Microcausality and Energy-Positivity in all frames imply Lorentz Invariance of dispersion laws

by

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Abstract. A new presentation of the Borchers-Buchholz result of the Lorentz-invariance of the energy-momentum spectrum in theories with broken Lorentz symmetry is given in terms of properties of the Green’s functions of microcausal Bose and Fermi-fields. Strong constraints based on complex geometry phenomenons are shown to result from the interplay of the basic principles of causality and stability in Quantum Field Theory: if microcausality and energy-positivity in all Lorentz frames are satisfied, then it is unavoidable that all stable particles of the theory be governed by Lorentz-invariant dispersion laws: in all the field sectors, discrete parts outside the continuum as well as the thresholds of the continuous parts of the energy-momentum spectrum, with possible holes inside it, are necessarily represented by mass-shell hyperboloids (or the light-cone). No violation of this geometrical fact can be produced by spontaneous breaking of the Lorentz symmetry.

1- Introduction

In a recent work [1], it has been advocated that the occurrence of spontaneous Lorentz and CPT violations in Quantum Field Theories governed by suitable non-local Lagrangians can very well generate non-Lorentz-invariant dispersion laws\textsuperscript{(1)} which avoid the problems with stability and causality. Such Lorentz violation effects produced at Planck scale might then in principle be observed at lower energies in particle physics. In support of their claim, the authors of [1] have produced examples of possible “non-local models” in which the quadratic part of the Lagrangian is supposed to yield a dispersion law $p_0 = \omega(\vec{p})$ enjoying the following properties:

a) The hypersurface $\mathcal{M}$ with equation $p_0 = \omega(\vec{p})$ differs from a Lorentz-invariant mass shell hyperboloid,

b) $\mathcal{M}$ is contained in the positive energy-momentum cone $\overline{\mathcal{C}}_+(p_0 \geq |\vec{p}|)$,

c) For every momentum $\vec{p}$, the “group velocity condition” $|\partial \omega(\vec{p})| \leq 1$ holds, which means that $\mathcal{M}$ admits a space-like (or light-like) tangent hyperplane at each of its points.

While condition b) expresses energy-positivity in all Lorentz frames, condition c) ensures that all wave-packets satisfying the dispersion law $p_0 = \omega(\vec{p})$ propagate “essentially”

\textsuperscript{(1)} We prefer keeping here the terminology of “dispersion law” (used traditionally e.g. in Thermal Quantum Field Theory) rather than adopting the new usage of “dispersion relation”, which is of course confusing in a domain where (Cauchy-type) dispersion relations relating the absorptive and dispersive parts of Feynman-type amplitudes remain a basic tool of frequent use.
with a *subluminal (or luminal)* velocity; essentially means “up to the quantum spreading of wave-packets, of the order of the Planck constant”, as it is the case for the solutions of the Klein-Gordon and Dirac equations.

However, we wish to stress that the latter condition c) should *by no means* be taken as a criterion of *microcausality* for the underlying Quantum Field Theory. Microcausality states that the commutator (resp. anticommutator) $[\Phi(x), \Phi(x')]_\mp$ of a boson (resp. fermion) field $\Phi(x)$ should vanish in the whole region of *relativistic spacelike separation* $\{(x, x'); (x-x')^2 < 0\}$. As we shall see below, the “group velocity condition” c) only appears as a necessary consequence of microcausality, but the converse is not true. This is why, in the various examples presented in [1], checking the validity of condition c) does *not* constitute a check of the validity of microcausality. On the contrary, the requirement of microcausality represents such a strong constraint that, when combined with energy positivity in all frames, it definitely implies the following properties:

i) any dispersion law describing particles generated by the field is Lorentz invariant, namely the corresponding hypersurface $\mathcal{M}$ is a sheet of hyperboloid with equation of the form $p_0 = \sqrt{\vec{p}^2 + m^2}$ (or the light cone $\partial V^+$ if $m = 0$).

ii) In all the sectors (or collision channels) of the space of states of the (interacting) field theory considered, the hypersurfaces which border the continuous part of the energy-momentum spectrum, including possible holes in the latter, are also Lorentz-invariant, namely sheets of hyperboloid of the form $p_0 = \sqrt{\vec{p}^2 + M_i^2}$ (or the light-cone).

It is the purpose of the present paper to give a hopefully elementary presentation of the latter facts, which have been established long ago in a general, although slightly different, framework by Borchers and Buchholz. As a matter of fact, the interest for the possible occurrence of Lorentz-symmetry breaking is not new and it has already been a subject of deep investigation in the framework of the basic principles of Quantum Field Theory (QFT): the latter two properties of Lorentz-invariance of the energy-momentum spectrum have indeed been proven in a paper by H.J. Borchers and D. Buchholz entitled “The Energy-Momentum Spectrum in Local Field Theories with Broken Lorentz-Symmetry”[2] completed by a paper by H.J. Borchers entitled “Locality and covariance of the spectrum”[3] in the general framework of Algebraic QFT (or “Local Quantum Physics”) [4]. In this deep analysis, generalizing similar results already obtained in [5] (see also [6] for a complete survey of the question), it was proven that the interplay of a weak form of microcausality, namely the commutativity of local observables attached to pairs of mutually space-like regions, together with energy-positivity in all Lorentz frames was sufficient to produce a Lorentz-invariant shape of the energy-momentum spectrum, even if the Lorentz-symmetry was broken in the considered physical representation of the field observables. In view of the always vivid interest of the community for the possible occurrence of some form of Lorentz-symmetry violation emerging from the spontaneous breaking at Planck scale of a “fundamental field or string theory” (see [1] and references therein), but also of its apparent unawareness of the results of [2,3], we think it useful to give a revival to these results in a way which we hope to be accessible to the current field-theorist reader. In fact, we wish to give here a new presentation of these unexpected properties of geometrical nature in energy-momentum space in terms of propagators and Green’s functions of microcausal Bose and Fermi-fields of usual type. We shall thus avoid using the more ab-
bstract formulation of Algebraic QFT, and will focus on the contrary on the phenomenons of complex geometry which play a basic role in this matter.

As in [2,3], the proof of properties i) and ii) which we give below is of general nature, i.e. non-perturbative and even independent of any Lagrangian formulation of the field model. We wish to stress that the somewhat surprising phenomenon of geometrical Lorentz-invariance produced in the present problem has to do with peculiar properties of complex geometry in several complex variables; such properties, which are also closely related to the Jost-Lehmann-Dyson (JLD) formula [7], have been thoroughly exploited in [2,3] precisely in the spirit of [7]. Here we propose to give a completely clear-cut and self-contained account of the previously stated properties, by exploiting the simplest geometrical situation, which is provided by propagators (as commented below, these are in fact the typical objects considered in [1]) and by indicating subsequently how and why the same phenomena still occur for the spectral properties of four-point (and general $n-$point) functions, which provide a complete framework for interacting fields. Under this respect, our presentation is in the spirit of the analyticity properties of Green’s functions in general QFT (see [8,9,17] and references therein) and therefore differs from that of [2,3] which always deals with the properties of expectation values of commutators in general states (with appropriate energy-momentum spectrum) in the JLD-way.

In the models considered in [1], the dispersion laws of particles are always associated with given quadratic parts of field Lagrangians incorporating explicit Lorentz-symmetry breaking coefficients of appropriate type. Such dispersion laws therefore correspond to particles which are “elementary” with respect to the field introduced in the Lagrangian, namely they appear as associated with poles of the propagator of this field in energy-momentum space. Another case of dispersion laws should also be considered, namely those which correspond to “composite” particles of the field: the latter appear as associated with poles of the four-point (or higher n-point) functions of the field in energy-momentum space; for example, this is the case for the hadronic particles if the fundamental fields are those of the standard model.

Here we shall show in detail the previously announced geometrical properties for the poles of propagators (corresponding to the case considered in [1]) and we shall also indicate the derivation of the corresponding equally valid results for the poles of four-point (or n-point) functions. The essential point is that we are only concerned here with stable particles, corresponding to discrete parts of the spectrum, not embedded in the continuum. The case of unstable particles corresponding to possible complex poles of the Green’s functions in unphysical sheets is excluded from our study.

In our section 2, we shall recall the basic analyticity properties of retarded and advanced two-point functions which express microcausality in the complexified energy-momentum space, and the procedure through which information on the energy-momentum spectrum is encoded in this framework. We then formulate three basic results of complex geometry, called Properties A, B and C, whose physical consequences in terms of admissible dispersion laws are derived in a straightforward way: Property A explains why the velocity group condition c) of dispersion laws is implied by microcausality under a weak requirement of energy-positivity. Properties B and C provide a proof of the previous statements of Lorentz-invariance for the dispersion laws of elementary particles and for the
thresholds (and possible holes) of the continuous spectrum, under the joint requirement of microcausality and energy-positivity in all frames. A complete proof of Properties A, B and C is given in this section. In section 3, it is shown that similar consequences of microcausality and (weak or strong) energy-positivity requirements can be formulated in terms of momentum-space analyticity properties of four-point (resp. more generally $2n$-point) Green’s functions established in [8,9] (resp. [17d,e]). The exact counterparts of Properties A,B,C, called respectively $A',B',C'$, are then described and these phenomenons of complex geometry are shown to imply the corresponding constraints for the dispersion laws of composite particles and for the thresholds (and possible holes) of the continuous spectrum in the channel considered. Section 4 gives concluding remarks.

2 Shape of the energy-momentum spectral supports for the two-point functions

Let $F^+(p)$ and $F^-(p)$ (with $p = (p_0, \vec{p})$) be respectively the Fourier transforms of the vacuum expectation values of the retarded and advanced (anti-)commutators of a general (fermion or boson) quantum field $\Phi(x)$, which we write formally

\begin{align}
F^+(p) &= \int e^{ip\cdot x} \theta(x_0) \left< \Phi\left(\frac{x}{2}\right), \Phi\left(-\frac{x}{2}\right) \right>_\pm dx_0 d\vec{x}, \\
F^-(p) &= -\int e^{ip\cdot x} \theta(-x_0) \left< \Phi\left(\frac{x}{2}\right), \Phi\left(-\frac{x}{2}\right) \right>_\pm dx_0 d\vec{x}.
\end{align}

For writing the latter, we have assumed as usual that the space of states in which the field is acting carries a representation of the group of spacetime translations and that the field is invariant under this representation; energy and momentum operators are the corresponding generators of this group. It is of current use to exploit the analyticity properties of $F^+(p)$ and $F^-(p)$ respectively in the upper and lower half-planes of the complexified energy variable $p_0$. However, the postulate of microcausality for the field $\Phi(x)$ implies much more. In fact, it requires that the retarded and advanced propagators occurring under the integrals at the r.h.s. of Eqs (1) and (2) have respectively their supports contained in the closed forward and backward cones $V^+(x_0 \geq |\vec{x}|)$ and $V^-(x_0 \leq -|\vec{x}|)$. It then follows that the integrals (1) and (2) remain convergent and define analytic functions of the complex energy-momentum vector $k = p + iq$, still denoted by $F^+(k)$ and $F^-(k)$, in the respective domains $\mathcal{T}^+$ ($p$ arbitrary, $q \in V^+$) and $\mathcal{T}^-$ ($p$ arbitrary, $q \in V^-$); $V^+ = -V^-$ is the open forward cone: $q_0 > |q|$. $\mathcal{T}^+$ and $\mathcal{T}^-$ are called the “forward and backward tubes”; they contain respectively the upper and lower half-planes in all their one-dimensional sections by (complexified) time-like straight lines, interpreted as energy variables in all possible Lorentz frames. $F^\pm(k)$ are the “Fourier-Laplace transforms” of the retarded and advanced propagators in complex energy-momentum space; their boundary values $F^\pm(p)$ on the reals from (respectively) $\mathcal{T}^\pm$ are the Fourier transforms themselves of these propagators.

\(\text{(2) The distribution character of the integrand at } x = 0 \text{ is treated rigorously by a standard mathematical procedure.}\)
So in the sector generated by “one-field vector-states” of the form $\int \varphi(x)\Phi(x)dx >$ (with $\varphi$ arbitrary in the Schwartz space of smooth and rapidly decreasing functions), microcausality is fully expressed by the analyticity of the pair of functions $(F^+, F^-)$ in the corresponding domains $T^+, T^-$. Now any usable information on the support of the energy-momentum spectrum of the theory in this sector amounts to specifying an open subset $\mathcal{R}$ of the (real) energy-momentum space in which the distributions $^{(3)} <\hat{\Phi}(p)\hat{\Phi}(-p)>$ and $<\hat{\Phi}(-p), \hat{\Phi}(p)>$ vanish simultaneously. In fact, such a support property implies the coincidence relation $F^+_|_{\mathcal{R}} = F^-|_{\mathcal{R}}$, since the expression $^{(3)}$

$$F^+(p) - F^-(p) = <[\hat{\Phi}(p), \hat{\Phi}(-p)]_\pm>$$

vanishes in $\mathcal{R}$. It then follows from a standard theorem of complex analysis, called the “edge-of-the-wedge theorem” (see [10] and references therein), that $F^+(k)$ and $F^-(k)$ then admit a common analytic continuation $F(k)$ which is analytic in the union of $T^+, T^-$ and of a complex neighborhood of the real set $\mathcal{R}$; in other words, $F^+$ and $F^-$ “communicate analytically” through $\mathcal{R}$, as functions of the set of complex variables $k = (k_0, \vec{k})$.

It is one of the basic phenomenons of Analysis and Geometry in several complex variables that arbitrary (connected) subsets of complex space $\mathbb{C}^n$ are not in general “natural” for the class of holomorphic functions: this means that for such a general subset $\Sigma$, all the functions holomorphic in $\Sigma$ admit an analytic continuation in a common larger domain $\hat{\Sigma}$, called the holomorphy envelope of $\Sigma$. This phenomenon, which does not exist in the single-variable case, involves exclusively geometrical properties of the set $\Sigma$ and the extension from $\Sigma$ to $\hat{\Sigma}$ can always be done in principle by an appropriate use of the Cauchy integral formula; this analytic completion procedure presents a strong analogy with the procedure of taking the convex hull $\hat{S}$ of a subset $S$ in the ordinary real space $\mathbb{R}^n$, the notion of a “natural holomorphy domain” in $\mathbb{C}^n$ being a certain generalisation of the notion of “convex domain” in $\mathbb{R}^n$ (see e.g. [11,12] and references therein). As a matter of fact, the most standard and useful result in this connection is the so-called “tube theorem” (see e.g. [12]) which we shall apply below: Any domain $D$ in $\mathbb{C}^n$ which is “tube-shaped”, i.e. of the form $\mathbb{R}^n + iB$ admits a holomorphy envelope which is the tube $\hat{D} = \mathbb{R}^n + i\hat{B}$, where $\hat{B}$ is the convex hull of $B$ in $\mathbb{R}^n$.

It turns out that sets of the form $\Sigma_{\mathcal{R}} = T^+ \cup T^- \cup \mathcal{R}$ are not natural and that, for various choices of $\mathcal{R}$ of physical interest, the corresponding holomorphy envelope $\hat{\Sigma}$ or parts of it can be computed and unexpectedly strong results then follow. Cases when $\mathcal{R}$ itself can be extended to a larger real region $\hat{\mathcal{R}}$ (namely $\hat{\mathcal{R}} = \hat{\Sigma} \cap \mathbb{R}^n \supset \mathcal{R}$) are specially interesting, since they correspond to enlarging the region on which the “spectral function” $<[\hat{\Phi}(p), \hat{\Phi}(-p)]_\pm>$ is proven to vanish, and therefore to refining our information on the support of the distribution $<\hat{\Phi}(p)\hat{\Phi}(-p)>$, called “spectral support”. Properties A and C given below are precisely of this type. Property B is a basic example of holomorphy envelope for a domain $\Sigma_{\mathcal{R}}$ which exactly corresponds to the case when energy-positivity is satisfied in all frames.

\(^{(3)}\) This “bracket notation” in terms of operator products and of (anti-)commutators is used purely for its suggestive content; no (infinite!) energy-momentum conservation $\delta-$function is involved in it.
2.1 Microcausality implies dispersion laws with subluminal velocities

If energy-positivity is required to hold only in privileged frames, such as the laboratory frame and a set of frames which have small velocities with respect to the latter \(^{(4)}\), there exists a maximal region \( \mathcal{R} \) of the form \(-\omega(\vec{p}) < p_0 < \omega(\vec{p}) \) (with \( \omega(\vec{p}) \geq \gamma |\vec{p}| \) for some positive constant \( \gamma \)) in which the (anti-)commutator function \( \langle [\hat{\Phi}(p), \hat{\Phi}(-p)] \rangle_{\pm} \) vanishes. We claim that, due to microcausality, the hypersurface with equation \( p_0 = \omega(\vec{p}) \) is not arbitrary: it has to be a space-like hypersurface. In fact, the geometry of the relativistic light-cone is deeply involved in the implications of microcausality, as it results from the following

Property A (“Double-cone theorem”):
Let \( \mathcal{R}_{a,b} \) be a neighborhood (in real \( p \)-space) of a given time-like segment \( [a, b] \) with endpoints \( a \) and \( b \) \((b \) in the future of \( a)\). Then any function \( F(k) \) holomorphic in \( \Sigma_{\mathcal{R}_{a,b}} \) admits an analytic continuation in a (complex) domain which contains the real region \( \overline{\mathcal{R}}_b^a \), where \( \overline{\mathcal{R}}_b^a \) is the “double-cone” defined as the set of all points \( p \) such that \( p \) is in the future of \( a \) and in the past of \( b \).

Interpretation of Property A:
Let \( \mathcal{M} \) \( (p_0 = \omega(\vec{p})) \) be the hypersurface bordering the vanishing region \( \mathcal{R} \) of the (anti-)commutator function of a certain field theory satisfying microcausality and energy-positivity in privileged frames. Then for each point \( b = (\omega(\vec{p}), \vec{p}) \) in \( \mathcal{M} \), there exists some interval of the form \( \omega(\vec{p}) - \epsilon < p_0 < \omega(\vec{p}) \) and some open neighborhood \( \mathcal{R}_{a,b} \) of the time-like segment \( [a, b] \) defined by this interval (i.e. \( a \equiv (\omega(\vec{p}) - \epsilon, \vec{p}) \)) which lies in \( \mathcal{R} \). It then follows from Property A that the propagator \( F(k) \) of this theory has to be analytic in the full double-cone \( \overline{\mathcal{R}}_b^a \), and therefore that the corresponding (anti-)commutator function must vanish in this double-cone: therefore, \( \overline{\mathcal{R}}_b^a \) belongs to \( \mathcal{R} \), and this argument holds for every point \( b \) of \( \mathcal{M} \), which shows that \( \mathcal{M} \) has to be a spacelike hypersurface.

Similarly, assume that the vanishing region \( \mathcal{R} \) is accompanied by another pair of maximal vanishing regions \( \mathcal{R}_1^2 \) of the form \( \omega(\vec{p}) < |p_0| < |\omega_1(\vec{p})| \) of the (anti-)commutator function. Then \( p_0 = \omega_1(\vec{p}) \) appears as the dispersion law of a particle corresponding to a pole \( \frac{Z(\vec{p})}{k_0 - \omega(\vec{p})} \) of the propagator \( F(k) \). So the previous argument shows in this case that both the hypersurface \( \mathcal{M} \) describing the dispersion law of the particle and the hypersurface \( \mathcal{M}_1 \) \( (p_0 = \omega_1(\vec{p})) \) bordering the region \( \mathcal{R}_1^2 \) have to be spacelike. The argument extends of course to the case of any (ordered) set of dispersion laws corresponding to several particles. Therefore, for every particle appearing with an energy gap in the propagator of the field considered, microcausality alone implies that condition c) (subluminal or luminal velocities) is satisfied by such a particle.

Proof of Property A:
This theorem, which can be seen as a generalisation of a similar property (corollary of the “mean value Asgeirsson theorem”) for the solutions of the wave-equation \([13]\), has been proved by Vladimirov \([14]\) and by Borchers \([15]\). The main geometrical idea is displayed by treating a typical case in two-dimensional energy-momentum space with coordinates \((p_0, p_1)\). We take for \( [a, b] \) the segment \( \delta = [-1, +1] \) of the time axis and for \( \mathcal{R}_{a,b} \) a thin

\(^{(4)}\) This refers to the notion of “concordant frames” introduced in \([1]\)
rectangle $\delta_\epsilon$ of the form: $|p_0| < 1$, $|p_1| < \epsilon$. The tubes $T^+, T^-$ in the complexified space with coordinates $(k_0 = p_0 + iq_0$, $k_1 = p_1 + iq_1)$ are defined respectively by the conditions $q_0 + q_1 > 0$, $q_0 - q_1 > 0$ and $q_0 + q_1 < 0$, $q_0 - q_1 < 0$, and we shall show that the real region obtained by analytic completion of $T^+ \cup T^- \cup \delta_\epsilon$ contains the “double-cone” $\diamondsuit$ (a square in this case!) defined by the inequalities: $|p_0 - p_1| < 1$, $|p_0 + p_1| < 1$. One introduces the family of complex curves $h_\lambda$ with equation $[k_0^2 - (k_1 - 1)^2] = \lambda[k_0^2 - (k_1 + 1)^2]$, where the parameter $\lambda$ varies in a complex neighborhood $\mathcal{V}$ of the real interval $[0, +\infty]$. Except for $h_1$ which is the (complexified) $p_0$-axis, all these curves are hyperbolae, and $\diamondsuit$ is generated by the (real) arcs $\hat{h}_\lambda$ of $h_\lambda$ parametrized by $-1 < p_0 < 1$ (with $|p_1| < 1$) when $\lambda$ varies from 0 to $+\infty$; in a subinterval of the form $|\lambda - 1| < \eta$ (for some $\eta$ determined by $\epsilon$), $\hat{h}_\lambda$ remains inside the rectangle $\delta_\epsilon$ (see fig 1).

One then checks that for any function $F(k_0, k_1)$ holomorphic in $T^+ \cup T^- \cup \delta_\epsilon$ the change of complex variables $(k_0, k_1) \rightarrow (k_0, \lambda)$ is admissible and allows one to define $F(k_0, \lambda) = F(k_0, k_1(k_0, \lambda))$ as an analytic function in the domain where $\lambda$ varies in $\mathcal{V}$ and $k_0$ varies in the unit disk $|k_0| < 1$ deprived from a neighborhood of a real interval of the form $-1 + \alpha(\lambda) \leq p_0 \leq 1 - \alpha(\lambda)$ (fig 2a). This comes from the fact that for $0 < \lambda < +\infty$, the
full upper (resp. lower) half-plane in the variable \( k_0 \) represents a set of points \((k_0, k_1)\) of \( h_\lambda \) in \( T^+ \) (resp. \( T^- \)) (5) and that these two half-planes are connected by small real intervals \([-1, -1 + \alpha[, ]1 - \alpha, 1[\) which represent points in \( \delta_\varepsilon \). Moreover, for \( 1 - \eta < \lambda < 1 + \eta \) the full unit disk \(|k_0| < 1\) is in the analyticity domain of \( F \) (fig 2b) since the corresponding arcs \( \tilde{h}_\lambda \) are all contained in \( \delta_\varepsilon \).

\[ a) \lambda \text{ arbitrary in } V \quad b) 1 - \eta < \lambda < 1 + \eta \]

Fig. 2. Initial analyticity domains of \( F(k_0, \lambda) \) in the \( k_0 \)-plane

Now consider the Cauchy integral

\[
I(k_0, \lambda) = \frac{1}{2i\pi} \oint_{|k'_0| = 1 - \frac{\alpha(\lambda)}{2}} \frac{F(k'_0, \lambda)}{k'_0 - k_0} dk'_0,
\]

which is a holomorphic function of \( k_0 \) and \( \lambda \) for \( k_0 \) varying in the unit disk and \( \lambda \) varying in \( V \); in view of the latter analyticity property of \( F \), one has \( I(k_0, \lambda) = F(k_0, \lambda) \) for \( 1 - \eta < \lambda < 1 + \eta \) and therefore \( I(k_0, \lambda) \) provides an analytic continuation of \( F(k_0, \lambda) \) itself inside the full unit disk \(|k_0| < 1\) and therefore on the real interval \([-1, +1[\) which represents the arc \( \tilde{h}_\lambda \) for all \( \lambda \) in the interval \([-\infty, +\infty[\). By coming back to the original variables \((k_0, k_1)\), this shows that \( F \) admits an analytic continuation in the full region \( \diamond \).

(5) To see this, one can e.g. rewrite the equation of \( h_\lambda \) as follows: \( \frac{U - 1}{U + 1} = \lambda \frac{V - 1}{V + 1} \) with \( U = k_0 + k_1, \ V = k_0 - k_1 \), which entails (for \( \lambda > 0 \)) the condition \( \Im m U \times \Im m V > 0 \), and therefore the fact that all complex points \((k_0, k_1) \equiv (U, V)\) in \( h_\lambda \) belong either to \( T^+ \) or to \( T^- \) according to whether \( \Im m k_0 \equiv \frac{1}{2}(\Im m U + \Im m V) \) is positive or negative.
In the most general version of the theorem in two dimensions, the neighborhood \( R_{a,b} \) of the given time-like segment \([a, b]\) is considered as a union of rectangles of the previous \( \delta_e \)-type, whose thickness \( \epsilon \) tends to zero while they tend to \([a, b] \): the double-cone (or square) \( \diamond_{a,|d=2}^{b} \) is then clearly obtained as the union of the corresponding squares \( \diamond \) obtained in the previous procedure of analytic completion. Finally the proof of the theorem in the \( d \)-dimensional case is obtained by applying the two-dimensional result in all the planar sections passing by \( a \) and \( b \), since i) the two-dimensional sections of the tubes \( T^\pm \) are the corresponding tubes of the (complexified) planar sections, and ii) \( \diamond_{a}^{b} \) is generated by the union of all double-cones of the previous type \( \diamond_{a,|d=2}^{b} \) in these planar sections.

2.2 Microcausality and energy-positivity in all frames imply Lorentz invariant spectral supports

A basic implication of microcausality together with energy-positivity in all frames is the fact that propagators \( F(k) \) of the underlying fields have to be holomorphic in a domain which is invariant under all complex Lorentz transformations, even if these propagators are not Lorentz invariant functions due to the fact that the Lorentz symmetry is broken in the representation of the fields under consideration. The key property which is at the origin of this peculiarity is the following

Property B (“Källen-Lehmann domain”):

Let \( \mathcal{R} = \mathcal{R}_0 \) be the set of all space-like energy-momentum vectors \( p = (p_0, \vec{p}) : |p_0| < |\vec{p}| \). Then any function \( F(k) \) holomorphic in \( \Sigma_{\mathcal{R}_0} = T^+ \cup T^- \cup \mathcal{R}_0 \) admits an analytic continuation in the domain \( \hat{\Sigma}_{\mathcal{R}_0} \) which is the set of all complex vectors \( k = (k_0, \vec{k}) \) such that \( k^2 \equiv k_0^2 - \vec{k}^2 \) is different from any positive number and from zero.

Interpretation of Property B:

Energy-positivity in all Lorentz frames implies that the distribution \( \langle \tilde{\Phi}(p)\tilde{\Phi}(-p) \rangle \) vanishes in the complement of \( \mathbb{V}^+ \) and therefore, in view of (3), that the coincidence relation \( F_{|\mathcal{R}_0}^+ = F_{|\mathcal{R}_0}^- \) is satisfied. Property B then implies the analyticity of the propagator \( F(k) \) in the full “cut-domain” \( \hat{\Sigma}_{\mathcal{R}_0} \). Our denomination of “Källen-Lehmann domain” for the latter is motivated by the fact that in the usual case when Lorentz invariance (or covariance) of the field is postulated, the analyticity domain \( \hat{\Sigma}_{\mathcal{R}_0} \) is directly obtained as a byproduct of the Källen-Lehmann integral representation of the propagator

\[
F(k) \equiv F(k^2) = \frac{1}{2i\pi} \int_0^\infty \frac{\rho(\sigma)}{k^2 - \sigma} d\sigma,
\]

since the image of \( \hat{\Sigma}_{\mathcal{R}_0} \) in the variable \( k^2 \) is the usual cut-plane domain \( \mathbb{C} \setminus \mathbb{R}^+ \). Here, however, this Lorentz-invariant domain (considered in the full complex \( k \)-space) is obtained without any assumption of Lorentz covariance and of boundedness of the functions, but purely on the basis of microcausality and energy-positivity.

Moreover, one will show that any further information on the spectral support which is superimposed to the conditions of Property B implies the Lorentz-invariant shape of all the components of the spectral support together with the invariance under complex Lorentz transformations of the corresponding analyticity domain of the propagator. This
is the purpose of the following property, whose statement in the present form is valid for any spacetime dimension \( d \geq 3 \); we postpone to the proof the corresponding statement for the two-dimensional case, which requires a little more care in view of the decomposition of the light-cone into two straight-lines (the so-called “left and right-movers”).

**Property C (Lorentz-invariance of the borders of the spectral supports); case \( d \geq 3 \):**

If \( \mathcal{R} \) is any real open set, not necessarily connected, containing \( \mathcal{R}_0 \) then every function \( F(k) \) holomorphic in \( \Sigma_{\mathcal{R}} = T^+ \cup T^- \cup \mathcal{R} \) admits an analytic continuation in the (Lorentz-invariant) set \( \hat{\mathcal{R}} \) of all real vectors \( p \) whose Minkowskian norm \( p^2 \) has a value already taken at some vector in \( \mathcal{R} \). Moreover the domain \( \hat{\Sigma}_{\mathcal{R}} \) in which every such function \( F(k) \) can be analytically continued is the set of vectors \( k \) such that \( k^2 \) takes all possible complex values and all real values taken by \( p^2 \) when \( p \) varies in \( \mathcal{R} \).

**Interpretation of Property C:**

It is easy to see that Property C (in its first part) implies that if microcausality and energy-positivity are satisfied, then the most general type of set \( \hat{\mathcal{R}} \) where the (anti-)commutator function (3) has to vanish is a set composed of one distinguished region \( \mathcal{R}_{M_0} \) of the form \(-\infty < p^2 < M_0^2 \), with \( M_0 \geq 0 \) and of zero, one or several disjoint Lorentz-invariant regions of the form \( M_i^2 < p^2 < M_{i+1}^2 \), where \( M'_1 \geq M_0 \) and \( M'_{i+1} \geq M_i \), \( i = 1, \ldots, l-1, M_l \leq \infty \). This implies in turn that the support of \( \langle \hat{\Phi}(p)\hat{\Phi}(-p) \rangle \) is exactly the union of all the “thick (or thin) hyperbolic shells” defined by \( M_i^2 \leq p^2 \leq M_{i+1}^2 \), \( p_0 \geq 0 \), \( (i = 0, 1, \ldots, l-1) \), \( p^2 \geq M_l^2 \) (and of the origin if \( \langle \hat{\Phi}(p) \rangle \neq 0 \)). The equality case \( M_i = M'_{i+1} \) corresponds to some “thin shell” \( p^2 = M_i^2 \). This thin shell situation occurs precisely when the distribution \( \langle \hat{\Phi}(p)\hat{\Phi}(-p) \rangle \) describes a particle with dispersion law \( p_0 = \sqrt{p^2 + M_i^2} \). No possibility is left for a Lorentz-symmetry breaking dispersion law. (Note that in this argument, the positivity of the Hilbert-space norm, implying the fact that the previous distribution is a positive measure factoring out a \( \delta(p^2 - M_i^2) \), has not been used).

The proofs of Properties B and C given below are based on purely geometrical arguments. Both of them rely on a standard analytic completion procedure of geometrical type, namely the “tube theorem” (stated at the beginning of this section); apart from the recourse to this piece of knowledge in complex geometry, these proofs are completely self-contained. The analytic completion procedure is actually at work in the two-dimensional case, which we treat at first, while the general \( d \)-dimensional case will be reducible to the latter.

For the two-dimensional case, Property C must be properly restated as follows:

**Property C (Lorentz-invariance of the borders of the spectral supports); case \( d = 2 \):**

If \( \mathcal{R} \) is any real open set, not necessarily connected, containing \( \mathcal{R}_0 \) then every function \( F(k) \) holomorphic in \( \Sigma_{\mathcal{R}} = T^+ \cup T^- \cup \mathcal{R} \) admits an analytic continuation in the (Lorentz-invariant) set \( \hat{\Sigma}_{\mathcal{R}} \) obtained by adding to \( \hat{\Sigma}_{\mathcal{R}_0} \) the set of all (real or complex) vectors \( k \) obtained by the action of real or complex Lorentz transformations on all vectors in \( \mathcal{R} \).

We note that in the \( d \)-dimensional case, the latter version of Property C is equivalent to the former. In fact, the set of all vectors \( k \) obtained from a given vector \( p = p_0 \neq 0 \) in \( \mathcal{R} \)
by real or complex Lorentz transformations is the full complexified hyperboloid $k^2 = p^2$ if $p^2 \neq 0$ or the full complexified light cone $k^2 = 0$ if $p^2 = 0$. However in the two-dimensional case, the latter statement differs from the former if $\mathcal{R}$ contains vectors $p$ such that $p^2 = 0$. In that case, the set of vectors $k$ obtained from such a vector $p$ by the action of real or complex Lorentz transformations is not the full light cone but only the complexified line of left or right-movers which the given vector $p$ itself belongs to. In other words, one of these two lines may very well be a singular set of the propagator, and therefore contribute to the spectral support, although the other line doesn’t; in such a case the parity symmetry of the spectral support is then broken but its Lorentz invariance is still preserved.

![Diagram](https://via.placeholder.com/150)

**Fig. 3.** The set $B$ (dark gray) and its convex hull $\hat{B}$ (light gray)
Proof of Properties B and C in the two-dimensional case:

We here consider the case when \( k = (k_0, k_1) \) varies in \( \mathbb{C}^2 \), corresponding to two-dimensional field-theory. In the complex variables \( (U = k_0 + k_1, \ V = k_0 - k_1) \), the domains \( T^\pm \) are described as \( T^+ : \exists m \bar{U} > 0, \exists m \bar{V} > 0, \ T^- : \exists m \bar{U} < 0, \exists m \bar{V} < 0, \) and \( \mathcal{R}_0 \) is the real set: \( p^2 = UV < 0 \). Let us then pass to the logarithmic variables \( u = \log \bar{U}, \ v = \log \bar{V} \) and use the fact that any function \( F(k) \equiv F(U,V) = F(e^u, e^v) \equiv f(u,v) \) is holomorphic and \( 2\pi \)-periodic with respect to the variables \( u \) and \( v \) in the image of \( T^+ \cup T^- \cup \mathcal{R}_0 \) in the space of these variables. One easily sees that the domain \( T^+ \) is one-to-one mapped (periodically) onto each one of the following (tube-shaped) domains \( \Theta^+_l = \mathbb{R}^2 + iB^+_l \) (integer) where \( B^+_l \) is the square \( 0 < \exists m u < 2\pi < \pi, 0 < \exists m v < 2\pi \) and similarly for \( T^- \) onto each one of the domains \( \Theta^-_l = \mathbb{R}^2 + iB^-_l \) (integer) where \( B^-_l \) is the square \(-\pi < \exists m u < 2\pi < \pi, \pi < \exists m v < 2\pi < 2\pi \). As seen on fig 3, the set of all squares \( B^+_l \) and \( B^-_l \) form a connected set if one adds to them the common boundary vertices represented by all the points \( b^+_l = (0, m u = l\pi, m v = (-l + 1)\pi) \), with \( l \) integer. But as one easily checks, the sets \( \theta^+_l = \mathbb{R}^2 + ib^+_l \) belong precisely to the (periodic) image of the set \( \mathcal{R}_0 \) \( (UV = e^u + v < 0; \ e^u, e^v \) real). The function \( f(u,v) \) is therefore holomorphic in the union of all the tube-shaped sets \( \Theta^+_l \), \( \Theta^-_l \) and even (view of the invariance of this edge-of-the-wedge configuration by all real translations in \( \mathbb{R}^2 \) in a connected open tube \( \Theta = \mathbb{R}^2 + iB \) such that \( B \) is the union of all sets \( B^+_l, B^-_l \) together with open neighborhoods of all the points \( b^+_l \). Then in view of the tube theorem, \( f(u,v) \) admits a (periodic) analytic continuation in the image of \( \hat{\Theta} = \mathbb{R}^2 + i\hat{B} \), where \( \hat{B} \), namely the convex hull of \( B \), is (as shown by fig 3) the domain \( \hat{B} : 0 < \exists m u + \exists m v < 2\pi \). \( F(k) \) therefore admits an analytic continuation in the inverse image of the tube \( \Theta \) in the original variables, which is the set of all \( k \equiv (U,V) \) such that \( 0 < \arg U + \arg V \equiv \arg k^2 < 2\pi \), namely the domain \( \hat{\Sigma}_{\mathcal{R}_0} \) described in Property B.

The domain \( \hat{\Sigma}_{\mathcal{R}_0} \) can also be seen as the union of all complex hyperbolae \( h_\zeta \) in \( \mathbb{C}^2 \) with equation \( k^2 = UV = \zeta \) such that \( \zeta \) belongs to the cut-plane \( \mathbb{C} \setminus \{0, \infty \} \). Let us now assume that in addition to \( \mathcal{R}_0 \), the set \( \mathcal{R} \) contains a given point \( p = (\bar{U}, \bar{V}) \) with \( \bar{p}^2 = \zeta \geq 0 \). To be specific, consider the case when one has: \( \bar{U} > 0 \) and \( \bar{V} \geq 0 \) and put \( \bar{U} = e^{\bar{u}} > 0, \bar{V} = \zeta e^{\bar{v}} \geq 0, \) with \( \bar{u}, \bar{v} \) real; the remaining cases would be treated similarly by i) exchanging the roles of \( U \) and \( V \) and ii) changing \( (U,V) \) into \((-U,-V) \) in the following. We now use the fact that any function \( F(k) \equiv F(U,V) \) analytic in \( \Sigma_{\mathcal{R}} = T^+ \cup T^- \cup \mathcal{R} \) is analytic in a complex neighborhood of \( p \) and therefore in particular in a set of the form \( \mathcal{N}(p) = \{k = (U,V); U = e^t, \ V = \zeta e^{-t}; (\zeta, t) \in S_1 \} \), where \( S_1 = \{ (\zeta, t); \zeta - \epsilon < \zeta < \zeta + \epsilon, |t - \bar{t}| < \rho \} \). It also follows from Property B that the image \( G_{\mathcal{R}}(U,V) \) in the space of complex variables \( (\zeta, t) \), namely \( G(\zeta, t) \equiv F(e^t, \zeta e^{-t}) \), is analytic in the set \( S_2 = \{ (\zeta, t); |\zeta - \zeta| < \epsilon, \exists m \zeta \neq 0; t \in \mathbb{C} \} \) (with periodicity with respect to the translations \( t \rightarrow t + 2\pi \)). Putting these two facts together, namely the analyticity of \( G(\zeta, t) \) in the union of the sets \( S_1 \) and \( S_2 \), and making the new change of variables

\[
\alpha = \log (\zeta - \zeta + \epsilon \zeta + \epsilon - \zeta), \quad \beta = i\log(t - \bar{t}),
\]

one checks that the function \( g(\alpha, \beta) \equiv G(\zeta + \epsilon e^\alpha \zeta - i, t + e^{-i\beta}) \) is holomorphic in the
following tube-shaped domain $\mathcal{T} = \mathbb{R}^2 + i\mathcal{B}$, where $\mathcal{B}$ is the union of the (disconnected) open set $\{(3m\alpha, 3m\beta); 0 < |3m\alpha| < \frac{\pi}{2}; 3m\beta \text{ arbitrary}\}$ with the “connection interval” $\{(3m\alpha, 3m\beta); 3m\alpha = 0; 3m\beta < \log \rho\}$ (see fig 4). Now since the convex hull of $\mathcal{B}$ is obviously the domain $\hat{\mathcal{B}} = \{(3m\alpha, 3m\beta); \frac{-\pi}{2} < 3m\alpha < \frac{\pi}{2}; 3m\beta \text{ arbitrary}\}$, the tube theorem implies that $g(\alpha, \beta)$ admits an analytic continuation in $\mathbb{R}^2 + i\hat{\mathcal{B}}$, and therefore that $G(\zeta, t)$ admits an analytic continuation in the set $\{((\zeta, t); |\zeta - \tilde{\zeta}| < \epsilon, t \in \mathbb{C}\}$ (with periodicity with respect to the translations $t \rightarrow t + 2i\pi$). Coming back to $F(U, V)$, this shows that $F$ admits an analytic continuation in a set which is the union of all complex curves parametrized by $U = e^t, \ V = \zeta e^{-t}; \ t \in \mathbb{C}$, for $\zeta$ varying in the disk $|\zeta - \zeta| < \epsilon$. These curves are complex hyperbolae except for the one corresponding to the value $\zeta = 0$, which is the straight-line $V = 0$, namely the (complexified) “right-mover” component of the light-cone. All these curves can be seen as generated by the action of all real or complex Lorentz transformations (parametrized by $t$) on the set $\mathcal{N}(p)$ and Property C is therefore established for the two-dimensional case.

![Fig. 4. The set $\mathcal{B}$ (gray)](image)

As a by-product of the latter, we stress the following result which is used below:

**Property B with masses:**

Let $\mathcal{R} = \mathcal{R}_\mu$ be the set of all real energy-momentum vectors $p$ such that $p^2 < \mu^2$. Then any function $F(k)$ holomorphic in $\Sigma_{\mathcal{R}_\mu} = T^+ \cup T^- \cup \mathcal{R}_\mu$ admits an analytic continuation
in the domain $\hat{\Sigma}_{\mathcal{R}_\mu}$, which is the set of all complex vectors $k$ such that $k^2$ belongs to the cut-plane $C \setminus [\mu^2, +\infty[$.

Remark It is sufficient that $\mathcal{R}$ is known to contain (neighborhoods of) one point $p$ on the line $V = 0$ and one point $p'$ on the line $U = 0$ (besides $\mathcal{R}_0$) in order to obtain an analyticity domain $\hat{\Sigma}_{\mathcal{R}}$ of the previous type $\hat{\Sigma}_{\mathcal{R}_\mu}$: in fact, Property C implies that both complex lines $U = 0$ and $V = 0$ are contained in the domain, except maybe for the point $U = V = 0$ which is not obtained by the previous analytic completion procedure. However, this point must also belong to the domain since an analytic function of two complex variables cannot be singular at an isolated point surrounded by its domain of analyticity (see e.g. [11]): it admits an analytic continuation at this isolated point defined by an appropriate Cauchy integral.

Proof of Properties B and C in the $d$-dimensional case:
The general case when $k = (k_0, \vec{k})$ varies in $\mathbb{C}^d$ (e.g. $d = 4$ for field theory in the physical Minkowskian space) will be treated by appropriately using the previous two-dimensional results in sections of $\mathbb{C}^d$ by (complexified) planes containing a time-direction.

Let $\vec{k} = p + iq$ be any vector in $\mathbb{C}^d$ such that $k^2 \in C \setminus [0, +\infty[. In the affine Minkowskian space $\mathbb{R}^d$ consider the point $P$ such that $|\vec{OP}| = p$ and the time-like plane $\Pi$ passing by $P$ and generated by $q$ and the unit vector $e_0$ of the time-axis (or choose one of these planes and call it $\Pi$ in the degenerate case when $q$ is along $e_0$ or is the null vector). Is a unique decomposition $p = p' + p_\perp$ such that $p'$ is parallel to $\Pi$ and $p_\perp$ is orthogonal to $\Pi$ and therefore spacelike, if not the null vector: $p_\perp^2 = -p'^2 \leq 0$. Introducing the complexified space $\Pi^{(c)}$ of $\Pi$ and the two-dimensional vector variable $k' = p' + iq$ such that every point $k = p + iq$ in $\Pi^{(c)}$ can be uniquely written as $k = k' + p_\perp$ with $k'$ orthogonal to $p_\perp$, one has: $k'^2 = k^2 + \rho^2$. In $\Pi^{(c)}$ the section of the domain $\Sigma_{\mathcal{R}_0} = T^+ \cup T^- \cup \mathcal{R}_0$ is represented in the vector-variable $k'$ as the union of the two-dimensional tubes $T'^+$ and $T'^-$ defined by $\exists m k'^2 > 0$ and respectively $\exists m k'_0 > 0, \exists m k'_1 < 0$, and of the real region defined by $\rho^2 = p^2 - p_\perp^2 = p'^2 + \rho^2 < \rho^2$. Therefore since the given vector $k = p' + p_\perp + iq \equiv k' + p_\perp$ is such that $k'^2 = k^2 + \rho^2 \in \mathbb{C} \setminus [\rho^2, +\infty[$, it follows from the two-dimensional Property B with masses, applied in $k'$-space to the restriction $F'(k') = F_{|\Pi^{(c)}}(k)$ of any function $F(k)$ analytic in $\Sigma_{\mathcal{R}_0}$, that $F'$ admits an analytic continuation at $k'$ and therefore that $F$ itself can be analytically continued at the given vector $\vec{k}$. This shows that Property B holds in the $d$-dimensional case.

Proof of Property C: let us assume that in addition to $\mathcal{R}_0$, the set $\mathcal{R}$ contains a given vector $\vec{p} = [OP]$ with $\vec{p}^2 \geq 0$. Considering at first the case $\vec{p}^2 > 0$, we know that the two-sheeted hyperboloid $H(P)$ with equation $p^2 = \vec{p}^2$ can be seen as the union of all the hyperbolae $h_\alpha(P)$ passing by $P$ which are the sections of $H(P)$ by all the two-dimensional planes $\Pi_\alpha$ containing the parallel to the time axis passing by $P$. In the complexified space of each (Minkowskian-type) plane $\Pi_\alpha$, the domain $\Sigma_{\mathcal{R}}$ admits a restriction represented by a two-dimensional domain of the form $\Sigma_{\mathcal{R}_\alpha}$, where $\mathcal{R}_\alpha$ contains $P$ in addition to a region of the form $p_\alpha^2 < \rho_\alpha^2$, corresponding to the intersection of $\mathcal{R}_0$ by $\Pi_\alpha$. Therefore, in view of Property C for the two-dimensional case the whole hyperbola $h_\alpha(P)$ (and even its complexified) belongs to the holomorphy envelope $\hat{\Sigma}_{\mathcal{R}_\alpha}$ of $\Sigma_{\mathcal{R}_\alpha}$. Since this is true for all hyperbolae $h_\alpha(P)$, the full hyperboloid $H(P)$ itself belongs to the holomorphy envelope
\( \Sigma_R \) of \( \Sigma_R \). In the case \( p^2 = 0 \) (with \( P \neq 0 \)), \( H(P) \) is the light-cone and the previous argument of analytic completion in the union of all hyperbolic sections by the planes \( \Pi_\alpha \) yields the whole light-cone deprived from the “light-ray” distinct from \([OP]\) and contained in the (unique) plane \( \Pi_{0_\alpha} \) passing by the origin. However, this exceptional light-ray can be recovered by replacing \( P \) by a neighbouring point \( P' \) also such that \([OP']^2 = 0\): this is always possible since \( R \) is an open set. (We also note that for the same reason one thus obtains in that case an open set \( \tilde{R} \) of the form \( p^2 < \epsilon^2 \), the isolated point \( p = 0 \) being also obtained according to the remark given at the end of the two-dimensional case). We have thus established the first part of Property C, namely the analytic completion at all real vectors \( p \) such that the value \( p^2 \) is taken by some vector \( \overrightarrow{OP} \) with \( P \in R \).

In order to establish the second part, we can now assume that \( R \) is the union of \( R_0 \) together with a set of hypersurfaces \( H_\mu \) of the form \( p^2 = \mu^2 \), with \( \mu \geq 0 \); then there remains to prove that all the points of the corresponding complex hypersurfaces \( H_\mu^{(c)} \) can be reached by the previous analytic completion procedure. Here again, one can proceed as in the proof of Property B, namely taking any given vector \( \overrightarrow{k} = \overrightarrow{p} + iq \) in \( H_\mu^{(c)} \), one considers the complex two-dimensional configuration in the corresponding plane \( \Pi^{(c)} \) (specified above in the proof of Property B). Now the section of \( H_\mu \) by the plane \( \Pi \) is a hyperbola contained in the region \( R_\Pi \) of the corresponding section, so that as a result of Property C in the two-dimensional case, the holomorphy envelope contains all the points of the corresponding complex hyperbola, which includes by construction the given point \( \overrightarrow{k} \). For the case \( \mu = 0 \), the same method still works, including the treatment of the vectors \( \overrightarrow{k} = \overrightarrow{p} + iq \) such that \( p^2 = q^2 = 0 \), which belong to complexified light-rays: the latter are again obtained by the two-dimensional version of Property C in the special case of the right and left movers (no complex light-ray can be excluded since each light-ray has all its real points in the analyticity domain). This ends the proof of Property C in the general case.

3 Shape of the energy-momentum spectral supports for the \( N \)-point functions

We shall now show that the previous study can be repeated for the sector generated by “two-field vector-states” of the form \( \int \varphi(x,x')\Phi(x)\Phi(x') dx dx' \). It is in fact possible to perform a similar treatment in complex momentum space, in which propagators of the fields are now replaced by four-point functions of the latter: the corresponding results on the form of dispersion laws will then apply to composite particles appearing as “two-field bound-states”. Subsequently, we shall indicate the existence of a similar treatment for the sectors of “\( n \)-field vector states” in terms of \( 2n \)-point Green’s functions with applications to dispersion laws of composite particles appearing as “\( n \)-field bound states”, with \( n \geq 3 \). The validity of such a general study relies in an essential way on the general formalism of the analytic Green’s functions of interacting fields in complex momentum space [17].

The basic fact is that there exists an analog of formula (3) for the four-point function, which can be written as follows (see again (3) for our use of the bracket notation):

\[
F^+(p; p_1, p_2) - F^-(p; p_1, p_2) = \langle [\tilde{R}(p_1, p - p_1), \tilde{R}(p_2, -p - p_2)]_\pm >
\]

where \( \tilde{R} \) denotes the Fourier transform of a retarded two-point field operator carrying the
total energy-momentum \( p \):

\[
\tilde{R}(p', p - p') = \int e^{ip\cdot x} e^{ip'\cdot (x' - x)} \theta(x' - x) \left[ \Phi(x'), \Phi(x) \right]_{\pm} \, dx \, dx'
\] (5)

and where \( F^+(p; p_1, p_2) \) and \( F^-(p; p_1, p_2) \) are distributions affiliated with the “generalized retarded four-point functions” (see [8,9]).

Here again, any usable information on the support of the energy-momentum spectrum of the theory in the corresponding two-field sector will amount to specifying an open subset \( \mathcal{R} \) in the space of energy-momentum vectors \( (p, p_1, p_2) \) whose boundary only depend on the total energy-momentum vector \( p \) in which the distributions \( < \tilde{R}(p_1, p - p_1) \tilde{R}(p_2, -p - p_2) > \) and \( < \tilde{R}(p_2, -p - p_2) \tilde{R}(p_1, p - p_1) > \) vanish simultaneously. In view of (4), such a support property (corresponding to the knowledge of the “intermediate states in the latter matrix elements”) then implies the coincidence relation \( F^+_{|\mathcal{R}} = F^-_{|\mathcal{R}} \).

Moreover, as in the case of propagators, the postulate of microcausality for the field \( \Phi(x) \) implies properties of analytic continuation of the previous objects in complex energy-momentum space, which play a crucial role. Even if the description of these properties is more complicated, due to the occurrence of three complex energy-momenta \( k = p + iq, k_1 = p_1 + iq_1, k_2 = p_2 + iq_2 \), the situation reproduces the case of propagators as far as the total energy-momentum \( p \) is concerned. In fact, \( F^+ \) and \( F^- \) are boundary values of holomorphic functions from tubes \( \mathcal{T}^+, \mathcal{T}^- \) whose projections onto the space of complex total energy-momentum \( k = p + iq \) are respectively \( \mathcal{T}^+ : \ q \in V^+ \) and \( \mathcal{T}^- : \ q \in V^- \), so that formula (4) still appears (like (3)) as a discontinuity formula: it indicates that the discontinuity between the two holomorphic functions \( F^+(k; k_1, k_2) \) and \( F^-(k; k_1, k_2) \) is known to vanish on the set \( \mathcal{R} \).

However we must describe more carefully the situation concerning the analyticity properties of these functions in the “internal momenta” \( k_1 \) and \( k_2 \). First, it is clear from formula (5) that in view of the support property of the retarded product \( (x' - x \) contained in \( \mathcal{V}^+ \)), \( \tilde{R} \) is the boundary value of an (operator-valued) analytic function \( \tilde{R}(k', p - k') \) from the tube \( k' = p' + iq' : \ q' \in V^+ \) for all real \( p \). Therefore the r.h.s. of Eq.(4) is the boundary value of a holomorphic function \( \Delta F(p; k_1, k_2) \) of \( (k_1, k_2) \) in the tube \( \Theta \) defined by the conditions \( q_1 \in V^+, \ q_2 \in V^+ \).

Now it is also shown [8,9] that the domains of analyticity of \( F^+, F^- \) implied by microcausality are the tubes \( \mathcal{T}^+ \) and \( \mathcal{T}^- \) defined by the following conditions:

\[
\mathcal{T}^+ : \ q \in V^+, \ q_1 \in V^+, \ q_2 \in V^+
\] (6)

\[
\mathcal{T}^- : \ -q \in V^+, \ q + q_1 \in V^+, \ q + q_2 \in V^+
\] (7)

and one easily checks that these two tubes admit precisely as their common boundary (at \( q = 0 \)) the tube \( \Theta \) for all real \( p \). On the latter, there holds the following discontinuity formula for the boundary values of \( F^+ \) and \( F^- \):

\[
\Delta F(p; k_1, k_2) = F^+(p; k_1, k_2) - F^-(p; k_1, k_2).
\] (8)
The main geometrical difference with respect to the case of propagators is that the tubes $\mathcal{T}^+$ and $\mathcal{T}^-$ in the big complex $(k,k_1,k_2)$-space are not opposite as it is the case for $T^+$ and $T^-$ in $k$-space. As a matter of fact, in view of (6) and (7), the union of the tubes $\mathcal{T}^+$ and $\mathcal{T}^-$ admits a convex hull $\tilde{\mathcal{T}}$ which is contained in the tube defined by the conditions $q_1 \in V^+$, $q_2 \in V^+$, $q + q_1 \in V^+$, $q + q_2 \in V^+$. Now in such a situation, and provided the coincidence relation $F^+_R = F^-_R$ holds true, there exists a generalized version of the edge-of-the-wedge theorem [10], which states that $F^+(k;k_1,k_2)$ and $F^-(k;k_1,k_2)$ still admit a common analytic continuation $F(k;k_1,k_2)$. The latter is analytic in the union of $\mathcal{T}^+$, $\mathcal{T}^-$ and of a complex set $\mathcal{N}(\mathcal{R})$ of the following form: $\mathcal{N}(\mathcal{R})$ is the intersection of a complex neighborhood of $\mathcal{R}$ with the convex hull $\tilde{\mathcal{T}}$ of $\mathcal{T}^+ \cup \mathcal{T}^-$; in other words, $F^+$ and $F^-$ “communicate analytically” through the complex set $\mathcal{N}(\mathcal{R})$ which is bordered by $\mathcal{R}$, although not being analytic anymore in $\mathcal{R}$ itself.

In the present situation, the open set $\mathcal{R}$ is always of the following “cylindric” form: $p_1$ and $p_2$ are arbitrary and $p$ varies in an open set $\mathcal{R}$ (namely the projection of $\mathcal{R}$ onto $p$-space). Then the equivalence of the following two statements (proved in [8,9]) deserves to be stressed:

a) the boundary values of $F^+(k;k_1,k_2)$ and $F^-(k;k_1,k_2)$ coincide on $\mathcal{R}$,

b) $\Delta F(p;k_1,k_2)$ vanishes as an analytic function of $(k_1,k_2)$ in $\Theta$ for all $p$ in $\mathcal{R}$.

Property b) means that the “bridge” in which $F^+$ and $F^-$ have a common analytic continuation contains not only the “small” set $\mathcal{N}(\mathcal{R})$ but the “large common face” defined by the conditions $(k_1,k_2)$ in $\Theta$ for all $p$ in $\mathcal{R}$.

As in Sec. 2, one is then led to make use of an analytic completion procedure in order to enlarge the primitive (“non-natural”) set $\Sigma_{\mathcal{R}} = \mathcal{T}^+ \cup \mathcal{T}^- \cup \mathcal{N}(\mathcal{R})$, in which $F(k;k_1,k_2)$ is known to be analytic. It turns out that one can obtain results very similar to those of Sec 2, which reproduce the corresponding physical interpretations. In fact, the Properties A’, B’ and C’ listed below can be seen as exact counterparts of the respective Properties A, B and C, since they involve identical regions (now called $\mathcal{R}$ and $\tilde{\mathcal{R}}$) in the space of the total energy-momentum $p$, while the additional analyticity properties with respect to the internal energy-momenta $k_1$ and $k_2$ are a remnant of microcausality in these variables.

i) Dispersion laws with subluminal velocities

Under the weak assumption that energy-positivity only holds in privileged Lorentz frames (see Sec 2-1 and (4)), microcausality implies that all the hypersurfaces $\mathcal{M}_i$ and $\mathcal{M}$ representing respectively dispersion laws $p_0 = \omega_i(\vec{p})$ of one-particle states and the border of the continuous energy-momentum spectrum of “intermediate states in the matrix elements” $< \tilde{R}(p_1,p-p_1)\tilde{R}(p_2,-p-p_2) >$ have to be space-like hypersurfaces. This follows from

Property A’:
Let $\mathcal{R}_{a,b}$ be the set of all points $(p,p_1,p_2)$ such that $p$ belongs to a neighborhood of a given time-like segment $[a,b]$ with end-points $p = a$ and $p = b$ (b in the future of a). Then any function $F(k;k_1,k_2)$ holomorphic in $\Sigma_{\mathcal{R}_{a,b}} = \mathcal{T}^+ \cup \mathcal{T}^- \cup \mathcal{N}(\mathcal{R}_{a,b})$ admits an analytic continuation in a (complex) domain which contains the set of all points $(p,k_1,k_2)$ such that $p$ belongs to the double-cone $\phi^+_a$ and $(k_1,k_2)$ varies arbitrarily in the tube $\Theta$.  

The argument of Sec 2-1, based on the consideration of time-like segments \(|a, b|\) with \(b\) contained in \(\mathcal{M}\) or \(\mathcal{M}_1\), then shows again the necessity of the space-like character of these hypersurfaces. In fact, for all such choices of \(|a, b|\), the conclusion of Property A' implies that the discontinuity \(\Delta F(p; k_1, k_2)\) of \(F\) vanishes for all \(p\) in \(\mathcal{O}_a^b\) and \((k_1, k_2)\) in \(\Theta\) and therefore that the distribution \(<|\hat{R}(p_1, p - p_1), \hat{R}(p_2, -p - p_2)|_\pm>\) vanishes for all \(p\) in \(\mathcal{O}_a^b\) and \((p_1, p_2)\) arbitrary.

ii) Lorentz invariance of dispersion laws

Under the (usual) strong assumption that energy-positivity holds in all Lorentz frames, microcausality implies (as in Sec 2-2) that all the hypersurfaces \(\mathcal{M}_i\) and \(\mathcal{M}\) representing respectively dispersion laws \(p_0 = \omega_i(\vec{p})\) of one-particle states and the border of the continuous energy-momentum spectrum of "intermediate states in the matrix elements" \(<\hat{R}(p_1, p - p_1)\hat{R}(p_2, -p - p_2)>\) have to be hyperboloid-shells with equations of the form \(p_0 = \sqrt{\vec{p}^2 + m_i^2}, p_0 = \sqrt{\vec{p}^2 + M^2}\). This follows from the applicability of

Property B' :
Let \(\mathcal{R} = \mathcal{R}_0\) be the set of all (real) configurations \((p, p_1, p_2)\) such that the total energy-momentum vector \(p = (p_0, \vec{p})\) belongs to the following region \(\mathcal{R}_0: |p_0| < |\vec{p}|\). Then any function \(F(k; k_1, k_2)\) holomorphic in \(\Sigma_{\mathcal{R}_0} = \mathcal{T}^+ \cup \mathcal{T}^- \cup \mathcal{R}_0\) admits an analytic continuation in the domain \(\Sigma_{\mathcal{R}_0}\) which is the set of all complex configurations \((k, k_1, k_2)\) belonging to the convex hull \(\hat{\mathcal{T}}\) of \(\mathcal{T}^+ \cup \mathcal{T}^-\) and such that \(k^2 \equiv k_0^2 - \vec{k}^2\) is different from any positive number and from zero, supplemented by

Property C' (Lorentz-invariance of the borders of the spectral supports):
If \(\mathcal{R}\) is any real open set, not necessarily connected, containing \(\mathcal{R}_0\) and of "cylindric form" \(p \in \mathcal{R}\), with \(\mathcal{R} \supset \mathcal{R}_0, p_1, p_2\) arbitrary, then every function \(F(k; k_1, k_2)\) holomorphic in \(\Sigma_{\mathcal{R}_0} = \mathcal{T}^+ \cup \mathcal{T}^- \cup \mathcal{R}\) admits an analytic continuation in the set of all configurations \((p, k_1, k_2)\) such that \((k_1, k_2)\) belongs to the tube \(\Theta\) and \(p\) varies in an open set \(\mathcal{R}\) defined as in Property C: it is (for \(d \geq 3\)) the set of all real vectors \(p\) whose Minkowskian norm \(p^2\) has a value already taken at some vector in \(\mathcal{R}\). Equivalently (but then including the case \(d = 2\)), it is the set of all vectors \(p\) obtained from vectors in \(\mathcal{R}\) by the action of a (real) Lorentz transformation.

The conclusion of Property C' implies that the discontinuity \(\Delta F(p; k_1, k_2)\) of the holomorphic function \(F(k; k_1, k_2)\) vanishes for all \(p\) in \(\mathcal{R}\) and \((k_1, k_2)\) in \(\Theta\) and therefore that the distribution \(<|\hat{R}(p_1, p - p_1), \hat{R}(p_2, -p - p_2)|_\pm>\) vanishes for all \(p\) in \(\mathcal{R}\) and \((p_1, p_2)\) arbitrary. It thus expresses the property of Lorentz invariance of the borders of the energy-momentum spectrum and therefore (according to the same analysis as in Sec 2-2) the results announced above follow.

A derivation of Properties A', B' and C' can be given along the same line as the proofs of Properties A, B and C presented above in Sec. 2. Let us only mention here that Property A' corresponds to a specific case of the double-cone theorem for tubes \(\mathcal{T}^+, \mathcal{T}^-\) in general (i.e. non-opposite) situations (see [16]) and that Property B' is exactly the statement given
in Theorem 1 of [8] for the case of \( n = 3 \) vector variables, with \( m = 0 \).

Remark: In the statements previously given under i) and ii), the constraints which were obtained concern the shape of the energy-momentum spectrum as it appears in the subspace of two-field states generated by retarded products of the following form
\[
R[\varphi] = \int \varphi(x, x') \theta(x_0' - x_0) \left[ \Phi(x'), \Phi(x) \right]_\pm > dxdx'.
\] (6) However, it is clear that the same treatment and results are valid as well for two-field states generated by the corresponding advanced products, and therefore for the subspace generated by all states of the form \( C[\varphi] = \int \varphi(x, x') \left[ \Phi(x'), \Phi(x) \right]_\pm > dxdx' \) (for all admissible test-functions \( \varphi \)).

The general case:

We shall now end this section by explaining why the previous treatment of spectral properties of the space of “two-field states” can be generalized to the spaces of “\( n \)-field states” for all \( n \geq 3 \). Although it is not here the right place for presenting this general treatment with all its technical details, it is still possible to indicate briefly how it works.

The formalism of generalized retarded operators (g.r.o.) [17] allows one to introduce 
\underline{generalized absorptive parts}: these are expectation values of (anti-) commutators of the following form \( \langle [\hat{R}_\alpha(p_i; i \in I), \hat{R}_{\alpha'}(\{p'_j; i' \in I'\})]_\pm \rangle \), where the operators \( \hat{R}_\alpha(p_i; i \in I) \) and \( \hat{R}_{\alpha'}(\{p'_j; i' \in I'\}) \) denote the Fourier transforms of \( n \)-point g.r.o. \( R_\alpha, R_{\alpha'} \) with supports contained in relevant corresponding salient cones \( C_\alpha \) and \( C_{\alpha'} \) in the space of differences \( x_j - x_k \) (resp. \( x'_j - x'_{k'} \)) of space-time vectors: these cones are (non-trivial) analogs of the supports of the usual retarded and advanced operators of the case \( n = 2 \) (i.e. \( x_1 - x_2 \in \hat{V}^\pm \)). In our notation, \( I \) and \( I' \) represent disjoint subsets of \( n \) elements \((|I| = |I'| = n)\) of the set \( \{1, 2, \ldots, 2n\} \) and the corresponding energy-momenta \( p_i, p'_j \) are linked by the energy-momentum conservation law \( p = \sum_{i \in I} p_i = -\sum_{i' \in I'} p'_i \), \( p \) being the total energy-momentum of the corresponding channel \( (I, I') \) of the \( 2n \)-point function of the fields considered; as previously (see (3)), it is understood that the distribution \( \delta((\sum_{i \in I} p_i) + (\sum_{i' \in I'} p'_i)) \) has been factored out in the brackets \( < > \).

We then claim that for each \( n \) and each \((I, I')\) there exists a complete set of g.r.o. \( R_\alpha, R_{\alpha'} \) whose Fourier transforms satisfy a discontinuity formula analogous to (4) of the following form
\[
F_{\alpha,\alpha'}^\pm(\{p_i; i \in I\}; \{p'_j; i' \in I'\}) - F_{\alpha,\alpha'}^-\pm(\{p_i; i \in I\}; \{p'_j; i' \in I'\}) =
\]
\[
\langle [\hat{R}_\alpha(\{p_i; i \in I\}), \hat{R}_{\alpha'}(\{p'_j; i' \in I'\})]\rangle_\pm >;
\]
(9)
in the latter, \( F_{\alpha,\alpha'}^\pm(\{p_i; i \in I\}; \{p'_j; i' \in I'\}) \) are distributions affiliated with the “generalized retarded \( 2n \)-point functions” which are boundary values of analytic functions (still denoted by) \( F_{\alpha,\alpha'}^\pm(\{k_i; i \in I\}; \{k'_j; i' \in I'\}) \) from respective tubes \( \tau_{\alpha,\alpha'}^+, \tau_{\alpha,\alpha'}^- \), in the space of complex vectors \( k_i = p_i + iq_i, k'_j = p'_j + iq'_j \), such that \( k = p + iq = \sum_{i \in I} k_i = -\sum_{i' \in I'} k'_{i'} \).

\[\text{(6)}\] Rigorously speaking, the passage from support properties of the “scalar” distribution
\[
< \hat{R}(p_1, p - p_1)\hat{R}(p_2, p - p_2) >
\]
to corresponding support properties of the vector-valued distribution \( \varphi \rightarrow R[\varphi] \) relies on a Hilbert-space-norm argument.
These pairs of tubes play the same role as the pair \((\mathcal{T}^+, \mathcal{T}^-)\) of the case of two-field states: all points in \(\mathcal{T}^+_{\alpha,\alpha'}\) (resp. \(\mathcal{T}^-_{\alpha,\alpha'}\)) satisfy the condition \(q = \Re mk \in V^+\) (resp. \(V^-\)). Micro-causality is a basic ingredient in the proof of the previous statement, which relies on the results of [17 d), e]).

We are again led to express the energy-momentum spectral assumptions of the theory in the corresponding \(n\)--field sector by specifying an open subset \(\mathcal{R}\) in the space of energy-momentum vectors \(p_i, p_i'\) whose boundary only depend on the total energy-momentum vector \(p = \sum_{i \in I} p_i\), in which the distributions \(< \tilde{R}_{\alpha}(\{p_i; i \in I\}) \tilde{R}_{\alpha'}(\{p_i'; i' \in I'\}) >\) and \(< \tilde{R}_{\alpha}(\{p_i'; i' \in I'\}) \tilde{R}_{\alpha}(\{p_i; i \in I\}) >\) vanish simultaneously. Here again, the edge-of-the-wedge theorem [10] implies that \(F^+_{\alpha,\alpha'}\) and \(F^-_{\alpha,\alpha'}\) have a common analytic continuation \(F_{\alpha,\alpha'}\) in a set of the form \(\Sigma_\mathcal{R} = \mathcal{T}^+_{\alpha,\alpha'} \cup \mathcal{T}^-_{\alpha,\alpha'} \cup \mathcal{N}(\mathcal{R})\).

One could then present the “\(n\)--field-state version” of Properties A', B' and C' in a way which closely parallels the two-field state case. For brevity, we shall not repeat the full statements and the corresponding physical interpretations which are identical to those listed above in paragraphs i) and ii) under the respective “weak” and “strong” forms of the energy-positivity condition. To exhibit the parallelism of the geometry of the \(n\)--field case with the one of the two-field case, it is sufficient to make a little more precise the description of the situation in the sets of energy-momentum vectors \(k_i\) and \(k_i'\) and the characterization of the domains \(\mathcal{T}^+_{\alpha,\alpha'}, \mathcal{T}^-_{\alpha,\alpha'}\), and of their common face in the subspace \(k = p\) real.

For \(p\) real, we introduce the sets of complex vectors \(K_I = \{k_i = k_i - \frac{p_i}{n}; i \in I\}\) and \(K'_I = \{k_i' = k_i' + \frac{p_i}{n}; i' \in I'\}\) linked by the relations \(\sum_{i \in I} k_i = \sum_{i' \in I'} k_i' = 0\); correspondingly \(Q_I = \Re mK_I\) (resp. \(Q'_I = \Re mK'_I\)) is the set of all \(q_i\) (resp. \(q_i'\)) such that \(\sum_{i \in I} q_i = 0\) (resp. \(\sum_{i' \in I'} q_i' = 0\)). Each of the sets of vectors \(K_i, K_i'\) (resp. \(Q_i, Q_i'\)) varies in a space of \((n-1)\) independent complex (resp. real) energy-momentum vectors.

By taking into account analogs of formula (5) for the operators \(\tilde{R}_{\alpha}\) and \(\tilde{R}_{\alpha'}\) together with linear identities between them (called “Steinmann relations” [17]), one can deduce from the support properties of \(R_{\alpha}\) and \(R_{\alpha'}\) (namely \(\text{supp } R_{\alpha} \subset C\alpha\), \(\text{supp } R_{\alpha'} \subset C\alpha'\)) the following analyticity property: the r.h.s. of Eq.(9) is for every real \(p\) the boundary value of an analytic function \(\Delta F_{\alpha,\alpha'}(p; K_I, K'_I)\) of \((K_I, K'_I)\), holomorphic in a well-defined tube \(\Theta_{\alpha,\alpha'}\) (playing the same role as \(\Theta\) in the case \(n = 2\)). This tube is specified by a set of conditions of the following type in the space of the imaginary parts \((Q_I, Q_I')\). There exists a set \(\Pi_\alpha\) of partitions \((J, L)\) of \(I\) and a set \(\Pi_{\alpha'}\) of partitions \((J', L')\) of \(I'\) such that the defining conditions for \(\Theta_{\alpha,\alpha'}\) are: \(q_J = -q_L \in V^+\) and \(q_{J'} = -q_{L'} \in V^+\) for all \((J, L)\) in \(\Pi_\alpha\) and all \((J', L')\) in \(\Pi_{\alpha'}\); in the latter the notation \(q_J\) (resp. \(q_{J'}\)) refers to the corresponding partial sum \(\sum_{i \in J} q_i\) (resp. \(\sum_{i' \in J'} q_i'\)). The sets \(\Pi_\alpha\) and \(\Pi_{\alpha'}\) are not arbitrary but must satisfy the so-called “cell-conditions” (see [17]) which express the fact that no linear subspace with equation \(q_M = 0\) or \(q'_{M'} = 0\), with \(M \subset I\) and \(M' \subset I'\) intersects the domain \(\Theta_{\alpha,\alpha'}\).

Now it can be shown that the tubes \(\mathcal{T}^+_{\alpha,\alpha'}\) and \(\mathcal{T}^-_{\alpha,\alpha'}\) in which the functions \(F^+_{\alpha,\alpha'}\) and \(F^-_{\alpha,\alpha'}\) are holomorphic are defined by the following conditions:
\[
\mathcal{T}^+_{\alpha,\alpha'}: \quad q \in V^+, \quad -q_L \in V^+ \quad \text{and} \quad q_J' \in V^+ \quad \text{(10)}
\]
for all \((J,L)\) in \(\Pi_\alpha\) and all \((J',L')\) in \(\Pi'_{\alpha'}\);

\[
\mathcal{T}^-_{\alpha,\alpha'}: \quad -q \in V^+, \quad qJ = -qL + q \in V^+ \quad \text{and} \quad qJ' = -qL' + q \in V^+
\]

for all \((J,L)\) in \(\Pi_\alpha\) and all \((J',L')\) in \(\Pi'_{\alpha'}\).

These two tubes admit as their common boundary (at \(q = 0\)) the tube \(\Theta_{\alpha,\alpha'}\) for all real \(p\). On the latter, there holds the following discontinuity formula for the boundary values of \(F^+_{\alpha,\alpha'}\) and \(F^-_{\alpha,\alpha'}\):

\[
\Delta F^+_{\alpha,\alpha'}(p; K_I, K'_I) = F^+_{\alpha,\alpha'}([k_i; i \in I]); \{k'_i; i' \in I')|_{q=0} - F^-_{\alpha,\alpha'}([k_i; i \in I]); \{k'_i; i' \in I')|_{q=0}.
\]

One easily checks that the defining conditions \((10), (11)\) of the tubes \(\mathcal{T}^+_{\alpha,\alpha'}\) and \(\mathcal{T}^-_{\alpha,\alpha'}\), are completely analogous to the defining conditions \((6), (7)\) of \(\mathcal{T}^+\) and \(\mathcal{T}^-\), up to the replacement of the two vector variables \(q_1, q_2\) by all the vector variables \(-qL, qJ\), corresponding to the sets of partitions \(\Pi_\alpha, \Pi'_{\alpha'}\).

As a matter of fact, it is known (see [17]) that it is sufficient to consider a subset of g.r.o. called “Steinmann monomials” \(R_\alpha, R_{\alpha'}\) for which each of the corresponding sets \(\Pi_\alpha, \Pi'_{\alpha'}\) contains exactly \(n - 1\) partitions (one also says that the corresponding cell-conditions are “simplicial”); in fact, the most general g.r.o. are linear combinations of these Steinmann monomials. It then turns out that in this restricted class of g.r.o. the analog of Property B’ coincides with Theorem 1 of [8] in its general \(n\)-vector form (with \(m = 0\)): this property states that any function holomorphic in \(\Sigma_{R_0} = \mathcal{T}^+_{\alpha,\alpha'} \cup \mathcal{T}^-_{\alpha,\alpha'} \cup \mathcal{N}(R_0)\) (with \(R_0\) now defined by the conditions \(|p_0| < |\vec{p}|\), \(K_I\) and \(K'_I\) real and arbitrary), admits an analytic continuation at all the points \((k, K_I, K'_I)\) in the convex hull of the tube \(\mathcal{T}^+_{\alpha,\alpha'} \cup \mathcal{T}^-_{\alpha,\alpha'}\) such that \(k^2 \equiv k_0^2 - \vec{k}^2\) is different from any positive number and from zero. Property C’ then follows from B’ as in the case \(n = 2\), while Property A’ corresponds again to the double-cone theorem in a geometrical situation of general type.

These considerations can be completed by a remark similar to the one given at the end of the case \(n = 2\) (including footnote \((^{(6)}\)): since the g.r.o. generate (by linear combinations of Steinmann monomials) all the multiple (anti-)commutators of \(n\) field operators, the constraints on the energy-momentum spectrum apply to the subspace generated by all states of the form \(C[\varphi] = \int \varphi(x_1, \ldots, x_{n-1}, x_n)[\Phi(x_1), \ldots, \Phi(x_{n-1}, \Phi(x_n))] \: dx_1 \ldots dx_n\) (for all admissible test-functions \(\varphi\)).

### 4 Concluding remarks

In this paper, we have displayed the geometrical constraints on the shape of the energy-momentum spectrum which result from microcausality together with (weak or strong) energy-positivity requirements in any (boson or fermion) interacting field theory. These results apply to field theories involving Lorentz symmetry breaking with a rather high degree of generality. This is due to the purely geometrical character of our method, based on analyticity properties in several complex variables, which has allowed a strict exploitation of the latter requirements in terms of Green’s functions of the fields: it is in terms of these objects that the spectral constraints are expressed. As a matter of fact, the Hilbert
space interpretation of these constraints can be done separately as for instance in our Remark in Sec. 3 (see our footnote \(^{(6)}\)). An advantage of the method is therefore the fact that the constraints obtained are still proven to hold in an indefinite-metric framework, as for example in the usual treatment of the QCD-fields with a gauge-fixing preserving the microcausality conditions for the Green’s functions.

Another feature of these geometrical results (linked again to the method) is the fact that they still remain true if the usual temperateness conditions at infinity in energy-momentum space are violated, provided the primitive analyticity domains of the Green’s functions expressing microcausality in that space are still valid: this includes cases when the fields have short-distance singularities which may be wilder than distributions but still allow a generalized form of microcausality to hold; in such cases, the Green’s functions may still enjoy a temperate behaviour at infinity in the Euclidean energy-momentum subspace (i.e. at purely imaginary energies) and therefore admit a corresponding perturbative treatment valid (by analytic continuation from the Euclidean subspace) in the usual analyticity domains considered. \(^{(7)}\)

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\(^{(7)}\) Note that the method also applies to the (opposite) case of Green’s functions enjoying a behaviour at infinite energy-momenta which is very regular at real energies but of exponential increase at purely imaginary energies: this is precisely what happens in the case of the fields generated (via space-time translations) by local observables in the “local quantum physics” framework of [4] which were considered in the original works of Borchers and Buchholz [2,3] on the present subject.
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