Active Phase for Activated Random Walk on $\mathbb{Z}^2$

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Abstract

We show that for small enough sleep rate, the critical density of the symmetric Activated Random Walk model on $\mathbb{Z}^2$ is strictly less than one.

1 Introduction

The Activated Random Walk (ARW) model is a system of interacting particles, which is believed to exhibit self-organized criticality. In this paper, we study the dynamics of the simple ARW on the square lattice $\mathbb{Z}^2$. The process starts with a random number of particles at each site $x \in \mathbb{Z}^2$ distributed as a spatially ergodic distribution with average density $\zeta > 0$. Each particle can be in one of two states, either active or sleepy. Initially all particles are active. Every active particle performs a continuous-time simple random walk at rate 1 until it falls asleep, which happens with sleep rate $\lambda$. A sleepy particle stays put until it becomes active, i.e. it gets reactivated whenever an active particle arrives at its site.

For every sleep rate $\lambda$, the system has an absorbing-state phase transition: if the initial density $\zeta$ of particles is below a critical value $\zeta_c(\lambda)$, all particles eventually sleep and the system fixates; above $\zeta_c(\lambda)$, each particle moves indefinitely and the system stays active. It is intuitively clear and was proved in [4, 9] that $\zeta_c(\lambda) \leq 1$ for all $\lambda$ in all dimensions.

Our main result answers an open question about the existence of a non-trivial active phase on $\mathbb{Z}^2$.

**Theorem 1.1.** There is a constant $C > 0$ such that for all $\lambda < 1$,

$$
\zeta_c(\lambda) \leq \frac{C}{\ln(1/\lambda)}.
$$

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In other words, the system stays active a.s. if \( \zeta > \frac{C}{\ln(1/\lambda)} \).

**Corollary 1.2.** For small enough \( \lambda \), the critical density \( \zeta_c(\lambda) < 1 \).

**Corollary 1.3.** The critical density \( \zeta_c(\lambda) \to 0 \) as \( \lambda \to 0 \).

Though we choose to demonstrate our proof in \( \mathbb{Z}^2 \), the same argument works in all dimensions.

In one dimension, similar results as Corollary 1.2 and 1.3 were first proved in the breakthrough work of [2]. The approach from [2] was later reformulated in [6] as based on two major ingredients: a mass balance equation between blocks and a single-block estimate. More recently, this reformulation has been adopted successfully in [1, 5] to find the right order of asymptotics in Corollary 1.3 and extend Corollary 1.2 for all \( \lambda > 0 \) respectively. All these works rely crucially on the topological convenience of \( \mathbb{Z} \) in order to maintain independence structures between blocks. However, we no longer enjoy such property in \( \mathbb{Z}^2 \), making the block arguments rather unwieldy. Similar results were also obtained in higher dimensions \( d \geq 3 \) [10, 12] or for biased ARW [11, 8], but these arguments only work for transient walks. See [6] for a more complete survey of results.

In this paper, we take a step back and consider the mass balance equations between sites instead of blocks. A straightforward energy-entropy calculation is then employed. Our main goal is to show that a similar line of arguments works in all dimensions and yields non-trivial results. The argument also provides simpler proofs and a handy framework for further understanding.

We note that another proof of the non-trivial active phase in two dimensions was obtained independently by Forien and Gaudillièr [3]. The asymptotic upper bound in Theorem 1.1 is comparable to the one from [3, Theorem 2].

In order to establish Theorem 1.1, it suffices to prove the following quantitative estimate for the **finite-volume dynamics**. For any positive integer \( N \), let \( B_N := \{-N, \ldots, N\}^2 \) be the finite box in \( \mathbb{Z}^2 \) and let \( |B_N| \) be the cardinality of \( B_N \). Consider the ARW dynamics on \( B_N \) where the walks are killed upon leaving \( B_N \). Without loss of generality, we only consider the initial configurations with at most \( |B_N| \) particles in \( B_N \). The finite-volume dynamics eventually stabilizes, that is when all particles that remain in \( B_N \) are sleeping. We call \( S(B_N) \) the number of sleeping particles in \( B_N \) after stabilization.
Theorem 1.4. There exist positive constants $C$ and $c$, such that for any integer $N$ and any initial configuration in $B_N$,

$$
P\left(S(B_N) \geq \frac{C}{\ln(1/\lambda)} |B_N| \right) \leq e^{-cN^2}.
$$

Theorem 1.4, combined with the criterion [6, Theorem 2.11] and the universality result [7], gives a proof of Theorem 1.1. The rest of the paper is devoted to the proof of 1.4.

2 Modified site-wise representation

Formally, the configuration of the system at time $t \geq 0$ is given by $\eta_t \in \{0, s, 1, 2, \ldots \}^{Z^2}$, where $s$ represents a single sleeping particle and we have $0 < s < 1$. Write $x \sim y$ if $x$ is a neighbor of $y$. In the continuous-time ARW, for $x \in Z^2$ such that $\eta_t - (x) \geq 1$, the system undergoes transitions $\eta_t = t_{x,y} \eta_t$ and $\eta_t = t_{x,s} \eta_t$ at rates $\frac{1}{2} \eta_t - (x)$ and $\lambda \eta_t - (x)$ respectively. For $x \sim y$, the movement transition $t_{x,y}$ is defined by

$$
t_{x,y}(\eta)(z) = \begin{cases} 
\eta(y) + 1, & z = y, \\
\eta(x) - 1, & z = x, \\
\eta(z), & \text{otherwise},
\end{cases}
$$

where we use the convention $s + 1 = 2$. The sleeping instruction is given by

$$
t_{x,s}(\eta)(z) = \begin{cases} 
s, & z = x \text{ and } \eta(x) = 1, \\
\eta(z), & \text{otherwise.}
\end{cases}
$$

It is useful to consider the site-wise representation of ARW. However, we will use a slight modification of the usual site-wise representation, cf.[6]. In our setting, each site $x \in Z^2$ is associated with one stack of i.i.d. movement instructions $(\xi^x_k)_{k \in N^+}$, together with another independent stack of i.i.d. geometric random variables $(g^x_k)_{k \in N}$. More specifically, each movement $\xi^x_k$ has the distribution of $t_{x,y_k}$, where $y_k$ is a uniformly random neighbor of $x$, while each geometric $g^x_k$ encodes the number of sleep instructions between consecutive movements $\xi^x_k$ and $\xi^x_{k+1}$, thus taking value in $\{0, 1, 2, \ldots \}$ with success probability $1/(1 + \lambda)$.

It has been proved [6 Section 11.2] that the stacks of instructions $\xi$ and $g$, with poisson clocks attached to every site, gives an explicit construction of the ARW process $(\eta_t)_{t \geq 0}$ on $Z^2$. We will not need the Abelian property
of the ARW in the remaining sections, but note that essential inputs of Theorem 1.1 including [6, Theorem 2.11], [7] and [6, Section 11.2], all use the Abelian property in their proofs.

For \( x \sim y \), let \( n_{x,y}(m) \) be the number of times that \( t_{x,y} \) appears in the first \( m \) movement instructions \( (\xi_k)_{k=1}^m \). Also let \( \chi_x(m) := 1\{g_m^x > 0\} \) be the Bernoulli random variable with parameter \( \lambda/(1 + \lambda) \).

### 3 Mass balance equation

In this section we focus on the finite-volume dynamics and use the mass balance equation to prove Theorem 1.4.

We start by introducing three random fields \( M = (M_x) \in \mathbb{N}^{\mathcal{B}_N} \), \( S = (S_x) \in \{0, 1\}^{\mathcal{B}_N} \) and \( \Phi = (\Phi_x) \in \mathbb{N}^{\partial \mathcal{B}_N} \). First, define the activity odometer field \( M_x \) which counts the number of movement instructions used at \( x \) until stabilization. Note that in \( M_x \), we do not count any sleep instruction. Secondly, let \( S_x \) be the indicator random variable of there being a sleeping particle at \( x \) after stabilization. Thirdly, define the exit measure \( \Phi_x \) to be the number of particles killed upon entering a site \( x \in \partial \mathcal{B}_N \). Here, for any \( A \subseteq \mathbb{Z}^2 \), we define the boundary set \( \partial A := \{ x \in A^c; \text{there exists } y \in A \text{ such that } y \sim x \} \).

Now suppose we’re given deterministic fields \( m = (m_x) \in \mathbb{N}^{\mathcal{B}_N} \), \( s = (s_x) \in \{0, 1\}^{\mathcal{B}_N} \) and \( \phi = (\phi_x) \in \mathbb{N}^{\partial \mathcal{B}_N} \). A tuple of fields \((m, s, \phi)\) is said to satisfy the mass balance equation if

(i) the mass balance equation holds at every \( x \in \mathcal{B}_N \):

\[
\eta_0(x) + \sum_{y \sim x} n_{y,x}(m_y) = m_x + s_x,
\]

where \( \eta_0 \) denotes the initial configuration;

(ii) the boundary condition is met at every \( x \in \partial \mathcal{B}_N \):

\[
n_{R(x),x}(m_{R(x)}) = \phi_x,
\]

where \( R(x) \) is the unique neighbor of \( x \) contained in \( \mathcal{B}_N \).

Note that the tuple of random fields \((M, S, \Phi)\) defined above always satisfies the mass balance equation.

To prove Theorem 1.4 we rely on the following entropy bound on all possible field configurations satisfying the mass balance equation. Denote by \( \mathcal{M}(x, m, s, \phi) \) the event where equation (1) holds at \( x \in \mathcal{B}_N \); for \( x \in \partial \mathcal{B}_N \),
we use the same notation for the event where equation (2) holds at \( x \). Also note that \( \|\Phi\|_\infty \leq |\eta_0| \), where \( |\eta_0| := \sum_{x \in B_N} \eta_0(x) \).

**Lemma 3.1.** There exists \( c_1 > 0 \) such that for any \( N \in \mathbb{N}^+ \) and initial configuration \( \eta_0 \) in \( B_N \) with \( |\eta_0| \leq |B_N| \),

\[
\sum_{m,s,\phi} \mathbb{P} \left( \bigcap_{x \in B_N \cup \partial B_N} \mathcal{M}(x, m, s, \phi) \right) \leq e^{c_1 |B_N|},
\]

(3)

where the summation is over \( \|\phi\|_\infty \leq |\eta_0| \) and arbitrary \( m, s \).

The proof of Lemma 3.1 will be given in the next section.

**Proof of Theorem 1.4.** Recall from Theorem 1.4 that \( S(B_N) \) represents the number of sleeping particles in \( B_N \) after stabilization, so we have

\[
S(B_N) = \sum_{x \in B_N} S_x.
\]

Note also that \( S_x \leq \chi_x(M_x) \), where \( \chi_x(m) \) is defined in Section 2. Thus in order to prove Theorem 1.4 it suffices to upper bound \( \mathbb{P}(\mathcal{A}(M)) \), where \( M \) is the odometer field, and for a fixed \( m \), we use \( \mathcal{A}(m) \) to denote the event that

\[
\sum_{x \in B_N} \chi_x(M_x) \geq \frac{C}{\ln(1/\lambda)} |B_N|.
\]

By decomposing over the fields we get \( \mathbb{P}(\mathcal{A}(M)) \) is equal to

\[
\sum_{m,s,\phi} \mathbb{P} \left( \mathcal{A}(m) \right) \mathbb{P} \left( \bigcap_{x \in B_N \cup \partial B_N} \mathcal{M}(x, m, s, \phi) \right)
\]

\[
\leq \sum_{m,s,\phi} \mathbb{P} \left( \mathcal{A}(m) \right) \sum_{x \in B_N \cup \partial B_N} \mathcal{M}(x, m, s, \phi)
\]

\[
\leq e^{-D_{KL} \left( \frac{C |B_N|}{\ln(1/\lambda)} \right) \left( \frac{\chi_x(M)}{1+\lambda} \right) |B_N|},
\]

(4)

where in the last inequality we use the independence between \( \chi_x(m) \)'s and \( n_{x,y}(m) \)'s. For any fixed \( m \), the probability of \( \mathcal{A}(m) \) is controlled via Chernoff bound by

\[
e^{-D_{KL} \left( \frac{C |B_N|}{\ln(1/\lambda)} \right) \left( \frac{\chi_x(M)}{1+\lambda} \right) |B_N|}.
\]

(5)
where

\[ D_{KL}(p_1 \| p_2) = p_1 \ln \frac{p_1}{p_2} + (1 - p_1) \ln \left( \frac{1 - p_1}{1 - p_2} \right) \]

is the Kullback-Leibler divergence between Bernoulli distributed random variables with parameters \( p_1 \) and \( p_2 \) respectively. By combining (4), (5) and Lemma 3.1 and picking a large enough \( C \), we complete the proof of Theorem 1.4.

4 Entropy bound

In this section we prove the entropy bound in Lemma 3.1. Partition \( B_N \) into a set of cycles \( B_N = \bigcup_{n=0}^{N} C_n \), where each \( C_n := \{ x \in \mathbb{Z}^2; \| x \|_\infty = N - n \} \).

Set \( C_{-1} := \partial B_N \). Recall from Section 2 that there is a stack of movement instruction \( (\xi_k)_{k \in \mathbb{N}^+} \) at every \( x \in B_N \). Consider the filtration \( (F_n)_{n=-1}^N \), where \( F_{-1} \) is the trivial \( \sigma \)-field and for \( n \geq 0 \), each \( F_n \) is the \( \sigma \)-field generated by the instructions \( \{ \xi_k^x; k \in \mathbb{N}^+, x \in C_i, 0 \leq i \leq n \} \) in the outermost \( n + 1 \) cycles.

**Lemma 4.1.** Fix \( N, \eta_0, s \) and \( \phi \). For every \( n = 0, \ldots, N \), a.s.,

\[ \mathbb{E} \left[ \sum_{m_n} \prod_{x \in C_{n-1}} 1_{M(x, m, s, \phi)} \left| F_{n-1} \right. \right] \leq 4^{\left| C_n \right|}, \quad (6) \]

where \( m_n := (m_x)_{x \in C_n} \) is the restriction of \( m \) to \( C_n \), and the summation is over all \( m_n \in \mathbb{N}^{C_n} \).

**Proof.** Define the backward neighbor function \( B \) that maps each \( y \in C_n \) to a neighbor of \( y \) in \( C_{n-1} \); whenever there are multiple such neighbors, just pick one arbitrarily. Note that \( B \) is a bijection from \( C_n \) to \( B(C_n) \). Rather than working with (6), it suffices to give an upper bound of a slightly larger expression

\[ \mathbb{E} \left[ \sum_{m_n} \prod_{y \in C_n} 1_{M(B(y), m, s, \phi)} \left| F_{n-1} \right. \right]. \quad (7) \]

By either the mass balance equation (1) when \( n \geq 1 \) or the boundary condition (2) when \( n = 0 \), the event \( M(B(y), m, s, \phi) \) above depends on \( m_n \) only through \( m_y \). Thus we may rewrite (7) as

\[ \mathbb{E} \left[ \prod_{y \in C_n} \sum_{m_y} 1_{M(B(y), m, s, \phi)} \left| F_{n-1} \right. \right]. \quad (8) \]
Now note that all randomness defining $M(B(y), m, s, \phi)$, except for $n_y, B(y)(m_y)$, are measurable with respect to $F_{n-1}$. Therefore, the sums over $m_y$ in (8), conditioned on $F_{n-1}$, are stochastically dominated by i.i.d. geometric random variables with parameter $1/4$. This implies that both (8) and (7) are controlled by $4|C_n|$, thus proving (6).

Proof of Lemma 3.1. We will inductively show that for $0 \leq n \leq N$,

$$E \left[ \sum_{m_0} \sum_{m_1} \cdots \sum_{m_n} \prod_{x \in A_{n-1}} 1_M(x, m, s, \phi) \right] \leq 4|A_n \setminus A_{n-1}|,$$

(9)

where $A_n := \bigcup_{i=1}^n C_i$. Assuming (9) for $n = N$, it follows that the left-hand side of (9) is bounded above by $4|B_N| \cdot 2|B_N| \cdot |B_N| |\partial \mathcal{B}_N|$. This would imply Lemma 3.1 by choosing $c_1$ large enough.

It remains to prove (9) by induction. The base case of $n = 0$ is nothing but Lemma 4.1 when $n = 0$. Suppose the inequality is true for $n - 1$. By conditioning on $F_{n-1}$ and using Lemma 4.1 we get

$$E \left[ \sum_{m_0} \sum_{m_1} \cdots \sum_{m_n} \prod_{x \in A_{n-1}} 1_M(x, m, s, \phi) \right] = E \left[ \sum_{m_0} \sum_{m_1} \cdots \sum_{m_{n-1}} \prod_{x \in A_{n-2}} 1_M(x, m, s, \phi) \times E \left[ \prod_{x \in C_{n-1}} 1_M(x, m, s, \phi) \mid F_{n-1} \right] \right] \leq 4|C_n| \cdot E \left[ \sum_{m_0} \sum_{m_1} \cdots \sum_{m_{n-1}} \prod_{x \in A_{n-2}} 1_M(x, m, s, \phi) \right] \leq 4|C_n| \cdot 4|A_{n-1} \setminus A_{n-2}| = 4|A_n \setminus A_{n-1}|.$$

\[ \square \]

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