Research Article
GSNA: A Novel Sparse Array Design Achieving Enhanced Degree of Freedom for Noncircular Sources

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To solve the dilemma that the existing sparse arrays only limitedly enhance the array degree of freedom (DOF) for noncircular sources, a novel nested array with graded spacing (GSNA) is advanced as a solution. The proposed GSNA fully exploits the characteristic of the noncircular sources and expands the virtual array based on the concept of sum and difference coarray (SDCA). In comparison with other sparse linear configurations, the GSNA enjoys the largest consecutive virtual array and the highest resolution and perfectly achieves the estimation of multiple directions of arrival (DOA) in underdetermined conditions. The closed-form expression of sensor distribution and the uniform DOF are derived. The detailed experimental simulations are conducted to validate the feasibility and superiority of the proposed configuration.

1. Introduction

Array direction finding techniques can realize high-resolution multitarget DOA estimation through statistical processing of the received data by a group of sensors arranged by specific rules in space. It is widely used in civil and military fields such as self-driving, medical imaging, seismic survey, and precision attack on battlefield targets.

The traditional uniform linear array (ULA) with \( N \) sensors can only resolve \( N - 1 \) individual signals, and it rarely meets the demands of the currently complex electromagnetic environment. By contrast, the layouts of sparse arrays [1] break through the limitation of the Nyquist sampling theorem and obtain signal information in a broader spatial range, thus achieving enhanced DOF, weaker mutual coupling, and higher resolution.

Abundant classic sparse configurations have been proposed in recent years. The minimum redundancy array (MRA) [2] and minimum hole array (MHA) [3] are supposed to be the optimum array configurations, which achieve the highest DOF and the minimum mutual coupling, respectively. However, both MRA and MHA require exhaustive search to obtain the optimum design, which is time-consuming when the number of original sensors increases. Nested array [4, 5] is nested by two uniform subarrays with different densities. The nested array generates a hole-free virtual array, but severe mutual coupling exists between the elements of the dense subarray. Coprime array [6, 7] is composed of two sparse ULAs whose interelement spacing satisfies the coprime condition, yet the holes reconstructed by coprime array in the virtual array fail to utilize all the information to operate the DOA estimation. And a latest sparse configuration termed Cantor Array [8] has a symmetrical structure, enhanced DOF, hole-free virtual array, and economic applicability at the same time.

Studies found that the time-domain character of the received signal can be regarded as prior information to enhance the performance of DOA estimation. For example, [9, 10] exploit the cyclic stationarity and non-Gaussian characteristics of signals, respectively, to expand the array DOF and improve the direction-finding accuracy. Noncircular signals including binary phase shift keying (BPSK), pulse amplitude modulation (PAM), and amplitude shift keying (ASK) are ubiquitous in practice. Distinguished from circular...
sources, the nonzero elliptic covariance matrix of noncircular sources provides bonus information for DOA parameter estimation, hence has become a hotspot in array signal processing [11]. The extended subspace DOA estimation algorithm based on the noncircular properties like NC-MUSIC [12], NC Root-MUSIC [13], and NC-ESPRIT [14] derives \(O(2(N−1))\) DOF with \(O(N)\) channels on ULA. Sparse arrays are introduced into the DOA estimation of noncircular signals to improve the array DOF further. Since noncircular signals have both nonzero covariance matrix and nonzero elliptic covariance matrix, the location set of virtual elements is jointly determined by the difference coarray (DCA) and sum coarray (SCA) of physical sensors. In other words, the virtual array constructed by noncircular signals is generated by the SDCA in the process of vectorizing covariance matrix. However, the aforementioned classical sparse arrays reconstruct virtual arrays only through DCA, which cannot take full advantage of noncircular signals to detect more sources. Therefore, an appropriate array configuration is in urgent need to achieve better-performed DOA estimation oriented toward noncircular signals.

Unfolder coprime array (UFCA) [15] expands the two subarrays of the conventional coprime array and achieves ambiguity-free DOA estimation. Nevertheless, the array aperture and DOF of UFCA are restricted by the existence of holes. A novel nested array (NNA) [16] making a virtue out of the prototype nested array is proposed to enhance DOF. Both the interelement spacing and subarray spacing of the original nested array are widened, which significantly increases the length of the central continuous virtual elements. Nested array with displaced subarray (NADiS) [17] achieves similar results to NNA by redefining the displacement between the subarrays and concluding the optimal selection strategy of the number of subarray elements. From the perspective of diminishing redundancy, translational nested array (TNA) [18] performs an appropriate displacement on the two subarrays of the prototype nested array, eliminating the aliasing elements generated by DCA and SCA and obtaining a hole-free virtual array as well as enhanced resolution. Yet, the physical array aperture of TNA is relatively small. Sparse array for noncircular sources (SANC) [19] largely resembles MRA for circular sources. It determines the optimal array arrangement strategy based on SDCA through exhaustive enumeration, which achieves the highest DOF under the same conditions. Yet SANC becomes inapplicable as the number of sensors grows owing to the lack of systematic expression.

These SDCA-led array configurations explore the noncircular property to boost the performance of DOA estimation to some extent but remain to be further developed. Therefore, we present a rearranged array configuration based on the nested array. Compared with state-of-the-art array configurations, the array aperture and uniform DOF of the proposed configuration are maximum for a fixed number of sensors. The expression of the sensor distribution and the uniform DOF is derived; in addition, the superiority of the proposed configuration is verified by experimental simulation and comparative analysis.

Our main contributions are as follows:

1. A novel sparse array configuration optimized nested array with graded spacing (GSNA) is designed for noncircular sources, which effectively extends the array aperture and improves the DOA estimation accuracy.

2. The construction method of the GSNA configuration is exhibited through the design drawing and mathematical formulas. The comparisons between the proposed configuration and the popular array configurations are given.

3. The system model is constructed, and the typical SS-MUSIC is employed to estimate the DOA under the sparse array configurations. Simulations prove that the proposed GSNA is suitable for detecting the noncircular sources even under such specific conditions as underdetermined estimation, low signal-to-noise ratio (SNR), and low snapshots.

The remainder of this paper is organized as follows. Section 2 reviews the signal model and SS-MUSIC algorithm. In Section 3, the configuration of GSNA is described. Simulation results are presented in Section 4. Section 5 concludes this paper.

2. Preliminary

2.1. Notation Conventions. Throughout the whole paper, the lowercase (uppercase) bold symbols represent vector (matrix). \((\cdot)^T\), \(\cdot^*\), and \(\cdot^H\) denote transpose, complex conjugate, and conjugate transpose, respectively. \(E(\cdot)\) is the mathematical expectation, \([\cdot]\) is the floor function, \(\text{vec}(\cdot)\) denotes the vectorization process, and \(\text{diag}(\cdot)\) is used to create the diagonal matrix. Finally, \(\emptyset\) stands for the Kronecker product, and \(\|\cdot\|_F\) represents the F-norm.

2.2. System Model. Assume \(D\) noncircular quasi-stationary targets impinge on the measuring array with \(N\) sensors from the direction of \(\theta_i, i=1,2,\cdots, D\), \(\theta_i \in [-\pi/2, \pi/2]\). Let \(d_i\) represents the location of \(n\)th source and \(S\) denotes the distribution set. The first marginal element is chosen to be the reference with appropriate generality, and \(d_1 = 0\),

\[
D_n \in S = \{xd, x = 1, 2, 3, \cdots, N-1\},
\]

where \(d = \lambda/2\), \(\lambda\) denotes the wavelength of the transmitting source. Hence, the \(k\)th \((k = 1, 2, \cdots K)\) snapshot of the received signal is expressed as follows:

\[
x(k) = \sum_{j=1}^{D} a(\theta_j)s_j(k) = As(k) + n(k),
\]

\[
\Lambda(\theta) = [a(\theta_1), a(\theta_2), \cdots a(\theta_D)],
\]

\[
a(\theta_j) = [1, \exp(j2\pi d_1 \sin(\theta_j)/\lambda), \cdots, \exp(j2\pi d_{N-1} \sin(\theta_j)/\lambda)]^T,
\]
where $A \in \mathbb{C}^{N \times D}$ denotes manifold matrix, $s \in \mathbb{C}^{D \times 1}$ denotes signal waveform vector, $a \in \mathbb{C}^{N \times 1}$ denotes the steering vector, and $n \in \mathbb{C}^{N \times 1}$ denotes the noise vector.

Owing to the prominent feature of noncircular sources, both the covariance matrix and elliptic covariance matrix of the incoming signals are nonzero:

\[
R_{ax} = E[x(t)x(t)^H] \neq 0, \\
R_{nx} = E[x(t)x(t)^T] \neq 0.
\]

Denote $r_{qp}$ as the entry at row $q$ and column $p$ of the sample covariance matrix, which is written as follows:

\[
r_{qp} = E[x_q(t)x_p^*(t)] = \begin{cases} 
\sum_{k=0}^{K-1} \sigma^2 q e^{2i\pi (d_q - d_p) \sin(\theta_k)} & q \neq p \\
\sum_{k=0}^{K-1} \sigma^2 q e^{2i\pi (d_q - d_p) \sin(\theta_k)} + \sigma^2 n & q = p
\end{cases}.
\]

Then, the interrelated statistic operation on the signals received by $d_q$ and $d_p$ can be regarded as the difference of a physical sensor positions in the exponential form. Analogous to the definition of steering vector, the exponential term represents the steering vector of the virtual array reconstructed by DCA. Similarly, a virtual array determined by SCA can be reconstructed based on the elliptic covariance matrix of the received signal. This is the principle of constructing virtual domain signal parameters through second-order statistics. The location set $D$ of virtual sensors is determined by the SDCA of the physical sensors:

\[
D = \{d_q - d_p, d_q + d_p \} \{d_q, d_p \in S\}.
\]

If we set $y(k)$ equals to:

\[
y(k) = \begin{bmatrix} x(k) \\ x^*(k) \end{bmatrix} = \begin{bmatrix} A \\ A^* \end{bmatrix} s(k) + \begin{bmatrix} n(k) \\ n^*(k) \end{bmatrix}.
\]

Then, the extended covariance matrix of vector $y(k)$ is expressed as follows:

\[
R_{yy} = E[y(k)y^H(k)] = \sum_{i=1}^{D} \sigma^2_i b(\theta_i) b^H(\theta_i) + \sigma^2 n I_N,
\]

where $B = \begin{bmatrix} A \\ A^* \end{bmatrix} = [b(\theta_1), b(\theta_2), \cdots b(\theta_D)]$, $b(\theta_i) = [a(\theta_i); a^*(\theta_i)]$, $\sigma^2_s$ is the source power and $\sigma^2_n$ is the noise power, $R_n = \text{diag}([\sigma^2_1, \sigma^2_2, \cdots \sigma^2_N])$ is the signal covariance matrix, and $I_N$ is the $N$-dimensional identity matrix. Considering that $R_{yy}$ is usually unavailable in real life, the statistical average is applied to replace mathematical average:

\[
R_{yy} = \frac{1}{K} \sum_{k=1}^{K} y(k)y^H(k).
\]

Equation (11) is vectorized as follows:

\[
z = \text{vec}(R_y) = \text{vec} \left( \sum_{i=1}^{D} \sigma^2_i b(\theta_i) b^H(\theta_i) \right) + \text{vec}(\sigma^2 n I) = Bp + \sigma^2 n i,
\]

where $z$ amounts to the received data of the reconstructed virtual array, $B(\theta_1, \theta_2, \cdots \theta_D) = [b(\theta_1) \otimes b(\theta_2), \cdots \otimes b(\theta_D)]$ corresponds to its manifold matrix, $p = [\sigma^2_1, \sigma^2_2, \cdots \sigma^2_N]$ and $i = \text{vec}(I_N)$.

Notice that $z$ is the single snapshot form of a high-dimensional array and the rank of its noise-free covariance matrix is 1 ($\text{rank} (R_{zz}) = \text{rank} (zz^H) = 1$). The single-snapshot DOA estimation method, i.e., spatial smoothing (SS) operation, is performed to deal with the rank-deficient $R_{zz}$. Since the SS technology is strictly restricted for use on ULA, only the longest uniform continuous part $U$ in the center of $D$ is selected for model matching. For more specific, $U$ is defined as follows:

\[
U = \{m|{-|m|}, \cdots , -0, 1, \cdots , |m| \} \subseteq D\}.
\]

After removing the information of discontinuous virtual elements and sorting the unrepeated rows of $B$, the new vector $\tilde{z}$ is expressed as follows:

\[
\tilde{z} = \tilde{B}p + \sigma^2 n i.
\]

2.3. DOA Estimation Method. Note that the second-statistics $\tilde{z}$ actually behaves like a single snapshot of $p$, and rank $\text{rank} (R_{zz}) = \text{rank} (zz^H) = 1$. To resolve the rank deficiency and obtain the true DOAs, the classic Spatial Smoothing Multiple Signal Classification (SS-MUSIC) algorithm is employed.

Define $L_c = (D + 1)/2$, and $\tilde{z} \in \mathbb{C}^{D|D|$ is divided into $L_c$ overlapping segments, and each $\tilde{z}_l \in \mathbb{C}^{l|l} (l = 1, 2, \cdots , L_c)$ contains the $(L_c + 1 - l)\text{th}$ to the $(2L_c - l)\text{th}$ elements in $\tilde{z}$. To successfully employ the subspace-based MUSIC, the smoothing times $L_c$ should satisfy $L_c > D$, that is:

\[
|D| > 2D - 1.
\]

Then, the smoothed full-rank covariance matrix is collected as follows:

\[
R_{\text{smooth}} = \frac{1}{L_c} \sum_{l=1}^{L_c} \tilde{z}_l \tilde{z}_l^H.
\]
The eigenvalue decomposition is conducted to $R_{\text{smooth}}$:

$$R_{\text{smooth}} = U_s \Sigma_s U_s^H + U_n \Sigma_n U_n^H,$$

where $\Sigma = \text{diag} ([\sigma_1^2, \sigma_2^2, \cdots, \sigma_n^2])$ is the signal eigenvalues, $U_s$ is the corresponding eigenvectors, $\Sigma_n$ is the noise eigenvalues, and $U_n$ is the corresponding eigenvectors.

Because the signal subspace and the noise subspace are orthogonal, and the steering vectors span the same vector space as signal space do, the DOAs can be obtained by searching the peaks of

$$P = \frac{1}{b_{ij}(\theta) U_u U_u^H b_{ij}(\theta)}.$$

3. Proposed Array Configuration

The pillar of configuration design for noncircular signals is to obtain a longer uniform continuous virtual array based on SDCAs through the reasonable arrangement of physical sensors. Inspired by nested array, we propose a novel sparse nested array based on the graded spacing (GSNA), which makes full use of SDCAs to produce a larger virtual array aperture.

Figure 1 exhibits the array configuration of the proposed GSNA. As shown, the dense subarray $S_d$ is a $N_1$ element ULA with the basic unit interelement spacing, while the sparse part $S_s$ has $N_2$ sensors with graded spacing. $N = N_1 + N_2$ is the total number of physical sensors. According to the paper [4], the optimum selection of $N_1$ and $N_2$ to maximize the uniform DOF can be verified as follows:

$$\begin{cases} 
N_1 = N_2 = \frac{N}{2} & \text{N is even}, \\
N_1 = \frac{N - 1}{2}, N_2 = \frac{N + 1}{2} & \text{N is odd}.
\end{cases}$$

$S_j$ is composed of three parts $S_j (j = 1, 2, 3)$, each containing $K_j (j = 1, 2, 3)$ sensors. The internal spacing of $S_j$ is $D_j d$ ($i = 1, 2, 3$).

$$\begin{align*}
K_1 &= 1 & D_1 &= 2N_1 + 1, \\
K_2 &= N_2 - 2 & D_2 &= 2N_1, \\
K_3 &= 1 & D_3 &= 2N_1 - 1.
\end{align*}$$

In this way, the sensor position set of GSNA with respect to an arbitrary given $N$ is expressed as follows:

$$S_{\text{GSNA}} = \{0, 1, \cdots, N_1 - 1, N_1 + D_1 - 1, N_1 + D_1 + D_2 - 1, \cdots, N_1 + D_1 + K_2D_2 - 1\}.$$

Equation (21) is simplified as follows:

$$S_{\text{GSNA}} = \{0, 1, 2, \cdots, N_1 - 1, 3N_1, 5N_1, 7N_1, \cdots, 2N_1N_2 - N_1, 2N_1N_2 - N_1 - 1\}.$$  

Lemma 1. The uniform DOF for GSNA with $N$ ($N = N_1 + N_2$) elements is

$$\text{DOF}_{\text{GSNA}} = 2N_1N_2 + 2N_1 - 2.$$  

Proof. Assuming that $S_j^2 (jj) (j = 1, 2, \cdots, K_2)$ is the position of the $jj$th sensor in $S_j^2$ and $S_j^2 (jj) = N_1 - 1 + D_j + jj * D_2$. Furthermore, the positions of $S_j^1$ and $S_j^0$ are expressed as $S_j^1 = N_1 + D_1 - 1$ and $S_j^0 = N_1 + D_1 + K_2D_2 + D_3 - 1$. According to the symmetry, only the nonnegative case of SDCAs is conducted. By definition, the $A(\pm)B$ is the nonnegative SDCA of $A$ and $B$, we get:

$$S_d(\pm)S_d = \{0 : 1 : 2(N_1 - 1)\} = \{0 : 1 : (2N_1 - 2)\},$$

$$S_d(\pm)S_s^1 = \{(S_s^1 - N_1 + 1) : 1 : (S_s^1 + N_1 - 1)\}$$

$$= \{(2N_1 + 1) : 1 : (4N_1 - 1)\},$$

$$S_d(\pm)S_s^2 (jj) = \{(S_s^2 (jj) - N_1 + 1) : 1 : (S_s^2 (jj) + N_1 - 1)\}$$

$$= \{(2jjN_1 + 2N_1 + 1) : 1 : (2jjN_1 + 4N_1 - 1)\},$$

$$S_d(\pm)S_s^3 = \{(S_s^3 - N_1 + 1) : 1 : (S_s^3 + N_1 - 1)\}$$

$$= \{2N_1N_2 + 1 : (2N_1N_2 + 2N_2 - 2)\}. $$

especially when $jj = 1$, $S_d(\pm)S_s^3 (1) = \{(6N_1 - 1) : 1 : (6N_1 - 1)\};$ $jj = K_2$, $S_d(\pm)S_s^3 (K_2) = \{(2N_1N_2 + 1) : (2N_1N_2 + 2N_2 - 1)\}$. Note that the segments of virtual elements constructed by the SDCAs of $S_d$ and $S_s$ are not completely continuous, and holes appear at the positions of $2N_1 - 1$ and $2(1 + jj)N_1$, ($jj = 0, 1, 2, \cdots, K_2$).

Next, we consider the virtual array constructed by the DCA of $S_j$ and itself:

$$S_j^1 - S_j^2 (K_2) = (N_1 + D_1 + K_2 D_2 + D_3 - 1)$$

$$- (N_1 + D_1 + K_2 D_2 - 1) = 2N_1 - 1,$$

$$S_j^0 - S_j^2 (1) = (2N_1N_2 + 1)$$

$$- (2N_1N_2 + 2N_2 - 2) = 2N_1 - 1.$$
Table 1: The physical sensor distribution and uniform DOF of GSNA.

| $N$ | $S_{GSNA}$                                      | $L$ | DOF |
|-----|-------------------------------------------------|-----|-----|
| 9   | [0, 1, 2, 3, 12, 20, 28, 36, 43]                | 91  | 46  |
| 10  | [0, 1, 2, 3, 4, 15, 25, 35, 45, 54]             | 115 | 58  |
| 11  | [0, 1, 2, 3, 4, 15, 25, 35, 45, 55, 64]         | 135 | 68  |
| 12  | [0, 1, 2, 3, 4, 5, 18, 30, 42, 54, 66, 77]      | 163 | 82  |
| 16  | [0, 1, 2, 3, 4, 5, 6, 7, 24, 40, 56, 72, 88, 104, 120, 135] | 283 | 142 |

Figure 2: The comparisons of array aperture and DOF.
\[ S_1^1 + S_2^2(K_2 - 2) = (N_1 + D_1 - 1) + (N_1 + D_1 + (K_2 - 2)D_2 - 1) \\
= 2N_1 + 2D_1 + K_2D_2 - 2D_2 - 2 \\
= 2N_1 + K_2D_2 = 2(1 + K_2)N_1, \]  
(26)

\[ S_2^2(K_2) - S_1^1 = (N_1 + D_1 + K_2D_2 - 1) - (N_1 + D_1 - 1) \\
= K_2D_2 = 2(1 + (K_2 - 1))N_1. \]  
(27)

Let \( jj = K_2 - ii - 1 \) and \( ii = 1, 2, 3, \cdots, K_2 - 1 \), then:

\[ S_1^1 (ii) = (N_1 + D_1 + K_2D_2 - 1) \\
- (N_1 + D_1 + iiD_2 - 1) = 2(K_2 - ii)N_1 \\
= 2(1 + jj)N_1 \quad jj = 0, 1, 2, \cdots, K_2 - 2. \]  
(28)

According to equation (25) to equation (28), the missing lags (holes) in equation (24) can be filled by the DCA of \( S_t \) and itself. In terms of the symmetry of SDCA, a total of \( L = 4N_1N_2 + 4N_1 - 3 \) consecutive lags from \( -(2N_1N_2 + 2N_2 - 2) \) to \( (2N_1N_2 + 2N_2 - 2) \) can be generated. Therefore, the uniform DOF of the proposed GSNA configuration is:

\[ \text{DOF}_\text{GSNA} = \frac{(L + 1)}{2} = 2N_1N_2 + 2N_1 - 2. \]  
(29)

As demonstrated in Table 1, the physical sensor distributions and uniform DOF of GSNA for certain sensor numbers are listed according to equation (21).

To exhibit the superiority of the proposed GSNA configuration more clearly, existing sparse arrays involving
nested array, UFCPA, NADiS, and NNA are chosen for comparisons. Varying $N$ from 10 to 34, the physical aperture and array DOF are compared in Figure 2. As depicted in Figure 2(a), the proposed GSNA owns the largest physical aperture, followed by NNA, NADiS, UFCPA, and nested array, which indicates that GSNA has the capability to receive signal information in a wider space. And Figure 2(b) demonstrates the DOF advantage of nested family arrays (all configurations except UFCPA), among which GSNA possesses the largest DOF. Hence, we conclude that GSNA is able to estimate more independent non-circular sources than any other sparse arrays used for comparison.

4. Experimental Simulation

In this section, the measuring indicators including spatial spectrum and root mean square error (RMSE) are used to evaluate the DOA estimation performance of the proposed GSNA.

4.1. Simulation 1: Spatial Spectrum. For illustrative purposes, the normalized spatial spectrum detected by GSNA under the underdetermined condition is provided in this simulation, and nested array is used for comparison. In both array configurations, we assume 10 physical sensors and SNR of $-5$ dB and $5$ dB and 25 narrowband sources. The incident sources are evenly distributed between $-40^\circ$ and $40^\circ$, and the number of snapshots is set to be 2000. The SS-MUSIC algorithm is conducted, and the search interval is chosen to be $0.005^\circ$. The results are acquired from 200 independent Monte Carlo trials.

According to equation (23), the maximum number of detectable sources by ten-element GSNA is $2N_1N_2 + 2N_1 - 2 = 2 \times 5 \times 5 + 2 \times 5 - 2 = 58$, while that by nested array is $N_1N_2 + N_1 + N_2 - 2 = 5 \times 5 + 5 + 5 - 2 = 33$. The massive difference in the length of continuous virtual elements exerts considerable influence on DOA estimation.

As presented in Figure 3, the red dotted lines represent the real DOAs, and the peaks of the solid blue line are supposed to be the estimated DOAs. Compare (a) with (b) and (c) with (d), the spectrum peaks go sharper at higher SNR. Clearly, the proposed GSNA can effectively estimate all the 25 sources no matter at high SNR or low SNR, which is benefited from the high DOF of GSNA. In contrast, nested array cannot accurately estimate the angles outside the range of $-15^\circ$ to $15^\circ$, where there are even serious spectral peak deviation or pseudo peaks. The unavailability of nested arrays to identify all signals below its DOF is due to the

![Figure 4: The comparison of RMSE versus SNR.](image-url)

Table 2: The physical sensor distributions of array configurations.

| Array type | Distribution |
|------------|--------------|
| Nested array | [0, 1, 2, 3, 4, 5, 11, 17, 23, 29] |
| UFCPA | [0, 5, 10, 15, 20, 25, 28, 31, 34, 37] |
| NADiS | [0, 1, 2, 3, 4, 5, 16, 27, 38, 49] |
| NNA | [0, 1, 2, 3, 4, 14, 23, 32, 41, 50] |
| GSNA | [0, 1, 2, 3, 4, 15, 25, 35, 45, 54] |
imperfect redundant sample covariance matrix, based on which the virtual array is defined.

4.2. Simulation 2: RMSE versus SNR. RMSE is a critical metric to reflect the magnitude of angle estimation bias, which is defined as follows:

\[
RMSE = \sqrt{\frac{1}{\mathcal{Y}D} \sum_{r=1}^{\mathcal{Y}} \sum_{i=1}^{D} \left( \hat{\theta}_i(r) - \theta_i \right)^2},
\]

where \( \mathcal{Y} \) is the number of Monte Carlo simulations and \( \hat{\theta}_i(r) \) denotes the estimation angle of the \( i \)th noncircular source in the \( r \)th Monte Carlo simulation.

Figure 4 draws the performance curves of RMSE versus SNR. Nested array, UFCPA, NADiS, and NNA are used in comparison with GSNA. The physical sensor distributions of the abovementioned configurations are listed in Table 2.

Consistent with expectations, as SNR increases, there is a gradual decrease of RMSE. It is evident that the blue star line always corresponds to the lowest RMSE, which indicates that the DOA estimation performance of the proposed GSNA is significantly better than that of other configurations. The NNA and NADiS share similar performance in RMSE, followed by UFCPA and nested array. Correct its causes, the proposed GSNA has the longest continuous virtual elements, hence exploiting more information on virtual array to enhance the accuracy of DOA estimation.

4.3. Simulation 3: RMSE versus Snapshot Number. The number of snapshots is another crucial factor to determine the accuracy of DOA estimation except for SNR. The Monte Carlo experiments study the comparison of RMSE versus snapshot number between the proposed GSNA and other comparisons. For the third simulation, the SNR is fixed at 0 dB, and other experimental parameters remain unchanged. As plotted in Figure 5, the performance curve of the RMSE is continuing to decline with the addition of snapshots. Apparently, the proposed GSNA can obtain the lowest RMSE for each snapshot number, followed by NNA, NADiS, UFCPA, and nested array.

5. Conclusion

A novel array configuration suitable for noncircular sources termed GSNA is designed. The interelement spacing of GSNA is systematically divided into different grades to maximize the consecutive virtual array based on SDCA. Compared with the existing sparse linear array, GSNA owns the largest DOF and the highest estimation precision and gives the closed-form expression of sensor distribution simultaneously. Exhaustive simulations confirm the superior performance of the GSNA configuration on DOA estimation.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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