Symmetric orbits arising from Figure-Eight for N-body problem

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Abstract

In this paper, we first describe how we can arrange any bodies on Figure-Eight without collision in a dense subset of $[0,T]$ after showing that the self-intersections of Figure-Eight will not happen in this subset. Then it is reasonable for us to consider the existence of generalized solutions and non-collision solutions with Mixed-symmetries or with Double-Eight constraints, arising from Figure-Eight, for N-body problem. All of the orbits we found numerically in Section 6 have not been obtained by other authors as far as we know. To prove the existence of these new periodic solutions, the variational approach and critical point theory are applied to the classical N-body equations. And along the line used in this paper, one can construct other symmetric constraints on N-body problems and prove the existence of periodic solutions for them.

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1 Introduction

N-body problem is a classical problem in mathematics and celestial mechanics. Since Euler’s collinear solutions and Lagrange’s equilateral triangle solutions, any more “clearly” (including the existence and the shape of the orbit) solutions have not been found except Figure-Eight solution, a planer three-body orbits with equal masses, which was proved strictly by Chenciner and Montgomery in [3] in 1999. This remarkable and interesting orbit arouse many authors’ curiosity in studying more choreography orbits or other interesting symmetric constraints acting on the orbits of N-body problems, see [1, 2, 4, 6, 10, 11, 12, 13], etc. Furthermore, some investigations of a possible star system in real space have been performed in practice (see [5]).

Among those authors’ studies, the variational approach and the theory of critical points perform an important role. By minimizing the Lagrangian action on a space of loops symmetry with respect to a well-chosen symmetry group, which also has some simple conditions on it, Ferrario and Terracini give a fairly general condition on symmetry groups of the loop space in [6]. But their results “are not suitable to prove the existence of those orbits that are found as minimizers in classes of paths characterized by homotopy conditions or a mixture of symmetry and homotopy conditions”. A further study based on [6] is carried by Shibayama to distinguish the solutions, which have been obtained by Shibayama in [11] and some of which by Chen in [2].

In this paper, we study the Figure-Eight constraint carefully in Section 3 and draw a conclusion that even bodies with equivariant shift on Figure-Eight will collide at least on the original point, and odd bodies will not collide in a dense subset of [0, T]. Then in Section 4 we compose Figure-Eight from simple to double (or more, that we do not describe in this paper) by setting odd bodies on them reasonably. And in Section 4.2 it is practical to mix Figure-Eight with other constraints on $q^{(3)}$ in space. Then along Gordon [7, 8] and Zelati’s [14] line, in Section 5 we have proved the existence of generalized solutions and non-linear solutions for these periodic orbits. The numerical study in Section 4 highly enlightens us that we can arrange many (even or odd) bodies on Figure-Eight by choosing an appropriate phase differences. Obviously, all the study is based on Figure-Eight constraint, hence this paper is a continued study on Chenciner and Montgomery’s significant work.
2 N-body problem and Figure-Eight

The periodic solutions for N-body problem is described by the following second order ordinary differential equations in general:

\[- m_i \ddot{q}_i = \nabla U(q), \quad i = 1, \cdots, N, \quad (1)\]

\[q(0) = q(T), \quad \dot{q}(0) = \dot{q}(T), \quad (2)\]

where \(m_i\) is the mass of \(i\)th body, \(q_i\) is the position of \(i\)th body in \(\mathbb{R}^d\), \(N\) is the number of bodies, \(T\) is the period, and generally the singular potential function \(U\) is

\[U(q) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} U_{ij}(q_j - q_i),\]

with assumptions: \(\forall 1 \leq i \neq j \leq N\) and \(\forall \xi \in \mathbb{R}^d \setminus \{0\}\):

\[U_{ij} \in C^2(\mathbb{R}^d \setminus \{0\}, \mathbb{R}), \quad U_{ij}(\xi) = U_{ji}(\xi),\]

\[U_{ij}(\xi) \to -\infty, \quad \text{as} \quad |\xi| \to 0,\]

\[U(q_1, \cdots, q_N) \leq 0, \quad \forall (q_1, \cdots, q_N) \in (\mathbb{R}^d)^N,\]

where \(U_{ij}\) is usually described by

\[U_{ij} = \frac{m_i m_j}{|q_i - q_j|^\alpha}. \quad (3)\]

When \(\alpha = 1\) in (3), it is called Newton N-body problem. The figures, as the numerical examples to illuminate the existence of the solutions in this paper, are all coming from Newton N-body problem. Poincaré has shown that it is impossible for \(N \geq 3\) to find an explicit expression for the general solution. So even for the three-body problem, it is impossible to describe all the solutions. Then the numerical approach and/or endowing orbits with some symmetric constraints are quite reasonable directions for the study in N-body problem. No matter which direction adopted by the authors it is, the Lagrangian

\[J : H^1(\mathbb{R}/T\mathbb{Z}, (\mathbb{R}^d)^N) \to \mathbb{R},\]

\[J(q) = \frac{1}{2} \sum_{i=1}^{N} m_i \int_0^T |\dot{q}_i|^2 dt - \int_0^T U(q) dt, \quad (4)\]
acts an important role in their study in finding the periodic solutions for N-body problems.

In one direction, finding some symmetric orbits, Chenciner and Montgomery have introduced a new “torsional” symmetric constraint to the possible orbits for Three-body problem with equal masses in 1999. The “torsional” symmetry is described by the following expressions:

Let \( q : (\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}^2, \), \( i = 1, 2, 3, \) the orbit of the \( i \)-th body is

\[
q_i = q_0 \left( t + (3-i)\frac{T}{3} \right),\]

\[
q(t) + q(t+\frac{T}{3}) + q(t + \frac{2T}{3}) = 0, \quad t \in [0, T],
\]

\[
q(\sigma(t)) = \sigma(q(t)), \quad q(\tau(t)) = \tau(q(t)).
\]

where \( q(t) = (q(1)(t), q(2)(t)) \), \( \sigma \) and \( \tau \) are generators of the action of Klein group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \) on \( \mathbb{R}/T\mathbb{Z} \) and on \( \mathbb{R}^2 \):

\[
\sigma(t) = t + \frac{T}{2}, \quad \tau(t) = -t + \frac{T}{2},
\]

\[
\sigma(q) = (-q(1), q(2)), \quad \tau(q) = (q(1), -q(2)).
\]

By the constraints (5)-(7), the existence is proved in [3] for three equal masses with the exclusion of collision. It is called simple choreography orbits, Figure-Eight solution. After that, many choreography/choreographies solutions are found numerically by many authors, e.g. Simó [12, 13], Arioli, Barutello and Terracini [1] etc.

### 3 Number of the bodies on Figure-Eight

One of the facile development to simple choreography is considering more bodies on the same Figure-Eight. We can develop (5) and (6) by

\[
q_i = q \left( t + (N-i)\frac{T}{N} \right), \quad i = 1, \ldots, N,
\]

\[
\sum_{i=1}^{N} q_i(t) = 0, \quad t \in [0, T].
\]
Hence, (8), (9) and (7) describe $N$ bodies on the same Figure-Eight with equivalent shift $\frac{T}{N}$. By (7), one can deduce that

\begin{align}
(q^{(1)}(t), q^{(2)}(t)) &= \left(-q^{(1)}(t + \frac{T}{2}), q^{(2)}(t + \frac{T}{2})\right), \\
(q^{(1)}(t), q^{(2)}(t)) &= \left(q^{(1)}(-t + \frac{T}{2}), -q^{(2)}(-t + \frac{T}{2})\right). \tag{10}
\end{align}

When we set $t = -t$,

\begin{align}
(q^{(1)}(-t), q^{(2)}(-t)) &= \left(q^{(1)}(t + \frac{T}{2}), -q^{(2)}(t + \frac{T}{2})\right) \\
&= -\left(q^{(1)}(t), q^{(2)}(t)\right).
\end{align}

It shows that $q(t)$ is an odd function. Let $t = 0$, one can deduce the following formulas from (10) and (11),

\begin{align}
(-q^{(1)}(0), q^{(2)}(0)) &= \left(q^{(1)}\left(\frac{T}{2}\right), q^{(2)}\left(\frac{T}{2}\right)\right), \\
(q^{(1)}(0), q^{(2)}(0)) &= \left(q^{(1)}\left(\frac{T}{2}\right), -q^{(2)}\left(\frac{T}{2}\right)\right),
\end{align}

that is

$$q(0) = q\left(\frac{T}{2}\right) = \theta.$$ 

This means the original point $\theta = (0, 0)$ is one of the points of self-intersections of $q(t)$. From the expression in (8), one has

\begin{align}
q_i(t) = q\left(t + (N - i)\frac{T}{N}\right) &= q\left(t - i\frac{T}{N}\right) \\
&= \left(q^{(1)}(t - i\frac{T}{N}), q^{(2)}(t - i\frac{T}{N})\right). \tag{12}
\end{align}

Now, in the case of even bodies on Figure-Eight, i.e. $N = 2k, k \in \mathbb{Z}^+$, the following is from (12),

\begin{align}
q_i(t) = \left(q^{(1)}\left(t - \frac{i}{2k}T\right), q^{(2)}\left(t - \frac{i}{2k}T\right)\right). \tag{13}
\end{align}
For all $i \leq k$, we consider
\[
q_{i+k}(t) = \left( q^{(1)} \left( t - \frac{i+k}{2k}T \right), q^{(2)} \left( t - \frac{i+k}{2k}T \right) \right)
\]
\[
= \left( q^{(1)} \left( t - \frac{i}{2k}T - \frac{T}{2} \right), q^{(2)} \left( t - \frac{i}{2k}T - \frac{T}{2} \right) \right).
\]

We notice this in (10)
\[
q_{i+k}(t) = \left( -q^{(1)} \left( t - \frac{i}{2k}T \right), q^{(2)} \left( t - \frac{i}{2k}T \right) \right).
\]
Together (13) with (14), it shows that when $t = \frac{i}{2k}T$,
\[
q_i \left( \frac{i}{2k}T \right) = q_{i+k} \left( \frac{i}{2k}T \right) = \theta.
\]
That is the $i$-th and $(i+k)$-th bodies will collide at the time $t = \frac{i}{2k}T$.

**Proposition 1.** With the assumption of (6), (8) and (9), even bodies on Figure-Eight orbit will collide on original point.

Since the constraint is more free, the shape of the possible orbit is more complex. It is reasonable to make a further strict assumption, a “simple” orbits constraint, on Figure-Eight. We set the further Simple Figure-Eight assumption:
\[
q(t_1) = q(t_2) \Rightarrow t_i \in \left\{ 0, \frac{T}{2} \right\}, \quad i = 1, 2.
\]
It means that the choreography orbit of $q$ has one and only one self-intersection on Figure-Eight.

Now we consider the case for the odd bodies on Figure-Eight. Let $N = 2k+1$, $k \in \mathbb{Z}^+$ and suppose that $q_i$ and $q_j$ collide at $t^*$, then
\[
q \left( t^* - \frac{i}{2k+1}T \right) = q \left( t^* - \frac{j}{2k+1}T \right).
\]
According to assumption (15), one has
\[
t^* - \frac{i}{2k+1}T, \quad t^* - \frac{j}{2k+1}T \in \left\{ 0, \frac{T}{2} \right\}.
\]
Without losing generality, set
\[ t^* - \frac{i}{2k+1} T = 0, \quad t^* - \frac{j}{2k+1} T = \frac{T}{2}, \]
which means
\[ \frac{i - j}{2k+1} T = \frac{T}{2}. \]
Since \( i, j, k \in \mathbb{Z}^+ \), the expression above is contradictory. Then we have

**Proposition 2.** With the assumption (6), (8), (9) and (15), odd bodies will not collide on simple figure-eight orbits.

It is obvious that assumption (15) can be generalized by

\[ q(t_1) = q(t_2) \Rightarrow t_i \in I \quad i = 1, 2, \quad (16) \]

where
\[ I = \{0\} \cup \left\{ \frac{(2r_1 - 1)T}{2r_2} \mid r_1, r_2 \in \mathbb{Z}^+, \frac{2r_1 - 1}{2r_2} \leq 1 \right\}. \]

**Theorem 1.** There is a dense subset in \([0, T]\), such that odd bodies on Figure-Eight without collision in it.

**Proof:** One can easily verify that \( I \) is dense in \([0, T]\), i.e., for \( \forall t_0 \in [0, T] \) and \( \forall \epsilon \), there exists \( t^* \in I \), such that \( |t_0 - t^*| \leq \epsilon \). Then \( I \) is at least one of the dense subset of \([0, T]\) we need. This completes the proof.

**Remark 1.** If we could make a further generalization for assumption (15) by

\[ q(t_1) = q(t_2) \Rightarrow t_i \in [0, T], \quad i = 1, 2, \]

we would have gotten a perfect result. But we can not get the generalization now. It means the Figure-Eight perhaps has some self-intersection points on time \( t^* \), which is not in \( I \), e.g. \( t^* \) is an irrational point.

**Remark 2.** It is obvious, when we record \( t \in \mathbb{R}/T\mathbb{Z} \), a loop space, we may have the same result by modifying

\[ I = \{0\} \cup \left\{ \frac{(2r_1 - 1)T}{2r_2} \mid r_1, r_2 \in \mathbb{Z}^+ \right\}. \]

**Figure 1** in Section 6 is one of our numerical examples for odd bodies on Figure-Eight.
4 Symmetries arising from Figure-Eight

4.1 From simple to double choreographies

We consider some two choreographies orbits, the Double-Eight orbits, which arises from Figure-Eight. With this idea, one can also figure out other more choreographies orbits. We describe the following Double-Eight orbits as an example for N-body problem.

Based on the aforementioned results, we set $N_1 \geq 3$ and $N_2 \geq 3$ are all odd in $\mathbb{Z}^+$, and the number of bodies is $N = N_1 + N_2$. Then we make a two-choreographies assumption: one Figure-Eight reclines on $x$-axis, the other on $y$-axis. That is, one Figure-Eight $q(t)$ satisfies

$$q_i(t) = q\left(t + (N_1 - i)\frac{T}{N_1} + \alpha_1\right), \quad i = 1, \cdots, N_1$$

$$\sum_{i=1}^{N_1} q_i(t) = 0,$$  \hspace{1cm} (17)

$$q(\sigma(t)) = \sigma(q(t)), \quad q(\tau(t)) = \tau(q(t)),$$  \hspace{1cm} (18)

the other Figure-Eight $Q(t)$ satisfies

$$Q_i(t) = Q\left(t + (N_2 - i)\frac{T}{N_2} + \alpha_2\right), \quad i = 1, \cdots, N_2,$$  \hspace{1cm} (19)

$$\sum_{i=1}^{N_2} Q_i(t) = 0,$$  \hspace{1cm} (20)

$$Q(\sigma(t)) = \tau(Q(t)), \quad Q(\tau(t)) = \sigma(Q(t)),$$  \hspace{1cm} (21)

where $\alpha_1$ and $\alpha_2$ can be regarded as the phase differences after equivariant shifts. In order to guarantee that any two bodies in two Figure-Eight separately will not collide on the origin point, specially, we set $\alpha_1 = 0$, $\alpha_2 = \frac{T}{K}$, where $K \in \mathbb{Z}^+$ is prime to $N_1$.

When we set $Q$ not satisfy (22) but (19), we will get the orbits with the constraint such that two Figure-Eights recline on the same $x$-axis (or $y$-axis). The Figures 2-22 in Section 6 are the numerical examples for Double-Eight orbits.
4.2 Mixed-symmetry constraints in $\mathbb{R}^3$

To construct some Mixed-symmetric constraints on simple choreography or two choreographies for $q = (q^{(1)}, q^{(2)}, q^{(3)}) \in \mathbb{R}^3$, we set $q^{(1)}$ and $q^{(2)}$ satisfy (17)-(19) on $x-y$ plane, and in $z$-coordinate, $q^{(3)}$, satisfy one of the following constraints:

- **Symmetry 1** (simple choreography orbit)

  $$q^{(3)}(t) = -q^{(3)} \left( t + \frac{T}{2} \right), \quad t \in [0, T].$$

- **Symmetry 2** (simple choreography orbit)

  $$q^{(3)}(t) = q^{(3)} \left( \frac{T}{2} - t \right), \quad t \in \left[ \frac{T}{4}, \frac{T}{2} \right],$$

  (23)

  and

  $$q^{(3)}(t) = -q^{(3)} \left( t + \frac{T}{2} \right), \quad t \in [0, T].$$

  (24)

- **Symmetry 3** (two choreographies orbits)

  Let $q = (q^{(1)}, q^{(2)}, q^{(3)})$ be a simple choreography and $Q = (Q^{(1)}, Q^{(2)}, Q^{(3)})$ another one, we set

  1. $q$ and $Q$ locate on the same Figure-Eight on $x-y$ plane;
  2. $q^{(3)}$ satisfy (23) and (24);
  3. $Q^{(3)} = -q^{(3)}$.

See Figures 23-30 for numerical examples.

4.3 Return to Figure-Eight

From the discussion in Section 3 and Section 4, we can choose many $\alpha_i$, $i = 1, \ldots, M, M \in \mathbb{Z}^+$, such that $M$ groups of odd bodies can be arranged on Figure-Eight without any collision in a dense subset of $[0, T]$.

**Theorem 2.** For all $N \geq 3$, $N \in \mathbb{Z}^+$, $N$ bodies can be arranged on Figure-Eight without any collision in a dense subset of $[0, T]$. 
Proof: If \( N \geq 3 \) is odd, by Theorem 1, the proof is as follows. If \( N \) is even, it can be decomposed by two odds \( N_1 \) and \( N_2 \), such that \( N = N_1 + N_2 \). Then we set

\[
q_i(t) = q \left( t + (N_1 - i) \frac{T}{N_1} + \alpha_1 \right), \quad i = 1, \ldots, N_1
\]

\[
Q_i(t) = q \left( t + (N_2 - i) \frac{T}{N_2} + \alpha_2 \right), \quad i = 1, \ldots, N_2,
\]

\[
\sum_{i=1}^{N_1} q_i(t) + \sum_{i=1}^{N_2} Q_i(t) = 0,
\]

\[
q(\sigma(t)) = \sigma(q(t)), \quad q(\tau(t)) = \tau(q(t)),
\]

\[
Q(\sigma(t)) = \sigma(Q(t)), \quad Q(\tau(t)) = \tau(Q(t)),
\]

\[
\alpha_1 = 0, \quad \alpha_2 = \frac{T}{K},
\]

where \( K \in \mathbb{Z}^+ \) is prime to \( N_1 \). Through the discussion in the counterpart of Section 4, one can easily verify that the theorem follows.

Figures 31–33 are numerical examples for Theorem 2.

5 Existence of symmetric orbits

In this section, we mainly consider the existence of generalized solutions and non-collision solutions for Mixed-symmetry and Double-Eight.

At first, we assemble some well-known facts from [7], [8], [9] and [14] about Sobolev space \( H^1(\mathbb{R}/TZ, (\mathbb{R}^d)^N) \) and the ordinary space \( C^2([0, T], (\mathbb{R}^d)^N) \).

- Let \( \xi_1, \xi_2 \in C^2([0, T], (\mathbb{R}^d)^N) \), then the inner product can be defined by

\[
\langle \xi_1, \xi_2 \rangle_0 = \int_0^T \langle \xi_1(t), \xi_2(t) \rangle dt,
\]

and the corresponding norm is \( \| \xi \|_0 \).

- Let \( \xi_1, \xi_2 \in H^1(\mathbb{R}/TZ, (\mathbb{R}^d)^N) \), then the inner product can be defined by

\[
\langle \xi_1, \xi_2 \rangle_1 = \langle \xi_1, \xi_2 \rangle_0 + \langle \dot{\xi}_1, \dot{\xi}_2 \rangle_0,
\]

and the corresponding norm is \( \| \xi \|_1 = \sqrt{\| \xi \|_0^2 + \| \dot{\xi} \|_0^2} \).
By Sobolev imbedding theorems, weak convergence in \( \| \cdot \|_1 \) implies uniform convergence, i.e., the weak \( \mathbb{H}^1 \) topology is stronger than the \( \mathbb{C}^0 \) topology.

Let \( J \) be a functional from \( \mathbb{H}^1 \) to \( \mathbb{R} \). A sequence \( \{k_q\}_{k=1}^{\infty} \) is called a \textit{minimizing sequence}, if

\[
J(k_q) \to \inf J, \quad \text{whenever} \quad k \to \infty.
\]

A functional \( J : \mathbb{H}^1 \to \mathbb{R} \) is \textit{weakly lower semi-continuous} if

\[
k_q \rightharpoonup q \quad \Rightarrow \quad \lim J(k_q) \geq J(q).
\]

A weakly lower semi-continuous functional \( J \) on \( \mathbb{H}^1 \) has a minimum in \( \mathbb{H}^1 \), if \( J \) has a bounded minimizing sequence.

Since it is difficult to exclude the collision without any more additional constraints for the orbits, we introduce the following two definitions, both of which partially exclude the collision in some sense.

We denote

\[
\Delta = \{ t \in [0, T] \mid q_i(t) = q_j(t), \text{ for some } i \neq j \}.
\]

**Definition 1.** We call \( q = (q_1, \cdots, q_N) \in \mathbb{C}^2([0, T], (\mathbb{R}^d)^N) \) a non-collision solution of (\ref{eq1}) and (\ref{eq2}), if \( \Delta = \phi \).

**Definition 2.** We call \( q = (q_1, \cdots, q_N) \in \mathbb{H}^1(\mathbb{R}/T\mathbb{Z}, (\mathbb{R}^d)^N) \) a generalized solution of (\ref{eq1}) and (\ref{eq2}), if

\begin{enumerate}
  \item \( \text{meas}(\Delta) = 0 \);
  \item \( q \) satisfy (\ref{eq1});
  \item
  \[
  \frac{1}{2} \sum_{i=1}^{N} m_i |\dot{q}_i(t)|^2 - U(q(t)) \equiv C,
  \]
\end{enumerate}

where \( t \in [0, T] \setminus \Delta \) and \( C \) is a constant.

In order to overcome the singularity of \( U \), the following condition is usually used as an efficient method in considering the existence of minimum of functional \( J \).
**Definition 3.** Let some \( V_{ij} \in C^1(R^d \setminus \{0\}, R) \), \( V_{ij} \to +\infty (\xi \to 0) \) and

\[
-U_{ij}(\xi) \geq |\nabla V_{ij}(\xi)|^2, \quad \forall \xi \in R^d \setminus \{0\},
\]

where \( |\xi| \) is small. Then we say \( U_{ij} \) satisfies Strong Force (SF) condition.

For the individuality of the Mixed-symmetric (the same as Double-Eight) solutions, we have to consider them respectively. But in the following theorem, we describe our result at one time for all the symmetric constraints, which have been mentioned therein.

**Theorem 3.** There are infinitely many generalized solutions for (1)-(2) with anyone of those symmetric constraints. Furthermore, if \( U \) satisfies (SF) condition, there will be infinitely many non-collision solutions.

The proof of theorem 3 is extremely along the line from Gordon [7, 8] and Zelati [14]. The key points locate on the estimation of \( |q(t)| \) and the construction of those nested intervals to cover \([0, T] \setminus \Delta\). In the ongoing proof, the existence of periodic solutions with symmetry 1 is proved as a candidate of those Mixed-symmetries and Double-Eight.

**Proof:** We set

\[
\Omega = \{ q \ | \ q = (q_1, \cdots, q_N) \in (\mathbb{R}^d)^N, q_i \neq q_j, \text{ for all } i \neq j \},
\]

\[
\Lambda = \{ q \ | \ q \in H^1(\mathbb{R}/T\mathbb{Z}, \Omega) \},
\]

\[
\Lambda_0 = \{ q \ | \ q \in \Lambda, \ q \text{ satisfies symmetry 1} \}.
\]

It’s easy to verify that the critical points of \( J \) on \( \Lambda \) are non-collision solutions of (1)-(2). And one can easily check that the critical points of \( J|_{\Lambda_0} \) are actually those of \( J \) on \( \Lambda \).

**Step1:** If \( U_{ij} \) does not satisfy (SF) condition, we can modify it by the following formula

\[
U_{ij}^\delta = U_{ij} - \frac{\phi(|q_j - q_i|)}{|q_j - q_i|^2},
\]

where \( \delta \) is small and

\[
\phi = \begin{cases}
0, & \text{when } |q_j - q_i| \leq \frac{\delta}{2}, \\
1, & \text{when } |q_j - q_i| \geq \delta.
\end{cases}
\]
Then the modified $U^\delta$ satisfies (SF) condition, and the corresponding Lagrangian functional is

$$J^\delta(q) = \frac{1}{2} \sum_{i=1}^{N} m_i \int_{0}^{T} |\dot{q}_i|^2 dt - \int_{0}^{T} U^\delta(q) dt,$$

where

$$U^\delta(q) = \frac{1}{2} \sum_{1 \leq i \neq j \leq N} U^\delta_{ij} (q_j - q_i).$$

**Step2:** In this step, we will show the existence of a minimum of $J$. Let the infimum of $J^\delta$ be

$$c_\delta = \inf \{ J^\delta(q) \mid q \in \Lambda_0 \}.$$

Consider a minimizing sequence $kq \in \Lambda_0$ such that $J^\delta(kq) \to c_\delta$ when $k \to +\infty$. Then, when $k$ is large enough, we have

$$J^\delta(kq(t)) \leq c_\delta + 1.$$

Hence for one term of $J^\delta$, we can deduce that

$$\int_{0}^{T} |k\dot{q}_i|^2 dt \leq \frac{2(c_\delta + 1)}{\bar{m}} \triangleq C, \quad \forall i,$$

where $C$ is a constance and $\bar{m} = \min\{m_1, \cdots, m_N\}$.

We notice that from (10)-(11), one can deduce the period of $q^{(2)}$ is $\frac{T}{2}$, and $q(t)$ is symmetrical about $q^{(2)}$-axis, then one has

$$q^{(2)}(t) = -q^{(2)} \left( t + \frac{T}{4} \right).$$

(26)
Hence we can estimate the norm of $q$,

$$
|q(t)| \leq |q^{(1)}(t)| + |q^{(2)}(t)| + |q^{(3)}(t)| \\
= \frac{1}{2} \left| q^{(1)}(t) - q^{(1)}(t + \frac{T}{2}) \right| + \frac{1}{2} \left| q^{(2)}(t) - q^{(2)}(t + \frac{T}{4}) \right| \\
+ \frac{1}{2} \left| q^{(3)}(t) - q^{(3)}(t + \frac{T}{2}) \right| \\
= \frac{1}{2} \left| \int_{t}^{t+\frac{T}{2}} \ddot{q}^{(1)}(s) ds \right| + \frac{1}{2} \left| \int_{t}^{t+\frac{T}{4}} \ddot{q}^{(2)}(s) ds \right| + \frac{1}{2} \left| \int_{t}^{t+\frac{T}{2}} \ddot{q}^{(3)}(s) ds \right| \\
\leq \frac{1}{2} \int_{t}^{t+\frac{T}{2}} \left| \ddot{q}^{(1)}(s) \right| ds + \frac{1}{2} \int_{t}^{t+\frac{T}{4}} \left| \ddot{q}^{(2)}(s) \right| ds + \frac{1}{2} \int_{t}^{t+\frac{T}{2}} \left| \ddot{q}^{(3)}(s) \right| ds \\
\leq \frac{1}{2} \left( \int_{t}^{t+\frac{T}{2}} \left| \dddot{q}^{(1)}(s) \right|^2 ds \right)^{\frac{1}{2}} \left( \frac{T}{2} \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \left( \int_{t}^{t+\frac{T}{4}} \left| \dddot{q}^{(2)}(s) \right|^2 ds \right)^{\frac{1}{2}} \left( \frac{T}{4} \right)^{\frac{1}{2}} \\
+ \frac{1}{2} \left( \int_{t}^{t+\frac{T}{2}} \left| \dddot{q}^{(3)}(s) \right|^2 ds \right)^{\frac{1}{2}} \left( \frac{T}{2} \right)^{\frac{1}{2}} \\
\leq \frac{\sqrt{T}}{4} \left( \left( \int_{0}^{T} \left| \dddot{q}^{(1)}(t) \right|^2 dt \right)^{\frac{1}{2}} + \left( \int_{0}^{T} \left| \dddot{q}^{(2)}(t) \right|^2 dt \right)^{\frac{1}{2}} \right) \\
+ \left( \int_{0}^{T} \left| \dddot{q}^{(3)}(t) \right|^2 dt \right)^{\frac{1}{2}} \\
\leq \frac{\sqrt{3T}}{4} \left( \int_{0}^{T} \left| \dddot{q}^{(1)}(t) \right|^2 dt + \int_{0}^{T} \left| \dddot{q}^{(2)}(t) \right|^2 dt + \int_{0}^{T} \left| \dddot{q}^{(3)}(t) \right|^2 dt \right)^{\frac{1}{2}} \\
= \frac{\sqrt{3T}}{4} \left( \int_{0}^{T} \left| \dddot{q}(t) \right|^2 dt \right)^{\frac{1}{2}}.
$$

Then for $\forall i$, $\forall k$ (sufficiently large), the norm for one of the minimizing sequence is

$$
\| k q \|_1 = \sqrt{\| k q \|_2^2 + \| k \dddot{q} \|_0^2} \leq \frac{1}{4} \sqrt{3TC(T+1)}.
$$
This implies the existence of $\bar{q}^\delta = (\bar{q}_1^\delta, \ldots, \bar{q}_N^\delta)$ with $\bar{q}_i^\delta \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^d)$, for all $i = 1, \ldots, N$ such that

\[ kq_i \to \bar{q}_i^\delta, \quad \forall \ kq_i \in H^1(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^d), \]

and

\[ kq_i \to \bar{q}_i^\delta, \quad \forall \ kq_i \in C^0(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^d). \]

On condition (SF), it follows that $J(kq) \to +\infty$ for every sequence $\{kq\}$ such that $kq \to \bar{q}$ weakly in $H^1$ and strongly in $C^0$ if $\bar{q} \in \partial\Lambda$. This proves that $\bar{q}^\delta \in \Lambda_0$ is a minimum for $J^\delta$ on $\Lambda_0$. This minimum is a non-collision solution of (1)-(2). Then we have proved that (1)-(2) has at least one non-collision solution if condition (SF) is satisfied.

**Step3:** This step is the same as the corresponding part of the proof for Theorem 1.1 in [14], so we only describe the sketch: We can find a generalized solution $q^*$ of (1)-(2) by constructing compact nested interval cover of $[0, T] \setminus \Delta$ to show $U^\delta \to U$ and $q^\delta \to q^*$ when $\delta \to 0$. This process shows that $q^*$ satisfies definition 2.

We can also prove $q_{T/z}$ is a $T$-periodic solution of (1)-(2) if $q$ is a $T$-periodic solution, where $z$ is a positive integer, then we can find infinitely many $T$-periodic solutions of (1)-(2). This completes the proof.

### 6 Numerical Examples

In this section, we along the line in Theorem 3, finding the critical points of $J$, to list some of numerical examples for those theoretic consideration. In the following figures, $m$ is the masses, some of which are divided into two parts with a semicolon to distinguish the masses of $N_1$ bodies and $N_2$ bodies, $\alpha_1$ and $\alpha_2$ as defined above. In those figures with sub-figures, the top left sub-figure is the projection of the orbits on $x$-$y$ plane, the top right is the projection of the orbits on $y$-$z$ plane, the bottom left is the projection of the orbits on $x$-$z$ plane, and the bottom right is the orbits in space.

During our numerical study, it shows some interesting phenomena. The followings are some of them for further consideration:

- When we set some other constraints on $q^{(3)}$ at will, e.g. the following symmetries 4-6, we have gotten some interesting figures numerically (see Figure 33).
\( q^{(3)}(t) = q^{(3)} \left( t + \frac{T}{4} \right), \quad t \in \left[ 0, \frac{T}{4} \right), \quad (27) \)
\[ q^{(3)}(t) = -q^{(3)} \left( t + \frac{T}{4} \right), \quad t \in \left[ \frac{T}{2}, \frac{3T}{4} \right]. \quad (28) \]

- symmetry 5
\[ q^{(3)}(t) = q^{(3)} \left( t + \frac{T}{4} \right), \quad t \in \left[ 0, \frac{T}{4} \right) \cup \left[ \frac{T}{2}, \frac{3T}{4} \right]. \quad (29) \]

- symmetry 6
\[ q^{(3)}(t) = q^{(3)} \left( t - \frac{3T}{4} \right), \quad t \in \left[ \frac{3T}{4}, T \right], \quad (30) \]
\[ q^{(3)}(t) = q^{(3)} \left( t - \frac{T}{4} \right), \quad t \in \left[ \frac{T}{2}, \frac{3T}{4} \right). \quad (31) \]

See Figures 34-36 for numerical examples.

- In Figure-Eight, we can arrange more bodies on it, but we can not find the “8” including more self-intersections, because the dense subset \( I \) is so dense that we can not separate those points from others by a modern computer.

- In Double-Eight, the shape of those two “8” depends on the proportion of the masses. The orbit with less masses looks more slim, and the bodies on it shuttle rapidly across the interspace of the bodies on the other orbit.

- In Double-Eight, the shape of the orbits also depends on the phase difference and the initial values of functional \( J \).

- In symmetry 1, 4, 5 and 6, there are two of the projections of the orbits become slimmer and slimmer. It seems that one of the constraints acting on three projections is “stronger” than the other two.

- In symmetry 2 and 3, Theorem 2 guarantees that we can arrange many bodies on those orbits, no matter even or odd.

The following figures are numerical examples for Theorem 2.
**Figure 1:** 13 bodies on Figure-Eight.

**Figure 2:** $m = \begin{bmatrix} 1, 1, 1, 1, 1 \end{bmatrix}$

$\alpha_1 = 0, \alpha_2 = \frac{T}{6}$

**Figure 3:** $m = \begin{bmatrix} 1, 1, 1, 10, 10, 10 \end{bmatrix}$

$\alpha_1 = 0, \alpha_2 = \frac{T}{6}$

**Figure 4:** $m = \begin{bmatrix} 1, 1, 1, 1, 1, 1, 1, 1 \end{bmatrix}$

$\alpha_1 = 0, \alpha_2 = \frac{T}{6}$

**Figure 5:** $m = \begin{bmatrix} 1, 1, 1, 10, 10, 10, 10, 10 \end{bmatrix}$

$\alpha_1 = 0, \alpha_2 = \frac{T}{6}$

**Figure 6:** $m = \begin{bmatrix} 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \end{bmatrix}$

$\alpha_1 = 0, \alpha_2 = \frac{T}{10}$

**Figure 7:** $m = \begin{bmatrix} 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \end{bmatrix}$

$\alpha_1 = 0, \alpha_2 = \frac{T}{10}$
Figure 8: $m=[1,1,1,1;1,1,1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{10}$.

Figure 9: $m=[1,1,1;1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{6}$.

Figure 10: $m=[1,1,1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{12}$.

Figure 11: $m=[1,1,1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{12}$.

Figure 12: $m=[1,1,5,5,5]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{6}$.

Figure 13: $m=[1,1,5,5,5]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{12}$.

Figure 14: $m=[1,1,1,1,1;1,1,1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{12}$.

Figure 15: $m=[1,1,1,1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{20}$.

Figure 16: $m=[1,1,1,1,1,1]$; 
$\alpha_1 = 0, \alpha_2 = \frac{T}{20}$.
Figure 17: \( m=\begin{bmatrix} 10,10,10;1,1,1,1,1 \end{bmatrix} \); \( \alpha_1 = 0, \alpha_2 = \frac{T}{10} \).

Figure 18: \( m=\begin{bmatrix} 10,10,10;1,1,1,1,1 \end{bmatrix} \); \( \alpha_1 = 0, \alpha_2 = \frac{T}{10} \).

Figure 19: \( m=\begin{bmatrix} 1,1,1;5,5,5,5,5 \end{bmatrix} \); \( \alpha_1 = 0, \alpha_2 = \frac{T}{10} \).

Figure 20: \( m=\begin{bmatrix} 1,1,1;5,5,5,5,5 \end{bmatrix} \); \( \alpha_1 = 0, \alpha_2 = \frac{T}{10} \).

Figure 21: \( m=\begin{bmatrix} 1,1,1;5,5,5,5,5 \end{bmatrix} \); \( \alpha_1 = 0, \alpha_2 = \frac{T}{10} \).

Figure 22: \( m=\begin{bmatrix} 1,1,1;5,5,5,5,5 \end{bmatrix} \); \( \alpha_1 = 0, \alpha_2 = \frac{T}{10} \).

Figure 23: 11 bodies with symmetry 1, \( m = [1,1,1,1,1,1,1,1,1,1,1] \).

Figure 24: The same orbit as in Figure 23, with different scale.
Figure 25: 3 bodies with symmetry 2: 
\( m = [1, 1, 1] \).

Figure 26: 5 bodies with symmetry 2: 
\( m = [1, 1, 1, 1, 1] \).

Figure 27: 9 bodies with symmetry 2: 
\( m = [1, 1, 1, 1, 1, 1, 1, 1, 1] \).

Figure 28: 6 bodies with symmetry 3: 
\( m = [1, 1, 1, 1, 1, 1], \alpha_1 = 0, \alpha_2 = \frac{T}{8} \).
Figure 29: 8 bodies with symmetry 3:
\[ m = [1, 1, 1; 1, 1, 1], \alpha_1 = 0, \alpha_2 = \frac{T}{8} \]

Figure 30: 8 bodies with symmetry 3:
\[ m = [1, 1, 1; 1, 1, 1], \alpha_1 = 0, \alpha_2 = \frac{T}{12} \]

Figure 31: 6 bodies on Figure-Eight:
\[ m = [1, 1, 1; 1, 1, 1], \alpha_1 = 0, \alpha_2 = \frac{T}{12} \]

Figure 32: 8 bodies on Figure-Eight:
\[ m = [1, 1, 1; 1, 1, 1, 1], \alpha_1 = 0, \alpha_2 = \frac{T}{20} \]

Figure 33: 12 bodies on Figure-Eight:
\[ m = [1, 1, 1; 1, 1, 1, 1, 1, 1], \alpha_1 = 0, \alpha_2 = \frac{T}{36} \]
Figure 34: 12 bodies with symmetry 4 for equal masses. (The left figure and right one are in different scales.)

Figure 35: 12 bodies with symmetry 5 for equal masses. (The left figure and right one are in different scales.)

Figure 36: 11 bodies with symmetry 6 for equal masses. (The left figure and right one are in different scales.)
A  The estimation of $|q|$ for other symmetric constraints

For Double-Eight:

$$|q(t)| \leq |q^{(1)}(t)| + |q^{(2)}(t)|$$

$$= \frac{1}{2} \left| q^{(1)}(t) - q^{(1)}(t + \frac{T}{2}) \right| + \frac{1}{2} \left| q^{(2)}(t) - q^{(2)}(t + \frac{T}{4}) \right|$$

$$= \frac{1}{2} \left| \int_{t}^{t+\frac{T}{4}} q^{(1)}(s)ds \right| + \frac{1}{2} \left| \int_{t}^{t+\frac{T}{4}} q^{(2)}(s)ds \right|$$

$$\leq \frac{1}{2} \int_{t}^{t+\frac{T}{4}} |q^{(1)}(s)| ds + \frac{1}{2} \int_{t}^{t+\frac{T}{4}} |q^{(2)}(s)| ds$$

$$\leq \frac{1}{2} \left( \int_{t}^{t+\frac{T}{4}} |\dot{q}^{(1)}(s)|^2 ds \right)^{\frac{1}{2}} \left( \frac{T}{2} \right)^{\frac{1}{2}}$$

$$+ \frac{1}{2} \left( \int_{t}^{t+\frac{T}{4}} |\dot{q}^{(2)}(s)|^2 ds \right)^{\frac{1}{2}} \left( \frac{T}{4} \right)^{\frac{1}{2}}$$

$$\leq \sqrt{\frac{T}{4}} \left( \left( \int_{0}^{T} |\dot{q}^{(1)}(t)|^2 dt \right)^{\frac{1}{2}} + \left( \int_{0}^{T} |\dot{q}^{(2)}(t)|^2 dt \right)^{\frac{1}{2}} \right)$$

$$\leq \sqrt{\frac{3T}{4}} \left( \int_{0}^{T} |\dot{q}^{(1)}(t)|^2 dt + \int_{0}^{T} |\dot{q}^{(2)}(t)|^2 dt \right)^{\frac{1}{2}}$$

$$= \sqrt{\frac{3T}{4}} \left( \int_{0}^{T} |\dot{q}(t)|^2 dt \right)^{\frac{1}{2}}.$$

For $Q(t)$ in Double-Eight, we have the same estimation.

For symmetry 2: Since (24) is the same condition as in symmetry 1, we can at least draw the same conclusion as in symmetry 1, i.e.

$$|q(t)| \leq \sqrt{\frac{3T}{4}} \left( \int_{0}^{T} |\dot{q}(t)|^2 dt \right)^{\frac{1}{2}}.$$

For symmetry 3: We set $X = (q, Q)$, it is easy to verify that

$$|X(t)| \leq |q(t)| + |Q(t)| \leq \sqrt{\frac{3T}{2}} \left( \int_{0}^{T} |\dot{q}(t)|^2 dt \right)^{\frac{1}{2}}.$$
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