequently, if \( T \) is sufficiently large the value of \( X(0) \) does not depend on \( T \) and its distribution is the law of the vector given by Relation (11). The proposition is proved.

**Proposition 3.** When the firing rate is constant and equal to \( \lambda \), the Laplace transform of the state of a node at equilibrium is given by, for \( \xi \geq 0 \),

\[
\mathbb{E}(e^{-\xi X_1}) = \int_0^{+\infty} \exp \left( -\lambda(N - 1) \int_0^x \left( 1 - \tilde{W}(\xi e^{-u}) \right) du \right) \lambda e^{-\lambda x} \, dx.
\]

where \( \tilde{W}(\xi) = \mathbb{E}(\exp(-\xi W)) \) is the Laplace transform of \( W \) at \( \xi \).

This formula gives \( \mathbb{P}(X_1 = 0) = 1/N \), which is simply the probability that the node is the last one which spiked. Similarly, the expected value at equilibrium is given by \( \mathbb{E}(X_1) = (N - 1)\mathbb{E}(W)\lambda/(\lambda + 1) \).

**Proof.** With the same notations as before, \( \mathcal{M} = \mathcal{N}_2 + \cdots + \mathcal{N}_N = (s_n) \) is a Poisson process with rate \( \lambda(N - 1) \), and the above proposition gives that

\[
\mathbb{E}(e^{-\xi X_1}) = \mathbb{E} \left( \exp \left( -\xi \sum_{n \geq 1} W_n e^{-s_n} \mathbb{1}_{\{s_n \leq t_{11}\}} \right) \right)
\]

\[
= \mathbb{E} \left( \mathbb{E} \left( \exp \left( -\xi \sum_{n \geq 1} W_n e^{-s_n} \mathbb{1}_{\{s_n \leq t_{11}\}} \right) \right| (s_n), t_{11}) \right)
\]

where \( (W_1(s), s \in \mathcal{M}) \) are i.i.d. with the same distribution as \( W \). Consequently, one obtains that

\[
\mathbb{E}(e^{-\xi X_1}) = \mathbb{E} \left( \prod_{s_n \leq t_{11}} \mathbb{E} \left( \exp \left( -\xi W_1 e^{-s_n} \right) \right| s_n, t_{11}) \right)
\]

\[
= \mathbb{E} \left( \prod_{s_n \leq t_{11}} \tilde{W}(\xi e^{-s_n}) \right) = \mathbb{E} \left( \exp \left( -\int_{0}^{t_{11}} g(u) \mathcal{M}(du) \right) \right)
\]

with

\[
g(u) \overset{\text{def.}}{=} -\log(\tilde{W}(\xi e^{-u})).
\]

This gives the relation

\[
\mathbb{E}(e^{-\xi X_1}) = \int_0^{+\infty} \mathbb{E} \left( \exp \left( -\int_0^x g(u) \mathcal{M}(du) \right) \right) \lambda e^{-\lambda x} \, dx,
\]

since \( t_{11} \) is exponentially distributed with parameter \( \lambda \).

The point process \( \mathcal{M} \) being Poisson with rate \( \lambda(N - 1) \), from a classical formula for its Laplace transform, see Proposition 1.5 of Robert [29] for example, one gets

\[
\mathbb{E} \left( \exp \left( -\int_0^x g(u) \mathcal{M}(du) \right) \right) = \exp \left( -\lambda(N - 1) \int_0^x \left( 1 - e^{-g(u)} \right) du \right).
\]

The Laplace transform of \( X_1 \) can thus be expressed as

\[
\mathbb{E}(e^{-\xi X_1}) = \int_0^{+\infty} \exp \left( -\lambda(N - 1) \int_0^x \left( 1 - \tilde{W}(\xi e^{-u}) \right) du \right) \lambda e^{-\lambda x} \, dx.
\]
The proposition is proved.

One concludes with a limiting regime which will be analyzed in a more general framework in the following. With little effort, it gives an idea of the results which can be obtained when the size of the network gets large, for example that the states of the nodes become independent in the limit. For this limiting regime, the size $N$ of the network goes to infinity and the rescaling is achieved through the values of spikes which are of the order of $1/N$. The proposition shows in fact the mean-field convergence of the invariant distribution of the state of the network. See Sznitman [35] for an introduction on this topic.

**Proposition 4** (A large network at equilibrium for constant firing rate). If the value of a spike is $V_1/N$ for some integrable random variable $V_1$ and $(X_i^N)$ is the vector whose distribution is the equilibrium distribution of the state of the network then

1. the sequence of random variables $(X_i^N)$ converges in distribution to a random variable $X_1^\infty$ whose distribution has the density
$$
\frac{1}{\mathbb{E}(V_1)} \left(1 - \frac{u}{\lambda \mathbb{E}(V_1)}\right)^{\lambda-1}
$$
for $u \in [0, \lambda \mathbb{E}(V_1)]$.

2. For $1 \leq i < j \leq N$, the random variables $X_i^N$ and $X_j^N$ are asymptotically independent when $N$ gets large.

*Proof.* By symmetry, one can take $i = 1$ and $j = 2$, for $\ell = 1, 2$, if
$$
Y_\ell^N \overset{\text{def}}{=} \sum_{j=3}^N \sum_{k \geq 1} \frac{V_{jk}^j}{N} e^{-t_\ell k} \mathbb{I}_{\{t_\ell k \leq t_{\ell 1}\}},
$$
then, as $N$ gets large, the distribution of $(X_1^N, X_2^N)$ is arbitrarily close to the distribution of $(Y_1^N, Y_2^N)$, just because the contribution of the spikes of node 1 to the state of node 2 are of the order of $1/N$ and vice versa. For $\xi_1, \xi_2 \geq 0$, by using the same method as in the proof of the above proposition, one gets that
$$
\mathbb{E}\left(e^{-\xi_1 Y_1^N - \xi_2 Y_2^N}\right) =
\mathbb{E}\left(\exp\left[-\lambda(N-2) \int_0^{+\infty} \left(1 - \hat{V}(\xi_1 N e^{-u} \mathbb{I}_{\{u \leq t_{11}\}}) \hat{V}(\xi_2 N e^{-u} \mathbb{I}_{\{u \leq t_{21}\}})\right) du\right]\right),
$$
where $\hat{V}$ denotes the Laplace transform of $V_1$. The equivalence $1 - \hat{V}(x) \sim x \mathbb{E}(V_1)$ when $x$ goes to 0 gives the relation
$$
\lim_{N \to +\infty} \mathbb{E}\left(e^{-\xi_1 Y_1^N - \xi_2 Y_2^N}\right) = H(\xi_1) H(\xi_2),
$$
with
$$
H(\xi) = \mathbb{E}\left(\exp\left[-\xi \lambda \mathbb{E}(V_1) \left(1 - e^{-t_{11}}\right) du\right]\right).
$$
This gives the asymptotic independence and the identification of the limit. The proposition is proved.

The analysis of the constant firing-rate model is instructive in many regards: it provides a completely solvable model for which no explosion of the firing rate is found. By comparison with the constant firing-rate case, coupling methods may allow to show that there is no explosion of the total firing-rate in the network...
(see [11 Appendix A.1]), i.e. that the probability of occurrence of a large number of spikes in a fixed interval is small. We will come back to this property in the forthcoming section. Let us just state that this is an important from the biological viewpoint: consistently with the actual firing of neurons and in contrast with what happens for the stochastic integrate-and-fire neuron, there is no explosion of the firing rate and non-explosion of the membrane potential.

4. Analysis of the McKean Vlasov process

We shall prove in particular in section 5 that \( (X^N(t)) \) converges in law towards the distribution of the stochastic process \( (X(t)) \) satisfying the SDE:

\[
\frac{dX(t)}{dt} = \left( \mathbb{E}(V)\mathbb{E}[b(X(t))] - X(t) \right) dt - X(t^-) \int_{\{0 \leq u \leq b(X(t^-))\}} N(du, dt, dz).
\]

The object of this section is to show that this McKean-Vlasov equation defines a unique process for all times. Section 6 will characterize its stationary solutions. Throughout this section, we denote for \( (U(t)) \) a locally bounded process and \( T > 0 \) the quantity \( ||U||_T \) defined as

\[
||U||_T = \sup \{ |U(t)|, 0 \leq t \leq T \}.
\]

Lemma 2. If \( b \) is non-decreasing \( C^1 \)-function on \( \mathbb{R}_+ \) and \( P \) is a Poisson process on \( \mathbb{R}_+^2 \) with rate 1, for any non-negative locally bounded Borelian function \( (u(t)) \) there exists a unique solution \( (Z_u(x,t)) \) of the SDE

\[
\frac{dZ_u(x,t)}{dt} = -Z_u(x,t) dt + u(t) dt - Z_u(x,t^-)P([0,b(Z_u(x,t^-))], dt)
\]

with initial condition \( x > 0 \). For any couple of non-negative locally bounded Borelian functions \( u \) and \( v \) on \( \mathbb{R}_+ \), for \( t \leq T \), the relation

\[
\mathbb{E} \left( ||Z_u - Z_v||_t \right) \leq e^{C_T t} \int_0^t ||u - v||_s \, ds
\]

holds with \( C_T = 1 + (x + T||u||_T)(1 + ||v||_T + T||u||_T) \).

Proof. For a non-negative Borelian function \( (u(t)) \), the existence and uniqueness of a solution to the SDE \( (13) \) is straightforward. Let \( u \) and \( v \) be non-negative locally bounded Borelian functions on \( \mathbb{R}_+ \). Note that, almost surely,

\[
||Z_u||_T \leq x + \int_0^T u(s) \, ds \leq x + T||u||_T.
\]

One has to estimate, for \( 0 \leq t \leq T \),

\[
\Delta(t) = \left| \int_0^t Z_u(s^-)P([0,b(Z_u(s^-))], ds) - \int_0^t Z_v(s^-)P([0,b(Z_v(s^-))], ds) \right|
\]

then

\[
\Delta(t) \leq \int_0^t |Z_u(s^-) - Z_v(s^-)|P([0,b(Z_u(s^-))], ds)
\]

\[
+ \int_0^t Z_v(u^-)P([b(Z_u(s^-)), b(Z_v(s^-))], ds)
\]
hence
\[
\mathbb{E}(\|\Delta\|_T) \leq \mathbb{E}\left( \int_0^t \|Z_u - Z_v\|_s \mathcal{P}([0, b(Z_u(s-))], ds) \right) \\
+ \mathbb{E}\left( \int_0^t Z_v(u-) \mathcal{P}([b(Z_u(s-)), b(Z_v(s-))], ds) \right) \\
\leq b(x + T\|u\|_T) \int_0^t \mathbb{E}(\|Z_u - Z_v\|_s) \, ds \\
+ (x + T\|u\|_T) \int_0^t \mathbb{E}(b(Z_v(s)) - b(Z_u(s))) \, ds.
\]

The SDE associated to \( u \) and \( v \) give, for \( 0 \leq t \leq T \) and a convenient constant \( C_T \),
\[
\mathbb{E}(\|Z_u - Z_v\|_T) \leq C_T \int_0^t \mathbb{E}(\|Z_u - Z_v\|_s) \, ds + \int_0^t \|u - v\|_s \, ds,
\]
with \( C_T \) as defined above. Gronwall’s Lemma completes the proof of the lemma. \( \square \)

In order to ensure boundedness of moments, we need some control on the divergence of the map \( b \). This is why we introduce the following assumption, that is actually the first part of our assumptions in section 5 (see (18)).

**MF (a):** The firing rate function \( x \mapsto b(x) \) is assumed to be \( C^1 \), non-decreasing and such that there exist \( \Lambda < 1/[5\mathbb{E}(V)] \) and \( \gamma > 0 \) such that
\[
b'(x) \leq \Lambda b(x) + \gamma
\]

**Lemma 3 (Integrability).** Any possible solution of the mean-field equation \( Z_t \) with integrable initial condition \( Z_0 \) is integrable for all times and has a bounded expectation. Moreover, under assumption **MF (a)**, the expectation \( \mathbb{E}[b^p(Z_t)] \) is bounded by a fixed quantity independent of time for any \( p \in \{1, \ldots , 5\} \) (as soon as \( \mathbb{E}[b^p(Z_0)] < \infty \)).

**Proof.** These properties are simple applications of Gronwall’s lemma. Let \( \mu_t = \mathbb{E}[Z_t] \). We have:
\[
\mu_t = \mu_0 + \int_0^t -\mu_s + \mathbb{E}(V)[b(Z_s)] - \mathbb{E}[Z_s b(Z_s)] \, ds = \mu_0 + \int_0^t -\mu_s + \mathbb{E}[\Phi(Z_s)] \, ds.
\]
with \( \Phi(x) = \mathbb{E}(V)b(x) - x b(x) \sim_{x \to \infty} -x b(x) \), and under the monotony assumption on \( b \), we can find \( \Lambda > 0 \) and \( \gamma > 0 \) such that \( \Phi(x) \leq -\Lambda x + \gamma \), allowing to conclude on the boundedness of the moment by Gronwall’s lemma.

Let us now show that we have bounded moments for \( b^p \) with \( p \in \{1, \ldots , 5\} \). Since, as we show in the proof of lemma 3 in the Appendix, the differential of \( b^p \) is also upperbounded \( \gamma b^p(x) + c \) with \( \gamma < 1/\mathbb{E}(V) \), we deal with the control of \( B_t = b(Z_t) \) and only assuming \( \Lambda < 1/\mathbb{E}(V) \). We have
\[
B_t = B_0 + \int_0^t \mathbb{E}[b'(Z_s)(-Z_s + \mathbb{E}(V)B_s) + (b(0) - b(Z_s))b(Z_s)].
\]
and by Cauchy-Schwarz' inequality using the fact that \( \Lambda \mathbb{E}(V) - 1 < 0 \):

\[
B_t \leq B_0 + \int_0^t \mathbb{E}[-b'(Z_s)Z_s + (b(0) - b(Z_s))b(Z_s)] + \mathbb{E}[b'(Z_s)]\mathbb{E}(V)B_s \, ds
\]

\[
\leq B_0 + \int_0^t \mathbb{E}[(b(0) - b(Z_s) + \gamma \mathbb{E}(V))b(Z_s) + \Lambda \mathbb{E}(V)b(Z_s)^2] \, ds
\]

\[
\leq B_0 + \int_0^t (b(0) + \mathbb{E}(V)\gamma)B_s + (\Lambda \mathbb{E}(V) - 1)B_s^2 \, ds.
\]

We conclude using lemma [1].

**Theorem 2** (Existence and Uniqueness of the McKean-Vlasov Process). If \( (b(y)) \) is non-decreasing \( C^1 \)-function on \( \mathbb{R}^+ \) satisfying assumption \textbf{MF} (a), then for any \( T > 0 \), there exists a unique càdlàg process \( (Z(t)) \) satisfying the stochastic differential equation

\[
dZ(t) = -Z(t) \, dt + \mathbb{E}(b(Z(t))) \, dt - Z(t-)\mathbb{P}([0, b(Z(t-))]) \, dt,
\]

and with initial condition \( x > 0 \).

**Proof.** Lemma 3 provided an a priori bound \( \tilde{C} \) on \( \mathbb{E}[b(Z_t)] \) for \( Z_t \), a solution of the McKean-Vlasov equation. Let us define \( C = \tilde{C} + 1 \). We define by induction the sequence of processes \( (Z_n(t)) \) by \( Z_0(t) = x \exp(-t) \), for \( t \geq 0 \), and

\[
\begin{cases}
  dZ_n(t) = -Z_n(t) \, dt + \mathbb{E}(V)[\mathbb{E}(b(Z_{n-1}(t))) \wedge C] \, dt - Z_1(t-\mathbb{P}([0, b(Z(t-))]) \, dt, \\
  Z_n(0) = x
\end{cases}
\]

for \( n \geq 1 \). It is easy to show that we have:

\[
Z_n(t) \leq xe^{-t} + C(1 - e^{-t}) \leq x + C.
\]

For \( T > 0 \), Lemma 2 and the above relation show that there exists a constant \( C_T \) independent of \( n \) such for \( 0 \leq t \leq T \),

\[
\mathbb{E}(\|Z_{n+1} - Z_n\|_t) \leq C_T \int_0^t \|u_n - u_{n-1}\|_s \, ds,
\]

with \( u_n(t) = \mathbb{E}(b(Z_n(t))) \). This implies that:

\[
\mathbb{E}(\|Z_{n+1} - Z_n\|_t) \leq C_T \int_0^t \mathbb{E}(\|b(Z) - b(Z_{n-1})\|_s) \, ds,
\]

and thanks to the deterministic bound on the sequence of processes \( (Z_n) \), we have:

\[
\mathbb{E}(\|Z_{n+1} - Z_n\|_t) \leq K_T \int_0^t \mathbb{E}(\|Z_n - Z_{n-1}\|_s) \, ds,
\]

with \( K_T = C_T \|b\|_{x+C} \).

We hence have:

\[
\mathbb{E}(\|Z_{n+1} - Z_n\|_t) \leq \frac{(K_T t)^n}{n!} \|z(x, \cdot)\|_T
\]

From this relation one gets that 1) the sequence of processes \( (Z_N(t), 0 \leq t \leq T) \) is converging almost surely uniformly on compact sets to a solution \( (Z(t), 0 \leq t \leq T) \).
of SDE (15). If \((\tilde{Z}(t), 0 \leq t \leq T)\) is another solution of this SDE starting from \(x\), then necessarily \(\tilde{Z}(t) \leq z(t)\) for all \(0 \leq t < T\), consequently the relation
\[
\mathbb{E}\left(\|Z_{n+1} - \tilde{Z}\|_t\right) \leq K_T \int_0^t \mathbb{E}\left(\|Z_n - \tilde{Z}\|_s\right) d s,
\]
holds, hence
\[
\mathbb{E}\left(\|Z - \tilde{Z}\|_t\right) \leq K_T \int_0^t \mathbb{E}\left(\|Z - \tilde{Z}\|_s\right) d s,
\]
so \(Z\) and \(\tilde{Z}\) are identical. We have therefore shown that there exists a unique solution to the equation:
\[
(16)
\begin{cases}
dZ(t) = -Z(t) \, dt + \mathbb{E}(V) \left[\mathbb{E}(b(Z(t))) \wedge C\right] \, dt - Z(t-)P([0, b(Z(t-))], dt), & t > 0 \\
Z(0) = x
\end{cases}
\]
Lemma 3 actually ensures that any solution of that equation is such that \(\mathbb{E}[b(Z_t)] < C\), hence these are also solution of (15), and reciprocally any solution of (15) is also solution of (16), which concludes the proof. □

5. MEAN-FIELD ASYMPTOTICS

In this section, one considers the asymptotic regime of these networks when the number \(N\) of nodes of the network goes to infinity. The interaction between nodes is as follows: for \(1 \leq i \neq j \leq N\), when node \(i\) fires the value of the state of node \(j\) is increased by \(W_{ij}^N = V_{ij}/N\). The variables \((V_{ij})\) are i.i.d. integrable random variables with distribution \(V(dz)\), with a slight abuse of notation \(V\) denotes in the following a random variable with such a distribution. The associated Markov process is denoted by \(X_i^N(t)\). We recall the SDE equations (5) in this context, for \(1 \leq i \leq N\), one has
\[
(17) \quad dX_i^N(t) = -X_i^N(t) \, dt + \frac{1}{N} \sum_{j \neq i} \int_{\mathbb{R}_+^2} z_i \mathbb{1}_{\{0 \leq u \leq b(X_i^N(t-))\}} \mathcal{N}_j(du, dz, dt)
- X_i^N(t-') \int_{\mathbb{R}_+^2} \mathbb{1}_{\{0 \leq u \leq b(X_i^N(t-'))\}} \mathcal{N}_i(du, dz, dt),
\]
where \(\mathcal{N}_i\) is a Poisson processes with intensity measure given by \(du \otimes V(dz) \otimes dt\). The Poisson processes \(\mathcal{N}_j, 1 \leq j \leq N\) are independent.

The empirical distribution is denoted by \((\Lambda_N(t))\), for any continuous function \(\phi\) on \(\mathbb{R}_+\),
\[
\langle \Lambda_N(t), \phi \rangle = \frac{1}{N} \sum_{i=1}^N \phi(X_i^N(t)).
\]

The main result of this section is that under appropriate conditions a mean-field convergence holds: The sequence \((\Lambda_N(t))\) of random measures valued processes converges in distribution to the distribution of the McKean-Vlasov process analyzed in Section 4.

The strategy of the proof is the following: first it is shown that the scaled second moment of the total firing rate of the network
\[
\frac{1}{N} \sum_{i=1}^N b^2(X_i^N(t))
\]
is, with high probability, bounded uniformly on any finite time interval. Then, by using the stochastic evolution equations of \((\Lambda_N(t))\), it is proved, via a coupling method, that for any continuous function \(\phi\) on \(\mathbb{R}_+\) with compact support, the sequence of processes \((\langle \Lambda_N(t), \phi \rangle)\) is tight for the topology of the uniform norm, in particular any of its limiting points is a continuous process.

Concerning the main parameters and the initial state of the network, the assumptions are given below.

**Assumptions MF**

(a) Growth Condition.

The firing rate function \(x \mapsto b(x)\) is assumed to be \(C^1\), non-decreasing and such that there exist \(\gamma < 1/\mathbb{E}(V)\) and \(c > 0\) such that

\[
\tag{18}
b'(x) \leq \gamma b(x) + c
\]

holds for any \(x \geq 0\).

(b) Bounded Support.

The distribution \(V(dx)\) has a bounded support, there exists some \(S_V > 0\) such that \(V([0, S_V]) = 1\).

(c) Initial Conditions.

The random variables \((X_i^N(0), 1 \leq i \leq N)\) are i.i.d. and \(m_0\) denotes their common distribution and

\[
\int_{\mathbb{R}_+} b(u)^2 \, m_0(du) < +\infty.
\]

Assumption (MF-a) implies that, for any \(a > 0\), the ratio \(b(x + a)/b(x)\) is bounded as \(x\) gets large, i.e. a slow growth at infinity. Note that polynomial functions satisfy this assumption. See the proof of Lemma [9] in the Appendix. Additionally, for convenience, it will be assumed that \(b(0) = 0\) in the following. It turns out that the case \(b(0) > 0\) is easier from the point of view of the mean-field analysis of this section. Indeed, in this case, the nodes are “refreshed” at a minimal positive rate, in this way there is a maximal, state independent, interval between two spikes of a given node.

5.1. **Stochastic Evolution Equations for the Empirical Distribution.** Let \(f\) a \(C^1\)-function on \(\mathbb{R}_+\) then, from Equation [17] and Proposition [12] of the Appendix, one gets that, for \(1 \leq i \leq N\),

\[
\tag{19}
f(X_i^N(t)) = f(X_i^N(0)) - \int_0^t X_i^N(u) f'(X_i^N(u)) \, du \\
+ \sum_{j \neq i} \int_0^t \int_{\mathbb{R}_+} \left( f \left( X_i^N(u) + \frac{v}{N} \right) - f(X_i^N(u)) \right) b(X_j^N(u)) \, du V(dv) \\
+ \int_0^t [f(0) - f(X_i^N(u))] \, b(X_i^N(u)) \, du + M_{f,i}^N(t),
\]
where \((M^N_{f,i}(t))\) is the local martingale defined by

\[
\int_{s=0}^{t} \int_{\mathbb{R}_+^2} \left[ f(0) - f(X^N_i(s-)) \right] \left[ \mathbb{I}_{\{0 \leq u \leq b(X^N_i(s-))\}} \mathcal{N}_j(du, dz, ds) - b(X^N_i(s)) \right] ds
\]

\[
+ \sum_{j \neq i} \int_{s=0}^{t} \int_{\mathbb{R}_+^2} \left( f(X^N_i(s) + \frac{z_i}{N}) - f(X^N_i(s-)) \right) \left[ \mathbb{I}_{\{0 \leq u \leq b(X^N_i(s-))\}} \mathcal{N}_j(du, dz, ds) - b(X^N_i(s)) \right] ds dv(dz_i).
\]

Provided that the local martingales \((M^N_i), \ i = 1, \ldots, N\) are locally square integrable, the associated previsible increasing processes are given by

\[
\langle M^N_{f,i}(t) \rangle = \int_0^t \left[ f(0) - f(X^N_i(s)) \right]^2 b(X^N_i(s)) ds
\]

and, for \(1 \leq i < j \leq N,\)

\[
\langle M^N_{f,i}, M^N_{f,j}(t) \rangle = \int_0^t \left[ f(0) - f(X^N_i(s)) \right] \left[ f(X^N_j(s) + \frac{u}{N}) - f(X^N_j(s)) \right] b(X^N_j(s)) ds
\]

\[
+ \sum_{k \neq (i,j)} \int_0^t \int_{\mathbb{R}_+^2} \left[ f(X^N_i(s) + \frac{u}{N}) - f(X^N_i(s)) \right] \left[ f(X^N_j(s) + \frac{v}{N}) - f(X^N_j(s)) \right] V(dv) b(X^N_i(s)) ds
\]

by Proposition 12 of the Appendix. Equation 19 gives therefore the following relation for the empirical measure

\[
\langle \Lambda_N(t, f) \rangle = \langle \Lambda_N(0, f) \rangle - \int_0^t \int_{\mathbb{R}_+^2} \langle \Lambda_N(u, \odot f' (\cdot)) \rangle du
\]

\[
+ N \int_0^t \int_{\mathbb{R}_+^2} \left( \langle \Lambda_N(u, f(\cdot) + \frac{v}{N} - f(\cdot)) \rangle - \langle \Lambda_N(u, b) \rangle \right) du V(dv)
\]

\[
- \int_0^t \left( \langle \Lambda_N(u, f(\cdot) + \frac{v}{N} - f(\cdot)) b(\cdot) \rangle - \langle \Lambda_N(u, (f(0) - f(\cdot)) b(\cdot)) \rangle \right) du + M^N_f(t),
\]

where \((M^N_f(t))\) is the martingale

\[
M^N_f(t) = \frac{1}{N} \sum_{i=1}^{N} M^N_{f,i}(t),
\]
The corresponding previsible increasing process is given by

\[
(24) \quad \langle M^N_f \rangle (t) = \frac{1}{N^2} \left( \sum_{i=1}^N \langle M^N_{f,i} \rangle (t) + 2 \sum_{i<j} \langle M^N_{f,i}, M^N_{f,j} \rangle (t) \right).
\]

5.2. Estimates for the scaled firing rate. In this section it is proved that the scaled second moment of the firing rate

\[
\langle \Lambda_N(t), b^p \rangle = \frac{1}{N} \sum_{i=1}^N b \left( X_i^N(t) \right)^2,
\]

remains with high probability within a compact set. One starts with a result on the boundedness of some of its moments.

**Lemma 4.** Under Assumptions (MF), the relation

\[
\sup_{N \geq 1} \sup_{t \geq 0} \mathbb{E}(\langle \Lambda_N(t), b^p \rangle) < +\infty,
\]

holds.

**Proof.** By Lemma 6 of the Appendix, Assumptions (MF) imply that

\[
(25) \quad b^\delta(x+a)-b^\delta(x) \leq a \left( \gamma_1 b^\delta(x) + c_1 \right), \forall a \in (0, \eta_b) \text{ and } \forall x \geq 0,
\]

for some \( \eta_b > 0 \) and with \( \gamma_1 < 1/\mathbb{E}(V) \).

For \( K > 0 \), let \( \tau_K = \inf\{t \geq 0 : \langle \Lambda_N(t), b^6 \rangle \geq K\} \). Holder’s Inequality shows that for all \( t \geq 0 \) the random variables \( \langle \Lambda_N(t \wedge \tau_K), b^p \rangle, 1 \leq p \leq 6 \), are integrable. By taking \( f = b^5 \) in Equation (22), the optional stopping theorem gives the relation

\[
\mathbb{E}\left( \langle \Lambda_N(t \wedge \tau_K), b^5 \rangle \right) \leq \langle \Lambda_N(0), b^5 \rangle
\]

\[+ N \int_0^t \int_{\mathbb{R}_+} \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b^5 \left( \cdot + \frac{v}{N} \right) - b^5(\cdot) \rangle \langle \Lambda_N(u \wedge \tau_K), b \rangle \right) d\nu(dv)
\]

\[ - \int_0^t \mathbb{E}(\langle \Lambda_N(u \wedge \tau_K), b^5 \rangle) \, du,
\]

Recall that \( S_V \) is an upper bound for the support of the distribution \( V \), \( N \) is chosen sufficiently large so that \( S_V/N \leq \eta_b \), from Inequality (25), one gets

\[
\mathbb{E}\left( \langle \Lambda_N(t \wedge \tau_K), b^5 \rangle \right) \leq \langle \Lambda_N(0), b^5 \rangle
\]

\[+ \gamma_1 \mathbb{E}(V) \int_0^t \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle \langle \Lambda_N(u \wedge \tau_K), b \rangle \right) \, du
\]

\[+ c_1 \mathbb{E}(V) \int_0^t \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b \rangle \right) \, du - \int_0^t \mathbb{E}(\langle \Lambda_N(u \wedge \tau_K), b^5 \rangle) \, du.
\]

Holder’s Inequality and the fact that \( \Lambda_N \) is a probability distribution give

\[
\langle \Lambda_N(u \wedge \tau_K), b \rangle \leq \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle^{1/5} \text{ and } \langle \Lambda_N(u \wedge \tau_K), b^6 \rangle \geq \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle^{6/5}
\]

hence

\[
\mathbb{E}\left( \langle \Lambda_N(t \wedge \tau_K), b^5 \rangle \right) \leq C_0 - (1 - \gamma_1 \mathbb{E}(V)) \int_0^t \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle^{6/5} \right) \, du
\]

\[+ c_1 \mathbb{E}(V) \int_0^t \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle^{1/5} \right) \, du,
\]

\[
\langle \Lambda_N(t \wedge \tau_K), b^6 \rangle \leq C_0 - (1 - \gamma_1 \mathbb{E}(V)) \int_0^t \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle^{6/5} \right) \, du
\]

\[+ c_1 \mathbb{E}(V) \int_0^t \mathbb{E}\left( \langle \Lambda_N(u \wedge \tau_K), b^5 \rangle^{1/5} \right) \, du,
\]
where
\[ C_0 \overset{\text{def}}{=} \sup_{N \geq 1} \mathbb{E}(\langle \Lambda_N(0), b^5 \rangle). \]

with again Holder’s Inequality and the fact that \( \gamma_1 \mathbb{E}(V) < 1 \), one gets finally
\[
\mathbb{E}(\langle \Lambda_N(t \wedge \tau_K), b^5 \rangle) \leq C_0 - (1 - \gamma_1 \mathbb{E}(V)) \int_0^t \left[ \mathbb{E}(\langle \Lambda_N(u \wedge \tau_K), b^5 \rangle)^{5/6} \right] du \\
+ c_1 \mathbb{E}(V) \int_0^t \left[ \mathbb{E}(\langle \Lambda_N(u \wedge \tau_K), b^5 \rangle) \right]^{1/5} du,
\]

By using the inequality \( \gamma_1 \mathbb{E}(V) < 1 \) and Proposition 11 of the Appendix, one gets that there exists a finite constant \( C_0 \) independent of \( K \) such that
\[
\sup_{N \geq 1} \sup_{t \geq 0} \mathbb{E}(\langle \Lambda_N(t \wedge \tau_K), b^5 \rangle) \leq C_0,
\]
on concludes the proof by letting \( K \) go to infinity. \( \square \)

**Proposition 5** (Control of the scaled firing rate). Under Assumptions (MF), for any \( T > 0 \) there exists some constant \( C_T \) such that the
\[
\lim_{N \to +\infty} \mathbb{P}\left( \sup_{0 \leq t \leq T} \langle \Lambda_N(t), b^2 \rangle \geq C_T \right) = 0.
\]

Proof. One first shows that there exists some constant \( C_T \) such that,
\[
\sup_{0 \leq t \leq T} \mathbb{E}(M_N^T(t)^2) = \mathbb{E}(\langle M_N^T(t) \rangle^2) \leq \frac{C_T}{N}.
\]
The relations (24) and (21) are used in the case \( f = b^2 \). The first term of Relation (21) for \( \mathbb{E}(\langle M_N^T(t) \rangle^2) \) in this case is the sum of
\[
\frac{1}{N^2} \sum_{i=1}^N \int_0^t \mathbb{E}\left[ b(X_i^N(u))^5 \right] du = \frac{1}{N} \int_0^t \mathbb{E}(\langle \Lambda_N(u), b^5 \rangle) du
\]
and of
\[
\frac{1}{N^2} \sum_{1 \leq i \neq j \leq N} \int_0^t \int_{\mathbb{R}_+} \mathbb{E}\left[ b(X_i^N(u)) \left( b \left( X_j^N(u) + \frac{u}{N} \right)^2 - b(X_i^N(u))^2 \right) \right] du dv.
\]
As before, if \( N \) is sufficiently large so that \( S_N/N \leq \eta_b \), Equation (43) of the Appendix gives that this last term is upper bounded by
\[
\frac{\mathbb{E}(V^2)}{N} \int_0^t \gamma_1 \mathbb{E}(\langle \Lambda_N(u), b^5 \rangle) + 2 \gamma_1 c_1 \mathbb{E}(\langle \Lambda_N(u), b^5 \rangle) + c_1 \mathbb{E}(\langle \Lambda_N(u), b \rangle) du.
\]

With similar arguments, analogous bounds can be obtained for the second term of Relation (21) for \( \mathbb{E}(\langle M_N^T(t) \rangle^2) \) involving the \( \mathbb{E}(\langle M_N^T, M_{N^2}^T \rangle(t)) \). Holder’s Inequality and Lemma 4 show that there exists a constant \( C_T \) such that the upper bound (27) holds. Define
\[
M_{b^2}^{N,*}(T) \overset{\text{def}}{=} \sup_{0 \leq s \leq T} M_{b^2}^N(s),
\]

Doob’s Inequality shows therefore that, for any \( \epsilon > 0 \), there exists \( N_0 \) such that if \( N \geq N_0 \) then \( \mathbb{P}(M_{b^2}^{N,*}(T) > 1) \leq \epsilon \). With \( f = b^2 \) in Equation (22), one gets that,
for $0 \leq t \leq T$,
\begin{equation}
\langle \Lambda_N(t), b^2 \rangle \leq \langle \Lambda_N(0), b^2 \rangle + M^{N^*}(T)
+ N \int_0^t \int_{\mathbb{R}^+} \langle \Lambda_N(u), b^2 \left( \cdot + \frac{v}{N} \right) \rangle \langle \Lambda_N(u), b \rangle \, d\mu_V(dv) \\
- \int_0^t \langle \Lambda_N(u), b^3 \rangle \, du.
\end{equation}

By the integrability condition of $b^2$ with respect to $m_0$ in Assumptions (MF), if $N$ is sufficiently large then, by the law of large numbers,
\[ P \left( \langle \Lambda_N(0), b^2 \rangle > 1 + \mathbb{E} \left[ b^2(X_1^0) \right] \right) < \varepsilon. \]

Let $C_0 \equiv 2 + \mathbb{E}(b^2(X_1^0)))$. On the event $\{M^{N^*}(t) \leq 1\}$, by using again Equation (13) of the Appendix and Relation (27), Relation (30) gives the inequality
\begin{equation}
\langle \Lambda_N(t), b^2 \rangle \leq C_0 + (\gamma_1 \mathbb{E}(V) - 1) \int_0^t \int_{\mathbb{R}^+} \langle \Lambda_N(u), b^2 \rangle^{3/2} \, d\mu_V(dv)
+ c_1 \int_0^t \langle \Lambda_N(u), b^2 \rangle^{1/2} \, du.
\end{equation}

One concludes with Proposition 11 of the Appendix. \qed

5.3. Mean-Field Convergence. If $(H(s))$ is some locally bounded process on $\mathbb{R}^+$, one denotes, for $t \geq 0$
\[ \|H\|_{2,t} = \mathbb{E} \left( \sup_{0 \leq s \leq t} |H(s)|^2 \right) \]
and, with a slight abuse of notation, if $(G(t)) = (G_i(t), 1 \leq i \leq N)$,
\[ \|G\|_{2,t} = \frac{1}{N} \sum_{i=1}^N \|G_i\|_{2,t}. \]

The main result of this section is the following theorem.

**Theorem 3** (Mean-Field Convergence). Under Assumptions (MF) the sequence of empirical processes $(\Lambda_N(t))$ converges in distribution to the law of the unique process $(Z(t))$ with initial distribution $m_0$ and solution of the SDE [15] with $\alpha = \mathbb{E}(V_1)$.

**Proof.** By Theorem 2 there exist stochastic processes $(Y^N_i(t), 1 \leq i \leq N)$ such that
\begin{equation}
Y^N_i(t) = Y^N_i(0) - \int_0^t Y^N_i(u) \, du + \mathbb{E}(V) \int_0^t \mathbb{E}[b(Y^N_i(u))] \, du \\
- \int_0^t Y^N_i(u) \int_{\mathbb{R}^+} \mathbb{1}(0 \leq v \leq b(Y^N_i(u-))) \mathcal{N}_i(dv, dz, du).
\end{equation}

with initial condition $Y^N_i(0) = X^N_i(0)$. The random variables $(X^N_i(0), 1 \leq i \leq N$ and the Poisson processes $(\mathcal{N}_i, 1 \leq i \leq N)$ being i.i.d. and independent, the processes $(Y^N_i(t), 1 \leq i \leq N$ are also i.i.d. Throughout this section we will denote
by $\Lambda_N(t)$ the empirical distribution associated to $(Y^N_i(t))$.\[ \text{RANDOM INTEGRATE-AND-FIRE NEURONS} \]
and, from Equation (21),

\[ X_i(t) = X_i(0) - \int_0^t X_i(u) \, du + \frac{E(V)}{N} \sum_{j \neq i} \int_0^t b(X_j(u)) \, du \]

\[ - \int_0^t X_i(u- \big) \int_{\mathbb{R}_+^2} 1_{\{0 \leq v \leq b(X_i(u-)} \big) J_i(dv, dz, ds) + M_{i,i}^N(t), \]

and, from Equation (21),

\[ \langle M_{i,i}^N \rangle(t) = \frac{E(V^2)}{N^2} \sum_{j \neq i} \int_0^t b(X_j(u)) \, du. \]

By substracting Equation (32) and (33), one gets, for \(0 \leq t \leq T\),

\[ \|X_i^N - Y_i^N\|_{2,t} \leq 5T \int_0^t \|X_i^N - Y_i^N\|_{2,u} \, du + 5E(V) [A_N(t) + TC_N] + B_{N,i}(t) + \|M_{i,i}^N\|_{2,t}, \]

with

\[ A_N(t) = E \left( \left( \int_0^t \frac{1}{N} \sum_{i=1}^N |b(X_i^N(s)) - b(Y_i^N(s))| \, ds \right)^2 \right) \]

\[ B_{N,i}(t) = E \left[ \left( \int_0^t \int_{\mathbb{R}_+^2} |Y_i^N(s-1) 1_{\{0 \leq u \leq b(Y_i^N(s-)} \big) - X_i^N(s-1) 1_{\{0 \leq u \leq b(X_i^N(s-)} \big) \right| J_i(du, dz, ds) \right)^2 \right] \]

\[ C_N = \int_0^T E \left[ \left( \frac{1}{N} \sum_{j \neq i} b(Y_j^N(s)) - E[b(Y_i^N(s))] \right)^2 \right] \, ds. \]

Estimates for these three sequences of processes are now derived. The monotonicity and differentiability properties of \(b(\cdot)\) and Assumption (18) of (MF) show that for \(x\) and \(y \geq 0\),

\[ |b(x) - b(y)| \leq (\gamma(b(x) + b(y)) + c)|x - y|. \]

by using repeatedly Cauchy-Shwartz’s Inequality, one deduces the relation

\[ \int_0^t \frac{1}{N} \sum_{i=1}^N |b(X_i^N(u)) - b(Y_i^N(u))| \, du \]

\[ \leq \int_0^t \sqrt{\frac{1}{N} \sum_{i=1}^N |X_i^N(u) - Y_i^N(u)|^2} \sqrt{\frac{1}{N} \sum_{i=1}^N (\gamma(b(X_i^N(u)) + b(Y_i^N(u)) + c)^2 \, du \]

\[ \leq \sqrt{\int_0^t \frac{1}{N} \sum_{i=1}^N |X_i^N(u) - Y_i^N(u)|^2} \, du \sqrt{\int_0^t \frac{1}{N} \sum_{i=1}^N (\gamma(b(X_i^N(u)) + b(Y_i^N(u))) + c)^2 \, du \]
Let
\[ \mathcal{E}_N = \left\{ \sup_{0 \leq t \leq T} \langle \Lambda_N(t), b^2 \rangle \leq C_0, \quad \sup_{0 \leq t \leq T} \langle \Lambda_N^X(t), b^2 \rangle \leq C_1 \right\}, \]
and \( \mathcal{E}_N^c \) denotes the complementary set, then, for \( 0 \leq t \leq T \),
\[
E \left[ \left( \int_0^t \frac{1}{N} \sum_{i=1}^N |b(X_i^N(u)) - b(Y_i^N(u))| \, du \right)^2 \right] \\
\leq E \left[ \int_0^t \frac{1}{N} \sum_{i=1}^N |X_i^N(u) - Y_i^N(u)|^2 \, du \int_0^t \frac{1}{N} \sum_{i=1}^N (\gamma(b(X_i^N(u)) + b(Y_i^N(u))) + c)^2 \, du \mathbb{1}_{\mathcal{E}_N^c} \right] \\
+ E \left[ \left( \int_0^t \frac{1}{N} \sum_{i=1}^N |b(X_i^N(u)) - b(Y_i^N(u))| \, du \right)^2 \mathbb{1}_{\mathcal{E}_N} \right].
\]
The first term of the right hand side of this relation is bounded by
\[
3T(\gamma^2(C_0 + C_1) + c) \int_0^t E \left[ \left( \langle \Lambda_N(u), b^2 \rangle + \langle \Lambda_N^X(u), b^2 \rangle \right) \mathbb{1}_{\mathcal{E}_N^c} \right] \, du,
\]
and the second term is bounded by
\[
(36) \quad 4T \int_0^T E \left[ \left( \langle \Lambda_N(t), b^2 \rangle + \langle \Lambda_N^X(t), b^2 \rangle \right) \mathbb{1}_{\mathcal{E}_N^c} \right] \, du,
\]
again by Holder’s Inequality, for \( 0 \leq t \leq T \),
\[
E \left[ \langle \Lambda_N(t), b^2 \rangle \mathbb{1}_{\mathcal{E}_N^c} \right] \leq \sqrt{P(\mathcal{E}_N)} \sqrt{E(\langle \Lambda_N(t), b^4 \rangle)}
\]
a similar identity also holds for \( \langle \Lambda_N^X(s), b^2 \rangle \). Lemma 4 and Proposition 5 show that the quantity \( \langle \Lambda_N^X(s), b^2 \rangle \) is arbitrarily small if \( N \) is sufficiently large. By gathering these results, one gets that there exists constants \( D_0 \) and \( D_1 \)
\[
(37) \quad A_N(t) \leq D_0 P(\mathcal{E}_N)^{1/3} + D_1 \int_0^t \frac{1}{N} \sum_{i=1}^N \| X_i^N - Y_i^N \|_{2,s} \, ds
\]
With the same procedure as for the estimation of \( A_N(t) \) and similar arguments as in the proof of Proposition 5, the quantity
\[
\frac{1}{N} \sum_{i=1}^N B_{N,i}(t)
\]
also satisfies an inequality of the form (37).
The processes \( (Y_i^N(t)), 1 \leq i \leq N \) being i.i.d. and their common distribution does not depend on \( N \), the quantity \( C_N \) is given by
\[
\frac{1}{N^2} \int_0^T ((N - 1) \text{Var} \left[ b(Y_i^1(s)) \right] + E \left[ b(Y_i^1(s)) \right]) \, ds,
\]
consequently, \( C_N \) converges to 0 as \( N \) gets large.
Concerning the last term of Relation (35)
\[
\frac{1}{N} \sum_{i=1}^N \| M_i^N \|_t \leq \frac{E(V^2)}{N} \int_0^t E[\langle \Lambda_N(s), b \rangle] \, ds \leq \frac{E(V^2)T}{N} \sup_{N \geq 1} \sup_{0 \leq t \leq T} E \left( \langle \Lambda_N^X(s), b \rangle \right)
\]
and the last term is finite according to Lemma 4.
To summarize, there is a constant $D$ such that for any $\varepsilon > 0$, there exists some $N_1 > 0$ such that if $N \geq N_1$ and
\[
\|X^N - Y^N\|_{2,t} \leq \varepsilon + D \int_0^t \|X^N - Y^N\|_{2,u} \, du, \quad 0 \leq t \leq T.
\]
With Gronwall’s Inequality, one deduces that
\[
\lim_{N \to +\infty} \|X^N - Y^N\|_{2,t} = 0, \quad 0 \leq t \leq T.
\]
Now let $\phi$ be a $C_1$-function with compact support on $\mathbb{R}_+$, then
\[
E \left[ \sup_{0 \leq t \leq T} \left( \langle \Lambda_N(t), \phi \rangle - \langle \Lambda_Y^N(t), \phi \rangle \right)^2 \right] \leq \|\phi'\|_{\infty} \|X^N - Y^N\|_{2,t}
\]
Theorem 3.7.1 of Dawson [10] shows that the sequence of measure-valued processes $(\langle \Lambda_N(t) \rangle)$ is tight and converging to the distribution of $(Y_1(t))$. The theorem is proved. □

6. Analysis of Invariant Distributions

This section is devoted to the analysis of the invariant distributions of the McKean-Vlasov process. We shall denote in this section $\pi$ a distribution on $\mathbb{R}_+$ which is invariant along the McKean-Vlasov evolution. When $X(0)$ has the distribution $\pi$, the law of $X(t)$ is $\pi$ for all times $t \geq 0$, and hence the map $t \mapsto E(b(X(t)))$ is constant. It is then clear that, denoting $\alpha = E(V)E(b(X(0)))$, the stationary process $(X(t))$ can be seen as the solution $(Y(t))$ of the SDE
\[
\text{(38)} \quad dY(t) = [\alpha - Y(t)] \, dt - Y(t-) \int \mathbb{1}_{\{0 \leq u \leq b(Y(t-))\}} \, \mathcal{N}(du, dt, dz).
\]
Define
\[
\tau = \inf\{t > 0 : Y(t) = 0\}
\]
and $x(t) = \alpha(1 - \exp(-t))$.

The variable $\tau$ is the instant of the first spike of the neuron. If $Y(0) = 0$, before time $\tau$ the evolution of $(Y(t))$ is deterministic, one has in fact $Y(t) = x(t)$ for $t < \tau$. The Poisson property gives that
\[
P(\tau \geq t) = \exp \left( - \int_0^t b(x(u)) \, du \right)
\]
The invariant distribution $\pi$ of $(Y(t))$ can then be expressed as
\[
\pi(f) = \frac{1}{E(\tau)} E \left( \int_0^\tau f(x(u)) \, du \right),
\]
if $f$ is some continuous function with compact support on $[0, \alpha)$. By Fubini’s Theorem
\[
E \left( \int_0^\tau f(x(u)) \, du \right) = \int_0^{+\infty} f(x(u)) P(\tau \geq u) \, du
\]
\[
= \int_0^{+\infty} f(x(u)) \exp \left( - \int_0^u b(x(v)) \, dv \right) du
\]
\[
= \int_0^\alpha \frac{f(u)}{\alpha - u} \exp \left( - \int_0^u \frac{b(v)}{\alpha - v} \, dv \right) du.
\]
The measure has a compact support $$[0, \alpha)$$, it has finite mass if and only if

$$\mathbb{E}(\tau) = \int_0^\alpha \frac{1}{\alpha - u} \exp \left( - \int_0^u \frac{b(v)}{\alpha - v} \, dv \right) \, du < +\infty.$$  

**Theorem 4.** The invariant distribution of the solution of SDE [12] has density

$$u \mapsto \frac{1}{C(\beta)(\beta \mathbb{E}(V) - u)} \exp \left( - \int_0^u \frac{b(v)}{\beta \mathbb{E}(V) - v} \, dv \right)$$

on $$[0, \beta \mathbb{E}(V))$$, where

$$C(\beta) = \int_0^{\beta \mathbb{E}(V)} \frac{1}{\beta \mathbb{E}(V) - u} \exp \left( - \int_0^u \frac{b(v)}{\beta \mathbb{E}(V) - v} \, dv \right) \, du$$

and $$\beta$$ is the solution of the equation

$$\beta C(\beta) = 1 - \exp \left( - \int_0^{\beta \mathbb{E}(V)} \frac{b(v)}{\beta \mathbb{E}(V) - v} \, dv \right).$$

**Remark.** In the case where $$\beta \mathbb{E}(V) > 0$$, the term $$\exp \left( - \int_0^{\beta \mathbb{E}(V)} \frac{b(v)}{\beta \mathbb{E}(V) - v} \, dv \right)$$ vanishes and the fixed point equation reduces to:

$$\beta C(\beta) = 1.$$

Moreover, the change of variable $$x = u/\beta \mathbb{E}(V)$$ and $$y = v/\beta \mathbb{E}(V)$$ yields the simplified formulation of $$C(\beta)$$:

$$C(\beta) = \frac{1}{\beta \mathbb{E}(V)} \int_0^1 \frac{1}{1 - x} \exp \left( - \int_0^x \frac{b(\beta \mathbb{E}(V)y)}{1 - y} \, dy \right) \, dx.$$  

We shall now analyze the behavior of the map $$\Psi : \beta \mapsto \beta C(\beta)$$ in order to characterize the number of possible stationary distributions.

**Proposition 6.** For any $$b$$ satisfying our assumptions, $$\lim_{\beta \to \infty} \Psi(\beta) = \infty$$. At $$\beta = 0$$, the behavior of the map $$\Psi$$ depends on the typical behavior of $$b(x)$$ at $$x = 0$$

- if $$\frac{b(x)}{x} \to \infty$$ at $$x = 0$$, $$\lim_{\beta \to 0} \Psi(\beta) = 0$$. This is for instance the case when $$b(0) > 0$$.
- if $$\frac{b(x)}{x} \to \lambda \in \mathbb{R}^*_+$$, $$\lim_{\beta \to 0} \Psi(\beta) = \frac{1}{\lambda \mathbb{E}(V)}$$
- if $$\frac{b(x)}{x} \to 0$$, $$\lim_{\beta \to 0} \Psi(\beta) = \infty$$

**Proof.** Let us start by analyzing the behavior of $$\Psi$$ at infinity. Because $$b$$ is a non-decreasing function, its primitive $$B(x) = \int_0^x b(y) \, dy$$ is upperbounded by $$xb(x)$$. Let us fix $$\delta \in (0, 1)$$, we have:

$$\Psi(\beta) \geq \beta \int_0^\delta \frac{1}{1 - x} \exp \left( - \frac{1}{1 - \delta} \frac{B(\beta \mathbb{E}(V)\delta)}{\beta \mathbb{E}(V)} \right) \, dx$$

$$\geq -\beta \log(1 - \delta) \exp \left( - \frac{\delta}{1 - \delta} b(\beta \mathbb{E}(V)\delta) \right)$$

and simply taking, for arbitrary $$C > 0$$, $$\delta = C/\beta$$ (for $$\beta > C$$), we obtain

$$\Psi(\beta) \geq -\beta \log(1 - C/\beta) \exp \left( - \frac{C}{\beta - C} b(C \mathbb{E}(V)) \right).$$

It is then easy to see that from this formula that:

$$\lim_{\beta \to \infty} \Psi(\beta) \geq C.$$
and since \( C \) is arbitrary, this precisely means that \( \Psi \to \infty \).

Let us now analyze the behavior of \( \Psi(\beta) \) at \( \beta = 0 \) as a function of the limit of \( b(x)/x \) at 0.

— If \( b(x)/x \to 0 \) at \( x = 0 \), then for any \( \delta > 0 \), there exists \( \beta(\delta) \) such that \( b(\beta \mathbb{E}(V))/\beta \mathbb{E}(V) \leq \delta \) for \( \beta \leq \beta(\delta) \), and therefore for such \( \beta \leq \beta(\delta) \) we have:

\[
\Psi(\beta) \geq \beta \int_0^1 \frac{1}{1-x} \exp \left( -\alpha \beta \mathbb{E}(V) \int_0^x \frac{y}{1-y} dy \right) dx \geq \frac{1}{\delta \mathbb{E}(V)}
\]

which proves that \( \Psi(\beta) \to \infty \) at \( \beta = 0 \).

— If \( b(x)/x \to \infty \) at \( x = 0 \), then for any \( \delta > 0 \) we have for \( \beta \) small enough \( b(\beta \mathbb{E}(V)y) \geq \delta \beta \mathbb{E}(V)y \), and therefore:

\[
\beta C(\beta) \leq \beta \int_0^1 (1-x)^{\delta \mathbb{E}(V)\beta-1} \exp(\delta \beta \mathbb{E}(V)x) dx
\]

\[
= \frac{1}{\delta \mathbb{E}(V)} + \beta \int_0^1 (1-x)^{\delta \mathbb{E}(V)\beta} e^{\delta \beta \mathbb{E}(V)x} dx
\]

and therefore we have \( \lim_{\beta \to 0} \beta C(\beta) \leq \frac{1}{\mathbb{E}(V)} \) for arbitrarily large \( \delta \), showing that \( \Psi(\beta) \to 0 \) at \( \beta = 0 \).

— When \( b(x)/x \to \lambda \) when \( x \to 0 \), it is easy to show using the same estimates as in the two previous cases that for any \( \delta > 0 \) small enough we have

\[
\frac{1}{(\lambda + \delta)\mathbb{E}(V)} \leq \lim_{\beta \to 0} \Psi(\beta) \leq \frac{1}{(\lambda - \delta)\mathbb{E}(V)},
\]

which ends the proof.

\[\square\]

These estimates allow to characterize the number of possible stationary solutions as a function of the behavior of \( b(x) \) at zero:

**Corollary 1.** The existence of stationary solutions of the mean-field equations depends on the behavior of the firing function \( b \) at zero:

— If \( b(x)/x \to \lambda \in [1/\mathbb{E}(V), \infty] \), then there always exists at least one non-trivial stationary solution for the mean-field equation.

— If \( b(x)/x \to \lambda \in (0, 1/\mathbb{E}(V)) \), the trivial fixed point is the unique stationary solution of the mean-field equations.

— If \( b(x)/x \to 0 \), then the trivial solution \( \delta_0 \) is always a stationary solution of the mean-field equations. Non-trivial stationary solutions exist if the minimal value of \( \Psi(\beta) \) is smaller than 1. The number of fixed points is generically odd (except for very specific values of the parameters where local minima are exactly equal to 1).

The mean-field equations can therefore present several stationary solutions. The question of the **stability** of these solutions (in a sense to be made more precise) therefore arises. The question behind this characterization is the following: we have seen in theorem 1 that in finite-sized networks, the only stationary solution is the trivial state in which no neuron spikes, and that the network converges towards this solution. Additional solutions arising in the mean-field limit, a deep question arising is the emergence of new stationary and attractive solutions.
6.1. Stability of the trivial solution. In the cases where \( b(0) = 0 \), the trivial process a.s. equal to zero for all times is an invariant distribution of the mean-field equations. We have seen that additional stationary solutions may appear depending on the behavior of the \( b(x) \) at zero, and this raises the question of which invariant solution is selected by the system. We now investigate here the stability of the trivial solution as a function of the local behavior of \( b(x) \) at zero. The main result of the section is the following:

**Proposition 7.** The stability of the trivial solution depends on the behavior of \( b(x) \) at zero. Denoting \( \Delta = \mathbb{E}(V) \lim_{x \to 0} \frac{b(x)}{x} \in [0, \infty) \), we have:

- if \( \Delta \in [0, 1) \), the trivial solution is almost surely exponentially stable. In detail, there exists \( \delta > 0 \) and \( A_\delta > 0 \) sufficiently small such that for any initial condition with support included in \([0, A_\delta]\),
  \[
  \limsup_{t \to \infty} \frac{\log(X_t)}{t} < -\delta \quad \text{a.s.}
  \]
- if \( \Delta \in (1, \infty] \), the trivial solution is unstable in probability. Specifically, there exists \( A > 0 \) such that for any initial condition \( X_0 \) with support included an interval \([0, A]\) and mean \( \mu_0 > 0 \), there exists a deterministic time \( \tau(\mu_0) \) such that
  \[
  \mathbb{P}[ \sup_{t \in [0, \tau(\mu_0)]} X_t > A ] > 0.
  \]

**Remark.** The result we demonstrate for \( \Delta < 1 \) is relatively strong: almost any trajectory converge exponentially fast towards 0 provided that the initial condition is chosen sufficiently close from \( \delta_0 \) (in the sense that its support is included in a small interval around 0).

However, the instability result part of the proposition for \( \Delta > 1 \) is slightly weaker: we show that whatever the initial condition, the probability of reaching in finite time a specified level away from 0 is strictly positive. From a pathwise viewpoint, the result is indeed less strong than the exponential stability result. But from the distribution viewpoint, this corresponds to an instability of the distribution \( \delta_0 \) in the sense of [16].

**Proof.** Let us first deal with the case \( \Delta < 1 \). Since jumps are all negative, any solution of the McKean-Vlasov equation has the upperbound:

\[
X_t \leq X_0 + \int_0^t (-X_s + \mathbb{E}(V)\mathbb{E}[b(X_s)]) \, ds.
\]

Moreover, if \( \Delta < 1 \), there exists \( \delta > 0 \) and \( A_\delta > 0 \) such that for any \( x < A_\delta \),

\[
-x + \mathbb{E}(V)b(x) \leq -\delta x
\]

We introduce \( X_t^* = \text{ess sup} X_t = \inf\{u > 0; \mathbb{P}[X_t > u] = 0\} \), and assume that \( X_0^* < A_\delta \). We show that along the evolution, the essential support of \( (X_t) \) never exceeds \( A_\delta \) and actually shrinks to 0, ensuring stability of the solution \( \delta_0 \). Indeed, using Gronwall’s lemma and the monotonicity of the map \( b \), we have:

\[
X_t \leq X_0 e^{-t} + \mathbb{E}(V) \int_0^t e^{t-s} \mathbb{E}[b(X_s)] \, ds \leq X_0^* e^{-t} + \mathbb{E}(V) \int_0^t e^{t-s} b(X_s^*) \, ds
\]
readily implying that:
\[ X_t^* \leq X_0 e^{-t} + \mathbb{E}(V) \int_0^t e^{-(t-s)} b(X_s^*) \, ds. \]

Let us now introduce the deterministic time:
\[ \tau = \inf\{ t > 0 ; X_t^* > A\delta \} \]
On the interval \([0, \tau)\), we have:
\[ X_t^* \leq X_0 e^{-t} + (1 - \delta) \int_0^t e^{-(t-s)} X_s^* \, ds \]
and by Gronwall’s lemma again, we obtain for any \( t \in [0, \tau) \):
\[ X_t^* \leq X_0^* e^{-\delta t} \leq A\delta e^{-\delta t}. \]
This implies that (i) \( \tau = \infty \) and (ii) \( X_t^* \) converges exponentially fast towards \( 0 \). We have therefore proved that for any initial condition with support sufficiently concentrated around \( 0 \), the process converges almost surely towards \( 0 \) when \( t \to \infty \), hence the solution \( \delta_0 \) is stable.

If \( \Delta > 1 \), we can find \( \delta > 0 \) and \( A\delta > 0 \) sufficiently small so that:
\[ -x + \mathbb{E}(V) b(x) - x b(x) \geq \delta x. \]
Let \( X_t \) be the solution of the McKean-Vlasov equation with initial condition \( X_0 \) such that \( X_0^* < A\delta \) (we recall that \( X_t^* \) denotes in this proof the essential supremum of \( X_t \)). Denoting \( \mu_t = \mathbb{E}[X_t] \), we have:
\[
\mu_t = \mu_0 + \int_0^t -\mu_s + \mathbb{E}(V) [b(X_s) - \mathbb{E}[X_s b(X_s)]] \, ds \\
= \mu_0 + \int_0^t \int_{\mathbb{R}} (-x + \mathbb{E}(V) b(x) - x b(x)) p_s(dx) \, ds
\]
with \( p_s \) is the probability measure of \( X_s \). Similarly to the previous case, let us denote \( \tau \) the deterministic time:
\[ \tau = \inf\{ t > 0 ; X_t^* < A\delta \}. \]
On the interval \([0, \tau)\), we have:
\[ \mu_t \geq \mu_0 + \delta \int_0^t \mu_s \, ds \]
i.e. \( \mu_t \geq \mu_0 e^{\delta t} \). This implies that necessarily
\[ t \leq \frac{1}{\delta} \log \left( \frac{A\delta}{\mu_0} \right) =: \tau(\mu_0). \]
We therefore conclude that the essential supremum of the solution exceeds \( A\delta \) whatever the initial condition, which means that
\[ \mathbb{P}[ \sup_{t \in [0, \tau(\mu_0)]} X_s > A\delta ] > 0. \]

6.2. Power firing functions. We now provide some specific examples, for power functions of the form \( b(x) = \lambda x^{\alpha} + \delta \). We distinguish the affine (\( \alpha = 1 \)), superlinear (\( \alpha > 1 \)) and sublinear (\( \alpha < 1 \)) cases.
6.2.1. Affine firing functions. We start by considering affine firing functions, and apply proposition 6 (or its corollary 1) and the characterization of the stability in proposition 7 to characterize the number of invariant distributions and their stability:

**Proposition 8.** For linear firing-rate functions \( b(x) = \lambda x + \delta \) with \( \lambda > 0 \), defining the excitation rate \( \rho = \lambda \mathbb{E}(V) \), we have for \( \delta = 0 \):

- For \( \rho < 1 \), \( \delta_0 \) is the unique stationary solution and it is almost surely exponentially stable.
- For \( \rho > 1 \), \( \delta_0 \) is unstable in probability, and there exists an additional solution to the mean-field equations.

For \( \delta > 0 \), there exists a unique, non-trivial, invariant distribution.

**Proof.** Affine firing functions allow analytical calculations for all quantities. Basic algebra yields

\[
C(\beta) = \frac{1}{\rho \beta + \delta} + \frac{\rho \beta}{\rho \beta + \delta} \int_0^1 (1 - x)^{\rho \beta + \delta} e^{\rho \beta x} \, dx.
\]

We change variables and define \( x = -1 + y - \log(y) \). The map \( y \mapsto -1 + y - \log(y) \) is strictly decreasing on \((0, 1)\) and its inverse is \( \phi(x) = \exp(-W(-e^{-1-x}) - 1 - x) \) where \( W \) is the first real branch of the Lambert \( W \) function (inverse function of \( xe^x \), see [9]). This allows to rewrite the mean-field equation as:

\[
\beta C(\beta) = \frac{\beta}{\rho \beta + \delta} \left( 1 - \mathbb{E}[\Phi(E_1/\rho \beta)] \right)
\]

where \( \Phi(x) = \phi'(x) \phi''(x) \). It is then easy to see that:

\[
\frac{d}{d\beta} \beta C(\beta) = \frac{\delta}{(\rho \beta + \delta)^2} \left( 1 - \mathbb{E}[\Phi(E_1/\rho \beta)] \right) + \frac{1}{\rho \beta + \delta} \mathbb{E}\left[ E_1 \phi'(E_1/\rho \beta) \right].
\]

The map \( \phi \) satisfies:

\[
\begin{align*}
\phi'(x) &= \frac{W(-e^{-x-1})}{1 + W(-e^{-x-1})} \\
\phi''(x) &= -\frac{W(-e^{-x-1})}{(1 + W(-e^{-x-1}))^3},
\end{align*}
\]

and therefore,

\[
\Phi'(x) = \frac{(-W(-e^{-x-1}))^{\delta+1}(1 + W(-e^{-x-1})) + 1}{(1 + W(-e^{-x-1}))^3}.
\]

For \( x \geq 0, -e^{-1} \leq -e^{-x-1} \leq 0 \) and hence \( W(-e^{-x-1}) \leq 0 \), ensuring that \( \phi'(x) < 0, \phi''(x) > 0 \) and eventually \( \Phi'(x) > 0 \). All these estimates put together prove that \( \beta \mapsto \beta C(\beta) \) is strictly increasing, and therefore that there exists a unique non-trivial solution to the mean-field equation \( \beta C(\beta) = 1 \) when \( \rho > 1 \) or when \( \delta > 0 \).

We therefore conclude that in the case \( b(0) > 0 \), there exists a unique stationary solution, which is non-trivial, as was also the case in the finite-sized networks.

In the case \( b(0) = 0 \), we have shown that the only stationary distribution of finite-sized networks is the trivial solution \( \delta_0 \). In the mean-field limit, this solution persists whatever the value of the parameters. However, we showed that for \( \rho > 1 \),
this solution is no more stable, and non-trivial solution appears when $\rho > 1$. This is what we observe in the simulations of the network (see Fig. 1) in the linear firing rate case $b(x) = x$ for varying values of $E(V)$: for $E(V) > 1$, the trivial solution no more attracts the network, and a new solution with a non-zero value of the firing rate emerges. The value of $\beta$ at this equilibrium can be computed numerically, and shows a very good agreement with the simulations of the finite-sized network, even if this finite-sized network will eventually extinct.

This phenomenon illustrates the presence of a phase transition in the system. For small coupling ($\rho < 1$), both finite-sized networks and their mean-field limit have a trivial stationary solution. In that case, the time of extinction remains small and do not dramatically depend on the network size. However, for $\rho > 1$, the trivial solution is no more stable for the mean-field limit and, in that limit, a sustained activity appears. The time of extinction shows a dramatic dependence on the network size. A pseudo-stationary solution emerges, which is meta-stable for any finite system in the sense that even though the system will eventually stop firing, the time during which the system supports this non-trivial stationary firing rate diverges as the network size increases (see Fig. 1(b)).

From the biological viewpoint, the non-trivial solution found corresponds to a self-sustained activity. In this regime, neurons fire independently as a Poisson processes with a common intensity. This regime is a natural regime of activity of large neuronal networks. It is a typical regime of the awake brain often referred to as the asynchronous irregular state [2].

6.2.2. Sub-linear power firing functions. We now consider sub-linear firing functions $b(x) = \lambda x^\alpha + \gamma$ with $0 < \alpha < 1$ and $\gamma \geq 0$. We show the following:
Proposition 9. For any $0 < \alpha < 1$ and $\gamma \geq 0$, there exists a unique non-trivial invariant distribution to the McKean-Vlasov equation with firing rate function $b(x) = \lambda x^\alpha + \gamma$. For $\gamma = 0$, the trivial solution is unstable.

Proof. For $\alpha < 1$, proposition 6 implies that the map $\beta \mapsto \beta C(\beta)$ tends to 0 at $\beta = 0$ and to infinity when $\beta \to \infty$, ensuring the existence of a non-trivial invariant distribution. Moreover, proposition 7 shows that the trivial solution is unstable. The only result that remains to be proved is the uniqueness of the invariant distribution. To this end, we show that the map $\beta C(\beta)$ is strictly increasing. This is done by rewriting the expression of $\beta C(\beta)$ noting $\rho = \lambda \beta^\alpha$ and using the expression:

$$\beta C(\beta) = 1 + \frac{1}{\rho \beta^\alpha - 1} \left[ 1 + E \left[ \psi_\gamma \left( \frac{E_1}{\rho \beta^\alpha} \right) \right] \right]$$

with

$$\phi(x) = \int_0^x \frac{y^\alpha - 1}{1 - y} dy - \log(1 - x).$$

This map is strictly increasing, tends to 0 when $x \to 0$ and to $\infty$ at $x = 1$. It is therefore invertible, and we denote $\varphi = \phi^{-1}$. Using the variable $z = \phi(x)$, we can express our equation as:

$$\beta C(\beta) = 1 + \frac{1}{\rho \beta^\alpha - 1} \left[ 1 + E \left[ \psi_\gamma \left( \frac{E_1}{\rho \beta^\alpha} \right) \right] \right]$$

where $E_1$ is an exponential random variable with parameter 1 and

$$\psi_\gamma = (1 - \varphi)^\gamma \frac{1 - \varphi^\alpha}{1 - \varphi} \varphi'.$$

We hence have:

$$\frac{d}{d\beta} \beta C(\beta) = (1 - \alpha) \frac{1}{\rho \beta^\alpha} \left[ 1 + E \left[ \psi_\gamma \left( \frac{E_1}{\rho \beta^\alpha} \right) \right] \right] - \frac{\alpha}{\rho^2 \beta^\alpha} E \left[ E_1 \psi\gamma' \left( \frac{E_1}{\rho \beta^\alpha} \right) \right].$$

The first term of this expression is clearly positive. The second term is handled by expressing the differential $\psi\gamma'$ and showing that it is strictly negative. In details, we have:

$$\psi\gamma' = -\gamma (1 - \varphi)^{\gamma - 1} \psi_0 \varphi' + (1 - \varphi)^\gamma \psi_0'$$

and therefore we only need to show that $\psi_0' < 0$. Straightforward calculations yield:

$$\psi_0' = \left( \frac{\varphi'}{1 - \varphi} \right)^2 \left( 1 - (\alpha \varphi^{\alpha - 1} + (1 - \alpha) \varphi^\alpha) \right) + \frac{1 - \varphi^\alpha}{1 - \varphi} \varphi''.$$

The first term is clearly positive, and the second term has the sign of $\varphi''$. Since we have:

$$\begin{cases} \varphi' = \frac{1}{\varphi^{\alpha - 1}}, \\ \varphi'' = -\frac{\varphi'}{\varphi^2} \varphi'' \circ \phi \\ \phi'(x) = \frac{x^\alpha}{1 - x} > 0, \\ \phi''(x) = \frac{\alpha x^{\alpha - 1}}{1 - x} + \frac{x^\alpha}{(1 - x)^2} > 0, \end{cases}$$

we conclude that $\varphi'' < 0$. This ensures that $\beta \mapsto \beta C(\beta)$ is strictly increasing, and therefore there exists a unique invariant distribution for the mean-field equations. \qed
6.2.3. Super-linear power firing functions. The case $b(x) = \lambda x^\alpha$ with $\lambda > 0$ and $\alpha > 1$ shows a more intricate behavior. Proposition 6 shows that the map $\beta \mapsto \beta C(\beta)$ diverges to infinity when $\beta \to 0$ or $\beta \to \infty$, which allowed to conclude that apart from the trivial invariant distribution, there either exist no other invariant distribution or generically an even number of non-trivial invariant distributions. We analyze the dependence of the number of non-trivial invariant distributions as a function of the parameters.

**Lemma 5.** Denoting by $\rho = \lambda \mathbb{E}(V)^\alpha$. For any $\beta > 0$, there exists a unique $\rho(\beta)$ solving the fixed point equation. Moreover, the map $\beta \mapsto \rho(\beta)$ reaches its minimum: there exists $\beta_c \in \mathbb{R}_+$ such that $\rho_c = \min_{\beta \in \mathbb{R}_+} \rho(\beta) = \rho(\beta_c) > 0$.

**Proof.** Simple algebraic manipulations allow to rewrite the fixed point equation as:

\begin{equation}
\beta C(\beta) = \frac{1}{\rho \beta^{\alpha - 1}} + \beta \int_0^1 (1 - x)^{\rho \beta^{\alpha - 1}} \frac{1 - x^\alpha}{1 - x} \exp \left( -\rho \beta^{\alpha - 1} \int_0^x y^{\alpha - 1} \frac{1 - y}{1 - y} \, dy \right) \, dx
\end{equation}

We define

$g(u) = \int_0^u \frac{v^\alpha}{1 - v}$

and

$\Psi(x) = \int_0^1 \frac{1}{1 - u} \exp(-xg(u)) \, du$.

Our fixed point equation simply reads

\begin{equation}
\beta \Psi(\rho \beta^{\alpha - 1}) = 1.
\end{equation}

With this expression, it is now relatively easy to show that for any $\beta > 0$ fixed, there exists a unique $\rho(\beta)$ such that equation (41) is satisfied. Indeed, it is clear from the expression (41) that $1/\rho(\beta)$ is strictly decreasing, tends to infinity at $\rho = 0$ and to 0 when $\rho \to \infty$.

Moreover, the map $\beta \mapsto \rho(\beta)$ has the following properties.

- Using equation (40) we observe that $\beta^{\alpha - 1} \rho(\beta) \to 1$ when $\beta \to 0$ hence $\rho(\beta) \to \infty$.
- $\rho(\beta) \to \infty$ when $\beta \to \infty$. Indeed, using equation (40) and the series representation of $1/(1 - y)$, we can show that

$$
\lim_{\beta \to \infty} \left( \frac{\beta}{\rho(\beta)} \right)^{\frac{1}{\alpha+1}} = \int_0^\infty e^{-z^{\alpha+1}/\alpha+1}.
$$

- Using equation (40) we observe that $\beta^{\alpha - 1} \rho(\beta) \to 1$ when $\beta \to 0$ hence $\rho(\beta) \to \infty$.

The quantity $\rho_c$ constitute a transition point in the system, and governs the number of invariant distributions:

**Proposition 10.** For $b(x) = \lambda x^\alpha$ with $\lambda > 0$ and $\alpha > 1$, the number of invariant distributions of the McKean-Vlasov equation depends on the value of $\rho$ compared to $\rho_c$:

- For any $\rho < \rho_c$, there is no non-trivial invariant distribution
- For any $\rho > \rho_c$, there exists at least two non-trivial invariant distribution corresponding to $\beta_- < \beta_c$ and $\beta_+ > \beta_c$ which correspond to the quantities such that $\rho(\beta_\pm) = \rho$
For any $\rho = \rho_c$, there exists a unique non-trivial invariant distribution corresponding to $\beta = \beta_c$.

This is a simple consequence of lemma 5.

Numerical computations of the fixed point equation (41) show that when $\rho > \rho_c$, there exists exactly two non-trivial invariant distributions. In order to prove this fact, we would need to show that the function $\beta \mapsto \rho(\beta)$ has a unique minimum on $\mathbb{R}_+$, i.e. that it is strictly decreasing on $[0, \beta_c]$ and increasing on $[\beta_c, \infty]$, or in other words that there exists a unique $\beta \in \mathbb{R}_+$ such that $\rho'(\beta) = 0$. These conditions yield the implicit equation:

$$\Psi(x^*) = -2x^*\Psi(x^*)$$

with $x^* = (\beta^*)^2\rho(\beta^*)$. Showing analytically uniqueness of the solutions of this implicit equations is actually very complicated even for simple firing functions such as $b(x) = x^2$. Extensive numerical simulations tend to show however that this is the case.

**Figure 2.** Quadratic firing function $b(x) = x^2$: numerical simulations of a 2000 neurons network for different values of $E(V)$, 30 initial conditions and 1000 realizations. Each blue dot corresponds an average firing rate for a given initial condition (see text). The stationary solution $\delta_0$ persists for large values of $E(V)$ and the additional non-trivial invariant distribution appears when increasing $E(V)$. Red dots correspond to the separatrix between initial conditions converging to the trivial solution and those going to the sustained state. The purple line is the numerical solution of the mean-field equation (39), and shows a good agreement with the non-trivial and separatrix points.

Let us for instance discuss in more detail the case $b(x) = x^2$ (see Fig. 2). We have shown that, depending of $E(V)$, either the trivial distribution is the unique
stationary distribution, or there exists two additional non-trivial equilibrium solutions. These solutions can be found numerically and are depicted in Fig. 2 (purple line). A phase transition arises, at a specific value of $E(V)$, in which two additional solutions emerge. Similarly to what we did for the linear network, we extensively simulated the network in order to characterize equilibria of the system. In contrast to the linear case, all trajectories do not go to the same state, and we do expect to find certain initial conditions converging towards the trivial solution and some towards the non-trivial equilibrium. In one-dimensional dynamical systems presenting multi-stability, one typically has, between two stable equilibria, one unstable equilibrium, which acts as a separatrix, in the sense that trajectories with an initial condition on one side of the unstable equilibrium converge to the stable equilibrium on that same side. Here, the system is much more complex, and in particular it is a priori infinite-dimensional. However, one may conjecture that the limiting dynamics collapses on a smaller dimensional system. In our simulations, we considered a network made of 2000 neurons, in the simple case in which the initial conditions of neurons are uniformly distributed around a value $v_0$, with a fixed standard deviation 0.2. For 30 fixed values of $v_0$, we simulated 1000 times the network. We observed that, except for values of $E(V)$ close from the phase transition, that trajectories converge either towards the trivial equilibrium or towards a state with non-zero voltage. The value of $v_0$ at which a switch occurs between those trajectories going to the trivial state and those going to the non-zero state has been recorded. The average value is depicted in red in Fig. 2 and shows very good agreement with the middle stationary solution of the mean-field equation. The end state is characterized by one quantity per initial condition, which corresponds to the average in time (in the time interval $[90, 100]$), over all neurons and over the different simulations, of the voltage variable. One point is therefore obtained for each of the 30 initial conditions, and is depicted as a blue circle in Fig. 2. The dynamics of the system, constrained to these precise initial conditions, is therefore highly similar to a one-dimensional dynamical system. Note that it is not rare that solutions of McKean-Vlasov systems reduce to low-dimensional systems. For instance in a model arising in neuroscience, it was shown that a specific, rate-based neuron model reduces exactly, in the mean-field limit, to a one-dimensional dynamical system. Although dynamics of the spiking neuron is much more complex, we conjecture that the dynamics of the firing rate is much simpler and characterized deterministically by a few statistical quantities.

**Appendix A. A Few Elementary Technical Results**

**Lemma 6.** We assume that $b$ is such that there exists $\gamma > 0$ and $c > 0$ such that

$$b'(x) \leq \gamma b(x) + c$$

Then there exists for any $\varepsilon > 0$ a constant $\gamma_1 < 5 + \varepsilon \gamma$, $c_1 > 0$ and a value $\eta_b > 0$ such for any $a \in (0, \eta_b)$ and $x \geq 0$,

$$b^p(x + a) - b^p(x) \leq a (\gamma_1 b^p(x) + c_1), \quad 1 \leq p \leq 5$$

**Proof.** Let us start by noting that the inequality is trivial for $b$ bounded. We will therefore assume in the rest of the proof that $b$ diverges at infinity. We also remark that for any $p \in \{1, \cdots, 5\}$, the map $b^p$ satisfies an inequality of type (42) where $\gamma$
is multiplied by \( p \). Indeed, for any \( \delta > 0 \), we can find \( c_\delta > 0 \) such that:

\[
\frac{db^p(x)}{dx} \leq p\gamma b^p(x) + pecb^{p-1}(x) \leq (p\gamma + \delta)b^p(x) + c_\delta.
\]

We will therefore demonstrate without loss of generality the proposition for \( p = 1 \), and i.e. control the modulus of continuity of \( \beta \) arbitrary

\[ x \]

We will therefore demonstrate without loss of generality the proposition for any

\[ \gamma \]

Therefore, there exists \( x_0 > 0 \) and any \( x \geq x_0 \), we have:

\[
\frac{b(x + a)}{b(x)} = \exp \left( \int_x^{x+a} \frac{b'(y)}{b(y)} dy \right) \leq e^{a\hat{\gamma}}
\]

with \( \hat{\gamma} = \gamma + \frac{c}{b(x_0)} \). We conclude that for \( x \geq x_0 \),

\[
b(x + a) - b(x) \leq (e^{a\hat{\gamma}} - 1)b(x).
\]

The map \( a \mapsto (e^{a\hat{\gamma}} - 1)/a \) is smooth, non-decreasing and tends to \( \hat{\gamma} \) at \( a = 0 \), which can be made arbitrarily close from \( \gamma \) for sufficiently large \( x_0 \) (since \( b \) is unbounded).

Therefore, there exists \( x_0 > 0 \) and \( \eta > 0 \) such that for any \( x \geq x_0 \) and \( a \in [0, \eta] \),

\[
b(x + a) - b(x) \leq a(1 + \varepsilon)\gamma b(x).
\]

Denoting \( c_1 \) the Lipschitz constant of \( b \) over the interval \([0, x_0 + \eta] \), we readily obtain (43) with \( \gamma_1 = \gamma(1 + \varepsilon) \). □

Another elementary property that is useful in our developments is the following:

**Proposition 11.** If \( (x(t)) \) is a non-negative \( C^1 \)-function on \( \mathbb{R}_+ \) and \( \kappa > \delta > 0 \) such that, for \( A, C \in \mathbb{R} \),

\[
x(t) \leq x(s) - \int_s^t x(u)\kappa \, du + A \int_s^t x(u)\delta \, du,
\]

holds for any \( 0 \leq s \leq t \) then

\[
\sup_{t \geq 0} x(t) \leq C_0 < +\infty,
\]

where \( C_0 = x(0) \wedge A^{\kappa-\delta} \).

The proof is elementary once noted that trajectories cannot exceed \( x \geq A^{k-\delta} \) (except if the initial condition does), value above which the map \( x \mapsto -x^{\kappa} + Ax^{\delta} \) becomes strictly negative.

The third elementary technical result used is related to the martingales associated to marked Poisson processes.

**Proposition 12.** If \( N \) is a Poisson process on \( \mathbb{R}_+^3 \) with intensity measure \( du \otimes V(dz) \otimes dt \), \( f \) is a continuous function on \( \mathbb{R}_+^3 \) and \( (Y(t) = (Y_1(t), Y_2(t)) \) is a cadlag adapted processes then the process \( (M(t)) \) defined by

\[
\left( \int_{s=0}^t \int_{\mathbb{R}_+^3} f(Y(s-z), z) \mathbb{1}_{[0\leq u \leq Y_1(s-z)]} N(du, dz, ds) - f(Y(s), z)Y_1(s) \, ds \, V(dz) \right)
\]

is a local martingale whose predictable increasing process is given by

\[
(\langle M \rangle(t)) = \left( \int_0^t f(Y(s), z)^2 Y_1(s) \, ds \, V(dz) \right)
\]
References

[1] Søren Asmussen, *Applied probability and queues*, John Wiley & Sons Ltd., Chichester, 1987.
[2] N. Brunel, *Dynamics of sparsely connected networks of excitatory and inhibitory spiking neurons*, Journal of Computational Neuroscience 8 (2000), 183–208.
[3] Nicolas Brunel, *Dynamics of networks of randomly connected excitatory and inhibitory spiking neurons*, Journal of Physiology-Paris 94 (2000), no. 5–6, 445 – 463.
[4] AN Burkitt, *A review of the integrate-and-fire neuron model: I. homogeneous synaptic input*, Biological cybernetics 95 (2006), no. 1, 1–19.
[5] ———, *A review of the integrate-and-fire neuron model: II. inhomogeneous synaptic input and network properties*, Biological cybernetics 95 (2006), no. 2, 97–112.
[6] Maria Caceres, Jose Carrillo, and Benoît Perthame, *Analysis of nonlinear noisy integrate & fire neuron models: blow-up and steady states*, The Journal of Mathematical Neuroscience (JMN) 1 (2011), no. 1 (English).
[7] Maria J Caceres and Benoît Perthame, *Beyond blow-up in excitatory integrate and fire neuronal networks: refractory period and spontaneous activity*, Journal of theoretical biology 350 (2014), 81–89.
[8] EJ Chichilnisky, *A simple white noise analysis of neuronal light responses*, Network: Computation in Neural Systems 12 (2001), no. 2, 199–213.
[9] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, *On the Lambert W function*, Advances in Computational Mathematics 5 (1996), no. 4, 329–359. MR 1414285 (98j:33015)
[10] Donald A. Dawson, *Measure-valued Markov processes*, École d’Été de Probabilités de Saint-Flour XXI—1991, Lecture Notes in Math., vol. 1541, Springer, Berlin, 1993, pp. 1–260.
[11] A. De Masi, A. Galves, E. Löcherbach, and E. Presutti, *Hydrodynamic limit for interacting neurons*, arXiv preprint arXiv:1401.4264, 2014.
[12] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Global solvability of a networked integrate-and-fire model of mckean-vlasov type*, arXiv preprint arXiv:1211.0299, 2014.
[13] Christine Fricker, Philippe Robert, Ellen Saada, and Danielle Tibi, *Analysis of some networks with interaction*, Annals of Applied Probability 4 (1994), no. 4, 1112–1128.
[14] Georges L. Gerstein and Benoit Mandelbrot, *Random walk models for the spike activity of a single neuron*, Biophysical Journal 4 (1964), 41–68.
[15] Carl Graham and Philippe Robert, *Interacting multi-class transmissions in large stochastic networks*, Annals of Applied Probability 19 (2009), no. 6, 2334–2361.
[16] RZ Has'minskii, *Stochastic stability of differential equations*, Kluwer Academic Pub, 1980.
[17] James Inglis and Denis Talay, *Mean-field limit of a stochastic particle system smoothly interacting through threshold hitting-times and applications to neural networks with dendritic component*, arXiv preprint arXiv:1409.8221 (2014).
[18] M. Kac, *Foundations of kinetic theory*, Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, vol. 1955, 1954, pp. 171–197.
[19] B. W. Knight, *Dynamics of encoding in a population of neurons*, J. Gen. Physiol. 59 (1972), 734–766.
[20] L Lapicque, *Recherches quantitatives sur l’excitation des nerfs traitées comme une polarisation*, J. Physiol. Paris 9 (1907), 620–635.
[21] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer, *Markov chains and mixing times*, American Mathematical Society, Providence, RI, 2009.
[22] R.M. Loynes, *The stability of queues with non independent inter-arrival and service times*, Proc. Cambridge Phil. Soc. 58 (1962), 497–520.
[23] Esa Nummelin, *General irreducible Markov chains and nonnegative operators*, Cambridge University Press, Cambridge, 1984.
[24] Khashayar Pakdaman, Benoît Perthame, and Delphine Salort, *Dynamics of a structured neuron population*, Nonlinearity 23 (2010), no. 1, 55.
[25] ———, *Relaxation and self-sustained oscillations in the time elapsed neuron network model*, SIAM Journal on Applied Mathematics 73 (2013), no. 3, 1260–1279.
[26] Khashayar Pakdaman, Benoît Perthame, Delphine Salort, et al., *Adaptation and fatigue model for neuron networks and large time asymptotics in a nonlinear fragmentation equation*, (2012).
[27] Jonathan W Pillow, Liam Paninski, Valerie J Uzzell, Eero P Simoncelli, and EJ Chichilnisky, Prediction and decoding of retinal ganglion cell responses with a probabilistic spiking model, The Journal of Neuroscience 25 (2005), no. 47, 11003–11013.

[28] Jonathan W Pillow, Jonathon Shlens, Liam Paninski, Alexander Sher, Alan M Litke, EJ Chichilnisky, and Eero P Simoncelli, Spatio-temporal correlations and visual signalling in a complete neuronal population, Nature 454 (2008), no. 7207, 995–999.

[29] Philippe Robert, Stochastic networks and queues, Stochastic Modelling and Applied Probability Series, vol. 52, Springer, New-York, June 2003.

[30] I Chris G Rogers and David Williams, Diffusions, markov processes, and martingales: Volume 1, foundations, Cambridge university press, 2000.

[31] ET Rolls and G Deco, The noisy brain: stochastic dynamics as a principle of brain function, Oxford university press, 2010.

[32] R. B. Stein, A theoretical analysis of neuronal variability, Biophysics Journal 5 (1965), 173–194.

[33] AS Sznitman, Nonlinear reflecting diffusion process, and the propagation of chaos and fluctuations associated, Journal of Functional Analysis 56 (1984), no. 3, 311–336.

[34] A.S. Sznitman, école d’été de saint flour, Lecture Notes in Maths, ch. Topics in propagation of chaos, pp. 167–243, Springer Verlag, 1989.

[35] ______, Topics in propagation of chaos, École d’Été de Probabilités de Saint-Flour XIX — 1989, Lecture Notes in Maths, vol. 1464, Springer-Verlag, 1991, pp. 167–243.

[36] Jonathan Touboul, Mean-field equations for stochastic firing-rate neural fields with delays: derivation and noise-induced transitions, Physica D: Nonlinear Phenomena 241 (2012), no. 15, 1223—1244.

[37] Jonathan Touboul, The propagation of chaos in neural fields, Annals of Applied Probability 24 (2014), no. 3, 1298–1328.

[38] ______, Spatially extended networks with singular multi-scale connectivity patterns, Journal of Statistical Physics 156 (2014), no. 3, 546–573 (English).

[39] Jonathan Touboul, Geoffroy Hermann, and Olivier Faugeras, Noise-induced behaviors in neural mean field dynamics, SIAM Journal on Applied Dynamical Systems 11 (2012), no. 1, 49–81.

E-mail address: Philippe.Robert@inria.fr
(Ph. Robert) INRIA PARIS—ROCQUENCOURT, DOMAINE DE VOLUCEAU, 78153 LE CHESNAY, FRANCE.
URL: http://team.inria.fr/rap/robert

E-mail address: jonathan.touboul@college-de-france.fr
(J. Touboul) MATHEMATICAL NEUROSCIENCE TEAM, CIRB - COLLEGE DE FRANCE and INRIA PARIS-ROCQUENCOURT, 11, PLACE MARCELIN BERTHELOT 75005 PARIS, FRANCE
URL: http://mathematical-neuroscience.net/team/jonathan/