The third boundary value problem for loaded differential Sobolev type equation and grid methods of their numerical implementation

M KH Beshtokov
Kabardino-Balkarian State University, Institute of physics and mathematics, Russia, Kabardino-Balkar Republic, Nalchik, 36000, 173 Chernyshevskogo St.

E-mail: beshtokov_murat@rambler.ru

Abstract. In this paper, we consider the third boundary value problem for loaded differential Sobolev type equation and grid methods of their numerical implementation. To solve the problem, a priori estimates in differential and difference settings are obtained. These a priori estimates imply uniqueness and stability of the solution with respect to the initial data and the right-hand side on the layer, as well as the convergence of the solution of each of the difference problem to the solution of the corresponding differential problem.

1. Introduction.
It is known that the solution of many practical problems arising while studying the liquid filtration process in fractured porous media [1], groundwater flow with a free surface in multi-media [2], moisture transfer [3], [4], the heat [5] and salts transfer [6] in porous media is all connected with the need to study boundary value problems for an equation of the third order

\[ u_t = (k(x,t)u_x)_x + (\eta(x,t)u_t)_x + r(x,t)u_x - q(x,t)u + f(x,t). \] (*)

Equation (*) is not a S.V.Kovalevskaya type equation and according to terminology [7] it is called a pseudo-parabolic equation. Equations of this type are also called Sobolev type equations or equations unsolved with respect to the derivative [8].

Interest in the loaded differential equations in partial derivatives is conditioned by the fact that linear loaded differential equations serve as mathematical models that describe the processes of long-term forecasting and regulation of groundwater and soil moisture levels, heat and mass transfer problems with a finite velocity, movement of a little compressible fluid, surrounded by a porous medium, the optimal management of agro-ecosystems.

2. Statement of the problem and a priori estimate in the differential form.
In the closed cylinder \( \mathcal{Q}_T = \{(x,t) : 0 \leq x \leq l, 0 \leq t \leq T\} \) we consider the third boundary value problem for a loaded equation of Sobolev type

\[ u_t = (k(x,t)u_x)_x + (\eta(x,t)u_t)_x + r(x,t)u_x(x_0,t) - q(x,t)u + f(x,t), \quad 0 < x < l, 0 < t \leq T, \] (1)

\[ \Pi(0,t) = \beta_1(t)u(0,t) - \mu_1(t), \quad 0 \leq t \leq T, \] (2)

\[ -\Pi(l,t) = \beta_2(t)u(l,t) - \mu_2(t), \quad 0 \leq t \leq T, \] (3)
where
\[ 0 < c_0 \leq \eta(x,t), k(x,t) \leq c_1, \quad |\eta, \eta_x, \eta_t, r, q, \beta_1, \beta_2| \leq c_2, \quad c_0, c_1, c_2 - \text{const} > 0, \]
\[ u(x,t) \in C^{4,3} (Q_T), \quad \eta(x,t) \in C^{3,3} (Q_T), \quad k(x,t) \in C^{3,2} (Q_T), \quad r, q, f \in C^{2,2} (Q_T), \]
\[ u_0(x) \in C^2 [0, l], \quad \Pi(x,t) = ku_x + (qu_x), \quad Q_T = \{(x,t) : 0 < x < l, 0 < t \leq T\}, \]
\[ \beta_1(t), \beta_2(t), \mu_1(t), \mu_2(t) \in C[0, T]. \]

Throughout the paper, we will use positive constants \( M_i, i = 1, 2, \ldots \), which depend only on the input data of problem (1) - (4).

Note that while constructing the difference scheme we require a higher smoothness of the solution and the coefficients of the equation to ensure the desired order of approximation of difference scheme [9].

**Theorem 1.** Let conditions (5), be satisfied, then for the solution of the differential problem (1) - (4) the following a priori estimate is valid
\[
\| u \|_0^2 + \| u_x \|_0^2 + \| u_{xx} \|_0^2 \leq M \left( \int_0^T \left( \| f \|_0^2 + \mu_1^2(t) + \mu_2^2(t) \right) dt + \| u_0(x) \|_0^2 + \| u'_0(x) \|_0^2 \right),
\]
wherein \( M \) - depends only on the input data of problem (1) - (4).

3. Stability and convergence of the difference scheme.

On the grid \( \bar{\omega}_{nt} \) to the differential problem (1) - (4) we assign a difference scheme with the order of approximation \( O(h^2 + \tau^2) \):
\[
y_{i,j} = \left( a y_{x,i}^{\sigma} \right)_{x} + \left( y_{x,j}^{\sigma} \right)_{x}, \quad r \left( y_{x,i}^{(\sigma)} \frac{x_{i+1} - x_i}{h} + y_{x,j}^{(\sigma)} \frac{x_{j+1} - x_j}{h} \right) - d_i y_i^{(\sigma)} + \varphi_i, \quad (x,t) \in \omega_{nt}, \quad (6)
\]
\[
a_i y_{x,0}^{x} + \left( y_{1,0}^{x} \right)_{x} + 0.5h q_0 \left( y_{x,i}^{(\sigma)} \frac{x_{i+1} - x_i}{h} + y_{x,i}^{(\sigma)} \frac{x_{i+1} - x_i}{h} \right) = \beta_i y_i^{(\sigma)} + \frac{h}{2} y_{i,0}^{x} - \mu_i, \quad t \in \omega_T, \quad (7)
\]
\[
- \left( a_N y_{x,N}^{x} + \left( y_{N,x,N}^{x} \right)_{x} \right) + 0.5h r_N \left( y_{x,i}^{(\sigma)} \frac{x_{i+1} - x_i}{h} + y_{x,i}^{(\sigma)} \frac{x_{i+1} - x_i}{h} \right) = \beta_N y_N^{(\sigma)} + \frac{h}{2} y_{i,N}^{x} - \mu_i, \quad t \in \omega_T, \quad (8)
\]
\[
y(x,0) = u_0(x), \quad x \in \bar{\omega}_h, \quad (9)
\]
where
\[
y_i^{(\sigma)} = \sigma y_i + (1 - \sigma) y, \quad a_i = k(x_i, t), \quad y_i = \eta(x_i, t), \quad d_i = q(x_i, t), \quad \varphi_i = f(x_i, t),
\]
\[
\tilde{\beta}_i^1 = \beta_i^1 + 0.5h d_i, \quad \tilde{\beta}_i^2 = \beta_i^2 + 0.5h d_N, \quad \tilde{\mu}_i^1 = \mu_i^1 + 0.5h \varphi_i, \quad \tilde{\mu}_i^2 = \mu_i^2 + 0.5h \varphi_N,
\]
\[
\tilde{y}_i = t_{i+0.5} - t_i + 0.5 \tau, \quad x_{i-0.5} = x_i - 0.5h, \quad h, \tau - \text{steps of the grid.}
\]
In order to get the a priori estimate we use the method of energy inequalities. Then problem (6) - (9) can be rewritten in another form, assuming \( \sigma = 0.5 \) and denoting \( Y = \tilde{y} + y \):
\[
y_i = 0.5 \tilde{y}_i + \tilde{y} + \Phi, \quad (10)
\]
\[
y(x,0) = u_0(x), \quad (11)
\]
\[ \tilde{A}Y = \begin{cases} \tilde{\lambda} h + n \left( \frac{y_{0} - y_{0}}{x_{0} - x_{0}} - \frac{y_{0} - y_{0}}{x_{0} - x_{0}} \right) - d_{i} Y, & \text{if } x = \omega_{h}, \\ a_{i} Y_{0} - \beta_{i} Y_{0} + 0.5h r_{i} \left( \frac{y_{0} - y_{0}}{x_{0} - x_{0}} + \frac{y_{0} - y_{0}}{x_{0} - x_{0}} \right), & \text{if } x = 0, \\ -a_{i} Y_{N} - \beta_{i} Y_{N} + 0.5h r_{i} \left( \frac{y_{0} - y_{0}}{x_{0} - x_{0}} + \frac{y_{0} - y_{0}}{x_{0} - x_{0}} \right), & \text{if } x = l. \end{cases} \]

\[ \Phi = \begin{cases} \varphi = \varphi_{0}, & x = \omega_{h}, \\ \varphi = \varphi_{0} + \frac{\varphi_{0}}{0.5h}, & x = 0, \\ \varphi = \varphi_{0} + \frac{\varphi_{0}}{0.5h}, & x = l. \end{cases} \]

We introduce the scalar product

\[ (u,v) = \sum_{i=1}^{N-1} u_{i} v_{i} h, \quad [u,v] = \sum_{i=1}^{N} u_{i} v_{i} h, \quad |u|_{0}^{2} = (u,u), \quad |u|_{0}^{2} = \sum_{i=1}^{N} u_{i}^{2} h = (u,u). \]

and the norm

\[ \|u\|_{0}^{2} = (u,v), \quad |u|_{0}^{2} = [u,v], \quad \|u\|_{0}^{2} = \sum_{i=1}^{N} u_{i}^{2} h = (u,u). \]

**Theorem 2.** Let conditions (5), be fulfilled, then there exist such \( \tau_{0}, \tau_{1}, \tau_{2}, \tau_{3}, \tau_{4} \), that if \( \tau \leq \min \{ \tau_{0}, \tau_{1}, \tau_{2}, \tau_{3}, \tau_{4} \} \), then with \( \sigma = 0.5 \) for the solution of the difference problem (10), (11) a priori estimate is valid.

\[ \|y^{0}_{i} + \|y^{0}_{i} + \|Y_{i} \|_{0} \|_{0} \leq \sum_{j=0}^{N} \left( \|\varphi\|_{0}^{2} + \mu_{1}^{2} + \mu_{2}^{2} \right) r + \|y^{0}_{i} \|_{0} + \|y^{0}_{i} \|_{0}^{2} + \|y^{0}_{i} \|_{0}^{2}, \]

where \( M \) - a positive constant, not depending on \( h \) and \( \tau \).

The resulting a priori estimate (12) implies the uniqueness and stability of the solution of the difference equation (6) - (9) with respect to the initial data and the right-hand side in the grid norm

\[ \|y^{0}_{i} \|_{0}^{2} + \|y^{0}_{i} \|_{0}^{2} + \|Y_{i} \|_{0} \|_{0} \] on a layer.

Let \( u(x,t) \) - be the solution of problem (1) - (4), \( y^{0}_{i} \) - be the solution of the difference problem (6) - (9), then we denote by \( z = y - u \) the error. Substituting \( y = z + u \) into (6) - (9) and assuming that \( u(x,t) \) is a given function, we obtain a problem for \( z \):

\[ z_{i,j} = \left( \frac{z_{i,j}^{(0)}}{x_{i,j}} + \frac{z_{i,j}^{(0)}}{x_{i,j}} \right) - d_{i} z_{i,j}^{(0)} + y_{i,j}, \]

where \( y_{i,j} \) is the right-hand side of the difference equation.
\[ a_i z(x,t) + \left( \frac{\partial}{\partial t} \right) z(x,t) + 0.5 h \left( \frac{z(x,t) - z(x,t+h)}{h} + \frac{\partial^2 z(x,t)}{\partial x^2} \right) + \beta_0 z(0,t) + \frac{h}{2} z_{t,0} - v_1, \quad t \in \omega_x, \quad \text{for } \sigma = 0.5, \]

\[ - \left( a_N z_N(x,t) + \left( \frac{\partial}{\partial t} \right) z_N(x,t) + 0.5 h \left( \frac{z_N(x,t) - z_N(x,t+h)}{h} + \frac{\partial^2 z_N(x,t)}{\partial x^2} \right) + \beta_N z_N(0,t) + \frac{h}{2} z_{t,N} - v_2, \quad t \in \omega_x, \quad \text{for } \sigma = 0.5, \]

where \( v_1 = O(h^2 + \tau^2) \), \( v_2 = O(h^2 + \tau^2) \) - are the approximation errors on the solution of the initial problem for every fixed \( t \) with \( \sigma = 0.5 \).

Applying a priori estimate (12) to the problem for the error, with \( \sigma = 0.5 \) we obtain the estimate

\[ \| z^{j+1}_i \|_0^2 + \| z^{j+1}_N \|_0^2 + \sum_{j=0}^1 \| Z_{x,i} \| \tau \leq M \sum_{j=0}^1 \left( |||\Psi||^2 + v^2_1 + v^2_2 \right) \tau, \]

where \( M \) - a positive constant, not depending on \( h \) and \( \tau \).

The a priori estimate (17) implies the convergence of the solution of the difference problem (6) - (9) to the solution of the differential problem (1) - (4) in the sense of the norm \( \| z^{j+1}_i \|_0^2 + \| z^{j+1}_N \|_0^2 + \sum_{j=0}^1 \| Z_{x,i} \| \tau, \) on each layer so that there exist such \( \tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \) that if \( \tau \leq \min \{ \tau_0, \tau_1, \tau_2, \tau_3, \tau_4 \} \), then with \( \sigma = 0.5 \) the following estimate is valid

\[ \| y^{j+1}_i - u^{j+1}_i \|_0^2 \leq M(h^2 + \tau^2). \]

In order to solve numerically the difference schemes obtained in the course of approximation of the loaded pseudo-parabolic third-order equations, it is convenient to use the parametric sweep method (see. [12, p.131]).

It is my pleasant duty to express my deep gratitude to my teacher Mohammed Habalovich Shhanukov-Lafishev.

4. The results of the numerical experiment

Consider the following example:

\[ u_t = (k(x,t)u_x)_x + \eta(x,t)u_x + r(x,t)u(x,t) - q(x,t)u + f(x,t), \quad 0 < x < l, 0 < t \leq T, \]

\[ \Pi(0,t) = \beta_0(t)u(0,t) - \mu_0(t), \quad 0 \leq t \leq T, \]

\[ \Pi(l,t) = \beta_2(t)u(l,t) - \mu_2(t), \quad 0 \leq t \leq T, \]

\[ u(x,0) = u_0(x), \quad 0 \leq x \leq l, \]

where \( l = 1, T = 1, \) \( k(x,t) = 1 + \cos(x+t), \) \( \eta(x,t) = 1 + \sin(x+t), \) \( r(x,t) = (0.5 - x)\cos(x+t), \) \( q(x,t) = e^x \sin(x+t), \) \( \beta_1 = \sin(t), \) \( \mu_1(t) = \sin(t)e^{-t} - e^t(1 + \cos(t)) - e^t(1 + \sin(t) + \cos(t)), \) \( u_0(x) = e^x, \) \( \beta_2 = e^t, \) \( \mu_2(t) = e^{-2t}(1 + \cos(t + l)) - e^{-t}(1 + \sin(t + l) + \cos(t + l)), \) \( x_0 = 0.325, \) \( f(x,t) = e^{x+t} - e^{-t}(2 + 3\cos(x+t) - \sin(x+t)) - (0.5 - x)e^{x+t}\cos(x+t) + e^{x+t}(\sin(x+t)). \)

The exact solution is \( u(x,t) = e^{-t}. \)

Table 1 compares the value of the numerical and exact solution of the problem for \( n = 10. \)

Table 2 shows that when \( h = \tau \) while the grid size is decreasing, then the maximum value of the error with \( \sigma = 0.5 \) is decreasing in accordance with the order of approximation of the difference scheme \( O(h^2 + \tau^2). \) The order of convergence is defined by the following formula:

The order of convergence = \( \log_{10} \frac{e^2}{e_1}. \)
Table 1. The difference between the numerical and exact solution with $t = 1, h = \tau = 0.1$

| $\chi_i$ | Numerical solution | Exact solution | Error solution |
|----------|-------------------|----------------|---------------|
| 0.0000   | 2.7170768         | 2.7182818      | 0.0012050     |
| 0.1000   | 3.0026299         | 3.0041660      | 0.0015361     |
| 0.2000   | 3.3182353         | 3.3201169      | 0.0018816     |
| 0.3000   | 3.6670599         | 3.6692967      | 0.0022368     |
| 0.4000   | 4.0526036         | 4.0552000      | 0.0025964     |
| 0.5000   | 4.4787343         | 4.4816891      | 0.0029548     |
| 0.6000   | 4.9497270         | 4.9530324      | 0.0033055     |
| 0.7000   | 5.4703066         | 5.4739474      | 0.0036408     |
| 0.8000   | 6.0456958         | 6.0496475      | 0.0039516     |
| 0.9000   | 6.6816678         | 6.6858944      | 0.0042266     |
| 1.0000   | 7.3846043         | 7.3890561      | 0.0044518     |

Table 2. The error alteration with decreasing grid size at $t=1$, where $h = \tau$

| $h$     | The maximum error | The order of convergence |
|---------|-------------------|-------------------------|
| 1/500   | 23.56             |                         |
| 1/1000  | 34.64             | 1.9755555               |
| 1/2000  |                   | 1.9830674               |

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