Abstract

We study whether a depth two neural network can learn another depth two network using gradient descent. Assuming a linear output node, we show that the question of whether gradient descent converges to the target function is equivalent to the following question in electrodynamics: Given $k$ fixed protons in $\mathbb{R}^d$, and $k$ electrons, each moving due to the attractive force from the protons and repulsive force from the remaining electrons, whether at equilibrium all the electrons will be matched up with the protons, up to a permutation. Under the standard electrical force, this follows from the classic Earnshaw’s theorem. In our setting, the force is determined by the activation function and the input distribution. Building on this equivalence, we prove the existence of an activation function such that gradient descent learns at least one of the hidden nodes in the target network. Iterating, we show that gradient descent can be used to learn the entire network one node at a time.

1 Introduction

Deep learning has resulted in major strides in machine learning applications including speech recognition, image classification, and ad-matching. The simple idea of using multiple layers of nodes with a non-linear activation function at each node allows one to express any function. To learn a certain target function we just use (stochastic) gradient descent to minimize the loss; this approach has resulted in significantly lower error rates for several real world functions, such as those in the above applications. Naturally the question remains: how close are we to the optimal values of the network weight parameters? Are we stuck in some bad local minima? While there are several recent works [CHM+15, DPG+14, Kaw16] that have tried to study the presence of local minima, the picture is far from clear.

There has been some work on studying how well can neural networks learn some synthetic function classes (e.g. polynomials [APVZ14], decision trees). In this work we study how well can neural networks learn neural networks with gradient descent? Our focus here, via the framework of
proper learning, is to understand if a neural network can learn a function from the same class (and hence achieve vanishing error).

Specifically, if the target function is a neural network with randomly initialized weights, and we attempt to learn it using a network with the same architecture, then, will gradient descent converge to the target function?

Experimental simulations (see Figure 1 and Section 5 for further details) show that for depth 2 networks of different widths, with random network weights, stochastic gradient descent of a hypothesis network with the same architecture converges to a squared $\ell_2$ error that is a small percentage of a random network, indicating that SGD can learn these shallow networks with random weights. Because our activations are sigmoidal from -1 to 1, the training error starts from a value of about 1 (random guessing) and diminishes quickly to under 0.002. This seems to hold even when the width, the number of hidden nodes, is substantially increased (even up to 125 nodes), but depth is held constant at 2.

In this paper, we attempt to understand this phenomenon theoretically. We prove that, under some assumptions, depth-2 neural networks can learn functions from the same class with vanishingly small error using gradient descent.

![Figure 1: Test Error of Depth 2 Networks of Varying Width.](image)

**Results and Contributions.** We theoretically investigate the question of convergence for networks of depth two. Our main conceptual contribution is that for depth 2 networks where the top node is a sum node, the question of whether gradient descent converges to the desired target function is equivalent to the following question in electrodynamics: Given $k$ fixed protons in $\mathbb{R}^d$, and $k$ moving electrons, with all the electrons moving under the influence of the electrical force of attraction from the protons and repulsion from the remaining electrons, at equilibrium, are all the electrons matched up with all the fixed protons, up to a permutation?

In the above, $k$ is the number of hidden units, $d$ is the number of inputs, the positions of each fixed charge is the input weight vector of a hidden unit in the target network, and the initial positions
of the moving charges are the initial values of the weight vectors for the hidden units in the learning network. The motion of the charges essentially tracks the change in the network during gradient descent. The force between a pair of charges is not given by the standard electrical force of $1/r^2$ (where $r$ is the distance between the charges), but by a function determined by the activation and the input distribution. Thus the question of convergence in these simplified depth two networks can be resolved by studying the equivalent electrodynamics question with the corresponding force function.

Theorem 1.1 (informal statement of Theorem 2.3). Applying gradient descent for learning the output of a depth two network with $k$ hidden units with activation $\sigma$, and a linear output node, under squared loss, using a network of the same architecture, is equivalent to the motion of $k$ charges in the presence of $k$ fixed charges where the force between each pair of charges is given by a potential function that depends on $\sigma$ and the input distribution.

Our main technical contribution is to prove the existence of an activation function such that the corresponding gradient descent dynamics under standard Gaussian inputs result in learning at least one of the hidden nodes in the target network. We then show that this allows us to learn the complete target network one node at a time. We leave open the problem of convergence for dynamics corresponding to more realistic activation functions. We assume the sample complexity is close to its infinite limit.

Theorem 1.2 (informal statement of Theorem 4.1). There is an activation function such that running gradient descent for minimizing the squared loss along with $\ell_2$ regularization for standard Gaussian inputs, at convergence, we learn at least one of the hidden weights of the target neural network.

We prove that the above result can be iterated to learn the entire network node-by-node using gradient descent (Theorem 4.6). Our algorithm learns a network with the same architecture and number of hidden nodes as the target network, in contrast with several existing improper learning results.

In the appendix, we show some weak results for more practical activations. For the sign activation, we show that for the loss with respect to a single node, the only local minima are at the hidden target nodes with high probability if the target network has a randomly picked top layer. For the polynomial activation, we derive a similar result under the assumption that the hidden nodes are orthonormal.

| Name of Activation | Potential ($\Phi(\theta, w)$) | Convergence? |
|--------------------|-------------------------------|--------------|
| Almost $\lambda$-harmonic | Complicated (see Lem 4.2) | Yes, Thm 4.6 |
| Sign | $1 - \frac{2}{\pi} \cos^{-1}(\theta^T w)$ | Yes for $d = 2$, Lem G.2 |
| Polynomial | $(\theta^T w)^m$ | Yes, for orthonormal $w_i$, Lem G.3 |

Table 1: Activation, Potentials, and Convergence Results Summary

**Intuition and Techniques.** Note that for the standard electric potential function given by $\Phi = 1/r$ where $r$ is the distance between the charges, it is known from Earnshaw’s theorem that an electrodynamic system with some fixed protons and some moving electrons is at equilibrium only when the moving electrons coincide with the fixed protons. Given our translation above
between electrodynamic systems and depth 2 networks (Section 2), this would imply learnability of depth 2 networks under gradient descent under $\ell_2$ loss, if the activation function corresponds to the electrostatic potential. However, there exists no activation function $\sigma$ corresponding to this $\Phi$.

The proof of Earnshaw’s theorem is based on the fact that the electrostatic potential is harmonic, i.e., its Laplacian (trace of its Hessian) is identically zero. This ensures that at every critical point, there is direction of potential reduction (unless the hessian is identically zero). We generalize these ideas to potential functions that are eigenfunctions of the Laplacians, $\lambda$-harmonic potentials (Section 3). However, these potentials are unbounded. Subsequently, we construct a non-explicit activation function such that the corresponding potential is bounded and is almost $\lambda$-harmonic, i.e., it is $\lambda$-harmonic outside a small sphere (Section 4). For this activation function, we show at a stable critical point, we must learn at least one of the hidden nodes. Gradient descent (possibly with some noise, as in the work of Ge et al. [GHJY15]) is believed to converge to stable critical points. However, for simplicity, we descend along directions of negative curvature to escape saddle points. Our activation lacks some regularity conditions required in [GHJY15]. We believe the results in [JGN+17] can be adapted to our setting to prove that perturbed gradient descent converges to stable critical points.

There is still a large gap between theory and practice. However, we believe our work can offer some theoretical explanations and guidelines for the design of better activation functions for gradient-based training algorithms. For example, better accuracy and training speed were reported when using the newly discovered exponential linear unit (ELU) activation function in [CUH15, SKS+16]. We hope for more theory-backed answers to these and many other questions in deep learning.

**Related Work.** If the activation functions are linear or if some independence assumptions are made, Kawaguchi shows that the only local minima are the global minima [Kaw16]. Under the spin-glass and other physical models, some have shown that the loss landscape admits well-behaving local minima that occur usually when the overall error is small [CHM+15, DPG+14]. When only training error is considered, some have shown that a global minima can be achieved if the neural network contains sufficiently many hidden nodes [SC16]. Recently, Daniely has shown that SGD learns the conjugate kernel class [Dan17]. Under simplifying assumptions, some results for learning ReLU’s with gradient descent are given in [Tia17, BG17]. Our research is inspired by [APVZ14], where the authors show that for polynomial target functions, gradient descent on neural networks with one hidden layer converges to low error, given a large number of hidden nodes, and under complex perturbations, there are no robust local minima.

Under worst case assumptions, there has been hardness results for even simple networks. A neural network with one hidden unit and sigmoidal activation can admit exponentially many local minima [AIW96]. Backprogration has been proven to fail in a simple network due to the abundance of bad local minima [BRSS9]. Training a 3-node neural network with one hidden layer is NP-complete [BR88]. But, these and many similar worst-case hardness results are based on worst case training data assumptions. However, by using a result in [KS06] that learning a neural network with threshold activation functions is equivalent to learning intersection of halfspaces, several authors showed that under certain cryptographic assumptions, depth-two neural networks are not efficiently learnable with smooth activation functions [LSSS14, ZLWJ15, ZLJ16].

Due to the difficulty of analysis of the non convex gradient descent in deep learning, many have turned to improper learning and the study of non-gradient methods to train neural networks. Janzamin et. al use tensor decomposition methods to learn the shallow neural network weights, provided access to the score function of the training data distribution [JSA15]. Eigenvector and tensor methods are also used to train shallow neural networks with quadratic activation functions in
Combinatorial methods that exploit layerwise correlations in sparse networks have also been analyzed provably in [ABGM14]. Kernel methods, ridge regression, and even boosting were explored for regularized neural networks with smooth activation functions in [SSSS11, ZLWJ15, ZLJ16]. Non-smooth activation functions, such as the ReLU, can be approximated by polynomials and are also amenable to kernel methods [GKKT16]. These methods however are very different from the simple popular SGD.

2 Deep Learning, Potentials, and Electron-Proton Dynamics

Preliminaries. We will work in the space \( M = \mathbb{R}^d \). We denote the gradient and Hessian as \( \nabla_{\mathbb{R}^d} f \) and \( \nabla^2_{\mathbb{R}^d} f \) respectively. The Laplacian is defined as \( \Delta_{\mathbb{R}^d} f = \text{Tr}(\nabla^2_{\mathbb{R}^d} f) \). If \( f \) is multivariate with variable \( x_i \), then let \( f_{x_i} \) be a restriction of \( f \) onto the variable \( x_i \) with all other variables fixed. Let \( \nabla_{x_i} f, \Delta_{x_i} f \) to be the gradient and Laplacian, respectively, of \( f_{x_i} \) with respect to \( x_i \). Lastly, we say \( x \) is a critical point of \( f \) if \( \nabla f \) does not exist or \( \nabla f = 0 \).

We focus on learning depth two networks with a linear activation on the output node. If the network takes inputs \( x \in \mathbb{R}^d \) (say from some distribution \( D \)), then the network output, denoted \( f(x) \) is a sum over \( k = poly(d) \) hidden units with weight vectors \( w_i \in \mathbb{R}^d \), activation \( \sigma(x, w) : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \), and output weights \( b_i \in \mathbb{R} \). Thus, we can write \( f(x) = \sum_{i=1}^k b_i \sigma(x, w_i) \). We denote this concept class \( \mathcal{C}_{\sigma,k} \). Our hypothesis concept class is also \( \mathcal{C}_{\sigma,k} \).

Let \( a = (a_1, ..., a_k) \) and \( \theta = (\theta_1, ..., \theta_k) \); similarly for \( b, w \) and our guess is \( \hat{f}(x) = \sum_{i=1}^k a_i \sigma(x, \theta_i) \). We define \( \Phi \), the potential function corresponding to the activation \( \sigma \), as

\[
\Phi(\theta, w) = \mathbb{E}_{X \sim D} [\sigma(X, \theta) \sigma(X, w)].
\]

We work directly with the true squared loss error \( L(a, \theta) = \mathbb{E}_{x \sim D} [(f - \hat{f})^2] \). To simplify \( L \), we re-parametrize \( a \) by \( -a \) and expand.

\[
L(a, \theta) = \mathbb{E}_{X \sim D} \left[ \left( \sum_{i=1}^k a_i \sigma(X, \theta_i) \right)^2 \right]
= \sum_{i=1}^k \sum_{j=1}^k a_i a_j \Phi(\theta_i, \theta_j) + 2a_i b_j \Phi(\theta_i, w_j) + b_i b_j \Phi(w_i, w_j),
\]

(1)

Given \( D \), the activation function \( \sigma \), and the loss \( L \), we attempt to show that we can use some variant of gradient descent to learn, with high probability, an \( \epsilon \)-approximation of \( w_j \) for some (or all) \( j \). Note that our loss is jointly convex, though it is quadratic in \( a \).

In this paper, we restrict our attention to translationally invariant activations and potentials. Specifically, we may write \( \Phi = h(\theta - w) \) for some function \( h(x) \). Furthermore, a translationally invariant function \( \Phi(r) \) is radial if it is a function of \( r = ||x - y|| \).

Remark: Translationally symmetric potentials satisfy \( \Phi(\theta, \theta) \) is a positive constant. We normalize \( \Phi(\theta, \theta) = 1 \) for the rest of the paper.

We assume that our input distribution \( D = \mathcal{N}(0, I_{d \times d}) \) is fixed as the standard Gaussian in \( \mathbb{R}^d \). This assumption is not critical and a simpler distribution might lead to better bounds. However, for arbitrary distributions, there are hardness results for PAC-learning halfspaces [KS06].

We call a potential function realizable if it corresponds to some activation \( \sigma \). The following theorem characterizes realizable translationally invariant potentials under standard Gaussian inputs. Proofs and a similar characterization for rotationally invariant potentials can be found in Appendix [B].
**Theorem 2.1.** Let $\mathcal{M} = \mathbb{R}^d$ and $\Phi$ is square-integrable and $\mathcal{F}(\Phi)$ is integrable. Then, $\Phi$ is realizable under standard Gaussian inputs if $\mathcal{F}(\Phi)(\omega) \geq 0$ and the corresponding activation is $\sigma(x) = (2\pi)^{d/4} e^{x^T x / 4 \mathcal{F}^{-1}(\sqrt{\mathcal{F}(\Phi)})(x)}$, where $\mathcal{F}$ is the generalized Fourier transform in $\mathbb{R}^d$.

### 2.1 Electron-Proton Dynamics

By interpreting the pairwise potentials as electrostatic attraction potentials, we notice that our dynamics is similar to electron-proton type dynamics under potential $\Phi$, where $w_i$ are fixed point charges in $\mathbb{R}^d$ and $\theta_i$ are moving point charges in $\mathbb{R}^d$ that are trying to find $w_i$. The total force on each charge is the sum of the pairwise forces, determined by the gradient of $\Phi$. We note that standard dynamics interprets the force between particles as an acceleration vector. In gradient descent, it is interpreted as a velocity vector.

**Definition 2.2.** Given a potential $\Phi$ and particle locations $\theta_1, ..., \theta_k \in \mathbb{R}^d$ along with their respective charges $a_1, ..., a_k \in \mathbb{R}$. We define **Electron-Proton Dynamics** under $\Phi$ with some subset $S \subseteq [k]$ of fixed particles to be the solution $(\theta_1(t), ..., \theta_k(t))$ to the following system of differential equations: For each pair $(\theta_i, \theta_j)$, there is a force from $\theta_j$ exerted on $\theta_i$ that is given by $F_i(\theta_j) = a_i a_j \nabla \theta_i \Phi(\theta_i, \theta_j)$ and

$$-\frac{d\theta_i}{dt} = \sum_{j \neq i} F_i(\theta_j)$$

for all $i \not\in S$, with $\theta_i(0) = \theta_i$. For $i \in S$, $\theta_i(t) = \theta_i$.

For the following theorem, we assume that $\theta$ is fixed.

**Theorem 2.3.** Let $\Phi$ be a symmetric potential and $L$ be as in [1]. Running continuous gradient descent on $\frac{1}{2} L$ with respect to $\theta$, initialized at $(\theta_1, ..., \theta_k)$ produces the same dynamics as Electron-Proton Dynamics under $2\Phi$ with fixed particles at $w_1, ..., w_k$ with respective charges $b_1, ..., b_k$ and moving particles at $\theta_1, ..., \theta_k$ with respective charges $a_1, ..., a_k$.

### 3 Earnshaw’s Theorem and Harmonic Potentials

When running gradient descent on a non-convex loss, we often can and do get stuck at a local minima. In this section, we use second-order information to deduce that for certain classes of potentials, there are no spurious local minima. The potentials In this section are often unbounded and un-realizable. However, in the next section, we apply insights developed here to derive similar convergence results for approximations of these potentials. Earnshaw’s theorem in electrodynamics shows that there is no stable local minima for electron-proton dynamics. This hinges on the property that the electric potential $\Phi(\theta, w) = \|\theta - w\|^{2-d}$, $d \neq 2$ is harmonic, with $d = 3$ in natural setting. If $d = 2$, we instead have $\Phi(\theta, w) = -\ln(\|\theta - w\|)$. First, we notice that this is a symmetric loss, and our usual loss in [4] has constant terms that can be dropped to further simplify.

**Definition 3.1.** $\Phi(\theta, w)$ is a harmonic potential on $\Omega$ if $\Delta \Phi(\theta, w) = 0$ for all $\theta \in \Omega$, except possibly at $\theta = w$. 

$$L(a, \theta) = 2 \sum_{i=1}^{k} \sum_{i<j} a_i a_j \Phi(\theta_i, \theta_j) + 2 \sum_{i=1}^{k} \sum_{j=1}^{k} a_i b_j \Phi(\theta_i, w_j)$$  \hspace{1cm} (2)
Then, if \( \Delta \) except when as in is a strict local minimum. WLOG, we can partition \( \text{Theorem 3.6.} \) \( \theta \) when we generalize harmonic potentials. \( \text{Fact 3.3.} \) \( H \) \( i \) \( \iff \) In the next section, we construct realizable potentials that are \( \text{Definition 3.5.} \) \( 3.1 \) \( \lambda \) note that this potential function goes to \( 1 \) \( \text{function} \) any non zero eigenvalue then at least one eigenvalue is negative. This idea results in the following \( \text{decreases} \) \( f \) \( \ast \) \( \min \) \( \lambda_\text{min}(\nabla^2 f(x^*)) \geq 0. \) \( \text{Moreover,} \) if \( \lambda_\text{min}(\nabla^2 f(x^*)) > 0, \) then \( x^* \) is a strict local minimum.

Note that if \( \lambda_\text{min}(\nabla^2 f(x^*)) < 0 \) then moving along the direction of the corresponding eigenvector decreases \( f \) locally. If \( \Phi \) is harmonic then it can be shown the trace of its Hessian is 0 so if there is any non zero eigenvalue then at least one eigenvalue is negative. This idea results in the following known theorem (see full proof in supplementary material) that is applicable to the electric potential function \( 1/r \) in 3-dimensions since is harmonic. It implies that a configuration of \( n \) electrons and \( n \) protons cannot be in a strict local minimum even if one of the mobile charges is isolated (however note that this potential function goes to \( \infty \) at \( r = 0 \) and may not be realizable).

\( \text{Theorem 3.4.} \) (Earnshaw’s Theorem. See \[AKNS85\]) Let \( M = \mathbb{R}^d \) and let \( \Phi \) be harmonic and \( L \) be as in \( 2 \). Then, \( L \) admits no differentiable strict local minima.

Note that the Hessian of a harmonic potential can be identically zero. To avoid this possibility we generalize harmonic potentials.

### 3.1 \( \lambda \)-Harmonic Potentials

In order to relate our loss function with its Laplacian, we consider potentials that are non-negative eigenfunctions of the Laplacian operator. Since the zero eigenvalue case simply gives rise to harmonic potentials, we restrict our attention to positive eigenfunctions.

\( \text{Definition 3.5.} \) A potential \( \Phi \) is \( \lambda \)-harmonic on \( \Omega \) if there exists \( \lambda > 0 \) such that for every \( \theta \in \Omega \), \( \Delta_\theta \Phi(\theta, w) = \lambda \Phi(\theta, w) \), except possibly at \( \theta = w \).

Note that there are realizable versions of these potentials; for example \( \Phi(a, b) = e^{-||a-b||^1} \) in \( \mathbb{R}^1 \).

In the next section, we construct realizable potentials that are \( \lambda \)-harmonic almost everywhere except when \( \theta \) and \( w \) are very close.

\( \text{Theorem 3.6.} \) Let \( \Phi \) be \( \lambda \)-harmonic and \( L \) be as in \( 1 \). Then, \( L \) admits no local minima \((a, \theta)\), except when \( L(a, \theta) = L(0, \theta) \) or \( \theta_i = w_j \) for some \( i, j \).

Proof. Let \((a, \theta)\) be a critical point of \( L \). On the contrary, we assume that \( \theta_i \neq w_j \) for all \( i, j \). WLOG, we can partition \([k]\) into \( S_1, \ldots, S_r \), such that for all \( u \in S_i, v \in S_j \), we have \( \theta_u = \theta_v \) iff \( i = j \). Let \( S_1 = \{\theta_1, \ldots, \theta_l\} \). We consider changing all \( \theta_1, \ldots, \theta_l \) by the same \( v \) and define \( H(a, v) = L(a, \theta_1 + v, \ldots, \theta_l + v, \theta_{l+1}, \ldots, \theta_k) \).

The optimality conditions on \( a \) are \( 0 = \frac{\partial L}{\partial a} = 2 \sum_j a_j \Phi(\theta_i, \theta_j) + 2 \sum_{j=1}^k b_j \Phi(\theta_i, w_j) \). Thus, by the definition of \( \lambda \)-harmonic potentials, we may differentiate as \( \theta_i \neq w_j \) and compute the Laplacian as

\[
\Delta_\theta H = \lambda \sum_{i=1}^l a_i \left( 2 \sum_{j=1}^k b_j \Phi(\theta_i, \theta_j) + 2 \sum_{j=l+1}^k a_j \Phi(\theta_i, \theta_j) \right) \\
= \lambda \sum_{i=1}^l a_i \left( -2 \sum_{j=1}^l a_j \Phi(\theta_i, \theta_j) \right) = -2\lambda \sum_{i=1}^l a_i \left( \sum_{j=1}^l a_j \right) = -2\lambda \left( \sum_{i=1}^l a_i \right)^2
\]
If $\sum_{i=1}^{d} a_i \neq 0$, then we conclude that the Laplacian is strictly negative, so we are not at a local minimum. Similarly, we can conclude that for each $S_i, \sum_{u \in S_i} a_u = 0$. In this case, since $\sum_{i=1}^{k} a_i \sigma(\theta_i, x) = 0$, $L(a, \theta) = L(0, \theta)$. \hfill $\square$

4 Realizable Potentials with Convergence Guarantees

In this section, we derive convergence guarantees for realizable potentials that are almost $\lambda$-harmonic, specifically, they are $\lambda$-harmonic outside of a small neighborhood around the origin. First, we prove the existence of activation functions such that the corresponding potentials are almost $\lambda$-harmonic. Then, we reason about the Laplacian of our loss, as in the previous section, to derive our guarantees. We show that at a stable minima, each of the $\theta_i$ is close to some $w_j$ in the target network. We may end up with a many to one mapping of the learned hidden weights to the true hidden weights, instead of a bijection. To make sure that $\|a\|$ remains controlled throughout the optimization process, we add a quadratic regularization term to $L$ and instead optimize $G = L + \|a\|^2$.

Our optimization procedure is a slightly altered version of gradient descent, where we incorporate a second-order method (which we call Hessian descent as in Algorithm 1) that is used when the gradient is small and progress is slow. The descent algorithm (Algorithm 2) allows us to converge to points with small gradient and small negative curvature. Namely, for smooth functions, in poly(1/\epsilon) iterations, we reach a point in $\mathcal{M}_{G, \epsilon}$, where

$$\mathcal{M}_{G, \epsilon} = \left\{ x \in \mathcal{M} \middle| \|\nabla G(x)\| \leq \epsilon \text{ and } \lambda_{\min}(\nabla^2 G(x)) \geq -\epsilon \right\}$$

We show that if $(a, \theta)$ is in $\mathcal{M}_{G, \epsilon}$ for $\epsilon$ small, then $\theta_i$ is close to $w_j$ for some $j$. Finally, we show how to initialize $(a^{(0)}, \theta^{(0)})$ and run second-order GD to converge to $\mathcal{M}_{G, \epsilon}$, proving our main theorem.

Algorithm 1 $x = HD(L, x_0, T, \alpha)$

Input: $L : \mathcal{M} \to \mathbb{R}$; $x_0 \in \mathcal{M}$; $T \in \mathbb{N}$; $\alpha \in \mathbb{R}$

Initialize $x \leftarrow x_0$

for $i = 1$ to $T$ do

\begin{align*}
\quad & \text{Find unit eigenvector } v_{\min} \text{ corresponding to } \lambda_{\min}(\nabla^2 f(x)) \\
\quad & \beta \leftarrow -\alpha \lambda_{\min}(\nabla^2 f(x)) \text{sign}(\nabla f(x)^T v_{\min}) \\
\quad & x \leftarrow x + \beta v_{\min}
\end{align*}

Algorithm 2 $x = SecondGD(L, x_0, T, \alpha, \eta, \gamma)$

Input: $L : \mathcal{M} \to \mathbb{R}$; $x_0 \in \mathcal{M}$; $T \in \mathbb{N}$; $\alpha, \eta, \gamma \in \mathbb{R}$

for $i = 1$ to $T$ do

\begin{align*}
\quad & \text{if } \|\nabla L(x_{i-1})\| \geq \eta \text{ then } x_i \leftarrow x_{i-1} - \alpha \nabla L(x_{i-1}) \\
\quad & \text{else } x_i \leftarrow HD(L, x_{i-1}, 1, \alpha) \\
\quad & \text{if } L(x_i) \geq L(x_{i-1}) - \min(\alpha \eta^2 / 2, \alpha^2 \gamma^3 / 2) \text{ then return } x_{i-1}
\end{align*}

Theorem 4.1. Let $\mathcal{M} = \mathbb{R}^d$ for $d \equiv 3 \mod 4$ and $k = \text{poly}(d)$. For all $\epsilon \in (0, 1)$, we can construct an activation $\sigma_\epsilon$ such that if $w_1, ..., w_k \in \mathbb{R}^d$ with $w_i \sim \mathcal{N}(0, O(d \log d)I_{d \times d})$ and $b_1, ..., b_k$ be randomly chosen at uniform from $[-1, 1]$, then with high probability, we can choose an initial point $(a^{(0)}, \theta^{(0)})$ such that after running SecondGD (Algorithm 2) on the regularized objective $G(a, \theta)$ for at most $(d/\epsilon)^{O(d)}$ iterations, there exists an $i, j$ such that $\|\theta_i - w_j\| < \epsilon$. 

8
We start by stating a lemma concerning the construction of an almost $\lambda$-harmonic function on $\mathbb{R}^d$. The construction is given in Appendix B and uses a linear combination of realizable potentials that correspond to an activation function of the indicator function of a $n$-sphere. By using Fourier analysis and Theorem 2.1, we can finish the construction of our almost $\lambda$-harmonic potential.

**Lemma 4.2.** Let $\mathcal{M} = \mathbb{R}^d$ for $d \equiv 3 \mod 4$. Then, for any $\epsilon \in (0, 1)$, we can construct a radial activation $\sigma_\epsilon(r)$ such that the corresponding radial potential $\Phi_\epsilon(r)$ is $\lambda$-harmonic for $r \geq \epsilon$.

Furthermore, we have $\Phi_\epsilon^{(d-1)}(r) \geq 0$ for all $r > 0$, $\Phi_\epsilon^{(k)}(r) = 0$, and $\Phi_\epsilon^{(k+1)}(r) \leq 0$ for all $r > 0$ and $d - 3 \geq k \geq 0$ even.

When $\lambda = 1$, $|\Phi_\epsilon^{(k)}(r)| \leq O((d/\epsilon)^{2d})$ for all $0 \leq k \leq d-1$. And when $r \geq \epsilon$, $\Omega(e^{-r}r^{2d}(d/\epsilon)^{-2d}) \leq \Phi_\epsilon(r) \leq O((1 + r)^d e^{-r}(r)^{2-d})$ and $\Omega(e^{-r}r^{1-d}(d/\epsilon)^{-2d}) \leq |\Phi_\epsilon'(r)| \leq O((d + r)(1 + r)^d e^{1-r}r^{1-d})$

Our next lemma use the almost $\lambda$-harmonic properties to show that at an almost stationary point of $G$, we must have converged close to some $w_j$ as long as our charges $a_i$ are not too small. The proof is similar to Theorem 3.6. Then, the following lemma relates the magnitude of the charges $a_i$ to the progress made in the objective function.

**Lemma 4.3.** Let $\mathcal{M} = \mathbb{R}^d$ for $d \equiv 3 \mod 4$ and let $G$ be the regularized loss corresponding to the activation $\sigma_\epsilon$ given by Lemma 4.2 with $\lambda = 1$. For any $\epsilon \in (0, 1)$ and $\delta \in (0, 1)$, if $(a, \theta) \in \mathcal{M}_G, \delta$, then for all $i$, either 1) there exists $j$ such that $\|\theta_i - w_j\| < \kappa$ or 2) $a_i^2 < 2d\delta$.

**Lemma 4.4.** Assume the conditions of Lemma 4.3. If $\sqrt{G(a, \theta)} \leq \sqrt{G(0, 0)} - \delta$ and $(a, \theta) \in \mathcal{M}_{G, \delta^2/(2k^2\delta)}$, then there exists some $i, j$ such that $\|\theta_i - w_j\| < \kappa$.

Finally, we guarantee that our initialization substantially decreases our objective function. Together with our previous lemmas, it will imply that we must be close to some $w_j$ upon convergence. This is the overview of the proof of Theorem 4.1 presented below.

**Lemma 4.5.** Assume the conditions of Theorem 4.1 and Lemma 4.3. With high probability, we can initialize $(a^{(0)}, \theta^{(0)})$ such that $\sqrt{G(a^{(0)}, \theta^{(0)})} \leq \sqrt{G(0, 0)} - \delta$ with $\delta = (d/\epsilon)^{-O(d)}$.

**Proof of Theorem 4.1.** Let our potential $\Phi_{\epsilon/k}$ be the one as constructed in Lemma 4.2 that is $1$-harmonic for all $r \geq \epsilon/k$ and as always, $k = \text{poly}(d)$. First, by Lemma 4.5 we can initialize $(a^{(0)}, \theta^{(0)})$ such that $\sqrt{G(a^{(0)}, \theta^{(0)})} \leq \sqrt{G(0, 0)} - \delta$ for $\delta = (d/\epsilon)^{-O(d)}$. If we set $\alpha = (d/\epsilon)^{-O(d)}$ and $\eta = \gamma = \delta^2/(2k^2d)$, then running Algorithm 2 will terminate and return some $(a, \theta)$ in at most $(d/\epsilon)^{O(d)}$ iterations. This is because our algorithm ensures that our objective function decreases by at least min$(\alpha^2/2, \alpha^2 \gamma^3/2)$ at each iteration, $G(0, 0)$ is bounded by $O(k)$, and $G \geq 0$ is non-negative.

Let $\theta = (\theta_1, \ldots, \theta_k)$. If there exists $\theta_i, w_j$ such that $\|\theta_i - w_j\| < \epsilon$, then we are done. Otherwise, we claim that $(a, \theta) \in \mathcal{M}_G, \delta^2/(2k^2\delta)$. For the sake of contradiction, assume otherwise. By our algorithm termination conditions, then it must be that after one step of gradient or Hessian descent from $(a, \theta)$, we reach some $(a', \theta')$ and $G(a', \theta') > G(a, \theta) - \min(\alpha^2/2, \alpha^2 \gamma^3/2)$.

Now, Lemma 4.2 ensures all first three derivatives of $\Phi_{\epsilon/k}$ are bounded by $O((dk/\epsilon)^{2d})$, except at $w_1, \ldots, w_k$. Furthermore, since there do not exist $\theta_i, w_j$ such that $\|\theta_i - w_j\| < \epsilon$, $G$ is three-times continuously differentiable within a $\alpha(dk/\epsilon)^{2d} = (d/\epsilon)^{-O(d)}$ neighborhood of $\theta$. Therefore, by Lemma D.1 and D.2 in the appendix, we must have $G(a', \theta') \leq G(a, \theta) - \min(\alpha^2/2, \alpha^2 \gamma^3/2)$, a contradiction. Lastly, since our algorithm maintains that our objective function is decreasing, so $\sqrt{G(a, \theta)} \leq \sqrt{G(0, 0)} - \delta$. Finally, we conclude by Lemma 4.4. 

Algorithm 3 Node-wise Descent Algorithm

Input: \((a, \theta) = (a_1, ..., a_k, \theta_1, ..., \theta_k), a_i \in \mathbb{R}, \theta_i \in \mathcal{M}; T \in \mathbb{N}; L; \alpha, \eta, \gamma \in \mathbb{R};\)

for \(i = 1\) to \(k\) do

Initialize \((a_i, \theta_i)\)

\((a_i, \theta_i) = \text{SecondGD} (L_{a_i, \theta_i}, (a_i, \theta_i), T, \alpha, \eta, \gamma)\)

return \(a = (a_1, ..., a_k), \theta = (\theta_1, ..., \theta_k)\)

4.1 Node-by-Node Analysis

We cannot easily analyze the convergence of gradient descent to the global minima when \(\theta_i\) are simultaneously moving since the pairwise interaction terms between the \(\theta_i\) present complications, even with added regularization. Instead, we run a greedy node-wise descent (Algorithm 3) to learn the hidden weights, i.e. we run a descent algorithm with respect to \((a_i, \theta_i)\) sequentially. The main idea is that after running SGD with respect to \(\theta_1, \theta_1\) should be close to some \(w_j\) for some \(j\). Then, we can carefully induct and show that \(\theta_2\) must be some other \(w_k\) for \(k \neq j\) and so on.

Let \(L_1(a_1, \theta_1)\) be the objective \(L\) restricted to \(a_1, \theta_1\) being variable, and \(a_2, ..., a_k = 0\) are fixed. The tighter control on the movements of \(\theta_1\) allows us to remove our regularization. While our previous guarantees before allow us to reach a \(\epsilon\)-neighborhood of \(w_j\) when running SGD on \(L_1\), we will strengthen our guarantees to reach a \((d/\epsilon)^{-O(d)}\)-neighborhood of \(w_j\), by reasoning about the first derivatives of our potential in an \(\epsilon\)-neighborhood of \(w_j\). By similar argumentation as before, we will be able to derive the following convergence guarantees for node-wise training.

Theorem 4.6. Let \(\mathcal{M} = \mathbb{R}^d\) and \(d \equiv 3 \mod 4\) and let \(L\) be as in (1) and \(k = \text{poly}(d)\). For all \(\epsilon \in (0, 1)\), we can construct an activation \(\sigma\) such that if \(w_1, ..., w_k \in \mathbb{R}^d\) with \(w_i\) randomly chosen from \(w_i \sim \mathcal{N}(0, O(d \log d)I_{d \times d})\) and \(b_1, ..., b_k\) be randomly chosen at uniform from \([-1, 1]\), then with high probability, after running nodewise descent (Algorithm 3) on the objective \(L\) for at most \((d/\epsilon)^{O(d)}\) iterations, \((a, \theta)\) is in a \((d/\epsilon)^{-O(d)}\) neighborhood of the global minima.

5 Experiments

For our experiments, our training data is given by \((x_i, f(x_i))\), where \(x_i\) are randomly chosen from a standard Gaussian in \(\mathbb{R}^d\) and \(f\) is a randomly generated neural network with weights chosen from a standard Gaussian. We run gradient descent (Algorithm 4) on the empirical loss, with stepsize around \(\alpha = 10^{-5}\), for \(T = 10^6\) iterations. The nonlinearity used at each node is sigmoid from -1 to 1, including the output node, unlike the assumptions in the theoretical analysis. A random guess for the network will result in a mean squared error of around 1. Our experiments (see Fig 1) show that for depth-2 neural networks, even with non-linear outputs, the training error diminishes quickly to under 0.002. This seems to hold even when the width, the number of hidden nodes, is substantially increased (even up to 125 nodes), but depth is held constant; although as the number of nodes increases, the rate of decrease is slower. This substantiates our claim that depth-2 neural networks are learnable.

However, it seems that for depth greater than 2, the test error becomes significant when width is high (see Fig 2). Even for depth 3 networks, the increase in depth impedes the learnability of the neural network and the training error does not get close enough to 0. It seems that for neural networks with greater depth, positive convergence results in practice are elusive. We note that we are using training error as a measure of success, so it’s possible that the true underlying parameters are not learned.
Figure 2: Test Error of Varying-Depth Networks vs. Width

Table 2: Test Error of Learning Neural Networks of Various Depth and Width

| Depth | Width 5  | Width 10 | Width 20 | Width 40 |
|-------|----------|----------|----------|----------|
| 2     | 0.0015   | 0.0017   | 0.0018   | 0.0019   |
| 3     | 0.0033   | 0.0264   | 0.1503   | 0.2362   |
| 5     | 0.0036   | 0.0579   | 0.2400   | 0.4397   |
| 9     | 0.0085   | 0.1662   | 0.4171   | 0.6071   |
| 17    | 0.0845   | 0.3862   | 0.4934   | 0.5777   |

References

[ABGM14] Sanjeev Arora, Aditya Bhaskara, Rong Ge, and Tengyu Ma. Provable bounds for learning some deep representations. In *ICML*, pages 584–592, 2014.

[AHW96] Peter Auer, Mark Herbster, and Manfred K Warmuth. Exponentially many local minima for single neurons. pages 316–322, 1996.

[AKNS5] Vladimir I Arnold, Valery V Kozlov, and Anatoly I Neishtadt. Mathematical aspects of classical and celestial mechanics. *Encyclopaedia Math. Sci.*, 3:1–291, 1985.

[APVZ14] Alexandr Andoni, Rina Panigrahy, Gregory Valiant, and Li Zhang. Learning polynomials with neural networks. In *International Conference on Machine Learning*, pages 1908–1916, 2014.

[BG17] Alon Brutzkus and Amir Globerson. Globally optimal gradient descent for a convnet with gaussian inputs. *arXiv preprint arXiv:1702.07966*, 2017.

[BR88] Avrim Blum and Ronald L. Rivest. Training a 3-node neural network is np-complete. pages 9–18, 1988.

[BRS89] Martin L Brady, Raghu Raghavan, and Joseph Slawny. Back propagation fails to separate where perceptrons succeed. *IEEE Transactions on Circuits and Systems*, 36(5):665–674, 1989.
[CHM+15] Anna Choromanska, Mikael Henaff, Michael Mathieu, Gérard Ben Arous, and Yann LeCun. The loss surfaces of multilayer networks. In AISTATS, 2015.

[CUH15] Djork-Arné Clevert, Thomas Unterthiner, and Sepp Hochreiter. Fast and accurate deep network learning by exponential linear units (elus). CoRR, abs/1511.07289, 2015.

[Dan17] Amit Daniely. Sgd learns the conjugate kernel class of the network. arXiv preprint arXiv:1702.08503, 2017.

[DFS16] Amit Daniely, Roy Frostig, and Yoram Singer. Toward deeper understanding of neural networks: The power of initialization and a dual view on expressivity. In Advances In Neural Information Processing Systems, pages 2253–2261, 2016.

[DPG+14] Yann N Dauphin, Razvan Pascanu, Caglar Gulcehre, Kyunghyun Cho, Surya Ganguli, and Yoshua Bengio. Identifying and attacking the saddle point problem in high-dimensional non-convex optimization. In Advances in neural information processing systems, pages 2933–2941, 2014.

[GHJY15] Rong Ge, Furong Huang, Chi Jin, and Yang Yuan. Escaping from saddle points online stochastic gradient for tensor decomposition. In COLT, pages 797–842, 2015.

[GKKT16] Surbhi Goel, Varun Kanade, Adam Klivans, and Justin Thaler. Reliably learning the relu in polynomial time. arXiv preprint arXiv:1611.10258, 2016.

[JGN+17] Chi Jin, Rong Ge, Praneeth Netrapalli, Sham M Kakade, and Michael I Jordan. How to escape saddle points efficiently. arXiv preprint arXiv:1703.00887, 2017.

[JSA15] Majid Janzamin, Hanie Sedghi, and Anima Anandkumar. Beating the perils of non-convexity: Guaranteed training of neural networks using tensor methods. CoRR, abs/1506.08473, 2015.

[Kaw16] Kenji Kawaguchi. Deep learning without poor local minima. In Advances in Neural Information Processing Systems, pages 586–594, 2016.

[KS06] Adam R Klivans and Alexander A Sherstov. Cryptographic hardness for learning intersections of halfspaces. In 2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS’06), pages 553–562. IEEE, 2006.

[LSSS14] Roi Livni, Shai Shalev-Shwartz, and Ohad Shamir. On the computational efficiency of training neural networks. In Advances in Neural Information Processing Systems, pages 855–863, 2014.

[SC16] Daniel Soudry and Yair Carmon. No bad local minima: Data independent training error guarantees for multilayer neural networks. CoRR, abs/1605.08361, 2016.

[SKS+16] Anish Shah, Eashan Kadam, Hena Shah, Sameer Shinde, and Sandip Shingade. Deep residual networks with exponential linear unit. In Proceedings of the Third International Symposium on Computer Vision and the Internet, pages 59–65. ACM, 2016.

[SSSS11] Shai Shalev-Shwartz, Ohad Shamir, and Karthik Sridharan. Learning kernel-based halfspaces with the 0-1 loss. SIAM Journal on Computing, 40(6):1623–1646, 2011.

[Tia17] Yuandong Tian. An analytical formula of population gradient for two-layered relu network and its applications in convergence and critical point analysis. arXiv preprint arXiv:1703.00560, 2017.

[ZLJ16] Yuchen Zhang, Jason D Lee, and Michael I Jordan. 11-regularized neural networks are improperly learnable in polynomial time. In International Conference on Machine Learning, pages 993–1001, 2016.

[ZLWJ15] Yuchen Zhang, Jason D. Lee, Martin J. Wainwright, and Michael I. Jordan. Learning halfspaces and neural networks with random initialization. CoRR, abs/1511.07948, 2015.
A Electron-Proton Dynamics

Theorem 2.3. Let $\Phi$ be a symmetric potential and $L$ be as in \[1\]. Running continuous gradient descent on $\frac{1}{2}L$ with respect to $\theta$, initialized at $(\theta_1, \ldots, \theta_k)$ produces the same dynamics as Electron-Proton Dynamics under $2\Phi$ with fixed particles at $w_1, \ldots, w_k$ with respective charges $b_1, \ldots, b_k$ and moving particles at $\theta_1, \ldots, \theta_k$ with respective charges $a_1, \ldots, a_k$.

Proof. The initial values are the same. Notice that continuous gradient descent on $L(a, \theta)$ with respect to $\theta$ produces dynamics given by

$$
\frac{d\theta_i(t)}{dt} = -\nabla_{\theta_i} L(a, \theta).
$$

Therefore,

$$
\frac{d\theta_i(t)}{dt} = -2 \sum_{j \neq i} a_i a_j \nabla_{\theta_i} \Phi(\theta_i, \theta_j) - 2 \sum_{j=1}^{k} a_i b_j \nabla_{\theta_i} \Phi(\theta_i, w_j)
$$

And gradient descent does not move $w_i$. By definition, the dynamics corresponds to Electron-Proton Dynamics as claimed. \qed

B Realizable Potentials

B.1 Activation-Potential Calculations

First define the dual of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined to be

$$
\widehat{f}(\rho) = \mathbb{E}_{X,Y \sim N(\rho)}[f(X)f(Y)],
$$

where $N(\rho)$ is the bivariate normal distribution with $X,Y$ unit variance and $\rho$ covariance. This is as in \[DFS16\].

Lemma B.1. Let $\mathcal{M} = S^{d-1}$ and $\sigma$ be our activation function, then $\widehat{\sigma}$ is the corresponding potential function.

Proof. If $u, v$ have norm 1 and if $X$ is a standard Gaussian in $\mathbb{R}^d$, then note that $X_1 = u^T X$ and $X_2 = v^T X$ are both standard Gaussian variables in $\mathbb{R}^1$ and the covariance is $E[X_1 X_2] = u^T v$.

Therefore, the dual function of the activation gives us the potential function.

$$
\mathbb{E}_{X} \sigma(u^T X) \sigma(v^T X) = \mathbb{E}_{X,Y \sim N(u^T v)}[\sigma(X)\sigma(Y)]
= \widehat{\sigma}(u^T v).
$$

By Lemma B.1, the calculations of the activation-potential for the sign, ReLU, Hermite, exponential functions are given in \[DFS16\]. For the Gaussian and Bessel activation functions, we can calculate directly. In both case, we notice that we may write the integral as a product of integrals in each dimension. Therefore, it suffices to check the following 1-dimensional identities.

$$
\int_{-\infty}^{\infty} \sqrt{2}e^{x^2/4}e^{-(x-\theta)^2} \int_{-\infty}^{\infty} \sqrt{2}e^{x^2/4}e^{-(x-w)^2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
= \sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-(x-\theta)^2} e^{-(x-w)^2} dx = e^{-(\theta-w)^2/2}
$$
The last equality follows by Fourier uniqueness and taking the Fourier transform of both sides, which are both equality $\sqrt{2/\pi}(\omega^2 + 1)^{-1}$.

### B.2 Characterization Theorems

**Theorem 2.1.** Let $\mathcal{M} = \mathbb{R}^d$ and $\Phi$ is square-integrable and $\mathfrak{F}(\Phi)$ is integrable. Then, $\Phi$ is realizable under standard Gaussian inputs if $\mathfrak{F}(\Phi)(\omega) \geq 0$ and the corresponding activation is $\sigma(x) = (2\pi)^{d/4}e^{x^2/4}\mathfrak{F}^{-1}((\mathfrak{F}(\Phi))(x))$, where $\mathfrak{F}$ is the generalized Fourier transform in $\mathbb{R}^d$.

**Proof.** Since $\Phi$ is square-integrable, its Fourier transform exists. Let $h(x) = \mathfrak{F}^{-1}((\sqrt{\mathfrak{F}(\Phi)}))(x)$ and this is well-defined since the Fourier transform was non-negative everywhere and the Fourier inverse exists since $\sqrt{\mathfrak{F}(\Phi)}(x)$ is square-integrable. Now, let $\sigma(x,w) = (2\pi)^{d/4}e^{x^2/4}h(x-w)$. Realizability follows by the Fourier inversion theorem:

\[
\mathbb{E}_{X \sim \mathcal{N}}[\sigma(X,w)\sigma(X,\theta)] = \int_{\mathbb{R}^n} h(x-w)h(x-\theta) \, dx \\
= \int_{\mathbb{R}^n} h(x)h(x-(\theta-w)) \, dx \\
= \mathfrak{F}^{-1}((\mathfrak{F}(h*h))(\theta-w)) \\
= \mathfrak{F}^{-1}((\mathfrak{F}(h))^2(\theta-w)) \\
= \mathfrak{F}^{-1}((\mathfrak{F}(\Phi))(\theta-w)) \\
= \Phi(\theta-w)
\]

Note that $*$ denotes function convolution. \hfill $\square$

When our relevant space is $\mathcal{M} = S^{d-1}$, we let $\Pi_\mathcal{M}$ be the projection operator on $\mathcal{M}$. The simplest way to define the gradient on $S^{d-1}$ is $\nabla_{S^{d-1}} f(x) = \nabla_{\mathbb{R}^d} f(x/\|x\|)$, where $\|\cdot\|$ denotes the $l_2$ norm and $x \in S^{d-1}$. The Hessian and Laplacian are analogously defined and the subscripts are usually dropped where clear from context.

We say that a potential $\Phi$ on $\mathcal{M} = S^{d-1}$ is rotationally invariant if for all $\theta,w \in S^{d-1}$, we have $\Phi = h(\theta^T w)$.

**Theorem B.2.** Let $\mathcal{M} = S^{d-1}$ and $\Phi(\theta,w) = f(\theta^T w)$. Then, $\Phi$ is realizable if $f$ has non-negative Taylor coefficients, $c_i \geq 0$, and the corresponding activation $\sigma(x) = \sum_{i=1}^{\infty} \sqrt{c_i} h_i(x)$ converges almost everywhere, where $h_i(x)$ is the $i$-th Hermite polynomial.

**Proof.** By [B.1] and due to the orthogonality of hermite polynomials, if $f = \sum_i a_i h_i$, where $h_i(x)$ is the $i$-th Hermite polynomial, then

\[
\hat{f}(\rho) = \sum_i a_i^2 \rho^i
\]

Therefore, any function with non-negative taylor coefficients is a valid potential function, with the corresponding activation function determined by the sum of hermite polynomials, and the sum is bounded almost everywhere by assumption. \hfill $\square$
B.3 Further Characterizations

To apply Theorem 2.1, we need to check that the Fourier transform of our function is non-negative. Not only is this not straightforward to check, many of our desired potentials do not satisfy this criterion. In this section, we would like to have a stronger characterization of realizable potentials, allowing us to construct realizable potentials that approximates our desired potential.

Definition B.3. Let \( \Phi \) be a positive semidefinite function if for all \( x_1, \ldots, x_n \), the matrix \( A_{ij} = \Phi(x_i - x_j) \) is positive semidefinite.

Lemma B.4. Let \( \mathcal{M} = \mathbb{R}^d \) and \( \Phi(\theta, w) = f(\theta - w) \) is realizable, then it is positive semidefinite.

Proof. If \( \Phi \) is realizable, then there exists \( \sigma \) such that \( \Phi(\theta, w) = \mathbb{E}_{X \sim \mathcal{N}}[\sigma(X, w)\sigma(X, \theta)] \). For \( x_1, \ldots, x_n \), we note that the quadratic form:

\[
\sum_{i,j} \Phi(x_i, x_j)v_i v_j = \sum_{i,j} \mathbb{E}_{X \sim \mathcal{N}}[\sigma(X, x_i)\sigma(X, x_j)] v_i v_j = \mathbb{E}_{X \sim \mathcal{N}} \left[ \left( \sum_i v_i \sigma(X, x_i) \right)^2 \right] \geq 0
\]

Since \( \Phi \) is translationally symmetric, we conclude that \( \Phi \) is positive semidefinite.

Definition B.5. A potential \( \Phi \) is \( \mathfrak{F} \)-integrable if it is square-integrable and \( \mathfrak{F}(\Phi(\omega)) \) is integrable, where \( \mathfrak{F} \) is the standard Fourier transform.

Lemma B.6. Let \( w(x) \geq 0 \) be a positive weighting function such that \( \int_{a}^{b} w(x) \, dx \) is bounded. If \( \Phi_x \) is a parametrized family of \( \mathfrak{F} \)-integrable realizable potentials, then, \( \int_{a}^{b} w(x)\Phi_x \) is \( \mathfrak{F} \)-integrable realizable.

Proof. Let \( \Phi = \int_{a}^{b} w(x)\Phi_x \). From linearity of the Fourier transform and \( \int_{a}^{b} w(x) \, dx \) is bounded, we know that \( \Phi \) is \( \mathfrak{F} \)-integrable. Since \( \Phi_x \) are realizable, they are positive definite by Lemma B.4 and by Bochner's theorem, their Fourier transforms are non-negative. And since \( w(x) \geq 0 \), we conclude by linearity and continuity of the Fourier transform that \( \mathfrak{F}(\Phi) \geq 0 \). By Theorem 2.1, we conclude that \( \Phi \) is realizable.

Lemma B.7. Let \( \mathcal{M} = \mathbb{R}^d \) for \( d \equiv 3 \mod 4 \). Then, for any \( \epsilon, t > 0 \), there exists a \( \mathfrak{F} \)-integrable realizable \( \Phi \) such that for \( t \geq r > \epsilon \), \( \Phi^{(d-1)}(r) = t - r \) and for \( r \leq \epsilon \), \( \Phi^{(d-1)}(r) = \frac{t^d}{\epsilon^d} r \). Furthermore, \( \Phi^{(k)}(r) = 0 \) for \( r > t \) for all \( 0 \leq k \leq d \).

Proof. Our construction is based on the radial activation function \( h_t(x, \theta) = 1_{\|\theta - x\| \leq t/2} \), which is the indicator in the disk of radius \( t/2 \). This function, when re-weighted correctly as \( \sigma_t(x, \theta) = (2\pi)^{1/4} \epsilon^{d/4}h_t(x, \theta) \) gives rise to a radial potential function that is simply the convolution of \( h_t \) with itself, measuring the volume of the intersection of two spheres of radius \( t \) centered at \( \theta \) and \( w \).

\[
\Phi_t(\theta, w) = \mathbb{E}_{X} [\sigma_t(X, \theta)\sigma_t(X, w)] = \left\{ \begin{array}{ll} C \int_{\|\theta - w\|/2}^{t/2} ((t/2)^2 - x^2)^{(d-1)/2} \, dx & \|\theta - w\| \leq t \\ 0 & \text{otherwise} \end{array} \right.
\]

Therefore, as a function of \( r = \|\theta - w\| \), we see that when \( r \leq t \), \( \Phi_t(r) = C \int_{r/2}^{t/2} ((t/2)^2 - x^2)^{(d-1)/2} \, dx \) and \( \Phi_t'(r) = -C'((t/2)^2 - (r/2)^2)^{(d-1)/2} \). Since \( d \equiv 3 \mod 4 \), we notice that \( \Phi_t' \) has a positive coefficient in the leading \( r^{d-1} \) term and since it is a function of \( r^2 \), it has a zero \( r^{d-2} \) term. Therefore, we can scale \( \Phi_t \) such that

\[
\Phi_t^{(d-1)}(r) = \left\{ \begin{array}{ll} r & r \leq t \\ 0 & \text{otherwise} \end{array} \right.
\]
\( \Phi_t \) is clearly realizable and now we claim that it is \( \mathcal{F} \)-integrable. First, \( \Phi_t \) is bounded on a compact set so it is square-integrable. Now, since \( \Phi_t = h_t * h_t \) can be written as a convolution, \( \mathcal{F}(\Phi_t) = \mathcal{F}(h_t)^2 \). Since \( h_t \) is square integrable, then by Parseval’s, \( \mathcal{F}(h_t) \) is square integrable, allowing us to conclude that \( \Phi_t \) is \( \mathcal{F} \)-integrable.

Now, for any \( \epsilon > 0 \), let us construct our desired \( \Phi \) by taking a positive sum of \( \Phi_t \) and then appealing to Lemma B.6 consider

\[
\Phi(r) = \int_{\epsilon}^{t} \frac{1}{x^2} \Phi_x(r) \, dx
\]

First, note that the total weight \( \int_{\epsilon}^{t} \frac{1}{x^2} \) is bounded. Then, when \( r \geq t \), since \( \Phi_x(r) = 0 \) for \( x \leq t \), we conclude that \( \Phi^{(k)}(r) = 0 \) for any \( k \). Otherwise, for \( \epsilon < r < t \), we can apply dominated convergence theorem to get

\[
\Phi^{(d-1)}(r) = \int_{\epsilon}^{r} \frac{1}{x^2} \Phi^{(d-1)}_x(r) \, dx + \int_{r}^{t} \frac{1}{x^2} \Phi^{(d-1)}_x(r) \, dx = 0 + \int_{r}^{t} \frac{r}{x^2} \, dx = 1 - r/t
\]

Scaling by \( t \) gives our desired claim. For \( r \leq \epsilon \), we integrate similarly and scale by \( t \) to conclude.

**Lemma B.8.** Let \( \mathcal{M} = \mathbb{R}^d \) for \( d \equiv 3 \mod 4 \) and let \( \Phi(r) \) be a radial potential. Also, \( \Phi^{(k)}(r) \geq 0 \) and \( \Phi^{(k+1)}(r) \leq 0 \) for all \( r > 0 \) and \( k \geq 0 \) even, and \( \lim_{r \to \infty} \Phi^{(k)}(r) = 0 \) for all \( 0 \leq k \leq d \).

Then, for any \( \epsilon > 0 \), there exists a \( \mathcal{F} \)-integrable realizable potential \( \Phi \) such that \( \Phi^{(k)}(r) = \Phi^{(k)}(r) \) for all \( 0 \leq k \leq d - 1 \) and \( r \geq \epsilon \). Furthermore, we have \( \Phi^{(d-1)}(r) \geq 0 \) for all \( r > 0 \) and \( \Phi^{(k)}(r) \geq 0 \) and \( \Phi^{(k+1)}(r) \leq 0 \) for all \( r > 0 \) and \( d - 3 \geq k \geq 0 \) even.

Lastly, for \( r < \epsilon \) and \( 0 \leq k \leq d - 1 \), \( |\Phi^{(d-1-k)}(r)| \leq |\Phi^{(d-1-k)}(\epsilon)| + \sum_{j=1}^{k} \frac{(r-\epsilon)^{k-j+1}}{(k-j+1)!} |\Phi^{(d-j)}(\epsilon)|

**Proof.** By Lemma B.7 we can find \( \Phi_t \) such that

\[
\Phi_t^{(d-1)} = \begin{cases} 
\frac{t-r}{\epsilon}r & 0 \leq r \leq \epsilon \\
\frac{t-r}{\epsilon}r & \epsilon < r \leq t \\
0 & r > t
\end{cases}
\]

Furthermore, \( \Phi_t^{(k)}(r) = 0 \) for \( r > t \) for all \( 0 \leq k \leq d \). Therefore, we consider

\[
\Phi(r) = \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) \Phi_x(r) \, dx
\]

Note that this is a positive sum with \( \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) \, dx = -\Phi^{(d)}(\epsilon) < \infty \). By the non-negativity of our summands, we can apply dominated convergence theorem and Fubini’s theorem to get

\[
\Phi^{(d-1)}(r) = \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) (\Phi_x^{(d-1)}(r)) \, dx
\]

\[
= \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) (\Phi_x^{(d-1)}(r)) \, dx
\]

\[
= \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) \int_{r}^{x} 1 \, dy \, dx
\]

\[
= \int_{\epsilon}^{\infty} \int_{r}^{\infty} \Phi^{(d+1)}(x) \, dx \, dy = \int_{r}^{\infty} -\Phi^{(d)}(y) \, dy
\]

\[
= \Phi^{(d-1)}(r)
\]
Now, since $\Phi^{(d-1)}(r) = \Phi^{(d-1)}(r)$ for $r \geq \epsilon$ and $\lim_{r \to \infty} \Phi^{(k)}(r) = \lim_{r \to \infty} \Phi^{(k)}(r) = 0$ for $0 \leq k \leq d - 1$, repeated integration gives us our claim.

Finally, for the second claim, notice that for $r \leq \epsilon$, we get

$$\Phi^{(d-1)}(r) = \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) \Phi^{(d-1)}(r) \, dx = r \int_{\epsilon}^{\infty} \Phi^{(d+1)}(x) \frac{x - \epsilon}{\epsilon} \, dx = C r$$

Note that our constant $C \geq 0$ since the summands are non-negative. Therefore, we conclude that $\Phi^{(d-1)}(r) \geq 0$ for all $r > 0$. Repeated integration and noting that $\lim_{r \to \infty} \Phi^{(k)}(r) = 0$ for $0 \leq k \leq d - 1$ gives us our claim.

Lastly, we prove the last claim of the theorem with induction on $k$. This holds trivially for $k = 0$ since $\Phi^{(d-1)}(r) \leq \Phi^{(d-1)}(\epsilon) = \Phi^{(d-1)}(\epsilon)$ for $r \leq \epsilon$. Then, assume we have the inequality for $k < d - 1$. By integration, we have

$$|\Phi^{(d-k-2)}(r)| \leq |\Phi^{(d-k-2)}(\epsilon)| + \int_{\epsilon}^{r} |\Phi^{(d-1-k)}(y)| \, dy$$

$$\leq |\Phi^{(d-k-2)}(\epsilon)| + \int_{\epsilon}^{r} \frac{(\epsilon - y)^{-k+1}}{(k - j + 1)!} |\Phi^{(d-j)}(\epsilon)| \, dy$$

$$+ \int_{r}^{\infty} \frac{(\epsilon - y)^{k-j+1}}{(k - j + 1)!} |\Phi^{(d-j)}(\epsilon)| \, dy$$

$$\leq |\Phi^{(d-k-2)}(\epsilon)| + \sum_{j=1}^{k+1} \frac{(\epsilon - y)^{k-j+2}}{(k - j + 2)!} |\Phi^{(d-j)}(\epsilon)|$$

Therefore, we conclude with induction. \qed

Lemma 4.2. Let $\mathcal{M} = \mathbb{R}^d$ for $d \equiv 3 \mod 4$. Then, for any $\epsilon \in (0, 1)$, we can construct a radial activation $\sigma_\epsilon(r)$ such that the corresponding radial potential $\Phi_\epsilon(r)$ is $\lambda$-harmonic for $r \geq \epsilon$.

Furthermore, we have $\Phi_\epsilon^{(d-1)}(r) \geq 0$ for all $r > 0$, $\Phi_\epsilon^{(k)}(r) \geq 0$, and $\Phi_\epsilon^{(k+1)}(r) \leq 0$ for all $r > 0$ and $d - 3 \geq k \geq 0$ even.

When $\lambda = 1$, $|\Phi_\epsilon^{(k)}(r)| \leq O((d/\epsilon)^{2d})$ for all $0 \leq k \leq d - 1$. And when $r \geq \epsilon$, $\Omega^{-r^2-2d}((d/\epsilon)^{-2d}) \leq \Phi_\epsilon(r) \leq O((1 + r^2)^{d/\epsilon} \Omega^{-r^2-2d})$.

Proof. This is a special case of the following lemma. \qed

Lemma B.9. Let $\mathcal{M} = \mathbb{R}^d$ for $d \equiv 3 \mod 4$. Then, for any $1 > \epsilon > 0$, we can construct a radial activation $\sigma_\epsilon(r)$ with corresponding normalized radial potential $\Phi_\epsilon(r)$ that is $\lambda$-harmonic when $r \geq \epsilon$.

Furthermore, we have $\Phi_\epsilon^{(d-1)}(r) \geq 0$ for all $r > 0$ and $\Phi_\epsilon^{(k)}(r) \geq 0$ and $\Phi_\epsilon^{(k+1)}(r) \leq 0$ for all $r > 0$ and $d - 3 \geq k \geq 0$ even.

Also, $|\Phi_\epsilon^{(k)}(r)| \leq 3(2d + \sqrt{\lambda})^{2d} \epsilon^{-2d} e^{\sqrt{\lambda}}$ for all $0 \leq k \leq d - 1$. And for $r \geq \epsilon$, $e^{-\sqrt{\lambda}r^2-2d}((d + \sqrt{\lambda})^{2d} \epsilon^{-2d}) \leq \Phi_\epsilon(r) \leq (1 + \sqrt{\lambda})^{2d} e^{-\sqrt{\lambda}(1-r)^2 d}$. Also for $r \geq \epsilon$, $e^{-\sqrt{\lambda}r^2-2d}((d + \sqrt{\lambda})^{2d} \epsilon^{-2d}) \leq |\Phi_\epsilon^{(k)}(r)| \leq \Phi_\epsilon^{(d-1)}(r) \leq 1 + \sqrt{\lambda}r$.

Proof. Consider a potential of the form $\Phi(r) = p(r)e^{-\sqrt{\lambda}r^2/r^d-2}$. We claim that there exists a polynomial $p$ of degree $k = (d-3)/2$ with non-negative coefficients and $p(1) = 0$ such that $\Phi$ is $\lambda$-harmonic. Furthermore, we will also show along the way that $p(r) \leq (1 + \sqrt{\lambda})^d$.

When $d = 3$, it is easy to check that $\Phi(r) = e^{(\sqrt{\lambda}(1-r))^2} \Phi$ is our desired potential. Otherwise, by our formula for the radial Laplacian in $d$ dimensions, we want to solve the following differential equation:

$$\Delta \Phi = \frac{1}{r^{d-1}} \frac{\partial}{\partial r} (r^{d-1} \frac{\partial \Phi}{\partial r}) = \lambda \Phi$$
Solving this gives us the following second-order differential equation on $p$

$$rp'' - (d - 3 + 2\sqrt{\lambda}r)p' + \sqrt{\lambda}(d - 3)p = 0$$

Let us write $p(r) = \sum_{i=0}^{k} a_ir^i$. Then, substituting into our differential equation gives us the following equations by setting each coefficient of $i$ to zero:

$$r^k: \ a_{i+1}(i+1)(i-(d-3)) = a_i\sqrt{\lambda}(2i-(d-3))$$

$$r^k: (-2k+d-3)a_k = 0$$

The last equation explains why we chose $k = (d-3)/2$, so that it is automatically zero. Thus, setting $a_0 = 1$ and running the recurrence gives us our desired polynomial. Note that the recurrence is valid and produces positive coefficients since $i < k = (d-3)/2$. Our claim follows and $\Phi$ is $\lambda$-harmonic. And furthermore, notice that $a_{i+1} \leq \sqrt{\lambda}a_i \leq (\sqrt{\lambda})^{i+1}$. Therefore, $p(r) \leq (1 + r\sqrt{\lambda})^d$.

Lastly, we assert that $\Phi^{(j)}(r)$ is non-negative for $j$ even and non-positive for $j$ odd. To prove our assertion, we note that it suffices to show that if $\Phi$ is of the form $\Phi(r) = p(r)e^{-\sqrt{\lambda}r}/r^d$ for some $p$ of degree $k < l$ and $p$ has non-negative coefficients, then $\Phi'(r) = -q(r)e^{-\sqrt{\lambda}r}/r^{d+1}$ for some $q$ of degree $k + 1$ with non-negative coefficients.

Differentiating $\Phi$ gives:

$$\Phi' = e^{-\frac{r}{d+1}}(rp'(r) - (l + \sqrt{\lambda}r)p(r))$$

It is clear that if $p$ has degree $k$, then $q(r) = (l + \sqrt{\lambda}r)p(r) - rp'(r)$ has degree $k + 1$, so it suffices to show that it has non-negative coefficients. Let $p_0, ..., p_k$ be the non-negative coefficients of $p$. Then, by our formula, we see that

$$q_0 = lp_0$$
$$q_i = lp_i - ip_i + \sqrt{\lambda}p_{i-1} = (l-i)p_i + \sqrt{\lambda}p_{i-1}$$
$$q_{k+1} = \sqrt{\lambda}p_k$$

Since $i \leq k < l$, we conclude that $q$ has non-negative coefficients. Finally, our assertion follows with induction since $\Phi^{(0)}(r)$ is non-negative and has our desired form with $k = (d-3)/2 < d - 2$.

By Lemma B.8, our primary theorem follows, we can construct a realizable radial potential $\Phi(r)$ that is $\lambda$-harmonic when $r \geq \epsilon$ and has alternating-signed derivatives.

Lastly, we prove the following preliminary bound on $\Phi^{(k)}(r)$ when $k \leq d$: $|\Phi^{(k)}(r)| \leq 3(2d + \epsilon\sqrt{\lambda})^{2d}e^{-2d}$ for all $0 \leq k \leq d - 1$. First, notice that by the results of Lemma B.8, $\Phi^{(k)}(r)$ is monotone and $\lim_{r \to 0} \Phi^{(k)}(r) = 0$. So, it follows that we just have to bound $|\Phi^{(k)}(0)|$. From our construction, $\Phi^{(k)}(\epsilon) = p_k(\epsilon)e^{-\sqrt{\lambda}\epsilon^{2d-2k}}$, for some polynomial $p_k$. Furthermore, from our construction, we have the recurrence $p_k(\epsilon) = (d-2+k+\sqrt{\lambda}e)p_{k-1}(\epsilon) - \epsilon p_{k-1}(\epsilon)$. Therefore, we conclude that for $k \leq d$, $p_k(\epsilon) \leq (2d + \sqrt{\lambda}e)^kp_0(\epsilon) \leq (2d + \sqrt{\lambda}e)^k(1 + \sqrt{\lambda}e)^d \leq (2d + \sqrt{\lambda}e)^{2d}$.

Therefore, we can bound $|\Phi^{(k)}(\epsilon)| \leq (2d + \sqrt{\lambda}e)^{2d}e^{-2d}$. Finally, by Lemma B.8

$$|\Phi^{(d-1-k)}(0)| \leq |\Phi^{(d-1-k)}(\epsilon)| + \sum_{j=1}^{k} \frac{e^{k-j+1}}{(k-j+1)!} |\Phi^{(d-j)}(\epsilon)|$$

$$\leq (2d + \sqrt{\lambda}e)^{2d}e^{-2d}(1 + \sum_{j=1}^{k} \frac{e^{k-j+1}}{(k-j+1)!})$$

$$\leq (2d + \sqrt{\lambda}e)^{2d}e^{-2d}e^\epsilon \leq 3(2d + \sqrt{\lambda}e)^{2d}e^{-2d}$$

And for $r \geq \epsilon$, we see that $|\Phi(\epsilon)| = |\Phi(r)| \leq |p(r)|\frac{e^{-\sqrt{\lambda}r}}{r^{d-2}} = (1 + r\sqrt{\lambda})^d e^{-\sqrt{\lambda}r^{2-d}}$. And $|\Phi'(r)| = |\Phi'(r)| \leq |p_1(\epsilon)|\frac{e^{-\sqrt{\lambda}r}}{r^d} \leq (d + \sqrt{\lambda}r)(1 + r\sqrt{\lambda})^de^{-\sqrt{\lambda}r^{1-d}}$. And
Finally, we consider the normalized potential: $\tilde{\Phi}_e = \Phi_e / \Phi_e(0)$. Note that since $\Phi_e$ is monotonically decreasing, we can lower bound $\Phi_e(0) \geq \Phi_e(\epsilon) \geq e^{-\sqrt{X}}$. Therefore, we can derive the following upper bounds: $|\Phi_e^\prime(r)| \leq 3(2d + \sqrt{X})^2e^{-2d/e\sqrt{X}}$ and for $r \geq \epsilon$, $|\Phi_e^\prime(r)| \leq (1 + r\sqrt{X})^d e^{\sqrt{X}(1-r)r^{2-d}}$ and its derivative is bounded by $|\Phi_e^\prime(r)| \leq (d + \sqrt{X})^d e^{\sqrt{X}(1-r)r^{1-d}}$.

And lastly, we derive some lower bounds on $\tilde{\Phi}_e$ and the first derivative when $r \geq \epsilon$, by using the upper bound on $\Phi_e(0)$: $\tilde{\Phi}_e^\prime(r) \geq |\Phi_e^\prime(r)| (2d + \sqrt{X})^{-2d^2/3} \geq e^{-\sqrt{X}/2}(2d + \sqrt{X})^{-2d^2/3}$. For the derivative, we get $|\tilde{\Phi}_e^\prime(r)| \geq e^{-\sqrt{X}r^{1-d}(2d + \sqrt{X})^{-2d^2/3}}$. \hfill \square

Lemma B.10. The $\lambda$-harmonic radial potential $\Phi(r) = e^{-r}/r$ in 3-dimensions is realizable by the activation $\sigma(r) = K_1(r)/r$.

Proof. The activation is obtained from the potential function by first taking its Fourier transform, then taking its square root, and then taking the inverse Fourier transform. Since the functions in consideration are radially symmetric the Fourier transform $F(y)$ of $f(x)$ (and inverse) are obtained by the Hankel Transform $yF(y) = \int_0^\infty x f(x) J_{1/2}(xy) \sqrt{xy} dx$. Plugging $f(x) = e^{-x}/x$, from the Hankel transform tables we get $yF(y) = cy/(1 + y^2)$, giving $F(y) = cy/(1 + y^2)$. So we wish to find the inverse Fourier transform for $1/\sqrt{1 + y^2}$. The inverse $f(x)$ is given by $xf(x) = \int_0^\infty y F(y) J_{1/2}(xy) \sqrt{xy} dy = cK_1(x)$. So $\sigma(r) = K_1(r)/r$. \hfill \square

C Earnshaw’s Theorem

Theorem 3.4. (Earnshaw’s Theorem. See [AKNS5]) Let $\mathcal{M} = \mathbb{R}^d$ and let $\Phi$ be harmonic and $L$ be as in (2). Then, $L$ admits no differentiable strict local minima.

Proof. If $(a, \theta)$ is a differentiable strict local minima, then for any $i$, we must have

$$\nabla_{\theta_i} L = 0, \text{ and } Tr(\nabla^2_{\theta_i} L) > 0.$$  

Since $\Phi$ is harmonic, we also have

$$Tr(\nabla^2_{\theta_i} L(\theta_1, \ldots, \theta_n)) = \Delta_{\theta_i} L = 2 \sum_{j \neq i} a_i a_j \Delta_{\theta_j} \Phi(\theta_i, \theta_j) + 2 \sum_{j=1}^k a_i b_j \Delta_{\theta_j} \Phi(\theta_i, w_j) = 0,$$

which is a contradiction. In the first line, there is a factor of 2 by symmetry. \hfill \square

D Descent Lemmas and Iteration Bounds

Algorithm 4 $x = GD(L, x_0, T, \alpha)$

Input: $L : \mathcal{M} \rightarrow \mathbb{R}; x_0 \in \mathcal{M}; T \in \mathbb{N}; \alpha \in \mathbb{R}$

Initialize $x = x_0$

for $i = 1$ to $T$ do

$x = x - \alpha \nabla L(x)$

$x = \Pi_{\mathcal{M}} x$

Lemma D.1. Let $f : \Omega \rightarrow \mathbb{R}$ be a thrice differentiable function such that $|f(y)| \leq B_0, \|\nabla f(y)\| \leq B_1, \|\nabla^2 f(y)\| \leq B_2, \|\nabla^3 f(z) - \nabla^3 f(L(y))\| \leq B_3\|z - y\|$ for all $y, z$ in a $(\alpha B_1)$-neighborhood of $x$. If $\|\nabla f(x)\| \geq \eta$ and $x'$ is reached after one iteration of gradient descent (Algorithm 4) with stepsize $\alpha \leq \frac{1}{2B_2}$, then $\|x' - x\| \leq \alpha B_1$ and $f(x') \leq f(x) - \alpha \eta^2/2$.  


Proof. The gradient descent step is given by \( x' = x - \alpha \nabla f(x) \). The bound on \( \| x' - x \| \) is clear since \( \| \nabla f(x) \| \leq B_1 \).

\[
f(x') \leq f(x) - \alpha \nabla f(x)^T \nabla f(x) + \alpha^2 \frac{B_2}{2} \| \nabla f(x) \|^2 \]

\[
\leq f(x) - (\alpha - \alpha^2 \frac{B_2}{2}) \eta^2
\]

For \( 0 \leq \alpha \leq \frac{1}{B_2} \), we have \( \alpha - \alpha^2 \frac{B_2}{2} \geq \alpha / 2 \), and our lemma follows. \( \square \)

Lemma D.2. Let \( f : \Omega \to \mathbb{R} \) be a thrice differentiable function such that \( |f(y)| \leq B_0, \| \nabla f(y) \| \leq B_1, \| \nabla^2 f(y) \| \leq B_2, \| \nabla^2 f(z) - \nabla^2 L(y) \| \leq B_3 \| z - y \| \) for all \( y, z \) in a \((\alpha B_2)\)-neighborhood of \( x \). If \( \lambda_{\min}(\nabla^2 f(x)) \leq -\gamma \) and \( x' \) is reached after one iteration of Hessian descent (Algorithm 1) with stepsize \( \alpha \leq \frac{1}{B_3} \), then \( \| x' - x \| \leq \alpha B_2 \) and \( f(x') \leq f(x) - \alpha^2 \frac{B_3}{2} \).

Proof. The gradient descent step is given by \( x' = x + \beta v_{\min} \), where \( v_{\min} \) is the unit eigenvector corresponding to \( \lambda_{\min}(\nabla^2 f(x)) \) and \( \beta = -\alpha \lambda_{\min}(\nabla^2 f(x)) \text{sgn}(\nabla f(x)^T v_{\min}) \). Our bound on \( \| x' - x \| \) is clear since \( |\lambda_{\min}(\nabla^2 f(x))| \leq B_2 \).

\[
f(x') \leq f(x) + \beta \nabla f(x)^T v_{\min} + \beta^2 v_{\min}^T \nabla^2 f(x) v_{\min} + \frac{B_3}{6} \beta^3 \| v_{\min} \|^3
\]

\[
\leq f(x) - |\beta|^2 \gamma + \frac{B_3}{6} |\beta|^3
\]

The last inequality holds since the sign of \( \beta \) is chosen so that \( \beta \nabla f(x)^T v_{\min} \leq 0 \). Now, since \( |\beta| = \alpha \gamma \leq \frac{\gamma}{B_3}, -|\beta|^2 \gamma + \frac{B_3}{6} |\beta|^3 \leq -\alpha^2 \gamma^3 / 2 \). \( \square \)

E Convergence of Almost \( \lambda \)-Harmonic Potentials

Lemma 4.3. Let \( M = \mathbb{R}^d \) for \( d \equiv 3 \mod 4 \) and let \( G \) be the regularized loss corresponding to the activation \( \sigma_\epsilon \), given by Lemma 4.2 with \( \lambda = 1 \). For any \( \epsilon \in (0, 1) \) and \( \delta \in (0, 1) \), if \( (a, \theta) \in M_{G, \delta} \), then for all \( i \), either 1) there exists \( j \) such that \( \| \theta_i - w_j \| < k \epsilon \) or 2) \( a_i^2 < 2 k d \delta \).

Proof. The proof is similar to Theorem 3.6. Let \( \Phi_\epsilon \) be the realizable potential in 4.2 such that \( \Phi_\epsilon(r) \) is \( \lambda \)-harmonic when \( r \geq \epsilon \) with \( \lambda = 1 \). Note that \( \Phi_\epsilon(0) = 1 \) is normalized. And let \( (a, \theta) \in M_{G, \delta} \).

WLOG, consider \( \theta_1 \) and a initial set \( S_0 = \{ \theta_1 \} \) containing it. For a finite set of points \( S \) and a point \( x \), define \( d(x, S) = \min_{y \in S} \| x - y \| \). Then, we consider the following set growing process. If there exists \( \theta_i, w_i \notin S_j \) such that \( d(\theta_i, S_j) < \epsilon \) or \( d(w_i, S_j) < \epsilon \), add \( \theta_i, w_i \) to \( S_j \) to form \( S_{j+1} \). Otherwise, we stop the process. We grow \( S_0 \) to until the process terminates and we have the grown set \( S \).

If there is some \( w_j \in S \), then it must be the case that there exists \( j_1, \ldots, j_q \) such that \( \| \theta_1 - \theta_{j_1} \| < \epsilon \) and \( \| \theta_{j_1} - \theta_{j_1+1} \| < \epsilon \), and \( \| \theta_{j_q} - w_j \| < \epsilon \) for some \( w_j \). So, there exists \( j \) such that \( \| \theta_1 - w_j \| < k \epsilon \). Otherwise, notice that for each \( \theta_i \in S, \| w_j - \theta_i \| \geq \epsilon \) for all \( j \), and \( \| \theta_i - \theta_j \| \geq \epsilon \) for all \( \theta_j \notin S \).

WLOG, let \( S = \{ \theta_1, \ldots, \theta_l \} \).

We consider changing all \( \theta_1, \ldots, \theta_l \) by the same \( v \) and define

\[
H(a, v) = G(a, \theta_1 + v, \ldots, \theta_l + v, \theta_{l+1}, \ldots, \theta_k).
\]

The optimality conditions on \( a \) are

\[
\left| \frac{\partial H}{\partial a_i} \right| = \left| 4a_i + 2 \sum_{j \neq i} a_j \Phi_\epsilon(\theta_i, \theta_j) + 2 \sum_{j=1}^k b_j \Phi_\epsilon(\theta_i, w_j) \right| \leq \delta
\]
Next, since \( \Phi(r) \) is \( \lambda \)-harmonic for \( r \geq \epsilon \), we may calculate the Laplacian of \( H \) as

\[
\Delta H = \sum_{i=1}^{l} \lambda \left( 2 \sum_{j=1}^{k} a_i b_j \Phi(\theta_i, w_j) + 2 \sum_{j=1}^{k} a_i a_j \Phi(\theta_i, \theta_j) \right)
\]

\[
\leq \sum_{i=1}^{l} \lambda \left( -4a_i^2 - 2 \sum_{j=1, j \neq i}^{l} a_i a_j \Phi(\theta_i, \theta_j) \right) + \delta \sum_{i=1}^{l} \lambda |a_i|
\]

\[
= -2\lambda \mathbb{E} \left( \sum_{i=1}^{l} a_i \sigma(\theta_i, X) \right)^2 - 2\lambda \sum_{i=1}^{l} a_i^2 + \delta \sum_{i=1}^{l} |a_i|
\]

The second line follows from our optimality conditions and the third line follows from completing the square. Since \((a, \theta) \in \mathcal{M}_{G, \delta} \), we have \( \Delta H \geq -2kd\delta \). Let \( S = \sum_{i=1}^{l} a_i^2 \). Then, by Cauchy-Schwarz, we have \(-2\lambda S + \delta \lambda \sqrt{k} \sqrt{S} \geq -2kd\delta \). When \( S \geq \delta^2 k \), we see that \(-\lambda S \geq -2\lambda S + \delta \lambda \sqrt{k} \sqrt{S} \geq -2kd\delta \). Therefore, \( S \leq 2kd\delta / \lambda \).

We conclude that \( S \leq \max(\delta^2 k, 2kd\delta / \lambda) \leq 2kd\delta / \lambda \) since \( \delta \leq 1 \leq 2d / \lambda \) and \( \lambda = 1 \). Therefore, \( a_i^2 \leq 2kd\delta \).

**Lemma 4.4.** Assume the conditions of Lemma 4.3. If \( \sqrt{G(a, \theta)} \leq \sqrt{G(0, 0)} - \delta \) and \((a, \theta) \in \mathcal{M}_{G, \delta^2/(2k^2 d)} \), then there exists some \( i, j \) such that \( \|\theta_i - w_j\| < k \epsilon \).

**Proof.** If there does not exist \( i, j \) such that \( \|\theta_i - w_j\| < k \epsilon \), then by Lemma 4.3, this implies \( a_i^2 < \delta^2 / k^2 \) for all \( i \). Now, for an integrable function \( f(x) \), \( \| f \|_X = \mathbb{E}_X |f(X)| \) is a norm. Therefore, if \( f(x) = \sum_i b_i \sigma(w_i, x) \) be our true target function, we conclude that by triangle inequality

\[
\sqrt{G(a, \theta)} \geq \left\| \sum_{i=1}^{k} a_i \sigma(\theta_i, x) - f(x) \right\|_X
\]

\[
\geq \| f(x) \|_X - \sum_{i=1}^{k} \| a_i \sigma(\theta_i, x) \|_X
\]

\[
\geq \sqrt{G(0, 0)} - \delta
\]

This gives a contradiction, so we conclude that there must exist \( i, j \) such that \( \theta_i \) is in a \( k \epsilon \) neighborhood of \( w_j \).

**Lemma 4.5.** Assume the conditions of Theorem 4.1 and Lemma 4.3. With high probability, we can initialize \((a^{(0)}, \theta^{(0)})\) such that \( \sqrt{G(a^{(0)}, \theta^{(0)})} \leq \sqrt{G(0, 0)} - \delta \) with \( \delta = (d / \epsilon)^{-O(d)} \).

**Proof.** Consider choosing \( \theta_1 = 0 \) and then optimizing \( a_1 \). Given \( \theta_1 \), the loss decrease is:

\[
G(a_1, 0) - G(0, 0) = \min_{a_1} 2a_1^2 + 2 \sum_{j=1}^{k} a_1 b_j \Phi(0, w_j) = -\frac{1}{2} \left( \sum_{j=1}^{k} b_j \Phi(0, w_j) \right)^2
\]

Because \( w_j \) are random Gaussians with variance \( O(d \log d) \), we have \( \| w_j \| \leq O(d \log d) \) with high probability for all \( j \). By Lemma 4.2, our potential satisfies \( \Phi(0, w_j) \geq (d / \epsilon)^{-O(d)} \). And since \( b_j \) are uniformly chosen in \([-1, 1] \), we conclude that with high probability over the choices of \( b_j \),

\[
-\frac{1}{2} \left( \sum_{j=1}^{k} b_j \Phi(\theta_1, w_j) \right)^2 \geq (d / \epsilon)^{-O(d)}
\]

by appealing to Chebyshev’s inequality on the squared term.
Therefore, we conclude that with high probability, \( G(a_1, 0) \leq G(0, 0) - \frac{1}{2}(d/\epsilon)^{-O(d)} \). Let \( \sqrt{G(a_1, 0)} = \sqrt{G(0, 0)} - \Delta \geq 0 \). Squaring and rearranging gives \( \Delta \geq \frac{1}{4\sqrt{G(0, 0)}}(d/\epsilon)^{-O(d)} \). Since \( G(0, 0) \leq O(k) = O(\text{poly}(d)) \), we are done.

\[ \Box \]

### E.1 Node by Node Analysis

The first few lemmas are similar to the ones proven before in the simultaneous case. The proof are presented for completeness because the regularization terms are removed. Note that our loss function is quadratic in \( a \). Therefore, let \( a_1^*(\theta_1) \) denote the optimal value of \( a_1 \) to minimize our loss.

**Lemma E.1.** Let \( M = \mathbb{R}^d \) for \( d \equiv 3 \mod 4 \) and let \( L_1 \) be the loss restricted to \( (a_1, \theta_1) \) corresponding to the activation function \( \sigma \) given by Lemma 4.2 with \( \lambda = 1 \). For any \( \epsilon \in (0, 1) \) and \( \delta \in (0, 1) \), we can construct \( \sigma \) such that if \( (a_1, \theta_1) \in M_{L_1, \delta} \), then for all \( i \), either 1) there exists \( j \) such that \( \|\theta_1 - w_j\| < \epsilon \) or 2) \( a_1^2 < 2\delta \).

**Proof.** The proof is similar to Lemma 4.3. Let \( \Phi_\epsilon \) be the realizable potential in 4.2 such that \( \Phi_\epsilon(r) \) is \( \lambda \)-harmonic when \( r \geq \epsilon \). Note that \( \Phi_\epsilon(0) = 1 \) is normalized. And let \( (a_1, \theta_1) \in M_{L_1, \delta} \). Assume that there does exist \( w_j \) such that \( \|\theta_1 - w_j\| < \epsilon \).

The optimality condition on \( a_1 \) is

\[
\left| \frac{\partial L}{\partial a_1} \right| = 2a_1 + 2 \sum_{j=1}^k b_j \Phi_\epsilon(\theta_1, w_j) \leq \delta
\]

Next, since \( \Phi_\epsilon(r) \) is \( \lambda \)-harmonic for \( r \geq \epsilon \), we may calculate the Laplacian of \( L \) as

\[
\Delta_{\theta_1} L = \lambda \left( 2 \sum_{j=1}^k a_1 b_j \Phi_\epsilon(\theta_1, w_j) \right) \leq -2\lambda a_1^2 + \delta |\theta_1|
\]

The inequality follows from our optimality conditions. Since \( (a_1, \theta_1) \in M_{L_1, \delta} \), we have \( \Delta_{\theta_1} L \geq -2d\delta \).

When \( a_1^2 \geq \delta^2 \), we see that \(-\lambda a_1^2 \geq -2\lambda a_1^2 + \delta |\theta_1| \geq -2d\delta \). Therefore, \( a_1^2 \leq 2d\delta/\lambda \). We conclude that \( a_1^2 \leq \text{max}(\delta^2, 2d\delta/\lambda) \leq 2d\delta/\lambda \) for \( \delta \leq 2d/\lambda \) since \( \lambda = 1 \). Therefore, \( a_1^2 \leq 2d\delta \).

**Lemma E.2.** Assume the conditions of Lemma E.1. If \( \sqrt{L_1(a_1, \theta_1)} \leq \sqrt{L_1(0, 0)} - \delta \) and \( (a_1, \theta_1) \in M_{G, 3d^2/(2d)} \), then there exists some \( j \) such that \( \|\theta_1 - w_j\| < \epsilon \).

**Proof.** The proof follows similarly from Lemma 4.4.

Now, our main observation is below, showing that in a neighborhood around \( w_j \), descending along the gradient direction will move \( \theta_1 \) closer to \( w_j \). Our tighter control of the gradient of \( \Phi_\epsilon \) around \( w_j \) will eventually allow us to show that \( \theta_1 \) converges to a small neighborhood around \( w_j \).

**Lemma E.3.** Assume the conditions of Theorem E.5 and Lemma E.1. If \( \|\theta_1 - w_j\| \leq d \) and \( |b_j| \geq 1/\text{poly}(d) \) and \( |a_1 - a_1^*(\theta_1)| \) is almost optimal and for \( i \), \( \|w_i - w_j\| \geq \Omega(d \log d) \), then \( -\nabla_{\theta_1} L_1 = \zeta \frac{w_j - \theta_1}{\|\theta_1 - w_j\|} + \xi \) with \( \zeta \geq \frac{1}{\text{poly}(d)} (d/\epsilon)^{-8d} \) and \( \xi \leq (d/\epsilon)^{-O(d)} \).

**Proof.** Through the proof, we assume \( k = \text{poly}(d) \). Now, our gradient with respect to \( \theta_1 \) is

\[
\nabla_{\theta_1} L_1 = 2a_1 b_j \Phi_\epsilon(\|\theta_1 - w_j\|) \frac{\theta_1 - w_j}{\|\theta_1 - w_j\|} + 2 \sum_{i \neq j} a_1 b_i \Phi_\epsilon(\|\theta_1 - w_i\|) \frac{\theta_1 - w_i}{\|\theta_1 - w_i\|}
\]
Since $\|\theta_1 - w_j\| \leq d$, we may lower bound $|\Phi'_i(\|\theta_1 - w_j\|)| \geq e^{-\sqrt{d}d^{1-d}(2d + \sqrt{d})-2d/3 \geq \Omega((d/\epsilon)^{-4d})}$. Similarly, $\Phi_i(\|\theta_1 - w_j\|) \geq \Omega((d/\epsilon)^{-4d})$. On the other hand since $\|w_i - w_j\| \geq \Omega(d \log d)$ for all $i \neq j$, we may upper bound $|\Phi_i(\|\theta_1 - w_i\|)| \leq (d/\epsilon)^{-O(d)}$ and $|\Phi'_i(\|\theta_1 - w_i\|)| \leq (d/\epsilon)^{-O(d)}$.

Together, we conclude that $\nabla_{a_1} L_1 = 2a_1 b_j \Phi'_i(\|\theta_1 - w_j\|) \frac{\theta_1 - w_j}{\|\theta_1 - w_j\|} + 2a_1 \xi$, where $|\xi| \leq (d/\epsilon)^{-O(d)}$.

By assumption, $|a_1 - a_1^*(\theta_1)| \leq (d/\epsilon)^{-O(d)}$, so

$$ \left| \frac{\partial L_1}{\partial a_1} \right| = 2|a_1 + 2b_j \Phi_i(\|\theta_1 - w_j\|) + 2 \sum_{i \neq j} b_i \Phi_i(\|\theta_1 - w_i\|) | \leq (d/\epsilon)^{-O(d)}$$

By a similar argument as on the derivative, we see that $a_1 = -b_j \Phi_i(\|\theta_1 - w_j\|) + (d/\epsilon)^{-O(d)}$. Therefore, the direction of $-\nabla_{a_1} L_1$ is moving $\theta_1$ closer to $w_j$ since

$$ -\nabla_{a_1} L_1 = b_j^2 \Phi_i(\|\theta_1 - w_j\|) \Phi'_i(\|\theta_1 - w_j\|) \frac{\theta_1 - w_j}{\|\theta_1 - w_j\|} + (d/\epsilon)^{-O(d)} $$

and we know $\Phi_i > 0$ and $\Phi'_i < 0$, thereby $-b_j^2 \Phi_i(\|\theta_1 - w_j\|) \Phi'_i(\|\theta_1 - w_j\|) \geq 1/(\text{poly}(d)(d/\epsilon)^{-8d})$.

**Lemma E.4** (Node-wise Initialization). Assume the conditions of Theorem E.3 and Lemma E.7. With high probability, we can initialize $(a_0^{(0)}, \theta_1^{(0)})$ such that $\sqrt{L(a_0^{(0)}, \theta_1^{(0)})} \leq \sqrt{L(0, 0)} - \delta$ with $\delta = \frac{1}{\text{poly}(d)}(d/\epsilon)^{-18d}$ in time $\text{log}(d)^{O(d)}$.

**Proof.** By our conditions, there must exist some $|b_j|$ such that $|b_j| \geq 1/\text{poly}(d)$ and for all $i$, $\|w_i - w_j\| \geq \Omega(d \log d)$. Note that if we randomly sample points in a ball of radius $O(d \log d)$, we will land in a $d$-neighborhood of $w_j$ with probability $\log(d)^{-O(d)}$ since $\|w_j\| \leq O(d \log d)$.

Let $\theta_1$ be such that $\|\theta_1 - w_j\| \leq d$ and then we can solve for $a_1 = a_1^*(\theta_1)$ since we are simply minimizing a quadratic in one variable. Then, by Lemma E.3 we see that $\|\nabla_{a_1} L_1\| \geq 1/(\text{poly}(d)(d/\epsilon)^{-8d})$. Finally, by Lemma E.4 we know that the Hessian is bounded by $\text{poly}(d)(d/\epsilon)^{2d}$. So, by Lemma D.4 we conclude by taking a stepsize of $\alpha = \frac{1}{\text{poly}(d)}(d/\epsilon)^{-2d}$ to reach $(a_1', \theta_1')$, we can guarantee that $L_1(a_1', \theta_1') \leq L_1(a_1^*(\theta_1), \theta_1) - \frac{1}{\text{poly}(d)}(d/\epsilon)^{-18d}$.

But since $L_1(a_1^*(\theta_1), \theta_1) \leq L_1(0, 0)$, we conclude that $L_1(a_1', \theta_1') \leq L_1(0, 0) - \frac{1}{\text{poly}(d)}(d/\epsilon)^{-18d}$. Let $\sqrt{L_1(a_1', \theta_1')} = \sqrt{L_1(0, 0)} - \Delta \geq 0$. Squaring and rearranging gives $\Delta \geq \frac{1}{4\sqrt{L_1(0, 0)}} \frac{1}{\text{poly}(d)}(d/\epsilon)^{-18d}$. Since $L_1(0, 0) \leq O(k) = O(\text{poly}(d))$, we are done.

**Lemma E.5.** Assume the conditions of Lemma E.4. Also, assume $b_1, ..., b_k$ are any numbers in $[-1, 1]$ and $w_1, ..., w_k \in \mathbb{R}^d$ satisfy $\|w_i\| \leq O(d \log d)$ for all $i$ and there exists some $|b_j| \geq 1/\text{poly}(d)$ with $\|w_i - w_j\| \geq \Omega(d \log d)$ for all $i$.

Then with high probability, we can choose an initial point $(a_0^{(0)}, \theta_1^{(0)})$ such that after running SecondGD (Algorithm 2) on the restricted regularized objective $L_1(a_1, \theta_1)$ for at most $(d/\epsilon)^{O(d)}$ iterations, there exists some $w_j$ such that $\|\theta_1 - w_j\| < \epsilon$. Furthermore, if $|b_j| \geq 1/\text{poly}(d)$ and $\|w_i - w_j\| \geq \Omega(d \log d)$ for all $i$, then $\|\theta_1 - w_j\| < (d/\epsilon)^{-O(d)}$ and $|a + b_j| < (d/\epsilon)^{-O(d)}$.

**Proof.** First, by Lemma E.4 we can initialize $(a_0^{(0)}, \theta_1^{(0)})$ such that $\sqrt{L_1(a_0^{(0)}, \theta_1^{(0)})} \leq \sqrt{L_1(0, 0)} - \delta$ for $\delta = \frac{1}{\text{poly}(d)}(d/\epsilon)^{-18d}$. If we set $\alpha = (d/\epsilon)^{-O(d)}$ and $\eta = \gamma = \lambda \delta^2/(2d)$, then running Algorithm 2 will terminate and return some $(a_1, \theta_1)$ in at most $(d/\epsilon)^{O(d)}$ iterations. This is because our algorithm
ensures that our objective function decreases by at least \(\min(\alpha n^2/2, \alpha^2 \gamma^3/2)\) at each iteration and \(G(0, 0)\) is bounded by \(O(k)\) and \(G \geq 0\) is non-negative.

Assume there does not exist \(w_j\) such that \(||\theta_1 - w_j\|| < (d/\epsilon)^{-O(d)}\). Then, we claim that \((a_1, \theta_1) \in \mathcal{M}_{L, \lambda^2/(2d)}\). For the sake of contradiction, assume otherwise. By our algorithm termination conditions, then it must be that after one step of gradient or Hessian descent from \((a_1, \theta_1)\), we reach some \((a', \theta')\) and \(L_1(a', \theta') > L_1(a_1, \theta_1) - \min(\alpha n^2/2, \alpha^2 \gamma^3/2)\). Now, Lemma 4.2 ensures all first three derivatives of \(\Phi\) are bounded by \((d/\epsilon)^{2d}\), except at \(w_1, ..., w_k\). Furthermore, since there does not exist \(w_j\) such that \(||\theta_1 - w_j\|| < (d/\epsilon)^{-O(d)}\), \(L_1\) is three-times continuously differentiable within a \(\alpha(d/\epsilon)^{2d} = (d/\epsilon)^{-O(d)}\) neighborhood of \(\theta_1\). Therefore, by Lemma D.1 and D.2 we know that \(L(a', \theta') \leq L_1(a', \theta') \leq L_1(a_1, \theta_1) - \min(\alpha n^2/2, \alpha^2 \gamma^3/2)\), a contradiction.

So, it must be \((a_1, \theta_1) \in \mathcal{M}_{L, \lambda^2/(2d)}\). Since our algorithm maintains that our objective function is decreasing, so \(\sqrt{L_1(a_1, \theta_1)} \leq \sqrt{L_1(0, 0)} - \delta\). So, by Lemma E.2 there must be some \(w_j\) such that \(||\theta - w_j\|| \leq \epsilon\).

Now, if \(|b_j| \geq 1/\text{poly}(d)\) and \(||w_i - w_j|| \geq \Omega(d \log d)\) for all \(i\), then since \((a, \theta) \in \mathcal{M}_{L, \lambda^2/(2d)}\) and \(||\theta - w_j|| \leq \epsilon\), by Lemma E.3 we have \(||\nabla_\theta L_1|| \geq 1/\text{poly}(d)(d/\epsilon)^{-8d} > \delta^2/(2d)\), a contradiction. Therefore, we must conclude that our original assumption was false and \(||\theta - w_j|| < (d/\epsilon)^{-O(d)}\) for some \(w_j\).

Finally, we see that the charges also converge since \(a = -2b_j \Phi_\epsilon(||\theta - w_j||) + O(d/\epsilon)^{-O(d)}\) and \(||\theta - w_j|| = (d/\epsilon)^{-O(d)}\). By noting that \(\Phi_\epsilon(0) = 1\) and \(\Phi_\epsilon\) is \((d/\epsilon)^{2d}\)-Lipschitz, we conclude.

Finally, we have our final theorem.

Theorem 4.6. Let \(\mathcal{M} = \mathbb{R}^d\) and \(d \equiv 3 \mod 4\) and let \(L\) be as in \([7]\) and \(k = \text{poly}(d)\). For all \(\epsilon \in (0, 1)\), we can construct an activation \(\sigma\) such that if \(w_1, ..., w_k \in \mathbb{R}^d\) with \(w_i\) randomly chosen from \(w_i \sim \mathcal{N}(0, O(d \log d)I_d)\) and \(b_1, ..., b_k\) be randomly chosen at uniform from \([-1, 1]\), then with high probability, after running nodewise descent (Algorithm 3) on the objective \(L\) for at most \((d/\epsilon)^{O(d)}\) iterations, \((a, \theta)\) is in a \((d/\epsilon)^{-O(d)}\) neighborhood of the global minima.

Proof. Let our potential \(\Phi_\epsilon\) be the one as constructed in Lemma 4.2 that is \(\lambda\)-harmonic for all \(r \geq \epsilon\) with \(\lambda = 1\). Let \((a_i, \theta_i)\) be the \(i\)-th node that is initialized and applied second order gradient descent onto. We want to show that the nodes \((a_i, \theta_i)\) will converge, in a node-wise fashion, to some permutation of \(\{b_1, w_1\}, ..., \{b_k, w_k\}\).

First, with high probability we know that \(1 - 1/\text{poly}(d) \geq |b_j| \geq 1/\text{poly}(d)\) and \(||w_i - w_j|| \leq O(d \log d)\) for all \(i, j\). By Lemma E.5 we know that with high probability \((a_1, \theta_1)\) will converge to some \((d/\epsilon)^{-O(d)}\) neighborhood of \((b_{\pi(1)}, w_{\pi(1)})\) for some function \(\pi : [k] \rightarrow [k]\). Now, we treat \(a_1, \theta_1\) as one of the fixed charges and note that \(||a_1|| \leq 1\) and \(||\theta_1|| \leq O(d \log d)\) and as long as \(k > 1\) (if \(k = 1\), we are done), then there exists \(|b_j| \geq 1/\text{poly}(d)\) with \(||w_i - w_j|| \geq O(d \log d)\) for all \(i\) and \(||\theta_1 - w_j|| \geq \Omega(d \log d)\).

Then, by Lemma E.4 we can initialize \((a_2(0), \theta_2(0))\) such that \(\sqrt{L_2(a_2(0), \theta_2(0))} \leq \sqrt{L_2(0, 0)} - \delta\), with \(\delta = 1/\text{poly}(d)(d/\epsilon)^{-8d}\). Then, by Lemma E.5 we know that \((a_2, \theta_2)\) will converge to some \(w_{\pi(2)}\) such that \(||\theta_2 - w_{\pi(2)}|| < \epsilon\) (or \(||\theta_2 - \theta_1|| < \epsilon\) but \(\theta_2\) is still \(\epsilon\)-close to \(w_{\pi(1)}\)). We claim that \(\pi(1) \neq \pi(2)\).

By optimality conditions on \(a_2\), we see that

\[ a_2^*(\theta_2) = a_1 \Phi_\epsilon(||\theta_2 - \theta_1||) + b_j \Phi_\epsilon(||\theta_1 - w_j||) + \sum_{i \neq j} b_j \Phi_\epsilon(||\theta_1 - w_i||) \]

If \(w_{\pi(1)} = w_{\pi(2)}\), then note that \(||\theta_1 - w_i|| \geq \Omega(d \log d)\) for all \(i \neq \pi(1)\). Therefore, \(2 \sum_{i \neq j} b_j \Phi_\epsilon(||\theta_1 - w_i||) = (d/\epsilon)^{-O(d)}\). And by our convergence guarantees and the \((d/\epsilon)^{2d}\)-Lipschitzness of \(\Phi_\epsilon\), \(a_1 \Phi_\epsilon(||\theta_2 - \theta_1||) + b_j \Phi_\epsilon(||\theta_1 - w_j||) \leq (d/\epsilon)^{-O(d)}\). Therefore, \(a_2^*(\theta_2) \leq (d/\epsilon)^{-O(d)}\).
However, we see that \( L_2(a_2, \theta_2) \geq L_2(a_2^*(\theta_2), \theta_2) = L_2(0, 0) - \frac{1}{2} a_2^*(\theta_2)^2 \geq L_2(0, 0) - (d/\epsilon)^{-O(d)} \). But since \( L_2 \) is non-increasing, this contradicts our initialization and therefore \( \pi(1) \neq \pi(2) \). Therefore, our claim is done and by Lemma 2.5, we see that since \( |b_{\pi(2)}| \geq 1/poly(d) \) and for all \( i, \|w_i - w_{\pi(2)}\| \geq \Omega(d \log d) \) and \( \|\theta_1 - w_{\pi(2)}\| \geq \Omega(d \log d) \), we conclude that (\( a_2, \theta_2 \)) is in a \((d/\epsilon)^{-O(d)}\) neighborhood of \((b_{\pi(2)}, \theta_{\pi(2)})\). Finally, we induct and by similar reasoning, \( \pi \) is a permutation. Now, our theorem follows.

\[ \Box \]

\section{Convergence of Almost Strictly Subharmonic Potentials}

\textbf{Definition F.1.} \( \Phi(\theta, w) \) is a \textit{strictly subharmonic} potential on \( \Omega \) if it is differentiable and \( \Delta_\theta \Phi(\theta, w) > 0 \) for all \( \theta \in \Omega \), except possibly at \( \theta = w \).

An example of such a potential is \( \Phi(\theta, w) = \|\theta - w\|^{2-d-\epsilon} \) for any \( \epsilon > 0 \). Although this potential is unbounded at \( \theta = w \) for most \( d \), we remark that it is bounded when \( d = 1 \). Furthermore, the sign of the output weights \( a_i, b_i \) matter in determining the sign of the Laplacian of our loss function. Therefore, we need to make suitable assumptions in this framework.

Under Assumption 1, we are working with an even simpler loss function:

\[ L(\theta) = 2 \sum_{i=1}^{k} \sum_{i<j} \Phi(\theta_i, \theta_j) - 2 \sum_{i=1}^{k} \sum_{j=1}^{k} \Phi(\theta_i, w_j) \]  

\textbf{Theorem F.2.} Let \( \Phi \) be a symmetric strictly subharmonic potential on \( \mathcal{M} \) with \( \Phi(\theta, \theta) = \infty \). Let Assumption 1 hold and let \( L \) be as in (3). Then, \( L \) admits no local minima, except when \( \theta_i = w_j \) for some \( i, j \).

\textbf{Proof.} First, let \( \Phi \) be translationally invariant and \( \mathcal{M} = \mathbb{R}^d \). Let \( \theta \) be a critical point. Assume, for sake of contradiction, that for all \( i, j, \theta_i \neq w_j \). If \( \theta_i \) are not distinct, separating them shows that we are not at a local minima since \( \Phi(\theta_i, \theta_j) = \infty \) and finite elsewhere.

The main technical detail is to remove interaction terms between pairwise \( \theta_i \) by considering a correlated movement, where each \( \theta_i \) are moved along the same direction \( v \). In this case, notice that our objective, as a function of \( v \), is simply

\[ H(v) = L(\theta_1 + v, \theta_2 + v, \ldots, \theta_k + v) \]

\[ = 2 \sum_{i=1}^{k} \sum_{i<j} \Phi(\theta_i + v, \theta_j + v) - 2 \sum_{i=1}^{k} \sum_{j=1}^{k} \Phi(\theta_i + v, w_j) \]

Note that the first term is constant as a function of \( v \), by translational invariance. Therefore,

\[ \nabla_v^2 H = -2 \sum_{i=1}^{k} \sum_{j=1}^{k} \nabla^2 \Phi(\theta_i, w_j) \]

By the subharmonic condition, \( \text{Tr}(\nabla_v^2 H) = -2 \sum_{i=1}^{k} \sum_{j=1}^{k} \Delta_{\theta_i} \Phi(\theta_i, w_j) < 0 \). Therefore, we conclude that \( \theta \) is not a local minima of \( H \) and \( L \). We conclude that \( \theta_i = w_j \) for some \( i, j \).

The above technique generalizes to \( \Phi \) being rotationally invariant case by working in spherical coordinates and correlated translations are simply rotations. Note that we can change to spherical coordinates (without the radius parameter) and let \( \theta_1, \ldots, \theta_k \) be the standard spherical representation of \( \theta_1, \ldots, \theta_k \).
We will consider a correlated translation in the spherical coordinate space, which are simply rotations on the sphere. Let \( v \) be a vector in \( \mathbb{R}^{d-1} \) and our objective is simply
\[
H(v) = L(\theta_1 + v, ..., \theta_k + v)
\]

Then, we apply the same proof since \( \Phi(\theta_1 + v, \theta_j + v) \) is constant as a function of \( v \) by rotationally invariance.

**Corollary F.3.** Assume the conditions of Theorem F.2 and \( \Phi(\theta, \theta) < \infty \). Then, \( L \) admits no local minima, except at the global minima.

**Proof.** From the same proof from theorem F.2 we conclude that there must exists \( i, j \) such that \( \theta_i = w_j \). Then, since \( \Phi(\theta, \theta) < \infty \), notice that \( \theta_i, w_j \) cancels each other out and by drop \( \theta_i, w_j \) from the loss function, we have a new loss function \( L \) with \( k-1 \) variables. Then, using induction, we see that \( \theta_i = w_{\pi(i)} \) at the local minima for some permutation \( \pi \).

For concreteness, we will focus on a specific potential function with this property: the Gaussian kernel \( \Phi(\theta, w) = \exp(-c||\theta - w||^2/2) \). In \( \mathbb{R}^d \), the Laplacian is \( \Delta \Phi = (||\theta - w||^2 - d) \exp(-||\theta - w||^2/2) \), which becomes positive when \( ||\theta - w||^2 \geq d \). Thus, \( \Phi \) is strictly subharmonic outside a ball of radius \( \sqrt{d} \). This informally implies that \( \theta_j \) converges to a \( \sqrt{d} \)-ball around some \( w_j \).

For concreteness, we will focus on a specific potential function with this property: the Gaussian kernel \( \Phi(\theta, w) = \exp(-c||\theta - w||^2/2) \), which corresponds to a Gaussian activation. In \( \mathbb{R}^d \), the Laplacian is \( \Delta \Phi = (c||\theta - w||^2 - d) \exp(-c||\theta - w||^2/2) \), which becomes positive when \( ||\theta - w||^2 \geq d/c \). Thus, \( \Phi \) is strictly subharmonic outside a ball of radius \( \sqrt{d/c} \). Note that Gaussian potential restricted to \( S^{d-1} \) gives rise to the exponential activation function, so we can show convergence similarly.

**Theorem F.4.** Let \( \mathcal{M} = \mathbb{R}^d \) and \( \Phi(\theta, w) = e^{-c||\theta - w||^2/2} \) and Assumption 1 holds. Let \( L \) be as in (3) and \( ||w|| \leq \text{poly}(d) \).

If \( c = O(d/\epsilon) \) and \((a, \theta) \in \mathcal{M} e^{-\text{poly}(d, 1/\epsilon)} \), then there exists \( i, j \) such that \( ||\theta_i - w_j||^2 \leq \epsilon \).

**Proof.** Consider again a correlated movement, where each \( \theta_i \) are moved along the same direction \( v \). As before, this drops the pairwise \( \theta_i \) terms. If for all \( i, j \) \( ||\theta_i - w_j||^2 \leq \epsilon \), then we see that \( \Delta \Phi = (c||\theta - w||^2 - d) \exp(-c||\theta - w||^2/2) > e^{-\text{poly}(d, 1/\epsilon)} \).

\[
\text{Tr}(\nabla^2 L) = -2 \sum_{i=1}^k \sum_{j=1}^k \Delta \Phi(\theta_i, w_j) < -e^{-\text{poly}(d, 1/\epsilon)}
\]

Therefore, \( \nabla^2 L \) must admit a strictly negative eigenvalue that is less than \( e^{-cd} \), which implies our claim (we drop the \( \text{poly}(d, k) \) terms).

**G Common Activations**

First, we consider the sign activation function. Under restrictions on the size of the input dimension or the number of hidden units, we can prove convergence results under the sign activation function, as it gives rise to a harmonic potential.

**Assumption 1.** All output weights \( b_i = 1 \) and therefore the output weights \( a_i = -b_i = -1 \) are fixed throughout the learning algorithm.
Lemma G.1. Let \( \mathcal{M} = S^1 \) and let Assumption 1 hold. Let \( L \) be as in (2) and \( \sigma \) is the sign activation function. Then \( L \) admits no strict local minima, except at the global minima.

We cannot simply analyze the convergence of GD on all \( \theta_i \) simultaneously since as before, the pairwise interaction terms between the \( \theta_i \) present complications. Therefore, we now only consider the convergence guarantee of gradient descent on the first node, \( \theta_1 \), to some \( w_j \), while the other nodes are inactive (i.e. \( a_2, ..., a_k = 0 \)). In essence, we are working with the following simplified loss function.

\[
L(a_1, \theta_1) = a_1^2 \Phi(\theta_1, \theta_1) + 2 \sum_{j=1}^{k} a_1 b_j \Phi(\theta_1, w_j) \tag{4}
\]

Lemma G.2. Let \( \mathcal{M} = S^1 \) and \( L \) be as in (4) and \( \sigma \) is the sign activation function. Then, almost surely over random choices of \( b_1, ..., b_k \), all local minima of \( L \) are at \( \pm w_j \).

For the polynomial activation and potential functions, we also can show convergence under orthogonality assumptions on \( w_j \). Note that the realizability of polynomial potentials is guaranteed in Section B.

Theorem G.3. Let \( \mathcal{M} = S^{d-1} \). Let \( w_1, ..., w_k \) be orthonormal vectors in \( \mathbb{R}^d \) and \( \Phi \) is of the form \( \Phi(\theta, w) = (\theta^T w)^l \) for some fixed integer \( l \geq 3 \). Let \( L \) be as in (4). Then, all critical points of \( L \) are not local minima, except when \( \theta_1 = w_j \) for some \( j \).

G.1 Convergence of Sign Activation

Lemma G.1. Let \( \mathcal{M} = S^1 \) and let Assumption 1 hold. Let \( L \) be as in (2) and \( \sigma \) is the sign activation function. Then \( L \) admits no strict local minima, except at the global minima.

Proof. We will first argue that unless all the electrons and protons have matched up as a permutation it cannot be a strict local minimum and then argue that the global minimum is a strict local minimum.

First note that if some electron and proton have merged, we can remove such pairs and argue about the remaining configuration of charges. So WLOG we assume there are no such overlapping electron and proton.

First consider the case when there is an isolated electron \( e \) and there is no charge diagonally opposite to it. In this case look at the two semicircles on the left and the right half of the circle around the isolated electron – let \( q_1 \) and \( q_2 \) be the net charges in the left and the right semi-circles. Note that \( q_1 \neq q_2 \) since they are integers and \( q_1 + q_2 = +1 \) which is odd. So by moving the electron slightly to the side with the larger charge you decrease the potential.

If there is a proton opposite the isolated electron the argument becomes simpler as the proton benefits the motion of the electron in either the left or right direction. So the only way the electron does not benefit by moving in either direction is that \( q_1 = -1 \) and \( q_2 = -1 \) which is impossible.

If there is an electron opposite the isolated electron then the combination of these two diagonally opposing electrons have a zero effect on every other charge. So it is possible rotate this pair jointly keeping them opposed in any way and not change the potential. So this is not a strict local minimum.

Next if there is a clump of isolated electrons with no charge on the diagonally opposite point then again as before if \( q_1 \neq q_2 \) we are done. If \( q_1 = q_2 \) then the electrons in the clump locally are unaffected by the remaining charges. So now by splitting the clump into two groups and moving them apart infinitesimally we will decrease the potential.
Now if there is only protons in the diagonally opposite position an isolated electron again we are done as in the case when there is one electron diagonally opposite one proton.

Finally if there is only electrons diagonally opposite a clump of electrons again we are done as we have found at least one pair of opposing electrons that can be jointly rotated in any way.

Next we will argue that a permutation matching up is a strict local minumum. For this we will assume that no two protons are diagonally opposite each other (as they can be removed without affecting the function). Now given a perfect matching up of electrons and protons, if we perturb the electrons in any way infinitesimally, then any isolated clump of electrons can be moved slightly to the left or right to improve the potential.

Lemma G.2. Let $M = S^1$ and $L$ be as in [4] and $\sigma$ is the sign activation function. Then, almost surely over random choices of $b_1, \ldots, b_k$, all local minima of $L$ are at $\pm w_j$.

Proof. In $S^1$, notice that the pairwise potential function is $\Phi(\theta, w) = 1 - 2 \cos^{-1}(\theta^T w) / \pi = 1 - 2 \alpha / \pi$, where $\alpha$ is the angle between $\theta, w$. So, let us parameterize in polar coordinates, calling our true parameters as $\tilde{w}_1, \ldots, \tilde{w}_k \in [0, 2\pi]$ and rewriting our loss as a function of $\tilde{\theta} \in [0, 2\pi]$.

Since $\Phi$ is a linear function of the angle between $\theta, w_j$, each $w_j$ exerts a constant gradient on $\tilde{\theta}$ towards $\tilde{w}_j$, with discontinuities at $\tilde{w}_j, \pi + \tilde{w}_j$. Almost surely over $b_1, \ldots, b_k$, the gradient is non-zero almost everywhere, except at the discontinuities, which are at $\tilde{w}_j, \pi + \tilde{w}_j$ for some $j$.

G.2 Convergence of Polynomial Potentials

Theorem G.3. Let $M = S^{d-1}$. Let $w_1, \ldots, w_k$ be orthonormal vectors in $\mathbb{R}^d$ and $\Phi$ is of the form $\Phi(\theta, w) = (\theta^T w)^l$ for some fixed integer $l \geq 3$. Let $L$ be as in [4]. Then, all critical points of $L$ are not local minima, except when $\theta_1 = w_j$ for some $j$.

Proof. WLOG, we can consider $w_1, \ldots, w_d$ to be the basis vectors $e_1, \ldots, e_d$. Note that this is a manifold optimization problem, so our optimality conditions are given by introducing a Lagrange multiplier $\lambda$, as in [GHJY15].

$$\frac{\partial L}{\partial a} = 2 \sum_{i=1}^d ab_i(\theta_i)^l + 2a = 0$$

$$(\nabla_\theta L)_i = 2ab_i(\theta_i)^{l-1} - 2\lambda \theta_i = 0$$

where $\lambda$ is chosen that minimizes

$$\lambda = \arg \min_\lambda \sum_i (ab_i(\theta_i)^l - \lambda \theta_i)^2 = \sum ab_i(\theta_i)^l$$

Therefore, either $\theta_i = 0$ or $b_i(\theta_i)^{l-2} = \lambda / (al)$. From [GHJY15], we consider the constrained Hessian, which is a diagonal matrix with diagonal entry:

$$(\nabla^2 L)_{ii} = 2ab_i(l-1)(\theta_i)^{l-2} - 2\lambda$$

Assume that there exists $\theta_i, \theta_j \neq 0$, then we claim that $\theta$ is not a local minima. First, our optimality conditions imply $b_i(\theta_i)^{l-2} = b_j(\theta_j)^{l-2} = \lambda / (al)$. So,

$$\nabla^2 L_{ii} = \nabla^2 L_{jj} = 2ab_i(l-1)(\theta_i)^{l-2} - 2\lambda = 2(l-2)\lambda = -2(l-2)la^2$$

28
Now, there must exist a vector \( v \in S^{d-1} \) such that \( v_k = 0 \) for \( k \neq i, j \) and \( v^T \theta = 0 \), so \( v \) is in the tangent space at \( \theta \). Finally, \( v^T (\nabla^2 L) v = -2(l - 2) a^2 < 0 \), implying \( \theta \) is not a local minima when \( a \neq 0 \). Note that \( a = 0 \) occurs with probability 0 since our objective function is non-increasing throughout the gradient descent algorithm and is almost surely initialized to be negative with \( a \) optimized upon initialization, as by observed before.

Under a node-wise descent algorithm, we can show polynomial-time convergence to global minima under orthogonality assumptions on \( w_j \) for these polynomial activations/potentials. We will not include the proof but it follows from similar techniques presented for nodewise convergence in Section E.

### H Proof of Sign Uniqueness

For the sign activation function, we can show a related result.

**Theorem H.1.** Let \( \mathcal{M} = S^{d-1} \) and \( \sigma \) be the sign activation function and \( b_2, ..., b_k = 0 \). If the loss \( (1) \) at \( (a, \theta) \) is less than \( O(1) \), then there must exist \( \theta_i \) such that \( w^T (\nabla^2 L) v = \frac{1}{\sqrt{1 - \theta_i^2}} \).

**Proof.** WLOG let \( w_1 = e_1 \). Notice that our loss can be bounded below by Jensen’s:

\[
\mathbb{E}_X \left[ \left( \sum_{i=1}^k a_i \sigma(\theta_i^T X) - \sigma(X_1) \right)^2 \right] \\
\geq \mathbb{E}_{X_1} \left[ \left( \mathbb{E}_{X_2,...,X_d} \left[ \sum_{i=1}^k a_i \sigma(\theta_i^T X) \right] - \sigma(X_1) \right)^2 \right],
\]

where \( X \) is a standard Gaussian in \( \mathbb{R}^d \).

\[
\mathbb{E}_{X_2,...,X_d} \left[ \sum_{i=1}^k a_i \sigma(\theta_i^T X) \right] = \sum_{i=1}^k a_i \mathbb{E}_{X_2,...,X_d} \left[ \sigma(\theta_{i1}X_1 + \sum_{j>1} \theta_{ij}X_j) \right] \\
= \sum_{i=1}^k \mathbb{E}_Y \left[ \sigma(\theta_{i1}X_1 + \sqrt{1 - \theta_i^2} Y) \right] \\
= \sum_{i=1}^k a_i \mathbb{E}_Y \left[ \sigma(\frac{\theta_{i1}}{\sqrt{1 - \theta_i^2}} X_1 + Y) \right],
\]

where \( Y \) is an independent standard Gaussian and for any small \( \delta \), if \( p(y) \) is the standard Gaussian density,

\[
\mathbb{E}_Y [\sigma(\delta + Y)] = \int_{-\delta}^{\delta} p(y) \, dy = 2p(0)\delta + O(\delta^2)
\]

If \( w^T \theta_i = \theta_{i1} < \epsilon \) for all \( i \), then notice that with high probability on \( X_1 \) (say condition on \( |X_1| \leq 1 \)),

\[
\mathbb{E}_Y \left[ \sigma(\frac{\theta_{i1}}{\sqrt{1 - \theta_i^2}} X_1 + Y) \right] = 2p(0) \frac{\theta_{i1}}{\sqrt{1 - \theta_i^2}} X_1 + O(\epsilon^2 X_1^2)
\]

29
Therefore, since $\epsilon < O(1/\sqrt{k})$,

$$
\mathbb{E}_{X_2,\ldots,X_d} \left[ \sum_{i=1}^{k} a_i \sigma(\theta_i^T X) \right] = X_1 \sum_{i=1}^{k} 2p(0)a_i \frac{\theta_i}{\sqrt{1-\theta_i^2}} + O(k\epsilon^2 X_1^2) \\
= cX_1 + O(1)
$$

Finally, our error bound is now

$$
\mathbb{E}_{X_1} \left[ \left( \mathbb{E}_{X_2,\ldots,X_d} \left[ \sum_{i=1}^{k} a_i \sigma(\theta_i^T X) \right] - \sigma(X_1) \right)^2 \right] \\
\geq \mathbb{E}_{|X_1|\leq 1} [(cX_1 + O(1) - \sigma(X_1))^2]
$$

And the final expression is always larger than some constant, regardless of $c$. \qed