SPINORIAL REPRESENTATION OF SUBMANIFOLDS IN A PRODUCT OF SPACE FORMS

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Abstract. We present a method giving a spinorial characterization of an immersion in a product of spaces of constant curvature. As a first application we obtain a proof using spinors of the fundamental theorem of immersion theory in that spaces. We also study special cases: we recover previously known results concerning immersions in $S^2 \times \mathbb{R}$ and we obtain new spinorial characterizations of immersions in $S^2 \times \mathbb{R}^2$ and in $H^2 \times \mathbb{R}$. We then study the theory of $H = 1/2$ surfaces in $S^2 \times \mathbb{R}$ using this spinorial approach, obtain new proofs of some of its fundamental results and give a direct relation with the theory of $H = 1/2$ surfaces in $\mathbb{R}^{1,2}$.

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Introduction

Characterizations of immersions in space forms using spinors have been widely studied, as for instance in [8, 9, 11, 14, 17, 18, 19, 20] and more recently in [6] (see also the references in these papers). It appears that spin geometry furnishes an elegant formalism for the description of the immersion theory in space forms, especially in low dimension and in relation with the Weierstrass representation formulas. See also the Weierstrass representation obtained in [2] for CMC hypersurfaces in some four-dimensional Einstein manifolds. We are interested here in spinorial characterizations of immersions in a product of space forms. Some special cases have been studied before, as immersions in the products $S^2 \times \mathbb{R}$, $H^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}^2$ and $S^3 \times \mathbb{R}$ [12, 15, 16]. We propose here a method allowing the treatment of an immersion in an arbitrary product of space forms. The dimension and the co-dimension of the immersion are moreover arbitrary. Let us note that the spinor bundle that we use in the paper is not the usual spinor bundle: in general it is a real bundle, and of larger rank. We used this idea in [6]. Let us also mention that even in low dimensions we obtain new results: the theory permits to recover in a unified way the previously known results and to complete them; in particular, we show how to recover the spinorial characterization of an immersion in $S^2 \times \mathbb{R}$ and we obtain new spinorial characterizations of immersions in $S^2 \times \mathbb{R}^2$ and in $H^2 \times \mathbb{R}$.

A first application of the general theory is a proof using spinors of the fundamental theorem of immersion theory in a product of space forms.

A second application concerns the theory of CMC surfaces with $H = 1/2$ in $H^2 \times \mathbb{R}$: we show that a component of the spinor field representing the immersion of such a surface is an horizontal lift of the hyperbolic Gauss map, for a connection which depends on the Weierstrass data of the immersion, and we deduce that there exists a two-parameter family of $H = 1/2$ surfaces in $H^2 \times \mathbb{R}$ with given hyperbolic
Gauss map and Weierstrass data, a result obtained in [7] using different methods. We finally study the spinorial representation of $H = 1/2$ surfaces in $\mathbb{R}^{1,2}$ and obtain a direct relation between the two theories.

In order to simplify the exposition we first consider immersions in a product of spheres $\mathbb{S}^m \times \mathbb{S}^n$ and in a product $\mathbb{S}^m \times \mathbb{R}^n$, and we then state without proof the analogous results for immersions in a product of hyperbolic spaces $\mathbb{H}_1^m \times \mathbb{H}_2^n$ and in $\mathbb{H}_1^m \times \mathbb{R}^n$. Using the same ideas it is possible to state analogous results for an arbitrary quantity of factors involving $\mathbb{S}^m$, $\mathbb{H}_2^n$ and $\mathbb{R}^p$, or for space forms with pseudo-riemannian metrics, but these general statements are not included in the paper.

The outline of the paper is as follows. We first study the immersions in a product of spheres $\mathbb{S}^m \times \mathbb{S}^n$ in Section 1 and the immersions in $\mathbb{S}^m \times \mathbb{R}^n$ in Section 2. We then state the analogous results for a product of hyperbolic spaces $\mathbb{H}_1^m \times \mathbb{H}_2^n$ and for $\mathbb{H}_1^m \times \mathbb{R}^n$ in Section 3. Finally the theory of $H = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ is studied in Section 4. Some useful auxiliary results are gathered in an appendice at the end of the paper.

1. Isometric immersions in $\mathbb{S}^m \times \mathbb{S}^n$

We are interested here in immersions in a product $\mathbb{S}^m \times \mathbb{S}^n$ of two spheres, of constant curvature $c_1, c_2 > 0$. We construct the suitable spinor bundle in Section 1.1, we consider the case of a manifold which is already immersed in $\mathbb{S}^m \times \mathbb{S}^n$ in Section 1.2, we state and prove the main theorem in Section 1.3 and the fundamental theorem in Section 1.4.

1.1. The suitable spinor bundle. Let $M$ be a $p$-dimensional riemannian manifold and $E \to M$ a vector bundle of rank $q$, with $p + q = m + n$, with a bundle metric and a connection compatible with the metric. Let $\mathcal{E}_2 = M \times \mathbb{R}^2 \to M$ be the trivial bundle, equipped with its natural metric and its trivial connection. Let us construct a spinor bundle on $M$ equipped with a Clifford action of $TM \oplus E \oplus \mathcal{E}_2$. We suppose that $M$ and $E$ are spin, with spin structures $Q_M : Q_M \to Q_M$, $Q_E : Q_E \to Q_E$ and set $\bar{Q} := Q_M \times_M Q_E$. Let us denote by $Spin(N)$ and $Cl(N)$ the spin group and the Clifford algebra of $\mathbb{R}^N$. Associated to the splitting $\mathbb{R}^{m+n+2} = \mathbb{R}^p \oplus \mathbb{R}^q \oplus \mathbb{R}^2$ we consider

$$Spin(p) \cdot Spin(q) \subset Spin(p+q) \subset Spin(m+n+2)$$

and define the representation

$$\rho : \quad Spin(p) \times Spin(q) \to GL(Cl(m+n+2))$$

$$\quad a := (a_p, a_q) \mapsto \rho(a) : (\xi \mapsto a_p \cdot a_q \cdot \xi)$$

with the bundles

$$\Sigma := \bar{Q} \times_{\rho} Cl(m+n+2), \quad U\Sigma := \bar{Q} \times_{\rho} Spin(m+n+2) \subset \Sigma.$$ Since the bundle of Clifford algebras constructed on the fibers of $TM \oplus E \oplus \mathcal{E}_2$ is

$$Cl(TM \oplus E \oplus \mathcal{E}_2) = \bar{Q} \times_{Ad} Cl(m+n+2)$$

with

$$Ad : \quad Spin(p) \times Spin(q) \to GL(Cl(m+n+2))$$

$$\quad a \mapsto Ad(a) : (\xi \mapsto a \cdot \xi \cdot a^{-1}),$$
there is a Clifford action
\[ \text{Cl}(TM \oplus E \oplus \mathcal{E}_2) \oplus \Sigma \rightarrow \Sigma \]
\[ (Z, \varphi) \rightarrow Z \cdot \varphi \]
similar to the usual Clifford action in spin geometry. Let us note that \( \Sigma \) is not the usual spinor bundle, since it is a real vector bundle, associated to a representation which is not irreducible: it is rather a (maybe large) sum of real spinor bundles. We nevertheless interpret the bundle \( \Sigma \) as the bundle of spinors, \( \Sigma \) as the bundle of unit spinors, and \( \text{Cl}(TM \oplus E \oplus \mathcal{E}_2) \) as the Clifford bundle acting on the bundle of spinors. There is a natural map
\[ \langle\langle \cdot, \cdot \rangle\rangle : \quad \Sigma \times \Sigma \rightarrow \text{Cl}(m+n+2) \]
\[ (\varphi, \varphi') \mapsto \langle\langle \varphi, \varphi' \rangle\rangle := \tau[\varphi'] [\varphi] \]
where \( \varphi = [\hat{s},[\varphi]] \) and \( \varphi' = [\hat{s},[\varphi']] \) in \( \Sigma = \hat{Q} \times_\rho \text{Cl}(m+n+2) \) and \( \tau \) is the involution of \( \text{Cl}(m+n+2) \) reversing the order of a product of vectors. Here and in all the paper we use the brackets \([\cdot]\) to denote the component in \( \text{Cl}(m+n+2) \) of an element of the spinor or the Clifford bundle in a given spinorial frame \( \hat{s} \). This map is such that, for all \( \varphi, \varphi' \in \Sigma \) and \( Z \in TM \oplus E \oplus \mathcal{E}_2 \),
\[ \langle\langle \varphi, \varphi' \rangle\rangle = \tau(\langle\langle \varphi', \varphi \rangle\rangle) \]
and
\[ \langle\langle Z \cdot \varphi, \varphi' \rangle\rangle = \langle\langle \varphi, Z \cdot \varphi' \rangle\rangle. \]
Moreover, it is compatible with the connection \( \nabla \) induced on \( \Sigma \) by the Levi-Civita connection on \( M \) and the given connection on \( E \):

**Lemma 1.1.** For all \( X \in TM \) and \( \varphi, \varphi' \in \Gamma(\Sigma) \),
\[ \partial_X(\langle\langle \varphi, \varphi' \rangle\rangle) = \langle\langle \nabla_X \varphi, \varphi' \rangle\rangle + \langle\langle \varphi, \nabla_X \varphi' \rangle\rangle \]
where on the left hand side \( \partial \) stands for the usual derivative.

A similar result is proved in [6, Lemma 2.2].

### 1.2. Spin geometry of a submanifold in \( S^m_1 \times S^n_2 \)
We assume in that section that \( M \) is a \( p \)-dimensional submanifold of \( S^m_1 \times S^n_2 \), with normal bundle \( E \) of rank \( q \) and second fundamental form \( B : TM \times TM \rightarrow E \), denote by \( \nu_1 : M \rightarrow \mathbb{R}^{m+1} \) and \( \nu_2 : M \rightarrow \mathbb{R}^{n+1} \) the vector fields such that \( \frac{\partial}{\partial X} \nu_1 \) and \( \frac{\partial}{\partial X} \nu_2 \) are the two components of the immersion \( M \rightarrow S^m_1 \times S^n_2 \) and consider the trivial bundle \( \mathcal{E}_2 = \mathbb{R}\nu_1 \oplus \mathbb{R}\nu_2 \rightarrow M \). We consider spin structures on \( TM \) and \( E \), and the bundles \( \Sigma, U\Sigma \) and \( \text{Cl}(TM \oplus E \oplus \mathcal{E}_2) \) constructed in the previous section. For a convenient choice of the spin structures on \( TM \) and \( E \), the bundle \( \Sigma \) identifies canonically with the trivial bundle \( M \times \text{Cl}(m+n+2) \), and two connections are defined on \( \Sigma \), the connection \( \nabla \) introduced above and the trivial connection \( \partial \). Since the second fundamental form of \( M \) in \( \mathbb{R}^{m+n+2} \) is
\[ (X, Y) \mapsto -\sqrt{c_1} \langle X_1, Y_1 \rangle \nu_1 - \sqrt{c_2} \langle X_2, Y_2 \rangle \nu_2 + B(X, Y) \]
where \( X = X_1 + X_2 \) and \( Y = Y_1 + Y_2 \) in the decomposition \( TM \subset T S^m_1 \oplus T S^n_2 \) and setting
\[ \frac{1}{2} B(X) := \frac{1}{2} \sum_{j=1}^{p} e_j \cdot B(X, e_j) \in \text{Cl}(TM \oplus E \oplus \mathcal{E}_2) \]
where \( e_1, \ldots, e_p \) is an orthonormal basis of \( TM \), they satisfy the following Gauss formula:

\[
\partial_X \varphi = \nabla_X \varphi - \frac{1}{2}(\sqrt{\epsilon_1} X_1 \cdot \nu_1 + \sqrt{\epsilon_2} X_2 \cdot \nu_2) \cdot \varphi + \frac{1}{2}B(X) \cdot \varphi
\]

for all \( \varphi \in \Gamma(\Sigma) \) and all \( X \in TM \); see [3] for the proof of a spinorial Gauss formula in a slightly different context. By formula (5), the constant spinor field \( \varphi = 1_{Cl(m+n+2)|M} \) satisfies, for all \( X \in TM \),

\[
\nabla_X \varphi = \frac{1}{2}(\sqrt{\epsilon_1} X_1 \cdot \nu_1 + \sqrt{\epsilon_2} X_2 \cdot \nu_2) \cdot \varphi - \frac{1}{2}B(X) \cdot \varphi.
\]

1.3. **The main theorem.** We assume that \( M \) and \( E \to M \) are abstract objects as in Section 1.1 (i.e. \( M \) is not a priori immersed in \( S^n_1 \times S^n_2 \)), and suppose moreover that there is a product structure on \( TM \oplus E \), i.e. a bundle map \( \mathcal{P} : TM \oplus E \to TM \oplus E \) such that \( \mathcal{P}^2 = id, \mathcal{P} \neq id \). Setting \( \mathcal{P}_1 := Ker(\mathcal{P} - id) \) and \( \mathcal{P}_2 := Ker(\mathcal{P} + id) \) we have \( TM \oplus E = \mathcal{P}_1 \oplus \mathcal{P}_2 \); the product structure \( \mathcal{P} \) is equivalent to a splitting of \( TM \oplus E \) into two subbundles \( \mathcal{P}_1 \) and \( \mathcal{P}_2 \), that we assume to be respectively of rank \( m \) and \( n \).

1.3.1. **Statement of the theorem.** Let \( B : TM \times TM \to E \) be a symmetric tensor. Let us fix two unit orthogonal and parallel sections \( \nu_1, \nu_2 \) of the trivial bundle \( E_2 = M \times \mathbb{R}^2 \to M \).

**Theorem 1.** The following statements are equivalent:

(i) There exist an isometric immersion \( F : M \to S^n_1 \times S^n_2 \) and a bundle map \( \Phi : TM \oplus E \to T(S^n_1 \times S^n_2) \) above \( F \) such that \( \Phi(X,0) = dF(X) \) for all \( X \in TM \), which preserves the bundle metrics, maps the connection on \( E \) and the tensor \( B \) to the normal connection and the second fundamental form of \( F \), and is compatible with the product structures.

(ii) There exists a section \( \varphi \in \Gamma(U\Sigma) \) solution of

\[
\nabla_X \varphi = \frac{1}{2}(\sqrt{\epsilon_1} X_1 \cdot \nu_1 + \sqrt{\epsilon_2} X_2 \cdot \nu_2) \cdot \varphi - \frac{1}{2}B(X) \cdot \varphi
\]

for all \( X \in TM \), where \( X = X_1 + X_2 \) is the decomposition in the product structure \( \mathcal{P} \) of \( TM \oplus E \), such that the map

\[
Z \in TM \oplus E \mapsto \langle (Z \cdot \varphi, \varphi) \rangle \in \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}
\]

commutes with the product structure \( \mathcal{P} \) and the natural product structure on \( \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \).

Moreover, the bundle map \( \Phi \) and the immersion \( F \) are explicitly given in terms of the spinor field \( \varphi \) by the formulas

\[
\Phi : TM \oplus E \to T(S^n_1 \times S^n_2), \quad Z \mapsto \langle (Z \cdot \varphi, \varphi) \rangle
\]

and

\[
F = \left( \frac{1}{\sqrt{\epsilon_1}} \langle (\nu_1 \cdot \varphi, \varphi) \rangle, \frac{1}{\sqrt{\epsilon_2}} \langle (\nu_2 \cdot \varphi, \varphi) \rangle \right) \in S^n_1 \times S^n_2.
\]

**Remark 1.** Formulas (7) and (8) can be regarded as a generalized Weierstrass representation formula.
1.3.2. Proof of Theorem 1. The proof of "(i) ⇒ (ii)" was obtained in Section 1.2: if \( M \) is immersed in \( S^m_1 \times S^n_2 \), the spinor field \( \varphi \) is the constant spinor field \( 1_{Cl(m+n+2)} \) restricted to \( M \). We prove "(ii) ⇒ (i)". We suppose that \( \varphi \in \Gamma(U\Sigma) \) is a solution of (7) and obtain (i) as a direct consequence of the following two lemmas:

**Lemma 1.2.** The map \( F \) defined by (8) satisfies

\[
dF(X) = \langle \langle X \cdot \varphi, \varphi \rangle \rangle,
\]

for all \( X \in TM \). It preserves the product structures and takes values in \( S^m_1 \times S^n_2 \).

**Proof.** Let us consider for \( i = 1, 2 \) the functions \( F_i = \frac{1}{\sqrt{c_i}} \langle \langle \nu_i \cdot \nabla X \varphi, \varphi \rangle \rangle \). Recalling the properties (1)-(3) of \( \langle \langle \cdot, \cdot \rangle \rangle \) and since \( \nu_1, \nu_2 \) are parallel sections of \( E_2 \) and \( \varphi \) satisfies (7), we have, for \( i = 1, 2 \),

\[
dF_i(X) = \frac{1}{\sqrt{c_i}} \langle \langle \nu_i \cdot \nabla X \varphi, \varphi \rangle \rangle = \frac{1}{\sqrt{c_i}} (\tau + id) \langle \langle \nu_i \cdot (-B(X) + \sqrt{c_1} X_1 + \sqrt{c_2} X_2 \cdot \nu_2) \cdot \varphi, \varphi \rangle \rangle
\]

(9) follows from (10). Let us see now that \( F_i \) is immersed in \( S^m_1 \times S^n_2 \) which implies that \( F_i \) preserves the product structures and takes values in \( TM \). Restricted to \( S^m_1 \times S^n_2 \), \( F_i \) takes respectively values in the spheres of \( \mathbb{R}^m+1 \) and \( \mathbb{R}^n+1 \) : since \( \varphi \) belongs to \( Spin(p+q+2) \), we have

\[
\langle \langle \nu_i \cdot \varphi, \varphi \rangle \rangle = Ad_{\varphi}(e_i^0) \quad \text{for} \quad i = 1, 2
\]

which implies that \( F_1 \) and \( F_2 \) take values in the spheres of \( \mathbb{R}^m+1 \) of radius \( 1/\sqrt{c_1} \) and \( 1/\sqrt{c_2} \) respectively. We then have to check that \( F_1 \) and \( F_2 \) take respectively values in \( \mathbb{R}^m+1 \) and \( \mathbb{R}^n+1 \) : since \( dF \) preserves the product structures, we have for \( X = X_1 + X_2 \in TM \) that

\[
dF(X) = dF(X_1) + dF(X_2) \quad \in \ \mathbb{R}^m+1 \oplus \mathbb{R}^n+1;
\]

since by (10) \( dF_1(X) = dF(X_1) \) and \( dF_2(X) = dF(X_2) \) we conclude that \( dF_1 \) and \( dF_2 \) take values in \( \mathbb{R}^m+1 \) and \( \mathbb{R}^n+1 \) respectively, and so do \( F_1 \) and \( F_2 \). \( \square \)

**Lemma 1.3.** The map

\[
\Phi : \ TM \oplus E \oplus E_2 \rightarrow \mathbb{R}^m+1 \times \mathbb{R}^n+1, \quad Z \mapsto \langle \langle Z \cdot \varphi, \varphi \rangle \rangle
\]

is a bundle map which preserves the metrics, identifies \( E \) with the normal bundle of the immersion \( F \) in \( S^m_1 \times S^n_2 \), and sends the connection on \( E \) and the tensor \( B \) to the normal connection and the second fundamental form of the immersion \( F \).
Proof. Let us first see that \( \Phi \) takes values in \( T(S^m_1 \times S^n_2) \). By Lemma 1.2, if \( X \in TM \), \( \Phi(X) \) belongs to \( T(S^m_1 \times S^n_2) \). For \( Z_1, Z_2 \in TM \oplus E \oplus \mathcal{E}_2 \), we have in \( Cl(m+n+2) \)

\[
\langle \Phi(Z_1), \Phi(Z_2) \rangle = -\frac{1}{2} (\Phi(Z_1) \Phi(Z_2) + \Phi(Z_2) \Phi(Z_1))
\]

\[
= -\frac{1}{2} \tau[\nu][Z_1][\nu][Z_2][\nu] + \tau[Z_2][\nu][Z_1][\nu]
\]

\[
= \tau[Z_1][Z_2][\nu]
\]

Thus, since \( \tau[\nu][\nu] = 1 \) and \( \langle Z_1, Z_2 \rangle = \langle [Z_1], [Z_2] \rangle \) belongs to \( \mathbb{R} \). Let us note that \( \Phi(\nu_1) \) and \( \Phi(\nu_2) \) are normal to \( T(S^m_1 \times S^n_2) \): for \( i = 1, 2 \), we have at \( p \in M \)

\[
\Phi(\nu_i(p)) = \langle \nu_i \cdot \nu, \nu \rangle_{F(p)} = \sqrt{c_i} F_i(p),
\]

which is the unit normal to \( S_i \) at \( p \). Thus, since \( \Phi \) preserves the metric, for \( Y \in E \), \( \Phi(Y) \) belongs to \( T(S^m_1 \times S^n_2) \). Let us now compute, for \( Z \in \Gamma(TM \oplus E) \),

\[
\nabla^m_1 \times \nabla^n_2 \Phi(Z) = p_{S^m_1 \times S^n_2} (\partial_X \langle Z \cdot \nu, \nu \rangle)
\]

where \( p_{S^m_1 \times S^n_2} \) is the projection \( \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \to T(S^m_1 \times S^n_2) \). We have

\[
\partial_X \langle Z \cdot \nu, \nu \rangle = \langle (\nabla_X Z \cdot \nu, \nu) \rangle + \langle (id + \tau)(\nu, Z \cdot \nabla_X \nu) \rangle.
\]

We focus on the second term. From the Killing type equation (7), we have

\[
\langle (\varphi, Z \cdot \nabla_X \varphi) \rangle = \frac{1}{2} \sum_{i=1}^{2} \sqrt{c_i} \langle (\varphi, Z \cdot X_i \cdot \nu_i \cdot \varphi) \rangle - \frac{1}{2} \langle (\varphi, Z \cdot B(X) \cdot \varphi) \rangle.
\]

For \( i = 1, 2 \), we have

\[
Z \cdot X_i \cdot \nu_i + \nu_i \cdot X_i \cdot Z = \nu_i \cdot (Z \cdot X_i + X_i \cdot Z) = -2 \langle X_i, Z \rangle \nu_i
\]

and thus

\[
(id + \tau) \langle (\varphi, Z \cdot X_i \cdot \nu_i \cdot \varphi) \rangle = \langle (\varphi, (Z \cdot X_i \cdot \nu_i + \nu_i \cdot X_i \cdot Z) \cdot \varphi) \rangle
\]

\[
= -2 \langle X_i, Z \rangle \langle (\varphi, \nu_i \cdot \varphi) \rangle.
\]

Moreover, we have

\[
(id + \tau) \langle (\varphi, Z \cdot B(X) \cdot \varphi) \rangle = \langle (\varphi, (Z \cdot B(X) - B(X) \cdot Z) \cdot \varphi) \rangle.
\]

Since

\[
\frac{1}{2} \langle B(X) \cdot Z - Z \cdot B(X) \rangle = B(X, Z_T) - B^*(X, Z_N)
\]

where \( B^* : TM \times E \to TM \) is so that \( \langle B(X, Y), Z \rangle = \langle Y, B^*(X, Z) \rangle \) for all \( X,Y \in TM \) and \( Z \in E \) (Lemma A.2 in Appendix A), we deduce that

\[
\partial_X \langle (Z \cdot \nu, \nu) \rangle = \langle (\nabla_X Z \cdot \nu, \nu) \rangle - \frac{2}{2} \sqrt{c_i} \langle X_i, Z \rangle \langle (\varphi, \nu_i \cdot \varphi) \rangle
\]

\[
+ \langle (\varphi, (B(X, Z_T) - B^*(X, Z_N) \cdot \varphi) \rangle,
\]

and

\[
\nabla^m_1 \times \nabla^n_2 \Phi(Z) = \Phi(\nabla_X Z + B(X, Z_T) - B^*(X, Z_N)).
\]

This formula implies the following expressions for the second fundamental form \( B^F \) and the normal connection \( \nabla^F \) of the immersion \( F : Z \in \Gamma(TM) \) is such that
\( \nabla Z = 0 \) at the point where we do the computations, then \( B^F(X, Z) = \Phi(B(X, Z)) \);
if \( Z \in \Gamma(E) \), then \( \nabla_X^Z(\Phi(Z)) = \Phi(\nabla_X^Z Z) \). This finishes the proof of the lemma (and of Theorem 1).

1.4. **The fundamental theorem in \( S^n_1 \times S^n_2 \)**. We give here a proof using spinors of the fundamental theorem of the immersion theory in \( S^n_1 \times S^n_2 \). This result has been proved independently by Kowalczyk [10] and Lira-Tojeiro-Vitório [13].

1.4.1. *Statement of the theorem*. Let \( P \) and \( P' \) be the product structures of \( TM \oplus E \) and \( \mathbb{R}^{m+n+2} = \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \). We define \( f : TM \to TM, h : TM \to E, s : E \to TM \) and \( t : E \to E \) such that

\[
P(X) = \left\{ \begin{array}{ll}
f(X) + h(X) & \text{if } X \in TM, \\
s(X) + t(X) & \text{if } X \in E.
\end{array} \right.
\]

We set, for \( U, V, W \in TM \oplus E \),

\[
(U \wedge V)W := \langle U, W \rangle V - \langle V, W \rangle U.
\]

We first write the compatibility equations necessary for the existence of a non-trivial spinor field solution of (7):

**Proposition 1.4.** Let \( \varphi \in \Gamma(U\Sigma) \) be a solution of (7) such that

\[
\Phi : TM \oplus E \to T(S^n_1 \times S^n_2), \quad Z \mapsto (\langle Z \cdot \varphi, \varphi \rangle)
\]

commutes with the product structures, i.e. satisfies

\[
(\Phi(P)(Z)) = P'(\Phi(Z))
\]

for all \( Z \in TM \oplus E \). If \( R^T \) stands for the curvature tensor of the Levi-Civita connection on \( M \) and \( R^N \) for the curvature tensor of the connection \( \nabla' \) on \( E \), the following fundamental equations hold: for all \( X, Y, Z \in TM \) and \( N \in E \),

\[
R^T(X, Y, Z) = B^*(X, B(Y, Z)) - B^*(Y, B(X, Z))
\]

\[
= \frac{1}{4}(c_1 + c_2)((X \wedge Y)Z + (f(X) \wedge f(Y))Z) - \frac{1}{4}(c_1 - c_2)((f(X) \wedge Y) + X \wedge f(Y))Z,
\]

\[
R^N(X, Y, N) = B(X, B^*(Y, N)) - B(Y, B^*(X, N)) + \frac{1}{4}(c_1 + c_2)(h(X) \wedge h(Y))N,
\]

\[
(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) = \frac{1}{4}(c_1 + c_2)((f(Y), Z)h(X) - (f(X), Z)h(Y))
\]

\[
- \frac{1}{4}(c_1 - c_2)((Y, Z)h(X) - \langle X, Z \rangle h(Y))
\]

where \( \nabla \) stands for the natural connection on \( T^* M \otimes T^* M \otimes E \). Moreover, if we use the same symbol \( \nabla \) to denote the natural connections on \( T^* M \otimes TM, T^* M \otimes E, E^* \otimes TM \) and \( E^* \otimes E \), we have, for \( X, Y \in TM \) and \( Z \in E \),

\[
(\nabla_X f)(X) = s(B(Y, X)) + B^*(Y, h(X)),
\]

\[
(\nabla_X h)(X) = t(B(Y, X)) - B(Y, f(X)),
\]

\[
(\nabla_X s)(Z) = -f(B^*(Y, Z)) + B^*(Y, t(Z)),
\]

\[
(\nabla_X t)(Z) = -h(B^*(Y, Z)) - B(Y, s(Z)).
\]
Equations (13), (14) and (15) are respectively the equations of Gauss, Ricci and Codazzi. Equations (16)-(19) are additional equations traducing that \( \Phi \) commutes with the product structures \( P \) and \( P' \), with \( P' \) parallel in \( \mathbb{R}^{m+n+2} \). All these equations are necessary for the existence of an immersion \( M \rightarrow S_1^m \times S_2^n \) with second fundamental form \( B \) and normal connection \( \nabla' \). It appears that they are also sufficient:

**Theorem 2.** Let us assume that \( B : TM \times TM \rightarrow E \) is symmetric and such that the Gauss, Ricci and Codazzi Equations (13), (14) and (15) hold together with (16)-(19). Let us moreover suppose that \( \dim \text{Ker}(P - id) = m \) and \( \dim \text{Ker}(P + id) = n \). Then there exists \( \varphi \in \Gamma(U \Sigma) \) solution of (7) such that the map

\[
\Phi : TM \oplus E \rightarrow \mathbb{R}^{m+n+2}, \quad Z \mapsto (\langle Z \cdot \varphi, \varphi \rangle)
\]

commutes with the product structures \( P \) and \( P' \). The spinor field \( \varphi \) is moreover unique up to the natural right action of \( \text{Spin}(m+1) \cdot \text{Spin}(n+1) \) on \( U \Sigma \). In particular, there is an isometric immersion \( F : M \rightarrow S_1^m \times S_2^n \) and a bundle isomorphism \( \Phi : TM \oplus E \rightarrow T(S_1^m \times S_2^n) \) above \( F \) identifying \( E, B \) and \( \nabla' \) to the normal bundle, the second fundamental form and the normal connection of \( F \) in \( S_1^m \times S_2^n \). The immersion is moreover unique up to the natural action of \( \text{SO}(m+1) \times \text{SO}(n+1) \) on \( S_1^m \times S_2^n \).

Section 1.4.2 is devoted to the proof of Proposition 1.4, and Section 1.4.3 to the proof of Theorem 2.

1.4.2. **Proof of Proposition 1.4.** We assume that \( X, Y \in \Gamma(TM) \) are such that \( \nabla X = \nabla Y = 0 \) at the point where we do the computations. A direct computation using (7) twice yields

\[
R_{XY} \varphi = \nabla^2_{X,Y} \varphi - \nabla^2_{Y,X} \varphi
= \frac{1}{2} \left( (\nabla Y B)(X) - (\nabla X B)(Y) \right) \cdot \varphi + \frac{1}{4} \left( B(Y) \cdot B(X) - B(X) \cdot B(Y) \right) \cdot \varphi
+ \frac{1}{4} \left( c_1(Y_1 \cdot X_1 - X_1 \cdot Y_1) + c_2(Y_2 \cdot X_2 - X_2 \cdot Y_2) \right) \cdot \varphi + (D + E) \cdot \varphi
\]

with

\[
D = \frac{\sqrt{c_1}}{4} \left( B(X) \cdot Y_1 \cdot \nu_1 - Y_1 \cdot \nu_1 \cdot B(X) - B(Y) \cdot X_1 \cdot \nu_1 + X_1 \cdot \nu_1 \cdot B(Y) \right)
+ \frac{\sqrt{c_2}}{4} \left( B(X) \cdot Y_2 \cdot \nu_2 - Y_2 \cdot \nu_2 \cdot B(X) - B(Y) \cdot X_2 \cdot \nu_2 + X_2 \cdot \nu_2 \cdot B(Y) \right)
\]

and

\[
E = \frac{1}{2} \left( \sqrt{c_1} (\nabla X Y_1 - \nabla Y X_1) \cdot \nu_1 + \sqrt{c_2} (\nabla X Y_2 - \nabla Y X_2) \cdot \nu_2 \right).
\]

**Lemma 1.5.** In local orthonormal frames \( \{e_j\}_{1 \leq j \leq p} \) of \( TM \) and \( \{n_r\}_{1 \leq r \leq q} \) of \( E \) we have

\[
A = \frac{1}{2} \sum_{j=1}^{p} e_j \cdot \left( (\nabla Y B)(X, e_j) - (\nabla X B)(Y, e_j) \right)
\]
These expressions respectively mean that $A \in TM \otimes E$ represents the transformation 

\[(22) \quad B = \frac{1}{2} \sum_{1 \leq j < k \leq p} \langle [B^*(X, B(Y, e_j)), e_k] - \langle B^*(Y, B(X, e_j)), e_k \rangle \rangle e_j \cdot e_k \\
+ \frac{1}{2} \sum_{1 \leq r < s \leq q} \langle \langle B(X, B^*(Y, n_r)), n_s \rangle - \langle B(Y, B^*(X, n_r)), n_s \rangle \rangle n_r \cdot n_s, \]

\[(23) \quad C = -\frac{1}{8} (c_1 + c_2) \sum_{1 \leq j < k \leq p} \langle (X \wedge Y + f(X) \wedge f(Y)) e_j, e_k \rangle e_j \cdot e_k \\
- \frac{1}{8} (c_1 + c_2) \sum_{1 \leq r < s \leq q} \langle (h(X) \wedge h(Y)) n_r, n_s \rangle n_r \cdot n_s \\
+ \frac{1}{8} (c_1 + c_2) \sum_{j = 1}^{p} e_j \cdot ((f(Y), e_j) h(X) - (f(X), e_j) h(Y)) \\
- \frac{1}{8} (c_1 - c_2) \sum_{1 \leq j < k \leq p} \langle (f(X) \wedge Y + X \wedge f(Y)) e_j, e_k \rangle e_j \cdot e_k \\
+ \frac{1}{8} (c_1 - c_2) \sum_{j = 1}^{p} e_j \cdot (\langle Y, e_j \rangle h(X) - \langle X, e_j \rangle h(Y)).\]

These expressions respectively mean that $A \in TM \otimes E$ represents the transformation 

\[(24) \quad Z \in TM \mapsto \tilde{\nabla}_Y B(X, Z) - \tilde{\nabla}_X B(Y, Z) \in E, \]

$B \in \Lambda^2 TM \oplus \Lambda^2 E$ represents the transformation 

\[(25) \quad Z \in TM \mapsto B^*(X, B(Y, Z)) - B^*(Y, B(X, Z)) \in TM \]

together with 

\[(26) \quad N \in E \mapsto B(X, B^*(Y, N)) - B(Y, B^*(X, N)) \in E \]

and $C \in \Lambda^2 TM \oplus \Lambda^2 E \oplus TM \otimes E$ represents 

\[(27) \quad -\frac{1}{4} (c_1 + c_2) (X \wedge Y + f(X) \wedge f(Y)) \in \text{End}(TM), \]

\[(28) \quad -\frac{1}{4} (c_1 + c_2) (h(X) \wedge h(Y)) \in \text{End}(E), \]

\[(29) \quad Z \in TM \mapsto \frac{1}{4} (c_1 + c_2) (\langle f(Y), Z \rangle h(X) - \langle f(X), Z \rangle h(Y)) \in E, \]

\[(30) \quad -\frac{1}{4} (c_1 - c_2) (f(X) \wedge Y + X \wedge f(Y)) \in \text{End}(TM) \]

and 

\[(31) \quad Z \in TM \mapsto \frac{1}{4} (c_1 - c_2) (\langle Y, Z \rangle h(X) - \langle X, Z \rangle h(Y)) \in E. \]

**Proof.** The expression (21) directly follows from the definition (4) of $B(X)$. For (22) we refer to [6, Lemma 5.2] where a similar computation is carried out. By Lemma A.1, formula (95), $A$ represents the transformation (24) and $B$ the transformations (25) and (26). We now prove (23). Using 

\[X_1 = \frac{1}{2} (X + f(X) + h(X)) \quad \text{and} \quad X_2 = \frac{1}{2} (X - f(X) - h(X)) \]
and the analogous expressions for $Y$, straightforward computations yield

\begin{equation}
(32) \quad C = \frac{1}{16} (c_1 + c_2) \{ (Y \cdot X - X \cdot Y) + (f(Y) \cdot f(X) - f(X) \cdot f(Y)) \}
\end{equation}

\begin{equation}
(33) \quad + \frac{1}{16} (c_1 + c_2) \{ (h(Y) \cdot h(X) - h(X) \cdot h(Y)) \}
\end{equation}

\begin{equation}
(34) \quad + \frac{1}{8} (c_1 + c_2) (f(Y) \cdot h(X) - f(X) \cdot h(Y))
\end{equation}

\begin{equation}
(35) \quad + \frac{1}{16} (c_1 - c_2) \{ Y \cdot f(X) - f(X) \cdot Y - X \cdot f(Y) + f(Y) \cdot X \}
\end{equation}

\begin{equation}
(36) \quad + \frac{1}{8} (c_1 - c_2) \{ Y \cdot h(X) - X \cdot h(Y) \}.
\end{equation}

By Lemma A.3 in the Appendix, the right hand terms (32) and (33) represent the transformations (27) and (28); the term (34) represents the transformation (29) since the commutator in the Clifford bundle

$$\alpha := \left[ \frac{1}{4} (f(Y) \cdot h(X) - f(X) \cdot h(Y)), Z \right]$$

is

\begin{align*}
\alpha &= \frac{1}{4} (f(Y) \cdot h(X) - f(X) \cdot h(Y)) \cdot Z - Z \cdot \frac{1}{4} (f(Y) \cdot h(X) - f(X) \cdot h(Y)) \\
&= -\frac{1}{4} (f(Y) \cdot Z + Z \cdot f(Y)) \cdot h(X) + \frac{1}{4} (f(X) \cdot Z + Z \cdot f(X)) \cdot h(Y) \\
&= \frac{1}{2} (\langle f(Y), Z \rangle h(X) - \langle f(X), Z \rangle h(Y)) ;
\end{align*}

similarly, the terms (35) and (36) represent the transformations (30) and (31).

Formula (23) then follows from Lemmas A.1 and A.2 in the Appendix. \qed

The curvature tensor of the spinorial connection on $TM \oplus E$ is given by

\begin{equation}
(37) \quad R_{XY}\varphi = \left( \frac{1}{2} \sum_{1 \leq j < k \leq p} \langle R^T(X,Y)e_j, e_k \rangle e_j \cdot e_k \right) \cdot \varphi \\
\quad + \left( \frac{1}{2} \sum_{1 \leq r < s \leq q} \langle R^N(X,Y)n_r, n_s \rangle n_r \cdot n_s \right) \cdot \varphi.
\end{equation}

Comparing Equations (20) and (37) and since $\varphi$ is represented in a frame $\tilde{Q}$ by an element of $\text{Spin}(m + n + 2)$, invertible in $\text{Cl}(m + n + 2)$, we deduce that

\begin{equation}
(38) \quad A + B + C = \frac{1}{2} \sum_{1 \leq j < k \leq p} \langle R^T(X,Y)e_j, e_k \rangle e_j \cdot e_k \\
\quad + \frac{1}{2} \sum_{1 \leq r < s \leq q} \langle R^N(X,Y)n_r, n_s \rangle n_r \cdot n_s
\end{equation}

and

\begin{equation}
(39) \quad D + E = 0.
\end{equation}

Now the right hand side of (38) represents the transformations $Z \in TM \mapsto R^T(X,Y)Z \in TM$ and $N \in E \mapsto R^N(X,Y)N \in E$. The equations (13)-(15) of Gauss, Ricci and Codazzi follow from this and Lemma 1.5. Let us now prove that Equations (16)-(19) are consequences of the fact that $\Phi$ commutes with the product
structures \( \mathcal{P} \) and \( \mathcal{P}' \), Equation (12), where the product structure \( \mathcal{P}' \) is parallel. We have by (12)
\[
\mathcal{P}'(\partial_Y \Phi(X)) = \partial_Y (\Phi(\mathcal{P}(X))) .
\]
Assuming that \( \nabla X = 0 \) at the point where we do the computations and recalling (11) we have
\[
\partial_Y \Phi(X) = -\sum_{i=1,2} \sqrt{c_i} \langle Y_i, X \rangle N_i + \Phi(B(Y, X_T) - B^*(Y, X_N))
\]
and the left hand side of (40) is given by
\[
\mathcal{P}'(\partial_Y \Phi(X)) = -\sum_{i=1,2} \sqrt{c_i} \langle Y_i, X \rangle \mathcal{P}'(N_i) + \Phi(\mathcal{P}(B(Y, X_T)) - \mathcal{P}(B^*(Y, X_N)));
\]
by (11) again, the right hand side of (40) is given by
\[
\partial_Y (\Phi(\mathcal{P}(X))) = \Phi(\nabla_Y \mathcal{P}(X)) - \sum_{i=1,2} \sqrt{c_i} \langle Y_i, \mathcal{P}(X) \rangle N_i + \Phi(B(Y, \mathcal{P}(X)_T) - B^*(Y, \mathcal{P}(X)_N))
\]
and since for \( i = 1, 2 \)
\[
\langle Y_i, \mathcal{P}(X) \rangle N_i = (\mathcal{P}(Y_i), X) N_i = (-1)^i + 1 \langle Y_i, X \rangle N_i = (Y_i, X) \mathcal{P}'(N_i)
\]
we deduce that for \( X \in TM \)
\[
\Phi(\mathcal{P}(B(Y, X))) = \Phi(\nabla_Y \mathcal{P}(X)) + \Phi(B(Y, f(X)) - B^*(Y, h(X))
\]
and for \( X \in E \)
\[
-\Phi(\mathcal{P}(B^*(Y, X))) = \Phi(\nabla_Y \mathcal{P}(X)) + \Phi(B(Y, s(X)) - B^*(Y, t(X))).
\]
Using that \( \Phi \) is injective on the fibers and decomposing \( \nabla_Y \mathcal{P}(X), \mathcal{P}(B(Y, X)) \) and \( \mathcal{P}(B^*(Y, X)) \) in their tangent and normal parts, we get
\[
(\tilde{\nabla}_Y f)(X) + (\tilde{\nabla}_Y h)(X) = s(B(Y, X)) + t(B(Y, X)) - B(Y, f(X)) + B^*(Y, h(X))
\]
if \( X \in TM \), and
\[
(\tilde{\nabla}_Y s)(X) + (\tilde{\nabla}_Y t)(X) = -f(B^*(Y, X)) - h(B^*(Y, X)) - B(Y, s(X)) + B^*(Y, t(X))
\]
if \( X \in E \). Finally, taking the tangent and the normal parts of each one of the last two equations we get (16)-(19).

**Remark 2.** Eq. (39) is in fact equivalent to the antisymmetric part of (16)-(19).

### 1.4.3. Proof of Theorem 2

Let us set, for \( X \in TM \) and \( \varphi \in \Gamma(US) \),
\[
\nabla'_X \varphi := \nabla_X \varphi - \frac{1}{2} (\sqrt{c_1} X_1 \cdot \nu_1 + \sqrt{c_2} X_2 \cdot \nu_2 - B(X)) \cdot \varphi .
\]
We consider \( US \to M \) as a principal bundle of group \( Spin(p + q + 2) \), where the action is the multiplication on the right
\[
\varphi = [\tilde{x}, [\varphi]] \mapsto \varphi \cdot a := [\tilde{x}, [\varphi] \cdot a]
\]
for all \( a \in Spin(p + q + 2) \). The connection \( \nabla' \) may be considered as given by a connection 1-form on this principal bundle, since so is \( \nabla \) and the term
\[
\mathcal{X}(\varphi) := \frac{1}{2} (\sqrt{c_1} X_1 \cdot \nu_1 + \sqrt{c_2} X_2 \cdot \nu_2 - B(X)) \cdot \varphi
\]
defines a vertical and invariant vector field on $U\Sigma$. The compatibility equations (13)-(19) imply that this connection is flat (the computations are similar to the computations in the previous section). Since it is flat and assuming moreover that $M$ is simply connected, the principal bundle $U\Sigma \to M$ has a global parallel section: this yields $\varphi \in \Gamma(U\Sigma)$ such that $\nabla \varphi = 0$, i.e. a non-trivial solution of (7). Let us verify that equations (16)-(19) imply that the map
\[
\Phi : TM \oplus E \to \mathbb{R}^{m+n+2} = \mathbb{R}^{m+1} \times \mathbb{R}^{n+1}, \quad X \mapsto \langle \langle X \cdot \varphi, \varphi \rangle \rangle
\]
is compatible with the product structures, i.e. verifies $\Phi(\mathcal{P}(X)) = \mathcal{P}'(\Phi(X))$ for all $X \in TM \oplus E$. The sum of (16) and (17) gives, for $X, Y \in TM$,
\[
\nabla_Y \mathcal{P}(X) = \mathcal{P}(B(Y, X) - B(Y, f(X)) + B^*(Y, h(X)).
\]
Similarly, for $X \in E$ and $Y \in TM$, (18) and (19) imply that
\[
\nabla_Y \mathcal{P}(X) = -\mathcal{P}(B^*(Y, X)) - B(Y, s(X)) + B^*(Y, t(X)).
\]
As in the proof of Theorem 1, $\Phi$ is a bundle map above the immersion
\[
F : M \to \mathbb{R}^{m+1} \times \mathbb{R}^{n+1} \quad \quad p \mapsto \left( \frac{1}{\sqrt{c_1}}(\langle \nu_1 \cdot \varphi, \varphi \rangle), \frac{1}{\sqrt{c_2}}(\langle \nu_2 \cdot \varphi, \varphi \rangle) \right).
\]
The product structure $\mathcal{P}$ on $TM \oplus E$ extends to a product structure $\tilde{\mathcal{P}}$ on $TM \oplus E \oplus \mathcal{E}_2$ by setting $\tilde{\mathcal{P}}(\nu_1) = \nu_1$ and $\tilde{\mathcal{P}}(\nu_2) = -\nu_2$. Let us consider the trivial connection $\partial$ induced on $TM \oplus E \oplus \mathcal{E}_2$ by the bundle isomorphism
\[
\tilde{\Phi} : TM \oplus E \oplus \mathcal{E}_2 \to F^* T \mathbb{R}^{m+n+2}.
\]

**Lemma 1.6.** We have for $Y \in \Gamma(TM \oplus E)$ and $X \in TM$
\[
(\partial - \nabla)_X Y = -\sum_{i=1,2} \sqrt{c_i} \langle X_i, Y \rangle \nu_i + B(X, Y_f) - B^*(X, Y_N)
\]
and $\partial_X \nu_1 = X_1, i = 1, 2$.

**Proof.** Assuming that $\nabla_X Y = 0$ at the point where we do the computations, we have by definition $\langle \partial - \nabla \rangle_X Y = \tilde{\Phi}^{-1}(\partial_X \Phi(Y))$, and the formula is a consequence of (11). Finally, $\partial_X \nu_1 = \partial_X \nu_1 = \Phi^{-1}(\partial_X N_1) = X_1$. □

**Lemma 1.7.** The product structure $\tilde{\mathcal{P}}$ is parallel with respect to $\partial$.

**Proof.** Using (45) twice, for $X, Y$ tangent to $M$ we have
\[
(\partial - \nabla)_X \mathcal{P}(Y) = (\partial - \nabla)_X (\mathcal{P}(Y)) - \mathcal{P}(\partial - \nabla)_X Y
\]
\[
= B(X, f(Y)) - B^*(X, h(Y)) - \tilde{\mathcal{P}}(B(X, Y))
\]
since $\langle Y_1, \tilde{\mathcal{P}}(X) \rangle = \langle \tilde{\mathcal{P}}(Y_1), \tilde{\mathcal{P}}(X) \rangle = \langle Y_1, X \rangle$ and $\langle Y_2, \tilde{\mathcal{P}}(X) \rangle = -\langle \tilde{\mathcal{P}}(Y_2), \tilde{\mathcal{P}}(X) \rangle = -\langle Y_2, X \rangle$, and we conclude with (43) that $(\partial_X \tilde{\mathcal{P}})(Y) = 0$. The computation for $Y \in \Gamma(E)$ is analogous. For $Y = \tilde{\nu}_1$ we have
\[
(\partial - \nabla)_X \mathcal{P}(\nu_1) = (\partial - \nabla)_X (\mathcal{P}(\nu_1)) - \mathcal{P}(\partial - \nabla)_X \nu_1
\]
\[
= (\partial - \nabla)_X \nu_1 - \mathcal{P}(X_1) = X_1 - \mathcal{P}(X_1) = 0
\]
which implies that $(\partial_X \tilde{\mathcal{P}})(\nu_1) = 0$ since
\[
(\nabla_X \tilde{\mathcal{P}})(\nu_1) = \nabla_X (\mathcal{P}(\nu_1)) - \tilde{\mathcal{P}}(\nabla_X \nu_1) = \nabla_X \nu_1 - \tilde{\mathcal{P}}(\nabla_X \nu_1) = 0.
\]
The computations for $X = \nu_2$ are analogous. 

Since $\tilde{\mathcal{P}}$ and $\mathcal{P}'|_M$ are parallel sections of endomorphisms of $TM \oplus E \oplus \mathcal{E}_2 \cong F^s(T\mathbb{R}^{m+n+2})$ and since $id + \tilde{\mathcal{P}}$ and $id - \tilde{\mathcal{P}}$ have rank $m+1$ and $n+1$, there exists $A \in O(m+n+2)$ such that

$$A \circ \tilde{\Phi} \circ \tilde{\mathcal{P}} \circ \tilde{\Phi}^{-1} \circ A^{-1} = \mathcal{P}'$$

on $F^s(T\mathbb{R}^{n+m+2})$. We consider $a \in Spin(m+n+2)$ such that $Ad(a) = A^{-1}$ and the spinor field $\varphi' := \varphi \cdot a \in U\Sigma :$ it is still a solution of (7) and $\tilde{\Phi}'(X) := \langle\langle (X \cdot \varphi', \varphi') \rangle \rangle$ is such that

$$\tilde{\Phi}'(X) = \tau[\varphi'][X][\varphi'] = \tau[\varphi \cdot a][X][\varphi \cdot a]$$

$$= a^{-1}[\varphi]^{-1}[X][\varphi] a = Ad(a^{-1})(\tilde{\Phi}(X)) = A \circ \tilde{\Phi}(X).$$

The map $\tilde{\Phi}'$ thus satisfies $\tilde{\Phi}' \circ \tilde{\mathcal{P}} \circ \tilde{\Phi}'^{-1} = \mathcal{P}'$ which implies that $\Phi' : TM \oplus E \to F^sT(S^{m_1}_1 \times S^{m_2}_2)$ is compatible with the product structures $\mathcal{P}$ and $\mathcal{P}'$. Finally, it is clear from the proof that if a solution $\varphi$ of (7) is such that $\tilde{\Phi} : X \mapsto \langle\langle X \cdot \varphi, \varphi \rangle \rangle$ commutes with the product structures, then the other solutions of (7) satisfying this property are of the form $\varphi \cdot a$ with $a \in Spin(m+n+2)$ such that $Ad(a)$ belongs to $SO(m+1) \times SO(n+1)$, i.e. with $a \in Spin(m+1) \cdot Spin(n+1)$.

2. Isometric immersions in $S^{m}_1 \times \mathbb{R}^n$

We now consider immersions in $S^{m}_1 \times \mathbb{R}^n$ where $S^{m}_1$ is a $m$-dimensional sphere of curvature $c_1 > 0$. After the statement of the main theorem in Section 2.1, we study the special cases $S^2 \times \mathbb{R}$ and $S^2 \times \mathbb{R}^2$ in Sections 2.2 and 2.3.

In that section $M$ still denotes a $p$-dimensional riemannian manifold and $E \to M$ a metric bundle of rank $q$ with $p+q = m+n$, equipped with a connection compatible with the metric. We consider here the trivial bundle $\mathcal{E}_1 := M \times \mathbb{R} \to M$, with its natural metric and the trivial connection, and fix a unit parallel section $\nu_1$ of $\mathcal{E}_1$. We finally consider the representation associated to the splitting $\mathbb{R}^{m+n+1} = \mathbb{R}^p \oplus \mathbb{R}^q \oplus \mathbb{R}$ $\rho : Spin(p) \times Spin(q) \to Spin(p) \cdot Spin(q) \subset Spin(m+n+1) \to Aut(Cl(m+n+1))$

(the last map is given by the left multiplication) and the bundles (associated to a spin structure $\tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E$ of $TM$ and $E$)

$$\Sigma := \tilde{Q} \times_\rho Cl(m+n+1), \quad U\Sigma := \tilde{Q} \times_\rho Spin(m+n+1)$$

and

$$Cl(TM \oplus E \oplus \mathcal{E}_1) := \tilde{Q} \times_{Ad} Cl(m+n+1).$$

We finally suppose that a product structure $\mathcal{P}$ is given on $TM \oplus E$ as in Section 1.3.

2.1. Statement of the theorem.

**Theorem 3.** We suppose that $M$ is simply connected. Let $B : TM \times TM \to E$ be a symmetric tensor. The following statements are equivalent:
(i) There exist an isometric immersion \( F : M \to S^m_1 \times \mathbb{R}^n \) and a bundle map \( \Phi : TM \oplus E \to T\mathbb{S}^m_1 \times \mathbb{R}^n \) above \( F \) such that \( \Phi(X,0) = dF(X) \) for all \( X \in TM \), which preserves the bundle metrics, maps the connection on \( E \) and the tensor \( B \) to the normal connection and the second fundamental form of \( F \), and is compatible with the product structures.

(ii) There exists a section \( \varphi \in \Gamma(U\Sigma) \) solution of

\[
\nabla_X \varphi = \frac{1}{2} \sqrt{c_1} X_1 \cdot \nu_1 \cdot \varphi - \frac{1}{2} B(X) \cdot \varphi
\]

for all \( X \in TM \), where \( X = X_1 + X_2 \) is the decomposition in the product structure \( \mathcal{P} \) of \( TM \oplus E \), such that the map

\[
Z \in TM \oplus E \mapsto \langle\langle Z \cdot \varphi, \varphi \rangle\rangle \in \mathbb{R}^{m+1} \times \mathbb{R}^n
\]

commutes with the product structures \( \mathcal{P} \) and \( \mathcal{P}' \).

Moreover, the bundle map \( \Phi \) and the immersion \( F \) are explicitly given in terms of the spinor field \( \varphi \) by the formulas

\[
\Phi : TM \oplus E \to T\mathbb{S}^m_1 \times \mathbb{R}^n, \ Z \mapsto \langle\langle Z \cdot \varphi, \varphi \rangle\rangle
\]

and \( F = (F_1, F_2) \in S^m_1 \times \mathbb{R}^n \) with

\[
F_1 = \frac{1}{\sqrt{c_1}} \langle\langle \nu_1 \cdot \varphi, \varphi \rangle\rangle.
\]

Brief indications of the proof: setting

\[
\Phi_2(X) = \langle\langle X_2 \cdot \varphi, \varphi \rangle\rangle \quad \text{and} \quad F_2 = \int \Phi_2
\]

and using (16)-(19) it is not difficult to see that \( \Phi_2 \) is a closed 1-form and \( F_2 \) is well defined if \( M \) is simply connected. Formulas (47) and (48) thus give an explicit expression for \( F = (F_1, F_2) \) in terms of the spinor field \( \varphi \), and the theorem may then be proved by direct computations as in the previous sections.

Here again, as in the case of a product of two spheres, we can obtain a spinorial proof of the fundamental theorem of immersions theory in \( S^m_1 \times \mathbb{R}^n \).

2.2. Surfaces in \( S^2 \times \mathbb{R} \). The aim is to recover the spinorial characterization of an immersion in \( S^2 \times \mathbb{R} \) given in [15]. Let us consider \( \Sigma_0 = Q \times_{\rho} Cl^0(4) \). If \( e_0^0, e_1^0, e_2^0, e_3^0 \) is an orthonormal basis of \( \mathbb{R}^4 \), where \( e_0^0 \) belongs to the second factor of \( S^2 \times \mathbb{R} \), we set \( \omega := -e_0^0 \cdot e_1^0 \cdot e_2^0 \cdot e_3^0 \), consider the two ideals \( I_1 := Cl^0(4) \cdot \frac{1}{2} (1 - \omega) \) and \( I_2 := Cl^0(4) \cdot \frac{1}{2} (1 + \omega) \) of \( Cl^0(4) \) and the splitting \( Cl^0(4) = I_1 \oplus I_2 \). It induces a decomposition

\[
\varphi = \varphi_1 + \varphi_2 \quad \in \quad \Sigma_1 \oplus \Sigma_2
\]

with \( \Sigma_1 = Q \times_{\rho} I_1 \) and \( \Sigma_2 = Q \times_{\rho} I_2 \). Let us consider the map

\[
u : \Sigma_2 \to \Sigma_1, \quad \varphi_2 \mapsto u(\varphi_2) = -\nu_1 \cdot \varphi_2 \cdot e_0^0
\]

to identify \( \Sigma_2 \) with \( \Sigma_1 \), and an identification

\[
\Sigma M \otimes \Sigma E \to \Sigma_1, \quad \psi \mapsto \psi^*
\]
such that \((X \cdot \psi)^* = X \cdot \nu_1 \cdot (\psi)^*\) for all \(X \in TM \oplus E\) and \(\psi \in \Sigma M \otimes \Sigma E\). We set \(\psi_1, \psi_2 \in \Sigma M \otimes \Sigma E\) such that \(\psi_1^* = \varphi_1\) and \(\psi_2^* = u(\varphi_2)\). Since \(\varphi_1\) and \(\varphi_2\) are both normalized solutions of

\[
\nabla_X \varphi = \frac{1}{2} X_1 \cdot \nu_1 \cdot \varphi - \frac{1}{2} S(X) \cdot N \cdot \varphi
\]

where \(N\) is a unit normal and \(S : TM \to TM\) is the corresponding shape operator of \(M\) in \(\mathbb{S}^2 \times \mathbb{R}\), \(\psi_1\) and \(\psi_2\) are so that \((50)\)

\[
\nabla_X \psi_1 = \frac{1}{2} X_1 \cdot \psi_1 - \frac{1}{2} S(X) \cdot N \cdot \psi_1
\]

and

\[
\nabla_X \psi_2 = -\frac{1}{2} X_1 \cdot \psi_2 - \frac{1}{2} S(X) \cdot N \cdot \psi_2
\]

with \(|\psi_1| = |\psi_2| = 1\). Now the condition expressing that \(\Phi\) commutes with the product structures gives the following:

**Lemma 2.1.** For a convenient choice of the unit section \(V \in \Gamma(TM \oplus E)\) generating the distinguished line \(P_2\) of the product structure \(P\) of \(TM \oplus E\), we have

\[
V \cdot \psi_1 = \psi_2.
\]

**Proof.** Choosing \(V \in P_2\) so that \(\Phi(V) = e_0^\nu\), we have

\[
\Phi(V) = \langle (V \cdot \varphi, \varphi) \rangle = \tau[V][\varphi] = e_0^\nu,
\]

that is \([V][\varphi] = [\varphi]e_0^\nu\). Writing \(\varphi = [\varphi_1] + [\varphi_2] \in I_1 \oplus I_2\) and since the right-multiplication by \(e_0^\nu\) exchanges the ideals \(I_1\) and \(I_2\) (since \(\omega \cdot e_0^\nu = -e_0^\nu \cdot \omega\), we deduce that \([V][\varphi_1] = [\varphi_2]e_0^\nu\) and \([V][\varphi_2] = [\varphi_1]e_0^\nu\). We thus have \([V][\nu_1][\varphi_1] = -[\nu_1][\varphi_2]e_0^\nu\) that is \(V \cdot \nu_1 \cdot \varphi_1 = -\nu_1 \cdot \nu_2 \cdot e_0^\nu\), which readily implies (52).

Equations (50) and (51) and the lemma imply that \(\psi_1\) and \(\psi_2\) satisfy

\[
\nabla_X \psi_1 = -\frac{1}{2} X_1 \cdot V \cdot \psi_2 - \frac{1}{2} S(X) \cdot N \cdot \psi_1.
\]

and

\[
\nabla_X \psi_2 = -\frac{1}{2} X_1 \cdot V \cdot \psi_1 - \frac{1}{2} S(X) \cdot N \cdot \psi_2.
\]

The spinor field \(\psi := \psi_1 - \psi_2 \in \Sigma M \otimes \Sigma E\) is a solution of

\[
\nabla_X \psi = \frac{1}{2} X_1 \cdot V \cdot \psi - \frac{1}{2} S(X) \cdot N \cdot \psi.
\]

By (52), we have \(\langle \psi_1, \psi_2 \rangle = \langle \psi_1, V \cdot \psi_1 \rangle = -\langle V \cdot \psi_1, \psi_1 \rangle = -\langle \psi_2, \psi_1 \rangle = 0\), that is \(\psi_1\) and \(\psi_2\) are orthogonal in \(\Sigma M \otimes \Sigma E\), which implies that \(|\psi| = \sqrt{2}\). Finally, since \(E = \mathbb{R}N\) there is an identification

\[
\Sigma M \to \Sigma M \otimes \Sigma E, \quad \psi \mapsto \psi^*
\]

such that \((X \cdot \psi)^* = X \cdot N \cdot (\psi)^*\). Using that \(X_1 = X - (X, V)V\) and \(V = T + fN\), we readily get from (53) that

\[
\nabla_X \psi = \frac{1}{2} X \cdot T \cdot \psi + \frac{1}{2} fX \cdot \psi + \frac{1}{2} (X, T)\psi - \frac{1}{2} S(X) \cdot \psi.
\]

This is the spinorial characterization of an immersion in \(\mathbb{S}^2 \times \mathbb{R}\) obtained in [15].
Remark 3. Similarly, it is possible to obtain as a consequence of Theorem 3 the characterizations in terms of usual spinor fields of immersions of surfaces or hypersurfaces in $\mathbb{S}^3 \times \mathbb{R}$, or of surfaces in $\mathbb{S}^2 \times \mathbb{R}^2$, obtained in [12, 16]. We rather focus below on the new case of hypersurfaces in $\mathbb{S}^2 \times \mathbb{R}^2$.

2.3. Hypersurfaces in $\mathbb{S}^2 \times \mathbb{R}^2$. Let us assume that $M$ is a 3-dimensional manifold. The aim is to obtain the characterization of an immersion of $M$ in $\mathbb{S}^2 \times \mathbb{R}^2$ in terms of usual spinor fields. Suppose that $\varphi \in \Gamma(U\Sigma)$ represents the immersion of $M$ in $\mathbb{S}^2 \times \mathbb{R}^2$, as in Theorem 3 (with $m = 2$, $n = 2$). Let us set $\Sigma_0 = \overline{Q} \times_\rho Cl^0(5)$. If $e^0_0, e^1_0, e^2_0, e^3_0, e^4_0$ is an orthonormal basis of $\mathbb{R}^5$, where $e^0_0, e^1_0$ is a basis of the second factor of $\mathbb{S}^2 \times \mathbb{R}^2$, we set $\omega := -e^0_0 \cdot e^1_0 \cdot e^2_0 \cdot e^3_0$, consider the two ideals $I_1 := Cl^0(5) \cdot \frac{1}{2} (1 - \omega)$ and $I_2 := Cl^0(5) \cdot \frac{1}{2} (1 + \omega)$ of $Cl^0(5)$ and the splitting $Cl^0(5) = I_1 \oplus I_2$. It induces a decomposition

$$\varphi = \varphi_1 + \varphi_2 \in \Sigma_1 \oplus \Sigma_2$$

with $\Sigma_1 = \overline{Q} \times_\rho I_1$ and $\Sigma_2 = \overline{Q} \times_\rho I_2$. As in the previous section we consider the map

$$u : \Sigma_2 \rightarrow \Sigma_1, \quad \varphi_2 \mapsto u(\varphi_2) = -\nu_1 \cdot \varphi_2 \cdot e^0_0$$

to identify $\Sigma_2$ with $\Sigma_1$, and for

$$\Sigma'_1 := \overline{Q} \times_\rho Cl(4) \cdot \frac{1}{2} (1 - \omega),$$

an identification

$$\Sigma'_1 \rightarrow \Sigma_1, \quad \psi \mapsto \psi^*$$

such that $(X \cdot \psi)^* = X \cdot \nu_1 \cdot \psi^*$ for all $X \in TM \oplus E$ and $\psi \in \Sigma'_1$. Let us set $\psi_1, \psi_2 \in \Gamma(\Sigma'_1)$ such that

$$\psi_1^* = \varphi_1 \quad \text{and} \quad \psi_2^* = u(\varphi_2).$$

They satisfy

$$\nabla_X \psi_1 = \frac{1}{2} X_1 \cdot \psi_1 - \frac{1}{2} B(X) \cdot \psi_1$$

and

$$\nabla_X \psi_2 = -\frac{1}{2} X_1 \cdot \psi_2 - \frac{1}{2} B(X) \cdot \psi_2.$$
$X_1 = X - \langle X, T_1 \rangle V_1 - \langle X, T_2 \rangle V_2$. Under the Clifford action of the volume element $-e_0 \cdot e_1 \cdot e_2 \cdot e_3 \in \text{Cl}(TM \oplus E)$ the bundle $\Sigma'_1$ splits into $\Sigma'_1 = \Sigma'_1^+ \oplus \Sigma'_1^-$. There is a $\mathbb{C}$-linear isomorphism
\[ \Sigma M \simeq \Sigma'_1^+ , \quad \Psi \mapsto \Psi^* \]
so that $(X \cdot \Psi)^* = X \cdot N \cdot \Psi^*$ for all $X \in TM$ and $\Psi \in \Sigma M$, where the complex structure on $\Sigma'_1^+$ is given by the right-action of $e_0^* \cdot e_1^*$. We write $\psi_1 = \psi_1^+ + \psi_1^-$ in $\Sigma'_1 = \Sigma'_1^+ \oplus \Sigma'_1^-$ and consider $\Psi_1, \Psi_2 \in \Gamma(\Sigma M)$ such that $\Psi_1^* = \psi_1^+$ and $\Psi_2^* = N \cdot \psi_1^-$. From (58) we have
\[ \nabla_X \Psi_1 = -\frac{1}{2} (X - \langle X, T_1 \rangle (T_1 - f_1) - \langle X, T_2 \rangle (T_2 - f_2)) \cdot \Psi_2 - \frac{1}{2} S(X) \cdot \Psi_1 \]
and
\[ \nabla_X \Psi_2 = -\frac{1}{2} (X - \langle X, T_1 \rangle (T_1 + f_1) - \langle X, T_2 \rangle (T_2 + f_2)) \cdot \Psi_1 + \frac{1}{2} S(X) \cdot \Psi_2, \]
then
\[ |\Psi_1|^2 + |\Psi_2|^2 = 1. \]
Moreover,
\[ (T_1 - f_1) \cdot (T_2 + f_2) \cdot \Psi_1 = i\Psi_1 \]
and
\[ (T_1 + f_1) \cdot (T_2 - f_2) \cdot \Psi_2 = i\Psi_2. \]

Conversely, the existence of two spinor fields $\Psi_1, \Psi_2 \in \Gamma(\Sigma M)$ solutions of (59)-(63) implies the existence of an isometric immersion of $M$ into $S^2 \times \mathbb{R}^2$; we may indeed construct $\varphi \in \Gamma(U\Sigma)$ solution of (46) from $\Psi_1$ and $\Psi_2$, just doing step by step the converse constructions; it is such that the map $\Phi : X \mapsto \langle (X \cdot \varphi, \varphi) \rangle$ commutes with the product structures.

**Remark 4.** Two non-trivial spinor fields $\Psi_1, \Psi_2 \in \Gamma(\Sigma M)$ solutions of (59)-(60) are in fact such that $|\Psi_1|^2 + |\Psi_2|^2$ is a constant, and may thus be supposed so that (61) holds.

### 3. Isometric immersions in $H^m_1 \times H^m_2$ and $H^m_1 \times \mathbb{R}^n$

We state here without proof the analogous results for immersions in $H^m_1 \times H^m_2$ and $H^m_1 \times \mathbb{R}^n$, where $H^m_1$ and $H^m_2$ are spaces of constant curvature $c_1, c_2 < 0$. Here $M$ still denotes a $p$-dimensional riemannian manifold and $E \to M$ a metric bundle of rank $q$ with a connection compatible with the metric and such that $p + q = m + n$. We suppose that a product structure $P$ is given on $TM \oplus E$ as in Section 1.3. We denote by $\mathbb{R}^{r,s}$ the space $\mathbb{R}^{r+s}$ with the metric with signature
\[ -\sum_{i=1}^{r} dx_i^2 + \sum_{j=r+1}^{r+s} dx_j^2, \]
by $\text{Cl}(r,s)$ its Clifford algebra and by $\text{Spin}(r,s)$ its spin group. For immersions in $H^m_1 \times H^m_2$ we consider the trivial bundle $E_2 := M \times \mathbb{R}^{2,0} \to M$, with the natural positive metric and the trivial connection, and two orthonormal and parallel sections $\nu_1, \nu_2$ of that bundle. We also consider the representation associated to the splitting $\mathbb{R}^{2,m+n} = \mathbb{R}^{2,0} \oplus \mathbb{R}^{0,p} \oplus \mathbb{R}^{0,q}$
\[ \rho : \text{Spin}(p) \times \text{Spin}(q) \to \text{Spin}(p) \cdot \text{Spin}(q) \subset \text{Spin}(2, m+n) \to \text{Aut}(\text{Cl}(2, m+n)) \]
and the bundles (associated to a spin structure \( \tilde{Q} := \tilde{Q}_M \times_M \tilde{Q}_E \) of \( TM \) and \( E \))

\[
\Sigma := \tilde{Q} \times_{\rho} \text{Cl}(2, m + n), \quad U\Sigma := \tilde{Q} \times_{\rho} \text{Spinc}(2, m + n)
\]

and

\[
\text{Cl}(TM \oplus E \oplus E_2) := \tilde{Q} \times_{\text{Ad}} \text{Cl}(2, m + n).
\]

**Theorem 4.** Let \( B : TM \times TM \to E \) be a symmetric tensor. The following statements are equivalent:

(i) There exist an isometric immersion \( F : M \to \mathbb{H}_1^m \times \mathbb{H}_2^n \) and a bundle map \( \Phi : TM \oplus E \to T(\mathbb{H}_1^m \times \mathbb{H}_2^n) \) above \( F \) such that \( \Phi(X, 0) = dF(X) \) for all \( X \in TM \), which preserves the bundle metrics, maps the connection on \( E \) and the tensor \( B \) to the normal connection and the second fundamental form of \( F \), and is compatible with the product structures.

(ii) There exists a section \( \varphi \in \Gamma(U\Sigma) \) solution of

\[
\nabla_X \varphi = -\frac{1}{2}(\sqrt{|c_1|} X_1 \cdot \nu_1 + \sqrt{|c_2|} X_2 \cdot \nu_2) \cdot \varphi - \frac{1}{2} B(X) \cdot \varphi
\]

for all \( X \in TM \), where \( X = X_1 + X_2 \) is the decomposition in the product structure \( P \) of \( TM \oplus E \), such that the map

\[
Z \in TM \oplus E \mapsto \langle\langle Z \cdot \varphi, \varphi \rangle\rangle \in \mathbb{R}^{1,m} \times \mathbb{R}^{1,n}
\]

commutes with the product structures \( P \) and \( P' \).

Moreover, the bundle map \( \Phi \) and the immersion \( F \) are explicitly given in terms of the spinor field \( \varphi \) by the formulas

\[
\Phi : TM \oplus E \to T(\mathbb{H}_1^m \times \mathbb{H}_2^n), \quad Z \mapsto \langle\langle Z \cdot \varphi, \varphi \rangle\rangle
\]

and

\[
F = \left( \frac{1}{\sqrt{|c_1|}} \langle\langle \nu_1 \cdot \varphi, \varphi \rangle\rangle, \frac{1}{\sqrt{|c_2|}} \langle\langle \nu_2 \cdot \varphi, \varphi \rangle\rangle \right) \in \mathbb{H}_1^m \times \mathbb{H}_2^n.
\]

For immersions in \( \mathbb{H}_1^m \times \mathbb{R}^n \) we consider \( E_1 := M \times \mathbb{R}^{1,0} \to M \), a parallel section \( \nu_1 \) of \( E_1 \) such that \( \langle\langle \nu_1, \nu_1 \rangle\rangle = -1 \) and the bundles

\[
\Sigma := \tilde{Q} \times_{\rho} \text{Cl}(1, m + n), \quad U\Sigma := \tilde{Q} \times_{\rho} \text{Spinc}(1, m + n)
\]

and

\[
\text{Cl}(TM \oplus E \oplus E_1) := \tilde{Q} \times_{\text{Ad}} \text{Cl}(1, m + n).
\]

**Theorem 5.** Let \( B : TM \times TM \to E \) be a symmetric tensor. The following statements are equivalent:

(i) There exist an isometric immersion \( F : M \to \mathbb{H}_1^m \times \mathbb{R}^n \) and a bundle map \( \Phi : TM \oplus E \to T\mathbb{H}_1^m \times \mathbb{R}^n \) above \( F \) such that \( \Phi(X, 0) = dF(X) \) for all \( X \in TM \), which preserves the bundle metrics, maps the connection on \( E \) and the tensor \( B \) to the normal connection and the second fundamental form of \( F \), and is compatible with the product structures.

(ii) There exists a section \( \varphi \in \Gamma(U\Sigma) \) solution of

\[
\nabla_X \varphi = -\frac{1}{2} \sqrt{|c_1|} X_1 \cdot \nu_1 \cdot \varphi - \frac{1}{2} B(X) \cdot \varphi
\]
for all \( X \in TM \), where \( X = X_1 + X_2 \) is the decomposition in the product structure \( P \) of \( TM \oplus E \), such that the map
\[
Z \in TM \oplus E \mapsto \langle\langle Z \cdot \varphi, \varphi \rangle\rangle \in \mathbb{R}^{1,m} \times \mathbb{R}^n
\]
commutes with the product structures \( P \) and \( P' \).
Moreover, the bundle map \( \Phi \) and the immersion \( F \) are explicitly given in terms of the spinor field \( \varphi \) by the formulas
\[
\Phi : TM \oplus E \rightarrow T \mathbb{H}^m \times \mathbb{R}^n, \quad Z \mapsto \langle\langle Z \cdot \varphi, \varphi \rangle\rangle.
\]
and \( F = (F_1, F_2) \in \mathbb{H}^m_1 \times \mathbb{R}^n \) with
\[
F_1 = \frac{1}{\sqrt{|c_1|}} \langle\langle \nu_1 \cdot \varphi, \varphi \rangle\rangle.
\]

As in the positive curvature case, it is possible to deduce a spinorial proof of the fundamental theorem of immersions theory in \( \mathbb{H}^m_1 \times \mathbb{H}^n_2 \) or \( \mathbb{H}^m_1 \times \mathbb{R}^n \).

4. CMC surfaces with \( H = 1/2 \) in \( \mathbb{H}^2 \times \mathbb{R} \)

We consider the immersion of a surface with \( H = 1/2 \) in \( \mathbb{H}^2 \times \mathbb{R} \subset \mathbb{R}^{1,2} \times \mathbb{R} \) represented by a spinor field \( \varphi \) as in Theorem 5 (with \( m = 2 \) and \( n = 1 \)). Let us first introduce some notation. We denote by \( N \) the unit vector normal to the surface and tangent to \( \mathbb{H}^2 \times \mathbb{R} \), it is of the form \((N', \nu)\) in \( \mathbb{R}^{1,2} \times \mathbb{R} \), and by \( \nu_1 \) the unit vector normal to \( \mathbb{H}^2 \times \mathbb{R} \) so that the immersion reads \( F = (\nu_1, h) \in \mathbb{H}^2 \times \mathbb{R} \). The function \( \nu \) is the angle function of the immersion, and we assume that it is always positive (the surface has regular vertical projection), and the function \( h : M \rightarrow \mathbb{R} \) is the height function of the immersion. We fix a conformal parameter \( z = x + iy \) of the surface, in which the metric reads \( \mu^2(dx^2 + dy^2) \). The matrix of the shape operator in the basis \( \partial_x/\mu, \partial_y/\mu \) reads
\[
S = \begin{pmatrix}
1/2 + \alpha & \beta \\
\beta & 1/2 - \alpha 
\end{pmatrix}
\]
and we set the following two important quantities
\[
Q_0 := -\frac{\mu^2}{2} (\alpha - i\beta) - h_z^2 \quad \text{and} \quad \tau_0 := \mu^2 \nu^2.
\]
Following [7] \((Q_0, \tau_0)\) are called the Weierstrass data of the immersion, and we will see below that they appear naturally in the equations satisfied by the spinor field representing the immersion in adapted coordinates. We will then compute the hyperbolic Gauss map in terms of these data (we will recall the definition below) and we will interpret geometrically the relation between the spinor field and the hyperbolic Gauss map. Using these observations we will show that conversely the hyperbolic Gauss map and its Weierstrass data determine a family of spinor fields (parameterized by \( \mathbb{C} \)), and thus a family of immersions, a result obtained in [7] by other methods. We will finally use this spinorial approach to describe directly the correspondence between the theories of \( H = 1/2 \) surfaces in \( \mathbb{H}^2 \times \mathbb{R} \) and in \( \mathbb{R}^{1,2} \).
4.1. The Clifford algebra and the Spin group of \( \mathbb{R}^{1,3} \). Let us consider the algebra of complex quaternions \( \mathbb{H}^\mathbb{C} := \mathbb{H} \otimes \mathbb{C} \). An element \( a \) of \( \mathbb{H}^\mathbb{C} \) is of the form
\[
a = a_0 + a_1I + a_2J + a_3K, \quad a_0, a_1, a_2, a_3 \in \mathbb{C},
\]
its complex norm is
\[
H(a, a) = a_0^2 + a_1^2 + a_2^2 + a_3^2 \in \mathbb{C}
\]
and its complex conjugate is the complex quaternion
\[
\widehat{a} = \overline{a}_0 I + \overline{a}_1 J + \overline{a}_2 K
\]
where \( \overline{a}_i \) denotes the usual complex conjugate of \( a_i \). Let us associate to
\[
x = x_0e_0^o + x_1e_1^o + x_2e_2^o + x_3e_3^o \in \mathbb{R}^{1,3}
\]
the complex quaternion
\[
X = ix_0 + x_1I + x_2J + x_3K \in \mathbb{H}^\mathbb{C}
\]
where \( JI = -IJ = -K \). Using the Clifford map
\[
x = x_0e_0^o + x_1e_1^o + x_2e_2^o + x_3e_3^o \in \mathbb{R}^{1,3} \mapsto \left( \begin{array}{c} 0 \\ X \end{array} \right) \in \mathbb{H}^\mathbb{C}(2)
\]
we easily obtain that
\[
Cl(1, 3) = \left\{ \left( \begin{array}{c} a \\ b \\ \overline{a} \end{array} \right), \ a, b \in \mathbb{H}^\mathbb{C} \right\}, \quad Cl^0(1, 3) = \left\{ \left( \begin{array}{c} a \\ 0 \\ \overline{a} \end{array} \right), \ a \in \mathbb{H}^\mathbb{C} \right\}
\]
and
\[
Spin(1, 3) = \left\{ \left( \begin{array}{c} a \\ 0 \\ \overline{a} \end{array} \right), \ a \in \mathbb{H}^\mathbb{C}, \ H(a, a) = 1 \right\},
\]
i.e. the identification
\[
Spin(1, 3) \simeq S^3_3 := \{ a \in \mathbb{H}^\mathbb{C} : \ H(a, a) = 1 \}.
\]
For the sake of simplicity, we will frequently use below the natural identifications of \( Cl^0(1, 3) \) and \( Cl^1(1, 3) \) with \( \mathbb{H}^\mathbb{C} \). We will moreover use the models
\[
\mathbb{H}^2 = \{ ix_0 + x_2J + x_3K, \ -x_0^2 + x_2^2 + x_3^2 = -1 \}, \ \mathbb{R} := \{ x_1I, \ x_1 \in \mathbb{R} \}
\]
and we will decompose the special direction \( I \) of the product structure in the form
\[
I = T + \nu N,
\]
where \( T \) is tangent and \( N \) is normal to the immersion, and \( \nu \) is the angle function.

4.2. The Killing type equation in adapted coordinates. In a fixed spinorial frame \( \tilde{s} \) above the orthonormal frame \( s = (\partial_x/\mu, \partial_y/\mu, N, \nu_1) \), the spinor field is represented by \( [\varphi] = g \in S^3_3 \), and we consider the components
\[
g_1 := \frac{1}{2}(1 + iI)g, \quad g_2 := \frac{1}{2}(1 - iI)g
\]
so that \( g = g_1 + g_2 \). Let us note that
\[
\frac{1}{2}(1 + iI)\frac{1}{2}(1 + iI) = \frac{1}{2}(1 + iI), \quad \frac{1}{2}(1 - iI)\frac{1}{2}(1 - iI) = \frac{1}{2}(1 - iI)
\]
and
\[
\frac{1}{2}(1 + iI)\frac{1}{2}(1 - iI) = 0.
\]
It will be convenient to consider the following norm on \( \frac{1}{2} (1 + i I) \mathbb{H}^2 \): writing an element \( g_1' \) belonging to \( \frac{1}{2} (1 + i I) \mathbb{H}^2 \) in the form

\[ g_1' = \frac{1}{2} (1 + i I) (a + b J) \]

for some (unique) \( a, b \in \mathbb{C} \), we define its hermitian norm \( |g_1'|^2 := |a|^2 - |b|^2 \).

**Proposition 4.1.** The component \( g_1' := \sqrt{\tau_0} g_1 \) satisfies

\[ dg_1' = (\log \sqrt{\tau_0}) dz g_1' + \frac{1}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} d\zeta \right) J g_1' I, \]

Moreover the compatibility conditions of the equation read

\[ (Q_0)_{\zeta} = 0 \quad \text{and} \quad (\log \sqrt{\tau_0})_{\zeta} = -\frac{1}{\tau_0} |Q_0|^2 + \frac{\tau_0}{16} \]

and \( g_1' \) is such that \( |g_1'|^2 = \sqrt{\tau_0} \).

The equation of \( g_1' \) only depends on the Weierstrass data \((Q_0, \tau_0)\). In the statement and in the rest of the section we use the following notation: \( z = x + I y, dz = dx + I dy, \partial_x = 1/2 (\partial_x - I \partial_y), \partial_{\zeta} = x - I y, d\zeta = dx - I dy \) and \( \partial_{\zeta} = 1/2 (\partial_x + I \partial_y) \), i.e. the complex parameter \( z \) is with respect to the complex structure \( I \).

**Remark 5.** The Abresch-Rosenberg differential is the quadratic differential \(-Q_0 dz^2\). It rather appears here as a 1-form.

We will need for the computations the following form of the compatibility equations for the product structure:

**Lemma 4.2.** In the frame \( \tilde{s} \) the product structure \( T + \nu N \) reads

\[ [T] + \nu [N] = \frac{2}{\mu} h_z J + \nu I, \]

and \( h_z, \mu \) and \( \nu \) satisfy the relations

\[ d(h_z) = \frac{2}{\mu} \mu_z h_z dz + \frac{\sqrt{\tau_0}}{2} \left( \frac{1}{2} d\zeta + (\alpha - I \beta) dz \right) \]

and

\[ \nu_z = -\frac{1}{2} h_z - (\alpha - I \beta) h_{\zeta}. \]

Moreover, the two components \( g_1 \) and \( g_2 \) of the spinor field are linked by

\[ g_2 = -\frac{1}{\nu} I \bar{g}_1 I + \frac{2 i}{\sqrt{\tau_0}} h_z J g_1. \]

**Proof of Lemma 4.2:** In \( \tilde{s} \), we represent the vectors \( \partial_x/\mu, \partial_y/\mu \) and \( N \) of the basis \( s \) by, respectively, \( J, JJ \) and \( I \in \mathbb{H}^2 \). Since \( h \) is the second component of the immersion we have \( dh(X) = \langle X, T \rangle \) and \( T = 1/\mu^2 (\partial_x h \partial_x + \partial_y h \partial_y) \), which gives

\[ [T] = \frac{1}{\mu} J(\partial_x h + I \partial_y h) = \frac{2}{\mu} h_z J \]

and (72). Writing that \( T + \nu N \) is parallel in \( \mathbb{H}^2 \times \mathbb{R} \) we obtain the two equations \( \nabla T - \nu S = 0 \) and \( \langle S, T \rangle + d\nu = 0 \). The first equation yields (73): the first term gives
\[ [\nabla T] = d[T] - aI[T] \] with \[ T = \frac{2}{\mu} h_z J \] and where \( a \) is the Levi-Civita connection form

\[(77)\quad a = -\frac{1}{\mu} \left( \partial_{\mu} \mu \ dx - \partial_x \mu \ dy \right) = \left( \frac{1}{\mu} d\mu - \frac{2}{\mu} \mu_z dz \right) I \]

and the second term gives

\[(78)\quad [S] = \mu \left( \frac{1}{2} d\tau + (\alpha - I\beta) dz \right) J \]

with \( \sqrt{\tau_0} = \mu \nu \). The second equation yields (74) by a computation of \( \langle S, T \rangle = -\frac{1}{2} (\langle S \rangle [T] + [T] \langle S \rangle) \) in \( \mathbb{H}^C \) using (76) and (78). Finally, since the spinor field preserves the product structure we have \( \langle (T + \nu N) \cdot \varphi, \varphi \rangle \rangle = I \), which reads \( \langle [T] + \nu [N] \rangle g = g ) \) with \( g = g_1 + g_2 \), \( [T] = 2/\mu \ h_z J \) and \([N] = I \); the component in \( 1/2(1 + iI) \mathbb{H}^C \) of that expression yields \( 2h_z/\mu \ J_{g1} + \nu I g_2 = \hat{g}_1 I \) and (75).

**Proof of Proposition 4.1:** Since \( X_1 = X - X_2 = X - (X, T)(T + \nu N) \) with \( [X] = \mu d\tau J, \langle X, T \rangle = dh(X) \) and \( [T] + \nu [N] \) given by (72), we have

\[ [X_1] = (\mu d\tau - \frac{2}{\mu} h_z dh) J - \frac{1}{\mu} dh \sqrt{\tau_0} I \]

and since

\[(79)\quad 2/\mu \ |h_z|^2 = \mu/2 \ |T|^2 = \mu/2 \ (1 - \nu^2) = \mu/2 - \tau_0/2\mu \]

we obtain

\[(80)\quad [X_1] = \left( \frac{\mu}{2} d\tau - \frac{2}{\mu} h_z^2 dz + \frac{\tau_0}{2\mu} d\tau \right) J - \frac{1}{\mu} dh \sqrt{\tau_0} I. \]

Using that \( [\nabla \varphi] = dg - \frac{1}{2} aIg, \) (78) and (80), the Killing type equation (66) with \( [\nu_1] = iI \) and \([B(X)] = [S(X)] \cdot [N] \) reads

\[
dg^{-1} - \frac{1}{2} aI = \frac{i}{2} [X_1] + \frac{\mu}{2} \left( Hd\tau + (\alpha - I\beta) dz \right) IJ \]

\[
= \frac{\mu}{2} \left( \frac{i}{2} + H1 \right) d\tau J - \frac{i}{2\mu} dh \sqrt{\tau_0} I + \frac{1}{\mu} \left( \frac{\mu^2}{2} (\alpha - I\beta) dz + ih_z^2 dz I - \frac{\tau_0}{4} d\tau \right) IJ. \]

We take \( H = 1/2 \) and we multiply both sides of the equation by \( 1/2(1 + iI) \) on the left: since

\[ \frac{1}{2} (1 + iI) I = -i \frac{1}{2} (1 + iI), \]

the first right-hand term vanishes and we get

\[
dg_1 - \frac{1}{2} aI g_1 = -\frac{1}{\mu} dh \sqrt{\tau_0} g_1 IJ g_2. \]

Using (75) we obtain

\[
dg_1 - \frac{1}{2} aI g_1 = ( -\frac{1}{2\mu} h_z \sqrt{\tau_0} + \frac{2}{\sqrt{\tau_0} \mu} Q_0 h_z) d\tau g_1 + \frac{1}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} d\tau \right) J \hat{g}_1 I \]

and using finally that

\[ -\frac{h_z}{2\mu} \sqrt{\tau_0} + \frac{2}{\sqrt{\tau_0} \mu} Q_0 h_z = \frac{1}{\nu} \nu_z \]

(this is a consequence of (74) and (79)) together with (77) we get

\[
dg_1 = ( -\frac{1}{2\mu} d\mu + (\log \sqrt{\tau_0}) d\tau) g_1 + \frac{1}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} d\tau \right) J \hat{g}_1 I \]
and \( g_1' = \sqrt{\mu} g_1 \) satisfies (70). A computation using (70) twice then shows that
\[
0 = d(dg_1') = \left(- (\log \sqrt{\tau_0})_{\tau_0} - \frac{1}{\tau_0} |Q_0|^2 + \frac{\tau_0}{16} \right) dz \wedge d\bar{z} \ g_1' - \frac{1}{\sqrt{\tau_0}} (Q_0)_{\tau_0} dz \wedge d\bar{z} \ Jg_1' I,
\]
which proves (71). For the last claim, \( g = a_01 + a_1 I + a_2 J + a_3 K \) with \( a_j \in \mathbb{C} \) is such that \( gIg = [T] + \nu[N] \) belongs to \( \mathbb{R} I \oplus \mathbb{R} J \oplus \mathbb{R} K \) (recall the last step of the proof of Lemma 4.2), i.e. the component of \( gIg \) on \( i1 \) is zero, which yields
\[
|g_1'|^2 = \mu(|a_0 - ia_1|^2 - |a_2 - ia_3|^2) = \mu(|a_0|^2 + |a_1|^2 - |a_2|^2 - |a_3|^2) = \nu = \sqrt{\tau_0}.
\]

4.3. The product structure and the second component of the spinor field.
It appears that the product structure (\( \nu \) and \( \text{h}_z / \mu \) in (72)) may be determined independently of \( g_1' \) and \( g_2' \). This relies on the following key lemma of [7]:

**Lemma 4.3.** [7, Lemma 10] The function \( \text{h}_z \) satisfies the system
\[
(h_z)_z = (\log \sqrt{\tau_0})_{\tau_0} \text{h}_z - Q_0 \sqrt{\tau_0 + 4|\text{h}_z|^2 / \tau_0} \nonumber
\]
\[
(h_z)_{\bar{z}} = 4 \sqrt{\tau_0 (\tau_0 + 4|\text{h}_z|^2)}. \nonumber
\]
If we fix \( \tau_0 \) and \( \theta_0 \in \mathbb{C} \), the system admits a unique globally defined solution \( \text{h}_z \) satisfying the initial condition \( \text{h}_z(z_0) = \theta_0 \).

The system is a consequence of the compatibility equations (73) and (74). Since the equations only depend on the Weierstrass data \( (Q_0, \tau_0) \), \( \text{h}_z \) only depends on these data and on the choice of the initial condition \( \text{h}_z(z_0) \in \mathbb{C} \). Moreover, since
\[
\mu = \sqrt{\tau_0 + 4|\text{h}_z|^2} \quad \text{and} \quad \nu = \sqrt{\tau_0 / \tau_0 + 4|\text{h}_z|^2},
\]
(by (76) and since \( |T|^2 = 1 - \nu^2 \) with \( \tau_0 = \mu^2 \nu^2 \)) \( \mu \) and \( \nu \) are also determined by \( (Q_0, \tau_0) \) and \( \text{h}_z(z_0) \). We finally observe that the other component \( g_2' := \sqrt{\mu} g_2 \) of the spinor field is given by
\[
g_2' = - \frac{1}{\nu} I g_1' J + \frac{2i}{\mu \nu} \text{h}_z Jg_1'
\]
(Equation (75)) and is thus determined by \( g_1' \) if \( \text{h}_z \) is known: so \( g_1' \) determines a family of spinor fields parametrized by \( \mathbb{C} \); the parameter corresponds to the choice of an initial condition for the determination of the product structure.
4.4. The hyperbolic Gauss map. If $N$ is normal to $M$ and tangent to $\mathbb{H}^2 \times \mathbb{R}$, $\nu_1$ is normal to $\mathbb{H}^2 \times \mathbb{R}$ in $\mathbb{R}^{1,3}$ and $\nu$ is the angle function of $M$ as above, it is defined in [7] as the map

$$G = \frac{1}{\nu}(N + \nu_1).$$

It belongs to the light-cone $\{X \in \mathbb{R}^{1,3} : |X|^2 = 0\}$, and since $\nu = \langle N, I \rangle$ it is of the form $G = G' + I$ where $G' : \mathbb{C} \to \mathbb{H}^2$ takes values in the model (69) of $\mathbb{H}^2$. We will frequently identify $G$ and $G'$ below. In terms of the spinor field representing the immersion, since $|N| = I$, $[nu] = i1$ and $[\nu] = g$ in $\mathfrak{sl}(2,\mathbb{C})$ it is written

$$G = \frac{1}{\nu}g(i + I)\bar{g} = \frac{2i}{|g|^2}g_1\gabar_1$$

and we see that it only depends on the component $g_1'$ of the spinor field. A direct computation using (70) shows that

$$dG = \sqrt{\tau_0}/4 \left\{ (1 + 4Q_0/\tau_0) dz + (1 + 4\overline{Q_0}/\tau_0) d\bar{z} \right\} u_1$$

$$- i \left\{ (1 - 4Q_0/\tau_0) dz - (1 - 4\overline{Q_0}/\tau_0) d\bar{z} \right\} u_2$$

where $(u_1, u_2)$ is the positively oriented orthonormal basis of $T_G\mathbb{H}^2$

$$H(u_1, u_1) = H(u_2, u_2) = 1$$

and $H(u_1, u_2) = 0$, we also readily get that

$$H(dG, dG) = Q_0 dz^2 + \left( \frac{\tau_0}{4} + \frac{4|Q_0|^2}{\tau_0} \right) d\bar{z} dz + Q_0 d\bar{z} d\bar{z}.$$

As observed in [7], since $Q_0$ is a holomorphic function, $G : M \to \mathbb{H}^2$ is harmonic, and following that paper we will say that $Q_0 : \mathbb{C} \to \mathbb{C}$ and $\tau_0 : \mathbb{C} \to (0, +\infty)$ such that (85) and (71) hold form the Weierstrass data of the map $G : \mathbb{C} \to \mathbb{H}^2$. At a regular point of $G$ (i.e. where $Q_0 \neq 0$), computations show that conversely (85) implies the existence of a unique positively oriented orthonormal basis $(u_1, u_2)$ of $T_G\mathbb{H}^2$ such that (83) holds. We will assume in the rest of the paper that the set of singular points of $G$ has empty interior.

4.5. Interpretation in terms of a principal bundle. If $g_1' = \frac{1}{2}(1 + iI)(a + bJ)$ and $g_1'' = \frac{1}{2}(1 + iI)(a' + b'J)$ belong to $\frac{1}{2}(1 + iI)\mathbb{H}^2\mathbb{C}$, we consider the hermitian product $\langle g_1', g_1'' \rangle = \bar{a}a' - b\bar{b}$ and the norm $|g_1'|^2 = |a|^2 - |b|^2$. We consider the set

$$\mathcal{V} := \{ g_1' \in \frac{1}{2}(1 + iI)\mathbb{H}^2\mathbb{C} | |g_1'|^2 > 0 \}$$

and the map

$$\pi : \mathcal{V} \to \mathbb{H}^2, \quad g_1' \mapsto \frac{2i}{|g_1'|^2}g_1\gabar_1.$$

This is the projection of a principal bundle, with group of structure $\mathbb{C}^*$ acting by multiplication. This bundle is moreover equipped with a natural invariant connection form $\omega_0$ given by

$$\omega_{0g_1}(v) = \frac{1}{|g_1'|^2} \langle v, g_1' \rangle.$$
for all $g_1' \in \mathcal{V}$ and $v$ tangent to $\mathcal{V}$ at $g_1'$: the horizontal distribution at $g_1'$ is the complex line orthogonal to the line $\mathbb{C}g_1'$ (the fiber of $\pi$) with respect to the hermitian product $\langle \cdot \rangle$ introduced above. We consider the bundle induced from the bundle $\pi : \mathcal{V} \to \mathbb{H}^2$ by the Gauss map $G : \mathbb{C} \to \mathbb{H}^2$

$$G^* \mathcal{V} = \{(z, g_1') \in \mathbb{C} \times \mathcal{V} : \pi(g_1') = G(z)\}$$

and the hypersurface

$$\mathcal{H} := \{(z, g_1') \in G^* \mathcal{V} : |g_1'|^2 = \sqrt{\tau_0(z)}\}$$

with the projection $p_1 : \mathcal{H} \to \mathbb{C}$, $(z, g_1') \mapsto z$; this is a $S^1$ principal bundle. If $p_2 : \mathcal{H} \to \mathcal{V}$, $(z, g_1') \mapsto g_1'$ is the other projection we consider the 1-form

$$(86) \quad \omega := -p_1^*(\log \tau_0) dz + p_2^* \omega_0.$$ 

It is a connection form on the $S^1$ principal bundle $\mathcal{H} \to \mathbb{C} :$

- it takes values in $S^1 = i \mathbb{R} : (U, V) \in T(z, g_1') \mathcal{H}$ satisfies $\Re \langle V, g_1' \rangle = \frac{1}{2} d(\log \tau_0)(U)$
- $\omega(z, g_1')(U, V) = -(\log \tau_0) U + \frac{1}{\sqrt{\tau_0}} \langle V, g_1' \rangle$

$$= -\frac{1}{2} \left( (\log \tau_0) U - (\log \tau_0) z U \right) + \frac{i}{\sqrt{\tau_0}} \Im \langle V, g_1' \rangle \in i \mathbb{R};$$

- it is $S^1$-invariant: $S^1$ only acts on the component $g_1'$ and $\omega_0$ is $S^1$-invariant;

- it is normalized on vertical vectors:

$$\omega \left( \frac{d}{d \theta} \bigg|_{\theta = 0} (z, e^{i\theta} g_1') \right) = \omega_0 g_1'(i g_1') = i.$$

**Proposition 4.4.** The component $g_1' : \mathbb{C} \to \mathcal{V}$ of the spinor field which represents the immersion naturally identifies to a section of $\mathcal{H} \to \mathbb{C}$, which is parallel with respect to the connection $\omega$.

**Proof.** $g_1'$ defines a section $\sigma : z \mapsto (z, g_1'(z))$ of $\mathcal{H} \to \mathbb{C}$ since $\pi(g_1') = G$ and $|g_1'|^2 = \sqrt{\tau_0}$. It satisfies

$$\sigma^* \omega = -\sigma^* p_1^*(\log \tau_0) dz + \sigma^* p_2^* \omega_0 = -\langle \log \tau_0 \rangle dz + g_1^* \omega_0 = 0$$

since $p_1 \circ \sigma = \text{id}$, $p_2 \circ \sigma = g_1'$ and by (70); indeed, the right-hand term of (70) is horizontal: if $g_1' = \frac{1}{2} (1 + i I)(a + b J)$, then $\hat{J}g_1' I = \frac{1}{2} (1 + i I)(\bar{b} + a J)$ is orthogonal to $g_1'$ with respect to the hermitian product $\langle \cdot, \cdot \rangle$. \hfill $\square$

**Remark 6.** The connection $\omega$ is flat since it admits a parallel section. In fact a computation shows that $\omega$ defined by (86) is flat if and only if $(Q_0, \tau_0)$ satisfy the conditions (71).

**Corollary 1.** $g_1'$ is determined by the Gauss map $G$ and its Weierstrass data $(Q_0, \tau_0)$ up to a sign.

**Proof.** A parallel section of $\mathcal{H} \to \mathbb{C}$ is unique up to the multiplication by a complex number $e^{i\theta} \in S^1$, and since $g_1'$ is a solution of (70) the section $g_1'' \equiv e^{i\theta} g_1'$ satisfies

$$(87) \quad dg_1'' = (\log \tau_0) dz g_1'' + e^{2i\theta} \frac{1}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} dz^2 \right) \hat{J} g_1'' I;$$

it is a solution of (70) if and only if $e^{i\theta} = \pm 1$. \hfill $\square$
4.6. **Surfaces with prescribed hyperbolic Gauss map.** We assume here that $G : \mathbb{C} \to \mathbb{H}^2$ is a given map with Weierstrass data $(Q_0, \tau_0)$ and that the set of singular points of $G$ has empty interior. The following result was obtained in [7].

**Corollary 2.** There exists a family of CMC surfaces with $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$ with hyperbolic Gauss map $G$ and Weierstrass data $(Q_0, \tau_0)$. The family is parameterized by $\mathbb{C} \times \mathbb{R}$.

**Proof.** Since the connection $\omega$ is flat (Remark 6), the principal bundle $\mathcal{H} \to \mathbb{C}$ admits a globally defined parallel section that we may consider as a map $\sigma : \mathbb{C} \to \mathcal{V}$ such that $\pi \circ \sigma = G$. At a regular point of $G$, since $d\sigma - (\log \sqrt{\tau_0})_\tau d\sigma$ is horizontal and projects onto $d\pi(d\sigma) = d(\pi \circ \sigma) = dG$ it satisfies

\[
d\sigma - (\log \sqrt{\tau_0})_\tau d\sigma = e^{2i\theta} \frac{1}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} dz \right) \ J\tilde{\sigma} I
\]

for some $\theta \in \mathbb{R}$ (indeed, the term $1/\sqrt{\tau_0} (Q_0 dz + \tau_0/4d\tilde{z}) J\tilde{\sigma} I$ is horizontal at $\sigma$ and a computation shows that its projection by $d\pi$ is of the form (83), with an orthonormal basis $(u_1', u_2')$ of $T_0 \mathbb{H}^2$ which is perhaps different to the basis $(u_1, u_2)$; taking $\theta$ such that $(u_1', u_2')$ matches with $(u_1, u_2)$, the right hand term of (88) is the horizontal lift of $dG$). The compatibility conditions of (88) imply that $\theta$ is a constant, and in view of (87) the section $g_1' := e^{-i\theta} \sigma$ is a solution of (70). $g_1'$ is uniquely determined up to a sign, as in Corollary 1. Note that this equation extends by continuity to the singular points of $G$. We then consider a product structure $h_z$ given by Lemma 4.3 (there is a family of solutions, depending on a parameter belonging to $\mathbb{C}$), the solution $g_2'$ given by (82) and set $g := 1/\sqrt{\mu} (g_1' + g_2')$ : it belongs to $S^3_0$ (since $H(g, g) = 1$ by a computation), and we consider the spinor field $\varphi$ whose component is $g$ in $\tilde{s}$, and the corresponding immersions into $\mathbb{H}^2 \times \mathbb{R}$; they depend on $\mathbb{C} \times \mathbb{R}$ since a last integration is required to obtain $h$ from $h_z$. □

**Remark 7.** The spinorial representation formula permits to recover the explicit representation formula of the immersion in terms of all the data: calculations from the representation formula $F = (i\sqrt{\mu} g, h)$ (formula (67)) lead to the expression of the immersion in terms of $G_z$, $Q_0$, $\tau_0$ and $h$ given in [7, Theorem 11].

4.7. **Link with $H = 1/2$ surfaces in $\mathbb{R}^{1,2}$.** We first describe with spinors the immersions of $H = 1/2$ surfaces in $\mathbb{R}^{1,2}$ and deduce that they are entirely determined by their Gauss map and its Weierstrass data. This is a proof using spinors of a result obtained in [1] with other methods. We then obtain a simple algebraic relation between the spinor fields representing these immersions and families (parametrized by $\mathbb{C}$) of spinor fields representing $H = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$: this gives a simple interpretation of the natural correspondence between $H = 1/2$ surfaces in $\mathbb{R}^{1,2}$ and in $\mathbb{H}^2 \times \mathbb{R}$. Since the proofs are very similar to proofs of the preceding sections, we will omit many details. We consider the model

$$Spin(1, 2) = \{ \alpha_0 I + \alpha_1 I + i\alpha_2 J + i\alpha_3 J I, \ \alpha_j \in \mathbb{R}, \ \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 = 1 \}$$

which is the subgroup of $Spin(1, 3) = S^3_0$ fixing $I \in \mathbb{R}^{1,3}$ (under the double covering $Spin(1, 3) \to SO(1, 3)$) and thus also leaving $\mathbb{R}^{1,2} := \{ ix_0 1 + x_2 J + x_3 J I, \ x_0, x_2, x_3 \in \mathbb{R} \} \subset \mathbb{R}^{1,3}$ globally invariant. By [4], if $\tilde{Q}$ is a spin structure of $M$ and $\rho : Spin(2) \to Spin(1, 2)$ a natural representation, a spacelike immersion of a surface in $\mathbb{R}^{1,2}$ is represented
by a spinor field \( \psi \in \tilde{Q} \times \rho Spin(1, 2) \) solution of the Killing type equation

\[
(89) \quad \nabla_X \psi = -\frac{1}{2} S(X) \cdot \nu_1 \cdot \psi
\]

for all \( X \in TM \), where \( \nu_1 \) is the upward vector normal to \( M \) in \( \mathbb{R}^{1,2} \) so that \( |\nu_1|^2 = -1 \) and \( S = \nabla \nu_1 : TM \to TM \) is the shape operator. Explicitly, the immersion is a primitive of the 1-form \( \xi(X) = \langle (X \cdot \psi, \psi) \rangle \). Fixing a conformal parameter \( z = x + iy \) of the surface in which the metric reads \( \mu^2(dx^2 + dy^2) \) and choosing a spinorial frame \( \tilde{s} \) above (\( \partial_x/\mu, \partial_y/\mu \)), the spinor field reads as a map \( v : \mathbb{C} \to Spin(1, 2) \). Assuming that the matrix of \( S \) reads as (68), we set here

\[
Q_0 = \frac{\mu^2}{2} (\alpha - I \beta) \quad \text{and} \quad \tau_0 = \mu^2.
\]

**Proposition 4.5.** The function \( v' := \sqrt{\mu} v \) satisfies

\[
(90) \quad dv' = \left\{ (\log \sqrt{\tau_0})_z \ d\zeta + \frac{i}{\sqrt{\tau_0}} \left( Q_0 d\zeta + \frac{\tau_0}{4} d\zeta \cdot J \right) \right\} v'.
\]

The compatibility conditions of the equation are (71). Moreover \( |v'|^2 := H(v', v') = \sqrt{\tau_0} \).

**Proof.** As in the proof of Proposition 4.1, in the spinorial frame \( \tilde{s} \) the Killing type equation (89) reads

\[
dv v^{-1} - \frac{1}{2} aI = \frac{\mu}{2} (H d\zeta + (\alpha - I \beta) d\zeta) iJ,
\]

which, for \( H = 1/2 \) and the definitions of \( Q_0 \) and \( \tau_0 \), gives

\[
(91) \quad dv v^{-1} = \frac{1}{2} aI + \frac{1}{\sqrt{\tau_0}} \left( \frac{\tau_0}{4} d\zeta + Q_0 d\zeta \right) iJ.
\]

Using (77), we obtain (90). Finally, \( |v'|^2 = H(v', v') = \mu H(v, v) = \mu = \sqrt{\tau_0} \). \( \square \)

We now consider the usual Gauss map \( G : \mathbb{C} \to \mathbb{H}^2 \) of the surface, still with the model (69). Since \( G = \langle [\nu_1, \psi, \bar{\psi}] \rangle \) with

\[
[\nu_1] = \begin{pmatrix} 0 & 1 \\ -i1 & 0 \end{pmatrix} \quad \text{and} \quad [\psi] = \begin{pmatrix} v & 0 \\ 0 & \bar{v} \end{pmatrix}
\]

it reads

\[
(92) \quad G = i \nabla \tilde{v}.
\]

The projection

\[
\pi' : Spin(1, 2) \to \mathbb{H}^2, \quad v \mapsto i \nabla \tilde{v}
\]

is a principal bundle of group of structure \( H = \{ \cos \theta + \sin \theta I, \theta \in \mathbb{R} \} \) (acting on the left) that we equip with a natural connection: we consider the decomposition of the Lie algebra \( Spin(1, 2) = \mathfrak{h} \oplus \mathfrak{m} \) with \( \mathfrak{h} = \mathbb{R}I \) and \( \mathfrak{m} = iJ(\mathbb{R}1 \oplus \mathbb{R}I) \), the projection \( p_1 \) onto the first factor \( \mathfrak{h} \) and the connection form

\[
\omega_c = p_1 \circ \omega_{MC} \in \Omega^1(Spin(1, 2), \mathfrak{h})
\]

where \( \omega_{MC} = d\sigma \sigma^{-1} \in \Omega^1(Spin(1, 2), Spin(1, 2)) \) is the Maurer-Cartan form. The bundle \( \pi' : Spin(1, 2) \to \mathbb{H}^2 \) and the Gauss map \( G : \mathbb{C} \to \mathbb{H}^2 \) induce a bundle

\[
G^*Spin(1, 2) := \{(z, v) \in \mathbb{C} \times Spin(1, 2) | G(z) = \pi'(v)\}
\]
Proposition 4.6. The component \( v : \mathbb{C} \to \text{Spin}(1, 2) \) of the spinor field representing the immersion is naturally a section of \( G^*\text{Spin}(1, 2) \to \mathbb{C} \). It is horizontal for the connection \( \omega \).

\begin{proof}
This is a traduction of (91), similar to Proposition 4.4.
\end{proof}

Corollary 3. \( v \) is determined by the Gauss map \( G \) and its Weierstrass data \((Q_0, \tau_0)\) up to a sign.

\begin{proof}
It is analogous to the proof of Corollary 1.
\end{proof}

We now suppose that a map \( G : \mathbb{C} \to \mathbb{H}^2 \) is given with Weierstrass data \((Q_0, \tau_0)\). We moreover suppose that the set of singular points of \( G \) has empty interior.

Corollary 4. [1] There exists a \( H = 1/2 \) surface in \( \mathbb{R}^{1,2} \) with Gauss map \( G \) and Weierstrass data \((Q_0, \tau_0)\). It is unique up to a translation in \( \mathbb{R}^{1,2} \).

\begin{proof}
As in the proof of Corollary 2, a horizontal section \( v \) of \( G^*\text{Spin}(1, 2) \to \mathbb{C} \) is a map such that
\[
dv^{-1} = \frac{1}{2}aI + e^{2i\theta'} \frac{1}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} d\bar{z} \right) iJ
\]
for some function \( \theta' : \mathbb{C} \to \mathbb{R} \) which has to be constant, and \( \bar{v} = e^{-i\theta'} v \) is a section solution of (91) such that \( \pi'(\bar{v}) = G \). Since such a solution is unique up to a sign, the spinor field \( \psi \) is determined up to a sign and the immersion is unique up to a translation (since the immersion in \( \mathbb{R}^{1,2} \) is finally obtained by the integration of the 1-form \( \xi(X) = \langle X \cdot \psi, \psi \rangle \)).
\end{proof}

Proposition 4.7. The correspondence
\[
g'_1 := \frac{1}{2}(1 + iI)v'
\]
transforms a solution \( v' \) of (90) such that \( \pi'(v'/|v'|) = G \) to a solution \( g'_1 \) of (70) such that \( \pi(g'_1) = G + I \), and vice-versa.

\begin{proof}
Assuming that (91) holds we compute
\[
dg'_1 = \frac{1}{2}(1 + iI)dv' = (\log \sqrt{\tau_0}) z \ dz + \frac{1}{2}(1 + iI)v' + \frac{i}{\sqrt{\tau_0}} \left( Q_0 dz + \frac{\tau_0}{4} d\bar{z} \right) J_2 \frac{1}{2}(1 - iI)v'.
\]
For the last term, since \( Iv' = \bar{v} I \) (\( v \) belongs to \( \text{Spin}(1, 2) \)) we observe that
\[
iJ_2 \frac{1}{2}(1 - iI)v' = J_2 \frac{1}{2}(1 - iI)Iv' = J_2 \frac{1}{2}(1 - iI)\bar{v} I = J_2 g'_1 I,
\]
which shows that \( g'_1 \) satisfies (70). We finally note that \( |g'_1|^2 = \mu \) and therefore
\[
\pi(g'_1) = \frac{2i}{|g'_1|^2} \bar{g}'_1 \bar{g}'_1 = 2i(\frac{1}{2}(1 - iI)\bar{v}) = i\bar{v} + \pi I \bar{v}.
\]
The first term \( i\bar{v} \) is the Gauss map \( G \). Since \( I \bar{v} = vI \) and \( \pi v = 1 \), the second term \( \pi I \bar{v} \) is the constant \( I \). So \( \pi(g'_1) = G + I \) and \( g'_1 \) lies above the same (hyperbolic)
Gauss map $G$. Conversely, if now $g'_1 \in 1/2(1 + iI)\mathbb{H}^C$ is a solution of (70) we consider the unique $v' = \alpha_01 + \alpha_1I + i\alpha_2J + i\alpha_3K$ such that (93) holds, and similar computations show that $v'$ is a solution of (90) above $G$, which proves the proposition.

**Corollary 5.** Immersions of CMC surfaces with $H = 1/2$ in $\mathbb{R}^{1,2}$ (up to translations) correspond to 2-parameter families of immersions of CMC surfaces with $H = 1/2$ in $\mathbb{H}^2 \times \mathbb{R}$ and regular vertical projection (up to vertical translations).

**Proof.** In the correspondence (93), $g'_1$ determines a family of spinor fields parametrized by $\mathbb{C}$, and therefore a 2-parameter family of immersions in $\mathbb{H}^2 \times \mathbb{R}$ up to a vertical translation, whereas $v'$ determines a spinor field and therefore an immersion in $\mathbb{R}^{1,2}$ up to a translation.

Note that the correspondence preserves the (hyperbolic) Gauss maps and their Weierstrass data.

**Appendix A. Skew-symmetric operators and Clifford algebra**

We consider $\mathbb{R}^N$ with its standard scalar product $\langle ., . \rangle$. If $\eta$ and $\eta'$ belong to the Clifford algebra $Cl(N)$, we set

$$[\eta, \eta'] = \eta \cdot \eta' - \eta' \cdot \eta,$$

where the dot $\cdot$ is the Clifford product. We denote by $(e_1, \ldots, e_N)$ the canonical basis of $\mathbb{R}^N$. The next two lemmas were proved in the Appendix A of [5]. We include the statements here for the convenience of the readers.

**Lemma A.1.** Let $u : \mathbb{R}^N \to \mathbb{R}^N$ be a skew-symmetric operator. Then the bivector

$$u = \frac{1}{4} \sum_{j=1}^{N} e_j \cdot u(e_j) \in \Lambda^2 \mathbb{R}^N \subset Cl(N)$$

represents $u$ in the sense that, for all $\xi \in \mathbb{R}^N$, $[u, \xi] = u(\xi)$. We also have the formula

$$u = \frac{1}{2} \sum_{1 \leq j < k \leq N} \langle u(e_j), e_k \rangle e_j \cdot e_k.$$

We now assume that $\mathbb{R}^N = \mathbb{R}^p \oplus \mathbb{R}^q$, $p + q = N$.

**Lemma A.2.** Let us consider a linear map $u : \mathbb{R}^p \to \mathbb{R}^q$ and its adjoint $u^* : \mathbb{R}^q \to \mathbb{R}^p$. Then the bivector

$$u = \frac{1}{2} \sum_{j=1}^{p} e_j \cdot u(e_j) \in \Lambda^2 \mathbb{R}^N \subset Cl(N)$$

represents

$$\left( \begin{array}{cc} 0 & -u^* \\ u & 0 \end{array} \right) : \mathbb{R}^p \oplus \mathbb{R}^q \to \mathbb{R}^p \oplus \mathbb{R}^q$$

in the sense that, for all $\xi = \xi_p + \xi_q \in \mathbb{R}^N$, $[u, \xi] = u(\xi_p) - u^*(\xi_q)$. Moreover we have

$$u = \frac{1}{4} \left( \sum_{j=1}^{p} e_j \cdot u(e_j) + \sum_{j=p+1}^{n} e_j \cdot (-u^*(e_j)) \right).$$
We moreover have the following

**Lemma A.3.** Consider, for \( U, V \in \mathbb{R}^N \), the skew-symmetric operator

\[
U \wedge V : \mathbb{R}^N \to \mathbb{R}^N
\]

\[
W \mapsto \langle U, W \rangle V - \langle V, W \rangle U.
\]

It is represented by

\[
\frac{1}{4} (U \cdot V - V \cdot U) \in \Lambda^2 \mathbb{R}^N \subset Cl(N),
\]

i.e., for all \( W \in \mathbb{R}^N \),

\[
U \wedge V (W) = \left[ \frac{1}{4} (U \cdot V - V \cdot U), W \right].
\]

**Proof.** Let us write

\[
(U \wedge V)(W) = \langle U, W \rangle V - \langle V, W \rangle U
= \frac{1}{2} \langle U, W \rangle V + \frac{1}{2} V \langle U, W \rangle - \frac{1}{2} \langle V, W \rangle U - \frac{1}{2} U \langle V, W \rangle.
\]

Using \( \langle U, W \rangle = -\frac{1}{2} (U \cdot W + W \cdot U) \) and \( \langle V, W \rangle = -\frac{1}{2} (V \cdot W + W \cdot V) \) we get

\[
(U \wedge V)(W) = -\frac{1}{4} (U \cdot W \cdot V + W \cdot U \cdot V + V \cdot U \cdot W + V \cdot W \cdot U)
+ \frac{1}{4} (V \cdot W \cdot U + W \cdot V \cdot U + U \cdot V \cdot W + U \cdot W \cdot V)
= -\frac{1}{4} (W \cdot U \cdot V + V \cdot U \cdot W - W \cdot V \cdot U - U \cdot V \cdot W).
\]

This last expression is

\[
\left[ \frac{1}{4} (U \cdot V - V \cdot U), W \right] = \frac{1}{4} (U \cdot V - V \cdot U) \cdot W - W \cdot \frac{1}{4} (U \cdot V - V \cdot U).
\]

□

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