CONSTRUCTION OF HYPERBOLIC HYPERSURFACES OF LOW DEGREE IN $\mathbb{P}^n(\mathbb{C})$

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Abstract

We construct families of hyperbolic hypersurfaces $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geq (\frac{n+3}{2})^2$.

Keywords: Kobayashi conjecture, hyperbolicity, Brody Lemma, Nevanlinna Theory

1 Introduction and the main result

It was conjectured by Kobayashi [12] in 1970 that a generic hypersurface $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ of sufficiently high degree $d \geq d(n) \gg 1$ is hyperbolic. According to Zaidenberg [20], the optimal degree bound should be $d(n) = 2n + 1$.

This conjecture, with nonoptimal degree bound in the assumption, was proved, in the case of surface in $\mathbb{P}^3(\mathbb{C})$, by Demailly and El Goul [6], and later, by Păun [14] with a slight improvement of the degree bound, and in the case of three-fold in $\mathbb{P}^4(\mathbb{C})$ [15], [8]. For arbitrary $n$, it was proved in [7] that any entire curve in generic hypersurface $X_d \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geq 2n^5$ must be algebraically degenerate. An improvement of the effective degree bound in this result was given in [1]. Recently, for any dimension $n$, a positive answer for generic hypersurfaces of degree $d \geq d(n) \gg 1$ very high was proposed by Siu [18], and a strategy which is expected to give a confirmation of this conjecture for very generic hypersurfaces of degree $d \geq 2n + 2$ was announced by Demailly [3].

Another direction on this subject is to construct examples of hyperbolic hypersurfaces of low degree. In low dimensional case, several examples of hyperbolic hypersurfaces were given. The first example of a hyperbolic surface in $\mathbb{P}^3(\mathbb{C})$ was constructed by Brody and Green [2]. In $\mathbb{P}^3(\mathbb{C})$, Duval [4] gave an example of a hyperbolic surface of degree 6, which is the lowest degree found up to date. Later, Ciliberto and Zaidenberg [5] gave a new construction of hyperbolic surface of degree 6 and their method works for all degree $d \geq 6$ (hence, this is the first time when a hyperbolic surface of degree 7 was created). In [11], we constructed families of hyperbolic hypersurfaces of degree $d = d(n) = 2n + 2$ for $2 \leq n \leq 5$ (the method works for all $d \geq 2n + 2$). The first examples in any dimension $n \geq 4$ were discovered by Masuda and Noguchi [13], with high degree. Improving this result, examples of hyperbolic hypersurfaces of lower degree asymptotic were given by Siu and Yeung [16] with $d(n) = 16n^2$, and by Shiffman and Zaidenberg [19] with $d(n) = 4n^2$.

In this note, using the technique of [11], we improve the result of Shiffman and Zaidenberg [19] by proving that a small deformation of a union of $q \geq (\frac{n+3}{2})^2$ hyperplanes in general position in $\mathbb{P}^{n+1}(\mathbb{C})$ is hyperbolic.

A family of hyperplanes $\{H_i\}_{1 \leq i \leq q}$ with $q \geq n + 1$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in general position if any $n + 1$ hyperplanes in this family have empty intersection, namely if

$$\cap_{i \in I} H_i = \emptyset, \quad \forall I \subset \{1, \ldots, q\}, |I| = n + 1.$$

Let $\{H_i\}_{1 \leq i \leq q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. A hypersurface $S$ in $\mathbb{P}^n(\mathbb{C})$ is said to be in general position with respect to $\{H_i\}_{1 \leq i \leq q}$ if it avoids all intersection points of $n$ hyperplanes, namely if

$$S \cap (\cap_{i \in I} H_i) = \emptyset, \quad \forall I \subset \{1, \ldots, q\}, |I| = n.$$

Main Theorem. Let $\{H_i\}_{1 \leq i \leq q}$ be a family of $q \geq (\frac{n+3}{2})^2$ hyperplanes in general position in $\mathbb{P}^{n+1}(\mathbb{C})$, where $H_i = \{h_i = 0\}$. Then there exists a hypersurface $S = \{s = 0\}$ of degree $q$ in general position with respect to $\{H_i\}_{1 \leq i \leq q}$ such that the hypersurface

$$\Sigma_c = \{cs + \prod_{i=1}^q h_i = 0\}$$
is hyperbolic for sufficiently small complex $\epsilon \neq 0$.

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## 2 Preparations

### 2.1 Brody Lemma and its applications

Let $X$ be a compact complex manifold equipped with a hermitian metric $\| \cdot \|$. An *entire curve* in $X$ is a nonconstant holomorphic map $f : \mathbb{C} \to X$. Such an $f : \mathbb{C} \to X$ is called a *Brody curve* if its derivative $\| f' \|$ is bounded. The following result [1] is a useful tool for studying complex hyperbolicity.

**Brody Lemma.** Let $f_k : \mathbb{D} \to X$ be a sequence of holomorphic maps from the unit disk to a compact complex manifold $X$. If $\| f'_k(0) \| \to \infty$ as $k \to \infty$, then there exist a point $a \in \mathbb{D}$, a sequence $(a_k)$ converging to $a$ and a decreasing sequence $(r_k)$ of positive real numbers converging to 0 such that the sequence of maps

$$z \to f_k(a_k + r_k z)$$

converges toward a Brody curve, after extracting a subsequence.

Consequently, we have a well-known characterization of Kobayashi hyperbolicity.

**Brody Criterion.** A compact complex manifold $X$ is Kobayashi hyperbolic if and only if it contains no entire curve.

The following form of the Brody Lemma shall be repeatedly used in the proof of the Main Theorem.

**Sequences of entire curves.** Let $X$ be a compact complex manifold and let $(f_k)$ be a sequence of entire curves in $X$. Then there exist a sequence of reparameterizations $r_k : \mathbb{C} \to \mathbb{C}$ and a subsequence of $(f_k \circ r_k)$ which converges toward an entire curve.

### 2.2 Stability of intersections

We recall here the following known complex analysis fact.

**Stability of intersections.** Let $X$ be a complex manifold and let $H \subset X$ be an analytic hypersurface. Suppose that a sequence $(f_k)$ of entire curves in $X$ converges toward an entire curve $f$. If $f(\mathbb{C})$ is not contained in $H$, then

$$f(\mathbb{C}) \cap H \subset \lim_{k \to \infty} f_k(\mathbb{C}) \cap H.$$  

### 2.3 Hyperbolicity of the complement of $2n + 1$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$

We also need the classical generalization of Picard’s theorem (case $n = 1$) [10].

**Theorem 2.1.** The complement of a collection of $2n + 1$ hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$ is hyperbolic.
3 Proof of the Main Theorem

Given a hypersurface $S$ of degree $q$ in general position with respect to the family $\{H_i\}_{1 \leq i \leq q}$, we would like to determine what conditions $S$ should satisfy for $\Sigma$ to be hyperbolic. Suppose that $\Sigma_{\varepsilon_k}$ is not hyperbolic for a sequence $(\varepsilon_k)$ converging to $0$. Then we can find entire curves $f_{\varepsilon_k} : \mathbb{C} \to \Sigma_{\varepsilon_k}$. By the Brody Lemma, after reparametrization and extraction, we may assume that the sequence $(f_{\varepsilon_k})$ converges to an entire curve $f : \mathbb{C} \to \bigcup_{i=1}^q H_i$. By uniqueness principle, the curve $f(\mathbb{C})$ lands in $\cap_{i \in I} H_i$, for some subset $I$ of the index set $Q := \{1, \ldots, q\}$ and does not land in any $H_j$ with $j \in Q \setminus I$.

Lemma 3.1. One has

$$|I| \leq n - 1.$$  

Proof. If on the contrary $|I| = n$, then for all $j \in Q \setminus I$, by stability of intersections, one has

$$f(\mathbb{C}) \cap H_j \subset \lim f_{\varepsilon_k}(\mathbb{C}) \cap H_j \subset \lim \Sigma_{\varepsilon_k} \cap H_j \subset S \cap H_j.$$  

Thus, $f(\mathbb{C}) \cap H_j \subset S \cap H_j \cap (\cap_{i \in I} H_i) = \emptyset$. Hence, $f(\mathbb{C}) \subset \cap_{i \in I} H_i \\setminus \{\cup_{j \in Q \setminus I} H_j\}$, which is a contradiction, since the complement of $q - |I| > 3$ points in a line is hyperbolic by Picard’s theorem.

By the above argument, $f(\mathbb{C}) \cap H_j$ is contained in $S$ for all $j \in Q \setminus I$. Therefore, the curve $f(\mathbb{C})$ lands in

$$\cap_{i \in I} H_i \\setminus \{\cup_{j \in Q \setminus I} H_j \setminus S\}.  \quad (3.1)$$  

So, the problem reduces to finding a hypersurface $S$ of degree $q$ such that all complements of the form $(3.1)$ are hyperbolic, where $I$ is an arbitrary subset of $Q$ having cardinality at most $n - 1$.

Such a hypersurface $S$ will be constructed by using the deformation method of Zaidenberg and Shiffman [17].

Starting point of the deformation process. Let $\{H_i\}_{1 \leq i \leq q}$ be a family of hyperplanes in general position in $\mathbb{P}^n(\mathbb{C})$. For some integer $0 \leq k \leq n - 1$ and some subset $I_k = \{i_1, \ldots, i_{n-k}\}$ of the index set $\{1, \ldots, q\}$ having cardinality $n - k$, the linear subspace $P_{k,I_k} = \cap_{i \in I_k} H_i \cong \mathbb{P}^k(\mathbb{C})$ will be called a subspace of dimension $k$. We will denote by $P^*_{k,I_k}$ the complement $P_{k,I_k} \\setminus \{\cup_{i \in I_k} H_i\}$, which we will call a star-subspace of dimension $k$. The process of constructing $S$ by deformation will start with the following result, which is an application of Theorem 2.1.

Starting Lemma. Let $\{H_i\}_{1 \leq i \leq q}$ be a family of $q \geq \left(\frac{n+3}{2}\right)^2$ hyperplanes in general position in $\mathbb{P}^{n+1}(\mathbb{C})$. Let $I$ and $J$ be two disjoint subsets of the index set $\{1, \ldots, q\}$ such that $1 \leq |I| \leq n - 1$, and $|J| = q + m + 1 - 2|I|$ with some $0 \leq m \leq |I| - 1$. Then all complements of the form

$$\cap_{i \in I} H_i \\setminus \{\cup_{j \in J} H_j \setminus A_{m,n+1-|I|}\}  \quad (3.2)$$  

are hyperbolic, where $A_{m,n+1-|I|}$ is a set of at most $m$ star-subspaces coming from the family of hyperplanes $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$ in the $(n + 1 - |I|)-$dimensional projective space $\cap_{i \in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$.

Proof. Suppose on the contrary that there exists an entire curve $f : \mathbb{C} \to \cap_{i \in I} H_i \\setminus \{\cup_{j \in J} H_j \setminus A_{m,n+1-|I|}\}$. Since each star-subspace in $A_{m,n+1-|I|}$ is constructed from at most $n + 1 - |I|$ hyperplanes in the family $\{\cap_{i \in I} H_i \cap H_j\}_{j \in J}$, the curve $f$ must avoid completely at least $|J| - m(n + 1 - |I|)$ hyperplanes in the projective space $\cap_{i \in I} H_i \cong \mathbb{P}^{n+1-|I|}(\mathbb{C})$. By the elementary estimate

$$|J| - m(n + 1 - |I|) = q + 1 - 2|I| - m(n - |I|)$$  

$$\geq 2(n + 1 - |I|) + 1 + \left[\left(\frac{n+3}{2}\right)^2 - 2(n + 1) - (|I| - 1)(n - |I|)\right]$$  

$$\geq 2(n + 1 - |I|) + 1,$$

and by using Theorem 2.1 we derive a contradiction.\]
Deformation lemma. For \(2 \leq l \leq n\), let \(\Delta_l\) be a finite collection of subspaces of dimension \(n + 1 - l\) coming from the family \(\{H_i\}_{1 \leq i \leq q}\) possibly with \(\Delta_l = \emptyset\), and let \(D_l \not\in \Delta_l\) be another subspace of dimension \(n + 1 - l\), defined as \(D_l = \cap_{i \in I_{D_l}} H_i\). For an arbitrary hypersurface \(S = \{s = 0\}\) in general position with respect to the family \(\{H_i\}_{1 \leq i \leq q}\) and for \(\epsilon \neq 0\), we set

\[
S_\epsilon = \{\epsilon s + \Pi_{i \not\in I_{D_l}} h_i^n = 0\},
\]

where \(n_i \geq 1\) are chosen (freely) so that \(\sum_{i \not\in I_{D_l}} n_i = q\). Then the hypersurface \(S_\epsilon\) is also in general position with respect to \(\{H_i\}_{1 \leq i \leq q}\). We denote by \(\Delta_l\) the family of all subspaces of dimension \(n + 1 - l\) \((2 \leq l \leq n + 1)\), with the convention \(\Delta_{n+1} = \emptyset\). We shall apply inductively the following lemma.

Lemma 3.2. Assume that all complements of the form

\[
\cap_{i \in I_l} H_i \setminus \left( \cup_{j \in J} H_j \setminus \left( (\Delta_l \cup \mathbf{\Delta}_{l+1}) \cap S \cup A_{m,n+1-|l|} \right) \right)
\]

are hyperbolic where \(I\) and \(J\) are two disjoint subsets of the index set \(\{1, \ldots, q\}\) such that \(1 \leq |I| \leq n - 1\), and \(|J| = q + m + 1 - 2|I|\) with some \(0 \leq m \leq |I| - 1\), and where \(A_{m,n+1-|l|}\) is a set of at most \(m\) star-subspaces coming from the family of hyperplanes \(\cap_{i \in I_l} H_i \cap H_j \cap J \cap J\) in \(\cap_{i \in I_l} H_i \simeq \mathbb{P}^{n+1-|I|}(\mathbb{C})\).

Then all complements of the form

\[
\cap_{i \in I_l} H_i \setminus \left( \cup_{j \in J} H_j \setminus \left( (\Delta_l \cup D_l \cup \mathbf{\Delta}_{l+1}) \cap S_\epsilon \cup A_{m,n+1-|l|} \right) \right)
\]

are also hyperbolic for sufficiently small \(\epsilon \neq 0\).

Proof. By the definition of \(S_\epsilon\), we see that \(S_\epsilon \cap (\cap_{m \in M} H_m) = S \cap (\cap_{m \in M} H_m)\) when \(M \cap (Q \setminus I_{D_l}) \neq \emptyset\), hence

\[
(\Delta_l \cup D_l \cup \mathbf{\Delta}_{l+1}) \cap S_\epsilon = ((\Delta_l \cup \mathbf{\Delta}_{l+1}) \cap S) \cup (D_l \cap S_\epsilon).
\]

When \(|I| \geq l\), using this, we observe that the two complements \((3.3), (3.4)\) coincide.

Assume therefore \(|I| \leq l - 1\). Suppose by contradiction that there exists a sequence of entire curves \(f_{\epsilon_k}(\mathbb{C})\), \(\epsilon_k \to 0\), contained in the complement \((3.3)\) for \(\epsilon = \epsilon_k\). By the Brody Lemma, we may assume that \((f_{\epsilon_k})\) converges to an entire curve \(f(\mathbb{C}) \subset \cap_{i \in I_l} H_i\). We are going to prove that the curve \(f(\mathbb{C})\) lands in some complement of the form \((3.3)\).

Let \(\cap_{i \in K} H_k\) be the smallest subspace containing \(f(\mathbb{C})\), so that \(I\) is a subset of \(K\). Take an index \(j\) in \(J \setminus K\). By stability of intersections, we have

\[
f(\mathbb{C}) \cap H_j \subset \lim f_{\epsilon_k}(\mathbb{C}) \cap H_j \\
\subset ((\Delta_l \cup \mathbf{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|l|} \cup \lim(D_l \cap S_\epsilon).
\]

If the index \(j\) does not belong to \(I_{D_l}\), then \(H_j \cap D_l \subset S_\epsilon \subset \mathbf{\Delta}_{l+1} \cap S\). It follows from \((3.5)\) that

\[
f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \mathbf{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|l|}.
\]

If the index \(j\) belongs to \(I_{D_l}\), noting that \(\lim(D_l \cap S_\epsilon)\) is contained in \(D_l \cap (\cup_{i \not\in I_{D_l}} H_i)\), hence from \((3.5)\)

\[
f(\mathbb{C}) \cap H_j \subset ((\Delta_l \cup \mathbf{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|l|} \cup (D_l \cap (\cup_{i \not\in I_{D_l}} H_i)).
\]

Assume first that \(K = I\). We claim that \((3.7)\) also holds when the index \(j \in J \setminus I\) belongs to \(I_{D_l}\). Indeed, for the supplementary part in \((3.7)\), we have

\[
f(\mathbb{C}) \cap H_j \cap (D_l \cup \cup_{i \not\in I_{D_l}} H_i) \subset \cup_{i \not\in I_{D_l}} f(\mathbb{C}) \cap H_j \cap H_i,
\]

so that \((3.6)\) applies here to all \(i \not\in I_{D_l}\). Hence, the curve \(f(\mathbb{C})\) lands inside

\[
\cap_{i \in I_l} H_i \setminus \left( \cup_{j \in J} H_j \setminus \left( ((\Delta_l \cup \mathbf{\Delta}_{l+1}) \cap S) \cup A_{m,n+1-|l|} \right) \right),
\]

contradicting the hypothesis.
Assume now that $I$ is a proper subset of $K$. Let us set

$$A_{m,n+1-|I|,K} = \{X \cap (\cap_{k \in K} H_k) | X \in A_{m,n+1-|I|}\}.$$ 

This set consists of star-subspaces of $\cap_{k \in K} H_k \cong \mathbb{P}^{n+1-|K|}$. Let $B_{m,K}$ be the subset of $A_{m,n+1-|I|,K}$ containing all star-subspaces of dimension $n-|K|$ (i.e. of codimension 1 in $\cap_{k \in K} H_k$), and let $C_{m,K}$ be the remaining part. A star-subspace in $B_{m,K}$ is of the form $(\cap_{k \in K} H_k \cap H_j)^*$ for some index $j \in J \setminus K$. Let then $R$ denote the set of such indices $j$, so that

$$|R| = |B_{m,K}|.$$ 

We consider two cases separately, depending on the dimension of the subspace $Y = \cap_{k \in K} H_k \cap D_I$.

**Case 1:** $Y$ is a subspace of dimension $n - |K|$. In this case, $Y$ is of the form $(\cap_{k \in K} H_k) \cap H_y$ for some index $y$ in $I_D$. It follows from (3.3), (3.6), (3.7) that the curve $f(C)$ lands inside

$$\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \setminus ((\Delta_1 \cup \Delta_{+1}) \cap S) \cup C_{m,K} \right).$$

To conclude that this set is of the form (3.3), we need to show that

1. $|(J \setminus K) \setminus (R \cup \{y\})| = q + m' + 1 - 2|K|$ with $|C_{m,K}| \leq m' \leq |K| - 1$;
2. $|K| \leq n - 1$.

Consider (1). We need to verify the corresponding required inequality between cardinalities

$$|C_{m,K}| \leq |(J \setminus K) \setminus (R \cup \{y\})| - q + 2|K| - 1 \leq |K| - 1.$$

The right inequality is equivalent to

$$|(J \setminus K) \setminus (R \cup \{y\})| \leq |\{1, \ldots, q\} \setminus K|,$$

which is trivial. The left inequality follows from the elementary estimates

$$|(J \setminus K) \setminus (R \cup \{y\})| - q + 2|K| - 1 \geq |J \setminus K| - |B_{m,K}| - q + 2|K| - 2$$

$$= |J| - |J \cap K| - |B_{m,K}| - q + 2|K| - 2$$

$$= (m - |B_{m,K}|) + (2|K| - 2|I| - |J \cap K| - 1)$$

$$\geq |C_{m,K}|,$$

where the last inequality holds because $I$ and $J$ are two disjoint sets and $I$ is a proper subset of $K$.

Consider (2). Suppose on the contrary that $|K| = n$. Since $S$ is in general position with respect to $\{H_i\}_{1 \leq i \leq 2n+2}$, we see that

$$\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \right) \setminus ((\Delta_1 \cup \Delta_{+1}) \cap S) \cup C_{m,K} = \cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus (R \cup \{y\})} H_j \right) \setminus C_{m,K}.$$ 

Since $|(J \setminus K) \setminus (R \cup \{y\})| \geq q + 1 - 2n + |C_{m,K}| \geq 3 + |C_{m,K}|$, the curve $f$ lands in a complement of at least 3 points in a line. By Picard’s Theorem, $f$ is constant, which is a contradiction.

**Case 2:** $Y$ is a subspace of dimension at most $n - |K| - 1$. In this case, the curve $f(C)$ lands inside

$$\cap_{k \in K} H_k \setminus \left( \cup_{j \in (J \setminus K) \setminus R} H_j \setminus ((\Delta_1 \cup \Delta_{+1}) \cap S) \cup C_{m,K} \cup Y^* \right),$$

which is also of the form (3.3), since

$$|(J \setminus K) \setminus R| \geq q - 2|K| + 1 + |C_{m,K} \cup Y^*|,$$

and since $|K| \leq n - 1$, by similar arguments as in Case 1.

The Lemma is thus proved. □
Inductive deformation process and end of the proof of the Main Theorem. We may begin by applying Lemma 3.2 for \( l = n \) (with \( \Delta_{n+1} = \emptyset \)), firstly with \( \Delta_{n} = \emptyset \), and with some \( D_{n} \in \Delta_{n} \), since \( (\Delta_{n} \cup \Delta_{n+1}) \cap S = \emptyset \), hence the assumption of this lemma holds by the Starting Lemma. Next, we reapply Lemma 3.2 inductively until we exhaust all \( D_{n} \in \Delta_{n} \). We get at the end a hypersurface \( S_{1} \) such that all complements of the forms

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( S_{1} \cup A_{m,n+1-|I|} \right) \right) \quad (|I| = n-1)
\]

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( (\Delta_{n} \cap S_{1}) \cup A_{m,n+1-|I|} \right) \right) \quad (|I| \leq n-2)
\]

are hyperbolic, since when \( |I| = n - 1 \), two components \( \cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( (\Delta_{n} \cap S_{1}) \cup A_{m,n+1-|I|} \right) \right) \) and \( \cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( S_{1} \cup A_{m,n+1-|I|} \right) \right) \) are equal. Considering this as the starting point of the second step, we apply inductively Lemma 3.2 for \( l = n - 1 \) and receive at the end a hypersurface \( S_{2} \) such that all complements of the forms

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( S_{2} \cup A_{m,n+1-|I|} \right) \right) \quad (n-2 \leq |I| \leq n-1)
\]

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( (\Delta_{n-1} \cap S_{2}) \cup A_{m,n+1-|I|} \right) \right) \quad (|I| \leq n-3)
\]

are hyperbolic, for the same reason as in above. Continuing this process, we get at the end of the \((n-1)^{th}\) step a hypersurface \( S = S_{n-1} \) such that all complements of the forms

\[
\cap_{i \in I} H_{i} \setminus \left( \cup_{j \in J} H_{j} \setminus \left( S_{n-1} \cup A_{m,n+1-|I|} \right) \right) \quad (1 \leq |I| \leq n-1)
\]

are hyperbolic. In particularly, by choosing \( m = |I| - 1 \), whence \( |J| = q - |I| \), and by choosing \( A_{m,n+1-|I|} = \emptyset \), all complements of the form (3.1) are hyperbolic for \( S = S_{n-1} \).

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