Invariant measures of the Milstein method for stochastic differential equations with commutative noise

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Abstract

In this paper, the Milstein method is used to approximate invariant measures of stochastic differential equations with commutative noise. The decay rate of the transition probability kernel generated by the Milstein method to the unique invariant measure of the method is observed to be exponential with respect to the time variable. The convergence rate of the numerical invariant measure to the underlying one is shown to be a one. Numerical simulations are presented to demonstrate the theoretical results.

Key words: the Milstein method, commutative noise, exponential decay, convergence rate of one, numerical invariant measure.

1 Introduction

In this paper, numerical approximations to invariant measures of stochastic differential equations (SDEs) of the Itô type are studied. Briefly speaking, an unique invariant measure exists for the SDE

\[ dx(t) = f(x(t))dt + g(x(t))dB(t), \]

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when the drift and diffusion coefficients, \( f \) and \( g \), satisfy certain conditions.

The explicit solutions to SDEs are hardly found, not to mention the invariant measures. Therefore, the approach that uses the numerical methods to SDEs has been attracting lots of attention recently. In this approach, one uses the numerical methods for SDEs to obtain the invariant measures of the numerical solutions and shows that the numerical one can converge to the underlying one when the step size tends to zero.

For SDEs, authors in [27] investigated the Euler-Maruyama method to approximate the invariant measure, the semi-implicit Euler method was studied in [13], and the stochastic theta method was discussed in [7]. A modified truncated Euler-Maruyama method was considered in [12] for SDEs with both the drift and diffusion coefficients growing superlinearly.

When the Markovian switching is combined with SDEs, authors in [17] and [2] studied the Euler-Maruyama method. The difference between those two works are that in the earlier one [17] all the SDEs in the switching system have invariant measures, but in the later one only some of SDEs in the switching system have invariant measures. Therefore, in the later paper some conditions on the Markovian switching are imposed such that the system as a whole stay at those SDEs with invariant measures longer than those without invariant measures. In a more recent paper [11], the authors investigated the backward Euler-Maruyama method.

All the above works discussed the Euler-type methods, which provide the convergence rate, a half, of the numerical invariant measure to the underlying one (see Theorem 3.2 in [2]).

To our best knowledge, there has been few works on the approximation to invariant measures using the Milstein-type method. Therefore, we will investigate the ability of the Milstein method to approximate the invariant measures of a class of SDEs in this paper. Other higher order methods were also discussed, for example [1, 23, 24], we just mention some of them here and refer the readers to the references therein.

The Milstein method was firstly proposed in [19]. Due to the higher convergence rate in the finite time, different kinds of the Milstein-type methods have been developed in recent years. The balanced Milstein method was introduced in [8]. A family of fully
implicit Milstein methods was developed in [26]. A double-parameter Milstein method that can preserves positivity was constructed in [4]. The tamed Milstein method was studied in [25] employing the taming technique introduced in [6]. The truncated Milstein method was investigated in [3] borrowing the idea of truncating functions proposed in [16]. For the detailed introduction to the Milstein method and other types of numerical methods for SDEs, we refer the readers to the monographs [9, 10, 20, 21].

Although many works have been devoted to the Milstein-type methods, most of them focused on the finite time convergence or the asymptotic behaviour with zero as the attracting point. Few papers have devoted themselves to the asymptotic behaviour in terms of distributions. Therefore, this paper could also be regarded as a complement to the fruitful studies on the Milstein-type methods.

The main contributions of this paper are threefold. Firstly, we observe that the transition probability kernels of the numerical solutions decay exponentially to the unique numerical invariant measure. Such a quick rate is important for finding the numerical invariant measure in relatively short time when the simulations are conducted. Secondly, the numerical invariant measure generated by the Milstein method is shown to be convergent to the invariant measure of the underlying SDE with the rate one. At the end, based on the observations of numerical simulations of SDEs beyond the scope of the theoretical results, we raise an open question on the convergence in distribution.

This paper is constructed as follows. Section 2 contains the mathematical preliminaries. Main Results and their proofs are presented in Section 3. Numerical simulations are displayed in Section 4. Section 5 concludes this paper.

2 Mathematical Preliminaries

In this paper, let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfying the usual conditions that it is right continuous and increasing while \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets. Let \(|\cdot|\) denote the Euclidean norm in \(\mathbb{R}^d\). Let \(\langle \cdot, \cdot \rangle\) denote inner product in \(\mathbb{R}^d\). The transpose of a vector or matrix, \(M\), is denoted by \(M^T\) and the trace norm of a matrix, \(M\), is denoted by \(|M| = \sqrt{\text{trace}(M^TM)}\). The maximum between \(a\) and \(b\)
is denoted by $a \lor b$, and the minimum between $a$ and $b$ is denoted by $a \land b$. Denote the family of all probability measures on $\mathbb{R}^d$ by $\mathcal{P}(\mathbb{R}^d)$.

For any $q \in (0, 2]$, define a metric $d_q(\cdot, \cdot)$ on $\mathbb{R}^d$ by

$$d_q(x, y) = |x - y|^q, \; x, y \in \mathbb{R}^d.$$  

For $q \in (0, 2]$, the Wasserstein distance between $\mu \in \mathcal{P}(\mathbb{R}^d)$ and $\mu' \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$W_q(\mu, \mu') = \inf_{E} \mathbb{E}(d_q(x, y)),$$

where the infimum is taken over all pairs of random variables $x$ and $y$ on $\mathbb{R}_d$ with respect to the laws $\mu$ and $\mu'$.

Let $B(t) = (B^1(t), B^2(t), \ldots, B^m(t))^T$ be an $m$-dimensional Brownian motion. We consider the $d$-dimensional stochastic differential equation of the Itô type

$$dx(t) = f(x(t))dt + \sum_{j=1}^{m} g_j(x(t))dB^j(t) \quad (2.1)$$

with initial value $x(0) = x_0 \in \mathbb{R}^d$, where

$$f : \mathbb{R}^d \to \mathbb{R}^d, \; g_j : \mathbb{R}^d \to \mathbb{R}^d, \; j = 1, 2, \ldots, m,$$

and $x(t) = (x^1(t), x^2(t), \ldots, x^d(t))^T$.

For the simplicity of the notations, we only consider the case of the commutative noise in this paper. Meanwhile, the case of the non-commutative noise is definitely interesting, but requires more careful analysis and more complicated notations. Due to the length of the paper, we will focus on the case of the commutative noise and report the more general case in the future work.

In some of the proofs, we need the more specified notation that $g_j = (g_{1,j}, g_{2,j}, \ldots, g_{d,j})^T$, with $g_{i,j} : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, 2, \ldots, d$ and $j = 1, 2, \ldots, m$.

For $j_1, j_2 = 1, \ldots, m$, define

$$L^{j_1}g_{j_2}(x) = \sum_{i=1}^{d} g_{i,j_1}(x) \frac{\partial g_{j_2}(x)}{\partial x^i}. \quad (2.2)$$

Denote the transition probability kernel induced by the underlying solution, $x(t)$, by $\bar{P}_t(\cdot, \cdot)$, with the notation $\delta_z \bar{P}_t$ emphasizing the initial value $z$. Recall that a probability
measure, $\pi(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, is called an invariant measure of $x(t)$, if

$$\pi(B) = \int_{\mathbb{R}^d} \mathbb{P}_t(x, B)\pi(dx)$$

holds for any $t \geq 0$ and any Borel set $B \subset \mathbb{R}^d$.

The following conditions are imposed on the drift and diffusion coefficients.

**Condition 2.1** Assume that there exists a positive constant $\alpha$ such that for any $x, y \in \mathbb{R}^d$

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq \alpha|x - y|^2.$$ 

**Condition 2.2** Assume that there exists a positive constant $\sigma$ such that for any $x, y \in \mathbb{R}^d$

$$2\langle x - y, f(x) - f(y) \rangle + |g(x) - g(y)|^2 \leq -\sigma|x - y|^2.$$ 

The next two conditions can be derived from Conditions 2.1 and 2.2 but with a little bit complicated coefficients. For the simplicity, we give two new conditions as follows.

**Condition 2.3** Assume that there exist positive constants $a$ and $b$ such that for any $x \in \mathbb{R}^d$

$$|f(x)|^2 \vee |g(x)|^2 \leq a|x|^2 + b.$$ 

**Condition 2.4** Assume that there exist positive constants $\mu$ and $c$ such that for any $x \in \mathbb{R}^d$

$$2\langle x, f(x) \rangle + |g(x)|^2 \leq -\mu|x|^2 + c.$$ 

**Condition 2.5** Assume that there exists a positive constant $\lambda$ such that for any $x \in \mathbb{R}^d$, $j = 1, 2, \ldots, m$ and $l = 1, 2, \ldots, d$

$$\left| \frac{\partial g_j(x)}{\partial x^l} \right| \leq \lambda.$$ 

The Milstein method to the SDE (2.1) is defined by

$$y_{k+1} = y_k + f(y_k)\Delta + \sum_{j=1}^{m} g_j(y_k)\Delta B^j_k + \frac{1}{2} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} L^{j_1} g_{j_2}(y_k)\Delta B^{j_1}_k\Delta B^{j_2}_k - \frac{1}{2} \sum_{j=1}^{m} L^j g_j(y_k)\Delta,$$ 

(2.3)
where $\Delta$ is the time step, $y_0 = x(0)$, and $\Delta B^j_k$ is the Brownian motion increment in the $j$th component, $j = 1, 2, \ldots, m$.

For any $x \in \mathbb{R}^d$ and any Borel set $B \subset \mathbb{R}^d$, define the one-step and the $k$-step transition probability kernels for the numerical solutions, respectively, by

$$P(x, B) := \mathbb{P}(y_1 \in B | y_0 = x) \quad \text{and} \quad P_k(x, B) := \mathbb{P}(y_k \in B | y_0 = x).$$

If $\Pi_\Delta(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ satisfies

$$\Pi_\Delta(B) = \int_{\mathbb{R}^d} P_k(x, B) \Pi_\Delta(dx)$$

for any $t \geq 0$ and any Borel set $B \subset \mathbb{R}^d$, then $\Pi_\Delta(\cdot)$ is called the numerical invariant measure of $y_k$.

# 3 Main Results

This section is divided into two parts. The first part sees the existence and uniqueness of the invariant measure of the Milstein method. The convergence of the numerical invariant measure to the underlying one is presented in the second part.

## 3.1 The existence and uniqueness of the numerical invariant measure

We firstly present our main theorem as follows and the proof is delayed to the end of this subsection.

**Theorem 3.1** Assume that Conditions $2.1$ to $2.5$ hold, then there exists a $\Delta^\# := \Delta^* \wedge \Delta^{**}$ such that for any given $\Delta \in (0, \Delta^\#)$ the numerical solution generated by the Milstein method $\{y_k\}_{k \geq 0}$ has a unique invariant measure $\Pi_\Delta$.

To prove this theorem, we need two ingredients. Briefly speaking, the first one is the second moment boundedness of the numerical solution $y_k$ for $k = 0, 1, 2, \ldots$, and the second one is that two numerical solutions starting from two different initial values will get arbitrary close in the mean square sense when the time variable gets large.
Assume Conditions 2.3, 2.4 and 2.5 hold, then there exists a $\Delta^* \in (0, 1)$ such that for any $\Delta \in (0, \Delta^*)$ the solution generated by the Milstein method (2.3) obeys

$$E|y_k|^2 \leq C_1, \ k = 1, 2, \ldots,$$

where $C_1$ is a constant that does not rely on $k$.

**Proof.** Taking squares and expectations on both sides of (2.3) and applying Conditions 2.4, 2.3 and 2.5 we have

$$E|y_{k+1}|^2 = E\left\{|y_k|^2 + f(y_k)^2 \Delta^2 + \sum_{j=1}^{m} g_j(y_k) \Delta B^j_k \right\}^2 + \frac{1}{4} \sum_{j=1}^{m} \sum_{j_2=1}^{m} L^{j_1} g_{j_2}(y_k) \Delta B^{j_1}_k \Delta B^{j_2}_k

- \sum_{j=1}^{m} L^j g_j(y_k) \Delta^2]^2 + 2 \langle y_k, f(y_k) \Delta \rangle \right\}

\leq E|y_k|^2 + f(y_k)^2 \Delta^2 + \sum_{j=1}^{m} g_j(y_k) \Delta B^j_k \right\}^2 + 2 \langle y_k, f(y_k) \Delta \rangle

+ 2^{m^2-2} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} [L^{j_1} g_{j_2}(y_k) \Delta B^{j_1}_k \Delta B^{j_2}_k]^2 + 2^{m^2-2} \sum_{j=1}^{m} [L^j g_j(y_k) \Delta^2]

\leq E|y_k|^2 + \Delta^2(aE|y_k|^2 + b) + \Delta(-\mu E|y_k|^2 + c)

+ 3 \times 2^{m^2-2} \lambda \Delta^2(aE|y_k|^2 + b) + 2^{m^2-2} \lambda \Delta^2(aE|y_k|^2 + b)

=(1 + \Delta^2 a - \mu \Delta + 3 \times 2^{m^2-2} \lambda \Delta^2 a + 2^{m^2-2} \lambda \Delta^2 a)E|y_k|^2

+ \Delta^2 b + c\Delta + 3 \times 2^{m^2-2} \lambda \Delta^2 b + 2^{m^2-2} \lambda \Delta^2 b,

where the facts that $E(\Delta B^j_k) = 0$ for $j = 1, 2, \ldots, m$, and $E(\Delta B^{j_1}_k \Delta B^{j_2}_k) = \Delta$ if $j_1 = j_2$, $E(\Delta B^{j_1}_k \Delta B^{j_2}_k) = 0$ if $j_1 \neq j_2$ for $j_1, j_2 = 1, 2, \ldots, m$ are used. Let

$$A1 := 1 - \mu \Delta + (a + 3 \times 2^{m^2-2} \lambda a + 2^{m^2-2} \lambda a)\Delta^2,$$

and

$$A2 := c\Delta + (b + 3 \times 2^{m^2-2} \lambda b + 2^{m^2-2} \lambda b)\Delta^2.$$

Due to $\mu > 0$, it is not hard to see that there exists a $\Delta^* \in (0, 1)$ such that $A_1 \in (0, 1)$ and $A_2 > 0$ if $\Delta \in (0, \Delta^*)$. By iteration, we have

$$E|y_{k+1}|^2 \leq A_1 E|y_k|^2 + A_2 \leq A_1^{k+1} E|y_0|^2 + A_2 \frac{1}{1 - A_1} \leq E|y_0|^2 + A_2 \frac{1}{1 - A_1} := C_1$$

This completes the proof. \[\square\]
Lemma 3.3 Let Conditions \ref{2.1} and \ref{2.2} hold. Then there exists a $\Delta^{**} \in (0, 1)$ such that for any $\Delta \in (0, \Delta^{**})$ and any two initial values $x, y \in \mathbb{R}^d$ with $x \neq y$, the solutions generated by the milstein method \ref{2.3} satisfy

$$E|y^x_k - y^y_k|^2 \leq C_2 E|x - y|^2,$$

where $C_2$ depends on $k$ with

$$\lim_{k \to +\infty} \frac{\log C_2}{k} < 0. \quad (3.1)$$

**Proof.** For the simplicity of the notations, we denote $y^x_k$ and $y^y_k$ by $x_k$ and $y_k$, respectively. From \ref{2.3}, we have

$$x_{k+1} - y_{k+1} = x_k - y_k + (f(x_k) - f(y_k))\Delta + \sum_{j=1}^{m} (g_j(x_k) - g_j(y_k))\Delta B^j_k$$

$$+ \frac{1}{2} \sum_{j_1=1}^{m} \sum_{j_2=1}^{m} (L^{j_1} g_{j_2}(x_k) \Delta B^{j_1}_k \Delta B^{j_2}_k - L^{j_1} g_{j_2}(y_k) \Delta B^{j_1}_k \Delta B^{j_2}_k) - \frac{1}{2} \sum_{j=1}^{m} (L^j g_j(x_k)\Delta - L^j g_j(y_k)\Delta). \quad (3.2)$$

Taking squares and expectations on both sides of \ref{2.3} and applying Conditions \ref{2.1} and \ref{2.2} in the similar manner as the proof of Lemma 3.2 we have

$$E|x_{k+1} - y_{k+1}|^2$$

$$\leq E|x_k - y_k|^2 + \Delta^2 \alpha E|x_k - y_k|^2 - \sigma \Delta E|x_k - y_k|^2$$

$$+ 3 \times 2^{m^2-2} \lambda \Delta^2 \alpha E|x_k - y_k|^2 + 2^{m^2-2} \lambda \Delta^2 \alpha E|x_k - y_k|^2$$

$$= (1 + \Delta^2 \alpha - \sigma \Delta + 3 \times 2^{m^2-2} \lambda \Delta^2 \alpha + 2^{m^2-2} \lambda \Delta^2 \alpha) E|x_k - y_k|^2.$$

Denote

$$A_3 := 1 - \sigma \Delta + (\alpha + 3 \times 2^{m^2-2} \lambda \alpha + 2^{m^2-2} \lambda \alpha) \Delta^2.$$

It is not hard see that there exists a $\Delta^{**} \in (0, 1)$ such that $A_3 \in (0, 1)$ for any $\Delta \in (0, \Delta^{**})$. Then by iteration, we have

$$E|x_{k+1} - y_{k+1}|^2 \leq C_2 E|x - y|^2,$$

where $C_2 = A_3^{k+1}$. This completes the proof. □
Remark 3.4 The inequality (3.1) in Lemma 3.3 also indicates that the transition probability kernel decays to the invariant measure in the exponential rate. Example 4.1 in Section 4 demonstrates such an observation (see the left plot in Figure 1).

Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1. For each integer $m \geq 1$ and any Borel set $B \subset \mathbb{R}^d$, define the measure

$$\mu_m(B) = \frac{1}{m} \sum_{k=0}^{m-1} P(y_k \in B).$$

Lemma 3.2 together with the Chebyshev inequality yields that the measure sequence $\{\mu_m\}_{m \geq 1}$ is tight. Then a subsequence that converges to an invariant measure can be extracted. This proves the existence of the numerical invariant measure.

Assume $\Pi_{\Delta,1}$ and $\Pi_{\Delta,2}$ are two different invariant measure of $y_k^x$, then

$$W_q(\Pi_{\Delta,1}, \Pi_{\Delta,2}) = W_q(\Pi_{\Delta,1}P_k, \Pi_{\Delta,2}P_k) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Pi_{\Delta,1}(dx)\Pi_{\Delta,2}(dy)W_q(\delta_xP_k, \delta_yP_k).$$

From Lemma 3.3 we have

$$W_q(\delta_xP_k, \delta_yP_k) \leq (C_2 E|x-y|^2)^{q/2} \to 0, \text{ as } k \to \infty.$$ 

Therefore, we have

$$\lim_{k \to \infty} W_q(\Pi_{\Delta,1}, \Pi_{\Delta,2}) = 0,$$

which indicates the uniqueness of the invariant measure.

3.2 The Convergence of the numerical invariant measure to the underlying one

In this subsection, the main result on the convergence of the numerical invariant measure to the invariant measure of the underlying SDE is presented. We also reveal that by proper choosing the step size and the number of iterations of numerical solutions the convergence rate of one can be obtained.
Theorem 3.5 Given Conditions 2.1 to 2.5 for any given $\Delta \in (0, \Delta^\#)$ there exists a constant $C_3$ such that

$$W_q(\pi, \Pi_\Delta) \leq C_3 \Delta^q,$$

where $q \in (0, 2]$.

Proof. Note that for any $q \in (0, 2]$

$$W_q(\delta_x \bar{P}_{k\Delta}, \pi) \leq \int_{\mathbb{R}^d} \pi(dy) W_q(\delta_x \bar{P}_{k\Delta}, \delta_y \bar{P}_{k\Delta}),$$

and

$$W_q(\delta_x \bar{P}_{k\Delta}, \Pi_\Delta) \leq \int_{\mathbb{R}^d} \Pi_\Delta(dy) W_q(\delta_x \bar{P}_{k\Delta}, \delta_y \bar{P}_{k\Delta}).$$

Due to the existence and uniqueness of the invariant measure for the underlying SDE (2.1) [2] and Theorem 3.1 for the given $\Delta \in (0, \Delta^\#)$, one can choose $k$ sufficiently large such that

$$W_q(\delta_x \bar{P}_{k\Delta}, \pi) \leq \frac{C_3}{3} \Delta^q \quad \text{and} \quad W_q(\delta_x \bar{P}_{k\Delta}, \Pi_\Delta) \leq \frac{C_3}{3} \Delta^q.$$

In addition, for the chosen $k$, it can be derived from [19] that

$$W_q(\delta_x \bar{P}_{k\Delta}, \delta_y \bar{P}_{k\Delta}) \leq \frac{C_3}{3} \Delta^q.$$

Therefore, the proof is completed by the triangle inequality.

4 Numerical simulations

This section is divided into two parts. The first part is devoted to numerical examples of those SDEs with coefficients satisfying the conditions required by the theoretical results in Section 3. The second part sees some simulations for SDEs that are not covered by the theorems in this paper.

It should be mentioned that although those SDEs in the second part are not included in the theoretical results in this paper. The numerical simulations still show that the Milstein method is a good candidate for the approximations to the invariant measures. We think it may be due to the fact that the convergence of the numerical invariant measure is a kind of convergence in the sense of distribution, which is even weaker than
the convergence in probability. Although the weak divergence of the Milstein method can be seen from [5], the weak sense in that paper is in terms of moment. Meanwhile, the convergence in probability of the Milstein method can be obtained from [15].

Therefore, it is still an open question that under what as weak as possible conditions the numerical invariant measure obtained from the Milstein method converges to the underlying invariant measure.

4.1 Examples within the scope of the theorems

We consider an example with the linear drift coefficient. Recall, from Example 3.5.1 in [14], that the underlying solution of (4.1) has the stationary distribution \( N(0, \sigma^2/(2\alpha)) \).

Example 4.1 We write the Itô type equation of the Langevin equation as

\[
dx(t) = -\alpha x(t)dt + \sigma dB(t) \quad \text{on} \quad t \geq 0,
\]

where \( \alpha > 0 \) and \( \sigma \in \mathbb{R} \).

By choosing \( \alpha = 2 \) and \( \sigma = 2 \), we test the numerical invariant measure against the standard normal distribution. The left plot in Figure 1 shows that the difference between the empirical distribution and the real distribution decays exponentially fast with time getting large. The right one shows that the convergence rate of the numerical invariant measure to the underlying invariant measure is one. Here the statistic of the Kolmogorov-Smirnov test (K-S test) [18] is used to measure the distance between two distributions. The K-S test also indicates that after \( t = 2 \), there is no significate difference between the numerical distribution and the underlying one with the 95% confidence.

4.2 Examples beyond the scope of the theorems

In this subsection, an example with super-linear drift coefficient is considered. It should be noted that this example can not be covered by Condition 2.1. In addition, the real invariant measure of this example is known.

Example 4.2

\[
dx(t) = -0.5(x + x^3)dt + dB(t).
\]
The difference between two distributions

Figure 1: Linear case. Left: The difference between the empirical distribution and the real distribution along the time line. Right: Errors against step sizes.

The real density of the stationary distribution of this SDE is known to be \[ p(x) = \frac{1}{I_{\frac{1}{8}}(\frac{1}{8}) + I_{-\frac{1}{8}}(\frac{1}{8})} \exp\left(\frac{1}{8} - \frac{1}{2}x^2 - \frac{1}{4}x^4\right), \]

where \( I_\nu(x) \) is a modified Bessel function of the first kind. Therefore, we compare the numerical results with the real ones. The left plot in Figure 2 shows that the difference between the empirical distribution and the real distribution also decays exponentially fast with time getting large for this super-linear example. In addition, the right plot in Figure 2 indicates the convergence rate in this case is close to one as well.

This example demonstrates that the Milstein method may also be a good candidate to approximate invariant measures for the super-linear SDEs, as we discussed at the beginning of this section.
Figure 2: Non-linear case. Left: The difference between the empirical distribution and the real distribution along the time line. Right: Errors against step sizes.

5 Conclusion

In this paper, the Milstein method is proposed to approximate the invariant measure of a class of SDEs.

The sufficient conditions on the drift and diffusion coefficients are provided to guarantee the convergence of the numerical invariant measure to the underlying one in the Wasserstein distance. The decay rate of the numerical transition probability kernel to the numerical invariant is observed to be exponential. The convergence rate of numerical invariant measure to the underlying one is shown to be one. Some numerical simulations are conducted to demonstrate the theoretical results.

At last, based on the observations of the numerical results, an open question on the
conditions upon the coefficients to guarantee the convergence in distribution is raised.

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