GABOR FRAME DECOMPOSITION OF EVOLUTION OPERATORS AND APPLICATIONS

BERRA MICHELE

Abstract. We compute the Gabor matrix for Schrödinger-type evolution operators. Precisely, we analyze the Heat Equation, already presented in [2], giving the exact expression of the Gabor matrix which leads to better numerical evaluations. Then, using asymptotic integration techniques, we obtain an upper bound for the Gabor matrix in one-dimension for the generalized Heat Equation, new in the literature. Using Maple software, we show numeric representations of the coefficients’ decay. Finally, we show the super-exponential decay of the coefficients of the Gabor matrix for the Harmonic Repulsor, together with some numerical evaluations. This work is the natural prosecution of the ideas presented in [2] and [4].

1. Introduction

The Gabor frame theory has been developed in the last fifty years in order to give a discrete time-frequency representation of a signal in the phase space. We shall use Gabor frames to give a discrete time frequency representation of operators. This new field of research, started in [2, 4] is the main topic of our study. For simplicity, we limit the study to Gabor frames defined on a regular lattice

\[ \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d, \alpha, \beta > 0, \]

but more general lattices may be considered.

Precisely, given a window function \( g \in \mathcal{S}(\mathbb{R}^d)\setminus\{0\} \), i.e. the Schwartz class, and a lattice \( \Lambda := \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) with \( \alpha, \beta > 0 \), a Gabor system is defined as

\[ \mathcal{G}(g, \alpha, \beta) := \{ g_{m,n} = M_{m}T_{n}g \} \]

where \( M_n g(x) = e^{2\pi inx} g(x) \) and \( T_m g(x) = g(x-m) \). \( \mathcal{G}(g, \alpha, \beta) \) is a Gabor frame for \( L^2(\mathbb{R}^d) \) if and only if there exist \( 0 < A \leq B < +\infty \):

\[ A\|f\|_2^2 \leq \sum_{(m,n) \in \Lambda} |\langle f, g_{m,n} \rangle|^2 \leq B\|f\|_2^2, \quad \forall f \in L^2(\mathbb{R}^d). \]

The previous inequality implies the representations

\[ f = \sum_{m,n} \langle f, g_{m,n} \rangle \gamma_{m,n} \quad \text{or} \quad \sum_{m,n} \langle f, \gamma_{m,n} \rangle g_{m,n}, \quad \forall f \in L^2(\mathbb{R}^d), \]

where \( \{ \gamma_{m,n} \} \) is a Gabor frame called dual frame and (1) yields with unconditional convergence in \( L^2(\mathbb{R}^d) \).

2010 Mathematics Subject Classification. 35S05, 42C15.

Key words and phrases. Pseudodifferential Operator, Gabor Frames, Metaplectic Operator, Heat Equation, Evolution operators.
Gabor frames turned out to be the appropriate tool for many problems in time-frequency analysis, especially signal processing and imaging problems with related numerical issues, see for example [1] [10], [18], [21] and the references therein. From the theoretical point of view, Gabor frames are used to investigate Fourier Integral Operators, see [2], [3] and [5], and in particular Pseudo Differential Operators [12], by representing such operators as infinite matrices and studying the decay of their entries.

Following the ideas presented in [2], we perform the Gabor decomposition of evolution operators by computing the Gabor matrix and giving numerical evidences of the coefficients’ decay.

Given a bounded operator acting on \( L^2(\mathbb{R}^d) \), say \( T : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d) \), the idea is to discretize the action of \( T \) on a given function \( f \in L^2(\mathbb{R}^d) \). Roughly speaking, we want to give a clear meaning to the following diagram:

\[
\begin{array}{ccc}
L^2(\mathbb{R}^d) & \xrightarrow{T} & L^2(\mathbb{R}^d) \\
\downarrow{\text{Analysis}} & & \downarrow{\text{Synthesis}} \\
\ell^2 & \xrightarrow{T_{m,n,m',n'}} & \ell^2 \\
\end{array}
\]

where \( T_{m,n,m',n'} \) is a matrix operator that we are about to discuss.

The first idea is to represent the signal \( f \) via Gabor frames and then apply the operator \( T \) on it. Precisely, given \( f = \sum_{m,n} \langle f, \gamma_{m,n} \rangle g_{m,n} \) we have

\[
Tf = T \left( \sum_{m,n} \langle f, \gamma_{m,n} \rangle g_{m,n} \right) = \sum_{m,n} \langle f, \gamma_{m,n} \rangle Tg_{m,n}.
\]

On the other hand, since \( Tf \in L^2(\mathbb{R}^d) \), we can decompose it as

\[
Tf = \sum_{m',n'} \langle Tf, g_{m',n'} \rangle \gamma_{m',n'},
\]

with \( \{g_{m',n'}\}_{m',n'} \) and \( \{\gamma_{m',n'}\}_{m',n'} \) Gabor frame and dual frame respectively. Putting together these relations, one gets:

\[
Tf = \sum_{m',n'} \langle Tf, g_{m',n'} \rangle \gamma_{m',n'} = \sum_{m',n'} \sum_{m,n} \langle f, \gamma_{m,n} \rangle Tg_{m,n}, g_{m',n'} \gamma_{m',n'}
\]

\[
(2) \quad = \sum_{m,n,m',n'} \langle Tg_{m,n}, g_{m',n'} \rangle \langle f, \gamma_{m,n} \rangle \gamma_{m',n'}.
\]

We have something similar to a “kernel operator”. Indeed, using the notations of the diagram above, the kernel is \( T_{m,n,m',n'} = \langle Tg_{m,n}, g_{m',n'} \rangle \) and \( \langle f, \gamma_{m,n} \rangle \gamma_{m',n'} \) is somehow similar to the expansion of \( f \) in terms of the dual frame \( \gamma \). Therefore, we can analyze the action of the operator \( T \) by studying the behavior of this infinite matrix which we call the Gabor matrix of \( T \).

The main theoretical topics on which we base our numerical evaluation are the sparsity results for the Gabor matrix of different classes of Fourier Integral Operator proved in [2] and [5]. According to [15], given a discrete
representation of the operator $T$ of the form
\[ T f_k = \sum_j T_{j,k} f_j, \]
where $(T_{j,k})_{j,k}$ is a $N \times N$ matrix, one requires $o(N^2)$ operations to calculate $T f_k$. If $T$ is diagonal, the computational effort reduces to $o(N)$.

In order to speed up the computations, we can select only the coefficients of the matrix that are relevant in the following sense: we select a small parameter $\varepsilon > 0$ and consider only the coefficients $T_{j,k}$ satisfying $|T_{j,k}| > \varepsilon$. Hence, when the matrix shows off-diagonal decay, for example
\[ |T_{j,k}| \leq C (1 + |j - k|^2)^{-\frac{N}{2}}, \]
there exists $R(\varepsilon) > 0$ such that $|T_{j,k}| > \varepsilon$ for $|j - k| < R(\varepsilon)$. In this case, the computational effort reduces to $o(N \cdot R(\varepsilon))$ and if the off-diagonal decay is heavy, then $R(\varepsilon)$ can be chosen small.

The arguments above can be adapted to a “well-organized” matrix in the sense of [4]. Precisely, given a canonical transformation $\chi : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ and an operator $T$ such that
\[ |T_{j,k}| \leq C (1 + |j - \chi(k)|^2)^{-\frac{N}{2}}, \]
we reach fast decay far from the graph of a function $\chi$. Hence, there exists $R_\chi(\varepsilon) > 0$ such that $|T_{j,k}| > \varepsilon$ for $|j - \chi(k)| < R_\chi(\varepsilon)$.

Thus, using (2), the more decay “away from the diagonal” the more efficient is the representation of the operator $T$. Therefore, it is important to give a precise expression for the Gabor matrix and provide estimates for the coefficients’ decay.

We calculate the Gabor matrix for three well-known PDEs, i.e. the Heat equation, the so-called generalized Heat equation and the Harmonic Repulsor. The Heat equation was formerly investigated in [2], we improve their estimates by giving the exact formulation of the Gabor matrix. This yield to an improvement of the coefficients’ decay rate, when compared with [2]. The analysis of the generalized Heat equation is more intriguing and completely new in the literature. We find an upper bound for the Gabor matrix exploiting all the constants that are significant from a numerical point of view. Finally, we analyze the case of the so-called Harmonic Repulsor giving the exact representation of the Gabor matrix. This new result yields to a super exponential decay for the coefficient that is similar to the ones of the Harmonic Oscillator, treated in [4].

The paper is organized as follows: we give a brief review of the main tools of time frequency analysis used in this paper, namely Gabor frames and Gelfand-Shilov classes. Using these arguments we recall the main result of [2] on the super exponential decay of the Gabor matrix for Pseudo Differential Operators. Then we introduce the definition of the metaplectic representation used to analyze the Cauchy problem for the Harmonic Repulsor. We conclude the preliminaries by proving an interesting result of classical asymptotic integration, precisely we give an estimate for the Fourier transform of
\[ f = e^{-\alpha x^2 k}, x \in \mathbb{R}, \alpha > 0, k \in \mathbb{Z}_+. \]
Section 3 is devoted to the Heat Equation and generalizations. We start from the Cauchy problem for the Heat equation, namely
\[
\partial_t u - \rho \Delta_x u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,
\]
\[
u(0, x) = u_0(x).
\]
(3)

The solution can be written as
\[
u(t, x) = \sigma_\rho(t, D)u_0(x),
\]
where \(\sigma_\rho(t, D)\) is the family of Fourier multipliers
\[
\sigma_\rho(t, D)f(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \omega} \sigma_\rho(t, \omega) \hat{f}(\omega) d\omega,
\]
with symbol
\[
\sigma_\rho(t, \omega) = e^{-4\rho t \pi^2 |\omega|^2}, \quad \omega \in \mathbb{R}^d,
\]
(5)

Using this relation, we obtain faster decay of the coefficient of those obtained in [2] as proved in Section 5 with some numerical implementation.

In the second part of the section, we find a bound for the decay of the coefficients of the Gabor matrix for the generalized Heat equation using the asymptotic theory developed in the preliminaries. We obtain results that are consistent with the ones proved in [2] but here we make explicit the constants involved in the calculations and give a more precise information about the decay of the coefficients. The main result of the section reads
\[
|\langle \sigma_\rho(t, D)\pi(m, n)g, \pi(m', n')g \rangle| \lesssim C_{t, k} e^{-\tilde{\epsilon}_{t, k} 2^{-\frac{k}{2}}},
\]
with \(s = \frac{2k}{2k-1}, \tilde{\epsilon}_{t, k} = (\frac{2k-1}{2k})^{\frac{1}{2k-1}} 2^{-\frac{k}{2k-1}} \) and \(C_{t, k} = |2k|\frac{1}{2k-1}\) and a suitable constant \(C_0\) that does not depend on \(t\) and \(k\). Again, \(\sigma_\rho(t, D)\) is the family of Fourier multiplier depending on \(t\) and \(k\) that solves
\[
\partial_t u - \Delta_x^k u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \quad k \geq 1,
\]
\[
u(0, x) = u_0(x).
\]
(9)

The symbol related to the Fourier multiplier \(\sigma_\rho(t, D)\) reads
\[
\sigma_\rho(\omega) = e^{(-1)^k t(2\pi \omega)^{2k}}, \quad \omega \in \mathbb{R}.
\]
(10)

Finally, the last equation to be considered is the Cauchy problem for the Harmonic Repulsor, or Harmonic Repulsive Oscillator [16], namely
\[
i \partial_t u - \frac{1}{4\pi} \Delta u + \pi |x|^2 = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d,
\]
\[
u(0, x) = u_0(x).
\]
(11)
The solution can be written as
\begin{equation}
(12) \quad u(t, x) = T_t u_0(x),
\end{equation}
where \( T_t \) is the family of metaplectic operators given by
\begin{equation}
(13) \quad T_t f(x) = \frac{1}{\cosh(t)} \int_{\mathbb{R}^d} e^{\pi i \tanh(t) x^2 - \frac{2 \pi i \omega}{\cosh(t)} - \pi i \tanh(t) \omega^2} f(\omega) d\omega.
\end{equation}

The treatment of the Harmonic Repulsor is similar to the Harmonic Oscillator, already treated in [6]. We perform a direct calculus of the Gabor matrix obtaining the following super-exponential decay
\begin{equation}
(14) \quad T_{m,n,m',n'} = C_t e^{-2 \left[ |m|^2 + |n|^2 + |m'|^2 + |n'|^2 + 2 \tanh(mn - m'n') - 2(mn - m'n') \right]},
\end{equation}
where \( C_t = \frac{|e^{it\psi}|}{2 \cosh(t)} \) and \(|e^{it\psi}| = 1\). The last section is entirely devoted to the numerical simulation in which we give an evidence of the theoretical results by showing Maple’s plot of the coefficients’ decay and the spectrogram of the Gabor matrices associated to each of the three problems.

Notations. The Schwartz class is denoted by \( S(\mathbb{R}^d) \), the space of tempered distributions by \( S'(\mathbb{R}^d) \). We use the brackets \( \langle f, g \rangle \) for the extension to \( S(\mathbb{R}^d) \times S'(\mathbb{R}^d) \) of the inner product \( \langle f, g \rangle = \int f(t) g(t) dt \) on \( L^2(\mathbb{R}^d) \). The scalar product on \( \mathbb{R}^d \) is given by \( xy \) for \( x, y \in \mathbb{R}^d \). The Fourier transform is normalized to be \( \hat{f}(\omega) = \mathcal{F} f(\omega) = \int f(t) e^{-2 \pi i t \omega} dt \). The Modulation and Translation Operators \( M \) and \( T \) are defined by \( M \omega g(t) = e^{2 \pi i t \omega} g(t) \) and \( T_x g(t) = g(t-x) \). Time Frequency-Shifts are denoted \( M \omega T_x g(t) = \pi(z) g(t) \), with \( z = (x, \omega) \in \mathbb{R}^{2d} \). The Euclidean norm of \( x \in \mathbb{R}^d \) is given by \(|x| = (x_1^2 + \cdots + x_d^2)^{1/2}\). Let \( \alpha \in \mathbb{Z}^d_+ \) be a multiindex. The length of \( \alpha \) is \(|\alpha| = \alpha_1 + \cdots + \alpha_d \). For \( x \in \mathbb{R}^d \) the power is represented by \( x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} \).

The operator of partial differentiation \( \partial \) is given by
\[ \partial^\alpha = \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d} \]
for \( x \in \mathbb{R}^d \) and \( \alpha \in \mathbb{Z}^d_+ \). The letter \( C \) denotes a positive constant, not necessarily the same at every appearance. Throughout the paper, we shall use the notation \( A \lesssim B \) to indicate \( A \leq cB \) for a suitable constant \( c > 0 \), whereas \( A \simeq B \) if \( A \leq cB \) and \( B \leq kA \), for suitable \( c, k > 0 \). We denote the \( \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \). We denote also \( \mathcal{M}(2d, \mathbb{R}) \) for the set of \( 2d \times 2d \) real-valued matrices while \( GL(2d, \mathbb{R}) \) is the general linear group over \( \mathcal{M}(2d, \mathbb{R}) \).

2. Preliminaries

2.1. Gabor Frames. We recall the basic concepts of the Gabor Frame theory and refer the reader to [11] and [13] for the details. The Gabor Frames are used to give a discrete Time-Frequency representation of signal and operators. Let \( \Lambda := \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \) be a lattice on the phase space, with \( \alpha, \beta > 0 \) lattice parameters. The set of time-frequency shifts
\[ G(g, \alpha, \beta) = \{ \pi(\lambda) g, \lambda \in \Lambda \}, \]
If (15) holds, then there exists a dual window \( \gamma \) is a (Gabor) frame for (15) for every \( \lambda \) \( \in \Lambda \). \( \lambda \) is a frame for (15).

Proposition 2.1. Let \( g = e^{-\pi |x|^2}, \ x \in \mathbb{R}^d \). Then \( G(g, \alpha, \beta) \) is a Gabor Frame for \( L^2(\mathbb{R}^d) \) if and only if \( \alpha \beta < 1 \).

See [11] Theorem 7.5.3 and [19][20] for the details.

2.2. Gelfand Shilov Spaces. The Gelfand-Shilov spaces are subspaces of the Schwartz class \( \mathcal{S}(\mathbb{R}^d) \) introduced in [2] to give information about how fast a function \( f \in \mathcal{S}(\mathbb{R}^d) \) and its derivatives decay at infinity. We recall only the main properties used in this paper, the main references on this topic are [5] and [17]. Given \( s, r \geq 0, \) \( f \) is in the Gelfand-Shilov type space \( \mathcal{S}^s_r(\mathbb{R}^d) \) if there exist constants \( A, B > 0 \) such that

\[
|x^n \partial^\beta f(x)| \lesssim A^{\alpha} B^{\beta} |(\alpha!)^r (\beta!)^s, \alpha, \beta, \in \mathbb{Z}^d_+.
\]

The space \( \mathcal{S}^s_r(\mathbb{R}^d) \) is non-trivial if and only if \( r + s > 1 \) or \( r + s = 1 \) and \( r, s > 0 \). It can be shown that the smallest non-trivial space with \( r = s \) is given by \( \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \). We observe that \( \mathcal{S}^{s_1}_{r_1}(\mathbb{R}^d) \subset \mathcal{S}^{s_2}_{r_2}(\mathbb{R}^d) \) for \( s_1 \leq s_2 \) and \( r_1 \leq r_2 \). Moreover, we have:

\[
f \in \mathcal{S}^s_r(\mathbb{R}^d) \iff \hat{f} \in \mathcal{S}^s_r(\mathbb{R}^d).
\]

Therefore the spaces \( \mathcal{S}^s_r(\mathbb{R}^d) \) are invariant under the action of the Fourier Transform. Functions of type \( f(x) = e^{-a|x|^2} \), with \( a > 0 \) belong to \( \mathcal{S}^{1/2}_{1/2}(\mathbb{R}^d) \).

An equivalent condition for \( f \in \mathcal{S}(\mathbb{R}^d) \) to be in \( \mathcal{S}^s_r(\mathbb{R}^d) \) is as follows:

**Proposition 2.2.** Let \( r, s > 0 \) and \( r + s \geq 1 \). For \( f \in \mathcal{S}(\mathbb{R}^d) \) the following conditions are equivalent:

a) \( f \in \mathcal{S}^{s}_{r}(\mathbb{R}^d) \);

b) There exist positive constants \( h, k \) such that

\[
\|fe^{h|x|^{1/r}}\|_{\infty} < \infty \quad \text{and} \quad \|\hat{f}e^{k|\omega|^{1/s}}\|_{\infty} < \infty.
\]

This result is contained in [17] Theorem 6.1.6.

The Gelfand-Shilow spaces were used in [2] as symbol space to characterize the behavior of the Gabor matrix of the corresponding pseudodifferential operators. Precisely, the un-weighted version of [2] Theorem 4.2] can be stated as follows:
Theorem 2.3. Let \( s \geq \frac{1}{2}, g \in S^s_s(\mathbb{R}^d) \) and \( \sigma \in \mathcal{C}^\infty(\mathbb{R}^d) \). Then the following properties are equivalent:

a) The symbol \( \sigma \) satisfies
\[
|\partial^\alpha \sigma(z)| \lesssim C|\alpha|^s, \quad \forall \ z \in \mathbb{R}^d, \forall \alpha \in \mathbb{Z}^{2d}_+,
\]

b) There exists \( \varepsilon > 0 \) such that
\[
|\langle \sigma^W(\pi(z)g, \pi(w)g) \rangle| \lesssim e^{-\varepsilon |w-z|^s}, \quad \forall \ z, w \in \mathbb{R}^d.
\]

where \( \sigma^W(x,D) \) indicates the Weyl quantization defined as
\[
\sigma^W(x,D)f(x) := \int e^{2\pi i(x-y)\omega} \sigma \left( \frac{x+y}{2}, \omega \right) f(y)dyd\omega.
\]

2.3. Metaplectic Operators. The Metaplectic representation is a powerful tool to study certain classes of PDEs. We briefly recall the main concepts and the results needed later in this paper. We refer to \[8,22\]. Let \( d \in \mathbb{N} \) being the dimension, then the Symplectic group is defined to be
\[
\text{Sp}(d, \mathbb{R}) = \left\{ g \in \text{GL}(2d, \mathbb{R}) : t^g J g = J \right\},
\]
where
\[
J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix}.
\]
Together with the group structure, it is useful to define the Symplectic algebra. Let \( d \in \mathbb{N} \), then the Symplectic algebra is defined to be
\[
\text{sp}(d, \mathbb{R}) = \{ A \in \mathcal{M}(2d, \mathbb{R}) / e^{tA} \in \text{Sp}(d, \mathbb{R}) \}.
\]
The metaplectic representation \( \mu \) is a unitary representation of (double cover of) the Symplectic group \( \text{Sp}(d, \mathbb{R}) \) on \( L^2(\mathbb{R}^d) \). We shall study the unitary operator \( \mu(t) \) giving the solution for the Harmonic Repulsor in Section 4. Precisely \[8, \text{Theorem 4.51}\] gives the following explicit representation for the unitary operator.

Proposition 2.4. Consider \( \mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{Sp}(d, \mathbb{R}) \) with \( A \neq 0 \) then
\[
\mu(A)f(x) = (\det A)^{-1/2} \int e^{-\pi x CA^{-1} x + 2\pi i \omega A^{-1} x + \pi i \omega A^{-1} B \omega} \hat{f}(\omega)d\omega
\]
For the general integral representation of a metaplectic operator see \[22\]. Consider now the following Cauchy problem
\[
\begin{cases}
   i\partial_t u - H_{\mathcal{A}} u = 0 \\
   u(0, x) = u_0(x),
\end{cases}
\]
where the Hamiltonian \( H_{\mathcal{A}} \) represents the Weyl quantization of a quadratic form on \( \mathbb{R}^{2d} \). Indeed every matrix \( \mathcal{A} \in \text{sp}(d, \mathbb{R}) \) defines a quadratic form \( \mathcal{P}(x, \omega) \in \mathbb{R}^{2d} \) as follows:
\[
\mathcal{P}(x, \omega) = -\frac{1}{2} \mathcal{A}^T(x, \omega) \mathcal{A}^* J(x, \omega).
\]
Setting \( \mathcal{A} = \begin{pmatrix} A & B \\ C & tA \end{pmatrix} \in \text{sp}(d, \mathbb{R}) \) we have
\[
\mathcal{P}(x, \omega) = \frac{1}{2} \omega B \omega - \omega A x - \frac{1}{2} x C x.
\]
Lemma 2.5. Let Asymptotic Integration can be used to give an estimate of its behavior. Thus, using Proposition 2.2, we would obtain that positive constants
\[ r C \]

From the Weyl quantization, the quadratic form in (21) corresponds to the Weyl operator \( P_A^\omega = P_A^\omega(D, X) \) defined by
\[ 2\pi P_A^\omega = -\frac{1}{4\pi} \sum_{j,k=1}^d B_{j,k} \frac{\partial^2}{\partial x_j x_k} + i \sum_{j,k=1}^d A_{j,k} x_k \frac{\partial}{\partial x_j} + i \frac{1}{2} \text{Tr}(A) - \pi \sum_{j,k=1}^d C_{j,k} x_j x_k. \]

The operator \( H_A = 2\pi P_A^\omega(D, X) \) is the Hamiltonian operator. The evolution operator which provides the solution to (20) is then given by
\[ e^{itH_A} = \mu(e^{iA}). \]

Hence, the solution to (20) is \( u(t) = e^{itH_A}u_0 = \mu(e^{iA})u_0 \)

2.4. Asymptotic Integration.
Throughout this section we will work in dimension \( d = 1 \). Consider the function \( f(x) = e^{-\alpha x^2k}, \alpha > 0, k \geq 1 \).

Let \( r = \frac{1}{2\pi} \); it is clear that there exists \( h > 0 \) such that \( \|f^h|x|^\frac{1}{2}\|_\infty < +\infty \).

We shall prove that given \( s = 1 - \frac{1}{2k} \) there exist \( \beta > 0 \) such that
\[ \|\hat{f}(\omega)e^{\beta|\omega|^\frac{2}{k}}\|_\infty < +\infty. \]

Thus, using Proposition 2.2 we would obtain that \( f(x) \in S^{1-\frac{1}{2k}}(\mathbb{R}) \) and consequently \( \hat{f} \in S^{1-\frac{1}{2k}}(\mathbb{R}) \).

We need to represent the Fourier transform of \( f \). The classic theory of Asymptotic Integration can be used to give an estimate of its behavior.

Lemma 2.5. Let \( \Omega = \mathbb{R}^+ \). Given \( n > 0 \) and \( y : \mathbb{R} \to \mathbb{C} \) such that \( y \in C^n(\mathbb{R}) \), consider \( y^{(n)} \) to be the \( n \)-th derivative of the function \( y \). Define \( r(t) = t^\alpha, q(t) = \pm t^\beta \) with \( \beta > -n \). Then, given a positive constant \( C \)
\[ y^{(n)}(t) - C q(t)y(t) = 0 \]

admits a fundamental system of solutions \( y_j, j = 1, \ldots, n \), such that
\[ y_j(t) \asymp |C(\pm t^\beta)|^{-\frac{n-1}{2\alpha}} e^{i\nu_j t} \sqrt{\frac{\pi}{2}}(\pm t)^{\frac{\beta}{2}} \]

for \( t > 0 \) and \( \nu_j \) are \( n \)-th roots of the unity.

The proof of this Lemma can be found in [23]. See also [7] and [14].

Theorem 2.6. Let \( f(x) = e^{-\alpha x^2}, \) with \( x \in \mathbb{R}, \alpha > 0 \) and \( k \geq 1 \). Then \( \hat{f} \) satisfies:
\[ |\hat{f}(\omega)| \leq C_{k,\alpha} e^{-\varepsilon_{k,\alpha}|\omega|^\frac{2k}{2k-1}}, \]

where \( C_{k,\alpha} = \frac{|2k\alpha|^{\frac{1}{2k-1}}}{(2\pi)^{\frac{1}{2k-1}}(2\alpha)^{\frac{1}{2k-1}}} \) and \( \varepsilon_{k,\alpha} = \delta_k(2\pi)^{\frac{2k}{2k-1}} \left( \frac{2k-1}{2k} \right) \left( \frac{1}{2\alpha} \right)^{\frac{1}{2k-1}}, \) for some positive constants \( \delta_k \), as determined in the subsequent proof.

Proof.

The idea is to use Lemma 2.5. Let us start by differentiating the function
Remark 2.7. This result prove what we claimed at the beginning of this section. Indeed, there exists \( k > 0 \) such that \( \| f e^{i\alpha x^2} \|_s \leq \infty \), for \( s = 1 - \frac{1}{2k} \). Using Proposition 2.2, we have that \( f(x) = e^{-\alpha x^2} \in S'_{\alpha}(\mathbb{R}) = S_{1 - \frac{1}{2k}}(\mathbb{R}) \).

Consequently \( \hat{f} \in S'_{\alpha}(\mathbb{R}) = S_{1 - \frac{1}{2k}}(\mathbb{R}) \).

Corollary 2.8. Let \( f(x) = e^{-\beta_k x^2} \), with \( x \in \mathbb{R} \), \( \beta_k = (-1)^{k-1} \alpha \), \( \alpha > 0 \) and \( k \geq 1 \). Then \( \hat{f} \) satisfies:

\[
|\hat{f}(\omega)| \leq C_k \alpha e^{-\varepsilon_k \alpha |\omega|^{\frac{2k}{2k-1}}},
\]
where $C_{k,\alpha} = \frac{|2k\alpha|^{k-1}}{(2\pi)^{2k-1}}$ and $\varepsilon_{k,\alpha} = (2\pi)^{2k} \left( \frac{2k-1}{2k} \right) \left( \frac{1}{2k\alpha} \right)^{\frac{1}{2k-1}}$.

Proof. Following the path of the proof above, one can easily obtain the analog of (28), that is

$$|\hat{f}(\omega)| \leq \frac{(2\pi)^{2k}(-1)^k \omega}{2k\beta_k} e^{-\frac{k-1}{2k-1} \Re \left\{ \nu_j \int_0^\infty \left[ \frac{(2\pi)^{2k}(-1)^k \omega}{2k\beta_k} \right] \frac{1}{2k-1} dt \right\}}$$

$$= \frac{(2\pi)^{2k} \omega}{2k\alpha} e^{-\frac{k-1}{2k-1} \Re \left\{ \nu_j \int_0^\infty \left[ \frac{(2\pi)^{2k} \omega}{2k\alpha} \right] \frac{1}{2k-1} dt \right\}}$$

$$= \frac{(2k\alpha)^{k-1}}{(2\pi)^{2k-1} |\omega|^{k-1}} e^{-\frac{k-1}{2k-1} \Re \left\{ \nu_j \int_0^\infty \left[ \frac{(2k\alpha)^{k-1}}{(2\pi)^{2k-1} |\omega|^{k-1}} \right] \frac{1}{2k-1} dt \right\}}$$

$$\leq C_a \frac{(2k\alpha)^{k-1}}{(2\pi)^{2k-1} |\omega|^{k-1}} e^{-\frac{k-1}{2k-1} \Re \left\{ \nu_j \int_0^\infty \left[ \frac{(2k\alpha)^{k-1}}{(2\pi)^{2k-1} |\omega|^{k-1}} \right] \frac{1}{2k-1} dt \right\}} = C_k, \quad \omega \in \Omega.$$

In this case $\delta_k = 1$, since we are in the “odd” case described in the proof of 2.6. Hence the result follows. □

3. Heat Equation

3.1. Heat Equation. We will focus on the numerical representation of solutions for the following Cauchy problem:

$$\begin{align*}
\partial_t u + (-\rho \Delta) u &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \\
u(0, x) &= u_0(x),
\end{align*}$$

(31)

where $u_0 \in \mathcal{S}(\mathbb{R}^d)$ and the parameter $\rho$ is the thermal diffusion. The solution $u(t, x)$ to this Cauchy Problem can be represented by the action of the families of Fourier multipliers $\sigma_\rho(t, D)$ on the initial datum as follows:

$$u(t, x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \omega} \sigma_\rho(t, \omega) \hat{u}_0(\omega) d\omega = \mathcal{F}^{-1} (\sigma_\rho(t, \omega) \hat{u}_0)(x),$$

(32)

where $\sigma_\rho(t, x) = e^{-4\pi^2 \rho t |x|^2}$. The super-exponential sparsity of the Gabor matrix for the Heat Equation was proved in [2], and some numerical simulations were shown in [2, Sec. 6]. Here we refine those numerical estimates presenting a new exact representation of the solution via Gabor frames.

Theorem 3.1. For every $t \in \mathbb{R}$ consider the operator $\sigma_\rho(t, D)$ defined in (32). Let $g(x, \alpha, \beta)$ be a Gabor frame with a Gaussian window function $g$. Let $(m, n), (m', n') \in \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ with $\alpha \beta < 1$; then the modulus of the Gabor matrix of the operator $\sigma_\rho(t, D)$ is equal to:

$$|\langle \sigma_\rho(t, D) \pi(m, n) g, \pi(m', n') g \rangle| = (2 + 4\pi \rho t)^{-\frac{d}{2}} e^{-\pi \left[ |m|^2 + |n|^2 + \frac{1}{\alpha^2 \beta^2} (|m-m'|^2 + |n-n'|^2) \right]}$$

(33)
Proof. The calculation of the Gabor matrix for the Fourier multiplier reduces to
\begin{equation}
|\langle \sigma_\rho(t, D)\pi(m, n)g, \pi(m', n')g \rangle| = \left| \mathcal{F}^{-1} \left( e^{-4\rho^2\pi^2|\cdot|^2} \mathcal{F}(\pi(m, n)g), \pi(m', n')g \right) \right|
\end{equation}
We recall that for \( a > 0 \)
\begin{equation}
e^{-ax^2}(k) = \sqrt{\frac{\pi}{a}} e^{-\frac{x^2k^2}{a}}
\end{equation}
Using Plancherel’s Theorem and Relation (35) above, Equation (34) can be restated as follows
\begin{align*}
&\left| \langle e^{-4\rho^2\pi^2|\cdot|^2} \mathcal{F}(\pi(m, n)g), \mathcal{F}(\pi(m', n')g) \rangle \right| \\
&= \left| \int_{\mathbb{R}^d} e^{-4\rho^2\pi^2|\cdot|^2} T_{\rho} M_{m} g(x) T_{\rho'} M_{m'} g(x) dx \right| \\
&= \left| \int_{\mathbb{R}^d} e^{-4\rho^2\pi^2|x|^2} e^{-2\pi i m(x-n)} e^{-\pi|x-n|^2} e^{2\pi i m'(x-n')} e^{-\pi|x-n'|^2} dx \right| \\
&= \int_{\mathbb{R}^d} e^{-2\pi i \xi[(m-m')+i(n+n')]} e^{-2\pi\rho^2 |x|^2} dx \\
&= \frac{1}{2\pi^{d/2}}\Gamma\left(\frac{d}{2}\right) e^{-\frac{1}{4\rho^2}((m-m')+i(n+n'))^2} \\
&= e^{-\pi((|n'|^2+|n|^2)(2+4\rho)|t|)} e^{-\pi((|m|^2+|m|'+|n|^2)+|m'-n'|^2)}.
\end{align*}
That is the claim. \( \square \)

Remark 3.2. We claim that our estimate is more accurate than the one in [2]. To figured this out, first one can check that
\begin{equation}
e^{-\frac{\pi}{2} \frac{r^2}{1+2\pi^2 r} (|n'|^2+|n|^2)} \leq e^{\pi \frac{r^2}{1+2\pi^2 r} (|n'|^2+|n|^2)}
\end{equation}
and thus
\begin{equation}
e^{-\pi((|n'|^2+|n|^2)} e^{\pi \frac{r^2}{1+2\pi^2 r} (|n'|^2+|n|^2)} = e^{-\frac{2\pi^2 r^2}{1+2\pi^2 r} (|n'|^2+|n|^2)} \leq e^{-\frac{2\pi^2 r^2}{1+2\pi^2 r} (|n'|^2+|n|^2)}.
\end{equation}
Then, it is easy to see that
\begin{align*}
e^{-\pi((|n'|^2+|n|^2)} (2+4\pi r) - d/2 e^{-\frac{\pi}{2} \frac{r^2}{1+2\pi^2 r} (|m|^2+|n|^2)+|n'|^2)} \\
\leq (2+4\pi r) - d/2 e^{-\frac{\pi}{2} \frac{r^2}{1+2\pi^2 r} (|m'-n'|^2)} \\
\leq (2+4\pi r) - d/2 e^{-\pi((|m|^2+|n'|^2)}.
\end{align*}
Hence (33) implies
\begin{equation}
|\langle \sigma_\rho(t, D)\pi(m, n)g, \pi(m', n')g \rangle| \leq e^{-\varepsilon(|m-m'|^2+|n-n'|^2)}
\end{equation}
which is the upper bound for the Gabor Matrix given in [3]. This proves that our estimate is more accurate, as expected.
3.2. Generalized Heat Equation. The Cauchy problem for the Generalized Heat Equation can be stated as follows:

\[
\begin{align*}
\partial_t u + (-\Delta^k) u &= 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \\
u(0, x) &= u_0(x),
\end{align*}
\]

with \( u_0 \in S(\mathbb{R}) \). The solution \( u(t, x) \) to this Cauchy Problem can be represented by the action of the families of Fourier multipliers \( \sigma_k(t, D) \) on the initial datum as follows:

\[
u(t, x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \omega} \sigma_k(t, \omega) \hat{u}_0(\omega) d\omega = \mathcal{F}^{-1}(\sigma_k(t, \omega) \hat{u}_0)(x),
\]

where \( \sigma_k(t, \omega) = e^{t(2\pi i)^{2k}} \sigma_k(t, (2\pi x)^{2k}) \).

We shall give an estimate from above of the Gabor matrix of \( \sigma_k(t, D) \).

First we need a preliminary result, which gives a bound for the convolution of super-exponential functions.

**Lemma 3.3.** Let \( w_{s,\varepsilon}(z) = e^{-\varepsilon|z|^\frac{1}{s}}, \quad z \in \mathbb{R}^d \) and \( s > \frac{1}{2} \). Then

\[
(w_{s,\varepsilon} * w_{s,\varepsilon})(z) \lesssim e^{-\varepsilon 2^{\frac{s-1}{s}}|z|^\frac{1}{s}}.
\]

**Proof.** Expanding the convolution product between the weight functions one has:

\[
(w_{s,\varepsilon} * w_{s,\varepsilon})(z) = \int_{\mathbb{R}^d} e^{-\varepsilon|x|^\frac{1}{s}} e^{-\varepsilon|z-x|^\frac{1}{s}} dx.
\]

Consider the set \( N_z := \{ x \in \mathbb{R}^d : |z - x| \leq \frac{|z|}{2} \} \). If \( x \in N_z \) then \( |x| \geq \frac{|z|}{2} \); therefore

\[
e^{-\varepsilon |x|^\frac{1}{s}} \lesssim e^{-\varepsilon 2^{-\frac{s-1}{s}}|z|^\frac{1}{s}}.
\]

Now, using the previous results:

\[
(w_{s,\varepsilon} * w_{s,\varepsilon})(z) = \int_{\mathbb{R}^d} e^{-\varepsilon |x|^\frac{1}{s}} e^{-\varepsilon |z-x|^\frac{1}{s}} dx = \int_{N_z} e^{-\varepsilon |z|^\frac{1}{s}} \left( \int_{N_z^c} e^{-\varepsilon |x|^\frac{1}{s}} dx + \int_{N_z^c} e^{-\varepsilon |z-x|^\frac{1}{s}} dx \right) \lesssim e^{-\varepsilon 2^{-\frac{s-1}{s}}|z|^\frac{1}{s}} = w_{s,\varepsilon 2^{-\frac{s-1}{s}}}.
\]

as desired. \( \square \)

**Theorem 3.4.** Let \( G(g, \alpha, \beta) \) be a Gabor Frame with a Gaussian window function \( g \). Let \( \lambda = (m, n), \nu = (m', n') \in \alpha \mathbb{Z} \times \beta \mathbb{Z} \) with \( \alpha \beta < 1 \). Then

\[
|\langle \sigma_k(t, D) \pi(\lambda) g, \pi(\nu) g \rangle| \lesssim C_{t, k} e^{-\tilde{c}_{k,t} 2^{-\frac{s}{2}} \nu_0^\frac{1}{2}},
\]

with \( s = \frac{2k}{2k-1} \), \( \tilde{c}_{k,t} = \left( \frac{2k-1}{2kt} \right)^{\frac{1}{2k-1}} 2^{-\frac{k}{2k-1}} \) and \( C_{t,k} = |2k| \frac{k-1}{2k-1} \).
Proof. Considering equality (37) and using the Plancherel’s Theorem, it follows that:

\[
|\langle \sigma_k(t, D) \pi(\lambda) g, \pi(\nu) g \rangle| = |\langle \sigma_k(t, \omega) \mathcal{F}(\pi(\lambda) g), \mathcal{F}(\pi(\nu) g) \rangle| = \left| \int_{\mathbb{R}^d} e^{i(-1)^k(2\pi\omega)^{2k}} \mathcal{F}\left( \pi(\lambda)e^{-\pi x^2} \right)(\omega) \overline{\mathcal{F}\left( \pi(\nu)e^{-\pi x^2} \right)(\omega)} d\omega \right|
\]

(39)

Recalling that \( \lambda = (m, n) \) and \( \nu = (m', n') \), then the Fourier transform of the time-frequency shift reads:

\[
\mathcal{F}(\pi(\lambda) g) = T_n M_{-m} g = e^{-2\pi i m(\omega-n)} g(\omega-n).
\]

Similar relations works for \( \nu \). Hence (39) becomes,

\[
\left| \int_{\mathbb{R}^d} e^{i(-1)^k(2\pi\omega)^{2k}} e^{-2\pi i m(\omega-n)} e^{2\pi i n'(\omega-n')} \right| d\omega.
\]

with \( \alpha = (-1)^{k-1} \left( 2\pi \right)^{2k} \) and \( \theta = ((m-m') + i(n+n')) \).

Now we want to use Corollary 2.8. Recall that (30) reads:

\[
\left| \mathcal{F}\left( e^{-\omega x^2} \right) \right| \leq C_{k,\alpha} e^{\epsilon_{k,\alpha} |\omega|^2},
\]

where \( C_{k,\alpha} = \frac{[2k\alpha]^{k-1}}{2\pi^{k+1}} \), \( \epsilon_{k,\alpha} = \pi \left( \frac{2k-1}{2k} \right) \left( \frac{1}{2k\alpha} \right) \frac{1}{2^{k+1}} \) and \( |\omega| \) is the modulus of \( \omega \in \mathbb{R} \).

Using \( \mathcal{F}(f \cdot g) = (\hat{f} \ast \hat{g}) \), and Lemma 3.3 the last integral can be restated as follows:

\[
\leq C_{k,\alpha} e^{-\pi(n^2+n'^2)} \left( e^{-\epsilon_{k,\alpha} |x|^2} \right) \left( e^{-\epsilon_{k,\alpha} |x'|^2} \right) \left( (m-m') + i(n+n') \right) \frac{1}{2^{k+1}}.
\]
It is well known that \( \forall a, b > 0, p > 0 \) the following inequality holds:

\[
(a + b)^p \leq 2^p (a^p + b^p).
\]

Thus, putting \( a = |m - m'|^2 \), \( b = |n - n'|^2 \) and \( p = \frac{k}{2k-1} \) it follows that

\[
((m - m')^2)^{\frac{k}{2k-1}} + ((n - n')^2)^{\frac{k}{2k-1}} \geq 2^{-\frac{k}{2k-1}} ((m - m')^2 + (n - n')^2)^{\frac{k}{2k-1}}.
\]

This inequality is useful in the calculation before to reach the desired estimate:

\[
|\langle \sigma_k(t, D)\pi(m, n)g, \pi(m', n')g \rangle| \\
\lesssim |C_{k,\alpha}e^{-\frac{2}{
u}(m'-n')^2}e^{-\varepsilon_{k,\alpha}}2^{-\frac{2k}{2k-1}}((m'-n')^2)^{\frac{k}{2k-1}}| \\
\lesssim |C_{k,\alpha}e^{-\varepsilon_{k,\alpha}}2^{-\frac{2k}{2k-1}}((m'-n')^2)^{\frac{k}{2k-1}}| \\
\lesssim |C_{k,\alpha}e^{-\varepsilon_{k,\alpha}}2^{-\frac{2k}{2k-1}}((m'-n')^2+(n'-n')^2)^{\frac{k}{2k-1}}| \\
\lesssim |C_{k,\alpha}e^{-\varepsilon_{k,\alpha}}2^{-\frac{2k}{2k-1}}(\nu)\lambda^{\frac{k}{2k-1}}|,
\]

where \( \lambda = (m, n) \) and \( \nu = (m', n') \).

If we exploit the constant \( \alpha = \alpha(k, t) = t(2\pi)^{2k} \) into \( C_{k,\alpha} \) and \( \varepsilon_{k,\alpha} \), we obtain:

\[
C_{k,\alpha} = \frac{|2k\alpha|}{(2\pi)^{2k(k-1)}} = \frac{|2kt(2\pi)^{2k}|}{(2\pi)^{2k(k-1)}} = |2kt|^{\frac{k+1}{2k-1}}.
\]

Analogously

\[
\varepsilon_{k,\alpha}2^{-\frac{k}{2k-1}} = (2\pi)^{\frac{k}{2k-1}} \left( \frac{2k-1}{2k} \right) \left( \frac{1}{2k(2\pi)^{2k}} \right)^{\frac{1}{2k-1}} 2^{-\frac{k}{2k-1}} \\
= \left( \frac{2k-1}{2k} \right) \left( \frac{1}{2k} \right)^{\frac{1}{2k-1}} 2^{-\frac{k}{2k-1}}.
\]

Finally, renaming the two constants

\[
C_{k,t} = |2kt|^{\frac{k+1}{2k-1}} \quad \text{and} \quad \tilde{\varepsilon}_{k,t} = \left( \frac{2k-1}{2k} \right) \left( \frac{1}{2k} \right)^{\frac{1}{2k-1}} 2^{-\frac{k}{2k-1}},
\]

the result follows.

\[\square\]

**Remark 3.5.** Thanks to Remark 2.7, it is clear that \( \sigma_k(x, t) = e^{-t(2\pi x)^{2k}} \) fulfills the hypothesis of Theorem 2.3 with \( s = \frac{2k-1}{2k} \). From Theorem 3.4, we get

\[
|\langle \sigma_k(t, D)\pi(\lambda)g, \pi(m', n')g \rangle| \lesssim C_{t,k}e^{-\tilde{\varepsilon}_{k,t}2^{-\frac{1}{2}}(m,n)\cdot(m', n')}^{\frac{1}{2}},
\]

with \( s = \frac{2k}{2k-1} \) and that is consistent with (18).
4. Schrödinger equation with Hamiltonian $\mathcal{H}_A = -\frac{1}{4\pi} \Delta u + \pi |x|^2$

The Cauchy problem for the Harmonic Repulsor can be stated as follows:

\[ i\partial_t u - \frac{1}{4\pi} \Delta u + \pi |x|^2 = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \]
\[ u(0, x) = u_0(x), \]

$u_0 \in \mathcal{S}(\mathbb{R}^d)$. The solution can be calculated using the metaplectic operator in Section 2.3. Here the Hamiltonian $H_A = -\frac{1}{4\pi} \Delta u + \pi |x|^2$ can be written as $H_A = 2\pi P_A^* A$, with $P_A^*$ with $B_{j,k} = C'_{j,k} = \delta_{j,k}$ and $A_{j,k} = D_{j,k} = 0$. Therefore, the symplectic matrix related to (41) is $A = \begin{pmatrix} 0 & I_d \\ I_d & 0 \end{pmatrix}$. We know that the solution of the Cauchy problem is given by $\mu(e^{\tau A})u_0(x)$, where $\mu$ is the metaplectic representation. So we have to calculate the exponential of the matrix $A$. The diagonal decomposition of $A$ is

\[ A = \frac{1}{2} \begin{pmatrix} I_d & -I_d \\ I_d & I_d \end{pmatrix} \begin{pmatrix} I_d & 0 \\ 0 & -I_d \end{pmatrix} \begin{pmatrix} I_d & I_d \\ -I_d & I_d \end{pmatrix}. \]

Then we have

\[ e^{\tau A} = \frac{1}{2} \begin{pmatrix} I_d & -I_d \\ I_d & I_d \end{pmatrix} \begin{pmatrix} e^{\tau I_d} & 0 \\ 0 & e^{-\tau I_d} \end{pmatrix} \begin{pmatrix} I_d & I_d \\ -I_d & I_d \end{pmatrix}. \]

It is easy to see that $e^{\pm \tau t} = e^{\pm t} \begin{pmatrix} I_d & 0 \\ 0 & I_d \end{pmatrix}$. Thus,

\[ e^{\tau A} = \begin{pmatrix} \frac{e^{\tau} + e^{-\tau}}{2} I_d & \frac{e^{\tau} - e^{-\tau}}{2} I_d \\ \frac{e^{\tau} - e^{-\tau}}{2} I_d & \frac{e^{\tau} + e^{-\tau}}{2} I_d \end{pmatrix}. \]

Using the definition of the hyperbolic sine and cosine, we obtain

\[ e^{\tau A} = \begin{pmatrix} \cosh(t)I_d & \sinh(t)I_d \\ \sinh(t)I_d & \cosh(t)I_d \end{pmatrix}. \]

Now, we can use (19) to calculate the solution of (41) with initial datum $u_0 = \pi(m, n)g$, that is

\[ u(t, x) = \mu(e^{\tau A})\pi(m, n)g \]
\[ = \frac{1}{\cosh(t)} \int_{\mathbb{R}^d} e^{\pi i \tanh(t) x^2 + \frac{2\pi i x \omega}{\cosh(t)} - \pi i \tanh(t) \omega^2} F(\pi(m, n)g) \omega d\omega. \]

Fix $t \in \mathbb{R}$, in order to calculate the Gabor matrix of the operator $T$, we compute first (42).

**Lemma 4.1.** Consider the metaplectic operator (19) and the time-frequency shifts of the Gaussian $\pi(m, n)g(x) = M_m T_n e^{-\pi |x|^2}$, where $(m, n) \in \Lambda$ with $\Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d$ and $\alpha, \beta < 1$. Then one has

\[ T\pi(m, n)g(x) = C_t e^{-\pi (m+i n)^2 \cosh(t) \cosh(t)} e^{-\pi x^2 \frac{\cosh(t) - i \sinh(t)}{\cosh(t) + i \sinh(t)}} + \frac{2\pi (m+i n)}{\cosh(t) + i \sinh(t)}, \]

where $C_t = \left( \frac{1}{\cosh(t) + i \sinh(t)} \right)^d e^{-\pi n^2 + 2\pi im \cdot n}$. 

Proof. We expand (42):

\[ u(t, x) = \left( \frac{1}{\cosh(t)} \right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{\pi i \tanh(t) x^2 + \frac{2 \text{Im} \omega}{\cosh(t)} - \pi i \tanh(t) \omega^2} \mathcal{F}(\pi(m, n) g)(\omega) d\omega = \left( \frac{1}{\cosh(t)} \right)^{\frac{d}{2}} e^{\pi i \tanh(t) x^2} \int_{\mathbb{R}^d} 2\pi i e^{\left[ \frac{\omega}{\cosh(t)} - \frac{\text{tanh}(t) \omega^2}{2} \right]} T_n M - \tilde{g}(\omega) d\omega = \left( \frac{1}{\cosh(t)} \right)^{\frac{d}{2}} e^{\pi i \tanh(t) x^2} \int_{\mathbb{R}^d} e^{-2\pi \text{Im}(\omega - n) - \pi |\omega - n|^2} d\omega = \left( \frac{1}{\cosh(t)} \right)^{\frac{d}{2}} e^{\pi i \tanh(t) x^2} e^{-\pi n^2 + 2\pi \text{Im} n}
\]

Using (35), we can restate (43) as follows

\[ C_t e^{\pi i \tanh(t) x^2} e^{-\pi \left[ m + \text{Im} - \frac{\pi}{\cosh(t)} \right]^2 \left( \frac{\cosh(t)}{\cosh(t) + \text{Im} \sinh(t)} \right)}, \]

with

\[ C_t = \left( \frac{1}{\cosh(t) + i \sinh(t)} \right)^{\frac{d}{2}} e^{-\pi n^2 + 2\pi \text{Im} n}. \]

Since

\[ u(t, x) = C_t e^{\pi i \tanh(t) x^2 - \pi \left[ m + \text{Im} - \frac{\pi}{\cosh(t)} \right]^2 \left( \frac{\cosh(t)}{\cosh(t) + \text{Im} \sinh(t)} \right)} \]

\[ = C_t e^{\pi i \tanh(t) x^2 - \pi \left[ m + \text{Im} - \frac{\pi}{\cosh(t)} \right]^2 \left( \frac{\cosh(t)}{\cosh(t) + \text{Im} \sinh(t)} \right)} - \pi \left[ -2(m + \text{Im}) \frac{\pi}{\cosh(t)} + \frac{x^2}{\cosh(t)} \right] \left( \frac{\cosh(t)}{\cosh(t) + \text{Im} \sinh(t)} \right) \]

\[ = C_t e^{-\pi x^2 \left[ \frac{1 - \text{Im} \sinh(t)}{\cosh(t)} \right] \frac{2\pi x (m + \text{Im})}{\cosh(t) + \text{Im} \sinh(t)}} \]

\[ \cdot e^{\frac{n (m^2 + 2\text{Im} n - n^2)}{\cosh(t) + \text{Im} \sinh(t)}} \]

\[ = C_t e^{-\pi x^2 \left[ \frac{\cosh(t)^2 - \text{Im} \sinh(t)}{\cosh(t) \cosh(t) + \text{Im} \sinh(t)} \right] \frac{2\pi x (m + \text{Im})}{\cosh(t) + \text{Im} \sinh(t)}} \]

\[ \cdot e^{\frac{n (m^2 + 2\text{Im} n - n^2)}{\cosh(t) + \text{Im} \sinh(t)}} \]

\[ = C_t e^{-\pi x^2 \left[ \frac{\cosh(t) - \text{Im} \sinh(t)}{\cosh(t) \cosh(t) + \text{Im} \sinh(t)} \right] \frac{2\pi x (m + \text{Im})}{\cosh(t) + \text{Im} \sinh(t)}} \]

\[ \cdot e^{\frac{n (m^2 + 2\text{Im} n - n^2)}{\cosh(t) + \text{Im} \sinh(t)}} \]

\[ = \tilde{C}_t e^{-\pi x^2 \left[ \frac{\cosh(t) - \text{Im} \sinh(t)}{\cosh(t) \cosh(t) + \text{Im} \sinh(t)} \right] \frac{2\pi x (m + \text{Im})}{\cosh(t) + \text{Im} \sinh(t)}}, \]

where

\[ \tilde{C}_t = \left( \frac{1}{\cosh(t) + i \sinh(t)} \right)^{\frac{d}{2}} e^{-\pi n^2 + 2\pi \text{Im} n - \frac{n (m^2 + 2\text{Im} n - n^2)}{\cosh(t) + \text{Im} \sinh(t)}}. \]
Hence the result is proved.  

The computation of the Gabor matrix \( T_{m,n,m',n'} = \langle T_\pi(m,n)g, \pi(m',n')g \rangle \) of the metaplectic operator \( T_t \), reduces now to compute the inner product above.

**Theorem 4.2.** Let \( T_t \) be the operator defined in \((42)\) and \((m,n),(m',n') \in \Lambda \) with \( \Lambda = \alpha \mathbb{Z}^d \times \beta \mathbb{Z}^d \), \( \alpha \beta < 1 \). Then

\[
(45) \quad T_{m,n,m',n'} = C_t e^{-\frac{\pi}{2} |m|^2 + \frac{\pi}{2} |n|^2 + \frac{\pi}{2} |m'|^2 + \frac{\pi}{2} |n'|^2 + 2 \tanh(mn - m'n') - 2(mm' - nn')} ,
\]

where \( C_t = \frac{e^{i\psi}}{(2 \cosh(t))^{\frac{d}{2}}} \) and \( |e^{i\psi}| = 1 \).

**Proof.** Using Lemma 4.1,

\[
T_{m,n,m',n'} = \langle \tilde{C}_t e^{-\frac{\pi}{2} x^2} \left[ \frac{\cosh(t) - i \sinh(t)}{\cosh(t) + \sinh(t)} \right] + \frac{2\pi x (m+in)}{\cosh(t) + \sinh(t)}, \pi(m',n') \rangle
\]

\[
= \tilde{C}_t \int_{\mathbb{R}^d} e^{-\frac{\pi}{2} x^2} \left[ \frac{\cosh(t) - i \sinh(t)}{\cosh(t) + \sinh(t)} \right] + \frac{2\pi x (m+in)}{\cosh(t) + \sinh(t)} M_{m'} M_{n'} e^{-\frac{\pi}{2} |x|^2} dx
\]

\[
= \tilde{C}_t e^{-\frac{\pi}{2} m'^2} \int_{\mathbb{R}^d} e^{-\frac{\pi}{2} x^2} \left[ \frac{\cosh(t) - i \sinh(t)}{\cosh(t) + \sinh(t)} \right] + \frac{2\pi x (m+in)}{\cosh(t) + \sinh(t)} dx
\]

\[
= \tilde{C}_t e^{-\frac{\pi}{2} m'^2} \int_{\mathbb{R}^d} e^{-\frac{\pi}{2} x^2} \left[ \frac{\cosh(t) - i \sinh(t)}{\cosh(t) + \sinh(t)} + 1 \right]
\]

\[
\cdot \frac{2\pi x (m+in)}{\cosh(t) + \sinh(t)} dx
\]

\[
= \tilde{C}_t e^{-\frac{\pi}{2} m'^2} \int_{\mathbb{R}^d} e^{-\frac{\pi}{2} x^2} \left[ \frac{2\cosh(t)}{\cosh(t) + \sinh(t)} \right]
\]

\[
\cdot \frac{2\pi x (m+in)}{\cosh(t) + \sinh(t)} dx
\]

Using \((33)\), we obtain:

\[
(46) \quad T_{m,n,m',n'} = \tilde{C}_t e^{-\frac{\pi}{2} m'^2} \left( \frac{\cosh(t) + i \sinh(t)}{2 \cosh(t)} \right)^{\frac{d}{2}}
\]

\[
\cdot e^{-\frac{\pi}{2} \left[ \frac{(im - n) + (im' + n')(\cosh(t) + i \sinh(t))}{\cosh(t) + \sinh(t)} \right]^2} .
\]

Expanding \( \tilde{C}_t \), \((46)\) becomes:

\[
(47) \quad T_{m,n,m',n'} = \frac{1}{(2 \cosh(t))^{\frac{d}{2}}} e^{-\frac{\pi}{2} \frac{1}{\cosh(t)} \left( \frac{1}{\cosh(t)} + \sinh(t) \right)} \Phi(m,n,m',n,t).
\]

Now we have to give a clear formulation of \( \Phi \). The calculations that follow do not take care of the imaginary part which is always contained in a real-valued
function $\psi$.

$$
\Phi = 2 \left[ (|m'|^2 + |n|^2) + 2imn \right] \cdot \left[ \cosh(t) \cdot \left( \cosh^2(t) + \sinh^2(t) \right) \right] \\
+ \left\{ 2(m + in)^2 \cosh^2(t) \right\} (\cosh(t) - i \sinh(t)), \\
+ \left\{ \left[ (im - n) + (im' + n')(\cosh(t) + i \sinh(t)) \right]^2 \right\} \\
\cdot (\cosh(t) - i \sinh(t)), \\
= 2(|m'|^2 + |n|^2) (\cosh^3(t) + \cosh(t) \sinh^2(t)) \\
+ 2(|m|^2 + 2imn - |n|^2) \cdot (\cosh^3(t) - i \cosh^2(t) \sinh(t)) \\
+ (im - n)^2 \cdot (\cosh(t) - i \sinh(t)) - 2(im - n)(im' + n') \\
\cdot (\cosh^2(t) + \sinh^2(t)) \\
+ (im' + n')^2 (\cosh(t) + i \sinh(t)) (\cosh^2(t) + \sinh^2(t)) + \psi \\
= 2(|m'|^2 + |n|^2) (\cosh^3(t) + \cosh(t) \sinh^2(t)) + 2(|m|^2 - |n|^2) \cosh^3(t) \\
+ 4mn \cosh^2(t) \sinh(t) + (\cosh^2(t) - 2m \sinh(t)) \\
- 2 \left( -mn' - mn' - imn' \right) \cosh^2(t) + \sinh^2(t)) \\
\cdot \cosh^2(t) + \sinh^2(t)) \\
- 2m'n' \sinh(t) (\cosh^2(t) + \sinh^2(t)) + \psi \\
= [(|m|^2 + |n|^2) \cosh(t) + 2mn \sinh(t) + 2(mn' + nn')] \\
\cdot (\cosh^2(t) + \sinh^2(t)) \\
+ [(|m'|^2 + |n'|^2) \cosh(t) - 2m'n' \sinh(t)] \cdot (\cosh^2(t) + \sinh^2(t)) + \psi.
$$

Finally, using $[17]$ the Gabor matrix can be expressed as

$$
T_{m,n,m',n'} = C_t e^{-\frac{\psi}{2}} [m^2 + |n|^2 + |n'|^2 + |m'|^2 + 2 \tanh(t)(mn - m'n') - 2(mn' - mn')] \\
\text{where } C_t = \frac{e^{i\psi}}{(2 \cosh(t))^\frac{3}{2}} \text{ and } |e^{i\psi}| = 1. \text{ Thus, the Theorem is proved.} \, \square
$$

5. Numerical Result

In this section we show numerical examples to test the fastness of the Gabor coefficients’ decay, in dimension $d = 1, 2, 3$. In dimension $d = 1$ we will show the magnitude of the Short-Time Fourier transform of the solutions, i.e. the spectrogram of the solutions to the Cauchy problems of the previous sections. The initial datum we will use is provided by $M_n T_m g$ with $g$ Gaussian function. We shall represent the behavior of the solution in the phase space at different instants of time. In all our examples we will use a lattice on $\mathbb{Z}^{2d}$ with parameters $\alpha = 1, \beta = \frac{1}{2}$. In this way the Gabor system $G(e^{-\pi |x|^2}, 1, \frac{1}{2})$ is a frame for $L^2(\mathbb{R}^d)$.

5.1. Heat Equation. Numerical evaluations of the Gabor matrix for this problem are already treated in $[2]$. Here, using the exact representation of
Dependence of the coefficients decay from the Thermal Diffusion.

![Figure 5.1.](image)

The dissipative effect caused by the thermal diffusion $\alpha$.

The Gabor matrix, namely

$$\left|\langle \sigma(t, D)\pi(m, n)g, \pi(m', n')g \rangle\right| = (2 + 4\pi \rho t)^{-\frac{d}{2}} e^{-\pi \left[ n^2 + n'^2 + \frac{1}{2 + 4\pi \rho t} \left( m - m' \right)^2 + \frac{1}{n + n'^2} \right]}.$$  

we obtain a faster decay. Moreover, the equation above clarify that the growth of the diffusion factor $\rho$ and the time $t$ cause the same diffusive effect on the solution. In fact, if we fix the time variable and we let $\rho$ increase, we see a diffusion in the space variable, as shown in Figure 5.1. Notice that it is equivalent to fix $\rho = 1$ and let the time $t$ grow, as shown by 5.1.

### 5.2. Generalized Heat Equation.

In dimension $d = 1$, the Gabor matrix associated to the Cauchy problem (36) fulfills the estimate:

$$\left|\langle \sigma_k(t, D)\pi(m, n)g, \pi(m', n')g \rangle\right| \lesssim C_{t,k} e^{-\tilde{\varepsilon}_{k,t} 2^{-\frac{1}{s}} (\lambda - \nu)^{\frac{1}{2}}}$$

with $s = \frac{2k}{2k - 1}$, $\tilde{\varepsilon}_{k,t} = \left( \frac{2k - 1}{4k} \right) \frac{1}{2k - 1} 2^{-\frac{1}{2k - 1}}$ and $C_{t,k} = |4\pi k t|^{\frac{1}{2k - 1}}$. Using this equation we show the coefficients’ decay in dimension $d = 1$ together with their dependence from $k$ and $t$.

### 5.3. Harmonic Repulsor.

The exact expression of the Gabor matrix related to the Harmonic Repulsor is given by equation (45), that is

$$|T_{m,n,m',n'}| = Ce^{-\frac{\pi}{2} \left[ m^2 + n^2 + m'^2 + m'^2 + 2 \tanh(t)(mn - m'n') - 2(mm' - nn') \right]}$$

with $C = \frac{1}{(2\cosh(t))^2}$. Figure 5.3 shows that using this expression we obtain huge decays of the coefficients. In dimension $d = 2$ and $d = 3$ we obtain results similar to those in [4] for the case of the harmonic oscillator.
Magnitude of the coefficient for the Heat Equation at different instants of time $t$, in dimension $d = 2$.

Figure 5.2. The dissipative effect grows together with the time $t$. Similar effects are observable by increasing the thermal diffusion $\rho$.

5.3 resemble the behavior of the Harmonic Repulsor in the phase space. The Gaussian approaches the origin from the south-east and then goes to north-east. As the picture shows, although the Gaussian spread in the spatial variable, it remains concentrated in the Time-Frequency domain.
Magnitude of the coefficient for the Heat Equation at different instants of time $t$, in dimension $d = 3$.

Figure 5.3.
Magnitude of the coefficient for the Generalized Heat Equation at different instants of time $t$ and Laplacian powers $k$, in dimension $d = 1$.

Figure 5.4. This picture shows that the decay obtained at different instant times and powers $k$ is almost identical.
Magnitude of the coefficient for the Harmonic Repulsor at different instants of time $t$, in dimension $d = 2$.

Figure 5.5. This picture shows that the decay obtained at different instants of time is almost identical.
Magnitude of the coefficient for the Harmonic Repulsor at different instants of time $t$, in dimension $d = 3$.

Figure 5.6. This picture shows that the decay obtained at different instants of time is almost identical.
Contourplots of the STFT of Harmonic Repulsor’s solution at different instants of time $t$.

Figure 5.7. Contour plot for the Short-Time Fourier Transform of the solution of (41) in dimension $d = 1$ at different instants of time, with initial datum and window $u(x) = g(x) = e^{-\pi|x|^2}$.
Acknowledgments

I am sincerely grateful to Professor E. Cordero for the fruitful discussion, valuable advices, constructive criticism and constant review of this work. I would like to thank Professors E. Cordero and L. Rodino for inspiring this paper. I also wish to thank M. Borsero for the useful suggestions, and the final review aimed at improving the readability of the paper. Finally, I am thankful for the enormous work of the anonymous reviewer who suggested important corrections that gave consistency to the paper.

References

[1] Ole Christensen, Hans G Feichtinger, and Stephan Paukner. Gabor analysis for imaging. In Handbook of Mathematical Methods in Imaging, pages 1271–1307. Springer, 2011.
[2] E. Cordero, F. Nicola, and L. Rodino. Gabor representations of evolution operators. ArXiv e-prints, September 2012.
[3] Elena Cordero, Karlheinz Gröchenig, Fabio Nicola, and Luigi Rodino. Wiener algebras of Fourier integral operators. J. Math. Pures Appl. (9), 99(2):219–233, 2013.
[4] Elena Cordero, Fabio Nicola, and Luigi Rodino. Sparsity of Gabor representation of Schrödinger propagators. Appl. Comput. Harmon. Anal., 26(3):357–370, 2009.
[5] Elena Cordero, Fabio Nicola, and Luigi Rodino. Time-frequency analysis of Fourier integral operators. Commun. Pure Appl. Anal., 9(1):1–21, 2010.
[6] Elena Cordero, Fabio Nicola, and Luigi Rodino. Time-frequency analysis of Schroedinger propagators. In Evolution Equations of Hyperbolic and Schrödinger Type, pages 63–85. Springer, 2012.
[7] M. S. P. Eastham. Asymptotic formulae of Liouville-Green type for higher-order differential equations. J. London Math. Soc. (2), 28(3):507–518, 1983.
[8] Gerald B. Folland. Harmonic analysis in phase space, volume 122 of Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1989.
[9] I. M. Gelfand and G. E. Shilov. Generalized functions. Vol. 2-3. Academic Press, New York, 1967.
[10] Loukas Grafakos and Christopher Sansing. Gabor frames and directional time–frequency analysis. Applied and Computational Harmonic Analysis, 25(1):47–67, 2008.
[11] Karlheinz Gröchenig. Foundations of time-frequency analysis. Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA, 2001.
[12] Karlheinz Gröchenig. Time-frequency analysis of Sjöstrand’s class. Rev. Mat. Iberoam., 22(2):703–724, 2006.
[13] Christopher Heil. A basis theory primer. Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, New York, expanded edition, 2011.
[14] Don B. Hinton. Asymptotic behavior of solutions of $(ry^{(m)})^{(k)} \pm qy = 0$. J. Differential Equations, 4:590–596, 1968.
[15] M. P. Lamoureux and G. F. Margrave. An introduction to numerical methods of pseudodifferential operators. In Pseudo-differential operators, volume 1949 of Lecture Notes in Math., pages 79–133. Springer, Berlin, 2008.
[16] P. G. L. Leach. Sl(3, R) and the repulsive oscillator. J. Phys. A, 13(6):1991–2000, 1980.
[17] Fabio Nicola and Luigi Rodino. Global pseudo-differential calculus on Euclidean spaces, volume 4 of Pseudo-Differential Operators. Theory and Applications. Birkhäuser Verlag, Basel, 2010.
[18] Darian M Onchis, Pedro Real, and Gilbert-Rainer Gillich. Gabor frames and topology-based strategies for astronomical images. 2010.
[19] Kristian Seip. Density theorems for sampling and interpolation in the Bargmann-Fock space. I. J. Reine Angew. Math., 429:91–106, 1992.
[20] Kristian Seip and Robert Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. J. Reine Angew. Math., 429:107–113, 1992.
[21] Thomas Strohmer. Pseudodifferential operators and banach algebras in mobile communications. *Applied and Computational Harmonic Analysis*, 20(2):237 – 249, 2006.

[22] Hennie ter Morsche and Patrick J. Oonincx. On the integral representations for metaplectic operators. *J. Fourier Anal. Appl.*, 8(3):245–257, 2002.

[23] Wolfgang Wasow. *Asymptotic expansions for ordinary differential equations*. Dover Publications Inc., New York, 1987. Reprint of the 1976 edition.

Dipartimento di Matematica “Giuseppe Peano”, Università degli Studi di Torino, Via Carlo Alberto 10, 10123 Torino (TO), Italy.

*E-mail address: michele.berra@unito.it*