DEHN FILLINGS CREATING ESSENTIAL SPHERES AND TORI

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Abstract. Let \( M \) be a simple 3-manifold with a toral boundary component. It is known that if two Dehn fillings on \( M \) along the boundary produce a reducible manifold and a toroidal manifold, then the distance between the filling slopes is at most three. This paper gives a remarkably short proof of this result.

Let \( M \) be a simple 3-manifold with a toral boundary component \( \partial_0 M \), that is, it contains no essential sphere, disk, torus or annulus. The slope of an essential, unoriented, simple closed curve on \( \partial_0 M \) is its isotopy class. We assume that \( \alpha \) and \( \beta \) are two slopes on \( \partial_0 M \) such that \( M(\alpha) \) is a reducible manifold and \( M(\beta) \) contains an essential torus. The goal is to measure the upper bound for \( \Delta(\alpha, \beta) \) (the minimal geometric intersection number of \( \alpha \) and \( \beta \)), and Oh [Oh1] and Wu [Wu] independently gave the optimum upper bound 3 for this case. In the present paper we give another short proof of this result.

**Theorem 1.** Let \( M \) be a simple 3-manifold such that \( M(\alpha) \) is reducible and \( M(\beta) \) is toroidal. Then \( \Delta(\alpha, \beta) \leq 3 \).

Assume \( \Delta(\alpha, \beta) \geq 4 \) for contradiction. Let \( \hat{Q} \) be a reducing sphere in \( M(\alpha) \) which intersects the filling solid torus \( V_\alpha \) in a family of meridian disks. We choose \( \hat{Q} \) so that \( q = |\hat{Q} \cap V_\alpha| \) is minimal (over all reducing spheres \( \hat{Q} \) in \( M(\alpha) \)). Similarly let \( \hat{T} \) be an essential torus in \( M(\beta) \) meeting in \( t = |\hat{T} \cap V_\beta| \) meridian disks, the number of which is minimal over all such tori. Let \( Q = \hat{Q} \cap M \) and \( T = \hat{T} \cap M \). By an isotopy of \( Q \), we may assume that \( Q \) and \( T \) intersect transversally, and \( Q \cap T \) has the minimal number of components. Then as described in [Oh1], we obtain graphs \( G_Q \) in \( \hat{Q} \) and \( G_T \) in \( \hat{T} \). We use the definitions and terminology of [Oh1, Section 2].

**Lemma 2.** [BZ, Lemmas 2.3, 4.1, 4.3] \( q \geq 3 \) and \( t \geq 3 \).

**Lemma 3.** [GL2, Theorem 2.4] \( G_T \) cannot contain Scharlemann cycles on distinct label pairs.

**Lemma 4.** [Wu, Lemmas 1.4, 1.8] \( G_Q \) cannot have more than \( \frac{t}{2} + 2 \) mutually parallel edges connecting parallel vertices. Furthermore, if \( G_Q \) has \( \frac{t}{2} + 2 \) mutually parallel edges connecting parallel vertices, then \( t \equiv 0 \) (mod 4).

**Theorem 5.** [JLOT, Theorem 1.1] If \( M(\alpha) \) and \( M(\beta) \) contain a projective plane and an essential torus respectively, then either \( \Delta(\alpha, \beta) \leq 2 \) or \( \Delta(\alpha, \beta) = 3 \) with \( t = 2 \).

Now we will prove Theorem 1.

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Proof. We distinguish two cases;

**Case I)** Suppose that there is a vertex \( x \) of \( G_Q \) such that more than \( \frac{3}{2}t \) edges connect \( x \) to antiparallel vertices.

This implies that there exist more than \( \frac{3}{2}t \) \( x \)-edges connecting parallel vertices in \( G_T \). Consider the subgraph of \( G_T \) consisting of all vertices and those all \( x \)-edges of \( G_T \). An Euler characteristic count gives that it contains more than \( \frac{1}{2}t \) disk faces whose boundaries are \( x \)-edge cycles of \( G_T \). By [HM, Proposition 5.1], each disk face contains a Scharlemann cycle. By Lemma 3, these all Scharlemann cycles are, say, 12-Scharlemann cycles.

Construct a graph \( \Gamma \) in \( \hat{T} \) as follows. Choose a dual vertex in the interior of the face \( G_T \) bounded by a 12-Scharlemann cycle, and let the vertices of \( \Gamma \) be the vertices of \( G_T \) together with these dual vertices. The edges of \( \Gamma \) are defined by joining each dual vertex to the vertices of the corresponding Scharlemann cycle in the obvious way. Let \( s \) be the number of 12-Scharlemann cycles. Then \( s > \frac{1}{2}t \). \( \Gamma \) has \( t + s \) vertices and at least \( 3s \) edges because each Scharlemann cycle has order at least 3 by Theorem 5. Note that if \( G_T \) contains an Scharlemann cycle of length two, then \( M(\alpha) \) contains a projective plane [GL]. Again an Euler characteristic count guarantees that \( \Gamma \) has a disk face \( E \). But \( E \) determines a 1-edge cycle bounding a disk face in \( E \) which, as long as \( q > 2 \), contains a Scharlemann cycle. This contradicts the definition of \( \Gamma \).

**Case II)** As the negation of case I, suppose that each vertex \( x \) of \( G_Q \) has at least \((\Delta - \frac{3}{2})t\) labels where edges connecting parallel vertices are incident.

Let \( G_Q^+ \) be the subgraph of \( G_Q \) consisting of all vertices and edges connecting parallel vertices of \( G_Q \). By the assumption, every vertex of \( G_Q^+ \) has valency at least \((\Delta - \frac{3}{2})t\). Let \( \overline{G}_Q^+ \) be the reduced graph of \( G_Q^+ \). By Lemma 4, any vertex has valency at least 3 in \( \overline{G}_Q^+ \). So, we can choose an innermost component \( H \) of \( \overline{G}_Q^+ \), and a block \( \overline{\Lambda} \) of \( H \) with at most one cut vertex of \( H \). Let \( \Lambda \) be the subgraph of \( G_Q^+ \) corresponding to \( \overline{\Lambda} \).

Note that there is a disk \( D \) in \( \hat{Q} \) such that \( D \cap G_Q^+ = \Lambda \). A vertex of \( \overline{\Lambda} \) is a boundary vertex if there is an arc connecting it to \( \partial D \) whose interior is disjoint from \( \overline{\Lambda} \), and an interior vertex otherwise. Remark that \( \overline{\Lambda} \) has an interior vertex by [WM, Lemma 3.2].

If \( \Lambda \) contains a cut vertex of \( H \), then let \( x \) be a label such that the cut vertex has an edge attached with label \( x \) there. Otherwise, let \( x \) be any label. Then each vertex of \( \Lambda \), except a cut vertex, has at least two edges attached with label \( x \). Thus we can find a great \( x \)-cycle. This implies that \( \Lambda \) has a Scharlemann cycle and so \( t \) is even by [BZ, Lemma 2.2(1)].

Let \( v, e \) and \( f \) be the numbers of vertices, edges and faces of its reduced graph \( \overline{\Lambda} \). Also say \( v_1, v_2 \) and \( v_c \) the numbers of interior vertices, boundary vertices and a cut vertex of \( \overline{\Lambda} \). Hence \( v = v_1 + v_2 \) and \( v_c = 0 \) or 1. Since each face of \( \overline{\Lambda} \) is a disk with at least 3 sides, we have \( 2e \geq 3f + v_2 \). Combined with \( 1 = \chi(\text{disk}) = v - e + f \), we get \( e \leq 3v - v_2 - 3 = 3v_1 + 2v_2 - 3 \).

Suppose that every interior vertex of \( \overline{\Lambda} \) has valency at least 6 and that every boundary vertex except a cut vertex has valency at least 4. Since a cut vertex has valency at least 2, we have \( 2e \geq 6v_1 + 4(v_2 - v_c) + 2v_c \). These two inequalities give us that \( 0 \leq v_c - 3 \), a contradiction.

Hence there are two cases. For the first case, assume that some boundary vertex \( y \) of \( \overline{\Lambda} \) has valency at most 3. There are at least \((\Delta - \frac{3}{2})t\) edges in \( G_Q \) which are incident to \( y \) and connect \( y \) to parallel vertices. By Lemma 4, \( 2 \leq (\Delta - \frac{3}{2})t \leq 3(\frac{t}{2} + 2) \), i.e. \( t \leq 6 \) and \( t \neq 6 \).
But in the case of $t = 4$ there are at least 4 mutually parallel edges in $\Lambda$, which implies that $G_\Lambda$ contains two S-cycles with disjoint label pairs. Then $M(\beta)$ contains a Klein bottle as in the proof of \cite[Lemma 3.10]{GL1}. (Note that $M(\beta)$ is irreducible \cite{GL2}. Hence the edges of an S-cycle cannot lie in a disk on $\hat{T}$.) This contradicts to \cite[Theorem 1.1]{Oh2}.

For the second case, assume that some interior vertex of $\Lambda$ has valency at most 5. Again we have $4t \leq \Delta t \leq 5\left(\frac{t}{2} + 2\right)$. Then we can use the same argument as above. □

References

\cite{BZ} S. Boyer and X. Zhang, Reducing Dehn filling and toroidal Dehn filling, Topology Appl. 68 (1996), 285–303.

\cite{CGLS} M. Culler, C. Gordon, J. Luecke and P. Shalen, Dehn surgery on knots, Ann. Math. 125 (1987), 237–300.

\cite{GL1} C. Gordon and J. Luecke, Dehn surgeries on knots creating essential tori, I, Communications in Analysis and Geometry 3 (1995), 597–644.

\cite{GL2} C. Gordon and J. Luecke, Reducible manifolds and Dehn surgery, Topology 35 (1996), 385–409.

\cite{GLi} C. Gordon and R. Litherland, Incompressible planar surfaces in 3-manifolds, Topology Appl. 18 (1984), 121–144.

\cite{HM} C. Hayashi and K. Motegi, Only single twists on unknots can produce composite knots, Trans. Amer. Math. Soc. 349 (1997), 4465–4479.

\cite{JLOT} G. T. Jin, S. Lee, S. Oh and M. Teragaito, $P^2$-reducing and toroidal Dehn fillings, preprint.

\cite{Oh1} S. Oh, Reducible and toroidal 3-manifolds obtained by Dehn fillings, Topology Appl. 75 (1997), 93–104.

\cite{Oh2} S. Oh, Dehn filling, reducible 3-manifolds, and Klein bottles, Proc. Amer. Math. Soc. 126-1 (1998), 289–296.

\cite{Wu} Y.Q. Wu, Dehn fillings producing reducible manifolds and toroidal manifolds, Topology 37 (1998), 95–108.