BRST ANALYSIS OF GAUGE THEORIES BASED ON
NONLINEAR ALGEBRAS IN 2d

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Abstract

Covariant quantization of theories based on nonlinear extensions of Lie algebras in 2d is studied by using a generalized Lagrangian BRST formalism. The quantum action is constructed to be invariant under the off–shell nilpotent BRST transformations by using a set of independent antifields as auxiliary, nonpropagating variables in the quantum theory. The general results are applied to the quantization of nonlinear gauge theory based on quadratic Poincaré algebra, which is closely related to 2d gravity with dynamical torsion.

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1. Introduction

General relativity is a successful theory of macroscopic gravitational phenomena, but all attempts to quantize the theory encounter serious difficulties. It seems natural to try to understand the structure of gravity on the basis of the concept of gauge symmetry, which has been very successful in describing other fundamental interactions in nature. The usual gauge theory is based on a symmetry group with local properties determined by a Lie algebra. Ikeda and Izawa [1,2] considered an interesting approach to the construction of gauge theories in two dimensions, based on a non-linear extension of the Lie algebra structure. It turns out that the resulting non-linear gauge theory is closely related to two-dimensional gravity. In case when the non-linear algebra is a quadratic extension of the Poincaré algebra, they obtained a gauge theory which is, in a certain region, equivalent to two-dimensional gravity with dynamical torsion [1,3], or to dilaton gravity [2,3]. In a similar way, the quadratic $W_3$ algebra is shown to lead to the $W_3$ theory of gravity [4,5].

One expects that investigations of two-dimensional theories of gravity may provide a better understanding of quantum properties of higher dimensional gravity, as well as a deeper understanding of string theory. For this reason non-linear gauge theories may be of importance in considerations related to quantum gravity.

In this paper we shall study the BRST quantization of theories based on non-linear extensions of Lie algebras in 2d. Ikeda and Izawa described the construction of the action for non-linear gauge theory, whose classical gauge symmetries are characterized by a gauge algebra which is closed only on shell [1-3]. The gauge theories of this type can be covariantly quantized by using the general Lagrangian method developed by Batalin and Vilkovisky (BV) [6]. The method is based on a generalization of the BRST approach, but the effective BRST transformations obtained in the process of gauge fixing are nilpotent only on shell. The BRST analysis of gauge theories based on the non-linear Poincaré algebra and the $W_3$ algebra have been done in references [1] and [5], respectively. In both cases the gauge–fixed theory is characterized by an on–shell nilpotent BRST symmetry. On the other hand, the generalization of the BV method, which yields an off shell nilpotent BRST symmetry of the gauge–fixed action, has been proposed in Ref. [7]. It has been successfully used to quantize Witten’s interacting bosonic string field theory, the superparticle in $d = 10$ and $d = 9$, the heterotic superstring and the simple supergravity [8]. In this approach, the gauge–fixed theory is realized on a set of fields containing a convenient set of antifields. The elimination of the antifields leads to the BV form of the theory, showing clearly that the essential role of the antifields is to ensure the off–shell nilpotency of the BRST transformations in the gauge–fixed action.
We shall begin our exposition in sect. 2 by considering the form of the classical gauge structure of the general nonlinear gauge theory [1-3]. The basic features of the Lagrangian gauge algebra are characterized by a set of structure functions, determined by the Poisson bracket algebra and the Jacobi identities of the gauge generators. In sect. 3 we find the form of the BRST generator as the solution of the master equation, by making use of the classical structure functions. Then, we find a BRST invariant extension of the classical action and define the quantum theory by an appropriate gauge-fixing procedure. The quantum action is constructed to be invariant under the off-shell nilpotent BRST transformations, which is achieved by introducing a set of antifields as auxiliary, nonpropagating variables in the quantum theory. The relation to the BV approach is clarified by observing that the elimination of the antifields yields an effective theory which coincides with the BV form. In sect. 4 we apply the general results of the previous section to quantize the gauge theory based on the quadratic extension of the Poincaré algebra. Section 5 is devoted to conclusions.

2. Classical theory and gauge symmetry

Let us consider a nonlinear extension of a Lie algebra with a basis $T_a$, the bracket product of which is given by the relation

$$[T_a, T_b] = W_{ab}(T),$$

(1a)

where $W_{ab}(T) = -W_{ba}(T)$ is an antisymmetric polynomial in $T_c$,

$$W_{ab}(T) = k_{ab} + f_{abc} T_c + V_{abcd} T_c T_d + \cdots,$$

and the coefficients of $k$, $f$, $V$, and so on, are the structure constants of the algebra. One also assumes that the bracket product is a derivation, eg. $[A, BC] = [A, B]C + B[A, C]$. The zeroth order term $k_{ab}$ is the central element of the algebra, the first order term $f_{abc}$ is the usual structure constant of the corresponding Lie algebra, and all other terms characterize nonlinear structure of the algebra. The antisymmetry of $W_{ab}(T)$ implies the following symmetry properties of the structure constants:

$$k_{ab} = -k_{ba}, \quad f_{abc} = -f_{bac}, \quad V_{abcd} = V_{cdab} = -V_{dabc}, \quad \cdots$$

The algebra (1) is clearly not a Lie algebra, but is often referred to as “nonlinear Lie algebra”.

The Jacobi identity for the bracket product (1a) implies

$$\frac{\partial W_{ab}}{\partial T_d} W_{cd} + (abc) = 0,$$

(1b)
where \((abc)\) denotes cyclic permutation of \(a\), \(b\) and \(c\). This is a generalization of the Jacobi identity for the usual Lie algebras. It implies a set of identities for the structure constants; in particular, \(f_{ab}^d f_{cd}^e + (abc) = 0\).

The problem of constructing a gauge theory based on the nonlinear extension of a Lie algebra has been studied by Ikeda and Izawa [1]. They introduced a set of gauge potentials \(A^a \mu\) and an additional set of scalar fields \(\phi_a\), with gauge transformations defined by

\[
\delta_0 A^a \mu = \partial_\mu \xi^a + W^a_{bc} A^b \mu \xi^c , \\
\delta_0 \phi_a = - W_{ab} \phi^b .
\]

Here, \(W_{ab}\) is a function of \(\phi_a\), \(W_{ab} = W_{ab}(\phi)\), and we introduce the notation

\[
W^c_{ab} \equiv \frac{\partial W_{ab}}{\partial \phi_c} , \quad W^{cd}_{ab} \equiv \frac{\partial^2 W_{ab}}{\partial \phi_c \partial \phi_d} .
\]

These transformations satisfy the following gauge algebra:

\[
[\delta_0(\xi_1), \delta_0(\xi_2)] A^a \mu = \delta(\xi_3) A^a \mu - \xi^c_1 \xi^d_2 W_{cd}^{ab} D_\mu \phi^b , \\
[\delta_0(\xi_1), \delta_0(\xi_2)] \phi_a = \delta(\xi_3) \phi_a ,
\]

where \(\xi^a_3 = W^a_{bc} \xi^b_1 \xi^c_2\), and \(D_\mu \phi_b = \partial_\mu \phi_b + W_{bc} A^c \mu\) is the covariant derivative of \(\phi_b\). If the above gauge transformations are to represent some Lagrangian symmetry, they should be closed at least on shell, i.e. the relation \(D_\mu \phi_b = 0\) should be an equation of motion.

The algebra (3) of the gauge transformations is based on the nonlinear algebra (1) in the sense that the functions \(W_{ab}(\phi)\) in (3) are antisymmetric and satisfy the identities \((\partial W_{ab}/\partial \phi_d) W_{cd} + (abc) = 0\), in complete agreement with the corresponding Jacobi identities for \(W_{ab}(T)\).

By considering the commutator of two covariant derivatives on \(\varphi_b\) one defines the curvature \(R^a \mu \nu \equiv \partial_\mu A^a \nu - \partial_\nu A^a \mu + W^a_{bc} A^b \mu A^c \nu\). Since the curvature does not transforms homogeneously, there is no standard prescription for constructing an action invariant under these gauge transformations. Ikeda and Izawa [1-3] found that the classical action \(I_0 = \int d^2 x \mathcal{L}\) is determined by

\[
\mathcal{L} = -\frac{1}{2} \epsilon^{\mu \nu} \left[ (R^a \mu \nu \phi_a + (W_{ab} - W_{ab} c \phi_c) A^a \mu A^b \nu ) \right] .
\]

The gauge invariance follows from \(\delta \mathcal{L} = -\partial_\mu [\epsilon^{\mu \nu} (W_{ab} - W_{ab} c \phi_c) A^a \nu \phi^b].\) Moreover, the equations of motion for \(A^a \mu\) and \(\phi_a\) are given by

\[
F^\mu_a \equiv - \epsilon^{\mu \nu} D_\nu \phi_a = 0 , \\
F^a \equiv -\frac{1}{2} \epsilon^{\mu \nu} R^a \mu \nu = 0 ,
\]
and, as a consequence, the gauge algebra (3) is closed on shell.

Let us now introduce a convenient notation

\[ \varphi^i = (A^a_{\mu}, \phi_b), \quad \xi^\alpha = (\xi^a), \]

and observe that the gauge transformations (2) can be rewritten in the form

\[ \delta_0 \varphi^i = \xi^\alpha T^i_{\alpha}(\varphi). \]

Gauge invariance of the classical action implies Noether identities \( T^i_{\alpha} F_i = 0 \), where \( F_i \) are the classical field equations, \( F_i \equiv \delta I_0/\delta \varphi^i = (F_a^{\mu}, F^b). \)

The structure of a classical theory with local symmetries is greatly clarified by the properties of its gauge algebra. Although the most natural framework for studying classical gauge algebra is the Hamiltonian approach, the related informations can also be obtained within the Lagrangian formalism. Since the change of a functional \( I[\varphi] \) under the gauge transformations (6) has the form \( \delta_0 I[\varphi] = \xi^\alpha T^i_{\alpha} \delta_i I[\varphi] \), we can introduce the Lagrangian generators of \( \xi^\alpha \) transformations by the relation \( \delta_0 \equiv \xi^\alpha \Gamma_{\alpha} \):

\[ \Gamma_{\alpha} = T^i_{\alpha} \delta_i. \]

The algebra of the generators is, in general, not closed:

\[ [\Gamma_{\alpha}, \Gamma_{\beta}] = \bar{f}_{\alpha\beta} \Gamma_{\gamma} + E_{\alpha\beta}^{ij} F_i \delta_j. \]

The explicit content of this relation in our case can be found by rewriting Eq.(3) in the form

\[ [\delta_0(\xi_1), \delta_0(\xi_2)] = \delta(\xi_3) + \xi^c_1 \xi^d_2 W^{ab}_{cd} \varepsilon_{\mu\nu} F^{\nu}_{b} \frac{\delta}{\delta A^{a}_{\mu}}, \]

whereupon one easily finds

\[ [\Gamma_{\alpha}, \Gamma_{\beta}] = W_{\alpha\beta}^{\gamma} \Gamma_{\gamma} + W^{cd}_{\alpha\beta} \varepsilon_{\mu\nu} F^{\nu}_{d} \frac{\delta}{\delta A^{c}_{\mu}}. \]

We see that the coefficients \( \bar{f} \) and \( E \) of the gauge algebra (8a) are not constants but depend on the fields \( \phi_a \).

The commutators (8) satisfy certain consistency requirements following from the related Jacobi identities. These requirements in general lead to a natural introduction of new structure functions [8]. However, explicit calculation shows that these structure functions vanish in this case.

Thus, the set of structure functions \((T, \bar{f}, E)\) represents a complete description of the classical gauge structure of the theory (4), based on the nonlinear algebra (1).
3. Generalized BRST quantization

We shall now use a generalized BRST method \[7,8\] to quantize the theory (4), whose gauge symmetry is determined by Eqs.(2) and (8b). It represents a generalization of the BV quantization procedure \[6\], and provides an off-shell nilpotent BRST symmetry of the gauge-fixed action.

**BRST transformations.** The first step in this approach is the construction of the BRST transformations. Let us start by introducing for each gauge parameter \( \xi^\alpha \) a ghost \( c^\alpha \); then, to each field \( \Phi^A = (\varphi^i, c^\alpha) \) we associate the antifield \( \Pi^A = (\varphi^{\ast i}, c^{\ast \alpha}) \). The Grassmann parities \( (\epsilon) \) and the ghost numbers of these variables are given in Table 1, where \( \epsilon_i = \epsilon(\varphi^i) \) and \( \epsilon_\alpha = \epsilon(\xi^\alpha) \).

Following Batalin and Vilkovisky \[6\] we define the BRST transformation \( s \) as

\[
sX = (S, X),
\]

where \((X, Y)\) is the antibracket of \( X \) and \( Y \),

\[
(X, Y) = \frac{\partial_R X \partial_L Y}{\partial \Phi^A \partial \Pi_A} - \frac{\partial_R X \partial_L Y}{\partial \Pi_A \partial \Phi^A},
\]

and the BRST generator \( S = S(\Phi^A, \Pi_A) \) is the solution of the master equation \((S, S) = 0\). The master equation is usually solved by expanding \( S \) in a number of antifields, \( S = S_0 + S_1 + S_2 + \cdots \), and using \( S = S_0 - c^\alpha T^i_{\alpha} \Pi_i + \cdots \) as the boundary condition, with \( S_0 = I_0 \). The off-shell nilpotency of the BRST transformations follows from \((S, S) = 0\).

Following the ideas of the Hamiltonian BRST formalism, we shall try to find the BRST generator by adding to \( S_0 \) all possible terms containing classical structure functions combined with \( \Phi^A \) and \( \Pi_A \) so that \( gh(S) = 0 \), while the coefficients of the various terms are determined by the master equation. Limiting our discussion to dynamical systems characterized by the structure functions \((T, \bar{f}, E)\) we obtain the result \[8\]

\[
S = S_0 + S_1 + S_2,
\]

\[
S_1 = -c^\alpha T^i_{\alpha} \Pi_i + \frac{1}{2}(-)^{\beta + 1} c^\beta c^\alpha f^\gamma_{\alpha\beta} \Pi_\gamma,
\]

\[
S_2 = \frac{1}{4}(-)^{i + \beta + 1} c^\beta c^\alpha F^{ij}_{\alpha\beta} \Pi_j \Pi_i.
\]

This compact form of \( S \) shows very clearly the connection of the BRST structure to the classical gauge algebra.

After introducing the component notation,

\[
c^\alpha = (c^a), \quad \Pi_A = (A^*_a, \phi^{*a}, c^{*}_a),
\]

\[6\]
a direct calculation based on Eq.(11) leads to the following component expression for $S$:

\[
S_1 = -(D_\mu c^a) A^*_a \mu + W_{ab} c^b \phi^* a - \frac{1}{2} W_{ab} c^b c^a c^c,
\]

\[
S_2 = -\frac{1}{4} W_{cd} \varepsilon_{\mu \nu} c^d c^c A^*_b A^*_a \mu \nu ,
\]  

(11)

where $D_\mu c^a \equiv \partial_\mu c^a - W^a_{eb} A_b^{\mu} c^c$. By using the above result for the BRST generator one can easily find the BRST transformations of all the fields,

\[
sA^a_\mu = D_\mu c^a + \frac{1}{2} W_{cd} \varepsilon_{\mu \nu} c^d c^c A^*_b A^*_a \nu ,
\]

\[
s\phi_a = -W_{ab} c^b,
\]

\[
s c^a = \frac{1}{2} W_{bc} c^c c^b .
\]  

(12a)

The transformations of the antifields are

\[
sA^*_a \mu = F^a_\mu - W^c_{ab} c^b A^*_c A^*_a \mu ,
\]  

(12b)

and similarly for $\phi^* a$ and $c^*_a$.

**Quantum theory.** The classical action (4) is not BRST invariant, as the gauge algebra is open. Our next step in the quantization procedure will be to find a BRST–invariant extension of $I_0$. The BRST invariant action is not unique: the BV choice is $I_{BRST} = S$, while we choose

\[
I_{BRST} = I_0 - S_2 ,
\]  

(13)

which is more convenient at the level of gauge-fixed theory, as we shall see.

It is important to note that the action (13) is degenerate. A complete understanding of this degeneracy is of central importance for the construction of the quantum, gauge-fixed theory. Since in our approach the antifields $\Pi$ will not be eliminated from the quantum action as in the BV formalism, we first note that $S_2$ is degenerate with respect to $\Pi$–variables. In fact, we see from Eq.(11) that $S_2$ is a function of only $A^*_a \mu$. The degeneracy of $S_2$ with respect to the sector $(\phi^* a, c^*_a)$ can be removed by fixing these components to zero.

The transition to the restricted set $(A^*_a \mu)$ of antifields resolves the problem of $\Pi$–degeneracy. One should also show that this restriction is consistent, i.e. that the new set of variables $(\Phi^A, A^*_a \mu)$ carries the representation of the off-shell nilpotent BRST transformations. This consistency follows from Eqs. (12a) and (12b): the restricted set of variables is seen to be closed under the off-shell nilpotent BRST transformations from the very beginning, as the transformations of this set are decoupled from the transformations of the removed set $(\phi^* a, c^*_a)$. 

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After removing the Π–degeneracy, there remains the usual degeneracy related to the classical gauge symmetry. The quantum action can be defined by introducing the gauge breaking term:

\[ I_q = I_{BRST} - s\Psi, \]  

(14a)

where Ψ is the so-called gauge fermion, \( gh(\Psi) = -1, \varepsilon(\Psi) = 1 \) [6], which satisfies certain regularity conditions. To construct Ψ we introduce additional fields: antighosts \( \bar{c}_a \) and multipliers \( B_a \). Their Grassmann parities and ghost numbers are defined as \( \varepsilon(\bar{c}_a) = 1, gh(\bar{c}_a) = -1, \varepsilon(B_a) = 0, gh(B_a) = 0 \), and their BRST transformations are

\[ s\bar{c}_a = B_a, \quad sB_a = 0. \]

We can choose Ψ to be a function on the restricted set \( (\Phi^A, A^*_\mu, \bar{c}_a, B_a) \). The simple choice \( \Psi = \Psi(\Phi^A, \bar{c}_a) \) leads to

\[ s\Psi = (s\Phi^A) \frac{\partial L\Psi}{\partial \Phi^A} + B_a \frac{\partial L\Psi}{\partial \bar{c}_a} \equiv I_{FP} + I_{GF}. \]  

(14b)

The complete quantum theory is now determined by the generating functional

\[ Z_\Psi = \int D\mu \exp[i(I_{BRST} - s\Psi)], \]  

\[ D\mu = D\Phi^A D\Phi^*_\mu D\bar{c}_a DB_a. \]  

(15)

Here, the measure is defined only over the independent variables, and \( Z_\Psi \) is nondegenerate by construction. Note that the quantum action \( I_q \) is invariant under the off-shell nilpotent BRST transformations (this implies Ψ–independence of \( Z_\Psi \) and, consequently, gauge invariance of the \( S \)–matrix). The condition \( I_q(\Pi = 0) = I_0 \) ensures the correct classical limit of the theory.

The simplest choice for Ψ is given by a bilinear combination of antighosts times gauge conditions (linear gauges):

\[ \Psi = \bar{c}_a M^a_i \varphi^i, \]

where \( \chi^a = M^a_i \varphi^i \) are gauge conditions defined on the classical fields \( \varphi^i \), and \( M \) is a field–independent matrix. The structure of \( M \) is determined by the regularity conditions imposed on \( \Psi \).

**Comparisson to the BV formalism.** It is instructive to compare our approach to the BV method [6]. The basic difference lies in the treatment of antifields.

In the BV approach the quantum action is obtained by the replacing \( \Pi_A \to \partial\Psi/\partial\Phi^A \) in \( S \):

\[ I_{BV} = S' - B_a \frac{\partial L\Psi}{\partial \bar{c}_a}, \quad S' \equiv S(\Phi^A, \Pi_A = \partial L\Psi/\partial\Phi^A). \]

8
We shall show that the integration over the antifields $A_{a}^{*\mu}$ in Eq.(15) yields an effective action that coincides with the BV result. To see that we introduce the notation $S_1 = \Lambda A^{* \mu} \Pi_A$, $S_2 = \Lambda^{AB} \Pi_B \Pi_A$, and observe that the quantum action (14) can be written in the form

$$I_q = I_0 + \Lambda A^{* \mu} \frac{\partial L}{\partial \Phi^A} + \Lambda^{AB} \frac{\partial L}{\partial \Phi^B} \frac{\partial L}{\partial \Phi^A} - B_a \frac{\partial \Psi}{\partial \bar{c}_a} - \Delta = I_{BV} - \Delta,$$

where

$$\Delta \equiv \Lambda^{AB} \left( \frac{\partial L}{\partial \Phi^B} (\Pi_B - \frac{\partial L}{\partial \Phi^B}) (\Pi_A - \frac{\partial L}{\partial \Phi^A}) \right) = -\frac{1}{4} W_{\mu\nu}^{cd} \epsilon^{b \mu \nu} c^b c^a \left( A_{d}^{* \nu} - \frac{\partial L}{\partial A^d_{\nu}} \right) \left( A_{c}^{* \mu} - \frac{\partial L}{\partial A^c_{\mu}} \right).$$

It is now evident that the integration over $A_{a}^{*\mu}$ in $Z_{\Psi}$ eliminates the term $\Delta$, and the resulting effective quantum action coincides with the BV expression, $I'_q = I_{BV}$.

The BRST transformations obtained in the BV formalism after fixing the gauge are nilpotent only on shell. Indeed, from the general relation $s' \Phi^A = (S, \Phi^A) \mid_{\Pi=\partial \Psi/\partial \Phi}$ we have

$$s'^2 A_{a}^{\mu} = -\frac{1}{2} W_{cd}^{ab} \epsilon_{\mu \nu} c^d c^e \mathcal{F}_{b \nu}^{e},$$

where $\mathcal{F}_{b \nu}$ is the equation of motion for $A_{b}^{\nu}$ following from $I_{BV}$:

$$\mathcal{F}_{b \nu} = F_{b \nu} - W_{bc}^{d} c^d \frac{\partial \Psi}{\partial A_{d \nu}^c} - B_{c} \frac{\partial^2 \Psi}{\partial A_{b \nu}^c \partial \bar{c}_c}.$$

Thus, $s'^2 = 0$ only on shell, as a consequence of the nonclosure of the classical gauge algebra ($W_{cd}^{ab} \neq 0$). BRST transformations of other variables are nilpotent off shell.

The essential role of the antifields $A_{b}^{\nu}$ in the quantum action (14) is to ensure the off–shell nilpotency of the BRST transformations in the gauge–fixed theory.

4. Gauge theory based on quadratically extended Poincaré algebra

Attempts to formulate the theory of gravity as a gauge theory led to considering the Poincaré gauge theory as a candidate for a consistent theory of gravity. We shall consider here, as an application of the previous general formalism, the quantization of two–dimensional gauge theory based on quadratically extended Poincaré algebra.

**Poincaré gauge theory.** The basic dynamical variables of this theory in 2d are the diad $b_{a}^{\mu}$ and the connection $A_{ab}^{\mu}$, associated with the translation and Lorentz subgroup of the Poincaré group, respectively. Here, $a, b, ... = 0, 1$ are the local Lorenz indeces, while $\mu, \nu, ... = 0, 1$ are the coordinate indeces. The structure of the Poincaré group is also reflected in the existence of two kinds of gauge field strengths: the torsion $T_{a \mu \nu}$, and the
curvature $R_{\mu\nu}^{ab}$. The most general action of the Poincaré gauge theory in 2d, which is at most quadratic in gauge field strengths, has the form [9,10]:

$$I_0' = \int d^2x \left( \frac{1}{16\alpha} R_{\mu\nu}^{ab} R_{\mu\nu}^{ab} - \frac{1}{8\beta} T_{\mu\nu}^a T_{\mu\nu}^a - \gamma \right),$$

(16)

where $b = \det(b_{\mu}^a)$, and $\alpha, \beta, \gamma$ are constants. In 2d the Lorenz connection $A_{\mu}^{ab}$ can be parametrized as $A_{\mu}^{ab} = \epsilon^{ab} A_{\mu}$, so that

$$R_{\mu\nu}^{ab} = \epsilon^{ab} R_{\mu\nu}, \quad R_{\mu\nu} \equiv \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu},$$

and $D_{\mu} b_{\nu}^a = \partial_{\mu} b_{\nu}^a + \epsilon^{ac} A_{\mu} b_{c\nu}^a$ is the covariant derivative of the diad field. The action (16) is invariant under the local Poincare transformations with parameters $\omega \epsilon^{ab}$ and $a^\lambda$:

$$\delta_0 b_{\mu}^a = \omega \epsilon^{ac} b_{c\mu}^a - a^\lambda \partial_{\lambda} b_{\mu}^a,$$

$$\delta_0 A_{\mu} = -\partial_{\mu} \omega - a^\lambda \partial_{\lambda} A_{\mu},$$

$$\delta_0 \varphi = -a^\lambda \partial_{\lambda} \varphi, \quad \delta_0 \varphi^a = \omega \epsilon^{ac} \varphi^c - a^\lambda \partial_{\lambda} \varphi.$$

**Nonlinear extension.** It has been shown [1,3] that the theory (16) in the region $b \neq 0$ is equivalent to a gauge theory based on the following quadratic extension of the Poincaré algebra:

$$[M, M] = 0, \quad [M, P_a] = -\epsilon^{ab} P_b,$$

$$[P_a, P_b] = \epsilon^{ab}(\alpha M^2 + \beta \eta^{ab} P_a P_b + \gamma).$$

(17)

Indeed, by using the general procedure of the previous section with $A_{\mu}^a \rightarrow (b_{\mu}^a, A_{\mu})$ and $\phi_a \rightarrow (\varphi, \varphi_a)$, one obtains the gauge invariant action in the form

$$I_0 = \int d^2x \left[ \frac{1}{2} \epsilon^{\mu\nu} (\varphi R_{\mu\nu} + \varphi_a T_{\mu\nu}^a) - b(\alpha \varphi^2 + \beta \varphi_a \varphi^a + \gamma) \right].$$

(18)

This expression represents the first–order formulation of the action (16), with $(\varphi, \varphi_a)$ playing the role of auxiliary fields. It is equivalent to (16) in the region $b \neq 0$, in the sense that (18) reduces to (16) when the auxiliary fields are eliminated. In the weak coupling limit, $\alpha, \beta, \gamma \rightarrow 0$, the theory (18) becomes the topological ISO(1, 1) gauge theory [11]. The clarification of the dynamical content of the theory (16) [10] helps us to understand this relationship more clearly.

The equations of motion for $b_{\mu}^a$, $A_{\mu}$, $\varphi$ and $\varphi_a$ are given by

$$F_{\mu}^a = \epsilon^{\mu\nu} D_\nu \varphi_a - bh_{\mu}^a (\alpha \varphi^2 + \beta \varphi_a \varphi^a + \gamma) = 0,$$

$$F^\mu = \epsilon^{\mu\nu} (\partial_\nu \varphi + \epsilon^{ab} \varphi_a b_{b\nu}) = 0,$$

$$F = \frac{1}{2} \epsilon^{\mu\nu} R_{\mu\nu} - 2\alpha b \varphi = 0,$$

$$F^a = \frac{1}{2} \epsilon^{\mu\nu} T_{\mu\nu}^a - 2\beta b \varphi^a = 0,$$

(19)
where \( h_a^\mu \) is the inverse diad field and \( D_\nu \varphi_a = \partial_\nu \varphi_a - \epsilon_a^c A_\nu \varphi_c \).

The action is invariant under the gauge transformations with parameters \((\xi, \xi^a)\):

\[
\begin{align*}
\delta_0 A_\mu &= \partial_\mu \xi + 2\alpha \varepsilon_{bc} \xi^b c^c \mu \varphi, \\
\delta_0 b^a_\mu &= \partial_\mu \xi^a + \varepsilon^{ab}(-\xi b^a_\mu + \xi b A_\mu) + 2\beta \varepsilon_{bc} \xi^b c^c \mu \varphi^a, \\
\delta_0 \varphi &= \varepsilon_{ab} \xi^a \varphi^b, \\
\delta_0 \varphi_a &= \varepsilon_{ab} \xi^a \varphi^b + (\alpha \varphi^2 + \beta \varphi c \varphi^c + \gamma).
\end{align*}
\] (20)

After redefining the parameters by \( \xi \to -\omega - a^\lambda A_\lambda, \xi^a \to -a^\lambda b^a_\lambda \), one easily finds that these transformations reduce, up to the equations of motion, to the standard local Poincaré transformations.

Now, we introduce the gauge generators \((\Gamma, \Gamma_a)\) corresponding to the \((\xi, \xi^a)\) transformations by \( \delta_q = \xi \Gamma + \xi^a \Gamma_a \). The algebra of the generators is closed only on shell:

\[
\begin{align*}
[\Gamma, \Gamma] &= 0, \\
[\Gamma, \Gamma_b] &= -\varepsilon_b c \Gamma_c, \\
[\Gamma_a, \Gamma_b] &= -2\varepsilon_{ab} (\alpha \varphi \Gamma + \beta \varphi^c \Gamma_c) + 2\varepsilon_{ab} \varepsilon_{\mu \nu} \left( \alpha F^\nu_\mu \frac{\delta}{\delta A_\mu} + \beta F^c_\nu \frac{\delta}{\delta b^c_\mu} \right).
\end{align*}
\] (21)

Equations (20) and (21) determine the structure functions \((T, f, E)\) of the classical gauge algebra.

**Quantization.** Let us now apply the general method developed in the previous section to study the quantization of the theory (18). The structure of classical fields, ghosts and antifields is given in Table 3.

The evaluation of the BRST generator on the basis of Eq.(11) leads to the result

\[
S_1 = -((\partial_\mu c + 2\alpha \varepsilon_{bc} c^b c^c \mu \varphi) A_*^{*\mu} \\
- [\partial_\mu c^a + \varepsilon^{ab} (-c b^a_\mu + c A_\mu)] + 2\beta \varepsilon_{bc} c^b c^c \mu \varphi^a b_*^{*\mu} \\
- \varepsilon_{ab} \varepsilon^{ab} \varphi^* - \varepsilon_{ab} (-c \varphi^b + c (\alpha \varphi^2 + \beta \varphi c \varphi^c + \gamma)) \varphi^* \\
- \varepsilon_{ab} [\alpha \varphi c^a c^b c^c + (\beta c^a c^b c^c - c^a \xi^b bc) c^c] \\
S_2 = \frac{1}{2} \varepsilon_{ab} \varepsilon_{\mu \nu} c^a c^b (\alpha A_*^{*\mu} A_*^{*\nu} + \beta b_*^{*\mu} b_*^{*\nu}).
\] (22)

From here one can easily find the form of the BRST transformations on the previous fields:

\[
\begin{align*}
sA_\mu &= \partial_\mu c + 2\alpha \varepsilon_{bc} c^b c^c \mu \varphi - \alpha \varepsilon_{bc} c^b c^c \varepsilon_{\mu \nu} A_*^{*\nu}, \\
sb^a_\mu &= \partial_\mu c^a + \varepsilon^{ab} (-c b^a_\mu + c A_\mu) + \beta \varepsilon_{bc} (2b^b c^c \mu \varphi^a - b^b \varepsilon_{\mu \nu} b_*^{*\nu}), \\
s\varphi &= \varepsilon_{ab} \varphi^a, \\
s\varphi_a &= \varepsilon_{ab} (-c \varphi^b + c (\alpha \varphi^2 + \beta \varphi c \varphi^c + \gamma)), \\
s c &= \alpha \varepsilon_{ab} \varphi^a c^b, \\
s c^a &= \beta \varepsilon_{bc} c^c \varphi^a + \varepsilon^a b^b c.
\end{align*}
\] (23a)
and the antifields,

\begin{align}
  sA^*\mu &= F^\mu - \varepsilon^{ab} c_b b_\alpha^* \mu , \\
  sb_\alpha^* \mu &= F_a^\mu + \varepsilon_{ab} (2\alpha c^\beta \varphi A^*\mu - cb_\beta b^\mu + 2\beta c^\beta \varphi c^a b^\mu) ,
\end{align}

(23b)

and similarly for other antifields. It should be observed that the set of all fields together with the restricted set of antifields \((A^*\mu, b_\alpha^*\mu)\) also carries the representation of the BRST transformations: these variables transform into each other under \(s\) and, moreover, \(s^2 = 0\) off-shell.

The BRST invariant action (13) is degenerate in the antifield sector, as \(S_2\) contains only \(A_\mu^*\) and \(b_\alpha^*\mu\). This degeneracy can be removed by imposing the following extra conditions: \(\varphi^* = \varphi^{*a} = c^* = c^{*a} = 0\). The restricted set of antifields \((A_\mu^*, b_\alpha^*\mu)\) is sufficient to define the representation of the off-shell nilpotent BRST transformations, so that the whole procedure is completely consistent.

The quantum action is now determined by the expression (14a). The final form of the quantum theory depends on the choice of gauge. The authors of Ref. [1] considered two gauge choices: the temporal gauge \((A_0 = 0, b^a_0 = 0)\), and the background–covariant gauge \([\varphi_a = 0, \partial_\mu (\bar{g}^{\mu \nu} A_\nu) = 0]\). The related gauge fermions are given by the expressions

\begin{enumerate}
  \item \(\Psi_1 = \bar{c} A_0 + \bar{c}_a b^{a}_0\) and
  \item \(\Psi_2 = \bar{c}^a \varphi_a + \bar{c} \partial_\mu (\bar{g}^{\mu \nu} A_\nu)\), while
\end{enumerate}


while the BRST transformations of antighosts and multipliers are of the standard forms. The above gauge choices, in conjunction with the equations of motion, imply that the diad field is degenerate, i.e. \(b = 0\). This is acceptable if we interpret the action (18) as describing a Yang–Mills theory. However, the correct quantization of the gravitational theory (16) requires a gauge condition consistent with \(b \neq 0\). c) The conformal gauge \(b_\alpha^*_\alpha = c^* \delta^*_a \) was used in [9] for classical calculations, but its nonlinearity makes it not so convenient in quantum theory [12]. d) The Landau–type gauge \((\partial_\mu A_\mu = 0, \partial_\mu b^a_\mu = 0)\), and e) the light cone gauge \((A_+ = 0, b^+_+ = b^-_+ = 0)\) were used in Ref. [12] in considerations related to the question of renormalizability of \(R^2 + T^2\) theory of gravity.

After choosing the gauge conditions, the quantum, nonlinear Poincaré gauge theory is defined by the generating functional

\[Z_\Psi = \int D\mu \exp[i(I_{BRST} - s\Psi)] ,
\]

\[D\mu = Db^a_\mu DA_\mu D\varphi_a D\varphi Dc Dc^a Db^*\mu DA^*\mu D\bar{c} DB .
\]

The structure of antighosts \(\bar{c}\) and multipliers \(B\) depends on the choice of gauge.
5. Concluding remarks

In this paper we have studied the covariant quantization of 2d gauge theories based on nonlinear extension of Lie algebras, in the generalized Lagrangian formalism. We first constructed the BRST invariant action $I_{BRST}$ by using the information on the classical gauge structure of the theory. The BRST symmetry of the gauge–fixed, quantum theory is off–shell nilpotent. It is realized on the set of variables $(A^a_{\mu}, \phi_a, c^a; A^*_a{}^{\mu})$ containing the restricted set of antifields $A^*_a{}^{\mu}$ as auxiliary variables with nonvanishing ghost number. The relation of our approach to the BV one is clarified.

The general results are then applied to study the covariant quantization of a specific nonlinear gauge theory in 2d, based on the quadratic extension of the Poincaré algebra. This theory is of particular interest for investigations of the quantum structure of gravity, as it represents an interesting connection between several possible formulations of 2d gravity.

The $W_3$ gravity and the dilaton gravity can be treated in a similar way.
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Table 1. The Grassmann parities and the ghost numbers of the fields and antifields

|        | $\varphi^i$ | $c^\alpha$ | $\Pi_i$ | $\Pi_\alpha$ |
|--------|-------------|------------|---------|--------------|
| $\varepsilon$ | $\varepsilon_i$ | $\varepsilon_\alpha + 1$ | $\varepsilon_i + 1$ | $\varepsilon_\alpha$ |
| $gh$ | 0 | 1 | $-1$ | $-2$ |

Table 2. The components of the fields and antifields

|        | $A^a_\mu$ | $\phi_a$ | $c^a$ | $A^{*a}_\mu$ | $\phi^{*a}$ | $c^{*a}$ |
|--------|-----------|----------|-------|--------------|-------------|----------|
| $\varepsilon$ | 0 | 0 | 1 | 1 | 1 | 0 |
| $gh$ | 0 | 0 | 1 | $-1$ | $-1$ | $-2$ |

Table 3. The fields and antifields for quadratically nonlinear Poincaré gauge theory

|        | $b^a_\mu$ | $A_\mu$ | $\varphi$ | $\varphi_a$ | $c$ | $c^a$ | $b^{*a}_\mu$ | $A^{*a}_\mu$ | $\varphi^*$ | $\varphi^{*a}$ | $c^*$ | $c^{*a}$ |
|--------|-----------|--------|----------|------------|-----|-------|--------------|--------------|-----------|-------------|------|-------|
| $\varepsilon$ | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |
| $gh$ | 0 | 0 | 0 | 0 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | $-2$ | $-2$ |