Harmonic analysis

Sharp weighted estimates involving one supremum

Estimations pondérées précisées associées à un seul supremum

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1. Introduction and main result

Our main object is the following so-called sparse operator:

$$A_S(f)(x) = \sum_{Q \in S} \langle f \rangle_Q \chi_Q(x),$$

where $S \subset D$ is a sparse family, i.e. for all (dyadic) cubes $Q \in S$, there exist $E_Q \subset Q$ which are pairwise disjoint and $|E_Q| \geq \gamma |Q|$ with $0 < \gamma < 1$, and $\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f$. We only consider the sparse operator, because it dominates the Calderón–Zygmund operator pointwisely, see [2,14,9,8,11].

We are going to study the sharp weighted bounds of $A_S$. Before that, let us recall

$$[w]_{A_p} = \sup_Q A_p(w, Q) := \sup_Q \langle w \rangle_Q \langle (w^{1-p})^p \rangle_Q^{-1},$$

$$[w]_{A_\infty} = \sup_Q A_\infty(w, Q) := \sup_Q \langle M(w^Q) \rangle_Q \langle w \rangle_Q.$$
In [6], Hytönen and Lacey proved the following estimate:
\[ \|A_S\|_{L^p} \leq C_n[w]_{A_p}^{\frac{1}{2}} [w]_{A_p}^{\frac{1}{2}} + [w^{1-p}]_{A_p}^{\frac{1}{2}}, \]
(1)
which generalizes the famous $A_2$ theorem, obtained by Hytönen in [5]. (We also remark that when $p = 2$, (1) was obtained by Hytönen and Pérez in [7].) It was also conjectured in [6] that
\[ \|A_S\|_{L^p} \leq C_n([w]_{A_p}^{\frac{1}{2}} [w^{1-p}]_{A_p}^{\frac{1}{2}} + [w^{1-p}]_{A_p}^{\frac{1}{2}}), \]
where
\[ [w]_{A_p}^{\alpha} := \sup_Q A_p(w, Q)^\alpha A_r(w, Q)^{\beta}. \]
This conjecture, which is usually referred to as the one supremum conjecture, is still open. Before this conjecture was formulated, Lerner [10] obtained the following mixed $A_p-A_r$ estimate:
\[ \|A_S\|_{L^p} \leq C_{n,p,r}([w]_{A_p^{\frac{1}{2}}} [w^{1-p}]_{A_r}^{\frac{1}{2}} + [w^{1-p}]_{A_p}^{\frac{1}{2}}), \]
which was further extended by Lerner and Moen [13] to the $r = \infty$ case with Hrusčëv $A_{\infty}$ constant:
\[ \|A_S\|_{L^p} \leq C_{n,p}(\|w\|_{A_p^{\frac{1}{2}}} (\exp(A_{\infty})^{\frac{1}{2}})^{-\frac{1}{p'}} + [w^{1-p}]_{A_p^{\frac{1}{2}}} (\exp(A_{\infty})^{\frac{1}{2}})^{\frac{1}{p'}}), \]
where $A_{\infty}^{\exp}(w, Q) = \langle w \rangle_Q \exp((\log w)^{-1})_Q$. Some further extension can also be found in [15]. Comparing this result with the one supremum conjecture, besides replacing the Fujii–Wilson $A_{\infty}$ constant by Hrusčëv $A_{\infty}$ constant, the main difference is that the power of $A_r$ constant is larger, leading to a result which is weaker than the one-supremum conjecture. However, there is also another idea, which is replacing $A_p$ by $A_q$, where $q < p$. Our main result follows from this idea and it is formulated as follows.

**Theorem 1.1.** Let $1 \leq q < p$ and $w \in A_q$. Then
\[ \|A_S\|_{L^p} \leq C_{n,p,q}[w]_{A_p^{\frac{1}{2}}} (\exp(A_{\infty})^{\frac{1}{2}})^{\frac{1}{p'}}. \]
This result was conjectured by Lerner and Moen, see [13, p.251]. It improves the previous result of Duoandikoetxea [3], i.e.
\[ \|A_S\|_{L^p} \leq C_{n,p,q}[w]_{A_q}. \]
proved by means of extrapolation. In the next section, we will give a proof for this theorem. Extensions to rough homogeneous singular integrals will be provided in Section 3.

2. Proof of Theorem 1.1

Before we state our proof, we would like to demonstrate our understanding of this $A_q$ condition, which allows us to avoid using extrapolation or interpolation completely. We can rewrite the $A_q$ condition in the following form:
\[ \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{\frac{1}{q-1}} = \langle w \rangle_Q (w^{(1-p')} \langle w \rangle_Q^{\frac{1}{q-1}}) \]
\[ := \langle w \rangle_Q (\sigma^{\frac{1}{p'}})_{A_q}, \]
where $\sigma(t) = t^{p'(p'-1)/(q-1)} = t^{\frac{p'}{1-p'}}$ and as usual, $\sigma = w^{1-p'}$. So we have seen that the $A_q$ condition is actually the power bumped $A_p$ condition! Now we are ready to present our proof. Without loss of generality, we can assume $f \geq 0$. By duality, we have
\[ \|A_S(f)\|_{L^p} = \sup_{\|g\|_{L^p(w)} = 1} \int A_S(f)g w \]
\[ = \sup_{\|g\|_{L^p(w)} = 1} \sum_{Q \in S} \langle f \rangle_Q \langle g \rangle_Q w(Q) \]
\[ \leq \sup_{\|g\|_{L^p(w)} = 1} \sum_{Q \in S} \langle f \rangle_{A,q} \langle w^{\frac{1}{p'}} \rangle_{A,q} \langle g \rangle_Q w(Q) |Q| \]
× \exp((\log w^{-1})_Q) \frac{1}{p} \exp((\log w)_Q) \frac{1}{p} \\
≤ \left[ w \right]_A^\frac{1}{p} (A_{\text{exp}}^\infty) \frac{1}{p} \sup_{q \in \mathcal{S}} \left( \sum_{Q \in \mathcal{S}} (f_{\mathcal{W}}^q)^{1/p} |Q| \right)^{1/p} \\
× \left( \sum_{Q \in \mathcal{S}} (w_{\mathcal{W}}^q)^{1/p} \exp((\log w)_Q) |Q| \right)^{1/p} \\
≤ c_n \gamma^{-1} p \|M_A\|_{L^p} \left[ w \right] \frac{1}{A_{\text{exp}}^\infty} \|f\|_{L^p(w)}.

where in the last step, we have used the sparsity and the Carleson embedding theorem.

3. Rough homogeneous singular integral operators

Recall that the rough homogeneous singular integral operator $T_\Omega$ is defined by

$$T_\Omega(f)(x) = \lim_{\Omega \to 0} \int_{\mathbb{R}^n} f(x - y) \frac{\Omega(y)}{|y|^n} dy,$$

where $\int_{S^{n-1}} \Omega = 0$. The quantitative weighted bound of $T_\Omega$ with $\Omega \in L^\infty$ has been studied in [8], based on refinement of the ideas in [4]; see also a recent paper by the author, Pérez, Rivera-Ríos and Roncal [16], relying upon the sparse domination formula established in [1].

Our main result in this section is stated as follows.

**Theorem 3.1.** Let $1 < q < p$, $w \in A_q$ and $\Omega \in L^\infty(S^{n-1})$. Then

$$\|T_\Omega\|_{L^p(w)} \leq c_{n,p,q} \left[ w \right] \frac{1}{A_{\text{exp}}^\infty} \frac{1}{p}.$$

**Proof.** The proof is again based on the sparse domination formula [1] (see also a very recent paper by Lerner [12]). It suffices to prove

$$\|A_{r,S} f\|_{L^p(w)} \leq c_{n,p,r,q} \left[ w \right] \frac{1}{A_{\text{exp}}^\infty} \frac{1}{p},$$

where $1 < r < \frac{p}{q}$ and

$$A_{r,S} f = \sum_{Q \in \mathcal{S}} (\|f\|_Q^r)^{\frac{1}{r}} X_Q.$$

Denote $p(t) = \frac{p(r-1)}{r-1} = \frac{p}{q}$. Again, we assume $f \geq 0$. By duality, we have

$$\|A_{r,S} f\|_{L^p(w)} = \sup_{\|g\|_{L^p(w)} = 1} \int A_{r,S} f g w$$

$$= \sup_{\|g\|_{L^p(w)} = 1} \sum_{Q \in \mathcal{S}} (f^r)^{\frac{1}{r}} \frac{w(Q)}{g} \frac{w(Q)}{g}$$

$$≤ \sup_{\|g\|_{L^p(w)} = 1} \sum_{Q \in \mathcal{S}} (f^r)^{\frac{1}{r}} a_{B,Q} (w^{-\frac{1}{r}}) (w^{-\frac{1}{r}}) B,Q \frac{w(Q)}{g} \frac{w(Q)}{g} |Q|$$

$$× \exp((\log w^{-1})_Q) \frac{1}{p} \exp((\log w)_Q) \frac{1}{p}$$

$$≤ \left[ w \right]_A^\frac{1}{p} (A_{\text{exp}}^\infty) \frac{1}{p} \sup_{\|g\|_{L^p(w)} = 1} \left( \sum_{Q \in \mathcal{S}} (f^r w^r)^{\frac{1}{r}} B,Q \frac{w(Q)}{g} \frac{w(Q)}{g} |Q| \right)^{\frac{1}{p}}$$

$$× \left( \sum_{Q \in \mathcal{S}} (w_{\mathcal{W}}^q)^{\frac{1}{r}} \exp((\log w)_Q) |Q| \right)^{\frac{1}{p}}$$

$$≤ c_n \gamma^{-1} p \|M_B\|_{L^p(w)} \left[ w \right] \frac{1}{A_{\text{exp}}^\infty} \|f\|_{L^p(w)},$$

where again, in the last step we have used the sparsity and the Carleson embedding theorem. \hfill \Box
Acknowledgements

The author is supported by Juan de la Cierva-Formación 2015 FJCI-2015-24547, the Basque Government through the BERC 2014–2017 program and Spanish Ministry of Economy and Competitiveness MINECO: BCAM Severo Ochoa excellence accreditation SEV-2013-0323. He would like to thank Dr. Simeng Wang for helping him to translate the title and the abstract into French. Thanks also go to the anonymous referees for helpful comments on the presentation.

References

[1] J.M. Conde-Alonso, A. Culiuc, F. Di Plinio, Y. Ou, A sparse domination principle for rough singular integrals, Anal. PDE 10 (5) (2017) 1255–1284.
[2] J.M. Conde-Alonso, G. Rey, A pointwise estimate for positive dyadic shifts and some applications, Math. Ann. 365 (2016) 1111–1135.
[3] J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds, J. Funct. Anal. 260 (2011) 1886–1901.
[4] J. Duoandikoetxea, J.L. Rubio de Francia, Maximal and singular integral operators via Fourier transform estimates, Invent. Math. 84 (1986) 541–561.
[5] T.P. Hytönen, The sharp weighted bound for general Calderón–Zygmund operators, Ann. of Math. (2) 175 (3) (2012) 1473–1506.
[6] T.P. Hytönen, M.T. Lacey, The $A_p$–$A_{\infty}$ inequality for general Calderón–Zygmund operators, Indiana Univ. Math. J. 61 (6) (2012) 2041–2092.
[7] T.P. Hytönen, C. Pérez, Sharp weighted bounds involving $A_{\infty}$, Anal. PDE 6 (4) (2013) 777–818.
[8] T.P. Hytönen, L. Roncal, O. Tapiola, Quantitative weighted estimates for rough homogeneous singular integrals, Isr. J. Math. 218 (2017) 133–164.
[9] M.T. Lacey, An elementary proof of the $A_2$ bound, Isr. J. Math. 217 (2017) 181–195.
[10] A.K. Lerner, Mixed $A_p$–$A_1$ inequalities for classical singular integrals and Littlewood–Paley operators, J. Geom. Anal. 23 (3) (2013) 1343–1354.
[11] A.K. Lerner, On pointwise estimates involving sparse operators, N.Y. J. Math. 22 (2016) 341–349.
[12] A.K. Lerner, A weak type estimate for rough singular integrals, preprint, available at arXiv:1705.07397.
[13] A.K. Lerner, K. Moen, Mixed $A_p$–$A_\infty$ estimates with one supremum, Stud. Math. 219 (3) (2013) 247–267.
[14] A.K. Lerner, F. Nazarov, Intuitive dyadic calculus: the basics, preprint, available at arXiv:1508.05639, 2015.
[15] K. Li, Two weight inequalities for bilinear forms, Collect. Math. 68 (2017) 129–144.
[16] K. Li, C. Pérez, I.P. Rivera-Ríos, L. Roncal, Weighted norm inequalities for rough singular integral operators, preprint, available at arXiv:1701.05170.