WEIGHTED COMPOSITION OPERATORS ON WEAK VECTOR-VALUED WEIGHTED BERGMAN SPACES AND HARDY SPACES

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Abstract. In this paper we investigate weighted composition operators between weak and strong vector-valued weighted Bergman spaces and Hardy spaces.

1. Introduction and Preliminaries

Weighted composition operators have been studied on different spaces of analytic functions. In [5], Contreras and Hernandez-Diaz have made a study of weighted composition operators on Hardy spaces whereas Mirzakarimi and Siddighi [12] have studied these operators on Bergman and Dirichlet spaces. On Bloch-type spaces, these operators are explored by MacCluer and Zhao [11]. Ohno [13], Ohno and Zhao [14] and Ohno, Stroethoff and Zhao [15]. In [8] Kumar studied weighted composition operators between spaces of Dirichlet type by using Carleson measures.

Recently these studies are about spaces of vector-valued analytic functions. For example, in [17], Wang presented some necessary and sufficient conditions for weighted composition operators to be bounded on vector-valued Dirichlet spaces and Laitila, Tylli and Wang [10] studied composition operators from weak to strong vector-valued Bergman spaces Hardy spaces. For some information about vector-valued Bergman spaces see [1, 3].

Let $X$ be a complex Banach space and $\mathbb{D}$ be the open unit ball of $\mathbb{C}$. We consider weight as a strictly positive bounded continuous function $v : \mathbb{D} \to \mathbb{R}^+$. Let $p \geq 1$ and $v$ be a weight. The vector-valued weighted Bergman space $A_p^v(X)$ consists of all analytic functions $f : \mathbb{D} \to X$ such that

$$
\|f\|_{A_p^v(X)} = \left( \int_{\mathbb{D}} \|f(z)\|_{X}^p v(z) dA(z) \right)^{\frac{1}{p}} < \infty,
$$

where $dA$ is the normalized area measure on $\mathbb{D}$. Also, the vector-valued weighted Hardy space $H_p^v(X)$ consists of all analytic functions $f : \mathbb{D} \to X$ for which

$$
\|f\|_{H_p^v(X)} = \sup_{0 < r < 1} \left( \int_{T} \|f(r\zeta)\|_{X}^p v(r\zeta) \, dm(\zeta) \right)^{\frac{1}{p}} < \infty,
$$

where $dm(\zeta)$ is the normalized Lebesgue measure on the unit circle $T = \partial \mathbb{D}$. In the case $X = \mathbb{C}$, we write $A_p^v(X) = A_p^v$ and $H_p^v(X) = H_p^v$. Also, if $v \equiv 1$, then we have $A_p^v(X) = A_p(X)$ and $H_p^v(X) = H_p(X)$. The following weak versions of these spaces were considered by e.g. Blasco [2] and Bonet, Domanski and Lindstrom [4]:

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the weak spaces $wA^p_v(X)$ and $wH^p_v(X)$ consist of all analytic functions $f : \mathbb{D} \to X$ for which

$$
||f||_{wA^p_v(X)} = \sup_{||x^*|| \leq 1} ||x^* \circ f||_{A^p_v}, \quad ||f||_{wH^p_v(X)} = \sup_{||x^*|| \leq 1} ||x^* \circ f||_{H^p_v},
$$

are finite, respectively. Here $x^* \in X^*$, the dual space of $X$.

Let $\varphi$ be an analytic self-map of $\mathbb{D}$; that is $\varphi(\mathbb{D}) \subset \mathbb{D}$, and $u$ a scaler-valued analytic function on $\mathbb{D}$. We can define the weighted composition operator $uC_{\varphi}$ on the space of analytic functions as follows:

$$
uC_{\varphi}(f)(z) = u(z)f(\varphi(z)).
$$

When $u(z) \equiv 1$, we just have the composition operator $C_{\varphi}$, defined by $C_{\varphi}(f) = f \circ \varphi$. Also if $\varphi = I$, the identity function, then we get the multiplication operator $M_u$ defined by $M_u(f)(z) = u(z)f(z)$. It is well known that for every analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, $C_{\varphi} : A^p(X) \to A^p(X)$ and $C_{\varphi} : H^p(X) \to H^p(X)$ are bounded, and also on $wA^p(X), wH^p(X)$. For complete discussion about composition operators we refer to [6, 16]. We consider the infinite dimensional complex Banach space $X$, since $wA^p(X) = A^p(X)$ and $wH^p(X) = H^p(X)$, for $\alpha > -1$ and any finite dimensional Banach space $X$.

But for the infinite dimensional complex Banach space $X$, $A^p(X) \neq wA^p(X)$ ($H^p(X) \neq wH^p(X)$) and $||.||_{wA^p(X)}$ is not equivalent to $||.||_{A^p(X)}$ on $A^p(X)$ (||.||_{wH^p(X)} is not equivalent to $||.||_{H^p(X)}$, see [10] Proposition 3.1 ([9] Example 15).

Our aim in this paper is to compute the norm of weighted composition operators between $wA^p_v(X)$ and $A^p_v(X)$, for $p \geq 2$ and also between $wH^p_v(X)$ and $H^p_v(X)$, for $p \geq 2$, where $v$ and $v'$ are weights.

Throughout the remainder of this paper, $c$ will denote a positive constant, the exact value of which will vary from one appearance to the next. The notation $A \equiv B$ means that there are positive constants $c_1$ and $c_2$ such that $c_1A \leq B \leq c_1A$.

## 2. Main Results

**Proposition 2.1.** Let $X$ be any complex Banach spaces, $v$ be a weight of the form $v = |\mu|$, where $\mu$ is an analytic function without any zeros on $\mathbb{D}$, $v'$ be a weight and $1 \leq p < \infty$. Then

$$(2.1) \quad ||uC_{\varphi} : wA^p_v(X) \to A^p_v(X)|| \leq \left( \int_{\mathbb{D}} \frac{|u(z)|^{p}\nu'(z)}{(1 - |\varphi(z)|^2)^{1/2}(\varphi(z))} \, dA(z) \right)^{1/p}, \text{ and}
$$

$$(2.2) \quad ||uC_{\varphi} : wH^p_v(X) \to H^p_v(X)|| \leq \sup_{0 < \zeta < 1} \left( \int_{\mathbb{T}} \frac{|u(r\zeta)|^{p}\nu'(r\zeta)}{(1 - |\varphi(r\zeta)|^2)^{1/2}(\varphi(r\zeta))} \, dm(\zeta) \right)^{1/p}.
$$

**Proof.** By Lemma 2.1 of [18] we have

$$
||f||_X \leq \frac{||f||_{A^p_v}}{(1 - |z|^2)^{1/2}v(z)},
$$

for any $f \in A^p_v$ and $z \in \mathbb{D}$. Thus, for $f \in wA^p_v(X)$, we have

$$
||f||_X = \sup_{||x^*|| \leq 1} ||x^* \circ f|| \leq \frac{1}{(1 - |z|^2)^2v(z)} \sup_{||x^*|| \leq 1} ||x^* \circ f||_{A^p_v},
$$

and

$$
||f||_{wA^p_v(X)} = \sup_{||x^*|| \leq 1} ||x^* \circ f||_{A^p_v},
$$

and

$$
||f||_{wH^p_v(X)} = \sup_{||x^*|| \leq 1} ||x^* \circ f||_{H^p_v}.
$$

By (2.1) and (2.2) we conclude the proof.

We now prove Proposition 2.1.
Hence
\[
\|uC_\varphi f\|_{A^p_\varphi(X)}^p = \int_D |u(z)|^p \|f(\varphi(z))\|_{X}^p v'(z) \, dA(z)
\]
\[
\leq \|f\|_{wA^p_\varphi(X)}^p \int_D \frac{|u(z)|^p v'(z)}{(1 - |z|^2)^2 u(\varphi(z))} \, dA(z).
\]

The proof of the theorem is complete. \(\square\)

For the next results we need the following Dvoretzky’s well-known theorem.

**Lemma 2.2.** Suppose that \(X\) is an infinite dimensional complex Banach space. Then for any \(\epsilon > 0\) and \(n \in \mathbb{N}\), there is a linear embedding \(T_n : l^2_n \rightarrow X\) such that

\[(1 + \epsilon)^{-1} \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq \left\| \sum_{j=1}^{n} a_j T_n e_j \right\|_X \leq \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \]

for any scalars \(a_1, a_2, \ldots, a_n\) and some orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(l^2_n\).

Now, we prove a lower bound for the operator \(uC_\varphi : wA^p_\varphi(X) \rightarrow A^p_\varphi(X)\), in the case \(2 \leq p < \infty\).

**Theorem 2.3.** Let \(X\) be any complex infinite-dimensional Banach space, \(v\) be a weight of the form \(v = |\mu|\), where \(\mu\) is an analytic function without any zeros on \(D\), \(v'\) be a weight and \(2 \leq p < \infty\). Then

\[(1 + \epsilon)^{-1} \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \leq \left\| \sum_{j=1}^{n} a_j T_n e_j \right\|_X \leq \left( \sum_{j=1}^{n} |a_j|^2 \right)^{1/2} \]

for any scalars \(a_1, a_2, \ldots, a_n\) and some orthonormal basis \(\{e_1, \ldots, e_n\}\) of \(l^2_n\).

**Proof.** We only prove there exists a positive constant \(c\) such that

\[\|uC_\varphi : wA^p_\varphi(X) \rightarrow A^p_\varphi(X)\| \geq c \left( \int_D \frac{|u(z)|^p v'(z)}{(1 - |\varphi(z)|^2)^2 v(\varphi(z))} \, dA(z) \right)^{1/p}.\]

Suppose that \(x \in X\) with \(\|x\| = 1\) and define \(g : D \rightarrow X\) by \(g(z) = \frac{1}{\mu(z)^p} x\). Then \(g\) is an analytic function on \(D\), and \(\|g\|_{wA^\infty_\varphi(X)} = 1\), so that

\[\|uC_\varphi\|^p \geq \|ug \circ \varphi\|_{A^p_{\varphiT}}^p = \int_D \frac{|u(z)|^p v'(z)}{v(\varphi(z))} \, dA(z).\]

Hence
\[
\int_{\{|z| \leq 1/2\}} \frac{|u(z)|^p v'(z)}{v(\varphi(z))} \, dA(z) \leq 4 \int_D \frac{|u(z)|^p v'(z)}{v(\varphi(z))} \, dA(z) \leq 4 \|uC_\varphi\|^p.
\]

So, it will be sufficient to prove that there exists a positive constant \(c\) such that

\[\|uC_\varphi\|^p \geq c \int_{\{|z| \geq 1/2\}} \frac{|u(z)|^p v'(z)}{v(\varphi(z))} \, dA(z).\]

Let \(\lambda_k = k^{2/p - 1/2}\), for any \(n \in \mathbb{N}\), we define functions \(f_n\) as follows

\[f_n(z) = \frac{1}{\mu(z)^p} \sum_{k=1}^{n} \lambda_k z^k T_n e_k,\]
where the linear embedding $T_n$ is the same as in Lemma 2.2 \[|T_n|| = 1 \text{ and } |T_n^{-1}| \leq (1 + \epsilon) \text{ and } (e_1, \ldots, e_n) \text{ is an orthonormal basis of } E^2.\] As in the proof of Theorem 3.2 [10], there exists $c > 0$ such that for $X^*$ with $||x^*|| \leq 1$, we have

$$||x^* \circ f_n||_{A^p} = ||\frac{1}{\mu(z)^p} \sum_{k=1}^{n} \lambda_k z^k x^* T_n e_k||_{A^p}$$

$$= ||\sum_{k=1}^{n} \lambda_k x^* (T_n e_k) z^k||_{A^p}$$

$$\leq c \left( \sum_{k=1}^{n} |x^* (T_n e_k)|^2 \right)^{1/2} \leq c.$$

It follows that $||f_n||_{wA^p(X)} \leq c$. Thus, Fatou’s lemma implies that

$$||u C_\varphi||_p \geq c^{-p} \limsup_{n \to \infty} ||u C_\varphi f_n||_{A^p,(X)}^p$$

$$= c^{-p} \limsup_{n \to \infty} \int_{\mathbb{D}} |u(z)|^p \left| \frac{1}{\mu(\varphi(z))^p} \sum_{k=1}^{n} \lambda_k \varphi(z)^k T_n e_k \right|^p dA(z)$$

$$= c^{-p} \limsup_{n \to \infty} \int_{\mathbb{D}} \left\| \sum_{k=1}^{n} \lambda_k \varphi(z)^k T_n e_k \right\| \left| \frac{u(z)|^p v'(z)}{v(\varphi(z))} \right| dA(z)$$

$$\geq \frac{c^{-p}}{(1 + \epsilon)^p} \limsup_{n \to \infty} \int_{\mathbb{D}} \left( \sum_{k=1}^{n} k^{4/p-1} |\varphi(z)|^{2k} \right)^{p/2} \left| \frac{u(z)|^p v'(z)}{v(\varphi(z))} \right| dA(z)$$

$$= \frac{c^{-p}}{(1 + \epsilon)^p} \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} k^{4/p-1} |\varphi(z)|^{2k} \right)^{p/2} \left| \frac{u(z)|^p v'(z)}{v(\varphi(z))} \right| dA(z)$$

$$\geq \frac{c_1 c^{-p}}{(1 + \epsilon)^p} \int_{\mathbb{D}} \left( \sum_{k=1}^{\infty} k^{4/p-1} |\varphi(z)|^{2k} \right)^{p/2} \left| \frac{u(z)|^p v'(z)}{v(\varphi(z))} \right| dA(z)$$

and the last inequality is derived by Lemma 2.3 [10]. As $\epsilon > 0$ was arbitrary, we obtain the desired lower bound estimate.

\[\square\]

**Theorem 2.4.** Let $X$ be any complex infinite-dimensional Banach space, $v$ be a weight of the form $v = |\mu|$, where $\mu$ is an analytic function without any zeros on $\mathbb{D}$, $v'$ be a weight and $2 \leq p < \infty$. Then

$$(2.5) \quad ||u C_\varphi : wH^p_v(X) \to H^p_v(X)|| \approx \left( \int_{\mathbb{T}} \left| \frac{u(\zeta)|^p v'(\zeta)}{(1 - |\varphi(\zeta)|^2) v(\varphi(\zeta))} \right| dm(\zeta) \right)^{1/p}.$$  

**Proof.** Similar to the proof of previous theorem, we only prove that there exists $c > 0$ such that

$$||uC_\varphi||_p \geq c \int_{\{\zeta \in \mathbb{T} : |\varphi(\zeta)|^2 \geq 1/2\}} \left| \frac{u(r\zeta)|^p v'(r\zeta)}{(1 - |\varphi(r\zeta)|^2) v(\varphi(r\zeta))} \right| dm(\zeta).$$

Let $\lambda_k = k^{1/p-1/2}$ and define

$$f_n(z) := \frac{1}{\mu(z)^p} \sum_{k=1}^{n} \lambda_k z^k T_n e_k.$$
where the linear embedding $T_n$ is the same as in Lemma 2.2. $|T_n| = 1$ and $|T_n^{-1}| \leq (1 + \epsilon)$ and $(e_1, \ldots, e_n)$ is an orthonormal basis of $e_2^\infty$. As in the proof of Theorem 2.2 [10], there exists $c > 0$ such that for $X^*$ with $||x^*|| \leq 1$, we have

$$\|x^* \circ f_n\|_{H_p^\infty} = \left\| - \frac{1}{\mu(z)} \sum_{k=1}^{n} \lambda_k z^{k-1} T_n e_k \right\|_{H_p^\infty}$$

$$= \left\| \sum_{k=1}^{n} \lambda_k x^* (T_n e_k) z^k \right\|_{H_p^\infty}$$

$$\leq c \left( \sum_{k=1}^{n} |x^* (T_n e_k)|^2 \right)^{1/2} \leq c.$$

Thus $\|f_n\|_{wH_p^\infty(X)} \leq c$ and by using Fatou’s lemma and Lemma 2.3 [10], we have

$$\|uC_\varphi^p \| \geq c^{-p} \limsup_{n \to \infty} \|uC_\varphi f_n\|_{H_p^\infty(x)}$$

$$= c^{-p} \limsup_{n \to \infty} \int_T |u(r\zeta)|^p \frac{1}{\mu(\varphi(r\zeta))} \sum_{k=1}^{n} \lambda_k \varphi(r\zeta)^k T_n e_k \|v'(r\zeta)\|_{X} \, dm(\zeta)$$

$$= c^{-p} \limsup_{n \to \infty} \int_T \| \sum_{k=1}^{n} \lambda_k \varphi(r\zeta)^k T_n e_k \|^p_X \frac{|u(r\zeta)|^p v'(r\zeta)}{\varphi(r\zeta)} \, dm(\zeta)$$

$$\geq c^{-p} \limsup_{n \to \infty} \int_T \left( \sum_{k=1}^{n} k^{2/p-1} |\varphi(r\zeta)|^2k \right)^{p/2} \frac{|u(r\zeta)|^p v'(r\zeta)}{\varphi(r\zeta)} \, dm(\zeta)$$

$$= c^{-p} \left( \sum_{k=1}^{n} k^{2/p-1} |\varphi(r\zeta)|^2k \right)^{p/2} \frac{|u(r\zeta)|^p v'(r\zeta)}{\varphi(r\zeta)} \, dm(\zeta)$$

$$\geq c_1 c^{-p} \left( \sum_{k=1}^{n} k^{2/p-1} |\varphi(r\zeta)|^2k \right)^{p/2} \frac{|u(r\zeta)|^p v'(r\zeta)}{\varphi(r\zeta)} \, dm(\zeta).$$

Take $\epsilon = 1$, then

$$\|uC_\varphi^p \| \geq c \int_T \frac{|u(r\zeta)|^p v'(r\zeta)}{1 - |\varphi(r\zeta)|^2 v(\varphi(r\zeta))} \, dm(\zeta).$$

As $r \to 1$,

$$\|uC_\varphi^p \| \geq c \limsup_{r \to 1} \int_T \frac{|u(r\zeta)|^p v'(r\zeta)}{1 - |\varphi(r\zeta)|^2 v(\varphi(r\zeta))} \, dm(\zeta)$$

$$\geq c \int_T \frac{|u(r\zeta)|^p v'(r\zeta)}{1 - |\varphi(r\zeta)|^2 v(\varphi(r\zeta))} \, dm(\zeta).$$

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