Thermoelastic-plastic fracture problems based on the Dugdale-Barenblatt crack model for functionally graded coatings

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Abstract. This paper is devoted to the problem for the thermal fracture of functionally graded coatings on a homogeneous substrate (FGC/H) with pre-existing systems of cracks with plasticity included in the model. For the elastic-plastic crack problem for FGCs the Dugdale-Barenblatt model is used. This work is a part of the semi-analytical model for the fracture analysis of FGC/H structures. As a preliminary analysis of the fracture problem for FGC/H structures, the crack interaction problem of a main crack (with plastic zones) and with small cracks is used. Approximate analytical solutions are obtained for vertical displacement jumps, which determine the crack tip opening displacement, the main fracture characteristic.

1. Introduction

Many engineering systems operate under elevated temperatures and mechanical loads, e.g. in aerospace and airplane industry, as well as in power engineering. In order to withstand such environment and to increase the effectiveness and durability of engineering components (e.g. turbine blades), new materials and new concepts for designing the materials are challenging tasks for researchers. Functionally graded materials (FGMs) are one of these concepts. FGMs are composites that consist of one or more component materials with spatial gradation with respect to material properties [1]. Ceramic/metal FGMs are used for thermal barrier coatings with ceramic materials on the heating surface of the structure and with a continuous (or layered) decrease in the volume fraction of the ceramic to the metal with the depth of coating. Due to the influence of thermal and mechanical loads on functionally graded coatings (FGCs), different fracture mechanisms are observed in FGCs. In this connection a fundamental understanding of fracture of FGCs is important for improving the fracture resistance of FGCs.

The paper is devoted to a theoretical study of the thermal fracture of FGCs on a homogeneous substrate (FGC/H) with pre-existing systems of cracks with plasticity included in the model. As mentioned in paper [2] devoted to Zener-Stroh cracks, “the crack propagation always starts from the sharp tip. By carrying out the plastic zone correction, failure analysis becomes more accurate thus able to avoid overestimation in composite structure design and enhance the service life of the structure.” In the present study for the elastic-plastic crack problem for FGCs an approach based on the Dugdale-Barenblatt model (DB-model) [3, 4] is used. There exist already many investigations for the DB-model or various modified DB-models for cracks, e.g. for a periodic array of parallel cracks in FGMs with possible crack closure under thermal loading [5], for three collinear cracks in a homogeneous...
material [6] and macro-microcrack interactions in homogeneous materials [7]. The crack closure effects (without plasticity) were also considered in a series of papers for the interaction problem of a main crack with systems of microcracks [8-11]. However, crack plasticity effects were not considered in the frame of the FGCs systems.

The paper is organized as follows. The formulation of the problem is presented in Section 2, where the geometry of the problem, the loading and models for FGMs are described, and also the boundary conditions and assumptions are formulated. Then, in Section 3, singular integral equations are written for the problem; and, afterwards, the solution of these equations is presented in the form of series (for a special case when one crack is much larger than the others). An illustrative example is presented in Section 4 to show the influence of a system of small cracks for different non-homogeneity parameters of thermal expansion coefficient on the opening displacements on the main crack. Some concluding remarks are also made in Section 4.

2. Formulation of the problem

2.1. Geometry of the problem

The geometry of the problem is depicted in figure 1, i.e. a FGC/H structure with pre-existing arbitrarily located cracks under thermo-mechanical loads. The layer of thickness \( h \) is made from a functionally graded material (FGM), and the semi-infinite substrate consists of a homogeneous material. The FGC contains arbitrary located cracks of length \( 2a_k \) \((k = 1, \ldots, N)\), which can be internal and/or edge cracks as well as an interface crack. The following systems of coordinates are chosen: the global coordinates \((x, y)\) with the \( x \)-axis located on the surface of the FGC/H structure and the local coordinates \((x_k, y_k)\) connected with cracks and with centers in the midpoint of the \( k \)-th crack. The crack positions are determined explicitly by the midpoint coordinates \( z_k^0 \) and the inclination angles \( \alpha_k \) to the \( x \)-axis (figure 1). The general case of multiple cracks with open zones \((-c_k, c_k)\) and coordinate of centers \( z_k^0 \) of these open zones and closed zones near crack tips is depicted in figure 2.

![Figure 1. An FGC/H structure with a system of cracks.](image1)

![Figure 2. System of partially closed cracks.](image2)

The FGC/H structure is subjected to thermal and/or mechanical loads. The loads could be the following: a thermal flux \( q \), tensile load and cooling by the temperature \( \Delta T \). In the case of cooling the tensile residual stresses are observed in the FGC [13].
2.2. Assumptions, loads and boundary conditions

Using the superposition principle, the remotely applied load (far from cracks) is reduced to the load applied to the crack surfaces (denoted \( p^{\text{appl}} \)), i.e. the problem is solved with boundary conditions prescribed to the crack surfaces. Besides, since the problem is considered in the frame of linear fracture mechanics (in exception of plastic zones at cracks which have local character), the results for each load can be superimposed.

In the case of an FGC/H under a heat flux of intensity \( q \) the problem is solved in two steps, first, the thermal problem for the FGC/H structure with cracks, and then the thermo-elastic problem for the same geometry. These resultant thermal residual stresses (denoted \( p^T \)), caused by the perturbation of heat flux due to the cracks (first problem), are taken into account in the thermo-elastic problem (second problem).

It is assumed that the thermal and mechanical properties of a functionally graded material in the FGC/H structure are continuous functions of the thickness coordinate \( y \). Furthermore, the non-homogeneity of the functionally graded material is revealed in the form of corresponding inhomogeneous stress distributions on the surfaces of the cracks [14, 15]. This method is approximate and used with the assumption, that the gradation of material properties of the FGC with the depth of the layer is not abrupt. Hence, these residual stresses should be added to the formulation of the problem. The stresses due to the non-homogeneity of Young’s modulus are denoted by \( p^e \); and those due to thermal loading through the temperature change by \( \Delta T \) are denoted by \( \sigma^T_n \).

In accordance with the above, the problem is solved with conditions on the crack surfaces. Let \( p_n \) denote the resulting stresses on the surfaces of the \( n \)-th crack.

The model chosen for the elastic-plastic problem for FGCs is based on the Dugdale-Barenblatt model (DB-model) [3, 4]. It means that plasticity is localized in the vicinity of crack tips, whereas at a certain distance from the crack the material can still be deformed elastically. The DB-model assumes that the crack tips are closed by stresses \( p \) and these stresses are equal to the yield stress \( \sigma_y \) of the material. The general system of cracks in an FGC with open and closed parts is depicted in figure 2. The boundary conditions on the \( n \)-th crack line is written as

\[
p_n(x_n) = \begin{cases} 
-p_n, & -c_{1n} < x_n < c_{2n} \\
-p_n + p_n - \sigma_y, & -c_{1n} \leq x_n \leq c_{1n} \text{ and } c_{2n} \leq x_n < a_n
\end{cases}
\]  

(1)

Here \( p_n \) are the stresses due to the remotely applied load and additional residual stresses, i.e.

\[
p_n = p^e + p^T + p^{\text{appl}}.
\]

(2)

In the frame of the formulated general problem consider the following one. A main crack contains an open part \([ -c_1, -c_2 ] \) and two plastic zones \([ -a_0, -c_1 ] \) and \([ c_2, a_0 ] \), which are closed by stresses \( p = \sigma_y \), figure 3. The other cracks are open. The boundary conditions on the main crack \( 2a_0 \) are defined by equation (1) with \( n = 0 \), and on the other cracks \( 2a_n \) – by equation (2).

2.3. Material modelling

The thermal and mechanical properties of an FGC are continuous functions of the thickness coordinate \( y \). As it was adopted in the previous works [10, 11], the exponential form of these properties is used:
The problem is formulated with respect to these complex potentials, i.e. the boundary conditions for the plane strain state and

\[ f(y) = f_1 \exp(\zeta_n (y + h)), \quad -h \leq y \leq 0, \]

\[ f = [k, \quad \alpha_t, \quad E], \quad f_1 = \{k_1, \quad \alpha_{t1}, \quad E_1\}, \quad \zeta_n = [\delta, \quad \epsilon, \quad \omega], \]

where \( k \) is the thermal conductivity, \( \alpha_t \) – thermal expansion coefficient and \( E \) – the Young’s modulus with non-homogeneity parameters \( \delta \), \( \epsilon \) and \( \omega \), respectively. \( f_1 \) are thermal and mechanical properties of the homogeneous substrate. The Poisson’s ratio is assumed to be constant and is equal to the value of the homogeneous substrate.

The values of the dimensionless graded parameters \( \zeta_n h \) (\( h \) is the thickness of the FGC, figure 1) are obtained from equation (2) as

\[ \zeta_n h = \ln(f_2 / f_1), \quad f_2 = f(y)_{|y=0} \quad f_1 = f(y)_{|y=-h} \]  

(4)

The FGMs inhomogeneity is accounted via the continuously varying residual stresses as following [14, 15]:

\[ \sigma^T_{xx}(y) = [\alpha_t(y) - \alpha_{t1}] \Delta T E(y), \quad \sigma^e_{xx}(y) = [E(y)/E_1 - 1]\sigma^0_{xx}, \quad \sigma^0_{xx} = p^{appl}, \]  

(5)

\( \alpha_t \) and \( E_1 \) are, respectively, the thermal expansion coefficient and the Young’s modulus of a homogeneous material and at the interface, i.e. in the region \( y \leq -h \); \( \alpha_t(y) \) and \( E(y) \) are defined by equation (3). In the common case of FGMs (thermally and elastically graded), the full load on the \( n \)-th crack consists of \( p_n \), \( \sigma^e_{xx} \) and \( \sigma^e_{yy} \) (the index “\( n \)” denotes that the functions are written in the local coordinate systems \((x_n, y_n)\) connected with the cracks), see equation (2). In the local coordinate system \((x_n, y_n)\) connected with each crack the material parameters \( f \) (equation (3)) take the form

\[ f(x_n/a) = f_1 \exp[\zeta_n a(h/a + y_n^0/a + x_n/a \sin \alpha_n)], \]  

(6)

where \( a = \max a_n \). In equation (6) the relation between global coordinate \((x, y)\) and local coordinate \((x_n, y_n)\) systems was used \( z = z_n^0 + z_n e^{i\alpha_n} \).

3. Formulation of the integral equations and solution

3.1. Formulation of equations

Because the plasticity is considered through the DB-model, the equations of the problem are formulated within the framework of the elasticity theory. For plane elasticity, the stress and displacement fields are expressed in terms of two potentials of complex variables, which are called Kolosov-Muskhelishvili potentials [16]. For crack modeling an additional function is introduced, namely, the jumps of displacements on the crack lines, and their derivatives \( g_n'(x) \) are used in further formulation

\[ g_n'(x) = -\frac{2\mu}{i(k+1)} \frac{\partial}{\partial x} (|u_n| + i|v_n|), \]  

(7)

where \([u_n]\) and \([v_n]\) are shear and vertical displacement jumps, respectively, on the \( n \)-th crack line, \( \mu = E/2(1+\nu) \) is the shear modulus, \( E \) - Young’s modulus, \( \nu \) - Poisson’s ratio, \( k = 3 - 4\nu \) for the plane strain state and \( \kappa = (3-\nu)/(1+\nu) \) for the plane stress state.

The problem is formulated with respect to these complex potentials, i.e. the boundary conditions are rewritten in terms of these potentials. Then, satisfying the boundary conditions, a series of Riemann-Hilbert problems on the crack lines are obtained. The solution of the Riemann-Hilbert problem for a single crack is defined by the Cauchy type integral. For a system of cracks the transformation of coordinate systems is used for obtaining the complex potentials. Finally, satisfying
all boundary conditions the system of differential equations reduces to a system of singular integral equations with respect to unknowns $g_n'(x)$ and is written as

$$\int_{-a_n}^{a_n} \frac{g_n'(t)}{t-x} \, dt + \sum_{k=1}^{N} \int_{-a_k}^{a_k} \left[ g_k'(t)R_{nk}(t,x) + \overline{g_k'(t)}S_{nk}(t,x) \right] dt = \pi p_n(x), \quad |x| < a_n \quad (n=0, 1, 2, \ldots, N), \quad (8)$$

$$\int_{-a_n}^{a_n} g_n'(t) \, dt = 0.$$

Details of the method for deriving equations similar to equation (8) can be found in [17].

In equations (8) $R_{nk}$ and $S_{nk}$ are regular functions containing the parameters of geometry of the problem, i.e. the coordinates of the centers of cracks and their angles of inclination; an overbar $\overline{...}$ denotes the complex conjugate. $N+1$ is the number of cracks.

The initial system (8) is divided into two sets of real and imaginary parts for determination of the shear $[u_n]$ and vertical $[v_n]$ displacement jumps on crack lines with corresponding vertical and shear loads, see [18]. It should be noted, that the crack closure is defined by the vertical displacements $[v_n]$.

### 3.2. Solution of the equation

The solution of the system (8) can be obtained numerically in the general case. Besides, for some special cases approximate analytical solutions are also possible. The crack interaction problem of a main crack (with plastic zones) and small cracks is used for preliminary analysis of the fracture problem for FGC/H structures.

As in previous papers [7, 10, 11], the solution of equation (8) is obtained in a series form as

$$g_n'(x) = g_{n0}'(x) + \lambda^2 g_{n2}'(x) \quad (9)$$

where the small parameter $\lambda$ is equal to the ratio of the typical length of the small cracks to the main crack size, i.e. $\lambda = a/a_0$ with $a = a_0$. Equation (9) refers to the main crack (with the index “0”). The crack closure is defined by the vertical displacements $[v_n]$, so we are interested in the real part of equation (9). After extracting this real part in (9) and integrating the expression, the solution is written as

$$v_0(x) = v_{00}(x) + \lambda^2 v_{02}(x), \quad v_{02}(x) = \sum_{k=1}^{N} f_k(x). \quad (10)$$

Here, $v_{00}$ is the solution for the main crack with plastic zones, and $v_{02}$ – the influence of other small cracks.

The crack opening displacement (COD) $\delta$ is defined by $v_0$, equation (10), for the main crack. The opening displacement at the crack tip (CTOD) is denoted by $\delta_c$.

### 3.3. Fracture characteristics

A suitable fracture criterion for a crack defined by the DB-model is based on the crack tip opening displacement (CTOD) [19, 20]. According to this criterion the crack propagation is predicted to occur when the magnitude of CTOD at a fixed distance behind the current crack tip reaches a critical value [20]. In our case CTOD = $\delta_c = v_0$. The crack starts to propagate when $\delta_c = \delta_{cr}$. Applying the CTOD criterion for FGCs special additional assumptions should be adopted, e.g. a rule for the determination of the intrinsic fracture toughness near crack tips. The critical value $\delta_{cr}$ is a material parameter which is obtained experimentally.

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4. Results and discussion

The solution is derived for the case of a thermo-mechanical load and the loading functions are used as defined in equations (2), (5) and (6). In [7] this model was applied for the problem of a homogeneous material for the interaction between a system of microcracks and a main crack with plastic zones.

In figure 4 an illustrative example is presented to show the influence of a system of five small cracks for different non-homogeneity parameters of thermal expansion coefficient $\epsilon a_0 = 0.2$ and 0.4 on the opening displacements $v_0(\chi)$ on the main crack, $\chi$ is the non-dimensional coordinate on the crack line, i.e. $\chi = x_k^0/a_0$. Function $v_{02}$, equation (10), is responsible for this influence for the small cracks. Figure 4a corresponds to the system of cracks with the coordinate centers $x_k^0/a_0 = y_k^0/a_0 = (1, \ldots, 5)$ and inclination angles $\alpha_k = (0, \ldots, \pi/2)$; and figure 4b – to the cracks with the coordinates $x_k^0/a_0 = (-1.5, \ldots, 1.5), y_k^0/a_0 = 0.9$ and inclination angles $\alpha_k = \pi/4$.

![Figure 4](image)

**Figure 4.** Opening displacements $v_{02}$ on the main crack as function of the non-dimensional crack coordinate $\chi$ as influenced by a system of five small cracks with the following coordinate centers and angles: (a) $x_k^0/a_0 = y_k^0/a_0 = (1, \ldots, 5), \alpha_k = (0, \ldots, \pi/2)$ and (b) $x_k^0/a_0 = (-1.5, \ldots, 1.5), y_k^0/a_0 = 0.9, \alpha_k = \pi/4$, for different $\epsilon a_0 = 0.2$ and 0.4.

This approach and the resulting explicit solution provide the possibility for an explicit parametric analysis of the problem. In the frame of the DB-model not only the main crack can be considered with closure parts, but also the other cracks could be closed by stresses associated with the yield stresses $\sigma_t$. It was shown in previous papers [10, 11], when the crack closure was considered, that with the assumption of a frictionless contact between the crack faces the crack closure affects the stress intensity factors (SIFs) Mode I $k_I$ at the main crack tips; and SIFs Mode II $k_{II}$ are calculated disregarding these contacts. Microcrack closure influences both $k_I$ and $k_{II}$, but not much. The SIFs $k_I$ and $k_{II}$ are calculated using corresponding normal and shear displacements, so that this conclusion is also valid for the functions COD $v_0$ and $u_0$ (normal and shear displacements), see equations (7), (9) and (10).

The problem contains geometric parameters, such as the sizes of the cracks and sizes of the closed part of the main crack, the coordinates of the crack centers and the inclination angles, the parameters of loads, i.e. main remote load, thermal residual stresses and additional stresses $p$ on the plastic zones, and the parameters of material non-homogeneity, determined by formula (2). The presented model allows analyzing the influence of these parameters on the main fracture characteristics of the problem.

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