Some Remarks on Fuzzy sb-Metric Spaces

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Abstract: Fuzzy strong \(b\)-metrics called here by fuzzy sb-metrics, were introduced recently as a fuzzy version of strong \(b\)-metrics. It was shown that open balls in fuzzy sb-metric spaces are open in the induced topology (as different from the case of fuzzy \(b\)-metric spaces) and thanks to this fact fuzzy sb-metrics have many useful properties common with fuzzy metric spaces which generally may fail to be in the case of fuzzy \(b\)-metric spaces. In the present paper, we go further in the research of fuzzy sb-metric spaces. It is shown that the class of fuzzy sb-metric spaces lies strictly between the classes of fuzzy metric and fuzzy \(b\)-metric spaces. We prove that the topology induced by a fuzzy sb-metric is metrizable. A characterization of completeness in terms of diameter zero sets in these structures is given. We investigate products and coproducts in the naturally defined category of fuzzy sb-metric spaces.

Keywords: fuzzy metric; fuzzy sb-metric; fuzzy \(b\)-metric

1. Introduction

The class of \(b\)-metric spaces, one of the generalizations of metric spaces, has been introduced by different authors under different names (\(b\)-metric by Czerwik [1], quasimetric by Bakhtin [2] and by Heinonen [3], \(NEM_r\) (nonlinear elastic matching) by Fagin et al. [4], metric type by Khamsi et al. [5]. To make our exposition homogeneous in the sequel we use the term \(b\)-metric in all these cases. The definition of a \(b\)-metric is obtained by replacing the triangular inequality in the definition of a metric space with the inequality \(d(x, z) \leq k \cdot (d(x, y) + d(y, z)), \forall x, y, z \in X\) for some \(k \geq 1\), so called “relaxed triangle inequality” in [4]. The topology induced by a \(b\)-metric has some “unpleasant” features. For instance, open balls may be not open [6,7], closed balls may be not closed [7] and a \(b\)-metric as a mapping may be not continuous in the induced topology [8]. To remedy these defects, Kirk and Shahzad [9] introduced the notion of an \(sb\)-metric by using the inequality \(d(x, z) \leq d(x, y) + k \cdot d(y, z), \forall x, y, z \in X\) for some \(k \geq 1\). Therefore, a metric is an \(sb\)-metric and an \(sb\)-metric is a \(b\)-metric. They also investigated the fixed point theory for mappings of these structures and complained about the absence of nontrivial examples of such spaces.

Recently, in [10], the authors of this paper presented a series of examples of \(sb\)-metric spaces that fail to be metric. They also considered these metric type spaces in the context when the ordinary sum operation is replaced by an extended \(t\)-conorm that is an operation satisfying certain conditions.

In the papers [11,12], a fuzzy version of an \(sb\)-metric was introduced and the basic properties of fuzzy strong \(b\)-metric spaces (renamed here as fuzzy sb-metric spaces) were studied. Besides, this notion can be viewed as a generalization of fuzzy metric spaces in the sense of George and Veeramani [13] since its definition is obtained by replacing the fuzzy triangularity axiom in the definition of a fuzzy metric with \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + k \cdot s)\) for some \(k \geq 1\). Thus the class of fuzzy sb-metric spaces lies between fuzzy metric spaces and fuzzy \(b\)-metric spaces [14–16].
As expected, fuzzy \( sb \)-metric spaces have useful properties in common with metric and fuzzy metric spaces such as openness of open balls whereas it is not true in general for \( b \)-metric and fuzzy \( b \)-metric spaces.

As one can expect \( sb \)-metric and fuzzy \( sb \)-metric spaces have important properties sharing with metric and fuzzy metric spaces but that are not valid for general \( b \)-metric and fuzzy \( b \)-metric spaces. Specifically, open balls in fuzzy \( sb \)-metric spaces are open in the topology induced by an \( sb \)-metric, but may fail, as our examples in [14,16] show, to be open in the topology of fuzzy \( b \)-metric spaces. Based on this fact we prove that fuzzy \( sb \)-metric is continuous, that fuzzy \( sb \)-metric spaces are metrizable and other important properties common with the properties of metric and fuzzy metric spaces but not sharing with \( b \)-metric spaces.

The problem of definition of a concept of a fuzzy metric has a very long and diverse theory. As the first works of definitions of fuzzy metric, in an essentially different way, were given by Deng [17], Kaleva and Seikkala [18] and Kramosil and Michalek [19]. In our work we follow Kramosil-Michalek approach to fuzzy metric in the form modified by George-Veeramani [13]. Therefore, in order to designate the place of our research, we give a very brief survey of some work done but different authors in the direction of generalization of George-Veeramani fuzzy metrics.

In [20], Park defined the notion of intuitionistic fuzzy metric spaces by using the idea of intuitionistic fuzzy sets and proved some known results such as Baire’s theorem and the Uniform limit theorem. In [21], the authors proposed a method for constructing a Hausdorff fuzzy metric on the set of the nonempty compact subsets for a given fuzzy metric space and discussed several important properties as completeness, completion and precompactness. By using complete lattices, Adibi et al. [22], introduced the notion of \( L \)-fuzzy metric space and studied common fixed point and coincidence point theorems on these structures. Sedghi et al. [23] introduced the concept of partial fuzzy metric as a fuzzy analogy of partial metric spaces and gave the topological structure and proved some fixed point results. In [24], a fuzzy version of cone metric spaces was introduced by Öner et al., and some topological properties and Banach’s fixed point theorem were given. By employing a control function, Sezen [25] defined controlled fuzzy metric spaces and established some fixed point results.

The aim of the present paper is to go further in the research of fuzzy \( sb \)-metric spaces. The structure of the paper is as follows. We recall some terminology related to fuzzy \( sb \)-metrics, present some examples and prove some properties of fuzzy \( sb \)-metric spaces in Section 2. Section 3 is devoted to the study of (complete) metrizability of a (complete) fuzzy \( sb \)-metric space. Here we also prove the metrizability of \( sb \)-metric spaces by using fuzzy \( sb \)-metrics. Diameter zero sets in fuzzy \( sb \)-metrics and related properties such as completeness are given in Section 4. Moreover, by using fuzzy \( sb \)-metrics we indirectly characterized the completeness of an \( sb \)-metric space in terms of diameter zero sets. Products and coproducts in the category of fuzzy \( sb \)-metric spaces are studied in Section 5. In the last section we sketch some directions where the research in the field of fuzzy \( sb \)-metrics can be continued and its results could find some applications.

2. Fuzzy \( sb \)-Metric Spaces: Basic Concepts and Properties

In the following, we collect the terminology concerning fuzzy metric and its generalizations that can be found in the literature (see e.g., [11–16,26–28]).

**Definition 1.** Let \( X \) be a nonempty set. A mapping \( M : X \times X \times (0, \infty) \to [0, 1] \) is called a fuzzy symmetric if it satisfies the following properties for all \( x, y, z \in X \) and all \( t > 0 \):

\[
(f_{m1}) \quad M(x, y, t) > 0,
(f_{m2}) \quad M(x, y, t) = 1 \text{ if and only if } x = y,
(f_{m3}) \quad M(x, y, t) = M(y, x, t),
(f_{m4}) \quad M(x, y, \cdot) : (0, \infty) \to [0, 1] \text{ is continuous.}
\]

The corresponding tuple \((X, M, \ast)\) is called a fuzzy symmetric space.
Definition 2. Given a continuous t-norm $*: [0, 1] \times [0, 1] \rightarrow [0, 1]$, a fuzzy symmetric $M : X \times X \rightarrow (0, \infty) \rightarrow [0, 1]$ is called

($f_{ms}$) a fuzzy metric if
\[
M(x, y, t) = M(y, z, s) \leq M(x, z, t + s) \forall x, y, z \in X, s, t > 0
\]
and the corresponding tuple $(X, M, *)$ is called a fuzzy metric space;

($f_{bms}$) a fuzzy b-metric (or fuzzy metric type) if there exists $k \geq 1$ such that
\[
M(x, y, t) = M(y, z, s) \leq M(x, z, k \cdot (t + s)) \forall x, y, z \in X, s, t > 0
\]
and the corresponding tuple $(X, M, *, k)$ is called a fuzzy b-metric space (or fuzzy metric type);

($f_{sbms}$) a fuzzy sb-metric if there exists $k \geq 1$ such that
\[
M(x, y, t) = M(y, z, s) \leq M(x, z, k \cdot (t + s)) \forall x, y, z \in X, s, t > 0
\]
and the corresponding tuple $(X, M, *, k)$ is called a fuzzy sb-metric space.

Remark 1. We prefer to call the notion of fuzzy strong b-metric as “fuzzy sb-metric” since the term a fuzzy strong b-metric may lead to a misunderstanding when it comes into a collision with the concepts of a strong fuzzy metric [29] that has a quite different meaning.

Remark 2. It is clear that every fuzzy metric is also a fuzzy sb-metric. Furthermore, for a fuzzy sb-metric $M$, since $M(x, y, t) : (0, \infty) \rightarrow [0, 1]$ is nondecreasing, we have
\[
M(x, y, t) = M(y, z, s) \leq M(x, z, k \cdot t) \forall x, y, z \in X, s, t > 0
\]
and it means that a fuzzy sb-metric is fuzzy b-metric.

For fuzzy metric-like structures, an open ball with center $x$, radius $r \in (0, 1)$ and $t > 0$ is defined as $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$. The induced topology denoted $\tau_M$ is defined by open balls as follows:
\[
\tau_M = \{A \subseteq X : \forall x \in A \exists r \in (0, 1), t > 0 \text{ such that } B(a, r, t) \subseteq A\}
\]
Contrary to fuzzy b-metric spaces, in fuzzy sb-metric spaces open balls are open sets and form a base for the induced topology.

In the following we will need the following.

Remark 3. Let a continuous t-norm $*$ be given. Then for any $r_1 > r_2$, we can find a $r_3$ such that $r_1 \ast r_3 \geq r_2$ and for any $r_4$ we can find a $r_5$ such that $r_5 \ast r_5 \geq r_4$ where $r_1, r_2, r_3, r_4, r_5 \in (0, 1)$. [13]

Definition 3. Let $X$ be a nonempty set and $k \geq 1$. A mapping $d : X \times X \rightarrow [0, \infty)$ is called an sb-metric if it satisfies the following properties for all $x, y, z \in X$ [9]

($m_1$) $d(x, y) = 0$ if and only if $x = y,$

($m_2$) $d(x, y) = d(y, x),$

($sb_3$) $d(x, z) \leq d(x, y) + k \cdot d(y, z).$

The corresponding tuple $(X, d, k)$ is called an sb-metric space.

As in the case of fuzzy metrics and fuzzy b-metrics the construction of the so called “standard” fuzzy sb-metric plays an important role. Namely, given an sb-metric $d$ on a set $X$ we define a mapping $M_d : X \times X \times (0, \infty) \rightarrow [0, 1]$ by setting $M_d(x, y, t) = \frac{1}{1+d(x, y)}$. In [11], see Example 2.2 and Proposition 2.15, it was proved that $M_d$ is a fuzzy sb-metric for the case of the product t-norm and the topologies induced on the set $X$ by sb-metric $d$ and the fuzzy sb-metric $M_d$ coincide. Here we extend this result for the case of the minimum t-norm $\wedge$. Similar results for fuzzy metric and fuzzy b-metric spaces can be found in [16,30], respectively.
Proposition 1. Let \((X,d,k)\) be an sb-metric space. Then \((X, M_d, \wedge, k)\) is a fuzzy sb-metric space where \(M_d(x,y,t) = \frac{t}{t + d(x,y)}\) and \(\tau_d = \tau_{M_d}\).

Proof. We prove only \((f sbm_5)\) since the other conditions are straightforward. To do this, first we need to prove the following inequality for \(t, s > 0\).

\[
\frac{d(x,z)}{t + k \cdot s} < \max \left\{ \frac{d(x,y)}{t}, \frac{d(y,z)}{s} \right\}
\]

(1)

Consider three cases:

Case 1: \(d(x,z) \leq d(x,y)\).

Case 2: \(d(x,z) \leq k \cdot d(y,z)\).

Case 3: \(d(x,z) > d(x,y)\) and \(d(x,z) > k \cdot d(y,z)\).

Without loss of generality we may assume that \(d(x,z) = d(x,y) + k \cdot d(y,z)\).

Since \(d(x,z) > d(x,y)\), there exist \(\beta \in (0,1)\) such that \(d(x,y) = \beta \cdot d(x,z)\). Hence, we have \(d(y,z) = \frac{(1-\beta)d(x,z)}{\beta}\). Therefore, Equation (1) become

\[
\frac{d(x,z)}{t + k \cdot s} < \max \left\{ \frac{\beta \cdot d(x,z)}{t}, \frac{(1-\beta) \cdot d(x,z)}{k \cdot s} \right\}
\]

and we should prove simply

\[
\frac{1}{t + k \cdot s} \leq \max \left\{ \frac{\beta}{t}, \frac{1-\beta}{k \cdot s} \right\}.
\]

To do this, consider the functions \(f(\beta) = \frac{\beta}{t}\) and \(g(\beta) = \frac{k \cdot s}{1-\beta}\). Since \(f\) is decreasing and \(g\) is increasing, the largest value of \(\min \left\{ \frac{t}{\beta}, \frac{k \cdot s}{1-\beta} \right\}\) is \(t + k \cdot s\) that is taken when \(f(\beta) = g(\beta)\) where \(\beta = \frac{t}{t + k \cdot s}\). Then

\[
t + k \cdot s = f\left( \frac{t}{t + k \cdot s} \right) \\
\leq \min \left\{ \frac{t}{\beta}, \frac{k \cdot s}{1-\beta} \right\}
\]

implies

\[
\frac{1}{t + k \cdot s} \leq \max \left\{ \frac{\beta}{t}, \frac{1-\beta}{k \cdot s} \right\}.
\]

If \(d(x,z) < d(x,y) + k \cdot d(y,z)\), then there exists \(a \in (0,1)\) such that

\[
\frac{1}{a} \cdot d(x,z) = d(x,y) + k \cdot d(y,z).
\]

Further, \(d(x,z) > d(x,y)\) and \(d(x,z) > k \cdot d(y,z)\) imply \(\frac{1}{a} \cdot d(x,z) > d(x,y)\) and \(\frac{1}{a} \cdot d(x,z) > k \cdot d(y,z)\). Hence by above case, we have

\[
\frac{1}{a} \cdot \frac{d(x,z)}{t + k \cdot s} \leq \max \left\{ \frac{d(x,y)}{t}, \frac{d(y,z)}{s} \right\}
\]

which implies

\[
\frac{d(x,z)}{t + k \cdot s} \leq \max \left\{ \frac{d(x,y)}{t}, \frac{d(y,z)}{s} \right\}.
\]

Now we are ready to prove \((f sbm_5)\). By Equation (1), we have
\[
1 + \frac{d(x, z)}{t + k \cdot s} \leq \max \left\{ 1 + \frac{d(x, y)}{t}, 1 + \frac{d(y, z)}{s} \right\} \Rightarrow \\
\frac{t + k \cdot s + d(x, z)}{t + k \cdot s} \leq \max \left\{ \frac{t + d(x, y)}{t}, \frac{s + d(y, z)}{s} \right\} \Rightarrow \\
\frac{t + k \cdot s}{t + k \cdot s + d(x, z)} \geq \min \left\{ \frac{t}{t + d(x, y)}, \frac{s}{s + d(y, z)} \right\} \Rightarrow \\
M(x, z, t + k \cdot s) \geq M(x, y, t) \land M(y, z, s).
\]

The proof of the equivalence of the induced topologies is the same as the case where \( * = - \) (Proposition 2.15 in [11]) since the axiom \( (fsbms) \) and t-norm do not effect it. \( \square \)

Our next two examples present fuzzy sb-metric spaces which fail to be fuzzy metric spaces. Since on the other hand from the fact that open balls are open in fuzzy sb-metric, but need not be open in fuzzy b-metric, these examples will show also that the class of fuzzy sb-metric spaces lies strictly between the class of fuzzy metrics and fuzzy b-metrics.

**Example 1.** Let \( X = X_a \cup X_b \cup X_c \) where \( X_i = \{ i \} \times [0, 1], i \in \{ a, b, c \} \) and \( d : X \times X \rightarrow [0, 5] \) be the distance function defined as follows:

\[
d(x, y) = d(y, x) = \begin{cases} 
| x - y |, & x, y \in X_i \\
1, & x \in X_a, y \in X_b \\
2, & x \in X_a, y \in X_c \\
5, & x, y \in X_b, y \in X_c
\end{cases}
\]

where \( x = \{ i \} \times x, x \in [0, 1] \). Then \( (X, d, 3) \) is an sb-metric space (see Example 7 in [10]). By the above example, \( (X, M_d, \land, 3) \) is a fuzzy sb-metric space that fails to be a fuzzy metric space as seen below:
Let \( x \in X_b, y \in X_a \) and \( z \in X_c \). For \( t = 2 \) and \( s = 3 \), we have

\[
M_d(x, z, t + s) = \frac{t + s}{t + s + d(x, z)} = \frac{1}{2} < \frac{3}{5} = \frac{2}{3} \land \frac{3}{5} = \frac{t}{t + d(x, y)} \land \frac{s}{s + d(y, z)} < M_f(y, z, s).
\]

**Example 2.** Let \( X \) be the unit disk in \( \mathbb{R}^2 \) with center \((0,0)\) and \( S^1 \) the corresponding unit circle and let the distance function \( d : X \times X \rightarrow \mathbb{R} \) be defined as follows: for \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \)

\[
d(x, y) = d(y, x) = \begin{cases} 
10, & x, y \in S' \\
d'(x, y), & \text{otherwise}
\end{cases}
\]

where \( d' \) is the post office metric that is \( d'(x, y) = \sqrt{x_1^2 + x_2^2} + \sqrt{y_1^2 + y_2^2} \). Then \( (X, d, 9) \) is an sb-metric space (see Example 8 and Remark 7 in [10]). Hence \( (X, M_d, \land, 9) \) is the standard fuzzy sb-metric space induced by \( d \). Let \( x, y \in S' \) and \( z \) be the origin. Then for \( t = s = 1 \) we have

\[
M(x, z, t) \cdot M(z, y, s) = \frac{t}{t + d'(x, z)} \cdot \frac{s}{s + d'(z, y)} = \frac{t}{t + d'(x, z)} \cdot \frac{s}{s + d'(z, y)} = \frac{t}{t + 1} \cdot \frac{s}{s + 1} = \frac{1}{4} \geq \frac{1}{6} = \frac{2}{2 + 10} = \frac{t + s}{t + s + d(x, y)} = M(x, y, t + s)
\]
and this means that \( M_d \) fails to be a fuzzy metric.

A fuzzy metric space \((X, M, \ast)\) is said to be \(F\)-bounded if there exist \( t > 0 \) and \( r \in (0, 1) \) such that \( M(x, y, t) > 1 - r \) for all \( x, y \in X \) [13]. We use the same definition for fuzzy \( sb\)-metrics. We shall obtain a \(F\)-bounded fuzzy \( sb\)-metric equivalent for a given one in the sense that they induce the same topologies. However, in order to construct this fuzzy \( sb\)-metric we need to recall the convergence of a sequence and its characterization.

**Definition 4.** Let \((X, M, \ast, k)\) be a fuzzy \( sb\)-metric space, \( x \in X \) and \( \{x_n\} \) be a sequence in \( X \). Then \([11]\)

(i) \( \{x_n\} \) is said to converge to \( x \) if for any \( t > 0 \) and any \( r \in (0, 1) \) there exists a natural number \( n_0 \) such that 

\[
M(x_n, x, t) > 1 - r \quad \text{for all} \quad n \geq n_0.
\]

We denote this by \( \lim_{n \to \infty} x_n = x \) or \( x_n \to x \) as \( n \to \infty \).

(ii) \( \{x_n\} \) is said to be a Cauchy sequence if for any \( r \in (0, 1) \) and any \( t > 0 \) there exists a natural number \( n_0 \) such that 

\[
M(x_m, x_n, t) > 1 - r \quad \text{for all} \quad n, m \geq n_0.
\]

(iii) \( (X, M, \ast, k) \) is said to be a complete fuzzy \( sb\)-metric space if every Cauchy sequence is convergent.

**Theorem 1.** Let \((X, M, \ast, k)\) be a fuzzy \( sb\)-metric space, \( x \in X \) and \( \{x_n\} \) be a sequence in \( X \). \( \{x_n\} \) converges to \( x \) if and only if 

\[
M(x_n, x, t) \to 1 \quad \text{as} \quad n \to \infty, \quad \text{for each} \quad t > 0. \quad [11]
\]

**Proposition 2.** Let \((X, M, \ast, k)\) be a fuzzy \( sb\)-metric space and \( \lambda \in (0, 1) \). Then \((X, N, \ast, k)\) is an \( F\)-bounded fuzzy \( sb\)-metric space where 

\[
N(x, y, t) = \max\{M(x, y, t), \lambda\} \quad \text{for} \quad x, y \in X \quad \text{and} \quad t > 0. \quad [11]
\]

**Proof.** We only show the \((fsbms)\) since the other conditions are immediate. We distinguish two cases:

Case 1: \( N(x, y, t) = \lambda \) or \( N(y, z, s) = \lambda \). Assume that \( N(x, y, t) = \lambda \). Then we have 

\[
N(x, z, t + k \cdot s) \geq \lambda = \lambda \ast 1 \geq \lambda \ast N(y, z, s) \\
\geq N(x, y, t) \ast N(y, z, s).
\]

Case 2: \( N(x, y, t) = M(x, y, t) > \lambda \) and \( N(y, z, s) = M(y, z, s) > \lambda \). Here we have 

\[
N(x, z, t + k \cdot s) \geq M(x, z, t + k \cdot s) \geq M(x, y, t) \ast M(y, z, s) \\
\geq N(x, y, t) \ast N(y, z, s).
\]

To establish the equivalence of the induced topologies, it is sufficient to show the equivalence of the convergence of sequences since induced topologies are first countable (Proposition 2.8. in [11]). Let \( (x_n) \) be a sequence in \( X \). For each \( t > 0 \), we have 

\[
N(x_n, x, t) \to 1 \iff \max\{\lambda, M(x_n, x, t)\} \to 1 \\
\iff M(x_n, x, t) \to 1
\]

and it means that the induced topologies are the same. \( \square \)

**Proposition 3.** Let \((X, M_1, \ast, k_1)\) and \((X, M_2, \ast, k_2)\) be fuzzy \( sb\)-metric spaces and define 

\[
N_1(x, y, t) = M_1(x, y, t) \ast M_2(x, y, t) \\
N_2(x, y, t) = \min\{M_1(x, y, t), M_2(x, y, t)\}.
\]

and \( k = \max\{k_1, k_2\} \). Then

(i) \((X, N_1, \ast, k)\) is a fuzzy \( sb\)-metric space.

(ii) \((X, N_2, \ast, k)\) is a fuzzy \( sb\)-metric space.

(iii) \( \tau_{N_1} = \tau_{N_2} \).
Theorem 2. Let

\[ (iii) \]

\begin{align*}
N_1(x, z, t + k \cdot s) &= M_1(x, z, t + k \cdot s) \ast M_2(x, z, t + k \cdot s) \\
&\geq M_1(x, z, t + k_1 \cdot s) \ast M_2(x, z, t + k_2 \cdot s) \\
&\geq M_1(x, y, t) \ast M_1(y, z, s) \ast M_2(x, y, t) \ast M_2(y, z, s) \\
&= M_1(x, y, t) \ast M_2(x, y, t) \ast M_1(y, z, s) \ast M_2(y, z, s) \\
&= N_1(x, y, t) \ast N_1(y, z, s).
\end{align*}

(ii)

\begin{align*}
N_2(x, z, t + k \cdot s) &= \min\{M_1(x, z, t + k \cdot s), M_2(x, z, t + k \cdot s)\} \\
&\geq \min\{M_1(x, z, t + k_1 \cdot s), M_2(x, z, t + k_2 \cdot s)\} \\
&\geq \min\{M_1(x, y, t) \ast M_1(y, z, s), M_2(x, y, t) \ast M_2(y, z, s)\} \\
&\geq \min\{M_1(x, y, t) \ast M_2(x, y, t)\} \ast \min\{M_1(y, z, s) \ast M_2(y, z, s)\} \\
&= N_2(x, y, t) \ast N_2(y, z, s).
\end{align*}

(iii) Let \( (x_n) \) be a sequence in \( X \). Then for all \( t > 0 \), we have

\[ N_1(x_n, x, t) \to 1 \iff M_1(x_n, x, t) \ast M_2(x_n, x, t) \to 1 \]

\[ \iff M_1(x_n, x, t) \to 1 \text{ and } M_2(x_n, x, t) \to 1 \]

\[ \iff \min\{M_1(x_n, x, t), M_2(x_n, x, t)\} \to 1 \]

\[ \iff N_2(x_n, x, t) \to 1 \]

and it means that the induced topologies are the same. \( \square \)

Example 3. Let \( X \) be a set and \( * \) be a continuous t-norm. By Example 7 in [31] and Lemma 3.1 in [30],

\[ M(x, y, t) = \begin{cases} 
1 & x = y \\
\lambda & x \neq y
\end{cases} \]

is a discrete fuzzy metric on \( X \) where \( \lambda \in (0, 1) \). Then \((X, M, *, k)\) can be considered as a fuzzy sb-metric for any \( k \geq 1 \). Hence, by Example 3, \((X, N_i, *, k), i = 1, 2, \) are also discrete fuzzy sb-metric spaces where

\[ N_1(x, y, t) = \begin{cases} 
1 & x = y \\
M_1(x, y, t) \ast \lambda & x \neq y
\end{cases} \]

\[ N_2(x, y, t) = \begin{cases} 
1 & x = y \\
\min\{M_1(x, y, t), \lambda\} & x \neq y
\end{cases} \]

and \((X, M_1, *, k)\) is an arbitrary fuzzy sb-metric space.

Theorem 2. Let \((X, M, *, k)\) be a fuzzy sb-metric space. Then \( M(-, -, t) : X \times X \to [0, 1] \) is continuous.

Proof. Since \((X, \tau_M)\) is first countable, it is also sequential. Moreover, product topology on \( X \times X \) is also first countable and sequential. So, it is enough to show that \( M(-, -, t) \) is sequentially continuous. Let two convergent sequences \((x_n)\) and \((y_n)\) in \( X \) be given such that \( x_n \to x \) and \( y_n \to y \) for some \( x, y \in X \). Hence \( \lim_{n \to \infty} M(x_n, x, t) \to 1 \) and \( \lim_{n \to \infty} M(y_n, y, t) \to 1 \) for any \( t > 0 \). For a given \( \varepsilon > 0 \) and \( t > 0 \),
\[
M(x, y, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \leq M\left(x_n, y, t + \frac{\varepsilon}{2}\right)
\]
\[
M(x, y, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right) \leq M\left(x_n, y, t + \frac{\varepsilon}{2}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right)
\]
\[
M(x, y, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right) \leq M(x, y_n, t + \varepsilon).
\]

As \(n \to \infty\), we have
\[
\lim_{n \to \infty} \left[ M(x, y, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right) \right] \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon)
\]
\[
\lim_{n \to \infty} M(x, y, t) \ast \lim_{n \to \infty} M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast \lim_{n \to \infty} M\left(y_n, y, \frac{\varepsilon}{2k}\right) \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon)
\]
\[
\lim_{n \to \infty} M(x, y, t) \ast 1 \ast 1 \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon)
\]
\[
M(x, y, t) \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon).
\]

Since \(\varepsilon > 0\) is arbitrary, we obtain
\[
M(x, y, t) \leq \lim_{n \to \infty} M(x_n, y_n, t).
\]

Similarly, for a given \(\varepsilon > 0\) and \(t > 0\),
\[
M(x_n, y_n, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \leq M\left(y_n, x, t + \frac{\varepsilon}{2}\right)
\]
\[
M(x_n, y_n, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right) \leq M\left(y_n, x, t + \frac{\varepsilon}{2}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right)
\]
\[
M(x_n, y_n, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right) \leq M(x, y, t + \varepsilon).
\]

As \(n \to \infty\), we have
\[
\lim_{n \to \infty} \left[ M(x_n, y_n, t) \ast M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast M\left(y_n, y, \frac{\varepsilon}{2k}\right) \right] \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon)
\]
\[
\lim_{n \to \infty} M(x_n, y_n, t) \ast \lim_{n \to \infty} M\left(x_n, x, \frac{\varepsilon}{2k}\right) \ast \lim_{n \to \infty} M\left(y_n, y, \frac{\varepsilon}{2k}\right) \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon)
\]
\[
\lim_{n \to \infty} M(x_n, y_n, t) \ast 1 \ast 1 \leq \lim_{n \to \infty} M(x_n, y_n, t + \varepsilon)
\]
\[
\lim_{n \to \infty} M(x_n, y_n, t) \leq M(x, y, t + \varepsilon).
\]

Since \(\varepsilon > 0\) is arbitrary, we obtain
\[
\lim_{n \to \infty} M(x_n, y_n, t) \leq M(x, y, t).
\]

Therefore, \(\lim_{n \to \infty} M(x_n, y_n, t) = M(x, y, t)\) and \(M(-, -, t) : X \times X \to [0, 1]\) is continuous.

**Lemma 1.** Let \((X, d, k)\) be an sb-metric space and \((X, M_d, \cdot, k)\) be the standard fuzzy sb-metric space.

1. \(x_n \to x\) in \((X, d, k)\) if and only if \(x_n \to x\) in \((X, M_d, \cdot, k)\).
2. \((x_n)\) is Cauchy in \((X, d, k)\) if and only if \((x_n)\) is Cauchy in \((X, M_d, \cdot, k)\).

**Proof.** (1) Let \(x_n \to x\) in \((X, d, k)\). Then \(d(x_n, x) \to 0\) and we have
\[
M_d(x_n, x, t) = \frac{t}{t + d(x_n, x)} \to 1
\]
which implies that \( x_n \to x \) in \((X, M_d)\). Converse is similar.

(2) Let \((x_n)\) be Cauchy in \((X, d, k)\) and \(r \in (0, 1)\) and \(t > 0\) are given. For \(\epsilon = \frac{tr}{1 - r}\), there exists \(n_0 \in \mathbb{N}\) such that \(d(x_n, x_m) < \epsilon\) for every \(n, m \geq n_0\). Then we have

\[
\begin{align*}
\frac{t + d(x_n, x_m)}{t + d(x_n, x_m)} &< t + \epsilon = t + \frac{tr}{1 - r} = \frac{t}{1 - r} \\
\frac{t}{t + d(x_n, x_m)} &> 1 - r \\
M_d(x_n, x_m, t) &> 1 - r
\end{align*}
\]

for every \(n, m \geq n_0\). Hence \((x_n)\) is Cauchy in \((X, M_d)\). On the other hand, let \((x_n)\) be Cauchy in \((X, M_d, k)\) and \(\epsilon > 0\) is given. Choose \(r = \frac{\epsilon}{t + \epsilon} \in (0, 1)\). Then for any \(t > 0\), there exists \(n_0 \in \mathbb{N}\) such that \(M_d(x_n, x_m, t) > 1 - r\) for every \(n, m \geq n_0\). It follows that

\[
M_d(x_n, x_m, t) = \frac{t}{t + d(x_n, x_m)} > 1 - r = 1 - \frac{\epsilon}{t + \epsilon} = \frac{t}{t + \epsilon}
\]

\[
d(x_n, x_m) < \epsilon
\]

for every \(n, m \geq n_0\) which implies \((x_n)\) is Cauchy in \((X, d, k)\).

\[
\square
\]

**Corollary 1.** An sb-metric space \((X, d, k)\) is complete if and only if the standard fuzzy sb-metric space \((X, M_d)\) is complete.

3. **Metrizability of Fuzzy sb-Metric Spaces**

Gregori et al. in [32] proved that every fuzzy metric space is metrizable. In the proof they use Kelly metrization lemma.

**Lemma 2 ([33]).** A \(T_1\) topological space \((X, \tau)\) is metrizable if and only if it admits a compatible uniformity with a countable base.

Applying the same lemma we prove here that fuzzy sb-metric spaces are metrizable as well.

**Theorem 3.** Fuzzy sb-metric spaces are metrizable.

**Proof.** Let \((X, M, \epsilon, k)\) be a fuzzy sb-metric space and \(\tau_M\) be the induced topology by \(M\). Firstly, we will show that

\[
B = \left\{ U_n : U_n = \left\{ (x, y) \in X \times X : M\left( x, y, \frac{1}{n} \right) > 1 - \frac{1}{n} \right\}, n \in \mathbb{N} \right\}
\]

is a base for a uniformity \(\mathcal{U}\) on \(X\). Then we will show that \(\tau_M\) and the topology induced by \(\mathcal{U}\) are the same. To show that \(B\) is a base for a uniformity \(\mathcal{U}\) on \(X\) we have to verify the following conditions:

(i) \(\Delta \subseteq U_n\).
(ii) \(U_n = U_{n+1}\).
(iii) \(U_{n+1} \subseteq U_n\).

Conditions (i) and (ii) are obvious for each \(n \in \mathbb{N}\). To show condition (iii) we are reasoning as follows. Let \((x, y) \in U_{n+1}\). Since \(n < n + 1\), we have

\[
1 - \frac{1}{n} < 1 - \frac{1}{n + 1} < M(x, y, \frac{1}{n + 1}) < M(x, y, \frac{1}{n})
\]
which implies \((x, y) \in U_n\).

Since \(\ast\) is continuous, referring to Remark 3 we can find sufficiently large \(m \in \mathbb{N}\) such that
\[
1 - (1/n) < (1 - (1/m)) \ast (1 - (1/m)).
\]

Let \((x, y) \in U_m\) and \((y, z) \in U_m\), \(1 + k/m < 1/n\) implies \(M(x, z, 1 + k/m) \leq M(x, z, 1/n)\) and we have
\[
M(x, z, 1/n) \geq M(x, z, 1 + k/m) \geq M(x, y, 1/m) \ast M(y, z, 1/m)
\]
and this means that \((x, z) \in U_n\) and \(U_m \circ U_m \subset U_n\). Hence \(\{U_n : n \in \mathbb{N}\}\) is a base for a uniformity \(U\) on \(X\). Moreover, since we have
\[
U_n(x) = \left\{ y \in X : M\left(x, y, \frac{1}{n}\right) > 1 - \frac{1}{n}\right\} = \left\{ B\left(x, \frac{1}{n}, \frac{1}{n}\right) : n \in \mathbb{N}\right\}
\]
for each \(n \in \mathbb{N}\) and \(x \in X\), the topology induced by \(U\) is the same as \(\tau_M\). By Lemma 2, \((X, \tau_M)\) is a metrizable. \(\Box\)

The topology \(\tau_d\) induced by an \(sb\)-metric \(d\) is same as the topology induced by the corresponding standard \(sb\)-metric \(M_d\). Hence, we have the following by the above theorem.

**Corollary 2.** Every \(sb\)-metric space is metrizable.

**Corollary 3.** A topological space is metrizable if and only if it admits a compatible fuzzy \(sb\)-metric.

**Theorem 4.** If \((X, M, \ast, k)\) is a complete fuzzy \(sb\)-metric space, then \((X, \tau_M)\) is completely metrizable.

**Proof.** By Theorem 3, \((X, M, \ast, k)\) is metrizable and there is a metric space \((X, d)\) whose induced topology and \(\tau_M\) are same. Moreover, the uniformity induced by \(d\) coincides with the uniformity \(U\); in its turn the topology induced by \(U\) coincides with the topology \(\tau_M\). Recalling that
\[
U_n = \left\{ (x, y) \in X \times X : M\left(x, y, \frac{1}{n}\right) > 1 - 1/n \right\}
\]
is a base for \(U\), we consider a Cauchy sequence \((x_n)\) in \((X, d)\). Then \((x_n)\) is a Cauchy sequence in \((X, U)\). For any \(r \in (0, 1)\) and \(t > 0\), we can choose \(s \in \mathbb{N}\) such that \(1/s \leq \min\{t, r\}\) and there is \(n_0 \in \mathbb{N}\) such that \((x_n, x_m) \in U_s\) for every \(n, m \geq n_0\). Therefore, for every \(n, m \geq n_0\),
\[
M(x_n, x_m, t) \geq M(x_n, x_m, 1/s) > 1 - \frac{1}{s} \geq 1 - r
\]
and this means that \((x_n)\) is a Cauchy sequence in \((X, M, \ast, k)\). Since \((X, M, \ast, k)\) is complete, it is convergent with respect to \(\tau_M\). Hence, \((X, d)\) is complete and \((X, \tau_M)\) is completely metrizable. \(\Box\)

For a precise characterization of completely metrizable spaces in terms of complete fuzzy \(sb\)-metrics, we need the following statement:

**Proposition 4.** A topological space is completely metrizable if and only if it admits a compatible complete fuzzy metric \([32]\).
Corollary 4. A topological space is completely metrizable if and only if it admits a compatible complete fuzzy sb-metric.

Proof. For a completely metrizable space \((X, \tau)\), by Proposition 4, it admits a compatible complete fuzzy metric. Since every fuzzy metric is a fuzzy sb-metric, \((X, \tau)\) admits a compatible complete fuzzy sb-metric. The converse follows from Theorem 4. \(\square\)

4. Diameter Zero Sets and Completeness in Fuzzy sb-Metric

Completeness is an important property of metric spaces. For instance, it is needed to establish Baire property in metric spaces; this is crucial in the investigation of the existence and uniqueness of fixed points for mappings of metric spaces and in many other important both from theoretical and practical points of view fields of research. In a similar way, completeness of fuzzy metric spaces showed to be crucial in the study of the analogous problems in fuzzy context. In particular, fuzzy versions of Baire theorem were given for fuzzy metrics in [13] and for fuzzy sb-metrics in [12]. In addition, a restricted version of Baire theorem for fuzzy \(b\)-metric spaces was proved in [16].

By patterning the results in [34], in this section we define and investigate diameter zero sets in fuzzy sb-metric spaces and characterize their completeness in terms of diameter zero sets.

Definition 5. Let \((X, M, *, k)\) be a fuzzy sb-metric space and \(\{A_n\}_{n \in \mathbb{I}}\) a collection of subsets of \(X\). \(\{A_n\}_{n \in \mathbb{I}}\) is said to have fuzzy sb-diameter zero if for each \(r \in (0, 1)\) and \(t > 0\) there exists \(n \in \mathbb{I}\) such that \(M(x, y, t) > 1 - r\) for all \(x, y \in A_n\).

Remark 4. One can easily notice that \(A \subseteq X\) is a nonempty subset in a fuzzy sb-metric space \((X, M, *, k)\) then \(A\) has fuzzy sb-diameter zero if and only if \(A\) is a singleton set.

Theorem 5. Let \((X, M, *, k)\) be a fuzzy sb-metric space. \((X, M, *, k)\) is complete if and only if every nested sequence of nonempty closed sets \(\{A_n\}_{n=1}^\infty\) with fuzzy sb-diameter zero has nonempty intersection.

Proof. Let \((X, M, *, k)\) be a complete fuzzy sb-metric space and \(\{A_n\}_{n=1}^\infty\) be a nested sequence of nonempty closed sets with fuzzy sb-diameter zero. We need to prove that \(\bigcap_{n=1}^\infty A_n\) is nonempty. For each \(n \in \mathbb{N}\), choose \(x_n\) in \(A_n\). Then for \(t > 0\) and \(r \in (0, 1)\), there is \(n_0 \in \mathbb{N}\) such that \(M(x_n, x_{n+1}, t) > 1 - r\) for all \(x, y \in A_{n_0}\). Hence, for all \(n, m \geq n_0\), we have \(M(x_n, x_m, t) > 1 - r\). Notice that \(x_n \in A_{n_0} \cap A^n_{n_0}\) and \(x_m \in \bigcap_{n=n_0}^\infty A_n\). Hence \(\{x_n\}\) is Cauchy and \(x_n \to x\) for some \(x \in X\). For each \(n, x_k \in A_n\) for all \(k \geq n\). Hence \(x_k \to x\) and \(x \in \overline{A}_n\) where \(\overline{A}_n\) is the closure of \(A_n\). Since \(A_n\) is closed \(\forall n\) means that \(x \in \bigcap_{n=1}^\infty A_n\).

Conversely, let every nested sequence of nonempty closed sets \(\{A_n\}_{n=1}^\infty\) with fuzzy sb-diameter zero have nonempty intersection. We shall show that \((X, M, *, k)\) is complete. Let a Cauchy sequence \(\{x_n\}\) in \((X, M, *, k)\) be given. We define \(B_0 = \{x_n, x_{n+1}, x_{n+2}, \ldots\}\) and \(A_n = \overline{B}_n\). Clearly, \(\{A_n\}_{n=1}^\infty\) is nested, nonempty and all sets \(A_n\) are closed. Now we shall show that \(\{A_n\}_{n=1}^\infty\) has fuzzy sb-diameter zero. Referring to Remark 3, for a given \(s \in (0, 1)\), we can find \(r \in (0, 1)\), such that

\[(1 - r) * (1 - r) * (1 - r) > (1 - s)\]

For \(r \in (0, 1)\) and \(t > 0\), there is \(n_0 \in \mathbb{N}\) such that for every \(m, n \geq n_0\), \(M(x_n, x_m, t/3k) > 1 - r\). Hence, for every \(x, y \in B_{n_0}\), \(M(x, y, t/3k) > 1 - r\). Let \(x, y \in A_{n_0}\). Since \(A_{n_0}\) is closed, there are sequences \(\{a'_n\}\) and \(\{b'_n\}\) in \(B_{n_0}\) such that \(a'_n\) converges to \(x\) and \(b'_n\) converges to \(y\). Therefore, \(a'_n \in B(x, r, t/3)\) and \(b'_n \in B(y, r, t/3k^2)\) for sufficiently large \(n\). Then we have
\[
M(x, a_n', t/3) \ast M(b_n', t/3k) \ast M(y, t/3k^2) \leq M(x, y, t)
\]
\[
(l - r) \ast (1 - r) \ast (1 - r) < 1 - s <
\]
and this means that \( \{ A_n \} \) has fuzzy sb-diameter zero. Hence by hypothesis \( \bigcap_{n=1}^{\infty} A_n \neq \emptyset \). Choose \( x \in \bigcap_{n=1}^{\infty} A_n \). For \( t > 0 \) and \( r \in (0, 1) \) there is \( n_1 \) such that for all \( n \geq n_1 \), \( M(x_n, x, t) > 1 - r \). This implies that for each \( t > 0 \), \( M(x_n, x, t) \to 1 \) as \( n \to \infty \). Therefore, \( x_n \to x \) and \( (X, M, *, k) \) is complete. 

**Remark 5.** It is clear that in the above theorem the element \( x \in \bigcap_{n=1}^{\infty} A_n \) is unique.

Now, by Theorem 5, we indirectly characterize the completeness of an sb-metric space in terms of diameter zero sets.

**Definition 6.** Let \( (X, d, k) \) be an sb-metric space and \( A \subseteq X \). Diameter of \( A \) is defined as \( \delta(A) = \sup \{ d(x, y) : x, y \in A \} \).

**Lemma 3.** Let \( (X, d, k) \) be an sb-metric space and \( (X, M_{d, *}, k) \) be the standard fuzzy sb-metric space. If \( \{ A_n \}_{n=1}^{\infty} \) is a nested sequence of nonempty closed sets with diameter tending to zero in \( (X, d, k) \), then \( \{ A_n \}_{n \in \mathbb{N}} \) has fuzzy sb-diameter zero in \( (X, M_{d, *}, k) \).

**Proof.** \( \delta(A_n) \to 0 \) implies that, for each \( 0 < r < 1 \) and \( t > 0 \) there exists \( n \in \mathbb{N} \) such that \( \delta(A_n) < \frac{tr}{1 - r} \).

Then we have \( d(x, y) < \frac{tr}{1 - r} \) which implies that \( M_d(x, y, t) > 1 - r \) for all \( x, y \in A_n \). Therefore, \( \{ A_n \} \) has fuzzy diameter zero. 

**Corollary 5.** An sb-metric space \( (X, d, k) \) is complete if and only if every nested sequence of nonempty closed sets \( \{ A_n \}_{n=1}^{\infty} \) with diameter tending to zero has nonempty intersection.

**Proof.** Assume that \( (X, d, k) \) is complete. Then \( (X, M_{d, *}, k) \) is also complete. By Lemma 3, every nested sequence of nonempty closed sets \( \{ A_n \} \) in \( (X, d, k) \) with diameter tending to zero has fuzzy sb-diameter zero in \( (X, M_{d, *}, k) \). Hence by Theorem 5, \( \bigcap_{n=1}^{\infty} A_n \) is nonempty.

On the other hand, assume that every nested sequence of nonempty closed sets \( \{ A_n \}_{n=1}^{\infty} \) in \( (X, d, k) \) with diameter tending to zero has nonempty intersection. By Lemma 3, \( \{ A_n \} \) has fuzzy sb-diameter zero and has a nonempty intersection in \( (X, M_{d, *}, k) \). Therefore, by Theorem 5 \( (X, M_{d, *}, k) \) is complete and by Corollary 1 \( (X, d, k) \) is also complete. 

5. **Category of Fuzzy sb-Metric Spaces.**

In this section we make a brief glance on fuzzy sb-metric spaces from the categorical point of view. First of all we must define the morphisms for this category, naturally called continuous mappings. Let * be a fixed t-norm.

**Definition 7.** Let \( (X, M_X, *, k_1) \) and \( (Y, N_Y, *, k_2) \) be fuzzy sb-metric spaces. A mapping \( f : (X, M_X) \to (Y, M_Y) \) is called continuous if it is continuous as a mapping \( f : (X, \tau_{M_X}) \to (Y, \tau_{M_Y}) \).

The following characterizations of a continuous function between fuzzy sb-metric spaces are obvious since open balls form a base for the corresponding first countable topologies.

**Theorem 6.** Let \( (X, M_X, *, k_1) \) and \( (Y, N_Y, *, k_2) \) be fuzzy sb-metric spaces. For a mapping \( f : (X, M_X) \to (Y, M_Y) \), the following are equivalent.

1. \( f : (X, M_X) \to (Y, M_Y) \) is continuous;
2. For every \( a \in X, r_1 \in (0, 1) \) and \( t_1 > 0 \), there exits \( r_2 \in (0, 1) \) and \( t_2 > 0 \) such that \( M_Y(f(a), f(x), t_1) > 1 - r_1 \) whenever \( M_X(a, x, t_2) > 1 - r_2 \).

3. If \( (x_n) \) is a sequence converging to a point \( x \) in \( (X, M_X, *, k_1) \), then the sequence \( (f(x_n)) \) converges to \( f(x) \) in \( (Y, M_Y, *, k_2) \).

Now, we define category of fuzzy sb-metric spaces.

**Definition 8.** The objects of the category \(*\text{-Fsb-Metr}\) of fuzzy sb-metric spaces are four-tuples \((X, M, *, k)\) where \( k \geq 1 \). The morphisms of the category \(*\text{-Fsb-Metr}\) are continuous mappings \( f : (X, M_X) \to (Y, M_Y) \).

By \(*\text{-Fsbk-Metr}\), we denote the full subcategory of the category \(*\text{-Fsb-Metr}\) whose objects are fuzzy sb-meric spaces \((X, M, *, k)\) where \( k \geq 1 \) is a fixed constant.

In the following, we investigate the products and coproducts of fuzzy sb-metric spaces. For the products, we distinguish finite and countable cases.

Let \( \{ (X_i, M_i, *, k_i) : i = 1, \ldots, n \} \) be a family of fuzzy sb-metric spaces and \( k = \max\{k_1, \ldots, k_n\} \).

We define \( X = \prod_{i=1}^n X_i \) and \( M : X \times X \times (0, \infty) \to [0, 1] \) by

\[
M(x, y, t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t) * \cdots * M_n(x_n, y_n, t)
\]

where \( t > 0 \) and \( x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in X \).

**Theorem 7.** \((X, M, *, k)\) is the product of the family \( \{ (X_i, M_i, *, k_i) : i = 1, \ldots, n \} \) in the category \(*\text{-Fsb-Metr}\). Moreover, the topology \( \tau_M \) induced by \( M \) coincides with the product of the topologies \( \tau_{M_i} \) induced by \( M_i \).

**Proof.** Showing that \( M \) is a fuzzy sb-metric on \( X \) can be done repeating the proof of Proposition 2.2 in [12]. Let \((Y, N, *, k')\) be a fuzzy sb-metric space and \( f_i : (Y, N) \to (X_i, M_i) \) be a continuous function for every \( i = 1, \ldots, n \). It is clear that, \( f : (Y, N) \to (X, M) \) defined by \( f(y) = (f_1(y), \ldots, f_n(y)) \) is a continuous function such that \( f_i = p_i \circ f \) where \( p_i : (X, M) \to (X_i, M_i) \) is the projection. Therefore, \((X, M, *, k)\) is the product of the family \( \{ (X_i, M_i, *, k_i) : i = 1, \ldots, n \} \) in the category \(*\text{-Fsb-Metr}\). Now, we show that projections are continuous. Let \( a \in X \) and \( B_M(a, r, t) \) be given for some \( r \in (0, 1) \) and \( t > 0 \). Consider the corresponding ball \( B_M(a, r, t) \) in the product space. Since \( * \) is monotone and 1 is its neutral element, for any \( y \in B_M(a, r, t) \), we have

\[
M(a, y, t) > 1 - r
\]

and \( y \in B_M(a, r, t) \) and this means that \( p_i \) is continuous. Therefore, the topology \( \tau_M \) induced by \( M \) is finer than the product of the topologies \( \tau_{M_i} \) induced by \( M_i \). On the other hand, let \( U \subset X \) be open in \( \tau_M \) and \( a \in U \). Then there exists \( r \in (0, 1) \) and \( t > 0 \) such that \( B_M(a, r, t) \subset U \). We fix \( s_1, s_2, \ldots, s_n \in (0, 1) \) such that \( s_1 > s' > 1 - r \). Then there exists \( s_2 \in (0, 1) \) such that \( s_1 * s_2 > s' > 1 - r \). Continuing in this way we can find \( s_1, s_2, \ldots, s_n \in (0, 1) \) such that \( s_1 * s_2 * \cdots * s_n > s' > 1 - r \). Now consider the ball \( B_M(a, r, t) \), where \( s_i = 1 - s_i \) for each \( i \). Then for \( x \in \bigcap_{i=1}^n P_i^{-1}(B_M(a_i, r_i, t)) \) we have

\[
M(a, x, t) = M_1(a_1, x_1, t) * \cdots * M_n(a_n, x_n, t)
\]

\[
> (1 - r_1) * \cdots * (1 - r_n)
\]

\[
> s_1 * \cdots * s_n
\]

\[
> 1 - r.
\]

This means that
\[ a \in \bigcap_{i=1}^{n} p_i^{-1}(B_{M_i}(a_i, r_i, t)) \subset B_M(a, r, t) \subset U \]

and therefore \( U \) is an open set in the product topology. \( \square \)

Coming to the problem of countable products for fuzzy \( sb \)-metric spaces, we restrict to the case when the constant \( k \) is fixed. Besides, in order to provide correctness of the definition of the fuzzy \( sb \)-metric space, we restrict to the cases when \( * = \cdot \) or \( * = \wedge \).

Let \( \{(X_i, M_i, \cdot, k) : i \in \mathbb{N}\} \) be a countable family of fuzzy \( sb \)-metric spaces and \( \{(X_i, N_i, \cdot, k) : i \in \mathbb{N}\} \) be the family of corresponding \( F \)-bounded fuzzy \( sb \)-metric spaces where \( N_i(x, y, t) = \max \{M_i(x, y, t), 1 - \frac{1}{2^i}\} \) (see Proposition 2). We define \( X = \prod_{i \in \mathbb{N}} X_i \) and \( M : X \times X \to (0, \infty) \) by

\[ M(x, y, t) = \prod_{i \in \mathbb{N}} N_i(x_i, y_i, t) \]

where \( t > 0, x, y \in X \) and \( x_i, y_i \) are \( i \)-th coordinates of \( x \) and \( y \) respectively.

**Theorem 8.** \((X, M, \cdot, k)\) is the product of the family \( \{(X_i, N_i, \cdot, k) : i \in \mathbb{N}\} \) in the category \( \text{--Fsbk-Metr} \). Moreover, the topology \( \tau_M \) induced by \( M \) coincides with the product of the topologies \( \tau_{N_i} \) induced by \( N_i \).

**Proof.** To make the definition of \( M \) meaningful, first we need to show that the infinite product \( \prod_{i \in \mathbb{N}} N_i(x_i, y_i, t) \) is convergent. Since \( 1 - \frac{1}{2^i} \leq N_i(x_i, y_i, t) \leq 1 \) the sequence of partial products \( \prod_{i=1}^{n} N_i(x_i, y_i, t) \) is decreasing and bounded. Therefore, it converges to the limit \( \prod_{i \in \mathbb{N}} N_i(x_i, y_i, t) \) and hence the product metric is defined correctly.

The validity of axioms \((fm_1)(fm_2),(fm_3),(fm_4)\) is obvious. We prove \((fsbm_5)\) for \( M \) as follows. Let \( x, y, z \in X \), then

\[ M(x, z, t + k \cdot s) = \prod_{i \in \mathbb{N}} N_i(x_i, z_i, t + k \cdot s) \geq \prod_{i \in \mathbb{N}} (N_i(x_i, y_i, t) \cdot N_i(y_i, z_i, s)) \]

\[ \geq \left( \prod_{i \in \mathbb{N}} (N_i(x_i, y_i, t)) \right) \cdot \left( \prod_{i \in \mathbb{N}} (N_i(y_i, z_i, s)) \right) \]

\[ \geq M(x, y, t) \cdot M(y, z, s). \]

Let \((Y, N, \cdot, k)\) be a fuzzy \( sb \)-metric space and \( f_i : (Y, N) \to (X_i, N_i) \) be a continuous function for every \( i \in \mathbb{N} \). It is clear that \( f : (Y, N) \to (X, M) \) defined by \( f(y) = (f_i(y) \in X_i)_{i \in \mathbb{N}} \) is a continuous function such that \( f_i = p_i \circ f \) where \( p_i : (X, M) \to (X_i, N_i) \) is the projection. Therefore, \((X, M, \cdot, k)\) is the product of the family \( \{(X_i, N_i, \cdot, k) : i \in \mathbb{N}\} \) in the category \( \text{--Fsbk-Metr} \). Now, we show that projections are continuous. Let \( a \in X \) and \( B_{N_i}(a_i, r, t) \) be given for \( r \in (0, 1) \) and \( t > 0 \). Consider \( B_{M}(a, r, t) \). Since the product \( \cdot \) is monotone and 1 is its neutral element, for any \( y \in B_{M}(a, r, t) \), we have

\[ M(a, y, t) > 1 - r \Rightarrow \prod_{i \in \mathbb{N}} N_i(a_i, y_i, t) > 1 - r \Rightarrow N_i(a_i, y_i, t) > 1 - r \]

and \( y_i \in B_{N_i}(a_i, r, t) \) and this means that \( p_i \) is continuous. Therefore, the topology \( \tau_M \) induced by \( M \) is finer than the product of the topologies \( \tau_{N_i} \) induced by \( N_i \). On the other hand, let \( U \subset X \) be open in \( \tau_M \) and \( a \in U \). Then there exist \( r \in (0, 1) \) and \( t > 0 \) such that \( B_{M}(a, r, t) \subset U \). We can find \( n \in \mathbb{N} \) such that \( \prod_{i=n+1}^{\infty} \left( 1 - \frac{1}{2^i} \right) = s' > 1 - r \) since \( \lim_{n \to \infty} \prod_{i=n+1}^{\infty} \left( 1 - \frac{1}{2^i} \right) = 1 \). We fix \( s'' \in (0, 1) \) such that \( s'' > s'' > 1 - r \). In addition, there exist \( s_1, s_2, \ldots, s_n \in (0, 1) \) such that \( s_1 \cdot s_2 \cdots s_n \cdot s'' > 1 - r \). For all \( i \leq n \), consider the balls \( B_{N_i}(a_i, r_i, t) \) where \( r_i = 1 - s_i \). Then for \( x \in \bigcap_{i=1}^{n} p_i^{-1}(B_{N_i}(a_i, r_i, t)) \) we have
where

\[ a \in \bigcap_{i=1}^{n} p_i^{-1}(B_{N_i}(a_i, r_i, t)) \subset B_M(a, r, t) \subset U \]

and therefore $U$ is an open set in the product topology. \(\square\)

**Corollary 6.** $(X, M, \tau, k)$ is the product of the family $\{(X_i, M_i, \tau, k) : i \in \mathbb{N}\}$ in the category $\mathbf{Fsbk-Metr}$.

Coming to the minimum $t$-norm case let $\{(X_i, M_i, \wedge, k) : i \in \mathbb{N}\}$ be a countable family of fuzzy $sb$-metric spaces and $\{(X_i, N_i, \wedge, k) : i \in \mathbb{N}\}$ be the family of corresponding $F$-bounded fuzzy $sb$-metric spaces where $N_i(x_i, y_i, t) = \max \{ M_i(x_i, y_i, t), 1 - \frac{1}{2^i}\}$. We define $X = \prod_{i\in\mathbb{N}} X_i$ and $M : X \times X \times (0, \infty)$ by

\[ M(x, y, t) = \bigwedge_{i\in\mathbb{N}} N_i(x_i, y_i, t) \]

where $t > 0, x, y \in X$ and $x_i, y_i$ are the $i^{th}$ coordinates of $x$ and $y$ respectively.

**Theorem 9.** $(X, M, \wedge, k)$ is the product of the family $\{(X_i, N_i, \wedge, k) : i \in \mathbb{N}\}$ in the category $\mathbf{Fsbk-Metr}$. Moreover, the topology $\tau_M$ induced by $M$ coincides with the product of the topologies $\tau_{N_i}$ induced by $N_i$.

**Proof.** The validity of axioms $(fm_1)(fm_2), (fm_3), (fm_4)$ is obvious. We prove $(fsbm_5)$ for $M$ as follows. Let $x, y, z \in X$, then

\[ M(x, z, t + k \cdot s) = \bigwedge_{i\in\mathbb{N}} N_i(x_i, z_i, t + k \cdot s) \geq \bigwedge_{i\in\mathbb{N}} (N_i(x_i, y_i, t) \wedge N_i(y_i, z_i, s)) \]

\[ \geq \left( \bigwedge_{i\in\mathbb{N}} (N_i(x_i, y_i, t)) \right) \wedge \left( \bigwedge_{i\in\mathbb{N}} (N_i(y_i, z_i, s)) \right) \]

\[ \geq M(x, y, t) \wedge M(y, z, s) \]

Let $(Y, N, \wedge, k)$ be a fuzzy $sb$-metric space and $f_i : (Y, N) \to (X_i, N_i)$ be a continuous function for every $i \in \mathbb{N}$. It is clear that the function $f : (Y, N) \to (X, M)$ defined by $f(y) = (f_i(y))_{i\in\mathbb{N}} \in X$ is a continuous function such that $f_i = p_i \circ f$ where $p_i : (X, M) \to (X_i, N_i)$ is the projection. Therefore, $(X, M, \wedge, k)$ is the product of the family $\{(X_i, N_i, \wedge, k) : i \in \mathbb{N}\}$ in the category $\mathbf{Fsbk-Metr}$. Now, we show that projections are continuous. Let $a \in X$ and $B_{N_i}(a_i, r, t)$ is given for $r \in (0, 1)$ and $t > 0$. Consider $B_M(a, r, t)$. Since $\wedge$ is monotone and 1 is its neutral element, for any $y \in B_M(a, r, t)$, we have

\[ M(a, y, t) > 1 - r \Rightarrow \bigwedge_{i\in\mathbb{N}} N_i(a_i, y_i, t) > 1 - r \Rightarrow N_i(a_i, y_i, t) > 1 - r \]

and $y_i \in B_{N_i}(a_i, r, t)$ and this means that $p_i$ is continuous. Therefore, the topology $\tau_M$ induced by $M$ is finer than the product of the topologies $\tau_{N_i}$ induced by $N_i$. On the other hand, let $U \subset X$ be open in $\tau_M$ and $a \in U$. Then there exists $r \in (0, 1)$ and $t > 0$ such that $B_M(a, r, t) \subset U$. We can find
\( n \in \mathbb{N} \) such that \( 1 - \frac{1}{2^i} > 1 - r \) whenever \( i > n \). For all \( i \leq n \), consider the ball \( B_{N_i}(a, r, t) \). Then for \( x \in \bigcap_{i=1}^{n} p_i^{-1}(B_{N_i}(a, r, t)) \) we have

\[
M(a, x, t) = \bigwedge_{i=1}^{n} N_i(a, x_i, t) \land \bigwedge_{i=n+1}^{\infty} N_i(a, x_i, t)
\]

\[
> \bigwedge_{i=1}^{n} (1 - r) \land \bigwedge_{i=n+1}^{\infty} (1 - \frac{\lambda}{2^i})
\]

\[
> \bigwedge_{i \in \mathbb{N}} (1 - r)
\]

\[
> 1 - r.
\]

This means that

\[
a \in \bigcap_{i=1}^{n} p_i^{-1}(B_{N_i}(a, r, t)) \subset B_M(a, r, t) \subset U
\]

and therefore \( U \) is an open set in the product topology. \( \square \)

**Corollary 7.** \((X, M, \land, k)\) is the product of the family \( \{(X_i, M_i, \land, k) : i \in \mathbb{N}\} \) in the category \( \land\text{-Fsbk-Metr} \).

Let \( \{(X_i, M_i, *, k) : i \in I\} \) be an arbitrary family of fuzzy sb-metric spaces and \( \{(X_i, N_i, *, k) : i \in I\} \) be the family of corresponding F-bounded fuzzy sb-metric spaces where \( N_i(x, y, t, \lambda) = \max\{M_i(x, y, t) - \lambda\} \) where \( \lambda \in (0, 1) \). We define \( X = \coprod_{i \in I} X_i \) and \( M : X \times X \times (0, \infty) \) by

\[
M(x, y, t) = \begin{cases} N_i(x, y, t), & x, y \in X_i \land \lambda \\ \lambda, & otherwise \end{cases}
\]

where \( x, y \in X \) and \( t > 0 \).

**Theorem 10.** \((X, M, *, k)\) is the coproduct of the family \( \{(X_i, N_i, *, k) : i \in I\} \) in the category \( \ast\text{-Fsbk-Metr} \).

Moreover, the topology \( \tau_M \) on \( X \) induced by \( M \) coincides with the coproduct (direct sum) of the topologies \( \tau_{N_i} \) on \( X_i \) induced by \( N_i \).

**Proof.** The validity of axioms \((fm_1)(fm_2), (fm_3), (fm_4)\) is obvious for \( M \). For \((fsbm_5)\), we distinguish 3 cases.

Case 1: If \( x, y, z \in X_i \), then it is obvious.

Case 2: If \( x, y \in X_i \) and \( z \in X_j, i \neq j \), then

\[
M(x, z, t + k \cdot s) = \lambda \geq N_i(x, y, t) \ast \lambda
\]

\[
\geq M(x, y, t) \ast M(y, z, s).
\]

Case 3: If \( x, z \in X_i \) and \( y \in X_j, i \neq j \), then

\[
M(x, z, t + k \cdot s) = N_i(x, z, t + k \cdot s) \geq \lambda \geq \lambda \ast \lambda
\]

\[
\geq M(x, y, t) \ast M(y, z, s).
\]

Therefore, \( M \) is a fuzzy sb-metric on \( X \). For all \( i \in I \), it is obvious that the inclusion mapping \( q_i : (X_i, N_i) \rightarrow (X, M) \) is continuous. Let \( (Y, M', *, k) \) be a fuzzy sb-metric space and \( f_i : (X_i, N_i) \rightarrow (Y, M') \) be continuous function for all \( i \in I \). By setting \( f(x) = f_i(x) \) if \( x \in X_i \), we obtain a continuous function \( f : (X, M) \rightarrow (Y, M') \) such that \( f \circ q_i = f_i \). Hence, \((X, M, *, k)\) is the coproduct of the family \( \{(X_i, N_i, *, k) : i \in I\} \) in the category \( \ast\text{-Fsbk-Metr} \). Finally, we show the equivalence of the topologies. It is easy to see that \( X_i \in \tau_M \) and \( \tau_M|X_i = \tau_{N_i} \) since \( M|X_i = N_i \) for all \( i \in I \). Therefore, \( X \) can be
represented as the union of a family of pairwise disjoint open subsets. By Proposition 2.2.4 in [33], the topology $\tau_M$ on $X$ induced by $M$ coincides with the coproduct of the topologies $\tau_{N_i}$ on $X_i$ induced by $N_i$. 

**Corollary 8.** $(X, M, *, k)$ is the coproduct of the family $\{(X_i, M_i, *, k) : i \in I\}$ in the category $\ast\text{-Fsbk-Metr}$.

6. Conclusions

The concept of a fuzzy $sb$-metric as a strengthening of the concept of a fuzzy $b$-metric on one side and as a fuzzy version of the notion of a strong $b$-metric was introduced in Reference [11,12]. In this work, we further develop the study of fuzzy $sb$-metrics. There are five main issues considered in the paper. We study continuity of a fuzzy $sb$-metric $M(x, y, -)$ and prove metrizability of the topology induced by a fuzzy $sb$-metric, investigate diameter zero sets in fuzzy $sb$-metric spaces and use them for characterization of completeness of fuzzy $sb$-metric spaces. Some examples showing that the class of fuzzy $sb$-metric spaces lies strictly between the classes of fuzzy $b$-metric spaces and fuzzy metric spaces are provided. In the last section we turn to the constructions of products and coproducts of fuzzy $sb$-metric spaces.

Concerning the further development of the research in the area of fuzzy and crisp $sb$-metrics we have vision of both theoretical aspects to be explored and possible practical applications. In the theoretical direction of the research of fuzzy metric spaces we distinguish the following two. First, to develop further the study of categorical properties of fuzzy $sb$-metric spaces. In this paper we have touched only the problem of products and coproducts in the category of fuzzy $sb$-metric spaces. It is interesting to study further the inner properties of this category, such as the existence of initial and final structures, special objects in this category, etc., as well as to investigate deeper the connections of this category with other related categories, such as fuzzy metric and fuzzy $b$-metric spaces from one side and with the category of crisp $sb$-metric spaces from the other. The second direction which is interesting from the theoretical point of view and can be important also in studying practical problems is the issue of fixed point property for mappings of such spaces. There are many published works developing methods for extending fixed point properties for continuous mappings from the “classical” crisp case to the case of fuzzy metric spaces. On the other hand, similar extension of most of such methods to the case of fuzzy $b$-metric spaces seems problematic because of the peculiarity of the induced topology of such spaces (“open” balls need not be open in the induced fuzzy topology, see comments in front of Remark 5 in our paper). We see some prospects for extending these methods to the case of mappings of fuzzy $sb$-metric spaces that are free from this disadvantage.

Another area where fuzzy and crisp $sb$-metrics could be helpful is image processing. Noticing that the full power of a metric in order to describe distance between the images in the problems of pattern matching is not needed and in some situations may be also onerous, Fagin et al. [4] turned to the use of special $b$-metrics (called nonlinear elastic matching in this paper) for measuring distance between sequences. We foresee, that (fuzzy) $sb$-metrics, being a special kind of (fuzzy) $b$-metrics, can also be useful in improving methods for color image filtering and pattern matching.

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