Steering between Bloch oscillation and dipole oscillation in parabolic optical waveguide arrays

Ming Jie Zheng, 1,* Yun San Chan, 1 and Kin Wah Yu 1,2

1 Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong, China

2 Institute of Theoretical Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong, China

*Corresponding author: mjzheng@phy.cuhk.edu.hk

We study the optical oscillations of supermodes in planar optical waveguide arrays with parabolically graded propagation constant in individual waveguide interacting through nearest neighbor couplings. In these arrays, we have identified a transition between a symmetric dipole oscillation (DO) and a symmetry-breaking Bloch oscillation (BO) under appropriate conditions. There exist obvious correspondences between gradon localization and various optical oscillations. By virtue of an analogue between the oscillation of optical system and that of a plane pendulum, we propose a shift of the graded profile to cause a transition from BO to DO. We confirm the optical transition by means of Hamiltonian optics, as well as by the field evolution of the supermodes. The results offer great potential applications in optical switching, which can be applied to design suitable optical devices. © 2010 Optical Society of America

OCIS codes: 130.2790, 130.4815, 230.7370, 350.5500
1. INTRODUCTION

The propagation and steering of light in optical waveguide arrays (OWAs) have attracted much interest [1], since OWAs are good candidates to realize the optical analogies of electron dynamics [1–5]. Among these, Bloch oscillation (BO) and dipole oscillation (DO) are two important types of optical oscillations. BO is the oscillatory motion of a particle in a periodic potential when a finite force is acting on it. Optical BOs are easier to realize than electronic BOs, since the coherence time of an optical wave packet is usually much longer than that of an electronic wave packet [6]. The optical equivalent of a finite force can be either a gradient in the propagation constant [2] or a geometric variation in the structure [7]. The former has been achieved by a temperature gradient in thermo-optic polymer waveguide arrays [8], or a gradient in the width [9] and/or refractive index [6]. For the latter many investigations have been conducted in helical OWAs [7], curved OWAs [10], chirped photonic crystals (PCs) [11], and other photonic heterostructures [12]. Tunable photonic BOs have been realized in nonlinear composite media with a tilted band structure [13]. If the band structure is parabolic, long-living photonic DOs can be achieved at the bottom of a parabolic band [14], while at the top of the parabolic band, BOs occur [15]. Since exact BO is defined in linear index gradient, the term BO used in parabolic band is just an approximation. DOs are distinguished from BOs through the different evolution patterns in the wavevector space. If we can combine the advantages of BOs and DOs in the OWAs, the tunability of light propagation will be improved significantly. The key issue is to realize the transition between BO and DO in the optical system. Although both BOs and DOs have been observed under different initial conditions of the cold atoms in parabolic optical lattices [15], the transition between them have not been investigated thoroughly. Thus we aim to study the transition between BO and DO in the parabolic optical waveguide arrays (POWAs) in this work.

The parabolic profile of the propagation constant in POWA can be obtained by the electro-optical effects and careful structural design. We find that there exists a mechanical analogue between the optical oscillations in POWA and the mechanical oscillations of a plane pendulum. Optical DO and BO in POWA are analogous to the libration and rotation of a plane pendulum, respectively [16]. The libration and rotation can be transformed to each other by applying an impulsive torque, which changes the angular momentum of the pendulum. Inspiring by this, we propose to shift the center of the parabolic index profile, which causes a lift of the propagation constant. Applying this lift-n-shift procedure, the transition between BO and DO can be studied. The proposed transition between BO and DO is demonstrated by Hamiltonian optics, and is further confirmed through the field-evolution analysis, which shows the propagation of a discrete Gaussian beam along the axis of waveguides. It is demonstrated that the tunable range of shift distance and phase through the BO-DO transition is wider than that of a single BO or DO process [1, 13, 14].

2
2. MODEL AND FORMALISM

The POWA consists of $N = 100$ waveguides, as shown schematically in Fig. 1. The array is divided into two zones ($0 \leq z \leq z_1$ and $z_1 < z \leq z_2$) along the longitudinal direction, where there are two parabolic index profiles $H_0$ and $H_1$, respectively. The centers of these two profiles are different so as to realize the BO-DO transition. A feasible experimental realization of the parabolic index profile is proposed based on the previous related experimental work and the improved structural design of OWAs. A linearly varied effective index profile has been realized in AlGaAs waveguide arrays [9]. Similarly, the parabolic index profile can also be obtained by carefully designing the rib width of each waveguide and the spacing between neighboring waveguides. The gradually varied rib width of individual waveguide corresponds to a graded on-site potential, while the varied spacing between neighboring waveguides may result in a constant or graded coupling constant of the array. The center of the index profile $H_0$ when $0 \leq z \leq z_1$ is on the central waveguide, while that of the index profile $H_1$ when $z_1 < z \leq z_2$ is shifted to the right by a certain amount. The shift of profile center from $H_0$ to $H_1$ can be realized by imposing an additional linear graded profile of propagation constants by using electro-optical effects. The size of each waveguide is in the micrometer scale. However, the real physical parameters should be calibrated through experiments. The plane wave input beam propagates along the axis of the waveguide array, that is, along the $z$ direction. The waveguide array is labeled by $n$ ($n = 1, 2, ..., N$) in the transverse direction.

According to the coupled-mode theory, the evolutionary equation of modal amplitude $a_n$ in the $n$th waveguide is written as

$$
\left[ i \frac{d}{dz} + V_n \right] a_n(z) + a_{n+1}(z) + a_{n-1}(z) = 0,
$$

where $V_n = \alpha[x(n) - S]^2 + \alpha$ is the “on-site potential”, in which $x(n) = 4[(n-1)/(N-1) - 1/2]$ is the rescaled position of the $n$th waveguide in the transverse direction. The replacement of $x$ instead of $n$ is for convenience in the following calculation. $S$ is the shift of the parabolic index profile relative to $x = 0$. While $\alpha = \Delta/C$ and $z = CZ$ are normalized quantities. Here $\Delta$ is the gradient factor of propagation constant, $C$ the coupling constant, and $Z$ the propagation distance of the beam along the axis of the waveguide. By careful designing the width of each waveguide and spacing between waveguides, $\alpha$ can be kept as a constant. Substituting the solution $a_n^m(z) = u_n^m \exp(i\beta_m z)$ into Eq. (1), we have

$$
\beta_m u_n^m = \left[ \alpha(x - S)^2 + \alpha \right] u_n^m + u_{n+1}^m + u_{n-1}^m,
$$

where $\beta_m$ means the wavenumber of the supermode $m$ and the transverse mode profile is given by a superposition of the mode amplitudes $u_n^m$ of the individual waveguides. Equation (2) is rewritten in the matrix form

$$
\beta | u \rangle = H | u \rangle,
$$

3
where the Hamiltonian matrix $H$ is defined as $H_{nn} = \alpha(x-S)^2 + \alpha$ and $H_{n,n-1} = H_{n,n+1} = 1$. The column vector $|u\rangle$ and $\beta$ denote the eigenvectors and eigenvalues of $H$, respectively. Using the Hamiltonian matrix $H$, Eq. (1) is written as a $z$-dependent equation

$$-i\frac{d}{dz}|u\rangle = H|u\rangle.$$  \hfill (4)

It is analogous to the Schrödinger equation in quantum system,

$$-\frac{i}{\hbar}\frac{d}{dt}|\phi\rangle = H|\phi\rangle.$$  \hfill (5)

Here $\hbar$ is taken to be unity. Thus the quantities $(\beta, z)$ in optical waveguide arrays corresponds to $(\omega, t)$ in quantum system, and we can refer to the functional dependence of $\beta$ on transverse wavenumber $k$ as the dispersion relation in periodic optical waveguide arrays. For graded arrays, we can divide the infinite waveguide arrays into a large number of sub-waveguide arrays in the transverse direction, each of which can be regarded as infinite in size and symmetric in translation. Based on the treatment of graded system, we can obtain the band structure approximately as follows. The solution satisfies the relation $u_{n+1} = u_n \exp(ik)$, where $k$ is the transverse wavenumber. Substituting this relation into Eq. (2), we obtain the position-dependent pseudo-dispersion relation

$$\beta(x,k,S) = 2(x-S)^2 + 2(1 + \cos k).$$ \hfill (6)

We have taken $\alpha = 2$ hereafter. Equation (6) resembles the Hamiltonian of a plane pendulum \cite{16},

$$H(p_\theta, \theta) = \frac{p_\theta^2}{2mL^2} + mgL(1 - \cos \theta),$$ \hfill (7)

where $m$ and $L$ are the mass and length of the pendulum, $p_\theta$ and $\theta$ are the angular momentum and angle of deflection, $g$ is the acceleration due to gravity. Comparing Eq. (6) and Eq. (7), we can see that $k$ is analogous to $\theta$ while $x$ to the angular momentum and $\beta$ to the total energy of the system. In this mechanical analogue, DO corresponds to the libration while BO to clockwise and anticlockwise rotations about the pivot.

3. RESULTS

3.A. Normal modes and their transitions

Let us take the original index profile centered at the central waveguide ($S = 0$) as an example to analyze the various normal modes and transitions in POWA. By diagonalizing the Hamiltonian matrix $H$, we obtain the eigenvalues and eigenvectors of the system. As described by Eq. (6), a tilted band is formed between the lower- and upper-limits $\beta(x,k,-\pi)$ and $\beta(x,k,0)$. The normal modes (called gradons) of POWA must be confined between the
classical turning points of the tilted band structure. The beating of a few normal modes of nearby eigenvalues gives rise to various oscillations between the classical turning points. These normal modes are localized at different positions with different propagation constants. There is a critical value of the propagation constant \( \beta_c = \beta(0, 0, 0) \). Separated by this critical line, there are three regions on the phase diagram as shown in Fig. 2(a), which represent three kinds of gradon modes. At the bottom of the parabolic band, the modes are nondegenerate and localized at the middle part of the array, which are called middle-nondegenerate gradons. At the upper branches of the parabolic bands, the modes are twofold degenerate due to the symmetry. We need to be careful to choose the correct form of the eigenmodes for the twofold degeneracy. In this sense, we apply a small perturbation to split the twofold degenerate eigenmodes, and then let the perturbation tend to zero to obtain the correct form of the eigenmodes. In the right branch, the modes are called right-degenerate gradons, while in the left branch, the modes are called left-degenerate gradons. The mode patterns of these three gradon modes and a critical mode are shown in the insets of Fig. 2(a). There exist obvious correspondences between gradon localization and various oscillations, which is similar to the findings in graded plasmonic chains [17] and graded OWAs [18]. When \( \beta < \beta_c \), we have DO between two symmetrical classical turning points, which comes from the the contribution of middle-nondegenerate gradons. When \( \beta > \beta_c \), right-degenerate gradons (left-degenerate gradons) undergo BO on the right (left) hand side of the array. A transition between DO and BO is possible when \( \beta \) is increased beyond \( \beta_c \). To demonstrate the BO-DO transition clearly, we define the mean position with respect to the index profile center as follows,

\[
\langle x - S \rangle = \langle u_m | x - S | u_m \rangle, (m = 1, 2, ..., N)
\]

where \( x - S \) is rescaled position relative to the center of parabolic profile in the transverse direction, and \( u_m \) is the \( m \)th eigenmode. The mean position \( \langle x - S \rangle \) versus eigenvalue \( \beta \) is plotted in Fig. 2(b). It can be seen that \( \langle x - S \rangle = 0 \) for middle-nondegenerate gradons \( (\beta < \beta_c) \), \( \langle x - S \rangle < 0 \) for left-degenerate gradons \( (\beta > \beta_c) \), and \( \langle x - S \rangle > 0 \) for right-degenerate gradons \( (\beta > \beta_c) \). The rapid variation of \( \langle x - S \rangle \) at \( \beta_c = 4 \) indicates the occurrence of BO-DO transition. When \( \beta < \beta_c \), the single branch of \( \langle x - S \rangle \) indicates the symmetric DOs. When \( \beta > \beta_c \), the two branches of \( \langle x - S \rangle \) indicates the symmetry-breaking BOs. This shows that \( \langle x - S \rangle \) is a viable parameter to show the BO-DO transition in POWA.

3.B. BO-DO transition

We propose a lift-n-shift procedure to shift the parabolic index profile, which causes the lift of the propagation constant. Applying this procedure, the transition between BO and DO can occur. Let us first sketch an example of BO-DO transition, as shown in Fig. 3(a). In the range \( 0 \leq z \leq z_1 \), the original index profile is centered at the central waveguide
(S = 0) with Hamiltonian $H_0 = \beta(x, k, 0)$. To make the occurrence of BO (DO), the input beam should be a combination of eigenmodes with different propagation constants. In the following explanation, the propagation constants $\beta_{\text{BO}}$ and $\beta_{\text{DO}}$ in Fig. 3 represent the central propagation constants of the beam which undergo BO and DO, respectively. The start point A at $z = 0$ has a transverse position $x = \sqrt{2}$ and transverse wavenumber $k = 0$. The dominant modes at point A are right-degenerate gradons, thus BO occurs at the right side of the array, whose period is $z_{\text{BO}}$. After a propagation distance $3/2z_{\text{BO}}$, the beam center reaches a point B, where $z = z_1$, $x = 2$, and $k = -\pi$. From the propagation distance $z = z_1$, the index profile is shifted to the right by an amount $S = 1$, that is, $H_1 = \beta(x, k, 1)$. As a consequence, the beam center is moved downward to a point C, where the dominant modes become middle-nondegernate gradons. Thus DO occurs between points C and D. Hence, the BO-DO transition is realized by shifting the center of the index profile. The lift-n-shift procedure can also be demonstrated in Fig. 3(b). The solid lines 1, 2, and 3 represent the phase space orbits for DO, critical motion, and BO respectively for $S = 0$. The dashed lines 1’, 2’, and 3’ denote the DO, critical motion, and BO respectively for the shift $S = 1$, marked by an arrow. The corresponding points of lift-n-shift procedure $\text{A} \rightarrow \text{B} \rightarrow \text{C} \rightarrow \text{D}$ are also marked on Fig. 3(b) accordingly. The mechanical analogue is useful to analyze the BO-DO transition graphically. The lift-n-shift procedure is a viable scheme in controlling the light propagation in optical waveguide arrays. Through the BO-DO transition, we can realize the optical steering in POWA, that is, the position of output signal is shifted with respect to the input signal.

3.C. Hamiltonian optics

The proposed switching procedure between BO and DO can be demonstrated by Hamiltonian optics approach, which is important to the quantum-optical-mechanical analogies. From the position-dependent dispersion relation Eq. (6), the evolution of the beam can be solved by using the equations of motion

$$\frac{dx}{dz} = \frac{\partial \beta(x, k, S)}{\partial k}, \quad \frac{dk}{dz} = -\frac{\partial \beta(x, k, S)}{\partial x}. \quad (9)$$

From these equations, the conjugate variables play analogous roles as action-angle variables in a plane pendulum, where $x$ and $k$ correspond to the angular momentum and angle, respectively. The equations of motion can be integrated in terms of Jacobi elliptic functions [16] and be calculated numerically as well. The numerical Hamiltonian optics results of the mean transverse position $\langle x \rangle$ and the mean transverse wavenumber $\langle k \rangle$ are shown by solid lines in Figs. 4(a) and 4(b), respectively. Separated by the line $z = z_1$, there are two zones ($0 \leq z \leq z_1$ and $z_1 < z \leq z_2$) in the whole range of propagation distances. When $0 \leq z \leq z_1$, Fig. 4(a) shows $\langle x \rangle$ periodically varies on the right side of POWA with
increasing propagation distance $z$, and Fig. 4(b) shows the reduced $\langle k \rangle$ in the first Brillouin zone which indicates that $\langle k \rangle$ increases in the negative direction with propagation distance $z$. Both features demonstrate that BO occurs in this range. When $z_1 < z \leq z_2$, both $\langle x \rangle$ and $\langle k \rangle$ varies periodically with the propagation distance $z$. These features indicate that DO takes place in this range. Therefore, the results confirm the occurrence of the BO-DO transition, which is realized by shifting the center of index profile. This shift leads to a lift of the propagation constant, that is, the jump of $\beta$ at $z_1$ is caused by the shift of index profile from $S = 0$ to $S = 1$ (figure not shown here).

3.D. Field-evolution analysis

The BO-DO transition is further confirmed through the field-evolution analysis. The analysis is performed with an input wave function at $z = 0$,

$$\psi(0) = \frac{1}{(2\pi\sigma^2)^{1/4}} e^{-\frac{(n-n_0)^2}{4\sigma^2}} e^{-ik_0(n-n_0)},$$

(10)

where $k_0$ is the input transverse wave number. The incoming field at $z \ (z < 0)$ is $\psi(z) = \psi(0) \exp(i\beta_0 z)$, where $\beta_0$ is the propagation constant of individual homogeneous channel. The intensity profile $|\psi(0)|^2$ has a discrete Gaussian distribution centered at the $n_0$th waveguide with spatial width $\sigma$. This input beam is a discrete Gaussian beam, whose intensity distribution is schematically shown as the circle in Fig. 1. The exponential factor $\exp[-ik_0(n-n_0)]$ denotes the phase differences between input beams excited on the $n$th and the $n_0$th waveguides. In this study, $k_0 = 0$ represents that the phase difference between input beams of different waveguides is zero, that is, the input beam is a plane wave, as shown in Fig. 1.

We expand the input wave function in terms of supermodes $|u_m\rangle$,

$$|\psi(0)\rangle = \sum_m A_m |u_m\rangle,$$

(11)

where $A_m = \langle u_m | \psi(0) \rangle$ is the constituent component of the input Gaussian beam. The subsequent wave function at propagation distance $z$ is

$$|\psi(z)\rangle = \sum_m A_m e^{i\beta_m z} |u_m\rangle.$$

(12)

At a certain propagation distance $z$, the wave function in the reciprocal space can be obtained by taking the following Fourier transform

$$|\phi(k, z)\rangle = \mathcal{F}[|\psi(x, z)\rangle].$$

(13)

By using these wave functions, the mean value of $x$ and $k$ are obtained as

$$\langle x \rangle = \frac{\langle \psi | x | \psi \rangle}{\langle \psi | \psi \rangle}, \quad \langle k \rangle = \frac{\langle \phi | k | \phi \rangle}{\langle \phi | \phi \rangle}.$$
Figures 4(a) and 4(b) show the comparison of Hamiltonian optics results (solid lines) with field-evolution analysis results (dashed lines) for $\langle x \rangle$ and $\langle k \rangle$, respectively. At a shorter propagation distance, results of $\langle x \rangle$ and $\langle k \rangle$ obtained from the two approaches are in good agreement. As the propagation distance increases, the discrepancies between them become larger. The evolution process is clearly demonstrated by the contour plots of beam intensity $|\psi(z)|^2$ in the real space and $|\phi(k)|^2$ in the reciprocal space, as shown by the contour plots in Figs. 4(c) and 4(d), respectively. Figure 4(c), that is, the contour plot of $|\psi(z)|^2$ as a function of the waveguide index $n$ and the propagation distance $z$ shows the BO-DO transition in the real space. When the propagation distance increases, there are spurious fields that cannot be lifted and shifted effectively to the required output channels. Since the beam contains many components of modes with different propagation constants, at which the spacing between two classical turning points of the parabolic band are slightly different, the oscillation periods for different modes are different. Another reason is that the force varies slightly in the transverse direction, the different parts of the beam propagate along different waveguides have different oscillation periods. Thus some parts of the beam deviate from the main path after some propagation distances. These are also the reasons for the discrepancies between Hamiltonian optics and field-evolution analysis results. However, the leaked energy in each waveguide is quite small, we can set a threshold for the detection of the output signal, which can avoid the disturbance of the spurious fields.

4. DISCUSSION AND CONCLUSION

Our proposed realization of BO-DO transition by lift-n-shift procedure is advantageous over a single BO or DO process, as the shift range of BO-DO transition is much larger. To achieve larger shift of the optical steering, we can combine several steps of BO-DO or DO-BO transitions. Through the BO-DO transition, we are able to realize the required position and phase for the output signal by using appropriate POWA with proper parameters and boundary conditions.

In summary, we studied the optical oscillations (BO and DO) and transitions between them in the POWA. The variety of gradon modes and transitions in POWA are identified and the interplay between gradon localization and various oscillations are elaborated. The mean position is applied to demonstrate the BO-DO transition in a set of eigenmodes. We proposed a lift-n-shift procedure to shift the center of parabolic index profile, which causes a lift of propagation constant, so that a transition between BO and DO can occur. The proposed switching procedure between BO and DO are confirmed by Hamiltonian optics approach and field-evolution analysis. The results from these two methodologies match with each other. Through this kind of switching mechanism, we can achieve the required position and phase for the output signal by using appropriate POWA structure with proper parameters and
boundary conditions. These findings have potential applications in the designing of optical switching devices.

ACKNOWLEDGMENTS

This work was supported by RGC General Research Fund of the Hong Kong SAR Government. We thank Prof. Yakubo for careful reading of the manuscript and for many useful discussion and helpful suggestions.

References

1. L. Verslegers, P. B. Catrysse, Z. Yu, and S. Fan, “Deep-subwavelength focusing and steering of light in an aperiodic metallic waveguide array,” Phys. Rev. Lett. 103, 033902 (2009).
2. U. Peschel, T. Pertsch, and F. Lederer, “Optical Bloch oscillations in waveguide arrays,” Opt. Lett. 23, 1701-1703 (1998).
3. H. Trompeter, W. Krolikowski, D. N. Neshev, A. S. Desyatnikov, A. A. Sukhorukov, Y. S. Kivshar, T. Pertsch, U. Peschel, and F. Lederer, “Bloch oscillations and Zener tunneling in two-dimensional photonic lattices,” Phys. Rev. Lett. 96, 053903 (2006).
4. A. Szameit, T. Pertsch, S. Nolte, A. Tunnermann, U. Peschel, and F. Lederer, “Optical Bloch oscillations in general waveguide lattices,” J. Opt. Soc. Am. B 24, 2632-2639 (2007).
5. S. Longhi, “Quantum-optical analogies using photonic structures,” Laser and Photon. Rev. 3, 243-261 (2008).
6. R. Sapienza, P. Costantino, D. Wiersma, M. Ghulinyan, C. J. Oton, and L. Pavesi, “Optical analogue of electronic Bloch oscillations,” Phys. Rev. Lett. 91, 263902 (2003).
7. S. Longhi, “Bloch dynamics of light waves in helical optical waveguide arrays,” Phys. Rev. B 76, 195119 (2007).
8. T. Pertsch, P. Dannberg, W. Elflein, A. Brauer, and F. Lederer, “Optical Bloch oscillations in temperature tuned waveguide arrays,” Phys. Rev. Lett. 83, 4752-4755 (1999).
9. R. Morandotti, U. Peschel, J. S. Aitchison, H. S. Eisenberg, and Y. Silberberg, “Experimental observation of linear and nonlinear optical Bloch oscillations,” Phys. Rev. Lett. 83, 4756-4759 (1999).
10. G. Lenz, I. Talanina, and C. M. de Sterke, “Bloch oscillations in an array of curved optical waveguides,” Phys. Rev. Lett. 83, 963-966 (1999).
11. V. Lousse and S. Fan, “Tunable terahertz Bloch oscillations in chirped photonic crystals,” Phys. Rev. B 72, 075119 (2005).
12. E. Istrate and E. H. Sargent, “Photonic crystal heterostructures and interfaces,” Rev. Mod. Phys. 78, 455-481 (2006).
13. G. Wang, J. P. Huang, and K. W. Yu, “Tunable photonic Bloch oscillations in electrically modulated photonic crystals,” Opt. Lett. 33, 2200-2202 (2008).

14. G. Wang, J. P. Huang, and K. W. Yu, “Long-living photonic dipole oscillations in photonic crystals,” Opt. Lett. 34, 1777-1779 (2009).

15. A. V. Ponomarev and A. R. Kolovsky, “Dipole and Bloch oscillations of cold atoms in a parabolic lattice,” Laser Phys. 16, 367-370 (2006).

16. H. Goldstein, Classical Mechanics, 2nd ed. (Addison-Wesley, California, 1980).

17. M. J. Zheng, J. J. Xiao, and K. W. Yu, “Tunable localization and oscillation of coupled plasmon waves in graded plasmonic chains,” J. Appl. Phys. 106, 113307 (2009).

18. M. J. Zheng, J. J. Xiao, and K. W. Yu, “Controllable optical Bloch oscillation in planar graded optical waveguide arrays”, Phys. Rev. A 81, 033829 (2010).
List of Figure Captions

Fig. 1. (Color online) Schematic diagram for the parabolic optical waveguide arrays and the input Gaussian beam. The light propagates along the axis of waveguide, that is, the $z$ direction. The waveguide array is labeled by $n$ ($n = 1, 2, ..., N$). The parabolic propagation constant is described by $\beta(x, k, S)$ as Eq. (6), $H_0 = \beta(x, k, 0)$ and $H_1 = \beta(x, k, 1)$ are applied in the corresponding ranges $[0, z_1]$ and $[z_1, z_2]$, respectively. The input Gaussian beam has the form as described in Eq. (10), whose cross section is denoted by the green circle. The parameters are $N = 100$, $n_0 = 86$, $k_0 = 0$, $\sigma = 1$, $z_1 = 1.39$, and $z_2 = 5.32$.

Fig. 2. (Color online) (a) Phase diagram for the parabolic optical waveguide arrays with $N = 100$ waveguides. Separated by the critical curve $\beta = \beta_c$, there are three regions representing three kinds of gradon modes, namely the right-degenerate gradons, the left-degenerate gradons, and the middle-nondegenerate gradons. Insets show the mode patterns of the three gradon modes and a critical mode, respectively. (b) The plot of mean position $\langle x - S \rangle$ versus eigenvalues $\beta$ ($S = 0$). The abrupt variation of $\langle x - S \rangle$ indicates the occurrence of BO-DO transition at $\beta_c = 4$.

Fig. 3. (Color online) (a) A possible BO-DO transition. The arrow marks the shift $S$. The lift-n-shift procedure is shown by the route $A \rightarrow B \rightarrow C \rightarrow D$. (b) The phase space orbits in POWA for the cases $S = 0$ (solid lines) and $S = 1$ (dashed lines). The solid (dashed) lines 1 (1’), 2 (2’), 3 (3’) are corresponding to DO, critical motion, and BO when $S = 0$ ($S = 1$), respectively. The shift $S$ is shown by an arrow. The points A, B, C, D are also marked accordingly.

Fig. 4. (Color online) Comparison of Hamiltonian optics results with field-evolution analysis results for (a) $\langle x \rangle$ and (b) $\langle k \rangle$ in BO-DO transition. Contour plots of field-evolution analysis results for (c) $|\psi(x)|^2$ as a function of the waveguide index $n$ and the propagation distance $z$ and (d) $|\phi(k)|^2$ as a function of the transverse wave vector $k$ and the propagation distance $z$. 
Fig. 1. (Color online) Schematic diagram for the parabolic optical waveguide arrays and the input Gaussian beam. The light propagates along the axis of waveguide, that is, the $z$ direction. The waveguide array is labeled by $n$ ($n = 1, 2, ..., N$). The parabolic propagation constant is described by $\beta(x, k, S)$ as Eq. (6), $H_0 = \beta(x, k, 0)$ and $H_1 = \beta(x, k, 1)$ are applied in the corresponding ranges $[0, z_1]$ and $[z_1, z_2]$, respectively. The input Gaussian beam has the form as described in Eq. (10), whose cross section is denoted by the green circle. The parameters are $N = 100$, $n_0 = 86$, $k_0 = 0$, $\sigma = 1$, $z_1 = 1.39$, and $z_2 = 5.32$. POWA.eps.
Fig. 2. (Color online) (a) Phase diagram for the parabolic optical waveguide arrays with $N = 100$ waveguides. Separated by the critical curve $\beta = \beta_c$, there are three regions representing three kinds of gradon modes, namely the right-degenerate gradons, the left-degenerate gradons, and the middle-nondegenerate gradons. Insets show the mode patterns of the three gradon modes and a critical mode, respectively. (b) The plot of mean position $\langle x - S \rangle$ versus eigenvalues $\beta$ ($S = 0$). The abrupt variation of $\langle x - S \rangle$ indicates the occurrence of BO-DO transition at $\beta_c = 4$. PhaseDiagram.eps.
Fig. 3. (Color online) (a) A possible BO-DO transition. The arrow marks the shift $S$. The lift-n-shift procedure is shown by the route $A \rightarrow B \rightarrow C \rightarrow D$. (b) The phase space orbits in POWA for the cases $S = 0$ (solid lines) and $S = 1$ (dashed lines). The solid (dashed) lines 1 (1'), 2 (2'), 3 (3') are corresponding to DO, critical motion, and BO when $S = 0$ ($S = 1$), respectively. The shift $S$ is shown by an arrow. The points A, B, C, D are also marked accordingly. BO-DO.eps.
Fig. 4. (Color online) Comparison of Hamiltonian optics results with field-evolution analysis results for (a) $\langle x \rangle$ and (b) $\langle k \rangle$ in BO-DO transition. Contour plots of field-evolution analysis results for (c) $|\psi(x)|^2$ as a function of the waveguide index $n$ and the propagation distance $z$ and (d) $|\phi(k)|^2$ as a function of the transverse wave vector $k$ and the propagation distance $z$. xkCP.eps.