Dynamic Assortment Selection under the Nested Logit Models

Xi Chen¹, Yining Wang², and Yuan Zhou³

¹Leonard N. Stern School of Business, New York University
²Machine Learning Department, Carnegie Mellon University
³Computer Science Department, Indiana University at Bloomington

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Abstract

We study a stylized dynamic assortment planning problem during a selling season of finite length $T$. At each time spot, the seller offers an arriving customer an assortment of substitutable products and the customer makes the purchase among offered products according to a discrete choice model. The goal of the seller is to maximize the expected revenue, or equivalently, to minimize the worst-case expected regret. One key challenge is that utilities of products are unknown to the seller, which need to be learned. Although dynamic assortment planning problem has received an increasing attention in revenue management, most existing work is based on the multinomial logit choice models (MNL). In this paper, we study the problem of dynamic assortment planning under a more general choice model—nested logit model, which models hierarchical choice behavior and is “the most widely used member of the GEV (generalized extreme value) family” (Train, 2009). By leveraging the revenue-ordered structure of the optimal assortment within each nest, we develop a novel upper confidence bound (UCB) policy with an aggregated estimation scheme. Our policy simultaneously learns customers’ choice behavior and makes dynamic decisions on assortments based on the current knowledge. It achieves the regret at the order of $\tilde{O}(\sqrt{MNT + MN^2})$, where $M$ is the number of nests and $N$ is the number of products in each nest. We further provide a lower bound result of $\Omega(\sqrt{MT})$, which shows the optimality of the upper bound when $T > M$ and $N$ is small. However, the $N^2$ term in the upper bound is not ideal for applications where $N$ is large as compared to $T$. To address this issue, we further generalize our first policy by introducing a discretization technique, which leads to a regret of $\tilde{O}(\sqrt{MT^{2/3} + MN^2T^{1/3}})$ with a specific choice of discretization granularity. It improves the previous regret bound whenever $N > T^{1/3}$. We provide numerical results to demonstrate the empirical performance of both proposed policies.

Keywords: dynamic assortment optimization, nested logit models, regret analysis, upper confidence bounds.

1 Introduction

Assortment planning has a wide range of applications in retailing and online advertising. Given a large number of substitutable products, the assortment planning problem refers to the selection of a subset of products (a.k.a., an assortment) offered to a customer such that the expected revenue is maximized. To model customers’ choice behavior when facing a set of offered products, discrete choice models have been widely used, which capture demand for each product as a function of the entire assortment. One of the most popular discrete choice models is the multinomial logit model (MNL), which is naturally resulted from

*Author names listed in alphabetical order.
the random utility theory where a customer’s preference of a product is represented by the mean utility of the product with a random factor (McFadden, 1974). An important extension of the MNL is the nested logit model (Williams, 1977; McFadden, 1980; Borch-Supan, 1990) that models a customer’s choice in a hierarchical way: a customer first selects a category of products (known as a nest), and then a product within the category. When the mean utilities of the products are given, the static assortment optimization problem under MNL or nested logit models can be efficiently solved (Talluri & van Ryzin, 2004; Davis et al., 2014).

In many scenarios, customers’ choice behavior (e.g., mean utilities of products) is not given as a priori and cannot be easily estimated well due to the insufficiency of historical data (e.g., fast fashion sale or online advertising). To address this challenge, dynamic assortment planning that simultaneously learns choice behavior and makes decisions on the assortment has received a lot of attentions (Caro & Gallien, 2007; Rusmevichientong et al., 2010; Saure & Zeevi, 2013; Agrawal et al., 2017a,b). More specifically, in a dynamic assortment planning problem, the seller offers an assortment (or a set of assortments for different nests in a nested logit model) to each arriving customer in a finite time horizon $T$; and the goal of the seller is to maximize the cumulative expected revenue over $T$ periods. In the literature, the regret is often adopted to measure the performance of a given dynamic assortment planning policy, which is defined as the gap between the expected revenue generated by the policy and the oracle expected revenue when the mean utility for each product is known as a priori.

In existing dynamic assortment literature, the underlying choice model is usually assumed to be the MNL (Rusmevichientong et al., 2010; Saure & Zeevi, 2013; Agrawal et al., 2017a,b). In this paper, we study this problem under a more general choice model — two-level nested logit model. Indeed, the nested logit model is considered as “the most widely used member of the GEV (generalized extreme value) family” and “has been applied by many researchers in a variety of situations” (see Chapter 4 from Train (2009)). It is well known that the standard MNL suffers from the independence of irrelevant alternatives (IIA), which implies proportional substitution across alternatives (see, Chapter 4 from Train (2009)). The nested logit model relaxes the IIA assumption on alternatives in different nests and thus provides a richer set of substitution patterns. Despite the importance of nested logit model, the dynamic assortment under nested logit models still remains an open problem in revenue management due to the complicated structure of nested logit models.

The main contribution of this paper is to develop computationally efficient policies for addressing this problem. Assume that there are $M$ nests and each nest has $N$ possible products to recommend. By leveraging the revenue-ordered structure of optimal assortment and the idea of aggregate estimation, we propose the first upper confidence bound (UCB) based policy, which leads to a non-asymptotic regret bound of $\tilde{O}(\sqrt{MNT} + MN^2)$ (see Corollary 1). Here, $\tilde{O}$ hides the logarithmic dependence on $T, N, M$. To understand the information-theoretical limitation of the problem, we further provide a lower bound on the regret $\Omega(\sqrt{MT})$ (see Theorem 2). First, this lower bound shows the optimality of the upper bound in the regime when $T > M$ (which is a common case in practice) and $N$ is a fixed constant. Second, this lower bound also demonstrates a fundamental difference between the nested logit models and standard (plain) MNL models. According to Wang et al. (2018), the standard MNL admits a tight lower bound of $\Omega(\sqrt{T})$, independent of other problem parameters (e.g., the number of products). In contrast, for nested logit models, our lower bound shows that, in addition to $\sqrt{T}$, dependency on the number of nests $M$ is unavoidable.

When the number of products in each nest $N$ is large as compared to $T$, the regret of our UCB policy is large due to the quadratic term on $N$ (i.e., $N^2$). The optimal dependence on $N$ is, however, a technically very challenging question and is beyond the scope of this paper. Nevertheless, we address the challenge of large $N$ cases by introducing a discretization technique, which provides a policy with a much improved dependence on $N$. Our discretization technique is a general approach that includes the first policy as a special case by setting a “discretization granularity” parameter to zero. By choosing another appropriate discretization granularity, we will obtain a policy with the regret of $\tilde{O}(\sqrt{MT^{2/3}} + MN^2T^{1/3})$. It is easy to check when $N > T^{1/3}$, the second regret improves over the previous one. Therefore, each policy has its
own advantage and a user can choose a suitable policy based on the knowledge of the relationship between \( N \) and \( T \). We will also demonstrate the effect of \( N \) in both policies via our numerical studies (see Section 6).

In addition, it is worthwhile noting that one advantage of our policies is that they are almost assumption-free. We only require that the revenue and mean utility for each product are upper bounded by constants. We do not make any separability condition (e.g., for any pair of candidate assortments, there is a gap between their expected revenues, see Proposition 8 in Rusmevichientong et al. (2010)) or the assumption that the no-purchase is the most frequent outcome as in Agrawal et al. (2017a,b).

The details of the proposed policies will be presented in the main paper and here we briefly highlight a few key technical novelty in the proposed policies:

1. Leveraging the revenue-ordered structure: For \( M N \) products in each nest, the total number of possible assortment combinations (i.e., the action space) will be \( (2^N)^M \), which is exponentially large. By leveraging the revenue-ordered structure of the optimal assortment within each nest (see Lemma 1 and Davis et al. (2014); Li et al. (2015)), the size of the action space can be effectively reduced to \( O(N^M) \). However, the size of this reduced action space is still too large if one directly applies existing bandit learning algorithms that treat each assortment in the action space independently, which will incur a regret related to \( N^M \). To address this challenge, we propose an aggregate estimation technique as follows.

2. Aggregate estimation: A key message of the paper is that estimating utility parameters for each individual product (that will incur a large regret) is unnecessary for dynamic assortment planning. Instead, we propose an aggregate estimation technique that only estimates the preference and revenue parameters on a nested-level. To implement this technique, we propose a simplified “nested singleton model”, which only allows a “single item” in each nest with a random revenue parameter. More specifically, each singleton corresponds to a level-set assortment and the random revenue corresponds to the aggregated revenue at a nested-level. This simplified model greatly facilitates the policy design and our regret analysis. For example, this singleton model suggests that estimation of exponent parameters \( \{\gamma_i\}_{i=1}^M \) (see Eq. (1) in the nested logit model specification) is not necessary. Instead, we directly estimate “nested-level utility” \( V_i (\cdot)^{\gamma_i} \) (see Eq. (7) and the discussions above Eq. (7)).

3. UCB policies: Based on this simplified “nested singleton model” and using an epoch-based strategy from Agrawal et al. (2017a), we propose a upper confidence bound algorithm (UCB), which leads to a worst-case expected regret of \( O(\sqrt{MNT + MN^2}) \). Although UCB has been a well-known technique for bandit problems, how to adopt this high-level idea to solve a problem with specific structures certainly requires technical innovations (e.g., how to build a confidence bound on a carefully designed parameter, see Lemma 4). We further note that our UCB policy generalizes the one in Agrawal et al. (2017a) because our singleton model has unknown reward parameters and the utility parameters could exceed one.

4. Discretization technique: To further improve the dependence on \( N \), we introduce a discretization technique to reduce the size of the action space to \( O((1/\delta)^M) \), where \( \delta \) is discretization granularity. Our first policy without discretization corresponds to a special case of \( \delta = 0 \). We are able to show that the proposed space reduction techniques lose very little in terms of optimal expected revenue, i.e., the gap of the optimal expected revenue between the all possible assortment combinations and the reduced action space is at most \( \delta \) (see Lemma 8). Using the aggregate estimation with an appropriate choice of \( \delta \approx T^{-1/3} \), we obtain our second UCB policy that achieves the regret of \( O(\sqrt{MT^{2/3} + MNT^{1/3}}) \). It is also worthwhile noting the adopted “nested singleton model” greatly simplifies our presentation of the second policy, which is almost the same as the first one (only with a different candidate singleton set).
To the best of our knowledge, our policies are the first policies for dynamic assortment planning under the nested logit model, which presents unique challenges compared to the standard MNL model as the nest-level revenues of assortment selections are not known and have to be estimated on the fly. It is also worthwhile noting that due to the complicated structure of the nested logit model, it is very challenging to derive a tight lower bound on the regret in terms of $N$. Since the main focus of the paper is to derive the first efficient policy for dynamic assortment planning under nested logit models, we leave to this challenging technical problem as a future work.

1.1 Related Works

Static assortment planning with known choice behavior has been an active research area since the seminal works by van Ryzin & Mahajan (1999); Mahajan & van Ryzin (2001). When the customer makes the choice according to the MNL model, Talluri & van Ryzin (2004); Gallego et al. (2004) proved an optimal assortment will belong to revenue-ordered assortments (a.k.a. nested-by-revenue assortments). An alternative proof is provided in Liu & van Ryzin (2008). This important structural result enables the efficient computation of static assortment planning under the MNL model, which reduces the number of candidate assortments from $2^N$ to $N$, where $N$ is the number of products. When there is a set constraint on the assortment set, an efficient polynomial-time algorithm (with running time $O(N^2)$) was proposed in Rusmevichientong et al. (2010). For nested logit models, Davis et al. (2014) proved an important structural result that the optimal assortment within each nest is revenue-ordered, which will also be used in designing our dynamic policy. Assuming that there are $M$ nests and $N$ products within each nest, Li & Rusmevichientong (2014) further proposed an efficient greedy algorithm to find an optimal assortment set with $O(NM \log M)$ time complexity. Kök & Xu (2011) considered the joint assortment optimization and pricing problems with a restricted number of nests. There are several extensions of assortment planning under nested logit models. For example, Gallego & Topaloglu (2014) studied constrained nested logit model; Li et al. (2015) extended the popular two-level nested logit model to $d$-level nested logit model with $d \geq 2$; and Zhang et al. (2017) studied the paired combinatorial nested logit model. In addition, there are extensive research on static assortment optimization for more complex choice models, e.g., a robust version of MNL (Rusmevichientong & Topaloglu, 2012), mixture of logits models (Bront et al., 2009; Méndez-Díaz et al., 2014; Rusmevichientong et al., 2014), Markov chain based choice models (Blanchet et al., 2016; Désir et al., 2015), Mallows based choice models (Désir et al., 2016), and a general class of choice models based on a distribution over permutations (Farias et al., 2013).

Due to increasing popularity of data-driven revenue management, researchers start to relax the assumption on fully available prior knowledge of customers’ choice behavior and investigate dynamic assortment planning. Motivated by fast-fashion retailing, the work by Caro & Gallien (2007) was the first to study dynamic assortment planning problem, which assumes that the demand for product is independent of each other. Rusmevichientong et al. (2010); Saure & Zeevi (2013); Agrawal et al. (2017a,b) incorporated choice models of MNL and capacitated MNL into dynamic assortment planning and formulated the problem into an online regret minimization problem. On the high-level, our paper is similar to Agrawal et al. (2017a) since both adopt the idea of UCB. However, to extend from the MNL in Agrawal et al. (2017a) to nested logit models, it requires several technical innovations. For example, we introduce a discretization technique. Moreover, instead of estimating utility parameters for each product, we estimate nest-level aggregated quantities (see more discussions in the introduction). Finally, we also note that to highlight our key idea and focus on the balance between information collection and revenue maximization, we study the stylized dynamic assortment planning problems following the existing literature (Rusmevichientong et al., 2010; Saure & Zeevi, 2013; Agrawal et al., 2017a,b), which ignore operations considerations such as price decisions and inventory replenishment.

There is another line of recent research on investigating personalized assortment planning, where each
arriving customer could have a different choice behavior. For example, Golrezaei et al. (2014); Chen et al. (2016) assumed that each customer’s choice behavior is known but the customers’ arriving sequence can be adversarially chosen, and took into account both the revenue and inventory levels. Since the arriving sequence can be arbitrary, there is no learning component in the problem and both Golrezaei et al. (2014) and Chen et al. (2016) adopted the competitive ratio as the performance evaluation metric.

Finally, it is also worthwhile mentioning that the problem of dynamic assortment planning is closely related to multi-armed bandit (MAB); both dynamic assortment and MAB need to carefully balance the exploration-exploration tradeoff. However, due to the complex structure of discrete choice models, one cannot directly apply the results from MAB (e.g., the UCB policies from Auer et al. (2002)). Indeed, a straightforward analogy between assortment planning and MAB is to treat each feasible assortment as an arm. But directly using this mapping will result in a large regret due to the very large number of possible assortments. Another way is to treat each problem as an arm and consider the combinatorial MAB framework (see, e.g., Chen et al. (2013)), where each pull can involve a subset of arms. However, existing works in combinatorial MAB assumes reward for different arms are generated independently, and thus cannot be applied to our problem.

1.2 Notations and paper organizations

Throughout the paper, we use \( f(\cdot) \lesssim g(\cdot) \) to denote that \( f(\cdot) = O(g(\cdot)) \). Similarly, by \( f(\cdot) \gtrsim g(\cdot) \) we denote \( f(\cdot) = \Omega(g(\cdot)) \). We also use \( f(\cdot) \asymp g(\cdot) \) for \( f(\cdot) = \Theta(g(\cdot)) \). In the paper, \( O(\cdot) \) is used to hide logarithmic factors on \( T, N, \) and \( M \). The rest of the paper is organized as follows. In Section 2, we first provide the background of nested logit models and introduce an important structural result on optimal assortments (Davis et al., 2014; Li et al., 2015). We further propose a nested singleton model in Section 2.3. In Section 3, we propose our UCB-policy and establish the corresponding regret bound. A lower bound on regret is provided in Section 4. In Section 5, we further propose a policy with an improved dependence on \( N \). The numerical results are provided in Section 6, followed by the conclusion in Section 7.

2 Model specifications and a nested singleton model

In this section we formally introduce the nested logit assortment choice model considered in this paper. We restrict ourselves to 2-level nested logit models, where items are organized as \( M \) known commodity nests and customers’ purchasing actions are modeled by a hierarchical multinomial logit model (more details given in Section 2.1). We also discuss existing structural results of the optimal assortment combinations, and introduce a singleton model that significantly simplified notations and presentation of our policies and analysis.

2.1 The nested logit model

We use \([M] = \{1, 2, \cdots, M\}\) to denote \( M \) nests. For each nest \( i \in [M] \), denote the items in nest \( i \) by \([N_i] = \{1, 2, \cdots, N_i\}\). Each item \( j \in [N_i] \) is associated with a known revenue parameter \( r_{ij} \) and an unknown mean utility parameter \( v_{ij} \). Without loss of generality we assume each nest has equal number of items, i.e., \( N_1 = \cdots = N_M = N \), because one can always add items with zero utility and revenue parameters. Let \( S_i = 2^{[N]} \) be the set of all possible assortment for nest \( i \). Further, let \( \{\gamma_i\}_{i \in [M]} \subseteq [0, 1] \) be a collection of unknown correlation parameters for different nests. Each parameter \( \gamma_i \) is a measure of the degree of independence among the items in nest \( i \): a larger value of \( \gamma_i \) indicates less correlation (see Chapter 4 of Train (2009)).

At each time period \( t \in \{1, 2, \cdots, T\} \), the retailer offers the arriving customer an assortment \( S^{(t)}_i \in S_i \) for every nest \( i \in [M] \), conveniently denoted as \( S^{(t)} = (S^{(t)}_1, \cdots, S^{(t)}_M) \). The retailer then observes a nest-
level purchase option \( i_t \in [M] \cup \{0\} \) and, if \( i_t \in [M] \), the purchased item \( j_t \in [N] \) within nest \( i_t \). Again, \( i_t = 0 \) means no purchase occurs at time \( t \). The probabilistic model for the purchasing option \( (i_t, j_t) \) can be formulated below:

\[
\Pr \left[ i_t = i | S^{(t)} \right] = \frac{V_i(S_i^{(t)})^{\gamma_i}}{V_0 + \sum_{i'=1}^{M} V_i'(S_i'^{(t)})^{\gamma_i'}}, \quad \text{where } V_0 \equiv 1, \quad V_i(S_i^{(t)}) = \sum_{j \in S_i^{(t)}} v_{ij} \quad \text{for } i \in [M];
\]

(1)

\[
\Pr \left[ j_t = j | i_t = i, S^{(t)} \right] = \frac{v_{ij}}{\sum_{j' \in S_i^{(t)}} v_{ij'}} \quad \text{for } i \in [M], \quad j \in S_i^{(t)}.
\]

(2)

Note that when \( \gamma_i = 1 \) for all \( i \in [M] \), the nested logit model reduces to the standard MNL model.

The retailer then collects revenue \( r_{i_t,j_t} \) provided that \( i_t \neq 0 \). The expected revenue \( R(S^{(t)}) \) given the assortment combination \( S^{(t)} \) can then be written as

\[
R(S^{(t)}) = \sum_{i=1}^{M} \Pr \left[ i_t = i | S^{(t)} \right] \sum_{j \in S_i^{(t)}} r_{ij} \Pr \left[ j_t = j | i_t = i, S^{(t)} \right]
\]

\[
= \sum_{i=1}^{M} R_i(S_i^{(t)}) V_i(S_i^{(t)})^{\gamma_i} \quad \text{where } R_i(S_i^{(t)}) = \frac{\sum_{j \in S_i^{(t)}} r_{ij} v_{ij}}{\sum_{j' \in S_i^{(t)}} v_{ij'}}.
\]

(3)

The objective of the seller to minimize expected regret defined as follows:

\[
\text{Regret} \left( \left( S^{(t)} \right)_{t=1}^{T} \right) := \sum_{t=1}^{T} R^* - \mathbb{E}^* \left[ R(S^{(t)}) \right], \quad \text{where } R^* = \max_{S \in S_1 \times \cdots \times S_M} R(S).
\]

(4)

Throughout the paper, we make the following boundedness assumptions on revenue and utility parameters.

(A1) \( 0 \leq r_{ij} \leq 1 \) for all \( i \in [M] \) and \( j \in [N] \).

(A2) \( 0 < v_{ij} \leq C_V \) for all \( i \in [M] \) and \( j \in [N] \) with some constant \( C_V \geq 1 \).

The first boundedness assumption on revenue parameters is standard in the literature (see, e.g., Theorem 1 in Agrawal et al. (2017a)). It is also worthwhile noting that assumption (A2) is different from and weaker than the common assumption that no-purchase (with \( V_0 = 1 \)) is the most frequent outcome (see, e.g., Agrawal et al. (2017a,b)).

### 2.2 Assortment space reductions

For nested logit models, the complete assortment selection space (a.k.a. action space) \( S = S_1 \times S_2 \times \cdots \times S_M \) is extremely large, consisting of an exponential number of candidate assortment selections (on the order of \( 2^N \)^M). Existing bandit learning approaches treating each assortment set in \( S \) independently would easily incur a regret also exponentially large. It is thus mandatory to reduce the number of candidate assortment sets in \( S \).

Fortunately, existing results on the structure of optimal \( S \) show that it suffices to consider level sets \( L_i(\theta_i) := \{ j \in [N] : r_{ij} \geq \theta_i \} \) for each nest \( i \). In other words, \( L_i(\theta_i) \) is the set of products in nest \( i \) with revenue larger than or equal to a given threshold \( \theta_i \geq 0 \). Define \( P_i := \{ L_i(\theta_i) : \theta_i \geq 0 \} \subseteq S_i \) to be all the possible level sets of \( S_i \) and let

\[
P := P_1 \times P_2 \times \cdots \times P_M \subseteq S.
\]

(5)
The following lemma from Davis et al. (2014); Li et al. (2015) shows that one can restrict the assortment selections to \( \mathbb{P} \) without loss of any optimality in terms of expected revenue.

**Lemma 1** (Davis et al. (2014); Li et al. (2015)). There exists level set threshold parameters \((\theta_1^*, \ldots, \theta_M^*)\) and \(S^* = (L_1(\theta_1^*), \ldots, L_M(\theta_M^*)) \in \mathbb{P}\) such that the following hold:

1. \( R(S^*) = \max_{S \in \mathbb{S}} R(S) = R^* \);
2. \( \theta_i^* \geq \gamma_i R^* + (1 - \gamma_i) R_i(S_i^*) \) for all \( i \in [M] \), where \( S_i^* = L_i(\theta_i^*) \).

The first item in Lemma 1 is an important structural result showing that the optimal assortments are “revenue-ordered” within each nest. The second item is a technical result, which will be used in the proof. Compared to the original action space \( \mathbb{S} \), the reduced “level set” space \( \mathbb{P} \) is much smaller, with each \( \mathbb{P}_i \) consisting of \( N \) instead of \( 2^N \) assortment candidates.

### 2.3 A nested singleton model

To facilitate the illustration of our idea, we introduce a “singleton” description of the original nested model which we name nested singleton models. In our singleton model, we treat each level set as a “singleton item” in the nested model with an aggregate random revenue (which corresponds to the nest-level revenue in (3)). The introduced “nested singleton model” not only helps simplify our algorithms’ descriptions and their analysis but also highlights our main idea of “aggregated estimation” on a nested level. Moreover, this nested singleton model will provide a unified description of a more sophisticated policy based on a discretization technique, which will be introduced in Section 5.2.

Recall that for each nest \( i \in [M] \), there will be only \((N + 1)\) distinct level sets \( \{L_i(\theta_i) : i \in [N]\} \cup \{L_i(\infty)\} \), where \( L_i(\infty) = \emptyset \) corresponds to the empty assortment set. To simplify the problem, we shall consider each level set as a singleton item, associated with a preference parameter and a mean revenue parameter. More specifically, each nest \( i \in [M] \) consists of \( N + 1 \) “singleton items”, each labeled as \( \theta \in \mathbb{K}_i := \{r_{ij} : j \in [N]\} \cup \{\infty\} \), where the singleton \( \theta \) in nest \( i \) corresponds to the assortment level set \( L_i(\theta) \). It should also be noted that \( \theta = \infty \) corresponds to the empty assortment. With this notation, each assortment combination \( S = (S_1, \ldots, S_M) \in \mathbb{P} \) can be equivalently written as

\[
\theta = (\theta_1, \ldots, \theta_M) \in \mathbb{K}_1 \times \cdots \times \mathbb{K}_M.
\]

Here \( \theta_i \in \mathbb{K}_i \) corresponds to level set \( S_i = L_i(\theta_i) = \{j \in [N] : r_{ij} \geq \theta_i\} \) being offered in nest \( i \). When it is necessary, we will also write \( S_i(\theta_i) \) to emphasize that the assortment in nest \( i \) depends on the singleton \( \theta_i \).

After presenting the customer with assortment combination \( S \in \mathbb{P} \) (or equivalently parameterized by \( \theta \)), the retailer observes \( i_t \in [M] \cup \{0\} \) indicating which nest is chosen (\( i_t = 0 \) means no purchase is made at time \( t \)) and collects revenue \( r_t \in [0,1] \), which is a random variable corresponding to the revenue \( r_{it,j} \) of item \( j \) in nest \( i \) (if \( i_t = 0 \) then \( r_t = 0 \) almost surely). In the nested singleton model we discard the item choice \( j \) within nest \( i \), and only record the nest-level selection \( i_t \) and revenue collected \( r_t \).

For any \( i \in [M] \) and \( \theta_i \in \mathbb{K}_i \), we define \( u_{i\theta_i} := V_i(S_i(\theta_i)) \gamma_i \) and \( \phi_{i\theta} := R_i(S_i(\theta_i)) \), where \( V_i(S_i(\theta_i)) \) and \( R_i(S_i(\theta_i)) \) are nest-level utility parameter and expected revenue associated with the level set \( S_i(\theta_i) = L_i(\theta_i) \) (see definitions of \( V_i \) and \( R_i \) in Eq (1) and (3), respectively). Given an assortment combination \( \theta^{(t)} = (\theta_1, \ldots, \theta_M) \) at time epoch \( t \), it is easy to verify that the random choice \( i_t \in [M] \cup \{0\} \), and corresponding random revenue \( r_t \in [0,1] \) satisfy the following:

\[
\Pr[i_t = i|\theta^{(t)}] = \frac{u_{i\theta_i}}{1 + \sum_{\theta_i'} u_{i'\theta_i'}}; \quad \mathbb{E}[r_t|i_t = i] = \phi_{i\theta_i}; \quad r_t = 0 \ a.s. \ if \ i_t = 0.
\]
The objective is to minimize the regret assortment in the classical plain MNL model with capacity constraints, any $M$ (Bubeck & Cesa-Bianchi, 2012) and repeated exploration of the same action until no-purchase events, which approximately equivalent to the original nested logit model as shown in Proposition 1.

For a given dynamic assortment selection policy $\pi$ under the original nested logit model simply by converting to policy $\pi'$; please see Algorithm 1 and the following Proposition 1 for more details.

Proposition 1. Suppose there exists a policy $\pi'$ that attains a regret of at most $\Delta$ on any instance of the nested singleton model with $|K_i| = K = N + 1$ for all $i \in [M]$ and $u_{i\theta_i} \in [0, U]$ with $U \leq NC_V$ for all $i \in [M]$. Then there is a meta-policy $\pi$ (see Algorithm 1) that produces an assortment combination sequence $\{S^{(t)}\}_{t=1}^T$ under the original nested choice model with regret (defined in Eq. (4)) at most $\Delta$.

Algorithm 1: The meta-policy $\pi$ built upon policy $\pi'$ under the nested singleton model.

In fact, Eq. (7) resembles the classical plain MNL model, with two important differences. First, the revenues collected on each purchased singleton $i_t$ are random instead of fixed, and the mean revenue parameters $\{\phi_{i\theta_i}\}$ are unknown and have to be estimated from random revenues collected from purchasing events. Second, it is constrained that at most one singleton $\theta_i \in K_i$ can be offered within each nest $i \in [M]$, where in the classical plain MNL model with capacity constraints, any $M$ items can be combined together as an assortment.

Define the expected revenue for a assortment combination $\theta^{(t)}$

$$R'(\theta^{(t)}) := \sum_{i=1}^M \Pr[i_t = i|\theta^{(t)}] \cdot \mathbb{E}[r_t|i_t = i] = \sum_{i=1}^M \frac{\phi_{i,\theta_i} u_{i,\theta_i}}{1 + \sum_{i=1}^M u_{i,\theta_i}}$$

The objective is to minimize the regret

$$\text{Regret}(\{\theta^{(t)}\}_{t=1}^T) := \mathbb{E} \sum_{t=1}^T R'(\theta^{(t)}) - R'(\theta^*) \quad \text{where} \quad R'(\theta^*) = \max_{\theta \in K_1 \times \cdots \times K_M} R'(\theta). \quad (8)$$

By our assumptions (A1) and (A2), it is easy to verify that $\phi_{i,\theta_i} \in [0, 1]$ and $u_{i,\theta_i} \in [0, (NC_V)^{\gamma_i}] \subseteq [0, NC_V]$ for all $i \in [M]$ and $\theta_i \in K_i$. We also note that $(NC_V)^{\gamma_i} \leq NC_V$ since $\gamma_i \in [0, 1]$ and $C_V \geq 1$.

For a given dynamic assortment selection policy $\pi'$ under this nested singleton model, it is easy to construct a policy $\pi$ under the original nested logit model simply by converting $\{\theta^{(t)}\}$ to their corresponding assortment combinations $\{S^{(t)}\}$. Please see Algorithm 1 and the following Proposition 1 for more details.

Proposition 1. Suppose there exists a policy $\pi'$ that attains a regret of at most $\Delta$ on any instance of the nested singleton model with $|K_i| = K = N + 1$ for all $i \in [M]$ and $u_{i\theta_i} \in [0, U]$ with $U \leq NC_V$ for all $i \in [M]$ and $\theta_i \in K_i$. Then there is a meta-policy $\pi$ (see Algorithm 1) that produces an assortment combination sequence $\{S^{(t)}\}_{t=1}^T$ under the original nested choice model with regret (defined in Eq. (4)) at most $\Delta$.

Proposition 1 is a simple consequence of Lemma 1 and we omit its proof.

### 3 UCB-based dynamic assortment planning policies

In this section we design dynamic planning policies under the nested logit model using an upper-confidence-bound (UCB) approach. We focus entirely on the simpler assortment model specified in Eq. (7), since it is (approximately) equivalent to the original nested logit model as shown in Proposition 1.

Our main policy is based on the idea of upper-confidence-bound (UCB) from classical bandit algorithms (Bubeck & Cesa-Bianchi, 2012) and repeated exploration of the same action until no-purchase events, which
Input: singleton sets $K_1, \ldots, K_M$, upper bound $U$ on $\{u_{i\theta}\}$, time horizon $T$.
Output: assortment sequences $\theta^{(1)}, \ldots, \theta^{(T)} \in K_1 \times \cdots \times K_M$.

1. Initialization: $\tau = 1$, $\{E_\tau\}_{\tau=1}^\infty = \emptyset$, $t = 1$; for every $i \in [M]$ and $\theta \in K_i$, set $T(i, \theta) = \emptyset$.

$T(i, \theta) = 0$, $\widehat{\phi}_{i\theta} = \phi_{i\theta} = 1$, $\widehat{\tau}_{i\theta} = \tau_{i\theta} = U$; for all $i \in [M]$ and $\theta \in K_i$ corresponding to the empty assortment (i.e., $L_i(\theta) = \emptyset$), set $\widehat{\phi}_{i\theta} = \phi_{i\theta} = \tau_{i\theta} = u_{i\theta} = 0$.

2. While $t \leq T$ do
   3. Find $\widehat{\theta}^{(\tau)} = \widehat{\theta} \leftarrow \arg\max_{\theta \in K_1 \times \cdots \times K_M} \widehat{R}'(\theta)$, where $\widehat{R}'(\theta) = [(\sum_{i=1}^M \widehat{\phi}_{i\theta} \tau_{i\theta})]/[1 + \sum_{i=1}^M \tau_{i\theta}]$
      ▶ This optimization problem can be solved in polynomial time; see Sec. 3.1;
   4. Repeat
      5. Pick $\theta^{(t)} = \theta$ and observe $i_t, r_t$ in Eq. (7) and update $E_{\tau} \leftarrow E_{\tau} \cup \{t\}$, $t = t + 1$;
   6. Until $i_{t-1} = 0$ or $t > T$;
   7. For each $i \in [M]$ do
      8. Compute $\widehat{\pi}_{i,\tau} = \sum_{i' \in E_{\tau}} \mathbb{1}[i' = i]$ and $\widehat{\tau}_{i,\tau} = \sum_{i' \in E_{\tau}} r_{i'}[i' = i]$
      9. Let $\theta = \hat{\theta}_t$ (for notational simplicity) and update: $T(i, \theta) \leftarrow T(i, \theta) \cup \{\tau\}$, $T(i, \theta) \leftarrow T(i, \theta) + 1$;
   10. Update the utility and mean revenue estimates and as well as their associated confidence bands: $\widehat{u}_{i\theta} = \frac{1}{T(i, \theta)} \sum_{t' \in T(i, \theta)} \widehat{\pi}_{i,\tau'}, \widehat{\phi}_{i\theta} = \frac{\sum_{t' \in T(i, \theta)} \tau_{i,\tau'}}{\sum_{t' \in T(i, \theta)} \pi_{i,\tau'}}$
      11. If $T(i, \theta) \geq 96 \ln(2MTK)$ then
      12. $\overline{u}_{i\theta} = \min\{U, \overline{u}_{i\theta} + \sqrt{96 \max(\overline{u}_{i\theta}, \overline{\tau}_{i\theta}) \ln(2MTK) / T(i, \theta)} + 144 \ln(2MTK) / T(i, \theta)}\}$,
      13. $\overline{\phi}_{i\theta} = \min\{1, \overline{\phi}_{i\theta} + \sqrt{\ln(2MTK) / T(i, \theta) \overline{u}_{i\theta}}\}$.
      14. Else
      15. $\overline{u}_{i\theta} = U$, $\overline{\phi}_{i\theta} = 1$
      16. End
      17. $\tau \leftarrow \tau + 1$;
   18. End

Algorithm 2: The Upper Confidence Bound (UCB) policy for dynamic assortment planning under the nested singleton model in Eq. (7).

was found to be very useful for assortment planning problems (Agrawal et al., 2017a) because it provides unbiased estimates of model parameters.

The pseudo-code of our proposed policy is given in Algorithm 2. We first explain a few notations used in the algorithm and then describe the details of the algorithm.

- $E_{\tau}$: all iterations in epoch $\tau$ where the same assortment combination $\theta$ is provided. Each epoch (corresponding to Steps 4-6 in Algorithm 2) terminates whenever the no-purchase action is observed. In other words, one and only one “no-purchase” action $i_t = 0$ appears at the last iteration of each epoch $E_{\tau}$.

- $T(i, \theta)$: the indices of epochs in which $\theta \in K_i$ is supplied in nest $i$; $T(i, \theta) = |T(i, \theta)|$ denotes the cardinality of $T(i, \theta)$;

- $\widehat{n}_{i,\tau}$: the number of iterations in the epoch $\tau$ (i.e., $E_{\tau}$) in which an item in nest $i$ is purchased;

- $\widehat{r}_{i,\tau}$: the total revenue collected for all iterations in $E_{\tau}$ in which an item in nest $i$ is purchased;
Our policy in Algorithm 2 involves a combinatorial optimization problem over all \( \hat{\theta} \) assortment combinations (provided that \( \hat{\theta} \) (see Step 3 in Algorithm 2). A brute-force algorithm that enumerates all such \( i \) computationally intractable even for small the nests \( i \) offered. It is worth noting that in those epochs, the offered singletons in nests other than the \( i \) are taken to be the sample averages of \( \hat{\theta} \) procedure to compute \( \) models.

one in Rusmevichientong et al. (2010) introduced for dynamic assortment optimization in capacitated MNL expectations of the number of iterations and total revenues collected in which nest \( i \) is purchased (denoted by \( \hat{n}_{i,\tau} \) and \( \hat{\tau}_{i,\tau} \) respectively in Algorithm 2) satisfies the following regardless of the other offered assortments \( \hat{\theta}_i \) for \( i' \neq i \):

1. \( \mathbb{E}[\hat{n}_{i,\tau}] = u_i \hat{\theta}_i \);
2. \( \mathbb{E}[\hat{\tau}_{i,\tau} | \hat{n}_{i,\tau}] = \hat{n}_{i,\tau} \hat{\theta}_i \).

The above properties motivate intuitive parameter estimators \( \hat{u}_i, \hat{\phi}_i \) of \( u_i, \phi_i \) for \( \theta = \hat{\theta}_i \), which are taken to be the sample averages of \( \hat{n}_{i,\tau} \) and \( \hat{\tau}_{i,\tau} \) over all epochs \( \mathcal{E}_\tau \) in which the item \( \hat{\theta}_i \) in nest \( i \) is offered. It is worth noting that in those epochs, the offered singletons in nests other than the \( i \)-th nest (i.e., the nests \( i' \) for \( i' \neq i \)) can be arbitrary since the distributions of \( \hat{n}_{i,\tau} \) and \( \hat{\tau}_{i,\tau} \) are independent of \( \hat{\theta}_i \) for \( i' \neq i \). This key independence property enables us to combine purchasing information of vastly different assortment combinations (provided that \( \hat{\theta}_i \) remain the same), which forms an important aggregation strategy that avoids exponentially large regret if assortment combinations are treated independently.

### 3.1 Efficient computation of \( \hat{\theta} \)

Our policy in Algorithm 2 involves a combinatorial optimization problem over all \( \theta \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M \) (see Step 3 in Algorithm 2). A brute-force algorithm that enumerates all such \( \theta \) takes \( O(K^M) \) time and is computationally intractable even for small \( M \) values. In this section we introduce a computationally efficient procedure to compute \( \hat{\theta} \) by using a binary search technique. The idea behind our procedure is similar to the one in Rusmevichientong et al. (2010) introduced for dynamic assortment optimization in capacitated MNL models.

For any \( \lambda \in [0, 1] \) and \( \theta = (\theta_1, \cdots, \theta_M) \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M \) define potential function

\[
\psi_\lambda(\theta) := \sum_{i=1}^M (\phi_{i,\theta_i} - \lambda) \psi_{i,\theta_i}.
\]  

Recall the definition of \( \overline{R}^\tau(\theta) = \frac{\sum_{i=1}^M \phi_{i,\theta_i} \psi_{i,\theta_i}}{1+\sum_{i=1}^M \psi_{i,\theta_i}} \) in Step 3 of Algorithm 2. The following lemma characterizes the properties of \( \psi_\lambda(\theta) \) and its relationship with \( \overline{R}^\tau = \max_{\theta \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M} \overline{R}^\tau(\theta) \):

**Lemma 3.** The following hold for all \( \lambda \in [0, 1] \):

1. If \( \overline{R}^\tau \geq \lambda \) then there exists a \( \theta \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M \) such that \( \psi_\lambda(\theta) \geq \lambda \); furthermore if \( \overline{R}^* > \lambda \) then the inequality is strict;
2. If $\overline{R}^t \leq \lambda$ then for all $\theta \in K_1 \times \cdots \times K_M$, $\psi_\lambda(\theta) \leq \lambda$; furthermore if $\overline{R}^t < \lambda$ then the inequalities are strict.

Based on Lemma 3, an efficient optimization algorithm computing the maximizer $\hat{\theta}^{(r)}$ can be designed by a binary search over $\lambda \in [0, 1]$. In particular, for each fixed value of $\lambda$, the $\theta^*(\lambda) = (\theta^*_1(\lambda), \cdots, \theta^*_M(\lambda)) \in K_1 \times \cdots \times K_M$ that maximizes $\psi_\lambda(\theta)$ can be found by setting $\theta^*_i(\lambda) \in \arg \max_{\theta_i \in K_i} (\phi_{i, \theta_i} - \lambda)\overline{u}_{i, \theta_i}$. If $\psi_\lambda(\theta^*(\lambda)) > \lambda$, then $\overline{R}^t > \lambda$ because otherwise it violates the second property in Lemma 3. Similarly, if $\psi_\lambda(\theta^*(\lambda)) \leq \lambda$ then $\overline{R}^t \leq \lambda$ because otherwise it violates the second part of the first property in Lemma 3 (note that since $\theta^*(\lambda)$ is the maximizer of $\psi_\lambda(\theta)$, $\psi_\lambda(\theta^*(\lambda)) \leq \lambda$ implies that $\psi_\lambda(\theta) \leq \lambda$ for all $\theta$). Thus, whether $\overline{R}^t > \lambda$ or $\overline{R}^t \leq \lambda$ can be determined by solely comparing $\psi_\lambda(\theta^*(\lambda))$ with $\lambda$.

We remark that each evaluation of $\psi_\lambda(\theta^*(\lambda))$ takes $O(MK)$ time, and the entire binary search procedure takes time $O(MKT \log(1/\epsilon))$ to approximate $\overline{R}^t$ up to arbitrarily small error $\epsilon$. This is much faster than the brute force algorithm that takes $O(K^M)$ time.

### 3.2 Regret analysis

Below is our main regret theorem for Algorithm 2.

**Theorem 1.** The assortment sequence $\{\theta^{(t)}\}_{t=1}^T$ produced by Algorithm 2 has regret (defined in Eq. (8)) upper bounded as

$$\text{Regret}(\{\theta^{(t)}\}_{t=1}^T) \leq \sqrt{MKT} \log(MKT) + MKU \log^2(MKT) + O(1),$$

where $K = |K|$ and $U = \max_{i \in [M]} \max_{\theta \in K_i} u_{i, \theta}$.

**Corollary 1.** If $K = |K_i| = N + 1$ (for any $i \in [M]$) and the meta-policy in Algorithm 1 is used to convert Algorithm 2 into a dynamic assortment planning algorithm for the original nested model, then

$$\text{Regret}(\{S^{(t)}\}_{t=1}^T) \leq \sqrt{MNT} \log(MNT) + MN^2C_\tau \log^2(MNT) + O(1)$$

$$= \tilde{O}(\sqrt{MNT} + MN^2)$$

We make several remarks on the regret upper bound in Corollary 1. First, when $T > M$ and the number of items per nest $N$ is small, the dominating term in Eq. (11) is $\tilde{O}(\sqrt{MNT})$. This matches the lower bound result $\Omega(\sqrt{MT})$ in Theorem 2 when $N$ is a constant. When the number of items per nest $N$ is large compared to the time horizon $T$, the dominating term in Eq. (11) is $\tilde{O}(MN^2)$. We will show later in Sec. 5 how regret can be improved by considering a “discretization” approach under such large $N$ settings.

In the rest of the section we sketch key steps and lemmas towards the proof of Theorem 1. The following lemma shows that the estimates $\hat{\phi}_{i, \theta}, \hat{u}_{i, \theta}$ concentrate around true values $\phi_{i, \theta}, u_{i, \theta}$ they estimate.

**Lemma 4.** Suppose $T(i, \theta) \geq 96 \ln(2MTK)$. With probability $1 - T^{-1}$ uniformly over all $i \in [M], \theta \in K_i$ and $t \in [T]$

$$|\hat{u}_{i, \theta} - u_{i, \theta}| \leq \min \left\{ U, \sqrt{\frac{96 \max(\hat{u}_{i, \theta}, \hat{u}^2_{i, \theta}) \ln(2MTK)}{T(i, \theta)}} + \frac{144 \ln(2MTK)}{T(i, \theta)} \right\};$$

(12)

$$|\hat{\phi}_{i, \theta} - \phi_{i, \theta}| \leq \min \left\{ 1, \sqrt{\frac{\ln(2MTK)}{T(i, \theta)\hat{u}_{i, \theta}}} \right\}.$$
The following corollary is an immediate consequence of Lemma 4:

**Corollary 2.** With probability $1 - T^{-1}$, $\overline{u}_{i,\theta} \geq u_{i,\theta}$ and $\overline{\phi}_{i,\theta} \geq \phi_{i,\theta}$ for all $i \in [M], \theta \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$.

Corollary 2 shows that (with high probability) $\overline{u}_{i,\theta}$ and $\overline{\phi}_{i,\theta}$ are valid upper bounds for $u_{i,\theta}$ and $\phi_{i,\theta}$.

Our next corollary shows that $\overline{R}$ is also an upper bound for $R'$ at maximizers of $\overline{R}'$ and $\overline{R}$. Recall that $\overline{R}'(\theta) = [\sum_{i=1}^{M} \overline{\phi}_{i,\theta} \overline{u}_{i,\theta}]/[1 + \sum_{i=1}^{M} \overline{u}_{i,\theta}]$ and $R'(\theta) = [\sum_{i=1}^{M} \phi_{i,\theta} u_{i,\theta}]/[1 + \sum_{i=1}^{M} u_{i,\theta}]$.

We defer its proof to the online supplement.

**Corollary 3.** With probability $1 - T^{-1}$, $\overline{R}(\hat{\theta}) \geq R(\hat{\theta})$ and $\overline{R}'(\hat{\theta}^{(r)}) \geq R'(\hat{\theta}^{(r)})$, where $\hat{\theta}, \hat{\theta}^{(r)} \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$ are maximizers of $\overline{R}$ and $R'$, respectively.

We are now ready to sketch the proof of Theorem 1. The first step is to use the classical regret decomposition for UCB type policies ($\mathcal{A}$ denotes the success event in Corollary 3):

\[
\text{Regret}(\{\hat{\theta}^{(r)}\}_{t=1}^T) = \mathbb{E} \sum_{t=1}^{T} R'(\theta^{*}) - R'(\hat{\theta}^{(r)})
\]

\[
\leq \mathbb{E} \left[ \sum_{t=1}^{T} R'(\theta^{*}) - R'(\hat{\theta}^{(r)}) \right] \mathbb{P}_t[A] + O(T) \cdot \mathbb{P}[\mathcal{A}^c]
\]

\[
\leq O(1) + \mathbb{E} \left[ \sum_{t=1}^{T} R'(\theta^{*}) - R'(\hat{\theta}^{(r)}) \right] \mathbb{P}_t[A] + O(T) \cdot \mathbb{P}[\mathcal{A}^c]
\]

\[
\leq O(1) + \mathbb{E} \left[ \sum_{t=1}^{T} R'(\theta^{(r)}) - R'(\hat{\theta}^{(r)}) \right] \mathbb{P}_t[A].
\]

\[
= O(1) + \mathbb{E} \left[ \sum_{\tau} |\mathcal{E}_{\tau}| \cdot (\overline{R}(\hat{\theta}^{(r)}) - R'(\hat{\theta}^{(r)})) \right] \mathbb{P}_t[A].
\]

Here, $\hat{\theta}^{(r)}$ denotes any $\theta^{(r)}$ in the $\tau$-th epoch $\mathcal{E}_{\tau}$. We also note that Eq. (14) holds because $\mathbb{P}_t[A] \leq T^{-1}$ and $\overline{R}'(\theta^{*}) \geq R'(\theta^{*})$; Eq. (15) holds because $\overline{R}'(\theta^{(r)}) \geq \overline{R}(\hat{\theta}^{(r)})$ since $\theta^{(r)}$ is the maximizer of $\overline{R}$ at time $t$.

It remains to upper bound the discrepancy between $\overline{R}'(\hat{\theta}^{(r)})$ and $R'(\hat{\theta}^{(r)})$ at every epoch $\tau$. This is accomplished by the following “aggregation lemma”, which is proved in the online supplement.

**Lemma 5.** With probability $1 - T^{-1}$, for all $t \in [T], i \in [M]$ and $\theta = (\theta_1, \ldots, \theta_M) \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M$,

\[
|\overline{R}'(\theta) - R'(\theta)| \leq \frac{1}{1 + \sum_{i=1}^{M} u_{i,\theta_i}} \left[ \sum_{i=1}^{M} \overline{u}_{i,\theta_i} - u_{i,\theta_i} \right] + \sum_{i=1}^{M} u_{i,\theta_i} (\overline{\phi}_{i,\theta_i} - \phi_{i,\theta_i})
\]

Note that $\mathbb{E}[\overline{\mathcal{E}}_{\tau}] = 1 + \sum_{i=1}^{M} \mathbb{E}[\overline{\mathcal{E}}_{i,\tau}] = 1 + \sum_{i=1}^{M} u_{i,\theta_i}$. Combining Lemma 5 with Eq. (16) we obtain

\[
\text{Regret}(\{\hat{\theta}^{(r)}\}_{t=1}^T) \leq O(1) + \sum_{\tau} \mathbb{E} \left[ \sum_{i=1}^{M} \overline{u}_{i,\hat{\theta}^{(r)}} - u_{i,\hat{\theta}^{(r)}} \right] + \sum_{i=1}^{M} u_{i,\hat{\theta}^{(r)}} (\overline{\phi}_{i,\hat{\theta}^{(r)}} - \phi_{i,\hat{\theta}^{(r)}}) \right] \mathbb{P}_t[A].
\]

\[
\leq O(1) + \sum_{\tau} \mathbb{E} \left[ \sum_{i=1}^{M} \overline{u}_{i,\hat{\theta}^{(r)}} - u_{i,\hat{\theta}^{(r)}} \right] + \sum_{i=1}^{M} u_{i,\hat{\theta}^{(r)}} (\overline{\phi}_{i,\hat{\theta}^{(r)}} - \phi_{i,\hat{\theta}^{(r)}}) \right] \mathbb{P}_t[A].
\]

1Recall that in Algorithm 2, $\theta^{(r)}$ does not change within the same epoch $\mathcal{E}_{\tau}$. We write $\hat{\theta}^{(r)}$ to highlight that $\hat{\theta}^{(r)}$ is the maximizer of $\overline{R}$ in the $\tau$-th epoch (see Step 3 of Algorithm 2).
We next upper bound the two terms in Eq. (18) separately. Define \( T_0(i, \theta) \) as the final size of \( T(i, \theta) \) when Algorithm 2 terminates (i.e., the total number of epochs in which singleton \( \theta \in K_i \) was offered in nest \( i \)). Define also \( \hat{u}_{i\theta}^{(r)} \) to be the estimate of \( u_{i,\theta}^{(r)} \) at epoch \( \tau \). Using Lemma 4, the expectation of the first term in Eq. (18) can be upper bounded by:

\[
\sum_{i=1}^{M} \sum_{\theta \in K_i} \sum_{\ell=0}^{T_0(i, \theta)} U \cdot 1\{\ell < 96 \ln(2MTK)\} + \sqrt{\frac{96 \max(\hat{u}_{i\theta}^{(r)}, |u_{i\theta}^{(r)}|^2) \ln(2MTK)}{\ell(1 + u_{i\theta})^2}} + \frac{144 \ln(2MTK)}{\ell} \cdot 1\{\ell \geq 96 \ln(2MTK)\}
\]

\[
\lesssim \sum_{i=1}^{M} \sum_{\theta \in K_i} U \log(MTK) + \sqrt{u_{i\theta} T_0(i, \theta) \log(MTK)} + \log T_0(i, \theta) \log(MTK)
\]

\[
\leq MKU \log^2(MTK) + \sum_{i=1}^{M} \sum_{\theta \in K_i} \sqrt{u_{i\theta} T_0(i, \theta) \log(MTK)}
\] (19)

Here Eq. (19) holds by plugging in upper bounds on \( |a_{i\theta}^{(r)} - u_{i\theta}| \) (Lemma 4) and noting that, \( \max\{a, a^2\}/(1 + a) \leq a, \), \( \sum_{\ell \leq T_0(i, \theta)} \ell^{-1/2} \leq T_0(i, \theta) \) and \( \sum_{\ell \leq T_0(i, \theta)} \ell^{-1} \leq \log T_0(i, \theta) \). Eq. (20) holds by replacing \( \log T_0 \) with \( \log(MTK) \).

Applying Cauchy-Schwartz inequality and the fact that \( E[\hat{n}_{i,\tau}] = u_{i,\tau}, E[\hat{\epsilon}_\tau] = 1 + \sum_{i=1}^{M} E[\hat{n}_{i,\tau}] \), the summation term in (20) can be further bounded by

\[
\sum_{i=1}^{M} \sum_{\theta \in K_i} \sqrt{u_{i\theta} T_0(i, \theta) \log(MTK)} \leq \sqrt{M|K|} \cdot \sum_{i=1}^{N} \sum_{\theta \in K_i} u_{i\theta} T_0(i, \theta) \log(MTK)
\]

\[
= \sqrt{M|K|} \cdot \sum_{i=1}^{N} \sum_{\theta \in K_i} \sum_{\tau \in T(i, \theta)} E[\hat{n}_{i,\tau}] \log(MTK)
\]

\[
\leq \sqrt{M|K|} \cdot \sqrt{\sum_{\tau} E[|\hat{\epsilon}_\tau|] \log(MTK)}
\]

\[
= \sqrt{MKT \log(MTK)}
\]

Subsequently,

\[
\sum_{i=1}^{M} \sum_{\tau = 1}^{\theta_{i,\tau}} \frac{\pi_{i,\theta}^{(r)} - u_{i,\theta}^{(r)}}{1 + u_{i,\theta}^{(r)}} \lesssim \sqrt{MKT \log(MTK)} + MKU \log^2(MTK).
\] (21)

We next consider the second term in Eq. (18). The following result is a corollary of Lemma 4 which gives a lower bound (with high probability) on \( T(i, \theta) \hat{u}_{i\theta} \) when \( u_{i\theta} \) is not too small. Its proof is given in the online supplement.

**Corollary 4.** With probability \( 1 - T^{-1} \) for all \( i \in [M], \theta \in K_i \) such that \( u_{i\theta} \geq 768 \ln(2MTK)/T(i, \theta) \) and \( T(i, \theta) \geq 96 \ln(2MTK) \), we have \( T(i, \theta) \hat{u}_{i\theta} \geq 0.5T(i, \theta) u_{i\theta} \).

Combining Corollary 4 with Lemma 4 and noting that \( |\hat{\phi}_{i\theta} - \phi_{i\theta}| \leq 1 \) always holds, the second term on
Table 1: Adversarial construction of two types of nests. The revenue parameter \( \rho \) is set to \( \rho = 9\sqrt{2}/(1 + \sqrt{2}) \approx 0.694774 \).

| Type A nest | Type B nest |
|-------------|-------------|
| Item 1 | Item 1 |
| Revenues \( r_{ij} \) | \( \rho \) |
| Preferences \( u_{ij} \) | \( (1 + \epsilon)/M^2 \) |
| Item 2 | Item 2 |
| Revenues \( r_{ij} \) | \( (1 - \epsilon)/M^2 \) |
| Preferences \( u_{ij} \) | \( 1/M^2 \) |
| Item 3 | Item 3 |
| Revenues \( r_{ij} \) | \( 1/\sqrt{2} \) |
| Preferences \( u_{ij} \) | \( (1 - \epsilon)/M^2 \) |

The right-hand side of Eq. (18) can be upper bounded by

\[
\sum_{i=1}^{M} \sum_{\theta \in \mathcal{K}_i} \sum_{\ell=0} T_0(i, \theta) U \{ \ell < 96 \ln(2MTK) \} + \left[ \frac{768 \ln(2MTK)}{\ell} + u_{i\theta} \sqrt{\frac{2 \ln(2MTK)}{T(i, \theta)u_{i\theta}}} \right] \cdot 1\{ \ell \geq 96 \ln(2MTK) \} \lesssim \sum_{i=1}^{M} \sum_{\theta \in \mathcal{K}_i} U \log(MKT) + \sqrt{u_{i\theta}T_0(i, \theta) \log(MKT)} + \log T_0(i, \theta) \log(MKT). \tag{22}
\]

Using similar derivation as in Eq. (21), we have

\[
\sum_{\tau} \sum_{i=1}^{M} u_{i\tau\bar{\theta}}(\phi_{i\tau\bar{\theta}} - \phi_{i\hat{\theta}\tau}) \lesssim \sqrt{MKT \log(MKT)} + MKU \log^2(MTK). \tag{23}
\]

Combining Eqs. (21,23) with Eq. (18) we complete the proof of Theorem 1.

4 A regret lower bound

We establish the following lower bound on the regret of any dynamic assortment planning policy under nested logit models.

**Theorem 2.** Suppose the number of nests \( M \) is divisible by 4 and \( \gamma_1 = \cdots = \gamma_M = 0.5 \). Assume also that (A1) and (A2) hold. Then there exists a numerical constant \( C_0 > 0 \) such that for any dynamic assortment planning policy \( \pi \),

\[
\sup_{\{r_{ij}, v_{ij}\}} \sum_{t=1}^{T} R^* - \mathbb{E}^{\pi} \left[ R(S^{(t)}) \right] \geq C_0 \sqrt{MT} \quad \text{where} \quad R^* = \max_{S \in \mathcal{S}} R(S). \tag{24}
\]

We note the condition that \( M \) is divisible by 4 is only a technical condition and does not affect the main message delivered in Theorem 2, which shows necessary dependency on \( M \) asymptotically when \( M \) is large. Our lower bound construction treats \( N \) as a constant. In particular, in our constructions of adversarial model parameters \( \{r_{ij}, v_{ij}\}_{i,j=1}^{M,N} \), each nest consists of \( N = 3 \) items. Since \( N \) is a constant, Eq. (24) cannot possibly be tight in terms of dependence on \( N \). The optimal dependence on \( N \) is a technically very challenging problem and we leave it as an open problem.

In the rest of this section we provide the proof of Theorem 2, while deferring proofs of several technical lemmas to the supplementary material.
4.1 Construction of adversarial model parameters

Let \( \epsilon > 0 \) be a small positive parameter depending on \( M \) and \( T \), which will be specified later. Each nest \( i \in [M] \) in our construction consists of \( N = 3 \) items and is classified into two categories: “type A” and “type B”, with parameter configurations detailed in Table 1. Note that regardless of which type a nest \( i \in [M] \) is, the three items in nest \( i \) have revenue parameters \( (1 + \epsilon)/M^2, (1 - \epsilon)/M^2 \) and \( 1/M^2 \). Hence it is impossible to decide the type of a nest without observations of customers’ purchasing actions. Given the model parameters in Table 1, it is easy to verify that for a type A nest, the optimal assortment is \{1, 2, 3\}, while for a type B nest, the optimal assortment is \{1, 2\}.

The following lemma shows that any assortment \( S_i \) that does not equal \{1, 2\} for type A nests or \{1, 2, 3\} for type B nests incurs an \( \Omega(\epsilon/M) \) regret. It is proved in the supplementary material.

**Lemma 6.** Let \( U \subseteq [M] \) be the set of type A nests, and by construction \([M] \setminus U\) are all type B nests. For any \( S = (S_1, \ldots, S_M) \in [N]^M \), define \( m_U^\ast(S) := \sum_{i \in U} 1\{S_i \neq \{1, 2\}\} + \sum_{i \notin U} 1\{S_i \neq \{1, 2, 3\}\} \).

Then there exists a numerical constant \( C > 0 \) such that for all \( S \), \( R(S^\ast) - R(S) \geq m_U^\ast(S) \cdot C\epsilon/M \), where \( S^\ast \in \arg\max_S R(S) \) is the optimal assortment combination under \( U \).

4.2 Reduction to average-case regret

Recall that for any policy \( \pi \), we want to show a lower bound on the worst-case regret

\[
\sup_{\{r_{ij}, v_{ij}\}} \sum_{t=1}^T R^\ast - \mathbb{E}^\pi \left[ R(S^{(t)}) \right].
\]

(25)

Let \( M_0 = M/4 \) be an integer (because \( M \) is divisible by 4) and \( S_{M_0} \) be all \( \binom{M}{M_0} \) subsets of \([M]\) with size \( M_0 \). Recall that in our adversarial construction, \( U \subseteq [M] \) denotes the set of all type A nests and the remaining nests \([M] \setminus U\) are of type B. The following inequalities show a reduction to average-case regret:

\[
\sup_{\{r_{ij}, v_{ij}\}} \sum_{t=1}^T R^\ast - \mathbb{E}^\pi \left[ R(S^{(t)}) \right] \geq \sup_{U \in S_{M_0}} \sum_{t=1}^T R^\ast - \mathbb{E}_U^\pi \left[ R(S^{(t)}) \right] \geq \frac{1}{|S_{M_0}|} \sum_{U \in S_{M_0}} \sum_{t=1}^T R^\ast - \mathbb{E}_U^\pi \left[ R(S^{(t)}) \right].
\]

(26)

Here we use the \( \mathbb{E}_U^\pi \) notation to emphasize that the distribution of \( \{S^{(t)}\} \) (and hence the expectation) depends on both the parameter setting (uniquely determined by the set of type A nests \( U \subseteq [M] \)) and the policy \( \pi \) itself.

For any \( i \in [M] \) and \( S \subseteq [N] \), denote \( n_S(i) := \sum_{t=1}^T 1\{S^{(t)} = S\} \) as the random variable of the number of times assortment \( S \) is offered in nest \( i \). Let \( \mathbb{E}_U^\pi[n_S(i)] \) be the expectation of \( n_S(i) \), with expectation taken under model parameters setting \( U \) (recall that \( U \) is the set of all type A nests) and policy \( \pi \). Invoking Lemma 6 and noting that \( \sum_{i=1}^M \sum_{S \subseteq [N]} \mathbb{E}_U^\pi[n_S(i)] = MT \) for any \( U \subseteq [M] \) and policy \( \pi \), the right-hand side of
Eq. (26) can be lower bounded by,

\[
\frac{1}{|S_{M_0}|} \sum_{U \in S_{M_0}} \sum_{i=1}^{T} \mathbb{E}^\pi \left[ m_U^\pi(S^{(i)}) \cdot C_e \right] = \frac{C_e}{M} \frac{1}{|S_{M_0}|} \sum_{U \in S_{M_0}} \left[ \sum_{i \in U} \sum_{S \neq \{1,2\}, S \neq \{1,2,3\}} \mathbb{E}^\pi_U[n_S(i)] + \sum_{i \notin U} \sum_{S \neq \{1,2,3\}} \mathbb{E}^\pi_U[n_S(i)] \right] = \frac{C_e}{M} \left( M T - \frac{1}{|S_{M_0}|} \sum_{U \in S_{M_0}} \left[ \sum_{i \in U} \sum_{S \neq \{1,2\}} \mathbb{E}^\pi_U[n_{\{1,2\}}(i)] + \sum_{i \notin U} \sum_{S \neq \{1,2,3\}} \mathbb{E}^\pi_U[n_{\{1,2,3\}}(i)] \right] \right)
\]

Here in the last identity we replace \( U \) in the second summation (that corresponding to \( n_{\{1,2,3\}}(i) \) with \( U^c := [M] \setminus U \).

To obtain a tight lower bound on the right hand side of (27), we re-structure it by considering subsets \( U' \subseteq [M] \) with size \( M_0 - 1 \), which facilitates the use of a KL divergence bound between the probability law with the type A nests \( U' \subseteq [M] \) and that with the type A nests \( U = U' \cup \{i\} \) (see Lemma 7 in the next subsection). More specifically, let \( S_{M_0-1} \) be the collection of all \( \binom{M}{M_0-1} \) subsets of \([M]\) with size \( M_0 - 1 \), and \( S_{M_0-1}^{(i)} := \{ U \in S_{M_0-1} : i \notin U \} \) be a subset of \( S_{M_0-1} \) excluding the \( i \)th nest. By swapping summation orders, Eq. (27) can be equivalently written as

\[
\frac{C_e}{M} \left( M T - \frac{1}{|S_{M_0}|} \sum_{U' \in S_{M_0-1}} \sum_{i=1}^{M} \sum_{U'' \in S_{M_0-1}^{(i)}} \mathbb{E}^\pi_{U' \cup \{i\}}[n_{\{1,2\}}(i)] + \mathbb{E}_{(U' \cup \{i\})^c}[n_{\{1,2,3\}}(i)] \right)
\]

\[
= \frac{C_e}{M} \cdot T - \frac{C_e}{|S_{M_0}|} \sum_{U' \in S_{M_0-1}} \sum_{i \notin U'} \left( \mathbb{E}_{U' \cup \{i\}}[n_{\{1,2\}}(i)] + \mathbb{E}_{(U' \cup \{i\})^c}[n_{\{1,2,3\}}(i)] \right).
\]

4.3 Pinsker’s inequality

We upper bound \( \mathbb{E}_{U' \cup \{i\}}[n_{\{1,2\}}(i)] \) and \( \mathbb{E}_{(U' \cup \{i\})^c}[n_{\{1,2,3\}}(i)] \) by comparing them with \( \mathbb{E}_{U'}[n_{\{1,2\}}(i)] \) and \( \mathbb{E}_{(U')^c}[n_{\{1,2,3\}}(i)] \). In particular, let \( P_U^\pi \) denote the law under \( U \) and policy \( \pi \). Then for any \( S \subseteq [N] \),

\[
|\mathbb{E}^\pi_U[n_S(i)] - \mathbb{E}^\pi_W[n_S(i)]| \leq \sum_{j=0}^{T} j \cdot |P_U^\pi[n_S(i) = j] - P_W^\pi[n_S(i) = j]| \leq T \cdot \sum_{j=0}^{T} |P_U^\pi[n_S(i) = j] - P_W^\pi[n_S(i) = j]|
\]

\[
= T \|P_U^\pi - P_W^\pi\|_{TV} \leq T \sqrt{\frac{1}{2} \min \{KL(P_U^\pi \| P_W^\pi), KL(P_W^\pi \| P_U^\pi)\}}
\]

\[
\leq T \sqrt{\frac{1}{2} \min \{\max_s KL(P_U(\cdot | S) \| P_W(\cdot | S)), \max_s KL(P_W(\cdot | S) \| P_U(\cdot | S))\}}.
\]

Here \( \|P - Q\|_{TV} \) and \( KL(P\|Q) \) denote the total variational distance and Kullback-Leibler divergence between two probability laws \( P \) and \( Q \), Eq. (29) is known as the Pinsker’s inequality (see e.g., Tsypakov (2009); Csiszar & Körner (2011)). Note that in the last term \( P_U \) and \( P_W \) do not have superscript \( \pi \), because conditioned on a particular assortment combination \( S \) the KL divergence no longer depends on \( \pi \).
The following lemma shows that if \( U \) and \( W \) differ by only one nest, then the KL divergence between \( P_U \) and \( P_W \) is small for all \( S = (S_1, \ldots, S_M) \).

**Lemma 7.** Suppose \( |U \Delta W| = 1 \), where \( U \Delta W = (U \backslash W) \cup (W \backslash U) \) denotes the symmetric difference between subsets \( U, W \subseteq [M] \). Then there exists a constant \( C' > 0 \) such that for any \( S = (S_1, \ldots, S_M) \),

\[
\min\{\KL(P_U(\cdot|S)||P_W(\cdot|S)), \KL(P_W(\cdot|S)||P_U(\cdot|S))\} \leq C'e^2/M.
\]

Invoking Lemma 7, the right-hand side of Eq. (30) can be further upper bounded by

\[
T \sqrt{\frac{T}{2} \cdot \frac{C'e^2}{M}} \leq T \sqrt{T\epsilon^2/M}.
\] (31)

Invoking Eq. (28) and noting that \( |S_{M_0-1}| = \binom{M}{M_0-1} = M - M_0 \), \( |S_{M_0}| \leq |S_{M_0}|/3 \) and \( \sum_{i=1}^{M} \mathbb{E}_{U}[n_S(i)] \leq MT \) for all \( U \subseteq [M] \) and \( S \subseteq [N] \), we have

\[
\frac{1}{|S_{M_0}|} \sum_{U \in S_{M_0}} \sum_{t=1}^{T} R^*_U - \mathbb{E}_U[R(S^{(t)})] \\
\geq C\epsilon \cdot T - \frac{C\epsilon}{|S_{M_0}|} \sum_{U' \in S_{M_0-1}} \frac{1}{M} \sum_{i \notin U'} \left( \mathbb{E}_{U' \cup \{i\}}[n_{1,2}(i)] + \mathbb{E}_{(U' \cup \{i\})c}[n_{1,2,3}(i)] \right) \\
\geq C\epsilon \cdot T - \frac{C\epsilon}{3} \sup_{U' \in S_{M_0-1}} \frac{1}{M} \sum_{i=1}^{M} \left( \mathbb{E}_{U'}[n_{1,2}(i)] + \left| (\mathbb{E}_{U' \cup \{i\}} - \mathbb{E}_{U'})[n_{1,2}(i)] \right| \right) \\
+ \left( \mathbb{E}_{(U')c}[n_{1,2,3}(i)] + \left| (\mathbb{E}_{U' \cup \{i\}} - \mathbb{E}_{(U')c})[n_{1,2,3}(i)] \right| \right) \\
\geq C\epsilon \cdot T - \frac{C\epsilon}{3} \left( \frac{1}{M} \cdot 2MT + \frac{1}{M} \cdot M \cdot O \left( T \sqrt{\frac{T\epsilon^2}{M}} \right) \right). \quad (32)
\]

Setting \( \epsilon = c_0 \sqrt{M/T} \) for some sufficiently small positive constant \( c_0 > 0 \) we complete the proof of Theorem 2.

## 5 Policies with an improved \( N \) dependency

Although we are not able to provide a lower bound on the dependence of \( N \) and derive an optimal policy, we provide a class of policies based on a discretizing technique. This class of policies generalizes our first policy since the first policy simply corresponds to a special case by setting the discretization granularity to zero. In addition, by choosing an appropriate non-zero discretization granularity, we obtain another policy with an improved regret dependence on \( N \) while sacrificing the dependence on \( T \).

### 5.1 Discretizing the singleton sets

In this section, we introduce a discretization technique to further reduce the size of the level set space \( \mathbb{P}_i \) in (5) (or equivalently, \( \mathcal{K}_i \) in the nested singleton model introduced in Sec. 2.3), Instead of considering level sets defined for thresholds \( \theta = r_{ij} \) for all \( j \in [N] \) so that \( |\mathcal{K}_i| = N + 1 \), we only include level sets whose thresholds are on a finite grid.

More specifically, let \( \delta \in (0, 1) \) be a granularity parameter to be optimized later. Recall the definition of the level set \( \mathcal{L}_i(\theta) = \{ j \in [N] : r_{ij} \geq \theta \} \) and we only consider level set threshold parameters \( \theta \) that are
Lemma 8 (Discretized reduction lemma). Fix an arbitrary $\delta \in (0, 1)$. Then

$$\max_{\theta \in \hat{K}_i^\delta} R'(\theta) - \max_{\theta \in \hat{K}_i^\delta} R'(\theta) \leq \delta,$$

where $R'(\theta) := \left[\sum_{i=1}^{M} \phi_{i,\delta, u_{i, \delta}}\right]/\left[1 + \sum_{i=1}^{M} u_{i, \delta}\right]$.

5.2 A discretization based meta-policy and regret analysis

We first present a meta-policy using the discretization technique in Algorithm 3, which connects a dynamic assortment planning policy under the singleton nested model to the original nested logit model. The following proposition upper bounds the regret of the proposed meta-policy, as consequences of Lemma 8 and the fact that $|\hat{K}^\delta_i| \leq \min\{N, [1/\delta] + 2\}$ for all $i \in [M]$ and $\delta \in [0, 1)$.

Proposition 2. Suppose there exists a policy $\pi'$ that attains a regret of at most $\Delta$ on any instance of the nested singleton model with $|\hat{K}^\delta_i| \leq \min\{N, [1/\delta] + 2\}$ for all $i \in [M]$ and $u_{i, \delta} \in [0, U]$ with $U \leq NC_V$ for all $i \in [M]$ and $\theta_i \in \hat{K}_i^\delta$. Then there is a meta-policy $\pi$ (see Algorithm 3) that produces an assortment combination sequence $\{S^{(t)}\}_{t=1}^{T}$ under the original nested choice model with regret (defined in Eq. (4)) at most $\Delta + \delta T$.

We note that the extra regret $\delta T$ comes from the loss of the discretization in Lemma 8.
Now for the nested singleton model, we invoke Algorithm 2 with the discretized singletons $\tilde{K}_\delta^i, \cdots, \tilde{K}_\delta^M$ as input to construct the policy $\pi'$. Then we obtain a class of polices parameterized by $\delta$. By replacing $K = |K_i|$ in Corollary 1 with $\tilde{K}_\delta^i \leq \min\{N, [1/\delta] + 2\} \leq \min\{N, [1/\delta] + 1\}$ and combing with Proposition 2, we obtain the following corollary on the regret under the original nested logit model.

**Corollary 5.** If the meta-policy in Algorithm 3 with discretization granularity $\delta$ is used to convert Algorithm 2 into a dynamic assortment planning algorithm for the original nested model then

$$\text{Regret}(\{S^{(t)}\}_{t=1}^T) \lesssim \sqrt{\min\{N, \delta^{-1}\}MT \log(MNT)} + \min\{N, \delta^{-1}\}MNC_{\delta} \log^2(MNT) + \delta T + O(1). \quad (34)$$

For example, by choosing $\delta \asymp T^{-1/3}$, we have $\text{Regret}(\{S^{(t)}\}_{t=1}^T) = \tilde{O}(\sqrt{MT^{2/3}} + MNT^{1/3})$.

We first note that Corollary 5 is a more general result, which includes Corollary 1 as a special case. Indeed, when $\delta = 0$ (i.e., $\min\{N, \delta^{-1}\} = N$), then $\tilde{K}_\delta^i = K_i$ and Eq. (34) automatically reduces to Eq. (11), which upper bounds the regret of our UCB policy without discretization. On the other hand, when $N$ is large compared with $T$, it is beneficial to set the discretization granularity parameter $\delta$ to be a non-zero value. In particular, by setting $\delta \asymp T^{-1/3}$ we obtain a regret upper bound of $\tilde{O}(\sqrt{MT^{2/3}} + MNT^{1/3})$, which is smaller than the regret bound in Corollary 1 whenever $N > T^{1/3}$ (Note that when $N > T^{1/3}$ the dominating term in Corollary 1 is $\tilde{O}(MN^2)$). In general, an optimal discretization granularity $\delta$ can be calculated with knowledge of the order of $M$, $N$ and $T$, by finding $\delta > 0$ that minimizes the regret upper bound in Eq. (34) and treating $C_{\delta}$ as a constant.

**6 Numerical results**

We present numerical studies of our proposed policies for dynamic nested assortment planning on synthetic data. The main focus of our simulation is the regret of our policies under various model parameter settings of $M$, $N$ and $T$, as well as the effect of the discretization granularity $\delta \in [0, 1]$ on the regret.

**6.1 Experimental settings**

For each nest $i \in [M]$, we generate the revenue parameters $\{r_{ij}\}_{j=1}^N$ independently and identically from the uniform distribution on $[0.2, 0.8]$ and the preference parameters $\{v_{ij}\}_{j=1}^N$ independently and identically from the uniform distribution on $[10/N(M - 1), 20/N(M - 1)]$, where $N$ is the number of items in each nest. The nest discounting parameters $\{\gamma_i\}_{i=1}^M$ are generated independently and identically from the uniform distribution on $[0.5, 1]$.

We consider the model parameter settings of $M \in \{5, 10\}$ for the number of nests, $N \in \{25, 1000\}$ for the number of items per nest, $T \in \{100, 10^7\}$ for the time horizon and $\delta \in \{0, 10^{-3}, 5 \times 10^{-3}, 10^{-2}, 5 \times 10^{-2}\}$ for the granularity parameter in the discretized policy. ($\delta = 0$ means that no discretization is carried out, and the policy reduces to Algorithms 1 and 2). For each $(M, N)$ settings, we generate model parameters $\{r_{ij}, v_{ij}, \gamma_i\}_{i,j=1}^{M,N}$ as described in the previous paragraph, and then run the dynamic assortment policy for 100 independent trials. The median and maximum accumulated regret over $T$ periods are reported.

**6.2 Results**

In Table 2 we compare the accumulated regret of our proposed policies with different granularity parameters $\delta$, under a range of different parameter settings of number of nests $M$, number of items per nest $N$ and time horizon $T$. From Table 2, the overall trend of the accumulated regret is to first decrease and then increase as the granularity parameter $\delta$ increases. Furthermore, we observe that the optimal granularity parameter selection $\delta$ achieving the smallest accumulated regret is larger when there are more items per nest (i.e., $N$
Table 2: Median (MED) and Maximum (MAX) accumulated regret (summation over $T$ periods) under various model and parameter settings.

| $(M, N)$ | $\delta = 0$ | $\delta = 10^{-3}$ | $\delta = 5 \times 10^{-3}$ | $\delta = 10^{-2}$ | $\delta = 5 \times 10^{-2}$ |
|----------|--------------|---------------------|-----------------------------|-------------------|-----------------------------|
|          | MED | MAX | MED | MAX | MED | MAX | MED | MAX | MED | MAX |
| $T = 100$: |     |     |     |     |     |     |     |     |     |     |
| (5,100)  | 5.5 | 6.4 | 5.5 | 6.0 | 3.8 | 4.1 | 3.2 | 4.3 | 5.4 | 8.5 |
| (10,100) | 4.8 | 6.2 | 5.4 | 5.5 | 4.7 | 6.5 | 2.3 | 3.9 | 5.8 | 7.0 |
| (5,250)  | 10.4 | 14.1 | 9.8 | 12.0 | 5.7 | 6.5 | 3.3 | 3.4 | 7.0 | 8.3 |
| (10,250) | 10.8 | 12.0 | 9.7 | 12.3 | 5.5 | 7.4 | 3.0 | 4.4 | 5.1 | 8.7 |
| (5,1000) | 22.0 | 25.3 | 16.0 | 18.2 | 6.2 | 7.5 | 3.2 | 5.0 | 6.9 | 10.9 |
| (10,1000)| 21.5 | 24.1 | 15.1 | 17.7 | 5.1 | 6.4 | 3.1 | 4.9 | 6.2 | 9.4 |

| $T = 500$: |     |     |     |     |     |     |     |     |     |     |
| (5,100)  | 14.3 | 18.5 | 18.3 | 22.6 | 26.8 | 30.9 | 31.9 | 35.3 | 33.3 | 34.3 |
| (10,100) | 15.7 | 23.0 | 16.5 | 22.1 | 28.4 | 28.9 | 35.4 | 36.5 | 35.0 | 36.5 |
| (5,250)  | 14.2 | 17.3 | 12.7 | 14.9 | 16.4 | 18.4 | 29.1 | 36.8 | 32.6 | 34.2 |
| (10,250) | 13.8 | 15.9 | 13.0 | 17.4 | 16.6 | 19.6 | 29.2 | 35.0 | 35.8 | 38.6 |
| (5,1000) | 41.1 | 46.1 | 22.7 | 25.7 | 14.1 | 17.3 | 29.4 | 37.4 | 33.0 | 35.8 |
| (10,1000)| 39.3 | 44.2 | 21.0 | 27.2 | 13.7 | 18.7 | 28.0 | 37.0 | 35.7 | 41.5 |

| $T = 10000$: |     |     |     |     |     |     |     |     |     |     |
| (5,100)  | 491.5 | 505.5 | 489.4 | 496.5 | 494.5 | 500.8 | 503.1 | 511.8 | 513.4 | 525.2 |
| (10,100) | 548.4 | 558.0 | 548.6 | 552.9 | 529.3 | 534.7 | 538.2 | 544.3 | 554.3 | 565.2 |
| (5,250)  | 534.4 | 543.7 | 529.7 | 543.9 | 523.4 | 536.1 | 519.7 | 525.5 | 526.1 | 532.2 |
| (10,250) | 551.0 | 560.5 | 554.5 | 563.3 | 547.4 | 555.2 | 548.6 | 555.1 | 571.6 | 578.4 |
| (5,1000) | 669.0 | 704.4 | 570.5 | 584.8 | 538.8 | 552.7 | 532.9 | 541.3 | 535.8 | 558.4 |
| (10,1000)| 703.5 | 738.2 | 613.1 | 633.6 | 555.7 | 566.3 | 549.9 | 559.5 | 567.2 | 578.6 |

Figure 1: Accumulated regret of our policy with $M = 5$ nests, varying number of items per nest $N$ and granularity parameters $\delta$.

is large). Both observations verify our theoretical findings. In Figure 1, we further plot the accumulated regret of our policies for time horizon when $T$ is large ($T$ between $10^5$ and $10^7$). When $N$ is small, a smaller discretization granularity leads to better empirical performance; while when $N$ is large, a larger discretization granularity is better.
7 Conclusions

In this paper, we consider the dynamic assortment planning problem under the nest logit models and we propose two UCB policies. When $N$ is a small constant and $T$ is large, the first policy has a better regret bound; while when $N$ is large and comparable to $T$, the second policy leads to a better regret bound.

There are several interesting future directions of the current work. The first technical open problem is to establish a lower bound for dynamic assortment under nested logit models and to further improve the UCB policy to be tight in terms of both $N$ and $T$. Second, it is interesting to further extend the current 2-level nested logit models to variants of nested models (e.g., constrained nested logit models (Gallego & Topaloglu, 2014), $d$-level nested logit models (Li et al., 2015), and paired combinatorial nested logit models (Zhang et al., 2017)). Third, the dynamic assortment planning is a relatively new topic in revenue management and the understanding of this problem is still limited. Therefore, most existing work (including this paper) focuses on the stylized models where the assortment is the only decision variable. One important future work is to incorporate other operations decisions and constraints, such as prices and inventory constraints.

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A Proofs of Statements

Proof of Lemma 2

**Lemma 9** (Lemma 2 (restated)). For each epoch $\mathcal{E}_\tau$ and nest $i \in [M]$, let $\hat{\theta}_i \in \mathcal{K}_i$ be the singleton provided in nest $i$. The expectations of the number of iterations and total revenues collected in which nest $i$ is purchased (denoted by $\hat{\nu}_{i,\tau}$ and $\hat{\nu}_{i,\tau}$ respectively in Algorithm 2) satisfies the following regardless of the other offered assortments $\hat{\theta}_{i'}$ for $i' \neq i$:

1. $E[\hat{\nu}_{i,\tau}] = u_{i,\hat{\theta}_i}$;
2. $E[\hat{\nu}_{i,\tau} | \hat{\nu}_{i,\tau}] = \hat{\nu}_{i,\tau} \phi_{i,\hat{\theta}_i}$;

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Proof. Simple calculations show that (see for example Corollary A.1 of Agrawal et al. (2017a))

\[
\Pr \left[ \hat{n}_{i, \tau} = k \right] = \left( \frac{u_{i, \hat{\theta}_i}}{1 + u_{i, \hat{\theta}_i}} \right)^k \left( \frac{1}{1 + u_{i, \hat{\theta}_i}} \right) \quad \text{for } k = 0, 1, 2, \ldots \tag{35}
\]

That is, \( \hat{n}_{i, \tau} \) is a geometric random variable with parameter \( 1/(1 + u_{i, \hat{\theta}_i}) \). Hence, \( \hat{n}_{i, \tau} \) is an unbiased estimator of \( u_{i, \hat{\theta}_i} \), meaning that \( \mathbb{E} \hat{n}_{i, \tau} = u_{i, \tau} \).

The distribution and expectation of \( \hat{\psi}_{i, \tau} \) can be similarly derived, using the property that \( \mathbb{E}[\hat{\psi}_{i, \tau} | i_t = i] = \phi_i \cdot \hat{\theta}_i \). \( \square \)

Proof of Lemma 3

Lemma 10 (Lemma 3 (restated.).) The following hold for all \( \lambda \in [0, 1] \):

1. If \( \bar{R}^* \geq \lambda \) then there exists \( \boldsymbol{\theta} \in K_1 \times \cdots \times K_M \) such that \( \psi_\lambda(\boldsymbol{\theta}) \geq \lambda \); furthermore if \( \bar{R}^* > \lambda \) then the inequality is strict;

2. If \( \bar{R}^* \leq \lambda \) then for all \( \boldsymbol{\theta} \in K_1 \times \cdots \times K_M \), \( \psi_\lambda(\boldsymbol{\theta}) \leq \lambda \); furthermore if \( \bar{R}^* < \lambda \) then the inequalities are strict.

Proof. Let \( \boldsymbol{\theta}^* = (\theta_1^*, \ldots, \theta_M^*) \in K_1 \times \cdots \times K_M \) be a maximizer of \( \bar{R}' \) (i.e., \( \bar{R}' = \bar{R}'(\boldsymbol{\theta}^*) \)). By definition, \( \sum_{i=1}^M (\bar{\psi}^*_i, \theta^*_i) \bar{u}_{i, \theta^*_i} = \bar{R}' \). If \( \bar{R}' \geq \lambda \), then \( \sum_{i=1}^M (\bar{\psi}^*_i, \theta^*_i) \bar{u}_{i, \theta^*_i} \geq \sum_{i=1}^M (\bar{\psi}^*_i, \theta^*_i) \bar{u}_{i, \theta^*_i} = \bar{R}' \geq \lambda \).

Therefore \( \psi_\lambda(\boldsymbol{\theta}^*) \geq \lambda \). Furthermore, if \( \bar{R}' > \lambda \) then the last inequality in the chain of inequalities is strict.

The first property is thus proved.

We next prove the second property. Assume by way of contradiction that there exists \( \boldsymbol{\theta} = (\theta_1, \ldots, \theta_M) \in K_1 \times \cdots \times K_M \) such that \( \psi_\lambda(\boldsymbol{\theta}) > \lambda \), meaning that \( \sum_{i=1}^M (\bar{\psi}_i, \theta_i) \bar{u}_{i, \theta_i} > \lambda \). Re-arranging terms and dividing both sides by \( (1 + \sum_{i=1}^M \bar{u}_{i, \theta_i}) \) we have \( \bar{R}'(\boldsymbol{\theta}) = \sum_{i=1}^M \bar{\psi}_i \bar{u}_{i, \theta_i} / [1 + \sum_{i=1}^M \bar{u}_{i, \theta_i}] > \lambda \). This contradicts the assumption that \( \bar{R}^* = \max_{\theta \in K_1 \times \cdots \times K_M} R'(\boldsymbol{\theta}) \leq \lambda \). To prove the second half of the second property, simply replace all occurrences of \( > \) by \( \geq \). \( \square \)

Proof of Lemma 4

Lemma 11 (Lemma 4 (restated.).) Suppose \( T(i, \theta) \geq 96 \ln(2MK) \). With probability \( 1 - O(T^{-2}) \) uniformly over all \( i \in [M], \theta \in \mathcal{K}_i \) and \( t \in [T] \)

\[
|\hat{u}_{i, \theta} - u_{i, \theta}| \leq \min \left\{ U, \sqrt{\frac{96 \max(\hat{u}_{i, \theta}, \hat{u}_{i, \theta}^2) \ln(2MK)}{T(i, \theta)}} + \frac{144 \ln(2MK)}{T(i, \theta)} \right\};
\]

\[
|\hat{\psi}_{i, \theta} - \psi_{i, \theta}| \leq \min \left\{ 1, \sqrt{\frac{\ln(2MK)}{T(i, \theta) \hat{u}_{i, \theta}}} \right\}.
\]

Proof. We first prove the upper bound on \( |\hat{u}_{i, \theta} - u_{i, \theta}| \) for fixed \( i \in [M] \) and \( \theta \in \mathcal{K}_i \).
Case 1: $u_{i,\theta} \leq 1$. Let $\delta > 0$ be a parameter to be specified later. Applying Lemma 18, we have

$$
\Pr [ |\tilde{u}_{i,\theta} - u_{i,\theta}| > \delta u_{i,\theta}] \leq \exp \left\{ -\frac{nu_{i,\theta}^2 \delta^2}{2(1 + \delta) \times 4} \right\} + \exp \left\{ -\frac{nu_{i,\theta}^2 \delta^2}{6 \times 4 \left( 3 - \frac{2\delta u_{i,\theta}}{2} \right)} \right\}
$$

$$
\leq \exp \left\{ -\frac{nu_{i,\theta}^2 \delta^2}{16 \max(1, \delta)} \right\} + \exp \left\{ -\frac{nu_{i,\theta}^2 \delta^2}{24 \left( 3 - \delta u_{i,\theta} \right)} \right\}
$$

Suppose in addition that $\delta u_{i,\theta} \leq 1$. Then

$$
\Pr [ |\tilde{u}_{i,\theta} - u_{i,\theta}| > \delta u_{i,\theta}] \leq 2 \exp \left\{ -\frac{nu_{i,\theta} \min\{\delta, \delta^2\}}{24} \right\}
$$

Equating the right-hand side of the above inequality with $1/MKT^2$ we have, we have

$$
\delta = \max \left\{ \sqrt{\frac{48 \ln(2MTK)}{u_{i,\theta} T(i, \theta)}}, \frac{48 \ln(2MTK)}{u_{i,\theta} T(i, \theta)} \right\}
$$

and applying the union bound over all $i \in [M], \theta \in \mathcal{K}_i$ and $t \in [T]$, with probability $1 - T^{-1}$

$$
|\tilde{u}_{i,\theta} - u_{i,\theta}| \leq \delta u_{i,\theta} \leq \sqrt{\frac{48 u_{i,\theta} \ln(2MTK)}{T(i, \theta)} + \frac{48 \ln(2MTK)}{T(i, \theta)}}.
$$

Note that if $T(i, \theta) \geq 48 \ln(2MTK)$ the condition $\delta u_{i,\theta} \leq 1$ is met. Replacing all occurrences of $u_{i,\theta}$ in Eq. (38) by $\tilde{u}_{i,\theta}$ and using the fact that $\sqrt{a + b} \leq \sqrt{a} + \sqrt{b}$, we have

$$
|\tilde{u}_{i,\theta} - u_{i,\theta}| \leq \sqrt{\frac{48 \tilde{u}_{i,\theta} \ln(2MTK)}{T(i, \theta)} + \frac{48 \ln(2MTK)}{T(i, \theta)}}.
$$

Case 2: $u_{i,\theta} > 1$. Let $\delta \in (0, 1]$ be a parameter to be specified later. Applying Lemma 18, we have

$$
\Pr [ |\tilde{u}_{i,\theta} - u_{i,\theta}| > \delta u_{i,\theta}] \leq \exp \left\{ -\frac{nu_{i,\theta}^2 \delta^2}{6 \times 4 u_{i,\theta}^2 \left( 3 - \frac{2\delta u_{i,\theta}}{2} \right)} \right\} + \exp \left\{ -\frac{nu_{i,\theta}^2 \delta^2}{2 \times 4 u_{i,\theta}^2} \right\}
$$

$$
\leq 2 \exp\{-n\delta^2/12\}.
$$

Equating the right-hand side of the above inequality with $1/MKT^2$ we have

$$
\delta = \sqrt{\frac{24 \ln(2MTK)}{T(i, \theta)}}.
$$
and applying the union bound over all \(i \in [M], \theta \in \mathcal{K}_i\) and \(t \in [T]\), with probability \(1 - T^{-1}\),

\[
|\hat{u}_{i\theta} - u_{i\theta}| \leq \delta u_{i\theta} \leq \sqrt{\frac{24u_{i\theta}^2 \ln(2MTK)}{T(i, \theta)}}. \tag{40}
\]

Note that \(\delta \leq 1\) holds if \(T(i, \theta) \geq 24 \ln(2MTK)\). In addition, if \(T(i, \theta) \geq 96 \ln(2MTK)\) we have \(|\hat{u}_{i\theta} - u_{i\theta}| \leq 0.5u_{i\theta}\) and hence \(\hat{u}_{i\theta} \geq 0.5u_{i\theta}\). Subsequently, Eq. (40) implies

\[
|\hat{u}_{i\theta} - u_{i\theta}| \leq \sqrt{\frac{96u_{i\theta}^2 \ln(2MTK)}{T(i, \theta)}}. \tag{41}
\]

Finally, combining Eqs. (39,41) we proved the upper bound on \(|\hat{u}_{i\theta} - u_{i\theta}|\).

We next prove the upper bound on \(|\hat{\phi}_{i\theta} - \phi_{i\theta}|\). Recall that for each \(\tau \in T(i, \theta), \tau_{ik}\) is the sum of \(\hat{n}_{ik}\) i.i.d. random variables with mean \(\phi_{i\theta}\) and within range \([0, 1]\) almost surely. Also note that \(\sum_{\tau' \in T(i, \theta)} \hat{n}_{i,k} = T(i, \theta)\hat{u}_{i\theta}\). Applying Hoeffding’s inequality (Lemma 16) we have for any \(\delta > 0\) that

\[
\Pr \left[ |\hat{\phi}_{i\theta} - \phi_{i\theta}| > \delta \right] \leq 2 \exp \left\{ -2\delta^2 \cdot T(i, \theta)\hat{u}_{i\theta} \right\}.
\]

Equating the right-hand side of the above inequality with \(1/M(K + 1)T^2\) and applying the union bound, we have with probability \(1 - T^{-1}\) uniformly over \(i \in [M], \theta \in \mathcal{K}_i\) and \(t \in [T]\) that

\[
|\hat{\phi}_{i\theta} - \phi_{i\theta}| \leq \sqrt{\frac{\ln(2MTK)}{T(i, \theta)\hat{u}_{i\theta}}}. \tag{42}
\]

\(\square\)

**Proof of Corollary 3**

**Corollary 6** (Corollary 3 (restated)). With probability \(1 - T^{-1}\), \(\overline{R}(\bar{\theta}) \geq \overline{R}(\bar{\theta})\) and \(\overline{R}(\theta^*) \geq \overline{R}(\theta^*)\), where \(\bar{\theta}, \theta^* \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M\) are maximizers of \(\overline{R}'\) and \(R'\), respectively.

**Proof.** We first prove \(\overline{R}'(\tilde{\theta}) \geq R'(\tilde{\theta})\). By definition, \(\sum_{i=1}^M (\tilde{\theta}_{i, \tilde{\theta}_i} - \overline{R}'(\tilde{\theta}))\tilde{\theta}_{i, \tilde{\theta}_i} = \overline{R}'(\tilde{\theta})\). In addition, because \(\overline{R}'(\tilde{\theta})\) is the maximizer of \(\overline{R}'\), setting \(\lambda = \overline{R}'(\tilde{\theta})\) and by the second property of Lemma 3 we know that \(\psi_\lambda(\theta) = \lambda\) and \(\psi_\lambda(\theta) \leq \lambda\) for all \(\theta \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M\), where \(\psi_\lambda(\theta) = \sum_{i=1}^M (\tilde{\phi}_{i, \tilde{\theta}_i} - \lambda)\tilde{\theta}_{i, \tilde{\theta}_i}\).

We claim that \(\tilde{\phi}_{i, \tilde{\theta}_i} \geq \overline{R}(\theta)\) whenever \(\tilde{\theta}_{i, \tilde{\theta}_i} > 0\). Assume the contrary, that \(\tilde{\phi}_{i, \tilde{\theta}_i} < \overline{R}(\theta)\) = \(\lambda\) and \(\tilde{\theta}_{i, \tilde{\theta}_i} > 0\) for some \(i \in [M]\). Consider \(\tilde{\theta}' = (\tilde{\theta}_1', \ldots, \tilde{\theta}_M')\) defined as \(\tilde{\theta}_i' = \infty\) and \(\tilde{\theta}_i' = \tilde{\theta}_i'\) for all \(i' \neq i\). Because \(\tilde{\theta}_i' = \infty\), we know that \(\tilde{\theta}_{i, \tilde{\theta}_i'} = u_{i, \tilde{\theta}_i} = 0\). Subsequently, \(\psi_\lambda(\tilde{\theta}') = \psi_\lambda(\tilde{\theta}) - (\tilde{\phi}_{i, \tilde{\theta}_i} - \lambda)\tilde{\theta}_{i, \tilde{\theta}_i} \geq \psi_\lambda(\tilde{\theta}) = \lambda\). This contradicts the condition that \(\psi_\lambda(\tilde{\theta}) \leq \lambda\) for all \(\theta \in \mathcal{K}_1 \times \cdots \times \mathcal{K}_M\).

Define \(\psi_\lambda(\tilde{\theta}) = \sum_{i=1}^M (\phi_{i, \theta_i} - \lambda)u_{i, \theta_i}\), which is similar to the definition of \(\psi_\lambda\) except all occurrences of \(\tilde{\phi}_{i, \tilde{\theta}_i}\) and \(\tilde{\theta}_{i, \tilde{\theta}_i}\) are replaced by their true values \(\phi_{i, \theta_i}\) and \(\theta_{i, \theta_i}\). Because \(\tilde{\phi}_{i, \tilde{\theta}_i} \geq \overline{R}(\theta)\) for all \(\tilde{\theta}_{i, \tilde{\theta}_i} > 0\) and \(\tilde{\phi}_{i, \tilde{\theta}_i}\) are upper bounds of \(\phi_{i, \theta_i}\) and \(\theta_{i, \theta_i}\), we conclude that \(\psi_\lambda(\tilde{\theta}) \leq \psi_\lambda(\tilde{\theta}) = \lambda\), implying that \(\sum_{i=1}^M (\phi_{i, \theta_i} - \lambda)u_{i, \theta_i} \leq \lambda\). Re-arranging terms we have \(R'(\tilde{\theta}) = [\sum_{i=1}^M \phi_{i, \theta_i}u_{i, \theta_i}]/[1 + \sum_{i=1}^M u_{i, \theta_i}] \leq \lambda = \overline{R}'(\tilde{\theta})\).

We next prove \(\overline{R}'(\theta^*) \geq R'(\theta^*)\). Recall that \(R'(\theta^*) = [\sum_{i=1}^M \phi_{i, \theta_i}u_{i, \theta_i}]/[1 + \sum_{i=1}^M u_{i, \theta_i}]\). Hence, \(\psi_\lambda(\theta^*) = \lambda\) for \(\lambda = R'(\theta^*)\), meaning that \(\sum_{i=1}^M (\phi_{i, \theta_i}^* - \lambda)u_{i, \theta_i}^* = \lambda\). By similar analysis, we know that \(\phi_{i, \theta_i}^* \geq \lambda\) for all \(u_{i, \theta_i}^* > 0\). Because \(\tilde{\phi}_{i, \tilde{\theta}_i}\), \(\tilde{\theta}_{i, \tilde{\theta}_i}\) are upper bounds of \(\phi_{i, \theta_i}\) and \(\theta_{i, \theta_i}\), the above conclusion holds for any \(\tilde{\theta}_{i, \tilde{\theta}_i} > 0\).
we have \( \psi_\lambda(\theta^*) = \sum_{i=1}^M (\phi_i,\theta_i^* - \lambda)\bar{u}_{i,\theta_i^*} \geq \sum_{i=1}^M (\phi_i,\theta_i^* - \lambda)u_{i,\theta_i^*} = \psi_R(\theta^*) = \lambda. \) This implies that \( \sum_{i=1}^M (\phi_i,\theta_i^* - R'(\theta^*))\bar{u}_{i,\theta_i^*} \geq R'(\theta^*), \) and therefore \( R'(\theta^*) = \frac{\sum_{i=1}^M \phi_i,\theta_i^*\bar{u}_{i,\theta_i^*}}{1 + \sum_{i=1}^M \bar{u}_{i,\theta_i^*}} \geq R'(\theta^*). \)

Proof of Lemma 5

Lemma 12 (Lemma 5 (restated).) With probability \( 1 - T^{-1}, \) for all \( t \in [T], i \in [M] \) and \( \theta = (\theta_1, \ldots, \theta_M) \in K_1 \times \cdots \times K_M, \) \[
|R'(\theta) - R'(\theta)| \leq \sum_{i=1}^M \frac{\bar{u}_{i,\theta_i} - u_{i,\theta_i} + \sum_{i=1}^M u_{i,\theta_i}(\phi_i,\theta_i - \phi_i,\theta_i)}{1 + u_{i,\theta_i}}.
\] (43)

Proof: To simplify notations, we shall abbreviation \( \phi_i = \phi_i,\theta_i, u_i = u_{i,\theta_i} \) and \( \bar{\phi}_i = \bar{\phi}_{i,\theta_i}, \bar{u}_i = \bar{u}_{i,\theta_i}. \) We also abbreviate \( R' = R'(\theta) \) and \( \bar{R}' = \bar{R}'(\theta). \)

By definition of \( R' \) and \( \bar{R}', \) we have \[
(1 + \sum_{i=1}^M u_i) |R' - \bar{R}'| = (1 + \sum_{i=1}^M u_i) \left[ \sum_{i=1}^M \frac{\bar{u}_{i,\theta_i}}{1 + \sum_{i=1}^M u_i} - \sum_{i=1}^M \frac{u_{i,\theta_i}}{1 + \sum_{i=1}^M u_i} \right]
= \sum_{i=1}^M \bar{\phi}_i \left( \frac{1 + \sum_{i' = 1}^M u_{i'}}{1 + \sum_{i' = 1}^M \bar{u}_{i'}} - u_i \right) + \sum_{i=1}^M u_i(\bar{\phi}_i - \phi_i)
\leq \sum_{i=1}^M \left( \frac{1 + \sum_{i' = 1}^M u_{i'}}{1 + \sum_{i' = 1}^M \bar{u}_{i'}} - u_i \right) + \sum_{i=1}^M u_i(\bar{\phi}_i - \phi_i) \quad (44)
\]

The first term on the right-hand side of Eq. (44) can be further upper bounded by \[
\sum_{i=1}^M \frac{1 + \sum_{i' = 1}^M u_{i'}}{1 + \sum_{i' = 1}^M \bar{u}_{i'}} - u_i = \sum_{i=1}^M \frac{\bar{u}_i(1 + \sum_{i' = 1}^M u_{i'}) - \sum_{i=1}^M u_i(1 + \sum_{i' = 1}^M u_{i'})}{1 + \sum_{i=1}^M \bar{u}_i} \leq \sum_{i=1}^M \frac{\bar{u}_i - u_i}{1 + u_i}. \quad (45)
\]

Here the last inequalities holds because the \( \sum_{i=1}^M u_{i} \bar{u}_{i'} \) term cancels out, and \( 1 + \sum_{i=1}^M \bar{u}_i \geq 1 + u_i. \) \( \square \)

Proof of Corollary 4

Corollary 7 (Corollary 4 (restated).) With probability \( 1 - T^{-1} \) for all \( i \in [M], \theta \in K_\theta \) such that \( u_{i,\theta} \geq 768 \ln(2MK)/T(i, \theta) \) and \( T(i, \theta) \geq 96 \ln(2MK), \) we have \( T(i, \theta)\hat{u}_{i,\theta} \geq 0.5T(i, \theta)u_{i,\theta}. \)

Proof: First consider the case of \( u_{i,\theta} \geq 1. \) By Eq. (40) in the proof of Lemma 4, if \( T(i, \theta) \geq 96 \ln(2MK) \) we have \( |\hat{u}_{i,\theta} - u_{i,\theta}| \leq 0.5u_{i,\theta} \) and therefore \( T(i, \theta)\hat{u}_{i,\theta} \geq 0.5T(i, \theta)u_{i,\theta}. \)

In the rest of the proof we consider the case of \( 768 \ln(2MK)/T(i, \theta) \leq u_{i,\theta} \leq 1. \) By Eq. (38) in the proof of Lemma 4, we have \[
T(i, \theta)\hat{u}_{i,\theta} \geq T(i, \theta)u_{i,\theta} - \sqrt{48T(i, \theta)u_{i,\theta} \ln(2MK)} - 48 \ln(2MK).
\]

Under the condition that \( u_{i,\theta} \geq 768 \ln(2MK)/T(i, \theta), \) the above inequality yields \( T(i, \theta)\hat{u}_{i,\theta} \geq 0.5T(i, \theta)u_{i,\theta}. \) \( \square \)

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Proof of Lemma 6

Lemma 13 (Lemma 6 (restated)). Let \( U \subseteq [M] \) be the set of type A nests, and by construction \([M] \setminus U \) are all type B nests. For any \( S = (S_1, \cdots, S_M) \in [N]^M \), define \( m_{U,S}^1(S) := \sum_{i \in U} 1\{S_i \neq \{1, 2\} \} + \sum_{i \notin U} 1\{S_i \neq \{1, 2, 3\} \} \). Then there exists a numerical constant \( C > 0 \) such that for all \( S \), \( R(S^*) - R(S) \geq m_{U,S}^1(S) \cdot C \epsilon / M \), where \( S^* \in \arg \max_{S} R(S) \) is the optimal assortment combination under \( U \).

Proof. For any \( U \subseteq [M] \), \( S \subseteq \{1, 2, 3\} \) and \( S = (S_1, \cdots, S_M) \in [3]^M \), define \( m_{U,S}^2(S) := \sum_{i \in U} 1\{S_i = S\} \) and similarly \( m_{U,S}^2(S) := \sum_{i \notin U} 1\{S_i = S\} \). Denote also \( S^* = (S_1^*, \cdots, S_M^*) \) as the optimal assortment combination, in which \( S_i = \{1, 2\} \) for all \( i \in U \) and \( S_i = \{1, 2, 3\} \) for all \( i \notin U \). Let also \( R_U(\cdot), V_U(\cdot), R_{U^c}(\cdot), V_{U^c}(\cdot) \) be revenue and preference of assortment selections in nests of type A \((R_U(\cdot)\) and \(V_U(\cdot)\)) or type B \((R_{U^c}(\cdot)\) and \(V_{U^c}(\cdot)\)), respectively. Recall that \( |U| = M/4 \) and \( |U^c| = 3M/4 \). We then have

\[
R(S^*) - R(S) = \frac{R_U(\{1, 2\}) V_U(\{1, 2\})^{1/2} \cdot M/4 + R_{U^c}(\{1, 2, 3\}) V_{U^c}(\{1, 2, 3\})^{1/2} \cdot 3M/4}{1 + V_U(\{1, 2\})^{1/2} \cdot M/4 + V_{U^c}(\{1, 2, 3\})^{1/2} \cdot 3M/4} - \frac{\sum_{S \subseteq \{1, 2, 3\}} m_{U,S}^2(S) \cdot R_U(S) V_U(S)^{1/2} + m_{U,S}^2(S) \cdot R_{U^c}(S) V_{U^c}(S)^{1/2}}{1 + \sum_{S \subseteq \{1, 2, 3\}} m_{U,S}^2(S) \cdot V_U(S)^{1/2} + m_{U,S}^2(S) \cdot V_{U^c}(S)^{1/2}}. \tag{46}
\]

We next list the values of \( V_U(\cdot), R_U(\cdot), V_{U^c}(\cdot) \) and \( R_{U^c}(\cdot) \) under our adversarial construction, shown in Table 1.

| \( S = \emptyset \) | \( S = \{1\} \) | \( S = \{2\} \) | \( S = \{3\} \) | \( S = \{1, 2\} \) | \( S = \{1, 3\} \) | \( S = \{2, 3\} \) | \( S = \{1, 2, 3\} \) |
|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( V_U(S) \)    | \( V_U(S) \)  | \( V_U(S) \)  | \( V_U(S) \)  | \( V_U(S) \)  | \( V_U(S) \)  | \( V_U(S) \)  | \( V_U(S) \)  |
| \( R_U(S) \)    | \( R_U(S) \)  | \( R_U(S) \)  | \( R_U(S) \)  | \( R_U(S) \)  | \( R_U(S) \)  | \( R_{U^c}(S) \) | \( R_{U^c}(S) \) |
| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)| \( V_{U^c}(S) \)|
| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)| \( R_{U^c}(S) \)|

Plugging the values of \( V_U(\cdot), R_U(\cdot), V_{U^c}(\cdot), R_{U^c}(\cdot) \) into \( R(S^*) - R(S) \), and taking \( \epsilon \to 0^+ \), by detailed algebraic calculations we proved the lemma.

\[\square\]

Proof of Lemma 7

Lemma 14 (Lemma 7 (restated)). Suppose \( |U \triangle W| = 1 \), where \( U \triangle W = (U \setminus W) \cup (W \setminus U) \) denotes the symmetric difference between subsets \( U, W \subseteq [M] \). Then there exists a constant \( C' > 0 \) such that for any \( S = (S_1, \cdots, S_M) \), \( \min\{KL(P_U(\cdot|S)||P_W(\cdot|S)), KL(P_W(\cdot|S)||P_U(\cdot|S))\} \leq C' \epsilon^2 / M \).

28
Proof. By symmetry we may assume without loss of generality that \( W = U \cup \{i_0\} \) for some \( i_0 \not\in U \). The random variables observable are \((i,j)\) where \( i \in [M] \cup \{0\} \) indicates the nest in which a purchase is made (if no purchase is made then \( i = 0 \)) and \( j \in [N] = \{1,2,3\} \) is the particular item purchased in nest \( i \) (if \( i = 0 \) simply define \( j = 0 \) with probability 1). The KL divergence \( KL(P_U(\cdot|S)||P_W(\cdot|S)) \) can then be written as

\[
KL(P_U(\cdot|S)||P_W(\cdot|S)) = -\mathbb{E}_U \left[ \log \frac{P_W(i,j|S)}{P_U(i,j|S)} \right] = -\mathbb{E}_U \left[ \log \frac{P_W(i|S)}{P_U(i|S)} \right] - \mathbb{E}_U \left[ \log \frac{P_W(j|i,S)}{P_U(j|i,S)} \right].
\]

(47)

We next upper bound the first term on the right-hand side of Eq. (47). By the nested model, the nest-level purchase action \( i \in [M] \cup \{0\} \) follows a categorical distribution of \( M + 1 \) categories, parameterized by probabilities \( p = (p_0, \ldots, p_M) \) under \( U \) and \( q = (q_0, \ldots, q_M) \) under \( W \). By elementary algebra (see for example Lemma 3 in (Chen & Wang, 2017)), \( KL(p||q) \) can be upper bounded as

\[
KL(p||q) = - \sum_{i=0}^{M} p_i \log \frac{q_i}{p_i} \leq \sum_{i=0}^{M} |p_i - q_i|^2.
\]

Note that \( U \) and \( W \) only differ in nest \( i_0 \). Using the nested model description and \( \gamma_i \equiv 0.5 \), it is easy to verify that \( |p_i - q_i| \leq \epsilon/M \) for \( i \in \{0, i_0\} \), \( |p_i - q_i| \leq \epsilon/M^2 \) if \( i \not\in \{0, i_0\} \), \( q_0 \geq \Omega(1) \) and \( q_i \geq 1/M \) for all \( i \geq 1 \). Subsequently,

\[
KL(p||q) \lesssim \epsilon^2/M.
\]

(48)

We proceed to upper bound the second term on the right-hand side of Eq. (47). Because \( U \) and \( W \) only differ in nest \( i_0 \), this term is non-zero only if \( i = i_0 \). Conditioned on \( i = i_0 \), it is easy to verify that \( KL(P_U(i_0|S_0)||P_W(i_0|S_0)) \lesssim \epsilon^2 \) for all \( S_0 \subseteq [N] \). In addition, \( \max \{P_U(i_0|S), P_W(i_0|S)\} \lesssim 1/M \). Subsequently,

\[
-\mathbb{E}_U \left[ \log \frac{P_W(j|i,S)}{P_U(j|i,S)} \right] = P_U(i_0|S) \cdot KL(P_U(\cdot|i_0,S_0)||P_W(\cdot|i_0,S_0)) \lesssim \epsilon^2/M.
\]

(49)

Combining Eqs. (48,49) we complete the proof of Lemma 7.

\[\square\]

Proof of Lemma 8

Lemma 15 (Lemma 8 (restated).). Fix an arbitrary \( \delta \in (0,1) \). Then

\[
\max_{\theta \in K_1 \times \ldots \times K_M} R'(\theta) - \max_{\theta \in \tilde{K}_1^\delta \times \ldots \times \tilde{K}_M^\delta} R'(\theta) \leq \delta,
\]

where \( R'(\theta) := [\sum_{i=1}^{M} \phi_i \theta_i u_i \theta_i]/[1 + \sum_{i=1}^{M} u_i \theta_i] \).

Proof. Let \( \theta^* = (\theta^*_1, \ldots, \theta^*_M) \in K_1 \times \ldots \times K_M \) be the assortment that maximizes \( R' \). Define \( \tilde{\theta}_i^* := [\theta^*_i/\delta] \cdot \delta \) for all \( i \in [M] \) and \( \tilde{\theta} := (\tilde{\theta}_1, \ldots, \tilde{\theta}_M) \). It is easy to verify that \( \tilde{\theta} \in \tilde{K}_1^{\delta} \times \ldots \times \tilde{K}_M^{\delta} \). Therefore, it suffices to prove that \( R'(\tilde{\theta}) \geq R^* - \delta \) where \( R^* = R'(\theta^*) \).

To simplify notations, abbreviate \( R_i = R_i(L_i(\theta^*_i)), V_i = V_i(L_i(\theta^*_i)), \tilde{R}_i = R_i(L_i(\tilde{\theta}^*_i)) \) and \( \tilde{V}_i = V_i(L_i(\tilde{\theta}^*_i)) \), where \( R_i(\cdot) \) and \( V_i(\cdot) \) are defined in Eqs. (1,3). Denote also that \( x_i := \tilde{V}_i - V_i \). By definition of
\( R_i \) and \( \tilde{R}_i \), we have \( R_i V_i = \sum_{i,j \geq \theta_i} r_{ij} v_{ij} \) and \( \tilde{R}_i \tilde{V}_i = \sum_{i,j \geq \tilde{\theta}_i} r_{ij} v_{ij} \). Subsequently,

\[
\tilde{R}_i \tilde{V}_i = R_i V_i + \sum_{\theta_i^* > r_{ij} \geq \tilde{\theta}_i^*} r_{ij} v_{ij} \geq R_i V_i + x_i (\theta_i^* - \delta). \tag{50}
\]

Here the last inequality holds because \(|\theta_i^* - \tilde{\theta}_i^*| \leq \delta\) and \(\sum_{\theta_i^* > r_{ij} \geq \tilde{\theta}_i^*} v_{ij} = \tilde{V}_i - V_i = x_i\). Subsequently,

\[
\tilde{V}_i^{\gamma_i} [\tilde{R}_i - (R^* - \delta)] = (V_i + x_i)^{\gamma_i} \left[ R_i V_i + x_i (\theta_i^* - \delta) \right] - R^* \geq (V_i + x_i)^{\gamma_i} \frac{R_i V_i + x_i \theta_i^*}{V_i + x_i} - R^*. \tag{51}
\]

In Eq. (51), we apply Eq. (50), and Eq. (53) holds because \(x_i / (V_i + x_i) \leq 1\).

**Proposition 3.** For \( i \in [M] \) define function \( h_i(\Delta) := (V_i + \Delta)^{\gamma_i} [R_i V_i + \Delta \theta_i^*/(V_i + \Delta) - R^*] \). Then \( h_i \) is monotonically non-decreasing in \( \Delta \) for \( \Delta \geq 0 \).

Invoking Proposition 3, we have that for all \( i \in [M] \),

\[
\tilde{V}_i^{\gamma_i} \left[ \tilde{R}_i - (R^* - \delta) \right] \geq (V_i + x_i)^{\gamma_i} \left[ \frac{R_i V_i + x_i \theta_i^*}{V_i + x_i} - R^* \right] \geq \tilde{V}_i^{\gamma_i} [R_i - R^*]. \tag{54}
\]

Summing over \( i \in [M] \) on both sides of the above inequality and using the definition that \( R^* = (\sum_{i \in [M]} R_i V_i^{\gamma_i}) / (1 + \sum_{i \in [M]} V_i^{\gamma_i}) \),

\[
\sum_{i \in [M]} \tilde{V}_i^{\gamma_i} \left[ \tilde{R}_i - (R^* - \delta) \right] \geq \sum_{i \in [M]} R_i V_i^{\gamma_i} - \left( \sum_{i \in [M]} V_i^{\gamma_i} \right) R^* = R^* \geq R^* - \delta. \tag{55}
\]

Re-organizing terms we have

\[
R'(\theta^*) = \frac{\sum_{i=1}^{M} \phi_i \tilde{R}_i \tilde{V}_i}{1 + \sum_{i=1}^{M} u_i \tilde{V}_i} = \frac{\sum_{i \in [M]} R_i (L_i(\theta_i^*)) V_i (L_i(\tilde{\theta}_i^*))^{\gamma_i}}{1 + \sum_{i \in [M]} V_i (L_i(\theta_i^*))^{\gamma_i}} \geq \frac{\sum_{i \in [M]} \tilde{R}_i \tilde{V}_i^{\gamma_i}}{1 + \sum_{i \in [M]} V_i^{\gamma_i}} \geq R^* - \delta,
\]

which completes the proof.

**Proposition 4 (Proposition 3 (restated)).** For \( i \in [M] \) define function \( h_i(\Delta) := (V_i + \Delta)^{\gamma_i} [(R_i V_i + \Delta \theta_i^*)/(V_i + \Delta) - R^*] \). Then \( h_i \) is monotonically non-decreasing in \( \Delta \) for \( \Delta \geq 0 \).

**Proof.** Note that \( h_i(\Delta) = (V_i + \Delta)^{\gamma_i-1} (R_i V_i + \Delta \theta_i^*) - (V_i + \Delta)^{\gamma_i} R^* \). Differentiating \( h_i \) with respect to \( \Delta \) we have

\[
h_i'(\Delta) = (\gamma_i - 1)(V_i + \Delta)^{\gamma_i-2} (R_i V_i + \Delta \theta_i^*) + \theta_i^* (V_i + \Delta)^{\gamma_i-1} - \gamma_i (V_i + \Delta)^{\gamma_i-1} R^*. \tag{56}
\]

\( \square \)
Using the second property of Lemma 1 that \( \theta_i^* \geq \gamma_i R_i + (1 - \gamma_i) R_i (S_i^*) \), we have for all \( \Delta \geq 0 \) that

\[
h_i'(\Delta) \geq (\gamma_i - 1)(V_i + \Delta)^{\gamma_i - 2} (R_i V_i + \Delta \theta_i^*) + [\gamma_i R_i^* + (1 - \gamma_i) R_i ] (V_i + \Delta)^{\gamma_i - 1}
- \gamma_i (V_i + \Delta)^{\gamma_i - 1} R_i^* \\
= (1 - \gamma_i)(V_i + \Delta)^{\gamma_i - 2} [R_i (V_i + \Delta) - R_i V_i - \Delta \theta_i^*] \\
= (1 - \gamma_i)(V_i + \Delta)^{\gamma_i - 2} \cdot (R_i - \theta_i^*) \Delta \geq 0.
\]

(57) (58) (59)

The proposition is then proved, because \( h_i'(\Delta) \geq 0 \) for all \( \Delta \geq 0 \).

\[\Box\]

**B References to some concentration inequalities**

**Lemma 16** (Hoeffding’s inequality (Hoeffding, 1963)). Suppose \( X_1, \cdots, X_n \) are i.i.d. random variables such that \( a \leq X_i \leq b \) almost surely. Then for any \( t > 0 \),

\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}X \right| > t \right] \leq 2 \exp \left\{ -\frac{2nt^2}{(b-a)^2} \right\}.
\]

**Lemma 17** (Bernstein’s inequality (Bernstein, 1924)). Suppose \( X_1, \cdots, X_n \) are i.i.d. random variables such that \( \mathbb{E}[(X_i - \mathbb{E}X_i)^2] \leq \sigma^2 \) and \( |X_i| \leq M \) almost surely. Then for any \( t > 0 \),

\[
\Pr \left[ \left| \frac{1}{n} \sum_{i=1}^{n} X_i - \mathbb{E}X \right| > t \right] \leq 2 \exp \left\{ -\frac{nt^2/2}{\sigma^2 + Mt/3} \right\}.
\]

The following result is cited from Theorem 5 of (Agrawal et al., 2017a).

**Lemma 18** (Concentration of geometric random variables (Agrawal et al., 2017a)). Suppose \( X_1, \cdots, X_n \) are i.i.d. geometric random variables with parameters \( p > 0 \), meaning that \( \Pr[X_i = k] = (1-p)^k p \) for \( k = 0, 1, 2, \cdots \). Define \( \mu := \mathbb{E}X_i = (1-p)/p \). Then

\[
\Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i > (1 + \delta) \mu \right] \leq \begin{cases} \exp \left\{ -\frac{n\mu \delta^2}{2(1+\delta)(1+\mu)^2} \right\}, & \text{if } \mu \leq 1, \\
\exp \left\{ -\frac{n\delta^2 \mu^2}{6(1+\mu)^2} \left( 3 - \frac{2\delta \mu}{1+\mu} \right) \right\}, & \text{if } \mu \geq 1, \delta \in (0, 1); \end{cases}
\]

\[
\Pr \left[ \frac{1}{n} \sum_{i=1}^{n} X_i < (1 - \delta) \mu \right] \leq \begin{cases} \exp \left\{ -\frac{n\mu \delta^2}{6(1+\mu)^2} \left( 3 - \frac{2\delta \mu}{1+\mu} \right) \right\}, & \text{if } \mu \leq 1, \\
\exp \left\{ -\frac{n\delta^2 \mu^2}{2(1+\mu)^2} \right\}, & \text{if } \mu \geq 1. \end{cases}
\]