A Positive Solution for a Nonlocal Schrödinger Equation

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Abstract

We provide an existence result of radially symmetric, positive, classical solutions for a nonlinear Schrödinger equation driven by the infinitesimal generator of a rotationally invariant Lévy process.

Key Words: nonlocal Schrödinger equation; positive solution; mountain pass theorem.

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1 Introduction

The purpose of this paper is to provide an existence result of radially symmetric, positive, classical solutions for the following problem,

\[
\begin{aligned}
-2Au + \lambda u &= |u|^{p-2}u \\
&\quad u \in H^1(\mathbb{R}^N),
\end{aligned}
\]

where \( \lambda > 0, 2 \leq N \leq 6, 2 < p < 2^* \) with \( 2^* := +\infty \) if \( N = 2 \) and \( 2^* := 2N/(N-2) \) if \( N > 2 \), and \( A \) is the infinitesimal generator of a rotationally invariant Lévy process.

Example 1.1. Consider the infinitesimal generator \( A \) of a Lévy process with jumps of normal distribution.

\[
Au(x) := \frac{1}{2}\Delta u(x) + \frac{1}{2} \int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x)) \varphi(y)dy,
\]

where \( \varphi(y) := (2\pi)^{-N/2} \exp(-|y|^2/2). \)

A basic motivation for the study of the problem (1.1) is the well known nonlinear Schrödinger equation driven by the infinitesimal generator of a Brownian motion,

\[
-\Delta u + \lambda u = |u|^{p-2}u.
\]

Many authors investigated Equation (1.2) (see [2, 4, 9, 10] etc.).

Note that the Brownian motion is a special rotationally invariant stable Lévy process. It is natural to consider the following equation,

\[
(-\Delta)^{\alpha/2} u + \lambda u = |u|^{p-2}u,
\]

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where $0 < \alpha \leq 2$, since $-(-\Delta)^{\alpha/2}$ is the infinitesimal generator of a rotationally invariant stable Lévy process with index $\alpha$. Equation (1.3) has been studied by many authors (see [5–8] etc.).

Naturally, we consider the following (nonlocal) Schrödinger equation,

$$-2Au + \lambda u = |u|^{p-2}u, \quad (1.4)$$

where $A$ is the infinitesimal generator of a rotationally invariant Lévy process. In the present paper, we assume that the Lévy process is of $N$ dimensions, where $2 \leq N \leq 6$, with nondegenerate diffusion terms and a finite Lévy measure.

Equation (1.4) also arises from looking for the standing waves of the following Schrödinger equation,

$$i\frac{\partial \psi}{\partial t} - 2A\psi = |\psi|^{p-2}\psi.$$

Before stating the main result of the present paper, let us make some comments on the operators $-(-\Delta)^{\alpha/2}$ and $A$. If $0 < \alpha < 2$, then the Lévy processes generated by $-(-\Delta)^{\alpha/2}$ are pure jump processes; in other words, these processes do not contain any diffusion term. In fact, the corresponding characteristics of them are given by $(0, 0, \mu)$ with

$$\mu(dx) = \frac{K(\alpha)dx}{|x|^{N+\alpha}} \text{ for some positive constant } K(\alpha).$$

Consequently, the Lévy measure $\mu$ is not finite. For the operator $A$, the corresponding characteristics are given by $(0, aI, \nu)$ for some positive number $a$ and some finite rotationally invariant Lévy measure $\nu$. Therefore, $-(-\Delta)^{\alpha/2}$ does not cover operators of type $A$ and vice versa; besides, Equation (1.4) is an extension of Equation (1.2).

Now we state the main result as follows.

**Theorem 1.2.** (1) Any weak solution of the problem (1.1) in $H^1(\mathbb{R}^N)$ is a $C^2$ continuous function.

(2) There exists a radially symmetric, positive, classical solution of problem (1.1).

(3) The values of any positive solution of the problem (1.1) at maximum points are not less than $\lambda^{p-2}$.

The rest of the paper is organized as follows. In Section 2 we present some preliminaries. The proof of Theorem 1.2 will be given in Section 3.

### 2 Some Preliminaries

This section serves as a preparation for the proof of Theorem 1.2. First, we state a compact embedding result. Second, a regularity result will be proved. Finally, we investigate the sign of solutions for a modified version of Equation (1.4).

Define

$$H^1_{O(N)}(\mathbb{R}^N) := \{ u \in H^1(\mathbb{R}^N) : u = gu, \ g \in O(N) \},$$

where $gu := u \circ g^{-1}$.

Then we have the following lemma.
Lemma 2.1 ([13, p.18, Corollary 1.26]). The following embedding is compact,

\[ H^{1}_{O(N)}(\mathbb{R}^N) \hookrightarrow L^{p}(\mathbb{R}^N), \; 2 < p < 2^{*}. \]

Lemma 2.2. If \( u \) is a weak solution of the equation

\[-2Au + \lambda u = (u^{+})^{p-1}\]

in \( H^{1}(\mathbb{R}^N) \), then \( u \in C^{2}(\mathbb{R}^N) \).

Proof. 1. Note that the symbol \( \sigma_{A} \) of \( A \) is given by

\[ \sigma_{A}(\xi) = -\frac{a}{2}|\xi|^{2} + \int_{\mathbb{R}^N}[\cos(\xi \cdot x) - 1]\nu(dx), \]

where \( a \) is a positive number and \( \nu \) is a finite \( O(N) \)-invariant Lévy measure (see [1, p.128, Exercise 2.4.23 and pp.163-164, Theorem 3.3.3]).

Let \( A_{2} \) be the operator with the symbol

\[ \sigma_{A_{2}}(\xi) = -\frac{a}{2}|\xi|^{2}, \]

and \( A_{0} \) be the operator with the symbol

\[ \sigma_{A_{0}}(\xi) = \int_{\mathbb{R}^N}[\cos(\xi \cdot x) - 1]\nu(dx). \]

Then we have

\[-2A_{2}u = h(\cdot)(1 + |u|), \]

where

\[ h(x) := \frac{2A_{0}u(x) + (u^{+}(x))^{p-1} - \lambda u(x)}{1 + |u(x)|} \quad \text{for } x \in \mathbb{R}^N. \]

2. For any \( u \in H^{1}(\mathbb{R}^N) \), we have

\[ \int_{\mathbb{R}^N}(1 + |\xi|^{2}) \left( \int_{\mathbb{R}^N}[\cos(\xi \cdot x) - 1]\nu(dx) \right)^{2} |\hat{u}(\xi)|^{2}d\xi < \infty, \tag{2.1} \]

where “\( \hat{\cdot} \)” denotes the Fourier transformation.

Thus \( A_{0} : H^{1}(\mathbb{R}^N) \rightarrow H^{1}(\mathbb{R}^N) \) is a bounded operator thanks to (2.1).

Furthermore, it follows that \( h \in L^{N/2}_{\text{loc}}(\mathbb{R}^N) \). Consequently, we have \( u \in L^{q}_{\text{loc}}(\mathbb{R}^N) \) for any \( q \in [1, +\infty) \) by Brézis-Kato theorem (see, for example, [12, p.270, B.3 Lemma]). Then, by the ellipticity of operator \( A \), we find \( u \in W^{2,q}_{\text{loc}}(\mathbb{R}^N) \) for any \( q \in [1, +\infty) \). Now Sobolev embedding theorem implies \( u \in C^{1}_{\text{loc}}(\mathbb{R}^N) \). Finally, also by the ellipticity of operator \( A \), it follows that \( u \in C^{2}(\mathbb{R}^N) \). \qed

Lemma 2.3. If \( u \in C^{2}(\mathbb{R}^N) \cap H^{1}(\mathbb{R}^N) \) is a nontrivial solution of the equation

\[-2Au + \lambda u = (u^{+})^{p-1}, \]

then \( u > 0 \).
Proof. 1. First we have

\[ \int \int (u(x) - u(x + y))(u^{-}(x) - u^{-}(x + y))\nu(dy)dx \]

where we have used

\[ \mathbb{R}^{2} = \{x : u(x) \geq 0\} \times \{y : u(y) \geq 0\} \cup \{x : u(x) \geq 0\} \times \{y : u(y) < 0\} \]

\[ \cup \{x : u(x) < 0\} \times \{y : u(y) \geq 0\} \cup \{x : u(x) < 0\} \times \{y : u(y) < 0\} \]

for the inequality.

Then it follows that

\[ (-2Au, -u^{-})_{L^{2}} = a\|\nabla u^{-}\|_{L^{2}} - \int \int (u(x) - u(x + y))(u^{-}(x) - u^{-}(x + y))\nu(dy)dx \geq 0. \]

Therefore, in light of \((-2Au, -u^{-})_{L^{2}} + \lambda\|u^{-}\|_{L^{2}}^{2} = 0\), we have \(u^{-} = 0\), which implies \(u \geq 0\).

2. Rewrite the equation \(-2Au + \lambda u = (u^{+})^{p-1}\) as

\[ -2A_{2}u + (\lambda + 2\nu(\mathbb{R}^{N}))u = (u^{+})^{p-1} + 2 \int_{\mathbb{R}^{N}} u(y)\nu(dy). \]

Then we find that

\[ -2A_{2}u + (\lambda + 2\nu(\mathbb{R}^{N}))u \geq 0. \]

It follows from the strong maximum principle that \(u > 0\).

Corollary 2.4. Assume that \(u \in C^{2}(\mathbb{R}^{N}) \cap H^{1}(\mathbb{R}^{N})\) is a nontrivial solution of the equation \(-2Au + \lambda u = (u^{+})^{p-1}\). If \(x_{0} \in \mathbb{R}^{N}\) is a maximum point of the function \(u\), then \(u(x_{0}) \geq \lambda^{\frac{1}{p-2}}\).

Proof. 1. Since \(x_{0}\) is a maximum point of the function \(u\), we have \(\Delta u(x_{0}) \leq 0\).

2. Note that Lemma 2.3 implies \(u(x_{0}) > 0\). It follows from the positive maximum principle (see, for example, [11, p.283, (1.5) proposition] or [1, p.181, Theorem 3.5.2]) that \(A_{0}u(x_{0}) \leq 0\). This and \(\Delta u(x_{0}) \leq 0\) imply \(Au(x_{0}) \leq 0\). Therefore,

\[ u(x_{0})^{p-1} - \lambda u(x_{0}) = -2Au(x_{0}) \geq 0. \]

So the inequality \(u(x_{0}) \geq \lambda^{\frac{1}{p-2}}\) holds.

3  Proof of Theorem 1.2

In this section, we provide a proof of Theorem 1.2 via the mountain pass theorem.

Observe that the operator \(-A\) is positively self-adjoint (see [1, p.178, Theorem 3.4.10 and p.190, Theorem 3.6.1]). We define a new inner product on \(H^{1}(\mathbb{R}^{N})\) by

\[ (v, w) := (-2Av, w)_{L^{2}} + \lambda(v, w)_{L^{2}} \text{ for any } v, w \in C_{0}^{\infty}(\mathbb{R}^{N}), \]

and denote the induced norm of it by \(\| \cdot \|\). Since the operator \(-A_{0}\) is also positively self-adjoint, it follows from \(A = A_{2} + A_{0}\) and (2.1) that the norm \(\| \cdot \|\) is equivalent to \(\| \cdot \|_{H^{1}}\).
Define a functional $E : H^1(\mathbb{R}^N) \to \mathbb{R}$ by
\[
E(u) := \frac{1}{2} \|u\|^2 - \frac{1}{p} \int_{\mathbb{R}^N} (u^+(x))^p \, dx.
\]
Then it follows from [13, p.11, Corollary 1.13] that $E \in C^2(H^1(\mathbb{R}^N), \mathbb{R})$. In addition, the critical points of the functional $E$ are weak solutions of the equation $-2Au + \lambda u = (u^+)^{p-1}$ in $H^1(\mathbb{R}^N)$, and vice versa.

**Lemma 3.1.** The functional $E$ is $O(N)$-invariant.

**Proof.** We only need to prove that the norm $\| \cdot \|$ is $O(N)$-invariant.

Note that the symbol $\sigma_A$ of $A$ is given by
\[
\sigma_A(\xi) = -\frac{a}{2} |\xi|^2 + \int_{\mathbb{R}^N} [\cos(\xi \cdot x) - 1] \nu(dx),
\]
where $a$ is a positive number and $\nu$ is a finite $O(N)$-invariant Lévy measure (see [1, p.128, Exercise 2.4.23 and pp.163-164, Theorem 3.3.3]). We find the symbol $\sigma_A$ of $A$ is $O(N)$-invariant.

Therefore, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $g \in O(N)$, we have
\[
\|g\varphi\|^2 = (-2A(g\varphi), g\varphi)_{L^2} + \lambda \|g\varphi\|^2_{L^2}
= (-2\sigma_A \cdot \hat{g\varphi}, \hat{g\varphi})_{L^2} + \lambda \|g\varphi\|^2_{L^2}
= (-2g^{-1}\sigma_A \cdot \hat{\varphi}, \hat{\varphi})_{L^2} + \lambda \|\varphi\|^2_{L^2}
= (-2\sigma_A \cdot \hat{\varphi}, \hat{\varphi})_{L^2} + \lambda \|\varphi\|^2_{L^2} = \|\varphi\|^2,
\]
which implies that the norm $\| \cdot \|$ is $O(N)$-invariant. \qed

We need the following Lemma 3.2 in the verification of the PS condition for the functional $E$ restricted to $H^1_{O(N)}(\mathbb{R}^N)$.

**Lemma 3.2** ([13, p.134, Theorem A.4]). Assume that $1 \leq p < \infty$, $1 \leq q < \infty$, and $g \in C(\mathbb{R}^N)$ such that
\[
|g(u)| \leq c |u|^{p/q} \text{ for some constant } c.
\]
Then the operator $L : L^p(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ defined by $u \mapsto g(u)$ is continuous.

**Lemma 3.3** (The PS condition for the functional $E$ restricted to $H^1_{O(N)}(\mathbb{R}^N)$). Any sequence $\{u_n\}_{n \in \mathbb{N}} \in H^1_{O(N)}(\mathbb{R}^N)$ such that
\[
d := \sup_{n \in \mathbb{N}} \{E(u_n)\} < \infty, \quad E'(u_n) \to 0 \text{ as } n \to \infty
\]
contains a convergent subsequence.

**Proof.** The proof is the same as that of [13, p.15, Lemma 1.20].

1. For $n$ large enough, we have
\[
d + \|u_n\| \geq E(u_n) - \frac{1}{p} \langle E'(u_n), u_n \rangle = \left(\frac{1}{2} - \frac{1}{p}\right) \|u_n\|^2.
\]
It follows that \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1_{O(N)}(\mathbb{R}^N) \) since \( p > 2 \).

2. Without loss of generality, we assume that \( u_n \rightharpoonup u \) in \( H^1_{O(N)}(\mathbb{R}^N) \). Then it follows from Lemma 2.1 that \( u_n \to u \) in \( L^p(\mathbb{R}^N) \). Consequently, by Lemma 3.2, we have \( (u_n^+)^{p-1} \to (u^+)^{p-1} \) in \( L^q(\mathbb{R}^N) \), where \( q := p/(p - 1) \).

Note that

\[
\|u_n - u\|^2 = (E'(u_n) - E'(u), u_n - u) + \int_{\mathbb{R}^N} (u_n^+(x)^{p-1} - u^+(x)^{p-1})(u_n(x) - u(x))dx.
\]  

(3.1)

For the first term of the right hand side of the above equality, we see that

\( (E'(u_n) - E'(u), u_n - u) \to 0 \) as \( n \to \infty \),

since \( E'(u_n) \to 0 \) as \( n \to \infty \) and \( \{u_n\}_{n \in \mathbb{N}} \) is bounded in \( H^1_{O(N)}(\mathbb{R}^N) \).

And for the second term, it follows from Hölder inequality that

\[
\int_{\mathbb{R}^N} (u_n^+(x)^{p-1} - u^+(x)^{p-1})(u_n(x) - u(x))dx \leq \|u_n^+(x)^{p-1} - u^+(x)^{p-1}\|_{L^p}\|u_n(x) - u(x)\|_{L^p} \to 0 \text{ as } n \to \infty,
\]

because \( u_n \to u \) in \( L^p(\mathbb{R}^N) \) and \( (u_n^+)^{p-1} \to (u^+)^{p-1} \) in \( L^q(\mathbb{R}^N) \).

Therefore, \( u_n \to u \) in \( H^1_{O(N)}(\mathbb{R}^N) \) as \( n \to \infty \) by (3.1).

Now we are at the position to give a proof of Theorem 1.2.

**Proof of Theorem 1.2.** 1. Consider the functional \( E \) restricted to \( H^1_{O(N)}(\mathbb{R}^N) \). Thanks to Lemma 2.1 or Sobolev imbedding theorem, there is a positive constant \( c \) such that \( \|u\|_{L^p} \leq c\|u\| \) for any \( u \in H^1_{O(N)}(\mathbb{R}^N) \). Then it follows from the definition of the functional \( E \) that

\[
E(u) \geq \frac{1}{2}\|u\|^2 - \frac{c^p}{p}\|u\|^p.
\]

Setting \( r := \left( \frac{p}{4c^p} \right)^{\frac{1}{p-2}} \), we have

\[
\inf_{\|u\|=r} E(u) \geq \frac{1}{4} \left( \frac{p}{4c^p} \right)^{\frac{2}{p-2}} > 0.
\]

2. Set \( w(x) := \exp(-|x|^2) \). Then \( w(x) \in H^1_{O(N)}(\mathbb{R}^N) \) and for any \( t \in [0, +\infty) \),

\[
E(tw) = \frac{t^2}{2}\|w\|^2 - \frac{t^p}{p}\|w\|_{L^p}^p.
\]

Note that \( p > 2 \). We can take a positive number \( t \) such that \( t\|w\| > r \) and \( E(tw) < 0 \).

3. Now by the mountain pass theorem, there is a nontrivial critical point \( u \) of the functional \( E \) restricted to \( H^1_{O(N)}(\mathbb{R}^N) \). Note that the functional \( E \) is \( O(N) \)-invariant. Thanks to the principle of symmetric criticality (see, for example, [13, p.18, Theorem 1.28]), it follows that the point \( u \) is also a critical point of the functional \( E \). Consequently, the point \( u \) is a weak solution of the equation \(-2Au + \lambda u = (u^+)^{p-1} \) in \( H^1(\mathbb{R}^N) \).

4. Finally, Lemma 2.2 and Lemma 2.3 complete the proof. 

\( \square \)
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