Magnetic oscillations of critical current in intrinsic Josephson-junction stacks

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A key phenomenon related to the Josephson effect is oscillations of different properties of superconducting tunneling junctions with magnetic field. We consider magnetic oscillations of the critical current in stacks of intrinsic Josephson junctions, which are realized in mesas fabricated from layered high-temperature superconductors. The oscillation behavior is very different from the case of a single junction. Depending on the stack lateral size, oscillations have either the period of half flux quantum per junction (wide-stack regime) or one flux quantum per junction (narrow-stack regime). We study in detail the crossover between these two regimes. Typical size separating the regimes is proportional to magnetic field meaning that the crossover can be driven by the magnetic field. In the narrow-stack regime the lattice structure experiences periodic series of phase transitions between aligned rectangular configuration and triangular configuration. Triangular configurations in this regime are realized only in narrow regions near magnetic-field values corresponding to integer number of flux quanta per junction.

I. INTRODUCTION

Layered high-temperature superconducting materials, such as Bi$_2$Sr$_2$CaCu$_2$O$_x$ (BSCCO), are composed of superconducting cuprate layers coupled by Josephson interaction. This system possesses the Josephson effects at the atomic scale (“Intrinsic Josephson Effect”). A rich spectrum of classical dc and ac Josephson phenomena have been observed in this system, see reviews [12].

In a bulky superconductor the magnetic field applied along the layers generates a triangular lattice of Josephson vortices. The anisotropy factor γ and the interlayer periodicity s set the important field scale, $B_{cr} = \Phi_0 / (2\pi s^2)$ (∼ 0.5 tesla for BSCCO). When the magnetic field exceeds $B_{cr}$ the Josephson vortices homogeneously fill all layers[3]. Strong coupling between the vortex arrays in neighboring layers mediated by the in-plane supercurrent[13] determines the static and dynamic properties of the lattice. Dynamic properties of the Josephson-vortex lattice in BSCCO have been extensively studied by several experimental groups (see, e.g., Refs. [5][6][7][8]).

When an external transport current flowing across the layers exceeds the critical current, the Josephson vortex lattice starts to move. In a homogeneous junction the critical current is determined by interaction with the boundaries. The simplest and most known case is a single small junction without inhomogeneities, where the field dependence of the critical current is given by the Fraunhofer dependence, $I_c(\Phi) = I_0 |\sin(\pi \Phi / \Phi_0)| / (\pi \Phi / \Phi_0)$, with $\Phi$ being the magnetic flux through the junction. Observation of this dependence has been considered as an important confirmation of the dc Josephson effect[9].

The same dependence is also expected for the junction stack with the lateral size smaller than the Josephson length[10]. In a single long junction the critical current has the rather complicated field dependence due to multiple coexistent states of the lattice[11].

In the previous paper [12] we considered the behavior of the critical current for the dense Josephson-vortex lattice in a homogeneous wide stack for which the critical current is caused by interaction with the boundaries. We found that the boundary induces an alternating deformation of the lattice. Averaging out the rapid phase oscillations, we obtained that the lattice deformation obeys the sine-Gordon equation and decays inside superconductor at the typical length $L_B / \sqrt{2}$, which is larger than the Josephson length, $\lambda_J = \gamma s$, and increases proportional to the magnetic field, $L_B = \lambda_J B / B_{cr}$[13]. The stack is in the wide-stack regime if its lateral width $L$ is larger than this typical length. In this situation the surface deformation and the total current flowing along the surface is uniquely determined by the lattice position far away from the boundaries. The surface current has oscillating dependence on the lattice displacement and, due to the triangular-lattice ground state, the period of this dependence is half the lattice spacing. The total current flowing through the stack is given by the sum of two independent surface currents flowing at the sample edges. The magnetic field determines the magnitude of the maximum surface current (it is inversely proportional to the field) and sets the phase shift between the oscillating dependences of the two surface currents on the lattice position. One can trace that, due to the half-lattice-spacing periodicity of the surface current, a full period change of this phase shift corresponds to the change of the magnetic flux through one junction, $\Phi$, equal to the half flux quantum, $\Phi_0 / 2$. As a consequence, the maximum current through the stack has oscillating field dependence, which resembles the Fraunhofer dependence: it has strong oscillations and overall $1 / B$ dependence. However, the period of these oscillations is two times smaller: it corresponds to adding one flux quantum per two junctions and the critical current has local maxima at $\Phi = k \Phi_0 / 2$.

Oscillations of the flux-flow voltage in BSCCO mesas at slow lattice motion have been observed by Ooi et al[8]. The oscillations have the period of $\Phi_0 / 2$ per junctions and are caused the size-matching effect described in the previous paragraph. These oscillations have been reproduced by numerical simulations[15]. More recently the
flux-flow oscillations in BSCCO mesas have been reproduced and studied in more details by several experimental groups. Similar oscillations also have been observed in underdoped YBa2Cu3O6+x. Size dependence of oscillations has been systematically studied in Refs. 15,17. It was found that at smaller lateral sizes and/or higher magnetic fields the crossover to the new oscillation regime takes place, in which the period becomes $\Phi_0$ per junction, as in a single junction.

Being motivated by recent experiments, in this paper we extend our consideration to the regime when the junction size $L$ is comparable with the length $L_B$ and the system crosses over from the wide-stack to narrow-stack regime. As the length scale $L_B$ increases with the magnetic field, it also sets the field scale $B_L = B_c L/\lambda_J$, at which this length becomes of the order of the junction length $L$. Therefore for a junction of a given size the crossover to the narrow-stack regime can be driven by the magnetic field, as it was observed experimentally. The critical-current oscillations hold until interactions between surface deformations can be neglected. This interaction becomes progressively stronger with decreasing the ratio $L/L_B$. Surface deformations at $L > L_B$ can be described as partial sine-Gordon solitons. The relative sign of two solitons at the opposite edges is determined by the magnetic field and the lattice positions. At the integer-flux-quantum points $\Phi = k\Phi_0$ the surface solitons always have the same sign and repel each other. As a consequence, the amplitude of surface deformations drops and the critical current decreases. At the half-integer-flux-quantum points $\Phi = (k + 1/2)\Phi_0$ situation is opposite: the surface solitons always have opposite signs and attract each other leading to enhancement of the surface deformations and increase of the critical current. Therefore the interaction between the surface solitons leads to the crossover between the $\Phi_0/2$-periodic oscillations of the critical current and $\Phi_0$-periodic oscillations. This crossover occurs via suppression of the current peaks at the points $\Phi = k\Phi_0$ and enhancement of the current peaks at the points $\Phi = (k + 1/2)\Phi_0$. Such behavior is consistent with recent studies of the oscillations of the flux-flow voltage in mesas with small lateral sizes. The crossover in the voltage oscillations also has been studied numerically.

In the region $L \sim L_B$ the lattice structure is determined by competition between two energies: the interaction with boundaries and the bulk shearing interaction between the Josephson-vortex planar arrays in neighboring layers. The interaction with the boundaries favors the aligned rectangular arrangement of the Josephson vortices while the local shearing interaction favors the triangular lattice. The boundary interactions decay slower with increasing field than the shearing interaction and become dominating at large fields. On the other hand, the boundary interaction energy has oscillating field dependence and vanishes at the points $\Phi = k\Phi_0$. At these points the shearing interaction is relevant at any magnetic field. In addition, the interaction with the boundary interactions between the Josephson-vortex planar arrays in neighbor-

\[ \Phi = \Phi_0/2 \]
of alternating from layer to layer solution. Averaging with respect to the rapidly changing phase oscillations, we obtain equation and boundary conditions for the slow lattice deformation. We also express the lattice energy and current flowing through the stack via this deformation. In Section III we obtain and analyze solution for the smooth phase in terms of elliptic integrals. We found that the problem can be reduced to solution of three nonlinear coupled equations for three unknowns, the boundary phases and elliptic parameter. In Section IV we derive a criterion for the transition into the rectangular-lattice state. In Appendix I we consider weak finite-size effects in the wide-stack regime and analytically compute exponentially small finite-size corrections to the critical current, which break the $\Phi_0/2$-periodicity of oscillations. In Section V we present results of numerical analysis of the crossover between the wide-stack and narrow-stack regimes with the increasing magnetic field. We obtain the oscillation patterns of the critical current for stacks with different lateral sizes and find location of the rectangular-lattice regions in the current-field plane. In Section VI we reanalyze in detail the narrow-stack regime using independent analytical approach. In Section VII we consider the voltage oscillations in the case of slowly moving lattice and relate these oscillations with the critical-current oscillations. We elaborate the recipe to extract the anisotropy factor from the voltage oscillations.

II. PHASE DISTRIBUTION AND ENERGY OF FINITE STACK ASSUMING ALTERNATING SOLUTION

We consider a Josephson-junction stack consisting of $N$ layers, $N \gg 1$, with lateral size equal to $L$ in a magnetic field $B > B_c$ applied along the layers. At high magnetic fields one can neglect screening effects. In this case the stack is described by the energy functional (per layer and per unit length in the field direction) of the layer phases $\varphi_n(x)$,

$$E[\varphi_n] = \frac{1}{N} \sum_n \int_0^L dx \left[ \frac{J}{2} \left( \frac{d\varphi_n}{dx} \right)^2 - E_J \cos \left( \varphi_{n+1} - \varphi_n - \frac{2 \pi s B}{\Phi_0} x \right) \right]. \quad (1)$$

where $J$ is the in-plane phase stiffness and $E_J$ is the Josephson energy per unit area. To simplify derivations, we introduce reduced coordinate, $u = x/\lambda_J$, with $\lambda_J = \sqrt{J/E_J}$, reduced magnetic field, $h = 2 \pi s B \lambda_J / \Phi_0$, and reduced energy $\mathcal{E} = E/(E_J \lambda_J)$. We also represent the phase variable in the form, which naturally describes the dense triangular lattice in the bulk in the limits $h \gg 1$ and $L/\lambda_J \gg h$, $\varphi_n(u) = \phi_n(u) + \alpha n + \pi n(n-1)/2$, where the phases $\phi_n(u)$ are assumed to be small and rapidly oscillating and the parameter $\alpha$ will describe lattice displacement. The the reduced energy per one junction and per unit length in the field direction can now be represented as

$$\mathcal{E}[\phi_n] = \frac{1}{N} \sum_n \int_0^L du \left[ \frac{1}{2} \left( \frac{d\phi_n}{du} \right)^2 - \cos (\phi_{n+1} - \phi_n - hu + \alpha + \pi n) \right]. \quad (2)$$

The oscillating behavior is determined by the reduced parameter $hL$ which is directly related to the total magnetic flux through one junction $\Phi = BLs$,

$$hL = 2\pi \Phi / \Phi_0,$$

We consider the stack containing a very large number of junctions, $N \gg 1$. This will allow us to focus on bulk behavior and neglect c-axis boundary effects coming from the top and bottom junctions, which give $1/N$ corrections to the bulk results. We will also not consider potentially interesting “parity effects”, small differences between stacks containing odd and even number of junctions, which have the same order. Our results are also not influenced by possible perturbations of the current distribution near the boundaries. Due to the large anisotropy of the material, the “bulk” current distribution usually is realized already in the second junction in the stack.

Following Ref. [12] we will assume the alternating phase distribution in the form $\phi_n = (-1)^n \phi$. This distribution describes both the deformed triangular lattice in the wide-stack regime and the transition to the rectangular lattice in narrow-stack regime. Substituting this presentation into Eq. (2), we represent the energy functional as

$$\mathcal{E}(\alpha; \phi) = \int_0^L du \left[ \frac{1}{2} \left( \frac{d\phi}{du} \right)^2 - \sin (2\phi) \sin (hu + \alpha) \right]. \quad (3)$$

The lattice energy as function of displacement, $\mathcal{E}(\alpha)$, is determined by the minimum of the functional $\mathcal{E}(\alpha; \phi(u))$ with respect to $\phi(u)$, $\mathcal{E}(\alpha) = \min_{\phi} \{\mathcal{E}(\alpha; \phi)\}$. As the energy functional has a symmetry property $\mathcal{E}(\alpha; \phi) = \mathcal{E}(\alpha + \pi; \phi)$, we can evaluate the energy distribution near the boundaries. Due to the large anisotropy of the material, the “bulk” current distribution usually is realized already in the second junction in the stack.

The ground-state phase distribution $\phi$ obeys the following equation

$$\frac{d^2 \phi}{du^2} + 2 \cos (2\phi) \sin (hu + \alpha) = 0, \quad (4)$$

which has to be solved with the boundary conditions

$$\frac{d\phi}{du} = 0, \text{ for } u = 0, L. \quad (5)$$

In the limit $h \gg 1$ further significant simplification is possible: we can average out rapid phase oscillations and derive a simplified equation for the smooth phase perturbation. We split the total phase into the smooth and rapidly-oscillating components

$$\phi(u) = \psi(u) + \tilde{\phi}(u), \quad (6)$$
where we assume $|\tilde{\phi}|, |dv/du| \ll 1$. The maximum value of $v(u), v_{\text{max}} = \pi/4$, corresponds to rectangular lattice. The rapidly-oscillating phase by definition obeys the equation

$$\frac{d^2\tilde{\phi}}{du^2} + 2\cos(2v)\sin(-hu + \alpha) = 0,$$  

(7)

which has the following approximate solution

$$\tilde{\phi} \approx \frac{2}{h^2}\cos(2v)\sin(-hu + \alpha).$$  

(8)

To the first order with respect to $\tilde{\phi}$, equation for $v(u)$ is given by

$$\frac{d^2v}{du^2} - 4\tilde{\phi}\sin(2v)\sin(-hu + \alpha) = 0.$$  

Substituting into this equation the oscillating phase $v(u)$ and averaging it with respect to rapid oscillations, we finally obtain the sine-Gordon equation for the smooth phase

$$\frac{d^2v}{du^2} - \frac{2}{h^2}\sin(4v) = 0.$$  

(9)

Computing the derivatives of the rapid phase $v(u)$ at the edges, we also derive the boundary conditions for the smooth phase,

$$\frac{dv}{du}(0) = \frac{2}{h}\cos(2v_0)\cos(\alpha),$$  

(10a)

$$\frac{dv}{du}(L) = \frac{2}{h}\cos(2v_L)\cos(-hL + \alpha)$$  

(10b)

with $v_0 \equiv v(0)$ and $v_L \equiv v(L)$. The local current density, $j(u) = j_J J_J \sin(\theta_{n,n+1}(u))$ with $j_J$ being the maximum Josephson current density, is determined by the gauge-invariant phase difference, $\theta_{n,n+1}(u) \equiv \phi_{n+1} - \phi_n - hu + \alpha + \pi n$, which is related to $v(u)$ as

$$\theta_{n,n+1} \approx -hu + \alpha + \pi n - (1)^n 2v - 4\frac{1}{h^2}\cos(2v)\sin(-hu + \alpha + \pi n).$$  

(11)

Substituting the phase presentation (6) and (8) into the energy (3) and averaging with respect to the rapid oscillations, we derive the energy functional in terms of the lattice shift $\alpha, \mathcal{E}(\alpha)$.

$$\mathcal{E}(\alpha; v) \approx \frac{1}{h}[\sin(2v_0)\cos(\alpha) - \sin(2v_L)\cos(-hL + \alpha)] - \int_0^L du \left[ \frac{1}{2}\left(\frac{dv}{du}\right)^2 + \frac{1}{2}\cos(4v)\right].$$  

(12)

To shorten notations, we omitted the arguments $h$ and $L$ in $\mathcal{E}(\alpha, h, L; v)$. Minimization of this energy functional with respect to $v(u)$ gives the energy as a function of the lattice shift $\alpha$, $\mathcal{E}(\alpha)$. Minimum of this energy with respect to $\alpha$ gives the ground state for given $h$ and $L$. Higher-energy states at other values of $\alpha$ typically carry a finite current. The total Josephson current flowing through the stack is proportional to $d\mathcal{E}/d\alpha$. Taking derivative of the functional (12) with respect to $\alpha$, assuming that at every $\alpha$ it is minimized with respect to $v(u)$, we obtain

$$J(\alpha) = \frac{1}{h}[\sin(2v_0)\sin\alpha + \sin(2v_L)\sin(-hL + \alpha)].$$  

(13)

The unit of current in this equation is $j_J J_J \lambda_J w$ where $j_J$ is the Josephson-current density and $w$ is the junction size in the field direction. An important consequence of this equation is that nonzero current exists only if the surface deformations $v_0$ and $v_L$ are finite. Further analysis is based on Eqs. (9), (10), (12), and (13).

### III. Solutions for Smooth Phase

A general solution of the sine-Gordon equation (9) can be found in terms of elliptic integrals. From the first integral of Eq. (9) we obtain

$$\frac{dv}{du} = \delta_d\sqrt{2}\int_0^\frac{\pi}{2}\sqrt{1 - \cos(2v)\cos(\alpha)}.$$  

(14)

with $\delta_d \equiv \text{sign}[dv/du] = \pm 1$ and $m$ is the elliptic parameter which has to be found from the boundary conditions. From this equation we obtain implicit equation for deformation $v(u)$,

$$\int_0^u \frac{dv}{\sqrt{1 - \cos(2v)\cos(\alpha)}} = \delta_d\sqrt{2}\alpha/h.$$  

(15)

To rewrite this equation using standard elliptic integrals, we introduce a new variable $\varphi$,

$$\varphi = \pi/2 + 2v.$$  

(16)

This variable has its own physical meaning: it describes the alternating deformation of the interlayer phase difference with respect to the rectangular-lattice state. In particular, $\varphi = 0$ corresponds to the rectangular lattice. Using these variables, we can rewrite Eq. (15) as

$$\sqrt{m}[F(\varphi, m) - F(\varphi_0, m)] = \delta_d\sqrt{8}\alpha/h.$$  

(17)

where

$$F(\varphi, m) \equiv \int_0^\varphi \frac{dx}{\sqrt{1 - m \sin^2 x}}$$

is the incomplete elliptic integral of the first kind.

In the limit of very large $L$ the deformation $\varphi$ has to vanish far away from edges meaning that $m \to 1$. For finite-size junctions, depending on $hL$ and $\alpha$, the solution $v(u)$ can be either monotonic or nonmonotonic. For the nonmonotonic solution the derivative $dv/du$ and the parameter $\delta_d$ change sign inside. The monotonic solution can either change sign ($v_0 v_L < 0, m < 1$) or not.
\( (\nu_0 \nu_L > 0) \). For large \( L \) the monotonic solution corresponds to the two surface partial solitons of the same sign and the nonmonotonic case corresponds to the surface solitons of opposite signs. A mathematical structure of solutions for these two cases is different and we will consider them separately.

First, we find some general relations between the boundary phases and the parameter \( m \). From the boundary conditions \((10a), (10b)\), and Eq. \((14)\) we obtain equations

\[
\delta_0 \sqrt{1/m - \cos^2(2\nu_0)} = \sqrt{2} \cos(2\nu_0) \cos(\alpha), \quad (18a)
\]

\[
\delta_L \sqrt{1/m - \cos^2(2\nu_L)} = \sqrt{2} \cos(2\nu_L) \cos(hL - \alpha) \quad (18b)
\]

with \( \delta_0 = \delta_{d}(0) \) and \( \delta_L = \delta_{d}(L) \) (for monotonic solution \( \delta_0 = \delta_L \) and for nonmonotonic solution \( \delta_0 = -\delta_L \)). As \( |\nu_0, L| < \pi/4 \), the inequality \( \cos(2\nu_0, L) > 0 \) always holds meaning that \( \delta_0 = \text{sign}[\cos \alpha] \) and \( \delta_L = \text{sign}[\cos(hL - \alpha)] \). From the conditions \((18)\) we obtain the following results for the boundary deformations

\[
\cos(2\nu_0) = \frac{1/m}{1 + 2 \cos^2(\alpha)}, \quad (19a)
\]

\[
\cos(2\nu_L) = \frac{1/m}{1 + 2 \cos^2(hL - \alpha)}, \quad (19b)
\]

or, in terms of variable \( \varphi \) \((16)\),

\[
\sin \varphi_0 = \frac{1/m}{1 + 2 \cos^2(\alpha)}, \quad (20a)
\]

\[
\sin \varphi_L = \frac{1/m}{1 + 2 \cos^2(hL - \alpha)}. \quad (20b)
\]

From these considerations one can conclude that the monotonic solution realizes for all \( \alpha \)'s for \( hL = 2k\pi \) (magnetic flux per junction equals to integer number of flux quanta, \( \Phi = k\Phi_0 \)) and the nonmonotonic solution realizes for \( hL = (2k + 1)\pi \) (magnetic flux equals to half-integer number of flux quanta, \( \Phi = (k + 1/2)\Phi_0 \)). If the magnetic flux through one junction is not close to these special values then the solution changes from monotonic to nonmonotonic depending on the lattice phase shift \( \alpha \). Location of different types of solution depending on \( hL \) and \( \alpha \) is illustrated in Fig. 2.

One can distinguish two important special cases corresponding to symmetric locations of the lattice with respect to the boundaries. The first case, \( \alpha = hL/2 + \pi k, \nu_L = -\nu_0 \), corresponds to the monotonic solution and the second case, \( \alpha = hL/2 + \pi/2 + \pi k, \nu_L = \nu_0 \), corresponds to the nonmonotonic solution. The energy always has extremums at these values of \( \alpha \). Moreover, a detailed study shows that the ground state is always realized at one of these values of \( \alpha \). At large \( L \) the system switches between these locations in the vicinity of the points \( hL = (2k + 1/2)\pi \), as it is illustrated in Fig. 2.

In the vicinity of these switching points the energy has minima at the both values of \( \alpha \).

In general, the conditions \((19a) \text{ and } (19b)\) are not sufficient to determine signs of the edge deformations \( \nu_0 \) and \( \nu_L \). In the limit of large \( L \) the deformation \( \nu(u) \) has to decay from the edges leading to relations \( \text{sign}[\nu_0] = -\delta_0 = -\text{sign}[\cos \alpha] \) and \( \text{sign}[\nu_L] = \delta_L = \text{sign}[\cos(hL - \alpha)] \). In this case we also obtain conditions

\[
\tan(2\nu_0) = -\delta_0 \sqrt{m (2 \cos^2(\alpha) + 1) - 1},
\]

\[
\tan(2\nu_L) = \delta_L \sqrt{m (2 \cos^2(hL - \alpha) + 1) - 1}
\]

which fix signs of \( \nu_0 \) and \( \nu_L \). For large values of \( L/h \) monotonic solution typically changes sign inside. However for finite \( L \) there are intermediate regions exist located near lines \( \alpha = \pi/2 + \pi k \) and \( \alpha = hL + \pi/2 + \pi k \) where solution is still monotonic but does not change sign. We now proceed with analyzing separately the monotonic and nonmonotonic solutions.
A. Monotonic solution

The monotonic solutions realize for ranges of \(\alpha\) where 
\[
\cos \alpha \cos (hL - \alpha) > 0
\]
(grey regions in Fig. 2). For such solution we obtain from Eq. (15) relation connecting the parameter \(m\) with the boundary deformations

\[
\int_{v_0}^{v_L} \frac{dv}{\sqrt{1/m - \cos^2(2v)}} = \text{sign} [\cos \alpha] \sqrt{2L}/h. \tag{21}
\]

Using previously introduced variable \(\varphi\) (10), we can rewrite Eq. (21) via elliptic integrals as

\[
\sqrt{m} [F(\varphi_L, m) - F(\varphi_0, m)] = \text{sign} [\cos \alpha] \sqrt{8L}/h \tag{22}
\]

This equation together with boundary conditions (20a) and (20b) have to be solved to find the three unknown constants \(\varphi_0, \varphi_L, \) and \(m\), which completely determine the solution. The boundary deformations \(v_0 = (\varphi_0 - \pi/2)/2\) and \(v_L = (\varphi_L - \pi/2)/2\) may have either the same sign or opposite signs. In Appendix A we find the boundary separating these two types of solution (boundaries between dark-grey regions and light-grey regions in Fig. 2).

The energy (12) and Josephson current (13) can be represented as

\[
E = - \frac{1}{h} \left[ \cos (\varphi_0) \cos (\alpha) - \cos (\varphi_L) \cos (hL - \alpha) \right]
+ \frac{1}{\sqrt{2m}h} \left| E(\varphi_L, m) - E(\varphi_0, m) \right| - \frac{L}{mh^2}, \tag{23}
\]

\[
J(\alpha) = \frac{1}{h} \left[ \cos (\varphi_0) \sin \alpha - \cos (\varphi_L) \sin (-hL + \alpha) \right], \tag{24}
\]

where \(E(\varphi, m) \equiv \int_0^{\varphi} \sqrt{1 - m \sin^2 x} \, dx\) is the incomplete elliptic integral of the second kind.

B. Nonmonotonic solution

The solution is nonmonotonic in the regions given by 
\[
\cos \alpha \cos (hL - \alpha) < 0
\]
(white regions in Fig. 2). In this case the function \(v(u)\) has extremum at some point \(u = u_m\), so that we can rewrite Eq. (14) as

\[
\frac{dv}{du} = \delta_d \sqrt{2 \frac{1/m - \cos^2(2v)}{h}}, \quad \text{for } u < u_m,
\]

\[
\frac{dv}{du} = -\delta_d \sqrt{2 \frac{1/m - \cos^2(2v)}{h}}, \quad \text{for } u_m < u < L
\]

with \(\delta_d = \text{sign} [\cos \alpha]\). As \(dv/du = 0\) at \(u = u_m\), we have \(m = 1/\cos^2(2u_m) > 1\) with \(v_m \equiv v(u_m)\). Integrating the first equation from 0 to \(u_m\) and the second equation from \(L\) to \(u_m\), we obtain

\[
\int_{v_0}^{v_m} \frac{dv}{\sqrt{1/m - \cos^2(2v)}} = \delta_d \sqrt{2} u_m / h,
\]

\[
\int_{v_L}^{v_m} \frac{dv}{\sqrt{1/m - \cos^2(2v)}} = \delta_d \sqrt{2} (L - u_m) / h.
\]

Adding these two equations, we obtain equation connecting \(v_m\) with the boundary deformations \(v_0\) and \(v_L\)

\[
\int_{v_0}^{v_m} \frac{dv}{\sqrt{\cos^2(2v) - \cos^2(2v)}} + \int_{v_L}^{v_m} \frac{dv}{\sqrt{\cos^2(2v) - \cos^2(2v)}} = \delta_d \sqrt{2L}/h. \tag{25}
\]

This equation together with boundary conditions (19a) and (19b), represents the full system for determination of three unknown constants \(v_0, v_L, \) and \(m = 1/\cos^2(2u_m)\). To rewrite these equations in terms of elliptic functions, we again transfer to variable \(\varphi\) (16). Then equation (25) can be rewritten as

\[
\sqrt{m} \left[ 2F(\varphi_m, m) - F(\varphi_0, m) - F(\varphi_L, m) \right] = \delta_d \sqrt{8L}/h. \tag{26}
\]

with \(\varphi_m = \pi/2 + 2v_m\). The boundary conditions in terms of \(\varphi_0\) and \(\varphi_L\) are again given by Eqs. (20a) and (20b). The elliptic parameter is related to \(\varphi_m\) as \(m = 1/\sin^2 \varphi_m\) leading to the following relation, \(F(\varphi_m, m) = K(1/m)/\sqrt{m}\). The energy for nonmonotonic solution can be represented as

\[
E = - \frac{1}{h} \left[ \cos \psi_v \cos (\alpha) - \cos \varphi_L \cos (-hL + \alpha) \right]
+ \frac{1}{\sqrt{2m}h} \left| 2E(\varphi_m, m) - E(\varphi_0, m) - E(\varphi_L, m) \right| - \frac{L}{mh^2} \tag{27}
\]

and the Josephson current is again given by Eq. (24). One can easily check that the nonmonotonic solution matches the monotonic solution at the boundaries. For example, for \(\cos \alpha = 0\) the extremum is located at the boundary \(u = 0\). In this case we have \(\phi_0 = \phi_m = \arcsin(1/\sqrt{m})\) (or \(\pi - \arcsin(1/\sqrt{m})\)) and Eq. (26) coincides with Eq. (22).

C. Alternative presentation of equations via elliptic integrals

Using known relations for the elliptic integrals

\[
F(\varphi, m) = \frac{1}{\sqrt{m}} F(\psi, 1/m) \quad \text{with } \sin \psi = \sqrt{m} \sin \varphi, \tag{28a}
\]

\[
E(\varphi, m) = \frac{1}{\sqrt{m}} \left[ (1-m)F(\psi, 1/m) + mE(\psi, 1/m) \right], \tag{28b}
\]

valid for \(\varphi < \pi/2\) and \(\sin \varphi < 1/\sqrt{m}\), one can rewrite equations (22) and (20) for the case of the same-sign monotonic solution in the following alternative form

\[
F(\psi_L, \tilde{m}) - F(\psi_0, \tilde{m}) = \text{sign} [\cos \alpha] \sqrt{8L}/h, \tag{29a}
\]

\[
\sin \psi_0 = \frac{1}{\sqrt{1 + 2 \cos^2(\alpha)}}, \tag{29b}
\]

\[
\sin \psi_L = \frac{1}{\sqrt{1 + 2 \cos^2(hL - \alpha)}}, \tag{29c}
\]
with \( \tilde{m} \equiv 1/m \). Correspondingly, the energy \( \mathcal{E} \) in this representation is given by

\[
\mathcal{E} = \frac{1}{\hbar} \left| \sqrt{1 - \tilde{m} \sin^2 \psi_0 \cos \alpha - \sqrt{1 - \tilde{m} \sin^2 \psi_L \cos (hL - \alpha)}} \right|
\]

\[
+ \frac{1}{\sqrt{2h}} \left| E(\psi_0, \tilde{m}) - E(\psi_L, \tilde{m}) - \frac{(2 - \tilde{m}) L}{h^2} \right|. \tag{30}
\]

In the case of nonmonotonic solution, using relations \( 28a \) and \( 28b \) for the elliptic integrals, one can rewrite equation \( 28a \) in the following equivalent form

\[
2K(\tilde{m}) - F(\psi_0, \tilde{m}) - F(\psi_L, \tilde{m}) = \delta \sqrt{8L} / h. \tag{31}
\]

where \( \tilde{m} \equiv 1/m \), \( \psi_0 \) and \( \psi_L \) are given by Eqs. \( 29b \) and \( 29c \), and represent the energy as

\[
\mathcal{E} = -\frac{1}{\hbar} \left[ \sqrt{1 - \tilde{m} \sin^2 \psi_0 |\cos \alpha|} + \sqrt{1 - \tilde{m} \sin^2 \psi_L |\cos (hL - \alpha)|} \right]
\]

\[
+ \frac{1}{\sqrt{2h}} |2E(\tilde{m}) - E(\psi_0, \tilde{m}) - E(\psi_L, \tilde{m}) - \frac{(2 - \tilde{m}) L}{h^2}|. \tag{32}
\]

This representation is especially useful in the case of large \( m \) (small \( \tilde{m} \)). In particular, it will allow us to study transition to the rectangular-lattice state corresponding to the limit \( m \to \infty \), which we will consider in the next section.

**IV. TRANSITION TO THE RECTANGULAR LATTICE**

An important particular case is the solution of Eq. \( \Phi = \pi/2 + \phi \) corresponding to rectangular lattice, \( v = \pm \pi/4 \) or \( \varphi = 0, \pi \). This case corresponds to the limit \( m \to \infty \). The energy of the rectangular lattice coincides with the well-known result for a single junction

\[
\mathcal{E}_{\text{rect}}(\alpha) = -\frac{2}{\hbar} \sin \left( \frac{hL}{2} \right) \sin \left( \alpha - \frac{hL}{2} \right) \tag{33}
\]

and has minimum \( \mathcal{E}_{\text{rect}} = -2|\sin(hL/2)|/h \) at \( \alpha = hL/2 + \delta \pi/2 \) with \( \delta = \text{sign} \left[ \sin(hL/2) \right] \).

To find condition for the transition to the rectangular lattice, we take the limit \( \tilde{m} \to 0 \) in Eq. \( 31 \) for nonmonotonic solution. Using relations \( K(\tilde{m}) = \pi/2 \) and \( F(\psi, 0) = \psi \), we obtain

\[
\pi - \psi_0 - \psi_L = \sqrt{8L} / h, \tag{34}
\]

where \( \psi_0 \) and \( \psi_L \) are given by Eqs. \( 29b \) and \( 29c \). Using these definitions, the condition for the rectangular lattice can be rewritten in an explicit form as

\[
\frac{\sqrt{2} \left[ |\cos(hL - \alpha)| + |\cos \alpha| \right]}{\sqrt{(1 + 2 \cos^2 \alpha)(1 + 2 \cos^2(hL - \alpha))}} < \sin \left( \frac{\sqrt{8L}}{h} \right). \tag{35}
\]

**FIG. 3: Typical lattice deformations at large \( L \) for \( \Phi = k\Phi_0 \) (upper plot) and \( \Phi = (k + 1/2)\Phi_0 \) (lower plot). In the former case the surface partial solitons have the same sign and repel each other while in the latter case they have the opposite signs and attract each other. The insets in both plots illustrate corresponding displacement fields of the Josephson-vortex lattice in the two neighboring layers.**

This equation gives the transition criterion in general case, including the current-carrying states. In particular, the rectangular lattice gives a local energy minimum at \( \alpha = hL/2 + \pi/2 \) in the regions \( |hL/2 - (k + 1/2)| < 1/4 \) if the inequality

\[
|\sin(hL/2)| < \tan \left( \frac{\sqrt{2}}{\sqrt{2}} \right) \tag{36}
\]

is satisfied. The rectangular lattice first appears in the ground state at points \( hL = (k + 1/2)2\pi \) for \( L/h \leq l_1 = \arctan(\sqrt{2}) / \sqrt{2} \approx 0.675 \). This value is marked in the right plot of Fig. \( 4 \).

**V. WIDE-STACK/NARROW-STACK CROSSOVER**

In this section we investigate in detail the crossover between the wide-stack and narrow-stack regimes. As this crossover is driven by the reduced parameter \( h/L \), for a junction with size \( L \) the crossover takes place with increasing magnetic field at size-dependent field \( B_L = L\Phi_0/(2\pi \gamma^2 s^3) \).

At large \( L, L \gg h \) or \( B \ll B_L \), the smooth alternating deformation has solutions in the form of two isolated surface solitons. The monotonic solution corresponds to the solitons of the same sign and the nonmonotonic solution corresponds to the solitons of opposite signs, as it is illustrated in Fig. \( 3 \). If one neglects the interaction between the solitons then the relative sign of surface soliton has no importance and the total Josephson current is given by the sum of two independent surface currents, which do not depend on the soliton signs.
sequence, the product $hJ_c$ has periodicity of half flux-quantum per junction and reaches maxima at $hL = \pi j$ ($\Phi = j\Phi_0/2$) with $hJ_c \approx 1.035$. At finite $L$ the interaction between the surface solitons disturbs such periodicity. At large $L$ one can derive analytically corrections to the infinite-$L$ results, see Appendix [15] for details. In particular, near the maxima $hL = \pi j$ ($\Phi = j\Phi_0/2$), we find the finite-size correction,

$$\delta J_c(h, \pi j) \approx -1.544 \frac{(-1)^j}{h} \exp \left( -\frac{\sqrt{8L}}{h} \right), \quad (37)$$

As we can see, the finite-size effects increase the critical current maxima at $\Phi = (k+1/2)\Phi_0$ ($j = 2k$) and $\Phi = k\Phi_0$ ($j = 2k$). In the wide-stack regime, however, these corrections are exponentially small, which explains nice $\Phi_0/2$-periodic oscillation of the flux-flow voltage observed in this regime [15].

In the whole range of fields and sizes we explore the phase diagram numerically. To find the ground state and the critical current at given $h$ and $L$, we study dependences of the lattice structure, energy, and Josephson current on the lattice phase shift $\alpha$. First, we have to find the boundary deformations $\varphi_0$, $\varphi_L$ and the elliptic parameter $m$ using the boundary conditions [20] together with either Eq. [22] for $\cos \alpha \cos(hL-\alpha) > 0$ or Eq. [20] for $\cos \alpha \cos(hL-\alpha) < 0$. Using obtained values, we compute the energy from Eq. [23] or [27] and the current from Eq. [24]. This procedure has been implemented in Mathematica. Figure 4 shows representative $\alpha$-dependences of the energy and current for $L = 4$ and three values of flux per junction $\Phi/\Phi_0 = 3.1$, 3.25, and 3.4 within one oscillation period. The minimum of the energy with respect to $\alpha$ determines the ground state and the maximum of the current determines the critical current.

Figure 5 shows the field dependences of the critical current for three different values of the junction size $L$, 3, 4, and 6. One can observe that with increasing field $\Phi_0/2$-periodic oscillations smoothly transform into $\Phi_0$-periodic oscillations. This occurs via suppression of the peaks at $\Phi = k\Phi_0$ and enhancement of the peaks at $\Phi = (k + 1/2)\Phi_0$. Such behavior of the critical current has been recently reproduced by numerical simulations by Irie and Oya [14]. Experimentally, the crossover between $\Phi_0/2$- and $\Phi_0$-periodic oscillations has been observed in the flux-flow voltage by Kakeya et al. [15]. The crossover field can be arbitrarily defined as a field at which a $k\Phi_0$-peak in the product $J_c \Phi$ drops below the half of a $(k + 1/2)\Phi_0$-peak. At larger $L$ the crossover takes place at larger field and larger $\Phi/\Phi_0$ but it always occurs at the same ratio $h/L$, $h/L \sim 1.6$. Important property of the system, discussed in Sec. [IV] is the transformation of the lattice into the rectangular state at sufficiently large $h/L$. This
FIG. 6: The field dependences of the critical current (solid lines) for sizes \( L = 2.5 \) and 4 for the same range of the ratio \( h/L, h/L \lesssim 3.5 \) shown on the top axes. Shaded areas show the regions of stable rectangular lattice. The rectangular-lattice regions first appear in the vicinity of points \( \Phi = (k + 1/2)\Phi_0 \) when \( h/L \) exceeds 1/1.48. When \( h/L \) exceeds 1/1.48 the rectangular lattice remains stable at these points up to the critical current.

The critical current at \( \Phi = (k + 1/2)\Phi_0 \) is given by \( J_{\text{max},1} = \max_\alpha J_1(\alpha) \). At small \( L, L/h < l_1 \), the maximum critical current is realized at the instability point of the rectangular lattice \( \cos(\alpha) \approx L/h \) giving

\[
J_{\text{max},1} \approx \frac{2}{h} \left( 1 - \frac{L^2}{2h^2} \right) .
\] (42)

It is always somewhat smaller than the “Fraunhofer” value \( 2/h \).

It was obtained in Section IV that the rectangular lattice is realized in ground state (\( \alpha = 0 \)) at points \( \Phi = (k + 1/2)\Phi_0 \) for \( L/h < l_1 = 0.675 \). If, however, \( L/h \) is only slightly smaller than this value, the rectangular lattice becomes unstable with increasing current and the configuration at the critical current still corresponds to the deformed lattice. We found that there is another typical value of the ratio \( L/h, L/h = l_2 \approx 0.484 \), below which the rectangular lattice remains stable up to the critical current. Both typical values of \( h/L, 1/l_1 \) and \( 1/l_2 \), are marked in Fig. 6. One can see that for both shown stack sizes, \( L = 2.5 \) and 4, the rectangular lattice first appears around points \( \Phi = (k + 1/2)\Phi_0 \) when \( h/L \) exceeds 1/1.48 and 1/1.2, and its stability range extends up to the critical current when \( h/L \) exceeds 1/1.2.

For integer flux quanta, \( \Phi = k\Phi_0 \), the changing-sign monotonic solution always realizes, \( \psi_L = -\sqrt{\Phi} \). In the case \( \cos \phi_0 > 0 \) this corresponds to \( \psi_0 = \pi - \varphi_L = \arcsin[m(1 + 2\cos^2(\phi))^{-1/2}] \) and Eq. (22) can be reduced to the form

\[
\sqrt{m} (K(m) - F(\phi_0, m)) = \sqrt{2}L/h .
\] (43)

Solving this equation with respect to \( m \), we can obtain the energy from Eq. (23) and current from Eq. (24).

\[
\begin{align*}
E_2 &= -\frac{2}{h} \cos \phi_0 \cos \alpha + \frac{\sqrt{2}}{\sqrt{m}h} [E(m) - E(\phi_0, m)] - \frac{L}{m^2 h} .
\end{align*}
\] (44)

\[
J_2(\alpha) = \frac{2}{h} \cos \phi_0 \sin \alpha
\] (45)
The critical current at $\Phi = (k+1/2)\Phi_0$ is given by $J_{\text{max},2} = \max_\alpha [J_2(\alpha)]$. At small $L$ inequality $\cos \phi_0 \ll 1$ holds. In this limit, using relation $F(\phi_0, m) \approx K(m) - (\pi/2 - \phi_0)/\sqrt{1-m}$, we can approximately rewrite Eq. (43) as $\pi/2 - \phi_0 = (\sqrt{2}L/h)\sqrt{1-1/m}$. As $\sin \phi_0$ is close to one, Eq. (20a) gives $1/m \approx 1 + 2\cos^2(\alpha)$ and $\phi_0 = \pi/2 - (2L/h)\cos \alpha$. Therefore we obtain for the $\alpha$-dependent current (45),
\[ J_2(\alpha) \approx \frac{2L}{h^2} \sin 2\alpha. \] (46)

The maximum is realized at $\alpha = \pi/4$ giving the following result for the critical current
\[ J_{\text{max},2} \approx \frac{2L}{h^2}, \] (47)
i.e., it decays at large $h$ as $1/h^2$ but never drops to zero as for usual Fraunhofer dependence. The behavior in the narrow-stack regime will be considered in more details in the next section. One can see that the critical currents at both maxima $J_{\text{max},\alpha} (\alpha = 1, 2)$ have the same scaling property: the product $hJ_{\text{max},\alpha}$ depends only on the ratio $L/h$. These scaling dependences are plotted in Fig. 5 in the plots for $L = 4$ and $6$.

VI. NARROW-STACK REGIME

Let’s consider in more detail the narrow-stack regime at $L/h \ll 1$. In this regime interaction with the boundaries is typically stronger than the bulk shearing interaction. As a consequence, the boundaries stabilize the rectangular lattice configuration in most part of the phase diagram. The exception is the narrow regions in the vicinity of the integer-flux-quanta points $\Phi = k\Phi_0$ where the interaction with the boundaries vanishes and the rectangular lattice looses its stability. The rectangular lattice also becomes unstable near the critical current. In this section we will study in details this behavior. Instead of using asymptotic behavior of elliptic integrals, it is more transparent to use as a starting point the equation for smooth alternating deformation (9), the boundary conditions (10) and the energy (12). It will be more convenient to use variable $\varphi$ given by Eq. (16) (instead of $v$) from the very beginning, because it vanishes in the rectangular-lattice state. We also introduce a new variable for the lattice phase shift,
\[ \beta \equiv \alpha + \pi/2 - hL/2, \]
which will facilitate a more compact presentation of results. In terms of the variables $\varphi(u)$ and $\beta$ the energy (12) can be rewritten as
\[ \mathcal{E}(\beta) \approx \frac{1}{h} \left[ \cos \varphi_0 \sin \left( \beta + \frac{hL}{2} \right) - \cos \varphi_L \sin \left( \beta - \frac{hL}{2} \right) \right] \]
\[ + \int_0^L du \left[ \frac{1}{8} \left( \frac{d\varphi}{du} \right)^2 - \frac{1 - \cos(2\varphi)}{2h^2} \right]. \] (48)

From this energy we obtain equation for $\varphi(u)$,
\[ \frac{d^2\varphi}{du^2} + \frac{4}{h^2} \sin(2\varphi) = 0, \] (49)
and the boundary conditions
\[ \frac{d\varphi}{du}(0) = \frac{4}{h} \sin(\varphi_0) \sin \left( \beta + \frac{hL}{2} \right), \] (50a)
\[ \frac{d\varphi}{du}(L) = \frac{4}{h} \sin(\varphi_L) \sin \left( \beta - \frac{hL}{2} \right). \] (50b)

For small $L$ Eq. (49) can be solved as expansion with respect to powers of $u - L/2$,
\[ \varphi = \varphi_a + a \left( u - \frac{L}{2} \right) - \frac{2(u - L/2)^2}{h^2} \sin(2\varphi_a) \] (51)
with $\varphi_a = \varphi(L/2)$. Boundary conditions (50) give two equations for two unknown variables, the midpoint phase $\varphi_a$ and the linear slope $a$. We obtain two types of solutions: (i) the rectangular-lattice solution $a = 0$, $\sin \varphi_a = 0$ and (ii) the deformed-lattice solution. In the leading order with respect to the small parameter $L/h$, the latter solution can be represented as
\[ a \approx \frac{4}{h} \sin(\varphi_a) \sin \beta \cos(hL/2), \] (52)
\[ \cos(\varphi_a) \approx \frac{h}{L} \frac{\sin(hL/2) \cos \beta}{1 + 2\sin^2 \beta \cos^2(hL/2)}. \] (53)

As follows from the last equation, the deformed-lattice solution does not exist if
\[ \frac{h}{L} \frac{\sin(hL/2) \cos \beta}{1 + 2\sin^2 \beta \cos^2(hL/2)} > 1. \] (54)

In this case the configuration must be the rectangular lattice. The solution (53) also includes the case of the ideal triangular lattice $\varphi_a = \pi/2$ which is always realized if either $\sin(hL/2) = 0$ or $\cos \beta = 0$. As we consider the region $L/h \ll 1$, both triangular and deformed lattices exist only in vicinity of these points.

For analysis of lattices, it is also useful to derive the energy as a function of the average lattice shift, $\beta$, and the relative phase shift between the neighboring layers, $\phi_a$. For that we substitute expansion (51) up to the linear order with the phase gradient given by Eq. (48) into the energy (48) and obtain
\[ \mathcal{E}(\varphi_a, \beta) \approx -\frac{2}{h} \cos(\varphi_a) \sin \left( \frac{hL}{2} \right) \cos \beta \]
\[ - \frac{L}{h^2} \sin^2(\varphi_a) \left[ 1 + 2\sin^2 \beta \cos^2 \left( \frac{hL}{2} \right) \right]. \] (55)

In particular, the result (53) corresponds to the minimum of this energy with respect to $\varphi_a$ when the condition (54) is satisfied. We will see that this relatively simple energy function of two variables, whose shape evolves with the
we conclude that the rectangular lattice is stable if

$$\Phi = 3k$$

For the deformed-lattice solution (53) the energy at fixed $\beta$ is given by

$$\mathcal{E}(\beta) \approx -\frac{1}{L} \frac{\sin^2 \frac{hL}{2} \cos^2 \beta}{1 + 2 \sin^2 \beta \cos^2 \frac{hL}{2}} - \frac{L}{h^2} \left(1 + 2 \sin^2 \beta \cos^2 \frac{hL}{2}\right).$$

This energy always has extrema at $\beta = 0, \pi, \text{and } \pi/2$. As follows from Eq. (53), the state at $\beta = \pi/2$ always corresponds to the triangular lattice, $\varphi_a = \pi/2$. We find now the conditions that the energy reaches minima at these values of $\beta$. Consider first the point $\beta = 0$. Expanding the energy near this point,

$$\mathcal{E}(\beta) \approx -\frac{1}{L} \frac{\sin^2 \frac{hL}{2}}{2} - \frac{L}{h^2}$$

we obtain that it corresponds to minimum if

$$\tan^2 \left(\frac{hL}{2}\right) \left[1 + 2 \cos^2 \left(\frac{hL}{2}\right)\right] > \frac{2L^2}{h^2}.$$  

As $L/h \ll 1$, this inequality is valid almost everywhere except in the vicinity of the integer-flux-quanta points at which $\sin(hL/2) \to 0$ and $\cos^2(hL/2) \to 1$ where this condition can be rewritten in an approximate simpler form,

$$\left|\sin \left(\frac{hL}{2}\right)\right| > \frac{\sqrt{2L}}{\sqrt{3h}}.$$  

Comparing this condition with the condition (56), we can see that the deformed lattice gives the energy minimum at $\beta = 0$ only within the narrow region given by

$$\sqrt{\frac{2}{3}} \frac{h}{L} \left|\sin \left(\frac{hL}{2}\right)\right| < 1.$$  

In this region the optimum value of $\varphi_a$ for $\beta = 0$ is given by $\cos (\varphi_a) = (h/L) \sin (hL/2)$.

To find if the triangular lattice at the point $\beta = \pi/2$ gives the local energy minimum, we expand the energy (58) near this point, $\beta = \pi/2 - \zeta$,

$$\mathcal{E}(\beta) \approx -\frac{L}{h^2} \left(1 + 2 \cos^2 \frac{hL}{2}\right)$$

$+ \zeta^2 \left(-\frac{1}{L} \frac{\sin^2 \frac{hL}{2}}{1 + 2 \cos^2 \frac{hL}{2}} + \frac{2L^2}{h^2} \cos^2 \frac{hL}{2}\right).$

We can see that the value $\beta = \pi/2$ corresponds to energy minimum if

$$\tan^2 \frac{hL}{2} \frac{1 + 2 \cos^2 \frac{hL}{2}}{1 + 2 \cos^2 \frac{hL}{2}} < \frac{2L^2}{h^2}.$$  

or, approximately, $|\sin(hL/2)| < \sqrt{6L/h}$. We can conclude that near the integer-flux points $\Phi = k\Phi_0$.

magnetic field, describes a surprisingly rich behavior in
the vicinity of the integer-flux-quanta points. The typical
energy landscapes for the cases of half-integer and integer
flux quanta per junction are illustrated in Fig. 7.

Let’s study zero-current ground states first. We
start with stability analysis for the rectangular lattice
which is realized at $\varphi_a = 0, \pi$ and gives ground
state in the most part of phase space. In this case
the energy is minimal with respect to $\beta$ either at $\beta = 0$, for $\cos (\varphi_a) \sin (hL/2) > 0$, or at $\beta = \pi$, for $\cos (\varphi_a) \sin (hL/2) < 0$, see, e.g., left part of Fig. 7 where $\sin (hL/2) = -1$. Expanding the energy with respect to $\varphi_a$ near the point $\varphi_a = 0$,

$$\mathcal{E}(\varphi, 0) \approx \frac{2}{h} \left|\sin \left(\frac{hL}{2}\right)\right| + \varphi_a^2 \frac{h}{2},$$

we conclude that the rectangular lattice is stable if

$$\left|\sin \left(\frac{hL}{2}\right)\right| > \frac{L}{h},$$

which coincides with the condition (54) at $\beta = 0, \pi$. In

real variables the condition (56) can be rewritten as

$$\frac{2\pi \Phi}{\Phi_0} \sin \left(\frac{\pi \Phi}{\Phi_0}\right) > \left(\frac{L}{\lambda_f}\right)^2.$$  

Therefore even at small $L/h$ the rectangular lattice is always unstable near the integer-flux-quanta points, $\Phi = k\Phi_0$. This is easy to understand: near these points the interaction with the boundaries vanishes and even small shearing interaction between the neighboring planar Josephson-vortex arrays becomes sufficient to induce instability with respect to the alternating deformations. Further analysis, however, will show that this instability takes place when the rectangular lattice does not give already the ground state, meaning that the system actually experiences a first-order transition.

FIG. 7: Examples of the energy landscape (55) as a function of
the average lattice shift $\beta$ and the amplitude of the alternating
phase deformation $\varphi_a$ for $L = 2$ and $\Phi/\Phi_0 = 3.5$ and 3. For $\Phi = 3.5\Phi_0$ the equivalent minima at $(\varphi_a, \beta) = (0, \pi)$ and $(\pi, 0)$ correspond to the rectangular lattice, while for $\Phi = 3\Phi_0$ the minima at $(\varphi_a, \beta) = (\pi/2, \pi/2)$ and $(\pi/2, 3\pi/2)$ correspond to the triangular lattice.
\( (hL = 2k\pi) \) the minimum location switches from \( \beta = 0 \) to \( \beta = \pi/2 \). In the intermediate region given approximately

\[
\sqrt{\frac{2}{3}} < \frac{h}{L} \left| \sin \left( \frac{hL}{2} \right) \right| < \sqrt{6} \tag{60}
\]

the energy has local minimums at both points, \( \beta = 0 \) and \( \pi/2 \). Moreover, in the region \( (h/L) |\sin(hL/2)| > 1 \) the minimum at \( \beta = 0 \) is realized by the rectangular lattice. This behavior indicates that switching between the rectangular and triangular lattices in the ground state occurs via a first-order phase transition.

To find the transition point, we compare the triangular-lattice energy,

\[
\mathcal{E}_{\text{trian}} = \mathcal{E} \left( \frac{\pi}{2}, \frac{\pi}{2} \right) = -\frac{L}{h^2} \left( 1 + 2 \cos^2 \frac{hL}{2} \right),
\]

with the rectangular-lattice energy,

\[
\mathcal{E}_{\text{rect}} = \mathcal{E}(0, 0) = -\frac{2}{h} \left| \sin \left( \frac{hL}{2} \right) \right|,
\]

and obtain that the triangular lattice wins if

\[
\left| \sin \left( \frac{hL}{2} \right) \right| < \frac{L}{2h} \left( 1 + 2 \cos^2 \left( \frac{hL}{2} \right) \right).
\]

As this only happens near the points where \( \cos^2(hL/2) \approx 1 \), the equation for the transition points, \( h_t \), can again be rewritten in a simpler form,

\[
\left| \sin \left( \frac{hL}{2} \right) \right| = \frac{3}{2} \frac{L}{h_t}, \tag{61}
\]

or in real units, in terms of flux per junction,

\[
\frac{2\pi \Phi_t}{\Phi_0} \left| \sin \left( \frac{\pi \Phi_t}{\Phi_0} \right) \right| = \frac{3}{2} \left( \frac{L}{\lambda_J} \right)^2. \tag{62}
\]

Comparing Eq. \( \text{(61)} \) with the stability criterion of the rectangular lattice \( \text{(56)} \), we indeed can see that before the rectangular lattice becomes unstable, it switches to the triangular lattice via a first-order phase transition. From this equation we can also obtain small shift of transition point with respect to the integer-flux-quanta point \( \Phi = k\Phi_0 \) as a function of the index \( k \). Writing \( \Phi_t = (k + f_{t,k}) \Phi_0 \), we compute

\[
|f_{t,k}| \approx \frac{3}{4\pi^2k} \left( \frac{L}{\lambda_J} \right)^2 \ll 1. \tag{63}
\]

The discussed behavior is illustrated in Fig. 8 in which the \( \beta \)-dependences of the energy and current are plotted for \( L = 2 \) and several values of \( \Phi \) above the point \( 3\Phi_0 \). The contour plot of energy at the transition point is also shown.

Let us investigate now current-carrying states and behavior of the critical current. In the region given by Eq. \( \text{(54)} \) the current in the rectangular-lattice state is

\[
J(\beta) = \frac{2}{h} \left| \sin \left( \frac{hL}{2} \right) \right| \sin \beta, \tag{64}
\]

where we assumed for definiteness that \( \cos \varphi, \sin(hL/2) > 0 \). If the condition \( \text{(54)} \) is violated then the deformed lattice is realized. In this case, differentiating the energy \( \text{(58)} \) with respect to \( \beta \), we obtain the current in the deformed-lattice state (including the triangular lattice)

\[
J(\beta) = \left( \frac{\sin^2 \frac{hL}{2} \left( 1 + 2 \cos^2 \frac{hL}{2} \right)}{\left( 1 + 2 \sin^2 \beta \cos^2 \frac{hL}{2} \right)^2 - \frac{2L \cos^2 \frac{hL}{2}}{h^2}} \right) \sin 2\beta, \tag{65}
\]

for

\[
\frac{h}{L} \left| \sin \frac{hL}{2} \cos \beta \right| < 1.
\]

In particular, at \( hL = 2\pi k \) this formula reproduces results \( \text{(46)} \) and \( \text{(47)} \) obtained from the elliptic-integral representation.

Consider first the region where the ground state is given by the rectangular lattice. Maximum current in
this state would be achieved at $\beta = \pi/2$ but the condition (54) always breaks down before that. From this condition we compute the value of $\beta$ at which the rectangular lattice becomes unstable

$$|\cos \beta_l| = \frac{2 L^2 (1 + 2 \cos^2 \frac{hL}{2})}{|\sin \frac{hL}{2}| + \sqrt{\sin^2 \frac{hL}{2} + \frac{8L^2}{\pi^2} \cos^2 \frac{hL}{2} (1 + 2 \cos^2 \frac{hL}{2})}}. \tag{66}$$

In a wide range of parameters, away from the regions given by Eq. (61), the maximum current is achieved at this instability point

$$J_c = \frac{2}{L^2} \left| \sin \left( \frac{hL}{2} \right) \right| \sin \beta_l. \tag{67}$$

In particular, in most part of the parameter space, for $|\tan(hL/2)| \gg L/h$ we obtain much simpler results

$$|\cos \beta_l| \approx \frac{L + 2 \cos^2 \frac{hL}{2}}{L/h}, \tag{68}$$

$$J_c \approx \frac{2}{h^2} \sin \left( \frac{hL}{2} \right) \left[ 1 - \left( \frac{L}{h} \right)^2 \left( 1 + 2 \cos^2 \frac{hL}{2} \right)^2 \right] \frac{1}{2 \sin^2 \frac{hL}{2}}. \tag{69}$$

In this region the critical current is only slightly smaller than the “Fraunhofer” result $(2/h) |\sin(hL/2)|$. At the half-integer-flux-quanta points $hL = (2k + 1)\pi$ these equations reproduce result (12) obtained from the elliptic-integrals representation. The property that the rectangular lattice is always unstable at some lattice displacement $\beta$ also has important dynamic consequences. It means that the lattice cannot maintain its static rectangular configuration when it starts to move.

The critical current has a nontrivial behavior in the vicinity of points $hL = 2\pi k$ where $|\sin(hL/2)| \ll 1$ and general formula (65) can be simplified as

$$J(\beta) \approx \frac{2L}{h^2} \left( 1 - \frac{3h^2 \sin^2 \left( \frac{hL}{2} \right)}{2L^2 (2 - \cos 2\beta)^2} \right) \sin 2\beta.$$

This gives the critical current near $hL = 2\pi k$

$$J_c(h, L) \approx \frac{2L}{h^2} \left( 1 - \frac{3h^2 \sin^2 \left( \frac{hL}{2} \right)}{2L^2} \right),$$

for $\sin(hL/2) \ll L/h$. This results shows that the dependence $J_c(\Phi)$ for junction stacks always has local maxima at $\Phi = k\Phi_0$, in contrast to the Fraunhofer dependence for which the critical current vanishes at these points. To find the critical current behavior in the whole field range in the region $h/L \gg 1$, we numerically found maximum of $J(\beta)$ with respect to $\beta$ and different $hL = 2\pi \Phi/\Phi_0$ and $L$. Figure 9 illustrates the field dependence of critical current and current dependence of the lattice structure within one oscillation period $2.5\Phi_0 < \Phi < 3.5\Phi_0$ for $L = 2\.\lambda_j$. To visualize the lattice structures, we represent the values of $|\cos \varphi_a|$ by grey level. In most part of the current-field diagram the rectangular lattice is realized shown by light grey ($|\cos \varphi_a| = 1$). The triangular lattice shown by black ($|\cos \varphi_a| = 0$) appears in the ground state only in vicinity of the point $\Phi = 3\Phi_0$. Exactly at this point the lattice remains triangular up to the critical current. Slightly away from this point the lattice deforms with increasing current. In the range of parameters given by Eq. (60) the dependence $|J(\beta)|$ has two maxima within $0 < \beta < \pi/2$ (see left plot in Fig. 8). As a consequence, the field dependence of the critical current has kinks related to switching between these maxima.

VII. SLOW DYNAMICS IN OVERDAMPED REGIME: OSCILLATIONS OF THE FLUX-FLOW VOLTAGE

When the external current flowing across the layers exceeds the critical current, the lattice starts to move. In general, dynamic behavior is quite complicated. A simple situation is realized only at slow lattice motion in the overdamped case when the lattice deformations have time to adjust to the current lattice position. In this case the lattice moves in the periodic potential given by its static energy (12) and one can use static results to predict the I-V dependences. On the other hand, one
FIG. 10: The representative field dependences of mean-squared average of current, \( \langle J^2(\alpha) \rangle \), with respect to the lattice displacements which determines the amplitude of relative voltage oscillations \( \delta U/U_{ff} \) at slow velocities via Eq. (72).

FIG. 11: The field dependence of the ratio of voltage-oscillation maxima at integer-flux-quanta points \( \delta U_{max,2}/\delta U_{max,1} \) and at half-integer-flux-quanta points \( \delta U_{max,2}/\delta U_{max,1} \). The inset illustrates definitions of \( \delta U_{max,1} \) and \( \delta U_{max,2} \) in the schematic voltage-field dependence. For comparison we also show plot of the ratio of the critical-current maxima squared \( (J_{max,2}/J_{max,1})^2 \). Extraction of the typical field \( \delta U/\delta U_{max} \) from the analysis of the voltage oscillations allows accurately evaluate the anisotropy factor \( \gamma \) using Eq. (73).

We can expect that the voltage oscillations are less sensitive to inhomogeneities than the critical-current oscillations, because the homogeneously moving lattice smears away disorder. The critical current is also smeared by thermal fluctuations. These are possible reasons why it is easier to observe and interpret the magnetic oscillations in the flux-flow resistivity than in the critical current. \[13\,16,17\]

In general, the dynamic behavior also depends on the dissipation mechanism. In BSCCO in a wide range of magnetic fields the flux-flow resistivity is mainly determined by the in-plane quasiparticle conductivity \( \sigma_{ab} \).\[25\]

Only when the magnetic field exceeds a typical value \( B_\sigma = \sqrt{\sigma_{ab}/\sigma_c \Phi_0}/(\sqrt{2}\pi \gamma^2 s^2) \), the c-axis conductivity, \( \sigma_c \), gives dominating contribution to the flux-flow dissipation. In this limit the flux-flow resistivity becomes field-independent. In BSCCO the field \( B_\sigma \) is typically several times larger than the crossover field \( B_{cr} \). An important feature of the in-plane dissipation regime at \( B < B_\sigma \) is that the lattice velocity at fixed applied current is very sensitive to lattice structure, the smallest velocity is realized for the triangular lattice and the largest velocity is realized for the rectangular lattice. As the lattice structure in the regime \( B \gtrsim B_L \) continuously changes with lattice displacement, the dynamic behavior in this regime is rather complicated. To avoid this complications, we limit ourself here by a simple case of dominating c-axis dissipation in the crossover region, \( B_L > B_\sigma \). In this case the viscous-friction coefficient weakly depends on lattice structure.

In the case of structure-independent viscous-friction coefficient \( \nu_{ff} \), time variation of the lattice phase shift obeys equation

\[ \nu_{ff} \frac{d\alpha}{dt} + J(\alpha) = J_{ext}, \]  

where \( J_{ext} \) is the external current, the current \( J(\alpha) \equiv J(\alpha, hL, h) \) is given by Eq. (13) (for brevity we again skip in equations its dependence on the magnetic field and size), and the viscosity coefficient, \( \nu_{ff} \), is related to the flux-flow resistance of the stack, \( R_{ff} \),

\[ \nu_{ff} = \frac{N\Phi_0}{2\pi c R_{ff}}, \]

where \( N \) is the number of junctions in the stack. The voltage drop per one junction \( U \) is related to \( d\alpha/dt \) by the Josephson relation

\[ U = \frac{\Phi_0}{2\pi c} \frac{d\alpha}{dt}. \]

Solution of Eq. (70) is given by the implicit relation

\[ \int_0^\alpha \frac{\nu_{ff} d\alpha'}{J_{ext} - J(\alpha')} = t, \]

from which we obtain the average phase change rate,

\[ \frac{d\alpha}{dt} = \left[ \frac{1}{\pi} \int_0^\pi \frac{\nu_{ff} d\alpha}{J_{ext} - J(\alpha)} \right]^{-1}, \]

and the flux-flow voltage

\[ \frac{U}{U_{ff}} = \left[ \frac{J_{ext}}{\pi} \int_0^\pi \frac{d\alpha}{J_{ext} - J(\alpha)} \right]^{-1} \]

with \( U_{ff} = R_{ff} J \) being the bare flux-flow voltage without the periodic potential. As the current \( J(\alpha) \equiv J(\alpha, hL, h) \) oscillates with the magnetic field, this flux-flow voltage will also experience similar field oscillations. In particular, when the external current significantly exceeds the critical current, \( J_{ext} \gg J_c \), we obtain weak oscillating correction to the flux-flow voltage,
\[ \delta U = U - U_{ff}, \]
\[ \delta U/U_{ff} \approx -\langle J^2(\alpha) \rangle / J_{ext}^2, \]  
(72)

where \( \langle f(\alpha) \rangle \equiv (1/\pi) \int_0^{\pi} f(\alpha) d\alpha \) is the average with respect to the lattice phase shift. As \( \delta U \propto \langle J^2(\alpha) \rangle \), the behavior of \( \delta U \) is overall similar to the behavior of the critical current but the amplitude of voltage oscillations roughly scales as the critical current squared. Figure 10 shows the field dependences of the average \( \langle J^2(\alpha) \rangle \), which determines the amplitude of weak voltage oscillations, for three junctions sizes, \( L = 3, 4, \) and \( 6 \).

Consider in more details behavior of \( \delta U \) at the points \( \Phi = (j/2)\Phi_0 \). To find the amplitude of the small voltage correction, \( \delta U_{max,1} \), at the half-integer flux quanta points, \( \Phi = (k+1/2)\Phi_0 \), we have to find the \( \alpha \)-average of \( J_f^2(\alpha) \) where the current \( J_1(\alpha) \) is given by Eq. (41) with the parameter \( \tilde{m} \) given by Eq. (39). Similarly, the amplitude of the voltage oscillation at the points \( \Phi = k\Phi_0 \), which we notate as \( \delta U_{max,2} \), is determined by \( \langle J_f^2(\alpha) \rangle \) where the current \( J_2(\alpha) \) is given by Eq. (43) with the parameter \( m \) given by Eq. (43). Figure 11 shows the computed field dependence of the ratio \( \delta U_{max,2}/\delta U_{max,1} \) together with the ratio \( (J_{max,2}/J_{max,1})^2 \). In practice, to extract the ratio \( \delta U_{max,2}/\delta U_{max,1} \), we should plot smooth curves via local maxima and two sets of local minima as it is illustrated in the inset of Fig. 11. Subtract the two minima voltages corresponding to integer number of flux quanta per junction. For slow lattice motion similar crossover can also be observed in the oscillations of the flux-flow resistivity. Quantitative study of the crossover allows for a very accurate evaluation of the anisotropy factor.

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**APPENDIX A: REGIONS OF MONOTONIC SAME-SIGN SOLUTIONS FOR \( v(u) \)**

One can distinguish two types of monotonic solutions depending on whether or not the smooth phase \( v(u) = (\varphi(u) - \pi/2)/2 \) changes sign inside the junction (see Fig. 2). For the changing-sign solution the condition \( 0 < m < 1 \) always holds. In this Appendix we find the boundary values of \( \alpha \) separating these two types of monotonic solutions. For definitiveness, we consider the region \( 2k\pi < hL < (2k+1)\pi \) and \( -\pi/2 + hL - 2k\pi < \alpha < \pi/2 \) (grey area in the lower part of the phase diagram in Fig. 2). For the monotonic changing-sign solution in this range we have \( 0 < \varphi_0 < \pi/2, \pi/2 < \varphi_L < \pi \), i.e.,

\[ \varphi_0 = \arcsin \sqrt{\frac{1/m}{1 + 2 \cos^2(\alpha)}}, \]
\[ \varphi_L = \pi - \arcsin \sqrt{\frac{1/m}{1 + 2 \cos^2(hL - \alpha)}}. \]

There are two boundaries in the region, one corresponding to the condition \( \varphi_0 = \pi/2 \) (\( v_0 = 0 \)) below \( \alpha = \pi/2 \) and another corresponding to \( \varphi_L = \pi/2 \) (\( v_L = 0 \)) above \( \alpha = -\pi/2 + hL - 2k\pi \) (see Fig. 2). For the first boundary,
\( \alpha_0(h, L) \), from Eqs. [20a] and [22] we obtain

\[
\sqrt{m_0} \left[ K(m_0) - F \left( \arcsin \sqrt{\frac{1/m_0}{1 + 2 \cos^2(hL - \alpha_0)}}, m_0 \right) \right] = \frac{\sqrt{8L}}{h},
\]

where \( K(m) = F(\pi/2, m) \) is the complete elliptic integral of the first kind and we used the identity \( F(\pi - \beta, m) - K(m) = K(m) - F(\beta, m) \). Analyzing similar equation for the second boundary, \( \alpha_L(h, L) \), we find that it is related to \( \alpha_0(h, L) \) as \( \alpha_L(h, L) = hL - 2k\pi - \alpha_0(h, L) \). One can check that \( \alpha_0(h, L) \to \pi/2 \) for \( hL \to 2k\pi, (2k + 1)\pi \) and for \( L \to \infty \), i.e., the region of the same-sign monotonic solution vanish in these limits.

**APPENDIX B: WEAK FINITE-SIZE EFFECTS AT LARGE \( L \)**

In this Appendix we derive finite-size corrections to the critical current due to interaction between the surface solitons. Consider for definiteness the case of monotonic solution. The nonmonotonic solution can be treated similarly. In the limit \( L/h \gg 1 \) the parameter \( m \) in Eq. [21] is close to one. Separating small correction, \( m = 1 - \eta/2 \) with \( \eta \ll 1 \), we evaluate the integral as

\[
\int_{v_0}^{v_L} \frac{dv}{\sqrt{1 + \eta/2 - \cos^2(2v)}} \approx \frac{1}{2} \ln \left( -\frac{32}{\eta} \tan v_0 \tan v_L \right).
\]

This gives the following result for \( \eta \)

\[
\eta \approx -32 \tan v_0 \tan v_L \exp \left( -\sqrt{8L}/h \right), \tag{B1}
\]

corresponding to the elliptic-function parameter \( m \approx 1 + 16 \tan v_0 \tan v_L \exp \left( -\sqrt{8L}/h \right) \). The boundary conditions can be represented as

\[
\cos(2v_0) \approx \frac{1 + \eta/4}{\sqrt{1 + 2 \cos^2(\alpha)}},
\]

\[
\cos(2v_L) \approx \frac{1 + \eta/4}{\sqrt{1 + 2 \cos^2(hL - \alpha)}}.
\]

One can neglect shift of \( \eta \) due to the finite-size corrections to \( v_0 \) and \( v_L \). Without interaction between edge deformations, the energy can be written as

\[
\mathcal{E}_0 = \frac{1}{\hbar} \int_{v_0}^{v_L} dv \left[ \sqrt{1 + \eta - \cos(4v)} - \sqrt{1 - \cos(4v)} \right] - \frac{L \eta}{2h^2}.
\]

where \( v_0 \) and \( v_L \) are the surface deformations neglecting finite-size correction,

\[
\tan v_0 = \frac{1 - \sqrt{1 + 2 \cos^2(\alpha)}}{\sqrt{2} \cos \alpha},
\]

\[
\tan v_L = -\frac{1 - \sqrt{1 + 2 \cos^2(\alpha - hL)}}{\sqrt{2} \cos(\alpha - hL)}.
\]

The finite-size correction to the energy change can now be estimated as

\[
\delta \mathcal{E} = \frac{1}{\hbar} \int_{v_0}^{v_L} dv \left[ \sqrt{1 + \eta - \cos(4v)} - \sqrt{1 - \cos(4v)} \right] - \frac{L \eta}{2h^2} \approx \frac{\eta}{h\sqrt{32}}.
\]

Therefore, the total energy can be written as

\[
\mathcal{E}(\alpha, h, hL) = \frac{L}{\hbar^2} \left[ \frac{1}{\hbar} \int_{v_0}^{v_L} dv \left( -\frac{\sin(2v_0)}{\sqrt{2 + \cos(2\alpha - \sqrt{8L}/h)}} \right) + \frac{8\sqrt{2}}{h} \left( \frac{\cos(\alpha) \cos(hL - \sqrt{8L}/h)}{1 + \sqrt{2 + \cos(2\alpha)}} \right) \right].
\]

The last correction term describes the exponentially small interaction energy between the surface solitons. It is important to note that the finite-size correction breaks \( \pi \)-periodicity with respect to parameter \( hL \) meaning that the states with \( \Phi = k\Phi_0 \) and \( \Phi = (k + 1/2)\Phi_0 \) are not equivalent any more.

For the Josephson current we obtain

\[
J(\alpha, h, hL) = \frac{1}{\sqrt{2\hbar}} \left( \frac{\sin 2(\alpha - hL)}{\sqrt{2 + \cos 2(\alpha - hL)}} + \frac{\sin 2\alpha}{\sqrt{2 + \cos 2(\alpha - hL)}} \right)
\]

\[
+ \frac{8\sqrt{2}}{h} \exp \left( -\sqrt{8L}/h \right) \frac{\partial}{\partial \alpha} \left[ \frac{\cos(\alpha) \cos(hL - \sqrt{8L}/h)}{1 + \sqrt{2 + \cos(2\alpha)}} \right]
\]

Assuming that without the finite-size correction the maximum current flows at \( \alpha = \alpha_m(hL) \), we obtain for the
finite-size correction to the critical current

$$\delta J_c(h, hL) = \frac{8\sqrt{2}}{h} \exp \left(-\frac{\sqrt{8}L}{h}\right) \frac{\partial}{\partial\alpha} \left[ \frac{\cos\alpha\cos(\alpha - hL)}{\left(1 + \sqrt{2 + \cos(2\alpha)}\right)\left(1 + \sqrt{2 + \cos(2(\alpha - hL))}\right)} \right]_{\alpha = \alpha_m(hL)}.$$

In particular, near the maxima $hL = \pi j$ ($\Phi = j\Phi_0/2$), using the result\(^{12}\) $\alpha_m(0) = 0.921$, we obtain result\(^{37}\).

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