Memory Effect and BMS-like Symmetries for Impulsive Gravitational Waves

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Abstract

Cataclysmic astrophysical phenomena can produce impulsive gravitational waves that can possibly be detected by the advanced versions of present-day detectors in the future. Gluing of two spacetimes across a null surface produces impulsive gravitational waves (in the phraseology of Penrose \cite{1}) having a Dirac Delta function type pulse profile along the surface. It is known that BMS-like symmetries appear as soldering freedom while we glue two spacetimes along a null surface. In this note, we study the effect of such impulsive gravitational waves on test particles (detectors) or geodesics. We show explicitly some measurable effects that depend on BMS-transformation parameters on timelike and null geodesics. BMS-like symmetry parameters carried by the gravitational wave leave some “memory” on test geodesics upon passing through them.

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1 Introduction

The “memory effect” for gravitational waves [2, 3, 4, 5, 6, 7] has attracted considerable attention in recent times due to i) its theoretical connection with asymptotic symmetries and soft-theorems [8, 9], and ii) possibility of its detection in the advanced detectors like aLIGO [10] or in LISA [11, 12, 13, 14, 15]. Gravitational memory effect is described as permanent displacement (or change in velocity) of a system of freely falling particles (idealized detectors), initially at relative rest, upon passing of a burst of gravitational radiation. In asymptotically flat spacetimes, in the far region, this change can be related to the diffeomorphisms that preserve the asymptotic structure, i.e. with the action of Bondi-van der Burg-Matzner-Sachs (BMS) group [16]. Further, it has been shown that, this displacement or change in metric components due to the passage of gravitational radiation is just the Fourier transform of the Ward identity satisfied by the infrared sector (soft) of quantum gravity’s S-matrix, corresponding to the BMS like symmetries (like Supertranslations). These findings and Hawking-Perry-Strominger’s [17, 18] proposal- that black holes may possess an infinite number of soft hairs corresponding to the spontaneously broken supertranslation symmetry have generated a lot of activities towards finding BMS like symmetries near the horizon of black holes. This, however, has already been achieved in a number of ways. BMS group and its extended version (including superrotation) [19] has been recovered by analysing the diffeomorphisms that preserve the near horizon asymptotic structure of black holes [20, 21]. Recently it has been shown that, the BMS symmetries can be recovered at any null hypersurface situated at some finite location of any spacetime [22]. In the context of soldering of two spacetimes across a null hypersurface (like black hole horizons), BMS-like symmetries have been recovered in [23, 24]. Now a natural question arises: whether some kind of memory effect can be detected near the horizon of a black hole or at a null surface situated in some finite location in spacetime just like what has been found in far region? In terms of the emergence of extended symmetries near the horizon of a black hole, BMS memory has been indicated in [25]. Aim of this note to study gravitational memory effect at a finite location of spacetime and obtain some detectable feature of it on test particles or geodesics.

Memory effect for plane gravitational waves has been extensively studied in recent times [26, 27]. The search of memory effect has also been extended to impulsive plane gravitational waves [28]. Memory effect for plane-fronted impulsive waves on null geodesic congruence (massless test particles) has been found in [30], where BMS supertranslation type memory has been

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1See [29] for clarifications on the calculation method adopted in [28].
obtained. We extend these studies for timelike geodesics (test particles or detectors). We show both supertranslation and superrotation like memory effect arise when a pulse of gravitational wave crosses these test particles. We also show there are jumps in expansion and shear for a null congruence encountering impulsive gravitational waves (IGW).

IGW is an artefact of Penrose’s ‘cut and paste’ method to glue two spacetimes [1]. Mathematically speaking, these are spacetimes having a Riemann tensor containing a singular part proportional to Dirac delta function whose support is a null or light-like hypersurface. This kind of signal can be represented unambiguously as a pure gravitational wave part and a shell or matter part. Such impulsive signals are produced during violent astrophysical phenomena like supernova explosion or coalescence of black holes. Naturally studying the effect of such IGW on detectors (placed nearby the source) would be important from astrophysical perspective also.

In this note, we consider null hypersurfaces that represent the history of the IGW, produced by gluing two space-times across the null surface and study the effect of these burst of radiation on geodesics crossing it. As shown in [23] and [24], the gluing is not unique and one can construct infinitely many thin null-shells related to each other via BMS like transformations (on either side of the surface). Due to this, the geodesics crossing the IGW suffer BMS symmetry induced memory effect.

In the next section (2), we review singular null-shells in General Relativity and show how BMS-like soldering freedom arises when one tries to glue two spacetimes across a null-hypersurface. In section (3), we study the memory effect for IGW on timelike geodesics (detectors). We show how the relative position of test particles changes due to the passage of such a pulse signal. Next, we study the effect of IGW on null congruences. Here we calculate the jump in optical properties of geodesic congruence namely shear and expansion parameters due to the passage of IGW indicating memory effect. Finally, we conclude with discussions of the results and future goals.

2 Emergence of BMS invariance on null hypersurfaces

Here we briefly review how BMS symmetries emerge while one tries to solder two spacetimes across a null hypersurface [23, 24]. It is intriguing to find supertranslation and superrotation like symmetries as soldering freedom [23, 24]. Throughout this note, we work in four dimensions. Let us consider two manifolds $\mathcal{M}_+$ and $\mathcal{M}_-$, with the corresponding intrinsic metrics $g_{\mu\nu}^+(x^\mu_+)$ and $g_{\mu\nu}^-(x^\mu_-)$. We will use Greek and Latin alphabets for the spacetime and hypersurface indices respectively. These two manifolds are bounded by two hypersurfaces $\Sigma_+$ and $\Sigma_-$ respectively. A common coordinate system $x^\mu$ can be installed across the common null boundary $\Sigma$ of these two manifolds $\mathcal{M}_\pm$ and $x^\mu|\Sigma = \zeta^\mu$, where $\Sigma$, is the surface across which the two manifolds will be joined. Let $e^a_\mu$ are tangent vectors to $\Sigma$. We also have to define the auxiliary vector $N^\pm_a$ to complete the basis such that,

$$N.N|_\pm = 0, n.N = -1.$$  \hspace{1cm} (2.1)

So together with $e^a_\pm\mu$ they form a complete basis.
Then the junction condition states:

\[
\Sigma_+ = \Sigma_- = \Sigma, \quad g_{ab} = g_{\mu\nu}^+ e_\alpha^a e_\beta^b | \Sigma_+ = g_{\mu\nu}^- e_\alpha^a e_\beta^b | \Sigma_- ,
\]  

(2.2)

Together with

\[
e^\mu_a | \Sigma_+ - e^\mu_a | \Sigma_- = 0, \quad n^\mu | \Sigma_+ - n^\mu | \Sigma_- = 0, \quad N^\mu | \Sigma_+ - N^\mu | \Sigma_- = 0
\]

(2.3)

Where,

\[
e^\pm_\alpha = \frac{\partial x^\mu}{\partial \xi^\alpha}.
\]

(2.4)

The normal vectors are defined as,

\[
n^\pm_\alpha = \chi \partial_\alpha (\Phi(x^\mu)) \quad \text{for both} \quad \Sigma_+ \quad \text{and} \quad \Sigma_- ; \quad \chi \text{is an arbitrary normalization.}
\]

These normals (we choose them to be future directing) are generators for the null congruences orthogonal to both the hypersurfaces \( \Phi(x^\mu) = 0 \).

The junction condition (2.2) enable us to use the distributional tensor calculus for solving Einstein equation. The metric \( g_{\mu\nu} \) for the entire spacetime \( M = M_+ \cup M_- \) can be written as,

\[
g_{\mu\nu} = g_{\mu\nu}^+ \mathcal{H}(\Phi) + g_{\mu\nu}^- \mathcal{H}(-\Phi),
\]

(2.5)

Where \( g^+ \) and \( g^- \) has been expressed in terms of the common chart \( x^\mu \) covering the total spacetime \( M \) and \( \mathcal{H}(\Phi) \) is the Heaviside step function. The Riemann tensor takes the following form,

\[
R^\alpha_{\beta\gamma\delta} = R^+_{\alpha\beta\gamma\delta} \mathcal{H}(\Phi) + R^-_{\alpha\beta\gamma\delta} \mathcal{H}(-\Phi) + \delta(\Phi) Q^\alpha_{\beta\gamma\delta},
\]

(2.6)

Where, \( Q^\alpha_{\beta\gamma\delta} = -\left( [\Gamma^\alpha_{\beta\delta} n_\gamma - \Gamma^\alpha_{\beta\gamma} n_\delta] \right) \). Where \( \left[ \right] \) denotes difference of quantities between + and − sides. Clearly if any component of the Riemann tensor proportional to Delta function is not zero, we will have impulsive gravitational wave supported on the null hypersurface \( \Sigma \).

Now following [31, 32, 33, 34] one can show that in general these spacetimes satisfy the Einstein equations with the following stress tensor,

\[
T_{\alpha\beta} = T^+_{\alpha\beta} \mathcal{H}(\Phi) + T^-_{\alpha\beta} \mathcal{H}(-\Phi) + S_{\alpha\beta}(\Phi),
\]

(2.7)

With \( S_{ab} = S_{\alpha\beta} e_\alpha^a e_\beta^b \) is denoted as the stress tensor of the null shell across which we are soldering \( M_\pm \). It takes the following form [31, 32, 33, 34],

\[
S^\alpha_{\beta} = \mu n^\alpha n^\beta + J^A (n^\alpha e^\beta_A + e^\alpha_A n^\beta) + p \sigma^{AB} e^\alpha_A e^\beta_B,
\]

(2.8)

Where \( \sigma^{AB} \) is the metric of the spatial slice of the surface of the null shell and \( A, B \) denote the spatial indices of the null surface, \( \mu, J^A \) and \( p \) are surface energy density, current and pressure of shell respectively. Note that, the null-hypersurface has now become a shell with some null matter supported on the surface \( \Sigma \). We identify various intrinsic quantities of the shell as follows,

\[
\mu = -\frac{1}{8\pi} \sigma^{AB} [\mathcal{K}_{AB}], \quad J^A = \frac{1}{8\pi} \sigma^{AB} [\mathcal{K}_{VB}], \quad p = -\frac{1}{8\pi} [\mathcal{K}_{VV}],
\]

(2.9)

Where,

\[
\gamma_{ab} = N^\alpha [\partial_\alpha g_{ab}] = 2[\mathcal{K}_{ab}].
\]

(2.10)
\( K_{ab} = e_a^\alpha e_b^\beta \nabla \alpha N_\beta \) is ‘oblique’ extrinsic curvature. Also,

\[
[K_{VV}] = \frac{1}{2} \gamma_{\alpha \beta} n^\alpha n^\beta, \quad [K_{VA}] = \frac{1}{2} \gamma_{\alpha \beta} e^\alpha_A n^\beta, \quad [K_{AB}] = \frac{1}{2} \gamma_{\alpha \beta} e^\alpha_A e^\beta_B. \tag{2.11}
\]

Note that jump in partial derivative of \( \gamma_{\mu \nu} \) is proportional to the normal covector of the surface \( \Sigma \):

\[
[\partial_\lambda g_{\mu \nu}] = \gamma_{\mu \nu} n_\lambda. \tag{2.12}
\]

Now let us recap how BMS like symmetries emerge as soldering freedom. We work in Kruskal coordinates \((U, V, x^A)\). Coordinates for the surface \( \Sigma \) are \( \zeta^a = \{V, x^A\} \), where \( V \) is the parameter along the hypersurface generating null congruences. In this coordinate system we will have,

\[
n^\alpha = (\partial_V)^\alpha. \tag{2.13}
\]

Now to find the all possible allowed coordinate transformations on either side of the shell preserving the junction condition (2.2), we need to solve the Killing equation for \( g_{ab} \), the metric on \( \Sigma \).

\[
\mathcal{L}_Z g_{ab} = 0. \tag{2.14}
\]

It was shown in [23], this equation, for a Killing horizon, has solutions leading to \( Z^V = F(V, x^A) \), i.e. an arbitrary function \( F \) depending on the coordinates on the null hypersurface. This indicates there exist infinitely many ways to solder two spacetimes without altering the junction condition. This is termed as soldering freedom. For the special case when \( F \) becomes \( V \) independent, this resembles the BMS-supertranslation like symmetry [23].

- **Supertranslation like freedom on Killing horizon:**

We first consider Schwarzschild black hole,

\[
ds^2 = -2G(r)dUdV + r^2(U, V)(d\theta^2 + \sin^2(\theta)d\phi^2), \tag{2.15}
\]

where

\[
G(r) = \frac{16m^3}{r}e^{-r/2m}, \quad UV = -\left(\frac{r}{2m} - 1\right)e^{r/2m}. \tag{2.16}
\]

The null shell is located at Killing horizon \( U = 0 \). Following [23] one can show that,

\[
Z^V \partial_V g_{AB} + Z^C \partial_C g_{AB} + \partial_A Z^C g_{BC} + \partial_B Z^C g_{AC} = 0, \tag{2.17}
\]

leading \( Z^V = F(V, \theta, \phi) \). Furthermore if one also demands \( \mathcal{L}_Z n^a = 0 \), then it gives,

\[
\partial_V Z^V = 0. \tag{2.18}
\]

This will further give,

\[
Z^V = T(\theta, \phi). \tag{2.19}
\]

It generates the following symmetry,

\[
V \rightarrow V + T(\theta, \phi) \tag{2.20}
\]

and this is strikingly similar to supertranslation found at null infinity of asymptotically flat spacetimes [23]. This construction was then extended for the rotating blackhole spacetimes in [24].
• **Superrotation like soldering freedom:**

If the non-degenerate subspace of a null surface is parametrized by some null parameters (e.g., null coordinates $U$ or $V$) then (2.17) offers a new kind of solution. This has been shown in [24]. This situation gives rise superrotations or conformal rescaling of the base space of null hypersurface. To see this, consider the following metric:

$$ds^2 = \frac{2(V - \eta U)^2 d\zeta d\bar{\zeta} + 2dUdV - 2\eta dU^2}{(1 + \frac{\Lambda U^6}{6}(V - \eta U))^2}. \quad (2.21)$$

This metric can represent de Sitter space when $\Lambda > 0, \eta = 1$, anti-de Sitter space when $\Lambda < 0, \eta = -1$ and Minkowski space when $\Lambda = 0, \eta = 0$. The null surface is at $U = 0$. For this case the following soldering freedoms are obtained [24]

$$V \rightarrow V(1 - \frac{\epsilon \hat{\Omega}(\zeta, \bar{\zeta})}{2}),$$

$$\zeta \rightarrow \zeta + \epsilon h(\zeta), \bar{\zeta} \rightarrow \bar{\zeta} + \epsilon \bar{h}(\bar{\zeta}), \quad (2.22)$$

where,

$$\hat{\Omega}(\zeta, \bar{\zeta}) = \frac{(1 + \eta \bar{\zeta}) (h'(\zeta) + \bar{h}'(\bar{\zeta})) - 2\eta \bar{\zeta} h(\zeta) - 2\eta \zeta \bar{h}(\bar{\zeta})}{1 + \eta \zeta \bar{\zeta}}. \quad (2.23)$$

$h(\zeta)$ and $\bar{h}(\bar{\zeta})$ are holomorphic and anti-holomorphic functions and $\epsilon$ is a small parameter. For simplicity we have only kept terms linear in $h(\zeta)$ and $\bar{h}(\bar{\zeta})$ but this can be generalized for all order of $h(\zeta)$ and $\bar{h}(\bar{\zeta})$. These kinds of local conformal transformations are usually referred as superrotation in the context of asymptotic symmetries. Emergence of such symmetries in the context of Penrose’s impulsive wave spacetime (introducing snapping cosmic string) has been discussed also in [35].

• **Null surface near Killing Horizon:**

There is another case where we can also get these superrotations, a null surface situated just a little away from horizon of a black hole [24]. We will restrict ourselves to four-dimensional Schwarzschild black hole. The metric on the horizon is of the form $r^2 d\Omega^2$ with $d\Omega^2 = \frac{d\zeta d\bar{\zeta}}{(1 + \frac{\xi}{\zeta})^{1/2}}$. $r^2$ takes the following form near the horizon in Kruskal like coordinates

$$r^2 = a + b (U V) + c (UV)^2 + \cdots \quad (2.24)$$

with, $a = 4m^2, b = -\frac{8m^2}{\epsilon}$. We consider a null shell located close to the horizon($U = 0$) but slightly away from it (i.e at $U = \epsilon$, where $\epsilon$ is very small but not zero), then the allowed transformations are,

$$V \rightarrow V(1 - \frac{\hat{\Omega}(\zeta, \bar{\zeta})}{2}) - \frac{a \hat{\Omega}(\zeta, \bar{\zeta})}{b \epsilon} + O(\epsilon),$$

$$\zeta \rightarrow \zeta + h(\zeta), \bar{\zeta} \rightarrow \bar{\zeta} + \bar{h}(\bar{\zeta}). \quad (2.25)$$

Here $\epsilon$ is small and we will only keep terms upto linear in $\epsilon, h(\zeta)$, and $\bar{h}(\bar{\zeta})$.

## 3 Memory Effect: Timelike congruence

In this section, we study the memory effect for IGW on test particles or on a timelike geodesics congruence. We must detect some observable effects on the relative motion of neighbouring test
particles. To see this effect we consider two test particles whose worldline (timelike geodesic) passing through a null hypersurface supporting IGW. The effect of IGW on the separation vector separating two such nearby test particles (detectors) can be captured by the use of geodesic deviation equation (GDE) and junction conditions. Theory of impulsive gravitational waves is a well-studied area of research [31, 36, 37, 38]. Usually, two different approaches are there to study the effect of IGW on geodesics. In [28, 36, 37, 38] etc, a discontinuous coordinate transformation is used and no null-shell is introduced. Here, we use a local coordinate system in which the metric tensor is continuous but its first derivative has discontinuity across the history of the signal and include a thin null-shell along with the wave signal. There are rigorous results reported in exact IGW spacetimes describing the existence, uniqueness and generic properties of geodesics across a null impulsive surface [39, 40, 41]. However our approach is somewhat different. We closely follow the construction given in [31, 32] and review the necessary parts here.

Let us consider a congruence having $T^\mu$ to be the tangent vector, where $T^\mu$ is a unit time-like vector field.

$$g_{\mu\nu}T^\mu T^\nu = -1$$  \hspace{1cm} (3.1)

The integral curves of $T^\mu$ happens to be time-like geodesics. Therefore,

$$\dot{T}^\mu = \nabla_\nu T^\mu T^\nu = 0.$$  \hspace{1cm} (3.2)

Dot denotes the covariant derivative of a tensor field in $T^\mu$ direction. This congruence is passing through the null shell located at $\Sigma$. Now we introduce a separation vector $X^\mu$ satisfying $g_{\mu\nu}T^\mu X^\nu = 0$ such that,

$$\dot{X}^\mu = \nabla_\nu T^\mu X^\nu.$$  \hspace{1cm} (3.3)

This vector $X^\mu$ satisfies the geodesic deviation equation,

$$\ddot{X}^\mu = -R^\mu_{\lambda\sigma\rho}T^\lambda X^\sigma T^\rho.$$  \hspace{1cm} (3.4)

$R^\mu_{\lambda\sigma\rho}$ is Riemann tensor of full space-time $\mathcal{M}$. With the junction conditions and (2.12) in mind one can plausibly set the jump in partial derivatives of $T^\mu$ and $X^\mu$ across $\Sigma$ are given by,

$$[T^\mu_\lambda] = P^\mu n_\lambda, \quad [X^\mu_\lambda] = W^\mu n_\lambda,$$  \hspace{1cm} (3.5)

for some vectors $P^\mu$ and $W^\mu$ defined on $\Sigma$. Let $\{E_a\}$ be a triad of vector fields defined along the time-like geodesics tangent to $T^\mu$ by parallel transporting $\{e_a\}$ along these geodesics. Therefore,

$$\dot{E}_a^\mu = 0,$$  \hspace{1cm} (3.6)

where, $E_a|_\Sigma = e_a$. Consequently the jump in the partial derivatives of $E_a^\mu$ should take the following form for some $F_a^\mu$ defined on $\Sigma$,

$$[E_a^\mu_\lambda] = F_a^\mu n_\lambda.$$  \hspace{1cm} (3.7)

Let $X^\mu_\mu (0)$ is the vector $X^\mu$ evaluated on $\Sigma$ i.e. the value of separation vector $X^\mu$ at the intersection point of the null hypersurface with the timelike congruence. Similarly let $T^\mu_\mu (0)$ denotes the tangent vector $T^\mu$ evaluated on $\Sigma$. We can write $X^\mu_\mu (0)$ in terms of components along the orthogonal and tangential vectors to the hypersurface in the following way,

$$X^\mu_\mu (0) = X_\mu (0)T^\mu_\mu (0) + X_a^\mu e_a^\mu,$$  \hspace{1cm} (3.8)
for some function $X_\mu(0)$, and $X^0_\mu(0)$ is the $X^\alpha$ evaluated at $\Sigma$. Now if $\Sigma$ is located at $U = 0$ (this happens typically in Kruskal-like coordinates), then using (3.5), expanding around $U = 0$ and keeping only the linear term we get,

$$X^\mu = X^\mu - U \theta(U) W^\mu$$

with $X^\mu$ in general depends on $U$ (in the $-$ side) such that when $U = 0$, $X^\mu = X^{\mu}_0$. It is convenient to calculate $X^\mu$ on the basis $\{E^\mu_a, T^\mu\}$. Its non-vanishing components,

$$X_a = g_{\mu\nu} X^\mu E^\nu_a.$$  

Using (3.8) and (3.9), we calculate the above expression for small $U > 0$ (in $+$ side),

$$X_a = (\tilde{g}_{ab} + \frac{1}{2} U \gamma_{ab}) X^b(0) + U V^a(0)$$

with $\gamma_{ab}$ defined via (2.10). Here $\gamma_{ab}$ is the pseudo-inverse of $g_{ab}$ with entries at the $V$-direction are zero defined as [33],

$$g^{ab}_{\ast} g_{bc} = \delta^a_c - n^a N_c, \quad g^{cd}_{\ast} \gamma_{cd} = g^{AB} \gamma_{AB}.$$  

Also, $\gamma^\dagger = \gamma_{ab} n^a n^b$ and $\gamma_{ab}$ is defined via (2.10). $\gamma_{ab}$ encodes the pure gravitational wave degrees of freedom as $\gamma_{ab} n^a = 0 = g^{ab}_{\ast} \gamma_{ab}$. On the other hand, $\gamma_{ab}$ encodes the degrees of freedom corresponding to the null matter of the shell. If $\gamma_{ab}$ vanishes then we will have a pure impulsive gravitational wave. Otherwise, the IGW will be accompanied by some null-matter.

Next, from (3.15) and using (2.9) in Kruskal type coordinate we get,

$$\tilde{\gamma}_{VB} = \tilde{\gamma}_{BV} = 16\pi g_{BC} S^{VC}, \quad \gamma_{AB} = -8\pi S^{VV} g_{AB}.$$  

Using (3.11), (3.12), (3.15) and (3.16) we get,

$$X_V = X^V(0) + \frac{1}{2} U \tilde{\gamma}_{VB} X^B(0) + U V^V(0),$$

$$X_A = g_{AB} X^B(0) + \frac{1}{2} U \gamma_{AB} X^B(0) + U V^V(0).$$
Now if we assume initially the test particles reside at the space like 2-dimensional surface (signal front) then we must have,

\[ X(0)V = V(0)V = 0, \quad (3.20) \]

leading to (using (3.18)),

\[ X_V = \frac{1}{2} U \tilde{\gamma}_{VB} X_B(0) = 8\pi U g_{BC} S^{VC} X_B(0), \]

\[ X_A = (1 - 4\pi U S^{VV})(g_{AB} + \frac{U}{2} \tilde{\gamma}_{AB}) X_B(0) + U V(0)_A, \quad (3.21) \]

Note that, the If \( S^{VC} \neq 0 \) then \( X_V \neq 0 \), which indicates the test particles will be displaced off the 2-dimensional surface. This is only possible if the anisotropic stress of the shell is non zero. Since \( \tilde{\gamma} \) does not affect \( X^V \), this component of deviation vector can only have non-vanishing change if the IGW is the history of a null-shell containing anisotropic stress. In the plane fronted wave we can have shell without any matter, in which this part will be zero [31]. However, for spherical-shells (such as horizon-shell in Schwarzschild spacetime) we can’t have a pure gravitational wave without matter [23, 24]. So we must investigate those cases carefully. For shells having \( S^{VC} = 0 \), we only have the spatial parts of the deviation vector non-vanishing,

\[ X_A = (1 - 4\pi U S^{VV})(g_{AB} + \frac{U}{2} \tilde{\gamma}_{AB}) X_B(0), \quad (3.22) \]

The second bracket in (3.22) with \( \tilde{\gamma}_{AB} \) given below describes the usual distortion effect of the wave part of the signal on the test particles in the signal front. Whereas there is an overall diminution factor in the first bracket of the above expression due to the presence of the null-shell. We now examine the expression of deviation vector components case by case.

### 3.1 Memory effect and supertranslations at the black hole horizon

We first consider horizon shell of Schwarzschild spacetimes (2.15). For this case following [23] we have,

\[ \tilde{\gamma}_{\theta\phi} = 2 \frac{\nabla^{(2)}_{\theta} \partial_{\phi} F}{F_V} \quad (3.23) \]

\[ \tilde{\gamma}_{\theta\theta} = 2 \left( \gamma_{\theta\theta} - \frac{1}{\sin^2 \theta} \gamma_{\phi\phi} \right) \quad (3.24) \]

If we focus on the case (2.20) for which we get supertranslation from the horizon soldering freedom, we get a shell having zero pressure \( p = 0 \) and current \( j^A = 0 \) i.e \( S^{VA} = 0 \) [23]. Then we will get,

\[ \tilde{\gamma}_{\theta\phi} = 2 \nabla^{(2)}_{\theta} \partial_{\phi} T(\theta, \phi), \tilde{\gamma}_{\theta\theta} = 2 \left( \nabla^{(2)}_{\theta} \partial_{\theta} T(\theta, \phi) - \frac{1}{\sin^2 \theta} \nabla^{(2)}_{\phi} \partial_{\phi} T(\theta, \phi) \right). \quad (3.25) \]

Also following [23] we get,

\[ S^{VV} = -\frac{1}{32m^2 \pi} (\nabla^2 T(\theta, \phi) - T(\theta, \phi)). \quad (3.26) \]
The shell allows impulsive gravitational waves along with some matter and the neighbouring test particles will encode this into the $X^A$ components of the deviation vector (3.21) ($X_V$ is zero):

$$X_\theta = \left(1 + \frac{U}{8m^2} \left(\nabla^2 T(\theta, \phi) - T(\theta, \phi)\right)\right) \left\{4m^2 + U \left(\nabla^{(2)}_\theta \partial_\phi T(\theta, \phi) - \frac{1}{\sin^2 \theta} \nabla^{(2)}_\phi \partial_\theta T(\theta, \phi)\right)\right\}X_\theta^{(0)}$$

$$+ U \nabla^{(2)}_\theta \partial_\phi T(\theta, \phi)X_\phi^{(0)}$$

(3.27)

and similarly for $X^\phi$ component.

Clearly, there is a distortion in the relative position of the particles after encountering the impulsive gravitational wave and the distortions are determined by the supertranslation parameter $T(\theta, \phi)$. The displacement is confined to 2-surface and for physical matter (See (A.14) of the appendix) having a positive energy density, there will be a diminishing effect on the test particles. Hence this is a reminiscence of BMS memory effect (velocity) at future null infinity. We may integrate this expression with respect to the parameter of the geodesics to get a shift in the spatial direction and obtain the displacement memory effect. For a slowly rotating spacetime we can get similar kind of memory effect following [24].

3.2 Memory effect and superrotation like transformations

To see the effect of superrotation like freedom of null shell on test particles we first focus on plane fronted wave signal introduced by Penrose. The analysis for constant negative and positive curvature spaces can be done similarly. Extending the transformation (2.22) off-shell we can compute the $\hat{\gamma}_{AB}$ and the stress tensor (2.8). The details of the computation is given in the appendix A. We simply quote the results below.

$$\hat{\gamma}_{\zeta\zeta} = \epsilon V h''(\zeta), \quad \hat{\gamma}_{\zeta\bar{\zeta}} = \epsilon V \bar{h}''(\bar{\zeta}).$$

(3.28)

Also we get,

$$S^{VV} = 0, \quad S^{V\zeta} = j^\zeta = \frac{\epsilon \bar{h}''(\bar{\zeta})}{16\pi V^2}, \quad S^{V\bar{\zeta}} = j^{\bar{\zeta}} = \frac{\epsilon h''(\zeta)}{16\pi V^2}$$

(3.29)

Using this we get,

$$X_\zeta = V \left(V X_\zeta^{(0)} + \epsilon \frac{U}{2} \bar{h}''(\zeta)X_\zeta^{(0)}\right), \quad X_{\bar{\zeta}} = V \left(V X_{\bar{\zeta}}^{(0)} + \epsilon \frac{U}{2} \bar{h}''(\bar{\zeta})X_{\bar{\zeta}}^{(0)}\right).$$

(3.30)

Here $\epsilon$ is an infinitesimal parameter introduced to emphasize the infinitesimal conformal transformations. Also note that unlike the previous case we have some non-zero anisotropic trace leading to,

$$X_V = \frac{\epsilon U}{2} \left(\bar{h}''(\bar{\zeta})X_{\bar{\zeta}}^{(0)} + h''(\zeta)X_\zeta^{(0)}\right).$$

(3.31)

Nevertheless, it is again evident from (3.30) that, there is a distortion of the particles after encountering the impulsive gravitational wave and the distortions are determined by superrotation parameter. Due to the presence of anisotropic stress, this distortion moves the test particles off the spatial 2-dimensional (as $X_V$ is non-zero) surface where they were initially situated at rest.
4 Memory Effect on Null geodesics

4.1 Setup

Here we shall consider a null congruence crossing the hypersurface $\Sigma$ orthogonally having a tangent vector $N$ and see the change in the optical parameters like shear, expansion, etc.. As already has been mentioned, we will be considering a coordinate system in which metric tensor is continuous across the null-shell. We will see the change in the expansion and shear corresponding to the congruence defined by this transverse vector $N$ as it crosses the IGW. To do so, we take a congruence $N_+$ to the $'+'$ side of null-shell and compute the components of it at the null surface incorporating the soldering conditions. When this vector is expressed in a continuous coordinate system, and evolved to the other side of the shell, we get jumps in the expansion and shear corresponding to the congruence. The setup is shown in the Fig. (1).

For the null geodesics, $N^\mu$ is already defined in (2.1), which can be thought as the tangent vector of some hypersurface orthogonal (locally with the null analogue of Gaussian normal coordinates) to null geodesics across $\Sigma$. We denote $N_-$ to be the tangent to the congruence to the $'-'$ side of $\Sigma$. Mathematically the effect of the congruence $N_+$ crossing the null-shell is obtained by applying a coordinate transformation between the coordinates $x^{+\alpha}$ to a common continuous coordinate system across the shell [30]. This relation between coordinates are just what we call as soldering transformations and consequently, the quantities evaluated will carry the soldering parameters indicating memory effect.

Let $N_0$ denotes the vector $N_+$ transformed in the common coordinate system and evaluated at the hypersurface $\Sigma$. We take coordinates $x^{\mu}$ coinciding with $x^{\alpha}$. For expressing $N_0$ in a common coordinate system needs the following transformation relation:

$$N_0^\alpha(x) = \left( \frac{\partial x^\beta}{\partial x^\alpha} \right)^{-1} N_+^\beta |_{\Sigma}$$ (4.1)

The inverse Jacobian matrix $\left( \frac{\partial x^\beta}{\partial x^\alpha} \right)^{-1}$ is to be evaluated using the “on-shell” version of coordi-
nate transformations \(^3\). The vector in the \(\prime-\prime\) side is obtained by taking components of \(N_0\) as initial values.

Next, we compute the “failure” tensor \(B\) for the vector \(N_0\) by taking the covariant derivative of it and projecting it on the hypersurface. This tensor encodes the amount of failure for the congruence to remain parallel to each other.

\[
B_{AB} = e^\alpha_A e^\beta_B \nabla_\beta N_0 \alpha
\]

Kinematical decomposition of this tensor leads two measurable quantities (assuming zero twist for hypersurface orthogonal congruence):

\[
\text{Expansion} : \Theta = \gamma^{AB} B_{AB}, \quad (4.2)
\]

\[
\text{Shear} : \Sigma_{AB} = B_{AB} - \frac{\Theta}{2} \gamma_{AB}, \quad (4.3)
\]

where \(\gamma_{AB}\) is the induced metric on the null shell. Then memory of the IGW or shell is captured through the jump of these quantities across \(\Sigma\). \(^4\) Note that if we use “on-shell” version of the soldering transformation, jump in these quantities will be captured for a geodesic crossing the shell originating from global + side of the shell.

\[
[\Theta] = \Theta \bigg|_{\Sigma^+} - \Theta \bigg|_{\Sigma^-},
\]

\[
[\Sigma_{AB}] = \Sigma_{AB} \bigg|_{\Sigma^+} - \Sigma_{AB} \bigg|_{\Sigma^-}. \quad (4.4)
\]

Next, we let the test geodesics to travel off-the shell in the other side of the shell, and we get additional contributions in jumps of the expansion and shear. This is achieved by transforming \(B\) tensor further to the \(\prime-\prime\) side of the shell via the following relation that is obvious from hypersurface orthogonal nature of the congruence:

\[
x^\mu_\prime \equiv x^\mu = x^\mu_0 + U N^\mu_0 (x^\mu) + \mathcal{O}(U^2), \quad (4.5)
\]

As already have been mentioned in the \(-\) side the coordinates \(x^\mu_\prime\) coincides with \(x^\mu\) and we denote \(x^\mu_0 = (x^\mu)\bigg|_{\Sigma}\). Note that all the additional contributions in expansion and shear are proportional to \(U\) (assuming Kruskal like coordinates), the parameter of off-shell extension and would coincide with the jumps when we restrict the transformations on the shell, i.e. for \(U = 0\).

Now after determining these off shell transformations we can evaluate the failure tensor. For determining \(B\) off the shell we first compute \(B_{AB} = e^\alpha_A e^\beta_B \nabla_\beta N_0 \alpha\) and then pull it back to a point infinitesimally away from the shell.

\[
\tilde{B}_{AB}(x^\mu) = \frac{\partial x^M_0}{\partial x^A} \frac{\partial x^N_0}{\partial x^B} B_{MN}(x^\mu_0) \quad (4.6)
\]

In the far region of the shell, \(B\) tensor is evaluated using \(N_+ = \lambda \partial_U\), and clearly it will be different from the expression we get from the above equation leading to the jumps in its different components. Next, we show these jumps for shells in Schwarzschild spacetime.

\(^3\)We call soldering freedom confined on the null hypersurface to be “on-shell” see (A.1) of appendix A. The “off-shell” version can be obtained from the “on-shell” version by extending the coordinate transformations off-the surface in the transverse direction, see (A.2) of appendix A .

\(^4\)We discard the possibility of generating a non-vanishing twist to the congruence, initially possessing a zero-twist, after IGW pass through it. This is ensured from the evolution equation [30] of twist.
4.2 Memory for supertranslation like transformations

We consider the case of the Schwarzschild black hole in Kruskal coordinate (2.15). For plane fronted waves a similar kind of memory effect has been reported in [30]. Here we extend that for spherical waves. For Schwarzschild spacetime the null shell is located at the killing horizon $U = 0$. For this case a supertranslation type symmetry arises due to the soldering freedom [23]. On + side we can perform the coordinate transformations mentioned in (2.17) keeping the junction condition intact. For supertranslation it will become the one mentioned in (2.20).

We next compute the failure tensor off-the shell. Using the inverse Jacobian of transformations given in (B.1) we get,

$$ N_0 = \frac{e}{8m^2} \left( \partial_U + \frac{1}{4F^2} \frac{F^2_{\phi}}{\sin^2 \theta} \partial_V - \frac{F_\theta}{2F V m^2 e} \partial_\theta - \frac{F_\phi}{2F V m^2 e \sin^2 \theta} \partial_\phi \right) \bigg| \Sigma. \quad (4.7) $$

With this and using (B.4) one can evaluate the failure tensor off-the shell:

$$ \tilde{B}_{AB} = \frac{1}{\det(J)^2} \begin{pmatrix} \left( 1 + \frac{2U}{e} T_{\phi \theta} \right)^2 + \frac{4U^2}{e^2} T^2_{\phi \theta} & -\frac{2U}{e} T_{\phi \theta} \left( 2 + \frac{2U}{e} T_{\phi \theta} + \frac{2U}{e} T_{\phi \phi} \right) \\ -\frac{2U}{e} T_{\phi \theta} \left( 2 + \frac{2U}{e} T_{\phi \theta} + \frac{2U}{e} T_{\phi \phi} \right) & \left( 1 + \frac{2U}{e} T_{\phi \theta} \right)^2 + \frac{4U^2}{e^2} T^2_{\phi \phi} \end{pmatrix} B_{MN}, \quad (4.8) $$

The failure tensor $B_{AB}$ for the congruence with $N_+ = \lambda \partial_U$ we get the following expression,

$$ B_{AB} = e^\alpha A B^\beta \nabla^\beta N_\alpha = -\Gamma^V_B A N_V \quad (4.9) $$

We find the $B$-tensor before and after the null congruence crosses the shell to be different which means there is a jump. Clearly for $U = 0$, the off-shell failure tensor $\tilde{B}$ reduces to $B$ evaluated at the shell $\Sigma$. Then focusing particularly on supertranslation, and using (2.20) and (B.2) one gets the following jumps across the shell:

$$ \begin{align*}
[\Theta] &= \frac{1}{(4m^2)^2} \left( T_{\theta \theta} + \csc(\theta)^2 T_{\phi \phi} + T_{\phi \theta} \cot(\theta) \right), \\
[\Sigma_{\theta \theta}] &= \frac{1}{8m^2} \left( T_{\theta \theta} - T_{\phi \phi} \sin^2(\theta) - T_{\phi \theta} \cot(\theta) \right), \\
[\Sigma_{\phi \phi}] &= \frac{1}{8m^2} \left( T_{\phi \phi} - T_{\theta \theta} \sin^2(\theta) + \frac{T_{\phi \theta} \sin(2\theta)}{2} \right), \\
[\Sigma_{\theta \phi}] &= \frac{1}{2m^2} \left( T_{\phi \phi} - T_{\phi \theta} \cot(\theta) \right). \quad \text{(4.10)}
\end{align*} $$

It is apparent from the above expressions that the jumps in different components of $B$ are related to supertranslation parameter $T(\theta, \phi)$ and we get a memory corresponding to this. We can also get jumps using the off-shell tensor $\tilde{B}$ will differ fom this by the terms proportional to $U$. It should be noted, the jump in the shear term is arising due to the presence of IGW while the expansion is due to the presence of shell stress tensor. So memory of IGW will be captured in the jump of shear.

\footnote{\( \lambda \) is arbitrary and only equal to \( \frac{e}{8m^2} \) on the shell}
4.3 Memory for superrotation type transformations

First we focus on the flat space which corresponds to $\eta = 0, \Lambda = 0$ for the metric mentioned in (2.21). For this case we will have superrotation like transformation mentioned in (2.22). We have,

$$N_0 = \left( \partial_U - \frac{\epsilon \bar{h}''(\bar{\zeta})}{2V} \partial_{\bar{\zeta}} - \frac{\epsilon h''(\zeta)}{2V} \partial_{\zeta} \right) + \mathcal{O}(\epsilon^2) \bigg|_\Sigma. \quad (4.11)$$

Doing the similar computation as done in section (4.2) and using (B.5) we get,

$$[\Theta] = \mathcal{O}(\epsilon^2),$$
$$[\Sigma_{\zeta\zeta}] = \frac{eVh''(\zeta)}{2} + \mathcal{O}(\epsilon^2), [\Sigma_{\bar{\zeta}\bar{\zeta}}] = \frac{eV\bar{h}''(\bar{\zeta})}{2} + \mathcal{O}(\epsilon^2), \quad (4.12)$$
$$[\Sigma_{\zeta\bar{\zeta}}] = \mathcal{O}(\epsilon^2), [\Sigma_{\bar{\zeta}\zeta}] = \mathcal{O}(\epsilon^2).$$

We have only kept the leading order terms in $\epsilon$. Again it is evident that the jumps in these tensors capture the superrotation type of transformation. We can perform the similar analysis for constant negative and positive curvature spaces which also described by the class of defined in (2.21).

4.4 Memory for null-shell near a black hole horizon

In this case the null surface is situated at $U = \epsilon$ surface for the Schwarzschild black hole. Then we will have symmetry transformations mentioned in (2.25). Next we proceed as before and compute the (4.4). The inverse Jacobian and other details are there in appendix B. Using (B.6) the tangent vector of past congruence can be written as,

$$N_0 = \frac{e - 2Ve}{8m^2} \left( \partial_U - B \partial_{\zeta} - \bar{B} \partial_{\bar{\zeta}} \right) + \mathcal{O}(h^2, \bar{h}^2, \epsilon h, \epsilon \bar{h}) \bigg|_\Sigma. \quad (4.13)$$

Doing the similar computation as done in section (4.2) we get,

$$[\Theta] = \frac{4e^{2-m^2}}{1 + \zeta^2} \left( (1 + \zeta \bar{\zeta}) (h'(\zeta) + \bar{h}'(\bar{\zeta})) - 2\zeta h(\zeta) - 2\bar{\zeta} \bar{h}(\bar{\zeta}) \right) + \mathcal{O}(h^2, \bar{h}^2, \epsilon h, \epsilon \bar{h}), \quad (4.14)$$
$$[\Sigma_{\zeta\zeta}] = -\frac{e}{4(1 + \zeta^2)^2} \partial_{\zeta} B + \mathcal{O}(h^2, \bar{h}^2, \epsilon h, \epsilon \bar{h}), [\Sigma_{\bar{\zeta}\bar{\zeta}}] = -\frac{e}{4(1 + \bar{\zeta}^2)^2} \partial_{\bar{\zeta}} \bar{B} + \mathcal{O}(h^2, \bar{h}^2, \epsilon h, \epsilon \bar{h}), \quad (4.15)$$
$$[\Sigma_{\zeta\bar{\zeta}}] = \mathcal{O}(h^2, \bar{h}^2, \epsilon h, \epsilon \bar{h}), [\Sigma_{\bar{\zeta}\zeta}] = \mathcal{O}(h^2, \bar{h}^2, \epsilon h, \epsilon \bar{h}). \quad (4.16)$$

$B$ and $\bar{B}$ are defined in (A.20).

5 Discussions

The motivation of this work is to find BMS memory effect near the vicinity of black hole horizons. For this, we have reviewed null-shells placed at the horizon of black holes that give rise BMS-like symmetries. It is well known that these shells also contain impulsive gravitational waves. We have studied the effect of this IGW on test particles. We extend the existing works concentrating on plane gravitational waves to spherical ones. Further, we also consider the effect of matter stress-tensor concentrated on a null hypersurface on test particles along with
the pure gravitational wave. We show how geodesics are getting affected by BMS like freedom appearing in the shell's stress-energy tensor and in IGW. For timelike test particles, we found supertranslation memory effect in the deviation vector corresponding to test geodesics. For spaces of constant curvature (including Minkowski space), we also get superrotation memory effect on test particles. If there is no current or anisotropic stress in the null-shell then the effect of IGW on test particles, initially situated at the 2-dimensional spatial surface, is to displace them relative to each other. This displacement happens due to the joint effect of the presence of matter on null-shell along with the IGW. In the case of flat space with Penrose's cut and paste form, we get non-zero anisotropic stress and consequently the test particles displaced from their initial plane or 2-surface. This time the deflection is governed by superrotation like parameters. A similar effect is expected to be exhibited by a null shell situated near to the event horizon of a black hole. Memory effect in the form of jumps in expansion and shear of a null congruence crossing a null-shell has also been shown for horizon-shell situated at the event horizon (or Killing horizon) of a black hole. These jumps are shown to be parametrized by supertranslation like parameters. Further, we also show for a null-shell placed just outside of black hole horizon can exhibit similarly superrotation like memory effect on the null observers. We know shock wave kind of gravitational radiation is generated in different astrophysical processes like black hole mergers, supernovae explosions. This work may offer some useful models to obtain some observable effects for upcoming advanced detectors.

The memory effect near the horizon of a black hole may offer many minute details of the symmetries appearing in the near horizon region. Analyzing this analogue of spin-memory effect in the near horizon of a black hole will be an interesting study. Memory effect for detectors placed near the horizon of a black hole, in a similar fashion like in far region, is yet to be shown explicitly. We expect to shed some light on these issues in the near future. The relation between BMS-like memories and the Penrose limit for exact impulsive wave spacetimes is an area to be explored in future[42]. As there exists some modified version of BMS algebra in a null-surface situated at a finite location of a Manifold [22], it will be interesting to investigate the role of those symmetries on gravitational memory effect. Role of BMS-like symmetries in the study of quantum fluctuations of these spherical impulsive wave background would be an interesting area of research [43, 44]. Finally, to understand the connection between the quantum vacuum structure of gravity and these memories are going to be an exciting field of study.

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A Off-shell extensions

Here we will give the necessary expressions useful for computation of stress tensor (2.8). We give the details case by case.
Killing Horizon

For this case we consider the case of Schwarzschild black hole mentioned in (2.15). The null shell located at the $U = 0$. On the $+$ side we can perform the following transformations,

$$ V_+ = F(V, \theta, \phi), \theta_+ = \theta, \phi_+ = \phi_+. \quad (A.1) $$

where again we choose the intrinsic coordinates $x^\mu$ covering the whole manifold as the coordinates on the $-$ side. Then we extend this symmetry transformations of the shell. It is done in [23]. Here we will briefly review the constructions of [23].

We start with following ansatz,

$$ V_+ = F(V, \theta, \phi) + U A(V, \theta, \phi), U_+ = U C(V, \theta, \phi), \quad \theta_+ = \theta + U B^\theta (V, \theta, \phi), \phi_+ = \phi + U B^\phi (V, \theta, \phi). \quad (A.2) $$

Demanding the continuity of full spacetime metric across the junction at leading order in $U$ we get,

$$ C = \frac{1}{\partial V F}, A = \frac{e}{4} \partial V F \left( \left( \frac{2}{e} \partial \theta F \right)^2 + \sin(\theta)^2 \left( \frac{1}{e} \sin(\theta)^2 \frac{\partial \phi F}{\partial V F} \right)^2 \right), \quad B^\theta = \frac{2}{e} \frac{\partial \theta F}{\partial V F}, B^\phi = \frac{1}{e} \sin(\theta)^2 \frac{\partial \phi F}{\partial V F}. \quad (A.3) $$

Also on $M_-$ we have,

$$ r(UV)^2 = 4m^2 - \frac{8m^2}{e} UV + \cdots. \quad (A.4) $$

and on $M_+$ we have,

$$ r(U_+ V_+)^2 = 4m^2 - \frac{8m^2}{e} U_+ V_+ + \cdots = 4m^2 - \frac{8m^2}{e} U F, \sin(\theta_+)^2 = \sin(\theta)^2 + 2UB^\theta \sin(\theta) \cos(\theta) + \cdots. \quad (A.5) $$

We now expand the tangential components of the metrics in $M_+$ and $M_-$ to linear order in $U$. For $M_-$ we have,

$$ g_{ab} dx^a dx^b = g_{0AB} dx^A dx^B - U \frac{8m^2}{e} \left( V d\theta^2 + V \sin(\theta)^2 d\phi^2 \right). \quad (A.6) $$

For $M_+$ we have,

$$ g_{ab} dx^a dx^b = g_{0AB} dx^A dx^B + 8m^2 U \left( \frac{2}{e} dA F + \sigma_{AB} dx^A (dB^B - (F/eF) d^B) + \sin(\theta) \cos(\theta) B^\theta d\phi^2 \right). \quad (A.7) $$

From (2.10) we get,

$$ \gamma_{ab} = N^\alpha [\partial_\alpha g_{ab}] = N^U [\partial_U g_{ab}] \quad (A.8) $$

with,

$$ N^U = \frac{e}{8m^2}. \quad (A.9) $$

Using (A.3) we get, on $M_+$,

$$ g_{ab} dx^a dx^b = g_{0AB} dx^A dx^B + 8m^2 U \left( \frac{2}{e} \left( \frac{\partial V F}{\partial V F} dV dx^a \right) + \frac{2}{e} \frac{\partial A \partial B F}{\partial V F} dV dx^A dx^B - \frac{4}{e} \cot(\theta) \frac{\partial \phi F}{\partial V F} d\theta d\phi \right) - \sigma_{AB} dx^A (F/eF) dx^B + \frac{2}{e} \sin(\theta) \cos(\theta) \frac{\partial \phi F}{\partial V F} d\phi^2. \quad (A.10) $$

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For $\mathcal{M}$ we have,

$$g_{ab}^{}dx^a^{}dx^b = g_{AB}^{}dx^A^{}dx^B^{} - U \frac{8m^2}{e} \left( Vd\theta^2 + V\sin(\theta)^2d\phi^2 \right). \quad \text{(A.11)}$$

So from these we have,

$$\gamma_{\alpha\beta} = 2\frac{\partial_\alpha F}{\partial V^{-} F}, \gamma_{\theta\phi} = 2\left( \frac{\nabla_\theta^{(2)} \partial_\phi F}{\partial V^{-} F} - \frac{1}{2} \left( F \frac{\partial V^{-} F}{\partial V^{-} F} - V \right) \right), \quad \text{(A.12)}$$

Using (A.12) we can write down the intrinsic energy momentum tensor off the shell using (2.9).

$$p = -\frac{1}{16\pi} \gamma_{VV} = -\frac{1}{8\pi} \frac{\partial_{\phi}^{\phi} F}{\partial V^{-} F}, j^A = \frac{1}{32m^2}\sigma^{AB}\partial_B \partial_V^{-} F, \quad \text{(A.13)}$$

If we focus on the case of supertranslation as mentioned in (2.20) which amounts to replacing $F(V, \theta, \phi)$ by $V + T(\theta, \phi)$. Then only $\mu$ is nonzero and we get,

$$\mu = -\frac{1}{32m^2\pi} \left( \nabla^{(2)} T(\theta, \phi) - T(\theta, \phi) \right). \quad \text{(A.14)}$$

**Constant curvature spacetimes**

Now we consider the class of metrics given by (2.21) and extend the symmetry transformations mentioned in (2.22). We proceed in the same way as before. We start with the following ansatz,

$$V^+ = V(1 - \frac{\epsilon \tilde{\Omega}(\zeta, \bar{\zeta})}{2}) + UA(V, \zeta, \bar{\zeta}), U^+ = UC(V, \zeta, \bar{\zeta}),$$

$$\zeta^+ = \zeta + \epsilon h(\zeta) + UB(V, \zeta, \bar{\zeta}), \bar{\zeta}^+ = \bar{\zeta} + \epsilon h(\bar{\zeta}) + UB(V, \zeta, \bar{\zeta}), \quad \text{(A.15)}$$

where $\tilde{\Omega}(\zeta, \bar{\zeta})$ is defined in (2.23) and again we will work in linear order in $\epsilon$. Again demanding the continuity of full spacetime metric across the junction at leading order in $U$ we get,

$$A = \frac{\epsilon \eta \tilde{\Omega}(\zeta, \bar{\zeta})}{2}, C = 1 + \frac{\epsilon \tilde{\Omega}(\zeta, \bar{\zeta})}{2},$$

$$B = \frac{\epsilon \left( (1 + \eta \zeta) (\bar{h}''(\bar{\zeta})(1 + \eta \zeta) - 2\eta \bar{\zeta}h'(\bar{\zeta})) + 2\eta \bar{\zeta}^2 h(\bar{\zeta}) - 2\eta h(\bar{\zeta}) \right)}{2V},$$

$$B = \frac{\epsilon \left( (1 + \eta \zeta) (\bar{h}''(\bar{\zeta})(1 + \eta \zeta) - 2\eta \bar{\zeta}h'(\bar{\zeta})) + 2\eta \bar{\zeta}^2 h(\bar{\zeta}) - 2\eta h(\bar{\zeta}) \right)}{2V}, \quad \text{(A.16)}$$

where we have used (2.23). Using this we can compute the $\gamma_{\alpha\beta}$ and component of stress tensor. For simplicity we write the results of flat space time i.e $\eta = 0, \Lambda = 0$. The $\eta, \Lambda \neq 0$ case can be done analogously.

So we for the flat space we get the following,

$$\gamma_{\nu\zeta} = h''(\zeta), \gamma_{\bar{\nu}\bar{\zeta}} = \bar{h}''(\bar{\zeta}), \gamma_{\zeta\bar{\zeta}} = V h''(\zeta), \gamma_{\bar{\zeta}\bar{\zeta}} = V \bar{h}''(\bar{\zeta}). \quad \text{(A.17)}$$

Consequently we will only have non-vanishing currents,

$$j^\zeta = \frac{\epsilon \bar{h}''(\bar{\zeta})}{16\pi V^2}, j^\bar{\zeta} = \frac{\epsilon h''(\zeta)}{16\pi V^2}. \quad \text{(A.18)}$$
Null surface near Killing Horizon

Lastly we present the off-shell extension of the symmetry transformations mentioned in (2.25). Again we keep only terms linear in \( h(\zeta), \bar{h}(\bar{\zeta}) \) and \( \epsilon \). We extend this symmetry off the shell \( U = \epsilon \).

\[
U_+ = (U - \epsilon) A, V_+ = -\frac{a \bar{\Omega}(\zeta, \bar{\zeta})}{b \epsilon} + V(1 - \bar{\Omega}(\zeta, \bar{\zeta})) + (U - \epsilon) C,
\]
\[
\zeta_+ = \zeta + h(\zeta) + (U - \epsilon) B, \bar{\zeta}_+ = \bar{\zeta} + \bar{h}(\bar{\zeta}) + (U - \epsilon) \bar{B}.
\]

\( a = 4m^2, b = -\frac{8m^2}{\epsilon} \) and \( \bar{\Omega}(\zeta, \bar{\zeta}) \) is defined in (2.23). Then demanding the continuity of the metric across the null shell upto \( O(U - \epsilon) \) we get,

\[
A = 1 + \frac{((\zeta + 1)(h'(\zeta)) - 2\zeta h(\zeta) - 2\zeta h(\zeta))}{1 + \zeta},
\]
\[
B = -2e^{1 - m^2} \left( 2h(\zeta) - (\zeta + 1)((\zeta + 1)h'(\zeta) - 2\zeta h'(\zeta)) - 2\zeta h(\zeta) \right),
\]
\[
\bar{B} = -2e^{1 - m^2} \left( 2h(\bar{\zeta}) - (\zeta + 1)((\zeta + 1)h'(\zeta) - 2\zeta h'(\zeta)) - 2\zeta h(\zeta) \right).
\]

### B Jacobi of transformations

Here we give the expression for \textit{Jacobian} matrix mentioned in (4.1) which is crucial for computation of the memory effect through (4.4). We use the expressions (A.2) and (A.3) to get

\[
\left( \frac{\partial x^\beta}{\partial x'^\alpha} \right)^{-1} \bigg|_{U=0} = \frac{1}{2m^2 e} \begin{pmatrix}
2m^2 e F_V & 0 & 0 \\
\frac{F_\theta^2}{F_V} + \frac{F_\phi^2}{\sin^2 \theta} & \frac{2m^2 e}{F_V} & -2m^2 e F_\phi \\
-F_\theta & 0 & 2m^2 e \\
-F_\phi & 0 & 2m^2 e
\end{pmatrix}.
\]

For the case of supertranslation it further simplifies.

\[
\left( \frac{\partial x^\beta}{\partial x'^\alpha} \right)^{-1} \bigg|_{U=0} = \frac{1}{2m^2 e} \begin{pmatrix}
n \frac{2m^2 e}{T_\theta^2 + \frac{F_\phi^2}{\sin^2 \theta}} & 0 & 0 \\
\frac{1}{2} \left( T_\theta^2 + \frac{F_\phi^2}{\sin^2 \theta} \right) & 2m^2 e & -2m^2 e T_\phi \\
-T_\theta & 0 & 2m^2 e \\
-T_\theta & 0 & 2m^2 e
\end{pmatrix}.
\]

For off-shell we already know that null congruence to the \(-\) side is related to the coordinates on the surface via the following expression,

\[
x^\alpha = x_0^\alpha + UN_0^\alpha
\]

The B-tensor in this mapping would give us the off shell extended version. The transformation can be written as,

\[
\tilde{B}_{AB} = \frac{\partial x_0^M}{\partial x^A} \frac{\partial x_0^N}{\partial x^B} B_{MN}
\]
\( B_{MN} \) is nothing but the components of \( B - \) tensor calculated on the shell. We would be considering here only BMS case.

\[
\frac{\partial x_0^M}{\partial x^A} = \frac{\delta^B_M}{(\delta^B_M - U \frac{\partial N_B}{\partial x^N})} = \frac{I}{J}
\]

and inverse of \( J \) matrix is then given by,

\[
(\delta^B_M - U \frac{\partial T^B}{\partial x^N})^{-1} = \frac{1}{\text{det}(J)} \begin{pmatrix}
1 + \frac{2U}{e} T_{\phi\phi} & -\frac{2U}{e} T_{\theta\phi} & 1 + \frac{2U}{e} T_{\theta\theta} \\
-\frac{2U}{e} T_{\phi\phi} & 1 + \frac{2U}{e} T_{\theta\theta} & \frac{2U}{e} T_{\phi\theta} \\
1 + \frac{2U}{e} T_{\theta\theta} & \frac{2U}{e} T_{\phi\theta} & 1 + \frac{2U}{e} T_{\phi\phi}
\end{pmatrix}.
\] (B.4)

where determinant of \( J \) is given by,

\[
\text{det}(J) = 1 + \frac{2U}{e} (T_{\theta\theta} + F_{\phi\phi}) - \frac{4U^2}{e^2} (T_{\theta\phi} - T_{\theta\theta} T_{\phi\phi})
\]

Next, using (A.15) and (A.16) for flat space we get,

\[
\left( \frac{\partial x^\beta}{\partial x^\alpha} \right)^{-1} \bigg|_{U=0} = \begin{pmatrix}
1 - \frac{\epsilon}{2} (h'(\zeta) + \bar{h}'(\bar{\zeta})) & 0 & 0 & 0 \\
0 & 1 + \frac{\epsilon}{2} (h'(\zeta) + \bar{h}'(\bar{\zeta})) & \frac{Ve h''(\zeta)}{2} & \frac{Ve h''(\bar{\zeta})}{2} \\
-\epsilon h(\zeta) & 0 & 1 - \epsilon h'(\zeta) & 0 \\
0 & -\epsilon h(\bar{\zeta}) & 0 & 1 - \epsilon \bar{h}'(\bar{\zeta})
\end{pmatrix} + \mathcal{O}(\epsilon^2).
\] (B.5)

For the null surface near Killing horizon using (A.19) and (A.20) we get, Jacobian matrix is given by,

\[
\left( \frac{\partial x^\beta}{\partial x^\alpha} \right)^{-1} \bigg|_{U=\epsilon} = \begin{pmatrix}
1 - \frac{(\zeta \bar{\zeta} + 1)(h'(\zeta) + \bar{h}'(\bar{\zeta})) - 2\zeta \bar{h}(\bar{\zeta}) - 2\bar{\zeta} h(\zeta)}{1 + \zeta \bar{\zeta}} & 0 & 0 & 0 \\
0 & 1 + \bar{\Omega}(\bar{\zeta}, \bar{\zeta}) & \frac{(a + b V e) \partial^\beta \bar{\Omega}(\zeta, \bar{\zeta})}{b \epsilon} & \frac{(a + b V e) \partial^\beta \bar{\Omega}(\zeta, \bar{\zeta})}{b \epsilon} \\
-B(\zeta, \bar{\zeta}) & 0 & 1 - \epsilon \bar{h}'(\bar{\zeta}) & 0 \\
-B(\bar{\zeta}, \zeta) & 0 & 0 & 1 - \epsilon h'(\zeta)
\end{pmatrix}.
\] (B.6)

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