On exceptional zeros of Garrett–Hida $p$-adic $L$-functions

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To Bernadette Perrin-Riou on the occasion of her 65th birthday.

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Abstract
This article proves a case of the $p$-adic Birch and Swinnerton-Dyer conjecture for Garrett $p$-adic $L$-functions of [6], in the exceptional zero setting of extended analytic rank 2.

Résumé
Cet article prouve un cas de la conjecture $p$-adique de Birch et Swinnerton-Dyer pour les fonctions $L_p$-adiques de Garrett formulée dans [6], dans le cadre de zéros exceptionnels de rang analytique étendu égal à 2.

Keywords Birch and Swinnerton-Dyer Conjecture · $p$-adic $L$-functions · Exceptional zeros

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Introduction

Let $A$ be an elliptic curve defined over $\mathbb{Q}$, having ordinary reduction at a rational prime $p > 3$. Let $\varrho_1$ and $\varrho_2$ be odd, irreducible, two-dimensional Artin representations of the absolute Galois group of $\mathbb{Q}$, which are unramified at $p$ and satisfy the self-duality condition

$$\det(\varrho_1) = \det(\varrho_2)^{-1}.$$ 

By modularity, the triple $(A, \varrho_1, \varrho_2)$ arises from a triple $(f, g, h)$ of cuspidal $p$-ordinary newforms of weights $w_o = (2, 1, 1)$. Let $f_\alpha$ be the ordinary $p$-stabilisation of $f$, and fix...
p-stabilisations $g_\alpha$ and $h_\alpha$ of $g$ and $h$ respectively. Set $\varrho = \varrho_1 \otimes \varrho_2$. In the recent paper [6] we proposed a $p$-adic analogue of the Birch and Swinnerton-Dyer conjecture for the leading term at $w_o$ of the 3-variable Garrett–Hida $p$-adic $L$-function $L_p^{\alpha\alpha}(A, \varrho) = L_p(f, g_\alpha, h_\alpha)$ associated with the triple $(f, g_\alpha, h_\alpha)$ of Hida families specialising to $(f_\alpha, g_\alpha, h_\alpha)$ at $w_\alpha$. In this article we verify our conjecture in the analytic rank-zero exceptional cases, viz. when the complex Garrett $L$-function $L(A, \varrho, s) = L(f \otimes g \otimes h, s)$ does not vanish at $s = 1$ and $L_p^{\alpha\alpha}(A, \varrho)$ has an exceptional zero at $w_\alpha$ in the sense of Mazur–Tate–Teitelbaum (cf. Theorem 2.1 and Sect. 2.1 below). Moreover, when $L(A, \varrho, 1) = 0$ and $L_p^{\alpha\alpha}(A, \varrho)$ has an exceptional zero, we propose a conjecture relating the value at $w_\alpha$ of the fourth partial derivative of $L_p^{\alpha\alpha}(A, \varrho)$ along the $f$-direction to the $p$-adic logarithms of two global points on $A$ rational over the number field cut out by $\varrho$ (cf. Conjecture 2.3).

1 Setting and notations

Fix algebraic closures $\bar{Q}$ and $\bar{Q}_p$ of $Q$ and $Q_p$ respectively, and field embeddings $i_p : \bar{Q} \hookrightarrow \bar{Q}_p$ and $i_\infty : \bar{Q} \hookrightarrow \mathbb{C}$. With the notations of the Introduction, let

$$\xi = \sum_{n \geq 1} a_n(\xi) \cdot q^n \in S_u(N_\xi, \chi_\xi)_{\bar{Q}}$$

denote one of the cuspidal newforms $f$, $g$ and $h$. Here $u$ and $N_\xi$ are the weight and the conductor of $\xi$ respectively, and $S_u(N_\xi, \chi_\xi)_F$ is the space of cuspidal modular forms of level $\Gamma_1(N_\xi)$, weight $u$, character $\chi_\xi$ and Fourier coefficients in the subfield $F$ of $\bar{Q}_p$. Fix a number field $Q(\varrho)$ containing for any $\xi$ the Fourier coefficients $a_n(\xi)$, as well as the roots $\alpha_\xi$ and $\beta_\xi$ of the $p$th Hecke polynomials $P_{\xi, p} = X^2 - a_p(\xi) \cdot X + \chi_\xi(p) \cdot p$. Let $V_{\varrho_i}$ be a two-dimensional $Q(\varrho)$-vector space affording the representation $\varrho_i$, and let $K_{\varrho}$ be a Galois number field such that $\varrho_i$ factors through $\text{Gal}(K_{\varrho}/Q)$. Set

$$V_\varrho = V_{\varrho_1} \otimes_{Q(\varrho)} V_{\varrho_2} \quad \text{and} \quad V_p(A, \varrho) = V_p(A) \otimes Q V_\varrho,$$

where $V_p(A) = H^1_{\varrho_1}(A_Q, Q_p(1))$ is the $p$-adic Tate module of $A$ with $Q_p$-coefficients. Throughout this note we make the following assumptions:

Assumption 1.1

1. (Self-duality) The characters $\chi_g$ and $\chi_h$ are inverse to each other.
2. (Local signs) The conductors $N_g$ and $N_h$ are coprime to $p \cdot N_f$.
3. (Étaleness) The forms $g$ and $h$ are cuspidal, $p$-regular and do not have RM by a real quadratic field in which $p$ splits.

The first condition is a reformulation of the self-duality condition mentioned in the Introduction, namely $\det(\varrho_1) = \det(\varrho_2)^{-1}$. Recall that the form $\xi$ is $p$-regular if $P_{\xi, p}$ has distinct roots. Moreover, one says that a weight-one eigenform has RM (real multiplication) if it is the theta series associated with a ray class character of a real quadratic field. Assumption 1.1.3 is equivalent to require that $V_{\varrho_i}$ is irreducible, not isomorphic to $\text{Ind}_K^Q \chi$ for a finite order character $\chi : G_K \to Q(\varrho)_* \otimes$ of a real quadratic field $K$ in which $p$ splits, and that an arithmetic Frobenius at $p$ acts on $V_{\varrho_i}$ with distinct eigenvalues. For $\xi = g, h$, this assumption guarantees that the $p$-adic Coleman–Mazur–Buzzard eigencurve of tame level $N_\xi$ is étale over the weight space at the points corresponding to the $p$-stabilisations of $\xi$ (cf. [2]). It is used in [6] to construct the Garrett–Nekovář height $\langle \cdot, \cdot \rangle_{f g_\alpha, h_\alpha}$ which appears in the main result of this note. To explain the relevance of Assumptions 1.1.1 and 1.1.2, let $\alpha_f$ be the unit root of $P_{f, p}$ and fix roots $\alpha_g$ and $\alpha_h$ of $P_{g, p}$ and $P_{h, p}$ respectively. Fix a finite extension

$\triangleright Springer$
$L$ of $\mathbb{Q}_p$ containing $\mathbb{Q}(\xi)$ and the roots of unity of order $\text{lcm}(N_f, N_g, N_h)$. Let $\xi$ be one of $f, g$ and $h$, and let $u_\alpha$ be the weight of $\xi$. According to the results of [2,10,18], there exists a unique Hida family

$$
\xi_\alpha = \sum_{n \geq 1} a_n(\xi_\alpha) \cdot q^n \in \mathcal{O}_\xi \{ q \}
$$

which specialises at $u_\alpha$ to the $p$-stabilised newform

$$
\xi_\alpha = \xi(q) - \frac{\chi_\xi(p)^{p-1}}{\alpha_\xi} \cdot \xi(q^p) \in S_{u_\alpha}(p \cdot M_\xi, \chi_\xi)_L.
$$

Here $M_\xi = N_\xi / p^{\text{ord}_p(N_\xi)}$ is the tame level of $\xi$ (so that $M_\xi = N_\xi$ if $\xi = g, h$), and $\mathcal{O}_\xi$ is the ring of bounded analytic functions on a (sufficiently small) connected open disc $U_\xi$ in the $p$-adic weight space over $L$. For each classical weight $u$ in $U_\xi \cap \mathbb{Z}_{\geq 3}$, the weight-$u$ specialisation $\xi_{w,u} = \sum_{n \geq 1} a_n(\xi_\alpha)(u) \cdot q^n \in L\{q\}$ of $\xi_\alpha$ is the $q$-expansion of the ordinary $p$-stabilisation of a newform $\xi_u$ in $S_u(M_\xi, \chi_\xi)_L$. Since $f$ has a unique $p$-ordinary $p$-stabilisation $f_\alpha$, we simply write $f$ for $f_\alpha$.

Assumption 1.1.1 guarantees that for each classical triple $w = (k, l, m)$ in the set

$$
\Sigma = U_f \times U_g \times U_h \cap \mathbb{Z}_{\geq 3}^2
$$

the complex Garrett $L$-function $L(f_k \otimes g_l \otimes h_m, s)$ admits an analytic continuation to all of $\mathbb{C}$ and satisfies a functional equation relating its values at $s$ and $k + l + m - 2 - s$, with root number $\varepsilon(w) = \prod_{\ell \leq \infty} \varepsilon_\ell(w)$ equal to $+1$ or to $-1$. Assumption 1.1.2 implies that all the local signs $\varepsilon_\ell(w)$ are equal to $+1$ for every $w$ in the $f$-unbalanced region $\Sigma_f = \{ w = (k, l, m) \in \Sigma : k \geq k + m \}$ (cf. [11]). Under these assumptions, [12] associates with $(f, g_\alpha, h_\alpha)$ an analytic function

$$
\mathcal{L}^{\alpha\alpha}_p(A, \varrho) = \mathcal{L}_p(f, g_\alpha, h_\alpha)
$$

in the ring $\mathcal{O}_f \otimes \mathcal{O}_L \otimes \mathcal{O}_g \otimes \mathcal{O}_h$, whose square

$$
L^{\alpha\alpha}_p(A, \varrho) = L_p(f, g_\alpha, h_\alpha) = \mathcal{L}_p(f, g_\alpha, h_\alpha)^2
$$

satisfies the following interpolation property. For each $w = (k, l, m)$ in $\Sigma_f$, the value of $L^{\alpha\alpha}_p(A, \varrho)$ at $w$ is an explicit non-zero complex multiple of

$$
\left( 1 - \frac{\beta_k \alpha_l \alpha_m}{p^{c_w}} \right)^2 \left( 1 - \frac{\beta_k \beta_l \beta_m}{p^{c_w}} \right)^2 \left( 1 - \frac{\beta_k \beta_l \beta_m}{p^{c_w}} \right)^2 \left( 1 - \frac{\beta_k \beta_l \beta_m}{p^{c_w}} \right)^2 \cdot L(f_k \otimes g_l \otimes h_m, c_w).
$$

(1)

Here $c_w = \frac{k + l + m - 2}{2}$, and for $\xi = f, g_\alpha, h_\alpha$ one denotes by $\alpha_u$ the unit root of $P_{\xi_u,p}$ and sets $\beta_u \cdot \alpha_u = \chi_\xi(p) \cdot p^{a-1}$, where $\chi_\xi(p)$ is the prime-to-$p$ part of $\chi_\xi$ (so that $\chi_\xi'(p) = \chi_\xi$ for $\xi = g, h$ and $\chi_f$ is the trivial character modulo $M_f$). We refer to Theorem A of loc. cit. for the precise interpolation formula. We call $L^{\alpha\alpha}_p(A, \varrho) = L_p(f, g_\alpha, h_\alpha)$ the Garrett–Hida $p$-adic $L$-function associated with $(A, \varrho)$ (or with $(f, g_\alpha, h_\alpha)$).

### 2 Exceptional zero formulae

The $p$-adic variant of the Birch and Swinnerton-Dyer conjecture formulated in [6] predicts that the leading term of $L_p^{\alpha\alpha}(A, \varrho)$ at $w_\alpha = (2, 1, 1)$ is encoded by the discriminant of the
Garrett–Nekovář height pairing

\[ \langle \cdot, \cdot \rangle_{fgh} : A^\dagger(K_\infty)^e \otimes \mathbb{Q}(\omega) \otimes \mathbb{Q}(h) \to \mathcal{S} / \mathcal{S}^2 \]  

(2)

constructed in Section 2 of loco citato, where \( \mathcal{S} \) is the ideal of functions in \( \mathcal{O}_{fgh} \) which vanish at \( w_i \) and the \( p \)-extended Mordell–Weil group \( A^\dagger(K_\infty)^e \) is defined as follows. When \( A \) has good reduction at \( p \), one sets \( A^\dagger(K_\infty)^e = A(K_\infty)^e \), where \( A(K_\infty)^e \) is a shorthand for the Galois \( (K_\infty/Q) \)-invariants of \( A(K_\infty) \otimes \mathbb{Z} V_p \). If \( A \) has multiplicative reduction at \( p \), then \( \alpha_f = a_p(f) = \pm 1 \) and the maximal \( p \)-unramified quotient \( V_p(A)^\circ \) of \( V_p(A) \) is a 1-dimensional \( \mathbb{Q}_p \)-vector space on which an arithmetic Frobenius acts as multiplication by \( \alpha_f \). Let \( q_A \) in \( p\mathbb{Z}_p \) be the \( p \)-adic Tate period of the base change \( A_{Q_p} \) of \( A \) to \( \mathbb{Q}_p \) (cf. Chapter V of [15]), and let \( Q_{p^2} \) be the quadratic unramified extension of \( \mathbb{Q}_p \). The Tate uniformisation yields a rigid analytic morphism

\[ \varrho_{Tate}: G_{m,Q_{p^2}} \to A_{Q_{p^2}} \]

with kernel \( q_A^p \) and unique up to sign. Set

\[ q(A) = p^{-\varepsilon_T}(\varrho_{Tate}(\sqrt[p]{q_A}))_{n \geq 1} \in V_p(A)^\circ, \]

where \( p^{-\varepsilon_T}(\varrho_{Tate}(\sqrt[p]{q_A}))_{n \geq 1} \) is any compatible system of \( p^n \)-th roots of \( q_A \), and define

\[ A^\dagger(K_\infty)^e = A(K_\infty)^e \oplus Q_p(A, \omega) \]

to be the direct sum of \( A(K_\infty)^e \) and the \( Q(\omega) \)-submodule

\[ Q_p(A, \omega) = H^0(Q_p, Q(\omega) \cdot q(A) \otimes \mathbb{Q}(\omega)) V_p(\mathbb{Q}) \]

of \( H^0(Q_p, V_p(A)^\circ \otimes \mathbb{Q} V_p) \). The Garrett–Nekovář height \( \langle \cdot, \cdot \rangle_{fgh} \) depends on the choice of suitably normalised \( G_Q \)-equivariant embeddings

\[ \gamma_R : V_{p1} \to V(\mathbb{g}) \quad \text{and} \quad \gamma_h : V_{p2} \to V(\mathbb{h}), \]

(3)

where \( V(\xi) = V(\xi^\omega) \otimes L \) (for \( \xi = g, h \)) is the weight-one specialisation of the big Galois representation \( V(\xi^\omega) \) associated with \( \xi^\omega \). (We refer to Sect. 3.1 below for precise definitions.) More precisely, denote by \( V(f) \) the \( f_p \)-isotypic component of the cohomology group \( H^0(Q_p, X_1(Nf, p)_Q, Q_p(1)) \), where \( X_1(Nf, p)_Q \) is the base change to \( \bar{Q} \) of the compact modular curve \( X_1(Nf, p) \) of level \( \Gamma_1(Nf) \cap \Gamma_0(p) \) over \( \bar{Q} \), and set

\[ V(f, g, h) = V(f) \otimes Q_p \otimes L \otimes V(h). \]

Section 2 of [6] constructs a canonical Garrett–Nekovář \( p \)-adic height pairing

\[ \langle \cdot, \cdot \rangle_{fgh} : \text{Sel}^\dagger(Q, V(f, g, h)) \otimes L \otimes \text{Sel}^\dagger(Q, V(f, g, h)) \to \mathcal{S} / \mathcal{S}^2 \]

(4)

on the naive extended Selmer group of \( V(f, g, h) \) over \( \mathbb{Q} \), defined as the direct sum of the Bloch–Kato Selmer group \( \text{Sel}(Q, V(f, g, h)) \) of \( V(f, g, h) \) over \( \mathbb{Q} \) and the module \( H^0(Q_p, V(f, g, h)^-) \) of \( G_{Q_p} \)-invariants of the maximal \( p \)-unramified quotient \( V(f, g, h)^- \) of \( V(f, g, h) \). (The definition of \( \langle \cdot, \cdot \rangle_{fgh} \) is briefly recalled in Sect. 3.2.3 below.) Fix a modular parametrisation \( \varphi_{\infty} : X_1(Nf, p) \to A \), under which one identifies \( V(f) \) and \( V_p(A) \). The embeddings \( \gamma_R \) and \( \gamma_h \) and the global Kummer map on \( A(K_\infty) \) then induce an embedding \( \gamma_{gh} : A^\dagger(K_\infty)^e \to \text{Sel}^\dagger(Q, V(f, g, h)) \). The pairing (2) is defined to be composition of the canonical Garrett–Nekovář height and \( \gamma_{gh}^2 \). The pairings (2) and (4) are skew-symmetric, and the discriminant of (2) in \( (\mathcal{S}^1(A, \omega) / \mathcal{S}^1(A, \omega)^+) / \mathbb{Q}(\omega)^{\otimes 2} \), where
If $\xi$ denotes either $g$ or $h$, then the restriction to $G_{Q_p}$ of the Artin representation $V(\xi)$ is the direct sum of the submodules $V(\xi)_{\alpha}$ and $V(\xi)_{\beta}$ on which an arithmetic Frobenius acts as multiplication by $\alpha_\xi$ and $\beta_\xi$ respectively (cf. Assumption 1.1.3). The $G_{Q_p}$-representation $V(f, g, h)^-$ then decomposes as the direct sum of the subspaces

$$V(f)_{ij} = V(f)^- \otimes_{Q_p} V(g)_{i} \otimes_{L} V(h)_{j},$$

where $(i, j)$ is a pair of elements of $[\alpha, \beta]$. If $\xi$ denotes either $g$ or $h$, Sect. 3.1.1 below recalls the definition of canonical weight-one differentials

$$\omega_\xi \in (V(\xi)_{\alpha} \otimes_{Q_p} Q_p^{nr})^{G_{Q_p}}$$

and

$$\eta_\xi \in (V(\xi)_{\beta} \otimes_{Q_p} Q_p^{nr})^{G_{Q_p}},$$

where $Q_p^{nr}$ is the maximal unramified extension of $Q_p$. If $A$ is multiplicative at $p$, set

$$q(f) = \varphi^{-1}(q(A)) \in V(f)^-,$$

where one denotes again by $\varphi_\infty : V(f)^- \simeq V_p(A)^-$ the isomorphism arising from the fixed modular parametrisation $\varphi_\infty : X_1(N, f, p) \longrightarrow A$.

Under the running assumptions, the $Q(\xi)$-module $Q_p(A, \xi)$ (resp., the $L$-module $H^0(Q_p, V(f, g, h)^-)$) is non-zero precisely $A$ is multiplicative at $p$ and

$$\alpha_f = \alpha_g \cdot \alpha_h \text{ or } \alpha_f = \beta_g \cdot \alpha_h,$$

in which case it has dimension 2 and one says that $(A, \xi)$ is exceptional at $p$. More precisely, note that $\alpha_g \neq \beta_g$ by Assumptions 1.1.3, hence only one of the previous identities can be satisfied. Moreover $\alpha_f = \alpha_g \cdot \alpha_h$ (resp., $\alpha_f = \beta_g \cdot \alpha_h$) if and only if $\alpha_f = \beta_g \cdot \beta_h$ (resp., $\alpha_f = \alpha_g \cdot \beta_h$) by Assumption 1.1.1. Fix an auxiliary integer $m_p$ such that $p$ splits (resp., is inert) in $Q[\sqrt{m_p}]$ if $\alpha_f = +1$ (resp., $\alpha_f = -1$), so that $G_{Q_p}$ acts trivially on $\sqrt{m_p}q(f)$ in $V(f)^- \otimes_{Q_p} Q_p^{nr}$. If $\alpha_f = \alpha_g \cdot \alpha_h$, then $G_{Q_p}$ acts trivially on $V(f)_{\alpha^2}$ and $V(f)_{\beta^2}$, hence the $p$-adic periods

$$q_{\alpha^2} = \sqrt{m_p} \cdot q(f) \otimes \omega_{\alpha_g} \otimes \omega_{\alpha_h} \text{ and } q_{\beta^2} = \sqrt{m_p} \cdot q(f) \otimes \eta_{\alpha_g} \otimes \eta_{\alpha_h}$$

can naturally be viewed as elements of $V(f)_{\alpha^2}$ and $V(f)_{\beta^2}$ respectively, which generate $H^0(Q_p, V(f, g, h)^-)$. Similarly, if $\alpha_f = \beta_g \cdot \alpha_h$, then the periods

$$q_{\alpha^2} = \sqrt{m_p} \cdot q(f) \otimes \omega_{\alpha_g} \otimes \eta_{\alpha_h}$$

and

$$q_{\beta^2} = \sqrt{m_p} \cdot q(f) \otimes \eta_{\alpha_g} \otimes \omega_{\alpha_h}$$

can naturally be viewed as generators of $H^0(Q_p, V(f, g, h)^-)$. Equation (1) shows that the value of the square-root Garrett–Hida $L$-function $L_{p}^{\alpha}(A, \xi)$ at $w_{\eta}$ is a non-zero multiple of

$$\left(1 - \frac{\alpha_g \alpha_h}{\alpha_f}\right) \left(1 - \frac{\beta_g \alpha_h}{\alpha_f}\right) \left(1 - \frac{\alpha_g \beta_h}{\alpha_f}\right) \left(1 - \frac{\beta_g \beta_h}{\alpha_f}\right) \cdot \sqrt{L(A, \xi, 1)},$$

where $L(A, \xi, s) = L(f \otimes g \otimes h, s)$. The previous discussion then shows that $(A, \xi)$ is exceptional at $p$ precisely if one of the Euler factors which appear in the previous expression is zero, id est if $L_{p}^{\alpha}(A, \xi)$ (or $L_{p}^{\alpha}(A, \xi)$) has an exceptional zero in the sense of Mazur–Tate–Teitelbaum [13]. In this case Lemma 9.8 of [7] proves that the restriction $L_{p}^{\alpha}(A, \xi)|_{L}$ of $L_{p}^{\alpha}(A, \xi)$ to the improving line $L$ defined by the equations $m = 1$ and $k = l + 1$ admits the factorisation

$$L_{p}^{\alpha}(A, \xi)|_{L} = \varepsilon_{f} \cdot \varepsilon_{g} \cdot L_{p}^{\alpha}(A, \xi)^*$$
in the ring \( \mathcal{O}(L) \) of analytic functions on \( L \), where 
\[
\varepsilon_f = 1 - \frac{a_p(f)}{a_p(g_o) \cdot a_p(h_o)} \bigg|_L \quad \text{and} \quad \varepsilon_g = 1 - \frac{a_p(g_o)}{a_p(f) \cdot a_p(h_o)} \bigg|_L .
\]
Moreover, the value at \( \omega_o \) of the improved \( p \)-adic \( L \)-function \( L_p^{\text{ar}}(A, \varrho) \) is an explicit algebraic number in \( \mathbb{Q}(\varrho) \), equal to zero precisely if \( L(A, \varrho, s) \) vanishes at \( s = 1 \). We refer to the proof of Proposition 8.3 of [12] for details.

The following is the main result of this note.

**Theorem 2.1** Assume that \( (A, \varrho) \) is exceptional at \( p \). Let \( (q_o, q_o) \) denote either the pair \( (q_o, q_o) \) or \( (q_o, q_o) \), depending on whether \( \alpha_f = \alpha_g \cdot \alpha_h \) or \( \alpha_f = \beta_g \cdot \alpha_h \) respectively. Then the following equality holds in \( \mathfrak{S} / \mathfrak{S}^2 \) up to sign.
\[
L_p^{\text{ar}}(A, \varrho) \pmod{\mathfrak{S}^2} = \frac{\deg(\varphi_{\infty}) \cdot (1 - \beta_h/\alpha_h)}{m_p \cdot \text{ord}_p(q_A)} \cdot L_p^{\text{ar}}(A, \varrho)^* (w_o) \cdot \langle q_o, q_o \rangle_{f_g h_a}
\]

Theorem 2.1 is proved in Sect. 4 below. More precisely, Sects. 3.3 and 3.4 below prove that the following equality holds in \( \mathfrak{S} / \mathfrak{S}^2 \) up to sign:
\[
\frac{2 \cdot \deg(\varphi_{\infty})}{m_p \cdot \text{ord}_p(q_A)} \cdot \langle q_o, q_o \rangle_{f_g h_a} = \left( \text{L}^{an}_f - \text{L}^{an}_g \right) \cdot (l - 1) + \varepsilon \cdot \left( \text{L}^{an}_f - \text{L}^{an}_h \right) \cdot (m - 1),
\]
where \( \varepsilon = +1 \) if \( \alpha_f = \alpha_g \cdot \alpha_h \) and \( \varepsilon = -1 \) if \( \alpha_f = \beta_g \cdot \beta_h \), and where
\[
- \frac{1}{2} \cdot \text{L}^{an}_x = d \log a_p(\xi)_{u = u_o}
\]
is the value at the centre \( u_o \) of \( U_\xi \) of the logarithmic derivative of the \( p \)-th Fourier coefficient of the Hida family \( \xi = f, g_o, h_o \). In Sect. 4 we then deduce Theorem 2.1 from Eq. (6) and the study carried out in [7, Section 9] of the linear term of \( L_p^{\text{ar}}(A, \varrho) \) at \( w_o \) in the exceptional case.

It should be possible to extend Theorem 2.1 (and Conjecture 2.3 below) to the case of \( p \)-new eigenforms of even weight \( k \geq 2 \) and trivial character (cf. Section 1.1 of [6]). We have not checked the details.

### 2.1 The rank-zero exceptional case of [6, Conjecture 1.1]

Assume in this section that \( (A, \varrho) \) is exceptional at \( p \), and that the Garrett complex \( L \)-function \( L(A, \varrho, s) = L(f \otimes g \otimes h, s) \) does not vanish at \( s = 1 \):
\[
L(A, \varrho, 1) \neq 0.
\]

According to the main result of [8] (see also Theorem B of [3]), one has
\[
A(K_0)^e = 0,
\]
hence \( A(K_0)^e = Q_p(A, \varrho) \). The Garrett–Nekovář \( p \)-adic regulator \( R_p^{\text{ar}}(A, \varrho) \), viz. the discriminant of the \( p \)-adic height \( \langle \cdot, \cdot \rangle_{f_g h_a} \) on \( A(K_0)^e \), is then given by
\[
R_p^{\text{ar}}(A, \varrho) = \det \left( \langle q_i, q_j \rangle_{f_g h_a} \right)_{1 \leq i, j \leq 2} = \langle q_1, q_2 \rangle_{f_g h_a}^2
\]
in \((\mathfrak{S}^2 / \mathfrak{S}^3) / \mathbb{Q}(\varrho)^{s^2} \), where \( (q_1, q_2) \) is a \( \mathbb{Q}(\varrho) \)-basis of \( Q_p(A, \varrho) \).
Let $γ_{gh} : V(Α, ϱ) \rightarrow V(f, g, h)$ be the $G_ℚ$-equivariant embedding defined by the tensor product of the isomorphism $V_p(Α) \simeq V(f)$ induced by $φ_∞, γ_φ$ and $γ_h$ (cf. Eq. (3)). The normalisation imposed on the embeddings $γ_φ$ and $γ_h$ (and described in Sect. 3.1.1 below) implies that the matrix $M$ in $GL_2(ℚ)$ defined by the identity $(q_1, q_2) : M = (γ_{gh}(q_1))γ_{gh}(q_2))$ has determinant in $ℚ(ϱ)^*$. In light of the above discussion, Theorem 2.1 then proves the following corollary, which together with Eq. (6) establishes [6, Conjecture 1.1] in the present setting.

**Corollary 2.2** If $L(A, ϱ, s)$ does not vanish at $s = 1$, then $A^1(K_ϱ)^0 = Q_p(A, ϱ)$ and the following equality holds in the quotient of $\mathcal{S}^2/\mathcal{S}^3$ by the action of $Q(ϱ)_{−2}$:

$$L_p^{αα}(A, ϱ) \pmod{\mathcal{S}^3} = R_p^{αα}(A, ϱ)$$

### 2.2 Exceptional zeros and rational points (cf. [14])

Assume in this section that $(A, ϱ)$ is exceptional at $p$, and that the Garrett complex $L$-function $L(A, ϱ, s)$ vanishes at the central critical point $s = 1$:

$$L(A, ϱ, 1) = 0.$$

Set $[α, β] = {αα, ββ}$ of $[α, β] = {αβ, βα}$, depending on whether

$$α_f = α_γ \cdot α_h \text{ or } α_f = β_γ \cdot α_h.$$

The $p$-adic $L$-function $L^{αα}_p(A, ϱ)$ belongs to $\mathcal{S}^2$ (cf. Theorem 2.1) and Conjecture 2.3 of [6] predicts that its image in $(\mathcal{S}^2/\mathcal{S}^3)/Q(ϱ)^*$ equals

$$(q_1, q_2)h_{fg}a \langle P, Q \rangle f_{g_α}h_a - ⟨q_1, P \rangle f_{g_α}h_a + ⟨q_2, Q \rangle f_{g_α}h_a + ⟨q_2, P \rangle f_{g_α}h_a$$

for two rational points $P$ and $Q$ in $A(K_ϱ)^0$. (Recall that the $p$-adic height $⟨·, ·⟩$ $f_{g_α}h_a$ is skew-symmetric, hence the previous expression is a square root of its discriminant on the $Q(ϱ)$-submodule of $A^1(K_ϱ)^0$ generated by $q_1, q_2, P$ and $Q$.) One has

$$⟨q_1, q_2⟩ f_{g_α}h_a (k, 1, 1) = 0$$

by Eq. (6). Moreover, Sect. 3.5 below proves that

$$⟨q_1, x⟩ f_{g_α}h_a (k, 1, 1) = \frac{1}{2} \cdot \log_p(\text{res}_p(x)) \cdot (k - 2)$$

for each Selmer class $x$ in $\text{Sel}(Q, V(f, g, h))$, where

$$\log_p = (\log_p(·), q_2) f_{gh} : H^1_{fin}(Q_p, V(f, g, h)) \rightarrow L.$$ 

Here $log_p : H^1_{fin}(Q_p, V(f, g, h)) \simeq D_{dR}(V(f, g, h))/\text{Fil}^0$ is the Bloch–Kato $p$-adic logarithm (cf. Lemma 9.1 of [7]), and $⟨·, ·⟩_{fg} : D_{dR}(V(f, g, h))^{⊗2} \rightarrow L$ is the pairing induced by the natural Kummer duality $π_{fg} : V(f, g, h)^{⊗2} \rightarrow L(1)$ defined in Sect. 3.1.1 below (cf. Eq. (11)). We are then led to the following

**Conjecture 2.3** Assume that $A(K_ϱ)^0$ is a 2-dimensional $Q(ϱ)$-vector space. Then for any $Q(ϱ)$-basis $(P, Q)$ of $A(K_ϱ)^0$, the equality

$$\frac{∂^2 L^{αα}(A, ϱ)}{∂k^2}(w_α) = \log_p(P) \cdot \log_p(Q) - \log_p(P) \cdot \log_p(Q)$$

holds in $L$ up to multiplication by a non-zero scalar in $Q(ϱ)^*$.  

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As explained in [5], the main result of [1] can be used to prove cases of Conjecture 2.3 when \( g \) and \( h \) are theta series associated with certain ray class characters of the same imaginary quadratic field in which \( p \) is inert (and \( P \) and \( Q \) are Heegner points). By combining this with an extension of the height computations carried out in [16,17], the article [4] proves instances of Conjecture 1.1 of [6] in this setting.

**Remark 2.4** In light of the aforementioned results of [5], Rivero proposes in [14, Conjecture 4.5] a variant of Conjecture 2.3. He also asks (cf. Question 5.3 of [14]) if one can expect a similar description of \( \frac{\partial^2 \tau_{\varphi}(A, q)}{\partial \tau_{\varphi}^2}(w_o) \) when \( A \) has good reduction at \( p \). The previous discussion places Rivero’s conjecture within a conceptual framework and sheds some light on this question.

### 3 Height computations

Throughout the rest of this note we assume that \((A, \varphi)\) is exceptional at \( p \). In particular \( A \) has multiplicative reduction at \( p \), i.e., \( p \) divides exactly \( N_f \).

#### 3.1 Setting and notations

This subsection briefly recalls the needed definitions and notations from our previous articles [6,7].

##### 3.1.1 Galois representations

Set \( N = \text{lcm}(N_f, N_g, N_h) \) and let \( G_{Q,N} \) be the Galois group of the maximal extension of \( Q \) contained in \( \overline{Q} \) and unramified outside \( N\infty \). If \( \xi \) denotes one of \( f, g_\alpha \) and \( h_\alpha \), let \( V(\xi) \) be the big Galois representation associated with \( \xi \) (cf. Section 5 of [7]). It is a free \( \mathcal{O}_\mathbb{Q} \)-module of rank two, equipped with a continuous linear action \( \chi_\xi \) of \( \text{Hom}(\text{Gal}(\overline{Q}/Q), \mathbb{Q}^\times) \), the base change \( \mathcal{O}_\xi \) and unramified outside \( Q\infty \). For each \( u \) in \( G_{Q,N} \), \( V(\xi)_u \) denotes one of the \( p \)-adic Deligne–Serre representation of \( \xi = g, h \) with coefficients in \( L \). In order to have a uniform notation, in this case one defines \( \rho_1 : V(\xi) \otimes_1 L \rightarrow V(\xi) \) to be the identity.

The restriction of \( V(\xi)_u \) to \( G_{Q,p} \) (via the embedding \( i_p \) fixed at the outset) fits into a short exact sequence of \( \mathcal{O}_\xi[G_{Q,p}] \)-modules \( V(\xi)^+ \rightarrow V(\xi) \rightarrow V(\xi)^- \rightarrow 0 \), free of rank one over \( \mathcal{O}_\xi \). More precisely, let \( \chi_\text{cyc} : \mathcal{O}_Q \rightarrow \mathbb{Z}_p^\times \) be the \( p \)-adic cyclotomic character, and let \( \tilde{a}_p(\xi) : G_{Q,p} \rightarrow \mathcal{O}_p^\times \) be the unramified character sending an arithmetic Frobenius to the \( p \)-th Fourier coefficients \( a_p(\xi) \) of \( \xi \). Then

\[
V(\xi)^+ \simeq \mathcal{O}_\xi(\chi_\text{cyc}^{-1} \cdot \chi_\xi \tilde{a}_p(\xi)^{-1}) \quad \text{and} \quad V(\xi)^- \simeq \mathcal{O}_\xi(\tilde{a}_p(\xi)),
\]

where \( \chi_\text{cyc}^{-1} : \mathcal{O}_Q \rightarrow \mathcal{O}_\xi^\times \) satisfies \( \chi_\text{cyc}(\sigma)^{-1} = \chi_\text{cyc}(\sigma)^u^{-1} \) for each \( u \) in \( U_{\mathbb{Q}} \cap \mathbb{Z} \). (The freeness of \( V(\xi)^\pm \) is guaranteed by Assumption 1.1.3, cf. Section 5 of [7].) If \( \xi = f \) and \( u = 2 \) the specialisation isomorphism \( \rho_2 \) identifies \( V(\varphi)^- \otimes_2 L \) with the maximal unramified quotient \( V(\varphi)^- \) of \( V(\varphi) \). If \( \xi = g_\alpha \) and \( u = 1 \) we set \( V(\xi)^+ = V(\xi) \otimes_1 L \).
and $V(\xi)_{\alpha} = V(\xi)^{-} \otimes_{1} L$. One has $V(\xi) = \alpha(\xi) \oplus V(\xi)_{\beta}$, where $V(\xi)_{\gamma} = (\xi)^{\text{Frob}_p = \gamma \xi}$ for $\gamma = \alpha, \beta$ is the submodule of $V(\xi)$ on which an arithmetic Frobenius $\text{Frob}_p$ acts as multiplication by $\gamma^* = \alpha \beta (c\text{f Assumption 1.1.3}).$

There is a natural $G_Q$-equivariant skew-symmetric perfect pairing

$$\pi_{\xi} : V(\xi) \otimes_{\mathcal{O}_\xi} V(\xi) \longrightarrow \mathcal{O}_\xi(V(\chi_\xi \cdot \chi_\xi^{-1}),$$

inducing perfect dualities $\pi_{\xi} : V(\xi) \otimes_{\mathcal{O}_\xi} V(\xi) = \mathcal{O}_\xi(V(\chi_\xi \cdot \chi_\xi^{-1})).$ (See Section 5 cf. [7] for the definitions).

Denote by $\mathbb{Z}_{fg} = \chi_{\text{cyc}}^{(-k-l-m)/2} : G_Q \longrightarrow \mathcal{O}_{\text{fg}}$ the character whose composition with evaluation at $(k, l, m)$ in $U_f \times U_g \times U_h \cap \mathbb{Z}^3$ on $\mathcal{O}_{fg}$ equals $\chi_{\text{cyc}}^{(4-k-l-m)/2}$. If $\cdot$ denotes one of the symbols $\otimes$, $+$, and $-$, define

$$V' = V(f) \hat{\otimes}_L V(g) \hat{\otimes} V(h) \otimes_{\mathcal{O}_{fg}} \mathbb{Z}_{fg}.$$

Then $V = V(f, g, h)$, resp. $V' = V(f, g, h)$ is a free $\mathcal{O}_{fg}$-module of rank 8, resp. 4, equipped with a continuous action of $G_{Q,N}$, resp. $G_{Q,p}$. As $\chi_{\xi} \cdot \chi_{h} = 1$ (cf. Assumption 1.1), the product of the perfect dualities $\pi_{\xi}$, for $\xi = f, g, h$, yields a perfect skew-symmetric Kummer duality $\pi : V \otimes_{\mathcal{O}_{fg}} V \longrightarrow \mathcal{O}_{fg}(1)$, inducing a perfect local Kummer duality $\pi : V' \otimes_{\mathcal{O}_{fg}} V' \longrightarrow \mathcal{O}_{fg}(1)$. After setting

$$V' = V(f, g, h) = V(f) \hat{\otimes}_L V(g) \hat{\otimes} L V(h)$$

and $w_\omega = (2, 1, 1)$, the product $\rho_{w_\omega} = \rho_2 \hat{\otimes} \rho_1 \hat{\otimes} \rho_1$ gives natural isomorphisms

$$\rho_{w_\omega} : V' \otimes_{w_\omega} L \simeq V'$$

(where $\cdot \otimes_{w_\omega} L$ denotes the base change along evaluation at $w_\omega$ on $\mathcal{O}_{fg}$). Let

$$\pi_{fg} : V \otimes_{L} V \longrightarrow L(1)$$

be the specialisation of $\pi$ via $\rho_{w_\omega}$, and define $\pi : V' \otimes_{L} V' \longrightarrow L(1)$ similarly.

**Weight one differentials** Define $D(\xi)^{-} = H^0(Q_p, V(\xi)^{-} \hat{\otimes} Q_p \hat{Q}_p^{nr})$, where $\hat{Q}_p^{nr}$ is the $p$-adic completion of the maximal unramified extension of $Q_p$ (and as usual $\xi$ denotes one of $f, g, h$ and $h$). For each $u$ in $U_\xi \cap \mathbb{Z}_{\geq 2}$ there is a natural comparison isomorphism between $D(\xi)^{-} \otimes_u L$ and the $\xi_u$-isotypic component of the space of cuspidal modular forms of weight $u$, level $\Gamma(1, \xi, p)$ and Fourier coefficients in $L$. Assumption 1.1.3 guarantees that $D(\xi)^{-}$ is free (of rank one) over $\mathcal{O}_\xi$, and admits a basis $\omega_\xi$ whose image in $D(\xi)^{-} \otimes_u L$ corresponds to $\xi_u$ under the aforementioned comparison isomorphism, for each $u$ in $U_\xi \cap \mathbb{Z}_{\geq 2}$. (We refer to Section 3.1 of [6] and the references therein for more details.)

For $\xi = g, h$, the *holomorphic weight-one differential*

$$\omega_{\xi} \in (V(\xi) \hat{\otimes} Q_p \hat{Q}_p^{nr})^{G_Q}$$

mentioned in Eq. (5) is defined to be the weight-one specialisation of $\omega_\xi$, viz. the image of $\omega_\xi$ in the quotient $D(\xi)^{-} \otimes_1 L = D(\xi)_{\alpha}$. The weight-one specialisation of $\pi_{\xi}$ yields a perfect $G_Q$-equivariant skew-symmetric pairing

$$\pi_{\xi} : V(\xi) \otimes_{L} V(\xi) \longrightarrow L(\chi_{\xi}).$$

Let $c$ be the common conductor of $\chi_g$ and $\chi_h$, and identify $(L(\chi_{\xi} \otimes Q_p \hat{Q}_p^{nr})^{G_Q}$ with $L$ via the Gauss sum $G(\chi_{\xi}) = (-c)^i z \sum_{a \in (\mathbb{Z}/c \mathbb{Z})^*} \chi_{\xi}(a)^{-1} \otimes e^{2\pi ia/c}$, where $i_g = 0$ and $i_h = 1$ (so
that $G(\chi_\ell) \cdot G(\chi_h) = 1$ by Assumption 1.1.1. The pairing $\pi_\xi$ then induces a perfect duality $\langle \cdot, \cdot \rangle_\xi : D(\xi)_\alpha \otimes_L D(\xi)_\beta \to L$, where $D(\xi)_\beta = (V(\xi)_\gamma \otimes \mathbb{Q}_p)^{\text{nr}}$. One defines the antiholomorphic weight-one differential (cf. Eq. (5))

$$\eta_{\xi_0} \in (V(\xi)_\beta \otimes \mathbb{Q}_p)^{\text{nr}} \mathbb{Q}_p$$

to be the dual of $\omega_{\xi_0}$ under $\langle \cdot, \cdot \rangle_\xi$, viz. the element satisfying $\langle \omega_{\xi_0}, \eta_{\xi_0} \rangle_\xi = 1$.

The embeddings $\gamma_g$ and $\gamma_h$ With the notations of Sect. 1, set $V_g = V_{\ell^1}$ and $V_h = V_{\ell^2}$. Let $\xi$ denote either $g$ or $h$. As recalled above, the Artin representation $V(\xi) = V(\xi) \otimes_1 L$ affords the dual of the $p$-adic Deligne representation of $\xi$ with coefficients in $L$, i.e., is isomorphic to $\pi_\xi \otimes_{\mathbb{Q}(q)} L$. Enlarging $L$ if necessary, we normalise the $G_{\mathbb{Q}}$-equivariant embedding $\gamma_\xi : V_\xi \to V(\xi)$ (introduced in Eq. (3)) by requiring that the composition $\pi_\xi \circ (\gamma_\xi \otimes \gamma_\xi)$ takes values in the number field $\mathbb{Q}(q)$ (via the embedding $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ fixed at the outset).

### 3.1.2 Selmer complexes

Let $\mathbf{R}\Gamma_f(Q, V)$ be the Nekovář Selmer complex associated with $(V, V^+)$ (cf. Section 2.2 of [6]). It is an element of the derived category $\mathbf{D}_{\text{ct}}^b(L)$ of cohomologically bounded complexes of $L$-modules with cohomology of finite type over $L$, sitting is exact triangle

$$\mathbf{R}\Gamma_{\text{cont}}(G_{Q,N}, V) \xrightarrow{p^{-}\text{res}_p} \mathbf{R}\Gamma_{\text{cont}}(G_{Q_p}, V^-) \to \mathbf{R}\Gamma_f(Q, V)[1], \tag{12}$$

where $\mathbf{R}\Gamma_{\text{cont}}(G, \cdot)$ is the complex of continuous non-homogeneous cochains of $G$ with values in $\cdot$, res$_p$ is the restriction map (induced by the embedding $i_p : \mathbb{Q} \hookrightarrow \mathbb{Q}_p$ fixed at the outset) and $p^-$ is the map induced by the projection $V \to V^-$. Denote by

$$\hat{H}_f(Q, V) = H^\cdot(\mathbf{R}\Gamma_f(Q, V))$$

the cohomology of $\mathbf{R}\hat{\Gamma}(Q, V)$, let $\text{Sel}(Q, V)$ be the Bloch–Kato Selmer group of $V$ over $Q$, and let $i^+ : V^+ \to V$ be the natural inclusion. Then there is a commutative and exact diagram of $L$-vector spaces (cf. loc. cit.)

$$\begin{array}{cccccc}
0 & \to & H^0(Q_p, V^-) & \xrightarrow{j} & \hat{H}^1_f(Q, V) & \to & \text{Sel}(Q, V) & \to & 0 \\
& & \downarrow{\cdot} & & \downarrow{\text{res}_p} & & \\
& & H^1(Q_p, V^+) & \xrightarrow{i^+} & H^1(Q_p, V) & & \\
\end{array} \tag{13}$$

where the first line arises from the exact triangle (12). In addition there is a unique section $t_{\text{ur}} : \text{Sel}(Q, V) \to \hat{H}^1_f(Q, V)$ of the above projection such that $t_{\text{ur}}(x)^+$ belongs to the Bloch–Kato finite subspace $H^1_{\text{fin}}(Q_p, V^+)$ for each $x$ in $\text{Sel}(Q, V)$. We often use $j$ and $t_{\text{ur}}$ to identify Nekovář’s extended Selmer group $\hat{H}^1_f(Q, V)$ with the naive extended Selmer group $\text{Sel}^\dagger(Q, V) = H^0(Q_p, V^-) \oplus \text{Sel}(Q, V)$ (cf. Sect. 1).

One similarly associates with $(V, V^+)$ a Selmer complex

$$\mathbf{R}\Gamma_f(Q, V) \in \mathbf{D}_{\text{ct}}^b(\mathcal{O}^{fgh})$$

sitting in an exact triangle analogous to (12). (We refer to loc. cit. for more details.)
3.2 Preliminary lemmas

This section gives a concrete description of the functionals $\langle q, \cdot \rangle_{fg, a, h_o} : \text{Sel}^I(Q, V) \to L$ for $q$ in $H^0(Q_p, V^-)$ (cf. Lemma 3.4 below).

3.2.1 Bockstein maps

Let $(C, C)$ denote one of the pairs

$$(R\Gamma_p(V^-), R\Gamma_p(V^-)), (R\Gamma(V), R\Gamma(V)) \text{ and } (R\tilde{\Gamma}_f(Q, V), R\tilde{\Gamma}_f(Q, V)),$$

where $R\Gamma_p(\cdot)$ and $R\Gamma(\cdot)$ are shorthands for $R\Gamma_{\text{cont}}(Q_p, \cdot) = R\Gamma_{\text{cont}}(G_{Q_p}, \cdot)$ and $R\Gamma_{\text{cont}}(G_{Q_N}, \cdot)$ respectively (cf. Sect. 3.1.2). The specialisation maps $\rho_{w_o}$ (cf. Eq. (10)) induce isomorphisms

$$\rho_{w_o} : C \otimes_{\partial_{fgh}, w_o} L \cong C \text{ and } \rho_{w_o} \otimes \text{id} : C \otimes_{\partial_{fgh}} \mathcal{I}/\mathcal{I}^2[1] \cong C \otimes_L \mathcal{I}/\mathcal{I}^2[1]. \tag{14}$$

Applying $C \otimes_{\partial_{fgh}} \cdot$ to the exact triangle

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \partial_{fgh}/\mathcal{I}^2 \longrightarrow L \longrightarrow \mathcal{I}/\mathcal{I}^2[1]$$

(arising from evaluation on $w_o$) then yields a derived Bockstein map

$$\beta_{C/C} : C \longrightarrow C \otimes_L \mathcal{I}/\mathcal{I}^2[1],$$

which in turn induces in cohomology a Bockstein map

$$\beta_{C/C} : H^i(C) \longrightarrow H^{i+1}(C) \otimes_L \mathcal{I}/\mathcal{I}^2.$$

If no risk of confusion arises, we simply write $\beta$ for $\beta_{C/C}$. Let

$$j : H^i(Q_p, V^-) \longrightarrow \tilde{H}^{i+1}_f(Q, V)$$

be the maps arising from the exact triangle (12).

**Lemma 3.1** The following diagram commutes.

$$
\begin{array}{ccc}
H^0(Q_p, V^-) & \xrightarrow{\beta} & H^1(Q_p, V^-) \otimes_L \mathcal{I}/\mathcal{I}^2 \\
j & & j \otimes \mathcal{I}/\mathcal{I}^2 \\
\tilde{H}^1_f(Q, V) & \xrightarrow{\beta} & \tilde{H}^2_f(Q, V) \otimes_L \mathcal{I}/\mathcal{I}^2
\end{array}
$$

**Proof** For $M = V$, $V$ one has an exact triangle (cf. Equation (12))

$$\Delta_M : R\Gamma_{\text{cont}}(G_{Q_N}, M)[-1] \xrightarrow{\cdot \text{res}_{p}} R\Gamma_{\text{cont}}(Q_p, M^)[-1] \xrightarrow{\cdot J_M} R\tilde{\Gamma}_f(Q, M).$$

Moreover $\Delta_V$ is obtained by applying $\cdot \otimes_{\partial_{fgh}, w_o} L$ to $\Delta_V$ (cf. Eq. (14)). It follows from the definition of the derived Bockstein maps $\beta^-$ and $\beta$ on $R\Gamma_{\text{cont}}(Q_p, V^-)$ and $R\tilde{\Gamma}(Q, V)$ respectively that $j \otimes \mathcal{I}/\mathcal{I}^2[1] \circ \beta^-$ is equal to $\beta \circ j_V$. Since by definition the maps $j$ are the ones induced in cohomology by $j_V$, the lemma follows. $\square$

The following lemma gives a concrete description of $\beta_{C/C}$.
Lemma 3.2 Let \((C, C)\) be as above, let \(z\) be a 1-cocycle in \(C\), let \(Z\) be a 1-cochain in \(C\), and let \(Z_k, Z_l, Z_m\) be 2-cochains in \(C\) such that
\[
\rho_{w_0}(Z) = z \quad \text{and} \quad dZ = Z_k \cdot (k - 2) + Z_l \cdot (l - 1) + Z_m \cdot (m - 1).
\]
Then \(z = \rho_{w_0}(Z)\) is a 2-cocycle for \(\cdot = k, l, m\), and one has the equality
\[
-\beta_{C/C} (cl(z)) = cl(z_k) \cdot (k - 2) + cl(z_l) \cdot (l - 1) + cl(z_m) \cdot (m - 1)
\]
in \(H^2(C) \otimes_L \mathcal{I}/\mathcal{I}^2\), where \(cl(\cdot)\) is the class in \(H^1(C)\) represented by the \(i\)-cocycle \(\cdot\).

Proof The proof is very similar to that of [16, Lemma 5.5]. We omit it.

3.2.2 Local and global duality

Nekovář’s generalised Poitou–Tate duality associates with the perfect duality \(\pi_{fgh}\) introduced in Eq. (11) a global cup-product pairing (cf. Section 2.4 of [6])
\[
\langle \cdot, \cdot \rangle_{\text{Nek}} : \tilde{H}^2_f(Q, V) \otimes_L \tilde{H}^1_f(Q, V) \longrightarrow L.
\] (15)
The pairing \(\pi_{fgh}\) induces a Kummer duality \(V^- \otimes_L V^+ \longrightarrow L(1)\) and we denote by
\[
\langle \cdot, \cdot \rangle_{\text{Tate}} : H^1(Q_p, V^-) \otimes_L H^1(Q_p, V^+) \longrightarrow L
\] (16)
the induced local Tate duality pairing. Recall finally the map
\[
\cdot^+ : \tilde{H}^1_f(Q, V) \longrightarrow H^1(Q_p, V^+)
\]
introduced in diagram (13).

Lemma 3.3 For each \(\zeta\) in \(H^1(Q_p, V^-)\) and \(\xi\) in \(\tilde{H}^1_f(Q, V)\) one has
\[
\langle j(\zeta), \xi \rangle_{\text{Nek}} = \langle \zeta, \xi^+ \rangle_{\text{Tate}}.
\]

Proof This is proved as in [16, Lemma 5.7].

3.2.3 The Garrett–Nekovář \(p\)-adic height pairing

Set
\[
\tilde{\beta}_{fgh} : \tilde{H}^1_f(Q, V) \longrightarrow \tilde{H}^2_f(Q, V) \otimes_L \mathcal{I}/\mathcal{I}^2.
\]
After identifying \(\tilde{H}^1_f(Q, V)\) with \(\text{Sel}^1(Q, V)\) (cf. Sect. 3.1.2), the canonical height \(\langle \cdot, \cdot \rangle_{fgh}\) introduced in Sect. is defined by (cf. [6, Section 2])
\[
\langle x, y \rangle_{fgh} = \langle \tilde{\beta}_{fgh}(x), y \rangle_{\text{Nek}}
\]
for each \(x\) and \(y\) in \(\tilde{H}^1_f(Q, V)\), where we write again \(\langle \cdot, \cdot \rangle_{\text{Nek}}\) for the \(\mathcal{I}/\mathcal{I}^2\)-base change of Nekovář’s cup-product (15). Lemmas 3.1 and 3.3 give the following

Lemma 3.4 For each \(q\) in \(H^0(Q_p, V^-)\) one has
\[
\langle \langle j(q), \cdot \rangle_{fgh} = \langle \beta_{fgh}(q), \cdot^+ \rangle_{\text{Tate}}
\]
as \(\mathcal{I}/\mathcal{I}^2\)-valued maps on \(\tilde{H}^1_f(Q, V)\), where \(\beta_{fgh} = \beta_{R\Gamma_p(V^-)/R\Gamma_p(V^-)}\) (and we write again \(\langle \cdot, \cdot \rangle_{\text{Tate}}\) for the \(\mathcal{I}/\mathcal{I}^2\)-base change of the local Tate pairing (16)).
3.3 Computation of $\langle q_{\beta \beta}, q_{\alpha \alpha} \rangle_{f_g a h_a}$

Assume in this subsection $\alpha_f = \alpha_g \cdot \alpha_h$, so that $H^0(Q_p, V^-)$ is generated over $L$ by the periods

$$q_{\alpha \alpha} = \sqrt{m_p} \cdot q(f) \otimes \omega_{\alpha \alpha} \otimes \omega_{h_a} \quad \text{and} \quad q_{\beta \beta} = \sqrt{m_p} \cdot q(f) \otimes \eta_{\alpha \alpha} \otimes \eta_{h_a}.$$  

Recall that $\chi_{\text{cyc}} : G_Q \rightarrow \mathbb{Z}^*$ denotes the $p$-adic cyclotomic character. Fix a lift $q_{\beta \beta}$ in $V^-$ of $q_{\beta \beta}$ under $\rho_{w_o}$. Since (cf. Sect. 3.1.1)

$$q_{\beta \beta} \in V(f)^- \otimes_{Q_p} V(g)_{\beta} \otimes_L V(h)_{\beta} \hookrightarrow V^-$$

and $V(\xi)_{\beta} = V(\xi_{\alpha})^+ \otimes_1 L$ for $\xi = g, h$, we can choose $q_{\beta \beta}$ in the $G_{Q_p}$-submodule

$$V(f)^- \otimes_{Q_p} V(g)^+ \otimes_{\ell \otimes q} \mathbb{Z} \hookrightarrow V^-$$

(cf. Sect. 3.1.1). By Eq. (9) one has

$$dq_{\beta \beta} = \Phi \cdot q_{\beta \beta}, \quad (17)$$

where $d$ denotes the differentials of the complex $R \Gamma_{\text{cont}}(Q_p, V^-)$ and

$$\Phi = \frac{\alpha_p(f)}{\alpha_p(g_a) \cdot \alpha_p(h_a)} \cdot \chi_{\text{cyc}} ((l + m - k)/2 - 1 : G_{Q_p} \rightarrow \ell \otimes q_{\beta \beta}.$$  

The assumption $\alpha_f = \alpha_g \cdot \alpha_h$ implies that $\Phi$ takes value in $\mathcal{F}$, and that its composition $\Phi'$ with the projection $\mathcal{F} \rightarrow \mathcal{F}/\mathcal{F}^2$ is of the form

$$\Phi' = \varphi_k \cdot (k - 2) + \varphi_l \cdot (l - 1) + \varphi_m \cdot (m - 1)$$

with $\varphi_u$ in $H^1(Q_p, Q_p)$ for $u = k, l, m$. Identify $H^1(Q_p, Q_p)$ with the $Q_p$-vector space $\text{Hom}(Q_p^*, Q_p)$ of continuous morphisms of groups from $Q_p^*$ to $Q_p$ via the local reciprocity map $\text{rec}_p : Q_p^* \rightarrow C_{Q_p}^{ab}$, normalised by requiring $\text{rec}_p(p^{-1})$ to be an arithmetic Frobenius. By local class field theory, for each $p$-adic unit $u$ one has

$$\varphi_k(u) = \frac{\partial}{\partial k} \left( \langle u \rangle^{(l + m - k)/2 - 1} \right) \bigg|_{u_o} = -\frac{1}{2} \cdot \log_p(u),$$

where $\langle \cdot \rangle : Z_p^* \rightarrow 1 + pZ_p$ denotes the projection to principal units, and

$$\varphi_k(p) = \frac{\partial}{\partial k} \left( \frac{a_p(g_a) \cdot a_p(h_a)}{a_p(f)} - 1 \right) \bigg|_{u_o} = \frac{1}{2} \cdot \varphi_{an} f$$

(cf. Eq. (7)). As a consequence $-2 \cdot \varphi_k$ is equal to

$$\log_f = \log_p - \varphi_{an} f \cdot \text{ord}_p \in H^1(Q_p, Q_p)$$

(where the $p$-adic valuation $\text{ord}_p : Q_p^* \rightarrow Q_p$ is normalised by $\text{ord}_p(p) = 1$). Similarly one shows that $2 \cdot \varphi_l$ and $2 \cdot \varphi_m$ are equal to the logarithms $\log_{g_a} = \log_p - \varphi_{an} g_a \cdot \text{ord}_p$ and $\log_{h_a} = \log_p - \varphi_{an} h_a \cdot \text{ord}_p$. It then follows from Eq. (17) and Lemma 3.2 that

$$2 \cdot \beta_{f_g a h_a} (q_{\beta \beta}) = \left( \log_f \cdot (k - 2) - \log_{g_a} \cdot (l - 1) - \log_{h_a} \cdot (m - 1) \right) \otimes q_{\beta \beta} \quad (18)$$

in $H^1(Q_p, V^-) \otimes_L \mathcal{F}/\mathcal{F}^2$, where (with the notations introduced in Sect. 3.2.1) one writes $\beta_{f_g a h_a}$ for the Bockstein map $\beta_{C/C}$ associated with $C = R \Gamma_p(V^-)$. Note that

$$V(f)_{\beta \beta} = V(f)^- \otimes_{Q_p} V(g)_{\beta} \otimes_L V(h)_{\beta}.$$
is the composition of $\xi = f, g_\alpha, h_\alpha$ belongs to the direct summand

$$H^1(Q_p, V(f)_{\bar{\beta}}^\alpha) = H^1(Q_p, Q_p) \otimes_{Q_p} V(f)_{\bar{\beta}}$$

of the local cohomology group $H^1(Q_p, V^-)$. Similarly

$$V(f)_{\alpha\alpha}^- = V(f)^+ \otimes_{Q_p} V(g_\alpha) \otimes_L V(h_\alpha)$$

is an $L[G_{Q_p}]$-direct summand of $V^+$ isomorphic to $Q_p(1)$, hence

$$H^1(Q_p, V(f)_{\alpha\alpha}^+) = H^1(Q_p, Q_p(1)) \otimes_{Q_p} V(f)_{\alpha\alpha}^+(-1)$$

is a direct summand of $H^1(Q_p, V^+)$.

Recall that we identify $\hat{H}^1_j(Q, V)$ introduced in Sect. 3.2.2 induces a perfect duality (denoted by the same symbol) between $H^1(Q_p, V(f)_{\bar{\beta}})$ and $H^1(Q_p, V(f)^+_{\alpha\alpha})$, and identifying $H^1(Q_p, Z_p(1))$ with the $p$-adic completion $\hat{Q}_p$ of $Q_p$ via the local Kummer map, local class field theory gives

$$\langle \varphi \otimes v^-, u \otimes v^+ \rangle_{\text{Tate}} = \varphi(u) \cdot \pi_{fgh}(-1)(v^+ \otimes v^-)$$

for each $\varphi$ in $H^1(Q_p, V_p)$, $u$ in $H^1(Q_p, Q_p(1))$, $v^-$ in $V(f)_{\bar{\beta}}$ and $v^+$ in $V(f)^+_{\alpha\alpha}$. Here

$$\pi_{fgh}(-1) : V(f)^+_{\alpha\alpha}(-1) \otimes_L V(f)_{\bar{\beta}} \longrightarrow L$$

is the composition of $\pi_{fgh} \otimes Q_p(-1)$ with the evaluation pairing $L(1) \otimes_L L(-1) \longrightarrow L$.

Recall that we identify $H^0(Q_p, V^-)$ with a submodule of $\hat{H}^1_j(Q, V)$ via the embedding $j$ introduced in Diagram (13). Lemma 3.4 and Eqs. (18) and (20) give

$$2 \cdot \langle q_{\beta\beta}, z \rangle_{f_{R_a}h_\alpha} \overset{\text{Lemma 3.8}}{=} 2 \cdot \langle \beta_{f_{R_a}h_\alpha}(q_{\beta\beta}), z^+ \rangle_{\text{Tate}}$$

$$\overset{\text{Equation (18)}}{=} \sum_{\xi} (-1)^{u_o} \cdot \langle \log_\xi \otimes q_{\beta\beta}, z^+ \rangle_{\text{Tate}} \cdot (u - u_o)$$

$$\overset{\text{Equation (20)}}{=} \sum_{\xi} (-1)^{u_o} \cdot \log_\xi(z_{\alpha\alpha}^+) \cdot (u - u_o)$$

for each $z$ in $\hat{H}^1_j(Q, V)$, where $\xi = f, g_\alpha, h_\alpha, u_o = 2, 1, 1$ is the centre of $U_\xi$, and

$$z_{\alpha\alpha}^+ \in H^1(Q_p, Q_p(1)) = Q_p^* \otimes_{Z_p} Q_p$$

is defined as follows. Let $pr_{\alpha\alpha}$ denote the projection onto the direct summand $H^1(Q_p, V(f)^+_{\alpha\alpha})$ of the local cohomology group $H^1(Q_p, V^+)$, and let $q_{\beta\beta}^*$ be the generator of $V(f)^+_{\alpha\alpha}(-1)$ dual to $q_{\beta\beta}$ under $\pi_{fgh}(-1)$, namely satisfying

$$\pi_{fgh}(-1)(q_{\beta\beta}^* \otimes q_{\beta\beta}) = 1.$$

Then $z_{\alpha\alpha}^+$ is defined (via the natural isomorphism (19)) by the identity

$$pr_{\alpha\alpha}(z^+) = z_{\alpha\alpha}^+ \otimes q_{\beta\beta}^*.$$

We now determine $z_{\alpha\alpha}^+$ for $z = j(q_{\alpha\alpha})$. By definition $j(q_{\alpha\alpha})$ is represented by

$$c_{\alpha\alpha} = (0, d \tilde{q}_{\alpha\alpha}, \tilde{q}_{\alpha\alpha}) \in \tilde{C}_f^1(Q, V),$$
where $\tilde{q}_{aa}$ in $V$ is a lift of $q_{aa}$ under the the projection $V \to V^-$, and where

$$d\tilde{q}_{aa} : G_{Q_p} \to V^+$$

is its image under the differential in $\mathbf{R}\Gamma_{\mathrm{cont}}(Q_p, V)$. By construction $d\tilde{q}_{aa}$ represents the class $q_{aa}^+ = j(q_{aa})^+$ in $H^1(Q_p, V^+)$. Since $V(\xi)$ is the direct sum of $V(\xi)_\alpha$ and $V(\xi)_\beta$ for $\xi = g, h$, we can (and will) choose $\tilde{q}_{aa}$ of the form

$$\tilde{q}_{aa} = \sqrt{m_p} \cdot \tilde{q}(f) \otimes \omega_{g_a} \otimes \omega_{h_a}$$

for a lift $\tilde{q}(f)$ of $q(f)$ under the projection $V(f) \to V(f)$, so that $d\tilde{q}_{aa}$ represents the image of $q_{aa}$ under the connecting morphism

$$\delta_{aa} : V(f)_{\alpha\alpha} \to H^1(Q_p, V(f)_{\alpha\alpha}^+)$$

arising from the short exact sequence of $G_{Q_p}$-modules

$$0 \to V(f)_{\alpha\alpha}^+ \to V(f)_{\alpha\alpha} \to V(f)_{\alpha\alpha}^- \to 0,$$

where $V(f)_{\alpha\alpha}$ is the $L[G_{Q_p}]$-direct summand $V(f)^- \otimes_{Q_p} V(g)_{\alpha} \otimes_{L} V(h)_{\alpha}$ of $V$. Let $q_A$ in $p\mathbb{Z}_p$ be the Tate period of $A_{Q_p}$. Tate’s theory gives a rigid analytic isomorphisms between the base change $E_{Q_p}$ of the Tate curve $E = G_{m, Q_p}/q^A_Z$ to the quadratic unramified extension $Q_{p^2}$ of $Q_p$ and $A_{Q_{p^2}}$. Set $V_p(E) = H^1_k(E_{Q_p}, Q_p(1))$ and let $\varphi_{\text{Tate}} : V_p(E) \simeq V_p(A)$ be the isomorphisms of $G_{Q_{p^2}}$-modules induced by the Tate uniformisation. There is a short exact sequence of $Q_p[G_{Q_p}]$-modules

$$0 \to Q_p(1) \xrightarrow{a} V_p(E) \xrightarrow{b} Q_p \to 0,$$

where $a(\xi) = (\xi^n \cdot q_A^n)_{n \geq 1}$ for each compatible system $\xi = (\xi^n)_{n \geq 1}$ of $p^n$-th roots of unity, and $b$ is the $Q_p$-linear extension of the inverse limit of (canonical) maps

$$b_n : E(\tilde{Q}_p/p^n) \to (\tilde{Q}_p/p^n)_{\alpha\alpha} \to \mathbb{Z}/p^n \mathbb{Z}$$

defined by $b_n(x \cdot q_A^n) = \frac{p^n \cdot \omega_{\text{ord}_p}(x)}{\text{ord}_p(q_A^n)} + p^n \cdot \mathbb{Z}$. By definition $q(A) = \varphi_{\text{Tate}}^{-1}(1)$, where $\varphi_{\text{Tate}}^{-1} \circ b$ is the composition of $\varphi_{\text{Tate}}$ and the projection $V_p(A) \to V_p(A)^-$ onto the maximal $G_{Q_{p^2}}$-unramified quotient, and

$$\tilde{q}(f) = \varphi_{\text{Tate}}^{-1} \circ \varphi_{\text{Tate}}(\sqrt{q_A})$$

is the image of a compatible system $\sqrt{q_A}$ of $p^n$-th roots of the Tate period $q_A$ under the composition of $\varphi_{\text{Tate}}$ and the inverse of the isomorphism $\varphi_{\infty} : V(f) \simeq V_p(A)$ induced by the fixed modular parametrisation $\varphi_{\infty} : X_1(N_f) \to A$. As a consequence 1 in $Q_p$ maps to $q_A \hat{\otimes} 1$ under the connecting map $Q_p \to H^1(Q_p, Q_p(1)) = Q_p^* \hat{\otimes} Q_p$ associated with the short exact sequence (23), hence

$$j(q_{aa}) = c \cdot (d\tilde{q}_{aa}) = \delta_{aa}(q_{aa}) = \sqrt{m_p} \cdot (\varphi_{\text{Tate}}^{-1})_*(q_A \otimes 1) \otimes \omega_{g_a} \otimes \omega_{h_a}$$

in

$$H^1(Q_p, V(f)_{\alpha\alpha}) = H^0(Gal(Q_{p^2}/Q), H^1(Q_{p^2}, V(f)^+) \otimes_{Q_p} V(g)_{\alpha} \otimes_{L} V(h)_{\alpha}),$$

where

$$(\varphi_{\text{Tate}}^{-1})_*(Q_{p^2}^* \hat{\otimes} Q_p) \simeq H^1(Q_{p^2}, V(f)^+)$$
is the map induced in cohomology by the composition of $\varphi_{\infty}^{-1}$ and

$$\varphi_{\text{Tate}}^{+} = \varphi_{\text{Tate}} \circ \alpha.$$  

If $A$ denotes either $A$ or $E$, denote by

$$\pi_A : V_p(A)(-1) \otimes \mathbb{Q}_p V_p(A) \rightarrow \mathbb{Q}_p$$

the composition of the evaluation pairing $\mathbb{Q}_p(1) \otimes \mathbb{Q}_p \mathbb{Q}_p(-1) \rightarrow \mathbb{Q}_p$ with the base change of the Weil pairing on $V_p(A)$ by $\mathbb{Q}_p(-1)$. Set

$$q(A)^* = \varphi_{\text{Tate}}^{+} (\zeta_{p^{\infty}}) \otimes \zeta_{p^{\infty}}^{*} \in V_p(A)^{+}(-1),$$

where $\zeta_{p^{\infty}}$ is a generator of $\mathbb{Q}_p(1)$ and $\zeta_{p^{\infty}}^{*}$ in $\mathbb{Q}_p(-1)$ is its dual basis, and set

$$q(f)^* = \deg(\varphi_{\infty}) \cdot \varphi_{\infty}^{-1}(q(A)^*) \in V(f)^{+}(-1).$$

As $\pi_E((a(y) \otimes z) \otimes x) = b(x) \cdot z(y)$ for each $x$ in $V_p(E)$, $y$ in $\mathbb{Q}_p(1)$ and $z$ in $\mathbb{Q}_p(-1)$, the functoriality of the Poincaré duality under finite morphisms yields

$$\pi_f(q(f)^* \otimes q(f)) = \pi_A(q(A)^* \otimes q(A)) = \pi_E((\alpha(\zeta_{p^{\infty}}) \otimes \zeta_{p^{\infty}}^{*}) \otimes p^{\infty}q_A) = 1,$$

then (by the definition of the weight-one differentials $\eta_{\hat{h}_a}$, cf. Sect. 3.1.1)

$$q_{\beta\beta}^{*} = \frac{1}{\sqrt{m_p}} \cdot q(f)^* \otimes \omega_{g_a} \otimes \omega_{h_a}.$$  

Together with Eq. (24) this gives

$$J(q_{\alpha\alpha})^+ = \frac{m_p}{\deg(\varphi_{\infty})} \cdot (q_A \hat{\otimes} 1) \otimes q_{\beta\beta}^*,$$

id est

$$J(q_{\alpha\alpha})^+_{\alpha\alpha} = \frac{m_p}{\deg(\varphi_{\infty})} \cdot q_A \hat{\otimes} 1.$$  

According to Theorem 3.18 of [9] $\omega_{f}^{\text{an}} = \frac{\log_p(q_A)}{\ord_p(q_A)}$, so that

$$- \frac{2 \cdot \deg(\varphi_{\infty})}{m_p \cdot \ord_p(q_A)} \cdot \langle q_{\beta\beta} \cdot q_{\alpha\alpha} \rangle_{f \mathbb{Q}_p \mathbb{Q}_p} h_a = (\omega_{f}^{\text{an}} - \omega_{g_a}^{\text{an}}) \cdot (l - 1) + (\omega_{f}^{\text{an}} - \omega_{h_a}^{\text{an}}) \cdot (m - 1)$$

by Eqs. (21) and (26).

### 3.4 Computation of $\langle q_{\alpha\beta} \cdot q_{\beta\alpha} \rangle_{f \mathbb{Q}_p \mathbb{Q}_p} h_a$

Assume in this subsection $\alpha_0 = \beta_\gamma \cdot \alpha_\delta$, so that $H^0(\mathbb{Q}_p, V^-)$ is generated by the $p$-adic periods

$$q_{\alpha\beta} = \sqrt{m_p} \cdot q(f) \otimes \omega_{g_a} \otimes \eta_{h_a} \quad \text{and} \quad q_{\beta\alpha} = \sqrt{m_p} \cdot q(f) \otimes \eta_{g_a} \otimes \omega_{h_a}.$$  

For $\gamma \mult 

\delta = \alpha \beta, \beta \alpha$ and $\cdot = \emptyset, \ast, \ast, \ast, \ast, \ast, \ast$, define $V(f)_{\gamma \delta} = V(f) \otimes \mathbb{Q}_p V(g)_{\gamma} \otimes V(h)_{\delta}$. Then

$$H^0(\mathbb{Q}_p, V^-) = V(f)_{\alpha\beta} \oplus V(f)_{\beta\alpha},$$

$G_{\mathbb{Q}_p}$ acts on $V(f)^{+}_{\alpha\beta}$ and $V(f)^{+}_{\beta\alpha}$ via the $p$-adic cyclotomic character, and the local Tate pairing $\langle \cdot, \cdot \rangle_{\text{Tate}}$ introduced in Sect. 3.2.2 induces a perfect duality (denoted by the same
symbol) between $H^1(Q_p, V(f)_{\alpha^\beta})$ and $H^1(Q_p, V(f)_{\beta^\alpha})$. The argument of the proof of Eq. (25) shows that

$$j(q_{\beta^\alpha})^+ = \frac{m_p}{\deg(\varphi_\infty)} \cdot (q_A \hat{\otimes} 1) \otimes q_{\alpha^\beta}^*$$

in the direct summand $H^1(Q_p, V(f)_{\beta^\alpha}) = Q_p^* \hat{\otimes} V(f)_{\beta^\alpha}^+(-1)$ of $H^1(Q_p, V^+)$, where

$$q_{\alpha^\beta}^* = \frac{1}{\sqrt{m_p}} \cdot q(f)^* \otimes \eta_{g_A} \otimes \omega_{h_A} \text{ satisfies } \pi_{fgh}(-1)(q_{\alpha^\beta}^* \otimes q_{\alpha^\beta}) = 1.$$  \hfill (29)

Let $pr_{\alpha^\beta} : H^1(Q_p, V^-) \to H^1(Q_p, Q_p) \otimes Q_p, V(f)_{\alpha^\beta}$ denote the projection, and write

$$pr_{\alpha^\beta} \otimes \mathcal{J}/\mathcal{J}^2 \circ \beta_{f_{g_A}h_A}(q_{\alpha^\beta}) = \sum_u \gamma_u \otimes q_{\alpha^\beta} \cdot (u - u_o)$$

with $\gamma_u$ in $H^1(Q_p, Q_p) = \text{Hom}(Q_p^*, Q_p)$ for $u = k, l, m$, where (with the notations introduced in Sect. 3.2.1) $\beta_{f_{g_A}h_A}$ is a shorthand for

$$\beta_{R\Gamma_{cont}(Q_p, V^-)/R\Gamma_{cont}(Q_p, V^-)} : H^0(Q_p, V^-) \to H^1(Q_p, V^-) \otimes_L \mathcal{J}/\mathcal{J}^2,$$

and $u_o = 2$ if $u = k$ and $u_o = 1$ if $u = l, m$. Then (cf. Eq. (21))

$$\langle q_{\beta^\alpha}, j(q_{\beta^\alpha})^+ \rangle_{f_{g_A}h_A} \overset{\text{Lemma 3.4}}{=} \langle \beta_{f_{g_A}h_A}(q_{\alpha^\beta}), j(q_{\beta^\alpha})^+ \rangle_{\text{Tate}}$$

Eqs. (28) and (30) = $$\frac{m_p}{\deg(\varphi_\infty)} \cdot \sum_u (\gamma_u \otimes q_{\alpha^\beta}, (q_A \hat{\otimes} 1) \otimes q_{\alpha^\beta}^*)_{\text{Tate}} \cdot (u - u_o)$$

= $$\frac{m_p}{\deg(\varphi_\infty)} \cdot \sum_u \gamma_u(q_A) \cdot (u - u_o),$$

\hfill (31)

where the last equality follows from Eq. (29) and the analogue of Eq. (20) obtained by replacing $\alpha^\alpha$ and $\beta^\beta$ with $\beta^\alpha$ and $\alpha^\beta$ respectively. It then remains to compute $\gamma_u$ for $u$ equal to $k, l$ and $m$.

For $\xi = f \cdot g_A, h_A$, fix $\mathcal{O}_\xi$-bases $b_{\xi}^\pm$ of $V(\xi)^\pm$. After identifying $V(\xi)$ with $\mathcal{O}_\xi \oplus \mathcal{O}_\xi$ via the $\mathcal{O}_\xi$-basis ($b_{\xi}^+, b_{\xi}^-$), the action of $G_{Q_p}$ on $V(\xi)$ is given by (cf. Eq. (9))

$$\begin{pmatrix}
\chi_{\xi} \cdot \hat{a}_p(\xi)^{-1} \cdot \chi_{\text{cyc}}^{-1} & c_{\xi} \\
0 & \hat{a}_p(\xi)
\end{pmatrix} : G_{Q_p} \to GL_2(\mathcal{O}_\xi)$$

for a continuous map $c_{\xi} : G_{Q_p} \to \mathcal{O}_\xi$. Without loss of generality, assume that

$$q_{\alpha^\beta} = b_{\xi}^+ \hat{\otimes} b_{g_A}^- \hat{\otimes} b_{h_A}^+ \otimes 1$$

in $V^- = V(f)^- \hat{\otimes} L V(g_A) \hat{\otimes} L V(h_A) \otimes_{\mathcal{O}_{fgh}} \mathcal{O}_{fgh}$ maps to

$$q_{\alpha^\beta}^* \in V(f)^-_{\alpha^\beta} = V(f)^- \otimes Q_p, V(g^\alpha) \otimes_L V(h)^\beta$$

under $\rho_{\psi} : V^- \to V^-$. (Recall that $V(\xi) = V(\xi_{\alpha}) \otimes 1 L$ is the direct sum of the modules $V(\xi_{\alpha}) = V(\xi_{\alpha})^- \otimes 1 L$ and $V(\xi)^\beta = V(\xi^\beta) + \otimes 1 L$ for $\xi = g, h$, cf. Sect. 3.1.1.) Then

$$dq_{\alpha^\beta} = \Gamma \cdot q_{\alpha^\beta} + \Delta \cdot q_{\beta^\alpha},$$

\hfill (32)

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where \( q_{\beta \beta} = b_f \hat{\otimes} b_g^+ \hat{\otimes} b_h^+ \otimes 1 \), where

\[
\Gamma = \frac{\tilde{a}_p(f) \cdot \tilde{a}_p(g_a)}{\tilde{a}_p(h_a)} \cdot \chi_{h} \cdot \chi_{\text{cyc}}^{(m-k-l+2)/2} - 1
\]

and where

\[
\Delta = \tilde{a}_p(f) \cdot \tilde{a}_p(h_a)^{-1} \cdot \chi_{h} \cdot \chi_{\text{cyc}}^{(m-k-l+2)/2} \cdot c_{g_a}.
\]

The exceptional zero condition \( \alpha = \beta \cdot \alpha_h \) and the self duality condition \( \chi_{\beta} \cdot \chi_{h} = 1 \) imply that \( \Gamma \) takes values in \( \mathcal{S} \). Moreover, since the \( G_{Q_p} \)-module \( V(g) = V(g_a)^{+} \otimes L \) splits as the direct sum of \( V(g)^{+} = V(g_a)^{+} \otimes L \) and \( V(g)^{-} = V(g_a)^{-} \otimes L \), the map \( c_{g_a} \) takes values in \( (l-1) \cdot \mathcal{O}_g \), hence \( \Delta \) takes values in \( \mathcal{S} \). Because by construction \( q_{\beta \beta} \) maps to an element of \( V(f)^{-} \), under the specialisation map \( \rho_{w_0} : V^{-} \longrightarrow V^{-} \), Lemma 3.2 and Eqs. (30) and (32) yield the identities

\[
\gamma_u = -\frac{\partial}{\partial u} \Gamma(\cdot(w_p)),
\]

hence (as in the previous subsection) a direct computation gives

\[
\gamma_k = \frac{1}{2} \cdot \log f, \quad \gamma_l = \frac{1}{2} \cdot \log g_a \quad \text{and} \quad \gamma_m = -\frac{1}{2} \cdot \log h_a.
\]

(33)

Recalling that \( \log f(q_A) = 0 \) by [9, Theorem 3.18], Eq. (31) finally proves

\[
\frac{2}{m_p \cdot \text{ord}_p(q_A)} \cdot \langle q_{\alpha \beta}, q_{\beta \alpha} \rangle_{g_a h_a} = (\Lambda_{f}^{\text{an}} - \Lambda_{g_a}^{\text{an}}) \cdot (l - 1) - (\Lambda_{f}^{\text{an}} - \Lambda_{h_a}^{\text{an}}) \cdot (m - 1).
\]

(34)

3.5 Proof of equation (8)

Assume in this subsection that \( (A, \vartheta) \) is exceptional at \( p \), and fix a Selmer class \( x \in \text{Sel}(Q, V(f, g, h)) \). Let

\[
\tilde{x} = i_{\text{tur}}(x) \in \tilde{H}_f^1(Q, V(f, g, h))
\]

be the corresponding extended Selmer class (cf. Sect. 3.1.2). By construction \( \tilde{x}^+ \) belongs to the finite subspace of \( H^1(Q_p, V^+) \), and its image under the natural map \( i^+ : H^1_{\text{fin}}(Q_p, V^+) \longrightarrow H^1_{\text{fin}}(Q_p, V) \) equals the restriction of \( x \) at \( p \):

\[
\text{res}_p(x) = i^+(\tilde{x}^+).
\]

(35)

The Galois group \( G_{Q_p} \) acts on \( V(f)^+_2 \) via the \( p \)-adic cyclotomic character, hence

\[
H^1_{\text{fin}}(Q_p, V(f)^+_2) = Z_p^* \otimes Z_p \cdot V(f)^+_2(-1)
\]

by Kummer theory. If \( q^+_2 \) in \( V(f)^+_2 \) denotes (as in the previous subsections) the dual basis of \( q_2 \) in \( V(f)^-_2 \) under the pairing \( \pi_{fgh} \), and if one writes

\[
\text{pr}_2(\tilde{x}^+) = \tilde{x}^+ \otimes q^+_2 \in H^1_{\text{fin}}(Q_p, V(f)^+_2)
\]

for some \( \tilde{x}^+_2 \) in \( Z_p^* \otimes Z_p \cdot L \), then Eq. (35) yields the equality

\[
\log_2(\text{res}_p(x)) = \langle \log_2^+(\tilde{x}^+), q_2 \rangle_{fgh} = \langle \log_2(\tilde{x}^+) \otimes q^+_2, q_2 \rangle_{fgh} = \log_p(\tilde{x}^+_2),
\]

(36)
where \( \log_p^+: H^1_{\text{fin}}(Q_p, V^+) \simeq D_{\text{dR}}(V^+) \) is the Bloch–Kato logarithm and (with a slight abuse of notation) we denote again by \( \log_p^+ : \mathbb{Z}_p^* \otimes \mathbb{Z}_p \to L \to L \) the \( L \)-linear extension of the \( p \)-adic logarithm. In the previous equation we used the functoriality of the Bloch–Kato logarithm and the fact that (by construction) the linear form \( \{ \cdot, q_b \}_{fgh} \) on \( D_{\text{dR}}(V^+) \) factors through the projection onto \( D_{\text{dR}}(V(f)^+_\xi) = V(f)^+_\xi(-1) \).

Assume \( (\alpha_f = \alpha_g = \alpha_h) \) and \( q_b = q_{\beta} \cdot \beta \). According to Eqs. (21) and (36)

\[
2 \cdot \langle q_{\beta}, x \rangle_{fga h_a} = \log_{\alpha\alpha}(\text{res}_p(x)) \cdot (k - l - m),
\]

thus proving Eq. (8) in this case.

Assume \( q_b = q_{\alpha} \cdot \alpha \). Since (with the notations of Section 3.4) \( \Delta \) takes values in \( (l - 1) \cdot \mathcal{O}_{fg h} \), it follows from Lemma 3.2 and Eqs. (32) and (33) that

\[
2 \cdot \beta^-_{fg\cdot h_a}(q_{\alpha}) = \sum_{\xi} \epsilon_{\xi} \cdot \log_{\xi} \otimes q_{\beta} \cdot (u - u_\alpha) + \vartheta \cdot (l - 1) - (38)
\]

for some cohomology class \( \vartheta \) in \( H^1(Q_p, V(f)_{\beta\beta}) \), where \( \epsilon_{h_a} = -1 \) and \( \epsilon_{\xi} = +1 \) for \( \xi = f \cdot g_\alpha \). One has then

\[
\langle q_{\alpha}, x \rangle_{fg\cdot h_a}(k, 1, 1) \quad \text{Lemma}\, 3.4
\]

\[
= \frac{1}{2} \cdot \langle \log f \otimes q_{\alpha}, \hat{x}^+_{\alpha\alpha} \otimes q_{\beta} \rangle_{\text{Tate}}(k, 1, 1)
\]

\[
= \frac{1}{2} \cdot \log f(\hat{x}^+_{\alpha\alpha}) \cdot \pi_{fgh}(q_{\beta} \otimes q_{\alpha}) \cdot (k - 2) - \text{Equation}\, (38)
\]

\[
= \frac{1}{2} \cdot \log_{\alpha\alpha}(\text{res}_p(x)) \cdot (k - 2)
\]

\[
\text{Equation}\, (36)
\]

(39)

thus proving Eq. (8) when \( q_b = q_{\alpha} \). Switching the roles of the Hida families \( g_\alpha \) and \( h_\alpha \), this also proves Eq. (8) when \( q_b = q_{\alpha} \).

Assume finally \( q_b = q_{\alpha} \). With the notations of Sect. 3.4, let \( (b_\alpha^+, b_\alpha^-) \) be \( \mathcal{O}_{\xi} \)-bases of \( V(\xi) \) such that \( q_{\alpha} = b_\alpha^+ \otimes b_\alpha^- \otimes 1 \) is a lift of \( q_{\alpha} \) under the specialisation map \( \rho_{w_\alpha} : V^- \to V^- \). Since \( c_\xi \) takes values in \( (u - u_\alpha) \cdot \mathcal{O}_{\xi} \) for \( \xi = g_\alpha \cdot h_\alpha \), one has

\[
dq_{\alpha} = \left( x_{\alpha\alpha}(\xi - \xi) - 1 \right) \cdot q_{\alpha \alpha} \quad \text{mod} (l - 1, m - 1, \cdot C_{\text{cont}}^1(Q_p, V^-)),
\]

henceLemma 3.2 and a direct computation give

\[
2 \cdot \beta^-_{fg\cdot h_a}(q_{\alpha}) = \log f \otimes q_{\alpha} \cdot (k - 2) + \vartheta \cdot (l - 1) + \vartheta' \cdot (m - 1)
\]

(40)

for some local cohomology classes \( \vartheta \) and \( \vartheta' \) in \( H^1(Q_p, V^-) \). As in (39) one deduces Eq. (8) for \( q_b = q_{\alpha} \) from Lemma 3.4 and Eqs. (36) and (40).

### 4 Proof of theorem 2.1

Let \( \Pi_f, \Pi_g \) and \( \Pi_h \) be the improving planes in \( U_f \times U_g \times U_h \) defined respectively by the equations \( k = l + m, k = l - m + 2 \) and \( k = m - l + 2 \). For \( \xi = f, g, h \) define

\[
\mathcal{E}_\xi = 1 - \chi_\xi(p) \cdot \frac{a_p(\xi)}{a_p(\xi^*) \cdot a_p(\xi^*)}
\]
in $O_{fgh}$, where $\{\xi, \xi', \xi''\} = \{f, g_\alpha, h_\alpha\}$. Lemma 9.8 of [7] implies that

$$L_p^{ac}(A, \varrho)|_{\Pi_k} = E_k|_{\Pi_k} \cdot L_p^{ac}(A, \varrho)_L^*$$

(41)

for an improved $p$-adic $L$-function $L_p^{ac}(A, \varrho)_L^*$ in $O(\Pi_k)$. Indeed loc. cit. (together with its analogue obtained by switching the roles of $g$ and $h$) proves that the meromorphic function $L_p^{ac}(A, \varrho)_L^*$ on $\Pi_k$ defined by the previous equation is (bounded, hence) regular at $w_\alpha$. Shrinking the discs $U_k$ if necessary, we then conclude that the improved $p$-adic $L$-function $L_p^{ac}(A, \varrho)_L^*$ is analytic on $\Pi_k$, as claimed.

Assume first $\alpha_f = \alpha_g \cdot \alpha_h$, so that

$$2 \cdot L_f \pmod{\mathcal{I}^2} = L_f^{an} \cdot (k - 2) - L_{g_\alpha}^{an} \cdot (l - 1) - L_{h_\alpha}^{an} \cdot (m - 1).$$

(42)

According to Theorem A and Proposition 9.3 of [7], the partial derivative of $L_p^{ac}(A, \varrho)$ with respect to $k$ vanishes at $w_\alpha$, hence

$$2 \cdot L_p^{ac}(A, \varrho) \pmod{\mathcal{I}^2}$$

is equal to

$$(L_f^{an} - L_{g_\alpha}^{an}) \cdot (l - 1) + (L_f^{an} - L_{h_\alpha}^{an}) \cdot (m - 1) \cdot L_p^{ac}(A, \varrho)_f^*(w_\alpha)$$

by Eqs. (41) and (42). Moreover, with the notations introduced before the statement of Theorem 2.1, one has $L = \Pi_f \cap \Pi_g$ and $\mathcal{E}_f = \mathcal{E}_f|_L$, thus

$$L_p^{ac}(A, \varrho)_{\mathcal{E}_f}^*(w_\alpha) = E_g(w_\alpha) \cdot L_p^{ac}(A, \varrho)_{\mathcal{E}_f}^*(w_\alpha).$$

Noting that $E_g(w_\alpha) = 1 - \beta_\alpha$, the previous discussion and Eq. (27) conclude the proof of Theorem 2.1 when $\alpha_f = \alpha_g \cdot \alpha_h$.

Assume now $\alpha_f = \beta_\alpha \cdot \alpha_h$. In this case, for $\xi = g, h$, one has

$$2 \cdot \mathcal{E}_\xi \pmod{\mathcal{I}^2} = L^{an}_{\xi_\alpha} \cdot (u - 1) - L_f^{an} \cdot (k - 2) - L^{an}_{\xi_\alpha} \cdot (u' - 1),$$

(43)

where $\{(\xi_\alpha, u), (\xi_\alpha', u')\} = \{(g_\alpha, l), (h_\alpha, m)\}$, and

$$- L_p^{ac}(A, \varrho)^*(w_\alpha) = L_p^{ac}(A, \varrho)^*_g(w_\alpha) = E_f(w_\alpha) \cdot L_p^{ac}(A, \varrho)^*(w_\alpha).$$

(44)

The second equality in the previous equation follows as above from the definitions, according to which $L = \Pi_f \cap \Pi_g$ and $\mathcal{E}_g = \mathcal{E}_g|_L$. The first equality follows by noting that the restrictions of $E_g$ and $E_h$ to the line $\Pi_g \cap \Pi_h$ satisfy

$$E_g|_{\Pi_g \cap \Pi_h} = \frac{-\bar{\chi}_g(p) \cdot a_p(g_\alpha)}{-a_p(f) \cdot a_p(h_\alpha)}|_{\Pi_g \cap \Pi_h} \cdot E_h|_{\Pi_g \cap \Pi_h}$$

(as $a_p(f)|_{\Pi_g \cap \Pi_h} = \alpha_f = \alpha_f^{-1}$ and $\chi_g \cdot \chi_h = 1$ by Assumption 1.1.1) with

$$\frac{-\bar{\chi}_g(p) \cdot a_p(g_\alpha)}{-a_p(f) \cdot a_p(h_\alpha)}(w_\alpha) = -1.$$

(In other words $E_g|_{\Pi_g \cap \Pi_h}$ and $-E_h|_{\Pi_g \cap \Pi_h}$ have the same leading term at $w_\alpha$, which together with the equality $E_g \cdot L_p^{ac}(A, \varrho)^*_g|_{\Pi_g \cap \Pi_h} = E_h \cdot L_p^{ac}(A, \varrho)^*_h|_{\Pi_g \cap \Pi_h}$ implies the first identity in Eq. (44).) Write

$$2 \cdot L_p^{ac}(A, \varrho) \pmod{\mathcal{I}^2} = a \cdot (k - 2) + b \cdot (l - 1) + c \cdot (m - 1).$$
with $a$, $b$ and $c$ in $L$. Equations (41) and (43) with $\xi = g$ and Eq. (44) give
\[
a + b = \mathcal{E}_f(w_o) \cdot (\mathcal{L}^{an}_{g_a} - \mathcal{L}^{an}_f) \cdot \mathcal{L}^\alpha_p(w_o) \quad \text{and} \quad c - a = \mathcal{E}_f(w_o) \cdot (\mathcal{L}^{an}_f - \mathcal{L}^{an}_h) \cdot \mathcal{L}^\alpha_p(w_o),
\]
where $\mathcal{L}^\alpha_p$ is a shorthand for $\mathcal{L}^{\alpha \alpha}_p(A, \varrho)^*$. Similarly
\[
b - a = \mathcal{E}_f(w_o) \cdot (\mathcal{L}^{an}_{g_a} - \mathcal{L}^{an}_f) \cdot \mathcal{L}^\alpha_p(w_o) \quad \text{and} \quad a + c = \mathcal{E}_f(w_o) \cdot (\mathcal{L}^{an}_f - \mathcal{L}^{an}_h) \cdot \mathcal{L}^\alpha_p(w_o)
\]
by Eqs. (41) and (43) with $\xi = h$ and Eq. (44). As a consequence
\[
-2 \cdot \mathcal{L}^{\alpha \alpha}_p(A, \varrho) \pmod{\mathfrak{g}^2}
\]
equals
\[
\mathcal{E}_f(w_o) \cdot ((\mathcal{L}^{an}_f - \mathcal{L}^{an}_{g_a}) \cdot (l - 1) - (\mathcal{L}^{an}_{f} - \mathcal{L}^{an}_h) \cdot (m - 1)) \cdot \mathcal{L}^{\alpha \alpha}_p(A, \varrho)^*(w_o).
\]
Noting that $\mathcal{E}_f(w_o) = 1 - \frac{\beta h_o}{\alpha h_o}$ (when $\alpha_f = \beta_{g} \cdot \alpha_h$), the previous discussion and Eq. (34) prove Theorem 2.1 when $\alpha_f = \beta_{g} \cdot \alpha_h$.

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