Trapping photons by a line singularity

M. Arik and O. Delice
Bogazici University, Department of Physics, Bebek, Istanbul, Turkey
E-mail: arikm@boun.edu.tr, odelice@boun.edu.tr

Abstract. We present cylindrically symmetric, static solutions of the Einstein field equations around a line singularity such that the energy momentum tensor corresponds to infinitely thin photonic shells. Positivity of the energy density of the thin shell and the line singularity is discussed. It is also shown that thick shells containing mostly radiation are possible by a numerical solution.

PACS numbers: 04.20, 11.27.+d

1. Introduction

The cylindrically symmetric solutions of the Einstein field equations have always been one of the more popular areas of research in general relativity. The static and rotating cylinders, collapsing and expanding cylindrical shells as well as their static counterparts have been studied in the literature. Although such objects are awaiting astrophysical discovery it is important to investigate different types of objects since we do not yet know which type may be realized in the cosmos. Some examples of these solutions are given in [1] - [12]. A review of static solutions has been given in [4]. [13] and [14] present solutions describing two counter-propagating beams of light confined to a cylindrical region. The energy momentum tensor is nonvanishing inside the cylinder and the solution is smoothly matched to the outside Levi-Civita metric. In this paper we will investigate solutions of the Einstein equations giving static cylindrical thin shells composed of massless particles around a line singularity. We will also present an approximate solution giving static cylindrical thick shell composed of equal amount of oppositely rotating photons along the angular dimension $\phi$ around a line singularity.

2. Properties of the Cylindrically Symmetric Static Spacetime

The general cylindrically symmetric static vacuum spacetime satisfying Einstein’s field equations is given by the Levi-Civita metric [3], which can be written in the form [3]

$$ds^2 = -\rho^{4\sigma} dt^2 + \rho^{4\sigma(2\sigma - 1)} (d\rho^2 + P^2 d\rho^2) + Q^2 \rho^{2(1 - 2\sigma)} dq^2$$

(1)

where $t$ and $\rho$ are the time and the radial coordinates with the range $-\infty < t < \infty$, $0 \leq \rho < \infty$ and $\sigma$, $P$ and $Q$ are real constants. The behaviour of the metric components determine the nature of the coordinates $p$ and $q$. Transforming the radius $\rho$ into a proper radius $r$ by defining

$$dr = \rho^{2\sigma(2\sigma - 1)} d\rho$$

(2)
Trapping photons by a line singularity

puts the Levi-Civita metric \( ds^2 = -R^{4/\sigma} dt^2 + dr^2 + P^2 R^{4(2\sigma-1)/\sigma} dp^2 + Q^2 R^{2(1-2\sigma)/\sigma} dq^2 \) (3)

where
\[ \rho = R^{1/N}, \quad R = Nr, \quad N = 4\sigma^2 - 2\sigma + 1. \] (4)

One of the coordinates \( p \) and \( q \) has to be interpreted as the axis of symmetry and the other as the angle about this axis. If we fix the angular coordinate to a finite range then one of the constants \( P \) and \( Q \) can be transformed away by a scale transformation depending on the behaviour of the coordinates \( p \) and \( q \). This leaves the metric with only two independent parameters. The parameter \( \sigma \) is related to the energy of the cylindrical source \( [2] \) whereas the surviving one of the parameters \( P \) and \( Q \) is related to the topology of spacetime \( [3] \).

It is generally accepted that for small values of the parameter \( \sigma \) \( (\sigma < \frac{1}{4}) \), this metric corresponds to cylindrically symmetric vacuum spacetime with \( p \) as \( z \) coordinate and \( q \) as angular coordinate. Recently, this limit was extended to the range \( 0 \leq \sigma < \frac{1}{2} \) \( [4] \). Finally it was shown in \( [5] \) and \( [6] \) that for \( 0 \leq \sigma < \infty \) the Levi-Civita metric describes the cylindrically symmetric vacuum spacetime provided that at \( \sigma = \frac{1}{2} \) the coordinates \( p \) and \( q \) interchange their roles, that is, for \( \sigma > \frac{1}{2} \), \( p \) becomes the angular coordinate and \( q \) becomes the \( z \) coordinate. For \( \sigma = \frac{1}{2} \), neither \( p \) nor \( q \) is entitled to be an angular coordinate, and the three coordinates \( (r, p, q) \) are better visualized as Cartesian coordinates \( x, y \) and \( z \) \( [5], [16] \). The \( \sigma < 0 \) case is isotropic to the \( \sigma > 0 \) case.

In this paper we will thus use the cylindrically symmetric vacuum metric in Kasner form. After rescaling the coordinates \( t \) and \( z \) and changing the constant in \( g_{\phi\phi} \) to give correct dimensions to the metric components we obtain
\[ ds^2 = -\left(\frac{r}{r_0}\right)^{2a} dt^2 + dr^2 + \left(\frac{r}{r_0}\right)^{2b} dz^2 + \left(\frac{r}{r_0}\right)^{2c} r_0^2 \alpha^2 d\phi^2, \] (5)

where \( \alpha, a, b, c \) are real constants, \( r \) is the radial coordinate, \( z \) is the axis of the line singularity and \( \phi \) is the angular coordinate with the range \( 0 \leq \phi \leq 2\pi \). The relation between parameters of the Levi-Civita form of the metric and its equivalent Kasner form is given by
\[ a = \frac{2\sigma}{N}, \quad b = \frac{-2\sigma(1-2\sigma)}{N}, \quad c = \frac{1-2\sigma}{N}, \quad N = 4\sigma^2 - 2\sigma + 1. \] (6)

The dependence of \( a, b \) and \( c \) on \( \sigma \) is shown in Figure 1.

This metric is cylindrically symmetric and represents the exterior field of the line singularity at \( r = 0 \) \( [5] \). The Einstein tensor for this metric, apart from the trivial solution \( (a = b = c = 0) \), gives vacuum solutions of Einstein equations with the constraints
\[ a + b + c = a^2 + b^2 + c^2 = 1, \] (7)

so only one of the parameters \( a, b, c \) is free. Choosing the free parameter as \( b \) than \( a \) and \( c \) are given by
\[ a, c = \frac{(1-b) \pm \sqrt{(1-b)(1+3b)}}{2} \] (8)

where \( -\frac{1}{3} \leq b \leq 1 \). For every value of \( b \), \( a \) and \( c \) can take two values. This is shown in Figure 2. In general, the choice of \( b \) as free parameter is arbitrary, so \( a, b, c \) can
Trapping photons by a line singularity

Figure 1. The dependence of $a, b, c$ on $\sigma$

Figure 2. The parameters $a$ and $c$ with $b$ as a free parameter

take values in between $(-\frac{1}{3} \leq a, b, c \leq 1)$ but only one of them can be negative at the same time. Note that, the Riemann tensor vanishes only for the case where one of the parameters is equal to one and the others are equal to zero.

The Levi-Civita metric (3) and its Kasner form (5) are equivalent. We thus discuss the relations between $\sigma$ and $a, b, c$. When $0 < \sigma < \frac{1}{2}$, $b < 0, c > 0$ and when $\sigma > \frac{1}{2}$, $b > 0, c < 0$. Since for $\sigma$ positive one always has $a \geq 0$, (Figure 1) one concludes that for the Kasner form of the metric (5), for positive $\sigma$, the coordinate corresponding to the parameter whose value is negative has the role of $z$ coordinate and whose value is positive has the role of angular coordinate of cylindrical coordinates. To avoid confusion, for the rest of the paper, either we should consider only $0 \leq \sigma \leq 1/2$ which $b$ is always negative or we should consider $0 \leq \sigma < \infty$ which $b$ and $c$ change sign at $\sigma = 1/2$ (Figure 1). We choose latter and unless stated explicitly, we write our expressions for $b < 0$, but we always have in mind that, for $b$ positive we have to replace $b$ with $c$ in all metrics, expressions and figures without writing them explicitly in our expressions. For negative $\sigma$, same change of the roles of the coordinates may occur since for small $\sigma$, $r^{2c} \approx r^2$ and for large $\sigma$, $r^{2b} \approx r^2$. 
Note that the Levi-Civita metric in Kasner form (5) has another physically relevant parameter $\alpha$ whose value is relevant to the topological defects in cylindrical coordinates. For this metric, for $\sigma$ vanishing $a$ and $b$ vanish and $c$ becomes unity. On the other hand for $\sigma = \infty$, $a$ and $c$ vanish and $b$ becomes unity. In these cases our metric becomes:

$$ds^2 = -dt^2 + dr^2 + dz^2 + \alpha^2 r^2 d\phi^2,$$

with $0 \leq \phi \leq 2\pi$. This metric is flat and locally Minkowskian but globally it has different topology and represents a conical spacetime for $\alpha \neq 1$. For $\alpha < 1$ this metric gives the exterior field of an infinitely long straight cosmic string lying along the $z$ direction. Only $\alpha = 1$ corresponds to flat and globally Minkowskian spacetime for $\sigma = 0$ or $\sigma = \infty$.

We now turn to discuss the Newtonian Limit of the cylindrically symmetric static Levi-Civita metric written in Kasner form (5). The Newtonian limit is given by $a = k \ll 1$. Using the constraints on $b$ & $c$ (7)

$$b + c = 1 - k$$

$$b^2 + c^2 = 1 - k$$

there are two limits for $b$, $c$ in which one of them goes to zero and the other to one. To identify the angular coordinate correctly we have to choose $b \to 0$ and $c \to 1$. Then

$$(r/r_0)^a = 1 + k \ln(r/r_0) + O(k^2)$$

$$(r/r_0)^b = 1 - k \ln(r/r_0) + O(k^2)$$

$$(r/r_0)^c = r/r_0 + O(k^2)$$

Hence one obtains

$$ds^2 = -(1 + k \ln(r/r_0))^2 dt^2 + dr^2 + (1 - k \ln(r/r_0))^2 dz^2 + \alpha^2 r^2 d\phi^2. (15)$$

Calculating the Einstein tensor for this metric verifies that it is zero up to order $k^2$. Thus one can identify the Newtonian potential per unit mass as $V = k \ln(r/r_0)$. For positive $k$ this corresponds to an attractive force field while for negative $k$ this corresponds to a repulsive force field $\vec{g}$,

$$\vec{g} = -\vec{\nabla}V = -k \frac{\vec{r}}{r^2}. (16)$$

A nonrelativistic particle in a circular orbit in this force field has centripetal acceleration

$$\frac{k}{r} = \frac{v^2}{r},$$

so that the particle moves with constant velocity independent of the radius. It is plausible that this generalizes to light which always moves with constant velocity.

Note that the Levi-Civita metric in Kasner form (5) is studied by Israel [8]. He showed that the singularity at the $r = 0$ represents the field of an infinite rod. He calculated the line energy-momentum tensor density and showed that the source has positive pressures and negative energy density. However the effective gravitational mass per unit length is calculated as $\frac{4\pi a}{r}$ in our notation and is positive for positive $a$ (or $\sigma$) and negative for negative $a$. His results are in accordance with the Newtonian
limit of the metric $\delta_r$ where when $a$ is small and positive the field is attractive while when $a$ is small and negative the force field is repulsive.

For general relativity the metric $\delta_r$ admits helical null geodesics for a certain range of $\sigma$ with the angular velocity $\omega$ and the pitch velocity $v$ (the constant velocity along $z$ direction of the particles following a helical path). They are given by

$$\omega^2 = \left(\frac{a-b}{c-b}\right) \frac{1}{\alpha^2 r_0^2} \left(\frac{r}{r_0}\right)^{2(a-c)},$$

$$v^2 = \left(\frac{c-a}{c-b}\right) \left(\frac{r}{r_0}\right)^{2(a-b)}.$$  \hspace{1cm} (18)

For $b < 0$ ($\sigma < 1/2$), since $|c| > |b|$, there exist helical null geodesics only for $c \geq a$ ($0 < \sigma \leq 1/4$) and for $a = c = 2/3$ ($\sigma = 1/4$) they become circular $\delta_r$. For $c < 0$, $b > 0$ ($\sigma > 1/2$) helical null geodesics, which are circular for $a = b = 2/3$ ($\sigma = 1$), exist for $b \geq a$ ($\sigma \geq 1$) (Figure 1). Motivated by this, in section 2 we will look for solutions of the Einstein equations with an infinitely thin shell at $r = r_1$. On the shell, the only nonvanishing components of the energy momentum tensor satisfy $T_{\mu}^{\mu} = 0$ so that this shell can be interpreted as radiation trapped in the gravitational field of the line singularity at $r = 0$. The metric which is Ricci flat for $0 < r < r_1$ and for $r > r_1$ is continuous at $r = r_1$ such that $T_{00}$ and $T_{\phi\phi}$, the only nonvanishing components of the energy-momentum tensor have Dirac $\delta$-function singularities at $r = r_1$. In section 4 we will present an approximate thick shell solution for the $T_{00} = T_{\phi\phi}$ case such that the Einstein tensor is finite and nonvanishing for $r_1 < r < r_2$. We find that $T_{00} = T_{\phi\phi}$ only approximately. The other diagonal components of the energy momentum tensor also pick up small values.

3. Infinitely thin photonic shells around the line singularity

3.1. General case: A thin shell with counter moving photons along a helical path.

There are several methods for calculating the Einstein tensor for thin shell sources but the conditions the metric should satisfy are the same. The metric should be continuous everywhere but its first derivatives may be discontinuous on the shell and these discontinuities give rise to an infinitely thin shell $\delta_r$. The interior and exterior regions of the infinitely long static thin shell with radius $r_1$. These are given by:

$$ds^2_- = -(\frac{r}{r_1})^{2a'} dt^2 + dr^2 + (\frac{r}{r_1})^{2b'} dz^2 + (\frac{r}{r_1})^{2c'} \alpha^2 r_1^2 d\phi^2 \quad (r < r_1)$$  \hspace{1cm} (20)

and

$$ds^2_+ = -(\frac{r}{r_1})^{2a} dt^2 + dr^2 + (\frac{r}{r_1})^{2b} dz^2 + (\frac{r}{r_1})^{2c} \alpha^2 r_1^2 d\phi^2 \quad (r > r_1).$$  \hspace{1cm} (21)

where the constant parameters $a, b, c$ and $a', b', c'$ satisfy the relations (18) so these metrics are both Ricci flat. As explained in the introduction we must have $b, b' < 0$ so that the $z$ coordinate is identified correctly.

We can combine (20) and (21) in the form:

$$ds^2 = -A^2(r)dt^2 + dr^2 + B^2(r)dz^2 + D^2(r)d\phi^2$$  \hspace{1cm} (22)
with
\[ A(r) = \frac{r}{r_1} a' \theta(r_1 - r) + \frac{r}{r_1} \theta(r - r_1), \quad (23) \]
\[ B(r) = \frac{r}{r_1} b' \theta(r_1 - r) + \frac{r}{r_1} \theta(r - r_1), \quad (24) \]
and
\[ D(r) = \left[ \frac{r}{r_1} c' \theta(r_1 - r) + \frac{r}{r_1} c \theta(r - r_1) \right] r_1 \delta, \quad (25) \]
where \( \theta(x - x_0) \) is the Heaviside step function with
\[ \theta(x - x_0) = 0, \quad x < x_0 \]
\[ \theta(x - x_0) = 1, \quad x \geq x_0 \quad (26) \]

At \( r = r_1 \) this metric is continuous but its first derivatives with respect to \( r \) are discontinuous. The nonzero components of the Einstein tensor for the metric (22) are given by
\[ G_{00} = -\left( \frac{B_{rr}}{B} + \frac{D_{rr}}{D} + \frac{B_r D_r}{B D} \right), \quad (27) \]
\[ G_{rr} = G_{11} = \frac{A_r B_r}{AB} + \frac{A_r D_r}{AD} + \frac{B_r D_r}{BD}, \quad (28) \]
\[ G_{zz} = G_{22} = \frac{A_r}{A} + \frac{D_{rr}}{D} + \frac{A_r D_r}{AD}, \quad (29) \]
\[ G_{\phi\phi} = G_{33} = \frac{A_r}{A} + \frac{D_{rr}}{B} + \frac{A_r B_r}{AB}, \quad (30) \]
where subscripts denote the partial derivatives. Since the interior and the exterior regions of the shell is vacuum, the only surviving terms are the terms which contain Dirac delta functions which give the energy momentum tensor of the shell.

So for \( b, (b') < 0 \) case the nonzero elements of \( G_{\mu\nu} \) are
\[ G_{00} = -\frac{b - b' + c - c'}{r_1} \delta(r - r_1) = \frac{a - a'}{r_1} \delta(r - r_1), \quad (31) \]
\[ G_{22} = \frac{a - a' + c - c'}{r_1} \delta(r - r_1) = \frac{b' - b}{r_1} \delta(r - r_1), \quad (32) \]
\[ G_{33} = \frac{a - a' + b - b'}{r_1} \delta(r - r_1) = \frac{c' - c}{r_1} \delta(r - r_1). \quad (33) \]

For the Einstein equation
\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \quad (34) \]
we can choose the energy momentum tensor of the shell in the form
\[ T_{\mu\nu} = \text{diag}(\rho, p_r, p_z, p_\phi) \quad (35) \]
where \( \rho \) is the energy density and \( p_i \ (i = 1, 2, 3) \) are the principal pressures. Since we have \( T_0^0 = -T_{00} \) and \( T_j^i = T_{ij} \ (i,j = r, z, \phi) \), using (3) one can show that the energy momentum tensor of the shell satisfies the condition \( T_{\mu}^\mu = 0 \) and this result can be interpreted as an infinitely long thin shell along the \( z \) direction with radius \( r_1 \) composed of equal amount of oppositely moving photons along a helical direction. This helical motion gives rise to pressures in the \( \phi \) and \( z \) directions with the equation of state \( \rho = p_z + p_\phi \). Thus if one chooses the interior and the exterior metrics of
the shell as Levi-Civita metrics in Kasner form \( \text{[20][21]} \) then the shell is necessarily composed of ordinary particles.

Energy per unit length of the shell is given by:

\[
\mu = \alpha(a - a').
\]  

(36)

For \((a - a') > 0\) the shell has positive energy density. Since the interior metric is of Kasner form, we have a line singularity at \(r = 0\). The singularity has positive effective mass density for \(a'\) positive and negative effective mass density for negative \(a'\). The only singularity free metric is flat Minkowski metric and if we take interior metric \(\text{[20]}\) as Minkowski metric \(\text{[9]}\) \((a' = b' = 0, c' = 1, a = 1)\) then our results are in accordance with \(\text{[5]}\) and \(\text{[12]}\). In this case the shell has positive energy density for \(a\) positive and satisfies all energy conditions for \(|a| > |b|\).

### 3.2. A thin shell with counter-rotating photons

Now we would like to discuss the case which may be called the photon cylinder with counter-rotating photons. We find a static cylindrical shell at \(r = r_1\) with \(T_{00} = \rho = p_\phi = T_{22}\) with other components of the energy momentum tensor vanishing. For comparison we note that for light in a laser tube and for two counter-propagating beams of light \(\text{[3][4]}\) the equation of state is \(\rho = p_\phi\). We take \(G_{22} = 0\) in \(\text{[22]}\), which gives:

\[
b = b'.
\]

(37)

Taking \(b\) as the free parameter, the parameters \(a, c, a'\) and \(c'\) are given by

\[
a, c = a', c' = \chi_+, \chi_- = \frac{(1 - b) \pm \sqrt{(1 - b)(1 + 3b)}}{2}.
\]

(38)

For this case to have positive energy density on the shell we need \(a - a' > 0\). Since \(a > a'\) leads to \(c' > c\) then we have to choose \(a' = c = \chi_-\) and \(c' = a = \chi_+\). Otherwise the shell disappears \((G_{00} = G_{33} = 0)\).

Applying these relations to \(\text{[20]}\) and \(\text{[21]}\) our interior and exterior metrics become

\[
ds_+^2 = -\left(\frac{r}{r_1}\right)^{2a} dt^2 + dr^2 + \left(\frac{r}{r_1}\right)^{2b} dz^2 + \left(\frac{r}{r_1}\right)^{2\alpha} r_1^2 d\phi^2 \quad (r < r_1) \tag{39}
\]

and

\[
ds_-^2 = -\left(\frac{r}{r_1}\right)^{2a} dt^2 + dr^2 + \left(\frac{r}{r_1}\right)^{2b} dz^2 + \left(\frac{r}{r_1}\right)^{2\alpha} r_1^2 d\phi^2 \quad (r > r_1) \tag{40}
\]

and the only nonzero elements of the Einstein tensor are

\[
G_{00} = G_{\phi\phi} = \frac{c' - c}{r_1} \delta(r-r_1) = \frac{a - a'}{r_1} \delta(r-r_1) = \frac{a - c}{r_1} \delta(r-r_1). \tag{41}
\]

It is clear that a shell with positive energy requires the condition \(a > c\) for \(c > 0\). Then our shell has positive energy density only for \(\frac{1}{3} < \sigma < 1\) (see figure(1)) or equivalently \(-1/3 < b < 2/3\) and \(a > c\) (see figure 2) for the external metric \(\text{[22]}\). For interior metric \(\text{[20]}\) these conditions correspond to \(0 < \sigma' < 1/4\) or \(\sigma' > 1\) since \(c' = a\) and \(a' = c\). For these values of \(\sigma\), one of the quantities \(b\) and \(c\) in \(\text{[2]}\) is negative. In accordance with the interpretation of the \(z\) coordinate we choose name the negative one as \(b\) in expressions above (For \(b > 0\) case see Section 4). The energy density of this cylindrical shell has a finite maximum value at \(b = 0\) (Figure 3). This is consistent with other static cylindrical shell solutions \(\text{[5]}\). Note that the interior metric \(\text{[33]}\) always admits helical null geodesics but exterior one \(\text{[40]}\) does not.
Figure 3. Changing of the energy density $b$ of the shell with counter-rotating photons with $b$.

The energy per unit length of this shell is:

$$\mu = \alpha (a - c).$$  \(42\)

As we mentioned before, unless we choose globally Minkowski metric inside the shell, at $r = 0$ there is a line singularity. When the shell satisfies positive energy condition the singularity at the origin has positive effective mass density. The singularity free configuration for the interior metric is given by $c = b = 0, a = 1, \alpha = 1$ which describes a cylindrical shell with flat interior. The exterior metric becomes

$$ds^2 = -(\frac{r}{r_1})^2 dt^2 + dr^2 + dz^2 + r_1^2 d\phi^2.$$  \(43\)

This metric is the Rindler metric \[16, 17\], which represents a uniform gravitational field and it is Riemann flat. The test particles are uniformly accelerated in this field. The gravitational field of a massive plane in Newtonian theory is also uniform. This extreme case happens when energy density of the cylinder becomes maximum (see Figure (3)). One can conclude that for this case the cylinder becomes an infinite plane.

3.3. A thin shell with counter moving photons in the $z$-direction

In this case we find a shell with the equation of state $\rho = p_z$. we choose $c = c'$ in \(33\) which leads to $a = b$ and $a' = b$ then the interior and exterior metrics become:

$$ds^-_z = -(\frac{r}{r_1})^{2b} dt^2 + dr^2 + (\frac{r}{r_1})^{2a} dz^2 + (\frac{r}{r_1})^{2c} r_1^2 d\phi^2 \quad (r < r_1)$$  \(44\)

and

$$ds^+_z = -(\frac{r}{r_1})^{2b} dt^2 + dr^2 + (\frac{r}{r_1})^{2a} dz^2 + (\frac{r}{r_1})^{2c} r_1^2 d\phi^2 \quad (r > r_1)$$  \(45\)

and the only nonzero elements of the Einstein tensor are

$$G_{00} = G_{zz} = \frac{b'}{r_1} - \frac{b}{r_1} \delta(r - r_1) = \frac{a - a'}{r_1} \delta(r - r_1) = \frac{a - b}{r_1} \delta(r - r_1).$$  \(46\)

To have a positive energy density of the shell we must have $a - b > 0$. Since when $a > 0, b (= a') < 0$, the shell satisfies all energy conditions. Note that for this solution,
Trapping photons by a line singularity

Figure 4. Changing of the energy density (46) of the shell with counter-moving photons with $b$.

Unlike the first two cases, the singularity at $r = 0$ has negative effective mass density according to [6] when the shell satisfies the positive energy conditions. The energy density is finite (Figure 3). The exterior metric (45) of the physically acceptable shell may admit helical null geodesics but interior metric (44) does not. The energy per unit length of the shell is $\mu = \alpha(a - b)$. If the interior is chosen as Riemann flat ($a = b = 0, c = 1$) then this shell disappears.

4. Thick cylindrical shell around the line singularity: An approximate solution

We try to replace the infinitely thin shell solution composed of rotating photons around a line singularity with a solution with a smooth hollow cylindrical material with inner radius $r_1$ and outer radius $r_2$. around a line singularity. We again choose the metrics of the inside and the outside of the cylinder to be the same as that of our thin shell solution (20), (21) except rescaling the coordinates and adding some constants to satisfy the continuity. In between we use a third metric which may represent the cylindrical material.

\[
\begin{align*}
\text{ds}_-^2 &= -r^2 c^2 dt^2 + dr^2 + r^2 b dz^2 + \alpha^2 r^2 d\phi^2 \quad (r < r_1), \\
\text{ds}_0^2 &= -e^2(r) dt^2 + dr^2 + f^2(r) dz^2 + \alpha^2 g(r)^2 d\phi^2 \quad (r_1 \leq r \leq r_2), \\
\text{ds}_+^2 &= -C^2 r^2 a dt^2 + dr^2 + r^2 b dz^2 + \alpha^2 F^2 r^2 c d\phi^2 \quad (r_2 < r).
\end{align*}
\]

We thus want the cylindrical material to be smooth so the metric and its first derivatives should be continuous at $r = r_1$ and $r = r_2$. Then we find following boundary conditions:

\[
\begin{array}{ccc}
  e(r) & f(r) & g(r) \\
  r = r_1 & r_1^+ & r_1^- \\
  r = r_2 & Cr_2^+ & Fr_2^- \\
  r = r_1 & Cr_1^+ & Fr_1^- \\
  r = r_2 & Car_2^+ & Fcr_2^- \\
\end{array}
\]
where $C$ and $F$ are constants to be determined from these boundary conditions. Since we have to match the metric and its first derivative at the boundaries we simply take $e(r)$ and $g(r)$ to be quadratic functions whose coefficients are determined by the boundary conditions. Since the $dz^2$ part of the metric is the same inside and outside we also choose $f = r^b$. We use the following ansatz for $e(r)$, $f(r)$, and $g(r)$ and we replace $r_1 = m$ and $r_2 = n$:

$$e(r) = cm^{c-1}r + A + \frac{(r - m)^2}{(n - m)^2}(Can^{a-1} - (cm^{c-1}n + A) + B),$$  

(51)

$$f(r) = r^b,$$  

(52)

$$g(r) = Fcn^{c-1}r + E + \frac{(r - n)^2}{(m - n)^2}G,$$  

(53)

where

$$A = m^c(1 - c),$$  

(54)

$$B = Cn^{a-1}(n - a),$$  

(55)

$$C = \frac{cm^{c-1}(n - m) + 2m^c}{an^{a-1}(m - n) + 2n^a},$$  

(56)

$$E = Fn^c(1 - c),$$  

(57)

$$G = -F(cn^{c-1}(m - n) + n^c) - m^a,$$  

(58)

and

$$F = -n\frac{2m^a + am^{a-1}(n - m)}{cn^{c-1}(n - m) - 2n^2}.$$  

(59)

Next we calculate the nonzero elements of the Einstein tensor between $m < r < n$ for $m = 1$ and $n = 2$. We refer the reader to the appendix for the explicit expressions. Here we present a series of graphs showing the components of the Einstein tensor as a function of $r$ for seven different values of $b$ (Figure 4-10). We consider the energy momentum tensor of this thick shell in the form (35). Note that for the thin shell solution we have only $\rho = p_\phi \neq 0$. For the thick shell case, since we present an approximate solution, the pressure in other directions also contribute to the energy-momentum tensor. We show that their contribution is small compared to the energy density and azimuthal stress. The radial pressure $p_r$ should vanish at the boundaries.

Note that for $-1/3 < b \leq -0.235$ and $0.315 \leq b \leq 1$ our solution satisfies only the weak energy condition since at some region $p_\phi > \rho$. For $-0.235 < b < 0.315$ all energy conditions are satisfied. As $b$ goes from $-1/3$ to 0 the energy density increases and reaches a finite maximum value at $b = 0$. Then as $b$ increases to 1 the energy decreases towards zero. Having a maximum finite energy density with $b$ is an expected behaviour since previous shell solutions \[4\] have this behaviour too. If we accept the dominant energy condition as physically relevant energy condition, because otherwise the local speed of sound can be greater than the speed of light \[24\] when $p_i > \rho$, then our solution is physically acceptable only for $-0.235 < b < 0.315$.\]
Figure 5. The nonzero components of the Einstein tensor for $b = -0.3$ of the metric (48).

Figure 6. The nonzero components of the Einstein tensor for $b = -0.23$ of the metric (48).

Figure 7. The nonzero components of the Einstein tensor for $b = -0.1$ of the metric (48).
5. CONCLUSION

In this paper, we investigated static, cylindrically symmetric solutions of Einstein equations for an infinitely thin cylindrical shell around a line singularity with $T^\mu_\mu = 0$ on the shell. We presented three cases which correspond to counter-propagating photons along a helical direction, along the circular direction and parallel to the cylinder axis. For the third case we found that the line singularity cannot have positive mass density. The interior and exterior regions of this cylindrically symmetric infinitely long shell are two different Levi-Civita vacuum spacetimes of Kasner form. The existence of circular and helical null geodesics for this metric reinforces our interpretation. Then we presented an approximate solution for the thick shell case with counter rotating photons around a line singularity where the other diagonal stresses also contribute a little to the energy-momentum tensor.

We have chosen the metric interior to the shell in Kasner form $\tilde{F}$ which has a line singularity at $r = 0$ $\tilde{F}$ and have shown that the Newtonian limit of this metric
Trapping photons by a line singularity corresponds to an attractive force field (16) for positive $a$ and repulsive force field for negative $a$. This result is consistent with Israel’s analysis of the same metric (8).

It has been demonstrated in [23], [24] and [27] that the singularity at the $r = 0$ of the cylindrically symmetric, static metric (6) can represent a superconducting cosmic string. The outside of the infinitely long straight ordinary cosmic string along the $z$ direction can be represented by a locally flat Minkowski metric with an angular deficit (15). In this case there is a boost invariance along the $z$ direction and this either corresponds to $a = b = 0$ and $c = 1$ or to the case $a = b = \frac{2}{3}, c = \frac{1}{2}$ which is also labeled as the external field of a static string (27). (18). However, considering (18) and (16) one realizes that $c$ cannot be negative if it represents the parameter related to $g_{\phi\phi}$ of the metric (6). Thus we exclude this solution from our considerations. For the superconducting straight string case it was shown that the effects due to current are negligible [24] and the external field can be represented by a metric in Kasner form (24) at least asymptotically. In this case the boost invariance along the $z$ direction is broken by the currents and in general we can take $a \neq b$. The deficit angle is not sensitive to the value of current but decreases when the string is about to switch from the superconducting phase to the ordinary one. Considering these arguments, we can say that the singularity at the $r = 0$ is consistent with a superconducting cosmic string. For the ordinary cosmic string the Riemann tensor vanishes but for the superconducting case it does not. The ordinary cosmic string does not have helical null geodesics but the superconducting one has for $\sigma \leq 1/4$ or $1 \leq \sigma$. For $\sigma = 1/4$ or 1 they become circular.

[28] investigated a line singularity such that the energy-momentum tensor is given by a Nielsen-Olesen [29] vortex. They concluded that there is no photon cylinder around their string solution which in our solution corresponds to the $a = b$ case where we find that there are no thin or thick shell solutions satisfying the energy conditions.

For $-1/3 < b < 2/3$ the thin shell solution with the equation of state $\rho = p_{\phi}$ is physically relevant since it obeys all energy conditions for this range of $b$. For the exterior metric (21) this range is equal to $1/4 < \sigma < 1$. For the thick shell solution the dominant energy condition is satisfied only for $-0.235 < b < 0.315$ and for this range of $b$ this solution can be physically acceptable. As we change the parameter $b$ form $-1/3$ to 1 the energy density of this thin shell solution first increases then reaches a finite maximum at $b = 0$ and then decreases (Figure 3). The energy density of the other two solutions are also finite.

This is the expected behaviour according to the Hoop conjecture since the cylindrical collapse does not produce a horizon due to the fact that the source is not bounded from every direction. It is also interesting that for the maximal energy density case $b = 0$, the ”cylinder” surrounding the line singularity becomes a flat plane when observed from the outside i.e. the cylindrical shell becomes a cosmic wall.

Appendix

Here we give the expressions of the nonzero components of the Einstein tensor of the metric (48) for $r_1 = 1$, $r_2 = 2$ for different values of the parameters $a, b, c$

\[ b = -0.3, \quad a = 0.83, \quad c = 0.47 \] (Figure 4)

\[ G_{00} = \frac{-1.8r^2 + 0.52r - 0.095}{r^2(r - 0.044)(r - 5.6)} \]
Trapping photons by a line singularity

\[ G_{rr} = \frac{2.8 (r + 2.1) (r - 1)(r - 2)}{r (r - 0.045) (r - 5.6) (r^2 + 3.3 r + 7)} \]

\[ G_{zz} = \frac{-4.2 - 9.2 r + 8 r^2}{(r^2 + 3.3 r + 7.0)(r - 0.044)(r - 5.6)} \]

\[ G_{\phi\phi} = \frac{2.6 + 0.29 r + 1.8 r^2}{r^2(r^2 + 3.3 r + 7.0)} \]  \hspace{1cm} \text{(A.1)}

\[ b = -0.23, \ a = 0.92, \ c = 0.32 \text{ (Figure 5)} \]

\[ G_{00} = \frac{-1.8 r^2 + 0.26 r - 0.22}{r^2 (r - 0.17)(r - 4.6)} \]

\[ G_{rr} = \frac{3.1 (r + 0.85) (r - 1) (r - 2)}{r (r - 0.17) (r - 4.6) (r^2 - 0.32 r + 4.6)} \]

\[ G_{zz} = \frac{8 (r - 1) (r - 1.5)}{(r - 0.17) (r - 4.6) (r^2 - 0.32 r + 4.6)} \]

\[ G_{\phi\phi} = \frac{25 - 0.34 r + 35 r^2}{(87 - 6 r + 19 r^2)r^2} \]  \hspace{1cm} \text{(A.2)}

\[ b = -0.1, \ a = 0.99, \ c = 0.11 \text{ (Figure 7)} \]

\[ G_{00} = \frac{-22 + 8.7 r - 430 r^2}{r^2 (200 - 940 r + 230 r^2)} \]

\[ G_{rr} = \frac{3.6 (r + 0.26)(r - 1)(r - 2)}{r (r - 0.25) (r - 3.9)(r^2 - 1.6 r + 4.1)} \]

\[ G_{zz} = \frac{8 (r - 1.4) (r - 1.5)}{(r - 0.25) (r - 3.9)(r^2 - 1.6 r + 4.1)} \]

\[ G_{\phi\phi} = \frac{13 - 0.47 r + 55 r^2}{r^2 (120 - 47 r + 29 r^2)} \]  \hspace{1cm} \text{(A.3)}

\[ b = 0, \ a = 1, \ c = 0 \text{ (Figure 8)} \]

\[ G_{00} = \frac{-2}{1 - 4 r + r^2} \]

\[ G_{rr} = \frac{4(r - 1)(r - 2)}{(4 - 2r + r^2)(1 - 4r + r^2)} \]

\[ G_{zz} = \frac{18 - 24r + 8r^2}{(4 - 2r + r^2)(1 - 4r + r^2)} \]

\[ G_{\phi\phi} = \frac{2}{4 + r^2 - 2r} \]  \hspace{1cm} \text{(A.4)}

\[ b = 0.5, \ a = 0.81, \ c = -0.31 \text{ (Figure 10)} \]

\[ G_{00} = \frac{-5.2 + 120 r - 360 r^2}{r^2 (13 - 1200 r + 200 r^2)} \]

\[ G_{rr} = \frac{2.8 (r + 2.8) (r - 1)(r - 2)}{r (r - 0.011)(r - 6) (r^2 + 5.5r + 8.5)} \]

\[ G_{zz} = \frac{8.2 (r + 1.3) (r - 1.5)}{(r - 0.022)(r - 6) (r^2 + 5.5 r + 8.5)} \]

\[ G_{\phi\phi} = \frac{1.8 r^2 + 0.52 r + 3.4}{r^2 (r^2 + 5.5 r + 8.5)} \]  \hspace{1cm} \text{(A.5)}
Trapping photons by a line singularity

Acknowledgment

We would like to thank C. Saclioglu for discussions.

References

[1] T. Levi-Civita, Rend. Acc. Lincei 28, 101 (1919).
[2] L. Marder, Proc. R. Soc. A 244, 524 (1958).
[3] W. B. Bonnor, J. Phys. A: Math. Gen. 12, 847 (1979).
[4] W. B. Bonnor, "The static cylinder in general relativity," contained in On Einstein’s path, edited by A. Harvey (New York, Springer, 1999), p. 113.
[5] L. Herrera, N. O. Santos, A. F. F. Teixeira and A. Z. Wang, 2001 Class. Quantum Grav. 18, 3847 (2001).
[6] A. Y. Miguelote, M. F. A. da Silva, A. Z. Wang and N. O. Santos, Class. Quantum Grav. 18, 4569 (2001).
[7] W. Israel, Phys. Rev. D 15, 935 (1976).
[8] P.R.C.T. Pereira and A. Z. Wang, Gen. Rel. Grav. 32, 2189 (2000).
[9] P.R.C.T. Pereira and A. Z. Wang, 2000 Phys. Rev. D 62, 124001 (2000).
[10] T. G. Philbin, Class. Quantum Grav. 13, 1217 (1996).
[11] G. Clément G and I. Zouzou, Phys. Rev. D 50, 7271 (1994).
[12] J. Bicák and M. Zofka, Class. Quantum Grav. 19 3653 (2002).
[13] D. Kramer, Class. Quantum Grav. 15, L73 (1998).
[14] U. Von der G masa and D. Kramer, Gen. Rel. Grav. 31, 349 (1999).
[15] A. Vilenkin and E.P.S. Shellard, Cosmic Strings and other Topological Defects (Cambridge University Press, Cambridge, 1994).
[16] M. F. A. da Silva, A. Z. Wang and N. O. Santos, Phys. Lett. A 244, 462 (1998).
[17] L. Herrera, J. Rui Fernández and N. O. Santos, Gen. Rel. Grav. 33, 515 (2001).
[18] W. Israel, 1966 Nuovo Cimento B44, 1 (1966).
[19] W. Israel, Nuovo Cimento B48, 463(E) (1967).
[20] G. Darmois, Mémorial des Sciences Mathématiques (Gauthier-Villars, Paris), Fasc 25.
[21] A. Lichnerowicz, Théories Relativistes de la Gravitation et de Électromagnétisme (Masson, Paris), p 61.
[22] A. Papapetrou A and A. Hamoi, Ann. Inst. Henri. Poincaré 9, 179 (1968).
[23] A. H. Taub, J Math. Phys. 21, 1423 (1980).
[24] S. W. Hawking and G. F. R. Ellis 1973 The Large Scale Structure of Spacetime (Cambridge University Press, Cambridge, 1973), p 88.
[25] P. Amsterdamski and P. Laguna-Castillo, Phys. Rev. D 37, 877 (1988).
[26] P. Peter and D. Puy, Phys. Rev. D 48, 5546 (1993).
[27] A. K. Raychaudhuri, Phys. Rev. D 41, 3041 (1989).
[28] C. C. Dyer and F. R. Marleau, Phys. Rev. D 52, 5588 (1995).
[29] H. B. Nielsen and P. Olesen, Nucl. Phys. B61, 45 (1973).