Generalizing the autonomous Kepler–Ermakov system in a Riemannian space

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Abstract

We generalize the two-dimensional autonomous Hamiltonian–Kepler–Ermakov dynamical system to three dimensions using the $sl(2,R)$ invariance of Noether symmetries and determine all three-dimensional autonomous Hamiltonian–Kepler–Ermakov dynamical systems which are Liouville integrable via Noether symmetries. Subsequently, we generalize the autonomous Kepler–Ermakov system in a Riemannian space which admits a gradient homothetic vector by the requirements (a) that it admits a first integral (the Riemannian Ermakov invariant) and (b) it has $sl(2,R)$ invariance. We consider both the non-Hamiltonian and the Hamiltonian systems. In each case, we compute the Riemannian–Ermakov invariant and the equations defining the dynamical system. We apply the results in general relativity and determine the autonomous Hamiltonian–Riemannian–Kepler–Ermakov system in the spatially flat Friedman–Robertson–Walker spacetime. We consider a locally rotational symmetric spacetime of class A and discuss two cosmological models. The first cosmological model consists of a scalar field with an exponential potential and a perfect fluid with a stiff equation of state. The second cosmological model is the $f(R)$-modified gravity model of $\Lambda_0$CDM. It is shown that in both applications the gravitational field equations reduce to those of the generalized autonomous Riemannian–Kepler–Ermakov dynamical system which is Liouville integrable via Noether integrals.

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1. Introduction

The Ermakov system has its roots in the study of the one-dimensional time-dependent harmonic oscillator

$$\ddot{x} + \omega^2(t)x = 0.$$ (1)
Ermakov [1] obtained the first integral $J$ of this equation by introducing the auxiliary equation
\[ \ddot{\rho} + \omega^2(t)\rho = \rho^{-3}, \tag{2} \]
eliminating the $\omega^2(t)$ term and multiplying with the integrating factor $\rho \dot{x} - \dot{\rho} x$:
\[ J = \frac{1}{2}[(\rho \dot{x} - \dot{\rho} x)^2 + (x/\rho)^2]. \tag{3} \]
The Ermakov system was rediscovered nearly a century after its introduction [2] and subsequently was generalized beyond the harmonic oscillator to a two-dimensional dynamical system which admits the first integral [3]. In a series of papers the Lie, Noether and dynamical symmetries of this generalized system have been studied. A short review of these studies and a detailed list of relevant references can be found in [4]. Earlier reviews of the Ermakov system and its numerous applications in divertive areas of physics can be found in [5, 6].

The general Ermakov system does not admit Lie point symmetries. The form of the most general Ermakov system which admits Lie point symmetries has been determined in [7] and it is called the Kepler–Ermakov system [8, 9]. It is well known that these Lie symmetries are a representation of the $sl(2, \mathbb{R})$ algebra.

In an attempt to generalize the Kepler–Ermakov system to higher dimensions, Leach [9] used a transformation to remove the time-dependent frequency term and then demanded that the autonomous ‘generalized’ Kepler–Ermakov system will posses two properties: (a) the first integral, the Ermakov invariant, and (b) $sl(2, R)$ invariance to Lie symmetries. It has been shown that the invariance group of the Ermakov invariant is richer than $sl(2, R)$ [10]. The purpose of this work is to use Leach’s proposal and generalize the autonomous Kepler–Ermakov system in two directions: (a) to higher dimensions using the $sl(2, R)$ invariance with respect to Noether symmetries (provided the system is Hamiltonian) and (b) in a Riemannian space which admits a gradient homothetic vector (HV).

The generalization of the autonomous Kepler–Ermakov system to three dimensions using Lie symmetries has been done in [9]. In the following we use the results of [11] to generalize the subset of autonomous Hamiltonian–Kepler–Ermakov systems to three dimensions via Noether symmetries. We show that there is a family of three-dimensional autonomous Hamiltonian Kepler–Ermakov systems parametrized by an arbitrary function $f$ which admits the elements of $sl(2, R)$ as Noether symmetries. Each member of this family admits two first integrals, the Hamiltonian and the Ermakov invariant.

We use this result in order to determine all three-dimensional Hamiltonian-Kepler–Ermakov systems which are Liouville integrable via Noether symmetries. To do this we need to determine all members of the family, that is, those functions, $f$, for which the corresponding system admits an additional Noether symmetry.

The results of [11] indicate that there are two cases to be considered, i.e. Noether point symmetries resulting from linear combinations of (a) translations and (b) rotations (elements of the $so(3)$ algebra). In each case we determine the functions $f$ and the required extra time independent first integral.

The above scenario can be generalized to an $n$-dimensional Euclidian space as Leach indicates in [9], however, at the cost of major complexity and number of cases to be considered. Indeed, as can be seen by the results of [11], the situation is complex enough even for the three-dimensional case.

We continue with the generalization of the Kepler–Ermakov system in a different and more drastic direction. We note that the Ermakov systems considered so far are based on the Euclidian space; therefore, we may call them Euclidian–Ermakov systems. Furthermore, the $sl(2, R)$ symmetry algebra of the autonomous Kepler–Ermakov system is generated by the trivial symmetry $\partial_t$ and the gradient $\text{HV}$ of the Euclidian two-dimensional space $E^2$. Using this observation, we generalize the autonomous Kepler–Ermakov system (not necessarily
Hamiltonian) in an \( n \)-dimensional Riemannian space which admits a gradient HV using either Lie or Noether symmetries. We call the Riemannian–Kepler–Ermakov system the new dynamical system. This generalization makes the application of the autonomous Kepler–Ermakov system in general relativity and, in particular, in cosmology possible.

Concerning general relativity we determine the four-dimensional autonomous Riemannian–Kepler–Ermakov system and the associated Riemannian–Ermakov invariant in the spatially flat Freedman–Robertson–Walker (FRW) spacetime; we use the results of [11, 12] to calculate the extra Noether point symmetries. The applications to cosmology concern two models for dark energy on a locally rotational symmetric (LRS) spacetime. The first model involves a scalar field with an exponential potential minimally interacting with a perfect fluid with a stiff equation of state. The second cosmological model is the \( f(R) \)-modified gravity model of \( \Lambda \)CDM. It is shown that in both models the gravitational field equations define an autonomous Riemannian–Kepler–Ermakov system which is integrable via Noether integrals.

The structure of the paper is as follows. In section 2, we review the main features of the two-dimensional autonomous Euclidian–Kepler–Ermakov system. In section 3, we discuss the general scheme of generalization of the two-dimensional autonomous Euclidian–Kepler–Ermakov system to higher dimensions and to a Riemannian space which admits a gradient HV. In section 4, we consider the generalization to the three-dimensional autonomous Euclidian–Hamiltonian–Kepler–Ermakov system by Noether symmetries and determine all such systems which are Liouville integrable. In section 5, we define the autonomous Riemannian–Kepler–Ermakov system by the requirements that it will admit (a) the first integral (the Ermakov invariant) and (b) posses \( sl(2, R) \) invariance. In section 5.1, we consider the non-conservative autonomous Riemannian–Kepler–Ermakov system and derive the Riemannian–Ermakov invariant, and in section 5.2 we repeat the same for the autonomous Hamiltonian–Riemannian–Kepler–Ermakov system. In the remaining sections, we discuss the applications of the autonomous Hamiltonian–Riemannian–Kepler–Ermakov system in general relativity and in cosmology. Finally, in section 8 we draw our conclusions.

2. The two-dimensional autonomous Kepler–Ermakov system

In [7], Hass and Goedert considered the most general two-dimensional Newtonian–Ermakov system to be defined by the equations

\[
\ddot{x} + \alpha^2(t, x, y, \dot{x}, \dot{y})x = \frac{1}{yx^2} f \left( \frac{y}{x} \right),
\]

\[
\ddot{y} + \alpha^2(t, x, y, \dot{x}, \dot{y})y = \frac{1}{xy^2} g \left( \frac{y}{x} \right).
\]

This system admits the Ermakov first integral

\[
I = \frac{1}{2} (\dot{x} \dot{y} - y \dot{y})^2 + \int_{y/\chi}^{y/\chi} f(\tau) \, d\tau + \int_{y/\chi}^{y/\chi} g(\tau) \, d\tau.
\]

If one considers the transformation:

\[
\Omega^2 = \alpha^2 - \frac{1}{xy^2} g \left( \frac{y}{x} \right)
\]

\[
F \left( \frac{y}{x} \right) = f \left( \frac{y}{x} \right) - \frac{x^2}{y^2} g \left( \frac{y}{x} \right),
\]

then equations (4) and (5) take the form

\[
\ddot{x} + \Omega^2(x, y, \dot{x}, \dot{y})x = \frac{1}{x^2 y} F \left( \frac{y}{x} \right)
\]
\[ \ddot{y} + \Omega^2(x, y, \dot{x}, \dot{y}) = 0. \]  

(8)

Due to the second equation, except for special cases, the new function \( \Omega \) is independent of \( t \) and depends only on the dynamical variables \( x, y \) and possibly on their derivative. The Ermakov first integral in the new variables is

\[ I = \frac{1}{2} (xy - y\dot{x})^2 + \int y/\dot{x} F(\lambda) \, d\lambda. \]  

(9)

The system of equations (7) and (8) defines the most general two-dimensional Ermakov system and produces all its known forms for special choices of the function \( \Omega \). For example, the weak Kepler–Ermakov system [9], defined by the equations [8]

\[ \ddot{x} + \omega^2(t)x + \frac{x}{r^3}H(x, y) - \frac{1}{x^3}f \left( \frac{y}{x} \right) = 0 \]  

(10)

\[ \ddot{y} + \omega^2(t)y + \frac{y}{r^3}H(x, y) - \frac{1}{x^3}g \left( \frac{y}{x} \right) = 0 \]  

(11)

where \( H, f \) and \( g \) are arbitrary functions of their argument, is obtained by the function

\[ \Omega^2(x, y) = \omega^2(t) + H(x, y)/r^3 \]  

(12)

with the Ermakov first integral

\[ I = \frac{1}{2} (xy - y\dot{x})^2 + \int \frac{1}{\rho^3} \{ \lambda f(\lambda) - \lambda^{-3}g(\lambda) \} \, d\lambda. \]  

(13)

The weak Kepler–Ermakov system does not admit Lie point symmetries, however the property of having a first integral prevails. The system of equations (10) and (11) admits the \( sl(2, \mathbb{R}) \) algebra of Lie point symmetries [13] only for \( H(x, y) = -\mu^2 r^3 + \frac{g(\lambda)}{r^2} \), where \( \mu \) is either real or pure imaginary number. This is the Kepler–Ermakov system defined by the equations

\[ \ddot{x} + (\omega^2(t) - \mu^2)x + \frac{1}{r^3}h \left( \frac{y}{x} \right) - \frac{1}{x^3}f \left( \frac{y}{x} \right) = 0 \]  

(14)

\[ \ddot{y} + (\omega^2(t) - \mu^2)y + \frac{1}{r^3}h \left( \frac{y}{x} \right) - \frac{1}{x^3}g \left( \frac{y}{x} \right) = 0. \]  

(15)

It is well known (see [13]) that the oscillator term \( \omega^2(t) - \mu^2 \) in (14) and (15) is removed if one considers new variables \( T, X \) and \( Y \) defined by the relations

\[ T = \int \rho^{-2} \, dt, \quad X = \rho^{-1}x, \quad Y = \rho^{-1}y, \]  

(16)

where \( \rho \) is any smooth solution of the time-dependent oscillator equation

\[ \dot{\rho} + (\omega^2(t) - \mu^2)\rho = 0. \]  

(17)

In [13], it is commented that ‘the effect of \( \mu^2 \) (\( C \) in the notation of [13]) is to shift the time-dependent frequency function’. However this is true as long as \( \omega(t) \neq 0 \). When \( \omega(t) = 0 \), one has the autonomous Kepler–Ermakov system whose Lie symmetries span the \( sl(2, \mathbb{R}) \) algebra with different representations for \( \mu = 0 \) and \( \mu \neq 0 \).

Before we justify the need for the consideration of the two cases \( \mu = 0 \) and \( \mu \neq 0 \), we note that by applying the transformation

\[ s = \int v^{-2} \, dT, \quad \dot{x} = v^{-1}X, \quad \dot{y} = v^{-1}Y, \]  

(18)

where \( v \) satisfies the Ermakov–Pinney equation

\[ \frac{d^2 v}{dT^2} + \frac{\mu^2}{v^3} = 0 \]  

(19)
to the transformed equations
\[ \frac{d^2X}{d\tau^2} + \frac{1}{R^3} h \left( \frac{Y}{X} \right) - \frac{1}{X^3} f \left( \frac{Y}{X} \right) = 0 \] (20)
\[ \frac{d^2Y}{d\tau^2} + \frac{1}{R^3} h \left( \frac{Y}{X} \right) - \frac{1}{X^3} g \left( \frac{Y}{X} \right) = 0 \] (21)
we retain the term \( \mu^2 \) and obtain the autonomous Kepler–Ermakov system of [14]
\[ \ddot{x} - \mu^2 x + \frac{1}{r^3} h \left( \frac{y}{x} \right) - \frac{1}{x^3} f \left( \frac{y}{x} \right) = 0 \] (22)
\[ \ddot{y} - \mu^2 y + \frac{1}{r^3} h \left( \frac{y}{x} \right) - \frac{1}{y^3} g \left( \frac{y}{x} \right) = 0 \] (23)
The above transformations show that the consideration of the autonomous Kepler–Ermakov system is not a real restriction.

We discuss now the need for the consideration of the cases \( \mu = 0 \) and \( \mu \neq 0 \). In [12], we have determined the Lie symmetries of the autonomous two-dimensional Kepler–Ermakov system and have found two cases. The first, case 1, concerns the autonomous Kepler–Ermakov system with \( \mu = 0 \) and has the Lie symmetry vectors
\[ X = (\tilde{c}_1 + \tilde{c}_2 2 \tau + \tilde{c}_3 \tau^2) \delta_\tau + (\tilde{c}_2 + \tilde{c}_3 \tau) r \delta_r \quad (\mu = 0). \] (24)
The second, case 2, concerns the same system with \( \mu \neq 0 \) and has the Lie symmetry vectors
\[ X = \left( c_1 + c_2 \frac{1}{\mu} e^{2 \mu t} - c_3 \frac{1}{\mu} e^{-2 \mu t} \right) \delta_\tau + \left( c_2 e^{2 \mu t} + c_3 e^{-2 \mu t} \right) r \delta_r \quad (\mu \neq 0), \] (25)
where in both cases \( r \delta_r = x \delta_x + y \delta_y \) is the gradient HV of the two-dimensional Euclidian metric. Each set of vectors in (25) and (24) is a representation of the \( sl(2, R) \) algebra and furthermore each set of vectors is constructed from the vector \( \delta_\tau \) and the gradient HV \( r \delta_r \) of the Euclidian two-dimensional space \( E^2 \).

The essence of the difference between the two representations is best seen in the corresponding first integrals. If a Kepler–Ermakov system is Hamiltonian, then the Lie symmetries are also Noether point symmetries; therefore, in order to find these integrals we determine the Noether invariants. The Noether symmetries of the Kepler–Ermakov system have been determined in [12]. For the convenience of the reader we repeat the relevant material.

Equations (22) and (23) follow from the Lagrangian [13]
\[ L = \frac{1}{2} (l^2 + r^2 \dot{\theta}^2) - \frac{\mu^2}{2} r^2 - \frac{C(\theta)}{2 r^2}, \] (26)
where \( C(\theta) = c + \sec^2 \theta f(\tan \theta) + \csc^2 \theta g(\tan \theta) \) provided the functions \( f \) and \( g \) satisfy the constraint:
\[ \sin^2 \theta f'(\tan \theta) + \cos^2 \theta g'(\tan \theta) = 0. \] (27)
The Ermakov invariant in this case is [13]
\[ J = r^4 \dot{\theta}^2 + 2C(\theta). \] (28)
Because the system is autonomous the first Noether integral is the Hamiltonian [12]
\[ E = \frac{1}{2} (l^2 + r^2 \dot{\theta}^2) + \frac{1}{2} \mu^2 r^2 + \frac{1}{r^2} F(\theta). \] (29)
In addition to the Hamiltonian, there exist two additional time-dependent Noether integrals as follows:
\( \mu = 0 \)

\[
I_1 = 2tE - \dot{r} \dot{r} \\
I_2 = t^2E - t\dot{r} + \frac{1}{2} r^2
\]  

(30)  

(31)  

\( \mu \neq 0 \)

\[
I_1' = \left( \frac{1}{\mu} E - \dot{r} + \mu r^2 \right) e^{2\mu t} \\
I_2' = \left( \frac{1}{\mu} E + \dot{r} + \mu r^2 \right) e^{-2\mu t}.
\]  

(32)  

(33)

We note that the Noether integrals corresponding to representation (24) are linear in \( t \), whereas the ones corresponding to representation (25) are exponential. Therefore, the consideration of the cases \( \mu = 0 \) and \( \mu \neq 0 \) is not spurious; otherwise, we loose important information. This latter fact is best seen in the applications of Noether symmetries to field theories where the main core of the theory is the Lagrangian. In these cases, the potential is given and, as has been shown in [12], a given potential admits certain Noether symmetries only; therefore, one has to consider all possible cases. We shall come to this situation in section 7 where it will be found that the potential selects representation (25).

To complete this section, we mention that for a Hamiltonian–Kepler–Ermakov system, the Ermakov invariant (28) is constructed [14] from the Hamiltonian and the Noether invariants (32) and (33) as follows:

\[
J = E^2 - I_1' I_2'.
\]

Finally, in [14] it is shown that the Ermakov invariant is generated by a dynamical Noether symmetry of the Lagrangian (26), a result which is also confirmed in [15].

3. Generalizing the autonomous Kepler–Ermakov system

We consider the generalization of the two-dimensional autonomous Kepler–Ermakov system [7, 9, 16–19] using a geometric point of view. From the results presented so far we have the following.

(i) Equations (4) and (5) which define the Ermakov system employ coordinates in the Euclidian two-dimensional space; therefore, the system is the Euclidian Ermakov system.

(ii) The autonomous two-dimensional Euclidian–Kepler–Ermakov system is defined by equations (22) and (23).

(iii) The Lie symmetries of the Kepler–Ermakov system span the \( sl(2, \mathbb{R}) \) algebra. These symmetries are constructed from the vector \( \partial_t \) and the gradient \( HV \) of the space \( E^2 \).

(iv) For the autonomous Hamiltonian–Kepler–Ermakov system the Lie symmetries reduce to Noether symmetries and the Ermakov invariant follows from a combination of the resulting three Noether integrals, two of which are time dependent. Furthermore, the Ermakov invariant is the Noether integral of a dynamical Noether symmetry.

The above observations imply that we may generalize the Kepler–Ermakov system in two directions:

(a) increase the number of dimensions by defining the \( n \)-dimensional Euclidian–Kepler–Ermakov system and/or

(b) generalize the background Euclidian space to be a Riemannian space and obtain the Riemannian–Kepler–Ermakov system.
Concerning the defining characteristics of the Kepler–Ermakov system we distinguish three different properties of reduced generality: the property of having the first integral, the property of admitting Lie/Noether symmetries and $sl(2, R)$ invariance and the property of being Hamiltonian and admitting $sl(2, R)$ invariance via Noether symmetries.

Following Leach [9], we generalize the autonomous Kepler–Ermakov system to higher dimensions by the following requirement: the generalized autonomous (Euclidian) Kepler–Ermakov system admits the $sl(2, R)$ algebra as a Lie symmetry algebra.

In order to exploit the significance of the $sl(2, R)$ invariance we refer to two theorems which relate the Lie and the Noether point symmetries of an autonomous dynamical system moving in a Riemannian space with the symmetries of the space (for a detailed statement of these theorems see [12]).

**Theorem 1.** The Lie point symmetries of the equations of motion of an autonomous dynamical system moving under the force $F_j(x^i)$ in a Riemannian space with metric $g_{ij}$, namely
\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = F^i, \] (34)
are given in terms of the generators of the special projective algebra of the metric $g_{ij}$.

If the equations of motion follow from the standard Lagrangian
\[ L(x^i, \dot{x}^i) = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V(x^i), \] (35)
where $V(x^i)$ is the potential, then the following theorem relates the Noether point symmetries of the Lagrangian with the symmetries of the metric.

**Theorem 2.** The Noether point symmetries of the Lagrangian (35) of an autonomous Hamiltonian system moving in a Riemannian space with metric $g_{ij}$ are generated from the homothetic algebra of $g_{ij}$.

### 4. The three-dimensional autonomous Euclidian–Kepler–Ermakov system

The generalization of the autonomous Euclidian–Kepler–Ermakov system using the $sl(2, R)$ invariance of Lie symmetries has been done in [9, 18, 19]. In this section, using the results of [11], we give the generalization of the autonomous Euclidian–Hamiltonian–Kepler–Ermakov system to three dimensions by demanding the $sl(2, R)$ invariance with respect to Noether symmetries. The reason for attempting this generalization is that it leads to the potentials for which the corresponding extended systems are Liouville integrable. Furthermore, it indicates the path to the $n$-dimensional Riemannian–Kepler–Ermakov system.

Depending on the value $\mu \neq 0$ or $\mu = 0$ we consider the three-dimensional Hamiltonian–Kepler–Ermakov systems of types I and II.

#### 4.1. The three-dimensional autonomous Hamiltonian–Kepler–Ermakov system of type I ($\mu \neq 0$)

For $\mu \neq 0$, the admitted Noether symmetries are required to be (see (25))
\[ X^1 = \partial_\mu, \quad X_{\pm} = \frac{1}{\mu} e^{\pm 2 \mu t} \partial_\mu \pm e^{\pm 2 \mu t} R \partial_R. \] (36)

From table 6, line 2 for $T = \frac{1}{\mu} e^{\pm 2 \mu t}$ of [11] we find that for these vectors the potential is $V(R, \phi, \theta) = -\frac{\mu^2}{2} R^2 + \frac{1}{\mu} f(\phi, \phi)$; hence the Lagrangian
\[ L = \frac{1}{2} (\dot{R}^2 + R^2 \dot{\phi}^2 + R^2 \sin^2 \phi \dot{\theta}^2) + \frac{\mu^2}{2} R^2 - \frac{1}{R^2} f(\phi, \phi). \] (37)
The equations of motion, that is, the equations defining the generalized dynamical system are

\[ \ddot{R} - R\dot{\phi}^2 - R\sin^2 \phi \dot{\theta}^2 - \mu^2 R - \frac{2}{R^3} f = 0 \]  
(38)

\[ \dot{\phi} + \frac{2}{R} R\dot{\phi} - \sin \phi \cos \phi \dot{\theta}^2 + \frac{1}{R^3} f_{,\phi} = 0 \]  
(39)

\[ \ddot{\theta} + \frac{2}{R} \ddot{R} + \cot \phi \dot{\phi} \dot{\theta} + \frac{1}{R^4 \sin^2 \phi} f_{,\theta} = 0. \]  
(40)

The Noether integrals corresponding to the Noether vectors are

\[ E = \frac{1}{2} (\dot{R}^2 + R^2 \dot{\phi}^2 + R^2 \sin^2 \phi \dot{\theta}^2) - \frac{\mu^2}{2} R^2 + \frac{1}{R^2} f (\theta, \phi) \]  
(41)

\[ I_+ = \frac{1}{\mu} e^{2\mu t} E - e^{2\mu t} R\ddot{R} + \mu e^{2\mu t} R^2 \]  
(42)

\[ I_- = \frac{1}{\mu} e^{-2\mu t} E + e^{-2\mu t} R\ddot{R} + \mu e^{-2\mu t} R^2, \]  
(43)

where \( E \) is the Hamiltonian. The Noether integrals \( I_\pm \) are time dependent. Following [14] we define the time-independent combined first integral

\[ J = E^2 - I_+ I_- = R^4 \dot{\phi}^2 + R^4 \sin^2 \phi \dot{\theta}^2 + 2 f (\theta, \phi). \]  
(44)

Using (44) the equation of motion (38) becomes

\[ \ddot{R} - \mu^2 R = \frac{J}{R^3}, \]  
(45)

which is the autonomous Ermakov–Pinney equation [20]. Therefore, \( J \) is the Ermakov invariant [9].

An alternative way to construct the Ermakov invariant (44) is to use dynamical Noether symmetries [22]. Indeed, one can show that the Lagrangian (37) admits the dynamical Noether symmetry

\[ X = R^2 (\dot{R} \partial R + \dot{\phi} \partial \phi + \dot{\theta} \partial \theta). \]  
(46)

From table 5, line 3 for \( T(t) = 1 \) and table 6, line 1 for \( T(t) = t \) of [11] we find that the potential \( V(R, \phi, \theta) = \frac{1}{R^2} f (\theta, \phi) \); hence the Lagrangian

\[ L' = \frac{1}{2} (\dot{R}^2 + R^2 \dot{\phi}^2 + R^2 \sin^2 \phi \dot{\theta}^2) - \frac{1}{R^2} f (\theta, \phi). \]  
(47)

The equations of motion are (38)–(40) with \( \mu = 0 \). The Noether invariants of the Lagrangian (47) are

\[ E = \frac{1}{2} (\dot{R}^2 + R^2 \dot{\phi}^2 + R^2 \sin^2 \phi \dot{\theta}^2) + \frac{1}{R^2} f (\theta, \phi) \]  
(48)

\[ I_\pm = 2t E' - R\ddot{R} \]  
(49)

4.2. The three-dimensional autonomous Hamiltonian–Kepler–Ermakov system of type II

(\( \mu = 0 \))

For \( \mu = 0 \), the Noether symmetries are required to be (see (24)) [9]

\[ X_1 = \partial_t, \quad X_2 = 2t \partial_t + R \partial_R, \quad X_3 = t^2 \partial_t + tR \partial_R. \]  
(46)

The equations of motion are (38)–(40) with \( \mu = 0 \). The Noether invariants of the Lagrangian (47) are

\[ E = \frac{1}{2} (\dot{R}^2 + R^2 \dot{\phi}^2 + R^2 \sin^2 \phi \dot{\theta}^2) + \frac{1}{R^2} f (\theta, \phi) \]  
(48)

\[ I_\pm = 2t E' - R\ddot{R} \]  
(49)
\[ I_2 = t^2 E^2 - tR \dot{R} + \frac{1}{2} R^2. \]  

(50)

We note that the time-dependent first integrals \( I_{1,2} \) are linear in \( t \), whereas the corresponding integrals \( I_\pm \) of the case \( \mu \neq 0 \) are exponential. From \( I_{1,2} \), we define the time-independent first integral \( J = 4I_2 E^2 - I_1^2 \), which is calculated to be

\[ J = R^4 \dot{\phi}^2 + R^2 \sin^2 \phi \dot{\theta}^2 + 2f(\theta, \phi). \]

(51)

Using (51) the equation of motion for \( R(t) \) becomes \( \ddot{R} - J' R^3 = 0 \), which is the one-dimensional Ermakov–Pinney equation; hence, \( J' \) is the Ermakov invariant [9]. As was the case with the three-dimensional Hamiltonian–Kepler–Ermakov system of type I, the Lagrangian (47) admits the dynamical Noether symmetry \( XD = R^2 (\dot{\phi} \partial_\phi + \dot{\theta} \partial_\theta) \) whose integral is (51).

4.3. Integrability of the three-dimensional autonomous Hamiltonian–Euclidian–Kepler–Ermakov system

The three-dimensional autonomous Hamiltonian–Euclidian–Kepler–Ermakov systems need three independent first integrals in involution in order to be Liouville integrable. As we have shown each system has the two Noether integrals \((E, J)\), therefore we need one more Noether symmetry. Such a symmetry exists only for special forms of the arbitrary function \( f(\theta, \phi) \).

From tables 5, 6 and A1–A3 of [11] we find that extra Noether symmetries are possible only for linear combinations of translations (i.e. vectors of the form \( \sum_{A=1}^{3} a_A \partial_A \)), where \( a_A \) are constants) and/or rotations (i.e. elements of \( so(3) \)).

4.3.1. Noether symmetries generated from the translation group. We determine the functions \( f(\theta, \phi) \) for which the three-dimensional autonomous Hamiltonian–Euclidian–Kepler–Ermakov systems admit extra Noether symmetries for linear combinations of the translation group.

The Lagrangian (37). In Cartesian coordinates the Lagrangian (37) is

\[ L(x^i, \dot{x}^i) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{\mu^2}{2}(x^2 + y^2 + z^2) - \frac{1}{\mu^2} f_I(\frac{y}{x}, \frac{z}{x}), \]

(52)

where \( f_I = (1 + \frac{\dot{x}^2}{\dot{y}^2} + \frac{\dot{z}^2}{\dot{x}^2})^{-1} \). From table 6, line 1 of [11] with \( m = -\mu^2, p = 0 \), we find that the Lagrangian (52) admits Noether symmetries which are produced from a linear combination of translations, if the function \( f_I(\frac{y}{x}, \frac{z}{x}) \) has the form

\[ f_I(\frac{y}{x}, \frac{z}{x}) = \frac{1}{(1 - \frac{\dot{y}^2}{\dot{z}^2})^2} F \left( \frac{b_2 - c_2}{(1 - \frac{\dot{y}^2}{\dot{z}^2})} \right). \]

(53)

In this case, the Lagrangian (52) admits at least the following two extra Noether symmetries (see table A3, line 1 of [11]):

\[ X_{\pm} = e^{\pm \mu t} \sum_{A=1}^{3} a_A \partial_A, \]

(54)

with corresponding Noether integrals

\[ I_{\pm} = e^{\pm \mu t} \left( \sum_{A=1}^{3} a_A x_A \right) \pm \mu e^{\pm \mu t} \left( \sum_{A=1}^{3} a_A x_A \right). \]

(55)

1 The linear combination of an element of \( so(3) \) with a translation does not give a potential, hence an additional Noether symmetry.
We note that the first integrals $I_k$ are time dependent; however, the first integral
\[ J_2 = I_1 I_2 = (ax + by + cz)^2 + \mu^2 (ax + by + cz)^2 \]  
(56)
is time independent. As was the case with the Ermakov invariant (44), it is possible to construct
the integral $J_2$ directly from the dynamical Noether symmetry $X'_D = K'_{[21]} x^i \partial_i$, where $K'_{[2]}$ is
a Killing tensor of the second rank [22, 23], with non-vanishing components
\[ K_{11} = a^2, \quad K_{22} = b^2, \quad K_{33} = c^2 \]
\[ K_{12} = 2ab, \quad K_{13} = 2ac, \quad K_{23} = 2bc, \]
so that the dynamical symmetry vector is
\[ X'_D = (a^2 + ab + ac) x_i \partial_i + (b^2 + ab + bc) y_i \partial_i + (c^2 + ac + bc) z_i \partial_i. \]
(57)
The Ermakov invariant $J$ (see (44)) in Cartesian coordinates is
\[ J = 2E(x^2 + y^2 + z^2) - (ax + by + cz)^2. \]
(58)
The first integrals $J$ and $J_2$ are not in involution. Using the Poisson brackets we construct new
first integrals and at some stage one of them will be in involution. These new first integrals
can also be constructed from corresponding dynamical Noether symmetries.

An example of a known Lagrangian of the form (52) is the three-body Calogero–Moser
Lagrangian [21, 24, 25]
\[ L = \frac{1}{2} (x^2 + y^2 + z^2) - \frac{\mu^2}{2} (x^2 + y^2 + z^2) - \frac{1}{(x - y)^2} - \frac{1}{(y - z)^2} - \frac{1}{(x - z)^2}. \]
(59)
The extra Noether symmetries of this Lagrangian are produced by the vector (54) for $a^k = (1, 1, 1)$.

*The Lagrangian (47).* In Cartesian coordinates, the Lagrangian (47) is
\[ L(x^i, \dot{x}^i) = \frac{1}{2} (x^2 + y^2 + z^2) - \frac{1}{x^2} \int_1^3 \left( \frac{y}{x}, \frac{z}{x} \right). \]
(60)
According to table A2, line 1 and table A3, line 1 of [11] (with $m = 0, p = 0$), the Lagrangian
(60) admits extra Noether point symmetries for a linear combination of translations if the
function $f$ is of the form (53). In this case, the corresponding Noether integrals are
\[ I_1 = \sum_{k=1}^3 a^k \dot{x}_k, \quad I_2 = \int \sum_{k=1}^3 a^k \dot{x}_k = \sum_{k=1}^3 a^k \dot{x}_k. \]
(61)
An example of such a Lagrangian is the Calogero–Moser Lagrangian [21] (without the
oscillator term)
\[ L = \frac{1}{2} (x^2 + y^2 + z^2) - \frac{1}{(x - y)^2} - \frac{1}{(x - z)^2} - \frac{1}{(y - z)^2}. \]
(62)
For the Lagrangian (62), we have the first integrals $E, J, I_1, I_2$. From the integrals $J$ and $I_1$ we
construct the first integral $\Phi = \{ I_1, \{ J, I_1 \} \}$. It is easy to show that the integrals $E, I_1, \Phi$ are
in involution; hence, the dynamical system is Liouville integrable. We remark that the first
integrals $E, J, I_1, I_2$ can also be computed by making use of the Lax pair tensor [25].

4.3.2. Noether symmetries generated from so(3). The elements of so(3) in spherical
coordinates are the three vectors $CK_{1,2,3}$,
\[ CK^1 = \sin \theta \partial_\theta + \cos \theta \cot \phi \partial_\phi, \quad CK^2 = -\sin \theta \cot \phi \partial_\phi - \sin \theta \cot \phi \partial_\phi, \quad CK^3 = \partial_\theta, \]
(63)
which are also the KVs of the Euclidian sphere.
In this case, the symmetry condition (equation (56) of [12]) becomes
$$L_{CK} \left[ \frac{1}{R^2} f (θ, φ) \right] + p = 0$$
(64)
or, equivalently
$$\frac{1}{R^2} (R^2 g_{ij} CK^i f^j) + p = 0 \Rightarrow g_{ij} CK^i \{ f^j \} + p = 0,$$
(65)
where $g_{ij}$ is the metric of the Euclidian sphere, that is,
$$dx^2 = dφ^2 + sin^2 φ dθ^2.$$  
(66)
We infer that the problem of determining the extra Noether point symmetries of Lagrangian (37) generated from elements of $so(3)$ is equivalent to the determination of the Noether point symmetries for motion on the two-dimensional sphere, a problem which has been answered in [12].

It is easy to show that the integrals of table 7 of [11] are in involution with the Hamiltonian symmetries for motion on the two-dimensional sphere, a problem which has been answered in [12].

The above results are extended to the case in which the system moves on the hyperbolic sphere, that is, it has Lagrangian
$$L = \frac{1}{2} (R^2 + R^2 \dot{θ}^2 + R^2 \sin^2 θ \dot{φ}^2) + \frac{μ^2}{2} R^2 - \frac{1}{R^2} g(θ, φ).$$
(67)
We reach the following conclusion.

**Proposition 3.** The three-dimensional autonomous Hamiltonian–Kepler–Ermakov system with Lagrangian (37) is Liouville integrable via Noether point symmetries which are generated from a linear combination of the three elements of the $so(3)$ algebra, if and only if the equivalent dynamical system in the fundamental hyperquadrics of the three-dimensional flat space is integrable.

We note that it is possible for a three-dimensional autonomous Kepler–Ermakov system to admit more Noether symmetries, which are due to the rotation group and the translation group (but not to a linear combination of elements from the two groups). For example the three-dimensional Kepler–Ermakov system with Lagrangian [27]
$$L = \frac{1}{2} (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{x^2 (1 - \frac{z}{x} - \frac{z}{x} )}$$
(68)
has the following extra Noether point symmetries (in addition to the elements of $sl(2, R)$):
$$Y^1 = \dot{a}_t + \dot{a}_z, Y^2 = \dot{a}_t + \dot{a}_z$$
$$Y^3 = t(\dot{a}_t + \dot{a}_z), Y^4 = t (\dot{a}_t + \dot{a}_z)$$
$$Y^5 = (y - z) \dot{a}_t - (x + z) \dot{a}_y + (x + y) \dot{a}_z.$$
The vectors $Y^{1,2}$ and $Y^{3,4}$ follow from (54) for $a^t = (1, 1, 0)$ and $a^z = (1, 0, 1)$, respectively, whereas $Y^5$ is a linear combination of the three elements of $so(3)$. The Noether integrals of the Noether symmetries $Y^{1-3}$ are, respectively,
$$I_{Y^1} = \dot{x} + \dot{y}, I_{Y^2} = \dot{x} + \dot{z}$$
(69)
$$I_{Y^3} = t (\dot{x} + \dot{y}) - (x + y), I_{Y^4} = t (\dot{x} + \dot{z}) - (x + z)$$
(70)
$$I_{Y^5} = (y - z) \dot{x} - (x + z) \dot{y} + (x + y) \dot{z}.$$  
(71)
It is clear that in order to extend the Kepler–Ermakov system to higher dimensions one needs to have the type of results of [11]; therefore, the remark made in [9] that the ‘notion is easily generalized to higher dimensions’ has to be understood as referring to the general scenario and not to the actual work.

See [26] for definitions.
5. The autonomous Riemannian–Kepler–Ermakov system

As has been noted in section 2 the Kepler–Ermakov systems considered so far in the literature are the Newtonian–Kepler–Ermakov systems. In this section, we make a drastic step forward and introduce the autonomous Riemannian–Kepler–Ermakov system of dimension \( n \).

**Definition 4.** The \( n \)-dimensional autonomous Riemannian–Kepler–Ermakov system is an autonomous dynamical system which

(a) is defined on a Riemannian space which admits a gradient \( HV \),

(b) admits the first integral, which we name the Riemannian–Ermakov first integral and it is characterized by the requirement that the corresponding equation of motion takes the form of the Ermakov–Pinney equation,

(c) is invariant at least under the \( sl(2, R) \) algebra, which is generated by the vector \( \partial_t \) and the gradient \( HV \) of the space.

There are two types of \( n \)-dimensional autonomous Riemannian–Kepler–Ermakov systems. The ones which are not Hamiltonian and admit the \( sl(2, R) \) algebra as Lie symmetries and the ones which are Hamiltonian and admit the \( sl(2, R) \) algebra as Noether symmetries.

5.1. The non-Hamiltonian autonomous Riemannian–Kepler–Ermakov system

Consider a \( n \)-dimensional Riemannian space which admits a gradient \( HV \). It is well known that the metric of this space can always be written in the form \[ ds^2 = du^2 + u^2 h_{AB} dy^A dy^B, \] where the Latin capital indices \( A, B, \ldots \) take the values \( 1, \ldots, n-1 \) and \( h_{AB} = h_{AB}(y^C) \) is the generic \( n-1 \) metric. The gradient \( HV \) of the metric is the vector \( H^i = u \partial_u (\psi = 1) \) generated from the function \( H = \frac{1}{2} u^2 \). We note the relation \[ h_{DA} \Gamma^A_{BC} = \frac{1}{2} h_{DB,C}, \] where \( \Gamma^A_{BC} \) are the connection coefficients of the \( n-1 \) metric \( h_{AB} \). In that space we consider a particle moving under the action of the force \( F^i = u \partial_u (\psi = 1) \). The equations of motion \( \frac{d}{ds} = F^i \) when projected along the direction of \( u \) and in the \( n-1 \) space give the equations

\[ u'' - u h_{AB} y^A y^B F^u = F^u \]
\[ y''^A + 2 \frac{u'}{u} y'^A + \Gamma^A_{BC} y'^B y'^C = F^A, \]

where \( u' = \frac{ds}{dt} \) and \( s \) is an affine parameter.

Because the system is autonomous, it admits the Lie point symmetry \( \partial_t \). Using the vector \( \partial_t \) and the gradient \( HV H^i = u \partial_u \), we construct two representations of \( sl(2, R) \) by means of the sets of vectors (see (36) and (46))

\[ \partial_s, 2 s \partial_s + u \partial_u, s^2 \partial_s + su \partial_u \quad \text{when } \mu = 0 \]
\[ \partial_s + \frac{1}{\mu} e^{\pm 2 \mu s} \partial_s \pm e^{\pm 2 \mu s} u \partial_u \quad \text{when } \mu \neq 0 \]
and require that the vectors in each set will be Lie symmetries of the system of equations (74) and (75). In appendix A, we show that the requirement of the invariance of the force under both representations (76) and (77) of sl(2, R) demands that the force is of the form

\[ F^i = \left( \mu^2 u + \frac{1}{u^3} G^w(y^C) \right) \partial_u + \frac{1}{u^4} G^A(y^C) \partial_A. \]  

(78)

Replacing \( F^i \) in the system of equations (74) and (75), we find

\[ u'' - u \partial_u h^A y^A = \mu^2 u + \frac{1}{u^3} G^w(y^C), \]  

(79)

\[ y''^A + \frac{2}{u} u' y^A + \Gamma^A_{BC} y^B y^C = \frac{1}{u^4} G^A. \]  

(80)

Multiplying the second equation with \( 2u^4 h^A y^B \) and using (73) we have

\[ u^4 \frac{d}{ds} \left( h^A y^B \right) + 4u^3 h^A u' y^B = 2G_D y^D. \]  

(81)

from which follows

\[ \frac{d}{ds} \left( u^4 h^A y^B \right) = 2G_D y^D. \]  

(82)

The rhs is a perfect differential if \( CD = -\Sigma_D \), where \( \Sigma(y^A) \) is a differentiable function. If this is the case, then we find the first integral

\[ J = u^4 h^A y^B + 2\Sigma(y^C). \]  

(83)

We note that \( J \) involves the arbitrary metric \( h^{AB} \) and the function \( \Sigma(y^A) \). Furthermore, equations (79) and (80) become

\[ u'' - u h^A y^A = \mu^2 u + \frac{1}{u^3} G^w(y^C), \]  

(84)

\[ y''^A + \frac{2}{u} u' y^A + \Gamma^A_{BC} y^B y^C = -\frac{1}{u^4} h^{AB} \Sigma(y^C),.B. \]  

(85)

These are the equations defining the \( n \)-dimensional autonomous Riemannian–Kepler–Ermakov system.

Using the first integral (83), the equation of motion (84) is written as follows:

\[ u'' = \mu^2 u + \frac{\tilde{G}(y^C)}{u^3}, \]  

(86)

where \( \tilde{G} = J + G^w(y^C) - 2\Sigma(y^C) \). This is the Ermakov–Pinney equation; hence, we identify (83) as the Riemannian–Ermakov integral of the autonomous Riemannian–Kepler–Ermakov system.

5.2. The autonomous Hamiltonian–Riemannian–Kepler–Ermakov system

In this section, we assume that the force is derived from the potential \( V(u, y^C) \). Then the equations of motion follow from the Lagrangian

\[ L = \frac{1}{2} \left( u'^2 + u^2 h^A y^A y^B \right) - V(u, y^C). \]  

(87)

The Hamiltonian is

\[ E = \frac{1}{2} \left( u'^2 + u^2 h^A y^A y^B \right) + V(u, y^C). \]  

(88)

The equations of motion are

\[ u'' - u h^A y^A + V_u = 0. \]  

(89)
The demand that Lagrangian (87) admits Noether point symmetries which are generated from the gradient HV leads to the following cases (see theorem 2 and for details see [12]).

**Case A.** The Lagrangian (87) admits the Noether point symmetries (76) if the potential is of the form

\[ V(u,y^C) = \frac{1}{u^2} V(y^C). \]  

(91)

The Noether integrals of these Noether point symmetries are

\[ E_A = \frac{1}{2} \left( u'^2 + u^2 h_{AB} y^A y^B \right) + \frac{1}{u^2} V(y^C) \]  

(92)

\[ I_1 = 2sE - uu' \]  

(93)

\[ I_2 = s^2 E - ssu' + \frac{1}{2} u^2 \]  

(94)

where \( E_A \) is the Hamiltonian.

**Case B.** The Lagrangian (87) admits the Noether point symmetries (77) if the potential is of the form

\[ V(u,y^C) = -\frac{\mu^2}{2} u^2 + \frac{1}{u^2} V'(y^C). \]  

(95)

The Noether integrals of these Noether point symmetries are

\[ E_B = \frac{1}{2} \left( u'^2 + u^2 h_{AB} y^A y^B \right) - \frac{\mu^2}{2} u^2 + \frac{1}{u^2} V'(y^C) \]  

(96)

\[ I_+ = \frac{1}{\mu} e^{2\mu} E - e^{2\mu} uu' + \mu e^{2\mu} u^2 \]  

(97)

\[ I_- = \frac{1}{\mu} e^{-2\mu} E + e^{-2\mu} uu' + \mu e^{-2\mu} u^2 \]  

(98)

where \( E_B \) is the Hamiltonian.

Using the Noether integrals we construct the Riemannian–Ermakov invariant \( J_G \), which is common for both cases A and B, as follows:

\[ J_G = u^2 h_{DB} y^D y^C + 2V'(y^C). \]  

(99)

This coincides with the invariant first integral defined in (83). We note that with the use of the first integral (99) the Hamiltonians (92) and (96) take the form

\[ E = \frac{1}{2} u'^2 - \frac{\mu^2}{2} u^2 + \frac{J}{2u^2}. \]  

(100)

which is the Hamiltonian for the Ermakov–Pinney equation.

As was the case with the Euclidian case of section 2, it can be shown that the Riemannian Ermakov invariant (99) is due to a dynamical Noether symmetry [22].

We collect the results in the following proposition.

**Proposition 5.** In a Riemannian space with metric \( g_{ij} \) which admits a gradient HV, the equations of motion of a Hamiltonian system moving under the action of the potential \( (\mu \epsilon C) \)

\[ V(u,y^C) = -\frac{\mu^2}{2} u^2 + \frac{1}{u^2} V'(y^C) \]  

(101)

admit the \( sl(2,R) \) invariance and also an invariant first integral, the Riemannian–Ermakov invariant. This latter quantity is also possible to be identified as the Noether integral of a dynamical Noether symmetry.

Without going into detail we state the following more general result.
Proposition 6. Consider an n-dimensional Riemannian space with an r-decomposable metric which in the Cartesian coordinates \( x_1, \ldots, x_r \) has the general form

\[
dx^2 = p_{ij} dx_i dx_j + h_{ij} dx_i dx_j, i, j = r + 1, \ldots, n, \Sigma = 1, \ldots, r,
\]

where \( p_{ij} \) is a flat non-degenerate metric (of arbitrary signature). If there exists a potential so that the vectors \( e^{\pm \mu} \sum_M a_M \partial_M \) are Noether symmetries, where \( a_M \) are constants, with Noether integrals

\[
I_{\pm} = e^{\pm \mu} \sum_M a_M z_M' \pm \mu e^{\pm \mu} \sum_M a_M z_M,
\]

the combined first integral \( I = I_+ I_- \) is time independent, and it is the result of a dynamical Noether symmetry.

In the remaining sections we will consider applications of the autonomous Riemannian–Kepler–Ermakov system in general relativity and in cosmology.

6. The autonomous Riemannian–Kepler–Ermakov system in general relativity

6.1. The autonomous Riemannian–Kepler–Ermakov system of four degrees of freedom in a FRW spacetime

Consider the spatially flat FRW spacetime with the metric

\[
dx^2 = du^2 - u^2 (dx^2 + dy^2 + dz^2).
\]

This metric admits the gradient HV \( \partial_u \) and six non-gradient KVs [30] which are the KVs of \( E^3 \).

We consider the autonomous Riemannian–Kepler–Ermakov system defined by the Lagrangian (see (101)) (\( \mu \in \mathbb{C} \)):

\[
L = \frac{1}{2} (u^2 - u^2 (x^2 + y^2 + z^2)) + \frac{\mu^2}{2} u^2 - \frac{1}{u^2} V(x, y, z).
\]

The Euler–Lagrange equations are

\[
u'' + u (x'' + y'' + z'') - \mu^2 u - \frac{2V(x, y, z)}{u^3} = 0
\]

\[
\sigma'' = -\frac{2}{u} \sigma' u - \frac{V' (x, y, z)}{u^3} = 0,
\]

where \( \sigma = 1, 2, 3 \). The Lagrangian (105) has the same form of the Lagrangian (87) for potential \( V(u, y^2) = -\frac{\mu^2}{2} u^2 + \frac{1}{u^2} V(x, y, z) \) hence according to proposition 5 possesses sl(2, R) invariance with Noether symmetries and for both representations (76) and (77). The two time-independent invariants are the Hamiltonian and the Riemannian–Ermakov invariant (proposition 5):

\[
E = \frac{1}{2} (u^2 - u^2 (x^2 + y^2 + z^2)) - \frac{\mu^2}{2} u^2 + \frac{1}{u^2} V(x, y, z)
\]

\[
J_{G_4} = u^4 (x^2 + y^2 + z^2) + 2V(x, y, z).
\]

We remark that if we had considered representation (76) only (that is we had set \( \mu = 0 \)), then we would have lost all information concerning the system defined for \( \mu \neq 0 \)! We emphasize that in application to physics the major datum is the Lagrangian and not the equations of motion; therefore, one should not make mathematical assumptions which restrict the physical generality.
To assure Liouville integrability we need one more Noether symmetry whose Noether integral is in involution with $E, J_{G_4}$. This is possible for certain forms of the potential $V(x, y, z)$. Using the general results of [11] where all three-dimensional potentials are given which admit extra Noether symmetries we find the following result.

**Proposition 7.** The Lagrangian (105) admits an extra Noether symmetry if and only if the potential $V(x, y, z)$ has the form given in table 5, lines: 1,2. table A1, lines :1–4 and table A2, lines: 1–4 of [11].

For example if $V(x, y, z) = (x^2 + y^2 + z^2)^n$, then the system admits three extra Noether symmetries which are the elements of $so(3)$. If $V(x, y, z) = V_0$, then the system admits six extra Noether symmetries (the KVs of the three-dimensional Euclidian space).

### 6.2. The autonomous Riemannian–Kepler–Ermakov system of three degrees of freedom in a three-dimensional spacetime

Consider the three-dimensional Lorenzian metric
\[
d s^2 = du^2 - u^2(dx^2 + dy^2),
\]
which admits the gradient $HV(u \partial_u$ and the three KVs of the Euclidian metric $E^2$. In that space consider the Lagrangian
\[
L' = \frac{1}{2}(u'^2 - u^2(x'^2 + y'^2)) + \frac{\mu^2}{2}u^2 - \frac{1}{u^2}V(x, y).
\]
According to proposition 5 this Lagrangian admits as Noether symmetries the elements of $sl(2, R)$. Then from proposition 5 we have that the Noether invariants of these symmetries are
\[
E = \frac{1}{2}(u'^2 - u^2(x'^2 + y'^2)) - \frac{\mu^2}{2}u^2 + \frac{1}{u^2}V(x, y)
\]
\[
J_{G_1} = u^4(x^2 + y^2) + 2V(x, y).
\]
The requirement that the Lagrangian admits an additional Noether symmetry leads to the condition $L'_{KV}V(x, y) + p = 0$; therefore, in that case we have a two-dimensional potential and we can use the results of [12]. If we demand the new Noether integral to be time independent ($p = 0$), then the potential $V(x, y)$ and the new Noether integrals are given in table B1 of appendix B.

Lagrangians with the kinetic term $T_k = \frac{1}{2}(u'^2 - u^2(x'^2 + y'^2))$ appear in cosmological models. In the following section we discuss such applications.

### 7. The autonomous Riemannian–Kepler–Ermakov system in cosmology

In a LRS spacetime, we consider two cosmological models for dark energy, a scalar field cosmology and an $f(R)$ cosmology.

#### 7.1. The case of scalar field cosmology

Consider the class A LRS spacetime
\[
dx^2 = -N^2(t)\, dt^2 + a^2(t)\, e^{-2\beta(t)}\, dx + a^2(t)\, e^\beta(t)\, (dy^2 + dz^2)
\]
which is assumed to contain a scalar field with the exponential potential $V(\phi) = V_0 e^{-\phi}$, $c \neq \sqrt{6k}$ and a perfect fluid with a stiff equation of state $\rho = \rho_0$, where $\rho$ is the pressure and $\rho_0$ is the energy density of the fluid. The conservation equation for the matter density gives
\[ \dot{\rho} + 6\rho \frac{\dot{a}}{a} = 0 \rightarrow \rho = \frac{\rho_0}{a^6}. \] (115)

Einstein field equations for the comoving observers $u^a = \frac{1}{\sqrt{1-\dot{c}^2}} \partial_a, \ u^a u_a = -1$ follow from the autonomous Lagrangian [31, 32]:
\[ L = -3a^2 + \left( \frac{3}{4} \beta^2 + \frac{k}{2} \phi^2 \right) a e^{-\phi} - k a^3 e^{-\phi} - N \rho_0. \] (116)

We set $N^2 = e^{\phi}$ and the Lagrangian becomes
\[ L = \left( -3a^2 + \frac{3}{4} \beta^2 + \frac{k}{2} \phi^2 \right) a e^{-\phi} - k a^3 e^{-\phi} - \frac{\rho_0}{a^3}. \] (117) The Hamiltonian is
\[ E = \left( -3a^2 + \frac{3}{4} \beta^2 + \frac{k}{2} \phi^2 \right) a e^{-\phi} + k a^3 e^{-\phi} + \frac{\rho_0}{a^3 e^{\phi}} = 0. \] (118)

If we consider the transformation
\[ a^3 = e^{\phi}, \quad \phi = \frac{1}{3} \sqrt{\frac{6k}{c}} (x - y), \] (119)
where
\[ x = \frac{1}{1 - \tilde{c}} \ln \left( \frac{1 - \tilde{c}}{\sqrt{2}} u e^{\phi} \right), \quad y = \frac{1}{1 + \tilde{c}} \ln \left( \frac{1 + \tilde{c}}{\sqrt{2}} u e^{-\phi} \right), \quad \tilde{c} = \frac{c}{\sqrt{6k}} \neq 1, \] (120)
then the Lagrangian (117) becomes
\[ L = -\frac{2}{3} \dot{a}^2 + u^2 \left( \frac{2}{3} \tilde{c}^2 - \frac{3}{8kV_0} \beta^2 \right) - \frac{\mu^2}{2} u^2 + \frac{kV_0 \rho_0}{\mu^2} \frac{1}{u^2}. \] (121)

We consider a two-dimensional Riemannian space with the metric defined by the kinematic terms of the Lagrangian, that is,
\[ ds^2 = (-6 du^2 + \frac{4}{3} a^2 d\beta^2 + ka^2 d\phi^2) a e^{-\phi}. \] (122)

We easily show that this metric admits the gradient $HV = \frac{4}{6k\tilde{c}^2} (ka\dot{\alpha} + c\dot{\phi})$, with the gradient function $H = \frac{4kV_0}{e^{-\phi} \tilde{c}^2} u e^{-\phi}$. Therefore, the Lagrangian (117) defines an autonomous Hamiltonian–Riemannian–Kepler–Ermakov system with the potential ($\mu \neq 0$)
\[ V(u, y) = -\frac{1}{2} \mu^2 u^2 + \frac{kV_0 \rho_0}{\mu^2} \frac{1}{u^2}. \] (123)
Because $\mu \neq 0$, this Lagrangian admits sl(2, $R$) invariance only for representation (76) (an additional result which shows the necessity for the consideration of the cases $\mu = 0$ and $\mu \neq 0$).

Using proposition 5 we write the Ermakov invariant
\[ J = u^4 \left( \frac{2}{3} \tilde{c}^2 + \frac{3}{8kV_0} \beta^2 \right) + \frac{kV_0 \rho_0}{\mu^2} \frac{1}{u^2}. \] (124)

The second invariant is the Hamiltonian $E$:
\[ E = -\frac{2}{3} \dot{u}^2 + u^2 \left( \frac{2}{3} \tilde{c}^2 + \frac{3}{8kV_0} \beta^2 \right) + \frac{\mu^2}{2} u^2 - \frac{kV_0 \rho_0}{\mu^2} \frac{1}{u^2}. \] (125)

From table 1 line 6 of [11], we find that the Lagrangian admits three more Noether symmetries
\[ \partial_{\beta}, \partial_{\phi}, \sqrt{a} \partial_{\beta} - \beta \partial_{\phi} \] (126)
with corresponding integrals
\[ I_1 = u^2 \dot{\beta}, \quad I_2 = u^2 \dot{z}, \quad I_3 = u^2 \left( \frac{3}{8kV_0} z \dot{\beta} - \frac{2}{3} \beta \dot{z} \right). \]
(127)

It is easy to show that three of the integrals are in involution; therefore, the system is Liouville integrable.

7.2. The case of \( f(R) \) cosmology

Consider the modified Einstein–Hilbert action
\[ S = \int d^4x \sqrt{-g} f(R), \]
(128)
where \( f(R) \) is a smooth function of the curvature scalar \( R \). The resulting field equations for this action in the metric variational approach are [33]
\[ f'R_{ab} - \frac{1}{2} f g_{ab} \Box f' - f'_{,ab} = 0, \]
(129)
where \( f' = \frac{df(R)}{dR} \) and \( f'' \neq 0 \). In the LRS spacetime (114) with \( N(t) = 1 \), these equations for comoving observers are the Euler–Lagrange equations of the Lagrangian
\[ L = (6af' \dot{a}^2 + 6a^2 f'' \dot{a} \dot{R} - \frac{3}{2} f a^3 \dot{\beta}^2) + a^3 (f' R - f). \]
(130)

The Hamiltonian is
\[ E = (6af' \dot{a}^2 + 6a^2 f'' \dot{a} \dot{R} - \frac{3}{2} f a^3 \dot{\beta}^2) - a^3 (f' R - f) = 0. \]
(131)

Again we consider the three-dimensional Riemannian space whose metric is defined by the kinematic part of the Lagrangian (130)
\[ ds^2 = 12af' da^2 + 12a^2 f'' da dR - 3a^3 f' d\beta^2. \]
(132)

This metric admits the gradient \( HV \),
\[ H' = \frac{1}{2} \left( a \partial_a + \frac{f'}{f} \partial_R \right), \]
with the gradient function \( H = 3a^3 f' \).

In order to determine the function \( f(R) \) we demand the geometric condition that Lagrangian (130) admits \( s(2, R) \) invariance via Noether symmetries (see [12]). Then for each representation (76), (77) we have a different function \( f(R) \), and hence a different physical theory.

Representation (76) in this context is
\[ \partial_t, 2r \partial_t + \frac{1}{2} \left( a \partial_a + \frac{f'}{f} \partial_R \right), r^2 \partial_t + \frac{1}{2} \left( a \partial_a + \frac{f'}{f} \partial_R \right). \]
(134)
The Noether conditions become
\[ -4a^3 f' R + \frac{3}{2} a^3 f + p = 0. \]
(135)

These vectors are Noether symmetries if \( p = 0 \) and
\[ f(R) = R^2; \]
(136)
however power law \( f(R) \) theories are not cosmologically viable [34].

The second representation (77) in this context gives the vectors
\[ \partial_t, \frac{1}{\mu} e^{2\mu t} \partial_t \pm \frac{1}{2} e^{2\mu t} \left( a \partial_a + \frac{f'}{f} \partial_R \right). \]
(137)
The Noether conditions give

\[-4a^3 f' R + \frac{7}{2} a^3 f + 3\mu^2 a^3 f' + p = 0. \quad (138)\]

These vectors are Noether symmetries if the constant \(p = 0\) and the function

\[f(R) = (R - 2\Lambda)^{\frac{7}{8}}, \quad (139)\]

where \(2\Lambda = 3\mu^2\). This model is the viable \(\Lambda\text{cdm}\)-like cosmological with \(b = 1, c = \frac{7}{8}\).

We note that had we not considered the latter representation, then we would have lost this interesting result.

For the function (139) the Lagrangian (130) becomes for both cases (if \(\Lambda = 0\) we have the power-law \(f(R) = R^{\frac{7}{8}}\))

\[L = \frac{21}{4} a (R - 2\Lambda)^{-\frac{1}{2}} a^2 - \frac{21}{16} a^2 (R - 2\Lambda)^{-\frac{3}{2}} a\dot{R} - \frac{21}{8} a^3 (R - 2\Lambda)^{-\frac{5}{2}} \dot{\beta}^2 - a^3 (R - 16\Lambda)^{\frac{1}{4}}. \quad (140)\]

Furthermore, there exist a coordinate transformation for which the metric (132) is written in the form of (87).

We introduce new variables \(u, v, w\) with the relations

\[a = \left(\frac{21}{4}\right)^{\frac{1}{2}} \sqrt{u e^v}, \quad R = 2\Lambda + \frac{e^{12v}}{u^2}, \quad \beta = \sqrt{2} w. \quad (141)\]

In the new variables, the Lagrangian (140) takes the form

\[L = \frac{1}{2} u^2 - \frac{1}{2} u^2 (\dot{v}^2 + \dot{w}^2) + \frac{\mu^2}{2} u^2 - \frac{1}{42} e^{12v}. \quad (142)\]

The Hamiltonian (131) in the new coordinates is

\[E = \frac{1}{2} u^2 - \frac{1}{2} u^2 (\dot{v}^2 + \dot{w}^2) - \frac{\mu^2}{2} u^2 + \frac{1}{42} e^{12v}. \quad (143)\]

The Lagrangian (142) defines a Hamiltonian–Riemannian–Kepler–Ermakov system with the potential

\[V(u, v) = -\frac{\mu^2}{2} u^2 + \frac{1}{42} e^{12v}\]

from which follows the potential \(V(v) = \frac{1}{42} e^{12v}\). In addition to the Hamiltonian, the dynamical system admits the Riemannian–Ermakov invariant

\[J_f = u^4 (\dot{v}^2 + \dot{w}^2) + \frac{1}{21} e^{12v}. \quad (144)\]

The Lagrangian (142) admits the extra Noether point symmetry \(\partial_w\) with the Noether integral \(I_w = u^2 \dot{w}\) (see table 1). The three integrals \(E, I_w\) and \(J_f\) are in involution and independent; therefore, the system is integrable.

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3 The importance of the result is due to the fact that it follows from a geometric assumption which is beyond and above the physical considerations. Furthermore, the assumption of Noether symmetries provides the Noether integrals which allow for an analytic solution of the model.
8. Conclusion

In this work we have considered the generalization of the autonomous Kepler–Ermakov dynamical system in the spirit of Leach [9], that is, using invariance wrt the \( sl(2,R) \) Lie and Noether algebra. We have generalized the autonomous Newtonian–Hamiltonian–Kepler–Ermakov system to three dimensions using Noether rather than Lie symmetries and have determined all such systems which are Liouville integrable via Noether symmetries. We introduced the autonomous Riemannian–Kepler–Ermakov system in a Riemannian space which admits a gradient HV. This system is the generalization of the autonomous Euclidian–Kepler–Ermakov system and opens new fields of applications for the autonomous Kepler–Ermakov system, especially in relativistic physics. Indeed, we have determined the autonomous Riemannian–Kepler–Ermakov system in a spatially flat FRW spacetime which admits a gradient HV. As a further application we have considered two types of cosmological models which are described by the autonomous Riemannian–Kepler–Ermakov system, i.e. scalar field cosmology with the exponential potential and \( f(R) \)-modified gravity in an LRS spacetime.

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Appendix A. Proof of the force in equations (74) and (75)

We require that the force admits two Lie symmetries which are due to the gradient HV \( H = u \partial_u \). (If we require the force to be invariant under three Lie symmetries which are due to the gradient HV, then it is reduced to the isotropic oscillator.) From theorem 1 (for details see [12]) we have the following cases.

(I) Case \( \mu = 0 \). In this case the Lie symmetries are

\[ \partial_u, 2s \partial_u + u \partial_u, s^2 \partial_u + su \partial_u. \]

The condition which the force must satisfy is

\[ LH F^i + d F^i = 0. \]

Replacing components we find the equations

\[ \left( \frac{\partial}{\partial u} F^u \right) u + (d - 1) F^u = 0 \]
\[ \left( \frac{\partial}{\partial u} F^A \right) u + d F^A = 0 \]

from which follows

\[ F^u = \frac{1}{u^{d-1}} F^u, \quad F^A = \frac{1}{u^d} F^A. \]

Because the HV is gradient, condition A2 of theorem 1 applies and gives the condition

\[ LH F^i + 4 F^i + a_1 H^i = 0 \]

from which follows \( a_1 = 0 \) and \( d = 4 \). Therefore,

\[ F^u = \frac{1}{u^4} G^u (\gamma^\ell), \quad F^A = \frac{1}{u^4} G^A (\gamma^\ell). \]
(II) Case \( \mu \neq 0 \). In this case the Lie symmetries are
\[
\partial_t, \quad \frac{1}{\mu} e^{\pm 2\mu \sigma} \partial_x \pm e^{\pm 2\mu \sigma} u \partial_u.
\]
The condition which the force must satisfy is
\[
L_H F^i + 4 F^j + a_1 H^i = 0.
\]
We demand \( a_1 \neq 0 \) and obtain the system of equations:
\[
\left( \frac{\partial}{\partial u} F^u \right) u + 3 F^u + a_1 u = 0
\]
\[
\left( \frac{\partial}{\partial u} F^A \right) u + 4 F^A = 0
\]
whose solution is
\[
F^u = \mu^2 u + \frac{1}{u} G^u, \quad F^A = \frac{1}{u^2} G^A
\]
where we have set \( a_1 = -4 \mu^2 \).

Appendix B

Table B1. Potentials for which the Lagrangian (111) admits extra Noether point symmetry.

| Noether symmetry | \( V(x, y) \) | Noether integral |
|------------------|---------------|-----------------|
| \( \partial_x \) | \( f(y) \)    | \( I_x = u^2 x' \) |
| \( \partial_y \) | \( f(x) \)    | \( I_y = u^2 y' \) |
| \( y \partial_x - x \partial_y \) | \( f(x^2 + y^2) \) | \( I_{xy} = u^2 (xy' - xy') \) |
| \( \partial_x + b \partial_y \) | \( f(y - bx) \) | \( I_{by} = u^2 (x' - by') \) |
| \( (a + y) \partial_x + (b - x) \partial_y \) | \( f \left( \frac{1}{2} \left( x^2 + y^2 \right) + ay - bx \right) \) | \( I_{by} = (a + y) u^2 x' + (b - x) u^2 y' \) |
| \( \partial_t, \partial_x, y \partial_x - x \partial_y \) | \( V_0 \) | \( I_x, I_y, I_{xy} \) |

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