Stochastic approach for a multivalued Dirichlet-Neumann problem

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Abstract

We prove the existence and uniqueness of a viscosity solution of the parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann-Dirichlet boundary condition:

\[
\begin{aligned}
&\partial u(t,x) \quad \frac{\partial}{\partial t} - L_t u(t,x) + \partial \varphi (u(t,x)) \ni f(t,x,u(t,x),\nabla u\sigma(t,x)), \quad t > 0, \quad x \in \mathcal{D}, \\
&\partial u(t,x) \quad \frac{\partial}{\partial n} + \partial \psi (u(t,x)) \ni g(t,x,u(t,x)), \quad t > 0, \quad x \in \partial \mathcal{D}, \\
u(0,x) = h(x), \quad x \in \mathcal{D},
\end{aligned}
\]

where \( \partial \varphi \) and \( \partial \psi \) are subdifferentials operators and \( L_t \) is a second differential operator given by

\[
L_t v(x) = \frac{1}{2} \sum_{i,j=1}^{d} (\sigma \sigma^*)_{ij}(t,x) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t,x) \frac{\partial v(x)}{\partial x_i}.
\]

The result is obtained by a stochastic approach. First we study the following backward stochastic generalized variational inequality:

\[
\begin{aligned}
&dY_t + F(t,Y_t,Z_t) \, dt + G(t,Y_t) \, dA_t \in \partial \varphi (Y_t) \, dt + \partial \psi (Y_t) \, dA_t + Z_t dW_t, \quad 0 \leq t \leq T, \\
&Y_T = \xi,
\end{aligned}
\]

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where \((A_t)_{t \geq 0}\) is a continuous one-dimensional increasing measurable process, and then we obtain a Feynman-Kač representation formula for the viscosity solution of the PVI problem.

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**Keywords:** Variational inequalities; Backward stochastic differential equations; Neumann-Dirichlet boundary conditions; Viscosity solutions; Feynman-Kač formula.

## 1 Introduction

Viscosity solutions were introduced by M.G. Crandall and P.L. Lions in [1], and then developed in the classical work of M.G. Crandall, H. Ishii, P.L. Lions [2], where are presented several equivalent ways to formulate the notion of such type solutions. The framework of this theory allows for merely continuous functions to be the solutions of fully nonlinear equations of second order which provides a very general existence and uniqueness theorems.

In 1992 E. Pardoux and S. Peng [10] introduced backward stochastic differential equations (BSDE) and supplied probabilistic formulas for the viscosity solutions of semilinear partial differential equations, both of parabolic and elliptic type in whole space. Elliptic equations with Dirichlet boundary condition have been treated by R.W.R. Darling and E. Pardoux in [3] and with a homogeneous Neumann boundary condition by Y. Hu in [4].

The parabolic (and elliptic) systems of partial differential equations (PVI without the subdifferential operator) with nonlinear Neumann boundary conditions was the subject of the paper E. Pardoux and S. Zhang [11]. The case of the systems of variational inequalities for partial differential equations in whole space was studied by L. Maticiu, E. Pardoux, A. Răşcanu and A. Zălinescu in [7].

The main idea for proving the existence of the viscosity solutions for PDE and PVI is the stochastic approach. Using a suitable BSDE, or backward stochastic variational inequality (BSVI) for the PVI case, one can obtain a generalizations of the Feynman-Kač formula (i.e. stochastic representation formula of the viscosity solution for deterministic problems).

The origin of our study goes from the PDE

\[
\begin{align*}
\frac{\partial u}{\partial t} - \mathcal{L}_t u &= f, \quad t > 0, \quad x \in \mathcal{D}, \\
\frac{\partial u}{\partial n} &= g, \quad t > 0, \quad x \in \partial \mathcal{D}, \\
u(0, x) &= h(x), \quad x \in \overline{\mathcal{D}},
\end{align*}
\]

which is a mathematical model for the evolution of a state \(u(t, x) \in \mathbb{R}\) of a diffusion dynamical system with sources \(f\) acting in the interior of the domain \(\mathcal{D}\) and \(g\) on the boundary of \(\mathcal{D}\).
In certain applications it is call upon to maintain the state \( u(t, x) \) in a interval \( \mathbb{I} \subset \mathbb{R} \) for all \( x \in \mathcal{D} \) and in a interval \( \mathbb{J} \subset \mathbb{R} \) for all \( x \in Bd(\mathcal{D}) \). Practically these can be realized adding the supplementary sources \( \partial \mathbb{I} (u(t, x)) \) and \( \partial \mathbb{J} (u(t, x)) \) on the system. These sources produce “inward pushes” that would keep the state process \( u(t, x) \) in \( \mathbb{I} \), \( \forall x \in \mathcal{D} \) and \( u(t, x) \) in \( \mathbb{J} \), \( \forall x \in \text{Bd}(\mathcal{D}) \) and do this in a minimal way (i.e. only when \( u(t, x) \) arrives on the boundary of \( \mathbb{I} \) and respectively \( \mathbb{J} \)). Hence \( \partial \mathbb{I} (u(t, x)) \) and \( \partial \mathbb{J} (u(t, x)) \) represent perfect feedback flux controls.

The aim of this paper is to treat the more general case of a parabolic variational inequality with mixed nonlinear multivalued Neumann-Dirichlet boundary condition. This requires the presence of a new terms in the associated BSVI considered, namely an integral with respect to a continuous increasing process.

The scalar BSDE with one-sided reflection, which provides a probabilistic representation for the unique viscosity solution of an obstacle problem for a nonlinear parabolic PDE, was considered by N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, M.C. Quenez in [7]. E. Pardoux and A. Răşcanu in [8] (and [9] for the generalization to the Hilbert spaces framework) studied the general case of BSVI and obtained probabilistic representation for the solutions of PVI in whole space.

The paper is organized as follows: In Section 2 we formulate the Neumann-Dirichlet PVI problem; we present the main results and we prove the uniqueness theorem. For the existence theorem we first study in Section 3 a certain BSVI. The solution of this backward equation gives us, via Feynman-Kač representation formula, a viscosity solution for the deterministic multivalued partial differential equation as it is shown in Section 4.

### 2 Main results

Let \( \mathcal{D} \) be a open connected bounded subset of \( \mathbb{R}^d \) of form

\[
\mathcal{D} = \{ x \in \mathbb{R}^d : \ell(x) < 0 \}, \quad Bd(\mathcal{D}) = \{ x \in \mathbb{R}^d : \ell(x) = 0 \},
\]

where \( \ell \in C^3_0(\mathbb{R}^d) \), \( |\nabla \ell(x)| = 1 \), for all \( x \in Bd(\mathcal{D}) \).

We define outward normal derivative by

\[
\frac{\partial v(x)}{\partial n} = \sum_{j=1}^{d} \frac{\partial \ell(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j} = \langle \nabla \ell(x), \nabla v(x) \rangle, \quad \text{for all } x \in Bd(\mathcal{D}).
\]

The aim of this paper is to study the existence and uniqueness of a viscosity solution for the following parabolic variational inequality (PVI) with a mixed nonlinear multivalued Neumann-Dirichlet boundary condition:
\[
\begin{aligned}
&\frac{\partial u(t, x)}{\partial t} - L_t u(t, x) + \partial \varphi(u(t, x)) \geq f(t, x, u(t, x), (\nabla u)\sigma(t, x)), \\
&\quad t > 0, x \in D, \\
&\frac{\partial u(t, x)}{\partial n} + \partial \psi(u(t, x)) \geq g(t, x, u(t, x)), \quad t > 0, x \in Bd(D), \\
&u(0, x) = h(x), \quad x \in \overline{D},
\end{aligned}
\]

where operator \( L_t \) is given by
\[
L_t v(x) = \frac{1}{2} \text{Tr} \left[ \sigma(t, x)\sigma^*(t, x)D^2v(x) \right] + \left( b(t, x), \nabla v(x) \right)
\]
\[
= \frac{1}{2} \sum_{i,j=1}^{d} (\sigma\sigma^*)_{ij}(t, x) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(t, x) \frac{\partial v(x)}{\partial x_i}.
\]

for \( v \in C^2(R^d) \).

We will make the following assumptions:

(1) Functions
\[
\begin{align*}
b &: [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^d, \\
\sigma &: [0, \infty) \times \mathbb{R}^d \to \mathbb{R}^{d \times d}, \\
f &: [0, \infty) \times \overline{D} \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}, \\
g &: [0, \infty) \times Bd(D) \times \mathbb{R} \to \mathbb{R}, \\
h &: \overline{D} \to \mathbb{R}
\end{align*}
\]

We assume that for all \( T > 0 \) there exist \( \alpha \in \mathbb{R} \) and \( L, \beta, \gamma \geq 0 \) (which can depend on \( T \)) such that \( \forall t \in [0, T] \), \( \forall x, \tilde{x} \in \mathbb{R}^d \):
\[
\left| b(t, x) - b(t, \tilde{x}) \right| + \left| \sigma(t, x) - \sigma(t, \tilde{x}) \right| \leq L \| x - \tilde{x} \|,
\]
and \( \forall t \in [0, T] \), \( \forall x \in \overline{D} \), \( u \in Bd(D) \), \( y, \tilde{y}, \in \mathbb{R} \), \( z, \tilde{z} \in \mathbb{R}^d \):
\[
(i) \quad (y - \tilde{y}) \left( f(t, x, y, z) - f(t, x, \tilde{y}, \tilde{z}) \right) \leq \alpha \| y - \tilde{y} \|^2,
\]
\[
(ii) \quad \left| f(t, x, y, z) - f(t, x, y, \tilde{z}) \right| \leq \beta |z - \tilde{z}|,
\]
\[
(iii) \quad \left| f(t, x, y, 0) \right| \leq \gamma (1 + |y|),
\]
\[
(iv) \quad (y - \tilde{y}) \left( g(t, u, y) - g(t, u, \tilde{y}) \right) \leq \alpha \| y - \tilde{y} \|^2,
\]
\[
(v) \quad |g(t, u, y)| \leq \gamma (1 + |y|).
\]

In fact, condition (1i) and (1iv) mean that, for all \( t \in [0, T] \), \( x \in \overline{D} \), \( u \in Bd(D) \), \( z \in \mathbb{R}^d \):
\[
y \mapsto \alpha y - f(t, x, y, z) : \mathbb{R} \to \mathbb{R}
\]
\[
y \mapsto \alpha y - g(t, u, y) : \mathbb{R} \to \mathbb{R}
\]
are increasing functions.
(II) With respect to functions $\varphi$ and $\psi$ we assume

$$
\begin{align*}
(i) & \quad \varphi, \psi : \mathbb{R} \to (-\infty, +\infty] \text{ are proper convex l.s.c. functions}, \\
(ii) & \quad \varphi(y) \geq \varphi(0) = 0 \text{ and } \psi(y) \geq \psi(0) = 0, \forall y \in \mathbb{R},
\end{align*}
$$

and there exists a positive constant $M$ such that

$$
\begin{align*}
(i) & \quad |\varphi(h(x))| \leq M, \forall x \in \overline{D}, \\
(ii) & \quad |\psi(h(x))| \leq M, \forall x \in \text{Bd}(D).
\end{align*}
$$

Remark 1 Condition (5-ii) is generally realized by changing problem (1) in an equivalent form, as example: if $(u_0, u_0^*) \in \partial \varphi$ we can replace $\varphi(u)$ by $\varphi(u + u_0) - \varphi(u_0) - \langle u_0^*, u \rangle$; a similar transformation one can do for $\psi$.

We denote

$$
\begin{align*}
\text{Dom} (\varphi) & = \{ u \in \mathbb{R} : \varphi(u) < \infty\}, \\
\partial \varphi (u) & = \{ u^* \in \mathbb{R} : u^* (v - u) + \varphi(u) \leq \varphi(v), \forall v \in \mathbb{R} \}, \\
\text{Dom} (\partial \varphi) & = \{ u \in \mathbb{R} : \partial \varphi(u) \neq \emptyset \}, \\
(u, u^*) & \in \partial \varphi \iff u \in \text{Dom} \partial \varphi, \ u^* \in \partial \varphi(u)
\end{align*}
$$

(for function $\psi$ we have the similar notations).

In every point $y \in \text{Dom} (\varphi)$ we have

$$
\partial \varphi(y) = \mathbb{R} \cap [\varphi'_-(y), \varphi'_+(y)],
$$

where $\varphi'_-(y)$ and $\varphi'_+(y)$ are left derivative and, respectively, right derivative at point $y$.

(III) We introduce the compatibility assumptions:

for all $\varepsilon > 0$, $t \geq 0$, $x \in \text{Bd}(D)$, $x_0 \in \overline{D}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$

$$
\begin{align*}
(i) & \quad \nabla \varphi_\varepsilon(y) g(t, x, y) \leq [\nabla \psi_\varepsilon(y) g(t, x, y)]^+, \\
(ii) & \quad \nabla \psi_\varepsilon(y) f(t, x, y, z) \leq [\nabla \varphi_\varepsilon(y) f(t, x, y, z)]^+.
\end{align*}
$$

where $a^+ = \max \{0, a\}$ and $\nabla \varphi_\varepsilon(y)$, $\nabla \psi_\varepsilon(y)$ are unique solutions $U$ and $V$, respectively, of equations

$$
\partial \varphi(y - \varepsilon U) \ni U \quad \text{and} \quad \partial \psi(y - \varepsilon V) \ni V.
$$
Remark 2  
A) Clearly, using the monotonicity of $\nabla \varphi_\varepsilon, \nabla \psi_\varepsilon$, we see that, if
$$y \cdot g(t, x, y) \leq 0 \text{ and } y \cdot f(t, \tilde{x}, y, z) \leq 0$$
for all $t \geq 0$, $x \in Bd(D)$, $\tilde{x} \in \overline{D}$, $y \in \mathbb{R}$ and $z \in \mathbb{R}^d$, then compatibility assumptions (7) are satisfied.

B) If $\varphi, \psi : \mathbb{R} \to (-\infty, +\infty]$ are convex indicator functions
$$\varphi(y) = \begin{cases} 0, & \text{if } y \in [a, \infty) \\ +\infty, & \text{if } y \notin [a, \infty) \end{cases}$$
and
$$\psi(y) = \begin{cases} 0, & \text{if } y \in (-\infty, b] \\ +\infty, & \text{if } y \notin (-\infty, b] \end{cases}$$
where $a \leq 0 \leq b$, then
$$\nabla \varphi_\varepsilon(y) = -\frac{1}{\varepsilon}(y - a)^- \quad \text{and} \quad \nabla \psi_\varepsilon(y) = \frac{1}{\varepsilon}(y - b)^+$$
and the compatibility assumptions become
$$g(t, x, y) \geq 0, \quad \text{for } y \leq a, \text{ and}$$
$$f(t, \tilde{x}, y, z) \leq 0, \quad \text{for } y \geq b.$$

We shall define now the notion of viscosity solution in the language of sub- and super-jets, see [2]. $\mathbb{S}\mathbb{R}^{d \times d}$ will denote below the set of $d \times d$ symmetric non-negative real matrices.

Definition 3 Let $u : [0, \infty) \times \overline{D} \to \mathbb{R}$ a continuous function, and $(t, x) \in [0, \infty) \times \overline{D}$. We denote by $\mathcal{P}^{2,+} u(t, x)$ (the parabolic superjet of $u$ at $(t, x)$) the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}\mathbb{R}^{d \times d}$ which are such that for all $(s, y) \in [0, \infty) \times \overline{D}$ in a neighbourhood of $(t, x)$:
$$u(s, y) \leq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2).$$

Similarly is defined $\mathcal{P}^{2,-} u(t, x)$ (the parabolic subjet of $u$ at $(t, x)$) as the set of triples $(p, q, X) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}\mathbb{R}^{d \times d}$ which are such that for all $(s, y) \in [0, \infty) \times \overline{D}$ in a neighbourhood of $(t, x)$:
$$u(s, y) \geq u(t, x) + p(s - t) + \langle q, y - x \rangle + \frac{1}{2} \langle X(y - x), y - x \rangle + o(|s - t| + |y - x|^2),$$
where $r \to o(r)$ is the Landau function i.e. $o : [0, \infty] \to \mathbb{R}$ is a continuous function such that $\lim_{r \to 0} \frac{o(r)}{r} = 0$. 


We can give now the definition of a viscosity solution of the parabolic variational inequality (1). We denote first
\[
V(t, x, p, q, X) \overset{\text{def}}{=} p - \frac{1}{2} \text{Tr}((\sigma\sigma^*)(t, x)X) - \langle b(t, x), q \rangle - f(t, x, u(t, x), q\sigma(t, x)).
\]

**Definition 4** Let \( u : [0, \infty) \times \overline{D} \rightarrow \mathbb{R} \) a continuous function, which satisfies \( u(0, x) = h(x), \forall \ x \in \overline{D} \),

(a) \( u \) is a viscosity subsolution of (1) if:
\[
\begin{align*}
&\quad u(t, x) \in \text{Dom}(\varphi), \quad \forall (t, x) \in (0, \infty) \times \overline{D}, \\
&\quad u(t, x) \in \text{Dom}(\psi), \quad \forall (t, x) \in (0, \infty) \times \text{Bd}(D), \\
\end{align*}
\]
and, at any point \((t, x) \in (0, \infty) \times \overline{D}, \) for any \((p, q, X) \in \mathcal{P}^2_+ u(t, x)\):
\[
\min \left\{ V(t, x, p, q, X) + \varphi_+(u(t, x)), \langle \nabla\ell(x), q \rangle - g(t, x, u(t, x)) + \psi_+(u(t, x)) \right\} \leq 0 \quad \text{if} \ x \in \text{Bd}(D).
\]

(b) the viscosity supersolution of (1) is defined in a similar manner as above, with \( \mathcal{P}^2_+ \) replaced by \( \mathcal{P}^2_- \), the left derivative replaced by the right derivative, \( \min \) by \( \max \), and the inequalities \( \leq \) by \( \geq \).

(c) a continuous function \( u : [0, \infty) \times \overline{D} \) is a viscosity solution of (1) if it is both a viscosity sub- and super-solution.

We now present the main results

**Theorem 5 (Existence)** Let assumptions (2)-(7) be satisfied. Then PVI (1) has a viscosity solution.

For the proof of the existence we shall study a certain backward stochastic generalized variational inequality (then we use a nonlinear representation Feynman-Kač type formula). We present this approach in the following section and after then the proof of Theorem 5 in Section 4.

**Theorem 6 (Uniqueness)** Let the assumptions of Theorem 5 be satisfied. If function \( r \rightarrow g(t, x, r) \) is decreasing for \( t \geq 0, \ x \in \text{Bd}(D), \)
and there exists a continuous function \( m : [0, \infty) \rightarrow [0, \infty), \ m(0) = 0, \) such that
\[
|f(t, x, r, p) - f(t, y, r, p)| \leq m(|x - y| (1 + |p|)),
\]
\( \forall \ t \geq 0, \ x, y \in \overline{D}, \ p \in \mathbb{R}^d, \)
then the viscosity solution is unique.
Proof. It is sufficient to prove the uniqueness on a fixed arbitrary interval \([0, T]\).

Also, it suffices to prove that if \(u\) is a subsolution and \(v\) is a supersolution such that \(u(0, x) = v(0, x) = h(x), x \in \mathcal{D}\), then \(u \leq v\).

Firstly, from definition of \(\mathcal{D}\), there exists a function \(\tilde{\ell} \in C^3_0(\mathbb{R}^d)\) such that \(\tilde{\ell}(x) \geq 0\) on \(\mathcal{D}\) with \(\nabla \tilde{\ell}(x) = \nabla \ell(x)\) for \(x \in \text{Bd}(\mathcal{D})\) (as example \(\tilde{\ell}(x) = \ell(x) + \sup_{y \in \mathcal{D}} |\ell(y)|\)).

For \(\lambda = |\alpha| + 1\) and \(\delta, \varepsilon, c > 0\) let
\[
\bar{u}(t, x) = e^{\lambda t}u(t, x) - \delta \tilde{\ell}(x) - c
\]
\[
\bar{v}(t, x) = e^{\lambda t}v(t, x) + \delta \tilde{\ell}(x) + c + \varepsilon /t.
\]
Denote
\[
\bar{f}(t, x, r, q, X) = \lambda r - \frac{1}{2} \text{Tr}\left[(\sigma \sigma^*) (t, x)X\right] - \langle b(t, x), q \rangle
\]
\[
- e^{\lambda t}f(t, x, e^{-\lambda t}r, e^{-\lambda t}q \sigma (t, x))
\]
and
\[
\bar{g}(t, x, r) = e^{\lambda t}g(t, x, e^{-\lambda t}r)
\]
Clearly \(r \rightarrow \bar{f}(t, x, r, q, X)\) is an increasing function for all \((t, x, q, X) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^{d \times d}\). Moreover, since
\[
M = \sup_{(t, x) \in [0, T] \times \mathcal{D}} \left\{ |\tilde{\ell}(x)| + |D\tilde{\ell}(x)| + |D^2\tilde{\ell}(x)| + |b(t, x)| + |\sigma(t, x)| \right\} < \infty,
\]
then we can choose \(c = c(\delta, M) > 0\) such that for \(\bar{u} = \bar{u}(t, x)\) and \(\tilde{\ell} = \tilde{\ell}(x)\):
\[
\bar{f}(t, x, \bar{u}, D\bar{u}, D^2\bar{u}) \leq \bar{f}(t, x, \bar{u} + \delta \tilde{\ell} + c, D\bar{u} + \delta D\tilde{\ell}, D^2\bar{u} + \delta D^2\tilde{\ell}).
\]
Using these properties, assumption \([9]\), and the fact that left and right derivative of \(\varphi, \psi\) are increasing we infer that function \(\bar{u}\) satisfy in the viscosity sense
\[
\min \left\{ \frac{\partial \bar{u}}{\partial t} (t, x) + \bar{f}(t, x, \bar{u}(t, x), D\bar{u}(t, x), D^2\bar{u}(t, x)) + e^{\lambda t} \varphi_+(e^{-\lambda t} \bar{u}(t, x)) \leq 0 \text{ if } x \in \mathcal{D}
\]
\[
\frac{\partial \bar{u}}{\partial t} (t, x) + \bar{f}(t, x, \bar{u}(t, x), D\bar{u}(t, x), D^2\bar{u}(t, x)) + e^{\lambda t} \varphi_+(e^{-\lambda t} \bar{u}(t, x)) + \langle \nabla \tilde{\ell}(x), D\bar{u}(t, x) \rangle + \delta
\]
\[
- \bar{g}(t, x, \bar{u}(t, x)) + e^{\lambda t} \psi_-(e^{-\lambda t} \bar{u}(t, x)) \leq 0 \text{ if } x \in \text{Bd}(\mathcal{D}).
\]
Analogously we see that \(\bar{v}\) satisfy in the viscosity sense:
\[
\frac{\partial \tilde{v}}{\partial t}(t, x) + \tilde{f}(t, x, \tilde{v}(t, x), D\tilde{v}(t, x), D^2\tilde{v}(t, x)) + e^{\lambda t} \varphi'_+(e^{-\lambda t}\tilde{v}(t, x)) - \varepsilon/t^2 \geq 0 \quad \text{if } x \in \mathcal{D}
\]
\[
\max \left\{ \frac{\partial \tilde{v}}{\partial t}(t, x) + \tilde{f}(t, x, \tilde{v}(t, x), D\tilde{v}(t, x), D^2\tilde{v}(t, x)) + e^{\lambda t} \varphi'_+(e^{-\lambda t}\tilde{v}(t, x)) - \varepsilon/t^2, \langle \nabla \tilde{\ell}(x), D\tilde{v}(t, x) \rangle - \delta \right\} 
\geq 0 \quad \text{if } x \in Bd(\mathcal{D}),
\]

For simplicity of notation we continue to write \( u, v \) for \( \tilde{u}, \tilde{v} \) respectively.

We assume now, to the contrary, that

\[
\max_{[0,T] \times \overline{\mathcal{D}}} (u - v)^+ > 0.
\]

Exactly as in Theorem 4.2 in [8] we have \((\hat{t}, \hat{x}) \in [0, T] \times Bd(\mathcal{D})\) where \((\hat{t}, \hat{x})\) is the maximum point, i.e.

\[
u(\hat{t}, \hat{x}) - v(\hat{t}, \hat{x}) = \max_{[0,T] \times \overline{\mathcal{D}}} (u - v)^+ > 0.
\]

We put now (see also the proof of the Theorem 7.5 in Crandall, Ishii, Lions [2])

\[
\Phi_n(t, x, y) = u(t, x) - v(t, y) - \rho_n(t, x, y), \quad \text{with } (t, x, y) \in [0, T] \times \overline{\mathcal{D}} \times \overline{\mathcal{D}},
\]

where

\[
\rho_n(t, x, y) = \frac{n}{2} |x - y|^2 + \tilde{g}(\hat{t}, \hat{x}, u(\hat{t}, \hat{x})) \langle \nabla \tilde{\ell}(\hat{x}), x - y \rangle + |x - \hat{x}|^4 + |t - \hat{t}|^4 - e^{\lambda t} \psi'_-(e^{-\lambda t}u(\hat{t}, \hat{x})) \langle \nabla \tilde{\ell}(\hat{x}), x - y \rangle.
\]

Let it be \((t_n, x_n, y_n)\) a maximum point of \(\Phi_n\).

We observe that \(u(t, x) - v(t, x) - |x - \hat{x}|^4 - |t - \hat{t}|^4\) has in \((\hat{t}, \hat{x})\) a unique maximum point. Then, by Proposition 3.7 in Crandall, Ishii, Lions [2], we have, as \(n \to \infty\)

\[
t_n \to \hat{t}, \quad x_n \to \hat{x}, \quad y_n \to \hat{x}, \quad n |x_n - y_n|^2 \to 0, \quad u(t_n, x_n) \to u(\hat{t}, \hat{x}), \quad v(t_n, x_n) \to v(\hat{t}, \hat{x}).
\]

But domain \(\mathcal{D}\) verify the uniform exterior sphere condition:

\[
\exists r_0 > 0 \text{ such that } S(x + r_0 \nabla \tilde{\ell}(x), r_0) \cap \mathcal{D} = \emptyset, \quad \text{for } x \in Bd(\mathcal{D})
\]

where \(S(x, r_0)\) denotes the closed ball of radius \(r_0\) centered at \(x\).

Then

\[
|y - x - r_0 \nabla \tilde{\ell}(x)|^2 > r_0^2, \quad \text{for } x \in Bd(\mathcal{D}), \quad y \in \overline{\mathcal{D}},
\]

\[9\]
or equivalent
\[
\langle \nabla \tilde{\ell}(x), y - x \rangle < \frac{1}{2r_0} |y - x|^2 \text{ for } x \in Bd(D), \ y \in \overline{D}.
\] (17)

If we denote
\[
B(t, x, r, q) = \langle \nabla \tilde{\ell}(x), q \rangle - \tilde{g}(t, x, r)
\]
then, if \( x_n \in Bd(D) \), we have, using the form of \( \rho_n \) given by (15) and (17), that
\[
B(t_n, x_n, u(t_n, x_n), D_x\rho_n(t_n, x_n, y_n)) = B(t_n, x_n, u(t_n, x_n), n(x_n - y_n) + \tilde{g}(t, x_n, n(t_n, x_n)) + 4|x_n - \hat{x}|^2(x_n - \hat{x}) - e^{\lambda t}y_\rho'(e^{-\lambda t}u(t, \hat{x}))\nabla \tilde{\ell}(\hat{x})
\]
\[
\geq -\frac{n}{2r_0} |x_n - y_n|^2 + \tilde{g}(t, \hat{x}, u(t, \hat{x}))\langle \nabla \tilde{\ell}(\hat{x}), \nabla \tilde{\ell}(x_n) \rangle - 4|x_n - \hat{x}|^2\langle \nabla \tilde{\ell}(x_n), x_n - \hat{x} \rangle - e^{\lambda t}y_\rho'(e^{-\lambda t}u(t, \hat{x})\langle \nabla \tilde{\ell}(\hat{x}), \nabla \tilde{\ell}(x_n) \rangle
\]

Then (16) implies for \( x_n \in Bd(D) \):
\[
\liminf_{n \to \infty} \left[ B(t_n, x_n, u(t_n, x_n), D_x\rho_n(t_n, x_n, y_n)) + \delta + e^{\lambda t}y_\rho'(e^{-\lambda t}u(t_n, x_n)) \right] > 0
\]

Analogously if \( y_n \in Bd(D) \) we infer
\[
\limsup_{n \to \infty} \left[ B(t_n, y_n, v(t_n, y_n), -D_y\rho_n(t_n, x_n, y_n)) - \delta + e^{\lambda t}y_\rho'(e^{-\lambda t}v(t_n, x_n)) \right] < 0.
\]

Then from (12), (13) we conclude that
\[
p + \tilde{f}(t_n, x_n, u(t_n, x_n), D_x\rho_n(t_n, x_n, y_n), X) + e^{\lambda t}y_\rho'(e^{-\lambda t}u(t_n, x_n)) \leq 0,
\]
for \((p, D_x\rho_n(t_n, x_n, y_n), X) \in \overline{\mathcal{D}}^2 u(t_n, x_n)\) (18)

and
\[
p + \tilde{f}(t_n, y_n, v(t_n, y_n), -D_y\rho_n(t_n, x_n, y_n), Y)
+ e^{\lambda t}y_\rho'(e^{-\lambda t}v(t_n, y_n)) \geq \frac{\varepsilon}{\ell^2},
\]
for \((p, -D_y\rho_n(t_n, x_n, y_n), Y) \in \overline{\mathcal{D}}^2 v(t_n, y_n)\) (19)

From Theorem 8.3 in Crandall, Ishii, Lions [2] (apply with \( k = 2, \mathcal{O}_1 = \mathcal{O}_2 = \overline{D}, u_1 = u, u_2 = -v, b_1 = p, b_2 = -p \)) we deduce that there exists
\[
(p, X, Y) \in \mathbb{R} \times \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d},
\]
such that
\[ (p, D_x \rho_n(t_n, x_n, y_n), X) \in \overline{P}^{2+} u(t_n, x_n) \]
\[ (p, -D_y \rho_n(t_n, x_n, y_n), Y) \in \overline{P}^{2-} v(t_n, y_n) \]
and
\[ - (n + ||A||) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \frac{1}{n} A^2, \tag{20} \]
where \( A = D_{x,y}^2 \rho_n(t_n, x_n, y_n) \). From (15) we have
\[ A = n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + O(|x_n - \hat{x}|^2), \]
\[ A^2 = 2n^2 \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + O(n |x_n - \hat{x}|^2 + |x_n - \hat{x}|^4), \]
where \( |O(h)| \leq C |h| \) (the Landau symbol). Then (20) become
\[ - (3n + \kappa_n) \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq 3n \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \kappa_n \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{21} \]
where \( \kappa_n \to 0 \). Now from (18) and (19)
\[ \frac{\varepsilon}{t^2} \leq \tilde{f}(t_n, y_n, v(t_n, y_n), -D_y \rho_n(t_n, x_n, y_n), Y) + e^{\lambda_n} \varphi'_+(e^{-\lambda_n} v(t_n, y_n)) \]
\[ - \tilde{f}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n), X) - e^{\lambda_n} \varphi'_-(e^{-\lambda_n} u(t_n, x_n)) \]
By (14) and (16) there exists \( N \geq 1 \) such that
\[ u(t_n, x_n) > v(t_n, y_n), \quad \forall \ n \geq N \]
and consequently
\[ e^{\lambda_n} \varphi'_+(e^{-\lambda_n} u(t_n, x_n)) \geq e^{\lambda_n} \varphi'_+(e^{-\lambda_n} v(t_n, y_n)) \]
and
\[ \tilde{f}(t_n, y_n, u(t_n, x_n), -D_y \rho_n(t_n, x_n, y_n), Y) \geq \tilde{f}(t_n, y_n, v(t_n, y_n), -D_y \rho_n(t_n, x_n, y_n), Y) \]
Then, by definition (11) of \( \tilde{f} \) and assumption (10), we have
\[ \frac{\varepsilon}{t^2} \leq \liminf_{n \to +\infty} \left[ \tilde{f}(t_n, y_n, u(t_n, x_n), -D_y \rho_n(t_n, x_n, y_n), Y) \right. \]
\[ \left. - \tilde{f}(t_n, x_n, u(t_n, x_n), D_x \rho_n(t_n, x_n, y_n), X) \right] \]
\[ \leq \frac{1}{2} \text{Tr} \left[ (\sigma\sigma^*)(t_n, x_n)X - (\sigma\sigma^*)(t_n, y_n)Y \right] \]
But from (21), \( \forall q, \tilde{q} \in \mathbb{R}^d \),

\[
\langle Xq, q \rangle - \langle Y\tilde{q}, \tilde{q} \rangle \leq 3n |q - \tilde{q}|^2 + (|q|^2 + |\tilde{q}|^2)\kappa_n.
\]

Hence

\[
\text{Tr} \left[ (\sigma^*)^T (t_n, x_n)X - (\sigma^*)^T (t_n, y_n)Y \right] \\
= \sum_{i=1}^d (X\sigma(t_n, x_n)e_i, \sigma(t_n, x_n)e_i) - (Y\sigma(t_n, y_n)e_i, \sigma(t_n, y_n)e_i) \\
\leq 3C n |x_n - y_n|^2 + (|\sigma(t_n, x_n)|^2 + |\sigma(t_n, y_n)|^2)\kappa_n,
\]

and consequently

\[
\frac{\varepsilon}{t^2} \leq 0
\]

that is a contradiction.

Then

\[ u(t, x) \leq v(t, x), \; \forall (t, x) \in [0, T] \times \overline{D}. \]

\[ \square \]

3 Backward stochastic variational inequalities

Let \( \{W_t : t \geq 0\} \) be a \( d \)-dimensional standard Brownian motion defined on some complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\). We denote by \( \{\mathcal{F}_t : t \geq 0\} \) the natural filtration generated by \( \{W_t : t \geq 0\} \) and augmented by \( \mathcal{N} \), the set of \( \mathbb{P} \)-null events of \( \mathcal{F} \):

\[ \mathcal{F}_t = \sigma\{W_r : 0 \leq r \leq t\} \cup \mathcal{N}. \]

Let \( \tau : \Omega \rightarrow [0, \infty) \) be an a.s. \( \mathcal{F}_t \)-stopping time and

\[ \{A_t : t \geq 0\} \text{ be a continuous one-dimensional increasing progressively measurable stochastic process (p.m.s.p.) satisfying } A_0 = 0. \]

We shall study the existence and uniqueness of a solution \((Y, Z)\) of the following backward stochastic variational inequality (BSVI) :

\[
\begin{cases}
  dY_t + F(t, Y_t, Z_t) \, dt + G(t, Y_t) \, dA_t \in \partial \phi (Y_t) \, dt + \partial \psi (Y_t) \, dA_t + Z_t \, dW_t, \\
  Y_\tau = \xi,
\end{cases}
\tag{22}
\]
3.1 Assumptions and results

Let $\lambda, \mu \geq 0$.

Let

$$
\mathcal{H}_k^{\lambda, \mu} \subset L^2(\mathbb{R}_+ \times \Omega, e^{\lambda s + \mu A_s} \mathbf{1}_{[0, \tau]} (s) \, ds \otimes d\mathbb{P}; \mathbb{R}^k)
$$

the Hilbert space of p.m.s.p. $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}^k$ such that

$$
\|f\|_{\mathcal{H}} = \left[ \mathbb{E} \left( \int_0^\tau e^{\lambda s + \mu A_s} |f(s)|^2 \, ds \right) \right]^{1/2} < \infty
$$

and

$$
\tilde{\mathcal{H}}_k^{\lambda, \mu} \subset L^2(\mathbb{R}_+ \times \Omega, e^{\lambda s + \mu A_s} \mathbf{1}_{[0, \tau]} (s) \, dA_s \otimes d\mathbb{P}; \mathbb{R}^k)
$$

the Hilbert space of p.m.s.p. $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}^k$ such that

$$
\|f\|_{\tilde{\mathcal{H}}} = \left[ \mathbb{E} \left( \int_0^\tau e^{\lambda s + \mu A_s} |f(s)|^2 \, dA_s \right) \right]^{1/2} < \infty.
$$

We also introduce the notation $\mathcal{S}_k^{\lambda, \mu}$ for the Banach space of p.m.s.p. $f : \Omega \times [0, \infty) \rightarrow \mathbb{R}^k$ such that

$$
\|f\|_{\mathcal{S}} = \left[ \mathbb{E} \left( \sup_{0 \leq t \leq \tau} e^{\lambda t + \mu A_t} |f(t)|^2 \right) \right]^{1/2} < \infty.
$$

With respect to BSVI (22) we formulate the following assumptions:

- Let $F : \Omega \times [0, \infty) \times \mathbb{R}^k \times \mathbb{R}^{k \times d} \rightarrow \mathbb{R}^k$, $G : \Omega \times [0, \infty) \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ satisfy that there exist $\alpha, \beta \in \mathbb{R}$, $L \geq 0$ and $\eta, \gamma : [0, \infty) \times \Omega \rightarrow [0, \infty)$ an p.m.s.p. such that for all $t \geq 0$, $y, y' \in \mathbb{R}^k$, $z, z' \in \mathbb{R}^{k \times d}$:

  (i) $F(\cdot, \cdot, y, z)$ is p.m.s.p.,

  (ii) $y \rightarrow F(\omega, t, y, z) : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is continuous, a.s.

  (iii) $\langle y - y', F(t, y, z) - F(t, y', z) \rangle \leq \alpha |y - y'|^2$, a.s. \hspace{1cm} (23)

  (iv) $|F(t, y, z) - F(t, y, z')| \leq L \|z - z'\|$, a.s.

  (v) $|F(t, y, z)| \leq \eta_t + L \left( |y| + \|z\| \right)$, a.s.

and

  (i) $G(\cdot, \cdot, y)$ is p.m.s.p.,

  (ii) $y \rightarrow G(\omega, t, y) : \mathbb{R}^k \rightarrow \mathbb{R}^k$ is continuous, a.s.

  (iii) $\langle y - y', G(t, y) - G(t, y') \rangle \leq \beta |y - y'|^2$, a.s. \hspace{1cm} (24)

  (iv) $|G(t, y)| \leq \gamma_t + L |y|$, a.s.
• Terminal date $\xi$ is an $\mathbb{R}^k$-valued $\mathcal{F}_\tau$-measurable random variable such that there exists $\lambda > 2\alpha + 2L^2 + 1$, $\mu > 2\beta + 1$:

$$M(\tau) \overset{\text{def}}{=} \mathbb{E}e^{\lambda\tau + \mu A_\tau} \left( |\xi|^2 + \varphi(\xi) + \psi(\xi) \right) + \mathbb{E} \int_0^\tau e^{\lambda s + \mu A_s} [ |\eta_s|^2 + |\gamma_s|^2 dA_s] < \infty. \quad (25)$$

• Let it be $\varphi, \psi$ such that

$$(i) \quad \varphi, \psi : \mathbb{R}^k \to (-\infty, +\infty] \text{ are proper convex l.s.c. functions},$$

$$(ii) \quad \varphi(y) \geq \varphi(0) = 0, \psi(y) \geq \psi(0) = 0,$$

The subdifferentials are defined by

$$\partial \varphi(x) = \{ v \in \mathbb{R}^k : \langle v, y - x \rangle + \varphi(x) \leq \varphi(y), \forall y \in \mathbb{R}^k \}$$

and similar for $\psi$.

The existence result for (22) will be obtained via Yosida approximations. Define for $\varepsilon > 0$ the convex $C^1$-function $\varphi_\varepsilon$ by

$$\varphi_\varepsilon(y) = \inf \left\{ \frac{1}{2\varepsilon} |y - v|^2 + \varphi(v) : v \in \mathbb{R}^k \right\}$$

(and similar for $\psi_\varepsilon$).

Denoting

$$J_\varepsilon y = (I + \varepsilon \partial \varphi)^{-1}(y) \quad \text{and} \quad \nabla \varphi_\varepsilon(y) = \frac{y - J_\varepsilon y}{\varepsilon}.$$

Hence $y \to \nabla \varphi_\varepsilon(y)$ is an monotone Lipschitz function and

$$\varphi_\varepsilon(y) = \frac{1}{2\varepsilon} |y - J_\varepsilon y|^2 + \varphi(J_\varepsilon y).$$

(Analog for $\psi_\varepsilon$).

• We introduce now the compatibility assumptions:

for all $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^k$ and $z \in \mathbb{R}^{k \times d}$

$$(i) \quad \langle \nabla \varphi_\varepsilon(y), \nabla \psi_\varepsilon(y) \rangle \geq 0,$$

$$(ii) \quad \langle \nabla \varphi_\varepsilon(y), G(t,y) \rangle \leq \langle \nabla \psi_\varepsilon(y), G(t,y) \rangle^+,$$

$$(iii) \quad \langle \nabla \psi_\varepsilon(y), F(t,y,z) \rangle \leq \langle \nabla \varphi_\varepsilon(y), F(t,y,z) \rangle^+. \quad (27)$$
Definition 7 \((Y, Z, U, V)\) will be called a solution of BSVI (22) if
\[
(a) \quad Y \in S^\lambda_{k,\mu} \cap H^\lambda_{k,\mu} \cap \tilde{H}^\lambda_{k,\mu}, \quad Z \in H^\lambda_{k,\mu},
\]
\[(b) \quad U \in H^\lambda_{k,\mu}, \quad V \in \tilde{H}^\lambda_{k,\mu},
\]
\[(c) \quad \mathbb{E} \int_0^\tau e^{\lambda s + \mu A_s} (\varphi(Y_s) + \psi(Y_s)) dA_s < \infty,
\]
\[(d) \quad (Y_t, U_t) \in \partial \varphi, \quad \mathbb{P} (d\omega) \otimes dt, \quad (Y_t, V_t) \in \partial \psi, \quad \mathbb{P} (d\omega) \otimes A(\omega, dt)
\quad \text{a.e. on } \Omega \times [0, \tau], \quad (28)
\]
\[(e) \quad Y_t + \int_{t \wedge \tau}^\tau U_s ds + \int_{t \wedge \tau}^\tau V_s dA_s = \xi + \int_{t \wedge \tau}^\tau F(s, Y_s, Z_s) ds
\quad + \int_{t \wedge \tau}^\tau G(s, Y_s) dA_s - \int_{t \wedge \tau}^\tau Z_s dW_s, \quad \text{for all } t \geq 0 \ a.s.
\]

In all that follows, \(C\) denotes a constant, which may depend only on \(\mu, \alpha, \beta\) and \(L\), which may vary from line to line.

Proposition 8 Let assumptions (23), (24), and (26). If \((Y, Z, U, V)\) and \((\tilde{Y}, \tilde{Z}, \tilde{U}, \tilde{V})\) are corresponding solutions to \(\xi\) and \(\tilde{\xi}\) which satisfy (27), then
\[
\mathbb{E} \int_0^\tau e^{\lambda s + \mu A_s} \left[ |Y_s - \tilde{Y}_s|^2 (ds + dA_s) + ||Z_s - \tilde{Z}_s||^2 ds \right]
+ \mathbb{E} \sup_{0 \leq t \leq \tau} e^{\lambda t + \mu A_t} |Y_t - \tilde{Y}_t|^2 \leq C \mathbb{E} \left[ e^{\lambda \tau + \mu A_\tau} |\xi - \tilde{\xi}|^2 \right]. \quad (29)
\]

Proof. From Itô's formula we have
\[
e^{\lambda(t \wedge \tau) + \mu A_{t \wedge \tau}} |Y_{t \wedge \tau} - \tilde{Y}_{t \wedge \tau}|^2 + \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} |Y_s - \tilde{Y}_s|^2 (\lambda ds + \mu dA_s)
+ 2 \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} \langle Y_s - \tilde{Y}_s, U_s - \tilde{U}_s \rangle ds + 2 \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} \langle Y_s - \tilde{Y}_s, V_s - \tilde{V}_s \rangle dA_s
+ \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} ||Z_s - \tilde{Z}_s||^2 ds
= e^{\lambda \tau + \mu A_\tau} |\xi - \tilde{\xi}|^2 + 2 \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} \langle Y_s - \tilde{Y}_s, F(s, Y_s, Z_s) - F(s, \tilde{Y}_s, \tilde{Z}_s) \rangle ds
+ 2 \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} \langle Y_s - \tilde{Y}_s, G(s, Y_s) - G(s, \tilde{Y}_s) \rangle dA_s
- 2 \int_{t \wedge \tau}^\tau e^{\lambda s + \mu A_s} \langle Y_s - \tilde{Y}_s, (Z_s - \tilde{Z}_s) dW_s \rangle
\]
Since
\[
\langle Y_s - \tilde{Y}_s, U_s - \tilde{U}_s \rangle ds \geq 0, \quad \langle Y_s - \tilde{Y}_s, V_s - \tilde{V}_s \rangle dA_s \geq 0,
\]
Consider the approximating equation

\[2\langle Y_s - \tilde{Y}_s, F(s, Y_s, Z_s) - F(s, \tilde{Y}_s, \tilde{Z}_s) \rangle \leq (2\alpha + 2L^2 + 1)|Y_s - \tilde{Y}_s|^2 + \frac{1}{2}||Z_s - \tilde{Z}_s||^2\]

and

\[2\langle Y_s - \tilde{Y}_s, G(s, Y_s) - G(s, \tilde{Y}_s) \rangle \leq (2\beta + 1)|Y_s - \tilde{Y}_s|^2\]

then (using also the Burkholder–Davis–Gundy’s inequality), inequality \((29)\) follows. □

The main result of this section is given by

**Theorem 9** Let assumptions \((23)-(27)\) be satisfied. Then there exists a unique solution \((Y, Z, U, V)\) for \((22)\).

### 3.2 BSVI - proof of the existence

Consider the approximating equation

\[Y_t^\varepsilon = \int_t^\tau \nabla \varphi_\varepsilon(Y_s^\varepsilon) \, ds + \int_t^\tau \nabla \psi_\varepsilon(Y_s^\varepsilon) \, dA_s = \xi + \int_t^\tau F(s, Y_s^\varepsilon, Z_s^\varepsilon) \, ds + \int_t^\tau G(s, Y_s^\varepsilon) \, dA_s - \int_t^\tau Z_s^\varepsilon dW_s, \forall t \geq 0, P - a.s.\] (30)

Since \(\nabla \varphi_\varepsilon, \nabla \psi_\varepsilon : \mathbb{R}^k \rightarrow \mathbb{R}^k\) are Lipschitz functions then, by a standard argument (Banach fixed point theorem when \(y \rightarrow F(t, y, z)\) and \(y \rightarrow G(t, y)\) are uniformly Lipschitz functions and Lipschitz approximations when \(y \rightarrow \alpha y - F(t, y, z)\) and \(y \rightarrow \beta y - G(t, y)\) are continuous monotone functions) (see also [1]), equation \((30)\) has a unique solution

\[(Y^\varepsilon, Z^\varepsilon) \in (\mathcal{S}^{\lambda,\mu}_k \cap \mathcal{H}^{\lambda,\mu}_k) \times \mathcal{H}^{\mu,\lambda}_{k_1d}\]

**Proposition 10** Let assumptions \((23)-(26)\) be satisfied. Then

\[\mathbb{E} \left[ \sup_{0 \leq t \leq \tau} e^{\lambda_t + \mu_A} |Y_t^\varepsilon|^2 + \int_0^\tau e^{\lambda_s + \mu_A} \left( |Y_s^\varepsilon|^2 + ||Z_s^\varepsilon||^2 \right) ds \right.\]

\[+ \left. \int_0^\tau e^{\lambda_s + \mu_A} |Y_s^\varepsilon|^2 dA_s \right] \leq CM(\tau)\] (31)

**Proof.** Itô’s formula for \(e^{\lambda_t + \mu_A} |Y_t^\varepsilon|^2\) yields

\[e^{\lambda(t+\tau) + \mu_A} |Y_{t+\tau}^\varepsilon|^2 + \int_{t+\tau}^{t+\tau} e^{\lambda_s + \mu_A} |Y_s^\varepsilon|^2 \left( \lambda ds + \mu dA_s \right) + \int_{t+\tau}^{t+\tau} e^{\lambda_s + \mu_A} ||Z_s^\varepsilon||^2 \, ds \]

\[+ 2 \int_{t+\tau}^{t+\tau} e^{\lambda_s + \mu_A} \left[ \langle Y_s^\varepsilon, \nabla \varphi_\varepsilon(Y_s^\varepsilon) \rangle \lambda ds + \langle Y_s^\varepsilon, \nabla \psi_\varepsilon(Y_s^\varepsilon) \rangle \mu dA_s \right] = e^{\lambda T + \mu A_T} |\xi|^2 \]

\[+ 2 \int_{t+\tau}^{t+\tau} e^{\lambda_s + \mu_A} \langle Y_s^\varepsilon, F(s, Y_s^\varepsilon, Z_s^\varepsilon) \rangle ds + 2 \int_{t+\tau}^{t+\tau} e^{\lambda_s + \mu_A} \langle Y_s^\varepsilon, G(s, Y_s^\varepsilon) \rangle dA_s\]
−2 \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \langle Y^\varepsilon_s, Z^\varepsilon_s dW_s \rangle.

But from Schwartz’s inequality and assumptions (23)-(25) we obtain

\[2 \langle Y_s, F(s, Y_s, Z_s) \rangle \leq 2\alpha |Y_s|^2 + 2L |Y_s| \|Z_s\| + 2 |Y_s| |F(s, 0, 0)| \leq (2\alpha + 2L^2 + 1) |Y_s|^2 + � \|Z_s\|^2 + |F(s, 0, 0)|^2\]

and

\[2 \langle Y_s, G(s, Y_s, Z_s) \rangle \leq 2\beta |Y_s|^2 + 2 |Y_s| |G(s, 0)| \leq (2\beta + 1) |Y_s|^2 + |G(s, 0)|^2\]

Hence, using also that \langle y, \nabla \varphi_\varepsilon (y) \rangle, \langle y, \nabla \psi_\varepsilon (y) \rangle \geq 0,

\[e^{\lambda(t \wedge \tau) + \mu A(t \wedge \tau)} |Y^\varepsilon_{t \wedge \tau}|^2 + \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} |Y^\varepsilon_s|^2 (\lambda - 2\alpha - 2L^2 - 1) ds + \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} |Y^\varepsilon_s|^2 (\mu - 2\beta - 1) dA_s + \frac{1}{2} \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \|Z^\varepsilon_s\|^2 ds \leq e^{\lambda s + \mu A_s} |\varepsilon|^2 + \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \left( |F(s, 0, 0)|^2 ds + |G(s, 0)|^2 dA_s \right) - 2 \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \langle Y^\varepsilon_s, Z^\varepsilon_s dW_s \rangle\]

that clearly yields (for \(\lambda > 2\alpha + 2L^2 + 1\) and \(\mu > 2\beta + 1\)):

\[E \int_{0}^{\tau} e^{\lambda s + \mu A_s} [ |Y^\varepsilon_s|^2 (ds + dA_s) + \|Z^\varepsilon_s\|^2 ds] \leq C M (\tau)\]

Since, by Burkholder–Davis–Gundy’s inequality,

\[E \sup_{0 \leq t \leq \tau} \left| \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \langle Y^\varepsilon_s, Z^\varepsilon_s dW_s \rangle \right| \leq 3E \left( \int_{0}^{\tau} e^{2(\lambda s + \mu A_s)} \langle Y^\varepsilon_s, Z^\varepsilon_s \rangle^2 ds \right)^{1/2} \leq \frac{1}{4} E \sup_{0 \leq t \leq \tau} e^{\lambda t + \mu A_t} |Y^\varepsilon_t|^2 + C E \int_{0}^{\tau} e^{\lambda s + \mu A_s} \|Z^\varepsilon_s\|^2 ds , \]

then it follows

\[E \sup_{0 \leq t \leq \tau} e^{\lambda t + \mu A_t} |Y^\varepsilon_t|^2 \leq C M (\tau) .\]

The proof is complete.
Proposition 11 Let assumptions (32), (33) be satisfied. Then there exists a positive constant $C$ such that for any stopping time $\theta \in [0, \tau]$:

\[
\begin{align*}
(a) & \quad \mathbb{E} \int_0^T e^{\lambda s + \mu A_s} \left( |\nabla \varphi_\varepsilon (Y_s^\varepsilon) |^2 + |\nabla \psi_\varepsilon (Y_s^\varepsilon) |^2 \right) dA_s \leq C \, M (\tau) , \\
(b) & \quad \mathbb{E} \int_0^T e^{\lambda s + \mu A_s} \left( \varphi (J_\varepsilon (Y_s^\varepsilon)) + \psi (\hat{J}_\varepsilon (Y_s^\varepsilon)) \right) dA_s \leq C \, M (\tau) , \\
(c) & \quad \mathbb{E} e^{\lambda t + \mu A_s} \left( |Y_\theta^\varepsilon - J_\varepsilon (Y_s^\varepsilon)|^2 + |Y_\theta^\varepsilon - \hat{J}_\varepsilon (Y_s^\varepsilon)|^2 \right) \leq \varepsilon \, C \, M (\tau) , \\
(d) & \quad \mathbb{E} e^{\lambda t + \mu A_s} \left( \varphi (J_\varepsilon (Y_s^\varepsilon)) + \psi (\hat{J}_\varepsilon (Y_s^\varepsilon)) \right) \leq C \, M (\tau) .
\end{align*}
\]

Proof. Essential for the proof is the stochastic subdifferential inequality introduced by Pardoux and Raşcanu in [8], 1998. We will use this inequality for our purpose. First we write the subdifferential inequality

\[
e^{\lambda s + \mu A_s} \varphi_\varepsilon (Y_s^\varepsilon) \geq \left( e^{\lambda s + \mu A_s} - e^{\lambda t + \mu A_s} \right) \varphi_\varepsilon (Y_s^\varepsilon) + e^{\lambda t + \mu A_s} \varphi_\varepsilon (Y_t^\varepsilon) + e^{\lambda t + \mu A_s} \langle \nabla \varphi_\varepsilon (Y_t^\varepsilon), Y_s^\varepsilon - Y_t^\varepsilon \rangle
\]

for $s = t_i + 1 \wedge \tau, r = t_i \wedge \tau$, where $t = t_0 < t_1 < t_2 < ... < t \wedge \tau$ and $t_{i+1} - t_i = \frac{1}{n}$, then summing up over $i$, and passing to the limit as $n \to \infty$, we deduce

\[
e^{\lambda t + \mu A_s} \varphi_\varepsilon (\xi) \geq e^{\lambda (t \wedge \tau) + \mu A_{t \wedge \tau}} \varphi_\varepsilon (Y_{t \wedge \tau}^\varepsilon) + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon), dY_s^\varepsilon \rangle + \int_{t \wedge \tau}^T \varphi_\varepsilon (Y_s^\varepsilon) d(e^{\lambda s + \mu A_s})
\]

We have the similar inequalities for function $\psi_\varepsilon$

If we summing and we use equation (30), we infer that for all $t \geq 0$:

\[
\begin{align*}
& e^{\lambda (t \wedge \tau) + \mu A_{t \wedge \tau}} \left( \varphi_\varepsilon (Y_{t \wedge \tau}^\varepsilon) + \psi_\varepsilon (Y_{t \wedge \tau}^\varepsilon) \right) + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} |\nabla \psi_\varepsilon (Y_s^\varepsilon)|^2 dA_s \\
& + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} |\nabla \varphi_\varepsilon (Y_s^\varepsilon)|^2 ds + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} (\varphi_\varepsilon (Y_s^\varepsilon) + \psi_\varepsilon (Y_s^\varepsilon)) (\lambda ds + \mu dA_s) \\
& + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} \left( \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon), \nabla \psi_\varepsilon (Y_s^\varepsilon) \rangle \right) (ds + dA_s) \\
& \leq e^{\lambda t + \mu A_s} (\varphi_\varepsilon (\xi) + \psi_\varepsilon (\xi)) + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon), F (s, Y_s^\varepsilon, Z_s^\varepsilon) \rangle ds \\
& + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon), G (s, Y_s^\varepsilon) \rangle dA_s \\
& + \int_{t \wedge \tau}^T e^{\lambda s + \mu A_s} \langle \nabla \psi_\varepsilon (Y_s^\varepsilon), F (s, Y_s^\varepsilon, Z_s^\varepsilon) \rangle ds
\end{align*}
\]
Proposition 12

\[
\begin{align*}
&+ \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} \langle \nabla \psi_{\epsilon} (Y_s^\epsilon), G (s, Y_s^\epsilon) \rangle dA_s \\
&- \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} \langle \nabla \phi_{\epsilon} (Y_s^\epsilon) + \nabla \psi_{\epsilon} (Y_s^\epsilon), Z_s^\epsilon dW_s \rangle \\
\end{align*}
\]

The result follows by combining this with (31), assumptions (27) and the following inequalities

\[
\begin{align*}
\frac{1}{2\epsilon} |y - J_{\epsilon} (y)|^2 &\leq \varphi_{\epsilon} (y) , \frac{1}{2\epsilon} |y - \tilde{J}_{\epsilon} (y)|^2 \leq \psi_{\epsilon} (y) \\
\varphi (J_{\epsilon} (y)) &\leq \varphi_{\epsilon} (y) , \psi (\tilde{J}_{\epsilon} (y)) \leq \psi_{\epsilon} (y) \\
\varphi_{\epsilon} (\xi) &\leq \varphi (\xi) , \psi_{\epsilon} (\xi) \leq \psi (\xi) \\
\langle \nabla \phi_{\epsilon} (y), F (s, y, z) \rangle &\leq \frac{1}{4} |\nabla \phi_{\epsilon} (y)|^2 + 3 (\eta_s^2 + L^2 |y|^2 + L^2 ||z||^2) \\
\langle \nabla \psi_{\epsilon} (y), G (s, y) \rangle &\leq \frac{1}{4} |\nabla \psi_{\epsilon} (y)|^2 + 2 (\gamma_s^2 + L^2 |y|^2) \\
\langle \nabla \psi_{\epsilon} (y), F (s, y, z) \rangle &\leq \langle \nabla \phi_{\epsilon} (y), F (s, y, z) \rangle^+ \\
&\leq \frac{1}{4} |\nabla \phi_{\epsilon} (y)|^2 + 3 (\eta_s^2 + L^2 |y|^2 + L^2 ||z||^2) \\
\langle \nabla \psi_{\epsilon} (y), G (s, y) \rangle &\leq \langle \nabla \psi_{\epsilon} (y), G (s, y) \rangle^+ \\
&\leq \frac{1}{4} |\nabla \psi_{\epsilon} (y)|^2 + 2 (\gamma_s^2 + L^2 |y|^2)
\end{align*}
\]

Proposition 12 Let assumptions (23)-(27) be satisfied. Then

\[
\mathbb{E} \int_0^T e^{\lambda t + \mu A_t} (|Y_s^\epsilon - Y_s^\delta|^2 (ds + dA_s) + ||Z_s^\epsilon - Z_s^\delta||^2 ds \\
\quad + \mathbb{E} \sup_{0 \leq t \leq \tau} e^{\lambda t + \mu A_t} |Y_t^\epsilon - Y_t^\delta|^2 \leq C (\epsilon + \delta) \quad M (\tau)
\]

Proof. By Itô's formula

\[
e^{\lambda (t\wedge \tau) + \mu A_{t\wedge \tau}} |Y_{t\wedge \tau}^\epsilon - Y_{t\wedge \tau}^\delta|^2 + \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} |Y_s^\epsilon - Y_s^\delta|^2 (\lambda ds + \mu dA_s)
\]

\[
+ 2 \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} \langle Y_s^\epsilon - Y_s^\delta, \nabla \phi_{\epsilon} (Y_s^\epsilon) - \nabla \phi_{\delta} (Y_s^\delta) \rangle ds
\]

\[
+ 2 \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} \langle Y_s^\epsilon - Y_s^\delta, \nabla \psi_{\epsilon} (Y_s^\epsilon) - \nabla \psi_{\delta} (Y_s^\delta) \rangle dA_s =
\]

\[
= 2 \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} \langle Y_s^\epsilon - Y_s^\delta, F (s, Y_s^\epsilon, Z_s^\epsilon) - F (s, Y_s^\delta, Z_s^\delta) \rangle ds
\]

\[
+ 2 \int_{t\wedge \tau}^T e^{\lambda s + \mu A_s} \langle Y_s^\epsilon - Y_s^\delta, G (s, Y_s^\epsilon) - G (s, Y_s^\delta) \rangle dA_s
\]

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\[- \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \| Z_s^\varepsilon - Z_s^\delta \|^2 \, ds - 2 \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \langle Y_s^\varepsilon - Y_s^\delta, (Z_s^\varepsilon - Z_s^\delta) \rangle dW_s \]

We have moreover,
\[
2 \langle Y_s^\varepsilon - Y_s^\delta, F(s, Y_s^\varepsilon, Z_s^\varepsilon) - F(s, Y_s^\delta, Z_s^\delta) \rangle \\
\leq (2\alpha + 2L^2) \| Y_s^\varepsilon - Y_s^\delta \|^2 + \frac{1}{2} \| Z_s^\varepsilon - Z_s^\delta \|^2
\]

\[
2 \langle Y_s^\varepsilon - Y_s^\delta, G(s, Y_s^\varepsilon) - G(s, Y_s^\delta) \rangle \leq 2\beta \| Y_s^\varepsilon - Y_s^\delta \|^2
\]

But from definition of \( \varphi_\varepsilon \) and the monotonicity of operator \( \partial \varphi \) we have
\[
0 \leq \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) - \nabla \varphi_\delta (Y_s^\delta), J_\varepsilon (Y_s^\varepsilon) - J_\delta (Y_s^\delta) \rangle \\
= \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) - \nabla \varphi_\delta (Y_s^\delta), Y_s^\varepsilon - Y_s^\delta \rangle - \varepsilon |\nabla \varphi_\varepsilon (Y_s^\varepsilon)|^2 - \delta |\nabla \varphi_\delta (Y_s^\delta)|^2 \\
+ (\varepsilon + \delta) \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \varphi_\delta (Y_s^\delta) \rangle
\]

then
\[
\langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) - \nabla \varphi_\delta (Y_s^\delta), Y_s^\varepsilon - Y_s^\delta \rangle \geq - (\varepsilon + \delta) \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \varphi_\delta (Y_s^\delta) \rangle
\]

and in the same manner
\[
\langle \nabla \psi_\varepsilon (Y_s^\varepsilon) - \nabla \psi_\delta (Y_s^\delta), Y_s^\varepsilon - Y_s^\delta \rangle \geq - (\varepsilon + \delta) \langle \nabla \psi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \psi_\delta (Y_s^\delta) \rangle
\]

and consequently
\[
e^{\lambda(t \wedge \tau) + \mu A(t \wedge \tau)} \| Y_{t \wedge \tau}^\varepsilon - Y_{t \wedge \tau}^\delta \|^2 + \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \| Y_s^\varepsilon - Y_s^\delta \|^2 (\lambda - 2\alpha - 2L^2) \, ds \\
+ \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \| Y_s^\varepsilon - Y_s^\delta \|^2 (\mu - 2\beta) \, dA_s + \frac{1}{2} \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \| Z_s^\varepsilon - Z_s^\delta \|^2 \, ds \\
\leq 2 (\varepsilon + \delta) \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \left[ \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \varphi_\delta (Y_s^\delta) \rangle \right] ds \\
+ \langle \nabla \psi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \psi_\delta (Y_s^\delta) \rangle \, dA_s - 2 \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \langle Y_s^\varepsilon - Y_s^\delta, (Z_s^\varepsilon - Z_s^\delta) \rangle dW_s
\]

Now, from (32a)
\[
2 (\varepsilon + \delta) \mathbb{E} \int_{t \wedge \tau}^{\tau} e^{\lambda s + \mu A_s} \left[ \langle \nabla \varphi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \varphi_\delta (Y_s^\delta) \rangle \right] ds \\
+ \langle \nabla \psi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \psi_\delta (Y_s^\delta) \rangle \, dA_s \leq C (\varepsilon + \delta) M (\tau)
\]

and clearly by standard calculus inequality (33) follows.

We give now the proof of Theorem 9

**Proof.** Uniqueness is a consequence of Proposition 8. The existence of solution \((Y, Z, U, V)\) is obtained as limit of \((Y_s^\varepsilon, Z_s^\varepsilon, \nabla \varphi_\varepsilon (Y_s^\varepsilon) \cdot \nabla \psi_\varepsilon (Y_s^\varepsilon))\)
From Proposition 12 we have

\[
\exists Y \in S_{k}^{λ^{μ}} \cap H_{k}^{λ^{μ}} \cap \tilde{H}_{k}^{λ^{μ}}, \quad \exists Z \in H_{k \times d}^{λ^{μ}}
\]

\[
\lim \epsilon Y_{S}^{ε} = Y \quad \text{in} \quad S_{k}^{λ^{μ}} \cap H_{k}^{λ^{μ}} \cap \tilde{H}_{k}^{λ^{μ}},
\]

\[
\lim \epsilon Z_{S}^{ε} = Z \quad \text{in} \quad H_{k \times d}^{λ^{μ}}.
\]

Also, from (32-a) and (32-c) we have

\[
\lim \epsilon \epsilon Y_{S}^{ε} = Y \quad \text{in} \quad H_{k}^{λ^{μ}}, \quad \lim \epsilon \epsilon Y_{S}^{ε} = Y \quad \text{in} \quad \tilde{H}_{k}^{λ^{μ}}
\]

\[
\lim \epsilon \epsilon e^{λ^{0}+μA_{ν}} |J_{ε}(Y_{θ}^{ε}) - Y_{θ}|^{2} = 0, \quad \lim \epsilon \epsilon e^{λ^{0}+μA_{ν}} |\hat{J}_{ε}(Y_{θ}^{ε}) - Y_{θ}|^{2} = 0,
\]

for any stopping time \( θ \), \( 0 ≤ θ ≤ τ \).

Using Fatou’s Lemma, from (32-b), (32-d) and the fact that \( ϕ \) is l.s.c. we obtained (28-c).

Denoting \( U^{ε} = \nabla ϕ_{ε}(Y^{ε}), V^{ε} = \nabla ψ_{ε}(Y^{ε}) \), from (32-a) it follows:

\[
\mathbb{E} \left[ \int_{θ}^{τ} e^{λ^{s}+μA_{ν}} \left( |U^{ε}|^2 ds + |V^{ε}|^2 dA_{s} \right) \right] \leq C M (τ)
\]

Hence there exists \( U \in H_{k}^{λ^{μ}} \) and \( V \in \tilde{H}_{k}^{λ^{μ}} \) such that for a subsequence \( ε_{n} \searrow 0 \)

\[
U^{ε_{n}} \rightharpoonup U, \quad \text{weakly in Hilbert space} \quad H_{k}^{λ^{μ}}
\]

\[
V^{ε_{n}} \rightharpoonup V, \quad \text{weakly in Hilbert space} \quad \tilde{H}_{k}^{λ^{μ}}
\]

and then

\[
\mathbb{E} \left[ \int_{θ}^{τ} e^{λ^{s}+μA_{ν}} \left( |U|^{2} ds + |V|^{2} dA_{s} \right) \right] \leq \lim \inf_{n \to \infty} \mathbb{E} \left[ \int_{θ}^{τ} e^{λ^{s}+μA_{ν}} \left( |U^{ε_{n}}|^{2} ds + |V^{ε_{n}}|^{2} dA_{s} \right) \right] \leq C M_{2} (θ, τ).
\]

Passing now to lim in (30) we obtain (28-c).

Let \( u \in H_{k}^{λ^{μ}}, \ v \in \tilde{H}_{k}^{λ^{μ}} \). Since \( \nabla ϕ_{ε}(y_{θ}^{ε}) \in ∂ϕ(J_{ε}(y_{θ}^{ε})) \) and \( \nabla ψ_{ε}(y_{θ}^{ε}) \in ∂ψ(\hat{J}_{ε}(y_{θ}^{ε})) \), \( ∀t ≥ 0, \) then as signed measures on \( Ω \times [0, τ] \)

\[
e^{λ^{s}+μA_{ν}} \left( J_{ε}(y_{θ}^{ε}) - u_{s} \right) \mathbb{P} (dω) ⊗ ds + e^{λ^{s}+μA_{ν}} \varphi(J_{ε}(y_{θ}^{ε})) \mathbb{P} (dω) ⊗ ds
\]

\[
\leq e^{λ^{s}+μA_{ν}} \varphi(u_{s}) \mathbb{P} (dω) ⊗ ds
\]

and

\[
e^{λ^{s}+μA_{ν}} \left( J_{ε}(y_{θ}^{ε}) - u_{s} \right) \mathbb{P} (dω) ⊗ A (ω, ds)
\]

\[
+ e^{λ^{s}+μA_{ν}} \psi(J_{ε}(y_{θ}^{ε})) \mathbb{P} (dω) ⊗ A (ω, ds)
\]

\[
\leq e^{λ^{s}+μA_{ν}} \psi(v_{s}) \mathbb{P} (dω) ⊗ A (ω, ds).
\]

Passing to lim inf in these last two inequalities we obtain (28-d). The proof is complete.
4 PVI - proof of the existence theorem

It follows from result in [6] that for each \((t, x) \in \mathbb{R}_+ \times \overline{D}\) there exists a unique pair of continuous \(F^{t,x}_s\)-p.m.s.p. \((X^{t,x}_s, A^{t,x}_s)_{s \geq 0}\), with values in \(\overline{D} \times \mathbb{R}_+\), solution of the reflected stochastic differential equation

\[
\begin{aligned}
X^{t,x}_s &= x + \int_t^s b(r, X^{t,x}_r) dr + \int_t^s \sigma(r, X^{t,x}_r) dW_r - \int_t^s \nabla \ell(X^{t,x}_r) dA^{t,x}_r, \\
s &\mapsto A^{t,x}_s \text{ is increasing} \\
A^{t,x}_s &= \int_t^s 1_{\{X^{t,x}_r \in Bd(\overline{D})\}} dA^{t,x}_r,
\end{aligned}
\] (35)

where

\[
\mathcal{F}_s^t = \sigma \{W_r - W_t : t \leq r \leq s\} \vee \mathcal{N}.
\]

Since \(\overline{D}\) is a bounded set, then

\[
\sup_{s \geq 0} |X^{t,x}_s| \leq M
\] (36)

and with similar calculus as in [11] we have for all \(\mu, T, p > 0\) there exists a positive constant \(C\) such that \(\forall t, t' \in [0, T], x, x' \in \overline{D}\):

\[
\mathbb{E} \sup_{s \in [0,T]} |X^{t,x}_s - X^{t',x'}_s|^p \leq C \left( |x - x'|^p + |t - t'|^{\frac{p}{2}} \right),
\] (37)

and

\[
\mathbb{E}[e^{\mu A^{t,x}_T}] < \infty.
\] (38)

Let \(T > 0\) be arbitrary fixed. Under assumptions (2)-(7), it follows from Theorem [9] with \(\tau\) replaced by \(T\) that for each \((t, x) \in [0, T] \times \overline{D}\) there exists a unique solution \((Y^{tx}, Z^{tx}, U^{tx}, V^{tx})\) of p.m.s.p. \(Y^{tx}_s \in S^{\lambda,\mu}_1 \cap \mathcal{H}^{\lambda,\mu}_1 \cap \tilde{\mathcal{H}}^{\lambda,\mu}_1\), \(Z^{tx} \in \mathcal{H}^{\lambda,\mu}_d\), \(U^{tx} \in \mathcal{H}^{\lambda,\mu}_1\), \(V^{tx} \in \tilde{\mathcal{H}}^{\lambda,\mu}_1\)

with \(Y^{tx}_s = Y^{t,x}_t, Z^{tx}_s = 0, U^{tx}_s = 0, V^{tx}_s = 0\), for all \(s \in [0, t]\)

solution of BSDE:

\[
\begin{aligned}
Y^{t,x}_s + \int_s^T U^{tx}_r dr + \int_s^T V^{tx}_r dA^{t,x}_r &= h \left( X^{t,x}_T \right) \\
+ \int_s^T 1_{[t,T]} (r) f \left( r, X^{t,x}_r, Y^{t,x}_r, Z^{tx}_r \right) dr \\
+ \int_s^T 1_{[t,T]} (r) g \left( r, X^{t,x}_r, Y^{t,x}_r \right) dA^{t,x}_r - \int_s^T Z^{tx}_r dW_r, & \text{for all } s \in [0, T] \text{ a.s.}
\end{aligned}
\]
such that $(Y_{s}^{t,x}, U_{s}^{t,x}) \in \partial \varphi$, $\mathbb{P}(d\omega) \otimes dt$, $(Y_{s}^{t,x}, V_{s}^{t,x}) \in \partial \psi$, $\mathbb{P}(d\omega) \otimes A(\omega, dt)$, a.e. on $\Omega \times [t, T]$.

We observe that function $f, g$ depends by $\omega$ only via function $X^{t,x}$.

**Proposition 13** Under assumptions (2)-(7), we have

$$
\mathbb{E} \sup_{s \in [0, T]} e^{\lambda s + \mu A_{r}} |Y_{s}^{t,x}|^2 \leq C(T)
$$

and

$$
\mathbb{E} \sup_{s \in [0, T]} e^{\lambda s + \mu A_{r}} |Y_{s}^{t,x} - Y_{s}^{t,x'}|^2 \leq \mathbb{E} \left[ e^{\lambda \tau + \mu A_{r}} |h(X_{T}^{t,x}) - h(X_{T}^{t,x'})|^2 + \int_{0}^{T} e^{\lambda \tau + \mu A_{r}} |1_{[t, T]}(r)f(r, X_{r}^{t,x}, Y_{r}^{t,x}, Z_{r}^{t,x}) - 1_{[v, T]}(r)f(r, X_{r}^{t,x'}, Y_{r}^{t,x}, Z_{r}^{t,x'})|^2 dr + \int_{0}^{T} e^{\lambda \tau + \mu A_{r}} |1_{[t, T]}(r)g(r, X_{r}^{t,x}, Y_{r}^{t,x}) - 1_{[v, T]}(r)g(r, X_{r}^{t,x'}, Y_{r}^{t,x'})|^2 dA_{r}^{t,x} \right]
$$

**Proof.** Inequality (39) follows from Theorem 9 using also (36), (38). Inequality (40) follows from (29) in Proposition 8.

We define

$$
u(t, x) = Y_{t}^{t,x}, \quad (t, x) \in [0, T] \times \mathcal{D}
$$

which is a determinist quantity since $Y_{t}^{t,x}$ is $\mathcal{F}_{t} \equiv \mathcal{N}$-measurable.

From Markov property we have

$$
u(s, X_{s}^{t,x}) = Y_{s}^{t,x}
$$

**Corollary 14** Under assumptions (2)-(7), function $\nu$ satisfies:

(a) $\nu(t, x) \in \text{Dom}(\varphi)$, $\forall (t, x) \in [0, T] \times \mathcal{D}$,

(b) $\nu(t, x) \in \text{Dom}(\psi)$, $\forall (t, x) \in [0, T] \times \text{Bd}(\mathcal{D})$,

(c) $\nu \in C([0, T] \times \mathcal{D})$.

**Proof.** Using (28)c we have $\varphi(\nu(t, x)) = \mathbb{E}\varphi(Y_{t}^{t,x}) < +\infty$ and similarly for $\psi$. Hence (13)a,b follows. Let $(t_{n}, x_{n}) \to (t, x)$. Then

$$
|\nu(t_{n}, x_{n}) - \nu(t, x)|^2 = \mathbb{E}|Y_{t_{n}}^{t_{n}, x_{n}} - Y_{t}^{t,x}|^2 \leq 2\mathbb{E} \sup_{s \in [0, T]} |Y_{s}^{t_{n}, x_{n}} - Y_{s}^{t,x}|^2 + 2\mathbb{E} |Y_{t_{n}}^{t,x} - Y_{t}^{t,x}|^2
$$

Using (10), (36), (37) and (38) we obtain $\nu(t_{n}, x_{n}) \to \nu(t, x)$ as $(t_{n}, x_{n}) \to (t, x)$. ■

We present now the proof of Theorem 5 (existence of the viscosity solutions).
**Proof.** It suffices to show the existence of the solution of PVI (1) on an arbitrary fixed interval \([0, T]\). Setting
\[
\tilde{u}(t, x) = u(T - t, x)
\]
then the existence for problem (1) it is equivalent with existence for (44)
\[
\begin{cases}
\frac{\partial \tilde{u}(t, x)}{\partial t} + \tilde{L}_t \tilde{u}(t, x) + \tilde{f}(t, x, \tilde{u}(t, x), (\nabla \tilde{u})\sigma(t, x)) \in \partial \varphi(\tilde{u}(t, x)), \\
-\frac{\partial \tilde{u}(t, x)}{\partial n} + \tilde{g}(t, x, \tilde{u}(t, x)) \in \partial \psi(\tilde{u}(t, x)), \quad t \in (0, T), \ x \in D, \\
\tilde{u}(T, x) = h(x), \ x \in \overline{D};
\end{cases}
\] (44)

where
\[
\tilde{f}(t, x, u, z) = f(T - t, x, u, z), \quad \tilde{g}(t, x, u) = g(T - t, x, u) \\
\tilde{\sigma}(t, x) = \sigma(T - t, x), \quad \tilde{b}(t, x) = b(T - t, x)
\]

and
\[
\tilde{L}_t v(x) = \frac{1}{2} \sum_{i,j=1}^{d} ((\tilde{\sigma} \tilde{\sigma}^*)_{ij}(t, x)) \frac{\partial^2 v(x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} \tilde{b}_i(t, x) \frac{\partial v(x)}{\partial x_i}.
\]

We denote also
\[
\tilde{V}(t, x, p, q, X) \overset{\text{def}}{=} -p - \frac{1}{2} \text{Tr}((\tilde{\sigma} \tilde{\sigma}^*)(t, x)X) - \langle \tilde{b}(t, x), q \rangle - \tilde{f}(t, x, \tilde{u}(t, x), q \tilde{\sigma}(t, x)).
\]

In the sequel, for simplicity we keep notations \(b, \sigma, u, f, g, L, V\) instead of \(\tilde{b}, \tilde{\sigma}, \tilde{u}, \tilde{f}, \tilde{g}, \tilde{L}, \tilde{V}\) and we shall prove that function \(u\) defined by (41) is a viscosity solution of parabolic variational inequality (44). We show only that \(u\) is a viscosity subsolution of (44) (the supersolution case is similar).

Let \((t, x) \in [0, T] \times \overline{D}\) and \((p, q, X) \in \mathcal{P}^{2,+} u(t, x)\).

1. The proof for the case \(x \in D\) is similar of that from [8].
2. Let \(x \in Bd(D)\). Suppose, contrary to our claim, that
\[
\begin{align*}
\min \left\{ V(t, x, p, q, X) + \varphi_+(u(t, x)), \right. \\
\left. \langle \nabla \ell(x), q \rangle - g(t, x, u(t, x)) + \psi_+(u(t, x)) \right\} > 0
\end{align*}
\]

and we will find a contradiction.
It follows by continuity of $f$, $g$, $u$, $b$, $\sigma$, $\ell$, left continuity and monotonicity of $\varphi'_-$ and $\psi'_-$ that there exists $\varepsilon > 0$, $\delta > 0$ such that for all $|s - t| \leq \delta$, $|y - x| \leq \delta$,

\[-(p + \varepsilon) - \frac{1}{2}\text{Tr}\left((\sigma\sigma^*)(s, y) (X + \varepsilon I)\right) - \langle b(s, y), q + (X + \varepsilon I) (y - x) \rangle \]

\[-f(s, y, u(s, y), (q + (X + \varepsilon I) (y - x)) \sigma(s, y)) + \varphi'_-(u(s, y)) > 0, \quad \text{if } x \in D \]

and

\[\langle \nabla \ell(y), q + (X + \varepsilon I) (y - x) \rangle - g(s, y, u(s, y)) + \psi'_-(u(s, y)) > 0, \quad \text{if } x \in B_d(D) \]

Now since $(p, q, X) \in \mathcal{P}^{2+} u(t, x)$ there exists $0 < \delta' \leq \delta$ such that

\[u(s, y) < \hat{u}(s, y), \]

for all $s \in [0, T]$, $s \neq t$, $y \in \overline{D}$, $y \neq x$ such that $|s - t| \leq \delta'$, $|y - x| \leq \delta'$, where

\[\hat{u}(s, y) = u(t, x) + (p + \varepsilon) (s - t) + \langle q, y - x \rangle + \frac{1}{2} \langle (X + \varepsilon I) (y - x), y - x \rangle \]

Let

\[\nu \overset{def}{=} \inf \{s > t : |X_{s,x}^t - x| \geq \delta'\} \]

We note that

\[
\begin{aligned}
\bar{Y}_{s}^{t,x} &= u(\nu, X_{\nu}^{t,x}) + \int_{s}^{\nu} (f(r, X_{r}^{t,x}, \bar{Y}_{r}^{t,x}, \bar{Z}_{r}^{t,x}) - U_{r}^{t,x}) dr - \int_{s}^{\nu} \bar{Z}_{r}^{t,x} dW_{r} \\
&\quad + \int_{s}^{\nu} (g(r, X_{r}^{t,x}, \bar{Y}_{r}^{t,x}) - V_{r}^{t,x}) dA_{r}^{t,x} \\
(Y_{s}^{t,x}, U_{s}^{t,x}) &\in \partial \varphi, \mathbb{P} (d\omega) \otimes dt, \quad (Y_{s}^{t,x}, V_{s}^{t,x}) \in \partial \psi, \mathbb{P} (d\omega) \otimes A (\omega, dt), \quad \text{a.e. on } \Omega \times [t, T].
\end{aligned}
\]

Moreover, it follows from Itô’s formula that

\[
\hat{Y}_{s}^{t,x} = \hat{u}(\nu, X_{\nu}^{t,x}) - \int_{s}^{\nu} \left[ \frac{\partial \hat{u}(r, X_{r}^{t,x})}{\partial t} + \mathcal{L}_{r} \hat{u}(r, X_{r}^{t,x}) \right] dr - \int_{s}^{\nu} \hat{Z}_{r}^{t,x} dW_{r} \\
&\quad + \int_{s}^{\nu} \langle \nabla_{x} \hat{u}(r, X_{r}^{t,x}), \nabla (X_{r}^{t,x}) \rangle dA_{r}^{t,x}
\]

satisfies
Let \((\tilde{Y}_{t,x}^{s}, \tilde{Z}_{t,x}^{s}) = (\hat{Y}_{t,x}^{s} - \bar{Y}_{t,x}^{s}, \hat{Z}_{t,x}^{s} - \bar{Z}_{t,x}^{s})\).

We have
\[
\tilde{Y}_{t,x}^{s} = [\hat{u}(\nu, X_{t,x}^{s}) - u(\nu, X_{t,x}^{s})] + \int_{s}^{\nu} \left[ - \frac{\partial \hat{u}(r, X_{t,x}^{s})}{\partial t} - L_r \hat{u}(r, X_{t,x}^{s}) \
-f(r, X_{t,x}^{s}, \hat{Y}_{t,x}^{s}, \hat{Z}_{t,x}^{s}) + U_{t,x}^{s} \right] dr - \int_{s}^{\nu} \hat{Z}_{t,x}^{s} dW_r 
+ \int_{s}^{\nu} \left[ \langle \nabla_x \hat{u}(r, X_{t,x}^{s}), \nabla \ell(X_{t,x}^{s}) \rangle - g(r, X_{t,x}^{s}, \hat{Y}_{t,x}^{s}) + V_{t,x}^{s} \right] dA_{t,x}^{s}.
\]

Let
\[
\beta_{s} = L_s \hat{u}(s, X_{t,x}^{s}) + f(s, X_{t,x}^{s}, \bar{Y}_{t,x}^{s}, \bar{Z}_{t,x}^{s}) \\
\hat{\beta}_{s} = L_s \hat{u}(s, X_{t,x}^{s}) + f(s, X_{t,x}^{s}, \hat{Y}_{t,x}^{s}, \hat{Z}_{t,x}^{s})
\]

Since \(|\hat{\beta}_{s} - \bar{\beta}_{s}| \leq C |\hat{Z}_{t,x}^{s} - \bar{Z}_{t,x}^{s}|\), there exists a bounded \(d\)-dimensional p.m.s.p. \(\{\zeta_{s}; 0 \leq s \leq \nu\}\) such that \(\hat{\beta}_{s} - \bar{\beta}_{s} = \langle \zeta_{s}, \tilde{Z}_{t,x}^{s} \rangle\)

Now
\[
\tilde{Y}_{t,x}^{s} = [\hat{u}(\nu, X_{t,x}^{s}) - u(\nu, X_{t,x}^{s})] 
+ \int_{s}^{\nu} \left[ - \frac{\partial \hat{u}(r, X_{t,x}^{s})}{\partial t} + \langle \zeta_{r}, \tilde{Z}_{t,x}^{s} \rangle - \hat{\beta}_{r} + U_{t,x}^{s} \right] dr 
+ \int_{s}^{\nu} \left[ \langle \nabla_x \hat{u}(r, X_{t,x}^{s}), \nabla \ell(X_{t,x}^{s}) \rangle - g(r, X_{t,x}^{s}, \hat{Y}_{t,x}^{s}) + V_{t,x}^{s} \right] dA_{t,x}^{s} 
- \int_{s}^{\nu} \hat{Z}_{t,x}^{s} dW_r
\]

It is easily to see that, for the process
\[
\Gamma_{t,x}^{s} = \exp \left[ -\frac{1}{2} \int_{t}^{s} |\zeta_{r}|^2 dr + \int_{t}^{s} \langle \zeta_{r}, dW_r \rangle \right],
\]
we have, from Itô’s formula,
\[
\Gamma_{t,x}^{s} = \Gamma_{t,x}^{t} + \int_{t}^{s} \langle \zeta_{r}, dW_r \rangle
\]
and so
\[
d(\tilde{Y}_{t,x}^{s}, \Gamma_{t,x}^{s}) = \Gamma_{t,x}^{s} \left[ \frac{\partial \hat{u}(s, X_{t,x}^{s})}{\partial t} + \hat{\beta}_{s} - U_{t,x}^{s} \right] ds + \Gamma_{t,x}^{s} \langle \hat{Z}_{t,x}^{s} + \tilde{Y}_{t,x}^{s} \zeta_{s}, dW_s \rangle 
+ \Gamma_{t,x}^{s} \left[ \langle \nabla_x \hat{u}(r, X_{t,x}^{s}), \nabla \ell(X_{t,x}^{s}) \rangle + g(s, X_{t,x}^{s}, \hat{Y}_{t,x}^{s}) - V_{t,x}^{s} \right] dA_{t,x}^{s}.
\]
Then
\[
\begin{align*}
\tilde{Y}_{t,x}^t &= E \left[ \Gamma_t \left( \hat{u}(\nu, X_{t,x}^\nu) - u(\nu, X_0^\nu) \right) \right] \\
&- E \left[ \int_t^\nu \Gamma_r \left[ \frac{\partial \hat{u}(r, X_r^{t,x})}{\partial t} + \hat{\beta}_r - U_r^{t,x} \right] dr \right] \\
&- E \left[ \int_t^\nu \Gamma_r \left(-\langle \nabla_x \hat{u}(r, X_r^{t,x}), \nabla \ell(X_r^{t,x}) \rangle + g(r, X_r^{t,x}, Y_r^{t,x}) - V_r^{t,x} \right) dA_r^{t,x} \right]
\end{align*}
\] (47)

We first note that \((Y_t, U_t) \in \partial \varphi, \ (Y_t, V_t) \in \partial \psi\) implies that
\[
\varphi'_{-}(u(s, X_{s}^{t,x})) ds \leq U_s^{t,x} ds, \quad \psi'_{-}(u(s, X_{s}^{t,x})) dA_s^{t,x} \leq V_s^{t,x} dA_s^{t,x}
\]
Moreover, the choice of \(\delta'\) and \(\nu\) implies that
\[
u(X_{s}^{t,x}) < \hat{u}(\nu, X_{t,x})
\]
From (45) and (46) it follows that
\[
-(p + \varepsilon) - \hat{\beta}_s + \varphi'_{-}(u(s, X_{s}^{t,x})) > 0, \text{ if } x \in D
\]
and
\[
\frac{\partial \hat{u}(s, X_{s}^{t,x})}{\partial n} - g(s, X_{s}^{t,x}, \bar{Y}_{s}^{t,x}) + \psi'_{-}(u(s, X_{s}^{t,x})) > 0, \text{ if } x \in \partial D
\]
All these inequalities and equation (47) imply that \(\tilde{Y}_{t,x}^t > 0\) and equivalent
\[
\hat{u}(t, x) > u(t, x),
\]
which is a contradiction with the definition of \(\hat{u}\). Hence we have
\[
\min \left\{ V(t, x, p, q, X) + \varphi'_{-}(u(t, x)), \langle \nabla \ell(x), q \rangle - g(t, x, u(t, x)) + \psi'_{-}(u(t, x)) \right\} \leq 0
\]
This proves that \(u\) is a viscosity subsolution of (44). Symmetric arguments show that \(u\) is also a supersolution; hence \(u\) is a viscosity solution of PVI (44). 

\begin{remark}
If \(b, \sigma, f\) and \(g\) do not depend on \(t\) then we have a directly a representation formula for the viscosity solution \(u\) of PVI (7):
\[
u(t, x) = Y_{0,x}^{0,t}
\]
where \((Y_{s}^{0,x}, Z_{s}^{0,x}, U_{s}^{0,x}, V_{s}^{0,x})_{0 \leq s \leq t}\) solution of BSVI
\[
Y_{s}^{0,x} + \int_s^t U_r^{0,x} dr + \int_s^t V_r^{0,x} dA_r^{0,x} = h(X_0^{0,x}) + \int_s^t f(X_r^{0,x}, Y_r^{0,x}, Z_r^{0,x}) dr
\]
\[
+ \int_s^t g(X_r^{0,x}, Y_r^{0,x}) dA_r^{0,x} - \int_s^t Z_r^{0,x} dW_r, \text{ for all } s \in [0, T] \text{ a.s.}
\]
\end{remark}
and \((X_{s}^{0,x},A_{s}^{0,x})_{0 \leq s \leq t}\) solves SDE

\[
\begin{aligned}
X_{s}^{0,x} &= x + \int_{0}^{s} b(X_{r}^{0,x})dr + \int_{0}^{s} \sigma(X_{r}^{0,x})dW_{r} - \int_{0}^{s} \nabla \ell(X_{r}^{0,x})dA_{r}^{0,x}, \\
A_{s}^{0,x} &= \int_{0}^{s} 1_{\{X_{r}^{0,x} \in \text{Bd}(\mathcal{D})\}}dA_{r}^{0,x}.
\end{aligned}
\]

**Corollary 16** We have

\[ u(t,x) \in \text{Dom}(\partial \varphi), \quad \forall (t,x) \in [0,T] \times \mathcal{D} \]

**Proof.** Let \((t,x)\) be fixed. We have two cases:

1) \(\text{Dom}(\partial \varphi) = \text{Dom}(\varphi)\), and so, from (43a), \(u(t,x) \in \text{Dom}(\partial \varphi)\).

2) \(\text{Dom}(\partial \varphi) \neq \text{Dom}(\varphi)\). Let \(b \in \text{Dom} \varphi \setminus \text{Dom}(\partial \varphi)\).

Then \(b = \sup(\text{Dom} \varphi)\) or \(b = \inf \text{Dom} \varphi\). If \(b = \sup(\text{Dom} \varphi)\) and \(u(t,x) = b\), then \((0,0,0) \in \mathcal{P}^{2,+}u(t,x)\) since

\[ u(s,y) \leq u(t,x) + o(|s-t| + |y-x|^{2}) \]

and from \([\text{N}]\) it follows \(\varphi_{-}'(b) = \varphi_{-}'(u(t,x)) < \infty\) and consequently \(b \in \text{Dom}(\partial \varphi)\); a contradiction which shows that \(u(t,x) < b\). Similarly for \(b = \inf(\text{Dom} \varphi)\). 

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