HARDY'S INEQUALITY FOR HERMITE EXPANSIONS
REVISITED

PENG CHEN AND JINSEN XIAO

Abstract. In this article, we give a short proof of Hardy’s inequality for
Hermite expansions of functions in the classical Hardy spaces \( H^p(\mathbb{R}^n) \), by using
an atomic decomposition of the Hardy spaces associated with the Hermite
operators. When the space dimension is 1, we obtain a new estimate of Hardy’s
inequality for Hermite expansions in \( H^p(\mathbb{R}) \) for the range \( 0 < p < 1 \).

1. Introduction

A function \( f(z) \) analytic in the unit disk \(|z| < 1\) for \( z \in \mathbb{C} \) is said to be of class
\( H^p(\mathbb{C}) \) where \( 0 < p < \infty \) if
\[
\lim_{r \to 1} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right\}^{1/p} < \infty.
\]
For \( f(z) = \sum_{k=0}^{\infty} a_k z^k \), it is interesting to find a condition on the Taylor coefficient
\( a_k \) which is both necessary and sufficient for \( f \) to be in \( H^p(\mathbb{C}) \) for some \( p \in (0, \infty) \).
If \( 1 < p < \infty \), the problem is equivalent to that of describing the Fourier coefficients
of \( L^p \) functions, as the M. Riesz theorem shows. For the case \( 0 < p \leq 1 \), the known
results can be described as Hardy’s inequality given by
\[
\sum_{k=0}^{\infty} \frac{|a_k|^p}{(k+1)^{2-p}} \leq C_p \|f\|^p_{H^p(\mathbb{C})},
\]
where the constant \( C_p \) depends only on \( p \) (see [2, Theorems 6.2]).

Analogues of Hardy’s inequality in the context of eigenfunction expansions have
been considered, see [1, 7, 8, 9, 10] and the references therein. Recall that the
Hermite operator \( L \) on \( \mathbb{R}^n \) is defined by
\[
L = -\Delta + |x|^2 = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + |x|^2, \quad x = (x_1, \cdots, x_n) \in \mathbb{R}^n.
\]
The operator \( L \) is non-negative and self-adjoint with respect to the Lebesgue measure on \( \mathbb{R}^n \). For \( k \in \mathbb{N} = \{0, 1, 2, \cdots\} \), the Hermite polynomials \( H_k(t) \) on \( \mathbb{R} \)
are defined by \( H_k(t) = (-1)^k e^{t^2} \frac{d^k}{dt^k} (e^{-t^2}) \), and the Hermite functions \( h_k(t) := (2^k k! \sqrt{\pi})^{-1/2} H_k(t)e^{-t^2}/2, k = 0, 1, 2, \cdots \) form an orthonormal basis of \( L^2(\mathbb{R}) \).
any multiindex $\mu \in \mathbb{N}^n$, the $n$-dimensional Hermite functions are given by tensor product of the one-dimensional Hermite functions:

$$\Phi_\mu(x) = \prod_{i=1}^{n} h_{\mu_i}(x_i), \quad \mu = (\mu_1, \ldots, \mu_n).$$

Then the functions $\Phi_\mu$ are eigenfunctions for the Hermite operator with eigenvalues $(2|\mu| + n)$ and $\{\Phi_\mu\}_{\mu \in \mathbb{N}^n}$ forms a complete orthonormal system in $L^2(\mathbb{R}^n)$ (see [13]). Thus, for every $f \in L^2(\mathbb{R}^n)$ we have the Hermite expansion

$$f(x) = \sum_{\mu} \langle f, \Phi_\mu \rangle \Phi_\mu(x) \quad \text{with} \quad \|f\|_{L^2(\mathbb{R}^n)}^2 = \sum_{\mu} |\langle f, \Phi_\mu \rangle|^2.$$

Kanjin [7] obtained the following Hardy’s inequality in the context of one-dimensional Hermite functions, namely,

$$\sum_{k=0}^{\infty} \frac{|\langle f, h_k \rangle|}{(k+1)^{\frac{3p}{2}}} \leq C\|f\|_{H^1(\mathbb{R})}.$$

Radha and Thangavelu [10] proved inequalities of Hardy’s type for $n$-dimensional Hermite expansions for $n \geq 2$ and $0 < p \leq 1$,

$$\sum_{\mu \in \mathbb{N}^n} \frac{|\langle f, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\frac{p}{2}(2-p)}} \leq C\|f\|_{H^p(\mathbb{R}^n)}^p.$$

However, their method does not work for the one-dimensional case. Kanjin in [8] obtained an improved form of (1.5) with $\frac{3}{4} + \epsilon$ for $\epsilon > 0$ in place of $\frac{3p}{2}$, and conjectured that the possible form should be

$$\sum_{k=0}^{\infty} \frac{|\langle f, h_k \rangle|}{(k+1)^{\frac{3}{4}}} \leq C\|f\|_{H^1(\mathbb{R})}.$$

Li, Yu and Shi [9] gave a positive answer to prove estimate (1.7) by using a different approach to evaluate the square integration of the Poisson integral associated to Hermite expansions of functions in $H^1(\mathbb{R})$.

The aim of this paper is to prove Hardy’s inequality for Hermite expansions of functions in a class Hardy spaces $H^p_L(\mathbb{R}^n)$ associated with the operator $L = -\Delta + |x|^2$ for $n \geq 1$ and $0 < p \leq 1$. Our main result is stated as follows.

**Theorem 1.1.** Let $n \geq 1$ and $0 < p \leq 1$. Then there exists a constant $C > 0$ such that for $f \in H^p_L(\mathbb{R}^n)$,

$$\sum_{\mu \in \mathbb{N}^n} \frac{|\langle f, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\frac{p}{2}(2-p)}} \leq C\|f\|_{H^p_L(\mathbb{R}^n)}^p.$$

As a consequence, we have that for any $f \in H^p(\mathbb{R}^n)$,

$$\sum_{\mu \in \mathbb{N}^n} \frac{|\langle f, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\frac{p}{2}(2-p)}} \leq C\|f\|_{H^p(\mathbb{R}^n)}^p.$$

We would like to mention that in the proof of Theorem 1.1, we do not need to estimate the Taylor expansion of the function $\sum_{|\mu| = k} \Phi_\mu(x) \Phi_\mu(y)$ as in [1, 7, 8, 9, 10]. Instead, our proof heavily depends on the atomic decomposition of the Hardy spaces $H^p_L(\mathbb{R}^n)$ in [4], see Section 3 below. Besides, when the space dimension is 1,
we obtain a new estimate of Hardy’s inequality for Hermite expansions in \( H^p(\mathbb{R}) \) for the range \( 0 < p < 1 \).

The paper is organized as follows. In Section 2, we recall the definition of the Hardy spaces \( H^p_L(\mathbb{R}^n) \) associated with \( L = -\Delta + |x|^2 \) and its atomic decomposition, and show that the classical Hardy space \( H^p(\mathbb{R}^n) \), \( 0 < p \leq 1 \), is a proper subspace of the space \( H^p_L(\mathbb{R}^n) \) associated with \( L \). Our main result, Theorem 1.1, will be proved in Section 3.

Throughout, the letters \( C \) and \( c \) will denote (possibly different) constants that are independent of the essential variables.

2. The space \( H^p(\mathbb{R}^n) \) is a proper subspace of \( H^p_L(\mathbb{R}^n) \) for \( 0 < p \leq 1 \)

Recall that Hardy space \( H^p(\mathbb{R}^n) \) can be defined in terms of the maximal function associated with the heat semigroup generated by the Laplace operator \( \Delta \) on \( \mathbb{R}^n \). Following [12], a distribution \( f \) is said to be in \( H^p(\mathbb{R}^n) \), \( 0 < p \leq 1 \) if

\[
\mathcal{M}_{\Delta} f(x) = \sup_{t > 0} |e^{t\Delta} f(x)|
\]

belongs to \( L^p(\mathbb{R}^n) \). If this is the case, then we set \( \|f\|_{H^p(\mathbb{R}^n)} = \|\mathcal{M}_{\Delta} f\|_{L^p(\mathbb{R}^n)} \).

The Hardy space associated with the Hermite operator \( L = -\Delta + |x|^2 \) has attracted much attentions in the last decades and has been a very active research topic in harmonic analysis, see, for example, [3, 4]. Following [3], we say that \( f \) is in the space \( H^p_L(\mathbb{R}^n) \) if

\[
\mathcal{M}_L f(x) := \sup_{t > 0} |e^{-tL} f(x)|
\]

is in \( L^p(\mathbb{R}^n) \). The quasi-norm in \( H^p_L \) is defined by

\[
\|f\|_{H^p_L(\mathbb{R}^n)} = \|\mathcal{M}_L f\|_{L^p}.
\]

When \( p > 1 \), \( H^p_L(\mathbb{R}^n) \cong L^p(\mathbb{R}^n) \).

Now, define an auxiliary function \( m(x) \) by

\[
m(x) = \sum_{\beta \leq (2, \ldots, 2)} |D^\beta |x|^2|^{-|\beta|+2},
\]

where \( |\beta| = |(\beta_1, \ldots, \beta_n)| = \sum_{i=1}^n \beta_i \). There exists a constant \( c > 0 \) such that \( m(x) > c \) for every \( x \in \mathbb{R}^n \). Set

\[
\mathcal{B}_0 = \{x | x \in \mathbb{R}^n, c \leq m(x) \leq 1 \};
\]

\[
\mathcal{B}_k = \{x | x \in \mathbb{R}^n, 2^{k-1} \leq m(x) \leq 2^k \}, k = 1, 2, 3, \ldots .
\]

We say that a function \( a \) is a \( p \)-atom for the space \( H^p_L(\mathbb{R}^n) \) associated to a ball \( B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r \} \) if

(i) \( \text{supp } a \subseteq B(x_0, r) \);

(ii) \( \|a\|_{L^\infty} \leq |B(x_0, r)|^{-1/p} \).
(iii) If $x_0 \in B_k$, then $r \leq 2^{1-\frac{k}{2}}$;
(iv) If $x_0 \in B_k$ and $r \leq 2^{1-\frac{k}{2}}$, then $\int x^\beta a(x)dx = 0$ for all $|\beta| \leq n(\frac{1}{p} - 1)$.

In [3, Theorem 1.12], Dziubański obtained the following atomic characterization of $H^p_L(\mathbb{R}^n)$.

**Proposition 2.1.** A distribution $f$ is in $H^p_L(\mathbb{R}^n)$, $0 < p \leq 1$ if and only if there exist $\lambda_j \in \mathbb{R}$ and $p$-atom $a_j$, $j = 0, 1, 2, \cdots$, such that

$$f(x) = \sum_{j=0}^{\infty} \lambda_j a_j(x)$$

and

$$C_1 \|f\|_{H^p_L}^p \leq \sum_{j=0}^{\infty} |\lambda_j|^p \leq C_2 \|f\|_{H^p_L}^p,$$

where constants $C_1, C_2$ depend only on $p$.

It follows from Proposition 2.1 and the properties of $B_k$ that $H^p(\mathbb{R}^n)$ is a proper subspace of the space $H^p_L(\mathbb{R}^n)$ (see [3, p.77]). We can decompose every element in $H^p_L(\mathbb{R}^n)$ into atoms that are supported on small balls, but some atoms may not have the moment condition. That is,

**Proposition 2.2.** Let $n \geq 1$ and $0 < p \leq 1$. Let $L = -\Delta + |x|^2$. Then we have that

$$H^p(\mathbb{R}^n) \subsetneq H^p_L(\mathbb{R}^n).$$

3. **Proof of Theorem 1.1**

To show Theorem 1.1, we need an $L$-atomic decomposition of $H^p_L(\mathbb{R}^n)$ (see [4, 11]). Let $n \geq 1$ and $0 < p \leq 1$. Assume that $M$ is an integer satisfying

$$M > \frac{n(2-p)}{4p}.$$

Let $\mathcal{D}(T)$ be the domain of an operator $T$ and $B = B(x_B, r_B)$ with the measure $V(B) = cr_B^n$.

Given $0 < p \leq 1$, a function $a \in L^2(\mathbb{R}^n)$ is called a $(p, L, M)$-atom associated with the operator $L$ if there exists a function $b \in \mathcal{D}(L^M)$ and a ball $B \subset \mathbb{R}^n$ such that

1. $a = L^M b$;
2. $\text{supp} L^k b \subset B$, $k = 0, 1, \cdots, M$;
3. $\| (r_B^2 L)^k b \|_{L^2(\mathbb{R}^n)} \leq r_B^{2M} V(B)^{1/2-1/p}$, $k = 0, 1, \cdots, M$. 


Following [11], for a function \( f \in L^2(\mathbb{R}^n) \), we will say that \( f = \sum_{j=0}^{\infty} \lambda_j a_j \) is an atomic \((p, L, M)\)-representation if \( \{\lambda_j\}_{j=0}^{\infty} \in \ell^p \), each \( a_j \) is a \((p, L, M)\)-atom, and the sum converges in \( L^2(\mathbb{R}^n) \). Set

\[
H^p_{L, at, M}(\mathbb{R}^n) := \{ f \in L^2(\mathbb{R}^n) : f \text{ has an atomic } (p, L, M)\text{-representation},
\]

with the norm \( \|f\|_{H^p_{L, at, M}(\mathbb{R}^n)} \) given by

\[
\inf \left\{ \left( \sum_{j=0}^{\infty} |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^{\infty} \lambda_j a_j \text{ is an atomic } (p, L, M)\text{-representation} \right\}.
\]

The atomic Hardy space \( H^p_{L, at, M}(\mathbb{R}^n) \) is then defined as the completion of \( H^p_{L, at, M}(\mathbb{R}^n) \) with respect to this norm. Then we have the following result.

**Proposition 3.1.** For \( 0 < p \leq 1 \), we have

\[
H^p_{L, at, M}(\mathbb{R}^n) \simeq H^p_{L, at, M}(\mathbb{R}^n).
\]

Moreover, for \( f \in H^p_{L, at, M}(\mathbb{R}^n) \), there exist \((p, L, M)\)-atoms \( \{a_j\}_{j=0}^{\infty} \) and \( \{\lambda_j\}_{j=0}^{\infty} \in \ell^p \), such that \( f = \sum_{j=0}^{\infty} \lambda_j a_j \) is in \( H^p_{L, at, M} \) and

\[
\sum_{j=0}^{\infty} |\lambda_j|^p \leq C \|f\|_{H^p_{L}}^p.
\]

**Proof.** For the proof of the first conclusion, we refer the reader to [4, Theorem 8.2] for \( p = 1 \) and [11, Theorem 1.3] for general \( 0 < p \leq 1 \). For the proof of the second conclusion, we refer the reader to [6, Corollary 4.1]. \( \square \)

Let \( \phi = L^M \nu \) be a function in \( L^2(\mathbb{R}^n) \), where \( \nu \in \mathcal{D}(L^M) \). Denote \( U_0(B) = B(0,1) \) and \( U_j(B) = B(0,2^j)\setminus B(0,2^{j-1}) \) for \( j = 1, 2, \cdots \). Following [4, 5, 6], for \( \epsilon > 0 \) and \( M \in \mathbb{N} \), we introduce the space

\[
\mathcal{M}^{M, \epsilon}(L) := \{ \phi = L^M \nu \in L^2(\mathbb{R}^n) : \|\phi\|_{\mathcal{M}^{M, \epsilon}(L)} < \infty \},
\]

where

\[
\|\phi\|_{\mathcal{M}^{M, \epsilon}(L)} := \sup_{j \in \mathbb{N}} \left\{ 2^j 2^{\alpha j(1/2+1/p-1)} \sum_{k=0}^{M} \|L^k \nu\|_{L^2(U_j(B))} \right\}.
\]

Then for any \( M \in \mathbb{N} \), define

\[
\mathcal{M}^M(L) := \bigcap_{\epsilon > 0} \mathcal{M}^{M, \epsilon}(L)^*.
\]

A functional \( f \in \mathcal{M}^M(L) \) is said to be in \( \text{Lip}^M_{p, L}(\mathbb{R}^n) \) if

\[
\|f\|_{\text{Lip}^M_{p, L}} := \sup_{B \subset \mathbb{R}^n} V(B)^{1-1/p} \left[ \frac{1}{V(B)} \int_B |(I - e^{-r L})^M f(x)|^2 \, dx \right]^{1/2} < \infty,
\]

where the supremum is taken over all ball \( B \) of \( \mathbb{R}^n \). Then by [6, Theorem 4.1], we have the following dual result.

**Proposition 3.2.** For \( 0 < p \leq 1 \), we have

\[
(H^p_{L}(\mathbb{R}^n))^* = \text{Lip}^M_{p, L}(\mathbb{R}^n).
\]

Now we are ready to prove Theorem 1.1.
**Proof of Theorem 1.1.** Let \( n \geq 1 \) and \( 0 < p \leq 1 \). Set

\[ \sigma(n, p) = \frac{3n}{4} (2 - p). \]

First, we claim that there exists a constant \( C > 0 \) such that for any \((p, L, M)\)-atom \( a \), supported in \( B(x_B, r) \),

\[ \sum_{\mu \in \mathbb{N}^n} \left| \langle a, \Phi_\mu \rangle \right|^p \left( \frac{2}{|\mu| + n} \right)^{\sigma(n, p)} \leq C < \infty. \]  

(3.1)

We decompose the summation over \( \mu \) by

\[ \sum_{\mu \in \mathbb{N}^n} \frac{|\langle a, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\sigma(n, p)}} = \sum_{j \geq -1} \sum_{2^j < 2|\mu| + n \leq 2^{j+1}} |\langle a, \Phi_\mu \rangle|^p \left( \frac{2}{|\mu| + n} \right)^{\sigma(n, p)} \]

\[ = \sum_{j: 2^{j+1} \geq 1} \sum_{2^j < 2|\mu| + n \leq 2^{j+1}} |\langle a, \Phi_\mu \rangle|^p \left( \frac{2}{|\mu| + n} \right)^{\sigma(n, p)} \]

\[ + \sum_{j: 2^{j+1} < 1} \sum_{2^j < 2|\mu| + n \leq 2^{j+1}} |\langle a, \Phi_\mu \rangle|^p \left( \frac{2}{|\mu| + n} \right)^{\sigma(n, p)} = I + II. \]

For the term I, we apply Hölder’s inequality, Plancherel type equality (1.4) and estimate (3) in the definition of \((p, L, M)\)-atom to get

\[ I = \sum_{j: 2^{j+1} \geq 1} \sum_{2^j < 2|\mu| + n \leq 2^{j+1}} \left| \langle a, \Phi_\mu \rangle \right|^2 \left( \frac{2}{|\mu| + n} \right)^{\sigma(n, p)} \left( \sum_{2^j < 2|\mu| + n \leq 2^{j+1}} \left( \frac{2}{|\mu| + n} \right)^{\frac{2\sigma(n, p)}{2-p}} \right)^{-\frac{2-p}{2}} \]

\[ \leq C \sum_{j: 2^{j+1} \geq 1} 2^{-j} \left( \frac{2}{|\mu| + n} \right)^{\sigma(n, p) - \frac{n(2-p)}{2}} \]

\[ = C r^{-\frac{n(p-2)}{2}} \sum_{j: 2^{j+1} \geq 1} 2^{-j} \left( \frac{2}{|\mu| + n} \right)^{-\frac{n(2-p)}{2}} \]

\[ \leq C. \]

For the term II, we notice that \( a = L^M b \) and obtain

\[ \langle a, \Phi_\mu \rangle = \langle L^M b, \Phi_\mu \rangle = \langle b, L^M \Phi_\mu \rangle = (2|\mu| + n)^M \langle b, \Phi_\mu \rangle. \]
By Hölder’s inequality, Plancherel type equality (1.4) and estiamte (3) in the definition of \((p, L, M)\)-atom,

\[
\begin{align*}
II &= \sum_{j : 2^{j/2} < 1} \sum_{2^{j} < 2|\mu| + n \leq 2^{j+1}} \frac{|\langle a, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\sigma(n,p)}} \\
&= \sum_{j : 2^{j/2} < 1} \sum_{2^{j} < 2|\mu| + n \leq 2^{j+1}} \frac{|\langle b, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{-Mp+\sigma(n,p)}} \\
&\leq \sum_{j : 2^{j/2} < 1} \left( \sum_{2^{j} < 2|\mu| + n \leq 2^{j+1}} |\langle b, \Phi_\mu \rangle|^2 \right)^{\frac{p}{2}} \left( \sum_{2^{j} < 2|\mu| + n \leq 2^{j+1}} (2|\mu| + n)^{\frac{2(Mp-\sigma(n,p))}{2-p}} \right)^{\frac{2-p}{2}} \\
&\leq C \|b\|_2^p \sum_{j : 2^{j/2} < 1} 2^{j(Mp-\sigma(n,p)+\frac{n(2-p)}{2})} \\
&\leq C r^{2Mp+\frac{n(2-p)}{2}} \sum_{j : 2^{j/2} < 1} 2^{j(Mp-\frac{n(2-p)}{2})} \\
&\leq C.
\end{align*}
\]

Thus we complete the proof of estimate (3.1). Now for \(f \in H^p_{\sigma}(\mathbb{R}^n)\), it follows from Proposition 3.1 that there exist \((p, L, M)\)-atoms \(\{a_j\}_{j=0}^\infty\) and \(\{\lambda_j\}_{j=0}^\infty \subseteq \ell^p\), such that \(f = \sum_{j=0}^{\infty} \lambda_j a_j\) in \(H^p_{\sigma}\) and

\[
\sum_{j=0}^{\infty} |\lambda_j|^p \leq C \|f\|_{H^p_{\sigma}}^p.
\]

A direct calculation shows that \(\Phi_\mu \in \text{Lip}_{p,L}^M(\mathbb{R}^n)\). Thus it follows from Proposition 3.2 that

\[
|\langle f, \Phi_\mu \rangle| = |\langle \sum_{j=0}^{\infty} \lambda_j a_j, \Phi_\mu \rangle| \\
\leq \lim_{N \to \infty} |\langle \sum_{j=0}^{N} \lambda_j a_j, \Phi_\mu \rangle| + \lim_{N \to \infty} \|f - \sum_{j=N+1}^{\infty} \lambda_j a_j\|_{H^p_L} \\
\leq \sum_{j=0}^{\infty} |\lambda_j| \|a_j, \Phi_\mu \| + \lim_{N \to \infty} C_\mu \|f - \sum_{j=N+1}^{\infty} \lambda_j a_j\|_{H^p_L} \\
\leq \sum_{j=0}^{\infty} |\lambda_j| |\langle a_j, \Phi_\mu \rangle|.
\]

This, together with Minkowski’s inequality and (3.1), shows that for \(0 < p \leq 1\),

\[
\sum_{\mu \in \mathbb{N}^n} \frac{|\langle f, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\sigma(n,p)}} \leq \sum_{j=0}^{\infty} |\lambda_j|^p \sum_{\mu \in \mathbb{N}^n} \frac{|\langle a_j, \Phi_\mu \rangle|^p}{(2|\mu| + n)^{\sigma(n,p)}} \\
\leq C \sum_{j=0}^{\infty} |\lambda_j|^p \leq C \|f\|_{H^p_{\sigma}}^p.
\]
Consequently, we have obtained estimate \((1.8)\). By Proposition \(2.2\), we see that \(H^p(\mathbb{R}^n)\) is a proper subspace of the space \(H^p_L(\mathbb{R}^n)\), and so \((1.9)\) follows readily. The proof of Theorem 1.1 is complete. \(\square\)

Acknowledgments. P. Chen is supported by NNSF of China (Grant No. 12171489). J. Xiao is supported by the Natural Science Foundation of Guangdong Province (Grant No. 2019A1515010955). The authors would like to thank Lixin Yan for helpful discussions.

References

[1] L. Colzani and G. Travaglini, Hardy-Lorentz spaces and expansions in eigenfunctions of the Laplace-Beltrami operator on compact manifolds. *Colloq. Math.*, 58 (1990), 305-316.

[2] P.L. Duren, *Theory of \(H^p\) spaces*. Academic Press, New York (1970).

[3] J. Dziubański, Atomic decomposition of \(H^p\) spaces associated with some Schrödinger operators. *Indiana Univ. Math. J.*, 47 (1998), 75–98.

[4] S. Hofmann, G.Z. Lu, D. Mitrea, M. Mitrea and L.X. Yan, Hardy spaces associated to nonnegative self-adjoint operators satisfying Davies-Gaffney estimates. *Mem. Amer. Math. Soc.*, 214 (2011), 78.

[5] S. Hofmann and S. Mayboroda, Hardy and BMO spaces associated to divergence form elliptic operators. *Math. Ann.*, 344 (2009), 37–116.

[6] R. Jiang and D. Yang, Orlicz-Hardy spaces associated with operators satisfying Davies-Gaffney estimates. *Comm. Contemp. Math.*, 13 (2011), 331–373.

[7] Y. Kanjin, Hardy’s inequalities for Hermite and Laguerre expansions. *Bull. London Math. Soc.*, 29 (1997), 331-337.

[8] Y. Kanjin, Hardy’s inequalities for Hermite and Laguerre expansions revisited. *J. Math. Soc. Japan*, 63 (2011), 753-767.

[9] Z. Li, Y. Yu and Y. Shi, The Hardy inequality for Hermite expansions. *J. Fourier Anal. Appl.*, 21 (2015), 267-280.

[10] R. Radha and S. Thangavelu, Hardy’s inequalities for Hermite and Laguerre expansions. *Proc. Amer. Math. Soc.*, 132 (2004), 3525-3536.

[11] L. Song and L.X. Yan, Maximal function characterizations for Hardy spaces associated with nonnegative self-adjoint operators on spaces of homogeneous type. *J. Evol. Equ.*, 18 (2018), 221-243.

[12] E.M. Stein, *Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals*. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993.

[13] S. Thangavelu, Hermite and special Hermite expansions revisited. *Duke Math. J.*, 94 (1998), 257-278.