Lightlike hypersurfaces in indefinite $S$-manifolds

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Abstract

In a metric $g,f,f$-manifold we study lightlike hypersurfaces $M$ tangent to the characteristic vector fields, and owing to the presence of the $f$-structure, we determine some decompositions of $TM$ and of a chosen screen distribution obtaining two distributions invariant with respect to the structure. We discuss the existence of a $g,f,f$-structure on a lightlike hypersurface and, under suitable hypotheses, we obtain an indefinite $S$-structure on the leaves of an integrable distribution. The existence of totally umbilical lightlike hypersurfaces of an indefinite $S$-space form is also discussed. Finally, we explicitly describe a lightlike hypersurface of an indefinite $S$-manifold.

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1 Introduction

In the wide field of semi-Riemannian geometry, the study of lightlike, or degenerate, submanifolds comes now to fill a gap in the general theory of submanifolds. In fact, while the geometry of submanifolds of Riemannian manifolds, since [1], has received powerful impulse, polarizing a lot of attention, with studies conducted in great generality and developing a great variety of techniques, on the contrary the study of degenerate geometry is a relatively new field of research. It rises within the semi-Riemannian context, due to the existence of the so-called causal character of geometrical objects: their spacelike, timelike or lightlike nature, in fact, implies the existence of three types of hypersurfaces and submanifolds. Among them, the spacelike and timelike cases have received a systematic exposition in the fundamental book [22]. We have to be looking forward 1990 to find the first studies about the lightlike case, when A. Bejancu and K.L. Duggal introduced the lightlike geometry ([12],[13],[14],[15]). At the moment lightlike hypersurfaces are studied in paraquaternionic Kähler manifolds’ contexts by S. Ianuș, R. Mazzocco and G. E. Vîlcu ([18]).

The primary difference between the lightlike submanifolds and non-degenerate submanifolds arises due to the fact that in the first case the normal vector bundle has non trivial intersection with the tangent vector bundle, and moreover in a lightlike hypersurface the normal vector bundle is contained in the tangent vector bundle. Thus, one fails to use the classical theory of non-degenerate submanifolds to define the induced geometrical objects (as linear connection, second fundamental form, Gauss and Weingarten equations) on a lightlike hypersurface.

The growing importance of lightlike geometry is motivated by its extensive use in mathematical physics, in particular in relativity. In fact, semi-Riemannian manifolds $(\bar{M}, \bar{g})$ with $\dim \bar{M} = n > 4$ are natural generalizations of spacetime of general relativity and lightlike hypersurfaces are models of different types of horizons separating domains of $(\bar{M}, \bar{g})$ with different physical properties.

There exist also reasons that motivate the study of the lightlike hypersurfaces of indefinite $g,f,f$-manifolds, in particular of indefinite $S$-manifolds. In [15], K.L. Duggal and A. Bejancu proved that a lightlike framed hypersurface of a Lorentz $C$-manifold, with an induced metric connection, is a Killing horizon. In a recent paper ([16]), K.L. Duggal and B. Sahin begin to work on lightlike
submanifolds of indefinite Sasakian manifolds because the contact geometry has a significant use in differential equations, optics and phase spaces of dynamical systems. Furthermore, in [13], K.L. Duggal shows that a globally hyperbolic spacetime and the de Sitter spacetime can carry a framed structure.

We begin with some basic information about indefinite $S$-manifolds and about lightlike hypersurfaces of a semi-Riemannian manifold. Afterwards, for a metric $g.f.f$-manifold we consider a lightlike hypersurface $M$ tangent to the characteristic vector fields, we introduce a particular screen distribution $S(TM)$, using the properties of the indefinite $S$-manifold. Then, finding other decompositions of $S(TM)$ and $TM$ yields two distributions $D_0$ and $D$ on $M$ that are studied in section 4. We discuss the existence of a $g.f.f$-structure on a lightlike hypersurface and we obtain an indefinite $S$-structure on the leaves of $D_0$, if $D_0$ is an integrable distribution. Section 5 deals with the existence of totally umbilical lightlike hypersurfaces of an indefinite $S$-space form. In the last section, we consider the three examples of indefinite $S$-manifolds given in [10]. For the first one we explicitely describe a lightlike hypersurface, to which we apply the previous results, while, for the other two examples, we prove that they do not admit lightlike hypersurfaces tangent to the characteristic vector fields.

All manifolds, tensor fields and maps are assumed to be smooth. We shall use the Einstein convention omitting the sum symbol for repeated indexes.

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2 Preliminaries

A manifold $\tilde{M}$ is called a globally framed $f$-manifold if it is endowed with a nonnull $(1,1)$-tensor field $\tilde{\varphi}$ of constant rank, such that $\ker \tilde{\varphi}$ is parallelizable i.e. there exist global vector fields $\tilde{\xi}_\alpha$, $\alpha \in \{1, \ldots, r\}$, with their dual 1-forms $\tilde{\eta}^\alpha$, satisfying $\tilde{\varphi}^2 = -I + \tilde{\eta}^\alpha \otimes \tilde{\xi}_\alpha$ and $\tilde{\eta}^\alpha(\tilde{\xi}_\beta) = \delta^\alpha_\beta$.

The $g.f.f$-manifold $(M^{2n+r}, \tilde{\varphi}, \tilde{\xi}_\alpha, \tilde{\eta}^\alpha, \tilde{g})$, $\alpha \in \{1, \ldots, r\}$ is said an indefinite metric $g.f.f$-manifold if $\tilde{g}$ is a semi-Riemannian metric, with index $\nu$, $0 < \nu < 2n + r$, satisfying the following compatibility condition

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = \tilde{g}(X, Y) - \varepsilon_\alpha \tilde{\eta}^\alpha(X) \tilde{\eta}^\alpha(Y)$$

for any $X, Y \in \Gamma T \tilde{M}$, being $\varepsilon_\alpha = \pm 1$ according to whether $\tilde{\xi}_\alpha$ is spacelike or timelike. Then, for any $\alpha \in \{1, \ldots, r\}$, one has $\tilde{\eta}^\alpha(X) = \varepsilon_\alpha \tilde{g}(X, \tilde{\xi}_\alpha)$.

An indefinite metric $g.f.f$-manifold is called indefinite $S$-manifold if it is normal and $d\tilde{\eta}^\alpha = \Phi$, for any $\alpha \in \{1, \ldots, r\}$, where $\Phi(X, Y) = \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y)$ for any $X, Y \in \Gamma (TM)$. The normality condition is expressed by the vanishing of the tensor field $N = \tilde{\eta} + 2d\tilde{\eta}^\alpha \otimes \tilde{\xi}_\alpha$, $\nabla \tilde{\xi}_\alpha$ being the Nijenhuis torsion of $\tilde{\varphi}$. Furthermore, as proved in [10], the Levi-Civita connection of an indefinite $S$-manifold satisfies:

$$\langle \nabla_{\tilde{\varphi}X} \tilde{\varphi}Y \rangle = \tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y)\tilde{\xi} + \tilde{\eta}(Y)\varphi^2(X) \tag{1}$$

where $\tilde{\xi} = \sum_{\alpha=1}^r \tilde{\xi}_\alpha$ and $\tilde{\eta} = \sum_{\alpha=1}^r \varepsilon_\alpha \tilde{\eta}^\alpha$. We recall that $\nabla_{\tilde{\varphi}X} \tilde{\xi}_\alpha = -\varepsilon_\alpha \tilde{\varphi}X$ and $\ker \tilde{\varphi}$ is an integrable flat distribution since $\nabla \tilde{\xi}_\alpha \tilde{\xi}_\beta = 0$. For more details we refer to [10].

Following [15], we recall some basic results about lightlike hypersurfaces of a semi-Riemannian manifold $(M, \tilde{g})$. Given a lightlike hypersurface $M$ of $\tilde{M}$, one can consider for any $p \in M$ the vector spaces:

$$T_p M^\perp = \{ E_p \in T_p \tilde{M} \mid \tilde{g}_p(E_p, W) = 0 \text{ for all } W \in T_p M \},$$

$$Rad(T_p M) = \{ V \in T_p M \mid g_p(V, W) = 0 \text{ for all } W \in T_p M \} = T_p M^\perp \cap T_p M.$$
Then, \( \text{Rad}(T_pM) = T_pM^\perp \subset T_pM \) and one has the 1-dimensional degenerate distribution on \( M \), called the radical distribution. A screen distribution on \( M \) is defined as a distribution complementary to the radical one, so that we have

\[
TM = \text{Rad}(TM) \perp S(TM) \\
T\bar{M}|_M = S(TM) \perp S(TM)^\perp
\]

where \( S(TM)^\perp \) is the complementary vector bundle to \( S(TM) \) in \( T\bar{M}|_M \) with respect to \( \bar{g} \). Obviously, there exist several screen distributions and they are non-degenerate.

We report the following theorem proved in [15], adapting it to our context.

**Theorem 2.1 ([15])** Let \( (M, g, S(TM)) \) be a lightlike hypersurface of an indefinite g.f.f.-manifold \( (\bar{M}, \bar{\varphi}, \bar{\xi}_n, \bar{\eta}^a, \bar{g}) \). Then there exists a unique one vector subbundle \( \text{ltr}(M) \) of \( T\bar{M} \), with base space \( M \), such that for any non-zero section \( E \) of \( \text{Rad}(TM) \) on a coordinate neighbourhood \( U \subset M \), there exists a unique section \( N \) of \( \text{ltr}(M) \) on \( U \) satisfying:

\[
\bar{g}(N, E) = 1, \quad \bar{g}(N, N) = 0, \quad \bar{g}(N, W) = 0 \quad \text{for all} \quad W \in \Gamma(S(TM)|_U).
\]

\( \text{ltr}(M) \) is called the lightlike transversal vector bundle of \( M \) with respect to \( S(TM) \).

One can consider the following decompositions

\[
S(TM)^\perp = \text{Rad}(TM) \oplus \text{ltr}(M),
\]

\[
T\bar{M}|_M = TM \oplus \text{ltr}(M).
\]

Let \( \bar{\nabla} \) be the Levi-Civita connection on \( \bar{M} \). Using \([6]\) we deduce

\[
\bar{\nabla}_XY = \nabla_XY + h(X, Y), \quad \bar{\nabla}_XV = -A_YX + \nabla^\text{ltr}_XV,
\]

for any \( X, Y \in \Gamma(TM) \) and \( V \in \Gamma(\text{ltr}M) \). Following [15], \( \nabla \) and \( \nabla^\text{ltr} \) are called the induced connections on \( M \) and \( \text{ltr}(M) \) respectively, and as in the classical theory of Riemannian hypersurfaces, \( h \) and \( A_V \) are called the second fundamental form and the shape operator, respectively. Further, the above equations are cited as the Gauss and Weingarten equation, respectively. Locally, let \( E, N \) and \( U \) be as in Theorem 2.1 then for any \( X, Y \in \Gamma(TM|_U) \), putting:

\[
B(X, Y) = \bar{g}(h(X, Y), E), \quad \text{and} \quad \tau(X) = \bar{g}(\nabla^\text{ltr}_YN, E),
\]

for any \( X, Y \in \Gamma(TM|_U) \), we can write:

\[
\bar{\nabla}_XY = \nabla_XY + B(X, Y)N, \quad \text{and} \quad \bar{\nabla}_XN = -A_XY + \tau(X)N,
\]

\( B \) is called the local second fundamental form of \( M \), because it determines \( h \) on \( U \). As proved in [3], the local second fundamental form of \( M \) on \( U \) is independent of the choice of the screen distribution. Moreover \( B \) is degenerate and \( B(X, E) = 0 \) for any \( X \in \Gamma(TM|_U) \).

The decomposition [2] allows to define a canonical projection \( P : \Gamma(TM) \rightarrow \Gamma(S(TM)) \).

Then, for any \( X, Y \in \Gamma(TM) \) and \( U \in \Gamma(\text{Rad}(TM)) \) we can write

\[
\nabla_XPY = \nabla^*_XPY + \hat{h}(X, PY), \quad \text{and} \quad \nabla_XU = -\hat{A}_U X + \nabla^*_XU,
\]

where \( \nabla \) and \( \nabla^* \) are linear connections on the bundles \( S(TM) \) and \( \text{Rad}(TM) \), respectively. Further, \( \hat{h} : \Gamma(TM) \times \Gamma(S(TM)) \rightarrow \Gamma(\text{Rad}(TM)) \) is \( \mathfrak{g}(M) \)-bilinear and \( \hat{A}_U : \Gamma(TM) \rightarrow \Gamma(S(TM)) \) is an \( \mathfrak{g}(M) \)-linear operator and they are called the second fundamental form and the shape operator of
the screen distribution, respectively. The equations in (7) are cited as the Gauss equation and the Weingarten equation. Locally, let $\mathcal{U}$ be a coordinate neighbourhood of $M$, and $E$, $N$ sections on $\mathcal{U}$, as in Theorem 2.1. Then, putting $C(X, PY) = \bar{g}(h(X, PY), N)$ for any $X, Y \in \Gamma(TM|_{\mathcal{U}})$, one has $\bar{h}(X, PY) = C(X, PY)E$, $\bar{g}(\nabla_X E, N) = -\tau(X)$ and, locally on $\mathcal{U}$, (7) becomes

$$\nabla_X PY = \nabla_X PY + C(X, PY)E, \quad \nabla_X E = -\bar{A}_E E - \tau(X)E.$$  

Finally, geometrical objects of the lightlike hypersurface and of the screen distribution are related as follows, for any $X, Y \in \Gamma(TM)$:

$$g(A_N X, PY) = C(X, PY), \quad g(\bar{A}_E X, PY) = B(X, PY), \quad \bar{g}(A_N X, N) = 0, \quad \bar{g}(\bar{A}_E X, N) = 0.$$  

Furthermore, one has: $\bar{A}_E E = 0$, $\nabla_E E = \nabla_E E = -\tau(E)E$.

**Remark 2.2** The induced connection $\nabla$ on $M$ does not depend on the choice of $S(TM)$ if and only if the second fundamental form $h$ of $M$ vanishes identically. Furthermore, $\nabla$ is not a metric connection, in fact it satisfies $(\nabla_X g)(Y, Z) = B(X, Y)\bar{g}(Z, N) + B(X, Z)\bar{g}(Y, N)$, for any $X, Y, Z \in \Gamma(TM|_{\mathcal{U}})$. However, if we choose $Y, Z \in \Gamma(S(TM))$, we get $(\nabla_X g)(Y, Z) = 0$, and using (7) we easily obtain that the linear connection $\nabla$ on $S(TM)$ is a metric connection. Finally, $S(TM)$ is an integrable distribution if and only if $h$ is symmetric on $S(TM)$.

### 3 Characteristic lightlike hypersurfaces of indefinite g.f.f-manifolds

Let $(\bar{M}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ be an indefinite g.f.f-manifold and $\bar{M}$ a lightlike hypersurface. From the existence of the $f$-structure, for any $Z \in \Gamma(\text{Rad}(TM))$, one has $\bar{g}(\bar{\varphi} Z, Z) = 0$, therefore $\bar{\varphi} Z \in \Gamma(TM)$, and we get a 1-dimensional distribution $\bar{\varphi}(\text{Rad}(TM))$ on $\bar{M}$.

Moreover, it is easy to state the following result.

**Proposition 3.1** Let $\bar{M}$ be a lightlike hypersurface of an indefinite g.f.f-manifold $(\bar{M}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$ such that the characteristic vector fields $\bar{\xi}_\alpha$ are tangent to $\bar{M}$. Let $E$ be a non zero section of $\text{Rad}(TM)$. Then there exists a screen distribution such that $\ker \bar{\varphi} \subset S(TM)$ and $\bar{\varphi}(E)$ belongs to $\Gamma(S(TM))$.

**Definition 3.2** A lightlike hypersurface $M$ of $\bar{M}$ is called characteristic if all the characteristic vector fields $\bar{\xi}_\alpha$ are tangent to $M$. A screen distribution $S(TM)$ will be called characteristic if $\ker \bar{\varphi} \subset S(TM)$ and $\bar{\varphi}(E) \in \Gamma(S(TM))$.

From now on, we shall write simply $(M, g, S(TM))$ to denote a characteristic lightlike hypersurface $(\bar{M}, \bar{g}, S(TM))$, together with the choices of a fixed non zero section $E$ of $\text{Rad}(TM)$, a fixed characteristic screen distribution $S(TM)$ and the ltr$(M)$ and $N$ as in Theorem 2.1. In these hypotheses $\bar{\varphi} N \in \Gamma(S(TM))$. Namely, the vector field $\bar{\varphi} N \in \Gamma(TM|_{\bar{M}})$, and using (7), we have that $\bar{\varphi} N$ is orthogonal to $S(TM)\perp$ since $\bar{g}(\bar{\varphi} N, E) = -\bar{g}(N, \bar{\varphi} E) = 0$, and obviously $\bar{g}(\bar{\varphi} N, N) = 0$. Moreover, from the compatibility condition, we obtain $\bar{g}(\bar{\varphi} N, \bar{\varphi} E) = 1$.

**Remark 3.3** Given an indefinite g.f.f-manifold $(\bar{M}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})$, the existence of a characteristic lightlike hypersurface implies that $\dim \bar{M} = 2n + r$ with $n \geq 2$, being $E, N, \bar{\varphi}(E)$ linearly independent.
Examples of characteristic hypersurfaces and characteristic screen distributions of indefinite $S$-manifolds are given in section [6]. It is easy to prove the following result ([9]).

**Proposition 3.4** Let $(M, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^a, \bar{g})$ be an indefinite $g.f.f$-manifold and $(M, g, S(TM))$ a characteristic lightlike hypersurface of $\bar{M}$. Then the rank 2 vector subbundle $\bar{\varphi}(\text{Rad}(TM)) \oplus \bar{\varphi}(\text{ltr}(M))$ of $S(TM)$ is non-degenerate.

Following [13], being $S(TM)$ and $\bar{\varphi}(\text{Rad}(TM)) \oplus \bar{\varphi}(\text{ltr}(M))$ non-degenerate, we can define the unique non-degenerate distribution $D_0$ such that

$$S(TM) = (\bar{\varphi}(\text{Rad}(TM)) \oplus \bar{\varphi}(\text{ltr}(M))) \perp D_0.$$  \hfill (10)

Then, each $\bar{\xi}_\alpha \in D_0$, $D_0$ is $\bar{\varphi}$-invariant and, using [2], [15] and [10], we can write:

$$TM = D_0 \perp F$$  \hfill (11)

$$TM|_M = D_0 \perp E$$  \hfill (12)

$$TM = D \oplus \bar{\varphi}(\text{ltr}(M))$$  \hfill (13)

where $F := (\bar{\varphi}(\text{Rad}(TM)) \oplus \bar{\varphi}(\text{ltr}(M)) \perp \text{Rad}(TM)$, $E := (\bar{\varphi}(\text{Rad}(TM)) \oplus \bar{\varphi}(\text{ltr}(M)) \perp \text{Rad}(TM)$, and $D := D_0 \perp \bar{\varphi}(\text{Rad}(TM)) \perp \text{Rad}(TM)$, is a $\bar{\varphi}$-invariant distribution.

Now, we look for a $g.f.f$-structure on $(M, g, S(TM))$. We consider the local lightlike vector fields $U := -\bar{\varphi}N \in \bar{\varphi}(\text{ltr}(M))$ and $V := -\bar{\varphi}E \in D$. From [1K] any $X \in \Gamma(TM)$ can be written as

$$X = SX + QX \quad \text{and} \quad QX = u(X)U,$$

where $S : TM \rightarrow D$ and $Q : TM \rightarrow \bar{\varphi}(\text{ltr}(M))$ are the canonical projection maps, and $u$ is a local 1-form on $M$ defined by $u(X) := g(X, V)$. We note that

$$u(U) = 1, \quad \forall Y \in \Gamma(D) \quad u(Y) = 0, \quad \bar{\varphi}^2 N = -N.$$  \hfill (14)

Then, applying $\bar{\varphi}$ to $X$, we obtain $\bar{\varphi}X = \bar{\varphi}(SX) + u(X)\bar{\varphi}U = \bar{\varphi}(SX) + u(X)N$. For any $X \in \Gamma(TM)$ we put $\varphi X := \bar{\varphi}(SX)$, obtaining a tensor field $\varphi$ of type (1,1) on $M$. From the above equality we get then

$$\bar{\varphi}X = \varphi X + u(X)N,$$  \hfill (15)

and applying again $\bar{\varphi}$, we have $-X + \bar{\eta}^a(\varphi X)\bar{\xi}_\alpha = \varphi\varphi X - u(X)U$. We note that if $X \in \Gamma(TM)$, then $SX \in D$, $\varphi X = \bar{\varphi}(SX) \in D$, so that $S(\varphi X) = \varphi X$. Furthermore, since $\bar{\varphi}(\varphi X) = \bar{\varphi}(\bar{\varphi}(S\varphi X)) = \bar{\varphi}S(\varphi X) = \varphi^2 X$, we can write $\varphi^2 X = -X + \bar{\eta}^a(X)\bar{\xi}_\alpha + u(X)U$. Finally, we have $\varphi U = 0$, since $U \in \bar{\varphi}(\text{ltr}(M))$, $\bar{\eta}^a \circ \varphi = 0$, and $u(\varphi X) = 0$, for any $X \in \Gamma(TM)$.

Thus, we can state the following theorem.

**Theorem 3.5** Let $(M, \varphi, \xi_\alpha, \eta^a, g)$ be an indefinite $S$-manifold and consider $(M, g, S(TM))$ a characteristic lightlike hypersurface of $\bar{M}$ such that $E$ and $N$ are globally defined on $M$. Then $(M, \varphi, \xi_\alpha, U, \eta^a, u)$ is a $g.f.f$-manifold.

For any $X, Y \in \Gamma(TM)$, we compute the field $(\nabla_X \varphi)Y$. Using [15] and [6], we get

$$(\nabla_X \varphi)Y = (\nabla_X \varphi)Y - u(Y)A_N X + B(X, Y)U + (B(X, \varphi Y) + (\nabla_X u)Y + u(Y)\tau(X))N,$$

then, from [11], comparing the components along $TM$ and $\text{ltr}(M)$, we have:

$$(\nabla_X \varphi)Y = u(Y)A_N X - B(X, Y)U + g(\varphi X, \varphi Y)\bar{\xi} + \bar{\eta}(\varphi^2 X), \quad \text{and} \quad (\nabla_X u)Y = -B(X, \varphi Y) - u(Y)\tau(X).$$  \hfill (16)
Theorem 3.7 Let \((M, \tilde{\xi}, \tilde{\eta}, \tilde{g})\) be an indefinite \(\tilde{S}\)-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface of \(M\). Then \(M\) is totally geodesic if and only if for any \(X \in \Gamma(TM)\) and for any \(Y \in \Gamma(D)\)

\[
\begin{align*}
(\nabla_X \varphi)Y &= \tilde{g}(\varphi X, \varphi Y)\tilde{\xi} + \tilde{\eta}(Y)\varphi^2 X, \quad (17) \\
A_X Y &= -\varphi(\nabla_X U) - \tilde{g}(X, U)\tilde{\xi}. \quad (18)
\end{align*}
\]

Proof. We suppose that \(M\) is totally geodesic, that is \(B(X, Y) = 0\) for any \(X, Y \in \Gamma(TM)\). Therefore, from (16), if \(Y \in \Gamma(D)\) we have \(u(Y) = 0\) and \((\nabla_X \varphi)Y = \tilde{g}(\varphi X, \varphi Y)\tilde{\xi} + \tilde{\eta}(Y)\varphi^2 X\). Again, writing (16) for \(U\), using (14), we have \((\nabla_X \varphi)U = A_X X + \tilde{g}(\varphi U, \varphi U)\tilde{\xi} + \tilde{\eta}(U)\varphi^2 X\), from which we obtain \(A_X X = -\varphi(\nabla_X U) - \tilde{g}(X, U)\tilde{\xi}\), since \(\varphi(U) = 0\) and \(\tilde{\eta}(U) = 0\).

Conversely, we suppose that the conditions (17) and (18) hold and we prove that the local second fundamental form \(B\) vanishes. If \(Y \in \Gamma(TM)\), using the decomposition (13), there exists \(\alpha \in \mathfrak{g}(\mathcal{U})\) such that \(Y = Y_d + \alpha U\), and, for any \(X \in \Gamma(TM)\) we obtain \(B(X, Y) = B(X, Y_d) + \alpha B(X, U)\). Using (16) and (17) with \(Y = Y_d\), we find \(B(X, Y_d) = u(Y_d)A_X X = 0\), which implies \(B(X, Y_d) = 0\). From (16), putting \(Y = U\) and using (18), we get \(B(X, U)U = 0\), which implies \(B(X, U) = 0\), concluding the proof. 

4 Distributions on a lightlike hypersurface of an indefinite \(\tilde{S}\)-manifold

4.1 The distribution \(D_0\)

The following Lemma can be easily proved.

Lemma 4.1 Let \((M, \varphi, \xi, \eta, \tilde{g})\) be an indefinite \(\tilde{S}\)-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface of \(M\). Let \(U \subset M\) be a coordinate neighbourhood as fixed in Theorem 2.1 then for any \(X, Y \in \Gamma(D_0)\)

\[
\tilde{g}((\nabla_X \varphi)Y, E) = 0, \quad \tilde{g}((\nabla_X \varphi)Y, N) = 0. \quad (19)
\]

Now, referring to the decomposition (11), for any \(X \in \Gamma(TM)\), \(Y \in \Gamma(D_0)\), we have:

\[
\nabla_X Y = \tilde{\nabla}_X Y + \tilde{h}(X, Y), \quad (20)
\]

where \(\tilde{\nabla}\) is a linear connection on the bundle \(D_0\), and \(\tilde{h} : \Gamma(TM) \times \Gamma(D_0) \to \Gamma(F)\) is \(\mathfrak{g}(\mathcal{U})\)-bilinear.

Let \(U \subset M\) be a coordinate neighbourhood as fixed in Theorem 2.1 and let \(X, Y \in \Gamma(D_0|_U)\). Then, using (11), (20) can be written (locally) in the following way:

\[
\nabla_X Y = \tilde{\nabla}_X Y + g(\nabla_X Y, \varphi N)\varphi E + g(\nabla_X Y, \varphi E)\varphi N + g(\nabla_X Y, N)E. \quad (21)
\]

Using Lemma 4.1, (9), being \(D_0\) \(\varphi\)-invariant and \(\tilde{\nabla}\tilde{g} = 0\), we get

\[
\begin{align*}
\tilde{g}(\nabla_X Y, \varphi N) &= -\tilde{g}(\varphi(\nabla_X Y), N) = -\tilde{g}(\varphi(\nabla_X Y - B(X, Y))N), N) \\
&= -\tilde{g}(\varphi(\nabla_X Y), N) = \tilde{g}((\nabla_X \varphi)Y, N) - \tilde{g}(\nabla_X (\varphi Y), N) \\
&= -\tilde{g}(\nabla_X (\varphi Y), N) = \tilde{g}(\varphi Y, \nabla_X N) = -g(A_N X, \varphi Y) = -C(X, \varphi Y).
\end{align*}
\]

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Again, using Lemma 4.1, (6), (8), (9), \( D_0 \) being \( \bar{\phi} \)-invariant and \( \bar{\nabla} \bar{g} = 0 \)

\[
g(\nabla_X Y, \bar{\phi} E) = -\bar{g}(\bar{\phi}(\nabla_X Y), E) = -\bar{g}(\bar{\phi}(\bar{\nabla}_X Y), E) + B(X, Y)\bar{g}(\bar{\phi} N, E) = -\bar{g}(\bar{\phi}(\bar{\nabla}_X Y), E)
\]

\[
= -\bar{g}(\bar{\nabla}_X (\bar{\phi} E), E) = \bar{g}(\bar{\phi} Y, \bar{\nabla}_X E + B(X, E) N) = \bar{g}(\bar{\phi} Y, \bar{\nabla}_X E) = -B(X, \bar{\phi} Y).
\]

For any \( X, Y \in \Gamma(D_0|\mathcal{U}) \), since \( X, Y \in \Gamma(S(TM)|\mathcal{U}) \), we know that \( \nabla_X Y = \bar{\nabla}_X Y + C(X, Y) E \), so we get \( g(\nabla_X Y, N) = C(X, Y) \). Therefore (21) becomes

\[
\nabla_X Y = \bar{\nabla}_X Y - C(X, \bar{\phi} Y)\bar{\phi} E - B(X, \bar{\phi} Y)\bar{\phi} N + C(X, Y) E,
\]

and the local expression of \( \bar{h} \) is

\[
\bar{h}(X, Y) = -C(X, \bar{\phi} Y)\bar{\phi} E - B(X, \bar{\phi} Y)\bar{\phi} N + C(X, Y) E.
\]

**Theorem 4.2** Let \((\bar{M}, \bar{\phi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})\) be an indefinite S-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface of \( \bar{M} \), with the induced g.f.f.-structure. The distribution \( D_0 \) on \( M \) is integrable if and only if for any \( X, Y \in \Gamma(D_0) \):

\[
C(X, Y) = C(Y, X), \quad C(X, \bar{\phi} Y) = C(\bar{\phi} X, Y), \quad B(X, \bar{\phi} Y) = B(\bar{\phi} X, Y).
\]

**Proof.** First of all, \( \nabla \) being a torsion-free connection, using (22), for any \( X, Y \in \Gamma(D_0) \) we get

\[
[X, Y] = \bar{\nabla}_X Y - \bar{\nabla}_Y X + (C(Y, \bar{\phi} X) - C(X, \bar{\phi} Y))\bar{\phi} E
\]

\[
+ (B(Y, \bar{\phi} X) - B(X, \bar{\phi} Y))\bar{\phi} N - (C(Y, X) - C(X, Y)) E.
\]

So, \( D_0 \) is integrable if and only if the components of \( [X, Y] \) with respect to \( \bar{\phi} E, \bar{\phi} N, E \) vanish, therefore if and only if \( C(X, Y) = C(Y, X), C(X, \bar{\phi} Y) = C(\bar{\phi} X, Y), B(X, \bar{\phi} Y) = B(\bar{\phi} X) \). \( \square \)

**Remark 4.3** Looking at (23) and using the above theorem, we deduce that \( \bar{\nabla} \) is a linear symmetric connection on the integral manifolds.

**Corollary 4.4** If \( D_0 \) is an integrable distribution, then for \( X, Y \in \Gamma(D_0) \) we have

\[
B(\bar{\phi} X, \bar{\phi} Y) = -B(X, Y), \quad C(\bar{\phi} X, \bar{\phi} Y) = -C(X, Y).
\]

Looking at the decomposition (11) and considering the symmetric connection \( \nabla \), we can define, as usual, the unsymmetrized second fundamental form of \( D_0 \), \( A^{D_0} \), setting

\[
A^{D_0}_X Y = p_\mathcal{F}(\nabla_{X_0} Y_0),
\]

for \( X, Y \in \Gamma(TM) \), where \( X_0, Y_0 \) are the projection of \( X \) and \( Y \) onto \( D_0 \) and \( p_\mathcal{F} : TM \rightarrow \mathcal{F} \) is the canonical projection on \( \mathcal{F} \). Then, using (20), since \( p_\mathcal{F}(\bar{\nabla}_{X_0} Y_0) = 0 \), we get \( A^{D_0}_X Y = \bar{h}(X_0, Y_0) \), and the symmetrized second fundamental form \( B^{D_0} \) is given by

\[
B^{D_0}(X, Y) = \frac{1}{2}\{\bar{h}(X_0, Y_0)) + \bar{h}(Y_0, X_0))\},
\]

for any \( X, Y \in \Gamma(TM) \). Furthermore the mean curvature vector of the distribution \( D_0 \), \( D_0 \) being integrable or not integrable, is defined as

\[
\mu^{D_0} = \frac{1}{\text{rank}(D_0)} \text{trace}(B^{D_0}),
\]

and \( D_0 \) is called minimal (respectively totally geodesic) if \( \mu^{D_0} \) (respectively \( B^{D_0} \)) vanishes.
Proposition 4.5 Let \((\tilde{M}, \tilde{\varphi}, \xi_\alpha, \bar{\eta}^\alpha, \tilde{g})\) be an indefinite \(S\)-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface of \(M\) such that the distribution \(D_0\) is integrable. Then, \(D_0\) is minimal with respect to the symmetric connection \(\nabla\) on \(M\) and all its integral manifolds are minimal submanifolds of \(M\) with respect to \(\nabla\).

Proof. We note that \(\text{rank}(D_0) = 2n + r - 4 = 2(n - 2) + r,\) and, \(D_0\) being integrable, \(\bar{\nabla}\) is symmetric and \(\mu^{D_0} = \frac{1}{2(n-2)+r}\text{trace}(B^{D_0}) = \frac{1}{2(n-2)+r}\text{trace}(\bar{h}).\) We consider an adapted frame in \(D_0, (X_a, \varphi X_a, \xi_\alpha)\) with \(a \in \{1, \ldots, n-2\}\) and \(\alpha \in \{1, \ldots, r\},\) and we have

\[
\text{trace}(\bar{h}) = \sum_{a=1}^{n-2} \varepsilon_a (\tilde{h}(X_a, X_a) + \bar{\varphi}X_a, \bar{\varphi}X_a)) + \sum_{\alpha=1}^{r} \varepsilon_\alpha \bar{h}(\xi_\alpha, \xi_\alpha).
\]

We get \(\tilde{h}(\xi_\alpha, \xi_\alpha) = -C(\xi_\alpha, \xi_\alpha)\varphi E - B(\xi_\alpha, \xi_\alpha)\bar{\varphi}N + C(\xi_\alpha, \xi_\alpha)E = C(\xi_\alpha, \xi_\alpha)E = 0,\) since \((\ref{eq:10})\) and \((\ref{eq:11})\) imply \(C(\xi_\alpha, \xi_\alpha) = g(A_N \xi_\alpha, \xi_\alpha) = \bar{g}(A_N \xi_\alpha, \xi_\alpha) = -\bar{g}(\nabla_{\xi_\alpha} N, \xi_\alpha) = \bar{g}(N, \nabla_{\xi_\alpha} \xi_\alpha) = 0.\)

Furthermore, we know that \(\bar{\varphi}^2 X_a = -X_a,\) therefore we deduce

\[
\bar{h}(X_a, X_a) + \bar{\varphi}X_a, \bar{\varphi}X_a) = -C(X_a, \varphi X_a)\varphi E - B(X_a, \varphi X_a)\bar{\varphi}N + C(X_a, X_a)E
\]

\[
+ C(\varphi X_a, X_a)\varphi E + B(\varphi X_a, X_a)\bar{\varphi}N + C(\varphi X_a, \varphi X_a)E
\]

\[
= -(C(X_a, X_a) + C(\varphi X_a, \varphi X_a))E = 0,
\]

since, using Corollary \((\ref{cor:4.1})\) we get \(C(X_a, X_a) + C(\varphi X_a, \varphi X_a) = 0.\) This completes the proof. \(\square\)

Now, we consider the decomposition \((\ref{eq:12})\) and for any \(X \in \Gamma(TM), Y \in \Gamma(D_0)\) and \(W \in \Gamma(E),\) we have

\[
\bar{\nabla}_XY = \bar{\nabla}_XY + \bar{h}(X,Y),
\]

where \(\bar{\nabla}\) is a linear connection on \(D_0\) and \(\bar{h} : \Gamma(TM) \times \Gamma(D_0) \to \Gamma(E)\) is \(\mathfrak{g}(M)\)-bilinear. Let \(U \subset M\) be a coordinate neighbourhood as fixed in Theorem \((\ref{thm:2.4})\) and \(X, Y \in \Gamma(D_0|U).\) Then, using \((\ref{eq:12})\), and putting \(F_1(X,Y) = \bar{g}(\bar{h}(X,Y), \varphi N), F_2(X,Y) = \bar{g}(\bar{h}(X,Y), \varphi E), F_3(X,Y) = \bar{g}(\bar{h}(X,Y), N),\) and \(F_4(X,Y) = \bar{g}(\bar{h}(X,Y), E),\) we can write locally:

\[
\bar{\nabla}_XY = \bar{\nabla}_XY + F_1(X,Y)\varphi E + F_2(X,Y)\bar{\varphi}N + F_3(X,Y)E + F_4(X,Y)N.
\]

Now, we express the \(F_i\)'s, \(i \in \{1, 2, 3, 4\},\) in terms of \(B\) and \(C.\) We begin to compute \(F_3\) and \(F_4.\)

For any \(X, Y \in \Gamma(D_0|U),\) from \((\ref{eq:25}),\) using \((\ref{eq:6}), (\ref{eq:8}), (\ref{eq:9}),\) and being \(\nabla\) a metric connection, we have

\[
F_3(X,Y) = -\bar{g}(Y, \bar{\nabla}_X N) = -\bar{g}(Y, -A_N X + \tau(N)X) = -\bar{g}(Y, -A_N X) = g(A_N X, Y) = C(X,Y),
\]

and, again from \((\ref{eq:25}),\) using \((\ref{eq:6}), (\ref{eq:8}), (\ref{eq:9}),\) we have

\[
F_4(X,Y) = -\bar{g}(Y, \bar{\nabla}_X E) = -\bar{g}(Y, \bar{\nabla}_X E + B(X,E)N) = -\bar{g}(Y, \bar{\nabla}_X E)
\]

\[
= -\bar{g}(Y, -A_E X - \tau(E)X) = g(A_E X, Y) = B(X,Y).
\]

For \(F_2,\) using Lemma \((\ref{lem:4.1}), (\ref{eq:6}), (\ref{eq:8}), (\ref{eq:9}),\) and \(D_0\) being a \(\varphi\)-invariant distribution, we have:

\[
F_2(X,Y) = \bar{g}(\bar{\nabla}_X Y, \varphi E) = -\bar{g}(\bar{\varphi} \nabla_X Y, E) = -\bar{g}(\bar{\nabla}_X (\varphi Y), E) = \bar{g}(\bar{\varphi} Y, \bar{\nabla}_X E) = \bar{g}(\bar{\varphi} Y, \bar{\nabla}_X E)
\]

\[
= -g(A_E X, \varphi Y) = -B(X, \varphi Y).
\]
Again, using Lemma \[4.1\] (6) and (9), also by the \(\bar{\varphi}\)-invariance of distribution \(D_0\), we compute:

\[
F_1(X, Y) = \bar{g}(\nabla_X Y, \bar{\varphi}N) = -\bar{g}(\bar{\varphi}(\nabla_X Y), N) = -\bar{g}(\nabla_X (\bar{\varphi}Y), N) = -g(A_N X, \bar{\varphi}Y) = -B(X, \varphi Y).
\]

Then, (25) becomes \(\bar{\nabla}_X Y = \bar{\nabla}_X Y - C(X, \bar{\varphi}Y)\). 

Again, using Lemma 4.1, (6) and (9), also by the \(\bar{\varphi}\)-invariance, we obtain

\[
\bar{\nabla}_X Y - C(X, \bar{\varphi}Y) \varphi E = B(X, \bar{\varphi}Y) + C(X, Y) E + B(X, Y)\bar{N}.
\]

and, locally, \(\bar{h}(X, Y) = -C(X, \bar{\varphi}Y) \varphi E = B(X, \bar{\varphi}Y) + C(X, Y) E + B(X, Y)\bar{N}.
\]

Proposition 4.6 Let \((\bar{M}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})\) be an indefinite \(S\)-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface of \(\bar{M}\). Supposing \(D_0\) integrable, we have \(\text{trace}(\bar{h}) = 0\), that is all the integral manifolds of \(D_0\) are minimal submanifolds of \(\bar{M}\) and \(D_0\) is minimal.

Proof. We have: \(\text{trace}(\bar{h}) = \sum_{\alpha=1}^{n-2} \epsilon_\alpha (\bar{h}(X_{\alpha}, X_{\alpha}) + \bar{h}(\bar{\varphi} X_{\alpha}, \bar{\varphi} X_{\alpha})) + \sum_{\alpha} \epsilon_\alpha \bar{h}(\bar{\xi}_\alpha, \bar{\xi}_\alpha).
\]

We note that \(\bar{h}(\bar{\xi}_\alpha, \bar{\xi}_\alpha) = \bar{h}(\bar{\xi}_\alpha, \bar{\xi}_\alpha) + B(\bar{\xi}_\alpha, \bar{\xi}_\alpha) = \bar{h}(\bar{\xi}_\alpha, \bar{\xi}_\alpha),\) since, using (9), (8) and (3):

\[
B(\bar{\xi}_\alpha, \bar{\xi}_\alpha) = \bar{g}(\bar{\xi}_\alpha, \bar{\xi}_\alpha) = \bar{g}(\bar{\xi}_\alpha, \bar{\xi}_\alpha) = -\bar{g}(\nabla_{\bar{\xi}_\alpha} E, \bar{\xi}_\alpha) = -\bar{g}(\nabla_{\bar{\xi}_\alpha} E, \bar{\xi}_\alpha) = 0.
\]

Proposition 4.7 Let \((\bar{M}, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})\) be a indefinite \(S\)-manifold of dimension \(2n + r\), \(n \geq 3\) and \((M, g, S(TM))\) a characteristic lightlike hypersurface of \(\bar{M}\). If \(D_0\) is an integrable distribution, then the leaves of \(D_0\) have an indefinite \(S\)-structure.

Proof. Let \(M_0\) be a leaf of \(D_0\), then for any \(p \in M_0\) we have \(T_p M_0 = (D_0)_p\) and \(\text{dim} M_0 = (n-2)\). If \(X_0 \in TM_0\), we have

\[
\bar{\varphi} X_0 = \bar{S} X_0 = \bar{\varphi} X_0,
\]

being \(S : \Gamma(TM) \rightarrow \Gamma(D)\) and \(D = D_0 \perp \bar{\varphi}(\text{Rad}(TM)) \perp \text{Rad}(TM)\). We put \(\bar{\varphi} := \bar{\varphi}|_{D_0}\), and for any \(\alpha \in \{1, \ldots, r\}\) \(\bar{\eta}^\alpha := \bar{\eta}^\alpha|_{D_0}\) so \(\bar{\varphi}\) defines an \((1, 1)\)-type tensor field on \(M_0\) because \(D_0\) is \(\bar{\varphi}\)-invariant.

Now we consider \((M_0, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, \bar{g})\) and check that this is an indefinite \(S\)-structure. We know that \(\varphi^2 = -X + \bar{\eta}^\alpha (X) \bar{\xi}_\alpha + u(X) U\), for any \(X \in \Gamma(TM)\), and that \(u(Y) = 0\) for any \(Y \in \Gamma(D)\), so we deduce

\[
\bar{\varphi}^2 X_0 = -X_0 + \bar{\eta}^\alpha (X_0) \bar{\xi}_\alpha,
\]

for any \(X_0 \in \Gamma(TM_0)\). For any \(\alpha, \beta \in \{1, \ldots, r\}\) we have \(\bar{\eta}^\alpha (\bar{\xi}_\beta) = \delta^\alpha_\beta\) and then \((M_0, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha)\) is a \(g, f, f\)-manifold. Now, to prove the compatibility between the \(g, f, f\)-structure and the metric \(g\) on \(M_0\), by the definition in [13], for any \(X_0, Y_0 \in \Gamma(TM_0)\) we have

\[
g(\bar{\varphi} X_0, \bar{\varphi} Y_0) = \bar{g}(\bar{\varphi} X_0, \bar{\varphi} Y_0) - u(Y_0)\bar{g}(\bar{\varphi} X_0, N) - u(X_0)\bar{g}(\bar{\varphi} Y_0, N) + u(X_0)u(Y_0)\bar{g}(N, N)
\]

\[
= g(X_0, Y_0) - \epsilon_\alpha \bar{\varphi}^\alpha (X_0) \bar{\eta}^\alpha (Y_0).
\]

Moreover, for any \(X_0, Y_0 \in \Gamma(TM_0)\), for any \(\alpha \in \{1, \ldots, r\}\), we get \(\bar{d}\bar{\eta}^\alpha (X_0, Y_0) = \bar{d}\bar{\eta}^\alpha (X_0, Y_0)\), and \(\bar{\Phi}(X_0, Y_0) = g(X_0, \bar{\varphi} Y_0) = \bar{g}(X_0, \bar{\varphi} Y_0) = \bar{d}\bar{\eta}^\alpha (X_0, Y_0) = \bar{d}\bar{\eta}^\alpha (X_0, Y_0)\). Finally, \((M_0, \bar{\varphi}, \bar{\xi}_\alpha, \bar{\eta}^\alpha, g)\) is an indefinite \(S\)-manifold, since \(N_\varphi + 2\bar{d}\bar{\eta}^\alpha \otimes \bar{\xi}_\alpha\) and \(N_\varphi + 2\bar{d}\bar{\eta}^\alpha \otimes \bar{\xi}_\alpha\) coincide on \(D_0\). Moreover,
\( \tilde{\nabla} \) is the Levi-Civita connection on \( M_0 \). In fact, by Remark 2.3 since \( D_0 \subset S(TM) \), for any \( X_0, Y_0, Z_0 \in \Gamma TM_0 \) we have

\[
(\tilde{\nabla}_{X_0}g)(Y_0, Z_0) = X_0(g(Y_0, Z_0)) - g(\tilde{\nabla}_{X_0}Y_0, Z_0) - g(Y_0, \tilde{\nabla}_{X_0}Z_0) = X_0(g(Y_0, Z_0)) - g(\tilde{\nabla}_{X_0}Y_0, Z_0) + g(h(X_0, Y_0), Z_0) - g(Y_0, \tilde{\nabla}_{X_0}Z_0) + g(Y_0, h(X_0, Z_0)) = (\nabla_{X_0}g)(Y_0, Z_0) = 0.
\]

Hence \( \tilde{\nabla} \) is a metric connection. By Remark 2.3 it is also symmetric, thus it is the Levi-Civita connection and, from (1), we have \( (\tilde{\nabla}_{X_0}\phi)(Y_0) = g(\tilde{\nabla}_{X_0}\phi Y_0, \phi Y_0) + \eta(Y_0)\phi^2(X_0). \) \( \square \)

### 4.2 The distribution \( D \)

By a direct computation, one can prove the following result.

**Lemma 4.8** Let \((M, \tilde{\varphi}, \xi_\alpha, \eta^\alpha, \tilde{\eta})\) be an indefinite \( S \)-manifold and consider \((M, g, S(TM))\) a characteristic lightlike hypersurface. Then the component of \((\tilde{\nabla}_X\phi)Y\) along \( ltr(TM) \) vanishes, for any \( X \in \Gamma(TM) \) and \( Y \in \Gamma(TM) \).

**Proposition 4.9** Let \((\tilde{M}, \tilde{\varphi}, \tilde{\xi}_\alpha, \tilde{\eta}^\alpha, \tilde{\eta})\) be an indefinite \( S \)-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface. Then \( D = D_0 \perp \tilde{\varphi}(\text{Rad}(TM)) \perp \text{Rad}(TM) \) is integrable if and only if \( B \) satisfies the following conditions:

a) \( B(X, \phi Y) = B(\phi X, Y) \), for any \( X, Y \in \Gamma(D_0) \)

b) \( B(X, V) = 0 \), for any \( X \in \Gamma(D_0) \)

c) \( B(V, V) = 0 \),

where \( V = -\tilde{\varphi}E \).

**Proof.** At first, for any \( X, Y \in \Gamma(D) \) we compute the component of \([X, Y]\) along \( \tilde{\varphi}(ltr(TM)) \)

\[
\tilde{g}([X, Y], \phi E) = -\tilde{g}(\phi \tilde{\nabla}_X Y, E) + \tilde{g}(\phi \tilde{\nabla}_Y X, E) = -\tilde{g}(\tilde{\nabla}_X (\phi Y), E) + \tilde{g}(\tilde{\nabla}_Y (\phi X), E) = \tilde{g}(\phi Y, \tilde{\nabla}_X E) - \tilde{g}(\phi X, \tilde{\nabla}_Y E) = -g(\phi Y, A_X E) + g(\phi X, A_Y E).
\]

From the definition of \( D \) we get \( X = \alpha E + \beta \phi E + X_0 \), and \( Y = \delta E + \gamma \phi E + Y_0 \). Using the previous expression of \( X, Y \in \Gamma(D) \), being \( D \phi \)-invariant and \( B(E, X) = 0 \) for any \( X \in \Gamma(TM) \), we have

\[
\tilde{g}([X, Y], \phi E) = (\gamma \alpha - \beta \delta)B(V, V) - \beta B(V, \phi X_0) - \alpha B(Y_0, V)
\]

\[
+ \beta B(V, \phi Y_0) + \delta B(X_0, V) + B(Y_0, \phi X_0) - B(\phi Y_0, X_0).
\]

So, if we suppose that \( D \) is integrable, being \( \phi E \), \( E \), \( X_0 \) and \( Y_0 \) sections of \( D \), then we get

\[
0 = \tilde{g}([\phi E, E], \phi E) = -\tilde{g}(\phi E, \phi E) - B(V, V). \]

Finally, if \( X \in \Gamma(D_0) \) we find

\[
0 = \tilde{g}([X, E], \phi E) = B(E, \phi X) - B(X, \phi E) = B(X, \phi E) + X_0, \eta \\phi X_0 \], and if \( X, Y \in \Gamma(D_0) \) we get

\[
0 = \tilde{g}([X, Y], \phi E) = B(\phi Y, X) - B(Y, \phi X). \]

Vice versa, using (26) and a), b), c), it is easy to check that \([X, Y]\) in \( \Gamma(D) \).

Now, a consequence is the following proposition.

**Proposition 4.10** Let \((\tilde{M}, \tilde{\varphi}, \tilde{\xi}_\alpha, \tilde{\eta}^\alpha, \tilde{\eta})\) be an indefinite \( S \)-manifold and \((M, g, S(TM))\) a characteristic lightlike hypersurface. If \((M, g, S(TM))\) is totally geodesic, then the following statements hold:

a) the distribution \( D \) is integrable;
b) the distribution D is parallel with respect to the induced connection ∇;

c) M is locally a product $M^* \times C$, where $M^*$ is a leaf of D and C is a lightlike curve tangent to the distribution $\tilde{\varphi}(ltr(TM))$.

Proof. Being $(M, g, S(TM))$ totally geodesic, a) follows from the previous proposition. To prove b), we need only to check $g(\nabla_X E, \varphi E) = 0$, $g(\nabla_X \varphi E, \varphi E) = 0$ and $g(\nabla_X Y_0, \varphi E) = 0$ for any $X \in \Gamma(TM)$ and $Y_0 \in \Gamma(D_0)$. Hence, using Lemma 4.8, we get

$$g(\nabla_X E, \varphi E) = \tilde{g}(\nabla_X E, \varphi E) = -\tilde{g}(\varphi \nabla_X E, \varphi E) = -B(X, \varphi E) = 0,$$

$$g(\nabla_X \varphi E, \varphi E) = \tilde{g}(\nabla_X \varphi E, \varphi E) = -\tilde{g}(\varphi \nabla_X \varphi E, E) = \tilde{g}(\nabla_X \varphi, E) = 0,$$

$$g(\nabla_X Y_0, \varphi E) = \tilde{g}(\nabla_X Y_0, \varphi E) = -\tilde{g}(\varphi \nabla_X Y_0, E) = \tilde{g}(\nabla_X \varphi) Y_0, E)$$

Finally, from a) we deduce that $D$ determines a foliation. Being $\tilde{\varphi}(ltr(TM))$ a 1-dimensional distribution, it defines a foliation. So, being $TM = D \oplus \tilde{\varphi}(ltr(M))$, we obtain c). \qed

5 Totally umbilical lightlike hypersurface and totally umbilical screen distribution

We begin to prove the following Lemma.

Lemma 5.1 Let $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}_a, \bar{\eta}^a, \bar{\tilde{g}})$ be an indefinite $S$-manifold and $(M, g, S(TM))$ a lightlike hypersurface. Then, the Riemannian $(0,4)$-type curvature tensor fields $\tilde{R}$ and $R$ of $\tilde{M}$ and $M$ verify the following relations, for any $X, Y, Z \in \Gamma(TM)$,

$$\tilde{R}(X, Y, Z, E) = -\{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\},$$

$$\tilde{R}(X, Y, Z, N) = R(X, Y, Z, N),$$

Proof. Only using the Gauss and Weingarten equations for lightlike hypersurfaces, we get

$$\tilde{R}(X, Y, Z, E) = -\{B(X, \nabla_Y Z) + X(B(Y, Z)) + \tau(X)B(Y, Z) - B(Y, \nabla_X Z)$$

$$- Y(B(X, Z)) - \tau(Y)B(X, Z) + B(\nabla_Y X, Z) - B(\nabla_X Y, Z)\}$$

$$= -\{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z)\},$$

$$\tilde{R}(X, Y, Z, N) = -\tilde{g}(\tilde{R}(X, Y, Z), N) = -\{\tilde{g}(\nabla_X \nabla_Y Z, N) + B(Y, Z)\tilde{g}(\nabla_X N, N)$$

$$- \tilde{g}(\nabla_Y \nabla_X Z, N) + B(X, Z)\tilde{g}(\nabla_Y N, N) - \tilde{g}(\nabla_{[X,Y]} Z, N)\}$$

$$= -\tilde{g}(\nabla_X \nabla_Y Z, N) - \tilde{g}(\nabla_Y \nabla_X Z, N) = R(X, Y, Z, N).$$

\qed

Definition 5.2 Let $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}_a, \bar{\eta}^a, \bar{\tilde{g}})$ be an indefinite $S$-manifold and $(M, g, S(TM))$ a lightlike hypersurface. Then $M$ is called totally umbilical if, for any coordinate neighbourhood $U$, there exists a function $\rho \in \mathfrak{g}(U)$ such that $B(X, Y) = \rho g(X, Y)$ for any $X, Y \in \Gamma(TM|_U)$.

Theorem 5.3 Let $(\tilde{M}, \tilde{\varphi}, \tilde{\xi}_a, \bar{\eta}^a, \bar{\tilde{g}})$ be an indefinite $S$-space form and $(M, g, S(TM))$ a characteristic lightlike hypersurface. If $(M, g, S(TM))$ is totally umbilical then $c = \varepsilon = \sum_{\alpha=1}^{r} \varepsilon_\alpha$. 

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Proof. Being $\bar{M}(c)$ an indefinite $S$-space form, the Riemannian curvature $\bar{R}$ is given by

$$\bar{R}(X, Y, Z, W) = -\frac{c + 3\varepsilon}{4} \left\{ \bar{g}(\varphi Y, \varphi Z)\bar{g}(\varphi X, \varphi W) - \bar{g}(\varphi X, \varphi Z)\bar{g}(\varphi Y, \varphi W) \right\}$$

$$- \frac{c - \varepsilon}{4} \left\{ \Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) + 2\Phi(X, Y)\Phi(W, Z) \right\}$$

$$- \{ \bar{\eta}(W)\bar{\eta}(X)\bar{g}(\varphi Z, \varphi Y) - \bar{\eta}(W)\bar{\eta}(Y)\bar{g}(\varphi Z, \varphi X) + \bar{\eta}(Y)\bar{\eta}(Z)\bar{g}(\varphi W, \varphi X)$$

$$- \bar{\eta}(Z)\bar{\eta}(X)\bar{g}(\varphi W, \varphi Y) \right\}$$

for any $X, Y, Z, W \in \Gamma(TM)$, (10). Since $\bar{g}(\varphi E, \varphi X) = 0$, for any $X \in \Gamma(TM)$, and $\bar{\eta}(E) = 0$, then for any $X, Y, Z \in \Gamma(TM)$, we get

$$\bar{R}(X, Y, Z, E) = -\{ (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)B(Y, Z) - \tau(Y)B(X, Z) \}.$$

So, replacing $X, Y, Z$ by $PX, E, PZ$ in the above two equations, $P$ being the projection on $S(TM)$ with respect to the decomposition (2), we find

$$\bar{R}(PX, E, PZ, E) = -\frac{c - \varepsilon}{4} \left\{ -\bar{g}(V, PX)\bar{g}(PZ, V) - 2\bar{g}(X, V)\bar{g}(V, Z) \right\} = \frac{3}{4}(c - \varepsilon)u(PZ)u(PX).$$

On the other hand we have

$$\bar{R}(PX, E, PZ, E) = -\{ -B(\nabla_{PX} E, PZ) - E(B(PX, PZ)) + B(\nabla_E PX, PZ) + B(PX, \nabla_E PZ) - \tau(E)B(PX, PZ) \}.$$

Then, comparing the two equations and using Gauss and Weingarten equations, we get

$$\frac{3}{4}(c - \varepsilon)u(PZ)u(PX) = -\{ \rho B(PX, PZ) - E(\rho)g(PX, PZ) - \rho(B(E, PX)\bar{g}(PZ, N)$$

$$- B(E, PZ)\bar{g}(PX, N)) - \rho\tau(E)g(PX, PZ) \}.$$

Therefore, being $B(X, E) = 0$ for any $X \in \Gamma(TM)$, we still find

$$\frac{3}{4}(c - \varepsilon)u(PZ)u(PX) = -(\rho^2 - E(\rho) - \rho\tau(E))\bar{g}(PX, PZ).$$

Choosing $X = Z = U \in S(TM)$, we have $PX = PZ = U$ and being $u(U) = 1$ and $g(U, U) = 0$, from the above equation we obtain $c = \varepsilon$. \qed

**Corollary 5.4** Let $(\bar{M}(c), \varphi, \bar{\xi}_a, \bar{\eta}_a, \bar{g})$ be an indefinite $S$-space form. If $c \neq \varepsilon$, then there exists no totally umbilical characteristic lightlike hypersurface.

**Definition 5.5** Let $(\bar{M}, \varphi, \bar{\xi}_a, \bar{\eta}_a, \bar{g})$ be an indefinite $S$-manifold and $(M, g, S(TM))$ a lightlike hypersurface. The screen distribution $S(TM)$ is called **totally umbilical** if for any coordinate neighbourhood $\mathcal{U}$ there exists a function $\lambda \in \mathfrak{X}(\mathcal{U})$ such that $C(X, PY) = \lambda g(X, PY)$, for any $X, Y \in \Gamma(TM|\mathcal{U})$.
Proposition 5.6 Let \((M, g, S(TM))\) be a characteristic lightlike hypersurface of an indefinite \(S\)-space form. If \(S(TM)\) is totally umbilical, then \(S(TM)\) is totally geodesic.

Proof. From Lemma 5.1 for any \(X, Y, Z \in \Gamma(TM)\) we get \(\overline{g}(\overline{\varphi}(X, Y, Z), N) = \overline{g}(R(X, Y, Z), N).\) Then, replacing \(Z\) by \(PZ\), we have

\[
\overline{g}(\overline{R}(X, Y, PZ), N) = \overline{g}(\nabla_X \nabla_Y PZ, N) - \overline{g}(\nabla_Y \nabla_X PZ, N) - \overline{g}(\nabla_{[X,Y]} PZ, N)
\]

Now, replacing \(X\) by \(E\) and \(Y, Z\) by \(U\) in the previous equation, we obtain

\[
\overline{g}(\overline{R}(E, U, U), N) = E(C(U, U)) - C(U, \nabla_E U) - C(\nabla_E U, U) - \tau(E)C(U, U)
\]

\[
- U(C(E, U)) + C(E, \nabla_U U) + C(\nabla_U E, U) + \tau(U)C(E, U)
\]

\[
= -\lambda(g(U, \nabla_E U) + g(\nabla_E U, U) + g(\nabla_U E, U)).
\]

Note that \(\nabla_E U = \nabla_E U - C(E, U) = \nabla_E U\) and \(\nabla_U U = \nabla_U U - C(U, U)E = \nabla_U U\). So, we find

\[
\overline{g}(\overline{R}(E, U, U), N) = -\lambda(g(U, \nabla_U E) + 2g(\nabla_E U, U)) - \lambda g(\nabla_U E, U) = -\lambda g(\nabla_U \varphi E, N) = -\lambda g(\nabla_U \varphi E, N)
\]

\[
= -\lambda g(\nabla_U \varphi E, N) = \lambda g(\nabla_U \varphi E, N)
\]

\[
= -\lambda g(U, \varphi E) = \lambda^2 g(U, U) = \lambda^2.
\]

Being \(\overline{\Delta}(c)\) an indefinite \(S\)-space form, for the Riemannian curvature tensor field we have

\[
\overline{g}(\overline{R}(E, U, U), N) = \frac{c - \varepsilon}{4} \{-\Phi(N, E)g(\varphi N, N) - \Phi(U, E)\overline{g}(U, U) + 2\Phi(E, U)\overline{g}(U, U)\} = 0.
\]

So, we obtain \(\lambda = 0\) and \(S(TM)\) is totally geodesic. \(\square\)

6 An example of a characteristic lightlike hypersurface

We begin with a general remark. Consider a hypersurface \(M\) of a semi-Riemannian manifold \((\overline{M}^m, \overline{g})\) locally given by

\[
x^A = f^A(u^1, \ldots, u^{m-1}), \quad \text{rank}(\frac{\partial f^A}{\partial u^a}) = m - 1, \quad A \in \{1, \ldots, m\}, \quad a \in \{1, \ldots, m - 1\}.
\]

A natural basis of the tangent bundle of \(M\) is \(\frac{\partial}{\partial u^a} = \frac{\partial f^A}{\partial u^a} \frac{\partial}{\partial x^A}\) and the induced metric on \(M\), denoted by \(g\), has components given by \(g_{ab} = \overline{g}_{AB} \frac{\partial f^A}{\partial u^a} \frac{\partial f^B}{\partial u^b}\). Obviously, a hypersurface \(M\) of \((\overline{M}^m, \overline{g})\) is lightlike if and only if \(\text{rank}(g_{ab}) \leq m - 2\), i.e.

\[
\Delta = \det(\frac{\partial f^A}{\partial u^a} \overline{g}_{AB} \frac{\partial f^B}{\partial u^b}) = 0.
\]
Proposition 6.1 ([22]) Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{jk}) \in M_{n,m}(\mathbb{R})$. If we consider $C = AB$, then we have

$$\det C = \sum_{1 \leq j_1 \leq j_2 \ldots \leq j_n \leq m} \begin{vmatrix} a_{1j_1} & a_{2j_1} & \cdots & a_{nj_1} \\ a_{1j_2} & a_{2j_2} & \cdots & a_{nj_2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1j_n} & a_{2j_n} & \cdots & a_{nj_n} \end{vmatrix} \begin{vmatrix} b_{1j_1} & b_{1j_2} & \cdots & b_{1j_1} \\ b_{1j_1} & b_{1j_2} & \cdots & b_{1j_1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1j_n} & b_{1j_n} & \cdots & b_{1j_n} \end{vmatrix}.$$ 

Now, we refer to the three examples of indefinite $S$-manifolds given in [10] to look for a lightlike hypersurface. We begin with Example 4.2 in [10], $\bar{M} = (\mathbb{R}^2, \varphi, \xi$, $\eta$, $\gamma$, $g)$, where $(x^1, x^2, y^1, y^2, z^1, z^2)$ are standard coordinates in $\mathbb{R}^6$, $\xi := \frac{\partial}{\partial x^2}$, $\eta := \frac{\partial}{\partial y^2}$, $\gamma := \frac{\partial}{\partial z^2}$, $\varphi$, $g$ are given by

$$F = \begin{pmatrix} 0 & -I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & 0 & Y \end{pmatrix}, \quad \text{where} \quad Y = \begin{pmatrix} -y^1 & y^2 \\ -y^1 & y^2 \end{pmatrix},$$

$$g = \sum_{\alpha=1}^{2} \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum_{i=1}^{2} \tau_i ((dx^i)^2 + (dy^i)^2),$$

respectively, being $\tau_i = \pm 1$ according to whether $i = 1$ or $i = 2$. Note that the Killing characteristic vector fields are both spacelike.

We compute all the non null determinants $M_{AB}$, obtained by taking the line $A$ and the column $B$ out of $G$, matrix of $g$,

$$M_{11} = M_{33} = -\frac{1}{8}, \quad M_{22} = M_{44} = \frac{1}{8}, \quad M_{55} = M_{66} = \frac{1}{16} + \frac{1}{8}(y^2)^2 - \frac{1}{8}(y^1)^2,$$

$$M_{15} = \frac{1}{8} y^1, \quad M_{16} = -\frac{1}{8} y^1, \quad M_{25} = -\frac{1}{8} y^2, \quad M_{26} = \frac{1}{8} y^2, \quad M_{56} = -\frac{1}{8}(y^2)^2 + \frac{1}{8}(y^1)^2.$$

Now, in terms of $D^A$ and $M_{AB}$, where $D = (\frac{\partial f_A}{\partial \bar{u}})$ and, for any $A \in \{1, \ldots, 6\}$, $D^A$ denotes the determinant of the matrix obtained by $D$ deleting the column of index $A$, we state the following result.

Proposition 6.2 Let $M$ be a hypersurface of $\bar{M}$. Then, $M$ is lightlike if and only if

$$\frac{1}{2}((-D)^2 + (D^2)^2) - ((D^3)^2 + (D^4)^2) + \frac{1}{2}((y^1)^2 - (y^2)^2)(D^5)^2 + (D^6)^2)$$

$$+ y^1(D^2 D^5 - D^1 D^6) - y^2(D^2 D^5 - D^2 D^6) + ((y^1)^2 - (y^2)^2)D^5 D^6 = 0.$$

Proof. From [27] we know that $M$ is a lightlike hypersurface if and only if $\Delta = \det(\frac{\partial f_A}{\partial \bar{u}} g_{AB} \frac{\partial f_B}{\partial \bar{u}}) = 0$, thus, using Proposition 6.1 we have

$$\Delta = -\frac{1}{8}((D^1)^2 + (D^3)^2) + \frac{1}{4} y^1(D^1 D^5 - D^1 D^6)$$

$$+ \frac{1}{4}((D^2)^2 + (D^4)^2) - \frac{1}{4}y^2(D^2 D^5 - D^2 D^6) + \frac{1}{4}((y^1)^2 - (y^2)^2)D^5 D^6$$

$$+ \frac{1}{8}(\frac{1}{2} + (y^2)^2 - (y^1)^2)((D^5)^2 + (D^6)^2).$$

This ends the proof. \qed

Now, we give a condition which ensures that the characteristic vector fields belong to $TM$. 

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Proposition 6.4 Let $W$ be a hypersurface of $\bar{M}$. Then, the characteristic vector fields are tangent to $M$ if and only if $D^5 = D^6 = 0$.

Proof. We suppose that, for any $\alpha \in \{1, 2\}$, $\xi_\alpha$ is tangent to $M$, then we can write $\xi_\alpha = X_\alpha \frac{\partial}{\partial u^\alpha}$. Being $\frac{\partial}{\partial u^\alpha} = \frac{\partial f^A}{\partial u^\alpha} \frac{\partial}{\partial x^A}$, we have $\xi_\alpha = X_\alpha \frac{\partial f^A}{\partial u^\alpha} \frac{\partial}{\partial x^A}$ and, on the other hand, we know $\xi_\alpha = \frac{\partial}{\partial u^\alpha}$. So, we obtain

$$\begin{align*}
& \text{I): } \begin{cases} 
X_1^a \frac{\partial f^A}{\partial u^a} = 0 & A \neq 5 \\
X_1^a \frac{\partial f^5}{\partial u^a} = 1 
\end{cases} \quad \text{and} \quad \text{II): } \begin{cases} 
X_2^a \frac{\partial f^A}{\partial u^a} = 0 & A \neq 6 \\
X_2^a \frac{\partial f^6}{\partial u^a} = 1 
\end{cases}
\end{align*}$$

We observe that the first equations in I) state that $(X_1^a)_{a \in \{1, \ldots, 5\}}$ is a non zero solution of a homogeneous system whose matrix has to be degenerate, i.e., $D^5 = 0$. Analogously from II) $D^6 = 0$ follows. Vice versa, if we suppose that $D^5 = 0$, then I) admits non trivial solutions and $\xi_1 = X_1^a \frac{\partial}{\partial u^a}$. Analogously, supposing $D^6 = 0$, from II) we obtain $\xi_2 = X_2^a \frac{\partial}{\partial u^a}$. □

The following is an immediate consequence of Proposition 6.3 and Proposition 6.4.

Proposition 6.4 Let $M$ be a hypersurface of $\bar{M}$. $M$ is a characteristic lightlike hypersurface if and only if $-(D^1)^2 - (D^2)^2 - (D^3)^2 - (D^4)^2 = 0$ and $D^5 = D^6 = 0$.

Now, we describe a lightlike hypersurface $M$ in $\bar{M}$ such that $\xi_1$, $\xi_2$ and $\varphi(E)$ belong to $S(TM)$. We consider

$$f(u^1, u^2, u^3, u^4, u^5) = (u^1 + u^5, u^2, u^3, u^1 + u^5, u^4, u^5).$$

It easy to check that this map describes a hypersurface in $\bar{M}$ and $D^1 = D^4 = 1, \quad D^2 = D^3 = D^5 = D^6 = 0$, hence such a hypersurface verifies Proposition 6.4. Then $M$ is a lightlike hypersurface and $\xi_1$ and $\xi_2$ belong to $\Gamma(TM)$. The natural basis of $TM$ is given by

$$\begin{align*}
U_1 &= \frac{\partial}{\partial u^1} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^2}, \\
U_2 &= \frac{\partial}{\partial u^2} = \frac{\partial}{\partial x^2}, \\
U_3 &= \frac{\partial}{\partial u^3} = \frac{\partial}{\partial y^1}, \\
U_4 &= \frac{\partial}{\partial u^4} = \frac{\partial}{\partial z^1} = \xi_1, \\
U_5 &= \frac{\partial}{\partial u^5} = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^2} + \xi_2.
\end{align*}$$

We deduce that $\xi_2 = U_5 - U_1$. Considering the vector field of $\bar{M}$

$$E = -\frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^2} + y^1 \xi_1 + y^1 \xi_2,$$

it is easy to check that $E$ belongs to $\text{Rad}(TM)$. In fact we have $g(E, U_\alpha) = 0$ for all $\alpha \in \{1, \ldots, 5\}$ and $g(E, E) = 0$. We construct $N \in \Gamma(TM)$ such that $g(N, N) = 0$, and $g(N, E) = 1$. To this aim, we consider $Z = \frac{\partial}{\partial x^1} - y^1 \xi_1 - y^1 \xi_2$, we get $g(Z, E) = \frac{1}{2} \neq 0$, and putting

$$N = \frac{1}{g(Z, E)} \left( Z - \frac{g(Z, Z)}{2g(Z, E)} E \right),$$

we have $N = \frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^1} - y^1 \xi_1 - y^1 \xi_2$.

Now, to give a basis of the screen distribution, we compute $\varphi(E)$ and $\varphi(N)$ and we have

$$\varphi(E) = -\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} - y^2 \xi_1 - y^2 \xi_2, \quad \varphi(N) = -\frac{\partial}{\partial x^2} - \frac{\partial}{\partial y^1} - y^2 \xi_1 - y^2 \xi_2.$$

Hence $\varphi(E) = -U_2 + U_3 - y^2 U_4 - y^2 (U_5 - U_1)$ and $\varphi(N) = -U_2 - U_3 - y^2 U_4 - y^2 (U_5 - U_1)$ are linear combinations of $U_\alpha$, so they are tangent to $M$. Thus, we define the characteristic screen
distribution as $S(TM) = \langle \xi_1, \xi_2, \varphi(E), \varphi(N) \rangle$, and it is easy to check that the restriction of $\tilde{g}$ (or $g$) to $S(TM)$ has index one and therefore it is a metric tensor field on $S(TM)$. Moreover, we find

$$S(TM)^\perp = \langle \frac{\partial}{\partial x^1} - y^1 \xi_1 - y^1 \xi_2, \frac{\partial}{\partial y^2} \rangle.$$ 

Finally, being $Z \in \Gamma(S(TM)^\perp)$, we deduce that $N$ is the vector field provided by Theorem 2.1.

Now, for the described lightlike hypersurface of $M$ we compute its geometrical object and write the Gauss and Weingarten equations for it and for the screen distribution.

Let $\nabla$ be the Levi-Civita connection of $M$ and $\{E, \xi_1, \xi_2, U = -\varphi E, V = -\varphi N\}$ a basis of $TM$. Then, reminding that $B(X, E) = 0$ for any $X \in \Gamma(TM)$ and that $B$ is symmetric, we compute $B$ with respect to the aforesaid basis and we get, being $\varepsilon_1 = \varepsilon_2 = 1$,

$$B(X, \xi_\alpha) = -\tilde{g}(\nabla_X \xi_\alpha, E) = -\tilde{g}(\varphi X, E) = \begin{cases} 0 & X = \xi_\beta, \quad \beta \in \{1, 2\} \\ \tilde{g}(\varphi^2 E, E) = 0 & X = U \\ \tilde{g}(\varphi^2 N, E) = -1 & X = V \end{cases}.$$

Hence, $M$ is neither totally geodesic nor totally umbilical since $\tilde{g}(V, \xi_\alpha) = -\tilde{g}(\varphi N, \xi_\alpha) = 0$.

We compute the Christoffel symbols of the Levi-Civita connection $\nabla$ in order to find $B(U, U)$ and $B(V, V)$. It is easy to check

$$\Gamma^i_{11} = 4y^1, \quad \Gamma^i_{12} = \Gamma^i_{21} = -2y^2, \quad \Gamma^i_{12} = \Gamma^i_{21} = 2y^1, \quad \Gamma^i_{22} = -4y^2,$$

$$\Gamma^i_{13} = \Gamma^i_{31} = \Gamma^i_{51} = \Gamma^i_{61} = \frac{1}{2} + 2(y^1)^2, \quad \Gamma^i_{13} = \Gamma^i_{31} = \Gamma^i_{14} = \Gamma^i_{41} = -2y^1,$$

$$\Gamma^i_{13} = \Gamma^i_{31} = \Gamma^i_{51} = \Gamma^i_{61} = \frac{1}{2} + 2(y^1)^2,$$

while the other $\Gamma^i_{ij}$ vanish. Being $U = \frac{\partial}{\partial x^1} - \frac{\partial}{\partial y^2} + y^2 \left( \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} \right)$, we have $\tilde{\nabla}_U U = 0$, hence, we deduce $B(U, U) = \tilde{g}(\tilde{\nabla}_U U, E) = 0$. Finally, being $V = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^1} + y^2 \left( \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^2} \right)$, we obtain $\tilde{\nabla}_V V = 0$ and $B(V, V) = \tilde{g}(\tilde{\nabla}_V V, E) = 0$ and $M$ is minimal.

Using (6), we compute $\tau$. To aim this, we find the value of $\tilde{\nabla}_X N$ for any $X \in \Gamma(TM)$. For any $\alpha \in \{1, 2\}$, it is easy to check $\tilde{\nabla}_E N = \frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2} + y^2 \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^1}, \quad \tilde{\nabla}_U N = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^1}$ and $\tilde{\nabla}_V N = 0$. Immediately we find

$$\tau(\xi_\alpha) = \tilde{g}(\tilde{\nabla}_\xi_\alpha N, E) = 0, \quad \tau(E) = \tilde{g}(\tilde{\nabla}_E N, E) = 0, \quad \tau(U) = \tilde{g}(\tilde{\nabla}_U N, E) = 0, \quad \tau(V) = \tilde{g}(\tilde{\nabla}_V N, E) = 0.$$

So, from (6), we have $A_N X = -\tilde{\nabla}_X N$, for any $X \in \Gamma(TM)$.

Again using (6), we compute $\tilde{\nabla}_E E = -\frac{\partial}{\partial y^2} + \frac{\partial}{\partial x^2} + y^2 \left( \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^1} \right), \quad \tilde{\nabla}_E E = 0, \quad \tilde{\nabla}_U E = 0$ and $\tilde{\nabla}_V E = \frac{\partial}{\partial x^1} + \frac{\partial}{\partial y^1}$. Using (6) and being $\tau(X) = 0$ for any $X \in \Gamma(TM)$, we have $A_E X = -\tilde{\nabla}_X E$, for any $X \in \Gamma(TM)$. Finally, recalling that $S(TM) = \langle \xi_1, \xi_2, U, V \rangle$ and $C(X, PY) = \tilde{g}(\tilde{h}(X, PY), N)$, we value $C$ and we find

$$C(X, PY) = \tilde{g}(\tilde{h}(X, PY), N) = \tilde{g}(\tilde{\nabla}_X PY, N) = \tilde{g}(\tilde{\nabla}_X PY, N).$$
For any $\alpha \in \{1, 2\}$, we have

$$C(X, \xi_\alpha) = \bar{g}(\nabla_X \xi_\alpha, N) = -\bar{g}(\varphi X, N) = \bar{g}(X, \varphi N) = \begin{cases} 0 & \text{for } X = \xi_\beta, \quad \beta \in \{1, 2\} \\ 0 & \text{for } X = V \\ -1 & \text{for } X = U \\ 0 & \text{for } X = E \end{cases}. \quad (28)$$

Being $\nabla_{\xi_\alpha} V = -\frac{\partial}{\partial y^\alpha} + y^1 \frac{\partial}{\partial z^1} + y^2 \frac{\partial}{\partial z^2}$, we obtain $C(\xi_\alpha, V) = \bar{g}(\nabla_{\xi_\alpha} V, N) = 0$. Analogously, $\nabla_U V = 0$ and $\nabla_V V = 0$ imply $C(U, V) = \bar{g}(\nabla_U V, N) = 0$, $C(V, V) = \bar{g}(\nabla_V V, N) = 0$. As $\nabla_E V = -\frac{\partial}{\partial y^1} + \frac{\partial}{\partial y^2}$, we get $C(E, V) = \bar{g}(\nabla_E V, N) = 0$. Since $\nabla_U U = 0$ and $\nabla_U U = 0$, we have $C(V, U) = \bar{g}(\nabla_U U, N) = 0$, $C(U, U) = \bar{g}(\nabla_U U, N) = 0$. Finally, $\nabla_E U = \frac{\partial}{\partial y^1} - \frac{\partial}{\partial y^2} - y^1 \frac{\partial}{\partial z^1} + y^2 \frac{\partial}{\partial z^2}$ for any $\alpha \in \{1, 2\}$, we find

$$C(E, U) = \bar{g}(\nabla_E U, N) = 0, \quad C(\xi_\alpha, U) = \bar{g}(\nabla_{\xi_\alpha} U, N) = -1.$$

We check some properties of $D_0$. To this aim, we note that the distribution $D_0 = \langle \xi_1, \xi_2 \rangle$ coincides with ker$(\bar{\varphi})$ so it is integrable, totally geodesic and flat with respect to $\nabla$ (10). It is also easy to check that $D_0$ verifies the conditions of Theorem 4.2. Moreover, from (23), we obtain

$$\hat{h}(\xi_\alpha, \xi_\beta) = -C(\xi_\alpha, \varphi \xi_\beta) = 0 = C(\xi_\alpha, \varphi \xi_\beta) = 0 + C(\xi_\alpha, \eta_\beta)E = 0.$$

and hence $\hat{h}|_{D_0 \times D_0}$ vanishes, while the second fundamental form $\hat{h} : \Gamma(TM) \times \Gamma(D_0) \to \Gamma(F)$ does not vanish, for example $\hat{h}(E, \xi_\alpha) = -\varphi E$, for any $\alpha \in \{1, 2\}$. Furthermore, for any $\alpha \in \{1, 2\}$, $\hat{h}(\xi_\alpha, \xi_\alpha) = 0$, then we have trace$(\hat{h}) = 0$ i.e. $D_0$ is a minimal distribution in $M_0$ with respect to $\nabla$.

Finally, we consider the indefinite $S$-manifold $\bar{M} = (\mathbb{R}^2, \varphi, \xi_1, \xi_2, \eta^1, \eta^2, g)$, described in Example 4.1 in [10] and given by $\xi_\alpha = \frac{\partial}{\partial z^\alpha}, \eta^\alpha = dz^\alpha - \sum_{i=1}^2 y^i dx^i, \alpha \in \{1, 2\}, g = -\sum_{\alpha=1}^2 \eta^\alpha \otimes \eta^\alpha + \frac{1}{2} \sum_{i=1}^2 ((dx^i)^2 + (dy^i)^2)$, and $\varphi$ given by the matrix

$$F = \begin{pmatrix} 0 & I_2 & 0 \\ -I_2 & 0 & 0 \\ 0 & Y & 0 \end{pmatrix}, \quad \text{where} \quad Y = \begin{pmatrix} y^1 & y^2 \\ y^1 & y^2 \end{pmatrix}.$$ 

In this case, the Killing characteristic vector fields are both timelike and we obtain

$$M_{11} = M_{22} = M_{33} = M_{44} = \frac{1}{8}, \quad M_{55} = M_{66} = -\frac{1}{16} + \frac{1}{8}(y^1)^2 + \frac{1}{8}(y^2)^2,$$

$$M_{15} = \frac{1}{8} y^1, \quad M_{16} = -\frac{1}{8} y^1, \quad M_{25} = -\frac{1}{8} y^2, \quad M_{26} = \frac{1}{8} y^2, \quad M_{56} = -\frac{1}{8}(y^1)^2 - \frac{1}{8}(y^2)^2,$$

while the other $M_{AB}$ vanish. Using (27) and Proposition 6.1 we can prove the following proposition omitting its proof because it is analogous to that of Proposition 6.2.

**Proposition 6.5** Let $M$ be a hypersurface of $\bar{M}$. Then, $M$ is a lightlike hypersurface if and only if

$$\frac{1}{2}((D^1)^2 + (D^2)^2 + (D^3)^2 + (D^4)^2 + (y^1)^2 + (y^2)^2)((D^5)^2 + (D^6)^2)$$

$$- y^2(D^2 D^5 - D^2 D^9) + (y^1)^2 + (y^2)^2)D^5 D^6 + y^1(D^1 D^5 - D^1 D^9) = 0.$$
Now, we note that the proof of Proposition 6.3 does not depend on the metric, therefore it also holds in this case. Thus, omitting the proof, we have

**Proposition 6.6** Let $M$ be a hypersurface of $\bar{M}$. $M$ is a characteristic lightlike hypersurface if and only if $(D^1)^2 + (D^2)^2 + (D^3)^2 + (D^4)^2 = 0$ and $D^5 = D^6 = 0$. Hence there are not characteristic lightlike hypersurfaces in $\bar{M}$.

As regards the Example 4.3 in [10] which describes a 4-dimensional Lorentzian $S$-manifold with two characteristic vector fields of different causal type Remark 3.3 ensures that there are not characteristic lightlike hypersurfaces, since $n = 1$.

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