ON THE MILNOR FIBRATION OF NEWTON NON-DEGENERATE ISOLATED COMPLETE INTERSECTION SINGULARITIES

TAT THANG NGUYEN†

Abstract. We prove that the Milnor fibrations over a same base of a family of Newton nondegenerate isolated singularity complete intersections which have the same Newton boundaries are isomorphic. As a consequence, we obtain that the Milnor number of a Newton nondegenerate complete intersection is an invariance of Newton boundaries.

1. Introduction

Let \( f : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) be an analytic mapping from a neighbourhood of the origin of \( \mathbb{C}^n \) to \( \mathbb{C}^p \) such that \( V := f^{-1}(0) \) is a germ of a complete intersection variety with an isolated singularity at 0. Let \( \epsilon_0 \) be a positive and sufficiently small real number such that the sphere \( S^{2n-1}_\epsilon \) intersects the variety \( V \) transversally for all \( \epsilon \leq \epsilon_0 \). Let \( U \) be an open neighbourhood of 0 in \( \mathbb{C}^p \) such that sphere \( S^{2n-1}_\epsilon \) intersect \( f^{-1}(c) \) transversally for any \( c \in U \). Let \( \mathbb{B}^{2n}_{\epsilon_0} \) be the closed ball of radius \( \epsilon_0 \) and \( D(f) \subset U \) be the set of the critical values (the discriminant set) of the restriction of \( f \) to \( X^* := f^{-1}(U \cap \mathbb{B}^{2n}_{\epsilon_0}) \). By the fibration theorem of Ehreshmann, the restriction

\[
\left. f \right|_{X^* \setminus f^{-1}(D(f))} : X^* \setminus f^{-1}(D(f)) \to U \setminus D(f)
\]

is a locally \( C^\infty \)-trivial fibration. This fibration is called the Milnor fibration and its fiber \( F_0(f) = f^{-1}(t) \cap \mathbb{B}^{2n}_{\epsilon_0} \) is called the Milnor fiber of \( f \) at the origin, where \( t \in U \setminus D(f) \) (see [K]). By a result of Hamm [2], the Milnor fiber \( F_0(f) \) is a non-singular analytic manifold which is homotopically equivalent to a bouquet of real spheres of dimension \( n - p \). The number of such spheres is called the Milnor number of \( f \) and is denoted by \( \mu_0(f) \). For each \( C \in \pi_1(U \setminus D(f)) \), the Milnor fibration generates a fibration:

\[
\left. f \right|_{f^{-1}(C) \cap \mathbb{B}^{2n}_{\epsilon_0}} : f^{-1}(C) \cap \mathbb{B}^{2n}_{\epsilon_0} \to C.
\]

We call this the Milnor fibration over \( C \), or the monodromy over \( C \) of \( f \).

Date: December 23, 2019.

1991 Mathematics Subject Classification. 14D05, 14D06, 14B05, 14B07.

Key words and phrases. isolated complete intersection singularity, monodromies, fibrations, Milnor fiber, Milnor number, Newton nondegenerate, Newton boundary.

The author is partially supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under Grant No 101.04-2019.305.
In this paper we prove that the Milnor fibration over a same base associated to a nondegenerate isolated complete intersection singularities is an invariance of the Newton boundaries. More precisely, we show that, if \( \{ f_t, t \in [0, 1] \} \) is a family of convenient analytic mappings defined in a neighbourhood of the origin of \( \mathbb{C}^n \) whose Newton boundaries are independent of \( t \) and zero sets \( f_t^{-1}(0) \) are Newton non-degenerate complete intersections (see Definition 2.1), then the monodromies of \( f_t \) over any common closed curve are isomorphic (see Theorem 4.1). As a corollary, we prove that the Milnor number of nondegenerate isolated complete intersection singularities is an invariant of Newton boundaries (see Theorem 4.2). Our result is the version for complete intersections of [11, Theorem 2.1] where the case \( p = 1 \) was studied. It also gives an analog of the \( \mu \)-constance theorem due to Le Dung Trang and C. P. Ramanujam ([7]). Similar observations for global settings were considered in [4], [3], [14], [15], [10].

2. Notations and Definitions

In this section we present some notations and definitions, which are used throughout this paper.

2.1. Notations. We suppose \( 1 \leq n \in \mathbb{N} \) and abbreviate \((x_1, \ldots, x_n)\) by \( x \). The inner product (resp., norm) on \( \mathbb{C}^n \) is denoted by \( \langle x, y \rangle \) for any \( x, y \in \mathbb{C}^n \) (resp., \( \|x\| := \sqrt{\langle x, x \rangle} \) for any \( x \in \mathbb{C}^n \)). The complex conjugate of a complex number \( c \in \mathbb{C} \) are denoted by \( \overline{c} \).

For each \( \epsilon > 0 \), we will write \( B_{2n}^{\epsilon} := \{ x \in \mathbb{C}^n : \|x\| \leq \epsilon \} \) for the closed ball and write \( S_{2n-1}^{\epsilon} := \{ x \in \mathbb{C}^n : \|x\| = \epsilon \} \) for the sphere.

Given nonempty sets \( I \subset \{1, \ldots, n\} \) and \( A \subset \mathbb{C}^n \), we define \( A^I := \{ x \in A : x_i = 0 \text{ for all } i \notin I \} \).

Let \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \) and we denote by \( \mathbb{Z}_+ \) the set of non-negative integer numbers. If \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n \), we denote by \( x^\alpha \) the monomial \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \).

The gradient of an analytic function defined in a neighbourhood of the origin \( h: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) is denoted by \( \nabla h \) as usual, i.e.,

\[
\nabla h(x) := \left( \frac{\partial h}{\partial x_1}(x), \ldots, \frac{\partial h}{\partial x_n}(x) \right),
\]

so the chain rule may expressed by the inner product \( \partial h/\partial v = \langle v, \nabla h \rangle \).

2.2. Newton polyhedra and non-degeneracy conditions. Let \( h: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be analytic function defined in a neighbourhood of the origin such that \( h(0) = 0 \). Suppose that \( h \) is written as \( h = \sum_\alpha a_\alpha x^\alpha \). Then the support of \( h \), denoted by \( \text{supp}(h) \), is defined as the set of those \( \alpha \in \mathbb{Z}_+^n \) such that \( a_\alpha \neq 0 \). The Newton polyhedron of \( h \), denoted by \( \Gamma_+(h) \), is defined as the convex hull in \( \mathbb{R}^n \) of the union of \( \{ \alpha + \mathbb{R}_+^n \} \) for \( \alpha \in \text{supp}(h) \). The Newton boundary of \( h \), denoted by \( \Gamma(h) \), is by definition the union of compact boundary of \( \Gamma_+(h) \). The function
h or its Newton boundary is said to be convenient if $\Gamma(h)$ intersects each coordinate axis in a point different from the origin 0 in $\mathbb{R}^n$. For each (compact) face $\Delta$ of $\Gamma_+(h)$, we will denote by $h_\Delta$ the polynomial $\sum_{\alpha \in \Delta} a_\alpha x^\alpha$; if $\Delta \cap \text{supp}(h) = \emptyset$ we let $h_\Delta := 0$.

Given a nonzero vector $q \in \mathbb{R}_{\geq 0}^n$, we define
\[
\begin{align*}
d(q, \Gamma_+(h)) &:= \min\{\langle q, \alpha \rangle : \alpha \in \Gamma_+(h)\}, \\
\Delta(q, \Gamma_+(h)) &:= \{\alpha \in \Gamma_+(h) : \langle q, \alpha \rangle = d(q, \Gamma_+(h))\}.
\end{align*}
\]

It is easy to check that for each nonzero vector $q \in \mathbb{R}_{\geq 0}^n$, $\Delta(q, \Gamma_+(h))$ is a closed face of $\Gamma_+(h)$. Conversely, if $\Delta$ is a closed face of $\Gamma_+(h)$ then there exists a nonzero vector $q \in \mathbb{R}_{\geq 0}^n$ such that $\Delta = \Delta(q, \Gamma_+(h))$.

**Remark 2.1.** The following statements follow immediately from definitions:

(i) For each nonempty subset $I$ of $\{1, \ldots, n\}$, if the restriction of $h$ on $C^I$ is not identically zero, then $\Gamma_+(h) \cap \mathbb{R}^I = \Gamma_+(h|_{C^I})$. Also, for every nonzero vector $q = (q_1, \ldots, q_n) \in \mathbb{R}^I$ with $q_i > 0$ if $i \in I$ and $\Delta := \Delta(q, \Gamma_+(h|_{C^I}))$, one can find a strictly positive vector $q' \in \mathbb{R}_{\geq 0}^n$ such that $\Delta = \Delta(q', \Gamma_+(h))$.

(ii) Let $\Delta := \Delta(q, \Gamma_+(h))$ for some nonzero vector $q := (q_1, \ldots, q_n) \in \mathbb{R}_{\geq 0}^n$. By definition, $h_\Delta = \sum_{\alpha \in \Delta} a_\alpha x^\alpha$ is a weighted homogeneous polynomial of type $(q, d := d(q, \Gamma_+(h)))$, i.e., we have for all $t$ and all $x \in C^n$,
\[
h_\Delta(t^{q_1}x_1, \ldots, t^{q_n}x_n) = t^d h_\Delta(x_1, \ldots, x_n).
\]

This implies the Euler relation
\[
\sum_{i=1}^n q_i x_i \frac{\partial h_\Delta}{\partial x_i}(x) = dh_\Delta(x).
\]

Now, let $f = (f^1, \ldots, f^p) \colon C^n \to C^p$ be an analytic mapping defined in a neighbourhood of the origin in $C^n$ such that $f(0) = 0$. The following definition of non-degeneracy is inspired from the work of Kouchnirenko [6], where the case $S = C^n$ was considered (see also [12], [13]).

**Definition 2.1.** We say that the variety $V := f^{-1}(0)$ is a (Newton) non-degenerate complete intersection variety at the origin 0 in $C^n$ if, for any strictly positive weight vector $q \in \mathbb{R}_{>0}^n$ the $p$-form $df^1_{\Delta_1} \wedge \ldots \wedge df^p_{\Delta_p}$ does not vanish on the set
\[
\{x \in C^n : f^1_{\Delta_1}(x) = \cdots = f^p_{\Delta_p}(x) = 0\};
\]
i.e., the system of gradient vectors $\nabla f^j_{\Delta_j}(x)$ for $j = 1, \ldots, p$ is $C$-linearly independent on this variety; where $\Delta_j := \Delta(q, \Gamma_+(f^j))$ for $j = 1, \ldots, p$. 

3
3. Milnor fibration

Let \( F(t, x) = (F^1(t, x), \ldots, F^p(t, x)) : [0, 1] \times \mathbb{C}^n \to \mathbb{C}^p \) be a mapping such that \( F \) is real analytic on \( t \) and for each \( t \in [0, 1] \) the map \( f_t(x) := F(t, x) \) is analytic in some neighbourhood of the origin in \( \mathbb{C}^n \) with \( f_t(0) = 0 \). For fixed \( t \in [0, 1] \) and each \( j = 1, \ldots, p \), we denote \( f^j_t \) the function \( f^j_t(x) = F^j(t, x) \).

**Definition 3.1.** We say that the positive number \( \epsilon_0 > 0 \) is a *uniform stable radius* for the Milnor fibration of the family \( \{ f_t \}_{t \in [0, 1]} \) if for each \( \epsilon \leq \epsilon_0 \), and each \( t \in [0, 1] \) the set \( f_t^{-1}(0) \) intersects transversally with the sphere \( S^{2n-1}_\epsilon \).

We have the following properties of Milnor balls.

**Lemma 3.1.** Suppose that the family \( \{ f_t \} \) satisfies:

(i) For each \( j = 1, \ldots, p \), the Newton boundary of \( f^j_t \) is convenient and does not depend on \( t \in [0, 1] \);

(ii) For each \( t \in [0, 1] \), the zero set \( f_t^{-1}(0) \) is a Newton non-degenerate complete intersection variety at the origin.

Then the family \( \{ f_t \}_{t \in [0, 1]} \) has a uniform stable radius.

**Proof.** Assume by contradiction that such a uniform stable radius does not exist, then there exist sequences \( \{ t_k \}_{k \in \mathbb{N}} \subset [0, 1] \), \( \{ \epsilon_k \}_{k \in \mathbb{N}} \to 0 \) such that the sets \( f_{t_k}^{-1}(0) \) do not intersect transversally with the spheres \( S^{2n-1}_{\epsilon_k} \). Therefore, there exist sequences \( \{ x^k \}_{k \in \mathbb{N}} \subset \mathbb{C}^n \) and \( \{ \lambda^k_j \}_{k \in \mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p+1 \), such that

(a1) \( \| x^k \| \to 0, \| x^k \| \neq 0 \) as \( k \to \infty \);

(a2) \( F(t^k, x^k) = f_{t_k}(x^k) = 0 \) for all \( k \in \mathbb{N} \);

(a3) \( \sum_{j=1}^p \lambda^k_j \nabla f^j_{t_k}(x^k) = \lambda^k_{p+1} x^k \);

(a4) The numbers \( \lambda^k_j, j = 1, \ldots, p+1 \) are not all zero for any \( k \in \mathbb{N} \).

By the Curve Selection Lemma (see [9]), there exist analytic curves \( \phi = (\phi_1, \ldots, \phi_n) : [0, \epsilon) \to \mathbb{C}^n \), \( t : [0, \epsilon) \to [0, 1] \) and \( \lambda_j : (0, \epsilon) \to \mathbb{C}, j = 1, \ldots, p + 1 \), such that

(a5) \( \| \phi(s) \| \to 0, \) as \( s \to 0 \) and \( \phi(s) \neq 0 \) for all \( s \in (0, \epsilon) \);

(a6) \( F(t(s), \phi(s)) = 0 \) for all \( s \in (0, \epsilon) \);

(a7) \( \sum_{j=1}^p \lambda_j(s) \nabla f^j_{t(s)}(\phi(s)) = \lambda_{p+1}(s) \phi(s) \) for all \( s \in (0, \epsilon) \);

(a8) \( \lambda_j(s), j = 1, \ldots, p + 1, \) are not all zero for any \( s \in (0, \epsilon) \).

Put \( I := \{ i : \phi_i \neq 0 \} \). By the condition (a5), \( I \neq \emptyset \). For \( i \in I \), we can write the curve \( \phi_i \) in terms of parameter as follows

\[
\phi_i(s) = x_i^0 s^n + \text{higher-order terms in } s,
\]
where \( x_i^0 \neq 0 \), and \( q_i \in \mathbb{Q} \). We have \( \min_{i \in I} q_i > 0 \), due to the condition (a5). We also write \( t(s) \) as

\[
t(s) = t^0 + t^1 s^q + \text{higher-order terms in } s,
\]

where \( t^0 = \lim_{s \to 0} t(s) \in [0, 1] \), \( t^1 \in \mathbb{R} \) and \( q > 0 \).

For each \( j = 1, \ldots, q \) and each \( t \in [0, 1] \), \( f_t^j(x) \) is convenient then \( f_t^j|_{C^j} \neq 0 \). Let \( d_j > 0 \) be the minimal value of the linear function \( \sum_{i \in I} \alpha_i q_i \) on \( \mathbb{R}^j \cap \Gamma_+(f_t^j) \) and \( \Delta_j \) be the maximal face of \( \mathbb{R}^j \cap \Gamma_+(f_t^j) \), where this linear function attains its minimum value. Remark that the Newton polyhedrons \( \Gamma_+(f_t^j) \) do not depend on \( t \). It is easy to check that

\[
F^j(t(s), \phi(s)) = (f_t^j)^{\Delta_j}(x^0) s^{d_j} + \text{higher-order terms in } s,
\]

where \( x^0 := (x_1^0, \ldots, x_n^0) \) with \( x_i^0 = 1 \) for \( i \not\in I \) and \( (f_t^j)^{\Delta_j} \) is the face function associated with \( f_t^j \) and \( \Delta_j \) which does not depend on the variables \( x_i \) if \( i \not\in I \). It implies from the condition (a6) that

\[
(f_t^j)^{\Delta_j}(x^0) = 0, \quad \text{for all } j = 1, \ldots, p.
\]

For \( i \in I \) and \( j \in \{1, \ldots, p\} \), we also have:

\[
\frac{\partial F^j(t(s), \phi(s))}{\partial x_i} = \frac{\partial (f_t^j)^{\Delta_j}}{\partial x_i}(x^0) s^{d_j} + \text{higher-order terms in } s.
\]

It follows from (a7) and (a8) that one of the functions \( \lambda_1(s), \ldots, \lambda_p(s) \) is not equal to zero.

For \( j \in \{1, \ldots, p\} \) which \( \lambda_j(s) \neq 0 \) we write

\[
\lambda_j(s) = c_j s^{\beta_j} + \text{higher-order terms in } s, \quad c_j \neq 0.
\]

Put

\[
e := \min \{ \beta_l + d_l : l \in \{1, \ldots, p\} \quad \text{which } \lambda_l(s) \neq 0 \}
\]

and

\[
J := \{ j : \lambda_j(s) \neq 0, \beta_j + d_j = e \}.
\]

Then the condition (a7) is equivalent to the following:

\[
\left( \sum_{j \in J} c_j \frac{\partial (f_t^j)^{\Delta_j}}{\partial x_i}(x^0) \right) s^{e-q_i} + \cdots = \lambda_{p+1}(s) \phi_i(s) \quad \text{for all } i \in I,
\]

where dots stand for higher-order terms in \( s \).

If \( \lambda_{p+1}(s) \equiv 0 \) : for all \( i \in I \) we get

\[
\sum_{j \in J} c_j \frac{\partial (f_t^j)^{\Delta_j}}{\partial x_i}(x^0) = 0.
\]

Hence

\[
\sum_{j \in J} c_j \nabla (f_t^j)^{\Delta_j}(x^0) = 0.
\]
Two equalities \( \square \) and \( \square \) imply the contradiction to the nondegeneracy of \( f_{i^0} \).

If \( \lambda_{p+1}(s) \neq 0 \) : we also write \( \lambda_{p+1}(s) \) as

\[
\lambda_{p+1}(s) = c_{p+1}s^{\beta_{p+1}} + \text{ higher-order terms in } s, \quad c_{p+1} \neq 0.
\]

The equation \( \square \) becomes

\[
\left( \sum_{j \in J} c_j \frac{\partial (f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0) \right) s^{e-q_i} + \cdots = c_{p+1}x_i^0s^{\beta_{p+1}+q_i} + \cdots \text{ for all } i \in I. \tag{4}
\]

Since \( c_{p+1} \) and \( x_i^0 \) are nonzero for all \( i \in I \), we get \( e - q_i \leq \beta_{p+1} + q_i \) for all \( i \in I \). If \( e - \beta_{p+1} < 2 \min_{i \in I} q_i \) then by the same argument as above we obtain a contradiction to the non-degeneracy condition of \( f_{i^0} \). Otherwise, if \( e - \beta_{p+1} = 2 \min_{i \in I} q_i > 0 \). Denote

\[
I_1 := \{ i \in I : q_i = \min_{i \in I} q_i \}.
\]

The equation \( \square \) gives us the following

\[
\sum_{j \in J} c_j \frac{\partial (f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0) = \begin{cases} 
  c_{p+1}x_i^0 & \text{if } i \in I_1, \\
  0 & \text{if } i \in I \setminus I_1, \\
  0 & \text{if } i \not\in I,
\end{cases}
\]

the last equation holds because for all \( i \not\in I \) and all \( j \in J \), the polynomial \( (f_{i^0}^j)_{\Delta_j} \) does not depend on the variable \( x_i \). Consequently,

\[
\sum_{i=1}^{n} \left( \sum_{j \in J} c_j \frac{\partial (f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0) \right) x_i^0q_i = \sum_{i \in I_1} \left( \sum_{j \in J} c_j \frac{\partial (f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0) \right) x_i^0q_i = \sum_{i \in I_1} c_{p+1}|x_i^0|^2 \frac{e - \beta_{p+1}}{2}.
\]

On the other hand, by the Euler relation, for all \( j \in J \), we have

\[
\sum_{i=1}^{n} \frac{(f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0)x_i^0q_i = d_j (f_{i^0}^j)_{\Delta_j}(x^0).
\]

Combining this equality and the equation \( \square \) we get

\[
\sum_{i=1}^{n} \left( \sum_{j \in J} c_j \frac{\partial (f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0) \right) x_i^0q_i = \sum_{j \in J} c_j \left( \sum_{i=1}^{n} \frac{\partial (f_{i^0}^j)_{\Delta_j}}{\partial x_i}(x^0)x_i^0q_i \right) = \sum_{j \in J} c_j d_j (f_{i^0}^j)_{\Delta_j}(x^0) = 0.
\]

Therefore \( \sum_{i \in I_1} c_{p+1}|x_i^0|^2 \frac{e - \beta_{p+1}}{2} = 0 \). This is a contradiction. \( \square \)
**Remark 3.1.** It implies from the proof of Lemma 3.1 that for each $t$ the zero set $f_t^{-1}(0)$ has only isolated singularity at the origin.

Now we work with the non-convenient case. Let $f(x) = (f^1, \ldots, f^p)(x) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a germ of analytic mapping such that $f(0) = 0$ and $V := f^{-1}(0)$ is a germ of a complete intersection with an isolated singularity at the origin.

Let $\mathcal{O}_{\mathbb{C}^n, 0}$ be the ring of germs of analytic functions at $0 \in \mathbb{C}^n$ and $\mathfrak{m}$ be its maximal ideal. Let $J_f$ be the ideal of $\mathcal{O}_{\mathbb{C}^n, 0}$ generated by $f^1, \ldots, f^p$ and determinants of maximal order minors of the Jacobian matrix of $f$. Since $f^{-1}(0)$ has only isolated singularity at the origin, by the Hilbert nullstellensatz ([5], Proposition 1.1.29) we have

$$\mathfrak{m} \subset \sqrt{J_f}.$$ 

Let $\mu \in \mathbb{N}$ be the smallest number such that $x_i^\mu \in J_f$, $\forall i = 1, \ldots, n$.

The following is an analog of Lemma 2.5 in [11].

**Lemma 3.2.** With the above notation, consider the family

$$f_t(x) := F(t, x) := (f^1(x) + tx^\nu, f^2(x), \ldots, f^p(x)), \quad t \in [0, 1],$$

where $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{N}^n$ satisfying $|\nu| = \sum_{i=1}^n \nu_i \geq \mu + 2$ and for each $i$, either $\nu_i = 0$ or $\nu_i \geq 2$. Then the family $\{f_t\}_{t \in [0, 1]}$ has a uniform stable radius for the Milnor fibration.

**Proof.** Suppose that such uniform stable radius does not exist. Then, by the same argument as in the proof of Lemma 3.1, we can find real analytic functions:

$$\phi = (\phi_1, \ldots, \phi_n) : [0, \epsilon) \to \mathbb{C}^n, \quad t : [0, \epsilon) \to [0, 1] \quad \text{and} \quad \lambda_j : (0, \epsilon) \to \mathbb{C}, \quad j = 1, \ldots, p + 1,$$

such that

1. $\|\phi(s)\| \to 0$, as $s \to 0$ and $\phi(s) \neq 0$ for all $s \in (0, \epsilon)$;
2. $F(t(s), \phi(s)) = 0$ for all $s \in (0, \epsilon)$;
3. $t(s)\lambda_1(s)(\nabla x^\nu)(\phi(s)) + \sum_{j=1}^p \lambda_j(s)(\nabla f^j)(\phi(s)) = \lambda_{p+1}(s)\phi(s)$ for all $s \in (0, \epsilon)$;
4. $\lambda_j(s), j = 1, \ldots, p + 1$, are not all zero for any $s \in (0, \epsilon)$.

We expand those functions as follows:

$$\phi_i(s) = x_i^0 s^{\beta_0^i} + \cdots, \quad i = 1, \ldots, n$$
$$t = t^0 + t^1 s^q + \cdots,$$
$$\lambda^j = c_j s^{\beta^j} + \cdots, \quad j = 1, \ldots, p + 1.$$
where \( q_i > 0 \) for all \( i \) (possibly \( q_i = \infty \)). For each \( i \) we have \( q_i = \infty \) if \( \phi_i \equiv 0 \), otherwise \( x^0_i \in \mathbb{C}^* \). We also see that \( t^0 \in [0, 1] \) and \( q > 0 \). Put

\[
a := \min_{i=1,\ldots,n} \{q_i\} > 0.
\]

Without loss of generality, we may assume that \( a = q_1 \).

Denote by \( F^j, j = 1, \ldots, p \) the component functions of \( F \). Take the derivative both sides of (2), we obtain that:

\[
\frac{\partial F^j}{\partial t} \frac{dt}{ds} + \left\langle \frac{d\phi}{ds}, \nabla F^j(\phi) \right\rangle = 0, \quad \text{for all } j = 1, \ldots, p.
\]

Combining these equations with the condition (3), we get:

\[
\lambda_1 \phi^\mu \frac{dt}{ds} + \lambda_{p+1} \left\langle \frac{d\phi}{ds}, \phi \right\rangle = 0.
\]

We consider the following two possibilities:

**Case 1:** \( \lambda_{p+1} = 0 \). If \( \lambda_1 \neq 0 \), then \( \phi^\mu(s) = 0 \). Since for each \( i \) either \( \nu_i = 0 \) or \( \nu_i \geq 2 \), it implies from (2) and (3) that \( f(\phi(s)) = 0 \) and \( \sum_{j=1,\ldots,p} \nabla f^j(\phi(s)) = 0 \). This means \( f^{-1}(0) \) has non-isolated singularities at the origin (contradiction). Otherwise, if \( \lambda_1 = 0 \) the the vectors \( \nabla f^2(\phi(s)), \ldots, \nabla f^p(\phi(s)) \) are linearly dependent. Furthermore \( x^\nu_1 \in J_f \) then there exist analytic functions \( g_j, h_k \) such that

\[
x^\nu_1 = \sum_{j=1,\ldots,p} g_j f^j + \sum_k h_k J_k
\]

where \( J_k \) are determinants of maximal order minors of the Jacobian matrix of \( f \). Substitute \( x = \phi(s) \) both sides of the above equation and remark that all the determinants \( J_k(\phi(s)) = 0 \), we get

\[
\phi^\mu_1(s) = -t(s)g_1(s)\phi^\mu(s).
\]

This is again a contradiction, since the order of the left hand side is \( a\mu \), while the right hand side’s order is not less than \( av \).

**Case 2:** \( \lambda_{p+1} \neq 0 \). It follows from the equation (5), by comparing the orders, that

\[
\beta_1 + a\nu + q = 2a + \beta_{p+1} > \beta_1 + a\nu,
\]

by the assumption \( \nu > \mu + 1 \), we get \( a + \beta_{p+1} - \beta_1 > a\mu \).

On the other hand, it is easy to check that

\[
J_f \subset J_{f^t} + m^\nu.
\]

Hence, due to \( x^\nu_1 \in m \), there exist analytic functions \( g'_j, h'_k, p_I \) such that

\[
x^\nu_1 = \sum g'_j f^j + \sum h'_k J'_k + \sum_{I=(i_1,\ldots,i_\nu) \subset\{1,\ldots,n\}} p_I x_{i_1} x_{i_2} \cdots x_{i_\nu}
\]
where \( f^j_t \) are component functions of \( f_t \) vanishing along \( \phi(s) \) and \( J'_k \) are determinants of maximal order minors of Jacobian matrix of \( f_t \). Substituting \( x \) by \( \phi(s) \) both sides of the equation gives us the following

\[
\phi^k_t = \sum h^j_k(\phi(s))J'_k(\phi(s)) + \sum_p p_t\phi_i\phi_{i_2}\ldots\phi_{i_w}.
\]

(6)

By the condition (3), the first row of the Jacobian matrix of \( f_t \) (the gradient vector of \( f^1_t \)) is the linear combination of the others and the vector \( \frac{\lambda_{p+1}(s)}{\lambda_1(s)}\phi(s) \). Thus the order of \( J'_k(\phi(s)) \) is not less than \( a + \beta_{p+1} - \beta_1 \). By comparing orders of both sides of the equation (6) we get the contradiction. \( \Box \)

One application of uniform stable radius is the following.

**Lemma 3.3.** Let \( f_t(x) = F(t, x), t \in [0, 1] \) be a family of germs of analytic functions at the origin in \( \mathbb{C}^n \) such that \( F(t, x) \) is real analytic on \( t \in [0, 1] \) and \( f_t(0) = 0 \) for all \( t \). Suppose that the family has a uniform stable radius \( \epsilon_0 \) for the Milnor fibration. Then, there exists a small neighbourhood \( U \) of the origin in \( \mathbb{C}^p \) such that for each \( c \in U \) and each \( t \in [0, 1] \) the set \( f_t^{-1}(c) \) intersects transversally with the sphere \( S^{2n-1}_{\epsilon_0} \).

**Proof.** Assume by contradiction that such neighbourhood \( U \) does not hold. This means there exist sequences \( \{t^k\}_{k \in \mathbb{N}} \subset [0, 1] \) and \( \{c^k\}_{k \in \mathbb{N}} \subset \mathbb{C}^p \) such that \( c^k \to 0 \) and the set \( f_{t^k}^{-1}(c^k) \) does not intersect the sphere \( S^{2n-1}_{\epsilon_0} \) transversally, for any \( k \in \mathbb{N} \). Then, there exist sequences \( \{x^k\}_{k \in \mathbb{N}} \subset S^{2n-1}_{\epsilon_0} \) and \( \{\lambda^j_k\}_{k \in \mathbb{N}} \subset \mathbb{C}, j = 1, \ldots, p + 1, \) such that

1. \( F(t^k, x^k) = f_{t^k}(x^k) = c^k \) for all \( k \in \mathbb{N} \);
2. \( \sum_{j=1}^p \lambda^j_k \nabla f^j_{t^k}(x^k) = \lambda^k_{p+1} x^k \);
3. The numbers \( \lambda^j_k, j = 1, \ldots, p + 1 \) are not all zero for any \( k \in \mathbb{N} \).

By the Curve Selection Lemma (see [9]), there exist analytic curves

\[
\phi: [0, \epsilon) \to S^{2n-1}_{\epsilon_0}, \quad t: [0, \epsilon) \to [0, 1] \quad \text{and} \quad \lambda_j: (0, \epsilon) \to \mathbb{C}, \quad j = 1, \ldots, p + 1,
\]

such that

1. \( \phi(s) \to x^0 \in S^{2n-1}_{\epsilon_0} \) as \( s \to 0 \);
2. \( F(t(s), \phi(s)) \to 0 \) as \( s \to 0 \);
3. \( \sum_{j=1}^p \lambda_j(s) \nabla f^j_{t(s)}(\phi(s)) = \lambda_{p+1}(s) \phi(s) \) for all \( s \in (0, \epsilon) \);
4. \( \lambda_j(s), j = 1, \ldots, p + 1, \) are not all zero for any \( s \in (0, \epsilon) \).

Denote

\[
J := \{j \in \{1, \ldots, p + 1\} : \lambda_j(s) \neq 0\},
\]

due to the condition (a7), \( J \neq \emptyset \). By dividing both sides of (a6) by \( s^a \), if necessary, where \( a \) is the lowest order of nonzero functions \( \lambda_j(s), j \in J \), we may assume that, for all \( j = 1, \ldots, p + 1 \)
there exist limits

\[ c_0^j := \lim_{s \to 0} \lambda_j(s) \]

and the numbers \( c_0^j, j = 1, \ldots, p + 1 \), are not all zero (by (a7)). Denote \( t^0 := \lim_{s \to 0} t(s) \in [0, 1] \).

Now, taking the limit when \( s \to 0 \) in the conditions (a5) and (a6) we get \( f_\nu(x_0) = F(t^0, x_0) = 0 \) and

\[ \sum_{j=1}^p c_0^j(s) \nabla f_{\nu_0}^j(x^0) = c_{p+1}^0 x^0. \]

This means that the set \( f_\nu^{-1}(0) \) does not intersect the sphere \( S_{\epsilon_0}^{2n-1} \) transversally. This is a contradiction.

\[ \square \]

For each \( t \in [0, 1] \) let \( D_t \) be the discriminant set of \( f_t : \mathbb{C}^n \to \mathbb{C}^p \). Then \( D_t \subset \mathbb{C}^p \) is a hypersurface of dimension \( p - 1 \) (see [8], Section 2.8). We have the following remark of Milnor fibration of the family \( f_t \).

**Proposition 3.1.** With the notations and assumption as in Lemma 3.3, let \( \epsilon_0 > 0 \) be the uniform stable radius of the family \( \{f_t\}_{t \in [0,1]} \) and \( U \subset \mathbb{C}^p \) be a neighbourhood of the origin in \( \mathbb{C}^p \) such that the conclusions of 3.3 is fulfill. Then, for each \( t \in [0, 1] \), the restriction

\[ f_t|_{f_t^{-1}(U \setminus D_t) \cap \mathbb{B}_{\epsilon_0}^{2n}} : f_t^{-1}(U \setminus D_t) \cap \mathbb{B}_{\epsilon_0}^{2n} \to U \setminus D_t \]

is a locally \( C^\infty \)-trivial fibration.

**Proof.** The proof is straightforward by the Ehresmann theorem.

\[ \square \]

### 4. INVARIANCE OF MILNOR FIBRATIONS

Let \( F(t, x) = (F^1(t, x), \ldots, F^p(t, x)) : [0, 1] \times \mathbb{C}^n \to \mathbb{C}^p \) be a mapping such that \( F \) is real analytic on \( t \) and for each \( t \in [0, 1] \) the map \( f_t(x) := F(t, x) \) is analytic in some neighbourhood of the origin in \( \mathbb{C}^n \) with \( 0 \in V_t := f_t^{-1}(0) \). For fix \( t \), we also denote \( f_t^j \) the function \( x \mapsto F^j(t, x) \). In this Section, we consider the Milnor fibration of the family over a common base. The main result is the following.

**Theorem 4.1.** With the above notation, suppose that the family \( \{f_t\}_{t \in [0,1]} \) has a uniform stable radius for the Milnor fibration \( \epsilon_0 \). Denote \( U \) the neighbourhood of the origin in \( \mathbb{C}^p \) as in Lemma 3.3. Let \( C \) be any closed subset of \( U \setminus (\cup_{t \in [0,1]} D_t) \). Then the Milnor fibration of \( f_t, t \in [0, 1] \) over \( C \) are isomorphic; i.e. there is \( C^\infty \)-diffeomorphism

\[ \Phi_t : f_0^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n} \to f_t^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n}, \quad t \in [0, 1], \]
which makes the following diagram commutes

\[
\begin{array}{c}
f^{-1}_0(C) \cap B_{\epsilon_0} \xrightarrow{\Phi_t} f^{-1}_t(C) \cap B_{\epsilon_0} \\
f_0 \downarrow & f_t \downarrow \\
C & C \\
\end{array}
\]

where id denotes the identity map.

In order to prove the theorem, we need the following lemma.

**Lemma 4.1.** With the assumption as in Theorem 4.1. Then, there exists \(0 < \delta(C) < \epsilon_0\) small enough such that, for any \(\epsilon_0 - \delta(C) < \epsilon \leq \epsilon_0\), \(t \in [0, 1] \) and \(c \in C\) the set \(f^{-1}_t(c)\) intersects transversally with the sphere \(S_{\epsilon}^{2n-1}\).

**Proof.** Assume by contradiction that the conclusion of the lemma does not hold. Similarly, by Curve Selection Lemma, there are analytic curves:

\[
\phi: (0, \epsilon') \to B_{\epsilon_0}, \quad t: (0, \epsilon') \to [0, 1] \quad \text{and} \quad \lambda_j: (0, \epsilon') \to C, \quad j = 1, \ldots, p + 1,
\]

such that

(a1) \(\phi(s) \to x^0 \in S_{\epsilon_0}^{2n-1}\) as \(s \to 0\);
(a2) \(F(t(s), \phi(s)) \in C\) for \(s \in (0, \epsilon')\);
(a3) \(\sum_{j=1}^p \lambda_j(s) \nabla f_j(t(s)) = \lambda_{p+1}(s) \phi(s)\) for \(s \in (0, \epsilon')\);
(a4) \(\lambda_j(s), j = 1, \ldots, p + 1,\) are not all zero for \(s \in (0, \epsilon')\).

By the same argument as in the proof of Lemma 3.3 we may assume that there exist limits

\[
c^0_j := \lim_{s \to 0} \lambda_j(s), j = 1, \ldots, p + 1
\]

which the numbers \(c^0_j, j = 1, \ldots, p + 1\) are not all zero. Let \(t^0 := \lim_{s \to 0} t(s) \in [0, 1]\).

Taking the limit when \(s \to 0\) in the conditions (a2) and (a3), we have \(f_{t^0}(x^0) = c \in C\) and

\[
\sum_{j=1}^p c^0_j \nabla f_j^{t^0}(x^0) = c^0_{p+1} x^0.
\]

That means the set \(f^{-1}_{t^0}(c)\) does not intersect the sphere \(S_{\epsilon_0}^{2n-1}\) transversally. This is a contradiction to the conclusion of Lemma 3.3.

\[\square\]

**Proof of theorem 4.1.** Denote

\[
X := \{(t, x) \in [0, 1] \times B_{\epsilon_0} : F(t, x) \in C\}.
\]

Let \(0 < \delta := \delta(C) \leq \epsilon_0\) as in Lemma 4.1. Since for all \(t \in [0, 1], C\) does not intersect discriminants of \(f_t\), all vector \(\nabla f_1^t(x), \ldots, \nabla f_p^t(x)\) are \(C\)-linear independent. Then, we can
find a smooth map

\[ v_1 : U_1 := X \cap \left\{ x : \|x\| < \epsilon_0 - \frac{\delta}{2} \right\} \longrightarrow \mathbb{C}^n \]

such that

\[ \langle v_1(t, x), \nabla f_t^j(x) \rangle = -\frac{\partial f_t^j}{\partial t}(x); \quad \text{for all } j = 1, \ldots, p. \]

Similarly, by Lemma 4.1, on the set

\[ U_2 := X \cap \{ x : \epsilon_0 - \delta < \|x\| \leq \epsilon_0 \} \]

all vectors \( \nabla f_t^1(x), \ldots, \nabla f_t^p(x), x \) are \( \mathbb{C} \)-linear independent. Then, we can find a smooth map:

\[ v_2 : U_2 \longrightarrow \mathbb{C}^n \]

such that

\begin{enumerate}
  \item[(a1)] \( \langle v_2(t, x), \nabla f_t^j(x) \rangle = -\frac{\partial f_t^j}{\partial t}(x) \) for all \( j = 1, \ldots, p; \)
  \item[(a2)] \( \langle v_2(t, x), x \rangle = 0 \).
\end{enumerate}

Now, fix a partition of unity \( \{ \theta_1, \theta_2 \} \) subordinated to the covering \( \{ U_1, U_2 \} \) of \( X \). We define a smooth vector field

\[ v : X \longrightarrow \mathbb{C}^n \]

as \( v = \theta_1 v_1 + \theta_2 v_2 \). We have the following:

\begin{enumerate}
  \item[(a3)] \( \langle v(t, x), \nabla f_t^j(x) \rangle = -\frac{\partial f_t^j}{\partial t}(x) \) for all \( j = 1, \ldots, p \) and \( (t, x) \in X; \)
  \item[(a4)] \( \langle v(t, x), x \rangle = 0 \) for all \( (t, x) \in X \) which \( \epsilon_0 - \delta < \|x\| \leq \epsilon_0. \)
\end{enumerate}

Finally, we can see that for each \( x \in f_t^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n} \), there exists a unique \( C^\infty \)-map \( \varphi : [0, 1] \rightarrow \mathbb{C}^n \) such that

\[ \varphi'(t) = v(t, \varphi(t)), \quad \varphi(0) = x. \]

Moreover, for each \( t \in [0, 1] \), the map

\[ \Phi_t : f_t^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n} \rightarrow f_t^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n}, \quad x \mapsto \varphi(t), \]

is well-defined and is a \( C^\infty \)-diffeomorphism, which makes the following diagram commutes

\[ \begin{array}{ccc}
  f_0^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n} & \xrightarrow{\Phi_t} & f_t^{-1}(C) \cap \mathbb{B}_{\epsilon_0}^{2n} \\
  \downarrow f_0 & & \downarrow f_t \\
  C & \xrightarrow{id} & C \\
\end{array} \]

where id denotes the identity map. The proof is complete.

\[ \square \]

**Corollary 4.1.** Assume that the family \( \{ f_t \} \) has uniform stable radius. Then the Milnor fibers of \( f_t, t \in [0, 1] \) are diffeomorphic to each other.
Proof. For each \( t \), the discriminant \( D_t \) of \( f_t \) is a hypersurface of dimension \( p - 1 \). Then, in any neighbourhood of the origin in \( \mathbb{C}^p \), there exists a point \( M \) in the complement of \( \bigcup_{t \in [0,1]} D_t \). Now applying Theorem 4.1 for \( C = \{ M \} \), we get the conclusion. \( \square \)

**Theorem 4.2.** Let \( f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) be two analytic mappings defined in some neighbourhood of the origin in \( \mathbb{C}^n \) such that the following conditions are fulfill:

i) The Newton boundaries of \( f \) and \( g \) are the same;

ii) The zero sets \( f^{-1}(0), g^{-1}(0) \) are Newton nondegenerate complete intersections with isolated singularity at the origin.

Then \( \mu_0(f) = \mu_0(g) \).

In other words, the Milnor number of a nondegenerate isolated singularity complete intersection is an invariant of the Newton boundaries.

**Proof.** By Lemma 3.2 and Corollary 4.1, we may assume furthermore that the Newton boundary of \( f \) and \( g \) are convenient.

Since the non-degeneracy is an open condition (see [11], Appendix), we can find a piecewise analytic family \( F(t, x) \) such that \( F(0, x) = f(x), F(1, x) = g(x) \), \( F \) is piecewise analytic on \( t \in [0,1] \), for each \( t \) the mapping \( f_t(x) := F(t, x) \) is analytic on some neighbourhood of the origin \( 0 \in \mathbb{C}^n \) and the following two conditions hold:

(a1) The Newton boundaries of component functions of \( f_t \) are convenient and independent of \( t \);

(a2) For each \( t \), the zero set \( f_t^{-1}(0) \) is Newton nondegenerate.

Then by Lemma 3.1 the family \( \{ f_t \} \) has a uniform stable radius. Therefore, it follows by the Corollary 4.1 that \( \mu_0(f) = \mu_0(g) \). \( \square \)

**Remark 4.1.**

1) In the case the Newton boundaries of the component functions of \( f = (f^1, \ldots, f^p) \) are similar, i.e. there are integer numbers \( d_i \) and a polyhedron \( \Gamma \) such that \( \Gamma_+(f^i) = d_i \Gamma \), then the Milnor number \( \mu_0(f) \) is computed in terms of \( \Gamma \) (see [12]). As a corollary it is an invariant of Newton boundaries.

2) In [11], with stronger nondegeneracy condition, the author also gives a formula for the Milnor number of a convenient isolated complete intersection singularities which depends only on the Newton polyhedra.

**References**

[1] C. Bivia-Ausina. Mixed Newton numbers and isolated complete intersection singularities, *Proc. Lond. Math. Soc.*, 94(3): 749–771, 2007.

[2] H. A. Hamm. Lokale topologische Eigenschaften komplexer Raume. *Math. Ann.*, 191: 235–252, 1971.

[3] H. V. H` a and T. S. Pha. Invariance of the global monodromies in families of polynomials of two complex variables. *Acta Math. Vietnam.*, 22(2):515–526, 1997.
[4] H. V. Hà and A. Zaharia. Families of polynomials with total Milnor number constant. *Math. Ann.*, 313:481–488, 1996.
[5] D. Huybrechts. *Complex Geometry: An Introduction*. Springer, 2005.
[6] A. G. Kouchnirenko. Polyhedres de Newton et nombre de Milnor. *Invent. Math.*, 32:1–31, 1976.
[7] D.T. Le and C.P. Ramanujam. Invariance of Milnors number implies the invariance of topological type. *Amer. J. Math.* 98: 67–78, 1976.
[8] E.J.N. Looijenga. *Isolated singular points on complete intersections*, London Mathematical Society lecture note series 77. Cambridge University Press London-New York, 1984.
[9] J. Milnor. *Singular points of complex hypersurfaces*, Annals of Mathematics Studies 61. Princeton University press, 1968.
[10] T.T.Nguyen, P. P. Pham and T. S. Pham. Bifurcation sets and global monodromies of Newton nondegenerate polynomials on algebraic sets. *Publ. RIMS Kyoto Univ.*, 55: 1–24, 2019.
[11] M. Oka. On the bifurcation of the multiplicity and topology of the Newton boundary. *J. Math. Soc. Japan*, 31: 435–450, 1979.
[12] M. Oka. Principal zeta-function of non-degenerate complete intersection singularity. *J. Fac. Sci. Univ. Tokyo*, 37: 11–32, 1990.
[13] M. Oka. *Non-degenerate complete intersection singularity*, Actualit'es Math'ematiques, Hermann, Paris, 1997.
[14] T. S. Pham. On the topology of the Newton boundary at infinity. *J. Math. Soc. Japan*, 60(4):1065–1081, 2008.
[15] T. S. Pham. Invariance of the global monodromies in families of nondegenerate polynomials in two variables. *Kodai Math. J.*, 33(2):294–309, 2010.

†Institute of Mathematics, 18, Hoang Quoc Viet Road, Cau Giay District 10307, Hanoi, Vietnam

E-mail address: ntthang@math.ac.vn