A unified approach to mean-field team: homogeneity, heterogeneity and quasi-exchangeability

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May 18, 2021

Abstract

This paper aims to systematically solve stochastic team optimization of large-scale system, in a rather general framework. Concretely, the underlying large-scale system involves considerable weakly-coupled cooperative agents for which the individual admissible controls: (i) enter the diffusion terms, (ii) are constrained in some closed-convex subsets, and (iii) subject to a general partial decentralized information structure. A more important but serious feature: (iv) all agents are heterogeneous with continuum instead finite diversity. Combination of (i)-(iv) yields a quite general modeling of stochastic team-optimization, but on the other hand, also fails current existing techniques of team analysis. In particular, classical team consistency with continuum heterogeneity collapses because of (i). As the resolution, a novel unified approach is proposed under which the intractable continuum heterogeneity can be converted to a more tractable homogeneity. As a trade-off, the underlying randomness is augmented, and all agents become (quasi) weakly-exchangeable. Such approach essentially involves a subtle balance between homogeneity v.s. heterogeneity, and left (prior-sampling)-v.s. right (posterior-sampling) information filtration. Subsequently, the consistency condition (CC) system takes a new type of forward-backward stochastic system with double-projections (due to (ii), (iii)), along with spatial mean on continuum heterogenous index (due to (iv)). Such system is new in team literature and its well-posedness is also challenging. We address this issue under mild conditions. Related asymptotic optimality is also established.

Key words: Continuum heterogeneity, Exchangeability, Homogeneity, Input constraints, LQG mean-field game, Partial decentralized information, Weak construction duality.

1 Introduction

The starting point of present work is the well-studied mean-field team (MT). In its standard form, a MT involves a large-scale system with considerable weakly-interactive but cooperative agents $\{A_i\}_{i=1}^N$. All agents are endowed with an individual (principal) state, cost functional and admissible decision set respectively in the following manner. The individual state dynamics of $A_i$ is formulated by a controlled Itô-type linear stochastic differential equation (LSDE):

$$\begin{cases}
    dx_i(t) = [A(t)x_i(t) + B(t)u_i(t) + F(t)x^{(N)}(t) + f_i]dt + \sigma_i dW_i(t), \\
    x_i(0) = \xi \in \mathbb{R}^n, \\ 1 \leq i \leq N,
\end{cases}$$

where $x^{(N)} := \frac{1}{N} \sum_{i=1}^N x_i$ is the weakly-coupled state-average across all agents, $W_i$ is a Brownian Motion (BM) that might be vector-valued (e.g., with a common noise). For each $A_i$, its principal cost $J_i$ (while we may call $\{J_j\}_{j \neq i}$ the marginal costs for $A_i$) is measured by the following quadratic functional:

$$J_i(u(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Q(t)(x_i(t) - H(t)x^{(N)}(t)), x_i(t) - H(t)x^{(N)}(t) \rangle + \langle R(t)u_i(t), u_i(t) \rangle \right] dt,$$

with admissible team strategy $u(\cdot) = (u_{i,op}(\cdot))^\top$. Note individual admissible $u_{i,op}(\cdot) \in U_{i,op}^{d,f} = L^2_{\mathcal{F}}(0, T; \mathbb{R}^m)$ with filtration $\mathcal{F}$ defined later, representing the decentralized open-loop information of $A_i$. 

A subtle point here is the distinction between centralized \( (\mathcal{U}_{cl}^{i,f}) \), and decentralized \( (\mathcal{U}_{cl,op}^{d,f}, \mathcal{U}_{cl,ol}^{d,f}) \) but of full information. This makes team-optimization differing from classical vector-optimization/control; superscripts "cl", "ol" denote the closed-loop and open-loop; "f" the full-information. We will address this point more detailed in Section 2. Hereafter, we may exchange the usage of \( \mathbf{u} = (u_1, \cdots, u_N) \in \mathbb{R}^{m \times N} \), \( \mathbf{u} = (u^T_1, \cdots, u^T_N)^T \in \mathbb{R}^N \) and \( \mathbf{u} = (u_i, u_{-i}) \in \mathbb{R}^{m \times N} \) with \( u_{-i} = (u_1, \cdots, u_{i-1}, u_{i+1}, u_N) \in \mathbb{R}^{m \times (N-1)} \) by noting all of them represent team profile among all agents, but only differ in formations. For simplicity, we focus on Lagrange problem only, and no essential difficulty to Bolza problem extension.

By mean-field “team”, we refer all weakly-coupled agents \( \{\mathcal{A}_i\}_{i=1}^N \) are cooperative aiming to optimize the following social (or, team) cost functional (the related optimal functional is called social optima):

\[
\mathcal{J}_{soc}^{(N)}(\mathbf{u}(\cdot)) = \sum_{i=1}^{N} \mathcal{J}_i(\mathbf{u}(\cdot)).
\]

Because of the cooperation nature, the analysis of MT should proceed very differently from that of mean-field game (e.g., [3, 5, 11, 13, 30]), especially in its analysis ingredients on variational decomposition and person-by-person optimality principle. For non-cooperate N-player game with interaction of mean field type, the objective of the players is to seek the Nash equilibria. Please refer [3, 11, 13, 29, 34] for the limit relation between mean-field games (MFG) and non-cooperate N-player games. The interested readers may refer e.g., [27, 33, 37], for detailed analysis comparison between MFG and MT, and [36, 38] for some recent MT study from various perspectives with different modeling variants. In particular, see [23] for social optima in mean field control problems with volatility uncertainty; see [20] for linear-quadratic-Gaussian (LQG) mean-field social optimization with a major player; and [33] for social optima in LQG models with Markov jump parameters.

Our work distinguishes itself from all above MT literature by the following fairly (even not the most) general formulation, in LQG context. Unlike (1), the individual dynamic of agent \( \mathcal{A}_i \) now takes:

\[
\begin{aligned}
\left\{
\begin{array}{l}
dx_i(t) = [A_i(t)x_i(t) + B_i(t)u_i(t) + F(t)x^{(N)}(t)]dt \\
+ [C(t)x_i(t) + D_i(t)u_i(t) + \tilde{F}(t)x^{(N)}(t)]dW_i(t),
\end{array}
\right.
\end{aligned}
\]

(3)

and the admissible strategy set for \( \mathcal{A}_i \) is now assumed to be

\[
\mathcal{U}^{d,p}_i = \{u_i(\cdot) | u_i(\cdot) \in L^2_{G^i}(0, T; \Gamma)\}
\]

(4)

where \( G^i \subseteq \mathbb{F}^i \) or \( \mathcal{G}^i \subseteq \mathbb{H}^i \) is a subfiltration representing the partial information; \( \Gamma \subseteq \mathbb{R}^m \) is a nonempty closed convex set representing the input constraint.

There are four main modeling features in formulation (3), (4):

(i) Weakly-coupled controlled-diffusion. It is remarkable that in (1), when \( D \neq 0 \) so control process enters diffusion terms of Itô-type LSDE (driven by \( W_i(\cdot) \)), and when \( \tilde{F} \neq 0 \) so all individual states are weakly-coupled in diffusion terms also. In this case, we may call (3) to be diffusion-controlled and weakly-coupled. This differs from [27] in modeling that is only drift-controlled and weakly-coupled. Such modeling difference also brings considerable analysis distinctions, for example, on the relevant study of Hamiltonian systems, as well as consistency condition (CC) (see more comparison details in Section 3 and Section 4). Without loss of generality, no forcing terms such as \( f, \sigma \) involve in (3).

(ii) Random diversity. Recall that (1) is homogenous since all agents are endowed with identical parameters thus they become symmetric. Subsequently, the (decentralized) optimal strategy and states, still denoted as \( \{u_i\}_{i=1}^{N} \) and \( \{x_i\}_{i=1}^{N} \), should turn to be exchangeable. By contrast, in (3), a random index \( \Theta_i \) is introduced in parameter \( A, D \) (also possible to be equipped on other parameters including the cost) to model the diversity across underlying large-scale system. All agents thereby become heterogenous. Although heterogenous large-scale system is already well addressed in such as [21, 25], we point out in these works, the heterogenous index is technically treated as some realization after random sampling, along with necessary ordinal arrangements within each sub-classes. Thus, essentially the index therein is some deterministic realization. This differs substantially from our random index treatment here along with related analysis, to be highlighted later. In addition, our index \( \Theta_i \) can assume a continuum support that distinguishes from most heterogenous literature with only finite/discrete support (see, e.g., [21, 25]). Moreover, although continuum heterogeneity is also discussed in e.g., [32], but analysis therein heavily relies upon the LQ structure with full input and resultant explicit representation. Such analysis collapses
in current formulation (iii), due to the intrinsic diffusion-controlled weakly-coupled feature introduced before, and an input constraint feature to be introduced below.

(iii) Input constraint. Note that a convex-closed set \( \Gamma \) is introduced in (i) denoting some point-wise constraint in control input. Recall that such pointwise input constraint is well documented in e.g., [13, 16, 22, 31]. A typical example is \( \Gamma = \mathbb{R}^+ \) representing the positive control, or no-shorting constraint in portfolio selection (31). Other examples may include subspace (114) or a general convex cone (22). We remark that point-wise input constraint is also studied in large-scale/large-population context such as [20] but in competitive mean-field-game setup, which differs from our cooperative mean-field team here.

(iv) Partial information. Last but not least, the admissible control set is confined on a partial information set \( \mathbb{L}^2_{\mathbb{G}}(0, T; \Gamma) \). LQG control with partial information is also well documented (e.g., 10). Also, partial information for large population system is also addressed recently (see [6, 7, 17, 24] for partial information/observation mean-field game). However, to our best knowledge, it is the first time to address partial information in mean-field team context. Notice that the partial information setting differs from that of partial observation (ii) for which some filtering method with innovation process should be invoked. We defer more detailed information structure in Section 2 after more rigorous formulation.

To certain content, our aim in current work is to solve LQG MT problem in a rather general setup, by combining aforementioned features (i)-(iv) together. Although we admit various effective techniques have been already proposed to tackle these features individually, however their combination brings much more technical hurdles, and makes the associated analysis rather challenging. For example, the continuum heterogenous large-scale system is well studied by [22] in mean-field game setup. Nevertheless, its parallel analysis variant to MT fails to work in current formulation because of the following reasoning. Due to controlled-diffusion feature (i), the related CC does not admit direct characterization because the adjoint process of some backward SDE should enter CC dynamics. Therefore, the direct augmented method in [38] fails to work here. Instead, some indirect embedding method [21, 36] becomes necessary in the presence of (i). Nevertheless, due to continuum heterogenous feature (ii), the classical embedding CC in [21, 36] no longer works since we have to construct an infinite-dimensional Brownian motion-driven system (on continuum-valued space) to replicate the empirical distribution generated by controlled large-scale system. Meanwhile, the method in [36] is also not infeasible since it mainly rely on some close-form representation of optimal state/cost. This becomes unavailable because of the input constraint (iii) imposed above. In nutshell, in case (i) or (iii) not combined together, we may still handle continuum heterogenous MT with (ii) by modifying existing methods in e.g., [36]. However, combination of (i), (ii), (iii) together make all such existing methods no longer workable.

Other examples include the person-by-person procedure due to continuum heterogeneous (ii), and tailor-made decentralized strategy in presence of both point-wise constraint (iii) and partial information constraint (iv). To circumvent these difficulties, we propose some novel analysis techniques such as weak construction duality and modified embedding representation, etc. More analysis details are illuminated in Section 3 and Section 5.

Our main contributions can be sketched as follows: (1) First, we devise a new framework to unify homogenous and heterogenous (discrete or continuum) setups in large-scale system. In particular, it is enabled to transform heterogenous setup into a homogenous one, with the tradeoff of an augmented randomness. (2) Second, under such new framework, we derive a modified embedding representation of CC system (a crux in MT analysis) to accommodate the continuum diversities. (3) Third, the input constraint and partial information constraint are tackled both, and a CC system with double projection operator is derived. Specifically, the CC system takes a coupled mean-field type forward backward stochastic differential equations (FBSDEs) involving both projection mapping and conditional expectation. This seems quite novel in large-scale literature. (4) Last, the well-posedness of CC system and asymptotic team optimality are established under mild conditions.

We would like to conclude above discussion by highlighting a literature comparison. Seemingly, the current work seems closely related to previous work [21]. However, the formulation of [21] is non-cooperative mean-field game with finite heterogenous diversity. By contrast, the current work focuses on a cooperative mean-field team with continuum random diversity index. In addition, current formulation includes partial information also, thus the CC condition here involves a double projection whereas [20, 21] only involves one single projection. Last but not least, other MT analysis ingredients also differ essentially from those in MG setup such as [21], owning to the intrinsic distinction between game and team.

The remaining of this paper is organized as follows. In Section 2, we give the formulation of LQG heterogeneous agents problem with input constraints and partial information pattern. In Section 3, we apply person-by-person optimality and weak construction duality to find the auxiliary control problem.
of the individual agent. The decentralized strategy and consistency condition is established in Section 4. Moreover, we also compare our framework with that in the current literature. Section 5 studies the well-posedness of CC system, asymptotic optimality of decentralized strategy is given in Section 6.

2 Problem formulation

We first introduce some standard notations used throughout this paper. Let $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with the inner product denoted by $\langle \cdot, \cdot \rangle$. $\mathbb{R}^{n \times m}$ is the space of all $(n \times m)$ matrices, endowed with the inner product $\langle M_1, M_2 \rangle = tr[M_1^T M_2^T]$, where $x^T$ denotes the transpose of a matrix (or vector) $x$ and $tr$ is the trace of a matrix. $M \in \mathbb{S}^n$ denotes the set of symmetric $n \times n$ matrices with real elements. $M > (\geq) 0$ denotes that $M \in \mathbb{S}^n$ which is positive (semi)definite, while $M \gg 0$ denotes that, $M - \varepsilon I \geq 0$ for some $\varepsilon > 0$.

Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space on which $\{W_i(t), 0 \leq t \leq T\}_{i=1}^N$ is a $N$-fold Brownian motion (note here $W_i$ might be vector-valued, say, including a common noise component $W_0$) and $\{\Theta_i\}_{i=1}^N$ is a sequence of independent random variables to represent diversity. In some sense, we may interpret $\{\Theta_i\}$ as some endogenous randomness, while $\{W_i\}$ some exogenous randomness for generic agent $A_i$. Moreover, we assume $\{\Theta_i\}_{i=1}^N$ are also independent of $\{W_i(s), 0 \leq s \leq t\}_{i=1}^N$. Let $\{\mathcal{F}_t^W\}_{0 \leq t \leq T}$ be the filtration generated by $\{W_i(s), 0 \leq s \leq t\}_{i=1}^N$ and define $\mathcal{F}_t^{W, \Theta} = \sigma(\Theta_i, 1 \leq i \leq N) \lor \mathcal{F}_t^W$. The set of null sets on $\Omega$ is defined by $\mathcal{N}_0 = \{M \in \Omega \mid \exists G \in \mathcal{F}_t^{W, \Theta} \text{ with } M \subset G \text{ and } \mathbb{P}(G) = 0\}$. Consider the augmented filtration $\mathbb{P} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ with $\mathcal{F}_t = \sigma(\mathcal{F}_t^{W, \Theta} \cup \mathcal{N}_0)$. Similarly, define $\mathcal{F}_t^{W_i}, \mathcal{F}_t^{W, \Theta}, \mathcal{F}_t^1$. As discussed below, they respectively denote the centralized and decentralized information.

For any Euclidean space $\mathcal{V}$, $1 \leq p < \infty$, and any $T > 0$, we introduce some spaces which will be used later:

- $L^p_{\mathcal{F}_T}(\Omega; \mathcal{V}) := \{\eta : \Omega \to \mathcal{V} \mid \eta \text{ is } \mathcal{F}_T\text{-measurable such that } \mathbb{E}|\eta|^p < \infty\}$.
- $L^\infty(0, T; \mathcal{V}) := \{\varphi(\cdot) : [0, T] \to \mathcal{V} \mid \text{esssup}_{0 \leq s \leq T}|\varphi(s)| < \infty\}$.
- $L^p(0, T; \mathcal{V}) := \{\varphi(\cdot) : [0, T] \to \mathcal{V} \mid \int_0^T |\varphi(s)|^p ds < \infty\}$.
- $L^p_\mathcal{F}(0, T; \mathcal{V}) := \{\varphi(\cdot) : \Omega \times [0, T] \to \mathcal{V} \mid \text{progressively measurable such that } \mathbb{E}\int_0^T |\varphi(s)|^p ds < \infty\}$.

We consider a weakly coupled large population system of heterogeneous agents $\{A_i : 1 \leq i \leq N\}$ with the dynamics of the agents given in (3), and cost functional (2). For sake of presentation, we restate them as follows:

$$
\begin{align*}
\begin{cases}
dx_i(t) = & [A_\Theta x_i + Bu_i + F x_i^{(N)}] dt + [C x_i + D_\Theta u_i + \bar{F} x_i^{(N)}] dW_i, \\
& \quad x_i(0) = \xi \in \mathbb{R}^n, \quad 1 \leq i \leq N, \\
& \quad J_i(u(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left[ \langle Q(x_i - H x_i^{(N)}), x_i - H x_i^{(N)} \rangle + \langle R u_i, u_i \rangle \right] dt.
\end{cases}
\end{align*}
$$

As mentioned before, state (3) and functional (2) formulate a weakly coupled large-scale system with heterogeneous agents $\{A_i : 1 \leq i \leq N\}$. The aggregate team functional of $N$ agents is

$$
J_{\text{soc}}^{(N)}(u(\cdot)) = \sum_{i=1}^N J_i(u(\cdot)).
$$

$(A_\Theta(\cdot), B(\cdot), C(\cdot), D_\Theta(\cdot), F(\cdot), \bar{F}(\cdot))$ are called the state-coefficient datum, while $(Q(\cdot), H(\cdot), R(\cdot))$ the cost weight datum. We explain more details for above datum. $F, \bar{F}$ are weakly-coupling coefficients on state-drift and state-diffusion respectively; $H$ is weakly-coupling coefficient on functional; $C, D_\Theta$ are diffusion state-dependence and diffusion control-dependence coefficients respectively. Note that $D_\Theta \neq 0$ represents the case when control enters diffusion terms alike the risky portfolio selection (e.g., [22, 31, 41]); $F, \bar{F} \neq 0$ denotes the agents are coupled in their dynamics such as the price formation problem (e.g., [19, 28]); $H \neq 0$ denotes the relative performance formulation (e.g., [14]).

Unlike state (1), we introduce $\{\Theta_i\}_{i=1}^N$ in (3) as some diversity index to characterize the possible heterogenous features among all agents in underlying large-scale system. We point out that $\Theta_i$ may
vector-valued on a Cartesian grid space, say $[a_1, b_1] \times [a_2, b_2]$ or $[a_1, b_1] \times \{1, \cdots, K\}$, to represent various feature dimensions, either in continuum space or discrete space, or in a hybrid manner.

**Remark 2.1** We remark that discrete- or finite-valued $\Theta_i$ might be transformed into continuum one by assigning uniform distribution on compact interval along with given partitions. Indeed, this is equivalent to simulate a given discrete random variable using quantile method by uniform distribution. Thus, hereafter we focus on vector-valued index $\Theta_i$ on Cartesian space $\mathbb{R}^k$.

For simplicity, we only assume that the coefficients $A$ and $D$ to be dependent on $\Theta_i$. Similar analysis can be generalized to the case when all other coefficients are also $\Theta_i$-dependent. Besides, all datum may depend on time variable $t$, in what follows the variable $t$ will usually be suppressed if no confusion occurs.

We now introduce the following assumption on distribution and coefficient datum set:

(A1) For $i=1, \cdots, N$, $\Theta_i : \Omega \to \mathcal{S}$ are independently identically distributed (i.i.d) with the distribution function $\Phi(\theta)$, i.e., $\int_{\mathcal{S}} d\Phi(\theta) = 1$, where $\mathcal{S}$ is a continuum subset in Cartesian space $\mathbb{R}^k$.

(A2) For any $\theta \in \mathcal{S}$, $A_{\theta}()$, $F_{\theta}()$, $C()$, $\tilde{F} \in L^\infty(0, T; \mathbb{R}^{n \times n})$, $B()$, $D_{\theta}() \in L^\infty(0, T; \mathbb{R}^{n \times m})$, $Q() \in L^\infty(0, T; \mathbb{S}^n)$, $H() \in L^\infty(0, T; \mathbb{S}^n)$, $R() \in L^\infty(0, T; \mathbb{S}^m)$.

(A3) $Q() \geq 0$, $R() \gg 0$.

Under assumptions (A1)-(A2), the state $[3]$ admits a unique strong solution

$$x(\cdot) = (x_1(\cdot), \cdots, x_N(\cdot)) \in L^2(0, T; \mathbb{R}^{N \times n}),$$

and the cost functional is well defined for each admissible control strategy $u(\cdot)$ on appropriate admissible space, to be detailed soon. Moreover, under assumption (A3), the cost functional is uniform convex, that is, there exists some $\delta > 0$ such that $J^\text{loc}(u) \geq \delta \mathbb{E} \int_0^T |u(s)|^2 ds$.

Given state $[2]$ and functional $[2]$, we can specify the associated information structures. Recall that in LQG MT, $\{x_i\}_{i=1}^N$ and $\{u_i\}_{i=1}^N$ denote states and controls of $\{A_i\}_{i=1}^N$ respectively. Because of interactive coupling by state-average $x^{(N)} := \frac{1}{N} \sum_{i=1}^N x_i$, $J_i(u_i, u_{-i})$ depends on total team-decision $u = (u_i, u_{-i})$. In this sense, $[3]$ exhibits the so-called weakly interactive coupling in decision when $N \to +\infty$. Again, by such interactive coupling, information structure of $[3]$ becomes more involved:

- **Centralized information**: consider the filtration $F^W_t = \sigma(W_t(s), 0 \leq s \leq t, i = 1, \cdots, N)$, $F^W_{t, \Theta} = \sigma(\Theta_i, 1 \leq i \leq N) \vee F^W_t, 0 \leq t < \infty$, as well as the set of null sets $\mathcal{N}^W_\Theta = \{M \in \Omega \exists G \in F^W_\infty, M \subset G \} \cap \mathbb{P}(G) = 0$, and create the augmented filtration $F = \{F_t\}_{0 \leq t \leq T}$ with $F_t = \sigma(F^W_t, \Theta) \cup \mathcal{N}^W_\Theta$. Then $F = \{F_t\}_{0 \leq t \leq T}$ represents the centralized information including all Brownian motions (BM) and diversity index components across all agents (principal and marginals).

- **Decentralized, open-loop information**: consider the filtration $F^W_t = \sigma(W_t(s), 0 \leq s \leq t)$, $F^W_{t, \Theta} = \sigma(\Theta_i) \vee F^W_t, 0 \leq t < \infty$, as well as the set of null sets $\mathcal{N}^W_\Theta = \{M \in \Omega \exists G \in F^W_\infty, M \subset G \} \cap \mathbb{P}(G) = 0$, and create the augmented filtration $F^t = \{F^t_{0 \leq t \leq T}\}$ with $F^t_t = \sigma(F^W_t, \Theta) \cup \mathcal{N}^W_\Theta$. Then $F^t$ represents the decentralized open-loop information that only includes the principal components for $A_i$. Note that $\{F^t_{0 \leq t \leq T}\}$ only depends on underlying $W^i$ and $\Theta_i$ instead of state $x_i$ itself, thus we call it open-loop (although it also differs from classical open-loop due to mean-field nature) information since it depends directly on underlying randomness.

- **Decentralized, closed-loop information**: denote by $\{\mathcal{H}^i_{0 \leq t \leq T}\}$ the filtration by individual state $x_i$ augmented by $\mathcal{N}^W_\Theta$, i.e., $\mathcal{H}^i_t = \sigma(x_i(s), 0 \leq s \leq t) \vee \mathcal{N}^W_\Theta$, then $\mathbb{H}^i := \{\mathcal{H}^i_{0 \leq t \leq T}\}$ represents decentralized closed-loop information; Note that $\{\mathcal{H}^i_{0 \leq t \leq T}\}$ only depends on underlying principal state $x_i$ itself, thus we call it closed-loop (although it also differs from classical closed-loop due to mean-field nature). We remark that $x_i$ is not adapted to $W^i$ and $\Theta_i$ due to weakly coupling.

- **Decentralized, partial information**: Let $\mathcal{G}^i_t \subseteq F^i_t$ be a sub-$\sigma$-field of $F^i_t$ (or, $\mathcal{G}^i_t \subseteq \mathcal{H}^i_t$ be a sub-$\sigma$-field of $\mathcal{H}^i_t$), then $\mathcal{G}^i = \{\mathcal{G}^i_{0 \leq t \leq T}\}$ represents the decentralized partial information (open-loop or closed-loop) available to $A_i$. 


Remark 2.2 For decentralized, partial information pattern, $\mathcal{G}_i^t$ is a given filtration representing the information available to $\mathcal{A}_i$ at time $t$. For example, $\mathcal{G}_i^t = \mathcal{F}_i^{t-\delta_+}$, or $\mathcal{G}_i^t = \mathcal{H}_i^{t-\delta_+}$, $t \in [0, T]$, where $\delta > 0$ denotes the fixed delay of information. In this case, $\mathcal{G}_i^t$ represents the partial information in open-loop or closed-loop sense, respectively. Another example is that $\tilde{W}_i = (\tilde{W}_i, \tilde{W}_0)$ takes vector-valued Brownian motion including a common noise component $\tilde{W}_0$, then $\mathcal{G}_i^t = \sigma\{\tilde{W}_i(s), \Theta_i, 0 \leq s \leq t\}$ denotes the partial information in open-loop. Also, in case $\Theta_i = (\Theta_{i1}, \Theta_{i2})$, then $\mathcal{G}_i^t = \sigma\{W_i(s), \Theta_{i1}, 0 \leq s \leq t\}$ denotes the partial information to underlying diversity.

Therefore, $\mathcal{B}_i^t = \mathcal{F}_i^t \lor \mathcal{H}_i^t$ and $\mathcal{B}^t := \{\mathcal{B}_i^t\}_{0 \leq t \leq T}$ represents (full) decentralized information. Then we have the following structure inclusion chart:

$$
\mathcal{G}_i^t \subset \{\mathcal{F}^t_{(\text{decentralized open-loop})}, \mathcal{H}^t_{(\text{decentralized closed-loop})}\} \subset \mathcal{B}^t_{(\text{decentralized})} \subset \mathcal{F} \ (\text{full}).
$$

Noticing due to state-average $x^{(N)}$, $x_i(t) \notin \mathcal{F}_i^t$, thus, NO inclusion relations between open-loop $\mathcal{F}^t = \{\mathcal{F}_i^t\}_{0 \leq t \leq T}$ and closed-loop $\mathcal{H}^t = \{\mathcal{H}_i^t\}_{0 \leq t \leq T}$. This is different to classical control where the open-loop information includes closed-loop information. Given information structure, we are ready to formulate the relevant admissible control sets:

- Centralized full-information admissibility set: $\mathcal{U}_i^{t,f} = \{u_i(\cdot)|u_i(\cdot) \in L^2_0(0, T; \Gamma)\}$.
- Decentralized full-information open-loop admissibility set:
  $$\mathcal{U}_i^{t,op} = \{u_i(\cdot)|u_i(\cdot) \in L^2_0(0, T; \Gamma)\}.$$
- Decentralized full-information closed-loop admissibility set:
  $$\mathcal{U}_i^{t,cl} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathcal{H}_i^t}(0, T; \Gamma)\}.$$
- Decentralized partial-information admissibility set: $\mathcal{U}_i^{t,p} = \{u_i(\cdot)|u_i(\cdot) \in L^2_{\mathcal{G}_i^t}(0, T; \Gamma)\}$.

We point out here $\mathcal{G}_i^t$ is general to include both open-loop or closed-loop partial information. Now we propose the following optimization problem:

**Problem LQG-MT.** Find a team strategy set $\tilde{u}(\cdot) = (\tilde{u}_1(\cdot), \cdots, \tilde{u}_N(\cdot))$ where $\tilde{u}_i(\cdot) \in \mathcal{U}_i^{t,f}, 1 \leq i \leq N$, such that

$$
\mathcal{J}_t^{(N)}(\tilde{u}(\cdot)) = \inf_{u_i \in \mathcal{U}_i^{t,f}, 1 \leq i \leq N} \mathcal{J}_t^{(N)}(u_1(\cdot), \cdots, u_i(\cdot), \cdots, u_N(\cdot)).
$$

Under some mild conditions on datum ($Q, R$) (e.g., (A3)), it is possible to ensure the existence and uniqueness of optimal mean-field team strategy in a centralized sense. This can be proceeded by classical vector-optimization or control method but in a high-dimension setting because of the existence of large number of weakly-coupled team agents. However, such strategy, from a computational viewpoint, turns to be intractable because of the information requirement to collect all agents’ states simultaneously. Instead, it is more tractable to consider some decentralized strategy for which only the local (distributed) information for given agent is needed. Moreover, considering the partial information pattern, we introduce the following definition on asymptotic social optimality.

**Definition 2.1** A strategy set $\tilde{u}(\cdot) = (\tilde{u}_1(\cdot), \cdots, \tilde{u}_N(\cdot))$ with $\{\tilde{u}_i \in \mathcal{U}_i^{t,p}\}_{i=1}^N$ is said to be $\varepsilon$-social optimal if there exists $\varepsilon = \varepsilon(N) > 0, \lim_{N \to +\infty} \varepsilon(N) = 0$ such that

$$
\frac{1}{N}\left(\mathcal{J}_t^{(N)}(\tilde{u}(\cdot)) - \inf_{u_i \in \mathcal{U}_i^{t,f}} \mathcal{J}_t^{(N)}(u(\cdot))\right) \leq \varepsilon.
$$

**Remark 2.3** In Remark 2.2, we emphasize $W_i$ might be vector-valued Brownian motion including a common noise component. For simplicity, in the following we assume that $W_i, \ i = 1, \cdots, N$ are independent one-dimensional Brownian motions. Note that for the case $W_i = (W_i, \tilde{W}_0)$ takes vector-valued Brownian motion including a common noise component $\tilde{W}_0$ and $W_i, \ i = 1, \cdots, N$ being independent one-dimensional Brownian motions, the procedures in Section 5 and Section 4 are still workable. However, in this case $E\alpha$ in (32) should be the conditional expectation $E[\alpha|\mathcal{F}_i^t]$ where $\mathcal{F}_i^t$ is the filtration generated by the common noise $\tilde{W}_0$. For this kind of consistency system, please refer [24] for more information.
3 Mean-field team analysis

As discussed above, the centralized strategy based on traditional vector optimization/control, turns to be inefficient to tackle the weakly-coupled but highly complex LQG MT. Alternatively, it is more desirable to construct some decentralized strategy using distributed information only. Such strategy construction might be proceeded using mean-field team analysis through the following steps:

(Step 1) applying person-by-person optimality to variational decomposition for generic agent;
(Step 2) constructing some auxiliary control problem using necessary (weakly) duality;
(Step 3) solving auxiliary control and determining limiting state-average by consistency condition;
(Step 4) verifying the asymptotic social optimality of derived decentralized team strategy.

We now proceed step by step to construct the distributed LQG-MT strategy.

3.1 Person-by-person optimality

As (Step 1), we would like to propose some variational decomposition for original around centralized strategy (although we prefer to avoid its direct computation). The person-by-person optimality principle is thus adopted for this purpose, from standpoint of a generic agent. More details are as below.

Let \( \{\bar{u}_i \in \mathcal{U}_i\}_{i=1}^N \) be centralized optimal team strategy (its existence can be ensured under some mild convexity conditions. But, as discussed above, such strategies are intractable for real computation purpose because of “curse of dimensionality”). Now consider the perturbation for given benchmark agent, say, \( A_i \) use the alternative strategy \( u_i \in \mathcal{U}_i\) and all other agents still apply the strategy \( \bar{u}_{-i} = (\bar{u}_1, \ldots, \bar{u}_{i-1}, \bar{u}_{i+1}, \ldots, \bar{u}_N) \). The realized state \( x \) corresponding to \((u_i, \bar{u}_{-i})\) and \((\bar{u}_i, \bar{u}_{-i})\) are denoted by \((x_1, \ldots, x_N)\) and \((\bar{x}_1, \ldots, \bar{x}_N)\), respectively. We denote agent index set as \( \mathcal{I} = \{1, \ldots, N\} \). To start the variation decomposition, it is helpful to present the following causal-relation flow-chart first:

\[
\begin{align*}
\delta u_i &= u_i - \bar{u}_i \quad \Rightarrow \quad \delta x_i = x_i(u_i) - \bar{x}_i(\bar{u}_i) \quad \Rightarrow \quad \delta x_j = x_j(x_i) - \bar{x}_j(\bar{u}_i) \\
\delta J_j(\delta u_i) &= \delta J_j(u_i, \bar{u}_{-i}) - \delta J_j(\bar{u}_i, \bar{u}_{-i}), \quad j = 1, \ldots, N, \\
\delta J^{(N)}_{soc}(\delta u_i) &= J^{(N)}_{soc}(u_i, \bar{u}_{-i}) - J^{(N)}_{soc}(\bar{u}_i, \bar{u}_{-i}), \\
\text{total cost variation}
\end{align*}
\]

where \( \delta u_i \) is the most basic variation “block” for other variation structures; we write \( x_i(u_i) \) to emphasize its dependence of \( x_i \) on \( u_i \), and similar for \( \bar{x}_i(\bar{u}_i) \); we call \( \delta x_i \) the principal intermediate variation as it depends indirectly on \( \delta u_i \) via \( \bar{x}_i(\bar{u}_i) \); similarly, \( \delta x_j \) marginal variations from point of \( A_i \); also \( x_j(x_i) \) depends on \( x_i \) via weak-coupling \( x^{(N)} \), similar to \( \bar{x}_j(\bar{x}_i) \). Moreover, from standpoint of \( A_i \), the variational equations for principal state \( x_i \), and marginal states \( \{x_j\}_{j \neq i} \) satisfy:

\[
d\delta x_i = [A_{\Theta_i} \delta x_i + B \delta u_i + F \delta x^{(N)}]dt + [C \delta x_i + D_{\Theta_i} \delta u_i + \bar{F} \delta x^{(N)}]dW_i, \quad \delta x_i(0) = 0, \quad j \neq i, \quad d\delta x_j = [A_{\Theta_j} \delta x_j + F \delta x^{(N)}]dt + [C \delta x_j + \bar{F} \delta x^{(N)}]dW_j, \quad \delta x_j(0) = 0.
\]

(6)

Denote \( \delta x_{-i} = \sum_{j \neq i} \delta x_j \) the aggregate variation of marginal agents (benchmark to \( A_i \)), so applying linear state-aggregation,

\[
d\delta x_{-i} = \left( \sum_{j \neq i} A_{\Theta_j} \delta x_j + (N-1)F \delta x^{(N)} \right) dt + \sum_{j \neq i} [C \delta x_j + \bar{F} \delta x^{(N)}]dW_j, \quad \delta x_{-i}(0) = 0.
\]

Similarly, we can also obtain the variation of cost functionals as follows. For principal cost of \( A_i \):

\[
\delta J_i(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q(\bar{x}_i - H \bar{x}^{(N)}) \rangle, \delta x_i - H \delta x^{(N)} \rangle + \langle R \delta u_i, \delta u_i \rangle \right] dt.
\]

For marginal costs of \( A_i \):

\[
\delta J_j(\delta u_i) = \mathbb{E} \int_0^T \langle Q(\bar{x}_j - H \bar{x}^{(N)}) \rangle, \delta x_j - H \delta x^{(N)} \rangle dt, \quad j \neq i.
\]
Therefore, the total variation of social cost, from the person-by-person variation of $A_i$ side, becomes

$$\delta J_{soc}^{(N)}(\delta u_i) = \mathbb{E} \int_0^T \left[ \sum_{j=1}^N (Q(\tilde{x}_j - H\bar{x}^{(N)}), \delta x_j - H\delta x^{(N)} + (R\bar{u}_i, \delta u_i) \right) dt. $$

We thus have the following variation decomposition on social cost differential:

$$\delta J_{soc}^{(N)}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q\tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH)\bar{x}, \delta x_i \rangle \right. $$

$$- \langle (QH + HQ - HQH)\bar{x}^{(N)}, \sum_{j \neq i} \delta x_j \rangle + \left. \sum_{j \neq i} \langle Q\bar{x}_j, \delta x_j \rangle + (R\bar{u}_i, \delta u_i) \right] dt $$

$$=: I_1 + I_2 + I_3 + I_4 + I_5. $$

There arise five decomposition terms in (9). Among them, $I_1, I_2$ depend on the principal basic variation $\delta u_i$, whereas $I_1, I_2$ depend on principal intermediate variation $\delta x_i$ that further depends on the basic $\delta u_i$. Moreover, $I_3, I_4$ depend on the marginal variations $\{\delta x_j\}_{j \neq i}$ that further depends on the principal ones $\delta x_i, \delta u_i$. We denote $||\delta x_i||_{L^2} = (\mathbb{E} \int_0^T |\delta x_i|^2 dt)^{1/2}$. By standard SDE estimation, $||\delta x_i||_{L^2} \leq (K + O(N^{-2}))||\delta u_i||_{L^2}$ where $K$ is independent on $N$, and only depends on coefficients of (3). Moreover, $||\delta x_j||_{L^2} = O(N^{-2}))||\delta u_i||_{L^2}$ for $j \neq i$. Also, we remark that in general, it is not true that $||\delta x_i||_{L^2} = O(||\delta u_i||_{L^2})$.

We aim to reformulate (9) into some variation differential based on principal terms $\delta u_i, \delta x_i$ only and some auxiliary control problem can thus be constructed in Step 2. We may realize this objective through the following procedures.

First, we need asymptote the empirical state-average $\bar{x}^{(N)}$ in variations $I_2, I_3$ of (9) by its mean-field limit using heuristic reasoning. Therefore, replacing $\bar{x}^{(N)}$ of $I_2, I_3$ in (9) by state-average $\hat{x}$ (to be determined later in Step 3) will yield

$$\delta J_{soc}^{(N)}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q\tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH)\hat{x}, \delta x_i \rangle - \langle (QH + HQ - HQH)\bar{x}, \delta x_i \rangle \right. $$

$$- \langle (QH + HQ - HQH)\bar{x}^{(N)}, \sum_{j \neq i} \delta x_j \rangle + \left. \sum_{j \neq i} \langle Q\bar{x}_j, N\delta x_j \rangle + (R\bar{u}_i, \delta u_i) \right] dt + \varepsilon_1 $$

$$=: I_1 + \hat{I}_2 + \hat{I}_3 + I_4 + I_5 + \varepsilon_1, $$

where

$$\varepsilon_1 = \mathbb{E} \int_0^T \langle (QH + HQ - HQH)(\hat{x} - \bar{x}^{(N)}), N\delta x^{(N)} \rangle dt. $$

Second, note that terms $I_1, \hat{I}_2, I_5$ in (10) already depend on the principal variations $\delta u_i$ or $\delta x_i$. Thus, we need only analyze the limiting behavior for term $\hat{I}_3$ and $I_4$. It is remarkable that $\hat{I}_3, I_4$ respectively involve components: $\delta x_{-i}$ and $\frac{1}{N} \sum_{j \neq i} \langle Q\bar{x}_j, N\delta x_j \rangle$ that both depend on principal basic $\delta u_i$ in rather implicit manner.

Note that for $j \neq i$, $||\delta x_j||_{L^2} = O(N^{-2}))||\delta u_i||_{L^2}$, so $\lim_{N \to \infty} ||\delta x_j||_{L^2} = 0$. Therefore, we need introduce some limiting term $x_j^*$ to replace the re-scaled $N\delta x_j$ in rate $||x_j^* - N\delta x_j|| = O(N^{-2}))||\delta u_i||_{L^2}$.

This helps us to deal with variation of $I_4$. In addition, we introduce limiting term $x^*$ to replace $\delta x_{-i}$ in rate that $||x^* - \delta x_{-i}|| = O(N^{-2}))||\delta u_i||_{L^2}$. This will help us to deal with variation $\hat{I}_3$.

Moreover, by the independence between $\{\Theta_j\}, \{W_j\}$ and heuristic mean-field arguments, we construct the following coupled limiting system:

\[
\begin{align*}
\begin{cases}
\frac{dx_j}{dt} = [A_{\theta_j} x_j + F\delta x_i + F \int_S x_{\theta}^* d\Phi(\theta)]dt + [C x_j^* + \tilde{F}\delta x_i + \tilde{F} \int_S x_{\theta}^* d\Phi(\theta)]dW_j, \\
\frac{dx_{\theta}}{dt} = [A_{\theta} x_{\theta}^* + F\delta x_i + F x_{\theta}^*]dt, \\
x_j^*(0) = 0, \quad j \neq i, \quad \theta \in S.
\end{cases}
\end{align*}
\]
Therefore,

\[
\delta J^{(N)}_{soc}(\delta u_i) = \mathbb{E} \int_0^T \left[ (Q\bar{x}_i, \delta x_i) - (QH + HQ - HQH)\bar{x}, \delta x_i) - (QH + HQ x^{**}) + \frac{1}{N} \sum_{j \neq i} (Q\bar{x}_j, x_j^*) + (R\bar{u}_i, \delta u_i) \right] dt + \sum_{i=1}^3 \varepsilon_i
\]

\[
= I_1 + I_2 + I_3 + I_4 + I_5 + \sum_{i=1}^3 \varepsilon_i.
\]

where

\[
\begin{align*}
\varepsilon_2 &= \mathbb{E} \int_0^T (QH + HQ - HQH)\bar{x}, x^* - \delta x_i dt, \\
\varepsilon_3 &= \mathbb{E} \int_0^T \frac{1}{N} \sum_{j \neq i} (Q\bar{x}_j, N\delta x_j - x_j^*) dt.
\end{align*}
\]

Noting \(I_4\) of \((12)\) connects to a sequence of exchangeable random variables \(\{\int_0^T (Q\bar{x}_j, x_j^*) dt\}\) \(\in L^2_{\mathbb{F}}(\Omega; \mathbb{R})\). By de Finetti theorem, they are conditionally independent identically distributed with respect to some tail sigma-algebra. Also, it is observable that such tail sigma-algebra should depend on \(\delta x_i\) in rather implicit way. Then, we may apply conditional law of large number to identify the related average. We present some weak duality approach to break away \(\delta J^{(N)}_{soc}(\delta u_i)\) from dependence on \(x_j^*\) and \(x^{**}\).

### 3.2 Weak construction duality

In order to break away \(\delta J^{(N)}_{soc}(\delta u_i)\) of \((12)\) from direct dependence on \(x_j^*\) and \(x^{**}\) (see \(I_3, I_4\)), we introduce the following adjoint equations \(\{y_1^i\}_{j \neq i}\) and \(y_2^\theta\) satisfying:

\[
\begin{align*}
dy_1^i &= \alpha_1^i dt + \beta_{1j}^i dW_j + \sum_{l=1, l \neq j}^N \beta_{lj}^i dW_l, \quad y_1^i(T) = 0, \quad j \neq i, \\
dy_2^\theta &= \alpha_2^\theta dt, \quad y_2^\theta(T) = 0, \quad \theta \in \mathcal{S}.
\end{align*}
\]

where \(\{W_i\}_{j \neq i}\) are some Brownian motion copies matching all marginal agents in large-scaled system, from the benchmark point of \(\mathcal{A}_i\). We remark that \(y_2^\theta\) is parameterized by diversity index in continuum support: \(\theta \in \mathcal{S}\), while \(y_1^i\) is parameterized by marginal agent index \(j \neq i \in \mathcal{I}\). Accordingly, the duality below should be some weak construction in distributional and agent-wise sense, respectively indexed by \(\theta \in \mathcal{S}\) and \(j \in \mathcal{I}\). To start, first apply Itô’s formula to \((y_1^i, x_j^*)\) for each marginal agent index \(j \neq i\), integrating from 0 to \(T\) and taking expectation, by countable agent-wise addition for all \(j \in \mathcal{I}\setminus i\),

\[
0 = \mathbb{E} \int_0^T \left[ \frac{1}{N} \sum_{j \neq i} (\alpha_1^j + A^\theta_j y_2^j + C^\theta \beta_{1j}^j, x_j^*) + \frac{1}{N} \sum_{j \neq i} \langle F^T y_1^j + \bar{F}^T \beta_{1j}^j, x^{**} \rangle \\
+ \frac{1}{N} \sum_{j \neq i} \langle F^T y_2^j + \bar{F}^T \beta_{1j}^j, \delta x_i \rangle \right] dt.
\]

(14)

Similarly, by distributed integral on all \(\theta \in \mathcal{S}\),

\[
0 = \int_0^T \left[ \int_S (\alpha_2^\theta + A^\theta_2 y_2^\theta + F^T y_2^\theta, x_0^{**}) d\Phi(\theta) + \int_S (F^T y_2^\theta, \delta x_i) d\Phi(\theta) \right] dt.
\]

(15)
Combing (14) and (15) with (12), we observe that the initial terms such as \( y \) are degenerated BSDE by noting \( \Theta \in \mathcal{F}_0 \). Also, it is not necessary to specify any dependence assumption between \( \Theta_j \) and \( \Theta \) since \( y^\Theta_j \) and \( y^\Theta_j \) get coupled only through expectation operator. In other words, their coupling here and further variant in consistency condition, only depend on the expectation in distribution sense. Again, this is why we term the resultant duality as weak-duality. Substituting (17) into (16), we have

\[
\delta \mathcal{J}^{(N)}_{soc}(\delta u_i) = \mathbb{E} \int_0^T \left[ \langle Q \tilde{x}_i, \delta x_i \rangle - \langle (QH + HQ - HQH) \tilde{x}, \delta x_i \rangle - \frac{1}{N} \sum_{j \neq i} \langle F^\top y_i^j + \tilde{F}^\top \beta^j_1, \delta x_i \rangle 
- \int_S \langle F^\top y^\Theta_2, \delta x_i \rangle d\Phi(\theta) + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \sum_{l=1}^3 \varepsilon_l,
\]

where

\[
\varepsilon_4 = \mathbb{E} \int_0^T \langle F^\top (\mathbb{E}[y_i^j] - 1) \sum_{j \neq i} y_i^j + \tilde{F}^\top (\mathbb{E}[\beta^j_1] - 1) \sum_{j \neq i} \beta^j_1, x^{**} \rangle dt.
\]

We observe that the initial terms such as \( \langle Q \tilde{x}_j, x^*_j \rangle \) in (12), is now reformulated with some inner product between principal intermediate variation \( \delta x_i \) and some quantities in terms by \( y^\Theta_j \) and \( y^\Theta_j \) in an agent-wise (i.e., \( j \neq i \)) manner. Then, we can identify the tail filtration for exchangeable \( \int_0^T \langle Q \tilde{x}_j, x^*_j \rangle dt \) based on \( \delta x_i \) with a degenerated filtration. So, applying conditional law of large number, and noticing \( \{ y_i^j, \} \)
\( j \neq i \) are identical distributed, we reach the following representation with expectation operator: 

\[
\delta J^{(N)}(\delta u_i) = \mathbb{E} \int_0^T \left[ (Q\hat{x}_i, \delta x_i) - \langle (QH + HQ - HQH)\hat{x} + F^T\mathbb{E}[y_1] + \tilde{F}^T\mathbb{E}[\beta_1^i] \right] \\
+ F^T \int_S y_2^\theta d\Phi(\theta, \delta x_i) + \langle R\bar{u}_i, \delta u_i \rangle dt + \sum_{l=1}^N \varepsilon_l, 
\]

where \( y_1 \) (depending on \( \bar{x}_1 \), that is the optimized state for generic agent) is some copy with same distribution for generic \( y_1^j \):

\[
\begin{cases}
  dy_1 = [Q\bar{x}_1 - A_{\theta}^T y_1 - C^T \beta_1]dt + \beta_1^i dW_i + \sum_{l=1, l \neq 1}^N \beta_1^l dW_l, \\
  dy_2^i = -(QH + HQ - HQH)\hat{x} - (F^T\mathbb{E}[y_1] + \tilde{F}^T\mathbb{E}[\beta_1]) - A_{\theta}^T y_2^i - F^T y_2^i dt, \\
  y_1(T) = 0, \quad y_2^i(T) = 0, \quad \theta \in S,
\end{cases}
\]

and

\[
\varepsilon_5 = \mathbb{E} \int_0^T (F^T (\mathbb{E}[y_1]) - \frac{1}{N} \sum_{j \neq i} y_1^j) + \tilde{F}^T (\mathbb{E}[\beta_1]) - \frac{1}{N} \sum_{j \neq i} \beta_1^j, \delta x_i) dt.
\]

We remark that \( y_1 \) has the same distribution with generic \( y_1^j \), thus we call above procedure as weak duality. We point out all variations terms in (18), are now directly depending only on principal (basic, or intermediate) variations. Thus, we now formulate a decentralized auxiliary cost differential \( \delta J_i(\delta u_i) \):

\[
\delta J_i(\delta u_i) = \mathbb{E} \int_0^T \left[ (Q\bar{x}_i, \delta x_i) - \langle (QH + HQ - HQH)\hat{x} + F^T\hat{y}_1 + \tilde{F}^T\hat{\beta}_1 \right] \\
+ F^T \int_S y_2^\theta d\Phi(\theta, \delta x_i) + \langle R\bar{u}_i, \delta u_i \rangle dt.
\]

Remark 3.1 There are four undetermined terms in (20) respectively:

- \( \hat{x} \) by (10) is the state-average limit;
- \( \hat{y}_1 = \mathbb{E}[y_1], \hat{\beta}_1 = \mathbb{E}[\beta_1^i], y_2^\theta \) is from (19) because of the weak construction duality procedure. All these terms, especially \( \hat{x} \), will be determined by consistency condition (CC) in Section 4.

Remark 3.2 In (20), we introduce the first variation of auxiliary cost functional \( \delta J_i(\delta u_i) \) and ignore the error term \( \varepsilon_l, l = 1, \cdots, 5 \). The convergence rate estimation of these terms and the rigorous proofs will be given in Section 7.

4 Auxiliary control problem and consistency condition

4.1 Auxiliary control with double-projection

By (20), we can introduce the following (auxiliary control (AC) problem) for a generic \( A_i \):

\[
\text{(AC): } \\
\begin{align*}
\text{Minimize } & J_i(u_i(\cdot)) = \frac{1}{2} \mathbb{E} \int_0^T \left[ (Qx_i, x_i) - 2\langle \Xi, x_i \rangle + \langle R\bar{u}_i, u_i \rangle \right] dt, \\
\text{subject to } & dx_i(t) = [A_{\theta}^T x_i + B_u + F\hat{x}]dt + [C_{\theta}^T + D_{\theta}^T u_i + \tilde{F}\hat{x}]dW_i(t), x_i(0) = \xi,
\end{align*}
\]

with

\[
\Xi(t; \hat{x}, y_2^\theta, \hat{y}_1, \hat{\beta}_1) = (QH + HQ - HQH)\hat{x} + F^T \hat{y}_1 + \tilde{F}^T \hat{\beta}_1 + F^T \int_S y_2^\theta d\Phi(\theta),
\]

where \( \hat{x} \) is the limiting state-average term introduced in (10); \( y_2^\theta, \hat{y}_1, \hat{\beta}_1 \) depends on \( \hat{x} \) satisfying dynamics (19). Also, we remark that \( \hat{y}_1 \) depends on optimal state \( \bar{x}_j \).

We will apply stochastic maximum principle to study Problem (AC). To this end, we introduce the following first-order adjoint equation:

\[
dp_i(t) = -[A_{\theta}^T p_i + Qx_i - \Xi + C^T q_i]dt + q_i dW_i(t), \quad p_i(T) = 0.
\]
Let $u_1^*$ be the optimal control and $(x_1^*, p_1^*, q_1^*)$ the corresponding state and adjoint state. For any $u_i \in L_{\mathcal{P}}^2((0,T;\mathbb{R}^m)$ such that $u_i^* + u_i \in \mathcal{U}_1^{op,p}$. Let $u_i^* := u_i^* + \epsilon u_i \in \mathcal{U}_i^{op,p}$. The corresponding state and adjoint state with respect to $u_i^*$ are denoted by $(x_i^*, p_i^*, q_i^*)$. Introduce the following variational equation

$$dy_i(t) = [A_{\Theta}, y_i + Bu_i]dt + [C_{\Theta} + D_{\Theta}, u_i]dW_i(t), \quad y_i(0) = 0.$$ 

Applying Itô’s formula to $(p_i, y_i)$, by the optimality of $u_i^*$ (i.e., $J_i(u_i^*) - J_i(u_i^*) \geq 0$), we have

$$\mathbb{E} \int_0^T \langle Ru_i^* + B^\top p_i + D_{\Theta}, q_i, u_i \rangle ds \geq 0.$$ 

For any $0 \leq t \leq T$ and $\mathcal{G}_t^i$-measurable random variable $\eta_i$, let

$$u_i^*(s) + u_i(s) = \begin{cases} u_i^*(s), & s \notin [t, t+\epsilon]; \\ \eta_i, & s \in [t, t+\epsilon]. \end{cases}$$

Therefore,

$$\frac{1}{\epsilon} \mathbb{E} \int_t^{t+\epsilon} \langle Ru_i^* + B^\top p_i + D_{\Theta}, q_i, \eta_i - u_i^* \rangle ds \geq 0.$$ 

Let $\epsilon \to 0$, we have

$$\mathbb{E}(R(t)u_i^*(t) + B^\top(t)p_i^*(t) + D_{\Theta}, q_i^*(t), \eta_i - u_i^*(t)) \geq 0, \quad t \in [0, T].$$ 

For any $v \in \Gamma$ and $A \in \mathcal{G}_t^i$, define $\eta_i = vI_A + u_i^*(t)I_{A^c}$, we have

$$\mathbb{E}(R(t)u_i^*(t) + B^\top(t)p_i^*(t) + D_{\Theta}, q_i^*(t), v - u_i^*(t))I_A \geq 0, \quad t \in [0, T].$$

Since $A \in \mathcal{G}_t^i$ is arbitrary, we have

$$\mathbb{E}((R(t)u_i^*(t) + B^\top(t)p_i^*(t) + D_{\Theta}, q_i^*(t), v - u_i^*(t))|\mathcal{G}_t^i) \geq 0, \quad t \in [0, T], \mathbb{P} - a.s.$$ 

i.e.,

$$(-R(t)u_i^*(t) + \mathbb{E}[-B^\top(t)p_i^*(t) - D_{\Theta}, q_i^*(t)|\mathcal{G}_t^i], v - u_i^*(t)) \leq 0, \quad t \in [0, T], \mathbb{P} - a.s. \quad (22)$$

Since $v \in \Gamma$ is arbitrary and $\Gamma$ is a closed convex set, it follows from the well-known results of convex analysis that (22) is equivalent to

$$u_i^*(t) = P_\Gamma[R^{-1}-\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta}, q_i^*(t)|\mathcal{G}_t^i]], \text{ a.e. } t \in [0, T], \mathbb{P} - a.s., \quad (23)$$

where $P_\Gamma[\cdot]$ is the projection mapping from $\mathbb{R}^m$ to its closed convex subset $\Gamma$ under the norm $\|v\|_\Gamma := (R^Tv, R^Tv)$. We point out that there involves two projections in (23), because of the input constraint and partial information constraint. This differs from [24] which include only single-projection on input set. Furthermore, the two projections are non-commutative due to above maximum principle arguments. In this case, the related Hamiltonian system for (AC) problem becomes

$$\begin{cases} \text{d}x_i^* = \left[ A_{\Theta}x_i^* + BP_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta}, q_i^*(t)|\mathcal{G}_t^i]] + F \right] dt \\ \text{d}p_i^* = \left[ Cx_i^* + D_{\Theta}P_\Gamma[R^{-1}\mathbb{E}[-B^\top p_i^*(t) - D_{\Theta}, q_i^*(t)|\mathcal{G}_t^i]] + F \right] dW_i(t), \end{cases} \quad (24)$$

which is a fully-coupled FBSDEs with double-projection: the mapping on input convex-closed set, and the filtering for partial information (i.e., conditional expectation on sub-space).
4.2 Consistency condition

In this section, we will characterize the undetermined processes, especially state-average limit \( \hat{x} \), in \([21]\) via some consistency matching scheme. Given the Hamiltonian system by \([24]\), all agents should apply some exchangeable team decisions \( \{u_i^*\}_i \) and the realized states should be as follows:

\[
\begin{align*}
\dot{x}_i^* &= \left[ A_{\Theta} x_i^* + B P_T [ R^{-1} E [ -B^\top p_i^* (t) - D_{\Theta} q_i^* (t) | G_i ] ] + F x^* (N) \right] dt \\
&\quad + \left[ C x_i^* + D_{\Theta} P_T [ R^{-1} E [ -B^\top p_i^* (t) - D_{\Theta} q_i^* (t) | G_i ] ] + \tilde{F} x^* (N) \right] dW_i (t), \\
x_i^* (0) &= \xi_i,
\end{align*}
\]

where \( x^* (N) = \frac{1}{N} \sum_{i=1}^{N} x_i^* \) and \( (p_i^*, q_i^*) \) is the solution of \([24]\). Making all such exchangeable strategies aggregated, and applying de Finetti theorem, we can obtain the limiting system by identifying \( \hat{x} = E x^* \),

\[
\begin{align*}
\dot{\bar{x}} &= \left[ A_{\Theta} \bar{x} + B P_T [ R^{-1} E [ -B^\top \bar{p} (t) - D_{\Theta} \bar{q} (t) | G_i ] ] + F E \bar{x} \right] dt \\
&\quad + \left[ C \bar{x} + D_{\Theta} P_T [ R^{-1} E [ -B^\top \bar{p} (t) - D_{\Theta} \bar{q} (t) | G_i ] ] + F E \bar{x} \right] dW (t), \\
\dot{\bar{p}} &= - \left[ A_{\Theta} \bar{p} + Q \bar{x} - (Q H + H Q H) E \bar{x} - F^T \bar{y}_1 - \tilde{F}^T \bar{\beta}_1 \right] \\
&\quad - F^T \int_S \bar{y}_2^\theta d \Phi (\theta) + C^T \bar{q} \] dt + \bar{q} dW (t), \\
\bar{x} (0) &= \xi, \quad \bar{p} (T) = 0,
\end{align*}
\]

where \( \Theta \) is a random variable with distribution defined in \((A1)\), \( W (t) \) is a generic Brownian motion independent of \( \Theta \), \( G \) is sub-filtration representing the partial information and \( (\bar{y}_1 = E [ y_1 ], \bar{\beta}_1 = E [ \beta_1 ], \bar{y}_2^\theta ) \) is from \([19]\). Note that we suppress subscript \( i \) in \([20]\) as all agents are statistically identical in the distribution sense. Combing with \([19]\), we will obtain consistency condition \((CC)\) of Problem LQG-MT. For simplicity, define

\[
E_r [ - B^\top \gamma - D_{\Theta} \theta ] = E [ - B^\top \gamma - D_{\Theta} \theta | G_i ] .
\]

Hence we have the following result.

**Theorem 4.1** The undetermined parameters of \([21]\) can be determined by

\[
(\hat{x}, \bar{y}_1, \bar{\beta}_1, \bar{y}_2^\theta) = (E \alpha, E \bar{y}_1, E \bar{\beta}_1, \bar{y}_2^\theta),
\]

where \((\alpha, \gamma, \bar{y}_1, \bar{\beta}_1, \bar{y}_2^\theta)\) is the solution of the consistency condition of Problem LQG-MT:

\[
\begin{align*}
\dot{\alpha} &= [ A_{\Theta} \alpha + B P_T [ R^{-1} E_r [ - B^\top \gamma - D_{\Theta} \theta ] ] + F E \alpha ] dt \\
&\quad + [ C \alpha + D_{\Theta} P_T [ R^{-1} E_r [ - B^\top \gamma - D_{\Theta} \theta ] ] ] + F E \alpha dW, \\
\dot{\gamma} &= [ - Q \alpha + (Q H + H Q H) E \alpha - A_{\Theta}^\top \gamma + F^T \int_S \bar{y}_2^\theta d \Phi (\theta) + F^T E \bar{y}_1 ] \\
&\quad - C^T \dot{\theta} + \tilde{F}^T E \bar{\beta}_1 \] dt + \dot{\theta} dW, \\
\dot{\bar{y}}_1 &= [ Q \alpha - A_{\Theta}^\top \bar{y}_1 - C^T \bar{\beta}_1 ] dt + \bar{\beta}_1 dW, \\
\dot{\bar{y}}_2^\theta &= [ - (Q H + H Q H) E \alpha - F^T E \bar{y}_1 - \tilde{F}^T E \bar{\beta}_1 - A_{\Theta}^\top \bar{y}_2^\theta - F^T \bar{y}_2^\theta ] dt, \\
\alpha (0) &= \xi, \quad \gamma (T) = 0, \quad \bar{y}_1 (T) = 0, \quad \bar{y}_2^\theta (T) = 0, \quad \theta \in S.
\end{align*}
\]

**Remark 4.1** \([20]\) is a new type of fully-coupled FBSDEs with double-projection (projection mapping on the convex-closed sub-set and partial-information sub-space). Moreover, both temporal variable \( t \) and spatial variable \( \theta \) appear in \([20]\). Considering this, we can rewrite \([20]\) in the following more compact
Note that by the independence between \( \Theta \) and \( W \), (27) can be viewed as defined on the product space \( \Omega_1 \times \Omega_2 \rightarrow \mathcal{S} \times \mathbb{R}^n \). This is a general system which includes many framework in current literature as special cases. For more information, please refer to Section 4.3.

The well-posedness of (28) will be studied in Section 5.

### 4.3 Literature comparison

We now present comparisons to some relevant literature.

#### 4.3.1 Homogeneous case without diversity

For the homogeneous case with \( S = \{s_1\} \) being singleton set, we have \( A_{\theta_i} = A_{s_1} := A \) and \( D_{\theta_i} = D_{s_1} := D \) for \( i = 1, \cdots, N \). In this case, we do not need to introduce \( x^*_{\theta} \) as in (11) when applying person-by-person optimality. We only need to introduce \( x^{**} \) to replace \( \delta x_{-i} \). In fact, in current case, \( x^{**} \) satisfies the following dynamics:

\[
dx^{**} = [(A + F)x^{**} + F\delta x_i]dt, \quad x^{**}(0) = 0.
\]

Moreover, CC in homogeneous case becomes

\[
\begin{align*}
\displaystyle{\frac{d\alpha}{dt}} &= [A\alpha + B^T\mathcal{E}_t[-B^T\gamma - D_{\theta}\vartheta]] + F\mathbb{E}\alpha||dt \\
&\quad + [C\alpha + D^T\mathcal{E}_t[-B^T\gamma - D_{\theta}\vartheta]] + F\mathbb{E}\alpha||dW, \\
d\gamma &= [Q_{\alpha} + (QH + HQ - HQH)\mathbb{E}\alpha - A_{\theta}\gamma + F^T\mathbb{E}\tilde{y}_1 - F^T\mathbb{E}\tilde{y}_2 - F^T\mathbb{E}\tilde{y}_2]dt \\
&\quad + F^T\mathbb{E}\tilde{y}_1||dW(t), \\
d\tilde{y}_1 &= [Q_{\alpha} - A_{\theta}\tilde{y}_1 - C^T\tilde{\beta}_1]dt + \tilde{\beta}_1dW, \\
d\tilde{y}_2 &= [-Q_{\alpha} + (QH + HQ - HQH)\mathbb{E}\alpha - F^T\mathbb{E}\tilde{y}_1 - F^T\mathbb{E}\tilde{y}_2 - F^T\mathbb{E}\tilde{y}_2]dt, \\
\alpha(0) &= \xi, \quad \gamma(T) = 0, \quad \tilde{y}_1(T) = 0, \quad \tilde{y}_2(T) = 0.
\end{align*}
\]

This is the special case of (28) with \( \Phi(\theta) \) being a Dirac distribution. Subsequently, our framework covers the homogeneous case as its special case. Furthermore, in case \( C = D = F = \tilde{F} = 0, \Gamma = \mathbb{R}^m \) and \( \mathcal{G}^T = F^T \), by applying expectation, \( \tilde{\alpha} = \mathbb{E}\alpha \) and \( \tilde{\gamma} = \mathbb{E}\gamma \) satisfy the dynamics:

\[
\begin{align*}
\displaystyle{\frac{d\tilde{\alpha}}{dt}} &= [A\tilde{\alpha} - BR^{-1}B^T\tilde{\gamma}]dt, \\
\displaystyle{\frac{d\tilde{\gamma}}{dt}} &= [(-Q + QH + HQ - HQH)\tilde{\alpha} - A^T\tilde{\gamma}]dt, \\
\tilde{\alpha}(0) &= \xi, \quad \tilde{\gamma}(T) = 0.
\end{align*}
\]

This is just the special case discussed in pp. 1742 of (27) (see (42),(43) therein). The only difference is that (29) is of open-loop (\( \tilde{\gamma} \) is the adjoint process) while (42) and (43) in (27) are of closed-loop (\( \Pi\tilde{x} + s \) is of feedback form).

#### 4.3.2 Heterogeneous case with finite diversities

Specifically, we assume that \( \Theta_i \) is deterministic (post-sampling) and assumes values in a finite discrete set \( \mathcal{S} = \{1, 2, \cdots, K\} \). For \( 1 \leq k \leq K \), introduce

\[
\mathcal{I}_k = \{i|\Theta_i = k, 1 \leq i \leq N\}, \quad N_k = |\mathcal{I}_k|,
\]

form:

\[
\begin{align*}
d\alpha &= [A\alpha + B^T\mathcal{E}_t[-B^T\gamma - D_{\theta}\vartheta]] + \mathbb{E}\alpha||dt \\
&\quad + [C\alpha + D^T\mathcal{E}_t[-B^T\gamma - D_{\theta}\vartheta]] + \mathbb{E}\alpha||dW, \\
d\gamma &= [Q_{\alpha} + (QH + HQ - HQH)\mathbb{E}\alpha - A_{\theta}\gamma + F^T\mathbb{E}\tilde{y}_1 - F^T\mathbb{E}\tilde{y}_2 - F^T\mathbb{E}\tilde{y}_2]dt \\
&\quad + F^T\mathbb{E}\tilde{y}_1||dW(t), \\
d\tilde{y}_1 &= [Q_{\alpha} - A_{\theta}\tilde{y}_1 - C^T\tilde{\beta}_1]dt + \tilde{\beta}_1dW, \\
d\tilde{y}_2 &= [-Q_{\alpha} + (QH + HQ - HQH)\mathbb{E}\alpha - F^T\mathbb{E}\tilde{y}_1 - F^T\mathbb{E}\tilde{y}_2 - F^T\mathbb{E}\tilde{y}_2]dt, \\
\alpha(0) &= \xi, \quad \gamma(T) = 0, \quad \tilde{y}_1(T) = 0, \quad \tilde{y}_2(T) = 0.
\end{align*}
\]
where \( N_k \) is the cardinality of index set \( \mathcal{I}_k \) (i.e., cardinality of set of \( k \)-type agents). For \( 1 \leq k \leq K \), let \( \pi_k^{(N)} = \frac{N_k}{N} \), then \( \pi^{(N)} = (\pi_1^{(N)}, \cdots, \pi_K^{(N)}) \) is a probability vector representing the empirical distribution of \( \Theta_1, \cdots, \Theta_N \). Suppose there exists a probability mass vector \( \pi = (\pi_1, \cdots, \pi_K) \) such that \( \lim_{N \to +\infty} \pi^{(N)} = \pi \) and \( \min_{1 \leq k \leq K} \pi_k > 0 \). Under these assumptions, the person-by-person procedure still proceeds as in Section 3.1.

Let \( \delta x_{(k)} = \sum_{j \in \mathcal{I}_k, j \neq i} \delta x_j \). By exchangeability of agents within same type, we need only consider a representative agent in each type when using a limit to approximate \( x \). Therefore, for \( k = 1, \cdots, K \), we should introduce the term \( x_k^* \) to replace \( \delta x_{(k)} \), where \( x_k^* \) satisfies the following dynamics:

\[
dx_k^* = \left[ A_k x_k^* + F \pi_k \delta x_i + F \pi_k \sum_{l=1}^K x_l^* \right] dt, \quad x_k^*(0) = 0, \quad k = 1, \cdots, K.
\]

Furthermore, if \( \mathcal{G}^t = \mathcal{F}^t \), CC of heterogeneous case with finite diversities becomes:

\[
\begin{align*}
    da_k &= [A_k a_k + B P_{\gamma_k} [R^{-1}_k (B^T \gamma_k + D^T_k \vartheta_k)] + F \sum_{l=1}^K \pi_l \mathbb{E} a_l] dt \\
    &\quad + [C a_k + D_k P_{\gamma_k} [R^{-1}_k (B^T \gamma_k + D^T_k \vartheta_k)] + \bar{F} \sum_{l=1}^K \pi_l \mathbb{E} a_l] dW_k(t), \\
    d\gamma_k &= [-Q a_k + (Q H + H Q - H Q H) \sum_{l=1}^K \pi_l \mathbb{E} a_l - A_k^T \gamma_k + F^T \sum_{l=1}^K \pi_l \mathbb{E} \gamma_l^2 + F^T \sum_{l=1}^K \pi_l \mathbb{E} \gamma_l^1] dt \\
    &\quad - C^T \gamma_k + \bar{F} \sum_{l=1}^K \pi_l \mathbb{E} \gamma_l^1 dt + \vartheta_k dW_k(t), \\
    d\gamma_k^1 &= [Q a_k - A_k^T \gamma_k^1 - C^T \gamma_k^1] dt + \beta_k^1 dW_k(t), \\
    d\gamma_k^2 &= [- (Q H + H Q - H Q H) \sum_{l=1}^K \pi_l \mathbb{E} a_l - \sum_{l=1}^K \pi_l (F^T \mathbb{E} \gamma_l^1 + \bar{F}^T \mathbb{E} \gamma_l^1) - A_k^T \gamma_k^2 - F^T \sum_{l=1}^K \pi_l \mathbb{E} \gamma_l^2] dt, \\
    a_k(0) &= \xi, \quad \gamma_k(T) = 0, \quad \gamma_k^1(T) = 0, \quad \gamma_k^2(T) = 0, \quad k = 1, \cdots, K.
\end{align*}
\]

is similar to the consistency condition in [21] (see (2.15) therein). [21] deals with mean-field game with heterogeneous case with finite diversities, hence the consistency condition only involves the Hamiltonian system of the auxiliary control problem. While for LQG-MT, besides the Hamilton system [25], CC also includes [19] by the person-by-person and weak-construction duality procedure.

### 4.3.3 Heterogeneous case with continuum diversities but without state-coupling

When \( F = \bar{F} = 0 \), i.e., there is no weakly-coupling in state, by (7) we have \( \delta x_j = 0 \) for \( j \neq i \), thus \( x_i^*, x_i^{**} \) both vanish in (11). The resulting cost variation (12) takes a rather simple form than (9):

\[
\delta J_{soc}^{(N)} = E \int_0^T \left[ \langle Q \bar{x}, \delta x_i \rangle - \langle (Q H + H Q - H Q H) \dot{x}, \delta x_i \rangle + \langle R \tilde{u}_i, \delta u_i \rangle \right] dt + \varepsilon_1,
\]

where

\[
\varepsilon_1 = E \int_0^T \langle (Q H + H Q - H Q H) (\dot{x} - \bar{x})^{(N)}, N \delta \bar{x}^{(N)} \rangle dt.
\]

From (31) we can obtain the auxiliary control problem directly, i.e., it becomes unnecessary to introduce the limit terms (11) and adjoint processes (13). This is similar to the case in Section IV.A of [27]. Note that in [27], there is no point-wise constraint or partial information constraint on the admissible control, hence the main focus is to find the optimal closed-loop control for the auxiliary control problem (see (32) therein). While with the above two constraints, we will obtain the optimal open-loop control for the auxiliary control problem (see [29]). In this case, (32) reduces to

\[
\begin{align*}
    da &= [A_{\theta} a + B \Phi_{\gamma} [R^{-1}_\gamma [-B^T \gamma - D^T \vartheta]]] dt \\
    &\quad + [C a + D_{\theta} \Phi_{\gamma} [R^{-1}_\gamma [-B^T \gamma - D^T \vartheta]]] dW, \\
    d\gamma &= [-Q a + (Q H + H Q - H Q H) \mathbb{E} a - A_{\theta}^T \gamma - C^T \vartheta] dt + \vartheta dW(t), \\
    a(0) &= \xi, \quad \gamma(T) = 0.
\end{align*}
\]
for which the well-posedness is much more easily to establish. Furthermore, if \( C = D = 0, \Gamma = \mathbb{R}^m \) and \( G^i = F^i \), by taking expectation to (32), the derived FBSDEs reduces to the case on pp. 1740 of [27].

By contrast, when \( F, \bar{F} \neq 0 \), variation functional \( \delta \mathcal{J}^{(N)}_t(\delta u_i) \) of (12) becomes rather involved depending on \( x_i^* \) and \( x_i^{**} \) both. Those two terms are some intermediate variation limits related to basic variation term \( \delta x_i \) in an indirect manner. Thus, the current representation (12) cannot lead a direct construction to an auxiliary control. Some duality method are required to remove dependence on these intermediate variations.

4.3.4 Other cases

For homogeneous case, [20] studies linear-quadratic mean-field games with control process constrained in a closed convex subset of full space \( \mathbb{R}^m \); [21] studies backward mean-filed linear-quadratic games with partial information. When there involves only constraints on the control or only partial information, our framework is the extension of [20] and [21] for social optima case.

4.4 Homogeneity and heterogeneity: a unified quasi-exchangeable approach

Recall that the mean-field theory has been extensively applied to study the large-scale weakly-coupled system along both (competitive) game and (cooperative) team directions, see e.g., [5, 11, 21, 25, 26, 30] for recent relevant studies for game; and [30, 35] for team. Essentially, such mean-field analysis is build on some exchangeability among all individual weakly-coupled agents. It can be proved that any exchangeable sequences should be conditional independent with respect to some tail-sigma algebra. Thus, applying de Finetti theorem, the original complex weakly-coupling structure can be replaced by a deterministic-or common-noise-driven process as agent number \( N \) tends to infinity. By this, all agents thus become asymptotically decoupled along with chaos propagation. Subsequently, original game or team can be reduced to low dimensional single agent optimization problem with some off-line quantities via consistency condition that matches the above exchangeable reasoning. In this sense, mean-field analysis connects closely to exchangeable game/team in random context, and further to symmetric game/team ([15]) in deterministic context. We remark that all agents in symmetric game are endowed with same underlying parameters and so become identical in analysis. So, the primal high-dimensional computation can be greatly reduced using “mirror” argument among all symmetric agents.

Regarding large-scale system, there exist three progressive levels of diversity relevant to aforementioned exchangeability: homogeneous, heterogenous with finite/discrete diversity, and heterogenous with continuum diversity. Among them, homogenous case is most special but tractable one because all agents are statistical identical and the designed optimal team strategies should also be exchangeable. Consequently, the resulting optimized states are thus exchangeable. We refer [30] for recent studies in such case for team, and [20] for game.

Compared with homogenous case, heterogenous case with finite/discrete diversity is more realistic. Virtually, most systems in reality demonstrate some diversities in their random behaviors. In this case, all agents, from whole system scale, are no longer identical because they are endowed with diversified parameters. However, all agents inside a sub-system with same diversity index, are still exchangeable in small scale. Thus, we can treat the large-scale system as some mixed combination of finite exchangeable sub-systems. The previous mean-field analysis to homogenous can be suitably modified to tackle such case, with some technical but straightforward arguments. We refer [2] for recent studies in such case for team in discrete time setup, and [25, 21] for game, where a similar partial exchangeability is introduced.

The heterogenous case with continuum diversity, as discussed in [27, 32], should be most realistic setup for practical large-scale system. Indeed, it is less possible that the diversity of real system, can only be limited on a finite or discrete support set. Instead, considerable statistical diversity demonstrate its support on a continuum set such as compact closed interval. On the other hand, such heterogenous case should be most difficult to be handled. One reason for the continuum heterogeneity to be analytically intractable, is that the sub-class exchangeability featured in finite heterogeneity case, will shrink to zero mass along with the continuum diversity support. For this reason, the relevant results for continuum heterogeneity seems few compared with homogeneous- or finite-heterogenous-case.

We remark [32] discussed mean-field analysis with continuum diversity in game setup, and [27] in team setup, using a direct state-aggregating method. However, the setting in both works are relatively simple, in particular, its weakly-coupled dynamics is only drift-controlled. This corresponds to our model with \( C = D = F = 0 \), and cannot cover various applications such as portfolio selection with
By contrast, the third system is also a stochastic mixture. It is obvious that above three systems:

\[ x_i(t) = \xi \in \mathbb{R}^n, \quad 1 \leq i \leq N, \]

can be reformulated as follows:

\[
\begin{align*}
\begin{cases}
 dx_i = [A(x_i) x_i + Bu_i + F(x_i) u_i + \bar{F} x_i] dt + [C x_i + D_{\theta_i} u_i + \bar{D} x_i] dW_i, \\
 x_i(0) = \xi \in \mathbb{R}^n, \quad 1 \leq i \leq N,
\end{cases}
\end{align*}
\]

In other words, initial weakly-coupled system with continuum diversity can be viewed as some quasi-exchangeable method. The main idea is as follows: first, note that the dynamics of conditional expectation, can be further written with some augmented state as

\[
d x_i = [A(x_i) x_i + Bu_i + F(x_i) u_i + \bar{F} x_i] dt + [C x_i + D_{\theta_i} u_i + \bar{D} x_i] dW_i, \quad x_i(0) = (\xi^T, \Theta^T)^T.
\]

For sake of illustration, we set \( \Theta \in \Lambda = \{\theta_1, \theta_2, \ldots, \theta_K\} \) with the mass \( m_1, \ldots, m_K \) to admit finite \( K \) diversity classes. Later, we will illustrate its possible extension to infinite continuum diversities. The second system is a stochastic mixture: \( \tilde{x} = \sum_{j=1}^K m_j \tilde{x}_j \) but driven by identical noise \( W \):

\[
P_2 : \quad d\tilde{x}_j = [A_{\theta_j} \tilde{x}_j + Bu_j + F \tilde{x}_j] dt + [C \tilde{x}_j + D_{\theta_j} u_j + \bar{D} \tilde{x}_j] dW_j, \quad \tilde{x}_j(0) = (\xi^T, \Theta_j^T)^T.
\]

By contrast, the third system is also a stochastic mixture \( \tilde{x} = \sum_{j=1}^K m_j \tilde{x}_j \) but driven by \( K \) i.i.d noises \( \{W_j\}_{j=1}^K \):

\[
P_3 : \quad d\tilde{x}_j = [A_{\theta_j} \tilde{x}_j + Bu_j + F \tilde{x}_j] dt + [C \tilde{x}_j + D_{\theta_j} u_j + \bar{D} \tilde{x}_j] dW_j, \quad \tilde{x}_j(0) = (\xi^T, \Theta_j^T)^T.
\]

It is obvious that above three systems: \( x, \tilde{x} \) and \( \tilde{x} \) are not of the same distributions. Actually, \( x \) has different initial distribution at \( t = 0 \) with \( \tilde{x}, \tilde{x} \) whereas \( \tilde{x} \) is driven by different noise with \( x, \tilde{x} \). Thus, all three systems are not equivalent in weak sense. However, they have same expectation dynamics, as verified using tower property of conditional expectation, \( \forall t \in [0, T] : E(x(t)) = E(E(x(t) | \Theta)) = \sum_{j=1}^K m_j E(\tilde{x}_j(t)) = E(\tilde{x}(t)) = \sum_{j=1}^K m_j E(\tilde{x}_j(t)) = E(\tilde{x}(t)) \). Besides, all three systems have different second-moment function, and other finite-dimensional distributions. For example:

\[
\begin{align*}
E|x(t)|^2 &= E(E(|x(t)|^2 | \Theta)) = \sum_{j=1}^K m_j E|\tilde{x}_j(t)|^2, \\
E|\tilde{x}(t)|^2 &= \sum_{j=1}^K m_j^2 E|\tilde{x}_j(t)|^2 + \sum_{1 \leq j < \ell \leq K} m_j m_\ell E[\tilde{x}_j(t) \tilde{x}_\ell(t)], \\
E|\tilde{x}(t)|^2 &= \sum_{j=1}^K m_j^2 E|\tilde{x}_j(t)|^2 + \sum_{1 \leq j < \ell \leq K} m_j m_\ell E[\tilde{x}_j(t) \tilde{x}_\ell(t)] = \sum_{j=1}^K m_j^2 E|\tilde{x}_j(t)|^2.
\end{align*}
\]
Noticing above expectation equivalence is special degenerated version of Jensen inequality, thanks to the underlying LQG context. Such property cannot be extended to nonlinear moments hence $x$, $\tilde{x}$ and $\hat{x}$ are with same expectation but different distributions.

Corresponding to $P_1$, $P_2$, $P_3$, we may construct three weakly-coupled systems $M_1, M_2, M_3$:

$$M_1 : d\tilde{x}_i = [A(x_i)\tilde{x}_i + Bu_i + Fx(N)]dt + [C\tilde{x}_i + D(x_i)u_i + \tilde{F}x(N)]dW_i, \quad \tilde{x}_i(0) = (\xi^T, \Theta)^T,$$

where $\tilde{x}(N) = \frac{1}{N} \sum_{i=1}^{N} x_i$. Another is weakly-coupled system $M_2 : \{\bar{x}_i\}_{i=1}^{N}$ with $\bar{x}_i = \sum_{j=1}^{K} m_{j} \tilde{x}_{i,j}$,

$$M_2 : d\bar{x}_{i,j} = [A_{\theta_j} \bar{x}_{i,j} + Bu_i + Fx(N)]dt + [C\bar{x}_{i,j} + D_{\theta_j} u_i + \tilde{F}x(N)]dW_i, \quad \bar{x}_{i,j}(0) = (\xi^T, \theta_j)^T,$$

where $\tilde{x}(N) = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i$. For $1 \leq j \leq K$, we can introduce $\tilde{M}_2 : \{\tilde{x}_{i,j}\}_{i=1}^{N}$ that is a homogeneous weakly-coupled system indexed by $\theta_j$. Abusing notation, we may write informally that $\bar{M}_2 = \sum_{j=1}^{K} m_{j} \tilde{M}_2$, in other words, $\bar{M}_2$ is a finite mixture of homogeneous systems $\{\tilde{M}_2\}_{j=1}^{K}$. Noticing for $\tilde{M}_2$, the driving BMs become $\{W_i\}_{i=1}^{N}$ which are same to that of $\tilde{M}_2$, for $j \neq j'$. Thus, totally there involve $N$ independent BMs for $\bar{M}_2$. Moreover, if we introduce a sampling sequence from $\{1, \ldots, K\}$ with $I_j = \{\theta_i = j, 1 \leq i \leq N\}$ and $\lim_{N \to +\infty} \frac{\text{Card} I_j}{N} = m_j$, $1 \leq j \leq K$. Then, $\bar{M}_2$ is equivalent in weak sense to stochastic $K$-heterogeneous weakly-coupled system introduced in [21 [23].

The third system is $M_3 : \{\tilde{x}_i\}_{i=1}^{N}$ with $\tilde{x}_i = \sum_{j=1}^{K} m_{j} \tilde{x}_{i,j}$,

$$M_3 : d\tilde{x}_{i,j} = [A_{\theta_j} \tilde{x}_{i,j} + Bu_i + Fx(N)]dt + [C\tilde{x}_{i,j} + D_{\theta_j} u_i + \tilde{F}x(N)]dW_{i,j}, \quad \tilde{x}_{i,j}(0) = (\xi^T, \theta_j)^T,$$

where $\tilde{x}(N) = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i$. For $1 \leq j \leq K$, we can introduce $\tilde{M}_3 : \{\tilde{x}_{i,j}\}_{i=1}^{N}$ that is a homogeneous weakly-coupled system indexed by $\theta_j$. Noticing for $\tilde{M}_3$, the driving BMs become $\{W_i\}_{i=1}^{N}$. So, totally there arise $N \times K$ independent BMs for $\tilde{M}_3$, or re-scale to $N$ BMs for each sub-system $\tilde{M}_3, 1 \leq j \leq K$. This is not problematic when $K$ is finite. Again, $M_3$ is finite mixture of homogeneous system $\{\tilde{M}_3\}_{j=1}^{K}$. We remark that $\tilde{M}_2$ and $\tilde{M}_3$ are driven by different BMs, but they are equivalent weak-coupled homogenous system in weak sense. This is because they share have same state-average limit by law of large numbers, although they are driven by different BMs systems.

Moreover, we can introduce an augmented state $y_i = (\tilde{x}_{i,1}^T, \ldots, \tilde{x}_{i,K}^T)^T$ and $\tilde{x}(N) = \frac{1}{N} \sum_{i=1}^{N} \tilde{x}_i$, it follows that

$$dy_i = \hat{A} y_i + \hat{B} u_i + Fy(N)dt + \sum_{j=1}^{K} (\hat{C}_j y_i + \hat{D}_j u_i + \hat{F}_j y(N))dW_{i,j}, \quad y_i(0) = (\xi^T, \theta_1^T, \ldots, \xi^T, \theta_K^T)^T,$$

where

$$\hat{A} = \begin{pmatrix} A_{\theta_1} & \cdots & 0 \\ \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{\theta_K} \end{pmatrix}_{(nK \times nK)}, \quad \hat{B} = \begin{pmatrix} B & \cdots & 0 \\ \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B \end{pmatrix}_{(nK \times mK)}, \quad \hat{u}_i = \begin{pmatrix} u_i \\ \vdots \\ u_{i,K} \end{pmatrix}_{(mK \times 1)},$$

$$\hat{F} = \begin{pmatrix} F_{m1} & \cdots & F_{mK} \\ \\ \vdots & \ddots & \vdots \\ F_{m1} & \cdots & F_{mK} \end{pmatrix}_{(nK \times nK)}, \quad \hat{C}_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}_{(nK \times nK)},$$

$$\hat{D}_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{(nK \times nK)}, \quad \hat{F}_j = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix}_{(nK \times nK)}.$$
on more intractable (finite) heterogenous system. As the trade-off, the associated Riccati or Hamiltonian system become augmented accordingly with coupled block structure due to $K$ diversity.

The above three weakly-coupled systems $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ have different distributions but always with the same asymptotic empirical state-average as $N \to +\infty$. In fact, they are generated from same underlying weakly-coupled stochastic systems but differs in filtration on given timing point. To be precise, all agents in $\mathcal{M}_1$ are exchangeable in quasi-sense (at filtration point $\mathcal{F}_0$) before the diversity sampling. In this case, $x_i(t) = \mathbb{E}(x_i(t)|\mathcal{F}_t) = \mathbb{E}(x_i(t)|\Theta, W_i(s), 0 \leq s \leq t, 1 \leq i \leq N)$. On the other hand, $\mathcal{M}_2$ is the same system but conditional on the pre-sampled diversity index $\Theta_i$. In this case, $\tilde{x}_i(t) = \mathbb{E}(E(x_i(t)|\Theta)|W_i(s), 0 \leq s \leq t, 1 \leq i \leq N)$. Last, $\mathcal{M}_3$ is same weak-coupled system but after the sampling of diversity $\Theta$ and $\tilde{\mathcal{M}}_3$ is just the re-labeled system with realization $\Theta = \theta_j$. In this sense, all three systems $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ characterize the same underlying dynamics but from different temporal section. Thus, they are equivalent for mean-field analysis because they share the same state-average limit (in formulation, and Step 1 for decomposition) and expectation operator (in Step 3 for CC).

To recap, we present the following diagram where “$\iff$” represents the equivalent expectation operator in first line, while asymptotic state-average operator in second line:

$$\begin{align*}
\text{single-agent: } & P_1 \iff P_2 \iff P_3, \\
\text{weakly-coupled agents: } & \mathcal{M}_1 \iff \mathcal{M}_2 \iff \mathcal{M}_3 \iff \mathcal{M} \text{ (stochastic $K$-heterogenous system)}, \\
\mathcal{M}_1 & : \text{homogenous but with random diversity index } \Theta, \text{ augmented randomness, pre-sampling} \\
\mathcal{M}_2 & : \text{mixture of } K \text{ homogenous system, pre-sampling} \\
\mathcal{M}_3 & : \text{homogenous system with (augmented) mixture of states, post-sampling} \\
\mathcal{M} & : \text{K heterogenous system defined by relative frequency of diversity sequence, post-sampling.}
\end{align*}$$

Above arguments in (33) are on the basis that $\Theta$ is finite-valued only. Now we present its generalization to case when $\Theta$ has continuum diversity support. In this case, we have

$$\begin{align*}
\mathcal{M}_1 & : \text{homogenous but with random diversity index } \Theta, \text{ augmented randomness, pre-sampling} \\
\mathcal{M}_2 & : \text{mixture of continuum homogenous system, pre-sampling} \\
\mathcal{M}_3 & : \text{homogenous system with (augmented) mixture of states, post-sampling} \\
\mathcal{M} & : \text{continuum heterogenous system defined by empirical distribution of diversity sequence, post-sampling.}
\end{align*}$$

$\mathcal{M}_1$ is still well-defined and we have already proceeded the analysis as in Section 3. On the other hand, $\mathcal{M}_3$ is no longer well defined since now we have to introduce continuum-valued BMs for $\tilde{\mathcal{M}}_3^\Theta$ to model the diversity. By contrast, $\mathcal{M}_2$ is still well defined since we need still only formulate countable BMs for each $\tilde{\mathcal{M}}_2^\Theta, \theta \in \mathcal{S}$, and in total, only countable BMs are still invoked. In this case, we may further set $\bar{x}_i = \int_\mathcal{S} x_i, \theta dF(\theta)$ and proceed the classical mean-field analysis as in [27]. However, classical mean-field analysis only works on $\mathcal{M}_2$ with $C = D = F = \bar{F} = 0$. In general case with $F, \bar{F} \neq 0$, such classical analysis fails because its CC system should invoke an embedding representation (see e.g., [24], and a continuum-valued BMs system will be required to replicate the distribution for a generic agent who is still continuum-heterogenous (diversified). Moreover, in [20], the continuum heterogeneity is defined through some limiting empirical distribution by Glivenko-Cantelli Lemma. Note that the continuum set therein is required to be compact when using Glivenko-Cantelli arguments, while in our framework of $\mathcal{M}_1$, such compactness is not required. Consequently, this paper can deal with general continuum diversity based on formulation $\mathcal{M}_1$, as summarized as follows.

First, we can verify that $\mathcal{M}_1, \mathcal{M}_2$ as well as $\mathcal{M}$ (note that $\mathcal{M}_3$ becomes infeasible to be defined) are still of the same asymptotic state-average limit. In this sense, the generic agents in $\mathcal{M}$ are quasi-exchangeable because although they are not exchangeable after diversity sampling, but $\mathcal{M}$ shares the same expectation and asymptotic state-average limit with $\mathcal{M}_1, \mathcal{M}_2$, and all agents of $\mathcal{M}_1$ and $\mathcal{M}_2$ are exchangeable before the sampling. Second, given such quasi-exchangeable property, the original $\mathcal{M}$ or $\mathcal{M}_2$ system with continuum heterogeneity can be converted to $\mathcal{M}_1$ that is a homogenous one but with augmented randomness $\{\Theta_i, W_i\}_{i=1}^N$ as trade-off. Third, as discussed in Section 3 some new type variation-decomposition and auxiliary control problem can thus be constructed, and CC condition can be represented via some weak-construction on continuum diversity support as in Theorem 1.
5 Wellposedness of consistency condition

This section continues to complete (Step 3) by establishing some well-posedness to consistency condition derived in Section 4. Note that (41) is fully-coupled FBSDEs involved with double projections whose well-posedness cannot be guaranteed by current literature. Moreover, as explained in Section 4, (41) is obtained by converting system with continuum heterogeneity to a homogenous one but with augmented randomness ($\{\Theta_i, W_i\}_{i=1}^N$) as trade-off. Based on this, we will apply the discounting method to study (41) which would provide some mild conditions to ensure the existence and uniqueness of fully-coupled FBSDEs as (41). Define $X = \alpha, Y = (\gamma^T, \hat{y}_1^T, (\hat{y}_2^T)^T)^T$ and $Z = (\theta^T, \hat{\beta}_1^T, 0)^T$. For simplicity, let $\mathcal{E}_i[Y] = \mathbb{E}[Y|\mathcal{G}_i]$ and $\mathcal{E}_i[Z] = \mathbb{E}[Z|\mathcal{G}_i]$, $\mathbb{E}[Y] = ((\int_S \gamma d\Phi(\theta))^T, (\int_S \hat{y}_1 d\Phi(\theta))^T, (\int_S \hat{y}_2^T d\Phi(\theta))^T)^T$, then (41) takes the following form:

$$
\begin{align*}
\begin{cases}
    dX = [A_0 X + F\mathbb{E}[X] + B_1(Y,Z)]dt + [C X + \tilde{F}\mathbb{E}[X] + D_\Theta(Y,Z)]dW,
    \\
    dY = [A_2 X + \tilde{A}_2 \mathbb{E}[X] + \tilde{B}_2 \mathbb{E}[Y] + \tilde{C}_2 Z + \tilde{C}_2 \mathbb{E}[Z]]dt + ZdW,
\end{cases}
\end{align*}
$$

where

$$
\begin{align*}
\tilde{B}_2 &= \begin{pmatrix} -Q & 0 \\ 0 & -F^T \end{pmatrix}, \tilde{C}_2 = \begin{pmatrix} 0 & 0 \\ 0 & -F^T \end{pmatrix},
\end{align*}
$$

and $0$ denotes the zero vector or zero matrix with suitable dimensions. Note that in (41), $\tilde{B}_2 \mathbb{E}[Y] = \mathbb{E}[\tilde{B}_2 Y] = \mathbb{E}[\tilde{B}_2 Y]$. To start, we first give some results for general nonlinear mean-field forward-backward system with double projections:

$$
\begin{align*}
\begin{cases}
    dX = b(t, X, \mathbb{E}[X], Y, \mathcal{E}_i[Y], Z, \mathcal{E}_i[Z])dt + \sigma(t, X, \mathbb{E}[X], Y, \mathcal{E}_i[Y], Z, \mathcal{E}_i[Z])dW, \\
    dY(t) = -f(t, X, \mathbb{E}[X], Y, \mathcal{E}_i[Y], \mathbb{E}[Y], Z, \mathcal{E}_i[Z])dt + ZdW,
\end{cases}
\end{align*}
$$

where $\mathbb{E}[\mathbb{E}[Y]] = \mathbb{E}[Y]$ and the coefficients satisfy the following conditions:

\textbf{(H1)} There exist $\rho_1, \rho_2 \in \mathbb{R}$ and positive constants $k_i, i = 1, \cdots, 17$ such that for all $t \in [0, T], x, x_1, x_2, \bar{x}, \bar{x}_1, x_2 \in \mathbb{R}^m, y, y_1, y_2, \tilde{y}, \tilde{y}_1, \tilde{y}_2, \tilde{y}_1, \tilde{y}_2, \tilde{y}_1, \tilde{y}_2 \in \mathbb{R}^m, z, z_1, z_2, \tilde{z}, \tilde{z}_1, \tilde{z}_2 \in \mathbb{R}^m$, a.s.,

$$
\begin{align*}
&b(t, x_1, \bar{x}, \tilde{y}, z, \tilde{z}) - b(t, x_2, \bar{x}, y, z, \tilde{z}), x_1 - x_2) \leq \rho_1 |x_1 - x_2|^2,
&|b(t, x_1, \bar{x}, \tilde{y}_1, z_1, \tilde{z}_1) - b(t, x, \bar{x}_2, y_2, z_2, \tilde{z}_2)| \\
&\leq k_1 |x_1 - x_2| + k_2 |y_1 - y_2| + k_3 |\tilde{y}_1 - \tilde{y}_2| + k_4 |z_1 - z_2| + k_5 |\tilde{z}_1 - \tilde{z}_2|,
&f(t, x, \tilde{x}_1, y, \tilde{y}, \tilde{z}) - f(t, x, \tilde{x}_2, \tilde{y}, \tilde{z}, y_1 - y_2, z_1 - z_2) \leq \rho_2 |y_1 - y_2|^2,
&|f(t, x_1, \bar{x}_1, \bar{y}_1, z_1, \tilde{z}_1) - f(t, x_2, \bar{x}_2, y_2, \tilde{y}_2, z_2, \tilde{z}_2)| \\
&\leq k_6 |x_1 - x_2| + k_7 |\bar{x}_1 - \bar{x}_2| + k_8 |\bar{y}_1 - \bar{y}_2| + k_9 |y_1 - y_2| + k_10 |z_1 - z_2| + k_11 |\tilde{z}_1 - \tilde{z}_2|,
&|\sigma(t, x_1, \bar{x}_1, \tilde{y}_1, z_1, \tilde{z}_1) - \sigma(t, x_2, \bar{x}_2, y_2, \tilde{y}_2, z_2, \tilde{z}_2)|^2 \\
&\leq k_{12}^2 |x_1 - x_2|^2 + k_{13}^2 |\bar{x}_1 - \bar{x}_2|^2 + k_{14}^2 |y_1 - y_2|^2 + k_{15}^2 |\tilde{y}_1 - \tilde{y}_2|^2 + k_{16}^2 |z_1 - z_2|^2 + k_{17}^2 |\tilde{z}_1 - \tilde{z}_2|^2.
\end{align*}
$$

\textbf{(H2)} \quad \mathbb{E} \int_0^T \left[ |b(t, 0, 0, 0, 0, 0, 0)|^2 + |\sigma(t, 0, 0, 0, 0, 0, 0, 0)|^2 + |f(t, 0, 0, 0, 0, 0, 0)|^2 \right] dt < \infty.
Comparing (35) with (34), we can check that the parameters of (H1) and (H2) can be chosen as follows:

Similar to [21] and [35], we have the following result of the solvability of (35). For the readers’ convenience, (A4)

Now we introduce the following assumption:

strategy derived in Section 4. Here we proceed our verification based on the assumption in Section 5,

6 Asymptotic ε-optimality

This section aims to complete (Step 4) so as to verify the asymptotic optimality of mean-field team strategy derived in Section 4. Here we proceed our verification based on the assumption in Section 5, i.e., (A4).

6.1 Representation of social cost

First, we give a quadratic representation of the team functional. Rewrite the large-population system (3) as follows:

\[ dx = (Ax + Bu)dt + \sum_{i=1}^{N} (Ci x + Di u) dW_i, \quad x(0) = \bar{\xi}, \] (36)

where

\[ A = \begin{pmatrix} A_{e_1} + \frac{F}{2} & \frac{F}{2} & \cdots & \frac{F}{2} \\ \frac{F}{2} & A_{e_2} + \frac{F}{2} & \cdots & \frac{F}{2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{F}{2} & \frac{F}{2} & \cdots & A_{e_N} + \frac{F}{2} \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad B = \begin{pmatrix} B_0 & 0 & \cdots & 0 \\ 0 & B_0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & B_0 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}, \]

\[ C_i = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad D_i = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad \bar{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_N \end{pmatrix} \]

Similarly, the social cost takes the following form:

\[ J_{soc}^{(N)}(u) = \frac{1}{2} \sum_{i=1}^{n} \mathbb{E} \int_{0}^{T} \left[ \langle Q(x_i - H x_i^{(N)}) , (x_i - H x_i^{(N)}) \rangle + \langle R u_i, u_i \rangle \right] dt \]

\[ = \frac{1}{2} \mathbb{E} \int_{0}^{T} \left[ \langle Q x, x \rangle + \langle R u, u \rangle \right] dt, \]
where
\[
Q = \begin{pmatrix}
\frac{1}{2}(H^\top QH - QH^\top H) & \frac{1}{2}(H^\top QH - QH^\top H) & \cdots & \frac{1}{2}(H^\top QH - QH^\top H) \\
\frac{1}{2}(H^\top QH - QH^\top H) & \frac{1}{2}(H^\top QH - QH^\top H) & \cdots & \frac{1}{2}(H^\top QH - QH^\top H) \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2}(H^\top QH - QH^\top H) & \frac{1}{2}(H^\top QH - QH^\top H) & \cdots & \frac{1}{2}(H^\top QH - QH^\top H)
\end{pmatrix},
R = \begin{pmatrix}
R & 0 & \cdots & 0 \\
0 & R & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & R
\end{pmatrix}.
\]

Next, by the variation of constant formula, we know that the strong solution of (36) admits the following representation:
\[
x(t) = \Phi(t)\xi + \Phi(t)\int_0^t \Phi(s)^{-1}[B - \sum_{i=1}^NC_i\Delta_i]u(s)ds + \sum_{i=1}^N\Phi(t)\int_0^t \Phi(s)^{-1}D_iu(s)dW_i(s),
\]
where
\[
d\Phi(t) = A\Phi(t)dt + \sum_{i=1}^NC_i\Phi(t)dW_i(t), \quad \Phi(0) = I.
\]

Define the following operators
\[
\begin{align*}
\phi(u)(\cdot) & := \Phi(\cdot) \left\{ \int_0^t \Phi(s)^{-1}[B - \sum_{i=1}^NC_i\Delta_i]u(s)ds + \sum_{i=1}^N\int_0^t \Phi(s)^{-1}D_iudW_i(s) \right\} \\
\tilde{\phi}(u) & := \phi(u)(T), \quad S(y)(\cdot) := \Phi(\cdot)\Phi^{-1}(0)\xi, \quad \tilde{S}(y) := S(y)(T),
\end{align*}
\]
then for any admissible control \( u \), we have
\[
x(\cdot) = \phi(u)(\cdot) + S(y)(\cdot), \quad x(T) = \tilde{\phi}(u) + \tilde{S}(y).
\]

Note that \( \phi(\cdot) : (L^2_0(0, T; \Gamma), \cdots, L^2_0(0, T; \Gamma)) \rightarrow (L^2_0(0, T; \mathbb{R}^n), \cdots, L^2_0(0, T; \mathbb{R}^n)) \) is a bounded linear operator, thus there exists a unique bounded linear operator \( \phi^*(\cdot) : (L^2_0(0, T; \mathbb{R}^n), \cdots, L^2_0(0, T; \mathbb{R}^n)) \rightarrow (L^2_0(0, T; \Gamma), \cdots, L^2_0(0, T; \Gamma)) \) such that for any \( u(\cdot) \in (L^2_0(0, T; \Gamma), \cdots, L^2_0(0, T; \Gamma)) \) and \( x(\cdot) \in (L^2_0(0, T; \mathbb{R}^n), \cdots, L^2_0(0, T; \mathbb{R}^n)) \),
\[
\mathbb{E}\int_0^T \langle \phi(u)(t), x(t) \rangle dt = \mathbb{E}\int_0^T \langle u(t), \phi^*(x)(t) \rangle dt.
\]

Hence, we can rewrite the cost functional as follows:
\[
2\mathcal{J}_{soc}(u) = \mathbb{E}\int_0^T \left[ (\phi^*Q\phi + R)u, u + 2\phi^*Qs(y), u + \langle Qs(y), S(y) \rangle \right] dt
\]
\[
:= (M_2(u)(\cdot), u(\cdot)) + 2\langle M_1, u(\cdot) \rangle + M_0,
\]
where we have used \( \langle \cdot, \cdot \rangle \) as inner products in different Hilbert spaces. Note that, \( M_2(\cdot) \) is a bounded self-adjoint positive semi-definite linear operator.

### 6.2 Agent \( A_i \) perturbation

Let \( \bar{u} = (\bar{u}_1, \cdots, \bar{u}_N) \) be decentralized strategy given by
\[
\bar{u}_i(t) = \varphi_{\Theta_i}(p_i(t), q_i(t)) := P_1[R(t)^{-1}\mathbb{E}[B(t)^\top p_i(t) + D\Theta_i(t)^\top q_i(t)|G^c_i]], \quad i = 1, \cdots, N,
\]
where \( (p_i, q_i) \) is the solution of
\[
\begin{align*}
dx_i &= [A_{\Theta_i}x_i + B\varphi_{\Theta_i}(p_i, q_i) + FE\alpha]dt + [Cx_i + D_{\Theta_i}\varphi_{\Theta_i}(p_i, q_i) + F\mathbb{E}\alpha]dW_i(t), \\
p_i &= [-Qx_i + (QH + HQH)\mathbb{E}\alpha - A_{\Theta_i}p_i + F^{\top}E\hat{y}_1]dt + q_idW_i(t), \\
x_i(0) &= \xi, \quad p_i(T) = 0, \quad i = 1, \cdots, N.
\end{align*}
\]
Here, $(\alpha, \vec{y}_1, \vec{y}_2^2)$ is the solution of \cite{20}. Correspondingly, the realized decentralized states $(\bar{x}_1, \cdots, \bar{x}_N)$ satisfy

\[
\begin{align*}
\dot{\bar{x}}_i &= [A_{\Theta_i} \bar{x}_i + B \varphi_{\Theta_i}(p_i, q_i) + F^{x(N)}]dt + [C \bar{x}_i + D_{\Theta_i} \varphi_{\Theta_i}(p_i, q_i) + \tilde{F}^{x(N)}]dW_i(t), \\
\bar{x}_i(0) &= \xi,
\end{align*}
\]  

(38)

and $\bar{x}^{(N)} = \frac{1}{N} \sum_{i=1}^N \bar{x}_i$.

Let us consider the case that the agent $A_i$ (without loss of generality, assume $i > 1$) uses an alternative strategy $u_i \in U_i^{N, i}$ while the other agents $A_j, j \neq i$ use the strategy $\bar{u}_{-i}$. The realized state with the $i$-th agent’s perturbation is

\[
\begin{align*}
\dot{x}_i &= [A_{\Theta_i} \bar{x}_i + Bu_i + F^{x(N)}]dt + [C \bar{x}_i + D_{\Theta_i} u_i + \tilde{F}^{x(N)}]dW_i, \\
\dot{x}_j &= [A_{\Theta_j} \bar{x}_j + B \varphi_{\Theta_j}(p_j, q_j) + F^{x(N)}]dt + [C \bar{x}_j + D_{\Theta_j} \varphi_{\Theta_j}(p_j, q_j) + \tilde{F}^{x(N)}]dW_j, \\
\dot{\bar{x}}_i(0) &= \xi, \quad \dot{\bar{x}}_j(0) = \xi, \quad 1 \leq j \leq N, \quad j \neq i,
\end{align*}
\]

where $\bar{x}^{(N)} = \frac{1}{N} \sum_{i=1}^N x_i$. For $j = 1, \cdots, N$, denote the perturbation

\[
\begin{align*}
\delta u_i &= u_i - \bar{u}_i, \quad \delta x_j = \dot{x}_j - \bar{x}_j, \quad \delta J_j = J_j(u_i, \bar{u}_{-i}) - J_j(\bar{u}_i, \bar{u}_{-i}).
\end{align*}
\]

Introducing the following frozen states

\[
\begin{align*}
\dot{\bar{y}}_j &= [A_{\Theta_j} \bar{y}_j + B \varphi_{\Theta_j}(p_j, q_j) + F^{E_{\alpha}}]dt + [C \bar{y}_j + D_{\Theta_j} \varphi_{\Theta_j}(p_j, q_j) + \tilde{F}^{E_{\alpha}}]dW_j(t), \\
\bar{y}_j(0) &= \xi, \quad j = 1, \cdots, N,
\end{align*}
\]

(39)

and

\[
\begin{align*}
\dot{\bar{l}}_i &= [A_{\Theta_i} \bar{l}_i + Bu_i + F^{E_{\alpha}}]dt + [C \bar{l}_i + D_{\Theta_i} u_i + \tilde{F}^{E_{\alpha}}]dW_i, \\
\dot{\bar{l}}_j &= [A_{\Theta_j} \bar{l}_j + B \varphi_{\Theta_j}(p_j, q_j) + F^{E_{\alpha}}]dt + [C \bar{l}_j + D_{\Theta_j} \varphi_{\Theta_j}(p_j, q_j) + \tilde{F}^{E_{\alpha}}]dW_j, \\
\bar{l}_i(0) &= \xi, \quad \bar{l}_j(0) = \xi, \quad 1 \leq j \leq N, \quad j \neq i.
\end{align*}
\]

Similar to the computations in Section 3.1, we have

\[
\delta J^{(N)}_{\text{soc}} = \mathbb{E} \int_0^T [(Q \bar{x}_i, \delta x_i) - (\Xi, \delta x_i) + (R \bar{u}_i, \delta u_i)] dt + \sum_{i=1}^5 \epsilon_i,
\]

where

\[
\begin{align*}
\epsilon_1 &= \mathbb{E} \int_0^T \langle (Q H + HQ - HQ H) (E_{\alpha} - \bar{x}^{(N)}), N \delta x^{(N)} \rangle dt, \\
\epsilon_2 &= \mathbb{E} \int_0^T \langle (Q H + HQ - HQ H) E_{\alpha}, x^{**} - \delta x_{-i} \rangle dt, \\
\epsilon_3 &= \mathbb{E} \int_0^T \frac{1}{N} \sum_{j \neq i} (Q \bar{x}_j, N \delta x_j - \bar{x}_j) dt, \\
\epsilon_4 &= \mathbb{E} \int_0^T \langle f^T (E[y'_1]) - \frac{1}{N} \sum_{j \neq i} y'_1(j) + \tilde{F}^T (E[\beta^{(1)}]) - \frac{1}{N} \sum_{j \neq i} \beta^{(1)}(j), \delta x_i \rangle dt, \\
\epsilon_5 &= \mathbb{E} \int_0^T \langle f^T (E[y'_1]) - \frac{1}{N} \sum_{j \neq i} y'_1(j) + \tilde{F}^T (E[\beta^{(1)}]) - \frac{1}{N} \sum_{j \neq i} \beta^{(1)}(j), x^{**} \rangle dt.
\end{align*}
\]

Therefore, we have

\[
\delta J^{(N)}_{\text{soc}} = \mathbb{E} \int_0^T [(Q \bar{y}_i, \delta l_i) - (\Xi, \delta l_i) + (R \bar{u}_i, \delta u_i)] dt + \sum_{i=1}^7 \epsilon_i,
\]

where

\[
\begin{align*}
\epsilon_6 &= \mathbb{E} \int_0^T \langle (\dot{l}_i - \dot{x}_i, \Xi) + (\tilde{l}_i - \bar{x}_i, \Xi) \rangle dt, \\
\epsilon_7 &= \mathbb{E} \int_0^T [(Q(\bar{x}_i - \bar{l}_i), \delta x_i) + (Q \tilde{l}_i, \dot{x}_i - \dot{l}_i) + (Q \bar{l}_i, \bar{x}_i - \bar{l}_i)] dt.
\end{align*}
\]
First, we need some estimations. In the proofs, $L$ will denote a constant whose value may change from line to line. Applying the same technique as in Lemma 5.1, we have

**Lemma 6.1** There exist two constants $L_1$ and $L_2$ independent of $N$ such that

\[
    \mathbb{E} \sup_{0 \leq t \leq T} \left[ |\alpha|^2 + |\gamma|^2 + |\bar{\alpha}|^2 + |\bar{\gamma}|^2 \right] + \sum_{j=1}^{N} \mathbb{E} \sup_{0 \leq t \leq T} \left[ |x_j|^2 + |p_j|^2 \right] + \mathbb{E} \int_0^T \left[ |\varphi|^2 + |\bar{\varphi}|^2 \right] dt + \sum_{j=1}^{N} \mathbb{E} \int_0^T \left[ |q_j|^2 + |\varphi_{ij}(p_j, q_j)|^2 \right] dt \leq L_1,
\]

and

\[
    \sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}_j(t)|^2 + \sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{y}_j(t)|^2 \leq L_2.
\]

**Lemma 6.2** There exists a constant $L_3$ independent of $N$ such that

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x^{(N)}|^2 + \sup_{1 \leq j \leq N, j \neq i} \mathbb{E} \sup_{0 \leq t \leq T} |\delta x_j|^2 \leq \frac{L_3}{N^2}.
\]

**Proof** Recall the equations (6), (7) and (8), we have

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_i|^2 \leq L + L \mathbb{E} \int_0^t |\delta x_i|^2 ds + L \mathbb{E} \int_0^t |\delta x^{(N)}|^2 ds,
\]

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_j|^2 \leq LE \int_0^t |\delta x_j|^2 ds + L \mathbb{E} \int_0^t |\delta x^{(N)}|^2 ds,
\]

and

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_{-i}|^2 \leq L \mathbb{E} \int_0^t |\delta x_{-i}|^2 ds + L N^2 \mathbb{E} \int_0^t |\delta x^{(N)}|^2 ds.
\]

Note that

\[
    \delta x^{(N)} = \frac{1}{N} \delta x_i + \frac{1}{N} \delta x_{-i},
\]

we have

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_i|^2 \leq L + L \mathbb{E} \int_0^t |\delta x_i|^2 ds + \frac{L}{N^2} \mathbb{E} \int_0^t |\delta x_{-i}|^2 ds,
\]

and

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_{-i}|^2 \leq L \mathbb{E} \int_0^t |\delta x_{-i}|^2 ds + L \mathbb{E} \int_0^t |\delta x_i|^2 ds.
\]

Therefore, it follows from Gronwall inequality that

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_i|^2 + \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_{-i}|^2 \leq L.
\]

Thus,

\[
    \mathbb{E} \sup_{0 \leq s \leq t} |\delta x^{(N)}|^2 \leq \frac{L}{N^2}.
\]

From (30), by Gronwall inequality again, we have

\[
    \sup_{1 \leq j \leq N, j \neq i} \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_j|^2 \leq \frac{L_3}{N^2}.
\]

**Lemma 6.3** There exists a constant $L_4$ independent of $N$ such that

\[
    \sup_{0 \leq t \leq T} \mathbb{E} |\bar{x}^{(N)}(t) - \bar{x}_0|^2 \leq \frac{L_4}{N}.
\]
Proof First, for any $\theta \in S$, let
\[
\begin{align*}
\frac{d\bar{x}_{\theta,j}}{dt} &= [A_{\theta}\bar{x}_{\theta,j} + B_{\theta}(p_j, q_j) + \Phi_{\theta}(N)dt] + [C\bar{x}_{\theta,j} + D_{\theta}\varphi_{\theta}(p_j, q_j) + \Phi_{\theta}(N)dt]dW_j(t), \\
\bar{x}_{\theta,j}(0) &= \xi,
\end{align*}
\]
where $\bar{x}_{\theta,j}^N = \frac{1}{N} \sum_{j=1}^{N} x_{\theta,j}$ and $\alpha_{\theta}$ is the solution of (26) corresponding to $\Theta = \theta$. By Cauchy-Schwartz inequality and Burkholder-Davis-Gundy inequality, we have
\[
E \sup_{0 \leq s \leq t} |\bar{x}_{\theta,j}(s) - \bar{\theta}_{\theta,j}(s)|^2 \leq LE \int_{0}^{t} |\bar{x}_{\theta,j}(s) - \bar{\theta}_{\theta,j}(s)|^2 + |\bar{x}_{\theta,j}(s) - E\alpha_{\theta}(s)|^2 ds.
\]
By Gronwall inequality, we have
\[
E \sup_{0 \leq s \leq t} |\bar{x}_{\theta,j}(s) - \bar{\theta}_{\theta,j}(s)|^2 \leq LE \int_{0}^{t} E|x_{\theta,j}(s) - E\alpha_{\theta}(s)|^2 ds.
\] (41)

Next, recalling the state equations (38) and (39), similarly we have
\[
E \sup_{0 \leq s \leq t} |x_{\theta,j}(s) - \bar{\theta}_{\theta,j}(s)|^2 \leq LE \int_{0}^{t} E|x_{\theta,j}(s) - E\alpha_{\theta}(s)|^2 ds.
\] (42)

Note that for any $t \in [0, T]$,
\[
E|\bar{x}_{\theta,j}^N(t) - E\alpha(t)|^2 \\
\leq 2E \left[ \frac{1}{N} \sum_{j=1}^{N} x_{\theta,j}(t) - \frac{1}{N} \sum_{j=1}^{N} \bar{x}_{\theta,j}(t) d\Phi(\theta) \right]^2 \\
+ 2E \left[ \frac{1}{N} \sum_{j=1}^{N} \bar{x}_{\theta,j}(t) d\Phi(\theta) - \int_{S} E[\alpha(t)|\Theta = \theta] d\Phi(\theta) \right]^2 \\
\leq \frac{6}{N} \sum_{j=1}^{N} E|x_{\theta,j}(t) - \bar{\theta}_{\theta,j}(t)|^2 + \frac{6}{N^2} \sum_{j=1}^{N} E|\bar{\theta}_{\theta,j}(t)|^2 - \int_{S} E[\alpha(t)|\Theta = \theta] d\Phi(\theta) \\
+ 12 \sum_{1 \leq j \neq k \leq N} (E[\bar{\theta}_{\theta,j}(t) - \int_{S} \bar{\theta}_{\theta,j}(t) d\Phi(\theta), E[\bar{\theta}_{\theta,k}(t) - \int_{S} \bar{\theta}_{\theta,k}(t) d\Phi(\theta)]) \\
+ 6E \left[ \frac{1}{N} \sum_{j=1}^{N} \bar{\theta}_{\theta,j}(t) d\Phi(\theta) - \frac{1}{N} \sum_{j=1}^{N} \bar{x}_{\theta,j}(t) d\Phi(\theta) \right]^2 \\
+ 2 \int_{S} E \left[ \frac{1}{N} \sum_{j=1}^{N} x_{\theta,j}(t) d\Phi(\theta) - E[\alpha(t)|\Theta = \theta] \right]^2 d\Phi(\theta).
\] (43)

Similar to Lemma (6.1) there exists a constant $L$ such that
\[
\sup_{\theta \in S} \sup_{1 \leq j \leq N} \sup_{0 \leq t \leq T} E|x_{\theta,j}(t)|^2 \leq L.
\]

Consequently,
\[
\frac{6}{N^2} \sum_{j=1}^{N} E|\bar{\theta}_{\theta,j}(t)|^2 - \int_{S} E[\alpha(t)|\Theta = \theta] d\Phi(\theta) \leq \frac{L}{N}.
\] (44)

From $E\alpha = \int_{S} E\alpha_{\theta} d\Phi(\theta)$ and $E(A\Theta, \bar{\theta}_{\theta,j}) = \int_{S} E(A\Theta \bar{\theta}_{\theta,j}) d\Phi(\theta)$, we have
\[
E(\bar{\theta}_{\theta,j}(t) - \int_{S} \bar{\theta}_{\theta,j}(t) d\Phi(\theta)) = 0.
\] (45)
It is easy to see that
\[ E\frac{1}{N} \sum_{j=1}^{N} \int_{S} \tilde{I}_{\theta,j}(t) d\Phi(\theta) - \frac{1}{N} \sum_{j=1}^{N} \int_{S} \bar{x}_{\theta,j}(t) d\Phi(\theta) \]
\[ = E\frac{1}{N} \sum_{j=1}^{N} \int_{S} (\tilde{I}_{\theta,j}(t) - \bar{x}_{\theta,j}(t)) d\Phi(\theta) \]
\[ \leq \frac{1}{N} \sum_{j=1}^{N} \int_{S} E(\tilde{I}_{\theta,j}(t) - \bar{x}_{\theta,j}(t))^2 d\Phi(\theta). \]

Substituting (41), (42), (44), (45), and (46) into (43), we have

There exist constants

Lemma 6.4

It is easy to see that

and

Therefore, there exists a constant \( L \) independent of \( t \) such that

By Gronwall inequality, we have

\[ E|\bar{x}^{(N)}(t) - \bar{\alpha}(t)|^2 \leq \frac{L}{N} e^{Lt}. \]

\[ \mathbb{E} \left[ \frac{1}{N} \sum_{j=1}^{N} \bar{x}_{\theta,j}(t) - \mathbb{E}[\alpha(t) | \Theta = \theta] \right]^2 \leq \frac{L}{N}, \]

and

\[ \mathbb{E} \int_{0}^{t} |\bar{x}^{(N)}(s) - \mathbb{E}[\alpha(s)]|^2 ds \leq \frac{L}{N}. \]

Therefore, there exists a constant \( L \) independent of \( t \) such that

\[ E|\bar{x}^{(N)}(t) - \mathbb{E}[\alpha]^2 \leq \frac{L}{N} \int_{0}^{t} |\bar{x}^{(N)}(s) - \mathbb{E}[\alpha(s)]|^2 ds + \frac{L}{N}. \]

By Gronwall inequality, we have

\[ E|\bar{x}^{(N)}(t) - \mathbb{E}[\alpha]^2 \leq \frac{L}{N} e^{Lt}. \]

\[ \text{Lemma 6.4} \quad \text{There exist constants} \; L_5, L_6 \; \text{independent of} \; N \; \text{such that} \]

\[ \sup_{0 \leq t \leq T} \mathbb{E}|x^{**} - \delta x_{-i}|^2 \leq \frac{L_5}{N}, \]  

(47)

and for \( j \neq i \),

\[ \mathbb{E} \sup_{0 \leq t \leq T} |N \delta x_j - x^*_j|^2 \leq \frac{L_6}{N}. \]  

(48)

\[ \text{Proof} \quad \text{Introduce the following equations} \]

\[ \begin{cases} d\delta \hat{x}_i = [A_{\theta,i} \delta \hat{x}_i + B_{\theta} u_i + \frac{F}{N} \delta x_i + \frac{F}{N} x^{**} + C_{\delta \hat{x}_i} + D_{\theta,i} \delta u_i + \frac{\bar{F}}{N} \delta x_i + \frac{\bar{F}}{N} x^{**}] dW_i, \\ j \neq i, \quad d\delta \hat{x}_j = [A_{\theta,j} \delta \hat{x}_j + B_{\theta} x_j + \frac{F}{N} \delta x_j + \frac{F}{N} x^{**}] dt + [C_{\delta \hat{x}_j} + D_{\theta,j} \delta u_i + \frac{\bar{F}}{N} \delta x_j + \frac{\bar{F}}{N} x^{**}] dW_j, \\ \delta \hat{x}_i(0) = 0, \delta \hat{x}_j(0) = 0. \end{cases} \]

Recalling (47), by Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, and Gronwall inequality, we have

\[ \mathbb{E} \sup_{0 \leq s \leq t} |\delta x_j(s) - \delta \hat{x}_j(s)|^2 \leq \frac{L}{N^2} \mathbb{E} \int_{0}^{t} |\delta x_i(s) - x^{**}(s)|^2 ds. \]  

(49)
For any \( \theta \in S \), let
\[
\begin{align*}
\left\{
\begin{array}{l}
d\delta x_{t,i} &= [A_0 \delta x_{t,i} + B \delta u_i + F \delta x_0^{(N)}]dt + [C \delta x_{t,i} + D_0 \delta u_i + \overline{F} \delta x_0^{(N)}]dW_i, \delta x_{t,i}(0) = 0, \\
\quad \quad j \neq i, \\
d\delta x_{t,j} &= [A_0 \delta x_{t,j} + F \delta x_0^{(N)}]dt + [C \delta x_{t,j} + \overline{F} \delta x_0^{(N)}]dW_j, \delta x_{t,j}(0) = 0,
\end{array}
\right.
\end{align*}
\]
and
\[
\begin{align*}
\left\{
\begin{array}{l}
d\delta \bar{x}_{t,i} &= [A_0 \delta \bar{x}_{t,i} + B \delta u_i + \frac{F}{N} \delta x_{t,i} + \frac{F}{N} x_0^{**}]dt + [C \delta \bar{x}_{t,i} + D_0 \delta u_i + \frac{\overline{F}}{N} \delta x_{t,i} + \frac{\overline{F}}{N} x_0^{**}]dW_i, \\
\quad \quad j \neq i, \\
d\delta \bar{x}_{t,j} &= [A_0 \delta \bar{x}_{t,j} + \frac{F}{N} \delta x_{t,j} + \frac{F}{N} x_0^{**}]dt + [C \delta \bar{x}_{t,j} + \frac{\overline{F}}{N} \delta x_{t,j} + \frac{\overline{F}}{N} x_0^{**}]dW_j, \\
\delta \bar{x}_{t,i}(0) &= 0, \delta \bar{x}_{t,j}(0) = 0, \quad j \neq i,
\end{array}
\right.
\end{align*}
\]
where \( \delta x_0^{(N)} = \frac{1}{N} \sum_{j=1}^{N} \delta x_{t,j} \). Similarly,
\[
\begin{align*}
\mathbb{E} \sup_{0 \leq s \leq t} |\delta x_{t,j}(s) - \delta \bar{x}_{t,j}(s)|^2 &\leq \frac{L}{N^2} \mathbb{E} \int_0^t |\sum_{j \neq i} \delta x_{t,j}(s) - x_{t,j}^{**}(s)|^2 ds.
(50)
\end{align*}
\]
For any \( t \in [0, T] \),
\[
\begin{align*}
\mathbb{E} |x^{**}(t) - \delta x_{-i}(t)|^2 &\leq 6(N - 1) \sum_{j \neq i} \mathbb{E} |\delta x_{t,j} - \delta \bar{x}_{t,j}|^2 + 6 \sum_{j \neq i} \mathbb{E} |\delta \bar{x}_{t,j} - \int_S \delta \bar{x}_{t,j} d\Phi(\theta)|^2 \\
&+ 12 \sum_{1 \leq j \neq k \leq N, j, k \neq i} \mathbb{E} \langle \delta \bar{x}_{t,j} - \int_S \delta \bar{x}_{t,j} d\Phi(\theta), \delta \bar{x}_{t,k} - \int_S \delta \bar{x}_{t,k} d\Phi(\theta) \rangle \\
&+ 6(N - 1) \int_S \mathbb{E} |\int_S \delta \bar{x}_{t,j} - \delta x_{t,j}|^2 d\Phi(\theta) + 2 \int_S \mathbb{E} |\sum_{j \neq i} \delta x_{t,j} - x_{t,j}^{**}|^2 d\Phi(\theta).
(51)
\end{align*}
\]
Similar to Lemma [63], we have
\[
\begin{align*}
\mathbb{E} |x^{**}(t) - \delta x_{-i}(t)|^2 &\leq \mathbb{E} \int_0^t |\delta x_{t-i}(s) - x^{**}(s)|^2 ds + \frac{L}{N} + \mathbb{E} \int_0^t |\sum_{j \neq i} \delta x_{t,j}(s) - x_{t,j}^{**}(s)|^2 ds d\Phi(\theta) \\
&+ 2 \int_S \mathbb{E} |\sum_{j \neq i} \delta x_{t,j} - x_{t,j}^{**}|^2 d\Phi(\theta).
\end{align*}
\]
Applying similar technique as homogeneous case (e.g., pp. 29 in [61]), we have
\[
\mathbb{E} \sup_{0 \leq s \leq t} |\sum_{j \neq i} \delta x_{t,j}(s) - x_{t,j}^{**}(s)|^2 \leq \frac{L}{N}.
\]
Therefore, there exists a constant \( L \) independent of \( t \) such that
\[
\mathbb{E} |x^{**}(t) - \delta x_{-i}(t)|^2 \leq \mathbb{E} \int_0^t |\delta x_{t-i}(s) - x^{**}(s)|^2 ds + \frac{L}{N}.
\]
By Gronwall inequality, we have
\[
\mathbb{E} |x^{**}(t) - \delta x_{-i}(t)|^2 \leq \frac{L}{N} e^{Lt}.
\]
Hence (47) follows. Note that
\[
\begin{align*}
\left\{
\begin{array}{l}
d(x_j^* - N \delta x_j) &= [A_{\theta,j}(x_j^* - N \delta x_j) + F(x^{**} - \delta x_{-i})]dt \\
&+ [C(x_j^* - N \delta x_j) + \overline{F}(x^{**} - \delta x_{-i})]dW_j, \\
(x_j^* - \delta x_j)(0) &= 0.
\end{array}
\right.
\end{align*}
\]
By (47), we have (48).
Lemma 6.5 There exists a constant $L_7$ independent of $N$ such that

$$\sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{l}_j - \bar{x}_j|^2 \leq \frac{L_7}{N}.$$  \hspace{1cm} (52)

Proof Note that

$$d(\bar{l}_j - \bar{x}_j) = [A_{\Theta_j}(\bar{l}_j - \bar{x}_j) + F(\Theta - \bar{x}(N))]dt + [C(\bar{l}_j - \bar{x}_j) + \bar{F}(\Theta - \bar{x}(N))]dW_j(t),$$

where $\bar{l}_j(0) - \bar{x}_j(0) = 0$. 

By Cauchy-Schwartz inequality, Burkholder-Davis-Gundy inequality, Gronwall inequality and Lemma 6.3, we have (52).

6.3 Asymptotic optimality

In order to prove asymptotic optimality, it suffices to consider the perturbations $u_i \in \mathcal{U}^c$ such that $J_{soc}^{(N)}(u_1, \ldots, u_N) \leq J_{soc}^{(N)}(\bar{u}_1, \ldots, \bar{u}_N)$. It is easy to check that

$$J_{soc}^{(N)}(\bar{u}_1, \ldots, \bar{u}_N) \leq LN,$$

where $L$ is a constant independent of $N$. Therefore, in the following we only consider the perturbations $u_i \in \mathcal{U}^c$ satisfying

$$\sum_{i=1}^N \mathbb{E} \int_0^T |u_i|^2dt \leq LN.$$

Therefore, similar to Lemma 6.3 and Lemma 6.5, we have

Lemma 6.6 There exist constants $L_8$ and $L_9$ independent of $N$ such that

$$\mathbb{E} \sup_{0 \leq t \leq T} |\bar{x}^{(N)}(t) - \Theta|^2 \leq \frac{L_8}{N},$$

and

$$\sup_{1 \leq j \leq N} \mathbb{E} \sup_{0 \leq t \leq T} |\bar{l}_j - \bar{x}_j|^2 \leq \frac{L_9}{N}.$$  

Let $\delta u_i = u_i - \bar{u}_i$, and consider a perturbation $u = \bar{u} + (\delta u_1, \ldots, \delta u_N) := \bar{u} + \delta u$. Then by Section 6.1, we have

$$2J_{soc}^{(N)}(\bar{u} + \delta u) = (M_2(\bar{u} + \delta u), \bar{u} + \delta u) + 2(M_1, \bar{u} + \delta u) + M_0$$

$$= 2J_{soc}^{(N)}(\bar{u}) + 2 \sum_{i=1}^{N} (M_2(\bar{u}) + M_1, \delta u_i) + (M_2(\delta u), \delta u),$$

where $M_2(\bar{u}) + M_1$ is the Fréchet differential of $J_{soc}^{(N)}$ on $\bar{u}$.

Theorem 6.1 Under the assumptions (A1)-(A5), $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_N)$ defined in (37) is a $(\sqrt{1/N})$-social optimal strategy for the agents.

Proof From Section 6.2, we have

$$\langle M_2(\bar{u}) + M_1, \delta u_i \rangle = \mathbb{E} \int_0^T \left[ \langle Q\bar{l}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R\bar{u}_i, \delta u_i \rangle \right]dt + \sum_{i=1}^{7} \varepsilon_i.$$ 

From the optimality of $\bar{u}$, we have

$$\mathbb{E} \int_0^T \left[ \langle Q\bar{l}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R\bar{u}_i, \delta u_i \rangle \right]dt \geq 0.$$ 

Suppose this is not true, then for $u_i$ such that $\bar{u}_i + u_i \in \mathcal{U}_i^{d,p}$, we have

$$\bar{u}_i + \rho u_i \in \mathcal{U}_i^{d,p}, \quad 0 < \rho < 1,$$
and
\[
\lim_{\rho \to 0} \frac{J_i(\overline{u}_i + \rho u_i, \overline{u}_{i-1}) - J_i(\overline{u}_i, \overline{u}_{i-1})}{\rho} < 0.
\]
Therefore,
\[
J_i(\overline{u}_i + \rho u_i, \overline{u}_{i-1}) < J_i(\overline{u}_i, \overline{u}_{i-1})
\]
for sufficiently small \( \rho \), which is a contradiction with the optimality of \( \overline{u}_i \). Moreover, combing Lemmas 6.3-6.6 with iteration analysis (e.g., [23]), we have
\[
\sum_{i=1}^{7} \varepsilon_i = O\left(\frac{1}{\sqrt{N}}\right).
\]
Therefore,
\[
\mathcal{J}_{soc}^{(N)}(\overline{u} + \delta u) = \mathcal{J}_{soc}^{(N)}(\overline{u}) + \sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ \langle Q \overline{l}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R \overline{u}_i, \delta u_i \rangle \right] dt + \sum_{i=1}^{N} \sum_{l=1}^{5} \varepsilon_l + \frac{1}{2} \langle M_2(\delta u), \delta u \rangle.
\]
Note that
\[
\sum_{i=1}^{N} \mathbb{E} \int_{0}^{T} \left[ \langle Q \overline{l}_i, \delta l_i \rangle - \langle \Xi, \delta l_i \rangle + \langle R \overline{u}_i, \delta u_i \rangle \right] dt + \frac{1}{2} \langle M_2(\delta u), \delta u \rangle \geq 0,
\]
and
\[
\sum_{i=1}^{N} \sum_{l=1}^{7} \varepsilon_l = O(\sqrt{N}),
\]
there exists a constant \( L \) independent of \( N \) such that
\[
\frac{1}{N} \left( \mathcal{J}_{soc}^{(N)}(\overline{u}) - \inf_{u \in U_t} \mathcal{J}_{soc}^{(N)}(u) \right) \leq \frac{L}{\sqrt{N}}.
\]

First, for any given \((Y, Z) \in L^2(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{R}^m)\) and \(0 \leq t \leq T\), the following SDE has a unique solution:
\[
X(t) = x + \int_{0}^{t} b(s, X, \mathbb{E}[X], Y, \mathcal{E}_t[Y], Z, \mathcal{E}_t[Z]) ds + \int_{0}^{t} \sigma(s, X, \mathbb{E}[X], Y, \mathcal{E}_t[Y], Z, \mathcal{E}_t[Z]) dW(s). \tag{53}
\]
Therefore, we can introduce a map \( \mathcal{M}_1 : L^2(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{R}^m) \to L^2(0, T; \mathbb{R}^n) \). Moreover, by the standard estimations of SDE, we have the following result:

**Lemma 7** Let \( X_i \) be the solution of \((53)\) corresponding to \((Y_i, Z_i) \), \( i = 1, 2 \) respectively. Then for all \( \rho \in \mathbb{R} \) and some constants \( l_1, l_2, l_3, l_4 > 0 \), we have
\[
\mathbb{E} e^{-\rho t} |\dot{X}(t)|^2 + \tilde{\rho}_1 \mathbb{E} \int_{0}^{t} e^{-\rho s} |\dot{X}(s)|^2 ds \leq (k_2 l_1 + k_3 l_2 + k_4 l_3 + k_5 l_4 + k_6 l_5) \mathbb{E} \int_{0}^{t} e^{-\rho s} |\dot{Y}(s)|^2 ds + (k_7 l_3 + k_8 l_4 + k_9 l_5 + k_{10} l_6 + k_{11} l_7) \mathbb{E} \int_{0}^{t} e^{-\rho s} |\dot{Z}(s)|^2 ds,
\]
and
\[
\mathbb{E} e^{-\rho t} |\dot{X}(t)|^2 \leq (k_2 l_1 + k_3 l_2 + k_4 l_3 + k_5 l_4 + k_6 l_5) \mathbb{E} \int_{0}^{t} e^{-\tilde{\rho}_1(s-t) - \rho s} |\dot{Y}(s)|^2 ds + (k_7 l_3 + k_8 l_4 + k_9 l_5 + k_{10} l_6 + k_{11} l_7) \mathbb{E} \int_{0}^{t} e^{-\tilde{\rho}_1(s-t) - \rho s} |\dot{Z}(s)|^2 ds,
\]
where \( \tilde{\rho}_1 = \rho - 2\rho_1 - 2k_1 - k_2l_1^{-1} - k_3l_2^{-1} - k_4l_3^{-1} - k_5l_4^{-1} - k_6l_5^{-1} = 0 \) and \( \Phi := \Phi_1 - \Phi_2, \Phi = X, Y, Z. \) Moreover,

\[
E \int_0^T e^{-\rho t} |\hat{X}(t)|^2 dt \leq (k_2l_1 + k_3l_2 + k_4^2 + k_5^2) \frac{1 - e^{-\tilde{\rho} T}}{\tilde{\rho}_1} E \int_0^T e^{-\rho s} |\hat{Y}(s)|^2 ds + (k_4l_3 + k_5l_4 + k_7^2 + k_8^2) E \int_0^T e^{-\rho s} |\hat{Z}(s)|^2 ds, 
\]

and

\[
e^{-\rho T} E|\hat{X}(T)|^2 \leq (1 \vee e^{-\tilde{\rho} T}) \{ (k_2l_1 + k_3l_2 + k_4^2 + k_5^2) E \int_0^T e^{-\rho s} |\hat{Y}(t)|^2 dt + (k_4l_3 + k_5l_4 + k_7^2 + k_8^2) E \int_0^T e^{-\rho s} |\hat{Z}(t)|^2 dt \}.
\]

Specially, if \( \tilde{\rho}_1 > 0, \)

\[
e^{-\rho T} E|\hat{X}(T)|^2 \leq (k_2l_1 + k_3l_2 + k_4^2 + k_5^2) E \int_0^T e^{-\rho s} |\hat{Y}(t)|^2 dt + (k_4l_3 + k_5l_4 + k_7^2 + k_8^2) E \int_0^T e^{-\rho s} |\hat{Z}(t)|^2 dt.
\]

Next, for any given \( X \in L^2_{\Phi}(0, T; \mathbb{R}^n), \) consider the following BSDE:

\[
Y(t) = \int_t^T f(s, X, E[X], Y, E[Y], \bar{E}[Y], Z, E[Z]) ds - \int_t^T Z(s) dW(s). \tag{54}
\]

**Proposition 1** \( \square \) admits a unique solution \( (Y, Z) \in L^2_{\Phi}(0, T; \mathbb{R}^m) \times L^2_{\Phi}(0, T; \mathbb{R}^m). \)

**Proof** For any fixed \( (y, z) \in L^2_{\Phi}(0, T; \mathbb{R}^m) \times L^2_{\Phi}(0, T; \mathbb{R}^m), \)

\[
Y(t) = \int_t^T f(s, X, E[X], Y, E[y], \bar{E}[y], z, E[z]) ds - \int_t^T Z(s) dW(s)
\]

admits a unique solution \( (Y, Z) \in L^2_{\Phi}(0, T; \mathbb{R}^m) \times L^2_{\Phi}(0, T; \mathbb{R}^m). \) Hence we can introduce the mapping \( \mathcal{N} : (y, z) \mapsto (Y, Z). \) For any \( (y, z), (y', z') \in L^2_{\Phi}(0, T; \mathbb{R}^m) \times L^2_{\Phi}(0, T; \mathbb{R}^m), \) denote \( (Y, Z) = \mathcal{N}(y, z) \) and \( (Y', Z') = \mathcal{N}(y', z'). \) Let \( \hat{(y', \hat{z}, \hat{Y}, \hat{Z})} = (y - y', z - z', Y - Y', Z - Z'). \) Applying Itô’s formula to \( e^{\delta s} |\hat{Y}(s)|^2, \) we have

\[
e^{\delta t} |\hat{Y}(t)|^2 + \int_t^T e^{\delta s} |\hat{Z}(s)|^2 ds + \int_t^T \delta e^{\delta s} |\hat{Y}(s)|^2 ds \\
\leq \int_t^T e^{\delta s} (2\rho_2 + 4k_2^2 + 4k_3^2 + 4k_4^2 + 4k_5^2 + 4k_6^2 + 4k_7^2 + 4k_8^2 + 4k_9^2) |\hat{Y}(s)|^2 ds \\
+ \frac{1}{2} \int_t^T e^{\delta s} (E[|\hat{y}|^2] + E[|\hat{y}'|^2] + |\hat{z}|^2) + E[|\hat{z}'|^2]) ds + 2 \int_t^T e^{\delta s} (\hat{Y}(s), \hat{Z}(s)) dW(s).
\]

Note that \( E[|\hat{y}|^2] = E[|\hat{y}'|^2], \) letting \( \delta = 2\rho_2 + 4k_2^2 + 4k_3^2 + 4k_4^2 + 4k_5^2 + 4k_6^2 + 4k_7^2 + 4k_8^2 + 4k_9^2 \) and taking expectation, we have

\[
E \int_t^T e^{\delta s} (|\hat{Y}(s)|^2 + |\hat{Z}(s)|^2) ds \leq \frac{1}{2} E \int_t^T e^{\delta s} (|\hat{y}(s)|^2 + |\hat{z}(s)|^2) ds,
\]

i.e., \( \mathcal{N} \) is a contraction mapping. Hence \( \square \) admits a unique solution \( (Y, Z) \in L^2_{\Phi}(0, T; \mathbb{R}^m) \times L^2_{\Phi}(0, T; \mathbb{R}^m). \)

Thus, we can introduce another map \( \mathcal{M}_2 : L^2_{\Phi}(0, T; \mathbb{R}^n) \to L^2_{\Phi}(0, T; \mathbb{R}^m) \times L^2_{\Phi}(0, T; \mathbb{R}^m). \) By the standard estimation of BSDE, we have the following result:
Lemma 8 Let \((Y_i, Z_i)\) be the solution of (5.4) corresponding to \(X_i, i = 1, 2, \) respectively. Then for all \(\rho \in \mathbb{R}\) and some constants \(l_5, l_6, l_7, l_8 > 0,\) we have

\[
\begin{align*}
\mathbb{E}e^{-\rho t}|\dot{Y}(t)|^2 + \check{\rho}_2 \mathbb{E} \int_t^T e^{-\rho s}|\dot{Y}(s)|^2 ds + (1 - k_{10}l_7 - k_{11}l_8)\mathbb{E} \int_t^T e^{-\rho s}|\dot{Z}(s)|^2 ds \\
\leq (k_6l_5 + k_7l_6)\mathbb{E} \int_t^T e^{-\rho s}|\dot{X}(s)|^2 ds,
\end{align*}
\]

and

\[
\begin{align*}
\mathbb{E}e^{-\rho t}|\dot{Y}(t)|^2 + (1 - k_7l_5 - k_8l_6)\mathbb{E} \int_t^T e^{-\rho s}|\dot{Z}(s)|^2 ds \\
\leq (k_4l_3 + k_5l_4)\mathbb{E} \int_t^T e^{-\check{\rho}_2(s-t)-\rho s}|\dot{X}(s)|^2 ds,
\end{align*}
\]

where \(\check{\rho}_2 = -\rho - 2\rho_2 - 2k_8 - 2k_9 - k_6l_5^{^{-1}} - k_7l_6^{^{-1}} - k_{10}l_7^{^{-1}} - k_{11}l_8^{^{-1}},\) and \(\check{\Phi} := \Phi - \Phi_2, \Phi = X, Y, Z.\) Moreover,

\[
\mathbb{E} \int_0^T e^{-\rho t}|\dot{Y}(t)|^2 dt \leq \frac{1 - e^{-\check{\rho}_2 T}}{\check{\rho}_2} (k_6l_5 + k_7l_6)\mathbb{E} \int_0^T e^{-\rho s}|\dot{X}(s)|^2 ds,
\]

and

\[
\mathbb{E} \int_0^T e^{-\rho t}|\dot{Z}(t)|^2 dt \leq \frac{(k_6l_5 + k_7l_6)(1 \vee e^{-\check{\rho}_2 T})}{(1 - k_{10}l_7 - k_{11}l_8)(1 \vee e^{-\check{\rho}_2 T})} \mathbb{E} \int_0^T e^{-\rho s}|\dot{X}(s)|^2 ds.
\]

Specially, if \(\check{\rho}_2 > 0,\)

\[
\mathbb{E} \int_0^T e^{-\rho t}|\dot{Z}(t)|^2 dt \leq \frac{k_6l_5 + k_7l_6}{1 - k_{10}l_7 - k_{11}l_8} \mathbb{E} \int_0^T e^{-\rho s}|\dot{X}(s)|^2 ds.
\]

Proof of Theorem 5.1. Define \(\mathcal{M} := \mathcal{M}_2 \circ \mathcal{M}_1,\) where \(\mathcal{M}_1\) is defined by (5.3) and \(\mathcal{M}_2\) is defined by (7.4). Thus \(\mathcal{M}\) is a mapping from \(L^2(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{R}^m)\) into itself. For \((U_1, V_1) \in L^2(0, T; \mathbb{R}^m) \times L^2(0, T; \mathbb{R}^m),\) let \(X_1 := \mathcal{M}_1(U_1, V_1)\) and \((Y_1, Z_1) := \mathcal{M}(U_1, V_1).\) Therefore,

\[
\begin{align*}
\mathbb{E} \int_0^T e^{-\check{\rho}_2} & |Y_1(t) - Y_2(t)|^2 dt + \mathbb{E} \int_0^T e^{-\check{\rho}_2} |Z_1(t) - Z_2(t)|^2 dt \\
\leq & \left[ \frac{1}{\check{\rho}_2} \frac{1}{1 \vee e^{-\check{\rho}_2 T}} \right] \left[ (k_6l_5 + k_7l_6) \frac{1 - e^{-\check{\rho}_2 T}}{\check{\rho}_2} \right] \\
& \left\{ (k_2l_1 + k_3l_2 + k_4l_3 + k_5l_4) \mathbb{E} \int_0^T e^{-\check{\rho}_2} |U_1(t) - U_2(t)|^2 dt \\
& + (k_4l_3 + k_5l_4 + k_7l_5 + k_8l_6) \mathbb{E} \int_0^T e^{-\rho s}|\dot{X}(s)|^2 ds \right\}.
\end{align*}
\]

Choosing suitable \(\rho,\) we get that \(\mathcal{M}\) is a contraction mapping.

Furthermore, if \(2\rho_1 + 2\rho_2 < -2k_1 - 2k_8 - 2k_9 - k_{10} - k_{11} - k_{12} - k_{13},\) we can choose \(\rho \in \mathbb{R}, 0 < k_{10}l_7 < \frac{1}{2}\) and \(0 < k_{11}l_8 < \frac{1}{2}\) and sufficient large \(l_1, l_2, l_3, l_4, l_5, l_6\) such that

\[
\check{\rho}_1 > 0, \quad \check{\rho}_2 > 0, \quad 1 - k_{10}l_7 - k_{11}l_8 > 0.\]

Therefore,

\[
\begin{align*}
\mathbb{E} \int_0^T e^{-\rho t}|Y_1(t) - Y_2(t)|^2 dt & + \mathbb{E} \int_0^T e^{-\rho t}|Z_1(t) - Z_2(t)|^2 dt \\
\leq & \left[ \frac{1}{\check{\rho}_2} + \frac{1}{1 \vee e^{-\check{\rho}_2 T}} \right] \frac{1}{\check{\rho}_1} (k_6l_5 + k_7l_6) \\
& \left\{ (k_2l_1 + k_3l_2 + k_4l_3 + k_5l_4) \mathbb{E} \int_0^T e^{-\rho s}|\dot{X}(s)|^2 ds \\
& + (k_4l_3 + k_5l_4 + k_7l_5 + k_8l_6) \mathbb{E} \int_0^T e^{-\rho s}|\dot{X}(s)|^2 ds \right\}.
\end{align*}
\]

Thus, the proof is complete. \(\square\)
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