The gravitational deflection of light in MOND

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Abstract. The deflection angle $\Delta \phi$ of light rays by the gravitational field of a spherical system $M(r)$ is calculated using the MOdified Newtonian Dynamics (MOND). It is shown that $\Delta \phi$ with an impact parameter $r_0$ can be expressed by the measured rotation velocity $v(r)$ as

$$\Delta \phi(r_0) = 2 \int_{r_0}^{\infty} \frac{v^2(r)}{c^2} \frac{r_0dr}{r^2 - r_0^2},$$

where

$$v(r) = \begin{cases} \left(\frac{Ga_0 M(r)}{r}\right)^{1/4}, & r_0 > r_c; \\ \left(\frac{GM(r)}{r}\right)^{1/2}, & r_0 \leq r_c, \end{cases}$$

and $r_c$ is the critical radius that is determined by the critical acceleration $a_0$. In the Newtonian limit of the gravitational acceleration $a \gg a_0$, $\Delta \phi$ approaches $\Delta \phi = \frac{2GM(r_0)}{c^2r_0}$ with the projected surface mass $m(r_0)$.

Whilst the asymptotic value of $\Delta \phi$ reaches a constant $\pi(v_\infty/c)^2$ in the low-acceleration limit of $a \ll a_0$. Taking the empirical correction of a factor of 2 from the theory of general relativity into account and utilizing the relation between rotation velocity $v$ and velocity dispersion $\sigma$, MOND results naturally in a constant deflection angle, $4\pi(\sigma/c)^2$, which has been widely used in the present-day study of gravitational lensing by galaxies and clusters of galaxies, implying that without introducing the massive halos acting as $r^{-2}$ for dark matter MOND has no difficulty in reproducing the known cases of gravitational lensing associated with galaxies and clusters of galaxies.

Key words: gravitation – gravitational lensing – dark matter

1. Introduction

Many attempts have been made for decades to account for the large discrepancies between the luminous matter seen from the galaxy radiations and the dynamical matter derived from galaxy rotation curves. A conventional way out of these mass discrepancies is to assume the existence of some kinds of unseen matter, which has been widely accepted today by the community of astronomers and physicists and many experiments have been devoted to searches for these dark matter despite that there have been no positive results obtained thus far. Whilst some alternatives to the puzzle are to modify the Newtonian inverse square law, among which the most successful of the empirically suggested models is the Milgrom’s MOdified Newtonian Dynamics (MOND) (Milgrom 1983a, b, c).

As a proposal to avoid introducing dark matter in astrophysics, MOND has naturally predicted the flat form of the rotation curves for spiral galaxies, the Tully-Fisher law and the luminosity-rotation velocity/velocity dispersion relations in spiral/elliptical galaxies. Although there have been some arguments, besides the unknown details of any fundamental physics, against MOND using the stellar kinematics in dwarf galaxies (Lake 1989; Gerhard & Spergel 1992) and the dynamical properties of X-ray selected clusters of galaxies (Gerbal et al. 1992), these claims still need to be further considered (Milgrom 1991; 1993).

As was pointed out recently by Sanders and Begeman (1994), the more fundamental problem with MOND is the observation of gravitationally distorted images of background galaxies by the gravitational potential of intervening clusters of galaxies. Indeed, the deflection of light rays by the gravitational field has remained to be an unsolved problem in MOND (Milgrom 1993, private communication). The presently known gravitational cases, such as the lensed quasar pairs, luminous arcs (see Surdej & Soucail 1993 for a summary), quasar-galaxy associations (see Wu 1994 for a summary), etc., associated with galaxies and clusters of galaxies where MOND would appear to be important can constitute a critical test for MOND.
This paper is then to calculate the deflection of light by the gravitational field of a spherical matter system in MOND. Due to the absence of a general theory of gravity in MOND comparable to General Relativity, the present paper deals with the motion of particles only in its classical way.

2. The equation of motion in MOND

In MOND the true gravitational acceleration $\mathbf{g}$ is related to the Newtonian gravitational acceleration $\mathbf{g}_N$ as

$$\mu(g/a_0)\mathbf{g} = \mathbf{g}_N,$$

where $a_0$ is a new fundamental constant or the critical acceleration parameter with the value of $a_0 \sim 10^{-8}$ cm s$^{-2}$ (Milgrom & Braun 1988), and $\mu$ is some function that has the asymptotic behaviour

$$\mu(x) = 1, \quad x \gg 1;$$
$$\mu(x) = x, \quad x \ll 1.$$  \hspace{1cm} (2)

In particular, the major properties of the results in MOND are insensitive to the choice of $\mu$ (Milgrom 1983b). For simplicity, we take $\mu(x) = 1$ when $x > 1$ and $\mu(x) = x$ when $x \leq 1$.

The motion of a test particle in the gravitational field of spherical symmetry is described by

$$\begin{cases}
\dot{r} - \mu \frac{\dot{\phi}}{\dot{r}} = -\frac{GM(r)}{r^2}, & r > r_c \\
\dot{\phi} = 0.
\end{cases}$$  \hspace{1cm} (3)

Here $a_r$ and $a_\phi$ represent the components of acceleration along the $r$- and $\phi$- direction, respectively. $M(r)$ is the total mass enclosed within the radius of $r$. The equation of motion of the test particle can then be specifically written as

$$\begin{cases}
\dot{r} - r\ddot{\phi} = -\sqrt{\frac{G_0 M(r)}{r^2}}, & r > r_c \\
r^2 \ddot{\phi} = c_1; \hspace{1cm} (4)
\end{cases}$$

and

$$\begin{cases}
\dot{r} - r\ddot{\phi} = -\frac{GM(r)}{r^2}, & r \leq r_c \\
r^2 \ddot{\phi} = c_2; \hspace{1cm} (5)
\end{cases}$$

where $r_c$ measures the radius separating Newtonian mechanics from MOND and is determined by

$$r_c = \sqrt{\frac{GM(r_c)}{\mu a_0}},$$  \hspace{1cm} (6)

c_1$ and $c_2$ are the integral constants, and the time derivative of the position of the test particle is denoted by “$\cdot$”.

The circular rotation velocity $v(r)$ resulting from these two different kinds of equations of motion reflects their significant difference. Setting $\dot{r} = \ddot{\phi} = 0$ yields

$$v(r) = \begin{cases}
(G_0 M(r))^{1/4}, & r > r_c; \\
\left(\frac{GM(r)}{r}\right)^{1/2}, & r \leq r_c.
\end{cases}$$  \hspace{1cm} (7)

For a pointlike mass or a finite mass distribution within a radius of $R$, a constant circular velocity instead of the decreasing form of $r^{-1/2}$ in the Newtonian mechanics is predicted by MOND when $r$ tends to infinity. This provides a natural explanation for the flatness of the rotation curves of spiral galaxies alternative to dark matter.

Some initial and/or boundary conditions are needed to find the solution to the orbit of a test particle from eqs. (4) and (5). Define an impact parameter $r_0$

$$\dot{r}(r_0) = 0$$  \hspace{1cm} (8)

so that the velocity of the test particle at $r = r_0$ can be written as

$$v_0 = r_0 \dot{\phi}(r_0)$$  \hspace{1cm} (9)

Moreover, the first-order time derivatives of position of the test particle at the critical radius $r_c$ are required to keep their continuities when the particle crosses the boundary of $r = r_c$ from either sides. An immediate consequence of this requirement is that the two integral constants $c_1$ and $c_2$ in eqs. (4) and (5) have the same value and

$$c_1 = c_2 = v_0 r_0$$  \hspace{1cm} (10)

In fact, this is the result of the conservation of angular momentum analogue to the one in Newtonian dynamics.

The equation of motion of the test particle, eqs. (4) and (5), can be unified to one simple equation by replacing the matter distribution $M(r)$ by the circular rotation velocity $v(r)$ of eq. (7)

$$\frac{d^2 (\frac{\dot{\phi}}{r})}{d\phi^2} + \frac{1}{r} \left[ 1 - \left(\frac{r}{r_0}\right)^2 \left(\frac{v(r)}{v_0}\right)^2 \right] = 0.$$  \hspace{1cm} (11)

The solution to this equation with the boundary condition of eq. (9) is

$$\frac{d\phi}{dr} = \pm \frac{r_0}{r \sqrt{r^2 - r_0^2}} \frac{1}{\sqrt{1 - \frac{r^2}{r^2 - r_0^2} \int_{r_0}^{r_c} \frac{2v(x)^2}{v_0^2} dx}}.$$  \hspace{1cm} (12)

Given the observed rotation curve $v(r)$ in and around the massive system or the matter distribution $M(r)$, the orbit of a test particle can be found from eq. (12).
3. The deflection of light in MOND

The deflection angle of light rays propagating from and to the distant universe with an impact parameter of \( r_0 \) from the massive body \( M \) is

\[
\Delta \phi = 2|\phi(r_0) - \phi(\infty)| - \pi. \tag{13}
\]

The test particle discussed in the above section is now replaced by the photons and hence, the speed of light \( c \) is used instead of \( v_0 \).

Eq.(12) can now be expanded in the series of \( (v(r)/c)^2 \):

\[
\phi(r) = \pm \left[ \int_{r_0}^{r} \frac{r_0 dr}{r} - \int_{r_0}^{r} \frac{\sqrt{v^2(r)} \, dr}{c^2} + O\left(\left(\frac{v(r)}{c}\right)^4\right) \right]. \tag{14}
\]

Note that the integral of \( \int_{r_0}^{r} \frac{\sqrt{v^2(r)} \, dr}{c^2} \) approaches infinity when \( r \to \infty \), which may then invalidate the expansion of eq.(12) into eq.(14). To see this, utilizing a constant circular rotation velocity \( v_\infty \) at a large distance \( r^* \) well beyond both the radius of the massive body and the MOND limit leads to

\[
\int_{r^*}^{r} \left(\frac{v_\infty}{c}\right)^2 \frac{dr}{r} = \left(\frac{v_\infty}{c}\right)^2 \ln \frac{r}{r^*}.
\]

However, the distance \( r \) that satisfies \( r \sim r^* \exp\left((c/v_\infty)^2\right) \) would exceed the whole visible size of the present universe for any realistic gravitationally bound systems like galaxies and clusters of galaxies. Therefore, eq.(14) is valid in the presently observable universe. The first term of the right-hand side of eq.(14) is the orbit of a straight line when the photon travels in an Euclidean space, and the rest terms represent the contributions of the gravitational field.

Finally, to the first order of \( (v(r)/c)^2 \) the deflection of light is

\[
\Delta \phi = 2 \int_{r_0}^{\infty} \frac{v^2(r)}{c^2} \frac{r_0}{r \sqrt{r^2 - r_0^2}} dr. \tag{15}
\]

3.1. Pointlike mass

In the case of pointlike mass, inserting eq.(7) with \( M(r) = M \) into eq.(15) yields:

(1) For \( r_0 > r_c \),

\[
\Delta \phi = \pi \frac{\sqrt{G_0 M}}{c^2} \tag{16}
\]

(2) For \( r_0 \leq r_c \),

\[
\Delta \phi = \frac{2GM}{c^2 r_0} \sqrt{\frac{r_c - r_0}{r_c + r_0}} + \frac{2GM}{c^2 r_c} \sqrt{\frac{r_c - r_0}{r_c + r_0}}.
\]

In particular, eq.(17) reads

\[
\Delta \phi = \frac{2GM}{c^2 r_0}, \quad r_c \to \infty, \tag{18}
\]

i.e., the result of Newtonian limit. Whilst taking \( r_c = r_0 \) in eq.(17) identifies the result in a non-Newtonian limit [eq.(16)].

The most interesting result appears in the low-acceleration limit or at large radii of galaxies and clusters of galaxies where MOND plays an important role. Eq.(16) can be simplified using the observable asymptotic rotation velocity \( v_\infty \)

\[
\Delta \phi = \pi \left(\frac{v_\infty}{c}\right)^2. \tag{19}
\]

The significant feature of the deflection of light in MOND is its constant angle of light bending, in accord with the flat rotation curves at large distances from the centers of galaxies and of clusters of galaxies. A relatively larger deflection is then provided by MOND than that by the Newtonian mechanics (see Fig.1).

Recall that the present theory of gravitational lensing uses indeed a constant deflection of light for the dark matter associated with galactic halos (Turner, Ostriker

Fig. 1. The deflecting angle of light \( \Delta \phi \) in unit of \( \phi_0 = 2\pi(v_\infty/c)^2 \) by a pointlike mass versus the impact parameter \( r_0 \) in unit of \( r_c \). Beyond the critical distance \( r_c \), MOND predicts a constant deflection of light instead of the declining form of \( 1/r \) derived from the Newtonian dynamics (the dotted line)
& Gott, 1984) and for matter distributions in clusters of galaxies (Wu & Hammer, 1993; Wu 1993). Essentially, these constant deflections of light can account for the statistical properties of the lensed quasar pairs and the giant luminous arcs. To quantitatively compare the result of MOND with the update theory of gravitational lensing that adopts the line-of-sight velocity dispersion for galaxies and clusters of galaxies, the circular velocity \( v_\infty \) in eq.(19) should be written in terms of the observed line-of-sight velocity dispersion. Unfortunately, there has not been set up an analog of the virial theorem in MOND (Milgrom 1983c). Additionally, the present discussion is confined to a pointlike mass rather than an extended system, although the point mass can be regarded as a good approximation to a galaxy seen at a large distance. Note that there is no massive dark halo at all surrounding the luminous arc. To quantitatively compare the result of MOND with the update theory of gravitational lensing associated luminous arcs. To quantitatively compare the result of MOND with the update theory of gravitational lensing associated luminous arcs.

3.2. Extended mass distribution: rotation velocity

For a galaxy whose circular velocity has been well measured, \( \Delta \phi \) can be straightforward obtained using eq.(15). A typical rotation curve exhibits an increasing form in the central region and tends to a constant \( v_\infty \) beyond a characteristic radius of \( r_h \). Two types of light orbit should be specified:

(1) For \( r_0 > r_h \),

\[
\Delta \phi = \pi \frac{v^2}{c^2};
\]

(2) For \( r_0 \leq r_h \),

\[
\Delta \phi = 2\pi \frac{v^2}{c^2} \left[ \sin^{-1} \frac{r_0}{r_h} + \int_{r_0}^{r_h} \left( \frac{v^2(r)}{v^2_\infty} \right) \frac{r_0 dr}{r \sqrt{r^2 - r_0^2}} \right].
\]

Figure 2 illustrates the variations of \( \Delta \phi \) with the impact parameters for three kinds of rotation curves. Note that utilizing the observed rotation curves in the calculation of \( \Delta \phi \) is independent of the assumed dynamics of whether it is the Newtonian one or the MOND.

\[
\begin{align*}
\Delta \phi &= \pi \frac{v^2}{c^2}, \\
\Delta \phi &= 2\pi \frac{v^2}{c^2} \left[ \sin^{-1} \frac{r_0}{r_h} + \int_{r_0}^{r_h} \left( \frac{v^2(r)}{v^2_\infty} \right) \frac{r_0 dr}{r \sqrt{r^2 - r_0^2}} \right].
\end{align*}
\]

**Fig. 2.** The deflecting angle of light \( \Delta \phi \) in unit of \( \phi_0 = \pi (v_\infty/c)^2 \) derived from the observed circular velocity. Three types of rotation curves in the central region of radius of \( r_h \) are assumed: (1) \( v(r) = v_\infty (r/r_h)^{1/3} \) (top); (2) \( v(r) = v_\infty (r/r_h) \) (middle); (3) \( v(r) = v_\infty (r/r_h)^2 \) (bottom). Beyond the radius of \( r_h \) the rotation velocity remains unchanged, leading to a constant deflection of light.

3.3. Extended mass distribution: surface brightness

Surface brightness (optical/X-ray) of galaxies and of clusters of galaxies are relatively easier to measure than their rotation velocities. Assuming that the light profile traces the mass distribution as argued By MOND, one can obtain the deflection of light from eqs.(7) and (15): (1) For \( x_c \leq x_0 \),

\[
\Delta \phi(x_0) = \frac{r_c x_0}{x_c} \int_{x_0}^{x_c} \frac{\dot{M}(x) dx}{\sqrt{M(x_c) x_c^2 \sqrt{x^2 - x_0^2}}}
\]
+\int_{x_{c}}^{\infty} \frac{\sqrt{M(x)}dx}{x \sqrt{x^2 - x_0^2}}, \quad (24)

(2) For \( x \gg x_{c} \),

\[
\frac{\Delta \phi(x_0)}{\phi_0} = x_0 \int_{x_0}^{\infty} \frac{\sqrt{M(x)}dx}{x \sqrt{x^2 - x_0^2}}.
\]

Here \( \phi_0 = 4\sqrt{G\rho_0 MT}/c^2 \), and \( M(x) \) is the dimensionless mass enclosed within \( x \) normalized to the total mass \( MT \). All the distances \( x \) are measured in unit of the so-called length scale in the model fit to the surface luminosity distribution. The subscripts “\( c \)” and “\( 0 \)” indicate the critical and the impact positions, respectively.

An example is shown in Fig.3 using the King model for the matter distribution. This model has been found to fit fairly well the surface brightness of the central parts of galaxies and of clusters of galaxies, and has been widely adopted today in the modeling of the luminosity distribution for these systems. The spatial mass distribution \( M(r) \) can be found to be

\[
M(x) = \frac{9b\sigma^2}{G} \left( \ln(x + \sqrt{1 + x^2}) - \frac{x}{\sqrt{1 + x^2}} \right),
\]

in which \( x = r/b \) and \( b \) is the core radius. The deflection of light resulting from this model shows a slight increasing form instead of the declining one of \( r^{-1} \) in the Newtonian limit, when the impact distance tends to infinity. This shows again that MOND provides a significantly large deflection of light beyond \( a_0 \).

4. Discussion and conclusions

Dark massive halos of galaxies and unseen matter excess in clusters of galaxies, if they were real, can be well described by a singular isothermal sphere with a power law of \( r^{-2} \). This model has been widely used in the study of gravitational lensing today. Certainly, this is due partially to the fact that it results in a constant deflection of light \( 4\pi(\sigma/c)^2 \), making the calculations much simplified. Nevertheless, there have been no any conclusive evidences, from the experiments that search for dark matter, for the existence of the hidden mass. Dynamical analysis of the rotational curves of galaxies on the basis of Newtonian mechanics is the only strong observational evidence that shows the mass discrepancies.

An alternative to the missing mass is to modify the Newtonian gravity. Though it is not conventional, Milgrom’s MOND has successfully predicted the flat rotation curves of galaxies, leading to a new sight into the dynamics on large scales. The present paper has computed one of the critical issues in MOND, the deflection of light rays by a spherical gravitational field. Surprisingly, MOND provides a constant deflecting angle at large distance from the center of the gravitational field, which is consistent with the the value presently used in the study of gravitational lensing by galaxies and clusters of galaxies. It is then likely that all the lensing cases can be equally reproduced in MOND without the massive dark matter in galaxies and in clusters of galaxies. Similar to the flatness of rotation velocity in galaxies predicted by MOND without assuming the massive halos, the constant deflection of light from MOND has the same effect as the \( r^{-2} \) halos did.

It has been shown that light bending is no more a critical argument against MOND today. Conversely, MOND predicts a reasonable deflection angle of light by large massive systems. Therefore, whether or not MOND reflects the nature of gravity needs to be further investigated using other astronomical methods.

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