Abstract

Recently Daviau showed the equivalence of ordinary matrix based Dirac theory –formulated within a spinor bundle $S_x \cong \mathbb{C}_4^+$, to a Clifford algebraic formulation within space Clifford algebra $\mathcal{O}(\mathbb{R}^3, \delta) \cong M_2(\mathbb{C}) \cong \mathbb{P} \cong \text{Pauli algebra (matrices)} \cong \mathbb{H} \oplus \mathbb{H} \cong \text{biquaternions}$. We will show, that Daviau’s map $\theta : \mathbb{C}_4^+ \rightarrow M_2(\mathbb{C})$ is an isomorphism. Furthermore it is shown that Hestenes’ and Parra’s formulations are equivalent to Daviau’s space Clifford algebra formulation, which however uses outer automorphisms. The connection between such different formulations is quite remarkable, since it connects the left and right action on the Pauli algebra itself viewed as a bi-module with the left (resp. right) action of the enveloping algebra $\mathbb{P}^e \cong \mathbb{P} \otimes \mathbb{P}$ on $\mathbb{P}$. The isomorphism established in this article and given by Daviau’s map does clearly show that right and left actions are of similar type. This should be compared with attempts of Hestenes, Daviau and others to interprete the right action as the iso-spin freedom.

MSCS: 15A66

Keywords: Daviau map, Dirac theory, Dirac-Hestenes equation, spinors, tensors, multi-vectors, Pauli algebra, space Clifford algebra, Maxwell-Dirac isomorphism

1 Introduction

There is a long quest on a geometric intuitive description of Dirac spinor fields. Only a few month after the publication of Dirac’s first paper [Dirac 1928] Charles Galton Darwin tried to re-express the strange new objects called half vectors by Pauli [Pauli 1933] and spinors due to Paul Ehrenfest –according to B.L. van der Waerden, see [Budinich et al. 1988]– with help of tensors [Darwin 1928]. He did not fully succeed in obtaining an equivalence by writing down complex tensor equations which yield Dirac’s theory “twice over” –with a doubling of degrees of freedom from complexification--; see Parra [Parra] for a detailed review on this topic. Madelung, trying the same transcription essentially reproduced Darwin’s results, most likely without knowing them [Madelung 1929]. Also in the thirties Fock and Ivanenko [Fock 1929] did very important work on the geometric relations behind the $\gamma$-algebra introduced by Dirac. De Broglie and his school developed a very valuable and complete picture of the Dirac fluid –a tensor description of the Dirac field– and its hydrodynamics [Yvon 1940, Takabayasi 1957].
This reasoning has a revival in recent times because of the improved tool of Clifford algebra now available [Rylov 1995].

But the historical development abandoned the attempt of trying to find a geometric –and thereby tensorial– description of the Dirac field. Firstly there seemed to be a tendency to concrete calculations which on the one hand were extremely successful and on the other hand could be performed without an elaborated interpretation by applying simply the rules of $\gamma$-algebra, see discussion in [Sham 1993]. Furthermore, quantum theory had (has?) to be interpreted within a statistical picture. It was simply out of the imagination of that time to search for such an explanation or even to connect geometry with spinor variables.

One has to wonder, but neither the physicists Pauli and Dirac nor the mathematicians Weyl, Jordan, von Neumann and others cited or seemed to have known substantially the work of Grassmann, Clifford, Klein, Cayley, Hamilton and other algebraists of the 19th century. If some of their formulas and results were acknowledged –the quaternions e.g. were well known to be isomorphic to Pauli matrices– this was done in a technical sense. The geometric origin of hypercomplex number systems was unknown or ignored and thus lost for a further development of the theory. One result of this missed opportunity –in the sense of Dyson [Dyson 1972]– was the thereby obtained “interpretation” of spinors, which became artificial objects in an abstract spin space or an inner spin space and had thusly no physical counterpart in the “real world”. However, from a technical point of view, dropping the interpretation, there was an extraordinary and fruitful development of spinor methods in physics.

The situation changes with the appearance of the writings of David Hestenes [Hestenes 1966], see references in [Hestenes 1995]. He recovered again the geometric origin of spinor objects and the formerly well known connection of (metric) space and certain algebras. The first time he gave a geometrical motivated treatment of real Dirac theory in his book “Space time algebra” [Hestenes 1966]. The reformulation of Dirac’s theory in real(!) space time algebra $\mathcal{C}(\mathbb{R}^4, \eta)$, $\eta = \text{diag}(1, -1, -1, -1)$ is the starting point of a host of new insights into the interplay between geometry, algebra and physics. Hestenes’ reformulation was also the very first starting point of Daviau’s consideration which lead to a space algebraic ($\simeq \mathcal{C}(\mathbb{R}^4, \eta)$) formulation of Dirac theory.

But even up to now, there is a discussion on the proper interpretation of spinorial objects in either geometrical or statistical settings. This lead to a large number of slightly different notations of spinors; e.g. spinor modules $S_x \simeq \mathbb{C}^{2n}$, operator or Hestenes spinors $\simeq \mathcal{C}_{p,q}^+$, ideal spinors $\simeq \mathcal{C}f$, an primitive idempotent element, algebraic spinors and the spin Clifford bundle –isomorphism classes of ideal spinors to geometrically equivalent idempotents– etc. If Clifford algebra provides us the universal language for mathematics and physics [Hestenes 1985] we have to give exact and unambiguous notations of physical objects and of their exact mathematical design.

Hestenes in succeeding to write down a real Dirac theory within $\mathcal{C}_{1,3}$ translated the non-geometrical $i = \sqrt{-1}$ into the right action of $\gamma_2\gamma_1$ –recall $(\gamma_2\gamma_1)^2 = -\gamma_1^2\gamma_2^2 = -1$–. But right actions mix different left ideals related to different idempotents, while left action remains in the same left ideal. Rodrigues et al. introduced therefore the spin Clifford bundle and algebraic spinors, in which spinors or even better algebraic spinors are defined to be equivalence classes of ideals which belong to geometrically equivalent idempotents [Rodrigues et al. 1996]. Such idempotents are conjugated to one another within the Clifford–Lipschitz group $\Gamma$ by $e' = ueu^*$, $u \in \Gamma$– the reversion map, and are therefore members of the same group orbit. To obtain a mathematical clear picture one should then translate the Dirac–Hestenes spinors into the quotient space $DH \simeq \mathcal{C}_{1,3}/\Gamma$ (as linear space) to be not troubled with the probably ill chosen representatives. This consideration should, however, be compared with the approach of Parra to Dirac–Hestenes spinors and his illuminating explanation of the equivalence classes and their relations to the Wigner definition of a particle as an irreducible representation of the Poincaré group [Parra].
In this paper, we want to study the map from ordinary Dirac matrix theory onto space Clifford algebra used by Daviau. This will be done in several steps. Starting with the definition of the Daviau map, we analyse afterwards the Hestenes formulation of Dirac theory. Then it is shown, that a special option of Parra’s formulation corresponds directly to Hestenes’ formulation, showing the well known correspondence between them. Finally the equivalence of Hestenes’, sic. Parra’s, formulation to the space Clifford algebraic formulation of Daviau is demonstrated. The correct identification to Parra’s options is given.

We can however not appreciate every work concerned with space Clifford algebraic formulations of Dirac theory for lack of space, one important paper may be added here for those [Baylis 1997].

Our analysis unmasks a close connection between the ordinary spinor module $S_x \cong \mathbb{C}_x^4$ which is equivalent to a formulation by ideal spinors in $\mathbb{C}_x^{4,1} \cong M_4(\mathbb{C})$ which is actually used by physicists. Daviau’s map furthermore shows up a correspondence of left actions on $\mathbb{C}_x^4$ spinors to homomorphisms of $P$, which can be written as $u x v$, $u, v, x \in P$. If one defines the enveloping algebra $P_e$ as in Hahn [Hahn 1994], $P_e \cong P \otimes P^T$, where $P^T$ denotes the right module or transposed module, it is easily seen, that the $P_e$ left action is equivalent to the $P$-bi-module structure by writing $P_e \cdot P \rightarrow P$, $x \otimes y^T \cdot z = xyz$. We have therefore to consider left and right actions on $P$, as Daviau did. This makes $P$ a $P$-bi-module. This bi-module structure is crucial for further investigations of the enveloping algebra $P_e$ of Clifford algebras, which will be given elsewhere, and for a thoughtful interpretation of left and right actions in Clifford algebras. There is a widespread thinking about the meaning of right actions, see [Hestenes 1967, Daviau 1998b, Fauser et al. 1999c].

2 The Daviau map $\theta : \mathbb{C}^4 \mapsto \mathbb{C}l_{3,0}$

2.1 Definition of the Daviau map

Daviau changed his notation and got rid of his cyclic permuted $\sigma$-matrices in a new work [Daviau 1998a], however, we stay with his old notations to be coherent.

We start according to Daviau with the Dirac equation in its standard matrix representation due to Bjorken & Drell [Bjorken et al. 1964]

$$-i \gamma^\mu \partial_\mu \Psi + q A^\mu \gamma_\mu \Psi + m \Psi = 0. \quad (1)$$

We have $m, q$ real constants, $i = \sqrt{-1}$ the usual complex unit, $\partial_\mu := \partial/\partial x^\mu$ the partial derivatives with respect to a local holonom coordinate system, $A^\mu$ real components of an external vector potential, $\Psi$ is the Dirac spinor of $\mathbb{C}^4$ valued functions of the (tangent) Minkowski space and finally $\gamma_\mu$ the Dirac matrices in Dirac representation

$$\gamma_0 = \gamma^0 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \gamma_k = -\gamma^k := \begin{pmatrix} 0 & -\sigma^k \\ \sigma^k & 0 \end{pmatrix}, \quad 1 := 1_{2 \times 2}$$

$$\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (2)$$

It is an easy task, to translate the Dirac equation into a set of eight real coupled differential equations, see also [Parra 1989]. From a mathematical point of view, this two sets of equations are identical. But in setting

$$\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix} := \begin{pmatrix} a + ic \\ -g - if \\ d + ih \\ b + ic \end{pmatrix} \quad (3)$$
with $a, \ldots, h : (M, \eta) \to \mathbb{R}$ real valued functions, one does no longer insist on the “spinorial” character of the object in favour for playing with components and forgetting about transformation properties – compare the analysis of Parra [Parra Parra 1983].

On the other hand, one has to consider the Pauli algebra or space Clifford algebra $\mathcal{O}_{3,0} \simeq \mathcal{P}$. This algebra is isomorphic to the full matrix algebra $M_2(\mathbb{C})$ and thus eight dimensional over the reals. A purely dimensional comparison yields $\dim \Psi = 8 = \dim \mathcal{P} = \dim_{\mathbb{R}} M_2(\mathbb{C})$.

The aim of the Daviau map is to give an isomorphism from $\mathcal{C}_4^+ \to$ co-ordinates $\to M_2(\mathbb{C})$ which is also a morphism of the algebraic structure. One could call such a map a Dirac-morphism.

Now, by letting

$$u := a + ib, \quad v := f + ib, \quad w := c + ig, \quad t := d + ie$$

$$\phi_1 := u + w, \quad \phi_2 := t + v, \quad \phi_3 := t - v, \quad \phi_4 := u - w$$

$$\phi_D = \begin{pmatrix} \phi_1 & \phi_3 \\ \phi_2 & \phi_4 \end{pmatrix} \in M_2(\mathbb{C}) \simeq \mathcal{P} \simeq C\ell_{3,0}, \quad (4)$$

we obtain a map $\theta : \mathbb{C}^4 \to M_2(\mathbb{C})$. Introducing then (note our indexing)

$$\nabla := \partial_0 + \bar{\partial}, \quad \bar{\partial} := \sigma_2 \partial_1 + \sigma_3 \partial_2 + \sigma_1 \partial_3$$

$$A := A_0 + \bar{A}, \quad \bar{A} := A^1 \sigma_2 + A^2 \sigma_3 + A^3 \sigma_1$$

$$\phi^* := \begin{pmatrix} \bar{\phi}_1 \\ -\bar{\phi}_3 \\ \bar{\phi}_2 \\ \bar{\phi}_1 \end{pmatrix} = \sigma_2 \bar{\phi} \sigma_2$$

$$i := \sigma_1 \sigma_2 \sigma_3, \quad [i, X] = 0 \quad \forall X \in \mathcal{P}, \quad (5)$$

we obtain the space Clifford or Pauli algebraic form of Dirac’s equation due to Daviau:

$$\nabla \phi i \sigma_1 = m \phi^* + q A \phi. \quad (6)$$

Daviau showed, that all transformation properties and requirements are fulfilled within this picture, making his map finally a Dirac-morphism preserving the algebraic structure of Dirac theory. A Lagrangian formulation is also possible. Using the above given representation of Pauli matrices [3] one can reconstruct an algebraic expression of the $M_2(\mathbb{C})$ matrix $\phi_D$. From [3] we find

$$\phi_D = \begin{pmatrix} u + w & t - v \\ t + v & u - w \end{pmatrix}$$

$$= \begin{pmatrix} a + c + i(h + g) & d - f + i(e - b) \\ d + f + i(e + b) & a - c + i(h - g) \end{pmatrix}$$

$$= a \mathbb{1} + d \sigma_1 + b \sigma_2 + c \sigma_3 + e i \sigma_1 - f i \sigma_2 + g i \sigma_3 + h i. \quad (7)$$

This form of the Daviau spinor will be used below to show the equivalence to other formulations.

### 2.2 Hestenes equation

We may further notice, that since $\dim \mathcal{O}_{1,3}^+ = 16$ and $\dim \mathcal{O}_{1,3}^{++} = 8$, $\mathcal{O}_{1,3}^+$ may also be used as a target for a map $H : \mathbb{C}^4 \to \mathcal{O}_{1,3}^+$. This algebra $\mathcal{O}_1^+$, called even subalgebra, consist of Dirac–Hestenes operator spinors and has in a natural manner a bimodul structure under the action of even elements. With the above choice of names for the real spinor components [3] we obtain the correspondence using $\gamma_{ij} := \gamma_i \gamma_j$, $\Sigma_i := \gamma_i \gamma_0$, $i := \Sigma_1 \Sigma_2 \Sigma_3 = \gamma_{0123}$:

$$\Psi_H = a + b \gamma_{10} + c \gamma_{20} + d \gamma_{30} + e \gamma_{21} + f \gamma_{23} + g \gamma_{13} + h \gamma_{0123}.$$
Where we have used the identities

\[ i\Sigma_1 = i\gamma_{10} = -\gamma_{23}, \quad i\Sigma_2 = i\gamma_{20} = \gamma_{13}, \quad i\Sigma_3 = i\gamma_{30} = \gamma_{21} \tag{9} \]

and anticipated the names of the variables in an appropriate manner to fit into the Daviau scheme. The translated Dirac equation reads

\[ \partial \Psi_{H} \gamma_{21} = m\Psi_{H} \gamma_{0} + qA\Psi_{H}, \tag{10} \]

which is the famous Dirac–Hestenes equation and representation free. The elements on the right hand side of \( \Psi_{H} \) describe the spin bivector \( S := \gamma_{21} \) and the “particles” (local) velocity \( v := \gamma_{0} \) –a time-like vector measuring proper-time– and do not fix a representation. For a discussion of the relation between quantum logic, measurement and the choice of a time-like direction in Dirac theory see [Haft 1996, Saller 1996].

Now, we may left multiply (10) by \(-\gamma_{0}\) which turns the equation (beside the mass term) into the space part of the algebra. Using (9) and

\[ \begin{align*}
-\gamma_{0}\partial &= -\gamma_{0}\gamma_{\mu}\partial_{\mu} = \Sigma_{\mu}\partial_{\mu} \\
-\gamma_{0}A &= -\gamma_{0}\gamma_{\mu}A_{\mu} = \Sigma_{\mu}A_{\mu}
\end{align*} \tag{11} \]

we remain with

\[ \begin{align*}
\Sigma_{\mu}\partial_{\mu}\Psi_{H}i\Sigma_{3} &= -m\gamma_{0}\Psi_{H}\gamma_{0} + q\Sigma_{\mu}A_{\mu}\Psi_{H} \\
\Sigma_{i}\partial_{i}\Psi_{H}i\Sigma_{3} &= -m\Psi_{H}^{\dagger} + q\Sigma_{\mu}A_{\mu}\Psi_{H},
\end{align*} \tag{12} \]

which is written now within the space sector only. The transformation \( \Psi_{H}^\dagger = \gamma_{0}\Psi_{H}\gamma_{0} \) represents the hermitian adjoint, which is not an inner automorphism of the Pauli algebra isomorphic to \( \mathbb{C}^{2\times 2} \), as indicated by the odd element \( \gamma_{0} \).

This form of the Dirac-Hestenes’ formulation will be needed in the proof of the isomorphy to Daviau’s formulation below.

### 2.3 Parra’s analysis of Dirac theory

Parra analyzed the Dirac equation also in terms of a real set of eight differential equations [Parra 1989]. Like Darwin and Madelung he afterwards tried to reinterpret this set of equations in terms of vector analysis, –spinors versus multi-vectors [Parra]–. The novelty of Parra’s approach is, that he succeeded in formulating tensorial equations without any complexification and thereby no doubling of degrees of freedom. This is achieved by a simple inspection of the resulting eight real equations. Under the assumption, that the real part \( \Re(\Psi_{1}) \) of \( \Psi_{1} \)–first component of the \( \mathbb{C}^{4}_{x} \) Dirac spinor– transforms as a scalar quantity, the full set of eight equations admits a vectorial character. The result is at first not satisfactory since some terms remain to be only third components of vectors. By introducing the spin vector \( \vec{n} = (0, 0, \hbar) (= -iS) \), one obtains a full \( \text{SO}(3) \) rotationally invariant set of vector equations. Denoting the two scalar quantities as \( \alpha, \lambda \) and the two vectorial quantities as \( \vec{E} = (E_{1}, E_{2}, E_{3}) \), \( \vec{B} = (B_{1}, B_{2}, B_{3}) \) one arrives at the Parra type \( \{0\} \) spinor

\[ \Psi_{\{0\}} = \begin{pmatrix} \alpha + iB_{3} \\ -B_{2} + iB_{1} \\ E_{3} + i\lambda \\ E_{1} + iE_{2} \end{pmatrix}. \tag{13} \]

Now, it is purely a matter of choice which type of vector component –scalar, first, second or third vector component– one asserts for \( \Re(\Psi_{1}) \). The other three possibilities yield by
the same procedure, also introducing the spin-vector $\vec{n}$, equally well suited spinor–tensor translations. A suitable choice of names for the involved scalars and vectors yields:

$$
\Psi_{(0)} = \begin{pmatrix}
\alpha + iB_3 \\
-B_2 + iB_1 \\
E_3 + i\lambda \\
E_1 + iE_2 \\
E_1 - iE_2
\end{pmatrix}, \\
\Psi_{(2)} = \begin{pmatrix}
\alpha - iB_3 \\
B_2 + iB_1 \\
E_3 + i\lambda \\
E_1 + iE_2 \\
E_1 - iE_2
\end{pmatrix}, \\
\Psi_{(1)} = \begin{pmatrix}
-B_3 + i\lambda \\
E_3 + i\lambda \\
B_2 + iB_1 \\
\alpha - iB_2
\end{pmatrix}, \\
\Psi_{(3)} = \begin{pmatrix}
-B_2 + iB_1 \\
B_3 + i\lambda \\
E_1 + iE_2 \\
\alpha + iB_3
\end{pmatrix}.
$$

(14)

If we now introduce a basis $\{e_i\}$ with Clifford algebraic relations $e_ie_j + e_je_i = 2\eta_{ij}$ and the above notations for $m$ and $q$, one obtains four different equations:

$$
\begin{align*}
\{2\} & \quad \nabla \Psi_{(2)}e_{21} + qA\Psi_{(2)} + m\Psi_{(2)}e_0 = 0 \quad e_+^+ \\
\{0\} & \quad -\nabla \Psi_{(0)}e_{21} + qA\Psi_{(0)} + m\Psi_{(0)}e_0 = 0 \quad e_-^+ \\
\{3\} & \quad \nabla \Psi_{(3)}e_{21} - qA\Psi_{(3)} + m\Psi_{(3)}e_0 = 0 \quad e_-^- \\
\{1\} & \quad -\nabla \Psi_{(1)}e_{21} - qA\Psi_{(1)} + m\Psi_{(1)}e_0 = 0 \quad e_+^-.
\end{align*}
$$

(15)

In the second column we give the identification –due to Parra– with “particles” associated with the corresponding equations. $±$ indicates electron or positron where $↑↓$ indicates spin up or down –this is a choice, one might exchange the meanings. The second of these equations –Parra option $\{2\}$– happens to be the Dirac-Hestenes equation [10] if we identify the $\{e_i\}$ and $\{\gamma_{\mu}\}$ bases, which thereby includes the spin explicitly. The other three equations are new. Even if they are similar in structure one is not able to remove the relative changes in sign if two or more of these equations are considered at the same time. Once more, we see the right action of the spin-bivector $e_{21}$ and of the velocity vector $e_0$. One should note, that proceeding from Dirac theory to quantum electrodynamics (QED), it became necessary to introduce particle and antiparticle creation and annihilation operators for each spin polarisation. While in QED the formalism takes care of the different types of spinors, a simple complex linear combination –as quite common in Dirac matrix theory!– intermingles the different Parra options without any chance to re-obtain them as different equations.

The Parra spinors can easily be put within a quaternion basis. Let $1, i_k := i e_k$ be a quaternion basis, then the spinors of $r$-option become $\Psi$ the velocity vector $e$ tum electrodynamics (QED), it became necessary to introduce particle and antiparticle without any chance to re-obtain them as different equations.

\[\begin{pmatrix}
\alpha + iB_3 \\
-B_2 + iB_1 \\
E_3 + i\lambda \\
E_1 + iE_2 \\
E_1 - iE_2
\end{pmatrix}, \quad \begin{pmatrix}
\alpha - iB_3 \\
B_2 + iB_1 \\
E_3 + i\lambda \\
E_1 + iE_2 \\
E_1 - iE_2
\end{pmatrix}, \quad \begin{pmatrix}
-B_3 + i\lambda \\
E_3 + i\lambda \\
B_2 + iB_1 \\
\alpha - iB_2
\end{pmatrix}, \quad \begin{pmatrix}
-B_2 + iB_1 \\
B_3 + i\lambda \\
E_1 + iE_2 \\
\alpha + iB_3
\end{pmatrix}.
\]

(14)

If we now introduce a basis $\{e_i\}$ with Clifford algebraic relations $e_ie_j + e_je_i = 2\eta_{ij}$ and the above notations for $m$ and $q$, one obtains four different equations:

\[\begin{align*}
\{2\} & \quad \nabla \Psi_{(2)}e_{21} + qA\Psi_{(2)} + m\Psi_{(2)}e_0 = 0 \quad e_+^+ \\
\{0\} & \quad -\nabla \Psi_{(0)}e_{21} + qA\Psi_{(0)} + m\Psi_{(0)}e_0 = 0 \quad e_-^+ \\
\{3\} & \quad \nabla \Psi_{(3)}e_{21} - qA\Psi_{(3)} + m\Psi_{(3)}e_0 = 0 \quad e_-^- \\
\{1\} & \quad -\nabla \Psi_{(1)}e_{21} - qA\Psi_{(1)} + m\Psi_{(1)}e_0 = 0 \quad e_+^-.
\end{align*}
\]

(15)

In the second column we give the identification –due to Parra– with “particles” associated with the corresponding equations. $\pm$ indicates electron or positron where $↑↓$ indicates spin up or down –this is a choice, one might exchange the meanings. The second of these equations –Parra option $\{2\}$– happens to be the Dirac-Hestenes equation [10] if we identify the $\{e_i\}$ and $\{\gamma_{\mu}\}$ bases, which thereby includes the spin explicitly. The other three equations are new. Even if they are similar in structure one is not able to remove the relative changes in sign if two or more of these equations are considered at the same time. Once more, we see the right action of the spin-bivector $e_{21}$ and of the velocity vector $e_0$. One should note, that proceeding from Dirac theory to quantum electrodynamics (QED), it became necessary to introduce particle and antiparticle creation and annihilation operators for each spin polarisation. While in QED the formalism takes care of the different types of spinors, a simple complex linear combination –as quite common in Dirac matrix theory!– intermingles the different Parra options without any chance to re-obtain them as different equations.

The Parra spinors can easily be put within a quaternion basis. Let $1, i_k := i e_k$ be a quaternion basis, then the spinors of $r$-option become $\Psi = q^r + i q^r$ where $\vec{m}$ means quaternion conjugation. Since Hestenes spinors are elements of $\mathbb{C}^{1,3} \supset \mathbb{C}^{1,3} \simeq \mathbb{M}_2(\mathbb{H})$, this can be extended to matrix spinors

\[\Psi_{\{r\}} = \begin{pmatrix}
q^r_1 & -q^r_2 \\
q^r_2 & q^r_1
\end{pmatrix}.
\]

(16)

The $2 \times 2$ matrix structure is a matrix representation of the complex structure $(1, i)$. Since the Hestenes equation is formulated within abstract algebra and not within a representation it is trivially representation independent. But a change of bases has to be not only an algebra isomorphism but moreover a Clifford algebra isomorphism. Only elements of the Clifford-Lipschitz group $\Gamma_{1,3}$ induce such transformations. Denoting the group of even such elements as $\Gamma_{1,3}^+$, we expect the quotient $D = \Gamma_{1,3}/\Gamma_{1,3}^+$ to be exactly the discrete group of transformations which connect the Parra options. Such transformations are beside the identity space inversion, charge conjugation and time reversal.

We would thus submit, that the spin Clifford bundle defined by Rodrigues et al. [Rodrigues et al. 1996] is a slightly to large structure, since it does not properly distinguish the different particle types of Parra. The “spin-particle” Clifford bundle should consist of equivalence classes of idempotents with respect to an even geometrical equivalence relation. The commutator relation and thus the Clifford structure can be seen
to be invariant under discrete –or more generally odd– transformation of the Clifford-
Lipschitz group [Crumeyrolle 1990].

Since we have thus established the equivalence of Parra’s equations –and the spin
Clifford bundle– in essence to the Hestenes formulation, we concentrate now on the
connection of Hestenes’ and Daviau’s space Clifford algebraic formulations. The Daviau
space Clifford algebra form of Dirac’s equation will correspond directly to Parra option {1} as will be shown below.

2.4 Equivalence of space Clifford and Hestenes formulation

We will calculate the action of the outer automorphism within the even algebra. Therefore we compare the $\gamma_0$ action with the action of $*$ introduced in (3) on the Daviau
spinor (4). Observe the relation:

$$
\phi_D^* = \sigma_2 \phi_D \sigma_2 = \begin{pmatrix}
  a - c + i(g - h) & -d + f + i(e + b) \\
  f - d + i(b - e) & a + c + i(-h - g)
\end{pmatrix}
= a \mathbb{1} - d\sigma_1 - b\sigma_2 - c\sigma_3 + e\sigma_1 - f\sigma_2 + g\sigma_3 - hi.
$$

(17)

Now, let us use the injection $\sigma_i \mapsto \sigma_i \otimes \mathbb{1}$, which gives a $4 \times 4$ representation of the space Clifford algebra, we are able to introduce a $\gamma_0$ in this representation, thereby identifying $\Sigma$ and $\sigma$ elements. However, this is no longer an element of the space Clifford algebra. We can calculate

$$
\gamma_0 \phi_D^* \gamma_0 = a \mathbb{1} + d\sigma_1 + b\sigma_2 + c\sigma_3 + e\sigma_1 - f\sigma_2 + g\sigma_3 + hi
= \phi_D = \phi_D^*,
$$

(18)

by comparing with (3). This might be rewritten as

$$
\phi_D^* = \gamma_0 \phi_D \gamma_0
$$

(19)

and used in the rewriting of the Dirac-Hestenes equation [12] which then yields the
Pauli or space Clifford algebraic equation

$$
\Sigma_\mu \partial_\mu \Psi_H i\Sigma_3 = -m \Psi_H^* + q \Sigma_\mu A_\mu \Psi_H.
$$

(20)

To obtain the full equivalence between this formulation of the Dirac-Hestenes theory to
the space Clifford algebraic version of Daviau, we have to perform two further steps.

The first is to explain the additionally minus sign in front of the mass term. Re-
defining the sign of charge and angular momentum measurement, i.e. $e \mapsto -e$, $h \mapsto -h$, results in the appropriate change. Of course, from a particle point of view this two
particles are not identical. They have a relation as a spin up electron to a spin down
positron and do correspond to different types of Parra options in rewriting Hestenes’
theory [Parra 1989]. Since no weak interactions are involved here, one can physically
not distinguish these options and there is no harm in this settings. However, one should
note that Daviau got four different equations within his calculations, and there may be
the chance that one of them fit exactly to Hestenes theory without changing the sign of
the mass term.

The second step is a relabelling of base elements in a cyclic way. This can be done by defining

$$
z : \sigma \mapsto \Sigma
z(\mathbb{1}) = \mathbb{1}
z(\sigma_i) = \Sigma_{i-1} \text{ cyclic.}
$$

(21)
The map $z$ can be extended as an outer-morphism, that is a grade preserving extension \cite{Hestens et al. 1984}, to the whole algebra by setting $z(\sigma_i \sigma_j) = z(\sigma_i) z(\sigma_j)$ etc. Since $z$ is a cyclic permutation, we have $z^3 = 1$ and $z^{-1} = z^2$. It is crucial to note, that even if in the definition of the $*$ morphism in \cite{Hestens et al. 1984} via complex conjugation followed by a transformation with $\sigma_3$, $*$ is not inner, it commutes with $z$. That is we have $z(\phi^*) = z(\phi)^*$. We obtain the following isomorphism noticing from \cite{Hestens et al. 1984} and \cite{Hestens et al. 1984} that $z^{-1}(\Psi_H) = \phi_D$ holds:

\[ \Sigma_\mu \partial_\mu \Psi_H i\Sigma_3 = -m\Psi_H^* + q\Sigma_\mu A_\mu \Psi_H \] (22)

acting by $m \mapsto -m$ and $z^{-1}$ results in

\[ (\sigma_0 \partial_0 + \sigma_2 \partial_1 + \sigma_3 \partial_2 + \sigma_1 \partial_3)\phi_D i\sigma_1 = m\phi_D^* + q(\sigma_0 A_0 + \sigma_2 A_1 + \sigma_3 A_2 + \sigma_1 A_3)\phi_D \] (23)

which results with \cite{Hestens et al. 1984} in

\[ \nabla \phi_D i\sigma_1 = m\phi_D^* + qA\phi_D. \] (24)

This proves the equivalence of Daviau’s space Clifford algebraic and Hestenes’ formulation of Dirac’s theory.

3 Related work

There seems to be a notorious revival of the transition between spinor and tensor descriptions of Dirac theory. As we mentioned Darwin and Madelung, there are far more also recent such approaches of which we will mention only two more. Based on ideas of Sallhofer \cite{Sallhofer 1991}, Simulik et al. \cite{Simulik et al. 1998} used extensively a spinor–tensor transition, called there Maxwell–Dirac isomorphism, in applications and some theoretical investigations. Since their formalism is a restriction of the approach developed by Parra, however not so detailed and pedagogical, we have nothing more to prove there. Our preference is however not intended to provide any priority claims.

A detailed thoughtful description of geometric electron theory with many citations and critical remarks can be found in Keller \cite{Keller 1993}. A further genuine and important approach to the spinor-tensor transition was developed starting probably with Crawford by P. Lounesto, \cite{Lounesto 1997} and references there. He investigated the question, how a spinor field can be reconstructed from known tensor densities. The major characterization is derived, using Fierz-Kofink identities, from elements called Boomerangs—because they are able to come back to the spinorial picture. Lounesto’s result is a characterization of spinors based on multi-vector relations which unveils a new unknown type of spinor.

However, we want to submit, that even the notion of a multi-vector is quite questionable in Dirac theory \cite{Fauser 1998} and in general \cite{Fauser 1999d}. The $\mathbb{Z}_n$-grading used to define multi-vectors is not a feature of Clifford algebra. One expects very different spinor structures if different $\mathbb{Z}_n$-gradings are properly implemented \cite{Fauser et al. 1999b, Fauser 1997}.

4 Conclusion

In this paper we discussed the isomorphism between spinor and multi-vector formulation of Dirac theory. We proved the equivalence of Daviau’s space Clifford algebraic and Hestenes’ operator spinor formulations of Dirac theory as their equivalence to different special options of Parra’s treatment. The important observation is, that in usual
formulations the spinor representations are made up from left actions, while Daviau’s formulation requires the bi-module structure of left and right actions. A detailed mathematical analysis of this fact will be given elsewhere [Fauser 1999a]. Regarding iso-spin, which was sometimes introduced as right action, our analysis shows that one should be very careful in doing so.

A further remarkable fact is that the Daviau spinor is of the most general form – most general element in the algebra – and utilizes the full Pauli algebra as representation space. This should be compared with the Hestenes even operator spinors and ideal or column spinors which span the representation space but not the algebra itself. It is peculiar at this point carefully to distinguish representations and abstract algebra. In this sense, Daviau’s formulation is the most compact formulation which can be found.

However, we gave some references which critically discussed the concept of multi-vectors or $\mathbb{Z}_n$-gradings in Clifford algebras. One knows that different $\mathbb{Z}_n$-gradings can produce quite different spinor modules. This fact renders the unquestioned multi-vector structure as a peculiar one. A careful study of the representation theory and their dependence on gradings in such cases is required.

Acknowledgements

This work was supported by the Deutsche Forschungsgemeinschaft DFG providing a travel grant to Zacatecas and Ixtapa Conferences in Mexico, June/July 1999. Suggestions provided by C. Daviau are gratefully acknowledged.

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