Convergence rate of EM algorithm for SDEs under integrability condition

Jianhai Bao\textsuperscript{b),} Xing Huang\textsuperscript{a),} Shao-Qin Zhang\textsuperscript{c)}

\textsuperscript{a)} Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
\texttt{xinghuag@tju.edu.cn}
\textsuperscript{b)} Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK
\texttt{Jianhai.Bao@Swansea.ac.uk}
\textsuperscript{c)} School of Statistics and Mathematics, Central University of Finance and Economics, Beijing 100081, China
\texttt{zhangsq@cufe.edu.cn}

September 11, 2020

Abstract

In this paper, by employing Gaussian type estimate of heat kernel, we establish Krylov’s estimate and Khasminskii’s estimate for EM algorithm. As applications, by taking Zvonkin’s transformation into account, we investigate convergence rate of EM algorithm for a class of multidimensional SDEs under integrability conditions, where the drifts need not to be piecewise Lipschitz and are much more singular in some sense.

AMS subject Classification: 60H10, 34K26, 39B72.

Keywords: Zvonkin’s transform, Euler-Maruyama approximation, integrable drift, Krylov’s estimate

1 Introduction and Main Results

Strong/weak convergence of numerical schemes for stochastic differential equations (SDEs for short) with regular coefficients have been investigated considerably; see monographs e.g. [15, 16]. As we know, (forward) Euler-Maruyama (EM for abbreviation) is the simplest algorithm to simulate SDEs whose coefficients are of linear growth. Whereas, EM scheme is invalid as long as the coefficients of SDEs involved are of nonlinear growth; see e.g. [12, 14] for some illustrative counterexamples. Whence the other variants of EM scheme were designed to deal with SDEs with non-globally Lipschitz condition; see e.g. [9, 10] for backward EM scheme, [2, 13, 31] as for tamed EM algorithm, and [5, 23] concerning truncated EM method, to name a few. Nowadays, convergence analysis of numerical algorithms for SDEs with irregular coefficients also receives much attention; see e.g. [7] for SDEs with Hölder continuous diffusions via Yamada-Watanabe approximation approach, [35] for

\textsuperscript{*}Supported in part by NNSFC (11801406).
SDEs whose drift terms are Hölder continuous with the aid of Meyer-Tanaka formula and estimates on local times, and [1, 29] for SDEs whose drifts enjoy Hölder(-Dini) continuity by the regularity of backward Kolmogorov equations. In the past few years, numerical approximations of SDEs with discontinuous drifts have also gained a lot of interest; see, for instance, [4, 8, 20, 21, 22, 25, 26]. Up to now, most of the existing literatures above on strong approximations of SDEs with discontinuous drift coefficients are implemented under the additional assumption that the drift term is piecewise Lipschitz continuous.

Since the pioneer work of Zvonkin [38], the wellposedness for SDEs under integrability conditions has been developed greatly in different manners; see e.g. [3, 6, 18, 34, 36, 37] for SDEs driven by Brownian motions or jump processes, and e.g. [11, 30] for McKean-Vlasov (or distribution-dependent or mean-field) SDEs. So far, there also exist a few of literatures upon numerical simulations of SDEs and/or mean-field) SDEs. Moreover, (1.1) has a unique strong solution \((X_t)_{t \geq 0}\); see, for instance, [11, Lemma 3.1]. Moreover, (A2) imposed is to reveal the convergence rate of EM scheme corresponding to (1.1), which is defined as below: for any \(\delta \in (0, 1)\),
with \( t_\delta := [t/\delta] \delta \), where \([t/\delta]\) denotes the integer part of \( t/\delta \). We emphasize that \((X^{(\delta)}_{k\delta})_{k \geq 0}\) is a homogeneous Markov process; see e.g. [24, Theorem 6.14]. For \( t \geq s \) and \( x \in \mathbb{R}^d \), denote \( p^{(\delta)}(s, t, x, \cdot) \) by the transition density of \( X^{(\delta)}_t \) with the starting point \( X^{(\delta)}_s = x \).

Our first main result in this paper is stated as follows.

**Theorem 1.1.** Assume (A1)-(A3). Then, for \( \beta \in (0, 2) \) and \( q > 2 \), there exist constants \( C_1, C_2 > 0 \) such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t - X^{(\delta)}_t|^\beta \right) \leq C_1 e^{C_2 (1 + \|b\|_{L^p}^\beta) \delta^{\frac{2p}{p(q-1)}}},
\]

where \( \gamma_0 := \frac{1}{1-1/q - d/2p} \).

Compared with [27], in Theorem 1.1 we get rid of the one-side Lipschitz condition. On the other hand, [28] is extended to the multidimensional setup. We point out that an \( \mathcal{A} \) approximation is given in advance in [27, 28] to approximate the drift term. So, with contrast to the assumption put in [27, 28], the assumption (A2) is much more explicit. Moreover, by a close inspection of the argument of Lemma 2.2, the assumption (A2) can be replaced by the other alternatives. For instance, (A2) may be taken the place of (A2') below.

(A2') There exist constants \( \beta, \theta > 0 \) such that

\[
\frac{1}{(rs)^{d/2}} \sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \times \mathbb{R}^d} |b(x) - b(y)|^2 e^{-\frac{|x-z|^2}{2r}} e^{-\frac{|y-z|^2}{2s}} dy dx \leq C r^\theta s^{\beta - 1}, \quad s, r > 0
\]

for some constant \( C > 0 \).

The drift \( b \) satisfying (1.6) is said to the Gaussian-Besov class with the index \((\beta, \theta)\), denoted by \( GB^2_{\beta, \theta}(\mathbb{R}^d) \). Remark that functions with the same order of continuity may enjoy different type continuity; see, for instance, \( f(x) = |x| \) with \((1/2, 1)\), and \( f(x) = 1_{[c,d]}(x) \), \( c, d \in \mathbb{R} \), with \((1/2, 1/2)\).

We refer to Example 4.2 below for the drift \( b \in GB^2_{\beta, \theta}(\mathbb{R}^d) \). For \( \theta \in (0, 1) \) and \( p \geq 1 \), let \( W^{\theta, p}(\mathbb{R}^d) \) be the fractional order Sobolev space on \( \mathbb{R}^d \). Nevertheless, \( W^{\theta, p}(\mathbb{R}^d) \not\subset GB^2_{1-\frac{d}{p}, \theta}(\mathbb{R}^d) \), \( \theta > 0, p \in [2, \infty) \cap (d, \infty) \); see Example

In Theorem 1.1, the integrable condition (i.e., \(|b|^2 \in L^p\)) seems to be a little bit restrictive, which rules out some typical examples, e.g., \( b(x) = 1_{[0, \infty)}(x) \). In the sequel, by implementing a truncation argument, the integrable condition can indeed be dropped. In such setup (i.e., without integrable condition), we can still derive the convergence rate of the EM algorithm, which is presented as below.

**Theorem 1.2.** Assume (A1)-(A3) without \(|b|^2 \in L^p\). Then, for \( \beta \in (0, 2) \), and \( p, q > 2 \) with \( \frac{2}{p} + \frac{1}{q} < 1 \), there exist constants \( C_1, C_2 > 0 \) such that

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X_t - X^{(\delta)}_t|^\beta \right) \leq C_1 \left[ e^{C_2 (-\frac{\beta}{2} (1+\frac{q}{2}) \log \delta) \frac{d_{ap}}{2p} + 1} \right] \delta^{\frac{2}{1+\frac{q}{2}}}.\]

We remark that the right hand side of (1.7) approaches zero since

\[
\lim_{\delta \to 0} e^{C_2 (-\frac{\beta}{2} (1+\frac{q}{2}) \log \delta) \frac{d_{ap}}{2p} \delta^{\frac{2}{1+\frac{q}{2}}}} = 0.
\]
due to the fact that \( \lim_{x \to \infty} e^{C x^\frac{d}{p}} \frac{dx}{e^x} = 0 \) whenever \( \frac{d}{p} + \frac{1}{q} < 1 \).

The remainder of this paper is organized as follows. In Section 2, by employing Zvonkin’s transform and establishing Krylov’s estimate and Khaminskii’s estimate for EM algorithm, which is based on Gaussian type estimate of heat kernel, we complete the proof of Theorem 1.1; In Section 3, we aim to finish the proof of Theorem 1.2 by adopting a truncation argument; In Section 4 we provide some illustrative examples to demonstrate our theory established; In the Appendix part, we reveal explicit upper bounds of the coefficients associated with Gaussian type heat kernel of the exact solution and the EM scheme.

## 2 Proof of Theorem 1.1

Before finishing the proof of Theorem 1.1, we prepare several auxiliary lemmas. Set

\[
\Lambda_1 := 2 \left( \frac{||b||_\infty}{\sqrt{\lambda_0}} + 2\sqrt{d}L_0(\lambda_0/\check{\lambda}_0)^2 + d^{2+1}d!(\lambda_0/\check{\lambda}_0)^dL_0 \right) e^{\frac{||b||_\infty^2}{2\lambda_0}}
\]

\[
(2.1)
\]

\[
\vee \left\{ 2\sqrt{\lambda_0}||b||_\infty + (||b||_\infty^2 + 2\check{\lambda}_0L_0\sqrt{d})(\sqrt{d} + 2) + 2^{m+1}\check{\lambda}_0^{-1}(L_0 + 2||b||_\infty) \right. \\
\times \left( (||b||_\infty^3 + (d\check{\lambda}_0)^{\frac{3}{2}}) + \check{\lambda}_0 \left( ||b||_\infty^2 + d\check{\lambda}_0 \right) \right) \right\} \frac{2^{d+1}}{(\lambda_0/\check{\lambda}_0)^d} e^{\frac{||b||_\infty^2}{2\lambda_0}}.
\]

and

\[
(2.2) \Lambda_2 := e^{\frac{||b||_\infty^2}{2\lambda_0}} \sum_{i=0}^{\infty} \frac{\Lambda_1^{\pi T}((1 + 24d)\lambda_0/\check{\lambda}_0)^i}{\Gamma(1 + \frac{d}{2})},
\]

where \( \Gamma(\cdot) \) denotes the Gamma function. Due to Stirling’s formula: \( \Gamma(z + 1) \sim \sqrt{2\pi z}(z/e)^z \), we have \( \Lambda_2 < \infty \).

The lemma below provides an explicit upper bound of the transition kernel for \( (X_t^{(\delta)})_{t \geq 0} \).

**Lemma 2.1.** Under \((A1)\) and \((A3)\),

\[
p^{(\delta)}(j\delta, t, x, y) \leq \frac{\Lambda_3 e^{-\frac{|y-x|^2}{2\lambda_0(t-j\delta)}}}{(2\pi \lambda_0(t-j\delta))^{d/2}}, \quad x, y \in \mathbb{R}^d, \quad t > j\delta, \quad \delta \in (0, 1),
\]

where

\[
(2.3)
\]

\[
\kappa_0 := 4(1 + 24d)\lambda_0, \quad \Lambda_3 := \Lambda_2 e^{\frac{\kappa_0^2}{2\lambda_0}} \left( \frac{\kappa_0}{2\lambda_0} \right)^{d/2}.
\]

**Proof.** For fixed \( t > 0 \), there is an integer \( k \geq 0 \) such that \( [k\delta, (k+1)\delta) \). By a direct calculation, it follows from (1.2) and (1.3) that

\[
p^{(\delta)}(k\delta, t, x, y) \leq \frac{e^{-\frac{|y-x|^2}{2\lambda_0(t-k\delta)}}}{(2\pi \lambda_0(t-k\delta))^{d/2}} \leq e^{\frac{||b||_\infty^2}{2\lambda_0}} \frac{e^{-\frac{|y-x|^2}{4\lambda_0(t-k\delta)}}}{(2\pi \lambda_0(t-k\delta))^{d/2}},
\]

\[
(2.5)
\]
where in the second inequality we used the basic inequality: \(|a - b|^2 \geq \frac{1}{2}|a|^2 - |b|^2, a, b \in \mathbb{R}^d\). Next, by invoking Lemma 5.2 below, one has

\[ p^{(d)}(j\delta, j'\delta, x, x') \leq \frac{\Lambda_2 e^{-|x - y|^2/\kappa_0(k'\delta - j\delta)}}{(2\pi \lambda_0(k'\delta - j\delta))^{d/2}}, \quad j' > j, \ x, x' \in \mathbb{R}^d, \]

where \(\Lambda_2, \kappa_0\) were given in (2.2) and (2.4), respectively. Subsequently, (2.3) follows immediately by taking advantage of the facts that

\[ p^{(d)}(j\delta, t, x, y) = \int_{\mathbb{R}^d} p^{(d)}(j\delta, [t/\delta]\delta, x, u)p^{(d)}([t/\delta]\delta, t, u, y)du, \]

which is due to the Chapman-Kolmogrov equation, and

\[ \int_{\mathbb{R}^d} \frac{e^{-|x - y|^2/\kappa_0(k\delta - j\delta)}}{(2\pi \lambda_0(k\delta - j\delta))^{d/2}} \frac{e^{-|v - u|^2/\kappa_0(t\delta - k\delta)}}{(2\pi \lambda_0(t\delta - k\delta))^{d/2}} du \leq \left( \frac{\kappa_0}{2\lambda_0} \right)^{d/2} \frac{e^{-|x - y|^2/\kappa_0(t\delta - k\delta)}}{(2\pi \lambda_0(t\delta - k\delta))^{d/2}}, \quad k > j. \]

\[ \square \]

**Lemma 2.2.** Under (A1)-(A3), for any \(T > 0\), there exists a constant \(C > 0\) such that

\[ \int_0^T \mathbb{E}|b(X^{(\delta)}_t) - b(X^{(\delta)}_{t_k})|^2 dt \leq C\delta^{1 + \frac{d}{2}}. \]

**Proof.** Observe that

\[ \int_0^T \mathbb{E}|b(X^{(\delta)}_t) - b(X^{(\delta)}_{t_k})|^2 dt = \int_0^\delta \mathbb{E}|b(X^{(\delta)}_t) - b(X^{(\delta)}_{t_k})|^2 dt + \sum_{k=1}^{\lfloor T/\delta \rfloor} \int_{k\delta}^{T\wedge(k+1)\delta} \mathbb{E}|b(X^{(\delta)}_t) - b(X^{(\delta)}_{k\delta})|^2 dt. \]

By \(\|b\|_\infty < \infty\) due to (A1), it follows that

\[ \int_0^\delta \mathbb{E}|b(X^{(\delta)}_t) - b(X^{(\delta)}_{t_k})|^2 dt \leq 4\|b\|_\infty^2 \delta. \]

For \(t \in [k\delta, (k+1)\delta)\), by taking the mutual independence between \(X^{(\delta)}_{k\delta}\) and \(W_t - W_{k\delta}\) into account and employing Lemma 2.1, we derive that

\[ \mathbb{E}|b(X^{(\delta)}_t) - b(X^{(\delta)}_{k\delta})|^2 \]

\[ = \mathbb{E}|b(X^{(\delta)}_{k\delta} + b(X^{(\delta)}_{k\delta})(t - k\delta) + \sigma(X^{(\delta)}_{k\delta})(W_t - W_{k\delta}))/k\delta - b(X^{(\delta)}_{k\delta})|^2 \]

\[ = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(y + z) - b(y)|^2 p^{(d)}(0, k\delta, x, y) \]

\[ \times \exp\left( -\frac{1}{2(t-k\delta)} \right) \left( ((\sigma^*)^{-1})(z - b(y)(t - k\delta)), (z - b(y)(t - k\delta)) \right) \]

\[ \times \frac{\sqrt{(2\pi)^d \det((t - k\delta)(\sigma^*)^{-1}(y))}}{\det((t - k\delta)) \sqrt{2\pi}^d} \] d\(\mu_{k\delta}^2\) dydz

\[ \leq \frac{C_1}{(k\delta(t - k\delta))^{d/2}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |b(y + z) - b(y)|^2 e^{-|z|^2/\kappa_0(t-k\delta)} e^{-|z - y|^2/\kappa_0 k\delta} dydz, \]
for some constant $C_1 > 0$, where $\kappa_0$ was introduced in (2.4). With the aid of the fact that

\begin{equation}
\sup_{x \geq 0} (x^{\gamma}e^{-\beta x^2}) = \left(\frac{\gamma}{2\beta}\right)^{\frac{\gamma}{2}}, \quad \gamma, \beta > 0,
\end{equation}

we therefore infer from (A2) and (2.9) that

\[
\mathbb{E}[|b(X_t^{(\delta)} - b(X_{k\delta}^{(\delta)})|^2 \leq \frac{C_2 \phi(\kappa_0 k\delta)}{(t - k\delta)^{d/2}} \int_{\mathbb{R}^d} |z|^{\alpha} e^{-\frac{|z|^2}{4\lambda_0 (t - k\delta)}} \, dz
\]

\[
\leq \frac{C_3 \phi(\kappa_0 k\delta) \delta^{\alpha/2}}{(t - k\delta)^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|z|^2}{8\lambda_0 (t - k\delta)}} \, dz \leq C_4 \phi(\kappa_0 k\delta) \delta^{\alpha/2}
\]

for some constants $C_2, C_3, C_4 > 0$. Whence, we arrive at

\begin{equation}
\sum_{k=1}^{[T/\delta]} \int_{k\delta}^{(k+1)\delta} \mathbb{E}[|b(X_t^{(\delta)} - b(X_{k\delta}^{(\delta)})|^2 dt \leq C_4 \delta^{\alpha/2} \int_0^T \phi(\kappa_0 [t/\delta]) dt.
\end{equation}

Thus, (2.7) holds true by combining (2.8) with (2.11) and by utilizing that $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is locally integrable and continuous. \hfill \square

For any $p, q \geq 1$ and $0 \leq S \leq T$, let $L^p_q(S,T)$ be the family of all Borel measurable functions $f : [S,T] \times \mathbb{R}^d \to \mathbb{R}^d$ endowed with the norm

\[\|f\|_{L^p_q(S,T)} := \left( \int_S^T \left( \int_{\mathbb{R}^d} |f_t(x)|^p dx \right)^{\frac{q}{p}} dt \right)^{\frac{1}{q}} < \infty.\]

For simplicity, in the sequel, we write $L^p_q(T)$ in place of $L^p_q(0,T)$. Set

\[\mathcal{K}_1 := \left\{(p,q) \in (1,\infty) \times (1,\infty) : \frac{d}{p} + \frac{2}{q} < 2 \right\}, \mathcal{K}_2 := \left\{(p,q) \in (1,\infty) \times (1,\infty) : \frac{d}{p} + \frac{2}{q} < 1 \right\}.
\]

Compared with (1.1), in (1.4) we have written the drift term as $b(X_{t\delta}^{(\delta)})$ in lieu of $b(X_t^{(\delta)})$ so that the classical Krylov estimate (see e.g. [6, 11, 18, 34, 36, 37]) is unapplicable directly. However, the following lemma manifests that $(X_t^{(\delta)})_{t \geq 0}$ still satisfies the Khasminskii estimate by employing Gaussian type estimate of heat kernel although the Krylov estimate for $(X_{t\delta}^{(\delta)})_{t \geq 0}$ is invalid as Remark (2.5) below describes.

**Lemma 2.3.** Assume (A1) and (A3). Then, for $f \in L^p_q(T)$ with $(p,q) \in \mathcal{K}_1$, the following Khasminskii type estimate

\begin{equation}
\mathbb{E} \exp \left( \lambda \int_0^T |f_t(X_t^{(\delta)})| dt \right) \leq 2^{1+T(2\lambda_0\|f\|_{L^p_q(T)})^{\gamma_0}}, \quad \lambda > 0
\end{equation}

holds, where $\gamma_0 := \frac{1}{1-1/q-d/2p}$ and

\begin{equation}
\alpha_0 := \frac{(1 - 1/p)^{\frac{2}{d}(1-1/p)}}{(\lambda_0(2\pi)^{\frac{d}{2}})^{\frac{2}{d}}} \left\{ \left( \frac{d}{2} \right)^{1-1/p} + \Lambda_3 \gamma_0 (1 - 1/q) \right\}.
\end{equation}
Proof. For $0 \leq s \leq t \leq T$, note that

$$
\mathbb{E}
\left(
\int_s^t |f_r(X_r^{(\delta)})| \, \mathrm{d}r \bigg| \mathcal{F}_s
\right)
= \mathbb{E}
\left(
\int_s^{t \wedge (s_\delta + \delta)} |f_r(X_r^{(\delta)})| \, \mathrm{d}r \bigg| \mathcal{F}_s
\right)
+ \mathbb{E}
\left(
\int_t^{t \wedge (s_\delta + \delta)} |f_r(X_r^{(\delta)})| \, \mathrm{d}r \bigg| \mathcal{F}_s
\right)
=: I_1(s, t) + I_2(s, t).
$$

Since

$$
X_r^{(\delta)} = X_{s_\delta}^{(\delta)} + b(X_{s_\delta}^{(\delta)})(r - s_\delta) + \sigma(X_{s_\delta}^{(\delta)})(W_s - W_{s_\delta}) + \sigma(X_{s_\delta}^{(\delta)})(W_r - W_s), \quad r \in [s, s_\delta + \delta),
$$

we derive from (1.2) and Hölder’s inequality that

$$
I_1(s, t)
= \int_s^{t \wedge (s_\delta + \delta)} \int_{\mathbb{R}^d} f_r(y_{x,w} + z)
\times \exp \left(-\frac{1}{2}r(y_{x,w})\right)|z\rangle \, \mathrm{d}z \bigg|_{x=X_{s_\delta}^{(\delta)}} \, \mathrm{d}r
$$

$$
\leq \|f\|_{L^p_{p}(T)} \left(\int_s^{t \wedge (s_\delta + \delta)} \left(\frac{1}{(2\pi(r - s))^d \det((\sigma\sigma^*)^p)}\right)^{\frac{p}{q}} \left|X_{s_\delta}^{(\delta)}\right\| \, \mathrm{d}r\right)
\leq \left((2\pi)^{-d/p}(p-1)/p\right)^{d/(q-1)} \left(\frac{\lambda_0^1}{\lambda_0^2}\right)^{d/2} (t - s) \gamma_0 \|f\|_{L^p_{p}(T)},
$$

where $y_{x,w} := x + b(x)(r - s_\delta) + \sigma(x)w, x \in \mathbb{R}^d, w \in \mathbb{R}^m$, and $\gamma_0 := \frac{1}{1-q-\frac{d-2}{2p}}$. For $r > k\delta$, let $X_{k\delta,r}^{(\delta),x} = x$. From the tower property of conditional expectation, one has

$$
I_2(s, t)
= \int_{s_\delta + \delta}^t \mathbb{E}
\left(
|f_r(X_r^{(\delta)})| \bigg| \mathcal{F}_s
\right) \, \mathrm{d}r
= \int_{s_\delta + \delta}^t \mathbb{E}
\left(
\mathbb{E}
\left(
|f_r(X_r^{(\delta)})| \bigg| \mathcal{F}_{s_\delta + s}
\right) \bigg| \mathcal{F}_s
\right) \, \mathrm{d}r
$$

In terms of Lemma 2.1, besides Hölder’s inequality, one obtains that

$$
\mathbb{E}
\left(
|f_r(X_{s_\delta + \delta,r}^{(\delta),x})| \bigg| \mathcal{F}_{s_\delta + \delta}
\right)
\leq \frac{\Lambda_3}{(2\pi \lambda_0(r - s_\delta - \delta))^{d/2}} \int_{\mathbb{R}^d} |f_r(y)| e^{-\frac{|x-y|^2}{\gamma_0(r - s_\delta - \delta)}} \, \mathrm{d}y
$$

$$
\leq \frac{\Lambda_3}{((2\pi)^{1/2} \lambda_0)^{d/2}} \left(\frac{\kappa_0 (p-1)}{2p}\right)^{d/(q-1)} \left(r - s_\delta - \delta\right)^{-\frac{d}{2p}} \left(\int_{\mathbb{R}^d} |f_r(y)|^p \, \mathrm{d}y\right)^{\frac{1}{p}}.
$$

This further yields by Hölder’s inequality that

$$
I_2(s, t)
\leq \frac{\Lambda_3}{((2\pi)^{1/2} \lambda_0)^{d/2}} \left(\frac{\kappa_0 (p-1)}{2p}\right)^{d/(q-1)} \left(r - s_\delta - \delta\right)^{-\frac{d}{2p}} \left(\int_{\mathbb{R}^d} |f_r(y)|^p \, \mathrm{d}y\right)^{\frac{1}{p}}
$$

$$
= \frac{\Lambda_3}{((2\pi)^{1/2} \lambda_0)^{d/2}} \left(\frac{\kappa_0 (1-1/q)}{2 \gamma_0}\right)^{d/(q-1)} \left(1 - \frac{1}{p}\right)^{d/(q-1)} (t - s) \gamma_0 \|f\|_{L^p_{p}(T)}.
$$
Hence, (2.14) and (2.15) imply
\begin{equation}
\mathbb{E} \left( \int_s^t |f_r(X^{(\delta)}_r)|dr \bigg| \mathcal{F}_s \right) \leq \alpha_0 \|f\|_{L^p_0(T)} (t-s)^{\frac{1}{\alpha}} , \quad 0 \leq s \leq t \leq T, \tag{2.16}
\end{equation}
in which \(\alpha_0 > 0\) was introduced in (2.13). For each \(k \geq 1\), applying inductively (2.16) gives
\begin{equation}
\begin{aligned}
\mathbb{E} \left( \left( \int_s^t |f_r(X^{(\delta)}_r)|dr \right)^k \bigg| \mathcal{F}_s \right) &= k! \mathbb{E} \left( \int_{\Delta_{k-1}(s,t)} |f_{r_1}(X^{(\delta)}_{r_1})| \cdots |f_{r_{k-1}}(X^{(\delta)}_{r_{k-1}})| dr_1 \cdots dr_{k-1} \bigg| \mathcal{F}_s \right) \\
&\leq \alpha_0 k! (t-s)^{\frac{k}{\alpha}} \|f\|_{L^p_0(T)} \times \mathbb{E} \left( \left( \int_{r_{k-1}}^t |f_{r_k}(X^{(\delta)}_{r_k})| dr_k \bigg| \mathcal{F}_{r_{k-1}} \right) \bigg| \mathcal{F}_s \right) \\
&\leq \cdots \leq k! (\alpha_0 (t-s)^{\frac{1}{\alpha}} \|f\|_{L^p_0(T)})^k, \quad 0 \leq s \leq t \leq T,
\end{aligned}
\end{equation}
where
\(\Delta_k(s, t) := \{(r_1, \cdots, r_k) \in \mathbb{R}^k : s \leq r_1 \leq \cdots \leq r_k \leq t\}\).

Taking \(\delta_0 = (2\alpha_0 \|f\|_{L^p_0(T)})^{-\gamma_0}\), one obviously has \(\lambda \alpha_0 \delta_0^{\gamma_0} \|f\|_{L^p_0(T)} = \frac{1}{2}\). With this and (2.17) in hand, we derive that
\begin{equation}
\mathbb{E} \left( \exp \left( \lambda \int_{(i-1)\delta_0}^{i\delta_0} |f_t(X^{(\delta)}_t)| dt \bigg| \mathcal{F}_{(i-1)\delta_0} \right) \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2, \quad i \geq 1,
\end{equation}
which further implies inductively that
\begin{equation}
\begin{aligned}
\mathbb{E} \exp \left( \lambda \int_0^T |f_t(X^{(\delta)}_t)| dt \right) &= \mathbb{E} \left( \exp \left( \lambda \sum_{i=1}^{\lfloor T/\delta_0 \rfloor} \int_{(i-1)\delta_0}^{i\delta_0} |f_t(X^{(\delta)}_t)| dt \right) \right) \\
&= \mathbb{E} \left( \exp \left( \lambda \int_{\lfloor T/\delta_0 \rfloor \delta_0}^T |f_t(X^{(\delta)}_t)| dt \right) \bigg| \mathcal{F}_{\lfloor T/\delta_0 \rfloor \delta_0} \right) \\
&\leq 2 \mathbb{E} \exp \left( \lambda \sum_{i=1}^{\lfloor T/\delta_0 \rfloor} \int_{(i-1)\delta_0}^{i\delta_0} |f_t(X^{(\delta)}_t)| dt \right) \\
&\leq \cdots \leq 2^{1+T/\delta_0}.
\end{aligned}
\end{equation}
Therefore, (2.12) is now available by recalling \(\delta_0 = (2\alpha_0 \|f\|_{L^p_0(T)})^{-\gamma_0}\).

The following lemma is concerned with Khasminskii’s estimate for the solution process \((X_t)_{t \geq 0}\), which is more or less standard; see, for instance, [6, 11, 18, 34, 36, 37]. Whereas, we herein state the Khasminskii estimate and provide a sketch of its proof merely for the sake of explicit upper bound.
Lemma 2.4. Assume (A1) and (A3). Then, for \( f \in L^p_q(T) \) with \((p,q) \in \mathcal{K}_1\) and \( \lambda > 0 \),
\[
\mathbb{E} \exp \left( \lambda \int_0^T |f_t(X_t)| \, dt \right) \leq 2^{1+T(2\lambda \alpha_0 \|f\|_{L^p_q(T)\gamma_0})},
\]
where
\[
\alpha_0 := (2\pi)^{-\frac{d}{2}} \beta_T \frac{(s-1)/p}{\hat{b}_T (\lambda_{0}^{1/q} / \lambda_0)^{d}}, \quad \beta_T := e^{\frac{\|f\|_{L^p_q(T)}}{2\lambda_0} \sum_{i=0}^\infty \frac{\beta_T^i}{\Gamma(1 + \frac{i}{z})}}
\]
with \( \beta_T \) being given in (5.2).

Proof. By (5.1) below, it follows from Hölder’s inequality and Markov property that
\[
\mathbb{E} \left( \int_s^t |f_r(X_r)| \, dr \mid \mathcal{F}_s \right) = \int_s^t \left( \mathbb{E} |f_r(X_r^t,x)| \right) \, dr \bigg|_{x=X_s}
\]
\[
\leq \hat{\beta}_T \int_s^t \int_{\mathbb{R}^d} \frac{|f_r(y)|}{(2\pi \lambda_0 (r-s))^{d/2}} \, dy \, dr \bigg|_{x=X_s}
\leq \hat{\alpha}_0 (t-s)^{1-\frac{d}{2} - \frac{1}{q}} \|f\|_{L^p_q(T)},
\]
where \( (X^t,x)_{t \geq s} \) stands for the solution to (1.1) with the initial value \( X^t,s = x \) and \( \hat{\beta}_T, \hat{\alpha}_0 > 0 \) were introduced in (2.21). Then, (2.20) follows immediately by utilizing (2.22) and by following the exact line to derive (2.19). \( \square \)

Remark 2.5. In (2.16), Krylov’s estimate for \( (X_t^\delta)_{t \geq s} \) instead of \( (X_t^{\delta_0})_{t \geq s} \) is available. Whereas, the Krylov estimate associated with \( (X_t^\delta)_{t \geq s} \) no longer holds true. Indeed, if we take \( s, t \in [k\delta, (k + 1)\delta) \) for some integer \( k \geq 1 \), we obviously have
\[
\mathbb{E} \left( \int_s^t |f_{r_3}(X_{r_3}^\delta)| \, dr \mid \mathcal{F}_s \right) = |f_{k\delta}(X_{k\delta}^\delta)(t-s), \ f \in L^p_q(T), \ (p,q) \in \mathcal{K}_1
\]
which is a random variable. Hence, it is impossible to control the quantity on the left hand side of (2.23) by \( \|f\|_{L^p_q(T)} \) up to a constant. Moreover, we would like to refer to e.g. [32] for more details.

Before we go further, we introduce some additional notation. For \( p \geq 1 \) and \( m \geq 0 \), let \( H^m_p \) be the usual Sobolev space on \( \mathbb{R}^d \) with the norm
\[
\|f\|_{H^m_p} := \sum_{k=0}^m \|\nabla^m f\|_{L^p},
\]
where \( \nabla^m \) denotes the \( m \)-th order gradient operator. For \( m \geq 0 \) and \( 0 \leq S \leq T \), let \( \mathbb{H}_p^{m,q}(S,T) = L^q(S,T; H^m_p) \) and \( \mathcal{H}_p^{m,q}(S,T) \) be the collection of all functions \( f : (S,T) \times \mathbb{R}^d \to \mathbb{R} \) such that \( u \in \mathbb{H}_p^{m,q}(S,T) \) and \( \partial_t f \in L^q_p(S,T) \). For a locally integrable function \( h : \mathbb{R}^d \to \mathbb{R} \), the Hardy-Littlewood maximal operator \( \mathcal{M}h \) is defined as follows
\[
(\mathcal{M}h)(x) = \sup_{r > 0} \frac{1}{|B_r(x)|} \int_{B_r(x)} h(y) \, dy, \quad x \in \mathbb{R}^d,
\]
where \(B_r(x)\) is the ball with the radius \(r\) centered at the point \(x\) and \(|B_r(x)|\) denotes the \(d\)-dimensional Lebesgue measure of \(B_r(x)\).

To make the content self-contained, we recall the Hardy-Littlewood maximum theorem (see e.g. [37, Lemma 5.4]), which is stated as the lemma below.

**Lemma 2.6.** For any continuous and weak differential function \(f : \mathbb{R}^d \rightarrow \mathbb{R}\), there exists a constant \(C > 0\) such that

\[
|f(x) - f(y)| \leq C|x - y|\{(\mathcal{M} |\nabla f|)(x) + (\mathcal{M} |\nabla f|)(y)\}, \quad \text{a.e. } x, y \in \mathbb{R}^d.
\]

Moreover, for any \(f \in L^p(\mathbb{R}^d), p > 1\), there exists a constant \(C_p\), independent of \(d\), such that

\[
\|\mathcal{M} f\|_{L^p} \leq C_p\|f\|_{L^p}.
\]

Now we are in position to complete

**Proof of Theorem 1.1.** For any \(\lambda > 0\), consider the following PDE for \(u^\lambda : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d:\)

\[
\partial_t u^\lambda + \frac{1}{2} \sum_{i,j=1}^{d} \langle \sigma \sigma^* e_i, e_j \rangle \nabla_x e_i \cdot \nabla_x e_j u^\lambda + b + \nabla_x u^\lambda = \lambda u^\lambda,
\]

where \((e_j)_{1 \leq j \leq d}\) stipulates the orthogonal basis of \(\mathbb{R}^d\). According to [34, Lemma 4.2], (2.26) has a unique solution \(u^\lambda \in \mathcal{H}^{2,2q}_{2p}(0, T)\) for the pair \((p, q)\) \(\in \mathcal{H}_1\) due to \(p > \frac{d}{2}\) satisfying

\[
(1 \land \lambda)^{1 - \frac{d}{2p}} \|\nabla u^\lambda\|_{T, \infty} + \|\nabla^2 u^\lambda\|_{L^2_{2q}(T)} \leq C_1\|b\|_{L^p}^2,
\]

for some constant \(C_1 > 0\), where \(\|\nabla u^\lambda\|_{T, \infty} := \sup_{0 \leq t \leq T, x \in \mathbb{R}^d} \|\nabla u^\lambda(x)\|_{HS}\). With the help of (2.27), there is a constant \(\lambda_0 \geq 1\) such that

\[
\|\nabla u^\lambda\|_{T, \infty} \leq \frac{1}{2}, \quad \lambda \geq \lambda_0.
\]

For \(u^\lambda \in \mathcal{H}^{2,2q}_{2p}(0, T)\), there exists a sequence \(u^{\lambda,k} \in C^{1,2}([0, T] \times \mathbb{R}^d)\) such that

\[
\lim_{k \to \infty} \|u^{\lambda,k} - u^\lambda\|_{\mathcal{H}^{2,2q}_{2p}(0, T)} = 0,
\]

where

\[
\|u\|_{\mathcal{H}^{2,2q}_{2p}(0, T)} := \|\partial_t u\|_{L^2_{2q}(0, T)} + \|u\|_{\mathcal{H}^{2,2q}_{2p}(0, T)}.
\]

Henceforth, we can apply directly Itô’s formula to \(u^\lambda \in \mathcal{H}^{2,2q}_{2p}(0, T)\) by adopting a standard approximation approach; see e.g. the arguments of [34, Theorem 2.1] and [37, Lemma 4.3] for more details. Set \(\theta_t^\lambda(x) := x + u^\lambda_t(x), x \in \mathbb{R}^d\), and \(Z_t^{(\delta)} := X_t - X_t^{(\delta)}\). By Itô’s formula, we obtain from (2.26) that

\[
\begin{align*}
\mathrm{d}\theta_t^\lambda(X_t) &= \lambda u^\lambda(X_t)\mathrm{d}t + \nabla\theta_t^\lambda(X_t)\sigma(X_t)\mathrm{d}W_t \\
\mathrm{d}\theta_t^{\lambda}(X_t^{(\delta)}) &= \left\{ \lambda u^\lambda(X_t^{(\delta)}) + \nabla\theta_t^\lambda(X_t^{(\delta)})(b(X_t^{(\delta)}) - b(X_t^{(\delta)})) + \frac{1}{2} \sum_{i,j=1}^{d} \langle \sigma \sigma^* (X_t^{(\delta)}), e_i, e_j \rangle \nabla e_i \nabla e_j u^\lambda(X_t^{(\delta)}) \right\}\mathrm{d}t + \nabla\theta_t^{\lambda}(X_t^{(\delta)})\sigma(X_t^{(\delta)})\mathrm{d}W_t.
\end{align*}
\]
Applying Itô’s formula once more, and taking advantage of

\[(2.29)\quad \frac{1}{4}|Z_t^{(\delta)}|^2 \leq |\theta^\lambda_t(X_t) - \theta^\lambda_t(X_t^{(\delta)})|^2 \leq \frac{5}{2}|Z_t^{(\delta)}|^2\]

due to (2.28) yields that

\[(2.30)\quad |Z_t^{(\delta)}|^2 \leq 8\lambda \int_0^t \langle \theta^\lambda_s(X_s) - \theta^\lambda_s(X_s^{(\delta)}), u^\lambda(X_s) - u^\lambda(X_s^{(\delta)}) \rangle ds

+ 8 \int_0^t \langle \theta^\lambda_s(X_s) - \theta^\lambda_s(X_s^{(\delta)}), \nabla \theta^\lambda_s(X_s^{(\delta)})(b(X_s^{(\delta)}) - b(X_s)) \rangle ds

+ \sum_{i,j=1}^d \int_0^t \|\nabla \theta^\lambda_s(X_s)^\sigma(X_s) - \nabla \theta^\lambda_s(X_s^{(\delta)})^\sigma(X_s^{(\delta)})\|_{HS}^2 ds + M_t

=: I_{1,\delta}(t) + I_{2,\delta}(t) + I_{3,\delta}(t) + I_{4,\delta}(t) + M_t,

where

\[M_t := 8 \int_0^t \left\langle \theta^\lambda_s(X_s) - \theta^\lambda_s(X_s^{(\delta)}), ((\nabla \theta^\lambda_s^\sigma)(X_s) - \nabla \theta^\lambda_s^\delta(X_s^{(\delta)})^\sigma(X_s^{(\delta)})\rangle dW_s \right\rangle.

By means of (2.28), we have

\[(2.31)\quad I_{1,\delta}(t) \leq 6\lambda \int_0^t |Z_s^{(\delta)}|^2 ds.

Also, by virtue of (2.28), besides (2.29), we find that there exists a constant \(C_2 > 0\) such that

\[(2.32)\quad I_{2,\delta}(t) \leq C_2 \left\{ \int_0^t |Z_s^{(\delta)}|^2 ds + \int_0^t |b(X_s^{(\delta)}) - b(X_s)|^2 ds \right\}.

Next, with the aid of (1.2), (1.3), and (2.29), we infer that

\[(2.33)\quad I_{3,\delta}(t) \leq C_3 \int_0^t |\sigma(X_s^{(\delta)}) - \sigma(X_s^{(\delta)})|_{HS} |Z_s^{(\delta)}| \cdot \|\nabla^2 u^\lambda(X_s^{(\delta)})\|_{HS} ds

\leq L_0 C_3 \int_0^t |X_s^{(\delta)} - X_s^{(\delta)}| |Z_s^{(\delta)}| \cdot \|\nabla^2 u^\lambda(X_s^{(\delta)})\|_{HS} ds

\leq \frac{L_0 C_3}{2} \int_0^t \left\{ \|\nabla^2 u^\lambda(X_s^{\delta})\|_{HS}^2 |Z_s^{(\delta)}|^2 + |X_s^{(\delta)} - X_s^{(\delta)}|^2 \right\} ds.

for some constant \(C_3 > 0\). Furthermore, thanks to (1.2), (1.3), (2.24) and (2.28), we derive from Hölder’s inequality that

\[(2.34)\quad I_{4,\delta}(t) \leq C_4 \int_0^t |Z_s^{(\delta)}|^2 \left\{ (\mathcal{M} \|\nabla^2 u^\lambda\|_{HS}^2)(X_s) + (\mathcal{M} \|\nabla^2 u^\lambda\|_{HS}^2)(X_s^{(\delta)}) \right\} ds

\quad + C_4 \int_0^t |X_s^{(\delta)} - X_s^{(\delta)}|^2 ds

11
for some constant $C_4 > 0$. As a result, plugging (2.31)-(2.34) into (2.30) gives that

$$|Z_t^{(δ)}|^2 \leq \int_0^t |Z_s^{(δ)}|^2 dA_s + \int_0^t \left\{ C_2 |b(X_s^{(δ)}) - b(X_s^{(δ)})|^2 + \frac{L_0C_3 + 2C_4}{2} |X_s^{(δ)} - X_s^{(δ)}|^2 \right\} ds + M_t,$$

in which, for some constant $C_5 > 0$,

$$A_t := C_5 \int_0^t \left\{ 1 + (\mathcal{M}\|\nabla^2 u_s^\lambda\|^2_{\text{HS}}(X_s) + (\mathcal{M}\|\nabla^2 u_s^\lambda\|^2_{\text{HS}}(X_s^{(δ)})) + \|\nabla^2 u_s^\lambda\|^2_{\text{HS}}(X_s^{(δ)}) \right\} ds.$$

Consequently, we deduce by stochastic Gronwall’s inequality (see e.g. [34, Lemma 3.8]) that, for $0 < \kappa' < \kappa < 1$,

$$\left( \mathbb{E} \left( \sup_{0 \leq s \leq t} |Z_t^{(δ)}|^{2\kappa'} \right) \right)^{1/\kappa'} \leq \left( \frac{\kappa}{\kappa - \kappa'} \right)^{1/\kappa'} \left( \mathbb{E} e^{A_t/((1-\kappa))} \right)^{(1-\kappa)/\kappa} \times \int_0^t \left\{ C_2 \mathbb{E}|b(X_s^{(δ)}) - b(X_s^{(δ)})|^2 + \frac{L_0C_3 + 2C_4}{2} \mathbb{E}|X_s^{(δ)} - X_s^{(δ)}|^2 \right\} ds.$$

This, together with (2.7) and

$$\sup_{0 \leq t \leq T} \mathbb{E}|X_t^{(δ)} - X_{t_0}^{(δ)}|^2 \leq C_0\delta$$

for some constant $C_6 > 0$, leads to

$$\left( \mathbb{E} \left( \sup_{0 \leq s \leq t} |Z_t^{(δ)}|^{2\kappa'} \right) \right)^{1/\kappa'} \leq C_7 \left( \mathbb{E} e^{A_t/((1-\kappa))} \right)^{(1-\kappa)/\kappa} \left( \delta + \delta^{\alpha'/2} \right)$$

for some constant $C_7 > 0$. By Hölder’s inequality, we deduce that for some constant $C_8 > 0$,

$$\mathbb{E} e^{\frac{\kappa A_t}{1-\kappa}} \leq e^{\frac{\kappa A_t}{1-\kappa}} \left( \mathbb{E} \left( C_8 \int_0^t (\mathcal{M}\|\nabla^2 u_s^\lambda\|^2_{\text{HS}}(X_s)) ds \right) \right)^{1/2} \times \left( \mathbb{E} \left( C_8 \int_0^t (\mathcal{M}\|\nabla^2 u_s^\lambda\|^2_{\text{HS}}(X_s^{(δ)})) ds \right) \right)^{1/4} \times \left( \mathbb{E} \left( C_8 \int_0^t \|\nabla^2 u_s^\lambda\|^2_{\text{HS}}(X_s^{(δ)}) ds \right) \right)^{1/4}.$$

This, in addition to (2.12), (2.20), (2.22) as well as (2.27), implies that

$$\mathbb{E} e^{\frac{\kappa A_t}{1-\kappa}} \leq \exp \left( C_9 \left( 1 + \|\nabla^2 u^\lambda\|^2_{\text{HS}} L_{L_u}^0(T) + \|\mathcal{M}\|\nabla^2 u^\lambda\|^2_{\text{HS}} L_{L_u}^0(T) \right) \right) \leq \exp \left( C_{10} \left( 1 + \|\nabla^2 u^\lambda\|^2_{L_{L_u}^2(T)} \right) \right) \leq \exp \left( C_{11} \left( 1 + \|b\|^2_{L_{L_u}^2} \right) \right)$$

for some constants $C_9, C_{10}, C_{11} > 0$. Thus, the assertion (1.5) follows from (2.35) and (2.36). □
3 Proof of Theorem 1.2

In this section, we aim to complete the proof of Theorem 1.2 by carrying out a truncation approach; see, for example, [1, 28] for further details.

Let \( \psi : \mathbb{R}_+ \to [0, 1] \) be a smooth function such that \( \psi(r) = 1, r \in [0, 1] \), and \( \psi(r) \equiv 0, r \geq 2 \). For each integer \( k \geq 1 \), let \( b_k(x) = b(x)\psi(|x|/k), x \in \mathbb{R}^d \), be the truncation function associated with the drift \( b \). A direct calculation shows that

\[
\|b_k\|_\infty \leq \|b\|_\infty \quad \text{and} \quad \|b_k\|^2_{L_p} \leq \left( \frac{2d^\frac{4}{p}}{T(\frac{4}{d}+1)} \right)^{1/p} k^\frac{4}{p} \|b\|^2_\infty.
\]

Consider the following truncated SDE corresponding to (1.1)

\[
dX_t^k = b_k(X_t^k)dt + \sigma(X_t^k)dW_t, \quad t \geq 0, \quad X_0^k = X_0.
\]

The EM scheme concerned with (3.2) is given by

\[
dX_t^{k,}\ = b_k(X_{ts}^{k,})dt + \sigma(X_{ts}^{k,})dW_t, \quad t \geq 0, \quad X_0^{k,} = X_0^{(k)}.
\]

For \( q \in (0, 2) \), observe that

\[
\mathbb{E}\|X - X^{(k)}\|_{T,\infty}^q \leq 3^{0(q-1)} \{ \mathbb{E}\|X - X^k\|_{T,\infty}^q + \mathbb{E}\|X^{(k)} - X^{(k)}\|_{T,\infty}^q \}
\]

\[
+ \mathbb{E}\|X^k - X^{k,}\|_{T,\infty}^q \}
\]

\[
=: 3^{0(q-1)} \{ I_1 + I_2 + I_3 \}.
\]

Via Hölder’s inequality and \( \{ X_t \neq X^k, 0 \leq t \leq T \} \subseteq \{ \|X\|_{T,\infty} \geq k \} \), it follows that

\[
I_1 = \mathbb{E}\left( \|X - X^k\|_{T,\infty}^q \mathbb{1}_{\{\|X\|_{T,\infty} \geq k\}} \right) \leq \left( \mathbb{E}\left( \|X - X^k\|_{T,\infty}^{2q} \right) \right)^{1/2} \left( \mathbb{P}(\|X\|_{T,\infty} \geq k) \right)^{1/2}.
\]

Since

\[
\|X\|_{T,\infty} \leq \|X\|_\infty T + \sup_{0 \leq t \leq T} |M_t|,
\]

in which \( M_t := \int_0^t \sigma(X_s)dw_s, t \geq 0 \), with the quadratic variation \( \langle M \rangle_T \leq d\lambda_0 T \), we derive from [33, Proposition 6.8, p147] that

\[
\mathbb{P}(\|X\|_{T,\infty} \geq k) \leq \mathbb{P}\left( \sup_{0 \leq t \leq T} |M_t| \geq k - \|X\|_\infty T, \langle M \rangle_T \leq d\lambda_0 T \right)
\]

\[
\leq 2d \exp \left( - \frac{(k - \|X\|_\infty T)^2}{4d^2\lambda_0 T} \right)
\]

\[
\leq 2d \exp \left( \frac{(|X| + \|X\|_\infty T)^2}{4d^2\lambda_0 T} \right) e^{-\frac{d^2a^2\lambda_0 T}{8d^2\lambda_0 T}},
\]

where in the last display we used the inequality: \( (a - b)^2 \geq a^2/2 - b^2, a, b \in \mathbb{R} \). (3.4), in addition to

\[
\mathbb{E}\|X\|_{T,\infty}^{2q} + \mathbb{E}\|X^k\|_{T,\infty}^{2q} \leq C_1
\]
for some constant $C_1$ yields
\[(3.5) \quad I_1 \leq C_2 \exp\left(\frac{(|x| + \|b\|_\infty T)^2}{8d^2\lambda_0 T}\right)e^{-\frac{k^2}{16d^2\lambda_0 T}}\]

for some constant $C_2 > 0$. Following a similar procedure to derive (3.5), we also derive that
\[(3.6) \quad I_2 \leq C_3 \exp\left(\frac{(|x| + \|b\|_\infty T)^2}{8d^2\lambda_0 T}\right)e^{-\frac{k^2}{16d^2\lambda_0 T}}\]

for some constant $C_3 > 0$. Moreover, for $p, q > 2$ with $\frac{d}{p} + \frac{1}{q} < 1$, according to Theorem 1.1, there exist constants $C_4, C_5 > 0$ such that
\[
E\|X^k_t - X^k_t(\delta)\|_{q, \infty} \leq C_4 e^{C_5 \|b_k\|^2_2 \delta_0^{\frac{2}{p} (1 + \frac{\alpha}{2})}}.
\]

This, together with (3.1), implies
\[(3.7) \quad E\|X^k_t - X^k_t(\delta)\|_{q, \infty} \leq C_4 e^{C_6 \|b\|^2_2 \delta_0^{\frac{2}{p} (1 + \frac{\alpha}{2})}}\]

for some constant $C_6 > 0$. As a consequence, from (3.5), (3.6), and (3.7), we arrive at
\[
E\left(\sup_{0 \leq t \leq T} |X^k_t - X^k_t(\delta)|^q\right) \leq C_8 \left\{ e^{-\frac{k^2}{16d^2\lambda_0 T}} + e^{C_7 \frac{d^{\alpha}}{T} \delta_0^{\frac{2}{p} (1 + \frac{\alpha}{2})}} \right\}
\]

for some constants $C_7, C_8 > 0$. Thereby, the desired assertion (1.7) follows by taking
\[
k = \left( -8qd^2\lambda_0 \left( 1 + \frac{\alpha}{2} \right) \log \delta \right)^\frac{1}{2}.
\]

4 Illustrative Examples

In this section, we intend to give examples to demonstrate that the assumption imposed on drift term holds true.

Example 4.1. Let $b(x) = 1_{[a_1, a_2]}(x), x \in \mathbb{R}$, for some constants $a_1 < a_2$. Apparently, $b$ is not continuous at all but $b^2 \in L_p$ for any $p \geq 1$. Observe that
\[
\lim_{\varepsilon \downarrow 0} \frac{-\varepsilon (b(a_1 - \varepsilon) - b(a_1))}{\varepsilon^2} = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} = \infty
\]

so that $b$ does not obey the one-side Lipschitz condition. Next we aim to show that $b$ given above satisfies (A2). By a direct calculation, for any $s > 0$ and $y \in \mathbb{R}$,
\[
\int_{-\infty}^{\infty} |b(x + y + z) - b(x + y)|^2 e^{-\frac{x^2}{2}} dx \leq \int_{-\infty}^{\infty} |b(x + z) - b(x)|^2 dx
\]
\[
= \int_{a_2 - z}^{a_2 - z} 1_{[a_1 - z, a_2 - z]}(x) dx + \int_{a_1}^{a_2 - z} 1_{[a_1, a_2]}(x) dx
\]
\[
=: I_1(z) + I_2(z).
\]
If \( z \geq 0 \), then
\[
I_1(z) = \int_{a_1-z}^{(a_2-z)^\wedge a_1} dx \leq |z| \quad \text{and} \quad I_2(z) = \int_{(a_2-z)^\wedge a_1}^{a_2} dx \leq |z|.
\]
On the other hand, for \( z < 0 \), we have
\[
I_1(z) = \int_{(a_1-z)^\wedge a_2}^{a_2-z} dx \leq |z| \quad \text{and} \quad I_2(z) = \int_{a_1}^{a_2 \wedge (a_1-z)} dx \leq |z|.
\]
In a word, we conclude that (A2) holds with \( \alpha = 1 \) and \( \phi(s) = s^{-\frac{1}{2}} \) therein.

**Example 4.2.** For \( \theta > 0 \) and \( p \in [2, \infty) \cap (d, \infty) \), if the Gagliardo seminorm
\[
[b]_{W^{p,\theta}} := \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|b(x) - b(y)|^p}{|x - y|^d + \theta} \, dx \, dy \right)^\frac{1}{p} < \infty,
\]
then \( b \in G\hat{B}^{2-\frac{d}{p}, \theta}_{\frac{d}{p}}(\mathbb{R}^d) \). Indeed, by Hölder’s inequality and (2.10), it follows that
\[
\frac{1}{(rs)^d/2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |b(x) - b(y)|^2 e^{-\frac{|x-z|^2}{s}} e^{-\frac{|y-s|^2}{r}} \, dy \, dx
\]
\[
= \frac{1}{(rs)^d/2} \int_{\mathbb{R}^d \times \mathbb{R}^d} |b(x) - b(y)|^2 \frac{|x-z|^2}{|x-y|^d+\theta s} e^{-\frac{|y-x|^2}{r}} \, dy \, dx
\]
\[
\leq C_1 |b|_{W^{p,\theta}}^2 \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{\frac{|x-z|^2}{(p-2)s}} e^{\frac{|y-x|^2}{(p-2)r}} \frac{|x-y|^{2(d+p\theta)}}{|p-2|} \, dy \, dx \right)^{\frac{p-2}{p}}
\]
\[
\leq C_2 |b|_{W^{p,\theta}}^2 \left( \frac{p-2}{p} \right)^{\frac{d(p-2)}{p}} \frac{d(p-2)}{p} \frac{s}{p}^{\frac{d(p-2)}{p}} \, dy \, dx
\]
\[
\leq C_3 |b|_{W^{p,\theta}}^2 \left( \frac{p-2}{p} \right)^{\frac{d(p-2)}{p}} \frac{d(p-2)}{p} \frac{s}{p}^{\frac{d(p-2)}{p}} \, dy \, dx
\]
for some constants \( C_1, C_2, C_3 > 0 \). On the other hand, if \( d = 1 \) and \( p = 2 \), we deduce from (4.1) that \( b \in G\hat{B}^{2}_{1,\theta}(\mathbb{R}) \) due to \( \lim_{x \to 0} x^x = 1 \).

**Example 4.3.** For \( 0 < a < b < \infty \), \( f(\cdot) := 1_{[a,b]}(\cdot) \in G\hat{B}^{2}_{\frac{1}{2},\theta}(\mathbb{R}^d) \) whereas \( f \notin W^{1,2}_{\frac{d}{2}}(\mathbb{R}) \). In fact, it is easy to see that
\[
f \in \cap_{0 \leq \theta < \frac{1}{2}} W^{\theta,2}, \quad \lim_{\theta \to \frac{1}{2}} [f]_{W^{\theta,2}} = \infty,
\]
which yields \( f \notin W^{1,2}_{\frac{d}{2}}(\mathbb{R}) \). On the other hand, since
\[
\frac{1}{(rs)^d/2} \int_{\mathbb{R}^2} |f(x) - f(y)|^2 e^{-\frac{|x-z|^2}{s}} e^{-\frac{|y-s|^2}{r}} \, dy \, dx \leq C s^{-\frac{1}{2}} r^{\frac{1}{2}}, \quad r, s > 0, z \in \mathbb{R}
\]
for some constant \( C > 0 \), we arrive at \( f \in G\hat{B}^{2}_{\frac{1}{2},\theta}(\mathbb{R}^d) \).
5 Appendix

The lemma below provides explicit estimates of the coefficients concerning Gaussian type estimate of transition density for the diffusion process \( (X_t)_{t \geq 0} \) solving (1.1).

Lemma 5.1. Under \( \|b\|_\infty < \infty \) and (A3), the transition density \( p \) of \( (X_t)_{t \geq s} \) satisfies

\[
(5.1) \quad p(s, t, x, x') \leq e^{\|b\|_L^2 \tau} \frac{\beta^i_T}{\Gamma(1 + \frac{i}{2})} p_0(t - s, x, x'), \quad 0 \leq s < t \leq T, x, x' \in \mathbb{R}^d,
\]

where \( \Gamma(\cdot) \) is the Gamma function, and

\[
(5.2) \quad \beta_T := 2^{3d+1} \frac{(\lambda_0)}{d+1} (\pi T)^{\frac{d-1}{2}} \left\{ \frac{\|b\|_L^2 \tau}{\sqrt{\lambda_0}} + L_0(d + 2\sqrt{d}) \right\} e^{\frac{\|b\|_L^2 \tau}{4\lambda_0}}, p_0(t, x, x') := \frac{e^{-\frac{(x-x')^2}{16\lambda_0 t}}}{(2\pi\lambda_0 t)^{d/2}}.
\]

Proof. The proof of Lemma 5.1 is based on the parametrix method [17, 19]. To complete the proof of Lemma 5.1, it suffices to refine the argument of [17, Lemma 3.2]; see also e.g. [19, p1660-1662] for further details. Under \( \|b\|_\infty < \infty \) and (A3), \( X_t \) admits a smooth transition density \( p(s, t, x, y) \) at the point \( y \), given \( X_s = x \), such that

\[
(5.3) \quad \partial_t p(s, t, x, y) = L^* p(s, t, x, y), \quad p(s, t, x, \cdot) = \delta_x(\cdot), \quad t \downarrow s,
\]

\[
\partial_s p(s, t, x, y) = -Lp(s, t, x, y), \quad p(s, t, \cdot, y) = \delta_y(\cdot), \quad s \uparrow t,
\]

where \( L \) is the infinitesimal generator of (1.1) and \( L^* \) is its adjoint operator. For \( t > s \) and \( x, x' \in \mathbb{R}^d \), let \( \tilde{X}_t^{s,x,x'} \) solve the frozen SDE

\[
(5.4) \quad d\tilde{X}_t^{s,x,x'} = b(x')dt + \sigma(x')dW_t, \quad t > s, \quad \tilde{X}_s^{s,x,x'} = x \in \mathbb{R}^d
\]

and \( \tilde{p}^{x'}(s, t, x, x') \) stand for its transition density at \( x' \), given \( X_s^{s,x,x'} = x \). Apparently, \( \tilde{p}^{x'} \) admits the explicit form

\[
\tilde{p}^{x'}(s, t, x, x') = \frac{e^{-\frac{1}{2}((\sigma\sigma^*)^{-1}(x') - (\sigma\sigma^*)^{-1}(x) - b(x') (t-s)))((x'-x) - b(x')(t-s))}}{(2\pi(t-s))^{d/2}\det((\sigma\sigma^*)^{-1})}. \]

A direct calculation yields

\[
(5.5) \quad \partial_s \tilde{p}^{x'}(s, t, x, x') = -\tilde{L}^{x'} \tilde{p}^{x'}(s, t, x, x'), \quad t > s, \quad \tilde{p}^{x'}(s, t, \cdot, x') \to \delta_{x'}(\cdot), \quad s \uparrow t,
\]

where \( \tilde{L}^{x'} \) is the infinitesimal generator of (5.4). By (5.3) and (5.4), we derive from [17, (3.8)] that

\[
(5.6) \quad p(s, t, x, x') = \tilde{p}^{x'}(s, t, x, x') + \int_s^t \int_{\mathbb{R}^d} p(s, u, x, z) H(u, t, z, x')dzdu,
\]

where

\[
(5.7) \quad H(s, t, x, x') := (L - \tilde{L}^{x'})(\tilde{p}^{x'}(s, t, x, x') + \frac{1}{2}((\sigma\sigma^*)(x) - (\sigma\sigma^*)(x'), \nabla \tilde{p}^{x'}(s, t, x, x'))_{HS}.
\]
In (5.6), iterating for \( p(s, u, x, z) \) gives

\[
p(s, t, x, x') = \sum_{i=0}^{\infty} (\bar{p}^i \otimes H(i))(s, t, x, x'),
\]

where \( \bar{p} \otimes H^{(0)} := \bar{p} \) and \( \bar{p}^i \otimes H^{(i)} := (\bar{p}^i \otimes H^{(i-1)}) \otimes H, i \geq 1, \) with

\[
(f \otimes g)(s, t, x, x') := \int_s^t \int_{\mathbb{R}^d} f(s, u, x, z)g(u, t, z, y)du \, dz.
\]

If we can claim that

\[
|p(s, t, x, x')| \leq \frac{\|b\|_{\infty}^{|x-x'|}}{\sqrt{\lambda_0 \sqrt{t-s}}} p_0(t-s, x, x'),
\]

in which \( \beta, p_0 \) were introduced in (5.2), then (5.1) follows from (5.8) and (5.9). Below it suffices to show that (5.9) holds true. By means of (2.10) and \(|a-b|^2 \geq \frac{1}{\beta}|a|^2 - |b|^2, a, b \in \mathbb{R}^d\), it follows from (1.2) and \( \|b\|_{\infty} < \infty \) that

\[
|\nabla \bar{p}|(s, t, x, x') \leq \sqrt{\lambda_0^2 + \lambda_0^2} \frac{p_0(t-s, x, x')}{\lambda_0 \sqrt{t-s}}
\]

\[
\|\nabla^2 \bar{p}\|_{HS}(s, t, x, x') \leq \frac{(\sqrt{\lambda_0^2 + \lambda_0^2} \|b\|_{\infty}^{|x-x'|})^2}{\lambda_0 (t-s)} p_0(t-s, x, x').
\]

Thus, combining (2.10) with (5.10), besides \( \|b\|_{\infty} < \infty \) and (1.3), enables us to obtain

\[
|H|(s, t, x, x') \leq \frac{2\lambda_0 \left(\frac{\|b\|_{\infty}^{|x-x'|}}{\sqrt{\lambda_0^2 + \lambda_0^2}} + L_0(d + 2\sqrt{d})\right) e^{\frac{\|b\|_{\infty}^{|x-x'|}}{\lambda_0^2}}}{\lambda_0 \sqrt{t-s}} p_0(t-s, x, x').
\]

By \( \int_s^t (t-u)^{-\frac{d}{2}} (u-s)^{\alpha} du = (t-s)^{\alpha+\frac{d}{2}} B(1+\alpha, 1/2), t > s, \alpha > -1, \) we have

\[
A_i(s, t) := \int_s^t \cdots \int_s^{u_i-1} (t-u_1)^{-\frac{d}{2}} \cdots (u_i-1-u_i)^{-\frac{d}{2}} du_1 \cdots du_i = \frac{(\pi(t-s))^{\frac{d}{2}}}{\Gamma(1+\frac{d}{2})}, \quad i \geq 1.
\]

Whence, taking advantage of \( \|b\|_{\infty} < \infty, (1.2), (5.11) \) as well as

\[
\int_{\mathbb{R}^d} p_0(u-s, x, z) p_0(t-u, y, z) \, dz = \left( \frac{\delta \lambda_0}{\lambda_0} \right)^d p_0(t-s, x, x'), \quad s < u < t
\]

yields (5.9).

For \( x, x' \in \mathbb{R}^d \) and \( j \geq 0, (X_{i\delta}^{(j), j, x, x'})_{i \geq j} \) solve the following frozen EM scheme associated with (1.1)

\[
X_{(i+1)\delta}^{(j), j, x, x'} = X_{i\delta}^{(j), j, x, x'} + b(x')\delta + \sigma(x')(W_{(i+1)\delta} - W_{i\delta}), \quad i \geq j, \quad X_{j\delta}^{(j), j, x, x'} = x.
\]

Write \( \bar{p}^{(j), x'}(j \delta, j' \delta, x, y) \) by the transition density of \( X_{j\delta}^{(j), j, x, x'} \) at the point \( y \), given \( X_{j\delta}^{(j), j, x, x'} = x \).

The following lemma reveals explicit upper bounds of coefficients with regard to Gaussian bound of the discrete-time EM scheme.

\[
17
\]
Lemma 5.2. Under $\|b\|_\infty < \infty$ and (A3), for any $0 \leq j < j' \leq |T/\delta|

\begin{align}
(5.12) \quad p^{(j)}(j\delta, j'\delta, x, x') &\leq \frac{\|b\|_\infty^\gamma}{2\lambda_0^\gamma} \sum_{k=0}^{\|b\|_\infty^\gamma} \frac{(\sqrt{\pi T} \hat{C}_T ((1 + 24d)\lambda_0/\lambda_0^d)^j e^{-\frac{|x-x'|^2}{2(1 + 24d)\lambda_0(j'-j)\delta}})}{\Gamma(1 + \frac{k}{2})} (2\pi \lambda_0^d)^{(j'-j)\delta}.
\end{align}

Proof. To obtain (5.12), we refine the proof of [19, Lemma 4.1]. For $\psi \in C^2(\mathbb{R}^d; \mathbb{R})$ and $j \geq 0$, set

$$(\mathcal{L}^{(j)}_{j \delta} \psi)(x) := \delta^{-1}\{E(\psi(X^{(j)}_{j+1}\delta)|X^{(j)}_{j \delta} = x) - \psi(x)\}.$$ \hspace{1cm} (5.13)

and

$$H^{(j)}(j\delta, j'\delta, x, x') := (\mathcal{L}^{(j)}_{j \delta} - \mathcal{L}^{(j)}_{j' \delta})p^{(j')}((j + 1)\delta, j'\delta, x, x').$$ \hspace{1cm} (5.12)

In what follows, let $0 \leq j < j' \leq |T/\delta|$. According to [17, Lemma 3.6], we have

\begin{align}
(5.13) \quad p^{(j)}(j\delta, j'\delta, x, x') &= \sum_{k=0}^{j'-j}(p^{(j)}(x') \otimes_\delta H^{(j)}(k))(j\delta, j'\delta, x, x'),
\end{align}

where $p^{(j)}(x') \otimes_\delta H^{(j)}(0) = p^{(j)}(x')$, $H^{(j)}(k) = H^{(j)} \otimes_\delta H^{(j)}(k-1)$ with $\otimes_\delta$ being the convolution type binary operation defined by

\begin{align}
(f \otimes_\delta g)(j\delta, j'\delta, x, x') &= \delta \sum_{k=j}^{j'-1} \int_{\mathbb{R}^d} f(j\delta, k\delta, x, u)g(k\delta, j'\delta, u, x')du.
\end{align}

If the assertion

\begin{align}
(5.14) \quad H^{(j)}(j\delta, j'\delta, x, x') &\leq \frac{\hat{C}_T}{\sqrt{(j'-j)\delta}} (2\pi \lambda_0^d)^{(j'-j)\delta} e^{-\frac{|x-x'|^2}{4(1 + 24d)\lambda_0(j'-j)\delta}}
\end{align}

holds true, where $\hat{C}_T$ was given in (2.1), then (5.12) follows due to (5.13) by an induction argument. So, in order to complete the proof of Lemma 5.2, it remains to verify (5.14). First of all, we show (5.14) for $j' = j + 1$. By the definition of $H^{(j)}$, observe from (1.2) that

$$H^{(j)}(j\delta, (j + 1)\delta, x, x') = \frac{1}{\delta} |p^{(j)}(j\delta, (j + 1)\delta, x, x')| \leq \frac{1}{\delta} \left\{ e^{-\frac{|x-x'|^2}{2(1 + 24d)\lambda_0(j'-j)\delta}}} \right\}$$

and

$$|H^{(j)}(j\delta, (j + 1)\delta, x, x')| \leq \frac{1}{\delta} \left\{ e^{-\frac{|x-x'|^2}{2(1 + 24d)\lambda_0(j'-j)\delta}}} \right\}$$

Next, we aim to estimate $A_1, A_2, A_3$, one-by-one. By $\|b\|_\infty < \infty$, (1.2) and (2.10), it follows from the first fundamental theorem of calculus that

\begin{align}
(5.15) \quad |A_1| \leq 2\sqrt{\delta/\lambda_0} \|b\|_\infty e^{-\frac{|x-x'|^2}{8\lambda_0\delta}}.
\end{align}
(1.2) and (1.3) imply
\[ \|(\sigma\sigma^*)^{-1}(x) - (\sigma\sigma^*)^{-1}(x')\|_{HS} \leq 2\lambda_0^{-2}\sqrt{d\lambda_0}L_0|x - x'|. \]

This, by invoking \(|a^\alpha - b^\beta| \leq e^{\alpha |b|}a - b|, a, b \in \mathbb{R}, and utilizing \(\|b\|_{\infty} < \infty\), (1.2) and (2.10), yields
(5.16)
\[ |A_2| \leq 4\sqrt{d\delta}L_0(\lambda_0/\lambda_0)\sqrt{d}\lambda_0 e^{-\frac{|x-x'|^2}{4|\lambda_0|}}. \]

Also, making use of \(\|b\|_{\infty} < \infty\), (1.2) and (2.10), in addition to
\[ |\det((\sigma\sigma^*)(x)) - \det((\sigma\sigma^*)(x'))| \leq 2d^{\frac{d+1}{2}}L_0|x - x'|, \]
due to (1.2) and (1.3), we arrive at
(5.17)
\[ |A_3| \leq \sqrt{2d\delta}L_0(\lambda_0/\lambda_0)^dL_0\sqrt{\delta}e^{-\frac{|x-x'|^2}{4|\lambda_0|}}. \]

We therefore conclude that (5.14) holds with \(j' = j + 1\) by taking (5.15)-(5.17) into account. In the sequel, we are going to show that (5.14) is still available for \(j' > j + 1\). According to the notion of \(H^{(\delta)}\),
\[
H^{(\delta)}(j\delta, j'\delta, x, x') = \frac{1}{\delta(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} \left\{ \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x + \Gamma_z(x), x') - \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x, x') \right\} dz
- \frac{1}{\delta(2\pi)^{m/2}} \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} \left\{ \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x + \Gamma_z(x), x') - \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x, x') \right\} dz,
\]
where \(\Gamma_z(x) := b(x)\delta + \sqrt{\delta}\sigma(x)z, x \in \mathbb{R}^d, z \in \mathbb{R}^m\). By Taylor’s expansion, we further have
\[
H^{(\delta)}(j\delta, j'\delta, x, x') = \frac{1}{\delta(2\pi)^{m/2}} \left\{ \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} \langle \nabla \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x, x'), \Gamma_z(x) - \Gamma_z(x') \rangle dx \right. \\
+ \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} \langle \nabla^2 \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x, x'), (\Gamma_z \Gamma_z^*) (x) - (\Gamma_z \Gamma_z^*) (x') \rangle_{HS} dx \\
+ \frac{1}{2\delta(2\pi)^{m/2}} \int_{\mathbb{R}^m} \int_0^1 (1 - \theta)^2 e^{-\frac{|x|^2}{2}} \left\{ \nabla^3 \Gamma_z(x) \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x + \theta \Gamma_z(x), x') \\
- \nabla^3 \Gamma_z(x) \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x + \theta \Gamma_z(x'), x') \right\} d\theta dx \right\}
= : \Pi_1 + \Pi_2 + \Pi_3,
\]
where \(\nabla^i\) means the \(i\)-th order gradient operator. Employing
\[
\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} \text{trace}(A\sigma(x)zz^*\sigma(x)) dx = \int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} z^*\sigma^*(x) A\sigma(x) z dx = (2\pi)^m/2 \text{trace}((\sigma^*(x)A\sigma(x))
\]
for a symmetric \(d \times d\)-matrix and \(\int_{\mathbb{R}^m} e^{-\frac{|x|^2}{2}} z dx = 0\) gives
\[
\Pi_1 + \Pi_2 = H((j + 1)\delta, j'\delta, x, x') + \frac{\delta}{2} \langle \nabla^2 \bar{p}^{(\delta), x'}((j + 1)\delta, j'\delta, x, x'), (bb^*) (x) - (bb^*) (x') \rangle_{HS},
\]
where $H$ was defined as in (5.7) with $p^{x'}$ replaced by $\bar{p}^{(\delta),x'}$. (5.10) and (5.11) enable us to obtain

$$\begin{equation}
|\Pi_1| + |\Pi_2| \leq \frac{2^{d+1} e^{\frac{1}{4d\lambda_0} T}}{\lambda_0^\delta (\delta, x, x') \lambda_0^\delta \lambda_0 (\lambda_0^\delta + 2\lambda_0 L_0 \sqrt{d}) (\sqrt{d} + 2)} d_p((\lambda' - \lambda)\delta, x, x').
\end{equation}$$

Note that $\Pi_3$ can be reformulated as below

$$\begin{align*}
\Pi_3 &= \frac{1}{2\delta (2\pi)^{m/2}} \int_{\mathbb{R}^m} \int_0^1 (1 - \theta)^2 e^{-\frac{|z|^2}{2}} \left\{ \nabla^3_{\Gamma_z(x') \bar{p}^{(\delta),x'}} ((j + 1)\delta, j\delta, x + \theta \Gamma_z(x'), x') \\
&- \nabla^3_{\Gamma_z(x) \bar{p}^{(\delta),x'}} ((j + 1)\delta, j\delta, x + \theta \Gamma_z(x), x') \right\} d\theta dz \\
&+ \frac{1}{2\delta (2\pi)^{m/2}} \int_{\mathbb{R}^m} \int_0^1 (1 - \theta)^2 e^{-\frac{|z|^2}{2}} \left\{ \nabla^3_{\Gamma_z(x) \bar{p}^{(\delta),x'}} ((j + 1)\delta, j\delta, x + \theta \Gamma_z(x), x') \\
&- \nabla^3_{\Gamma_z(x') \bar{p}^{(\delta),x'}} ((j + 1)\delta, j\delta, x + \theta \Gamma_z(x'), x') \right\} d\theta dz =: \Pi_{31} + \Pi_{32}.
\end{align*}$$

By means of (1.2), (1.3) and (2.10), it follows that

$$\begin{equation}
|\Pi_{31}| \leq \frac{2^{m+\frac{d+1}{2}} L_0 + 2\|b\|_\infty (\|b\|_\infty^2 + d\lambda_0) \left( 1 + \sqrt{2(1 + 4d\lambda_0)} \right) e^{\frac{\lambda_0^2 T}{8d\lambda_0}}}{\lambda_0^\delta ((\lambda' - \lambda)\delta)^{1/2}}
\end{equation}$$

\begin{equation}
\times e^{-\frac{|z'|^2}{8(1 + 4d\lambda_0)(\lambda' - \lambda)\delta}} \left( \frac{2\pi \lambda_0 (\lambda' - \lambda)\delta)^{d/2}}{2\pi \lambda_0 (\lambda' - \lambda)\delta)^{d/2}}. \right.
\end{equation}

Also, by exploiting (1.2), and (2.10), we infer from Taylor expansion

$$\begin{equation}
|\Pi_{32}| \leq \frac{2^{m+\frac{d+1}{2}} L_0 + 2\|b\|_\infty (\|b\|_\infty^3 + (d\lambda_0)^2) \left( 1 + \sqrt{2(1 + 24d\lambda_0)} \right) e^{\frac{(61b)^2 + \|b\|_\infty T}{24d\lambda_0}}}{\lambda_0^\delta ((\lambda' - \lambda)\delta)^{1/2}}
\end{equation}$$

\begin{equation}
\times e^{-\frac{|z'|^2}{4(1 + 24d\lambda_0)(\lambda' - \lambda)\delta}} \left( \frac{2\pi \lambda_0 (\lambda' - \lambda)\delta)^{d/2}}{2\pi \lambda_0 (\lambda' - \lambda)\delta)^{d/2}}. \right.
\end{equation}

Consequently, (5.14) follows from (5.18), (5.19), and (5.20).

\[\Box\]

References

[1] Bao, J., Huang, X., Yuan, C., Convergence rate of Euler–Maruyama Scheme for SDEs with Hölder–Dini continuous drifts, *J. Theoret. Probab.*, 32 (2019), 848–871.

[2] Dareiotis, K., Kumar, C., Sabanis, S., On tamed Euler approximations of SDEs driven by Lévy noise with applications to delay equations, *SIAM J. Numer. Anal.*, 54 (2016), 1840–1872.

[3] Flandoli, M., Gubinelli, M., Priola, E., Flow of diffeomorphisms for SDEs with unbounded Hölder continuous drift, *Bull. Sci. Math.*, 134 (2010), 405–422.

[4] Gottlich, S., Lux, K., Neuenkirch, A., The Euler scheme for stochastic differential equations with discontinuous drift coefficient: A numerical study of the convergence rate, arXiv:1705.04562.
[5] Guo, Q., Mao, X., Yue, R., The truncated Euler-Maruyama method for stochastic differential delay equations, Numer. Algorithms, 78 (2018), 599–624.
[6] Gyöngy, I., Martinez, T., On stochastic differential equations with locally unbounded drift, Czechoslovak Math. J., 51 (2001), 763–783.
[7] Gyöngy, I., Rásonyi, M., A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients, Stoch. Process. Appl., 121 (2011), 2189–2200.
[8] Halidias, N., Kloeden, P. E., A note on the Euler-Maruyama scheme for stochastic differential equations with a discontinuous monotone drift coefficient, BIT, 48 (2008), 51–59.
[9] Higham, D. J., Mao, X., Stuart, A. M., Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM J. Numer. Anal., 40 (2002), 1041–1063.
[10] Higham, D. J., Mao, X., Yuan, C., Almost sure and moment exponential stability in the numerical simulation of stochastic differential equations, SIAM J. Numer. Anal., 45, 592–609.
[11] Huang, X., Wang, F.-Y., Distribution Dependent SDEs with Singular Coefficients, to appear in Stoch. Process. Appl., https://doi.org/10.1016/j.spa.2018.12.012.
[12] Hutzenthaler, M., Jentzen, A., Kloeden, P. E., Strong and weak divergence in finite time of Euler’s method for stochastic differential equations with non-globally Lipschitz continuous coefficients, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci., 467 (2011), 1563–1576.
[13] Hutzenthaler, M., Jentzen, Arnulf, Kloeden, P. E., Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients, Ann. Appl. Probab., 22 (2012), 1611–1641.
[14] Jentzen, A., Müller-Gronbach, T., Yaroslavtseva, L., On stochastic differential equations with arbitrary slow convergence rates for strong approximation, Commun. Math. Sci., 14 (2016), 1477–1500.
[15] Kloeden, P. E., Platen, E., Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1992.
[16] Kloeden, P. E., Platen, E., Schurz, H., Numerical solution of SDE through computer experiments, Springer-Verlag, Berlin, 1994.
[17] Konakov, V., Mammen, E., Local limits theorems for transition densities of Markov chains converging to diffusions, Probab. Theory Relat. Fields, 117 (2000), 551-587.
[18] Krylov, N. V., Röckner, M., Strong solutions of stochastic equations with singular time dependent drift, Probab. Theory Related Fields, 131 (2005), 154–196.
[19] Lemaire, V., Menozzi, S., On some non asymptotic bounds for the Euler scheme, Electron. J. Probab., 15 (2010), 1645–1681.
[20] Leobacher, G., Szölgyenyi, M., A numerical method for SDEs with discontinuous drift, BIT, 56 (2016), 151–162.
[21] Leobacher, G., Szölgyenyi, M., A strong order 1/2 method for multidimensional SDEs with discontinuous drift, Ann. Appl. Probab., 27 (2017), 2383–2418.
[22] Leobacher, G., Szölgyenyi, M., Convergence of the Euler-Maruyama method for multidimensional SDEs with discontinuous drift and degenerate diffusion coefficient, Numer. Math., 138 (2018), 219–239.
[23] Mao, X., The truncated Euler-Maruyama method for stochastic differential equations, J. Comput. Appl. Math., 290 (2015), 370–384.
[24] Mao, X., Yuan, C., Stochastic differential equations with Markovian switching, Imperial College Press, London, 2006.
[25] Müller-Gronbach, T., Yaroslavtseva, L., On the performance of the Euler-Maruyama scheme for SDEs with discontinuous drift coefficient, arXiv:1809.08423.
[26] Neuenkirch, A., Szölgyenyi, M., Szpruch, L., An adaptive Euler-Maruyama scheme for stochastic differential equations with discontinuous drift and its convergence analysis, SIAM J. Numer. Anal., 57 (2019), 378–403.
[27] Ngo, H-L., Taguchi, D., Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients, Math. Comp., 85 (2016), 1793–1819.
[28] Ngo, H.-L., Taguchi, D., On the Euler–Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients, IMA J. Numer. Anal., 37 (2017), 1864–1883.
[29] Pamen, O.M., Taguchi, D.: Strong rate of convergence for the Euler–Maruyama approximation of SDEs with Hölder continuous drift coefficient. arXiv: 1508.07513v1

[30] Röckner, M., Zhang, X., Well-posedness of distribution dependent SDEs with singular drifts, arXiv:1809.02216.

[31] Sabanis, S., Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients, Ann. Appl. Probab., 26 (2016), 2083–2105.

[32] Shao, J., Weak convergence of Euler-Maruyama’s approximation for SDEs under integrability condition, arXiv:1808.07250.

[33] Shigekawa, I.: Stochastic Analysis, Translations of Mathematical Monographs, 224, Iwanami Series in Modern Mathematics. American Mathematical Society, Providence (2004)

[34] Xie, L., Zhang, X., Ergodicity of stochastic differential equations with jumps and singular coefficients, arXiv:1705.07402.

[35] Yan, L., The Euler scheme with irregular coefficients, Ann. Probab., 30 (2002), 1172–1194.

[36] Zhang, X., Strong solutions of SDEs with singular drift and Sobolev diffusion coefficients, Stoch. Process. Appl., 115 (2005), 1805–1818.

[37] Zhang, X., Stochastic homeomorphism flows of SDEs with singular drifts and Sobolev diffusion coefficients, Electron. J. Probab., 16 (2011), 1096–1116.

[38] Zvonkin, A. K., A transformation of the phase space of a diffusion process that removes the drift, Math. Sb., 93 (1974), 129-149.