Crosstalk in multicore fibers with randomness: gradual drift vs. short-length variations

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Abstract: Random perturbations play an important role in the crosstalk of multicore fibers, and can be captured by statistical coupled-mode calculations. In this approach, phase matching contributes a multiplicative factor to the average crosstalk, depending on the perturbation statistics and any intentional heterogeneity of neighboring cores. The impact of perturbations is shown to be qualitatively different depending on whether they are gradually varying, or have short-length (centimeter-scale) variations. This insight implies a novel crosstalk suppression strategy: fast modulation of a bend perturbation by spinning the fiber can disrupt the bend-induced phase matching.

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1. Introduction

Multicore fibers (MCF) are a major focus of recent fiber research [1–11]. They offer one of the few remaining avenues for dramatic scaling of the capacity-per-fiber. At the same time, they face significant technical challenges; the potential advantages and viability of this technology compared to a multiple single-core alternative are unclear. One significant potential obstacle is crosstalk and the limitations on core density implied by a tradeoff with crosstalk, effective area and cutoff. It is important to use the right models in order to understand crosstalk: oversimplified models have given crosstalk estimates orders of magnitude too small in some cases. We have presented statistical models including random fiber perturbations [12]. These display power-coupling dynamics consistent with empirical measurements [7] and can be used to understand the impact of fiber layout: For example, a fiber measured on a spool may show significantly different crosstalk than the same fiber cabled and deployed in a realistic telecommunications link.

In this paper, we extend our previous statistical crosstalk models, focusing on whether the variations in a particular fiber are sufficiently “gradual” to be approximated by a quasi-static approximation. A formula for the quasi-static limit is derived, and has the intuitive interpretation that intermittent resonant coupling gives crosstalk in proportion to the likelihood of a phase-matching event. We further show that some realistic fiber variations are not gradual, and so deviate from this quasi-static approximation. As an example, we calculate that when fiber is spun with a period on the order of centimeters, crosstalk can be a highly structured function of core-index mismatch. The added structure is related to the spin periodicity, not the static likelihood of phase-matching, and could add significant variability to the crosstalk of nearly-identical fibers. Finally, we show that for fast spin, of order 1 turn/cm, it may be possible to mitigate crosstalk by disrupting bend-induced phase-matching.

2. Basic model

The basic coupled-mode model includes diagonal terms D and coupling terms K:

$$\frac{d}{dz} a = i(D + K)a$$

(1)

This is illustrated in Fig. 1 in the simplest case of a 2-core coupler. It is useful to use the “interaction picture” formulation, where we switch variables:

$$a = Pu$$

(2)

$$\frac{d}{dz} P = iDP.$$  

(3)

Each element of $a$ is an amplitude of a local mode that accumulates an explicit phase according to the corresponding element of $D$, while in the representation $u$, these phases are implicit. We obtain a modified evolution equation

$$\frac{d}{dz} u = iP^{-1}KPu$$

(4)
As in [12], our fundamental starting point is the coupled-mode description, including any random variations explicitly. The model is used to correctly average over these random variations—which may occur on a distance of centimeters or several meters—so that we can derive a power-coupling model, more useful for larger-scale telecommunications links (10s to thousands of kilometers). The coupled mode equation [Eq. (1)] is our basic starting point, but is actually non-trivial, since generally the local modes of each core are not orthogonal and the coupling coefficients are not symmetrical [13]. The impact of this asymmetry is an interesting question for further work, and is discussed, for example, in [14], but is set aside here.

The solution to Eq. (4) is a transfer matrix
\[
U(z = L) = UU(z = 0)
\]
If the coupling is not too large, then we can use a simple 1st-order perturbative expression for the transfer matrix
\[
U = 1 + dU
\]
with:
\[
\int \left[ P^{-1} KP \right]_{m,n} \, dz
\]
\[
\int \left[ \exp \left( \int \delta \beta_{m,n} (\zeta) \right) \right] \, dz
\]
Here, \( \delta \beta_{m,n} = D_{n,n} - D_{m,m} \) gives the differential phase between the cores and \( \kappa_{m,n} \) is the \( m,n \) element of \( K \). The notation is slightly different than used previously.

A variety of random perturbations and imperfections can be included in \( \delta \beta_{m,n} \) and its variation with fiber length. This could include variability in the fiber itself, temperature or strain effects, etc. In [9], we showed that including random variations is essential to a realistic model: even if other variations could be eliminated with perfect fabrication, bend perturbations are sufficiently large to make the deterministic coupled-mode model grossly unrealistic. The power transfer matrix \( M \) is extracted from the coupled-mode model by taking the average over the random perturbations:
\[
M_{m,n} = \left\langle |dU_{m,n}|^2 \right\rangle
\]
In addition, significant length-variation is unavoidable in a realistic fiber link, unless the orientation of the fiber relative to the bend is strictly controlled along with all other perturbations. Orientation drift and bend perturbations lead to intermittent resonant coupling as illustrated in Fig. 2.
Bends and other length-varying perturbations shift the index mismatch between cores, and lead to intermittent resonant coupling.

The incremental crosstalk between two cores over length $L$ is $|dU_{m,n}|^2$; the magnitude squared of Eq. (6) gives the double-integral:

$$\left|dU_{m,n}\right|^2 = \int_0^L dz \int_0^L d\zeta \kappa_{m,n}(\zeta) \exp \left[ i \zeta \delta \beta_{n,m}(\zeta) \right]$$

This equation is analogous to Eq. (11) of [12], but appears more compact because, in our current notation, $\delta \beta_{n,m}$ includes both the length-varying and non-varying parts. If we neglect length variation of the coupling coefficient themselves, then

$$\left\langle |dU_{m,n}|^2 \right\rangle = \kappa_{m,n}^2 \int_0^\infty dz \int_0^\infty d\zeta \exp \left[ i \zeta \delta \beta_{n,m}(\zeta) \right]$$

The statistical average in the integral is the correlation of the accumulated differential phase between positions $z$ and $z'$. This correlation is clearly 1 when $z = z'$. It may be approximately sinusoidal in $z-z'$ over short lengths, but if there is sufficient disorder (e.g. sufficient orientation drift of the fiber), the integrand will be zero for $|z'-z| > \zeta$. If we choose the increment $L$ of fiber length much larger than this correlation length, then the bounds of the second integral can be ignored. Now assuming the variations are statistically stationary:

$$\left\langle |dU_{m,n}|^2 \right\rangle \approx \kappa_{m,n}^2 \int_0^\infty d\zeta \exp \left[ i \zeta \delta \beta_{n,m}(\zeta) \right]$$

The separation of length scales is reasonable, at least in some relevant cases: Phase oscillations occur on length scale $2\pi/\delta \beta$, typically centimeters. The correlation length for perturbations is much longer, perhaps on the order of 1 meter. The segment length $L$ can be chosen somewhat arbitrarily; if we choose a length of order 100m-1km, then the segment is much larger than the correlation length, as required.

Previously [12] we derived,

$$\left\langle |dU_{m,n}(\omega)|^2 \right\rangle = \kappa_{m,n}^2 LS_{ff}(\delta \beta_{n,m}^0)$$

Here $\delta \beta^0$ is the non-length-varying component (e.g. due to the intentional index mismatch between two cores) $S_{ff}$ is the power-spectral density (PSD) of $f$, and $f$ is defined as the accumulated phase of the length varying part of $\delta \beta$: 

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\[ f(z) = \exp \left[ i \int_0^z d\zeta \delta \beta_{m,m}^\text{eff}(\zeta, \omega) \right]. \tag{12} \]

3. Gradual variations

It is interesting to look at the limit where \( \delta \beta \) varies very gradually. In the limit that the rate of variation is truly negligible, then we can treat \( \delta \beta \) as be a constant, \( b \), in the integrals. The statistical average could be calculated directly using the probability distribution \( p_{\delta \beta} \):

\[
\left\langle |dU_{m,n}|^2 \right\rangle \to L\left| \kappa_{m,n} \right|^2 \int db p_{\delta \beta}(b) e^{i b(z - z)}
\tag{13}
\]

If we naively evaluate this expression, the integral in \( z \) gives \( 2\pi \delta(b) \), and so,

\[
\left\langle |dU_{m,n}|^2 \right\rangle \to 2\pi L\left| \kappa_{m,n} \right|^2 p_{\delta \beta}(0)
\tag{14}
\]

That is, in the quasi-static limit, the integral in \( z \) gives \( 2\pi \delta(b) \), and so, the probability density measuring how often phase matching \( (\delta \beta = 0) \) is achieved during the process of gradual intermittent phase matching illustrated in Fig. 2. This reinforces the intuition of previous detailed calculations: if there is an intentional index mismatch between cores, this will suppress crosstalk only insofar as random perturbations cannot cancel the deterministic component. The probability density \( p_{\delta \beta}(0) \) has units of \( 1/\delta \beta \). If there is no systematic index mismatch, \( p_{\delta \beta}(b) \) will typically have a peak value at \( b = 0 \) proportional to one over the spread \( \Delta \delta \beta \) in \( \delta \beta \) (or the corresponding spread in random index perturbations \( \Delta n_{\text{eff}} = \Delta \beta \lambda / 2\pi \)):

\[
\left\langle |dU_{m,n}|^2 \right\rangle \sim 2\pi L\left| \kappa_{m,n} \right|^2 / \Delta \beta \sim L\left| \kappa_{m,n} \right|^2 \lambda / \Delta n_{\text{eff}}
\tag{15}
\]

This inverse-proportionality to the bend perturbation gives an intuitive explanation for the previous numerical predictions \[12\], which have been confirmed experimentally in \[7\].

The above derivation is oversimplified: if the rate of variation were truly zero, we would not be able to assume ergodicity; the statistical average might differ significantly from the actual crosstalk, even over long lengths. Worse, the correlation length would be much larger than \( L \), violating the assumptions leading to Eq. (10). However, the final result is correct: in the Appendix, we show that same expression results if we allow \( \delta \beta \) to drift at a small but finite rate along the fiber length.

4. Gradual variations: numerical examples

Many of the power-spectral density (PSD) calculations previously presented \[12,15\] illustrate the regime of gradual variations just discussed. This is illustrated in Fig. 3, which compares the PSD calculation and quasi-static approximation for a bend perturbation with slow orientation drift. Several bend radii are shown. For simplicity, the curvature in each case is assumed perfectly constant. For the PSD calculations, we used 40m long randomly generated processes, and averaged 16 realizations to smooth out fluctuations. For the quasi-static formula, constant curvature implies \( b = b_0 \cos(\theta) \) with bend orientation \( \theta \) uniformly distributed over all angles and

\[
b_0 = 2\pi \gamma n_{\text{core}} a / R_{\text{bend}}.
\tag{16}
\]

Here \( a \) is the core spacing, and \( \gamma \) includes a stress correction to the purely geometrical \((\gamma = 1)\) bend perturbation model. The probability density function for \( b \) is then derived with the usual change-of-variable formulas (recognizing that each \( b \) corresponds to two angles, \( \theta \) and \(-\theta\):
Clearly the quasi-static expression (dashed black) approximates the PSD calculations very well. The dashed distributions $\sim 1/\sin(\theta)$ have peaks with infinite probability density followed by an abrupt drop to zero probability. These features are naturally smoothed by the PSD calculations, since the correlation length and the spectral estimation method have finite resolution. The shapes of the curves are characteristic of bend perturbations with well-defined radius. A more realistic link would include uncertainty in the bend radius and other sources of fiber variation, and may typically have a more bell-shaped curve. This is illustrated in [15], where we compared crosstalk for bent fibers with different degrees of non-bend perturbations.

Fig. 3. The phase-matching factor of crosstalk is calculated as a power spectral density (solid) and compared to the quasi-static approximation (dashed black) for the simple case where the index perturbation is due to constant bend radius and gradual orientation drift.

5. Variations that are not gradual: spin

Intuitively, we would guess that gradual variations will play an important role in crosstalk for realistic telecom fibers. Preform variations will generally manifest on long lengths of fiber. Changes in random orientation drift (which modulates the bend perturbation) likely occur on something like a 1m scale. Cables are typically designed to protect fibers from abrupt kinks and stresses, and so many other perturbations may similarly vary on a length scale much larger than $1/\Delta\beta$. However, short-length variations are certainly realistic: for example, fiber can be intentionally spun so that its orientation varies on a centimeter length scale. The PSD formulation allows us to cover gradual and short-length variations on an equal footing.

Figure 4 shows the calculated crosstalk for a perturbation consisting of a constant-radius bend for a spun fiber. The spin rate has deterministic part 10turns/m plus a random drift with standard deviation 1turn/m. The PSD traces (solid) clearly show fine structure related to the periodicity of the bend perturbation, and differ substantially from the quasi-static approximation (dashed). A perfectly periodic process with period $\Lambda_{\text{spin}}$ would have discrete PSD samples at index mismatch multiples of $\lambda/\Lambda = 0.155 \times 10^{-4}$. The grid lines on the plot are
The fine structure remains even if there is some variability in both the curvature and spin rate. As long as \( R_{\text{bend}} \) and spin rate drift slowly, they seem constant on a length scale shorter than this drift; that is, locally

\[
f(z) \approx \exp \left( \frac{iayn_{\text{core}}\Lambda_{\text{spin}}}{\lambda R_{\text{bend}}} \sin(2\pi z / \Lambda_{\text{spin}}) \right)
\]

and so the PSD reflects a quasi periodicity with period \( \Lambda_{\text{spin}} \). This is illustrated in Fig. 5, where the same calculation is repeated, but where there is now considerable random variation in the bend radius: the curvature is now equal to the nominal value (indicated in the legend) \( 1/R_{\text{bend}} \) plus a random component with standard deviation \( 1/R_{\text{bend}} \). The tail of the PSD is now much larger, since the random curvature variation can lead to large bend perturbations. The PSD curves are generally smoother, since there is more disorder in the system. However, the dominant peaks clearly still show the structure of the quasi-periodicity, falling at multiples of \( \lambda/\Lambda_{\text{spin}} \).
Fig. 5. Power spectral density is calculated for the same parameters as Fig. 4, but with much larger variance in the bend radius. Structure imposed by the spin periodicity is still clearly visible.

The highly structured spectrum could potentially have significant consequences for system performance. We can model the fiber as having random preform perturbations $\delta \beta_0$ that are constant with length (on a ~1km scale), in addition to the bend perturbations. The bend perturbations may have identical statistics for the different core pairs, and for each kilometer of fiber, but each core pair and each spliced fiber has a different random $\delta \beta_{nm}$. The highly structured PSD means that each time the PSD is sampled at a different random $\delta \beta_{nm}$, the total crosstalk will be very different, and there is thus a high variability of crosstalk from between core pairs, or from one fiber to a spliced (nearly identical fiber). This is true even if the fabrication tolerances are quite good: if the $\delta \beta_{nm}$ values are controlled to index precision $\sim 10^{-5}$.

We note that spin is shown to have an impact for these particular cases because the spin rate is well-defined and fast enough that the structure in the PSD matters. Slower, random spin was considered previously [7,12], but its role was essentially to ensure that all orientations were randomly sampled in a long length of fiber. For slow, random spin, it was correctly concluded that the precise spin rate does not impact crosstalk. The structure related to periodicity is interesting because in part because it points out an additional way in which we can impact crosstalk.

6. Crosstalk reduction: fast spin

The results of Figs. 4 and 5 do not suggest successful crosstalk reduction. For example, comparing the solid (spun) and dashed (un-spun) curves of the same bend radius (colors), we see that spin sometimes increases and sometimes decreases crosstalk. In contrast, Fig. 6 shows an analogous calculation where the spin rate, 100turns/meter, is fast relative to the magnitude of the bend-induced index perturbations. For this fast spin rate, the quasi-periodicity means that the PSD peaks are spaced by $2\pi/\Lambda_{spin}$ in $\delta \beta$ units or $\lambda/\Lambda_{spin}$ on the index mismatch axis. For 1550nm wavelength this is $1.55 \times 10^{-4}$. The calculated crosstalk is suppressed by orders of magnitude by the spin for cores with index mismatch far from these peaks. For example, cores with index mismatch of $(0.8 \pm 0.5) \times 10^{-4}$ show large calculated crosstalk suppression.
Fig. 6. For a very fast, well-controlled spin (1 turn/cm) spin periodicity leaves large gaps in the power spectral density: Crosstalk in this calculation is dramatically reduced for cores with index mismatch in between $0$ and $1.55 \times 10^{-4}$.

The calculation of Fig. 6 is highly idealized in that only bend perturbations are included. Figure 7 repeats the calculation with gradually-varying non-bend perturbations included, and with a random length-varying component to the bend radius itself. It is assumed that the spin modulates the bend perturbation only, and so the non-bend perturbations do not have any periodicity imposed by the spin. Non-bend perturbations are normal-distributed with standard deviations $2 \times 10^{-5}$ (left) or $4 \times 10^{-5}$ (right). From the results, we see that the spun (solid) crosstalk falls orders of magnitude below the un-spun (dashed) crosstalk as long as the non-bend perturbations are not too large. The larger the non-bend perturbations are, the more tightly the index mismatch control must be to obtain large crosstalk suppression. Successful crosstalk suppression can then be accomplished by simultaneously: generating spin with very short and well-defined spin period, reducing the non-bend length-variation index perturbations, and arranging for the index mismatch between neighboring cores to fall in-between the peaks of the PSD, which occur at multiples of $\lambda/\Lambda_{\text{spin}}$. Fast spin (with pitch $\sim 1$ mm) has been demonstrated experimentally [16], but clearly adds a fabrication challenge to the difficulty of strict index-profile control; prospects for low-cost manufacturing will need further study.
Fig. 7. While fast spin can disrupt bend-mediated phase matching between cores, other perturbations may not share the spin periodicity. Low-crosstalk regimes remain as long as non-bend perturbations are not too large.

7. Conclusions

Previously, we have intuitively described the phase-matching contribution of crosstalk as measuring the frequency of intermittent resonant coupling events [15]. Here, we have shown that this can be derived in the limit that perturbations vary gradually with fiber length. This provides simple intuition, and confirms many of the trends reported previously. For example, for quasi-homogeneous fibers, it gives a simple explanation for trends shown numerically [12] and experimentally [7], that crosstalk varies inversely with the bend perturbation.

This derivation also illustrates that there can be more to crosstalk than simply the probability of phase-matching. For perturbations that vary on a centimeter length scale, the approximation of gradual variation breaks down. The power spectral density captures the phase matching contribution in the general case, including any effects analogous to quasi-phase matching that arise from short-length variations.

Spin modulates the bend perturbation, and is an example where variations can reasonably have a length scale of a few centimeters or smaller. Our calculations show that spin could contribute significant sensitivity to small changes in core index mismatch, and thus variability to the crosstalk of nearly-identical fibers.

We also describe a regime where a very short-period spin is used to drastically reduce crosstalk, and discuss the fabrication tolerances (control of non-bend perturbations) needed to realize such a regime. This approach would be technically challenging to realize at low cost, but illustrates how improved crosstalk models can lead to novel crosstalk mitigation approaches.

Appendix

In the oversimplified derivation of Eq. (14), we assumed that $\delta \beta_{n,m}$ equaled a constant $b$ over the entire segment length $L$. We can more realistically assume that $\delta \beta_{n,m}$ drifts slowly along the length. That is, to evaluate Eq. (10), we say that the random $\delta \beta_{n,m}$ has some random value $b$ at the center of the interval $\zeta = (z+z')/2$, and is drifting at some random rate of drift $b'$. Then $\delta \beta_{n,m}(\zeta) = b + b'(\zeta-(z+z'/2))$, and so

$$\int_{z}^{z'} d\zeta \delta \beta_{n,m}(\zeta) = b(z-z') + b'(z-z')^2 / 2$$

Equation (10) becomes
Strictly speaking, the right hand side should be averaged over the statistics of $b'$, but we can treat $b'$ as a deterministic value in anticipation that it will soon drop out. The integral is now of a well-known form: the Fourier transform of a Gaussian.

$$\left\langle \left| dU_{m,n}\right|^2 \right\rangle \approx L|\kappa_{m,n}|^2 \int db p_{0\beta}(b) \int_0^\infty d\varepsilon e^{ib(\varepsilon - \varepsilon')} e^{i\varepsilon(\varepsilon - \varepsilon')^2/2}$$  \hspace{1cm} (21)

If $b'$ is small, the function sampling the probability distribution is delta-like:

$$\sqrt{\frac{i}{2b' \pi}} e^{-ib'/(2b')} \rightarrow \delta(b)$$  \hspace{1cm} (24)

So that, again, we get Eq. (14)

$$\left\langle \left| dU_{m,n}\right|^2 \right\rangle \approx 2\pi L |\kappa_{m,n}|^2 p_{0\beta}(0)$$  \hspace{1cm} (25)