A convergent algorithm for the hybrid problem of reconstructing conductivity from minimal interior data

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Abstract

We consider the hybrid problem of reconstructing the isotropic electric conductivity of a body Ω from interior current density imaging data obtainable using MRI measurements. We only require knowledge of the magnitude |J| of one current for a given voltage f on the boundary ∂Ω. As previously shown, the corresponding voltage potential u in Ω is a minimizer of the weighted least gradient problem

\[ u = \arg\min_{u \in H^1(\Omega), u|_{\partial\Omega} = f} \left\{ \int_{\Omega} a(x)|\nabla u|^2 : u \right\}, \]

with \( a(x) = |J(x)| \). In this paper, we present an alternating split Bregman algorithm for treating such least gradient problems, for \( a \in L^2(\Omega) \) non-negative and \( f \in H^{1/2}(\partial\Omega) \). We give a detailed convergence proof by focusing to a large extent on the dual problem. This leads naturally to the alternating split Bregman algorithm (which is a re-interpretation of the alternating direction method of multipliers adapted to \( L^1 \) problems). The dual problem also turns out to yield a novel method to recover the full vector field J from knowledge of its magnitude and of the voltage f on the boundary. We then present several numerical experiments that illustrate the convergence behavior of the proposed algorithm.

1. Introduction

The classical electrical impedance tomography (EIT) problem seeks to obtain quantitative information on the electrical conductivity \( \sigma \) of a body from multiple measurements of voltages
and corresponding currents at its surface. The extensive study of this problem has led to major mathematical advances on uniqueness and reconstruction methods for inverse problems with boundary data. See the excellent reviews [4, 5, 16]. However, by now, it is well understood that the problem is severely ill-posed, yielding images of low resolution away from the boundary [18, 22].

A new class of inverse problems seeks to significantly improve both the accuracy and the resolution of traditional inverse boundary value problems by using data which can be determined in the interior of the object. These have been dubbed ‘hybrid methods’, as they usually involve the interaction of two kinds of physical fields, and several recent advances are presented in the current issue of the journal.

In this paper, we continue our study of the current density impedance imaging (CDII) problem of reconstructing the conductivity of a body based on measurement of currents in its interior. Such measurements have been possible since the early 1990s due to the pioneering work of Joy’s group at the University of Toronto [19, 20, 33]. The idea was to use magnetic resonance imaging (MRI) in a novel way, to determine the magnetic flux density \( B \) induced by an applied current. It is important to note that in the problem addressed in this paper, we only require the knowledge of the magnitude \( |J| \) of one current generated while maintaining a given voltage \( f \) on the boundary \( \partial \Omega_1 \). The analytic and numerical methods presented here do not necessarily depend on MRI. Since the results only require knowledge of the magnitude of one current, they may lead to simpler physical methodologies to obtain such data (see, e.g., [41]).

The problem of recovering the isotropic conductivity \( \sigma \) of an object from knowledge of the magnitude of one current density \( |J| \) in the interior was studied in [25–28, 30]. See [29] for a review and for numerous references to other hybrid approaches to conductivity imaging. In this paper, we present an alternating split Bregman algorithm for the numerical solution of this problem, along with a convergence proof.

Let \( \sigma \) be the isotropic conductivity of an object \( \Omega \subset \mathbb{R}^n, n \geq 2 \), and let \( J \) be the current density vector field determined by a given boundary voltage \( f \). Then, the corresponding voltage potential \( v \) satisfies the elliptic equation

\[
\nabla \cdot (\sigma \nabla v) = 0, \quad v|_{\partial \Omega} = f. \tag{1}
\]

By Ohm’s law, \( J = -\sigma \nabla v \). Hence, the voltage potential \( v \) satisfies the degenerate elliptic equation

\[
\nabla \cdot \left( \frac{|J|}{|\nabla v|} \nabla v \right) = 0, \quad v|_{\partial \Omega} = f. \tag{2}
\]

In general, there is no uniqueness for viscosity solutions [8] of the above equation [27, 37]. However, in [27] and [25], the authors proved that if the potential \( v \) satisfies equation (1), then it minimizes the energy functional \( E(v) = \int_{\Omega} |J| |\nabla v| \) associated with equation (2) and moreover that this functional has a unique minimizer in \( H^1(\Omega) \). Thus, the conductivity is uniquely determined by the magnitude of the current generated by a given boundary voltage and the corresponding voltage potential is the unique solution of the (infinite-dimensional) minimization problem

\[
\arg\min \left\{ \int_{\Omega} |J| |\nabla v| : v \in H^1(\Omega), \ v|_{\partial \Omega} = f \right\}. \tag{3}
\]

A simple iterative procedure to solve the above problem was given in [27]. This method was only defined when the Dirichlet problems such as (1) for successive approximations of \( \sigma \) were guaranteed to produce solutions with non-vanishing gradients in \( \Omega \). For planar regions, this led to the requirement that the given boundary voltage be almost two-to-one. (See [27] for the precise definition.)
In this paper, we present a convergent alternating split Bregman algorithm to find the minimizer of (3) for any given boundary voltage \( f \in H^{1/2} (\partial \Omega) \). More generally, let \( \Omega \) be a bounded region in \( \mathbb{R}^n, n \geq 2 \). Also, let \( a \in L^2(\Omega) \) be a non-negative function and let \( f \in H^{1/2} (\partial \Omega) \). Consider the minimization problem
\[
\arg\min_{v \in H^1(\Omega), v|_{\partial \Omega} = f} \int_{\Omega} a(x)|\nabla v|, (4)
\]
and assume that it has an optimal solution in \( H^1(\Omega) \). The algorithm that we present in this paper will converge to a minimizer of (4).

Problem (4) belongs to a general class of problems of the form
\[
\arg\min_{u \in H_1, d \in H_2} G(u) + F(Lu), (5)
\]
where \( L : H_1 \to H_2 \) is a bounded linear operator, the functions \( G : H_1 \to \mathbb{R} \cup [\infty] \) and \( F : H_2 \to \mathbb{R} \cup [\infty] \) are proper, convex and lower semi-continuous, and \( H_1 \) and \( H_2 \) are real Hilbert spaces. To see this, let \( H_1 = H_1(\Omega), H_2 = (L^2(\Omega))^n \) and \( Lu = \nabla u; \) fix \( u_f \in H^1(\Omega) \) with \( u_f|_{\partial \Omega} = f \) and define \( F : (L^2(\Omega))^n \to \mathbb{R} \) and \( G : H^1_0(\Omega) \to \mathbb{R} \) as follows:
\[
F(d) := \int_{\Omega} a(|d + \nabla u_f|) \, dx, \quad G(u) \equiv 0. (6)
\]
One approach to problem (5) is to write it as a constrained minimization problem
\[
\arg\min_{u \in H_1, d \in H_2} G(u) + F(d) \quad \text{subject to} \quad Lu = d, (7)
\]
which leads to the unconstrained problem
\[
\arg\min_{u \in H_1, d \in H_2} G(u) + F(d) + \frac{\lambda}{2} \| Lu - d \|^2. (8)
\]
To solve the above problem, Goldstein and Osher [15] introduced the split Bregman method:
\[
(u^{k+1}, d^{k+1}) = \arg\min_{u \in H_1, d \in H_2} \left\{ G(u) + F(d) + \frac{\lambda}{2} \| b^k + Lu - d \|^2 \right\}, (9)
\]
Since the joint minimization problem (9) in both \( u \) and \( d \) could sometimes be hard or expensive to solve exactly, Goldstein and Osher [15] proposed the following alternating split Bregman algorithm to solve problem (5):
\[
\begin{align*}
  u^{k+1} &= \arg\min_{u \in H_1} \left\{ G(u) + \frac{\lambda}{2} \| b^k + Lu - d^k \|^2 \right\}, \quad (10) \\
  d^{k+1} &= \arg\min_{d \in H_2} \left\{ F(d) + \frac{\lambda}{2} \| b^k + Lu^{k+1} - d \|^2 \right\}, \quad (11) \\
  b^{k+1} &= b^k + Lu^{k+1} - d^{k+1}. \quad (12)
\end{align*}
\]
As pointed out by Esser [10] and Setzer [36], the above idea to minimize alternatingly was first presented for the augmented Lagrangian algorithm by Gabay and Mercier [12] and Glowinski and Marroco [13]. The resulting algorithm is called the alternating direction method of multipliers (ADMM) [11] and is equivalent to the alternating split Bregman algorithm. The convergence of ADMM in finite-dimensional Hilbert spaces was established by Eckstein and Bertsekas [14]. This in particular implies convergence of the alternating split Bregman algorithm in finite-dimensional Hilbert spaces. Cai et al [7] and Setzer [35, 36] also independently presented convergence results for the alternating split Bregman algorithm, under the assumption that \( H_1 \) and \( H_2 \) are finite dimensional. Motivated by the infinite-dimensional problem (4), in [24], the authors recently presented general convergence results for the alternating split Bregman algorithm in infinite-dimensional Hilbert spaces. In this paper, we will study the following alternating split Bregman algorithm for the Dirichlet problem (4).
Algorithm 1 (Alternating split Bregman algorithm for weighted least gradient Dirichlet problems).

Let \( u_f \in H^1(\Omega) \) with \( u_f|_{\partial\Omega} = f \) and initialize \( b^0, d^0 \in (L^2(\Omega))^n \). For \( k \geq 1 \)

1. Solve
\[
\Delta u^{k+1} = \nabla \cdot (d^k(x) - b^k(x)), \quad u^{k+1}|_{\partial\Omega} = 0.
\]

2. Compute
\[
d^{k+1} := \begin{cases}
\max(|\nabla u^{k+1} + \nabla u_f + b^k|, 0), & \text{if } |\nabla u^{k+1}(x) + \nabla u_f + b^k(x)| \\ -\nabla u_f, & \text{if } |\nabla u^{k+1}(x) + \nabla u_f + b^k(x)| = 0.
\end{cases}
\]

3. Let
\[
b^{k+1}(x) = b^k(x) + \nabla u^{k+1}(x) - d^{k+1}(x).
\]

The following theorem, which we will prove in section 3, guarantees convergence of algorithm 1.

**Theorem 1.1.** Let \( a \in L^2(\Omega) \) be a non-negative function and \( f \in H^{1/2}(\partial\Omega) \). Then, for any \( u_f \in H^1(\Omega) \) and any \( b^0, d^0 \in (L^2(\Omega))^n \), the sequences \( \{b^k\}_{k \in \mathbb{N}}, \{d^k\}_{k \in \mathbb{N}} \) and \( \{u_f\}_{k \in \mathbb{N}} \) produced by algorithm 1 converge weakly to some \( \hat{b} \), \( \hat{d} \) and \( \hat{u} \). Moreover, \( \{\nabla u^{k+1} - d^k\}_{k \in \mathbb{N}} \) converges strongly to zero and
\[
\sum_{k=0}^\infty \| \nabla u^{k+1} - d^k \|^2 < \infty.
\]

Furthermore, \( \tilde{u} := \hat{u} + u_f \) is a solution of the minimization problem (4), \( \hat{d} = \nabla \tilde{u}, \nabla \cdot \hat{b} \equiv 0 \) and
\[
\hat{b} = \frac{1}{\lambda} \left( a \frac{\nabla \tilde{u}}{|\nabla \tilde{u}|} \right) \in \Omega \setminus \{x : |\nabla \tilde{u}(x)| = 0, \quad a(x) \neq 0\}.
\]

Note that convergence of algorithm 1 does not require uniqueness of the minimizers of problem (4). In the case of the hybrid inverse problem, where \( a(x) \) is the magnitude of the current density vector field \( J \) for some unknown conductivity \( \sigma \), we can say more.

**Theorem 1.2.** Let \( \Omega \) be a bounded region in \( \mathbb{R}^n \) with connected \( C^{1,\alpha} \) boundary. Assume that \( a = |J| > 0 \) a.e. in \( \Omega \), where \( J \in (L^2(\Omega))^n \) is the current density vector field determined by an unknown conductivity \( \sigma \in C^{1,\alpha}(\Omega) \) and voltage \( f \) on \( \partial\Omega \). Then, the corresponding voltage potential \( \tilde{u} \) is the unique minimizer of (4), and the sequences \( b^k, d^k + \nabla u_f \) and \( u^k + u_f \) produced by algorithm 1 converge weakly to \( -J/\lambda, \nabla \tilde{u} \) and \( \tilde{u} \), respectively.

**Remark 1.3.** A generalization of the above theorem holds when \( \Omega \) contains perfectly conducting \( U_C \) or insulating \( U_I \) inclusions [25]. The sequences \( b^k, d^k + \nabla u_f \) and \( u^k + u_f \) produced by algorithm 1 will converge weakly in \( \Omega \setminus (U_C \cup U_I) \) to \( -J/\lambda, \nabla \tilde{u} \) and \( \tilde{u} \), respectively.

Our approach to (5) is to start by working on the dual problem, which is well suited to a Douglas–Rachford splitting algorithm. (For an excellent presentation of up-to-date results on the Douglas–Rachford algorithm, we refer to [1].) This leads naturally to the alternating split Bregman algorithm. In the process, we discovered that for the inverse conductivity problem of interest here, the dual problem has a unique solution, which is in fact (up to a constant) the
current J! (See corollary 2.3.) This shows that the alternating split Bregman algorithm is quite natural for the hybrid problem of CDII.

This paper is organized as follows. In section 2, we study the dual problem. Based on the analysis of the dual problem, in section 3, we present our convergence results and prove theorems 1.1 and 1.2. In section 4, we will prove that algorithm 1 is stable with respect to possible errors in solving the Poisson equations (38) at each step. In section 5, we give a more general relaxed form of the alternating split Bregman algorithm, which we derive from a corresponding result for the Douglas–Rachford algorithm. In section 6, we present several numerical experiments to demonstrate the convergence behavior of our algorithms, in particular, when the boundary function is not two-to-one.

2. The dual problem

The starting point for our proof of theorem 1.1 is the Rockafellar–Fenchel duality [32]. In this section, we study the relation between problem (4) and its dual problem. To begin with, fix $u_f \in H^1(\Omega)$ with $u_f|_{\partial\Omega} = f$ and let $a$ be a non-negative function in $L^2(\Omega)$. Define $F : (L^2(\Omega))^n \to \mathbb{R}$ and $G : H^1_0(\Omega) \to \mathbb{R}$ as follows:

$$F(d) := \int_\Omega a|d + \nabla u_f| \, dx, \quad G(u) \equiv 0.$$  \hspace{1cm} (15)

Then, problem (4) can be written as

$$(P) \quad \min_{u \in H^1_0(\Omega)} G(u) + F(\nabla u).$$

By the Fenchel duality [32], the dual problem corresponding to the problem (P) can be written as

$$(D) - \min_{b \in (L^2(\Omega))^n} \{ G^*( - \nabla \cdot b) + F^*(b) \}. $$

Recall that the Legendre–Fenchel transform $F^*$ of a functional $F$ on a Hilbert space $H$ is the function on $H^*$ given by

$$F^*(b) = \sup\{ \langle d, b \rangle - F(d) : \ d \in H \}.$$

One can easily compute $G^* : H^{-1}(\Omega) \to \mathbb{R}$:

$$G^*(u) = \begin{cases} 0, & \text{if } u \equiv 0, \\ \infty, & \text{if } u \not\equiv 0. \end{cases}$$

The following lemma provides a formula for $F^* : (L^2(\Omega))^n \to \mathbb{R}$.

Lemma 2.1. Let $a \in L^2(\Omega)$ be non-negative, and let $F$ be defined as in equation (15), with $u_f \in H^1(\Omega)$ and $u_f|_{\partial\Omega} = f$. Then,

$$F^*(b) = \begin{cases} -\langle \nabla u_f, b \rangle, & \text{if } |b(x)| \leq a(x) \ a.e. \ in \ \Omega, \\ \infty, & \text{otherwise}. \end{cases}$$  \hspace{1cm} (16)

Proof. Assume

$$|b(x)| \leq a(x)$$

$$|b(x)| > a(x)$$
on a subset \( U \) of \( \Omega \) with a positive Lebesgue measure. Then,

\[
F^*(b) = \sup_{d \in (L^2(\Omega))^p} \langle d, b \rangle - \int_\Omega a|d + \nabla u_f| \\
= -\langle \nabla u_f, b \rangle + \sup_{d \in (L^2(\Omega))^p} \left( \langle d, b \rangle - \int_\Omega a|d| \right) \\
\geq -\langle \nabla u_f, b \rangle + \sup_{\lambda \in \mathbb{R}} \int_U (|b|^2 - a(x)|b|) \ dx = \infty,
\]

where for the last inequality we choose \( d(x) = \lambda b(x) \) for \( x \in U \), and \( d(x) = 0 \) otherwise. Now assume

\[ |b(x)| \leq a(x), \quad \text{a.e.} \]

and then

\[
F^*(b) = -\langle \nabla u_f, b \rangle + \sup_{d \in (L^2(\Omega))^p} \left( \langle d, b \rangle - \int_\Omega a|d| \right) \\
= -\langle \nabla u_f, b \rangle + \sup_{d \in (L^2(\Omega))^p} \int_\Omega (b \cdot d - a|d|) \ dx \\
\leq -\langle \nabla u_f, b \rangle + \sup_{d \in (L^2(\Omega))^p} \int_\Omega |d(x)|(|b(x)| - a(x)) \ dx \\
\leq -\langle \nabla u_f, b \rangle. \tag{17}
\]

Finally, note that taking \( d \equiv 0 \) in (17) yields

\[ F^*(b) \geq -\langle b, \nabla u_f \rangle. \]

It follows from (16) that the dual problem can be explicitly written as

\[
\max\{\langle \nabla u_f, b \rangle : b \in (L^2(\Omega))^p, \ |b(x)| \leq a(x) \text{ a.e. and } \nabla \cdot b \equiv 0\}. \tag{18}
\]

Now let \( u \in H_0^1(\Omega) \), \( b \in (L^2(\Omega))^p \) with \( \nabla \cdot b \equiv 0 \), and \( |b| \leq a \) a.e. in \( \Omega \). Then,

\[
\int_\Omega a|\nabla u + \nabla u_f| \ dx \geq \int_\Omega |b||\nabla u + \nabla u_f| \ dx \geq \int_\Omega b \cdot (\nabla u + \nabla u_f) \ dx = \langle \nabla u_f, b \rangle. \tag{19}
\]

Hence, if we denote the optimal values of the primal and dual problem by \( v(P) \) and \( v(D) \), respectively, then \( v(P) \geq v(D) \). This is a general fact. Moreover, since both of the functions \( F \) and \( G \) are continuous, it follows from theorem 4.1 in [9] that strong duality holds, i.e. \( v(P) = v(D) \) and the dual problem (D) has an optimal solution.

The algorithm we propose seeks to construct a solution of the dual problem (D). In the rest of this section, we show how this leads to the solution of the primal problem (P) as well.

**Lemma 2.2.** Let \( F^* \) be defined as in (16). Then,

\[
\partial F^*(b) = \begin{cases} -\nabla u_f + \left[ p : p \in (L^2(\Omega))^p, \ p = mb \text{ on } \{x : a(x) > 0\}, \ m \in M_b \right], & \text{if } |b| \leq a \text{ a.e.} \\ \emptyset, & \text{otherwise}, \end{cases}
\]

where

\[ M_b := \{m : \Omega \to [0, \infty) : m \text{ is measurable and } m(x) = 0 \text{ if } |b(x)| < a(x)\}. \]
Proof. Assume $|b| \leq a$ a.e. in $\Omega$. Let $m \in M_b$ and consider $d \in (L^2(\Omega))^n$ with $|d| \leq a$ a.e in $\Omega$. We have
\[
\langle mb, d \rangle \leq \int_{\Omega} m|d||b| \, dx = \int_{\{|b|=0\}} m|d| |b| \, dx = \int_{\{|b|=0\}} m|b|^2 \, dx = \int_{\Omega} m|b|^2 \, dx = \langle mb, b \rangle,
\]
and hence,
\[
F^*(d) - F^*(b) = -\langle \nabla u_f, d \rangle + \langle \nabla u_f, b \rangle \geq -\langle \nabla u_f + mb, d - b \rangle.
\]
Note that the points where $a(x) = 0$ (hence also $b(x) = d(x) = 0$) do not contribute to any of the terms above. On the other hand,
\[
F^*(d) - F^*(b) \geq -\langle \nabla u_f + mb, d - b \rangle
\]
trivially holds if $|d| > a$ on a set of positive measure, as $F^*(d) = \infty$ in that case. Therefore,
\[
-\nabla u_f + mb \in \partial F^*(b).
\]
Now, let $p \in \partial F^*(b)$ and define $\tilde{p} := \nabla u_f + p$. Then, for any $d \in (L^2(\Omega))^n$ with $|d| \leq a$ a.e in $\Omega$,
\[
F^*(d) - F^*(b) = -\langle \nabla u_f, d \rangle + \langle \nabla u_f, b \rangle \geq \langle p, d - b \rangle.
\]
Consequently,
\[
\langle \tilde{p}, d \rangle \leq \langle \tilde{p}, b \rangle, \tag{20}
\]
for all $d \in (L^2(\Omega))^n$ with $|d(x)| \leq a(x)$. In particular, if we let
\[
d = \begin{cases} \frac{a}{|p|}, & |\tilde{p}| \neq 0, \\ 0, & \text{otherwise}, \end{cases}
\]
then it follows from (20) that
\[
\langle \tilde{p}, b \rangle \geq \langle \tilde{p}, d \rangle = \int_{\Omega} a|\tilde{p}| \, dx \geq \int_{\Omega} |b| |\tilde{p}| \, dx \geq \int_{\Omega} b \cdot \tilde{p} \geq \langle \tilde{p}, b \rangle. \tag{21}
\]
Therefore, all inequalities in (21) are equalities. Hence, there exists a non-negative function $m$, such that
\[
\tilde{p}(x) = \begin{cases} 0, & \text{if } |b(x)| < a(x), \\ mb, & \text{if } |b(x)| = a(x) \neq 0. \end{cases}
\]
This completes the proof. \hfill \square

Now we are ready to prove the following proposition that is a special case of the Rockafellar–Fenchel duality theorem [32] in convex analysis. Given one solution of the dual problem (D), this result gives a description of all the solutions of the primal problem (P).

Proposition 2.1. Let $\hat{b}$ be an optimal solution of the dual problem (D). Then, $\hat{u} \in H^{1}_{0}(\Omega)$ is an optimal solution of the primal problem (P) if and only if
\[
\nabla \hat{u} \in \partial F^*(\hat{b}). \tag{22}
\]

Proof. Let $\hat{u}$ be a solution of the primal problem. Then, as we saw in (19)
\[
\int_{\Omega} a|\nabla \hat{u} + \nabla u_f| \, dx \geq \int_{\Omega} |\hat{b}||\nabla \hat{u} + \nabla u_f| \, dx \geq \int_{\Omega} \hat{b} \cdot (\nabla \hat{u} + \nabla u_f) \, dx = \langle \hat{b}, \nabla u_f \rangle.
\]
Remark 2.4. Furthermore, \( \hat{\mathbf{b}}(x) = a(x) \frac{\nabla \hat{u} + \nabla u_f}{|\nabla \hat{u} + \nabla u_f|} \), if \( |\nabla \hat{u} + \nabla u_f| \neq 0 \). \hfill (23)

Therefore, for \( x \) with \( a(x) \neq 0 \),

\[ \nabla \hat{u}(x) = -\nabla u_f + m(x) \hat{b}(x), \]

where

\[ m(x) = \begin{cases} \frac{|\nabla u(x) + \nabla u_f(x)|}{a(x)}, & \text{if } |\nabla u + \nabla u_f| \neq 0, \\ 0, & \text{otherwise}. \end{cases} \]

In view of lemma 2.2, we conclude \( \nabla \hat{u} \in \partial F^*(\hat{b}) \).

Now assume \( \nabla \hat{u} \in F^*(\hat{b}) \). Then, by lemma 2.2, there exists \( m \in M_b \) such that for \( x \) with \( a(x) \neq 0 \), \( \nabla \hat{u}(x) = -\nabla u_f + m(x) \hat{b}(x) \). Therefore,

\[ \langle \hat{b}, \nabla u_f \rangle = \int_{\Omega} \hat{b} \cdot (\nabla \hat{u} + \nabla u_f) \, dx = \int_{\Omega} |\hat{b}| |\nabla \hat{u} + \nabla u_f| \, dx = \int_{\Omega} a|\nabla \hat{u} + \nabla u_f| \, dx, \]

which means \( \hat{a} \) is a minimizer of the primal problem (P). \( \square \)

In the next section, we shall see that the above relation (22) between optimal solutions of the primal and dual problems is at the heart of algorithm 1. The relation (23) shows that \( \hat{b} \) is determined on the set where \( |\nabla \hat{u}(x) + \nabla u_f(x)| \) does not vanish. We record this fact as a separate partial uniqueness result.

**Proposition 2.2.** Let \( \hat{u} \) be an optimal solution of the primal problem, and assume that \( \hat{b}_1 \) and \( \hat{b}_2 \) are two optimal solutions for the dual problem (D). Then,

\[ \hat{b}_1 \equiv \hat{b}_2 \text{ in } \Omega \setminus \{ x : |\nabla \hat{u}(x) + \nabla u_f(x)| = 0, a(x) \neq 0 \}. \hfill (24) \]

**Proof.** By lemma 2.1, \( \nabla \hat{u} \in \partial F^*(\hat{b}_1) \cap \partial F^*(\hat{b}_1) \). Hence, it follows from lemma 2.2 that there exist \( m_1 \in M_b \), and \( m_2 \in M_b \), such that

\[ \nabla \hat{u} = -\nabla u_f + m_1 \hat{b}_1 \quad \text{and} \quad \nabla \hat{u} = -\nabla u_f + m_2 \hat{b}_2. \]

Thus, \( m_1 \hat{b}_1 \equiv m_2 \hat{b}_2 \). Since \( \hat{b}_1 \) and \( \hat{b}_2 \) are both optimal, by (23)

\[ |\hat{b}_1(x)| = |\hat{b}_2(x)| = a(x) \text{ on } \Omega \setminus \{ x : |\nabla \hat{u}(x) + \nabla u_f(x)| = 0, a(x) \neq 0 \}. \]

Hence, the uniqueness relation (24) follows. \( \square \)

If \( a(x) \) is the magnitude of the current corresponding to a conductivity \( \sigma \), the above proposition yields uniqueness of solutions to the dual problem.

**Corollary 2.3.** Let \( \Omega \) be a bounded region in \( \mathbb{R}^d \) with connected \( C^{1,\alpha} \) boundary. Assume that \( \sigma = |J| > 0 \) a.e. in \( \Omega \), where \( J \in (C^0(\Omega))^d \) is the current density vector field corresponding to an unknown conductivity \( \sigma \in C^0(\Omega) \) for the given voltage \( f \) on \( \partial \Omega \). Then, the interior voltage potential \( \hat{u} \) is the unique minimizer of (4) and the dual problem (D) has a unique solution \( b \). Furthermore, \( \hat{b} = -J \).

**Proof.** The proof follows from proposition 2.2, the uniqueness theorem 1.3 in [27] and equation (23). \( \square \)

**Remark 2.4.** The above corollary generalizes to the case when \( \Omega \) contains perfectly conducting \((U_p)\) and/or perfectly insulating \(U_i\) inclusions. Proposition 2.2 together with the uniqueness result in [25] show that a solution \( \hat{b} \) of the dual problem (D) equals the current \( J \) in \( \Omega \setminus U_p \).

We omit the details.
3. Convergence analysis

In this section, we present proofs of theorems 1.1 and 1.2. The proofs rely on the representation of solutions of the primal problem (22) in proposition 2.1. Note that the dual problem (D) can be written in the form of an inclusion problem

\[ 0 \in A(\hat{b}) + B(\hat{b}), \]

where \( A := \partial(G^*o(-V^*)) \) and \( B := \partial F^* \) are maximal monotone operators on \((L^2(\Omega))^n\).

If we can compute a solution \( \hat{b} \) of problem (25) as well as \( \hat{d} \in B(\hat{b}) = \partial F^*(\hat{b}) \) then, by proposition 2.1, \( \hat{u} = \nabla^{-1}(\hat{d}) \) will be a solution of the primal problem (P). The Douglas–Rachford splitting method in convex analysis yields precisely such a pair \((\hat{b}, \hat{d})\). Following this route to the primal problem leads naturally to the alternating split Bregman algorithm, as we explain below.

Let \( H \) be a real Hilbert space and let \( A, B : H \rightarrow 2^H \) be two maximal monotone (set valued) operators. For a set valued function \( P : H \rightarrow 2^H \), let \( J_P \) denote its resolvent:

\[ J_P = (I + P)^{-1}. \]

The sub-gradient of a convex, proper, lower semi-continuous function is maximal monotone, and if \( P \) is maximal monotone, then \( J_P \) is single valued [2, 32]. Lions and Mercier [21] showed that for any general maximal monotone operators \( A, B \) and any initial element \( x_0 \), the sequence defined by the Douglas–Rachford recursion

\[ x_{k+1} = (J_A(2J_B - I) + I)A_k \]

converges weakly to some point \( \hat{x} \in H \), such that \( \hat{p} = J_B(\hat{x}) \) solves the inclusion problem (25). More recent results also prove weak convergence of the sequence \( p_k = J_{\beta B}(x_k) \) to \( \hat{p} \) (see [39], and chapters 25 and 27 in [1]). This fact will be important for our problem. The following theorem describes the Douglas–Rachford splitting algorithm and summarizes these known convergence results.

**Theorem 3.1.** Let \( H \) be a Hilbert space and let \( A, B : H \rightarrow 2^H \) be maximal monotone operators and assume that a solution of (25) exists. Then, for any initial elements \( x_0 \) and \( p_0 \) and any \( \lambda > 0 \), the sequences \( p_k \) and \( x_k \) generated by the following algorithm:

\[
\begin{align*}
x_{k+1} &= J_A(2J_B - I)x_k + x_k - p_k, \\
p_{k+1} &= J_B(x_{k+1}),
\end{align*}
\]

converge weakly to some \( \hat{x} \) and \( \hat{p} \), respectively. Furthermore, \( \hat{p} = J_{\lambda B}(\hat{x}) \) and \( \hat{p} \) satisfies

\[ 0 \in A(\hat{p}) + B(\hat{p}). \]

We wish to apply the Douglas–Rachford splitting algorithm to the operators \( A := \partial(G^*o(-V^*)) \) and \( B := \partial F^* \). We need an efficient way to evaluate the resolvents \( J_A(2p_k - x_k) \) and \( J_B(x_{k+1}) \) at each iteration. We shall use the following lemma, a proof of which can be found in [35, 36].

**Lemma 3.2.** Let \( H_1 \) and \( H_2 \) be two Hilbert spaces, \( f : H_1 \rightarrow \mathbb{R} \cup \{\infty\} \) and a bounded linear operator \( L : H_1 \rightarrow H_2 \). Assume that \( \hat{v} \) is a solution of

\[ \hat{v} = \arg\min_{v \in H_1} \left\{ \frac{\lambda}{2} \| Lv + q \|^2 + f(p) \right\}. \]

Then,

\[ \lambda(L\hat{v} + q) = J_{\lambda(\partial f(\cdot(-V^*)))}(\lambda q). \]

(28)
Given $x_k$ and $p_k$, let $u^{k+1}$ and $d^k$ be the minimizers of the functionals

$$I_1(u) = \| (2p_k - x_k)/\lambda + \nabla u \|^2$$

and

$$I_2(d) = F(d) + \frac{\lambda}{2} \| x_k/\lambda - d \|^2,$$

respectively. Then, by the above lemma

$$J_{\lambda A}(2p_k - x_k) = \lambda \nabla u^{k+1} + 2p_k - x_k$$

and

$$p_k = J_{\lambda B}(x_k) = x_k - \lambda d^k.$$

For $x_{k+1}$ and $p_{k+1}$ defined by (27), we have

$$x_{k+1} := x_k + \{ p_k - x_k + \lambda \nabla u^{k+1} \} = x_k + \lambda \{ \nabla u^{k+1} - d^k \}.$$

Hence,

$$x_k = x_0 + \lambda \sum_{i=1}^{k-1} (\nabla u^i - d^i) + \lambda \nabla u^i, \quad p_k = x_0 + \lambda \sum_{i=1}^k (\nabla u^i - d^i)$$

and

$$2p_k - x_k = x_0 + \lambda \sum_{i=1}^k (\nabla u^i - d^i) - \lambda d^k.$$  

If we define $b^k = x_0 + \sum_{i=0}^k (\nabla u^i - d^i)$, then we can write

$$x_k = \lambda (b^k + d^k), \quad p_k = \lambda b^k, \quad k \geq 0. \quad (29)$$

Thus, to evaluate $J_{\lambda A}(2p_k - x_k)$ and $J_{\lambda B}(x_{k+1})$ in (27) for all $k \geq 0$, it suffices to find the minimizers $u^{k+1}$ and $d^{k+1}$ of the functionals

$$I^{+1}_1(u) = \| \nabla u + b^k - d^k \|^2, \quad (30)$$

and

$$I^{+1}_2(d) = \int_{\Omega} a |d + \nabla u_f| \, dx + \frac{\lambda}{2} \| b^k + \nabla u^{k+1} - d \|^2, \quad (31)$$

and set $b^{k+1} = b^k + \nabla u^{k+1} - d^{k+1}$. Finding the minimizer of (30) amounts to solving a Poisson equation

$$\Delta u^{k+1} = \nabla \cdot (d^k - b^k), \quad u^{k+1}|_{\partial \Omega} = 0.$$  

Also, the minimizer of the functional $I^{+1}_2(d)$ can be computed explicitly as follows:

$$d^{k+1} := \begin{cases} \max \{ |\nabla u^{k+1} + \nabla u_f + b^k| - \frac{a}{2}, 0 \} & \\ \frac{\nabla u^{k+1} + \nabla u_f + b^k}{|\nabla u^{k+1} + \nabla u_f + b^k|} - \nabla u_f, & \text{if } |\nabla u^{k+1}(x) + \nabla u_f + b^k(x)| \neq 0, \\ -\nabla u_f, & \text{if } |\nabla u^{k+1}(x) + \nabla u_f + b^k(x)| = 0. \end{cases}$$

We are thus led to algorithm 1 for simultaneously finding solutions of both the primal problem and the dual problem.

To prove the strong convergence of the series (13), we will need the following simple lemma on firmly non-expansive operators.

It is well known that the operator $T := J_A(2I_B - I_d) + I_d - J_B$ is firmly non-expansive, i.e.

$$T = \frac{1}{2} I_d + \frac{1}{2} R,$$

such that $R$ is non-expansive:

$$\| R x - R y \| \leq \| x - y \| \quad \text{for all } x, y \in H.$$
Lemma 3.3. If $T : H \to H$ is a firmly non-expansive operator and $x_{k+1} = T(x_k)$ with $x_0 \in H$, then
\[ \| x_{k+1} - \hat{x} \|^2 + \| x_{k+1} - x_k \|^2 \leq \| x_k - \hat{x} \|^2. \]

Proof. Since $T$ is firmly non-expansive, $R = 2T - I$ is a non-expansive operator. Hence,
\[ \| Rx_k - R\hat{x} \|^2 = -\| x_k - \hat{x} \|^2 + 2 \| Tx_k - T\hat{x} \|^2 - 2 \| (I - T)x_k - (I - T)\hat{x} \|^2. \]
Therefore, we have
\[ \frac{1}{2} (\| x_k - \hat{x} \|^2 - \| Rx_k - R\hat{x} \|^2) = \| x_k - \hat{x} \|^2 - \| x_{k+1} - \hat{x} \|^2 - \| x_{k+1} - x_k \|^2. \]
Since $R$ is non-expansive, the left-hand side of the above inequality is non-negative, which completes the proof.

Proof of theorem 1.1. By interpreting algorithm 1 as a Douglas–Rachford splitting algorithm as detailed above, weak convergence of the sequences $d_k$ and $b_k$ follows immediately from theorem 3.1. To prove the estimate (13), let $T = J_{\lambda A}(2J_{\lambda B} - I) + I - J_{\lambda B}$. Since $T$ is firmly non-expansive, by lemma 3.3 we have
\[ \| x_{k+1} - \hat{x} \|^2 + \| x_{k+1} - x_k \|^2 \leq \| x_k - \hat{x} \|^2, \tag{32} \]
where $\hat{x}$ is the weak limit of $x_k$ with $T(\hat{x}) = \hat{x}$. By the above inequality, we have
\[ \sum_{k=0}^{\infty} \| x_{k+1} - x_k \|^2 < \infty. \tag{33} \]

Now observe that
\[ x_{k+1} - x_k = \lambda (b^{k+1} + d^{k+1} - b^k - d^k) = \lambda (\nabla u^{k+1} - d^k), \]
and hence, (13) follows.

By theorem 3.1, $\hat{b} = \lambda \hat{b}$ is a minimizer of the dual problem and $J_{\lambda \partial F^*}(\lambda (\hat{d} + \hat{b})) = \lambda \hat{b}$. Therefore,
\[ \lambda \hat{b} + \lambda \partial F^*(\lambda \hat{b}) = \lambda (\hat{d} + \hat{b}) \Leftrightarrow \hat{d} \in \partial F^*(\lambda \hat{b}). \]

In view of (13), the sequence $\{u_k\}_{k \in \mathbb{N}}$ is bounded in $H^1_0(\Omega)$, and therefore, it has a weakly converging subsequence. Let $\hat{u}$ be a weak cluster point of the sequence $\{u_k\}$. Then, $\nabla \hat{u} = \hat{d} \in \partial F(\hat{b})$, and hence, by proposition 2.1, $\hat{u}$ is a solution of the primal problem (P). On the other hand, $\nabla \hat{u} = \hat{d}$ for every weak cluster point $\hat{u}$ of the sequence $\{u_k\}_{k \in \mathbb{N}}$. Since $\nabla$ is injective on $H^1_0(\Omega)$, $\{u_k\}_{k \in \mathbb{N}}$ has at most one weak cluster point. In view of proposition 2.2, the proof is now complete.

Proof of theorem 1.2. The proof follows from theorem 1.1, combined with the uniqueness theorem 1.3 in [27], and our corollary 2.3.

4. Approximate alternating split Bregman algorithm

In this section, we show that the alternating split Bregman algorithm converges to the correct solutions even in the presence of possible errors at each step in solving Poisson equations. The proof relies on the following theorem about the Douglas–Rachford splitting algorithm.
Theorem 4.1. (Svaiter [39]). Let $\lambda > 0$, and let $\{a_k\}_{k\in\mathbb{N}}$ and $\{\beta_k\}_{k\in\mathbb{N}}$ be sequences in a Hilbert space $H$. Suppose $0 \in \text{ran}(A + B)$ and $\sum_{k\in\mathbb{N}}(\|a_k\| + \|\beta_k\|) < \infty$. Take $x_0 \in H$ and set

$$x_{k+1} = x_k + J_{\beta_k}(2(J_{\lambda_B}x_k + \beta_k) - x_k) + \alpha_k - (J_{\lambda_B}x_k + \beta_k), \quad k \geq 1. \quad (34)$$

Then, $x_k$ and $p_k = J_{\lambda_B}x_k$ converge weakly to $\hat{x} \in H$ and $\hat{p} \in H$, respectively, and $\hat{p} = J_{\lambda_B}\hat{x} \in (A + B)^{-1}(0)$.

The proof of the above theorem in infinite-dimensional Hilbert spaces is due to Svaiter [39] (see also [6]). It suggests the following approximate version of our algorithm.

4.1. Approximate alternating split Bregman algorithm

Let $u_f \in H^1(\Omega)$ with $u_f|_{\partial\Omega} = f$ and initialize $b^0, d^0 \in (L^2(\Omega))^n$. For $k \geq 1$

1. Find an approximate solution $u^k$ of

$$\Delta u^{k+1} = \nabla \cdot (d^k(x) - b^k(x)), \quad u^{k+1}|_{\partial\Omega} = 0,$$

with $\|\nabla u^k - \nabla u^k_{\alpha}\| \leq \alpha_k$, where $u^k_{\alpha}$ is the exact solution of the above problem.

2. Compute

$$d^{k+1} := \begin{cases} \max\{|\nabla u^{k+1} + \nabla u_f + b^k| - \frac{\alpha}{\lambda}, 0\} & \text{if } |\nabla u^{k+1}(x) + \nabla u_f + b^k(x)| \neq 0, \\ \frac{|\nabla u^{k+1} + \nabla u_f + b^k| - \nabla u_f}{|\nabla u^{k+1} + \nabla u_f + b^k|} & \text{if } |\nabla u^{k+1}(x) + \nabla u_f + b^k(x)| = 0. \end{cases}$$

3. Set

$$b^{k+1}(x) := b^k(x) + \nabla u^{k+1}(x) - d^{k+1}(x).$$

By theorem 4.1 and an argument similar to that of theorem 1.1, we can prove the following theorem about convergence of the sequences $u^k, d^k,$ and $b^k$ produced by the above algorithm.

Theorem 4.2. Let $a \in L^2(\Omega)$ be a non-negative function and $f \in H^{1/2}(\Omega)$. If

$$\sum_{k=1}^{\infty} a_k < \infty,$$

then for any $u_f \in H^1(\Omega)$ and any $b^0, d^0 \in (L^2(\Omega))^n$ the sequences $\{b^k\}_{k\in\mathbb{N}}, \{d^k\}_{k\in\mathbb{N}},$ and $\{u^k\}_{k\in\mathbb{N}}$ produced by the approximate alternating split Bregman algorithm converge weakly to some $\hat{b}, \hat{d}$ and $\hat{u}$. Moreover, $\|\nabla u^{k+1} - d^k\|_{L^2(\Omega)}$ converges strongly to zero and

$$\sum_{k=0}^{\infty} \|\nabla u^{k+1} - d^k\|^2 < \infty.$$

Furthermore, $\hat{u} := \hat{u} + u_f$ is a solution of the minimization problem (4), $\hat{b}$ is a solution of the dual problem (D), $\hat{d} = \nabla \hat{u}_a, \nabla \cdot \hat{b} \equiv 0,$ and

$$\hat{b} = \frac{1}{\lambda} \left( a \frac{\nabla \hat{u}}{|\nabla \hat{u}|} \right) \text{ in } \Omega \setminus \{x : |\nabla \hat{u}(x)| = 0, \ a(x) \neq 0\}. \quad (35)$$

There is also an analog to theorem 1.2 for the approximate alternating split Bregman algorithm, which we omit.
5. Relaxed alternating split Bregman algorithm

In this section, we present a relaxed form of the alternating split Bregman algorithm. It is based on the following theorem about the Douglas–Rachford splitting algorithm (see theorem 25.6 in [1]).

**Theorem 5.1.** Let $A$ and $B$ be maximal monotone operators from $H$ to $2^H$, such that $\text{zero}(A + B) \neq \emptyset$, let $\rho_n$ be a sequence in $[0, 2]$, such that $\sum_{n \in \mathbb{N}} \rho_n (2 - \rho_n) = \infty$, and let $\lambda \in (0, \infty)$ and $x_0 \in H$. Set

$$
p_k = J_{\lambda B} (x_k),
$$

$$
x_{k+1} = x_k + \rho_k (J_{\lambda A} (2p_k - x_k) - p_k). 
$$

(36)

Then, for any $x_0 \in H$, there exists $x \in H$, such that $J_{\lambda B} x \in \text{zero}(A + B)$, \{x_{k+1} - x_k\}_{k \in \mathbb{N}} converges strongly to zero and $p_k$ converges weakly to $J_{\lambda B} x$.

As in the proof of theorem 1.1, let $A = G^*(-\nabla)$ and $B = F^*$. Given $x_k$ and $p_k$, let $d_{k+1}$ and $u_{k+1}$ be the minimizers of the functionals

$$
I_1 (d) = F(d) + \frac{\lambda}{2} \| x_k/\lambda - d \|^2
$$

and

$$
I_2 (u) = \| (2p_k - x_k)/\lambda + \nabla u \|^2.
$$

Then, by lemma 3.2,

$$
p_k = J_{\lambda B} (x_k) = x_k - \lambda d_{k+1}
$$

and

$$
J_{\lambda A} (2p_k - x_k) = \lambda \nabla u_{k+1} + 2p_k - x_k.
$$

For $x_{k+1}$ and $p_{k+1}$ defined by (36), we then have

$$
x_{k+1} := x_k + \rho_k [p_k - x_k + \lambda \nabla u_{k+1}] = x_k + \rho_k [\nabla u_{k+1} - d_{k+1}].
$$

Hence,

$$
x_k = x_0 + \lambda \sum_{i=1}^{k} \rho_i (\nabla u^i - d^i), \quad p_k = x_0 + \lambda \sum_{i=1}^{k} \rho_i (\nabla u^i - d^i) - \lambda d_{k+1}
$$

and

$$
2p_k - x_k = x_0 + \lambda \sum_{i=1}^{k} \rho_i (\nabla u^i - d^i) - 2\lambda d_{k+1}.
$$

This suggests the following algorithm to find a minimizer of (4).

**Algorithm 2 (Relaxed alternating split Bregman algorithm).**

Let $u_f \in H^1(\Omega)$ with $u_f|_{\partial \Omega} = f$ and let $\rho_n$ be a sequence in $[\epsilon, 2]$, such that $\sum_{n \in \mathbb{N}} \rho_n (2 - \rho_n) = \infty$, where $\epsilon > 0$. Initialize $z^0$. For $k \geq 0$

(1) Compute

$$
d_{k+1} := \begin{cases} 
\max (|\nabla u_f + z^k| - \frac{\lambda}{2}, 0) \frac{\nabla u_f + z^k}{|\nabla u_f + z^k|} - \nabla u_f, & \text{if } |\nabla u_f + z^k| \neq 0, \\
-\nabla u_f, & \text{if } |\nabla u_f + z^k| = 0.
\end{cases}
$$
functions

The split Bregman algorithm proposed in this paper does not require the $\ell_1$ norm with the zero boundary conditions. For the range of conductivity values described above, the inverse problem is well defined, this algorithm requires the solution produced by the algorithm to have non-vanishing gradients in $\Omega$. To be well defined, this algorithm requires the solution $u$ of the minimization problem

$$\min_{u \in H^1(\Omega)} \left\{ \frac{1}{2} \| f - \nabla u \|^2_{L^2(\Omega)} + \frac{\lambda}{2} \| \nabla \cdot \sigma \cdot \nabla u \|^2_{L^2(\Omega)} \right\}$$

Moreover, the sequence $\{d_{k+1}\}$ of discrete values of $d_k$ converges strongly to zero and $\hat{u} = \hat{u} + u_f$ is a solution of the minimization problem (4), $\hat{d} = \nabla \hat{u}, \nabla \cdot (\hat{z} - \hat{d}) \equiv 0, \lambda \hat{b} = \lambda (\hat{z} - \hat{d})$ is a solution of the dual problem (D), and

$$\begin{align*}
\hat{b} = \frac{1}{\lambda} \left( a \frac{\nabla \hat{u}}{\|
abla \hat{u}\|} \right) \quad &\text{in } \Omega \setminus \{ x : |\nabla \hat{u}(x)| = 0, \ a(x) \neq 0 \}.
\end{align*}$$

We performed several numerical experiments using the relaxed alternating split Bregman algorithm for different choices of the sequences $\{\rho_n\}$. We observed that large choices of $\{\rho_n\}$ yield slightly better results. The detailed results are presented in section 6 and table 4.

6. Numerical study

In this section, we study numerically the convergence behavior of the proposed alternating split Bregman algorithm. In a model problem, we seek to reconstruct the conductivity from knowledge of the magnitude of one current density $|J|$ given inside the unit square $\Omega = (0, 1) \times (0, 1)$ and the corresponding voltage potential $f$ on $\partial \Omega$. In [27], the authors presented a simple iterative algorithm to recover the conductivity $\sigma$ from the knowledge of $(|J|, f)$. To be well defined, this algorithm requires the solution $\hat{u}$ and all intermediate functions $v^k$ produced by the algorithm to have non-vanishing gradients in $\Omega$. The alternating split Bregman algorithm proposed in this paper does not require $|\nabla \hat{u}| > 0$ in $\Omega$ and converges (by theorem 1.1) to the minimizer of (4). We perform several numerical experiments to illustrate this convergence behavior.

6.1. Data simulation

To simulate the internal data $|J|$, we use an abdominal human CT image rescaled to a realistic range of tissue conductivity, with values varying from 1 to 1.8 $\text{S m}^{-1}$. The scaled conductivity distribution, on a uniform grid $128 \times 128$, is shown in figure 1.

Given the above conductivity distribution, we first numerically solve the Dirichlet problem

$$\nabla \cdot \sigma \nabla v = 0, \quad v|\partial \Omega = f,$$

on the grid $128 \times 128$. To provide high accuracy of the numerical solution, we look for a solution of the form $v = u_h + u$, where $u_h$ is the harmonic function satisfying the Dirichlet boundary condition and $u$ is the solution to the Poisson equation

$$-\nabla \cdot \sigma \nabla u = \nabla \cdot \sigma \nabla u_h$$

with the zero boundary conditions. For the range of conductivity values described above, the norm $|u_h|$ is small compared to that of $|u_h|$. Such a representation is also helpful in algorithm 3; the function $u_h$ needs to be computed just once. As forward solvers, we use the FORTRAN software FISHPACK and MUDPACK for elliptic problems (see [38]). We run those routines
Figure 1. The original conductivity distribution used in the data simulation.

with double precision. Comparison of the numerical solutions with an analytical one (in the case of constant conductivity) verified that the relative $l^2$-error does not exceed $10^{-6}$. Note that if such an error were to exceed $10^{-5}$, then it may severely affect the quality of the reconstructed images. We combined the above solvers with the numerical differentiation via the three- or five-point Lagrangian interpolation to preserve the high accuracy needed in the reconstruction algorithms. Once the solution $u$ is computed, the magnitude of the current density in $\Omega$, which is the data for our problem, is simulated as $|J| = \sigma |\nabla u|$.

6.2. Reconstruction algorithms

For reconstruction, we use the alternating split Bregman algorithm, i.e. algorithm 1, choosing for $u_f$ the harmonic extension $u_h$ of the boundary voltage $f$. This leads to a simpler algorithm. For comparison purpose, we will also apply the so-called simple iterations algorithm from [27]. Below, we outline both algorithms for convenience.

Algorithm 3 (Simplified alternating split Bregman algorithm).

Let $u_h$ be the harmonic extension of $f$ to $\Omega$ and initialize $d^0, b^0 \in (L^2(\Omega))^n$.

(1) Beginning with $k = 0$, numerically solve

$$\Delta u = \nabla \cdot (d^k - b^k), \quad u|_{\partial \Omega} = 0,$$

and let $v^{k+1} = u + u_h$.

(2) Compute

$$d^{k+1}(x) := \begin{cases} 
\max\{ |\nabla v^{k+1}(x) + b^k(x)| - \frac{\sigma(x)}{\lambda}, 0\} & \text{if } |\nabla v^{k+1}(x) + b^k(x)| \neq 0, \\
\frac{\nabla v^{k+1}(x) + b^k(x)}{|\nabla v^{k+1}(x) + b^k(x)|}, & \text{if } |\nabla v^{k+1}(x) + b^k(x)| = 0.
\end{cases}$$
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(3) Let
\[ b^{k+1} = b^k + \nabla u^{k+1} - d^{k+1}. \]

(4) If \[ \| d_{k+1} \| / \| d_k \| \leq \text{Tol}, \] then compute
\[ \sigma = \frac{|J|}{|d_{k+1}|}. \]
Otherwise, \( k := k + 1 \) and repeat the process.

The simple iterations algorithm

Let \( u_h \) be the harmonic extension of \( f \) in \( \Omega \) and initialize \( u_0 = u_f, \sigma_1 = \frac{|J|}{\nabla u_h} \).

(1) Beginning with \( k = 1 \), solve the problems
\[ -\Delta u = \nabla \cdot (\sigma_k \nabla u_h), \quad u_{|\partial \Omega} = 0, \]
and \( v_k = u + u_h \).

(2) Update
\[ \sigma_{k+1} = \frac{|J|}{|\nabla v_k|}. \]

(3) If \[ \| \sigma_{k+1} - \sigma_k \| / \| \sigma_{k+1} \| \leq \text{Tol}, \] then STOP. Otherwise, let \( k := k + 1 \) and repeat the process.

In both algorithms, the harmonic part \( u_h \) of the solution is computed via the over-relaxation method, and the Dirichlet problem for the Poisson equation is solved numerically by using the implicit conjugate-gradient method (see [34]) in which the iterative process is constructed by minimizing the error of each approximation in the energy norm. The correction vector from the Krylov space is determined on each iteration. By virtue of the implicit method, a five-diagonal matrix is inverted on each iteration using a preconditioner.

6.3. Numerical reconstructions

We use \( \lambda = 1 \) in all the numerical experiments with the alternating Bregman algorithm. In our first experiment we choose the almost two-to-one boundary voltage \( f(x, y) = y \). The results obtained by applying the split Bregman algorithm with \( N = 1, 5, 10, 30, 50, 100 \) iterations are shown in figure 2. A larger image of the conductivity reconstructed using the alternating Bregman algorithm with \( N = 60 \) is shown in figure 3 (\( N = 60 \) is chosen because of the best accuracy). This image may be compared with the original image in figure 1.

For comparison, we repeat the above experiment with the same almost two-to-one boundary condition using the simple iterations algorithm. Figure 4 shows the resulting conductivity for different number of iterations. For two-to-one boundary data, the results of the two algorithms are similar, although the simple iterations method outperforms the alternating Bregman algorithm for \( N > 50 \).

Tables 1 and 2 give the numerical errors of experiments with alternating split Bregman and simple iterations algorithms for different levels of tolerance, and the computation times on a Dell Precision T5400 workstation with an Intel Xeon 64 bit 2 core processor.

Recall that the simple iteration algorithm requires \( |\nabla u^k| > 0 \) on \( \Omega \) for all \( k \geq 1 \). In general, if the boundary data \( f \) are not two-to-one, then there may exist \( x \in \Omega \) such that \( \nabla \hat{u}(x) = 0 \). This may lead to divergence of the simple iterations algorithm. For the next experiments, we choose the boundary voltage \( f(x, y) = y + 2 \sin(7\pi y) \), which is not two-to-one. As shown in figure 5 the resulting surface \( z = |J| \) touches the \( xy \)-plane. For these boundary data, the
Conductivity reconstruction using the alternating split Bregman algorithm with $N = 1, 5, 10, 30, 50, 100$ iterations (shown from the left upper corner to the right lower corner) for the almost two-to-one boundary condition $f(x, y) = y$.

Conductivity reconstructed using the alternating split Bregman algorithm with $N = 60$ iterations for the almost two-to-one boundary condition $f(x, y) = y$.

The simple iterations algorithm breaks down. However, as shown in figure 6, the alternating split Bregman algorithm converges.

In figure 7, we show the rate of convergence of the alternating split Bregman (solid curve) and simple iterations (dashed curve) algorithms for two-to-one boundary data $f(x, y) = y$ (left) and non-two-to-one data $f(x, y) = y + 2 \sin(7\pi y)$. There is no dashed curve on the left, as the errors rapidly exceed the scale in the figure. Our numerical experiments indicate that
Figure 4. Reconstruction using the simple iterations algorithm with \( N = 1, 5, 10, 30, 50, 100 \) iterations (shown from the left upper corner to the right lower corner) for the almost two-to-one boundary condition \( f = y \).

Table 1. Numerical errors and elapsed times for the alternating split Bregman algorithm.

| Tolerance \( \times 10^{-5} \) | Relative \( l^2 \)-error versus EXACT | Elapsed time (s) | Total number of iterates |
|-------------------------------|---------------------------------|-----------------|------------------------|
| \( 5 \times 10^{-5} \)        | 0.0156                          | 703             | 122                    |
| \( 1 \times 10^{-4} \)        | 0.0148                          | 574             | 99                     |
| \( 2 \times 10^{-4} \)        | 0.0075                          | 443             | 76                     |
| \( 5 \times 10^{-4} \)        | 0.0166                          | 276             | 47                     |

Table 2. Numerical errors and elapsed times for simple iterations.

| Tolerance \( \times 10^{-5} \) | Relative \( l^2 \)-error versus EXACT | Elapsed time (s) | Total number of iterates |
|-------------------------------|---------------------------------|-----------------|------------------------|
| \( 5 \times 10^{-5} \)        | 0.0030                          | 672             | 110                    |
| \( 1 \times 10^{-4} \)        | 0.0030                          | 583             | 99                     |
| \( 2 \times 10^{-4} \)        | 0.0137                          | 446             | 73                     |
| \( 5 \times 10^{-4} \)        | 0.0141                          | 264             | 43                     |

simple iterations outperform the alternating split Bregman algorithm for two-to-one boundary data. However, for non-two-to-one boundary data, simple iterations may diverge, while the alternating split Bregman always converges (see figure 7).

In [25], the more general problem of recovering an isotropic conductivity outside some perfectly conducting or insulating inclusions was considered. The data were, as before, the magnitude of one current density field \( |J| \) in the interior of \( \Omega \). We proved that (except in some exceptional cases) the conductivity outside the inclusions, and the shape and position of the perfectly conducting and insulating inclusions are uniquely determined by the magnitude of the current generated by a given boundary voltage. Since the relevant minimization problem is still of the form (3) in this case, the split Bregman algorithm can be applied. Figure 8 shows
the conductivity constructed using 100 iterations of the alternating split Bregman algorithm in the presence of perfectly conducting (right) and insulating (left) inclusions.

In additional experiments, we examined the effect of noise in our reconstruction. The noise model that we used is a simple stochastic model $|J|_n = |J| + \gamma \ast R$, where $R$ is the normally
Figure 7. Rate of convergence for alternating split Bregman (solid curve) and simple iterations (dashed curve) for two-to-one boundary data $f(x, y) = y$ (left) and non-two-to-one data $f(x, y) = y + 2\sin(7\pi y)$.

Figure 8. Reconstruction in the presence of the perfectly conducting (right) and insulating (left) inclusions

distributed pseudo-random matrix of the order as $|J|$ with mean zero and standard deviation of 1, and $\gamma > 0$ is the model standard deviation chosen as $\gamma = \delta * \|J\|/\|R\|$, where $\delta$ is the noise level, i.e. $(|J_n| - |J|)/|J|$. In figure 9, we show reconstructed images obtained by 100 iterations of the alternating split Bregman algorithm for three different levels of noise. Indeed since $R$ is a pseudo-random matrix, it is generated 20 times and reconstruction is performed for each realization. Next, a mean is calculated. The numerical values of the $l_2$-relative errors versus the exact solution for figure 9 are shown in table 3.
Figure 9. Low noise (left), moderate noise (middle) and higher noise (right).

Table 3. Numerical error for the alternating split Bregman algorithm.

|                | Low noise (level = 0.01) | Moderate noise (level = 0.035) | Higher noise (level = 0.06) |
|----------------|--------------------------|--------------------------------|-----------------------------|
| Error          | 0.026                    | 0.080                          | 0.152                       |

Table 4. Numerical errors for $N=59$ iterations of the relaxed alternating split Bregman algorithm with different choices of $\{\rho_k\}$.

| $\rho_k$ | Relative error |
|----------|----------------|
| $2 - \frac{1}{k}$ | 0.0339 |
| $2 - \frac{1}{\ln k}$ | 0.0340 |
| 1        | 0.03401        |
| $\frac{1}{\sqrt{k}}$ | 0.0344 |

Finally, we performed several numerical experiments using the relaxed alternating split Bregman algorithm (algorithm 2) for different choices of the sequences $\{\rho_k\}$. We observed that large choices of $\{\rho_k\}$ yield slightly better results. The results are presented in table 4.

7. Conclusions

This paper presents a convergent alternating split Bregman algorithm for least gradient problems with Dirichlet boundary data. This, in particular, leads to a convergent algorithm for recovering isotropic conductivity $\sigma$ from knowledge of the magnitude $|J|$ of one current generated by a given voltage $f$ on the boundary. Duality plays an essential role in our convergence proof and leads to a novel method to recover the full vector field $J$ from knowledge of its magnitude and of the voltage $f$ on the boundary. The alternating split Bregman algorithm presented here converges for non-two-to-one boundary data as well as two-to-one boundary data. The convergence of the proposed algorithm has been successfully demonstrated in several numerical experiments.

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