ON THE MACKOWIAK-TYMCHATYN THEOREM

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Abstract. In this paper we give new proofs of the theorem of Mackowiak and Tymchatyn that every metric continuum is a weakly-conuent image of some one-dimensional hereditarily indecomposable continuum of countable weight. The rst is a model-theoretic argument; the second is a topological proof inspired by the rst.

1. Introduction

In [5] Mackowiak and Tymchatyn proved that every metric continuum is the continuous image of a one-dimensional hereditarily indecomposable continuum by a weakly conuent map. In [3] this result was extended to general continua, with two proofs, one topological and one model-theoretic. Both proofs made essential use of the metric result.

The original purpose of this paper was to (re)prove the metric case by model-theoretic means. After we found this proof we realized that it could be combined with any standard proof of the completeness theorem of rst-order logic (see e.g., Hodges [4, 6.1]) to produce an inverse-limit proof of the general form of the Mackowiak-Tymchatyn result. We present both proofs. The model-theoretic argument occupies sections 3 and 4, and the inverse-limit approach appears in section 5.

We want to take this opportunity to point out some connections with work of Bankston [1], who dualized the model-theoretic notions of existentially closed structures and existentially m aps to those of co-existentially closed compacta and co-existentially maps. He proves that co-existentially closed continua are one-dimensional and hereditarily indecomposable, and that every continuum is the continuous image of a co-existentially closed one. The map can in general not be chosen co-existential, because co-existentially maps preserve indecomposability and do not raise dimension.

2. Preliminaries

2.1. Mackowiak-Tymchatyn theorem. The theorem of Mackowiak and Tymchatyn, we are dealing with in this paper states that every metric continuum is a weakly conuent image of a one-dimensional hereditarily indecomposable continuum of countable weight.

A continuum is decomposable if it can be written as a union of two proper subcontinua, it is called indecomposable if this is not the case. We call a continuum hereditarily indecomposable if every subcontinuum is indecomposable. This is

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equivalent to saying that every two subcontinua that meet, one is contained in the other. As in we can extend this notion for arbitrary compact Hausdor spaces. So a compact Hausdor space is hereditarily indecomposable if for every two subcontinua that meet, one is contained in the other. We call a continuous mapping between two continua weakly confluent if every subcontinuum in the range is the image of a subcontinuum in the domain.

Theorem 1 (Mackowiak and Tymchatyn [5]). Every metric continuum is a weakly confluent in age of some one-dimensional hereditarily indecomposable continuum of the same weight.

In Hart, van Mill and Pol showed that the Mackowiak and Tymchatyn result above implies the theorem for the non-metric case using model-theoretic means.

2.2. Wallman space. In the proof we will consider the lattice of closed sets of our metric continuum $X$ and try to nd, through model-theoretic means, another lattice in which we can embed our lattice of closed sets of $X$. This new lattice will be a model for some sentences which will make sure that its Wallman representation is a continuum with certain properties. So at the base of the proof is Wallman's generalization, to the class of distributive lattices, of Stone's representation theorem for Boolean algebras. Wallman's representation theorem is as follows.

Theorem 2 ([1]). If $L$ is a distributive lattice, then there is a compact $T_1$ space $X$ with a base for its closed sets that is a homomorphic image of $L$. If $L$ is also disjunctive then we can nd a base for its closed sets that is an isomorphic image of $L$.

We call the space $X$ a Wallman space of $L$ or a Wallman representation of $L$, notation: $WL$.

A lattice $L$ is disjunctive if it models the sentence

(1) $8ab9x[ (a u b = a) ! ( (a u x = x) ^ (b u x = 0))]$;

Furthermore, the space $X$ in theorem 2 is Hausdor if and only if the lattice $L$ is a normal lattice. We call a lattice normal if it models the sentence

(2) $8ab9xy[ (a u b = 0) ! ( (a u x = 0) ^ (b u y = 0) ^ (x t y = 1))]$;

Note that, if we start out with a compact Hausdor space $X$ and look at a base for its closed subsets which is closed under finite unions and intersections, i.e., a (normal, disjunctive and distributive) lattice, then the Wallman space of this lattice is just the space $X$.

Remark 1. From now on we refer to a base for the closed subsets of some topological space which is closed under finite unions and intersections as a lattice base for the closed sets of the space $X$.

The following theorem shows how to create an onto mapping from maps between lattices. In this theorem $2^X$ denotes the family of all closed subsets of the space $X$.

Theorem 3 ([1]). Let $X$ and $Y$ be compact Hausdor spaces and let $C$ be a base for the closed subsets of $Y$ that is closed under finite unions and intersections. Then $Y$ is a continuous image of $X$ if and only if there is a map $C : 2^X$ such that

1. $(C) = X$ and if $F \in C$; then $(F) \in C$
2. if $F \cup G = Y$ then $(F \cup G) = X$.
3. If \( F_1 \subseteq F \); then \( (F_1) \), \( \cap F = \).

So \( Y \) is certainly a continuous image of \( X \) if there is an embedding of some lattice base of the closed sets of \( Y \) into \( 2^X \).

2.3. Translation of properties. Our model-theoretic proof of theorem \( \# \) will be as follows. Given a metric continuum \( X \), we will construct a lattice \( L \) such that some lattice base of \( X \) is embedded into \( L \), the W hall representation \( wL \) of \( L \) is a one-dimensional hereditarily indecomposable continuum and that for every subcontinuum in \( X \) there exists a subcontinuum of \( wL \) that is mapped onto it.

For this we need to translate things like being hereditarily indecomposable, being of dimension less than or equal to one, and being connected in terms of closed nonempty subsets.

To translate hereditarily indecomposable we use the following characterization, due to Krasinkiewicz and Minc.

**Theorem 4 (Krasinkiewicz and Minc).** A compact Hausdorff space is hereditarily indecomposable if and only if it is crooked between every pair of disjoint closed nonempty subsets.

Which the authors translated in \( [4] \) into terms of closed sets only as follows.

**Theorem 5.** A compact Hausdorff space \( X \) is hereditarily indecomposable if and only if whenever four closed sets \( C, D, F \) and \( G \) in \( X \) are given such that \( C \setminus D = C \setminus G = F \setminus D = ; \) one can write \( X \) as the union of three closed sets \( X_0, X_1 \) and \( X_2 \) such that \( C \setminus X_0, D \setminus X_2, X_0 \setminus X_1 \setminus G = ; \) \( X_0 \setminus X_2 = ; \) and \( X_1 \setminus X_2 = F = ; \).

So a compact Hausdorff space is hereditary indecomposable if the lattice \( 2^X \) models the sentence

(3) \[ 8abcd 9xyz[(a \land b = 0)^ \land (a \land c = 0)^ \land (b \land d = 0)]^!
\]

\[ ! (a \land (y \land z) = 0)^ \land (b \land (x \land t \land y) = 0)^ \land (x \land u \land y = 0)^ \land \]
\[ \land (x \land u \land y \land d = 0)^ \land (y \land u \land z \land c = 0)^ \land (x \land t \land y \land t \land z = 1)); \]

A space \( X \) is of dimension less than or equal to one if the lattice \( 2^X \) models the sentence

(4) \[ 8abcd 9xyz[(a \land b \land c = 0)]^!
\]

\[ ! (a \land x = a)^ \land (b \land y = b)^ \land (c \land z = c)^ \land \]
\[ \land (x \land u \land y \land z = 0)^ \land (x \land t \land y \land t \land z = 1)); \]

A space \( X \) is connected if the lattice \( 2^X \) models the sentence \( \text{conn}(1) \), where \( \text{conn}(a) \) is shorthand for the formula \( 8xyz[(x \land u \land y = 0)^ \land (x \land t \land y = a)]^! \) (\( x = a \land (x = 0) \)).

**Remark 2.** For the next two sections, section \( \#5 \) and section \( \#6 \) we use the previous metric continuum \( X \) and we will show there exists a hereditarily indecomposable one-dimensional continuum \( Y \) of weight \( w(X) \) such that \( X \) is weakly connected in age of \( Y \).

3. A continuous image of an hereditarily indecomposable one-dimensional continuum of the same weight

Using theorem \( \#5 \) and \( \#6 \) of the previous section we see that to get a hereditarily indecomposable one-dimensional continuum of weight \( w(X) \) that maps onto \( X \) we
must be a countable distributive, disjointive normal lattice $L$ such that $L$ is a model of the sentences $\forall a;b \exists c \forall d \exists e (a \neq b \lor c = d \lor e = 0)$. If there are constants $k_i$ in $L$, model of the sentences $\exists x;y;z \forall a;b \exists c \forall d \exists e (a \neq b \lor c = d \lor e = 0)$, and further one countable lattice base for the closed sets of $X$ is embedded into this lattice $L$.

For some countable set of constants $K$ we will construct a set of sentences in the language $f(u;t;0;lg \{ K \}$. We will make sure that is a consistent set of sentences such that, if we have a model $A = (A;I)$ for then

$$L(A) = I \setminus K$$

is the universe of some model $A$ in the language $f(u;t;0;lg \{ K \}$. We will make sure that $B$ is a model of the sentences $\exists x;y;z \forall a;b \exists c \forall d \exists e (a \neq b \lor c = d \lor e = 0)$, and make sure that there are constants in $K$ representing the elements of $B$. The interpretations of $u,t,0$ and $1$ are given by the interpretations under $I$ in the model $A$.

Let $K$ be the following countable set of constants

$$K = \{ K_n = \{ f_{k_1,\ldots,k_n} : m < n ! g \} \mid 1 < n ! < n \}$$

We will define the sentences of in an !-recursion. So will be the set $S_{n < ! n}$. For definiteness we define $K_1 = B$ and $0 = 4_B$, the diagram of $B$.

31. Construction of in $f(u;t;0;lg \{ K \}$. Suppose we already defined the sentences up to $5_n$.

1. $5_{n+1}$ will be a set of sentences that will make sure that the supremum and minimum of any pair of constants in $5_n K_m$ are defined, using the new constants from $5_n K_m$.

2. $S_{5n+1}$ will be a set of sentences that will make sure that for every $a;b \in 5_n K_m$ there exists $c$ such that the formula that is sentence without quantifiers will hold for these $a,b,c$.

3. $S_{5n+2}$ will be a set of sentences that will make sure that for every $a;b \in 5_n K_m$ there exist $c,d \in 5_n K_m$ such that the formula that is sentence without quantifiers will hold for these $a,b,c,d$.

4. $S_{5n+3}$ will be a set of sentences that will make sure that the according to the elements of $5_{n+1} K_m$ the dimension of the Wallman space of $L(A)$ for any model $A$ of will be less than or equal to one.

5. $S_{5n+4}$ will be a set of sentences that will make sure that for any $a;b;c \in 5_n K_m$ there exist $x;y;z \in 5_{n+1} K_m$ such that the formula, which is sentence of without quantifiers, holds for this $a,b,c$ and $x;y;z$.

We now show how to define the sentences of $f(u;t;0;lg \{ S_{5n+4} K_m \}$ as described in $S_{5n+5} K_m$.

We have a natural order / on the set $K = \{ K_n \mid m < n \}$ de ned by

$$k_{m,n} / k_{m',n'} \iff (m < n) \land ((m = n) \land (m < t)):
Let \( fp_1^g \) be an enumeration of
\[
\begin{align*}
0_{5n+1} &= f \left( \bigcup \left\{ p_1 \mid k_{5n+1} p_1 : 0 < g \right\} \right) \\
1_{5n+1} &= f \left( p_1 \mid k_{5n+1} p_1 : 0 < g \right) \\
2_{5n+1} &= f(a = a; u a a = a : a 2 \mid K_m g) \mid 5n \\
3_{5n+1} &= f(a b c = (a b) c; a b c = (a b) c : a b c 2 \mid K_m g) \mid 5n \\
4_{5n+1} &= f(a b c = (a b) c; a b c = (a b) c : a b c 2 \mid K_m g) \mid 5n \\
5_{5n+1} &= f(a = a; u a (a b) = a : a b 2 \mid K_m g) \mid 5n \\
6_{5n+1} &= f((a b = 1) \mid (a b = 0)) \mid (a = 0 \mid (a = 1)) : a b 2 \mid K_m g) \mid 5n
\end{align*}
\]

Define \( 5n+1 \) by
\[
5n+1 = \left[ \begin{array}{c}
\vdots \\
1_{5n+1} \\
\vdots
\end{array} \right]
\]

This set of sentences will make sure that any model of \( \mathcal{L} \) in the language \( \{ f u t ; 0 ; 1 g \} \) \( K \) will be a distributive lattice and also a model of the sentence conn. (1).

\[
\begin{align*}
0_{5n+2} &= f((\max p_1 u m) \mid p_1 = 0) \mid (\max p_1 u \mid k_{5n+1} p_1 = 0) \\
\quad & \land (\max p_1 u \mid k_{5n+2} p_1 = 0) \\
\quad & \land (k_{5n+2} p_1 t \mid k_{5n+2} p_1 = 1)) : 1 < ! g
\end{align*}
\]

This set of sentences will make sure that any lattice model of \( \mathcal{L} \) in the language \( \{ f u t ; 0 ; 1 g \} \) \( K \) will be normal.

The following set of sentences makes sure that any model of \( \mathcal{L} \) in the language \( \{ f u t ; 0 ; 1 g \} \) \( K \) which is also a lattice is a disjunctive lattice.

\[
\begin{align*}
0_{5n+3} &= f((\max p_1 u m) \mid p_1 = 0) \mid (\max p_1 u \mid k_{5n+1} p_1 = k_{5n+2} p_1) \\
\quad & \land (\max p_1 u \mid k_{5n+2} p_1 = 0) \\
\quad & \land (k_{5n+2} p_1 t \mid k_{5n+2} p_1 = 1) : 1 < ! g
\end{align*}
\]

And define \( 5n+3 \) by
\[
5n+3 = \left[ \begin{array}{c}
\vdots \\
1_{5n+3} \\
\vdots
\end{array} \right]
\]

Let \( g \) denote the following formula in \( f u t ; 0 ; 1 g \)
\[
(a b c x y z) = (a \mid u b c = 0) \mid (a u x = a) \land (b u y = b) \land (c u z = c) \\
\quad \land (k u y u z = 0) \land (x u t y z = 1))
\]

Let \( fp_1^g \) be an enumeration of the set
\[
\begin{align*}
0_{5n+1} &= f \left( \bigcup \left\{ p_1 \mid k_{5n+1} p_1 : 0 < g \right\} \right) \\
1_{5n+1} &= f \left( p_1 \mid k_{5n+1} p_1 : 0 < g \right) \\
2_{5n+1} &= f(a = a; u a a = a : a 2 \mid K_m g) \mid 5n \\
3_{5n+1} &= f(a b c = (a b) c; a b c = (a b) c : a b c 2 \mid K_m g) \mid 5n \\
4_{5n+1} &= f(a b c = (a b) c; a b c = (a b) c : a b c 2 \mid K_m g) \mid 5n \\
5_{5n+1} &= f(a = a; u a (a b) = a : a b 2 \mid K_m g) \mid 5n \\
6_{5n+1} &= f((a b = 1) \mid (a b = 0)) \mid (a = 0 \mid (a = 1)) : a b 2 \mid K_m g) \mid 5n
\end{align*}
\]
For every $l < ! \text{ write } q_l = f(q_l(0); q_l(1); q_l(2))g.$

Now define $5n + 4$ by

$$5n + 4 = f(q_0(q_0(1); q_0(2)); k_{5n + 4}; l; k_{5n + 4}; l; k_{5n + 4}; l; k_{5n + 4}; l; 2 : l < ! g.$$ 

This will make sure that the Walm an space of any lattice model of will be at most one-dimensional.

For making sure that the Walm and space of any model of will be hereditarily indecomposable we introduce the following formulas in the language $f_u t_0; 0; 1 g$:

$$(a_b_c_d) = ((a_{u b} = 0) ^ (a_u d = 0) ^ (b_u c = 0))$$

$$(a_b_c_d; x; y; z) = ((x_t y_t z = 1) ^ (x_u z = 0) ^ (a_u (y_t z) = 0) ^ (x_u (x_t y) = 0) ^ (x_u y_u d = 0) ^ (y_u z u c = 0))$$

$$(a_b_c_d; x; y; z) = (a_b_c_d) \land (a_b_c_d; x; y; z)$$

Let $f_0 g_{k^n}$ be an enumeration of the set

$$f_r 2^d[K_m]; \text{ran}(X) n \quad m 5n \quad m 5\{n + 1\}$$

Let $5\{n + 1\}$ be the set of sentences defined by:

$$5\{n + 1\} = f(q_0(0); q_0(1); q_0(2); q_0(3)); k_{5\{n + 1\}; 3}; k_{5\{n + 1\}; 3}; k_{5\{n + 1\}; 3}; k_{5\{n + 1\}; 3}; k_{5\{n + 1\}; 3}; k_{5\{n + 1\}; 3}; 2 : l < ! g$$

Here the formula is as in equation $5\varphi^2$. 

32. Consistency of in $f_u t_0; 0; 1 g [K].$ In this section we show that is a consistent set of sentences.

We will nd for $0^2[\varphi]^{<1}$ a metric space $X (\varphi)$ and an interpretation function $I : K ! 2^X (\varphi)$ such that $X (\varphi); I \not\equiv [4\varphi].$ The interpretations of $u, t, 0$ and 1 will always be \', [ (the normal set intersection and union), ; and $X (\varphi)$ respectively.

For $0^1 = ;$ we let $X (\varphi) = X$ and we interpret every constant from $K_1$ as its corresponding base element in $B$. Extend the interpretation function by assigning the empty set to all constants of $K \cap K_1.$ It is obvious that $X (\varphi); I \not\equiv [4\varphi].$

Remark 3. As the interpretation of $u$ and $t$ in the metric continuum $X (\varphi)$ will always be the normal set intersection and set union, all the sentences in $5\{n + 1\}$ for some $n < !$ and $i2 f3; 4; 5; 6g$ are true in the model $2^X (\varphi); I).$ So we can ignore these sentences and for the remainder of this section concentrate on the remaining sentences of $\varphi$.

We can define a well order $\varnothing$ on the set $n f 5\{n + 1\}; n < !$ and $i2 f3; 4; 5; 6g$ by stating that \$ if and only if there are $n < m < !$ such that $2_n$ and $2_m$ or there are $k < 1 < !$ and $n < !$ such that $2_n$ and $1$ is a sentence that mentions $p_k$ ($q_k$ or $r_k$ respectively) and \$ is a sentence that mentions $p_1$ ($q_1$ or $r_1$ respectively).

Suppose $\varnothing$ is a finite subset of $\varnothing$ such that for all of its proper subsets $\varnothing$ there exists a metric continuum $X (\varphi)$ and an interpretation function $I : K ! 2^X (\varphi)$ such that $X (\varphi); I \not\equiv [4\varphi].$
Let be the \( \Theta \)-maximal sentence in \( \inf_{m+1}^m : n < \) and \( 12 f3;4;5;6g \). We will show that there exists a metric space \( X ( \Theta ) \) and an interpretation function \( I : K ! 2^X ( \Theta ) \) such that \( X ( \Theta ; I ) \not\equiv 0 [4_B] \).

Let \( \omega = \inf g \).

3.2.1. \( \forall m < 1 \exists f m+1 \mid 2 m+2 \mid 2 m+3 g \). We can simplify let \( X ( \Theta ) = X ( \Theta ) \) and either (re)interpret the new constant as the intersection or union of two closed sets in \( X ( \Theta ) \) if is in some \( m+1 \) or, if is an element of some \( m+2 \) or \( m+3 \), using the fact that the space \( X ( \Theta ) \) is normal and (re)interpretations for the newly added constants, in an obvious way.

3.2.2. \( \forall f m+4 : m < ! g \). Suppose the preamble of \( \Theta \) is true in the model \( (X ( \Theta ); I ) \), where \( \Theta \) is the following sentence

\[
\Theta = \{(a \cup b \cup c = 0)! \ (a \cup x = a) \cup (a \cup y = b) \cup (c \cup z = c) \cup (x \cup y \cup z = 0) \cup (x \times y \times z = 1)\}
\]

If \( \Theta \) has a zero interpretation then we can choose \( x = 0, y = 1 \) and \( z = 1 \), and this interpretation of \( x, y \) and \( z \) makes sure that \( \Theta \) holds in the model \( (X ( \Theta ); I ) \). So we may assume that \( a, b \) and \( c \) have non-zero interpretations.

As the space \( X ( \Theta ) \) is metric, we can assume that we have a metric on \( X ( \Theta ) \). Moreover, we can assume that \( X \) is bounded by 1.

Consider the following function \( f \) from \( X ( \Theta ) \) to \( R^3 \)

\[
f(x) = (a(x); b(x); c(x));
\]

where \( \Theta : X ( \Theta ) \not\equiv [0;1] \) is defined by

\[
a(x) = \frac{(x; a) + (x; b) + (x; c)}{0};
\]

and \( b \) and \( c \) are like \( a \), but with a 'interchange' with \( b \) and \( c \) respectively. Then \( f \) is a subset of the triangle \( T = f(t_1; t_2; t_3) \) \( R^3 : t_1 + t_2 + t_3 = 1 \) and \( t_1 \leq t_2 \leq t_3 \)

The space \( X ( \Theta ) \) is embedded in the space \( X ( \Theta ) \) \( T \) by the graph of \( f \) (in other words the embedding is defined by \( x \in \Theta (x; f(x)) \)). Let us denote this embedding by \( g \).

Consider the space \( \Theta \setminus [0;1] \), where \( \Theta = T \cap \Theta (T) \) in \( R^3 \). Let \( h \) be the map from \( \Theta \setminus [0;1] \) onto \( T \) defined by

\[
h(x; t) = x \times (1 - t) + t(\frac{1}{3}; \frac{1}{3}; \frac{1}{3});
\]

The map \( h \) restricted to \( \Theta \setminus [0;1] \) is a homeomorphism between \( \Theta \setminus [0;1] \) and \( \Theta \setminus [0;1] \).

We define \( X ( \Theta ) \) as the space

\[
X ( \Theta ) = (\epsilon; h) \uparrow \Theta (X ( \Theta ));
\]

Let us (re)interpret the constants \( k \) in \( K \) in the following way:

\[
I'(k) = I(k) \ (\Theta \setminus [0;1] \cap X ( \Theta )) = (\epsilon; h)^1 \Theta (I(k));
\]
Remark 4. For future reference we note that, as the inverse images of points \((x; t_1, t_2, t_3)\) under the map \(id \times T\) are points for \((x; t_1, t_2, t_3)\) in \(X (\alpha)\) so for which \((t_1, t_2, t_3) \in \binom{2}{2} \binom{2}{2}\) and equal to \(f^T \circ f^I\) for those \((x; t_1, t_2, t_3)\) in \(X (\alpha)\). For \(T\) for which \((t_1, t_2, t_3) = \binom{2}{2} \binom{2}{2}\), we have that the map \(id \times T\) is monotone. Further one it is also closed.

We did nothing to disturb the truth or falsity of the sentences \(\alpha\) in the model \(2^X (\alpha; I)\) as \(f^{-1} A \setminus B = f^{-1} A \setminus f^{-1} B\) and \(f^{-1} A \cap B = f^{-1} A \setminus f^{-1} B\) for any function \(f\) and any sets \(A\) and \(B\).

So we have that \(2^X (\alpha; I)\) is a model for \(\alpha\).

Let \(A\) be the line segments between \((0;1,0)\) and \((0;0,1)\), \(B\) the line segment between \((1;0,0)\) and \((0;0,1)\) and \(C\) the line segment between \((1;0,0)\) and \((0;1,0)\). Now we (re)interpret \(x, y\), and \(z\) as follows

\[
\begin{align*}
I(x) &= X (\alpha) \setminus \{0;1\} \setminus X (\beta) \\
I(y) &= X (\alpha) \setminus \{B;1\} \setminus X (\beta) \\
I(z) &= X (\alpha) \setminus \{C;1\} \setminus X (\beta)
\end{align*}
\]

As is easily seen, this interpretation of the constants \(x, y\), and \(z\) makes the sentence a true sentence in the model \(2^X (\beta; I)\). So \(C^X (\beta; I) \not\models \beta\)

3.23. \(\exists f \in \mathcal{B}(m + 1) : m < ! g\). Suppose the preamble of \(\alpha\) is true in the model \(2^X (\alpha; I)\), where \(\alpha\) is the sentence

\[
\alpha = (a; b;c;d) ! (a; b;c;d;x;y;z);
\]

as in equation \(\beta\).

If the interpretation of \(a\) is zero we can simply take \(x = y = 0\) and \(z = 1\) to make \(2^X (\alpha; I)\) a model of \(\alpha\). Again we may assume that the interpretations of \(a, b, c\), and \(d\) are non-zero.

To show that \(\beta\) is a consistent set of sentences we are going to use an idea from \(\beta\).

With the aid of Urysohn’s lemma we can nd a continuous function \(f : X (\alpha) \rightarrow [0;1]\) such that \(f(I(a)) \circ f_0, f(I(b)) \circ f_1, f(I(c)) \circ f_2, f(I(d)) \circ f_3\) and \(f(I(0)) \circ f_4\).

Let \(P\) denote the (closed and connected) subset of \([0;1]\) \(\cap [0;1]\) given by

\[
P = f_1 [1/4, 2/3] \cap [1/2, 1/4] \cap f_2 [1/3, 2/3] \cap [1/3, 2/3] \cap f_3 [1/3, 4/3] \cap [1/3, 3/4]
\]

Let \(X (\alpha) \rightarrow [0;1]\) denote the pre-image of the set \(P\) under the function \(f\):

\[
X^+ = f(t;x) \cap [0;1] \setminus X (\alpha) ! (t;f(x)) \cap P^g
\]

As \(P\) is closed and \(id \times f\) is continuous we have that \(X^+\) is compact metric space. Define the (continuous) map \(X^+ \rightarrow (\alpha)\) by \(f(t;x) = x\) for every \((t;x) \in X^+\).

Lemma 1. There exists a unique component \(C\) of \(X^+\) such that \(\exists x \in X (\alpha)\).
Proof. Suppose we have closed sets $F$ and $G$ such that $X^+ = F + G$. Define subsets $A_i;B_i$ of $X$, where $i$ is $0;1;2$, by

$$
A_0 = \text{fix } F X (0) : \frac{1}{4};x) 2 FG; B_0 = \text{fix } F X (0) : \frac{1}{4};x) 2 G G \\
A_1 = \text{fix } F X (0) : \frac{1}{2};x) 2 FG; B_1 = \text{fix } F X (0) : \frac{1}{2};x) 2 G G \\
A_2 = \text{fix } F X (0) : \frac{3}{4};x) 2 FG; B_2 = \text{fix } F X (0) : \frac{3}{4};x) 2 G G 
$$

It is clear that $A_i \setminus B_i = ;$ for every $i$ is $0;1;2$.

**Claim 1.** The following holds

1. For every $x \in (A_0 \setminus B_1) \cap (B_0 \setminus A_1)$ we have $f(x) < \frac{2}{3}$.
2. For every $x \in (A_1 \setminus B_2) \cap (B_1 \setminus A_2)$ we have $f(x) > \frac{1}{3}$.

Proof. As the proofs of the statements are very similar we will only prove the first statement.

If $x \in 2 A_0 \setminus B_1$ or $x \in 2 B_0 \setminus A_1$ then $f(x) = \frac{2}{3}$. As $f(x) = \frac{2}{3}$ is impossible, we are done.

Let us define the following closed sets $A$ and $B$ of $X (0)$ by

$$
A = \{ \text{fix } f^1 (0); \frac{1}{3} ] \setminus A_0; f^1 (\frac{2}{3};1] \setminus A_2; A_0 \setminus A_1 \setminus A_2; \}
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
As maps C onto X (\(\mathcal{C}\)) we have that \(\mathcal{C}(\mathcal{C},\mathcal{I})\) is a model of \(\mathcal{O}\), as the truth or falsity of sentences in \(\mathcal{O}\) are not affected by the new interpretation of the constants.

4. The Mackow–Tymchatyn theorem

Apart from the weakly \(\mathcal{C}\) property of the continuous map we have proven the Mackow–Tymchatyn theorem, theorem 2. To make sure that the continuous map following from the previous section is weakly \(\mathcal{C}\), we must consider all the subcontinua of the space X.

We let \(K\) be the following set

\[ K = [K_n : n < !], \]

where \(K_n : n < !\) are some normal distributive and disjunctive lattice such that

1. \(L(\mathcal{C})\) is a model of the sentences \(\exists j \land \text{conn}(1)\),
2. the lattice \(\mathcal{C}(\mathcal{C},\mathcal{I})\) is embedded into \(L(\mathcal{C})\) so there exists a continuous map \(f\) from \(wL(\mathcal{C})\) onto \(X\),
3. for every subcontinuum of \(X\) there exists a subcontinuum of \(wL(\mathcal{C})\) that is mapped onto it by \(f\).

4.1. A weakly \(\mathcal{C}\) map. We let \(K_{1} = [K_{1} : n < !] < \mathcal{C}(\mathcal{C},\mathcal{I})\) correspond to the set \(2^{X} = \{fx : < \mathcal{C}(\mathcal{C},\mathcal{I}) \text{ in such a way that the set } C(X) \text{ of all the subcontinua of } X \text{ corresponds to the set } fx : < \mathcal{C}(\mathcal{C},\mathcal{I}) \text{ for some ordinal numbe}r < \mathcal{C}(\mathcal{C},\mathcal{I})\). Let the set of sentences \(\bar{L}_{0} \in \{f \cup t, 0; 1\} \mathcal{C}(\mathcal{C},\mathcal{I})\) correspond to \(4_{2}\), the diagram of the lattice \(2^{X}\).

We want to define a set of sentences \(\bar{L}_{1} \in \{f \cup t, 0; 1\} \mathcal{C}(\mathcal{C},\mathcal{I})\) that will make sure that if \(A\) is a model of \(\mathcal{C}(\mathcal{C},\mathcal{I})\) in the language \(f \cup t, 0; 1\) \(\mathcal{C}(\mathcal{C},\mathcal{I})\) then we have for every subcontinuum in \(X\) a subcontinuum of \(wL(\mathcal{C})\) that will be mapped onto it by the continuous onto map we get by the fact that \(2^{X}\) is embedded in the lattice \(L(\mathcal{C})\).

\[
\begin{align*}
^0_{1} &= f(\text{conn}(k_{2};)) \bar{L}_{0} (k_{2}, k_{1}; k_{2};) : < g \\
^1_{1} &= f(\text{conn}(k_{2};)) \bar{L}_{0} (k_{2}; k_{1}; k_{2};) \\
^2_{1} &= f(k_{1}; k_{1}; k_{1};) : \bar{L}_{0} (k_{1}; k_{1}; k_{1};) : < i : \bar{L}_{0} (k_{1}; k_{1}; k_{1};) \\
0_{1} &= f(k_{1}; k_{1}; k_{1};) : \bar{L}_{0} (k_{1}; k_{1}; k_{1};) : < i : \bar{L}_{0} (k_{1}; k_{1}; k_{1};)
\end{align*}
\]

And define the set of sentences \(\bar{L}_{1}\) as

\(\bar{L}_{1} = \bar{L}_{0} [\bar{L}_{1} [\bar{L}_{2}]]\).

Suppose \(A\) is a model of \(\mathcal{C}(\mathcal{C},\mathcal{I})\). The set \(\bar{L}_{0}\) will make sure that for every subcontinuum \(C\) of \(X\) there is some subcontinuum \(\bar{C}(\mathcal{C},\mathcal{I})\) of \(wL(\mathcal{C})\) that is mapped into \(C\) by the continuous onto map \(f\) we get from theorem 2, and the fact that \(2^{X}\) is embedded...
into \( wL(A) \). The set \(^1A\) will then make sure that \( C^0 \) is in fact mapped onto \( C \) by the map \( f \).

Let us further construct the sets \(^nA\) for \( 0 < n < ! \). In the same manner as we have constructed the set \( A \) in the previous section. So that if we have a model \( A \), the lattice \( L(A) \) will be a normal distributive and disjunctive lattice that models the sentences \( \neg \neg \) and \( \neg \neg \).

To prove the consistency of \(^i\) it is enough to prove the following lemma.

Lemma 2. For every \( n \in \{ 0, 2 \} \) there is a metric continuum \( X (0) \), and an interpretation function \( I : 2^X \to 2^X \) such that \( Q^X (0) ; I = 0 \).

Proof. Suppose we have a metric continuum \( X (0) \) for every subset \( 0 \) of a given \( 0 \) with \( \{ 0 \} \) such that \( Q^X (0) ; I = 0 \) such that \( Q^X (0) ; I = 0 \). We want to show that there exists a metric continuum \( X (0) \) and an interpretation function \( I : 2^X \to 2^X \) such that \( Q^X (0) ; I = 0 \).

Let be an \( \neg \neg \) in a sentence in \( 0 \) that is of interest (see remark \( 3 \)). If it is an element of \( X (0) \) and \( \neg \neg \) or \( X (0) \) then we can choose \( X (0) \) and redefine the interpretation function \( I \) in a natural way to obtain the wanted result.

So let us suppose that \( \neg \neg \) is an element of \(^i\), \( X (0) \) or \( X (0) \) for some \( n < ! \).

If \( \neg \neg \) is an element of \(^i\), then, as \( \neg \neg \) is the \( \neg \neg \) in a sentence in \( 0 \) of interest, no \( 0 \) is an element of \(^i\) for any \( n < ! \). As \( 2^X \) is a normal distributive and disjunctive lattice and as \( X \) is a continuum, we have that the lattice \( 2^X \) with an obvious interpretation function \( I \) is even a model for \( 2^X \).

Suppose now that \( \neg \neg \) is an element of \(^i\) for some \( n < ! \). If we look at the construction in subsection \( 3.2.2 \), we know that the function \( h \) is a closed monotone map from \( X (0) \) onto \( X (0) \). So in every intervals of connected sets are connected and all the sentences of \(^i\) in \( 0 \) that were true (false) in the model \( Q^X (0) ; I = 0 \), stay true (resp. false) in the model \( Q^X (0) ; I = 0 \).

Finally, suppose that \( \neg \neg \) is an element of \( X (0) \). Let's take a look at the construction of \( X (0) \) in subsection \( 3.2.2 \). Let \( h \) be the map of \( X (0) \) onto \( X (0) \) as given in subsection \( 3.2.2 \). Consider the following lemma.

Claim 2. For every connected subset \( A \) of \( X (0) \) there exists a connected set \( C(A) \) such that \( C(A) = A \).

Proof. Suppose we have a \( X (0) \) connected. If we look at the image of \( A \) under the function \( f \) there are a number of possibilities:

1. \( f[A] \cup \{ \emptyset, A \} \) and \( f[A] \cup \{ \emptyset, A \} \) ; \( f[A] \cup \{ \emptyset, A \} \) and \( f[A] \cup \{ \emptyset, A \} \) ;
2. \( f[A] \cup \{ \emptyset, A \} \)
3. \( f[A] \cup \{ \emptyset, A \} \) ; \( f[A] \cup \{ \emptyset, A \} \) ; \( f[A] \cup \{ \emptyset, A \} \)

In case \( 1 \) we have \( A \) connected. In case \( 2 \) we have \( A \) connected. In case \( 3 \) we can, as above, assume \( A \) connected. In case \( 4 \) we can, as above, assume \( A \) connected.
This ends the proof of the claim.

We have that $G^X(\alpha); I \vDash 0^X \land 1$, we now define a new interpretation function $I$ on $K$ to $2^X(\alpha)$ such that $G^X(\alpha); I$ will be a model for $0$. Note that the set of constants that are mentioned in the set $0$ is a finite subset of $K$, and let $K(\beta)$ denote this finite subset. We will define the interpretation under $I$ of the constants in $K(\beta)$ "from the bottom up".

By the other sentence in $\alpha$, for every $k \mid 2^2$, $K(\beta)$ such that $C = I \vDash (k \mid 2^2)$ is a connected subset of $X(\alpha)$ we can nd a connected subset $C$ of $X(\alpha)$ that maps onto $C$ by the map $I$. Let the interpretation of the constant $k \mid 2^2$ be this connected subset $C$ in $X(\alpha)$.

For all those $k$ in $K(\beta) \setminus K(\alpha)$ that have no connected interpretation in $X(\alpha)$ and for all the constants $k$ in $K(\beta) \setminus K(\alpha)$, the interpretation under $I$ will be the same as the interpretation under $I$. So for those $k \mid 2^2$, we have

$I(k) = I(k \mid 2^2) = 1[I \vDash (k \mid 2^2)] \setminus X(\alpha)$.

The interpretations of the rest of the constants in $K(\beta)$ will follow from the interpretations of the constants we have just de ned, because the interpretation of every constant depends on just a finite set of other constants and we just have to make sure that we define their interpretation in the right order.

As $I(k)$, $I(k)$ for all $k \mid 2^2$ and for all $k \mid 2^2$, such that $I(k)$ is connected $I(k)$ is also connected we have that all the sentences of $\land 1$ true (false) in $G^X(\alpha); I = 1$ are true (false) in $G^X(\alpha); I$. The true or falsehood of the other sentences in $\alpha$ not have not been affected by the new interpretation function $I$, and we have completed the proof.

42. The Mackenson-Łukasiewicz theorem. As we have seen in the previous section the set of sentences $\land$ in the language $\mu$ is consistent. Let $A$ be a model for $\land$. This model gives us a normal distributive and disjunctive lattice $L(A)$ which models the sentences $3 \land 4$ and conn $(1)$. There also exists an embedding of $2^2$ into this lattice $L(A)$ (remember that we showed that with $K_2$ an enumeration of the lattice $2^X$, the set is a consistent set of sentences in the language $\mu$; $1 \land 1$). All this implies that the Wallman space $WL(A)$, is a one-dimensional hereditarily indecomposable continuum which admits a weakly connected surjection onto the metric continuum $X$.

Now we only have to make sure that there exists such a space that is of countable weight to complete the proof of the Mackenson-Łukasiewicz theorem.

Theorem 6. Let $f : Y \to X$ be a continuous surjection between compact Hausdorff spaces. Then $f$ can be factored as $h \circ g$, where $Y \cong Z \cong X$ and $Z$ has the same weight as $X$ and shares many properties with $Y$ (for instance, if $Y$ is one-dimensional so is $X$ or if $Y$ is hereditarily indecomposable, so is $X$).

Proof. Let $B$ be a minimal-sized lattice-base for the closed sets of $X$, and identify it with its copy $f(C)[B] : 2^B$ in $2^X$. By the Lowenheim-Skolem theorem there is an elementary sublattice of $2^X$, of the same cardinality as $B$ such that $B \subseteq 2^B$. The space $\omega D$ is as required.

Applying this theorem to the space $WL(A)$ and the weakly connected map $f : WL(A) \to X$ we get a one-dimensional hereditarily indecomposable continuum $\omega D$.
which admits a weakly continuous map onto the space \( X \) and moreover the weight of the space \( wD \) equals the weight of the space \( X \). This is exactly what we were looking for.

5. A topological proof of the Mackiewicz-Tymchatyn theorem

5.1. The Mackiewicz-Tymchatyn theorem. After the above proof was found we realized that it could be transformed into a purely topological proof, which we shall now describe.

Let \( X \) be a metric continuum. We are going to define a reverse sequence of metric continua with onto bonding maps \( \phi X_n; f_n i : n < ! g, \)

\[
X = X_0 f_1 X_1 f_2 \ldots f_n X_n f_{n+1};
\]

in such a way that the inverse limit space \( X \) is

\[
X_1 = \lim \phi X_n; f_n i : n < ! g,
\]

is a hereditarily indecomposable one-dimensional continuum of weight \( w(X) \) such that \( 0 : X_1 \not\to X \) is a weakly continuous and onto. Here, for every \( n < ! \) the continuous function \( n \) is defined by \( n = \text{proj}_1 X : X_1 \not\to X_n \), where \( \text{proj}_1 : X_1 \not\to X_n \) is the projection.

Let us further define one bonding map \( f_m : X_n \not\to X_m \) for \( m < n \) as

\[
f_m = \begin{cases} 
  f_{m+1} & \text{if } m + 1 < n \\
  f_{m+1} & \text{if } m + 1 = n
\end{cases}
\]

The following lemma is well known.

Lemma 3. The family of all sets of the form \( n^{-1}(F) \), where \( F \) is a closed subset of the space \( X_n \) and \( n \) runs over a subset \( N \) of natural numbers, is a base for the closed sets of the limit of the inverse sequence \( \phi X_n; f_n i : n < ! g \). Moreover, if for every \( n < ! \) and every \( g \) a base \( B_n \) for the closed sets of space \( X_n \) is fixed, then the family of all sets \( n^{-1}(F) \) for which \( F \not= B_n \) also is a base for the closed sets of \( X_1 \).

To make sure that the space \( X_1 \) is one-dimensional, it is sufficient to show that \( f_k^{-1}(F) : F \not= B_k \) and \( k < ! g \) is a model of sentence \( w(X) \).

Let \( \psi \) be an onto map in such a way that for every \( n,m < ! \) we have \( s^{-1}(mn) = m \max n \) and given \( g(n) = h_p g r i e \). We denote \( s(n) \) by

\[
s(n) = \begin{cases} 
  h_p g r i e & \text{if } n \\max n \wedge g r y \\
  h_j, 0 i & \text{otherwise}
\end{cases}
\]

Let \( X_0 = X \) and suppose we have dened the pairs \( hX_m; f_m i \) and the bases \( B_m \) for every \( m < n \). And suppose that we have also dened an enumeration of all the triples of \( B_m \) that have empty intersection for every \( m < n \). Let \( F^k \) be an enumeration of the set \( F \not= B_k \) : \( G = \{ i : g \} \) for every \( m < n \), write \( G_m = \max k \); \( c^n \gamma \). The way we now define the space \( X_n \) and the onto map \( f_n : X_n \not\to X_1 \) will be as follows.

Suppose \( s(n) = h_k m i \), we consider the closed sets \( (f_k^{-1}) (a_k^m), (f_k^{-1}) (b_k^m) \) and \( (f_k^{-1}) (c_k^m) \) of \( X_n \). They have empty intersection.
If there exist sets $x$, $y$ and $z$ in $2^{X_n-1}$ such that
\[(f^m_k)^1(x^k_m) x; (f^m_k)^1(y^k_m) y \text{ and } (f^m_k)^1(z^k_m) z;\]
\[x \setminus y \setminus z = \emptyset \text{ and } x \setminus \{y \mid z = X_n\};\]
then we let $X_n = X_{n-1}$, $f_n = \text{id}_{X_n}$ and we choose a countable base $B_n$ for the closed sets of $X_n$ such that $B_n \subseteq \{f \times y \cup z \mid B_n\}$.

If there do not exist such sets $x$, $y$ and $z$ in $2^{X_n}$, then we use the construction in subsection 3.2.2 to nd a (metric) continuum $X_n$ and a continuous onto map $f_n : X_n \to X_{n-1}$, such that in $X_n$ there are closed sets $x$, $y$ and $z$ in $X_n$ such that
\[(f^m_k)^1(x^k_m) x; (f^m_k)^1(y^k_m) y; (f^m_k)^1(z^k_m) z;\]
and $x \setminus y \setminus z = \emptyset$.

Let $B_n$ be some countable base for the closed sets of $X_n$ such that $f(f^m_k)^1(F) : F \subseteq B_n \to B_n$ and $xy; z \subseteq B_n$.

After we have chosen the base $B_n$ we can choose some enumeration of all the triples of $B_n$ that have empty intersection.

We do not get into trouble by considering base elements of some base for the closed sets of $X_n$ which have not yet been de ned, because this will not happen by the way the function $s$ is de ned and the bases $B_n$ are chosen.

The limit $X_1(s)$ of the inverse sequence $fX_n; f_n i : n < ! g$ is a continuum, as all the spaces $X_n$ are continua, moreover, as the base $f^n_1(F^k_1) : k < !$ of the space $X_1(s)$ models the sentence $\varphi$ we have that $X_1(s)$ is one-dimensional. As all the spaces $X_n$ are compact and all the bonding maps $f_n$ are onto, we have that
\[0 : X_1(s) \to \text{a continuum onto map.}\]

In a similar way we can construct a function $t : ! ! ! ! !$ onto a map in such a way that for all $k; n < !$ we have $t^1(X; t) = \text{max}_{k; n < !} g$, and use $t$ together with the construction in subsection 3.2.2 to de ne, given $X_0 = X_1(s)$, so that $X_1(t)$, the inverse limit of the sequence $fX_n; f_n i : n < ! g$ is a hereditarily indecomposable continuum which admits a continuous onto map $0 : X_1(t)$ onto the space $X_1$.

We can combine these two constructions by dening the function $r$ by letting $r(2n)$ equal $s(n)$ and $r(2n + 1)$ equal $t(n)$ for every $n < !$. Define $X_2 = X_1$ and use the construction in subsection 3.2.2 if $n$ is even and the construction in subsection 3.2.3 if $n$ is odd to construct $X_2$ and $f_2$.

Let $X_1(t)$ be the inverse limit of the inverse sequence $fX_n; f_n i : n < ! g$ we have constructed with the aid of the function $r$ as described in the previous paragraph.

As $B = f^n_1(F^k_1) : n < ! g$ is a base for the closed sets of $X_1(t)$ we see that $w(X_1(t)) = w(X_1) = \emptyset_0$. The space $X_2(t)$ is a one-dimensional hereditarily indecomposable continuum as, by construction $B$ is a model of the sentences $\varphi$ and $\varphi'$. So by the following claim we have proven the Mackowsk-Fym Chatyn theorem.

Claim 3. The map $0 : X_1(t)$ onto $X_1$.

Proof. Suppose we have a subcontinuum $C$ of the space $X_1(t)$, we want to nd a subcontinuum $C^0$ of $X_1(t)$ such that $0(C^0) = C$. As in construction, the $f_2m$'s are monotone closed maps from $X_{2m}$ onto $X_{2m+1}$, and the $f_2m$'s are monotone, we can den an inverse sequence $fY_n; g_n i : n < ! g$ such that $Y_0 = C$ and $Y_n \subseteq C$ for every $n$, some subcontinuum of $f_2m(Y_n)$, such is mapped onto $Y_{n+1}$ by the map $f_1$ and the map $g_n$ is the restriction of the map $f_1$ to the subspace $Y_n$ of $X_n$. Let $C^0$ be the inverse limit of the inverse sequence $fY_n; g_n i : n < ! g$, so
We have that $C^0$ is a closed subspace of the space $X$; if $\xi$ is a continuum as it is an inverse limit of continua, so $\xi$ is a subcontinuum of $X$; if $\eta$. As $\eta$ maps $C^0$ onto the $Y_n$'s we have proven the claim.

5.2. The extended Mackiw-Tymchatyn theorem. Given a continuum $X$ we will construct an inverse sequence $\mathfrak{f}_X$; $\mathfrak{i}: \leq \omega$ such that the inverse limit space $Y$ is a hereditarily indecomposable continuum of weight $w(X)$ and $\dim(Y) = 1$ and there exists a weakly continuous map of the space $X$ onto $X$. This is a somewhat different proof than is given in the paper of Hart, Van Mill and Pol (see [3]).

\[ X = X_0 \mathfrak{f}_1 X_1 \mathfrak{f}_2 X \cdots \mathfrak{f}_n ( < \omega; X) ; \]

We are going to make sure that every $\xi$ is a continuum of weight $w(X)$ and that there exists some base $B$ for the closed sets of $X$ of cardinality $w(X)$ such that the base $f^{-1}(B) : B \rightarrow B; < \omega(w(X))$ for the closed sets of the space $Y$ will show that $Y$ is the desired continuum.

For $< \omega$ a limit ordinal we let $X$ be the inverse limit of the sequence $\mathfrak{f}_X$; $\mathfrak{i}: \leq \omega$, and we let $B$ be the set $\mathfrak{f}( )^{-1}(B) : B \rightarrow B; < \omega$. This is a base for the closed sets of $X$ and $\mathfrak{f}$ is $\omega$ for the closed sets of $X$. Further we $X$ is a continuum as it is an inverse limit of continua.

Suppose we have defined the continua $X$ for $\xi$ for $< \omega(X)$, as well as the bases $B$ for the closed sets of these spaces and for every we also have defined an enumeration $f: \omega(X)$ such that for every $\xi < \omega(X)$ we have $s^{-1}(\xi; i) = \omega(X)$ such that for every $\xi < \omega(X)$ we have $s^{-1}(\xi; i) = \omega(X)$ such that for every $\xi < \omega(X)$ we have $s^{-1}(\xi; i) = \omega(X)$ such that for every $\xi < \omega(X)$ we have $s^{-1}(\xi; i) = \omega(X)$ such that for every $\xi < \omega(X)$ we have $s^{-1}(\xi; i) = \omega(X)$ such that for every $\xi < \omega(X)$. To nd $X_{+1}$ and $f$ we do almost the same thing as we have done in the previous section. If $s() = \xi$ we consider the closed sets $a = f(a), b = f(b)$ and $c = f(c)$ of the space $X$.

If there exist $x, y$ and $z$ in $2^X$ such that $a \leq x, b \leq y, c \leq z, x \setminus y \setminus z = ;$ and $x \{ y \{ z = X$ then we let $X_{+1} = X$ and $f_{+1} = \mathfrak{f}_{+1} X$.

If there are no such $x, y$ and $z$ in $2^X$ then we will do as in subsection 2.2.2 but as in that section we used a metric for $X$ we have to slightly alter the proof there. As $X$ is nonalad a $\mathfrak{f} \mathfrak{b} \mathfrak{c} = ;$ we can nd a continuous function $f_a : X \rightarrow [0; 1]$ such that $f_a(a) = f_{0g}$ and $f_a(b \setminus c) = f_{0g}$. Now, as $f_a^{-1}(f_{0g}) \setminus b \setminus c = ;$ we can nd a continuous function $f_a : X \rightarrow [0; 1]$ such that $f_a(a) = f_{0g}$ and $f_a(f_a^{-1}(f_{0g}) \setminus c)= f_{0g}$. Finally, since $f_a^{-1}(f_{0g}) \setminus f_a^{-1}(f_{0g}) \setminus c = ;$ we can nd a continuous function $f_a : X \rightarrow [0; 1]$ such that $f_a(c) = f_{0g}$ and $f_a(f_a^{-1}(f_{0g}) \setminus f_a^{-1}(f_{0g})) = f_{0g}$. Now we define the function $f : X \rightarrow R$ by

\[ f(\xi) = (a(\xi); b(\xi); c(\xi)); \]

where $a : X \rightarrow [0; 1]$ is defined by

\[ a(\xi) = \frac{f_a(\xi)}{f_a(\xi) + f_b(\xi) + f_c(\xi)}; \]
and \( b \) and \( c \) are likewise defined. The function \( f \) maps \( X \) into the triangle that is the convex hull of the points \( f(0;0;1);(0;1;0);(1;0;0)g \) in \( \mathbb{R}^3 \) just as in subsection 3.2.2 and from this point on we can follow the method in subsection 3.2.2 to nd a continuum \( X^+ \) and a continuous onto map \( f^+ : X^+ \rightarrow X \) such that there exist \( x; y; z \in 2^X \) such that \( f^+(a) = x, f^+(b) = y \) and \( f^+(c) = z \), \( x \setminus y \setminus z = ; \) and \( x \cup \{ x = X \} \). Now let \( B^+ \) be a base for the closed sets of \( X^+ \) such that \( f(f^+)^{-1}(B) : B \rightarrow B \) \( f(x; y; z) \in B^+ \) and \( B \cup \{ x \} \) \( w \) \( X \). Enumerate the set of triples of \( B^+ \) \( n; g \) with empty intersection as \( f(G)^+ : < \) \( +1 \), where \( +1 \) is some ordinal number less than or equal to \( w \) \( X \).

In a similar way we can nd an (trans nite) inverse sequence such that the inverse \( \lim \) is a hereditarily indecomposable continuum of the same weight as \( X \) and for which the map \( \alpha \) is a continuous onto map between the \( \lim \) and the space \( X \).

As in the previous section we can combine these two (we take care of the hereditary indecomposability at even ordinal stages and we take care that the dimension of the \( \lim \) space will not exceed one at the odd ordinal stages), to nd a trans - nite inverse sequence such that the inverse \( \lim \) is a one-dimensional hereditarily indecomposable continuum that admits a continuous map onto the space \( X \). After some thought, as in the previous section we see that this continuous map is in fact a weakly continuous map.

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