AMOEBA FINITE BASIS DOES NOT EXIST IN GENERAL

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Abstract. We show that the amoeba of a generic complex algebraic variety of codimension $1 < r < n$ do not have a finite basis. In other words, it is not the intersection of finitely many hypersurface amoebas. Moreover we give a geometric characterization of the topological boundary of hypersurface amoebas refining an earlier result of F. Schroeter and T. de Wolff [SW-13].

1. Introduction

Tropical geometry combines aspects of algebraic geometry, discrete geometry, computer algebra, mirror symmetry and symplectic geometry. This geometry can be seen as a limiting regime of algebraic geometry, where some of its interesting varieties are the limit of the so-called amoebas. Amoebas of algebraic (or analytic) varieties are their image under the logarithm with base a real number $t$. In many cases, a tropical variety is the limit of these amoebas as $t$ goes to infinity (e.g., the case of tropical hypersurfaces). In other words, tropical objects are some how, the image of a classical objects under the logarithm with base infinity, they are called non-Archimedean amoebas (this last naming comes from another view of tropical geometry, which coincides with the limiting view in the case of hypersurfaces, see for example [IMS-07]).

Given an algebraically closed field $K$ endowed with a non-trivial real valuation $\nu : K \to \mathbb{R} \cup \{\infty\}$, the tropical variety $\text{Trop}(I)$ of an ideal $I \subset K[x_1, \ldots, x_n]$ is defined as the topological closure of the set

$$\nu(V(I)) := \{(\nu(x_1), \ldots, \nu(x_n)) \mid (x_1, \ldots, x_n) \in V(I)\} \subset \mathbb{R}^n,$$

where $V(I)$ denotes the zero set of $I$ in $(K^*)^n$ (see for example [MS-09]). A tropical basis for $I$ is a generating set $\mathcal{B} = \{g_1, \ldots, g_l\}$ of $I$ such that

$$\text{Trop}(I) = \bigcap_{j=1}^l \text{Trop}(I_{g_j}),$$

where $I_{g_j}$ denotes the principal ideal generated by the polynomial $g_j$. Bogart, Jensen, Speyer, Sturmfels, and Thomas initiated the computational investigation of tropical bases [BGSS-07] by providing Gröbner bases techniques for computing tropical bases as well as by lower bounds on the size of such bases.

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bases when \( K \) is the field of Puiseux series \( \mathbb{C}((t)) \) and the ideal \( \mathcal{I} \) is linear with constant coefficients. Dropping the assumption on the degree of the polynomials, Hept and Theobald showed that there always exists a small tropical basis for a prime ideal \( \mathcal{I} \) (see [HT-09]).

We will use logarithm with base \( e \), so that the Archimedean amoeba of a subvariety of the complex torus \( \mathbb{C}^* \) is its image under the coordinate-wise logarithm map. Amoebas were introduced by Gelfand, Kapranov, and Zelevinsky in 1994 [GKZ-94]. The coamoeba of a subvariety of \( \mathbb{C}^* \) is its image under the coordinate-wise argument map to the real torus \( S^1 \). Coamoebas were introduced by Passare in a talk in 2004 (see e.g., [NS-11] for more details about coamoebas).

A variety \( V \subset \mathbb{C}^* \) of codimension \( r \) is generic if it contains a point \( p \) such that the Jacobian of the logarithmic map restricted to \( V \) at the point \( p \) has maximal rank i.e., equal to \( \min\{n, 2(n - r)\} \). If the defining ideal \( \mathcal{I}(V) \) of \( V \) is generated by the set of polynomials \( \{g_i\}_{i=1}^l \) with the following properties:

(i) \( \mathcal{A}(V) = \bigcap_{i=1}^l \mathcal{A}(V_{g_i}) \);
(ii) \( \mathcal{A}(V) \subseteq \bigcap_{i \in \{1, \ldots, l\} \setminus s} \mathcal{A}(V_{g_i}) \) for every \( 1 \leq s \leq l \),

then we said that \( \{g_i\}_{i=1}^l \) is an amoeba basis of \( \mathcal{A}(V) \).

The aim of this paper is to show that the main result of Hept and Theobald in [HT-09] does not have an analogue for Archimedean amoebas of generic complex varieties of positive dimension and not hypersurfaces.

**Theorem 1.1.** If \( V \) is a generic complex algebraic variety of codimension \( r \) with \( 1 < r < n \), then its amoeba cannot have a finite basis.

This paper is organized as follows. In Section 2, we prove our main Theorem 1.1; in Section 3 we describe the example of a generic line in the space. In Section 4, we give a geometric characterization of the topological boundary of hypersurface amoebas and prove Theorem 4.1 which refines the main theorem of F. Schroeter and T. de Wolff in [SW-13].

### 2. Non existence of finite amoeba basis in general

Let \( V \subset \mathbb{C}^* \) be an algebraic variety of codimension \( r \) with defining ideal \( \mathcal{I}(V) \) and amoeba \( \mathcal{A}(V) \). It was shown by Purbhoo [P-08] (a short proof for both amoebas and coamoebas can be found in [NP-11]) that the amoeba \( \mathcal{A}(V) \) of \( V \) is equal to the intersection of all hypersurface amoebas with defining polynomial in the ideal \( \mathcal{I}(V) \), i.e.,

\[
\mathcal{A}(V) = \bigcap_{f \in \mathcal{I}(V)} \mathcal{A}(V_f),
\]

where \( V_f \) is the hypersurface with defining polynomial \( f \). One naturally ask: Is the amoeba \( \mathcal{A}(V) \) the intersection of a finite number of hypersurface amoebas? In [P-08], Purbhoo expects a negative answer to this question in general, but he does not give a formal proof. We give a negative answer to this question when the codimension of our variety is different than 1 and \( n \).
In [SW-13], Schroeter and de Wolff give a positive answer to this question when \( V \) is the zero-dimensional solution set of a generic linear system of \( n \) equations.

Let us fix some notation and definitions. Let \( V \subset (\mathbb{C}^*)^n \) be an algebraic variety of codimension \( 1 < r < n \) with defining ideal \( \mathcal{I}(V) \) and amoeba \( \mathcal{A}(V) \). To prove Theorem 1.1 we will deal with two cases, the one where \( \min\{n, 2(n-r)\} = n \) and the other case where \( \min\{n, 2(n-r)\} = 2(n-r) \). Namely, the cases where the dimension of the ambient space is less or equal to twice the dimension of \( V \), and the case where the dimension of the ambient space is strictly greater than twice the dimension of \( V \). In the first case, the amoeba \( \mathcal{A}(V) \) has dimension \( n \), and then it necessarily has a boundary. In the second case, it may be without boundary, as we will see in some examples.

Let \( V \subset (\mathbb{C}^*)^n \) be a generic algebraic variety of codimension \( r \) such that \( n \leq 2(n-r) \). Moreover, assume that the set of polynomials \( \{g_i\}_{i=1}^l \) is an amoeba basis of \( \mathcal{A}(V) \).

**Claim A:** With the above hypotheses, let \( x \) be a point in \( \partial \mathcal{A}(V) \). Then there exists a vector direction \( v \) such that \( (x + \varepsilon v) \notin \mathcal{A}(V) \) for all small positive real numbers \( \varepsilon \).

**Proof of Claim A:** First of all, the set of vector direction around \( x \) can be identified to the \((n-1)\)-dimensional sphere. Assume on the contrary that for all vector direction \( v \) and any \( \eta > 0 \) there exists \( \varepsilon \) with \( 0 < \varepsilon \leq \eta \) and \( (x + \varepsilon v) \in \mathcal{A}(V) \). This means that \( x \in \mathcal{A}(V_{g_i}) \) for all \( 1 \leq i \leq l \), where \( \mathcal{A}(V_{g_i}) \) denotes the interior of the hypersurface amoeba \( \mathcal{A}(V_{g_i}) \). In fact, if there exists \( s \) with \( 1 \leq s \leq l \) such that \( x \in \partial \mathcal{A}(V_{g_i}) \), then by the convexity of the complement components of the hypersurface amoeba \( \mathcal{A}(V_{g_i}) \), we can find a vector direction \( w \) such that \( (x + \nu w) \) is outside \( \mathcal{A}(V_{g_i}) \) for all small positive numbers \( \nu \). Hence, \( (x + \nu w) \) is outside \( \mathcal{A}(V) \) for all small positive number \( \nu \). This is in contradiction with our hypotheses. Let \( d_i \) be the distance between \( x \) and the boundary of \( \mathcal{A}(V_{g_i}) \), and \( \rho = \min\{d_i\}_{i=1}^l \). It is claire that the ball of center \( x \) and radius \( \rho \) is contained in the amoeba \( \mathcal{A}(V) \) (because it is contained in all the hypersurface amoebas \( \mathcal{A}(V_{g_i}) \)). This contradict the fact that \( x \) is in the boundary of the amoeba \( \mathcal{A}(V) \). Hence, there exists a vector direction \( v \) such that \( (x + \varepsilon v) \notin \mathcal{A}(V) \) for all small positive real numbers \( \varepsilon \).

We can remark that for any point \( x \in \partial \mathcal{A}(V) \) there exists an open subset of unit vector directions for which the property of Claim A is true. Indeed, if there exists a vector direction \( v \) such that \( (x + \varepsilon v) \notin \mathcal{A}(V) \) for all small positive real numbers \( \varepsilon \), and as our amoeba is the intersection of a finite number of hypersurface amoeba, then there exists \( s \) with \( 1 \leq s \leq l \) such that \( x \in \partial \mathcal{A}(V_{g_i}) \). Since any complement component of the complement of \( \mathcal{A}(V_{g_i}) \) is convex, then there exists an open neighborhood \( V_{\varepsilon} \) of unit vector directions of \( v \) such that \( (x + \varepsilon V_{\varepsilon}) \cap \mathcal{A}(V_{g_i}) \) is empty for all small positive real number \( \varepsilon \). Hence, \( (x + \varepsilon V_{\varepsilon}) \cap \mathcal{A}(V) \) is empty for all small positive real numbers \( \varepsilon \).

We will use the following definitions:
Definition 2.1. An analytic subset $\mathcal{I}$ in $\mathbb{R}^n$ of codimension 1 is said to be locally convex if and only if for any point $x \in \mathcal{I}$ there exists an open $n$-dimensional ball $B(x, \rho) \subset \mathbb{R}^n$ of radius $\rho$ and center $x$ and a connected component of $B(x, \rho) \setminus \mathcal{I}$ which is convex. As the convexity is a local property, this means that $\mathcal{I}$ is the boundary of a convex subset in $\mathbb{R}^n$.

Recall that an $n$-dimensional subset $\mathcal{V} \subset \mathbb{R}^n$ is said to be locally convex if for any point $x \in \mathcal{V}$ there exists an $n$-dimensional ball $B(x, \mu)$ of center $x$ and radius $\mu$ contained in $\mathcal{V}$.

A subset $X$ of $\mathbb{R}^n$ is convex if for any affine line $\pi$ in $\mathbb{R}^n$, the intersection $X \cap \pi$ has at most one connected component. In other words, the intersection $X \cap \pi$ contains all intervals with boundary in $X \cap \pi$. If the points of $X$ are viewed as 0-cycles, then the convexity of $X$ means that if $a$ and $b$ are two 0-cycles homologous in $X$, then they are also homologous in $X \cap \pi$ where $\pi$ is the line containing $a$ and $b$. Write $\tilde{H}_s(X, \mathbb{Z})$ for the reduced integral homology of a space $X$ with integral coefficients. This is the kernel of the map $\deg: \tilde{H}_s(X, \mathbb{Z}) \to \tilde{H}_s(pt, \mathbb{Z})$ induced by the map $X \to pt$ to a point.

Definition 2.2. A subset $X$ of a vector space $V$ is $k$-convex if for any affine $(k+1)$-plane $\pi$, the maps $\tilde{H}_k(\pi \cap X, \mathbb{Z}) \to \tilde{H}_k(X, \mathbb{Z})$ induced by the inclusions are injective.

This global statement generalizing convexity was found by André Henriques [H-03]. Moreover, Henriques showed that the complement of the amoeba of a codimension $r$ variety is weakly $(r-1)$-convex. Namely, a non-negative $(r-1)$-cycle non homologue to zero in the intersection of a $r$-plane with the complement of the amoeba is also an $(r-1)$-cycle non homologue to zero in the complement of the amoeba itself (see Theorem 4.1 [H-03]).

Lemma 2.1. Let $V \subset (\mathbb{C}^*)^n$ be a generic algebraic variety of codimension $r$ such that $n \leq 2(n-r)$. Assume there exists a finite number of polynomials $\{g_i\}_{i=1}^l$ such that $\mathcal{A}(V) = \bigcap_{i=1}^l \mathcal{A}(V_{g_i})$. Then, for any point $x \in \partial \mathcal{A}(V)$, there exist a connected open neighborhood $U_x \subset \partial \mathcal{A}(V)$ of $x$, $g_s$ with $1 \leq s \leq l$, and a connected component $\mathcal{C}$ of $\partial \mathcal{A}(V_{g_s})$ such that $U_x \subset \mathcal{C}$.

Proof. As the variety $V$ is generic and its codimension $r$ satisfies the inequality $n \geq 2r$, then $\mathcal{A}(V)$ necessarily has a boundary of dimension $n-1$, which is the same dimension as the boundaries of all the hypersurface amoebas $\mathcal{A}(V_{g_i})$. As the point $x$ is in the boundary $\partial \mathcal{A}(V)$, by Claim A, we know that there is a vector direction (which we can assume unit) such that $(x+\varepsilon v) \cap \mathcal{A}(V)$ is empty for all small positive real numbers $\varepsilon$. By the remark made in the same claim, there exists $s$ with $1 \leq s \leq l$ such that $x \in \partial \mathcal{A}(V_{g_s})$. We claim that there exists an open neighborhood $U_x \subset \partial \mathcal{A}(V)$ of $x$ such that $U_x$ is also contained in the boundary of $\mathcal{A}(V_{g_s})$. Assume on the contrary that for any open neighborhood $U_x$ of $x$ in $\partial \mathcal{A}(V)$, the set $U_x$ is not contained in $\partial \mathcal{A}(V_{g_s})$ (i.e., $U_x$ intersect the interior of $\mathcal{A}(V_{g_s})$). Then for any point $y$ close to $x$ and contained in $\mathcal{A}(V_{g_s}) \cap \partial \mathcal{A}(V)$, there exists an
open $n$-dimensional ball $B(y, \rho_y) \subset \mathbb{R}^n$ with center $y$ such that

$$B(y, \rho_y) \cap \mathcal{A}(V) = B(y, \rho_y) \cap \left( \bigcap_{i \in \{1, \ldots, l\} \setminus s} \mathcal{A}(V_{g_i}) \right).$$

Namely, the set $\{g_1, \ldots, g_s, \ldots, g_l\}$ is a local basis of $\mathcal{A}(V)$ at $y$. Now, by the same reasoning as in Claim A, there exists $u$ with $1 \leq u \leq l$ and $u \neq s$ such that $y \in \partial \mathcal{A}(V_{g_u})$. Using induction on $l$ (more precisely, induction on the number of hypersurface amoebas), we can assume that for any point $y \in \partial \mathcal{A}(V)$ close to $x$ there exists an open neighborhood $U_y$ of $y$ in $\partial \mathcal{A}(V)$ with $U_y \subset \partial \mathcal{A}(V_{g_u})$ for some $u \leq l$ and $u \neq s$. Indeed, if this property is not satisfied for some $y$, by the same reasoning done for the point $x$ we can drop the number of local basis functions by one until we arrive to a hypersurface amoeba. It means that there exists an open neighborhood $W_x$ of $x$ in $\partial \mathcal{A}(V)$ (may be smaller than $U_x$) such that $W_x \setminus \{x\}$ is covered by at most $l$ open subsets where each of them is contained in a hypersurface amoeba (their number cannot exceed $l$ because of the convexity of a hypersurface amoeba complement). As $\partial \mathcal{A}(V_{g_u})$ are convex in the sense of Definition 2.1 and $l$ is finite, there exists an open $n$-dimensional ball $B(x, \nu) \subset \mathbb{R}^n$ with center $x$ and a hyperplane $\mathcal{H}_x$ containing $x$ such that $B(x, \nu) \cap \mathcal{A}(V)$ is contained in only one side of $\mathcal{H}_x$. This contradict the fact that the complement of the amoeba $\mathcal{A}(V)$ is $(r - 1)$-convex. Hence, there exists an open neighborhood $U_x \subset \partial \mathcal{A}(V)$ of $x$ such that $U_x$ is also contained in the boundary of $\mathcal{A}(V_{g_u})$ for some $s$.

\[ \square \]

Proof of Theorem 1.1 for $n \geq 2r$. With the same notation as above, assume that $r > 1$. Then $\partial \mathcal{A}(V)$ has only one noncompact connected component $\mathcal{C}(V)$ (i.e., unbounded connected component). Indeed, by Bergman [B-71] (see also Bieri and Groves [BG-84]) the logarithmic limit set of an algebraic variety of codimension $r$ is $(n - r - 1)$-dimensional (i.e., its dimension is strictly less than $n - 2$). So, the complement of the amoeba $\mathcal{A}(V)$ in $\mathbb{R}^n$ has only one noncompact connected component. Assume $x$ is contained in the noncompact connected component $\mathcal{C}(V)$ of the boundary of $\mathcal{A}(V)$. Lemma 2.1 shows that $\mathcal{C}(V)$ is locally convex viewed as the graph of a function (this is a fact of the convexity of the complement components of hypersurface amoebas). As the convexity is a local property, i.e., a subset of a vector space is globally convex if and only if it is locally convex, this implies that $\mathcal{C}(V)$ is globally convex. This contradict the fact that the logarithmic limit set of an algebraic variety of codimension $r$ is $(n - r - 1)$-dimensional. Also, this contradict the higher convexity of the complement components of the amoeba $\mathcal{A}(V)$. In fact, if a subset of $\mathbb{R}^n$ is $k$-convex with nontrivial $k$-homology, then it can never be convex. Namely, we know that the homology of degree $(r - 1)$ of the unbounded complement of the amoeba $\mathcal{A}(V)$ is nontrivial. This is a consequence of the injection of the $(r - 1)$-homology of the complement of its logarithmic set in the sphere $S^{n-1}$ into the $(r - 1)$-homology of the complement of the amoeba.
Lemma 2.2. Let \( V \subset (\mathbb{C}^*)^n \) be a generic algebraic variety of codimension \( r \) such that \( n > 2(n - r) \). Assume there exists a finite number of polynomials \( \{g_i\}_{i=1}^{l} \) such that \( \mathcal{A}(V) = \bigcap_{i=1}^{l} \mathcal{A}(V_{g_i}) \). Then, the complement of the amoeba \( \mathcal{A}(V) \) contains a component which is not \((r - 1)\)-convex.

Proof. In this case, the amoeba \( \mathcal{A}(V) \) may have or may not have a boundary. The variety \( V \) is generic, and its codimension \( r \) satisfies the inequality \( n < 2r \), means that \( \mathcal{A}(V) \) is \( 2(n - r) \)-dimensional i.e., its dimension is strictly less than the dimension of the ambient space \( \mathbb{R}^n \). Let \( x \) be a point in \( \mathcal{A}(V) \), and for simplicity assume that \( x \) is a smooth point of the amoeba. The amoeba \( \mathcal{A}(V) \) is \( 2(n - r) \)-dimensional, and \( 2(n - r) < n \) implies that for a small open neighborhood \( U_x \) of \( x \) in \( \mathcal{A}(V) \) there exists a unit vector direction \( v \) not in the tangent space of \( \mathcal{A}(V) \) at \( x \) such that \( (U_x + \varepsilon v) \cap \mathcal{A}(V) \) is empty for all small positive numbers \( \varepsilon \). The same reasoning as in Lemma 2.1 shows that there exists \( s \) with \( 1 \leq s \leq l \) such that \( (U_x + \varepsilon v) \cap \mathcal{A}(V_{g_s}) \) is empty for all small positive numbers \( \varepsilon \). Hence, \( U_x \) is contained in the boundary of the hypersurface amoeba \( \mathcal{A}(V_{g_s}) \). As the \( \partial \mathcal{A}(V_{g_s}) \) is locally convex in the sense of Definition 2.1, then there exists a hyperplane \( H_x \subset \mathbb{R}^n \) passing throughout the point \( x \) such that for a small ball \( B(x, \rho) \subset \mathbb{R}^n \) the set \( \partial \mathcal{A}(V_{g_s}) \cap B(x, \rho) \) is contained in only one side of \( H_x \). Let \( L_x \subset H_x \) be an \( r \)-dimensional plane containing \( x \). Let \( \mathcal{A}^c(V) := \mathbb{R}^n \setminus \mathcal{A}(V) \) i.e., the complement of the amoeba in \( \mathbb{R}^n \). Now it is claire that there exists an \((r - 1)\)-cycle \( \gamma \) in \( L_x \) (which we can assume positive in the sense of Henriques’s Definition 3.3 in [H-03] non homologue to zero in \( L_x \cap \mathcal{A}^c(V) \). In fact, take a small \((r - 1)\)-dimensional sphere in \( L_x \) centered at \( x \). As \( r < n \) (i.e., \( V \) is not a set of points), the cycle \( \gamma \) bounds an \( r \)-chain in \( \mathcal{A}^c(V) \) (because a small neighborhood of \( x \) in the amoeba \( \mathcal{A}(V) \) is contained in only one side of the hyperplane \( H_x \)). This means that the homology class of \( \gamma \) in \( H_{r-1}(\mathcal{A}^c(V)) \) is trivial. This contradict the higher convexity of the complement of the amoeba \( \mathcal{A}(V) \) (see [H-03]).

\( \square \)

If the codimension \( r \) of the variety \( V \) satisfies \( n > 2(n - r) \), then Theorem 1.1 is a consequence of Lemma 2.2.

Remark 2.1.

(a) Let \( \{g_j\}_{j=1}^{r} \) be a generator set of the defining ideal \( \mathcal{I}(V) \) of an algebraic variety \( V \). As a consequence of the proof of Theorem 1.1 in [NP-11], the degree of the polynomials \( f \in \mathcal{I}(V) \) such that \( \mathcal{A}(V) = \bigcap_{f \in \mathcal{I}(V)} \mathcal{A}(V_f) \) can always be bounded by \( 2\max_{j=1}^{r} \{\deg(g_j)\} \).

(b) Using the higher convexity of coamoeba complements proved by Sottile and I in [NS-13], the statement of Theorem 1.1 is valid if we replace amoebas by coamoebas.

3. Example of a generic affine line in \((\mathbb{C}^*)^3\)

The amoeba of a generic line \( L \) in \((\mathbb{C}^*)^3\) is a surface with Figure 1 or without boundary Figure 2 (it depends if it is real or not real, see [NP-11] for more details). By Lemma 2.2 if the amoeba \( \mathcal{A}(L) \) of \( L \) is the intersection of a finite number of hypersurface amoeba, then it is locally a convex surface.
because it is locally contained in the boundary of a hypersurface amoeba \( A(V_{g_s}) \). Namely, if \( x \) is a point in \( A(L) \), then there exists \( 1 \leq s \leq l \) and an open neighborhood \( U_x \) of \( x \) in \( A(L) \) which is contained in \( \partial A(V_{g_s}) \). Hence, the convexity of the complement of the hypersurface amoeba \( A(V_{g_s}) \) implies that there exists a 1-cycle \( \gamma \subset \mathcal{H}_x \) such that the class of \( \gamma \) in \( H_1(A(L)^c \cap \mathcal{H}_x, \mathbb{Z}) \) is different than zero, where \( \mathcal{H}_x \) is a hyperplane in \( \mathbb{R}^3 \) which separate locally at \( x \) the boundary \( \partial A(V_{g_s}) \) of the amoeba \( A(V_{g_s}) \) (i.e., locally in a small neighborhood of \( x \), the boundary \( \partial A(V_{g_s}) \) is situated in only one side of \( \mathcal{H}_x \)). But \( \gamma \) bounds a topological disk in \( A(L)^c \). This contradict the 1-convexity of \( A(L)^c \) in \( \mathbb{R}^3 \). Hence, the amoeba of a generic line in the space can never be the intersection of a finite number of hypersurface amoebas.

\[
\text{Figure 1.} \quad \text{The amoeba of the real line in } (\mathbb{C}^*)^3 \text{ given by the parametrization } \rho(z) = (z, z + \frac{1}{2}, z - \frac{3}{2}). \quad \text{In this case, the amoeba is topologically the closed disk without four points of its boundary.}
\]

\[
\text{Figure 2.} \quad \text{The amoeba of non real line in } (\mathbb{C}^*)^3 \text{ given by the parametrization } \rho(z) = (z, z + 1, z - 2i). \quad \text{In this case, the amoeba is topologically the Riemann sphere without four points.}
\]
4. Characterization of hypersurface amoeba boundaries

Given a smooth algebraic hypersurface $V \subset (\mathbb{C}^*)^n$, F. Schroeter and T. de Wolff give a characterization of the boundary of hypersurface amoeba $\mathcal{A}(V)$ up to singular points of the set of critical values of the logarithmic map restricted to $V$ (see Theorem 1.3 in [SW-13]). In this section, we will refine their theorem and give a very short proof of it. Let us start by giving some definitions and notation.

Let us denote by $\text{Log}$ the coordinatewise logarithmic map, i.e., the map from the complex algebraic torus into $\mathbb{R}^n$ defined as follows:

$$
\text{Log} : (\mathbb{C}^*)^n \rightarrow \mathbb{R}^n \quad (z_1, \ldots, z_n) \mapsto (\log |z_1|, \ldots, \log |z_n|).
$$

We denote by $\text{Crit}_p(\text{Log}|_V)$ (resp. $\text{Crit}_v(\text{Log}|_V)$) the set of critical points (resp. critical values) of the logarithmic map restricted to $V$.

Let $V \subset (\mathbb{C}^*)^n$ be a complex algebraic hypersurface defined by a polynomial $f$ and nowhere singular. The logarithmic Gauss map of the hypersurface $V$ is a rational map from all $V \setminus \text{Crit}_v(\text{Log}|_V)$ to $\mathbb{P}^{n-1}$ defined as follows:

$$
\gamma : V \setminus \text{Crit}_v(\text{Log}|_V) \rightarrow \mathbb{P}^{n-1} \\
z \mapsto \gamma(z) = [z_1 \frac{\partial f}{\partial z_1}(z) : \cdots : z_n \frac{\partial f}{\partial z_n}(z)].
$$

We have the following commutative diagram:

$$
\begin{array}{ccc}
V & \xrightarrow{\gamma} & \mathbb{P}^{n-1} \\
\cup & \uparrow & \uparrow \\
\text{Crit}_p(\text{Log}|_V) & \xrightarrow{\gamma_c} & \mathbb{R}^{n-1} \\
\text{Log} & \downarrow & \downarrow \\
\text{Crit}_v(\text{Log}|_V), & \xrightarrow{g} & \\
\end{array}
$$

where $\cup$ denotes the natural inclusion, $g$ is the usual Gauss map defined on the smooth part of $\text{Crit}_v(\text{Log}|_V)$, and $\gamma_c = \gamma|_{\text{Crit}_p(\text{Log}|_V)}$ (i.e., the restriction of $\gamma$ to the set of critical points $\text{Crit}_p(\text{Log}|_V)$ of the logarithmic map).

**Definition 4.1.** A point $x$ in $\text{Crit}_v(\text{Log}|_V)$ is called regular if and only if $\text{Log}^{-1}(x) \cap V$ is contained in the set of regular points of the restriction of the logarithmic Gauss map to $\text{Crit}_p(\text{Log}|_V)$ (i.e., $\text{Log}^{-1}(x) \cap V$ contains no critical point of $\gamma$).

**Remark 4.1.**

(i) A singular point of $\text{Crit}_v(\text{Log}|_V)$ can be a regular point in the sense of Definition 4.1. Moreover, a non regular point in the sense of Definition 4.1 is necessarily a singular point of $\text{Crit}_v(\text{Log}|_V)$;

(ii) The degree of the extension of $\gamma$ to the compactification $\overline{V}$ of $V$ in the projective space $\mathbb{C}P^n$ is equal to $n!\text{Vol}(\Delta_f)$, i.e., the cardinality
of $\gamma^{-1}(y)$ for a generic point $y$ is finite and equal to $n! \text{Vol}(\Delta_f)$ (see [M-00]);

(iii) The inverse image of a regular point $x \in \text{Crit} V$ by the logarithmic map is a finite number of points. But the inverse image by the logarithmic map of a non regular point can be of positive dimension (see the example of the hyperbola in Section 5).

(iv) In the case of plane curves, the logarithmic Gauss map $\nabla V \to \mathbb{C}P^1$ is a branched covering where the branching points are the points of logarithmic inflection (in other words, inflection after taking the holomorphic logarithm); see [M-00].

Let $x \in \text{Crit} V$ be a regular point contained in the image by the logarithmic map of the local holomorphic branches $B_1(x), \ldots, B_s(x)$, and we denote by $C_1(x), \ldots, C_s(x)$ their corresponding real branches of critical values passing through $x$. Recall that the $C_i(x)$’s are the image of the critical points inside the corresponding holomorphic branches and $C_i(x)$ can be empty if the local branch $B_i(x)$ is regular (i.e., does not intersect $\text{Crit} \log |V|$). In general, if $V$ is not smooth, it can happen that a critical value branch $C_i(x)$ has dimension strictly less than $(n-1)$ and then $x$ is necessarily contained in the interior of the amoeba. So, throughout this section, we assume that the dimension of $C_i(x)$ is equal to $(n-1)$ for all $i$ (i.e., we can assume $V$ singular but we consider only the set of smooth points of $V$). We denote by $v_i(x)$ the normal vector to $C_i(x)$ (if it is nonempty) pointed inside the local amoeba $\mathcal{A}(B_i(x))$ of $B_i(x)$ (the existence of $v_i(x)$ is assured by the regularity of $x$). We have the following:

Lemma 4.1. Let $V$ be a complex algebraic hypersurface. Let $x$ be a point in the boundary of the amoeba $\partial \mathcal{A}(V)$. Then the set $(\log^{-1}(x) \cap V)$ is contained in $\text{Crit} \log |V|$.

Proof. Assume there exists a component $C_x$ of $\log^{-1}(x) \cap V$ which is not critical. This means that there exists $z \in V$ such that $\log(z) = x$ and $z$ is a regular point of the logarithmic map (i.e., the Jacobian $\text{Jac}(\log |V|)_z$ of the logarithm map restricted to $V$ at the point $z$ has maximal rank). The fact that the set of regular points of the logarithmic map is an open subset of $V$, implies that there exists an open subset $U_z$ in $V$ containing $z$, such that $\log |V|_z$ is a submersion. Hence, the point $x$ must be in the interior of the amoeba and not in its boundary. This contradict our hypothesis on $x$. □

Lemma 4.2. Let $V$ be a complex algebraic hypersurface. Let $x$ be a regular point of $\text{Crit} V$. Then with the above notation, the following statements are equivalent:

(i) The point $x$ is in $\partial \mathcal{A}(V)$;

(ii) The convex hull of the vectors $\{v_i(x)\}_{i=1}^s$ does not contain the origin and the intersection of each holomorphic branch $B_i(x)$ with $\log^{-1}(x)$ is contained in $\text{Crit} \log |V|$.

Proof. (i) $\implies$ (ii). As $x \in \partial \mathcal{A}(V)$, then there exists a vector direction $v$ such that $(x + \varepsilon v) \notin \mathcal{A}(V)$ for all small strictly positive numbers $\varepsilon$ (see Claim A). This is equivalent to the fact that there exists $\eta$ such that for
any strictly positive number $\varepsilon \leq \eta$ the vector $(x + \varepsilon v)$ is outside the convex hull of $\{x, (x + v_i(x))_{i=1}^s\}$. Indeed, if there exists a sequence $\varepsilon_m$ such that $(x + \varepsilon_m v)$ is contained in the convex hull of $\{x, (x + v_i(x))_{i=1}^s\}$, then the fact that the number of branches is finite (because $V$ is algebraic), implies that for a small $\varepsilon$, the vector $(x + \varepsilon v)$ is contained in the local amoeba of some local holomorphic branch $B_i(x)$, and then $(x + \varepsilon v)$ is contained in the amoeba itself. This contradict the choice of $v$, and then $x$ must be a vertex of the convex hull of $\{x, (x + v_i(x))_{i=1}^s\}$. This means that the convex hull of the vectors $\{v_i(x)\}_{i=1}^s$ does not contain the origin. Finally, the fact that all the local holomorphic branches $B_i(x)$ intersect $\text{Crit}(\log |V|)$ with $x \in \mathcal{C}_i(x)$ is a consequence of Lemma \ref{lem:amoeba}.

$(ii) \implies (i)$. All the local holomorphic branches $B_i(x)$ intersect $\text{Crit}(\log |V|)$ with $x \in \mathcal{C}_i(x)$ and the convex hull of the vectors $\{v_i(x)\}_{i=1}^s$ does not contain the origin means that $x$ is not in the convex hull of the vectors $\{(x + v_i(x))\}_{i=1}^s$. This implies that there exists a vector direction $v$ not in the convex hull of $\{v_i(x)\}_{i=1}^s$ such that $(x + \varepsilon v) \notin \mathcal{A}(V)$ for any positive number $\varepsilon$. In fact, if for any vector direction $v$, the vector $(x + \varepsilon v)$ is contained in $\mathcal{A}(V)$ means that for any $v$ there exists a local holomorphic branch $B_i(x)$ such that $(x + \varepsilon v)$ is contained in the local amoeba of $B_i(x)$ for any small strictly positive number $\varepsilon$. As the number of local holomorphic branches is finite, then the point $x$ must be in the interior of the convex hull of $\{(x + v_i(x))\}_{i=1}^s$. This contradict our hypothesis and then the point $x$ is in the boundary of the amoeba of $V$.  

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**Theorem 4.1.** Let $V$ be a complex algebraic hypersurface and $x$ be a regular point in $\text{Crit}(\log |V|)$ with $(\log^{-1}(x) \cap V) \subset \text{Crit}(\log |V|)$. Then the convex hull of the vectors $\{v_i(x)\}_{i=1}^s$ does not contain the origin if and only if $x \in \partial \mathcal{A}(V)$. In other words, the convex hull of the vectors $\{v_i(x)\}_{i=1}^s$ contains the origin if and only if the point $x$ is contained in the interior of the amoeba.

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**Proof.** Let $x$ be a point of $\text{Crit}(\log |V|)$ such that the set $(\log^{-1}(x) \cap V) \subset \text{Crit}(\log |V|)$ and suppose the convex hull of $\{(x + v_i(x))\}_{i=1}^s$ contains $x$. This means that for any unit vector direction $v$ and for all small positive numbers $\varepsilon$ we have $(x + \varepsilon v) \in \mathcal{A}(V)$, which is equivalent to the fact that $x$ is contained in the interior of the amoeba $\mathcal{A}(V)$. If the convex hull of the vectors $\{v_i(x)\}_{i=1}^s$ does not contain the origin, by Lemma \ref{lem:amoeba} and the hypothesis of our theorem, the point $x$ is in the boundary of the amoeba $\partial \mathcal{A}(V)$.

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5. Example of a non regular point in $\text{Crit}(\log |V|)$

Let $\mathcal{H}$ be the real algebraic plane curve (hyperbola) parametrized as follows:

$$
\rho : \mathbb{C}^* \setminus \{-1, -\frac{1}{6}\} \longrightarrow (\mathbb{C}^*)^2
$$

$$
z \longmapsto \rho(z) = -\frac{z + \frac{1}{6}}{z + 1}.
$$

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Its amoeba has a non regular critical value $x_0$, called a pinching point by Mikhalkin (Remark 10, [M-00]). The inverse image of the point $x_0 = (-\log \frac{3}{2}, \log |\sqrt{\frac{3}{8}}|)$ by the logarithmic map in $\mathcal{H}$ is a non geodesic circle (i.e., $\mathcal{H} \cap \text{Log}^{-1}(x_0)$ is a circle but not geodesic in the flat torus $(S^1)^2 = \text{Log}^{-1}(x_0)$). As the set of critical points of the logarithmic map and the argument maps coincides (see [M-00]), we can check the fact that $\mathcal{H} \cap \text{Log}^{-1}(x_0)$ is a circle by looking to the coamoeba of $\mathcal{H}$. The set of critical values of the argument map is a non geodesic circle $\mathcal{C}$ which has two different real points (i.e., intersect the finite real subgroup $(\mathbb{Z}_2)^2$ of the hall real torus in two points) union the isolated point $(\pi, \pi)$. The circle $\mathcal{C}$ is also critical for the logarithmic Gauss map $\gamma_c$. More precisely, there are two real branches $\mathcal{B}_1$ and $\mathcal{B}_2$ of critical points intersecting $\mathcal{C}$ in two different real points contained in two different quadrants of $(\mathbb{R}^*)^2$, namely the quadrants $(+, -)$ and $(-, +)$. The image of each branch by the logarithmic map has an inflection point at $x_0$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{amoeba.png}
\caption{the amoeba and the coamoeba of the real hyperbola in $(\mathbb{C}^*)^2$ with defining polynomial $f(z, w) = \frac{1}{6} + z + w + zw$.}
\end{figure}

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