Queueing process with excluded-volume effect

Chikashi Arita

MI 2009-33

(Received October 2, 2009)
Queueing process with excluded-volume effect

Chikashi Arita
Faculty of Mathematics, Kyushu University, Ito 819-0395 JAPAN

We introduce an extension of the M/M/1 queueing process with a spatial structure and excluded-volume effect. The rule of particle hopping is the same as for the totally asymmetric simple exclusion process (TASEP). A stationary state solution is constructed in a slightly arranged matrix product form of the open TASEP. We obtain the critical line that separates the parameter space depending on whether the model has the stationary state. We calculate the average length of the model and the number of particles and show the monotonicity of the probability of the length in the stationary state. We also consider a generalization of the model with backward hopping of particles allowed and an alternate joined system of the M/M/1 queueing process and the open TASEP.

PACS numbers: 02.50.−r, 05.70.Ln

I. INTRODUCTION

The queueing process is a typical example of a Markov process [1]. One of the simplest queueing processes is of the so-called M/M/1 type, where the arrival of customers and their service obey the Poisson point process. This model’s stationary state is the geometric distribution that varies with the ratio of the arrival rate to the service rate. The M/M/1 queueing process has no spatial structure and particles do not interact with each other.

On the other hand, the asymmetric simple exclusion process (ASEP) on a one dimensional lattice is one of the simplest Markov processes with interacting particles [2]. In the ASEP, each site can be occupied by at most one particle and each particle can hop to a nearest neighbor site if it is empty. The ASEP admits exact analyses of non-equilibrium properties by the matrix product ansatz and the Bethe ansatz [3]. The matrix product form of the stationary state was firstly found in the totally ASEP with open boundaries (open TASEP), where each particle enters at the left end, hops forward (rightward) in the bulk and exits at the right end [4]. Similar results have been obtained in various generalized ASEPs and similar models in one dimension with both open and periodic boundary conditions [5].

In this paper, we introduce an extension of the M/M/1 queueing process on a semi-infinite chain with the excluded-volume effect (hard-core repulsion) as in the open TASEP. Each particle enters the chain at the left site next to the leftmost occupied site, hops and exits following the same rule as for the open TASEP. The stationary state solution is given by a slightly arranged matrix product form of the open TASEP. The normalization constant is given by the generating function of that of the open TASEP.

This paper is organized as follows. In Section 2, we briefly review the M/M/1 queueing process and the open TASEP. In Section 3, we define the model. In Section 4, we find a stationary state of the model. This will be constructed in a slightly modified matrix product form of the open TASEP. We obtain the critical line which separates the parameter space into the regions with and without the stationary state. The critical line will be written in terms of the stationary current of the open TASEP. We also calculate the average length of the system and average number of particles on the assumption of the uniqueness of the stationary state. We also show the monotonicity of the probability of the length (i.e. the position of the leftmost particle). In Section 5, we generalize the model by allowing particles to hop backward. It is fair to say that almost every calculation in Section 4 and 5 will be performed by using known formulae in studies of the open TASEP and the open partially ASEP (PASEP). In Section 6, we introduce an alternate joined system of the queueing process and the open TASEP.

II. REVIEW OF THE M/M/1 QUEUEING PROCESS AND THE OPEN TASEP

A. M/M/1 queueing process

Let us consider the simplest queueing process, that is the M/M/1 queueing process, as in Fig. 1, where N denotes the number of particles. Particles enter the system with rate α and exit the system with rate β. (Customers arrive at the queue with rate α and receive service with rate β at one server.) The M/M/1 queueing process does not have spatial structure and is characterized only by

\[ \alpha \rightarrow \begin{array}{c} \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \beta \\[N] \end{array} \]

FIG. 1: M/M/1 queueing process.

*Electronic address: airta@math.kyushu-u.ac.jp
the number of particles.

The system is encoded by a Markov process on the state space $\mathbb{Z}_{\geq 0}$ and governed by the following master equation for the probability $P(N)$ that the number of particles is $N$:

$$
\frac{d}{dt} P(0) = \beta P(1) - \alpha P(0),
$$

$$
\frac{d}{dt} P(N) = \alpha P(N-1) + \beta P(N+1) - (\alpha + \beta) P(N),
$$

for $N \in \mathbb{N}$. The M/M/1 queuing process is equivalent to a continuous-time random walk on $\mathbb{Z}_{\geq 0}$ whose jump rates to right and left directions are $\alpha$ and $\beta$, respectively, with reflection at 0.

A unique stationary-state solution is easily obtained as

$$
P(N) = \frac{1}{Z} \left( \frac{\alpha}{\beta} \right)^N, \quad Z = \sum_{N=0}^{\infty} \left( \frac{\alpha}{\beta} \right)^N.
$$

We should note, however, that the normalization constant $Z$ does not always converge, in other words, the stationary state does not always exist. If $\alpha < \beta$, $Z$ actually converges to $\beta/(\beta - \alpha)$ and the system has the stationary state. (Otherwise, $Z$ diverges and the system has no stationary state.) Thus the critical line is $\alpha = \beta$.

The average number of particles can be easily calculated as

$$
\langle N \rangle_{M/M/1} = \sum_{N=0}^{\infty} N P(N) = \frac{\alpha}{\beta - \alpha}.
$$

In the stationary state, the current of particles through the server is nothing but the arrival rate $\alpha$:

$$
\beta \sum_{N=1}^{\infty} P(N) = \alpha.
$$

**B. TASEP with open boundaries**

Let us consider an interacting particle system, the totally asymmetric simple exclusion process on the $L$-site chain with open boundaries (open TASEP), see Fig. 2. Each site can be occupied by at most one particle. Each particle enters the chain at the left end with rate $\alpha$ if it is empty, hops to its right nearest neighbor site in the bulk with rate $p$ if it is occupied, and exits at the right end with rate $\beta$. Let us write $\tau_j = 0$ if $j$th site is empty and $\tau_j = 1$ if it is occupied by a particle. The system is formulated by a Markov process on the state space $\{0,1\}^L$. The master equation on the probability $P(\tau_1, \ldots, \tau_L)$ is as follows:

$$
\frac{d}{dt} P(\tau_1, \ldots, \tau_L)
= \alpha (2\tau_1 - 1) P(0, \tau_2, \ldots, \tau_L)
+ p \sum_{j=1}^{L-1} (\tau_{j+1} - \tau_j) P(\tau_1, 1, 0, \ldots, \tau_L)
+ \beta (1 - 2\tau_L) P(\tau_1, \ldots, \tau_{L-1}, 1).
$$

In contrast to the M/M/1 queueing process, the state space of the open TASEP is finite. Moreover the open TASEP is irreducible, therefore it always has a unique stationary state. Derrida et al found the stationary-state solution to the open TASEP in the following simple form [4]:

$$
P(\tau_1, \ldots, \tau_L) = \frac{1}{Z_L(\alpha, \beta, p)} (W(\frac{\alpha}{p}))^{X_{\tau_1} \cdots X_{\tau_L}} (V(\frac{\beta}{p}))
$$

where $X_0 = E$ and $X_L = D$ are matrices, $\langle W(u) \rangle$ and $\langle V(v) \rangle$ are row and column vectors, respectively, and $Z_L(\alpha, \beta, p)$ is the normalization constant

$$
Z_L(\alpha, \beta, p) = \langle W(\frac{\alpha}{p}) (D + E)^L V(\frac{\beta}{p}) \rangle.
$$

The matrices and the vectors should satisfy the following relation so that the matrix product form (7) actually gives the stationary-state probability:

$$
DE = D + E,
$$

$$
\langle W(u) \rangle E = \frac{1}{u} \langle W(u) \rangle,
$$

$$
\langle V(v) \rangle D = \frac{1}{v} \langle V(v) \rangle.
$$

Set $\langle W(u) \rangle \langle V(v) \rangle = 1$ without loss of generality. The following representation satisfies the algebra (9):

$$
D = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
\vdots & \ddots & \ddots & \ddots
\end{pmatrix},
E = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{pmatrix},
$$

$$
\langle W(u) \rangle = \kappa (1, a, a^2, \ldots),
\langle V(v) \rangle = \kappa (1, b, b^2, \ldots)
$$

where $\kappa = \sqrt{(w + v - 1)/ww}, a = (1 - w)/w$ and $b = (1 - v)/v$. Note that, in the original paper [4], the bulk

![FIG. 2: TASEP with open boundaries.](image)
The hopping rate $p$ was set to be 1 and other representations of the matrices and vectors were found. For simplicity in what follows, however, we choose the representation (10)-(11) so that only the vectors depend on $\alpha, \beta$ and $p$. By using the algebraic relation (9), we can calculate the normalization constant as follows [4]:

$$Z_L(\alpha, \beta, p) = \sum_{j=0}^{L} \frac{j(2L - j - 1)! (\frac{p}{\alpha})^{j+1} - (\frac{p}{\beta})^{j+1}}{L!(L-j)!} \frac{p}{\alpha - \frac{p}{\beta}}.$$  

(12)

The stationary current, for example, can be written in terms of the normalization constant $Z_L(\alpha, \beta, p)$ as

$$J_L(\alpha, \beta, p) = p \frac{Z_{L-1}(\alpha, \beta, p)}{Z_L(\alpha, \beta, p)}.$$  

(13)

In the limit $L \to \infty$, the phase diagram of the current $J_{\infty}(\alpha, \beta, p)$ consists of three regions which are called the maximal current (MC) phase, the low density (LD) phase and the high density (HD) phase (see Fig. 3):

$$J_{\infty}(\alpha, \beta, p) = \begin{cases} \frac{p}{2} & \alpha, \beta \geq \frac{p}{2} \\ \alpha(1 - \alpha/p) & \alpha < \min(\beta, \frac{p}{2}) \\ \beta(1 - \beta/p) & \beta < \min(\alpha, \frac{p}{2}) \end{cases} \quad \text{(MC)}$$

$$J_{\infty}(\alpha, \beta, p) = \begin{cases} \frac{p}{2} & \alpha, \beta \geq \frac{p}{2} \\ \alpha(1 - \alpha/p) & \alpha < \min(\beta, \frac{p}{2}) \\ \beta(1 - \beta/p) & \beta < \min(\alpha, \frac{p}{2}) \end{cases} \quad \text{(LD)}$$

$$J_{\infty}(\alpha, \beta, p) = \begin{cases} \frac{p}{2} & \alpha, \beta \geq \frac{p}{2} \\ \alpha(1 - \alpha/p) & \alpha < \min(\beta, \frac{p}{2}) \\ \beta(1 - \beta/p) & \beta < \min(\alpha, \frac{p}{2}) \end{cases} \quad \text{(HD)}$$

(14)

The line $\alpha = \beta < \frac{p}{2}$ is called the coexistence line, where a shock between a low density segment and a high density segment exhibits a random walk [6]: $J_{\infty}(\alpha, \alpha, p) = \alpha(1 - \alpha/p)$.

FIG. 3: Phase diagram of the open TASEP.

Note that no particle can enter the chain if the leftmost site is occupied by another particle and thus the stationary current is not equal to $\alpha$. This means that the open TASEP is a “call-loss system.” Recall the stationary current of the M/M/1 queueing process, which is not a call-loss system, is $\alpha$.

III. MODEL

Let us introduce a new model which is an extended M/M/1 queueing process on a semi-infinite chain with the excluded-volume effect as in the TASEP. Figure 4 shows the model, where each site is numbered from right to left and $L$ denotes the leftmost occupied site. Each site can be occupied by at most one particle. Each particle hops to its right nearest neighbor site with rate $p$ in the bulk if it is empty and exits with rate $\beta$ at the right end. The rules of the bulk hopping and the exit are the same as for the open TASEP. However, the rules for the entering is different. Each particle enters the chain at the immediately left of the leftmost occupied site (or at site 1 if there is no particle on the chain). One can put this model into the TASEP in the semi-infinite chain with a new boundary condition.

The system is encoded by a Markov process on the state space

$$S := \emptyset \cup \bigcup_{L \in \mathbb{N}} \left( \{1\} \times \{0,1\}^{L-1} \right)$$

$$= \{\emptyset, (1), (1, 0), (1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1), (1, 0, 0, 0), \ldots \}.$$  

(15)

The leftmost particle of each element of $S$ except $\emptyset$ can always be specified. Configurations such as $(\ldots, 1, 1, 1)$ and $(\ldots, 1, 0, 1, 0, 1, 0)$ do not appear in $S$.

Let $P(1, \tau_{L-1}, \ldots, \tau_1)$ be the probability of finding a configuration $(1, \tau_{L-1}, \ldots, \tau_1)$ with the leftmost particle at $L$th site and $P(\emptyset)$ be the probability of finding no particle on the chain. The master equation governing the model is as follows:

$$\frac{d}{dt} P(\emptyset) = \beta P(1) - \alpha P(\emptyset),$$  

(16)

$$\frac{d}{dt} P(1) = \alpha P(\emptyset) + pP(1,0) - (\alpha + \beta)P(1),$$  

(17)
\[
\frac{d}{dt} P(1, \tau_{L-1}, \ldots, \tau_1) \\
= p P(1, 0, \tau_{L-1}, \ldots, \tau_1) \\
+ \alpha \tau_{L-1} P(1, \tau_{L-2}, \ldots, \tau_1) - \alpha P(1, \tau_{L-1}, \ldots, \tau_1) \\
+ p(\tau_{L-1} - 1) P(1, 0, \tau_{L-2}, \ldots, \tau_1) \\
+ p \sum_{j=1}^{L-2} (\tau_j - \tau_j+1) P(1, \tau_{L-1}, \ldots, 1, j+1, 0, \ldots, \tau_1) \\
+ \beta(1 - 2\tau_1) P(1, \tau_{L-1}, \ldots, \tau_2, 1).
\] (18)

For example,

\[
\frac{d}{dt} P(1, 1, 0, 1, 0, 1) \\
= \alpha P(1, 0, 1, 0, 1) + p P(1, 0, 1, 0, 1, 0) \\
+ p P(1, 1, 1, 0, 1, 0) + p P(1, 1, 0, 1, 0, 1) \\
- (\alpha + 2p + \beta) P(1, 1, 0, 1, 0, 1),
\] (19)

\[
\frac{d}{dt} P(1, 0, 0, 1, 0, 1, 0) \\
= p P(1, 0, 0, 1, 0, 0, 1, 0) + p P(1, 0, 1, 0, 1, 0, 1) \\
+ p P(1, 0, 0, 1, 1, 0, 0) + \beta P(1, 0, 0, 1, 0, 1, 0) \\
- (\alpha + 3p) P(1, 0, 0, 1, 1, 0, 1, 0).
\] (20)

We write the right-hand side of the master equation as \((HP)(\tau_1, \ldots, \tau_L)\) with the generator matrix \(H\) acting on the probability vector \(P = (P(\emptyset), P(1), P(1, 0), \ldots)^T\). Note that \(H\) is an infinite dimensional matrix.

**IV. STATIONARY STATE**

**A. Matrix product form**

The problem is how to find the solution to \(HP = 0\). We can see that a slightly arranged matrix product form

\[
P(\emptyset) = \frac{1}{Z(\alpha, \beta, p)},
\]

\[
P(1) = \frac{1}{Z(\alpha, \beta, p)} \alpha = \frac{1}{Z(\alpha, \beta, p)} \alpha (W(1)|V(\frac{\beta}{p})|
\]

\[
P(\tau_L = 1, \ldots, \tau_1) = \frac{1}{Z(\alpha, \beta, p)} \alpha^L \left( W(1)|X_{\tau_L-1}, \ldots, X_{\tau_1}|V(\frac{\beta}{p}) \right) \\
\text{(for } L \geq 2) \\
\]

(21)

gives a stationary-state solution. (This idea can be applicable to a discrete-time version of the model \([7]\).) Here \(Z(\alpha, \beta, p)\) is the normalization constant which can be written as a special case of the generating function of

the normalization constant of the open TASEP:

\[
Z(\alpha, \beta, p) = \sum_{L=0}^{\infty} \frac{\alpha^L}{\beta p^{L-1}} (W(1)|D + E)^{L-1} |V(\frac{\beta}{p})\]

\[
= \sum_{L=0}^{\infty} \frac{\alpha^L}{\beta p^{L-1}} Z_{L-1}(p, \beta, p)
\] (22)

with \(Z_{-1}(p, \beta, p) = \beta/p\). See Appendix A for some stationary probabilities calculated by using the algebraic relation (9). That the form (21) gives a stationary-state solution (i.e. \(HP = 0\)) can be proved by a similar canceling to that for the open TASEP. Let us use a short-hand notation

\[
(W(1)|\cdots|((\frac{\beta}{p})) = (\cdots).
\] (23)

Substituting the form (21) into \(HP(1, \tau_{L-1}, \ldots, \tau_1)\) and multiplying it by \(\frac{\beta p^{L-1}}{\alpha^L} Z(\alpha, \beta, p)\), we find

\[
\frac{\beta p^{L-1}}{\alpha^L} Z(\alpha, \beta, p)(HP)(1, \tau_{L-1}, \ldots, \tau_1)
\]

\[
= p \left( EX_{\tau_L-1} \cdots X_{\tau_1} \right)
\]

\[
+ \alpha \tau_{L-1} \frac{p}{\alpha} (X_{\tau_L-2} \cdots X_{\tau_1}) - \alpha (X_{\tau_L-1} \cdots X_{\tau_1}) 
\] (25)

\[
+ p(\tau_{L-1} - 1) (EX_{\tau_L-2} \cdots X_{\tau_1})
\]

\[
+ p \sum_{j=1}^{L-2} (\tau_j - \tau_{j+1}) (X_{\tau_L-1} \cdots X_{j+1}) 
\] (26)

\[
+ \beta(1 - 2\tau_1) (X_{\tau_L-1} \cdots X_{\tau_1} D)
\]

\[
+ p \left( - (1 - 2\tau_{L-1}) (X_{\tau_L-2} \cdots X_{\tau_1}) 
\]

\[
+ \sum_{j=1}^{L-2} (1 - 2\tau_{j+1}) (X_{\tau_L-1} \cdots X_{j+1} X_{\tau_{j+1}} \cdots X_{\tau_1})
\]

\[
+ \sum_{j=1}^{L-2} (1 - 2\tau_j) (X_{\tau_L-1} \cdots X_{\tau_j+1} X_{\tau_{j+1}} \cdots X_{\tau_1})
\]

\[
+ (1 - 2\tau_1) (X_{\tau_L-1} \cdots X_{\tau_2}) \right) 
\] (27)

In going from (25) to (26), we applied the algebraic relation (9):

\[
(\cdots D E \cdots) = (\cdots D \cdots) + (\cdots E \cdots),
\]

\[
(E \cdots) = (\cdots), \quad (\beta(\cdots D) = p(\cdots).
\] (29)

In our argument throughout this section, we assume that (21) is a unique stationary state of the model if \(Z(\alpha, \beta, p)\) converges and there is no stationary state if \(Z(\alpha, \beta, p)\) diverges.

The following function will be useful in the next section:

\[
Z(p\xi, \beta, p; \zeta)
\]

\[
= 1 + \sum_{L=1}^{\infty} \frac{p}{\beta} \xi^L \zeta^(W(1)|(\zeta D + E)^{L-1} |V(\frac{\beta}{p})\]

\] (30)
with a fugacity $\zeta$. This can be regarded as a special case of the generating function of the normalization constant of the TASEP with a single defect particle (see section 4.3 in [5]). The case where $\zeta = \alpha/p$ and $\zeta = 1$ corresponds to the normalization constant:

$$Z(\alpha, \beta, p) = Z(p\alpha/p, \beta, p; 1).$$

(31)

B. Critical line

The asymptotic behavior of $Z_L(\alpha, \beta, p)$ as $L \to \infty$ is as follows [4]:

$$Z_L(\alpha, \beta, p) \sim \begin{cases} \frac{4^\alpha}{\sqrt{\pi L \beta}} \left( \frac{2\beta}{2\beta-p} \right)^{2} \beta > \frac{p}{2} \\ \frac{4^\alpha}{\sqrt{\pi L \beta}} \left( \frac{1-2\beta/p}{1-\beta/p} \right)^{L} \beta < \frac{p}{2} \end{cases}$$

(32)

Thanks to this asymptotic form, we see that if the condition

$$\begin{cases} \alpha \leq \frac{p}{2} \\ \alpha < \beta(1-\beta/p) \end{cases}$$

(33)

is satisfied, then $Z(\alpha, \beta, p)$ converges. In other words, the critical line is given by

$$\alpha = \alpha_c = \begin{cases} \frac{p}{2} \\ \beta(1-\beta/p) \end{cases}$$

(34)

see Fig. 5. Note that the area (33), where the normaliza-
tion constant $Z(\alpha, \beta, p)$ converges, is embedded in that of the usual M/M/1 queueing process ($\alpha < \beta$). We remark that the critical line can be written in terms of the stationary current (14) of the open TASEP with $\alpha = p$:

$$\alpha_c = J_{\infty}(p, \beta, p).$$

(35)

Turning to the stationary current through the right end of the chain, we see that this must be the arrival rate $\alpha$, because the model is not a call-loss system. In fact, one can see

$$\beta P(1) + \sum_{L=2}^{\infty} \sum_{\tau_{L-1}=0,1} P(1, \tau_{L-1}, \ldots, \tau_2, 1)$$

$$= \frac{\alpha}{Z(\alpha, \beta, p)} + \frac{1}{Z(\alpha, \beta, p)} \sum_{L=2}^{\infty} p^{L-1} (W(1)(D + E)^{L-2}D|V(\frac{\beta}{p})|)$$

$$= \alpha$$

$$= \frac{\alpha}{Z(\alpha, \beta, p)} \left( 1 + \sum_{L=2}^{\infty} \frac{\alpha^{L-1}}{\beta p^{L-2}} (W(1)(D + E)^{L-2}D|V(\frac{\beta}{p})|) \right)$$

(36)

C. Average values

In this subsection and the next subsection, we assume that the condition (33) is satisfied. According to the formula (4.27) in [5] we find

$$Z(p\xi, \beta, p; \zeta) = 1 + \frac{p\xi}{\beta (1 - \xi (\sqrt{\zeta} - 1)^2)} (1 - \eta) (1 - \eta \mu),$$

(37)

where

$$\mu = 1 + \sqrt{\zeta} (p/\beta - 1),$$

(38)

$$\eta = \frac{1}{2} \left( 1 - \frac{1 - (1 + \sqrt{\zeta}) \xi}{\sqrt{1 - 2(1 + \xi)\xi + (1 - \xi)^2\xi^2}} \right).$$

(39)

In particular,

$$Z(\alpha, \beta, p) = \frac{2\beta}{2\beta - p(1 - r)}$$

(40)

with $r = \sqrt{1 - 4\alpha/p}$. The average length $\langle L \rangle$ of (the average position of the leftmost particle) and the average number $\langle N \rangle$ of particles on the chain can be calculated by differentiating $Z(p\xi, \beta, p; \zeta)$ as

$$\langle L \rangle = \xi \frac{\partial}{\partial \xi} \ln Z(p\xi, \beta, p; 1) \bigg|_{\xi = \alpha/p} = \frac{2\alpha/p}{r(-1 + r + 2\beta/p)},$$

(41)

$$\langle N \rangle = \frac{\partial}{\partial \zeta} \ln Z(\alpha, \beta, p; \zeta) \bigg|_{\zeta = 1} = \frac{2\alpha(1 + r - 3r\alpha/p)}{r(1 + r)((1 + r)\beta - 2\alpha)}$$

(42)
By the excluded-volume effect, these values are greater than the average length of the usual M/M/1 queueing process:

\[ (L) > \langle N \rangle > \langle N \rangle_{M/M/1} = \frac{\alpha}{\beta - \alpha}. \]  
(43)

Actually, one can find

\[ (N) - \langle N \rangle_{M/M/1} = \frac{2\alpha^2 ((\beta - \alpha) + r(\alpha + \beta))}{pr(1 + r)(\beta(1 + r) - 2\alpha)(\beta - \alpha)} > 0, \]  
(44)

whereas the first inequality in (43) is true by definition. \((L)\) and \((N)\) are expanded with respect to \(1/p\) as

\[ \langle L \rangle \sim \frac{\alpha}{\beta - \alpha} + \frac{\alpha^2 (2\beta - \alpha)}{(\beta - \alpha)^2} \frac{1}{p} + O \left((1/p)^2\right), \]  
(45)

\[ \langle N \rangle \sim \frac{\alpha}{\beta - \alpha} + \frac{\alpha^2 \beta}{(\beta - \alpha)^2} \frac{1}{p} + O \left((1/p)^2\right) \]  
(46)

and we can see a natural result that they approach \(\langle N \rangle_{M/M/1}\) in the usual-M/M/1-queueing-process limit \(p \to \infty\). Note that

\[ \langle L \rangle = \infty, \quad \langle N \rangle = \infty \]  
(47)

on the critical line \(\alpha = \frac{p}{2}\) and \(\beta > \frac{p}{2}\).

D. Monotonicity of the length

Let us consider the probability \(\lambda_L\) that the length is \(L\) (the leftmost particle is at site \(L\)):

\[ \lambda_L = \frac{1}{Z(\alpha, \beta, p)} \sum_{\tau_i=0,1} P(1, \tau_{L-1}, \ldots, \tau_1) \]  
(48)

\[ = \frac{1}{Z(\alpha, \beta, p)} \frac{p}{\beta} \left(\frac{\alpha}{p}\right)^L Z_{L-1}(p, \beta, p) \]

For \(L = 0\), we set \(\lambda_0 = P(0)\).

Thanks to the asymptotic form again (21), we can see that \(\lambda_L\) decays as \(L \to \infty\) as

\[ \lambda_L \sim \frac{1}{Z(\alpha, \beta, p)} \frac{p}{\beta} \times \left(1 + \frac{\beta}{\sqrt{\pi} L} \left(\frac{2\beta}{\pi \sqrt{p}}\right)^{L/2} \left(\frac{4\alpha}{p}\right)^L \right)^{\beta > \frac{p}{2}} \]  
(49)

When \(L\) is finite, \(\lambda_L\) possesses the property of the monotonicity with respect to \(L\):

\[ \cdots < \lambda_2 < \lambda_1 < \lambda_0. \]  
(50)

The rightmost inequality \(\lambda_1 < \lambda_0\) is clearly true. We devote the rest of this subsection to the proof of \(\lambda_{L+1} < \lambda_L\) for \(L \geq 1\). Let us use short-hand notations

\[ Z_L := Z_L(p, \beta, p), \quad x := \frac{p}{\beta}. \]  
(51)

and the following alternate expression which can be obtained by transforming (12):

\[ Z_L = \sum_{j=0}^{L} a_{L,j} x^j, \]  
(52)

where \(a_{L,j} = \frac{(j+1)(2L-j)!}{(L+1)!j!(L-j)!} \). (53)

1. Case when \(\beta \leq p/2\)

Under the assumption \(\alpha < \alpha_c = p/x(1-1/x)\), we find that

\[ \frac{\alpha}{p} Z_L - Z_{L-1} < \frac{1}{x} \left(1 - \frac{1}{x}\right) Z_L - Z_{L-1} = -\frac{C_L}{x} < 0, \]  
(53)

where \(C_L = \frac{(2L)!}{(L+1)!L!}\) is the Catalan number. Thus, we have \(\lambda_{L+1} < \lambda_L\).

2. Case when \(\beta > p/2\)

The proof of the monotonicity for \(\beta > p/2\) (i.e. \(x < 2\)) will be somewhat more technical. Our goal is to show \(Z_L < 4Z_{L-1}\), this implies that \(\frac{2}{p} Z_L \leq \frac{1}{4} Z_L < Z_{L-1}\) and thus \(\lambda_{L+1} < \lambda_L\).

Before proving this in the general \(L \geq 1\) case, we demonstrate it for \(L = 6\):

\[ Z_6 = 132 + 132x + 90x^2 + 48x^3 + 20x^4 + 6x^5 + x^6 \]
\[ < 132 + 132x + 90x^2 + 48x^3 + 20x^4 + 8x^5 \]
\[ < 132 + 132x + 90x^2 + 48x^3 + 28x^4 + 4x^5 \]
\[ < 132 + 132x + 90x^2 + 64x^3 + 20x^4 + 4x^5 \]
\[ < 132 + 132x + 106x^2 + 56x^3 + 20x^4 + 4x^5 \]
\[ < 168 + 168x + 112x^2 + 56x^3 + 20x^4 + 4x^5 \]
\[ = 4Z_5, \]  
(54)

where we used \(x^6 < 2x^5, x^5 < 2x^4, \text{ etc.}\)

Let us go back to the general \(L \geq 1\) case and introduce a sequence \(\{b_{L,j}\}_{1 \leq j \leq L-1}\) defined by the following recursion relation:

\[ b_{L,j} = 2(b_{L,j+1} - 4a_{L-1,j+1}) + a_{L,j} \]  
(55)

with \(a_{L-1,1} = b_{L-1,1} = 0\). One can find that

\[ b_{L,j} = \frac{(2L-j-2)!}{(L+1)!(L-j)!} \times (4jL^2 - 2(j^2 - 4j - 3)L - j(j+1)(j+7)). \]  
(56)
As long as $b_{L,j} > 4a_{L-1,j}$,
\[
\begin{align*}
& a_{L,j-1}x^{j-1} + b_{L,j}x^j \\
= & a_{L,j-1}x^{j-1} + (b_{L,j} - 4a_{L-1,j})x^j + 4a_{L-1,j}x^j \\
< & a_{L,j-1}x^{j-1} + 2(b_{L,j} - 4a_{L-1,j})x^j + 4a_{L-1,j}x^j \\
= & b_{L,j-1}x^{j-1} + 4a_{L-1,j}x^j.
\end{align*}
\] (57)

Let $k$ be an integer such that
\[
b_{L,k} \leq 4a_{L-1,k} \quad \text{and} \quad b_{L,j} > 4a_{L-1,j} \quad (\forall j > k). \quad (58)
\]

This is equivalent to
\[
k(k + 3) \leq 2L < (k + 1)(k + 4) \quad (59)
\]
and $k$ is determined uniquely. For example, $k = 2$ for $L = 6$. Using the inequality (57) repeatedly while $j > k$, we get
\[
\begin{align*}
Z_L &= a_{L,0} + \cdots + a_{L,L-2}x^{L-2} + a_{L,L-1}x^{L-1} + a_{L,L}x^L \\
&< a_{L,0} + \cdots + a_{L,L-2}x^{L-2} + b_{L,L}x^{L-1} \\
&< a_{L,0} + \cdots + b_{L,L-2}x^{L-2} + 4a_{L-1,L-1}x^{L-1} \\
&\cdots \\
&< \sum_{j=0}^{k-1} a_{L,j}x^j + b_{L,k}x^k + \sum_{j=k+1}^{L-1} 4a_{L-1,j}x^j.
\end{align*}
\] (60)

The coefficients $\{a_{L,j}\}_{0 \leq j \leq k-1}$ in the first summation of the last line of (60) satisfy $a_{L,j} < 4a_{L-1,j}$:
\[
\begin{align*}
\therefore \quad a_{L,j} - 4a_{L-1,j} \\
&= \frac{(j+1)(2L - j - 2)! (j^2 + 5j - 6L)}{(L+1)! (L-j)!} \\
&\leq \frac{(j+1)(2L - j - 2)! (2j - 4L)}{(L+1)! (L-j)!} < 0,
\end{align*}
\] (61)
where we used $j^2 + 3j < k^2 + 3k \leq 2L$ (see (59)).

Finally, we achieve
\[
\begin{align*}
Z_L &< \sum_{j=0}^{k-1} 4a_{L-1,j}x^j + 4a_{L-1,k}x^k + \sum_{j=k+1}^{L-1} 4a_{L-1,j}x^j \\
&= 4Z_{L-1}.
\end{align*}
\] (62)

V. GENERALIZATION

In this section, we generalize the model by allowing particles to hop backward with rate $pq$ ($q > 0$), see Fig. 6. We will construct a stationary state and derive the critical line, arranging the matrix product form of the partially ASEP (PASEP) with the open boundary condition as in Fig. 7.

The matrix product stationary state of the open PASEP is as follows:
\[
P(\tau_1, \ldots, \tau_L) = \frac{1}{Z_L(\alpha, \beta, p, q)} \langle W(\frac{\alpha}{p}), X_{\tau_1} \cdots X_{\tau_L} | V(\frac{\beta}{p}) \rangle
\]
where the matrices $X_0 = E_q$ and $X_1 = D_q$, the row vector $\langle W(w) \rangle$ and the column vector $| V(v) \rangle$ satisfy
\[
D_q E_q - qE_q D_q = D_q + E_q,
\]
\[
\langle W(w) | E_q = \frac{1}{w} \langle W(w) |,
\]
and $Z_L(\alpha, \beta, p, q)$ is the normalization constant:
\[
Z_L(\alpha, \beta, p, q) = \langle W(\frac{\alpha}{p}) | (D_q + E_q)^L | V(\frac{\beta}{p}) \rangle
\]
(63)

Some representations of the matrices and the vectors can be found in [5, 8, 9]. The notation used in (64) was chosen for convenience. However, we stress that, despite appearances, we have found no representation such that the matrices depend only on $q$ and the vectors are independent of $q$.

For $q < 1$, the normalization constant $Z_L(\alpha, \beta, p, q)$ can be written in the following integral form [8, 9]:
\[
Z_L(\alpha, \beta, p, q) = \frac{\langle q, ab; q \rangle_\infty}{4 \pi i} \int_K \frac{dz}{z} \cdot (az, a/z, b, b/z; q)_\infty \left( \frac{2 + z + z^{-1}}{1 - q} \right)^L
\]
where $a = \frac{p(1-q)}{\alpha} - 1$, $b = \frac{p(1-q)}{\alpha} - 1$ and $(x_1, \ldots, x_m; q)_\infty = \prod_{1 \leq n \leq m} \prod_{0 \leq i \leq \infty} (1 - x_n q^i)$ is the
in terms of the normalization constant as

\[ P_{\text{ASEP}} \text{ in the limit asymptotic form of the normalizing constant of the open PASEP in the limit } L \to \infty \text{ has been obtained by applying the saddle point method to the integral form (66) [8, 9]:} \]

\[
Z_L(\alpha, \beta, p, q) \sim \left\{ \begin{array}{ll}
\frac{4(ab;q)_\infty(q;q)_\infty}{\sqrt{\pi(a, b; q)_\infty L^2}} \left( \begin{array}{c}
\frac{1}{1-q}
\end{array} \right)^L & \alpha, \beta > \frac{p(1-q)}{2} \\
\frac{2}{\sqrt{\pi(a, b; q)_\infty L^2}} \left( \begin{array}{c}
\frac{1}{1-q}
\end{array} \right)^L & \alpha = \frac{p(1-q)}{2} < \beta \\
\frac{2}{\sqrt{\pi(a, b; q)_\infty L^2}} \left( \begin{array}{c}
\frac{1}{1-q}
\end{array} \right)^L & \beta = \frac{p(1-q)}{2} < \alpha \\
\frac{(2a+1)}{(a+1)(a-1)q^2; q^2}_\infty L^2 \left( \begin{array}{c}
\frac{1}{1-q}
\end{array} \right)^L & \alpha = \beta = \frac{p(1-q)}{2} \\
\frac{(a+1)}{(a+1)(a-1)q^2; q^2}_\infty L^2 \left( \begin{array}{c}
\frac{1}{1-q}
\end{array} \right)^L & \alpha < \min(\beta, \frac{p(1-q)}{2}) \\
\frac{(2a+1)}{(a+1)(a-1)q^2; q^2}_\infty L^2 \left( \begin{array}{c}
\frac{1}{1-q}
\end{array} \right)^L & \beta < \min(\alpha, \frac{p(1-q)}{2})
\end{array} \right. \\
\right.
\]

(67)

The stationary current of the open PASEP can be written in terms of the normalization constant as

\[ J_L(\alpha, \beta, p, q) = pZ_{L-1}(\alpha, \beta, p, q)/Z_L(\alpha, \beta, p, q). \]

(68)

Noting the asymptotic form (67), we have

\[ J_\infty(\alpha, \beta, p, q) = \left\{ \begin{array}{ll}
\frac{p(1-q)}{2} & \alpha, \beta \geq \frac{p(1-q)}{2} \\
\alpha \left( 1 - \frac{\alpha}{p(1-q)} \right) & \beta \leq \min(\alpha, \frac{p(1-q)}{2}) \\
\beta \left( 1 - \frac{\beta}{p(1-q)} \right) & \alpha \leq \min(\beta, \frac{p(1-q)}{2})
\end{array} \right. \]

(69)

In the symmetric case \( q = 1 \), the normalization constant has the following simple form [10]:

\[ Z_L(\alpha, \beta, p, q) = (\gamma + 1)(\gamma + 2) \cdots (\gamma + L) \]

(70)

where \( \gamma = \frac{p}{n} + \frac{p}{2} - 1 \).

In the reverse-bias case \( q > 1 \), the normalization constant behaves in the limit \( L \to \infty \) as

\[ Z_L(\alpha, \beta, p, q) \sim Aq^{ \frac{p}{2} } \left( \frac{ab}{1-q} \right)^L, \]

(71)

where \( A \) is a constant independent of \( L \) [9].

Let us go back to the generalized queueing process with forward and backward hopping (Fig. 6), which is governed by the following master equation:

\[
d\frac{d}{dt} P(0) = \beta P(1) - \alpha P(0),
\]

(72)

\[
d\frac{d}{dt} P(1) = \alpha P(0) + pP(0, 1) - (\alpha + \beta + pq)P(1),
\]

(73)

\[
d\frac{d}{dt} P(1, \tau_1, \ldots, \tau_1) = pP(1, 0, \tau_1, \ldots, \tau_1) - pqP(1, \tau_1, \ldots, \tau_1) + \alpha \tau_1 P(1, \tau_1, \ldots, \tau_1) - \alpha P(1, \tau_1, \ldots, \tau_1)
\]

(74)

A stationary-state solution to this equation is given by the following form, which can be proved in the same way as (24)–(27):

\[
P(0) = \frac{1}{Z(\alpha, \beta, p, q)}, \quad P(1) = \frac{1}{Z(\alpha, \beta, p, q) \beta},
\]

(75)

\[
P(1, \tau_1, \ldots, \tau_1) = \frac{1}{Z(\alpha, \beta, p, q) \beta \mu_{L-1}} (W(\frac{\alpha}{\alpha+pq})|X_{\tau_1} \cdots X_{\tau_1}|V^{(\frac{\beta}{p})}),
\]

where \( Z(\alpha, \beta, p, q) \) is the normalization constant:

\[
Z(\alpha, \beta, p, q) = \sum_{L=0}^{\infty} \alpha^L \beta \mu^{-L} Z_{L-1}(\frac{\alpha p}{\alpha+pq}, \beta, p, q)
\]

(76)

with \( Z_{-1}(\alpha p/(\alpha+pq), \beta, p, q) = \beta/p \).

We now obtain the critical line assuming the uniqueness of the stationary state. In view of (70) and (71), we find that the normalization constant \( Z(\alpha, \beta, p, q) \) converges only if the hopping ratio \( q < 1 \). Moreover, using the asymptotic form (67), we find that the condition for the model to have the stationary state is for \( 0 < q < \frac{1}{3} \):

\[
\left\{ \begin{array}{ll}
\alpha \leq \alpha_c & \beta > \frac{p(1-q)}{2} \\
\alpha < \alpha_c & \beta \left( 1 - \frac{\beta}{p(1-q)} \right) \leq \frac{p(1-q)}{2}
\end{array} \right.
\]

(77)

for \( \frac{1}{3} \leq q < 1 \),

\[
\alpha < \alpha_c = \left\{ \begin{array}{ll}
\frac{p(q-3)}{2} & \beta > \frac{p(q+2)}{2} \\
\beta \left( 1 - \frac{\beta}{p(1-q)} \right) & \beta \leq \frac{p(q+2)}{2}
\end{array} \right.
\]

(78)

where \( s = \sqrt{q(4-3q)} \). Note that the critical line \( \alpha_c \) is just the solution to

\[
\alpha_c = J_\infty(\frac{\alpha p}{\alpha+pq}, \beta, p, q).
\]

(79)
VI. ALTERNATE MODEL

We introduce here an alternate joined system of the M/M/1 queueing process and the open TASEP. This new system consists of a queue part and a TASEP part, see Fig. 9. Each particle enters the system with rate $\alpha$ and joins the queue part. The queue part has no spacial structure, and is characterized by the number of particles $N$. Each particle leaves the queue part and enters the TASEP part with rate $\alpha'$. After entering the TASEP part, particles follow the same rule as in the usual open TASEP. This is a model of, for example, a production line with a material inventory.

The state space of the Markov process encoding the model is $\mathbb{Z}_{\geq 0} \times \{0, 1\}^L$. The master equation governing the probability $P(N, \tau_1, \ldots, \tau_L)$ of finding the configuration $(N, \tau_1, \ldots, \tau_L)$ is

$$\frac{d}{dt} P(N, \tau_1, \ldots, \tau_L) = \alpha (1 - \delta_{N0}) P(N - 1, \tau_1, \ldots, \tau_L) - \alpha P(N, \tau_1, \ldots, \tau_L) + \alpha' \tau_1 P(N + 1, 0, \tau_2, \ldots, 1, 0, \ldots, \tau_L) - \alpha' (1 - \tau_1) P(N, 0, \tau_2, \ldots, 1, 0, \ldots, \tau_L) + p \sum_{j=1}^{L-1} (\tau_{j+1} - \tau_j) P(N, \tau_1, \ldots, \tau_{j-1}, 0, \tau_{j+2}, \ldots, \tau_L) + \beta (1 - 2 \tau_L) P(N, \tau_1, \ldots, \tau_{L-1}, 1).$$

(80)

For example, with $L = 4$,

$$\frac{d}{dt} P(0, 1, 1, 0, 1) = \alpha' P(1, 0, 1, 0, 1) + p P(0, 1, 1, 0, 1) - (\alpha + p + \beta) P(0, 1, 1, 0, 1),$$

(81)

$$\frac{d}{dt} P(5, 0, 1, 0, 0) = \alpha P(4, 0, 1, 0, 0) + p P(5, 1, 0, 0, 0) + \beta P(5, 0, 1, 0, 1) - (\alpha + \alpha' + p) P(5, 0, 1, 0, 0).$$

(82)

It is difficult to find an exact stationary state of this model. One can expect, however, the critical line separating the parameter space to take the form

$$\alpha = J_L(\alpha', \beta, p),$$

(83)

because

- In a stationary state, the current must be $\alpha$.
- If there is no stationary state and the queue part continues to grow, the TASEP part can be regarded as the open TASEP with a particle reservoir in the left end. Thus, the current must be $J_L(\alpha', \beta, p)$ in this case.
- These two values should be equal on the critical line.

In fact, the critical lines for $L = 1, 2, 3, 4$ with $\alpha' = p$ were calculated (although not rigorously) and found to agree with (83) [11].
Acknowledgments

The author thanks R. Jian, R. Nishi, K. Nishinari, S. Saito and T. Shirai for fruitful discussion. He is also grateful to M. Hay for his critical reading of the manuscript. This work is supported by Global COE Program “Education and Research Hub for Mathematics-for-Industry.”

[1] P. Robert, *Stochastic Network and Queues*, Springer (1999)
[2] T. M. Liggett: *Stochastic Interacting Systems: Contact, Voter and Exclusion Processes*, Springer (1999)
[3] G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, *Phase transitions and critical phenomena*, Vol. 19, C. Domb and J. Lebowitz eds., Academic (2001)
[4] B. Derrida, M. R. Evans, V. Hakim and V. Pasquier, J. Phys. A 26, 1493 (1993)
[5] R. A. Blythe and M. R. Evans, J. Phys. A 40, R333 (2007)
[6] A. B. Kolomeisky, G. M. Schütz, E. B. Kolomeisky and J. P. Staley, J. Phys. A 31, 6911 (1998)
[7] C. Arita and D. Yanagisawa (in progress)
[8] T. Sasamoto, J. Phys. A 32, 7109 (1999)
[9] R. A. Blythe, M. R. Evans, F. Colaiori and F. H. L. Essler, J. Phys. A 33, 2313 (2000)
[10] T. Sasamoto, S. Mori and M. Wadati, J. Phys. Soc. Jpn. 65 2000 (1996)
[11] C. Arita, RIAM Symposium No. 20 ME-S7 Mathematics and Physics in Nonlinear waves, 165 (2009) (This article was written in Japanese but readers can follow the calculation in it. URL: https://qir.kyushu-u.ac.jp/dspace/bitstream/2324/14300/1/Article_No28.pdf)

APPENDIX A: EXAMPLE

The stationary probabilities (21) for some configurations are listed here:

\[ Z_P(1,0) = \frac{\alpha^2}{\beta^3}, \quad Z_P(1,1) = \frac{\alpha^2}{\beta^2}, \quad Z_P(1,0,0) = \frac{\alpha^3}{\beta^3}, \]
\[ Z_P(1,0,1) = \frac{\alpha^3}{\beta^2}, \quad Z_P(1,1,0) = \frac{\alpha^3(p+\beta)}{p^2\beta^2}, \]
\[ Z_P(1,1,1) = \frac{\alpha^3}{\beta^3}, \quad Z_P(1,0,0,0) = \frac{\alpha^4}{p^3\beta^3}, \]
\[ Z_P(1,0,0,1) = \frac{\alpha^4}{p^3\beta^3}, \quad Z_P(1,0,1,0) = \frac{\alpha^4(p+\beta)}{p^3\beta^2}, \]
\[ Z_P(1,0,1,1) = \frac{\alpha^4}{p^3\beta^3}, \quad Z_P(1,1,0,0) = \frac{\alpha^4(p+2\beta)}{p^3\beta^2}, \]
\[ Z_P(1,1,0,1) = \frac{\alpha^4(p+\beta)}{p^2\beta^3}, \quad Z_P(1,1,1,0) = \frac{\alpha^4}{p^3\beta^3}, \]
\[ Z_P(1,1,1,1) = \frac{\alpha^4(p+\beta)}{p^3\beta^3}. \]

where \( Z = Z(\alpha, \beta, p) \). These were calculated by using the algebraic relation (9).
List of MI Preprint Series, Kyushu University
The Global COE Program
Math-for-Industry Education & Research Hub

MI

MI2008-1 Takahiro ITO, Shuichi INOKUCHI & Yoshihiro MIZOGUCHI
Abstract collision systems simulated by cellular automata

MI2008-2 Eiji ONODERA
The initial value problem for a third-order dispersive flow into compact almost Hermitian manifolds

MI2008-3 Hiroaki KIDO
On isosceles sets in the 4-dimensional Euclidean space

MI2008-4 Hirofumi NOTSU
Numerical computations of cavity flow problems by a pressure stabilized characteristic-curve finite element scheme

MI2008-5 Yoshiyasu OZEKI
Torsion points of abelian varieties with values in infinite extensions over a p-adic field

MI2008-6 Yoshiyuki TOMIYAMA
Lifting Galois representations over arbitrary number fields

MI2008-7 Takehiro HIROTsu & Setsuo TANIGUCHI
The random walk model revisited

MI2008-8 Silvia GANDY, Masaaki KANNO, Hirokazu ANAI & Kazuhiro YOKOYAMA
Optimizing a particular real root of a polynomial by a special cylindrical algebraic decomposition

MI2008-9 Kazufumi KIMOTO, Sho MATSUMOTO & Masato WAKAYAMA
Alpha-determinant cyclic modules and Jacobi polynomials
MI2008-10 Sangyeol LEE & Hiroki MASUDA
Jarque-Bera Normality Test for the Driving Lévy Process of a Discretely Observed Univariate SDE

MI2008-11 Hiroyuki CHIHARA & Eiji ONODERA
A third order dispersive flow for closed curves into almost Hermitian manifolds

MI2008-12 Takehiko KINOSHITA, Kouji HASHIMOTO and Mitsuhiro T. NAKAO
On the $L^2$ a priori error estimates to the finite element solution of elliptic problems with singular adjoint operator

MI2008-13 Jacques FARAUT and Masato WAKAYAMA
Hermitian symmetric spaces of tube type and multivariate Meixner-Pollaczek polynomials

MI2008-14 Takashi NAKAMURA
Riemann zeta-values, Euler polynomials and the best constant of Sobolev inequality

MI2008-15 Takashi NAKAMURA
Some topics related to Hurwitz-Lerch zeta functions

MI2009-1 Yasuhide FUKUMOTO
Global time evolution of viscous vortex rings

MI2009-2 Hidetoshi MATSUI & Sadanori KONISHI
Regularized functional regression modeling for functional response and predictors

MI2009-3 Hidetoshi MATSUI & Sadanori KONISHI
Variable selection for functional regression model via the $L_1$ regularization

MI2009-4 Shuichi KAWANO & Sadanori KONISHI
Nonlinear logistic discrimination via regularized Gaussian basis expansions

MI2009-5 Toshiro HIRANOUCHI & Yuichiro TAGUCHII
Flat modules and Groebner bases over truncated discrete valuation rings
MI2009-6 Kenji KAJIWARA & Yasuhiro OHTA
Bilinearization and Casorati determinant solutions to non-autonomous 1+1 dimensional discrete soliton equations

MI2009-7 Yoshiyuki KAGEI
Asymptotic behavior of solutions of the compressible Navier-Stokes equation around the plane Couette flow

MI2009-8 Shohei TATEISHI, Hidetoshi MATSUI & Sadanori KONISHI
Nonlinear regression modeling via the lasso-type regularization

MI2009-9 Takeshi TAKAISHI & Masato KIMURA
Phase field model for mode III crack growth in two dimensional elasticity

MI2009-10 Shingo SAITO
Generalisation of Mack’s formula for claims reserving with arbitrary exponents for the variance assumption

MI2009-11 Kenji KAJIWARA, Masanobu KANEKO, Atsushi NOBE & Teruhisa TSUDA
Ultradiscretization of a solvable two-dimensional chaotic map associated with the Hesse cubic curve

MI2009-12 Tetsu MASUDA
Hypergeometric \( \mathcal{H} \)-functions of the q-Painlevé system of type \( E_8^{(1)} \)

MI2009-13 Hidenao IWANE, Hitoshi YANAMI, Hirokazu ANAI & Kazuhiro YOKOYAMA
A Practical Implementation of a Symbolic-Numeric Cylindrical Algebraic Decomposition for Quantifier Elimination

MI2009-14 Yasunori MAEKAWA
On Gaussian decay estimates of solutions to some linear elliptic equations and its applications

MI2009-15 Yuya ISHIHARA & Yoshiyuki KAGEI
Large time behavior of the semigroup on \( L^p \) spaces associated with the linearized compressible Navier-Stokes equation in a cylindrical domain
MI2009-16  Chikashi ARITA, Atsuo KUNIBA, Kazumitsu SAKAI & Tsuyoshi SAWABE
Spectrum in multi-species asymmetric simple exclusion process on a ring

MI2009-17  Masato WAKAYAMA & Keitaro YAMAMOTO
Non-linear algebraic differential equations satisfied by certain family of elliptic functions

MI2009-18  Me Me NAING & Yasuhide FUKUMOTO
Local Instability of an Elliptical Flow Subjected to a Coriolis Force

MI2009-19  Mitsunori KAYANO & Sadanori KONISHI
Sparse functional principal component analysis via regularized basis expansions and its application

MI2009-20  Shuichi KAWANO & Sadanori KONISHI
Semi-supervised logistic discrimination via regularized Gaussian basis expansions

MI2009-21  Hiroshi YOSHIDA, Yoshihiro MIWA & Masanobu KANEKO
Elliptic curves and Fibonacci numbers arising from Lindenmayer system with symbolic computations

MI2009-22  Eiji ONODERA
A remark on the global existence of a third order dispersive flow into locally Hermitian symmetric spaces

MI2009-23  Stjepan LUGOMER & Yasuhide FUKUMOTO
Generation of ribbons, helicoids and complex scherk surface in laser-matter Interactions

MI2009-24  Yu KAWAKAMI
Recent progress in value distribution of the hyperbolic Gauss map

MI2009-25  Takehiko KINOSHITA & Mitsuhiro T. NAKAO
On very accurate enclosure of the optimal constant in the a priori error estimates for $H^2_0$-projection
MI2009-26 Manabu YOSHIDA
   Ramification of local fields and Fontaine’s property (Pm)

MI2009-27 Yu KAWAKAMI
   Value distribution of the hyperbolic Gauss maps for flat fronts in hyperbolic
   three-space

MI2009-28 Masahisa TABATA
   Numerical simulation of fluid movement in an hourglass by an energy-stable
   finite element scheme

MI2009-29 Yoshiyuki KAGEI & Yasunori MAEKAWA
   Asymptotic behaviors of solutions to evolution equations in the presence of
   translation and scaling invariance

MI2009-30 Yoshiyuki KAGEI & Yasunori MAEKAWA
   On asymptotic behaviors of solutions to parabolic systems modelling chemotaxis

MI2009-31 Masato WAKAYAMA & Yoshinori YAMASAKI
   Hecke’s zeros and higher depth determinants

MI2009-32 Olivier PIRONNEAU & Masahisa TABATA
   Stability and convergence of a Galerkin-characteristics finite element scheme
   of lumped mass type

MI2009-33 Chikashi ARITA
   Queueing process with excluded-volume effect