Formation of $\pi\mu$ atoms in $K_{\mu 4}$ decay

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Abstract

We derive the decay rate of $\pi\mu$ atom formation in $K_{\mu 4}$ decay. Using the obtained expressions we calculate the decay rate of atom formation and point out that considered decay can give a noticeable contribution as a background to the fundamental decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$.

1 Introduction

The elementary atoms formation in particles collisions and decays can give an unique information on strong interaction dynamics. The determination of the pionium atom lifetime [1] allows one to get information on $\pi\pi$ scattering lengths, whose knowledge are crucial for verification of Chiral Perturbation Theory predictions [2]. The accuracy of scattering lengths determination from nonleptonic decays $K^\pm \rightarrow \pi^\pm \pi^0 \pi^0$ also depends on effects caused by the possibility of $\pi\pi$ bound state formation [4]. The creation of positronium atoms in $\pi^0$ Dalitz decay [6] or its photoproduction on extended target [7, 8] can give information on dependence of interaction on the spin state of the system and on mechanism of bound state formation.

Since the basic works of Nemenov [9] which stimulated the search of elementary atoms, the $\pi\mu$ atom has been discovered [10] in the decays of neutral kaons $K_L \rightarrow \pi^+ \mu^- \nu$.

In the present work we point out the importance of investigation of the $\pi\mu$ atom formation in the decay

$$K^+ \rightarrow \pi^+ + \pi^- + \mu^+ + \nu$$

(1)

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The decay rate of the $\pi\mu$ atom formation

To obtain the decay rate of the $\pi\mu$ atom formation in $K_{\mu4}$ decay

$$K^+ \rightarrow \pi^+ + A_{\mu\pi} + \nu$$  \hspace{1cm} (2)

we begin from the well known \cite{17, 18} matrix element of the decay (1) written in the form of the product of the lepton and hadron currents

$$M = \frac{G_F}{\sqrt{2}} V_{us}^* i_\lambda J^\lambda = \frac{G_F}{\sqrt{2}} V_{us}^* \bar{u}(k_1) \gamma_\lambda (1 - \gamma_5) v(k_2)(V^\lambda - A^\lambda)$$  \hspace{1cm} (3)

where the axial $A^\lambda$ and vector $V^\lambda$ hadronic currents:

$$A^\lambda = -i \frac{1}{m_K} \left((p_1 + p_2)^\lambda F + (p_1 - p_2)^\lambda G + (k_1 + k_2)^\lambda R \right);$$

$$V^\lambda = -H \frac{1}{m_K^3} \epsilon^{\nu\rho\sigma}(k_1 + k_2)_\nu (p_1 + p_2)_\rho (p_1 - p_2)_\sigma$$  \hspace{1cm} (4)

Here and later on $k, p_1, p_2, k_1, k_2$ are the invariant momenta of kaon, pions, muon and neutrino; $m_K, m_\pi, m_\mu$ are the relevant masses.

Confining as usually by s and p waves and assuming the same p-wave phase $\delta_p$ for different form factors one has

$$F = F_s e^{i\delta_s} + F_p e^{i\delta_p}; \quad G = G_p e^{i\delta_p}; \quad H = H_p e^{i\delta_p}; \quad R = R_p e^{i\delta_p}$$  \hspace{1cm} (5)

The main goal of experimental investigation \cite{19, 20, 21} is measurements of the quantities $F_s, F_p, G_p, H_p, R_p, \delta = \delta_s - \delta_p$ as a function of three invariant combinations of pions and leptons momenta $s_\pi = (p_1 + p_2)^2, s_l = (k_1 + k_2)^2$.

\footnote{At the moment the six events are reported by CKM collaboration \cite{12}.}
and \( \Delta = -k(p_1 + p_2) \) \[18\].

From the other hand to make up the \( \pi \mu \) atom in the decay (1) the negative pion and muon should have a similar velocities. For such kinematic only two variables are at work, which we choose as \( s_\pi, s_\mu \).

Since the binding energy of the ground state of \( \pi \mu \) atom is small \[22\] \( \varepsilon = 1.6KeV \) the atom is a nonrelativistic system. According to the general rules of quantum mechanics, the amplitude of the decay (2) can be written as the product of the matrix element of the decay (1) taken at equal velocities of muon and negative pion and the square of the Coulomb wave function at the origin

\[
M(K^+ \rightarrow \pi^+ A_{\pi\mu}\nu) = \frac{\Psi(r = 0)}{\sqrt{2\mu}} M(K^+ \rightarrow \pi^+\pi^-\mu^+\nu)_{v_\pi = v_\mu} \tag{6}
\]

The square of the Coulomb wave function evaluated at the origin summed over principal quantum number \[11\]

\[
|\Psi(r = 0)|^2 = \sum_{n=1} |\Psi_n(r = 0)|^2 = \frac{1.2}{\pi} (\alpha \mu)^3 \tag{7}
\]

with \( \alpha = \frac{1}{137} \) the fine structure constant and \( \mu = \frac{m_\pi m_\mu}{m_\pi + m_\mu} \) reduced mass.

Using the well known rules, we obtain for the decay rate of (2)

\[
\Gamma = \frac{1}{(4\pi)^3 m_\pi m_K} |\Psi(r = 0)|^2 \int |M(K^+ \rightarrow \pi^+\pi^-\mu^+\nu)_{v_\pi = v_\mu}|^2 dE_\nu dE_\pi \tag{8}
\]

Integrations in this expression are going over neutrino \( E_\nu \) and positive pion \( E_\pi \) energies.

To calculate the square of matrix element in (8) we take advantage of the fact that the bilinear form of lepton current \( t_{\alpha\beta} = j_\alpha j_\beta \) can be written in the well known form (see e.g. \[23\])

\[
t_{\alpha\beta} = 8 \left( k_1^\alpha k_2^\beta + k_2^\alpha k_1^\beta - (k_1 k_2) g_{\alpha\beta} + i\epsilon_{\alpha\beta\rho\sigma} k_1^\rho k_2^\sigma \right) \tag{9}
\]

This expression has to be contracted with the relevant form of hadronic current \( T_{\alpha\beta} \). As an example let us consider the convolution of lepton tensors (9) with the square of the first term of axial hadronic current in (4)

\[
\sum t_{\alpha\beta} T_{\alpha\beta} = \frac{8}{m_K^2} \left( 2(p_1 k_1 + p_2 k_1)(p_1 k_2 + p_2 k_2) - (p_1 + p_2)^2 (k_1 k_2) \right) |F|^2 \tag{10}
\]

Accounting that muon and negative pion which compose atom should have the equal velocities let us express their momenta through the atom momentum \( p_a \) and mass \( m_a \): \( p_2 = \frac{m_\pi}{m_a} p_a \); \( k_1 = \frac{m_\mu}{m_a} p_a \) and introduce the following
Lorentz invariant combinations

\[ q_1 = 2p_1 k_2 = m_K^2 + m_a^2 - m_\pi^2 - 2m_K E_a \]
\[ q_2 = 2p_1 p_a = m_K^2 - m_a^2 - m_\pi^2 - 2m_K E_\nu \]
\[ q_3 = 2p_a k_2 = m_K^2 - m_a^2 + m_\pi^2 - 2m_K E_\pi \] (11)

As the atom energy in the kaon rest frame is \( E_a = m_K - E_\pi - E_\nu \) the decay (2) describes by two independent variables, which in our case are the positive pion energy \( E_\pi \) and neutrino energy \( E_\nu \).

The expression (10) can be rewritten through the above invariants

\[ \sum \alpha \beta T_{\alpha \beta} = \frac{4m_\nu}{m_a m_K} q_1 (q_2 + 2m_\pi m_a) | F |^2 \] (12)

Calculating in the same way all terms in the contraction of square of axial and vector form factors with lepton part we obtain for the atom formation decay rate

\[ \Gamma(K^+ \to \pi^+ A_{\pi \mu \nu}) = \frac{G_F^2 V_{us}^2}{m_\pi(4\pi m_K)^3} \frac{1.2\alpha^3 \mu^3}{\pi} \int \Phi(E_\pi, E_\nu) dE_\pi dE_\nu \]
\[ \Phi(E_\pi, E_\nu) = q_1 (q_2 + 2m_\pi m_a) | F |^2 + q_1 (q_2 - 2m_\pi m_a) | G |^2 + m_\nu q_3 | R |^2 
+ 2(q_1 q_2 - 2m_\pi^2 m_3) \text{Re}(FG^*) + 2m_\nu (m_\pi q_1 + m_\pi q_3) \text{Re}(FR^*) 
+ 2m_\mu (m_\pi q_1 - m_\pi q_3) \text{Re}(RG^*) + \frac{m_\pi^2}{m_a^2} (4E_\pi E_\nu q_1 - q_3^2 - 4m_\pi^2 E_\nu^2) 
\times \left( \frac{q_3}{m_\pi^2} | H |^2 - 2\frac{m_a}{m_\pi} \text{Re}(GH^* + FH^*) \right) \] (13)

The integration in this expression is going in the limits

\[ \frac{m_K^2 + m_\pi^2 - m_a^2 - 2m_K E_\pi}{2(m_K - E_\pi + \sqrt{E_\pi^2 - m_\pi^2})} \leq E_\nu \leq \frac{m_K^2 + m_\pi^2 - m_a^2 - 2m_K E_\pi}{2(m_K - E_\pi - \sqrt{E_\pi^2 - m_\pi^2})} \]
\[ m_\pi \leq E_\pi \leq \frac{m_K^2 + m_\pi^2 - m_a^2}{2m_K} \] (14)

The expression (13) is the main result of the present work. It allows one to calculate not only the decay rate of atom formation in \( K_{\mu4} \) decay, but also the differential decay rate \( \frac{d\Gamma}{dE_\nu} \), whose knowledge is important for estimation of background in the basic decay \( K^+ \to \pi^+ \nu \bar{\nu} \).

### 3 Numerical analysis

To calculate the atom formation decay rate using expression (13) one has to know the hadronic form factors. Accounting that hadronic form factors in
$K_{\mu 4}$ and $K_{e 4}$ decays are the same we take for three form factors $F$, $G$ and $H$ the standard parametrization \cite{19, 20, 21} with parameters\cite{20} from \cite{20}.

The axial hadronic form factor $R$ can not be extracted from the experimental data on $K_{e 4}$ decay \cite{3}. For this quantity we use the theoretical prediction \cite{24} $R = \frac{2}{3} F$. Substituting these parametrization in (13) and using the value of $K_{\mu 4}$ decay rate from \cite{25} we obtain for the atom formation probability in the $K_{\mu 4}$ decay

$$\frac{\Gamma(K^+ \rightarrow \pi^+ + A_{\pi \mu} + \nu)}{\Gamma(K^+ \rightarrow \pi^+ + \pi^- + \mu^+ + \nu)} \approx 3.7 \times 10^{-6}.$$  

This probability would be compared with the probability of $\pi \mu$ atom creation in $K_{\mu 3}$ decay \cite{11} $\sim 4 \times 10^{-7}$ and $\pi \pi$ atom formation in the nonleptonic decay \cite{26} $K^+ \rightarrow \pi^+ \pi^+ \pi^- \sim 8 \times 10^{-6}$.

As was mentioned above the atom formation in $K_{\mu 4}$ decay can serves as background to the rare decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ in the relevant kinematical region. The Standard Model predicts for the branching decay rate (see e. g. \cite{27}) $Br(K^+ \rightarrow \pi^+ \nu \bar{\nu}) \approx (0.85 \pm 0.07) \times 10^{-10}$ whereas the branching ratio for $\pi \mu$ atom formation considered in the present work turn out to be $Br(K^+ \rightarrow \pi^+ A_{\pi \mu} \nu) \approx 0.5 \times 10^{-10}$. Thus the branching ratio of the decay (2) considered in present work is comparable with branching ratio of basic decay $K^+ \rightarrow \pi^+ \nu \bar{\nu}$ and so has to be considered as a possible background to this decay\cite{4}.

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\footnote{The precision of the experimental data \cite{21} are better than in \cite{20}, but unfortunately only relative parameters determining form factors are cited.}

\footnote{The term with $R$ in $K_{e 4}$ decay rate is proportional to the square of electron mass and can be neglected.}

\footnote{The corresponding consideration will be done elsewhere.}
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