ON SUMS OF FIGURATE NUMBERS BY USING TECHNIQUES OF POSET REPRESENTATION THEORY

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We use representations and differentiation algorithms of posets, in order to obtain results concerning unsolved problems on figurate numbers. In particular, we present criteria for natural numbers which are the sum of three octahedral numbers, three polygonal numbers of positive rank or four cubes with two of them equal. Some identities of the Rogers-Ramanujan type involving this class of numbers are also obtained.

Keywords: cubic number, differentiation algorithm, diophantine equation, octahedral number, partition, polygonal number, poset, tetrahedral number.

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1. Introduction

The representation theory of posets was developed in the 70’s by Nazarova and Roiter [16]. The main aim of their study was to determine indecomposable matrix representations of a given poset \( P \) over a fixed field \( k \).

An important role in poset representation theory is played by some differentiation algorithms, reducing the study of matrix representations of a given dimension type to representations of smaller dimension type. One of the algorithms is the algorithm of differentiation with respect to a pair of points D-I due to Zavadskij which associates to any poset \( P \) with a suitable pair \((a, b)\) of elements \(a, b \in P\) a poset \( P'_{(a,b)} \). The algorithms are successfully applied in determining the representation type (finite or infinite) of posets and in the classification of indecomposable poset representations.

Soon after the discovery of matrix representations of a poset \( P \), Gabriel introduced the concept of a filtered \( k \)-linear representation of \( P \) in connection with the investigation of oriented graphs having finitely many isomorphism classes of indecomposable linear representations. In this case a \( k \)-linear representation is a system \( U = (U_0 \mid U_x \mid x \in P) \), where \( U_0 \) is a finite dimensional \( k \)-vector space and for each \( x \in P \), \( U_x \) is a subspace \( U_x \subseteq U_y \) if \( x \leq y \) in \( P \). These two concepts of poset representations can be connected by defining the associated matrix of a \( k \)-linear representation.

The purpose of this paper is to link the poset representation theory to the additive number theory, by adapting the tools mentioned above in order to solve different types of diophantine equations which involve figurate numbers, where a figurate number is a number that can be represented by a regular geometrical arrangement of equally
spaced points. If the arrangement forms a regular polygon the number is called a polygonal number [2,5].

The \( n \)-th polygonal number of order or rank \( k \), \( p_n^k \), is given by the formula (often, 0 is included as a polygonal number)

\[
p_n^k = \frac{1}{2}[(n - 2)k^2 - (n - 4)k].
\]

Perhaps the most remarkable result concerning polygonal numbers was stated by Fermat as early as 1638, who made the statement that every number is expressible as the sum of at most \( t \) \( t \)-gonal numbers, \( t \geq 3 \) [5,9]. A Fermat’s proof of this fact has not been found yet. Meanwhile for triangular numbers, Gauss proved (1796), that every number is expressible as the sum of three triangular numbers (i.e., every number \( m \) is expressible as the sum of three numbers of the form \( \frac{k(k+1)}{2} \)). Gauss’s statement is equivalent to the statement that every number of the form \( 8m + 3 \) can be expressible as the sum of three odd squares. About this particular fact, Legendre (1798) published first that a number of the form \( 4k(8m + 7), k, m \geq 0 \) cannot be expressible as the sum of three squares and the proof of this result was given by Gauss (1801) in his Disquisitiones Arithmeticae [5].

Euler (1772) stated that at least \( a + 2n - 2 \) terms are necessary to express every number as a sum of figurate numbers

\[
1, n + a, \frac{(n+1)(n+2a)}{12}, \frac{(n+1)(n+2)(n+3a)}{12}, \ldots
\]

Furthermore, Euler proved that if \( n = \frac{k(k+1)}{2} \) is a triangular number then \( 9n + 1, 25n + 3, 49n + 6 \) and \( 81n + 10 \) are also triangular numbers [7]. In fact, for every odd number \( m = 2j + 1 \) the number \( m^2n + \frac{m^2-1}{8} \) is a triangular number and each sum of the form \( n = \frac{k(k+1)}{2} + \frac{k(k+1)}{2} = p_j^3 + p_k^3 \) gives raise to an infinite sequence of this type of sums for an odd number \( m \), in such a way that

\[
m^2n + \frac{m^2-1}{4} = (m^2p_j^3 + \frac{m^2-1}{8}) + (m^2p_k^3 + \frac{m^2-1}{8}).
\]

Moreover, \( n \) is the sum of two triangular numbers exactly when

\[
2(4n + 1) = (2j + 1)^2 + (2k + 1)^2.
\]

That every number is a sum of four squares was proved by Lagrange (1772) with the help of Euler’s work. Finally Cauchy (1813) gave the first proof of the Fermat’s assertion on polygonal numbers [5].

On the number of representations of a number \( n \) as the sum of polygonal numbers, Dirichlet gave a formula for the number of ways in which \( n \) can be expressed as the sum of three triangular numbers, and Jacobi gave the formulas for two or four squares [5,9,10].

Recently, Ewell (1992) stated that the number of representations of \( n \) as the sum of two triangular numbers is \( d_1(4n+1) - d_3(4n+1) \) (where \( d_i(n) \) is the number of divisors of \( n \) which are congruent to \( i \)) [6]. Conway and Schneeberger (1993) proved (the fifteen
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theorem) that if a positive integer-matrix quadratic form represents each of 1, 2, 3, 5, 6, 7, 10, 14, 15, then it represents all positive integers [5]. Bhargava and Hanke (2005) proved the 290-theorem (see [5]) and Farkas (2006) proved that every positive integer can be written as the sum of two squares plus one triangular number and that every positive integer can be written as the sum of two triangular numbers plus one square. Furthermore Farkas gave formulas for the number of ways in which \( n \) can be expressed as the sum of three triangular numbers or three squares (by using theta functions) [8].

Although the Fermat’s theorem has been proved, there are still open problems concerning this theorem and others figurate numbers. In particular, we will investigate the following problems proposed by Guy in [9,10,11]:

1. What theorems are there, stating that all numbers of a suitable shape are expressible as the sum of three (say) squares of numbers of a given shape? For instance, can all sufficiently large numbers be expressed as the sum of three pentagonal (hexagonal, heptagonal) numbers of nonnegative rank? Equivalently, is every sufficiently large number of shape \( 24n + 3 \) \((8n + 3, 40n + 27)\) expressible as the sum of three squares of numbers of shape \( 6r - 1 \) \((4r - 1, 10r + 3)\)?

2. There are theorems giving the number of representations of a number \( n \), as the sum of triangular or square numbers. Can we find corresponding results for any of the other polygonal numbers?

3. The Pollock octahedral numbers conjecture claims that every number is the sum of at most seven octahedral numbers, where the \( n \)-th octahedral number \( O_n \) is given by the formula \( n(2n^2 + 1) \). The Corresponding conjecture for tetrahedral numbers claims that every number is the sum of at most five tetrahedral numbers, where the \( n \)-th tetrahedral number \( \rho_n \) is given by the formula \( \frac{n(n+1)(n+2)}{6} \). In this case Chou and Deng believe that all numbers greater than 343867 are expressible as the sum of four tetrahedral numbers [11].

4. Is every number of the form \( 9n \pm 4 \) the sum of four cubes? Deshouillers, Hennecart, Landreau and Purnaba believe that 7373170279850 is the largest integer which cannot be expressed as the sum of four nonnegative integral cubes. Actually more demanding is to ask if every number is the sum of four cubes with two of them equal [4,11,12,13].

2. Preliminaries

2.1. Posets

An ordered set (or partially ordered set or poset) is an ordered pair of the form \((P, \leq)\) of a set \( P \) and a binary relation \( \leq \) contained in \( P \times P \), called the order (or the partial order) on \( P \), such that \( \leq \) is reflexive, antisymmetric and transitive [3,15]. The elements of \( P \) are called the points of the ordered set. In fact we shall assume in this work that \( P \neq \emptyset \) and there is a bijective map (not necessary order-preserving) from a subset not empty of the set of all positive integers \( \mathbb{N} \), to the set of points of \( P \).
An ordered set $C$ is called a chain (or a totally ordered set or a linearly ordered set) if and only if for all $p, q \in C$ we have $p \leq q$ or $q \leq p$ (i.e., $p$ and $q$ are comparable).

Given an arbitrary point $a \in \mathcal{P}$, we define $a^\uparrow = \{ x \in \mathcal{P} \mid a \leq x \}$, $a_\downarrow = \{ x \in \mathcal{P} \mid x \leq a \}$.

In general for $A \subset \mathcal{P}$, $A^\uparrow = \bigcup_{a \in A} a^\uparrow$, $A_\downarrow = \bigcup_{a \in A} a_\downarrow$.

Let $\mathcal{P}$ be an ordered set and let $x, y \in \mathcal{P}$ we say $x$ is covered by $y$ if $x < y$ and $x \leq z < y$ implies $z = x$.

Let $\mathcal{P}$ be a finite ordered set. We can represent $\mathcal{P}$ by a configuration of circles (representing the elements of $\mathcal{P}$) and interconnecting lines (indicating the covering relation). The construction goes as follows.

1. To each point $x \in \mathcal{P}$, associate a point $p(x)$ of the Euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $p(x)$.
2. For each covering pair $x < y$ in $\mathcal{P}$, take a line segment $l(x, y)$ joining the circle at $p(x)$ to the circle at $p(y)$.
3. Carry out (1) and (2) in such a way that
   (a) if $x < y$, then $p(x)$ is lower than $p(y)$,
   (b) the circle at $p(z)$ does not intersect the line segment $l(x, y)$ if $z \neq x$ and $z \neq y$.

A configuration satisfying (1)-(3) is called a Hasse diagram or diagram of $\mathcal{P}$. In the other direction, a diagram may be used to define a finite ordered set; an example is given below.

![Fig. 1](image)

We have only defined diagrams for finite ordered sets. It is not possible to represent the whole of an infinite ordered set by a diagram, but if its structure is sufficiently regular it can often be suggested diagrammatically.

Let $\mathcal{P}$ be a poset and $S \subset \mathcal{P}$. Then $a \in S$ is a maximal element of $S$ if $a \leq x$ and $x \in S$ imply $a = x$. We denote the set of maximal elements of $S$ by Max $S$. If $S$ (with the order inherited from $\mathcal{P}$) has a top element, $\top_S$ (i.e., $s \leq \top_S$ for all $s \in S$), then Max $S = \{ \top_S \}$; in this case $\top_S$ is called the greatest (or maximum) element of $S$, and we write $\top_S = \max S$. A minimal element of $S \subset \mathcal{P}$ and min $S$, the least (or minimum) element of $S$ (when these exist) are defined dually, that is reversing the order.
Suppose that $\mathcal{P}_1$ and $\mathcal{P}_2$ are (disjoint) ordered sets. The disjoint union $\mathcal{P}_1 + \mathcal{P}_2$ of $\mathcal{P}_1$ and $\mathcal{P}_2$ is the ordered set formed by defining $x \leq y$ in $\mathcal{P}_1 + \mathcal{P}_2$ if and only if either $x, y \in \mathcal{P}_1$ and $x \leq y$ in $\mathcal{P}_1$ or $x, y \in \mathcal{P}_2$ and $x \leq y$ in $\mathcal{P}_2$. A diagram for $\mathcal{P}_1 + \mathcal{P}_2$ is formed by placing side by side diagrams for $\mathcal{P}_1$ and $\mathcal{P}_2$.

### 2.2. Partitions

A partition of a positive integer $n$ is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \ldots, \lambda_r$ such that $\sum_{i=1}^{r} \lambda_i = n$. The $\lambda_i$ are called the parts of the partition [1]. A composition is a partition in which the order of the summands is considered.

Often the partition $\lambda_1, \lambda_2, \ldots, \lambda_r$ will be denoted by $\lambda$ and we sometimes write $\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \ldots)$ where exactly $f_i$ of the $\lambda_j$ are equal to $i$. Note that $\sum_{i \geq 1} i f_i = n$.

The partition function $p(n)$ is the number of partitions of $n$. Clearly $p(n) = 0$ when $n$ is negative and $p(0) = 1$, where the empty sequence forms the only partition of zero. Let $H$ be a set of positive integers, we denote $p(H, n)$ the number of partitions of $n$ that have all their parts in $H$, where $H$ is the set of all partitions whose parts lie in $H$. We let $p(H \leq d)$ denote the set of all partitions in which no part appears more than $d$ times and each part is in $H$.

Thus if $\mathbb{N}$ is the set of all positive integers then $p(\mathbb{N} \leq 1), n) = p(\mathbb{D}, n)$ where $\mathbb{D}$ is the set of all partitions with distinct parts.

Euler stated the following fact. If $\mathbb{H}_0$ is the set of all odd positive integers then $p(\mathbb{H}_0, n) = p(\mathbb{D}, n)$. In order to obtain the proof of this theorem is necessary to consider the generating function $f(q)$ for a sequence $a_0, a_1, a_2, a_3, \ldots$ defined as the power series $f(q) = \sum_{n \geq 0} a_n q^n$. Concerning this result Andrews asks for subsets of positive integers $S, T$ such that $p(S, n) = p(T, n - 1)$ [1].

Given a partition $\lambda = (1^{f_1} 2^{f_2} 3^{f_3} \ldots)$ with parts in a set $H$. In [14] it was defined the derivative of substitution of $\lambda$, $\lambda_i(u) = \frac{\partial \lambda}{\partial u_i}(u)$ in such a way that if $u \in H$ then $\lambda_i(u)$ is a new partition with parts in $H$ such that $\lambda_i(u) = (1^{f_1} 2^{f_2} 3^{f_3} \ldots (i-1)^{f_{i-1}} u_i^{f_i-1} (i+1)^{f_{i+1}} \ldots)$. Furthermore, we can make such a substitution with several parts at the same time. For instance if $u_1, u_2, \ldots, u_k \in H$ then

$$\frac{\partial^k \lambda}{\partial u_1 \partial u_2 \ldots \partial u_k}(u_1, u_2, \ldots, u_k) = (1^{f_1} \ldots u_t^{f_t-1} \ldots m^{f_m}), \quad 1 \leq t \leq k.$$ 

and

$$\frac{\partial^{n} \lambda}{\partial u^n}(u) = (1^{f_1} \ldots (i-1)^{f_{i-1}} u_i^{f_i-n} (i+1)^{f_{i+1}} \ldots m^{f_m}), \quad n \leq f_i.$$ 

In the sequel we will enunciate the famous Rogers- Ramanujan identities.

**Theorem 1.** The partitions of an integer $n$ in which the difference between any two parts is at least 2 are equinumerous with the partitions of $n$ into parts $\equiv 1$ or $4 \mod 5$.

**Theorem 2.** The partitions of an integer $n$ in which each part exceeds 1 and the difference between any two parts is at least 2 are equinumerous with the partitions of $n$ into parts $\equiv 2$ or $3 \mod 5$. 

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One of the main goals of the present paper, is to give some identities of these types for some classes of figurate numbers.

3. Partitions and Representations of posets over $\mathbb{N}$

Let $(\mathbb{N}, \leq)$ be the set of all positive integers endowed with its natural order and $(\mathcal{P}, \leq')$ a poset. A representation of $\mathcal{P}$ over $\mathbb{N}$ [14] is a system of the form

$$\Lambda = (\Lambda_0 ; (n_x, \lambda_x) \mid x \in \mathcal{P}),$$

(1)

where $\Lambda_0 \subset \mathbb{N}$, $\Lambda_0 \neq \emptyset$, $n_x \in \mathbb{N}$, $\lambda_x \in \Lambda_0$ and $|\lambda_x|$ is the size of the partition $\lambda_x$. Further

$$x \leq' y \Rightarrow n_x \leq n_y, \quad |\lambda_x| \leq |\lambda_y|, \quad \text{and} \quad \max\{\lambda_x\} \leq \max\{\lambda_y\}.$$  

(2)

Let $(\mathcal{P}, \leq')$ be a poset and $(a, b)$ be a pair of its incomparable points. If $\mathcal{P} = a + b + C$, where $C = \{c_1 <' c_2 \cdots <' c_n\} \neq \emptyset$ is a chain and $a <' c_1$, $b <' c_1$ then the pair $(a, b)$ is called L-suitable or suitable for differentiation L [14], and the derivative poset of the poset $\mathcal{P}$ with respect to the pair $(a, b)$ is a poset $\mathcal{P}'_{(a,b)} = a^- + \mathcal{P}/(a^+) + C^- + C^+$, where $C^- = \{c_1^- < \cdots < c_n^-\}$, $C^+ = \{c_1^+ < \cdots < c_n^+\}$ are two new chains (replacing the chain $C$) with the relations $a^- < c_1 = a$, $c_i^- < c_i^+$, $1 \leq i \leq n$. It is assumed that each of the points $a^-$, $c_i^-$ ($c_i^+$) inherits all order relations of the point $a$ ($c_i$), with the points of the poset $\mathcal{P}\backslash C$. Fig. 2 provides a Hasse diagram of this differentiation.

In [14], it is assumed that if $\Lambda$ is a representation of a poset $\mathcal{P}$ with a pair $(a, b)$ L-suitable, $t$ fixed, and $k_x \in \mathbb{N}$ for all $x \in \mathcal{P}$, then

$$\lambda_x = ((k_x)^t), \quad \text{for all} \ x \in b^+ + B,$$

$$\frac{\partial^{[\lambda_u]}[\lambda_x]}{\partial k_x^{[\lambda_u]}}(\lambda_u) = ((I_x)(J_x)), \quad \text{if} \ x \in C,$$

$$I_x = \lambda_u,$$

$$|\lambda_u| < |\lambda_{c_1}|,$$

$$0 < n_{c_1} - n_u - |J_{c_1}|c_{c_1} < n_u,$$

$$\alpha = n_{c_1} - n_u - |J_{c_1}|c_{c_1} \leq \max\{\lambda_u\}.$$  

(3)
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The representation $\Lambda'$ of the derived poset $\mathcal{P}'_{a,b}$ is given by the following formulas, $(2 \leq j \leq i)$,

\[
\Lambda_0' = \Lambda_0,
\]

\[
n_{c_{-}}' = \alpha, \quad \lambda_{c_{-}}' = ((\alpha)^{1}),
\]

\[
n_{c_{i}}' = n_{a}, \quad \lambda_{c_{i}}' = \lambda_{a},
\]

\[
n_{e_{i}}' = |I_{c_{i}}|k_{c_{i}}, \quad \lambda_{e_{i}}' = ((k_{c_{i}})^{|I_{c_{i}}|}),
\]

\[
n_{d_{i}}' = n_{c_{i}}, \quad \lambda_{d_{i}}' = ((\alpha)^{1}\lambda_{c_{i}}'(|I_{c_{i}}|)(k_{c_{i}} - k_{c_{i-1}}))^{i_{c_{i}}}|I_{c_{i}}|^{\lambda_{c_{i}}'}),
\]

\[
(n_{x}', \lambda_{x}')(h_{x}, (h_{x}^{1})), \quad \text{for all } x \in b_{b}, \text{ if } B^{x} \cap \{a^{-}\} \neq \emptyset,
\]

\[
h_{x} = \min(\alpha, \min\{n_{a} - n_{x} \mid x \in B\}),
\]

\[
(n_{x}', \lambda_{x}')(n_{x}, \lambda_{x}), \quad \text{for all } x \in b_{b}, \text{ if } B^{x} \cap \{a^{-}\} = \emptyset.
\]

\[
\delta_{i} = \begin{cases} 1, & \text{if } i \geq 2, \\ 0, & \text{otherwise}. \end{cases}
\]

(if $i = 1$ then the formula for $\lambda_{c_{i}}'$ is given by $\lambda_{c_{i}}'$ by deleting the part corresponding to $|I_{c_{i}}|(k_{c_{i}} - k_{c_{i-1}})$, $n_{c_{i}}' = n_{c_{i}}$).

Let $\mathcal{P}_{c_{i}}$ be the family of posets with a Hasse diagram of the form

\[
\begin{array}{c}
\text{Fig. 3}
\end{array}
\]

We represent $\mathcal{P}_{c_{i}}$ over $\mathbb{N}$ in such a way that $\Lambda_{0} \supset \{t \in \mathbb{N} \mid t = p_{i}^{n_{0}}\}$ (i.e., $\Lambda_{0}$ contains the set of the $n_{0}$-gonal numbers with $n_{0} \geq 5$, fixed).

\[
(n_{b_{1}}, \lambda_{b_{1}}) = (n_{0}p_{1}^{n_{0}}, ((p_{1}^{n_{0}})^{n_{0}})),
\]

\[
(n_{b_{2}}, \lambda_{b_{2}}) = (n_{0}p_{2}^{n_{0}}, ((p_{2}^{n_{0}})^{n_{0}})),
\]

\[
(n_{a_{1}}, \lambda_{a_{1}}) = (p_{2}^{n_{0}} + p_{3}^{n_{0}}, ((p_{2}^{n_{0}})^{1}(p_{3}^{n_{0}})^{1})),
\]

\[
(n_{c_{i}}, \lambda_{c_{i}}) = (n_{0}p_{i+2}^{n_{0}}, ((p_{i+2}^{n_{0}})^{n_{0}})), \quad i \geq 1.
\]

(4)

For the corresponding representation of the poset $\mathcal{P}_{c_{i}}$, $2 \leq i \leq n$, we choose the same $\Lambda_{0}, n_{a_{i}} = p_{i+2}^{n_{0}} + p_{i}^{n_{0}}, \lambda_{a_{i}} = ((p_{i+2}^{n_{0}})^{1}(p_{i}^{n_{0}})^{1}), \quad 2 \leq i < i + 2$, and leave the formulas (4) without changes for each $x \in b_{b_{i}}$. 

The next results were obtained in [14], with the help of the differentiation L as well as of the representations of the posets \( P_{\overline{c}} \) over \( N \) given above.

**Theorem 3.** If \( p_0^3 = p_{-1}^3 = 0, \ p_1^3 = 1, \) and \( i,j \geq 1 \) then the entries of the matrices

\[
R = (r_{j,i}), \quad S = (s_{j,i}) \quad \text{and} \quad T = (t_{j,i})
\]

satisfy the following identities

\[
\begin{align*}
24(r_{j,i} - 2p_{j+1}^3) + 3 &= (6i + 5)^2 + (6j + 11)^2 + (12j + 29)^2, \\
8(s_{j,i} - 3p_{j+1}^3) + 3 &= (4i + 3)^2 + (4j + 7)^2 + (8j + 19)^2, \\
40(t_{j,i} - 4p_{j+1}^3) + 27 &= (10i + 7)^2 + (10j + 17)^2 + (20j + 47)^2
\end{align*}
\]

where

\[
\begin{align*}
r_{j,i} &= 97 + 57(j - 1) + 15p_{j-2}^3 + 9p_{j-2}^3 + 21(i - 1), \\
s_{j,i} &= 130 + 75(j - 1) + 20p_{j-2}^3 + 16p_{j-2}^3 + 36(i - 1), \\
t_{j,i} &= 165 + 93(j - 1) + 25p_{j-2}^3 + 25p_{j-2}^3 + 55(i - 1)
\end{align*}
\]

**Theorem 4.** If \( i,j \geq 1 \) then \( x = -(4j + 8), \ y = -(2j + 4), \) and \( z = 2j + i + 8 \) is a solution of the diophantine equation

\[ x^3 + y^3 + 2z^3 = m_{j,i} - n_{j,i}, \]

where

\[ m_{j,i} = 508 + (j + 1)891 + 690p_{j-3} + (507 + 306j + 72p_{j-1}^3)(i + 1) + (144 + 36j)p_{j}^3 + 18p_{j-1}^3 + 198p_{j-2}^3, \]

and

\[ n_{j,i} = 4068 + 4193(j - 1) + 1167(j - 2)(j - 1) + 89(j - 3)(j - 2)(j - 1) + (397 + 150(j - 1) + 12(j - 2)(j - 1)(i - 1) + 6(5 + j)(i - 1) + (i - 3)(i - 2)(i - 1). \]

That is,

\[
M = \begin{bmatrix}
4786 & 5977 & 7384 & \cdots \\
8047 & 9688 & 11581 & \cdots \\
12538 & 14701 & 17152 & \cdots \\
18457 & 21214 & 24295 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}, \quad N = \begin{bmatrix}
4068 & 4465 & 4934 & \cdots \\
8261 & 8808 & 9439 & \cdots \\
14788 & 15509 & 16326 & \cdots \\
24183 & 25102 & 26129 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{bmatrix}
\]

**Proof.** If \( q_k \) is the \( k \)-th positive cube, each term \( m_{j,i}, \ n_{j,i} \) can be obtained by making the substitutions

\[
\frac{\partial^2 \lambda_k}{\partial q_{k+2}^2 \partial q_{k+2}} (q_i, q_{j+1}), \quad \frac{\partial^2 \lambda_{ij}}{\partial q_{j+1+i}^2 \partial q_{j+1+i}^2} (q_{j+1}, q_{j+1+i}). \quad s \geq 4, \ 2s - 1 \leq k,
\]

where \( \lambda_k, \lambda_{ij} \) are partitions such that

\[
\begin{align*}
\lambda_k &= (((q_{k+2})^3), \\
\lambda_{ij} &= (((q_{j+1})^3)(q_{j+1+i})^3).
\end{align*}
\]

**Corollary 5.** If \( i,j \geq 1 \), then \( x = j + 3, \ y = 2j + 7 \) and \( z = 2j + i + 8 \) is a solution of the diophantine equation

\[ x^3 + y^3 + 2z^3 = m_{j,i} - q_{j+i+1}. \]
Corollary 6. If \( i, j \equiv 2 \pmod{3}, h \equiv 1 \pmod{3}, k = 6 + 45l, m = 2 + 9l, h, i, j, l \geq 1 \) then

\[
\begin{align*}
    m(j,i) - q_{2j+i+8} &\equiv -4 \pmod{9}, \\
    n(j,i) - q_{4j+8} &\equiv -4 \pmod{9}, \\
    m(j,i) - 2q_{2j+i+8} + q_{2j+4} &\equiv -4 \pmod{9}, \\
    m(j,k) - 3q_{2j+h+8} + 2q_{2j+h+k} &\equiv -4 \pmod{9}, \\
    m(j,i) - 3q_{2j+i+8} + 2q_{2j+i+m} &\equiv -4 \pmod{9}
\end{align*}
\]

and each number \( m(j,i) - q_{2j+i+8}, n(j,i) - q_{4j+8}, m(j,i) - 2q_{2j+i+8} + q_{2j+4} \) and \( m(j,i) - 3q_{2j+i+8} + 2q_{2j+i+n}, 8 \neq n \geq 1 \) is the sum of four positive cubes.

3.1. The associated graph

Given a representation \( \Lambda \), sometimes it is possible to associate it a suitable graph, having as set of vertices the points of \( P \), and containing all information about parts and partitions of the numbers \( n_x \). In this case we must attach to each vertex of the graph, either a number \( n_x \) given by the representation or one part of a partition of some \( n_y \) representing some \( y \in P \) such that \( x \leq y \). The theorem \( \square \) given more ahead shows the strength of this concept. To see this we must consider the (infinite) poset \( M \) with Hasse diagram of the form

\[ \text{Fig. 4} \]

and a representation \( \Delta \) of \( M \) over \( \mathbb{N} \) such that, \( \Delta_0 \) contains all sum of the form \( O_l + O_j + O_k \) (where \( O_l \) is the \( l \)-th octahedral number) and for \( i, j \geq 1 \), the pair \((n_{ij}, \lambda_{ij}) = (O_{ij} k_0, (O_i)^1 (O_j)^1 (O_{k_0})^1)\) represents \( v_{ij} \in P \), where \( O_{ij} k_0 = O_i + O_j + O_{k_0} \) and \( k_0 \geq 1 \) is a fixed number. It is concluded the theorem \( \square \) due to that \( \Delta \) induces the following partitions.
If \( n = O_{ijk} \), then we note \( P^{ijk}_O(n) \) the number of partitions (of type \( O \)) of \( n \) with the form

\[
n = O_{rsk} + z_1 + \cdots + z_t, \tag{10}
\]

where if \( z_{2j-1} \neq 0 \) for \( 1 \leq j < \frac{t+1}{2} \) then \( z_{2j} \neq 0, z_t \neq 0 \), and

\[
z_{2j-1} = p^5_{i_j}, \quad z_{2j} = p^5_{k(j+1)}, \quad |\{z_m \mid 1 \leq m \leq t\}| \leq 2(j + i - 2)
\]

for some \( k_j \). Further, for each partial sum of the form

\[
O_{rsk} + z_1 + \cdots + z_h, \quad z_{h-1} = p^5_{k(h-1)}, \quad z_h = p^5_{k(h-1)+1}, \quad h \leq t
\]

there exists \( O_{rsk} \leq O_{pqk} \leq O_{ijk} \) satisfying the condition

\[
O_{pqk} = O_{rsk} + z_1 + \cdots + z_h, \quad r \leq p \leq i, \quad s \leq q \leq j.
\]

We denote now, \( P^{ijk+\sigma}_S(n) \) the number of partitions (of type \( \sigma \)) of \( n = O_{ijk} \) with the shape

\[
n = P^5_{rsk} + \sigma_k(i-1) + y_1 + \cdots + y_t, \tag{11}
\]

where \( P^5_{ijk} = p^5_i + p^5_j + p^5_k \), (i.e., is a sum of three pentagonal numbers)

\[
\sigma_k(i) = \sum_{m=0}^{k-1} p^3_{2m-1} + 12i + 18p^3_{i-3} - 10, \quad p^3_k = 0 \text{ if } k \leq 0.
\]

If \( y_j \neq 0 \) then \( y_j = 1 + 3\nu \), for some \( \nu \geq 1 \), \( |\{y_j \mid 1 \leq j \leq t\}| \leq 2(i-1) \), and for each partial sum of the form \( P^5_{rsk} + \sigma_k(i-1) + y_1 + \cdots + y_h \), there exists \( P^5_{rsk} \leq P^5_{pqk} \leq P^5_{ijk} \) satisfying the condition

\[
P^5_{pqk} + \sigma_k(i-1) = P^5_{rsk} + \sigma_k(i-1) + y_1 + \cdots + y_h.
\]

For example the partition \( 3 + 1^2 + 2^2 + 2^2 + 3^2 + 1^2 + 2^2 + 2^2 + 3^2 \) is a partition of type \( O \) for \( 39 = O_{331} \), and \( 3 + 14 + 4 + 7 + 4 + 7 \) is a partition of type \( \sigma \).

**Theorem 6.** \( P^{ijk}_O(n) = P^{ijk+\sigma}_S(n) \).

**Proof.** If \( M \) is the poset described above represented either with three octahedral or three pentagonal numbers, and \( \Gamma_M \) is an associated graph with the shape

\[
\begin{array}{c}
v_{11} \circ \circ v_{22} \circ \circ v_{33} \circ \circ v_{44} \circ \circ \cdots \\
\downarrow \nearrow \downarrow \nearrow \downarrow \nearrow \\
v_{21} \circ \circ \circ \circ \\
\Gamma_M = v_{31} \downarrow \nearrow \downarrow \nearrow \cdots \\
v_{41} \circ \circ \circ \\
\end{array}
\]
On sums of figurate numbers...  

then the conclusion is obtained taking into account that $P_{O}^{ijk}(n)$ is the number of paths in $\Gamma_{M}$, with initial vertex of the form $v_{rs}$, $r \leq i$, $s \leq i$, and final vertex $v_{ni}$ (counting $v_{ni}$ as a path). This fact, because any partition of type $O$ ($\sigma$) can be obtained of a path $P = v_{rs}v_{ni} \in \Gamma_{M}$, by attaching the number $O_{rsk}(p_{ik}^{r} + \sigma_{k}(i-1))$ to the vertex $v_{rs}$ and a suitable number of the form $z_j = p_{kj}^{i} (y_j = 1 + 3\nu)$ to the rest of the vertices $v_{ij} \in P$.

The proof of the corollary given below, can be obtained fixing $q_k$ the $k$-th positive cube and attaching to each $v_{ij} \in M$ the pair $(n_{ij}, \lambda_{ij}) = (O_{ijk}, (q_k)^{1}(q_k)^{1}(q_k)^{2})$, where $O_{ijk} = q_i + q_i + 2q_k$.

**Corollary 7.** $P_{O}^{ijk}(n) = P_{O}^{ijk+\tau}(n)$. \qed

Where $P_{O}^{ijk}(n)$ is the number of partitions (of type $O$) of $n = O_{ijk}$ with the form,

$$n = O_{rsk} + y_{a_1} + \cdots + y_{a_t}, \quad a_i < a_{i+1}, \quad t \geq 1. \quad (12)$$

If $y_{a_1} \neq 0$ then $y_{a_1} = \frac{p_{5k+1}^{2} + 1}{2}$, for some $\nu \geq 1$, \quad $|\{y_{a_j} \mid 1 \leq j \leq t\}| \leq 2(i-1)$, and for each partial sum of the form $O_{rsk} + y_{a_1} + \cdots + y_{a_t}$, there exists $O_{rsk} \leq O_{pqk} \leq O_{ijk}$ satisfying the condition

$$O_{pqk} = O_{rsk} + y_{a_1} + \cdots + y_{a_t}, \quad r \leq p \leq i, \quad s \leq q \leq i.$$

$\tau_k(i) = p_{2k+1}^{4}(i-1) + 4p_{2k+1}^{2} + 4p_{2k+1}^{3}$, and $P_{O}^{ijk+\tau}(n)$ is the number of partitions of type $\tau$ for $n = O_{ijk}$ (i.e., partitions of type $O$ for $O_{ijk}$ with the additional term $\tau_k(i)$ such that the sum $n = O_{ijk} = O_{rsk} + \tau_k(i) + z_1 + \cdots + z_t$ is defined as the sum $p_{5k+1}^{2} + \sigma_{k}(i-1) + y_1 + \cdots + y_t$ of three octahedral numbers).

For example $4 + \frac{p_{5k+2}^{2}}{2} + \frac{p_{5k+3}^{2}}{2} + \frac{p_{5k+2}^{2}}{2} + \frac{p_{5k+3}^{2}}{2} = 4 + 7 + 19 + 7 + 19$ is a partition of type $O$ for $Q_{331} = 56$ and $3 + 17 + (1^2 + 2^2) + (2^2 + 3^2) + (1^2 + 2^2) + (2^2 + 3^2)$ is a partition of type $\tau$.

The following list shows the partitions of type $O$ for $39$.

39,

\begin{itemize}
  \item $13 + 2^2 + 3^2 + 2^2 + 3^2,$
  \item $26 + 2^2 + 3^2,$
  \item $21 + 2^2 + 2^2 + 2^2 + 3^2,$
  \item $8 + 2^2 + 3^2 + 1^2 + 2^2 + 2^2 + 3^2,$
  \item $8 + 1^2 + 2^2 + 2^2 + 3^2 + 2^2 + 3^2,$
  \item $3 + 1^2 + 2^2 + 1^2 + 2^2 + 2^2 + 3^2 + 2^2,$
  \item $3 + 1^2 + 2^2 + 2^2 + 3^2 + 1^2 + 2^2 + 2^2 + 3^2.$
\end{itemize}
If $\xi_k(i, j) = p_{i-1}^k + \sum_{m=j}^{i} p_m^k + \sum_{h=j}^{i} p_h^k$, $j \leq i$, $k \geq 1$ fixed, then a partition of type $\xi$ for $n = O_{ijk}$, it is obtained of a partition $\sigma$ for $O_{iik}$, by replacing in (11) the term $\xi_k(i, j)$ for $\sigma_k(i-1)$ and leaving the others terms and conditions without changes, taking into account that in this case $|\{y_s : 1 \leq s \leq t\}| \leq (j + i - 2)$, and $r \leq p \leq i$, $s \leq q \leq j$. We note $P_{ijk}^{\xi+k}(n)$ the number of partitions of type $\xi$ for $n = O_{ijk}$.

**Theorem 8.** $P_{ijk}^{\xi+k}(n) = P_{iik}^{\xi+k}(n)$.

**Proof.** It is enough to observe the number of paths in $\Gamma_M$ (see theorem (6)), with initial vertex $v_{rs}$ and final vertex $v_{ij}$, $r \leq i$, $s \leq j$.

Given a number with the shape $O_{(i+s)jk}$, $s \geq 0$, now we can dedicate our efforts in finding a formula for its partitions of type $\Theta$ (For a partition of type $O$, we assume $|\{z_m : 1 \leq m \leq t\}| \leq 2(2j + s - 2)$). To do this, we give the following two definitions.

Let $\Lambda$ be a representation over $\mathbb{N}$ for a poset $P$, then the number

$$W_\Lambda(P) = \sum_{x \in P} n_x,$$

is the weight of the representation $\Lambda$. For example, if $N_i$ is a poset with Hasse diagram of the form

![Diagram](Fig. 5)

and representation over $\mathbb{N}$, $\Lambda$ given by the formulas

$$(n_{c_j}, \lambda_{c_j}) = (j, (j)^1), \text{ if } 1 \leq j \leq i,$$

$$(n_{c_{i+1}}, \lambda_{c_{i+1}}) = (i, (i)^1),$$

$$(n_{d_1}, \lambda_{d_1}) = (i, (i)^1)$$

then $W_\Lambda(N_i) = p_i^3 + 2i$.

If $X_{ij} = P_{ijk}^{\xi+k}(n)$ is the number of partitions of type $\Theta$ for $O_{ijk}$, where $k$ is a fixed number and $j \geq 2$ then we write
Proof. The following list presents the values of $P_{ij}^{(j+i-1)k}(n)$, where

\[ P_{ij}^{(j+i-1)k}(n) = \sum_{h=0}^{j} b_h(s) \delta(h+j)s, \]

where $\delta_{ij} = X_{ij}$ if $j \geq 2$.

For example, $\delta_{22} = 3$, $\delta_{33} = 8$, $\delta_{44} = 22$.

It is easy to see that if $j = 1$, $s \geq 0$, then $X_{(s+1)1} = s + 1$. While for $j \geq 2$ we have the next result:

**Corollary 9.** If $s \geq 1$ then $P_{ij}^{(j+s-1)k}(n) = \sum_{h=0}^{j} b_h(s) \delta(h+j)s$, where

\[ b_0(s) = 1, b_1(s) = s - 1, \text{ and } b_h(s) = p_{s-h} if j = 2. \]

\[ b_0(s) = 1, b_1(s) = s - 1, b_h(s) = p_{s-h}^3 if j = 3. \]

**Proof.** If $\Delta$ is the representation over $\mathbb{N}$ for the poset $M$, and $\Gamma_M$ is the corresponding associated graph as before, then the relations

\[ P_{ij}^{(j+s-1)k}(n) = P_{ij}^{(j+s-1)k}(n) + P_{ij}^{(j+i)(j-1)k}(n) + 1, \quad s \geq 0, \]

\[ P_{ij}^{(j+s-2)k}(n) = W_{ij}(N_{s+1}), \]

\[ P_{ij}^{(j+s-1)k}(n) = P_{ij}^{(j)(j-1)k}(n) + 1 \]

observed in the representation of $\Gamma_M$, allow us to obtain both the terms $X_{(j+s)j}$, $0 \leq s \leq j + 2$, and the finite sequence $d_0 = X_{jj}$, $d_s = \{X_{(j+s+1)j} - X_{(j+s)j} \mid 0 \leq s \leq j + 1\}$ which give raise to the terms $\delta_{ij}^{(j+s)j}$, $b_h(s)$ of $P_{ij}^{(j+s)k}(n) = X_{(j+s)j}$.

**Remark 10.** $P_{ij}^{(j+s-2)k}(n) = W_{ij}(N_{s+1}) = 3 + 4s + p_{s-1}^3, \quad s \geq 0.$

The following list presents the values of $P_{ij}^{(j+k)}(n)$, for $2 \leq i \leq 11, 2 \leq j \leq 5$ and a fixed index $k$.

\[ P_{i2}^{(j+k)}(n) = 3, \quad P_{i3}^{(j+k)}(n) = 8, \quad P_{i4}^{(j+k)}(n) = 22, \quad P_{i5}^{(j+k)}(n) = 64. \]

\[ P_{i2}^{(j+k)}(n) = 7, \quad P_{i3}^{(j+k)}(n) = 21, \quad P_{i4}^{(j+k)}(n) = 63, \quad P_{i5}^{(j+k)}(n) = 195. \]

\[ P_{i2}^{(j+k)}(n) = 12, \quad P_{i3}^{(j+k)}(n) = 40, \quad P_{i4}^{(j+k)}(n) = 130, \quad P_{i5}^{(j+k)}(n) = 427. \]

\[ P_{i2}^{(j+k)}(n) = 18, \quad P_{i3}^{(j+k)}(n) = 66, \quad P_{i4}^{(j+k)}(n) = 231, \quad P_{i5}^{(j+k)}(n) = 803. \]

\[ P_{i2}^{(j+k)}(n) = 25, \quad P_{i3}^{(j+k)}(n) = 100, \quad P_{i4}^{(j+k)}(n) = 375, \quad P_{i5}^{(j+k)}(n) = 1376. \]

\[ P_{i2}^{(j+k)}(n) = 33, \quad P_{i3}^{(j+k)}(n) = 143, \quad P_{i4}^{(j+k)}(n) = 572, \quad P_{i5}^{(j+k)}(n) = 2210. \]

\[ P_{i2}^{(j+k)}(n) = 42, \quad P_{i3}^{(j+k)}(n) = 196, \quad P_{i4}^{(j+k)}(n) = 833, \quad P_{i5}^{(j+k)}(n) = 3381. \]

\[ P_{i2}^{(j+k)}(n) = 52, \quad P_{i3}^{(j+k)}(n) = 260, \quad P_{i4}^{(j+k)}(n) = 1170. \]
Now, we consider an infinite sum of infinite chains pairwise incomparable $R$ in such a way that $R = \sum_{i=0}^{\infty} C_i$, where $C_j$ is a chain such that $C_j = v_{0j} < v_{1j} < v_{2j} < \ldots$.

It is defined a representation over $N$ for $R$, by fixing a number $n \geq 3$ and assigning to each $v_{ij}$ the pair $(n_{ij}, \lambda_{ij}) = (3 + (n - 2)i + (n - 1)j, 3 + (n - 2)i + (n - 1)j)$, we note $R_n$ this representation, and write $v_{ij} \in R_n$ whenever it is assigned the number $n_{ij} = 3 + (n - 2)i + (n - 1)j$ to the point $v_{ij} \in R$ in this representation. Fig. 6 below suggests the Hasse diagram for this poset with its associated graph $\Gamma_{p_n^k}$ which attaches to each vertex $v_{ij}$ the number $3 + (n - 2)i + (n - 1)j$.

The representations of $R$ and $\Gamma_{p_n^k}$, allow us to observe that for $n_0$ fixed, each natural number $n \geq n_0$ represents at least one point in $R$, and that the numbers $n_{p_{i-1}^3}$
representing the vertices \( v_{p^3_{(i-1)i}} \) \( \in \mathcal{R} \), of the left boundary path, \( l.b.p \) have the form \( n_{p^3_{i-1}} = p^1_i + p^1_i + p^1_{i+1} \), \( i \) is a fixed index. Furthermore if \( i_0 \geq 0 \) is a fixed number, and \( v_{p^2_{(i-1)i_0}} \in l.b.p \) then \( p^k_{i_0+1} + p^k_{i+1} + p^k_1 \), represents the vertex\(^*\) \( v_{(p^2_{i_0-1}+p^2_{i-1})/(i_0+k)} \) in the path \( P = v_{p^2_{(i-1)i_0}} \| | (v_{(p^2_{i_0-1}+p^2_{i-1})/i_0+j}), 0 \leq k \leq j \), and \( p^k_{i_0+1} + p^k_{j+1} + p^k_1 \), represents the vertex \( v_{(p^2_{i_0-1}+p^2_{j-1}+p^2_{j-1}+p^2_{j-1})/(i_0+j+l)} \in P' = (v_{(p^2_{i_0-1}+p^2_{j-1})/i_0+j}) \| | (v_{(p^2_{i_0-1}+2p^2_{j-1})/(i_0+2)}), \) \( 0 \leq l \leq j \) (note that the vertices of the form \( v_{((2+7i+3k)(2i+k)+3)}, i, k \geq 0 \), do not lie on any non-trivial path of \( \Gamma_{p^n_k} \)). These facts prove the following theorem:

**Theorem 11.** A number \( m \in \mathbb{N} \) is the sum of three \( n \)-gonal numbers of positive rank if and only if \( m \) represents a vertex \( v_{ij} \in \mathcal{R}_n \) in a non-trivial component of \( \Gamma_{p^n_k} \).

\[
\Gamma_{p^n_k} \rightarrow \mathcal{R}.
\]

\[
v_{ij} \rightarrow 3 + 3i + 4j
\]

In the example given above (Fig. 7) each \( v_{ij} \) \( \in \mathcal{R} \). That is, \( n_{ij} = 3 + 3i + 4j \), \( i, j \geq 0 \), and the number associated to each vertex in a non-trivial component of \( \Gamma_{p^n_k} \), is the sum of three pentagonal numbers of positive rank.

\(^*\)\(v_{rs} \in P; (P') if and only if there exists k, (l_0), 0 \leq k \leq j, (0 \leq l_0 \leq j) such that \( v_{rs} = v_{(p^2_{i_0-1}+p^2_{j-1})/(i_0+k)} \| | (v_{(p^2_{i_0-1}+p^2_{j-1}+p^2_{j-1})/(i_0+j+l_0)}).\)
Corollary 12. The number \( m \) representing \( v_{ij} \in \mathcal{R}_3 \) is the sum of three triangular numbers \( \geq 1 \) if and only if the number \( n \) representing \( v_{ij} \in \mathcal{R}_4 \) is the sum of three \( t \)-gonal numbers of positive rank.

Note that, the structure of the l.b.p gives the general form of the graph \( \Gamma_{p_0^k} \). Thus, it is enough to change the form of such left boundary path, to build different graphs of this type. For instance, we note \( \Gamma_0 \) a graph associated to \( \mathcal{R} \) which \( v_{00}, v_{01}, v_{(p_0^{[2i-1]}+1)(i+1)} \), \( i \geq 1 \) are the locations of the vertices of its left boundary path. In this case the representation over \( N \) for \( \mathcal{R} \) is such that the number \( n_{ij} = 3k_i + 5j \), \( i,j \geq 0 \) represents the point \( v_{ij} \) (we note \( \mathcal{R}_0 \) this representation, and write \( v_{ij} \in \mathcal{R}_0 \) in this situation). Hence if \( v_{ij} \) is a vertex in a non-trivial component of \( \Gamma_0 \) then \( n_{ij} \) can be expressed as the sum of three octahedral numbers (in particular \( n_{(p_0^{[2i-1]}+1)(i+1)} = \Omega_{i+2} + 2\Omega_1 \), \( i \geq 1 \)).

Now, we represent the poset \( \mathcal{R} \) in such a way that to each point \( v_{ij} \), it is associated the number \( n_{ij} = 4 + 6i + 7j \) (\( v_{ij} \in \mathcal{R}_0 \), see \( \mathcal{R}_0 \)), thus the numbers \( n_{(p_0^{[2i-1]}+1)(i+1)} \) or \( (n_{(p_0^{[2i-1]}+1)(i+1)} \), \( i \geq 1 \), \( s \geq 2 \), representing the vertices in the l.b.p can be presented in the form \( q_i + q_{i+2} + 2q_1 \) (if \( s = 0 \) then \( n_{(p_0^{[2i-1]}+1)(i+1)} = n_{00} = 4 \), represents the vertex \( v_{00} \), and \( n_{01} = 11 \) represents the vertex \( v_{01} \) if \( s = 1 \)). These facts and the theorem \([11]\) prove the next theorems.

Theorem 13. A number \( m \in N \) is the sum of three octahedral numbers if and only if \( m \) represents a vertex \( v_{ij} \in \mathcal{R}_0 \) in a non-trivial component of the graph \( \Gamma_0 \). \( \square \)

Theorem 14. A number \( m \in N \) is the sum of four positive cubes with two of them equal if and only if \( m \) represents a vertex \( v_{ij} \in \mathcal{R}_0 \) in a non-trivial component of the graph \( \Gamma_0 \). \( \square \)

Theorem 15. The number \( m \) representing \( v_{ij} \in \mathcal{R}_0 \) is the sum of three octahedral numbers if and only if the number \( n \) representing \( v_{ij} \in \mathcal{R}_0 \) is the sum of four positive cubes with two of them equal. \( \square \)

The corollaries \([10,17]\) and \([15]\) of the theorems \([11,13]\) and \([14]\) respectively are also consequences, of the Gauss’s theorem for three triangular numbers, and of the structures of the graphs \( \Gamma_{p_0^k} \) and \( \Gamma_0 \). Furthermore these corollaries can be interpreted as algorithms which solve diophantine equations of the form, \( n = x^2 + y^2 + z^2 \), \( n = x^3 + y^3 + 2z^3 \), and \( n = x(2x^2 + 1)/3 + y(2y^2 + 1)/3 + z(2z^2 + 1)/3 \), \( x, y, z, n \geq 0 \).

Corollary 16. If \( n \in N \) is the sum of three square numbers of positive rank then there exists \( v_{ij} \in \mathcal{R}_4 \), and \( k_0 \geq 0 \) such that

\[
\begin{align*}
\alpha(k_0), \beta(k_0), \gamma(k_0) & \geq -1. \\
n_{ij} & = n, \\
i - 3k_0 & = p_{\alpha(k_0)}^3 + p_{\beta(k_0)}^3 + p_{\gamma(k_0)}^3, \\
j + 2k_0 - 3 & = \alpha(k_0) + \beta(k_0) + \gamma(k_0), \\
 & \alpha(k_0), \beta(k_0), \gamma(k_0) \geq -1. \end{align*}
\]

Therefore

\[
n = p_{\alpha(k_0)+2}^4 + p_{\beta(k_0)+2}^4 + p_{\gamma(k_0)+2}^4,\]

where

\[
\begin{align*}
n & = p_{\alpha(k_0)+2}^5 + 2p_{\alpha(k_0)+2}^4, \quad \text{if} \quad p_{\beta(k_0)}^3 = p_{\gamma(k_0)}^3 = 0, \\
n & = p_{\alpha(k_0)+2}^4 + p_{\beta(k_0)+2}^4 + 1, \quad \text{if} \quad p_{\gamma(k_0)}^3 = 0.
\end{align*}
\]
If the poset \( \mathcal{R} \) is represented in such a way that \( n_{ij} = 2i + j \), for each \( i, j \geq 0 \) and a number \( n \in \mathbb{N} \) is not of the form \( 4^k(8m + 7) \), \( k, m \geq 0 \) then there are \( i, j \geq 0 \), and \( k_0 \geq 0 \) such that

\[
\begin{align*}
n_{ij} &= n, \\
i - 5k_0 &= p_{\alpha(k_0)} + p_{\beta(k_0)} + p_{\gamma(k_0)} - (\alpha(k_0) + \beta(k_0) + \gamma(k_0)), \\
j + 6k_0 &= \alpha(k_0) + \beta(k_0) + \gamma(k_0) \\
&+ 2(2)
\end{align*}
\]

\( \alpha(k_0), \beta(k_0), \gamma(k_0) \geq -1 \). Therefore

\[
n = p_{\alpha(k_0)+1}^4 + p_{\beta(k_0)+1}^4 + p_{\gamma(k_0)+1}^4,
\]

where

\[
n = p_{\alpha(k_0)+1}^4, \quad \text{if} \quad p_{\beta(k_0)} = p_{\gamma(k_0)} = 0, \\
n = p_{\alpha(k_0)+1}^4 + p_{\beta(k_0)+1}^4, \quad \text{if} \quad p_{\gamma(k_0)} = 0.
\]

For example \( n_{(24)8} = 75 \in \mathcal{R}_4 \) in this case \( i = 24, j = 8 \) and \( 24 - 3(2) = 6 + 6 + 6 = 3p_3^2 \). Hence \( k_0 = 2 \), and \( 9 = 8 + 2(2) - 3 \). Thus, \( 75 = p_3^2 + p_3^2 + p_3^2 = 25 + 25 + 25 \).

**Corollary 17.** If \( n \in \mathbb{N} \) is the sum of three octahedral numbers then there exists \( \nu_{ij} \in \mathcal{R}_3 \), and \( k_0 \geq 0 \) such that

\[
\begin{align*}
n_{ij} &= n, \\
i - 5k_0 &= \rho_{\alpha(k_0)} + \rho_{\beta(k_0)} + \rho_{\gamma(k_0)} - (\alpha(k_0) + \beta(k_0) + \gamma(k_0)), \\
j + 6k_0 &= \alpha(k_0) + \beta(k_0) + \gamma(k_0)
\end{align*}
\]

\( \alpha(k_0), \beta(k_0), \gamma(k_0) \geq -1 \), and \( \rho_0 = \rho_{-1} = 0 \). Thus

\[
n = \bigcirc_{\alpha(k_0)+1} + \bigcirc_{\beta(k_0)+1} + \bigcirc_{\gamma(k_0)+1},
\]

where

\[
n = \bigcirc_{\alpha(k_0)+1} + 2, \quad \text{if} \quad \rho_{\beta(k_0)} = \rho_{\gamma(k_0)} = 0, \\
n = \bigcirc_{\alpha(k_0)+1} + \bigcirc_{\beta(k_0)+1} + 1, \quad \text{if} \quad \rho_{\gamma(k_0)} = 0.
\]

For example \( n_{(7)(4)} = 51 \in \mathcal{R}_3 \), \( 7 = (p_3 - 3) + (p_1 - 1) \), and \( 4 = 3 + 1 \). Therefore, \( 51 = \bigcirc_4 + \bigcirc_2 + 1 = 44 + 6 + 1 \).

**Corollary 18.** If \( n \in \mathbb{N} \) is the sum of four positive cubes with two of them equal then there exists \( \nu_{ij} \in \mathcal{R}_4 \), and \( k_0 \geq 0 \) such that

\[
\begin{align*}
n_{ij} &= n, \\
i - 7k_0 &= \rho_{\alpha(k_0)} + \rho_{\beta(k_0)} + \rho_{\gamma(k_0)} - (\alpha(k_0) + \beta(k_0) + \gamma(k_0)), \\
j + 6k_0 &= \alpha(k_0) + \beta(k_0) + \gamma(k_0)
\end{align*}
\]

thus

\[
n = q_{\alpha(k_0)+1} + q_{\beta(k_0)+1} + 2q_{\gamma(k_0)+1},
\]

where

\[
n = q_{\alpha(k_0)+1} + 3, \quad \text{if} \quad \rho_{\beta(k_0)} = \rho_{\gamma(k_0)} = 0, \\
n = q_{\alpha(k_0)+1} + q_{\beta(k_0)+1} + 2, \quad \text{if} \quad \rho_{\gamma(k_0)} = 0.
\]
Let us to illustrate the corollary 18 by considering the vertex \( v_{(14)6} \in \mathbb{R}_2 \). In this case \( n_{(14)6} = 130 \), \( k_0 = 0 \), \( \alpha(k_0) = \beta(k_0) = 3 \), \( \gamma(k_0) = 0 \), and \( 14 = 2\rho_3 - 6 \), thus \( 130 = q_4 + q_4 + 2q_1 \).

Remark 19. Since \( k_0 = 0 \) is one of such values of \( k_0 \) in the corollaries, an interesting problem consists in finding all the values of \( k_0 \) satisfying the requirements.

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