Compatibility conditions on local and global spectra for $n$-mode Gaussian states

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Abstract

Compatibility conditions between the (global) spectrum of an $n$-mode Gaussian state and the spectra of the individual modes are presented, making optimal use of beam splitter and (two-mode) squeezing transformations. An unexpected bye-product of our elementary approach is the result that every two-mode Gaussian state is uniquely determined, modulo local transformations, by its global spectrum and local spectra – a property shared not even by a pair of qubits.

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The quantum marginal problem has attracted considerable interest in quantum information theory [1, 2, 3, 4, 5, 6, 7, 8, 9]. Given a multipartite system, it asks: what kind of spectra for the subsystem density operators are consistent with a given spectrum for the density operator of the full system? The Gaussian quantum marginal problem (detailed below) has been solved recently [10, 11] (As noted in Ref. [11], the three-mode case was known earlier [12]). Our approach to this problem makes effective use of beam splitter and two-mode squeezing transformations. In the case of two modes it is shown that every Gaussian state is uniquely determined, modulo local canonical transformations, by its global spectrum and local spectra; in particular, the entanglement is fully determined by these spectra.

Consider a Gaussian state of a system of \( n \) modes, represented by density operator \( \hat{\rho} \). The mean values of the position and momentum variables \( q_j, p_j \) have no role to play in our considerations, and so we assume that these mean values vanish. Such a zero-mean Gaussian state is fully described by its \( 2n \times 2n \) covariance matrix \( V \).

The reduced state \( \hat{\rho}_j \) of the \( j \)th mode, obtained by tracing out from \( \hat{\rho} \) all other modes, is also a zero-mean Gaussian state. With the phase space variables assumed arranged in the order \( q_1, p_1; q_2, p_2; \cdots; q_n, p_n \) the \( j \)th \( 2 \times 2 \) block along the leading diagonal of \( V \) represents precisely the covariance matrix of the reduced state \( \hat{\rho}_j \). Through (independent) local canonical transformations \( \in Sp(2, R) \) on each mode we make all the \( 2 \times 2 \) blocks along the diagonal of \( V \) multiples of identity. The covariance matrix of the \( j \)th mode will then be of the form \( \text{diag}(m_j, m_j) \). It corresponds to a thermal state, with temperature \( T(m_j) \) which is a monotone increasing function of \( m_j \). Being thermal, \( \hat{\rho}_j \) has the spectral resolution \( \hat{\rho}_j = [1 - \xi(m_j)] \sum_{k=0}^{\infty} \xi(m_j)^{n_{jk}} |n_{jk}\rangle \langle n_{jk}| \). The parameter \( \xi(m_j) \) is another monotone increasing function of \( m_j \), and \( |n_{jk}\rangle \)'s are the energy eigenstates of the \( j \)th oscillator. Clearly, the eigenvalue spectra of the \( \hat{\rho}_j \)'s are determined by, and determine, the local spectral parameters \( m_j \).

Using an appropriate (nonlocal) canonical transformation \( S \in Sp(2n, R) \) the covariance matrix \( V \) can be decoupled and brought into the canonical form \( V^{(0)} \) of independent oscillators in thermal states [13]: \( V^{(0)} = SVS^T = \text{diag}(\kappa_1, \kappa_1; \kappa_2, \kappa_2; \cdots; \kappa_n, \kappa_n) \). The associated density operator \( \hat{\rho}^{(0)} \) thus has the spectral decomposition

\[
\hat{\rho}^{(0)} = \prod_{j=1}^{n} [1 - \xi(\kappa_j)] \sum_{k=0}^{\infty} \xi(\kappa_j)^{n_{jk}} |n_{jk}\rangle \langle n_{jk}|. \tag{1}
\]

Since \( \hat{\rho}^{(0)} \) and the original \( \hat{\rho} \) are unitarily related, the spectrum of \( \hat{\rho} \) is the same as that
of $ρ^{(0)}$. It is clear that this global spectrum and the $n$-tuple of global spectral parameters $(κ_1, κ_2, \cdots, κ_n)$ determine each other.

We may now ask what are the constraints connecting the global spectrum of a Gaussian state to its local spectra. In view of the invertible relationships just noted this Gaussian quantum marginal problem is equivalent to seeking the compatibility constraints between the global spectral parameters $\{κ_j\}$ and the local spectral parameters $\{m_j\}$. Interestingly, the answer can be given in the form of necessary and sufficient conditions.

**Theorem:** Let $m = (m_1, m_2, m_3, \cdots, m_n)$ and $κ = (κ_1, κ_2, \cdots, κ_n)$ be the local and global spectral parameters of an $n$-mode Gaussian state, written in nondecreasing order. These are compatible iff

\[
\sum_{j=1}^{k} m_j \geq \sum_{j=1}^{k} κ_j, \quad k = 1, 2, \cdots, n, \tag{2}
\]

\[
m_n - \sum_{j=1}^{n-1} m_j \leq κ_n - \sum_{j=1}^{n-1} κ_j. \tag{3}
\]

**Remarks:** What this claim means can be clarified by stating it in two parts. Suppose a Gaussian state is given. Its local spectral parameters $m_1, m_2, \cdots, m_n$, and global spectral parameters $κ_1, κ_2, \cdots, κ_n$ are certain to meet these inequalities (with $κ_1 \geq 1$). Conversely, given a set of local and global spectral parameters meeting these inequalities (with $κ_1 \geq 1$), we can certainly construct a physical Gaussian state with these parameters.

The first part of the theorem was essentially proved by Hiroshima [10]. But the full theorem in this form was formulated by Eisert et al. [11] who presented an inductive proof for the second part. Our proof of both parts will be seen to be constructive, consistent with the elementary nature of the theorem, and it rests in an essential manner on a fuller appreciation of the two-mode situation.

Given two vectors $m, κ \in R^n$, we will say $κ$ dominates $m$ if $m$ and $κ$, after their components are rearranged in the nondecreasing order, obey the set of $n + 1$ inequalities (2), (3). This definition is such that permutation of the components of $m$ or $κ$ does not affect dominance. Thus $(9, 7, 8, 6, 12, 11, 10)$ is dominated by $(5, 2, 18, 4, 1, 12, 3)$, since $(1, 2, 3, 4, 5, 12, 18)$ manifestly dominates $(6, 7, 8, 9, 10, 11, 12)$. Further, dominance so defined is transitive: $κ$ dominates $m$, and $m$ dominates $m'$, together imply $κ$ dominates $m'$.

In the Schur-Horn case [14] wherein $m$ corresponds to the diagonal entries of a hermitian matrix and $κ$ to its eigenvalues, the last inequality in (2) becomes an equality. It is clear
that (3) is subsumed by (2) in that case.

*The case of two modes:* This case is of interest in its own right. Further, it possesses an aspect which seems to be unique, not shared by any other system. Finally, our analysis of the \( n \)-mode case relies critically on repeated applications of the two-mode result. Hence we begin with a direct proof of the theorem in the two-mode case.

**Lemma:** The parameters \( m_1 \leq m_2 \) and \( 1 \leq \kappa_1 \leq \kappa_2 \) are compatible for two-mode Gaussian states iff

\[
\begin{align*}
    m_1 + m_2 &\geq \kappa_1 + \kappa_2, \\
    m_2 - m_1 &\leq \kappa_2 - \kappa_1.
\end{align*}
\]

(4)

Note that the condition \( m_1 \geq \kappa_1 \) is subsumed by (4).

**Proof of Lemma:** The covariance matrix can, through local unitary (canonical) transformation \( \in Sp(2, R) \times Sp(2, R) \), be brought to the form

\[
V = \begin{pmatrix}
    m_1 & 0 & k_x & 0 \\
    0 & m_1 & 0 & k_p \\
    k_x & 0 & m_2 & 0 \\
    0 & k_p & 0 & m_2
\end{pmatrix}.
\]

(5)

The global spectral parameters \( \kappa_1, \kappa_2 \) are related to the local \( m_1, m_2 \) through the symplectic invariants\[13\]

\[
\frac{1}{2} \text{tr}(\Omega V \Omega^T V) = \kappa_1^2 + \kappa_2^2 = m_1^2 + m_2^2 + 2k_x k_p,
\]

\[
\det V = (\kappa_1 \kappa_2)^2 = (m_1 m_2 - k_x^2)(m_1 m_2 - k_p^2).
\]

(6)

These immediately imply

\[
\begin{align*}
    \kappa_1 \kappa_2 &\leq m_1 m_2, \\
    \kappa_1^2 + \kappa_2^2 &\geq m_1^2 + m_2^2, \text{ if } k_x k_p \geq 0, \\
    \kappa_1^2 + \kappa_2^2 &\leq m_1^2 + m_2^2, \text{ if } k_x k_p \leq 0,
\end{align*}
\]

(7)

equality in the first inequality holding if \( k_x = 0 = k_p \). These inequalities imply

\[
\begin{align*}
    \kappa_2 - \kappa_1 &\geq m_2 - m_1, \text{ when } k_x k_p \geq 0, \\
    \kappa_2 + \kappa_1 &\leq m_2 + m_1, \text{ when } k_x k_p \leq 0.
\end{align*}
\]

(8)
This much is immediate from the symplectic invariants. What remain to be proved are:

\[ \kappa_2 - \kappa_1 \geq m_2 - m_1 \] when \( k_x k_p \leq 0 \) and \( \kappa_2 + \kappa_1 \geq m_2 + m_1 \) when \( k_x k_p \geq 0 \).

To prove these we reinterpret (6) as simultaneous expressions for \( k_x, k_p \) in terms of \( \kappa_1, \kappa_2; m_1, m_2 \):

\[
\begin{align*}
  k_x k_p &= \frac{[(\kappa_1^2 + \kappa_2^2) - (m_1^2 + m_2^2)]}{2}, \\
  k_x^2 + k_p^2 &= \frac{1}{m_1 m_2} [m_1^2 m_2^2 - \kappa_1^2 \kappa_2^2 + k_x^2 k_p^2].
\end{align*}
\]

It is clear that real solutions for \( k_x \) and \( k_p \) will exist iff \( k_x^2 + k_p^2 \geq 2\left| k_x k_p \right| \). That is, iff

\[
m_1 m_2 - \left| k_x k_p \right| \geq \kappa_1 \kappa_2.
\]

With use of (9) for \( k_x k_p \), this last condition reads

\[
\begin{align*}
  \kappa_2 - \kappa_1 &\geq m_2 - m_1, \text{ when } k_x k_p \leq 0, \\
  \kappa_2 + \kappa_1 &\leq m_2 + m_1, \text{ when } k_x k_p \geq 0.
\end{align*}
\]

Proof of the Lemma is thus complete.

Two types of simple transformations on any pair of modes characterised by annihilation operators \( a_j, a_k \) deserve particular mention; they play a key role in our proof of the theorem. The first, \( S_\theta \), corresponds to the compact transformations \( a_j \rightarrow \cos \theta a_j + \sin \theta a_k, a_k \rightarrow -\sin \theta a_j + \cos \theta a_k \), and therefore is represented by \( S_\theta = \cos \theta \sigma_0 \otimes \sigma_0 + \sin \theta i \sigma_2 \otimes \sigma_0 \in Sp(4, R) \), \( 0 \leq \theta < 2\pi \), where \( \sigma_0 \) is the 2 \( \times \) 2 unit matrix and \( \sigma_2 \) is the antisymmetric Pauli matrix. Physically, \( S_\theta \) is a beam splitter with transmissivity \( \cos^2 \theta \). The second one, \( S_\mu \), is noncompact and corresponds to squeezing transformations \( a_j \rightarrow \cosh \mu a_j + \sinh \mu a_j^\dagger, a_k \rightarrow \cosh \mu a_k + \sinh \mu a_k^\dagger \), and is represented by \( S_\mu = \cosh \mu \sigma_0 \otimes \sigma_0 + \sinh \mu \sigma_1 \otimes \sigma_3 \in Sp(4, R) \), \( 0 \leq \mu < \infty \).

It is easily verified that when the covariance matrix \( V \), Eq. (5), has \( k_\mu = k_x \equiv k \), it can be diagonalized by the beam splitter transformation \( V \rightarrow S_\theta V S_\theta^T \), with \( \theta \) fixed through \( \tan 2\theta = 2k/(m_2 - m_1) \). And \( \kappa_2 + \kappa_1 \) will precisely equal \( m_2 + m_1 \) in this case. Similarly, if \( k_p = -k_x = k > 0 \), then \( V \) is diagonalized by the squeezing transformation \( V \rightarrow S_\mu V S_\mu^T \), with \( \tanh 2\mu = 2k/(m_2 + m_1) \), and one will find \( \kappa_2 - \kappa_1 = m_2 - m_1 \) in this case.

Conversely, suppose we start with the canonical form \( V^{(0)} = \text{diag}(\kappa_1, \kappa_1; \kappa_2, \kappa_2) \), and we wish to achieve through symplectic congruence \( V^{(0)} \rightarrow SV^{(0)}S^T \), \( S \in Sp(4, R) \), a covariance matrix with diagonals \( m_1, m_2 \). If \( m_1 < m_2 \) are such that \( m_2 < \kappa_2 \) and \( \kappa_2 + \kappa_1 = m_2 + m_1 \),
such a redistribution of $\kappa_1, \kappa_2$ among $m_1, m_2$ can always be achieved through a beam splitter transformation $S_\theta$. Under $S_\theta$ we have $m_2 + m_1 = \kappa_2 + \kappa_1$ and $m_2 - m_1 = \cos 2\theta (\kappa_2 - \kappa_1)$. On the other hand, if $m_2 > \kappa_2$ and $\kappa_2 - \kappa_1 = m_2 - m_1$, so that $\kappa_1$ and $\kappa_2$ are enhanced by equal amounts to $m_1, m_2$, this can be achieved through a squeezing transformation $S_\mu$. Under $S_\mu$ we have $m_2 - m_1 = \kappa_2 - \kappa_1$ and $m_2 + m_1 = \cosh 2\mu (\kappa_2 + \kappa_1)$.

Our Lemma is similar to Lemma 5 of Ref. [11], but our proof is direct and constructive. There is an important distinction in content as well: while theirs claims that $m_2 - m_1 = \kappa_2 - \kappa_1$ iff $m_2 = \kappa_2$ and $m_1 = \kappa_1$, we have just demonstrated that if $m_2 - m_1 = \kappa_2 - \kappa_1$ then $m_2 + m_1$ could equal $\cosh 2\mu (\kappa_2 + \kappa_1)$ for any $0 \leq \mu < \infty$, not just $\mu = 0$. Indeed, this distinction is central to Stage 2 of our proof of the second part of the main theorem, the part which distinguishes the present symplectic situation from the Schur-Horn case.

Returning to Eq. (10), if we are given values for the expressions ‘$a^2 + b^2$’ and ‘$ab$’ with $a^2 + b^2 \geq 2|ab|$, the solution for $(a, b)$ is unique [(a, b) and (b, a) are not distinct solutions for our purpose]. This innocent looking observation leads to a surprising conclusion.

**Proposition:** Specification of the local and global spectra of a two-mode Gaussian state determines uniquely the state itself, modulo local unitary transformations.

States of a pair of qubits share a similarity with two-mode Gaussian states in important respects. For instance, positivity under partial transpose is a necessary and sufficient condition for separability and nondistillability in both cases. But a statement analogous to the above proposition is not true for a pair of qubits!

**Proof of main theorem:** Assume we are given a (zero-mean) Gaussian state, or equivalently, an acceptable covariance matrix $V$, the $2 \times 2$ blocks along the leading diagonal of $V$ being of the form $\text{diag}(m_j, m_j)$. The global spectral parameters $\{ \kappa_j \}$ are immediately defined by $V$ [13]. It is assumed that $m = (m_1, m_2, \ldots, m_n)$ and $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n)$ are arranged in nondecreasing order. Let $P_\kappa$ denote the product $\kappa_1 \kappa_2 \cdots \kappa_n$ and let $P_m = m_1 m_2 \cdots m_n$. Clearly, $P_\kappa = \det V \leq P_m$, equality holding iff $V$ is diagonal, i.e., iff $m_j = \kappa_j, j = 1, 2, \ldots, n$. Our task is to prove that $\kappa$ dominates $m$.

Choose a pair $1 \leq j < k \leq n$ such that the $2 \times 2$ block (in the off-diagonal location) connecting the $j^{th}$ and $k^{th}$ modes is nonzero. We can arrange (through local rotations) this block to be diagonal. Let us ‘diagonalize’ this $4 \times 4$ part of the covariance matrix using an appropriate two-mode canonical transformation $\in Sp(4, R)$, so that $m_j$ and $m_k$ are transformed to $\tilde{m}_j$ and $\tilde{m}_k$ respectively, the other diagonal parameters remaining unaffected.
It is be noted that the new \( m \) dominates the original \( m \). That this is so follows, in the case \( k < n \), from the facts \( \tilde{m}_j < m_j \) and \( \tilde{m}_j + \tilde{m}_k \leq m_j + m_k \). In the case \( k = n \) it follows from the additional fact that if \( \tilde{m}_k \) is less that \( m_k \) it is so by a magnitude which does not exceed the magnitude by which \( \tilde{m}_j \) is less than \( m_j \) \((\tilde{m}_k - \tilde{m}_j \geq m_k - m_j)\). Further, \( \tilde{m}_j \tilde{m}_k < m_j m_k \).

Denote by \( m' \) the new diagonal \( m \)-parameters arranged in nondecreasing order by correspondingly permuting the oscillators. Since \( \tilde{m}_j \tilde{m}_k < m_j m_k \) we have \( P_{m'} < P_m \).

For purpose of clarity, let us carry out this process one more time. The parameters \( m' \) will then go to \( m'' \) dominating \( m' \), with \( P_{m''} < P_{m'} \). It follows from the transmitivity of dominance that \( m'' \) dominates \( m \).

It is now clear that when this process is iterated, \( m \) goes through a sequence of intermediate values, the value at every stage dominating the previous value, and correspondingly \( P_m \) steadily decreasing, until \( P_m \) reaches \( P_\kappa \) or, equivalently, until \( V \) becomes diagonal. This completes proof of the first part of the theorem.

The elementary nature of our proof may be compared with that of Ref. \[10\]. \( P_m \) played the role of ‘profit function’ monitoring progress of this diagonalization process.

To prove the second part assume, conversely, that we are given the global and local spectral parameters \( \kappa, m \in R^n \). Assume that these are compatible: i.e., \( \kappa \) dominates \( m \), with \( \kappa_1 \geq 1 \). Our task is to construct a Gaussian state with these properties. In other words we have to present a canonical transformation \( S \in Sp(2n, R) \) which acting on a covariance matrix \( V = \text{diag}(\kappa_1, \kappa_1; \kappa_2, \kappa_2; \cdots; \kappa_n, \kappa_n) \) will produce a covariance matrix \( SVS^T \) with the target diagonal values \( m \). We build such an \( S \) as a product of \( n - 1 \) specific two-mode transformations, evolving \( m^{(0)} \equiv \kappa \) successively through a sequence of intermediates \( m^{(1)}, m^{(2)}, \cdots \) to finally \( m^{(n-1)} = m \). It will be manifestly clear that \( m^{(k)} \) dominates \( m^{(k+1)} \) at each stage. For clarity, this process is implemented through four elementary stages.

**Stage 1:** Since \( m^{(0)} \equiv \kappa \) dominates \( m \), we have \( m_1 \geq m_1^{(0)} = \kappa_1 \). Suppose \( m_1 = m_1^{(0)} + \epsilon_1 \), \( \epsilon_1 > 0 \) (one will move to the next step if \( m_1 = m_1^{(0)} \)). Let \( j_1 \) be the least integer \( < n \) such that \( m_1^{(0)} \geq m_1 \). Carry out a beam splitter transformation \( S_{j_1} \) between the first and \( j_1 \)th mode so that the corresponding diagonal elements \( (m_1^{(0)}, m_{j_1}^{(0)}) \) get redistributed to \((m_1^{(0)} + \epsilon_1, m_{j_1}^{(0)} - \epsilon_1) = (m_1, m_{j_1}^{(0)} - \epsilon_1)\), with no change in the other diagonal entries: \( m^{(0)} = (m_1^{(0)}, m_2^{(0)}, \cdots, m_n^{(0)}) \rightarrow m^{(1)} = (m_1, m_2^{(0)}, \cdots, m_{j_1}^{(0)} - \epsilon_1, \cdots, m_n^{(0)}) \equiv (m_1, m_2^{(1)}, m_3^{(1)}, \cdots, m_n^{(1)}) \).

We can repeat this process. Let \( m_2 = m_2^{(1)} + \epsilon_2 \). By hypothesis \( \epsilon_2 \geq 0 \) (this is so even if
\(j_1\) had equallled 2). Assume \(\epsilon_2 > 0\) (if \(\epsilon_2 = 0\), one moves to the next step). Let \(j_2\) be the smallest integer < \(n\) such that \(m_{j_2}^{(1)} \geq m_2\) [Clearly, \(j_2\) can be as small as \(j_1\), but not any smaller]. Carry out a beam splitter transformation on the 2\(^{\text{nd}}\) and \(j_2\)\(^{\text{th}}\) modes so that the corresponding diagonal elements \((m_2^{(1)}, m_{j_2}^{(1)})\) get redistributed to \((m_2, m_{j_2}^{(1)} - \epsilon_2)\) to produce \(m^{(2)}\), leaving the other diagonals unaffected.

If we are able to repeat this process only \(\ell\) times we have, at the end of it,

\[
m^{(\ell)} = (m_1, m_2, \ldots, m_\ell, m_{\ell+1}^{(\ell)}, m_{\ell+2}^{(\ell)}, \ldots, m_n^{(\ell)}),
\]

with \(m_j^{(\ell)} < m_j\), \(\forall \ell + 1 \leq j \leq n - 1\), and \(m_n^{(\ell)} = m_n^{(0)} = \kappa_n\). What we have done so far is identical to what one would have done in the Schur-Horn situation. Clearly, the beam splitter transformations carried out so far affected neither the sum of the diagonal entries of \(m^{(\ell)}\) nor its \(n^{\text{th}}\) entry. Consequently, the difference \(m_n^{(k)} - \sum_{j=1}^{n-1} m_j^{(k)}\) has remained the same for all \(0 \leq k \leq \ell\).

**Stage 2:** Define \(\delta^{(k)} = \sum_{j=1}^{n} m_j - \sum_{j=1}^{n} m_j^{(k)}\). It is clear that \(\delta^{(k)} = \delta^{(0)}\), for \(k = 1, 2, \ldots, l\).

In the Schur-Horn situation \(\delta^{(0)}\) vanishes by hypothesis. We will now employ two-mode squeezing transformations \(S_\mu\) to rectify this ‘departure’ from the Schur-Horn situation.

We know that \(\delta^{(\ell)} = \delta^{(0)}\) is nonnegative. Assume \(\delta^{(0)} > 0\) (if \(\delta^{(0)} = 0\), one will move directly to Stage 4, as will become evident below). Define \(\epsilon_{\ell+1} = m_{\ell+1} - m_{\ell+1}^{(\ell)}\). Assume \(\delta^{(\ell)} \geq 2\epsilon_{\ell+1}\) (if this is not the case one will move to Stage 3). Carry out a two-mode squeezing transformation \(S_\mu\) between the \((\ell + 1)^{\text{th}}\) and \(n^{\text{th}}\) modes, raising the corresponding diagonal entries \(m_{\ell+1}^{(\ell)}, m_n^{(\ell)} = m_n^{(0)} = \kappa_n\) by equal magnitude to \(m_{\ell+1}, m_n^{(\ell)} + \epsilon_{\ell+1}\) with no change in the other diagonal entries, so that

\[
m^{(\ell+1)} = (m_1, \ldots, m_{\ell+1}, m_{\ell+2}^{(\ell+1)}, \ldots, m_n^{(\ell+1)}),
\]

\[
m_j^{(\ell+1)} = m_j^{(\ell)}, \quad \forall \ell + 2 \leq j \leq n - 1,
\]

\[
m_n^{(\ell+1)} = m_n^{(\ell)} + \epsilon_{\ell+1} = \kappa_n + \epsilon_{\ell+1}.
\]

We can now repeat this kind of two-mode squeezing transformation between the \((\ell + 2)^{\text{th}}\) mode and the \(n^{\text{th}}\) mode, and so on. Assume we are able to carry out this process only \(r\) times. We will have, at the end of it,

\[
m^{(\ell+r)} = (m_1, \ldots, m_{\ell+r}, m_{\ell+r+1}^{(\ell+r)}, \ldots, m_n^{(\ell+r)}),
\]

\[
m_j^{(\ell+r)} = m_j^{(\ell)}, \quad \forall \ell + r + 1 \leq j \leq n - 1,
\]

\[
m_n^{(\ell+r)} = \kappa_n + \epsilon_{\ell+1} + \epsilon_{\ell+2} + \cdots + \epsilon_{\ell+r}.
\]
so that $\delta^{(\ell+r)} = \delta^{(0)} - 2(\epsilon_{\ell+1} + \epsilon_{\ell+2} + \cdots + \epsilon_{\ell+r})$. Clearly, $0 \leq \delta^{(\ell+r)} < 2\epsilon_{\ell+r+1} = 2(m_{\ell+r+1} - m^{(\ell+r)}_{\ell+r+1})$ (the last inequality encodes the fact that we could not carry out the Stage 2 operation one more time).

Stage 3: Assume $\delta^{(\ell+r)} > 0$ (if $\delta^{(\ell+r)} = 0$, we move directly to Stage 4). Carry out a two-mode canonical transformation between the $(\ell + r + 1)^{th}$ mode and the $n^{th}$ mode, taking the corresponding diagonal entries $m^{(\ell+r)}_{\ell+r+1}, m^{(\ell+r)}_n$ to $m_{\ell+r+1} = m^{(\ell+r)}_{\ell+r+1} + \epsilon_{\ell+r+1}$ and $m^{(\ell+r+1)}_n = m^{(\ell+r)}_n + \delta^{(\ell+r)} - \epsilon_{\ell+r+1}$ respectively, leaving the other diagonals invariant, so that we have

$$m^{(\ell+r+1)} = (m_1, \ldots, m_{\ell+r+1}, m^{(\ell+r+1)}_{\ell+r+2}, \ldots, m^{(\ell+r+1)}_n),$$
$$m^{(\ell+r+1)}_j = m_j < m_j, \quad \forall \ell + r + 2 \leq j \leq n - 1,$$
$$\sum_{j=\ell+r+2}^{n} m^{(\ell+r+1)}_j = \sum_{j=\ell+r+2}^{n} m_j.$$  \hspace{1cm} (16)

i.e., the situation in respect of the remaining $n - (\ell + r + 1)$ (or $n - \ell - r$ if $\delta^{(\ell+r)} = 0$) modes is precisely of the Schur-Horn type, suggesting that we deploy the beam splitter transformation $n - \ell - r - 2$ (or $n - \ell - r - 1$) times.

Stage 4: Note that at the end of Stage 3 we have $m^{(\ell+r+1)}_n$ larger than $m_n$ precisely by the sum of the amounts by which $m^{(\ell+r+1)}_{\ell+r+1+j}$, for $1 \leq j \leq n - \ell - r - 2$, are less than $m_{\ell+r+1+j}$. Therefore, for each value of $j$ in this range, we effect a beam splitter transformation connecting the $(\ell + r + 1 + j)^{th}$ mode to the $n^{th}$ mode, raising $m^{(\ell+r+1)}_{\ell+r+1+j}$ to $m_j$ and correspondingly pulling $m^{(\ell+r+1+j)}_n$ down by an equal amount. It is clear that at the end of these $n - \ell - r - 2$ (or $n - \ell - r - 1$) redistributions, the diagonals will be precisely $m$. That is, $m^{(n-1)} = m$. This completes proof of the theorem.

We have taken maximal advantage of the simpler two-mode transformations $S_\theta$, $S_\mu$. The former was deployed $r$ times in Stage 1 and $n - \ell - r - 2$ (or $n - \ell - r - 1$) times in Stage 4, and the latter $\ell$ times in Stage 2. The more general two-mode transformation was deployed (at the most) once in Stage 3.

As illustration, and for comparison with Ref. [11], we apply our procedure to the example noted after the statement of the theorem. The difference between $\sum_{j=1}^{7} m_j = 63$ and $\sum_{j=1}^{7} \kappa_j = 45$ indicates the amount of squeezing that will have to be deployed at Stages 3 and 4. We have $m^{(0)} = \kappa = (1, 2, 3, 4, 5, 12, 18); m^{(1)} = (6, 2, 3, 4, 5, 7, 18); m^{(2)} = (6, 7, 3, 4, 5, 2, 18); m^{(3)} = (6, 7, 8, 4, 5, 2, 23); m^{(4)} = (6, 7, 8, 9, 5, 2, 26); m^{(5)} =
(6, 7, 8, 9, 10, 2, 21); and $m^{(6)} = (6, 7, 8, 9, 10, 11, 12) = m$. The number of two-mode transformations required at the four stages are 2, 1, 1, and 2 respectively. Note that $m^{(k)}$ dominates $m^{(k+1)}$, for $k = 0, 1, \cdots, 5$.

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