Existence of a solution to a nonlinear equation

A.G. Ramm
Mathematics Department, Kansas State University,
Manhattan, KS 66506-2602, USA
ramm@math.ksu.edu

Abstract

Equation \((-\Delta + k^2)u + f(u) = 0\) in \(D\), \(u|_{\partial D}=0\), where \(k = \text{const} > 0\) and \(D \subset \mathbb{R}^3\) is a bounded domain, has a solution if \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a continuous function in the region \(|u| \geq a\), piecewise-continuous in the region \(|u| \leq a\), with finitely many discontinuity points \(u_j\) such that \(f(u_j \pm 0)\) exist, and \(uf(y) \geq 0\) for \(|u| \geq a\), where \(a \geq 0\) is an arbitrary fixed number.

1 Introduction

Let \(D \subset \mathbb{R}^3\) be a bounded domain with Lipschitz boundary \(S\), \(k = \text{const} > 0\), \(f : \mathbb{R} \rightarrow \mathbb{R}\) be a function such that
\[ uf(u) \geq 0, \quad \text{for} \quad |u| \geq a \geq 0, \quad (1.1) \]
where \(a\) is an arbitrary fixed number, and \(f\) is continuous in the region \(|u| \geq a\), and bounded and piecewise-continuous with at most finitely many discontinuity points \(u_j\), such that \(f(u_j + 0)\) and \(f(u_j - 0)\) exist, in the region \(|u| \leq a\).

Consider the problem
\[ (-\Delta + k^2)u + f(u) = 0 \quad \text{in} \quad D, \quad (1.2) \]
\[ u = 0 \quad \text{on} \quad S. \quad (1.3) \]

There is a large literature on problems of this type. Usually it is assumed that \(f\) does not grow too fast or \(f\) is monotone (see e.g. [1] and references therein).

The novel point in this note is the absence of monotonicity restrictions on \(f\) and of the growth restrictions on \(f\) as \(|u| \rightarrow \infty\), except for the assumption (1.1).

Math subject classification: 35J65
key words: nonlinear elliptic equations
This assumption allows an arbitrary behavior of \( f \) inside the region \( |u| \leq a \), where \( a \geq 0 \) can be arbitrary large, and an arbitrarily rapid growth of \( f \) to \( +\infty \) as \( u \to +\infty \), or arbitrarily rapid decay of \( f \) to \( -\infty \) as \( u \to -\infty \).

Our result is:

**Theorem 1.** Under the above assumptions problem (1.2)–(1.3) has a solution \( u \in H^2(D) \cap \hat{H}^1(D) := H^2_0(D) \).

Here \( H^\ell(D) \) is the usual Sobolev space, \( \hat{H}^1(D) \) is the closure of \( C_0^\infty(D) \) in the norm \( H^1(D) \). Uniqueness of the solution does not hold without extra assumptions.

The ideas of our proof are: first, we prove that if \( \sup_{u \in \mathbb{R}} |f(u)| \leq \mu \), then a solution to (1.2)–(1.3) exists by the Schauder's fixed-point theorem. Here \( \mu \) is a constant. Secondly, we prove an a priori bound \( \|u\|_\infty \leq a \). If this bound is proved, then the solution to problem (1.2)–(1.3) with \( f \) replaced by

\[
F(u) := \begin{cases} 
  f(u), & |u| \leq a \\
  f(a), & u \geq a \\
  f(-a), & u \leq -a
\end{cases}
\]

has a solution, and this solution solves the original problem (1.2)–(1.3). The bound \( \|u\|_\infty \leq a \) is proved by using some integral inequalities. An alternative proof of this bound is also given. This proof is based on the maximum principle for elliptic equation (1.2).

In Section 2 proofs are given. We use some ideas from [2].

# 2 Proofs.

If \( u \in L^\infty := L^\infty(D) \), then problem (1.2)–(1.3) is equivalent to the integral equation:

\[
u = -\int_D G(x, y) f(u(y)) dy := T(u),
\]

where

\[
(-\Delta + k^2)G = -\delta(x - y) \quad \text{in} \quad D, \quad g |_{x \in S} = 0.
\]

By the maximum principle,

\[
0 \leq G(x, y) < g(x, y) := \frac{e^{-k|x-y|}}{4\pi|x-y|}, \quad x, y \in D.
\]

The map \( T \) is a continuous and compact map in the space \( C(D) := X \), and

\[
\|u\|_{C(D)} := \|u\| \leq \mu \sup_x \int_D \frac{e^{-k|x-y|}}{4\pi|x-y|} dy \leq \mu \int_{\mathbb{R}^d} \frac{e^{-k|y|}}{4\pi|y|} dy \leq \frac{\mu}{k^2}
\]
This is an a priori estimate of any bounded solution to (1.2)–(1.3) for a bounded non-linearity $f$ such that $\sup_{u \in \mathbb{R}} |f(u)| \leq \mu$. Thus, Schauder’s fixed-point theorem yields the existence of a solution to (2.1), and consequently to problem (1.2)–(1.3), for bounded $f$. Indeed, if $B$ is a closed ball of radius $\frac{\mu}{k^2}$, then the map $T$ maps this ball into itself by (2.4), and since $T$ is compact, the Schauder principle is applicable. Thus, the following lemma is proved.

**Lemma 1.** If $\sup_{u \in \mathbb{R}} |f(u)| \leq \mu$, then problems (2.1) and (1.2)–(1.3) have a solution in $C(D)$, and this solution satisfies estimate (2.4).

Let us now prove an a priori bound for any solution $u \in C(D)$ of the problem (1.2)–(1.3) without assuming that $\sup_{u \in \mathbb{R}} |f(u)| < \infty$.

Let $u_+ := \max(u, 0)$, $u_- = \max(-u, 0)$. Multiply (1.2) by $(u - a)_+$, integrate over $D$ and then by parts to get

$$0 = \int_D [\nabla u \cdot \nabla (u - a)_+ + k^2 u (u - a)_+ + f(u)(u - a)_+] dx,$$

where we have integrated by parts and the boundary integral vanishes because $(u - a)_+ = 0$ on $S$ for $a \geq 0$. Each of the terms in (2.5) is nonnegative, the last one due to (1.1). Thus (2.5) implies

$$u \leq a.$$

(2.6)

Similarly, using (1.1) again, and multiplying (1.2) by $(-u - a)_+$, one gets

$$-a \leq u.$$

(2.7)

We have proved:

**Lemma 2.** If (1.1) holds, then any solution $u \in H^2_0(D)$ to (1.2)–(1.3) satisfies the inequality

$$|u(x)| \leq a.$$

(2.8)

Consider now equation (2.1) in $C(D)$ with an arbitrary continuous $f$ satisfying (1.1). Any $u \in C(D)$ which solves (2.1) solves (1.2)–(1.3) and therefore satisfies (2.8) and belongs to $H^2_0(D)$. This $u$ solves problem (1.2)–(1.3) with $f$ replaced by $F$ defined in (1.4), and vice-versa. Since $F$ is a bounded nonlinearity, equation (2.1) and problem (1.2)–(1.3) (with $f$ replaced by $F$) has a solution by Lemma 1.

Theorem 1 is proved.

An alternative proof of the estimate (2.8):

Let us sketch an alternative derivation of the inequality (2.8) using the maximum principle. Let us derive (2.6). The derivation of (2.7) is similar.

Assume that (2.6) fails. Then $u > a$ at some point in $D$. Therefore at a point $y$ at which $u$ attains its maximum value one has $u(y) \geq u(x)$ for all $x \in D$ and $u(y) > a$. The function $u$ attains its maximum value, which is positive, at some point in $D$, because
$u$ is continuous, vanishes at the boundary of $D$, and is positive at some point of $D$ by the assumption $u > a$. At the point $y$, where the function $u$ attains its maximum, one has $-\Delta u \geq 0$ and $k^2 u(y) > 0$. Moreover, $f(u(y)) > 0$ by the assumption (1.1), since $u(y) > a$. Therefore the left-hand side of equation (1.2) is positive, while its left-hand side is zero. Thus we have got a contradiction, and the estimate (2.6) is proved. Similarly one proves estimate (2.7). Thus, (2.8) is proved. 

References

[1] Berger, M., Nonlinearity and functional analysis, Acad. Press, New York, 1977.

[2] Ramm, A. G., Stationary regimes in passive nonlinear networks, in “Nonlinear Electromagnetics”, Ed. P.L.E. Uslenghi, Acad. Press, N. Y., 1980, pp. 263-302.