MODULI SPACES OF SHEAVES OVER NON-PROJECTIVE K3 SURFACES

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Abstract. We study moduli spaces of sheaves over non-projective K3 surfaces. More precisely, let \( \omega \) be a Kähler class on a K3 surface \( S \), \( r \geq 2 \) an integer, \( L \) a line bundle on \( S \). If \( \omega \) is generic, \( r \) is prime with \( L \) and the moduli space of \( \mu_\omega \)-stable sheaves of rank \( r \), determinant \( L \) and fixed second Chern class is not empty, then we show that it is an irreducible holomorphically symplectic manifold which is deformation equivalent to a Hilbert scheme of points on a projective K3 surface.

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1. Introduction

Moduli spaces of sheaves on projective K3 surfaces have been studied since the '80s. In [8] Fujiki considered the Hilbert scheme $Hilb^2(S)$ of 2 points on a K3 surface $S$; his result was widely generalized by Beauville in [3], who studied $Hilb^n(S)$ for any $n \in \mathbb{N}$, showing that it is an irreducible holomorphically symplectic manifold, i.e. a compact Kähler manifold which is simply connected, holomorphically symplectic and has $h^{2,0} = 1$.

Moduli spaces of $\mu$–stable sheaves are a generalization of Hilbert schemes of points, and they have been extensively studied when the base surface $S$ is a projective K3 surface. In [23] Mukai showed that on the moduli space $M$ of simple sheaves of Mukai vector $v = (r, c_1(L), a)$ (i.e. of rank $r$, determinant $L$ and second Chern character $a - r$), there is a natural holomorphic symplectic form associated to the one on $S$. This $M$ is a non-separated scheme containing as a smooth open subset the moduli space $M^\mu(S, H)$ of $\mu_H$–stable sheaves (with respect to some ample line bundle $H$ on $S$) of Mukai vector $v$; Mukai’s construction thus produces a holomorphic symplectic form on $M^\mu(S, \omega)$.

If $H$ is generic and $r$ and $L$ are prime to each other, then $M^\mu(S, H)$ is a projective holomorphically symplectic manifold. Moreover, it is an irreducible holomorphically symplectic manifold deformation equivalent to a Hilbert scheme of points on $S$ (see [25] and [37]).

If $S$ is a non-projective K3 surface and $\omega$ is a Kähler class on it, one still defines the notion of $\mu_\omega$–stable sheaf and constructs the moduli space $M^\mu(S, \omega)$ of $\mu_\omega$–stable sheaves of Mukai vector $v$. In [31] it is shown that $M^\mu(S, \omega)$ is a smooth complex manifold carrying a holomorphic symplectic form. If $\omega$ is generic and $r$ is prime with $c_1(L)$, then $M^\mu(S, \omega)$ is even compact (see subsection 2.2 for the precise notion of genericity we use for Kähler classes, which we call $v$–genericity in analogy to the projective case).

It is then natural to ask if $M^\mu(S, \omega)$ is an irreducible holomorphically symplectic manifold, and in this case what is its deformation class. The aim of the present paper is to answer these questions. More precisely, we show the following:

Theorem 1.1. Let $S$ be a K3 surface, $v = (r, \xi, a) \in H^*(S, \mathbb{Z})$ where $\xi \in NS(S)$, $r > 1$ is prime with $\xi$ and $v^2 \geq 0$.

1. If $\omega$ is a $v$–generic Kähler class on $S$, then $M^\mu(S, \omega)$ is an irreducible holomorphically symplectic manifold of dimension $v^2 + 2$ which is deformation equivalent to a Hilbert scheme of points on a projective K3 surface.
The moduli space \( M^\mu_v(S, \omega) \) is naturally endowed with a Kähler metric which is compatible with the holomorphic symplectic structure and which extends the Weil-Petersson metric from the open subset of \( M^\mu_v(S, \omega) \) parametrizing locally free sheaves.

The condition \( v^2 \geq 0 \) implies even \( M^\mu_v(S, \omega) \neq \emptyset \) (see [2], [28], [18]). As recalled above, this result is known to hold if \( S \) is projective and \( \omega = c_1(H) \) for a generic ample line bundle \( H \). To prove the theorem, we study the two remaining cases: projective K3 surfaces with \( \omega \notin NS(S) \), and non-projective K3 surfaces.

We first study the case of a projective surface \( S \) on which we choose a Kähler class \( \omega \) which is not the first Chern class of an ample line bundle: we show that there is a \( v \)-generic ample line bundle \( H \) such that \( M^\mu_v(S, \omega) = M^\mu_v(S, H) \). This is done by showing that the \( v \)-chamber in which \( \omega \) lies always intersects the ample cone, and that moving the polarization inside a \( v \)-chamber does not affect the moduli space.

We then turn to the case of a non-projective surface \( S \), where the main problem is to prove that \( M^\mu_v(S, \omega) \) is Kähler. We prove the existence of the desired Kähler metric on \( M^\mu_v(S, \omega) \) by using a result of Hitchin, Karlhede, Lindström and Roček (in [11]): roughly speaking, they show that hyperkähler manifolds are characterized by the existence of a twistor space. This result was used by Tyurin in [34] to show that the moduli space of \( \mu_\omega \)-stable sheaves with trivial determinant is Kähler: the twistor space of \( M^\mu_v(S, \omega) \) is the relative moduli space of stable sheaves over the twistor line of \((S, \omega)\).

In the cases we consider, the determinant of the sheaves is not trivial, hence in general they do not deform along a twistor line. To avoid this problem, we pass to twisted sheaves: after recalling the notion of stability for twisted sheaves (for projective manifolds this was already introduced by Yoshioka in [38] and Lieblich in [19]) we show that the relative moduli space of stable twisted sheaves along the twistor line of \( S \) given by \( \omega \) is a twistor space for \( M^\mu_v(S, \omega) \).

The connectedness of \( M^\mu_v(S, \omega) \) is proved by showing that we can find a finite sequence of twistor lines connecting the surface \( S \) (with polarization \( \omega \)) to a projective K3 surface \( S' \) (with a \( v \)-generic polarization \( \omega' \)), along which the moduli space \( M^\mu_v(S, \omega) \) deforms (in a smooth and proper family) to \( M^\mu_v(S', \omega') \), which is connected.

To conclude, we present a description of the Hodge and lattice structures on \( H^2(M^\mu_v, \mathbb{Z}) \), generalizing to non-projective surfaces the results which are available for projective surfaces (see [24], [25], [37]). As a corollary of this description, we show that \( M^\mu_v(S, \omega) \) is projective if and only if the base surface \( S \) is projective.
2. The moduli spaces of stable sheaves

In the following $S$ will be a K3 surface, possibly non-projective. If $\mathcal{F}$ is a coherent sheaf on $S$, we let the Mukai vector of $\mathcal{F}$ be

$$v(\mathcal{F}) := \text{ch}(\mathcal{F}) \cdot \sqrt{td(S)} \in H^{2*}(S, \mathbb{Z}).$$

Then if $v_i(\mathcal{F})$ is the component of $v(\mathcal{F})$ in $H^{2i}(S, \mathbb{Z})$, we have $v_0(\mathcal{F}) = \text{rk}(\mathcal{F})$, $v_1(\mathcal{F}) = c_1(\mathcal{F})$ and $v_2(\mathcal{F}) = \text{ch}_2(\mathcal{F}) + \text{rk}(\mathcal{F})$, which will be viewed as an integer (i.e. we fix an isomorphism $H^4(S, \mathbb{Z}) \cong \mathbb{Z}$).

We recall that on $H^{2*}(S, \mathbb{Z})$ we have a pure weight-two Hodge structure and a lattice structure with respect to the Mukai pairing (see Definitions 6.1.5 and 6.1.11 of [14]): the obtained lattice will be referred to as Mukai lattice, and we will write $v^2$ for the square of $v \in H^{2*}(S, \mathbb{Z})$ with respect to the Mukai pairing. Explicitly,

$$v^2 = v_1^2 - 2v_0v_2.$$

When $v_0 \neq 0$ we define the discriminant of $v$, or respectively of $\mathcal{F}$ in case $v = v(\mathcal{F})$, as

$$\Delta(v) := \frac{1}{2v_0^2}v^2 + 1,$$

This coincides with the definition of [2] for instance, where

$$\Delta(\mathcal{F}) = \Delta(v(\mathcal{F})) = \frac{1}{\text{rk}(\mathcal{F}) - 1} \left( c_2(\mathcal{F}) - \frac{\text{rk}(\mathcal{F}) - 1}{2\text{rk}(\mathcal{F})}c_1^2(\mathcal{F}) \right).$$

2.1. The stability condition. Let $g$ be a Kähler metric on $S$ and $\omega$ the associated Kähler class, that will be called a polarization on $S$. If $\mathcal{F} \in \text{Coh}(S)$ has positive rank, the slope of $\mathcal{F}$ with respect to $\omega$ is

$$\mu_\omega(\mathcal{F}) := \frac{c_1(\mathcal{F}) \cdot \omega}{\text{rk}(\mathcal{F})}.$$

Definition 2.1. A torsion-free coherent sheaf $\mathcal{F}$ is $\mu_\omega$–stable if for every coherent subsheaf $\mathcal{E} \subseteq \mathcal{F}$ such that $0 < \text{rk}(\mathcal{E}) < \text{rk}(\mathcal{F})$ we have $\mu_\omega(\mathcal{E}) < \mu_\omega(\mathcal{F})$. If $\mu_\omega(\mathcal{E}) \leq \mu_\omega(\mathcal{F})$ for all such subsheaves $\mathcal{E}$, then we say that $\mathcal{F}$ is $\mu_\omega$–semistable.

The family of $\mu_\omega$–stable sheaves of Mukai vector $v$ admits a moduli space $M^v_\mu(S, \omega)$. If $S$ is projective and $\omega$ is the first Chern class of an ample line bundle $H$, then $M^v_\mu(S, H)$ is the moduli space $M^v_\mu(S, H)$ of $\mu_H$–stable sheaves on $S$ with Mukai vector $v$. We have the following proposition dealing also with the non-projective case (see [31]).

Proposition 2.2. Let $S$ be a K3 surface, $v \in H^{2*}(S, \mathbb{Z})$ a Mukai vector and $\omega$ a polarization on $S$. The moduli space $M^v_\mu(S, \omega)$ is a smooth, holomorphically symplectic manifold (possibly non-compact) and, if it is not empty, its dimension is $v^2 + 2$. 
In the following we will restrict to the case of those $M_\mu^v(S,\omega)$ which are non-empty and compact. We introduce in the next section some hypothesis on $v$ and $\omega$ under which $M_\mu^v(S,\omega)$ is compact. We now present a condition which guarantees the non-emptiness, and even the existence of a stable vector bundle with respect to any polarization.

Over any non-algebraic surface there exist non-filtrable holomorphic rank two vector bundles (see [2], [29] p.18). By definition they do not admit coherent subsheaves of rank one, hence they are stable with respect to any polarization.

We now extend this type of result to arbitrary rank in the case of Kähler surfaces. Following [2] we say that a coherent sheaf on the surface $S$ is irreducible if its only coherent subsheaf of lower rank is the zero sheaf. In particular, an irreducible sheaf is stable with respect to any polarization. We have the following result, about the existence of locally free irreducible vector bundles.

**Proposition 2.3.** Let $S$ be a Kähler non-algebraic compact complex surface, $r$ a positive integer and $\xi \in NS(S)$. Then there exists a bound $b := b(r,\xi) \in \mathbb{Z}$ depending on $r$ and on $\xi$ such that for any integer $c \geq b$ there is on $S$ an irreducible locally free sheaf $\mathcal{F}$ of rank $r$, $c_1(\mathcal{F}) = \xi$ and $c_2(\mathcal{F}) = c$.

**Proof.** If $r = 2$, a statement of this type is proved in [2] and in [29] without the Kähler assumption. The idea there was to look at the versal deformation space of a rather arbitrary coherent sheaf $\mathcal{F}$ and show that if $c_2 \gg 0$ then $\mathcal{F}$ must contain irreducible objects. For $r > 2$ we shall this time consider deformations of suitably chosen coherent sheaves and make essential use of the fact that $S$ is Kähler. In this way we shall reduce ourselves to the argument used by Bănică and Le Potier in the case when the algebraic dimension of $S$ is zero, [2, Théorème 5.3].

We proceed by induction on $r$. The statement is trivial for $r = 1$ and already proven for $r = 2$. Let then $r \geq 3$ and suppose that the statement is true for rank $r - 1$. Take an irreducible locally free sheaf $\mathcal{E}$ on $S$ of rank $r - 1$, $c_1(\mathcal{E}) = \xi$ and $c_2(\mathcal{E}) = c$. Consider an irreducible component $B$ of the versal deformation space of $\mathcal{F}_0 := \mathcal{O}_S \oplus \mathcal{E}$ and the corresponding family $\mathcal{F}$ of coherent sheaves over $S \times B$.

We show that if $c \gg 0$, the relative Douady space $D_{(X \times B)/B}(\mathcal{F}, k)$ of flat quotients of rank $k$ of $\mathcal{F}$ over $B$ does not cover $B$ for $1 \leq k \leq r - 1$. Let $b : D_{(X \times B)/B}(\mathcal{F}, k) \to B$ be the natural morphism and $Q \subset B$ a relatively compact subdomain of $B$ containing the origin $0 \in B$. Fujiki proved in [9] that any irreducible component of $b^{-1}(Q)$ is proper over
Q. By another result of Fujiki in [7], there are countably many such components.

In particular very general neighbours of $\mathcal{F}_0$ are not in the image of $D_{(X \times B)/B}(\mathcal{F}, k)$ for $2 \leq k \leq r - 2$. Remark that if $\mathcal{F}_b$ is such a neighbour sitting in a short exact sequence

$$0 \to F' \to \mathcal{F}_b \to F'' \to 0$$

with $F''$ torsion-free, then $F'$ and $F''$ are irreducible of different ranks, hence $\text{Hom}(F', F'') = 0 = \text{Hom}(F'', F')$. This remark makes the arguments in the proof of [2, Théorème 5.3] work by replacing the corresponding inequality in loc. cit. Lemme 5.12. Hence our statement. □

2.2. The $v$–genericity for Kähler forms. Let $S$ be a K3 surface and $\mathcal{H}_S$ its Kähler cone, which is an open and convex cone in $H^{1,1}(S)$. For $v = (r, \xi, a)$ with $r \geq 2$ and $\xi \in NS(S)$, we define a system of hyperplanes in $H^{1,1}(S)$, which is locally finite in $\mathcal{H}_S$ and has the property that for any $\omega \in \mathcal{H}_S$ not lying on such hyperplanes, a torsion free sheaf $\mathcal{F}$ on $S$ with $v(\mathcal{F}) = v$ is $\mu_\omega$-stable if and only if it is $\mu_\omega$-semistable. Polarizations verifying this will be called $v$–generic.

2.2.1. The notion of $v$–genericity. We set

$$W_v := \{ D \in NS(S) \mid -\frac{r^4}{2} \Delta(v) \leq D^2 < 0 \}$$

and for every $\alpha \in H^{1,1}(S)$ we write

$$\alpha^\perp := \{ \beta \in H^{1,1}(S) \mid \alpha \cdot \beta = 0 \}.$$

When $\alpha \neq 0$, the set $\alpha^\perp$ is a hyperplane in $H^{1,1}(S)$. Using the same argument of Lemma 4.C.2 of [14], one shows that if $\beta \in H^{1,1}(S)$, then there is an open neighbourhood $U$ of $\beta$ in $H^{1,1}(S)$ such that $U \cap D^\perp \neq \emptyset$ for at most a finite number of $D \in W_v$.

Definition 2.4. For every $D \in W_v$, the hyperplane $D^\perp \cap \mathcal{H}_S$ will be called $v$–wall in the Kähler cone of $S$. A connected component of $\mathcal{H}_S \setminus \bigcup_{D \in W_v} D^\perp$ is an open convex cone called $v$–chamber in the Kähler cone of $S$. A Kähler class in a $v$–chamber of $\mathcal{H}_S$ is called $v$–generic polarization.

The ample cone of $S$ is $\text{Amp}(S) = \mathcal{H}_S \cap NS_\mathbb{R}(S)$ (where $NS_\mathbb{R}(S) = NS(S) \otimes \mathbb{R}$); if $\mathcal{C} \subseteq \mathcal{H}_S$ is a $v$–chamber in the Kähler cone of $S$, then $\mathcal{C} \cap NS_\mathbb{R}(S)$ is a $v$–chamber in the ample cone of $S$ in the usual terminology: if $H$ is an ample line bundle on $S$, then $c_1(H)$ is a $v$–generic polarization if and only if $H$ is $v$–generic as in [13].
2.2.2. Compactness of $M_{\mu}^v(S, \omega)$ when $\omega$ is $v$–generic. Using the same proof as in the projective case (see Theorem 4.C.3 of [14]), we show that $v$–generic polarizations enjoy the above stated property concerning the existence of properly semistable sheaves.

**Lemma 2.5.** Let $\omega$ be a Kähler class on a K3 surface $S$, and $F$ a $\mu_\omega$–semistable sheaf of Mukai vector $v = (r, \xi, a)$. Suppose that there is $E \subseteq F$ of rank $0 < s < r$, first Chern class $\zeta$ and such that $\mu_\omega(E) = \mu_\omega(F)$. Then $D := r\zeta - s\xi$ is such that

$$-\frac{r^4}{2} \Delta(v) \leq D^2 \leq 0,$$

and $D^2 = 0$ if and only if $D = 0$.

**Proof.** We can suppose that $E$ is saturated, so that $G := F/E$ is torsion free, $\mu_\omega$–semistable and of rank $r - s$. Notice that as $\mu_\omega(E) = \mu_\omega(F)$, we have $D \cdot \omega = 0$. As $\omega$ is a Kähler class, from the Hodge Index Theorem we then have $D^2 \leq 0$, and $D^2 = 0$ if and only if $D = 0$. We then just need to show that $D^2 \geq -\frac{r^4}{2} \Delta(v)$.

By definition of the discriminant, it follows that

$$\Delta(F) - \frac{s}{r} \Delta(E) - \frac{r - s}{r} \Delta(G) = -\frac{D^2}{2s(r - s)r^2}.$$

Now, recall that the Bogomolov inequality is surely satisfied by $E$ and $G$, so that $\Delta(E), \Delta(G) \geq 0$. But this implies that

$$-D^2 \leq 2s(r - s)r^2 \Delta(F) = 2s(r - s)r^2 \Delta(v) \leq \frac{r^4}{2} \Delta(v),$$

and we are done. \qed

Using the main result of [31] we then get the following:

**Proposition 2.6.** Let $r \geq 2$ an integer and $\xi \in NS(S)$ such that $(r, \xi) = 1$. Let $a \in \mathbb{Z}$, $v := (r, \xi, a)$ and $\omega$ a $v$–generic polarization. If $M_{\mu}^v(S, \omega) \neq \emptyset$, then it is a smooth, compact, holomorphically symplectic manifold.

**Proof.** The statement follows from the main result of [31] if $S$ is non-algebraic. When $S$ is projective we shall show in 3.1 that there exists some integer ample class $H$ in the same $v$–chamber as $\omega$. The (semi)stability with respect to $\omega$ or with respect to $H$ will then come down to the same thing and $M_{\mu}^v(S, \omega)$ will coincide with the Gieseker moduli space $M_v(S, H)$ of $H$–semistable sheaves, which is known to be smooth, projective and holomorphically symplectic (see [14]). \qed
3. Projective K3 surfaces with non-ample polarizations

In this section we prove that if $S$ is a projective K3 surface, $v = (r, \xi, a)$ is a Mukai vector with $(r, \xi) = 1$ and $\omega$ is a $v$–generic polarization, then $M_v^{\mu}(S, \omega)$ is an irreducible holomorphically symplectic manifold, deformation equivalent to a Hilbert scheme of points on $S$.

The strategy of the proof is the following: we first show that any $v$–chamber in $\mathcal{K}_S$ intersects $\text{Amp}(S)$ along a $v$–chamber. Then we show that $M_v^{\mu}(S, \omega)$ does not depend on $\omega$, but only on the $v$–chamber in which $\omega$ lies.

3.1. Chambers in the Kähler cone and in the ample cone. We start by showing that if $C$ is a $v$–chamber in $\mathcal{K}_S$, then $C \cap \text{Amp}(S) \neq \emptyset$. This is done in three steps: first we show that any intersection of $v$–walls in $\mathcal{K}_S$ has to intersect the ample cone; then we show that an intersection of $v$–walls in $\mathcal{K}_S$ cannot be contained in $\text{Amp}(S)$; finally we show the claim.

**Proposition 3.1.** Let $S$ be a projective K3 surface, $v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z})$ be such that $r \geq 2$ and $\xi \in NS(S)$, and let $D_1, \ldots, D_n \in W_v$. If $(D_1^+ \cap \ldots \cap D_n^+) \cap \mathcal{K}_S \neq \emptyset$, then $(D_1^+ \cap \ldots \cap D_n^+) \cap \text{Amp}(S) \neq \emptyset$.

**Proof.** Let $A := D_1^+ \cap \ldots \cap D_n^+$, and suppose that $A \cap \mathcal{K}_S \neq \emptyset$. We may suppose that $D_1, \ldots, D_n$ are linearly independent. If $\rho$ is the Picard number of $S$, we then have $n \leq \rho$.

If $n = \rho$ then $D_1, \ldots, D_n$ form a basis of $NS_R(S)$. If $H$ is an ample line bundle on $S$, there are then $\lambda_i \in \mathbb{R}$ such that $c_1(H) = \sum_{i=1}^{n} \lambda_i D_i$. As $A \cap \mathcal{K}_S \neq \emptyset$, there is $\omega \in \mathcal{K}_S$ such that $\omega \cdot D_i = 0$ for every $i = 1, \ldots, n$. Hence $\omega \cdot H = 0$, which is impossible as $\omega$ is a Kähler form and $H$ is ample. It follows that $n < \rho$, meaning that $A' := A \cap NS_R(S)$ is a linear subspace of $NS_R(S)$ of dimension $\rho - n$.

Suppose now $A \cap \text{Amp}(S) = \emptyset$, which means $A' \cap \text{Amp}(S) = \emptyset$. Let $\beta := \sum_{i=1}^{n} b_i D_i$ for some $b_1, \ldots, b_n \in \mathbb{R}$, and $B := \beta^\perp$, so that $A' \subset B$. Moreover, let

$$B_+ := \{ \alpha \in (NS(S) \otimes \mathbb{R}) \setminus B \mid \beta \cdot \alpha > 0 \}.$$ 

As $A' \cap \text{Amp}(S) = \emptyset$ and $\text{Amp}(S)$ is an open, convex cone in $NS_R(S)$, we can choose $\beta$ so that $\text{Amp}(S) \subseteq B_+$. This means that for every ample line bundle $H$ on $S$ we have $\beta \cdot c_1(H) > 0$, so that $\beta$ is a pseudo-effective class.

Now, let

$$\tilde{B} := \{ \alpha \in H^{1,1}(S) \mid \beta \cdot \alpha = 0 \},$$

which is a hyperplane in $H^{1,1}(S)$ such that $\tilde{B} \cap NS_R(S) = B$, and

$$\tilde{B}_+ := \{ \alpha \in H^{1,1}(S) \mid \beta \cdot \alpha > 0 \},$$
Lemma 3.2. Let \( A \) be a projective K3 surface with \( \text{NS}_R(S) \neq H^{1,1}(S) \). Let \( v = (r, \xi, a) \in H^2(S, \mathbb{Z}) \) be such that \( r \geq 2 \) and \( \xi \in \text{NS}(S) \), \( D_1, ..., D_n \in W_v \) and \( A := D_1^+ \cap ... \cap D_n^+ \). Suppose that \( A \cap \mathcal{K}_S \neq \emptyset \). Then the following equality of codimensions holds:

\[
\text{codim}_{H^{1,1}(S)} A = \text{codim}_{\text{NS}_R(S)} (A \cap \text{NS}_R(S)).
\]

In particular \( A \cap \mathcal{K}_S \) is not contained in \( \text{NS}_R(S) \).

Proof. If

\[
\text{codim}_{H^{1,1}(S)} A = \text{codim}_{\text{NS}_R(S)} (A \cap \text{NS}_R(S))
\]

holds and \( A \subseteq \text{NS}_R(S) \), then we will get \( \text{NS}_R(S) = H^{1,1}(S) \), which is excluded from the hypothesis.

To show the equality of codimensions, we proceed by induction on \( n \).

Suppose that \( n = 1 \): then \( \text{codim}_{H^{1,1}(S)} (D_1^+) = 1 \), and as \( D_1 \in \text{NS}(S) \), we have \( \text{codim}_{\text{NS}_R(S)} (D_1^+ \cap \text{NS}_R(S)) \leq 1 \). The codimension is 0 if and only if \( \text{NS}_R(S) \subseteq D_1^+ \), which is not possible as the intersection form on \( \text{NS}_R \) is non-degenerate.

Suppose now that \( A = D_1^+ \cap ... \cap D_n^+ \) is such that

\[
c := \text{codim}_{H^{1,1}(S)} A = \text{codim}_{\text{NS}_R(S)} (A \cap \text{NS}_R(S)).
\]

Then \( c \leq n \), and we can suppose that \( A = D_1^+ \cap ... \cap D_c^+ \).

Now, let \( D_{n+1} \in W_v \) be such that \( A \cap D_{n+1}^+ \neq 0 \). If the intersection of \( A \) and \( D_{n+1}^+ \) is not transversal, then \( A \cap D_{n+1}^+ = A \): in this case we also have \( A \cap D_{n+1}^+ \cap \text{NS}_R(S) = A \cap \text{NS}_R(S) \) so

\[
\text{codim}_{H^{1,1}(S)} (A \cap D_{n+1}^+) = \text{codim}_{\text{NS}_R(S)} (A \cap D_{n+1}^+ \cap \text{NS}_R(S))
\]

by the induction hypothesis.

If the intersection of \( A \) and \( D_{n+1}^+ \) is transversal, it follows that

\[
\text{codim}_{H^{1,1}(S)} (A \cap D_{n+1}^+) = c + 1.
\]

Hence, \( D_{n+1} \) is linearly independent of \( D_1, ..., D_c \): as these are all classes in \( \text{NS}_R(S) \), they are linearly independent in \( \text{NS}_R(S) \), hence \( \text{codim}_{\text{NS}_R(S)} (A \cap D_{n+1}^+ \cap \text{NS}_R(S)) = \text{codim}_{\text{NS}_R(S)} (A \cap \text{NS}_R(S)) + 1 = c + 1 \), since the intersection form on \( \text{NS}_R(S) \) is non-degenerate. \( \square \)
We are now ready to prove the main result of this section, which is the following:

**Proposition 3.3.** Let \( S \) be a projective K3 surface and \( v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z}) \) such that \( r \geq 2 \) and \( \xi \in NS(S) \). If \( C \) is a \( v \)-chamber in the Kähler cone of \( S \), then \( C \cap \text{Amp}(S) \neq \emptyset \).

**Proof.** The statement is clear if \( \rho < h^{1,1}(S) \). The chamber \( C \) is cut by linear inequalities defining a polyhedron \( P \) in \( H^{1,1}(S) \). Let \( F \) be the set of the faces of \( P \cap \mathcal{K}_S \), \( F' \subseteq F \) be the subset of faces of minimal dimension, and let \( A \in F' \).

Write \( A = D_1^+ \cap \ldots \cap D_n^+ \cap \mathcal{K}_S \) for \( D_1, \ldots, D_n \in W_v \). By minimality we can choose \( D_1, \ldots, D_n \) such that inside a neighbourhood \( U \) of \( A \) in \( \mathcal{K}_S \) the chamber \( C \) is the intersection of the half-spaces of type \( D_i^+ := \{ \omega \in H^{1,1}(S) \mid \omega D_i > 0 \} \). We may also assume that \( n \) is minimal, i.e. \( \text{codim}_{H^{1,1}(S)} \cap_i D_i^+ = n \).

By Proposition 3.1 there is \( \omega \in A \cap \text{Amp}(S) \), and by Lemma 3.2 we have \( \text{codim}_{NS(S)}(\cap_i (D_i^+ \cap NS(R(S))) = \text{codim}_{H^{1,1}(S)} \cap_i D_i^+ = n \). It follows that the \( D_i \)-s cut hyperplanes on a neighbourhood of \( \omega \) in \( \text{Amp}(S) \), whose equations on \( NS(R(S)) \) are linearly independent. Thus \( \text{Amp}(S) \cap \cap_i D_i^+ \cap U \neq \emptyset \). In particular \( \text{Amp}(S) \cap C \supset \text{Amp}(S) \cap C \cap U = \text{Amp}(S) \cap \cap_i D_i^+ \cap U \neq \emptyset \) and we are done. \( \square \)

**Remark 3.4.** The above arguments show in fact that every “chamber within an intersection of walls” of \( \mathcal{K}_S \) will intersect \( \text{Amp}(S) \).

### 3.2. Conclusion for Projective K3 Surfaces.

The following adaptation of Lemma 4.C.5 from [14] to the case of Kähler polarizations works also on Kähler manifolds; see [10] Lemma 6.2.

**Lemma 3.5.** Let \( \omega, \omega' \) be two Kähler classes on a compact Kähler manifold \( X \) and \( \mathcal{F} \) be a torsion free sheaf on \( X \) which is \( \mu_\omega \)-stable but not \( \mu_{\omega'} \)-stable. Denote by 

\[
[\omega, \omega'] := \{ \omega_t := t\omega' + (1-t)\omega \mid t \in [0, 1] \}
\]

the segment from \( \omega \) to \( \omega' \). Then there is a Kähler class \( \omega_t \in [\omega, \omega'] \) such that \( \mathcal{F} \) is properly \( \mu_{\omega_t} \)-semistable.

We now show that changing the polarization inside a \( v \)-chamber does not affect the moduli space. This is well-known for \( v \)-generic ample line bundles, and requires the same proof.

**Proposition 3.6.** Let \( S \) be a projective K3 surface and \( v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z}) \) such that \( r \geq 2 \) and \( \xi \in NS(S) \). Let \( C \) be a \( v \)-chamber in the Kähler cone of \( S \), and \( \omega, \omega' \in C \). Then \( M^\mu_v(S, \omega) = M^\mu_v(S, \omega') \).
Theorem 3.7. Let \( S \) be a projective K3 surface and \( v = (r, \xi, a) \in H^{2s}(S, \mathbb{Z}) \) such that \( r \geq 2, \xi \in NS(S) \) and \( (r, \xi) = 1 \). If \( \omega \) is \( v \)-generic and \( M^\mu_v(S, \omega) \neq \emptyset \), then \( M^\mu_v(S, \omega) \) is a projective irreducible holomorphically symplectic manifold which is deformation equivalent to a Hilbert scheme of points on \( S \).

Proof. Let \( \mathcal{C} \) be the \( v \)-chamber where \( \omega \) lies. By Proposition 3.3 we have \( \mathcal{C}' := \mathcal{C} \cap \text{Amp}(S) \neq \emptyset \). Hence, there is \( \omega' \in \mathcal{C} \cap \text{Amp}(S) \), and \( M^\mu_v(S, \omega) = M^\mu_v(S, \omega') \) by Proposition 3.6.

As \( \mathcal{C}' \) is open in \( \text{Amp}(S) \), there is \( \epsilon > 0 \) such that the ball \( B_\epsilon(\omega') \subseteq \text{Amp}(S) \) of radius \( \epsilon \) and centred at \( \omega' \) is contained in \( \mathcal{C}' \). Let \( \omega'' \in B_\epsilon(\omega') \cap H^2(S, \mathbb{Q}) \); by Proposition 3.6 we have \( M^\mu_v(S, \omega') = M^\mu_v(S, \omega'') \).

As \( \omega'' \in H^2(S, \mathbb{Q}) \cap H^{1,1}(S) \), there are \( p \in \mathbb{N} \) and \( H \in \text{Pic}(S) \) such that \( p\omega'' = c_1(H) \). As \( \omega'' \in \mathcal{C}' \) and \( \mathcal{C}' \) is a cone, we have \( c_1(H) \in \mathcal{C}' \); hence \( H \) is a \( v \)-generic ample line bundle, and \( M^\mu_v(S, \omega'') = M^\mu_v(S, H) \).

By [25] and [37], the moduli space \( M^\mu_v(S, H) \) is an irreducible holomorphically symplectic manifold which is deformation equivalent to a Hilbert scheme of points, and we are done. \( \square \)

4. Non-projective K3 surfaces with generic polarization

We now turn to the case of non-projective K3 surfaces. The first result we show is that \( M^\mu_v(S, \omega) \) is a compact hyperkähler (hence, in particular, Kähler) manifold. This will be accomplished using a result of Hitchin, Karlhede, Lindström and Roček in [11] roughly saying that a complex manifold is hyperkähler if and only if it has a twistor space.

The main difficulty here is to produce a candidate to be the twistor space for \( M^\mu_v(S, \omega) \). We proceed as follows: first, we consider the twistor space \( Z(S) \rightarrow \mathbb{P}^1 \) of the K3 surface \( S \) associated with the polarization \( \omega \). The sheaves in \( M^\mu_v(S, \omega) \) do not, in general, deform along
the twistor line as coherent sheaves, but still they deform as coherent twisted sheaves. We then construct the relative moduli space $\mathcal{M}$ of simple twisted sheaves and introduce stability for twisted sheaves: the open subset of $\mathcal{M}$ parametrizing stable twisted sheaves will be shown to be the twistor space of $M^\mu_v(S, \omega)$.

To conclude the proof of Theorem 1.1 we connect the K3 surface $S$ to a projective K3 surface $S'$ using twistor lines: this will induce a smooth and proper deformation of $M^\mu_v(S, \omega)$ to the moduli space $M^\mu_v(S', \omega')$, where $\omega'$ is $v$-generic. As all the fibres of the deformation are Kähler manifolds, it follows that $M^\mu_v(S, \omega)$ is irreducible symplectic and deformation equivalent to a Hilbert scheme of points on $S'$.

4.1. A result of Hitchin, Karlhede, Lindström and Roček. We first recall the result of Hitchin, Karlhede, Lindström and Roček (Theorem 3.3 of [11]) we need in the following.

A word on terminology before we proceed: the antipodal map on $\mathbb{P}^1$ is the map $\tau : \mathbb{P}^1 \to \mathbb{P}^1, \tau(\zeta) = -\bar{\zeta}^{-1}$. If $Z$ is a complex manifold, a real structure $\rho$ on $Z$ is an anti-holomorphic involution on $Z$. If $p : Z \to \mathbb{P}^1$ is a holomorphic map and $\rho$ is a real structure on $Z$, the real sections of $p$ are the sections $s$ satisfying $\rho \circ s = s \circ \tau$. We let $T^*_p$ denote the relative cotangent bundle of $p$.

**Theorem 4.1. (Hitchin-Karlhede-Lindström-Roček).** Let $Z$ be a complex manifold of dimension $2n + 1$ such that:

A) there is a holomorphic submersion $p : Z \to \mathbb{P}^1$.

B) there is a holomorphic section of $\wedge^2 T^*_p \otimes p^* \mathcal{O}_{\mathbb{P}^1}(2)$ defining a holomorphic symplectic form on each fibre;

C) there is a family of holomorphic sections $s : \mathbb{P}^1 \to Z$ of $p$, each with normal bundle $N_s$ isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}_{\mathbb{P}^1}(1)$.

D) there is a real structure $\rho : Z \to Z$ compatible with A), B) and C), and inducing the antipodal map on $\mathbb{P}^1$.

Then the parameter space of real sections of $p$ is a $4n$-dimensional manifold with a natural hyperkähler metric, whose twistor space is $Z$.

In the following, we will use Theorem 4.1 to show that $M^\mu_v(S, \omega)$ is hyperkähler. To do so, consider a K3 surface $S$ with a hyperkähler metric $g$ and a compatible complex structure $I$, and let $\omega$ be the corresponding Kähler form.

Let $\pi : Z(S) \to \mathbb{P}^1$ be the twistor space of $g$: every $t \in \mathbb{P}^1$ corresponds to a complex structure $I_t$ on $S$, which is compatible with $g$ and whose corresponding Kähler form is $\omega_t$. We write $S_t := (S, I_t)$. The parameter space of real sections of $\pi$ is identified with $S$, and we suppose that $(S, I)$ is the fibre over 0.
We will provide a smooth complex manifold $\mathcal{M}$ together with a holomorphic map $p: \mathcal{M} \to \mathbb{P}^1$ verifying conditions A), B), C) and D) of Theorem 4.1. The natural candidate for $\mathcal{M}$ would be the relative moduli space of $\mu$-stable sheaves: however a stable sheaf $E \in \mathcal{M}_{\mu}(S, \omega)$ does not necessarily deform to a coherent sheaf on $S_t$, as in general $c_1(E) \not\in NS(S_t)$. Anyway, it deforms as a sheaf twisted by the class defined by $c_1(E)$ in the Brauer group $Br(S_t)$ of $S_t$: we are then naturally led to deal with stable twisted sheaves.

4.2. Twisted sheaves and stability. Before going on, we recall some basic facts about twisted sheaves on a complex manifold $X$ (we refer the interested reader to [5] or [19] for more details).

There are several definitions of twisted sheaves, giving equivalent categories. We use three of them: the first one is due to Căldăraru [5], and presents twisted sheaves as a twisted gluing of local coherent sheaves on $X$; the second one (to be found again in [5]) presents twisted sheaves as modules over an Azumaya algebra on $X$; the last one, due to Yoshioka [38], presents twisted sheaves as a full subcategory of the category of coherent sheaves on some projective bundle over $X$.

We begin by recalling these definitions. As our aim are moduli spaces of stable twisted sheaves, we need to introduce several notions: first, we recall the Chern character and the slope of a twisted sheaf (for projective manifolds, this was done in [16] and [38]); we then introduce $\mu_\omega$-stability for twisted sheaves (with respect to a Kähler form $\omega$), and discuss genericity for polarizations (for projective manifolds this was done in [38]).

4.2.1. Twisted sheaves following Căldăraru. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be an open covering of $X$, and let $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$. Choose a 2-cocyle $\{\alpha_{ijk}\}$, where $\alpha_{ijk} \in \mathcal{O}_X^\times(U_{ijk})$, defining a class $\alpha \in H^2(X, \mathcal{O}_X^\times)$. A (coherent) sheaf twisted by $\{\alpha_{ijk}\}$ is a collection $\{\mathcal{F}_i, \phi_{ij}\}$, where $\mathcal{F}_i \in \text{Coh}(U_i)$ for every $i \in I$, and for every $i, j \in I$

- (1) $\phi_{ii} = id_{\mathcal{F}_i}$ for every $i \in I$;
- (2) $\phi_{ij} = \phi_{ji}^{-1}$ for every $i, j \in I$;
- (3) $\phi_{ij} \circ \phi_{jk} \circ \phi_{ki} = \alpha_{ijk} \cdot id$ for every $i, j, k \in I$.

By a morphism of $\{\alpha_{ijk}\}$-twisted sheaves

$$f: \mathcal{F} = \{\mathcal{F}_i, \phi_{ij}\} \to \mathcal{I} = \{\mathcal{I}_i, \psi_{ij}\}$$

we mean a collection $f = \{f_i\}$ of morphisms $f_i: \mathcal{F}_i \to \mathcal{I}_i$ of $\mathcal{O}_{U_i}$-modules such that $\psi_{ij} \circ f_j = f_i \circ \phi_{ij}$ for every $i, j \in I$. 

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We then form the abelian category \( \text{Coh}(X, \{ \alpha_{ijk} \}) \) of coherent \( \{ \alpha_{ijk} \} \)-twisted sheaves. If \( \{ \alpha_{ijk} \} \) and \( \{ \alpha'_{ijk} \} \) are two representatives of the same class \( \alpha \in H^2(X, \mathcal{O}_X^*) \), then there is an equivalence between \( \text{Coh}(X, \{ \alpha_{ijk} \}) \) and \( \text{Coh}(X, \{ \alpha'_{ijk} \}) \), so that we can speak of the category \( \text{Coh}(X, \alpha) \) of coherent \( \alpha \)-twisted sheaves.

If \( \mathcal{F} \in \text{Coh}(X, \alpha) \) and \( \mathcal{G} \in \text{Coh}(X, \beta) \), we can define in a natural way \( \mathcal{F} \otimes \mathcal{G} \) and \( \text{Hom}(\mathcal{F}, \mathcal{G}) \): the first one is a coherent sheaf twisted by \( \alpha \cdot \beta \), while the second is a coherent sheaf twisted by \( \alpha^{-1} \cdot \beta \).

We now recall an important definition: a sheaf \( \mathcal{A} \) of \( \mathcal{O}_X \)-modules is said to be an Azumaya algebra if it is a sheaf of \( \mathcal{O}_X \)-algebras whose generic fibre is a central simple algebra. Equivalence classes of Azumaya algebras form a group \( Br(X) \), the Brauer group of \( X \), which has an injection into \( H^2(X, \mathcal{O}_X^*) \). One of the main properties we will use in the following is (see Theorem 1.3.5 of [5]):

**Proposition 4.2.** Let \( X \) be a complex manifold and \( \alpha \in Br(X) \). Then there exist a locally free \( \alpha \)-twisted sheaf on \( X \).

For the rest of this section, we suppose \( \alpha \in Br(X) \). Using this, we are able to introduce the definition of twisted Chern character and twisted Mukai vector for \( \alpha \)-twisted sheaves. More precisely, let \( \mathcal{F} \) be an \( \alpha \)-twisted coherent sheaf on \( X \) and \( \mathcal{E} \) a locally free \( \alpha \)-twisted coherent sheaf. The Chern character of \( \mathcal{F} \) with respect to \( \mathcal{E} \) is

\[
\text{ch}_E(\mathcal{F}) := \frac{\text{ch}(\mathcal{F} \otimes \mathcal{E}^\vee)}{\sqrt{\text{ch}(\mathcal{E} \otimes \mathcal{E}^\vee)}}.
\]

The Mukai vector of \( \mathcal{F} \) with respect to \( \mathcal{E} \) is

\[
v_E(\mathcal{F}) := \text{ch}_E(\mathcal{F}) \cdot \sqrt{\text{td}(X)}.
\]

The slope of a torsion-free \( \alpha \)-twisted sheaf \( \mathcal{F} \) with respect to \( \mathcal{E} \) and to a Kähler class \( \omega \) is

\[
\mu_{E,\omega}(\mathcal{F}) := \frac{c_{E,1}(\mathcal{F}) \cdot \omega}{rk(\mathcal{F})},
\]

where \( c_{E,1}(\mathcal{F}) \) is the component of \( \text{ch}_E(\mathcal{F}) \) lying in \( H^2(S, \mathbb{Q}) \).

We collect some explicit formulas when \( X = S \) is a K3 surface. Let \( r := rk(\mathcal{F}) \), \( s := rk(\mathcal{E}) \), \( \xi := c_1(\mathcal{F} \otimes \mathcal{E}^\vee) \), \( a := \text{ch}_2(\mathcal{F} \otimes \mathcal{E}^\vee) \) and \( b := \text{ch}_2(\mathcal{E} \otimes \mathcal{E}^\vee) \). Then

\[
\text{ch}_E(\mathcal{F}) = (r, \xi/s, (2as - rb)/2s^2),
\]
\[
v_E(\mathcal{F}) = (r, \xi/s, r + (2as - rb)/2s^2)
\]
so that

\[
\mu_{E,\omega}(\mathcal{F}) = \frac{\xi \cdot \omega}{rs} = \frac{c_1(\mathcal{F} \otimes \mathcal{E}^\vee) \cdot \omega}{rk(\mathcal{F})rk(\mathcal{E})} = \mu(\mathcal{F} \otimes \mathcal{E}^\vee)
\]
and
\[ v_E^2(\mathcal{F}) = \frac{\xi^2}{s^2} - \frac{2ra}{s} + \frac{r^2b}{s^2} - 2r^2. \]
If \( \alpha = 0 \), then one easily sees that \( \mu_{E,\omega}(\mathcal{F}) = \mu_{\omega}(\mathcal{F}) - \mu(\omega) \) and that
\[ v_E^2(\mathcal{F}) = v^2(\mathcal{F}). \]
If \( \mathcal{F} \) is a torsion free \( \alpha \)-twisted sheaf on \( S \), we let
\[ ch_\alpha(\mathcal{F}) := ch_{\mathcal{F} \otimes \mathcal{A}}, \quad v_\alpha(\mathcal{F}) := v_{\mathcal{F} \otimes \mathcal{A}}, \]
called twisted Chern character and twisted Mukai vector of \( \mathcal{F} \). The twisted slope of \( \mathcal{F} \) with respect to \( \omega \) is
\[ \mu_{\alpha,\omega}(\mathcal{F}) := \frac{c_{\alpha,1}(\mathcal{F}) \cdot \omega}{rk(\mathcal{F})}, \]
where \( c_{\alpha,1}(\mathcal{F}) \) is the component of \( ch_\alpha(\mathcal{F}) \) in \( H^2(S, \mathbb{Q}) \).

Using twisted slopes, we introduce the notion of stability for twisted sheaves. Fix \( \alpha \in Br(X) \) and \( E \) and \( \alpha \)-twisted locally free sheaf.

**Definition 4.3.** We say that a torsion-free \( \mathcal{F} \in Coh(X, \alpha) \) is \( \mu_{E,\omega} \)-stable if for every \( \alpha \)-twisted coherent subsheaf \( \mathcal{E} \subseteq \mathcal{F} \) such that \( 0 < rk(\mathcal{E}) < rk(\mathcal{F}) \) we have \( \mu_{E,\omega}(\mathcal{E}) < \mu_{E,\omega}(\mathcal{F}) \). If \( \mu_{E,\omega}(\mathcal{E}) \leq \mu_{E,\omega}(\mathcal{F}) \) for every such subsheaf, we say that \( \mathcal{F} \) is \( \mu_{E,\omega} \)-semistable. A \( \mu_{\mathcal{F} \otimes \mathcal{A},\omega} \)-(semi)stable sheaf will be called \( \mu_{\alpha,\omega} \)-(semi)stable.

**4.2.2. Twisted sheaves as \( \mathcal{A} \)-modules.** Let again \( X \) be a complex manifold and \( \mathcal{A} \) an Azumaya algebra on \( X \). We let \( Coh(X, \mathcal{A}) \) be the abelian category of coherent sheaves on \( X \) having the structure of \( \mathcal{A} \)-module. The following is Proposition 1.3.6 of [5]:

**Proposition 4.4.** Let \( X \) be a complex manifold, \( \mathcal{A} \) an Azumaya algebra on \( X \) and \( \alpha \) its class in \( Br(X) \). If \( E \) is a locally free \( \alpha \)-twisted coherent sheaf such that \( \mathcal{E}nd(\mathcal{E}) \simeq \mathcal{A} \), we have an equivalence
\[ Coh(X, \alpha) \longrightarrow Coh(X, \mathcal{A}), \quad \mathcal{F} \mapsto \mathcal{F} \otimes E^\vee. \]

We now define Chern characters, Mukai vectors and slopes for the objects of \( Coh(X, \mathcal{A}) \), which allow us to define a notion of stability. For \( \mathcal{F} \in Coh(X, \mathcal{A}) \) we define
\[ ch_\mathcal{A}(\mathcal{F}) := \frac{ch(\mathcal{F})}{\sqrt{ch(\mathcal{A})}}, \quad v_\mathcal{A}(\mathcal{F}) := ch_\mathcal{A}(\mathcal{F}) \cdot \sqrt{td(X)}, \]
and if \( \omega \) is a Kähler class and \( \mathcal{F} \) is torsion-free, we let
\[ \mu_{\mathcal{A},\omega}(\mathcal{F}) := \frac{c_{\mathcal{A},1}(\mathcal{F}) \cdot \omega}{rk(\mathcal{F})}, \]
where $c_{A,1}(\mathcal{F})$ is the component of $ch_A(\mathcal{F})$ in $H^2(S, \mathbb{Q})$. We now introduce the notion of stability for $A$–modules:

**Definition 4.5.** A torsion-free $\mathcal{F} \in \text{Coh}(X, \mathcal{A})$ is $\mu_{A,\omega}$–stable if for every $\mathcal{E} \subseteq \mathcal{F}$ in $\text{Coh}(X, \mathcal{A})$ such that $0 < \text{rk}(\mathcal{E}) < \text{rk}(\mathcal{F})$, we have $\mu_{A,\omega}(\mathcal{E}) < \mu_{A,\omega}(\mathcal{F})$. If $\mu_{A,\omega}(\mathcal{E}) \leq \mu_{A,\omega}(\mathcal{F})$ for every such subobject, we say that $\mathcal{F}$ is $\mu_{A,\omega}$–semistable.

We notice that

**Remark 4.6.** We notice that $\Lambda := (\mathcal{O}_X, \mathcal{A})$ is a sheaf of rings of differential operators following the definition of [27], and $\text{Coh}(X, \mathcal{A})$ is the category of $\Lambda$–modules (always in the sense of [27]). Moreover, $\mu_{A,\omega}$–stable $\mathcal{A}$–modules are exactly $\mu$–stable $\Lambda$–modules (always in the sense of [27]). Even if the definitions of [27] are given only in the projective context, they can immediately be extended to the non-projective one.

### 4.2.3. Twisted sheaves following Yoshioka

In [38] Yoshioka introduces twisted sheaves as a full subcategory of the category of coherent sheaves on a projective bundle.

More precisely, let $X$ be a complex manifold, $\alpha \in Br(X)$ and $E$ a locally free $\alpha$–twisted sheaf. On an open cover $\mathcal{U} = \{U_i\}_{i \in I}$ we represent $E$ by $\{E_i, \phi_{ij}\}_{i,j \in I}$. Let $\mathbb{P}_i := \mathbb{P}(E_i)$, together with the map $\pi_i : \mathbb{P}_i \rightarrow U_i$. The twisted gluing data $\phi_{ij}$ turn to a gluing data $\phi_{ij}$ of the $\mathbb{P}_i$ and of the $\pi_i$, getting a projective bundle $\pi : \mathbb{P} \rightarrow X$ (whose class in $Br(S)$ is $\alpha$).

As shown in Lemma 1.1 of [38], we have $\text{Ext}^1(T_{\mathbb{P}/X}, \mathcal{O}_{\mathbb{P}}) = \mathbb{C}$, hence, up to scalars, there is a unique non-trivial extension

$$0 \rightarrow \mathcal{O}_{\mathbb{P}} \rightarrow G \rightarrow T_{\mathbb{P}/X} \rightarrow 0.$$  

The vector bundle $G$ can be described in another way. Fix a tautological line bundle $\mathcal{O}(\lambda_i)$ over $\mathbb{P}_i$, so that the twisted gluing data $\phi_{ij}$ give isomorphisms $\psi_{ij} : \mathcal{O}(\lambda_i) \rightarrow \mathcal{O}(\lambda_j)$, and $L := \{\mathcal{O}(\lambda_i), \psi_{ij}\}$ is an $\alpha$–twisted line bundle on $\mathbb{P}$. Then the vector bundles $\pi_i^*E_i(\lambda_i)$ glue together to give a locally free sheaf which is isomorphic to $G$.

**Definition 4.7.** A coherent sheaf $\mathcal{F}$ on $\mathbb{P}$ is called $\mathbb{P}$–sheaf if the canonical morphism $\pi^*\pi_*(G^\vee \otimes \mathcal{F}) \rightarrow G^\vee \otimes \mathcal{F}$ is an isomorphism. We let $\text{Coh}(\mathbb{P}, X)$ be the full subcategory of $\text{Coh}(\mathbb{P})$ given by $\mathbb{P}$–sheaves.
Lemma 1.5 of [38] shows that $\mathcal{F} \in \text{Coh}(\mathbb{P}, X)$ if and only if $\mathcal{F}|_{\mathbb{P}_i} \simeq \pi^* \mathcal{E}|_{U_i} \otimes \mathcal{O}(\lambda_i)$ for some $\mathcal{E} \in \text{Coh}(U_i)$. Using this, one shows:

**Proposition 4.8.** Let $X$ be a complex manifold and $\pi : \mathbb{P} \to X$ a projective bundle whose class in $\text{Br}(X)$ is $\alpha$. Then there is an equivalence of categories

$$P : \text{Coh}(\mathbb{P}, X) \to \text{Coh}(X, \alpha), \quad P(\mathcal{F}) := \pi_*(\mathcal{F} \otimes L^\vee).$$

Following Yoshioka, we have a definition of Chern character, Mukai vector and slope of a $\mathbb{P}$–sheaf $\mathcal{F}$. More precisely, we have

$$\text{ch}_P(\mathcal{F}) := \frac{\text{ch}(p_*(G^\vee \otimes \mathcal{F}))}{\sqrt{\text{ch}(p_*(G^\vee \otimes G))}}$$

so that

$$v_\mathcal{P}(\mathcal{F}) = \text{ch}_\mathcal{P}(\mathcal{F}) \cdot \sqrt{\text{td}(S)}, \quad \mu_{\mathcal{P}, \omega}(\mathcal{F}) := \frac{c_{\mathcal{P}, 1}(\mathcal{F}) \cdot \omega}{\text{rk}(\mathcal{F})},$$

where $c_{\mathcal{P}, 1}(\mathcal{F})$ is the component of $\text{ch}_\mathcal{P}(\mathcal{F})$ in $H^2(S, \mathbb{Q})$. We now introduce the notion of stability for $\mathbb{P}$–sheaves.

**Definition 4.9.** We say that a torsion-free $F \in \text{Coh}(\mathbb{P}, X)$ is $\mu_{\mathcal{P}, \omega}$–stable if for every subobject $\mathcal{E}$ of $F$ in $\text{Coh}(\mathbb{P}, X)$ such that $0 < \text{rk}(\mathcal{E}) < \text{rk}(F)$, we have $\mu_{\mathcal{P}, \omega}(\mathcal{E}) < \mu_{\mathcal{P}, \omega}(F)$. If $\mu_{\mathcal{P}, \omega}(\mathcal{E}) \leq \mu_{\mathcal{P}, \omega}(F)$ for every such subobject, we say that $\mathcal{F}$ is $\mu_{\mathcal{P}, \omega}$–semistable.

If $\mathbb{P} = \mathbb{P}(E)$ for some locally free $\alpha$–twisted sheaf $E$, the equivalence $P$ gives

$$\text{ch}_\mathcal{P}(\mathcal{F}) = \text{ch}_E(P(\mathcal{F})), \quad v_\mathcal{P}(\mathcal{F}) = v_E(P(\mathcal{F})), \quad \mu_{\mathcal{P}, \omega}(\mathcal{F}) = \mu_{E, \omega}(P(\mathcal{F})).$$

It follows that $\mathcal{F} \in \text{Coh}(\mathbb{P}, X)$ is $\mu_{\mathcal{P}, \omega}$–stable if and only if $P(\mathcal{F})$ is $\mu_{E, \omega}$–stable.

4.2.4. **Properties of stability.** In this section and in the following, we let $S$ be a K3 surface and $\alpha \in \text{Br}(S)$. We prove some properties about stable twisted sheaves in $\text{Coh}(S, \alpha)$; by the previous equivalences, similar properties hold in the other categories. First we show that the $\mu_{E, \omega}$–stability does not depend on $E$.

**Lemma 4.10.** Let $\alpha \in \text{Br}(S)$, $\mathcal{F} \in \text{Coh}(S, \alpha)$ and $\omega \in \mathcal{H}_S$. If $E', E \in \text{Coh}(S, \alpha)$ are locally free, then $\mathcal{F}$ is $\mu_{E, \omega}$–stable if and only if it is $\mu_{E', \omega}$–stable. In particular, $\mathcal{F}$ is $\mu_{E, \omega}$–stable if and only if it is $\mu_{\alpha, \omega}$–stable. If $\alpha = 0$, the sheaf $\mathcal{F}$ is $\mu_{0, \omega}$–stable if and only if it is $\mu_{\omega}$–stable.
Proof. Let $\mathcal{F} \in \text{Coh}(S, \alpha)$, $\mathcal{G}$ an $\alpha$–twisted coherent subsheaf of $\mathcal{F}$, and $H$ a locally free $\alpha$–twisted coherent sheaf. Then

$$\text{rk}(H)\text{rk}(\mathcal{F})c_1(\mathcal{G} \otimes H') - \text{rk}(H)\text{rk}(\mathcal{G})c_1(\mathcal{F} \otimes H') =$$

$$c_1(\mathcal{G} \otimes \mathcal{F}') \otimes H \otimes H') = \text{rk}^2(H)c_1(\mathcal{G} \otimes \mathcal{F}').$$

Suppose now that $\mathcal{F}$ is $\mu_{E, \omega}$–stable but not $\mu_{E', \omega}$–stable. Hence there is an $\alpha$–twisted coherent subsheaf $\mathcal{G}$ of $\mathcal{F}$ of rank $0 < s < \text{rk}(\mathcal{F})$ such that $\mu_{E', \omega}(\mathcal{G}) \geq \mu_{E', \omega}(\mathcal{F})$. By $\mu_{E, \omega}$–stability of $\mathcal{F}$ we even have $\mu_{E, \omega}(\mathcal{G}) < \mu_{E, \omega}(\mathcal{F})$. Writing these two inequalities explicitly we have

$$\omega \cdot (\text{rk}(E')c_1(\mathcal{G} \otimes (E')^\vee) - \text{rk}(E')sc_1(\mathcal{F} \otimes (E')^\vee)) \geq 0,$$

$$\omega \cdot (\text{rk}(E)rsc_1(\mathcal{G} \otimes \mathcal{F}^\vee) - \text{rk}(E)sc_1(\mathcal{F} \otimes \mathcal{F}^\vee)) < 0.$$

Using equation (4) for $H = E'$, equation (5) becomes $\omega \cdot c_1(\mathcal{G} \otimes \mathcal{F}^\vee) \geq 0$. Using equation (4) for $H = E$, equation (6) becomes $\omega \cdot c_1(\mathcal{G} \otimes \mathcal{F}^\vee) < 0$, getting a contradiction. □

4.2.5. Genericity for polarizations. If $\mathcal{F}$ is an $\alpha$–twisted coherent sheaf, we call discriminant of $\mathcal{F}$ with respect to $E$ the number

$$\Delta_E(\mathcal{F}) := \frac{1}{2}\text{rk}^2(\mathcal{F})\nu_E(\mathcal{F}) + 1.$$

If $\alpha = 0$, this is just $\Delta(\mathcal{F})$ by equation (3). More generally, the discriminant does not depend on $E$, as shown in the following:

Lemma 4.11. Let $\alpha \in Br(S)$ and $\mathcal{F} \in \text{Coh}(S, \alpha)$. If $E_1, E_2 \in \text{Coh}(S, \alpha)$ are locally free, then $\Delta_{E_1}(\mathcal{F}) = \Delta_{E_2}(\mathcal{F})$.

Proof. Let $E \in \text{Coh}(S, \alpha)$ be locally free of rank $s$, and pose $r := \text{rk}(\mathcal{F})$, $\xi := c_1(\mathcal{F} \otimes E^\vee)$, $a := c_2(\mathcal{F} \otimes E^\vee)$ and $b := c_2(E \otimes E^\vee)$. By equation (2) we have

$$\Delta_E(\mathcal{F}) = \frac{1}{2r^2} \left( \frac{\xi^2}{s} - \frac{2ra}{s} + \frac{r^2b}{s^2} - 2r^2 \right) + 1.$$

An easy computation shows that

$$\frac{\xi^2}{s} - \frac{2ra}{s} + \frac{r^2b}{s^2} = -\frac{c_2(\mathcal{F} \otimes \mathcal{F}' \otimes E \otimes E^\vee)}{s^2} + \frac{r^2c_2(E \otimes E^\vee)}{2s^2} =$$

$$\quad = -c_2(\mathcal{F} \otimes \mathcal{F}^\vee),$$

so that

$$\Delta_E(\mathcal{F}) = \frac{1}{2r^2}(-c_2(\mathcal{F} \otimes \mathcal{F}^\vee) - 2r^2) + 1,$$

which does not depend on $E$, implying the statement. □
For \( v \in H^{2*}(S, \mathbb{Q}) \) and \( \alpha \in Br(S) \), we let
\[
\Delta_\alpha(v) := \Delta_{\mathcal{F} \vee v}(\mathcal{F}),
\]
where \( \mathcal{F} \in \text{Coh}(S, \alpha) \) is torsion free and \( v_\alpha(\mathcal{F}) = v \). By Lemma 4.11 this is well defined and if \( \alpha = 0 \), then \( \Delta_0(v) = \Delta(v) \). Now, let
\[
W_{\alpha,v} := \{ D \in NS(S) | -\frac{r^4}{2} \Delta_\alpha(v) \leq D^2 < 0 \}.
\]
If \( \alpha = 0 \), we have \( W_{0,v} = W_v \).

**Definition 4.12.** If \( D \in W_{\alpha,v} \), we call the hyperplane \( D^\perp \) an \((\alpha,v)\)-wall. A connected component of \( \mathcal{H}_S \setminus \bigcup_{D \in W_{\alpha,v}} D^\perp \) is called \((\alpha,v)\)-chamber. A polarization \( \omega \in \mathcal{H}_S \) is \((\alpha,v)\)-generic if it lies in a \((\alpha,v)\)-chamber.

A polarization \( \omega \) is \((0,v)\)-generic if and only if it is \( v \)-generic.

### 4.3. Moduli space of stable twisted sheaves.

We now introduce (relative) moduli spaces of stable twisted sheaves. On projective manifolds these were constructed by Yoshioka in [38] (viewing twisted sheaves as \( P^- \)sheaves, and using a GIT construction), and by Lieblich in [19] (viewing twisted sheaves as sheaves on some \( O^\ast \)-gerbe).

Here we first provide a relative moduli space of simple twisted sheaves by viewing them as simple \( P^- \)sheaves. The relative moduli space of stable sheaves will then be an open subset of it.

#### 4.3.1. The relative moduli space of simple twisted sheaves.

Consider a smooth and proper morphism \( \pi : \mathcal{X} \longrightarrow T \) such that for every \( t \in T \) the fibre \( X_t \) over \( t \) is a K3 surface. We assume for simplicity that \( T \) is a complex manifold, although the constructions work over complex spaces as well. Suppose moreover that we are given a complex manifold \( \mathcal{P} \) together with a morphism \( f : \mathcal{P} \longrightarrow \mathcal{X} \) of \( T \)-complex spaces such that for every \( t \in T \), the morphism \( f_t : P_t \longrightarrow X_t \) is a projective bundle, where \( P_t = f^{-1}(X_t) \).

For every \( t \in T \) the projective bundle \( P_t \longrightarrow X_t \) defines a class \( \alpha_t \) in the Brauer group \( Br(X_t) \), and we have an equivalence between \( \text{Coh}(P_t, X_t) \) and \( \text{Coh}(X_t, \alpha_t) \).

Now, let \( f' := \pi \circ f \), so that we get a map \( f' : \mathcal{P} \longrightarrow T \). By Theorem (2.2) of [17], there is a complex space \( \mathcal{M}(\mathcal{P}/T) \) together with a holomorphic surjective map
\[
\phi : \mathcal{M}(\mathcal{P}/T) \longrightarrow T
\]
which is a relative moduli space of simple coherent sheaves on \( \mathcal{P} \): for every \( t \in T \) the fibre \( \mathcal{M}_t \) of \( \phi \) over \( t \) is the moduli space of simple coherent sheaves on \( P_t \).
Now, \( \mathcal{F} \in \text{Coh}(P_t) \) is simple if and only if \( \text{End}(\mathcal{F}) \simeq \mathbb{C} \). As \( \text{Coh}(P_t, X_t) \) is a full subcategory of \( \text{Coh}(P_t) \), a \( P_t \)-sheaf \( \mathcal{F} \) is simple in \( \text{Coh}(P_t, X_t) \) if and only if it is simple in \( \text{Coh}(P_t) \). Hence simple \( P_t \)-sheaves form a subset \( \mathcal{M}^s(\mathcal{P}/T) \) of \( \mathcal{M}(\mathcal{P}/T) \).

As the condition defining \( \mathbb{P} \)-sheaves is open (see Lemma 1.5 of [38]), it follows that \( \mathcal{M}^s(\mathcal{P}/T) \) is open in \( \mathcal{M}(\mathcal{P}/T) \), hence it is a complex space together with a holomorphic map \( \psi : \mathcal{M}^s(\mathcal{P}/T) \to T \). This is the relative moduli space of stable twisted sheaves on \( \mathcal{P} \).

The relative projective bundle \( f : \mathcal{P} \to \mathcal{X} \) corresponds to the existence of a relative Azumaya algebra \( \mathcal{A} \) on \( \mathcal{X} \): for every \( t \in T \), we have \( P_t = \mathbb{P}(E_t) \) for some locally free \( \alpha_t \)-twisted sheaf on \( X_t \), and we let \( \mathcal{A}_t := E_t \otimes E_t^\vee \). The previous equivalence of categories of twisted sheaves then shows that \( \mathcal{M}^s(\mathcal{P}/T) \) is the relative moduli space of simple \( \mathcal{A} \)-modules on \( \mathcal{X} \) or, equivalently, the relative moduli space of simple twisted sheaves on \( \mathcal{X} \).

4.3.2. The relative moduli space of stable twisted sheaves. We now produce out of \( \psi : \mathcal{M}^s(\mathcal{P}/T) \to T \) the relative moduli space of stable twisted sheaves. Choose \( v = (v_0, v_1, v_2) \in H^2(S, \mathbb{Q}) \) such that \( v_1 \in \text{NS}(S_t) \) for every \( t \in T \), and \( v_0 \geq 2 \). We let \( \mathcal{M}^s_v(\mathcal{P}/T) \) be the component of \( \mathcal{M}^s(\mathcal{P}/T) \) parametrizing simple \( \mathbb{P} \)-sheaves of Mukai vector \( v \), and we still write \( \psi : \mathcal{M}^s_v(\mathcal{P}/T) \to T \) for the induced morphism.

In order to define the moduli space of stable twisted sheaves of Mukai vector \( v \), we need a section \( \tilde{\omega} \in H^2(\pi_* \mathcal{A}) \) such that \( \omega_t := \tilde{\omega}|_{X_t} \) is a Kähler class for every \( t \in T \), which is used to define stability on every fibre. As stable twisted sheaves are simple, we let \( \mathcal{M}^s_v(\mathcal{P}/T, \tilde{\omega}) \) be the subset of \( \mathcal{M}^s_v(\mathcal{P}/T) \) whose fibre over \( t \in T \) is given by the simple \( P_t \)-sheaves which are \( \mu_{P_t, \omega_t} \)-stable and whose Mukai vector is \( v \). We then have a natural map (of sets)

\[ p : \mathcal{M}^s_v(\mathcal{P}/T, \tilde{\omega}) \to T. \]

The following can be proved as in Lemma 3.7 of [27].

**Proposition 4.13.** Let \( \pi : \mathcal{X} \to T \), \( f : \mathcal{P} \to \mathcal{X} \), \( v \) and \( \tilde{\omega} \) be as before. Then \( \mathcal{M}^s_v(\mathcal{P}/T, \tilde{\omega}) \) is an open subset of \( \mathcal{M}^s_v(\mathcal{P}/T) \). Hence it is a complex manifold, and the map \( p \) is holomorphic.

Indeed, if \( \mathcal{F} \in \text{Coh}(S, \alpha) \) and \( F := \mathcal{F}^{\vee \vee}, \) then \( \mathcal{F} \) is \( \mu_{\alpha, \omega} \)-stable if and only if \( \mathcal{F} \otimes F^{\vee} \) is \( \mu_{\mathcal{A}, \omega} \)-stable in \( \text{Coh}(S, \mathcal{A}) \), where \( \mathcal{A} = F \otimes F^{\vee} \). Moreover, the openness of stability in the analytic case may be proved...
in the usual way, by using boundedness results which are contained in [32] and [33].

If \( T \) is reduced to a point, we use the notation \( M_{\alpha,v}^\mu(S,\omega) \), where \( \alpha \) is the class of \( P \to S \) in \( Br(S) \), and we refer to this complex manifold as moduli space of \( \mu_{\alpha,\omega} \)-stable \( \alpha \)-twisted sheaves with twisted Mukai vector \( v \).

**Remark 4.14.** Suppose that \( \alpha = 0 \) and let
\[
\gamma := \frac{\text{ch}(F)}{\sqrt{\text{ch}(F \otimes F^\vee)}}
\]
for \( F \in M_{0,v}^\mu(S,\omega) \). Then \( v_0(F) = v \) if and only if \( v(F) = v/\gamma \), so that \( M_{0,v}^\mu(S,\omega) \simeq M_{v/\gamma}^\mu(S,\omega) \). We even notice that \( \omega \) is \((0,v)\)-generic if and only if it is \( v/\gamma \)-generic.

If \( F \) is a \( \mu_{\alpha,\omega} \)-stable sheaf of twisted Mukai vector \( v \), then \( P^{-1}(F) \) is a \( \mu_{P,\omega} \)-stable \( \mathbb{P} \)-sheaf, where \( P \to S \) is a projective bundle associated to \( \alpha \). As \( \text{Coh}(\mathbb{P},S) \) is a full subcategory of \( \text{Coh}(\mathbb{P}) \), classical arguments of deformation theory in [4] show that
\[
T_{[\mathcal{F}]} M_{\alpha,v}^\mu(S,\omega) \simeq \text{Ext}_1^{\text{Coh}(\mathbb{P},S)}(P^{-1}(\mathcal{F}),P^{-1}(\mathcal{F})) \simeq \text{Ext}_1^{\text{Coh}(\mathbb{P},\alpha)}(\mathcal{F},\mathcal{F}),
\]
and the obstruction for the existence of deformations of \( \mathcal{F} \) lie in
\[
\text{Ext}_2^{\text{Coh}(\mathbb{P},S)}(P^{-1}(\mathcal{F}),P^{-1}(\mathcal{F})) \simeq \text{Ext}_2^{\text{Coh}(\mathbb{P},\alpha)}(\mathcal{F},\mathcal{F}).
\]

Following [4] one then shows that if \( p : \mathcal{M} := \mathcal{M}_v^\mu(\mathcal{P}/\mathcal{X},\tilde{\omega}) \to T \) is the relative moduli space of twisted stable sheaves, then for every \( \mathcal{F} \in p^{-1}(t) \) we have an exact sequence
\[
\text{Ext}_1^{\text{Coh}(\mathcal{P},S)}(P^{-1}(\mathcal{F}),P^{-1}(\mathcal{F})) \to T_{[\mathcal{F}]} \mathcal{M} \to T_t \mathcal{M} \to \text{Ext}_2^{\text{Coh}(\mathcal{P},\mathcal{F})}(\mathcal{F},\mathcal{F})_0.
\]

We conclude this section with the following result about the existence of a quasi-universal family; cf. [1] for the absolute untwisted case.

**Proposition 4.15.** Let \( \pi : \mathcal{X} \to T, f : \mathcal{P} \to \mathcal{X}, v = (v_0,v_1,v_2) \) and \( \tilde{\omega} \) be as before. Let \( \mathcal{A} \) be a relative Azumaya algebra corresponding to \( \mathcal{P} \), and for every \( t \in T \) let \( \alpha_t \in Br(X_t) \) be the class of \( \mathcal{A}_t \). Suppose that there is a locally free \( \mathcal{A} \text{-module } V \) verifying the two following properties for every \( t \in T \):

1. the restriction \( V_t \) of \( V \) to \( X_t \) is \( \mu_{\alpha_t,\omega_t} \)-stable;
2. the twisted Mukai vector of \( V_t \) is \((v_0,v_1,w_2)\), where \( w_2 < v_2 \).

Then there is a quasi-universal family on \( \mathcal{M}_v^\mu(\mathcal{P}/T,\tilde{\omega}) \times_T \mathcal{X} \).
Proof. Let $\mathcal{M} := \mathcal{M}_s^u(\mathcal{P}/T, \tilde{\omega})$. As for stable coherent sheaves, there is an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of $\mathcal{M}$ given by analytic subsets endowed with universal $\mathcal{A}$-modules $\mathcal{F}_i$.

Let $p_i : U_i \times_T \mathcal{X} \to U_i$ and $q_i : U_i \times_T \mathcal{X} \to \mathcal{X}$ denote the two projections. We put $E_i := q_i^* p_i^* (\mathcal{F} \otimes \mathcal{O}_{\mathcal{X}}(\nabla))$. By the choice of $\nabla$ we have $R^0 p_i^* E_i = 0 = R^2 p_i^* E_i$ and $W_i := R^1 p_i^* E_i$ is a non-trivial locally free $\mathcal{O}_{U_i}$-module whose rank is independent of $i$.

It is easy to check now that the $\mathcal{A}$-modules $\mathcal{F}_i \otimes \mathcal{O}_{U_i}(\nabla)$ glue together to give the desired quasi-universal family. □

4.4. Deformation of stable twisted sheaves along twistor lines.

In this subsection we describe and generalize a construction used by several authors in the case of stable locally free sheaves of slope zero, cf. [35], [22], [30].

Let $(S, I, \omega)$ be a polarized K3 surface and $\pi : Z(S) \to \mathbb{P}^1$ its twistor family. We suppose that the fibre over 0 is $S_0 = (S, I)$, and we write $S_t = (S, I_t)$ for the fibre over $t$. Here $I = I_0$ and $I_t$ denote the complex structures on $S$. Recall that the choice of $\omega$ on $(S, I)$ is equivalent to the choice of a Riemannian metric $g$ which is compatible with $I$ and whose associated Kähler class is $\omega$. Along the twistor line the metric $g$ remains compatible with $I_t$, the associated class $\omega_t$ is Kähler, and we get a section $\tilde{\omega}$ of $R^2 \pi_* \mathcal{O}$ which is $\omega_t$ on $S_t$. Slope stability on $S_t$ will be considered with respect to $\omega_t$.

Before we introduce deformations of sheaves along twistor lines we make an observation on $(1,1)$-forms on the twistor space of $S$. Recall that, as a differentiable manifold, $Z(S)$ is the product $S \times \mathbb{P}^1$, which is endowed with a complex structure in the following way (see [31]): cover $\mathbb{P}^1$ by two charts (each isomorphic to $\mathbb{C}$) and take $\zeta$ the complex coordinate function one one of them and $\zeta^{-1}$ on the other. Further, let $I, J, K$ be the complex structures on $S$ which make it into a hyperkähler manifold. If $I_{\mathbb{P}^1}$ is the complex structure on $\mathbb{P}^1$ then put the following complex structure to act on the tangent space $T_S \times T_{\mathbb{P}^1}$ of $S \times \mathbb{P}^1$:

$$\mathfrak{J} := \left( 1 - \frac{\zeta \bar{\zeta}}{1 + \zeta \bar{\zeta}} I + \frac{\zeta + \bar{\zeta}}{1 + \zeta \bar{\zeta}} J + i \frac{\zeta - \bar{\zeta}}{1 + \zeta \bar{\zeta}} K, I_{\mathbb{P}^1} \right).$$

With respect to this complex structure the projection $q : S \times \mathbb{P}^1 \to S$ is not holomorphic but only $C^\infty$.

**Lemma 4.16.** Let $\psi$ be a $(1,1)$-form on $(S, I, \omega)$. Its pull-back $q^* \psi$ is a $(1,1)$-form on $Z(S)$ if and only if $\psi$ is anti-self-dual on $(S, I, \omega)$.

**Proof.** Let $\Psi := q^* \psi$. It is a 2-form on $Z(S)$, so it is of type $(1,1)$ if and only if $\Psi(\mathfrak{J} v, \mathfrak{J} w) = \Psi(v, w)$ for any pair $(v, w)$ of real tangent vectors.
at a point of $Z(S)$. As $I$ preserves the horizontal and the vertical directions on $Z(S) = S \times \mathbb{P}^1$, and as $\Psi(v, w) = 0$ if one of the tangent vectors $v$ or $w$ is horizontal, it suffices to check $\Psi(Iv, Iw) = \Psi(v, w)$ only on vertical vectors, meaning that the restrictions of $\Psi$ to the fibres of $\pi : Z(S) \to \mathbb{P}^1$ are of type $(1, 1)$.

Suppose that $\psi$ is anti-self-dual. This property only depends on $g$ and on the orientation of $S$: as $g$ is compatible with each complex structure $I_t$, it follows that the restriction of $\Psi$ to each fibre of $\pi$ is then anti-self-dual. In particular, it is of type $(1, 1)$, hence also $\Psi$ is of type $(1, 1)$ on $Z(S)$.

Conversely, if $\psi$ is not anti-self-dual, then it decomposes as $\psi = \psi^{SD} + \psi^{ASD}$, where the self-dual part is of the form $\psi^{SD} = f\omega_I$ for some non-zero function $f$. But $\omega_I$ is not of type $(1, 1)$ with respect to $J$ so neither will be $\Psi$.

We now turn to deformations of sheaves along the twistor line.

4.4.1. Deformation of a locally free polystable sheaf with trivial slope. Let $E_0$ be a polystable vector bundle on $S_0$ whose slope is zero, and denote by $E^\infty$ the underlying $C^\infty$–vector bundle of $E_0$. The Kobayashi-Hitchin correspondence provides $E^\infty$ with an anti-self-dual connection. By Lemma 4.16 the curvature of the connection is of $(1, 1)$-type on each $S_t$. We therefore obtain holomorphic structures $E_t$ on $E^\infty$ over each $S_t$, induced by the structure $E_0$ in a canonical way. In fact we even get a holomorphic structure on $q^*E^\infty$. As $E_t$ is holomorphic and carries an anti-self-dual connection, it is polystable for every $t \in \mathbb{P}^1$. It is easy to see that if $E_0$ is stable, then $E_t$ is stable for every $t \in \mathbb{P}^1$.

4.4.2. Deformation of a non-locally free polystable sheaf with trivial slope. Suppose now that $\mathcal{E}$ is a polystable torsion-free sheaf of slope zero on $S_0$ and let $E := \mathcal{E}^{\vee\vee}$, which is locally free, polystable and of slope zero. Let $\mathcal{C}$ be the sheaf of $C^\infty$ complex functions on $S$: put $\mathcal{E}^\infty := \mathcal{E} \otimes_{\mathcal{O}} \mathcal{C}$ and $E^\infty := E \otimes_{\mathcal{O}} \mathcal{C}$.

Since $\mathcal{C}$ is faithfully flat over $\mathcal{O}$, we get natural inclusions $\mathcal{E}^\infty \subset E^\infty$ and $\mathcal{E} \subset \mathcal{E}^\infty$ by means of which we have $\mathcal{E} = \mathcal{E}^\infty \cap E$ inside $E^\infty$. Now, as seen in section 4.4.1, there is a holomorphic structure $E_t$ on $E^\infty$ and $E_t$ is polystable, for every $t \in \mathbb{P}^1$. We put $\mathcal{E}_t := \mathcal{E}^\infty \cap E_t$ in $E^\infty$, which is then a polystable sheaf on $S_t$ with trivial slope. We notice that if $\mathcal{E}$ is stable, then $\mathcal{E}_t$ is stable for every $t \in \mathbb{P}^1$.

4.4.3. Deformation of an Azumaya algebra. Let now $\mathcal{A}_0$ be an Azumaya algebra on $S_0$, and let $\alpha_0$ be its class in $Br(S_0)$. Choose a locally free $\alpha_0$–twisted sheaf $E_0$ such that $\mathcal{A}_0 \simeq \mathcal{E}nd(E_0)$. We will suppose that $E_0$ is $\mu_{\alpha_0, \omega_0}$–stable.
The Kobayashi-Hitchin correspondence for twisted sheaves established by Wang in \[36\] shows that \( \mathcal{A}_0 \) is polystable. Notice that \( \mu_{\omega_0}(\mathcal{A}_0) = 0 \), hence by section 4.4.1 the vector bundle \( \mathcal{A} := q^*\mathcal{A}_0 \) carries a holomorphic structure, and for every \( t \in \mathbb{P}^1 \) its restriction \( \mathcal{A}_t \) to the fibre \( S_t \) is a polystable vector bundle with trivial slope. We need to show that \( \mathcal{A}_t \) is an Azumaya algebra.

To do so, we argue as in the proof of Lemma 6.5 in \[22\], point (3). The Azumaya algebra structure on \( \mathcal{A}_0 \) is given by a holomorphic map \( m_0 : \mathcal{A}_0 \otimes \mathcal{A}_0 \rightarrow \mathcal{A}_0 \) verifying some identities among holomorphic sections. This means that \( m_0 \) is a holomorphic section of the vector bundle \( \mathcal{H}om(\mathcal{A}_0 \otimes \mathcal{A}_0, \mathcal{A}_0) \). But this is a polystable sheaf as \( \mathcal{A}_0 \) is, hence it carries an ASD-connection, and \( m_0 \) is parallel with respect to it.

As a consequence, \( m_0 \) defines a parallel section of \( \mathcal{H}om(\mathcal{A}_t \otimes \mathcal{A}_t, \mathcal{A}_t) \), hence a holomorphic map \( m_t : \mathcal{A}_t \otimes \mathcal{A}_t \rightarrow \mathcal{A}_t \). Hence \( \mathcal{A}_t \) is an Azumaya algebra: as the same identities among sections which are verified on \( \mathcal{A}_0 \) are verified even on \( \mathcal{A}_t \), it follows that \( \mathcal{A}_t \) is an Azumaya algebra.

4.4.4. Deformation of a stable twisted sheaf. Let \( \alpha_0 \in \text{Br}(S_0) \) and \( \mathcal{F}_0 \) an \( \alpha_0 \)-twisted sheaf which is \( \mu_{\alpha_0,\omega_0} \)-stable. Choose and \( \alpha_0 \)-twisted locally free sheaf \( E_0 \) which is \( \mu_{\alpha_0,\omega_0} \)-twisted in such a way that \( c_{E_0,1}(\mathcal{F}_0) = 0 \).

We let \( \mathcal{G}_0 := \mathcal{F}_0 \otimes E_0^\vee \) and \( \mathcal{A}_0 := E_0 \otimes E_0^\vee \): then \( \mathcal{A}_0 \) is an Azumaya algebra, and as we saw in section 4.4.3 it is a polystable sheaf. Moreover, \( \mathcal{G}_0 \) is a coherent sheaf of trivial slope and it has the structure of \( \mathcal{A}_0 \)-module. The Kobayashi-Hitchin correspondence for twisted sheaves in \[36\] shows that \( \mathcal{G}_0 \) is a polystable sheaf.

Following section 4.4.3, \( q^*\mathcal{A}_0 \) is a holomorphic vector bundle, and for every \( t \in \mathbb{P}^1 \) its restriction \( \mathcal{A}_t \) to \( S_t \) is a polystable sheaf having the structure of Azumaya algebra. We let \( \alpha_t \) be its class in \( \text{Br}(S_t) \).

By section 4.4.2 the polystable sheaf \( \mathcal{G}_0 \) gives rise, for every \( t \in \mathbb{P}^1 \) to a polystable sheaf \( \mathcal{G}_t \) with trivial slope. The same argument used in

\[1\] If \( E_0 \) is an untwisted sheaf, we can give a more direct proof. The multiplication of two holomorphic sections \( \phi_1, \phi_2 \) of \( \mathcal{A}_t \) remains holomorphic (hence \( \mathcal{A}_t \) is a sheaf of algebras on \( S_t \)): this is a consequence of the formula \( \hat{\nabla}(\phi_1 \circ \phi_2) = \hat{\nabla}\phi_1 \circ \phi_2 + \phi_1 \circ \hat{\nabla}\phi_2 \), where \( \hat{\nabla} \) is the connection induced by \( D \) on \( \mathcal{A}_0 \).

By \[31\] Thm. 1.1.6 we just need to show that \( \mathcal{A}_t \) is locally of the form \( \mathcal{E}nd(F) \) for some locally free sheaf \( F \) of \( \mathcal{O}_{S_t} \)-modules. To do so, consider the self-dual part \( R_{SD} \) of the curvature \( R \) of \( D \). We have \( R_{SD} = c \cdot \text{Id} \cdot \omega_0 \) for a suitable constant \( c \).

By solving the equation \( dd^c \phi = -\hat{\nabla}\omega_0 \) on a open subset \( U \), we find a holomorphic hermitian line bundle \( (L, h) \) on \( U \) whose curvature is \( -\hat{\nabla}\omega_0 \). Hence \( F^\infty := E_0 \otimes L \) is a rank \( r \) vector bundle on \( U \) with a Hermite-Einstein connection, and \( A^\infty \cong \mathcal{E}nd(F^\infty) \) as ASD-vector bundles. Hence on \( F^\infty \) we have a holomorphic structure \( F_t \) compatible with the corresponding \( I_t \), and \( \mathcal{A}_t \cong \mathcal{E}nd(F_t) \).
section 4.4.3 to show that $\mathcal{A}_t$ is an Azumaya algebra, applied this time to $m_t : \mathcal{A}_t \otimes \mathcal{G}_t \to \mathcal{G}_t$, shows that the sheaf $\mathcal{G}_t$ has the structure of an $\mathcal{A}_t$–module.

As $\mathcal{G}_t$ is an $\mathcal{A}_t$–module, it corresponds to an $\alpha_t$–twisted sheaf $\mathcal{F}_t$ on $S_t$. In particular $E_t$ gives rise to an $\alpha_t$–twisted locally free sheaf $E_t$ on $S_t$ such that $\mathcal{E}nd(E_t) \simeq \mathcal{A}_t$ and $\mathcal{F}_t \otimes E_t^\vee \simeq \mathcal{G}_t$.

**Lemma 4.17.** The sheaves $\mathcal{F}_t$ and $E_t$ are $\mu_{\alpha_t,\omega_t}$–stable.

**Proof.** We show that $E_t$ is $\mu_{\alpha_t,\omega_t}$–stable. The proof for $\mathcal{F}_t$ is similar. Suppose that $E_t$ is not $\mu_{\alpha_t,\omega_t}$–stable, and let $\mathcal{E}_t \subseteq E_t$ in $\text{Coh}(S_t, \alpha_t)$ with $\mu_{E_t,\omega_t}(\mathcal{E}_t) \geq \mu_{E_t,\omega_t}(E_t)$. We suppose that $\mathcal{E}_t$ is $\mu_{\alpha_t,\omega_t}$–stable.

We let $\mathcal{H}_t := \mathcal{E}_t \otimes E_t^\vee$, which is an $\mathcal{A}_t$–module and we have $\mathcal{H}_t \subseteq \mathcal{A}_t$. The inequality $\mu_{E_t,\omega_t}(\mathcal{E}_t) \geq \mu_{E_t,\omega_t}(E_t)$ gives $\mu_{\mathcal{A}_t,\omega_t}(\mathcal{H}_t) \geq \mu_{\mathcal{A}_t,\omega_t}(\mathcal{A}_t)$, so that $\mu_{\omega_t}(\mathcal{H}_t) \geq \mu_{\omega_t}(\mathcal{A}_t) = 0$. As $\mathcal{A}_t$ is $\mu_{\omega_t}$–polystable, this implies that $\mu_{\omega_t}(\mathcal{H}_t) = 0$, and that it is a direct summand of $\mathcal{A}_t$. In particular, it is $\mu_{\omega_t}$–polystable.

Using the same argument given before, the sheaf $\mathcal{H}_t$ gives rise to a $\mu_{\omega_0}$–polystable sheaf $\mathcal{H}_0$ on $S_0$, which is contained in $\mathcal{A}_0$, has the structure of $\mathcal{A}_0$–module, and $\mu_{\omega_0}(\mathcal{H}_0) = \mu_{\omega_0}(\mathcal{A}_0) = 0$. The equivalence between $\text{Coh}(S_0, \alpha_0)$ and $\text{Coh}(S_0, \mathcal{A}_0)$ given by tensoring with $E_0^\vee$ produces then a subsheaf $\mathcal{E}_0$ of $E_0$ such that $\mu_{E_0,\omega_0}(\mathcal{E}_0) = \mu_{E_0,\omega_0}(E_0)$. But this is not possible as $E_0$ is $\mu_{\alpha_0,\omega_0}$–stable. In conclusion, the sheaf $E_t$ is $\mu_{\alpha_t,\omega_t}$–stable. \qed

### 4.5. Relative moduli space of twisted sheaves on twistor lines.

In this section we show that the relative moduli space of stable twisted sheaves over the twistor family of a polarized K3 surface verifies all the conditions of Theorem 4.1.

We let $S$ be a K3 surface, $v = (r, 0, a) \in H^{2*}(S, \mathbb{Z})$ with $r \geq 2$, $\alpha \in Br(S)$ and $\omega$ a $v$–generic polarization. The Kähler class $\omega$ corresponds to the choice of a Riemannian metric $g$ which is compatible with the complex structure $I$ of $S$, and whose associated Kähler class is $\omega$. Let $\pi : Z(S) \longrightarrow \mathbb{P}^1$ be the twistor family of $g$: we denote $S_t$ the fibre of $\pi$ over $t$, which corresponds to a complex structure $I_t$ on $S$ associated with $t$. The metric $g$ is compatible with $I_t$, the associated class $\omega_t$ is Kähler, and $v$ is a Mukai vector on $S_t$ for every $t \in \mathbb{P}^1$.

Choose now a $\mu_{\alpha,\omega}$–stable $\alpha$–twisted sheaf $\mathcal{E}$ on $S$ of rank $r$, and let $E := \mathcal{E}^{\vee \vee}$: this is a $\mu_{\alpha,\omega}$–stable $\alpha$–twisted vector bundle of rank $r$, and we let $\mathcal{A}_0 := \mathcal{E}nd(E)$ the corresponding Azumaya algebra. We suppose that $v_{E}(\mathcal{E}) = v$. By section 4.4.3, there is holomorphic vector bundle $\mathcal{A}$ on $Z(S)$ whose restriction $\mathcal{A}_t$ to $S_t$ is an Azumaya algebra on...
S_t for every t ∈ P^1. We let α_t ∈ Br(S_t) be its class and A_t ≃ End(E_t), where E_t is the deformation of E along the twistor line.

By section 4.3.2 there is then a relative moduli space of stable twisted sheaves p : M → P^1 such that for every t ∈ P^1 the fibre over t is the moduli space M_{µ, α, ω_t}^µ of µ, α, ω-t-stable twisted sheaves whose twisted Mukai vector with respect to E_t is v.

Remark 4.18. On M × P^1 Z(S) we have a quasi-universal family: if F ∈ M_{µ, α, ω}^µ(S, ω), let F := F ∨ ∨ and V_0 := F ⊗ E^v. We let V in Proposition 4.15 be V := q^∗V_0.

The aim of this section is to prove the following:

Proposition 4.19. Let S be a K3 surface, v = (r, 0, a) ∈ H^2(S, Z) with r ≥ 2, α ∈ Br(S) and ω a v-generic polarization. The relative moduli space p : M → P^1 of stable twisted sheaves verifies the following properties:

(1) the morphism p is submersive;
(2) for every E ∈ π^−1(t) there is a canonical holomorphic section s_E : P^1 → M of p passing through E (given by the deformation of E along the twistor line), whose normal bundle N_{s_E} is isomorphic to

\[ N_{s_E} ∼= \bigoplus_{i=1}^{2n} O_{P^1}(1), \]

where 2n is the dimension of a fibre. As a consequence, there is a diffeomorphism (over P^1) f : M → M_{µ, α, ω}^µ(S, ω) × P^1;
(3) if T^*_p denotes the relative cotangent bundle of p, there is a holomorphic global section of \( \wedge^2 T^*_p \otimes O_{P^1}(2) \) whose restriction to any fibre is a holomorphic symplectic form;
(4) there is a real structure ρ on M which is compatible with the previous data, and inducing the antipodal map on P^1.

Proof. We divide the proof in several parts.

Step 1: submersivity. As every E ∈ M_t is simple and the canonical bundle of a K3 surface is trivial, we have Ext^2(E, E)_0 = 0. The exact sequence [ ] implies then that the map p is submersive, so that condition (1) of the statement is proved.

Step 2: sections. We now construct a family of holomorphic sections. Let t_0 ∈ P^1, and choose E ∈ M_{µ, α, ω}^µ(S_{t_0}, ω_{t_0}). As we saw in section 4.4.4, the sheaf E gives rise to a sheaf E_t ∈ M_{µ, α, ω}^µ(S_t, ω_t) for every t ∈ P^1. This produces a section

\[ s_E : P^1 → M, \quad s_E(t) := E_t \]
of $p$, which is holomorphic. If we let $E_t$ be the $\alpha_t$ – twisted $\mu_{\alpha_t,\omega_t}$ – stable sheaf such that $A_t = \mathcal{E}nd(E_t)$ (an Azumaya algebra on $S_t$ whose class in $Br(S_t)$ is $\alpha_t$), and $G_t := \mathcal{E}_t \otimes E_t^\vee$, we let $G := q^*G_t$, which is a coherent $\mathcal{O}_{Z(S)}$–module. The restriction of the relative tangent bundle $T_p$ of $p$ to the section $s$ is

$$s^*T_p \simeq R^1\pi_*\mathcal{E}nd(G).$$

Moreover, it is easy to see that the normal bundle $N_s$ of $s$ is isomorphic to $s^*T_p$.

We remark that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are not isomorphic, then the corresponding sections do not intersect over any $t \in \mathbb{P}^1$. It follows that the map

$$\mathcal{M} \longrightarrow \mathbb{P}^1 \times M^\mu_{\alpha,v}(S,\omega), \quad \mathcal{E} \mapsto (t, s_{\mathcal{E}}(t)),$$

where $t = p(\mathcal{E})$, is a diffeomorphism.

**Step 3: relative symplectic form.** We prove that the condition (3) is fulfilled. We first notice that for every $t \in \mathbb{P}^1$ the restriction $T_p|_{\alpha_t}$ of $T_p$ to $M_t$ is the tangent bundle of $M^\mu_{\alpha_t,v}(S_t,\omega_t)$, and similarly the restriction $(T_p)|_t$ of $T_p$ to $M_t$ is the cotangent bundle of $M^\mu_{\alpha_t,v}(S_t,\omega_t)$. As on $M^\mu_{\alpha_t,v}(S_t,\omega_t)$ we have a holomorphic symplectic form (if $S_t$ is projective, this is done in [38]; the proof in the general case is similar), we get an isomorphism $T_{\alpha_t}(T_p)|_t \simeq (T_p)|_t$.

This implies the existence of a line bundle $\mathcal{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$ together with an isomorphism $T_p \longrightarrow T_p^* \otimes p^*\mathcal{O}_{\mathbb{P}^1}(d)$. We then just need to show that $d = 2$. To do so, consider a locally free sheaf $F \in M_0$: as seen in Step 2 we have a holomorphic section $s : \mathbb{P}^1 \longrightarrow \mathcal{M}$ of $p$, and

$$s^*T_p \simeq R^1p_*\mathcal{E}nd(G)$$

where $G = q^*(F \otimes E_0^\vee)$. By the relative Serre duality we get

$$R^1p_*\mathcal{E}nd(G) \simeq (R^1p_*\mathcal{E}nd(G)^* \otimes K_\pi)^*,$$

where $K_\pi$ is the relative canonical bundle of $\pi : Z(S) \longrightarrow \mathbb{P}^1$.

Now, as $G$ is locally free, we have $\mathcal{E}nd(G) \simeq \mathcal{E}nd(G)^*$. Moreover, $K_\pi \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$, hence

$$R^1p_*\mathcal{E}nd(G) \simeq R^1p_*\mathcal{E}nd(G)^* \otimes \mathcal{O}_{\mathbb{P}^1}(2).$$

In conclusion,

$$s^*T_p \simeq s^*T_p^* \otimes \mathcal{O}_{\mathbb{P}^1}(2).$$

As $s^*T_p \simeq s^*T_p^* \otimes \mathcal{O}_{\mathbb{P}^1}(d)$, it follows $d = 2$. This shows that condition (3) is fulfilled.

**Step 4: normal bundle of a section.** We now calculate the normal bundle of a section. Let $s : \mathbb{P}^1 \longrightarrow \mathcal{M}$ be a section of $p$. First, we suppose that $s$ passes through a locally free sheaf $F$ in $M^\mu_{\alpha,v}(S,\omega)$. The
normal bundle $N_s$ is a vector bundle of rank $2n$ on $\mathbb{P}^1$, hence there are $d_1, \ldots, d_{2n} \in \mathbb{Z}$ such that

$$N_s = \bigoplus_{i=1}^{2n} \mathcal{O}_{\mathbb{P}^1}(d_i).$$

We need to show that $d_1 = \ldots = d_{2n} = 1$.

We first show that $N_s$ is globally generated. Recall by Step 2 that $N_s \simeq R^1 p_* \mathcal{E}nd(\mathcal{G})$, where $\mathcal{G} = q^*(F \otimes E_d^\vee)$. A standard argument using a Leray spectral sequence shows that we have an isomorphism

$$H^0(\mathbb{P}^1, R^1 p_* \mathcal{E}nd(\mathcal{G})) \simeq H^1(Z(S), \mathcal{E}nd(\mathcal{G})).$$

Thus the restriction to the fibre at $t \in \mathbb{P}^1$ of the evaluation morphism

$$ev : H^0(\mathbb{P}^1, R^1 p_* \mathcal{E}nd(\mathcal{G})) \otimes \mathcal{O}_{\mathbb{P}^1} \to R^1 p_* \mathcal{E}nd(\mathcal{G})$$

may be interpreted as the canonical restriction morphism

$$r_t : H^1(Z(S), \mathcal{E}nd(\mathcal{G})) \to H^1(S_t, \mathcal{E}nd(\mathcal{G}_t)).$$

Surjectivity of $ev$ follows if we are able to show that for every $t$ the $\mathbb{C}$-linear map $r_t$ admits a natural section

$$\Phi_t : H^1(S_t, \mathcal{E}nd(\mathcal{G}_t)) \to H^1(Z(S), \mathcal{E}nd(\mathcal{G})).$$

To do so, recall that the vector bundle $\mathcal{E}nd(\mathcal{G}_t)$ has a Hermite-Einstein metric with associated anti-self-dual connection $B_t$. We pull back the metric and the connection to $\mathcal{E}nd(\mathcal{G})$ through $q : Z(S) \to S_t$, getting a Hermitian connection $B$ on $\mathcal{E}nd(\mathcal{G})$ by Lemma 4.16.

We represent an element of $H^1(S_t, \mathcal{E}nd(\mathcal{G}_t))$ in Dolbeault cohomology by a section $\sigma$ in $\mathcal{E}nd(\mathcal{G}_t) \otimes \mathcal{A}_t^{0,1}$ which is $\bar{\partial}_B$-closed. Here $\mathcal{A}_t^{0,1}$ denotes the sheaf of $C^\infty$ forms of type $(0,1)$ on $S_t$. In order to see that this procedure gives a well defined map $H^1(S_t, \mathcal{E}nd(\mathcal{G}_t)) \to H^1(Z(S), \mathcal{E}nd(\mathcal{G}))$, we choose $\sigma$ to be harmonic.

We now show that the $(0,1)$-part of $q^* \sigma$ is $\bar{\partial}_B$-closed. The condition $\bar{\partial}_B \sigma = 0$ means that $d_B \sigma$ is of type $(1,1)$. By Lemma 4.16 again, if $d_B \sigma$ is anti-self-dual, we will have also $d_B(q^* \sigma)$ of type $(1,1)$ and thus $\bar{\partial}_B(q^* \sigma)^{0,1} = 0$. But by standard facts in hermitian geometry [20, 7.1 (6) and (17)] we get

$$d_B \sigma \wedge \omega_t = \bar{\partial}_B \sigma \wedge \omega_t = \bar{\partial}_B(\sigma \wedge \omega_t) = -i \bar{\partial}_B(\ast \sigma) = i \ast \bar{\partial}_B \sigma = 0.$$

It follows that $\bar{\partial}_B(q^* \sigma) = 0$.

In order to get an element in $H^1(Z(S), \mathcal{E}nd(\mathcal{G}))$ we consider the $(0,1)$-part $(q^* \sigma)^{0,1}$ of $q^* \sigma$. This is $\bar{\partial}_B$-closed and we set $\Phi_t(\sigma)$ to be its class in Dolbeault cohomology. Thus the surjectivity of $r_t$ and hence the surjectivity of $ev$ are proved. From this, it follows that the vector bundle $N_s \simeq R^1 p_* \mathcal{E}nd(\mathcal{G})$ is globally generated.
As already remarked in Step 2, we have $N_s \simeq s^* T_p$. By the isomorphism of Step 3, it follows that
\[
\bigoplus_{i=1}^{2n} \mathcal{O}_{\mathbb{P}^1}(d_i) \simeq N_s \simeq N_s^* \otimes \mathcal{O}_{\mathbb{P}^1}(2) \simeq \bigoplus_{i=1}^{2n} \mathcal{O}_{\mathbb{P}^1}(2 - d_i).
\]

The global generatedness of $N_s$ implies that $d_1, \ldots, d_{2n} \in \{0, 1, 2\}$. Indeed, note that with respect to the opposite complex structure $-I$, the section $\sigma$ is of pure type $(1, 0)$. If $\tau : \mathbb{P}^1 \to \mathbb{P}^1$ is the antipodal map on $\mathbb{P}^1$, then the opposite complex structure $-I$ is precisely the complex structure of $S_{\tau(t)}$ (see [11]). So $\Phi_t(\sigma)$, viewed as a section in $H^0(\mathbb{P}^1, R^1 p_* \mathcal{E}nd(H))$, vanishes at $\tau(t) \in \mathbb{P}^1$. As a component of type $\mathcal{O}_{\mathbb{P}^1}$ cannot be generated by sections admitting zeros on $\mathbb{P}^1$, it follows that $d_1, \ldots, d_{2n} \neq 0$. Hence $d_1, \ldots, d_{2n} = 1$, and we are done for sections passing through a locally free sheaf.

We are left with the calculation of the normal bundle of a section passing through a non-locally free sheaf. This follows exactly the same procedure described in [34], section 4, hence we refer the reader to it.

Step 5: the real structure. We conclude the proof of the Proposition by giving a real structure on $\mathcal{M}$: we let
\[
\rho : \mathcal{M} \to \mathcal{M}, \quad \rho(\mathcal{E}) := s_\mathcal{E}(\tau(t)),
\]
where $t = p(\mathcal{E})$. This map sends the complex structure on $\mathcal{M}_t$ to the conjugate of the one on $\mathcal{M}_{\tau(t)}$, hence it is a real structure on $\mathcal{M}$.

**Remark 4.20.** Theorem 4.1 may now be applied to the above situation showing that $p$ is a twistor space for $M_{\mu}^\nu$. In particular $M_{\mu}^\nu$ is endowed with the Kähler metric constructed in the proof of Theorem 3.3 of [11]. A direct verification shows that it coincides with the Weil-Petersson metric on the open part of $M_{\mu}^\nu$ parametrizing locally free sheaves.

4.6. **Proof of Theorem 1.1 for non-projective K3 surfaces.** We now conclude the proof of Theorem 1.1. We first recall some basic facts about periods and twistor lines for K3 surfaces. Let $\Omega$ be the period domain of K3 surfaces, i.e.
\[
\Omega = \{ x \in \mathbb{P}(\Lambda_{K3} \otimes \mathbb{C}) \mid q(x, x) = 0, \ q(x, \overline{x}) > 0 \},
\]
where $\Lambda_{K3}$ is the lattice of a K3 surface and $q$ is the intersection form on it, which has signature $(3, 19)$. Each class corresponds to the choice of a complex structure on $S$, i.e. to a Hodge decomposition of $H^2(S, \mathbb{C})$.

As well-known, $\Omega$ is diffeomorphic to $Gr^{po}(2, \Lambda_{K3} \otimes \mathbb{R})$, the Grassmannian of oriented positive planes in $\Lambda_{K3} \otimes \mathbb{R}$, via the following correspondence: to the period $x$ of a K3 surface $(S, I)$ we associate the
oriented positive plane $P(x) = \langle Re(\sigma), Im(\sigma) \rangle \subseteq H^2(S, \mathbb{R})$, where $\sigma$ is the generator of $H^{2,0}(S)$. Conversely, to the plane $\langle a, b \rangle \subseteq H^2(S, \mathbb{R})$ we associate the projective class of $a + ib$, viewed as the generator of $H^{2,0}(S)$.

If $W \subseteq \Lambda_{K3} \otimes \mathbb{R}$ is a positive 3-dimensional subspace, it defines a twistor line $T_W := \Omega \cap \mathbb{P}(W \otimes \mathbb{C})$, and every twistor line arises in this way. A point $x \in \Omega$ is on $T_W$ if and only if $P(x) \subseteq W$.

The twistor line $T_W$ is called generic if $W^\perp \cap \Lambda_{K3} = 0$, which is equivalent to the existence of $w \in W$ such that $w^\perp \cap \Lambda_{K3} = 0$. If $T_W$ is generic, then the generic point of $T_W$ (i.e. outside a countable subset) corresponds to a K3 surface whose Néron-Severi is trivial.

Finally, notice that a twistor line $T$ through $x \in \Omega$ corresponds to the choice of a Riemannian metric $g$ which is compatible with the complex structure $I$ corresponding to $x$, and whose associated form is Kähler. For every $t \in T$, the metric $g$ is compatible with the complex structure $I_t$ corresponding to $t$, and the associated form is Kähler. The set of Kähler classes giving rise to generic twistor lines is a dense open subset of the Kähler cone (see Remark 2.9 of [15]).

We now complete the proof of Theorem 1.1 by showing:

**Theorem 4.21.** Let $S$ be a K3 surface, $r, a \in \mathbb{Z}$, $\xi \in NS(S)$ prime with $r$ and $v = (r, \xi, a) \in H^{2r}(S, \mathbb{Z})$. If $\omega \in \mathcal{K}_S$ is a $v$-generic polarization and $M^\nu_v(S, \omega) \neq \emptyset$, then $M^\nu_v(S, \omega)$ is an irreducible holomorphically symplectic manifold which is deformation equivalent to a Hilbert scheme of points on a projective K3 surface.

**Proof.** Let $x \in \Omega$ be the period of $S$, and $T$ a twistor line $T$ corresponding to $\omega$. Let $F \in M^\nu_v(S, \omega)$ and

$$w := v \cdot \frac{\sqrt{ch(F^\vee)}}{\sqrt{ch(F \otimes F^\vee)}} = (r, 0, a - \xi^2/2r).$$

By section 4.3.2, we then have a family $p : \mathcal{M} \rightarrow T$ whose fibre over $t$ is the moduli space $M^\mu_{\alpha, \nu}(S_t, \omega_t)$.

Let now $y \in \Omega \cap \xi^\perp$, and $S_y$ the corresponding K3 surface. Hence $\xi \in NS(S_y)$ and $v$ is a Mukai vector on $S_y$. Suppose that there is $y \in \Omega \cap \xi^\perp$ whose corresponding K3 surface $S_y$ is projective, and that on $S_y$ there is a $v$-generic polarization $\omega_y$ for which there is a finite number $T_0, ..., T_n$ of twistor lines verifying:

1. $T_0 = T$, so that $x \in T_0$;
2. $y \in T_n$, and $T_n$ is the twistor line corresponding to $\omega_y$;
3. for $i = 0, ..., n-1$ we have $T_i \cap T_{i+1} = \{x_i\}$ where $x_i$ corresponds to a K3 surface $S_i$ with trivial Néron-Severi group.
Let us now prove that if this is the case, the statement follows. For every \( i = 0, \ldots, n \) let \( p_i : \mathcal{M}_i \rightarrow T_i \) be the corresponding relative moduli space of stable twisted sheaves of Mukai vector \( w \) (see section 4.5). Let \( \alpha_i \in Br(S_i) \) be the class defined by \( \xi \): the twistor lines \( T_i \) and \( T_{i+1} \) correspond to two polarizations \( \omega_i \) and \( \omega_{i+1} \) on \( S_i \), and the fibre of \( p_i \) (resp. \( p_{i+1} \)) over \( x_i \) is the moduli space \( M^\mu_{\alpha_i,w}(S_i, \omega_i) \) (resp. \( M^\mu_{\alpha_i,w}(S_i, \omega_{i+1}) \)) of \( \mu_{\alpha_i,\omega_i} \)-stable (resp. \( \mu_{\alpha_i,\omega_{i+1}} \)-stable) \( \alpha_i \)-twisted sheaves with twisted Mukai vector \( w \).

As \( NS(S_i) = 0 \), we have \( M^\mu_{\alpha_i,w}(S_i, \omega_i) = M^\mu_{\alpha_i,w}(S_i, \omega_{i+1}) \). By point 2 of Proposition 4.19, all the fibres of \( p_i \) are diffeomorphic to \( M^\mu_{\alpha_i,w}(S_i, \omega_i) \), and all the fibres of \( p_{i+1} \) are diffeomorphic to \( M^\mu_{\alpha_i,w}(S_i, \omega_{i+1}) \). As this holds for every \( i = 0, \ldots, n-1 \), it follows that all the fibres of \( p_0, \ldots, p_{n-1} \) are diffeomorphic to those of \( p_n \). In particular, they are diffeomorphic to \( p_n^{-1}(y) \cong M^\mu_v(S_y, \omega_y) \).

Now, notice that by Proposition 4.19 we can apply Theorem 4.1 to \( p_0, \ldots, p_n \), hence all the fibres of \( p_0, \ldots, p_n \) are hyperkähler manifolds. By Theorem 3.7 we know that \( M^\mu_v(S_y, \omega_y) \) is an irreducible hyperkähler manifold which is deformation equivalent to a Hilbert scheme of points on \( S_y \): it follows that all the fibres of \( p_0, \ldots, p_n \) are compact, connected, simply connected hyperkähler manifolds.

To show that they are irreducible hyperkähler manifolds it is then enough to show that their \( h^{2,0} \) is 1. By point 1 of Proposition 4.19 the morphisms \( p_0, \ldots, p_n \) are all submersive. As their fibres are compact and smooth, the Proposition in section 1 of [6] shows that \( p_0, \ldots, p_n \) are all smooth and proper morphisms. Now, all their fibres are Kähler: hence the Hodge numbers are constant (as they are upper-semicontinuous), and as \( h^{2,0}(M^\mu_v(S_y, \omega_y)) = 1 \) every fibre of \( p_0, \ldots, p_n \) is an irreducible hyperkähler manifold. In particular, \( M^\mu_v(S, \omega) \) is irreducible hyperkähler and deformation equivalent to \( M^\mu_v(S_y, \omega_y) \).

In conclusion, we only need to prove the existence of \( y \in \Omega \cap \xi^\perp \) whose corresponding K3 surface \( S_y \) is projective, of a \( v \)-generic polarization \( \omega_y \) on \( S_y \) and of a finite number \( T_0, \ldots, T_n \) of twistor lines verifying conditions (1), (2) and (3) above.

To do so, let \( B \subseteq \Omega \) be a ball centred at \( x \). Then \( B \cap \xi^\perp \neq \emptyset \): by density of projective K3 surfaces in \( \xi^\perp \), for every ball \( B \) of this type there is \( y \in B \cap \xi^\perp \) corresponding to a projective K3 surface \( S_y \). Choose now a \( v \)-generic polarization \( \omega_y \) on \( S_y \).

Recall that the set of generic twistor lines is open; moreover, by Proposition 3.6 we know that moving the polarization inside a \( v \)-chamber does not change the moduli space. It follows that the twistor lines \( T \) and \( T' \) corresponding to \( \omega \) and \( \omega_y \) respectively can be
supposed to be generic. We then let \( T_0 := T \) and \( T_n := T' \) (we show in the following that \( n = 4 \)).

Now, let \( x_1 \in B \cap T \) and \( y_1 \in B \cap T' \) correspond to \( 0 \) surfaces with trivial \( \text{Néron-Severi} \) group. The claim follows if we show that there are three generic twistor lines \( T_1, T_2 \) and \( T_3 \) such that

(i) \( x_1 \in T_1 \) and \( y_1 \in T_3 \)

(ii) \( T_1 \cap T_2 = \{x_2\} \), \( T_2 \cap T_3 = \{y_2\} \) and \( x_2, y_2 \) correspond to \( 0 \) surfaces having trivial \( \text{Néron-Severi} \) group.

To show this, we argue as in Proposition 3.7 of [13]. Let \( P(x_1) = \langle a, b \rangle \) and \( P(y_1) = \langle a', b' \rangle \). As \( x_1 \) and \( y_1 \) lie in a neighbourhood of \( x \) in \( \Omega \), the planes \( P(x_1) \) and \( P(y_1) \) lie in a neighbourhood of \( P(x) \) in \( Gr^p(2, \Lambda_{K_3} \otimes \mathbb{R}) \).

Up to shrinking \( B \), we can then suppose that there is \( c \in \Lambda_{K_3} \otimes \mathbb{R} \) such that \( W_1 := \langle a, b, c \rangle \) and \( W_3 := \langle a', b', c \rangle \) are two positive linear subspaces of \( \Lambda_{K_3} \otimes \mathbb{R} \) of dimension 3, and \( c^\perp \cap \Lambda_{K_3} = 0 \). Let \( W_2 := \langle a, b', c \rangle \) and \( T_i := T_{W_i} \) for \( i = 1, 2, 3 \). As \( c \in T_i \) and \( c^\perp \cap \Lambda_{K_3} = 0 \), the twistor line \( T_i \) is generic. Moreover, \( P(x_2) = \langle a, c \rangle \) and \( P(y_2) = \langle b', c \rangle \).

Let \( S_{x_2} \) be the \( K_3 \) surface corresponding to \( x_2 \). As \( P(x_2) = \langle a, c \rangle \), we have \( H^{2,0}(S_{x_2}) = \mathbb{C} \cdot (a + ic) \). If \( \gamma \in NS(S_{x_2}) \), then \( q(\gamma, a + ic) = 0 \), where \( q \) is the Beauville form. It follows that \( q(\gamma, a) = -iq(\gamma, c) \). As \( q \) is a real form and \( a, c, \gamma \in \Lambda_{K_3} \otimes \mathbb{R} \) we then have \( q(\gamma, a) = q(\gamma, c) = 0 \). In conclusion \( \gamma \in c^\perp \cap \Lambda_{K_3} = 0 \), so that \( NS(S_{x_2}) = 0 \). Similarly \( NS(S_{y_2}) = 0 \), and we are done. \( \square \)

5. Projectivity

This last section is devoted to the Beauville form of \( M^p_0(S, \omega) \): we show that there is a Hodge isometry between \( H^2(M^p_0(S, \omega), \mathbb{Z}) \) and \( v^\perp \) if \( v^2 > 0 \), and with \( v^\perp / \mathbb{Z} \cdot v \) if \( v^2 = 0 \). As an application, we show that \( M^p_0(S, \omega) \) is projective if and only if \( S \) is projective.

We introduce some notations. If \( v \in H^{2*}(S, \mathbb{Z}) \), we let \( v^\perp \) be the orthogonal of \( v \) with respect to the Mukai pairing. If \( v = (r, \xi, a) \) and \( \xi \in NS(S) \), then the pure weight-two Hodge structure on \( H^{2*}(S, \mathbb{Z}) \) induces a pure weight-two Hodge structure on \( v^\perp \): namely, a class \( \alpha = (\alpha_0, \alpha_1, \alpha_2) \in v^\perp \) is of \( (1, 1) \)-type if and only if \( \alpha_1 \in NS(S) \).

If \( \alpha = (\alpha_0, \alpha_1, \alpha_2) \in H^{2*}(S, \mathbb{Q}) \), we write \( \alpha^\vee := (\alpha_0, -\alpha_1, \alpha_2) \). If \( \alpha = ch(F) \) for some locally free sheaf \( F \), then \( \alpha^\vee = ch(F^\vee) \). It is immediate to see that if \( \alpha, \beta \in H^{2*}(S, \mathbb{Q}) \), then \( (\alpha \cdot \beta)^\vee = \alpha^\vee \cdot \beta^\vee \). In particular, this implies that \( (\beta/\alpha)^\vee = \beta^\vee/\alpha^\vee \) and \( (\sqrt{\alpha})^\vee = \sqrt{\alpha^\vee} \), whenever these expressions make sense.

We now introduce a morphism associating to any class in \( v^\perp \) a rational cohomology class on the moduli space of stable (twisted)
sheaves. The construction is inspired from the similar morphism which is used in the projective case (see [25], [37], [21], [26]). Let \( \alpha \in Br(S) \), \( w \in H^{2*}(S, \mathbb{Q}) \) a Mukai vector and \( \omega \) a \( w \)-generic polarization. Suppose moreover that \( M^\mu_{\alpha,w}(S, \omega) \) is compact, and let \( p : M^\mu_{\alpha,w}(S, \omega) \times S \longrightarrow M^\mu_{\alpha,w}(S, \omega) \) and \( q : M^\mu_{\alpha,w}(S, \omega) \times S \longrightarrow S \) be the projections.

Choosing a quasi-universal family \( \mathcal{E} \) on \( M^\mu_{\alpha,w}(S, \omega) \times S \) of similitude \( \rho \) (which exists by Remark 4.18), we define a morphism
\[
\lambda_{S,\alpha,w} : w^\perp \longrightarrow H^2(M^\mu_{\alpha,w}(S, \omega), \mathbb{Q})
\]
by letting
\[
\lambda_{S,\alpha,w}(\beta) := \frac{1}{\rho} [p_*(q^*(\beta^\vee \cdot \sqrt{td(S)}) \cdot ch(\mathcal{E}'))]_1,
\]
where \([\cdot]_1\) is the part lying in \( H^2(M^\mu_{\alpha,w}(S, \omega), \mathbb{Q}) \). As \( \beta \in w^\perp \), the class \( \lambda_{S,\alpha,w}(\beta) \) does not depend on the chosen quasi-universal family. If \( \alpha = 0 \) we simply write \( \lambda_{S,w} \) for \( \lambda_{S,0,w} \). Moreover, one sees immediately that \( \lambda_{S,\alpha,w} \) is a Hodge morphism.

We now show the following, which is a generalization of known results in the projective case (see [24], [25], [37]):

**Proposition 5.1.** Let \( S \) be a K3 surface, \( v = (r, \xi, a) \in H^{2*}(S, \mathbb{Z}) \) where \( r \geq 2 \), \( \xi \in NS(S) \), \( (r, \xi) = 1 \) and \( v^2 \geq 0 \). Moreover, let \( \omega \) be a \( v \)-generic polarization. Then the image of \( \lambda_{S,v} \) is contained in \( H^2(M^\mu_v(S, \omega), \mathbb{Z}) \), and

1. if \( v^2 = 0 \), then \( \lambda_{S,v} \) defines a Hodge isometry
   \[
   \overline{\lambda}_{S,v} : v^\perp / \mathbb{Z} \cdot v \longrightarrow H^2(M^\mu_v(S, \omega), \mathbb{Z});
   \]
2. if \( v^2 > 0 \), then \( \lambda_{S,v} \) is a Hodge isometry.

**Proof.** If \( v^2 > 0 \), we just need to show the following properties:

a) the image of \( \lambda_{S,v} \) is contained in \( H^2(M^\mu_v(S, \omega), \mathbb{Z}) \);

b) the morphism \( \lambda_{S,v} \) is bijective;

c) the morphism \( \lambda_{S,v} \) is an isometry.

Let \( \mathcal{E} \) be a quasi-universal family of similitude \( \rho \) on \( M^\mu_v(S, \omega) \times S \), and fix a locally free \( \mu_\omega \)-stable vector bundle \( F \) of Mukai vector \( v \). Let \( w := v_F(F) = (r, 0, a - \xi^2/2r) \) and
\[
  f : M^\mu_v(S, \omega) \longrightarrow M^\mu_{0,w}(S, \omega), \quad f_*(\mathcal{F}) := \mathcal{F} \otimes F^\vee
\]
which is an isomorphism (see Remark 4.14). The sheaf \( (f \times id_S)_*\mathcal{E} \) is a quasi-universal family of similitude \( \rho \) on \( M^\mu_{0,w}(S, \omega) \times S \). Moreover, as \( f \) is an isomorphism of irreducible symplectic manifolds, the morphism
\[
f_* : H^2(M^\mu_v(S, \omega), \mathbb{Z}) \longrightarrow H^2(M^\mu_{0,w}(S, \omega), \mathbb{Z})
\]
is a Hodge isometry.

Now, we let

\[ h : H^{2*}(S, \mathbb{Z}) \rightarrow H^{2*}(S, \mathbb{Q}), \quad h(\beta) := \frac{\beta \cdot ch(F^\vee)}{\sqrt{ch(F \otimes F^\vee)}}. \]

We let \((\cdot, \cdot)_S\) be the Mukai pairing on \(S\) and \([\cdot]\) the part lying in \(H^4(S, \mathbb{Q})\). If \(\beta \in v^\perp\) we have

\[
(h(\beta), w)_S = -\left[ \frac{\beta^\vee \cdot ch(F)}{\sqrt{ch(F \otimes F^\vee)}} \cdot v_F(F) \right]_2 = -[\beta^\vee \cdot ch(F) \cdot \sqrt{td(S)}]_2 = (\beta, v)_S = 0,
\]

so that

\[ h : v^\perp \rightarrow w^\perp. \]

The same argument shows that it is an isometry, and it is easy to see that it is a Hodge morphism. We even have

\[
f_*(\lambda_{S,w}(\beta)) = \lambda_{S,\tilde{w}}(h(\beta)).
\]

Indeed

\[
f_*(\lambda_{S,w}(\beta)) = \frac{1}{\rho} [f_* p_* (q^*(\beta^\vee \sqrt{td(S)}) ch(\mathcal{E}))]_1 = \frac{1}{\rho} [p_* ((f \times id_S)_* q^*(\beta^\vee \sqrt{td(S)}) ch((f \times id_S)_* \mathcal{E}))]_1 = \frac{1}{\rho} [p_* (q^*(h(\beta)^\vee \sqrt{td(S)}) ch((f \times id_S)_* \mathcal{E}))]_1 = \lambda_{S,w}(h(\beta)).
\]

In conclusion, we see that \(\lambda_{S,w}\) verifies the properties a), b) and c) above if and only if \(\lambda_{S,w}\) verifies them.

Now, choose a projective K3 surface \(S'\) such that \(\xi \in NS(S')\), and take \(\omega'\) a \(v\)-generic polarization on \(S'\). Following the proof of Theorem 1.1, we can suppose that there are generic twistor lines \(T, T_1, T_2, T_3\) and \(T'\) such that \(T\) passes through \(S\) and corresponds to \(\omega\), \(T'\) passes through \(S'\) and corresponds to \(\omega'\), and the intersection points \(T \cap T_1\), \(T_1 \cap T_2, T_2 \cap T_3\) and \(T_3 \cap T'\) correspond to K3 surfaces with trivial Néron-Severi group.

As we can define \(\lambda_{S,w}\) in a relative way using relative quasi-universal families (which exist by Remark 4.18), properties a), b) and c) above are verified on a fibre if and only if they are verified all along the twistor lines. Moreover, as the intersection points of twistor lines correspond to K3 surfaces with trivial Néron-Severi group, it follows that the properties a), b) and c) are preserved when passing from a twistor line to the following: indeed, in these cases the moduli space and the quasi-universal family do not change when changing twistor line.

It follows that \(\lambda_{S,w}\) verifies a), b) and c) if and only \(\lambda_{S',w}\) verifies them. As we saw before, this is the case if and only if \(\lambda_{S',w}\) verifies them.
As $S'$ is projective, $v$ is primitive and $\omega'$ is $v$–generic, by Theorem [3.7 and [25, 37] the morphism $\lambda_{S',v}$ verifies a), b) and c). Hence $\lambda_{S,v}$ is a Hodge isometry between $v^\perp$ and $H^2(M^\mu_v(S,\omega), \mathbb{Z})$.

If $v^2 = 0$, the proof is similar: the only difference is about the fact that $\mathbb{Z} \cdot v$ is the kernel of $\lambda_{S,v}$, which holds in the general case as it holds over a projective K3 surface (see [24]).

As a corollary we get the following:

**Corollary 5.2.** Let $S$ be a K3 surface, $v = (r, \xi, a) \in H^2(S, \mathbb{Z})$ where $\xi \in NS(S)$, $r \geq 2$, $(r, \xi) = 1$ and $v^2 \geq 0$. Moreover, let $\omega$ be a $v$–generic polarization. The moduli space $M^\mu_v(S,\omega)$ is projective if and only if $S$ is projective.

**Proof.** If $S$ is projective, then $M^\mu_v(S,\omega)$ is projective by Theorem [3.7]. Suppose then that $S$ is not projective, and write $\xi^2 = -2k$ for $k \geq 0$. Recall that $v^2 = \xi^2 - 2ra$: as $v^2 \geq 0$, we get $k \leq -ra$, so that $a \leq 0$.

Notice that $a = 0$ if and only if $v^2 = 0$ and $k = 0$.

As $M^\mu_v(S,\omega)$ is an irreducible holomorphically symplectic manifold, by [12] it is projective if and only if there is $L \in Pic(M^\mu_v(S,\omega))$ such that $q(L) > 0$, where $q$ is the Beauville form. By Proposition [5.1] it then suffices to show that if $\alpha \in (v^\perp)^{1,1}$, then $(\alpha, s) \leq 0$.

Let $\alpha = (s, \zeta, b)$: as $\alpha \in (v^\perp)^{1,1}$, we get $\zeta \in NS(S)$ and $\xi \cdot \zeta = rb + sa$. If $v^2 = 0$ and $\xi^2 = 0$, we then have $a = 0$ and $rb = \xi \cdot \zeta$. As 
\[ r^2b^2 = (\xi \cdot \zeta)^2 \leq \xi^2 \zeta^2 = 0, \]
it follows that $b = 0$, and $\alpha = (s, \zeta, 0)$. Hence $(\alpha, s) \leq 0$.

We are left with $v^2 = 0$ and $\xi^2 < 0$, or $v^2 > 0$. In both cases we have $a < 0$. Notice that 
\[ (\alpha, s) = \zeta^2 - 2s \frac{\xi \cdot \zeta}{r} - \frac{sa}{r} = \zeta^2 - 2s \frac{\xi \cdot \zeta}{r} + 2s^2 \frac{a}{r} = p(s), \]
and the coefficient of $s^2$ is negative. To show that $(\alpha, s) \leq 0$ it then suffices to show that if $\Delta$ is the discriminant of $p(s)$, then $\Delta \leq 0$. But 
\[ \Delta = \frac{(\xi \cdot \zeta)^2}{r^2} - 2\xi^2 \frac{a}{r} \leq \frac{\xi^2 \zeta^2}{r^2} - 2\xi^2 \frac{a}{r} = \frac{\zeta^2}{r} \left( -\frac{2k}{r} - 2a \right), \]
as $(\xi \cdot \zeta)^2 \leq \xi^2 \zeta^2$. As $k \leq -ra$ and $\zeta^2 \leq 0$ (since $S$ is not projective), we get $\Delta \leq 0$, and we are done. \qed

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