A HOMOTOPY DECOMPOSITION OF THE FIBRE OF THE SQUARING MAP ON $\Omega^3 S^{17}$

STEVEN AMELOTTI

Abstract. We use Richter’s 2-primary proof of Gray’s conjecture to give a homotopy decomposition of the fibre $\Omega^3 S^{17}(2)$ of the $H$-space squaring map on the triple loop space of the 17-sphere. This induces a splitting of the mod-2 homotopy groups $\pi_*(S^{17};\mathbb{Z}/2)$ in terms of the integral homotopy groups of the fibre of the double suspension $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$ and refines a result of Cohen and Selick, who gave similar decompositions for $S^9$ and $S^{10}$. We relate these decompositions to various Whitehead products in the homotopy groups of mod-2 Moore spaces and Stiefel manifolds to show that the Whitehead square $[i_{2n}, i_{2n}]$ of the inclusion of the bottom cell of the Moore space $P^{2n+1}(2)$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16.

1. Introduction

For a based loop space $\Omega X$, let $\Omega X\{k\}$ denote the homotopy fibre of the $k^{th}$ power map $k : \Omega X \to \Omega X$. In [14] and [15], Selick showed that after localizing at an odd prime $p$, there is a homotopy decomposition $\Omega^2 S^{2p+1}\{p\} \simeq \Omega^2 S^3(3) \times W_p$, where $S^3(3)$ is the 3-connected cover of $S^3$ and $W_p$ is the homotopy fibre of the double suspension $E^2 : S^{2n-1} \to \Omega^2 S^{2n+1}$. Since $\Omega^2 S^{2p+1}\{p\}$ is homotopy equivalent to the pointed mapping space $\text{Map}_*(P^3(p), S^{2p+1})$ and the degree $p$ map on the Moore space $P^3(p)$ is nullhomotopic, an immediate consequence is that $p$ annihilates the $p$-torsion in $\pi_*(S^3)$ when $p$ is odd. In [16], Ravenel’s solution to the odd primary Arf–Kervaire invariant problem [12] was used to show that, at least for $p \geq 5$, similar decompositions of $\Omega^2 S^{2n+1}\{p\}$ are not possible if $n \neq 1$ or $p$.

The 2-primary analogue of Selick’s decomposition, namely that there is a 2-local homotopy equivalence $\Omega^2 S^5\{2\} \simeq \Omega^2 S^3(3) \times W_2$, was later proved by Cohen [4]. Similarly, since $\Omega^2 S^5\{2\}$ is homotopy equivalent to $\text{Map}_*(P^3(2), S^5)$ and the degree 4 map on $P^3(2) \simeq \Sigma RP^2$ is nullhomotopic, this product decomposition gives a “geometric” proof of James’ classical result that 4 annihilates the 2-torsion in $\pi_*(S^3)$. Unlike the odd primary case however, for reasons related to the divisibility of the Whitehead square $[i_{2n-1}, i_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$, the fibre of the squaring map on $\Omega^2 S^{2n+1}$ admits nontrivial product decompositions for some other values of $n$.

First, in their investigation of the homology of spaces of maps from mod-2 Moore spaces to spheres, Campbell, Cohen, Peterson and Selick [1] found that if $2n + 1 \neq 3, 5, 9$ or 17, then $\Omega^2 S^{2n+1}\{2\}$ is atomic and hence indecomposable. Following this, it was shown in [5] that after localization at the prime 2, there is a homotopy decomposition $\Omega^2 S^9\{2\} \simeq BW_2 \times W_4$ and $W_4$ is a retract of $\Omega^3 S^{17}\{2\}$. Here $BW_n$ denotes the classifying space of $W_n$ first constructed by Gray [6]. Since $BW_1$ is known to be homotopy equivalent to $\Omega^2 S^3(3)$, the pattern suggested by the decompositions of $\Omega^2 S^5\{2\}$ and $\Omega^2 S^9\{2\}$ led Cohen and Selick to conjecture that $\Omega^2 S^{17}\{2\} \simeq BW_4 \times W_8$.

In this note we prove this is true after looping once. (This weaker statement was also conjectured in [3].)

Theorem 1.1. There is a 2-local homotopy equivalence $\Omega^3 S^{17}\{2\} \simeq W_4 \times \Omega W_8$. 

In addition to the exponent results mentioned above, decompositions of $\Omega^n S^{2n+1}(p)$ also give decompositions of homotopy groups of spheres with $\mathbb{Z}/p\mathbb{Z}$ coefficients. Recall that the mod-$p$ homotopy groups of $X$ are defined by $\pi_k(X; \mathbb{Z}/p\mathbb{Z}) = [P^k(p), X]$.

**Corollary 1.2.** $\pi_k(S^{17}; \mathbb{Z}/2\mathbb{Z}) \cong \pi_{k-4}(W_4) \oplus \pi_{k-3}(W_8)$ for all $k \geq 4$.

In section 3 we relate the problem of decomposing $\Omega^2 S^{2n+1}(2)$ to a problem considered by Mukai and Skopenkov in [11] of computing a certain summand in a homotopy group of the mod-2 Moore space $P^{2n+1}(2)$—more specifically, the problem of determining when the Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ of the inclusion of the bottom cell $i_{2n} : S^{2n} \to P^{2n+1}(2)$ is divisible by 2. The indecomposability result for $\Omega^2 S^{2n+1}(2)$ in [11] (see also [2]) was proved by showing that for $n > 1$ the existence of a spherical homology class in $H_{4n-3}(\Omega^2 S^{2n+1}(2))$ imposed by a nontrivial product decomposition implies the existence of an element $\theta \in \pi_{2n-2}^S$ of Kervaire invariant one such that $\theta \eta$ is divisible by 2, where $\eta$ is the generator of the stable 1-stem $\pi_1^S$. Such elements are known to exist only for $2n = 4, 8, 16$. We show that the divisibility of the Whitehead square $[i_{2n}, i_{2n}]$ similarly implies the existence of such Kervaire invariant elements to obtain the following.

**Theorem 1.3.** The Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16.

This will follow from a preliminary result (Proposition 3.1) equating the divisibility of $[i_{2n}, i_{2n}]$ with the vanishing of a Whitehead product in the mod-2 homotopy of the Stiefel manifold $V_{2n+1,2}$, i.e., the unit tangent bundle over $S^{2n}$. It is shown in [11] that there do not exist maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ extending the inclusions of the bottom cell $S^{2n-1}$ and bottom Moore space $P^{2n}(2)$ if $2n \neq 2, 4, 8$ or 16. When $2n = 2, 4$ or 8, the Whitehead product obstructing an extension is known to vanish for reasons related to Hopf invariant one, leaving only the boundary case $2n = 16$ unresolved. We find that the Whitehead product is also trivial in this case.

2. The decomposition of $\Omega^3 S^{17}(2)$

The proof of Theorem 1.1 will make use of the 2-primary version of Richter’s proof of Gray’s conjecture, so we begin by reviewing this conjecture and spelling out some of its consequences. In his construction of a classifying space of the fibre $W_n$ of the double suspension, Gray [6] introduced two $p$-local homotopy fibrations

$$S^{2n-1} \xrightarrow{E^2} \Omega^2 S^{2n+1} \xrightarrow{\nu} BW_n$$

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi} S^{2np-1}$$

with the property that $j \circ \nu \simeq OH$, where $H : \Omega S^{2n+1} \to \Omega S^{2np+1}$ is the $p$th James–Hopf invariant. In addition, Gray showed that the composite $BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\nu} \Omega^2 S^{2np+1}$ is nullhomotopic and conjectured that the composite $\Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1} \xrightarrow{E^2} \Omega^2 S^{2np+1}$ is homotopic to the $p$th power map on $\Omega^2 S^{2np+1}$. This was recently proved by Richter in [14].

**Theorem 2.1 ([14]).** For any prime $p$, there is a homotopy fibration

$$BW_n \xrightarrow{j} \Omega^2 S^{2np+1} \xrightarrow{\phi_n} S^{2np-1}$$

such that $E^2 \circ \phi_n \simeq p$. 
For odd primes, it was shown in \cite{21} that there is a homotopy fibration $\Omega W_{np} \to BW_n \to \Omega^2 S^{2np+1} \{p\}$ based on the fact that a lift $\overline{S} : BW_n \to \Omega^2 S^{2np+1} \{p\}$ of $j$ can be chosen to be an $H$-map when $p$ is odd. One consequence of Theorem 2.1 is that this homotopy fibration exists for all primes and can be extended one step to the right by a map $\Omega^2 S^{2np+1} \{p\} \to W_{np}$.

**Lemma 2.2.** For any prime $p$, there is a homotopy fibration

$$BW_n \longrightarrow \Omega^2 S^{2np+1} \{p\} \longrightarrow W_{np}.$$  

*Proof.* The homotopy pullback of $\phi_n$ and the fibre inclusion $W_{np} \to S^{2np-1}$ of the double suspension defines a map $\Omega^2 S^{2np+1} \{p\} \to W_{np}$ with homotopy fibre $BW_n$, which can be seen by comparing fibres in the homotopy pullback diagram

$$
\begin{array}{c}
\begin{array}{c}
BW_n \\
\downarrow j \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega^2 S^{2np+1} \\
\downarrow \phi_n \\
S^{2np-1} \\
\downarrow p \\
\Omega^2 S^{2np+1} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
W_{np} \\
\downarrow \\
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega^2 S^{2np+1} \\
\downarrow \nu \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega^2 S^{4n+1} \\
\end{array}
\end{array}
\end{array}

\square

Looping once, we obtain a homotopy fibration

$$W_n \longrightarrow \Omega^3 S^{2np+1} \{p\} \longrightarrow \Omega W_{np}$$

which we will show is split when $p = 2$ and $n = 4$. We now fix $p = 2$ and localize all spaces and maps at the prime 2. Homology will be taken with mod-2 coefficients unless otherwise stated.

The next lemma describes a factorization of the looped second James–Hopf invariant, an odd primary version of which appears in \cite{21}. By a well-known result due to Barratt, $\Omega H : \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1}$ has order 2 in the group $[\Omega^2 S^{2n+1}, \Omega^2 S^{4n+1}]$ and hence lifts to a map $\Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1} \{2\}$. Improving on this, a feature of Richter’s construction of the map $\phi_n$ is that the composite $\Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1} \{2\}$ is nullhomotopic \cite{13} Lemma 4.2. This recovers Gray’s fibration $S^{2n-1} \to \Omega^2 S^{2n+1} \nu \to BW_n$ and the relation $j \circ \nu \simeq \Omega H$ since there then exists a lift $\nu : \Omega^2 S^{2n+1} \to BW_n$ making the diagram

$$
\begin{array}{c}

\begin{array}{c}
BW_n \\
\downarrow j \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\Omega^2 S^{2n+1} \\
\downarrow \Omega H \\
\Omega^2 S^{4n+1} \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\nu \\
\end{array}
\end{array}
\end{array}

$$

 commute up to homotopy. Since $j$ factors through $\Omega^2 S^{4n+1} \{2\}$, by composing the lift $\nu$ with the map $BW_n \to \Omega^2 S^{4n+1} \{2\}$ we obtain a choice of lift $S : \Omega^2 S^{2n+1} \to \Omega^2 S^{4n+1} \{2\}$ of the looped James–Hopf invariant. Hence we have the following consequence of Richter’s theorem.
Lemma 2.3. There is a homotopy commutative diagram

\[ \Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{4n+1} \{2\} \]

\[ \downarrow \nu \quad \downarrow \]

\[ BW_n \xrightarrow{\tau} \Omega^2 S^{4n+1} \{2\} \]

where \( S \) is a lift of the looped second James–Hopf invariant \( \Omega H : \Omega^2 S^{2n+1} \rightarrow \Omega^2 S^{4n+1} \) and the map \( BW_n \rightarrow \Omega^2 S^{4n+1} \{2\} \) has homotopy fibre \( \Omega W_{2n} \).

The following homological result was proved in [1] and used to obtain the homotopy decompositions of [4] and [5].

Lemma 2.4 ([1]). Let \( n \geq 2 \) and let \( f : X \rightarrow \Omega^2 S^{2n+1} \{2\} \) be a map which induces an isomorphism on the module of primitives in degrees \( 2n-2 \) and \( 4n-3 \). If the mod-2 homology of \( X \) is isomorphic to that of \( \Omega^2 S^{2n+1} \{2\} \) as a coalgebra over the Steenrod algebra, then \( f \) is a homology isomorphism.

Theorem 2.5. There is a homotopy equivalence \( \Omega^3 S^{17} \{2\} \simeq W_4 \times \Omega W_8 \).

Proof. Let \( \tau_n \) denote the map \( BW_n \rightarrow \Omega^2 S^{4n+1} \{2\} \) appearing in Lemma 2.2. By [1], \( \tau_n \) is a lift of \( j \), implying that \( \tau_n \) is nonzero in \( H_{4n-2} \{2\} \) by naturality of the Bockstein since \( j \) is nonzero in \( H_{4n-1} \{2\} \). We can therefore use the maps \( \tau_n \) in place of the (potentially different) maps \( \sigma_n \) used in [5] to obtain product decompositions of \( \Omega^2 S^{4n+1} \{2\} \) for \( n = 1 \) and \( 2 \), the advantage being that \( \tau_n \) has fibre \( \Omega W_{2n} \). Explicitly, for \( n = 2 \) this is done as follows. By [5 Corollary 2.1], there exists a map \( g : \Omega^3 S^{17} \{2\} \rightarrow \Omega^2 S^9 \{2\} \) which is nonzero in \( H_{13} \{2\} \). Letting \( \mu \) denote the loop multiplication on \( \Omega^2 S^9 \{2\} \), it follows that the composite

\[ \psi : BW_4 \times W_4 \xrightarrow{\tau_2 \times (g \circ \Omega f_4)} \Omega^2 S^9 \{2\} \times \Omega^2 S^9 \{2\} \xrightarrow{\mu} \Omega^2 S^9 \{2\} \]

induces an isomorphism on the module of primitives in degrees 6 and 13. Since \( H_* (BW_2 \times W_4) \) and \( H_* (\Omega^2 S^9 \{2\}) \) are isomorphic as coalgebras over the Steenrod algebra, the map above is a homology isomorphism by Lemma 2.4 and hence a homotopy equivalence.

Now the map \( \Omega f_4 \) fits in the homotopy fibration

\[ W_4 \xrightarrow{\Omega f_4} \Omega^3 S^{17} \{2\} \longrightarrow \Omega W_8 \]

and has a left homotopy inverse given by \( \pi_2 \circ \psi^{-1} \circ g \) where \( \psi^{-1} \) is a homotopy inverse of \( \psi \) and \( \pi_2 : BW_2 \times W_4 \rightarrow W_4 \) is the projection onto the second factor. (Alternatively, composing \( g : \Omega^3 S^{17} \{2\} \rightarrow \Omega^2 S^9 \{2\} \) with the map \( \Omega^2 S^9 \{2\} \rightarrow W_4 \) of Lemma 2.2 yields a left homotopy inverse of \( \Omega f_4 \).) It follows that the homotopy fibration above is fibre homotopy equivalent to the trivial fibration \( W_4 \times \Omega W_8 \rightarrow \Omega W_8 \).

\[ \square \]

Corollary 2.6. \( \pi_k (S^{17}; \mathbb{Z}/2\mathbb{Z}) \cong \pi_{k-4} (W_4) \oplus \pi_{k-3} (W_8) \) for all \( k \geq 4 \).

One consequence of the splitting of the fibration \( W_n \rightarrow \Omega^3 S^{4n+1} \{p\} \rightarrow \Omega W_{2n} \) when \( n \in \{1, 2, 4\} \) is a corresponding homotopy decomposition of the fibre of the map \( f \) appearing in Lemma 2.3. As in [18], we define the space \( Y \) and the map \( t \) by the homotopy fibration

\[ Y \xrightarrow{t} \Omega^2 S^{2n+1} \xrightarrow{S} \Omega^2 S^{4n+1} \{2\} \]

This space and its odd primary analogue play a central role in the construction of Anick’s fibration in [18], [21] and the alternative proof given in [20] of Cohen, Moore and Neisendorfer’s determination
of the odd primary homotopy exponent of spheres. Unlike at odd primes, the lift $S$ of $\Omega H$ cannot be chosen to be an $H$-map. Nevertheless, the corollary below shows that its fibre has the structure of an $H$-space in cases of Hopf invariant one.

**Corollary 2.7.** There is a homotopy fibration $S^{2n-1} \overset{f}{\to} Y \overset{g}{\to} \Omega W_{2n}$ with the property that the composite $S^{2n-1} \overset{f}{\to} Y \overset{g}{\to} \Omega E^2 S^{2n+1}$ is homotopic to the double suspension $E^2$. Moreover, if $n = 1, 2$ or 4 then the fibration splits, giving a homotopy equivalence

$$Y \simeq S^{2n-1} \times \Omega W_{2n}.$$

**Proof.** By Lemma 2.3, the homotopy fibration defining $Y$ fits in a homotopy pullback diagram

$$
\begin{array}{ccc}
S^{2n-1} & \overset{f}{\longrightarrow} & S^{2n-1} \\
| & | & | \\
f \downarrow & \simeq & \downarrow E^2 \\
\scriptstyle Y \overset{t}{\longrightarrow} \Omega^2 S^{2n+1} & \overset{\nu}{\longrightarrow} & \Omega^2 S^{4n+1} \{2\} \\
| & | & | \\
g \downarrow & & \downarrow \\
\Omega W_{2n} & \longrightarrow & BW_n \longrightarrow \Omega^2 S^{4n+1} \{2\},
\end{array}
$$

which proves the first statement. Note that when $n = 1, 2$ or 4, the map $\Omega W_{2n} \to BW_n$ is nullhomotopic by Theorem 1.1, hence $t$ lifts through the double suspension. Since any choice of a lift $Y \to S^{2n-1}$ is degree one in $H_{2n-1}$, it also serves as a left homotopy inverse of $f$, which implies the asserted splitting.

**Remark 2.8.** The first part of Corollary 2.7 and an odd primary version are proved by different means in [18] and [20], respectively (see Remark 6.2 of [18]). At odd primes, there is an analogous splitting for $n = 1$:

$$Y \simeq S^1 \times \Omega W_p \simeq S^1 \times \Omega^3 T^{2p^2+1}(p)$$

where $T^{2p^2+1}(p)$ is Anick’s space (see [19]).

3. Relations to Whitehead products in Moore spaces and Stiefel manifolds

The special homotopy decompositions of $\Omega^3 S^{2n+1} \{2\}$ discussed in the previous section are made possible by the existence of special elements in the stable homotopy groups of spheres, namely elements of Arf–Kervaire invariant one $\theta \in \pi_{2n-2}^s$ such that $\theta \eta$ is divisible by 2. In this section, we give several reformulations of the existence of such elements in terms of mod-2 Moore spaces and Stiefel manifolds.

Let $i_{n-1} : S^{n-1} \to P^n(2)$ be the inclusion of the bottom cell and let $j_n : P^n(2) \to P^n(2)$ be the identity map. Similarly, let $i'_{2n-1} : S^{2n-1} \to V_{2n+1,2}$ and $j'_{2n} : P^{2n}(2) \to V_{2n+1,2}$ denote the inclusions of the bottom cell and bottom Moore space, respectively.

**Proposition 3.1.** The Whitehead product $[i'_{2n-1}, j'_{2n}] \in \pi_{4n-2}(V_{2n+1,2} ; \mathbb{Z}/2\mathbb{Z})$ is trivial if and only if the Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ is divisible by 2.

**Proof.** Let $\lambda : S^{4n-2} \to P^{2n}(2)$ denote the attaching map of the top cell in $V_{2n+1,2} \simeq P^{2n}(2) \cup e^{4n-1}$ and note that $[i'_{2n-1}, j'_{2n}] = j'_{2n} \circ [i_{2n-1}, j_{2n}]$ by naturality of the Whitehead product. The map $[i_{2n-1}, j_{2n}]$:

$\footnote{Note that we index these maps by the dimension of their source rather than their target, so the element of $\pi_{4n-1}(P^{2n+1}(2))$ we call $[i_{2n}, i_{2n}]$ is called $[i_{2n+1}, j_{2n+1}]$ in [17].}$
$P^{4n-2}(2) \to P^{2n}(2)$ is essential since its adjoint is a Samelson product with nontrivial Hurewicz image $[u, v] \in H_{4n-3}(\Omega P^{2n}(2))$, where $H_*(\Omega P^{2n}(2))$ is isomorphic as an algebra to the tensor algebra $T(u, v)$ with $|u| = 2n - 2$ and $|v| = 2n - 1$ by the Bott–Samelson theorem. Since the homotopy fibre of the inclusion $j_2^n : P^{2n}(2) \to V_{2n+1}$ has $(4n - 2)$-skeleton $S^{4n-2}$ which maps into $P^{2n}(2)$ by the attaching map $\lambda$, it follows that $[i'_{2n-1}, j'_{2n}]$ is trivial if and only if $[i_{2n-1}, j_{2n}]$ is homotopic to the composite

\[ P^{4n-2}(2) \xrightarrow{q'} S^{4n-2} \xrightarrow{\lambda} P^{2n}(2) \]

where $q$ is the pinch map.

To ease notation let $P^n$ denote the mod-2 Moore space $P^n(2)$ and consider the morphism of $EHP$ sequences

\[
\begin{array}{ccccccc}
[S^{4n}, P^{2n+1}] & \xrightarrow{H} & [S^{4n}, \Sigma P^{2n} \land P^{2n}] & \xrightarrow{P} & [S^{4n-2}, P^{2n}] & \xrightarrow{E} & [S^{4n-1}, P^{2n+1}] \\
\downarrow q' & & \downarrow q' & & \downarrow q' & & \downarrow q' \\
[P^{4n}, P^{2n+1}] & \xrightarrow{H} & [P^{4n}, \Sigma P^{2n} \land P^{2n}] & \xrightarrow{P} & [P^{4n-2}, P^{2n}] & \xrightarrow{E} & [P^{4n-1}, P^{2n+1}]
\end{array}
\]

induced by the pinch map. A homology calculation shows that the $(4n)$-skeleton of $\Sigma P^{2n} \land P^{2n}$ is homotopy equivalent to $P^{4n} \vee S^{4n}$. Let $k_1 : P^{4n} \to \Sigma P^{2n} \land P^{2n}$ and $k_2 : S^{4n} \to \Sigma P^{2n} \land P^{2n}$ be the composites $P^{4n} \hookrightarrow P^{4n} \vee S^{4n} \simeq \text{sk}_4(S^{4n} \land P^{2n}) \hookrightarrow \Sigma P^{2n} \land P^{2n}$ and $S^{4n} \hookrightarrow P^{4n} \vee S^{4n} \simeq \text{sk}_4(S^{4n} \land P^{2n}) \hookrightarrow \Sigma P^{2n} \land P^{2n}$ defined by the left and right wedge summand inclusions, respectively. Then we have that $\pi_{4n}(\Sigma P^{2n} \land \Sigma P^{2n}) = \mathbb{Z}/4\mathbb{Z}\{k_2\}$ and $P(k_2) = \pm 2\lambda$ by [9] Lemma 12. It follows from the universal coefficient exact sequence

\[ 0 \to \pi_{4n}(\Sigma P^{2n} \land P^{2n}) \otimes \mathbb{Z}/2\mathbb{Z} \to \pi_{4n}(\Sigma P^{2n} \land P^{2n}; \mathbb{Z}/2\mathbb{Z}) \to \text{Tor}(\pi_{4n-1}(\Sigma P^{2n} \land P^{2n}), \mathbb{Z}/2\mathbb{Z}) \to 0 \]

that

\[ \pi_{4n}(\Sigma P^{2n} \land P^{2n}; \mathbb{Z}/2\mathbb{Z}) = [P^{4n}, \Sigma P^{2n} \land P^{2n}] = \mathbb{Z}/2\mathbb{Z}\{k_1\} \oplus \mathbb{Z}/2\mathbb{Z}\{k_2 \circ q\} \]

and that the generator $k_2 \circ q$ is in the kernel of $P$ since $P(k_2) = \pm 2\lambda$ implies

\[ P(k_2 \circ q) = P(q^*(k_2)) = q^*(P(k_2)) = \pm \lambda \circ q = 0 \]

by the commutativity of the above diagram and the fact that $q : P^{4n-2} \to S^{4n-2}$ and $2 : S^{4n-2} \to S^{4n-2}$ are consecutive maps in a cofibration sequence. Therefore $[i_{2n-1}, j_{2n}] = P(k_1)$ since the suspension of a Whitehead product is trivial. On the other hand, $\Sigma \lambda$ is homotopic to the composite $S^{4n-1} \xrightarrow{[i_{2n}, j_{2n}]} S^{2n} \xrightarrow{2} P^{2n+1}$ by [9], which implies $E(\lambda \circ q) = i_{2n} \circ [i_{2n}, j_{2n}] \circ q = [i_{2n}, i_{2n}] \circ q$ is trivial in $[P^{4n-1}, P^{2n+1}]$ precisely when $[i_{2n}, i_{2n}]$ is divisible by 2. Hence $[i_{2n}, i_{2n}]$ is divisible by 2 if and only if $\lambda \circ q = P(k_1) = [i_{2n-1}, j_{2n}] \in [P^{4n-2}, P^{2n}]$, and the proposition follows.

We use Proposition 5.1 in two ways. First, since the calculation of $\pi_{31}(P^{17}(2))$ in [10] shows that $[i_{16}, i_{16}] = 2\tilde{\sigma}_{16}^2$ for a suitable choice of representative $\tilde{\sigma}_{16}$ of the Toda bracket $\{\sigma_{16}^2, 2i_{16}, i_{16}\}$, it follows
that the Whitehead product $[i_{15}, j_{16}] : P^{30}(2) \to V_{17,2}$ is nullhomotopic and hence there exists a map $S^{15} \times P^{16}(2) \to V_{17,2}$ extending the wedge of skeletal inclusions $S^{15} \vee P^{16}(2) \to V_{17,2}$. This resolves the only case left unsettled by Theorem 3.2 of [17].

In the other direction, note that such maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ restrict to maps $S^{2n-1} \to V_{2n+1,2}$ which exist only in cases of Kervaire invariant one by [22], Proposition 2.27, so Proposition 3.1 shows that when $2n \neq 2^k$ for some $k \geq 1$ the Whitehead square $[i_{2n}, i_{2n}]$ cannot be divisible by 2 for the same reasons that the Whitehead square $[i_{2n-1}, i_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$ cannot be divisible by 2. Moreover, since maps $S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ extending the inclusions of $S^{2n-1}$ and $P^{2n}(2)$ are shown not to exist for $2n > 16$ in [17], Proposition 3.1 implies that the Whitehead square $[i_{2n}, i_{2n}]$ is divisible by 2 if and only if $2n = 2, 4, 8$ or 16. In all other cases it generates a $\mathbb{Z}/2\mathbb{Z}$ summand in $\pi_{4n-1}(P^{2n+1}(2))$. This improves on the main theorem of [11] which shows by other means that $[i_{2n}, i_{2n}]$ is not divisible by 2 when $2n$ is not a power of 2.

These results are summarized in Theorem 3.3 below. First we recall the following well-known equivalent formulations of the Kervaire invariant problem.

**Theorem 3.2 ([2], [22]).** The following are equivalent:

(a) The Whitehead square $[i_{2n-1}, i_{2n-1}] \in \pi_{4n-3}(S^{2n-1})$ is divisible by 2;
(b) There is a map $P^{4n-2}(2) \to \Omega S^{2n}$ which is nonzero in homology;
(c) There exists a space $X$ with mod-2 cohomology $\tilde{H}^i(X) \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 2n$, $4n - 1$, $4n$ and zero otherwise with $Sq^{2n} : H^{2n}(X) \to H^{2n+1}(X)$ and $Sq^1 : H^{4n+1}(X) \to H^{4n}(X)$ isomorphisms;
(d) There exists a map $f : S^{2n-1} \times S^{2n-1} \to V_{2n+1,2}$ such that $f|_{S^{2n-1} \times S^{2n-1}}$ is the inclusion of the bottom cell;
(e) $n = 1$ or there exists an element $\theta \in \pi_{2n-2}^S$ of Kervaire invariant one.

The above conditions hold for $2n = 2, 4, 8, 16, 32$ and 64, and the recent solution to the Kervaire invariant problem by Hill, Hopkins and Ravenel [8] implies that, with the possible exception of $2n = 128$, these are the only values for which the conditions hold. Mimicking the reformulations above we obtain the following.

**Theorem 3.3.** The following are equivalent:

(a) The Whitehead square $[i_{2n}, i_{2n}] \in \pi_{4n-1}(P^{2n+1}(2))$ is divisible by 2;
(b) There is a map $P^{4n}(2) \to \Omega P^{2n+2}(2)$ which is nonzero in homology;
(c) There exists a space $X$ with mod-2 cohomology $\tilde{H}^i(X) \cong \mathbb{Z}/2\mathbb{Z}$ for $i = 2n + 1, 2n + 2, 4n + 1, 4n + 2$ and zero otherwise with $Sq^{2n} : H^{2n+1}(X) \to H^{2n+2}(X)$ and $Sq^1 : H^{4n+2}(X) \to H^{4n+1}(X)$ isomorphisms;
(d) There exists a map $f : S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2}$ such that $f|_{S^{2n-1} \times P^{2n}(2)}$ are the skeletal inclusions of $S^{2n-1}$ and $P^{2n}(2)$, respectively;
(e) $n = 1$ or there exists an element $\theta \in \pi_{2n-2}^S$ of Kervaire invariant one such that $\eta \theta$ is divisible by 2;
(f) $2n = 2, 4, 8$ or 16.

**Proof.** (a) is equivalent to (b). In the $n = 1$ case, $[i_2, i_2] = 2\eta_2$ implies $[i_2, i_2] = 0$, and since $\eta_3 \in \pi_4(S^3)$ has order 2 its adjoint $\tilde{\eta}_3 : S^3 \to \Omega S^3$ extends to a map $P^4(2) \to \Omega S^3$. If this map desuspends, then $\tilde{\eta}_3$ would be homotopic to a composite $S^3 \to P^4(2) \to S^2 \xrightarrow{E} \Omega S^3$, a contradiction since $\pi_3(S^3) \cong \mathbb{Z}$ implies...
that any map \( S^3 \to S^2 \) that factors through \( P^4(2) \) is nullhomotopic. Hence the map \( P^4(2) \to \Omega S^3 \) has nontrivial Hopf invariant in \([P^4(2), \Omega S^3]\) from which it follows that \( P^4(2) \to \Omega S^3 \) is nonzero in \( H_4(\cdot) \). Composing with the inclusion \( \Omega S^3 \to \Omega P^4(2) \) gives a map \( P^4(2) \to \Omega P^4(2) \) which is nonzero in \( H_4(\cdot) \).

Now suppose \( n > 1 \) and \([i_{2n}, i_{2n}]\) is a multiple of \( 2\alpha \) for some \( \alpha \in \pi_{4n-1}(P^{2n+1}(2)) \). Then \( \Sigma \alpha \) has order 2 so there is an extension \( P^{2n+1}(2) \to P^{2n+2}(2) \) whose adjoint \( f : P^{4n}(2) \to \Omega P^{2n+2}(2) \) satisfies \( f|_{S^{4n-1}} = E \circ \alpha \). Since \( \Omega \Sigma(P^{2n+1}(2) \wedge P^{2n+2}(2)) \) has 4n-skeleton \( S^{4n} \), to show that \( f_* \) is nonzero on \( H_4n(P^{4n}(2)) \) it suffices to show that \( H_2 \circ f \) is nontrivial in \([P^{4n}(2), \Omega \Sigma(P^{2n+1}(2) \wedge P^{2n+2}(2))] \) where \( H_2 : \Omega P^{2n+2}(2) \to \Omega \Sigma(P^{2n+1}(2) \wedge P^{2n+2}(2)) \) is the second James–Hopf invariant. If \( H_2 \circ f \) is nullhomotopic, then there is a map \( g : P^{4n}(2) \to P^{2n+2}(2) \) making the diagram

\[
P^{2n+1}(2) \xrightarrow{E} \Omega P^{2n+2}(2) \xrightarrow{H_2} \Omega \Sigma(P^{2n+1}(2) \wedge P^{2n+2}(2))
\]

commute. But then \( \alpha - g|_{S^{4n-1}} \) is in the kernel of \( E_* : \pi_{4n-1}(P^{2n+1}(2)) \to \pi_{4n}(P^{2n+2}(2)) \) which is generated by \([i_{2n}, i_{2n}]\), so \( \alpha - g|_{S^{4n-1}} \) is a multiple of \([i_{2n}, i_{2n}]\). Since \([i_{2n}, i_{2n}]\) has order 2 and clearly \( 2g|_{S^{4n-1}} = 0 \), it follows that \([i_{2n}, i_{2n}] = 2\alpha = 0 \), a contradiction. Therefore \( f_* \) is nonzero on \( H_{4n}(P^{4n}(2)) \).

Conversely, assume \( n > 1 \) and \( f : P^{4n}(2) \to \Omega P^{2n+2}(2) \) is nonzero in \( H_{4n}(\cdot) \). Since the restriction \( f|_{S^{4n-1}} \) lifts through the \((4n-1)\)-skeleton of \( \Omega P^{2n+2}(2) \), there is a homotopy commutative diagram

\[
\begin{array}{ccc}
S^{4n-1} & \xrightarrow{g} & P^{4n}(2) \\
\downarrow & & \downarrow \quad f \\
P^{2n+1}(2) & \xrightarrow{E} & \Omega P^{2n+2}(2)
\end{array}
\]

for some map \( g : S^{4n-1} \to P^{2n+1}(2) \). Since \( E \circ 2g \) is nullhomotopic, \( 2g \) is a multiple of \([i_{2n}, i_{2n}]\). But if \( 2g = 0 \), then \( g \) admits an extension \( e : P^{4n}(2) \to P^{2n+1}(2) \) and it follows that \( f - E \circ e \) factors through the pinch map \( g : P^{4n}(2) \to S^{4n} \). This makes the Pontrjagin square \( u^2 \in H_{4n}(\Omega P^{2n+2}(2)) \) a spherical homology class, and this is a contradiction which can be seen as follows. If \( u^2 \) is spherical, then the 4n-skeleton of \( \Omega P^{2n+2}(2) \) is homotopy equivalent to \( P^{2n+1}(2) \vee S^{4n} \). On the other hand, it is easy to see that the attaching map of the 4n-cell in \( \Omega P^{2n+2}(2) \) is given by the Whitehead square \([i_{2n}, i_{2n}]\) which is nontrivial as \( n > 1 \), whence \( P^{2n+1} \cup_{[i_{2n}, i_{2n}]} \mathbb{C}^4 \not\simeq P^{2n+1}(2) \vee S^{4n} \).

\([a]\) is equivalent to \([d]\). Since the Whitehead product \([i_{2n-1}', j_{2n}'] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z}) \) is the obstruction to extending \( i_{2n-1}' \vee j_{2n}' : S^{2n-1} \vee P^{2n}(2) \to V_{2n+1,2} \) to \( S^{2n-1} \times P^{2n}(2) \), this follows immediately from Proposition 2.1.

As described in [17], applying the Hopf construction to a map \( f : S^{2n-1} \times P^{2n}(2) \to V_{2n+1,2} \) as in \([d]\) yields a map \( H(f) : P^{4n}(2) \to \Sigma V_{2n+1,2} \) with \( S^q \) acting nontrivially on \( H^{2n}(C_{H(f)}) \). Since \( \Sigma^2 V_{2n+1,2} \simeq P^{2n+2}(2) \vee S^{2n-1} \), composing the suspension of the Hopf construction \( H(f) \) with a retract \( \Sigma^2 V_{2n+1,2} \to P^{2n+2}(2) \) defines a map \( g : P^{4n+1}(2) \to P^{2n+2}(2) \) with \( S^q \) acting nontrivially on \( H^{2n+1}(C_g) \), so \([d]\) implies \([e]\).

By the proof of [17] Theorem 3.1, \([e]\) implies \([e]\) and \([e]\) implies \([f]\). The triviality of the Whitehead product \([i_{2n-1}', j_{2n}'] \in \pi_{4n-2}(V_{2n+1,2}; \mathbb{Z}/2\mathbb{Z}) \) when \( n = 1, 2 \) or \( 4 \) is implied by [17] Theorem 2.1, for
example, and Proposition 3.1 implies $[i'_1, i'_6] \in \pi_30(V_{17,2}; \mathbb{Z}/2\mathbb{Z})$ is trivial as well since $[i_1, i_6] \in \pi_31(P^{17}(2))$ is divisible by 2 by [10, Lemma 3.10]. Thus (f) implies (d) \hfill \Box

4. A LOOP SPACE DECOMPOSITION OF $J_3(S^2)$

In this section, we consider some relations between the fibre bundle $S^{4n-1} \rightarrow V_{4n+1,2} \rightarrow S^{4n}$ defined by projection onto the first vector of an orthonormal 2-frame in $\mathbb{R}^{4n+1}$ (equivalently, the unit tangent bundle over $S^{4n}$) and the fibration $BW_n \rightarrow \Omega^2 S^{4n+1}\{2\} \rightarrow W_{2n}$ of Lemma 2.2. Letting $\partial : \Omega S^{4n} \rightarrow S^{4n-1}$ denote the connecting map of the first fibration, we will show that there is a morphism of homotopy fibrations

\[
\begin{array}{ccc}
\Omega^2 S^{4n} & \xrightarrow{\Omega \partial} & \Omega S^{4n-1} \\
| & | & |
\Omega W_{2n} & \xrightarrow{\partial} & \Omega^2 S^{4n+1}\{2\}
\end{array}
\]

(2)

from which it will follow that for $n = 1, 2$ or 4, $\Omega \partial$ lifts through $\Omega \phi_n : \Omega^3 S^{4n+1} \rightarrow \Omega^3 S^{4n-1}$. If this lift can be chosen to be $\Omega^2 E$, then it follows that there is a homotopy pullback diagram

\[
\begin{array}{ccc}
\Omega^2 V_{4n+1,2} & \xrightarrow{\Omega \partial} & \Omega^3 S^{4n} \\
| & | & | \\
W_n & \xrightarrow{\partial} & \Omega S^{4n-1}
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
| & | & | \\
\Omega^3 S^{8n+1} & \xrightarrow{\Omega^2 H} & \Omega^3 S^{8n+1}
\end{array}
\]

(3)

which identifies $\Omega^2 V_{4n+1,2}$ with $\Omega M_3(n)$ where $\{ M_k(n) \}_{k \geq 1}$ is the filtration of $BW_n$ studied in [7] beginning with the familiar spaces $M_1(n) \simeq \Omega S^{4n-1}$ and $M_2(n) \simeq S^{4n-1}\{2\}$. (Spaces are localized at an odd prime throughout [7] but the construction of the filtration works in the same way for $p = 2$.) We verify this (and deloop it) for $n = 1$ since it leads to an interesting loop space decomposition which gives isomorphisms $\pi_k(V_{5,2}) \cong \pi_k(J_3(S^2))$ for all $k \geq 3$.

In his factorization of the $4^\text{th}$-power map on $\Omega^2 S^{2n+1}$ through the double suspension, Theriault constructs in [18] a space $A$ and a map $\mathcal{E} : A \rightarrow \Omega S^{2n+1}\{2\}$ with the following properties:

(a) $H_4(A) \cong \Lambda (x_{2n-1}, x_{2n})$ with Bockstein $\beta x_{2n} = x_{2n-1}$;
(b) $\mathcal{E}$ induces a monomorphism in homology;
(c) There is a homotopy fibration $S^{2n-1} \rightarrow A \rightarrow S^{2n}$ and a homotopy fibration diagram

\[
\begin{array}{ccc}
S^{2n-1} & \xrightarrow{E^2} & A \\
| & | & |
\Omega^2 S^{2n+1} & \xrightarrow{\mathcal{E}} & \Omega S^{2n+1}\{2\}
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
| & | & | \\
\Omega^2 S^{2n+1} & \xrightarrow{\mathcal{E}} & \Omega S^{2n+1}\{2\}.
\end{array}
\]

Noting that the homology of $A$ is isomorphic to the homology of the unit tangent bundle $\tau(S^{2n})$ as a coalgebra over the Steenrod algebra, Theriault raises the question of whether $A$ is homotopy equivalent to $\tau(S^{2n}) = V_{2n+1,2}$. Our next proposition shows this is true for any space $A$ with the properties above.
Proposition 4.1. There is a homotopy equivalence $A \simeq V_{2n+1,2}$.

Proof. First we show that $A$ splits stably as $P^{2n} \vee S^{4n-1}$. As in [18], let $Y$ denote the $(4n-1)$-skeleton of $\Omega S^{2n+1}\{2\}$. Consider the homotopy fibration

$$\Omega S^{2n+1}\{2\} \longrightarrow \Omega S^{2n+1} \overset{2}{\longrightarrow} \Omega S^{2n+1}$$

and recall that $H_*(\Omega S^{2n+1}\{2\}) \cong H_*(\Omega S^{2n+1}) \otimes H_*(\Omega^2 S^{2n+1})$. Restricting the fibre inclusion to $Y$ and suspending once we obtain a homotopy commutative diagram

$$\begin{array}{cccc}
\Sigma Y & \longrightarrow & \Sigma \Omega S^{2n+1}\{2\} & \longrightarrow & \Sigma \Omega S^{2n+1} \\
\downarrow \ell & & \downarrow \Sigma \ell & & \downarrow \Sigma \ell \\
S^{2n+1} & \longrightarrow & \Sigma \Omega S^{2n+1}\{2\} & \longrightarrow & \Sigma \Omega S^{2n+1}
\end{array}$$

where $2$ is the degree 2 map, the vertical maps are inclusions of the bottom cell of $\Sigma \Omega S^{2n+1}$ and a lift $\ell$ inducing an isomorphism in $H_{2n+1}(\ )$ exists since $\Sigma Y$ is a 4n-dimensional complex and $\text{sk}_{4n}(\Sigma \Omega S^{2n+1}) = S^{2n+1}$. It follows from the James splitting $\Sigma \Omega S^{2n+1} \cong \bigvee_{i=1}^\infty S^{2n+i+1}$ and the commutativity of the diagram that $2 \circ \ell$ is nullhomotopic, so in particular $\Sigma \ell$ lifts to the fibre $S^{2n+2}\{2\}$ of the degree 2 map on $S^{2n+2}$. Since $H_*(S^{2n+2}\{2\}) \cong \mathbb{Z}/2[\mu_{2n+1}] \otimes \Lambda(v_{2n+2})$ with $\beta v_{2n+2} = u_{2n+1}$, this implies $\Sigma \ell$ factors through a map $r : \Sigma^2 Y \rightarrow P^{2n+2}\{2\}$ which is an isomorphism in homology by naturality of the Bockstein, and hence $P^{2n+2}\{2\}$ is a retract of $\Sigma^2 Y$. (Alternatively, $r$ can be obtained by suspending a lift $\Sigma Y \rightarrow S^{2n+1}\{2\}$ of $\ell$ and using the well-known fact that $\Sigma S^{2n+1}\{2\}$ splits as a wedge of Moore spaces.) Now since $\mathbb{F} : A \rightarrow \Omega S^{2n+1}\{2\}$ factors through $Y$ and induces a monomorphism in homology, composing $\Sigma^2 A \rightarrow \Sigma^2 Y$ with the retraction $r$ shows that $\Sigma^2 A \simeq \Sigma^2 (P^{2n}(2) \vee S^{4n-1})$.

Next, let $E^\infty : A \rightarrow QA$ denote the stabilization map and let $F$ denote the homotopy fibre of a map $g : QP^{2n}(2) \rightarrow K(\mathbb{Z}/2, 4n-2)$ representing the mod-2 cohomology class $u_{2n-1}^2 \in H^{4n-2}(QP^{2n}(2))$. A homology calculation shows that the $(4n-1)$-skeleton of $F$ is a three-cell complex with homology isomorphic to $A(x_{2n-1}, x_{2n})$ as a coalgebra. The splitting $\Sigma^2 A \cong \Sigma^2 (P^{2n}(2) \vee S^{4n-1})$ gives rise to a map $\pi_1 : QA \simeq QP^{2n}(2) \times QS^{4n-1} \rightarrow QP^{2n}(2)$ inducing isomorphisms in $H_{2n-1}(\ )$ and $H_{2n}(\ )$, and since the composite $g \circ \pi_1 = E^\infty : A \rightarrow K(\mathbb{Z}/2, 4n-2)$ is nullhomotopic, there is a lift $A \rightarrow F$ inducing isomorphisms in $H_{2n-1}(\ )$ and $H_{2n}(\ )$. The coalgebra structure of $H_*(A)$ then implies this lift is a $(4n-1)$-equivalence and the result follows as $V_{2n+1,2}$ can similarly be seen to be homotopy equivalent to the $(4n-1)$-skeleton of $F$.

The homotopy commutative diagram (2) is now obtained by noting that the composite $\Omega S^{4n-1} \rightarrow \Omega V_{4n+1,2} \overset{\Omega E^2}{\longrightarrow} \Omega^2 S^{4n+1}\{2\}$ is homotopic to $\Omega S^{4n-1} \overset{\Omega E^2}{\longrightarrow} \Omega^3 S^{4n+1} \rightarrow \Omega^2 S^{4n+1}\{2\}$, which in turn is homotopic to a composite $\Omega S^{4n-1} \rightarrow BW_n \rightarrow \Omega^2 S^{4n+1}\{2\}$ since by Theorem 2.1 there is a homotopy fibration diagram

$$\begin{array}{cccc}
\Omega S^{4n-1} & \longrightarrow & BW_n & \longrightarrow & \Omega^2 S^{4n+1} \\
\downarrow \Omega E^2 & & \downarrow \phi_n & & \downarrow 2 \\
\Omega^3 S^{4n+1} & \longrightarrow & \Omega^2 S^{4n+1}\{2\} & \longrightarrow & \Omega^2 S^{4n+1}
\end{array}$$
Specializing to the case \( n = 1 \), the proof of Proposition \( \ref{prop:homotopy-fibration} \) will show that \( \Omega V_{5,2} \) fits in a delooping of diagram \( \ref{fig:delooping} \) and hence that \( \Omega V_{5,2} \simeq M_{3}(1) \). We will need the following cohomological characterization of \( V_{5,2} \).

**Lemma 4.2.** Let \( E \) be the total space of a fibration \( S^{3} \to E \to S^{4} \). If \( E \) has integral cohomology group \( H^{1}(E; \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \) and mod-2 cohomology ring \( H^{*}(E) \) an exterior algebra \( \Lambda(u, v) \) with \( |u| = 3 \) and \( |v| = 4 \), then \( E \) is homotopy equivalent to the Stiefel manifold \( V_{5,2} \).

**Proof.** As shown in \([22, \text{Theorem 5.8}]\), the top row of the homotopy pullback diagram

\[
\begin{array}{ccc}
X^{4} & \longrightarrow & P^{4}(2) \longrightarrow BS^{3} \\
\downarrow & & \downarrow q \\
S^{7} & \longrightarrow & S^{4} \longrightarrow BS^{3}
\end{array}
\]

induces a split short exact sequence

\[
0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \longrightarrow \pi_{6}(P^{4}(2)) \longrightarrow \pi_{5}(S^{3}) \longrightarrow 0
\]

from which it follows that \( \pi_{6}(P^{4}(2)) = \mathbb{Z}/4\mathbb{Z}\{\lambda\} \oplus \mathbb{Z}/2\mathbb{Z}\{\eta_{2}^{3}\} \) where \( \lambda \) is the attaching map of the top cell of \( V_{5,2} \) and \( \eta_{2}^{3} \) maps to the generator \( \eta_{2}^{3} \) of \( \pi_{5}(S^{3}) \). It follows from the cohomological assumptions that \( E \simeq P^{4}(2) \cup e^{7} \), where \( f = a\lambda + b\eta_{2}^{3} \) for some \( a \in \mathbb{Z}/4\mathbb{Z}, b \in \mathbb{Z}/2\mathbb{Z} \), and that \( H_{*}(\Omega E) \) is isomorphic to a polynomial algebra \( \mathbb{Z}/2\mathbb{Z}[u_{2}, v_{3}] \). Since the looped inclusion \( \Omega P^{4}(2) \to \Omega E \) induces the abelianization map \( T(u_{2}, v_{3}) \to \mathbb{Z}/2\mathbb{Z}[u_{2}, v_{3}] \) in homology, it is easy to see that the adjoint \( f : S^{5} \to \Omega P^{4}(2) \) of \( f \) has Hurewicz image \( [u_{2}, v_{3}] = u_{2} \otimes v_{3} + v_{3} \otimes u_{2} \) and hence \( f \) is not divisible by 2. Moreover, since \( E \) is an \( S^{3} \)-fibration over \( S^{4} \), the pinch map \( q : P^{4}(2) \to S^{4} \) must extend over \( E \). This implies the composite \( S^{6} \xrightarrow{f} P^{4}(2) \xrightarrow{q} S^{4} \) is nullhomotopic and therefore \( b = 0 \) by the commutativity of the diagram above. It now follows that \( f = \pm \lambda \) which implies \( E \simeq V_{5,2} \). \( \square \)

**Proposition 4.3.** There is a homotopy fibration

\[
V_{5,2} \longrightarrow J_{3}(S^{2}) \longrightarrow K(\mathbb{Z}, 2)
\]

which is split after looping.

**Proof.** Let \( h \) denote the composite \( \Omega S^{3}(3) \to \Omega S^{3} \xrightarrow{H} \Omega S^{5} \) and consider the pullback

\[
\begin{array}{ccc}
P & \longrightarrow & S^{4} \\
\downarrow & & \downarrow E \\
\Omega S^{3}(3) & \xrightarrow{h} & \Omega S^{5}.
\end{array}
\]

Since \( h \) has homotopy fibre \( S^{3} \), so does the map \( P \to S^{4} \). Next, observe that \( P \) is the homotopy fibre of the composite \( \Omega S^{3}(3) \xrightarrow{h} \Omega S^{5} \xrightarrow{H} \Omega S^{9} \) and since \( \Omega S^{9} \) is 7-connected, the inclusion of the 7-skeleton of \( \Omega S^{3}(3) \) lifts to a map \( \text{sk}_{7}(\Omega S^{3}(3)) \to P \). Recalling that \( H^{1}(\Omega S^{3}(3); \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \) and \( H_{*}(\Omega S^{3}(3)) \cong \Lambda(u_{3}) \otimes \mathbb{Z}/2\mathbb{Z}[v_{4}] \) with generators in degrees \( |u_{3}| = 3 \) and \( |v_{4}| = 4 \), it follows that this lift must be a homology isomorphism and hence a homotopy equivalence. So \( P \) is homotopy equivalent to the total space of a fibration satisfying the hypotheses of Lemma \( \ref{lem:homotopy-equivalence} \) and there is a homotopy equivalence \( P \simeq V_{5,2} \).
It is well known that the iterated composite of the \( p \)-th James-Hopf invariant \( H^{ok} : \Omega S^{2n+1} \to \Omega S^{2np+1} \) has homotopy fibre \( J_{p^k - 1}(S^{2n}) \), the \( (p^k - 1)^{st} \) stage of the James construction on \( S^{2n} \). The argument above identifies \( V_{5,2} \) with the homotopy fibre of the composite
\[
\Omega S^3(3) \longrightarrow \Omega S^3 \xrightarrow{H^2} \Omega S^5 \xrightarrow{H} \Omega S^9,
\]
so there is a homotopy pullback diagram
\[
\begin{array}{ccc}
V_{5,2} & \longrightarrow & J_3(S^2) \longrightarrow K(\mathbb{Z}, 2) \\
\downarrow & & \downarrow \\
\Omega S^3(3) & \longrightarrow & \Omega S^3 \longrightarrow K(\mathbb{Z}, 2) \\
\downarrow & & \downarrow \\
\Omega S^9 & \longrightarrow & \Omega S^9 \\
\end{array}
\]
where the maps into \( K(\mathbb{Z}, 2) \) represent generators of \( H^2(J_3(S^2); \mathbb{Z}) \cong \mathbb{Z} \) and \( H^2(\Omega S^3; \mathbb{Z}) \cong \mathbb{Z} \). To see that the homotopy fibration along the top row splits after looping, note that the connecting map \( \Omega K(\mathbb{Z}, 2) = S^1 \to V_{5,2} \) is nullhomotopic since \( V_{5,2} \) is simply-connected. Therefore the looped projection map \( \Omega J_3(S^2) \to S^1 \) has a right homotopy inverse producing a splitting \( \Omega J_3(S^2) \simeq S^1 \times \Omega V_{5,2} \). \( \square \)

**Corollary 4.4.** \( \pi_k(J_3(S^2)) \cong \pi_k(V_{5,2}) \) for all \( k \geq 3 \).

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