Probabilizing Parking Functions

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Abstract
We explore the link between combinatorics and probability generated by the question “What does a random parking function look like?” This gives rise to novel probabilistic interpretations of some elegant, known generating functions. It leads to new combinatorics: how many parking functions begin with \( i \)? We classify features (e.g., the full descent pattern) of parking functions that have exactly the same distribution among parking functions as among all functions. Finally, we develop the link between parking functions and Brownian excursion theory to give examples where the two ensembles differ.

1 Introduction

Parking functions are a basic combinatorial object with applications in combinatorics, group theory, and computer science. This paper explores them by asking the question, “What does a typical parking function look like?” Start with \( n \) parking spaces arranged in a line ordered left to right, as:

1 2 3 \cdots n

There are \( n \) cars, each having a preferred spot. Car \( i \) wants \( \pi_i \), \( 1 \leq i \leq n \), with \( 1 \leq \pi_i \leq n \). The first car parks in spot \( \pi_1 \), then the second car tries to park in spot \( \pi_2 \); if this is occupied, it takes the first available spot to the right of \( \pi_2 \). This continues; if at any stage \( i \) there is no available spot at \( \pi_i \) or further right, the process aborts.
**Definition.** A parking function is a sequence \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \) with \( 1 \leq \pi_i \leq n \) so that all cars can park. Let \( \text{PF}_n \) be the set of parking functions.

**Example.** \((1, 1, \ldots, 1)\) is a parking function. A sequence with two \( n \)'s is not. When \( n = 3 \), there are 16 parking functions,

\[
111, 112, 121, 211, 113, 131, 311, 122, 221, 123, 132, 213, 231, 312, 321.
\]

Using the pigeonhole principle, it is easy to see that a parking function must have at least one \( \pi_i = 1 \), it must have at least two values \( \leq 2 \), and so on; \( \pi \) is a parking function if and only if

\[
\# \{ k : \pi_k \leq i \} \geq i, \quad 1 \leq i \leq n.
\]

Dividing through by \( n \) (both inside and outside) this becomes

\[
F^n(x) \geq x \quad \text{for} \quad x = \frac{i}{n}, \quad 1 \leq i \leq n,
\]

with \( F^n(x) = \frac{1}{n} \cdot \left( \# \{ k : \pi_k \leq nx \} \right) \) the distribution function of the measure

\[
\frac{1}{n} \sum_{i=1}^{n} \delta_{\pi_i/n}.
\]

Following a classical tradition in probabilistic combinatorics we ask the following questions for the distribution of features of a randomly chosen \( \pi \in \text{PF}_n \):

- What is the chance that \( \pi_i = j \)?
- What is the distribution of the descent pattern in \( \pi \)?
- What is the distribution of the sum \( \sum_{i=1}^{n} \pi_i \)?

It is easy to generate a parking function randomly on the computer. Figure 1 shows a histogram of the values of \( \pi_1 \) based on 50,000 random choices for \( n = 100 \). There seems to be a smooth curve. How does it behave? Notice that because of invariance under permutations (see (1.1)), the answer gives the distribution of any coordinate \( \pi_i \). Section 2 gives a closed formula as well as a simple large-\( n \) approximation.

In Section 3 we mine the treasure trove of elegant generating functions derived by the combinatorics community, leaning on Yan (2015). These translate into classical binomial, Poisson, and central limit theorems for features like

- the number of \( i \) with \( \pi_i = 1 \);

\[
\text{(1.3)} \quad \text{the number of repeats } \pi_i = \pi_{i+1};
\]

- the number of lucky cars, where car \( i \) gets spot \( \pi_i \).
Figure 1: This histogram gives the distribution of values of $\pi_1$ (the first preference) in 50,000 parking functions of size 100 chosen uniformly at random.

One unifying theme throughout may be called the *equivalence of ensembles*. Let $F_n = \{ f : [n] \to [n] \}$ (where $[n]$ is the standard shorthand for $\{1, \cdots, n\}$) so $|F_n| = (n)^n$. Of course $PF_n \subseteq F_n$ and

$$|PF_n|/|F_n| = (1 + 1/n)^{n-1}/n \sim e/(n).$$

A random parking function is a random function in $F_n$ conditioned on being in $PF_n$. In analogy with equivalence of ensembles in statistical mechanics, it is natural to expect that for some features, the distribution of the features in the “micro-canonical ensemble” ($PF_n$) should be close to the features in the “canonical ensemble” ($F_n$). As we shall see, there is a lot of truth in this heuristic—it holds for the features in (1.3) among others. Further, its failures are interesting.

Section 4 develops classes of features where the heuristic is exact. This includes the full descent theory, the up/down pattern in $\pi$. It explains some of the formulas behind features (1.3) and gives new formulas.

Section 5 uses some more esoteric probability. For a random $\pi$, it is shown that $[F^\pi(x) - x]_{0 \leq x \leq 1}$ converges to a Brownian excursion process. This allows us to
determine the large-$n$ behavior of
\[\begin{align*}
\bullet \ & \# \{k : \pi_k \leq i\}; \\
\bullet \ & \max_{1 \leq k \leq n} \{k : \pi_k \leq i\} - i; \\
\bullet \ & \sum_{i=1}^{n} \pi_i.
\end{align*}\] (1.4)

These distributions differ from their counterparts for a random \( f \in F_n \). For example, \( (\sum_{i=1}^{n} f_i - n^2/2)/n^{3/2} \) has a limiting normal distribution but \( \sum_{i=1}^{n} (\pi_i - n^2/2)/n^{3/2} \) has a limiting Airy distribution.

Section 6 tries to flesh out the connection between parking functions and Macdonald polynomials; it may be read now for further motivation. Section 7 briefly discusses open problems (e.g., cycle structure) and generalizations.

Literature review
Parking functions were introduced by [Konheim and Weiss (1966)] to study the widely used storage device of hashing. They showed there are \((n+1)^n-1\) parking functions, e.g., 16 when \( n = 3 \). Since then, parking functions have appeared all over combinatorics; in the enumerative theory of trees and forests [Chassaing and Marckert (2001)], in the analysis of set partitions [Stanley (1997b)], hyperplane arrangements [Stanley (1998)] [Shi (1986)], polytopes [Stanley and Pitman (2002)] [Chebikin and Postnikov (2010)], chip firing games [Cori and Rossin (2000)], and elsewhere.

Yan (2015) gives an accessible, extensive survey which may be supplemented by [Stanley (1999a)], [Beck et al. (2015)], and [Armstrong et al. (2015)]. Our initial interest in parking functions comes through their relation to the amazing Macdonald polynomials [Haiman (2002)] [Macdonald (2015)].

Generating a random parking function
As previously mentioned, it is quite simple to generate a parking function uniformly at random. To select \( \pi \in \text{PF}_n \):

1. Pick an element \( \pi \in \mathbb{Z}/(n+1)\mathbb{Z}^n \), where here (as in later steps) we take the equivalence class representatives \( 1, \cdots, (n+1) \).
2. If \( \pi \in \text{PF}_n \) (i.e. if \( \pi' = \text{sort}(\pi) \) is such that \( \pi'_i \leq i \) for all \( i \)), return \( \pi \).
3. Otherwise, let \( \pi := \pi + (1, \cdots, 1) \) (working mod \( n+1 \)). Return to (2).

In fact, for every \( \pi \in (\mathbb{Z}/(n+1)\mathbb{Z})^n \), there is exactly one \( k \in \mathbb{Z}/(n+1)\mathbb{Z} \) such that \( \pi + k(1, \cdots, 1) \) is a parking function. This process is suggested by Pollak’s original proof of the number of parking functions as given in [Foata and Riordan (1974)].
2 Coordinates of Random Parking Functions

Let \( \text{PF}_n \) be the set of parking functions \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \). This section gives exact and limiting approximations for the distribution of \( \pi_1 \) and, by symmetry, \( \pi_i \), when \( \pi \in \text{PF}_n \) is chosen uniformly. For comparison, consider \( \mathcal{F}_n = \{f : [n] \to [n]\} \). Then \( \text{PF}_n \subseteq \mathcal{F}_n \), \( |\text{PF}_n|/|\mathcal{F}_n| = (1 + 1/n)^n - 1/n \), and a natural heuristic compares a random \( \pi \) with a random \( f \). Section 2.1 looks at a single coordinate, while Section 2.2 looks at the joint distribution of \( k \) coordinates.

2.1 Single Coordinates

For random \( f \),
\[
P(f_1 = j) = \frac{1}{n}, \quad E(f_1) \sim \frac{n}{2},
\]
The results below show that for \( j \) fixed and \( n \) large,
\[
P(\pi_1 = j) \sim \frac{1 + Q(j)}{n}, \quad P(\pi_1 = n - j) \sim \frac{1 - Q(j + 2)}{n}, \quad E(\pi_1) \sim \frac{n}{2},
\]
with \( Q(j) = P(X \geq j), \ P(X = j) = e^{-j}j^{j-1}/j! \), the Borel distribution on \( j = 1, 2, 3, \ldots \) The extremes are \( P(\pi_1 = 1) \sim 2/n, \ P(\pi_1 = n) \sim 1/en \), with \( P(\pi_1 = j) \) monotone decreasing in \( j \). Since \( Q(j) \sim \sqrt{2/(\pi j)} \) when \( j \) is large, the bulk of the values are close to uniform. Figure 1 in the introduction shows that \( n \) must be large for this “flatness” to take hold.

The arguments depend on a combinatorial description of the parking functions that begin with \( k \). Let
\[
A_{\pi_2, \ldots, \pi_n} = \{j : (j, \pi_2, \pi_3, \ldots, \pi_n) \in \text{PF}_n\}.
\]
If \( k \in A_{\pi_2, \ldots, \pi_n} \) then \( k_1 \in A_{\pi_2, \ldots, \pi_n} \) for all \( 1 \leq k_1 \leq k \) so \( A_{\pi_2, \ldots, \pi_n} = [k] \) for some fixed value of \( k \) (or empty) and we need only determine the set’s maximal element.

**Definition** (Parking Function Shuffle). Say that \( \pi_2, \ldots, \pi_n \) is a parking function shuffle of \( \alpha \in \text{PF}_{k-1} \) and \( \beta \in \text{PF}_{n-k} \) (write \( \pi_2, \ldots, \pi_n \in \text{Sh}(k-1, n-k) \)) if \( \pi_2, \ldots, \pi_n \) is a shuffle of the two words \( \alpha \) and \( \beta + (k-1, \ldots, k-1) \).

**Example.** With \( n = 7 \) and \( k = 4 \), \( (2, 4, 1, 5, 7, 2, 6) \) is a shuffle of \( (2, 1, 2) \) and \( (1, 2, 4, 3) \). The main result is

**Theorem 1.** \( A_{\pi_2, \ldots, \pi_n} = [k] \) if and only if \( (\pi_2, \ldots, \pi_n) \in \text{Sh}(k-1, n-k) \).
Corollary 1. The number of $\pi \in \text{PF}_n$ with $\pi_1 = k$ is
\[\sum_{s=0}^{n-k} \binom{n-1}{s} (s + 1)^{n-1}(n-s)^{n-s-2} \cdot (s-1)^{s-2}.\]

Note that this quantity decreases as $k$ increases as we have fewer resulting summands.

Corollary 2. Let $P(X = j) = e^{-j} j^{j-1} / j!$, so that $X$ has a Borel distribution. Then for any parking function $\pi$ uniformly chosen of size $n$, fixed $j$ and $k$ and $0 < j \leq k < n$,
\[P(\pi_1 = j \text{ and } A_{\pi_2,\ldots,\pi_n} = [k]) \sim \frac{1}{n} P(X = k).\]

If $k$ is close to $n$, by contrast, and $0 < j < n - k$
\[P(\pi_1 = j \text{ and } A_{\pi_2,\ldots,\pi_n} = [k]) \sim \frac{1}{n} P(X = n - k + 1).\]

The explicit formulae and asymptotics derived above allow several simple probabilistic interpretations. Let $\pi$ be a parking function. This determines $\pi_2, \ldots, \pi_n$ and so define $K_\pi$ such that $[K_\pi] = A_{\pi_2,\ldots,\pi_n}$, that is, $K_\pi$ is the largest possible first element consistent with the last $n - 1$. This $K_\pi$ is a random variable and we may ask about it’s distribution and about the conditional distribution of $\pi_1$ given that $K_\pi = k$. The following argument shows that with probability tending to 1, $K_\pi$ is close to $n$ AND the conditional distribution is uniform on $1, 2, \ldots, K_\pi$.

Corollary 3. Let $q(j) = e^{-j} j^{j-1} / j!$ be the Borel distribution on $1, 2, \ldots$. For fixed $j$, as $n$ tends to infinity, $P(K_\pi = n - j)$ tends to $q(j)$. Further, the conditional distribution of $\pi_1$ given $K_\pi = n - j$ is close to the uniform distribution on $1, \ldots, n - j$ in the Levy metric.

Corollary 4. For $\pi$ uniformly chosen in $\text{PF}_n$, $j \geq 1$ fixed, and $n$ large relative to $j$,
\[P(\pi_1 = n - j) \sim \frac{P(X \leq j + 1)}{n} \quad \text{with} \quad P(X = j) = e^{-j} j^{j-1} / j!.\]

In particular, $P(\pi_1 = n) \sim 1/(en)$. 

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Corollary 5. For $\pi$ uniformly chosen in $PF_n$, $j \geq 1$ fixed, and $n$ large,
\[ P(\pi_1 = j) \sim \frac{1 + P(X \geq j)}{n}. \]

In particular, $P(\pi_1 = 1) \sim \frac{2}{n}$.

Corollaries 3 and 4 show that “in the corners” the distribution of $\pi_1$ and $f_1$ are different. However, for most $j$, $P(\pi_1 = j) \sim P(f(1) = j) = \frac{1}{n}$. Indeed, this is true in a fairly strong sense. Define the total variation distance between two measures $P$ and $\hat{P}$ on $\{1, 2, \ldots, n\}$ to be
\begin{equation}
\|P - \hat{P}\| := \max_{A \subseteq \{n\}} |P(A) - \hat{P}(A)| = \frac{1}{2} \sum_{j=1}^{n} |P(j) - \hat{P}(j)| = \frac{1}{2} \max_{\|f\|_{\infty} \leq i} |P(f) - \hat{P}(f)|.
\end{equation}
The first equality above is a definition; the others are easy consequences.

Corollary 6. Let $P_n(j)$ and $\hat{P}_n(j)$ be the probabilities associated with $\pi_1$, $f_1$, from uniformly chosen $\pi \in PF_n$, $f \in F_n$. Then,
\[ \|P_n - \hat{P}_n\| \xrightarrow{n \to \infty} 0. \]

Remark. With more work, a rate of convergence should follow in Corollary 6. Preliminary calculations suggest $\|P_n - \hat{P}_n\| \leq \frac{c}{\sqrt{n}}$ for $c$ universal. We conjecture that, in variation distance, the joint distribution of $\pi_1, \ldots, \pi_k$ is close to the joint distribution of $f_1, \ldots, f_k$ ($k$ fixed, $n$ large).

The final result gives an asymptotic expression for the mean $E(\pi_1) = \sum_{j=1}^{n} jP(\pi_1 = j)$. The correction term is of interest in connection with the area function of Section 5.

Theorem 2. For $\pi$ uniformly chosen in $PF_n$,
\[ E(\pi_1) = \frac{n}{2} - \frac{\sqrt{2\pi}}{4} n^{1/2} (1 + o(1)). \]

Note that $E(\pi_1) \sim E(f_1)$.

The proof of Theorem 1 is broken into four short, easy lemmas. These are given next, followed by proofs of the corollaries and Theorem 2. The results make nice use of Abel’s extension of the binomial theorem.
Lemma 1. If \((\pi_2, \ldots, \pi_n) \in \text{Sh}(k - 1, n - k)\) then \(\pi = (k, \pi_2, \ldots, \pi_n)\) is a parking function.

Proof. Certainly, \(#\{i : \pi_i \leq j\} \geq j\) for \(j < k\). Thus
\[
#\{i : \pi_i \leq k\} = #\{i : \pi_i \leq k - 1\} + 1 \geq k.
\]
Finally, for \(j > k\),
\[
#\{i : \pi_i \leq j\} = #\{i : \pi_i \leq k\} + #\{i : k < \pi_i \leq j\} \geq k + j - k = j,
\]
where the last inequality comes from the fact that the cars greater than \(k\) come from a parking function \(\beta\).

Lemma 2. If \((\pi_2, \ldots, \pi_n) \in \text{Sh}(k - 1, n - k)\) then \(\pi = (k + 1, \pi_2, \ldots, \pi_n)\) is not a parking function.

Proof. Assume not. Then \(#\{i : \pi_i > k\} = n - k + 1\) and thus \(#\{i : \pi_i \leq k\} \leq k - 1\).

Lemma 3. If \(\pi = (k, \pi_2, \ldots, \pi_n)\) is a parking function, some subsequence of \(\pi\) must be a parking function of length \(k - 1\).

Proof. Take a subsequence formed by the (first) \(k-1\) cars with value at most \(k-1\).

Lemma 4. If \(\pi = (k, \pi_2, \ldots, \pi_n)\) is a parking function but \(\pi' = (k + 1, \pi_2, \ldots, \pi_n)\) is not, some subsequence of \(\pi\) is of the form \(\beta + (k, \ldots, k)\) where \(\beta \in \text{PF}_{n-k}\).

Proof. It must be that \(\pi\) has exactly \(k\) cars less than or equal to \(k\) (including the first car). Thus call \(\beta'\) the subsequence with exactly \(n - k\) cars of value greater than \(k\) and let \(\beta = \beta' - (k, \ldots, k)\). Since
\[
#\{i : k \leq \pi_i \leq k + j\} \geq j,
\]
\(\beta'\) is a parking function as desired.

Proof of (Corollary 1) Note that, if \(\pi_1 = j\), then \(A_{j, \ldots, j} = [a]\) for some \(a \geq j\). Thus the number of parking functions with \(\pi_1 = j\) is
\[
\sum_{a=j}^{n} \binom{n-1}{a-1} a^{a-2} (n - a + 1)^{n-a-1} = \sum_{s=0}^{n-j} \binom{n-1}{s} (s+1)^{s-1} (n-s)^{n-s-2}.
\]
Here \(\binom{n-1}{a-1}\) accounts for the positions of the smaller parking functions in the shuffle, \(a^{a-2}\) is the number of smaller parking functions, and \((n - a + 1)^{n-a-1}\) is the number of larger parking functions in a possible shuffle. The equality follows by changing \(a\) to \(n - s\).

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Proof of Corollary 2. For fixed \( k \) and large \( n \), we have

\[
\frac{1}{n+1} \binom{n-1}{k-1} \frac{n-k-1}{(k-1)(n+1)^{n-1}} \sim \frac{(n-1)^{k-1}k^{-2}(n-k+1)^{n-k-1}}{(k-1)! (n+1)^{n-1}}
\]

\[
\sim \frac{k^{k-2}(n-k+1)^{n-k-1}}{(k-1)! (n+1)^{n-k}}
\]

\[
\sim \frac{k^{k-1}e^{-k}}{k!(n-k+1)}
\]

\[
= \frac{1}{n-k+1} P(X = k)
\]

The proof for \( k \) close to \( n \) follows similarly.

Proof of Corollary 3. From the formulas the chance that \( K_\pi = k \) is

\[
k \binom{n-1}{k-1} \frac{n-k-1}{(k-1)(n+1)^{n-1}} = \frac{1}{n-k+1} P(X = k)
\]

For \( k \) of form \( n-j \), the asymptotic used above shows that this tends to \( q(j) \). For any fixed \( \pi_1 \) from \( 1, 2, \ldots, n-j \) the chance of \( \pi_1 \) and \( K_\pi \) is asymptotic to \( q(j)/(n-j) \) by a similar calculation (\( j \) fixed and \( n \) tending to infinity). The result follows.

Proof of Corollary 4. When \( j = 0 \), from Corollary 1

\[
P(\pi_1 = n) = \frac{n^{n-2}}{(n+1)^{n-1}} = \frac{1}{n} \frac{1}{(1+1/n)^{n-1}} \sim \frac{1}{en} = \frac{P(X \leq 2)}{n}
\]

The proof for large \( n \) and fixed \( j \) follows similarly.

Proof of Corollary 5. Abel’s generalization of the binomial theorem (Pitman 2002; Riordan 1968) gives

\[
\sum_a \binom{n}{a} (x+a)^{a-1}(y+n-a)^{n-a-1} = (x^{-1} + y^{-1})(x+y+n)^{n-1}
\]

Apply this to Corollary 1 when \( n \to n-1, x \to 1 \) and \( y \to 1 \), and to see that the number of parking functions with \( \pi_1 = 1 \) is \( 2(n+1)^{n-2} \). Now standard asymptotics gives

\[
P(\pi_1 = 1) = \frac{2}{n+1} \sim \frac{1 + P(X \geq 2)}{n}
\]

The proof for large \( n \) and any fixed \( j \) follows similarly.
Proof of Corollary 6. From Corollary 1, $P(\pi_1 = j)$ is monotone decreasing in $j$. Now Corollary 4 and Corollary 5 show that, for large $n$,

\[ P(\pi_1 = j) = \frac{1 + P(X \geq j + 1)}{n}(1+o(1)), \quad P(\pi_1 = n-j) = \frac{P(X \leq j + 1)}{n}(1+o(1)), \]

with $Q_n$ tending to zero. Thus

\[ \|P_n - Q_n\| = \frac{1}{2} \sum_{j=1}^{n} |P(\pi_1 = j) - P(f_1 = j)| = \frac{1}{2} \sum_{j=1}^{n} |nP(\pi = j) - 1| \frac{1}{n}. \]

The terms in absolute value are bounded and are uniformly small for $j^* \leq j \leq n - j^*$ for appropriately chosen $j^*$. Any bounded number of terms when multiplied by $1/n$ tend to zero. The result follows.

Proof of Theorem 2. From Corollary 1

\[ \sum_{t=1}^{n} t \# \{\pi \in PF_n : \pi_1 = t\} = \sum_{t=1}^{n} \sum_{s=0}^{n-t} \binom{n-1}{s}(s+1)^{s-1}(n-s)^{n-s-2} \]

\[ = \sum_{s=0}^{n-1} \binom{n-1}{s}(n-s)^{n-s-2}(s+1)^{s-1} \sum_{t=1}^{n-s} 1 \]

\[ = \frac{1}{2} \sum_{s=0}^{n-1} \binom{n-1}{s}(n-s)^{n-s-1}(s+1)^{s-1} + (n-s+1) \]

\[ = \frac{1}{2} \sum_{s=0}^{n} \binom{n-1}{s}(n-s)^{n-s-1}(s+1)^{s-1} \]

\[ = \frac{1}{2} \sum_{s=0}^{n} \binom{n-1}{s}(n-s)^{n-s}(s+1)^{s-1} \]

\[ = \frac{1}{2}(I + II). \]

Use Abel’s identity above to see

\[ I = (n + 1)^{n-1}. \]
A variant of Abel’s identity gives

\[ II = \sum_{k=0}^{n-1} \binom{n-1}{k} (n+1)^k (n-k-1)! (n-k) \]

\[ = (n-1)! \sum_{k=0}^{n-1} \frac{(n+1)^k}{k!} (n-k) \]

\[ = (n-1)! \left[ n \sum_{k=0}^{n-1} \frac{(n+1)^k}{k!} - \sum_{k=1}^{n-1} \frac{(n+1)^k}{(k-1)!} \right] \]

\[ = n(n+1)^n - (n-1)! \sum_{k=0}^{n-2} \frac{(n+1)^k}{k!}. \]

Combining terms and dividing by \((n+1)^{n-1}\),

\[ E(\pi_1) = 1 + \frac{n}{2} + \frac{n-1}{2(n+1)^{n-1}} - \frac{(n-1)!}{2(n+1)^{n-1}} \sum_{k=0}^{n-2} \frac{(n+1)^k}{k!}. \]

From Stirling’s formula,

\[ \frac{(n-1)!}{(n+1)^{n-1}} \sim \sqrt{2\pi n^{1/2}} e^{-n-1}. \]

Finally,

\[ e^{-(n+1)} \sum_{k=0}^{n-2} \frac{(n+1)^k}{k!} \]

equals the probability that a Poisson random variable with parameter \(n+1\) is less than or equal to \(n-2\). This is asymptotic to \(1/2\) by the law of large numbers.

Remarks. • [Eu et al. (2005)] also enumerate parking functions by leading terms (in a more general case) using a bijection to labeled rooted trees. Knuth gives a generating function for the area of a parking function which he justifies by a nice argument using the final value of the parking function [Knuth (1998b)]. Not surprisingly, since any rearrangement of a parking function is also a parking function, he also relies on a shuffle of parking functions in the sense described above to get essentially the same formula in a slightly different language. [Foata and Riordan (1974)] give the same generating function and several others based on the values of the entries of \(\pi\).
• We find it somewhat mysterious that the Borel distribution sums to one:

\[ \sum_{j=1}^{\infty} \frac{e^{-j} j^{-1}}{j!} = 1. \]

The usual proof of this interprets the summands as the probability that a classical Galton-Watson process with Poisson(1) births dies out with \( j \) total progeny. The argument makes nice use of the Lagrange inversion theorem. Our colleague Kannan Soundararajan produced an elementary proof of a more general fact, which while new to us, Richard Stanley kindly pointed out, also occurs in Stanley (1999b, p.28). For \( 0 \leq x \leq 1 \),

\[ \sum_{j=1}^{\infty} \frac{e^{-xj} (x j)^{-1}}{j!} = 1. \]

Indeed, for \( k \geq 1 \), the coefficient of \( x^k \) on the left hand side is

\[ \sum_{j=1}^{k+1} \frac{j^{k-1} (-j)^{k-j+1}}{j! (k-j+1)} = (-1)^k + 1 \sum_{j=1}^{k+1} (-1)^j j^k \binom{k+1}{j}. \]

This last is zero for all \( k \geq 1 \) since

\[ \sum_{j=0}^{n} (-1)^j j^k \binom{n}{j} = 0 \]

for all fixed \( k \) from 0 to \( n - 1 \) as one sees by applying the transformation \( T(f(x)) = xf'(x) \) to the function \((1 - x)^n\) and setting \( x = 1 \).

2.2 Many coordinates of a random parking function are jointly uniform.

Corollary 6 shows that any single coordinate is close to uniform in a strong sense (total variation distance). In this section we show that any \( k \) coordinates are close to uniform, where \( k \) is allowed to grow with \( n \) as long as \( k \ll \sqrt{\frac{n}{\log(n)}} \). The argument gives insight into the structure of parking functions so we give it in a sequence of simple observations summarized by a theorem at the end. The argument was explained to us by Sourav Chatterjee.
Fix $k$ and $i_1, i_2, \ldots, i_k$ in $[n]$. Let $x_j = i_j/n$. For $\pi \in \text{PF}_n$, let $F^\pi(x) = \frac{1}{n} \sum_{j=1}^{n} \delta_{\pi_j/n \leq x}$ be the empirical distribution function of $\{\pi_i/n\}$. By symmetry, (2.3) \[ P \left( \pi : \frac{\pi_1}{n} \leq x_1, \ldots, \frac{\pi_k}{n} \leq x_k \right) = E \left[ \frac{1}{n(n-1) \ldots (n-k+1)} \sum_{j_1, \ldots, j_k \text{ distinct}} \delta_{\frac{\pi_{j_1}}{n} \leq x_1, \ldots, \frac{\pi_{j_k}}{n} \leq x_k} \right] \]

The expectation in (2.3) is over $\pi \in \text{PF}_n$. The expression inside the expectation has the following interpretation. Fix $\pi$, pick $k$-coordinates at random from $\{\pi_1, \ldots, \pi_n\}$, sampling without replacement. Because sampling without replacement is close to sampling with replacement, Theorem 13 in Diaconis and Freedman (1980) shows (2.4) \[ \frac{1}{n(n-1) \ldots (n-k+1)} \sum_{j_1, \ldots, j_k \text{ distinct}} \delta_{\frac{\pi_{j_1}}{n} \leq x_1, \ldots, \frac{\pi_{j_k}}{n} \leq x_k} = F^\pi(x_1) \ldots F^\pi(x_k) + O \left( \frac{k(k-1)}{n} \right) \]

The error term in (2.4) is uniform in $\pi, k, n$. Since $0 \leq F^\pi(x_i) \leq 1$, by (Billingsley, 2012, 27.5) (2.5) \[ \left| \prod_{i=1}^{n} F^\pi(x_i) - \prod_{i=1}^{n} x_i \right| \leq \sum_{i=1}^{k} |F^\pi(x_i) - x_i| \]

Let $f = (f_1, \ldots, f_n)$ be a random function from $[n]$ to $[n]$ with $G^f(x) = \frac{1}{n} \sum \delta_{f_i/n \leq x}$. By Hoeffding’s inequality for the binomial distribution, for any $\epsilon > 0$ and any $x \in [0, 1]$, (2.6) \[ P \left( |G^f(x) - x| > \epsilon \right) \leq 2e^{-2\epsilon^2 n}. \]

Since a random parking function is a random function, conditional on being in $\text{PF}_n$, (2.7) \[ P \left( |F^\pi(x) - x| > \epsilon \right) = P \left( \left| G^f(x) - x \right| > \epsilon | f \in \text{PF}_n \right) = \frac{P \left( \left| G^f(x) - x \right| > \epsilon \text{ and } f \in \text{PF}_n \right)}{P(f \in \text{PF}_n)} \leq 2e^{-2\epsilon^2 n} \left( 1 - \frac{1}{n+1} \right)^{n-1} < 2ne^{-2\epsilon^2 n}. \]

By a standard identity (Billingsley, 2012, 21.9), (2.8) \[ E[F^\pi(x) - x] \leq 2\sqrt{\frac{\log(n)}{n}} \text{ for } n \geq 4 \text{ uniformly in } x. \]
Combining bounds gives

**Theorem 3.** For $\pi$ uniformly chosen in $\text{PF}_n$ and $f$ uniformly chosen in $\mathcal{F}_n$, for all $n \geq 4$,

\[
\sup_{x_1,\ldots,x_k} \left| P \left( \frac{\pi_1}{n} \leq x_1, \ldots, \frac{\pi_k}{n} \leq x_k \right) - P \left( \frac{f_1}{n} \leq x_1, \ldots, \frac{f_k}{n} \leq x_k \right) \right| \leq 2k \sqrt{\frac{\log(n)}{n}} + \frac{k(k-1)}{n}.
\]

The sup is over $x_j = \frac{i_j}{n}$ with $i_j \in [n]$, $1 \leq j \leq k$.

**Remark.** We are not at all sure about the condition $k \ll \sqrt{n/(\log(n))}$. For the distance used in Theorem 3, the bound in (2.4) for sampling without replacement can be improved to $k/n$ using a result of Bobkov (Bobkov, 2005). The only lower bound we have is $k \ll n$ from events such as having more than one $n$, more than two $n$s or $n-1$s, etc. In particular, the probability that a randomly chosen function $f: [k] \rightarrow [n]$ has more than $i$ entries more than $n-i+1$ for some $i$ (and thus cannot possibly be the first $k$ entries of a parking function) is

\[
1 - \frac{(n-k+1)(n+1)^{k-1}}{n^k}.
\]

If $k = sn$ for some constant $0 < s < 1$, the probability goes to $1 - (1-s)e^s > 0$, so $k \ll n$.

### 3 Voyeurism (using the literature)

In this section we enjoy ourselves by looking through the literature on parking functions, trying to find theorems or generating functions which have a simple probabilistic interpretation. We took Catherine Yan’s wonderful survey and just began turning the pages. The first “hit” was her Corollary 1.3, which states

\[
(3.1) \sum_{\pi \in \text{PF}_n} q^{\text{car}\{i: \pi_i=\pi_{i+1}\}} = (q + n)^{n-1}.
\]

This looks promising; what does it mean? Divide both sides by $(n+1)^{n-1}$. The left side is $E\{q^{R(\pi)}\}$ with $R(\pi) = \text{car}\{i: \pi_i=\pi_{i+1}\}$ the number of repeats in $\pi$. The
right side is the generating function of \( S_{n-1} = X_1 + \cdots + X_{n-1} \), with \( X_i \) independent and identically distributed, with

\[
X_1 = \begin{cases} 1 & \text{probability } 1/(n+1) \\ 0 & \text{probability } n/(n+1). \end{cases}
\]

A classical theorem (Feller, 1971, p. 286) shows that \( P_{n-1}(j) = P(X_1 + \cdots + X_{n-1} = j) \) has an approximate Poisson distribution \( Q(j) = 1/e^j \),

\[
P_n(j) \sim \frac{1}{(e^j)!}.
\]

Indeed,

\[
\|P_n - Q\|_{TV} \leq \frac{9}{8} \frac{(n-1)}{(n+1)^2}.
\]

Putting things together gives

**Theorem 4.** Pick \( \pi \in \text{PF}_n \) uniformly. Let \( R(\pi) \) be the number of repeats in \( \pi \) reading left to right. Then for fixed \( j \) and \( n \) large,

\[
P\{R(\pi) = j\} \sim \frac{1}{(e^j)!}.
\]

**Remark.** How does our analogy with random functions play out here? Almost perfectly. If \( f = (f_1, \ldots, f_n) \) is a function \( f : [n] \to [n] \), let

\[
R(f) = \sum_{i=1}^{n-1} Y_i(f) \quad \text{with } Y_i(f) = \begin{cases} 1 & \text{if } f_i = f_{i+1} \\ 0 & \text{otherwise.} \end{cases}
\]

It is easy to see that if \( f \) is chosen uniformly at random then the \( Y_i \) are independent with

\[
Y_i(f) = \begin{cases} 1 & \text{probability } 1/n \\ 0 & \text{probability } (n-1)/n. \end{cases}
\]

The Poisson limit theorem says

**Theorem 5.** For \( f \in \mathcal{F}_n \) chosen uniformly, for fixed \( j \) and large \( n \),

\[
P\{R(f) = j\} \sim \frac{1}{e^j!}.
\]

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The next result that leapt out at us was in Yan (2015, Sect. 1.2.2). If \( \pi \in \text{PF}_n \) is a parking function, say that car \( i \) is “lucky” if it gets to park in spot \( \pi_i \). Let \( L(\pi) \) be the number of lucky cars. Gessel and Seo (2006) give the generating function

\[
\sum_{\pi \in \text{PF}_n} q^{L(\pi)} = q \prod_{i=1}^{n-1} [i + (n - i + 1)q].
\]

As above, any time we see a product, something is independent. Dividing both sides by \((n + 1)^{n-1}\), the right side is the generating function of \( S_n = 1 + \sum_{i=1}^{n-1} X_i \) with \( X_i \) independent,

\[
X_i = \begin{cases} 
0 & \text{probability } i/(n+1) \\
1 & \text{probability } 1 - i/(n+1). 
\end{cases}
\]

By elementary probability, the mean and variance of \( S_n \) are

\[
\mu_n = 1 + \sum_{i=1}^{n-1} \left( 1 - \frac{i}{n+1} \right) = n - \frac{1}{2} \frac{n(n-1)}{n+1} \sim \frac{n}{2},
\]

\[
\sigma_n^2 = \sum_{i=1}^{n-1} \frac{i}{n+1} \left( 1 - \frac{i}{n+1} \right) \sim \frac{n}{6}.
\]

Now, the central limit theorem (Feller, 1971, p. 262) applies.

**Theorem 6.** For \( \pi \in \text{PF}_n \) chosen uniformly, let \( L(\pi) \) be the number of lucky \( i \). Then for any fixed real \( x, -\infty < x < \infty \), if \( n \) is large,

\[
P \left\{ \frac{L(\pi) - n/2}{\sqrt{n/6}} \leq x \right\} \sim \int_{-\infty}^{x} e^{-t^2/2} \sqrt{2\pi} \, dt.
\]

One more simple example: Let \( N_1(\pi) = \text{car}\{i : \pi_i = 1\} \). Yan (2015, Cor. 1.16) gives

\[
\sum_{\pi \in \text{PF}_n} q^{N_1(\pi)} = q(q + n)^{n-1}.
\]

Dividing through by \((n + 1)^{n-1}\) shows that \( N_1(\pi) \) has exactly the same distribution as of \( 1 + X \) where \( X \) has a binomial\((n - 1, 1/(n+1))\) distribution. As above, a Poisson limit holds.

**Theorem 7.** Let \( \pi \in \text{PF}_n \) be chosen uniformly. Then \( Z(\pi) \), the number of ones in \( \pi \), satisfies

\[
P\{Z(\pi) = 1 + j\} \sim \frac{1}{e j!}
\]

for \( j \) fixed and \( n \) large.
Remark. Any \( \pi \in \text{PF}_n \) has \( N_1(\pi) \geq 1 \) and \( \pi_i \equiv 1 \) is a parking function. The theorem shows that when \( n \) is large, \( N_1(\pi) = 1 \) with probability \( \frac{1}{e} \) and

\[
E(N_1(\pi)) = \frac{2n}{(n+1)} \sim 2.
\]

A random function has \( N_1(f) \) having a limiting Poisson(1) distribution; just 1 off \( N_1(\pi) \). A parking function can have at most one \( i \) with \( \pi_i = n \). Let \( N_i(\pi) \) be the number of \( i \)'s in \( \pi \). An easy argument shows \( P(N_n(\pi) = 1) \sim \frac{1}{e} \), \( P(N_n(\pi) = 0) \sim 1 - \frac{1}{e} \). A random function has \( N_i(f) \) with an approximate Poisson(1) distribution for any \( i \). Interestingly, note that this is already an example of a distinct difference between \( \text{PF}_n \) and \( \text{F}_n \): while \( P(N_n(f) = 1) \sim \frac{1}{e} \), \( P(N_n(f) = 0) = \frac{1}{e} \) in contrast to the parking function case.

4 Equality of ensembles

Previously we have found the same limiting distributions for various features \( T(\pi) \) and \( T(f) \) with \( \pi \) random in parking functions \( \text{PF}_n \) and \( f \) random in all functions \( \text{F}_n \). This section gives collections of features where the two ensembles have exactly the same distribution for fixed \( n \). These have a rich classical structure involving a determinantal point process, explained below.

To get an exact equivalence it is necessary to slightly change ensembles. Let \( \tilde{\text{F}}_n = \{ f : [n] \to [n+1] \} \). Thus \( |\tilde{\text{F}}_n| = (n+1)^n \), \( \text{PF}_n \subseteq \tilde{\text{F}}_n \), and \( |\text{PF}_n|/|\tilde{\text{F}}_n| = 1/(n+1) \). One might ask here why we ever consider \( \tilde{\text{F}}_n \) rather than the more seemingly natural \( \text{F}_n \). Elements of \( \text{PF}_n \) are naturally selected uniformly at random by first uniformly selecting element from \( \tilde{\text{F}}_n \) and then doing a bit of additional work to find a true parking function. In practice, this means that while we can get asymptotic results using \( \text{PF}_n \) and \( \text{F}_n \), we can generally get precise results for all \( n \) using \( \tilde{\text{F}}_n \). In practice, this also tells us about the asymptotics when we compare \( \text{PF}_n \) and \( \text{F}_n \), since comparing \( \text{F}_n \) and \( \tilde{\text{F}}_n \) is generally quite easy.

For \( f \in \tilde{\text{F}}_n \) let \( X_i(f) = 1 \) if \( f_{i+1} < f_i \) and 0 otherwise, so \( X_1, X_2, \ldots, X_{n-1} \) give the descent pattern in \( f \). The following theorem shows that the descent pattern is the same on \( \text{PF}_n \) and \( \tilde{\text{F}}_n \).

Theorem 8. Let \( \pi \in \text{PF}_n \) and \( f \in \tilde{\text{F}}_n \) be uniformly chosen. Then

\[
P \{ X_1(\pi) = t_1, \ldots, X_{n-1}(\pi) = t_{n-1} \} = P \{ X_1(f) = t_1, \ldots, X_{n-1}(f) = t_{n-1} \}
\]

for all \( n \geq 2 \) and \( t_1, \ldots, t_{n-1} \in \{0, 1\} \).
The same theorem holds with $X_i$ replaced by $Y_i(f) = \begin{cases} 1 & \text{if } f_{i+1} = f_i \\ 0 & \text{otherwise,} \end{cases}$

$W_i(f) = \begin{cases} 1 & \text{if } f_{i+1} \leq f_i \\ 0 & \text{otherwise,} \end{cases}$

or for the analogs of $X_i, W_i$ with inequalities reversed.

For equalities, the $\{Y_i(f)\}_{i=1}^{n-1}$ process is independent and identically distributed with $P(Y_i = 1) = \frac{1}{f(n+1)}$. Thus any fluctuation theorem for independent variables holds for $\{Y_i(\pi)\}_{i=1}^{n-1}$. In particular, this explains (3.1) above.

The descent pattern in a random sequence is carefully studied in [Borodin et al. 2010]. Here is a selection of facts, translated to the present setting. Throughout, $X_i = X_i(\pi), 1 \leq i \leq n - 1$, is the descent pattern in a random function in $\tilde{F}_n$ (and thus by the previous theorem a descent pattern in a random parking function).

**Single descents**

\begin{equation}
P(X_i = 1) = \frac{1}{2} - \frac{1}{2(n+1)}. \tag{4.1}
\end{equation}

**Run of descents** For any $i, j$ with $1 \leq i + j \leq n$,

\begin{equation}
P(X_i = X_{i+1} = \cdots = X_{i+j-1} = 1) = \binom{n+1}{j} / (n+1)^j. \tag{4.2}
\end{equation}

In particular,

\[
\text{Cov}(X_iX_{i+1}) = E(X_iX_{i+1}) - E(X_i)E(X_{i+1}) = -\frac{1}{12} \left( 1 - \frac{1}{(n+1)^2} \right).
\]

**Stationary one-dependence** The distribution of $\{X_i\}_{i \in [n-1]}$ is stationary: for $J \subseteq [n-1], i \in [n-1]$ with $i + J \subseteq [n-1]$, the distribution of $\{X_j\}_{j \in J}$ is the same as the distribution of $\{X_j\}_{j \in i + J}$. Further, the distribution of $\{X_i\}_{i \in [n-1]}$ is one-dependent: if $J \subseteq [n-1]$ has $j_1, j_2 \in J \Rightarrow |j_1 - j_2| > 1$, then $\{X_j\}_{j \in J}$ are jointly independent binary random variables with

\begin{equation}
P(X_j = 1) = \frac{1}{2} - \frac{1}{2(n+1)}. \tag{4.3}
\end{equation}

The following central limit theorem holds.
Theorem 9. For \( n \geq 2 \), \( S_{n-1} = X_1 + \cdots + X_{n-1} \) has

\[
\text{mean } (n - 1) \left( \frac{1}{2} - \frac{1}{2(n+1)} \right),
\]

\[
\text{variance } \frac{(n+1)}{12} \left( 1 - \frac{1}{(n+1)^2} \right),
\]

and, normalized by its mean and variance, \( S_{n-1} \) has a standard normal limiting distribution for large \( n \).

**k-point correlations** For \( A \subseteq [n-1] \),

\[
P(X_i = 1 \text{ for } i \in A) = \prod_{i=1}^{k} \left( \frac{n+1}{a_i+1} \right) / (n+1)^{a_i+1}
\]

if \( A = \bigcup A_i \), with \( |A_i| = a_i \) and \( A_i \) disjoint, nonempty consecutive blocks, e.g., \( A = \{2,3,5,6,7,11\} = \{2,3\} \cup \{5,6,7\} \cup \{11\} \). (It has \( a_1 = 2 \), \( a_2 = 3 \), and \( a_3 = 1 \).)

**Determinant formula** Let \( \epsilon_1, \epsilon_2, \ldots, \epsilon_{n-1} \in \{0,1\} \) have exactly \( k \) ones in positions \( s_1 < s_2 < \cdots < s_k \).

\[
P\{X_1 = \epsilon_1, \ldots, X_n = \epsilon_n\} = \frac{1}{(n+1)^n} \det \left( \frac{s_{j+1} - s_i + n}{n} \right).
\]

The determinant is of a \((k+1) \times (k+1)\) matrix with \((i,j)\) entry \( \left( \frac{s_{j+1} - s_i + n}{n} \right) \) for \( 0 \leq i, j \leq n \) with \( s_0 = 0 \), \( s_{k+1} = n \).

In [Borodin et al. (2010)](Borodin), these facts are used to prove that \( \{X_i\}_{i \in [n-1]} \) is a determinantal point process and a host of further theorems are given.

The proof of Theorem 8 proves somewhat more. Let \( P \) be a partial order on \([n]\) formed from a disjoint union of chains:

\[
P = \begin{pmatrix}
3 & 4 \\
5 & 6 \\
1 & 2
\end{pmatrix}
\]

**Definition.** A function \( f \in \tilde{F}_n \) is \( P \)-monotone if \( i <_P j \) implies \( f(i) < f(j) \).
Theorem 10. Let $P$ be a poset on $[n]$ formed from disjoint chains. Then, for $\pi \in \text{PF}_n$, $f \in \tilde{F}_n$ uniformly chosen,

$$P\{\pi \text{ is } P\text{-monotone}\} = P\{f \text{ is } P\text{-monotone}\}.$$ 

Proof. Let $f \in \tilde{F}_n$ be $P$-monotone. Consider the set

$$(4.6) \quad S(f) = \{f +_{n+1} k(1, \ldots, 1)\}$$

when $+_{n+1}$ indicates addition mod $n + 1$ and representatives of each equivalence class are chosen from $[n]$. Note that $f' \in S(f)$ need not be $P$-monotone, e.g., the addition mod $n + 1$ may cause the largest element to be smallest. Simply selecting the numbers corresponding to each chain in $P$ and reordering them to be $P$-monotone within each chain results in a new element of $\tilde{F}_n$ consistent with $P$. Let $S'(f)$ be the reordered functions. Note that if $k$ in (4.6) is known, $f$ can be uniquely reconstructed from any element in $S'(f)$. Finally observe that exactly one element of $S'(f)$ is in $\text{PF}_n$. Since $|\tilde{F}_n| = (n + 1)|\text{PF}_n|$, this implies the result. \qed

Example. With $n = 4$,

$$P = \begin{array}{ccc}
1 & \ & 2 \\
& 3 & \\
& & 4
\end{array}$$

and $f = (3, 4, 5, 1)$, $f$ is $P$-monotone but $S(f)$ contains $(4, 5, 1, 2)$ which is not $P$-monotone. However, $(1, 4, 5, 2)$ is $P$-monotone, and would be used as a new element of $\tilde{F}_n$ consistent with $P$.

Theorem 10 can be generalized. For every chain $c$ in a poset $P$ of disjoint chains, let $<_c$ be one of $<, >, \leq, \geq, =$. Say $a <_P b$ if $a <_P b$ and $a, b \in c$. Say $f \in \tilde{F}_n$ is $P$-chain monotone if for all chains $c$ in $P$ and $i <_P j$ implies $f(i) <_c f(j)$. Thus we assign to disjoint subsets of $[n]$ the requirement that $f$ is either (weakly) decreasing or (weakly) increasing on the subset in some order. The argument for Theorem 10 gives
**Theorem 11.** Let $P$ be a poset on $[n]$ formed from a union of disjoint chains with order $P_c$ specified on each chain as above. Let $\pi \in \text{PF}_n$, $f \in \tilde{\mathcal{F}}_n$ be uniformly chosen. Then

$$P\{\pi \text{ is } P\text{-chain monotone}\} = P\{f \text{ is } P\text{-chain monotone}\}.$$ 

**Remarks.** 1. Theorems 9, 10 imply Theorem 7: For $f$ in $\tilde{\mathcal{F}}_n$ let

$$\text{Des}(f) = \{i \in [n-1] | f(i+1) < f(i)\}.$$ 

By inclusion-exclusion, it is enough to show that for any $S$ contained in $[n-1]$, $P(S \subseteq \text{Des}(f)) = P(S \subseteq \text{Des}(\pi))$. For every $i$ in $S$, say that $i-1 <_P i+1$. These form the covering relations of the poset $P$; if $S = \{1, 3, 4, 7, 9, 10\}$ the poset is the following:

```
1 2 3 4 5 6 7 8 9 10
1. 2. 3. 4. 5. 6. 7. 8. 9. 10.
```

The $f$'s that are $P_S$ monotone (with respect to $>$) are exactly the $f$'s with $S \subseteq \text{Des}(f)$.

2. Theorem 11 can be combined with inclusion/exclusion to give further results. For example, considering

```
P_1: 1 \prec_2 3 \prec 3 \prec 3 \\
P_2: 1 \prec 3 \prec_2 3 \\
```

shows that the chance of $f(1) < f(2) \geq f(3)$ as the same for both $\text{PF}_n$ and $\tilde{\text{PF}}_n$. Call such an occurrence a “weak peak at position 2”. Weak peaks at position $i$ are the same for randomly chosen $f$ and $\pi$. 

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3. Results for longest consecutive increasing sequences have the same distribution in $\mathcal{PF}_n$ and $\mathcal{F}_n$. For example:

\[
\begin{array}{cccccc}
 & & & < & & \\
& & & 4 & & \\
 &3 & &5 & &6 \\
 &2 & & & 7 & \\
 &1 & & & & \\
\end{array}
\quad
\begin{array}{cccccc}
 & & & > & & \\
& & & 5 & & \\
 &3 & &6 & &8 \\
 &2 & & & 7 & \\
 &1 & & & & \\
\end{array}
\]

$P_1: \quad 1 \cdot 5 \cdot 6 \quad P_2: \quad 3 \cdot 6 \cdot 8$

Functions consistent with $(P_1 <_c)$ and not $(P_2 <_c)$ have longest increasing sequences of length 4, e.g., $f = (1, 2, 3, 4, 4, 5, 6, 5)$.

4. The descent patterns above appear as local properties but they also yield global properties such as the total number of descents (or equalities, . . .) being equidistributed.

5. We tried a number of generalizations of the patterns above which failed to be equidistributed. This includes peaks, $f(1) < f(2) > f(3)$, or mixed orderings within a chain, $f(1) \leq f(2) < f(3)$. Equidistribution also failed for $P$ made from non-disjoint chains, such as

\[
\begin{array}{ccc}
 & & \\
\cdot 3 & \cdot 5 & \\
\cdot 2 & & \\
\cdot 1 & \cdot 4 & \\
\end{array}
\]

or forced differences larger than one, e.g., $f(1) < f(2) - 1$. Of course, asymptotic equidistribution may well hold in some of these cases.
Remark. Note that Corollary 4 already generally shows that we should not expect an equality of ensembles for statistics (such as the number of \( \pi_i = 1 \)) that are computed based on the values of the cars, even in the limit. Theorem 11 gives us that we should generally expect an equality of ensembles for certain types of statistics that are based on the relative values of cars for every \( n \). We do not have an example of a statistic that cannot be derived from Theorem 11 that does “show an equivalence of ensembles,” either for every \( n \) or in the limit, and would be quite interested in such an example. An open question is simply to describe in some natural way the set of statistics that can be shown to demonstrate our desired equality of ensembles by Theorem 11.

There is another way to see the above equivalence of ensembles, as kindly remarked by Richard Stanley.

Definition (species). Let \( \mu_i \) be the number of entries in a function \( f : [n] \to [n+1] \) which occur exactly \( i \) times. (In particular, let \( \mu_0 \) give the number of values in \([n+1]\) which do not occur in \( f \), so that \( n+1 - \mu_0 \) gives the number of distinct values taken on by \( f \).) Then call \( \mu(f) = (\mu_0, \mu_1, \ldots, \mu_n) \) the species of \( f \).

Example. Let \( f : [6] \to [7] \) such that \( (f[1], \ldots, f[6]) = (2, 1, 2, 1, 1, 1) \). The species of \( f \) is \( (5, 0, 1, 0, 1, 0, 0) \).

Note that the species of \( \pi \) is the same as the species of \( \pi +_{n+1} (1, \ldots, 1) \). Thus by a similar argument to above, the distribution of species of functions in \( PF_n \) and \( \tilde{F}_n \) are equidistributed. (Stanley kindly noted this can also be derived from the one variable Frobenius characteristic of the diagonal harmonics, when expressed in terms of the homogeneous basis. The relationship to this space will be explained in Section 6.) Whether or not a function fits a poset \( P \) of disjoint chains (as defined above) depends only on its species. This follows from the theory of P-partitions, since in this language, functions which fit a poset \( P \) are order preserving or order reversing maps of \( P \) (depending on the choice of \( \leq \) or \( \geq \)) or the strictly ordering preserving or order reversing (depending on the choice of \( < \) or \( > \)) maps of \( P \). See Stanley (1997a, p. 211) for an introduction to P-partitions.

There has been extensive work on the distribution of species counts in a random function. The following list gives a brief description of available distributional results. They are stated for random functions but of course, they apply to random parking functions.

Let \( b \) balls be allocated, uniformly and independently, into \( B \) boxes. Let \( \mu_r(b, B) = \)
\( \mu_r \) be the number of boxes containing \( r \) balls. Thus

\[ (4.7) \quad \sum_{r=0}^{b} \mu_r = B \text{ and } \sum_{r=1}^{b} r \mu_r = b \]

The joint distribution of \( \{ \mu_r \}_{r=0}^{b} \) is a classical subject in probability. It is the focus of Kolchin et al. (1978, Ch. 2) which contains extensive historical references. The following basics give a feeling for the subject. Throughout, we take \( B = n + 1 \) and \( b = n - 1 \) as this is the object of interest for \( \text{PF}_n \) and \( \tilde{\mathcal{F}}_n \).

1. The joint distribution is specified exactly by Kolchin et al. (1978, p.36):

\[
P \{ \mu_n = m_n, 0 \leq r \leq b \} = \begin{cases} \frac{B! (b)_r}{B^r r! m_r!} & \text{if } (4.7) \text{ is satisfied} \\ 0 & \text{else} \end{cases}
\]

2. The means and covariances are:

\[
E(\mu_r) = B \left( \frac{b}{r} \right) \frac{1}{B^r} \left( 1 - \frac{1}{B} \right)^{b-r}
\]

\[
E(\mu_r^2) = E(\mu_r) + B(B-1) \frac{b(2r)}{r!^2B^{2r}} \left( 1 - \frac{2}{B} \right)^{b-2r}
\]

\[
E(\mu_r \mu_t) = B(B-1) \frac{b^{r+t}}{r!t!N^{r+t}} \left( 1 - \frac{2}{N} \right)^{b-r-t}
\]

where \( x^r = x(x-1) \ldots (x-r+1) \).

3. Asymptotically, as \( n \to \infty \), for fixed \( r, t \),

\[
E(\mu_r) = \frac{n}{er!} + \frac{1}{er!} \left( r - \frac{1}{2} \binom{r}{2} \right) + O \left( \frac{1}{n} \right)
\]

\[
\text{Cov}(\mu_r, \mu_t) \sim n\sigma_{rt},
\]

\[
\sigma_{rr} = \frac{1}{er!} \left( 1 - \frac{1}{er!} - \frac{1}{er!} (1-r)^2 \right), \quad \sigma_{rt} = \frac{1}{e^2r!t!} (1 + (1-r)(1-t)).
\]

4. For any fixed \( s \) and \( 0 \leq r_1 < r_2 \ldots < r_s \), the random vector

\[
X = \left( \frac{\mu_{r_1} - E(\mu_{r_1})}{\sqrt{n}}, \frac{\mu_{r_2} - E(\mu_{r_2})}{\sqrt{n}}, \ldots, \frac{\mu_{r_s} - E(\mu_{r_s})}{\sqrt{n}} \right)
\]

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has limiting covariance matrix \( \Sigma = (\sigma_{ij})_{1 \leq i,j \leq s} \) with \( \sigma_{ij} \) as defined just above. This matrix can be shown to be positive definite and Kolchin et al. [1978, p. 54] imply that the vector has a limiting normal approximation.

If \( D^2 = \det(\Sigma) \), they show

\[
P(X \in G) = \frac{1}{(2\pi)^{s/2}} \int_G e^{-\frac{1}{2D^2} \sum_{i,j=1}^s \Sigma_{i,j} \mu_i \mu_j} d\mu_1 \ldots d\mu_s + o(1).
\]

The also give local limit theorems and rates of convergence.

5. For \( n - 1 \) balls dropped into \( N + 1 \) boxes, the individual box counts are well approximated by independent Poisson(1) random variables. In particular, the maximal box count \( m_n \) is distributed as the maximum of \( n \) independent Poisson(1) variables. This is well known to concentrate on one of two values with slowly oscillating probabilities. Kolchin [1986, Ch. 2] show that if \( r = r(n) \) is chosen so that \( r > 1 \) and \( \frac{n}{e^r} \to \lambda \) where \( \lambda > 0 \), then

\[
P(\max m_n = r - 1) \to e^{-\lambda}, \quad P(m_n = r) \to 1 - e^{-\lambda}
\]

Very roughly, \( m_n \approx \frac{\log n}{\log \log n} \). See Briggs et al. [2009]. This determines the largest \( r \) such that \( \mu_r > 0 \).

6. There are many further properties of random functions \( f : [n] \to [n+1] \) known. See Aldous and Pitman [1994] and Kolchin et al. [1978]. We mention one further result, consider such a random function and let \( l(f) \) be the length of the longest increasing subsequence as \( f \) is read in order \( f(1), f(2), \ldots, f(n) \). This has the same asymptotic distribution as the length of the longest increasing subsequence in a random permutation. Indeed the asymptotics of the shape of the tableau of \( f \) under the RSK algorithm matches that of a random permutation. In particular,

\[
P\left( \frac{l(f) - 2\sqrt{n}}{n^{1/6}} \leq x \right) \to F(x)
\]

with \( F(x) \) the Tracy-Widom Distribution. See Baik et al. [1999].

7. One particularly well known statistic that is equidistributed on \( \tilde{F}_n \) and PF\(_n\) and clearly follows from their equidistribution of species is the number of inversions: \( \{ f(i) > f(j) : i < j \} \).

\[\text{\textsuperscript{1}}\text{We switch to function values here, for a moment, which are of course not equidistributed between } \tilde{F}_n \text{ and PF}_n, \text{ but it will be translated into species in a moment.}\]
5 From probability to combinatorics

For \( \pi \in \text{PF}_n \) and \( 0 \leq x \leq 1 \), let

\[
F^\pi(x) = \frac{1}{n}, \#\{i : \pi_i \leq nx\}.
\]

From the definitions, \( F^\pi(x) \geq x \) for \( x = \frac{i}{n}, \ 0 \leq i \leq n \). The main result of this section studies \( \{F^\pi(x) - x\}_{0 \leq x \leq 1} \) as a stochastic process when \( \pi \) is chosen uniformly. It is shown that \( \sqrt{n}\{F^\pi(x) - x\}_{0 \leq x \leq 1} \) converges to the Brownian excursion process \( \{E_x\}_{0 \leq x \leq 1} \). This last is well studied and the distribution of a variety of functions are available. One feature of these results: they show a deviation between parking functions \( \pi \) and all functions \( f \). For a random function, \( \sqrt{n}\{F^f(x) - x\}_{0 \leq x \leq 1} \) converges to the Brownian bridge \( \{B_x^0\}_{0 \leq x \leq 1} \). This has different distributions for the functions of interest. We state results for three functions of interest followed by proofs.

**Theorem 12** (Coordinate counts). For \( 0 < x < 1 \) fixed,

\[
\frac{\#\{i : \pi_i < nx\} - nx}{\sqrt{n}} \Rightarrow G_x,
\]

with \( G_x \) a random variable on \([0, \infty)\) having

\[
P\{G_x \leq t\} = \frac{1}{\sqrt{2\pi x^3(1-x)^3}} \int_0^t e^{-y^2/2x(1-x)}y^2 dy.
\]

**Remark.** \( G_x \) is the square of a Gamma(3) random variable scaled by \( x(1-x) \). For a random function \( f \), a similar limit theorem holds with \( G_x \) replaced by a normal random variable.

**Theorem 13** (Maximum discrepancy).

\[
\max_{1 \leq k \leq n} \frac{\#\{i : \pi_i \leq k\} - k}{\sqrt{n}} \Rightarrow M,
\]

with

\[
P\{M \leq t\} = \sum_{-\infty < k < \infty} (1 - 4k^2t^2)e^{-2k^2t^2}.
\]
Remark. The random variable $M$ has a curious connection to number theory. Its moments are

$$E(M) = \sqrt{\frac{\pi}{2}}, \quad E(M^s) = 2^{-s/2}s(s-1)\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(s), \quad 1 < s < 2.$$ 

The function $\xi(s)$ was introduced by Riemann. It satisfies the functional equation $\xi(s) = \xi(1-s)$. See Edwards (1974) or Smith and Diaconis (1988). For a random function $f$, a similar limit theorem holds with $M$ replaced by $M_1$, where

$$P\{M_1 \leq t\} = 1 - e^{-2t^2}.$$ 

Theorem 14 (Area).

$$(5.3) \quad \frac{1}{\sqrt{n}} \left(\frac{n^2}{2} - \sum_{i=1}^{n} \pi_i\right) \Longrightarrow A,$$

where $A$ has density function

$$f(x) = \frac{2\sqrt{6}}{x^{10/3}} \sum_{k=1}^{\infty} e^{-b_k/x^2}b_k^{2/3}U(-5/6, 4/3, b_k/x^2)$$

where $U(a, b, z)$ is the confluent hypergeometric function, $b_k = -2\alpha_k^3/27$, and $\alpha_k$ are the zeros of the Airy function

$$\text{Ai}(x) = \frac{1}{\pi} \int_{0}^{\infty} \cos\left(\frac{t^3}{3} + xt\right) dt,$$

$$a_1 = -2.3381, a_2 = -4.0879, a_3 = -5.5204, \ldots, a_j \sim \left(\frac{3\pi}{2}\right)^{2/3} j^{2/3},$$

Remarks. A histogram of the area based on 50,000 samples from PF$_{100}$ is shown in Figure 2, while Figure 3 shows (a scaled version of) the limiting approximation. Motivation for studying area of parking functions comes from both parking functions and Macdonald polynomials; see Section 6.

- The density $f(x)$ is called the Airy density. It is shown to occur in a host of further problems in Majumdar and Comtet (2005) which also develops its history and properties. Specific commentary on area of parking functions and the Airy distribution is available also in Flajolet et al. (1998).

- For a random function $f$, the right side of the theorem is replaced by $A_1$, with $A_1$ a normal $(0, 1/12)$ random variable.
Motivational notes for area  For a parking function \( \pi = (\pi_1, \pi_2, \ldots, \pi_n) \), the function \( \binom{n+1}{2} - (\pi_1 + \pi_2 + \cdots + \pi_n) \) has been called the area of the parking function. This appears all over the subject. Consider first the inconvenience \( I(\pi) \), the number of extra spaces each car is required to travel past their desired space. For example, if \( n = 5 \) the parking function \((1, 3, 5, 3, 1)\) results in the parking

\[
\begin{array}{cccccc}
1 & 1 & 3 & 3 & 5 \\
\end{array}
\]

with \( I(\pi) = 2 \): the last two cars (the second preferring the space 1 and the second preferring 3) both parked one space off. We show

(5.4) \[ I(\pi) = \binom{n+1}{2} - (\pi_1 + \cdots + \pi_n). \]

When \( n = 5 \), for \( \pi \) given above,

\[ I(\pi) = \binom{6}{2} - (1 + 3 + 5 + 3 + 1) = 2. \]

To motivate the name area — and to prove (5.4) — it is helpful to have a coding of PF\(_n\) as Dyck paths in an \( n \times n \) grid, as shown in Figure 4. The construction, as first suggested by Garsia, is, from \( \pi \):
1. Label the boxes at the start of column 1 with the position of the ones in $\pi$.

2. Moving to the next highest label, say $i'$, label the boxes in column $i'$ with the positions of the $i'$'s, starting at height $#1^s + 1$.

3. Continue with $i''$, $i'''$, . . .

4. Draw a Dyck path, by connecting the left edges of each number in the lattice.

Clearly $\pi$ can be reconstructed from this data. The condition $\#\{k : \pi_k \leq i\} \geq i$ insures that a Dyck path (having no edge below the main diagonal) results. Define

---

**Figure 3:** The Airy distribution. A rescaling gives the limiting approximation of the area shown in [Figure 2](#).

---

**Figure 4:** Dyck path for $\pi = (1, 3, 5, 3, 1)$.  

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area(π) as the number of labeled boxes strictly above the main diagonal, so area(π) = 2 in the figure.

**Lemma 5.** In a parking function π,

\[
\text{area} = \text{inconvenience distance} = \left( \frac{n+1}{2} \right) - \sum_{i=1}^{n} \pi_i.
\]

**Proof.** The proof is easiest if \( \pi_1 \leq \pi_2 \leq \cdots \leq \pi_n \). In this case the resulting lattice diagram has car \( i \) in row \( i \) (numbered from the bottom) for all \( i \). Then the distance that any driver \( i \) drives past his desired space is \( i - \pi_i \), since a driver will always park in the next available space, i.e., the \( i \)th. Moreover, in the lattice diagram, there are \( i-1 \) complete squares in the \( i \)th row above the main diagonal, and all but the first \( \pi_i - 1 \) of them are under the Dyck path. Thus complete squares in the \( i \)th row correspond to spaces the \( i \)th driver passes and wants to park in.

In the general case, let \( \alpha \) give the permutation formed by reading the cars in the lattice diagram from the bottom row to the top row. Let \( \beta \) give the permutation formed by the order in which the cars finally park. Similarly, the number of complete squares under the Dyck path in row \( i \) is \( i - \pi_{\alpha_i} \). The distance any driver drives past his desired space is \( \beta_i - \pi_i \). Relying throughout on the fact that we may sum over \( \{1, \ldots, n\} \) in the order prescribed by any permutation, the area is

\[
\sum_{i} i - \pi_{\alpha_i} = \left( \frac{n+1}{2} \right) - \sum_{i} \pi_i = \sum_{i} \beta_i - \pi_i,
\]

which is exactly the inconvenience distance of the parking function. \( \square \)

**Remark.** We can use what we know about \( E(\pi_1) \) to determine that average area is \( \sqrt{\frac{2\pi}{3}} n^{3/2} \) plus lower order terms. Kung and Yan (2003a) computes this and higher moments of the average area of parking functions (and their generalizations.) Interestingly, Knuth (1998a, volume 3, third edition, p.733) shows \( \sum_{k} \pi \binom{n}{k} \) is the number of connected graphs with \( n + k \) edges on \( n + 1 \) labeled vertices.

The previous theorems all flow from the following.

**Theorem 15.** For \( \pi \) uniformly chosen in PF\(_n\) and \( F^\pi \) defined by (Section 4),

\[
\sqrt{n} [F^\pi(x) - x]_{0 \leq x \leq 1} \Rightarrow (E_x)_{0 \leq x \leq 1}
\]

(weak convergence of processes on \([0,1]\)), with \( E_x \) the Brownian excursion (Itô and McKean, 1965, p. 75).
Proof. Chassaing and Marckert (2001, Prop. 4.1) give a 1:1 correspondence \( \pi \leftrightarrow J(\pi) \) between parking functions and labeled trees, rooted at 0, with \( n + 1 \) vertices. The bijection satisfies \( y_k(J(\pi)) = \hat{y}_k(\pi) \) for \( k = 0, 1, 2, \ldots, n \). Here for a tree \( t \), \( y_k(t) \) denotes the length of the queue lengths in a breadth-first search of \( t \) and \( \hat{y}_k(\pi) = \hat{a}_0(\pi) + \cdots + \hat{a}_k(\pi) - k \) with \( \hat{a}_k(\pi) = \# \{ k : \pi_k = i \} \) (their parking functions start at 0). Thus \( \hat{y}_k(\pi) = \# \{ i : \pi_i \leq k \} - k \) and

\[
\frac{\hat{y}_k(\pi)}{n} = F_{\pi} \left( \frac{k}{n} \right) - \frac{k}{n}.
\]

In Chassaing and Marckert (2001, Sect. 4.1) Chassaing and Marckert prove that for a uniformly chosen tree,

\[
\left\{ \frac{y(t)}{n \cdot x} \right\} \Rightarrow \{ E_x \}, \quad 0 \leq x \leq 1.
\]

This implies the result.

Theorems 11, 12, and 13 now follow from the continuity theorem; the distribution of \( E_x \) is well known to be the same as

\[
\sqrt{B_0^0(x)^2 + B_2^0(x)^2 + B_3^0(x)^2},
\]

with \( B_i^0(x) \) independent Brownian bridges (Itô and McKean, 1965, p. 79) and so the square root of a scaled Gamma(3) variate. The distribution of \( \max_{0 \leq x \leq 1} E_x \) is given by Kaigh (1978). The distribution of \( \int_0^1 E_x \, dx \) has a long history; see Janson (2013) for a detailed development.

6 Parking functions and representation theory

This section sketches out the connection between parking functions and Macdonald polynomials. It is not intended as a detailed history of either set of objects, which would require many pages. For more details, we suggest Macdonald (2015) for the polynomials, Haglund (2008) for the parking functions, or for a longer informal history, the afterward by Adriano Garsia in the second author’s Ph. D. thesis (Hicks, 2013).

The intersection between parking functions and Macdonald Polynomials is an exciting part of current algebraic combinatorics research, advanced by the work of Bergeron, Loehr, Garsia, Procesi, Haiman, Haglund, Remmel, Carlsson, Mellit, and many
others. The story starts with symmetric functions and Macdonald’s two-parameter family of bases \( P_\lambda(x; q, t) \). After several transformations to get to a basis \( \tilde{H}_\lambda(x; q, t) \) — which should be Schur positive, the proof of which involves diagonal harmonics — the Shuffle Theorem results in

\[
\text{Char}(DH_n) = \sum_{\pi \in \text{PF}_n} t^{\text{area}(\pi)} q^{\text{dinv}(\pi)} F_{\text{ides}(\pi)}.
\]

Our parking functions and area are on the right. Motivating and defining the other terms is our goal.

A healthy part of combinatorics is unified by symmetric function theory. Recall that \( f \in \mathbb{Q}[x_1, \ldots, x_n] \) is symmetric if \( f(x_1, \ldots, x_n) = f(z_{\sigma_1}, \ldots, x_{\sigma_n}) \) for all \( \sigma \in \mathfrak{S}_n \). For example, \( p_i(x) = \sum_{j=1}^{n} x_j^i \) is symmetric, and if \( \lambda \) is a partition of \( N \) (write \( \lambda \vdash N \)) with \( n_i(\lambda) \) parts of size \( i \), then

\[
p_\lambda(x) = \prod_i p_i^{n_i(\lambda)}
\]

is symmetric. A fundamental theorem says that as \( \lambda \) ranges over partitions of \( N \), \( \{p_\lambda(x)\}_{\lambda \vdash N} \) is a basis for the homogeneous symmetric polynomials of degree \( N \).

The other classical bases are: \( \{m_\lambda\} \) the monomial symmetric functions; \( \{e_\lambda\} \) the elementary symmetric functions; \( \{h_\lambda\} \) the homogeneous symmetric functions; and \( \{s_\lambda\} \) the Schur functions. The change of bases matrices between these families code up an amazing amount of combinatorics. For example, going from the power sums to the Schur basis,

\[
p_\rho = \sum_{\lambda} \chi^\lambda \rho s_\lambda,
\]

with \( \chi^\lambda \rho \) the \( \lambda \)th irreducible character of the symmetric group at the \( \rho \)th conjugacy class.

All of this is magnificently told in [Macdonald (2015)] and [Stanley (1999a)]. Because of (6.2) there is an intimate connection between symmetric function theory and the representation theory of the symmetric group \( \mathfrak{S}_n \). Let \( R^n \) be the space of class functions on \( \mathfrak{S}_n \) and \( R = \bigoplus_{n=0}^{\infty} R^n \). Let \( \Lambda \) be the ring of all symmetric functions. Then \( R \) is isomorphic to \( \Lambda \) via the characteristic map (also known as the Frobenius map). For \( f \in R^n \) taking values \( f_\rho \) at cycle type \( \rho \),

\[
\text{Ch}(f) = \sum_{|\rho|=n} z_\rho^{-1} f_\rho p_\rho \quad \text{with} \quad z_\rho = \prod_i i^{n_i(\rho)} n_i(\rho)!. \]
The linear extension of $Ch$ to all of $R$ is an isometry for the usual inner product on $S_n$ and the Hall inner product $\langle p_\lambda|p_\rho \rangle = \delta_{\lambda\rho}z_\rho$.

Following the classical bases, a host of other bases of $\Lambda$ began to be used. Statisticians used zonal polynomials to perform analysis of covariance matrices of Gaussian samples. Group theorists developed Hall–Littlewood polynomials to describe the subgroup structure of Abelian groups and the irreducible characters of $GL_n(F_q)$. An amazing unification was found by Ian Macdonald. He introduced a two-parameter family of bases $P_\lambda(x; q, t)$. These can be defined by deforming the Hall inner product to

$$\langle p_\lambda|p_\mu \rangle = \delta_{\lambda\mu}z_\lambda \prod_i \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \quad (p_\lambda, p_\mu \text{ are power sums}).$$

Then $P_\lambda(x; q, t) = P_\lambda$ is uniquely defined by

1. orthonormality, i.e., $\langle P_\lambda|P_\mu \rangle = \delta_{\lambda\mu}$;

2. triangularity, i.e., $P_\lambda$ is upper triangular in the $\{m_\mu\}$ basis.

Remarkably, specializing parameters gave essentially all the other bases: when $t = q$ the Schur functions emerge; when $q = 0$ we have the Hall–Littlewoods; setting $q = t^2$ and letting $t \to 1$ gives the Jack symmetric functions $J^2_{2\alpha}$; and setting $\alpha = 2$ gives zonal polynomials.

Early calculations of $P_\lambda$ showed that it was not polynomial in $q$ and $t$; multiplication by a factor fixed this, giving $J_\lambda$ [Macdonald, Sect. 8.1]. Further calculation showed these were almost Schur positive and, expressed in a $t$-deformation $s_{\tau}(x; t)$ of the Schur functions, Macdonald’s Schur positivity conjecture [Macdonald Sect. 8.18] suggests that $J_\lambda$ expands as a positive integral polynomial in $q$ and $t$.

Adriano Garsia (Garsia and Haiman, 1993) suggested a further modification using a homomorphism of $\Lambda$ (plethysm) to get modified Macdonald polynomials $\tilde{H}_\lambda[x; q, t]$. These restrict to classical bases and have the advantage of being (conjecturally) simply Schur positive.

Early attempts to prove Schur positivity and to find more convenient formulas for $\tilde{H}_\lambda$ were representation-theoretic using the characteristic map: if $\tilde{H}_\lambda$ was Schur positive, it must be the image of a representation of $\mathfrak{S}_n$. The $n!$ conjecture (Garsia and Haiman 1993), proved nearly a decade later (Haiman 2001), states that $\{H_\lambda\}$ is the image of the representation given by $\mathcal{L}[\partial x\partial y\Delta_\lambda]$ the linear span of the derivatives of $\Delta_\lambda$ (span $\partial x^a\partial y^b\Delta_\lambda$) where $\Delta_\lambda$ is a polynomial based on the diagram of $\lambda$. Each (irreducible) representation of the symmetric group occuring in $\mathcal{L}[\partial x\partial y\Delta_\lambda]$ is thus sent by characteristic map to a Schur function; when summed together (along with $t$ and $q$ to give the degree in the $x$ or $y$ variables) they give an expression for $\tilde{H}_\lambda$, and thus Macondald’s Schur positivity conjecture.
Example. For $\lambda = (4, 3, 2)$,

\[
\begin{array}{ccc}
02 & 12 \\
01 & 11 & 21 \\
00 & 10 & 20 & 30
\end{array}
\]

$\rightarrow \Delta_{432} = \det(x_i^{p_j} y_i^{q_j})_{1 \leq i,j \leq n}$.

where $(p_j, q_j)$ run over the indices in the diagram $[(0, 0), \ldots, (1, 2)]$.

Combinatorial methods for studying the irreducible representations of this space, and thus giving a directly computable definition for the Macdonald polynomials eluded researchers for a number of years. They naturally began studying a larger space which contained $L[\partial_x \partial_y \Delta_\lambda]$ for all partitions $\lambda$ of $n$; the resulting space was called the diagonal harmonics and there, the Frobenius image proved easier (at least conjecturally) to compute.

\[DH_n = \left\{ f \in \mathbb{Q}[x, y] : n \sum_{i=1}^{n} \partial_{x_i}^{r} \partial_{y_i}^{s} f(x, y) = 0 \text{ for all } r, s \geq 0, r + s > 0 \right\} .\]

This space (or rather an isomorphic space, the diagonal coinvariants) is also be seen as as the space of polynomials $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ moded out by the ideal generated by polynomials invariant under the diagonal action $(x_1, \ldots, x_n, y_1, \ldots, y_n)^\sigma = (x_{\sigma_1}, \ldots, x_{\sigma_n}, y_{\sigma_1}, \ldots, y_{\sigma_n})$. Of course, the symmetric group acts on this quotient. The characteristic image of this representation was conjecturally identified (the shuffle conjecture) in the 90s (Haglund et al., 2005) and recently proved by Carlsson and Mellit (2015) to be describable as a weighted sum of parking functions,

\[\text{Char}(DH_n) = \sum_{\pi \in \text{PF}_n} t^\text{area}(\pi) q^\text{dinv}(\pi) F_{\text{ides}(\pi)} .\]

with $\text{area}(\pi)$ as in Section 5, $\text{dinv}(\pi)$ a second statistic, and $F$ the quasi-symmetric function (Stanley, 1999a). This is known to be a positive integral combination of Schur functions. Finally, ides is closely related to the number of weak descents in $\pi$, introduced in Section 4. (In fact, equivalent formulations of the theorem use the same precise characterization.) The original shuffle conjecture led to a further conjecture of Haglund (Haglund, 2004) and proof by Haglund and Haiman (Haglund et al., 2005) of a combinatorial description of the Macdonald polynomials (with statistics related to the parking function statistics, but much more complicated).

In summary, the study of the area and descent structure of parking functions led directly to the discovery of a direct and combinatorial formula for the Macdonald Polynomials, arguably the most important symmetric function basis. Moreover,
facts about area and descent structure can be translated into information about the
degrees of polynomials that occur in any irreducible representations of $\mathfrak{S}_n$ occurring
in $\text{DH}_n$.

7 Some open problems

There is one feature of $f \in \mathcal{F}_n$ that we have not explored for $\text{PF}_n$; this is the cycle
structure under iteration. For $f \in \mathcal{F}_n$, iteration produces a disjoint union of directed
cycles with trees coming into the cycles. There is remarkable and detailed knowledge
of the properties of these graphs for uniformly chosen $f$: a typical cycle has length
about $\sqrt{n}$ as does a typical tree. Maxima and joint distributions are also carefully
studied. See [Kolchin (1986)] or [Harris (1960)] for the classical theory of random
mappings. Jenny Hansen’s many contributions here are notable; see [Hansen (1989)].
The paper by [Aldous and Pitman (1994)] relates natural features of a mapping to
natural features of the Brownian bridge. It is understandable to expect a parallel
coupling between $\text{PF}_n$ and Brownian excursion but this has not been worked out.

There are three other developments connected to parking functions where the
program outlined in our paper can be attempted. The first is so-called rational
parking functions or $(m,n)$ parking functions. See [Gorsky et al. (2016)], [Hikita
(2014)], [Bergeron et al. (2015)] and their many references. These are at the forefront
of current research with applications to things like the cohomology of affine springer
fibers. There are $m^{n-1}$ of these things: what does a typical one look like?

Our parking functions have a close connection to the symmetric group. There
are interesting natural analogs for other reflection groups. For definitions and basic
properties, see [Armstrong et al. (2015)]. We have not seen any probabilistic develop-
ment in this direction, although certain special cases overlap with [Kung and Yan
(2003b)].

The third development is to $G$-parking functions ([Postnikov and Shapiro, 2004]).
Here $G$ is a general graph. There is a general definition which specializes to $\text{PF}_n$
for the complete graph $K_n$. These $G$-parking functions have amazing connections to
algebraic geometry, Riemann surface theory, and much else through their connection
to chip firing games and sand pile models ([López, 1997]). One way to get started in
this area is to ask “What does a random $G$-parking function look like?”

Of course, our probabilistic path is just one thread in an extremely rich fabric.
We have found it useful in getting started and hope that our readers will, too.

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