A New L-Stable Third Derivative Hybrid Method for Solving First Order Ordinary Differential Equations

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Author’s contribution
The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, an L-stable third derivative multistep method has been proposed for the solution of stiff systems of ordinary differential equations. The continuous hybrid method is derived using interpolation and collocation techniques of power series as the basis function for the approximate solution. The method consists of the main method and an additional method which are combined to form a block matrix and implemented simultaneously. The stability and convergence properties of the block were investigated and discussed. Numerical examples to show the efficiency and accuracy of the new method were presented.

Keywords: Third derivative; L-stable; continuous hybrid methods; block matrix; stiff system.

1 Introduction

Mathematical models of some real-life phenomena are modeled in differential equations. A differential equation is an equation that relates some functions with its derivatives.

Differential equations can be classified into ordinary and partial differential equations. Ordinary differential equations (ODEs) are usually used in various fields, such as Geology, Economics, Biology, Physics, and many
branches of Engineering and even social sciences. Many of these Ordinary differential equations are known as stiff ODEs. Stiffness occurs when the stability requirement rather than accuracy constraints the step length for numerical integration,[1] Lambert.

Stiff initial value problems (IVPs) in ODEs are difficult to solve since some of the numerical methods have absolute stability restriction on the step size, [2].

In this research work, our aim is to develop an efficient third derivative Linear Multistep Methods (LMMs) which can solve stiff initial value problems in ODEs of the form

$$y' = f(x, y(x)), y(x_0) = y_0$$

(1)

where

$$f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m, \quad y \in \mathbb{R}^m \text{ and } x \in \mathbb{R}$$

Several works have been done to develop more advanced and efficient methods for the numerical integration of stiff systems. Consequent upon this, a wide variety of approaches have been proposed Cash [3,4,5].

A potentially good numerical method for the solution of stiff systems of ordinary differential equation must have good accuracy and some reasonably wide region of absolute stability, [6]. This stability property is known as A-stability [7].

A great deal of multistep formulas for integrating stiff initial value problems (IVPs) has been developed. This includes the Backward Differentiation Formulas (BDF) [8]. The backward differentiation formulae uses the first derivative of the solution of (1) [5]. Several modifications have been done in this regard to include the Extended Backward Differentiation Formulas (EBDF) introduced by [3]. According to [5], the EBDF is A-stable and have higher order of accuracy and better stability characteristics than the conventional BDF when applied to (1). Other improved classes of LMMs are the second and third derivatives of the solution of (1) [5,9,10]. Some well known second derivative multistep methods for the solution of stiff system are the k-step second order derivative multistep method (SDMM) of order of $p = k + 2$ which is A-stable for $p \leq 4$ [11] and the k-step second order derivative multistep method (SDMM) of order of $p = k + 2$ which is also A-stable for $p \leq 6$ [4].

A family of third derivative multistep methods (TDMM) for integrating (1) was developed by [5]. This method was developed to overcome the Dahlquist order barrier.

In this paper, we shall derive a continuous hybrid third derivative method with one off-grid point. The method contains one main method and an additional method which is obtained from the third derivative of the continuous linear multistep (clmms). These are then combined and implemented simultaneously as block hybrid method.

The paper is organized as follows: Section 2 discusses the derivation of the method, section 3 the properties of the block were analyzed and presented. In section 4 some numerical examples were presented to prove the accuracy and suitability of the method.

2 Derivation of The Methods

In this section we shall derived a $k$ — step multistep method of the form

$$y_{n+k} + \alpha_r y_{n+r} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h^2 \beta_k g_{n+k} + h^3 \beta_k^3 p_{n+k}$$

(2)
Let the exact solution of one of (1) be $Y(x)$ in the range $[x_n, x_{n+h}]$. We shall represent the exact solution by the approximation solution by the form

$$Y(x) = \sum_{j=0}^{4k+2} a_j x^j$$  

(3)

where $a_j$'s are the coefficients to be determined. We shall develop our method by imposing the following conditions on (3).

$$Y(x_n) = y_n$$  

(4)

$$Y'(x) = \sum_{j=0}^{4k+2} ja_j x^{j-1} = f_{n+j}, j = 0(1)k$$  

(5)

$$Y''(x) = \sum_{j=0}^{4k+2} j(j-1)a_j x^{j-2} = g_{n+j}, j = k$$  

(6)

$$Y'''(x) = \sum_{j=0}^{4k+2} j(j-1)(j-2)a_j x^{j-3} = p_{n+j}, j = k$$  

(7)

Equations 4—6 leads to a system of equations of the form

$$AX = B$$  

(8)

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}h & \frac{1}{4}h^2 & \frac{1}{8}h^3 & \frac{1}{16}h^4 & \frac{1}{32}h^5 & \frac{1}{64}h^6 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & h & h^2 & \frac{1}{2}h^3 & \frac{5}{16}h^4 & \frac{3}{16}h^5 \\ 0 & 1 & 2h & 3h^2 & 4h^3 & 5h^4 & 6h^5 \\ 0 & 0 & 2 & 6h & 12h^2 & 20h^3 & 30h^4 \\ 0 & 0 & 0 & 6 & 24h & 60h^2 & 120h^3 \end{bmatrix}$$

and

$$B = \begin{bmatrix} y_n \end{bmatrix}$$

On solving the system yield the value of $a_j$, $s, j = 0(1)6$. By substituting the values of $a_j$, $s, j = 0(1)6$ into equation (3), the continuous schemes is obtained as follows

$$Y(t) = \sum_{j=0}^{k-1} \alpha_j(t)y_{n+j} + \sum_{j=0}^{k} \beta_j(t)f_{n+j} + h^2\phi_k(t)g_{n+k} + h^3\delta_k(t)p_{n+k}$$  

(9)
Where \( t = \frac{x - x_n}{h} \) and \( \alpha_j(t), \beta_j(t), \phi_k(t), \delta_k(t) \) are the continuous coefficients.

\[
\begin{bmatrix}
\alpha_0(t) \\
\alpha_{\frac{1}{2}}(t)
\end{bmatrix} =
\begin{bmatrix}
1 & -\frac{80}{3} & 800 & -120 & 224 & -160 \\
0 & \frac{80}{3} & -800 & 120 & -224 & 160
\end{bmatrix} t^0 t^1 t^2 t^3 t^4 t^5 t^6
\]

\[
\begin{bmatrix}
\beta_0(t) \\
\beta_{\frac{1}{2}}(t) \\
\beta_1(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & -\frac{79}{12} & \frac{299}{28} & \frac{161}{8} & -\frac{161}{6} & -\frac{498}{18} \\
0 & 1 & -\frac{28}{9} & \frac{376}{22} & \frac{136}{3} & -\frac{104}{9} \\
0 & 1 & -\frac{31}{9} & \frac{251}{209} & \frac{119}{6} & -\frac{97}{18}
\end{bmatrix} t^0 t^1 t^2 t^3 t^4 t^5 t^6
\]

\[
\begin{bmatrix}
\phi_0(t) \\
\phi_{\frac{1}{2}}(t) \\
\phi_1(t)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{3}{4} & \frac{25}{6} & \frac{65}{8} & \frac{13}{2} & \frac{11}{6} \\
0 & 0 & 0 & -\frac{5}{72} & \frac{43}{108} & \frac{13}{16} & \frac{25}{36}
\end{bmatrix} t^0 t^1 t^2 t^3 t^4 t^5 t^6
\]

Evaluating (8) at \( x = x_{n+1} \) gives the main scheme

\[
y_{n+1} = \frac{1}{9} y_n + \frac{8}{9} y_{\frac{n+1}{2}} + h \left( \frac{1}{72} f_n + \frac{2}{9} f_{\frac{n+1}{2}} + \frac{23}{72} f_{n+1} \right) - h^2 \left( \frac{1}{24} g_{n+1} \right) + h^3 \left( \frac{1}{432} p_{n+1} \right)
\]

The additional method is obtained from the third derivative of (8) given by

\[
h^3 \tilde{Y}'''(t) = \sum_{j=0}^{k} \alpha_j(t) y_{n+j} + h \sum_{j=0}^{k} \beta_j(t) f_{n+j} + h^2 \phi_k(t) g_{n+k} + h^3 \delta_k(t) p_{n+k}
\]

Evaluating (10) at \( x = x_{\frac{n+1}{2}} \) to give

Where
3 Analysis of the Block Method

3.1 Order and error constant

The block matrix finite difference form of the new methods is given by

\[ A^{(0)}Y_m = A^{(0)}Y_{m-1} + h[B^{(0)}F_m + B^{(0)}F_{m-1}] + h^2[C^{(0)}G_m + C^{(0)}G_{m-1}] + h^3[D^{(0)}H_m + D^{(0)}H_{m-1}] \]

The linear operator associated with the matrix finite difference form of the new methods in (10) and (12) is defined as

\[ L[y(x); h] = A^{(0)}y_m - A^{(0)}y_{m-1} - h[B^{(0)}F_m + B^{(0)}F_{m-1}] - h^2[C^{(0)}G_m + C^{(0)}G_{m-1}] - h^3[D^{(0)}H_m + D^{(0)}H_{m-1}] \]

The Taylor series expansion of methods (14) above yield

\[ C^{(0)} = \begin{bmatrix} \frac{160}{3} & \frac{1}{2} & \frac{31}{6} & -\frac{104}{3} & \frac{1}{2} & \frac{79}{6} & \frac{1}{3} & 0 & 0 \end{bmatrix} \left( \begin{array}{c} 0 \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2} \end{array} \right) \]

Thus, we have that \( c_0 = c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0, c_7 \neq 0 \),


Thus, the analysis of methods (10 and 12) shows they are of order \( p = 6,6 \) with error constants given as

\[ C_7 = C_{p+1} = \begin{bmatrix} \frac{1}{160} & \frac{1}{3024} & \frac{1}{1451520} \end{bmatrix} \]

3.2 Stability analysis of method

With equations (10) and (12) above combined we have the block form of the one step block hybrid third derivative method as given in (13):
\[ A^{(1)}Y_m = A^{(0)}Y_{m-1} + h \left[ B^{(1)}F_m + B^{(0)}F_{m-1} \right] + h^2 \left[ C^{(1)}G_m + C^{(0)}G_{m-1} \right] + h^3 \left[ D^{(1)}H_m + D^{(0)}H_{m-1} \right] \]

Where

\[ Y_m = \begin{bmatrix} y_{n+\frac{1}{2}} \\ y_n \end{bmatrix}, \quad Y_{m-1} = \begin{bmatrix} y_{n-\frac{1}{2}} \\ y_n \end{bmatrix}, \quad F_m = \begin{bmatrix} f_{n+\frac{1}{2}} \\ f_{n+1} \end{bmatrix}, \quad F_{m-1} = \begin{bmatrix} f_{n-\frac{1}{2}} \\ f_n \end{bmatrix}, \quad G_m = \begin{bmatrix} g_{n+\frac{1}{2}} \\ g_{n+1} \end{bmatrix}, \quad H_m = \begin{bmatrix} p_{n+\frac{1}{2}} \\ p_{n+1} \end{bmatrix} \]

\[ A^{(1)} = \begin{bmatrix} -160 & 0 \\ -8 & 1 \end{bmatrix}, \quad A^{(0)} = \begin{bmatrix} 0 & -160 \\ 0 & 1 \end{bmatrix}, \quad B^{(1)} = \begin{bmatrix} 104 & -79 \\ 2 & 23 \end{bmatrix}, \quad B^{(0)} = \begin{bmatrix} 0 & 31 \\ 0 & 1 \end{bmatrix}, \quad C^{(1)} = \begin{bmatrix} 0 & -5 \\ 0 & -1 \end{bmatrix} \]

\[ D^{(1)} = \begin{bmatrix} -1 & -5 \\ 0 & 1 \end{bmatrix} \]

### 3.3 Zero stability of method

The zero stability property is concerned with the stability of difference system in (13) in the limit as \( h \to 0 \). As \( h \to 0 \) (11) becomes

\[ A^{(1)}Y_\alpha - A^{(0)}Y_{\alpha-1} = 0 \] (15)

The characteristic polynomial of the above is given by

\[ \rho(R) = \det \left( RA^{(1)} - A^{(0)} \right) \] (16)

Thus \( \rho(R) = \det \left( RA^{(1)} - A^{(0)} \right) = -\frac{160}{3} R + \frac{7}{9} \)

The block method (11) is zero stable since from (15); if \( \rho R = 0 \) satisfies \( |R| \leq 1, i = 1, 2 \)

### 3.4 Consistency of the method

The method (11) is consistent as it has order \( p > 1 \), hence according to[12], we can say that the proposed method is convergent.

### 3.5 Linear stability

In the spirit of [7,13], we shall consider the following test equations

\[ y' = \lambda y, \quad y'' = \lambda^2 y, \quad y''' = \lambda^3 y \] (17)

Which when applied to (11), yield
The stability requirement of the method for solving stiff system is that the method is A-stable. For method (13), this can only be achieved if the left – half complex plane is contained in Q. The stability function above is used to plot the region of absolute stability (RAS) of the method as shown below.

The proposed is method also L-stable as the limit of the stability function \( R(z) \) as \( z \to \infty \) is zero.

The stability plot of the method is therefore given below in Fig.1

\[
Y_\alpha = M(z)Y_{\alpha-1}, \ z = \lambda h
\]  
(18)

Where the matrix \( M(z) \) is given as

\[
M(z) = (A^{(0)} + B^{(1)}z + C^{(1)}z^2 + D^{(1)}z^3)^{-1}(A^{(0)} + B^{(0)}z)
\]  
(19)

From (16), we shall obtain the stability function is given as

\[
R(z) = \frac{9z^4 + 72z^3 - 432z^2 + 13632z - 26880}{3z^6 - 54z^5 + 478z^4 - 976z^3 + 10560z^2 + 82176z + 69120}
\]  
(20)

The region of absolute stability of the method is therefore defined as

\[
Q = \{z \in \mathbb{C}: R(z) \leq 1\}
\]  
(21)

Fig. 1. Stability region of the block hybrid third derivative method
3.5 Numerical experiments

We shall consider the following problems in order to examine the accuracy and efficiency of our method. The computation was done using MAPLE 18 software.

**Problem 4.1.** We consider the linear stiff problem

\[
\begin{align*}
y' &= -y + 95z, \quad y(0) = 1, \quad x \in [0, 1] \\
z' &= -y - 97z, \quad z(0) = 1
\end{align*}
\]

**Exact solution**

\[
\begin{align*}
y(x) &= \frac{1}{47} \left( 95e^{-2x} - 48e^{-96x} \right) \\
z(x) &= \frac{1}{47} \left( 48e^{-96x} - e^{-2x} \right)
\end{align*}
\]

Source: Abhulimen and Ukpebor [14]

| Step      | Method                  | \(y(x)\) | \(z(x)\) |
|-----------|-------------------------|-----------|-----------|
| \(1/8\)   | New method              | 0.27355004(1.61313\times10^{-10}) | -0.0028794741(1.688\times10^{-12}) |
| \(1/16\)  | Abhulimen & Ukpebor(2019) | 0.27354997(3.0\times10^{-8}) | -0.0028794741(1.4\times10^{-9}) |
| \(1/32\)  | New method              | 0.27355004(3.12445\times10^{-10}) | -0.0028794741(2.8709\times10^{-12}) |
| \(1/32\)  | Abhulimen & Ukpebor(2019) | 0.27354997(3.0\times10^{-8}) | -0.0028794741(1.4\times10^{-9}) |
| \(1/32\)  | New method              | 0.27355004(2.28712\times10^{-10}) | -0.0028794741(1.90027\times10^{-12}) |
| \(1/32\)  | Exact solution          | 0.273550040584643 | -0.00287947410542666 |

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\end{align*}
\]

Source: Abhulimen and Ukpebor [14]

\[
\begin{align*}
y_1' &= -100y_1 + 9.901y_2, \quad y_1(0) = 1 \\
y_2' &= 0.1y_1 - y_2, \quad y_2(0) = 10
\end{align*}
\]

**Exact solution:**

\[
y_1(x) = e^{-0.99x}, \quad y_2(x) = 10e^{-0.99x}
\]

Source: Sabo et al [15]
### Table 2. Errors in the Results of problem 4.2 compared with errors in the results of Sabo et al [15]

| X Value | Sabo et al. (2018) K=2, p=8 | New method (K=1,p=8) |
|---------|-----------------------------|----------------------|
|         | $y_1(x)$                    | $y_2(x)$             | $y_1(x)$                    | $y_2(x)$             |
| 0.1     | 5.00×10^{-10}               | 4.00×10^{-9}         | 6.7452×10^{-11}            | 4.8562×10^{-10}     |
| 0.2     | 1.50×10^{-9}                | 7.00×10^{-9}         | 1.05030×10^{-10}           | 8.7970×10^{-10}     |
| 0.3     | 1.50×10^{-9}                | 9.00×10^{-9}         | 1.34970×10^{-10}           | 1.19518×10^{-9}     |
| 0.4     | 2.20×10^{-9}                | 1.30×10^{-8}         | 1.58333×10^{-10}           | 1.44338×10^{-9}     |
| 0.5     | 2.20×10^{-9}                | 1.50×10^{-8}         | 1.76093×10^{-10}           | 1.63416×10^{-9}     |
| 0.6     | 2.30×10^{-9}                | 1.30×10^{-8}         | 1.89097×10^{-10}           | 1.77615×10^{-9}     |
| 0.7     | 1.30×10^{-9}                | 1.20×10^{-8}         | 1.98085×10^{-10}           | 1.87686×10^{-9}     |
| 0.8     | 2.10×10^{-8}                | 1.40×10^{-8}         | 2.03700×10^{-10}           | 1.94281×10^{-9}     |
| 0.9     | 1.70×10^{-9}                | 1.50×10^{-8}         | 2.06496×10^{-10}           | 1.97965×10^{-9}     |
| 1.0     | 2.40×10^{-9}                | 1.70×10^{-8}         | 2.05494×10^{-10}           | 1.98495×10^{-9}     |

**Problem 4.3**

\[
y_1 = -8y_1 + 7y_2, \quad y_1(0) = 1
\]

\[
y_2 = 42y_1 - 43y_2, \quad y_2(0) = 1
\]

**Exact solution:** \( y_1(x) = 2e^{-x} - e^{-50x}, \quad y_2(x) = 2e^{-x} + 6e^{-50x} \)

Source: Sabo et al [16]

### Table 3. Errors in the Results of problem 4.3 compared with the errors in the results of Sabo et al [16]

| X Value | Sabo et al. (2019) (k=2, p=8) | New method (K=1,p=8) |
|---------|-----------------------------|----------------------|
|         | $y_1(x)$                    | $y_2(x)$             | $y_1(x)$                    | $y_2(x)$             |
| 0.1     | 3.87000×10^{-4}             | 8.32000×10^{-2}      | 1.122459×10^{-08}           | 5.132424×10^{-08}   |
| 0.2     | 6.00000×10^{-6}             | 5.81000×10^{-4}      | 2.23865×10^{-08}           | 1.0212513×10^{-07}  |
| 0.3     | 6.90000×10^{-8}             | 5.33000×10^{-6}      | 3.33297×10^{-08}           | 1.5240662×10^{-07}  |
| 0.4     | 1.22000×10^{-7}             | 1.36000×10^{-7}      | 4.429816×10^{-08}          | 2.0217264×10^{-07}  |
| 0.5     | 4.50000×10^{-8}             | 1.12000×10^{-6}      | 5.512501×10^{-08}          | 2.5142706×10^{-07}  |
| 0.6     | 1.48000×10^{-7}             | 1.36000×10^{-7}      | 6.585420×10^{-08}          | 3.0017377×10^{-07}  |
| 0.7     | 8.08000×10^{-8}             | 9.66000×10^{-7}      | 7.648647×10^{-08}          | 3.4841658×10^{-07}  |
| 0.8     | 1.62000×10^{-7}             | 1.52000×10^{-7}      | 8.702249×10^{-08}          | 3.9615932×10^{-07}  |
| 0.9     | 1.03000×10^{-7}             | 8.25000×10^{-7}      | 9.746296×10^{-08}          | 4.4340579×10^{-07}  |
| 1.0     | 1.66000×10^{-7}             | 1.58000×10^{-7}      | 1.0780858×10^{-07}         | 4.9015973×10^{-07}  |

\[
y_1 = 198y_1 + 199y_2, \quad y_1(0) = 1
\]

**Problem 4.4**

\[
y_2 = 398y_1 - 399y_2, \quad y_2(0) = -1
\]

**Exact solution:** \( y_1(x) = e^{-x}, \quad y_2(x) = -e^{-x} \)
Source: Sabo et al [16]

Table 4. Errors in the results of problem 4.4 compared with the errors in Sabo et al. [16]

| X Value | Sabo et al. (2019) (k=2, p=8) | New method (K=1, p=8) |
|---------|-----------------------------|------------------------|
|         | $y_1(x)$                    | $y_2(x)$               | $y_1(x)$ | $y_2(x)$ |
| 0.1     | $2.60000 \times 10^{-6}$    | $9.05000 \times 10^{-6}$ | $8.4359600 \times 10^{-9}$ | $7.035960 \times 10^{-9}$ |
| 0.2     | $2.95000 \times 10^{-7}$    | $2.81000 \times 10^{-7}$ | $1.6489016 \times 10^{-8}$ | $1.5178152 \times 10^{-8}$ |
| 0.3     | $4.840000 \times 10^{-7}$   | $4.33000 \times 10^{-8}$ | $2.2971485 \times 10^{-8}$ | $2.1783977 \times 10^{-8}$ |
| 0.4     | $4.95000 \times 10^{-6}$    | $4.83000 \times 10^{-6}$ | $2.8072070 \times 10^{-8}$ | $2.6997518 \times 10^{-8}$ |
| 0.5     | $6.250000 \times 10^{-6}$   | $5.84000 \times 10^{-6}$ | $3.1993899 \times 10^{-8}$ | $3.1021605 \times 10^{-8}$ |
| 0.6     | $6.120000 \times 10^{-6}$   | $6.05000 \times 10^{-6}$ | $3.4915095 \times 10^{-8}$ | $3.4035320 \times 10^{-8}$ |
| 0.7     | $6.99000 \times 10^{-6}$    | $6.660000 \times 10^{-6}$ | $3.6900563 \times 10^{-8}$ | $3.6194517 \times 10^{-8}$ |
| 0.8     | $6.710000 \times 10^{-6}$   | $6.63000 \times 10^{-6}$ | $3.8354837 \times 10^{-8}$ | $3.7634545 \times 10^{-8}$ |
| 0.9     | $7.260000 \times 10^{-6}$   | $6.98000 \times 10^{-6}$ | $3.9124472 \times 10^{-8}$ | $3.8472724 \times 10^{-8}$ |
| 1.0     | $6.880000 \times 10^{-6}$   | $6.820000 \times 10^{-6}$ | $3.9269302 \times 10^{-8}$ | $3.8735696 \times 10^{-8}$ |

Problem 4.5

$y'_1 = -y_1 - 15y_2 + 15e^{-x}, \quad y_1(0) = 1$

$y'_2 = 15y_1 - y_2 - 15e^{-x}, \quad y_2(0) = 1$

Exact solution: $y_1(x) = e^{-x}, \quad y_2(x) = e^{-x}$

Source: Sabo et al [16]

Table 5. Errors in the results of problem 4.5 using the new method

| X Value | Computed Solution | Error in our new method |
|---------|-------------------|-------------------------|
|         | $y_1(x)$          | $y_2(x)$                | $y_1(x)$ | $y_2(x)$ |
| 0.1     | 0.99990000504     | 0.999900030504          | 4.27570 \times 10^{-11} | 1.65170 \times 10^{-11} |
| 0.2     | 0.998800201      | 0.998800220             | 8.54830 \times 10^{-11} | 3.30950 \times 10^{-11} |
| 0.3     | 0.997700450      | 0.997700450             | 1.28174 \times 10^{-10} | 4.97350 \times 10^{-11} |
| 0.4     | 0.9966000800     | 0.9966000800            | 1.70830 \times 10^{-10} | 6.64350 \times 10^{-11} |
| 0.5     | 0.9950000125     | 0.9950000125            | 2.13453 \times 10^{-10} | 8.31960 \times 10^{-11} |
| 0.6     | 0.9940000180     | 0.9940000180            | 2.56044 \times 10^{-10} | 1.00017 \times 10^{-10} |
| 0.7     | 0.993002450      | 0.993002450             | 2.98602 \times 10^{-10} | 1.16899 \times 10^{-10} |
| 0.8     | 0.992003200      | 0.992003200             | 3.41138 \times 10^{-10} | 1.33841 \times 10^{-10} |
| 0.9     | 0.991004050      | 0.991004050             | 3.83616 \times 10^{-10} | 1.50844 \times 10^{-10} |
| 1.0     | 0.990005002      | 0.990005002             | 4.68944 \times 10^{-10} | 1.85029 \times 10^{-10} |

4. Conclusion

An L-stable block hybrid third derivative method has been developed for solving stiff system of ordinary differential equations. The method is self-starting and has good accuracy. The method is also found to have good stability properties appropriate for solving stiff systems. The accuracy of the method was tested on some
stiff systems. The results show that the method is efficient and compared favorably with some existing methods in the literature.

**Competing Interests**

Author has declared that no competing interests exist.

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