ABSTRACT. In the present paper we describe the technology for translating algorithmic
descriptions of discrete functions to SAT. The proposed technology is aimed at applications
in algebraic cryptanalysis. We describe how cryptanalysis instances are reduced to SAT
in such a way that it should be perceived as natural by the cryptographic community.
Therefore, in the theoretical part of the paper we justify the main principles of general
reduction to SAT for discrete functions from a class containing the majority of functions
employed in cryptography. Based on these principles we describe the Transalg software
system, developed with SAT-based cryptanalysis specifics in mind. We show the results of
applications of Transalg to construction of a number of attacks on various cryptographic
functions. Some of the corresponding attacks are state of the art. We also compare the
functional capabilities of the proposed system with that of other software systems that can
be used to reduce cryptanalysis instances to SAT, and also with the CBMC system widely
employed in symbolic verification. In the paper we also present vast experimental data,
obtained using the SAT-solvers that took first places at the SAT-competitions in the recent
several years.

2012 ACM CCS: [Theory of computation]: Logic—Constraint and logic programming, [Theory of
computation]: Logic—Automated reasoning, [Software and its engineering]: Software notations and
tools—Context specific languages— Domain specific languages, [Security and privacy]: Cryptography—
Symmetric cryptography and hash functions, [Security and privacy]: Cryptography—Cryptanalysis and
other attacks.

Key words and phrases: SAT, SAT-based cryptanalysis, symbolic execution.
Introduction

The state-of-the-art algorithms for solving Boolean Satisfiability Problem (SAT) are successfully used in many practical areas including symbolic verification, scheduling and planning, bioinformatics, network science, etc. In the recent years there is observed a growth of interest to applications of SAT in cryptanalysis. The corresponding results are covered in [Mas99, MM00, MZ06, DKV07, CB07, EM08, Bar09, SNC09, EM09, SZBP11, CGS12, Nos12, Cou13, Cou15, SZ16, Sto16, NNS+17, NLG+17, SZO+18], etc.

SAT-based cryptanalysis implies two stages: on the first stage a SAT encoding of a considered cryptanalysis problem is constructed. On the second stage the obtained SAT instance is solved using some SAT solving algorithm. The success on the second stage is not guaranteed because SAT is an NP-hard problem, and also due to the fact that the hardness of cryptanalysis instances is usually preserved during their translation to SAT form [CM97]. Despite an impressive progress in the development of applied algorithms for solving SAT many cryptanalysis problems remain too hard even for the cutting edge SAT-solvers. Meanwhile, the first stage is guaranteed to be effective for the majority of cryptanalysis problems, at least in theory. It follows from the fact that cryptographic transformations are usually designed to be performed very fast. To reduce some cryptanalysis problem to SAT one has to perform the so-called propositional encoding of a corresponding function. In practice, this task is quite nontrivial because modern cryptographic algorithms are constructed from basic primitives, the total number of which is quite large. Often, the researcher has to carry out a substantial amount of manual work to construct a SAT encoding for a considered cipher due to some of its constructive features or because of specific requirements of the implemented attack on this cipher. Nevertheless there are several possible ways to reduce a considered cryptanalysis instance to SAT.

First, one can use the generic systems for symbolic verification for this purpose, e.g. the CBMC system [CKL04, Kro09] or LLBMC system [SMF12]. However, it seems that the authors no longer support the latter system as of 2019. In Section 4, in our computational experiments we study in detail the question whether the SAT instances constructed by CBMC can be used in SAT-based cryptanalysis. The short summary is that the CBMC system actually is able to solve the majority of problems for which we and other researchers designed and used domain specific systems. From our point of view, in this context the main disadvantage of the generic verification systems is that the originally cryptographic problems must be "translated" into symbolic verification language first. Also note that some of the important actions comprising the cryptographic attacks are often implemented in a more natural way in domain specific systems.

Among the domain specific systems for reducing cryptanalysis instances to SAT we would like to mention the Cryptol system [EM08, EM09], the authors of which claim that it is designed specifically for problems of information security, including cryptanalysis. Also, with some additional steps the cryptanalysis problems can be reduced to SAT using the URSA system [Jan12]. Finally, it is possible to reduce the cryptanalysis of a cryptographic functions from a limited class (formed by keystream generators based on feedback shift registers) to SAT using the Grain-of-Salt [Soo10] tool. However, from our point of view, the propositional encoding methods which are employed by aforementioned systems do not take into account a number of issues which are specific for cryptanalysis. In particular, in many cases knowing auxiliary information about particular variables in a SAT encoding makes it possible to significantly improve the effectiveness of a cryptographic attack. It is
critically important, for example, when constructing guess-and-determine attacks [Bar09]. Another example is provided by SAT variants of attacks on cryptographic hash functions, described in [Dob98, WLF+05, WY05]. When implementing the cited attacks it is necessary to impose additional constraints on specific variables in the propositional encoding. The mentioned domain-specific systems do not allow to do it in a straightforward manner.

In the present paper we introduce a new software system designed to encode algorithms that specify cryptographic functions to SAT. It is named Transalg (from TRANslation of ALGorithms). During its development we made an effort to avoid the disadvantages of the previously mentioned domain specific tools. In the final part of the paper we show the capabilities of the Transalg system in application to cryptanalysis of several cryptographic systems which are currently used or have been used in recent past.

Let us give an outline of the paper. In Section 1 we briefly touch the basics of the Boolean satisfiability problem (SAT) and cite the major algorithms for its solving. In Section 2 we give the theoretical foundations of SAT-based cryptanalysis. Here we discuss several features of the procedures for translating the programs defining discrete functions to SAT. As we show below, they are particularly important in the context of cryptographic applications. Also in the same section we discuss the main theoretical results that form the basis of the software system that performs effective reductions of inversion problems of discrete functions to SAT. This software system named Transalg is described in Section 3. In Section 4 we compare the functionality of Transalg with that of other software systems for encoding problems to SAT (CBMC, Cryptol, URSA, Grain-of-Salt). In Section 5 we show that thanks to the Transalg’s features it is possible to solve inversion problems for several relevant cryptographic functions. Some of the SAT-based cryptographic attacks from Section 5 are currently the best known. Section 6 contains a brief review of related works.

The present paper is an extended version of the report [OSG+16] presented at the ECAI 2016 conference. The sources of the Transalg software system are available at [OGS]. The examples of Transalg programs for various cryptographic primitives can be found at [OGZS]. All the instances considered in Section 4 are also available online at [OGZ+].

1. The Boolean Satisfiability Problem and Algorithms for its Solving Used in Cryptanalysis

The Boolean Satisfiability Problem (SAT) is a decision problem, in which for an arbitrary Boolean formula $F$ over Boolean variables forming a set $X$ it is necessary to decide whether there exists such truth assignment for variables from $X$ on which formula $F$ takes the value of True. Hereinafter, let us denote values True and False by 1 and 0, respectively.

It can be shown that SAT for an arbitrary Boolean formula $F$ can be effectively (in polynomial time in the size of description of $F$) reduced to SAT for a formula in a Conjunctive Normal Form (CNF). Hereinafter, we will consider SAT exactly in this sense. Also, below we view SAT not only as a decision problem but also as the corresponding search problem: if CNF $C$ is satisfiable to find any truth assignment that satisfies $C$.

The decision variant of SAT is NP-complete [Coo71]. Nevertheless, the wide spectrum of combinatorial problems which can be effectively reduced to SAT makes the development of practical SAT solving algorithms a relevant and important direction of research. Detailed information on various algorithms for solving SAT can be found in the book [BHvMW09].
In the present paper, we consider the applications of SAT solving algorithms to cryptanalysis problems, in particular to the problem of finding a preimage of a cryptographic function given its image. We will refer to this problem as to the inversion problem for the corresponding function. The specialists in cryptography know that it can be viewed as a problem of finding solutions of a system of algebraic equations which interconnect the ciphering algorithm steps. Lately, the direction of research in which a cryptanalysis problem is viewed in the general context of the problem of solving algebraic equations is often referred to as algebraic cryptanalysis [Bar09]. As we show below, from an algebraic system or even from the algorithm defining such function one can effectively transition to SAT. There is a number of examples when valuable results in algebraic cryptanalysis were obtained thanks to the use of SAT solvers: [MZ06, CGS12, CB07, Cou13, DKV07, Bar09, SNC09, SZBP11] and several others. The particular area of algebraic cryptanalysis which employs the SAT-solvers is known as SAT-based cryptanalysis.

There are no known theoretical limitations on the application of various SAT-solving algorithms to cryptanalysis instances. However, based on a large number of papers (both cited above and below) one can conclude that the CDCL SAT solvers [MSLM09] suit best for solving such problems.

The construction of the first effective CDCL SAT solvers was the result of a deep modernization of the well-known DPLL algorithm [DLL62, DP60], which was undertaken in [MSS96, MSS99, MMZ+01, ZMMM01, ES04a]. After this, the CDCL-based SAT solvers became the de facto algorithmic tools for solving computational problems in a number of areas, first and foremost, in symbolic verification [BCCZ99, BCC+99, PBG05, Mar08], etc. Approximately in the middle of 2000-s the computational potential of CDCL in application to cryptanalysis problems was realized. As it was noted above, to the present day there have been published a lot of papers in which the CDCL SAT solvers were applied to cryptanalysis instances. The short review of the most prominent of them will be given in the final part of the paper.

2. Theoretical Foundations of SAT-based Cryptanalysis

In this section we describe the basic theoretical principles of reducing inversion problems for a class of functions, containing the vast majority of functions used in cryptography, to SAT. We give the theoretical foundations of SAT-based cryptanalysis in such a way that the described principles can naturally be implemented in the form of a software system. We would like to note in advance that hereinafter we try to follow the ideology of symbolic execution, because from our point of view it best fits the problem of constructing SAT encodings for inversion of arbitrary discrete functions in general, and for cryptographic functions in particular.

Remind that Symbolic Execution [Kin76] is a technique that associates with a program for a computer or for an abstract machine some symbolic expression, usually a Boolean formula. By analyzing this formula one can conclude about the properties of a function defined by an original program.

Hereinafter, denote by \( \{0, 1\}^n \) the set of all possible binary words of length \( n \). By \( \{0, 1\}^* \) we denote the set of all binary words of length \( n = 1, 2, \ldots \). Let us consider functions of the kind

\[
f : \{0, 1\}^* \rightarrow \{0, 1\}^*,
\]  

(2.1)
i.e. functions that map arbitrary binary words into binary words. Additionally, we assume that each function of the kind (2.1) is defined everywhere on \( \{0, 1\}^* \) (i.e. is total) and is given by a Turing machine program \( A(f) \), the complexity of which is bounded by a polynomial in the length of an input word. A program \( A(f) \) defines an infinite family of functions of the kind
\[
f_n : \{0, 1\}^n \rightarrow \{0, 1\}^*, n = 1, 2, \ldots
\]  
(2.2)
It is clear that for an arbitrary \( n = 1, 2, \ldots \) it follows that \( \text{Dom} f_n = \{0, 1\}^n \). Hereinafter, to functions (2.1) and (2.2) we refer as discrete functions.

**Definition 2.1.** For a discrete function \( f \) of the kind (2.1) the problem of its inversion consists in the following. Given \( A(f) \) for an arbitrary \( n = 1, 2, \ldots \) and arbitrary \( y \in \text{Range} f_n \) to find such \( x \in \{0, 1\}^n \) that \( f_n(x) = y \).

Below we consider a number of cryptanalysis problems from the point of view of inversion of functions of the kind (2.1) and (2.2). The main terms related to cryptography that we use below can be found, for example in [MVO96].

In our first example, suppose that given a secret key \( x \in \{0, 1\}^n \), \( f_n \) generates a pseudorandom sequence (generally speaking, of an arbitrary length), that is later used to cipher some plaintext via bit-wise XOR. Such a sequence is called a keystream. Knowing some fragment of plaintext lets us know the corresponding fragment of keystream, i.e. some word \( y \) for which we can consider the problem of finding such \( x \in \{0, 1\}^n \), that \( f_n(x) = y \). Regarding cryptographic keystream generators, this corresponds to the so called known plaintext attack.

Let us give another example. Total functions of the kind \( f : \{0, 1\}^* \rightarrow \{0, 1\}^c \), where \( c \) is some constant, are called hash functions. If \( n \) is the length of an input message, and \( n > c \), then there exist such \( x_1, x_2, x_1 \neq x_2 \), that \( f_n(x_1) = f_n(x_2) \). Such a pair \( x_1, x_2 \) is called a collision of a hash function \( f \). A cryptographic hash function is considered compromised if one is able to find collisions of that function in reasonable time.

For an arbitrary function of the kind (2.1) there exists an effective in theory procedure for reducing the problem of its inversion to SAT. Essentially, it follows from the Cook-Levin theorem, and to prove it one can use any known technique, e.g. from [Coo71] or [Gol08]. Below we briefly review the main techniques used to prove the statements of such a kind. They play the crucial role in understanding the basic principles that serve as a foundation of the software system for translating algorithmic descriptions of discrete functions to SAT, which we describe in the following sections.

So, let us fix \( n \) and consider an arbitrary function of the kind (2.2). Since the runtime of a program defining the corresponding function is finite for any \( x \in \{0, 1\}^n \), we can consider this function in the following form:
\[
f_n : \{0, 1\}^n \rightarrow \{0, 1\}^m.
\]  
(2.3)
This function can be specified by a Boolean circuit \( S(f_n) \) over an arbitrary complete basis. Hereinafter, we use the basis \( \{\neg, \land\} \). On the current stage, assume that we are given a circuit \( S(f_n) \). Note that it has \( n \) inputs and \( m \) outputs. With each input of \( S(f_n) \) we associate a Boolean variable. We denote the obtained set of variables by \( X = \{x_1, \ldots, x_n\} \). We will say that \( X \) encodes the input of function \( f_n \). Similarly, let us encode the output of \( f_n \) via the Boolean variables forming the set \( Y = \{y_1, \ldots, y_m\} \).

For the circuit \( S(f_n) \) in linear time on the number of nodes in \( S(f_n) \) we can construct a CNF denoted by \( C(f_n) \). The corresponding algorithm traverses each inner node of a
circuit exactly once. With each gate \( G \) it associates an auxiliary variable \( v(G) \) from the set \( V : V \cap X = \emptyset \). For an arbitrary \( v(G) \) a CNF \( C(G) \) is constructed which uses at most 3 Boolean variables. The exact representation of \( C(G) \) depends on the gate \( G \). The result of this process is the CNF:

\[
C(f_n) = \bigwedge_{G \in S(f_n)} C(G). \quad (2.4)
\]

The described technique of constructing a CNF for a circuit \( S(f_n) \) is known as \textit{Tseitin transformations} \cite{Tse68}.

**Definition 2.2.** To CNF \( C(f_n) \) of the kind (2.4) we further refer as \textit{template CNF} encoding the algorithm that implements function \( f_n \), or in short \textit{template CNF for} \( f_n \).

Let \( u \) be an arbitrary Boolean variable. Below we will use the following notation: by \( l_\lambda(u) \), \( \lambda \in \{0, 1\} \) we denote the literal \(-u\) if \( \lambda = 0 \), and literal \( u \) if \( \lambda = 1 \). Let \( C(f_n) \) be \textit{template CNF} for \( f_n \). Now let \( x = (\alpha_1, \ldots, \alpha_n) \) and \( y = (\beta_1, \ldots, \beta_m) \) be arbitrary truth assignments from \( \{0, 1\}^n \) and \( \{0, 1\}^m \), respectively. In other words, let us consider \( x = (\alpha_1, \ldots, \alpha_n) \) as an assignment of variables from \( X = \{x_1, \ldots, x_n\} \), and \( y = (\beta_1, \ldots, \beta_m) \) as an assignment of variables from \( Y = \{y_1, \ldots, y_m\} \). Consider the following two CNFs:

\[
\begin{align*}
C(x, f_n) &= l_{\alpha_1}(x_1) \wedge \ldots \wedge l_{\alpha_n}(x_n) \wedge C(f_n), \\
C(f_n, y) &= C(f_n) \wedge l_{\beta_1}(y_1) \wedge \ldots \wedge l_{\beta_m}(y_m).
\end{align*}
\]

For many practical applications of SAT and, in particular, to describe many cryptographic attacks studied in algebraic cryptanalysis (see, e.g., \cite{SZO18}) the following fact plays a very important role.

\textit{The application of only the Unit Propagation rule} \cite{DG84} to CNF \( C(x, f_n) \) for a particular \( x = (\alpha_1, \ldots, \alpha_n) \) results in the derivation of values for all remaining variables, including that of variables \( Y : y_1 = \beta_1, \ldots, y_m = \beta_m \), such that \( f_n(x) = y, y = (\beta_1, \ldots, \beta_m) \).

This property was several times used in other papers (see e.g. \cite{JJ99, JB12, SZ15} and others). Its proof in a very similar formulation can be found in \cite{BKNW09}. Essentially, this property follows from the fact that the set \( X \) in \( C(f_n) \) is a Strong Unit Propagation Backdoor Set (SUPBS) \cite{WGS03}.

The following statement is essentially a variant of the Cook-Levin theorem in the context of the problem of inverting functions of the kind (2.3). The basic steps of its proof are standard and can be found, for example, in \cite{Gol08}. However, from our point of view there are several technical issues that should be clarified for better understanding of how the software system described below works with data. That is why we present the short proof of this statement detailing only the features that play an important role in the context of this study.

**Theorem 2.3.** There exists an algorithm \( A' \) such that given as an input a program \( A(f) \), a number \( n \) (in unary form), and a word \( y \in \{0, 1\}^* \), in polynomial time constructs CNF \( C(f_n, y) \) with the following properties:

1. For \( y \notin \text{Range}_n f \) the CNF \( C(f_n, y) \) is unsatisfiable.
2. For \( y \in \text{Range}_n f \) the CNF \( C(f_n, y) \) is satisfiable and from any of its satisfying assignments one can extract such a word \( x \in \{0, 1\}^n \) that \( f_n(x) = y \).

**Sketch proof.** Let us give a sketch proof. Assume that the program \( A(f) \) is executed on the Turing machine, described in \cite{GJ79}. Remind that it works only with binary data. By algorithm \( A' \) we mean the informal procedure that constructs a circuit \( S(f_n) \) based on the
text of a program $A(f)$ and a number $n$. Note only that the transition function in the
machine from [GJ79] looks as follows:

$$\delta : (q, \alpha) \rightarrow (q', \alpha', s),$$

(2.5)

where $\alpha \in \{0, 1, ,0\}$ is an input symbol, $q \in Q$ an arbitrary state, $s$ is a variable that defines
the direction in which the head is going to shift, i.e. $s \in \{-1, 0, +1\}$. The function $(2.5)$
describes the execution of one elementary command.

Let us consider the execution of program $A(f)$ as a sequence of time moments, such that
during the transition from one time moment to another exactly one elementary command is
executed. The moment $t = 0$ corresponds to a starting configuration. With each moment $t$
we associate the set of Boolean variables $X_t$, $X_i \cap X_j = \emptyset$ if $i \neq j$. With the transition from
$t$ to $t + 1$ we associate a formula

$$\Psi_{t \rightarrow t+1} = \bigvee_{(q, \alpha)} \Phi_{t \rightarrow t+1}^{(q, \alpha)}.$$  

(2.6)

An arbitrary formula $\Phi_{t \rightarrow t+1}^{(q, \alpha)}$ is a formula of the kind

$$\phi_t(q, \alpha) \Rightarrow \phi_{t+1}(q', \alpha'),$$

(here by $\Rightarrow$ we denote logical implication), which is constructed in the following manner.
The formula $\phi_t(q, \alpha)$ is a conjunction of literals over the set $X_t$ that encodes a particular
pair $(q, \alpha)$ and also the state of the head at the moment $t$. The formula $\phi_{t+1}(q', \alpha')$ is the
conjunction of literals over the set of Boolean variables $X_{t+1}$ that encodes a pair $(q', \alpha')$
and the state of the head corresponding to the triple $(q', \alpha', s)$. The correspondence between
$(q, \alpha)$ and $(q', \alpha', s)$ is defined by the transition function $(2.5)$. In (2.6) the disjunction is
performed over all possible pairs $(q, \alpha)$ in the program $A(f)$.

It is very important to note that the power of $Q$ does not depend on $n$. Therefore, the
size of the formula $\Psi_{t \rightarrow t+1}$ is a constant that does not depend on $n$. Omitting some details,
let us note that from the above a formula $\Psi_{t \rightarrow t+1}$ defines a function

$$F_{t \rightarrow t+1} : \{0, 1\}^{\left| X_t \right|} \rightarrow \{0, 1\}^{\left| X_{t+1} \right|},$$

which can also be defined using Boolean circuit $S(F_{t \rightarrow t+1})$ over the basis $\{\neg, \land\}$, which has
$\left| X_t \right|$ inputs and $\left| X_{t+1} \right|$ outputs. The sets $X_t$ and $X_{t+1}$ are the sets of input and output
variables of a circuit $S(F_{t \rightarrow t+1})$, respectively, and $X_{t+1}$ is the set of input variables of circuit
$S(F_{t+1 \rightarrow t+2})$.

Let $t(n)$ be the upper bound on the runtime of the program $A(f_n)$ over all inputs from
$\{0, 1\}^n$. By combining the circuits $S(F_{0 \rightarrow t_1}), \ldots, S(F_{t(n)-1 \rightarrow t(n)})$ according to the above we
construct the circuit, for which it is easy to see that it specifies the function $f_n$ of the kind
(2.3). This circuit is $S(f_n)$.

Let us construct the template CNF $C(f_n)$ for the circuit $S(f_n)$. Let $y$ be an arbitrary
assignment from $\{0, 1\}^m$. Consider a CNF $C(f_n, y)$. Now we use the property mentioned
above and conclude that $X$ is a SUPBS in $C(f_n, y)$. Using this fact it is easy to see that the points (1) and (2) from the Theorem formulation are valid.

Based on the Theorem 2.3, let us formulate the general concept of SAT-based crypt-
analysis. Assume that we have a function $f$ of the kind (2.1), and for a fixed $n$ there is a
problem of finding a preimage of a particular $y \in \text{Range} f_n$. Then, using Theorem 2.3 we
construct CNF $C(f_n, y)$. If SAT for $C(f_n, y)$ can be solved, then from the obtained solution
it is easy to extract such $x \in \{0, 1\}^n$ that $f_n(x) = y$. 

\end{proof}
As a concluding remark we would like to note that in the procedure used in the proof of Theorem 2.3 to transition from a program $A(f)$ to a template CNF $C(f_n)$ an input word from $\{0, 1\}^n$ is not constrained in any way. In fact, this procedure takes as an input Boolean variables $x_1, \ldots, x_n$ and outputs $C(f_n)$, thus essentially performing symbolic execution.

3. **Transalg: software system for encoding algorithmic descriptions of discrete functions to SAT**

In the present section we describe the Transalg software system that can be viewed as programmatic implementation of the translation procedure for transforming the algorithmic descriptions of functions (2.3) to SAT, which was outlined in Theorem 2.3. The only conceptual difference is that instead of the Turing machine Transalg uses an abstract machine with random access to memory cells.

Thus, in Transalg the process of computing a value of a discrete function $f_n$ is represented as a sequence of elementary operations with memory cells of an abstract machine. Each memory cell contains one bit of information. Any elementary operation $o$ over data in memory cells is essentially a Boolean function of arity $k$, $k \geq 1$, where $k$ is some constant that does not depend on the size of input (strictly speaking, it is possible to consider $k \in \{1, 2\}$). For example, if $o$ has the arity of 2 then the result of application of $o$ to data in memory cells $c_1$ and $c_2$ is one bit that is written to memory cell $c_3$. However, during the construction of propositional encoding Transalg does not use the real data. Instead, it associates with the cells $c_1, c_2, c_3$ the Boolean variables $v_1, v_2$ and $w$, respectively. Then it associates with operation $o$ the Boolean formula

$$w \equiv \phi_o(v_1, v_2),$$

(3.1)

where $\phi_o(v_1, v_2)$ is the Boolean formula specifying a function $o$. For an arbitrary formula of the kind $\phi_o(v_1, v_2)$ we represent the corresponding function as a Boolean circuit over the basis $\{\wedge, \neg\}$. Thus, Transalg if necessary can represent a considered function as a Boolean circuit over $\{\wedge, \neg\}$.

To describe discrete functions the Transalg system uses the domain specific language called the TA-language. The TA-language has a C-like syntax and block structure. An arbitrary block (composite operator) is essentially a list of instructions, and has its own (local) scope. In the TA-language one can use nested blocks with no limit on depth. During the analysis of a program Transalg constructs a scope tree with the global scope at its root. Every identifier in a TA-program belongs to some scope. Variables and arrays declared outside of any block and also all functions belong to the global scope and therefore can be accessed in any point of a program.

A TA-program is a list of functions. The **main** function is the entry point and, thus, must exist in every program. The TA-language supports basic constructions used in procedural languages (variable declarations, assignment operators, conditional operators, loops, function calls, etc.), various integer operations and bit operations including bit shifting and comparison. The main data type in the TA-language is the **bit** type. Transalg uses this type to establish links between variables used in a TA-program and Boolean variables included into a corresponding propositional encoding. It is important to distinguish between these two sets of variables. Below we will refer to variables that appear in a TA-program as **program variables**. All variables included in a propositional encoding are called **encoding variables**. Given a TA-program $A(f_n)$ as an input, Transalg constructs the propositional
encoding of the function \( f_n \). Below we will refer to this process as to the \emph{translation} of the TA-program \( A(f_n) \).

Upon the translation of an arbitrary instruction that contains a program variable of a \texttt{bit} type, \textsc{Transalg} links this program variable with a corresponding encoding variable. \textsc{Transalg} establishes such links only for program variables of the \texttt{bit} type. Variables of other types, in particular \texttt{int} and \texttt{void} are used only as service variables, e.g. as loop counters or to specify functions that do not return value.

Declarations of global \texttt{bit} variables can have the \_\texttt{in} or the \_\texttt{out} attribute. The \_\texttt{in} attribute marks variables that contain input data for an algorithm. The \_\texttt{out} attribute marks variables that contain an output of an algorithm. Local \texttt{bit} variables cannot be declared with these attributes. The translation of a TA-program has two main stages. At the first stage, \textsc{Transalg} parses the source code of the TA-program and constructs a syntax tree using standard techniques of the compilation theory. At the second stage, the system performs symbolic execution of the TA-program to construct the corresponding propositional encoding.

In order to reduce the number of auxiliary variables in a resulting propositional encoding, \textsc{Transalg} can use Boolean functions with arity \( k > 2 \) in the role of elementary operations over data in memory cells of its abstract machine. Therefore, as a result of each elementary step a new encoding variable \( v \) is introduced and the following Boolean formula is constructed
\[
v \equiv \phi(\tilde{v}_1, \ldots, \tilde{v}_k),
\]
(3.2)
in which \( \tilde{v}_1, \ldots, \tilde{v}_k \) are some encoding variables introduced earlier. The propositional encoding of a TA program is a set of formulas of the kind (3.2).

Cryptographic algorithms often use various bit shifting operators and also copy bits from one cell to another without changing their value. During the symbolic execution of such operators there may appear elementary steps, producing the formulas of the kind \( v \equiv \tilde{v} \). However, we do not really need such formulas in the propositional encoding since it is evident that without the loss of correctness we can replace an arbitrary formula of the kind \( v' \equiv \phi(v, \ldots) \) by a formula \( v' \equiv \phi(\tilde{v}, \ldots) \). In other words it is not necessary to introduce the encoding variable \( v \). \textsc{Transalg} tracks such situations using special data structures. Let us consider the following example.

**Example 3.1.** Consider an encoding of a linear feedback shift register (LFSR) \cite{MVO96} with \textsc{Transalg}. In Figure 1 we show the TA-program for the LFSR with feedback polynomial \( P(z) = z^{19} + z^{18} + z^{17} + z^{14} + 1 \) over \textit{GF}(2) (here \( z \) is a formal variable). Let us view the process of executing this TA-program as a sequence of data modifications in a memory of an abstract computing machine at moments \( \{1, \ldots, e\} \). At every moment \( t \in \{0, 1, \ldots, e\} \), \textsc{Transalg} links a set \( V^t \) of encoding variables with program variables of the \texttt{bit} type. Denote \( \hat{V} = \bigcup_{t=0}^e V^t \). Let us separately denote by \( V^{in} \) and \( V^{out} \) the sets formed by encoding variables which encode input and output data, respectively. Note that during the translation of transition from moment \( t \) to moment \( t + 1 \) it is not necessary to create new encoding variables for every cell of the register. If we copy data from one register cell to another, then we can use the same encoding variable to encode corresponding data value at moments \( t \) and \( t + 1 \). Therefore, at each moment \textsc{Transalg} creates only one new encoding variable and links it with program variable \texttt{reg[0]}. All the other program variables get linked with encoding variables created at previous moments.

In accordance with the above, the set of encoding variables corresponding to the initial values of the register is \( V^{in} = V^0 = \{v_1, \ldots, v_{19}\} \). After each shift we encode values of
define e 100;
__in bit reg[19];
__out bit output[e];

bit shift_reg(){
   bit u = reg[18];
   bit v = reg[18] ^ reg[17] ^ reg[16] ^ reg[13];
   reg = reg >> 1;
   reg[0] = v;
   return u;
}

void main(){
   for(int i = 0; i < e; i = i+1)
      output[i] = shift_reg();
}

\textbf{Figure 1. TA-program for LFSR}

The register’s cells with sets

\[ V^1 = \{v_2, v_3, \ldots, v_{20}\}, \quad V^2 = \{v_3, v_4, \ldots, v_{21}\}, \ldots, \quad V^e = \{v_{e+1}, v_{e+2}, \ldots, v_{e+19}\}. \]

Note that \( V^{\text{out}} = \{v_1, \ldots, v_e\} \). Thus the set of encoding variables for this program is \( V = \{v_1, v_2, \ldots, v_{e+19}\} \), and the corresponding variables are connected between each other by the following Boolean formulas (here \( \oplus \) stands for addition modulo 2):

\[
\begin{align*}
   v_{20} &\equiv v_1 \oplus v_2 \oplus v_3 \oplus v_6 \\
   \ldots \\
   v_{e+19} &\equiv v_e \oplus v_{e+1} \oplus v_{e+2} \oplus v_{e+5}.
\end{align*}
\]

\textsc{Transalg} supports encoding conditional operators to SAT with any depth of nesting. On the level of ideas the corresponding solutions do not differ from that employed in symbolic verification systems, such as CBMC (see e.g. [Kro09]). Briefly, the techniques in question are based on the following considerations. Each conditional operator of the kind if then else is associated with two arrays \( R_1 \) and \( R_2 \) in the memory of the abstract computing machine. The contents of these arrays represent two alternatives for data that will be in the memory of the machine after executing the conditional operator. With the cells of arrays \( R_1 \) and \( R_2 \) we first associate the encoding variables. Each encoding variable encodes the Boolean value which is the result of execution of this conditional operator.

Similar to the majority of symbolic execution systems, \textsc{Transalg} processes loops via unwinding. Only loops with fixed length are supported, since they are most common in cryptographic algorithms.

In the process of symbolic execution of algorithms (defined by TA-programs) \textsc{Transalg} uses several techniques that often result in significantly reduced size of the SAT encoding. First, note that many cryptographic algorithms can be represented in form of sequences of procedures which are simple and very similar to each other. Therefore, during the symbolic execution it is possible that the same Boolean formulas will be generated multiple times. Taking this fact into account, for each new formula \textsc{Transalg} first checks whether it is
already present in the database. If the answer is "no" then the newly constructed formula is added to the database and associated with a new encoding variable. Otherwise, on the following steps the variable associated with the existing formula from the database is used.

The next step implies using the Boolean minimization methods for constructing the final CNF encoding from formulas corresponding to separate steps of the process of translation of a considered algorithm. For this purpose we embedded into Transalg the well known Espresso library [BSVMH84]. Note that in the process of translation of an algorithm computing a function $f_n$ we often can view this function as a composition of functions which are more complex than the elementary ones from the basis $\{\land, \neg\}$. In other words, it is possible to represent $f_n$ over any complete basis with arbitrarily complex basis functions. To translate each of these functions to CNF we use Espresso. Here the arity of a minimized function is often a very serious limitation: for functions with more than 20 inputs the performance of Espresso is beginning to have a significant impact on the time of SAT encoding construction. To counter this issue we incorporated into Transalg the special mechanism that allows the researcher to flexibly react to the excessive increase in complexity of basis functions. It is achieved via the special commands that direct the system to introduce auxiliary variables in order to split a large formula into a desired number of simpler subformulas. As it will be shown in the results of computational experiments, in a number of cases the described solutions make it possible to significantly reduce the size of propositional encodings of specific algorithms in terms of the number of variables and clauses.

4. THE COMPARISON OF SYSTEMS FOR CONSTRUCTING SAT ENCODINGS OF CRYPTANALYSIS INSTANCES

In the present section we compare the functional capabilities of several software tools that can be used to translate the algorithms defining cryptographic functions into SAT form.

First, it is important to note that one can use for this purpose the systems designed for software verification. The CBMC tool (Bounded Model Checking for ANSI C [CKL04]) is one of the well known and powerful systems of this kind. As it will be shown below, CBMC actually makes it possible to solve the majority of problems related to encoding cryptographic functions to SAT, which are usually solved via domain specific systems. However, it is important to note that representing an original cryptographic instance in the notation system acceptable in symbolic verification does not always look natural. For example to consider the problem of inversion of some function via CBMC one has to formulate this problem as a problem of proving some hypothesis. The counterexample for this hypothesis will contain the satisfying assignment (the model), from which a sought function's preimage can be extracted effectively. It is possible that the fact that such formulations are quite far from cryptography prevented the specialists in algebraic cryptanalysis from using known systems for symbolic verification in their research.

We are aware of several domain specific software systems (besides Transalg) that can be used to encode the cryptographic algorithms into SAT. Below we provide their brief description.

The Grain-of-Salt system [Soo10] is designed to produce SAT encodings only for cryptographic keystream generators based on the shift registers. Unfortunately, Grain-of-Salt doesn’t support propositional encoding of many standard operations and therefore cannot be used to encode a number of cryptographic functions. URSA (a system for Uniform Reduction to SAT [Jan12]) is a propositional encoding tool that is applicable
to a wide class of combinatorial problems, varying from CSP (Constraint Satisfaction Problem) to cryptography. To describe these problems, URSA uses a proprietary domain specific language. To solve the produced SAT instances URSA uses two embedded solvers: ARGOSAT and CLASP. The Cryptol system [ECW09, EM08, EM09] is designed as a tool for analysis of cryptographic specifications using Satisfiability Modulo Theories (SMT) solvers. It uses functional Haskell-like domain specific language to describe algorithms. SAW (Software Analysis Workbench [CFH13]) allows one to produce SAT and SMT encodings for cryptographic problems described in Cryptol language. Cryptol+SAW uses the following SMT solvers: Boolector, CVC4, Mathsat, Z3 and Yices.

To compare the effectiveness of propositional encodings produced by the aforementioned tools we chose several cryptographic keystream generators. Here they are, ordered by the resistance to SAT-based cryptanalysis (from the weakest to the strongest): Geffe [Gef73], Wolfram [Wol86], Bivium [Can06] and Grain [HJM07]. The Geffe generator is a simple generator, which is not resistant to many cryptographic attacks including the correlation attack proposed in [Sie06]. We considered the strengthened Geffe generator (which we further refer to as S\_Geffe), which is a particular case of the threshold generator [Bru84]. It turned out that the S\_Geffe with a secret key length up to 160 bits is not resistant to SAT-based cryptanalysis (when implementing the known plaintext attack). We considered the variant of the S\_Geffe generator that uses three LFSRs defined by the following primitive polynomials:

\[
\begin{align*}
X^{52} + X^{49} + 1; \\
X^{53} + X^{52} + X^{38} + X^{37} + 1; \\
X^{55} + X^{31} + 1.
\end{align*}
\]

Thus, the considered generator has a secret key of 160 bits.

Unlike many other generators, the Wolfram generator does not use shift registers. It is based on a one-dimensional cellular automaton [von51]. This generator is not resistant to the attack proposed in [MS91] if its secret key length is less than 500 bits. Meanwhile, the cryptanalysis of the Wolfram generator with the secret key length of 128 bits is not a hard problem for state-of-the-art SAT solvers. This version of the generator is studied further. The Bivium generator [Can06] is a popular object for the SAT-based and algebraic cryptanalysis. Bivium is a weakened version of the Trivium generator. Nevertheless, SAT-based cryptanalysis of Bivium is quite a hard problem which, as the estimations show [SZ16], can be solved in reasonable time in powerful distributed computing environments. Finally, we considered the Grain generator [HJM07]. For the v1 version of this generator (considered below) there are no known attacks which are significantly faster than the brute force attack.

For each mentioned generator the SAT-based variant of the known plaintext attack was studied. It means that the following problem was considered: to invert a function of the form \( g : \{0,1\}^n \to \{0,1\}^m \), where \( n \) is the amount of bits of registers’ state, which produces the analyzed keystream, and \( m \) is the length of the analyzed keystream. Thus, using Cryptol+SAW, Grain-of-Salt, URSA, CBMC and Transalg we built propositional encodings for the following functions:

\[
\begin{align*}
g^{S\_Geffe} & : \{0,1\}^{160} \to \{0,1\}^{250}, \\
g^{Wolfram} & : \{0,1\}^{128} \to \{0,1\}^{128}, \\
g^{Bivium} & : \{0,1\}^{177} \to \{0,1\}^{200}, \\
g^{Grain} & : \{0,1\}^{160} \to \{0,1\}^{160}.
\end{align*}
\]
It should be noted that because Grain-of-Salt operates only with shift registers, it is not possible to construct a SAT encoding for the Wolfram generator via this tool.

In Table 1 the obtained encodings are compared by the amount of variables, clauses and literals.

Table 1. The parameters of SAT encodings.

|                | Grain-of-Salt | URSA   | Cryptol+SAW | Transalg | CBMC |
|----------------|---------------|--------|-------------|----------|------|
| Vars           | 1 910         | 2 394  | 1 883       | 1 000    | 2 668|
| Clauses        | 8 224         | 8 436  | 6 891       | 6 474    | 9 514|
| Literals       | 28 976        | 23 308 | 19 793      | 25 226   | 26 536|

|                | S. Geffe      | Wolfram|             |          |      |
|----------------|---------------|--------|-------------|----------|------|
| Vars           | -             | 24 704 | 24 620      | 12 544   | 32 904|
| Clauses        | -             | 86 144 | 85 784      | 74 112   | 114 830|
| Literals       | -             | 233 600| 232 811     | 246 400  | 311 460|

|                | Bivium        |        |             |          |      |
|----------------|---------------|--------|-------------|----------|------|
| Vars           | 842           | 1 637  | 1 432       | 1 172    | 1 985|
| Clauses        | 6 635         | 5 975  | 5 308       | 7 405    | 7 044|
| Literals       | 29 455        | 16 995 | 15 060      | 29 745   | 19 866|

|                | Grain         |        |             |          |      |
|----------------|---------------|--------|-------------|----------|------|
| Vars           | 4 546         | 9 279  | 4 246       | 1 945    | 10 088|
| Clauses        | 74 269        | 37 317 | 16 522      | 34 165   | 40 596|
| Literals       | 461 069       | 105 925| 46 402      | 190 388  | 115 178|

At the second stage, we used the constructed encodings for implementing the known plaintext attack on the described generators. As it was mentioned above, the inversion problems for $g^{S. Geffe} : \{0, 1\}^{160} \rightarrow \{0, 1\}^{250}$ and $g^{Wolfram} : \{0, 1\}^{128} \rightarrow \{0, 1\}^{128}$ are not hard even for sequential SAT solvers. The inversion problem for $g^{Bivium} : \{0, 1\}^{177} \rightarrow \{0, 1\}^{200}$ is hard even for the best parallel SAT solvers. Meanwhile, the inversion problem for $g^{Grain} : \{0, 1\}^{160} \rightarrow \{0, 1\}^{160}$ is extremely hard and cannot be solved in reasonable time in any modern distributed computing system that we are aware of. That is why we studied weakened variants of the last two problems. In particular, SAT solvers were given some information about unknown registers state. In the case of Bivium, the last 30 bits of the second register were assumed to be known. In the case of Grain, 102 (out of 160) bits were known: the whole 80-bit linear register and the last 22 bits of the 80-bit nonlinear register. The constructed SAT instances are denoted by Bivium30 and Grain102.

For each considered generator, a set of 100 cryptanalysis instances was constructed by generating random values of corresponding registers’ states, which were used to produce keystreams. For every cryptanalysis instance several SAT instances (using all compared tools) and two SMT instances were constructed. The SMT encodings were built by Cryptol+SAW and CBMC. All constructed SAT and SMT instances are available online [OGZ*].

In the experiments we employed SAT solvers that took prizes on the last SAT Competitions and also several SAT solvers which have shown good results on SAT-based cryptanalysis problems:

- MapleLcmDistChronoBt [NR18];
We also used SMT solvers that took prizes on the last SMT Competitions: CVC4 [BCD+11], Z3 [dMB08] and Yices [Dut14]. In all experiments described below we employed the HPC-cluster “Academician V.M. Matrosov” [mat], each node of which is equipped by two 18-core Intel Xeon E5-2695 CPUs. Each solver was launched in sequential mode (on 1 CPU core). The time limit of 5000 seconds was used, as it is a standard value on SAT Competitions.

In Table 2 for each considered pair (generator, tool) only the results obtained by the best solver are shown. In this table the following abbreviations are used:

- MaChr for MapleLcmDistChronoBt;
- MaCom for MapleCOMSPS_LRB_VSIDS_2;
- MaLcm for Maple_LCM_Dist;
- Cr-SAT for Cryptol+SAW_SAT;
- Cr-SMT for Cryptol+SAW_SMT;
- Cb-SAT for CBMC_SAT;
- Cb-SMT for CBMC_SMT;
- TrAlg for Transalg.

For each generator the best result is marked with bold. In Figure 2, Figure 3, Figure 4 and Figure 5 the corresponding cactuses are shown. The detailed data for all considered solvers is shown in Appendix A. All cactus plots in the present paper were made by the mkplot script [Ign].

Let us discuss the obtained results. On the considered problems the SAT solvers significantly outperform the SMT solvers. Among the latter, Yices showed the best results. As for SAT encodings, Transalg encodings showed the best results on Wolfram and the weakened Grain, Cryptol+SAW outperformed the competitors on the weakened Bivium, while CBMC showed the best results on Geffe.

We would like to specifically mention several Transalg features that prove to be quite useful when implementing various SAT-based cryptographic attacks. The distinctive feature of Transalg is that it can construct and explicitly output a template CNF $C(f_n)$ (see Definition 2.2). When Transalg constructs $C(f_n)$, it employs the concept of symbolic execution of the program $A(f_n)$ fully reflecting this process in the memory of abstract computing machine. As a result, in a template CNF $C(f_n)$ all elementary operations with the memory of abstract machine are represented in the form of a set of Boolean formulas. Transalg makes it possible to work with these variables directly, thus providing a number of useful features for cryptanalysis. In particular, we can quickly generate families of cryptographic instances: to make certain SAT instance for function inversion it is sufficient to add to a template CNF the unit clauses encoding the corresponding output. The fact that
Table 2. Solving cryptanalysis instances for S\textsubscript{Geffe}, Wolfram, Bivium and Grain. For each considered pair (keystream generator, tool) only results obtained by the best solver are shown. Time is shown in seconds.

|        | TrAlg | GoS  | URSA | Cr-SAT | Cr-SMT | Cb-SAT | Cb-SMT |
|--------|-------|------|------|--------|--------|--------|--------|
| S\textsubscript{Geffe} |       |      |      |        |        |        |        |
| Solver | MINISAT | MINISAT | MINISAT | MINISAT | YICES | MINISAT | YICES |
| Solved | 100/100 | 100/100 | 100/100 | 100/100 | 100/100 | 100/100 | 100/100 |
| Avg. time | 4.05 | 4.61 | 4.64 | 3.98 | 11.17 | **3.26** | 13.61 |

|        |       |      |      |        |        |        |        |
|--------|-------|------|------|--------|--------|--------|--------|
| Wolfram |       |      |      |        |        |        |        |
| Solver | **MaCom** | - | **MaCom** | rokk | yices | **MaCom** | 23 |
| Solved | **100/100** | - | 100/100 | 100/100 | 76/100 | 100/100 | 78/100 |
| Avg. time | **442** | - | 931 | 614 | 1631 | **536** | 1844 |

|        |       |      |      |        |        |        |        |
|--------|-------|------|------|--------|--------|--------|--------|
| Bivium30 |       |      |      |        |        |        |        |
| Solver | **MaLCm** | **MaLCm** | rokk | **MaChr** | yices | **MaChr** | - |
| Solved | 100/100 | 100/100 | 100/100 | **100/100** | 13/100 | 100/100 | 0/100 |
| Avg. time | 725 | 781 | 788 | **695** | 1759 | 995 | - |

|        |       |      |      |        |        |        |        |
|--------|-------|------|------|--------|--------|--------|--------|
| Grain102 |       |      |      |        |        |        |        |
| Solver | **rokk** | rokk | rokk | rokk | yices | rokk | - |
| Solved | **97/100** | 83/100 | 85/100 | 85/100 | 3/100 | 95/100 | 0/100 |
| Avg. time | **1407** | 2290 | 2038 | 2364 | 3090 | 1737 | - |

Figure 2. Comparison of $S\textsubscript{Geffe}$ encodings on best solvers
Figure 3. Comparison of Wolfram encodings on best solvers

Figure 4. Comparison of Bivium30 encodings on best solvers
in template CNFs there are outlined the variables corresponding to the function input makes it possible to use them for implementing the partitioning strategy [Hyv11] in distributed computing environment (see for example [SZ15, SZ16]). Also a number of cryptographic attacks directly employ some of the properties of template CNFs. In particular that the variables encoding an input of a corresponding function form a Strong Unit Propagation Backdoor Set (SUPBS) [WGS03] in this template CNF. We will touch these questions in more detail in Section 5.

In the template CNFs constructed by TRANSALG the variables are represented in the order in which they are introduced during the translation process. It means that the first variables correspond to a function’s input and the last variables — to its output. Thanks to this, testing the SAT encodings for correctness, as well as the analysis of SAT solvers output become easier.

When translating an algorithm to SAT, the data structures employed by TRANSALG preserve all the connections between the introduced Boolean variables and the corresponding cells of abstract machine’s memory. This fact allows one to effectively write auxiliary constraints on arbitrary subsets of program variables. For this purpose it is possible to introduce the corresponding conditions in a TA-program: they will then be correctly transferred into resulting SAT encoding. This feature is very important when implementing the SAT variants of the so-called differential attacks on cryptographic hash functions [WLF+05, WY05]. Among all the systems tested in the present paper, the CBMC system is the only one besides TRANSALG that supports similar actions. In the next section we present a number of examples of cryptographic attacks, which were constructed in a natural manner thanks to the features outlined here.
We would like to specifically note that almost all of the functional capabilities available in Transalg are present in CBMC. However, as we mentioned above, sometimes the original cryptographic instance does not look natural when it is put into symbolic verification context, despite the fact that usually such transition does not cause any difficulties. As it can be seen from the results of experiments, the encodings produced by CBMC are typically as good as the ones produced by other systems. This may seem surprising in view of the fact that CBMC does not actually take into account the cryptographic specifics.

In Table 3 we compare all considered systems with respect to criteria outlined above. Note that technically, URSA can produce the mapping between the variables from a program in its domain specific language and the Boolean variables in a constructed SAT encoding. This information then can be used to construct a template CNF, but with additional processing. The CBMC constructs template CNFs directly, but in order to do so one has to add to the corresponding program an empty hypothesis. For all practical purposes the obtained SAT encoding is a template CNF (despite having unit clauses induced by empty hypothesis), but this step does not look entirely natural.

Table 3. Main functional abilities of Grain-of-Salt, URSA, Cryptol+SAW, CBMC and Transalg.

| System                        | GoS | URSA | Cryptol+SAW | CBMC | Transalg |
|-------------------------------|-----|------|-------------|------|----------|
| Encodes conditional operators | -   | +    | +           | +    | +        |
| Encodes cellular automatons   | -   | +    | +           | +    | +        |
| Has procedural language       | -   | +    | +           | +    | +        |
| Has embedded solvers          | -   | +    | +           | +    | -        |
| Constructs SMT encodings      | -   | -    | +           | +    | -        |
| Outlines sets of input and output variables | + | + | - | + | + |
| Constructs template CNFs      | -   | -    | -           | +/-  | +        |
| Adds auxiliary conditions on variables inside a program | - | - | - | + | + |

5. Inversion of Real World Cryptographic Functions Using their Translation to SAT

In this section we present our results on SAT-based cryptanalysis of several ciphering systems. Some of these systems were used in practice in recent past, and some of them are still employed at the present moment. In all cases considered below we used the encodings produced by the Transalg system.

5.1. Using Transalg to Construct SAT-based Guess-and-determine Attacks on Several Ciphers. Guess-and-determine is a general cryptanalysis strategy that can be used to evaluate the cryptographic weakness of various ciphers. The number of such attacks proposed in the recent two decades is hard to enumerate. Here we would like to cite the book [Bar09], the major part of which studies guess-and-determine approach in the context of algebraic cryptanalysis.
The basic idea of guess-and-determine strategy can be described as follows. Let $F$ be an arbitrary cipher that works with binary data, and let $E(F)$ be a system of Boolean equations that corresponds to some cryptanalysis problem for $F$. For example, for a given known pair $(x, y)$, where $x$ is a plaintext and $y$ is a ciphertext, $E(F)$ can be a system of Boolean equations from which one can extract the key $z$ such that $F(x, z) = y$. Let us denote by $X$ the set of all variables from $F$. Let $\tilde{X}, \tilde{X} \subseteq X$ be such a set of $l$ Boolean variables, that by assigning values to all variables from $\tilde{X}$ the problem of solving $E(F)$ becomes trivial. The simplest example in this context is when $\tilde{X}$ consists of variables corresponding to a secret key of a cipher $F$. Also $\tilde{X}$ can be formed by variables corresponding to the internal state of a cipher at some time moment, for example, to an internal state of keystream generator registers at some fixed step. In these cases by checking all possible assignments for variables from $\tilde{X}$, i.e., by performing exhaustive search over the set $\{0, 1\}^l$, we perform a brute force attack on a cipher $F$. For some ciphers it is possible to find a set $B, B \subseteq X$, $|B| = s$, which has the following property. Let us set values, corresponding to all their possible assignments, to variables from $B$ into a system $E(F)$ and spend on solving each constructed weakened system by some fixed algorithm at most $t$ seconds (or elementary operations). Assume that if we search over the whole $\{0, 1\}^s$ in such a way, we find the solution for a considered cryptanalysis instance for $F$, and spend on this process at most $T = 2^s \cdot t$. In this case we can say that there is a guess-and-determine attack with complexity $T$. The set $B$ is called a set of guessed bits. Guess-and-determine attacks with complexity significantly less than that of brute force attack are of particular interest. Usually, for such attacks $s \ll l$.

In SAT-based cryptanalysis the problems of finding solutions for systems of equations of the type $E(F)$ are reduced to SAT. To construct a guess-and-determine attack in this case one needs to know additional information about the variables contained in a corresponding CNF. In particular it is very useful to know which CNF variables correspond to secret key bits. The ideology of constructing template CNFs, employed in Transalg, allows us to naturally outline the sets of guessed bits in CNF which encodes an original cryptanalysis problem and thus, construct guess-and-determine attacks. Below we will briefly describe guess-and-determine attacks on several ciphers, for which the SAT-encodings were produced by Transalg.

In [SZBP11] there was constructed a SAT-based guess-and-determine attack on a well known A5/1 keystream generator. In that attack the set of guessed bits of size 31 was used. The attack from [SZBP11] was later verified in the SAT@home volunteer computing project (several dozens of cryptanalysis instances were solved in it using the technique from [SZBP11]). Later in [SZ15, SZ16] there was described the automatic method for finding sets of guessed bits via optimization of a special function, the value of which for a particular set of guessed bits is the runtime estimation of a corresponding guess-and-determine attack. Using this method in [SZ16] there was constructed a guess-and-determine attack on the Bivium cipher [Can06] the estimated runtime of which makes it realistic for state-of-the-art distributed computing systems. In [ZK17] using the algorithms from [SZ16] there were constructed guess-and-determine attacks on several variants of the Alternating Step Generator (ASG). For ASG-72 and ASG-96 the corresponding SAT-based guess-and-determine attacks were implemented on a computing cluster.

In [SZO+18] there was described a new class of SAT-based guess-and-determine cryptographic attacks, which is based on the so-called Inverse Backdoor Sets (IBS). IBS is the modification of the notion of Strong Backdoor Set for Constraint Satisfaction Problems

\[\text{TRANSALG}\]
(including SAT), introduced in [WGS03]. IBS is oriented specifically on problems of SAT-based cryptanalysis. In more detail, a Strong Backdoor Set for CNF $C$ with respect to algorithm $A$ is such a subset $B$ of a set of variables $X$ of this CNF, that setting values to variables from $B$ in $C$ in any possible way results in a CNF, for which the satisfiability problem is solved by a polynomial algorithm $A$. This definition in its original form does not suit well to cryptanalysis problems. However, in [SZO+18] we modified it in the context of discrete functions inversion problems. Conceptually, the modification consists in the following. Instead of demanding that $A$ is polynomial, we limit its runtime by some value $t$. An arbitrary $B \subseteq X$ we use in the role of the set of guessed bits and demand that algorithm $A$ in time $\leq t$, using the assignments of variables from $B$ as hints, can invert some portion of outputs of function $f_n$ constructed for random inputs from $\{0, 1\}^n$. The portion of inverted outputs is the probability of a particular random event. If it is relatively large and the power of $B$ is relatively small, then, as it was shown in [SZO+18] $B$ can be used to construct on its basis a nontrivial guess-and-determine attack on $f_n$. The set $B$ defined in such a way is called Inverse Backdoor Set (IBS). The effectiveness of guess-and-determine attacks based on IBS can be evaluated using the Monte Carlo method [MU49]. The problem of finding IBS with good runtime estimation of a corresponding guess-and-determine attack in [SZO+18] was reduced to a problem of optimization of a black box function over Boolean hypercube. Using IBSs in [SZO+18] there were constructed record or close to record guess-and-determine attacks for several ciphers. In particular, the runtime estimation of the constructed attack using 2 known plaintext (2KP) on AES-128 with 2.5 rounds is several dozen times better than that of the attack from [BDF11] and requires little to no memory, while the attack from [BDF11] need colossal amounts of it.

Let us once again emphasize the important features of translating algorithms to SAT, which are advantageous in the context of construction of guess-and-determine attacks. Here we first and foremost mean an ability to provide information about the interconnection between the variables in propositional encodings with corresponding elementary operations performed in an original cryptographic algorithm. For example, the ability to outline the variables that encode an input of a considered function for attacks described in [SZO+18] allows us to use template CNFs for effective generation of large random samples containing simplified CNFs. Each of them is formed by applying Unit Propagation to template CNF augmented by the values corresponding to the known function input. When implementing guess-and-determine attacks with realistic runtime estimations, the Transalg features make it possible to naturally mount such attack using the incremental SAT technique [ES03]. In some cases it can lead to significant performance gains (see e.g. [ZK17]).

5.2. SAT-based Cryptanalysis of Hash Functions from MD Family. In this section we present examples of application of the Transalg system to cryptanalysis of cryptographic hash functions from the MD family. It should be noted that these functions are still considered to be interesting among cryptanalysts and the first successful examples of application of SAT-based cryptanalysis to real world cryptosystems are related specifically to hash functions from the MD family [MZ06]. The present section is split into several subsections: first we consider the problems of finding collisions for MD4 and MD5 functions. Then we construct preimage attacks on truncated variant of the MD4 function. In all cases we used the SAT encodings constructed with the help of the Transalg system.
5.2.1. Finding Collisions for MD4 and MD5. Let \( f \{0,1\}^* \rightarrow \{0,1\}^c \) be some cryptographic hash function, that works with messages split into blocks of length \( n \), \( n > c \). It defines the function of the kind \( f_n : \{0,1\}^n \rightarrow \{0,1\}^c \). To produce the SAT encoding for the problem of finding collisions of this function we essentially translate the program describing \( f_n \) twice using disjoint sets of Boolean variables. Let \( C_1 \) and \( C_2 \) be the corresponding CNFs in which the sets of input and output variables are denoted by \( X^1 = \{x^1_1, \ldots, x^1_n\} \), \( X^2 = \{x^2_1, \ldots, x^2_n\} \) and \( Y^1 = \{y^1_1, \ldots, y^1_k\} \), \( Y^2 = \{y^2_1, \ldots, y^2_k\} \) respectively. Then finding collisions of \( f_n \) is reduced to finding an assignment that satisfies the following Boolean formula:

\[
C_1 \land C_2 \land (y^1_1 \equiv y^2_1) \land \ldots \land (y^1_k \equiv y^2_k) \land ((x^1_1 \oplus x^2_1) \lor \ldots \lor (x^1_n \oplus x^2_n)).
\]  

(5.1)

Below we consider the problems of constructing collisions for the MD4 and MD5 hash functions, which were actively used up until 2005. Let us first briefly remind the features of the Merkle-Damgård construction [Mer89, Dam89], which serves as a basis of many cryptographic hash functions. In accordance with this construction, in MD4 and MD5 the process of computing a hash value is viewed as a sequence of transformations of data stored in a special 128-bit register, to which we will refer as a hash register. The hash register is split into four 32-bit cells. At the initial stage a message called Initial Value (IV), which is specified in the algorithm’s standard, is put into hash register. Then the contents of the hash register are mixed with a 512-bit block of a hashed message by means of iterative transformations called steps. In MD4 there are 48 steps and in MD5 — 64 steps. At each step with an arbitrary cell of a hash register a 32-bit variable is associated. Such variables are called chaining variables. The transformations of data in a hash register which were defined above, specify the so-called compression function, which transforms a 512-bit message (block) into 128-bit hash. We will denote the compression functions used in MD4 and MD5 as \( f^{MD4} \) and \( f^{MD5} \), respectively. The MD4 and MD5 algorithms can be used to construct hash values for messages of an arbitrary length. For this purpose an original message is split into 512-bit block (if necessary, the last block is padded using a specific procedure). Let \( M = M_1, \ldots, M_k \) be a \( k \)-block message, where \( M_i, i \in \{1, \ldots, k\} \) are 512-bit blocks. A hash value for message \( M \) is constructed iteratively according to the following recurrence relations: \( \chi_0 = IV \), \( \chi_i = f^{MD}(\chi_{i-1}, M_i), i = 1, \ldots, k \) (here \( MD \) is either MD4 or MD5). A word \( \chi_k \) is the resulting hash value of a message \( M \). If hash values of two different \( k \)-block messages coincide, then the corresponding messages form \( k \)-block collision of a considered hash function.

The MD4 and MD5 algorithms were completely compromised with respect to finding collisions in [WLF+05, WY05]. The cryptanalytic methods used in mentioned papers belong to a class of the so-called differential attacks. In attacks from [WLF+05, WY05] a considered hash function is applied to two different messages. The processes of constructing hash values for these messages correspond to transformations of the contents of two hash registers. There are imposed additional constraints on the chaining variables associated with the corresponding cells of hash registers on fixed steps in the form of integer differences modulo \( 2^{32} \). These constraints form the so-called differential path. In [WLF+05, WY05] there were proposed differential paths that make it possible to effectively construct single-block collisions for MD4 and two-block collisions for MD5.

As we already mentioned the first SAT-variants of attacks from [WLF+05, WY05] were constructed in [MZ06]. To obtain a corresponding propositional encoding, it is first necessary to construct a formula of the kind (5.1) and then transform it to CNF using the Tseitin transformations. However, the resulting CNF turns out to be extremely hard even for
state-of-the-art SAT solvers. The realistic runtime of cryptanalysis is achievable only by adding to a constructed CNF the clauses which encode a differential path. The features of the Transalg system discussed above make it possible to implement this step quite easily. We would like to additionally note that the constraints defining a non-zero differential path eliminate the need for constraints of the kind 
\[(x_1^1 \oplus x_2^1) \lor \ldots \lor (x_n^1 \oplus x_n^2)\]
in (5.1) which indicate the difference between sets of values of the input variables (since only the different inputs of a hash function can lead to a non-zero differential path).

In Table 4 we compare the SAT-encodings of differential attacks for finding collisions for MD4 and MD5 used in [MZ06] with that constructed by Transalg.

### Table 4. The parameters of SAT encodings for finding collisions of the MD4 and MD5 hash functions with differential paths from [WLF+05, WY05].

|        | SAT encodings from [MZ06] | Transalg encodings |
|--------|---------------------------|--------------------|
| MD4    | variables 53228 clauses 221440 | variables 18095 clauses 187033 |
| MD5    | variables 89748 clauses 375176 | variables 34181 clauses 295773 |

For the problem of finding single block collisions of the MD4 hash function we managed to find about 1000 MD4 collisions within 200 seconds on one core of Intel i7-3770K (16 Gb RAM) using the SAT encodings produced by Transalg and the Cryptominisat solver [SNC09]. Note that, in [MZ06] it took about 500 seconds to construct one single block collision for MD4.

After this we studied the problem of finding two-block collisions of MD5. This process consists of two stages. In the first stage we search for two 512-bit blocks \(M_1\) and \(M'_1\), which satisfy the differential path from [WY05]. We denote \(\chi_1 = f^{MD5}(IV, M_1)\), \(\chi'_1 = f^{MD5}(IV, M'_1)\). In the second stage we look for second 512-bit blocks \(M_2\) and \(M'_2\) such that \(f^{MD5}(\chi_1, M_2) = f^{MD5}(\chi'_1, M'_2)\).

For the problem of finding first message blocks pair \((M_1, M'_1)\), which turned out to be quite hard, we used a computing cluster [mat] on which we ran state-of-the-art SAT solvers working in multi-threaded mode (36 threads). In particular we used pelingeling, treengeling (versions from the SAT competition 2014 [Bie14]) and pelingeling, treengeling [Bie17], painless [FBSK17], glucose-syrup [AS17] from the SAT competition 2017. Surprisingly, only treengeling 2014 managed to solve corresponding SAT instances within a time limit (30 hours).

During these experiments several message blocks with a lot of zeros in the beginning were found. A more detailed analysis showed that the maximum number of first message bytes, which can be set to 0 simultaneously in \(M_1\) and \(M'_1\) is 10 bytes. Assignment of the 11-th byte to 0 in \(M_1\) and \(M'_1\) makes the corresponding SAT instance unsatisfiable (which can be proven quickly).

Thus we outlined the class of message blocks pairs that satisfy the differential path from [WY05] and both blocks have first 10 zero bytes. The problem of finding a pair of such blocks is relatively simple and can be solved using a number of SAT solvers (compared to the situation when the first 10 bytes are not set to zero). For the corresponding SAT instance we ran four different SAT solvers (painless, glucose-syrup, treengeling 2014, treengeling 2017) each working in multi-threaded mode on one cluster node (36 processor
cores, Intel Xeon E5-2695). In 24 hours each solver managed to find several message blocks pairs, except for TREENGEILING 2014 SAT solver, which found only one. The corresponding results are given in Table 5.

| SAT solver       | Solved instances | Avg. time (s) |
|------------------|------------------|---------------|
| PAINLESS         | 3                | 32327         |
| GLUCOSE-SYRUP    | 3                | 38302         |
| TREENGEILING 2014| 1                | 54335         |
| TREENGEILING 2017| 3                | 33357         |

For the obtained pairs of first blocks, the problem of constructing such pairs \((M_2, M'_2)\) that the messages \(M_1|M_2\) (the concatenation of two 512-bit blocks \(M_1\) and \(M_2\)) and \(M'_1|M'_2\) form the two-block collision for MD5 turned out to be much simpler: on average one such pair \((M_2, M'_2)\) was found by TREENGEILING 2014 solver in 500 seconds on one cluster node.

An example of the two-block collision of the described kind is shown in Table 6.

| Hash            | e22664780a9766ceb57065eba36af06b |
|-----------------|----------------------------------|
| \(M_1|M_2\)      | 00 00 00 00 00 00 00 00 00 00 20 74 67 a6 f5 48 |
|                 | cb c1 6d a5 3e f7 b8 bc 67 a3 8d d9 3c 9b f5 b8 |
|                 | 55 ed 32 06 06 0a 74 a3 0f b6 84 87 47 cf 91 d0 |
|                 | db 4c 6f 43 ef 64 f0 8d a4 1d 50 c6 26 df 95 fe |
|                 | ff d1 2e c9 a0 90 aa b3 7d e7 e5 bc f2 3a 4e ab |
|                 | 24 b8 d4 13 4c cc 7b 1b 00 29 eb f5 53 7a 0d d1 |
|                 | 5d 1f b7 79 af 36 ce 08 1e 44 a2 d0 51 ec 91 fb |
|                 | c5 4c a2 89 75 b3 a3 84 ac 97 7f f2 7e 50 d4 56 |

In conclusion we would like to once more point out the features of Transalg system that made it possible to obtain the presented results. It is mainly thanks to the translation concept of Transalg that allows one to directly work with variables encoding each elementary step of a considered algorithm. That is why we can effectively reflect in SAT encoding any additional constraints, such as, for example, the ones that specify a differential path. In similar software systems it requires a significant amount of work to be implemented.

It should be noted that at the current stage SAT-based cryptanalysis is less effective in application to the collision search problems for cryptographic hash functions in comparison with specialized methods [SSA+09, SWOK07]. On the other hand, the use of new SAT
encodings and state-of-the-art SAT solvers makes it possible to find collisions for MD4 hash function about 1000 times faster than it was done in [MZ06]. From our perspective, the potential for further improvements in this direction is far from being exhausted. It should be also mentioned that in relation to the preimage attacks on cryptographic hash functions the SAT-based cryptanalysis is, apparently, the most effective tool for their solving. In the next section, we build a new preimage attack on the 39-step version of the MD4 hash function using the Transalg system.

5.2.2. Preimage Attacks on Truncated Variants of MD4. Despite the fact that MD4 is compromised with respect to the collision finding, the problem of finding preimages for this function is still considered to be extremely hard. And while it is believed that MD4 is not highly resistant to preimage attacks, all the arguments of this kind are mostly theoretical [Leu08]. Currently there are no papers in which in reasonable time there would be solved the inversion problem of full-round MD4. As far as we are aware, until recently, the paper [DKV07] was considered to be the best practical attack on MD4 since it made it possible to invert a truncated variant of MD4 with 39 (out of 48) steps. The attack from [DKV07] is a SAT-based variant of the attack proposed by H. Dobbertin in [Dob98]. Let us briefly review the results from these papers.

In fact in [Dob98] it was shown that the problem of inverting the first two rounds of MD4 (i.e. 32 steps) is not computationally hard. The main idea of that paper was to add some additional constraints on several chaining variables. These constraints significantly weaken the system of equations corresponding to the process of filling the hash register of MD4 during the first two rounds.

In more detail, H. Dobbertin in [Dob98] considered a system of equations that describe the process of computing MD4 hash value. He suggested to fix the values of some of the chaining variables by a random constant $K$. In some cases such substitutions can lead to propagation of the values of a large number of other variables. The main interest in this context pose the variables that represent the unknown preimage of a known hash. For the first two rounds of MD4 Dobbertin proposed to fix with constant $K$ the values of 12 chaining variables and showed that choosing the value of $K$ at random with high probability leads to a consistent system which can be easily solved. Thus the problem of inverting MD4-32 is not computationally hard.

In [DKV07] there was proposed a SAT-based attack on MD4 that used ideas of [Dob98]. More precisely, in [DKV07] the constant $K$ was fixed to 0. Also, the authors of [DKV07] rejected one of the constraints from [Dob98]. Thus, in [DKV07] there were used 11 (instead of 12) constraints. The constraints of ”Dobbertin” type were added to the propositional encoding of the MD4 algorithm in the form of unit clauses. Hereinafter we refer to additional constraints of the ”Dobbertin” type on chaining variables as relaxation constraints.

In [DKV07] apart from the two-round variant of MD4 there were considered preimage attacks on truncated MD4 variants with $k$ steps, up to and including $k = 39$. For an arbitrary $k < 48$ we will refer to a corresponding truncated variant of MD4 as to MD4-$k$. The best result presented in [DKV07] was the successful solving of inversion of MD4-39 for several hash values of a special kind. To solve each of such problems it took about 8 hours of Minisat SAT solver. It is surprising that the computational results achieved in [DKV07] remained state-of-the-art for 10 years. In [GS18] we significantly improved them. It was the result of using a special technique which reduced the problem of finding promising relaxation constraints to the problem of optimization of a black box function over Boolean
hypercube. In all experiments in [GS18] we used the SAT encodings constructed by the TRANSLG system. Below let us briefly review results obtained in [GS18].

To automatically take into account information about relaxation constraints the ability to work with so-called switching variables was added to TRANSLG. The main idea of this approach consists in the following. Let $C$ be a CNF that encodes the inversion of some function and $X$ be a set of Boolean variables from $C$. Assume that we need to add to $C$ new constraints that specify some predicate over variables from a set $\tilde{X}$, $\tilde{X} \subseteq X$. Let $R(\tilde{X})$ be a formula specifying this predicate. Now let us introduce new Boolean variable $u$, $u \notin X$. Consider the formula $C' = C \land (\neg u \lor R(\tilde{X}))$. It is clear that the constraint $R(\tilde{X})$ will be inactive when $u = 0$ and active when $u = 1$. Let us refer to variables similar to $u$ as switching variables.

In application to preimage attacks on MD4-$k$ the switching variables make it possible to reduce the problem of finding effective relaxation constraints (of the "Dobbertin" type) to the optimization problem over Boolean hypercube. With that purpose in [GS18] we introduce a special measure $\mu$ that heuristically evaluates the effectiveness of a considered set of relaxation constraints. Each particular set of relaxation constraints is defined by assignments of switching variables. The measure $\mu$ is function of a black box type. The relaxation constraints for which the value of $\mu$ lies in a particular range are considered to be promising. Thus, the arguments of the considered function are the switching variables and its values are the values of $\mu$ on the corresponding sets of relaxation constraints. The function defined that way is maximized over Boolean hypercube, each point of which represents an assignment of switching variables. Since the constructed function does not have an analytical representation, it is sensible to use metaheuristic methods for its maximization. In particular in [GS18] we used an algorithm from the tabu search [GL97] class. We view as promising such sets of relaxation constraints, the activation of which results in derivation by the Unit Propagation rule of a relatively large number of variables corresponding to hashed message in a SAT encoding (the number of such variables gives the value of function $\mu$). Similar to [DKV07] in the role of relaxation constraints we used the constraints meaning that the corresponding chaining variables should take the value $K = 0$. As a result we managed to find new relaxation constraints, which make it possible to invert the MD4-39 hash function much faster than in [DKV07]. Let us briefly mention the results of computational experiments from [GS18].

Let us note here that the structure of the MD4-39 hash function makes it impossible to impose constraints on the first four and the last (preceding the calculation of the final hash value) four steps of the MD4-39 algorithm. According to this, the sets of new relaxation constraints were selected (using the values of the corresponding switching variables) from the set of power 31. Thus, the problem of maximization of a function described above over Boolean hypercube $\{0, 1\}^{31}$ was considered. The details of experiments can be found in [GS18]. As a result we found new sets of relaxation constraints presented below in the form of assignments of corresponding switching variables (the assignments are denoted as $\rho_1$ and $\rho_2$):

$$\begin{align*}
\rho_1 & : \quad 000000000110110111011101000000 \\
\rho_2 & : \quad 000000000101111011110110100000
\end{align*}$$

For example, vector $\rho_1$ specifies the set of 12 relaxation constraints: chaining variables on steps numbers: 14, 15, 17, 18, 19, 21, 22, 23, 25, 26, 27, 29 are assigned the value $K = 0$. The application of the relaxation constraints specified by $\rho_1$ and $\rho_2$ allows one to find preimages of the MD4-39 hash function for known hash values $0^{128}$ and $1^{128}$ within one minute of
MINISAT2.2 runtime (whereas using constraints from [DKV07] the solution of the preimage finding problem for $1^{128}$ requires about 2 hours, and the preimage finding problem for $0^{128}$ cannot be solved in 8 hours). Corresponding results are presented in Table 7, where $\rho_{De}$ denotes the set of relaxation constraints described in [DKV07] and $\rho_{Dobbertin}$ denotes the variant of Dobbertin’s constraints from [Dob98] with constant $K = 0$. Below these relaxation constraints are specified by the vectors of values of switching variables from $\{0, 1\}^{31}$ (in the notation similar to that of $\rho_1$ and $\rho_2$):

$$
\rho_{Dobbertin} : \begin{array}{c}
000000000110111011100000000
\end{array}
$$

$$
\rho_{De} : \begin{array}{c}
000000000110111011100000000
\end{array}
$$

What is particularly interesting is that the application of new sets of relaxation constraints $\rho_1$ and $\rho_2$ also allows one to find preimages of MD4-39 for randomly generated 128-bit Boolean vectors persistently. To obtain this result, we considered a test set of 500 randomly generated vectors from $\{0, 1\}^{128}$. Regarding each of these vectors we assumed that it is a hash value of MD4-39. After that the preimage finding problem for this value was solved using constraints specified by vectors $\rho_1$ and $\rho_2$. For the prevailing part of the tasks (65-75%) the solutions were successfully found using the MINISAT2.2 SAT solver. The average time of finding one preimage was less than 1 minute. The rest (25-35% of the tasks) corresponded to 128-bit vectors for which there were no MD4-39 preimages under constraints specified by $\rho_1$ and $\rho_2$ (this fact was proven by the SAT solver in under 1 minute on average). These results are presented in Table 8. Note that even in a few hours we did not manage to solve the preimage finding problem for any vector from the test set using constraints from [DKV07] or [Dob98].

**Table 7.** Finding the MD4-39 preimages for hash values $0^{128}$ and $1^{128}$.

| Relaxation constraints | Result / Solving time (s) | $\chi = 0^{128}$ | $\chi = 1^{128}$ |
|------------------------|---------------------------|------------------|------------------|
| $\rho_1$               | SAT / 20                  | SAT / 10         |
| $\rho_2$               | SAT / 60                  | UNSAT / < 1      |
| $\rho_{Dobbertin}$     | SAT / 20                  | Unknown / > 30000|
| $\rho_{De}$            | Unknown / > 30000         | SAT / 7000       |

**Table 8.** Finding the MD4-39 preimages for 500 randomly generated 128-bit Boolean vectors.

| Relaxation constraints | Avg. solving time (s) | Max. solving time (s) | Solved instances (in % of total number of instances) |
|------------------------|-----------------------|-----------------------|-----------------------------------------------------|
|                        |                       |                       | with preimages (SAT) with no preimages (UNSAT)       |
| $\rho_1$               | 12                    | 80                    | 65                                                   |
| $\rho_2$               | 46                    | 250                   | 75                                                   

Let us summarize the results of the present section. From our point of view in this section we convincingly demonstrated the power of SAT-based cryptanalysis methods. We showed how applying the propositional encoding procedures that take into account the
specifics of cryptographic functions makes it possible to construct successful, and sometimes performance-winning attacks on relevant cryptosystems. We believe that future ideas both in the area of reduction to SAT and in algorithms of state-of-the-art SAT-solvers will make it possible to increase the effectiveness of SAT-based cryptanalysis and extend the spectrum of its applications.

6. Related Works

The ideas of using general purpose combinatorial algorithms to solve cryptanalysis problems can be found in many papers starting from 90-th years of XX-th century. Apparently the first to propose using SAT solving algorithms in cryptanalysis were S.A. Cook and D.G. Mitchell in \cite{CM97}. The first example of propositional encoding of a cryptographic problem (in particular of DES cryptanalysis) was given in \cite{Mas99, MM00}. In \cite{JJ05} the problems of finding collisions of a number of cryptographic hash functions were reduced to SAT. In \cite{MZ06} there were also constructed propositional encodings for hash functions from the MD family. However, the authors of \cite{MZ06} added to these encodings the constraints that encode the differential paths introduced in \cite{WLFWY05, WY05}. It made it possible to persistently construct single-block collisions for MD4 (it took about 10 minutes per collision). Thus, the paper \cite{MZ06} can be considered to be the first paper in which SAT-based cryptanalysis was successfully applied to relevant cryptographic algorithms. In the book \cite{Bar09} it is the SAT-solvers that are considered to be the primary tool for solving problems of algebraic cryptanalysis. It should be noted that in all aforementioned papers to construct propositional encodings of the considered functions no automated system was used.

In the present paper we described in detail the principles of constructing propositional encodings for discrete functions with the focus on functions employed in cryptography. We also compared several different systems that can produce such SAT encodings in an automatic mode. First it is the well known \textsc{CBMC} system for symbolic verification \cite{CKL04, Kro09}, which have been developed for more than 15 years. \textsc{CBMC} is a generic system and it is not designed with cryptanalysis instances in mind. However, as we show in Section 4, it allows one to perform the majority of actions available to a limited number of considered domain specific systems. Here we mean \textsc{Cryptol}, \textsc{Grain-of-Salt}, \textsc{URSA} and \textsc{Transalg}.

The \textsc{Cryptol} software system \cite{EM08, ECW09, DFH16} was developed by Galois inc. In 2010 there appeared the \textsc{Grain-of-Salt} system \cite{Soo10}. Approximately at this time we started the development of the \textsc{Transalg} software system, which we describe in the present paper (\textsc{Transalg} was first mentioned in papers in Russian in 2011). In 2012 there appeared the \textsc{URSA} system \cite{Jan12}, aimed at reducing to SAT various constraint programming problems. It can be applied to construct encodings for cryptographic functions as well. Note that \textsc{Cryptol}, \textsc{URSA} and \textsc{Transalg} can encode to SAT algorithmic descriptions of a very wide class of functions working with binary data. Meanwhile, \textsc{Grain-of-Salt} is designed to work only with keystream generators based on shift registers. We considered the pros and cons of all mentioned systems in detail in Section 4.

In the basis of \textsc{CBMC}, \textsc{Transalg} and other systems lies the ideology of symbolic execution of a program specifying a considered function. Note that the idea to transform programs to Boolean formulas was first proposed by S.A. Cook in his paper \cite{Coo71} which led to the creation and development of the theory of NP-completeness. The notion "Symbolic Execution" first appeared in the paper \cite{Kin76} by J.C. King, where it is defined as a process of interpretation of a program in a special extended semantics, within the context of which
it is allowed to take symbols as input and put formulas as output. Currently, symbolic
execution using SAT is actively used in software verification (see for example [Kro09]).

As we mentioned above, SAT-based cryptanalysis is still actively developing. The
cryptographic attacks that employ SAT-solvers show very good results for a number of
keystream generators: Geffe, Wolfram (present paper), Crypto-1, Hitag2 (see [SNC09]). In
[SZBP11] there was described a successful SAT-based attack on the widely known A5/1
cryptographic keystream generator, which has been used in GSM networks to cipher traffic.
Later several dozen cryptanalysis instances for this cipher were solved in the SAT@home
volunteer computing project [PSZ12]. This result together with other attacks on A5/1 (see
[BSZK18, GKN+08, Noh10]) provides an exhaustive argument towards not using A5/1 any
more. The Bivium keystream cipher [Can06] is a popular object of algebraic and SAT-based
cryptanalysis [MCP07, EPV08, SNC09, EVP10]. In [SZ16] there was constructed a SAT-
based guess-and-determine attack on Bivium the estimated runtime of which is realistic
for modern distributed computing systems. In [ZK17] there were described SAT-based
guess-and-determine attacks on several variants of the alternating step generator.

In [SZO+18] there was described a new class of SAT-based guess-and-determine attacks,
in which the notion of Inverse Backdoor Set (IBS) is used. IBS is a modification of a well
known notion of Strong Backdoor Set [WGS03]. It allowed to construct record or close to
that guess-and-determine attacks on several ciphers. For example, the attack on 2.5-round
AES-128 with 2 Known Plaintexts, presented in [SZO+18], is significantly better than the
previously best known attack on this cipher proposed in [BDF11]. Note that the function
introduced in [SZO+18] to associate with a particular IBS the estimation of effectiveness of a
Corresponding guess-and-determine attack is a concretization of the notion of SAT-immunity,
introduced by N. Courtois in [Cou15, CGS12, Con13].

As we already noted, [MZ06] was the first paper to demonstrate the applicability of SAT-
based cryptanalysis to relevant cryptographic algorithms. In that paper using the MINISAT
solver [ES04b] it was possible to quite effectively find single-block collisions for MD4. Using
new propositional encoding methods (in particular, the TRANSALG system) and state-of-the-
art SAT solvers it is possible to find preimages for MD4 and MD5 several hundred times faster
than it was done in [MZ06]. Nevertheless, on the current stage SAT-based cryptanalysis
is less effective than specialized methods (see, for example [SWOK07, Ste12, SKP16]) on
problems of finding collisions of cryptographic hash functions. However, as far as we know,
it is the SAT-based approach that yields best known preimage attacks on truncated variants
of hash functions [DKV07, Nos12, NNS+17]. In application to MD4-39 for a long time the
SAT-based preimage attack from [DKV07] was considered to be the best. In [GS18] we
significantly improved the results from [DKV07]. It was possible for the large part thanks
to functional capabilities of the TRANSALG system.

7. Conclusion

The presented paper contains a detailed description of the procedures that guarantee the
effective encoding of problems of inversion of discrete function from a wide class to the
Boolean Satisfiability Problem. The main focus of the paper is on the functions employed in
cryptography. We constructed our paper in accordance with the concept from theoretical
foundations through implementation to practical applications. In the beginning of the paper
we provide strict mathematical basis of SAT-based cryptanalysis, define the corresponding
problems in the context of the general problem of finding preimages (inversion) of discrete
functions from a wide class. For a considered class of functions we prove that the problems of their inversion can be effectively (in polynomial time) be reduced to SAT.

Based on the obtained theoretical results, we construct a domain-specific system designed to be employed in algebraic cryptanalysis. This system was called Transalg. When developing Transalg we took into account various issues which are specific to cryptanalysis. We compared Transalg with several software systems that can be used to construct propositional encodings of cryptographic functions (CBMC, CRYPTOL+SAW, URSA, GRAIN-OF-SALT). Overall, the considered systems showed comparable results on cryptanalysis instances, however in most cases the Transalg encodings are slightly better. Also, Transalg possesses a number of additional features which make it a useful tool for constructing algebraic attacks. For example, the ability to directly output template CNFs with pre-defined numbering of input and output variables makes Transalg a very convenient tool when implementing SAT-based cryptographic attacks from the guess-and-determine class.

In the final part of the paper we provide a lot of examples where the described methods for translating algorithmic descriptions of cryptographic functions into SAT were used to construct algebraic attacks on considered cryptographic functions. Some of these attacks are state of the art.

We would like to specifically remark that the detailed study of CBMC revealed that this system is able to solve the majority of problems that we solved using Transalg. The employed SAT solvers show the comparable performance on Transalg and CBMC encodings, in some cases the former is better, in some — the latter. This fact is quite surprising since CBMC does not specifically target cryptanalysis applications.

As a final comment, we would like to once more emphasize the theoretical and practical importance of SAT-based cryptanalysis and note that the corresponding problems can be viewed as interesting challenges for researchers and they stimulate the development of new algorithms and SAT solving techniques. We believe that the results presented in this paper will be useful in that context.

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A. Detailed Comparison of all Considered Solvers

Figure 6. Solving \textit{S}\textsubscript{\textit{G}}effe cryptanalysis instances using different encodings (see Section 4). Part 1.
Figure 7. Solving $S_{\text{Geffe}}$ cryptanalysis instances using different encodings (see Section 4). Part 2.
Figure 8. Solving Wolfram cryptanalysis instances using different encodings (see Section 4). Part 1.
Figure 9. Solving Bivium30 cryptanalysis instances using different encodings (see Section 4).
Figure 10. Solving Grain102 cryptanalysis instances using different encodings (see Section 4)