WEAK APPROXIMATION FOR ISOTRIVIAL FAMILIES

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Abstract. We study weak approximation for isotrivial families of rationally connected varieties over a smooth projective curve. In the case where the family is formed by taking a quotient of a cyclic group action, we find a necessary and sufficient condition for weak approximation to hold. In most other cases, there is also a sufficient condition. We also check that this condition is satisfied for all smooth Fano hypersurfaces in projective spaces.

1. Introduction

We will work over an algebraically closed field of characteristic 0 unless otherwise stated. A variety $X$ is called rationally connected if there exists an irreducible component $V$ of $\text{Hom}(\mathbb{P}^1, X)$ and a family of maps $f : \mathbb{P}^1 \times V \rightarrow X$ such that $f^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times V \rightarrow X \times X$ is dominant. In [HT06] Hassett and Tschinkel proposed the following weak approximation conjecture for flat families of rationally connected varieties over a curve.

Conjecture 1.1. Let $\pi : \mathcal{X} \rightarrow B$ be a flat surjective morphism from a smooth projective variety to a smooth projective curve such that a general fiber is rationally connected (such a map $\mathcal{X} \rightarrow B$ is called a model of the generic fiber). Then the morphism $\pi$ satisfies weak approximation. That is, for every finite sequence $(b_1, \ldots, b_m)$ of distinct closed points of $B$, for every sequence $(\hat{s}_1, \ldots, \hat{s}_m)$ of formal power series sections of $\pi$ over $b_i$, and for every positive integer $N$, there exists a regular section $s$ of $\pi$ which is congruent to $\hat{s}_i$ modulo $m_{B,b_i}^N$ for every $i = 1, \ldots, m$.

In this article we consider the case where the family $\mathcal{X} \rightarrow B$ is an isotrivial family. There is a way of producing such families as described below. Let $X$ be a smooth projective rationally connected variety, $G$ be a cyclic group of order $l$, and $C \cong \mathbb{P}^1$ be a smooth proper curve. Let $G$ act on both $X$ and $C$. Then $G$ acts diagonally on $X \times C$. Thus we can form the quotient $q : X \times C \rightarrow \mathcal{X} = X \times C / G$ and $B = C / G \cong \mathbb{P}^1$. Then we have a projection $\pi : \mathcal{X} \rightarrow C / G$ and the following commutative diagram.

\[
\begin{array}{ccc}
X \times C & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow \pi \\
C & \longrightarrow & B
\end{array}
\]

Notice that the weak approximation property is in fact a property of the generic fiber. So we can choose different birational models, in particular, models whose total spaces are singular. Therefore, even though the total space $\mathcal{X}$ constructed above is in general not smooth, it still makes sense to talk about weak approximation.

The following is the first theorem.

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Theorem 1.2. Weak approximation holds for the isotrivial family $\pi : \mathcal{X} \to B$ constructed above if and only if for every pair of fixed points $(x, y)$ of $\mathcal{X}$, there exists a $G$-equivariant map $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(\infty) = y$.

Our proof also gives a sufficient criteria in the following more general situations. Consider an isotrivial family of rationally connected varieties $\mathcal{X} \to B$, where $\mathcal{X}$ is smooth and projective, and $B$ is a smooth projective curve. A general fiber is isomorphic to a rationally connected variety $X$. Let $b \in B$ be a closed point such that the fiber $X_b$ is not smooth. Let $K$ be the fraction field $\text{Frac}(\widehat{O}_{b,B})$ of the completed local ring of $B$ at $b$ and $X_K$ be the base change to $\text{Spec } K$. We make the following hypothesis:

**Hypothesis 1.3.** Assume that there is a Galois extension $\tilde{K}/K$ with a cyclic Galois group $G_b$ such that $\tilde{X} = X_K \times_{\text{Spec } K} \text{Spec } \tilde{K}$ is isomorphic to $X \times_k \text{Spec } \tilde{K}$. Also assume that the action of $G_b$ extends to an action on $X \times_{\text{Spec } \tilde{O}}$, where $\tilde{O}$ is the ring of integers in $\tilde{K}$.

Under this assumption, $G_b$ acts on $X$. Then weak approximation holds for this isotrivial family if for every such $b \in B$ and the corresponding Galois group $G_b$, and every pair of fixed points $x, y$ in $X$ under the action of $G_b$, there is a $G_b$-equivariant map $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(\infty) = y$.

By Theorem 1.2, if the weak approximation conjecture is true, then for every finite cyclic subgroup $H$ of $\text{Aut}(X)$, and for every pair of fixed points of $H$, there has to be an $H$-equivariant rational curve passing through them. Conversely, if such equivariant curves always exist, weak approximation is true for isotrivial families whose general fiber is isomorphic to $X$ and satisfies the above assumption. Therefore it is natural to analyze equivariant rational curves in a rationally connected variety admitting a finite cyclic group action. We do this in the case of smooth Fano hypersurfaces in $\mathbb{P}^n$.

Theorem 1.4. Let $X$ be a smooth degree $d$ hypersurface in $\mathbb{P}^n$ ($n \geq d$). Let $G$ be a cyclic group of order $l$ and act on $X$. Then for every pair of fixed points in $X$, there exist equivariant maps from $\mathbb{P}^1$ to $X$ connecting them. In particular, weak approximation holds for any isotrivial family of smooth Fano hypersurfaces satisfying Hypothesis 1.3.

The key observation here is that the existence of equivariant rational curves is an open and closed condition (Lemma 1.1). Therefore we use a specialization argument and try to find such curves on some special smooth hypersurfaces. The method used here should be applicable to complete intersections in a Grassmannian/toric variety, etc.

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2. Preliminary results

In this section we collect some preliminary results for later reference. The results proved here are slightly stronger than what are actually needed. So in this section we will assume that $G$ is a finite group except in Lemma 2.3 and $k$ is an algebraically closed field whose characteristic is not divisible by the order of $G$. 

First, we are concerned with the following infinitesimal lifting problem. Let $S$ and $R$ be $k$-algebras with a $G$-action and $f : S \to R$ be an algebra homomorphism compatible with the action. Also assume that $R$ is a finite type $S$-algebra via the map $f$. Let $A$ be an Artinian $k$-algebra with a $G$-action, $I \subset A$ an invariant ideal such that $I^2 = 0$. Consider the following commutative diagram, where $p$ is a $G$-equivariant $k$-algebra homomorphism.

\[
\begin{array}{ccc}
S & \xrightarrow{f} & R \\
\downarrow & & \downarrow p \\
A & \xrightarrow{\pi} & A/I & \longrightarrow & 0
\end{array}
\]

We want to know when can one find a $G$-equivariant lifting $h : R \to A$. The following lemma completely answers this question.

**Lemma 2.1.** If we can lift the map $p$ to a $k$-algebra homomorphism $h : R \to A$ such that $\pi \circ h = p$, then we can find an equivariant lifting $h : R \to A$ with the same property.

**Proof.** For every element $g$ in $G$, define a map $h_g : R \to A$ by $h_g(r) = g \cdot h(g^{-1} \cdot r)$. This is an $S$-algebra homomorphism and also a lifting of the map $p : R \to A/I$. The map $h$ is $G$-equivariant if and only if $h_g(r) = h(r)$ for every $g \in G$ and every $r \in R$. The difference of any two such liftings is an element in $\text{Hom}(\Omega_{R/S}, I)$, where $\Omega_{R/S}$ is the module of relative differentials. Therefore one has $\theta(g)(r) = h_g(r) - h(r)$ in $\text{Hom}(\Omega_{R/S}, I)$. Notice that $\text{Hom}(\Omega_{R/S}, I)$ is naturally a $G$-module with the action of $G$ on $\text{Hom}(\Omega_{R/S}, I)$ given by

\[
G \times \text{Hom}(\Omega_{R/S}, I) \to \text{Hom}(\Omega_{R/S}, I)
\]

\[
(g, \eta) \mapsto g \cdot \eta = (\omega \mapsto g \cdot \eta(g^{-1} \cdot \omega)).
\]

It is easy to check that

\[
\theta(gh) = g \cdot \theta(h) + \theta(g)
\]

Thus $\theta$ defines an element $[\theta]$ in $H^1(G, \text{Hom}(\Omega_{R/S}, I))$. The existence of an equivariant lifting is equivalent to the existence of an element $\Theta \in \text{Hom}(\Omega_{R/S}, I)$ such that $g\Theta - \Theta = \theta$, i.e, the class defined by $\theta$ is zero in $H^1(G, \text{Hom}(\Omega_{R/S}, I))$. But the characteristic of the field is relatively prime to the order of $G$, so the higher cohomology of $G$ vanishes ([Wei94], Proposition 6.1.10, Corollary 6.5.9).

**Corollary 2.2.** Let $X$ and $Y$ be two $k$-schemes with a $G$-action and $f : X \to Y$ be a finite type morphism compatible with the action. Let $x \in X$ be a fixed point, and $y = f(x)$ (hence also a fixed point). Assume that $f$ is smooth at $x$. Then there exists a $G$-equivariant section $s : \text{Spec } \mathcal{O}_{y,Y} \to X$.

**Proof.** Let $S$ be the local ring at $y$, and $R$ be the local ring at $x$. There is an obvious $G$ action on both of these $k$-algebras. We start with the section $s_0 : \text{Spec } k(y) \to f^{-1}(y), \text{Spec } k(y) \to x$, which is clearly $G$-equivariant. By the smoothness assumption, a section from $\text{Spec } (\mathcal{O}_{y,Y}/m_y^n)$ always lifts to a section from $\text{Spec } (\mathcal{O}_{y,Y}/m_y^{n+1})$. Now apply Lemma 2.1 inductively to finish the proof.
Finally, we need some results of deforming a morphism in a $G$-equivariant setting. Recall that a morphism $f : \mathbb{P}^1 \to X$ is called very free if $f^* TX \cong \oplus_i \mathcal{O}_{\mathbb{P}^1}(a_i)$ with $a_i \geq 1$. A smooth projective variety $X$ is separably rationally connected if and only if there is a very free morphism from $\mathbb{P}^1$ to $X$.

**Lemma 2.3.** Let $X$ be a smooth projective separably rationally connected variety and $G$ be a cyclic group acting on $X$. Fix a faithful action of $G$ on $\mathbb{P}^1$.

1. Let $f : \mathbb{P}^1 \to X$ be a $G$-equivariant map. Then there exists a $G$-equivariant map $\tilde{f} : \mathbb{P}^1 \to X$ such that $\tilde{f}(0) = f(0)$, $\tilde{f}(\infty) = f(\infty)$, and $\tilde{f}$ is very free.
2. Let $f_i : C_i \to X, 1 \leq i \leq n$ be a chain of equivariant maps, i.e. for each $i$, $C_i \cong \mathbb{P}^1$, and $f_i$ is a $G$-equivariant map such that $f_i(\infty) = f_{i+1}(0)$ for $1 \leq i \leq n - 1$. Then there is a $G$-equivariant map $f : \mathbb{P}^1 \to X$ such that $f(0) = f_1(0)$ and $f(\infty) = f_n(\infty)$.

**Proof.** We may assume the equivariant map $f$ is an embedding and $\dim X \geq 3$ by replacing $X$ with $X \times \mathbb{P}^M$ for some large $M$. Let $C$ be the equivariant curve corresponding to $f$ and $x, y$ be the image of 0 and $\infty$. We can attaching very free curves $C_i$ at general points $p_i \in C$ along general tangent directions at $p_i$. Let $D$ be the nodal curve assembled in this way. It follows from [GHS03] that after attaching enough such curves, the twisted normal sheaf $\mathcal{N}_{D/X}(-x - y)$ is globally generated and $H^1(D, \mathcal{N}_{D/X}(-x - y)) = 0$. In other word, we can smooth the nodes and get a very free curve. Now we can attach all the curves that are $G$-conjugate to $C_i$. The new nodal curve is again denoted by $D$ (we may choose $C_i$’s such that the $G$-orbits do not intersect each other). Then $D$ is invariant. Also $\mathcal{N}_{D/X}(-x - y)$ is globally generated and $H^1(D, \mathcal{N}_{D/X}(-x - y)) = 0$. The problem is that there may not be a $G$-equivariant deformation that smooths all the nodes. But we can continue attaching $G$-conjugate very free curves to $D$ at general points and general tangent directions until we get $G$-equivariant deformations that smooths all the nodes of $D$. Then we get a $G$-equivariant very free curve.

For the second part, we may assume that all the $f_i$’s are very free. Let $f$ be the $G$-equivariant map obtained by gluing together all the $f_i$’s. Let $(S, p)$ be a pointed smooth curve with a trivial $G$-action. Let $\tilde{C} = \mathbb{P}^1 \times S$ with the natural diagonal action. There are two $G$-equivariant sections, $s_0, s_\infty$. Now blow up the point $s_\infty(p)$ and still denote the strict transforms of the two sections by $s_0$ and $s_\infty$. The $G$-action extends to the blow up. We can make the fiber over $p \in S$ a chain of rational curves with $n$ irreducible components by repeating this operation. Then we get a smooth surface $C$ with the action of $G$. Let $h_0 : s_0 \to X \times S$ and $h_\infty : s_\infty \to X \times S$ be $S$-morphisms such that $h_0(s_0) = f_1(0) \times S$ and $h_\infty(s_\infty) = f_n(\infty) \times S$. Consider the scheme $\text{Hom}_S(C, X \times S, h_0, h_\infty)$ parameterizing $S$-morphisms from $C$ to $X \times S$ fixing $h$. It has a natural $G$-action and the map $\mu : \text{Hom}_S(C, X \times S, h_0, h_\infty) \to S$ is $G$-equivariant. Now $\mu$ is smooth at $f$. Then the action of $G$ on the tangent space at $f$ has invariant directions that are mapped onto the tangent space of $S$ at $p$. So there are $G$-equivariant deformations that smooth the map $f$.

\[\square\]

3. Proof of Theorem 1.2

The proof of this theorem uses the notion of pseudo-ideal sheaves.

**Definition 3.1.** Let $Y$ be an algebraic space and $f : X \to Y$ be a flat, locally finitely presented, proper algebraic stack over $Y$. For every morphism of algebraic
spacess \( g : Y' \to Y \), a flat family of pseudo-ideal sheaves of \( X/Y \) over \( Y' \) is a pair \((\mathcal{F}, u)\) consisting of

(i) a \( Y' \)-flat, locally finitely presented, quasi-coherent \( \mathcal{O}_{X'/Y} \)-module \( \mathcal{F} \), and

(ii) an \( \mathcal{O}_{X'/Y} \)-homomorphism \( u : \mathcal{F} \to \mathcal{O}_{X'/Y} \),

such that the following induced morphism is zero,

\[
u' : \bigwedge^2 \mathcal{F} \to \mathcal{F}, \quad f_1 \wedge f_2 \mapsto u(f_1)f_2 - u(f_2)f_1.
\]

Let \( D \) be an effective Cartier divisor in \( X \), considered as a closed subscheme of \( X \), and assume \( D \) is flat over \( Y \). Denote by \( \mathcal{I}_D \) the pullback

\[ \mathcal{I}_D := \mathcal{I} \otimes_{\mathcal{O}_X} \mathcal{O}_D \]

on \( \text{Hilb}_{X/Y} \times_X D \). And denote by

\[ u_D : \mathcal{I}_D \to \mathcal{O}_{\text{Hilb}_{X/Y} \times_X D} \]

the restriction of \( u \). Then we have:

**Proposition 3.2.** \([\text{RS09}]\) The locally finitely presented, quasi-coherent sheaf \( \mathcal{I}_D \) is flat over \( \text{Hilb}_{X/Y} \). Thus the pair \((\mathcal{I}_D, u_D)\) is a flat family of pseudo-ideal sheaves of \( D/Y \) over \( \text{Hilb}_{X/Y} \).

Denote by

\[ \iota_D : \text{Hilb}_{X/Y} \to \text{Pseudo}_{D/Y} \]

the 1-morphism associated to the flat family \((\mathcal{I}_D, u_D)\) of pseudo-ideal sheaves of \( D/Y \) over \( \text{Hilb}_{X/Y} \). This is the divisor restriction map.

We have the following theorem, due to M. Roth and J. Starr.

**Theorem 3.3.** \([\text{RS09}]\) If both

\[ H^1(C_\kappa, \mathcal{O}_X(-D) \cdot \text{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}^2_\kappa, \mathcal{O}_{C_\kappa})) \]

and

\[ H^1(C_\kappa, \text{Tor}_{\mathcal{O}_X}(\mathcal{O}_{C_\kappa}, \mathcal{O}_D) \cdot \text{Hom}_{\mathcal{O}_{C_\kappa}}(\mathcal{I}_\kappa/\mathcal{I}^2_\kappa, \mathcal{O}_{C_\kappa})) \]

equal 0, then \( \iota_D \) is smooth at \([C_\kappa]\).

**Proof of Theorem 3.3.** If weak approximation holds, one can choose two fixed points \((x, y)\) and ask for a section \( s : \mathbb{P}^1 \to \mathcal{X} \) which satisfies \( s(0) = q(x, 0) \), and \( s(\infty) = q(y, \infty) \), where \( q : X \times C \to \mathcal{X} \) is the quotient map. Then this section gives a \( G \)-equivariant section \( \tilde{s} \) of \( \pi_2 : X \times C \to C \) such that \( \tilde{s}(0) = (x, 0) \), \( \tilde{s}(\infty) = (y, \infty) \). Then the projection onto \( X \) gives the desired curve.

Conversely, assume that every pair of fixed points in \( X \) is connected by an equivariant map from \( \mathbb{P}^1 \). We may assume \( \dim X \geq 3 \) by replacing \( X \) with \( X \times \mathbb{P}^M \) for some large \( N \) and Let \( G \) act on \( \mathbb{P}^M \) trivially. It is easy to see that weak approximation is true for the original family if and only if it is true for the new family. Instead of working with the model whose total space is singular, we will work with the smooth Deligne-Mumford stack \([X \times C/G]\) together with the projection to the stacky curve \([C/G]\). Then \( \mathcal{X} \) and \( B \) are coarse moduli spaces of \([X \times C/G]\) and \([C/G]\). Notice that a (formal) section from \( C/G \cong \mathbb{P}^1 \) induces a \( G \)-equivariant (formal) section of \( p : X \times C \to C \). The latter (formal) section in turn induces a (formal) section of \( \pi : [X \times C/G] \to [C/G] \). Conversely a (formal) section of \( \pi : [X \times C/G] \to [C/G] \) gives a (formal) section of \( \mathcal{X} \to B \). So it suffices to prove weak approximation for this stack model.
We have the following Cartesian diagram.

\[
\begin{array}{ccc}
X \times C & \xrightarrow{q} & [X \times C/G] \\
\downarrow p & & \downarrow \pi \\
C & \xrightarrow{q'} & [C/G]
\end{array}
\]

It is proved in [HT06] that weak approximation is satisfied if the points are chosen such that the fiber over them are smooth. So it suffices to prove weak approximation at 0 and \(\infty\). We will focus on weak approximation at 0. The proof for weak approximation at \(\infty\) is the same. Let \(\hat{s}\) be a given formal section from the formal neighborhood of 0 and \(s\) be a regular section (\(s\) exists by [GHS03]).

Here is the idea of the proof. We first show that we can deform the section \(s\) to a section which agrees with the formal \(\hat{s}\) on any pre-specified order in a formal neighborhood of 0. In particular, this deformation gives a deformation of the corresponding pseudo-ideal sheaves. Then we use Theorem 3.3 to show that this deformation of pseudo-ideal sheaves lifts to a deformation of global sections.

There exists a Galois extension of the field \(K \subset \hat{K} = K[t]/(t^n - t)\) with Galois group \(G \cong \mathbb{Z}/l\mathbb{Z}\) such that \(X_K = X_K \otimes_K \text{Spec} \hat{K} \cong X \times \text{Spec} \hat{K}\). In other words, we have the following Cartesian diagram.

\[
\begin{array}{ccc}
X_K & \longrightarrow & X_K \\
\downarrow & & \downarrow \\
\text{Spec} \hat{K} & \longrightarrow & \text{Spec} K
\end{array}
\]

There is a natural action of \(G\) on \(X\) and \(\text{Spec} \hat{K}\). \(s\) and \(\hat{s}\) give two \(K\) points of \(X_K\) which induces two \(\hat{K}\) points \(s, \hat{s}\) of \(X_K \otimes_K \text{Spec} \hat{K}\), invariant under the action of the Galois group. The \(G\)-action on \(X_K\) extends to an action on the constant family \(X \times_k \text{Spec} \hat{O}_{0,C}\) and the projection onto the second factor is \(G\)-equivariant.

By the valuative criterion of properness, we have two \(\hat{O}_{0,C} \cong k[t]\) points of \(X \times \text{Spec} \hat{O}_{0,C}\), still denoted by \(s, \hat{s}\). Let \(s_0, \hat{s}_0\) be the points of these two sections contained in the central fiber. Since \(s, \hat{s}\) are invariant under the Galois group, \(s_0\) and \(\hat{s}_0\) are fixed points of the \(G\)-action on \(X\).

By Lemma 2.3 we may assume that there exists an equivariant very free curve \(f: (\mathbb{P}^1, 0, \infty) \to (X, s_0, \hat{s}_0)\). Moreover, we may assume that the map is a closed immersion since \(\dim X \geq 3\). Then \(H^1(\mathbb{P}^1, f^*T_X(-0 - \infty)) = 0\). Thus the map

\[
p: \text{Hom}(\mathbb{P}^1 \times \text{Spec} \hat{O}_{0,C}, X \times \text{Spec} \hat{O}_{0,C}, f \mid_{0} = s, f \mid_{\infty} = \hat{s}) \to \text{Spec} \hat{O}_{0,C}
\]

is smooth at \([f]\), where 0 means the 0 section \(\text{Spec} \hat{O}_{0,C} \to 0 \times \text{Spec} \hat{O}_{0,C}\) and similarly for \(\infty\). Notice that both spaces have a natural \(G\) action such that this map is equivariant and \([f]\) is a fixed point of the action. So by Corollary 2.2 there is an equivariant section

\[
\sigma: \text{Spec} \hat{O}_{0,C} \to \text{Hom}(\mathbb{P}^1 \times \text{Spec} \hat{O}_{0,C}, X \times \text{Spec} \hat{O}_{0,C}).
\]

i.e., we have a \(G\)-equivariant map:

\[
f: \mathbb{P}^1 \times \text{Spec} \hat{O}_{0,C} \to X \times \text{Spec} \hat{O}_{0,C}
\]

such that

\[
f \mid_{0 \times \text{Spec} \hat{O}_{0,C}} = s, f \mid_{\infty \times \text{Spec} \hat{O}_{0,C}} = \hat{s}
\]
This map is a closed immersion since the central fiber is a closed immersion. This gives a closed immersion:
\[ i : [\mathbb{P}^1 \times \text{Spec } \hat{\mathcal{O}}_{0,C/G}] \to [X \times \text{Spec } \hat{\mathcal{O}}_{0,C/G}]. \]

Moreover, the section \( s \) and the formal section \( \hat{s} \) restrict to two sections on the germ of ruled surface:
\[ [\mathbb{P}^1 \times \text{Spec } \hat{\mathcal{O}}_{0,C/G}] \to [\text{Spec } \hat{\mathcal{O}}_{0,C/G}]. \]

Let \( D \) be the divisor over \( 0 \) in \([X \times C/G]\) and \( E \) its restriction to this ruled surface. It is easy to see that there is a ruled surface
\[ \pi : \mathbb{P} = [\mathbb{P}^1 \times C/G] \to [C/G] \]

together with two sections \( \infty_P \) and \( 0_P \) such that the base change of this ruled surface to \( \text{Spec } \hat{\mathcal{O}}_{0,[C/G]} \) is isomorphic to
\[ [\mathbb{P}^1 \times \text{Spec } \hat{\mathcal{O}}_{0,C/G}] \to [\text{Spec } \hat{\mathcal{O}}_{0,C/G}]. \]

and the base change of these two sections are \( s \mid_{\text{Spec } \hat{\mathcal{O}}_{0,[C/G]}} \) and \( \hat{s} \). On \( \mathbb{P} \) consider the invertible sheaf \( \mathcal{L} = \mathcal{O}_P(\infty_P - 0_P - \pi^*(N \cdot E)) \), where \( N \) is the order to which we want the regular section to agree with the given formal section. Since the restriction to the generic fiber of \( P \to [C/G] \) is a degree 0 invertible sheaf on \([C/G]\), it has a 1-dimensional space of global sections. Thus the push-forward of \( \mathcal{L} \) to \([C/G]\) is a torsion-free, coherent \( \mathcal{O}_{[C/G]} \)-module of rank 1, i.e., it is an invertible sheaf. Thus there exists an effective divisor \( \Delta \) in \( C \), not intersecting \( E \) such that \( \mathcal{O}_P(\infty_P - 0_P + \pi^*\Delta - \pi^*(N \cdot E)) \) has a global section. In other words, there is an effective divisor \( F \) in \( P \), necessarily vertical, such that
\[ \infty_P + \pi^*\Delta = 0_P + \pi^*(N \cdot E) + F. \]

Let the curves \( G_t \) be the members of the pencil spanned by \( 0_P + \pi^*(N \cdot E) + F \) and \( \infty_P + \pi^*\Delta \). All but finitely many members of this pencil are comb-like curves with handle a section of \( P \to [C/G] \). Since the base locus of the pencil contains \( \infty_P \cap \pi^*(N \cdot E) \), these section curves agree with \( \infty_P \) over \( (N \cdot E) \). Restricting to \((N \cdot E)\) we get a one parameter family of pseudo-ideal sheaves in \( N \cdot E \), hence also in \( N \cdot D \) such that they agree with the given formal section \( s \) to a given order.

Denote the stacky curve in \( D \) by \( C_0 \). Let \( C_s \) be the union of the curve \( C_0 \) and the section \( s([C/G]) \). Let \( C_s \) be a comb obtained by attaching very free rational curves in general fibers along general normal directions to \( C_s \). Notice that the pseudo-ideal sheaf obtained by restricting \( C_s \) to \( N \cdot D \) is the same as that obtained by restricting \( C_s \). The sheaf
\[ \mathcal{O}_{[X \times C/G]}(-N \cdot D) \cdot \text{Hom}_{\mathcal{O}_{C_s}}(\mathcal{I}_s/T^2_s, \mathcal{O}_{C_s}) \]
is supported in the union of the section \( s([C/G]) \) and the very free curves since \( C_0 \) is contained in \( D \). If we attach enough very free curves, we can make
\[ H^1(C_s, \mathcal{O}_{[X \times C/G]}(-N \cdot D) \cdot \text{Hom}_{\mathcal{O}_{C_s}}(\mathcal{I}_s/T^2_s, \mathcal{O}_{C_s})) \]
zero (c.f. [GHS03]).

Now we want to show that
\[ H^1(C_s, Tor_{\mathcal{O}_X}(\mathcal{O}_{C_s}, \mathcal{O}_D) \cdot \text{Hom}_{\mathcal{O}_{C_s}}(\mathcal{I}_s/T^2_s, \mathcal{O}_{C_s})) \]
is also 0. First, using the exact sequence
\[ 0 \to \mathcal{O}(-N \cdot D) \to \mathcal{O}_{[X \times \mathbb{P}^1/G]} \to \mathcal{O}_{N \cdot D} \to 0 \]
we see that

\[ \text{Tor}_{\mathcal{O}_{X \times C/G}}(\mathcal{O}_{C_{\kappa}}, \mathcal{O}_{N \cdot D}) \cong \mathcal{O}_{C_0}(-p), \]

where \( p \) is the intersection of section and the stacky curve \( C_0 \) in \( D \). Thus the sheaf is supported on \( C_0 \). We have the short exact sequence of sheaves

\[ 0 \to N_{C_0/\{X \times C/G\}}(-p) \to N_{C_{\kappa}/\{X \times C/G\}} \to Q \to 0 \]

where \( Q \) is a torsion sheaf supported at \( p \). Thus it suffices to show that

\[ H^1(C_0, N_{C_{\kappa}/\{X \times C/G\}}(-p)) \]

is 0. Let \( C_0', C_{\kappa}' \), and \( p' \) be the preimages of \( C_0, C_{\kappa}, p \) in \( X \times B \) and \( q : C \to C_0 \)
be the restriction of the quotient map \( X \times B \to [X \times C/G] \). Then the sheaf

\[ N_{C_0'/\{X \times C/G\}}(-p) \]

is a direct summand of

\[ q_*(q^*N_{C_0/\{X \times C/G\}}(-p)) = q_*(N_{C_{\kappa}'/\{X \times C\}}(-p')) \]

Therefore, the cohomology groups is also direct summand of the corresponding cohomology. But we have chosen the curve \( C_{\kappa}' \) to be a very free curve in the fiber. So \( H^1(C_0', N_{C_{\kappa}'/\{X \times C/G\}}(-p')) \) is 0 and so is

\[ H^1(C_{\kappa}, N_{C_{\kappa}/\{X \times C/G\}})(\mathcal{O}_{C_{\kappa}}, \mathcal{O}_{D}) \cdot \mathcal{H}om_{\mathcal{O}_{C_{\kappa}}}(\mathcal{I}_p/\mathcal{T}_{\kappa}, \mathcal{O}_{C_{\kappa}})). \]

Therefore the divisor restriction map \( \iota_{N \cdot D} : \text{Hilb}_{\{X \times C/G\}} \to \text{Pseudo}_{N \cdot D} \) is smooth at \( [C_{\kappa}] \). So we get a one parameter family of section curves whose restriction to the divisor \( N \cdot D \) gives the deformation of pseudo-ideal sheaves constructed above. In particular, we get a section of the actual isotrivial family which agrees with the formal section to the given order \( N \).

\[ \square \]

**Remark 3.4.** For a general isotrivial family satisfying Hypothesis \[\text{[13]}\] we can construct quotient stacks locally near the singular fibers and glue them to the family over the complement of the points with singular fibers. Then our argument also works in this case.

4. **EQUIVARIANT RATIONAL CURVES ON HYPERSURFACES**

We will consider smooth hypersurfaces in projective spaces and use the following conventions/notations.

- \( X \), a smooth hypersurface of degree \( d \) in \( \mathbb{P}^n \), \( 3 \leq d \leq n \).
- \( G \), a cyclic group of order \( l \) acting on \( X \) faithfully.
- \( w(X_i), i = 0, 1, \ldots, n, 0 \leq w(X_i) \leq p-1 \), the weights of the \( G \)-action. That is, this action can be lifted to an action on \( \mathbb{P}^n \) and the linearization of the action on \( \mathcal{O}_{\mathbb{P}^n}(1) \) is given by

\[ g \cdot X_i = \zeta^{w(X_i)} X_i, \]

where \( g \) is a generator of \( G \), \( \zeta \) is a primitive \( l \)th root of unity.
- \( x_i, 0 \leq i \leq n \), the points whose coordinates are all zero except the \( i \)th one. These are fixed points of the action on \( \mathbb{P}^n \). Given two fixed points in \( X \), we will always assume that they are \( x_0 \) and \( x_1 \) by making a linear change of coordinates and keeping the coordinates to be eigenspaces of the action.
- \( F(X_0, X_1, \ldots, X_n) \), the defining equation of \( X \). This also gives a section of \( H^0(\mathbb{P}^n, \mathcal{O}(d)) \).
Example 4.2. Let \( p \geq 7 \) be a prime and \( G = \mathbb{Z}/p\mathbb{Z} \) a cyclic group of order \( p \). Let the action of \( G \) on \( \mathbb{P}^{p-1} \) be given by

\[
G \times \mathbb{P}^{p-1} \to \mathbb{P}^{p-1}
\]

\[
(\zeta, [X_0, \ldots, X_{p-1}]) \mapsto [X_0, \zeta X_1, \ldots, \zeta^{p-1} X_i, \ldots, \zeta^{p-1} X_{p-1}].
\]

Notice that there is a linear change of coordinates of \( \mathbb{P}^{p-1} \) such that the action is just a cyclic permutation of coordinates. And the Fermat hypersurface of degree \( d \leq p - 1 \) (in the new coordinates) is a smooth invariant hypersurface. Therefore a general invariant hypersurface of degree \( d \leq p - 1 \) (in the given coordinates) is smooth. It is easy to see that every such hypersurface contains the same set of fixed points, namely, \( x_1, x_2, \ldots, x_{p-1} \). For simplicity, we consider invariant hypersurfaces of degree \( p - 1 \).

We first show that there are smooth invariant hypersurfaces containing the line \( L_{1,p-1} \) spanned by \( x_1 \) and \( x_{p-1} \). Indeed, if a monomial of the form \( X_i X_{p-1}^{p-1} \) is invariant, then \( i + (p - 1)(p - 1 - i) \equiv 0 \pmod{p} \). So there is only one solution, \( i = \frac{p-1}{2} \). The base locus (with reduced scheme structure) of the sublinear system consisting of invariant polynomials vanishing on \( L_{1,p-1} \) is just the union of \( L_{1,p-1} \) and \( x_j, j = 1, \ldots, p - 1 \). It is easy to check that a general member of this sub-linear system is smooth in the base locus. So there are smooth invariant hypersurfaces containing the line \( L_{1,p-1} \). Then we get equivariant maps connecting \( x_1 \) and \( x_{p-1} \).
The same argument will fail when we try to find equivariant curves connecting $x_1$ and $x_2$. The only invariant monomial of the form $X_1^{p-2}X_i$ is $X_1^{p-2}X_2$. So any invariant hypersurface whose defining equation does not have this monomial is singular at $x_1$. However, we can find out other type of invariant curves. Notice that if we can find a smooth hypersurface whose intersection with the plane $p_{012}$ is a curve defined by $X_0^{p-1} + \lambda X_1^{p-2}X_2 = 0$, then we are done since the normalization of this singular curve is isomorphic to $\mathbb{P}^1$. Again, we consider the sub-linear system which does not contain monomials $X_0^aX_1^bX_2^{d-a-b}$ other than $X_1^{p-2}X_2, X_0^d$. The base locus is just $\{x_j, j = 1, \ldots, p-1\}$. So a general member of this sub-linear system satisfies our requirement.

There are two other ways to see that we can find equivariant curves connecting $x_1$ and $x_2$. First, observe that we can find equivariant curves connecting $x_2, x_{p-1}$ using the same argument as in the $x_1, x_{p-1}$ case. So there are chains of equivariant curves connecting $x_1$ and $x_2$. Then we are done by Lemma 2.3. We can also try to find a smooth hypersurface whose intersection with the plane $p_{1,2,p-4}$ is the union of the line $L_{1,p-4}$ and the curve defined by $X_1^{p-2} + \lambda X_2^{p-3}X_{p-4} = 0$. By considering the appropriate sub-linear system and analyzing the base locus, we see that this is also possible (c.f. the proof of Lemma 4.3).

With the same strategy, one can easily show that any pair of fixed points are connected by equivariant curves.

The difficulty in carrying out the above argument in general is that we know very little about the action. Basically all we know is that there are smooth hypersurfaces with a $G$-action. So it is much more difficult to see if a general member of some linear system is smooth. However, as we will show in the sequel, we can construct equivariant curves with such limited information, using the strategy above.

We first state a criteria that only depends on $V$.

**Lemma 4.3.** There is a hypersurface in $\mathcal{H}$ containing an equivariant curve connecting $x_0$ and $x_1$ if $V$ contains one of the following sets of monomials:

1. $X_0^{d-1}X_2, X_1^{d-1}X_3$,
2. $X_0^{d-1}X_1, X_1^{d-1}X_2, X_3^d$,
3. $X_0^{d-i}X_1, X_0^{d-1}X_2, X_1^{d-1}X_2, X_3^d$,
4. $X_0^{d-1}X_1, X_1^{d-1}X_2, X_2^{d-1}X_3$.

**Proof.** Let $B$ be the base locus of the linear system $\mathbb{P}(V)$ with the reduced scheme structure. It is easy to see that $B$ is the union of linear spaces of the form $X_{i_1} = \ldots = X_{i_k} = 0$.

In the first case, assume the line $L_{01}$ is not in $B$ otherwise we are done. Let $W$ be the subspace of $V$ consisting of the sections vanishing on line $L_{01}$. Denote by $B_W$ the base locus of the linear system $\mathbb{P}(W)$. If there is a point $p$ in $B_W$ but not in $B \cup L_{01}$, then there is a coordinate of $p$ other than $X_0, X_1$ is nonzero. Assume it is $X_2$. Since $W$ contains every monomial in $V$ except those of the form $X_0^aX_1^{d-a}$, monomials $X_0^aX_2^{d-a}$ and $X_1^aX_2^{d-a}$ are not in $V$. So all hypersurfaces in $\mathcal{H} \subset \mathbb{P}(V)$ contain the lines $L_{02}$ and $L_{12}$. Thus $x_0$ and $x_1$ are connected by a chain of lines, hence an equivariant curve.

So from now on we will assume $B_W = B \cup L_{01}$. Since $W$ contains monomials of the form $X_0^{d-1}X_2$ and $X_1^{d-1}X_3$, a general member of $\mathbb{P}(W)$ is smooth along the line $L_{01}$. It suffices to show that for each irreducible component of $B$, there are hypersurfaces in $\mathbb{P}(W)$ smooth on that component. Choose a general smooth
hypersurface $Y$ in $\mathbb{P}(V)$. Write the defining equation of $Y$ as

$$
\sum_{j=1}^{d-1} a_j X_0^j X_1^{d-j} + \sum_{k=0}^{n} X_k G_k(X_0, \ldots, X_n),
$$

where monomials $X_0^j X_1^{d-j}$ are not contained in $\sum_{k=0}^{n} X_k G_k(X_0, \ldots, X_n)$. Let $Y'$ be the hypersurface defined by $\sum_{k=2}^{n} X_k G_k(X_0, \ldots, X_n)$. If $a_1 = a_{d-1} = 0$ or the irreducible component is contained in $X_0 = X_1 = 0$, then it is easy to see that $Y'$ is smooth along that component. Assume at least one of $a_1, a_{d-1}$ is nonzero. Then the only components along which $Y'$ is possibly not smooth are components not contained in $X_0 = X_1 = 0$. But only one of $X_0$ and $X_1$ is nonzero in that component since it is contained in $X_0^n X_1^{d-a} = 0$. Assume it is $X_0$ (the other case is the same). We may also assume $a_{n-1}$ is nonzero otherwise $Y'$ is smooth along that component. This implies that $w(X_1) = w(X_2)$. Observe that the component is contained in $X_1 = X_2 = 0$. Assume it is $X_1 = X_2 = \ldots = X_k = 0$. $Y$ is smooth along this component if and only if

$$
a_{d-1} X_0^{d-1} + G_1 = G_2 = \ldots = G_k = 0
$$

has no common solution along the component. Since $w(X_1) = w(X_2)$ and $Y$ is general, we may assume that $G_2$ contains monomial $b_{d-1} X_0^{d-1}$, $b_{d-1} \neq 0$. So

$$
b_{d-1} G_1 - a_{d-1} G_2 = G_2 = \ldots = G_k = 0
$$

has no common solution in that component. This shows that the hypersurface defined by

$$
X_1(b_{d-1} G_1 - a_{d-1} G_2) + \sum_{k=2}^{n} X_k G_k(X_0, \ldots, X_n),
$$

is smooth along the component. By Bertini’s theorem, a general member of $\mathbb{P}(W)$ is smooth outside of $B_W = B \cup L_{01}$. We have shown that for each irreducible component of $B_W$, there are hypersurfaces in $\mathbb{P}(W)$ smooth along that component. Therefore a general member in $\mathbb{P}(W)$ is smooth.

For the second case, assume $X_0^d X_3$ and $X_1^{d-1} X_3$ are not in $V$ otherwise the statement follows from the first part. Consider sub-linear system that contains no monomials of the form $X_0^n X_1^b X_3^{d-a-b}$ except $X_0^{d-1} X_1 X_3$. If $X_0 X_1^{d-1}$ is not in $V$, it is easy to see a general member in this sub-linear system is smooth. If $X_0 X_1^{d-1}$ is in $V$, we only need to consider the components on which $X_1$ does not vanish. But then it is contained in $X_0 = X_2 = 0$. We also have $w(X_0) = w(X_2)$. Therefore the same argument as in the proof of the first part shows that a general member is smooth.

In the third case, we may again assume $X_0^{d-1} X_3$ and $X_1^{d-1} X_3$ are not in $V$. Now consider the sub-linear system that contains no monomials of the form $X_0^n X_1^b X_3^{d-a-b}$ except $X_0^i X_1^{d-i} X_3^d$. If the monomials $X_0 X_1^{d-1}$ and $X_0^{d-1} X_1$ are not in $V$, then a general member is smooth by the same argument as in the proof of the first part. Then the intersection with the plane $P_{013}$ is a singular invariant rational curve. If one of them is in $V$, we are in the second case.

In the last case, we may assume $X_2^d$ is not in $V$ otherwise $w(X_2) = w(X_3)$. So $X_2^d$ is also in $V$ and we are in the second case. So $x_2$ is in the base locus $B$. Similarly, we may assume $X_2^{d-1} X_1$ is not in $V$ otherwise $w(X_3) = w(X_1)$. Then we have $X_0^{d-1} X_3, X_1^{d-1} X_2 \in V$ and the statement follows from the first part.
Consider the sub-linear system which does not contain monomials \( X_0^a X_1^b X_2^{d-a-b} \) except \( X_0^{d-1} X_1, X_1^{d-1} X_2 \). The base locus contains the base locus \( B \) of \( V \) and the line \( L_{02} \).

If there is a point \( p \) in the base locus which is not in \( B \) or \( L_{02} \), then there is a coordinate, say \( X_4 \), such that there are no monomials of the form \( X_0^a X_4^{d-a}, X_2^a X_4^{d-a}, 0 \leq a \leq d \). Thus the line \( L_{04} \) spanned by \( x_0 \) and \( x_4 \) is in the base locus. So \( x_0 \) and \( x_4 \) are connected by equivariant curves. If there are monomials of the form \( X_4^{d-1} X_i \) for some \( i \neq 1 \), then \( x_1 \) and \( x_4 \) are connected by an equivariant curve by the first part of this lemma. So now assume that the only monomial of the form \( X_4^{d-1} X_i \) is \( X_4^{d-1} X_1 \). Now we consider the sub-linear system which does not contain monomials of the form \( X_1^a X_2^b X_4^{d-a-b} \) except \( X_4^{d-1} X_1, X_1^{d-1} X_2 \). The base locus of this sub-linear system is the same as \( B \). We want to find a smooth hypersurface containing an invariant curve connecting \( x_1 \) and \( x_4 \). The proof is the same as below of the case where the base locus is \( B \cup L_{02} \).

Now assume the base locus of the sub-linear system is \( B \cup L_{02} \) (\( L_{01} \) may be contained in \( B \)). Let \( C \) be an irreducible component of \( B \cup L_{02} \). There are three cases:

1. \( C \cap \{ X_0 X_2 \neq 0 \} \) is non-empty. Assume \( C = \{ X_1 = X_3 = X_4 = \ldots = X_k = 0 \} \) Choose a general hypersurface in \( \mathbb{P}(W) \) whose defining equation is

   \[ G = X_1 G_1 + X_3 G_3 + \ldots + X_k G_k , \]

   where \( G_1 = \lambda X_0^{d-1} + \ldots \) and does not contain monomials of the form \( X_0^a X_2^{d-1-a} \). Since a general hypersurface in \( \mathbb{P}(V) \) is smooth, the equations

   \[ G_1 + \sum \mu_a X_0^a X_2^{d-1-a} = G_3 = \ldots = G_k = 0 \]

   has no solution along \( C \). Therefore the set \( \{ G_3 = \ldots = G_k = 0 \} \) is a finite set in \( C \). Let \( D = \{ G_3 = \ldots = G_k = 0 \} \) \cap \{ \{ X_0 = 0 \} \) and \( \bar{E} \) be the complement of \( D \). Then \( G_1 \mid_{D} = G_1 + \sum \mu_a X_0^a X_2^{d-1-a} \mid_{D} \neq 0 \) (observe that we do not have \( X_2^{d-1} \) here by assumption) and \( G_1 \mid_{E} = \lambda X_0^{d-1} + \ldots \). We may choose a general \( \lambda \) such that \( G_1(p) \neq 0 \) for any \( p \in E \) since \( E \) is a finite set.

2. \( C \cap \{ X_0 \neq 0 \} \) is non-empty and contained in \( \{ X_2 = 0 \} \) or \( C \cap \{ X_2 \neq 0 \} \) is non-empty and contained in \( \{ X_0 = 0 \} \). The proof proceeds in the same way as the first part.

3. \( C \cap \{ X_1 \neq 0 \} \) is non-empty. Then

   \[ C = \{ X_0 = X_2 (= X_3) = X_4 = \ldots = X_k = 0 \}. \]

   Similar to the first part, there are no hypersurfaces smooth along \( C \) only if \( X_1^{d-1} X_0 \) is in \( V \). But in that case, \( w(X_0) = w(X_2) \). So the same argument as in the first part shows that this will not cause any problem.

4. \( C \subset \{ X_0 = X_1 = X_2 = 0 \} \). It is easy to see that there are hypersurfaces smooth along \( C \) since we are only dropping monomials \( X_0^a X_2^{d-a-b} \) and a general hypersurface in \( \mathbb{P}(V) \) is smooth.

So there are smooth hypersurfaces in \( \mathbb{P}(W) \). And the intersection of a general hypersurface with the plane \( P_{012} \) is the union of an invariant line (\( L_{02} = \{ X_1 = 0 \} \)) and an invariant singular rational curve (\( X_0^{d-1} + \nu X_2^{d-2} X_2 = 0 \)). We are done in this case.

□
Remark 4.4. By the above lemma, if there are no equivariant rational curves connecting $x_0$ and $x_1$, then one of the following cases will happen.

1. There is only one $i \neq 0,1$ such that $X^{d-1}_0 X_i$ and $X^{d-1}_1 X_i$ belongs to $V$. In this case we have

   $$(d - 1) w(X_0) + w(X_i) \equiv (d - 1) w(X_1) + w(X_i) \pmod{l}$$

   and

   $$(d - 1)(w(X_0) - w(X_1)) \equiv 0 \pmod{l}$$

2. The only monomial of the form $X^{d-1}_0 X_i$ in $V$ is $X^{d-1}_0 X_1$ and $X^{d-1}_1 X_i$ is in $V$ for some $i \neq 0,1$.

3. The only monomials of the form $X^{d-1}_0 X_i$ and $X^{d-1}_1 X_j$ in $V$ are $X^{d-1}_0 X_1$ and $X^{d-1}_1 X_0$. In this case, we have

   $$(d - 2)(w(X_0) - w(X_1)) \equiv 0 \pmod{l}.$$ Again, by the lemma, if there are monomials of the form $X_i^d$ for some $i \geq 3$ and we are in the first two cases, there are still equivariant rational curves connecting $x_0$ and $x_1$. Therefore if there are no equivariant rational curves, we get very strong constraints on the possible weights of the action.

It does happen that sometimes we cannot find a smooth hypersurface in $\mathcal{H}$ which contains an invariant rational curve as described in the above lemma. However, it seems like this only happens in some special cases. We discuss three examples in the sequel. Fortunately, we are able to construct an equivariant curve explicitly and will use these constructions later when proving Theorem 1.2.

Example 4.5. Let $G = \mathbb{Z}/l\mathbb{Z}$ and act on $\mathbb{P}^l$ by

$$\zeta[X_0, \ldots, X_p] = [X_0, X_1, \zeta X_2, \zeta^2 X_3, \ldots, \zeta^{p-1} X_p]$$

where $\zeta$ is a primitive $l$th root of unity.

Consider a smooth hypersurface $X$ defined by $Q(X_0, X_1) + X_3^3 + X_3 + \ldots + X_n = 0$, where $Q(X, Y)$ is a homogeneous polynomial of degree $l$ in two variables and. It is invariant under the action of $G$ and the only fixed points are the intersection points with the line $L: [s, t, 0, 0, \ldots, 0]$. It is easy to see that we cannot use the specialization argument. So we need to construct the invariant curve directly.

We may choose another homogeneous coordinate such that $F$ does not contain $X_0^l$ and $X_1^l$. Let the linearization of the action of $G$ on $\mathbb{P}^l$ be

$$\zeta[S, T] = [\zeta^a S, \zeta^b T]$$

with $a - b$ invertible in $\mathbb{Z}/l\mathbb{Z}$. Define $f: \mathbb{P}^1 \rightarrow \mathbb{P}^l$ by:

$$[S, T] \mapsto [F_0(S, T), \ldots, F_l(S, T)]$$

where $F_i(S, T)$ are homogeneous polynomials of degree $l$ and satisfies

$$F_0 = S^l, \quad F_1 = T^l, \quad F_i(0, 1) = F_i(1, 0) = 0, i \geq 2$$

$$Q(F_0(S, T), F_1(S, T)) + F_2(S, T) + \ldots + F_l(S, T) = 0$$

If we want the map to be equivariant, then we must have

$$F_i(\zeta^a S, \zeta^b T) = \zeta^{i-1} F_i(S, T), i \geq 2.$$
It is easy to see that under these conditions, each $F_i(S, T), i \geq 2$ is of the form $\mu_j S^l T^{l-j}$ for a unique $0 \leq j \leq l-1$ and every such $j$ only appears once. Notice that $Q(F_0(S, T), F_1(S, T)) = H(S^l, T^1)$, where $H(X, Y)$ is a homogeneous polynomial of degree $l$ and does not have $X^i, Y^l$. Then it is obvious that there is only one solution for the $\mu_j$'s.

**Example 4.6.** Let $k, q$ be integers such that $q \geq 2, d = kq + 2, l = dq$. Let $G$ a cyclic group of order $l$. Let $n$ be $d$ or $d + 1$. Let $X$ be the hypersurface defined by $X_0^{d-1}X_1 + X_1^{d-1}X_0 + X_2^d + \ldots + X_n^d = 0$. Let the weights of the $G$-action on $\mathbb{P}^n$ be such that $w(X_0) = x, w(X_1) = (1 - d)x$ and $w(X_i)$ distinct multiples of $q$ for every $2 \leq i \leq n$. Let $G$ acts on $\mathbb{P}^1$ by

$$\zeta \cdot [S, T] \mapsto [S, \zeta T]$$

where $\zeta$ is a primitive $l$th root of unity.

We may choose a different linearization of $\mathcal{O}_{\mathbb{P}^n}(1)$ such that $x \leq q - 1$ by subtracting a suitable multiple of $q$. Let $m = d - 1 - kq$. It follows from the assumptions that $d - 2 \geq m = 1 + k(q - 1) \geq 2$. If the set $\{q, mq\}$ (resp. $\{2q, (m - 1)q\}, \{3q, (m - 2)q\}$) is in $\{w(X_i), i \geq 2\}$ and consists of 2 elements, then by the same argument in Example 4.5, there is an equivariant map from $\mathbb{P}^1$ to $X$ such that $X_0 = S^{d(q - 1)}, X_1 = T^{d(q - x)}$ (resp. $X_0 = S^{d(2q - x)}, X_1 = T^{d(2q - x)}, X_0 = S^{d(3q - x)}, X_1 = T^{d(3q - x)}$).

If $n = d + 1$, then the weights $\{w(X_i), i \geq 2\}$ are exactly all the weights $\{0, q, 2q, \ldots, (d - 1)q\}$. And we are done. If $n = d$, then the weights $\{w(X_i), i \geq 2\}$ contain all but one of the weights $\{0, q, 2q, \ldots, (d - 1)q\}$. It is easy to see that if $m \neq 3$ then at least one of the above sets consists of 2 distinct elements and is contained in $\{w(X_i), i \geq 2\}$. If $m = 3$, then $d$ is at least 5. We can also consider the set $\{4q, 0\}$ and $\{q, 3q\}$. One of them is contained in $\{w(X_i), i \geq 2\}$. So again we are done.

**Example 4.7.** Let $G$ be a cyclic group of order 4 and act on $\mathbb{P}^3$ by

$$\zeta \cdot [X_0, X_1, X_2, X_3] \mapsto [\zeta X_0, \zeta^2 X_1, X_2, \zeta X_3].$$

Let $X$ be the hypersurface defined by $X_0^6X_1 + X_1^6X_2 + X_2^3X_3 + X_3^7X_1 + X_0X_2X_3 = 0$. It is easy to see that $X$ is smooth. There are three fixed points in $X$, namely, $x_0, x_1, x_3$. The line $L_{03} = \{X_1 = X_2 = 0\}$ is in $X$. Thus $x_0$ and $x_3$ are connected by equivariant rational curves. It suffices to find an equivariant curve connecting $x_0$ and $x_1$. Let $[S, T]$ be the homogeneous coordinate on $\mathbb{P}^1$ and the action on $\mathbb{P}^1$ be given by

$$\zeta \cdot [S, T] \mapsto [S, \zeta T].$$

Consider the following map:

$$f : \mathbb{P}^1 \to \mathbb{P}^3$$

$$[S, T] \mapsto [S^5 + \lambda ST^4, aT^5, bS^2T^3, cS^3T^2].$$

It is easy to see that this is an equivariant map. The map factors through the inclusion $i : X \to \mathbb{P}^3$ if and only if the following condition is satisfied:

$$S^{10}T^5 + 2\lambda S^6T^9 + \lambda^2 S^2T^{13} + \lambda^2 bS^2T^{13} + \lambda^3 S^6T^9 + cS^6T^9 + bcS^{10}T^5 + \lambda bcS^6T^9 = 0.$$
Equivalently,
\[
2\lambda + b^3 + c^2a + \lambda bc = 0 \\
bc + 1 = 0 \\
\lambda^2 + a^2b = 0.
\]
There are solutions to these equations such that \( a \neq 0 \). For example, \( \lambda = a = \frac{1}{2}, b = -1, c = 1 \). So there are equivariant curves connecting \( x_0 \) and \( x_1 \).

We now prove several lemmas.

**Lemma 4.8.** If there is a positive dimensional subvariety \( S \) of \( X \) consisting of fixed points of the same weight, then any two fixed points in \( S \) are connected by an equivariant curve.

**Proof.** The fixed point set \( S \) is the intersection of \( X \) with some linear subspace of \( \mathbb{P}^n \). Thus it is either isomorphic to \( \mathbb{P}^m \) or a positive dimensional hypersurface in \( \mathbb{P}^m \). So \( S \) is connected. Since there are equivariant curves such that both 0 and \( \infty \) are mapped to the same fixed point in \( \mathbb{C} \) (e.g. the constant map), by Lemma 2.3 we get an equivariant very free curve with 0 and \( \infty \) mapping to the same fixed points. Then we can deform this very free curve such that 0 and \( \infty \) are mapped to nearby fixed points in \( S \). Again, specialization to some special points may break the curve into a chain of equivariant curves, but we can smooth them into an irreducible equivariant curve. Thus every pair of fixed points in \( S \) are connected by an equivariant curve.

\( \square \)

We also notice that in some extremal cases, one can always find equivariant curves.

**Lemma 4.9.** If all the fixed points of \( X \) have the same weight, then for every pair of fixed points, there exist equivariant curves connecting them.

**Proof.** The proposition is true if the set of fixed points is at least one dimensional by Lemma 4.8.

Assume the set of fixed points is a discrete set. If it is a single point, then the constant map is equivariant. If it contains at least two elements, we may assume that they are \( x_0 \) and \( x_1 \). Thus \( X_0 \) and \( X_1 \) have the same weight. We may assume the line \( L_{01} \) does not lie in the hypersurface. Then monomials \( X_0^d X_1^{d-a} \) are in \( V \). Also the monomials \( X_i^d, 2 \leq i \leq n \) are in \( V \). So the hypersurface defined by \( X_0^{d-1} X_1 + X_1^{d-1} X_0 + X_2^d + X_3^d + \ldots + X_n^d = 0 \) is in \( H \), contains these two fixed points. By Lemma 4.1, it suffices to find equivariant curves in this hypersurface which connect \( x_0 \) and \( x_1 \). Notice that \( X_1, X_2, \ldots, X_n \) have different weights by assumption. Thus \( l \geq n \). Let \( m \) be the greatest common divisor of \( l \) and \( d \), \( l = mp \), and \( d = mq \). We may choose a linearization of \( \mathcal{O}(1)_{\mathbb{P}^n} \) such that \( w(X_0) = 0 \). Then \( w(X_i) \) is a multiple of \( p \) for every \( i, 0 \leq i \leq n \). Therefore \( p \) is 1 since the action is assumed to be faithful. Thus \( l \leq d \leq n \). So \( d = l = n \). Then this is the hypersurface in Example 4.5. And we are done.

\( \square \)

**Lemma 4.10.** If there are only two fixed points of the action in \( X \), they are connected by equivariant curves.
Proof. Assume that $x_0$ and $x_1$ are the only fixed points in $X$. By Lemma 4.10 and Remark 4.13 it suffices to find equivariant curves on the hypersurface defined by one of the following equations:

1. $X_0^a X_1^{d-a} + X_0^{d-1} X_2 + X_1^{d-1} X_2 + X_2^d + X_3^d + \ldots + X_n^d = 0.$
2. $X_0^d X_1 X_2^{d-1} X_3^d + \ldots + X_n^d = 0.$
3. $X_0^a X_1^{d-1} X_0 + X_1^d + \ldots + X_n^d = 0.$

The statement follows from Lemma 4.3 in the first two cases.

In the last case, we have

$$\begin{align*}
(d - 1)w(X_0) + w(X_1) & \equiv 0 \pmod{l}, \\
(d - 1)w(X_1) + w(X_0) & \equiv 0 \pmod{l}.
\end{align*}$$

Thus

$$\begin{align*}
(d - 1)^2 w(X_0) & \equiv w(X_0) \pmod{l}, \\
(d - 1)^2 w(X_1) & \equiv w(X_1) \pmod{l}, \\
(d - 2)(w(X_0) - w(X_1)) & \equiv 0 \pmod{l}.
\end{align*}$$

All the linear coordinates have different weights otherwise the line spanned by two points with the same weight will intersect $X$ at at least $d \geq 3$ fixed points. Therefore $l \geq n + 1 \geq d + 1$. Let $l = mq, d = mp$, where $m$ is the greatest common divisor of $l$ and $d$. Choose a linearization of $O_{\mathbb{P}^n}(1)$ such that the weight of $X_2$ is zero. Then the weights of $X_2, \ldots, X_n$ are in $\{0, q, \ldots, (m - 1)q\}$. Thus $m \geq n - 1$.

But $m = d$ or $m \leq d/2$. If $m \leq d/2 \leq n/2$, then $n \leq 2$. So this is impossible. Thus $m = d$ and is a divisor of $d$. And $w(X_i)$ is a multiple of $q$ for all $i \geq 2$.

With the above notations, we see that $q \mid (m - 2)w(X_0)$ and $q \mid (m - 2)w(X_1)$. So $m - 2$ is divisible by $q$ otherwise there is a divisor $a \leq 2$ of $q$ such that $a \mid w(X_0)$, $a \mid w(X_1)$ and the action is not faithful. Thus this is just Example 4.6. And we see that there are equivariant curves.

\[\square\]

**Lemma 4.11.** If the fixed points in $X$ have only two different weights, then every pair of fixed points is connected by equivariant curves.

**Proof.** By Lemma 4.10, we may assume $x_0, x_1, x_2$ are contained in $X$ and $w(X_0) = w(X_1) \neq w(X_2)$. By Remark 4.13, we cannot find a smooth hypersurface in $\mathcal{H}(x, y)$ containing the line spanned by $x_0$ and $x_2$ only if one of the following happens.

1. There exist monomials of the form $X_0^a X_2^{d-i}, X_0^{d-1} X_3, X_2^{d-1} X_3$ in $F$ and no other coordinates $X_i, i \neq 2$, have the same weight as $X_3$.
2. There exists $X_0^d - 1 X_2$ in $F$ and no other coordinates have the same weight as $X_2$.

Observe that in any case, monomials of the form $X_0^a X_1^{d-a}$ are not in $V$ otherwise we only have $w(X_0) = w(X_1) = w(X_3)$ or $w(X_0) = w(X_1) = w(X_2)$. So the line $L_{01}$ lies in the every hypersurface.

In the first case, the monomials $X_0^a X_1^{d-1-a} X_2$ are in $V$ otherwise $X$ is singular along $L_{01}$. Then we could have a smooth hypersurface whose defining equation does not contain $X_0^a X_1^{d-a} X_2^{d-a-b}$ except $X_0^a X_1^{d-1-a} X_2$. Now look at the intersection of $X$ with the plane $F_{012}$. It is defined by the equation $X_2 F(X_0, X_1)$, hence a union of invariant curves. And we are done.

In the second case, it is easy to see that $X$ is singular along $L_{01}$.

\[\square\]
Now we are ready to prove Theorem 1.4.

**Proposition 4.12.** Assume that $d - 1$ is invertible in $\mathbb{Z}/l\mathbb{Z}$. Then for every pair of fixed points, there exists an equivariant curve connecting them.

**Proof.** Thanks to Lemma 4.9, we may assume that there are fixed points with different weights in $X$. Then we only need to check that when the three cases listed in Remark 4.4 happen, we can find a smooth hypersurface containing a (chain of) equivariant rational curve(s) connecting the two fixed points. Because of the assumption that $d - 1$ are invertible in $\mathbb{Z}/l\mathbb{Z}$ and $x_0$ and $x_1$ have different weights, the only cases listed in Remark 4.4 that could happen are the second and the third case.

In the second case, the only monomial of the form $X_0^{d-1}X_i$ in $V$ is $X_0^{d-1}X_1$ and $X_1^{d-1}X_i$ is in $V$ for some $i \neq 0, 1$ and

$$(d - 1)w(X_0) + w(X_1) \equiv (d - 1)w(X_1) + w(X_i) \pmod{l}$$

for some $i \neq 0, 1$. Without loss of generality, assume $i$ is 2.

Notice that the monomial $X_2^d$ is not in $V$ otherwise $w(X_1) = w(X_2) = w(X_0)$. So $x_2$ is contained in $X$. And there are monomials of the form $X_2^{d-1}X_i$ for some $i \neq 2$ in $V$.

If $X_2^{d-1}X_i$ is in $V$ for some $i \geq 3$, then the statement follows from Lemma 4.3.

If $X_2^{d-1}X_1$ is in $V$, then $w(X_2) = w(X_0)$. So $X_1^{d-1}X_0$ is in $V$. Then we are in the third case of Remark 4.4. This is discussed below.

Now assume the only monomial of the form $X_2^{d-1}X_i$ is $X_2^{d-1}X_0$. If there is an $X_j$ for some $j \geq 3$ such that $X_j^d$ is in $V$, then the statement follows from Lemma 4.3.

From now on assume that all the $x_j$'s are contained in $X$. We may also assume the fixed points of $X$ have at least 3 different weights by Lemma 4.11. If $X_3$ has a weight different than the weights of $X_0, X_1$ and $X_2$, then $F$ does not contain monomials of the form $X_0^{d-1}X_3$, $X_1^{d-1}X_3$, $X_3^{d-1}X_k$, for $k = 0, 1, 2$. And we are done by Lemma 4.3.

But for $k \geq 3$, if $X_k$'s all have the same weight as $X_2$ then the plane $\{X_0 = X_1 = 0\}$ is in $X$. This is possible only if $X$ is a cubic surface in $\mathbb{P}^3$. But in that case, the defining equation is $X_0F_0(X_2, X_3) + X_1^2F_1(X_2, X_3) + X_0X_1F_2(X_0, X_1, X_2, X_3)$. Then the surface is singular along the line $\{X_0 = X_1 = 0\}$. So there are coordinates $X_k, k \geq 3$ such that $w(X_k) = w(X_0)$ or $w(X_1)$. But this will be a contradiction to the assumptions we made.

If the third case in Remark 4.3 happens, we have $X_0^{d-1}X_1$ and $X_1^{d-1}X_0$ in $V$. We may assume that there is a third fixed point $x_2$ in $X$ with a different weight than $x_0$ and $x_1$ otherwise the statement is true by Lemma 4.11. Then there are no monomials of the form $X_0^{d-1}X_2$, $X_1^{d-1}X_2$, $X_2^{d-1}X_0$, and $X_2^{d-1}X_1$ otherwise $X_2$ has the same weight as $X_0$ or $X_1$. Thus $x_0, x_2$ (resp. $x_1, x_2$) are connected by equivariant curves by Lemma 4.3. So are $x_0$ and $x_1$. \qed

**Proposition 4.13.** If $d$ is congruent to 1 modulo $l$, then there exist equivariant curves through every pair of fixed points.

**Proof.** There are monomials of the form $X_j^{d-1}X_i$ in $V$ if the point $x_i$ is in $X$. We may choose a linearization of $\mathcal{O}_{x_i}(1)$ such that $w(X_j) = 0$. Then the defining equation is invariant. Thus if $w(X_k) \neq 0$, $x_k$ is in the hypersurface $X$. 

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Now we show that there are at least two coordinates whose weights are zero. If not, let \( X_i \) be the only coordinate with weight zero. Since \( d \geq l + 1 \), there exists two coordinates \( X_i \) and \( X_j \) for some \( i \) and \( j \) with the same nonzero weight. So the line spanned by the two points \( x_i \) and \( x_j \) is contained in \( X \). It is easy to see that \( X \) is singular along the line.

Now for the two fixed points \( x_0 \) and \( x_1 \), if none of them has weight 0, then they are connected by equivariant curves by Lemma 4.3

If only one of them has weight 0 and the eigenspace of weight 0 eigenvectors has dimension at least 3, the statement follows from Lemma 4.3. Assume there are only two linear coordinates with weight 0 (assume they are \( X_n \) and \( X_{n-1} \)). Then there are linear coordinates \( X_i \) and \( X_j \) such that \( w(X_i) = w(X_j) \neq 0 \) since \( n \geq d \geq l + 1 \). So monomials \( X_i^p X_j^{d-a} \) are not in \( V \). That is, the line spanned by \( x_i \) and \( x_j \) is in the base locus of \( \mathbb{P}(V) \). Furthermore, every point on this line is a fixed point of the action. Now by the same argument as in Lemma 4.3 there are smooth hypersurfaces in \( H \) whose intersection with the plane \( P_{nij} \) is a degree \( d \) curve defined by the equation \( X_n F_{d-1}(X_i, X_j) \), where \( F_{d-1}(X_i, X_j) \) is a homogeneous polynomial of degree \( d - 1 \) in two variables. Thus the intersection curve is the union of the line \( X_n = 0 \) and a cone over finite number of points. Each irreducible component is an invariant rational curve. Therefore, we can always connect the fixed points of weight 0 with another fixed points with nonzero weight by an equivariant curve. And any pair of fixed points with nonzero weights can be connected by an equivariant curve.

So the proposition is proved. \( \square \)

**Proposition 4.14.** Let \( d = mp + 1 \) and \( l = mq \) be such that \( m, q \geq 2 \) and \( p, q \) are relatively prime. Then for any pair of fixed points, there are equivariant curves connecting them.

**Proof.** In the proof we will sometimes use the notation \( \gcd(a, b) \), which means the greatest common divisor of \( a \) and \( b \). The proof is long and a little technical but the idea is straightforward. One only need to observe that, by Lemma 4.3 and Remark 4.4, if there are no equivariant curves, then the vector space \( V \) has to be rather special and it is possible to get a quite explicit description of it. Once we have the description, we are likely to show the existence of equivariant curves or conclude that this will actually make every hypersurface in \( \mathbb{P}(V) \) singular, hence impossible.

First of all, we may choose a linearization of \( \mathcal{O}_{\mathbb{P}^n}(1) \) such that the total weight of the defining equation is divisible by \( m \) (e.g. choose \( w(X_i) = 0 \) if there is a monomial of the form \( X_j^{d-1}X_i \)). Then any fixed point in \( \mathbb{P}^n \) whose weight is not divisible by \( m \) is in \( X \). So there are fixed points in \( X \) whose weight is not divisible by \( m \). Then it suffices to show that given any such fixed point, and any other fixed point, there are equivariant curves connecting them.

Let \( x_0 \) be a fixed point whose weight is not divisible by \( m \) and \( x_1 \) another fixed point. If they are not connected by an equivariant curve, one of the following happens:

1. There is only one \( i \neq 0, 1 \) such that \( X_0^a X_1^{d-a} \) (for some \( 1 \leq a \leq d - 1 \)), \( X_0^d X_i \) and \( X_1^{d-1}X_i \) belong to \( V \). Assume \( i = 2 \).
2. The only monomial of the form \( X_0^{d-1}X_i \) in \( V \) is \( X_0^{d-1}X_1 \) and \( X_1^{d-1}X_i \) is in \( V \) for some \( i \neq 0, 1 \). Assume \( i = 2 \). And there are \( X_2^{d-1}X_i, i \geq 3 \) in \( V \).
3. The same as case (2) except that there are \( X_2^d \) in \( V \).
(4) The same as case (2) except that there are $X_i^{d-1} X_1$ in $V$.

Notice that we cannot have $X_2^{d-1} X_0$ in $V$ since the total weight is divisible by $m$ and $w(X_0)$ is not by assumption.

In all the above cases, $w(X_2)$ is divisible by $m$. So we may choose a linearization such that $w(X_2) = 0$ and the total weight is still divisible by $m$.

If the first case happens,

$$aw(X_0) + (d - a)w(X_1) \equiv (d - 1)w(X_1) + w(X_2) \pmod{mq}$$

and

$$w(X_1) + a(w(X_0) - w(X_1)) \equiv 0 \pmod{mq}.$$ 

But

$$(d - 1)w(X_0) + w(X_2) \equiv (d - 1)w(X_1) + w(X_2) \pmod{mq},$$

so

$$q \mid w(X_0) - w(X_1).$$

These imply that $q \mid w(X_1)$. Therefore the total weight of the defining equation is 0. So $X_j^d$ is in $V$.

If there are monomials of the form $X_i^d$ of $X_i^{d-1} X_j$ for some $i, j \geq 3$, then there are chains of equivariant curves connecting $x_0$ and $x_1$. So we are done. Otherwise all the points $x_i, i = 0, \ldots, n$ are in $X$ and we only have $X_i^{d-1} X_1$ or $X_i^{d-1} X_2$ for every $3 \leq i \leq n$.

Notice that we cannot have $X_i^{d-1} X_0$ in $V$ since the total weight is divisible by $m$ and $w(X_0)$ is not. If we have $X_i^{d-1} X_2$ for all $i \geq 2$, then $q \mid w(X_i)$ and the action is not faithful. This also shows that $w(X_1) \neq w(X_2)$. So for each $i = 0, 1, \ldots, n$, one and only one of $X_i^{d-1} X_1$ and $X_i^{d-1} X_2$ is in $V$. Without loss of generality, assume that we have $X_0^{d-1} X_2, X_1^{d-1} X_2, X_3^{d-1} X_2, \ldots, X_k^{d-1} X_2$ and $X_{k+1}^{d-1} X_1, \ldots, X_n^{d-1} X_1$.

Now notice that $x_i$ and $x_j$ are connected by equivariant curves for $i = 0, 3, 4, \ldots, k$ and $j = k + 1, \ldots, n$. Thus for every pair of fixed points $x$ and $y$, if neither of them is $x_1$, then they are connected by equivariant curves. If there are no monomials of the form $X_i^{d-1} X_1$, $1 \leq i \leq k$ in $V$, then the line spanned by $x_1$ and $x_i$ is in the base locus. Therefore $x_1$ and $x_0$ are connected by equivariant curves. So assume there are monomials of the form $X_i^{d-1} X_1$, $1 \leq i \leq k$ in $V$. If there are no monomials of the form $X_j^{d-1} X_1$, for some $1 \leq i \leq k, k + 1 \leq j \leq n$, then there are smooth hypersurfaces whose defining equation does not contain monomials of the form $X_i^d X_j^d$, except $X_i^{d-1} X_1$, $X_j^{d-1} X_2$, $a \leq d - 2$. Thus the intersection of this hypersurface with the plane spanned by $x_1, x_i, x_j$ is a line and an irreducible singular rational curve $(X_j^{d-1} + \lambda X_i^{d-1} X_1^d - a = 0)$.

Now assume there are monomials $X_i^d X_j^{d-a}$. Under this assumption, one easily checks that the base locus of the sub-linear system, which contains no monomials of the form $X_0^c X_1^b X_{k+1}^{d-a-b}$ except $X_0^c X_1^{d-c}$ for a fixed $c \geq 2$ and $X_{k+1}^{d-1} X_1$, is the union of the base locus of $V$ and the line $L_{0k+1}$. And a general member is smooth along the base locus, hence smooth. Then the intersection with the plane $P_{01k+1}$ is a line and an irreducible singular rational curve $(X_{k+1}^{d-1} + \lambda X_0^d X_1^{d-1-a} = 0)$.

So the statement is true in the first case.

The second case follows from Lemma 4.3.

In the remaining cases, we may assume that there are no monomials of the form $X_i^d, X_i^{d-1} X_j$, for some $i, j \geq 3$, otherwise the statement follows from Lemma 4.3.
In the third case, for any \( i \geq 3 \), there are no monomials of the form \( X_i^{d-1}X_j \), except \( j = 1 \) or \( 2 \), in \( V \). It is easy to see that if \( X_i^{d-1}X_1 \) (resp. \( X_i^{d-1}X_2 \)) is in \( V \), then \( q \mid w(X_i) - w(X_0) \) (resp. \( q \mid w(X_i) - w(X_0) \)). Notice the total weight is
\[
w(X_0^d) \equiv 0 \pmod{mq}.
\]
So if \( X_i^{d-1}X_2 \) is in \( V \), we have
\[
w(X_i) \equiv q \pmod{mq}.
\]
In particular, \( w(X_1) \) is divisible by \( q \).
Also notice that
\[
w(X_0^{d-1}X_1) \equiv 0 \pmod{mq}.
\]
So \( q \mid mw(X_0) \). Now \( w(X_i) \) is divisible by \( \gcd(q, w(X_0)) \) for all \( i \). So \( \gcd(q, w(X_0)) = 1 \) and \( q \mid m \).
If \( w(X_1) = 0 \), then
\[
w(X_0^{d-1}X_1) \equiv (d - 1)w(X_0) \pmod{mq}.
\]
For similar reasons, we see that the only monomials of the form \( X_i^aX_j^{d-1-a}X_k \) are \( X_i^aX_j^{d-1-a}X_2 \). Then every member in the linear system is singular along the line spanned by \( L_{ij} \). So this is impossible.
If we have \( X_i^{d-1}X_1 \) for all \( i \geq 3 \). Then \( \gcd(q, w(X_i)) = 1, i \neq 1, 2 \). So \( X_i^{a_1}X_j^{a_2} \cdots X_{i_c}^{a_c} \) is not in \( V \) for \( i_j \neq 1, 2, 1 \leq j \leq i_c, \sum a_j = d \), otherwise
\[
\sum a_j w(X_j) = dw(X_{i_1}) + \sum_{j \geq 2} a_j (w(X_{i_j}) - w(X_1)),
\]
and \( q \mid m, q \mid (w(X_i) - w(X_j)), i, j \geq 3 \) and \( d = mp + 1 \). So \( q \mid w(X_{i_k}) \). This is a contradiction. If there are monomials of the form \( X_i^aX_j^{d-1-a}X_k \), then \( w(X_k) \) is divisible by \( q \). So \( k = 1 \) or \( 2 \). But \( k \) cannot be \( 1 \) otherwise every hypersurface in the linear system is singular at some points in the line spanned by \( x_i \) and \( x_j \). So \( k = 2 \).
Now notice that the \( (n - 2) \)-plane \( \{X_1 = X_2 = 0\} \) is contained in \( X \). Since \( X \) is smooth, this is only possible when \( n = 3 \). In this case, it is easy to see that the one smooth member of the linear system is given by
\[
X_0^2X_1 + X_1^2X_2 + X_2^2X_3 + X_3^2X_1 + X_0X_2X_3 = 0.
\]
Also \( d = 3, p = 1, m = 2, q = 2, \) and \( l = 4 \). And the action is
\[
\zeta \cdot [X_0, X_1, X_2, X_3] \mapsto [\zeta^3X_0, \zeta^2X_1, X_2, \zeta X_3],
\]
where \( \zeta \) is a primitive 4th root of unity. This is just Example 4.7 And we know there are equivariant curves.
Finally, if for each \( i \neq 2 \), we have one and only one of \( X_i^{d-1}X_1 \) and \( X_i^{d-1}X_2 \) and they both appear in \( V \) for some \( i \). Notice that after a reordering of index, this is just the last part of the first case discussed above. So we are done.

In the fourth case, we have

\[ w(X_0^{d-1}X_1) \equiv w(X_2^{d-1}X_1) \pmod{mq}. \]

So

\[ q \mid w(X_0) = w(X_0) - w(X_2). \]

Similarly,

\[ w(X_1^{d-1}X_2) \equiv w(X_2^{d-1}X_1) \pmod{mq}. \]

\[(d - 2)w(X_1) \equiv (mp - 1)w(X_1) \equiv 0 \pmod{mq} \]

So

\[ m \mid w(X_1). \]

By assumption, for any \( i \geq 3 \), there are no monomials of the form \( X_i^{d-1}X_j \) except \( j = 1 \) or \( 2 \). It is easy to see that \( w(X_1) \neq w(X_2) \) otherwise \( q \mid w(X_i), 0 \leq i \leq n \) by the above argument and the action is not faithful. Also \((m, q) \mid w(X_i), 1 \leq i \leq n \). So \((m, q) = 1\).

A general hypersurface is smooth along the line \( L_{02} \), but we only have monomials \( X_0^{d-1}X_1, X_2^{d-1}X_1 \). So at least one of the monomials \( X_0^aX_2^{d-a}, X_0^{d-a}X_2^{a-1}X_3 \) is in \( V \). But if \( X_0^aX_2^{d-a} \) is in \( V \), then

\[ w(X_0^aX_2^{d-a}) \equiv w(X_0^{d-1}X_1) \pmod{mq}. \]

So

\[ aw(X_0) \equiv w(X_1) \pmod{mq}. \]

We know that \( q \mid w(X_0) \). Thus \( q \mid w(X_1) \). Furthermore, \( q \mid w(X_i), i = 0, 1, \ldots, n \). This is a impossible since the action is assumed to be faithful.

So \( X_0^aX_2^{d-a} \) is not in \( V \). Then the line \( L_{02} \) is contained in every hypersurface in \( \mathbb{P}(V) \). And \( X_0^{d-a}X_1^{a-1}X_3 \) is in \( V \). Notice that \( w(X_3) \) is not divisible by \( q \) otherwise

\[ w(X_1) \equiv w(X_2^{d-1}X_1) \equiv w(X_0^{d-a}X_1^{a-1}X_3) \pmod{mq} \]

to find if \( q \mid w(X_1) \) is divisible by \( q \). So is \( w(X_1), 0 \leq i \leq n \).

Thus there are no monomials of the form \( X_3^{d-1}X_i, i = 0, 1, 3, \ldots, n \). So \( X_3^{d-1}X_2 \) is in \( V \) since a general hypersurface contains \( x_3 \) and is smooth. Then monomials \( X_1^aX_3^{d-a} \) are not in \( V \) otherwise

\[ w(X_3^{d-1}X_2) \equiv w(X_1^aX_3^{d-a}) \pmod{mq}. \]

That is,

\[ w(X_3) + a(w(X_1) - w(X_3)) \equiv 0 \pmod{mq}. \]

Notice that \( q \mid w(X_1) - w(X_3) \) since

\[ w(X_3^{d-1}X_2) \equiv w(X_3^{d-1}X_2) \pmod{mq}. \]

So \( q \mid w(X_3) \). A contradiction.

To summarize, monomials \( X_1^aX_3^{d-a} \) are not in \( V \). And \( X_0^{d-1}X_1, X_3^{d-1}X_2 \) are both in \( V \). Thus we see that both pair \( x_1, x_3 \) and \( x_0, x_3 \) are connected by equivariant curves. So are \( x_0 \) and \( x_1 \).

\( \square \)
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