Centrifugal (centripetal), Coriolis’ velocities, accelerations and Hubble’s law in spaces with affine connections and metrics

Sawa Manoff

Institute for Nuclear Research and Nuclear Energy
Department of Theoretical Physics
Bld. Tzarigradsko Chaussee 72
1784 Sofia - Bulgaria

E-mail address: smanov@inrne.bas.bg

Abstract

The notions of centrifugal (centripetal) and Coriolis’ velocities and accelerations are introduced and considered in spaces with affine connections and metrics \((\mathcal{T}_n, g)\)-spaces as velocities and accelerations of flows of mass elements (particles) moving in space-time. It is shown that these types of velocities and accelerations are generated by the relative motions between the mass elements. They are closely related to the kinematic characteristics of the relative velocity and relative acceleration. The null (isotropic) vector fields are considered and their relations with the centrifugal (centripetal) velocity are established. The centrifugal (centripetal) velocity is found to be in connection with the Hubble law and the generalized Doppler effect in spaces with affine connections and metrics. The centrifugal (centripetal) acceleration could be interpreted as gravitational acceleration as it has been done in the Einstein theory of gravitation. This fact could be used as a basis for working out of new gravitational theories in spaces with affine connections and metrics.

1 Introduction

1. The relative velocity and the relative acceleration between particles or mass elements of a flow are important characteristics for describing the evolution and the motion of a dynamic system. On the other side, the kinematic characteristics related to the relative velocity and the relative acceleration (such as deformation velocity and acceleration, shear velocity and acceleration, rotation velocity and acceleration, and expansion velocity and acceleration) characterize specific
relative motions of particles and/or mass elements in a flow \([11, 12]\). On the basis of the links between the kinematic characteristics related to the relative velocity and these related to the relative acceleration the evolution of a system of particles or mass elements of a flow could be connected to the geometric properties of the corresponding mathematical model of a space or space-time \([11]\). Many of the notions of classical mechanics or of classical mechanics of continuous media preserve their physical interpretation in more comprehensive spaces than the Euclidean or Minkowskian spaces, considered as mathematical models of space or space-time \([6]\). On this background, the generalizations of the notions of Coriolis’ and centrifugal (centripetal) accelerations \([7]\) from classical mechanics in Euclidean spaces are worth to be investigated in spaces with affine connections and metrics \((\mathcal{L}_n, g)-\)spaces \([8, 12]\).

2. Usually, the Coriolis’ and centrifugal (centripetal) accelerations are considered as apparent accelerations, generated by the non-inertial motion of the basic vector fields determining a co-ordinate system or a frame of reference in an Euclidean space. In Einstein’s theory of gravitation (ETG) these types of accelerations are considered to be generated by a symmetric affine connection (Riemannian connection) compatible with the corresponding Riemannian metric \([5]\). In both cases they are considered as corollaries of the non-inertial motion of particles (i.e. of the motion of particles in non-inertial co-ordinate system).

3. In the present paper the notions of Coriolis’ and centrifugal (centripetal) accelerations are considered with respect to their relations with the geometric characteristics of the corresponding models of space or space-time. It appears that accelerations of these types are closely related to the kinematic characteristics of the relative velocity and of the relative acceleration. The main idea is to be found out how a Coriolis’ or centrifugal (centripetal) acceleration acts on a mass element of a flow or on a single particle during its motion in space or space-time described by a space of affine connections and metrics. In Section 1 the notions of centrifugal (centripetal) and Coriolis’ velocities are introduced and considered in \((\mathcal{L}_n, g)-\)spaces. The relations between the kinematic characteristics of the relative velocity and the introduced notions are established. The null (isotropic) vector fields are considered and their relations with the centrifugal (centripetal) velocity are established. The centrifugal (centripetal) velocity is found to be in connection with the Hubble law and the generalized Doppler effect in spaces with affine connections and metrics. In Section 2 the notions of centrifugal (centripetal) and Coriolis’ accelerations are introduced and their relations to the kinematic characteristics of the relative acceleration are found. In Section 3 the interpretation of the centrifugal (centripetal) acceleration as gravitational acceleration is given and illustrated on the basis of the Einstein theory of gravitation and especially on the basis of the Schwarzschild metric in vacuum.

4. The main results in the paper are given in details (even in full details) for these readers who are not familiar with the considered problems. The definitions and abbreviations are identical to those used in \([9, 11, 12]\). The reader is kindly asked to refer to them for more details and explanations of the statements and results only cited in this paper.
2 Centrifugal (centripetal) and Coriolis’ velocities

Let us now recall some well known facts from kinematics of vector fields over spaces with affine connections and metrics \((\mathcal{T}_n, g)\) - spaces, considered as models of space or space-time \([1] \div [12]\).

Every vector field \(\xi\) could be represented by the use of a non-isotropic (non-null) vector field \(u\) and its corresponding contravariant and covariant projective metrics \(h^u\) and \(h_u\) in the form

\[
\xi = \frac{1}{e} \cdot g(u, \xi) \cdot u + \overline{g}[h_u(\xi)],
\]

where

\[
h^u = \overline{g} - \frac{1}{e} \cdot u \otimes u, \quad h_u = g - \frac{1}{e} \cdot g(u) \otimes g(u),
\]

\[
\overline{g} = g^{ij} \cdot e_i \cdot e_j, \quad g^{ij} = g^{ji}, \quad e_i \cdot e_j = \frac{1}{2} \cdot (e_i \otimes e_j + e_j \otimes e_i),
\]

\[
g = g_{ij} \cdot e^i \cdot e^j, \quad g_{ij} = g_{ji}, \quad e^i \cdot e^j = \frac{1}{2} \cdot (e^i \otimes e^j + e^j \otimes e^i),
\]

\[
e = g(u, u) = g_{ij} \cdot u^i \cdot u^j = u^k \cdot u^k = \nabla^u : \neq 0,
\]

\[
g(u, \xi) = g_{ij} \cdot u^i \cdot \xi^j, \quad e_i = \partial_i, \quad e^j = dx^j \text{ in a co-ordinate basis.}
\]

By means of the representation of \(\xi\) the notions of relative velocity and relative acceleration are introduced \([9], [11]\) in \((\mathcal{T}_n, g)\) - spaces. By that, the vector field \(\xi\) has been considered as vector field orthogonal to the vector field \(u\), i.e. \(g(u, \xi) = 0, \xi = \xi_\perp = \overline{g}[h_u(\xi)]\). Both the fields are considered as tangent vector fields to the corresponding co-ordinates, i.e. they fulfil the condition \([8]\)

\[
\nabla \xi = -\nabla u, \xi = [u, \xi] = 0, \text{ where } [u, \xi] = u \circ \xi - \xi \circ u.
\]

Under these preliminary conditions the relative velocity \(\text{rel} v\) could be found in the form \([9], [11]\)

\[
\text{rel} v = \overline{g}[d(\xi_\perp)],
\]

where

\[
d = \sigma + \omega + \frac{1}{n - 1} \cdot \theta \cdot h_u.
\]

The tensor \(d\) is the deformation velocity tensor; the tensor \(\sigma\) is the shear velocity tensor; the tensor \(\omega\) is the rotation velocity tensor; the invariant \(\theta\) is the expansion velocity invariant. The vector field \(\text{rel} v\) is interpreted as the relative velocity of two points (mass elements, particles) moving in a space or space-time and having equal proper times \([1] \div [4]\). The vector field \(\xi_\perp\) is orthogonal to \(u\) and is interpreted as deviation vector connecting the two mass elements (particles) (if considered as an infinitesimal vector field).
Let us now consider the representation of the relative velocity by the use of a non-isotropic (non-null) vector field $\xi_\perp$ (orthogonal to $u$) and its corresponding projective metrics

$$h_{\xi_\perp} = \mathcal{F} - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp \otimes \xi_\perp,$$  

(7)

$$h_{\xi_\perp} = g - \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot g(\xi_\perp) \otimes g(\xi_\perp).$$  

(8)

Then a vector field $v$ could be represented in the form

$$v = \frac{g(v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \mathcal{F}[h_{\xi_\perp}(v)].$$  

(9)

Therefore, the relative velocity $\text{rel} v$ could be now written in the form

$$\text{rel} v = \frac{g(\text{rel} v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \mathcal{F}[h_{\xi_\perp}(\text{rel} v)] = v_z + v_c,$$  

(10)

where

$$v_z = \frac{g(\text{rel} v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp, \quad v_c = \mathcal{F}[h_{\xi_\perp}(\text{rel} v)].$$  

(11)

The vector field $v_z$ is collinear to the vector field $\xi_\perp$. If the factor (invariant) before $\xi_\perp$ is positive, i.e. if

$$g(\text{rel} v, \xi_\perp) > 0$$  

(12)

the vector field $v_z$ is called centrifugal velocity. If the factor (invariant) before $\xi_\perp$ is negative, i.e. if

$$g(\text{rel} v, \xi_\perp) < 0$$  

(13)

the vector field $v_z$ is called centripetal velocity.

The vector field $v_c$ is called Coriolis’ velocity.

### 2.1 Centrifugal (centripetal) velocity

**Properties of the centrifugal (centripetal) velocity**

(a) Since $v_z$ is collinear to $\xi_\perp$, it is orthogonal to the vector field $u$, i.e.

$$g(u, v_z) = 0.$$  

(14)

(b) The centrifugal (centripetal) velocity $v_z$ is orthogonal to the Coriolis velocity $v_c$

$$g(v_z, v_c) = 0.$$  

(15)

(c) The length of the vector $v_z$ could be found by means of the relation

$$g(v_z, v_z) = v_z^2 = \frac{g(\text{rel} v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp, \quad g(\text{rel} v, \xi_\perp) = \frac{g(\text{rel} v, \xi_\perp)^2}{g(\xi_\perp, \xi_\perp)^2} \cdot g(\xi_\perp, \xi_\perp) = \frac{|g(\text{rel} v, \xi_\perp)|^2}{\xi_\perp^2}. $$  

(16)
From the last expression we can conclude that the square of the length of the centrifugal (centripetal) velocity is in general in inverse proportion to the length of the vector field $\xi_\perp$.

**Special case:** $M_n := E_n$, $n = 3$ (3-dimensional Euclidean space): $\xi_\perp = \vec{r}$,

$g(\xi_\perp, \xi_\perp) = r^2$

$$v_z^2 = \frac{[g(\vec{r}, r\cdot \vec{v})]^2}{r^2}, \quad l_{v_z} = \frac{g(\vec{r}, r\cdot \vec{v})}{r}.$$  \hspace{1cm} (17)

If the relative velocity $r_{rel}v$ is equal to zero then $v_z = 0$.

(d) The scalar product between $v_z$ and $r_{rel}v$ could be found in its explicit form by the use of the explicit form of the relative velocity $r_{rel}v$

$$r_{rel}v = \vec{g}[d(\xi_\perp)] = \vec{g}[\sigma(\xi_\perp)] + \vec{g}[\omega(\xi_\perp)] + \frac{1}{n-1} \cdot \theta \cdot \xi_\perp,$$  \hspace{1cm} (18)

and the relations

$$g(\xi_\perp, r_{rel}v) = g(\xi_\perp, \vec{g}[\sigma(\xi_\perp)]) + g(\xi_\perp, \vec{g}[\omega(\xi_\perp)]) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp),$$  \hspace{1cm} (19)

$$g(\xi_\perp, \vec{g}[\sigma(\xi_\perp)]) = \sigma(\xi_\perp, \xi_\perp),$$  \hspace{1cm} (20)

$$\vec{g}(\xi_\perp, \vec{g}[\omega(\xi_\perp)]) = \omega(\xi_\perp, \xi_\perp) = 0,$$  \hspace{1cm} (21)

$$g(\xi_\perp, r_{rel}v) = \sigma(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp).$$  \hspace{1cm} (22)

For

$$g(v_z, r_{rel}v) = g(g(\xi_\perp, r_{rel}v), \xi_\perp) = g(\xi_\perp, r_{rel}v) = \frac{g(\xi_\perp, r_{rel}v)}{g(\xi_\perp, \xi_\perp)} = v_z^2,$$  \hspace{1cm} (23)

it follows that

$$g(v_z, r_{rel}v) = g(v_z, v_z) = v_z^2.$$  \hspace{1cm} (24)

**Remark:** $g(v_z, r_{rel}v) = g(v_z, v_z) = v_z^2$ because of $g(v_z, r_{rel}v) = g(v_z, v_z + v_z) = g(v_z, v_z) + g(v_z, v_z) = g(v_z, v_z)$.

On the other side we can express $v_z^2$ by the use of the kinematic characteristics of the relative velocity $\sigma$, $\omega$, and $\theta$

$$v_z^2 = \frac{[g(\xi_\perp, r_{rel}v)]^2}{g(\xi_\perp, \xi_\perp)} = \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot \{[\sigma(\xi_\perp, \xi_\perp)]^2 + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot [g(\xi_\perp, \xi_\perp)]^2 + \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp) \cdot g(\xi_\perp, \xi_\perp) \},$$  \hspace{1cm} (25)
Special case: $\theta := 0$ (expansion-free relative velocity)

$$v_z^2 = \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} = \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{\mp l_{\xi_\perp}^2},$$  \hfill (26)

$$g(\xi_\perp, \xi_\perp) = \pm l_{\xi_\perp}^2,$$  \hfill (27)

$$g(\xi_\perp, \xi_\perp) = -l_{\xi_\perp}^2 \quad \text{if} \quad g(u, u) = e = +l_u^2,$$  \hfill (28)

$$g(\xi_\perp, \xi_\perp) = +l_{\xi_\perp}^2 \quad \text{if} \quad g(u, u) = e = -l_u^2.$$  \hfill (29)

Special case: $\sigma := 0$ (shear-free relative velocity)

$$v_z^2 = 1 \cdot \frac{\theta^2 \cdot g(\xi_\perp, \xi_\perp)}{n - 1}.$$  \hfill (30)

(e) The explicit form of $v_z$ could be found as

$$v_z = \frac{g(\text{rel}, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot [\sigma(\xi_\perp, \xi_\perp) + \frac{1}{n - 1} \cdot \theta \cdot g(\xi_\perp, \xi_\perp)] \cdot \xi_\perp =$$

$$= \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} + \frac{1}{n - 1} \cdot \theta \cdot \xi_\perp,$$  \hfill (31)

$$v_z = \left[ \frac{1}{n - 1} \cdot \theta + \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \right] \cdot \xi_\perp.$$  \hfill (31)

If

$$\frac{1}{n - 1} \cdot \theta + \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} > 0,$$  \hfill (32)

we have a centrifugal (relative) velocity.

If

$$\frac{1}{n - 1} \cdot \theta + \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} < 0,$$  \hfill (33)

we have a centripetal (relative) velocity.

Special case: $\theta := 0$ (expansion-free relative velocity)

$$v_z = \frac{\sigma(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp.$$  \hfill (34)

Special case: $\sigma := 0$ (shear-free relative velocity)

$$v_z = \frac{1}{n - 1} \cdot \theta \cdot \xi_\perp.$$  \hfill (35)

If the expansion invariant $\theta > 0$ we have centrifugal (or expansion) (relative) velocity. If the expansion invariant $\theta < 0$ we have centripetal (or contraction) (relative) velocity. Therefore, in the case of a shear-free relative velocity $v_z$ is proportional to the expansion velocity invariant $\theta$. 

6
If we introduce the vector field $n_\perp$, normal to $u$ and normalized, as

$$
n_\perp : = \frac{\xi_\perp}{l_{\xi_\perp}} , \quad g(\xi_\perp, \xi_\perp) = \mp l_{\xi_\perp}^2 , \quad \xi_\perp = l_{\xi_\perp} \cdot n_\perp ,
$$

(36)

$$
l_{\xi_\perp} = |g(\xi_\perp, \xi_\perp)|^{1/2} , \quad g(n_\perp, n_\perp) = \mp 1 .
$$

(37)

then the above expressions with $\sigma(\xi_\perp, \xi_\perp)/ g(\xi_\perp, \xi_\perp)$ could be written in the forms:

$$
v_z = \left[ \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot \xi_\perp ,
$$

(38)

$$
\theta := 0 : \quad v_z = \mp \sigma(n_\perp, n_\perp) \cdot \xi_\perp ,
$$

(39)

The centrifugal (centripetal) relative velocity $v_z$ could be also written in the form

$$
v_z = \frac{g(\text{rel}v, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \mp \frac{g(\text{rel}v, \xi_\perp)}{l_{\xi_\perp}^2} \cdot \xi_\perp = \mp g(\text{rel}v, \xi_\perp) \cdot n_\perp .
$$

(40)

On the other side, on the basis of the relations

$$
\text{rel}v = \mathcal{F}[d(\xi_\perp)] = \mathcal{F}[\sigma(\xi_\perp)] + \mathcal{F}[\omega(\xi_\perp)] + \frac{1}{n-1} \cdot \theta \cdot \xi_\perp ,
$$

$$
\mathcal{F}[h_u(\xi)] = \xi_\perp , \quad \mathcal{F}[h_u(\xi_\perp)] = \mathcal{F}[h_u(\mathcal{F}[h_u(\xi)])] = \mathcal{F}[h_u(\xi)] = \xi_\perp ,
$$

$$
g(\mathcal{F}[\sigma(\xi_\perp)], n_\perp) = \sigma(n_\perp, \xi_\perp) = \sigma(n_\perp, l_{\xi_\perp} \cdot n_\perp) = l_{\xi_\perp} \cdot \sigma(n_\perp, n_\perp) ,
$$

$$
g(\mathcal{F}[\omega(\xi_\perp)], n_\perp) = \omega(n_\perp, \xi_\perp) = \omega(n_\perp, l_{\xi_\perp} \cdot n_\perp) = l_{\xi_\perp} \cdot \omega(n_\perp, n_\perp) = 0 ,
$$

$$
g(\xi_\perp, n_\perp) = l_{\xi_\perp} \cdot g(n_\perp, n_\perp) = \mp l_{\xi_\perp} ,
$$

it follows for $g(\text{rel}v, n_\perp)$

$$
g(\text{rel}v, n_\perp) = [\sigma(n_\perp, n_\perp) \mp \frac{1}{n-1} \cdot \theta] \cdot l_{\xi_\perp} ,
$$

(41)

and for $v_z$

$$
v_z = \mp g(\text{rel}v, n_\perp) \cdot n_\perp = [\mp \sigma(n_\perp, n_\perp) + \frac{1}{n-1} \cdot \theta] \cdot l_{\xi_\perp} \cdot n_\perp ,
$$

$$
v_z = \left[ \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot l_{\xi_\perp} \cdot n_\perp .
$$

(42)

Since $l_{\xi_\perp} > 0$, we have three different cases:

(a)

$$
v_z > 0 : \mp \sigma(n_\perp, n_\perp) + \frac{1}{n-1} \cdot \theta > 0 ;
$$

$$
\theta > \pm (n-1) \cdot \sigma(n_\perp, n_\perp) , \quad n-1 > 0 .
$$
(b) 
\[ v_z < 0 : \mp \sigma(n_\perp, n_\perp) + \frac{1}{n - 1} \cdot \theta < 0 ; \]
\[ \theta < \pm (n - 1) \cdot \sigma(n_\perp, n_\perp), \quad n - 1 > 0 . \]

(c) 
\[ v_z = 0 : \mp \sigma(n_\perp, n_\perp) + \frac{1}{n - 1} \cdot \theta = 0 ; \]
\[ \theta = \pm (n - 1) \cdot \sigma(n_\perp, n_\perp), \quad n - 1 > 0 . \]

Special case: \( \sigma := 0 \) (shear-free relative velocity)
\[ v_z = \mp \sigma(n_\perp, n_\perp) \cdot l_{\xi_\perp} \cdot n_\perp . \]

For \( v_z > 0 : \theta > 0 \) we have an expansion, for \( v_z < 0 : \theta < 0 \) we have a contraction, and for \( v_z = 0 : \theta = 0 \) we have a stationary case. 

Special case: \( \theta := 0 \) (expansion-free relative velocity)
\[ v_z = \mp \sigma(n_\perp, n_\perp) \cdot l_{\xi_\perp} \cdot n_\perp . \]

For \( v_z > 0 : \mp \sigma(n_\perp, n_\perp) > 0 \) we have an expansion, for \( v_z < 0 : \mp \sigma(n_\perp, n_\perp) < 0 \) we have a contraction, and for \( v_z = 0 : \sigma(n_\perp, n_\perp) = 0 \) we have a stationary case.

In an analogous way we can find the explicit form of \( v_z^2 \) as
\[ v_z^2 = \pm [\sigma(n_\perp, n_\perp)]^2 \mp \frac{1}{(n - 1)^2} \cdot \theta^2 \mp 2 \cdot \frac{1}{n - 1} \cdot \theta \cdot \sigma(n_\perp, n_\perp) \cdot l_{\xi_\perp}^2 = (43) \]
\[ = \pm [\sigma(n_\perp, n_\perp)] \mp \frac{1}{n - 1} \cdot \theta^2 \cdot l_{\xi_\perp}^2 . \] (44)

The expression for \( v_z^2 \) could be now written in the form
\[ v_z^2 = \mp H^2 \cdot l_{\xi_\perp}^2 = \mp v_{v_z}^2 , \] (45)
where
\[ H^2 = [\sigma(n_\perp, n_\perp) \mp \frac{1}{n - 1} \cdot \theta]^2 . \] (46)

Then
\[ l_{v_z}^2 = H^2 \cdot l_{\xi_\perp}^2 , \quad l_{v_z} = |H| \cdot l_{\xi_\perp} , \quad l_{v_z} > 0 . \] (47)
\[ H = \mp [\sigma(n_\perp, n_\perp) \mp \frac{1}{n - 1} \cdot \theta] = \] (48)
\[ = \frac{1}{n - 1} \cdot \theta \mp \sigma(n_\perp, n_\perp) = \] (49)
and
\[ v_z = \pm l_{v_z} \cdot n_\perp = \pm \|H\| \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp . \] (50)
It follows from the last expression that the centrifugal (centripetal) relative velocity \( v_z \) is collinear to the vector field \( \xi_\perp \). Its absolute value \( l_{v_z} = |H| \cdot l_{\xi_\perp} \) and the expression for the centrifugal (centripetal) relative velocity \( v_z = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp \) represent generalizations of the Hubble law \[15\]. The function \( H \) could be called Hubble function (some authors call it Hubble coefficient \[15\]). Since \( H = H(x^k(\tau)) = H(\tau) \), for a given proper time \( \tau = \tau_0 \) the function \( H \) has at the time \( \tau_0 \) the value \( H(\tau_0) = H(\tau = \tau_0) = \text{const} \). The Hubble function \( H \) is usually called Hubble constant.

**Remark.** The Hubble coefficient \( H \) has dimension \( \text{sec}^{-1} \). The function \( H^{-1} \) with dimension \( \text{sec} \) is usually denoted in astrophysics as Hubble’s time \[15\].

The Hubble function \( H \) could also be represented in the forms \[12\]

\[
H = \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) = \\
= \frac{1}{2}[(n_\perp)(h_u)(\nabla_u \mathcal{g} - \mathcal{L}_u \mathcal{g})(h_u)(n_\perp)] . \tag{51}
\]

In the Einstein theory of gravitation (ETG) the Hubble coefficient is considered under the condition that the centrifugal relative velocity is generated by a shear-free relative velocity in a cosmological model of the type of Robertson-Walker \[15\], i.e.

\[
v_z = H \cdot l_{\xi_\perp} \cdot n_\perp = H \cdot \xi_\perp \quad \text{with} \quad H = \frac{1}{n-1} \cdot \theta \quad \text{and} \quad \sigma = 0 . \tag{52}
\]

**Special case:** \( U_n \)- or \( V_n \)-space: \( \nabla_u \mathcal{g} = 0, \mathcal{L}_u \mathcal{g} := 0 \) (the vector field \( u \) is a Killing vector field in the corresponding space)

\[
H = 0 .
\]

Therefore, in a (pseudo) Riemannian space with or without torsion (\( U_n \)- or \( V_n \)-space) the Hubble function \( H \) is equal to zero if the velocity vector field \( u (n = 4) \) of an observer is a Killing vector field fulfilling the Killing equation \( \mathcal{L}_u \mathcal{g} = 0 \). This means that the condition \( \mathcal{L}_u \mathcal{g} = 0 \) is a sufficient condition for the Hubble function to be equal to zero.

**Special case:** \((\mathcal{L}_n, g)\)-space with \( \nabla_u \mathcal{g} - \mathcal{L}_u \mathcal{g} := 0 \) \[16\]. For this case, the last expression appears as a sufficient condition for the vanishing of the Hubble function \( H \), i.e. \( H = 0 \) if \( \nabla_u \mathcal{g} = \mathcal{L}_u \mathcal{g} \).

**Special case:** \((\mathcal{L}_n, g)\)-spaces with vanishing centrifugal (centripetal) velocity: \( v_z := 0. \)

\[
v_z : = 0 : H = 0 : \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp) = 0 ,
\]

\[
\theta = \pm (n-1) \cdot \sigma(n_\perp, n_\perp) .
\]

**Remark.** The Hubble function \( H \) is introduced in the above considerations only on a purely kinematic basis related to the notions of relative velocity and centrifugal (centripetal) relative velocity. Its dynamic interpretation in a theory of gravitation depends on the structure of the theory and the relations between the field equations and the Hubble function.
2.2 Generalized Hubble’s law and centrifugal (centripetal) relative velocity

2.2.1 Null (isotropic) vector fields in spaces with affine connections and metrics

The null (isotropic) vector fields are used in all cases where a radiation process and its corresponding radiation has to be described in relativistic mechanics and electrodynamics as well as in Einstein’s theory of gravitation [5]. The null (isotropic) vector fields are related to the wave vector of the classical electrodynamics and used for an invariant description of wave propagation in different types of spaces or space-times.

Definition. A null (isotropic) vector field is a contravariant vector field $k \neq 0 \in T(M)$ with zero length $l_k = |g(k,k)|^{1/2} = 0$.

Remark. Since $k^2 = g(k,k) = 0 = \pm l_k^2$ we have $l_k = |k| = |g(k,k)|^{1/2} = 0$.

As a contravariant vector field the null vector field $k$ could be represented by its projections along and orthogonal to a given non-isotropic (non-null) vector field $u$

$$k = \frac{1}{e} \cdot g(u,k) \cdot u + \frac{1}{2} [h_u(k)] ,$$

where

$$e = g(u,u) = \pm l_u^2 :\neq 0 .$$

The invariant (scalar product of $u$ and $k$) $g(u,k) := \omega$ is interpreted as the frequency $\omega$ of the radiation related to the null vector $k$ and detected by an observer with velocity $u$

$$g(u,k) = \omega := 2 \cdot \pi \cdot \nu . \quad (53)$$

The contravariant non-isotropic (non-null) vector field $u$ and its corresponding vector field $\xi_\perp$, orthogonal to $u \ [g(u,\xi_\perp) = 0]$, could be written in the forms

$$u = l_u \cdot n_\parallel , \quad l_u > 0 , \quad (54)$$
$$g(u,u) = l_u^2 \cdot g(n_\parallel,n_\parallel) = \pm l_u^2 , \quad (55)$$
$$g(n_\parallel,n_\parallel) = \pm 1 , \quad (56)$$
$$\xi_\perp = l_{\xi_\perp} \cdot n_\perp , \quad l_{\xi_\perp} > 0 , \quad (57)$$
$$g(\xi_\perp,\xi_\perp) = l_{\xi_\perp}^2 \cdot g(n_\perp,n_\perp) = \mp l_{\xi_\perp}^2 , \quad (58)$$
$$g(n_\perp,n_\perp) = \mp 1 . \quad (59)$$

The vector fields $n_\parallel$ and $n_\perp$ are normalized unit vector fields orthogonal to each other, i.e.

$$g(n_\parallel,n_\perp) = 0 . \quad (60)$$
Proof. From \( g(u, \xi \perp) = 0 = g(l_u \cdot n_i, l_{\xi \perp} \cdot n_{\perp}) = l_u \cdot l_{\xi \perp} \cdot g(n_i, n_{\perp}) \) and \( l_u \neq 0, l_{\xi \perp} \neq 0 \), it follows that \( g(n_i, n_{\perp}) = 0 \).

By the use of the unit vector fields \( n_i \) and \( n_{\perp} \) the null vector field \( k \) could be written in the form

\[
k = k_i + k_{\perp}, \quad g(k_i, k_{\perp}) = 0,
\]

where

\[
k_i = \frac{\omega}{c} \cdot u = \pm \frac{\omega}{l_u} \cdot l_u \cdot n_i = \pm \frac{\omega}{l_u} \cdot n_i,
\]

\[
k_{\perp} = \mathcal{g} \{ h_u(k) \} = k \mp \frac{\omega}{l_u} \cdot n_i = \mp l_{k_{\perp}} \cdot n_{\perp},
\]

\[
g(k_{\perp}, k_{\perp}) = \mp l_{k_{\perp}}^2 = l_{k_{\perp}}^2 \cdot g(n_{\perp}, n_{\perp}),
\]

\[
g(k, k) = g(k_{\perp}, k_{\perp}) + g(k_i, k_i) = \mp l_{k_{\perp}}^2 + \frac{\omega^2}{l_u^2} \cdot g(n_i, n_i) = \mp l_{k_{\perp}}^2 \mp \frac{\omega^2}{l_u^2} = 0,
\]

\[
k_{\perp} = \mp \frac{\omega^2}{l_u^2}, \quad l_{k_{\perp}}^2 = \frac{\omega^2}{l_u^2}, \quad l_{k_{\perp}}^2 = \frac{\omega}{l_u}, \quad l_{k_{\perp}} = \frac{\omega}{l_u},
\]

\[
k = \frac{\omega}{l_u} \cdot (\pm n_i \mp n_{\perp}), \quad g(k, k) = 0.
\]

### 2.2.2 Centrifugal (centripetal) relative velocity and null vector fields

The frequency \( \omega \) of the radiation related to the null vector field \( k \) and detected by an observer with velocity vector field \( u \) has been determined as the projection of the non-isotropic (non-null) velocity vector field \( u \) at the vector field \( k \) (or vice versa): \( \omega = g(u, k) \). In an analogous way, the projection of the centrifugal (centripetal) relative velocity \( v_z \) at the part of \( k \), orthogonal to \( u \), (or vice versa) could be interpreted as the change of the frequency \( \omega \) of the emitter under the motion of the observer described by the centrifugal (centripetal) relative velocity \( v_z \). By the use of the relations

\[
g(u, k) = \omega, \quad g(n_{\perp}, k_{\perp}) = \frac{\omega}{l_u},
\]

\[
v_z = \frac{1}{n - 1} \cdot \theta \mp \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}} \cdot n_{\perp},
\]

the projection \( g(v_z, k_{\perp}) \) of \( v_z \) at \( k_{\perp} \) could be found in the form

\[
_{\text{rel}} \omega = g(v_z, k_{\perp}) = \frac{1}{n - 1} \cdot \theta \pm \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}} \cdot g(n_{\perp}, k_{\perp}) = \frac{\omega}{l_u} \cdot \frac{1}{n - 1} \cdot \theta \pm \sigma(n_{\perp}, n_{\perp}) \cdot l_{\xi_{\perp}}.
\]
If we introduce the abbreviation for the Hubble function

$$H = \frac{1}{n-1} \cdot \theta \mp \sigma(n_\perp, n_\perp)$$  \hspace{1cm} (71)

then the expression for $\text{rel}\omega$ could be written in the form

$$\text{rel}\omega = H \cdot \frac{k_\perp}{l_u} \cdot \omega$$  \hspace{1cm} (72)

As mentioned above, the frequency $\omega$ is the frequency detected by the observer with the velocity vector field $u$. The frequency of the emitter $\overline{\omega}$ could be found by the use of the relation between $\omega$ and $\overline{\omega}$ on the basis of the expression of the change of $\omega$ under the centrifugal (centripetal) motion [described by the centrifugal (centripetal) velocity $v_\perp$] as

$$\overline{\omega} = \omega + \text{rel}\omega \quad , \quad \omega = \overline{\omega} - \text{rel}\omega .$$  \hspace{1cm} (73)

Therefore, the radiation with frequency $\overline{\omega}$ by the emitter could be expressed as

$$\overline{\omega} = \omega + \text{rel}\omega = \omega + H \cdot \frac{k_\perp}{l_u} \cdot \omega = (1 + H \cdot \frac{k_\perp}{l_u}) \cdot \omega$$  \hspace{1cm} (74)

and it will be detected by the observer as the frequency $\omega$. The relative difference between both the frequencies (emitted $\overline{\omega}$ and detected $\omega$)

$$\frac{\overline{\omega} - \omega}{\omega} = \frac{\triangle \omega}{\omega}$$

appears in the form

$$\frac{\triangle \omega}{\omega} = \frac{\overline{\omega} - \omega}{\omega} = H \cdot \frac{k_\perp}{l_u} .$$  \hspace{1cm} (75)

If we introduce the abbreviation

$$z := H \cdot \frac{k_\perp}{l_u}$$  \hspace{1cm} (76)

we obtain the relation between the emitted frequency $\overline{\omega}$ and the frequency detected by the observer in the form

$$\frac{\overline{\omega} - \omega}{\omega} = z \quad , \quad \overline{\omega} = (1 + z) \cdot \omega .$$  \hspace{1cm} (77)

The quantity $z$ could be denoted as observed shift frequency parameter.

If $z = 0$ then $H = 0$ and there will be no difference between the emitted and detected frequencies: $\overline{\omega} = \omega$, i.e. for $z = 0$ $\overline{\omega} = \omega$. This will be the case if

$$\theta = \pm (n-1) \cdot \sigma(n_\perp, n_\perp) .$$  \hspace{1cm} (78)

If $z > 0$ the observed shift frequency parameter is called red shift. If $z < 0$ the observed shift frequency parameter is called blue shift. If $\overline{\omega}$ and $\omega$ are known
the observed shift frequency parameter \( z \) could be found. If \( w \) and \( z \) are given then the corresponding \( \omega \) could be estimated.

On the other side, from the explicit form of \( z \)

\[
z = H \cdot \frac{l_{\xi \perp}}{l_u} = \left[ \frac{1}{n - 1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot \frac{l_{\xi \perp}}{l_u}
\]

we could find the relation between the observed shift frequency parameter \( z \) and the kinematic characteristics of the relative velocity such as the expansion and shear velocities.

**Special case**: \((L_n, g)-spaces\) with shear-free relative velocity: \(\sigma := 0\).

\[
z = \frac{1}{n - 1} \cdot \theta \cdot \frac{l_{\xi \perp}}{l_u}, \quad H = \frac{1}{n - 1} \cdot \theta.
\]

**Special case**: \((L_n, g)-spaces\) with expansion-free relative velocity: \(\theta := 0\).

\[
z = \mp \sigma(n_\perp, n_\perp) \cdot \frac{l_{\xi \perp}}{l_u}, \quad H = \mp \sigma(n_\perp, n_\perp).
\]

On the grounds of the observed shift frequency parameter \( z \) the distance (the length \( l_{\xi \perp} \) of \( \xi \perp \)) between the observer [with the world line \( x'(\tau) \) and velocity vector field \( u = \frac{dx'}{d\tau} \)] and the observed object (at a distance \( l_{\xi \perp} \) from the observer) emitted radiation with null vector field \( k \) could be found as

\[
l_{\xi \perp} = z \cdot \frac{l_u}{H} = \frac{\mathcal{F} - \omega}{\omega} \cdot l_u = \frac{\mathcal{F} - \omega}{\omega} \cdot l_u.
\]

On the other side, if \( z, H, \) and \( l_{\xi \perp} \) are known the absolute value \( l_u \) of the velocity vector \( u \) could be found as

\[
l_u = \frac{H}{z} \cdot l_{\xi \perp} = \frac{H \cdot \omega}{\mathcal{F} - \omega} \cdot l_{\xi \perp}.
\]

**Remark.** In the Einstein theory of gravitation (ETG) the absolute value of \( u \) is usually normalized to 1 or \( c \), i.e. \( l_u = 1, c \). Then the last expression could be used for experimental check up of the velocity \( c \) of light in vacuum if \( z, H, \) and \( l_{\xi \perp} \) are known

\[
c = \frac{H}{z} \cdot l_{\xi \perp} = \frac{H \cdot \omega}{\mathcal{F} - \omega} \cdot l_{\xi \perp}.
\]

Since

\[
z = \left[ \frac{1}{n - 1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot \frac{l_{\xi \perp}}{l_u} = \frac{\mathcal{F} - \omega}{\omega}
\]

it follows that

\[
l_u = \frac{\omega}{\mathcal{F} - \omega} \cdot \left[ \frac{1}{n - 1} \cdot \theta \mp \sigma(n_\perp, n_\perp) \right] \cdot l_{\xi \perp}.
\]

Analogous expression we can find for the length \( l_{\xi \perp} \) of the vector field \( \xi \perp \)

\[
l_{\xi \perp} = (n - 1) \cdot \left( \frac{\mathcal{F}}{\omega} - 1 \right) \cdot \frac{l_u}{\theta \mp (n - 1) \cdot \sigma(n_\perp, n_\perp)}.
\]
Special case: \((L_n, g)\)-space with shear-free relative velocity: \(\sigma := 0\).

\[
l_u = \frac{\omega}{\overline{\omega} - \omega} \cdot \frac{1}{n - 1} \cdot \theta \cdot l_{\perp},
\]

\[
l_{\perp} = (n - 1) \cdot \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot \frac{l_u}{\theta}. \tag{88}
\]

Special case: \((L_n, g)\)-space with expansion-free relative velocity: \(\theta := 0\).

\[
l_u = \overline{\omega} - \omega \cdot \sigma(n_\perp, n_\perp) \cdot l_{\perp}, \tag{89}
\]

\[
l_{\perp} = \overline{\omega} - 1 \cdot \frac{l_u}{\sigma(n_\perp, n_\perp)}. \tag{90}
\]

By the use of the relation between the Hubble function \(H\) and the observed shift parameter \(z\) we can express the centrifugal (centripetal) velocity by means of the frequencies \(\overline{\omega}\) and \(\omega\). From

\[
v_z = H \cdot l_{\perp} \cdot n_\perp, \quad H = z \cdot \frac{l_u}{l_{\perp}} = \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot \frac{l_u}{l_{\perp}}, \tag{91}
\]

it follows that

\[
v_z = z \cdot l_u \cdot n_\perp = \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u \cdot n_\perp. \tag{92}
\]

Then

\[
g(v_z, v_z) = \mp \left( \frac{\overline{\omega}}{\omega} - 1 \right)^2 \cdot l_u^2 = \mp l_{v_z}^2, \tag{93}
\]

\[
l_{v_z} = \pm \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u, \quad \pm l_{v_z} = \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u, \tag{94}
\]

where (since \(l_{v_z} > 0\))

\[
\overline{\omega} > \omega : l_{v_z} = \left( \frac{\overline{\omega}}{\omega} - 1 \right) \cdot l_u, \tag{95}
\]

\[
\overline{\omega} < \omega : l_{v_z} = \left( 1 - \frac{\overline{\omega}}{\omega} \right) \cdot l_u. \tag{96}
\]

On the other side, we can express the relation between \(\overline{\omega}\) and \(\omega\) by means of the last relations:

\[
\frac{\overline{\omega}}{\omega} = 1 \pm \frac{l_{v_z}}{l_u}, \quad \overline{\omega} = \left( 1 \pm \frac{l_{v_z}}{l_u} \right) \cdot \omega, \tag{97}
\]

\[
\overline{\omega} > \omega : \overline{\omega} = \left( 1 + \frac{l_{v_z}}{l_u} \right) \cdot \omega, \tag{98}
\]

\[
\overline{\omega} < \omega : \overline{\omega} = \left( 1 - \frac{l_{v_z}}{l_u} \right) \cdot \omega. \tag{99}
\]
If we express the frequencies as \( \omega = 2 \cdot \pi \cdot \nu \) and \( \omega = 2 \cdot \pi \cdot \nu \), we obtain

\[
\nu > \nu : \nu = (1 + \frac{l_v}{l_u}) \cdot \nu ,
\]

(99)

\[
\nu < \nu : \nu = (1 - \frac{l_v}{l_u}) \cdot \nu .
\]

(100)

The last relations represent a generalization of the Doppler effect in \((T_n, g)\)-spaces.

It should be stressed that the generalized Doppler effect is a result of pure kinematic considerations of the properties of a null (isotropic) vector field by means of the kinematic characteristics of the relative velocity in spaces with affine connections and metrics. The Hubble function \( H \), the observed shift frequency parameter \( z \) are kinematic characteristics related to the centrifugal (centripetal) relative velocity. For different classic field theories they could have different relations to the dynamic variables of the corresponding theory.

### 2.3 Coriolis’ velocity

The vector field

\[
v_c = g[h_{\xi} \cdot (rel v)] = g^{ij} \cdot (h_{\xi})_{ik} \cdot rel v^k \cdot \partial_i
\]

(101)

is called Coriolis’ velocity.

**Properties of the Coriolis’ velocity**  
(a) The Coriolis velocity is orthogonal to the vector field \( u \), interpreted as velocity of a mass element (particle), i.e.

\[
g(u, v_c) = 0 .
\]

(102)

Proof: From the definition of the Coriolis velocity, it follows

\[
g(u, v_c) = g(u, g[h_{\xi} \cdot (rel v)] = g_{ij} \cdot u^i \cdot g^{jk} (h_{\xi})_{ik} \cdot rel v^l =
\]

\[
= g_{ij} \cdot g^{jk} \cdot u^i \cdot (h_{\xi})_{ik} \cdot rel v^l = g_i^k \cdot u^i \cdot (h_{\xi})_{kl} \cdot rel v^l = (103)
\]

Since

\[
(u)(h_{\xi}) = (h_{\xi})_{kl} \cdot u^k \cdot dx^l = (u)(g) - \frac{1}{g(\xi, \xi)} \cdot (u)[g(\xi)] \cdot g(\xi) =
\]

\[
= g(u) - \frac{1}{g(\xi, \xi)} \cdot g(u, \xi) \cdot g(\xi) = g(u) ,
\]

(104)

\[
g(u, \xi) = 0 ,
\]

then \((u)(h_{\xi})_{(rel v)} = [g(u)]_{(rel v)} = g(u_{rel v}) = 0\). Because of \( g(u_{rel v}) = 0 \), it follows that \( g(u, v_c) = 0 \).
(b) The Coriolis velocity $v_c$ is orthogonal to the centrifugal (centripetal) velocity $v_z$
\[ g(v_c, v_z) = 0 . \]  
(105)

c) The length of the vector $v_c$ could be found by the use of the relations:
\[
\begin{align*}
  v_c &= \mathcal{F}[h_{\xi_\perp}(v)] , \\
  h_{\xi_\perp}(v) &= g_{(relv)} - \frac{1}{g_{(\xi_\perp, \xi_\perp)}} \cdot [g(t_{\xi_\perp})](relv) \cdot g_{(\xi_\perp)} = \\
  &= g_{(relv)} - \frac{g(t_{\xi_\perp, relv})}{g_{(\xi_\perp, \xi_\perp)}} \cdot g_{(\xi_\perp)} , \\
  [g(t_{\xi_\perp})](relv) &= g_{(\xi_\perp, relv)} ,
\end{align*}
\]
\[ g(v_c, v_c) = v_c^2 = g_{(relv, relv)} - \frac{[g(t_{\xi_\perp, relv})]^2}{g_{(\xi_\perp, \xi_\perp)}} . \]  
(106)

On the other side, because of $(u)(h_u) = h_u(u) = 0$, and $relv = \mathcal{F}[h_u(\nabla u \xi_\perp)]$, it follows that
\[ g(u, v) = h_u(u, \nabla u \xi_\perp) = (u)(h_u)(\nabla u \xi_\perp) , \\
  g(u, relv) = (u)(h_u)(\nabla u \xi_\perp) = 0 . \]  
(107)

Since
\[
\begin{align*}
  g_{(relv, relv)} &= (relv)^2 = g_{(\mathcal{F}[d(\xi_\perp)], \mathcal{F}[d(\xi_\perp)])} = \\
  &= \mathcal{F}(d(\xi_\perp), d(\xi_\perp)) , \quad L_u \xi_\perp = 0 , \\
  \mathcal{F}(d(\xi_\perp), d(\xi_\perp)) &= \mathcal{F}(\sigma(\xi_\perp) + \omega(\xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g_{(\xi_\perp)}, \sigma(\xi_\perp) + \omega(\xi_\perp) + \frac{1}{n-1} \cdot \theta \cdot g_{(\xi_\perp)})
\end{align*}
\]
we obtain
\[ g(\text{rel}v, \text{rel}v) = \text{rel}v^2 = g(d(\xi_\perp), d(\xi_\perp)) = g(\sigma(\xi_\perp), \sigma(\xi_\perp)) + g(\omega(\xi_\perp), \omega(\xi_\perp)) + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) + 2 \cdot g(\sigma(\xi_\perp), \omega(\xi_\perp)) + \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp). \quad (111) \]

**Special case:** \( \sigma := 0, \theta := 0 \) (shear-free and expansion-free velocity).

\[ \text{rel}v^2 = g(\omega(\xi_\perp), \omega(\xi_\perp)). \quad (112) \]

**Special case:** \( \omega := 0 \) (rotation-free velocity).

\[ \text{rel}v^2 = g(\sigma(\xi_\perp), \sigma(\xi_\perp)) + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) + \frac{2}{n-1} \cdot \theta \cdot \sigma(\xi_\perp, \xi_\perp). \quad (113) \]

**Special case:** \( \theta := 0, \omega := 0 \) (expansion-free and rotation-free velocity).

\[ \text{rel}v^2 = g(\sigma(\xi_\perp), \sigma(\xi_\perp)). \quad (114) \]

**Special case:** \( \sigma := 0, \omega := 0 \) (shear-free and rotation-free velocity).

\[ \text{rel}v^2 = \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp). \quad (115) \]

**Special case:** \( \theta := 0 \) (expansion-free velocity).

\[ \text{rel}v^2 = g(\sigma(\xi_\perp), \sigma(\xi_\perp)) + \frac{1}{(n-1)^2} \cdot \theta^2 \cdot g(\xi_\perp, \xi_\perp) + 2 \cdot g(\sigma(\xi_\perp), \omega(\xi_\perp)) + 2 \cdot g(\sigma(\xi_\perp), \omega(\xi_\perp)). \quad (116) \]

The square \( v_c^2 \) of \( v_c \) could be found on the basis of the relation \( v_c^2 = \text{rel}v^2 - v_z^2 \)

\[ v_c^2 = \text{rel}v^2 - v_z^2 = g(\sigma(\xi_\perp), \sigma(\xi_\perp)) - \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)} + 2 \cdot g(\omega(\xi_\perp), \omega(\xi_\perp)) + 2 \cdot g(\sigma(\xi_\perp), \omega(\xi_\perp)). \quad (117) \]

Therefore, the length of \( v_c \) does not depend on the expansion velocity invariant \( \theta \).

**Special case:** \( \sigma := 0 \) (shear-free velocity).

\[ v_c^2 = g(\omega(\xi_\perp), \omega(\xi_\perp)). \quad (118) \]

**Special case:** \( \omega := 0 \) (rotation-free velocity).

\[ v_c^2 = g(\sigma(\xi_\perp), \sigma(\xi_\perp)) - \frac{[\sigma(\xi_\perp, \xi_\perp)]^2}{g(\xi_\perp, \xi_\perp)}. \quad (119) \]

Therefore, even if the rotation velocity tensor \( \omega \) is equal to zero \( (\omega = 0) \), the Coriolis (relative) velocity \( v_c \) is not equal to zero if \( \sigma \neq 0 \).
(d) The scalar product between $v_c$ and $\text{rel} v$ could be found in its explicit form by the use of the relations:

$$g(v_c, \text{rel} v) = h_{\perp} (\text{rel} v, v_c) = \text{rel} v^2 - \frac{[g(\xi_{\perp}, \text{rel} v)]^2}{g(\xi_{\perp}, \xi_{\perp})} = \text{rel} v^2 - v^2_c = v^2_c. \quad (120)$$

(e) The explicit form of $v_c$ could be found by the use of the relations

$$v_c = g[h_{\perp} (\text{rel} v)] = \text{rel} v - g(\xi_{\perp}, \text{rel} v) \cdot \xi_{\perp} = \text{rel} v - v_z,$$

$$v_c = g[\sigma(\xi_{\perp})] - \frac{\sigma(\xi_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} + g[\omega(\xi_{\perp})]. \quad (121)$$

Therefore, the Coriolis velocity does not depend on the expansion velocity invariant $\theta$.

Special case: $\sigma := 0$ (shear-free velocity).

$$v_c = g[\omega(\xi_{\perp})]. \quad (122)$$

Special case: $\omega := 0$ (rotation-free velocity).

$$v_c = g[\sigma(\xi_{\perp})] - \frac{\sigma(\xi_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp}. \quad (123)$$

(f) The Coriolis velocity $v_c$ is orthogonal to the deviation vector $\xi_{\perp}$, i.e. $g(v_c, \xi_{\perp}) = 0$.

Proof: From $g(v_c, \xi_{\perp}) = g[g[h_{\perp} (\text{rel} v)]], \xi_{\perp}) = h_{\perp} (\xi_{\perp}, \text{rel} v) = (\xi_{\perp})(h_{\perp})(\text{rel} v)$ and $(\xi_{\perp})(h_{\perp}) = 0$, it follows that

$$g(v_c, \xi_{\perp}) = 0. \quad (124)$$

3 Centrifugal (centripetal) and Coriolis’ accelerations

In analogous way as in the case of centrifugal (centripetal) and Coriolis’ velocities, the corresponding accelerations could be defined by the use of the projections of the relative acceleration $\text{rel} a = g[h_u(\nabla_u \nabla u \xi_{\perp})]$ along or orthogonal to the vector field $\xi_{\perp}$

$$\text{rel} a = \frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} + g[h_{\perp} (\text{rel} a)] = a_z + a_c, \quad (125)$$

where

$$a_z = \frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp}, \quad (126)$$

$$a_c = g[h_{\perp} (\text{rel} a)]. \quad (127)$$
If
\[
\frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} > 0
\]
the vector field \( a_z \) is called (relative) centrifugal acceleration. If
\[
\frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} < 0
\]
the vector field \( a_z \) is called (relative) centripetal acceleration.

The vector field \( a_c \) is called (relative) Coriolis’ acceleration.

### 3.1 Centrifugal (centripetal) acceleration

The relative acceleration \( \text{rel} a \) is orthogonal to the vector field \( u \).

**Proof:** From \( g(u, \text{rel} a) = g(u, \mathbf{g}[h_u(\nabla_u \nabla_u \xi_{\perp})]) = (u)(h_u)(\nabla_u \nabla_u \xi_{\perp}) \) and \( (u)(h_u) = h_u(u) = 0 \), it follows that
\[
g(u, \text{rel} a) = 0 .
\]  

(a) The centrifugal (centripetal) acceleration \( a_z \) is orthogonal to the vector field \( u \), i.e.
\[
g(u, a_z) = 0.
\]

**Proof:** From
\[
g(u, a_z) = g(u, \frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp}) =
\]
\[
= \frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot g(u, \xi_{\perp}) ,
\]
\[
g(u, \xi_{\perp}) = 0 ,
\]
it follows that \( g(u, a_z) = 0 \).

(b) The centrifugal (centripetal) acceleration \( a_z \) is orthogonal to the Coriolis acceleration \( a_c \), i.e.
\[
g(a_z, a_c) = 0.
\]

**Proof:** From
\[
g(\frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp}, \mathbf{g}[h_{\xi_{\perp}}(\text{rel} a)]) = \frac{g(\xi_{\perp}, \text{rel} a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot g(\xi_{\perp}, \mathbf{g}[h_{\xi_{\perp}}(\text{rel} a)]) ,
\]
\[
g(\xi_{\perp}, \mathbf{g}[h_{\xi_{\perp}}(\text{rel} a)]) = (\xi_{\perp})(h_{\xi_{\perp}}(\text{rel} a) ,
\]
\[
(h_{\xi_{\perp}}(\xi_{\perp}) = 0 ,
\]
it follows that \( g(a_z, a_c) = 0 \).

(c) The length of the vector \( a_z \) could be found on the basis of the relations
\[
a_z^2 = g(a_z, a_z) = \frac{[g(\xi_{\perp}, \text{rel} a)]^2}{g(\xi_{\perp}, \xi_{\perp})} .
\]
Therefore, in general, the square $a_z^2$ of the length of the centrifugal (centripetal) acceleration $a_z$ is reverse proportional to $\xi_\perp^2 = g(\xi_\perp, \xi_\perp)$.

Special case: $M_n = E_n$, $n = 3$ (3-dimensional Euclidean space).

$$\xi_\perp : = \overrightarrow{r} \cdot g(\xi_\perp, \xi_\perp) = r^2$$  
$$a_z^2 = \frac{|g(\overrightarrow{r}, \overrightarrow{a})|^2}{r^2}$$  
$$n_\perp : = \frac{\overrightarrow{r}}{r} \cdot g(n_\perp, n_\perp) = \frac{g(\overrightarrow{r}, \overrightarrow{r})}{r^2} = n_\perp^2 = 1$$  

(138)

The length $l_a = |g(a_z, a_z)|^{1/2}$ of the centrifugal (centripetal) acceleration $a_z$ is equal to the projection of the relative acceleration $\overrightarrow{rel \ a}$ at the unit vector field $n_\perp$ along the vector field $\xi_\perp$. If $l_a = g(n_\perp, \overrightarrow{rel \ a}) = 0$, i.e. if the relative acceleration $\overrightarrow{rel \ a}$ is orthogonal to the radius vector $\overrightarrow{r}$, then $a_z = 0$.

If the relative acceleration $\overrightarrow{rel \ a}$ is equal to zero then the centrifugal (centripetal) acceleration $a_z$ is also equal to zero.

(d) The scalar product $g(\xi_\perp, \overrightarrow{rel \ a})$ could be found in its explicit form by the use of the explicit form of $\overrightarrow{rel \ a}$:

$$\overrightarrow{rel \ a} = \overrightarrow{g}[A(\xi_\perp)] = \overrightarrow{g}[sD(\xi_\perp)] + \overrightarrow{g}[W(\xi_\perp)] + \frac{1}{n-1} \cdot \overrightarrow{U} \cdot \xi_\perp$$  

Then

$$g(\xi_\perp, \overrightarrow{rel \ a}) = sD(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \overrightarrow{U} \cdot g(\xi_\perp, \xi_\perp)$$  

$$a_z^2 = \frac{|g(\overrightarrow{D}, \overrightarrow{rel \ a})|^2}{g(\xi_\perp, \xi_\perp)} = \frac{1}{g(\xi_\perp, \xi_\perp)} \cdot [sD(\xi_\perp, \xi_\perp) + \frac{1}{n-1} \cdot \overrightarrow{U} \cdot g(\xi_\perp, \xi_\perp)]^2 = \frac{|sD(\xi_\perp, \xi_\perp)|^2}{g(\xi_\perp, \xi_\perp)} + \frac{1}{(n-1)^2} \cdot U^2 \cdot g(\xi_\perp, \xi_\perp) + \frac{2}{n-1} \cdot \overrightarrow{U} \cdot sD(\xi_\perp, \xi_\perp)$$  

(141)

Special case: $sD := 0$ (shear-free acceleration).

$$a_z^2 = \frac{1}{(n-1)^2} \cdot U^2 \cdot g(\xi_\perp, \xi_\perp)$$  

(142)

Special case: $U := 0$ (expansion-free acceleration).

$$a_z^2 = \frac{|sD(\xi_\perp, \xi_\perp)|^2}{g(\xi_\perp, \xi_\perp)}$$  

(143)

(e) The explicit form of $a_z$ could be found in the form

$$a_z = \frac{g(\overrightarrow{D}, \overrightarrow{rel \ a})}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp = \frac{sD(\xi_\perp, \xi_\perp)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp + \frac{1}{n-1} \cdot \overrightarrow{U} \cdot \xi_\perp$$  

20
\[
\mathbf{g} \left( \mathbf{v}, \mathbf{a}_\perp \right) = \mathbf{a}_\perp \cdot \mathbf{v} = (144)
\]

\[
\mathbf{g} \left( \mathbf{v}, \mathbf{a}_\perp \right) = \mathbf{a}_\perp \cdot \mathbf{v} = \frac{1}{n-1} \cdot \mathbf{U} \pm \mathbf{s} \mathbf{D} \left( \mathbf{n}_\perp, \mathbf{n}_\perp \right) \cdot \mathbf{a}_\perp . (145)
\]

If
\[
\frac{1}{n-1} \cdot \mathbf{U} + \frac{\mathbf{s} \mathbf{D} \left( \mathbf{\xi}_\perp, \mathbf{\xi}_\perp \right)}{\mathbf{g} \left( \mathbf{\xi}_\perp, \mathbf{\xi}_\perp \right)} > 0 \quad (146)
\]
a\_z is a centrifugal (relative) acceleration. If
\[
\frac{1}{n-1} \cdot \mathbf{U} + \frac{\mathbf{s} \mathbf{D} \left( \mathbf{\xi}_\perp, \mathbf{\xi}_\perp \right)}{\mathbf{g} \left( \mathbf{\xi}_\perp, \mathbf{\xi}_\perp \right)} < 0 \quad (147)
\]
a\_z is a centripetal (relative) acceleration.

In a theory of gravitation the centripetal (relative) acceleration could be interpreted as gravitational acceleration.

Special case: \( \mathbf{s} \mathbf{D} := 0 \) (shear-free relative acceleration).
\[
a\_z = \frac{1}{n-1} \cdot \mathbf{U} \cdot \mathbf{\xi}_\perp \quad (148)
\]

If the expansion acceleration invariant \( U > 0 \) the acceleration \( a\_z \) is a centrifugal (or expansion) acceleration. If \( U < 0 \) the acceleration \( a\_z \) is a centripetal (or contraction) acceleration. Therefore, in the case of a shear-free relative acceleration the centrifugal or the centripetal acceleration is proportional to the expansion acceleration invariant \( U \).

### 3.2 Coriolis’ acceleration

The vector field \( a\_c \), defined as
\[
a\_c = \bar{g} \left[ h\xi_\perp (rela) \right],
\]
is called Coriolis’ (relative) acceleration. On the basis of its definition, the Coriolis acceleration has well defined properties.

(a) The Coriolis acceleration is orthogonal to the vector field \( u \), i.e.
\[
g(u, a\_c) = 0. \quad (149)
\]

Proof: From
\[
g(u, a\_c) = g(u, \bar{g} \left[ h\xi_\perp (rela) \right]) = (u)(h\xi_\perp) (rela) = h\xi_\perp (u, rela)
\]
and
\[
(u)(h\xi_\perp) = (h\xi_\perp)(u) = g(u), \quad g(u, rela) = 0 \quad (150)
\]
it follows that
\[
g(u, a\_c) = [g(u)] (rela) = g(u, rela) = 0.
\]
(b) The Coriolis acceleration $a_c$ is orthogonal to the centrifugal (centripetal) acceleration $a_z$, i.e.

$$g(a_c, a_z) = 0 .$$  \hfill (151)

(c) The Coriolis acceleration $a_c$ is orthogonal to the deviation vector $\xi_\perp$, i.e.

$$g(\xi_\perp, a_c) = 0 .$$  \hfill (152)

(d) The length $\sqrt{|a_c|^2} = \sqrt{|g(a_c, a_c)|}$ of $a_c$ could be found by the use of the relations

\[
\begin{align*}
    a_c &= \overline{g}[h_{\xi_\perp(\text{rel} a)}] , \\
    g(\xi_\perp)(\text{rel} a) &= g(\xi_\perp, \text{rel} a) , \\
    h_{\xi_\perp(\text{rel} a)} &= g(\text{rel} a) - \frac{g(\xi_\perp, \text{rel} a)}{g(\xi_\perp, \xi_\perp)} , \\
    a_c &= \overline{g}[h_{\xi_\perp(\text{rel} a)}] = \text{rel} a - a_z = \\
          &= \text{rel} a - \frac{g(\xi_\perp, \text{rel} a)}{g(\xi_\perp, \xi_\perp)} \cdot \xi_\perp . \hfill (155)
\end{align*}
\]

For $a_c^2$ we obtain

\[
\begin{align*}
    a_c^2 &= g(a_c, a_c) = g(\text{rel} a - a_z, \text{rel} a - a_z) = \\
          &= g(\text{rel} a, a_c) + g(a_z, a_z) - 2 \cdot g(a_z, \text{rel} a) . \hfill (157)
\end{align*}
\]

Since

\[
\begin{align*}
    g(a_z, a_z) &= g(\text{rel} a, a_z) = \frac{[g(\xi_\perp)(\text{rel} a)]^2}{g(\xi_\perp, \xi_\perp)} , \\
    a_c^2 &= g(\text{rel} a, a_c) - g(a_z, a_z) = \text{rel} a^2 - a_z^2 , \\
    g(\text{rel} a, \text{rel} a) &= \text{rel} a^2 = g(\overline{g}[A(\xi_\perp)], \overline{g}[A(\xi_\perp)]) = \\
                        &= \overline{g}[A(\xi_\perp), A(\xi_\perp)] , \\
    A &= sD + W + \frac{1}{n-1} \cdot U \cdot h_u , \\
    A(\xi_\perp) &= sD(\xi_\perp) + W(\xi_\perp) + \frac{1}{n-1} \cdot U \cdot g(\xi_\perp) , \\
    h_u(\xi_\perp) &= g(\xi_\perp) , \\
    \overline{g}(sD(\xi_\perp), h_u(\xi_\perp)) &= sD(\xi_\perp, \xi_\perp) , \hfill (160) \\
    \overline{g}(W(\xi_\perp), h_u(\xi_\perp)) &= W(\xi_\perp, \xi_\perp) = 0 , \hfill (161) \\
    \overline{g}(h_u(\xi_\perp), h_u(\xi_\perp)) &= h_u(\xi_\perp, \xi_\perp) = g(\xi_\perp, \xi_\perp) , \hfill (162)
\end{align*}
\]
it follows for \( \text{rel}a^2 \)

\[
\text{rel}a^2 = \frac{g(\text{rel}a, \text{rel}a)}{g(\xi_{\perp}, \xi_{\perp})} = \frac{g(sD(\xi_{\perp}), sD(\xi_{\perp})) + 2 \cdot g(sD(\xi_{\perp}), W(\xi_{\perp})) + 2 \cdot sD(\xi_{\perp}, \xi_{\perp}) + 1}{(n-1)^2} \cdot U \cdot g(\xi_{\perp}, \xi_{\perp}) .
\]  

On the other side,

\[
a_z^2 = \frac{[sD(\xi_{\perp}, \xi_{\perp})]^2}{g(\xi_{\perp}, \xi_{\perp})} + \frac{1}{(n-1)^2} \cdot U^2 \cdot g(\xi_{\perp}, \xi_{\perp}) + \frac{2}{n-1} \cdot U \cdot sD(\xi_{\perp}, \xi_{\perp}) .
\]

Therefore,

\[
a_c^2 = \text{rel}a^2 - a_z^2 = \frac{g(sD(\xi_{\perp}), sD(\xi_{\perp})) - \frac{[sD(\xi_{\perp}, \xi_{\perp})]^2}{g(\xi_{\perp}, \xi_{\perp})}}{g(\xi_{\perp}, \xi_{\perp})} + \frac{\bar{g}(W(\xi_{\perp}), W(\xi_{\perp}))}{g(\xi_{\perp}, \xi_{\perp})} + \frac{2}{n-1} \cdot g(sD(\xi_{\perp}), W(\xi_{\perp})) .
\]  

\textit{Special case:} \( sD := 0 \) (shear-free acceleration).

\[
a_c^2 = \bar{g}[sD(\xi_{\perp}), W(\xi_{\perp})] .
\]  

\textit{Special case:} \( W := 0 \) (rotation-free acceleration).

\[
a_c^2 = \bar{g}[sD(\xi_{\perp}), sD(\xi_{\perp})] - \frac{[sD(\xi_{\perp}, \xi_{\perp})]^2}{g(\xi_{\perp}, \xi_{\perp})} \cdot g(\xi_{\perp}, \xi_{\perp}) + \frac{\bar{g}(W(\xi_{\perp}), W(\xi_{\perp}))}{g(\xi_{\perp}, \xi_{\perp})} .
\]  

The explicit form of \( a_c \) could be found by the use of the relations:

\[
a_c = \bar{g}[h_{\xi_{\perp}}(\text{rel}a)] = \text{rel}a - \frac{g(\xi_{\perp}, \text{rel}a)}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} = \text{rel}a - a_z = \frac{\bar{g}[sD(\xi_{\perp})\cdot sD(\xi_{\perp})]}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} + \bar{g}[W(\xi_{\perp})] = \frac{\bar{g}[sD(\xi_{\perp})]}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} + \bar{g}[W(\xi_{\perp})] .
\]  

Therefore, the Coriolis acceleration \( a_c \) does not depend on the expansion acceleration invariant \( U \).

\textit{Special case:} \( sD := 0 \) (shear-free acceleration).

\[
a_c = \bar{g}[W(\xi_{\perp})] .
\]  

\textit{Special case:} \( W := 0 \) (rotation-free acceleration).

\[
a_c = \bar{g}[sD(\xi_{\perp})] - \frac{sD(\xi_{\perp}, \xi_{\perp})}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} = \frac{\bar{g}[sD(\xi_{\perp})]}{g(\xi_{\perp}, \xi_{\perp})} \cdot \xi_{\perp} .
\]  

The Coriolis acceleration depends on the shear acceleration \( sD \) and on the rotation acceleration \( W \). These types of accelerations generate a Coriolis acceleration between particles or mass elements in a flow.
4 Centrifugal (centripetal) acceleration as gravitational acceleration

1. The main idea of the Einstein theory of gravitation (ETG) is the identification of the centripetal acceleration with the gravitational acceleration. The weak equivalence principle stays that a gravitational acceleration could be compensated by a centripetal acceleration and vice versa. From this point of view, it is worth to be investigated the relation between the centrifugal (centripetal) acceleration and the Einstein theory of gravitation as well as the possibility for describing the gravitational interaction as result of the centrifugal (centripetal) acceleration generated by the motion of mass elements (particles).

The structure of the centrifugal (centripetal) acceleration could be considered on the basis of its explicit form expressed by means of the kinematic characteristics of the relative acceleration and the relative velocity. The centrifugal (centripetal) acceleration is written as

\[ a_z = \left[ \frac{1}{n - 1} \cdot U + \frac{D(\xi_\perp \cdot \xi_\perp)}{g(\xi_\perp \cdot \xi_\perp)} \right] \cdot \xi_\perp . \]

The vector field \( \xi_\perp \) is directed outside of the trajectory of a mass element (particle). If the mass element generates a gravitational field the acceleration \( a_z \) should be in the direction to the mass element. If the mass element moves in an external gravitational field caused by another gravitational source then the acceleration \( a_z \) should be directed to the source. The expansion (contraction) invariant \( U \) could be expressed by means of the kinematic characteristics of the relative acceleration or of the relative velocity in the forms [3], [9]

\[ U = U_0 + I = F U_0 - T U_0 + I, \quad U_0 = F U_0 - T U_0 , \]

where \( F U_0 \) is the torsion-free and curvature-free expansion acceleration, \( T U_0 \) is the expansion acceleration induced by the torsion and \( I \) is the expansion acceleration induced by the curvature, \( U_0 \) is the curvature-free expansion acceleration

\[ F U_0 = g[b] - \frac{1}{c} \cdot g(u, \nabla u a) , \quad F U_0 = a^k \cdot k - \frac{1}{c} \cdot g(P) \cdot u^k \cdot a^l \cdot m \cdot u^m , \]

\[ U_0 = g[b] - g[\sigma] - g(Q(\phi)\sigma \cdot \theta_1 - \frac{1}{n - 1} \cdot \theta_1 \cdot \theta - \frac{1}{e} \cdot [g(u, T(a, u)) + g(u, \nabla u a)] , \]

\[ I = R_{ij} \cdot u^i \cdot u^j , \quad g[b] = g[b] \cdot b^i \cdot b^j = g[b] \cdot a^i \cdot m \cdot g[a] = a^i \cdot m . \]

2. In the ETG only the term \( I \) is used on the basis of the Einstein equations. The invariant \( I \) represents an invariant generalization of Newton’s gravitational
law \[13\]. In a $V_n$-space ($n = 4$) of the ETG, a free moving spinless test particle with $a = 0$ will have an expansion (contraction) acceleration $U = I$ ($U_0 = 0$) if $R_{ij} \neq 0$ and $U = I = 0$ if $R_{ij} = 0$. At the same time, in a $V_n$-space (the bars over the indices should be omitted)

\[ sD = sM , \quad sD_0 = 0 , \quad M = h_u(K_s)h_u , \quad (177) \]

\[ M_{ij} = h_{ik} R_{kl}^{\text{R}} h_{lj} = \frac{1}{2} h_{ik} (K_s^{kl} + K_s^{lk}) h_{lj} = \]

\[ = \frac{1}{2} h_{ik} (R^k_{\text{mnr}} u^m u^n g^{rl} + R^l_{\text{mnr}} u^m u^n g^{kr}) h_{lj} = \]

\[ = \frac{1}{2} u^m u^n (h_{ik} R^k_{\text{mnr}} g^{rl} h_{lj} + h_{ik} R^l_{\text{mnr}} g^{kr} h_{lj}) = \]

\[ = \frac{1}{2} u^m u^n [h_{ik} R^k_{\text{mnr}} g^{rl} (g_{lj} - \frac{1}{e} u_t u_j) + \]

\[ + h_{lj} R^k_{\text{mnr}} g^{kr} (g_{ij} - \frac{1}{e} u_t u_i) . \quad (178) \]

Since

\[ h_{ik} R^k_{\text{mnr}} g^{rl} (g_{lj} - \frac{1}{e} u_t u_j) = \]

\[ = (g_{ik} - \frac{1}{e} u_i u_k) \cdot R^k_{\text{mnr}} g^{rl} g_{lj} - \]

\[ - \frac{1}{e} (g_{ik} - \frac{1}{e} u_i u_k) \cdot R^k_{\text{mnr}} g^{rl} u_t u_j , \]

\[ R^k_{\text{mnr}} g^{rl} u_t = R^k_{\text{mnr}} u^l , \quad u_t R^k_{\text{mnr}} = g_{ik} u^s R^k_{\text{mnr}} , \]

we have

\[ h_{ik} R^k_{\text{mnr}} g^{rl} h_{lj} = g_{ik} R^k_{\text{mnr}} g^{rl} g_{lj} - \]

\[ - \frac{1}{e} u_i g_{ik} u^s R^k_{\text{mnr}} g^{rl} g_{lj} - \]

\[ - \frac{1}{e} g_{ik} R^k_{\text{mnr}} g^{rl} g_{lj} u^s u_j + \]

\[ + \frac{1}{e^2} u_i g_{ik} u^s R^k_{\text{mnr}} g^{rl} g_{lj} u^g u_j . \quad (179) \]

**Special case:** $V_n$-space: $S := C$.

\[ M_{ij} = R_{imnj} u^m u^n . \quad (180) \]

\[ sD_{ij} = sM_{ij} = M_{ij} - \frac{1}{n-1} I \cdot h_{ij} = \]

\[ 25 \]
\[ R_{lmn} \cdot u^m \cdot u^n - \frac{1}{n-1} \cdot R_{mn} \cdot u^m \cdot u^n \cdot h_{ij}, \]

\[ sD_{ij} = (R_{lmn} - \frac{1}{n-1} \cdot R_{mn} \cdot h_{ij}) \cdot u^m \cdot u^n = sM_{ij}, \tag{181} \]

\[ U = I = R_{ij} \cdot u^i \cdot u^j. \]

In a \( V_n \)-space the components \( a_z^i \) of the centrifugal (centripetal) acceleration \( a_z \) have the form

\[ a_z^i = \left( \frac{1}{n-1} \cdot U + \frac{sD_{jk} \cdot \xi_j \cdot \xi_k}{g_{rs} \cdot \xi_r \cdot \xi_s} \right) \cdot \xi_i = \]

\[ = \left[ \frac{1}{n-1} \cdot R_{mn} \cdot u^m \cdot u^n + \right] \]

\[ + \frac{1}{g_{rs} \cdot \xi_r \cdot \xi_s} \cdot (R_{jmnk} - \frac{1}{n-1} \cdot R_{mn} \cdot h_{jk}) \cdot u^m \cdot u^n \cdot \xi_j \cdot \xi_k \cdot \xi_i. \]

3. If the Einstein equations in vacuum without cosmological term (\( \lambda_0 = 0 \)) are valid, i.e. if

\[ R_{ij} = 0, \quad \lambda_0 = \text{const.} = 0, \quad n = 4, \tag{183} \]

are fulfilled then

\[ a_z^i = \frac{1}{g_{rs} \cdot \xi_r \cdot \xi_s} \cdot R_{jmnk} \cdot u^m \cdot u^n \cdot \xi_j \cdot \xi_k \cdot \xi_i = \]

\[ = R_{jmnk} \cdot u^m \cdot u^n \cdot n^j \cdot n^k \cdot \xi_i = g \cdot \xi_i, \tag{185} \]

where

\[ n^j = \frac{\xi_j}{\sqrt{|g_{rs} \cdot \xi_r \cdot \xi_s|}}, \tag{186} \]

\[ g = R_{jmnk} \cdot u^m \cdot u^n \cdot n^j \cdot n^k. \tag{187} \]

If \( a_z \) is a centripetal acceleration interpreted as gravitational acceleration for a free spinless test particles moving in an external gravitational field (\( R_{ij} = 0 \)) then the condition \( g < 0 \) should be valid if the centripetal acceleration is directed to the particle. If the centripetal acceleration is directed to the gravitational source (in the direction \( \xi_i \) ) then \( g > 0 \).

From a more general point of view as that in the ETG, a gravitational theory could be worked out in a \( (T_n, g) \)-space where \( a_z \) could also be interpreted as gravitational acceleration of mass elements or particles generating a gravitational field by themselves and caused by their motions in space-time.

4. If we consider a frame of reference in which a mass element (particle) is at rest then \( u^i = g_1^i \cdot u^4 \) and \( \xi_i = g_2^i \cdot \xi_1^a := g_1^i \cdot \xi_1^a \). The centrifugal (centripetal) acceleration \( a_z^i \) could be written in the form

\[ a_z^i = R_{1441} \cdot u^4 \cdot u^4 \cdot n^1 \cdot n^1 \cdot \xi_i = R_{1441} \cdot (u^4)^2 \cdot (n^1)^2 \cdot \xi_i. \tag{188} \]
For the Schwarzschild metric
\[ ds^2 = \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 \left( d\theta^2 + \sin^2 \theta \cdot d\varphi^2 \right) - \left( 1 - \frac{r_g}{r} \right) \cdot (dx^4)^2 , \]  
(189)
\[ r_g = \frac{2 \cdot k \cdot M_0}{c^2} , \]  
(190)
the component \( R_{1441} = g_{11} \cdot R^1_{1441} \) of the curvature tensor has the form
\[ R_{1441} = -g_{11} \cdot (\Gamma^1_{44,1} - \Gamma^1_{14,4} + \Gamma^1_{11} \cdot \Gamma^1_{44} + \Gamma^1_{14} \cdot \Gamma^4_{44} - \Gamma^1_{14} \cdot \Gamma^1_{44} - \Gamma^1_{44} \cdot \Gamma^4_{41}) . \]  
(191)
After introducing in the last expression the explicit form of the metric and of the Christoffel symbols \( \Gamma^i_{jk} \), it follows for \( R_{1441} \)
\[ R_{1441} = \frac{r_g}{r^3} . \]  
(192)
Then
\[ a^i_z = \frac{r_g}{r^3} \cdot (u^4)^2 \cdot (n^1)^2 \cdot \xi^i_\perp . \]  
(193)
If the co-ordinate time \( t = x^4/c \) is chosen as equal to the proper time \( \tau \) of the particle, i.e. if \( t = \tau \) then
\[ u^4 = \frac{dx^4}{d\tau} = c \cdot \frac{d\tau}{d\tau} = c , \quad n^1 = 1 , \quad \xi^1_\perp = g^i_1 \cdot \xi^i_\perp , \]  
(194)
\[ a^i_z = \frac{r_g}{r^3} \cdot c^2 \cdot g^i_1 \cdot \xi^i_\perp , \quad \frac{r_g}{r^3} \cdot c^2 > 0 . \]  
(195)
For the centrifugal (centripetal) acceleration we obtain
\[ a^i_z = \frac{2 \cdot k \cdot M_0}{r^3} \cdot \xi^i_\perp , \]  
(196)
which is exactly the relative gravitational acceleration between two mass elements (particles) with co-ordinates \( x_{c1} = r \) and \( x_{c2} = r + \xi^i_\perp \) [14].
Therefore, the centrifugal (centripetal) acceleration could be used for working out of a theory of gravitation in a space with affine connections and metrics as this has been done in the Einstein theory of gravitation.

5 Conclusions

In the present paper the notions of centrifugal (centripetal) and Coriolis’ velocities and accelerations are introduced and considered in spaces with affine connections and metrics as velocities and accelerations of flows of mass elements (particles) moving in space-time. It is shown that these types of velocities and accelerations are generated by the relative motions between the mass elements. The null (isotropic) vector fields are considered and their relations with the centrifugal (centripetal) velocity are established. The centrifugal (centripetal) velocity is found to be in connection with the Hubble law and the generalized
Doppler effect in spaces with affine connections and metrics. The accelerations are closely related to the kinematic characteristics of the relative velocity and relative acceleration. The centrifugal (centripetal) acceleration could be interpreted as gravitational acceleration as it has been done in the Einstein theory of gravitation. This fact could be used as a basis for working out of new gravitational theories in spaces with affine connections and metrics.

References

[1] Manoff S., Mechanics of continuous media in $(\mathcal{L}_n, g)$-spaces. 1. Introduction and mathematical tools. E-print gr-qc /0203016

[2] Manoff S., Mechanics of continuous media in $(\mathcal{L}_n, g)$-spaces. 2. Relative velocity and deformations. E-print gr-qc /0203017

[3] Manoff S., Mechanics of continuous media in $(\mathcal{L}_n, g)$-spaces. 3. Relative accelerations. E-print gr-qc /0204003

[4] Manoff S., Mechanics of continuous media in $(\mathcal{L}_n, g)$-spaces. 4. Stress (tension) tensor. E-print gr-qc /0204004

[5] Stephani H., Allgemeine Relativitätstheorie (VEB Deutscher Verlag d. Wissenschaften, Berlin, 1977), pp. 75-76

[6] Ehlers J., Beitraege zur relativistischen Mechanik kontinuierlicher Medien. Abhandlungen d. Mainzer Akademie d. Wissenschaften, Math.-Naturwiss. Kl. Nr. 11 (1961)

[7] Laemmerzahl Cl., A Characterisation of the Weylian structure of Space-Time by Means of Low Velocity Tests. E-print gr-qc /0103047

[8] Bishop R. L., Goldberg S. I., Tensor Analysis on Manifolds (The Macmillan Company, New York, 1968)

[9] Manoff S., Kinematics of vector fields. In Complex Structures and Vector Fields. eds. Dimiev St., Sekigawa K. (World Sci. Publ., Singapore, 1995), pp. 61-113

[10] Manoff S., Spaces with contravariant and covariant affine connections and metrics. Physics of elementary particles and atomic nucleus (Physics of Particles and Nuclei) [Russian Edition: 30 (1999) 5, 1211-1269], [English Edition: 30 (1999) 5, 527-549]

[11] Manoff S., Geometry and Mechanics in Different Models of Space-Time: Geometry and Kinematics. (Nova Science Publishers, New York, 2002)

[12] Manoff S., Geometry and Mechanics in Different Models of Space-Time: Dynamics and Applications. (Nova Science Publishers, New York, 2002)
[13] Manoff S., *Einstein’s theory of gravitation as a Lagrangian theory for tensor fields*. Intern. J. Mod. Phys. A 13 (1998) 12, 1941-1967

[14] Manoff S., *About the motion of test particles in an external gravitational field*. Exp. Technik der Physik 24 (1976) 5, 425-431 (in German)

[15] Misner Ch. W., Thorne K. S., Wheeler J. A., *Gravitation*. (W. H. Freeman and Company, San Francisco, 1973). Russian translation: Vol 1., Vol. 2., Vol. 3. (Mir, Moscow, 1977)

[16] Manoff S., *Flows and particles with shear-free and expansion-free velocities in (L_\eta, g)- and Weyl’s spaces*. Class. Quantum Grav. 19 (2002) 16, 4377-4398. E- print gr-qc/02 07 060