DYNAMICS OF $L^p$ MULTIPLIERS ON HARMONIC MANIFOLDS

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Abstract. Let $X$ be a complete, simply connected harmonic manifold with sectional curvatures $K$ satisfying $K \leq -1$. In [Bis18], a Fourier transform was defined for functions on $X$, and a Fourier inversion formula and Plancherel theorem were proved. We use the Fourier transform to investigate the dynamics on $L^p(X)$ for $p > 2$ of certain bounded linear operators $T : L^p(X) \to L^p(X)$ which we call "$L^p$-multipliers" in accordance with standard terminology. These operators are required to preserve the subspace of $L^p$ radial functions. A notion of convolution with radial functions was defined in [Bis18], and these operators are also required to be compatible with convolution in the sense that

$$T\phi \ast \psi = \phi \ast T\psi$$

for all radial $C^\infty$-functions $\phi, \psi$. They are also required to be compatible with translation of radial functions. Examples of $L^p$-multipliers are given by the operator of convolution with an $L^1$ radial function, or more generally convolution with a finite radial measure. In particular elements of the heat semigroup $e^{t\Delta}$ act as multipliers. Given $2 < p < \infty$, we show that for any $L^p$-multiplier $T$ which is not a scalar multiple of the identity, there is an open set of values of $\nu \in \mathbb{C}$ for which the operator $\frac{1}{\nu} T$ is chaotic on $L^p(X)$ in the sense of Devaney, i.e. topologically transitive and with periodic points dense. Moreover such operators are topologically mixing. We also show that there is a constant $c_p > 0$ such that for any $c \in \mathbb{C}$ with $\text{Re} \, c > c_p$, the action of the shifted heat semigroup $e^{ct} e^{t\Delta}$ on $L^p(X)$ is chaotic. These results generalize the corresponding results for rank one symmetric spaces of noncompact type and negatively curved harmonic NA groups (or Damek-Ricci spaces).

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1. Introduction

The study of chaos in linear dynamics originated in the work of Godefroy and Shapiro [GS91]. The dynamics of a linear operator $T$ on a Frechet space $X$ is said
to be chaotic (in the sense of Devaney) if $T$ is hypercyclic (i.e. has a dense orbit, equivalently is topologically transitive), and has a dense set of periodic points. There is now an extensive literature on chaotic and hypercyclic operators, of which a summary may be found in the books [BM09], [GEM11].

In a geometric context, linear chaos has been investigated for the heat semigroup $e^{t\Delta}$ acting on the Lebesgue spaces $L^p(X)$, for certain complete Riemannian manifolds $X$ (where $\Delta = \text{div} \text{ grad}$ is the Laplace-Beltrami operator on $X$). Ji and Weber considered finite volume locally symmetric spaces of rank one in [JW10a], where they showed that for $p \in (1, 2)$ there is a constant $c_p \in \mathbb{R}$ such that for $c > c_p$ the shifted semigroup $e^{t(\Delta + c)}$ is subspace chaotic on $L^p(X)$, i.e. there is a closed, invariant subspace such that the semigroup restricted to the subspace is chaotic.

In [JW10a], Ji and Weber investigated the case of symmetric spaces of noncompact type, and showed that in this setting for $p \in (2, \infty)$ there is a constant $c_p \in \mathbb{R}$ such that for $c > c_p$ the shifted semigroup $e^{t(\Delta + c)}$ is subspace chaotic on $L^p(X)$.

In [Sar13], Sarkar improved on the result of Ji and Weber for rank one symmetric spaces, by showing that for the Damek-Ricci spaces (these are certain solvable Lie groups equipped with a left-invariant metric, which include as a particular case rank one symmetric spaces of noncompact type [DR92]), for $p \in (2, \infty)$ there is a constant $c_p \in \mathbb{R}$ such that for $c > c_p$ the shifted semigroup $e^{t(\Delta + c)}$ is subspace chaotic on $L^p(X)$.

In [Bis18], a study of harmonic analysis on noncompact harmonic manifolds in terms of eigenfunctions of the Laplace-Beltrami operator $\Delta$ was initiated. We briefly describe the results from [Bis18] which we will be needing:
Let $X$ be a complete, simply connected, harmonic manifold with sectional curvatures $K$ satisfying $K \leq -1$. Then $X$ is a CAT(-1) space, and can be compactified by adjoining a boundary at infinity $\partial X$, given by equivalence classes of geodesic rays $\gamma : [0, \infty) \to X$ in $X$. The Busemann cocycle $B : X \times X \times \partial X \to \mathbb{R}$ is defined by

$$B(x, y, \xi) := \lim_{z \to X, z \to \xi} (d(x, z) - d(y, z))$$

Given $x \in X$ and $\xi \in \partial X$, the Busemann function at $\xi$ based at $x$ is defined by $B_{\xi,x}(y) := B(y, x, \xi)$. The Busemann functions $B_{\xi,x}$ are $C^2$ convex functions, and their level sets are called horospheres based at $\xi$. As $X$ is harmonic, $X$ is also asymptotically harmonic, i.e. all horospheres have constant mean curvature, so there is a constant $h \in \mathbb{R}$ such that $\Delta B_{\xi,x} \equiv h$ for all $x \in X, \xi \in \partial X$. Since $X$ is negatively curved, in fact $h > 0$. We let

$$\rho = \frac{1}{2} h$$

Then in [Bis18] it is shown that for all $\lambda \in \mathbb{C}$, the function $e^{(i\lambda - \rho) B_{\xi,x}}$ is an eigenfunction of $\Delta$ with eigenvalue $-(\lambda^2 + \rho^2)$. Given $f \in C^\infty(X)$ and $x \in X$, the Fourier transform of $f$ based at $x$ is the function $\hat{f}^x$ on $\mathbb{C} \times \partial X$ defined by

$$\hat{f}^x(\lambda, \xi) := \int_X f(y)e^{-(i\lambda - \rho) B_{\xi,x}(y)}\,dvol(y)$$

Given $x \in X$, a function $f$ on $X$ is said to be radial around $x$ if $f$ is constant on geodesic spheres centered at $x$. In [Bis18] it is shown that for any $\lambda \in \mathbb{C}$, there is a unique eigenfunction $\phi_{\lambda,x}$ of $\Delta$ for the eigenvalue $-(\lambda^2 + \rho^2)$ which is radial around $x$ and satisfies $\phi_{\lambda,x}(x) = 1$. For $p > 2$, the functions $\phi_{\lambda,x}$ are in $L^p(X)$ for $\lambda$ in the strip

$$S_p := \{ \lambda \in \mathbb{C} | | \Im \lambda | < (1 - 2/p)\rho \}$$

Let $1 \leq q < 2$ be such that $1/p + 1/q = 1$. The spherical Fourier transform based at $x$ of a function $f \in L^q(X)$ is the function $\hat{f}^x$ on $\mathbb{R}$ defined by

$$\hat{f}^x(\lambda) := \int_X f(y)\phi_{\lambda,x}(y)dvol(y)$$

The spherical Fourier transform $\hat{f}^x$ extends to a holomorphic function on the strip $S_p$.

When $X$ is a rank one symmetric space of noncompact type, an $L^p$-multiplier is a bounded operator $T : L^p(X) \to L^p(X)$ which is translation invariant. Examples of $L^p$-multipliers are given by convolution on the right with radial $L^1$-functions, or more generally convolution on the right with finite radial measures. For a general harmonic manifold as in our case, a notion of convolution of functions with radial functions was defined in [Bis18] as follows: given a function $g$ radial around a point $x$ and another point $y \in X$, the $y$-translate of $g$ is the function $\tau_y g$ radial around $y$ defined by requiring that the value of $\tau_y g$ on a geodesic sphere of radius $r$ around $y$ equals the value of $g$ on the geodesic sphere of radius $r$ around $x$. The convolution of a function $f$ with $g$ is the function $f \ast g$ defined by

$$(f \ast g)(y) := \int_X f(z)\tau_y g(z)dvol(z)$$
Fixing a basepoint \( o \in X \), convolution with an \( L^1 \)-function radial around \( o \) gives rise to a bounded operator \( T : L^p(X) \to L^p(X) \) for all \( p \in [1, +\infty] \) satisfying the following properties (see section 2.4):

1. \( T \) preserves the subspace of \( L^p \)-functions radial around \( o \).
2. \( T \tau_x \phi = \tau_x T \phi \) for all \( \phi \in C_c^\infty(X) \) radial around \( o \) and for all \( x \in X \).
3. \( T \phi * \psi = \phi * T \psi \) for all \( \phi, \psi \in C_c^\infty(X) \) radial around \( o \).

This motivates the following definition: an \( L^p \)-multiplier is a bounded operator \( T : L^p(X) \to L^p(X) \) which satisfies properties (1)-(3) above. Examples of \( L^p \)-multipliers are given by convolution with radial \( L^1 \)-functions, or more generally convolution with radial complex measures of finite total variation (see section 2.4).

The terminology “multiplier” is motivated by the following: for \( p > 2 \), if \( T : L^p(X) \to L^p(X) \) is an \( L^p \)-multiplier, then there exists a holomorphic function \( m_T \) on the strip \( S_p \), called the symbol of \( T \), such that for all \( C_c^\infty \)-functions \( \phi \) radial around \( o \), the spherical Fourier transform of \( T \phi \) is given by

\[
\hat{T} \phi (\lambda) = m_T(\lambda) \hat{\phi}(\lambda), \lambda \in S_p
\]

Moreover if \( T \) is not a scalar multiple of the identity, then the function \( m_T \) is a nonconstant holomorphic function. We can now state our main theorem:

**Theorem 1.1.** Let \( X \) be a complete, simply connected, harmonic manifold with sectional curvature \( K \leq -1 \). Let \( 2 < p < \infty \) and let \( T : L^p(X) \to L^p(X) \) be an \( L^p \)-multiplier with symbol \( m_T \) such that \( T \) is not a scalar multiple of the identity. Then for all \( \lambda \in S_p \) such that \( m_T(\lambda) \neq 0 \), for any \( \nu \in \mathbb{C} \) such that \(|\nu| = |m_T(\lambda)|\) the dynamics of the operator \( \frac{1}{p}T \) on \( L^p(X) \) is topologically mixing with periodic points dense, in particular the dynamics is chaotic in the sense of Devaney.

A particular case of multipliers is given by the heat semigroup \( e^{t\Delta} \) on \( X \). For a simply connected harmonic manifold, the heat kernel \( H_t(x, y) \) is radial, i.e. there exists an \( L^1 \) function \( h_t \) radial around \( o \) such that \( H_t(x, y) = (\tau_x h_t)(y) \) (see [Sza90]). The action of \( e^{t\Delta} \) is thus given by convolution with the radial \( L^1 \) function \( h_t \), so \( e^{t\Delta} \) is an \( L^p \)-multiplier for all \( p \in [1, +\infty] \). We determine the symbol of \( e^{t\Delta} \) and then apply the previous theorem to obtain the following corollary:

**Corollary 1.2.** Let \( X \) be a complete, simply connected, harmonic manifold with sectional curvature \( K \leq -1 \), and let \( 2 < p < \infty, 1 < q < 2 \) be such that \( 1/p + 1/q = 1 \). There exists a constant \( c_p = \frac{4p^2}{pq} \) such that the action of the shifted heat semigroup \( (e^{ct\Delta})_{t \geq 0} \) on \( L^p(X) \) is chaotic in the sense of Devaney for all \( c \in \mathbb{C} \) with \( \text{Re } c > c_p \). In fact for any \( t_0 > 0 \), the operator \( e^{ct_0\Delta} \) on \( L^p(X) \) is chaotic for all \( c \in \mathbb{C} \) with \( \text{Re } c > c_p \).

In section 2 we recall some basic facts about eigenfunctions of the Laplacian, the Fourier transform, and convolution on harmonic manifolds, show that convolution with a radial measure of finite variation is an \( L^p \)-multiplier, and prove existence of the symbol of a multiplier. In section 3 we prove the main theorem. We also prove the corollary by determining the symbol of the multiplier \( e^{t\Delta} \).
2. Preliminaries

In this section we briefly recall the facts about the Fourier transform on harmonic manifolds which we will require. For details the reader is referred to [Bis18]. Throughout, $X$ will denote a complete, simply connected harmonic $n$-manifold with sectional curvatures $K$ satisfying $K \leq -1$. We fix a basepoint $o \in X$.

2.1. CAT(-1) spaces and Busemann functions. In this case, $X$ is a CAT(-1) space, and we can define a boundary at infinity $\partial X$ of the space $X$, defined as the set of equivalence classes of geodesic rays $\gamma : [0, \infty)$ in $X$, where two rays are equivalent if they stay at bounded distance from each other. There is a natural topology on $X := X \cup \partial X$ called the cone topology for which $X$ becomes a compactification of $X$ (for details on CAT($\kappa$) spaces we refer to [BH99]).

Given a point $x \in X$, let $\lambda _x$ be normalized Lebesgue measure on the unit tangent sphere $T^1_x X$, i.e. the unique probability measure on $T^1_x X$ invariant under the orthogonal group of the tangent space $T_x X$. For $v \in T^1_x X$, let $\gamma _v : [0, \infty) \to X$ be the unique geodesic ray with initial velocity $v$. Then we have a homeomorphism $p_x : T^1_x X \to \partial X, v \mapsto \gamma _v(\infty)$. The visibility measure on $\partial X$ (with respect to the basepoint $x$) is defined to be the push-forward $(p_x)_* \lambda _x$ of $\lambda _x$ under the map $p_x$; for notational convenience, we will however denote the visibility measure on $\partial X$ by the same symbol $\lambda _x$.

The Busemann cocycle $B : X \times X \times \partial X$ is defined by

$$B(x, y, \xi) := \lim_{z \to \xi} (d(x, z) - d(y, z))$$

Given a point $x \in X$ and a boundary point $\xi \in \partial X$, the Busemann function at $\xi$ based at $x$ is defined by

$$B_{\xi, x}(y) := B(y, x, \xi)$$

The Busemann functions $B_{\xi, x}$ are $C^2$ convex functions, and their level sets are called horospheres based at $\xi$.

2.2. Radial and horospherical eigenfunctions of the Laplacian. Let $\Delta$ denote the Laplace-Beltrami operator of $X$, or Laplacian. As $X$ is harmonic, $X$ is also asymptotically harmonic, i.e. all horospheres have constant mean curvature, so there is a constant $h$ such that $\Delta B_{\xi, x} \equiv h$ for all $\xi \in \partial X, x \in X$. Since $X$ is negatively curved, in fact $h > 0$. We let

$$\rho := \frac{1}{2} h$$

A function $f$ on $X$ is called radial around a point $x \in X$ if $f$ is constant on geodesic spheres centered at $x$. For any $x \in X$ and $\lambda \in \mathbb{C}$, there is a unique eigenfunction $\phi_{\lambda, x}$ of $\Delta$ for the eigenvalue $-(\lambda ^2 + \rho ^2)$ which is radial around $x$ and satisfies $\phi_{\lambda, x}(x) = 1$. Moreover for any fixed $y \in Y, \lambda \mapsto \phi_{\lambda, x}(y)$ is an entire function of $\lambda$. The functions $\phi_{\lambda, x}$ are real-valued for $\lambda \in \mathbb{R} \cup i \mathbb{R}$, and bounded by 1 for $|\text{Im} \lambda| \leq \rho$. Given $p > 2$, for all $\lambda$ in the strip $S_p := \{ |\text{Im} \lambda| < (1 - 2/p)\rho \}$, the function $\phi_{\lambda, x}$ is in $L^p(X)$. 
For any \( x \in X, \xi \in \partial X \) and \( \lambda \in \mathbb{C} \), the function \( e^{(i\lambda - \rho)B_{\xi,x}} \) is an eigenfunction of \( \Delta \) for the eigenvalue \(- (\lambda^2 + \rho^2)\). Note that this eigenfunction is constant on horospheres based at \( \xi \).

2.3. The spherical and Helgason Fourier transforms. Let \( f \in L^1(X) \). Given a point \( x \in X \), the spherical Fourier transform of \( f \) based at \( x \) is the function \( \hat{f}^x \) on \( \mathbb{R} \) defined by pairing \( f \) with the radial eigenfunctions \( \phi_{\lambda,x} \):

\[
\hat{f}^x(\lambda) := \int_X f(y) \phi_{\lambda,x}(y) \, d\text{vol}(y), \quad \lambda \in \mathbb{R}
\]

There exists a function \( c \) on \( \mathbb{C} - \{0\} \) satisfying, for some constants \( C, K > 0 \), the estimates

\[
\frac{1}{C}|\lambda| \leq |c(\lambda)|^{-1} \leq C|\lambda|, \quad 0 < |\lambda| \leq K
\]

\[
\frac{1}{C}|\lambda|^{(n-1)/2} \leq |c(\lambda)|^{-1} \leq C|\lambda|^{(n-1)/2}, \quad |\lambda| \geq K
\]

such that the following inversion formula for the spherical Fourier transform from [Bis18] holds:

**Theorem 2.1.** Let \( f \in C^\infty_c(X) \) be radial around \( x \). Then

\[
f(y) = \int_0^\infty \hat{f}^x(\lambda) \phi_{\lambda,x}(y)|c(\lambda)|^{-2} \, d\lambda
\]

for all \( y \in X \).

Given \( 1 \leq q < 2 \), if \( p > 2 \) is the conjugate exponent such that \( 1/p + 1/q = 1 \), then using the fact that the functions \( \phi_{\lambda,0} \) are in \( L^p(X) \) for \( \lambda \) in the strip \( S_p \), we have the following proposition from [Bis18]:

**Proposition 2.2.** Let \( 1 \leq q < 2 \) and \( p > 2 \) be such that \( 1/p + 1/q = 1 \). Then for any \( x \in X \) and \( f \in L^q(X) \), the spherical Fourier transform of \( f \) based at \( x \) is well-defined and extends to a holomorphic function on the strip \( S_p \).

Let \( f \in C^\infty_c(X) \). Given \( x \in X \), the Helgason Fourier transform of \( f \) based at \( x \) is the function \( \tilde{f}^x : \mathbb{C} \times \partial X \to \mathbb{C} \) defined by

\[
\tilde{f}^x(\lambda, \xi) := \int_X f(y)e^{-i\lambda \rho)B_{\xi,x}(y)} \, d\text{vol}(y), \quad \lambda \in \mathbb{C}, \xi \in \partial X
\]

We have the following relation between the Helgason Fourier transforms based at two different basepoints \( o, x \in X \):

\[
\tilde{f}^x(\lambda, \xi) = e^{i(\lambda+\rho)B_{\xi,o}(x)} \tilde{f}^o(\lambda, \xi)
\]

If \( f \) is radial around the point \( x \) then the Helgason Fourier transform reduces to the spherical Fourier transform,

\[
\tilde{f}^x(\lambda, \xi) = \hat{f}^x(\lambda), \lambda \in \mathbb{C}, \xi \in \partial X
\]
From [Bis18] we have the following inversion formula for the Helgason Fourier transform:

**Theorem 2.3.** Let \( x \in X \) and let \( f \in C_c^\infty(X) \). Then

\[
f(y) = \int_0^\infty \int_{\partial X} \tilde{f}^x(\lambda, \xi) e^{i(\lambda-\rho)B_t(y)} d\lambda_x(\xi) |c(\lambda)|^{-2} d\lambda
\]

for all \( y \in X \).

### 2.4. Convolution operators and \( L^p \) multipliers.

For a point \( x \in X \), let \( d_x \) denote the distance function from the point \( x \), defined by \( d_x(y) := d(x, y), y \in X \).

Given a function \( f \) on \( X \) radial around a point \( x \), let \( u \) be a function on \([0, \infty)\), such that \( f = u \circ d_x \). Given a point \( y \) in \( X \), the \( y \)-translate of \( f \) is the function \( \tau_yf \) radial around \( y \) defined by \( \tau_yf := u \circ d_y \). It follows from the fact that \( X \) is harmonic that \( ||\tau_yf||_p = ||f||_p \) for all \( p \in [1, +\infty] \). Moreover if \( f \) is also in \( L^1 \), then the spherical Fourier transforms satisfy

\[
\hat{\tau_yf^y}(\lambda) = \tilde{f}^x(\lambda)
\]

We note also from [Bis18] that there is an even \( C^\infty \) function on \( \mathbb{R} \) which we denote by \( \phi_\lambda \) such that \( \phi_{\lambda,x} = \phi_\lambda \circ d_x \). Thus the \( x \)-translate of the eigenfunction \( \phi_{\lambda,o} \) radial around \( o \) is the eigenfunction \( \phi_{\lambda,x} \) radial around \( x \), \( \tau_x \phi_{\lambda,o} = \phi_{\lambda,x} \).

For simplicity, in the sequel, unless otherwise mentioned, by ”radial function” we will mean a function which is radial around the basepoint \( o \). Likewise, by ”spherical Fourier transform” we will mean the spherical Fourier transform based at \( o \), unless otherwise mentioned.

Given \( f, g \in L^1(X) \) with \( g \) radial, the convolution of \( f \) with \( g \) is the function \( f \ast g \) on \( X \) defined by

\[
(f \ast g)(x) = \int_X f(y) \tau_x g(y) d\text{vol}(y)
\]

The integral above converges for a.e. \( x \), and satisfies

\[
||f \ast g||_1 \leq ||f||_1 ||g||_1
\]

We note that if \( f \in L^\infty(X) \) and \( g \in L^1(X) \) with \( g \) radial, then the integral defining \( (f \ast g)(x) \) converges for all \( x \) and satisfies

\[
||f \ast g||_\infty \leq ||f||_\infty ||g||_1
\]

It follows by interpolation that for any \( p \in [1, +\infty] \), convolution with a radial \( L^1 \) function \( g \) defines a bounded linear operator on \( L^p(X) \) satisfying

\[
||f \ast g||_p \leq ||f||_p ||g||_1
\]

for all \( f \in L^p(X) \).

A standard argument using the above inequality and density of \( C_c^\infty(X) \) in \( L^p(X) \) gives that if \( \{ \phi_n \} \) is an approximate identity, i.e. \( \phi_n \geq 0, \int_X \phi_n d\text{vol} = 1 \) and \( \int_{B(o,r)} \phi_n d\text{vol} \to 1 \) for any \( r > 0 \), then for any \( f \in L^p(X) \),

\[
||f \ast \phi_n - f||_p \to 0
\]

as \( n \to \infty \).
In [Bis18] it is shown that for $\phi, \psi \in C_c^\infty(X)$ with $\psi$ radial, the Helgason Fourier transform of the convolution $\phi * \psi$ satisfies

$$\hat{\phi * \psi}(\lambda, \xi) = \hat{\phi}(\lambda, \xi) \hat{\psi}(\lambda) , \lambda \in \mathbb{C}, \xi \in \partial X$$

In particular, if both $\phi, \psi$ are radial, then

$$\hat{\phi * \psi}(\lambda) = \hat{\phi}(\lambda) \hat{\psi}(\lambda)$$

We also have from [Bis18] that the radial $L^1$ functions form a commutative Banach algebra under convolution. It follows, using density of radial $C_c^\infty$-functions in radial $L^p$ functions, that for a radial $L^1$ function $g$ the convolution operator $T_g : f \mapsto f * g$ on $L^p(X)$ preserves the subspace of radial $L^p$ functions and satisfies, for all radial $C_c^\infty$-functions $\phi, \psi$,

$$T_g\phi * \psi = \phi * T_g\psi$$

In fact for any $x \in X$ the convolution operator $T_g$ preserves the subspace of $L^p$ functions radial around $x$. This is a consequence of the following lemma:

**Lemma 2.4.** Let $\phi, \psi$ be radial $C_c^\infty$-functions. Then for any $x \in X$,

$$\tau_x \phi * \psi = \tau_x(\phi * \psi)$$

**Proof:** We compute Helgason Fourier transforms:

$$\tau_x \hat{\phi * \psi}(\lambda, \xi) = \tau_x \hat{\phi}(\lambda, \xi) \hat{\psi}(\lambda)$$

$$= e^{-i(\lambda + \rho)B_{\xi,o}(x)} \tau_x \hat{\phi}(\lambda) \hat{\psi}(\lambda)$$

$$= e^{-i(\lambda + \rho)B_{\xi,o}(x)} \tau_x \hat{\phi}(\lambda) \hat{\psi}(\lambda)$$

$$= e^{-i(\lambda + \rho)B_{\xi,o}(x)} \tau_x (\phi * \psi)(\lambda)$$

$$= e^{-i(\lambda + \rho)B_{\xi,o}(x)} \tau_x (\phi * \psi)(\lambda, \xi)$$

$$= \tau_x (\phi * \psi)(\lambda, \xi)$$

It follows from the Fourier inversion formula (Theorem 2.3) that $\tau_x \phi * \psi = \tau_x (\phi * \psi)$. 

Now given $g$ a radial $L^1$ function and $\phi \in C_c^\infty(X)$, let $\{\psi_n\}$ be a sequence of radial $C_c^\infty$-functions converging to $g$ in $L^1$. Given $x \in X$, since $\phi$ and $\tau_x \phi$ are in $L^\infty$, it follows that $\phi * \psi_n$ and $\tau_x \phi * \psi_n$ converge pointwise to $\phi * g$ and $\tau_x \phi * g$ respectively, so $\tau_x (\phi * \psi_n)$ converges pointwise to $\tau_x (\phi * g)$. Applying the previous Lemma, we obtain $\tau_x \phi * g = \tau_x (\phi * g)$. Thus the convolution operator $T_g$ satisfies

$$T_g \tau_x \phi = \tau_x T_g \phi$$

for all radial $C_c^\infty$ functions $\phi$ and all $x \in X$.

This leads us to the following definition:
Definition 2.5. (L^p-multipliers) For \( p \in [1, +\infty] \), an L^p-multiplier is a bounded operator \( T : L^p(X) \to L^p(X) \) such that:

1. \( T \) preserves the subspace of radial L^p functions.
2. For all radial \( C_c^\infty \)-functions \( \phi, \psi \) we have
   \[ T\phi \ast \psi = \phi \ast T\psi \]
3. For all radial \( C_c^\infty \)-functions \( \phi \) and all \( x \in X \) we have
   \[ T\tau_x \phi = \tau_x T\phi \]

Thus convolution operators given by radial \( L^1 \) functions are \( L^p \) multipliers for all \( p \in [1, +\infty] \). For more general examples of \( L^p \)-multipliers we can consider convolution with radial complex measures \( \mu \) of finite total variation, which is defined as follows:

We say that a complex measure \( \mu \) on \( X \) is radial around \( o \) if there exists a complex measure \( \bar{\mu} \) on \( [0, \infty) \) such that for any continuous bounded function \( f \) on \( X \) we have
\[
\int_X f(x) d\mu(x) = \int_0^\infty \left( \int_{S(o,r)} f(y) d\lambda_{o,r}(y) \right) d\bar{\mu}(r)
\]
where \( S(o,r) \) denotes the geodesic sphere of radius \( r \) around \( o \) and \( \lambda_{o,r} \) denotes the volume measure on \( S(o,r) \) induced from the metric on \( X \). For \( x \in X \), the \( x \)-translate of such a measure \( \mu \) is the measure \( \tau_x \mu \) radial around \( x \) defined by requiring that
\[
\int_X f(y) d\tau_x \mu(y) = \int_0^\infty \left( \int_{S(x,r)} f(y) d\lambda_{x,r}(y) \right) d\bar{\mu}(r)
\]
for all continuous bounded functions \( f \) on \( X \) (where \( S(x,r) \) is the geodesic sphere of radius \( r \) around \( x \) and \( \lambda_{x,r} \) is the volume measure on \( S(x,r) \)).

For an \( L^1 \) function \( f \) on \( X \) and a radial complex measure \( \mu \) on \( X \) of finite variation, the convolution \( f \ast \mu \) is the function on \( X \) defined by
\[
(f \ast \mu)(x) := \int_X f(y) d\tau_x \mu(y)
\]
We note that any \( L^1 \) function \( g \) which is radial around \( o \) gives a complex measure \( \mu = gdvol \) which is radial around \( o \) and satisfies \( ||\mu|| = ||g||_1 \) (where \( ||\mu|| \) is the total variation norm of \( \mu \)), and \( f \ast \mu = f \ast g \), so convolution with finite variation radial measures generalizes convolution with \( L^1 \) radial functions.

Given a finite variation radial measure \( \mu \), we can approximate \( \mu \) in the weak-* topology by measures \( g_n \) where \( g_n \)'s are radial \( L^1 \) functions such that \( ||g_n||_1 \to ||\mu|| \), then for any \( f \in C_c^\infty(X) \) we have \( f \ast g_n \to f \ast \mu \) pointwise, and an application of Fatou’s Lemma then leads to the inequality
\[
||f \ast \mu||_1 \leq ||f||_1 ||\mu||
\]
valid for all \( f \in C_c^\infty(X) \) and all finite variation radial measures \( \mu \). The inequality then continues to hold for all \( f \in L^1(X) \) by density of \( C_c^\infty(X) \) in \( L^1(X) \).
Moreover for \( f \in L^\infty(X) \) and \( \mu \) a finite variation radial measure, it is straightforward to see that the integral defining \( f * \mu \) exists for all \( x \) and satisfies
\[
||f * \mu||_\infty \leq ||f||_\infty ||\mu||
\]
Thus by interpolation for any \( p \in [1, +\infty] \), convolution with a finite variation radial measure \( \mu \) defines a bounded operator on \( L^p(X) \) satisfying
\[
||f * \mu||_p \leq ||f||_p ||\mu||
\]
for all \( f \in L^p(X) \).

**Proposition 2.6.** Let \( \mu \) be a radial complex measure of finite total variation. Then for any \( p \in [1, +\infty] \), the operator \( T_\mu : f \mapsto f * \mu \) is an \( L^p \) multiplier.

**Proof:** Fix \( p \in [1, \infty] \). Let \( \{g_n\} \) be a sequence of radial \( L^1 \) functions such that \( g_n \text{divol} \to \mu \) in the weak-* topology and such that \( ||g_n||_1 \to ||\mu|| \). Then for any radial \( C^\infty \)-function \( \phi \), the functions \( \phi * g_n \) are radial and converge to \( \phi * \mu \) pointwise, so \( \phi * \mu \) is radial. It follows that \( T_\mu \) preserves the subspace of radial \( L^p \) functions.

Let \( \phi, \psi \) be radial \( C^\infty \)-functions. Then
\[
||\phi * g_n||_\infty \leq ||\phi||_\infty ||g_n||_1 \leq C||\phi||_\infty
\]
for some constant \( C > 0 \), so for any \( x \in X \) the functions \( \phi * g_n \) are uniformly bounded on the support of \( \tau_x \psi \), and converge to \( \phi * \mu \) pointwise, so it follows from dominated convergence that \( (\phi * g_n) * \psi(x) \to (\phi * \mu) * \psi(x) \) for all \( x \in X \). A similar argument gives that \( \phi * (\psi * g_n)(x) \to \phi * (\psi * \mu)(x) \) for all \( x \in X \). Since \( (\phi * g_n) * \psi = \phi * (\psi * g_n) \) for all \( n \), it follows that \( (\phi * \mu) * \psi = \phi * (\psi * \mu) \).

Let \( \phi \) be a radial \( C^\infty \)-function and let \( x \in X \). Then \( \phi * g_n \) and \( \tau_x \phi * g_n \) converge to \( \phi * \mu \) and \( \tau_x \phi * \mu \) respectively, so \( \tau_x (\phi * g_n) \) converges pointwise to \( \tau_x (\phi * \mu) \). Since \( \tau_x (\phi * g_n) = \tau_x (\phi * g_n) \) for all \( n \), it follows that \( \tau_x (\phi * \mu) = \tau_x (\phi * \mu) \).

Let \( 1 \leq q < 2 \) and \( p > 2 \) such that \( 1/p + 1/q = 1 \). Let \( f \) be a radial \( L^q \) function, then the spherical Fourier transform \( \hat{f} \) is holomorphic in the strip \( S_p \), and it turns out that for any radial \( C^\infty \)-function \( \psi \), we have
\[
\hat{f} * \psi(\lambda) = \hat{f}(\lambda) \hat{\psi}(\lambda), \lambda \in S_p
\]
This can be seen as follows: let \( \{\phi_n\} \) be a sequence of radial \( C^\infty \)-functions converging to \( f \) in \( L^q(X) \), then since \( \phi_{\lambda,\varphi} \in L^p(X) \) for \( \lambda \in S_p \), it follows from H"older's inequality that \( \hat{\phi}_n(\lambda) \to \hat{f}(\lambda) \) for \( \lambda \in S_p \). Moreover, since \( \psi \in L^1(X) \), \( \phi_n * \psi \) converges to \( f * \psi \) in \( L^q(X) \), so as before \( \hat{\phi}_n * \hat{\psi}(\lambda) \to \hat{f} * \hat{\psi}(\lambda) \) for \( \lambda \in S_p \). The desired equality follows by passing to the limit in the equality \( \phi_n * \hat{\psi}(\lambda) = \hat{\phi}_n(\lambda) \hat{\psi}(\lambda) \).

Other examples of \( L^p \)-multipliers can be obtained by using the Kunze-Stein phenomenon proved in [Bis18]. This asserts that if \( 1 \leq q < 2 \), then there is a constant \( C_q > 0 \) such that for all \( C^\infty \)-functions \( f, g \) with \( g \) radial, we have
\[
||f * g||_2 \leq C_q ||f||_2 ||g||_q.
\]
Combining this with the trivial estimate
\[
||f * g||_1 \leq ||f||_1 ||g||_1
\]
it follows from interpolation that for any \( p > 2 \), if \( 1 \leq r < 2 \) is such that \( 1/r < 1 + 1/p \), then there is a constant \( C_p > 0 \) such that

\[
||f * g||_p \leq C_p ||f||_p ||g||_r.
\]

The above inequality then implies that convolution with a radial \( L^r \)-function \( g \) defines an \( L^p \)-multiplier \( T_g : L^p(X) \to L^p(X) \).

The following proposition justifies the use of the term "multiplier":

**Proposition 2.7.** Let \( 1 \leq q < 2 \) and \( p > 2 \) be such that \( 1/p + 1/q = 1 \). Let \( T : L^p(X) \to L^p(X) \) be an \( L^p \)-multiplier. Then there exists a holomorphic function \( m_T \) on the strip \( S_p \) such that, for any radial \( C_c^\infty \)-function \( \phi \), we have \( T\phi \in L^q(X) \), and

\[
\hat{T}\phi(\lambda) = m_T(\lambda)\hat{\phi}(\lambda), \lambda \in S_p
\]

**Proof:** We first show that given a radial \( C_c^\infty \) function \( \phi \), \( T\phi \in L^q(X) \). For any radial \( C_c^\infty \)-function \( \psi \), we have

\[
\left| \int_X T\phi(x)\psi(x)dvol(x) \right| = |T\phi * \psi(o)|
\]

\[
= |\phi \ast T\psi(o)|
\]

\[
= \left| \int_X \phi(x)T\psi(x)dvol(x) \right|
\]

\[
\leq ||\phi||_q ||T\psi||_p
\]

\[
\leq (||T||||\phi||_q)||\psi||_p
\]

Since \( T\phi \) is radial and the above inequality holds for all radial \( C_c^\infty \)-functions \( \psi \), it follows that \( ||T\phi||_q \leq ||T||||\phi||_q < +\infty \).

Thus for any radial \( C_c^\infty \)-function \( \phi \) which is not identically zero, \( \hat{T}\phi \) is a holomorphic function in the strip \( S_p \), and we can define a meromorphic function \( m_\phi \) on \( S_p \) by

\[
m_\phi := \frac{\hat{T}\phi}{\hat{\phi}}
\]

If \( \psi \) is another radial \( C_c^\infty \)-function which is not identically zero, then the equality \( T\phi \ast \psi = \phi \ast T\psi \) implies \( \hat{T}\phi \ast \hat{T}\psi = \hat{\phi} \hat{T}\psi \) on \( S_p \) and hence \( m_\phi = m_\psi \). Thus the meromorphic function \( m_\phi \) is independent of the choice of \( \phi \), and we may denote it by \( m_T \).

It suffices to show that \( m_T \) is in fact holomorphic in \( S_p \). For this it is enough to show that given any \( \lambda_0 \in S_p \), there is a radial \( C_c^\infty \)-function \( \phi \) such that \( \hat{\phi}(\lambda_0) \neq 0 \), since then \( m_T = \hat{T}\phi/\hat{\phi} \) will be holomorphic near \( \lambda_0 \). If \( \hat{\phi}(\lambda_0) = 0 \) for all radial \( C_c^\infty \)-functions \( \phi \), then

\[
\int_X \phi(x)\phi_{\lambda_0,o}(x)dvol(x) = 0
\]

for all such \( \phi \), and since \( \phi_{\lambda_0,o} \) is radial this implies that \( \phi_{\lambda_0,o} \equiv 0 \), a contradiction. Thus \( m_T \) is holomorphic in \( S_p \) and by definition satisfies \( \hat{T}\phi = m_T\hat{\phi} \) for all radial \( C_c^\infty \)-functions \( \phi \). \( \diamond \)
Remark. If for $1 \leq q < 2$ we have an $L^q$-multiplier $T$, then by definition $T \phi \in L^q$ for $\phi$ a radial $C_\infty^\infty$-function, and then the proof of the above proposition applies to show that for any $L^q$-multiplier $T$ there is a function $m_T$ holomorphic in the strip $S_p$ such that $\hat{T}\phi(\lambda) = m_T(\lambda)\hat{\phi}(\lambda)$ for $\lambda \in S_p$ and $\phi$ a radial $C_\infty^\infty$-function. Thus the conclusion of the proposition holds in fact for all $L^p$-multipliers with $p \neq 2$.

We will call the holomorphic function $m_T$ given by the above proposition the symbol of the $L^p$-multiplier $T$. Note that if $T$ is given by convolution with a radial $L^1$-function $g$, then the symbol $m_T$ equals the spherical Fourier transform $\hat{g}^\phi$ of $g$, since $\hat{\phi} * \hat{g}^\phi = \hat{\phi}^\phi \hat{g}^\phi$ for all radial $C_\infty^\infty$-functions $\phi$.

**Proposition 2.8.** Let $1 \leq q < 2$ and $p > 2$ be such that $1/p + 1/q = 1$. Let $T : L^p(X) \to L^p(X)$ be an $L^p$-multiplier. Then for all $\lambda \in S_p$ and $x \in X$, we have

$$T \phi_{\lambda,x} = m_T(\lambda) \phi_{\lambda,x}$$

**Proof:** Let $\lambda \in S_p$ and let $\{\phi_n\}$ be a sequence of radial $C_\infty^\infty$-functions converging to $\phi_{\lambda,o}$ in $L^p(X)$. Then $T\phi_n$ converges to $T\phi_{\lambda,o}$ in $L^p(X)$. For any radial $C_\infty^\infty$-function $\psi$, since $\psi \in L^q(X)$ it follows from Holder’s inequality that

$$\int_X T\phi_n(x)\psi(x)dvol(x) \to \int_X T\phi_{\lambda,o}(x)\psi(x)dvol(x)$$

as $n \to \infty$. On the other hand, again using Holder’s inequality and the fact that $\phi_n$ converges to $\phi_{\lambda,o}$ in $L^q(X)$, we have

$$\int_X T\phi_n(x)\psi(x)dvol(x) = T\phi_n * \psi(o)$$

$$= \phi_n * T\psi(o)$$

$$= \int_X \phi_n(x)T\psi(x)dvol(x)$$

$$\to \int_X \phi_{\lambda,o}T\psi(x)dvol(x)$$

$$= \hat{T}\psi(\lambda)$$

$$= m_T(\lambda)\hat{\psi}(\lambda)$$

$$= m_T(\lambda)\int_X \phi_{\lambda,o}(x)\psi(x)dvol(x)$$

Thus

$$\int_X T\phi_{\lambda,o}(x)\psi(x)dvol(x) = m_T(\lambda)\int_X \phi_{\lambda,o}(x)\psi(x)dvol(x)$$

for all radial $C_\infty^\infty$-functions $\psi$, so it follows that $T\phi_{\lambda,o} = m_T(\lambda)\phi_{\lambda,o}$.
Now given \( x \in X \) and \( \lambda \in S_p \), the functions \( \tau_x \phi_n \) converge to \( \phi_{\lambda,x} \) in \( L^p(X) \), and so

\[
T\phi_{\lambda,x} = \lim_{n \to \infty} T\tau_x \phi_n \\
= \lim_{n \to \infty} \tau_x T\phi_n \\
= \tau_x T\phi_{\lambda,o} \\
= m_T(\lambda) \tau_x \phi_{\lambda,o} \\
= m_T(\lambda) \phi_{\lambda,x}
\]

\[ \diamond \]

3. Dynamics of \( L^p \) multipliers

3.1. General multipliers. We show in this section that the dynamics of appropriately scaled \( L^p \)-multipliers is chaotic in the sense of Devaney if \( 2 < p < \infty \). The following lemma is the key to the results which follow:

**Lemma 3.1.** Let \( 1 < q < 2 \) and \( 2 < p < \infty \) be such that \( 1/p + 1/q = 1 \). Let \( K \subset S_p \) be a subset such that \( K \) has a limit point in \( S_p \). Then the subspace

\[
V_K := \text{Span}\{ \tau_x \phi_{\lambda,o} | x \in X, \lambda \in K \}
\]

is dense in \( L^p(X) \).

**Proof:** It suffices to show that if \( f \in L^q(X) \) is such that \( \int_X f(y) \tau_x \phi_{\lambda,o}(y) \text{dvol}(y) = 0 \) for all \( x \in X, \lambda \in K \), then \( f = 0 \). Given such an \( f \in L^q(X) \), the hypothesis on \( f \) means that for any \( x \in X \), the spherical Fourier transform of \( f \) based at \( x \) vanishes on the set \( K \). By Proposition 2.2, \( \hat{f}^x \) is holomorphic in \( S_p \) and \( K \) has a limit point in \( S_p \), thus \( \hat{f}^x \) vanishes identically in \( S_p \), in particular on \( \mathbb{R} \). Thus for all \( x \in X \) and \( \lambda \in \mathbb{R} \), we have

\[
(f * \phi_{\lambda,o})(x) = \int_X f(y) \phi_{\lambda,x}(y) \text{dvol}(y) = \hat{f}^x(\lambda) = 0
\]

Let \( \phi \) be a radial \( C^\infty_c \)-function, then by the Fourier inversion formula (Theorem 2.1) we have

\[
\phi(y) = \int_0^\infty \hat{\phi}(\lambda) \phi_{\lambda,o}(y) |c(\lambda)|^{-2} d\lambda
\]

for all \( y \in X \), so it follows from Fubini’s theorem that

\[
(f * \phi)(x) = \int_0^\infty (f * \phi_{\lambda,o})(x) \hat{\phi}(\lambda) |c(\lambda)|^{-2} d\lambda = 0
\]

for all \( x \in X \). Thus \( f * \phi = 0 \) for all radial \( C^\infty_c \)-functions \( \phi \). Now letting \( \{ \phi_n \} \) be a sequence of radial \( C^\infty_c \)-functions which forms an approximate identity, we have \( f * \phi_n = 0 \) for all \( n \), and \( f * \phi_n \) converges to \( f \) in \( L^q(X) \), thus \( f = 0 \). \( \diamond \)

We will also need the following lemma:

**Lemma 3.2.** Let \( 2 < p < \infty \) and let \( T : L^p(X) \to L^p(X) \) be an \( L^p \)-multiplier. Suppose \( T \) is not a scalar multiple of the identity. Then the symbol \( m_T \) is a nonconstant holomorphic function in the strip \( S_p \).
Proof: Suppose to the contrary that \( m_T \equiv C \) for some constant \( C \in \mathbb{C} \). By Proposition 2.8 we then have \( T \phi_{\lambda,x} = C \phi_{\lambda,x} \) for all \( \lambda \in S_p \) and \( x \in X \). Thus \( T = C \text{Id} \) on the subspace \( V = \text{Span}\{\phi_{\lambda,x}|\lambda \in S_p, x \in X\} \), which is dense by the previous Lemma, hence \( T = C \text{Id} \) on \( L^p(X) \), a contradiction. \( \diamond \)

The main tool to prove that the dynamics of \( L^p \) multipliers is chaotic is the following criterion of Godfrey-Shapiro (GEMI, Theorem 3.1):

**Theorem 3.3. (Godfrey-Shapiro criterion)** Let \( X \) be a separable Banach space and let \( T: X \to X \) be a bounded operator. Suppose the subspaces \( X^+, X^- \) defined by

\[
X^+ = \text{Span}\{v \in X | Tv = \lambda v \text{ for some } \lambda \in \mathbb{C} \text{ such that } |\lambda| < 1\}
\]
\[
X^- = \text{Span}\{v \in X | Tv = \lambda v \text{ for some } \lambda \in \mathbb{C} \text{ such that } |\lambda| > 1\}
\]

are dense in \( X \). Then the dynamics of \( T \) on \( X \) is topologically mixing, i.e. for any two nonempty open sets \( U, V \subset X \), there exists \( N \geq 1 \) such that \( T^n U \cap V \neq \emptyset \) for all \( n \geq N \).

We can now prove Theorem 1.1.

**Proof of Theorem 1.1.** Let \( \lambda_0 \in S_p \) be such that \( m_T(\lambda_0) \neq 0 \), let \( \nu \in \mathbb{C} \) be such that \( |\nu| = |m_T(\lambda)| \) and set \( \alpha = m_T(\lambda)/\nu \in S^1 \). Let \( D_0 = \{z \in \mathbb{C}||z| < 1\} \) and \( D_\infty = \{z \in \mathbb{C}||z| > 1\} \). Let \( U \subset S_p \) be an open neighbourhood of \( \lambda_0 \), then since \( \alpha \in S^1 \) and by Lemma 3.2 \( m_T \) is a nonconstant holomorphic function, there are nonempty open subsets \( U^+, U^- \subset U \) such that \( \{m_T(\lambda)/\nu | \lambda \in U^+\} \subset D_0 \) and \( \{m_T(\lambda)/\nu | \lambda \in U^-\} \subset D_\infty \). By Proposition 2.8 for all \( \lambda \in U \) and \( x \in X \), the function \( \phi_{\lambda,x} \in L^p(X) \) is an eigenfunction of the operator \( \frac{1}{p}T \) with eigenvalue \( m_T(\lambda)/\nu \). By Lemma 3.1 the subspaces \( V^+ = \{\phi_{\lambda,x} | \lambda \in U^+, x \in X\} \) and \( V^- = \{\phi_{\lambda,x} | \lambda \in U^-, x \in X\} \) are dense in \( L^p(X) \). It follows from the Godfrey-Shapiro criterion that the dynamics of \( \frac{1}{p}T \) is topologically mixing.

It remains to show that the periodic points of \( \frac{1}{p}T \) are dense in \( L^p(X) \). Since \( m_T \) is a nonconstant holomorphic function and \( m_T(\lambda_0)/\nu \in S^1 \), we can choose sequences \( \{\lambda_n\} \subset U \) and \( \{p_n/q_n\} \subset \mathbb{Q} \) such that \( m_T(\lambda_n)/\nu = e^{2\pi ip_n/q_n} \) and \( \lambda_n \to \lambda_0 \) as \( n \to \infty \). Then by Lemma 3.1 the subspace \( V = \text{Span}\{\phi_{\lambda_n,x}|x \in X,n \geq 1\} \) is dense in \( L^p(X) \). It thus suffices to show that each element of \( V \) is a periodic point of \( \frac{1}{p}T \). Any element \( \phi \in V \) can be written as \( \phi = \sum_{j=1}^N a_j \phi_{\lambda_j,x_j} \) for some \( N \geq 1, a_1, \ldots, a_N \in \mathbb{C} \) and \( x_1, \ldots, x_N \in X \). Since \( \phi_{\lambda_j,x_j} \) is an eigenvector of \( \frac{1}{p}T \) with eigenvalue \( e^{2\pi ip_j/q_j} \), letting \( q = \prod_{j=1}^N q_j \) it follows that \( (\frac{1}{p}T)^q \phi_{\lambda_j,x_j} = \phi_{\lambda_j,x_j} \) for all \( j \), thus \( (\frac{1}{p}T)^q \phi = \phi \) and \( \phi \) is a periodic point of \( \frac{1}{p}T \). \( \diamond \)

### 3.2. The heat semigroup

We recall some basic facts about the heat semigroup and heat kernel on a complete Riemannian manifold \( X \). Denote by \( \Delta_X = \text{div grad} \) the Laplacian acting on \( C_c^\infty(X) \subset L^2(X) \), then this is an essentially self-adjoint operator, and so its closure \( \Delta_{X,2} \) is a self-adjoint operator on \( L^2(X) \). Since \( \Delta_{X,2} \) is negative, it generates a semigroup \( e^{t\Delta_{X,2}} \) on \( L^2(X) \) by the spectral theorem for unbounded self-adjoint operators. The operators \( e^{t\Delta_{X,2}} \) are positive, leave \( L^1(X) \cap L^\infty(X) \subset L^2(X) \) invariant, and may be extended to a positive contraction
semigroup $e^{t\Delta X,p}$ on $L^p(X)$ for any $p \in [1, +\infty]$, which is strongly continuous for $p \in [1, +\infty]$ ([Dav90]). In the sequel we will write simply $e^{t\Delta}$ for the semigroup $e^{t\Delta X,p}$ on $L^p(X)$. From [Str83] we have the following:

There exists a $C^\infty$ function $H_t(x,y)$ on $\mathbb{R}^+ \times X \times X$, the heat kernel, such that for all $t > 0$ and $x \in X$ the function $H_t(x,.)$ is positive and in $L^p$ for all $p \in [1, +\infty]$, and for all $f \in L^p(X)$,

$$e^{t\Delta} f(x) = \int_X f(y)H_t(x,y)dvol(y)$$

and

$$\frac{\partial}{\partial t}e^{t\Delta} f(x) = \Delta e^{t\Delta} f(x)$$

Moreover, it is shown in [Sza90] that for a $X$ a simply connected harmonic manifold, the heat kernel is radial, i.e. there exists a function $h_t$ radial around the basepoint $o$ such that $H_t(x,y) = (\tau_x h_t)(y)$. Thus the action of the heat semigroup on $L^p(X)$ is given in our case by convolution with the radial $L^1$ function $h_t$,

$$e^{t\Delta} f = f \ast h_t$$

for all $f \in L^p(X)$, so $e^{t\Delta}$ is an $L^p$-multiplier for all $p \in [1, +\infty]$. The symbol of the multiplier $e^{t\Delta}$ is given by the following proposition:

**Proposition 3.4.** For any $t > 0$, the spherical Fourier transform of the heat kernel is given by

$$\hat{h}_t^o (\lambda) = e^{-t(\lambda^2 + \rho^2)}, \lambda \in S_\infty$$

**Proof:** Let $p \in (2, \infty)$ and let $\lambda \in S_p$. Then $\phi_{\lambda,o} \in L^p(X)$, and using the fact that the operators $\Delta, e^{t\Delta}$ on $L^p(X)$ commute and $\Delta \phi_{\lambda,o} = -(\lambda^2 + \rho^2)\phi_{\lambda,o}$, we have

$$\frac{\partial}{\partial t} e^{t\Delta} \phi_{\lambda,o} = \Delta e^{t\Delta} \phi_{\lambda,o}$$

$$= e^{t\Delta} \Delta \phi_{\lambda,o}$$

$$= -(\lambda^2 + \rho^2) e^{t\Delta} \phi_{\lambda,o}$$

Thus $t \mapsto e^{t\Delta} \phi_{\lambda,o} \in L^p(X)$ satisfies the first order linear ODE

$$\frac{\partial}{\partial t} e^{t\Delta} \phi_{\lambda,o} = -(\lambda^2 + \rho^2) e^{t\Delta} \phi_{\lambda,o}$$

and $e^{t\Delta} \phi_{\lambda,o} \to \phi_{\lambda,o}$ in $L^p(X)$ as $t \to 0$, hence

$$e^{t\Delta} \phi_{\lambda,o} = e^{-t(\lambda^2 + \rho^2)} \phi_{\lambda,o}$$

for all $t > 0$. Evaluating both sides above at the point $o$ gives

$$\hat{h}_t^o (\lambda) = \int_X \phi_{\lambda,o}(x)h_t(x)dvol(x)$$

$$= e^{t\Delta} \phi_{\lambda,o}(o)$$

$$= e^{-t(\lambda^2 + \rho^2)} \phi_{\lambda,o}(o)$$

$$= e^{-t(\lambda^2 + \rho^2)}$$
We can now prove the result on the chaotic dynamics of shifted heat semigroups:

**Proof of Corollary 1.2.** Given \(2 < p < \infty\) and \(1 < q < 2\) such that \(1/p + 1/q = 1\), let \(c_p = 4\rho^2/(pq)\). Let \(c \in \mathbb{C}\) be such that \(\text{Re } c > c_p\), and let \(t_0 > 0\). Let \(T = e^{-t_0\Delta}\) and \(\nu = e^{-ct_0}\). By Proposition 3.4 above, the symbol of \(T\) is given by \(m_T(\lambda) = e^{-t_0(\lambda^2 + \rho^2)}\). In order to show that the operator \(e^{ct_0}e^{-t_0\Delta} = \frac{1}{2}T\) is chaotic, it suffices by Theorem 1.1 to show that there exists \(\lambda \in S_p\) such that \(|\nu| = |m_T(\lambda)|\).

Letting \(\lambda = s + it \in S_p\), the equality \(|\nu| = |m_T(\lambda)|\) is equivalent to

\[s^2 - t^2 + \rho^2 = \text{Re } c\]

Let \(t\) be such that \(t = (1 - 2/p)\rho - \epsilon\) where \(\epsilon > 0\) is small, then we have

\[
\text{Re } c + t^2 - \rho^2 = (\text{Re } c - c_p) + c_p + (1 - 2/p)^2 \rho^2 + O(\epsilon)
\]

\[= (\text{Re } c - c_p) + (4(1/p)(1 - 1/p) - 4/p + 4/p^2)\rho^2 + O(\epsilon)
\]

\[= (\text{Re } c - c_p) + O(\epsilon)
\]

\[> 0
\]

for \(\epsilon\) small enough such that \(\text{Re } c - c_p > 0\). Thus we can choose \(t\) with \(0 < t < (1 - 2/p)\rho\) such that \(\text{Re } c + t^2 - \rho^2 > 0\), so we can then choose \(s \in \mathbb{R}\) such that \(s^2 = \text{Re } c + t^2 - \rho^2\), or \(s^2 - t^2 + \rho^2 = \text{Re } c\), as required. \(\diamondsuit\)

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