Long time existence of smooth solutions to 2D compressible Euler equations of Chaplygin gases with non-zero vorticity

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Abstract

For the 2D compressible isentropic Euler equations of polytropic gases with an initial perturbation of size \( \varepsilon \) of a rest state, it has been known that if the initial data are rotationally invariant or irrotational, then the lifespan \( T_\varepsilon \) of the classical solutions is of order \( O(\frac{1}{\varepsilon^2}) \); if the initial vorticity is of size \( \varepsilon^{1+\alpha} \) \((0 \leq \alpha \leq 1)\), then \( T_\varepsilon \) is of \( O(\frac{1}{\varepsilon^{1+\alpha}}) \). In the present paper, for the 2D compressible isentropic Euler equations of Chaplygin gases, if the initial data are a perturbation of size \( \varepsilon \), and the initial vorticity is of any size \( \delta \) with \( 0 < \delta \leq \varepsilon \), we will establish the lifespan \( T_\delta = O(\frac{1}{\delta}) \). For examples, if \( \delta = e^{-\frac{1}{\varepsilon^2}} \) or \( \delta = e^{-e^{\frac{1}{\varepsilon^2}}} \) are chosen, then \( T_\delta = O(e^{\frac{1}{\varepsilon^2}}) \) or \( T_\delta = O(e^{e^{\frac{1}{\varepsilon^2}}} \) although the perturbations of the initial density and the divergence of the initial velocity are only of order \( O(\varepsilon) \). Our main ingredients are: finding the null condition structures in 2D compressible Euler equations of Chaplygin gases and looking for the good unknown; establishing a new class of weighted space-time \( L^\infty - L^\infty \) estimates for the solution itself and its gradients of 2D linear wave equations; introducing some suitably weighted energies and taking the \( L^p \) \((1 < p < \infty) \) estimates on the vorticity.

Keywords. Compressible Euler equations, Chaplygin gases, vorticity, null condition, weighted \( L^\infty - L^\infty \) estimates, ghost weight, \( A_p \) weight.

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1 Introduction

The 2D compressible isentropic Euler equations are

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P &= 0,
\end{align*}
\]

(Conservation of mass),

\[\text{(Conservation of momentum),} \tag{1.1}\]

where \((t, x) = (t, x^1, x^2) \in \mathbb{R}^{1+2}_+ = [0, \infty) \times \mathbb{R}^2, \nabla = (\partial_1, \partial_2) = (\partial_{x^1}, \partial_{x^2}), \text{ and } u = (u_1, u_2), \rho, P \text{ stand for the velocity, density, pressure respectively. In addition, the pressure } P = P(\rho) \text{ is a smooth function of } \rho \text{ when } \rho > 0, \text{ moreover, } P'(\rho) > 0 \text{ for } \rho > 0.

For the polytropic gases (see [9]),

\[P(\rho) = A \rho^\gamma, \tag{1.2}\]

where \(A \) and \(\gamma \) (\(1 < \gamma < 3\)) are some positive constants.

For the Chaplygin gases (see [9] or [12]),

\[P(\rho) = P_0 - \frac{B}{\rho}, \tag{1.3}\]

where \(P_0 > 0 \) and \(B > 0\) are constants.

If \((\rho, u) \in C^1\) is a solution of (1.1) with \(\rho > 0\), then (1.1) is equivalent to the following form

\[
\begin{align*}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u + \frac{c^2(\rho)}{\rho} \nabla \rho &= 0,
\end{align*}
\]

(1.4)

where the sound speed \(c(\rho) := \sqrt{P'(\rho)}\).

Consider the initial data of (1.1) as follows

\[(\rho(0, x), u(0, x)) = (\bar{\rho} + \rho^0(x), u^0(x)), \tag{1.5}\]

where \(\bar{\rho} > 0\) is a constant, \(\bar{\rho} + \rho^0(x) > 0\, \text{ and } \rho^0(x), u^0(x) = (u_1^0(x), u_2^0(x)) \in C^\infty_0. \) When

\[\text{curl } u^0(x) := \partial_1 u_2^0 - \partial_2 u_1^0 \equiv 0, \tag{1.6}\]
as long as \((\rho, u) \in C^1\) for \(0 \leq t \leq T_0\), then \(\text{curl } u \equiv 0\) always holds for \(0 \leq t \leq T_0\). In this case, one can introduce the potential function \(\phi\) such that \(u = \nabla \phi\), then the Bernoulli's law implies \(\partial_t \phi + \frac{1}{2}|\nabla \phi|^2 + h(\rho) = 0\) with \(h'(\rho) = \frac{c^2(\rho)}{\rho}\) and \(h(\bar{\rho}) = 0\). By the implicit function theorem due to \(h'(\rho) > 0\) for \(\rho > 0\), then the density function \(\rho\) can be expressed as

\[
\rho = h^{-1} \left( -\partial_t \phi - \frac{1}{2}|\nabla \phi|^2 \right) =: H(\partial \phi),
\]

where \(\partial = (\partial_t, \nabla)\). Substituting (1.7) into the mass conservation equation in (1.1) yields that

\[
\partial_t (H(\partial \phi)) + \sum_{i=1}^{2} \partial_i (H(\partial \phi) \partial_i \phi) = 0.
\]

For any \(C^2\) solution \(\phi\), (1.8) can be rewritten as the following second order quasilinear equation

\[
\partial_t^2 \phi + 2 \sum_{k=1}^{2} \partial_k \phi \partial_k^2 \phi - c^2(\rho) \Delta \phi + \sum_{i,j=1}^{2} \partial_i \phi \partial_j \phi \partial_{ij}^2 \phi = 0,
\]

where \(c(\rho) = c(H(\partial \phi))\), and the Laplace operator \(\Delta := \sum_{i=1}^{2} \partial_i^2\). Without loss of generality and for simplicity, \(c(\bar{\rho}) = 1\) can be supposed, and then \(c^2(\rho) = 1 - 2\bar{\rho} c'(\bar{\rho}) \partial_t \phi + O(|\partial \phi|^2)\). Especially, in the case of the Chaplygin gases, (1.9) is

\[
\partial_t^2 \phi - \Delta \phi + 2 \sum_{k=1}^{2} \partial_k \phi \partial_k \partial \phi - 2 \partial_t \phi \Delta \phi + \sum_{i,j=1}^{2} \partial_i \phi \partial_j \phi \partial_{ij}^2 \phi - |\nabla \phi|^2 \Delta \phi = 0.
\]

When \(\|\rho^0(x)\|_{H^4} + \|u^0(x)\|_{H^4} \leq \bar{\varepsilon}\) and \(\bar{\varepsilon} > 0\) is sufficiently small, if follows from Theorem 6.5.3 of [15] and equation (1.9) that the lifespan \(T_{\bar{\varepsilon}}\) of smooth solution \((\rho, u)\) to (1.4) fulfills \(T_{\bar{\varepsilon}} \geq \frac{C_{\bar{\varepsilon}}}{\bar{\varepsilon}}\), where \(C > 0\) is a constant depending only on the initial data. In addition, for the polytropic gases, since the first null condition does not hold for equation (1.9), then \(T_{\bar{\varepsilon}} \leq \frac{C_{\bar{\varepsilon}}}{\bar{\varepsilon}}\) holds for suitably positive constant \(\bar{C}\) (see [4] and [21]); for the Chaplygin gases, note that both the first null condition and the second null condition hold for equation (1.10), then \(T_{\bar{\varepsilon}} = +\infty\) holds (see [5]).

When

\[
\text{curl } u^0(x) \neq 0,
\]

if

\[
\text{curl } u^0(x) = O(\bar{\varepsilon}^{1+\alpha}),
\]

where \(\alpha \geq 0\) is a constant, then by Theorem 1 and Theorem 2 of [27] that the lifespan \(T_{\bar{\varepsilon}}\) of smooth solution \((\rho, u)\) to (1.4) satisfies

\[
T_{\bar{\varepsilon}} \geq \frac{C}{\varepsilon^{\min\{1+\alpha,2\}}}.
\]

Note that \(T_{\bar{\varepsilon}}\) in (1.12) is also optimal for the polytropic gases, i.e., \(T_{\bar{\varepsilon}} \leq \frac{C}{\varepsilon^{\min\{1+\alpha,2\}}}\) holds for some constant \(\bar{C} > C\), one can see [1]-[2], [21] and [27]. With respect to more results on the blowup or the blowup mechanism of (1.1) for polytropic gases, the papers [7]-[8], [11], [14], [23], [25], [26], [29] can
be referred. In the present paper, we intend to study the long time existence of smooth solutions to (1.4) for the Chaplygin gases with initial data (1.5) and

$$\text{curl } \sigma \neq 0, \quad \text{curl } \sigma = O(\delta),$$

(1.13)

where \( \delta > 0 \) and \( \delta = o(\varepsilon) \). For this end, we introduce the following quantities on the initial data and vorticity

$$\varepsilon = \sum_{k \leq N} \||x|\nabla|^k \sigma\|_{L^2_{\rho}} + \sum_{k \leq N-1} \||x|\nabla|^k \Delta^{-1} \text{div } \sigma\|_{L^2_{\rho}},$$

and

$$\delta = \sum_{k \leq N-1} \||x|\nabla|^k \text{curl } \sigma\|_{L^2_{\rho}} + \sum_{k \leq N_1} \sum_{\rho = \frac{10}{9}, \frac{10}{11}} \||x|\nabla|^k \text{curl } \sigma\|_{L^2_{\rho}},$$

(1.14)

(1.15)

where \( \|x| = \sqrt{1 + |x|^2} \), the integers \( N \) and \( N_1 \) fulfill \( N_1 \geq 13 \) and \( N_1 + 2 \leq N \leq 2N_1 - 11 \). In addition, the state equation in (1.3) is conveniently written as

$$P(\rho) = P_0 - \frac{\rho^2}{\rho}.$$

(1.16)

**Theorem 1.1.** There exists three constants \( \varepsilon_0, \delta_0, \kappa > 0 \) such that when the initial data \((\rho^0, \sigma^0)\) satisfies \( \varepsilon \leq \varepsilon_0 \) and \( \delta \leq \delta_0 \), then (1.1) with (1.16) admits a solution \((\rho - \rho, u) \in C([0,T\delta]; H^N(\mathbb{R}^2)) \) with \( T\delta = \frac{\varepsilon}{\varepsilon_0} \).

**Remark 1.1.** In Theorem 1.1, when \( \delta = \varepsilon^{1+\alpha} \) with \( 0 \leq \alpha \leq 1 \) and \( 0 < \varepsilon \leq \varepsilon_0 \), then \( T\delta = \frac{\varepsilon}{\varepsilon_0} \) has been shown in Theorem 2 of [27]. Hence we can assume \( \delta \leq \varepsilon^{\frac{1}{p}} \) in Theorem 1.1 without loss of generality.

**Remark 1.2.** If we choose \( \delta = \varepsilon^\ell \) with \( \ell \geq 2 \) or \( e^{-\frac{1}{n}} \) with \( p > 0 \) in Theorem 1.1, then the existence time of smooth solution \((\rho, u)\) to (1.1) for the Chaplygin gases is larger than \( \frac{\varepsilon}{\varepsilon_0} \) or \( \kappa \varepsilon^{\frac{1}{p}} \).

**Remark 1.3.** If \((\rho^0(\cdot, x), \sigma^0(\cdot, x)) \in C_0^\infty(B(0,R)) \), where \( B(0,R) \) is a ball with the center at the origin and the radius \( R > 0 \), then \( \||x|\nabla|^k \Delta^{-1} \text{curl } \sigma \|_{L^2_{\rho}} \) and \( \||x|\nabla|^k \Delta^{-1} \text{div } \sigma \|_{L^2_{\rho}} \) in (1.14) can be replaced by

\[
\||x|\nabla|^k \Delta^{-1} \text{curl } \sigma \|_{L^2(B(0,R))}, \quad \text{and} \quad \||x|\nabla|^k \Delta^{-1} \text{div } \sigma \|_{L^2(B(0,R))},
\]

\[
\text{respectively. The reasons are:}
\]

- **from** the Helmholtz decomposition of initial velocity \( \sigma \), one has \( \sigma = P_1 \sigma + P_2 \sigma \), where \( P_1 \sigma = -\nabla(-\Delta)^{-1} \text{div } \sigma \), \( P_2 \sigma = -\nabla(-\Delta)^{-1} \text{curl } \sigma \) and \( \nabla = (-\partial_{x_2}, \partial_{x_1}) \). When \( \text{supp } \sigma \subset B(0,R) \), we choose a smooth cut-off function \( \eta (x) \) such that \( \eta|_{\text{supp } \sigma} = 1 \) and \( \text{supp } \eta \subset B(0,R) \), and then \( \sigma = \eta P_1 \sigma \). In this case, the related term \( \Delta^{-1} \text{div } \sigma \) can be thought to be supported in \( B(0,R) \).

- **from** \( (A.2) \) in Appendix A, we have \( \partial_t \phi^0 + \frac{1}{2} |\sigma^0|^2 + \sigma^0 - \frac{1}{2} (\sigma^0)^2 = -(-\Delta)^{-1} \text{curl } \sigma \text{ curl } \sigma \), where \( P_1 \sigma = \nabla \phi^0 \) and \( \sigma^0 = \frac{\rho^0}{\rho^0 + \rho} \). When \( \text{supp } \sigma^0 \subset B(0,R) \), then \( \text{supp } \phi^0, \text{supp } \sigma^0 \subset B(0,R) \). As in the above, \( \Delta^{-1} \text{curl } (\sigma^0 \text{ curl } \sigma^0) \) can be thought to be supported in \( B(0,R) \).
Remark 1.4. Consider the 2D full compressible Euler equations of Chaplygin gases
\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0 & \text{(Conservation of mass)}, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P = 0 & \text{(Conservation of momentum)}, \\
\partial_t (\rho e + \frac{1}{2} \rho |u|^2) + \text{div}((\rho e + \frac{1}{2} \rho |u|^2 + P)u) = 0 & \text{(Conservation of energy)}, \\
\rho(0, x) = \bar{\rho} + \varepsilon \rho^0(x), u(0, x) = \varepsilon u^0(x), S(0, x) = \bar{S} + \varepsilon S^0(x),
\end{cases}
\]

where \( P = P(\rho, S) \), \( e = e(\rho, S) \), \( S \) stand for the pressure, inner energy and entropy respectively. In addition, \( \varepsilon > 0 \) is small, and \( (\rho^0(x), u^0(x), S^0(x)) \in C^0_\infty(\mathbb{R}^2) \). If \( \rho^0(x) = \rho^0(r), S^0(x) = \bar{S}(r) \) and \( u^0(x) = f^0(r) \frac{x}{r} + g^0(r) \frac{x^\perp}{r} \) with \( x^\perp = (-x^2, x^1) \) and \( r = |x| \), then we have shown (1.17) has a global smooth solution \( (\rho, u, S) \) in [17] and [18] (when \( g^0(r) \equiv 0 \), the global existence of smooth symmetric solutions \( (\rho, u, S) \) to 2D and 3D systems (1.17) has been established in [10] and [12] respectively). By combining the methods in the paper and [18], we can actually establish that for the perturbed problem of (1.17)
\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P = 0, \\
\partial_t (\rho e + \frac{1}{2} \rho |u|^2) + \text{div}((\rho e + \frac{1}{2} \rho |u|^2 + P)u) = 0, \\
\rho(0, x) = \bar{\rho} + \varepsilon \rho^0(x) + \delta \rho^1(x), u(0, x) = \varepsilon u^0(x) + \delta u^1(x), \\
S(0, x) = \bar{S} + \varepsilon S^0(x) + \delta S^1(x),
\end{cases}
\]

where \( \text{curl } u^0(x) \equiv 0 \) (or \( \rho^0(x) = \rho^0(r), S^0(x) = \bar{S}(r) \) and \( u^0(x) = f^0(r) \frac{x}{r} + g^0(r) \frac{x^\perp}{r} \), \( \delta = o(\varepsilon) \), and \( (\rho^1(x), u^1(x), S^1(x)) \in C^0_\infty(\mathbb{R}^2) \), then (1.18) admits a smooth solution \( (\rho, u, S) \) for \( t \in [0, T_\delta] \) with \( T_\delta = \frac{C}{\varepsilon} \) as in Theorem 1.1.

Remark 1.5. We mention the interesting works on the Euler-Maxwell system, which are related our result. The global smooth, small-amplitude, irrotational solution to the Euler-Maxwell two-fluid system was proved in [13]. Considering the influence of the vorticity, Ionescu and Lie in [20] have shown that the existence time is larger than \( \frac{C}{\varepsilon} \), where \( \delta \) is the size of initial vorticity and \( C \) is some positive constant.

Remark 1.6. It is well known that the compressible Euler system is symmetric hyperbolic with respect to the time \( t \) when the vacuum does not appear. A. Majda posed the following conjecture on Page 89 of [24]:

**Conjecture.** *If the multidimensional nonlinear symmetric system is totally linearly degenerate, then it typically has smooth global solutions when the initial data are in \( H^s(\mathbb{R}^n) \) with \( s > \frac{n}{2} + 1 \) unless the solution itself blows up in finite time.*

Note that the compressible Euler system of Chaplygin gases is totally linearly degenerate, A. Majda’s conjecture together with the opinion in [3] can yield the following open question:

**Open question.** *For the \( n \)-dimensional \( (n \geq 2) \) full compressible Euler equations of Chaplygin gases*
\[
\begin{cases}
\partial_t \rho + \text{div}(\rho u) = 0, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla P = 0, \\
\partial_t (\rho e + \frac{1}{2} \rho |u|^2) + \text{div}((\rho e + \frac{1}{2} \rho |u|^2 + P)u) = 0, \\
\rho(0, x) = \bar{\rho} + \varepsilon \rho^0(x), u(0, x) = \varepsilon u^0(x), S(0, x) = \bar{S} + \varepsilon S^0(x),
\end{cases}
\]
where \( \varepsilon > 0 \) is small, and \((\rho^0(x), u^0(x), S^0(x)) \in C_0^\infty(\mathbb{R}^2)\) with \( u^0(x) = (u_1^0(x), \cdots, u_n^0(x)) \), then \( T_\varepsilon = +\infty \).

Although this open question has not been solved so far, our result in Theorem 1.1 illustrates that the order of lifespan \( T_\varepsilon \) is only essentially influenced by the initial vorticity.

We next give some comments on the proof of Theorem 1.1. From now on, \( \bar{\rho} = 1 \) is always assumed. Introducing the perturbed sound speed \( \sigma(t, x) = 1 - \frac{1}{\bar{\rho}(t, x)} \) as the new unknown, then (1.1) is reduced to

\[
\begin{cases}
\partial_t \sigma + \text{div} u = Q_1 := \sigma \text{div} u - u \cdot \nabla \sigma, \\
\partial_t u + \nabla \sigma = Q_2 := \sigma \nabla \sigma - u \cdot \nabla u,
\end{cases}
\tag{1.19}
\]

where the \( i \)-th component of the vector \( Q_2 \) is \( Q_{2i} = \sigma \partial_i \sigma - u \cdot \nabla u_i \) \((i = 1, 2)\). In addition, we also define the good unknown \( g \) in the region \(|x| > 0\) as follows

\[
g := (g_1, g_2) = u - \omega \sigma \quad \text{with} \quad g_i = u_i - \omega_i \sigma, \quad i = 1, 2,
\tag{1.20}
\]

where \( \omega = (\omega_1, \omega_2) := (\frac{1}{|x|}, \frac{x}{|x|}) \in S^1 \). We point out that the introduction of \( g \) is motivated by the second order wave equations although (1.1) admits the non-zero and small higher order vorticity: by the potential equation (1.10), then \( u_i = \partial_i \phi \) and \( \sigma = -\partial_t \phi + \text{higher order error terms of } \partial \phi \), which derives \( g_i = (\partial_t + \omega_i \partial_t) \phi + \text{higher order error terms of } \partial \phi \). It is well known that \((\partial_t + \omega_i \partial_t) \phi \) is the good derivative in the study of the nonlinear wave equation (1.10) (see [6]) since \((\partial_t + \omega_i \partial_t) \phi \) will admit more rapid space-time decay rates. By some ideas and methods dealing with the null condition structures in [17–19], we obtain the better \( L^\infty \) space-time decay rates of \( g \). Based on this, the elementary energy \( E_N(t) \) can be estimated (see Lemma 5.1 in Section 5.1). Nevertheless, we have to overcome other essential difficulties which are arisen by the interaction between the irrotational part of the velocity and the vorticity. To solve the resulting difficulties, our ingredients are:

- By the transport equation \((\partial_t + u \cdot \nabla)(\text{curl} \frac{u}{\rho}) = 0\) and the careful analysis, we can derive that the key influence of the vorticity is concentrated in the interior of the outgoing light cone.
- Near the outgoing light conic surface, our first observation is that the system (1.1) can be changed into the second order potential flow equation. However, the optimal time-decay rate of solutions to the 2D free wave equation is merely \((1 + t)^{-\frac{1}{2}}\), which is far to derive the existence time \( T_\delta = \frac{1}{\varepsilon} \) in Theorem 1.1. The reason is due to: for example, when \( \delta = e^{-\frac{1}{\varepsilon^2}} \) is chosen, then the integral \( \int_0^{T_\delta} \frac{dt}{\sqrt{1 + t}} = O(e^{\frac{1}{\varepsilon^2}}) \) is sufficiently large as \( \varepsilon \to 0 \), which leads to that the related energy \( E(t) \) can not be controlled well by the corresponding energy inequality \( E(t) \leq E(0) + C \varepsilon \sqrt{1 + t} E(t) \). To overcome this difficulty, our second observation is that the velocity is the gradient of the potential and the potential satisfies a second order quasilinear wave equation with the first and the second null conditions. By establishing a new type of weighted \( L^\infty-L^\infty \) estimate for the derivatives of the potential, the better space-time decay rate of \( u \) can be obtained (see Corollary 4.9 in Section 4).

Based on the key estimates in the above, we eventually get the \( L^2 \) and other \( L^p \) (for some suitable numbers \( p \) with \( p \neq 2 \)) energy estimates of the vorticity and further close the bootstrap assumptions in Section 2.

This paper is organized as follows. In Section 2, we will introduce the basic bootstrap assumptions, Helmholtz decomposition and some elementary pointwise estimates. The estimates of the good unknown \( g \) and some auxiliary energies are derived in Section 3. In Section 4, by establishing a new type of the weighted \( L^\infty-L^\infty \) for the 2D wave equations, the required pointwise estimates of the solution \((\sigma, u)\)
with suitable space-time decay rates are derived. Based on the pointwise estimates in Section 4, the Hardy inequality and the ghost weight method in [5], we get the related energy estimates in Section 5. In Section 6, with the previous energy inequalities and Gronwall’s inequalities, the proof of Theorem 1.1 is finished.

2 Some preliminaries

2.1 The vector field and bootstrap assumptions

Define the spatial rotation vector field

$$\Omega := x_1 \partial_2 - x_2 \partial_1.$$  

For the vector-valued function $U = (U_1, U_2)$, denote

$$\tilde{\Omega}U := \Omega U - U^\perp = (\Omega U_1 + U_2, \Omega U_2 - U_1).$$

Let $\tilde{\Omega} U_k = (\tilde{\Omega} U)_k$ be the $k$-component of $\tilde{\Omega} U$ rather than the operator $\tilde{\Omega}$ act on the component $U_k$.

According to the definitions of $\Omega$ and $\tilde{\Omega}$, it is easy to check that for the scalar function $f$ and the vector-valued functions $U, V$,

$$\begin{align*}
\Omega \, \text{div} \, U &= \text{div} \, \tilde{\Omega} U, \\
\Omega \, \text{curl} \, U &= \text{curl} \, \tilde{\Omega} U, \\
\tilde{\Omega} \nabla f &= \nabla \Omega f, \\
\tilde{\Omega} \left( U \cdot \nabla V \right) &= U \cdot \nabla (\tilde{\Omega} V) + (\tilde{\Omega} U) \cdot \nabla V, \\
\tilde{\Omega} \left( U \cdot V \right) &= (\tilde{\Omega} U) \cdot V + U \cdot \tilde{\Omega} V.
\end{align*}$$

The spatial derivatives can be decomposed into the radial and angular components for $r = |x| \neq 0$:

$$\nabla = \omega \partial_r + \frac{\omega^\perp}{|x|} \Omega,$$

where $\omega^\perp := (-\omega_2, \omega_1)$. For convenience and simplicity, we often denote this decomposition as

$$\partial_i = \omega_i \partial_r + \frac{1}{|x|} \Omega_i.$$  

(2.2)

For the multi-index $a$, set

$$S := t \partial_t + r \partial_r, \quad \Gamma^a = S^{a_z} Z^{a_z}, \quad Z \in \{\partial, \Omega\}, \quad \tilde{\Gamma}^a = S^{a_z} \tilde{Z}^{a_z}, \quad \tilde{Z} \in \{\partial, \tilde{\Omega}\}.$$  

(2.3)

By acting $(S + 1)^{a_z} Z^{a_z}$ and $(S + 1)^{a_z} \tilde{Z}^{a_z}$ on the equations in (1.19), respectively, we then have

$$\begin{align*}
\begin{cases}
\partial_t \Gamma^a \sigma + \text{div} \, \tilde{\Gamma}^a u &= Q_1^a := \sum_{b+c=a} C_{bc}^a Q_{1b}^c, \\
\partial_t \tilde{\Gamma}^a u + \nabla \Gamma^a \sigma &= Q_2^a := \sum_{b+c=a} C_{bc}^a Q_{2b}^c, 
\end{cases}
\end{align*}$$

(2.4)

where $C_{bc}^a$ are constants ($C_{a0}^a = C_{0a}^a = 1$) and

$$\begin{align*}
Q_{1b}^c &:= \Gamma^b \sigma \text{div} \, \tilde{\Gamma}^c u - \tilde{\Gamma}^b u \cdot \nabla \Gamma^c \sigma, \\
Q_{2b}^c &:= \Gamma^b \sigma \nabla \Gamma^c \sigma - \tilde{\Gamma}^b u \cdot \nabla \tilde{\Gamma}^c u.
\end{align*}$$

(2.5)
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It is convenient to introduce the specific vorticity

\[ w := \frac{\text{curl} \, u}{\rho} = (1 - \sigma) \text{curl} \, u \]  

(2.6)

since \((\partial_t + u \cdot \nabla)w = 0\) holds. For integer \(m \in \mathbb{N}\), we define

\[
E_m(t) := \sum_{|a| \leq m} \| (\tilde{\Gamma}^a u, \Gamma^a \sigma)(t, x) \|_{L^2_t},
\]

\[
\mathcal{X}_m(t) := \sum_{|a| \leq m} \| \langle |x| - t \rangle (\partial_t \tilde{\Gamma}^a u, \text{div} \tilde{\Gamma}^a u, \nabla \Gamma^a \sigma, \partial_t \Gamma^a \sigma)(t, x) \|_{L^2_t},
\]

\[
W_m(t) := \sum_{|a| \leq m} \| \langle |x| \rangle \Gamma^a u(t, x) \|_{L^2_x},
\]

\[
W_m(t) := \sum_{|a| \leq m} \| \langle |x| \rangle \Gamma^a \text{curl} \, u(t, x) \|_{L^2_x},
\]

\[
\mathcal{W}_m(t) := \sum_{|a| \leq m} \left\{ \| \langle |x| \rangle \Gamma^a \text{curl} \, u(t, x) \|_{L^2_x} + \sum_{p = \frac{10}{9}, \frac{10}{7}} \| \langle |x| \rangle \Gamma^a w(t, x) \|_{L^p_x} \right\},
\]

\[
\mathcal{W}_m(t) := \sum_{|a| \leq m} \left\{ \| \langle |x| \rangle \Gamma^a w(t, x) \|_{L^2_x} + \sum_{p = \frac{10}{9}, \frac{10}{7}} \| \langle |x| \rangle \Gamma^a w(t, x) \|_{L^p_x} \right\}
\]

and \(\mathcal{X}_0(t) = 0\).

Throughout the whole paper, we will make the following bootstrap assumptions:

\[
\begin{align*}
\text{for} \quad t \delta \leq \kappa, \\
E_N(t) + \mathcal{X}_N(t) &\leq M \varepsilon (1 + t)^{M'}, \\
E_{N_1-4}(t) + \mathcal{X}_{N_1-4}(t) &\leq M \varepsilon, \\
W_{N_1-1}(t) + \mathcal{W}_{N_1-1}(t) + \mathcal{W}_{N_1}(t) + \mathcal{W}_{N_1-4}(t) &\leq M \delta (1 + t)^{M'}, \\
W_{N_1-4}(t) + \mathcal{W}_{N_1-4}(t) + \mathcal{W}_{N_1-4}(t) + \mathcal{W}_{N_1-4}(t) &\leq M \delta,
\end{align*}
\]

(2.8)

\[ \delta \leq \varepsilon^{\tilde{\delta}}, \quad M(\varepsilon + \kappa) \leq 1, \quad M \geq 1, \quad M' > 0, \]

where the constant \(M \geq 1\) will be chosen, \(M' > 0\) is some fixed constant. In Section 6, we will prove that the constant \(M\) on the right hands of the first four lines in (2.8) can be improved to \(\frac{3}{4} M\).

2.2 The Helmholtz decomposition and commutator

For the rapidly decaying vector \(U = (U_1, U_2)\) with respect to the space variable \(x\), we divide it into the curl-free part \(P_1 U\) (irrotational) and the divergence-free part \(P_2 U\) (solenoidal), which is called the Helmholtz decomposition

\[
P_1 U = \nabla \Phi, \quad P_2 U = \nabla^\perp \Psi, \quad \Delta \Phi = \text{div} \, U, \quad \Delta \Psi = \text{curl} \, U,
\]

\[
U = P_1 U + P_2 U = -\nabla (-\Delta)^{-1} \text{div} \, U - \nabla^\perp (-\Delta)^{-1} \text{curl} \, U,
\]

(2.9)

where \(\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})\) and \(\Delta\) is the Laplacian operator. It is easy to know

\[
\|U\|_{L^2}^2 = \|P_1 U\|^2_{L^2} + \|P_2 U\|^2_{L^2}.
\]

Next, we give some divergence-curl inequalities.
Lemma 2.1. For any vector function $U$, $1 < p < \infty$ and $0 \leq \beta < 2(p - 1)$, it holds that

\begin{align}
\|\langle |x| \rangle^\beta \nabla U\|_{L^p} &\lesssim \|\langle |x| \rangle^\beta \div U\|_{L^p} + \|\langle |x| \rangle^\beta \curl U\|_{L^p}, \\
\|\langle |x| - t \rangle \nabla U\|_{L^2} &\lesssim \|U\|_{L^2} + \|\langle |x| - t \rangle \div U\|_{L^2} + \|\langle |x| - t \rangle \curl U\|_{L^2}.
\end{align}

(2.11)

Proof. From the second line of (2.9), we get

$$\nabla U = -\nabla^2(-\Delta)^{-1} \div U - \nabla \nabla^\perp(-\Delta)^{-1} \curl U,$$

where $\nabla^2(-\Delta)^{-1}$ and $\nabla \nabla^\perp(-\Delta)^{-1}$ are the bounded operators from $L^p$ to $L^p$ ($1 < p < \infty$). On the other hand, $\langle |x| \rangle^\beta$ belongs to $A_p$ class with $0 \leq \beta < 2(p - 1)$ (see [28]). Therefore, the first inequality in (2.11) is derived.

Since the second inequality in (2.11) follows from the direct integration by parts, we omit the details here. \qed

Lemma 2.2 (Commutator). For the vector fields $\Gamma$ defined in (2.3), one has $[\Gamma, P_1] := \Gamma P_1 - P_1 \Gamma = 0$, $[\Gamma, P_2] = 0$.

Proof. We only prove $\Gamma P_1 U = P_1 (\Gamma U)$, since it always holds that $[\Gamma, P_2] = [\Gamma, \Id - P_1] = 0$. Note that $[\Gamma, P_1] = 0$ is obvious for $\Gamma \in \{\partial\}$, we now focus on the case of $\Gamma \in \{\Omega, S\}$.

According to (2.1) and the first line of (2.9), one has

$$-\Delta P_1 (\hat{\Omega} U) = -\nabla \div \hat{\Omega} U = -\hat{\Omega} \nabla \div U = \hat{\Omega} (-\Delta) P_1 U = -\Delta (\hat{\Omega} P_1 U).$$

This derives $P_1 (\hat{\Omega} U) = \hat{\Omega} P_1 U$ by the uniqueness of the solution to $\Delta$ since $U$ fulfills the rapid decay assumption and $P_1 (\Omega U), \hat{\Omega} P_1 U$ decay for the space variable.

Analogously, $P_1 (SU) = SP_1 U$ follows from

$$-\Delta P_1 (SU) = -\nabla \div SU = -(S + 2) \nabla \div U = (S + 2)(-\Delta) P_1 U = -\Delta (SP_1 U).$$

\qed

2.3 The elementary pointwise estimates

Lemma 2.3. Let $f(t, x)$ be a scalar function, then it holds that

\begin{align}
\langle |x| \rangle^{\frac{1}{p}} |f(t, x)| &\lesssim \sum_{j=0}^{\infty} \sum_{|a|=0}^{2-j} \|\nabla^a \Omega^j f(t, y)\|_{L^p_y}, \quad 1 < p < \infty, \\
\langle |x| \rangle^{\frac{1}{2}} \langle |x| - t \rangle |f(t, x)| &\lesssim \sum_{j=0}^{\infty} \sum_{|a|=0}^{2-j} \|\langle |y| - t \rangle \nabla^a \Omega^j f(t, y)\|_{L^2_y}, \\
\langle |x| \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{2}} \langle f(t, x) \rangle &\lesssim \sum_{j=0}^{\infty} \left\{ \|\Omega^j f(t, y)\|_{L^2_y} + \sum_{|a|=1}^{2-j} \|\langle |y| - t \rangle \nabla^a \Omega^j f(t, y)\|_{L^2_y} \right\}, \\
\|f(t, x)\|_{L^\infty_y} &\lesssim \ln^{\frac{1}{2}} (2 + t) \|\nabla f(t, y)\|_{L^2_y} + \langle t \rangle^{-1} \|\nabla^2 f(t, y)\|_{L^2_y} + \|\nabla^2 f(t, y)\|_{L^2_y}, \\
\langle |x| \rangle^{\frac{1}{p}} |f(t, x)| &\lesssim \|\langle |y| \rangle \nabla \Omega^{\leq 1} f(t, y)\|_{L^p_y} + \sum_{p=\frac{1}{p}, \frac{1}{p}+1} \|\nabla \Omega^{\leq 1} f(t, y)\|_{L^p_y}, \\
\langle |x| \rangle |f(t, x)| &\lesssim \|f(t, y)\|_{H^2_y} + \|\Omega^{\leq 1} f(t, y)\|_{L^2_y} + \|\langle y \rangle \nabla \Omega^{\leq 1} f(t, y)\|_{L^2_y},
\end{align}

(2.12) - (2.17)

where $\Omega^{\leq 1} f$ stands for $\sum_{j\leq 1} \Omega^j f$. 
Proof. The inequality (2.12) with \( p = 2 \) is just (3.1) of [27]. We now deal with the general case of \( p \in (1, \infty) \). Note that (2.12) in the region \( |x| \leq 1 \) follows from the Sobolev embedding \( W^{2,p}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \).

For \( |x| \geq 1 \), by the Sobolev embedding on the unit circle \( W^{1,p}(S^1) \hookrightarrow L^\infty(S^1) \) and the Newton-Leibniz formula in the radial direction, we arrive at

\[
|f(x)|^p = |f(t, |x|)|^p \lesssim |x| \int_{S^1} |\mathcal{L}^\leq_1 f(t, |x|)|^p d\omega \\
\lesssim |x| \int_{|x|}^\infty \int_{S^1} |\mathcal{L}^\leq_1 f(t, r\omega)|^{p-1} |\partial_r \mathcal{L}^\leq_1 f(t, r\omega)| d\omega dr \\
\lesssim \|\mathcal{L}^\leq_1 f(t, y)\|^p_{L^p_y} + \|\nabla \mathcal{L}^\leq_1 f(t, y)\|^p_{L^p_y},
\]

which yields (2.12).

For the inequalities (2.13)–(2.15), see (3.2), (3.4) of [27] and (3.4) of [22], respectively.

Next, we start to prove (2.16). By the Sobolev embedding \( W^{1,5,5}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \) and \( W^{1,4,5}(\mathbb{R}^2) \hookrightarrow L^5(\mathbb{R}^2) \), we have

\[
|f(t, x)| \lesssim \|f(t, y)\|_{L^5_y} + \|\nabla f(t, y)\|_{L^5_y} \\
\lesssim \|\nabla f(t, y)\|_{L^4_y} + \|f(t, y)\|_{L^5_y},
\]

this implies (2.16) when \( |x| \leq 1 \).

For \( |x| \geq 1 \), it follows from the Sobolev embedding \( W^{1,3}(S^1) \hookrightarrow L^\infty(S^1) \) that

\[
|f(t, x)|^3 = |f(t, |x|)|^3 \lesssim \int_{S^1} |\mathcal{L}^\leq_1 f(t, |x|)|^3 d\omega \\
\lesssim \int_{|x|}^\infty \int_{S^1} |\mathcal{L}^\leq_1 f(t, r\omega)|^3 |\partial_r \mathcal{L}^\leq_1 f(t, r\omega)| d\omega dr.
\]

Multiplying this inequality by \( |x|^8 \) and then applying the Hölder inequality infer

\[
|x|^8 |f(t, x)|^3 \lesssim \|\langle y \rangle^7 \nabla \mathcal{L}^\leq_1 f(t, y)\|_{L^5_y} \|\mathcal{L}^\leq_1 f(t, y)\|^2_{L^5_y} \\
\lesssim \|\langle y \rangle^7 \nabla \mathcal{L}^\leq_1 f(t, y)\|_{L^5_y} \|\mathcal{L}^\leq_1 f(t, y)\|^2_{L^5_y} \\
\lesssim \|\langle y \rangle^7 \nabla \mathcal{L}^\leq_1 f(t, y)\|_{L^5_y} \|\nabla \mathcal{L}^\leq_1 f(t, y)\|^2_{L^5_y},
\]

where we have used the Sobolev embedding \( W^{1,\frac{10}{3}}(\mathbb{R}^2) \hookrightarrow L^\frac{5}{2}(\mathbb{R}^2) \). Therefore, we achieve (2.16).

Finally, we turn to the proof of (2.17). For the case of \( |x| \leq 1 \), (2.17) is a direct result of the Sobolev embedding \( H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2) \).

For \( |x| \geq 1 \), similarly to the proof of (2.16), applying \( W^{1,2}(S^1) \hookrightarrow L^\infty(S^1) \) instead leads to

\[
|x|^2 |f(t, x)|^2 = |x|^2 |f(t, |x|)|^2 \lesssim |x|^2 \int_{S^1} |\mathcal{L}^\leq_1 f(t, |x|)|^2 d\omega \\
\lesssim |x|^2 \int_{|x|}^\infty \int_{S^1} |\mathcal{L}^\leq_1 f(t, r\omega)| |\partial_r \mathcal{L}^\leq_1 f(t, r\omega)| d\omega dr \\
\lesssim \int |\mathcal{L}^\leq_1 f(t, y)| \|\langle y \rangle\nabla \mathcal{L}^\leq_1 f(t, y)\| dy.
\]

Thus, we derive the desired inequality (2.17). ∎
Lemma 2.4. For any multi-indices $a, b$ with $|a| \leq N - 3$ and $|b| \leq N - 2$, it holds that
\[ \langle |x| \rangle^{\frac{5}{2}} |\Gamma^a w(t, x)| \lesssim W_{|a|+2}(t), \quad \langle |x| \rangle^{\frac{5}{2}} |\Gamma^a \text{curl } u(t, x)| \lesssim W_{|a|+2}(t), \quad \text{(2.18)} \]
\[ \langle |x| \rangle^{\frac{5}{2}} |x - t|^\frac{7}{2} (|\tilde{\Gamma}^b P_1 u(t, x)| + |\Gamma^b \sigma(t, x)|) \lesssim E_{|b|+2}(t) + \mathcal{X}_{|b|+2}(t), \quad \text{(2.19)} \]
Furthermore, for $|x| \leq 3(t/4)$,
\[ |\tilde{\Gamma}^b P_1 u(t, x)| + |\Gamma^b \sigma(t, x)| \lesssim (t^{-1} \ln \frac{2}{t}) (2 + t) \{ E_{|b|+2}(t) + \mathcal{X}_{|b|+2}(t) \}, \quad \text{(2.20)} \]

Proof. The proof of (2.18) follows from (2.12) with $p = 2$ directly. The inequalities in (2.19) can be concluded from (2.10), (2.11), (2.13) and (2.14).

To achieve (2.20), we introduce the cutoff function $\chi(s) \in C^\infty$ such that
\[ 0 \leq \chi(s) \leq 1, \quad \chi(s) = \begin{cases} 1, & s \leq 3/4, \\ 0, & s \geq 4/5. \end{cases} \quad \text{(2.21)} \]
Choosing $f(t, x) = \chi(|x| \frac{3}{4}) \tilde{\Gamma}^b P_1 u(t, x)$ and $\chi(|x| \frac{3}{4}) \Gamma^b \sigma(t, x)$ in (2.15), we then get (2.20) by (2.10) and (2.11).

Lemma 2.5. For any multi-indices $a, b$ with $|a| \leq N_1 - 1$ and $|b| \leq N - 2$, it holds that
\[ \langle |x| \rangle^{\frac{5}{2}} |P_2 \tilde{\Gamma}^a u(t, x)| \lesssim W_{|a|+1}(t), \quad \text{(2.22)} \]
\[ \langle |x| \rangle^{\frac{5}{2}} |\Gamma^a \text{curl } u(t, x)| \lesssim W_{|a|+1}(t), \quad \text{(2.23)} \]
\[ \langle |x| \rangle^{\frac{7}{2}} |\nabla \Gamma^a P_2 u(t, x)| \lesssim W_{|a|+1}(t), \quad \text{(2.24)} \]
\[ \langle |x| \rangle |P_2 \tilde{\Gamma}^b u(t, x)| \lesssim E_{|b|+2}(t) + W_{|b|+1}(t). \quad \text{(2.25)} \]

Proof. Applying (2.16) to $P_2 U$ yields that
\[ \langle |x| \rangle^{\frac{5}{2}} |P_2 U(t, x)| \lesssim \|\langle y \rangle^{\frac{7}{2}} \nabla \Omega^{\leq 1} P_2 U(t, y)\|_{L^5_y} + \sum_{p = \frac{10}{11}, \frac{10}{9}} \|\nabla \Omega^{\leq 1} P_2 U(t, y)\|_{L^5_y}, \]
\[ \lesssim \langle |y| \rangle^{\frac{7}{2}} \nabla \Omega^{\leq 1} \text{curl } U(t, y)\|_{L^5_y} + \sum_{p = \frac{10}{11}, \frac{10}{9}} \|\Omega^{\leq 1} \text{curl } U(t, y)\|_{L^5_y}, \]

where we have used (2.11). Subsequently, by choosing $U = \tilde{\Gamma}^a u$, (2.22) is then derived.

The inequality (2.23) is a direct result of the Sobolev embedding $W^{r,5}(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$.

Analogously, we can conclude from (2.11) that
\[ \langle |x| \rangle^{\frac{7}{2}} |\nabla \Gamma^a P_2 u(t, x)| \lesssim \|\langle y \rangle^{\frac{7}{2}} \nabla \Omega^{\leq 1} \Gamma^a P_2 u(t, y)\|_{L^5_y} \]
\[ \lesssim \|\langle y \rangle^{\frac{7}{2}} \nabla \Omega^{\leq 1} \text{curl } u(t, y)\|_{L^5_y} \lesssim W_{|a|+1}(t), \]
which yields (2.24).

At last, we turn to the proof of (2.25). Let $f(t, x) = P_2 \tilde{\Gamma}^b u(t, x)$ in (2.17) and then it concludes from (2.11) that
\[ \langle |x| \rangle |P_2 \tilde{\Gamma}^b u(t, x)| \lesssim E_{|b|+2}(t) + \|\nabla P_2 \Omega^{\leq 1} \tilde{\Gamma}^b u(t, x)\|_{L^2} \lesssim E_{|b|+2}(t) + \|\nabla \Omega^{\leq 1} \tilde{\Gamma}^b u(t, x)\|_{L^2}, \]
which implies (2.25).
Combining Lemma 2.4 and 2.5, we obtain the following pointwise estimates.

**Corollary 2.6.** For any multi-indices $a, b$ with $|a| \leq N_1 - 1$ and $|b| \leq N - 2$, it holds that

$$
|\Gamma^a \sigma(t, x)| + |\tilde{\Gamma}^a u(t, x)| \lesssim \langle |x| \rangle^{-\frac{1}{2}} \langle |x| - t \rangle^{-\frac{1}{2}} \{ E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) \} + \langle |x| \rangle^{-\frac{5}{2}} \mathcal{W}_{|a|+1}(t),
$$

(2.26)

$$
|\nabla \Gamma^a \sigma(t, x)| + |\nabla \tilde{\Gamma}^a u(t, x)| \lesssim \langle |x| \rangle^{-\frac{1}{2}} \langle |x| - t \rangle^{-\frac{1}{2}} \{ E_{|a|+3}(t) + \mathcal{X}_{|a|+3}(t) \} + \langle |x| \rangle^{-7} \mathcal{W}_{|a|+1}(t),
$$

and

$$
|\Gamma^b \sigma(t, x)| + |\tilde{\Gamma}^b u(t, x)| \lesssim \langle |x| \rangle^{-\frac{1}{2}} \langle |x| - t \rangle^{-\frac{1}{2}} \{ E_{|b|+2}(t) + \mathcal{X}_{|b|+2}(t) \} + \langle |x| \rangle^{-1} \{ E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) \}.
$$

(2.27)

Furthermore, for $|x| \leq 3\langle t \rangle / 4$,

$$
|\Gamma^a \sigma(t, x)| + |\Gamma^a u(t, x)| \lesssim \langle |x| \rangle^{-\frac{5}{2}} \mathcal{W}_{|a|+1}(t) + \langle t \rangle^{-1} \ln^2(2 + t) \{ E_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t) \}.
$$

(2.28)

## 3 Estimates of the good unknown and auxiliary energies

### 3.1 Estimates of the good unknown $g$

In this subsection, several estimates of the good unknown $g$ will be established.

**Lemma 3.1.** For $m \leq N - 1$, it holds that

$$
G_m(t) \lesssim E_{m+1}(t) + \mathcal{X}_{m+1}(t) + \mathcal{W}_m(t) + \sum_{|b| + |c| \leq m} \| \langle |x| \rangle Q_1^{b,c} \|_{L^2(|x| \geq \langle t \rangle / 8)},
$$

(3.1)

where

$$
G_m(t) := \sum_{|a| \leq m} \| \langle |x| \rangle \nabla \Gamma^a g(t, x) \|_{L^2(|x| \geq \langle t \rangle / 8)}^2.
$$

(3.2)

**Proof.** Note that $r \partial_r \tilde{\Gamma}^a g_i = x_j \partial_j \tilde{\Gamma}^a (u - \sigma \omega)_i$, where the Einstein summation convention is used. It follows from direct computation that there exist the bounded smooth functions $f_{i,j}^{a,b}(x)$ and $f_i^{a,b}(x)$ in $|x| \geq 1/8$ such that

$$
\tilde{\Gamma}^a (\sigma \omega)_i = \omega_i \Gamma^a \sigma + \frac{1}{|x|} \sum_{b+c \leq a} f_{i,j}^{a,b}(x) \Gamma^c \sigma
$$

and

$$
\partial_j \tilde{\Gamma}^a (\sigma \omega)_i = \omega_i \partial_j \Gamma^a \sigma + \frac{1}{|x|} \sum_{b+c \leq a} \left[ f_{i,j}^{a,b}(x) \partial_j \Gamma^c \sigma + f_i^{a,b}(x) \Gamma^c \sigma \right].
$$

(3.3)

Then we have

$$
r \partial_r \tilde{\Gamma}^a g_i + \omega_j \sum_{b+c \leq a} [f_{i,j}^{a,b}(x) \partial_j \Gamma^c \sigma + f_i^{a,b}(x) \Gamma^c \sigma]
$$

$$
= x_j \partial_j \tilde{\Gamma}^a u_i - \omega_i x_j \partial_j \tilde{\Gamma}^a \sigma
$$

$$
= x_j (\partial_j \tilde{\Gamma}^a u_i - \partial_i \tilde{\Gamma}^a u_j) + (x_j \partial_i - x_i \partial_j) \tilde{\Gamma}^a u_j + x_i \partial_i \tilde{\Gamma}^a u_i - \omega_i \partial_i \Gamma^a \sigma
$$

$$
= x_j \epsilon_{ij} \Gamma^a \text{curl } u + \epsilon_{ij} \Omega(\tilde{\Gamma}^a u_j) + x_i Q_1^a + \omega_i (t - |x|) \partial_i \Gamma^a \sigma - \omega_i \partial_i \Gamma^a \sigma,
$$

(3.4)
where the volume form \( \epsilon_{ij} \) is the sign of the arrangement \( \{ji\} \) and we have used the facts of \( \partial_j U_i = \epsilon_{ij} \text{curl} U \) and \( x_i \partial_i - x_j \partial_j = \epsilon_{ij} \Omega \). Taking the \( L^2(|x| \geq \langle t \rangle /8) \) norm on the both sides of (3.4) yields

\[
\sum_{|a| \leq m} \| \langle |x| \rangle \partial_i \Gamma^a g(t,x) \|_{L^2(|x| \geq \langle t \rangle /8)} \lesssim E_{m+1}(t) + \mathcal{X}_{m+1}(t) + W_m(t)
\]

\[
+ \sum_{|b|+|c| \leq m} \| \langle |x| \rangle Q^b c \|_{L^2(|x| \geq \langle t \rangle /8)}.
\]

In addition, it follows from (2.2) that

\[
G_m(t) \lesssim \sum_{|a| \leq m} \| \langle |x| \rangle \partial_i \Gamma^a g \|_{L^2(|x| \geq \langle t \rangle /8)} + E_{m+1}(t).
\]

Collecting (3.5) and (3.6) together leads to (3.1).

**Lemma 3.2.** For \(|a| \leq N - 1, |b| \leq N - 2\) and \(|x| \geq \langle t \rangle /8\), it holds that

\[
\langle |x| \rangle |\Gamma^a g(t,x)| \lesssim G_{|a|+1}(t) + E_{|a|+1}(t),
\]

\[
\langle |x| \rangle \frac{3}{2} |\nabla \Gamma^b g(t,x)| \lesssim G_{|b|+2}(t).
\]

**Proof.** Applying the Sobolev embedding on the unit circle and the Newton-Leibnitz formula in the radial direction derive that

\[
\langle |x| \rangle^{1+\ell} |U(t,x)|^2 \lesssim \langle |x| \rangle^{1+\ell} \int_{\mathbb{S}^1} |\tilde{\Omega}^{1,0} U(t,|x|\omega)|^2 d\omega
\]

\[
\lesssim \langle |x| \rangle^{1+\ell} \int_{|x|}^{\infty} \int_{\mathbb{S}^1} |\tilde{\Omega}^{1,0} U(t,r\omega)\partial_r \tilde{\Omega} \Omega^{1,0} U(t,r\omega)| rdrd\omega.
\]

Choosing \( U(t,x) = \tilde{\Gamma}^a g(t,x), \nabla \tilde{\Gamma}^b g(t,x) \) in the above equality with \( \ell = 1, 2 \), respectively, we then get that for \(|x| \geq \langle t \rangle /8\),

\[
\langle |x| \rangle^{2} |\tilde{\Gamma}^a g(t,x)|^2 \lesssim \langle \tilde{\Omega} \rangle^{1,0} g(t,y)\|L^2(|y| \geq \langle t \rangle /8) + \| \langle |y| \rangle \nabla \tilde{\Omega} \Omega^{1,0} g(t,y)\|L^2(|y| \geq \langle t \rangle /8),
\]

\[
\langle |x| \rangle^{3} |\nabla \tilde{\Gamma}^b g(t,x)|^2 \lesssim \| \langle |y| \rangle \nabla \Omega^{1,0} g(t,y)\|L^2(|y| \geq \langle t \rangle /8).
\]

This completes the proof of Lemma 3.2.

Based on Lemma 3.1 and 3.2, we have the following estimates.

**Lemma 3.3.** Under bootstrap assumptions (2.8), it holds that for \( m \leq N - 1 \),

\[
\sum_{|b|+|c| \leq m} \| \langle |x| \rangle (|Q^b c_1| + |Q^b c_2|) \|_{L^2(|x| \geq \langle t \rangle /8)} \lesssim M \varepsilon G_m(t) + E_{m+1}(t)[1 + G_{N-5}(t)].
\]

**Proof.** According to (1.20) and equalities (3.3), one easily gets

\[
\tilde{\Gamma}^b u_i = \tilde{\Gamma}^b g_i + \omega_i \Gamma^b \sigma + \frac{1}{|x|} \sum_{b_1+b_2 \leq b} f_i^{b,b_1}(x) \Gamma^{b_2} \sigma,
\]

\[
\partial_j \tilde{\Gamma}^c u_i = \partial_j \tilde{\Gamma}^c g_i + \omega_i \partial_j \Gamma^c \sigma + \frac{1}{|x|} \sum_{c_1+c_2 \leq c} f_i^{c,c_1}(x) \partial_j \Gamma^{c_2} \sigma + f_i^{c,c_1}(x) \Gamma^{c_2} \sigma.
\]
Substituting (2.2) and (3.9) into (2.5) yields

\[
Q_{1i}^{bc} = \Gamma^b \sigma \left\{ \partial_t \tilde{\Gamma}^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} \left[ f_{i}^{c_1 c_2} (x) \partial_t \Gamma^{c_2} \sigma + f_{i i}^{c_1} (x) \Gamma^{c_2} \sigma \right] \right\} \\
- \partial_t \Gamma^c \sigma \left\{ \tilde{\Gamma}^b g_i + \frac{1}{|x|} \sum_{b_1 + b_2 \leq b} f_{i}^{b_{1} b_{2}} (x) \Gamma^{b_2} \sigma \right\} ,
\]

and

\[
Q_{2i}^{bc} = \frac{1}{|x|} \Gamma^b \sigma \Omega^{c} \sigma - \omega \partial_j \Gamma^c \sigma \left\{ \tilde{\Gamma}^b g_j + \frac{1}{|x|} \sum_{b_1 + b_2 \leq b} f_{j}^{b_{1} b_{2}} (x) \Gamma^{b_2} \sigma \right\} \\
- \tilde{\Gamma}^b u_j \left\{ \partial_j \tilde{\Gamma}^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} \left[ f_{i}^{c_1} (x) \partial_j \Gamma^{c_2} \sigma + f_{i j}^{c_1} (x) \Gamma^{c_2} \sigma \right] \right\} .
\]

By applying the pointwise estimates (2.26) to the terms that containing the factor \( \frac{1}{|x|} \) in (3.10) and (3.11) directly, we obtain that

\[
\| |x| |Q_1^{bc} | + |Q_2^{bc} | \|_{L^2 (|x| \geq t/8)} \lesssim E_{m+1} (t) + \| |x| |\nabla \tilde{\Gamma}^c g |(\Gamma^b \sigma | + | \tilde{\Gamma}^b u |) \|_{L^2 (|x| \geq t/8)} + \| |x| |\tilde{\Gamma}^b g \nabla \Gamma^c \sigma \|_{L^2 (|x| \geq t/8)} .
\]

At last, by the virtue of the estimates in Lemma 3.1 and 3.2, we will deal with the remaining terms in (3.10) and (3.11).

Due to \(|b| + |c| \leq m \leq N - 1 \leq 2N - 12\), then \(|b| \leq N_1 - 6\) or \(|c| \leq N_1 - 7\) holds. For \(|b| \leq N_1 - 6\), it follows from (2.8) and (2.26) that

\[
\| |x| \tilde{\Gamma}^b g \nabla \Gamma^c \sigma \|_{L^\infty (|x| \geq t/8)} \lesssim \langle t \rangle^{-\frac{3}{2}} \{ E_{|b|+2} (t) + X_{|b|+2} (t) + \mathbb{W}_{|b|+1} (t) \} \lesssim M \epsilon .
\]

This together with (3.7) implies that

\[
\| |x| \tilde{\Gamma}^b g \nabla \Gamma^c \sigma \|_{L^2 (|x| \geq t/8)} + \| |x| |\nabla \tilde{\Gamma}^c g |(\Gamma^b \sigma | + | \tilde{\Gamma}^b u |) \|_{L^2 (|x| \geq t/8)} \lesssim E_{|c|+1} (t) [G_{|b|+1} (t) + E_{|b|+1} (t)] + M \epsilon G_{|c|} (t)
\lesssim M \epsilon [G_{m} (t) + E_{m+1} (t)] [1 + G_{N_1-5} (t)].
\]

For \(|c| \leq N_1 - 7\), by (2.19), we have

\[
\| \nabla \Gamma^c \sigma \|_{L^\infty (|x| \geq t/8)} \lesssim \langle x \rangle^{-\frac{3}{2}} \langle x \rangle^{-1} \{ E_{|c|+3} (t) + X_{|c|+3} (t) \}
\lesssim M \epsilon \langle x \rangle^{-\frac{3}{2}} \langle x \rangle^{-1} .
\]

Therefore,

\[
\| |x| \tilde{\Gamma}^b g \nabla \Gamma^c \sigma \|_{L^2 (|x| \geq t/8)} + \| |x| |\nabla \tilde{\Gamma}^c g |(\Gamma^b \sigma | + | \tilde{\Gamma}^b u |) \|_{L^2 (|x| \geq t/8)} \lesssim E_{|b|} (t) [G_{|c|+2} (t) + E_{|c|+2} (t)] + E_{|b|} (t) + M \epsilon \| |x|^{\frac{1}{2}} (|x| - t)^{-1} \| \tilde{\Gamma}^b g \|_{L^2 (|x| \geq 4(t)/5)} .
\]

For the last term in (3.14), performing the integration by parts for the radial direction yields

\[
\| |x|^{\frac{1}{2}} (|x| - t)^{-1} \tilde{\Gamma}^b g \|_{L^2 (|x| \geq 4(t)/5)} \lesssim \int_0^\infty \int_{S^1} \tilde{\Gamma}^b g (t, r \omega)^2 \langle r \rangle \left[ 1 - \chi (r \langle t \rangle) \right] d \omega d \arctan (r - t) \lesssim G_{|b|}^2 (t) + E_{|b|}^2 (t),
\]

where the cutoff function \( \chi \) is defined by (2.21).

Substituting (3.13)–(3.15) into (3.12) derives (3.8). This completes the proof of Lemma 3.3. \( \square \)

Combining Lemma 3.1–3.3, we obtain the following result.

**Corollary 3.4.** Under bootstrap assumptions (2.8), for \( |a| \leq N - 1, \, |b| \leq N - 2 \) and \( |x| \geq \langle t \rangle / 8 \), it holds that

\[
\langle |x| \rangle \langle |\tilde{\Gamma}^a g(t, x)| \rangle \lesssim E_0 + \mathcal{X}_2(t) + \mathcal{W}_1(t),
\]

where the cutoff function \( \chi \) is defined by (2.21).

Moreover, for any integer \( m \) with \( 0 \leq m \leq N - 1 \), it holds that

\[
\sum_{|b|+|c| \leq m} \| \langle |x| \rangle \langle |Q_1^{bc}| + |Q_2^{bc}| \rangle \|_{L^2(|x| \geq \langle t \rangle / 8)} \lesssim E_{m+1}(t) + M\varepsilon[\mathcal{X}_{m+1}(t) + \mathcal{W}_m(t)].
\]

**Proof.** It concludes from (3.1) and (3.8) with \( m = N_1 - 5, \, (2.8) \) and the smallness of \( M\varepsilon \) that

\[
\mathcal{G}_{N_1-5}(t) \lesssim E_{N_1-4}(t)[1 + \mathcal{G}_{N_1-5}(t)] + \mathcal{X}_{N_1-4}(t) + \mathcal{W}_{N_1-5}(t),
\]

which implies \( \mathcal{G}_{N_1-5}(t) \lesssim M\varepsilon \). Together with (3.1) and (3.8) again, this yields

\[
\mathcal{G}_m(t) \lesssim E_{m+1}(t) + \mathcal{X}_{m+1}(t) + \mathcal{W}_m(t).
\]

Substituting (3.18) into (3.7) and (3.8) completes the proof of Corollary 3.4. \( \square \)

### 3.2 Estimates of the auxiliary energy \( \mathcal{X}_m(t) \)

**Lemma 3.5** (Weighted \( \tilde{H}_1^1 \)). Under bootstrap assumptions (2.8), for any integer \( m \) with \( 1 \leq m \leq N \), it holds that

\[
\mathcal{X}_m(t) \lesssim E_m(t) + \mathcal{W}_{m-1}(t).
\]

**Proof.** For \( |a| \leq m - 1 \), it follows from direct computations and equations (2.4) that

\[
(|x|^2 - t^2) \partial_t \tilde{\Gamma}^{a} u_i = |x|^2 (Q_{a2i} - \partial_i \Gamma^a \sigma) - t \tilde{\Gamma}^a u_i + tx_j \partial_j \tilde{\Gamma}^{a} u_i
\]

\[
= |x|^2 Q_{a2i} - x_j (x_j \partial_i - x_i \partial_j) \Gamma^a \sigma + tx_i \partial_i \Gamma^a \sigma - t \tilde{\Gamma}^a u_i
\]

\[
+ tx_j (\partial_j \tilde{\Gamma}^{a} u_i - \partial_i \tilde{\Gamma}^{a} u_j) + t (x_j \partial_i - x_i \partial_j) \tilde{\Gamma}^a u_j + tx_i \text{div} \tilde{\Gamma}^a u_i
\]

\[
= |x|^2 Q_{a2i} - x_j \varepsilon_{ji} \Omega \tilde{\Gamma}^a \sigma - x_i \tilde{\Gamma}^a \sigma + tx_i Q_{a2i}^1 - t \tilde{\Gamma}^a u_i
\]

\[
+ tx_j \varepsilon_{ji} \tilde{\Gamma}^a \text{curl} u + t \varepsilon_{ji} \Omega (\tilde{\Gamma}^a u_j).
\]

Here we point out that the main difference between (3.20) and the analogous equality of \( \partial_i P_i \tilde{\Gamma}^a u_i \) in [27] lies in the presence of the vorticity \( \tilde{\Gamma}^a \text{curl} u \) in (3.20).

On the other hand, we can obtain

\[
(|x|^2 - t^2) \partial \tilde{\Gamma}^a \sigma = |x|^2 Q_{a} - x_j \varepsilon_{ji} \Omega (\tilde{\Gamma}^a u_i) - x_i \tilde{\Gamma}^a \sigma + tx_i Q_{a2i}^1 - t \tilde{\Gamma}^a u_i
\]

\[
+ tx_j \varepsilon_{ji} \tilde{\Gamma}^a \text{curl} u + t \varepsilon_{ji} \Omega (\tilde{\Gamma}^a u_j),
\]

\[
(|x|^2 - t^2) \text{div} \tilde{\Gamma}^a u = x_j \varepsilon_{ji} \Omega (\tilde{\Gamma}^a u_i) + x_i \tilde{\Gamma}^a u_i - tx_i Q_{a2i}^1 - t^2 Q_{a2i}^1 + t \tilde{\Gamma}^a \sigma.
\]
In view of $\langle |x| - t \rangle \lesssim 1 + | |x| - t |$, by dividing $|x| + t$ and then taking $L^2$ norm on the both sides of (3.20) and (3.21), we arrive at

$$X_m(t) \lesssim E_m(t) + W_{m-1}(t) + \sum_{|b|+|c| \leq m-1} \|\langle |x| + t \rangle (|Q_{1}^{bc}| + |Q_{2}^{bc}|)\|_{L^2},$$

(3.22)

where $Q_{1}^{bc}, Q_{2}^{bc}$ are defined in (2.5).

We next investigate the $L^2$ norms of $Q_{1}^{bc}$ and $Q_{2}^{bc}$, which are divided into the two parts of $|x| \geq \langle t \rangle / 8$ and $|x| \leq \langle t \rangle / 8$.

It is easy to deduce from (3.17) that

$$\sum_{|b|+|c| \leq m-1} \|\langle |x| + t \rangle (|Q_{1}^{bc}| + |Q_{2}^{bc}|)\|_{L^2(|x| \geq \langle t \rangle / 8)} \lesssim E_m(t) + W_{m-1}(t) + M \varepsilon X_m(t).$$

(3.23)

We now deal with $\|\langle |x| - t \rangle (|Q_{1}^{bc}| + |Q_{2}^{bc}|)\|_{L^2(|x| \leq \langle t \rangle / 8)}$. In fact, only $\tilde{\Gamma}^{c}u \cdot \nabla \Gamma^{c}u$ requires to be treated since the treatments on the other terms $\tilde{\Gamma}^{c}\sigma \div \Gamma^{c}u$, $\tilde{\Gamma}^{b}u \cdot \nabla \Gamma^{c}u$, $\tilde{\Gamma}^{b}\sigma \div \Gamma^{c}u$ in $Q_{1}^{bc}$ and $Q_{2}^{bc}$ are analogous.

Similarly to Lemma 3.3, it always holds that $|b| \leq N_1 - 5$ or $|c| \leq N_1 - 7$. For the case of $|c| \leq N_1 - 7$, applying (2.26) to $\nabla \Gamma^{c}u$ leads to

$$\|\langle |x| - t \rangle \tilde{\Gamma}^{b}u \cdot \nabla \Gamma^{c}u\|_{L^2(|x| \leq \langle t \rangle / 8)} \lesssim E_{|b|}(t)\langle t \rangle W_{|c|+1} + E_{|c|+3}(t) + X_{|c|+3}(t) \| \lesssim E_m(t),$$

(3.24)

where we have used assumptions (2.8).

For the case of $|b| \leq \min\{N_1 - 5, m - 2\}$, by utilizing (2.28) to $\tilde{\Gamma}^{b}u$, we can see that

$$\|\langle |x| - t \rangle \tilde{\Gamma}^{b}u \cdot \nabla \Gamma^{c}u\|_{L^2(|x| \leq \langle t \rangle / 8)} \lesssim \|\tilde{\Gamma}^{b}u\|_{L^\infty(|x| \leq \langle t \rangle / 8)} \|\langle |x| - t \rangle \chi\left(\frac{|x|}{\langle t \rangle}\right) \nabla \Gamma^{c}u\|_{L^2}$$

$$\lesssim \|\langle |x| - t \rangle \chi\left(\frac{|x|}{\langle t \rangle}\right) \nabla \Gamma^{c}u\|_{L^2} \left\{W_{|b|+1}(t) + \langle t \rangle^{-1} \ln 2(2 + t)[E_{|b|+2}(t) + X_{|b|+2}(t)]\right\}$$

$$\lesssim \|\langle |x| - t \rangle \chi\left(\frac{|x|}{\langle t \rangle}\right) \nabla \Gamma^{c}u\|_{L^2} \left\{W_{N_1 - 4}(t) + \langle t \rangle^{-1} \ln 2(2 + t)[E_m(t) + X_m(t)]\right\}. $$

(3.25)

In addition, it follows from (2.11) that

$$\|\langle |x| - t \rangle \chi\left(\frac{|x|}{\langle t \rangle}\right) \nabla \Gamma^{c}u\|_{L^2} \lesssim E_{|c|}(t) + \|\langle |x| - t \rangle \nabla \left(\chi\left(\frac{|x|}{\langle t \rangle}\right) \Gamma^{c}u\right)\|_{L^2}$$

$$\lesssim E_{|c|}(t) + \|\langle |x| - t \rangle \nabla \Gamma^{c}u\|_{L^2} + \langle t \rangle \|\langle |x| \rangle \text{curl } \Gamma^{c}u\|_{L^2} $$

$$\lesssim E_{|c|}(t) + X_{|c|+1}(t) + \langle t \rangle W_{|c|}(t)$$

$$\lesssim E_{m}(t) + \varepsilon X_{m}(t) + \langle t \rangle W_{m-1}(t). $$

(3.26)

Substituting (3.26) into (3.25) derives

$$\|\langle |x| - t \rangle \tilde{\Gamma}^{b}u \cdot \nabla \Gamma^{c}u\|_{L^2(|x| \leq \langle t \rangle / 8)} \lesssim \left\{E_{m}(t) + \varepsilon X_{m}(t) + \langle t \rangle W_{m-1}(t)\right\} \{M \delta + \langle t \rangle^{-1} \ln 2(2 + t)[E_m(t) + X_m(t)]\}$$

$$\lesssim E_{m}(t) + M \varepsilon X_{m}(t) + W_{m-1}(t),$$

(3.27)
In this subsection, by the virtue of the weighted identities (2.8).
Collecting (3.24) and (3.27) yields that

$$
\sum_{|b|+|c| \leq m-1} \| \langle |x| + t \rangle (|Q_1^{bc}| + |Q_2^{bc}|) \|_{L^2(|x| \leq (t/8)} \lesssim E_m(t) + M\varepsilon\mathcal{X}_m(t) + \mathcal{W}_{m-1}(t).
$$

(3.28)

By combining (3.23) and (3.28), we eventually achieve

$$
\sum_{|b|+|c| \leq m-1} \| \langle |x| + t \rangle (|Q_1^{bc}| + |Q_2^{bc}|) \|_{L^2} \lesssim E_m(t) + M\varepsilon\mathcal{X}_m(t) + \mathcal{W}_{m-1}(t).
$$

(3.29)

Plugging (3.29) into (3.22) with the smallness of $M\varepsilon$, then (3.19) is proved. 

4 Improved pointwise estimates

Note that the decay rate of the irrotational part $P_t u$ of the velocity $u$ is merely $\varepsilon(t)^{-1}$ away from the light cone (see Lemma 2.4). This is far to achieve the desired lifespan $T_\delta = O(1)$, for examples, when $\delta = e^{-\frac{1}{2t}}$ or $\delta = e^{-e^{-\frac{1}{2t}}}$ are chosen, whose reason has been explained in Section 1. It is required to improve the related pointwise estimates in Section 3.

4.1 Improved pointwise estimates of the first order derivatives of $(\sigma, u)$

In this subsection, by the virtue of the weighted identities (3.4), (3.20) and (3.21), the pointwise estimates of $\nabla \tilde{\Gamma}^a u$, $\nabla \tilde{\Gamma}^a \sigma$ in (2.26), (2.28) and $\nabla \tilde{\Gamma}^a g$ in (3.16) can be improved as follows.

Lemma 4.1. Under bootstrap assumptions (2.8), if $|a| \leq N_1 - 1$, then for $|x| \geq \langle t \rangle / 8$, it holds that

$$
|\partial_t \tilde{\Gamma}^a u(t, x)| + |\partial_t \tilde{\Gamma}^a \sigma(t, x)| + |\text{div} \tilde{\Gamma}^a u(t, x)| + |\nabla \tilde{\Gamma}^a \sigma(t, x)| \lesssim M\varepsilon \langle |x| \rangle^{2M\varepsilon^{-\frac{1}{2}}} \langle |x| - t \rangle^{-\frac{3}{2}},
$$

$$
|\nabla \tilde{\Gamma}^a u(t, x)| \lesssim M\varepsilon \langle |x| \rangle^{2M\varepsilon^{-\frac{1}{2}}} \langle |x| - t \rangle^{-\frac{1}{2}}.
$$

(4.1)

and

$$
|\nabla \tilde{\Gamma}^a g(t, x)| \lesssim M\varepsilon \langle |x| \rangle^{2M\varepsilon^{-\frac{1}{2}}} \langle |x| - t \rangle^{-\frac{1}{2}}.
$$

(4.2)

On the other hand, for $|x| \leq 3\langle t \rangle / 4$, we have

$$
|\partial_t \tilde{\Gamma}^a u(t, x)| + |\partial_t \tilde{\Gamma}^a \sigma(t, x)| \lesssim M\delta(t)^{M\varepsilon^{-1}} + M\varepsilon(t)^{2M\varepsilon^{-2}} \ln(2 + t),
$$

$$
|\text{div} \tilde{\Gamma}^a u(t, x)| + |\nabla \tilde{\Gamma}^a \sigma(t, x)| \lesssim M\delta(t)^{M\varepsilon^{-1}} + M\varepsilon(t)^{2M\varepsilon^{-2}} \ln(2 + t),
$$

$$
|\langle |x| \rangle|\nabla \tilde{\Gamma}^a u(t, x)| \lesssim M\delta(t)^{M\varepsilon} + M\varepsilon(t)^{2M\varepsilon^{-1}} \ln(2 + t).
$$

(4.3)

Proof. For $|x| \geq \langle t \rangle / 8$, it follows from (2.23), (2.26), (3.20) and (3.21) that

$$
\langle |x| - t \rangle \langle |\partial_t \tilde{\Gamma}^a u(t, x)| + |\partial_t \tilde{\Gamma}^a \sigma(t, x)| + |\text{div} \tilde{\Gamma}^a u(t, x)| + |\nabla \tilde{\Gamma}^a \sigma(t, x)| \rangle
\lesssim \sum_{|b| \leq |a| + 1} (|\Gamma^a \sigma| + |\tilde{\Gamma}^a u| + \langle |x| \rangle|\Gamma^a \text{curl} u| + \langle |x| \rangle) \sum_{b+c \leq a} (|Q_1^{bc}| + |Q_2^{bc}|)
\lesssim \mathbb{W}_{|a|+1}(t)|x|^{-\frac{3}{2}} + \langle |x| \rangle^{-\frac{1}{2}} \langle |x| - t \rangle^{-\frac{1}{2}} \left\{ E_{|a|+3}(t) + \mathcal{X}_{|a|+3}(t) \right\}
+ \mathbb{W}_{|a|+1}(t)|x|^{-\frac{3}{2}} + \langle |x| \rangle \sum_{b+c \leq a} (|Q_1^{bc}| + |Q_2^{bc}|).
$$

(4.4)
Applying (2.26) and (3.16) to (3.10) and (3.11) yields

\[
|Q_{1}^{bc}| + |Q_{2}^{bc}| \lesssim \langle |x| \rangle^{-1} \left( M\delta \langle |x| \rangle^{M'\varepsilon - \frac{\delta}{2}} + M\varepsilon \langle |x| \rangle^{M'\varepsilon - \frac{\delta}{2}} \langle |x| - t \rangle^{-\frac{1}{2}} \right)^{2} + \langle \tilde{\Gamma}^b g \nabla \Gamma^c \sigma \rangle + \langle \nabla \tilde{\Gamma}^c g \langle |\tilde{\Gamma}^b \sigma| + |\tilde{\Gamma}^b u| \rangle \right) 
\lesssim M\delta \langle |x| \rangle^{2M'\varepsilon - \frac{\delta}{4}} + M^2 \varepsilon^2 \langle |x| \rangle^{2M'\varepsilon - \frac{\delta}{2}} \langle |x| - t \rangle^{-1},
\]

(4.5)

where we have used the bootstrap assumptions (2.8). Substituting (4.5) into (4.4) infers

\[
\langle |x| - t \rangle \langle |\partial_t \tilde{\Gamma}^a u(t, x)| + |\partial_t \tilde{\Gamma}^a \sigma(t, x)| + |\text{div} \tilde{\Gamma}^a u(t, x)| \rangle \nabla \tilde{\Gamma}^a \sigma(t, x) \rangle 
\lesssim M\delta \langle |x| \rangle^{2M'\varepsilon - \frac{\delta}{4}} + M\varepsilon \langle |x| \rangle^{M'\varepsilon - \frac{\delta}{2}} \langle |x| - t \rangle^{-\frac{1}{2}} + M^2 \varepsilon^2 \langle |x| \rangle^{2M'\varepsilon - \frac{\delta}{2}} \langle |x| - t \rangle^{-1}
\lesssim M\varepsilon \langle |x| \rangle^{2M'\varepsilon - \frac{\delta}{4}} \langle |x| - t \rangle^{-\frac{1}{2}}.
\]

This leads to the first inequality in (4.1).

Next, we turn to the proof of the second inequality in (4.1). By (2.2), we have

\[
\langle |x| \rangle \nabla \tilde{\Gamma}^a u(t, x) \rangle \lesssim |\Gamma^{\leq 1} \tilde{\Gamma}^a u(t, x) + |r\partial_t \tilde{\Gamma}^a u(t, x)| \rangle \lesssim |\Gamma^{\leq 1} \tilde{\Gamma}^a u(t, x)| + |t\partial_t \tilde{\Gamma}^a u(t, x)|.
\]

This, together with the estimate of \( \partial_t \tilde{\Gamma}^a u(t, x) \) in (4.1), yields the second inequality in (4.1).

It is not hard to conclude from (3.4) that for \( |x| \geq \langle t \rangle/8 \),

\[
\langle |x| \rangle \nabla \tilde{\Gamma}^a g(t, x) \rangle \lesssim \langle |x| \rangle |\Gamma^a \text{curl} u| + \sum_{|b| \leq |a| + 1} (|\tilde{\Gamma}^b g| + |\Gamma^b \sigma| + |\tilde{\Gamma}^b u|) + \langle |x| \rangle |Q_1^{bc}| + \langle |x| - t \rangle |\partial_t \Gamma^a \sigma|
\lesssim \langle |x| \rangle^{\frac{\delta}{2}} \mathbb{W}_{|a|+1}(t) + \langle |x| \rangle \langle |x| - t \rangle^{-\frac{1}{2}} \left\{ E_{|a|+3}(t) + \mathcal{X}_{|a|+3}(t) \right\}
\]

(4.7)

where we have used (2.23) and (2.26).

Combining (4.7) with (4.1) and (4.5) yields (4.2).

Finally, we turn to the proof of (4.3). For \( |x| \leq 3\langle t \rangle/4 \), by using (2.23) and (2.28) to (3.20) and (3.21) directly, we arrive at

\[
\langle t \rangle \langle |\partial_t \tilde{\Gamma}^a u(t, x)| + |\partial_t \Gamma^a \sigma(t, x)| + |\text{div} \tilde{\Gamma}^a u(t, x)| + |\nabla \Gamma^a \sigma(t, x)| \rangle 
\lesssim \sum_{|b| \leq |a| + 1} (|\tilde{\Gamma}^b g| + |\tilde{\Gamma}^b u|) + \langle |x| \rangle |\Gamma^a \text{curl} u| + \langle t \rangle \sum_{b+c \leq a} (|Q_1^{bc}| + |Q_2^{bc}|) 
\lesssim \mathbb{W}_{|a|+1}(t) + \langle t \rangle^{-1} \ln \langle 2 + t \rangle \left\{ E_{|a|+3}(t) + \mathcal{X}_{|a|+3}(t) \right\}
\]

(4.8)

\[
+ \langle t \rangle^{-1} \ln (2 + t) \left( E_{|a|+3}(t) + \mathcal{X}_{|a|+3}(t) \right)^2 
\lesssim M\delta \langle t \rangle^{M'\varepsilon} + M\varepsilon \langle t \rangle^{2M'\varepsilon - 1} \ln (2 + t).
\]

Then we can achieve the first two inequalities in (4.3). Combining (4.6) with the estimates \( \partial_t \tilde{\Gamma}^a u(t, x) \) in (4.3), we get the third inequality in (4.3).
4.2 Weighted $L^\infty$-$L^\infty$ estimates for the linear wave equation

In this subsection, we will establish some weighted $L^\infty$-$L^\infty$ estimates for the solutions to the linear wave equations. Consider the following Cauchy problem

$$\Box \varphi := \partial_t^2 \varphi - \Delta \varphi = \mathcal{F}, \quad (\varphi, \partial_t \varphi)|_{t=0} = (\varphi_0, \varphi_1).$$

(4.8)

Then $\varphi = \varphi_{\text{hom}} + \varphi_{\text{inh}}$, where

$$\Box \varphi_{\text{hom}} = 0, \quad (\varphi_{\text{hom}}, \partial_t \varphi_{\text{hom}})|_{t=0} = (\varphi_0, \varphi_1),$$

(4.9)

and

$$\Box \varphi_{\text{inh}} = \mathcal{F}, \quad (\varphi_{\text{inh}}, \partial_t \varphi_{\text{inh}})|_{t=0} = (0, 0).$$

Lemma 4.2. [Proposition 4.1 and 4.2 of [16]] Let $0 < \nu < \frac{1}{2}$ and $\mu > 0$, then it holds that

$$\langle |x| + t \rangle^{\frac{1}{2}} \langle |x| - t \rangle^\nu |\varphi_{\text{inh}}(t, x)| \lesssim \tilde{M}_{\mu + \nu}(\mathcal{F})(t),$$

(4.10)

$$\langle |x| \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{1+\nu} |\nabla \varphi_{\text{inh}}(t, x)| \lesssim \sum_{|a| + j \leq 1} \tilde{M}_{\mu + \nu}(\nabla^a \Omega^j \mathcal{F})(t),$$

(4.11)

where

$$\tilde{M}_{\mu + \nu}(\mathcal{F})(t) = \sup_{(s, y) \in \Lambda_0(t)} \{ \langle |y| \rangle^{\frac{3}{2}} \langle |y| + s \rangle^{1+\mu+\nu} |\mathcal{F}(s, y)| \}$$

(4.12)

and

$$\Lambda_1(t) = \{(s, y) \in [0, t] \times \mathbb{R}^2 : |y| - s \leq s/3, |y| \geq 1\},$$

$$\Lambda_0(t) = [0, t] \times \mathbb{R}^2 \setminus \Lambda_1(t) = \{(s, y) \in [0, t] \times \mathbb{R}^2 : |y| - s \geq s/3, \text{ or } |y| \leq 1\}.$$  

(4.13)

Remark 4.1. The notation $\tilde{M}_{\mu + \nu}(\mathcal{F})(t)$ on the right hand side of (4.10) and (4.11) is slightly different from that in [16], in which is $M_{\nu}(\mathcal{F})(t)$ with

$$M_{\nu}(\mathcal{F})(t) = \sup_{(s, y) \in \Lambda_0(t)} \{ \langle |y| \rangle^{\frac{3}{2}} \langle |y| + s \rangle^{1+\mu+\nu} |\mathcal{F}(s, y)| \} + \sup_{(s, y) \in \Lambda_1(t)} \{ \langle s \rangle^{\frac{3}{2}} + \mu + \nu \langle |y| - s \rangle |\mathcal{F}(s, y)| \}.$$

Unfortunately, Lemma 4.2 can not be applied directly for our problem. We next give a modified version as follows.

Lemma 4.3. Let $0 < \mu_1, \nu < \frac{1}{2}$ and $\mu > 0$, then for $|x| \leq 2(t)$, it holds that

$$\langle |x| + t \rangle^{\frac{1}{2}} - \mu_1 \langle |x| - t \rangle^\nu |\varphi_{\text{inh}}(t, x)| \lesssim M_{\mu - \mu_1}(\mathcal{F})(t),$$

(4.14)

$$\langle |x| \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{1+\nu} |\nabla \varphi_{\text{inh}}(t, x)| \lesssim \sum_{|a| + j \leq 1} M_{\mu + \nu}(\nabla^a \Omega^j \mathcal{F})(t),$$

(4.15)

where

$$M_{\mu + \nu}(\mathcal{F})(t) = \sup_{(s, y) \in \Lambda_0(t), |y| \leq 3(t)} \{ \langle |y| \rangle^{\frac{3}{2}} \langle |y| + s \rangle^{1+\mu+\nu} |\mathcal{F}(s, y)| \}$$

(4.16)

$$+ \sup_{(s, y) \in \Lambda_1(t)} \{ \langle s \rangle^{\frac{3}{2}} + \mu + \nu \langle |y| - s \rangle |\mathcal{F}(s, y)| \}.$$
Proof. Recall the Poisson formula
\[ \varphi_{inh}(t, x) = \frac{1}{2\pi} \int_0^t \int_{|y-x| \leq t-s} \frac{\mathcal{F}(s, y) dy ds}{\sqrt{(t-s)^2 - |y-x|^2}}. \]

In the domain \( \{(y, s) : |y - x| \leq t - s\} \), one has \( \langle |y| + s \rangle \lesssim \langle |x| + t \rangle \) and \( |y| \leq |x| + t \leq 3\langle t \rangle \). Therefore, we obtain
\[ \langle |x| + t \rangle^{-\mu_1} \varphi_{inh}(t, x) \lesssim \int_0^t \int_{|y-x| \leq t-s} \frac{\langle |y| + s \rangle^{-\mu_1} |\mathcal{F}(s, y)| dy ds}{\sqrt{(t-s)^2 - |y-x|^2}}. \]

Applying (4.10) to the above integration yields (4.14). The proof of (4.15) is analogous.

Next, we study the pointwise estimates of \( \varphi_{hom} \).

**Lemma 4.4 (Estimates of \( \varphi_{hom} \)).** Let \( \varphi_{hom} \) be defined by (4.9). It holds that
\[ \langle |x| + t \rangle^{\frac{3}{2}} \langle |x| - t \rangle^{rac{1}{2}} |\varphi_{hom}(t, x)| \lesssim \|\langle |y| \rangle \varphi_0(y)\|_{W^{0,1}_y} + \|\langle |y| \rangle \varphi_1(y)\|_{W^{1,1}_y}, \tag{4.17} \]
\[ \langle |x| + t \rangle^{\frac{3}{2}} \langle |x| - t \rangle^{rac{3}{2}} |\nabla \varphi_{hom}(t, x)| \lesssim \|\langle |y| \rangle \varphi_0(y)\|_{W^{2,1}_y} + \|\langle |y| \rangle \varphi_1(y)\|_{W^{2,1}_y}. \tag{4.18} \]

**Proof.** The inequality (4.17) is just Lemma 3.2 of [19].

Next, we derive (4.18) by (4.17). In fact, for any vector field \( \tilde{Z} \in \{\partial, \mathcal{S}, \Omega, t\partial_i + x_i \partial_t, i = 1, 2\} \), one has \( \Box \tilde{Z} \varphi_{hom} = 0 \). Applying (4.17) to \( \tilde{Z} \varphi_{hom} \) yields
\[ \langle |x| + t \rangle^{\frac{3}{2}} \langle |x| - t \rangle^{rac{3}{2}} |\tilde{Z} \varphi_{hom}(t, x)| \lesssim \|\langle |y| \rangle \tilde{Z} \varphi_{hom}(0, y)\|_{W^{2,1}_y} + \|\langle |y| \rangle \partial_i \tilde{Z} \varphi_{hom}(0, y)\|_{W^{2,1}_y} \lesssim \|\langle |y| \rangle \varphi_0(y)\|_{W^{2,1}_y} + \|\langle |y| \rangle \varphi_1(y)\|_{W^{2,1}_y}. \]

This, together with
\[ \langle |x| - t \rangle |\nabla \varphi_{hom}(t, x)| \lesssim \sum_{\tilde{Z} \in \{\partial, \mathcal{S}, \Omega, t\partial_i + x_i \partial_t, i = 1, 2\}} |\tilde{Z} \varphi_{hom}(t, x)|, \]
derives Lemma 4.4.

\[ \Box \varphi = F := \partial_t \mathcal{A} - (2 - \sigma)Q_1 - u \cdot Q_2, \tag{4.20} \]

where the nonlocal term \( \mathcal{A} \) is defined by
\[ \mathcal{A} := -(-\Delta)^{-1} \text{curl}(u \text{ curl } u), \quad \lim_{|x| \to \infty} \mathcal{A}(t, x) = 0. \tag{4.21} \]

**4.3 Improved pointwise estimates of \( u \)**

This subsection is devoted to improve the pointwise estimates of \( u \) by the weighted \( L^\infty-L^\infty \) estimates in subsection 4.2. For this purpose, we need to find the related wave equation hidden in the equations (1.19).

By the Helmholtz decomposition in subsection 2.2, there exists a potential function \( \phi(t, x) \) such that
\[ u = P_1 u + P_2 u = \nabla \phi + P_2 u. \tag{4.19} \]

The inherent wave equation of \( \phi \) can be directly deduced from (1.19), see appendix A for details.
In addition,
\[ \sigma = -\partial_t \phi + A - \frac{1}{2}(|u|^2 - \sigma^2), \]  
which is achieved in appendix A.

Acting \((\mathcal{S} + 2)^{\alpha_s} Z^{\alpha_s}\) on (4.20), we can get the equation of \(\Gamma^a \phi\):
\[ \Box \Gamma^a \phi = F^a := \sum_{b \leq a} C^a_b \Gamma^b \partial_t A + \sum_{b + c \leq a} C^a_{bc} Q^c_{1} + \sum_{b + c + d \leq a} C^a_{bcd} \left\{ Q^b_1 \Gamma^d \sigma - Q^b_2 \tilde{\Gamma}^d_{i} u_i \right\}, \]  
(4.23)
where \(C^a_{\alpha \beta \gamma}\) are some suitable constants.

At first, we deal with the pointwise estimates of the nonlocal term \(A\).

**Lemma 4.5** (Estimates of \(A\).) Under bootstrap assumptions (2.8), for \(|d'| \leq N_1 - 2\), it holds that
\[ |\Gamma^a \cdot A(t, x)| \lesssim \langle |x| \rangle^{-\frac{8}{7}} M \delta(t)^{2M'} \varepsilon \left\{ M \delta + M \varepsilon(t)^{-1} \ln(2 + t) \right\}. \]  
(4.24)

**Proof.** In view of (2.16), (4.24) can be directly achieved by the following \(L^p\) estimates
\[ \sum_{|a| \leq |a'| + 1} \left\{ \| \langle |x| \rangle^7 \nabla \Gamma^a A(t, x) \|_{L^5} + \sum_{p = \frac{10}{9}, \frac{10}{7}} \| \nabla \Gamma^a A(t, x) \|_{L^p} \right\} \lesssim M \delta(t)^{2M'} \varepsilon \left\{ M \delta + M \varepsilon(t)^{-1} \ln(2 + t) \right\}. \]  
(4.25)

Next we focus on the proof of (4.25). Note that
\[ \Gamma^a A = -\sum_{b + c \leq a} C^a_{bc} (-\Delta)^{-1} \text{curl}(\tilde{\Gamma}^b u \Gamma^c \text{curl} u), \]  
(4.26)
which can be achieved by applying the following equality repeatedly
\[ \Delta \Gamma A = \text{curl}(u \Gamma \text{curl} u + \tilde{\Gamma} u \text{curl} u). \]

Indeed, for \(\Gamma = \Omega\) (the other case of \(\Gamma\) is analogous), direct computation yields
\[ \Delta \Omega A = \Omega \Delta A = \Omega \text{curl}(u \text{curl} u) = \text{curl} \tilde{\Omega}(u \text{curl} u) = \text{curl}\{\Omega(u \text{curl} u) - u^1 \text{curl} u\} = \text{curl}\{\Omega u \text{curl} u + u \text{curl} u - u^1 \text{curl} u\} = \text{curl}\{\tilde{\Omega} u \text{curl} u + u \text{curl} u\}. \]

It concludes from the \(L^p\) boundedness of the Riesz operators and (4.26) that
\[ \sum_{p = \frac{10}{9}, \frac{10}{7}} \| \nabla \Gamma^a A(t, x) \|_{L^p} \lesssim \sum_{p = \frac{10}{9}, \frac{10}{7}} \sum_{b + c \leq a} \| \tilde{\Gamma}^b u \Gamma^c \text{curl} u \|_{L^p} \lesssim \sum_{p = \frac{10}{9}, \frac{10}{7}} \sum_{b + c \leq a} \| \langle |x| \rangle^{-1} \tilde{\Gamma}^b u \|_{L^p} \| \langle |x| \rangle \Gamma^c \text{curl} u \|_{L^p} \lesssim \mathcal{W}_{|a|}(t) \{ \mathcal{W}_{|a|+1}(t) + \langle t \rangle^{-1} \ln(2 + t) [\mathcal{E}_{|a|+2}(t) + \mathcal{X}_{|a|+2}(t)] \} \lesssim M \delta(t)^{2M'} \varepsilon \left\{ M \delta + M \varepsilon(t)^{-1} \ln(2 + t) \right\}, \]
2.28

Moreover, according to [28], one knows that \( |x|^7 \) belongs to \( A_5 \) class. Thereafter, we have

\[
\| |x|^7 \nabla \Gamma^a A(t, x) \|_{L^\infty} \lesssim \sum_{b+c \leq a} \| |x|^7 \tilde{\Gamma}^b u \nabla \Gamma^c \nabla \|_{L^\infty}
\]

\[
\lesssim \sum_{b+c \leq a} \| |x|^{-1} \tilde{\Gamma}^b u \|_{L^\infty} \| |x|^8 \nabla \Gamma^c \nabla u \|_{L^\infty}
\]

\[
\lesssim M \delta \langle t \rangle^{2M \varepsilon} \{ M \delta + M \varepsilon \langle t \rangle^{-1} \ln(2 + \langle t \rangle) \}.
\]

Thus, we have proved (4.25).

Secondly, we focus on the pointwise estimates of the potential function \( \Gamma^a \phi \), where \( \phi \) is defined in (4.19).

**Lemma 4.6 (Estimates of \( \Gamma^a \phi \)).** Under bootstrap assumptions (2.8), for \( |a| \leq N_1 - 3 \) and \( |x| \leq 2\langle t \rangle \), it holds that

\[
\langle |x| + t \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{16}} \| \Gamma^a \phi(t, x) \| \lesssim M \delta \langle t \rangle^{\frac{1}{2}} + M \varepsilon.
\]

**Proof.** By applying the weighted \( L^\infty - L^\infty \) estimates (4.14) and (4.17) with \( \mu_1 = \frac{1}{8} \) and \( \nu = \mu = \frac{1}{16} \) to \( \Gamma^a \phi \) in (4.23), we obtain

\[
\langle |x| + t \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{16}} \| \Gamma^a \phi(t, x) \|
\lesssim \| \langle |y| \rangle \Gamma^a \phi(0, y) \|_{W^{2,1}} + \| \langle |y| \rangle \partial_t \Gamma^a \phi(0, y) \|_{W^{1,1}} + M_{-\frac{1}{8}}(F^a)(t).
\]

According to the definition of \( F^a \) in (4.23), we arrive at

\[
M_{-\frac{1}{8}}(F^a)(t) \lesssim \sum_{b \leq a} M_{-\frac{1}{8}}(\Gamma^b \partial_t A)(t) + \sum_{b+c \leq a} M_{-\frac{1}{8}}(Q_{bc}^d)(t)
\]

\[
+ \sum_{b+c+d \leq a} M_{-\frac{1}{8}}(\langle Q_{bc}^d \rangle + |Q_{bc}^d \partial_t u|)(t).
\]

Note that it only suffices to deal with the two terms on the right hand side of the first line in (4.29), since the treatment on the cubic nonlinearities in the second line are much easier.

It follows from the definition (4.16) of \( M_{-\frac{1}{8}}(F)(t) \) and (4.24) that

\[
\sum_{b \leq a} M_{-\frac{1}{8}}(\Gamma^b \partial_t A)(t) \lesssim \sum_{b \leq a} \sup_{s \leq \langle t \rangle} \langle |y| \rangle^{\frac{3}{2}} \langle |y| + s \rangle^{\frac{7}{8}} \| \Gamma^b \partial_t A(s, y) \|
\]

\[
\lesssim M \delta \sup_{s \leq \langle t \rangle} \langle |y| + s \rangle^{\frac{7}{8}} \langle |y| + s \rangle^{-\frac{7}{8}} \{ M \delta + M \varepsilon \langle s \rangle^{-1} \ln(2 + s) \}^{2M \varepsilon}
\]

\[
\lesssim M \delta.
\]

The control of \( M_{-\frac{1}{8}}(Q_{bc}^d)(t) \) will be divided into three parts corresponding to the domains \( \Lambda_1(t) \), \( D_1 \) and \( D_2 \), where

\[
D_1 := \Lambda_0(t) \cap \{(s, y) : |y| \leq 2\langle s \rangle / 3 \},
\]

\[
D_2 := \Lambda_0(t) \cap \{(s, y) : 4\langle s \rangle / 3 \leq |y| \leq 3\langle t \rangle \},
\]

and the definitions of \( \Lambda_0(t) \) and \( \Lambda_1(t) \) see (4.13).
In $D_1$, applying (2.26) to (2.5) yields that
\[
\sum_{b+c \leq a} \sup_{(s,y) \in D_1} \langle |y| \rangle^{\frac{3}{4}} \langle |y| + s \rangle^{\frac{3}{2}} |Q^{bc}_1(s,y)| \lesssim \sup_{s \leq t} \langle s \rangle^{\frac{11}{8} + 2M\varepsilon} \left( M\delta + M\varepsilon \langle s \rangle^{-\frac{1}{2}} \right) \left( M\delta + M\varepsilon \langle s \rangle^{-1} \right) \lesssim M\delta \langle t \rangle^{\frac{1}{2}} + M\varepsilon. \tag{4.31}
\]

In $D_2$, by using (2.26) again to (2.5), we deduce that
\[
\sum_{b+c \leq a} \sup_{(s,y) \in D_2} \langle |y| \rangle^{\frac{3}{4}} \langle |y| + s \rangle^{\frac{3}{2}} |Q^{bc}_1(s,y)| \lesssim \sup_{|y| \leq 3(t), s \leq t} \langle |y| \rangle^{\frac{11}{8} + 2M\varepsilon} \left( M\delta + M\varepsilon \langle |y| \rangle^{-\frac{1}{2}} \right) \left( M\delta + M\varepsilon \langle |y| \rangle^{-1} \right) \lesssim M\delta \langle t \rangle^{\frac{1}{2}} + M\varepsilon. \tag{4.32}
\]

In $\Lambda_1(t)$, the null condition structure in (3.10) will play a crucial rule (see Section 5 below). Applying (2.26) to the terms containing the factor $\frac{1}{|z|}$ in (3.10), we then see that
\[
\sum_{b+c \leq a} \sup_{(s,y) \in \Lambda_1(t)} \langle s \rangle^{\frac{11}{8}} \langle |y| - s \rangle \langle Q^{bc}_1(s,y) \rangle \lesssim \sup_{|y| \leq 4s/3 \leq 4t/3} \langle s \rangle^{\frac{3}{2}} \left( M\delta \langle s \rangle^{M\varepsilon - \frac{3}{2}} \langle |y| - s \rangle \right)^{\frac{1}{2}} + M\varepsilon \langle s \rangle^{M\varepsilon - \frac{3}{2}} \right)^{\frac{1}{2}} \tag{4.33}
\]
\[+ \sum_{b+c \leq a} \sup_{(s,y) \in \Lambda_1(t)} \langle s \rangle^{\frac{11}{8}} \langle |y| - s \rangle (|\nabla \Gamma^b \sigma \tilde{\Gamma}^c g| + |\Gamma^b \sigma \nabla \tilde{\Gamma}^c g|).
\]

By employing (2.26) and (3.16) to $\nabla \Gamma^b \sigma \tilde{\Gamma}^c g$ and $\Gamma^b \sigma \nabla \tilde{\Gamma}^c g$, one has
\[
\sum_{b+c \leq a} \sup_{(s,y) \in \Lambda_1(t)} \langle s \rangle^{\frac{11}{8}} \langle |y| - s \rangle (|\nabla \Gamma^b \sigma \tilde{\Gamma}^c g| + |\Gamma^b \sigma \nabla \tilde{\Gamma}^c g|) \lesssim M\delta + M\varepsilon. \tag{4.34}
\]

Collecting (4.31)–(4.34) together, we eventually arrive at
\[
\sum_{b+c \leq a} M_{-\frac{5}{8}}(Q^{bc}_1(t)) \lesssim M\delta \langle t \rangle^{\frac{1}{2}} + M\varepsilon. \tag{4.35}
\]

At last, we turn to estimate the initial data on the right hand side of (4.28). It is deduced from (1.14) and (4.22) that
\[
\| (|y|) \Gamma^a \phi(0,y) \|_{W^{0,1}_0} + \| (|y|) \partial_t \Gamma^a \phi(0,y) \|_{W^{1,1}_0} \lesssim \varepsilon \lesssim M\varepsilon. \tag{4.36}
\]

Substituting (4.29), (4.30), (4.35) and (4.36) into (4.28) yields (4.27).

Next, the pointwise estimates of the good unknown $g$ can be improved.

**Lemma 4.7** (Improved estimates of $\tilde{\Gamma}^a g$). Under bootstrap assumptions (2.8), for $|a| \leq N_1 - 4$ and $\langle t \rangle/8 \leq |x| \leq 2 \langle t \rangle$, it holds that
\[
|\tilde{\Gamma}^a g(t, x)| \lesssim M\delta \langle t \rangle^{-\frac{3}{4}} + M\varepsilon \langle t \rangle^{-\frac{1}{4}} + M\varepsilon \langle t \rangle^{M\varepsilon - \frac{3}{2}} (|x| - t)^{\frac{1}{2}}. \tag{4.37}
\]
Proof. The key idea to achieve (4.37) is to make full use of the potential function \( \phi \) with the relations \( u = \nabla \phi + P_2 u \) and (4.22). It follows from the definition (1.20), (3.9) and tedious computation that

\[
\begin{align*}
\hat{t} \hat{\Gamma}^a g_i &= (t - |x|) \hat{\Gamma}^a u_i + |x| \hat{\Gamma}^a u_i - t \omega_i \hat{\Gamma}^a \sigma - \frac{t}{|x|} \sum_{b+c \leq a} f_{i}^{a,b}(x) \Gamma^c \sigma \\
&= (t - |x|) \hat{\Gamma}^a u_i + |x| P_2 \hat{\Gamma}^a u_i - |x| \hat{\Gamma}^a \hat{\partial}_i \phi + t \omega_i \hat{\Gamma}^a \hat{\partial}_i \phi - t \omega_i \hat{\Gamma}^a A \\
&\quad - \frac{t}{|x|} \sum_{b+c \leq a} f_{i}^{a,b}(x) \Gamma^c \sigma + \frac{1}{2} t \omega_i \hat{\Gamma}^a (|u|^2 - \sigma^2) \\
&= (t - |x|) \hat{\Gamma}^a u_i + |x| P_2 \hat{\Gamma}^a u_i + \sum_{b \leq a} C_{i}^{a,b} (|x| \hat{\partial}_i + t \omega_i \hat{\partial}_i) \hat{\Gamma}^b \phi - t \omega_i \hat{\Gamma}^a A \\
&\quad - \frac{t}{|x|} \sum_{b+c \leq a} f_{i}^{a,b}(x) \Gamma^c \sigma + t \omega_i \sum_{b+c \leq a} C_{i}^{a,b} (\hat{\Gamma}^b u_j \hat{\Gamma}^c u_j - \Gamma^b \phi \Gamma^c \sigma) .
\end{align*}
\]

By utilizing (2.2) and the first line of (3.9) to the last summation in (4.38), we arrive at

\[
\begin{align*}
\hat{t} \hat{\Gamma}^a g_i &= (t - |x|) \hat{\Gamma}^a u_i + |x| P_2 \hat{\Gamma}^a u_i + \sum_{b \leq a} C_{i}^{a,b} (|x| \hat{\partial}_i \tilde{\Gamma}^b \phi + \Omega \tilde{\Gamma}^b \phi) - t \omega_i \hat{\Gamma}^a A \\
&\quad - \frac{t}{|x|} \sum_{b+c \leq a} f_{i}^{a,b}(x) \Gamma^c \sigma + t \omega_i \sum_{b+c \leq a} C_{i}^{a,b} (\tilde{\Gamma}^b u_j \tilde{\Gamma}^c u_j - \tilde{\Gamma}^b \phi \tilde{\Gamma}^c \sigma) \\
&\quad + \frac{t \omega_i}{|x|} \sum_{b_1 + b_2 + c \leq a} f_{j}^{b_1 b_2}(x) \Gamma^{b_2 \sigma} \tilde{\Gamma}^c u_j + \frac{t \omega_i}{|x|} \sum_{b+c_1 + c_2 \leq b} f_{j}^{c_1 c_2}(x) \omega_j \tilde{\Gamma}^b \sigma \tilde{\Gamma}^{c_2} \sigma .
\end{align*}
\]

Applying (2.22), (2.26), (4.24) and (4.27) to (4.39) leads to

\[
\begin{align*}
\langle t \rangle |\hat{\Gamma}^a g_i| &\lesssim M \delta(t)^{M \varepsilon - \frac{1}{2}} + M \varepsilon \langle t \rangle^{M \varepsilon - \frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{2}} + \langle t \rangle^{- \frac{1}{2}} \{ M \delta(t)^{\frac{1}{2}} + M \varepsilon \}
\end{align*}
\]

Applying (4.40) with the smallness of \( M \varepsilon \) derives (4.37). \( \square \)

Finally, we turn to the estimates of the velocity \( u = \nabla \phi + P_2 u \).

**Lemma 4.8 (Estimates of \( \nabla \Gamma^a \phi \)).** Under bootstrap assumptions (2.8), for \( |a| \leq N_1 - 4 \) and \( |x| \leq 2 \langle t \rangle \), it holds that

\[
\langle |x| - t \rangle^{\frac{1}{2}} \langle \nabla \Gamma^a \phi(t, x) \rangle \lesssim M \delta(t)^{\frac{1}{2}} + M \varepsilon .
\]

**Proof.** Applying (4.15) and (4.18) to (4.23) with \( \nu = \frac{1}{8} \) and \( \mu = \frac{1}{27} \) yields

\[
\langle |x| \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{2}} \langle \nabla \Gamma^a \phi(t, x) \rangle \lesssim \| \langle |y| \rangle^{2} \Gamma^a \phi(0, y) \|_{W^{3,1}_y} + \| \langle |y| \rangle^{2} \partial_t \Gamma^a \phi(0, y) \|_{W^{2,1}_y} + \sum_{|b| \leq |a| + 1} M_{\frac{1}{8}}(F^b(t)) .
\]

Similarly to Lemma 4.6, we can obtain

\[
\sum_{|b| \leq |a| + 1} M_{\frac{1}{8}}(F^b(t)) \lesssim \sum_{|b| \leq |a| + 1} M_{\frac{1}{8}}(\Gamma^b \partial_t \mathcal{A}(t)) + \sum_{|b| + |c| \leq |a| + 1} M_{\frac{1}{8}}(Q_{1}^{bc}(t))
\]

\[+ \sum_{|b| + |c| + |d| \leq |a| + 1} M_{\frac{1}{8}}(|Q_{2}^{bc} \Gamma^d \sigma| + |Q_{21}^{bc} \Gamma^d u_i|)(t) .
\]

(4.43)
It follows from the definition (4.16) of $\mathcal{M}_b(F)(t)$ and (4.24) that
\[
\sum_{|b| \leq |a| + 1} \mathcal{M}_b(\Gamma^b \partial_t A)(t) \lesssim \sum_{|b| \leq |a| + 1} \sup_{s \leq t,y} \langle |y| \rangle^{\frac{5}{2}} \langle |y| + s \rangle^{\frac{7}{5}} |\Gamma^b \partial_t A(s, y)| \\
\lesssim M \delta \sup_{s \leq t,y} \langle |y| + s \rangle^{\frac{7}{5}} \langle |y| \rangle^{-\frac{7}{5}} \langle s \rangle^{2M' \varepsilon} \{M \delta + M \varepsilon \langle s \rangle^{-1} \ln(2 + s)\} \tag{4.44} \\
\lesssim M \delta \langle t \rangle^{\frac{3}{4} + 2M' \varepsilon} \ln(2 + t) \lesssim M \delta \langle t \rangle^{\frac{3}{4}}.
\]

In $\mathcal{D}_1$, applying (2.28) and (4.3) to (2.5) implies
\[
\sum_{b+c \leq a} \sup_{(s,y) \in \mathcal{D}_1} \langle |y| \rangle^{\frac{5}{2}} \langle |y| + s \rangle^{\frac{7}{5}} |Q_{bc}^b(s, y)| \\
\lesssim \sup_{s \leq t} \langle s \rangle^{\frac{7}{5}} \left(M \delta \langle s \rangle^{M' \varepsilon} + M \varepsilon \langle s \rangle^{2M' \varepsilon - 1} \ln(2 + s)\right)^{2} \tag{4.45} \\
\lesssim M \delta \langle t \rangle^{\frac{3}{4}} + M \varepsilon.
\]

In $\mathcal{D}_2$, by using (2.26), (4.1) to (2.5), we get
\[
\sum_{b+c \leq a} \sup_{(s,y) \in \mathcal{D}_2} \langle |y| \rangle^{\frac{5}{2}} \langle |y| + s \rangle^{\frac{7}{5}} |Q_{bc}^b(s, y)| \\
\lesssim \sup_{|y| \leq 2s/3 \leq 4t/3} \langle s \rangle^{\frac{7}{5}} \left(M \delta \langle s \rangle^{M' \varepsilon} \langle |y| - s \rangle^{\frac{1}{2}} + M \varepsilon \langle s \rangle^{M' \varepsilon - \frac{1}{2}}\right)^{2} \tag{4.46} \\
+ \sum_{b+c \leq a} \sup_{(s,y) \in \Lambda_1(t)} \langle s \rangle^{\frac{5}{3}} \langle |y| - s \rangle \left\{\nabla \Gamma^b \sigma \tilde{\Gamma}^c g + |\Gamma^b \sigma \nabla \tilde{\Gamma}^c g|\right\}.
\]

On the other hand, by using (4.1) and (4.37) to $\nabla \Gamma^b \sigma \tilde{\Gamma}^c g$, one has
\[
\sum_{b+c \leq a} \sup_{(s,y) \in \Lambda_1(t)} \langle s \rangle^{\frac{5}{3}} \langle |y| - s \rangle |\nabla \Gamma^b \sigma \tilde{\Gamma}^c g| \\
\lesssim \sup_{s \leq t} \left\{M \delta \langle s \rangle^{2M' \varepsilon - \frac{5}{3} - \frac{1}{2} - \frac{3}{4}} + M \varepsilon \langle s \rangle^{2M' \varepsilon + \frac{5}{3} - \frac{1}{2} - \frac{3}{4}}\right\} \tag{4.48} \\
\lesssim M \delta \langle t \rangle^{\frac{3}{4}} + M \varepsilon.
\]

In addition, employing (2.19), (4.2) to $\Gamma^b \sigma \nabla \tilde{\Gamma}^c g$ yields that
\[
\sum_{b+c \leq a} \sup_{(s,y) \in \Lambda_1(t)} \langle s \rangle^{\frac{5}{3}} \langle |y| - s \rangle |\Gamma^b \sigma \nabla \tilde{\Gamma}^c g| \lesssim M \varepsilon. \tag{4.49}
\]
Collecting (4.45)-(4.49) together, we eventually achieve
\[
\sum_{b+c\leq a} M_{\frac{1}{b}}(Q_1^b)(t) \lesssim M\delta(t)^{\frac{3}{4}} + M\varepsilon. \tag{4.50}
\]
Note that similarly to (4.36), we have
\[
\|\langle|y|\rangle^2\Gamma^a \phi(0, y)\|_{W^{\frac{3}{2},1}} + \|\langle|y|\rangle^2\partial_t \Gamma^a \phi(0, y)\|_{W^{\frac{3}{2},1}} \lesssim M\varepsilon. \tag{4.51}
\]
Therefore, substituting (4.43), (4.44), (4.50) and (4.51) into (4.42) derives (4.41).

Combining (4.41) and (2.22) with the decomposition \( \tilde{\Gamma}^a u = \nabla \Gamma^a \phi + P_2 \tilde{\Gamma}^a u \), we arrive at

**Corollary 4.9.** Under bootstrap assumptions (2.8), for \( |a| \leq N_1 - 4 \) and \( |x| \leq 2\langle t \rangle \), it holds that
\[
|\tilde{\Gamma}^a u(t, x)| \lesssim M\delta(|x|)^{-\frac{3}{4}} + \langle |x| - t \rangle^{-\frac{3}{4}} \left\{ M\delta(t)^{\frac{3}{4}} + M\varepsilon \right\}. \tag{4.52}
\]

## 5 Energy estimates

### 5.1 Elementary energy estimates

This subsection is aimed to establish the elementary energy estimates for \( E_m(t) \), which is defined by (2.7).

**Lemma 5.1.** Under bootstrap assumptions (2.8), we have that
\[
E^2_{N}(t') \lesssim E^2_{N}(0) + \int_0^{t'} E^2_{N}(t) \left\{ M\delta + M\varepsilon(t)^{\frac{1}{4}} \right\} dt
\]
\[
+ \int_0^{t'} E_N(t) M\delta(t)^{2M\varepsilon} \left\{ M\delta + M\varepsilon(t)^{\frac{3}{4}} \right\} dt, \tag{5.1}
\]
\[
E^2_{N_1-4}(t') \lesssim E^2_{N_1-4}(0) + \int_0^{t'} E^2_{N_1-4}(t) \left\{ M\delta + M\varepsilon(t)^{\frac{3}{4}} \right\} dt
\]
\[
+ \int_0^{t'} E_{N_1-4}(t) M\delta(t)^{2M\varepsilon} \left\{ M\delta + M\varepsilon(t)^{\frac{3}{4}} \right\} dt. \tag{5.2}
\]

**Proof.** For the multi-index \( a \) with \( |a| = m \leq N \), multiplying (2.4) by \( 2e^q \Gamma^a \sigma \) and \( 2e^q \tilde{\Gamma}^a u \), respectively, yields the following equality
\[
\partial_t \{ e^q (|\Gamma^a \sigma|^2 + |\tilde{\Gamma}^a u|^2) \} + 2 \text{div} \left\{ e^q (1 - \sigma) \Gamma^a \sigma \tilde{\Gamma}^a u \right\} + \text{div} \left\{ e^q u (|\Gamma^a \sigma|^2 + |\tilde{\Gamma}^a u|^2) \right\}
\]
\[
+ \frac{e^q}{\langle |x| - t \rangle^{\frac{3}{4}}} \sum_{i=1}^{2} \left\{ |\tilde{\Gamma}^a u_i - \omega_i \Gamma^a \sigma |^2 - u_i \omega_i (|\Gamma^a \sigma|^2 + |\tilde{\Gamma}^a u|^2) + 2\sigma \omega_i \Gamma^a \sigma \tilde{\Gamma}^a u_i \right\}
\]
\[
= e^q (|\Gamma^a \sigma|^2 + |\tilde{\Gamma}^a u|^2) \text{div} u - 2e^q \Gamma^a \sigma \tilde{\Gamma}^a u \cdot \nabla \sigma + \sum_{b+c=a, \atop \langle b \prec c \rangle} 2e^q C_{bc}^a (Q_1^{bc} \Gamma^a \sigma + Q_2^{bc} \cdot \tilde{\Gamma}^a u),
\]
where the ghost weight \( e^q = e^q(|x| - t) \) with \( q(|x| - t) = \int_{-\infty}^{t} s^{-\frac{1}{4}} ds \) was introduced in [5] for treating the global small solution problem of the second order 2D quasilinear wave equation. Integrating
the above equality over \([0,t'] \times \mathbb{R}^2\) derives that

\[
E_{[a]}^2(t') + \sum_{i=1}^{2} \int_{0}^{t'} \int \frac{1}{\langle |x|-t \rangle^2} |\tilde{\Gamma}^a u_i - \omega_i \Gamma^a \sigma|^2 dxdt \\
\lesssim E_{[a]}^2(0) + \int_{0}^{t'} \left\{ |I^a| + \sum_{b+c=a, \ c < a} (|Q_{1}^{bc} \Gamma^a \sigma| + |Q_{2}^{bc} \cdot \tilde{\Gamma}^a u|) \right\} dxdt,
\]

(5.3)

where

\[
I^a := (|\Gamma^a \sigma|^2 + |\tilde{\Gamma}^a u|^2) \text{ div } u - 2\Gamma^a \sigma \tilde{\Gamma}^a u \cdot \nabla \sigma \\
+ \frac{1}{\langle |x|-t \rangle^2} \sum_{i=1}^{2} \left\{ u_i \omega_i (|\Gamma^a \sigma|^2 + |\tilde{\Gamma}^a u|^2) - 2\sigma \omega_i \Gamma^a \sigma \tilde{\Gamma}^a u_i \right\}.
\]

(5.4)

As in Lemma 3.5, the space domain is still divided into two parts of \(|x| \leq \langle t \rangle/8\) and \(|x| \geq \langle t \rangle/8\).

Firstly, we deal with \(I^a\), \(Q_{1}^{bc}\) and \(Q_{2}^{bc}\) in the region \(|x| \leq \langle t \rangle/8\). For \(I^a\), it concludes from (2.26) and (4.3) that

\[
\int_{|x| \leq \langle t \rangle/8} |I^a| dx \lesssim E_{m}^2(t) \left\{ M \varepsilon(t)^{-\frac{3}{2}} + M \delta \right\},
\]

(5.5)

In view of \(|b| + |c| \leq N \leq 2N_1 - 11\), then \(|b| \leq N_1 - 5\) or \(|c| \leq N_1 - 7\) holds.

If \(|b| \leq N_1 - 1\), applying (2.11) to \(\nabla \tilde{\Gamma}^c u\) and applying (2.28) to \(\Gamma^b \sigma\), \(\tilde{\Gamma}^b u\), respectively, infer that

\[
\int_{|x| \leq \langle t \rangle/8} (|Q_{1}^{bc} \Gamma^a \sigma| + |Q_{2}^{bc} \cdot \tilde{\Gamma}^a u|) dx \\
\lesssim E_{m}(t) \left\{ X_{m}(t) + E_{m-1}(t) + \langle t \rangle W_{m-1}(t) \right\} \left\{ M \varepsilon(t)^{M^{\varepsilon} - 2 \ln(2 + t) + M \delta(t)^{M^{\varepsilon} - 1}} \right\}
\]

(5.6)

where we have also used (2.8) and (3.19).

If \(|c| \leq N_1 - 5\), by using (4.3) and (4.52) to \(\nabla \tilde{\Gamma}^c \sigma\) and \(\nabla \tilde{\Gamma}^c u\), respectively, we can obtain

\[
\int_{|x| \leq \langle t \rangle/8} (|Q_{1}^{bc} \Gamma^a \sigma| + |Q_{2}^{bc} \cdot \tilde{\Gamma}^a u|) dx \lesssim E_{m}(t) \left\{ M \varepsilon(t)^{-\frac{3}{2}} + M \delta \right\}.
\]

(5.7)

Now, we turn to the treatments of \(I^a\), \(Q_{1}^{bc}\) and \(Q_{2}^{bc}\) in the region \(|x| \geq \langle t \rangle/8\). For \(I^a\) defined by (5.4), it follows from the definition of the good unknown (1.20) that

\[
I^a = \sum_{i=1}^{2} \left\{ \text{ div } u |\tilde{\Gamma}^a u_i - \omega_i \Gamma^a \sigma|^2 + 2\Gamma^a \sigma \tilde{\Gamma}^a u_i (\omega_i \text{ div } u - \partial_i \sigma) \right\} \\
+ \frac{1}{\langle |x|-t \rangle^2} \sum_{i,j=1}^{2} \left\{ u_i \omega_i |\tilde{\Gamma}^a u_j - \omega_j \Gamma^a \sigma|^2 + 2g_i \omega_j \omega_j \Gamma^a \sigma \tilde{\Gamma}^a u_j \right\}.
\]

(5.8)

It follows from (2.2) and the second equality of (3.9) that

\[
\omega_i \text{ div } u - \partial_i \sigma = \omega_i \left\{ \partial_j g_j + \frac{1}{|x|} [f_j^0(x) \partial_j \sigma + f_{jj}^0(x) \sigma] \right\} + \frac{1}{|x|} \Omega \sigma.
\]

(5.9)
By applying (2.26) and (3.16) to (5.8) and (5.9), we see that

$$\int_{|x| \geq \langle t \rangle / 8} |I|^2 dx \lesssim \sum_{i=1}^{2} \int \frac{M\varepsilon}{|x| - t} |\tilde{u}_i - \omega_i \Gamma^a u_i|^2 dx + E^2_m(t) \{M\delta + M\varepsilon \langle t \rangle^{-1}\}, \quad (5.10)$$

where we have used the Cauchy-Schwartz inequality.

Next, we focus on the treatments of $Q^{bc}_{2i}$ and $Q^{bc}_{2}$ in the region $|x| \geq \langle t \rangle / 8$.

The case of $|c| \leq N_1 - 7$: By the second equality of (3.9), we get

$$Q^{bc}_{2i} = -\tilde{G}^b u_j \partial_j \Gamma^c u_i + \Gamma^b \sigma \partial_i \Gamma^c \sigma$$

$$= -\partial_j \Gamma^c u_i (\tilde{G}^b u_j - \omega_j \Gamma^b \sigma) + \Gamma^b \sigma (\partial_i \Gamma^c \sigma - \omega_j \partial_j \tilde{G}^c u_i)$$

$$= -\partial_j \Gamma^c u_i (\tilde{G}^b u_j - \omega_j \Gamma^b \sigma) + \frac{1}{|x|} \Gamma^b \sigma \Omega \Gamma^c \sigma$$

$$- \omega_j \Gamma^b \sigma \left\{ \partial_j \tilde{G}^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} \left[ f^{c_1}_{i} (x) \partial_j \Gamma^{c_2} \sigma + f^{c_1}_{ij} (x) \Gamma^{c_2} \sigma \right] \right\}, \quad (5.11)$$

and

$$Q^{bc}_{1} = \Gamma^b \sigma \left\{ \partial_i \tilde{G}^c g_i + \frac{1}{|x|} \sum_{c_1 + c_2 \leq c} \left[ f^{c_1}_{i} (x) \partial_i \Gamma^{c_2} \sigma + f^{c_1}_{ii} (x) \Gamma^{c_2} \sigma \right] \right\}$$

$$- \partial_i \Gamma^c \sigma (\tilde{G}^b u_i - \omega_i \Gamma^b \sigma). \quad (5.12)$$

According to (2.26) with $|c| \leq N_1 - 7$, we arrive at

$$|\nabla \Gamma^c \sigma (t, x)| + |\nabla \tilde{G}^c u(t, x)| \lesssim \langle t \rangle^{-7} W_{|a|+1}(t) + \langle t \rangle^{-\frac{7}{2}} \langle |x| - t \rangle^{-1} \left\{ E_{|a|+3}(t) + X_{|a|+3}(t) \right\}$$

$$\lesssim M\delta + M\varepsilon \langle t \rangle^{-\frac{7}{2}} \langle |x| - t \rangle^{-1}. \quad (5.13)$$

Substituting this inequality and (3.16) into (5.11) and (5.12) leads to

$$\sum_{b+c=a, |c|\leq N_1-7} \int_{|x| \geq \langle t \rangle / 8} (|Q^{bc}_{1} \Gamma^a \sigma| + |Q^{bc}_{2} \cdot \tilde{G}^a u|) dx \lesssim \sum_{|b| \leq |a|} \sum_{i=1}^{2} \int \frac{M\varepsilon}{|x| - t} |\tilde{G}^b u_i - \omega_i \Gamma^b \sigma|^2 dx + E^2_m(t) \{M\delta + M\varepsilon \langle t \rangle^{-1}\}. \quad (5.13)$$

The case of $|b| \leq N_1 - 4$ and the lower order energy: By applying (3.18), (3.19), (4.37) to (3.10) and (3.11), we obtain

$$\sum_{b+c=a, |c|\leq N_1-4} \int_{|x| \leq 2} (|Q^{bc}_{1} \Gamma^a \sigma| + |Q^{bc}_{2} \cdot \tilde{G}^a u|) dx \lesssim E_m(t) \left\{ E_m(t) + W_{m-1}(t) \right\} \{M\delta + M\varepsilon \langle t \rangle^{-\frac{7}{4}}\}$$

$$\lesssim E_m(t) \left\{ E_m(t) + \delta(t)^M \varepsilon \right\} \{M\delta + M\varepsilon \langle t \rangle^{-\frac{5}{4}}\}, \quad (5.14)$$
Next, we pay our attention to the related treatments in the region $|x| \geq 2\langle t \rangle$. For this purpose, we need only to deal with $\|\nabla \tilde{T}^c u\|_{L^2(|x| \geq 2\langle t \rangle)}$. Choose the cutoff function $\chi(s) \in C^\infty$ which taking values in $[0, 1]$ and satisfying

$$\chi(s) = \begin{cases} 1, & s \geq 2, \\ 0, & s \leq 3/2. \end{cases}$$

Thereafter, it is easy to check that

$$\langle x \rangle \|\nabla \tilde{T}^c u\|_{L^2(|x| \geq 2\langle t \rangle)} \leq E|x|_c(t) + \frac{\|\langle x \rangle - t\|}{\nabla \left( \left( \frac{|x|}{t} \right) \bar{T}^c u \right)\|_{L^2}}. \tag{5.15}$$

By using (2.11) to the last term on the right hand side of (5.15), we then get

$$\left\| \left( \frac{|x| - t}{\nabla \left( \left( \frac{|x|}{t} \right) \bar{T}^c u \right)\right. \right\|_{L^2} \leq E|x|_c(t) + \mathcal{X}_{|x|+1}(t) + \mathcal{W}_{|x|}(t), \tag{5.16}$$

Consequently, it concludes from (2.26), (3.19), (5.15) and (5.16) that

$$\sum_{b_i, c_i \leq a, \epsilon \ll 1, |b| \leq N_1 - 4} \int_{|x| \geq 2\langle t \rangle} (|Q^{b_i}_{c_i}\Gamma^a_0\sigma| + |Q^{b_i}_{c_i} \cdot \bar{T}^a u|)dx$$

$$\leq E_m(t) \langle t \rangle^{M'\epsilon - 1} \left\{ M\delta(t)^{-3} + M\varepsilon(t)^{-1} \right\} \left\{ E_m(t) + \mathcal{X}_{|x|}(t) + M\delta(t)^{M'\epsilon} \right\} \tag{5.17}$$

$$\left\{ M\delta(t)^{-3} + M\varepsilon(t)^{-1} \right\} + E_m(t) M\delta(t)^{2M'\epsilon - 1}(M\delta(t)^{-3} + M\varepsilon(t)^{-1})$$

$$\leq E_m(t) \left\{ M\delta + M\varepsilon(t)^{-\frac{3}{2}} \right\} + E_m(t) M^2\varepsilon\delta(t)^{-1}.$$

For all $m \leq N$, substituting (5.5)-(5.7), (5.10), (5.13), (5.14) and (5.17) into (5.3) yields (5.1).

Finally, we are dedicated to the lower energy estimates of $I^a$ in the region $|x| \geq \langle t \rangle/8$ with $|a| = m \leq N_1 - 4$. In the region $|x| \geq 2\langle t \rangle$, applying (2.26) and (3.16) to (5.8) shows that

$$\int_{|x| \geq 2\langle t \rangle} |I^a|dx \lesssim E^2_{N_1-4}(t) \left\{ M\delta + M\varepsilon(t)^{-\frac{3}{2}} \right\}. \tag{5.18}$$

Substituting the first equality in (3.9) into (5.8) infers

$$I^a = \sum_{i=1}^{2} \left\{ \text{div } u \left| \Gamma^a g_i \right| + \frac{1}{|x|} \sum_{a_1 + a_2 \leq a} f^{a, a_1}_{i}(x) |\Gamma^a_2\sigma|^2 \right\}$$

$$+ 2 \sum_{i,j=1}^{2} \omega_i \Gamma^a \sigma \tilde{T}^a u_i \left( \partial_j g_j + \frac{1}{|x|} \left[ f^0_j(x) \partial_j \sigma + f^0_{j2}(x) \sigma \right] + \frac{1}{|x|} \Omega \sigma \right) \tag{5.19}$$

$$+ \frac{1}{\langle |x| - t \rangle^{\frac{1}{2}}} \sum_{i,j=1}^{2} \left\{ u_i \omega_i \left| \Gamma^a g_j \right| + \frac{1}{|x|} \sum_{a_1 + a_2 \leq a} f^{a, a_1}_{j}(x) |\Gamma^a_2\sigma|^2 \right\} + 2 g_{i} \omega_{i} \omega_{j} \Gamma^a \sigma \tilde{T}^a u_{j} \right\}.$$
5.2 Energy estimates of the vorticity

Before taking the estimates of the vorticity, we will establish some useful lemmas. Recalling the definition of the specific vorticity (2.6), then it is easy to check that

\[(\partial_t + u \cdot \nabla)w = 0.\]  

(5.21)

Lemma 5.2. Under bootstrap assumptions (2.8), for \(m \leq N - 2\) and \(k \leq N_1 - 4\), it holds that

\[
\sum_{|a| \leq m} \langle |x| \rangle |\nabla \Gamma^a w(t, x)| \lesssim \sum_{|a'| \leq m+1} |\Gamma^{a'} w(t, x)| + t \langle |x| \rangle^{-\frac{3}{2}} \mathcal{W}_{N_1-4}(t) \sum_{|b| \leq m} |\tilde{\Gamma}^b u(t, x)|, \tag{5.22}
\]

and

\[
\sum_{|a| \leq k} \langle |x| \rangle |\nabla \Gamma^a w(t, x)| \lesssim \sum_{|a'| \leq k+1} |\Gamma^{a'} w(t, x)|. \tag{5.23}
\]

Furthermore, for \(|x| \geq \langle t \rangle/8\), it holds that

\[
\sum_{|a| \leq m} \langle |x| \rangle |\nabla \Gamma^a w(t, x)| \lesssim \sum_{|a'| \leq m+1} |\Gamma^{a'} w(t, x)|. \tag{5.24}
\]

Proof. Similarly to the derivation of (2.4), acting \((S + 1)^a Z^a\) on (5.21) derives

\[(\partial_t + u \cdot \nabla)\Gamma^a w = \sum_{b+c=a, \atop c \prec a} C_{bc} \dddot{\Gamma}^a w := - \sum_{b+c=a, \atop c \prec a} C_{bc} \tilde{\Gamma}^b u \cdot \nabla \Gamma^c w, \tag{5.25}
\]

where \(\Gamma^a = S^a Z^a\). Symbolically, we can see that

\[
\langle |x| \rangle |\nabla \Gamma^a w = \sum_{\tilde{\Gamma} \in \{\nabla, \Omega\}} \tilde{\Gamma} \Gamma^a w + r \partial_t \Gamma^a w = \sum_{\tilde{\Gamma} \in \{\nabla, \Omega, \mathcal{S}\}} \tilde{\Gamma} \Gamma^a w - t \partial_t \Gamma^a w
\]

\[
= \Gamma^{\geq 1} \Gamma^a w + t \sum_{b+c=a, \atop b \prec c} C_{bc} \tilde{\Gamma}^b u \cdot \nabla \Gamma^c w, \tag{5.26}
\]

By using (2.18) to \(\nabla \Gamma^c w\) with \(|c| \leq N_1 - 7\), we arrive at

\[
\sum_{b+c=a, \atop |c| \leq N_1 - 7} |\tilde{\Gamma}^b u \cdot \nabla \Gamma^c w| \lesssim \langle |x| \rangle^{-\frac{3}{2}} \mathcal{W}_{N_1-4}(t) \sum_{|b| \leq m} |\tilde{\Gamma}^b u|. \tag{5.27}
\]

If \(|c| \geq N_1 - 6\), then \(|b| \leq N_1 - 4\). It follows from (2.8), (2.26) and (4.52) that

\[
\sum_{|b| \leq N_1 - 4} t |\tilde{\Gamma}^b u \cdot \nabla \Gamma^c w| \lesssim \sum_{|b| \leq N_1 - 4} \frac{t |\tilde{\Gamma}^b u|}{\langle |x| \rangle} |\langle |x| \rangle |\nabla \Gamma^c w| \lesssim M \varepsilon |\langle |x| \rangle |\nabla \Gamma^c w|. \tag{5.28}
\]

Thereafter, substituting (5.27) and (5.28) into (5.26) with the smallness of \(M \varepsilon\) implies (5.22).

By plugging (5.28) into (5.26), we can achieve (5.23).

For the proof of (5.24), by using (2.27) to \(\tilde{\Gamma}^b u\) directly, yields

\[
t |\tilde{\Gamma}^b u \cdot \nabla \Gamma^c w| \lesssim |\tilde{\Gamma}^b u| |\langle |x| \rangle |\nabla \Gamma^c w|
\]

\[
\lesssim \langle t \rangle^{-\frac{3}{2}} |\langle |x| \rangle |\nabla \Gamma^c w| |\{ E_{|b|+2}(t) + \mathcal{X}_{|b|+2}(t) + \mathcal{W}_{|b|+1}(t) \}
\]

\[
\lesssim M \varepsilon |\langle |x| \rangle |\nabla \Gamma^c w|. \tag{5.29}
\]

This, together with (5.26), implies (5.24). □
**Lemma 5.3.** Under bootstrap assumptions (2.8), we have

\[ W_{N_1-4}(t) \lesssim W_{N_1-4}(t), \quad W_{N-1}(t) \lesssim W_{N-1}(t). \quad (5.29) \]

**Proof.** Note that

\[ \Gamma^a \text{curl} \ u = \Gamma^a w + \sum_{b+c=a} C^a_{bc} \Gamma^b \sigma \Gamma^c \text{curl} \ u. \]

Multiplying the above equality by \( \langle |x| \rangle \) and taking \( L^2 \) norm lead to

\[
\| \langle |x| \rangle \Gamma^a \text{curl} \ u \|_{L^2} \lesssim \| \langle |x| \rangle \Gamma^a w \|_{L^2} + \sum_{b+c=a, |b| \leq N-2} \| \Gamma^b \sigma \|_{L^\infty} \| \langle |x| \rangle \Gamma^c \text{curl} \ u \|_{L^2} + \sum_{b+c=a, |b| \geq N-1} \| \langle |x| \rangle \Gamma^b \sigma \Gamma^c \text{curl} \ u \|_{L^2}. \quad (5.30)\]

In addition, it follows from (2.19) that

\[
\sum_{b+c=a, |b| \leq N-2} \| \Gamma^b \sigma \|_{L^\infty} \| \langle |x| \rangle \Gamma^c \text{curl} \ u \|_{L^2} \lesssim \sum_{c \leq a} \| \langle |x| \rangle \Gamma^c \text{curl} \ u \|_{L^2}. \quad (5.31)\]

For all \( |a| \leq N_1 - 4 \), substituting (5.31) into (5.30) yields the first inequality of (5.29).

Next, we deal the second line of (5.30). It is easy to check that

\[
\| \langle |x| \rangle \Gamma^b \sigma \Gamma^c \text{curl} \ u \|_{L^2} \lesssim \langle t \rangle^{-\frac{1}{3}} \| \Gamma^b \sigma \|_{L^2(|x| \geq \frac{a}{4})} \| \langle |x| \rangle \frac{1}{2} \Gamma^c \text{curl} \ u \|_{L^\infty(|x| \geq \frac{a}{4})} + \langle t \rangle^{-\frac{1}{3}} \| \langle |x| \rangle \frac{1}{2} \Gamma^c \text{curl} \ u \|_{L^\infty(|x| \leq \frac{a}{4})}. \quad (5.32)\]

Applying the Hardy inequality infers

\[
\| \langle |x| \rangle \langle |x| - t \rangle \Gamma^b \sigma \|_{L^2} \lesssim \| \langle |x| \rangle \frac{1}{2} \Gamma^b \sigma \|_{L^\infty} \lesssim E_{|b|}(t) + \chi_{|b|+1}(t) \lesssim M \varepsilon(t)^{M\varepsilon}. \quad (5.33)\]

From \( |b| + |c| \leq N - 1 \) and \( |b| \geq N - 1 \), then \( |c| \leq N - 3 \). By plugging (2.18) and (5.33) into (5.32), we derive

\[
\sum_{b+c=a, |c| \leq N-3} \| \langle |x| \rangle \Gamma^b \sigma \Gamma^c \text{curl} \ u \|_{L^2} \lesssim M \varepsilon \sum_{|c| \leq N-3} \mathcal{W}_{|c|+2}(t) \lesssim M \varepsilon W_{N-1}(t). \quad (5.34)\]

For all \( |a| \leq N - 1 \), combining (5.30), (5.31), (5.34) with the smallness of \( M \varepsilon \) yields the second inequality of (5.29).

With these lemmas, we begin to take the estimates of the vorticity.

**Lemma 5.4 (L^2 estimates).** Under bootstrap assumptions (2.8), for \( W_m(t) \) defined by (2.7), it holds that

\[
W_{N_1-4}^2(t') \lesssim W_{N_1-4}(t'') + \left[ \mathcal{W}_{N_1-4}(t) \left\{ M \delta + M \varepsilon(t)^{-\frac{2}{5}} \right\} dt, \quad (5.35)\right.
\]
\[ W_{N-1}^2(t') \leq W_{N-1}^2(0) + \int_0^{t'} W_{N-1}^2(t) \left\{ M\delta + M\varepsilon(t)^{-1} \right\} dt + \int_0^{t'} M^2\varepsilon(t)^{M'M'} W_{N-1}(t) dt. \] (5.36)

**Proof.** Multiplying (5.25) by \(2\langle |x| \rangle^2 e^q |x| \Gamma^a w\) infers

\[
\partial_t \left( \frac{e^q \langle |x| \rangle^2 e^q |x| \Gamma^a w|^2}{\langle |x| - t \rangle^{\frac{3}{2}}} \right) + \text{div} \left( e^q |x| \Gamma^a w \right) = e^q |x| \Gamma^a w \left( \text{div} u + u \cdot \nabla q \right) + u \cdot \nabla \left( \langle |x| \rangle^2 e^q |x| \Gamma^a w \right)^2 + \sum_{b+c=a, c < a} 2e^q \langle |x| \rangle^2 C_{bc}^a \Gamma^a w J_{bc}.
\]

Integrating the above equality over \([0, t'] \times \mathbb{R}^2\) leads to

\[
W_{|a|}^2(t') + \int_0^{t'} \int \frac{\langle |x| \rangle^2 e^q |x| \Gamma^a w|^2}{\langle |x| - t \rangle^{\frac{3}{2}}} dx dt \lesssim W_{|a|}^2(0) + \int_0^{t'} \int \langle |x| \rangle^2 e^q |x| \Gamma^a w J(u) dx dt + \sum_{b+c=a, c < a} \int_0^{t'} \int \langle |x| \rangle^2 |x| \Gamma^a w ||J_{bc}| dx dt,
\] (5.37)

where

\[
J(u) := \text{div} u + \frac{|u|}{\langle |x| - t \rangle^{\frac{3}{2}}} + \frac{|u|}{\langle |x| \rangle}. \quad (5.38)
\]

The integral domain is still decomposed into two parts of \(|x| \geq \langle t \rangle / 8\) and \(|x| \leq \langle t \rangle / 8\) as before.

The case \(|b| \leq N_1 - 4\) in the region \(|x| \leq \langle t \rangle / 8\): By plugging the improved pointwise estimates (4.52) into \(J(u)\) and \(\Gamma^b u\), we derive

\[
\int_{|x| \leq \langle t \rangle / 8} \langle |x| \rangle^2 e^q |x| \Gamma^a w J(u) dx + \sum_{b+c=a, c < a, |b| \leq N_1-4} \int_{|x| \leq \langle t \rangle / 8} \langle |x| \rangle^2 |x| \Gamma^a w ||J_{bc}| dx \lesssim W_{|a|}^2(t) \left\{ M\delta + M\varepsilon(t)^{-\frac{9}{8}} \right\}.
\] (5.39)

The case \(|b| \leq N_1 - 4\) in the region \(|x| \geq \langle t \rangle / 8\): Note that \(|c| \leq |a| - 1 \leq N - 2\). Then, from (2.26) and (5.24), we arrive at

\[
\int_0^{t'} \int_{|x| \geq \langle t \rangle / 8} \langle |x| \rangle^2 e^q |x| \Gamma^a w J(u) dx dt + \sum_{b+c=a, c < a, |b| \leq N_1-4} \int_0^{t'} \int_{|x| \geq \langle t \rangle / 8} \langle |x| \rangle^2 |x| \Gamma^a w ||J_{bc}| dx dt \lesssim M\varepsilon \sum_{|a'| \leq |a|} \int_0^{t'} \int \frac{\langle |x| \rangle^2 e^q |x| \Gamma^a w|^2}{\langle |x| - t \rangle^{\frac{3}{2}}} dx dt + \int_0^{t'} W_{|a|}^2(t) \left\{ M\delta + M\varepsilon(t)^{M'M'} - \frac{3}{2} \right\} dt,
\] (5.40)

where we have used Young’s inequality.

For all \(|a| \leq N_1 - 4\), substituting (5.39) and (5.40) into (5.37) yields (5.35).

In the rest part, we will focus on the case \(|b| \geq N_1 - 3\). From \(|b| + |c| \leq N - 1 \leq 2N_1 - 10\), one has \(|c| \leq N_1 - 7 \leq N - 4\).
In the region $|x| \geq (t/8)$: Applying (2.18) and (5.24) to $\nabla \Gamma^c w$ infers
\[
\int_{|x| \geq (t/8)} \langle |x|^2 |\nabla \Gamma^c w| J_{bc} \rangle dx \lesssim \langle t \rangle^{-\frac{1}{2}} W_{|a|}(t) E_{|b|}(t) W_{|c|+3}(t) \lesssim M \varepsilon \langle t \rangle^{M' \varepsilon - \frac{3}{2}} W_{|a|}(t) W_{N-1}(t).
\]
(5.41)

In the region $|x| \leq (t/8)$: We deduce from (2.18), (3.19), (5.23), (5.29) and the Hardy inequality that
\[
\|\langle |x| \rangle J_{bc} \|_{L^2(|x| \leq \omega(\varepsilon))} \lesssim \|\langle |x| \rangle^{\frac{3}{2}} \nabla \Gamma^c w \|_{L^\infty(|x| \leq \omega(\varepsilon))} \left\| \frac{\chi_{\Omega} b^u}{|x| \log |x|} \right\|_{L^2} \lesssim \langle t \rangle^{-1} W_{|c|+3}(t) \{ E_{|b|+1}(t) + \mathcal{W}_{|b|}(t) \}
\lesssim \langle t \rangle^{-1} W_{N-4}(t) \{ E_N(t) + W_{N-1}(t) \}
\lesssim M \delta W_{N-1}(t) + M^2 \varepsilon \delta \langle t \rangle^{M' \varepsilon - 1}.
\]
(5.42)

For all $|a| \leq N-1$, by plugging (5.39)–(5.42) into (5.37), we obtain (5.36).

\[\square\]

**Lemma 5.5** ($L^p$ estimates). Under bootstrap assumptions (2.8), set $\mathcal{W}_{p,m}(t) := \sum_{|a| \leq m} \|\langle |x| \rangle^{\theta(p)} \Gamma^a w(t, x)\|_{L^p_x}$, where
\[
\theta(p) := \begin{cases} 1, & p = \frac{10}{9}, \frac{10}{7}, \\ 8, & p = 5. \end{cases}
\]
Then, for $p = \frac{10}{9}, \frac{10}{7}, 5$, it holds that
\[
\mathcal{W}_{p,N-4}^p(t') \lesssim \mathcal{W}_{p,N-4}^p(0) + \int_0^{t'} W_{p,N-4}^p(t) \left\{ \frac{M \delta + M \varepsilon \langle t \rangle^{-\frac{3}{2}}}{} \right\} dt,
\]
(5.43)
\[
\mathcal{W}_{p,N-3}^p(t') \lesssim \mathcal{W}_{p,N-3}^p(0) + \int_0^{t'} W_{p,N-3}^p(t) \left\{ \frac{M \delta + M \varepsilon \langle t \rangle^{-1}}{} \right\} dt
\]
\[
+ \int_0^{t'} M \delta \langle t \rangle^{M' \varepsilon} \left\{ \frac{M \delta + M \varepsilon \langle t \rangle^{-1}}{} \right\} \mathcal{W}_{p,N-3}^{p-1}(t) dt.
\]
(5.44)

**Proof.** Multiplying (5.25) by $p \langle |x| \rangle^{\theta(p)} e^{q|(|x|-t)|} |\Gamma^a w|^p - 2 \Gamma^a w$ shows that
\[
\partial_t \left( e^{q \langle |x| \rangle^{\theta(p)} \Gamma^a w} \right) + e^{q \langle |x| \rangle^{\theta(p)} \Gamma^a w} + \text{div} \left( e^{q \langle |x| \rangle^{\theta(p)} \Gamma^a w} \right)
\]
\[
= e^{q \langle |x| \rangle^{\theta(p)} \Gamma^a w} \left( \text{div} u + u \cdot \nabla q \right) + u \cdot \nabla \left( e^{q \langle |x| \rangle^{\theta(p)}} \right) e^{q \langle |x| \rangle^{\theta(p)}} |\Gamma^a w|^p
\]
\[
+ \sum_{b+c=a, \ c < a} \sum_{b+c=a, \ c < a} \partial_t (e^{q \langle |x| \rangle^{\theta(p)} |\Gamma^a w|^{p-2}} C_{bc} \Gamma^a w J_{bc}.
\]
Integrating the above equality over \([0, t'] \times \mathbb{R}^2\) leads to

\[
\| \langle |x| \rangle^{\theta(p)} \Gamma^a w(t', x) \|_{L^p}^p + \int_0^{t'} \int \frac{|\langle |x| \rangle^{\theta(p)} \Gamma^a w|^p}{\langle |x| - t \rangle^{\frac{3}{2}}} \, dx \, dt \\
\lesssim \| \langle |x| \rangle^{\theta(p)} \Gamma^a w(0, x) \|_{L^p}^p + \int_0^{t'} \int |\langle |x| \rangle^{\theta(p)} \Gamma^a w|^p J(u) \, dx \, dt
\]

(5.45)

\[+ \sum_{b+c=a} \int_0^{t'} \int |\langle |x| \rangle^{\theta(p)} |\Gamma^a w|^{p-1}|J_{bc}| \, dx \, dt,
\]

see (5.38) for the definition of \(J(u)\).

Next, we deal with the two integrals on the right hand side of (5.45) in the two different parts \(|x| \geq \langle t \rangle / 8\) and \(|x| \leq \langle t \rangle / 8\), separately.

In the region \(|x| \geq \langle t \rangle / 8\), with the help of (5.24), applying (2.27) to \(\tilde{\Gamma}^b u\) yields that

\[
\int_{|x| \geq \langle t \rangle / 8} \left| \langle |x| \rangle^{\theta(p)} \Gamma^a w \right|^p J(u) \, dx + \sum_{b+c=a} \int_{|x| \geq \langle t \rangle / 8} \langle |x| \rangle^{\theta(p)} \left| \Gamma^a w \right|^{p-1} |J_{bc}| \, dx \\
\lesssim M \varepsilon \sum_{|a'| \leq |a|} \int_0^{t'} \int \frac{|\langle |x| \rangle^{\theta(p)} \Gamma^{a'} w|^p}{\langle |x| - t \rangle^{\frac{3}{2}}} \, dx \, dt + \int_0^{t'} \mathcal{W}_{p,a}^p(t) \left\{ M \delta + M \varepsilon \langle t \rangle^{M \varepsilon - \frac{5}{8}} \right\} dt.
\]

(5.46)

For \(|b| \leq N_1 - 4\), in the region \(|x| \leq \langle t \rangle / 8\), similarly to (5.39), we derive

\[
\int_{|x| \leq \langle t \rangle / 8} \left| \langle |x| \rangle^{\theta(p)} \Gamma^a w \right|^p J(u) \, dx + \sum_{b+c=a} \int_{|x| \leq \langle t \rangle / 8} \langle |x| \rangle^{\theta(p)} \left| \Gamma^a w \right|^{p-1} |J_{bc}| \, dx \\
\lesssim \mathcal{W}_{p,a}^p(t) \left\{ M \delta + M \varepsilon \langle t \rangle^{-\frac{5}{8}} \right\}.
\]

(5.47)

For all \(|a| \leq N_1 - 4\), plugging (5.46) and (5.47) into (5.45) implies (5.43).

Finally, we focus on the proof of (5.44). In fact, it only suffices to treat the case of \(N_1 - 3 \leq |b| \leq N_1\) in the region \(|x| \leq \langle t \rangle / 8\). At this time, \(|c| \leq N_1 - |b| \leq 3 \leq N_1 - 7\) holds.

For \(p = \frac{10}{9}\), \(10^{\frac{10}{9}}\), by the Hölder inequality, we have

\[
\| \langle |x| \rangle^3 \nabla c_w \|_{L^p} \lesssim \| \langle |x| \rangle^8 \nabla c_w \|_{L^2} \left\| \langle |x| \rangle^{-5} \| \right\|_{L^{\frac{2p}{5}}} \lesssim \mathcal{W}_{|c|+1}(t) \lesssim \mathcal{W}_{N_1-4}(t) \lesssim M \delta.
\]

(5.48)

On the other hand, we can deduce from \(N_1 + 2 \leq N\), the standard Sobolev embedding and Hardy inequality that

\[
\| \langle |x| \rangle^{-\frac{5}{8}} \tilde{\Gamma}^b u \|_{L^\infty} \lesssim \| \langle |x| \rangle^{-\frac{5}{8}} \tilde{\Gamma}^b u \|_{L^2} + \| \langle |x| \rangle^{-\frac{5}{8}} \nabla \nabla^{\leq 1} \tilde{\Gamma}^b u \|_{L^2} \\
\lesssim \left\| \frac{\tilde{\Gamma}^b u}{|x| \log |x|} \right\|_{L^2} + \langle t \rangle^{-1} E_{|b|+2}(t) + \mathcal{W}_{|b|+1}(t) \\
\lesssim \langle t \rangle^{-1} E_{|b|+2}(t) + \mathcal{W}_{|b|+1}(t) \\
\lesssim \langle t \rangle^{M \varepsilon} \left\{ M \delta + M \varepsilon \langle t \rangle^{-1} \right\}.
\]

(5.49)
where we have used (2.11) and (3.19).

Collecting (5.48) and (5.49) infers

$$
\int_{|x| \leq (t)/8} (|x|^p |\Gamma^n w|^p |J_{bc}| dx \lesssim \mathcal{W}_{p,a}^{p-1}(t) M^\delta (t)^{M^\varepsilon} \{ M^\delta + M^\varepsilon (t)^{-1} \}. \quad (5.50)
$$

Now, we deal with the case $p = 5$. It is concluded from (2.12), (5.23) and (5.49) that

$$
\| |x|^5 J_{bc} \|_{L^5(|x| \leq (t)/8)}^5 \lesssim \left( \mathcal{W}_{|c|+3}^{p=5}(t) + M^\delta \right)^5 \int \frac{|\chi_{*}^{\delta} u|^5}{|x|^6} dx
$$

$$
\lesssim \left( \mathcal{W}_{N_1-4}^{p=5}(t) + M^\delta \right)^5 \left( \| |x|^{-\frac{3}{2} \Gamma^n u} \|_{L^2}^2 \right) \left( \frac{\chi_{*}^{\delta} u}{|x| \log |x|} \right) \left( \frac{\chi_{*}^{\delta} u}{|x| \log |x|} \right)
$$

$$
\lesssim \langle t \rangle^{5M^\varepsilon}(M^\delta + M^\varepsilon (t)^{-1})^5. \quad (5.51)
$$

Consequently, substituting (5.46), (5.47), (5.50) and (5.51) into (5.45) yields (5.44).

\[ \square \]

\textbf{Lemma 5.6.} \textit{Under bootstrap assumptions (2.8), we have}

$$
\mathcal{W}_{N_1-4}(t) \lesssim \sum_{p=p_{10}}^{p_{10}} \mathcal{W}_{p,N_1-4}(t), \quad \mathcal{W}_{N_1}(t) \lesssim \sum_{p=p_{10}}^{p_{10}} \mathcal{W}_{p,N_1}(t). \quad (5.52)
$$

\textit{Proof.} The proof is the same as Lemma 5.3, we omit the details here. \[ \square \]

\section{6 \ Proof of Theorem 1.1}

Before starting the proof Theorem 1.1, we do some preparations.

\textbf{Lemma 6.1 (Standard Gronwall’s inequality).} \textit{Let $A(t)$ be an amplitude function which verifying that $A(0) = 0, \partial_t A(t) \geq 0$. If}

$$
h(t) \leq C \{ \mathcal{R} + \int_0^t h(s) \partial_t A(s) ds \},
$$

\textit{holds for some positive constants $C$ and $\mathcal{R}$, then}

$$
h(t) \leq C \mathcal{R} (1 + e^{CA(t)}).
$$

\textbf{Lemma 6.2 (Gronwall’s inequality).} \textit{For $t \delta \leq \kappa$ and two positive constants $c$ and $C \geq 1$, if}

$$
h(t) \leq C \left\{ h(0) + cM^\delta (1 + t)^{M^\varepsilon} + \int_0^t h(s)(M^\delta + M^\varepsilon (1 + s)^{-1}) ds \right\}
$$

\textit{holds, then for $M^\prime \geq 3CM$ and $\kappa \leq \frac{1}{2CM^\prime}$, we have}

$$
h(t) \leq C (1 + t)^{M^\varepsilon} (1 + e)(h(0) + cM^\delta).
$$

\textit{Proof.} Let $H(t) := \int_0^t h(s)(M^\delta + M^\varepsilon (1 + s)^{-1}) ds$ and $A(t) := M^\delta t + M^\varepsilon \ln(1 + t)$, then we find

$$
\partial_t A(t) = M^\delta + M^\varepsilon (1 + t)^{-1} \geq 0.
$$

Thus, we get

$$
\partial_t H(t) = h(t) \partial_t A(t) \leq C \partial_t A(t) \left\{ h(0) + cM^\delta (1 + t)^{M^\varepsilon} + H(t) \right\},
$$
which implies
\[ \partial_t(e^{-CA(t)} H(t)) \leq C \partial_t A(t) e^{-CA(t)} \left\{ h(0) + cM\delta(1 + t)^{M'\varepsilon} \right\}. \] (6.1)

Note that
\[
C \int_0^t \partial_t A(s) e^{-CA(s)}(1 + s)^{M'\varepsilon} ds
= C \int_0^t \left( M\delta + M\varepsilon(1 + s)^{-1} \right) e^{-CM\delta s}(1 + s)^{M'\varepsilon-CM\varepsilon} ds
\leq CM\delta t(1 + t)^{M'\varepsilon-CM\varepsilon} + CM\varepsilon \int_0^t (1 + s)^{M'\varepsilon-CM\varepsilon-1} ds
\leq (1 + t)^{M'\varepsilon-CM\varepsilon} \left\{ CM\kappa + \frac{CM}{M' - CM} \right\}
\leq (1 + t)^{M'\varepsilon-CM\varepsilon}.
\] (6.2)

By integrating (6.1) and plugging (6.2) into the resulted inequality, we arrive at
\[
H(t) \leq e^{CM\delta t}(1 + t)^{M\varepsilon} \left\{ h(0) + cM\delta(1 + t)^{M'\varepsilon-M\varepsilon} \right\}
\leq e(1 + t)^{M'\varepsilon}(h(0) + cM\delta).
\]

This completes the proof of Lemma 6.2. □

Next, we begin to prove the main theorem of this paper.

**Proof of Theorem 1.1.** Let \( \tilde{E}_m(t) := \sup_{0 \leq s \leq t} E_m(s) \). Analogously, we can also define \( \tilde{W}_m(t) \) and \( \mathcal{W}_{p,m}(t) \). But for convenience, we still denote \( \tilde{E}_m(t), \tilde{W}_m(t), \mathcal{W}_{p,m}(t) \) as \( E_m(t), W_m(t), \mathcal{W}_{p,m}(t) \). By these notations, (5.1), (5.2), (5.35), (5.36), (5.43) and (5.44) with assumption \( 2M'\varepsilon \leq \frac{1}{8} \) can be reduced to

\[
E_N(t') \leq C_1 \left\{ E_N(0) + M^2\varepsilon^\frac{3}{2}K^{\frac{1}{2}} + M^2\delta^\frac{7}{8}K^{\frac{3}{8}} \right\}
+ \int_0^{t'} E_N(t) \left\{ M\delta + M\varepsilon(1 + t)^{-\frac{3}{2}} \right\} dt,
\]
\[
E_{N_1-4}(t') \leq C_1 \left\{ E_{N_1-4}(0) + M^2\varepsilon^\frac{3}{2}K^{\frac{1}{2}} + M^2\delta^\frac{7}{8}K^{\frac{3}{8}} \right\}
+ \int_0^{t'} E_{N_1-4}(t) \left\{ M\delta + M\varepsilon(1 + t)^{-\frac{3}{2}} \right\} dt,
\]
\[
W_{N_1-4}(t') \leq C_1 \left\{ W_{N_1-4}(0) \right\}
+ \int_0^{t'} W_{N_1-4}(t) \left\{ M\delta + M\varepsilon(1 + t)^{-\frac{3}{2}} \right\} dt,
\]
\[
W_{N-1}(t') \leq C_1 \left\{ W_{N-1}(0) \right\}
+ \int_0^{t'} W_{N-1}(t) \left\{ M\delta + M\varepsilon(1 + t)^{-\frac{3}{2}} \right\} dt,
\]
\[
\mathcal{W}_{p,N_1-4}(t') \leq C_1 \left\{ \mathcal{W}_{p,N_1-4}(0) \right\}
+ \int_0^{t'} \mathcal{W}_{p,N_1-4}(t) \left\{ M\delta + M\varepsilon(1 + t)^{-\frac{3}{2}} \right\} dt,
\]
\[
\mathcal{W}_{p,N_1}(t') \leq C_1 \left\{ \mathcal{W}_{p,N_1}(0) \right\}
+ \int_0^{t'} \mathcal{W}_{p,N_1}(t) \left\{ M\delta + M\varepsilon(1 + t)^{-\frac{3}{2}} \right\} dt,
\]

where the positive constant \( C_1 > 1 \) is assumed to be suitably large.

Applying the Gronwall inequalities in Lemma 6.1 and 6.2 to the above equalities with amplitude function \( A(t) = M\delta t + M\varepsilon \ln(1 + t) \) and \( A(t) = M\delta t + 8M\varepsilon(1 + t)^{-\frac{1}{3}} \), respectively, we then
obtain

\[ E_N(t) \leq C_1(1 + e^{C_1 M\kappa}(1 + t)^{C_1 M\kappa})(E_N(0) + M^2\varepsilon^2\kappa^\frac{1}{M} + M^2\varepsilon K^\frac{1}{M}), \]

\[ E_{N_1-4}(t) \leq C_1(1 + e^{C_1 M(\kappa+8\varepsilon)})(E_{N_1-4}(0) + M^2\varepsilon^2\kappa^\frac{1}{M} + M^2\varepsilon K^\frac{1}{M}), \]

\[ \mathcal{W}_{p,N_1}(t) \leq C_1\mathcal{W}_{p,N_1}(0)(1 + e^{C_1 M(\kappa+8\varepsilon)}), \]

where \( \kappa \leq \frac{1}{2C_1 M} \) and \( M' \geq 3C_1 M \). If \( \varepsilon_0 \leq \frac{1}{C_1 M} \), then it concludes from the above inequalities with (1.14), (1.15), (3.19), (5.29) and (5.52) that there exists positive constants \( C_2 > 1 \) and \( C_3 > 0 \) such that

\[ E_N(t) + \mathcal{X}_N(t) \leq C_1 C_2(1 + t)^{M'\varepsilon}(C_3\varepsilon + M\varepsilon(M\delta)^\frac{1}{M} + M\varepsilon K^\frac{1}{M}), \]

\[ E_{N_1-4}(t) + \mathcal{X}_{N_1-4}(t) \leq C_1 C_2(1 + t)^{M'\varepsilon}(C_3\varepsilon + M\varepsilon(M\delta)^\frac{1}{M} + M\varepsilon K^\frac{1}{M}), \]

\[ \mathcal{W}_{p,N_1}(t) \leq C_1\mathcal{W}_{p,N_1}(0)(1 + e^{C_1 M(\kappa+8\varepsilon)}), \]

Choosing \( M = 4C_1 C_2 M', M' = 4C_1 C_2 M, \kappa = \min\left\{ \frac{1}{(4C_1 C_2)^2}, \frac{1}{2C_1 M} \right\}, \varepsilon_0 = \frac{1}{10M}, \delta_0 = \min\left\{ \frac{1}{(4C_1 C_2)^2}, \frac{1}{2C_1 M}, \varepsilon_0 \right\} \), we eventually achieve that for \( t\delta \leq \kappa \),

\[ E_N(t) + \mathcal{X}_N(t) \leq \langle t \rangle^{M'\varepsilon}(1 + \frac{1}{4} M\varepsilon + \frac{1}{4} M\varepsilon + \frac{1}{4} M\varepsilon) \leq \frac{3}{4} M\varepsilon \langle t \rangle^{M'\varepsilon}, \]

\[ E_{N_1-4}(t) + \mathcal{X}_{N_1-4}(t) \leq \frac{3}{4} M\varepsilon + \frac{1}{4} M\varepsilon + \frac{1}{4} M\varepsilon \leq \frac{3}{4} M\varepsilon, \]

\[ \mathcal{W}_{p,N_1}(t) \leq \frac{3}{4} M\delta \langle t \rangle^{M'\varepsilon}, \]

This, together with the local existence of classical solution to (1.19), yields that (1.19) admits a unique solution \( (\sigma, u) \in C([0, \frac{T}{2}], H^N(\mathbb{R}^2)) \). Thus, it completes the proof Theorem 1.1. \( \Box \)

A Derivation of the wave equation for the potential function \( \phi \)

Lemma A.1. We have

\[ \text{div}(u \cdot \nabla u) = \frac{1}{2} \Delta |u|^2 - \text{curl}(u \cdot \text{curl} u). \]  

(A.1)

Proof. At first, it is obvious to see that

\[ \text{div}(u \cdot \nabla u) = u \cdot \nabla \text{div} u + \partial_i u_j \partial_j u_i. \]

\[ ^1 \text{More precisely, } \varepsilon_0 \text{ should be the minor one of } \frac{1}{10M} \text{ and the smallness of } M\varepsilon_0 \text{ which has been used in the previous parts of this paper.} \]
Note that \( \text{curl} \, u = \epsilon_{ij} \partial_i u_j \), where the volume form \( \epsilon_{ij} \) is the sign of the arrangement \( \{ij\} \). Then we find that

\[
\begin{align*}
    u \cdot \nabla \text{div} \, u &= u_j \partial_j \partial_i u_i = u_j \partial_i (\partial_j u_i - \partial_i u_j) + u_j \Delta u_j \\
    &= u_j \partial_i (\epsilon_{ji} \text{curl} \, u) + u_j \Delta u_j \\
    &= \partial_i (u_j \epsilon_{ji} \text{curl} \, u) - \epsilon_{ji} \partial_i u_j \text{curl} \, u + u_j \Delta u_j \\
    &= -\text{curl}(u \text{curl} \, u) + (\text{curl} \, u)^2 + u_j \Delta u_j.
\end{align*}
\]

On the other hand, one has

\[
\frac{1}{2} \Delta |u|^2 = u_j \Delta u_j + \partial_i u_j \partial_i u_j.
\]

Therefore, we achieve

\[
\text{div}(u \cdot \nabla u) = -\text{curl}(u \text{curl} \, u) + (\text{curl} \, u)^2 + \frac{1}{2} \Delta |u|^2 + \partial_i u_j (\partial_j u_i - \partial_i u_j)
\]

Next we derive the wave equation (4.20) of the potential function \( \phi \). By taking divergence of the velocity equation in (1.19), we arrive at

\[
0 = \partial_t \text{div} \, u + \text{div}(u \cdot \nabla u) + \Delta (\sigma - \frac{1}{2} \sigma^2)
\]

where we have used (A.1). Hence,

\[
\partial_t \phi + \frac{1}{2} |u|^2 + \sigma - \frac{1}{2} \sigma^2 = A = (-\Delta)^{-1} \text{curl}(u \text{curl} \, u),
\]

and then (4.22) is obtained. Acting \( \partial_t \) on the both sides of (A.2) yields

\[
\partial_t^2 \phi + \frac{1}{2} \partial_t (|u|^2 - \sigma^2) + \partial_t \sigma = \partial_t A.
\]

On the other hand, from the first equation in (1.19), we get

\[
\partial_t \sigma = -\Delta \phi + Q_1.
\]

Substituting this into (A.3) derives

\[
\Box \phi + \frac{1}{2} \partial_t (|u|^2 - \sigma^2) + Q_1 = \partial_t A.
\]

By using (1.19) to \( \partial_t (|u|^2 - \sigma^2) \), one has

\[
\frac{1}{2} \partial_t (|u|^2 - \sigma^2) = u \cdot \partial_t u - \sigma \partial_t \sigma = u \cdot Q_2 - u \cdot \nabla \sigma - \sigma Q_1 + \sigma \text{div} \, u = u \cdot Q_2 + (1 - \sigma)Q_1.
\]

Substituting (A.5) into (A.4) yields (4.20).


References

[1] S. Alinhac, *Une solution approchée en grand temps des équations d’Euler compressibles axisymétriques en dimension deux*, Comm. Partial Differential Equations 17 (1992), no. 3-4, 447–490.

[2] S. Alinhac, *Temps de vie des solutions régulières des équations d’Euler compressibles axisymétriques en dimension deux*, Invent. Math. 111 (1993), 627–670.

[3] S. Alinhac, *Blowup for nonlinear hyperbolic equations*. Progress in Nonlinear Differential Equations and their Applications, 17. Birkhäuser Boston, Inc., Boston, MA, 1995.

[4] S. Alinhac, *Blowup of small data solutions for a class of quasilinear wave equations in two space dimensions. II*, Acta Math. 182 (1999), no. 1, 1–23.

[5] S. Alinhac, *The null condition for quasilinear wave equations in two space dimensions I*, Invent. Math. 145 (2001), no. 3, 597–618.

[6] S. Alinhac, *Geometric analysis of hyperbolic differential equations: an introduction*. London Mathematical Society Lecture Note Series, 374. Cambridge University Press, Cambridge, 2010. x+118 pp.

[7] D. Christodoulou, *The formation of shocks in 3-dimensional fluids*, EMS Monogr. Math., Eur. Math. Soc., Zürich, 2007.

[8] D. Christodoulou, Miao Shuang, *Compressible flow and Euler’s equations*, Surveys of Modern Mathematics, 9, International Press, Somerville, MA; Higher Education Press, Beijing, 2014.

[9] R. Courant, K. O. Friedrichs, *Supersonic flow and shock waves*. Interscience Publishers Inc., New York, 1948.

[10] Ding Bingbing, Ingo Witt, Yin Huicheng, *The global smooth symmetric solution to 2-D full compressible Euler system of Chaplygin gases*, J. Differential Equations 258 (2015), no. 2, 445–482.

[11] P. Godin, *The lifespan of a class of smooth spherically symmetric solutions of the compressible Euler equations with variable entropy in three space dimensions*, Arch. Ration. Mech. Anal. 177 (2005), no. 3, 479–511.

[12] P. Godin, *Global existence of a class of smooth 3D spherically symmetric flows of Chaplygin gases with variable entropy*, J. Math. Pures Appl. 87 (2007), 91–117.

[13] Guo Yan, A.D. Ionescu, B. Pausader, *Global solutions of the Euler-Maxwell two-fluid system in 3D*, Ann. of Math. (2) 183 (2016), 377–498.

[14] G. Holzegel, S. Klainerman, J. Speck, W.W.-Y. Wong, *Small-data shock formation in solutions to 3d quasilinear wave equations: An overview*, Journal of Hyperbolic Differential Equations 13 (2016), no. 01, 1–105.

[15] L. Hörmander, *Lectures on nonlinear hyperbolic differential equations*. Mathématiques & Applications (Berlin) [Mathematics & Applications], 26. Springer-Verlag, Berlin, 1997.

[16] A. Hoshiga, H. Kubo, *Global solvability for systems of nonlinear wave equations with multiple speeds in two space dimensions*, Differential Integral Equations 17 (2004), no. 5-6, 593–622.
[17] Hou Fei, Yin Huicheng, Global smooth axisymmetric solutions to 2D compressible Euler equations of Chaplygin gases with non-zero vorticity, J. Differential Equations 267 (2019), no. 5, 3114–3161.

[18] Hou Fei, Yin Huicheng, On global axisymmetric solutions to 2D compressible full Euler equations of Chaplygin gases, Discrete Contin. Dyn. Syst. 40 (2020) no. 3, 1435–1492.

[19] Hou Fei, Yin Huicheng, Global small data smooth solutions of 2-D null-form wave equations with non-compactly supported initial data, J. Differential Equations 268 (2020), no. 2, 490–512.

[20] A.D. Ionescu, V. Lie, Long term regularity of the one-fluid Euler-Maxwell system in 3D with vorticity, Adv. Math. 325 (2018), 719–769.

[21] F. John, Nonlinear wave equations, formation of singularities. Seventh Annual Pitcher Lectures delivered at Lehigh University, Bethlehem, Pennsylvania, April 1989. University Lecture Series, 2, American Mathematical Society, Providence, RI, 1990.

[22] Lei Zhen, Global well-posedness of incompressible elastodynamics in two dimensions, Comm. Pure Appl. Math. 69 (2016), no. 11, 2072–2106.

[23] J. Luk, J. Speck, Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity, Invent. Math. 214 (2018), no. 1, 1–169.

[24] A. Majda, Compressible fluid flow and systems of conservation laws in several space variables, Applied Mathematical Sciences, 53, Springer-Verlag, New York, 1984.

[25] M.A. Rammaha, Formation of singularities in compressible fluids in two-space dimensions, Proc. Am. Math. Soc. 107 (1989), 705–714.

[26] T.C. Sideris, Formation of singularities in three-dimensional compressible fluids, Comm. Math. Phys. 101 (1985), 475–485.

[27] T.C. Sideris, Delayed singularity formation in 2D compressible flow, Amer. J. Math. 119 (1997), 371–422.

[28] E.M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, 1993.

[29] Yin Huicheng, Formation and construction of a shock wave for 3-D compressible Euler equations with the spherical initial data, Nagoya Math. J., 175 (2004), 125–164.