Warps and grids for double and triple vector bundles

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Abstract

A triple vector bundle is a cube of vector bundle structures which commute in the (strict) categorical sense. A grid in a triple vector bundle is a collection of sections of each bundle structure with certain linearity properties. A grid provides two routes around each face of the triple vector bundle, and six routes from the base manifold to the total manifold; the warps measure the lack of commutativity of these routes.

In this paper we first prove that the sum of the warps in a triple vector bundle is zero. The proof we give is intrinsic and, we believe, clearer than the proof using decompositions given earlier by one of us. We apply this result to the triple tangent bundle $T^3 M$ of a manifold and deduce (as earlier) the Jacobi identity.

We further apply the result to the triple vector bundle $T^2 A$ for a vector bundle $A$ using a connection in $A$ to define a grid in $T^2 A$. In this case the curvature emerges from the warp theorem.
1 Introduction

1.1 Double vector bundles, grids and warps

Double vector bundles arise naturally in Poisson geometry, in the connection theory of vector bundles, and generally in the study of geometric objects with two compatible structures. Double vector bundles have been “floating around” since at least Dieudonné’s treatment of connection theory, but the first systematic and general treatment was provided by Pradines. A recent account with references is [10, Chap. 9]. We briefly recall the necessary facts.

A double vector bundle is a square of vector bundles as shown in the first figure of (1). There are two vector bundle structures on $D$, with bases $A$ and $B$, each of which is itself a vector bundle on base $M$; the two structures on $D$ commute in the categorical sense (see below), and the map $D \rightarrow A \times_M B$ formed by the two bundle projections is a surjective submersion.

The second and third figures in (1) show two standard examples arising from an arbitrary vector bundle $A$. If $A$ has a Poisson structure then it is linear if and only if the associated map $T^*A \rightarrow TA$ is a morphism of double vector bundles (in an obvious sense) of the structures above. When this is so, $A$ is the dual of a Lie algebroid [3]. The third structure was introduced in global form in [13]. We give more details on this double vector bundle in Example 4. Double vector bundles also arise in the Lie theory of double Lie groupoids [11]; we will not consider this theory here.

Each element $d$ of a double vector bundle $D$ may be represented in outline by the diagram in (2) which shows the projections of $d$ under the two bundle projections. Given another element $d'$ as shown, the sum over $A$ has the outline shown in the third figure.

The statement that the two vector bundle structures on $D$ ‘commute in the categorical sense’ implies for the additions that

$$ (d_1 + d_2) + (d_3 + d_4) = (d_1 + d_3) + (d_2 + d_4), \quad (3) $$

where $(d_i; a_i, b_i; m)$, $i = 1, \ldots, 4$, have $a_1 = a_2$, $a_3 = a_4$, $b_1 = b_3$ and $b_2 = b_4$. There are similar conditions involving the scalar multiplications.

It follows that for elements $d$ which project to zeros under both bundle projections, the two additions, and the scalar multiplications, coincide. Under these operations the set of such elements forms a vector bundle over $M$, called the core of $D$ [13], usually denoted $C$.

The core $C$ is a submanifold of $D$; every element of $C$ is an element of $D$. When working with examples, the core can usually be identified with a familiar vector bundle and it can be important
to distinguish between elements of this bundle and the corresponding element of the double vector bundle. For example, the core of the double vector bundle $TA$, the middle diagram in \[1\], can be identified with $A$ itself, \[10.9.1.7\]. Therefore, an element $a$ of the core $A$, can be viewed either as an element of $A$, or as an element of $TA$. In the latter case, we denote it by $\mathfrak{r} \in TA$. For general double vector bundles and triple vector bundles, this distinction is usually not necessary, so in sections 2, 3, and 4 we will not write bars over core elements. This distinction will be made clearly in section 5.

Now suppose that $(d; a, b; m)$ and $(d'; a', b'; m)$ have $a = a'$ and $b = b'$. Then there is a unique $c \in C$ such that

$$d = d' + (c + \tilde{0}_a^A) = d' + (c + \tilde{0}_b^B).$$

(4)

In equations of this type, what is usually important is that $d - d'$, calculated in either structure, gives the core element $c$ plus an appropriate zero. We will indicate this by $d - d' \triangleright c$. We use this notation from subsection 2.2 onwards.

We now describe the original motivating example for the concepts of grid and warp.

In the 1988 edition of their book \[1\] p.297, Abraham, Marsden and Raţiu gave the following formula for the Lie bracket of vector fields $X$ and $Y$ on a manifold $M$,

$$T(Y)(X(m)) - \tilde{X}(Y(m)) = ([X, Y](m))^\uparrow(Y(m)),$$

(5)

where $\tilde{X}$ is the complete lift of $X$ to a vector field on $TM$ and the uparrow denotes the vertical lift to $TM$ of the vector $[X, Y](m)$ to $Y(m)$. The complete lift, or tangent lift, $\tilde{X}$ is $J \circ T(X)$ where $J: T^2M \rightarrow T^2M$ is the canonical involution which interchanges the two bundle structures on $T^2M$.

The double vector bundle $T^2M$ is a special case of the middle diagram of \[1\], where $A = TM$, and its core vector bundle is yet a third copy of $TM$. The left hand side of (5) is encapsulated in (6).

If we look at the elements $T(Y)(X(m))$ and $\tilde{X}(Y(m))$, we see that they have the same outlines

$$T(Y)(X(m)) \rightarrow X(m) \quad \tilde{X}(Y(m)) \leftarrow X(m)$$

$$Y(m) \leftarrow m, \quad Y(m) \rightarrow m.$$

The two elements therefore determine a core element $c \in TM$. Taking $d = T(Y)(X(m))$ and $d' = \tilde{X}(Y(m))$ in \[4\], we have

$$T(Y)(X(m)) - \tilde{X}(Y(m)) = \mathfrak{r} + \tilde{0}_{Y(m)}^{T(p)}.$$
where the subtraction on the left is the usual subtraction of vectors which are tangent to $TM$ at $Y(m)$, and the addition on the right is addition in $T(p)$: $T^2M \to TM$. That is, $\tau + \tilde{0}_{Y(m)}$ is the vertical lift of $c$ to $Y(m)$ and so, by (5), $c = [X, Y](m)$.

A comment on the notation of the last equation. In the case of a general double vector bundle $D$, the two additions $+_A$ and $+_B$ are distinct. In the case of $T^2M$ however, both side bundles are copies of $TM$. To distinguish between the two additions, we use the projection maps, for example, addition in $T^2M \to TM$ will be denoted by $+_T$. We adopt this notation whenever necessary, especially in sections 5 and 6.

Note that (5) needs to be proved in local coordinates, or in terms of the action of vector fields on functions. The use of (6) expresses the result in a compact conceptual way.

We now express these results in the terms which will be used throughout the paper. Consider a double vector bundle $D$ as in (1).

**Definition 1.** A pair of sections $X \in \Gamma A$ and $\xi \in \Gamma_B D$ form a linear section of $D$ if $\xi$ is a morphism of vector bundles over $X$.

A grid on $D$ is a pair of linear sections $(\xi, X)$ and $(\eta, Y)$ as shown in (7).

For each $m \in M$, $\xi(Y(m))$ and $\eta(X(m))$ have the same outline. They therefore determine an element of the core $C$ and, as $m$ varies, a section of $C$ which we denote $w(\xi, \eta)$. Precisely,

$$\xi(Y(m)) - \eta(X(m)) = w(\xi, \eta)(m) + \tilde{0}_{X(m)}, \quad \xi(Y(m)) - \eta(Y(m)) = w(\xi, \eta)(m) + \tilde{0}_{Y(m)}.$$ (8)

**Definition 2.** The warp of the grid consisting of $(\xi, X)$ and $(\eta, Y)$ is $w(\xi, \eta) \in \Gamma C$.

Note that $w(\xi, \eta)$ changes sign if $\xi$ and $\eta$ are interchanged. Our convention gives the positive sign to the counterclockwise composition $\xi \circ Y$.

The question of signs — or orientations — will haunt us throughout the paper. Later on, we will see that there are various rules that, in many cases, determine which difference to take as the positive warp. These rules generally follow from established conventions of differential geometry.

Equation (5) can now be expressed as saying that the warp of (6) is $[X, Y]$.

Before proceeding, we give two further examples of grids and warps in double vector bundles, and an alternative formula.

**Example 3.** Consider the double vector bundle $TA$, the middle diagram in (1), where $(A, q, M)$ is a vector bundle, and let $\nabla$ be a connection in $A$.

For a vector field $Z$ on $M$, denote by $Z^H$ the horizontal lift of $Z$ to $A$; the word ‘horizontal’ here has its standard meaning in connection theory, and does not refer to the structures in $TA$. Then $Z^H$ is a linear vector field over $Z$. 
Take any \( \mu \in \Gamma A \) and form the grid shown in (10). Then the warp of the grid is \( \nabla_Z \mu \); that is, for \( m \in M \),
\[
T(\mu)(Z(m)) - Z^H(\mu(m)) = ((\nabla_Z \mu)(m))^\uparrow(\mu(m)),
\]
where the right hand side is the vertical lift of \( (\nabla_Z \mu)(m) \in T_m M \) to \( T_{\mu(m)}A \).

For details see [10, §3.4]. This example is central to §5.

Some historical remarks may be in order.

The most usual global language for working with connections in vector bundles is that of covariant derivatives, nowadays with the \( \nabla \) notation. This formulation goes back to early work on surfaces and Riemannian geometry. It was formalized as a general abstract concept by Koszul in lectures given in 1960 [8].

For connections in principal bundles a similarly convenient formulation is not available. Kobayashi and Nomizu [7] gave two global definitions of a connection in a principal bundle \( P(M,G) \): as a suitable \( g \)-valued 1-form on \( P \) and as an invariant horizontal distribution on \( P \). The latter defines, and is equivalent to, a lifting of vector fields on \( M \) to invariant horizontal vector fields on \( P \).

In the 1970s this lifting formulation was applied to connections in vector bundles. In modern language this approach defines a connection in \( (A,q,M) \) to be a map \( \mathfrak{X}(M) \to \mathfrak{X}(A) \), \( Z \mapsto Z^H \), which preserves addition, which sends \( fZ \), for \( f \in C^\infty(M) \), to \( (f \circ q)Z^H \), and which is such that \( Z^H \) is a linear vector field over \( Z \).

Dieudonné [4, XVII.16] and others expressed such a lifting in terms of a suitably linear right-inverse to the map \( TA \to TM \times_M A \) which combines the two projections. Such a map also defines a corresponding left-inverse \( TA \to A \) into the core, and this is the formulation which Besse [2] used. Equation (9) may be discerned on page 38 of [2] and is a special case of (17.17.2.1) in [4].

The important fact is that the usual definition of a connection in a vector bundle as a covariant derivative is equivalent to a suitable lifting of vector fields from the base to the total space. We will comment on curvature in §5.

**Example 4.** Consider the double vector bundle \( T^*A \), the third diagram in (1), where \( (A,q,M) \) is a vector bundle.

Given a section \( \varphi \in \Gamma A^* \), denote by \( \ell_\varphi \) the corresponding linear function \( A \to \mathbb{R} \). Then the 1-form \( d\ell_\varphi \) is a linear section over \( \varphi \).

Likewise given \( \mu \in \Gamma A \), we obtain a 1-form \( d\ell_\mu \) on \( A^* \). Composing with the canonical diffeomorphism \( R: T^*(A^*) \to T^*A \) [13, [10, §9.5], we obtain a linear section of \( T^*A \to A^* \) over \( \mu \).

It was proved in [13] that
\[
R(d\ell_\mu(\varphi(m))) - d\ell_\varphi(\mu(m)) = -q^*(d\langle \varphi, \mu \rangle)(\mu(m)).
\]
This shows that the warp of the grid consisting of \((d\ell_\phi, \phi)\) and \((R \circ (d\ell_\mu), \mu)\) is \(-d\langle\phi, \mu\rangle\).

An alternative formula for the warp of a grid is given in an appendix, §7.

### 1.2 Outline of the paper

In a previous paper [12] one of us used a grid in the triple vector bundle \(T^3M\) to express the Jacobi identity as a statement about the warps of the grids in the constituent double vector bundles. The proof given in that paper relied on a decomposition of \(T^3M\) into seven copies of \(TM\), and it was not clear whether the apparatus of grids and warps had provided a proof of the Jacobi identity or merely a formulation of it. One purpose of the present paper is to give an intrinsic proof of a general result for triple vector bundles and to resolve this question.

Section 2 reviews the basic setup and notation for triple vector bundles. In subsection 2.2 we formulate the main theorem of the paper on the warps of a grid in a triple vector bundle.

In a double vector bundle two elements with the same outline determine a core element (up to sign). In a triple vector bundle there are intermediate levels at which two elements may have the same outline, and there is more than one notion of core. Section 3 is concerned with describing the core elements determined by pairs of elements for which some levels of the outlines are equal.

Section 4 gives the proof of the warp theorem, Theorem 8. This states, roughly, that given a grid in a triple vector bundle \(E\), the sum of the ultrawarps is zero. The ultrawarps are the warps of the grids induced in the core double vector bundles by the grid on \(E\). The proof is intrinsic and does not rely on a decomposition of \(E\).

In section 5 we consider a vector bundle \(A\) and the triple vector bundle \(T^2A\). A connection in \(A\) induces a grid in \(T^2A\) and we show that the concept of curvature arises from the warp theorem.

Section 6 presents the example of \(T^3M\) and the deduction of the Jacobi identity from the warp theorem.

In an appendix, section 7, we show that the duality properties of double vector bundles give an alternative formula for the warp of a grid.

In a future paper we will consider in detail the application of the warp theorem to triple vector bundles with compatible bracket structures.

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2 Triple vector bundles and the warp theorem

In this section we do two things. First, we set up everything we need from triple vector bundles, in order to formulate the warp theorem (Theorem 8). Secondly, we describe the original formulation \cite{12} of the theorem, and outline the steps which lead to an intrinsic proof.

2.1 Basics on triple vector bundles

We review the basic structure of triple vector bundles from \cite{9,5,12}.

Definition 5. A \textit{triple vector bundle} is a cube of vector bundle structures, as in (11), such that each face is a double vector bundle, such that the vector bundle operations in the upper faces are morphisms of double vector bundles, and such that the decomposition condition described on page 8 below is satisfied.

\[
E_{1,2,3} \rightarrow E_{1,3} \\
E_{2,3} \rightarrow E_3 \\
E_{1,2} \rightarrow E_{1} \\
E_2 \rightarrow M.
\]  

(11)

In the rest of this section we work with a single triple vector bundle $E$.

By an \textit{upper face} we mean a face which has $E_{1,2,3}$ as total space. The \textit{lower faces} are the three faces which have $M$ as base manifold. We refer to the faces by the names

Back, Front, Left, Right, Up, Down.

The total space of a triple vector bundle should be denoted, for consistency with the labelling scheme, by $E_{1,2,3}$ but we will usually denote it by $E$.

How do we add elements in $E$? If $e, f \in E$ lie over the same point of $E_{2,3}$, their sum has the outline shown in (12).

\[
e \rightarrow e_{1,3} \\
e_{1,2} \rightarrow e_1 \\
edownarrow \quad \downarrow \quad \downarrow 2,3 \\
e_{2,3} \rightarrow e_3 \\
+ \quad \rightarrow \\
f \rightarrow f_{1,3} \\
f_{1,2} \rightarrow f_1 \\
2,3 \downarrow \quad \downarrow \quad \downarrow \\
e_{2,3} \rightarrow e_3 \\
e + f 2,3 \rightarrow e_{1,3} + f_{1,3} \\
e_{1,2} + f_{1,2} \rightarrow e_1 + f_1 \\
E_{2} \rightarrow E_{3} \\
E_{2} \rightarrow M.
\]  

(12)

The outlines for scalar multiplication are similar.
Warps and grids . . .

Core double vector bundles and the ultracore

Since each face of $E$ is a double vector bundle, each face has a core vector bundle.

The cores of the lower faces $E_{i,j}$ are denoted $E_{ij}$ with the comma removed. The core of the upper face with base manifold $E_k$ is denoted $E_{ij,k}$. (This convention comes from [6].)

Focus on the core vector bundles of the Up and of the Down faces. The Up face projects to the Down face via the double vector bundle morphism which consists of the bundle projections $E_{1,2,3} \rightarrow E_{1,2}$, $E_{2,3} \rightarrow E_2$, $E_{1,3} \rightarrow E_1$ and $E_3 \rightarrow M$. The restriction of $E_{1,2,3} \rightarrow E_{1,2}$ to $E_{12,3}$ goes into $E_{12}$ and inherits the vector bundle structure of $E_{1,2,3} \rightarrow E_{1,2}$. Together with the vector bundle structures on the cores of the Up face and the Down face, this yields another double vector bundle, with total space $E_{12,3}$, which we call the (U-D) core double vector bundle.

Of course this can also be done for the other two pairs of parallel faces. So there are three core double vector bundles, shown in (13).

\[
\begin{align*}
E_{23,1} & \rightarrow E_{23} \\
E_{13,2} & \rightarrow E_{13} \\
E_{12,3} & \rightarrow E_{12} \\
E_1 & \rightarrow M, \\
E_2 & \rightarrow M, \\
E_3 & \rightarrow M.
\end{align*}
\]

Elements of the core of $E_{12,3}$ project to zeros in the Down face. In the Up face they project to zeros over the zero in $E_3$. It follows that an element of the core of $E_{12,3}$ projects to zero in every bundle structure. Equally the cores of the (B-F) and (L-R) double vector bundles consist of the elements of $E_{1,2,3}$ which project to zeros in every bundle structure. Thus each double vector bundle in (13) has the same core. This is denoted $E_{123}$ (without commas) and called the ultracore of $E$.

From the interchange laws it follows that the three additions on $E$, namely $+_{1,2}$, $+_{1,3}$, and $+_{2,3}$, coincide on the ultracore and give it the structure of a vector bundle over $M$.

We can now state the decomposition condition needed in Definition 5.

First consider a double vector bundle $D$ as in (1), with core $C$. From $A$, $B$ and $C$ it is possible to define a double vector bundle structure on the pullback manifold $\overline{D} = A \times_M B \times_M C$ by defining $\overline{D} \rightarrow A$ to be the pullback of $B \oplus C$ across $q_A$: $A \rightarrow M$ and $\overline{D} \rightarrow B$ to be the pullback of $A \oplus C$ across $q_B$: $B \rightarrow M$. This double vector bundle has the same side bundles and same core as $D$, and the condition that the map $D \rightarrow A \times_M B$ be a surjective submersion implies that there are isomorphisms of double vector bundles from $D$ to $\overline{D}$ which are the identity on $A$, $B$ and $C$.

Now given a triple vector bundle $E$, it is possible to build a triple vector bundle $\overline{E}$ from $E_1$, $E_2$, $E_3$, $E_{12}$, $E_{23}$, $E_{13}$ and $E_{123}$ by taking pullbacks of Whitney sums in a similar way. The decomposition condition referred to in Definition 5 is that there exists an isomorphism of triple vector bundles from $E$ to $\overline{E}$ which is the identity on the seven bundles listed above. See [5] for full details.

This condition ensures that in a triple vector bundle $E$ (nontrival) grids, as defined below always exist.

Notation for zero sections

The zero section of $E_1$ is denoted by $0^{E_1}: M \rightarrow E_1$, $m \mapsto 0^{E_1}_m$, with similar notations for $E_2$ and $E_3$. 
The zero section of $E_{1,2} \rightarrow E_1$ is denoted by $\tilde{0}^{1,2} : E_1 \rightarrow E_{1,2}$, $e_1 \mapsto \tilde{0}^{1,2}_{e_1}$. The double zero of $E_{1,2}$ is denoted by $\odot_{m}^{1,2}$, with similar notations for the other vector bundle structures.

The zero section of $E \rightarrow E_{1,2}$ is denoted by $\tilde{0} : E_{1,2} \rightarrow E$, $e_{1,2} \mapsto \tilde{0}_{e_{1,2}}$. Note that the subscripts of the element $e_{1,2}$ are enough to indicate that this is the zero section of $E$ over $E_{1,2}$; there is no need for superscripts on $\tilde{0}$.

Finally, the triple zero of $E$ is denoted by $\odot_{m}^{3}$. This is the zero of the ultracore vector bundle.

**Grids in triple vector bundles**

A grid in a double vector bundle constitutes two linear sections. In a triple vector bundle the concept of grid requires what we call linear double sections.

**Definition 6.** A *down-up linear double section of $E$* is a collection of sections

$$Z_{1,2} : E_{1,2} \rightarrow E_{1,2,3}, \quad Z_1 : E_1 \rightarrow E_{1,3}, \quad Z_2 : E_2 \rightarrow E_{2,3}, \quad Z : M \rightarrow E_3,$$

which form a morphism of double vector bundles from the Down face to the Up face.

The core morphism of $Z_{1,2}$ defines a vector bundle morphism from the core of the Down face to the core of the Up face. We denote this by $Z_{12} : E_{12} \rightarrow E_{12,3}$. It is a linear section over $Z : M \rightarrow E_3$.

In a similar fashion we define *right-left* and *front-back linear double sections* of $E$.

**Definition 7.** A *grid on $E$* is a set of three linear double sections, one in each direction, as shown in (14).

That nontrivial grids always exist follows from the decomposability condition.

We note the following equations for future reference. They follow from the fact that the double sections are morphisms of double vector bundles.

For $e_{1,2}, e'_{1,2}$ over the same point of $E_1$,

$$Z_{1,2}(e_{1,2} - e'_{1,2}) = Z_{1,2}(e_{1,2}) - Z_{1,2}(e'_{1,2}). \quad (15)$$

For $e_{1,2}, e'_{1,2}$ over the same point of $E_2$,

$$Z_{1,2}(e_{1,2} - e'_{1,2}) = Z_{1,2}(e_{1,2}) - Z_{1,2}(e'_{1,2}). \quad (16)$$
2.2 Formulation of the warp theorem

We now have everything we need in order to describe the original formulation of the theorem, as given in [12].

Start with a grid on $E$. Focus on the Up face of the triple vector bundle. We see that $(Y_{13}, Y_3)$ and $(X_{23}, X_3)$ define a grid on the Up face. Denote its warp by $w_{up}$. This is a section of the core vector bundle of the Up face, that is, $w_{up} : E_3 \to E_{12,3}$. Similarly for the Down face, its warp $w_{down}$ is a section of the core vector bundle of the Down face, so $w_{down} : M \to E_{12}$. It follows that $(w_{up}, w_{down})$ is a linear section of the (U-D) core double vector bundle. Recall that the core morphism $Z_{12}$ of the linear double section $Z_{1,2}$ defines another linear section of the (U-D) core double vector bundle. Therefore, we have the following grid on $E_{12,3}$:

We call the warp of this grid the *Up-Down ultrawarp* and denote it by $u_{UD}$. It is a section of the ultracore $E_{123}$.

Of course we can also build corresponding grids on the other two core double vector bundles. We therefore have three ultrawarps, as shown in (17).

We take the ultrawarps with the orientations opposite to (17); that is, using a rough notation,

$$u_{BF} := w_{back} \circ X - X_{23} \circ w_{front}, \quad u_{LR} := w_{left} \circ Y - Y_{13} \circ w_{right}, \quad u_{UD} := w_{up} \circ Z - Z_{12} \circ w_{down}. \quad (18)$$

We can now state the main theorem about grids in triple vector bundles.

**Theorem 8** (Warp Theorem). *Given a triple vector bundle $E$ and a grid in $E$ as in (14),

$$u_{BF} + u_{LR} + u_{UD} = 0. \quad (19)$$

To give an intrinsic proof, we need to describe the ultrawarps in an alternative way.

So far, the only equation we have seen that describes the warp of a grid on a double vector bundle is (8). Focus on the ultrawarp $u_{UD}$. From the grid on the (U-D) core double vector bundle, for $m \in M$, by (8) we have that

$$(w_{up} \circ Z)(m) - (Z_{12} \circ w_{down})(m) = \hat{0}_{Z(m)} + u_{UD}(m). \quad (20)$$
How can we express \((w_{up} \circ Z)(m)\) and \(Z_{12}(w_{down}(m))\) in a more useful way? About \(w_{up}\), for any \(e_3 \in E_3\), again from \([8]\) we have that

\[
Y_{1,3}(X_3(e_3))_{1,3} X_{2,3}(Y_3(e_3)) = \hat{0}_{X_3(e_3)} + w_{up}(e_3).
\]

Putting \(e_3 = Z(m)\), we have

\[
Y_{1,3}(X_3(Z(m)))_{1,3} X_{2,3}(Y_3(Z(m))) = \hat{0}_{X_3(Z(m))} + w_{up}(Z(m)). \tag{21}
\]

We introduce a more succinct notation, for use in calculations.

\[
\begin{align*}
Z_{XY} &= Z_{1,2}(Y_1(X(m))), && Y_{ZX} = Y_{1,3}(Z_1(X(m))), && XZ = X_{2,3}(Z_2(Y(m))),
Z_{YX} &= Z_{1,2}(X_2(Y(m))), && YX = Y_{1,3}(X_3(Z(m))), && XY = X_{2,3}(Y_3(Z(m))).
\end{align*}
\]

Now \((21)\) becomes

\[
YXZ - XYZ = \hat{0}_{e_{1,3}} + \lambda_3, \tag{23}
\]

where \(e_{1,3}' = X_3(Z(m))\) and \(\lambda_3 = w_{up}(Z(m))\). Note the following. In the case of a double vector bundle, we can rewrite \([8]\) as

\[
w(\xi, \eta)(m) = (\xi \circ Y)(m) - (\eta \circ X)(m) - 0_{X(m)}^B.
\]

In the case of a triple vector bundle, this is also possible. If we tried a similar calculation on \((23)\), since \(\hat{0}_{e_{1,3}'} \neq 0_{e_{1,3}}\), we would have

\[
(YXZ - XYZ)_{2,3} \hat{0}_{e_{1,3}'} = \hat{0}_{e_3} + \lambda_3.
\]

Since \(\hat{0}_{e_3}\) plays the role of the double zero of the Up face, over \(e_3\), we have \(\hat{0}_{e_3} + \lambda_3 = \lambda_3\). So in total, we can rewrite \((23)\) as

\[
\lambda_3 = (YXZ - XYZ)_{2,3} \hat{0}_{e_{1,3}'}^{e_1}. \tag{24}
\]

About \((Z_{1,2} \circ w_{down})(m)\), first write \(w_{down}(m)\) out using \([8]\) as

\[
Y_1(X(m))_{E_1} X_2(Y(m)) = \hat{0}_{X(m)}^1 + w_{down}(m).
\]

Apply \(Z_{1,2}\) to this, and using \((15)\) and \((16)\), it follows that

\[
Z_{1,2}(Y_1(X(m)))_{1,3} Z_{1,2}(X_2(Y(m))) = \hat{0}_{Z_1(X(m))} + Z_{12}(w_{down}(m))
\]

Again, for reasons of economy of space, rewrite this as

\[
Z_{XY} - Z_{XY} = \hat{0}_{e_{1,3}} + k_3,
\]

where \(e_{1,3} = Z_1(X(m))\) and \(k_3 = Z_{12}(w_{down}(m))\). Alternatively, as we did for \(\lambda_3\)

\[
k_3 = (Z_{XY} - Z_{XY})_{2,3} \hat{0}_{e_{1,3}}^{e_1}. \tag{25}
\]

Let us go back to \((20)\). We can rewrite this as

\[
\lambda_3 = \hat{0}_{e_3} + w_{BD}(m),
\]
and using (24) and (25), we have that
\[ \left( \frac{YXZ - XYZ}{2,3} - \hat{0}_{1,3} \right) \frac{ZXY - ZYX}{2,3} = \hat{0}_{2,3} + u_{UD}(m) \]
or, more elegantly, using interchange laws,
\[
\left( \frac{YXZ - XYZ}{2,3} - \hat{0}_{1,3} \right) \frac{ZXY - ZYX}{2,3} = \left( \frac{ZXY - ZYX}{2,3} - \hat{0}_{1,3} \right) + \left( \frac{ZXY - ZYX}{2,3} - \hat{0}_{1,3} \right),
\]
\[ \hat{0}_{1,3} + \hat{0}_{1,3} + u_{UD}(m). \] (26)

In calculations it is generally preferable to use equations of the form (23), and to avoid equations of the form (24).

Therefore, in order to describe ultrawarps such as \( u_{UD}(m) \), we will use equations of the form (26) and we will often use the abbreviated notation
\[ (YXZ - XYZ) - (ZXY - ZYX) \triangleright u_{UD}(m), \]
introduced after (4).

It is worth emphasizing that the above arguments rely on the fact that core and ultracore elements are uniquely determined by equations such as (8).

There are similar abbreviated equations for the other two ultrawarps. Altogether we have
\[
\begin{align*}
(YXZ - YZX) - (XZ - XYZ) & \triangleright u_{BF}(m), \quad (27a) \\
(XZ - ZXY) - (YXZ - YZX) & \triangleright u_{LR}(m), \quad (27b) \\
(YXZ - XYZ) - (ZXY - ZYX) & \triangleright u_{UD}(m), \quad (27c)
\end{align*}
\]
and from now on we will use a further shortening of the notation
\[ u_{BF}(m) = u_1, \quad u_{LR}(m) = u_2, \quad u_{UD}(m) = u_3. \]

The main difficulty in proving (19) is that we cannot simply add and subtract the expressions in (27), since the operations are over different vector bundle structures. The apparatus of the next section overcomes this difficulty.

**Orientation**

A further problem arises from the fact that the warp of a grid in a double vector bundle is only defined up to sign. We now need to consider how to choose these signs consistently for a grid in a triple vector bundle. This is a question of fixing orientations.

We choose to orient each upper face so that the positive term in the formula for the warp defines the outward normal by the right-hand rule. We then take the positive and negative terms in the opposite lower face to match those in the upper face; that is, we orient the lower faces so that the positive term in the warp defines the inward normal.

Thus the orientation of the Up face determines the signs in the first subtraction in (27c) below and the orientation of the Down face determines the signs in the second subtraction.

The “middle subtractions” in (27), that is, the orientations of the core double vector bundles, is an independent choice, equivalent to the choice of signs in (18). What matters here is consistency: if we took all three warps with the opposite signs, that would also be fine.
3 Preliminaries for the proof of the warp theorem

This section contains the main technical work needed for the proof of the warp theorem. We first describe our approach.

We want to find how the three ultrawarps are related, more specifically, for \( m \in M \), we want to find a relation between three ultracore elements, \( u_1, u_2, u_3 \). The best way to do this, where slightly easier calculations are involved, is to manipulate the differences of the six elements, (27a), (27b), and (27c). That is exactly what we do in this and the following section. In this section we obtain formulas expressing the difference of two elements of a triple vector bundle in terms of core elements and zeros. If two elements of a triple vector bundle can be subtracted then their outlines must have at least one face in common. Cases where the outlines have two or more faces in common arise repeatedly in the rest of the paper. Each of these cases needs individual treatment.

Looked at from another point of view, two elements of a triple vector bundle which can be subtracted may admit exactly one, or two, or all three, of the subtractions \( _{1,2} - _{1,3} \), and \( _{2,3} - _{1,3} \).

3.1 First case: two elements that have the same outline

Let \( e \) and \( e' \) have exactly the same outline

All three differences \( e - e', e - e', e - e' \) are defined.

**Step 1.** Focus on the Back faces of \( e \) and \( e' \)

Then, from double vector bundle theory, we can write

\[
e - e' = k_1 + \hat{0}_{e_{1,2}}, \quad e - e' = k_1 + \hat{0}_{e_{1,3}},
\]

where \( k_1 \in E_{23,1} \), the core of the Back face, with outline

\[
E_{23,1} \ni k_1 \quad \rightarrow e_1 \quad \downarrow \quad \quad E_{23} \ni w_{23} \quad \rightarrow m.
\]
Step 2. Show that $w_{23} = \odot_{m}^{2,3}$.

Use the morphism $q_{2,3} : E \to E_{2,3}$. We know that $q_{2,3}(e - e') = 0_{e_{3}}^{2,3}$ and

$$q_{2,3}(k_{1} + \hat{0}_{e_{1,3}}) = q_{2,3}(k_{1}) + q_{2,3}(\hat{0}_{e_{1,3}}) = w_{23} + 0_{e_{3}}^{2,3}.$$ 

Therefore

$$w_{23} + 0_{e_{3}}^{2,3} = 0_{e_{3}}^{2,3},$$

and, from double vector bundle theory, we have that $w_{23} = 0_{e_{2}}^{2,3} = \odot_{m}^{2,3}$. So $k_{1}$ has the outline

\[
\begin{array}{c}
k_{1} \\
\odot_{m}^{2,3} \\
\end{array}
\begin{array}{c}
\longrightarrow e_{1} \\
\downarrow \\
\longrightarrow m.
\end{array}
\]

Step 3. Applying double vector bundle theory again, we get

$$k_{1} = u_{1} + \hat{0}_{e_{1}},$$

where $u_{1}$ is an ultracore element.

Step 4. Apply the same procedure to Left and Up faces of $e$ and $e'$.

Focus on the Left faces of $e$ and $e'$

$$e - e' = k_{2} + \hat{0}_{e_{2,3}}, \quad e - e' = k_{2} + \hat{0}_{e_{1,2}},$$

where $k_{2} \in E_{13,2}$, core of Left face, with outline

\[
\begin{array}{c}
k_{2} \\
\odot_{m}^{1,3} \\
\end{array}
\begin{array}{c}
\longrightarrow e_{2} \\
\downarrow \\
\longrightarrow m.
\end{array}
\]

So, we can write

$$k_{2} = u_{2} + \hat{0}_{e_{2}},$$

where $u_{2}$ is an ultracore element.

Similarly for the Up faces

$$e - e' = k_{3} + \hat{0}_{e_{1,3}}, \quad e - e' = k_{3} + \hat{0}_{e_{2,3}},$$

where $k_{3} \in E_{12,3}$, core of Up face, with outline

\[
\begin{array}{c}
k_{3} \\
\odot_{m}^{1,2} \\
\end{array}
\begin{array}{c}
\longrightarrow e_{3} \\
\downarrow \\
\longrightarrow m.
\end{array}
\]
so
\[ k_3 = u_3 + \hat{0}_{e_3} \]
with \( u_3 \) an ultracore element.

**Step 5.** Show that \( u_1 = u_2 = u_3 \).

We show that \( u_1 = u_3 \). So far, we have two expressions for \( e - e' \), namely:

\[ k_{1,2} + \hat{0}_{e_{1,3}} = k_{2,3} + \hat{0}_{e_{1,3}}. \] (28)

Expand the left hand side of (28), mimicking the double vector bundle case:

\[
\hat{0}_{e_{1,3}} + (\hat{0}_{e_1} + u_1) = (\hat{0}_{e_{1,3}} + \hat{0}_{e_3} + \hat{0}_{e_1} + u_1) = (\hat{0}_{e_{1,3}} + \hat{0}_{e_1} + \hat{0}_{e_3} + u_1) = \hat{0}_{e_{1,3}} + \hat{0}_{e_3} + u_1.
\]

Therefore, we see that (28) can be rewritten as:

\[
\hat{0}_{e_{1,3}} + (\hat{0}_{e_3} + u_1) = \hat{0}_{e_{1,3}} + \hat{0}_{e_3} + u_1,
\]
from where it follows that \( u_1 = u_3 \). Similarly, we can show that \( u_2 = u_3 \).

At this point write \( u_1 = u_2 = u_3 \) to be \( u \).

**Step 6.** We obtain six formulas for the differences between \( e \) and \( e' \).

**Proposition 9.** With the above notation, two elements \( e \) and \( e' \) which have the same outline are related by

\[
e - e' = \hat{0}_{e_{1,3}} + (\hat{0}_{e_1} + u) = \hat{0}_{e_{1,3}} + (\hat{0}_{e_3} + u),
e - e' = \hat{0}_{e_{1,2}} + (\hat{0}_{e_1} + u) = \hat{0}_{e_{1,2}} + (\hat{0}_{e_2} + u),
e - e' = \hat{0}_{e_{2,3}} + (\hat{0}_{e_3} + u) = \hat{0}_{e_{2,3}} + (\hat{0}_{e_2} + u). \] (29)

What is important here is that the subtraction with respect to each structure results in the same ultracore element \( u \).

We will use the following special case later on.

**Example 10.** If \( e, e' \) are in one of the core double vector bundles, for example if \( e, e' \in E_{23,1} \), with outline

\[
e, e' \quad \xrightarrow{w_{23}} \quad \hat{0}_{e_1} \quad \xrightarrow{0_{E_3}} \quad \hat{0}_{e_{1,2}} \quad \xrightarrow{e_1} \quad 0_{E_2} \quad \xrightarrow{m} \quad m,
\]
then from (29) we have
\[ e^2,3 = \hat{w}_{23} + (\hat{0}_{e_3} + u) = \hat{w}_{23} + (\hat{0}_{e_3} + u) = \hat{w}_{23} + u \]

and

\[ e^2,3 = \hat{w}_{23} + (\hat{0}_{e_2} + u) = \hat{w}_{23} + (\hat{0}_{e_2} + u) = \hat{w}_{23} + u, \]

and therefore

\[ \hat{w}_{23} + u = \hat{w}_{23} + u. \tag{30} \]

### 3.2 Second case: two elements that have two lower faces in common

What happens if \( e \) and \( e' \) have only two of the lower faces in common? Then only two of the three subtractions are defined. There are three cases to consider, each of which arises later.

**If \( e \) and \( e' \) have the same Front and Right face**

Since \( e \) and \( e' \) have the same \( e_{1,3} \) and \( e_{2,3} \), it follows that they have the same \( e_1, e_2 \) and \( e_3 \). However \( e \) and \( e' \) will differ at \( e_{1,2} \) and \( e'_{1,2} \), and these will differ by a core element \( w_{12} \in E_{12} \) of the core of the Down face, that is

\[ e_{1,2} - e'_{1,2} = w_{12} + \hat{0}_{e_1}^{1,2}, \quad e_{1,2} - e'_{1,2} = w_{12} + \hat{0}_{e_2}^{1,2}. \]

It is useful to write out the outlines of these differences

\[
\begin{array}{c}
\xymatrix{
  e_{1,3} & e_{2,3} \\
  e_3 & e_2 \\
  e_1 & e_2 \\
  0_{E_2} & 0_{m} \\
  e_{1,2} & e'_{1,2} \\
  \hat{0}_{e_3}^{2,3} & \hat{0}_{e_3}^{2,3} \\
  \hat{0}_{e_3}^{1,2} & \hat{0}_{e_3}^{1,2} \\
 0_{E_2} & 0_{m} \\
 e_{1,3} & e_{2,3} \\
  e_3 & e_2 \\
  e_1 & e_2 \\
  0_{E_2} & 0_{m} \\
  e_{1,2} & e'_{1,2} \\
  \hat{0}_{e_3}^{2,3} & \hat{0}_{e_3}^{2,3} \\
  \hat{0}_{e_3}^{1,2} & \hat{0}_{e_3}^{1,2} \\
 0_{E_2} & 0_{m} \\
 e_{1,3} & e_{2,3} \\
  e_3 & e_2 \\
  e_1 & e_2 \\
  0_{E_2} & 0_{m} \\
  e_{1,2} & e'_{1,2} \\
  \hat{0}_{e_3}^{2,3} & \hat{0}_{e_3}^{2,3} \\
  \hat{0}_{e_3}^{1,2} & \hat{0}_{e_3}^{1,2} \\
 0_{E_2} & 0_{m} \\
}
\end{array}
\]

Since \( e \) and \( e' \) have the same Up face, again by applying double vector bundle theory, we can write

\[ e - e' = k + \hat{0}_{e_{1,3}}, \quad e - e' = k + \hat{0}_{e_{2,3}}, \tag{31} \]

where \( k \in E_{23,1} \), the core of the Up face.

Also, using the morphism \( q_{1,2} : E \to E_{1,2} \), we show that \( q_{1,2}(k) = w_{12} \). First,

\[ q_{1,2}(e_{1,2} - e'_{1,2}) = e_{1,2} - e'_{1,2} = w_{12} + \hat{0}_{e_1}^{1,2}, \]

and

\[ q_{1,2}(k + \hat{0}_{e_{1,3}}) = q_{1,2}(k) + \hat{0}_{e_1}^{1,2}. \]
hence \( q_{1,2}(k) = w_{12} \), where \( k \) has outline

\[
\begin{array}{ccc}
E_{12,3} \ni k & \rightarrow & e_3 \\
\downarrow & & \downarrow \\
E_{12} \ni w_{12} & \rightarrow & m.
\end{array}
\]

If \( e \) and \( e' \) have the same Front and Down face

In this case, the elements \( e_{1,3} \) and \( e'_{1,3} \) differ by a core element \( w_{13} \) of \( E_{13} \)

\[
e_{1,3} - e'_{1,3} = w_{13} + \hat{0}_{e_1}^{1,3}, \quad e_{1,3} - e'_{1,3} = w_{13} + \tilde{0}_{e_1}^{1,3}.
\]

As before, we can write

\[
e - e' = k + \hat{0}_{e_{1,2}}, \quad e - e' = k + \tilde{0}_{e_{1,2}}, \quad (32)
\]

with \( k \) an element of the core of the Left face with outline

\[
\begin{array}{ccc}
E_{13,2} \ni k & \rightarrow & e_2 \\
\downarrow & & \downarrow \\
E_{13} \ni w_{13} & \rightarrow & m.
\end{array}
\]

If \( e \) and \( e' \) have the same Right and Down face

In this case, \( e_{2,3} \) and \( e'_{2,3} \) will differ by an element \( w_{23} \in E_{23} \) of the core of the Front face

\[
e_{2,3} - e'_{2,3} = w_{23} + \hat{0}_{e_2}^{2,3}, \quad e_{2,3} - e'_{2,3} = w_{23} + \tilde{0}_{e_2}^{2,3},
\]

and as before

\[
e - e' = k + \hat{0}_{e_{1,3}}, \quad e - e' = k + \tilde{0}_{e_{1,3}}, \quad (33)
\]

where \( k \) is an element of the core of the Back face with outline

\[
\begin{array}{ccc}
E_{23,1} \ni k & \rightarrow & e_1 \\
\downarrow & & \downarrow \\
E_{23} \ni w_{23} & \rightarrow & m.
\end{array}
\]

### 3.3 Special case: differences between zero elements

Using the fact that \( \hat{0} \) is the zero section

\[
\hat{0}_{e_{2,3}} + \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} + e'_{2,3}, \quad \hat{0}_{e_{2,3}} + \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} + e'_{2,3}.
\]
and \((-1) \cdot \hat{0}_{e_{2,3}} = \hat{0}_{f_{2,3}}\) where \(f_{2,3} = -e_{2,3}\).

If we have two elements \(e_{2,3}\) and \(e'_{2,3}\) of \(E_{2,3}\) that differ by a core element \(w_{23} \in E_{23}\), then

\[
e_{2,3} - e'_{2,3} = w_{23} + \bar{0}^{2,3}_{e_{2,3}}, \quad e_{2,3} - e'_{2,3} = w_{23} + \bar{0}^{2,3}_{e_{3}},
\]

and the differences we are interested in are

\[
\hat{0}_{e_{2,3}} - \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} + \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} - e'_{2,3} = \hat{0}_{w_{23}} + \bar{0}^{2,3}_{e_{2}}, \quad (34)
\]

and

\[
\hat{0}_{e_{2,3}} - \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} + \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} - e'_{2,3} = \hat{0}_{w_{23}} + \bar{0}^{2,3}_{e_{3}}, \quad (35)
\]

and

\[
\hat{0}_{e_{2,3}} - \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} + \hat{0}_{e'_{2,3}} = \hat{0}_{e_{2,3}} - e'_{2,3} = \hat{0}_{w_{23}} + \bar{0}^{2,3}_{e_{2}}, \quad (36)
\]

There are similar formulas in the other two cases.
4 Proof of the warp theorem

4.1 Notation

In this section we prove Theorem 8. We will use the notation of (22). We further simplify the notation for elements of the lower faces and edges, as follows

\[ X(m) := e_1, \quad Y(m) := e_2, \quad Z(m) := e_3, \]
\[ Z_1(X(m)) := e_{1,3}, \quad X_3(Z(m)) := e'_{1,3}, \quad Z_2(Y(m)) := e_{2,3}, \]
\[ Y_3(Z(m)) := e'_{2,3}, \quad Y_1(X(m)) := e_{1,2}, \quad X_2(Y(m)) := e'_{1,2}. \]

The outlines of the elements in (22) are now written as follows

We will need the following relations for the core elements of the lower faces in detailed form.

\[ e_{2,3} - e'_{2,3} = \tilde{0}^{2,3}_{e_2} + w_{23}, \quad e_{2,3} - e'_{2,3} = \tilde{0}^{2,3}_{e_3} + w_{23}, \]  
(37)

\[ e'_{1,3} - e_{1,3} = \tilde{0}^{1,3}_{e_1} + w_{13}, \quad e'_{1,3} - e_{1,3} = \tilde{0}^{1,3}_{e_3} + w_{13}, \]  
(38)

\[ e_{1,2} - e'_{1,2} = \tilde{0}^{1,2}_{e_1} + w_{12}, \quad e_{1,2} - e'_{1,2} = \tilde{0}^{1,2}_{e_2} + w_{12}, \]  
(39)

where \( w_{23} \in E_{23}, \ w_{13} \in E_{13} \) and \( w_{12} \in E_{12} \).
For the zeros of these \( w \) elements, the diagrams are

\[
\begin{align*}
\hat{0}_{w_{23}} & \rightarrow \circlearrowright_{m}^{1,3} \rightarrow w_{23} \rightarrow 0_{E_{3}}^{E_{m}} \rightarrow m, \\
\hat{0}_{w_{13}} & \rightarrow \circlearrowright_{m}^{2,3} \rightarrow w_{13} \rightarrow 0_{E_{3}}^{E_{m}} \rightarrow m, \\
\hat{0}_{w_{12}} & \rightarrow \circlearrowright_{m}^{1,3} \rightarrow w_{12} \rightarrow 0_{E_{3}}^{E_{m}} \rightarrow m.
\end{align*}
\]

Core and ultracore elements arising from the grid

We collect here for reference the definitions and outlines of the core and ultracore elements arising from the grid.

- \( \lambda_{1}, k_{1} \) and \( u_{1} \)

The elements \( ZYX \) and \( YZX \) have the same Right and Back faces, and so their differences define an element \( \lambda_{1} \in E_{23,1} \) with outline

\[
\begin{align*}
\lambda_{1} & \rightarrow \hat{0}_{e_{1}}^{1,3} \rightarrow w_{23} \rightarrow 0_{E_{3}}^{E_{m}} \rightarrow m, \\
\hat{0}_{e_{1}}^{1,2} & \rightarrow e_{1} \rightarrow 0_{E_{2}}^{E_{m}} \rightarrow m.
\end{align*}
\]

Using (33) the defining equations are

\[
\begin{align*}
ZYX_{1,2} - YZX = \hat{0}_{e_{1,2}}^{1,3} + \lambda_{1}, \\
ZYX_{1,3} - YZX = \hat{0}_{e_{1,3}}^{1,3} + \lambda_{1}.
\end{align*}
\]

If we look at \( XZY \) and \( XYZ \), we see that they also have two faces in common, and their differences define a \( k_{1} \in E_{23,1} \), with outline

\[
\begin{align*}
k_{1} & \rightarrow \hat{0}_{e_{1}}^{1,3} \rightarrow w_{23} \rightarrow 0_{E_{3}}^{E_{m}} \rightarrow m, \\
\hat{0}_{e_{1}}^{1,2} & \rightarrow e_{1} \rightarrow 0_{E_{2}}^{E_{m}} \rightarrow m.
\end{align*}
\]

The two differences are, again using (33),

\[
\begin{align*}
XZY_{1,2} - XYZ = \hat{0}_{e_{1,2}}^{1,3} + k_{1}, \\
XZY_{1,3} - XYZ = \hat{0}_{e_{1,3}}^{1,3} + k_{1}.
\end{align*}
\]
We see that $\lambda_1$ and $k_1$ have the same outlines so they differ by an ultracore element $u_1 \in E_{123}$,

\[ \begin{align*}
\lambda_1 - k_1^1_{1,3} &= \hat{0}_{e_1} + u_1, \\
\lambda_1 - k_1^1_{1,2} &= \hat{0}_{e_1} + u_1, \\
\lambda_1 - k_1^1_{2,3} &= \hat{0}_{w_{23}} + u_1 = \hat{0}_{w_{23}} + u_{1,2}.
\end{align*} \tag{41a} \tag{41b} \tag{41c} \]

There are four ways of describing the warp

\[ \begin{align*}
(\text{ZYX}_{1,2} - \text{YZX}_{1,3}) - (\text{XZY}_{1,2} - \text{XYZ}_{1,3}) &= \hat{0}_{e_1} + (\hat{0}_{w_{12}} + \hat{0}_{w_{23}} + u_1), \\
(\text{ZYX}_{1,2} - \text{YZX}_{2,3}) - (\text{XZY}_{1,2} - \text{XYZ}_{1,3}) &= (\hat{0}_{w_{12}} + \hat{0}_{w_{23}}) + (\hat{0}_{e_1} + u_{1,2}), \\
(\text{ZYX}_{1,3} - \text{YZX}_{1,2}) - (\text{XZY}_{1,3} - \text{XYZ}_{1,2}) &= \hat{0}_{e_1} + (\hat{0}_{w_{13}} + u_{1,2}), \\
(\text{ZYX}_{1,3} - \text{YZX}_{2,3}) - (\text{XZY}_{1,3} - \text{XYZ}_{2,3}) &= (\hat{0}_{w_{13}} + \hat{0}_{w_{23}}) + (\hat{0}_{e_1} + u_{1,2}).
\end{align*} \tag{42a} \tag{42b} \tag{42c} \tag{42d} \]

Here we write $\hat{0}_{w_{12}} + u_1$ to denote $\hat{0}_{w_{12}} + u_{1,3} = \hat{0}_{w_{12}} + u_{1,2}$.

- **$\lambda_2$, $k_2$ and $u_2$**

The same procedure can be applied to $\text{XZY}$ and $\text{ZXY}$; they have the same Front and Down faces, so their differences will define an element $\lambda_2 \in E_{13,2}$

\[ \begin{align*}
\lambda_2 - w_{13} &= \hat{0}_{e_1} + \hat{0}_{w_{13}} + \hat{0}_{e_2}, \\
\hat{0}_{e_2} &= \hat{0}_{e_1} + \hat{0}_{w_{13}} + \hat{0}_{e_2}.
\end{align*} \tag{32} \]

The corresponding equations, using (32), are

\[ \begin{align*}
\text{XZY}_{1,2} - \text{ZXY}_{1,3} &= \hat{0}_{e_1} + \hat{0}_{w_{13}} + \lambda_2, \\
\text{XZY}_{1,3} - \text{ZXY}_{2,3} &= \hat{0}_{w_{13}} + \hat{0}_{w_{23}} + \hat{0}_{e_1} + \lambda_2.
\end{align*} \]

If we look at $\text{YXZ}$ and $\text{YZX}$, their differences define a $k_2 \in E_{13,2}$, with outline

\[ \begin{align*}
k_2 - w_{13} &= \hat{0}_{e_1} + \hat{0}_{w_{13}} + \hat{0}_{e_2}, \\
\hat{0}_{e_2} &= \hat{0}_{e_1} + \hat{0}_{w_{13}} + \hat{0}_{e_2}.
\end{align*} \tag{32} \]

The corresponding equations, using (32), are

\[ \begin{align*}
\text{XZY}_{1,2} - \text{ZXY}_{1,3} &= \hat{0}_{e_1} + \hat{0}_{w_{13}} + \lambda_2, \\
\text{XZY}_{1,3} - \text{ZXY}_{2,3} &= \hat{0}_{w_{13}} + \hat{0}_{w_{23}} + \hat{0}_{e_1} + \lambda_2.
\end{align*} \]
and the differences defined are, due to (32),

\[ Y_{1,2}Z_{2,3} = \hat{0}_{e_{1,3}} + k_2, \quad Y_{2,3}Z_{1,2} = \hat{0}_{e_{2,3}} + k_2. \]

Since \( \lambda_2 \) and \( k_2 \) have the same outlines, they differ by an ultracore element \( u_2 \in E_{123} \),

\[
\begin{align*}
\lambda_2 - k_2 &= \hat{0}_{w_{13}} + u_2, \quad (43a) \\
\lambda_2 - k_2 &= \hat{0}_{w_{13}} + u_2, \quad (43b) \\
\lambda_2 - k_2 &= \hat{0}_{w_{13}} + u_2. \quad (43c)
\end{align*}
\]

Again there are four ways of describing the warp

\[
\begin{align*}
(X_{1,2}Y_{2,3}Z_{1,2}) - (X_{1,2}Z_{2,3}Y_{1,2}) &= \hat{0}_{w_{13}} + u_2, \quad (44a) \\
(X_{1,2}Z_{2,3}Y_{1,2}) - (X_{1,2}Y_{2,3}Z_{1,2}) &= \hat{0}_{w_{13}} + u_2, \quad (44b) \\
(X_{2,3}Y_{1,2}Z_{2,3}) - (X_{2,3}Z_{1,2}Y_{2,3}) &= \hat{0}_{w_{13}} + u_2, \quad (44c) \\
(X_{2,3}Z_{1,2}Y_{2,3}) - (X_{2,3}Y_{1,2}Z_{2,3}) &= \hat{0}_{w_{13}} + u_2. \quad (44d)
\end{align*}
\]

• \( \lambda_3, k_3 \) and \( u_3 \)

Likewise \( YXZ \) and \( XYZ \) define \( \lambda_3 \in E_{123} \) with outline

\[
\begin{align*}
\lambda_3 \quad &\quad \rightarrow \quad \hat{0}_{e_{1,3}}^{1,3} \\
\quad &\quad \downarrow \quad \hat{0}_{e_{1,3}}^{2,3} \quad \rightarrow \quad e_3 \\
\quad &\quad \downarrow \quad \hat{0}_{e_{1,3}}^{1,3} \quad \rightarrow \quad E_{13}. \\
\quad &\quad \downarrow \quad \hat{0}_{e_{1,3}}^{2,3} \quad \rightarrow \quad m.
\end{align*}
\]

The corresponding relations are, due to (31),

\[ Y_{1,3}X_{2,3}Z_{1,2} = \hat{0}_{e_{1,3}} + k_3, \quad Y_{2,3}X_{1,3}Z_{2,3} = \hat{0}_{e_{2,3}} + k_3. \]

Likewise \( ZYX \) and \( ZXY \) define \( k_3 \in E_{123} \) with outline

\[
\begin{align*}
k_3 \quad &\quad \rightarrow \quad \hat{0}_{e_{1,3}}^{1,3} \\
\quad &\quad \downarrow \quad \hat{0}_{e_{1,3}}^{2,3} \quad \rightarrow \quad e_3 \\
\quad &\quad \downarrow \quad \hat{0}_{e_{1,3}}^{1,3} \quad \rightarrow \quad E_{13}. \\
\quad &\quad \downarrow \quad \hat{0}_{e_{1,3}}^{2,3} \quad \rightarrow \quad m.
\end{align*}
\]
The differences defined are, due to (41),
\[ ZYX_{1,3} - ZXY = \hat{0}_{e_{1,3}} + k_3, \quad ZYX_{2,3} - ZXY = \hat{0}_{e_{2,3}} + k_3. \]

The ultracore element \( u_3 \in E_{123} \) defined by \( \lambda_3 \) and \( k_3 \) satisfies
\[ \lambda_3 - k_3 = \hat{0}_{e_{1,2}} + u_3, \quad (45a) \]
\[ \lambda_3 - k_3 = \hat{0}_{u_{212}} + u_3, \quad (45b) \]
\[ \lambda_3 - k_3 = \hat{0}_{e_{2,3}} + u_3. \quad (45c) \]

The four relations in this case are
\[ (YZX_{1,3} - XYZ)_{2,3} - (ZYX_{1,3} - ZXY) = \hat{0}_{\omega_{1,3}} + (\hat{0}_{u_{1,2}} + u_3), \quad (46a) \]
\[ (YZX_{1,3} - XYZ)_{1,2} - (ZYX_{1,3} - ZXY) = (\hat{0}_{\omega_{2,3}} + \hat{0}_{\omega_{1,2}} + \hat{0}_{e_{1,2}} + u_3), \quad (46b) \]
\[ (YZX_{2,3} - XYZ)_{1,3} - (ZYX_{2,3} - ZXY) = \hat{0}_{e_{1,2}} + (\hat{0}_{u_{2,3}} + u_3), \quad (46c) \]
\[ (YZX_{2,3} - XYZ)_{1,2} - (ZYX_{2,3} - ZXY) = (\hat{0}_{u_{2,3}} + \hat{0}_{u_{1,2}} + \hat{0}_{e_{1,2}} + u_3). \quad (46d) \]

### 4.2 Proof of the warp theorem

We will show that \( u_1 + u_2 + u_3 = \odot_{3}^{3} \) by showing that \( u_1 = -u_2 - u_3 \). There are four steps.

**Step 1.** Rewrite (42b)
\[ (ZYX_{1,2} - YZX)_{2,3} - (XZY_{1,2} - XYZ) \]
as
\[ (ZYX_{2,3} - XZY)_{1,2} - (YXZ_{2,3} - XYZ), \]
using the interchange law for the double vector bundle formed by the Left face. We know from (42b) that the ultracore element defined by the first expression is \( u_1 \), therefore, the ultracore element of the latter expression will also be \( u_1 \). We will show that the second expression has \(-u_2 - u_3\) as its ultracore element, and this will show that \( u_1 = -u_2 - u_3 \).

**Step 2.** First, write \( ZYX_{2,3} - XZY \) as
\[ ZYX_{2,3} - XZY = (ZYX_{2,3} - ZXY)_{2,3} - (XZY_{2,3} - ZXY), \]
where we have that
\[ ZYX_{2,3} - ZXY = \hat{0}_{e_{2,3}} + k_3, \quad XZY_{2,3} - ZXY = \hat{0}_{e_{2,3}} + \lambda_2. \quad (47) \]

**Step 3.** Similarly, write \( YZX_{2,3} - XYZ \) as
\[ YZX_{2,3} - XYZ = (YXZ_{2,3} - XYZ)_{2,3} - (YXZ_{2,3} - YZX), \quad (48) \]
and we have
\[
YXZ - XYZ = \hat{\nu}_{2,3}^3 + \lambda_3, \quad YXZ - YZX = \hat{\nu}_{2,3}^3 + k_2.
\]

**Step 4.** Since our convention is that \(\lambda_3 - k_3\) defines \(u_3\), it follows that \(k_3 - \lambda_3\) defines \(-u_3\). It is essential to follow these conventions.

We are finally able to complete the proof of Theorem 8.

\[
(ZYX - XZY)_{2,3}^{1,2} - (YZX - XYZ)_{2,3}^{1,2} = [(ZYX - ZXY)_{2,3}^{1,2} - (XYZ - XYZ)_{2,3}^{1,2} - (YXZ - YZX)_{2,3}^{1,2} - (XZY - ZXY)_{2,3}^{1,2}]
\]

(1) [(using operations in single vector bundles)]

\[
= [(\hat{\nu}_{e2,3} + k_3)_{2,3}^{1,2} - (\hat{\nu}_{e2,3} + \lambda_3)_{1,2} - (\hat{\nu}_{e2,3} + k_2)_{2,3}^{1,2}]
\]

(2) [(using (17) and (48)]

\[
= [(\hat{\nu}_{e2,3} + k_3)_{2,3}^{1,2} - (\hat{\nu}_{e2,3} + \lambda_3)_{1,2} - (\hat{\nu}_{e2,3} + k_2)_{2,3}^{1,2}]
\]

(3) [(interchange law in Left face applied to outer operations)]

\[
= [(\hat{\nu}_{e2,3} + k_3)_{2,3}^{1,2} - (\hat{\nu}_{e2,3} + \lambda_3)_{1,2} - (\hat{\nu}_{e2,3} + k_2)_{2,3}^{1,2}]
\]

(4) [(interchange law in Back face for first term)]

\[
= [(\hat{\nu}_{w23} + \hat{\nu}_{e2})_{2,3}^{1,2} - (\hat{\nu}_{w12} - u_3)_{2,3}^{1,2} - (\hat{\nu}_{u2} + \hat{\nu}_{e2} + u_3)_{2,3}^{1,2}]
\]

(5) [(first term in each [] by (35], then use (45b) and (43b)]

\[
= [(\hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3)_{2,3}^{1,2} - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)_{2,3}^{1,2}]
\]

(6) [(second group by interchange law in Back face)]

\[
= [(\hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3)_{2,3}^{1,2} - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)_{2,3}^{1,2}]
\]

(7) [(zeros in final group are over base of the addition)]

\[
= [(\hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3)_{2,3}^{1,2} - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)_{2,3}^{1,2}]
\]

(8) [(second group is in ordinary vector bundle)]

\[
= [(\hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3)_{2,3}^{1,2} - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)_{2,3}^{1,2}]
\]

(9) [(interchange law in Up face)]

\[
= \hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3 - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)
\]

(10) [(zeros are zeros over base of addition; then interchange law in Up face)]

\[
= \hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3 - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)
\]

(11) [(zeros are zeros over base of addition)]

\[
= \hat{\nu}_{w23} + \hat{\nu}_{e2} + \hat{\nu}_{w12} - u_3 - (\hat{\nu}_{w2} + \hat{\nu}_{e2} + \hat{\nu}_{w1} - u_3)
\]

(12) [(using an equation of the form (30)]

from which we obtain \(-(u_3 + u_2)\) as the ultracore element.

Comparing this with (121),

\[
(ZYX - YZX)_{2,3}^{1,2} - (XZY - XYZ)_{2,3}^{1,2} = (\hat{\nu}_{w12} + \hat{\nu}_{w23})_{2,3}^{1,2} + \hat{\nu}_{e2} + u_1,
\]
we have $u_1 = -(u_3 + u_2)$ as desired.

This completes the proof of the warp theorem.

The strategy of this proof deserves some commentary.

What should the warp of a grid on a triple vector bundle be? Or, in other words, why are we interested in the ultrawarps of a grid of a triple vector bundle?

The warp of a grid in the double case is a section of the core vector bundle, and measures the non-commutativity of the two routes defined by the grid.

So far, we have seen that all operations on a triple vector bundle are iterations of operations defined in double vector bundles. The ultracore, for example, is the core of the core double vector bundles.

For these reasons, we would want the warp of a grid in the triple case to be a section of the ultracore vector bundle, and to measure the non-commutativity of routes defined by the grid.

Pick an upper face of $E$, for example the Up face. If we compare the two routes defined by the grid in this face, then we obtain an element of the (U-D) core double vector bundle, which we denoted by $\lambda_3$. Similarly for the other upper faces, the non-commutativity of the corresponding routes defines $\lambda_1$ and $\lambda_2$. The three $\lambda$’s are elements of different spaces; therefore, if we tried to compare them, or indeed perform any sort of operation with them (such as adding them or subtracting them), we would see that such an operation could be algebraically possible but would not be geometrically meaningful.

The same applies for the three $k_i$ defined by the comparison of the routes for the lower faces.

The $\lambda_i$’s and the corresponding $k_i$’s however, are elements of the same spaces, therefore, comparing them is a possibility, and indeed the only sensible operation. And by comparing them, we measure the non-commutativity of four routes, instead of two.

This can be done for the three pairs of $\lambda_i$ and $k_i$, and so we obtain the three ultrawarps.

So what does the warp theorem tell us?

Each ultrawarp measures the non-commutativity of four routes. In total, a grid on a triple vector bundle provides six different routes from $M$ to $E$. The sum of the three ultrawarps takes into account each route twice, once with a positive and once with a negative sign. The warp theorem tells us that these add up to zero, a result that seems reasonable. The different vector bundle structures over which the operations take place however, are the main obstacle here — as soon as one realizes that simple operations like addition and subtraction in the triple vector bundle setting are no longer simple.
5 The triple vector bundle $T^2A$ and connections in $A$

In this section and the next we examine two typical instances of the warp theorem. In this section we consider the triple vector bundle $T^2A$ where $A \to M$ is a vector bundle, and grids which arise in it from connections in $A$. In the following section we consider $T^3M$, the triple tangent bundle of a manifold $M$.

For a vector bundle $(A, q, M)$ there is a triple vector bundle structure on $T^2A$ as shown in (49).

Here the Down face is the usual double vector bundle $TA$ and the Up face is the tangent prolongation of this; that is, it is obtained by applying the tangent functor to each structure in $TA$. Each vertical vector bundle in (49) is a standard tangent bundle.

5.1 The core double vector bundles of $T^2A$

The three core double vector bundles are shown in (50), in the usual order (B-F), (L-R), and (U-D), and arranged as in (13).

These core double vector bundles are the same as abstract double vector bundles but are embedded differently in $T^2A$.

An element $\xi \in TA$ determines core elements of the Back, Left and Up faces. We denote these by $\xi^B$, $\xi^L$ and $\xi^U$, respectively. Since they are elements of $T(TA)$ they can be represented as tangent vectors to curves in $TA$.

The Back face is the tangent double vector bundle for the tangent prolongation bundle $T(q): TA \to TM$. For the core of the Back face we have

$$\bar{\xi}^B = \frac{d}{dt}(t \cdot \xi)^{TM}_{|t=0}$$

(51)

where $t \cdot \xi$ denotes scalar multiplication in $T(q): TA \to TM$. 
The Left face is the double tangent bundle of the manifold $A$. Given $\xi \in TA$ the core element is
\[
\xi^L = \left. \frac{d}{dt}(t\xi) \right|_{t=0}
\] (52)
where the scalar multiplication is in the usual tangent bundle $TA \to A$.

For the core of the Up face, first write $\xi = \left. \frac{d}{dt} a_t \right|_{t=0}$ where $a_t$ is a curve in $A$. Write $a_t \in TA$ for the core element corresponding to $a_t$. Then
\[
\xi^U = \left. \frac{d}{dt}(a_t) \right|_{t=0}.
\] (53)

5.2 The canonical involution on $T^2A$

The canonical involution $J_A: T^2A \to T^2A$ for the manifold $A$ is an isomorphism from the double vector bundle $T^2A$ to its flip. In what follows we will need to use it as a map of triple vector bundles.

**Proposition 11.** The map $J_A$ is an isomorphism of triple vector bundles as shown in (54).

In (54) the Left faces are the double tangent bundles of the manifold $A$ and $J_A$ maps the Left face of the domain to its flip. It interchanges the Up and Back faces. The Right faces are the double tangent bundles of $M$ and $J_A$ induces $J_M: T^2M \to T^2M$ which maps the Right face of the domain to its flip. The Front and Down faces are interchanged.

The proof of Proposition 11 relies on the following two lemmas. The first is the naturality property of the canonical involution.

**Lemma 12.** Let $M$ and $N$ be smooth manifolds, and $F: M \to N$ a smooth map. Then $T^2(F) \circ J_M = J_N \circ T^2(F)$, where $T^2(F) = T(T(F))$ is the tangent of the tangent map $T(F)$.

**Lemma 13.** Given $\Phi_1, \Phi_2 \in T^2A$, over the same $\xi \in T^2M$, we have
\[
J_A \left( \frac{\Phi_1}{T^2(q)} + \frac{\Phi_2}{T^2(q)} \right) = J_A(\Phi_1) + J_A(\Phi_2).
\]

Consider now the maps which $J_A$ induces on the cores.

Take an element $\xi \in TA$ in the core of the Back face. Regarded as an element of $T^2A$ this is $\xi^B$, with outline shown on the left of (55).
It follows from (51) and (53) that
\[ J_A(\xi^B) = \xi^U. \]  
(56)

Since \( J^2_A \) is the identity, we also have
\[ J_A(\xi^J) = \xi^B. \]  
(57)

Since the Left faces in (54) are the double tangent bundle \( T^2 A \), the map on the cores of the Left faces is the identity and so
\[ J_A(\xi^L) = \xi^L. \]  
(58)

### 5.3 Grids on \( T^2 A \)

Now consider a connection \( \nabla \) in \( A \). Recall that Example 3 gave a construction of a grid in \( TA \) for which the warp is \( \nabla_X \mu \). We now extend this idea to define a grid in \( T^2 A \).

Let \( X, Z \in \mathfrak{X}(M) \), and \( \mu \in \Gamma A \). Define the following three double linear sections:

- From Front to Back face: \( (T(X^H); X^H, T(X); X) \).
- From Right to Left face: \( (T^2(\mu); T(\mu), T(\mu); \mu) \).
- From Down to Up face: \( (\tilde{Z}^H_A; Z^H, \tilde{Z}; Z) \).

Here \( \tilde{Z} = J_M \circ T(Z) \) is the complete (or tangent) lift of \( Z \) to a vector field on \( TM \). Likewise \( \tilde{Z}^H_A \) is the complete lift of \( Z^H \in \mathfrak{X}(A) \) to a vector field on \( TA \). The grid is shown in (59).

The core morphisms of the linear double sections will be needed later:
For \((T(X^H); X^H, T(X); X)\) the core morphism is \((X^H, X)\).

For \((T^2(\mu); T(\mu), T(\mu); \mu)\) the core morphism is \((T(\mu), \mu)\).

For \((\tilde{Z}^H A; Z^H, \tilde{Z}; Z)\) the core morphism is \((Z^H, Z)\).

The first two cases are instances of the general fact that given a morphism \((\varphi, f)\) of vector bundles, the core morphism of the double vector bundle map \((T(\varphi); T(f), \varphi; f)\) is \((\varphi, f)\).

To calculate the core morphism of \((\tilde{Z}^H A; Z^H, \tilde{Z}; Z)\), focus on (60). At this point we investigate this linear double section further; it is a double vector bundle morphism from the Down face to the Up face of \(T^2 A\). Note that (60) is not a triple vector bundle.

Take an element \(a \in A\). As an element of the core of the Down face of \((59)\) it is \(\bar{\pi} = \left. \frac{d}{dt} t a \right|_{t=0} \in TA\).

Using the fact that \((\tilde{Z}, Z)\) is a vector bundle map, we have that \(\tilde{Z}(0^T M) = T(0^T M)(Z(m))\).

Similarly, using the fact that \((Z^H, Z)\) is a vector bundle map, we have that \(Z^H(0^A m) = T(0^A)(Z(m))\).

Finally,

\[
\tilde{Z}^H A (\bar{\pi}) = J_A T(Z^H(\bar{\pi})) = J_A \left( \left. \frac{d}{dt} Z^H(ta) \right|_{t=0} \right) = J_A \left( \left. \frac{d}{dt} Z^H (a) \right|_{t=0} \right) = J_A \left( Z^H(a)^B \right).
\]

Note the following. Initially, \(\bar{\pi} \in TA\) is in the core of the Down face of \(T^2 A\). The canonical involution \(J_A\) maps the Down face to the Front face (see (54)). Therefore, in \(T(Z^H)(\bar{\pi})\), \(\bar{\pi}\) is now an element of the core of the Front face. The maps \((T(Z^H); Z^H, T(Z); Z)\) form a double vector bundle morphism from the Front to the Back face of (49), with core morphism \((Z^H, Z)\) as usual. Therefore, \(T(Z^H)(\bar{\pi}) = Z^H(a)^B\) is now in the core of the Back face. And by (56), it follows that \(J_A \left( Z^H(a)^B \right) \).  

This completes the proof that the core morphism of \((\tilde{Z}^H A; Z^H, \tilde{Z}; Z)\) is \((Z^H, Z)\).

### 5.4 The warp of the Back face

The warp of the Back face is given by

\[
T^2(\mu)(\tilde{Z} (X(m))) \mid_{\text{PTA}} \tilde{Z}^H A (T(\mu)(X(m))).
\]
The outlines of the two elements are

\[
\begin{align*}
T^2(\mu)(\tilde{Z}(X(m))) & \rightarrow \tilde{Z}(X(m)) \\
& \downarrow \quad T(\mu)(Z(m)) \rightarrow Z(m) \\
T(\mu)(X(m)) & \rightarrow X(m) \\
& \downarrow \quad \mu(m) \rightarrow m,
\end{align*}
\]

\[
\begin{align*}
\tilde{Z}^H_A(T(\mu)(X(m))) & \rightarrow \tilde{Z}(X(m)) \\
& \downarrow \quad Z^H(\mu(m)) \rightarrow Z(m) \\
T(\mu)(X(m)) & \rightarrow X(m) \\
& \downarrow \quad \mu(m) \rightarrow m,
\end{align*}
\]

(compare with the general triple outlines of the elements XYZ and ZYX, of subsection 4.1).

Writing the complete lifts as \( \tilde{Z}^H_A = J_A \circ T(Z^H) \) and \( \tilde{Z} = J_M \circ T(Z) \), and using the naturality of \( J \)-maps (Lemma 12), we have that

\[
T^2(\mu)(\tilde{Z}(X(m))) \xrightarrow{p_{TA}} \tilde{Z}^H_A(T(\mu)(X(m))) = T^2(\mu)(J_M(T(Z)(X(m))))) \xrightarrow{p_{TA}} J_A(T(Z^H)(T(\mu)(X(m)))))
\]

\[
= J_A(T^2(\mu)(T(Z)(X(m)))) \xrightarrow{p_{TA}} J_A(T(Z^H)(T(\mu)(X(m)))))
\]

\[
= J_A(\mu)(T(Z)(X(m)))) \xrightarrow{p_{TA}} J_A(T(Z^H)(T(\mu)(X(m))))).
\]  \tag{61}

Since \( J_A \) interchanges the structures \( p_{TA} \) and \( T(p_{A}) \), we can rewrite the last expression in (61) as

\[
J_A \left( T^2(\mu)(T(Z)(X(m))) \xrightarrow{T(p_{A})} T(Z^H)(T(\mu)(X(m)))) \right).
\]

Focus on \( T^2(\mu)(T(Z)(X(m))) \xrightarrow{T(p_{A})} T(Z^H)(T(\mu)(X(m))) \). We can rewrite this as

\[
T(T(\mu) \circ Z)(X(m)) \xrightarrow{T(p_{A})} T(Z^H \circ \mu)(X(m)). \tag{62}
\]

At this point, we use Proposition 14 below, that the warp of the tangent of a grid is the tangent of the warp. We apply this to the grid on the Down face which is as given in (10). From Example 3 the
warp of (10) is $\nabla_Z(\mu)$ (see (9)). The tangent of (10) is

$$T^2A \xleftarrow{T^2(\mu)} \xrightarrow{T^2(q)} T^2M$$

$$TA \xleftarrow{T(q)} \xrightarrow{T(\mu)} TM,$$

and so its warp is given, for any $x \in T_mM$, by Proposition 14,

$$(T^2(\mu) \circ T(Z))(x) = T(\nabla_Z(\mu))(x) U + T(\tilde{0}^TA)(T(\mu)(x)).$$

Here we have denoted by $\tilde{0}^TA$ the zero section of $TA \to A$. For $x = X(m)$, the right hand side of (63) is equal to (62). Therefore, (62) is equal to

$$T(\nabla_Z(\mu))(X(m)) U + T(\tilde{0}^TA)(T(\mu)(X(m))).$$

We return now to our calculation of (61). Applying $J_A$ to (64), we have that (61) is

$$J_A \left( T(\nabla_Z(\mu))(X(m)) U \right) + J_A \left( T(\tilde{0}^TA)(T(\mu)(X(m))) \right).$$

The addition over $T^2(q)$ does not change under $J_A$, by Lemma 13. From (57) we have

$$J_A \left( T(\nabla_Z(\mu))(X(m)) U \right) + J_A \left( T(\tilde{0}^TA)(T(\mu)(X(m))) \right) = T(\nabla_Z(\mu))(X(m)) U + 0^{T^2A}(T(\mu)(X(m))),$$

This completes the calculation of the warp of the Back face; taking into consideration the orientation of the Back face, the warp is $-T(\nabla_Z(\mu)) \in \Gamma_{TM}TA$.

**Proposition 14.** Let $(\xi, X)$ and $(\eta, Y)$ be a grid on a double vector bundle $D$ with warp $w(\xi, \eta) \in \Gamma C$. Then $(T(\xi), T(X))$ and $(T(\eta), T(Y))$ form a grid on the double vector bundle $TD$ in (65) below and the warp of the tangent grid $(T(\xi), T(X))$, $(T(\eta), T(Y))$ is $T(w(\xi, \eta)) \in \Gamma_{TM}(TC)$. 

$$D \xleftarrow{\xi} B \xrightarrow{T(\xi)} TB \xleftarrow{T \eta} Y \xrightarrow{T(\eta)} T(Y)$$

Here we have denoted by $\tilde{0}^TA$ the zero section of $TA \to A$. For $x = X(m)$, the right hand side of (63) is equal to (62). Therefore, (62) is equal to

$$T(\nabla_Z(\mu))(X(m)) U + T(\tilde{0}^TA)(T(\mu)(X(m))).$$

We return now to our calculation of (61). Applying $J_A$ to (64), we have that (61) is

$$J_A \left( T(\nabla_Z(\mu))(X(m)) U \right) + J_A \left( T(\tilde{0}^TA)(T(\mu)(X(m))) \right).$$

The addition over $T^2(q)$ does not change under $J_A$, by Lemma 13. From (57) we have

$$J_A \left( T(\nabla_Z(\mu))(X(m)) U \right) + J_A \left( T(\tilde{0}^TA)(T(\mu)(X(m))) \right) = T(\nabla_Z(\mu))(X(m)) U + 0^{T^2A}(T(\mu)(X(m))),$$

This completes the calculation of the warp of the Back face; taking into consideration the orientation of the Back face, the warp is $-T(\nabla_Z(\mu)) \in \Gamma_{TM}TA$.

**Proof.** We leave the verification that $TD$ is a double vector bundle and that $(T(\xi), T(X))$ and $(T(\eta), T(Y))$ are linear sections of $TD$ to the reader.
The warp \( w(\xi, \eta) \in \Gamma C \) of \((\xi, X), (\eta, Y)\) is given as usual by
\[
(\xi \circ Y)(m) \circ A (\eta \circ X)(m) = 0^D_X(m) + w(\xi, \eta)(m), \quad (\xi \circ Y)(m) \circ B (\eta \circ X)(m) = 0^D_Y(m) + w(\xi, \eta)(m).
\]

We calculate the warp of the tangent grid. From the definition of a warp, for \( x \in T_M N \),
\[
(T(\xi) \circ T(Y))(x) \circ A (T(\eta) \circ T(X))(x) = T(0^D \circ X)(x) + w(T(\xi), T(\eta))(x).
\]

Write \( x = \frac{4}{3} m_t\big|_{t=0} \), for \( m_t \) a curve in \( M \) with tangent vector \( x \) at \( t = 0 \). Then, for \( F \in C^\infty(D) \),
\[
\left( T(\xi) \circ T(Y))(x) \right)(F) = \left( T(\eta) \circ T(X)((\xi \circ Y)(m_t) \circ A (\eta \circ X)(m_t)) \right)_{t=0}
= \frac{d}{dt} F(\xi \circ \varphi^1_A + \xi \circ \varphi^2_A)
= \frac{d}{dt} F(0^D_X(m_t) + w(\xi, \eta)(m_t))_{t=0}
= \frac{d}{dt} F(0^D \circ X(m_t) + w(\xi, \eta)(m_t))_{t=0}
= \left( T(0^D \circ X)(x) + T(w(\xi, \eta))(x) \right)(F). \tag{66}
\]

Here we used the formula
\[
(T(\xi)(\Phi_1) + T(\xi)(\Phi_2))(F) = \frac{d}{dt} F(\xi \circ \varphi^1_A + \xi \circ \varphi^2_A)_{t=0} \tag{67}
\]
for the addition in \( TA \), and the corresponding formula for scalar multiplication. In \( F \in C^\infty(D) \), the \( \Phi_i \) are elements of \( TB \) and the \( \varphi^i_A \) are curves in \( B \) with
\[
\Phi_i = \frac{d}{dt} \varphi^i_A \big|_{t=0}, \quad i = 1, 2.
\]

Given \( T(q_B)(\Phi_1) = T(q_B)(\Phi_2) \) we can arrange that \( q_B(\varphi_1) = q_B(\varphi_2) \) for \( t \) near zero.

By uniqueness of the core element, it follows from \( w(\xi, \eta)(x) = T(w(\xi, \eta))(x) \).
\( \square \)

### 5.5 The three ultrawarps

We now focus on the grids defined on the core double vector bundles of \( T^2 A \). We present a table with the results here, and outline the calculations in the following subsections.

**Back-Front**

The Back face is the tangent double vector bundle of the prolonged bundle \( TA \rightarrow TM \) and by the results of subsection \( \ref{sec:5.4} \) we obtain
\[
T^2(\mu) \circ Z - Z^H_A \circ T(\mu) \triangleright T(\nabla Z \mu). \tag{68}
\]

Taking into account the orientation of the Back face, the warp is \(-T(\nabla Z \mu) \in \Gamma_{TM}(TA) \). For the Front face, with the appropriate orientation, the warp is \(-\nabla Z \mu \in A \), by Example \( \ref{ex:5.3} \). Therefore the ultrawarp for the Back-Front core double vector bundle (first row of Table\( \ref{table:5.1} \)) is, again using Example \( \ref{ex:5.3} \)
\[
-T(\nabla Z \mu) \circ X + X^H(\nabla Z \mu) \triangleright -\nabla X \nabla Z \mu.
\]
Left-Right

The Left face is the tangent double vector bundle $T^2A$ for the manifold $A$. We therefore apply (5).

Taking into account the orientation of the Left face, we have

$$T(X^H) \circ Z^H - \tilde{Z}^H \circ X^H \triangleright [Z^H, X^H].$$

The Right face is $T^2M$ so the warp is $[Z, X] \in \mathfrak{X}(M)$. So the warp of the core double vector bundle in the second row of Table 1 is defined by

$$T(\mu) \circ [Z, X] - [Z^H, X^H] \circ \mu.$$  \hspace{1cm} (69)

This grid is of a new type; we encounter it here for the first time.

First, what is $[Z^H, X^H]$? When a connection in a vector bundle is formulated in terms of a lifting of vector fields from the base to the total space (see Example 3), the difference $[Z^H, X^H] - [Z, X]^H$ is related to the curvature of the connection. This is analogous to describing the curvature of a connection.
in a principal bundle $P(M, G)$ as the failure of the horizontal distribution to be involutive. In [4] and other works of that period, it is stated that this expression reduces to the endomorphism $R_{\nabla}(X, Y)$. We deduce this now from the warp theorem.

Both $[Z^H, X^H]$ and $[Z, X]^H$ project to $[Z, X]$ and therefore their difference is a linear and vertical vector field on $A$.

At this point we take a closer look at linear sections $(\eta, 0^B)$, where $\eta \in \Gamma_A D$ and $0^B \in \Gamma_B$, of a general double vector bundle $D$. It can be shown that such a section $\eta$ induces a vector bundle map $\varphi : A \to C$, over $M$, and for $a \in A$,

$$\eta(a) = \varphi(a) + 0^D_B a.$$ 

We denote such linear sections by $(\varphi^\sharp, 0^B)$, where $\varphi$ is the induced vector bundle map. It follows immediately that the warp of a horizontal linear section $(\xi, X)$ and of a vertical linear section over the zero section $(\varphi^\sharp, 0^B)$ is

$$w(\xi, \varphi^\sharp) = -\varphi \circ X. \quad (70)$$

Back to (69). We will need the following Proposition.

**Proposition 15.** Given two grids $(\xi, X)$, $(\eta, Y)$, and $(\xi, X)$, $(\varphi^\sharp, 0^B)$ on a double vector bundle $D$,

$$w(\xi, \eta + \varphi^\sharp) = w(\xi, \eta) + w(\xi, \varphi^\sharp).$$

We leave the proof to the reader.

We can now rewrite the grid on the (L-R) core double vector bundle as the sum of

$$T_A T_M \left[ [Z, X]^\mu \right] \left( T(\mu) \circ [Z, X] - [Z, X]^H \circ \mu \right) + \left( T(\mu) \circ 0^T_M - (R_{\nabla}(Z, X)^\sharp) \circ \mu \right).$$

From Example 8

$$T(\mu) \circ [Z, X] - [Z, X]^H \circ \mu \nabla_{[Z, X]} \mu,$$
and from (70)

\[ T(\mu) \circ 0^{TM} - (R_{\nabla}(Z, X)^{\circ}) \circ \mu \triangleright - R_{\nabla}(Z, X)(\mu). \]

So in total, the warp of this core double vector bundle will be

\[ \nabla_{[Z,X]} \mu - R_{\nabla}(Z, X)(\mu). \]

Taking into consideration the orientation of the core double vector bundle, take the opposite sign

\[ -\nabla_{[Z,X]} \mu + R_{\nabla}(Z, X)(\mu). \]

**Up-Down**

The warp of the Down face is, again by Example 3, \( \nabla_X \mu \). For the warp of the Up face, we use Proposition 14 and obtain \( T(\nabla_X \mu) \in \Gamma_{TM}(TA) \). Therefore the warp of the grid in the third row of Table 1 is

\[ \nabla_Z \nabla_X \mu. \]

This completes the exposition of Table 1.

The warp theorem now gives us that

\[ -\nabla_X \nabla_Z \mu - \nabla_{[Z,X]} \mu + R_{\nabla}(Z, X)(\mu) + \nabla_Z \nabla_X \mu = 0. \] (71)

This is the definition of the curvature of \( \nabla \) via differential operators. Therefore, we see that if we start with the concept of a connection \( \nabla \), and apply the warp theorem to the grid in \( T^2A \), we are led to define the quantity \( R_{\nabla}(Z, X)(\mu) \) in this way.

### 6 The triple tangent bundle \( T^3M \) and the Jacobi identity

In this section we consider the triple tangent bundle \( T^3M \) of a manifold \( M \) and construct a grid on it, for which the Jacobi identity emerges as a consequence of the warp theorem. A version of this approach was given by Mackenzie [12]. We present here a clearer and more detailed calculation.

Take \( E \) to be \( T^3M \), the triple tangent bundle. This is a special case of \( T^2A \), for \( A = TM \):

\[
\begin{array}{ccc}
  T^3M & \xrightarrow{T^{(p)}} & T^2M \\
  p_{T^2M} & \xrightarrow{T(p_{TM})} & T^2M \\
  & \xrightarrow{p_{TM}} & T^2M \\
  & \xrightarrow{T(p)} & TM \\
  & \xrightarrow{p} & M.
\end{array}
\]

The three lower faces are copies of \( T^2M \). The Left face is the double tangent bundle of the manifold \( TM \). The Back face is not a double tangent bundle; it is the tangent double vector bundle of \( T^2M \xrightarrow{T^{(p)}} TM \). The Up face is obtained by applying the tangent functor to \( T^2M \).
Starting with three vector fields $X$, $Y$, and $Z$, each a section of one of the three copies of $TM$, one can build a grid on $T^3M$ as follows; see (72) below.

- The front-back linear double section $(T(X); T(X), \tilde{X}; X)$. Take the complete lift of $X$ across the Down face, and apply the tangent functor to the linear section $(\tilde{X}, X)$.

- The right-left linear double section $(T^2(Y); T(Y), T(Y); Y)$. Apply the tangent functor to $Y$ and then to $T(Y)$.

- The down-up linear double section $(\tilde{Z}; \tilde{Z}, \tilde{Z}; Z)$. Take the complete lift of $Z$ across the Front face, and the complete lift of this across the Left face. Likewise take the complete lift of $Z$ across the Right face. One does need to check that $(\tilde{Z}, \tilde{Z})$ is indeed a linear section of the Back face.

Equation (72) shows the entire grid.

We now calculate the three ultrawarps defined by this grid. To do this, we calculate the core morphisms of the three linear double sections, and the warps of the six faces.

First, the core morphisms. These follow in an analogous way as in the example of $T^2A$,

- The core morphism of $(T(X); T(X), \tilde{X}; X)$ is $(\tilde{X}, X)$.

- The core morphism of $(T^2(Y); T(Y), T(Y); Y)$ is $(T(Y), Y)$.

- The core morphism of $(\tilde{Z}; \tilde{Z}, \tilde{Z}; Z)$ is $(\tilde{Z}, Z)$.

To calculate the warps of the six faces, we take into consideration the orientation of the faces of a triple vector bundle. For the lower faces, by (5):

- For the Front face: $\tilde{Z}(Y) - T(Y)(Z) \triangleright [Y, Z]$.

- For the Right face: $T(X)(Z) - \tilde{Z}(X) \triangleright [Z, X]$.

- For the Down face: $T(Y)(X) - \tilde{X}(Y) \triangleright [X, Y]$.

We now calculate the warps of the upper faces.
Back face

The warp of the Back face, for \( x \in TM \), is given by

\[
\tilde{Z} \circ T(Y)(x) \xrightarrow{T^2(p)} T^2(Y) \circ \tilde{Z}(x) = w_{\text{back}}(x) + \hat{0}_{\tilde{Z}(x)}.
\] (73)

As we mentioned, the Back face is the tangent double vector bundle of \( T^2M \xrightarrow{T(p)} TM \). Apply \( T(J) \) to it, the tangent of the canonical involution \( J : T^2M \to T^2M \). The resulting double vector bundle is now the double tangent bundle of \( TM \). In fact, \( T(J) \) is a triple vector bundle morphism, and maps the Back face of \( T^3M \) to the double tangent bundle of \( TM \) as shown in (74).

![Diagram of vector bundles](image)

Also, the core morphism of (74) is \((J, \text{id})\). Hence, applying \( T(J) \) to (73),

\[
T(J) \left( \tilde{Z} \circ T(Y)(x) \xrightarrow{T^2(p)} T^2(Y) \circ \tilde{Z}(x) \right) = J(w_{\text{back}}(x)) + \hat{0}_{\tilde{Z}(x)}. \tag{75}
\]

Note that \( T(J) \) changes the vector bundle structure over which the subtraction of the left hand side takes place, and \( \xrightarrow{T^2(p)} \) will become \( \xrightarrow{T(p, TM)} \). Applying \( T(J) \) to the grid of the Back face yields the following grid on the double tangent bundle of \( TM \).

![Diagram of vector bundles](image)

Therefore, expanding the left hand side of (75),

\[
T(J) \left( \tilde{Z} \circ T(Y)(x) \xrightarrow{T^2(p)} T^2(Y) \circ \tilde{Z}(x) \right) = T(J)((\tilde{Z} \circ T(Y))(x)) \xrightarrow{T(p, TM)} T(J)((T^2(Y) \circ \tilde{Z})(x))
\]

\[
= (\tilde{Z} \circ \tilde{Y})(x) \xrightarrow{T(p, TM)} (T(\tilde{Y}) \circ \tilde{Z})(x) = -[\tilde{Z}, \tilde{Y}](x) + \hat{0}_{\tilde{Z}(x)} = [\tilde{Y}, \tilde{Z}](x) + \hat{0}_{\tilde{Z}(x)}.
\]

Therefore, expanding the left hand side of (75),

\[
T(J) \left( \tilde{Z} \circ T(Y)(x) \xrightarrow{T^2(p)} T^2(Y) \circ \tilde{Z}(x) \right) = T(J)((\tilde{Z} \circ T(Y))(x)) \xrightarrow{T(p, TM)} T(J)((T^2(Y) \circ \tilde{Z})(x))
\]

\[
= (\tilde{Z} \circ \tilde{Y})(x) \xrightarrow{T(p, TM)} (T(\tilde{Y}) \circ \tilde{Z})(x) = -[\tilde{Z}, \tilde{Y}](x) + \hat{0}_{\tilde{Z}(x)} = [\tilde{Y}, \tilde{Z}](x) + \hat{0}_{\tilde{Z}(x)}.
\]
Substituting this into (75),

\[ [Y, Z](x)_{\text{p}_{T^2M}} + \hat{0}_{\tilde{Z}(x)} = J(w_{\text{back}}(x))_{\text{p}_{T^2M}} + \hat{0}_{\tilde{Z}(x)}, \]

and using that \( J^2 = \text{id} \), we obtain

\[ w_{\text{back}} = T([Y, Z]). \]

**Left face**

The Left face is the double tangent bundle of \( TM \), so we simply apply (5) for the grid \( (T(\tilde{X}), \tilde{X}), (\tilde{Z}, \tilde{Z}) \),

\[ T(\tilde{X}) \circ \tilde{Z} - \tilde{Z} \circ \tilde{X} \triangleright [\tilde{Z}, \tilde{X}] = [\tilde{Z}, \tilde{X}], \]

so \( w_{\text{left}} = [\tilde{Z}, \tilde{X}] \).

**Up face**

For the Up face, using Proposition 14 it follows directly that \( w_{\text{up}} = T([X, Y]) \).

**The three ultrawarps**

The three core double vector bundles are all copies of \( T^2M \), and their ultracore is \( TM \to M \).

The three core double vector bundles in the usual order (B-F), (L-R), and (U-D), with the induced grids from the original grid on \( T^3M \),

Finally, by (5), the ultracore elements are

\[ w_{\text{back}} \circ X - \tilde{X} \circ w_{\text{front}} = T([Y, Z]) \circ X - \tilde{X} \circ [Y, Z] \triangleright [X, [Y, Z]], \]

\[ w_{\text{left}} \circ Y - T(Y) \circ w_{\text{right}} = [\tilde{Z}, X] \circ Y - T(Y) \circ [Z, X] \triangleright - [[Z, X], Y] = [Y, [Z, X]], \]

\[ w_{\text{up}} \circ Z - \tilde{Z} \circ w_{\text{down}} = T([X, Y]) \circ Z - \tilde{Z} \circ [X, Y] \triangleright [Z, [X, Y]]. \]

We see that in this way we have formulated the three terms of the Jacobi identity. And applying the warp theorem, we obtain a conceptual proof of the Jacobi identity.
7 Appendix: warps and duality

We include here an alternative formula for the warp which relies on the duality of double vector bundles. We will need only a few basics from duality theory; for further details see [10, §9.2].

Start with a double vector bundle $D$ and take the usual dual of $D$ with respect to $A$. This defines another double vector bundle $D \mathbin{\ltimes} A$, with core $B^*$. In a similar way we can define the double vector bundle $D \mathbin{\ltimes} B$, with core $A^*$. These duals are shown in (76).

$$D \xrightarrow{q_B} B \quad \quad \quad \quad \quad D \mathbin{\ltimes} A \xrightarrow{\gamma_A^C} C^* \quad \quad \quad \quad \quad D \mathbin{\ltimes} B \xrightarrow{\gamma_B^C} B$$

(76)

There exists a nondegenerate pairing between $D \mathbin{\ltimes} A$ and $D \mathbin{\ltimes} B$ over $C^*$ [10, 9.2.2], which is natural up to sign. For $\Phi \in D \mathbin{\ltimes} A$, and $\Psi \in D \mathbin{\ltimes} B$, with outlines $(\Phi; a, \kappa; m)$ and $(\Psi; \kappa, b; m)$ respectively, and for any $d \in D$ with outline $(d; a, b; m)$,

$$\| \Phi, \Psi \|_{C^*} = \langle \Phi, d \rangle_A - \langle \Psi, d \rangle_B. \quad (77)$$

This pairing induces two double vector bundle isomorphisms, namely,

$$Z_A : D \mathbin{\ltimes} A \to D \mathbin{\ltimes} B \mathbin{\ltimes} C^* , \quad \langle Z_A(\Phi), \Psi \rangle_{C^*} = \| \Phi, \Psi \|_{C^*}, \quad (78)$$

and

$$Z_B : D \mathbin{\ltimes} B \to D \mathbin{\ltimes} A \mathbin{\ltimes} C^* , \quad \langle Z_B(\Psi), \Phi \rangle_{C^*} = \| \Phi, \Psi \|_{C^*}. \quad (79)$$

The following material on linear sections and duality can be found in more detail in [11].

A linear section $(\eta, Y)$ of $D$ as in (7) induces a linear map $\ell_\eta : D \mathbin{\ltimes} A \to \mathbb{R}$. This map is automatically linear with respect to $D \mathbin{\ltimes} A \to A$ but is also linear with respect to the other vector bundle structure, $D \mathbin{\ltimes} A \to C^*$. It therefore induces a section of $D \mathbin{\ltimes} A \mathbin{\ltimes} C^* \to C^*$ and this is linear over $Y$. Using (77), (78), and (79), we will define a new pairing between sections of $D \mathbin{\ltimes} A \mathbin{\ltimes} C^* \to C^*$ and sections of $D \mathbin{\ltimes} B \mathbin{\ltimes} C^* \to C^*$. In the case where we begin with a grid $(\eta, Y)$ and $(\xi, X)$ on $D$ as usual, we will show that this new pairing provides us with a different way of expressing the warp $w(\xi, \eta)$.

How does a linear section $(\eta, Y)$ of $D$ define a linear section $(\eta^\gamma, Y)$ of $D \mathbin{\ltimes} A \mathbin{\ltimes} C^*$? Given $\kappa \in C^*$ we define $\eta^\gamma(\kappa) \in D \mathbin{\ltimes} A \mathbin{\ltimes} C^* |_{\kappa}$ by defining its pairing with $\Phi \in D \mathbin{\ltimes} A |_{\kappa}$ to be

$$\langle \eta^\gamma(\kappa), \Phi \rangle_{C^*} := \ell_{\eta}(\Phi) = \langle \Phi, \eta(\gamma_A^C(\Phi)) \rangle_{A}. \quad (80)$$

(We use notations such as $|_{\kappa}$ on double vector bundles when the symbol for the base point makes clear which structure is meant.)

That $(\eta^\gamma, Y)$ is a linear section of $D \mathbin{\ltimes} A \mathbin{\ltimes} C^* \to C^*$ follows immediately. The corresponding linear function is

$$\ell_{\eta^\gamma} : D \mathbin{\ltimes} A \to \mathbb{R}, \quad \Phi \mapsto \langle \eta^\gamma(\gamma_A^C(\Phi)), \Phi \rangle_{C^*},$$

and of course $\ell_{\eta^\gamma} = \ell_\eta$. The following proposition states that $\ell_\eta = \ell_{\eta^\gamma}$ is also linear with respect to $C^*$; see [11, 3.1].
Proposition 16. If $(\eta,Y)$ is a linear section, then $\ell_\eta : D \uparrow A \rightarrow \mathbb{R}$ defined by

$$\Phi \mapsto \langle \Phi, \eta(\gamma_A(\Phi)) \rangle$$

is linear with respect to $C^*$ as well as $A$, and the restriction of $\ell_\eta$ to the core of $D \uparrow A$ is $\ell_Y : B^* \rightarrow \mathbb{R}$.

Therefore, we see that there exists a one-to-one correspondence between linear sections $(\eta,X)$ of $D \rightarrow A$ and linear sections $(\eta^\gamma, X)$ of $D \uparrow A \uparrow C^* \rightarrow C^*$.

Similarly, given $(\xi,X)$, define a section $\xi\uparrow$ of $D \uparrow B \uparrow C^* \rightarrow C^*$ by defining the pairing of $\xi\uparrow(\kappa)$, for $\kappa \in C^*$, with $\Psi \in D \uparrow B|_\kappa$ to be

$$\langle \xi\uparrow(\kappa), \Psi \rangle_{C^*} := \ell_{\xi}(\Psi) = \langle \Psi, \xi(\gamma_B(\Psi)) \rangle_B.$$

Now we want to define a pairing between $\xi\uparrow \in \Gamma_{C^*}(D \uparrow B \uparrow C^*)$ and $\eta\uparrow \in \Gamma_{C^*}(D \uparrow A \uparrow C^*)$:

$$\llbracket \xi\uparrow, \eta\uparrow \rrbracket(\kappa) = \langle \xi\uparrow(\kappa), \eta\uparrow(\kappa) \rangle$$

and it will follow that $\llbracket \xi\uparrow, \eta\uparrow \rrbracket \in C^\infty(C^*)$. Since $D \uparrow A \uparrow C^*$ and $D \uparrow A$ are dual vector bundles over $C^*$, we have the usual pairing between them. We will use it to define a pairing between $D \uparrow A \uparrow C^*$ and $D \uparrow B \uparrow C^*$ over $C^*$. Using the map $Z_A^{-1} : D \uparrow B \uparrow C^* \rightarrow D \uparrow A$ we define

$$\llbracket \xi\uparrow(\kappa), \eta\uparrow(\kappa) \rrbracket = \langle \eta\uparrow(\kappa), Z_A^{-1}(\xi\uparrow(\kappa)) \rangle_{C^*}.$$  (82)

The following outlines are useful to help us keep track of the various calculations,

\[
\begin{array}{ccc}
D \uparrow A \ni Z_A^{-1}(\xi\uparrow(\kappa)) & \rightarrow & \kappa \\
\downarrow & & \downarrow \\
-X(m) & \rightarrow & m, \\
\downarrow & & \downarrow \\
-\xi(\gamma_B(Y(m))) & \rightarrow & Y(m) \\
\downarrow & & \downarrow \\
-\xi(\gamma_B(X(m))) & \rightarrow & m.
\end{array}
\]

Note that the minus sign on $-X(m)$ of $Z_A^{-1}(\xi\uparrow(\kappa))$ comes from the fact that $Z_A$ induces $-\text{id}_A : A \rightarrow A$ over $M$.

We can now begin calculations. Start with (82),

$$\llbracket \xi\uparrow(\kappa), \eta\uparrow(\kappa) \rrbracket = \langle \eta\uparrow(\kappa), Z_A^{-1}(\xi\uparrow(\kappa)) \rangle_{C^*} \overset{(80)}{=} \ell_{\eta}(Z_A^{-1}(\xi\uparrow(\kappa))) = -\langle Z_A^{-1}(\xi\uparrow(\kappa)), \eta(\gamma_B(X(m))) \rangle_A.$$  

Now use (77), with $\Phi = Z_A^{-1}(\xi\uparrow(\kappa))$, $\Psi = Z_B^{-1}(\eta\uparrow(\kappa))$, and $d = -\eta(\gamma_B(X(m)))$,

$$\|Z_A^{-1}(\xi\uparrow(\kappa)), Z_B^{-1}(\eta\uparrow(\kappa))\|_{C^*} = \langle Z_A^{-1}(\xi\uparrow(\kappa)), -\eta(\gamma_B(X(m))) \rangle_A - \langle Z_B^{-1}(\eta\uparrow(\kappa)), -\eta(\gamma_B(X(m))) \rangle_B \Rightarrow -\langle Z_A^{-1}(\xi\uparrow(\kappa)), \eta(\gamma_B(X(m))) \rangle_A = \|Z_A^{-1}(\xi\uparrow(\kappa)), Z_B^{-1}(\eta\uparrow(\kappa))\|_{C^*} - \langle Z_B^{-1}(\eta\uparrow(\kappa)), \eta(\gamma_B(X(m))) \rangle_B.$$
Returning to the previous calculations

\[
\begin{align*}
\langle \xi^\diamondsuit(\kappa), \eta^\diamondsuit(\kappa) \rangle & = -(Z_A^{-1}(\xi^\diamondsuit(\kappa)), \eta(X(m)))_A \\
& = \|Z_A^{-1}(\xi^\diamondsuit(\kappa), Z_B^{-1}(\eta^\diamondsuit(\kappa)))\|_{C^*} - \langle Z_B^{-1}(\eta^\diamondsuit(\kappa), \eta(X(m)))_B \\
& \text{(83)}
\end{align*}
\]

\[
\begin{align*}
\langle \xi^\diamondsuit(\kappa), Z_B^{-1}(\eta^\diamondsuit(\kappa)) \rangle_{C^*} - \langle Z_B^{-1}(\eta^\diamondsuit(\kappa), \eta(X(m)))_B \\
& \text{(83)}
\end{align*}
\]

\[
\begin{align*}
& = \langle Z_B^{-1}(\eta^\diamondsuit(\kappa)), \xi(Y(m))_B - \eta(X(m))_B \\
& = \langle Z_B^{-1}(\eta^\diamondsuit(\kappa)), w(\xi, \eta)(m) + 0^D_{Y(m)} B \rangle.
\end{align*}
\]

By equation (16) from [10, p. 348], since \( \gamma^B_C (Z_B^{-1}(\eta^\diamondsuit(\kappa))) = \kappa \), we can rewrite the last expression as

\[
\langle Z_B^{-1}(\eta^\diamondsuit(\kappa)), w(\xi, \eta)(m) + 0^D_{Y(m)} B \rangle = \langle \kappa, w(\xi, \eta)(m) \rangle = \ell_{w(\xi, \eta)}(\kappa).
\]

In other words, we have shown that, for \( \kappa \in C^* \),

\[
[[\xi^\diamondsuit, \eta^\diamondsuit]](\kappa) = \ell_{w(\xi, \eta)}(\kappa).
\]

We state this result formally.

**Proposition 17.** Let \((\xi, X)\) and \((\eta, Y)\) be linear sections forming a grid on a double vector bundle \( D \). Then

\[
[[\xi^\diamondsuit, \eta^\diamondsuit]] = \ell_{w(\xi, \eta)}.
\]

An equivalent way of defining the pairing (82) is via the map \( Z_B^{-1} : D \tens A \tens C^* \to D \tens B \), as follows

\[
[[\xi^\diamondsuit(\kappa), \eta^\diamondsuit(\kappa)]] = \langle \xi^\diamondsuit(\kappa), Z_B^{-1}(\eta^\diamondsuit(\kappa)) \rangle_{C^*}.
\]

(84)

It is easy to check that both (82) and (84) define the same pairing. And comparing (84) with (83), we see that

\[
\langle Z_B^{-1}(\eta^\diamondsuit(\kappa)), \eta(X(m))_B \rangle = 0.
\]

Indeed, to see this, let \( \Phi \in D \tens A \), with outline \((\Phi; X(m), \kappa; m)\). Then, via (77), rewrite (83) as follows

\[
\begin{align*}
\langle Z_B^{-1}(\eta^\diamondsuit(\kappa)), \eta(X(m))_B \rangle & = \langle \Phi, \eta(X(m))_A \rangle - \|\Phi, Z_B^{-1}(\eta^\diamondsuit(\kappa))\|_{C^*} \\
& \text{(80)}
\end{align*}
\]

\[
\begin{align*}
& = \langle \Phi, \eta(X(m))_A \rangle - \langle \eta^\diamondsuit(\kappa), \Phi \rangle_{C^*} \\
& = \langle \Phi, \eta(X(m))_A \rangle - \langle \Phi, \eta(X(m))_A \rangle \\
& = 0.
\end{align*}
\]

**7.1 Example with \( T^2M \)**

In case of \((\tilde{X}, X)\) and \((T(Y), Y)\), what are the corresponding \( \tilde{X}^\diamondsuit \) and \( T(Y)^\diamondsuit \)? Starting with \((T(Y), Y)\), we will calculate \( T(Y)^\diamondsuit \) using \( \ell_{T(Y)} \),

\[
\ell_{T(Y)} : T^\bullet(TM) \to \mathbb{R}, \quad T^\bullet(TM) \ni \xi \mapsto \langle \xi, T(Y)(x) \rangle_{TM},
\]

where \( x \in TM \) and \( T^\bullet(TM) \) is the fibre of \( T^\bullet(TM) \) over \( x \in TM \). Here \( T^\bullet(TM) \) is the dual of \( T(T(p)) : T^2M \to TM \) (see [10] p. 355 for the notation). The function \( \ell_{T(Y)} \) is linear with respect to
both $TM$ and $T^*M$ (see Proposition 10). Its linearity with respect to $TM$ will lead us back to $T(Y)$. We are interested in its linearity with respect to $T^*M$. This will define a linear section of the dual of the vector bundle $T^*(TM) \to T^*M$, that is, of $T^*(TM) \times T^*M \to T^*M$. However, this is very awkward to work with.

We use the map $I : T(T^*M) \to T^*(TM)$, see [10 9.3.2], a double vector bundle isomorphism that induces the identity map on both side bundles and on the core vector bundle.

Take the function

$$\ell_{T(Y)} \circ I : T(T^*M) \to \mathbb{R}.$$ 

It follows directly that this is also linear with respect to $T^*M$.

Therefore, it will define a linear section $\mathfrak{g}$ of the dual of the vector bundle $T(T^*M) \to T^*M$, that is, of the vector bundle $T^*(TM) \to T^*M$.

Consider $\mathfrak{g}(\varphi) \in T^*(T^*M)$ for $\varphi \in T^*M$. Pair this with a $\xi \in T(T^*M)$ with outline $(\xi; \varphi, x; m)$, where $x \in TM$. Using (86),

$$\langle \mathfrak{g}(\varphi), \xi \rangle_{T^*M} = (\ell_{T(Y)} \circ I)(\xi) = \langle I(\xi), T(Y)(x) \rangle_{TM}.$$ 

By [10 9.3.2] and [10 3.4.6], it follows that

$$\langle I(\xi), T(Y)(x) \rangle_{TM} = \langle d\ell_Y(\varphi), \xi \rangle.$$ 

This is true for any such $\xi \in T(T^*M)$ so it follows that

$$\mathfrak{g}(\varphi) = (d\ell_Y)(\varphi),$$

and the linear section in question, $(T(Y)^\dagger, Y)$, can be identified with $(d\ell_Y, Y)$.

Now $\tilde{X}$ defines the linear function

$$\ell_{\tilde{X}} : T^*(TM) \to \mathbb{R}.$$ 

The function $\ell_{\tilde{X}}$ is linear with respect to both $TM$ and $T^*M$ (see Proposition 10). Similarly as before, we elaborate on $\ell_{\tilde{X}}$ being linear with respect to $T^*M$. This defines a section of the dual of the vector bundle $T^*(TM) \to T^*M$, that is, of $T^*(TM) \times T^*M \to T^*M$. Again, this is not easy to work with, and in this case we use the reversal isomorphism $R : T^*(T^*M) \to T^*(TM)$. It follows that

$$\ell_{\tilde{X}} \circ R : T^*(T^*M) \to \mathbb{R}$$

is also linear with respect to $T^*M$, and defines a section $\tilde{X}$ of the dual of the vector bundle $T^*(T^*M) \to T^*M$, that is, of the vector bundle $T(T^*M) \to T^*M$.

Then for $\varphi \in T^*M$, and any $\mathfrak{f} \in T^*(T^*M)$ with outline $(\mathfrak{f}; x, \varphi; m)$, with $x \in TM$, using (86),

$$\langle \tilde{X}(\varphi), \mathfrak{f} \rangle_{T^*M} = (\ell_{\tilde{X}} \circ R)(\mathfrak{f}) = \ell_{\tilde{X}}(R(\mathfrak{f})) = \langle R(\mathfrak{f}), \tilde{X}(x) \rangle_{TM}.$$ 

At this point, we use that $R = J^* \circ I \circ (d\nu)^\sharp$, [10 p. 442], where $(d\nu)^\sharp$ is the map associated to the canonical symplectic structure $d\nu$ on $T^*M$.

$$\begin{array}{ccc}
T^*(T^*M) & \xrightarrow{R} & T^*(TM) \\
\downarrow{(d\nu)^\sharp} & & \uparrow{J^*} \\
T(T^*M) & \xrightarrow{I} & T^*(TM)
\end{array}$$

\[ \text{Equation (87)} \]
Note that \( \mathcal{S} \) is a commutative diagram of isomorphisms of double vector bundles. Therefore, we can write
\[
\langle R(\mathcal{S}), X(x) \rangle_{TM} = \langle J^*(I((dv)^2(\mathcal{S}))), X(x) \rangle_{TM} = \langle I((dv)^2(\mathcal{S})), J(X(x)) \rangle_{TM} = \langle I((dv)^2(\mathcal{S})), T(X(x)) \rangle_{TM}.
\]
As before, using \( \mathcal{S} \),
\[
\langle I((dv)^2(\mathcal{S})), T(X(x)) \rangle_{TM} = (dv)^2(\mathcal{S})(\ell_X) = \langle d\ell_X, (dv)^2(\mathcal{S}) \rangle = \langle (dv)^2(d\ell_X), \mathcal{S} \rangle,
\]
so we see that \( X = -(dv)^2(d\ell_X) \); that is, it is the Hamiltonian vector field for the function \( \ell_X \). Denote it by \( H_{\ell_X} \). Finally,
\[
\llangle T(Y) \rrangle = \langle d\ell_Y, H_{\ell_X} \rangle = \ell_{[X,Y]}.
\]

### 7.2 Example with \( TA \)

In the case of Example 3 what are the corresponding \( T(\mu) \) and \( (X^H) \) defined?

Just as in the case of \( T^2M, (T(\mu), \mu) \) can be identified with \( (d\ell_\mu, \mu) \).

We elaborate a bit more on \( (X^H, X) \). Again, we use \( \ell_XH \circ R : T^*A^* \to \mathbb{R} \), and its linearity with respect to \( A^* \). This will define a section of the dual of \( T^*A^* \to A^* \), that is, of \( TA^* \to A^* \). Denote this by \( \Phi \).

Then for \( \kappa \in A^*, \Phi(\kappa) \in TA^*. \) Pair it with a \( \Psi \in T^*A^* \) with outline \( (\Psi; \kappa, a; m) \).

Then, for suitable \( \mathcal{X} \in TA^* \), and using \([10, 9.5.1]\), it follows that
\[
\langle \Phi(\kappa), \Psi \rangle_{A^*} = (\ell_{XH} \circ R)(\Psi) = (R(\Psi), X^H(a))_A = \llangle \mathcal{X}, X^H(a) \rrangle - \langle \Psi, \mathcal{X} \rangle_{A^*}.
\]

The outlines for the elements involved are:

\[
\begin{array}{ccc}
T^*A \ni R(\Psi) & \longrightarrow & a \in A \\
\downarrow & & \downarrow \\
A^* \ni \kappa & \longrightarrow & m,
\end{array}
\begin{array}{ccc}
TA \ni X^H(a) & \longmapsto & X(m) \in TM \\
\downarrow & & \downarrow \\
A \ni a & \longrightarrow & m,
\end{array}
\begin{array}{ccc}
T^*A^* \ni \Psi & \longrightarrow & a \in A \\
\downarrow & & \downarrow \\
A^* \ni \kappa & \longrightarrow & m,
\end{array}
\begin{array}{ccc}
TA^* \ni \mathcal{X} & \longmapsto & X(m) \in TM \\
\downarrow & & \downarrow \\
A^* \ni \kappa & \longrightarrow & m.
\end{array}
\]

Now use \([10, 3.4.6]\) for the first term of the last equation. Choose a \( \varphi \in \Gamma A^* \) with \( \varphi(m) = \kappa \), and \( \mu \in \Gamma A \) with \( \mu(m) = a \). We can also make the following choice; linear vector fields of a vector bundle \( A \) are in bijective correspondence with linear vector fields on its dual bundle \( A^* \) (see \([10, 3.4.5]\)). Therefore, take \( \mathcal{X} \) to be \( X^{H*}(\varphi(m)) \), where \( X^{H*} \) is the corresponding linear vector field to \( X^H \). Then we can write
\[
\langle \llangle X^{H*}(\varphi(m)), X^H(a) \rrangle \rangle - \langle \Psi, X^{H*}(\varphi(m)) \rangle_{A^*} = X^{H*}(\varphi(m))(\ell_{\mu}) + X^H(\mu(m))(\ell_{\varphi}) - X(m)(\llangle \varphi, \mu \rrangle) - \langle \Psi, X^{H*}(\varphi(m)) \rangle_{A^*}.
\]

At this point, we recall that for \( \varphi \in \Gamma A^* \), and for \( \mu \in \Gamma A \),
\[
X^H(\ell_{\varphi}) = \ell_{\nabla_X(\varphi)} \in C^\infty(E), \quad X^{H*}(\ell_{\mu}) = \ell_{\nabla_X(\mu)} \in C^\infty(E^*),
\]
and of course the relation between $\nabla$ and $\nabla^*$,

$$\langle \nabla^*(\varphi), \mu \rangle = X(\langle \varphi, \mu \rangle) - \langle \varphi, \nabla X(\mu) \rangle,$$

and the latter equation can be rewritten as

$$\ell_{\nabla^*(\varphi)} \circ \mu = X(\langle \varphi, \mu \rangle) - \ell_{\nabla(\mu)} \circ \varphi.$$

Returning to (88),

$$\langle \langle X^H(\varphi(m)), X^H(a) \rangle \rangle - \langle \Psi, X^H(\varphi(m)) \rangle_{A^*} = \ell_{X(\mu)}(\varphi(m)) + \ell_{\nabla^*(\varphi)}(\mu(m)) - X(m)(\langle \varphi, \mu \rangle) - \langle \Psi, X^H(\varphi(m)) \rangle_{A^*} = -\langle \Psi, X^H(\varphi(m)) \rangle_{A^*}.$$

Finally, the pairing between $(d\ell_{\mu}, \mu)$ and $(X^H, X)$ is,

$$\langle X^H, d\ell_{\mu} \rangle = X^H(\ell_{\mu}) = \ell_{X(\mu)}.$$

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