Transverse spectral instabilities in Konopelchenko–Dubrovin equation

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Abstract
We study the transverse spectral stability of the one-dimensional small-amplitude periodic traveling wave solutions of the (2+1)-dimensional Konopelchenko–Dubrovin (KD) equation. We show that these waves are transversely unstable with respect to two-dimensional perturbations that are periodic in both directions with long wavelength in the transverse direction. We also show that these waves are transversely stable with respect to perturbations which are either mean-zero periodic or square-integrable in the direction of the propagation of the wave and periodic in the transverse direction with finite or short wavelength. We discuss the implications of these results for special cases of the KD equation—namely, KP-II and mKP-II equations.

KEYWORDS
perturbation theory, spectral stability, transverse instability, water waves

1 INTRODUCTION

We consider the (2+1)-dimensional Konopelchenko–Dubrovin (KD) equation\textsuperscript{1,2}

\[
\begin{cases}
  u_t - u_{xxx} - 6\rho uu_x + \frac{3}{2}\phi^2 u^2 u_x - 3v_y + 3\phi uu_x v = 0, \\
  u_y = v_x,
\end{cases}
\]

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where, \( u = u(x, y, t) \), \( v = v(x, y, t) \), the subscripts denote partial differentiation, \( \rho \) and \( \phi \) are real parameters, defining the magnitude of nonlinearity in wave propagation, modeled for stratified shear flow, the internal and shallow-water waves, and the plasmas,\(^3\) can also be regarded as combined KP and modified KP equation,\(^4\) or generalized (2+1)D Gardner equation.\(^5\)

### 1.1 Models

The \((1+1)\)-dimensional reduction of the KD equation (1) is the Gardner equation\(^6,7\)

\[
u_t - u_{xxx} - 6\rho uu_x + \frac{3}{2}\phi^2u^2u_x = 0,
\]

which is example of the generalized Korteweg–de-Vries (gKdV) equation,\(^8\) that is,

\[
u_t + (g(u) - u_{xx})_x = 0,
\]

where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is a smooth real function. Gardner equation (2) can be reduced to KdV and modified KdV equations for 

\( \phi = 0 \) and \( \rho = 0 \), respectively.

For \( \phi = 0 \), (1) is Kadomtsev–Petviashvili (KP) equation with negative dispersion\(^9\)

\[
(u_t - u_{xxx} - 6\rho uu_x)_x - 3u_{yy} = 0,
\]

which is also known as KP-II equation. Modified KP-II\(^10\) (say, mKP-II) reads from (1) for \( \rho = 0 \),

\[
\left( u_t - u_{xxx} + \frac{3}{2}\phi^2u^2u_x \right)_x - 3u_{yy} = 0.
\]

### 1.2 Integrability

The KD equation (1) is integrable.\(^1,4,11\) Integrability is an useful property to have for evolution equations, especially in higher dimensions. It gives sufficient freedom to explore the equation through different aspects. It also helps significantly to observe nonlinear coherent structures like rogue waves, breathers, solitons, and elliptic waves in the systems.\(^12–16\) Some well-known integrable water-wave models are classical KdV, KP, and Schrödinger equations. The KD equation (1), like similar evolution equations in \((2+1)\) dimension, for example, KP equation, Davey–Stewartson equation, and the three-wave equations, is solvable through Inverse Scattering Transform (IST).\(^1\) It is among the few nonlinear evolution equations which are completely integrable in different settings. Notably, the considered KD equation is also integrable in the Painlevé sense and solvable through IST.\(^1,3,5\)

### 1.3 Dispersion relation

Assuming a plane-wave solution of the form

\[
u(x, y, t) = e^{i(kx-\Omega t+\gamma y)},
\]
for the linear part

$$(u_t - u_{xxx})_x - 3u_{yy} = 0$$

of the KD equation (1), we arrive at the dispersion relation

$$\Omega(k) = k^3 - \frac{3\gamma^2}{k}.$$ 

### 1.4 Small-amplitude periodic traveling waves

The $y$-independent periodic traveling wave solutions of the KD equation (1) that are also solutions of the Gardner equation (2), are of the form

$$\begin{pmatrix} u(x, y, t) \\ v(x, y, t) \end{pmatrix} = \begin{pmatrix} u(x - ct) \\ v(x - ct) \end{pmatrix}$$

for some $c \in \mathbb{R}$. Under this assumption, we arrive at

$$\begin{cases} -cu_x - u_{xxx} + \frac{\phi^2}{2} (u^3)_x - 3\rho (u^2)_x + 3\phi u_x v = 0, \\ v_x = 0, \end{cases}$$

which implies $v = b_1$, where $b_1$ is an arbitrary constant. Substituting $v = b_1$ and integrating, (5) is reduced to

$$-cu - u_{xx} + \frac{\phi^2}{2} u^3 - 3\rho u^2 + 3\phi ub_1 = b_2,$$

where $b_1, b_2 \in \mathbb{R}$. Let $u$ be a $2\pi/k$-periodic function of its argument, for some $k > 0$. Then, $w(z) := u(x)$ with $z = kx$, is a $2\pi$-periodic function in $z$, satisfying

$$-cw - k^2 w_{zz} + \frac{\phi^2}{2} w^3 - 3\rho w^2 + 3\phi wb_1 = b_2,$$

For a fixed $\phi$ and $\rho$, let $F : H^2(\mathbb{T}) \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to L^2(\mathbb{T})$ be defined as

$$F(w, c; k, b_1, b_2) = -cw - k^2 w_{zz} + \frac{\phi^2}{2} w^3 - 3\rho w^2 + 3\phi wb_1 - b_2.$$ We try to find a solution $w \in H^2(\mathbb{T})$ of

$$F(w, c; k, b_1, b_2) = 0.$$

For any $c \in \mathbb{R}, k > 0, b_1, b_2 \in \mathbb{R}$, and $|b_1|, |b_2|$ sufficiently small, note that

$$w_0(c, k, b_1, b_2) = -\frac{1}{c} b_2 + O((b_1 + b_2)^2)$$
make a constant solution of (8). Note that \( w_0 \equiv 0 \) if \( b_1 = b_2 = 0 \). If nonconstant solutions of (8) bifurcate from \( w_0 \equiv 0 \) for some \( c = c_0 \) then \( \ker(\partial_w F(0, c_0; k, 0, 0)) \) is nontrivial. Note that
\[
\ker(\partial_w F(0, c_0; k, 0, 0)) = \ker(-c_0 - k^2 \partial_z^2) = \text{span}\{e^{\pm i z}\},
\]
provided that \( c_0 = k^2 \).

The periodic traveling waves of (2) exist (see Ref.\([17, \text{Section 2}]\)), and by following the Lyapunov–Schmidt procedure, their small-amplitude expansion (see Ref.\([18, \text{Appendix A}]\)) is as follows.

**Theorem 1.** For any \( k > 0, b_1, b_2 \in \mathbb{R} \) and \( |b_1|, |b_2| \) sufficiently small, a one parameter family of solutions of (1), denoted by
\[
\begin{pmatrix}
    u(x, t) \\
    v(x, t)
\end{pmatrix} = \begin{pmatrix}
    w(a, b_1, b_2)(z) \\
    v(z)
\end{pmatrix},
\]
where \( z = k(x - c(a, b_1, b_2)t) \), \( |a| \) sufficiently small, \( w(a, b_1, b_2)(z) \) is smooth, even and 2\( \pi \)-periodic in \( z \) and \( c \) is even in \( a \), is given by

\[
\begin{cases}
    w(a, b_1, b_2)(z) = -\frac{1}{k^2} b_2 + a \cos z + a^2 (A_0 + A_2 \cos 2z) + a^3 A_3 \cos 3z + O(a^4 + a^2 (b_1 + b_2)^2), \\
    v(z) = b_1,
\end{cases}
\]

\[
c(a, b_1, b_2) = k^2 + 3\phi b_1 + \frac{3\rho}{k^2} b_2 + a^2 c_2 + O(a^4 + a^2 (b_1 + b_2)^2),
\]

where
\[
A_0 = -\frac{3\rho}{2k^2}, \quad A_2 = \frac{\rho}{2k^2}, \quad A_3 = -\frac{\phi^2}{64k^2} + \frac{3\rho^2}{16k^4}, \quad \text{and} \quad c_2 = \frac{3\phi^2}{8} + \frac{15\rho^2}{2k^2}.
\]

### 1.5 Transverse stability

For the KD equation (1), the solution and integrability aspects have been studied thoroughly, see Refs. 4, 19–21 for instance. However, to the best of the authors’ knowledge, any result on the stability aspects of the periodic traveling waves or solitary waves of the KD equation (1) has not been discussed so far. However, the transverse instability of periodic traveling waves has been studied for many similar equations, for instance, for the KP equation in Refs. 22–25, for Zakharov–Kuznetsov (ZK) equation in Refs. 26, 27. Transverse instability of solitary wave solutions of various water-wave models has also been explored by several authors (see Refs. 28–31). Motivated by the importance of nonlinear waves propagation and its stability, we investigate the transverse spectral instability of the KD equation.

**Definition 1** (Transverse stability). A periodic traveling wave is said to be transversely spectrally stable if it remains spectrally stable when subjected to perturbations along its direction of propagation (one-dimensional perturbations) as well as perturbations perpendicular to its direction of propagation (two-dimensional perturbations).
We aim to study the (in)stability of the $y$-independent, that is, $(1+1)$-dimensional periodic traveling waves $(10)$ of $(1)$ with respect to two-dimensional perturbations which are either periodic or nonperiodic in the $x$-direction and always periodic in the $y$-direction. The periodic nature of the perturbations in the $y$-direction is classified into two categories: long wavelength and finite or short-wavelength transverse perturbations. The (in)stabilities that occur due to long-wavelength transverse perturbations are termed as modulational transverse (in)stabilities. Furthermore, we use the term high-frequency transverse (in)stabilities to refer to those transverse (in)stabilities that are occurring due to finite or short-wavelength transverse perturbations. Moreover, depending on the periodic or nonperiodic nature of perturbations in the direction of propagation of the one-dimensional wave, we term the resulting instability as transverse instability with respect to periodic or nonperiodic perturbations, respectively. Throughout the article, we assume that the small-amplitude periodic traveling waves $(10)$ of $(1)$ are spectrally stable with respect to one-dimensional perturbations along its direction of propagation.

Our main results are the following theorems depicting the transverse stability and instability of small-amplitude periodic traveling waves $(10)$ of $(1)$.

**Theorem 2** (Transverse stability). Assume that small-amplitude periodic traveling waves $(10)$ of $(1)$ are spectrally stable in $L^2(\mathbb{T})$ as a solution of the corresponding $y$-independent one-dimensional equation. Then, for any a sufficiently small, $\rho \in \mathbb{R}$, $\phi \in \mathbb{R}$, and $k > 0$, periodic traveling waves $(10)$ of $(1)$ are transversely stable with respect to two-dimensional perturbations which are either mean-zero periodic or nonperiodic (localized or bounded) in the direction of propagation and finite or short wavelength in the transverse direction.

**Theorem 3** (Transverse instability). For a fixed $\rho \in \mathbb{R}$ and $\phi \neq 0$, sufficiently small-amplitude periodic traveling waves $(10)$ of KD equation suffers modulational transverse instabilities with respect to periodic perturbations if

$$k > 2\left|\frac{\rho}{\phi}\right| \quad \text{and} \quad |\gamma| < k|a| \sqrt{\left|\frac{\phi^2}{4} - \frac{\rho^2}{k^2}\right|} + O(a(\gamma + a)).$$

As a consequence of these theorems, for all $k > 0$, periodic traveling waves $(10)$ of the mKP-II equation suffers modulational transverse instability with respect to periodic perturbations, which is in accordance with results in Ref. 24.

Also, in the limit $\phi \rightarrow 0$, there is no modulational transverse instability for KP-II equation by Theorem 3 which again agrees with results in Refs. 22, 24, 25, 32, 33. From Theorem 2, KP-II does not possess any high-frequency transverse instability. Moreover, we have the following stability result for the mKP-II equation using Theorem 2.

**Corollary 1.** For all $k > 0$, sufficiently small amplitude periodic traveling waves $(10)$ of the mKP-II equation does not possess any high-frequency transverse instability with respect to both mean-zero periodic and nonperiodic perturbations.

In Section 2, we linearize the equation and formulate the problem. In Section 3, we list all potentially unstable nodes. In Sections 4 and 5, we provide transverse instability analysis to investigate modulational and high-frequency transverse instabilities with respect to periodic and nonperiodic perturbations.
1.6 Notations

Throughout the article, we have used the following notations. Here, $L^2(\mathbb{R})$ is the set of Lebesgue measurable, real, or complex-valued functions over $\mathbb{R}$ such that

$$
\|f\|_{L^2(\mathbb{R})} = \left( \int_{\mathbb{R}} |f(x)|^2 \, dx \right)^{1/2} < +\infty,
$$

and, $L^2(\mathbb{T})$ denote the space of $2\pi$-periodic, measurable, real or complex-valued functions over $\mathbb{R}$ such that

$$
\|f\|_{L^2(\mathbb{T})} = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^2 \, dx \right)^{1/2} < +\infty.
$$

Here, $L^2_0(\mathbb{T})$ is the space of square-integrable functions with of zero-mean,

$$
L^2_0(\mathbb{T}) = \left\{ f \in L^2(\mathbb{T}) : \int_0^{2\pi} f(z) \, dz = 0 \right\}. \quad (12)
$$

The space $C_b(\mathbb{R})$ consists of all bounded continuous functions on $\mathbb{R}$, normed with

$$
\|f\| = \sup_{x \in \mathbb{R}} |f(x)|.
$$

For $s \in \mathbb{R}$, let $H^s(\mathbb{R})$ consists of tempered distributions such that

$$
\|f\|_{H^s(\mathbb{R})} = \left( \int_{\mathbb{R}} (1 + |t|^2)^s |\hat{f}(t)|^2 \, dt \right)^{1/2} < +\infty,
$$

and

$$
H^s(\mathbb{T}) = \{ f \in H^s(\mathbb{R}) : f \text{ is } 2\pi\text{-periodic} \}.
$$

We define $L^2(\mathbb{T})$-inner product as

$$
\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(z)\overline{g(z)} \, dz = \sum_{n \in \mathbb{Z}} \hat{f}_n \overline{\hat{g}_n},
$$

where $\hat{f}_n$ are Fourier coefficients of the function $f$ defined by

$$
\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(z)e^{inz} \, dz.
$$

Throughout the article, $\Re(\mu)$ represents the real part of $\mu \in \mathbb{C}$.
LINEARIZATION AND THE SPECTRAL PROBLEM SETUP

Linearizing (1) about its one-dimensional periodic traveling wave solution \( \begin{bmatrix} w \\
v \end{bmatrix} \) given in (10), and considering the perturbations to \( \begin{bmatrix} w \\
v \end{bmatrix} \) of the form

\[
\begin{bmatrix} w \\
v \end{bmatrix} + \varepsilon \begin{bmatrix} \xi \\
\psi \end{bmatrix} + O(\varepsilon^2) \quad \text{for} \quad 0 < |\varepsilon| \ll 1
\]

we arrive at,

\[
\begin{cases}
\xi_t - kc \xi_z - k^3 \xi_{zzz} - 6k \rho (w \xi)_z + \frac{3}{2} \phi^2 k (w^2 \xi)_z - 3\psi_y + 3\phi kw_z \psi + 3\phi k b_1 \xi_z = 0, \\
\xi_y - k \psi_z = 0.
\end{cases}
\tag{14}
\]

We seek a solution of the form \( \begin{bmatrix} \xi(z, t, y) \\
\psi(z, t, y) \end{bmatrix} = e^{\mu t + i\gamma y} \begin{bmatrix} \xi(z) \\
\psi(z) \end{bmatrix}, \mu \in \mathbb{C}, \gamma \in \mathbb{R}, \) of (14), which leads to

\[
\begin{cases}
\mu \xi - kc \xi_z - k^3 \xi_{zzz} - 6k \rho (w \xi)_z + \frac{3}{2} \phi^2 k (w^2 \xi)_z - 3i\gamma \psi + 3\phi kw_z \psi + 3\phi k b_1 \xi_z = 0, \\
i\gamma \xi - k \psi_z = 0.
\end{cases}
\tag{15}
\]

We can reduce this system of equations into

\[
Q_{a,b_1,b_2}(\mu, \gamma) \psi :=
\left( k \left( \mu - kc \partial_z - k^3 \partial_z^3 - 6k \rho \partial_z (w \cdot) + \frac{3}{2} \phi^2 k \partial_z (w^2 \cdot) \right) \partial_z + 3\gamma^2 + 3\phi (i\gamma w_z + k b_1 \partial_z^2) \right) \psi = 0.
\tag{16}
\]

**Definition 2.** (Transverse (in)stability) Assuming that the \( 2\pi/k \)-periodic traveling wave solution \( \begin{bmatrix} u(x, y, t) \\
v(x, y, t) \end{bmatrix} = \begin{bmatrix} w(k(x - ct)) \\
v(k(x - ct)) \end{bmatrix} \) of (1) is a stable solution of the one-dimensional equation (2) where \( w, v, \) and \( c \) are as in (10), we say that the periodic wave \( \begin{bmatrix} w \\
v \end{bmatrix} \) in (10) is transversely spectrally stable with respect to two-dimensional periodic perturbations (respectively, nonperiodic (localized or bounded perturbations)) if the KD operator \( Q_{a,b_1,b_2}(\mu, \gamma) \) acting in \( L^2(\mathbb{T}) \) (respectively, \( L^2(\mathbb{R}) \) or \( C_b(\mathbb{R}) \)) is invertible, for any \( \mu \in \mathbb{C}, R(\mu) > 0 \), and any \( \gamma \neq 0 \) otherwise it is deemed transversely spectrally unstable.

We split the study of invertibility of \( Q_{a,b_1,b_2}(\mu, \gamma) \) into periodic (\( L^2(\mathbb{T}) \)) and nonperiodic perturbations (\( L^2(\mathbb{R}) \) or \( C_b(\mathbb{R}) \)). In further study, we assume \( b_1 = b_2 = 0 \). For nonzero \( b_1 \) and \( b_2 \), one may explore in like manner. However, the calculation becomes lengthy and tedious.
2.1 Periodic perturbations

Here, we are considering perturbations which are periodic in $z$, that is, in the direction of the propagation of the wave. We check the invertibility of the operator $Q_{a,b_1,b_2}(\mu, \gamma)$ acting in $L^2(\mathbb{T})$ for any $\mu \in \mathbb{C}$, $\Re(\mu) > 0$, and any $\gamma \neq 0$. We use the notation $Q_a(\mu, \gamma)$ for $Q_{a,b_1,b_2}(\mu, \gamma)$ for simplicity. We convert the invertibility problem

$$Q_a(\mu, \gamma)\psi = 0; \quad \psi \in L^2(\mathbb{T})$$

into a spectral problem which requires invertibility of $\partial_z$. Since $\partial_z$ is not invertible in $L^2(\mathbb{T})$, we restrict the problem to mean-zero subspace $L^2_0(\mathbb{T})$, defined in (12), of $L^2(\mathbb{T})$. Since $L^2_0(\mathbb{T}) \subset L^2(\mathbb{T})$ if the operator $Q_a(\mu, \gamma)$ is not invertible for some $\mu \in \mathbb{C}$ in $L^2_0(\mathbb{T})$ implies that the operator $Q_a(\mu, \gamma)$ is not invertible in $L^2(\mathbb{T})$ as well for the same $\mu \in \mathbb{C}$.

The operator $Q_a(\mu, \gamma)$ acting on $L^2_0(\mathbb{T})$ has a compact resolvent so that the spectrum consists of isolated eigenvalues with finite multiplicity. Therefore, $Q_a(\mu, \gamma)$ is invertible in $L^2_0(\mathbb{T})$ if and only if zero is not an eigenvalue of $Q_a(\mu, \gamma)$. Using this and the invertibility of $\partial_z$ in $L^2_0(\mathbb{T})$, we have the following result.

**Lemma 1.** The operator $Q_a(\mu, \gamma)$ is not invertible in $L^2_0(\mathbb{T})$ for some $\mu \in \mathbb{C}$ if and only if $\mu \in \text{spec}_{L^2_0(\mathbb{T})}(H_a(\gamma))$, that is, $L^2_0(\mathbb{T})$-spectrum of the operator, where

$$H_a(\gamma) := ck\partial_z + k^3\partial_z^3 + 6k\rho \partial_z (w \cdot) - \frac{3}{2} \phi^2 k \partial_z (w^2 \cdot) - \frac{3y^2}{k} \partial_z^{-1} - i3\phi \gamma w_z \partial_z^{-1}.$$ 

**Proof.** The operator $Q_a(\mu, \gamma)$ is not invertible in $L^2_0(\mathbb{T})$ for some $\mu \in \mathbb{C}$, if and only if zero is an eigenvalue of $Q_a(\mu, \gamma)$. Moreover, for a $\varphi \in L^2_0(\mathbb{T})$, $Q_a(\mu, \gamma)\varphi = 0$ if and only if $H_a(\gamma)\varphi = \mu \varphi$. The proof follows trivially. •

Next, we analyze the spectrum of the operator $H_a(\gamma)$ acting in $L^2_0(\mathbb{T})$ with domain $H^3(\mathbb{T}) \cap L^2_0(\mathbb{T})$. Since $w$ is an even function, $w_z$ is an odd function and therefore, the spectrum of $H_a(\gamma)$ is not symmetric with respect to the reflection through the origin. Moreover, the operator $H_a(\gamma)$ is not real, therefore the spectrum of $H_a(\gamma)$ is not symmetric with respect to the real axis as well. The spectrum of $H_a(\gamma)$ inherits the following symmetry property.

**Lemma 2.** The spectrum of $H_a(\gamma)$ is symmetric with respect to the reflection through the imaginary axis.

**Proof.** We consider $R$ to be the reflection through the imaginary axis defined as follows:

$$R \psi(z) = \overline{\psi(-z)}.$$ 

Assume $\mu$ is the eigenvalue of $H_a(\gamma)$ with an associated eigenvector $\varphi$, then we have

$$H_a(\gamma)\varphi = \mu \varphi. \quad (17)$$
Since
\[
(H_a(\gamma)R\psi)(z) = H_a(\gamma)(R\psi(z)) = H_a(\gamma)\overline{\psi(-z)} = -(H_a(\gamma)\overline{\psi})(-z) = -(R H_a(\gamma)\psi)(z),
\]
therefore, \(H_a(\gamma)\) anticommutes with \(R\). Using (17), we arrive at
\[
H_a(\gamma)R\varphi = -R H_a(\gamma)\varphi = -\overline{\mu} R\varphi.
\]
We conclude from here that if \(\mu\) is an eigenvalue of \(H_a(\gamma)\) with associated eigenvector \(\varphi\), then \(-\overline{\mu}\) is also an eigenvalue of \(H_a(\gamma)\) with associated eigenvector \(R\varphi\).

\[\square\]

### 2.2 Nonperiodic perturbations

With respect to these perturbations, we aim to study the invertibility of \(Q_a(\mu, \gamma)\) acting in \(L^2(\mathbb{R})\) or \(C_b(\mathbb{R})\) (with domain \(H^4(\mathbb{R})\) or \(C^4_b(\mathbb{R})\)), for \(\mu \in \mathbb{C}, \Re(\mu) > 0, \) and \(\gamma \in \mathbb{R}, \gamma \neq 0\). In \(L^2(\mathbb{R})\) or \(C_b(\mathbb{R})\), the operator \(Q_a(\mu, \gamma)\) has no longer point isolated spectrum, rather it has continuous spectrum. Thus, we rely upon the Floquet Theory such that all solutions of (16) in \(L^2(\mathbb{R})\) or \(C_b(\mathbb{R})\) are of the form \(\psi(z) = e^{i\tau z}\Psi(z)\) where \(\tau \in (-\frac{1}{2}, \frac{1}{2}]\) is the Floquet exponent and \(\Psi(z)\) is a \(2\pi\)-periodic function, see Ref. 34 for a similar situation. By following same arguments as in the proof of Ref. [34, Proposition A.1], we can infer that the study of the invertibility of \(Q_a(\mu, \gamma)\) in \(L^2(\mathbb{R})\) or \(C_b(\mathbb{R})\) is equivalent to the invertibility of the linear operators \(Q_{a,\tau}(\mu, \gamma)\) in \(L^2(\mathbb{T})\) with domain \(H^4(\mathbb{T})\), for all \(\tau \in (-\frac{1}{2}, \frac{1}{2}]\), where
\[
Q_{a,\tau}(\mu, \gamma) = (\mu - kc(\partial_z + i\tau)) - k^3(\partial_z + i\tau)^3 - 6k\rho(\partial_z + i\tau)(w)(k(\partial_z + i\tau)) + \left(\frac{3}{2}\phi^2k(\partial_z + i\tau)(w^2)\right)(k(\partial_z + i\tau)) + 3\gamma^2 + i3ky\phi w_z.
\]

Since \(\tau = 0\) corresponds to the periodic perturbations we have already investigated, we would now restrict ourselves to the case of \(\tau \neq 0\). The \(L^2(\mathbb{T})\)-spectra of the operators \(Q_{a,\tau}(\mu, \gamma)\) consist of eigenvalues of finite multiplicity. Therefore, \(Q_{a,\tau}(\mu, \gamma)\) is invertible in \(L^2(\mathbb{T})\) if and only if zero is not an eigenvalue of \(Q_{a,\tau}(\mu, \gamma)\). We have the following result using this and the invertibility of \(\partial_z + i\tau\).

**Lemma 3.** The operator \(Q_{a,\tau}(\mu, \gamma)\) is not invertible in \(L^2(\mathbb{T})\) for some \(\mu \in \mathbb{C}\) and \(\tau \neq 0\) if and only if \(\mu \in \text{spec}_{L^2(\mathbb{T})}(H_a(\gamma, \tau))\), \(L^2(\mathbb{T})\)-spectrum of the operators,
\[
H_a(\gamma, \tau) := kc(\partial_z + i\tau) + k^3(\partial_z + i\tau)^3 + 6k\rho(\partial_z + i\tau)(w)
\]
\[
- \frac{3}{2}\phi^2k(\partial_z + i\tau)(w^2) - \left(\frac{3\gamma^2}{k} + i3\phi y w_z\right)(\partial_z + i\tau)^{-1}.
\]

**Proof.** The proof is similar to Lemma 1.  
\[\square\]
We will study the $L^2(\mathbb{T})$-spectra of linear operators $H_\alpha(\gamma, \xi)$ for $|a|$ sufficiently small, and for $|\tau| > \delta > 0$ since the operator $(\partial_z + i\tau)^{-1}$ becomes singular, as $\tau \to 0$. Note that the spectrum of $H_\alpha(\gamma, \tau)$ is not symmetric with respect to the reflection through real axis or origin. Instead, we have the following symmetry.

**Lemma 4.** The spectrum of $H_\alpha(\gamma, \tau)$ is symmetric with respect to the reflection through the imaginary axis for all $\tau \in (-\frac{1}{2}, \frac{1}{2}] \setminus \{0\}$.

**Proof.** The proof is similar to Lemma 2. \[\square\]

### 3 | CHARACTERIZATION OF THE UNPERTURBED SPECTRUM

#### 3.1 | Periodic perturbations

As a consequence of the symmetry of the spectrum obtained in Lemma 2, we obtain instability if there is an eigenvalue of $H_\alpha(\gamma)$ off the imaginary axis.

We start by talking about

$$H_0(\gamma) = k^3(\partial_z + \partial_z^3) - \frac{3\gamma^2}{k} \partial_z^{-1}$$

(18)

corresponding to the linearization of (1) about the zero solution and $c = k^2$ (see (10)). Since $H_0(\gamma)$ is a differential operator with constant coefficients, we can use Fourier analysis to compute its spectrum.

A straightforward calculation reveals that

$$H_0(\gamma)e^{inz} = i\Omega_{n,\gamma}e^{inz} \quad \text{for all} \quad n \in \mathbb{Z}^* := \mathbb{Z} \setminus \{0\},$$

(19)

where

$$\Omega_{n,\gamma} = k^3n(1 - n^2) + \frac{3\gamma^2}{kn}.$$  

(20)

The set of functions \{\(e^{inz} ; n \in \mathbb{Z}^*\}\} forms an orthonormal basis for $L^2_0(\mathbb{T})$. Since, this orthonormal basis is also a set of eigenfunctions for $H_0(\gamma)$ as calculated in (19), \{\(i\Omega_{n,\gamma} ; n \in \mathbb{Z}^*\}\} is the complete set of spectrum of $H_0(\gamma)$. Therefore, the $L^2_0(\mathbb{T})$-spectrum of $H_0(\gamma)$ is given by

$$\text{spec}_{L^2_0(\mathbb{T})}(H_0(\gamma)) = \{i\Omega_{n,\gamma} ; n \in \mathbb{Z}^*\},$$

which implies $\text{spec}_{L^2_0(\mathbb{T})}(H_0(\gamma))$ consists of purely imaginary eigenvalues of finite multiplicity. This is because the coefficients of the operator $H_0(\gamma)$ are real, which should be the case, since zero-amplitude solutions are spectrally stable.

**Remark 1.** For the case $b_1 \neq 0$ in (16), the $L^2_0(\mathbb{T})$-spectrum of $H_0(\gamma)$ is given by

$$\text{spec}_{L^2_0(\mathbb{T})}(H_0(\gamma)) = \{i\Omega_{n,\gamma} ; n \in \mathbb{Z}^*\} \quad \text{where} \quad \Omega_{n,\gamma} = k^3n(1 - n^2) + \frac{3\gamma^2}{kn} + 3n\phi k b_1.$$
Spectra of $\mathcal{H}_a(\gamma)$ and $\mathcal{H}_0(\gamma)$ remain close for $|a|$ small as

$$||\mathcal{H}_a(\gamma) - \mathcal{H}_0(\gamma)|| \rightarrow 0$$

as $a \rightarrow 0$ in the operator norm. Due to the symmetry in Lemma 2, for $|a|$ sufficiently small, bifurcation of eigenvalues of $\mathcal{H}_a(\gamma)$ from imaginary axis can happen only when a pair of eigenvalues of $\mathcal{H}_0(\gamma)$ collide on the imaginary axis.

Let $n \neq m \in \mathbb{Z}^*$, a pair of eigenvalues $i\Omega_{n,\gamma}$ and $i\Omega_{m,\gamma}$ of $\mathcal{H}_0(\gamma)$ collide for some $\gamma = \gamma_c$ when

$$\Omega_{n,\gamma_c} = \Omega_{m,\gamma_c}. \quad (21)$$

We list all the collisions in the following lemma.

**Lemma 5.** For a fix $\Delta \in \mathbb{N}$, eigenvalues $\Omega_{n,\gamma}$ and $\Omega_{n+\Delta,\gamma}$ of the operator $\mathcal{H}_0(\gamma)$ collide for all $n \in (-\Delta,0) \cap \mathbb{Z}$ at some $\gamma = \gamma_c(k)$. All such collisions take place away from the origin in the complex plane except when $\Delta$ is even and $n = -\Delta/2$ in which case eigenvalues $\Omega_{n,\gamma}$ and $\Omega_{-n,\gamma}$ collide at the origin.

**Proof.** Without any loss of generality, consider $m > n$ and $m = n + \Delta$ with $\Delta \in \mathbb{N}$ in the collision condition (21) then we obtain

$$3\gamma_c^2 = k^4 n(n+\Delta)(-3n^2 - 3n\Delta - \Delta^2 + 1), \quad (22)$$

which can be rewritten as

$$3\gamma_c^2 = -k^4[3n^2(n+\Delta)^2 + n(n+\Delta)(\Delta^2 - 1)]. \quad (23)$$

The above equation implies that collision between $n$ and $n + \Delta$ takes place if only if $n(n + \Delta) < 0$, that is, $-\Delta < n < 0$.

Observe that $\Omega_{n,\gamma_c} = \Omega_{-n,\gamma_c} = 0$ for $\gamma_c = \sqrt{\frac{k^4 n^2(n^2 - 1)}{3}}$. Therefore, $\Omega_{n,\gamma_c}$ and $\Omega_{n+\Delta,\gamma_c}$ collide at the origin when $\Delta$ is even and $n = -\Delta/2$. All other collisions are away from origin. Hence, the lemma. ■

From (23), assume $\gamma^2 = \frac{k^4}{3} f(n)g(n)$, where $f(n) = n(n + \Delta)$ and $g(n) = 3n^2 + 3n\Delta + \Delta^2 - 1$. For a fixed $\Delta \in \mathbb{N}$, $f(n) > f(-\Delta/2)$ and $g(n) > g(-\Delta/2)$ for all $n \in (-\Delta, 0) \cap \mathbb{Z}$. And $f(n)g(n) \leq f(-\Delta/2)g(-\Delta/2)$ for all $n \in (-\Delta, 0) \cap \mathbb{Z}$. Also, $f(n)g(n) \leq -\frac{(\Delta^2 - 1)^2}{12}$. Therefore, $\frac{k^4}{36}\Delta^2(\Delta^2 - 4) \leq \gamma^2 \leq \frac{k^4}{48}(\Delta^2 - 1)^2$. Collision for $\{n, n + \Delta\} = \{-1, 1\}$ occur at $\gamma = 0$ and all other collision mentioned in Lemma 5 occur for $\gamma^2 \in [\frac{k^4}{48}(\Delta^2 - 4), \frac{k^4}{36}(\Delta^2 - 1)^2]$ with $\frac{k^4}{48}\Delta^2(\Delta^2 - 4) > 0$. This shows that for each $k > 0$, there exist $\gamma_0 \neq 0$ such that all the collisions stated in Lemma 5 occur for $|\gamma| > |\gamma_0|$, except $\{n, n + \Delta\} = \{-1, 1\}$. 
3.2 Nonperiodic perturbations

A standard perturbation argument assures that the $L^2(\mathbb{T})$-spectrum of $\mathcal{H}_a(y, \tau)$ and $\mathcal{H}_0(y, \tau)$ will stay close for $|a|$ sufficiently small. Therefore, in order to locate the spectrum of $\mathcal{H}_a(y, \tau)$, we need to determine the spectrum of $\mathcal{H}_0(y, \tau)$. A simple calculation yields that

$$\mathcal{H}_0(y, \tau)e^{inz} = i\Omega_{n,y,\tau}e^{inz}, \quad n \in \mathbb{Z},$$

where

$$\Omega_{n,y,\tau} = k^3(n + \tau)(1 - (n + \tau)^2) + \frac{3y^2}{k(n + \tau)}.$$

Therefore, the $L^2(\mathbb{T})$-spectrum of $\mathcal{H}_0(y, \tau)$ is given by

$$\text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(y, \tau)) = \{i\Omega_{n,y,\tau}; n \in \mathbb{Z}, \tau \in (-1/2, 1/2] \setminus \{0\}\}.$$  \hspace{1cm} (24)

**Remark 2.** For the case $b_1 \neq 0$, the $L^2(\mathbb{T})$-spectrum of $\mathcal{H}_0(y, \tau)$ is given by

$$\text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(y, \tau)) = \{i\Omega_{n,y,\tau}; n \in \mathbb{Z}, \tau \in (-1/2, 1/2] \setminus \{0\}\},$$

where

$$\Omega_{n,y,\tau} = k^3(n + \tau)(1 - (n + \tau)^2) + \frac{3y^2}{k(n + \tau)} + 3(n + \tau)\phi k b_1.$$

Since if $\mu \in \text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(y, \tau))$ then $\bar{\mu} \in \text{spec}_{L^2(\mathbb{T})}(\mathcal{H}_0(y, -\tau))$, therefore, it is enough to consider $\tau \in (0, 1/2]$. Let $n \neq m \in \mathbb{Z}$, a pair of eigenvalues $i\Omega_{n,y,\tau}$ and $i\Omega_{m,y,\tau}$ of $\mathcal{H}_0(y, \tau)$ collide for some $y = y_c$ and $\tau \in (0, 1/2]$ when

$$\Omega_{n,y_c,\tau} = \Omega_{m,y_c,\tau}.$$  \hspace{1cm} (25)

We list all the collisions in the following lemma.

**Lemma 6.** For a fix $\Delta \in \mathbb{N}$, eigenvalues $\Omega_{n,y,\tau}$ and $\Omega_{n+\Delta,y,\tau}$ of the operator $\mathcal{H}_0(y, \tau)$ collide for all $n \in [-\Delta, -1] \cap \mathbb{Z}$ along a curve $y = y_c(\tau)$, $\tau \in (0, 1/2]$; except $\{n, n + \Delta\} = \{-1, 0\}$. All such collisions take place away from the origin in the complex plane except when $\Delta$ is odd and $n = -(\Delta + 1)/2$ in which case eigenvalues $\Omega_{n,y,\tau}$ and $\Omega_{-n-1,y,\tau}$ collide at the origin for $y = y_c(1/2)$.

**Proof.** Without loss of generality, assume that $m > n$ and $m = n + \Delta$ with $\Delta \in \mathbb{N}$. Then from collision condition (25), we obtain

$$3y^2 = -k^4[3(n + \tau)^2(n + \tau + \Delta)^2 + (n + \tau)(n + \tau + \Delta)(\Delta^2 - 1)].$$  \hspace{1cm} (26)

This implies that collision between $n$ and $n + \Delta$ takes place only if $(n + \tau)(n + \tau + \Delta) < 0$, that is, $-\Delta \leq n < 0$. In order to check for which $n \in [-\Delta, 0]$, there is indeed a collision, assume $n = -t$, 

\( t \in \mathbb{N} \) such that \(-t + \tau + \Delta > 0\). From collision condition (25), we get

\[
3\gamma^2 \left( \frac{1}{t - \tau} + \frac{1}{-t + \tau + \Delta} \right) = k^4 [(t - \tau)((t - \tau)^2 - 1) + (-t + \tau + \Delta)((-t + \tau + \Delta)^2 - 1)].
\]

(27)

There exist such \( \gamma \) satisfying (25) for all \( t \) and \(-t + \Delta \), except \{\(-1, 0\)\}. Hence the lemma.

Note that \( \Omega_{n, \gamma, \tau} = 0 \) at \( \gamma^2 = -\frac{k^4}{3} \frac{(n + \tau)^2}{(n + \tau + \Delta)^2} \). \( \Omega_{n, \gamma_c, \tau} = \Omega_{n+\Delta, \gamma_c, \tau} = 0 \) for a fixed \( \gamma_c \) is possible only for \( \Delta = -2n - 1, \tau = 1/2 \). Therefore, \( \Omega_{n, \gamma_c, \tau} \) and \( \Omega_{n+\Delta, \gamma_c, \tau} \) collide at the origin for \( n = -(\Delta + 1)/2 \), for all \( n \in [-\Delta, -1] \cap \mathbb{Z}, \tau = 1/2 \), and \( \gamma_c^2 = \frac{k^4 (2n + 1)^2 (4n^2 + 4n - 3)}{48} \); except the pair \{\(n, n + \Delta\)\} = \{-1, 0\}. All other collisions are away from origin. From (26), assume \( \gamma^2 = -\frac{k^4}{3} d(n) h(n) \), where \( d(n) = (n + \tau)(n + \tau + \Delta) \) and \( h(n) = 3(n + \tau)^2 + 3(n + \tau)\Delta + \Delta^2 - 1 \).

\[
d(n) = (n + \tau)(n + \tau + \Delta) = (n + \tau)^2 + \Delta(n + \tau) = \left( n + \tau + \frac{\Delta}{2} \right)^2 - \frac{\Delta^2}{4},
\]

(28)

\[
h(n) = 3(n + \tau)^2 + 3(n + \tau)\Delta + \Delta^2 - 1 = 3 \left( n + \tau + \frac{\Delta}{2} \right)^2 + \frac{\Delta^2 - 4}{4}.
\]

(29)

From (28) and (29), for a fixed \( \Delta \in \mathbb{N} \), \( f(n) \geq -\frac{\Delta^2}{4} \), and \( g(n) \geq \frac{\Delta^2 - 4}{4} \) for all \( n \in \mathbb{Z} \). Collision for \( \Delta = 2 \) occur for \( \gamma^2 \geq \frac{k^4}{4} \frac{\Delta^2 (\Delta^2 - 4)}{2 - \tau} > 0 \) and all other collision mentioned in Lemma 6 occur for \( \gamma^2 \geq \frac{k^4}{48} \frac{\Delta^2 (\Delta^2 - 4)}{2 - \tau} > 0 \). Also \( \gamma^2 \leq \frac{k^4}{36} (\Delta^2 - 1)^2 \) for all \( \Delta \in \mathbb{N} \). Therefore, collision for \( \Delta = 2 \) occur for \( \frac{k^4}{4} \frac{\Delta^2 (\Delta^2 - 4)}{2 - \tau} \leq \gamma^2 \leq \frac{k^4}{36} \) and all other collisions occur for \( \gamma^2 \in \left[ \frac{k^4}{48} \frac{\Delta^2 (\Delta^2 - 4)}{2 - \tau}, \frac{k^4}{36} \right) \) with \( \frac{k^4}{48} \frac{\Delta^2 (\Delta^2 - 4)}{2 - \tau} > 0 \). This shows that for each \( k > 0 \), there exist \( \gamma_0 \neq 0 \) such that all the collisions stated in Lemma 6 occur for \( |\gamma| > |\gamma_0| \).

Since if \( \mu \in \text{spec}_{L^2(\mathbb{T})}(H_0(\gamma, \tau)) \) then \( \bar{\mu} \in \text{spec}_{L^2(\mathbb{T})}(H_0(\gamma, -\tau)) \), there will be collision between conjugate of eigenvalues mentioned in Lemma 6, for all \( \tau \in (-1/2, 0) \).

More specifically, collisions for \{-\Delta, 0\} occur for all \( \tau \in (0, 1/2) \), for \{0, \Delta\} occur for all \( \tau \in (-1/2, 0) \), and the remaining collisions occur for all \( \tau \in (-1/2, 1/2) \). The perturbation analysis for the collisions mentioned in Lemma 6 will be performed with respect to finite or short wavelength perturbations.

### 4 | MODULATIONAL TRANSVERSE (IN)STABILITIES

Throughout this subsection, we work in the regime \(|\gamma| \ll 1\), that is, with respect to long-wavelength perturbations. From Lemma 5, when \( \gamma = 0 \), there is a collision among the eigenvalues \( i\Omega_{1,0} \) and \( i\Omega_{-1,0} \) at the origin, while all other eigenvalues, on the other hand, remain simple and purely imaginary. Also, in the regime \(|\gamma| \ll 1\), there is no collision with respect to nonperiodic
perturbations. Since
\[ \| \mathcal{H}_a(\gamma) - \mathcal{H}_0(\gamma) \| = O(|a|) \]
as \( a \to 0 \) uniformly in the operator norm. A standard perturbation argument assures that the spectrum of \( \mathcal{H}_a(\gamma) \) and \( \mathcal{H}_0(\gamma) \) will stay close for \( |a| \) and \( |\gamma| \) small.\(^{35}\) Therefore, we may write that
\[ \text{spec}(\mathcal{H}_a(\gamma)) = \text{spec}_0(\mathcal{H}_a(\gamma)) \cup \text{spec}_1(\mathcal{H}_a(\gamma)) \]
for \( a \) and \( \gamma \) sufficiently small where \( \text{spec}_0(\mathcal{H}_a(\gamma)) \) contains two eigenvalues bifurcating continuously in \( a \) from \( i\Omega_{1,0} \) and \( i\Omega_{-1,0} \) while \( \text{spec}_1(\mathcal{H}_a(\gamma)) \) consists of infinitely many simple eigenvalues (see Ref. 35 and references therein). Furthermore, we investigate if the pair of eigenvalues in \( \text{spec}_0(\mathcal{H}_a(\gamma)) \) bifurcate away from the imaginary axis and contribute to modulational transverse instabilities.

For \( a = 0 \), \( \text{spec}_0(\mathcal{H}_0(\gamma)) = \{ i\Omega_{-1,0}, i\Omega_{1,0} \} \) with eigenfunctions \( \{ e^{-iz}, e^{iz} \} \). We choose the real basis \( \{ \cos z, \sin z \} \). We calculate expansion of a basis \( \{ \psi_1, \psi_2 \} \) for the eigenspace corresponding to the eigenvalues of \( \text{spec}_0(\mathcal{H}_a(\gamma)) \) in \( L^2_0(\mathbb{T}) \) by using expansions of \( w \) and \( c \) in (10), for small \( a \) and \( \gamma \) as
\[ \psi_1(z) = \cos z + 2aA_2 \cos 2z + 3a^2A_3 \cos 3z + O(a^4), \]
\[ \psi_2(z) = \sin z + 2aA_2 \sin 2z + 3a^2A_3 \sin 3z + O(a^4). \]

We have the following expression for \( \mathcal{H}_a(\gamma) \) after expanding and using \( w \) and \( c \)
\[ \mathcal{H}_a(\gamma) = \mathcal{H}_0(\gamma) + k a^2 \left( c_2 + 6 \rho A_0 - \frac{3}{4} \phi^2 \right) \partial_z + \left( 6k \rho a - 3 \phi^2 k A_0 a^3 - \frac{3}{2} \phi^2 k A_2 a^2 \right) \partial_z(\cos z) + \]
\[ k a^2 \left( 6 \rho A_2 - \frac{3}{4} \phi^2 \right) \partial_z(\cos 2z) + \left( 6 k \rho a^3 A_3 - \frac{3}{2} \phi^2 k a^3 A_2 \right) \partial_z(\cos 3z) + i 3 \gamma \phi (a \sin z + \]
\[ 2 a^2 A_2 \sin 2z + 3 a^3 A_3 \sin 3z) \partial_z^{-1} + O(a^4). \]

In order to locate the bifurcating eigenvalues for \( |a| \) sufficiently small, we calculate the action of \( \mathcal{H}_a(\gamma) \) on the extended eigenspace \( \{ \psi_1(z), \psi_2(z) \} \), namely.
\[ T_a(\gamma) = \begin{bmatrix} \langle \mathcal{H}_a(\gamma) \psi_i(z), \psi_j(z) \rangle \\ \langle \psi_i(z), \psi_j(z) \rangle \end{bmatrix}_{i,j=1,2} \quad \text{and} \quad I_a = \begin{bmatrix} \langle \psi_i(z), \psi_j(z) \rangle & \langle \psi_i(z), \psi_j(z) \rangle \\ \langle \psi_i(z), \psi_j(z) \rangle & \langle \psi_i(z), \psi_j(z) \rangle \end{bmatrix}_{i,j=1,2}. \]

We use expansion of \( \mathcal{H}_a(\gamma) \) in (30) to find actions of \( \mathcal{H}_a(\gamma) \) and identity operator on \( \{ \psi_1, \psi_2 \} \), and arrive at
\[ T_a(\gamma) = \begin{pmatrix} 0 & -3y^2/k + 3a^2k \left( \frac{\phi^2}{4} - \frac{\rho^2}{k^2} \right) \\ 3y^2/k & 0 \end{pmatrix} + O(a^2(\gamma + a)). \]

To locate where these two eigenvalues are bifurcating from the origin, we analyze the characteristic equation \( |T_a(\gamma) - \mu I| = 0 \), where \( I_a \) is \( 2 \times 2 \) identity matrix. From which we conclude
that
\[ \mu = \pm \frac{3|\gamma|}{k} \sqrt{\Lambda + O(a(\gamma + a))}, \]  
(32)

where
\[ \Lambda = -\gamma^2 + a^2k^2 \left( \frac{\phi^2}{4} - \frac{\rho^2}{k^2} \right) + O(a^2(\gamma + a)). \]  
(33)

For \( \gamma = a = 0 \), we get zero as a double eigenvalue, which agrees with our calculation. For \( \gamma \) and \( a \) sufficiently small, we obtain two eigenvalues which have nonzero real part with opposite sign when
\[ \gamma^2 < a^2k^2 \left( \frac{\phi^2}{4} - \frac{\rho^2}{k^2} \right) + O(a^2(\gamma + a)), \]  
(34)

which is possible only for
\[ k > 2 \frac{\rho}{\phi}. \]

Hence, Theorem 3.

5 HIGH-FREQUENCY TRANSVERSE (IN)STABILITIES

As discussed in Subsections 3.1 and 3.2, all the collisions occur for \( |\gamma| > |\gamma_0| > 0 \), therefore, here we work in the regime \( |\gamma| > |\gamma_0| > 0 \), that is, with respect to finite or short wavelength perturbations. Note that, there is no collision for \( \Delta = 1 \) and 2 among all collisions mentioned in Lemma 5. From Lemma 6, there are collisions for \( \Delta = 2 \) with respect to nonperiodic perturbations. For each \( \Delta \geq 3 \), there are collisions for both periodic as well as nonperiodic perturbations mentioned in Lemma 5 and 6, respectively.

(In)stability analysis for \( \Delta = 2 \).

For \( \Delta = 2 \), we have three pairs of colliding eigenvalues \( \{\Omega_{-1,\gamma,\tau}, \Omega_{1,\gamma,\tau}\}, \{\Omega_{0,\gamma,\tau}, \Omega_{-2,\gamma,\tau}\}, \) and \( \{\Omega_{0,0,\gamma,\tau}, \Omega_{2,\gamma,\tau}\} \). We further check if these pairs lead to instability.

Let \( i\Omega_{n,\gamma,\tau} \) and \( i\Omega_{n+2,\gamma,\tau} \) be such two eigenvalues for some \( n \in \mathbb{Z} \). Assume that these eigenvalues collide at \( \gamma = \gamma_c \), that is
\[ 0 \neq \Omega_{n,\gamma_c,\tau} = \Omega_{n+2,\gamma_c,\tau} = \Omega(\text{say}). \]  
(35)

That is, \( i\Omega \) is an eigenvalue of \( H_0(\gamma_c, \tau) \) of multiplicity two with an orthonormal basis of eigenfunctions. Let \( i\Omega + i\nu_{a,\tau} \) and \( i\Omega + i\nu_{a+2,\tau} \) be the eigenvalues of \( H_0(\gamma, \tau) \) bifurcating from \( i\Omega_{n,\gamma_c,\tau} \) and \( i\Omega_{n+2,\gamma_c,\tau} \) respectively, for \( |a| \) and \( |\gamma - \gamma_c| \) small. Let \( \{\varphi_{a,n}(z), \varphi_{a,n+2}(z)\} \) be a orthonormal basis for the corresponding eigenspace. We assume the following expansions:
\[ \varphi_{a,n}(z) = e^{inz} + a\varphi_{n,1} + a^2\varphi_{n,2} + O(a^3), \]  
(36)

\[ \varphi_{a,n+2}(z) = e^{i(n+2)z} + a\varphi_{n+2,1} + a^2\varphi_{n+2,2} + O(a^3). \]  
(37)
We use orthonormality of \( \varphi_{a,n,\gamma} \) and \( \varphi_{a,n+2,\gamma} \) to find that
\[
\varphi_{n,1} = \varphi_{n,2} = \varphi_{n+2,1} = \varphi_{n+2,2} = 0.
\]

Next, we calculate the action of \( H_a(\gamma, \tau) \) on the eigenspace \( \{\varphi_{a,n}(z), \varphi_{a,n+2}(z)\} \) for \( |\gamma - \gamma_c| \) and \( |a| \) small. We arrive at
\[
\mathcal{T}_a(\gamma, \tau) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} + O(a^3(y^2 + a^2)),
\]
where
\[
T_{11} = i\Omega + \frac{i3\varepsilon}{k(n+\tau)} - i(n+\tau)a^2k\left(\frac{3}{8}\phi^2 + \frac{3\rho^2}{2k^2}\right),
\]
\[
T_{12} = i(n+2+\tau)a^2k\left(\frac{3\rho^2}{2k^2} - \frac{3}{8}\phi^2\right) - \frac{i\alpha^23\gamma A_2}{(n+\tau)},
\]
\[
T_{21} = i(n+\tau)a^2k\left(\frac{3\rho^2}{2k^2} - \frac{3}{8}\phi^2\right) + \frac{i\alpha^23\gamma A_2}{(n+2+\gamma)},
\]
\[
T_{22} = i\Omega + \frac{i3\varepsilon}{k(n+2+\gamma)} - i(n+2+\gamma)a^2k\left(\frac{3}{8}\phi^2 + \frac{3\rho^2}{2k^2}\right),
\]
and \( \varepsilon = \gamma^2 - \gamma_c^2 \), sufficiently small. Furthermore, we obtained the equation \( \det(\mathcal{T}_a(\gamma, \tau) - (i\Omega + iv)I_a) = 0 \), where \( I_a \) is the \( 2 \times 2 \) identity matrix, and concluded the discriminant \( \text{disc}_a(\varepsilon) \) as
\[
\text{disc}_a(\varepsilon) = \frac{36\varepsilon^2}{k^2(n+\tau)^2(n+2+\tau)^2} - \frac{36\gamma_c^2\phi^2a^4A_2^2}{(n+\tau)(n+2+\tau)^2} + 9a^4k^2(n+\tau+1)^2\left(\frac{\rho^4}{k^4} + \frac{\phi^4}{16}\right)
\]
\[
+ \frac{9a^4\rho^2\phi^2}{2k^2}(1 - (n+\tau)(n+2+\tau)) + O(a^2|\varepsilon| + |a|^5).
\]

Note that all the collisions stated in Lemma 6 for \( \Delta = 2 \) have \( (n+\tau)(n+2+\tau) < 0 \) which implies that for \( |\varepsilon| \) and \( |a| \) sufficiently small, the leading term in the discriminant is always positive for all \( \rho \) and \( \phi \). Therefore, we do not get any instability for \( \Delta = 2 \) case for sufficiently small-amplitude parameter \( a \).

\textbf{(In)stability analysis for} \( \Delta \geq 3 \).

For some \( n \in \mathbb{Z}^* \) and a fixed \( \Delta \geq 3 \), we have
\[
i\Omega_{n,\Delta,\gamma,\tau} = i\Omega_{n+\Delta,\gamma,\tau} = i\Omega, \quad \tau \in (-1/2, 1/2].
\]
\( \tau = 0 \) corresponds to the periodic case and \( \tau \neq 0 \) corresponds to the nonperiodic case.

\[
H_a(\gamma) = H_0(\gamma) + (\beta_2a^2 + \beta_4a^4 + \ldots)(\partial_z + i\tau) + \alpha_1a(\partial_z + i\tau)(\cos z)
\]
\[
+ \ldots + \alpha_\Delta a^\Delta(\partial_z + i\tau)(\cos(\Delta z)) + (i\delta_1a \sin z + \ldots + i\delta_\Delta a^\Delta \sin(\Delta z))(\partial_z + i\tau)^{-1}.
\]

To explicitly obtain the values of all unknown coefficients in the expansion of \( H_a(\gamma, \tau) \), we require coefficients of higher powers of \( a \) in the expansion of solution \( w \). Calculating higher coefficients is
difficult as the coefficients of the solution do not seem to have any apparent symmetry. Therefore, we pursue the instability analysis without calculating the unknown coefficients explicitly.

Following the same steps as in the previous subsection, we arrive at

\[ T_a(\gamma, \tau) = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} + O(a^{\Delta+1}), \]

where

\[ T_{11} = i\Omega + \frac{i3\varepsilon}{k(n + \tau)} + i(n + \tau)(\beta_2 a^2 + \beta_4 a^4 + ...), \]

\[ T_{12} = \frac{i\alpha^\Delta}{2} \left( (n + \Delta + \tau)\alpha - \frac{\delta_\Delta}{n + \tau} \right), \]

\[ T_{21} = \frac{i\alpha^\Delta}{2} \left( (n + \tau)\alpha + \frac{\delta_\Delta}{n + \Delta + \tau} \right), \]

\[ T_{22} = i\Omega + \frac{i3\varepsilon}{k(n + \Delta + \tau)} + i(n + \Delta + \tau)(\beta_2 a^2 + \beta_4 a^4 + ...). \]

The resulting discriminant of the characteristic equation \( \det(T_a(\gamma, \tau) - (i\Omega + iv)I_a) = 0 \) is

\[ \text{disc}_a(\varepsilon) = \frac{9\Delta^2 \varepsilon^2}{k^2(n + \tau)^2(n + \Delta + \tau)^2} + \Delta^2 \beta_2^2 a^4 + O(a^2(|\varepsilon| + |a^3|)), \]

which is positive for sufficiently small \( |\varepsilon| \) and \( |a| \) which implies that no eigenvalue of \( H_a(\gamma) \) is bifurcating from the imaginary axis due to collision. Hence, Theorem 2.

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**DATA AVAILABILITY STATEMENT**

Data sharing not applicable to this article as no data sets were generated or analyzed during the current study.

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