1. Introduction

A-twisted gauged linear sigma models are 2-dimensional topological field theories introduced by Witten [Wit93]. An A-twisted gauged linear sigma model is specified by a reductive algebraic group $G$ (or its compact real form) called the gauge group, an affine space $W$ with a linear action of $G \times \mathbb{G}_m$ called the matter, and an element $\xi$ of the dual $\mathfrak{g}^*$ of the center of the Lie algebra of $G$ called the Fayet–Iliopoulos parameter. The weights of the $\mathbb{G}_m$-actions are called $R$-charges. One can also introduce a superpotential in the theory, which is a $G$-invariant function on $W$ of $R$-charge 2. The correlation functions, which are quantities of primary interest, do not depend on the potential.

An A-twisted gauged linear sigma model with a suitable Fayet–Iliopoulos parameter is expected to be equivalent to the topological sigma model whose target is the classical vacuum subspace of the symplectic reduction $W/\!/\xi G$. This comes from a stronger expectation that the low-energy limit of a gauged linear sigma model should give the non-linear sigma model whose target is the symplectic reduction $W/\!/\xi G$. 
A prototypical example is the case $G = \mathbb{G}_m$ and $W = \mathbb{A}^6$, with the action

$$G \times \mathbb{G}_m \ni (\alpha, \beta) : (z_1, \ldots, z_5, P) \mapsto (\alpha z_1, \ldots, \alpha z_5, \alpha^{-5} \beta^2 P)$$

and a potential $Pf$, which is the product of the variable $P$ and a homogeneous polynomial $f$ in $z_1, \ldots, z_5$ of degree 5. The symplectic reduction $W/\xi G$ for the positive $\xi$ gives the total space of the bundle $\mathcal{O}_{\mathbb{P}^4}(-5)$. The R-charge of the $P$-field indicates that the target space should be considered not as a manifold but as a supermanifold, where the parity of the fiber is odd.

One candidate for a mathematical theory of A-twisted gauged linear sigma models is symplectic vortex invariants \cite{CGS00, CGMiRS02, MiR03, MiRT09, GW13, Zil14} and their generalizations incorporating potentials \cite{TX, FJR18}. Another candidate is quasimap theory, which is an intersection theory on moduli spaces of maps to the quotient stacks $[W/G]$. A review of the latter theory, with historical remarks and extensive references, can be found in \cite{CFK}. These two approaches should be related by Hitchin–Kobayashi correspondence for vortices \cite{Bra91, MiR00, VW16}.

When the gauge group is abelian, quasimap theory as a mathematical theory of A-twisted gauged linear sigma models goes back to \cite{MP95}. The relation with the Yukawa coupling of the mirror is formulated as toric residue mirror conjecture in \cite{BM02, BM03} and proved in \cite{SV04, Bor05, Kar05, SV06}.

Quasimap theory in the special case of projective hypersurfaces is also studied in the insightful paper \cite{Giv95a}, where a heuristic relation with semi-infinite homologies of loop spaces is discussed. This eventually leads to Givental’s proof of classical mirror symmetry \cite{CdLOGP91} for the quintic 3-fold. This has been extended to toric complete intersections in \cite{Giv98}.

The correlation functions of A-twisted gauged linear sigma models in the cases when gauge groups are not necessarily abelian are computed in \cite{BZ15, CCP15} using supersymmetric localization of path integrals. The result is given in terms of Jeffrey–Kirwan residues, and reproduces the results of \cite{MP95} in abelian cases.

The aim of this paper is twofold. One is to give an expository account of quasimap theory and its relation to other subjects such as instantons and integrable systems. The other is to formulate Conjecture \ref{conj:main} which states that the correlation function defined in \eqref{eq:correlation} in terms of residues coincides with the generating function of quasimap invariants defined in \eqref{eq:quasimap}, and prove it for Grassmannians in Section \ref{sec:grassmannians}. This can be considered as a generalization of toric residue mirror symmetry to Grassmannians. We also show in Section \ref{sec:toric} that a slightly weakened version of toric residue mirror conjecture follows from Givental’s mirror theorem. Nothing else in this paper is new.

This paper is organized as follows: In Section \ref{sec:background}, we recall the description of correlation functions of A-twisted gauged linear sigma models given in \cite{BZ15, CCP15}. In Section \ref{sec:quasimaps}, we recall the definition of the quasimap spaces $Q(\mathbb{P}^{n-1}; d)$. They are compactifications of the spaces of holomorphic maps of degrees $d$ from $\mathbb{P}^1$ to $\mathbb{P}^{n-1}$, and play an essential role in Givental’s homological geometry \cite{Giv95a, Giv95b}. In Section \ref{sec:toric}, we recall toric residue mirror symmetry for Calabi–Yau complete intersections in projective spaces. In Section \ref{sec:cone}, we discuss quasimap invariants of concave bundles. In Section \ref{sec:grassmannians}, we recall classical mirror symmetry for toric hypersurfaces proved in \cite{Giv98}. The exposition in Section \ref{sec:grassmannians} follows \cite{Iri11} closely. In Section \ref{sec:quasimap}, we briefly recall the definition of quasimap spaces for toric varieties due to \cite{MP95}. In Section \ref{sec:toric} we show that a slightly weakened version of toric residue mirror conjecture for CY hypersurfaces follows from classical mirror symmetry. In Section \ref{sec:approximation} we recall a theorem of Martin which relates integration on a symplectic quotient by a compact Lie group to that on the quotient by a maximal torus. In Section \ref{sec:proofs} we
we recall the definition of quasimap spaces to GIT quotients, which are called *quasimap graph spaces* in [CFK14]. The quasimap spaces come with the universal $G$-bundle and the canonical virtual fundamental classes, which allow us to define numerical invariants. We formulate Conjecture 10.10 which states that correlation functions of A-twisted gauged linear sigma models given in (2.2.4) are generating functions of quasimap invariants. There is a natural $\mathbb{G}_m$-action on the quasimap graph space coming from the $\mathbb{G}_m$-action on the domain curve. There is a distinguished connected component of the fixed locus of this action, which is used to define the *I*-function. In Section 11 the quasimap spaces and the $I$-functions for Grassmannians are recalled from [BCFK05]. In Section 12 we prove Conjecture 10.10 for Grassmannians. For this purpose, we introduce *abelianized quasimap spaces* for Grassmannians, which allows us to relate quasimap invariants for Grassmannians with correlations function in (2.2.4). In Section 13, we discuss the relation between gauged linear sigma models and Bethe ansatz following [NS09]. In Sections 14, 15, and 16, we recall the relations of quasimaps with instantons, monopoles, and vortices respectively.

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### 2. Correlation functions of A-twisted gauged linear sigma models

#### 2.1. Let $G$ be a reductive algebraic group of rank $r$, and $W$ be a representation of $G \times \mathbb{G}_m$. The center of $G$ and its Lie algebra will be denoted by $Z(G)$ and $\frak{z}$. Fix a maximal torus $T$ of $G$, and let $\frak{t}$ be its Lie algebra. The set of roots, its subset of positive roots, and the Weyl group will be denoted by $\Delta$, $\Delta^+$, and $W := N(T)/T$. Let $W = \bigoplus_{i=1}^N W_i$ be the weight space decomposition of $W$ with respect to the action of $T \times \mathbb{G}_m$. The weight of $W_i$ will be denoted by $(\rho_i, r_i) \in \frak{t}^\vee \oplus \mathbb{Z}$, and $r_i$ will be called the *R-charge*. If $W$ admits an action of another torus $H$ commuting with the action of $G \times \mathbb{G}_m$, then one can introduce the *twisted mass* $\lambda \in \frak{h}$ in the theory, which corresponds to the equivariant parameter for the $H$-action. The $T \times \mathbb{G}_m \times H$-weight of $W_i$ will be denoted by $(\rho_i, r_i, \nu_i) \in \frak{t}^\vee \oplus \mathbb{Z} \oplus \frak{h}^\vee$. We also introduce the *complexified Fayet–Illiopoulos parameter* $t' \in \frak{z}^\vee \otimes \mathbb{C}$, which corresponds to the complexified Kähler form of the symplectic quotient. Here, we save the unprimed symbol $t$ for the indeterminate in the generating function of quasimap invariants (see 3.4.1), (3.11.1), (4.2.4) and (12.7.4).

For $d \in \frak{t}$ and $t' \in \frak{z}^\vee$, the composition of the surjection $\frak{t}^\vee \rightarrow \frak{z}^\vee$ dual to the inclusion $\frak{z} \hookrightarrow \frak{t}$ and the evaluation $\frak{t}^\vee \times t \rightarrow \mathbb{C}$ will be denoted by $t' \cdot d$ or $t'(d)$.

#### 2.2. For $d \in \frak{t}$ and $x \in \frak{t}$, let

\begin{equation}
Z_d(x) := Z_d^{\text{vec}}(x)Z_d^{\text{mat}}(x)
\end{equation}

be the product of

\begin{equation}
Z_d^{\text{vec}}(x) := \prod_{\alpha \in \Delta_+} (-1)^{\alpha(d)+1} \alpha^2(x)
\end{equation}

3
\[ Z_d^{\text{mat}}(x) := \prod_{i=1}^{N} \left( \rho_i(x) + \nu_i(\lambda) \right)^{r_i - \rho_i(d) - 1}. \]

Here the superscripts ‘vec’ and ‘mat’ stands for the vector multiplet and the matter chiral multiplet respectively. According to \cite{BZ15, CCP15}, the correlation function of a \( \mathcal{W} \)-invariant polynomial \( P(x) \in \mathbb{C}[t]^{\mathcal{W}} \) on a 2-sphere is given, up to sign introduced by hand, by

\[ \langle P(x) \rangle_{\text{GLSM}} = \prod_{d \in \mathbb{P}^+} e^{t^d} \left( \frac{\mathcal{N}}{|\mathcal{W}|} \mathcal{K}_d(Z_d(x)P(x)) \right). \]

Here \( \mathbb{P}^+ \) is the coweight lattice of \( G \) and \( \mathcal{K}_d \) is the Jeffrey–Kirwan residue defined in \cite[Section 2]{SV04} (cf. also \cite{BV99}). The cone \( \mathfrak{c} \subset \mathfrak{z}^\vee \) is the ample cone of the GIT quotient determined by the Fayet–Iliopoulos parameter \( \eta \).

2.3. One can introduce a variable \( z \) associated with the background value of an auxiliary gauge field in the gravity multiplet. This corresponds to the equivariant parameter for the \( \mathbb{G}_m \)-action on the domain curve. This turns (2.2.1) into the product

\[ Z_d(x; z) := Z_d^{\text{vec}}(x; z)Z_d^{\text{mat}}(x; z) \]

of

\[ Z_d^{\text{vec}}(x; z) := \prod_{\alpha \in \Delta^+} (1 - \alpha(x + zd)) \]

and

\[ Z_d^{\text{mat}}(x; z) := \prod_{i=1}^{N} \prod_{l = -\infty}^{l - \rho_i(d) - r_i} \left( \rho_i(x) + \nu_i(\lambda) - (l + \frac{r_i}{2}) z \right), \]

and the correlation function of \( P(x) \in \mathbb{C}[t]^{\mathcal{W}} \) is given by

\[ \langle P(x) \rangle_{\text{GLSM}}^{H \times \mathbb{G}_m} = \prod_{d \in \mathbb{P}^+} e^{t^d} \mathcal{K}_d(Z_d(x; z)P(x)). \]

2.4. Another quantity of interest is the effective twisted superpotential on the Coulomb branch, or the effective potential for short. It is defined as the sum

\[ W_{\text{eff}}(x; t') := W_{\text{FI}}(x; t') + W_{\text{vec}}(x) + W_{\text{mat}}(x) \]

of the Fayet-Iliopoulos term

\[ W_{\text{FI}}(x; t') := t' \cdot x, \]

the vector multiplet term

\[ W_{\text{vec}}(x) := -\pi \sqrt{-1} \sum_{\alpha \in \Delta^+} \alpha(x), \]

and the matter term

\[ W_{\text{mat}}(x) := -\sum_{i=1}^{N} (\rho_i(x) + \nu_i(\lambda)) (\log(\rho_i(x) + \nu_i(\lambda)) - 1). \]
3. Quasimap spaces for projective spaces

3.1. A holomorphic map \( u: \mathbb{P}^1 \to \mathbb{P}^{n-1} \) of degree \( d \) is given by a collection \( (u_i(z_1, z_2))_{i=1}^n \) of \( n \) homogeneous polynomials of degree \( d \) satisfying the following condition:

\[ (3.1.1) \text{ There exists no } (z_1, z_2) \in \mathbb{A}^2 \setminus \{0\} \text{ such that } u(z_1, z_2) = 0 \in \mathbb{A}^n. \]

Two collections \( (u_i(z_1, z_2))_{i=1}^n \) and \( (u'_i(z_1, z_2))_{i=1}^n \) define the same map if and only if there exists \( \alpha \in \mathbb{G}_m \) such that \( u_i(z_1, z_2) = \alpha u'_i(z_1, z_2) \) for all \( i \in \{1, \ldots, n\} \). It follows that the space

\[ (3.1.2) \mathcal{M}(\mathbb{P}^{n-1}; d) := \{ u: \mathbb{P}^1 \to \mathbb{P}^{n-1} \mid \deg u = d \} \]

of holomorphic maps of degree \( d \) from \( \mathbb{P}^1 \) to \( \mathbb{P}^{n-1} \) can be compactified to the projective space of dimension \( n(d+1) - 1 \), whose homogeneous coordinate is given by the coefficients \( (a_{ij})_{i,j} \) of the collection \( (u_i(z_1, z_2))_{i=1}^n \) of homogeneous polynomials of degree \( d \);

\[ (3.1.3) u_i(z_1, z_2) = \sum_{j=0}^d a_{ij} z_1^i z_2^j, \quad i = 1, \ldots, n. \]

This compactification is called the quasimap space and denoted by \( \mathbf{Q}(\mathbb{P}^{n-1}; d) \). An element of the quasimap space is called a quasimap.

3.2. A point \( [z_1: z_2] \in \mathbb{P}^1 \) is a base point (or singularity) of a quasimap \( u \) if \( u(z_1, z_2) = 0 \).

A quasimap is a genuine map outside of the base locus. If the degree of the base locus is \( d' \), then a quasimap can be considered as a genuine map of degree \( d - d' \). However, it is more convenient to think of a quasimap as a morphism to the quotient stack \( \mathbb{A}^n / \mathbb{G}_m \). By definition, a morphism from \( \mathbb{P}^1 \) to \( \mathbb{A}^n / \mathbb{G}_m \) is a principal \( \mathbb{G}_m \)-bundle \( P \) over \( \mathbb{P}^1 \) and a \( \mathbb{G}_m \)-equivariant morphism \( \tilde{u}: P \to \mathbb{A}^n \). It is a quasimap if the generic point of \( P \) is mapped to the semi-stable locus \( \mathbb{A}^n \setminus \{0\} \).

3.3. Let \( x \in H^2(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{Z}) \) be the ample generator of the cohomology ring of \( \mathbf{Q}(\mathbb{P}^{n-1}; d) \cong \mathbb{P}^{n(d+1) - 1} \), so that

\[ (3.3.1) H^*(\mathbf{Q}(\mathbb{P}^{n-1}; d); \mathbb{Z}) \cong \mathbb{Z}[x] / (x^{n(d+1)}) \]

Given a polynomial \( P(x) \in \mathbb{C}[x] \), we set

\[ (3.3.2) \langle P(x) \rangle_{\mathbb{P}^{n-1}} := \sum_{d=0}^{\infty} q^d \langle P(x) \rangle_{\mathbb{P}^{n-1}, d} \in \mathbb{C}[q], \]

where

\[ (3.3.3) \langle P(x) \rangle_{\mathbb{P}^{n-1}, d} := \int_{\mathbf{Q}(\mathbb{P}^{n-1}, d)} P(x) \]

is the integration over the quasimap space. It follows from

\[ (3.3.4) \langle x^k \rangle_{\mathbb{P}^{n-1}, d} = \begin{cases} 1 & k = n(d+1) - 1, \\ 0 & \text{otherwise} \end{cases} \]

that

\[ (3.3.5) \langle x^k \rangle_{\mathbb{P}^{n-1}} = \begin{cases} q^d & k = n(d+1) - 1 \text{ for some } d \in \mathbb{Z}^\geq 0, \\ 0 & \text{otherwise}. \end{cases} \]
3.4. If we set $G := \mathbb{G}_m$ and $W := \mathbb{C}^n$ with the action $G \times \mathbb{G}_m \ni (\alpha, \beta) : (w_1, \ldots, w_n) \mapsto (\alpha w_1, \ldots, \alpha w_n)$, then we have $Z^d_3(x) = 1$ and $Z^{\text{mat}}_d(x) = (x^{-d-1})^n$, so that (2.2.4) gives the same result as (3.3.5) under the identification

$$q = e^\nu.$$

3.5. The small quantum cohomology of $\mathbb{P}^{n-1}$ is the free $\mathbb{C}[q]$-module

$$(3.5.1) \quad \text{QH}(\mathbb{P}^{n-1}) := H^*(\mathbb{P}^{n-1}; \mathbb{C}[q])$$

equipped with multiplication given by

$$x^i \circ x^j := \sum_{k=0}^n \sum_{d=0}^\infty q^d \langle I_{0,3,d} \rangle (x^i, x^j, x^k) x^{n-k-1}. $$

Here

$$\langle I_{0,3,d} \rangle (a, b, c) := \int_{\overline{\mathcal{M}}_{g,3}(\mathbb{P}^{n-1};d)} \text{ev}_1^* a \cup \text{ev}_2^* b \cup \text{ev}_3^* c$$

is the 3-point Gromov-Witten invariant. It is an associative commutative deformation of the classical cohomology ring;

$$(3.5.4) \quad \text{QH}(\mathbb{P}^{n-1}) / (q) \cong H^*(\mathbb{P}^{n-1}; \mathbb{C}).$$

Since the virtual dimension of the moduli space of stable maps is given by

$$(3.5.5) \quad \text{virt.dim}\overline{\mathcal{M}}_{g,k}(X; d) = (1 - g)(\dim X - 3) + \langle c_1(X), d \rangle + k$$

in general, one has

$$(3.5.6) \quad \text{virt.dim}\overline{\mathcal{M}}_{0,3}(\mathbb{P}^{n-1}; d) = nd + n - 1.$$

The 3-point Gromov-Witten invariant in (3.5.2) is non-zero only if

$$(3.5.7) \quad \text{virt.dim}\overline{\mathcal{M}}_{0,3}(\mathbb{P}^{n-1}; d) = i + j + k.$$

Since $0 \leq i, j, k \leq n - 1$, one has (3.5.7) only if $d = 0$, $i + j + k = n - 1$ or $d = 1$, $i + j + k = 2n - 1$. This shows that $x^i \circ x^j = x^{i+j}$ for $i + j \leq n - 1$. Since there is a unique line passing through two points on $\mathbb{P}^{n-1}$ in general position, and this line intersects a hyperplane at one point, one has $x \circ x^{n-1} = q$. Hence the ring structure of the quantum cohomology of $\mathbb{P}^{n-1}$ is given by

$$(3.5.8) \quad \text{QH}(\mathbb{P}^{n-1}) \cong (\mathbb{C}[q])[x]/(x^n - q).$$

We write the ring homomorphism $\mathbb{C}[x] \to \text{QH}(\mathbb{P}^{n-1})$ sending $x$ to $x$ as $P(x) \mapsto \hat{P}(x)$.

**Theorem 3.6.** For any $P(x) \in \mathbb{C}[x]$, one has

$$\langle P(x) \rangle_{\mathbb{P}^{n-1}} = \int_{\mathbb{P}^{n-1}} \hat{P}(x).$$

**Proof.** Since both sides of (3.6.1) are linear in $P(x) \in \mathbb{C}[x]$, it suffices to show

$$\langle x^k \rangle_{\mathbb{P}^{n-1}} = \int_{\mathbb{P}^{n-1}} x^{\hat{k}}$$

for any $k \in \mathbb{N}$, which is obvious from (3.3.5) and (3.5.8). \qed

Theorem 3.6 is equivalent to the Vafa-Intriligator formula [Vaf91, Int91].
Corollary 3.7 (Vafa-Intriligator formula for projective spaces). For any $P(x) \in \mathbb{C}[x]$, one has

\[(3.7.1) \quad \int_{\mathbb{P}^{n-1}} P(x) = \frac{1}{n} \sum_{\lambda = q} \frac{P(\lambda)}{\lambda^{n-1}}, \]

where the sum is over $\lambda \in \mathbb{C}[q^{1/n}]$ satisfying $\lambda^n = q$.

**Proof.** Since the integration over the projective space can be written by residue as

\[(3.7.2) \quad \int_{\mathbb{P}^{n-1}} x^k = \delta_{r-1,k} = \text{Res} \frac{x^k \mathrm{d}x}{x^r}, \]

one has

\[(3.7.3) \quad \int_{\mathbb{P}^{n-1}} \hat{P}(x) = \langle P(x) \rangle_{\mathbb{P}^{n-1}} \]

\[(3.7.4) \quad \sum_{d=0}^{\infty} q^d \int Q(\mathbb{P}^{n-1},d) P(x) \]

\[(3.7.5) \quad = \sum_{d=0}^{\infty} q^d \text{Res} \frac{P(x) \mathrm{d}x}{x^{n(d+1)}} \]

\[(3.7.6) \quad = \text{Res} \frac{x^{-n}P(x)}{1-qx^{-n}} \]

\[(3.7.7) \quad = \text{Res} \frac{P(x)}{x^n - q} \]

\[(3.7.8) \quad = \frac{1}{n} \sum_{\lambda = q} \frac{P(\lambda)}{\lambda^{n-1}}, \]

and (3.7.1) is proved. \(\square\)

3.8. The projective space $\mathbb{P}^{n-1}$ has a natural action of $\text{GL}_n$, which restricts to the action of the diagonal maximal torus $H$. The *equivariant cohomology* is defined as the ordinary cohomology $H^*_H(\mathbb{P}^{n-1}) := H^*(\mathbb{P}^{n-1})$ of the Borel construction $\mathbb{P}^{n-1} := \mathbb{P}^{n-1} \times_H E\mathbb{H}$, where $E\mathbb{H}$ is the product of $n$ copies of the total space of the tautological bundle $\mathcal{O}_{\mathbb{P}^\infty}(-1)$ over $B\mathbb{G}_m = \mathbb{P}^\infty$. It follows that $\mathbb{P}^{n-1}_H$ is the projectivization $\mathbb{P}(E)$ of the vector bundle $E := \bigoplus_{i=1}^n \pi_i^* \mathcal{O}_{\mathbb{P}^\infty}(-1)$ of rank $n$ over $\mathbb{P}^\infty$. A standard result on the cohomology of a projective bundle (see e.g. [GH78 page 606]) shows that $H^*(\mathbb{P}^{n-1}_H)$ is generated over $H^*_H(\text{pt}) = H^*((\mathbb{P}^\infty))^n \cong \mathbb{C}[\lambda_1, \ldots, \lambda_n]$ by $x := -c_1(\mathcal{O}_E(-1))$ with one relation

\[(3.8.1) \quad (x \lambda_i)^n - c_1(E)(x \lambda_i)^{n-1} + c_2(E)(x \lambda_i)^{n-2} + \cdots + (-1)^n c_n(E) = 0. \]

Since $c_i(E) = (-1)^i \sigma_i(\lambda_1, \ldots, \lambda_n)$, one obtains

\[(3.8.2) \quad H^*_H(\mathbb{P}^{n-1}) \cong \mathbb{C}[x, \lambda_1, \ldots, \lambda_n] / \prod_{i=1}^n (x - \lambda_i). \]

The $H$-fixed locus $(\mathbb{P}^{n-1})^H$ consists of $n$ points $\{p_i\}_{i=1}^n$, where $p_i$ is the point $[z_1 : \cdots : z_n] \in \mathbb{P}^{n-1}$ with $z_i = 1$ and $z_j = 0$ for $i \neq j$. Since the tautological bundle $\mathcal{O}_{\mathbb{P}(E)}(-1)$ restricts to $\pi_i^* \mathcal{O}_{\mathbb{P}^\infty}(-1)$ on $(p_i)_T = (\mathbb{P}^\infty)^n$, one has

\[(3.8.3) \quad \iota^*_i x = \lambda_i. \]
The push-forward

\[(3.8.4) \int_{\mathbb{P}^{n-1}}^H H^*_H(\mathbb{P}^{n-1}) \to H^*_H(\text{pt}) \cong \mathbb{C}[\lambda_1, \ldots, \lambda_n]\]

along the natural map \((\mathbb{P}^{n-1})_H \to (\text{pt})_H \cong BH\) is called the equivariant integration. The localization theorem \([\text{AB84}]\) shows

\[(3.8.5) \int_{\mathbb{P}^{n-1}}^H P(x) = \sum_{i=1}^n \frac{\iota_i^* P(x)}{\text{Eul}^H(Nx_{/\mathbb{P}^{n-1}})} = \sum_{i=1}^n \frac{P(\lambda_i)}{\prod_{j \neq i} (\lambda_i - \lambda_j)} = \text{Res} \frac{P(x)dx}{\prod_{i=1}^n (x - \lambda_i)}\]

for any \(P(x) \in H^*_H(\mathbb{P}^{n-1})\).

3.9. The quasimap space \(Q(\mathbb{P}^{n-1}; d)\) has a natural action of \(H \times \mathbb{G}_m\) given by

\[(3.9.1) H \times \mathbb{G}_m \ni (\alpha, \ldots, \alpha_n, \beta) : (u_1(z_1, z_2))_{i=1}^n \mapsto (\alpha_i u_i(z_1, \beta z_2))_{i=1}^n.\]

The equivariant cohomology of \(Q(\mathbb{P}^{n-1}; d)\) with respect to this torus action is given by

\[(3.9.2) H^*_{H \times \mathbb{G}_m}(Q(\mathbb{P}^{n-1}; d); \mathbb{C}) \cong \mathbb{C}[x, \lambda_1, \ldots, \lambda_n, z]/\left(\prod_{i=1, j=0}^n (x - \lambda_i - jz)\right).\]

The \(H \times \mathbb{G}_m\)-equivariant integration

\[(3.9.3) \langle \cdot \rangle_{\mathbb{P}^{n-1}, d}^{H \times \mathbb{G}_m} : H^*_{H \times \mathbb{G}_m}(Q(\mathbb{P}^{n-1}; d); \mathbb{C}) \to H^*(B(H \times \mathbb{G}_m); \mathbb{C})\]

is given by

\[(3.9.4) \langle P(x) \rangle_{\mathbb{P}^{n-1}, d}^{H \times \mathbb{G}_m} = \text{Res} \frac{P(x)dx}{\prod_{i=1}^n \prod_{j=0}^d (x - \lambda_i - jz)} = \sum_{i=1}^n \sum_{j=0}^d \frac{P(\lambda_i + jz)}{\prod_{(k,l) \neq (i,j)} ((\lambda_i + jz) - (\lambda_k + l z))}.\]

The \(H \times \mathbb{G}_m\)-equivariant correlator is given by

\[(3.9.6) \langle P(x) \rangle_{\mathbb{P}^{n-1}}^{H \times \mathbb{G}_m} := \sum_{d=0}^\infty q^d \langle P(x) \rangle_{\mathbb{P}^{n-1}, d}^{H \times \mathbb{G}_m}.\]

The \(H\)-equivariant correlator \(\langle P(x) \rangle_{\mathbb{P}^{n-1}}^H\) and the \(\mathbb{G}_m\)-equivariant correlator \(\langle P(x) \rangle_{\mathbb{P}^{n-1}}^{\mathbb{G}_m}\) are obtained by setting \(z = 0\) and \(\lambda = (\lambda_1, \ldots, \lambda_n) = 0\) respectively.

3.10. The fixed point of the \(\mathbb{G}_m\)-action on \(Q(\mathbb{P}^{n-1}; d)\) is the disjoint union

\[(3.10.1) Q(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m} = \coprod_{i=0}^d Q(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m}_i\]

of \(d + 1\) connected components

\[(3.10.2) Q(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m}_i := \{ [a_1 z_1^{d-i}, \ldots, a_n z_1^{d-i}] \in Q(\mathbb{P}^{n-1}; d) \mid [a_1, \ldots, a_n] \in \mathbb{P}^{n-1} \}.\]
Each of these connected components is isomorphic to $\mathbb{P}^{n-1}$, and the base locus is $i0 + (d - i)\infty$. The connected component $Q(\mathbb{P}^{n-1}; d)_{Q}^{m}$ will be denoted by $Q_n(\mathbb{P}^{n-1}; d)_{Q}$. There is a natural map $ev: Q_n(\mathbb{P}^{n-1}; d) \to \mathbb{P}^{n-1}$ called the evaluation map, and one has

$$Q(\mathbb{P}^{n-1}; d)^{Q_m} \cong \prod_{d_1 + d_2 = d} Q_n(\mathbb{P}^{n-1}, d_1) \times_{\mathbb{P}^{n-1}} Q_n(\mathbb{P}^{n-1}, d_2). \tag{3.10.3}$$

The normal bundle of $Q_n(\mathbb{P}^{n-1}; d)$ in $Q(\mathbb{P}^{n-1}; d)$ is given by $\mathcal{O}_{\mathbb{P}^{n-1}}(1)^{\oplus nd}$, whose equivariant Euler class is given by

$$\text{Eul}^{H \times \mathbb{C}_m} \left( N_{Q_n(\mathbb{P}^{n-1}; d)/Q(\mathbb{P}^{n-1}; d)} \right) = \prod_{i=1}^{d} \prod_{l=1}^{m} (x - \lambda_i + lz). \tag{3.10.4}$$

The equivariant $I$-function is defined by

$$I^H\mathbb{P}^{n-1}(t; z) := e^{tx/z} \sum_{d=0}^{\infty} e^{dt} I^H_d \tag{3.10.5}$$

where

$$I^H_d(z) := \text{ev}_* \left( \frac{1}{\text{Eul}^{H \times \mathbb{C}_m} \left( N_{Q_n(\mathbb{P}^{n-1}; d)/Q(\mathbb{P}^{n-1}; d)} \right)} \right) \tag{3.10.6}$$

$$= \frac{1}{\prod_{i=1}^{n} \prod_{l=1}^{m} (x - \lambda_i + lz)}. \tag{3.10.7}$$

The non-equivariant $I$-function is defined similarly, and given by setting $\lambda = 0$ in (3.10.5);

$$I_{\mathbb{P}^{n-1}}(t; z) := e^{tx/z} \sum_{d=0}^{\infty} \prod_{l=1}^{m} (x + lz)^n. \tag{3.10.8}$$

3.11. Let $(\mathbb{C}[e^t])[t]$ be the polynomial ring in $t$ with the ring $\mathbb{C}[e^t]$ of formal power series in $e^t$ as a coefficient. The equivariant $I$-function in (3.10.5) is an element of $H^*_H(\mathbb{P}^{n-1}; \mathbb{C}) \otimes_{\mathbb{C}} (\mathbb{C}[e^t])[t]$, and the variable $t$ is related to the variable $q$ appearing in the correlator by

$$q = e^t. \tag{3.11.1}$$

The equivariant $I$-function can also be considered as a $H^*_H(\mathbb{P}^{n-1}; \mathbb{C})$-valued analytic function, which is multi-valued as a function of $q$ and single-valued as a function of $t = \log q$.

3.12. There is a $\mathbb{G}_m$-equivariant evaluation map $ev_0: Q(\mathbb{P}^{n-1}; d) \to [\mathbb{C}^n/\mathbb{G}_m]$ at the point $0 \in \mathbb{P}^1$. By abuse of notation, we also let $x$ denote the $\mathbb{G}_m$-equivariant Euler class of the line bundle $ev_0^*(\mathcal{O}_{[\mathbb{C}^n/\mathbb{G}_m]}(1))$. Here $\mathcal{O}_{[\mathbb{C}^n/\mathbb{G}_m]}(1)$ is the line bundle $[([\mathbb{C}^n \times \mathbb{C}]/\mathbb{G}_m)]$ on the quotient stack with weights $(1, \ldots, 1), 1)$.

Let $t_i: Q(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m} \to Q(\mathbb{P}^{n-1}; d)$ be the inclusion of the $i$-th connected component (3.10.2). Since $t_i^*(x) = x + iz$ (under the identification $Q(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m} = \mathbb{P}^{n-1}$) and

$$\frac{1}{\text{Eul}^{\mathbb{G}_m} \left( N_{Q(\mathbb{P}^{n-1}; d)^{\mathbb{G}_m}/Q(\mathbb{P}^{n-1}; d)} \right)} = I_i(z) \cup I_{d-i}(-z), \tag{3.12.1}$$
localization with respect to the $\mathbb{G}_m$-action shows that

$$
\sum_{d=0}^{\infty} e^{\epsilon_1} \left\langle \frac{e^{(t-\tau)x/z}}{\tau} \right\rangle_{\mathbb{P}^{n-1},d}^{\mathbb{P}^n, \mathbb{G}_m} = \sum_{d=0}^{\infty} e^{\epsilon_1} \int_{Q_{\mathbb{P}^{n-1},d}^{\mathbb{G}_m}} \left( \frac{e^{(t-\tau)x/z}}{\tau} \right) \text{Eul}_{\mathbb{G}_m} \left( N_{Q_{\mathbb{P}^{n-1},d}^{\mathbb{G}_m}/Q_{\mathbb{P}^{n-1},d}} \right)
$$

$$
= \sum_{d=0}^{\infty} e^{\epsilon_1} \sum_{i=0}^{d} \int_{\mathbb{P}^{n-1}} e^{(t-\tau)(x+i\tau)/z} \cup I_i(z) \cup I_{d-i}(-z)
$$

$$
= \sum_{d=0}^{\infty} \sum_{i=0}^{d} \int_{\mathbb{P}^{n-1}} e^{tx/z} e^{t_i} I_i(z) \cup e^{-\tau x/z} e^{(d-i)\tau} I_{d-i}(-z)
$$

(3.12.2)

$$
= \int_{\mathbb{P}^{n-1}} I_{\mathbb{P}^{n-1}}^H(t; z) \cup I_{\mathbb{P}^{n-1}}^H(\tau; -z).
$$

The factorization of the $H \times \mathbb{G}_m$-equivariant correlator is proved similarly as

$$
\sum_{d=0}^{\infty} e^{\epsilon_1} \left\langle \frac{e^{(t-\tau)x/z}}{\tau} \right\rangle_{\mathbb{P}^{n-1},d}^{H \times \mathbb{G}_m}
$$

$$
= \sum_{d=0}^{\infty} \text{Res} \frac{e^{\epsilon_1} e^{(t-\tau)x/z} dx}{\prod_{i=1}^{n} \prod_{l=0}^{d} (x - \lambda_i - l z)}
$$

$$
= \sum_{d=0}^{\infty} \sum_{m=0}^{d} \sum_{j=1}^{n} \text{Res}_{x=\lambda_j + mz} \frac{e^{\epsilon_1} e^{(t-\tau)x/z} dx}{\prod_{i=1}^{n} \prod_{l=0}^{d} (x - \lambda_i - l z)}
$$

$$
= \sum_{d=0}^{\infty} \sum_{m=0}^{d} \sum_{j=1}^{n} \text{Res}_{x=\lambda_j} \frac{e^{tx/z} e^{tm} dx}{\prod_{i=1}^{n} \prod_{l=0}^{d} (x - \lambda_i - (l-m)z)}
$$

$$
= \sum_{d=0}^{\infty} \sum_{m=0}^{d} \sum_{j=1}^{n} \text{Res}_{x=\lambda_j} \frac{e^{tx/z} e^{tm} dx}{\prod_{i=1}^{n} \prod_{l=0}^{d} (x - \lambda_i - (l-m)z)}
$$

$$
= \int_{\mathbb{P}^{n-1}} I_{\mathbb{P}^{n-1}}^H(t; z) \cup I_{\mathbb{P}^{n-1}}^H(\tau; -z).
$$

This can also be regarded as a purely combinatorial proof.

3.13. Let $\psi: \overline{\mathcal{M}}_{0,1}(\mathbb{P}^{n-1},d) \to \mathbb{P}^{n-1}$ be the evaluation map from the moduli space of stable maps of genus 0 and degree $d$ with 1 marked point, and $\psi$ be the first Chern class of the line bundle over $\overline{\mathcal{M}}_{0,1}(\mathbb{P}^{n-1},d)$ whose fiber at a stable map $\varphi: (C, x) \to \mathbb{P}^{n-1}$ is the cotangent line $T_x \mathcal{C}$ at the marked point. The equivariant $J$-function [Giv96] is a $H^*(\mathbb{P}^{n-1}; \mathbb{C})$-valued hypergeometric series given by

(3.13.1) \[ J_{\mathbb{P}^{n-1}}^H(t; z) := e^{tx/z} \sum_{d=0}^{\infty} e^{dt} J_d \]

where

(3.13.2) \[ J_d := \psi_* \left( \frac{1}{z(z - \psi)} \right). \]
3.14. The graph space is defined by $G(\mathbb{P}^{n-1}; d) := \mathcal{M}_{0,0}(\mathbb{P}^{n-1} \times \mathbb{P}^1; (d, 1))$. The source of any map $\varphi : C \to \mathbb{P}^{n-1} \times \mathbb{P}^1$ in $G(\mathbb{P}^{n-1}; d)$ has a distinguished irreducible component $C$ which maps isomorphically to $\mathbb{P}^1$. Let $G(\mathbb{P}^{n-1}; d)_0$ be the open subspace of $G(\mathbb{P}^{n-1}; d)$ consisting of stable maps without irreducible components mapping constantly to $0 \in \mathbb{P}^1$. There is a map $e_0 : G(\mathbb{P}^{n-1}; d)_0 \to \mathbb{P}^{n-1}$ sending $\varphi : C \to \mathbb{P}^{n-1} \times \mathbb{P}^1$ to $\text{pr}_1 \circ \varphi \left( (\text{pr}_2 \circ \varphi)^{-1}(0) \right)$. The fixed locus of the natural $\mathbb{G}_m$-action on $G(\mathbb{P}^{n-1}; d)_0$ can be identified with $\mathcal{M}_{0,1}(\mathbb{P}^{n-1}; d)$. Since the natural morphism $G(\mathbb{P}^{n-1}; d)_0 \to Q(\mathbb{P}^{n-1}; d)_0$ is a $\mathbb{G}_m$-equivariant birational morphism which commutes with the evaluation maps, the push-forwards $J_d$ and $I_d$ of $e_0 : G(\mathbb{P}^{n-1}; d)_0 \to \mathbb{P}^{n-1}$ and $e_0 : Q(\mathbb{P}^{n-1}; d)_0 \to \mathbb{P}^{n-1}$ are equal, and hence $I_{n-1}(t; z) = J_{n-1}(t; z)$.

3.15. The effective potential (2.4.1) is given by

\[(3.15.1) \quad W_{\text{eff}}(x; t) = tx - \sum_{i=1}^{n} (x - \lambda_i) (\log (x - \lambda_i) - 1).\]

One can easily see

\[(3.15.2) \quad \frac{\partial W_{\text{eff}}}{\partial x} = t - \sum_{i=1}^{n} \log(x - \lambda_i),\]

\[(3.15.3) \quad e^{\partial_x W_{\text{eff}}} = q \prod_{i=1}^{n} (x - \lambda_i)^{-1},\]

so that

\[(3.15.4) \quad (P(x))^H_{\mathbb{P}^{n-1}} = \text{Res} \frac{P(x)dx}{\prod_{i=1}^{n} (x - \lambda_i)(1 - e^{\partial_x W_{\text{eff}}})}.\]

Note that the equation

\[(3.15.5) \quad e^{\partial_x W_{\text{eff}}} = 1\]

gives the relation

\[(3.15.6) \quad \prod_{i=1}^{n} (x - \lambda_i) = q\]

in the equivariant quantum cohomology of $\mathbb{P}^{n-1}$.

4. Projective complete intersections

4.1. Let $f_1(w_1, \ldots, w_n), \ldots, f_r(w_1, \ldots, w_n) \in \mathbb{C}[w_1, \ldots, w_n]$ be homogeneous polynomials of degrees $l_1, \ldots, l_r$ satisfying the Calabi–Yau condition

\[(4.1.1) \quad l_1 + \cdots + l_r = n.\]

Assume that $f_1, \ldots, f_r$ are sufficiently general so that

\[(4.1.2) \quad \mathcal{Y} := \{ [w_1, \cdots, w_n] \in \mathbb{P}^{n-1} \mid f_1(w_1, \ldots, w_n) = \cdots = f_r(w_1, \ldots, w_n) = 0 \}\]

is a smooth complete intersection of dimension $n - r - 1$, whose Poincaré dual is

\[(4.1.3) \quad v := \prod_{i=1}^{r} (l_i x).\]
Define the quasimap space $Q(Y; d)$ as the subset of $Q(\mathbb{P}^{n-1}; d)$ consisting of $[\varphi_1(z_1, z_2), \ldots, \varphi_n(z_1, z_2)]$ satisfying

\[ f_i(\varphi_1(z_1, z_2), \ldots, \varphi_n(z_1, z_2)) = 0 \in \mathbb{C}[z_1, z_2] \text{ for any } i \in \{1, \ldots, r\}. \]

Since $f_i(\varphi_1(z_1, z_2), \ldots, \varphi_n(z_1, z_2)) \in \mathbb{C}[z_1, z_2]$ is a homogeneous polynomial of degree $d l_i$ in $z_1$ and $z_2$, it contains $d l_i + 1$ terms, each of which is a homogeneous polynomial of degree $l_i$ in $(a_{kl})_{k,l}$. With this in mind, the Morrison-Plesser class is defined by

\[ \Phi(Y; d) := \prod_{i=1}^{r} (l_i x)^{l_i d} \in H^*(Q(\mathbb{P}^{n-1}; d); \mathbb{Z}), \]

so that $\Phi(Y; d) \cup v$ is the Poincaré dual of $[Q(Y; d)]^\text{virt} \in H_*(Q(\mathbb{P}^{n-1}; d); \mathbb{Z})$. If we set

\[ \langle P(x) \rangle_{Y,d} := \int_{Q(\mathbb{P}^{n-1}; d)} P(x) \cup \Phi(Y; d) \cup v \]

and

\[ \langle P(x) \rangle_Y := \sum_{d=0}^{\infty} q^d \langle P(x) \rangle_{Y,d} \]

for $P(x) \in \mathbb{C}[x]$, then we have

\[ \langle x^{n-r-1} \rangle_Y = \sum_{d=0}^{\infty} q^d \text{Res} \frac{x^{n-r-1} \Phi(Y, d) v dx}{x^{n(d+1)}} \]

\[ = \sum_{d=0}^{\infty} q^d \text{Res} \frac{x^{n-r-1} \prod_{i=1}^{r} (l_i x)^{l_i d+1} dx}{x^{n(d+1)}} \]

\[ = \sum_{d=0}^{\infty} q^d \prod_{i=1}^{r} (l_i)^{l_i d+1} \]

\[ = \frac{\prod_{i=1}^{r} l_i}{1 - q \prod_{i=1}^{r} (l_i)^{l_i}}. \]

4.2. The gauged linear sigma model for $Y$ is obtained from the gauged linear sigma model for $\mathbb{P}^{n-1}$ by adding $r$ fields of $G = \mathbb{C}_m$-charge $-l_1, \ldots, -l_r$ and R-charge 2. One has $Z^\text{rec}_d(x) = 1$ and $Z^\text{gau}_d(x) = (x^{-d-1})^n \cdot \prod_{i=1}^{r} (-l_i x)^{l_i d+1}$ in this case, so that \([2.2.4]\) gives

\[ \sum_{d=0}^{\infty} e^{t'd} \text{Res} (x^{-d-1})^n \prod_{i=1}^{r} (-l_i x)^{l_i d+1} x^{n-r-1} = \sum_{d=0}^{\infty} e^{t'd} \prod_{i=1}^{r} (-l_i)^{l_i d+1} \]

\[ = \sum_{d=0}^{\infty} (-1)^{\sum_{i=1}^{r} l_i d} e^{t'd} \prod_{i=1}^{r} (l_i)^{l_i d+1} \]

\[ = \sum_{d=0}^{\infty} (-1)^{n d} e^{t'd} \prod_{i=1}^{r} (l_i)^{l_i d+1}, \]

which coincides with \([4.1.8]\) under the identification

\[ q = (-1)^{n e^t}. \]
4.3. The mirror $\hat{Y}$ of $Y$ is a compactification of a complete intersection in $\mathbb{C}^n$ defined by
\begin{align}
(4.3.1) \quad & \hat{f}_1 := 1 - (a_1 \hat{y}_1 + \cdots + a_l \hat{y}_l), \\
(4.3.2) \quad & \hat{f}_2 := 1 - (a_{l+1} \hat{y}_{l+1} + \cdots + a_{l+t} \hat{y}_{l+t}), \\
(4.3.3) \quad & \vdots \\
(4.3.4) \quad & \hat{f}_r := 1 - (a_{1+\ldots+l_{r-1}} \hat{y}_{1+\ldots+l_{r-1}+1} + \cdots + a_n \hat{y}_n), \\
(4.3.5) \quad & \hat{f}_0 := \hat{y}_1 \cdots \hat{y}_n - 1.
\end{align}

The complex structure of $\hat{Y}$ depends not on the individual $a_i$ but only on $\alpha = a_1 \cdots a_n$. The *Yukawa $(n{-}2)$-point function* is defined by
\begin{equation}
(4.3.6) \quad \mathcal{Y}(\alpha) := \frac{(-1)^{(n-1)(n-2)/2}}{(2\pi)^{n-1}} \int_Y \Omega \wedge \left(\frac{\partial}{\partial \alpha}\right)^{n-2} \Omega,
\end{equation}
where
\begin{equation}
(4.3.7) \quad \Omega := \text{Res} \left( \frac{d\hat{y}_1 \wedge \cdots \wedge d\hat{y}_n}{\hat{f}_0 \hat{f}_1 \cdots \hat{f}_r} \right)
\end{equation}
is the holomorphic volume form on $\hat{Y}$. The computation in [BvS95 Proposition 5.1.2] shows
\begin{equation}
(4.3.9) \quad \mathcal{Y}(\alpha) = \left. \left( x^{n-r-1} \right) \right|_{q=\alpha}.
\end{equation}

A generalization of (4.3.9) to toric complete intersections is *toric residue mirror symmetry* conjectured in [BM02, BM03] and proved in [SV04, Bor05, Kar05, SV06].

5. Concave bundles on projective spaces

5.1. Let $l_1, l_2, \cdots, l_r$ be positive integers and
\begin{equation}
(5.1.1) \quad Y := \text{Spec}_{\mathbb{P}^{n-1}}(\text{Sym}^* \mathcal{E}^\vee)
\end{equation}
be the total space of the vector bundle associated with the locally free sheaf
\begin{equation}
(5.1.2) \quad \mathcal{E} := \mathcal{O}_{\mathbb{P}^{n-1}}(-l_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^{n-1}}(-l_r)
\end{equation}
on $\mathbb{P}^{n-1}$. Since any holomorphic map from $\mathbb{P}^1$ to $Y$ of positive degree $d$ factors through the zero-section $\mathbb{P}^{n-1} \to Y$, we define the quasimap space to $Y$ as
\begin{equation}
(5.1.3) \quad \mathcal{Q}(Y; d) := \mathcal{Q}(\mathbb{P}^{n-1}; d).
\end{equation}

5.2. To equip $\mathcal{Q}(Y; d)$ with a natural obstruction theory, we identify $\mathcal{Q}(Y; d)$ with an open substack of the mapping stack $\text{Map}(\mathbb{P}^1, Y)$ to the quotient stack
\begin{equation}
(5.2.1) \quad \mathcal{Y} := [(A^n \times A^r) / \mathbb{G}_m]
\end{equation}
of $A^n \times A^r$ by the $\mathbb{G}_m$-action given by
\begin{equation}
(5.2.2) \quad \mathbb{G}_m \ni \alpha: (x_1, \ldots, x_n, z_1, \ldots, z_r) \mapsto (\alpha x_1, \ldots, \alpha x_n, \alpha^{-l_1} z_1, \ldots, \alpha^{-l_r} z_r).
\end{equation}
A morphism $\mathbb{P}^1 \to Y$ consists of a line bundle $\mathcal{L}$ on $\mathbb{P}^1$ and sections
\[(\varphi_i)_{i=1}^n, (\psi_j)_{j=1}^r \in \left((H^0(\mathcal{L}))^n \times \prod_{j=1}^r H^0(\mathcal{L}^{\otimes(-l_j)})\right),\]
whose degree is defined as the degree of $\mathcal{L}$.

5.3. Recall from [BF97, Definition 4.4] that an obstruction theory for a Deligne–Mumford stack $\mathcal{X}$ is a morphism $\phi: E \to L_X$ from an object $E$ of the derived category of quasicoherent sheaves on $\mathcal{X}$ satisfying
\[(1)\ h^i(E) \cong 0 \text{ for } i > 0, \quad (2)\ h^i(E) \text{ is coherent for } i = 0, -1\]
to the cotangent complex $L_X$ such that
\[(1)\ h^0(\phi) \text{ is an isomorphism, and} \quad (2)\ h^{-1}(\phi) \text{ is an epimorphism}.\]
It is said to be perfect if $E$ is locally isomorphic to a two-term complex of locally free sheaves of finite rank [BF97, Definition 5.1].

5.4. A perfect obstruction theory produces the virtual fundamental cycle $[\mathcal{X}]_{\text{virt}}$ in the Chow group $A_{\text{virt}, \dim \mathcal{X}}(\mathcal{X})$ of degree
\[(5.4.1)\ \text{virt. dim } \mathcal{X} = \text{rank } h^0(E) - \text{rank } h^{-1}(E).\]
When $\mathcal{X}$ is a smooth scheme, then the cotangent complex $L_X$ is isomorphic to the sheaf $\Omega_X$ of Kähler differentials, and the virtual fundamental cycle is the Euler class of $h^{-1}(E)$.

5.5. The derived mapping stack $\mathbb{R}Map(\mathcal{S}, \mathcal{T})$ from a proper scheme $\mathcal{S}$ to a derived Artin stack $\mathcal{T}$ is a derived Artin stack (see e.g. [Toë14, Corollary 3.3]) whose tangent complex is given by
\[(5.5.1)\ T_{\mathbb{R}Map(\mathcal{S}, \mathcal{T})} \cong \mathbb{R}\pi_* (\text{Lev}^* T_T),\]
where
\[(5.5.2)\ \pi: \mathbb{R}Map(\mathcal{S}, \mathcal{T}) \times \mathcal{S} \to \mathbb{R}Map(\mathcal{S}, \mathcal{T})\]
is the first projection and
\[(5.5.3)\ \text{ev}: \mathbb{R}Map(\mathcal{S}, \mathcal{T}) \times \mathcal{S} \to \mathcal{T}\]
is the evaluation morphism. It is a derived thickening of the mapping stack $\text{Map}(\mathcal{S}, \mathcal{T})$, and the pull-back
\[(5.5.4)\ j^*: j^* L_{\mathbb{R}Map(\mathcal{S}, \mathcal{T})} \to L_{\text{Map}(\mathcal{S}, \mathcal{T})}\]
by the structure morphism
\[(5.5.5)\ j: \text{Map}(\mathcal{S}, \mathcal{T}) \to \mathbb{R}Map(\mathcal{S}, \mathcal{T})\]
gives an obstruction theory on $\text{Map}(\mathcal{S}, \mathcal{T})$.

5.6. The restriction of the natural obstruction theory for $\text{Map}(\mathbb{P}^1, Y)$ to the open sub-stack $Q(Y; d)$ gives an obstruction theory for $Q(Y; d)$ with $E = j^* L_{\mathbb{R}Map(\mathbb{P}^1, Y)}|_{Q(Y; d)}$ and $\phi = j^! Q(Y; d)$. 

14
5.7. Since Pic($\mathbb{A}^n \times \mathbb{A}^r$) is trivial, the Picard group Pic $\mathcal{Y} \cong \text{Pic}^{G_m}(\mathbb{A}^n \times \mathbb{A}^r)$ can be identified with the group of characters of $G_m$, which is non-canonically isomorphic to $\mathbb{Z}$. We fix an isomorphism in such a way that $\bigoplus_{i=0}^{\infty} H^0(\mathcal{O}_Y(a))$ is the coordinate ring of $\mathbb{A}^n$, where $\mathcal{O}_Y(a)$ is the line bundle associated with $a \in \mathbb{Z} \cong \text{Pic} \mathcal{Y}$. Since $\mathcal{Y}$ is the quotient stack of $\mathbb{A}^n \times \mathbb{A}^r$ by the action of $G_m$, the tangent complex $T_{\mathcal{Y}}$ satisfies
\[(5.7.1) \quad \varpi^* T_{\mathcal{Y}} \cong \text{Cone} \left( \mathcal{O}_{\mathbb{A}^n \times \mathbb{A}^r} \otimes \text{Lie}(\mathbb{G}_m) \to T_{\mathbb{A}^n \times \mathbb{A}^r} \right)\]
where $\varpi: \mathbb{A}^n \times \mathbb{A}^r \to \mathcal{Y}$ is the quotient morphism. This in turn implies that
\[(5.7.2) \quad T_{\mathcal{Y}} \cong \text{Cone} \left( \mathcal{O}_Y \to \mathcal{O}_Y(1)^{\oplus n} \oplus \bigoplus_{i=1}^{r} \mathcal{O}_Y(-l_i) \right).\]

5.8. We write the restriction of the evaluation morphism $\text{Map}(\mathbb{P}^1, \mathcal{Y}) \times \mathbb{P}^1 \to \mathcal{Y}$ to the open substack $\mathcal{Q}(Y; d) \subset \text{Map}(\mathbb{P}^1, \mathcal{Y})$ as
\[(5.8.1) \quad \text{ev}: \mathcal{Q}(Y; d) \times \mathbb{P}^1 \to \mathcal{Y}\]
again by abuse of notation. We have
\[(5.8.2) \quad \text{ev}^* \mathcal{O}_Y(1) \cong \mathcal{O}_{\mathcal{Q}(Y; d)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d)\]
equipped with the trivial perfect obstruction theory, so that
\[(5.8.3) \quad \text{ev}^* \mathcal{O}_Y(1) \cong \mathcal{O}_{\mathcal{Q}(Y; d)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d)\]
where $\varpi: \mathbb{A}^n \times \mathbb{A}^r \to \mathcal{Y}$ is the quotient morphism. This in turn implies that
\[(5.8.4) \quad h^0(E^\vee) \cong R^0 \pi_* \text{Cone} \left( \mathcal{O}_{\mathcal{Q}(Y; d) \times \mathbb{P}^1} \to \left( \mathcal{O}_{\mathcal{Q}(Y; d)}(1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(d) \right)^{\oplus n} \right)\]
\[(5.8.5) \quad \cong \text{Cone} \left( \mathcal{O}_{\mathcal{Q}(Y; d)} \to \mathcal{O}_{\mathcal{Q}(Y; d)}(1)^{\oplus n(d+1)} \right)\]
gives the Euler sequence
\[(5.8.6) \quad 0 \to \mathcal{O}_{\mathcal{Q}(Y; d)} \to \mathcal{O}_{\mathcal{Q}(Y; d)}(1)^{\oplus n(d+1)} \to T_{\mathcal{Q}(Y; d)} \to 0\]
on $\mathcal{Q}(Y; d) \cong \mathbb{P}^{n(d+1)-1}$. One has
\[(5.8.7) \quad h^1(E^\vee) \cong R^1 \pi_* \left( \bigoplus_{i=1}^{r} \mathcal{O}_{\mathcal{Q}(Y; d)}(-l_i) \boxtimes \mathcal{O}_{\mathbb{P}^1}(-l_id) \right)\]
\[(5.8.8) \quad \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathcal{Q}(Y; d)}(-l_i) \otimes H^1(\mathcal{O}_{\mathbb{P}^1}(-l_id))\]
\[(5.8.9) \quad \cong \bigoplus_{i=1}^{r} \mathcal{O}_{\mathcal{Q}(Y; d)}(-l_i)^{\oplus (l_id-1)}\]
equipped with the trivial perfect obstruction theory, so that
\[(5.8.10) \quad [\mathcal{Q}(Y; d)]^{\text{virt}} = [\mathcal{Q}(Y; d)] \cap \text{Eul} \left( h^1(E^\vee) \right) = [\mathcal{Q}(Y; d)] \cap \prod_{i=1}^{r} (-l_i x_i)^{l_id-1}.\]

5.9. When the degree is zero, the quasimap space $\mathcal{Q}(Y; 0)$ is naturally isomorphic to $Y$ equipped with the trivial perfect obstruction theory, so that
\[(5.9.1) \quad [\mathcal{Q}(Y; 0)]^{\text{virt}} = [Y].\]
5.10. For any $P(x) \in \mathbb{C}[x]$, we define

\begin{equation}
\langle P(x) \rangle_{Y,d} := \int_{[Q(Y,d)]_{\text{virt}}} P(x)
\end{equation}

and

\begin{equation}
\langle P(x) \rangle_Y := \sum_{d=0}^{\infty} q^d \langle P(x) \rangle_{Y,d}.
\end{equation}

It follows that

\begin{equation}
\langle P(x) \rangle_Y = \sum_{d=0}^{\infty} q^d \langle P(x) \rangle_{Y,d}.
\end{equation}

\begin{equation}
\langle P(x) \rangle_{Y,d} := \int_{\mathbb{P}^{n(d+1)-1}} P(x) \prod_{i=1}^{r} (-l_i x)^{l_i d-1}
\end{equation}

\begin{equation}
= \sum_{d=0}^{\infty} q^d \text{Res}_{x^n(d+1)} (x^{-d} \prod_{i=1}^{r} (-l_i x)^{l_i d-1})
\end{equation}

5.11. The gauged linear sigma model for $Y$ is obtained from the gauged linear sigma model for $P_n - 1$ by adding $r$ fields of $G = \mathbb{G}_{m}$-charge $-l_1, \ldots, -l_r$ and R-charge 0. One has $Z_{\text{vac}}(x) = 1$ and $Z_{\text{mat}}(x) = (x^{-d-1})^n \prod_{i=1}^{r} (-l_i x)^{l_i d-1}$ in this case, so that (2.2.4) gives

\begin{equation}
\langle P(x) \rangle_{\text{GLSM}} = \sum_{d=0}^{\infty} e^d \text{Res}_{x^n(d+1)} (x^{-d-1})^n \prod_{i=1}^{r} (-l_i x)^{l_i d-1} P(x),
\end{equation}

which coincides with (5.10.4) under the identification

\begin{equation}
q = e^d.
\end{equation}

5.12. If $(l_1, \ldots, l_r)$ satisfies the Calabi–Yau condition

\begin{equation}
l_1 + \cdots + l_r = n,
\end{equation}

then (5.10.4) gives

\begin{equation}
\langle x^k \rangle_Y = \begin{cases}
\frac{1}{\prod_{i=1}^{r} (-l_i)} (1 - q \prod_{i=1}^{r} (-l_i)^{l_i}) & k = n + r, \\
0 & \text{otherwise},
\end{cases}
\end{equation}

which matches the Yukawa coupling of the mirror (see e.g. [KM10, Example 6.15]).

6. Classical mirror symmetry for toric hypersurfaces

6.1. Let $\mathcal{N} := \mathbb{Z}^n$ be a free abelian group of rank $n$ and $\mathcal{M} := \mathcal{N}^* := \text{Hom}(\mathcal{N}, \mathbb{Z})$ be the dual group. Let further $(\Delta, \tilde{\Delta})$ be a polar dual pair of reflexive polytopes in $\mathcal{M}$ and $\mathcal{N}$.

6.2. Recall that the fan polytope of a fan is defined as the convex hull of primitive generators of one-dimensional cones. Let $(\Sigma, \tilde{\Sigma})$ be a pair of smooth projective fans whose fan polytopes are $\Delta$ and $\tilde{\Delta}$. The associated toric varieties will be denoted by $X := X_\Sigma$ and $\tilde{X} := X_{\tilde{\Sigma}}$. 
6.3. The set of primitive generators of one-dimensional cones of the fan $\Sigma$ will be denoted by
\begin{equation}
B := \{b_1, \ldots, b_m\} \subset N.
\end{equation}
Assume that $B$ generates $N$. One has the \textit{fan sequence}
\begin{equation}
0 \to L \to \mathbb{Z}^m \xrightarrow{b} N \to 0
\end{equation}
and the \textit{divisor sequence}
\begin{equation}
0 \to M \xrightarrow{b^\vee} \mathbb{Z}^m \to \mathbb{L} \to 0,
\end{equation}
where $b$ sends the $i$-th coordinate vector $e_i \in \mathbb{Z}^m$ to $b_i$. Recall that
\begin{equation}
\mathbb{L} \cong \text{Pic}(X) \cong H^2(X; \mathbb{Z}), \quad \text{Eff}(X) \subset L \subset \mathbb{Z}^m,
\end{equation}
where Eff($X$) denotes the semigroup of the effective curves (see [BM02, §3]). We write the group ring of $M$ as $\mathbb{C}[M]$ and define $T := \mathbb{N}_{\mathbb{G}_m} := \text{Spec} \mathbb{C}[M]$. We also set $\mathbb{T} := \text{Spec} \mathbb{C}[N]$, $\mathbb{L} := \text{Spec} \mathbb{C}[L]$, and $L := \text{Spec} \mathbb{C}[\mathbb{L}]$. The fan sequence induces the exact sequences
\begin{equation}
1 \to L \xrightarrow{\chi} (\mathbb{G}_m)^m \to T \to 1
\end{equation}
and
\begin{equation}
1 \to \mathbb{T} \to (\mathbb{G}_m)^m \to \mathbb{L} \to 1
\end{equation}
of algebraic tori. We write the $i$-th components of the map $\chi: L \to (\mathbb{G}_m)^m$ in (6.3.5) as $\chi_i$, and the affine line $\mathbb{A}^1$ equipped with the action of $L$ through $\chi_i$ as $\mathbb{A}_i$. Then one has
\begin{equation}
X \cong \left( \prod_{i=1}^{m} \mathbb{A}_i \right) / _{\theta} \mathbb{L}
\end{equation}
for a suitable choice of a character $\theta \in \mathbb{L} \cong \text{Hom}(\mathbb{L}, \mathbb{G}_m)$. The right-hand side of (6.3.7) denotes the GIT quotient with respect to the linearization determined by $\theta$.

6.4. We define a graded ring $S_\Delta := \bigoplus_{k=0}^\infty S^k_\Delta$ by
\begin{equation}
S^k_\Delta := \bigoplus_{m \in M \cap (k\Delta)} \mathbb{C} \cdot y_0^k y^m,
\end{equation}
which is a subalgebra of the semigroup ring
\begin{equation}
\mathbb{C}[N \times M] = \mathbb{C}[y_0, y_1^{\pm 1}] := \mathbb{C}[y_0, y_1^{\pm 1}, \ldots, y_n^{\pm 1}]
\end{equation}
of $\mathbb{N} \times M$. It is the anti-canonical ring of $X$, so that one has $X \cong \text{Proj} S_\Delta$ if and only if $X$ is Fano. The ring $S_\Delta$ is Cohen-Macaulay with the dualizing module $I_\Delta := \bigoplus_{k=0}^\infty I^k_\Delta$ given by
\begin{equation}
I^k_\Delta := \bigoplus_{m \in M \cap \text{Int}(k\Delta)} \mathbb{C} \cdot y_0^k y^m,
\end{equation}
where Int($k\Delta$) is the interior of $k\Delta$. 

17
6.5. For \( \alpha = (\alpha_1, \ldots, \alpha_m) \in (\mathbb{G}_m)^m \) (this \((\mathbb{G}_m)^m \) can be naturally considered as the dual torus of the big torus of \( \Sigma \)), we define an element of the group ring \( \mathbb{C}[N] \) by

\[
\tilde{W}_\alpha(y) := \sum_{i=1}^m \alpha_i y^b_i \in \mathbb{C}[N].
\]

An element \( \tilde{f} \in \mathbb{C}[N] \) is said to be \( \Delta \)-regular if

\[
\tilde{F} := (\tilde{F}_0, \tilde{F}_1, \ldots, \tilde{F}_n) := (\tilde{g}_0 \tilde{f}, \tilde{y}_0 \tilde{g}_1 \tilde{f}, \ldots, \tilde{y}_0 \tilde{g}_n \tilde{f})
\]

is a regular sequence in \( S_\Delta \). We write

\[
((\mathbb{G}_m)^m)^{\text{reg}} := \{ \alpha \in (\mathbb{G}_m)^m \mid \tilde{f}_\alpha := 1 - \tilde{W}_\alpha(y) \text{ is } \Delta \text{-regular} \}.
\]

6.6. Let \( \tilde{\varphi} : \tilde{\mathcal{Y}} \to ((\mathbb{G}_m)^m)^{\text{reg}} \) be the second projection from

\[
\tilde{\mathcal{Y}} = \{ (\tilde{y}, \alpha) \in \tilde{T} \times ((\mathbb{G}_m)^m)^{\text{reg}} \mid \tilde{W}_\alpha(y) = 1 \}.
\]

Assume that \( X \) is Fano. Any fiber \( \tilde{Y}_\alpha := \tilde{\varphi}^{-1}(\alpha) \) is an uncompactified mirror of a general anti-canonical hypersurface \( Y \subset X \). The closure of \( \tilde{Y}_\alpha \) in \( \tilde{X} \) is a smooth anti-canonical Calabi–Yau hypersurface, which is the compact mirror of \( Y \). The quotient of the family \( \tilde{\varphi} : \tilde{\mathcal{Y}} \to ((\mathbb{G}_m)^m)^{\text{reg}} \) by the free \( \tilde{T} \)-action

\[
\tilde{T} \ni \tilde{y} : (\tilde{y}, (\alpha_1, \ldots, \alpha_m)) \mapsto (\tilde{y}^{-1} \tilde{y}', (\tilde{y}^b_1 \alpha_1, \ldots, \tilde{y}^b_m \alpha_m))
\]

will be denoted by \( \check{\varphi} : \tilde{\mathcal{Y}} \to \tilde{\mathcal{L}}^{\text{reg}} \), where \( \tilde{\mathcal{Y}} := \tilde{\mathcal{Y}}/\tilde{T} \) and \( \tilde{\mathcal{L}}^{\text{reg}} := ((\mathbb{G}_m)^m)^{\text{reg}}/\tilde{T} \).

6.7. Choose an integral basis \( p_1, \ldots, p_r \) of \( \tilde{L} \cong \text{Pic } X \) such that each \( p_i \) is nef. This gives the corresponding coordinate \( q = (q_1, \ldots, q_r) \) on \( \tilde{L} \). Let \( \tilde{U}' \subset \tilde{\mathcal{L}}^{\text{reg}} \) be a neighborhood of \( q_1 = \cdots = q_r = 0 \), and \( \tilde{U} \) be the universal cover of \( \tilde{U}' \).

6.8. We write the image of the Poincaré residue as

\[
H_{\text{res}}^{n-1}(\tilde{Y}_\alpha) := \text{Im} (\text{Res} : H^0(\tilde{X}, \Omega^n_{\tilde{X}}(\ast \tilde{Y}_\alpha)) \to H^{n-1}(\tilde{Y}_\alpha)).
\]

Let \( H_B \) be the pull-back to \( \tilde{U} \) of the local system \( \text{gr}^W_{n-1} R^{n-1} \tilde{\varphi}^* \mathcal{C}_{\tilde{\mathcal{Y}}} \) on \( \tilde{U}' \), and \( H_B^{\text{res}} \) be the sub-system with stalks \( H_{\text{res}}^{n-1}(\tilde{Y}_\alpha) \). Here \( \text{gr}^W_{n-1} \) is the weight \( n - 1 \) piece of Deligne’s mixed Hodge structure. The \textit{residual B-model VHS} \( (H_B, \nabla_B, \mathcal{F}_B, Q_B) \) on \( \tilde{U} \) consists of the locally free sheaf \( H_B := H_B^{\text{res}} \otimes_{\mathcal{O}_{\tilde{U}}} \mathcal{O}_{\tilde{U}} \), the Gauss–Manin connection \( \nabla_B \), the Hodge filtration \( \mathcal{F}_B \), and the polarization \( Q_B : H_B \otimes_{\mathcal{O}_{\tilde{U}}} H_B \to \mathcal{O}_{\tilde{U}} \) given by

\[
Q_B(\omega_1, \omega_2) := (-1)^{(n-1)(n-2)/2} \int_{\tilde{Y}_\alpha} \omega_1 \cup \omega_2.
\]

6.9. On the A-model side, let

\[
H^{\text{amb}}_\alpha(Y; \mathbb{C}) := \text{Im}(\iota^* : H^\bullet(X; \mathbb{C}) \to H^\bullet(Y; \mathbb{C}))
\]

be the subspace of \( H^\bullet(Y; \mathbb{C}) \) coming from the cohomology classes of the ambient toric variety, and set

\[
U := \{ \tau = \beta + \sqrt{-1} \omega \in H^2_{\text{amb}}(Y; \mathbb{C}) \mid \langle \omega, d \rangle \gg 0 \text{ for any non-zero } d \in \text{Eff}(Y) \}.
\]

This open subset \( U \) is considered as a neighborhood of the large radius limit point. Let \( (\tau_i)_{i=1}^r \) be the coordinate on \( H^2_{\text{amb}}(Y; \mathbb{C}) \) dual to the basis \( \{ p_i \}_{i=1}^r \) so that \( \tau = \sum_{i=1}^r \tau_i p_i \).
6.10. The ambient A-model VHS \((H_A, \nabla_A, \mathcal{F}_A, Q_A)\) consists (\cite{irrill} Definition 6.2, cf. also \cite{ck99} Section 8.5) of the locally free sheaf \(H_A = H^*_\text{amb}(Y; \mathbb{C}) \otimes_\mathcal{O}_U\), the connection

\[(6.10.1) \quad \nabla_A = d + \sum_{i=1}^r (p_i \circ r) \, dr_i : H_A \to H_A \otimes \Omega^1_U,\]

the Hodge filtration

\[(6.10.2) \quad \mathcal{F}^p_A := H^{\leq 2(n-1-p)} \otimes_\mathcal{O}_U,\]

and the pairing

\[(6.10.3) \quad Q_A : H_A \otimes_\mathcal{O}_U H_A \to \mathcal{O}_U, \quad (\alpha, \beta) \mapsto (2\pi \sqrt{-1})^{n-1} \int_Y (-1)^{\deg \alpha/2} \alpha \cup \beta,\]

which is \((-1)^{n-1}\)-symmetric and \(\nabla_A\)-flat.

6.11. Let \(u_i \in H^2_\text{amb}(Y; \mathbb{Z})\) be the first Chern class of the line bundle on \(Y\) corresponding to the one-dimensional cone \(\mathbb{R}_{\geq 0} \cdot b_i \in \Sigma\) and \(\nu = u_1 + \cdots + u_m\) be the restriction of the anti-canonical class of \(X\). Denote \(t := \sum_{i=1}^r t_i p_i\). Givental’s \(I\)-function is defined as the series

\[(6.11.1) \quad I_Y(t; z) = e^{t/z} \sum_{d \in \mathbb{N}(X)} e^{d t} \prod_{k=-\infty}^0 (u_j + k z) \prod_{j=1}^m \prod_{k=-\infty}^0 (u_j + k z) = 1,\]

which is a multi-valued map from \(\hat{U}\) (or a single-valued map from \(U\)) to the classical cohomology group \(H^*_\text{amb}(Y; \mathbb{C}[z^{-1}])\). The \(J\)-function is defined by

\[(6.11.2) \quad J_Y(\tau; z) = L_Y(\tau; z)^{-1}(1),\]

where \(L_Y(\tau, z)\) is the fundamental solution of the quantum differential equation defined explicitly by using the Gromov–Witten invariants as in \cite{irrill} Equation (2.3) with \(c\) set to 1. If we write

\[(6.11.3) \quad I_Y(t; z) = F(t) 1 + \frac{G(t)}{z} + O(z^{-2}),\]

then Givental’s mirror theorem \cite{giv98} states that

\[(6.11.4) \quad I_Y(t; z) = F(t) \cdot J_Y(\varsigma(t); z),\]

where the mirror map \(\varsigma : \hat{U} \to H^2_\text{amb}(Y; \mathbb{C})\) is defined by

\[(6.11.5) \quad \varsigma(t) = t^* \left( \frac{G(t)}{F(t)} \right).\]

The relation between \(\tau = \varsigma(t)\) and \(\sigma = \beta + \sqrt{-1} \omega\) is given by \(\tau = 2\pi \sqrt{-1} \sigma\), so that \(\text{Im}(\sigma) \gg 0\) corresponds to \(\exp(\tau) \sim 0\). The functions \(F(t)\) and \(G(t)\) satisfy the Picard–Fuchs equations, and give periods for the B-model VHS \((H_B, \nabla_B, \mathcal{F}_B, Q_B)\).

6.12. \((6.11.4)\) implies the existence of an isomorphism

\[(6.12.1) \quad \text{Mir}_Y : \varsigma^*(H_A, \nabla_A, \mathcal{F}_A, Q_A) \cong (H_B, \nabla_B, \mathcal{F}_B, Q_B)\]

of variations of polarized Hodge structures, which sends \(F(t) 1\) on the left-hand side to

\[(6.12.2) \quad \Omega := \text{Res} \left( \frac{1}{f} \, d\bar{y}_1 \wedge \cdots \wedge d\bar{y}_n \right)\]
on the right-hand side. A stronger statement, which gives an isomorphism of the \( \hat{\Gamma} \)-integral structure on the A-side and the natural integral structure on the B-side, is proved in [Iri11, Theorem 6.9].

7. QUASIMAP CORRELATION FUNCTIONS FOR ANTI-CANONICAL HYPERSURFACES IN TORIC VARIETIES

7.1. For \( d \in \text{Eff}(X) \) and \( i \in \{1, \ldots, m\} \), we set

\[
(7.1.1) \quad k_i := \begin{cases} \langle u_i, d \rangle & \langle u_i, d \rangle \geq 0, \\ -1 & \langle u_i, d \rangle < 0. \end{cases}
\]

and define the quasimap space of degree \( d \) by

\[
(7.1.2) \quad Q(X; d) := \left( \prod_{i=1}^{m} \mathbb{A}_i^{k_i+1} \right) /_\theta \mathbb{L}
\]

with \((6.3.7)\) in mind. An argument parallel to that in Section 5.1 shows that \( Q(X; d) \) is a compactification of the space of holomorphic maps \( \mathbb{P}^1 \to X \) of degree \( d \). Later in Section 10.4, we will introduce the moduli spaces \( Q_P(d) \) in [Iri11, Theorem 6.9].

8. TORIC RESIDUE MIRROR SYMMETRY

8.1. Let \( \hat{G} = (\hat{G}_0, \ldots, \hat{G}_n) \) be a regular sequence in \( S_\Delta \). If we set \( I_G := I_\Delta / (\hat{G}_0, \ldots, \hat{G}_n) I_\Delta \), then the graded piece \( I_G^{n+1} \) is one-dimensional and spanned by \( J_G := \det (\hat{g}_{ij})_{i,j=0}^{n} \). The toric residue [Cox96] is the map \( \text{Res}_G : I_\Delta^{n+1} \to \mathbb{C} \) sending \((\hat{G}_0, \ldots, \hat{G}_n) I_\Delta \) to zero and \( J_G \) to the normalized volume \( \text{vol}(\hat{\Delta}) \), i.e., \( n! \) times the Euclidean volume of \( \hat{\Delta} \). For \( \alpha \in \mathbb{L}^{\text{reg}} \), we define \( \hat{F}_\alpha \) as in \((6.5.2)\) and write \( \text{Res}_{\hat{F}_\alpha} := \text{Res}_{\hat{F}_\alpha} \). Theorem 8.2 below is introduced in [BM02, Conjecture 4.6] and proved in [SV04, Bor05].
Theorem 8.2. For any homogeneous polynomial \( P(\alpha_1, \ldots, \alpha_m) \in \mathbb{C}[\alpha_1, \ldots, \alpha_m] \) of degree \( n \), the generating function \((8.1.5)\) gives the Laurent expansion of the toric residue
\[
\langle P(u_1, \ldots, u_m) \rangle_{X,Y} = (-1)^n \text{Res}_{\alpha} (\tilde{y}_0^{n+1} P(\alpha_1 \tilde{y}^{b_1}, \ldots, \alpha_m \tilde{y}^{b_m})) \]
around the large radius limit point associated with the fan \( \Sigma \).

[BM02 Conjecture 4.6] is generalized to toric complete intersections in [BM03 Conjecture 4.6] and proved in [Kar05, SV06].

8.3. The family \( \varphi : \tilde{Y} \to \mathbb{L}^{reg} \) of Calabi–Yau manifolds comes with the holomorphic volume form
\[
\Omega := \text{Res} \left( \frac{1}{f_\alpha} \frac{d\tilde{y}_1}{\tilde{y}_1} \wedge \cdots \wedge \frac{d\tilde{y}_n}{\tilde{y}_n} \right) \in H^0(\mathcal{H}_B).
\]
For a homogeneous polynomial \( Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m] \) of degree \( n - 1 \), the \( Q \)-Yukawa \((n - 1)\)-point function is defined in [BM02 Definition 9.1] by
\[
Y_Q(\alpha) := (-1)^{(n-1)(n-2)/2} \left( \frac{1}{(2\pi)^{n-1}} \right) \int_{Y_\alpha} \Omega \wedge Q \left( \alpha_1 \frac{\partial}{\partial \alpha_1}, \ldots, \alpha_m \frac{\partial}{\partial \alpha_m} \right) \Omega,
\]
where the differential operators \( \alpha_1 \frac{\partial}{\partial \alpha_1}, \ldots, \alpha_m \frac{\partial}{\partial \alpha_m} \) act on \( \mathcal{H}_B \) by the Gauss–Manin connection.

8.4. For \( Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m] \), we set
\[
P(\alpha_1, \ldots, \alpha_m) := (\alpha_1 + \cdots + \alpha_m)Q(\alpha_1, \ldots, \alpha_m) \in \mathbb{Q}[\alpha_1, \ldots, \alpha_m].
\]
By [BM02 Theorem 9.7], which is attributed to [Mav00], one has an equality
\[
Y_Q(\alpha) = (-1)^n \text{Res}_{\alpha} (\tilde{y}_0^n P(\alpha_1 \tilde{y}^{b_1}, \ldots, \alpha_m \tilde{y}^{b_m}))
\]
of the Yukawa \((n - 1)\)-point function and the toric residue.

8.5. Assume that the unstable locus of the \( \mathbb{L} \)-action on \( \mathbb{A}^m \) with respect to \( \theta \) has codimension strictly greater than 1. Then one has \( H^2(X_S) = \text{Pic}(X_S) = \text{Pic}^1(\mathbb{A}^m) \) so that the class \( p_i \) corresponds to a one-dimensional representation \( C_{p_i} \) of \( \mathbb{L} \). By abuse of notation, we let \( p_i \) denote the \( \mathbb{G}_m \)-equivariant Euler class of the pull-back of the line bundle \([\mathbb{A}^m \times C_{p_i}/\mathbb{G}_m] \) by the evaluation map \( \text{ev}_0 : X_d \to [\mathbb{A}^m/\mathbb{G}_m] \) at 0 \( \in \mathbb{P}^1 \). Denote \( v := \sum_{i=1}^m u_i \).

If we set
\[
\Phi(t, \tau; z) := \sum_{d \in \text{Eff}(X)} e^{\tau d} \int_{Q(\mathbb{X}, d)} e^{(t-\tau)/2} \Phi_d v,
\]
then for any polynomial \( R(t_1, \ldots, t_r) \in \mathbb{Q}[t_1, \ldots, t_r] \), one has
\[
\left. R \left( z \frac{\partial}{\partial t_1}, \ldots, z \frac{\partial}{\partial t_r} \right) \Phi(t, \tau; z) \right|_{\tau=t} = \sum_{d \in \text{Eff}(X)} e^{t d} \int_{Q(\mathbb{X}, d)} R(p_1, \ldots, p_r) \Phi_d v.
\]
In addition, one has
\[
\Phi(t, \tau; z) = \int_{Y} I(t; -z) \cup I(\tau; z),
\]
by [Giv98 Proposition 6.2]. This is the toric hypersurface version of \((8.12.2)\). Note that \( I(t; 1) \) is convergent for large enough \(- \text{Re} \, t\) by ratio test on the series \((6.11.1)\) without the prefactor. By specializing to \( z = 1 \) and using the definition of \( Q_\Lambda \), one obtains
\[
\Phi(t, \tau; 1) = Q_\Lambda (I(t; 1), I(\tau; 1)).
\]
By combining (8.5.4) with (6.11.4), one obtains
\begin{equation}
\Phi(t, \tau; 1) = Q_A \left( L^{-1}(t; 1)F(t)1, L^{-1}(\tau; 1)F(\tau)1 \right).
\end{equation}

Since \( L \) is the fundamental solution for the flat connection \( \nabla_B \), the function \( \Phi(t, \tau; 1) \) is obtained by parallel-transporting \( F(t)1 \in (H_B)_t \) and \( F(\tau)1 \in (H_B)_\tau \) to the fiber at the same point and taking the pairing \( Q_B \) at that point (the result does not depend on the choice of the point since \( Q_B \) is \( \nabla_B \)-parallel). By sending (8.5.5) by (6.12.1), one obtains
\begin{equation}
(2\pi)^{-n} \int_I \int_{\mathbb{Y}} (I(t; 1)I(\tau; 1) = (-1)^{(n-1)(n-2)/2} \int_{\mathbb{Y}} \Omega_t \wedge \Omega_{\tau}.
\end{equation}

Assume that \( P(\alpha_1, \ldots, \alpha_m) = (\alpha_1 + \cdots + \alpha_m)Q(\alpha_1, \ldots, \alpha_m) \) for a polynomial \( Q \) and take
\[ R(t_1, \ldots, t_r) := Q(\sum_{i=1}^{r} a_i t_{i_1}, \ldots, \sum_{i=1}^{r} a_i t_{i_r}) \]
where \( a_{i,j} \) are integers uniquely satisfying \( \chi_j = \sum_{i=1}^{r} a_{i,j}p_i \). By differentiating (8.5.6) by \( R(t_1, \ldots, t_r) \) and setting \( \tau = t \), we obtain toric residue mirror symmetry for polynomials of the form \( P(\alpha_1, \ldots, \alpha_m) = (\alpha_1 + \cdots + \alpha_m)Q(\alpha_1, \ldots, \alpha_m) \).

9. Martin’s formula

9.1. We use the same notations \( G, T, W, \) and \( \Delta \) for a reductive algebraic group, a maximal torus, the Weyl group, and the set of roots as in Section 2. Let \( W \) be an affine scheme with \( G \)-action, and fix a character \( \theta \) of \( G \). We write the line bundle on \( W//T \) associated with \( \alpha \in \Delta \) as \( L_{\alpha} \), and set
\begin{equation}
\epsilon := \prod_{\alpha \in \Delta} c_1(L_{\alpha}) \in H^{2|\Delta|}(W//T; \mathbb{Z}).
\end{equation}

We write the natural projection and inclusion as and say that \( \tilde{a} \in H^*(W//T) \) is a lift of \( a \in H^*(W//G) \) if \( \pi* a = \epsilon \tilde{a} \).

**Theorem 9.2** (Martin [Mar]). If \( \tilde{a} \) is a lift of \( a \), then one has
\begin{equation}
\int_{W//G} a = \frac{1}{|W|} \int_{W//T} \tilde{a} \cup \epsilon.
\end{equation}

9.3. Let \( \text{Mat}(r, n) \cong \mathbb{A}^{r \times n} \) be the space of \( n \times r \) matrices, which is considered as the space of linear maps from an \( r \)-dimensional vector space to an \( n \)-dimensional vector space. It has a natural action of \( \text{GL}_r \), and the GIT quotient \( \text{Gr}(r, n) := \text{Mat}(r, n)//\text{GL}_r \) is the Grassmannian of \( r \)-spaces in an \( n \)-space.

9.4. When \( W = \text{Mat}(r, n) \) and \( G = \text{GL}_r \), one has
\begin{equation}
W//G \cong \text{Gr}(r, n),
\end{equation}
\begin{equation}
W//T \cong (\mathbb{P}^{n-1})^r
\end{equation}
and
\begin{equation}
H^*(\text{Gr}(r, n)) \cong \mathbb{C}[\sigma_1, \ldots, \sigma_r]/(h_{n-r+1}, \ldots, h_n),
\end{equation}
\begin{equation}
H^*((\mathbb{P}^{n-1})^r) \cong \mathbb{C}[x_1, \ldots, x_r]/(x_1^n, \ldots, x_r^n),
\end{equation}
where \( \sigma_i = \sigma_i(x_1, \ldots, x_r) \in \mathbb{C}[x_1, \ldots, x_r]^{S_r} \) are elementary symmetric functions and \( h_i = h_i(x_1, \ldots, x_r) \in \mathbb{C}[x_1, \ldots, x_r]^{S_r} = \mathbb{C}[\sigma_1, \ldots, \sigma_r] \) are complete symmetric functions.
martin’s formula in this case gives

\[(9.4.5)\]
\[
\int_{\text{Gr}(r,n)} P(x_1, \ldots, x_r) = \frac{1}{r!} \int_{(\mathbb{P}^{n-1})^r} \prod_{i \neq j} (x_i - x_j) P(x_1, \ldots, x_r)
\]

\[(9.4.6)\]
\[
= \frac{(-1)^{r(r-1)/2}}{r!} \int_{(\mathbb{P}^{n-1})^r} \Delta^2 \cup P(x_1, \ldots, x_r)
\]

for any \(P(x_1, \ldots, x_r) \in \mathbb{C}[x_1, \ldots, x_r]^{\sigma_r}\) where \(\Delta := \prod_{1 \leq i < j \leq r} (x_i - x_j)\).

9.5. The equivariant cohomology ring of \(\text{Gr}(r, n)\) with respect to the natural action of the diagonal maximal abelian subgroup \(H \subset \text{GL}_n\) is presented as

\[(9.5.1)\]
\[
H^*_H(\text{Gr}(r, n); \mathbb{C}) \cong \mathbb{C}[\sigma_1, \ldots, \sigma_r, \lambda_1, \ldots, \lambda_n] / (h_{n-r+1}(\sigma, \lambda), \ldots, h_n(\sigma, \lambda)),
\]

where \(h_i\) is the degree \(2i\) part of \(c^H(S)c^H(Q) - \prod_{i=1}^n (1 + \lambda_i)\). Here \(S\) and \(Q\) are the tautological bundle and the universal quotient bundle respectively, and \(c^H(\cdot)\) stands for the \(H\)-equivariant total Chern class. Note that \(\sigma_i := c_i^H(S)\) is the elementary symmetric function of the \(H\)-equivariant Chern roots \(x_1, \ldots, x_r\) of \(S\), and \(c_i^H(Q)\) for \(i = 1, \ldots, n-r\) are expressed in terms of \(\sigma_1, \ldots, \sigma_r\) and \(\lambda_1, \ldots, \lambda_n\) by the condition \(h_1 = \cdots = h_{n-r} = 0\).

Martin’s formula gives

\[(9.5.2)\]
\[
\int_{\text{Gr}(r,n)} P(\sigma_1, \ldots, \sigma_r) = \sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \text{Res}_{x=(\lambda_{i_1}, \ldots, \lambda_{i_r})} P(\sigma_1, \ldots, \sigma_r) \prod_{i \neq j} (x_i - x_j) \frac{dx_1 \wedge \cdots \wedge dx_r}{\prod_{i=1}^r \prod_{j=1}^n (x_i - \lambda_j)}.
\]

10. Quasimap Spaces for GIT Quotients

10.1. Let \(G\) be a reductive algebraic group acting on an affine variety \(W\) and fix a character \(\theta\) of \(G\). In this paper, we will always assume the following:

1. Semi-stability implies stability.
2. The semi-stable locus \(W^{ss}\) is smooth and non-empty.
3. The \(G\)-action on \(W^{ss}\) is free (however, see [CCFK15] for allowing finite non-trivial stabilizers).
4. The codimension of the unstable locus \(W \setminus W^{ss}\) is greater than one.

The GIT quotient is defined by \(W/G := W^{ss}/G\), which is an open substack of \([W/G]\).

10.2. A map \(u: \mathbb{P}^1 \to [W/G]\) to the quotient stack \([W/G]\) is pair \((P, \tilde{u})\) of a principal \(G\)-bundle \(P \to \mathbb{P}^1\) and a \(G\)-equivariant map \(\tilde{u}: P \to W\). It is called a quasimap if the generic point of \(\mathbb{P}^1\) is mapped to \(W/G \subset [W/G]\). A point in the inverse image of the unstable locus will be called a base point.

10.3. For a quasimap \(u: \mathbb{P}^1 \to [W/G]\) and a \(G\)-equivariant line bundle \(L\) on \(W\), the pull-back \(\tilde{u}^*L\) is a \(G\)-equivariant line bundle on \(P\), which descends to a line bundle \(u^*L\) on \(\mathbb{P}^1\). The degree of a quasimap \(u: \mathbb{P}^1 \to [W/G]\) is the map \(d: \text{Pic}^G W \to \mathbb{Z}\) sending \(L \in \text{Pic}^G W\) to \(\text{deg} u^* L\).
10.4. An isomorphism of quasimaps \( u = (P, \overline{u}) \) and \( u' = (P', \overline{u}') \) is an isomorphism \( \varphi: P \to P' \) of principal \( G \)-bundles such that \( \overline{u} = \overline{u}' \circ \varphi \). By [CFKM14, Theorem 7.1.6], the moduli functor for quasimaps of degree \( d \) is representable by a Deligne-Mumford stack, which will be denoted by \( \mathcal{Q}(W/G; d) \). This stack is denoted by \( \text{Qmap}_{0,0}(W/G, d; \mathbb{P}^1) \) in [CFKM14, §7.2] and \( \mathcal{Q}G_{0,0,d}(W/G) \) in [CFK14] Section 4.1. Note that \( \mathcal{Q}(W/G) \) depends not only on \( W/G \) and \( d \) but also on \( W \), \( G \), and \( \theta \).

10.5. Let \( \mathcal{Q}_*(W/G; d) \subset \mathcal{Q}(W/G; d) \) be the substack parametrizing quasimaps such that \( u|_{\mathbb{P}^1 \setminus \{0\}} \) is a constant map to \( W/G \). This implies that \( 0 \in \mathbb{P}^1 \) is a base point of length \( d(\theta) \) by [CFKM14, Lemma 7.1.2]. This stack is denoted by \( \mathcal{Q}_{0,0,*}(W/G, d) \) in [CFK14] Section 4.1]. There is a natural map \( \text{ev}: \mathcal{Q}_*(W/G; d) \to W/G \), called the evaluation map, which sends \( u \in \mathcal{Q}_*(W/G; d) \) to \( u(\infty) \in W/G \).

10.6. There is a natural \( \mathbb{G}_m \)-action on \( \mathcal{Q}(W/G; d) \) coming from the standard \( \mathbb{G}_m \)-action on \( \mathbb{P}^1 \). As described in [CFK14 Section 4.1], the fixed locus of this action is identified with the coproduct

\[
\prod_{d_1 + d_2 = d} \mathcal{Q}_*(W/G; d_1) \times_{W/G} \mathcal{Q}_*(W/G; d_2)
\]

of fiber products with respect to the evaluation map.

10.7. If \( W \) has at worst lci singularity, then \( \mathcal{Q}(W/G; d) \) has a canonical perfect obstruction theory, which allows one to define the virtual fundamental cycle. The canonical perfect obstruction theory is \( (\mathbb{R}\pi_* \text{ev}^* T_{W/G})^\vee \), where \( T_{W/G} \) is the tangent complex of \( W/G \), \( \text{ev}: \mathcal{Q}(W/G; d) \times \mathbb{P}^1 \to W/G \) is the evaluation map, and \( \pi: \mathcal{Q}(W/G; d) \times \mathbb{P}^1 \to \mathcal{Q}(W/G; d) \) is the first projection; see Theorem 7.2.2 of [CFK14] or Section 5. The virtual fundamental cycle is an element of the Chow group of \( \mathcal{Q}(W/G; d) \) whose degree is given by the virtual dimension

\[
\text{virt.dim} \mathcal{Q}(W/G; d) = \langle d, \det T_W \rangle + \text{dim} W/G.
\]

10.8. Since the stack \( \mathcal{Q}_*(W/G; d) \) is the union of connected components of the fixed locus of the \( \mathbb{G}_m \)-action, it has a perfect obstruction theory inherited from \( \mathcal{Q}(W/G; d) \). The virtual push-forward

\[
\text{ev}_*^\text{virt}(-) := \text{PD} \left( \text{ev}_* \left( (-) \cap [\mathcal{Q}_*(W/G; d)]^{\text{virt}} \right) \right)
\]

along the evaluation map \( \text{ev}: \mathcal{Q}_*(W/G; d) \to W/G \) allows one to define the \( I \)-function

\[
I(t; z) := e^{p t/z} \sum_{d \in \text{Eff}(W/G)} e^{d t} I_d
\]

by

\[
I_d := \text{ev}_*^\text{virt} \left( \frac{1}{\text{Eul}_{\mathbb{G}_m}^{\text{virt}} \left( \frac{N_{\text{virt}}(\mathcal{Q}_*(W/G; d) / \mathcal{Q}(W/G; d))}{\mathcal{Q}(W/G; d)} \right)} \right),
\]

where the denominator is the \( \mathbb{G}_m \)-equivariant Euler class of the virtual normal bundle. Here \( p \) is a basis of \( H^2(W/G) \), and \( t \) is the coordinate of \( H^2(W/G) \) corresponding to \( p \).

An \( H \)-action on \( W \) commuting with the \( G \)-action induces an \( H \)-action on \( \mathcal{Q}_*(W/G; d) \), which allows one to define the \( H \)-equivariant \( I \)-function of \( W/G \).
10.11. By taking \( \mathcal{P}' \) to be the fiber over a fixed point of the natural \( \mathbb{G}_m \)-action on the domain curve \( \mathbb{P}^1 \), one can define \( \mathbb{G}_m \)-equivariant quasimap invariants \( \langle P \rangle_{\mathbb{G}_m}^{\mathbb{G}_m} \). If \( W \) has an action of an algebraic torus \( H \) commuting with the action of \( G \), then one can define \( H \times \mathbb{G}_m \)-equivariant quasimap invariants \( \langle P \rangle_{\mathbb{G}_m}^{H \times \mathbb{G}_m} \).

11. Quasimap spaces for Grassmannians

11.1. The quasimap space \( Q(\text{Gr}(r, n); d) \) classifies pairs \((P, u)\) of a principal \( GL_r \)-bundle \( P \) and a \( GL_r \)-equivariant map \( u \). The choice of a principal \( GL_r \)-bundle \( P \) is equivalent to the choice of a vector bundle \( S \) of rank \( r \), and the choice of a \( GL_r \)-equivariant map \( u \) is equivalent to the choice of a map \( S \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \), which is a sheaf injection since the generic point must go to the semi-stable locus (but not necessarily a morphism of vector bundles).

The choice of a sheaf injection \( S \to \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \) is equivalent to the choice of a surjection \( \mathcal{O}_{\mathbb{P}^1}^{\oplus n} \to Q \), where \( Q \) is a coherent sheaf whose Hilbert polynomial is \( d + (n - r)(t + 1) \). This is the same as the Hilbert polynomial of a locally free sheaf of rank \( n - r \) and degree \( d \), and one has an isomorphism

\[
Q(\text{Gr}(r, n); d) \cong \text{Quot}_{\mathbb{P}^1, d}(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, n - r).
\]

11.2. It is shown in [BCFK05] Lemma 1.2] that the subspace \( Q_*(\text{Gr}(r, n); d) \) of \( Q(\text{Gr}(r, n); d) \) is decomposed into connected components as

\[
Q_*(\text{Gr}(r, n); d) = \bigsqcup_{|d| = d} Q_*(\text{Gr}(r, n); d),
\]

where \( d = (d_1, \ldots, d_r) \) runs over elements of \( \mathbb{N}^r \) satisfying \(|d| := d_1 + \cdots + d_r = d, \ d_1 \leq d_2 \leq \ldots \leq d_r \) and each connected component is isomorphic to the partial flag manifold

\[
Q_*(\text{Gr}(r, n); d) \cong \text{Fl}(m_1, \ldots, m_k, r, n),
\]

where the \( \text{Fl}(m_1, \ldots, m_k, r, n) \) denotes the partial flag manifold of degree \( n \) and dimension \( r \) with \( m_i \) copies of \( \mathbb{C}^1 \) in the \( i \)-th position.
where \( 1 \leq m_1 < m_2 < \cdots < m_k = r \) denote the jumping indices;

\[
0 \leq d_1 = \cdots = d_{m_1} < d_{m_1 + 1} = \cdots = d_{m_2} < \cdots.
\]

Let \( x_1, \ldots, x_r \) be the Chern roots of the dual of the universal subbundle on \( \text{Gr}(r, n) \). We also define \( |x| := \sum_{i=1}^r x_i \) and \( |d| := \sum_{i=1}^r d_i \) for \( d = (d_1, \ldots, d_r) \). The \( I \)-function can be computed by localization as

\[
I_{\text{Gr}(r, n)}(t; z) = \sum_{d \in \mathbb{N}^r} (-1)^{|d|} e^{(|d|+|x|)/z} t^{|d|} I_d(z) \tag{11.2.3}
\]

where

\[
I_d(z) = \prod_{1 \leq i < j \leq r} (x_i - x_j + (d_i - d_j)z) \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{j=1}^n \prod_{i=1}^{d_i} (x_i + iz). \tag{11.2.4}
\]

As shown in [BCFK05, page 109], the \( I \)-function and the \( J \)-function agrees for \( \text{Gr}(r, n) \) just as in the case of projective spaces.

11.3. The Hori–Vafa conjecture [HV] proved in [BCFK05] shows that the \( I \)-functions of \((\mathbb{P}^{n-1})^r \) and \( \text{Gr}(r, n) \) are related by

\[
I_{\text{Gr}(r, n)}(t; z) = e^{-\sigma_1 (r-1)\sqrt{-1}/z} \frac{\bar{D} I_{(\mathbb{P}^{n-1})^r}(t; \bar{z})}{\Delta} \bigg|_{t_i = t + (r-1)\sqrt{-1}} \tag{11.3.1}
\]

where

\[
D := \prod_{1 \leq i < j \leq r} \left( z \frac{\partial}{\partial t_i} - z \frac{\partial}{\partial t_j} \right). \tag{11.3.2}
\]

11.4. As shown in [BCFK05], the equivariant \( I \)-function with respect to the natural action of \( H = (\mathbb{G}_m)^n \) on \( \text{Mat}(r, n) \) is given by

\[
I^H_{\text{Gr}(r, n)}(t; z) = e^{\sigma_1 / z} \sum_{d \in \mathbb{N}^r} (-1)^{|d|} e^{(|d|+|x|)/z} \prod_{1 \leq i < j \leq r} (x_i - x_j + (d_i - d_j)z) \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{j=1}^n \prod_{i=1}^{d_i} (x_i - \lambda_j + iz), \tag{11.4.1}
\]

and the factorization gives

\[
\sum_{d=0}^{\infty} e^{d\tau} \left< e^{(t-\tau)\sigma_1 / z} \right>_{\text{Gr}(r, n), d}^H = \int_{\text{Gr}(r, n)} I^H_{\text{Gr}(r, n)}(t; z) \text{d} \nu G = \int_{\text{Gr}(r, n)} I^H_{\text{Gr}(r, n)}(t; z) \text{d} \nu G \tag{11.4.2}
\]

Here \( \sigma_1 = \sum_{i=1}^r x_i \) is the \( H \)-equivariant first Chern class of the vector bundle

\[
S^\vee = (\text{Mat}(r, n) \times \mathbb{C}^r) \sslash G \tag{11.4.3}
\]

on \( \text{Gr}(r, n) \), where the \( G \)-action on \( \mathbb{C}^r \) is the defining representation.

11.5. Let \( V \) be an equivariant vector bundle on \( \text{Gr}(r, n) \) associated with a representation \( V \) of \( \text{GL}_r \). If \( V \) is globally generated and \( \det V \cong \omega_{\text{Gr}(r, n)}^r \), then the zero \( Y := s^{-1}(0) \) of a general section \( s \in H^0(V) \) is a smooth Calabi–Yau manifold by a generalization of the theorem of Bertini [Muk92, Theorem 1.10].
11.6. Let \([\text{Mat}(r, n) / \text{GL}_r]\) be the quotient stack containing \(\text{Gr}(r, n)\) as an open substack. The complete intersection \(Y \subset \text{Gr}(r, n)\) is an open substack of \(\mathcal{Y} := [Z / \text{GL}_r]\), where \(Z \subset \text{Mat}(r, n)\) is the zero of the map \(\bar{s}: \text{Mat}(r, n) \to V\) underlying \(s\). Indeed, \(Y\) has a GIT quotient description \(Y = Z / \text{GL}_r\), which allows us to define \(Q(Y; d)\) and its virtual fundamental cycle as in Section 11. Let \(S^r_Y\) be the vector bundle on \(\mathcal{Y}\) associated with the defining representation of \(\text{GL}_r\). Any point \(p \in \mathbb{P}^1\) determines a map \(\text{ev}_p: Q(Y; d) \to \mathcal{Y}\) sending \(f: \mathbb{P}^1 \to \mathcal{Y}\) to \(f(p) \in \mathcal{Y}\), and the Chern classes

\[
\sigma_i := c_i (\text{ev}^*_p S^r_Y), \quad i = 1, \ldots, r
\]

does not depend on the choice of \(p \in \mathbb{P}^1\). For \(P(\sigma_1, \ldots, \sigma_r) \in \mathbb{C}[\sigma_1, \ldots, \sigma_r]\), we set

\[
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_Y := \int_{(Q(Y; d))^\text{virt}} P(\sigma_1, \ldots, \sigma_r)
\]

and

\[
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_Y := \sum_{d=0}^{\infty} e^{dt} \langle P(\sigma_1, \ldots, \sigma_r) \rangle_Y.
\]

11.7. The equivariant \(I\)-function of \(Y\) is given by

\[
I^H_Y(t; z) = \sum_{d \in \mathbb{N}^r} e^{(d+\chi(z)) \cdot t} I_d(t; z) \bigg|_{t_i = t + (r-1)\pi^\sqrt{-1}},
\]

where

\[
I_d(t; z) := \frac{\prod_{\delta \in \Delta(V)} \prod_{i=1}^{\delta, d} (\langle \delta, x \rangle + iz) \prod_{1 \leq i < j \leq r} (x_i - x_j + (d_i - d_j)z)}{\prod_{1 \leq i < j \leq r}(x_i - x_j) \prod_{i=1}^{n} \prod_{j=1}^{d_i} (x_i - \lambda_j + iz)}
\]

where \(\Delta(V)\) denotes the set of weights of \(V\) and \(\langle \delta, x \rangle\) denotes the first Chern class associated to the weight \(\delta\) (expressed in terms of the fundamental weights \(x_1, \ldots, x_r\) of the maximal diagonal torus of \(G\)). Localization with respect to the natural \(\mathbb{G}_m\)-action on \(Q(\text{Gr}(r, n); d)\) shows

\[
\langle e^{(t-\tau)\sigma_1/z} \rangle_{\mathcal{Y}}^{H \times \mathbb{G}_m} = \int_Y^{H} I^H(t; z) \cup I^H(\tau; -z)
\]

just as in (8.5.3).

12. RESIDUE MIRROR FOR GRASSMANNIANS

12.1. We define the abelianized quasimap space for \(\text{Gr}(r, n)\) by

\[
Q^{ab}(\text{Gr}(r, n); d) := \prod_{|d| = d} Q^{ab}(\text{Gr}(r, n); d),
\]

\[
Q^{ab}(\text{Gr}(r, n); d) := Q(\mathbb{P}^{n-1}; d_1) \times \cdots \times Q(\mathbb{P}^{n-1}; d_r),
\]

where \(d\) runs over \(\bar{d} = (d_1, \ldots, d_r) \in \mathbb{N}^r\) such that \(|\bar{d}| := d_1 + \cdots + d_r = d\). An abelianized quasimap

\[
\varphi(z_1, z_2) = \left( \varphi_{i1}(z_1, z_2), \ldots, \varphi_{in}(z_1, z_2) \right) \in Q(\mathbb{P}^{n-1}; d_i)
\]

where
defines a genuine map of degree $d$ if the matrix $(\varphi_{ij}(z_1, z_2))_{i,j}$ has rank $r$ for any $(z_1, z_2) \neq 0$. For $P(\sigma_1, \ldots, \sigma_r) \in \mathbb{C}[\sigma_1, \ldots, \sigma_r]$, we set
\begin{equation}
(12.1.4)
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n), d}^{\text{ab}} := \frac{1}{r!} \int_{\mathbb{C}[\sigma_1, \ldots, \sigma_r]} \prod_{i \neq j} \langle x_i - x_j \rangle P(\sigma_1(x_1, \ldots, x_r), \ldots, \sigma_r(x_1, \ldots, x_r)),
\end{equation}
\begin{equation}
(12.1.5)
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n), d}^{\text{ab}} := \sum_{|d| = d} \langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n), d}^{\text{ab}},
\end{equation}
\begin{equation}
(12.1.6)
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n), d}^{\text{ab}} := \sum_{d=0}^{\infty} (-1)^{(r-1)d} q^d \langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n), d}^{\text{ab}}.
\end{equation}
\begin{equation}
(12.2.1)
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{GLSM}} = \langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n), d}^{\text{ab}}.
\end{equation}

If we set $G := \text{GL}_r$ and $V := \text{Mat}(r, n)$, where $G$ acts naturally on $V$ and $\mathbb{G}_m$ acts trivially on $V$, then we have $Z_d^{\text{vec}}(x) = \prod_{i \neq j} (x_i - x_j)$ and $Z_d^{\text{mat}}(x) = \prod_{i=1}^{r} (x_i^{d_i-1})$, so that (2.2.4) gives the same result as (12.1.4).

(12.3) We write the ring homomorphism $\mathbb{C}[\sigma_1, \ldots, \sigma_r] \to \text{QH}(\text{Gr}(r,n))$ sending $\sigma_i \in \mathbb{C}[\sigma_1, \ldots, \sigma_r]$ to $\sigma_i \in H^*(\text{Gr}(r,n); \mathbb{C}) \cong \mathbb{C}[\sigma_1, \ldots, \sigma_r]/(h_{n-r+1}, \ldots, h_n)$ as $P(\sigma_1, \ldots, \sigma_r) \mapsto \hat{P}(\sigma_1, \ldots, \sigma_r)$ just as in the case of $\mathbb{P}^{n-1}$.

\textbf{Theorem 12.4.} For any $P(\sigma_1, \ldots, \sigma_r) \in \mathbb{C}[\sigma_1, \ldots, \sigma_r]$, one has
\begin{equation}
(12.4.1)
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n)}^{\text{ab}} = \int_{\text{Gr}(r,n)} \hat{P}(\sigma_1, \ldots, \sigma_r).
\end{equation}

\textbf{Proof.} It follows from (5.7.2) that
\begin{equation}
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{\text{Gr}(r,n)}^{\text{ab}} = \frac{1}{r!} \sum_{d_1, \ldots, d_r = 0} \int \left( (-1)^{r-1} q^{d_1 + \cdots + d_r} \text{Res} \prod_{i \neq j} (x_i - x_j) P(\sigma_1(x_1, \ldots, x_r), \ldots, \sigma_r(x_1, \ldots, x_r)) \frac{dx_1}{x_1^{d_1+1}} \cdots \frac{dx_r}{x_r^{d_r+1}} \right)
\end{equation}
\begin{equation}
= \frac{1}{r!} \text{Res} \prod_{i \neq j} (x_i - x_j) P(\sigma_1(x_1, \ldots, x_r), \ldots, \sigma_r(x_1, \ldots, x_r)) \frac{dx_1}{x_1^{r-1+1} + (-1)^{r-1} q} \cdots \frac{dx_r}{x_r^{r-1+1} + (-1)^{r-1} q}
\end{equation}
\begin{equation}
= \frac{1}{r! n^r} \sum_{x_i = (-1)^{r-1} q_i} \cdots \sum_{x_r = (-1)^{r-1} q_r} \prod_{i \neq j} (x_i - x_j) P(\sigma_1(x_1, \ldots, x_r), \ldots, \sigma_r(x_1, x_r))
\end{equation}
\begin{equation}
= \int_{\text{Gr}(r,n)} \hat{P}(\sigma_1, \ldots, \sigma_r),
\end{equation}
where the last equality is the Vafa-Intriligator formula [ST97, Theorem 4.6].

(12.5) Theorem 12.4 is related to intersection theory on the moduli space of vector bundles on a Riemann surface through a theorem of Witten [Wit95], which states the existence of a ring isomorphism $\text{QH}(\text{Gr}(r,n))/q - 1 \sim \text{R(U(r)))}_{n-r,n}$ from the quantum cohomology of $\text{Gr}(r,n)$ at $q = 1$ and the Verlinde algebra of $\text{U(r)}$ at $\text{SU(r)}$ level $n - r$ and $\text{U(1)}$ level $n$.  

28
12.6. We define the $G_m$-equivariant correlator of $P(\sigma_1, \ldots, \sigma_r) \in \mathbb{C}[\sigma_1, \ldots, \sigma_r]$ by

\begin{equation}
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{Gr(r,n)}^{ab,G_m} := \sum_{d \in \mathbb{N}^r} e^{dt} \langle P(\sigma_1, \ldots, \sigma_r) \rangle_{Gr(r,n),d}^{ab,G_m} \bigg|_{t_i = t + (r-1)\pi \sqrt{-1}}
\end{equation}

where

\begin{equation}
\langle P(\sigma_1, \ldots, \sigma_r) \rangle_{Gr(r,n),d}^{ab,G_m} := \int_{Q(\mathbb{P}^{n-1})^r;d} \prod_{1 \leq i < j \leq n} \left( x_i - x_j \right) \left( x_j - x_i + (d_j - d_i)z \right) P(\sigma_1(x_1, \ldots, x_r), \ldots, \sigma_r(x_1, \ldots, x_r)).
\end{equation}

Since $Q((\mathbb{P}^{n-1})^r;d)^{G_m} = \prod_{i=1}^r Q(\mathbb{P}^{n-1};d_i)^{G_m}$ under $Q((\mathbb{P}^{n-1})^r;d) = \prod_{i=1}^r Q(\mathbb{P}^{n-1};d_i)$, we have a straightforward generalization of (12.6.2):

\begin{equation}
\sum_{d \in \mathbb{N}^r} e^{dt} \left( \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{1 \leq i < j \leq r} (x_j - x_i + (d_j - d_i)z) \cdot e^{(t-\tau)z/x} \right)_{\mathbb{P}^{n-1}} \int_{Q(\mathbb{P}^{n-1})^r} I_{\mathbb{P}^{n-1}}(t; z) \cup I_{\mathbb{P}^{n-1}}(\tau; -z),
\end{equation}

By acting $D_1 := \prod_{1 \leq i < j \leq r} (z\partial_{x_i} - z\partial_{x_j})$ and $-D_r := \prod_{1 \leq i < j \leq r} (-z\partial_{x_n} + z\partial_{x_r})$ on both sides of (12.6.3) one obtains

\begin{equation}
\sum_{d \in \mathbb{N}^r} e^{dt} \left( \prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{1 \leq i < j \leq r} ((x_j + d_jz) - (x_i + d_iz)) \cdot e^{(t-\tau)z/x} \right)_{\mathbb{P}^{n-1}}
\end{equation}

on the left hand side and

\begin{equation}
\int_{\mathbb{P}^{n-1}} D_1 I_{\mathbb{P}^{n-1}}(t; z) \cup (-D_r) I_{\mathbb{P}^{n-1}}(\tau; -z)
\end{equation}

on the right hand side. By setting $t_i = t + (r-1)\pi \sqrt{-1}$, $\tau_i = \tau + (r-1)\pi \sqrt{-1}$ and using (11.3.1), one obtains

\begin{equation}
\langle e^{(t-\tau)\sigma_1/z} \rangle_{Gr(r,n)}^{ab,G_m} = \frac{1}{r!} \int_{\mathbb{P}^{n-1}} \Delta \cup I_{Gr(r,n)}(t; z) \cup \Delta \cup I_{Gr(r,n)}(\tau; -z)
\end{equation}

where the last equality is Martin’s formula (9.5.2). On the other hand, localization with respect to the natural $G_m$-action on the domain curve gives the factorization

\begin{equation}
\langle e^{(t-\tau)\sigma_1/z} \rangle_{Gr(r,n)}^{G_m} = \int_{Gr(r,n)} I_{Gr(r,n)}(t; z) \cup I_{Gr(r,n)}(\tau; -z).
\end{equation}

Together with (12.6.7), this gives the equality

\begin{equation}
\langle e^{(t-\tau)\sigma_1/z} \rangle_{Gr(r,n)}^{ab,G_m} = \langle e^{(t-\tau)\sigma_1/z} \rangle_{Gr(r,n)}^{G_m}
\end{equation}

of the abelianized correlator and the ordinary correlator.
For any \( P(x) \in \mathbb{C}[x_1, \ldots, x_r]^{S_r} \), the same argument gives
\[
\langle P(x)e^{(t-\tau)x/z}\rangle_{\text{Gr}(r,n)_{\mathbb{G}_m}}^{\text{ab,}\mathbb{G}_m} = \int_{\text{Gr}(r,n)} \left( \sum_{d \in \mathbb{N}^r} P(x + dz)I_{\text{Gr}(r,n), d}(t, z) \right) \cup \left( \sum_{d \in \mathbb{N}^r} I_{\text{Gr}(r,n), d}(\tau ; -z) \right)
\]
\[
= \langle P(x)e^{(t-\tau)x/z}\rangle_{\text{Gr}(r,n)_{\mathbb{G}_m}}^{\text{Gr}(r,n)_{\mathbb{G}_m}}
\]
where \( t_\ell = t + (r - 1)\pi \sqrt{-1} \) and \( \tau_\ell = \tau + (r - 1)\pi \sqrt{-1} \). By setting \( t = \tau \) in (12.6.10), one obtains
\[
\langle P(x)\rangle_{\text{Gr}(r,n)_{\mathbb{G}_m}}^{\text{ab,}\mathbb{G}_m} = \langle P(x)\rangle_{\text{Gr}(r,n)_{\mathbb{G}_m}}^{\text{Gr}(r,n)_{\mathbb{G}_m}}.
\]
Together with (12.2.1), this proves Conjecture (10.10) for Grassmannians.

12.7. Let \( Y \subset \text{Gr}(r, n) \) be the zero locus of a general section of a globally-generated vector bundle \( \mathcal{V} \) on \( \text{Gr}(r, n) \) associated with a representation \( V \) of \( GL_r \). We define the abelianized \( \mathbb{G}_m \)-equivariant Morrison-Plesser class of \( Y \) by
\[
\Phi_d^{ab,\mathbb{G}_m}(Y, z) := \prod_{\delta \in \Delta(V)} \prod_{\ell=1}^{(\delta, d)} (\delta, x) + l z.
\]
For \( P \in \mathbb{C}[\sigma_1, \ldots, \sigma_r] \), we set
\[
\langle P(\sigma_1, \ldots, \sigma_r)\rangle_{Y_{\mathbb{G}_m}}^{ab,\mathbb{G}_m} = \sum_{d \in \mathbb{N}^r} q^{|d|} \langle P(\sigma_1, \ldots, \sigma_r)\Phi_d^{ab,\mathbb{G}_m}(Y, z)\rangle_{\text{Gr}(r,n)_{\mathbb{G}_m}}^{ab,\mathbb{G}_m}
\]
where \( v := \prod_{\delta \in \Delta(V)} (\delta, x) \) is the Euler class of the normal bundle of \( Y \) in \( \text{Gr}(r, n) \). By the same reasoning as in Section [12.6] with the insertion of the abelianized Morrison-Plesser class, one obtains
\[
\langle P(\sigma_1, \ldots, \sigma_r)\rangle_{Y} = (-1)^{|\Delta(V)|}\langle P(\sigma_1, \ldots, \sigma_r)\rangle_{\text{GLSM}}
\]
Here, the identification between \( q \) and the Fayet–Illiopoulos parameter \( t' \) is given by
\[
q = (-1)^{\sum_{\delta \in \Delta(V)} (\delta, 1)} e^{t'}
\]
where \( 1 := (1, \ldots, 1) \in \mathbb{N}^r \).

12.8. As an example, consider the vector bundle of rank 3 on \( \text{Gr}(3, 5) = \text{Mat}(3, 5)/U(3) \) associated with the representation of \( U(3) \) determined by the Young diagram
\[
\lambda = \begin{array}{|c|c|c|c|}
\hline
& & & \\
\hline
\end{array}
\]
This vector bundle is the tensor product \( \wedge^2 Q(1) \) of the second exterior power \( \wedge^2 Q \) of the universal quotient bundle \( Q \) on \( \text{Gr}(2, 5) \cong \text{Gr}(3, 5) \) and the ample generator \( O(1) \) of the Picard group. One can immediately see from the Young diagram that the restriction of the representation of \( U(3) \) associated with \( \lambda \) to the diagonal maximal torus \( T \cong (\mathbb{G}_m)^3 \) is the direct sum \( \rho_{1,2,2} \oplus \rho_{2,1,2} \oplus \rho_{2,2,1} \). The associated line bundle on the abelian quotient \( (\mathbb{P}^4)^3 \) is given by \( O(1, 2, 2) \oplus O(2, 1, 2) \oplus O(2, 2, 1) \).

The complete intersection in \( \text{Gr}(3, 5) \) defined by \( \wedge^2 Q(1) \) is a Calabi–Yau 3-fold of Picard number 1, which will be denoted by \( Y \) henceforth. The Euler class of the normal bundle of \( Y \) is
\[
v := (x_1 + 2x_2 + 2x_3)(2x_1 + x_2 + 2x_3)(2x_1 + 2x_2 + x_3),
\]
the abelianized Morrison-Plesser class is

\[
(12.8.3) \quad \Phi^{ab}(Y; d) := (x_1 + 2x_2 + 2x_3)^{d_1 + 2d_2 + 2d_3} (2x_1 + 2x_2 + 2x_3)^{2d_1 + d_2 + 2d_3} (2x_1 + 2x_2 + x_3)^{2d_1 + 2d_2 + d_3},
\]

and the generating function for \(\sigma^3\) is

\[
(12.8.4) \quad \langle \sigma^3 \rangle^{ab}_Y = \frac{1}{6} \sum_{d_1=0}^{\infty} \sum_{d_2=0}^{\infty} \sum_{d_3=0}^{\infty} q^{d_1 + d_2 + d_3} \text{Res}(x_1 + x_2 + x_3)^3 (x_1 - x_2)^2(x_1 - x_3)^2(x_2 - x_3)^2 \Phi^{ab}(Y; d) \frac{dx_1}{x_1^{n(d_1+1)}} \wedge \frac{dx_2}{x_2^{n(d_2+1)}} \wedge \frac{dx_3}{x_3^{n(d_3+1)}}
\]

\[
(12.8.5) \quad = \frac{25(1 - q)}{(1 + q)(1 - 123q + q^2)}.
\]

This matches the Yukawa coupling of the mirror computed by Miura [Min13 §5.2].

12.9. When \(V\) is a direct sum of line bundles, the mirror of \(Y\) is constructed by toric degenerations [BCFKvS98, BCFKvS00]. It is an interesting problem to compare the generating function (11.6.3) with the Yukawa coupling of this mirror.

13. Bethe/gauge correspondence

13.1. Let \(V_1\) and \(W_1\) be Hermitian vector spaces of dimensions \(r\) and \(n\). The unitary group \(U(r)\) acts naturally on \(V_1\) and trivially on \(W_1\), inducing an action on \(T^* \text{Hom}(V_1, W_1) \cong \text{Hom}(V_1, W_1) \oplus \text{Hom}(W_1, V_1)\). The real and complex moment maps for this action are given by

\[
(13.1.1) \quad \mu_R : \text{Hom}(W_1, V_1) \oplus \text{Hom}(V_1, W_1) \to \text{End}(V_1), \quad (i_1, j_1) \mapsto \sqrt{-1} (i_1 i_1^* - j_1^* j_1),
\]

\[
(13.1.2) \quad \mu_C : \text{Hom}(W_1, V_1) \oplus \text{Hom}(V_1, W_1) \to \text{End}(V_1), \quad (i_1, j_1) \mapsto i_1 j_1.
\]

If \((i_1, j_1) \in \mu^{-1}_R(\zeta \sqrt{-1} \text{id}_{V_1})\) for \(\zeta < 0\), then \(j_1\) is injective. If \((i_1, j_1) \in \mu^{-1}_C(0)\), then \(i_1\) descends to a map \(W_1/\text{Im} j_1 \to V_1\). It follows that the hyperKähler quotient is isomorphic to \(T^* \text{Gr}(r, n)\):

\[
(13.1.3) \quad (\zeta^{-1}_R (\zeta \sqrt{-1} \text{id}_{V_1}) \cap \mu^{-1}_C(0)) \subseteq U(r) \cong T^* \text{Gr}(r, n).
\]

This suggests that the gauged linear sigma model with the gauge group \(U(r)\) and the representation \(V := \text{Hom}(W_1, V_1) \oplus \text{Hom}(V_1, W_1) \oplus \text{End}(V_1)\) describes the quantum cohomology of \(T^* \text{Gr}(r, n)\). Here \(\text{End}(V_1)\) is the Lagrange multiplier for the complex moment map equation, and the potential is given by

\[
(13.1.4) \quad V \ni (i_1, j_1, P) \mapsto \text{tr}(P i_1 j_1).
\]

Let \(H := H_1 \times H_2\) be the product of

- the diagonal maximal torus \(H_1\) of \(U(n)\), acting on \(\text{Hom}(W_1, V_1)\) and \(\text{Hom}(V_1, W_1)\) through the natural action on \(W_1\), and trivially on \(\text{End}(V_1)\), and
- the group \(H_2 = U(1)\) acting trivially on \(\text{Hom}(W_1, V_1)\), by scalar multiplication on \(\text{Hom}(V_1, W_1)\), and by inverse scalar multiplication on \(\text{End}(V_1)\).
One has

\begin{align}
Z_{\text{vec}}^d (x) &= \prod_{1 \leq i \neq j \leq r} (x_i - x_j), \\
Z_{\text{mat}}^d (x) &= \prod_{j=1}^{n} \prod_{i=1}^{r} (x_i - \lambda_j)^{-d_i - 1} \\
&\quad \times \prod_{j=1}^{n} \prod_{i=1}^{r} (-x_i + \lambda_j - \mu)^{-(d_i) - 1} \\
&\quad \times \prod_{1 \leq i \neq j \leq r} (x_i - x_j + \mu)^{2-(d_i-d_j) - 1},
\end{align}

so that the $H$-equivariant correlator of $P \in \mathbb{C}[x_1, \ldots, x_r]^\otimes_r$ is given by

\begin{equation}
\langle P \rangle_{\text{GLSM}}^H = \frac{1}{r!} \sum_{d_1=0}^{\infty} \cdots \sum_{d_r=0}^{\infty} \left( (-1)^{r-1} \right)^{d_1 + \cdots + d_r} \text{Res} \left[ \frac{\prod_{1 \leq i \neq j \leq r} (x_i - x_j)}{\prod_{1 \leq i,j \leq r} (x_i - x_j + \mu)^{d_i - d_j - 1}} \frac{\prod_{i=1}^{n} \prod_{j=1}^{r} (-x_i + \lambda_j - \mu)^{d_i - 1}}{\prod_{i=1}^{n} \prod_{j=1}^{r} (x_i - \lambda_j)^{d_i + 1}} P \right] \right],
\end{equation}

where $\text{Res}$ denotes the sum of residues at the points where $x_i$ is one of $\lambda_j$ for $i = 1, \ldots, r$ and $j = 1, \ldots, n$ (there are $n^r$ such points). This can formally be regarded as an equivariant integration over the projective space of dimension $\sum_{i=1}^{r} (d_i + 1) - 1$, and it is an interesting problem to give a geometric interpretation.

The effective potential \([2.4.1]\) of this gauged linear sigma model is given by

\begin{align}
W_{\text{eff}}(x; t) &= W_{\text{FI}}(x; t') + W_{\text{vec}}(x) + W_{\text{mat}}(x), \\
W_{\text{FI}}(x; t) &= t(x_1 + \cdots + x_r), \\
W_{\text{vec}}(x) &= -\pi \sqrt{-1} \sum_{1 \leq i < j \leq r} (x_j - x_i) \\
&= -\pi \sqrt{-1} \sum_{i=1}^{r} (2i - r - 1)x_i, \\
W_{\text{mat}}(x) &= -\sum_{i=1}^{r} \sum_{j=1}^{n} (x_i - \lambda_j) (\log (x_i - \lambda_j) - 1) \\
&\quad - \sum_{i=1}^{r} \sum_{j=1}^{n} (-x_i + \lambda_j - \mu) (\log (-x_i + \lambda_j - \mu) - 1) \\
&\quad - \sum_{i=1}^{r} \sum_{j=1}^{r} (x_i - x_j + \mu) (\log (x_i - x_j + \mu) - 1),
\end{align}
where $\lambda_j$ and $\mu$ are equivariant parameters for the actions of $H_1$ and $H_2$ respectively. Note that

\begin{equation}
(13.1.14) \quad e^{iW_{\text{eff}}/\partial x_i} = e^{t} \cdot (-1)^{2i-r-1} \prod_{j=1}^{n} (x_i - \lambda_j)^{-1} \prod_{j=1}^{n} (-x_i + \lambda_j - \mu) \prod_{j \neq i} \frac{x_j - x_i + \mu}{x_i - x_j + \mu},
\end{equation}

\begin{equation}
(13.1.15) \quad = e^{t+n\pi\sqrt{-1}} \prod_{j=1}^{n} \frac{x_i - \lambda_j + \mu}{x_i - \lambda_j} \prod_{j \neq i} \frac{x_i - x_j - \mu}{x_i - x_j + \mu},
\end{equation}

so that the equations $e^{i\partial_x W_{\text{eff}}} = 1, i = 1, \ldots, r$ gives

\begin{equation}
(13.1.16) \quad \prod_{j=1}^{n} \frac{x_i - \lambda_j}{x_i - \lambda_j + \mu} = e^{t+n\pi\sqrt{-1}} \prod_{j \neq i} \frac{x_i - x_j - \mu}{x_i - x_j + \mu}.
\end{equation}

By taking the sum over $d_i$ just as in the proof of Corollary B.7, one obtains

\begin{equation}
(13.1.17) \quad \langle P \rangle_{\text{GLSM}}^{H} = \frac{1}{r!} \text{Res} \left[ \frac{1}{\prod_{i=1}^{r} \left( (1 - e^{i\partial_x W_{\text{eff}}}) \prod_{j=1}^{n} (x_i - \lambda_j) \right)} \frac{\prod_{1 \leq i < j \leq r} (x_i - x_j) \prod_{1 \leq i, j \leq r} (x_i - x_j + \mu)}{\prod_{i=1}^{r} \prod_{j=1}^{n} (-x_i + \lambda_j - \mu)} Pdx_1 \wedge \cdots \wedge dx_r \right]
\end{equation}

where Res denotes the sum of residues at the roots of the equations (13.1.16).

13.2. The Heisenberg model, also known as the homogeneous $XXX_\frac{1}{2}$ model, is the SU(2) spin chain model with Hamiltonian

\begin{equation}
(13.2.1) \quad H = \sum_{i=1}^{n} S_i \cdot S_{i+1},
\end{equation}

where $S_i = (S_i^x, S_i^y, S_i^z) = (\sigma_i^x/2, \sigma_i^y/2, \sigma_i^z/2)$ are halves of Pauli matrices acting on the $i$-th factor of the Hilbert space $\mathcal{H} := (\mathbb{C}^2)^{\otimes n}$ and

\begin{equation}
(13.2.2) \quad S_i \cdot S_{i+1} := S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z.
\end{equation}

The total spin

\begin{equation}
(13.2.3) \quad S^z := \sum_{i=1}^{n} S_i^z
\end{equation}

clearly commutes with the Hamiltonian, and we restrict to the $S^z$-eigenspace $\mathcal{H}_r \subset \mathcal{H}$ with eigenvalue $(-n + r)/2$. We impose the quasi-periodicity condition

\begin{equation}
(13.2.4) \quad S_{r+1} = e^{\sqrt{-1} \theta S_i^z} S_i e^{-\sqrt{-1} \theta S_i^z}.
\end{equation}

Introduce variables $x = (x_1, \ldots, x_r)$ related to quasi-momenta $p = (p_1, \ldots, p_r)$ by

\begin{equation}
(13.2.5) \quad e^{\sqrt{-1} p_i} = \frac{x_i + \sqrt{-1}}{x_i - \frac{\sqrt{-1}}{2}}.
\end{equation}

Then $H$-eigenspaces in $\mathcal{H}_r$ correspond bijectively to solutions of the Bethe equation

\begin{equation}
(13.2.6) \quad \left( \frac{x_i + \sqrt{-1}}{x_i - \frac{\sqrt{-1}}{2}} \right)^n = e^{\sqrt{-1} \theta} \prod_{j \neq i} \frac{x_i - x_j + \sqrt{-1}}{x_i - x_j - \sqrt{-1}}.
\end{equation}
with eigenvalues \( n - 2r + 2 \sum_{i=1}^{n} \cos p_i \). The integrability comes from factorization of many-body S-matrix into the product of the 2-body S-matrix given by

\[
S(p_i, p_j) = 1 - 2e^{-\sqrt{-1}p_j} + e^{-\sqrt{-1}(p_i+p_j)}.
\]

(13.2.7)

See e.g. [Sta12] and references therein for Bethe ansatz for the quasi-periodic Heisenberg model. The Bethe equation (13.2.6) coincides with (13.1.16) under \( \lambda_j = \sqrt{-1} \), \( j = 1, \ldots, n \), \( \mu = -\sqrt{-1} \), and \( \vartheta = -\sqrt{-1}t + n/2 \). This observation and its generalizations is called Bethe/gauge correspondence [NS09]. The relation between classical/quantum cohomology of Grassmannians and integrable systems is studied in [BMO11, MO19, GRTV13, Oko17].

14. Quasimaps and Instantons

14.1. As explained in [FR14] Section 2.3, the moduli space of framed instantons on \( \mathbb{C} \times [\mathbb{C}/(\mathbb{Z}/n\mathbb{Z})] \) is isomorphic to the Nakajima quiver variety associated with the chainsaw quiver shown in Figure 14.1.

14.2. Representations of the chainsaw quiver satisfying \( \dim V_n = 0 \) are in one-to-one correspondence with representations of the handsaw quiver shown in Figure 14.2. It is shown in [FR14] Section 2.3] (see also [Nak12] Section 3] for an exposition) that the Nakajima quiver variety associated with the handsaw quiver is isomorphic to the parabolic Laumon space parametrizing flags

\[
0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = W \otimes \mathcal{O}_{\mathbb{P}^1}
\]

(14.2.1)

of locally free sheaves on \( \mathbb{P}^1 \) such that \( \text{rank} E_i = \sum_{j \leq i} \dim W_j \), \( \text{deg} E_i = -\dim V_i \), and the flag at \( \infty \in \mathbb{P}^1 \) is equal to the standard flag \( 0 \subset W_1 \subset W_1 \oplus W_2 \subset \cdots \subset W_1 \oplus \cdots \).
$W_2 \oplus \cdots \oplus W_{n-1} \subset W$. This coincides with the space of based quasimaps to partial flag varieties, i.e., quasimaps with specified value at infinity.

15. Quasimaps and monopoles

15.1. Let $G$ be a compact Lie group with a maximal torus $H$. A monopole on $\mathbb{R}^3$ is a pair $(A, \Phi)$ of a connection $A$ on a principal $G$-bundle $P$ and a section $\Phi$ of $P \times_G \mathfrak{g}$ satisfying the Bogomolny equation

\[ F_A = *dA\Phi. \]  

(15.1.1)

In order for the curvature to have a finite $L^2$-norm, it is natural to demand that the restriction of $\Phi$ to a sphere with large radius tends to a map to a fixed adjoint orbit $O \cong G/H \cong G_C/P$. The homotopy class $k \in \pi_2(G_C/P)$ of the resulting map is called the charge of the monopole.

15.2. A choice of a gauge satisfying a certain boundary condition at infinity is called a framing of the monopole. The framed moduli space is a principal $H$-bundle over the unframed moduli space. The framed moduli space has a natural hyperKähler structure coming from the dimensional reduction of the anti-self-dual equation in dimension 4.

15.3. Monopoles on $\mathbb{R}^3$ are related to

1. spectral curves on $TP^1$,
2. Nahm’s equation

\[ \frac{dT_i}{ds} = \epsilon_{ijk}[T_j, T_k], \quad i = 1, 2, 3 \]  

(15.3.1)

for $T_i \in C^\infty((0, 2), \text{Mat}(k,k; \mathbb{C}))$, and
3. based quasimaps from $P^1$ to $G_C/P$ of degree $k$.

(1) comes from the twistor correspondence [Hit82, Hit83], and (2) comes from Nahm transform [Nah82]. (3) is proved for SU(2) in [Don84], and the general case can be found in [Jar98b, Jar98a] and references therein.

16. Quasimaps and vortices

16.1. Let $X$ be a Kähler manifold, $(E, h)$ be a Hermitian vector bundle on $X$, and $\tau$ be a positive real number. The Yang–Mills–Higgs functional sends a pair $(A, \phi)$ of a unitary connection $d_A$ of $(E, h)$ and a section $\phi$ of $E$ to

\[ \mathcal{YMH}(A, \phi) = \|F_A\|_{L^2}^2 + \|d_A\phi\|_{L^2}^2 + \frac{1}{4} \|\phi \otimes \phi^* - \tau\|_{L^2}^2. \]  

(16.1.1)

By [Bra90] Proposition 2.1, one has

\[ \mathcal{YMH}(A, \phi) = 4 \|F_{0,2}\|_{L^2}^2 + 2 \|T_A\phi\|_{L^2}^2 + \left\| \sqrt{-1} \Lambda F + \frac{1}{2} \phi \otimes \phi^* - \frac{\tau}{2} \right\|_{L^2}^2 \]  

(16.1.2)

+ $\tau \int_X \sqrt{-1} \text{tr} F \wedge \omega^{n-1} + \int_X \text{tr} F \wedge F \wedge \omega^{n-2}$.

where $\omega^k := \omega^k/(k!)$ and $\Lambda$ is the dual Lefschetz operator.
16.2. Assume that $X$ is a projective curve, so that

\[(16.2.1)\quad \deg(E) = \frac{\sqrt{-1}}{2\pi} \tr F.\]

Then (16.1.2) immediately implies the Bogomolny–Prasad–Sommerfield inequality

\[(16.2.2)\quad \mathcal{H}(A, \phi) \geq 2\pi \tau \deg(E),\]

and the equality holds if and only if the vortex equation

\[(16.2.3)\quad F^{0,1} = 0,\]

\[(16.2.4)\quad \bar{\partial}_A \phi = 0,\]

\[(16.2.5)\quad -\sqrt{-1} \Lambda F = 1/2 (\phi \otimes \phi^* - \tau \id_E)\]

is satisfied. (16.2.3) and (16.2.4) are holomorphicities for $E$ and $\phi$, and (16.2.5) is a generalization of the constant central curvature equation.

16.3. By taking the trace of (16.2.5) and integrating over $X$, one obtains

\[(16.3.1)\quad -2\pi \deg(E) = 1/2 \|\phi\|^2_{L^2} - 1/2 \tau \rank(E) \vol(X),\]

so that the condition

\[(16.3.2)\quad \tau \geq \frac{4\pi \deg(E)}{\rank(E) \vol(X)}\]

is necessary for (16.2.5) to have a solution.

16.4. The \textit{slope} of a holomorphic vector bundle $E$ is defined by

\[(16.4.1)\quad \mu(E) = \frac{\deg(E)}{\rank(E)}.\]

For a holomorphic section $\phi$ of $E$, we set

\[
\hat{\mu}(E) := \sup \{\mu(E') \mid E' \text{ is a reflexive subsheaf of } E \text{ of rank less than } E\},
\]

\[
\mu_M(E) := \max \{\hat{\mu}(E), \mu(E)\},
\]

\[
\mu_m(E, \phi) := \inf \left\{ \frac{\rank(E) \mu(E) - \rank(E') \mu(E')}{\rank(E) - \rank(E')} \right\}
\]

$E'$ is a reflexive subsheaf of $E$ such that $\rank(E') < \rank(E)$ and $\phi \in \Gamma(E')$.

A pair $(E, \phi)$ of a holomorphic vector bundle $E$ and its holomorphic section $\phi$ is said to be \textit{stable} if

\[(16.4.2)\quad \mu_M(E) < \mu_m(E, \phi).\]

\textbf{Theorem 16.5} ([Bra91, Theorem 2.1.6]). Let $(E, \phi)$ be a pair of a holomorphic vector bundle and its holomorphic section. If there exists a Hermitian metric on $E$ satisfying the vortex equation, then one has either of the following:

(i) $(E, \phi)$ is stable and satisfies

\[(16.5.1)\quad \mu_M(E) < \frac{\tau \Vol(X)}{4\pi} < \mu_m(\phi).\]

(ii) $E$ has a direct sum decomposition $E = E_\phi \oplus E'$, $\phi$ is an element of $H^0(E_\phi) \subset H^0(E)$, $(E_\phi, \phi)$ satisfies (i) above, and $E'$ is the direct sum of stable vector bundles of slope $\tau \Vol(X)/4\pi$. 

36
Theorem 16.6 ([Bra91, Theorem 3.1.1]). Let \((E, \phi)\) be a stable pair of a holomorphic vector bundle and its holomorphic section. Then for any real number \(\tau\) satisfying \((16.5.1)\), there exists a Hermitian metric on \(E\) satisfying \((16.2.5)\).

Bradlow proved these results not only for projective curves but also for compact Kähler manifolds.

16.7. Vortex equation \((16.2.5)\) admits the following generalization, which also contains Hitchin’s self-duality equation \([\text{Hit87}]\) as a special case. Let \(Q = (Q_0, Q_1, s, t)\) be a quiver and \(M = (M_a)_{a \in Q_1}\) be a collection of vector bundles on \(X\) labeled by \(Q_1\). An \(M\)-twisted \(Q\)-sheaf on \(X\) is a pair \(R = (E_v)_{v \in Q_0}, (\phi_a)_{a \in Q_1}\) of a collection \((E_v)_{v \in Q_0}\) of vector bundles labeled by \(Q_0\) and a collection \((\phi_a)_{a \in Q_1}\) of morphisms labeled by \(Q_1\).

Given a collection \((E_v)_{v \in Q_0}\) of holomorphic vector bundles on a Kähler manifold \(X\), another collection \((M_a)_{a \in Q_1}\) of holomorphic vector bundles on \(X\), a collection \(\sigma = (\sigma_v)_{v \in Q_0}\) of positive real numbers, and a collection \(\tau = (\tau_v)_{v \in Q_0}\) of real numbers, the equation

\[
\sigma_v \sqrt{-1} \Lambda F_v + \sum_{t(a) = v} \phi_a \circ \phi_a^* - \sum_{s(a) = v} \phi_a^* \circ \phi_a = \tau_v \text{id}_{E_v}
\]

for Hermitian metrics on \((E_v)_{v \in Q_0}\) is called the \(M\)-twisted quiver \((\sigma, \tau)\)-vortex equation.

The \((\sigma, \tau)\)-degree and the \((\sigma, \tau)\)-slope of an \(M\)-twisted \(Q\)-sheaf \(R\) is defined by

\[
\deg_{\sigma, \tau}(R) = \sum_{v \in Q_0} (\sigma_v \deg E_v - \tau_v \text{rank } E_v),
\]

\[
\mu_{\sigma, \tau}(R) = \frac{\deg_{\sigma, \tau}(R)}{\sum_{v \in Q_0} \sigma_v \text{rank } E_v}.
\]

A \(Q\)-sheaf is stable if one has \(\mu_{\sigma, \tau}(R') < \mu_{\sigma, \tau}(R)\) for any proper subsheaf \(R'\). A \(Q\)-sheaf is polystable if it is the direct sum of stable \(Q\)-sheaf of the same slope.

Theorem 16.8 ([ACGP03, Theorem 3.1]). A \(Q\)-sheaf \(R\) with \(\deg_{\sigma, \tau}(R) = 0\) admits a Hermitian metric satisfying the quiver vortex equation \((16.7.2)\) if and only if \(R\) is \((\sigma, \tau)\)-polystable. This Hermitian metric is unique up to a multiplication by a positive constant for each stable summand.

Quasimaps to \(\text{Mat}(r, n) / \text{GL}_r\) corresponds to the case when the quiver \(Q = (1 \rightarrow 2)\) consists of two vertices and one arrow between them, \(M_1\) and \(M_2\) are the structure sheaves, rank \(E_1 = r\), and \(E_2\) is the trivial bundle of rank \(n\).

16.9. Note that the map \(V \rightarrow \text{End}(V), \phi \mapsto \phi \otimes \phi^*\) appearing in \((16.2.5)\) is the moment map for the natural action of the unitary group \(U(V)\) on \(V\). With this in mind, a generalization

\[
*F_A + \mu(\Phi) = \tau \text{id}_E
\]

of the vortex equation \((16.2.5)\) to the case where one has a Hamiltonian action of a compact group \(G\) on a Kähler manifold \(X\) is given in [MIR00, CGS00]. Here \(A\) is a connection on a principal \(G\)-bundle on a curve \(C\), \(\Phi\) is a holomorphic section of \(P \times_G X\), and \(\mu : X \rightarrow \mathfrak{g}\) is the moment map. They are used to define invariants of a symplectic manifold with a Hamiltonian group action [CGS00, MIR03, CGMIRS02].
which are closely related to the Gromov–Witten invariants of the symplectic quotient [GS05, Zil14, Woo15a, Woo15b, Woo15c]. [CS06] use wall-crossing in vortex invariants to study quantum cohomology of monotone toric varieties with minimal Chern number greater than or equal to 2.

16.10. Let $X$ be a Kähler manifold with a Hamiltonian action of a compact connected Lie group $G$. We assume that $X$ is either compact or equivariantly convex at infinity with a proper moment map. We fix an invariant inner product to identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and write the moment map as $\mu: X \to \mathfrak{g}$.

An affine vortex is a pair $(A, u)$ of a connection $A$ on the principal bundle $P = \mathbb{C} \times G$ and a holomorphic section $u: C \to P \times_G X$ satisfying the vortex equation

\[(16.10.1) \quad *F_A + \mu(u) = 0.\]

A gauged holomorphic map to $X$ with respect to the complex Lie group $G_C$ acting on $X$ is a map to the quotient stack $[X/G_C]$. In other words, a gauged holomorphic map from a scheme $C$ to $X$ is a pair $(F, u)$ of a principal $G_C$-bundle $P$ over $C$ and a $G_C$-equivariant holomorphic map $u: P \to X$.

If the $G_C$-action on $X^{ss}$ is free, then by [VW16, Theorem 1.1], there is a natural bijection between the set of affine $K$-vortices with target $X$ up to gauge equivalence and the set of pairs gauged holomorphic maps such that $u(\infty) \in X^{ss}$. This is an open substack of the set of quasimaps such that $\infty$ is not contained in the base locus.

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Korea Institute for Advanced Study, 85 Hoegi-ro Dondaemun-gu Seoul 02455, Republic of Korea
E-mail address: bumsig@kias.re.kr

Korea Institute for Advanced Study, 85 Hoegi-ro Dondaemun-gu Seoul 02455, Republic of Korea
E-mail address: batistuta@kaist.ac.kr

Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba Meguro-ku Tokyo 153-8914 Japan.
E-mail address: kazushi@ms.u-tokyo.ac.jp

Kavli IPMU (WPI), UTIAS, University of Tokyo Kashiwa, Chiba 277-8583, Japan
E-mail address: yyyyyosida@gmail.com