Generalized support vector regression: duality and tensor-kernel representation.

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Abstract

In this paper we study the variational problem associated to support vector regression in Banach function spaces. Using the Fenchel-Rockafellar duality theory, we give explicit formulation of the dual problem as well as of the related optimality conditions. Moreover, we provide a new computational framework for solving the problem which relies on a tensor-kernel representation. This analysis overcomes the typical difficulties connected to learning in Banach spaces. We finally present a large class of tensor-kernels to which our theory fully applies: power series tensor kernels. This type of kernels describe Banach spaces of analytic functions and include generalizations of the exponential and polynomial kernels as well as, in the complex case, generalizations of the Szegö and Bergman kernels.

Keywords: support vector regression, regularized empirical risk, reproducing kernel Banach spaces, tensors, Fenchel-Rockafellar duality.

1 Introduction

Support vector regression is a kernel-based estimation technique which allows to estimate a function belonging to an infinite dimensional function space based on a finite number of pointwise observations [7, 21, 23, 24]. The (primal) problem is classically formulated as an empirical risk minimization on a reproducing kernel Hilbert space of functions, the regularization term being the square of the Hilbert norm. This infinite dimensional optimization problem is approached through its dual problem which turns out to be finite dimensional, quadratic (possibly constrained), and involving the kernel function only, evaluated at the available data points [7, 20, 24]. Therefore, the knowledge of the kernel suffices to completely describe and
solve the dual problem as well as to compute the solution of the primal (infinite dimensional) problem. This is what it is known as the kernel trick and makes support vector regression effective and so popular in applications [21].

Learning in Banach spaces of functions is an emerging area of research which in principle permits to consider learning problems with more general types of norms than Hilbert norms [5, 10, 27]. The main motivation for this generalization comes from the need of finding more effective sparse representations of data or for feature selection. To that purpose, several types of alternative regularization schemes have been proposed in the literature, and we mention, among others, \( \ell^1 \) regularization (lasso), elastic net, and bridge regression [8, 11]. Moreover, the statistical consistency of such more general regularization schemes have been addressed in [5, 6, 8, 15]. However, moving to Banach spaces of functions and Banach norms pose serious difficulties from the computational point of view [22]. Indeed, even though, in this more general setting, it is still possible to introduce appropriate reproducing kernels [27], they fail to properly represent the solution of the dual and primal problem, so that the dual approach becomes cumbersome. For this reason, the above mentioned estimation techniques are often implemented by directly tackling the primal problem and therefore, as a matter of fact, reduces to a finite dimensional estimation methods (that is to parametric models).

In this work we address support vector regression in Banach function spaces and we provide a new computational framework for solving the associated optimization problem, overcoming the difficulties we discussed above. Our model is described in the primal by means of an appropriate feature map in Banach spaces of features and a general regularizer. We first study, in great generality, the interplay between the primal and the dual problem through the Fenchel-Rockafellar duality. We obtain an explicit formulation of the dual problem, as well as of the related optimality conditions, in terms of the feature map and the subdifferentials of the loss function and of the regularizer. As a byproduct we also provide a general representer theorem.

Next, we consider the setting of a linear model described through a countable dictionary of functions with the regularization term being a function of the \( \ell^r \)-norm of the related coefficients, with \( r = m/(m - 1) \) and \( m \) an even integer. This choice allows \( r > 1 \) to be close to 1 and hence to approximate \( \ell^1 \) regularization, possibly keeping the stability properties of the \( \ell^2 \) regularization based estimation. Then we introduce a new type of kernel function which turns to be a symmetric positive-definite tensor of order \( m \), and we prove that it allows to formulate the dual problem without any reference to the underlying feature map as well as to evaluate the optimal regression function at any point in the input space. In this way, the dual problem becomes a finite dimensional convex homogeneous \( m \)-degree-polynomial minimization problem which can be solved by standard smooth optimization algorithms, e.g., Newton-type methods. In the end, we show that the kernel trick can be fully extended to tensor-kernels and makes the dual approach in the Banach setting still viable for computing the solution of the primal (infinite dimensional) problem. Finally, we illustrate the theoretical framework above by presenting an entire class of tensor-kernel functions, that is power series tensor-kernels, which are extensions of the analogue matrix-type power series kernels considered in [29]. We show that this class includes kernels of exponential and polynomial type as well as, in the complex case, generalizations of the Szegö and Bergman kernels.
The rest of the paper is organized as follows. Section 2 gives basic definitions and facts. Section 3 presents the dual framework for SVR in general Banach spaces of features. In Section 4 we introduce tensor kernels and explain their role in making Banach space problems more practical numerically. Section 5 treats tensor kernels of power series type, which give rise to a general class of function Banach spaces to which the theory applies. Finally Section 6 contains conclusions.

2 Basic definitions and facts

Let \( F \) be a real Banach space. We denote by \( F^* \) its dual space and by \( \langle \cdot, \cdot \rangle \) the canonical pairing between \( F \) and \( F^* \), meaning that, for every \((w, w^*) \in F \times F^*\), \( \langle w, w^* \rangle = w^*(w) \). We denote by \( \|\cdot\| \) the norm of \( F \) as well as the norm of \( F^* \). Let \( F: \mathcal{F} \to [0, +\infty] \). The domain of \( F \) is \( \text{dom} F = \{w \in \mathcal{F} | F(w) < +\infty\} \) and \( F \) is proper if \( \text{dom} F \neq \emptyset \). Suppose that \( F \) is proper and convex. The subdifferential of \( F \) is the set-valued operator \( \partial F: \mathcal{F} \to 2^{F^*} \) such that

\[
(\forall w \in \mathcal{F}) \quad \partial F(w) = \left\{ w^* \in F^* \mid (\forall v \in \mathcal{F}) \ F(w) + \langle v - w, w^* \rangle \leq F(v) \right\},
\]

and its domain is \( \text{dom} \partial F = \{w \in \mathcal{F} | \partial F(w) \neq \emptyset \} \). The Fenchel conjugate of \( F \) is the function \( F^*: F^* \to [0, +\infty] \): \( w^* \in F^* \mapsto \sup_{w \in \mathcal{F}} \langle w, w^* \rangle - F(w) \). We denote by \( \Gamma_0(\mathcal{F}) \) the set of proper, convex, and lower semicontinuous functions on \( \mathcal{F} \). If \( C \subset \mathcal{F} \), we denote by \( \iota_C \) the indicator function of \( C \), that is \( \iota_C: \mathcal{F} \to [0, +\infty] \), such that, for every \( w \in \mathcal{F} \), \( \iota_C(w) = 0 \) if \( w \in C \), and \( \iota_C(w) = +\infty \) if \( w \notin C \). Let \( F \in \Gamma_0(\mathcal{F}) \). Then the following duality relation between the subdifferentials of \( F \) and its conjugate \( F^* \) holds [26, Theorem 2.4.4(iv)]

\[
(\forall (w, w^*) \in \mathcal{F} \times F^*) \quad w^* \in \partial F(w) \iff w \in \partial F^*(w^*). \tag{2.1}
\]

Let \( r \in [1, +\infty] \). The conjugate exponent of \( r \) is \( r^* \in [1, +\infty] \) such that \( 1/r + 1/r^* = 1 \). If \((\mathcal{Z}, \mathfrak{A}, \mu)\) is a finite measure space, we denote by \( \langle \cdot, \cdot \rangle_{r, r^*} \) the canonical pairing between the Lebesgue spaces \( L^r(\mu) \) and \( L^{r^*}(\mu) \), i.e., \( \langle f, g \rangle_{r, r^*} = \int_{\mathcal{Z}} fg\,d\mu \). If \( \mathbb{K} \) is a countable set, we define the sequence space

\[
\ell^r(\mathbb{K}) = \left\{ (w_k)_{k \in \mathbb{K}} \in \mathbb{R}^\mathbb{K} \mid \sum_{k \in \mathbb{K}} |w_k|^r < +\infty \right\}
\]

endowed with the norm \( \|w\|_r = \left( \sum_{k \in \mathbb{K}} |w_k|^r \right)^{1/r} \). The pairing between \( \ell^r(\mathbb{K}) \) and \( \ell^{r^*}(\mathbb{K}) \) is

\[
\langle w, w^* \rangle_{r, r^*} = \sum_{k \in \mathbb{K}} w_k w_k^*.
\]

The Banach space \( \mathcal{F} \) is called smooth [4] if, for every \( w \in \mathcal{F} \) there exists a unique \( w^* \in \mathcal{F}^* \) such that \( \|w^*\| = 1 \) and \( \langle w, w^* \rangle = 1 \). The smoothness property is equivalent to the Gâteaux differentiability of the norm on \( \mathcal{F} \setminus \{0\} \). We say that \( \mathcal{F} \) is strictly convex if, for every \( w \) and every \( v \) in \( \mathcal{F} \) such that \( \|w\| = \|v\| = 1 \) and \( w \neq v \), one has \( \|(w + v)/2\| < 1 \). Let \( \mathcal{F} \) be a reflexive, strictly convex and smooth real Banach space and let \( r \in [1, +\infty] \). Then the \( r \)-duality map of \( \mathcal{F} \) is the mapping [4]

\[
J_r: \mathcal{F} \to \mathcal{F}^* \quad \text{such that} \quad (\forall w \in \mathcal{F}) \quad \langle w, J_r(w) \rangle = \|w\|^r \quad \text{and} \quad \|J_r(w)\| = \|w\|^{r-1}. \tag{2.2}
\]
This map is a bijection from $F$ onto $F^*$ and its inverse is the $r^*$-duality map of $F^*$. Moreover, for every $w \in F$ and every $\lambda \in \mathbb{R}_+$, $J_r(\lambda w) = \lambda^{-1} J_r(w)$ and $J_r(-w) = -J_r(w)$. The mapping $J_2$ is called the normalized duality map of $F$. The Banach space $\ell^r(\mathbb{K})$ is reflexive, strictly convex, and smooth, and, it is immediate to verify from (2.2) that, its $r$-duality map is

$$J_r: \ell^r(\mathbb{K}) \to \ell^r(\mathbb{K}): w = (w_k)_{k \in \mathbb{K}} \mapsto (|w_k|^{-r} \text{sign}(w_k))_{k \in \mathbb{K}}.$$

(2.3)

Moreover, $J_r^{-1}: \ell^r(\mathbb{K}) \to \ell^r(\mathbb{K})$ is the $r^*$-duality map of $\ell^r(\mathbb{K})$, hence it has the same form as (2.3) with $r$ replaced by $r^*$.

**Fact 2.1** ([1, Example 13.7]). Let $F$ be a reflexive, strictly convex, smooth, and real Banach space, let $r \in ]1, +\infty[$, and let $\varphi \in \Gamma_0(\mathbb{K})$ be even. Then $(\varphi \circ ||\cdot||)^* = \varphi^* \circ ||\cdot||$ and

$$(\forall w \in F) \quad \partial(\varphi \circ ||\cdot||)(w) = \begin{cases} \partial\varphi(||w||) J_r(w) & \text{if } w \neq 0 \\ \{w^* \in F^* \mid ||w^*|| \in \partial\varphi(0)\} & \text{if } w = 0. \end{cases}$$

**Fact 2.2** (Fenchel-Rockafellar duality [26, Corollary 2.8.5 and Theorem 2.8.3(vi)]). Let $F$ and $B$ be two real Banach spaces. Let $F \in \Gamma_0(F)$, let $\Psi \in \Gamma_0(B)$, and let $B: F \to B$ be a bounded linear operator. Suppose that $0 \in \text{int}(B(\text{dom } F) - \text{dom } \Psi)$. Then the dual problem

$$\min_{y^* \in B^*} F^*(-B^*y^*) + \Psi^*(y^*)$$

admits solutions and strong duality holds, that is

$$\inf_{x \in F} F(x) + \Psi(Bx) = -\min_{y^* \in B^*} F^*(-B^*y^*) + \Psi^*(y^*).$$

Moreover, for every $(\bar{x}, \bar{y}^*) \in F \times B^*$, $\bar{x}$ is a minimizer for $F + \Psi \circ B$ and $\bar{y}^*$ is a solution of (2.4) if and only if $-B^*\bar{y}^* \in \partial F(\bar{x})$ and $\bar{y}^* \in \partial \Psi(B \bar{x})$.

**Fact 2.3** (Generalized Cauchy-Schwarz inequality [2, Corollary 2.11.5]). Let $\mathbb{K}$ be a nonempty set. Let $m \in \mathbb{N}$ and let $a_1, a_2, \ldots, a_m \in \ell_1^m(\mathbb{K})$. Then $a_1a_2 \cdots a_m \in \ell_1^m(\mathbb{K})$ and

$$\sum_{k \in \mathbb{K}} a_1[k]a_2[k] \cdots a_m[k] \leq \left(\sum_{k \in \mathbb{K}} a_1[k]^m\right)^{1/m} \left(\sum_{k \in \mathbb{K}} a_2[k]^m\right)^{1/m} \cdots \left(\sum_{k \in \mathbb{K}} a_m[k]^m\right)^{1/m}.$$

### 3 General SVR in Banach spaces of features.

Support vector regression aims at learning a nonlinear relation between an input space $\mathcal{X}$ and an output space $\mathcal{Y} \subset \mathbb{R}$ based on a given set of input-output pairs $(x_i, y_i)_{1 \leq i \leq n} \in (\mathcal{X} \times \mathcal{Y})^n$, called the training set. The input-output relation is sought in a hypothesis function space of the following form

$$\left\{f: \mathcal{X} \to \mathbb{R} \mid (\exists(w, b) \in \mathcal{H} \times \mathbb{R})(\forall x \in \mathcal{X}) f(x) = \langle w, \Phi(x) \rangle + b\right\},$$
where $\Phi: \mathcal{X} \rightarrow \mathcal{H}$ is a nonlinear map (the feature map) from the input space to a Hilbert space (the feature space). Then, support vector regression is formulated as the following minimization problem

$$\min_{(w,b) \in \mathcal{H} \times \mathbb{R}} \frac{\gamma}{n} \sum_{i=1}^{n} L_\varepsilon(y_i - \langle w, \Phi(x_i) \rangle - b) + \frac{1}{2} \|w\|^2,$$  \hspace{1cm} (3.1)

where, $L_\varepsilon(t) = \max\{0, |t| - \varepsilon\}$ is the Vapnik’s $\varepsilon$-insensitive loss [24] and $\gamma > 0$ is the regularization parameter. The optimization problem (3.1) has the drawback that often it has to be solved in an high or even infinite dimensional Hilbert space. In such case, it may be more convenient to approach its dual problem (see [7, Proposition 6.21] and [12, 20, 24])

$$\begin{align*}
\min_{u \in \mathbb{R}^n} & \frac{1}{2n^2} \sum_{i,j=1}^{n} K(x_i, x_j) u_i u_j - \frac{1}{n} \sum_{i=1}^{n} y_i u_i + \frac{\varepsilon}{n} \sum_{i=1}^{n} |u_i| \\
\text{subject to} & \sum_{i=1}^{n} u_i = 0 \quad \text{and} \quad |u_i| \leq \gamma \quad \text{for every} \quad i \in \{1, \ldots, n\},
\end{align*}$$  \hspace{1cm} (3.2)

where $K(x, x') = \langle \Phi(x), \Phi(x') \rangle$ is the kernel function. Once a solution $u$ of the dual problem (3.2) is obtained, a solution $(w, b)$ of the primal problem\(^1\) (3.1) is computed by means of the representer theorem [12, 19]

$$w = \frac{1}{n} \sum_{i=1}^{n} u_i \Phi(x_i),$$  \hspace{1cm} (3.3)

and by choosing $b$ so that $y_j - \langle w, \Phi(x_j) \rangle - b = \text{sign}(u_j) \varepsilon$, for any $j$ with $0 < |u_j| < \gamma$. Moreover, and more importantly, the regression function $f = \langle w, \Phi(\cdot) \rangle + b$ can be evaluated at a new input point $x$ by using the kernel function only

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} u_i K(x_i, x) + b = \frac{1}{n} \sum_{i=1}^{n} u_i (K(x_i, x) - K(x_i, x_j)) + y_j - \text{sign}(u_j) \varepsilon.$$  \hspace{1cm} (3.4)

The significance of this theory stands in the fact that it provides a viable computational framework for solving SVR, which is a nonparametric (infinite dimensional) estimation technique: indeed given the kernel function (without knowing the feature map $\Phi$ itself) one can formulate the dual optimization problem and evaluate the resulting regression function — this is the kernel trick and constitutes the key idea of kernel methods [21]. We stress that, even in the case that $\mathcal{H}$ is finite dimensional, going through the dual is still convenient if $\text{dim}\mathcal{H} \gg n$.

The goal of the present work is to extend the above theory to the case of more general regularizers and more general hypothesis function spaces. Popular estimation techniques that require regularizers different from the square of Hilbert norms, are the lasso and the bridge regression, which can be formulated in a unifying manner as

$$\min_{(w,b) \in \ell^r(\mathcal{H}) \times \mathbb{R}} \frac{\gamma}{n} \sum_{i=1}^{n} \left( y_i - \langle w, \Phi(x_i) \rangle - b \right)^2 + \frac{1}{r} \|w\|^r, \quad (1 \leq r < 2).$$  \hspace{1cm} (3.5)

\(^1\)Since $|\cdot| \leq L_\varepsilon + \varepsilon$, the objective function in (3.1) is coercive in $(w,b)$, hence a solution exists.
These techniques aim at finding the most relevant features $w_k$'s in the representation of the regression function $f = \langle w, \Phi(\cdot) \rangle + b = \sum_{k \in \mathbb{K}} w_k \phi_k + b$, when this representation is known to be sparse. They are grounded on the fact that $\|\cdot\|^r$ preserves sparsity for $r > 1$ but close to 1 [15]. However, even though the dual of the optimization problem (3.5) is in principle finite dimensional, the presence of the non-Hilbertian norm $\|\cdot\|^r$ breaks the quadratic structure of the dual problem and does not allow to define any useful kernel function describing the dual problem as well as the regression function. Indeed the kernels defined in [5, 27, 28], in the setting of reproducing kernel Banach spaces, are not suitable for that purpose (see Remark 3.5).

In the next section we show that the estimation technique (3.5) can be naturally kernelized for certain choices of $r$, provided that one enlarges the concept of kernel functions.

In the following we consider duality for a continuous version of the support vector regression problem (3.1) and for general convex regularizers and loss functions. So we address the optimization problem

$$
\min_{(w, b) \in F \times \mathbb{R}} \gamma \int_{\mathcal{X} \times \mathcal{Y}} L(y - \langle w, \Phi(x) \rangle - b) \, dP(x, y) + G(w),
$$

(3.6)

where the following assumptions are made:

**A1** $\mathcal{X}$ and $\mathcal{Y}$ are two nonempty sets such that $\mathcal{Y} \subset \mathbb{R}$. $P$ is a probability distribution on $\mathcal{X} \times \mathcal{Y}$, defined on some underlying $\sigma$-algebra $\mathfrak{A}$ on $\mathcal{X} \times \mathcal{Y}$. $F$ is a real separable reflexive Banach space and $\Phi: \mathcal{X} \to F^*$ is a measurable function. The function $L: \mathbb{R} \to \mathbb{R}_+$ is positive and convex, $^2$ $p \in [1, +\infty]$, $\gamma \in \mathbb{R}_+$, and $G: F \to ]-\infty, +\infty]$ is proper, lower semicontinuous, and convex.

**A2** $(\exists (a, b) \in \mathbb{R}^2_+) (\forall t \in \mathbb{R}) \quad L(t) \leq a + b|t|^p$.

**A3** $\int_{\mathcal{X} \times \mathcal{Y}} |y|^p \, dP(x, y) < +\infty$ and $\int_{\mathcal{X} \times \mathcal{Y}} \|\Phi(x)\|^p \, dP(x, y) < +\infty$.

In this context $F$ and $\Phi$ are respectively the feature space and the feature map, and $L$ is the loss function [5, 27]. If $P$ is chosen as a discrete distribution, say $P = (1/n) \sum_{i=1}^n \delta_{(x_i, y_i)}$, for some sample $(x_i, y_i)_{1 \leq i \leq n} \in (\mathcal{X} \times \mathcal{Y})^n$, then (3.6) reduces to the optimization problem

$$
\min_{(w, b) \in F \times \mathbb{R}} \frac{1}{n} \sum_{i=1}^n L(y_i - \langle w, \Phi(x_i) \rangle - b) + G(w),
$$

which encompasses problems (3.1) and (3.5). Assumption A2 corresponds to an upper growth condition for the loss $L$, whereas assumption A3 includes a moment condition for the distribution $P$ and an integrable condition for the feature map $\Phi$, with respect to $P$. They are both standard assumptions in support vector machines [21] and ensure that the integral in (3.6) is

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$^2$ Usually one requires that $L$ is also even. In that case it is easy to see that necessarily $0$ is a minimizer of $L$ and that $L$ is increasing on $\mathbb{R}_+$. Indeed for every $t \in \mathbb{R}_+$, we have $-t \leq 0 \leq t$, and hence $0 = (1-\alpha)(-t) + \alpha t$, for some $\alpha \in [0, 1]$. Then, by convexity $L(0) \leq (1-\alpha)L(-t) + \alpha L(t) = L(t)$, for $L(-t) = L(t)$. Moreover, for every $s, t \in \mathbb{R}$, with $0 \leq s \leq t$, we have $s = (1-\alpha)s + \alpha t$, for some $\alpha \in [0, 1]$, and hence $L(s) \leq (1-\alpha)L(0) + \alpha L(t)$ which yields $L(s) - L(0) \leq \alpha(L(t) - L(0)) \leq L(t) - L(0)$.
finite for every \((w, b) \in \mathcal{F} \times \mathbb{R}\). In the following we consider the Lebesgue space

\[
L^p(P) = \left\{ u : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} \mid u \text{ is } \mathfrak{A}\text{-measurable and } \int_{\mathcal{X} \times \mathcal{Y}} |u(x, y)|^p dP(x, y) < +\infty \right\}.
\]

Problem (3.6) is a convex optimization problem of a composite form. The following result first recasts the problem in a constrained form, as done in [7, 23], then presents its dual problem, with respect to the Fenchel-Rockafellar duality, and the related optimality conditions (Fact 2.2).

**Theorem 3.1.** Let assumptions A1, A2, and A3 hold. Then problem (3.6) is equivalent to

\[
\begin{aligned}
\min_{(w, b, e) \in \mathcal{F} \times \mathbb{R} \times L^p(P)} & \gamma \int_{\mathcal{X} \times \mathcal{Y}} L(e(x, y)) dP(x, y) + G(w), \\
\text{subject to } & y - \langle w, \Phi(x) \rangle - b = e(x, y) \text{ for } P\text{-a.a. } (x, y) \in \mathcal{X} \times \mathcal{Y}
\end{aligned}
\]

and its dual is

\[
\begin{aligned}
\min_{u \in L^p(P)} & G^\ast \left( \int_{\mathcal{X} \times \mathcal{Y}} u(x, y) \Phi(x) dP(x, y) \right) \\
& + \gamma \int_{\mathcal{X} \times \mathcal{Y}} L^\ast \left( \frac{u(x, y)}{\gamma} \right) dP(x, y) - \int_{\mathcal{X} \times \mathcal{Y}} y u(x, y) dP(x, y)
\end{aligned}
\]

\[
\text{subject to } \int_{\mathcal{X} \times \mathcal{Y}} u dP = 0.
\]

Moreover, the dual problem \((\mathcal{D})\) admits solutions, strong duality holds, and for every \((w, b, e) \in \mathcal{F} \times \mathbb{R} \times L^p(P)\) and every \(u \in L^p(P)\), we have that \((w, b, e)\) is a solution of \((\mathcal{P})\) and \(u\) is a solution of \((\mathcal{D})\) if and only if the following optimality conditions hold

\[
\begin{aligned}
w & \in \partial G^\ast \left( \int_{\mathcal{X} \times \mathcal{Y}} u(x, y) \Phi(x) dP(x, y) \right) \\
& \int_{\mathcal{X} \times \mathcal{Y}} u dP = 0 \\
\frac{u(x, y)}{\gamma} & \in \partial L(e(x, y)) \text{ for } P\text{-a.a. } (x, y) \in \mathcal{X} \times \mathcal{Y} \\
y - \langle w, \Phi(x) \rangle - b & = e(x, y) \text{ for } P\text{-a.a. } (x, y) \in \mathcal{X} \times \mathcal{Y}
\end{aligned}
\]

**Remark 3.2.**

(i) If \(L\) and \(G\) are coercive, that is \(\lim_{|t| \to +\infty} L(t) = +\infty\) and \(\lim_{\|w\| \to +\infty} G(w) = +\infty\) (e.g., this is the case of (3.1)), then the primal problem \((\mathcal{P})\) admits solutions. Moreover, if in addition \(L\) and \(G\) are strictly convex (as for (3.5)), the solution is unique.

(ii) If the offset \(b = 0\), condition \(\int_{\mathcal{X} \times \mathcal{Y}} u dP = 0\) in (3.7) can be omitted. Moreover, the coercivity (resp. the strictly convexity) of \(G\) only suffices to get the existence (resp. uniqueness) of solutions of problem \((\mathcal{P})\).
Remark 3.3.

(i) The form \((\mathcal{P})\) resembles the way the problem of support vector machines for regression is often formulated \([23, \text{eq. (3.51)}]\) and the optimality conditions \((3.7)\) are the continuous versions of the one stated in \([23, \text{eq. (3.52)}]\) for RKHS, differentiable loss functions, and square norm regularizers.

(ii) Let \(u\) be a solution of the dual problem \((\mathcal{D})\). Then the (possibly empty) set of solutions of the primal problem \((\mathcal{P})\) is composed by the \((w, b, e)\)'s satisfying the optimality conditions \((3.7)\). Equivalently, recalling \((2.1)\), the solutions of the primal problem, in the form \((3.6)\), are given by

\[
\begin{align*}
    w & \in \partial G^* \left( \int_{\mathcal{X} \times \mathcal{Y}} u(x, y) \Phi(x) \, dP(x, y) \right) \\
    b & \in y - \langle w, \Phi(x) \rangle - \partial L^* \left( \frac{u(x, y)}{\gamma} \right) \quad \text{for } P\text{-a.a. } (x, y) \in \mathcal{X} \times \mathcal{Y}. 
\end{align*}
\]  \((3.8)\)

(iii) If \(G\) is strictly convex on every convex subset of \(\text{dom} \, \partial G\) and \(\text{int}(\text{dom} \, G^*) = \text{dom} \, \partial G^*\) \([1, \text{Proposition 18.9}]\) and, if \((w, b)\) is a solution of the primal problem \((3.6)\), then the first of \((3.8)\) yields

\[
w = \nabla G^* \left( \int_{\mathcal{X} \times \mathcal{Y}} u \Phi \, dP \right). \tag{3.9}\]

This constitutes a general \emph{nonlinear representer theorem}, since the solution of problem \((\mathcal{P})\) is expressed in terms of the values of the feature map \(\Phi\). When \(P\) is the discrete distribution \(P = (1/n) \sum_{i=1}^n \delta_{(x_i, y_i)}\), for some sample \((x_i, y_i)_{1 \leq i \leq n} \in (\mathcal{X} \times \mathcal{Y})^n\), then \((3.9)\) becomes

\[
w = \nabla G^* \left( \frac{1}{n} \sum_{i=1}^n u_i \Phi(x_i) \right). \tag{3.10}\]

(iv) In the special case that \(\mathcal{F}\) is a Hilbert space, \(\mathcal{F}\) is isomorphic to its dual and the pairing reduces to the inner product in \(\mathcal{F}\). If \(b = 0\) and \(G = (1/2)\|\cdot\|_2^2\), then \(\nabla G^* = \text{Id}\) and the optimality conditions \((3.7)\) reduce to the equations

\[
w = \int_{\mathcal{X} \times \mathcal{Y}} u \Phi \, dP \quad \text{and} \quad \frac{u(x, y)}{\gamma} \in \partial L(y - \langle w, \Phi(x) \rangle) \quad \text{for } P\text{-a.a. } (x, y) \in \mathcal{X} \times \mathcal{Y},
\]

which were obtained in \([9, \text{Corollary 3}]\). If additionally, \(P\) is discrete, then \((3.10)\) turns to \((3.3)\).

(v) If \(\mathcal{F} = \ell^2(\mathbb{K})\) and \(G = \|\cdot\|_1 + 1/(2\tau)\|\cdot\|_2^2\) (elastic net regularization), then \(G^*\) is strongly convex, \(\nabla G^* = \text{prox}_{\tau\|\cdot\|_1}(\tau \cdot)\) \([1]\), and \((3.9)\) and \((3.10)\) turn respectively to the following representation formulas

\[
w = \text{prox}_{\tau\|\cdot\|_1} \left( \tau \int_{\mathcal{X} \times \mathcal{Y}} u \Phi \, dP \right) \quad \text{and} \quad w = \text{prox}_{\tau\|\cdot\|_1} \left( \frac{\tau}{n} \sum_{i=1}^n u_i \Phi(x_i) \right),
\]

where \(\text{prox}_{\tau\|\cdot\|_1}\) acts component-wise as a soft-thresholding operation with threshold \(\tau\).
The optimality conditions (3.7) in Theorem 3.1 yield the following continuous representer theorem in Banach space setting and for regularizers that are function of the norm.

**Corollary 3.4** (Continuous representer theorem). Let assumptions A1, A2, and A3 hold. Suppose that \( \mathcal{F} \) is strictly convex and smooth and let \( r \in [1, +\infty[ \). In problem \((P)\), suppose that \( G = \varphi \circ \| \cdot \| \), for some convex and even function \( \varphi : \mathbb{R} \to \mathbb{R}_+ \) such that \( \text{argmin} \varphi = \{ 0 \} \). Let \((w, b)\) be a solution of problem \((P)\). Then \( w \) admits the following representation

\[
J_r(w) = \int_{X \times Y} c(x, y) \Phi(x) \, dP(x, y),
\]

for some function \( c \in L^p(P) \), where \( J_r : \mathcal{F} \to \mathcal{F}^* \) is the \( r \)-duality map of \( \mathcal{F} \).

**Remark 3.5.** If in Corollary 3.4, \( r = 2 \) and \( P \) is a discrete measure, say \( P = (1/n) \sum_{i=1}^n \delta(x_i, y_i) \), for some sample \((x_i, y_i)_{1 \leq i \leq n} \in (\mathcal{X} \times \mathcal{Y})^n \), then (3.11) becomes

\[
J_2(w) = \sum_{i=1}^n c_i \Phi(x_i), \quad (c_i)_{1 \leq i \leq n} \in \mathbb{R}^n,
\]

where \( J_2 \) is the normalized duality map of \( \mathcal{F} \). Formula (3.12) is the way the representer theorem is usually presented in reproducing kernel Banach spaces [10, 27, 28]. Here it is a simple consequence of the more general Theorem 3.1 and Corollary 3.4. Moreover, we stress that our derivation of (3.12) relies on convex analysis arguments only, while in the above cited literature it is proved as a consequence of a representer theorem for function interpolation, ultimately using different techniques and stronger hypotheses. We note also that in Banach space setting [27, 28], the kernel is defined as

\[
K(x, x') = \langle J_2^{-1}(\Phi(x)), \Phi(x') \rangle,
\]

but this kernel function is inadequate for describing the dual problem and evaluating the regression function \( \langle w, \Phi(x) \rangle + b \) at a new point \( x \) [22].

**Example 3.6.** We consider the case of the \( \varepsilon \)-insensitive loss [20, 24]. Let \( \varepsilon > 0 \) and define

\[
L_\varepsilon : \mathbb{R} \to \mathbb{R}_+ : t \mapsto \max\{0, |t| - \varepsilon\}.
\]

This loss clearly satisfies A2 for every \( p \geq 1 \). We note that (3.13) is the distance function from the set \([-\varepsilon, \varepsilon]\), that is, using the notation in [13], we have \( L_\varepsilon = d_{[-\varepsilon, \varepsilon]} \). Then, the Fenchel conjugate of \( L_\varepsilon \) is (see [13, Example 13.24(i)])

\[
L_\varepsilon^* = \sigma_{[-\varepsilon, \varepsilon]} + \iota_{[-1, 1]} = \varepsilon |\cdot| + \iota_{[-1, 1]}.
\]

Therefore, for the loss (3.13), the dual problem \((D)\) becomes

\[
\begin{aligned}
\min_{u \in L^p(P)} & \quad G^* \left( \int_{X \times Y} u(x, y) \Phi(x) \, dP(x, y) \right) \\
& \quad + \varepsilon \int_{X \times Y} |u(x, y)| \, dP(x, y) - \int_{X \times Y} y u(x, y) \, dP(x, y) \\
\text{subject to} & \quad \int_{X \times Y} u \, dP = 0 \quad \text{and} \quad |u(x, y)| \leq \gamma \quad \text{for} \quad P\text{-a.a.} \, (x, y) \in \mathcal{X} \times \mathcal{Y}.
\end{aligned}
\]
This is a continuous version of the dual problem (3.2), where here we have a general regularizer and a Banach feature space.

**Remark 3.7.** Let us consider the case that $\mathcal{F}$ is a Hilbert space. Moreover, suppose that $G = (1/2)\|\cdot\|^2$, that $L = (1/2)\|\cdot\|^2$, and that $b = 0$, so that in (3.7) the condition $\int_{\mathcal{X} \times \mathcal{Y}} u \, dP = 0$ can be neglected. Then it follows from (3.7) that

$$w = \int_{\mathcal{X} \times \mathcal{Y}} u \Phi \, dP, \quad u = e$$

and hence

$$\langle w, \Phi(x) \rangle = \int_{\mathcal{X} \times \mathcal{Y}} u(x', y') \langle \Phi(x'), \Phi(x) \rangle \, dP(x', y').$$

Thus, the last of (3.7) yields the following integral equation

$$(\forall (x, y) \in \mathcal{X} \times \mathcal{Y}) \quad \frac{u(x, y)}{\gamma} + \int_{\mathcal{X} \times \mathcal{Y}} u(x', y') \langle \Phi(x'), \Phi(x) \rangle \, dP(x', y') = y.$$

### 4 Tensor-kernel representation

In this section we study a framework that includes SVR problems of type (3.5) (for certain choices of $r$) and that provides a new tensorial kernelization of the dual problem (3). For clarity we consider separately the real and complex case. We describe the real case with full details, whereas in the complex case we provide results with sketched proofs only.

#### 4.1 The real case

Let $\mathcal{F} = \ell^r(\mathbb{K})$, with $\mathbb{K}$ a countable set and $r = m/(m-1)$ for some even integer $m \geq 2$. Thus, we have $r^* = m$. Let $(\phi_k)_{k \in \mathbb{K}}$ be a family of measurable functions from $\mathcal{X}$ to $\mathbb{R}$ such that, for every $x \in \mathcal{X}$, $(\phi_k(x))_{k \in \mathbb{K}} \in \ell^r(\mathbb{K})$ and define the feature map as

$$\Phi: \mathcal{X} \to \ell^r(\mathbb{K}): x \mapsto (\phi_k(x))_{k \in \mathbb{K}}.$$  \hspace{1cm} (4.1)

Thus, we consider the following linear model

$$(\forall (w, b) \in \ell^r(\mathbb{K}) \times \mathbb{R}) \quad f_{w, b} = \langle w, \Phi(\cdot) \rangle_{r, r^*} + b = \sum_{k \in \mathbb{K}} w_k \phi_k + b \text{ (pointwise)},$$

where $\langle \cdot, \cdot \rangle_{r, r^*}$ is the canonical pairing between $\ell^r(\mathbb{K})$ and $\ell^{r^*}(\mathbb{K})$. The space

$$\mathcal{B} = \left\{ f: \mathcal{X} \to \mathbb{R} \mid (\exists (w, b) \in \ell^r(\mathbb{K}) \times \mathbb{R})(\forall x \in \mathcal{X}) \left( f(x) = \sum_{k \in \mathbb{K}} w_k \phi_k(x) + b \right) \right\}$$

is a reproducing kernel Banach space with norm

$$(\forall f \in \mathcal{B}) \quad \|f\|_\mathcal{B} = \inf \left\{ \|w\|_r + |b| \mid (w, b) \in \ell^r(\mathbb{K}) \times \mathbb{R} \text{ and } f = \sum_{k \in \mathbb{K}} w_k \phi_k + b \text{ (pointwise)} \right\}.$$
meaning that, with respect to that norm, the point-evaluation operators are continuous [5, 27]. We also consider the following regularization function

\[ G(w) = \varphi(\|w\|_r), \quad (4.4) \]

for some convex and even function \( \varphi \colon \mathbb{R} \to \mathbb{R}_+ \), such that \( \text{argmin} \varphi = \{0\} \), and we set

\[ P = \frac{1}{n} \sum_{i=1}^{n} \delta_{(x_i, y_i)}, \quad (4.5) \]

for some given sample \((x_i, y_i)_{1 \leq i \leq n} \in (\mathcal{X} \times \mathcal{Y})^n\).

**Remark 4.1.** Consider the reproducing kernel Banach space

\[ \mathcal{B} = \left\{ f : \mathcal{X} \to \mathbb{R} \mid (\exists w \in \ell^r(\mathbb{K})) (\forall x \in \mathcal{X}) (f(x) = \sum_{k \in \mathbb{K}} w_k \phi_k(x)) \right\} \]

endowed with norm \( \|f\|_\mathcal{B} = \inf \{ \|w\|_r \mid w \in \ell^r(\mathbb{K}) \text{ and } f = \sum_{k \in \mathbb{K}} w_k \phi_k \text{ (pointwise)} \} \) Let \( f \in \mathcal{B} \) and let \((w_k)_{k \in \mathbb{K}} \in \ell^r(\mathbb{K})\) be such that \( f = \sum_{k \in \mathbb{K}} w_k \phi_k \text{ pointwise} \). Then, for every finite subset \( \mathbb{J} \subset \mathbb{K} \) we have \( f - \sum_{k \in \mathbb{J}} w_k \phi_k = \sum_{k \in \mathbb{K} \setminus \mathbb{J}} w_k \phi_k \text{ pointwise} \); hence, by definition

\[ \|f - \sum_{k \in \mathbb{J}} w_k \phi_k\|_\mathcal{B} \leq \| (w_k)_{k \in \mathbb{K} \setminus \mathbb{J}} \|_{\ell^r} = \left( \sum_{k \in \mathbb{K} \setminus \mathbb{J}} |w_k|^r \right)^{1/r} \to 0 \quad \text{as } |\mathbb{J}| \to +\infty. \]

Thus, the family \((w_k \phi_k)_{k \in \mathbb{K}}\) is summable in \((\mathcal{B}, \|\cdot\|_\mathcal{B})\) and it holds \( f = \sum_{k \in \mathbb{K}} w_k \phi_k \) in \((\mathcal{B}, \|\cdot\|_\mathcal{B})\). Therefore, if the family of functions \((\phi_k)_{k \in \mathbb{K}}\) is pointwise \( \ell^r \)-independent, in the sense that

\[ (\forall (w_k)_{k \in \mathbb{K}} \in \ell^r(\mathbb{K})) \sum_{k \in \mathbb{K}} w_k \phi_k = 0 \text{ (pointwise)} \Rightarrow (w_k)_{k \in \mathbb{K}} \equiv 0, \quad (4.6) \]

then \((\phi_k)_{k \in \mathbb{K}}\) is an unconditional Schauder basis of \(\mathcal{B}\). Indeed if \( \sum_{k \in \mathbb{K}} w_k \phi_k = 0 \) in \((\mathcal{B}, \|\cdot\|_\mathcal{B})\), since the evaluation operators on \(\mathcal{B}\) are continuous, we have \( \sum_{k \in \mathbb{K}} w_k \phi_k = 0 \) pointwise, and hence, by (4.6), \((w_k)_{k \in \mathbb{K}} \equiv 0\). We finally note that when \((\phi_k)_{k \in \mathbb{K}}\) is a (unconditional) Schauder basis of \(\mathcal{B}\), then \(\mathcal{B}\) is isometrically isomorphic to \(\ell^r(\mathbb{K})\).

In the setting (4.1)–(4.5), the primal and dual problems considered in Theorem 3.1 turn into

\[
\begin{align*}
\left[ \begin{array}{c}
\min_{(w, b, e) \in \ell^r(\mathbb{K}) \times \mathbb{R} \times \mathbb{R}^m} & \frac{\gamma}{n} \sum_{i=1}^{n} L(e_i) + \varphi(\|w\|_r), \\
\text{subject to} & y_i - \langle w, \Phi(x_i) \rangle_{r^*} - b = e_i, \quad \text{for every } i \in \{1, \ldots, n\}
\end{array} \right] \quad (\mathcal{P}_n)
\end{align*}
\]

and, since \( G^* = \varphi^* \circ \|\cdot\|_{r^*} \) (Fact 2.1),

\[
\begin{align*}
\left[ \begin{array}{c}
\min_{u \in \mathbb{R}^m} & \frac{\gamma}{n} \sum_{i=1}^{n} u_i \Phi(x_i) \left\| r^* \right\| + \frac{1}{n} \sum_{i=1}^{n} L^* \left( \frac{u_i}{\gamma} \right) - \frac{1}{n} \sum_{i=1}^{n} y_i u_i \\
\text{subject to} & \sum_{i=1}^{n} u_i = 0.
\end{array} \right] \quad (\mathcal{D}_n)
\end{align*}
\]
Moreover, Fact 2.1 and (3.7) yield that \((w, b, e)\) solves \((P_n)\) and \(u\) solves \((D_n)\) if and only if

\[
\begin{align*}
    w &\in \partial J^* \left( \frac{1}{n} \left\| \sum_{i=1}^{n} u_i \Phi(x_i) \right\|_{r^*} \right) J^* \left( \sum_{i=1}^{n} u_i \Phi(x_i) \right) \\
    \sum_{i=1}^{n} u_i &= 0 \\
    u_i / \gamma &\in \partial L(e_i) \quad \text{for every } i \in \{1, \ldots, n\} \\
    y_i - \langle w, \Phi(x_i) \rangle_{r^*,r^*} - b &= e_i \quad \text{for every } i \in \{1, \ldots, n\},
\end{align*}
\]

where \(J^* : \ell^\ast(\mathbb{K}) \to \ell^\ast(\mathbb{K}) \colon u \mapsto (|u_k|^{r^*-1} \text{sign}(u_k))_{k \in \mathbb{N}}\) and we assumed that in the first equation of (4.7), the right hand side has to be interpreted as \(\{0\}\) when \(\sum_{i=1}^{n} u_i \Phi(x_i) = 0\).

The dual problem \((D_n)\) is a convex optimization problem and it is finite dimensional, since it is defined on \(\mathbb{R}^n\). Once \((D_n)\) is solved, expressions in (4.7) in principle allow to recover the primal solution \((w, b)\) and eventually to compute the estimated regression function \(\langle w, \Phi(x) \rangle + b\) at a generic point \(x\) of the input space \(\mathcal{X}\). However, if \(\mathbb{K}\) is an infinite set, that procedure is not feasible in practice, since it relies on the explicit knowledge of the feature map \(\Phi\) and on the computation of an infinite dimensional scalar product. In the following we show that, in the dual problem \((D_n)\), we can actually get rid of the feature map \(\Phi\) and use instead a new type of kernel function evaluated at the sample points \((x_i)_{1 \leq i \leq n}\). This will ultimately provide a new and effective computational framework for treating support vector regression in Banach spaces of type (4.3).

Now we are ready to define a tensor-kernel associated to the feature map (4.1) and give its main properties.

**Proposition 4.2.** In the setting (4.1) described above, with \(r^* = m\) even integer, the following function is well-defined

\[
K : \mathcal{X}^m = \mathcal{X} \times \cdots \times \mathcal{X} \rightarrow \mathbb{R} : (x'_1, \ldots, x'_m) \mapsto \sum_{k \in \mathcal{K}} \phi_k(x'_1) \cdots \phi_k(x'_m),
\]

and the following hold.

(i) For every \((x'_1, \ldots, x'_m) \in \mathcal{X}^m\), and for every permutation \(\sigma\) of the indexes \(\{1, \ldots, m\}\),

\[
K(x'_{\sigma(1)} \ldots x'_{\sigma(m)}) = K(x'_1, \ldots, x'_m).
\]

(ii) For every \((x_i)_{1 \leq i \leq n} \in \mathcal{X}^n\)

\[
(\forall u \in \mathbb{R}^n) \sum_{i_1, \ldots, i_m = 1}^{n} K(x_{i_1}, \ldots, x_{i_m}) u_{i_1} \ldots u_{i_m} \geq 0.
\]

\(^3\text{Since } G^* = \varphi^* \circ \| \cdot \|_{r^*} \text{ and } \{0\} = \text{argmin } \varphi = \partial \varphi^*(0), \text{ Fact 2.1 yields } \partial (\varphi^* \circ \| \cdot \|_{r^*})(w^*) = \frac{\partial \varphi^*(\|w^*\|_{r^*})}{\|w^*\|_{r^*}} J^* (w^*) \text{ if } w^* \neq 0, \text{ and } \partial (\varphi^* \circ \| \cdot \|_{r^*})(w^*) = \{0\} \text{ if } w^* = 0.\)
(iii) For every \((x_i)_{1 \leq i \leq n} \in X^n\)

\[
\begin{align*}
    u \in \mathbb{R}^n \mapsto \left\| \sum_{i=1}^n u_i \Phi(x_i) \right\|_{r^*}^* &= \sum_{i_1, \ldots, i_m=1}^n K(x_{i_1}, \ldots, x_{i_m}) u_{i_1} \ldots u_{i_m} \tag{4.9}
\end{align*}
\]

is a homogeneous polynomial form of degree \(m\) on \(\mathbb{R}^n\).

(iv) For every \(x \in X\), \(K(x, \ldots, x) \geq 0\).

(v) For every \((x'_1, \ldots, x'_m) \in X^m\)

\[
|K(x'_1, \ldots, x'_m)| \leq |K(x'_1, \ldots, x'_1)|^{1/m} \cdots |K(x'_m, \ldots, x'_m)|^{1/m}.
\]

Proof. Since \((\phi_k(x'_1))_{k \in K}, (\phi_k(x'_2))_{k \in K}, \ldots, (\phi_k(x'_m))_{k \in K} \in \ell^m(K)\), it follows from Fact 2.3 that \((\phi_k(x'_1)\phi_k(x'_2) \cdots \phi_k(x'_m))_{k \in K} \in \ell^1(K)\) and

\[
\sum_{k \in K} |\phi_k(x'_1) \cdots \phi_k(x'_m)| \leq \left( \sum_{k \in K} |\phi_k(x'_1)|^m \right)^{1/m} \cdots \left( \sum_{k \in K} |\phi_k(x'_m)|^m \right)^{1/m}. \tag{4.10}
\]

This shows that definition (4.8) is well-posed and moreover, since \(m\) is even we can remove the absolute values in the right hand side of (4.10) and get (v). Properties (i) and (iv) are immediate from the definition of \(K\). Finally, since \(r^* = m\) is even, for every \(u \in \mathbb{R}^n\), we have

\[
\begin{align*}
    \left\| \sum_{i=1}^n u_i \Phi(x_i) \right\|_{r^*}^* &= \sum_{k \in K} \left( \sum_{i=1}^n u_i \phi_k(x_i) \right)^m \\
    &= \sum_{k \in K} \sum_{i_1, \ldots, i_m=1}^n \phi_k(x_{i_1}) \cdots \phi_k(x_{i_m}) u_{i_1} \ldots u_{i_m} \\
    &= \sum_{i_1, \ldots, i_m=1}^n \left( \sum_{k \in K} \phi_k(x_{i_1}) \cdots \phi_k(x_{i_m}) \right) u_{i_1} \ldots u_{i_m}. \tag{4.11}
\end{align*}
\]

Therefore, recalling the definition of \(K\), (ii) and (iii) follow.

Remark 4.3. Let \((x_i)_{1 \leq i \leq n} \in X^n\). Then \(K := (K(x_{i_1}, \ldots, x_{i_m}))_{i \in \{1, \ldots, n\}^m}\) defines a tensor of degree \(m\) on \(\mathbb{R}^n\). Then, properties (i) and (ii) establish that the tensor is symmetric and positive definite: they are natural generalization of the defining properties of standard positive (matrix) kernels.

Because of Proposition 4.2(v), tensor kernels, as defined in (4.8), can be normalized as for the matrix kernels.

Proposition 4.4 (normalized tensor kernel). Let \(K\) be defined as in (4.8) and suppose that, for every \(x \in X\), \(K(x, \ldots, x) > 0\). Define

\[
\tilde{K}: X^m \to \mathbb{R},
\]

\[
(x'_1, \ldots, x'_m) \mapsto \frac{K(x'_1, \ldots, x'_m)}{K(x'_1, \ldots, x'_1)^{1/m} \cdots K(x'_m, \ldots, x'_m)^{1/m}}. \tag{4.12}
\]

Then \(\tilde{K}\) is still of type (4.8), for some family of functions \((\tilde{\phi}_k)_{k \in K}: \Phi : X \to \mathbb{R}\), and the following hold.
(i) For every $x \in \mathcal{X}$, $\tilde{K}(x, \ldots, x) = 1$.

(ii) For every $(x_1', \ldots, x_m') \in \mathcal{X}^m$, $|\tilde{K}(x_1', \ldots, x_m')| \leq 1$.

Proof. Just note that, for every $x \in \mathcal{X}$, $\|\Phi(x)\|_m^m = K(x, \ldots, x) > 0$. Then define $\phi_k(x) = \phi_k(x)/\|\Phi(x)\|_m^m$.

Remark 4.5. The homogeneous polynomial form (4.9) can be written as follows

$$\sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \binom{m}{\alpha} K(x_1, \ldots, x_1, \ldots, x_n, \ldots, x_n) u^\alpha \quad (4.13)$$

where, for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ and for every vector $u \in \mathbb{R}^n$, we used the standard notation $u^\alpha = u_1^{\alpha_1} \cdots u_n^{\alpha_n}$, $|\alpha| = \sum_{i=1}^n \alpha_i$, and the multinomial coefficient

$$\binom{m}{\alpha} = \binom{m}{\alpha_1, \ldots, \alpha_n} = \frac{m!}{\alpha_1! \cdots \alpha_n!}. \quad (4.14)$$

Indeed it follows from (4.11) and the multinomial theorem [3, Theorem 4.12] that

$$\left\| \sum_{i=1}^n u_i \Phi(x_i) \right\|_{r^*}^{r^*} = \sum_{k \in \mathcal{K}} \left( \sum_{i=1}^n u_i \phi_k(x_i) \right)^m = \sum_{k \in \mathcal{K}} \sum_{\alpha \in \mathbb{N}^n \atop |\alpha| = m} \binom{m}{\alpha} \phi_k(x_1)^{\alpha_1} \cdots \phi_k(x_n)^{\alpha_n} u^\alpha \quad (4.15)$$

Thus (4.13) follows from (4.8).

We present the main result of the section, which is a direct consequence of Theorem 3.1 and Proposition 4.2.

Theorem 4.6. Under the setting (4.1)-(4.5) described above, with $r^* = m$ even integer, let $K$ be defined as in (4.8). Then the dual problem $(\mathcal{D}_n)$ can be written as the following finite dimensional optimization problem

$$\begin{aligned}
& \min_{u \in \mathbb{R}^n} \varphi^\ast \left( \frac{1}{n} \left( \sum_{i_1, \ldots, i_m=1}^n K(x_{i_1}, \ldots, x_{i_m}) u_{i_1} \cdots u_{i_m} \right)^{1/r^*} \right) + \frac{\gamma}{n} \sum_{i=1}^n L^\ast \left( \frac{u_i}{\gamma} \right) - \frac{1}{n} \sum_{i=1}^n y_i u_i \\
& \text{subject to } \sum_{i=1}^n u_i = 0.
\end{aligned} \quad (4.15)$$
Moreover, in the related optimality conditions (4.7), the first equation turns to

\[ w \in \frac{\partial \varphi^*(\frac{1}{n}K[u]^{1/r})}{K[u]^{1/r}} J_m \left( \sum_{i=1}^n u_i \Phi(x_i) \right), \] (4.16)

where \( K[u] := \sum_{i_1, \ldots, i_m=1}^n K(x_{i_1}, \ldots, x_{i_m})u_{i_1} \cdots u_{i_m}, \) \( J_m : \ell^m(\mathbb{K}) \to \ell^r(\mathbb{K}) : u \mapsto (u_k^{m-1})_{k \in \mathbb{N}}, \) and the right hand side (4.16) is meant to be \{0\} when \( K[u] = 0. \)

**Remark 4.7.**

(i) Problem (4.15) is a convex optimization problem. If the tensor kernel \( K \) is explicitly computable by means of (4.8), the dual problem (4.15) is a very finite dimensional problem, in the sense that it does not involve the feature map anymore. This is exactly how the kernel trick works within the matrix kernels.

(ii) Once a solution \( u \in \mathbb{R}^n \) of the dual problem (4.15) is computed, the solutions of the primal problem (\( P_n \)) are given by

\[
\begin{align*}
(\forall k \in \mathbb{N}) & \quad w_k = \xi(u) \left( \sum_{i=1}^n u_i \phi_k(x_i) \right)^{m-1}, \quad \xi(u) \in \frac{\partial \varphi^*(\frac{1}{n}K[u]^{1/r^*})}{K[u]^{1/r}}, \\
& \quad b \in \bigcap_{j=1}^n \left[ y_j - \langle w, \Phi(x_j) \rangle_{r,r^*} - \partial L^* \left( \frac{u_j}{\gamma} \right) \right],
\end{align*}
\] (4.17)

where \( \xi(u) = 0 \) if \( K[u] = 0. \)

(iii) If there exists \( j \) such that \( L^* \) is differentiable at \( u_j/\gamma \), the second of (4.17) uniquely determines \( b \) as \( b = y_j - \langle w, \Phi(x_j) \rangle_{r,r^*} - (L^*)'(u_j/\gamma). \)

**Corollary 4.8.** In Theorem 4.6, let \( \varphi = (1/r)|\cdot|^r \) (which gives \( G = (1/r)|\cdot|^r \)). Then the dual problem (4.15) becomes

\[
\begin{align*}
\min_{u \in \mathbb{R}^n \cap r^*n^{r^* \gamma}} & \quad \sum_{i_1, \ldots, i_m=1}^n K(x_{i_1}, \ldots, x_{i_m})u_{i_1} \cdots u_{i_m} + \frac{\gamma}{n} \sum_{i=1}^n L^* \left( \frac{u_i}{\gamma} \right) - \frac{1}{n} \sum_{i=1}^n y_i u_i \\
\text{subject to} & \quad \sum_{i=1}^n u_i = 0.
\end{align*}
\] (4.18)

Moreover, if \( u \) is a solution of (4.18) and there exists \( j \) such that \( L^* \) is differentiable at \( u_j/\gamma \), then the primal problem (\( P_n \)) has a unique solution which is given by

\[
(\forall k \in \mathbb{N}) \quad w_k = \frac{1}{n^{m-1}} \left( \sum_{i=1}^n u_i \phi_k(x_i) \right)^{m-1} \quad \text{and} \quad b = y_j - \langle w, \Phi(x_j) \rangle_{r,r^*} - (L^*)'(u_j/\gamma). \] (4.19)

**Proof.** Just note that \( \varphi^* = (1/r^*)|\cdot|^r \) and apply Theorem 4.6 and Remark 4.7(iii).

**Remark 4.9.**
The first term in the objective function in (4.18) is a positive definite homogeneous polynomial of order \(m\). So, if the function \(L^*\) is smooth, which occurs when \(L\) is strictly convex, then the dual problem (4.18) is a smooth convex optimization problem with a linear constraint and can be approached by standard optimization techniques such as Newton-type or gradient-type methods — in the case of square loss, the dual problem (4.18) is a convex polynomial optimization problem and possibly more appropriate optimization methods may be employed.

\[(ii)\] When (4.18) is specialized to the case of \(\varepsilon\)-insensitive loss (see Example 3.6) we obtain
\[
\begin{align*}
\min_{u \in \mathbb{R}^n} & \frac{1}{mn^m} \sum_{i_1, \ldots, i_m=1}^n K(x_{i_1}, \ldots, x_{i_m}) u_{i_1} \cdots u_{i_m} + \frac{\varepsilon}{n} \sum_{i=1}^n |u_i| - \frac{1}{n} \sum_{i=1}^n y_i u_i \\
\text{subject to } & \sum_{i=1}^n u_i = 0 \quad \text{and } |u_i| \leq \gamma \text{ for every } i \in \{1, \ldots, n\},
\end{align*}
\] (4.20)

which is clearly a generalization of (3.2).

The next issue is to evaluate the regression function corresponding to \((w, b)\) at a general input point, without the explicit knowledge of the feature map but relying on the tensor-kernel \(K\) only. The following proposition shows that a tensor-kernel representation holds and hence the kernel trick is fully viable in our more general situation.

**Proposition 4.10.** Under the assumptions (4.1)-(4.5), let \(K\) be defined as in (4.8). Suppose that the primal problem \((\mathcal{P}_n)\) has solutions. Let \(u \in \mathbb{R}^n\) be a solution of the dual problem (4.15) and let \((w, b)\) be a solution of \((\mathcal{P}_n)\) determined as in (4.17). Then,
\[
\begin{align*}
\langle w, \Phi(x) \rangle_{r^*, r^*} &= \xi(u) \sum_{i_1, \ldots, i_{m-1}=1}^n K(x_{i_1}, \ldots, x_{i_{m-1}}, x) u_{i_1} \cdots u_{i_{m-1}}, \quad \forall x \in \mathcal{X}, \\
b &\in \sum_{j=1}^n \left[ y_j - \xi(u) \sum_{i_1, \ldots, i_{m-1}=1}^n K(x_{i_1}, \ldots, x_{i_{m-1}}, x_j) u_{i_1} \cdots u_{i_{m-1}} - \partial L^* \left( \frac{u_j}{\gamma} \right) \right],
\end{align*}
\] (4.21)

where \(\xi(u) \in K[u]^{-1/r^*} \partial \varphi^*(\frac{1}{r} K[u]^{1/r^*})\) if \(K[u] \neq 0\) and \(\xi(u) = 0\) if \(K[u] = 0\), and \(K[u]\) is defined as in Theorem 4.6.

**Proof.** Let \(x \in \mathcal{X}\). Then, we derive from (4.17) that
\[
\langle w, \Phi(x) \rangle_{r^*, r^*} = \sum_{k \in \mathcal{K}} w_k \phi_k(x)
\]
\[
= \xi(u) \sum_{k \in \mathcal{K}} \left( \sum_{i=1}^n u_i \phi_k(x_i) \right)^{m-1} \phi_k(x)
\]
\[
= \xi(u) \sum_{k \in \mathcal{K}} \sum_{i_1, \ldots, i_{m-1}=1}^n \phi_k(x_{i_1}) \cdots \phi_k(x_{i_{m-1}}) \phi_k(x) u_{i_1} \cdots u_{i_{m-1}}
\]
\[
= \xi(u) \sum_{i_1, \ldots, i_{m-1}=1}^n K(x_{i_1}, \ldots, x_{i_{m-1}}, x) u_{i_1} \cdots u_{i_{m-1}},
\]
where we used the definition (4.8) of $K$. The second equation in (4.21) follows from the second equation in (4.17).

**Remark 4.11.** In the case treated in Corollary 4.8, assuming that there exists $j$ such that $L^*$ is differentiable at $u_j/\gamma$, (4.21) yields the following representation formula

$$\langle w, \Phi(x) \rangle_{r,r^*} + b = \frac{1}{n^m - 1} \sum_{i_1, \ldots, i_{m-1} = 1}^n (K(x_{i_1}, \ldots, x_{i_{m-1}}, x) - K(x_{i_1}, \ldots, x_{i_{m-1}}, x_j))u_{i_1} \cdots u_{i_{m-1}}$$

$$+ y_j - (L^*)'(\frac{u_j}{\gamma}),$$

which generalizes (3.4). Moreover, if in model (4.2) we assume no offset ($b = 0$), then we can avoid the requirement of the differentiability of $L^*$ and the representation formula becomes

$$\langle w, \Phi(x) \rangle_{r,r^*} = \frac{1}{n^m - 1} \sum_{i_1, \ldots, i_{m-1} = 1}^n K(x_{i_1}, \ldots, x_{i_{m-1}}, x)u_{i_1} \cdots u_{i_{m-1}}.$$

Concluding we have shown that, the estimated regression function can be evaluated at every point of the input space by means of a finite summation formula, provided that the tensor-kernel $K$ is explicitly available: we will show in Section 5 several significant examples in which this occurs.

### 4.2 The complex case

In this section we give the complex version of the theory developed in Section 4.1. Therefore, we let $\mathcal{F} = \ell^r(\mathbb{K}; \mathbb{C})$, with $\mathbb{K}$ a countable set and $r = m/(m-1)$ for some even integer $m \geq 2$. Let $(\phi_k)_{k \in \mathbb{K}}$ be a family of measurable functions from $\mathcal{X}$ to $\mathbb{C}$ such that, for every $x \in \mathcal{X}$, $(\phi_k(x))_{k \in \mathbb{K}} \in \ell^r(\mathbb{K}; \mathbb{C})$. The feature map is now defined as

$$\Phi: \mathcal{X} \rightarrow \ell^r(\mathbb{K}; \mathbb{C}): x \mapsto (\phi_k(x))_{k \in \mathbb{K}}, \quad (4.22)$$

which generates the model

$$(\forall w \in \ell^r(\mathbb{K}; \mathbb{C}))(\forall b \in \mathbb{C}) \quad x \mapsto \langle w, \Phi(x) \rangle_{r,r^*} + b = \sum_{k \in \mathbb{K}} w_k \phi_k(x) + b, \quad (4.23)$$

where $\langle w, w^* \rangle_{r,r^*} = \sum_{k \in \mathbb{N}} w_k \overline{w_k}$ is the canonical sesquilinear form between $\ell^r(\mathbb{K}; \mathbb{C})$ and $\ell^{r*}(\mathbb{K}; \mathbb{C})$. This case can be treated as a vector-valued real case by identifying complex functions with $\mathbb{R}^2$-valued functions and the space $\ell^r(\mathbb{K}; \mathbb{C})$ with $\ell^r(\mathbb{K}; \mathbb{R}^2)$. Moreover, it is not difficult to generalize the dual framework presented in Section 3 to the case of vector-valued (and specifically to $\mathbb{R}^2$-valued) functions. Then, the (complex) feature map (4.22) defines an underlying real vector-valued feature map on $\ell^r(\mathbb{K}; \mathbb{R}^2)$ [5], that is

$$\Phi_{\mathbb{R}}: \mathcal{X} \rightarrow \mathcal{L}(\mathbb{R}^2, \ell^r(\mathbb{K}; \mathbb{R}^2)) \cong \ell^r(\mathbb{K}; \mathbb{R}^{2 \times 2}): x \mapsto (\phi_{\mathbb{R},k}(x))_{k \in \mathbb{K}}, \quad (4.24)$$

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where \( \mathcal{L}(\mathbb{R}^2, \ell^r(\mathbb{K}; \mathbb{R}^2)) \) is the spaces of linear continuous operators from \( \mathbb{R}^2 \) to \( \ell^r(\mathbb{K}; \mathbb{R}^2) \) (which is isomorphic to \( \ell^r(\mathbb{K}; \mathbb{R}^{2 \times 2}) \)) and

\[
(\forall x \in \mathcal{X})(\forall k \in \mathbb{K}) \quad \phi_{R,k}(x) = \begin{bmatrix} \Re \phi_k(x) & \Im \phi_k(x) \\ -\Im \phi_k(x) & \Re \phi_k(x) \end{bmatrix} \in \mathbb{R}^{2 \times 2}.
\]

(4.25)

This way, denoting, for every \( x \in \mathcal{X} \), by \( \phi_{R,k}(x)^* \) the transpose of the matrix \( \phi_{R,k}(x) \), we have

\[
(\forall x \in \mathcal{X})(\forall k \in \mathbb{K})(\forall w_k \in \mathbb{R}^2 \cong \mathbb{C}) \quad \phi_{R,k}(x)^* w_k = w_k \phi_k(x),
\]

hence \( \Phi_R(x)^* w = \langle w, \Phi(x) \rangle_{r,r'} \). Moreover

\[
(\forall x \in \mathcal{X})(\forall u \in \mathbb{R}^2 \cong \mathbb{C}) \quad \Phi_R(x) u = (\phi_{R,k}(x) u)_{k \in \mathbb{K}} = (\overline{w_k(x)})_{k \in \mathbb{K}} = u \Phi(x).
\]

(4.27)

Then, problems \((\mathcal{P}_n)\) and \((\mathcal{D}_n)\) become

\[
\begin{align*}
&\min_{(w,b,e) \in \ell^r(\mathbb{K}; \mathbb{C}) \times \mathbb{C} \times \mathbb{C}^n} \sum_{i=1}^{n} L(e_i) + \varphi(\|w\|_r), \\
&\text{subject to } y_i - \langle w, \Phi(x_i) \rangle_{r,r'} - b = e_i, \quad \text{for every } i \in \{1, \ldots, n\}
\end{align*}
\]

(\(\mathcal{P}_n(\mathbb{C})\))

and

\[
\begin{align*}
&\min_{u \in \mathbb{C}^n} \varphi^* \left( \left\| \frac{1}{n} \sum_{i=1}^{n} u_i \Phi(x_i) \right\|_{r'} \right) + \frac{\gamma}{n} \sum_{i=1}^{n} L^* \left( \frac{u_i}{\gamma} \right) - \frac{1}{n} \sum_{i=1}^{n} \Re(u_i y_i) \\
&\text{subject to } \sum_{i=1}^{n} u_i = 0,
\end{align*}
\]

(\(\mathcal{D}_n(\mathbb{C})\))

where, \( L^*: \mathbb{C} \to \mathbb{R}: z^* \mapsto \sup_{z \in \mathbb{C}} \Re(z z^*) - L(z) \). Moreover, assuming that \( w \neq 0 \), the optimality conditions (4.7) still hold, where now \( J_r: \ell^r(\mathbb{K}; \mathbb{C}) \to \ell^r(\mathbb{K}; \mathbb{C}) : w^* \mapsto (|w_k|^r - |w_k^\ast|^r)^{\frac{1}{r-1}} |w_k^\ast| w_k^\ast^{-1} \) \( k \in \mathbb{K} \), and

\[
(\forall e \in \mathbb{C}) \quad \partial L(e) = \{ z^* \in \mathbb{C} \mid (\forall z \in \mathbb{C}) \quad L(z) \geq L(e) + \Re(\overline{z}^r(z - e)) \}.
\]

In the following we give the result corresponding to Proposition 4.2.

**Proposition 4.12.** In the setting described above, suppose that \( m \) is even and set \( q = m/2 \). Then, the following function is well-defined

\[
K: \mathcal{X}^q \times \mathcal{X}^q \to \mathbb{C}: (x'_1, \ldots, x'_q; x''_1, \ldots, x''_q) \mapsto \sum_{k \in \mathbb{K}} \phi_k(x'_1) \cdots \phi_k(x'_q) \overline{\phi_k(x''_1)} \cdots \overline{\phi_k(x''_q)},
\]

(4.28)

and the following hold.

(i) For every \( (x'_1, \ldots, x'_q; x''_1, \ldots, x''_q) \in \mathcal{X}^q \times \mathcal{X}^q \), and for every permutation \( \sigma' \) and \( \sigma'' \) of the indexes \( \{1, \ldots, q\} \),

\[
K(x'_{\sigma(1)} \cdots x'_{\sigma(q)}; x''_{\sigma'(1)} \cdots x''_{\sigma'(q)}) = K(x'_1, \ldots, x'_q; x''_1, \ldots, x''_q).
\]
(ii) For every \((x'; x'') \in X^q \times X^q\) \(K(x'; x'') = \overline{K(x''; x')}\);

(iii) For every \((x_i)_{1 \leq i \leq n} \in X^n\)
\[
(\forall u \in \mathbb{C}^n) \sum_{i_1, \ldots, i_q=1 \atop j_1, \ldots, j_q=1}^n K(x_{j_1}, \ldots, x_{j_q}; x_{i_1}, \ldots, x_{i_q})u_{i_1} \cdots u_{i_q} \overline{u_{j_1}} \cdots \overline{u_{j_q}} \geq 0.
\]

(iv) For every \((x_i)_{1 \leq i \leq n} \in X^n\)
\[
u \in \mathbb{C}^n \mapsto \left\| \sum_{i=1}^n u_i \Phi(x_i) \right\|_{r^*} = \sum_{i_1, \ldots, i_q=1 \atop j_1, \ldots, j_q=1}^n K(x_{j_1}, \ldots, x_{j_q}; x_{i_1}, \ldots, x_{i_q})u_{i_1} \cdots u_{i_q} \overline{u_{j_1}} \cdots \overline{u_{j_q}}
\]

is a positive homogeneous polynomial form of degree \(m\) on \(\mathbb{C}^n\).

(v) For every \((x'_1, \ldots, x'_q) \in X^q\), \(K(x'_1, \ldots, x'_q; x'_1, \ldots, x'_q) \geq 0\);

(vi) For every \((x'_1, \ldots, x'_q; x''_1, \ldots, x''_q) \in X^q \times X^q\),
\[
|K(x'_1, \ldots, x'_q; x''_1, \ldots, x''_q)| \leq K(x'_1, \ldots, x'_1; x'_1, \ldots, x'_1)^{1/m} \cdots K(x''_q, \ldots, x''_q; x''_q, \ldots, x''_q)^{1/m}.
\]

**Remark 4.13.** Item (iii) states that \(K := (K(x'_i, \ldots, x''_i))_{i \in \{1, \ldots, n\}^m}\) is a positive-definite tensor of degree \(m\).

As in the real case, the dual problem \((D_u)\) reduces to
\[
\begin{bmatrix}
\min_{\nu \in \mathbb{C}^n} \varphi^*
\left( \frac{1}{n} \left( \sum_{i_1, \ldots, i_q=1 \atop j_1, \ldots, j_q=1}^n K(x_{j_1}, \ldots, x_{j_q}; x_{i_1}, \ldots, x_{i_q})u_{i_1} \cdots u_{i_q} \overline{u_{j_1}} \cdots \overline{u_{j_q}} \right)^{1/r^*} \right)
+ \frac{\gamma}{n} \sum_{i=1}^n \left| \frac{u_i}{\gamma} \right| - \frac{1}{n} \Re \sum_{i=1}^n y_i \nu_i
\end{bmatrix}
\]
subject to \(\sum_{i=1}^n u_i = 0\)

and the homogeneous polynomial form in Proposition 4.12(iv) can be written as follows
\[
\sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n \atop |\alpha|=q, |\beta|=q} \left( \begin{array}{c}
q \\
\alpha
\end{array} \right) \left( \begin{array}{c}
q \\
\beta
\end{array} \right) K(x_{\alpha_1}, \ldots, x_{\alpha_q}, \ldots, x_{\alpha_n}, \ldots, x_{\alpha_1}, \ldots, x_{\alpha_q}, \ldots, x_{\beta_1}, \ldots, x_{\beta_n}) u_\alpha \overline{u_\beta}.
\]

Finally, in the setting of Proposition 4.10, defining
\[
K[u] = \sum_{i_1, \ldots, i_q=1 \atop j_1, \ldots, j_q=1}^n K(x_{j_1}, \ldots, x_{j_q}, x_{i_1}, \ldots, x_{i_q})u_{i_1} \cdots u_{i_q} \overline{u_{j_1}} \cdots \overline{u_{j_q}},
\]
for every $x \in \mathcal{X}$, the following representation formulas hold

$$\langle w, \Phi(x) \rangle_{r,r^*} = \xi(u) \sum_{i_1, \ldots, i_q=1}^{n} K(x_{j_1}, \ldots, x_{j_{q-1}}, x; x_{i_1}, \ldots, x_{i_q}) u_{i_1} \cdots u_{i_q} \overline{u_{j_1}} \cdots \overline{u_{j_{q-1}}}$$

$$\xi(u) \in \frac{\partial \varphi^* (\frac{1}{n} K[u]^{1/r^*})}{K[u]^{1/r}}$$

$$b = y_j - \langle w, \Phi(x_j) \rangle_{r,r^*} - \nabla L^* \left( \frac{u_j}{\gamma} \right).$$

where we assumed that $L^*$ is differentiable, as a function from $\mathbb{R}^2$ to $\mathbb{R}$, at some $x_j/\gamma$.

**Remark 4.14.** In view of Proposition 4.12(iv), definitions (4.22) and (4.28) correspond to those given in [21, Lemma 4.2] and the concept of positive definiteness stated in (iii) is a natural generalization of the analogue notion given in [21, Definition 4.15].

## 5 Power series tensor-kernels

In this section we consider reproducing kernel Banach spaces of complex analytic functions which are generated through power series. We show that, for such spaces, the corresponding tensor kernel, defined according to (4.8), admits an explicit expression. We provide also representation formulas. In this section we assume, for simplicity, that $\varphi = (1/r)|x|^r$, with $r = m/(m - 1)$ for some even integer $m \geq 2$. therefore we address the support vector regression problem

$$\min_{(w,b) \in \ell^p(\mathbb{R}^d; \mathbb{C}) \times \mathbb{C}} \frac{2}{n} \sum_{i=1}^{n} L(y_i - \langle w, \Phi(x_i) \rangle_{r,r^*} - b) + \frac{1}{r} \|w\|_{r^*}^r,$$

for a specific choice of the feature map (4.22).

We first need to set special notation for multi-index powers of complex vectors. Let $d \in \mathbb{N}$ with $d \geq 1$. We will denote the component of a vector $x \in \mathbb{C}^d$, by $x_t$, with $t \in \{1, \ldots, d\}$. For every $x \in \mathbb{C}^d$ and every $\nu \in \mathbb{N}^d$ we set

$$x^\nu = \prod_{t=1}^{d} x_t^{\nu_t}, \quad |x| = (|x_1|, \ldots, |x_d|), \quad \nu! = \prod_{t=1}^{d} t^{\nu_t}!$$

so that $\forall \nu \in \mathbb{N}^d$ we have $|x^\nu| = \prod_{t=1}^{d} |x_t|^{\nu_t} = |x|^\nu$. Moreover, when the exponent of the vector $x \in \mathbb{C}^d$ is an index (not a multi-index), say $m \in \mathbb{N}$, we consider $m$ as a constant multi-index, that is $(m, m, \ldots)$, so that $x^m$ means $\prod_{t=1}^{d} x_t^m$. Finally, we define the binary inner operation of component-wise multiplication in $\mathbb{C}^d$. For every $x, x' \in \mathbb{C}^d$, we define $x \odot x' \in \mathbb{C}^d$ such that, for every $t \in \{1, \ldots, d\}$, $(x \odot x')_t = x_t x'_t$. Let $m \in \mathbb{N}$ and $x \in \mathbb{C}^d$. We set $x^\odot m = x \odot \cdots \odot x$ ($m$-times), so that $x^\odot m \in \mathbb{C}^d$ and, for every $t \in \{1, \ldots, d\}$, $(x^\odot m)_t = x_t^m$.

Let $\rho = (\rho_\nu)_{\nu \in \mathbb{N}^d}$ be a multi-sequence in $\mathbb{R}_+$ and let $\mathcal{D}_\rho$ be the domain of (absolute) convergence of the power series $\sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu$, i.e., the interior of the set $\{ z \in \mathbb{C}^d \mid \sum_{\nu \in \mathbb{N}^d} \rho_\nu |z^\nu| < \infty \}$. For $\nu \in \mathbb{N}^d$ we set $\nu! = \prod_{t=1}^{d} t^{\nu_t}!$ and

$$\mathcal{D}_\rho = \{ z \in \mathbb{C}^d \mid \sum_{\nu \in \mathbb{N}^d} \rho_\nu |z^\nu| < \infty \}. $$

Finally, we introduce the set $\mathcal{C}_\rho$ of $M$-radially bounded sequences of complex numbers $a = (a_\nu)_{\nu \in \mathbb{N}^d}$, that is $\sup_{\nu \in \mathbb{N}^d} |a_\nu| |\nu!| < \infty$.
and we assume that \( \mathcal{D}_\rho \neq \{0\} \). Let \( \kappa: \mathcal{D}_\rho \to \mathbb{C} \) be the sum of the series \( \sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu \), that is

\[
(\forall z \in \mathcal{D}_\rho) \quad \kappa(z) = \sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu.
\]

Clearly \( \kappa \) is an analytic function on \( \mathcal{D}_\rho \subset \mathbb{C}^d \). Set

\[
\mathcal{D}_{\rho}^{\odot 1/m} = \{ x \in \mathbb{C}^d \mid x^{\odot m} = (x_1^m, \ldots, x_d^m) \in \mathcal{D}_\rho \},
\]

let \( \mathcal{X} \subset \mathcal{D}_{\rho}^{\odot 1/m} \), and define the dictionary

\[
(\forall \nu \in \mathbb{N}^d) \quad \phi_\nu: \mathcal{X} \to \mathbb{C}: x \mapsto \rho_\nu^{1/m} x^\nu.
\]

Then, for every \( x \in \mathcal{X} \), since \( x^{\odot m} \in \mathcal{D}_\rho \), we have

\[
\sum_{\nu \in \mathbb{N}^d} |\phi_\nu(x)|^m = \sum_{\nu \in \mathbb{N}^d} \rho_\nu |x^{\odot m}|^\nu < +\infty,
\]

hence \( (\phi_\nu(x))_{\nu \in \mathbb{N}^d} \in \ell^m(\mathbb{N}^d; \mathbb{C}) \). Thus, we are in the framework described at the beginning of Section 4.2. We define

\[
B^{r}_{\rho,b}(\mathcal{X}) = \left\{ f \in \mathbb{C}^X \mid (\exists (c_\nu)_{\nu \in \mathbb{N}^d} \in \ell^r(\mathbb{N}^d; \mathbb{C}))(\exists b \in \mathbb{C})(\forall x \in \mathcal{X}) \left( f(x) = \sum_{\nu \in \mathbb{N}^d} c_\nu \phi_\nu(x) + b \right) \right\},
\]

which is a reproducing kernel Banach spaces with norm

\[
\|f\|_{B^{r}_{\rho,b}(\mathcal{X})} = \inf \left\{ \|c\|_r + |b| \mid (c_\nu)_{\nu \in \mathbb{N}^d} \in \ell^r(\mathbb{N}^d; \mathbb{C}) \text{ and } f = \sum_{\nu \in \mathbb{N}^d} c_\nu \rho_\nu^{1/m} x^\nu + b \text{ (pointwise)} \right\}.
\]

Suppose now that \( b = 0 \) and that, for every \( \nu \in \mathbb{N}^d, \rho_\nu > 0 \). Then, defining the weights

\[
(\eta_\nu)_{\nu \in \mathbb{N}^d} = (\rho_\nu^{r/m})_{\nu \in \mathbb{N}^d}
\]

and the corresponding weighted \( \ell^r \) space

\[
\ell^r_\eta(\mathbb{N}^d; \mathbb{C}) = \left\{ (a_\nu)_{\nu \in \mathbb{N}^d} \in \mathbb{C}^{\mathbb{N}^d} \mid \sum_{\nu \in \mathbb{N}^d} \frac{1}{\rho_\nu^{r/m}} |a_\nu|^r < +\infty \right\},
\]

we can express the space \( B^r_{\rho,0}(\mathcal{X}) \) in the form of a weighted Hardy-like space \([17, 25]\)

\[
B^r_{\rho,0}(\mathcal{X}) = \left\{ f \in \mathbb{C}^X \mid (\exists (a_\nu)_{\nu \in \mathbb{N}^d} \in \ell^r_\eta(\mathbb{N}^d; \mathbb{C}))(\forall x \in \mathcal{X}) \left( f(x) = \sum_{\nu \in \mathbb{N}^d} a_\nu x^\nu \right) \right\}.
\]

Moreover, for every \((x_1', \ldots, x_q', x_1'', \ldots, x_q'') \in \mathcal{X}^q \times \mathcal{X}^q\),

\[
K(x_1', \ldots, x_q', x_1'', \ldots, x_q'') = \sum_{\nu \in \mathbb{N}^d} \rho_\nu x_1'^\nu \cdots x_q'^\nu x_1''^\nu \cdots x_q''^\nu = \kappa(x_1' \odot \cdots \odot x_q' \odot x_1'' \odot \cdots \odot x_q''),
\]

\[\text{(5.2)}\]

\footnote{It means that if \( z \in \mathcal{D}_\rho \), then \( \mathcal{D}_\rho \) contains the polydisk \( \{ t \in \mathbb{C}^d \mid (\forall j \in \{1, \ldots, d\}) |t_j| \leq |z_j| \} \).}
Remark 5.1. Suppose that $\rho_{\nu} > 0$, for every $\nu \in \mathbb{N}^d$. Then $\sum_{\nu \in \mathbb{N}^d} c_{\nu} \rho_{\nu}^{1/m} x^\nu = 0$ (pointwise) implies $c_{\nu} \rho_{\nu}^{1/m} = 0$, for every $\nu \in \mathbb{N}^d$ and hence $c_{\nu} = 0$, for every $\nu \in \mathbb{N}^d$. Thus, in virtue of Remark 4.1 this yields that $(\phi_{\nu})_{\nu \in \mathbb{N}^d}$ is an unconditional Schauder basis of $B^r_{\rho,0}(X)$ and that $B^r_{\rho,0}(X)$ is isometric to $\ell^r(\mathbb{N}^d; \mathbb{C})$.

Proposition 5.2. Under the notation and assumption above, suppose that $X$ is a compact subset of $D^{1/m}_{\rho}$ and that, for every $\nu \in \mathbb{N}^d$, $\rho_{\nu} > 0$. Then $B^r_{\rho,0}(X)$ is dense in $\mathcal{C}(X; \mathbb{C})$, the space of continuous functions on $X$ endowed with the uniform norm.

Proof. It is enough to note that $B^r_{\rho,0}(X)$ contains the set

$$\mathcal{A} = \text{span}\{\phi_{\nu} \mid \nu \in \mathbb{N}\} = \left\{\sum_{\nu \in I} c_{\nu} x^\nu \mid I \subset \mathbb{N}^d \text{ and } I \text{ finite } (c_{\nu})_{\nu \in I} \in \mathbb{C}^I\right\}$$

which is the algebra of polynomials on $X$ in $d$ variables with complex coefficients. Thus the statement is a consequence of the Stone-Weierstrass theorem.

In the sequence we also assume that the offset $b$ is zero. Because of (5.2), the representation given in (4.29) yields the following homogenous polynomial form

$$u \in \mathbb{C}^n \mapsto \left\| \sum_{i=1}^n u_i \Phi(x_i) \right\|_{r^*}^r = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n} \left( \begin{array}{c} q \\ \alpha \\ \beta \end{array} \right) \kappa(x_1^{\circ \alpha_1} \cdots x_n^{\circ \alpha_n} \odot \cdots \odot x_1^{\circ \beta_1} \cdots x_n^{\circ \beta_n}) u^{\alpha} \overline{w}^\beta,$$

where $(x_i)_{1 \leq i \leq n} \in X^n$ is the training set and, according to the convention established at the beginning of the section, $x_i^{\circ \alpha_i} = (x_{i,1}, \ldots, x_{i,d})$. Moreover, in this case, recalling (4.31) and (5.2), for every $x \in X$, we have

$$\langle w, \Phi(x) \rangle_{r^*, r} = \frac{1}{n^{m-1}} \sum_{i_1, \ldots, i_q=1}^n \kappa(x_{j_1}^{\circ \alpha} \cdots x_{j_q}^{\circ \alpha} x^{\circ \alpha} \odot x_{j_1}^{\circ \beta} \cdots x_{j_q}^{\circ \beta} \odot x^{\circ \beta}) u_{i_1} \cdots u_{i_q} \overline{w}_{j_1} \cdots \overline{w}_{j_q}. \quad (5.4)$$

We now treat two special cases of power series tensor-kernels built from a power series of one complex variable. Let $(\gamma_k)_{k \in \mathbb{N}} \in \mathbb{R}^N_+$ and suppose that the power series

$$\sum_{k \in \mathbb{N}} \gamma_k \zeta^k \quad (\zeta \in \mathbb{C}) \quad (5.5)$$

has radius of convergence $R_\gamma > 0$ ($R_\gamma = 1/\limsup_k \gamma_k^{1/k} > 0$). We denote by $D(R_\gamma) = \{\zeta \in \mathbb{C} \mid |\zeta| < R_\gamma\}$ and by $\psi: D(R_\gamma) \to \mathbb{R}$ respectively the disk of convergence and the sum of the power series (5.5).

Case 1. We set

$$\gamma_{|\nu|} = \gamma_{|\nu|} \left( \frac{|\nu|!}{\nu!} \right) = \gamma_{|\nu|} \frac{|\nu|!}{\nu_1! \cdots \nu_d!}. \quad (5.6)$$
Then, the domain of absolute convergence of the series \( \sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu \) is the strip

\[
\mathcal{D}_\rho = \left\{ z \in \mathbb{C}^d \left| \sum_{t=1}^d z_t < R_\gamma \right. \right\}
\]

and, it follows from the multinomial theorem [3, Theorem 4.12] that, for every \( z \in \mathcal{D}_\rho \),

\[
\kappa(z) = \sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu = \sum_{k \in \mathbb{N}^d} \gamma_k \sum_{\nu \in \mathbb{N}^d} \frac{k!}{\nu_1! \cdots \nu_d!} z^\nu = \sum_{k \in \mathbb{N}^d} \gamma_k \left( \sum_{t=1}^d z_t \right)^k = \psi \left( \sum_{t=1}^d z_t \right). \tag{5.7}
\]

Note also that \( \mathcal{D}_{\rho_1}^{1/m} = \{ z \in \mathbb{C}^d \mid \|z\|^m_\rho < R_\gamma \} \). Thus, it follows from (5.2) that

\[
K(x'_1, \ldots, x'_q; x''_1, \ldots, x''_q) = \kappa(x'_1 \circ \cdots \circ x'_q \circ x''_1 \circ \cdots \circ x''_q) = \psi \left( \sum_{t=1}^d x'_{1,t} \cdots x'_{q,t} x''_{1,t} \cdots x''_{q,t} \right), \tag{5.8}
\]

for every \( (x'_1, \ldots, x'_q, x''_1, \ldots, x''_q) \in \mathcal{X}^q \times \mathcal{X}^q \). For \( q = 1 \), the right hand side of (5.8) reduces to

\[
K(x', x'') = \psi(\langle x' \mid x'' \rangle) = \sum_{k \in \mathbb{N}} \gamma_k \langle x' \mid x'' \rangle^k,
\]

where \( \langle \cdot \mid \cdot \rangle \) is the Euclidean scalar product in \( \mathbb{R}^d \). These kind of kernels have been also called Taylor kernels in [21]. Thus, in virtue of (5.8), (5.3) takes the form

\[
u \in \mathbb{C}^n \mapsto \left\| \sum_{t=1}^n u_t \Phi(x_t) \right\|_{r^*} = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^q} \left( \begin{array}{c} q \\ \alpha \end{array} \right) \left( \begin{array}{c} q \\ \beta \end{array} \right) \psi \left( \sum_{t=1}^d \overline{x}_{1,t} \cdots \overline{x}_{\alpha,1,t} x_{1,t}^{\beta_1} \cdots x_{\beta_n,t} \right) u^\alpha \overline{u}^\beta,
\]

where we put, for every \( t \in \{1, \ldots, d\} \), \( x_{i,t} = (x_{1,t}, \ldots x_{n,t}) \in \mathbb{C}^n \).\(^5\) The representation formula (5.4) turns to

\[
\langle w, \Phi(x) \rangle_{r^*} = \frac{1}{n^{m-1}} \sum_{\begin{array}{c} i_1, \ldots, i_q = 1 \\ j_1, \ldots, j_{q-1} = 1 \end{array}}^{n} \psi \left( \sum_{t=1}^d x_{i_1,t} \cdots x_{i_q,t} x_{j_1,t} \cdots x_{j_{q-1},t} \right) u_{i_1} \cdots u_{i_q} \overline{u}_{j_1} \cdots \overline{u}_{j_{q-1}}.
\]

**Case 2.** We set

\[
(\forall \nu \in \mathbb{N}^d) \quad \rho_\nu = \prod_{t=1}^d \gamma_{\nu_t} \tag{5.9}
\]

\(^5\) If we consider the matrix of the data \( X = (x_{i,t})_{1 \leq i \leq n} \in \mathbb{C}^{n \times d} \), having the training set \((x_i)_{1 \leq i \leq n}\) as rows, the vectors \( x_{i,t} \) are the columns of \( X \).
Then the domain of absolute convergence of the series \( \sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu \) is
\[
D_\rho = \left\{ z \in \mathbb{C}^d \mid (\forall t \in \{1, \ldots, d\}) |z_t| < R_\gamma \right\}
\]
and
\[
(\forall z \in D_\rho) \quad \kappa(z) = \sum_{\nu \in \mathbb{N}^d} \rho_\nu z^\nu = \sum_{\nu \in \mathbb{N}^d} \prod_{t=1}^d \gamma_{v_t} z_t^\nu = \prod_{t=1}^d \gamma_k z_t^k = \prod_{t=1}^d \psi(z_t).
\]
In this case \( D_\rho^{1/m} = \{ z \in \mathbb{C}^d \mid (\forall t \in \{1, \ldots, d\}) |z_t| < R_\gamma^{1/m} \} \) and (5.2) becomes,
\[
K(x_1', \ldots, x_q'; x_1'', \ldots, x_q'') = \kappa(x_1' \odot \cdots \odot x_q' \odot \overline{x_1''} \odot \cdots \odot \overline{x_q''})
\]
\[
= \prod_{t=1}^d \psi(x_1', \ldots, x_q'; x_1, \ldots, x_q) = 1
\]
for every \( (x_1', \ldots, x_q', x_1'', \ldots, x_q'') \in \mathcal{X}^q \times \mathcal{X}^q \). Thus, as done before, relying on (5.10) we can obtain the corresponding expression for the homogeneous polynomial form (5.3)
\[
\langle u, \Phi(x) \rangle_{r, r^*} = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n} \frac{1}{n^{m-1}} \sum_{i_1, \ldots, i_q=1}^n \prod_{t=1}^d \psi(x_{j_1, t} \cdots x_{j_q, t}) u_{i_1} \cdots u_{i_q} v_{j_1} \cdots v_{j_q-1}.
\]
(5.12)

**Example 5.3.** We list significant examples of power series tensor kernels and for each one we provide the corresponding representation formulas.

(i) In (5.9) set \( (\gamma_k)_{k \in \mathbb{N}} \equiv 1 \), hence \( (\rho_\nu)_{\nu \in \mathbb{N}^d} \equiv 1 \) too. Then \( R_\gamma = 1 \) and \( \psi(\zeta) = 1/(1 - \zeta) \). Therefore, relying on (5.10), we obtain the tensor-Szegö kernel
\[
K(x_1', \ldots, x_q'; x_1'', \ldots, x_q'') = \frac{1}{\prod_{t=1}^d (1 - x_{1, t} \cdots x_{q, t})}. \]
This kernel generates a reproducing kernel Banach space of multi-variable analytic functions [17, 25]
\[
B_{\rho, 0}(\mathcal{X}) = \left\{ f \in \mathbb{C}^X \mid (\exists (c_\nu)_{\nu \in \mathbb{N}^d} \in l^r(\mathbb{N}^d; \mathbb{C}))(\forall x \in \mathcal{X})\left( f(x) = \sum_{\nu \in \mathbb{N}^d} c_\nu x^\nu \right) \right\}
\]
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with norm \( \|f\|_{B_{\rho,b}(\mathcal{X})} = \|c\|_r \), where \((c_{\nu}, \nu \in \mathbb{N}^d) \in \ell^r(\mathbb{N}^d; \mathbb{C})\) is such that \( f = \sum_{\nu \in \mathbb{N}^d} c_{\nu} x^\nu \) (pointwise). This space reduces to the Hardy space when \( r = 2 \). Moreover, (5.11) yields the following homogenous polynomial form

\[
\| \sum_{i=1}^n u_i \Phi(x_i) \|_{r^*} = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n} (q) (q) \prod_{t=1}^d (1 - (x_{t})^\alpha (x_{t})^\beta) \frac{w^\alpha \bar{w}^\beta}{\prod_{t=1}^d (1 - (x_{t})^\alpha (x_{t})^\beta)}.
\]

Finally, in view of (5.12), we have the following tensor-kernel representation

\[
\langle w, \Phi(x) \rangle_{r^*} = \frac{1}{n^{m-1}} \sum_{i_1, \ldots, i_q = 1}^n \frac{u_{i_1} \cdots u_{i_q} \bar{u}_{j_1} \cdots \bar{u}_{j_{q-1}}}{\prod_{t=1}^d (1 - x_{t}^1 \cdots x_{t}^q)}.
\]

(ii) Set \((\gamma_k)_{k \in \mathbb{N}} \equiv ((k + 1)/\pi)_{k \in \mathbb{N}}\) in (5.9). Then \( R_\gamma = 1 \) and \( \psi(\zeta) = 1/(\pi (1 - \zeta)^2) \). We then obtain the following Taylor type tensor kernel

\[
K(x_1^1, \ldots, x_1^q, x_1, \ldots, x_q^q) = \frac{1}{\pi^d \prod_{t=1}^d (1 - x_{t}^1 \cdots x_{t}^q)}.
\]

This kernel gives rise to a reproducing kernel Banach space of analytic functions which reduces to the Bergman space when \( m = 2 \). Proceeding as in the previous point, the expression of the corresponding homogeneous polynomial form and the representation formula can be obtained.

(iii) Let \((\gamma_k)_{k \in \mathbb{N}} = (1/k!)_{k \in \mathbb{N}}\) in (5.9). Then \( R_\gamma = +\infty \) and \( \psi(\zeta) = e^\zeta \). Hence, by (5.10),

\[
K(x_1^1, \ldots, x_1^q, x_1, \ldots, x_q^q) = \prod_{t=1}^d e^{x_{t}^1 \cdots x_{t}^q} = e^{\sum_{\gamma = 1}^d (x_1^\gamma, x_1^\gamma)_{r^*}}
\]

which is the tensor-exponential kernel and the form (5.11) becomes

\[
\| \sum_{i=1}^n u_i \Phi(x_i) \|_{r^*} = \sum_{\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^n} (q) (q) e^{\sum_{\gamma = 1}^d (x_1^\gamma, x_1^\gamma)} \frac{w^\alpha \bar{w}^\beta}{\prod_{t=1}^d (1 - (x_{t})^\alpha (x_{t})^\beta)}.
\]

The corresponding tensor representation is

\[
\langle w, \Phi(x) \rangle_{r^*} = \frac{1}{n^{m-1}} \sum_{i_1, \ldots, i_q = 1}^n \prod_{t=1}^d e^{x_{t}^1 \cdots x_{t}^q}.
\]

(iv) Let \( \alpha > 0 \), set

\[
(\forall k \in \mathbb{N}) \quad \gamma_k = \binom{-\alpha}{k} (-1)^k \prod_{i=1}^k \frac{\alpha + i - 1}{i} > 0,
\]

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and define \((\rho_\nu)_{\nu \in \mathbb{N}^d}\) according to (5.6). Then \(R_\gamma = 1\) and \(\psi(z) = (1 - \zeta)^{-\alpha}\) and formula (5.8) yields the following tensorial version of the binomial kernel [21]

\[
K(x'_1, \ldots, x'_q; x''_1, \ldots, x''_q) = \frac{1}{\left(1 - \sum_{t=1}^d x'_{1,t} \cdots x'_q, x''_{1,t} \cdots x''_q\right)^s}.
\]

(v) Let \(s \in \mathbb{N}\), set

\[
(\forall k \in \mathbb{N}) \quad \gamma_k = \begin{cases} \binom{s}{k} & \text{if } k \leq s \\ 0 & \text{if } k > s, \end{cases}
\]

and define \((\rho_\nu)_{\nu \in \mathbb{N}^d}\) according to (5.6). Then \(R_\gamma = +\infty\) and \(\psi(\zeta) = (1 + \zeta)^s\). This way, by (5.8), we have

\[
K(x'_1, \ldots, x'_q; x''_1, \ldots, x''_q) = \left(1 + \sum_{t=1}^d x'_{1,t} \cdots x'_q, x''_{1,t} \cdots x''_q\right)^s,
\]

which is the polynomial tensor-kernel of order \(s\). By (5.6) we have that \(\rho_\nu > 0\) if \(|\nu| \leq s\) and \(\rho_\nu = 0\) if \(|\nu| > s\). Therefore, recalling (5.1), we have that

\[
B_{\rho,0}^r(\mathcal{X}) = \left\{ f \in \mathbb{C}^\mathcal{X} \left| (\exists (c_\nu)_{\nu \in \mathbb{N}^d} \in \ell^r(\mathbb{N}^d; \mathbb{C})) (\forall x \in \mathcal{X}) \left( f(x) = \sum_{\nu \in \mathbb{N}^d} c_\nu \phi_\nu(x) \right) \right\},
\]

is the space of polynomials in \(d\) variables with coefficients in \(\mathbb{C}\) of degree up to \(s\).

6 Conclusion

In this work we first provided a complete duality theory for support vector regression in Banach function spaces with general regularizers. Then, we specialized the analysis to reproducing kernel Banach spaces that admit a representation in terms of a (countable) dictionary of functions with \(\ell^r\)-summable coefficients and regularization terms of type \(\varphi(\|\cdot\|_r)\), being \(r = m/(m-1)\) and \(m\) an even integer. In this context we showed that the problem of support vector regression can be explicitly solved through the introduction of a new type of kernel of tensorial type (with degree \(m\)) which completely encodes the finite dimensional dual problem as well as the representation of the corresponding infinite dimensional primal solution (the regression function). This can provide a new and effective computational framework for solving support vector regression in Banach space setting. We finally study a whole class of reproducing kernel Banach spaces of analytic functions to which the theory applies and show significant examples which can become useful in applications.

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A Proofs of section 3

proof of Theorem 3.1. The Banach spaces $L^p(P)$ and $L^{p'}(P)$ are put in duality by means of the pairing

$$
\langle \cdot, \cdot \rangle_{p,p'} : L^p(P) \times L^{p'}(P) \to \mathbb{R} : (e, u) \mapsto \int_{X \times Y} e(x,y)u(x,y)\,dP(x,y). \tag{A.1}
$$

In virtue of A3, the following linear operator

$$
A : \mathcal{F} \times \mathbb{R} \to L^p(P) \text{ s.t. } (\forall (w,b) \in \mathcal{F} \times \mathbb{R}) \ A(w,b) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R} : (x,y) \mapsto \langle w, \Phi(x) \rangle + b \tag{A.2}
$$
is well-defined and the function
\[
\text{pr}_2: \mathcal{X} \times \mathcal{Y} \to \mathbb{R}: (x, y) \mapsto y,
\]
is in \(L^p(P)\). Then problem (3.6) can be written in the following constrained form
\[
\min_{(w, b, e) \in \mathcal{F} \times \mathbb{R} \times L^p(P)} \gamma \int_{\mathcal{X} \times \mathcal{Y}} L(e(x, y)) \, dP(x, y) + G(w),
\]
subject to \(\text{pr}_2 - A(w, b) = e\)
\[(A.3)\]
— where, in the constraint, the equality is meant to be in \(L^p(P)\) — and hence \((P)\) follows.

Now, define the following integral functional
\[
R_P: L^p(P) \to \mathbb{R}: e \mapsto \int_{\mathcal{X} \times \mathcal{Y}} L(e(x, y)) \, dP(x, y),
\]
the linear operator
\[
B: \mathcal{F} \times \mathbb{R} \times L^p(P) \to L^p(P): (w, b, e) \mapsto A(w, b) + e,
\]
and the functional
\[
F: \mathcal{F} \times \mathbb{R} \times L^p(P) \to \mathbb{R}: (w, b, e) \mapsto \gamma R_P(e) + G(w).
\]
We note that the functional \(R_P\) is well defined, convex, and continuous. This follows from the convexity and continuity of \(L(\cdot)\) and from the fact that, because of \(A1\), for every \((x, y) \in \mathcal{X} \times \mathcal{Y}\), \(L(e(x, y)) \leq a + b|e(x, y)|^p\). Then, problem \((A.3)\) can be equivalently written as
\[
\min_{(w, b, e) \in \mathcal{F} \times \mathbb{R} \times L^p(P)} F(w, b, e) + \iota_{\{\text{pr}_2\}}(-B(w, b, e)).
\]
(A4)
This form of problem \((3.6)\) is amenable by the Fenchel-Rockafellar duality theory. In view of Fact 2.2 we need only to check that \(0 \in \text{int}\{ -B(\text{dom} F) + \text{dom} \iota_{\{\text{pr}_2\}}\}\). This is almost immediate. Indeed, since \(\text{dom} F = \text{dom} G \times \mathbb{R} \times L^p(P)\), we have
\[
B(\text{dom} F) = \{A(w, b) + e \mid (w, b) \in \text{dom} G \times \mathbb{R} \text{ and } e \in L^p(P)\} = L^p(P).
\]
Now we compute the dual of \((A.4)\). We have
\[
(\forall u \in L^p(P)) \quad (\iota_{\{\text{pr}_2\}})^*(u) = \langle \text{pr}_2, u \rangle_{p,p^*}
\]
and, for every \((w^*, b^*, u) \in \mathcal{F}^* \times \mathbb{R} \times L^p(P)\),
\[
F^*(w^*, b^*, u) = \sup_{(w, b, e) \in \mathcal{F} \times \mathbb{R} \times L^p(P)} \langle (w, b, e), (w^*, b^*, u) \rangle - F(w, b, e)
\]
\[
= \sup_{w \in \mathcal{F}} \sup_{b \in \mathbb{R}} \sup_{e \in L^p(P)} \langle w, w^* \rangle - G(w) + \langle u, e \rangle_{p,p^*} - \gamma R_P(e) + bb^*
\]
\[(A.6)\]
\[
= \begin{cases} 
G^*(w^*) + \gamma R_P^*(u/\gamma) & \text{if } b^* = 0 \\
+\infty & \text{if } b^* \neq 0.
\end{cases}
\]
Moreover, we need also to compute $A^*: L^p(P) \to \mathcal{F} \times \mathbb{R}$ and $B^*: L^p(P) \to \mathcal{F} \times \mathbb{R} \times L^p(P)$. To that purpose, we note that for every $(w, b, e) \in \mathcal{F} \times \mathbb{R} \times L^p(P)$ and every $u \in L^p(P)$,

$$
\langle B(w, b, e), u \rangle_{p,p^*} = \langle A(w, b) + e, u \rangle_{p,p^*} = \langle (w, b), A^* u \rangle + \langle e, u \rangle_{p,p^*}
$$

and

$$
\langle (w, b), A^* u \rangle = \langle A(w, b), u \rangle_{p,p^*} = \int_{X \times Y} (\langle w, \Phi(x) \rangle + b) u(x, y) \, dP(x, y)
$$

which yields

$$
A^* u = \left( \int_{X \times Y} u \Phi \, dP, \int_{X \times Y} u \, dP \right) \quad (A.7)
$$

and

$$
B^* u = \left( A^* u, u \right) = \left( \int_{X \times Y} u \Phi \, dP, \int_{X \times Y} u \, dP, u \right), \quad (A.8)
$$

where, for brevity, we put $\int_{X \times Y} u \Phi \, dP = \int_{X \times Y} u(x, y) \Phi(x) \, dP(x, y)$. Thus, taking into account (A.6),(A.7), and (A.8), we have that, for every $u \in L^p(P)$,

$$
F^*(B^* u) = F^*(A^* u, u) = \begin{cases} 
G^* \left( \int_{X \times Y} u \Phi \, dP \right) + \gamma R^*_P(u/\gamma) & \text{if } \int_{X \times Y} u \, dP = 0 \\
+\infty & \text{otherwise}
\end{cases}
$$

Moreover, it follows from [18, Theorem 21(a)] that the Fenchel conjugate of $R_P$ is still an integral operator, more precisely

$$
(\forall u \in L^p(P)) \quad R^*_P(u/\gamma) = \int_{X \times Y} L^*(u(x, y)/\gamma) \, dP(x, y).
$$

Therefore, recalling (A.5), the final form (D) is obtained. The corresponding optimality conditions for problem (A.4) and its dual (D) are (see Fact 2.1)

$$
B^* u \in \partial F(w, b, e) = \partial G(w) \times \{0\} \times \gamma \partial R(e) \quad \text{and} \quad B(w, b, e) = pr_2. \quad (A.9)
$$

Now, recalling (A.8), conditions (A.9) can be gathered together as follows

$$
\begin{cases} 
\int_{X \times Y} u \Phi \, dP \in \partial G(w) \\
\int_{X \times Y} u \, dP = 0 \\
u/\gamma \in \partial R(e) \\
y - \langle w, \Phi(x) \rangle - b = e(x, y) \quad \text{for } P\text{-a.a. } (x, y) \in X \times Y.
\end{cases}
$$

(A.10)
Thus, subdifferentiating under the integral sign [18, Theorem 21(c)] and recalling (2.1), (3.7) follows.

proof of Corollary 3.4. Let $t > 0$. We first note that, since 0 is the unique minimizer of $\varphi$ and $t > 0$, then $0 \notin \partial \varphi(t)$; moreover, for every $\xi \in \partial \varphi(t)$, we have $\xi t \geq \varphi(t) - \varphi(0) > 0$, hence, $\xi > 0$. Now, if $w = 0$, then (3.11) holds trivially. Suppose that $w \neq 0$. Then, it follows from Fact 2.1 that

$$\partial G(w) = \frac{\partial \varphi(\|w\|)}{\|w\|^{r-1}} J_r(w).$$

Therefore, it follows from the first of (3.7) and (2.1) that

$$\int_{X \times Y} w \Phi \, dP = \frac{\xi}{\|w\|^{r-1}} J_r(w), \quad \xi \in \partial \varphi(\|w\|).$$

Hence, since $\xi > 0$,

$$J_r(w) = \frac{\|w\|^{r-1}}{\xi} \int_{X \times Y} u \Phi \, dP$$

and the statement follows. \qed