Running Gauge Coupling and Quark-Antiquark Potential in Non-SUSY Gauge Theory at Finite Temperature from IIB SG/CFT correspondence

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abstract

We discuss the non-constant dilaton deformed AdS$_5 \times$ S$_5$ solutions of IIB supergravity where AdS sector is described by black hole. The investigation of running gauge coupling (exponent of dilaton) of non-SUSY gauge theory at finite temperature is presented for different regimes (high or low $T$, large radius expansion). Running gauge coupling shows power-like behavior on temperature with stable fixed point. The quark-antiquark potential at finite $T$ is found and possibility of confinement is established. It is shown that non-constant dilaton affects the potential, sometimes reversing its behavior if we compare it with the constant dilaton case ($\mathcal{N} = 4$ super Yang-Mills theory). Thermodynamics of obtained backgrounds is studied. In particular, next-to-leading term to free energy $F$ is evaluated as $F = -\frac{\bar{V}_3}{4\pi^2} \left( \frac{N^2(\pi T)^4}{2} + \frac{5c^2}{768g_5^6y^2N\alpha'^4(\pi T)^4} \right)$. Here $\bar{V}_3$ is the volume of the space part in the boundary of AdS, $c$ is the parameter coming from the non-constant dilaton and $N$ is the number of the coincident D3-branes.

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1 Introduction.

One of the unsolved problems in AdS/CFT correspondence [1] (for an excellent review, see [2]) is how to obtain non-SUSY gauge theory with typical running coupling as the boundary side. The related question is about confinement in such theory. It is desirable to answer these questions from supergravity (SG) side as it gives strong coupling regime of boundary quantum field theory (QFT).

There are different proposals to get running gauge coupling in non-SUSY theory: using Type 0 string theory approach [3], deforming $\mathcal{N} = 4$ theory [15] (also via AdS orbifolding [16]) or making non-constant dilaton deformations of AdS$_5 \times S_5$ vacuum in IIB SG [4, 7, 8, 9, 10, 17]. In the last case non-constant dilaton breaks conformal invariance and (a part of) supersymmetry of the boundary $\mathcal{N} = 4$ super YM theory. (In the presence of axion (RR-scalar), a part of supersymmetry may be unbroken but dilaton is still non-trivial [14]). Then, exponent of dilaton actually describes the running gauge coupling with a power-law behavior and UV-stable fixed point. Within such picture the indication to the possibility of confinement is also found. The features of running and confinement depend on the axion [8], vectors [9], worldvolume scalar [10] or curvature of four-dimensional space [17].

From another side, it is also realized that planar AdS$_5$ BH is dual to a thermal state of $\mathcal{N} = 4$ super YM theory. The corresponding coupling constant dependence has been studied in ref. [18, 24] based on earlier study of SG side free energy in ref. [19]. Spherical AdS BH shows the finite temperature phase transition [20] which may be used to realize the confinement in large $\mathcal{N}$ theory at low temperatures [21].

In this paper, we attempt to combine these two approaches, i.e. to find the deformation of IIB SG AdS$_5 \times S_5$ background with non-trivial dilaton where AdS sector is described by BH (hence, temperature appears). Then, running gauge coupling of gauge theory at non-zero temperature is given by exponent of dilaton. We present the class of approximate solutions of IIB SG\footnote{These solutions presumably describe thermal states of non-SUSY gauge theory which descends from $\mathcal{N} = 4$ super YM after breaking of SUSY and conformal invariance.} with such properties where running coupling shows power-like behavior in the temperature (in the expansion on radius). The quark-antiquark potential for these solutions is also found and possibility of confinement at

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non-zero temperature is established. Corrections to position of horizon (in near horizon regime) and to the temperature are calculated. Thermodynamics of obtained solutions is also investigated.

The paper is organized as follows. In the next section we present the approximate solution of IIB supergravity. It represents dilatonic perturbation of zero mass hyperbolic AdS BH. The temperature dependence of running gauge coupling (exponent of dilaton) and of corresponding beta-function is derived in different regimes. Quark-antiquark potential which is repulsive unlike to constant dilaton case is analyzed. Section 3 is devoted to the study of the same questions for background representing dilatonic deformation of (non)planar non-zero mass AdS BH. The temperature dependence of running gauge coupling is different from the situation in previous section. Confinement is possible as it follows from the study of quark-antiquark potential. In section 4, we investigate thermodynamic properties of our AdS backgrounds. Free energy, mass and entropy are found with account of non-trivial temperature corrections due to dilaton. This is compared with the leading behaviour of free energy in \( N = 4 \) super Yang-Mills theory. Some outlook is given in the last section.

## 2 Perturbative solutions of IIB supergravity, running gauge coupling and potential: zero mass BH case

We start from the action of dilatonic gravity in \( d + 1 \) dimensions:

\[
S = -\frac{1}{16\pi G} \int d^{d+1}x \sqrt{-G} \left( R - \Lambda - \alpha G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right). \tag{1}
\]

In the following, we assume \( \lambda^2 \equiv -\Lambda \) and \( \alpha \) to be positive. The action (1) contains the effective action of type IIB string theory. In the type IIB supergravity, which is the low energy effective action of the type IIB string theory, we can consider bosonic background where anti-self-dual five-form is given by the Freund-Rubin-type ansatz and the topology is \( M_5 \times S^5 \) with the manifold \( M_5 \) which is asymptotically AdS. If dilaton only depends on the coordinates in \( M_5 \), by integrating five coordinates on \( S^5 \), we obtain the effective five dimensional theory, which corresponds to \( d = 4 \) and \( \alpha = \frac{1}{2} \) case in (1). This will be the case under consideration in this work.
From the variation of the action (1) with respect to the metric $G^{\mu\nu}$, we obtain

$$0 = R_{\mu\nu} - \frac{1}{2} G_{\mu\nu} R + \frac{\Lambda}{2} G_{\mu\nu} - \alpha \left( \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} G_{\mu\nu} G^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi \right)$$ \hspace{1cm} (2)

and from that of dilaton $\phi$

$$0 = \partial_\mu \left( \sqrt{-G} G^{\mu\nu} \partial_\nu \phi \right).$$ \hspace{1cm} (3)

We now assume the $(d + 1)$-dimensional metric is given by

$$ds^2 = -e^{2\rho} dt^2 + e^{2\sigma} dr^2 + r^2 \sum_{i,j=1}^{d-1} g_{ij} dx^i dx^j.$$ \hspace{1cm} (4)

Here $g_{ij}$ does not depend on $r$ and it is the metric in the Einstein manifold, which is defined by

$$\hat{\mathcal{R}}_{ij} = k g_{ij}.$$ \hspace{1cm} (5)

Here $\hat{\mathcal{R}}_{ij}$ is Ricci tensor defined by $g_{ij}$ and $k$ is a constant, especially $k > 0$ for sphere, $k = 0$ for Minkowski space and $k < 0$ for hyperboloid. We also assume $\rho, \sigma$ and $\phi$ only depend on $r$. Then the equations (2) when $\mu = \nu = t$, $\mu = \nu = r$ and $\mu = i$ and $\nu = j$ give, respectively,

$$0 = \frac{(d-1)k e^2 \sigma}{2r^2} + \frac{(d-1)\sigma'}{r} - \frac{(d-1)(d-2)}{2r^2}$$

$$+ \frac{\lambda^2}{2} e^{2\sigma} - \frac{\alpha}{2} (\phi')^2$$ \hspace{1cm} (6)

$$0 = -\frac{(d-1)k e^2 \sigma}{2r^2} + \frac{(d-1)\rho'}{r} + \frac{(d-1)(d-2)}{2r^2}$$

$$- \frac{\lambda^2}{2} e^{2\sigma} - \frac{\alpha}{2} (\phi')^2.$$ \hspace{1cm} (7)

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4 The conventions of curvatures are given by

$$R = G^{\mu\nu} R_{\mu\nu}$$

$$R_{\mu\nu} = -\Gamma^\lambda_{\mu\lambda,\kappa} + \Gamma^\lambda_{\mu\kappa,\lambda} - \Gamma^\eta_{\mu\lambda} \Gamma^\lambda_{\kappa\eta} + \Gamma^\eta_{\mu\kappa} \Gamma^\lambda_{\lambda\eta}$$

$$\Gamma^\eta_{\mu\lambda} = \frac{1}{2} G^{\eta\nu} \left( G_{\mu\nu,\lambda} + G_{\lambda\nu,\mu} - G_{\mu\lambda,\nu} \right).$$
\[
0 = \frac{-(d - 3)ke^{2\sigma}}{2r^2} + \rho'' + (\rho')^2 - \rho'\sigma' + \frac{(d - 2)(d - 3)}{2r^2} - \frac{\lambda^2}{2}e^{2\sigma} + \frac{\alpha}{2}(\phi')^2 . \tag{8}
\]

Here \(\equiv \frac{d}{dr}\). Other components give identities. Eq. (3) has the following form
\[
0 = \left(r^{d-1}e^{\rho - \sigma}\phi\right)' , \tag{9}
\]
which can be integrated to give
\[
r^{d-1}e^{\rho - \sigma}\phi = c . \tag{10}
\]
Combining (3) and (7) and substituting (10), we obtain
\[
0 = \left(\frac{(d - 1)ke^{2\sigma}}{r} - \frac{\alpha c^2e^{2\sigma - 2\rho}}{r^{2d-2}}\right) - \frac{(d - 1)(\sigma' - \rho')}{r^2} - \frac{(d - 1)(d - 2)}{r^2} + \lambda^2e^{2\sigma} . \tag{12}
\]
If we introduce new variables \(U\) and \(V\) by
\[
U \equiv e^{\rho + \sigma} , \quad V \equiv r^{d-2}e^{\rho - \sigma} , \tag{13}
\]
Eqs. (3), (12) and (10) are rewritten as follows
\[
0 = (d - 1)U' - \frac{\alpha c^2}{rV^2}U \tag{14}
\]
\[
0 = \left\{\frac{(d - 1)k}{r^2} + \frac{\lambda^2}{r}\right\}U - \frac{(d - 1)}{r^{d-1}}V' \tag{15}
\]
\[
\phi' = \frac{c}{rV} \tag{16}
\]
Deleting \(U\) from (14) and (15), we obtain
\[
0 = V'' + \left[-\frac{d - 3}{r} - \frac{2\lambda^2r}{(d - 1)k + \lambda^2r^2} \right]V' - \frac{\alpha c^2V'}{(d - 1)rV^2} . \tag{17}
\]

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When $c = 0$ the solution is given by
\begin{align*}
U &= 1 \\
V &= V_0 \\
&\equiv \frac{kr^{d-2}}{d-2} + \frac{\lambda^2}{d(d-1)}r^d - \mu.
\end{align*}
(18)

Here $\mu$ corresponds to the mass of the black hole. $k = 0$, positive or negative corresponds to planar, spherical or hyperbolic AdS BH, respectively. Using (18), Eq.(15) and (17) can be rewritten as follows:
\begin{align*}
U &= \frac{V'}{V_0} \\
0 &= \left(\frac{V'}{V_0}\right)' - \frac{\alpha c^2 V'}{(d-1)rV_0^2V^2}.
\end{align*}
(19)

When $\mu = 0$, the solution is isomorphic to AdS. If we choose $k < 0$, the metric has the following form:
\begin{equation}
ds^2 = -(r^2 - r_0^2)\frac{l^2}{l^2}dt^2 + l^2\frac{l^2}{(r^2 - r_0^2)}dr^2 + r^2\sum_{i,j=1}^{d-1}g_{ij}dx^idx^j.
\end{equation}
(20)

Here
\begin{equation}
l^2 \equiv \frac{d(d - 1)}{\lambda^2}, \quad r_0 \equiv l\sqrt{-\frac{k}{d - 2}}.
\end{equation}
(21)

The obtained AdS metric has a horizon at $r = r_0$. When $r \sim r_0$, the metric behaves as
\begin{equation}
ds^2 \sim \frac{2r_0(r - r_0)}{l^2}dt^2 + \frac{l^2}{2r_0(r - r_0)}dr^2 + \cdots.
\end{equation}
(22)

Then if we define a new coordinate $\rho$ by
\begin{equation}
\rho = l\sqrt{\frac{2(r - r_0)}{r_0}}.
\end{equation}
(23)

the metric has the following form:
\begin{equation}
ds^2 \sim -\frac{r_0}{l^4}\rho^2dt^2 + d\rho^2 + \cdots.
\end{equation}
(24)
Therefore when we Wick-rotate $t$ by $t = i\tau$, $\tau$ has a period of $\frac{2\pi l^2}{r_0}$, whose inverse gives a temperature $T$:

$$T = \frac{r_0}{2\pi l^2} = \frac{1}{2\pi l} \sqrt{-\frac{k}{d-2}}. \tag{25}$$

We now consider the perturbation with respect to $c$. We will concentrate on the case of type IIB SG in $d = 4$, by putting $\alpha = \frac{1}{2}$. Note that in this approximation the radius is away from horizon. Near-horizon regime will be discussed independently.

For $\mu = 0$ and $k < 0$ case, the leading term for the dilaton $\phi$ is given by substituting $V_0$ in (18) into (16)

$$\phi = \phi_0 + c l^2 \left\{ \frac{1}{2r_0^4} \ln \left( 1 - \frac{r_0^2}{r^2} \right) + \frac{1}{2r_0^2 r^2} \right\}$$

$$= \phi_0 + c \left\{ \frac{1}{2l^6(2\pi T)^4} \ln \left( 1 - \frac{l^4(2\pi T)^2}{r^2} \right) + \frac{1}{2l^2(2\pi T)^2 r^2} \right\}. \tag{26}$$

which gives the temperature dependent running dilaton. We should note that there is a singularity in the dilaton field at the horizon $r = r_0 = 2\pi l^2 T$. The fact that dilaton may become singular at IR has been mentioned already in two-boundaries AdS solution of IIB SG in ref.[4]. It is also interesting that when $r$ is formally less than $r_0$ then dilaton (and also running coupling) becomes imaginary.

Since the string coupling is given by

$$g = g_s e^{\phi} \ (g_s : \text{constant}), \tag{27}$$

we find the behaviour when $r$ is large and $c$ is small as

$$g \sim g_s \left\{ 1 + c l^2 \left( -\frac{1}{2r^4} - \frac{(2\pi l^2 T)^2}{3r^6} + O \left( r^{-8} \right) \right) + O(c^2) \right\}. \tag{28}$$

Here $\phi_0$ has been absorbed into the redefinition of $g_s$. Since $r$ is the length scale corresponding to the radius of the boundary manifold, $r$ can be regarded as the energy scale of the field theory on the boundary [11]. Therefore the beta-function is given by

$$\beta(g) = r \frac{dg}{dr} = -4 (g - g_s) + \frac{2\hat{\chi}}{3} (2\pi T)^2 l^3 g_s \left( \frac{g_s - g}{c g_s} \right)^{\frac{3}{2}}. \tag{29}$$
The first term is usual and universal \cite{5, 8}. The second term defines the temperature dependence.

Let us comment on the case of high $T$. As we consider the behavior near the boundary, first we take $r$ to be large. After that we consider the case of high $T$. In this case $r \gg Tl^2$ and we can consider the large $T$ case in the expression (29). The problem might happen when $r \sim Tl^2$. In this case, we need to solve Eq. (26) with respect to $r$ as a function of $T$ and $\phi$ or coupling: $r = r(g, T)$. Then from (26) and (27), we find the following expression of the beta-function:

$$\beta(g) \sim r \frac{dg}{dr} \bigg|_{r=r(g, T)} = \frac{g_sc^2}{r(g, T)^4 \left(1 - \frac{1}{r(g, T)^2} \right)} .$$

(30)

In case $r$ is large, the above equation reproduces (29). We can also consider the case that the last term in (26) is larger than the second term which contains $\ln(\cdots)$. In this case, the coupling is given by

$$g \sim g_s \left(1 + \frac{c}{2l^2 (2\pi T)^2 r^2} \cdots \right) ,$$

(31)

which changes the leading behavior of the beta-function:

$$\beta(g) \sim -2 \left(g - g_s\right) + \cdots .$$

(32)

This beta-function presumably defines strong coupling regime of non-SUSY gauge theory at high temperature. It is interesting to note that in perturbative gauge theory at non-zero temperature the running gauge coupling contains not only standard logarithms of $T$ but also terms linear on $T$ (see ref. \cite{22} and references therein). Of course, in our case we have not AF theory but the one with stable fixed point.

Now we consider the correction for $V$ and $U$, writing them in the following form:

$$V = V_0 + c^2 v , \quad U = 1 + c^2 u .$$

(33)

Substituting (33) and neglecting the higher orders in $c^2$, we obtain

$$u = \frac{v'}{V_0'} ,$$

$$0 = \left( \frac{v'}{V_0'} \right)' - \frac{1}{6rV_0^2} .$$

(34)
With $\mu = 0$ and $k < 0$ in the above equations one gets,

$$
\begin{align*}
  u &= \frac{4}{3k^4l^4} \left\{ -\frac{1}{2s^2} - \frac{2}{s} \\
    &\quad -3 \ln \left( 1 - \frac{1}{s} \right) - \frac{1}{(s-1)} + c_1 \right\} \\
  v &= \frac{2}{3k^2l^2} \left\{ -\frac{1}{2} (3s^2 - 3s + 1) \ln \left( 1 - \frac{1}{s} \right) \\
    &\quad -\frac{3s}{2} + \frac{3}{4} - \frac{1}{4s} + \frac{c_1}{2} (s^2 - s) + c_2 \right\}
\end{align*}
$$

(35)

Here

$$
  s = -\frac{2r^2}{kl^2}
$$

(37)

and $c_1$ and $c_2$ are constants of the integration, which should vanish if we require $u, v \to 0$ when $r \to \infty$.

From (33) and (34), we find that $U$ and $V$ or $e^{2\rho}$ and $e^{2\sigma}$ have the singularity at the unperturbative horizon corresponding to $s = 1$. Eq. (26) tells also that the dilaton field is also singular there. In other words, the expansion with respect to $e^2$ breaks down when $s \sim 0$. Therefore the singularity in $U, V$ would not be real one.

In order to investigate the behavior in near-horizon regime we assume that the radius of the horizon is large and use $\frac{1}{r}$ expansion:

$$
V = \frac{r^4}{l^2} + \frac{kr^2}{2} + \frac{a}{r^4} + \mathcal{O} \left( r^{-6} \right).
$$

(38)

We put the constant term to be zero assuming that the black hole mass vanishes. The absence of $\frac{1}{r^2}$ term can be found from (19). Eq. (19) also tells that

$$
a = \frac{c^2l^2}{48}
$$

(39)

and $e^\phi, V$ and $U$ have the following forms:

$$
\begin{align*}
  e^\phi &= e^{\phi_0} \left( 1 - \frac{cl^2}{4r^4} + \mathcal{O} \left( r^{-6} \right) \right) \\
  V &= \frac{r^4}{l^2} + \frac{kr^2}{2} + \frac{c^2l^2}{48r^4} + \mathcal{O} \left( r^{-6} \right) \\
  U &= 1 - \frac{c^2l^4}{192r^8} + \mathcal{O} \left( r^{-10} \right).
\end{align*}
$$

(40)
From the equation $V = 0$ we find the position of the horizon

$$r = r_h \equiv l \sqrt{-\frac{k}{2} \left(1 - \frac{c^2}{6k^4l^4}\right)}, \quad (41)$$

which gives the correction to the temperature:

$$T = \frac{1}{2\pi l} \sqrt{-\frac{k}{2} - \frac{c^2 \left(-\frac{k}{2}\right)}{192l^5}}. \quad (42)$$

Let us turn now to the analysis of the potential between quark and anti-quark[6]. We evaluate the following Nambu-Goto action

$$S = \frac{1}{2\pi} \int d\tau d\sigma \sqrt{\det \left(g_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu\right)}. \quad (43)$$

with the “string” metric $g_{\mu\nu}$, which could be given by multiplying a dilaton function $e^\phi$ to the metric tensor in (4). We consider the static configuration $x^0 = \tau$, $x^1 \equiv x = \sigma$, $x^2 = x^3 = \cdots = x^{d-1} = 0$ and $r = r(x)$. Choose the coordinates on the boundary manifold so that the line given by $x^0 =$constant, $x^1 \equiv x$ and $x^2 = x^3 = \cdots = x^{d-1} = 0$ is geodesic and $g_{11} = 1$ on the line. Substituting the configuration into (43), we find

$$S = \frac{T}{2\pi} \int dx e^\phi(r) \sqrt{U(r)V(r) \left(\frac{U(r)}{V(r)} (\partial_x r)^2 + 1\right)}. \quad (44)$$

Here $T$ is the length of the region of the definition of $\tau$ and we choose $\phi_0 = 0$ for simplicity. The orbit of $r$ can be obtained by minimizing the action $S$ or solving the Euler-Lagrange equation $\delta S / \delta r - \partial_\sigma \left(\delta S / \delta (\partial_\sigma r)\right) = 0$. The Euler-Lagrange equation tells that

$$E_0 = e^\phi(r) \sqrt{U(r)V(r) \left(\frac{U(r)}{V(r)} (\partial_x r)^2 + 1\right)}, \quad (45)$$

is a constant. If we assume $r$ has a finite minimum $r_{\text{min}}$, where $\partial_x r|_{r=r_{\text{min}}} = 0$, $E_0$ is given by

$$E_0 = e^{\phi(r_{\text{min}})} \sqrt{U(r_{\text{min}})V(r_{\text{min}})}. \quad (46)$$

Introducing a parameter $t$, we parametrize $r$ by

$$r = r_{\text{min}} \cosh t. \quad (47)$$
Then we find
\[
\frac{dx}{dt} = \frac{l}{r_{\text{min}} \cosh^2 t \left( \cosh^2 t + 1 \right)^{\frac{3}{2}}} 
\times \left\{ 1 + \frac{kl^2}{4r_{\text{min}}^2} \cosh^4 t - \cosh^2 t - 1 \right\} + \mathcal{O} \left( r_{\text{min}}^{-4} \right) .
\] (48)

Taking \( t \to +\infty \), we find the distance \( L \) between ”quark” and ”anti-quark”
\[
L = \frac{LA}{r_{\text{min}}} + \frac{kl^3 B}{4r_{\text{min}}^3} + \mathcal{O} \left( r_{\text{min}}^{-5} \right) .
\] (49)

As one sees the next-to-leading correction to distance depends on the curvature of space-time\(^{[17]}\) or temperature.

Eq.(49) can be solved with respect to \( r_{\text{min}} \) and we find
\[
r_{\text{min}} = \frac{LA}{L} + \frac{kl^3 B}{4r_{\text{min}}^3} L^2 + \mathcal{O} \left( L^3 \right) .
\] (50)

Using (45), (47) and (49), we find the following expression for the action \( S \)
\[
S = \frac{T}{2\pi} E(L)
\] (51)
\[
E(L) = \int_{-\infty}^{\infty} dt \cosh^2 t \left( \cosh^2 t + 1 \right)^{\frac{3}{2}} \left\{ 1 + \frac{kl^2}{4r_{\text{min}}^2} \cosh^2 t \left( \cosh^2 t + 1 \right)^2 + \mathcal{O} \left( r_{\text{min}}^{-4} \right) \right\}.
\]

Here \( E(L) \) expresses the total energy of the “quark”-“anti-quark” system.

The energy \( E(L) \) in (51), however, contains the divergence due to the self energies of the infinitely heavy “quark” and “anti-quark”. The sum of their self energies can be estimated by considering the configuration \( x^0 = \tau, x^1 = x^2 = x^3 = \ldots = x^{d-1} = 0 \) and \( r = r(\sigma) \) (note that \( x_1 \) vanishes here) and
the minimum of $r$ is $r_D$, where branes would lie: $r_D \gg r_{\min}$. We divide the region for $r$ to two ones, $\infty > r > r_{\min}$ and $r_{\min} < r < r_D$. Using the parametrization of (47) for the region $\infty > r > r_{\min}$, we find the following expression of the sum of self energies:

$$E_{\text{self}} = 2r_{\min} \int_0^\infty dt \sinh t + 2 (r_{\min} - r_D) + O \left( r_{\min}^{-3} \right). \quad (52)$$

Then the finite potential between “quark” and “anti-quark” is given by

$$E_{q\bar{q}}(L) \equiv E(L) - E_{\text{self}} = r_{\min} \left( C + \frac{kl^2 D}{4r_{\min}^2} + O \left( r_{\min}^{-4} \right) \right)$$

$$= \frac{lAC}{L} + \frac{kl}{4} \left( \frac{BC}{A^2} + \frac{D}{A} \right) L + O \left( L^3 \right) \quad (53)$$

$$C = 2 \int_0^\infty dt \left\{ \frac{\cosh^2 t}{(\cosh^2 t + 1)^{\frac{3}{2}}} - \sinh t \right\} - 2 = -1.19814...$$

$$D = 2 \int_0^\infty \frac{dt}{(\cosh^2 t + 1)^{\frac{3}{2}}} = 0.711959.$$

Here we neglected the $r_{\min}$ or $L$ independent term. We should note that next-to-leading term is linear in $L$, which might be relevant to the confinement. For the confinement, it is necessary that the quark-antiquark potential behaves as

$$E_{q\bar{q}} \sim aL \quad (54)$$

with some positive constant $a$ for large $L$. For high temperature, it is usually expected that there occurs the phase transition to the deconfinement phase, where the potential behaves as Coulomb force,

$$E_{q\bar{q}} \sim \frac{d'}{L}. \quad (55)$$

Since $\frac{BC}{A^2} + \frac{D}{A} > 0$ and $k < 0$, the contribution from next-to-leading term in the potential is repulsive. The leading term expresses the repulsive but shows
the Coulomb like behavior. The next-leading-term tells that the repulsive force is long-range than Coulomb force.

The expression (53) is correct even at high temperature if $L$ is small or $r_{\text{min}}$ is large. If $r_{\text{min}}$ is small and the orbit of string approaches to the horizon and/or enters inside the horizon, the expression would not be valid. Since the horizon is given by (21), the expression (53) would be valid if

$$r_{\text{min}} \gg r_0 = l \sqrt{-\frac{k}{2}}$$

or using (25) and (50),

$$L \ll A \sqrt{-\frac{2}{k}} = 2\pi A l T.$$  (57)

The above condition (57) makes difficult to evaluate the potential quantitively by the analytic calculation when $L$ is large and numerical calculation would be necessary. In order to investigate the qualitative behavior of the potential when $L$ is large, we consider the background where the dilaton is constant $\phi = \phi_0$, which would tell the effect of the horizon or finite temperature. As $c = 0$ when the dilaton is constant, we can use the solution in (18). Then by the calculation similar to (53) but without assuming $L$ is small or $r_{\text{min}}$ is large, we obtain the following expression of the quark-antiquark potential:

$$E_{q\bar{q}} = r_{\text{min}} \int_{-\infty}^{\infty} dt \sinh t \left\{ \left( 1 - \frac{1}{\cosh^2 t} \cdot \frac{1 - \frac{r_0^2}{r_{\text{min}}^2}}{\cosh^2 t - \frac{r_0^2}{r_{\text{min}}^2}} \right) - 1 \right\}$$

$$+ 2 (r_D - r_{\text{min}}).$$  (58)

Constant $-1$ in $\{ \}$ and the last term correspond to the subtraction of the self-energy. The integration in (58) converges and the integrand is monotonically decreasing function of $\frac{1}{r_{\text{min}}}$ if $r_{\text{min}}$ is larger than the radius of the horizon $r_0 : r_{\text{min}} > r_0$ and vanishes in the limit of $r_{\text{min}} \to r_0$. Therefore if $r_{\text{min}}$ decreases and approaches to $r_0$ when $L$ is large, which seems to be very natural, the potential $E_{q\bar{q}}$ approaches to a constant $E_{q\bar{q}} \to 2 (r_D - r_0)$ and do not behaves as a linear function of $L$. This tells that the quark is not confined. This effect would corresponds to deconfining phase of QCD in the finite temperature.
We can also evaluate the potential between monopole and anti-monopole using the Nambu-Goto action for $D$-string instead of (43) (cf. ref. [13]):

$$S = \frac{1}{2\pi} \int d\tau d\sigma e^{-2\phi} \sqrt{\text{det} \left( g_{\mu\nu} \partial_{\alpha} x^\mu \partial_{\beta} x^\nu \right)} . \quad (59)$$

For the static configuration $x^0 = \tau$, $x^1 \equiv x = \sigma$, $x^2 = x^3 = \cdots = x^{d-1} = 0$ and $y = y(x)$, we find, instead of (44)

$$S = \frac{T}{2\pi} \int dx e^{-\phi(r)} \sqrt{U(r)V(r) \left( \frac{U(r)}{V(r)} (\partial_x r)^2 + 1 \right)} . \quad (60)$$

Since $\phi$ is proportional to $c$ and $V$ and $U$ contain $c$ in the form of its square $c^2$, the potential between monopole and anti-monopole is given by changing $c$ by $-c$ in the potential between quark and anti-quark. Since the expression (53) does not contain $c$ in the given order, the potential $E_{m\bar{m}}(L)$ for monopole and anti-monopole is identical with that of quark and anti-quark in this order:

$$E_{m\bar{m}}(L) = E_{q\bar{q}}(L) . \quad (61)$$

Hence, we showed that non-constant dilaton deformation of IIB SG vacuum changes the structure of potential and confinement is becoming non-realistic.

3 **Running coupling and quark-antiquark potential at finite temperature: non-zero mass BH case**

In this section we consider another interesting case that $k = 0$ and $\mu \neq 0$, which corresponds to the throat limit of D3-brane [18, 19] and $V_0$ has the following form:

$$V_0 = \frac{r^4}{l^2} - \mu . \quad (62)$$

In ref. [18] $\alpha'$-corrections to leading term ($T^4$) of free energy for above AdS BH have been derived. The temperature was actually fixed. In the case under discussion we consider dilatonic deformation of such AdS BH using tree level bosonic sector of IIB SG. Thus, we define the corrections (next-to-the leading term on the temperature) to solution (and free energy).
\(e^{2\rho}\) and \(e^{2\sigma}\) have the following form:

\[
e^{2\rho} = e^{-2\sigma} = \frac{1}{r^2} \left( \frac{r^4}{l^2} - \mu \right)
\]

(63)

Therefore when \(c = 0\), the horizon is given by

\[
r = \mu^{\frac{1}{4}} l^{\frac{1}{2}}
\]

(64)

and the black hole temperature is

\[
T = \frac{\mu^{\frac{1}{4}}}{\pi l^{\frac{1}{2}}}
\]

(65)

In a way similar to \(k < 0\) and \(\mu = 0\) case, we obtain

\[
\phi = \phi_0 + \frac{c}{4\mu} \ln \left( 1 - \frac{1}{q^2} \right)
\]

\[u = -\frac{12}{\mu^2} \left\{ \frac{1}{q^2 - 1} + \ln \left( 1 - \frac{1}{q^2} \right) + c'_1 \right\}
\]

\[v = \frac{1}{12\mu} \left\{ -q^2 \ln \left( 1 - \frac{1}{q^2} \right) - 1 + \frac{c'_1 q^2}{2} + c'_2 \right\}
\]

(66)

Here

\[q \equiv \frac{r^2}{l\sqrt{\mu}}
\]

(67)

and \(c'_1\) and \(c'_2\) are constants of the integration, which should vanish if we require \(u, v \to 0\) when \(r \to \infty\). The approximation when \(r\) is far from horizon is again employed.

Using (63) and (66), we find the behaviour of the string coupling (27) when \(r\) is large and \(c\) is small (\(\phi_0\) is absorbed into the redefinition of \(g_s\)):

\[
g = g_s \left\{ 1 + \frac{c^2}{4} \left( -\frac{1}{r^4} - \frac{(\pi T)^4}{r^8} + O \left( r^{-12} \right) \right) + O \left( c^2 \right) \right\}
\]

(68)

The behavior of the second term is characteristic for \(k = 0\) case since the second term behaves as \(O(r^{-6})\) for \(k \neq 0\). Eq.(68) gives the following

\(^{6}\) The case that the boundary is the Einstein manifold with \(k \neq 0\) has been discussed in [17].
The beta-function
\[ \beta(g) = r \frac{d}{dr} \left( -4(g - g_s) + \frac{8(\pi T)^4}{g_s c} (g - g_s)^2 + \cdots \right). \] (69)

The first term is universal one \[3, 8\] but the behavior of the second temperature dependent term is characteristic for \( k = 0 \).

We now consider the high temperature and \( r \sim T l^2 \) case. For this purpose, we write the coupling as follows:
\[ g = g_s \left( 1 - \frac{l^2 \mu}{r^4} \right)^{\frac{4}{\mu^2}}. \] (70)

Here we used (66). Eq. (70) can be solved with respect to \( \frac{1}{r^4} \):
\[ \frac{l^2 \mu}{r^4} = 1 - \left( \frac{g}{g_s} \right)^{\frac{4}{\mu}}. \] (71)

On the other hand, Eq. (70) gives
\[ r \frac{d}{dr} = \frac{g_s c l^2}{r^4} \left( 1 - \frac{1}{r^4} \right)^{-\frac{4}{\mu^2} - 1}. \] (72)

Substituting (71) into (72), we obtain the following expression:
\[ \beta(g) = r \frac{d}{dr} = \frac{gc}{\mu} \left( \frac{g}{g_s} \right)^{-\frac{4}{\mu}} \left( 1 - \left( \frac{g}{g_s} \right)^{\frac{4}{\mu}} \right). \] (73)

Using (55), we find the following expression of the temperature dependent beta-function:
\[ \beta(g) = \frac{gc}{\pi^2 l^6 T^4} \left( \frac{g}{g_s} \right)^{-\frac{4a^2 l^6 T^4}{c}} \left( 1 - \left( \frac{g}{g_s} \right)^{\frac{4a^2 l^6 T^4}{c}} \right). \] (74)

Since \( T \) always appears in the combination of \( \frac{c}{T^4} \), the high temperature is consistent with the small \( c \). As one can see \( T \)-dependence is quite complicated. It is qualitatively different from the case of low temperature.
In order to investigate the corrections to the position of the horizon and the temperature, we use $\frac{1}{r}$ expansion as in the previous section assuming the black hole is large. Then we find

\[ e^\phi = e^{\phi_0} \left( 1 - \frac{c^2 l^2}{4 r^4} + O \left( r^{-8} \right) \right) \]

\[ U = 1 - \frac{c^2 l^1}{48 r^8} + O \left( r^{-12} \right) \]

\[ V = \frac{r^4}{l^2} - \mu + \frac{c^2 l^2}{48 r^4} + O \left( r^{-8} \right). \]  

(75)

Then corrections to the position of the horizon and the temperature are given as

\[ r = r_h \equiv l^{\frac{1}{2}} \mu^{\frac{3}{4}} \left( 1 - \frac{c^2}{192 \mu^2} \right) \]

\[ T = \frac{\mu^{\frac{3}{4}}}{\pi l^{\frac{3}{2}}} \left( 1 - \frac{5c^2}{192 \mu^2} \right). \]  

(76)

The discussion of potential between quark and anti-quark for above case may be done similarly to the situation when $k < 0$ and $\mu = 0$. Instead of Eqs.(48), (49), (50), we obtain

\[ \frac{dx}{dt} = \frac{l}{r_{\min} \cosh^2 t \left( \cosh^2 t + 1 \right)^{\frac{3}{2}}} \times \left\{ 1 - \frac{\mu^2}{2 r_{\min}^3} \cosh^4 t - \frac{c^2}{4 r_{\min}^4} \cosh^4 t + O \left( r_{\min}^{-8} \right) \right\} \]  

(77)

\[ L = \frac{lA}{r_{\min}} - \frac{\mu^3 B_1}{2 r_{\min}^5} - \frac{c^2 B_2}{4 r_{\min}^5} + O \left( r_{\min}^{-9} \right) \]  

(78)

\[ B_1 \equiv \int_{-\infty}^{\infty} \frac{dt}{\sinh^2 t \left( \cosh^2 t + 1 \right)^{\frac{1}{2}}} = 0.479256... \]

\[ B_2 \equiv \int_{-\infty}^{\infty} \frac{dt}{\cosh^6 t \left( \cosh^2 t + 1 \right)^{\frac{1}{2}}} = 1.91702... \]  

(79)

\[ r_{\min} = \frac{lA}{L} - \frac{\mu B_1 L^3}{2 l A^4} - \frac{c^2 B_2 L^3}{4 l A^4} + O \left( L^7 \right). \]  

(80)
Then we find the finite potential between "quark" and "anti-quark" is given by

\[ E_{\bar{q}q}(L) = r_{\text{min}} \left\{ C + \frac{l^2 A}{r_{\text{min}}^4} \left( \frac{\mu}{2} - \frac{5c}{12} \right) + \mathcal{O} \left( r_{\text{min}}^8 \right) \right\} \]

\[ = \frac{lAC}{L} + \frac{L^3}{lA^3} \left\{ \mu \left( \frac{A}{2} - \frac{CB_1}{2} \right) + c \left( -\frac{5A}{12} - \frac{CB_2}{8} \right) \right\} + \mathcal{O} \left( L^7 \right) \]

\[ = \frac{lAC}{L} + \frac{L^3}{lA^3} \left\{ l^6 (\pi T)^4 \left( \frac{A}{2} - \frac{CB_1}{2} \right) + c \left( -\frac{5A}{12} - \frac{CB_2}{8} \right) \right\} + \mathcal{O} \left( L^7 \right) \]

Here we choose \( \phi_0 = 0 \) and neglect \( r_{\text{min}} \) or \( L \) independent terms, again. The behavior of the potential is qualitatively identical with that in [23] except \( L^3 \)-term in potential (next-to-leading term) contains the contribution from dilaton. Since

\[ \frac{A}{2} - \frac{CB_1}{2} = 0.886178... \quad , \quad -\frac{A}{4} - \frac{CB_2}{2} = 0.649204... \quad , \]

the \( L^3 \) potential becomes attractive if \( l^6 (\pi T)^4 > \gamma c \) (high temperature or small dilaton) and repulsive if \( l^6 (\pi T)^4 < \gamma c \) (low temperature or large dilaton). Here

\[ \gamma \equiv \frac{5A}{12} + \frac{CB_2}{2} - \frac{CB_1}{2} = -0.732589... \quad . \]

Hence, we proved the possibility of confinement at finite temperature. The potential (81) is valid if \( r_{\text{min}} \) is much larger than the radius of the horizon:

\[ r_{\text{min}} \gg \mu \frac{l^\frac{1}{2}}{\pi} \]

or

\[ L \ll \frac{Al^\frac{1}{2}}{\mu^\frac{1}{2}} = \pi lT . \]

The potential between monopole and anti-monopole is given by changing \( c \) into \( -c \) in (81):

\[ E_{m\bar{m}}(L) = \frac{lAC}{L} + \frac{L^3}{lA^3} \left\{ l^6 (\pi T)^4 \left( \frac{A}{2} - \frac{CB_1}{2} \right) - c \left( -\frac{5A}{12} - \frac{CB_2}{8} \right) \right\} + \mathcal{O} \left( L^7 \right) \]
Therefore the $L^3$ potential becomes attractive if $l^6 (\pi T)^4 > -\gamma c$ and repulsive if $l^6 (\pi T)^4 < -\gamma c$. In other words, the behavior of monopole-antimonopole potential is reversed.

We now consider more general case where either $k$ or $\mu$ do not vanish. If we define $r^2_\pm$ by

$$r^2_\pm \equiv \frac{k l^2}{4} \left( -1 \pm \sqrt{1 + \frac{16 \mu}{k^2 l^2}} \right) \tag{86}$$

$V_0$ has the following form:

$$V_0 = \frac{1}{l^2} \left( r^2 - r^2_+ \right) \left( r^2 - r^2_- \right) \tag{87}$$

Since $\mu$ corresponds the black hole mass, $\mu$ should not be negative. If $\mu > 0$, $r^2_+$ is positive and $r^2_-$ is negative when $k > 0$ and $r^2_-$ is positive and $r^2_+$ is negative when $k < 0$. Then $r = r_+$ corresponds to the horizon for $k > 0$ in $c = 0$ case and $r = r_-$ corresponds to the horizon for $k < 0$. Therefore there is only one horizon when $\mu > 0$. On the other hand, when $\mu < 0$ although it might look unphysical, there are two horizons corresponding to $r = r_\pm$ when $k < 0$.

When $c = 0$, the temperature corresponding to the horizon at $r = r_\pm$ is given by

$$T = \pm \frac{r^2_+ - r^2_-}{2 \pi r^2_\pm} \tag{88}$$

When $c$ is small but does not vanish, the leading behavior of $\phi$ is given by,

$$\phi = \phi_0 + \frac{c l^2}{2} \left\{ -\frac{1}{r^2_+ r^2_-} \ln r^2 + \frac{1}{r^2_+ (r^2_+ - r^2_-)} \ln \left( r^2 - r^2_+ \right) - \frac{1}{r^2_- (r^2_+ - r^2_-)} \ln \left( r^2 - r^2_- \right) \right\} \tag{89}$$

Then the behavior of the string coupling (27) when $r$ is large and $c$ is small ($\phi_0$ is absorbed into the redefinition of $g_s$) :

$$g = g_s \left\{ 1 + \frac{c l^2}{2} \left( -\frac{1}{2 r^4} - \frac{r^2_+ + r^2_-}{3 r^6} + \mathcal{O} \left( r^{-8} \right) \right) + \mathcal{O} \left( c^2 \right) \right\} \tag{90}$$
and the beta-function is given by

$$\beta(g) = r \frac{dg}{dr} = -4 \left( g - g_s \right) + \frac{8}{3} \left( r_+^2 + r_-^2 \right) \frac{c g_s}{l} \left( \frac{g - g_s}{c g_s} \right)^{\frac{3}{2}}. \quad (91)$$

Note that the second term vanishes when $k = 0$, which is the reason why the behavior of the next-to-leading term in $k = 0$ is different from that in $k \neq 0$. Eq. (88) gives the temperature dependence in the coupling (90) and the beta-function (91). The next-to-leading term shows again power-like behavior on $g$ as it happened already in IIB SG solutions of refs. [4, 5, 6, 8, 10, 17] (no temperature) and in GUTs with large internal dimensions [12]. Note that two of the parameters $k$, $\mu$ and $T$ are independent. If we consider the high temperature regime by fixing $k$, $\mu$ becomes large and the behavior approaches to $k = 0$ case in (73). On the other hand, if we consider the high temperature regime by fixing $\mu$, $k$ is positive and becomes large and the behavior approaches to $k < 0$ and $\mu = 0$ case in (30). One can also find quark-antiquark potential which looks very complicated so we do not write it explicitly.

The corrections to $U$ and $V$ coming from the non-trivial dilaton are given by

$$u = c_1'' + \frac{l^4}{(r_+^2 - r_-^2)} \left\{ \left( \frac{1}{r_+^2} - \frac{1}{r_-^2} \right)^2 \ln r^2 \right. $$

$$- \frac{1}{r_+^2} \frac{1}{r_-^2} - \frac{3 r_+^2 - r_-^2}{r_+^2 (r_+^2 - r_-^2)} \ln \left( r^2 - r_+^2 \right) $$

$$\left. - \frac{1}{r_-^2} \frac{1}{r_-^2} + \frac{3 r_-^2 - r_+^2}{r_-^2 (r_+^2 - r_-^2)} \ln \left( r^2 - r_-^2 \right) \right\} \quad (92)$$

$$v = \frac{1}{l^2} \left[ c_2'' + c_1'' \left\{ r^4 - \left( r_+^2 + r_-^2 \right) r^2 \right\} + \frac{l^4}{(r_+^2 - r_-^2)^2} \left\{ - \left( \frac{1}{r_+^2} + \frac{1}{r_-^2} \right) r^2 \right. $$

$$+ \left( \frac{-3 r_+^2 - r_-^2}{r_+^4 (r_+^2 - r_-^2)} \right) \left( r^2 - r_+^2 \right)^2 $$

$$\left. - \frac{3 r_+^2 - r_-^2}{r_-^4 (r_+^2 - r_-^2)} \left( r^2 - r_+^2 \right) - \frac{r_+^2 - r_-^2}{r_-^4 r_+^2} \right\} \ln \left( 1 - \frac{r_+^2}{r^2} \right) $$

$$+ \left( \frac{3 r_-^2 - r_+^2}{r_-^4 (r_+^2 - r_-^2)} \right) \left( r^2 - r_-^2 \right)^2 \right\}$$

20
\[ -\frac{3r_+^2 - r_0^2}{r_+^4} \left( r_+^2 - r_0^2 \right) + \frac{r_+^2 - r_0^2}{r_+^2} \ln \left( 1 - \frac{r_0^2}{r_+^2} \right) \right] . \] (93)

Here \( c_1'' \) and \( c_2'' \) are constants of the integration, which should vanish if we require \( u, v \to 0 \) when \( r \to \infty \).

## 4 Thermodynamics of approximate AdS backgrounds of IIB supergravity

In the present section we will be interesting in the thermodynamical quantities like free energy. After Wick-rotating the time variables by \( t \to i\tau \), the free energy \( F \) can be obtained from the action \( S \) in (1) where the classical solution is substituted:

\[ F = \frac{1}{T} S . \] (94)

Multiplying \( G^{\mu\nu} \) with the equation of motion in (2), we find

\[ R - \frac{1}{2} G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = \frac{5}{3} \Lambda . \] (95)

Here we only consider the case of \( d = 4 \) and \( \alpha = \frac{1}{2} \). Substituting (95) into (1), we find after the Wick-rotation

\[ S = \frac{1}{2\pi G l^2} \int d^5 x \sqrt{G} . \] (96)

Here we used (21). From the expressions of the metric \( G_{\mu\nu} \) in (1) and (13), Eq.(96) is rewritten as follows:

\[ S = \frac{1}{2\pi G l^2} V_3 \int_{r_h}^{\infty} drr^3 U . \] (97)

Here \( V_3 \) is the volume of 3d Einstein manifold and \( r_h \) is the radius of the horizon and we assume \( \tau \) has a period of \( \frac{1}{T} \). Since \( U \) has a singularity at \( r = r_h \) in the expansion with respect to \( c \), we use \( \frac{1}{r} \) expansion. Furthermore the expression of \( S \) contains the divergence coming from large \( r \). In order to subtract the divergence, we regularize \( S \) in (97) by cutting off the integral at a large radius \( r_{\text{max}} \). After that we subtract the divergent part. In case of
\( k < 0 \) and \( \mu = 0 \), we subtract it by using the extremal solution with \( c = 0 \) \((U = 1)\):

\[
S_{\text{reg}} = \frac{1}{2\pi G l^2 T} \left( \int_{r_{h}}^{r_{\text{max}}} drr^3 U - \sqrt{\frac{V(r = r_{\text{max}})}{V_{\text{ex}}(r = r_{\text{max}})}} \int_{r_{h}}^{r_{\text{ex}}} drr^3 \right). \tag{98}
\]

Here

\[
V_{\text{ex}} \equiv \frac{1}{l^2} \left( r^2 - r_{h}^{\text{ex}} \right)^2, \quad r_{h}^{\text{ex}} = \frac{l\sqrt{-k}}{2}, \tag{99}
\]

which corresponds to the extremal solution (the solution has negative mass parameter \( \mu = -\frac{k^2 l^2}{4} \)). The factor \( \sqrt{\frac{V(r = r_{\text{max}})}{V_{\text{ex}}(r = r_{\text{max}})}} \) is chosen so that the proper length of the circle which corresponds to the period \( \frac{1}{T} \) in the Euclidean time at \( r = r_{\text{max}} \) coincides with each other in the two solutions with \( \mu = 0 \) and \( k < 0 \) one and extremal one in (99). Then we obtain

\[
F = \frac{V_3}{2\pi G l^2} \left( -\frac{5k^2 l^4}{128} + \frac{c^2}{48k^2} \right). \tag{100}
\]

With the help of (102), we find the following expression

\[
F = -\frac{V_3}{2\pi G l^2} \left( \frac{5l^8 (2\pi T)^4}{32} + \frac{c^2}{768l^4 (2\pi T)^4} \right). \tag{101}
\]

In order to get the entropy \( S \), we need to know \( T \) dependence of \( V_3 \) although \( V_3 \) is infinite. Since \( k \) is proportional to the curvature, \( V_3 \) would be proportional to \( k^{-\frac{7}{2}} \). Then we find

\[
\frac{dV_3}{dT} = \frac{1}{k} \frac{dk}{dT} - \frac{1}{6l^2 (2\pi T)^8} + \cdots \tag{102}
\]

Therefore the entropy \( S \) and mass (energy) \( E \) are given by

\[
S = -\frac{dF}{dT} = \frac{V_3}{2\pi G l^2 T} \left( \frac{25l^8 (2\pi T)^4}{64} + \frac{49c^2}{1536l^4 (2\pi T)^4} \right)
\]

\[
E = F + TS = \frac{V_3}{2\pi G l^2} \left( \frac{15l^8 (2\pi T)^4}{64} + \frac{47c^2}{1536l^4 (2\pi T)^4} \right). \tag{103}
\]
In terms of string theory correspondence [18], the parameters $G$ and $l$ are given by

$$
l^4 = 2g_{YM}^2 N \alpha'^2
$$

$$
Gl = \frac{\pi g_s^2 Y M}{N} \alpha'^2 .
$$

(104)

Here the Yang-Mills coupling $g_{YM}$ is given by the string coupling $g_s$: $g_{YM}^2 = 2\pi g_s$ and $N$ is the number of the coincident D3-branes. As $V_3$ is now dimensionless, we multiply $l^3$ with $V_3$:

$$
\tilde{V}_3 \equiv l^3 V_3 .
$$

(105)

Then Eqs. (101) and (103) can be rewritten as follows:

$$
F = -\frac{\tilde{V}_3}{2\pi^2} \left( \frac{5N^2 (2\pi T)^4}{16} + \frac{5c^2}{3072l^4 g_{YM}^6 N \alpha'^6 (2\pi T)^4} \right)
$$

$$
S = \frac{\tilde{V}_3}{2\pi^2 T} \left( \frac{25N^2 (2\pi T)^4}{32} + \frac{49c^2}{6144g_{YM}^6 N \alpha'^6 (2\pi T)^4} \right)
$$

$$
E = \frac{\tilde{V}_3}{2\pi G l^2} \left( \frac{15N^2 (2\pi T)^4}{32} + \frac{47c^2}{6144g_{YM}^6 N \alpha'^6 (2\pi T)^4} \right) .
$$

(106)

For $k = 0$ and $\mu > 0$ case, we can obtain the thermodynamical quantities in a similar way using Eqs. (75) and (76) in $\frac{1}{T}$ expansion. We regularize $S$ in (77) by subtracting the solution with $\mu = 0$ and $c = 0$ ($U = 1$):

$$
S_{\text{reg}} = \frac{1}{2\pi G l^2 T} \left( \int_{r_h}^{r_{\text{max}}} dr r^3 U - \frac{V(r = r_{\text{max}})}{V(r = r_{\text{max}}, \mu = 0)} \int_0^{r_{\text{max}}} dr r^3 \right) .
$$

(107)

We can assume here that $V_3$ does not depend on $T$ since $k$ is fixed to vanish. Then we obtain for the case

$$
F = -\frac{V_3}{4\pi G l^2} \left( \frac{l^8 (\pi T)^4}{4} + \frac{5c^2}{192l^4 (\pi T)^4} \right)
$$

$$
S = \frac{V_3}{4\pi G l^2 T} \left( \frac{l^8 (\pi T)^4}{4} - \frac{5c^2}{48l^4 (\pi T)^4} \right)
$$

$$
E = \frac{V_3}{4\pi G l^2} \left( \frac{3l^8 (\pi T)^4}{4} - \frac{25c^2}{192l^4 (\pi T)^4} \right) .
$$

(108)
Note that the leading term in $S$ is the volume of 3d manifold at horizon \( \frac{V_3 r_h^3}{\pi^2} \) divided by $4G$. Then by using (104) and (105), we find

\[
F = -\frac{\tilde{V}_3}{4\pi^2} \left( \frac{N^2 (\pi T)^4}{2} + \frac{5c^2}{768 g_{YM}^6 N\alpha'^6 (\pi T)^4} \right)
\]

\[
S = \frac{\tilde{V}_3}{4\pi^2 T} \left( 2N^2 (\pi T)^4 - \frac{5c^2}{192 g_{YM}^6 N\alpha'^6 (\pi T)^4} \right)
\]

\[
E = \frac{\tilde{V}_3}{4\pi^2} \left( \frac{3N^2 (\pi T)^4}{2} - \frac{25c^2}{768 g_{YM}^6 N\alpha'^6 (\pi T)^4} \right).
\]

The leading behaviours of $F$ and $S$ are consistent with [18]. As we used $\frac{1}{r}$ expansion, the second terms in (109) become dominant when the radius of horizon $r_h$ is large and the parameter $c$ specifying non-trivial dilaton is not very small. Notice that in other temperature regimes (or using another schemes for approximated solutions of gravitational equations) one can get also qualitatively different thermodynamical next-to-leading terms (on temperature). One has to remark that leading term in above free energy describes the strong coupling regime free energy for $\mathcal{N} = 4$ super YM theory with the usual mismatch factor $3/4$ if compare with perturbative free energy (for a detailed discussion of this case, see [13]).

5 Discussion

We studied the approximate (dilaton perturbed) solutions of IIB SG near BH-like AdS$_5 \times$ S$_5$ background. Thanks to presence of dilaton, the running gauge coupling of non-SUSY gauge theory at finite temperature may be extracted from these solutions. It is interesting that corresponding strong coupling regime beta-function may depend on the temperature in the complicated way (mainly, power-like behavior). We also estimated quark-antiquark (and monopole-antimonopole) potential at finite temperature from SG side. Its comparison with the potential of $\mathcal{N} = 4$ super YM theory at finite temperature is also done. It is remarkable that confinement depending on features of geometry and dilaton may occur.

As our IIB SG solutions are approximate (actually large radius expansion) it is clear that one is able to develop other schemes to search for similar
solutions. Unfortunately, it is not yet clear how to identify explicitly boundary non-SUSY thermal gauge theory corresponding to these solutions. One possibility is to calculate free energy from SG side (with non-trivial dilaton) and compare it with free energy of perturbative thermal gauge theories with running gauge coupling. The last quantity is available in QFT.

Note also that one can generalize our solutions via adding RR-scalar(axion) to bosonic sector of IIB SG as it was discussed in refs.\[14\]. As it follows from results of refs.\[8, 17\] the structure of running gauge coupling changes drastically in this case. In particular, part of supersymmetries may be unbroken \[14\] but asymptotic freedom may be realized in strong coupling phase \[8, 17\]. We expect that at finite temperature this property may survive.

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