Poisson (co)homology of truncated polynomial algebras in two variables

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Abstract

We study the Poisson (co)homology of the algebra of truncated polynomials in two vari-
ables viewed as the semi-classical limit of a quantum complete intersection studied by Bergh
and Erdmann. We show in particular that the Poisson cohomology ring of such a Poisson alge-
bra is isomorphic to the Hochschild cohomology ring of the corresponding quantum complete
intersection.

Key words: Poisson cohomology, truncated polynomial algebra, quantum complete intersections.

1 Introduction

Given a Poisson algebra, its Poisson cohomology—as introduced by Lichnerowicz in [13]—provides
important informations about the Poisson structure (the Casimir elements are reflected by the de-
gree zero cohomology, Poisson derivations modulo Hamiltonian derivations by the degree one,...).
It also plays a crucial role in the study of deformations of the Poisson structure. A classical prob-
lem in algebraic deformation is to compare the Poisson (co)homology of a Poisson algebra with
the Hochschild (co)homology of its deformation. Although these homologies are known to behave
similarly in smooth cases (see e.g. [7, 9, 10, 12]), the singular case seems more complicated to
deal with. There are many examples in which the trace groups already do not match in [1]. Even
though the dimension of the homology spaces in degree zero match, as for Kleinian singularities
in [2], it might not be the case in higher degree (see [3] and [14]).

Our aim in this paper is to provide an example of a singular Poisson algebra such that
its Poisson cohomology ring is isomorphic as a graded commutative algebra to the Hochschild
cohomology ring of a natural deformation. Namely for two integers \(a, b \geq 2\) we consider the
truncated polynomial algebra \(\Lambda(a, b) := \mathbb{C}[X, Y]/(X^a, Y^b)\) with the Poisson bracket given by
\(\{X, Y\} = XY\). This algebra is the semi-classical limit of the quantum complete intersection
\(\Lambda_q(a, b)\) studied by Bergh and Erdmann in [5]. The homological properties of this class of non-
commutative finite-dimensional algebras have been extensively studied recently (see for instance
[8, 4, 5, 6]). In particular it is proved in [8] that the generic quantum exterior algebra (cor-
responding to the case \(a = b = 2\)) provides a counter-example to Happel’s question. That is,
the global dimension of \(\Lambda_q(2, 2)\) is infinite whereas its Hochschild cohomology groups vanish in
degree greater than 2. Recently the Hochschild cohomology ring of \(\Lambda_q(a, b)\) has been described

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in [5] when \( q \) is not a root of unity. It is a five-dimensional graded algebra isomorphic to the fibre product \( \mathcal{F} := \mathbb{C}[U]/(U^2) \times \mathbb{C}(V, W)/(V^2, VW + W^2, W^2) \), with \( U \) in degree zero and \( V \) and \( W \) in degree one. In this paper we obtain the same description for the Poisson cohomology ring of \( \Lambda(a, b) \), giving rise to

**Theorem 1.1.** For all \( a, b \geq 2 \) one has \( H^p(\Lambda(a, b)) \simeq \mathcal{F} \simeq H^p(\Lambda_q(a, b)) \) as graded commutative algebras, where \( H^p(\Lambda(a, b)) \) is endowed with the Poisson cup product induced by the exterior product on skew-symmetric multiderivations.

Note that \( \Lambda(a, b) \) is not unimodular (see for instance [11] for a recent survey on the modular class of a Poisson manifold). Indeed, there is no Poincaré duality between \( H^p(\Lambda(a, b)) \) and \( H_{2-k}(\Lambda(a, b)) \). For instance \( H^2(\Lambda(a, b)) \) has dimension 1 whereas \( H_0(\Lambda(a, b)) \) has dimension \( a + b - 1 \). One cannot even expect a twisted Poincaré duality as in [12] (see the paragraph below Lemma 3.1). However, the Nakayama automorphism \( \nu \) coming from the Frobenius structure of the quantum complete intersection \( \Lambda_q(a, b) \) (see [5]) allows us to construct a Poisson module \( M_\nu \) such that

**Theorem 1.2.** \( H^k(\Lambda(a, b)) \simeq H^k(\Lambda(a, b), M_\nu) \) for all nonnegative integer \( k \).

## 2 Poisson cohomology of truncated polynomials

The algebra \( \Lambda(a, b) \) has dimension \( ab \), with basis \( \{e_{ij} := X^i Y^j \mid 0 \leq i \leq a - 1, 0 \leq j \leq b - 1\} \). One easily checks that the following formulas hold in \( \Lambda(a, b) \) for all \( i, j \):

\[
\{X^i Y^j, X\} = -jX^{i+1}Y^j; \quad \{X^i Y^j, Y\} = iX^i Y^{j+1}. \tag{1}
\]

Recall that the complex computing the Poisson cohomology is \( (\chi^k, \delta_k) \), with \( \chi^k \) the space of skew-symmetric \( k \)-derivations of \( \Lambda(a, b) \). Interestingly, these spaces vanish for \( k \geq 3 \), since \( \Lambda(a, b) \) is 2-generated. The Poisson coboundary operator \( \delta_k : \chi^k \rightarrow \chi^{k+1} \) is defined by

\[
\delta_k(P)(f_0, \ldots, f_k) := \sum_{i=0}^{k} (-1)^i \left\{ f_i, P(f_0, \ldots, \hat{f}_i, \ldots, f_k) \right\}
+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} P \left( \{f_i, f_j\}, f_0, \ldots, \hat{f}_i, \ldots, \hat{f}_j, \ldots, f_k \right)
\]

for all \( P \in \chi^k \) and \( f_0, \ldots, f_k \in \Lambda(a, b) \). It is easy to check that \( \delta_k(P) \) belongs indeed to \( \chi^{k+1} \) and that \( \delta_{k+1} \circ \delta_k = 0 \). The \( k \)-th Poisson cohomology space of \( \Lambda(a, b) \), denoted by \( H^k(\Lambda(a, b)) \), is the \( k \)-th cohomology space of this complex. The Poisson cohomology ring is the space \( H^\ast(\Lambda(a, b)) := \bigoplus_{k=0}^{\infty} H^k(\Lambda(a, b)) \). Endowed with the cup product induced by the exterior product on \( \chi^* := \bigoplus k \chi^k \) it becomes a graded commutative algebra. We start by describing the spaces \( H^k(\Lambda(a, b)) \) for \( k = 0, 1, 2 \) as it is clear that \( H^k(\Lambda(a, b)) = 0 \) for \( k \geq 3 \).

Let \( \lambda = \sum \lambda_{ij} e_{ij} \in \Lambda(a, b) \), and assume it satisfies \( \{\lambda, X\} = \{\lambda, Y\} = 0 \). Then equations [11] lead successively to \( \lambda_{ij} = 0 \) for all \( i \neq a - 1 \) and all \( j \neq 0 \), and then to \( \lambda_{ij} = 0 \) for all \( i \neq 0 \) and \( j \neq b - 1 \). Hence

**Proposition 2.1.** The Poisson centre \( H^0(\Lambda(a, b)) \) is equal to \( \mathbb{C} \oplus \mathbb{C} X^{a-1} Y^{b-1} = \mathbb{C} e_{0,0} \oplus \mathbb{C} e_{a-1, b-1} \).
Any derivation \( d \in \chi^1 \) is uniquely determined by the values of \( d(X) = \sum \lambda_{i,j} e_{i,j} \) and \( d(Y) = \sum \lambda_{i,j} e_{i,j} \). Moreover, \( d \) must satisfy the relations \( d(X^a) = d(Y^b) = 0 \), that is \( X^{a-1} d(X) = Y^{b-1} d(Y) = 0 \), since \( d \) is a derivation. From that one easily deduces that \( \lambda_{0,j} = \lambda_{i,0} = 0 \) for all \( i, j \). Hence the space \( \chi^1 = \text{Der}(A, A) \) has basis \( \{d_{i,j}\} \cup \{d'_{i,j}\} \), where:

1. for \( 1 \leq i < a \) and \( 0 \leq j < b \), the derivation \( d_{ij} \) is defined by \( d_{ij}(X) = X^i Y^j \) and \( d_{ij}(Y) = 0 \);
2. for \( 0 \leq i < a \) and \( 1 \leq j < b \), the derivation \( d'_{ij} \) is defined by \( d'_{ij}(X) = 0 \) and \( d'_{ij}(Y) = X^i Y^j \).

In particular, \( \text{dim}(\chi^1) = b(a - 1) + a(b - 1) \). Let \( d = \sum_{i \neq 0} \alpha_{i,j} d_{i,j} + \sum_{j \neq 0} \beta_{i,j} d'_{i,j} \in \chi^1 \). Then \( d \in \text{Ker}\delta_1 \) if and only if it satisfies \( d(\{X, Y\}) = \{d(X), Y\} \cup \{X, d(Y)\} \), that is:

\[
\sum_{j \neq 0} \beta_{i,j} X^{i+1} Y^j + \sum_{i \neq 0} \alpha_{i,j} X^i Y^{j+1} = \sum_{i \neq 0} i \alpha_{i,j} X^i Y^{j+1} + \sum_{j \neq 0} j \beta_{i,j} X^{i+1} Y^j.
\]

Identifying the coefficients in front of \( X^i Y^j \) leads to

\[
d \in \text{Ker}\delta_1 \iff (1 - j) \beta_{i,j - 1} + (1 - i) \alpha_{i,j - 1} = 0 \quad \forall 1 \leq i \leq a - 1, 1 \leq j \leq b - 1. \tag{2}
\]

**Proposition 2.2.** \( HP^1(\Lambda(a, b)) = \mathbb{C} d_{1,0} \oplus \mathbb{C} d'_{0,1} \).

**Proof.** Let \( d = \sum_{i \neq 0} \alpha_{i,j} d_{i,j} + \sum_{j \neq 0} \beta_{i,j} d'_{i,j} \) \( \in \text{Ker}\delta_1 \). Set \( \lambda := \sum_{j \neq 0} \frac{\alpha_{i,j}}{i - 1} X^i Y^j \in \Lambda(a, b) \). From [1] one deduces that \( d_1 = d + \{\lambda, -\} \) is a Poisson derivation satisfying \( d_1(X) = \sum_{i \geq 1} \alpha_{i,0} X^i \). Then one deduces from [2] that \( \alpha_{i,0} = 0 \) for all \( i \neq 1 \), that is \( d_1(X) = \alpha_{1,0} X^1 \), and \( d_1(Y) = \sum_{i = 0}^{a - 2} \beta_{1,i} X^i Y + \sum_{j = 1}^{b - 1} \beta_{i,1} X^i Y^{j - 1} \). Now set \( \mu := \sum_{i = 1}^{a - 2} \frac{\beta_{i,1}}{i - 1} X^i + \sum_{j = 1}^{b - 1} \frac{\beta_{1,i}}{i - 1} X^i Y^{j - 1} \) and \( d_2 := d_1 - \{\mu, -\} \). From the construction we get that \( d_2 = \alpha_{1,0} d_{1,0} + \beta_{0,1} d'_{0,1} \), so that the images of \( d_{1,0} \) and \( d'_{0,1} \) span \( HP^1(\Lambda(a, b)) \). One deduces from [1] that they actually form a basis of \( HP^1(\Lambda(a, b)) \). \(\square\)

The complex computing the Poisson cohomology is vanishing after \( \chi^2 \), so we use the Euler-Poincaré principle to compute the dimension of \( HP^2(\Lambda(a, b)) \). First note that a skew-symmetric derivation \( f \in \chi^2 \) is determined by \( f(X \wedge Y) = \sum c_{i,j} e_{i,j} \), with \( a X^{a-1} f(X \wedge Y) = b Y^{b-1} f(X \wedge Y) = 0 \), so that \( c_{0,j} = c_{i,0} = 0 \) for all \( i, j \). Thus \( \chi^2 \) has dimension \( (a - 1)(b - 1) \).

**Proposition 2.3.** \( HP^2(\Lambda(a, b)) = \mathbb{C} f_{1,1}, \) with \( f_{1,1} : X \wedge Y \mapsto XY \).

**Proof.** We first prove that \( HP^2(\Lambda(a, b)) \) has dimension 1. From the Euler-Poincaré principle we get \( \dim(HP^2(\Lambda(a, b))) = \dim \chi^2 - \text{rg} \delta_1 = (a - 1)(b - 1) - (\dim \chi^1 - \dim \text{Ker}\delta_1) = (a - 1)(b - 1) - [a(b - 1) + b(a - 1)] - (\dim HP^1(\Lambda(a, b)) + \text{rg} \delta_0) = 1 - ab + (2 + \dim \Lambda(a, b) - \dim \text{Ker}\delta_0) = 3 - ab + ab - 2 = 1 \). Now all that remains is to check that \( X \wedge Y \mapsto XY \) is not a Poisson coboundary. Any \( P \in \chi^1 \) satisfies \( \delta_1(P)(X \wedge Y) = \{X, P(Y)\} - \{Y, P(X)\} - P(XY) \). Moreover one has \( P(XY) = P(X) Y + P(Y) X \), and it results straight from formulas [1] that one cannot have \( \delta_1(P)(X \wedge Y) = XY \). \(\square\)

**Proof of Theorem [1]** We have already proved that the graded commutative algebra \( HP^*(\Lambda(a, b)) \) is five-dimensional with basis \( (e_{0,0}, e_{a-1,b-1}, d_{1,0}, d'_{0,1}, f_{1,1}) \) with degree respectively \( (0,0,1,1,2) \). One can easily check that \( e_{a-1,b-1} \) annihilates \( d_{1,0}, d'_{0,1} \) and \( f_{1,1} \), and that \( d_{1,0} \sim d'_{0,1} = f_{1,1} \). Thus we have \( HP^*(\Lambda(a, b)) \cong \mathbb{F}, \) and one concludes using [5] Theorem 3.3]. \(\square\)

Note that the basis \( (e_{0,0}, e_{a-1,b-1}, d_{1,0}, d'_{0,1}, f_{1,1}) \) of \( HP^*(\Lambda(a, b)) \) corresponds to the basis \( (1, y^{b-1} x^{a-1}, g, h, hg) \) considered in the proof of [5] Theorem 3.3].
3 (Twisted) Poisson homology

Let $M$ be a right Poisson module over $\Lambda(a,b)$ (the reader is referred to [12] Section 3.1 for the definition of a Poisson module). We denote by $\{-,-\}_M : M \otimes \Lambda(a,b) \rightarrow M$ its external bracket. Then one defines a chain complex on the $\Lambda(a,b)$-modules $M \otimes_{\Lambda(a,b)} \Omega^k$, where $\Omega^k$ denotes the so-called Kähler differential $k$-forms of $\Lambda(a,b)$, as follows. The boundary operator $\partial_k : M \otimes_{\Lambda(a,b)} \Omega^k \rightarrow M \otimes_{\Lambda(a,b)} \Omega^{k-1}$ is defined by

$$\partial_k(m \otimes da_1 \wedge \cdots \wedge da_k) = \sum_{i=1}^k (-1)^{i+1} \{m, a_i\} M \otimes da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge da_k$$

$$+ \sum_{1 \leq i < j \leq k} (-1)^{i+j} m \otimes d\{a_i, a_j\} \wedge da_1 \wedge \cdots \wedge \widehat{da_i} \wedge \cdots \wedge \widehat{da_j} \wedge \cdots \wedge da_k,$$

where we have removed the expressions under the hats in the previous sums and $d$ denotes the exterior differential. The homology of this complex is denoted by $\text{HP}_*(\Lambda(a,b), M)$ and called the Poisson homology of the Poisson algebra $\Lambda(a,b)$ with values in the Poisson module $M$. Before computing the Poisson homology spaces for a specific module, we describe the spaces $\Omega^k$. By definition $\Omega^k$ is a $\Lambda(a,b)$-module generated by the wedge products of length $k$ of the 1-differential forms $dX, dY$. In particular $\Omega^k = 0$ for $k \geq 3$. In the remaining cases the torsion coming from the relations $X^a = Y^b = 0$ leads to the following spaces:

$$\Omega^0 = \Lambda(a,b), \quad \Omega^1 = \bigoplus_{0 \leq i \leq a-2, 0 \leq j \leq b-1} \mathbb{C}X^iY^j dX \oplus \bigoplus_{0 \leq i \leq a-1, 0 \leq j \leq b-2} \mathbb{C}X^iY^j dY,$$

and

$$\Omega^2 = \bigoplus_{0 \leq i \leq a-2, 0 \leq j \leq b-2} \mathbb{C}X^iY^j dX \wedge dY.$$

Their dimensions are respectively $ab, (a-1)b + b(a-1), (a-1)(b-1)$. As the bracket $\{\lambda, \mu\}$ belongs to the ideal generated by $XY$ for any $\lambda, \mu \in \Lambda(a,b)$, one easily checks that:

**Lemma 3.1.** $\text{HP}_0(\Lambda(a,b))$ has dimension $a+b-1$.

As $a+b-1 \geq 3$ for any $a,b \geq 2$, and $\text{HP}^2(\Lambda(a,b))$ has dimension 1, this lemma shows that there is no Poincaré duality between $\text{HP}^k(\Lambda(a,b))$ and $\text{HP}_{2-k}(\Lambda(a,b))$.

We may ask now if there is a twisted duality similar to the one obtained in [12]. The Poisson algebra $\Lambda(a,b)$ is the semi-classical limit of the quantum complete intersection $\Lambda_q(a,b)$ which is the $\mathbb{C}$-algebra generated by $x,y$ with relations $xy = qyx, x^a = 0, y^b = 0$ (see [3]). Any diagonal automorphism $\sigma$ of $\Lambda_q(a,b)$ defined by $\sigma(x) = q^\alpha x$ and $\sigma(y) = q^\beta y$ gives rise to a Poisson module $M_\sigma$ of $\Lambda(a,b)$ via a semi-classical limit process (see [12] Section 3.1 for details). As a vector space $M_\sigma$ is equal to $\Lambda(a,b)$, and the external Poisson bracket is defined by:

$$\{X^iY^j, X\}_{M_\sigma} = -(j + \alpha)X^{i+1}Y^j \quad \text{and} \quad \{X^iY^j, Y\}_{M_\sigma} = (i - \beta)X^iY^{j+1}. \quad (3)$$

From these formulas one easily deduces that if $\alpha < -b+1$ and $\beta > a-1$, then $\text{HP}_0(\Lambda(a,b), M_\sigma) = \mathbb{C}$. So one might be tempted to use such a Poisson module to restore a twisted Poincaré duality between $\text{HP}^k(\Lambda(a,b))$ and $\text{HP}_{2-k}(\Lambda(a,b), M_\sigma)$ as in [12] Theorem 3.4.2. Unfortunately, these hypotheses on $\alpha$ and $\beta$ also lead to $\text{HP}_2(\Lambda(a,b), M_\sigma) = 0$, so that $\text{HP}_2(\Lambda(a,b), M_\sigma)$ is not isomorphic to $\text{HP}^0(\Lambda(a,b))$. 

4
The Nakayama automorphism \( \nu \) of \( \Lambda_q(a,b) \) coming from the Frobenius algebra structure of \( \Lambda_q(a,b) \) is defined by \( \nu(x) = q^{1-b}x \) and \( \nu(y) = q^{a-1}y \). This automorphism was used in \cite[Section 3]{5} to link the twisted Hochschild homology and the Hochschild cohomology of \( \Lambda_q(a,b) \) in each degree. We end this paper by computing the dimensions of the Poisson homology spaces of \( \Lambda(a,b) \) with value in the Poisson module \( M_\nu \), corresponding to the Nakayama automorphism \( \nu \).

**Proposition 3.2.** The twisted Poisson homology spaces \( HP_k(\Lambda(a,b), M_\nu) \) have dimension 2,2,1 for \( k = 0,1,2 \) respectively, and are null if \( k \geq 3 \).

**Proof.** From \( \partial_1(X^iY^j \otimes dX) = \{X^iY^j, X\}_M \) and \( \partial_1(X^iY^j \otimes dY) = \{X^iY^j, Y\}_M \) one easily gets that \( HP_0(\Lambda(a,b), M_\nu) = \Lambda(a,b) / \text{Im} \partial_1 \) is generated as a \( \mathbb{C} \)-vector space by the classes of 1 and \( X^aY^b \) modulo \( \text{Im} \partial_1 \).

We compute \( \partial_2(X^iY^j \otimes dX \wedge dY) = \{X^iY^j, X\}_M \otimes dY - \{X^iY^j, Y\}_M \otimes dX - X^iY^j \otimes (XdY + YdX) = -(j - b + 1)X^{i+1}Y^j \otimes dY - (i - a + 2)X^iY^{j+1} \otimes dX \), from which one easily sees that \( \text{Ker} \partial_2 = HP_2(\Lambda(a,b), M_\nu) \) is the \( \mathbb{C} \)-vector space generated by \( X^{a-2}Y^{b-2} \otimes dX \wedge dY \).

We use the Euler-Poincaré principle to conclude. There is only one dimension missing, all the other ones have the same value as for the Poisson cohomology of \( \Lambda(a,b) \), so we end up with the same value for \( \text{dim} HP_1(\Lambda(a,b), M_\nu) \), that is 2. \( \square \)

We deduce from Proposition 3.2 and Theorem 1.2 a duality similar to the one appearing in \cite{5}, that is \( HP^k(\Lambda(a,b)) \cong HP_k(\Lambda(a,b), M_\nu) \) for all nonnegative integer \( k \) as claimed in Theorem 1.2.

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