Quantum clock synchronization with one qubit

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Abstract

The clock synchronization problem is to determine the time difference $T$ between two spatially separated parties. We improve on I. Chuang’s quantum clock synchronization algorithm and show that it is possible to obtain $T$ to $n$ bits of accuracy while communicating only one qubit in one direction and using an $O(2^n)$ frequency range. We also prove a quantum lower bound of $\Omega(2^n)$ for the product of the transmitted qubits and the range of frequencies, thus showing that our algorithm is optimal.

1 Introduction

Clock synchronization is a well studied problem with many practical and scientific applications. In the special theory of relativity there are two standard methods for synchronizing a pair of spatially separated clocks, Einstein Synchronization [Ein89] and Eddington’s Slow Clock Transport [Edd24].

Recently, two new quantum protocols have been proposed for synchronizing remote clocks. The first one uses prior quantum bit entanglement between the two parties and was proposed by Jozsa et al [JADW00]. This protocol is based on the assumption that the entanglement can be achieved without any relative phase error. However, the validity of this assumption has been discussed and questioned in a number of papers [BES00, YD00, GN00]. Once and if this entanglement can be obtained, their algorithm determines the time difference $T$ between the clocks by essentially monitoring the oscillation of a function $f(T) \sim \cos(\omega T)$, and thus requires $O(2^{2n})$ shared singlets.

The second protocol was proposed by I. Chuang [Chu00], and obtains $T$ to $n$ bits of accuracy by communicating only $O(n)$ qubits and using an $O(2^n)$ range of frequencies. After communicating the bits according to his protocol, they are in the state corresponding to the Fourier Transform (over $\mathbb{Z}_{2^n}$) of the state $|\omega T\rangle$, for some fixed and known $\omega$. As a result, one can apply an inverse Fourier Transform and subsequently measure the value of $\omega T$ and hence $T$.

In this paper, we improve significantly on Chuang’s result by presenting an algorithm that is able to calculate $T$ to $n$ bits of accuracy while communicating only one qubit in one direction and using an $O(2^n)$ range of frequencies. Further, we prove that, under our computational model, the product of the frequency range and the number of transmitted qubits must be $\Omega(2^n)$,
and conclude that our algorithm is optimal in this model.

2 The computational model

In our protocol Alice sends a photon $|\psi\rangle$ to Bob with some tick rate $\omega$. The state of the received photon is $e^{i\omega tZ}|\psi\rangle$, where $t$ is the time the photon spent in transit and $Z$ is the Pauli matrix

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$ 

Even though we only use one-way communication from Alice to Bob, for the purposes of proving computational lower bounds we assume an even stronger model, where the two parties can exchange photons back and forth. The information they get about the time difference $T$ between the two clocks comes from a phase change in the state of the qubit, which depends on $T$ and the tick rate $\omega$. Let’s now define this procedure and see how we can actually implement it.

The input to this procedure will be a quantum register which holds the tick rate $k$ and a qubit $|\psi\rangle$. The output is a state that has a phase which depends on $T$ and $k$.

**Definition 1** Let $\text{TQH}$ be a black box quantum procedure defined by the equation

$$\text{TQH}(|k\rangle|\psi\rangle) = |k\rangle e^{2\pi ik\omega_0 T Z} |\psi\rangle$$

where $T$ is the time difference between the two parties and $\omega_0$ is a known base tick rate.

This is a very reasonable and powerful model, since we know that all the information one can get about the time difference via such photon communications is in the form of a relative phase change. Here the first register handles the tick rate of the photon to be transmitted and the second register is the photon that Alice communicates to Bob (or Bob to Alice).

The implementation of this black box is based on the ticking qubit handshake protocol (TQH) described in I. Chuang’s paper [Chu99]. Suppose Alice wants to create the state $e^{2\pi ik\omega_0 T Z} |\psi\rangle$. She first sends the qubit $|\psi\rangle$ to Bob with ticking rate $(-2\pi k\omega_0)$. Along a classical channel she also tells him her time $t_A$ at the moment of the quantum communication. Bob receives at time $t_B$ (according to Bob’s clock) a quantum state $e^{-2\pi ik\omega_0 t_B Z} |\psi\rangle$, where $t_B$ is the time the qubit spent in transit. Finally, Bob applies a phase change $e^{2\pi ik\omega_0 (t_B - t_A) Z}$ and thus the final state of the qubit is $e^{2\pi ik\omega_0 (t_B - t_A - t_{tr}) Z} |\psi\rangle = e^{2\pi ik\omega_0 T Z} |\psi\rangle$.

3 An optimal Quantum Algorithm

We are going to describe a protocol for synchronizing two remote clocks by communicating one photon. In this algorithm, Alice starts by preparing a register $R$ of $n$ qubits in a certain superposition. Then she sends a photon to Bob with the superposition of tick rates specified in $R$. Bob measures the received photon and Alice obtains $T$ to $n$ bits of accuracy by processing a phase estimation on $R$.

In more detail,

1. Alice starts with a register of $n$ qubits initialized to $|0\rangle$, and after applying a Fourier Transform to them she obtains

$$\frac{1}{\sqrt{2^n}} \sum_{k \in \mathbb{Z}_2^n} |k\rangle.$$ 

She also prepares a photon with polarization state $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
2. Alice now transmits the prepared photon with the tick rate described by her first register. If the photon had a definite tick rate $k$ and polarization state $|\psi\rangle$ the final state would be $e^{2\pi i k \omega_0 T} |\psi\rangle$. Since the register described in step 1 is in a superposition of tick rates, the outcome will be in a superposition of states $\frac{1}{\sqrt{2^n+1}} \sum_{k \in \mathbb{Z}_{2^n}} |k\rangle e^{2\pi i (k \cdot \omega_0) T} |\psi\rangle$.

3. Bob measures the received photon. Assuming without loss of generality that the outcome is $|0\rangle$, Alice’s register $R$ becomes $\frac{1}{\sqrt{2^n}} \sum_{k \in \mathbb{Z}_{2^n}} e^{2\pi i k \omega_0 T} |k\rangle$.

4. Alice then applies an inverse Fourier Transform, obtaining the state $|\omega_0 T\rangle$ in $R$.

It is easy to see that this algorithm is an application of the general procedure known as \textit{phase estimation}. In this procedure, we assume a unitary operator $U$ with an eigenvector $|u\rangle$ and eigenvalue $e^{2\pi i \phi}$. The goal is to estimate $\phi$ to $n$ bits of accuracy. To perform the estimation we start with two registers, the first one in a uniform superposition over all states in $\mathbb{Z}_{2^n}$ and the second one in the state $|u\rangle$. Then we apply the unitary operation $U$ to the second register $j$ times, where $|j\rangle$ is the content of the first register. By analyzing the performance of this procedure it can be seen that our algorithm obtains $T$ to $n$ bits of accuracy with constant probability $4/\pi^2$. We can boost the probability of success to $1 - \delta$ by increasing the size of the first register to $n + \log(2 + \frac{1}{\delta})$. Further analysis can be found in [NC01], page 221.

4 A lower bound on frequency range $\times$ number of qubits

In this section we will prove a lower bound on the product of the range of tick rates (frequencies) we use and the number of qubits we communicate.

\textbf{Theorem 1} Any quantum algorithm which determines $T$ to $n$ bits of accuracy, using a range $F$ of frequencies and communicating $Q$ qubits between the two parties, must have that $F \cdot Q = \Omega(2^n)$.

In order to prove this theorem we are going to use the following lemma:

\textbf{Lemma 1} If a quantum algorithm in the TQH model makes only queries to the black box with a single tick rate $\omega$, then it must make a total number of $\Omega(2^n)$ queries in order to obtain $T$ to $n$ digits of accuracy.

\textbf{Proof:} By making a query to the black box, the input $|\psi\rangle \equiv \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ will become $e^{2\pi i \omega T} |\psi\rangle$; after applying a Hadamard transform we obtain the quantum state $\cos(2\pi \omega T)|0\rangle + \sin(2\pi \omega T)|1\rangle$.

From this we see that the problem of determining $T$ is equivalent to estimating the amplitude of $|0\rangle$ (or $|1\rangle$). The problem of estimating the amplitude of a quantum state, which is equivalent to the problem of counting the number of solutions to a quantum problem, is well-studied [BHMT00, NW99]. In [NW99] they prove that $\Omega(\sqrt{N/\Delta} + \sqrt{t(N-t)/\Delta})$ queries are required for a $\Delta$-approximate count, where $t$ is the number of solutions, $N$ is the set of possible inputs and $\Delta$ defines the closeness of the approximation.
If we use this lower bound for the case of amplitude estimation, we get a lower bound of \( \Omega(\sqrt{N/\Delta} + N\sqrt{a(1-a)/\Delta}) \), since the amplitude is \( a = t/N \). In our case, \( a \) can take any value in \((0,1)\), \( \Delta \) must be less than 1 and \( N = 2^n \), so we obtain the lower bound of \( \Omega(2^n) \) qubits.

Now we are ready to prove Theorem 1.

**Proof of theorem:** Suppose we are able to query the black box with frequencies in the range \([\omega,F\omega]\). We claim that this black box can be simulated by a black box with only one tick rate \( \omega \) at the cost of replacing each query with at most \( F \) queries. This can be done since one query to the \([\omega,F\omega]\) black box with tick rate \((k\omega), 1 \leq k \leq F\) is equivalent to \( k \) consecutive queries with tick rate \( \omega \), using the output of one query as the input to the next. Notice that a superposition of queries to the \([\omega,F\omega]\) black box does not pose any challenge to the simulation, since we can also query the one-tick rate black box in a superposition of times. For such an input, the number of queries is defined to be the maximum over all states of the superpositions.

Since in all cases we query the black box at most \( F \) times, this means that the one-tick rate version will run with at most \( F \cdot Q \) queries. Now, in Lemma 1 we have already proved that when we use only one tick rate, we need to communicate at least \( \Omega(2^n) \) qubits, and therefore \( F \cdot Q = \Omega(2^n) \).

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