Calculating the pion decay constant from $\alpha_s(M_Z)$

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We revisit the analysis of the improved ladder Schwinger-Dyson (SD) equation for the dynamical chiral symmetry breaking in QCD with emphasizing the importance of the scale ambiguity. Previous calculation done so far naively used one-loop $\overline{\text{MS}}$ coupling in the improved ladder SD equation without examining the scale ambiguity. As a result, the calculated pion decay constant $f_\pi$ was less than a half of its experimental value $f_\pi = 92.4\text{MeV}$ once the QCD scale is fixed from the high energy coupling $\alpha_s(M_Z)$. In order to settle the ambiguity in a proper manner, we adopt here in the present paper the next-to-leading-order effective coupling instead of a naive use of the $\overline{\text{MS}}$ coupling. The pion decay constant $f_\pi$ is then calculated from high energy QCD coupling strength $\alpha_s(M_Z) = 0.1172 \pm 0.0020$. Within the Higashijima-Miransky approximation, we obtain $f_\pi = 85-106\text{MeV}$ depending on the value of $\alpha_s(M_Z)$ which agrees well with the experimentally observed value $f_\pi = 92.4\text{MeV}$. The validity of the improved ladder SD equation is therefore ascertained more firmly than considered before.

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I. INTRODUCTION

The improved ladder approximation of the Schwinger-Dyson (SD) equation [1, 2] has been used widely for the analyses of the dynamical chiral symmetry breaking in QCD [3, 4, 5, 6] and of models of dynamical electroweak symmetry breaking [7, 8]. In the improved ladder approximation, the tree-level one-gluon-exchange diagram, with its QCD coupling strength being replaced by the running $\overline{\text{MS}}$ one, is assumed to give a dominant contribution in the attractive force between quark and anti-quark ($q\bar{q}$).

The method actually succeeded in explaining many quantitative relations among the low energy hadronic data in QCD [3, 4, 5, 6, 9] and established its qualitative validity. Recently, it has also been adopted to predict properties of QCD under extreme conditions (hot and/or dense QCD) [10, 11, 12, 13].

It is known, however, that a naive use of this approximation, combined with the high energy QCD coupling strength such as $\alpha_s(M_Z)$, predicts the pion decay constant $f_\pi$ significantly smaller than its experimentally observed value $f_\pi = 92.4\text{MeV}$. (See, for example, our Figure 4, where we obtained $f_\pi$ of order $30\text{MeV}$ for the one-loop running of the $\overline{\text{MS}}$ coupling.) Note that the decay constant $f_\pi$ determines the scale of chiral phase transition in QCD. Does this discrepancy indicate the existence of substantial non-ladder contribution in the driving force of the dynamical chiral symmetry breaking? If this is so, the success of the improved ladder approximation should be regarded as an accidental coincidence, without understanding deeply the nature of the driving force of the dynamical chiral symmetry breaking.

Failing to explain the experimental value of $f_\pi$ from the high energy QCD coupling strength, the improved ladder SD equation is often regarded as a viable phenomenological model of low energy QCD, in which we can freely tune its coupling strength so as to make $f_\pi$ consistent with the experimental value. We emphasize, however, in order to perform trustful analysis of QCD under extreme conditions, the low energy models need to be connected smoothly with the high energy QCD. The discrepancy of $f_\pi$ thus causes a serious trouble in the analysis of the critical behavior of hot and/or dense QCD.

In this paper, we point out that the previous calculations so far made for $f_\pi$ overlooked the existence of scale ambiguity [13] in the improved ladder SD equation. In order to settle the ambiguity, we introduce a concept of effec-
tive coupling \cite{10, 17} in the analysis of the improved ladder SD equation of QCD. (The idea similar to our effective coupling was proposed in the analysis of SD equation by Ref.\cite{18, 19} for different purpose to ours. It was also adopted by Ref.\cite{18, 19} for the analysis of SD equation in gauge theories with extra dimensions.) The effective coupling is calculated using the background field method \cite{20} in this paper. We next calculate numerically the improved ladder SD equation and obtain the value of the pion decay constant \( f_\pi \) using the high energy QCD coupling \( \alpha^\text{MS}_{\pi} (M_Z) \) as an input parameter of the analysis. Within Higashijima-Miransky approximation \cite{1}, we obtain \( f_\pi = 85 - 106 \text{MeV} \) depending on the value of \( \alpha^\text{MS}_{\pi} (M_Z) = 0.1172 \pm 0.0020 \) \cite{21}. The agreement of the calculated \( f_\pi \) with the experimentally observed value \( f_\pi = 92.4 \text{MeV} \) (or the value in the chiral limit \( m_u = m_d = m_s = 0 \), \( f_\pi = 86 \text{MeV} \) \cite{22}) is rather impressing. The discrepancy of \( f_\pi \) is resolved in our analysis. We thus ascertained the validity of the improved ladder SD equation more firmly than considered before.

We organize the present paper as follows: In Section \textbf{II} we give a brief derivation of the SD equation and point out the problem of the scale ambiguity. In Section \textbf{III} the concept of effective coupling is introduced. The behavior of QCD coupling and its regularization are discussed in Section \textbf{IV}. In Section \textbf{V} we give our results of numerical analysis within the Higashijima-Miransky approximation to the angular integral of the SD equation. A numerical analysis without the Higashijima-Miransky approximation is performed in Section \textbf{VI} using the non-local gauge fixing parameter method. Section \textbf{VII} is devoted for summary and discussions.

\section{Ladder Schwinger-Dyson Equation}

We outline here a derivation of the ladder SD equation based on Ref.\cite{23, 24, 25}. We first integrate out the gluon field from the QCD Lagrangian at the tree level. We then obtain a bi-local interaction model,

\begin{align}
S_{BL} = & \int d^4x \bar{\psi}i\gamma^\mu\partial_\mu \psi + \frac{1}{2} \int d^4x_1 d^4x_2 \times \\
& \times (\bar{\psi}T^a \gamma^\mu \psi)_{x_1} (\bar{\psi}T^a \gamma^\mu \psi)_{x_2} \tilde{D}_{\mu\nu}(x_1 - x_2),
\end{align}

with gluon propagator \( \tilde{D}_{\mu\nu}(x) \) being given by

\begin{align}
\tilde{D}_{\mu\nu}(x) = & \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \frac{g^2}{k^2} \left( g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2} \right) \quad (2)
\end{align}

Here \( g_s \) and \( \xi \) are the QCD coupling constant and the gauge parameter, respectively. In Eq.\cite{1} we have neglected six or higher quark interaction terms. The ladder SD equation for the dynamical chiral symmetry breaking can be derived from Eq.\cite{1} (bi-local four fermion interaction) in a similar manner to the gap equation in the NJL model (local four fermion interaction). We obtain

\begin{align}
A(-p^2) = & 1 + C_F \int \frac{d^4q}{(2\pi)^4} \times \\
& \times \frac{A(-q^2)}{-A^2(-q^2)q^2 + B^2(-q^2)q^2} \left[ (1 + \xi) \frac{p \cdot q}{(p - q)^2} + 2(1 - \xi) \frac{p \cdot (p - q) \cdot q \cdot (p - q)}{(p - q)^4} \right],
\end{align}

\begin{align}
B(-p^2) = & C_F \int \frac{d^4q}{(2\pi)^4} \times \\
& \times \frac{B(-q^2)}{-A^2(-q^2)q^2 + B^2(-q^2)q^2} \frac{3 + \xi}{(p - q)^2},
\end{align}

with \( C_F \equiv T_a T_a = (N_c^2 - 1)/(2N_c) = 4/3 \) being the Casimir of the fundamental representation of \( SU(N_c = 3) \). The functions \( A(-p^2) \) and \( B(-p^2) \) in Eq.\cite{1} are quark wave-function and mass-function, respectively. The quark propagator \( S \) is given in terms of these functions,

\begin{align}
iS^{-1}(p) = pA(-p^2) - B(-p^2).
\end{align}

Nonvanishing mass-function \( B \neq 0 \) implies the dynamical chiral symmetry breaking in QCD. It is therefore regarded as an order parameter of this system.

Performing the Wick rotation and the angular integrals, we obtain

\begin{align}
A(x) = & 1 + \frac{C_F}{x} \int_0^\Lambda^2 dy \frac{y A(y)}{y A^2(y) + B^2(y)} K_A(x, y),
\end{align}

\begin{align}
B(x) = & C_F \int_0^\Lambda^2 dy \frac{y B(y)}{y A^2(y) + B^2(y)} K_B(x, y).
\end{align}

Here we have introduced the ultraviolet (UV) cutoff \( \Lambda \) in the SD equation. It is known that
the result of the SD equation is insensitive to the cutoff Λ if we take sufficiently large Λ in QCD. The integral kernels $K_A$ and $K_B$ of the SD equation Eq. (4) are given by

$$K_A(x, y) = \frac{1}{2\pi^2} \int_0^\pi d\theta \sin^2 \theta \times \alpha_s \left[ (3 - \xi) \sqrt{xy} \cos \theta \frac{z}{\pi} \right] - 2(1 - \xi) \sqrt{xy} \sin^2 \theta, \quad (6a)$$

$$K_B(x, y) = \frac{1}{2\pi^2} \int_0^\pi d\theta \sin^2 \theta \frac{z}{\pi} \left[ (3 + \xi) \alpha_s \right], \quad (6b)$$

with $z$ and $\alpha_s$ being the square of the gluon momentum,

$$z \equiv x + y - 2\sqrt{xy} \cos \theta \quad (7)$$

and the QCD coupling

$$\alpha_s \equiv \frac{g_\pi^2}{4\pi}, \quad (8)$$

respectively.

The ladder SD equation Eq. (4) combined with the kernel Eq. (3) does not take account of the running of QCD coupling strength, however. In order to make the ladder SD equation consistent with the renormalization group (RG) consideration, the bi-local four-fermion interaction in Eq. (4) needs to be modified to include its running effects. A widely adopted prescription for such a purpose is to replace the QCD coupling $g_\pi$ with the $\overline{\text{MS}}$ renormalized one $g_\pi(\mu)$ where $\mu$ is identified with the gluon momentum $\sqrt{z}$. [25] [26] [27] [28]

The prescription is not unique, however. Actually, we can use $g_\pi^2(\mu = c\sqrt{z})$ ($c \neq 1$, $c > 0$) instead of the conventional choice $g_\pi^2(\mu = \sqrt{z})$ (scale ambiguity [15]). Even if we adopt $c \neq 1$, the solution of the SD equation is shown to be consistent with the RG. Moreover, as we stated previously, naive analysis based on the improved ladder SD equation with $c = 1$ predicts $f_\pi$ much lower than its experimentally observed value $f_\pi = 92.4$ MeV. [25] [26]

In order to calculate the pion decay constant $f_\pi$ from the high energy QCD coupling $\alpha_s(M_Z)$, we thus need to resolve the scale ambiguity in the context of the improved ladder SD equation. We consider this problem in the next section by introducing the concept of effective coupling.

### III. EFFECTIVE COUPLING IN QCD

Before investigating the scale ambiguity of the improved ladder SD equation of QCD, we first consider a simpler model, the strongly coupled QED with $N$ flavor of massless fermions. [23] [24].

In this model, ladder SD equation can be derived from a bi-local four-fermion interaction

$$\langle \bar{\psi} \gamma^\mu \psi \rangle_x \langle \bar{\psi} \gamma^\nu \psi \rangle_y \tilde{D}^{\text{QED}}_{\mu\nu}(x - y), \quad (9)$$

which is induced by the photon propagator

$$\tilde{D}^{\text{QED}}_{\mu\nu}(x) \equiv \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} g_\pi^2 k^2 (g_{\mu\nu} + \text{(gauge fixing)}), \quad (10)$$

with $g_\pi$ being the QED coupling constant.

Thanks to the Ward-Takahashi (WT) identities of QED, it is enough to calculate vacuum polarization function (loop correction to the photon propagator) for the evaluation of the bi-local four-fermion interaction Eq. (9). At the one-loop level, the $g_\pi^2/k^2$ part of Eq. (10) is then replaced as

$$\frac{g_\pi^2}{k^2} \rightarrow \frac{g_E^2}{k^2} \left( 1 - g_E^2 \Pi^{\text{QED}}(k^2) \right) k^2$$

$$\quad \quad \quad = \frac{1}{g_E^2(\mu) - \Pi^{\text{QED}}(k^2, \mu) k^2}, \quad (11)$$

with $g_E$ being the $\overline{\text{MS}}$ renormalized QED coupling. The $\overline{\text{MS}}$ renormalization is performed in the second line of Eq. (11). It is straightforward to calculate the $\overline{\text{MS}}$ regularized vacuum polarization function $\Pi^{\text{QED}}_{\overline{\text{MS}}}$:

$$\Pi^{\text{QED}}_{\overline{\text{MS}}}(k^2, \mu) = \frac{4}{3} N \left[ \ln \left( \frac{-k^2}{\mu^2} \right) - \frac{5}{3} \right]. \quad (12)$$

We compare Eq. (11) with the prescription of the naive improved ladder approximation,

$$\frac{g_E^2}{k^2} \rightarrow \frac{g_E^2}{k^2(\mu = \sqrt{|k^2|})}. \quad (13)$$

The naively improved formula Eq. (13) differs from our one-loop formula Eq. (11) due to the existence of nonvanishing finite part in the vacuum polarization,

$$\Pi^{\text{QED}}(k^2, \mu = \sqrt{|k^2|}) = -\frac{20}{9} N, \quad -k^2 \geq 0. \quad (14)$$
One method to resolve the scale ambiguity is to use the renormalization scale $\mu$ so as to minimize the finite corrections. In the present model, we find that the choice

$$\mu = \sqrt{|k^2|} \exp \left( \frac{5}{6} \right)$$  \hspace{1cm} (15)

eliminates the finite part

$$\Pi_{\text{MS}}^{\text{QED}}(k^2, \mu = \sqrt{|k^2|} \exp \left( \frac{5}{6} \right)) = 0. \hspace{1cm} (16)$$

The $\mu$-dependence in the RHS of Eq. (17) cancels at the leading-order. This method has an advantage compared with the former one that it can easily deal with massive particles in loop. We therefore adopt the effective coupling method hereafter.

We now turn to the problem of the scale ambiguity of the improved ladder SD equation of QCD. As we have done in the case of QED, we need to calculate loop corrections to the bi-local four-fermion interaction Eq. (15) in QCD. A difficulty arises, however, in the case of QCD, where Slavnov-Taylor (ST) identities hold instead of the QED-like WT identities. We therefore adopt a slightly different definition of the PT effective coupling. The effective coupling is defined by

$$\frac{1}{g_{\text{eff}}^2(-k^2, \mu)} \equiv \frac{1}{g_{\text{MS}}^2(\mu)} - \Pi_{\text{MS}}^{\text{QCD}}(k^2, \mu). \hspace{1cm} (17)$$

The $\mu$-dependence in the RHS of Eq. (17) cancels at the leading-order. This method has an advantage compared with the former one that it can easily deal with massive particles in loop. We therefore adopt the effective coupling method hereafter.

We now turn to the problem of the scale ambiguity of the improved ladder SD equation of QCD. As we have done in the case of QED, we need to calculate loop corrections to the bi-local four-fermion interaction Eq. (15) in QCD. A difficulty arises, however, in the case of QCD, where Slavnov-Taylor (ST) identities hold instead of the QED-like WT identities. Due to the lack of the ST identities, evaluation of the vacuum polarization function is not enough to renormalize the QCD coupling constant.

Pinch technique (PT) method developed by Cornwall and Papavassiliou [31, 32] may be a hopeful candidate to resolve the scale ambiguity of the improved ladder SD equation of QCD. The effective coupling in the (S-matrix) PT is extracted from the on-shell $q\bar{q}$ scattering amplitude. Since the renormalization scale dependence should cancel in the on-shell amplitude, the scale ambiguity in the $q\bar{q}$ scattering amplitude is automatically resolved in the PT. In addition, it does not depend on the choice of the gauge parameter. We note, however, that these great features of the PT effective coupling is assured only when contributions of the pinch part of the ladder type diagram Figure 1 is included in the amplitude. The pinch part of Figure 1 gives non-zero amplitude in gauges other than the Feynman gauge $\xi = 1$. The ladder type diagrams are resummed to all orders in the ladder SD equation. So, some portion of the ladder contribution will be doubly counted if we simply adopt the PT effective coupling in the analysis of the improved ladder SD equation with $\xi \neq 1$.

![FIG. 1: Ladder-like diagram which is included in the definition of the PT effective coupling.](image)

We therefore adopt here a slightly different choice, the background gauge fixing method [20], where the QED-like WT identities hold even in the case of QCD. Thanks to the naive QED-like WT identities, the QCD coupling can be easily renormalized by the calculation of the vacuum polarization function in this method

$$\frac{1}{g_{\text{eff}}^2(-k^2, \mu)} \equiv \frac{1}{g_{\text{MS}}^2(\mu)} - \Pi_{\text{MS}}^{\text{QED}}(k^2, \mu), \hspace{1cm} (18)$$

with $\Pi_{\text{MS}}^{\text{QCD}}(k^2, \mu)$ being the $\overline{\text{MS}}$ regularized vacuum polarization function. The $\overline{\text{MS}}$ renormalized QCD coupling is denoted by $g_{\text{MS}}$. Using the one-loop finite part of $\Pi_{\text{MS}}^{\text{QCD}}(k^2, \mu)$, it is easy to show that the $\mu$-dependence cancels in the Eq. (18) at the leading-order. We note that the coupling Eq. (18) can be free from the double counting problem in the ladder SD equation, since it does not include contributions from the ladder type diagrams.

After a straightforward calculation, we obtain

$$\Pi_{\text{MS}}^{\text{QCD}}(k^2, \mu) = C_G \left[ 4I^0_R(k^2, \mu) - I^R_I(k^2, \mu) - 2 \frac{(1 - \xi_{bg})}{(4\pi)^2} + \frac{1}{4} \frac{(1 - \xi_{bg})^2}{(4\pi)^2} \right] - 8T_R \sum_{I=1}^{N_F} I^R_I(k^2, m^2_I, \mu). \hspace{1cm} (19)$$
with $C_G = N_c = 3$, $T_R = 1/2$ for QCD ($SU(N_c = 3)$ gauge theory) and $\xi_{bg}$ being the gauge fixing parameter in the background gauge. The number of flavors is denoted by $N_F$. Here the functions $I^R_0$, $I^R_1$ are given by

\[ (4\pi)^2 I^R_0(k^2, \mu^2) = -\ln \frac{-k^2}{\mu^2} + 2, \quad (20a) \]

\[ (4\pi)^2 I^R_1(k^2, \mu^2) = -\frac{1}{3} \ln \frac{-k^2}{\mu^2} + \frac{5}{9}. \quad (20b) \]

The function $I^R_2$ represents the quark loop contribution. We obtain

\[ (4\pi)^2 I^R_2(k^2, m_f^2) = -\frac{1}{6} \ln \frac{m_f^2}{\mu^2} \]

\[ + \frac{1}{6} \left\{ \frac{1}{X_f^2} \left[ \frac{1}{X_f} \tanh^{-1} X_f - 1 - \frac{1}{3} X_f^2 \right] - 3 \right\}, \quad (21) \]

with

\[ X_f = \sqrt{-\frac{k^2}{4m_f^2 - k^2}}. \quad (22) \]

For a massless quark $m_f = 0$, the expression Eq.\,(21) reads

\[ (4\pi)^2 I^R_2(k^2, m_f^2 = 0) = -\frac{1}{6} \ln \frac{-k^2}{\mu^2} + \frac{5}{18}. \quad (23) \]

Note here that the Eq.\,(21) depends on the choice of the gauge fixing parameter $\xi_{bg}$. It is also known that the effective coupling Eq.\,(19) with $\xi_{bg} = 1$ coincides with the PT effective coupling.\,[33, 34]

Such a $\xi_{bg}$-dependence cancels with the ladder-type contributions and other vertex corrections in the one-loop amplitude. The ladder-type diagrams are resummed in the analysis of the ladder SD equation with a particular gauge parameter $\xi$. It is therefore plausible to take $\xi_{bg} = \xi$. As we will show in section IV, the most convenient gauge parameter is $\xi = 0$ in the analysis of the ladder SD equation with Higashijima-Miransky approximation. We will therefore take $\xi_{bg} = 0$ in the analysis of following sections. We will also use $\xi_{bg} = 1$ (the PT effective coupling) in order to make a comparison between them.

We comment on the choice of the renormalization scale $\mu$ which minimizes the finite correction Eq.\,(19). It is easy to find, for $N_F = 0$, the finite correction vanishes with the renormalization scale $\mu = \sqrt{|k^2|} \exp(-205/264)$ for $\xi_{bg} = 0$ ($\mu = \sqrt{|k^2|} \exp(-67/66)$ for $\xi_{bg} = 1$).

IV. BEHAVIOR OF RUNNING COUPLING

We next evaluate numerically the behavior of the running couplings of QCD. Since we are interested in the dynamical chiral symmetry breaking, we consider the running below the scale of b-quark mass $m_b = 4.3$GeV.

In our analysis, we use the PDG average\,[21] of the high energy QCD coupling $\alpha_s^\text{MS}(\mu = M_Z)$,

\[ \alpha_s^\text{MS}(M_Z) = 0.1172 \pm 0.0020, \quad N_F = 5, \quad (24) \]

where the $\overline{\text{MS}}$ coupling is defined in QCD with $N_F = 5$ flavor of quarks. Eq.\,(24) corresponds to $\alpha_s^\text{MS}(m_b) = 0.2197 \pm 0.0075$, $N_F = 4$, (25) at the next-to-next-to-leading-order (NNLO), and

\[ \alpha_s^\overline{\text{MS}}(m_b) = 0.2188 \pm 0.0074, \quad N_F = 4, \quad (26) \]

at the next-to-leading-order (NLO) approximation. We note that the difference between NLO and NNLO is of negligible order at $\mu = m_b$.

The renormalization group equation (RGE) of $\overline{\text{MS}}$ coupling is solved numerically below the scale of $m_b$ using Eq.\,(24) as its boundary condition. As we noted before, the difference between Eq.\,(24) and Eq.\,(25) is insignificant. We assume $m_u = m_d = m_s = 0$ for three light quarks. The $N_F = 4$ $\overline{\text{MS}}$ RGE is adopted for $\mu \geq m_c = 1.3$GeV, while $N_F = 3$ is used for $\mu \leq m_c$. Figure 2 shows the running behavior of one- and two-loop $\overline{\text{MS}}$ couplings. Note here that the two-loop effects become important for $\mu \lesssim 1$GeV.

We emphasize here that we do not use Eq.\,(24) directly as a boundary condition of the one-loop RGE. Actually, the two-loop RGE effect between $M_Z$ and $m_b$ is of non-negligible order. If we adopt directly Eq.\,(24) as the RGE boundary condition, the one-loop RGE leads to the value of $\alpha_s^\overline{\text{MS}}(m_b)$ sizably smaller than the values of Eq.\,(25) and Eq.\,(24). The behavior of one-loop $\overline{\text{MS}}$ coupling
with this boundary condition is then quite different from the result of Figure 2.

We next investigate the behavior of the effective coupling. By using Eqs. (18–21, 23) it is easy to show

\[
\frac{4\pi}{\alpha_s^\text{MS}(\mu)} = \frac{4\pi}{\alpha_s^\text{eff}(-k^2, \mu \geq m_c)} - \Pi_{\text{light}} - \Pi_{\text{charm}} - 4T_R \ln \frac{m_c^2}{\mu^2}. \tag{27}
\]

Here \(\Pi_{\text{light}}\) and \(\Pi_{\text{charm}}\) are defined as

\[
\Pi_{\text{light}} = -C_G \left[ \frac{11}{3} \ln \frac{-k^2}{\mu^2} - \frac{67}{9} \right] - C_G \left[ 2(1 - \xi_{\text{bg}}) - \frac{1}{4} (1 - \xi_{\text{bg}})^2 \right] + \frac{4}{3} T_R N_f \left[ \ln \frac{-k^2}{\mu^2} - \frac{5}{3} \right], \tag{28a}
\]

\[
\Pi_{\text{charm}} = \frac{4}{3} T_R \left\{ 3 \left[ \frac{1}{X_c} \tanh^{-1} X_c - 1 \right] - \frac{1}{X_c^2} \left[ \frac{1}{X_c} \tanh^{-1} X_c - 1 - \frac{1}{3} X_c \right] \right\}, \tag{28b}
\]

with \(N_f\) being the number of light flavors \(N_f = 3\). The parameter \(X_c\) is defined as

\[
X_c = \sqrt{-k^2 - m_c^2}. \tag{29}
\]

The \(\overline{\text{MS}}\) coupling \(\alpha_s^{\overline{\text{MS}}}(\mu)\) in Eq. (27) is the coupling at the scale \(\mu \geq m_c\) and thus defined in 4 flavor QCD. For \(\alpha_s^{\overline{\text{MS}}}(\mu \leq m_c)\) (3 flavor QCD), the \(\ln m_c^2/\mu^2\) term is just missing,

\[
\frac{4\pi}{\alpha_s^\text{eff}(-k^2, \mu < m_c)} = \frac{4\pi}{\alpha_s^{\overline{\text{MS}}}(\mu)} - \Pi_{\text{light}} - \Pi_{\text{charm}}. \tag{30}
\]

By using the expansion of \(\tanh^{-1} X_c\) around \(X_c = 0\),

\[
\tanh^{-1} X_c = \frac{1}{2} \ln \left( \frac{1 + X_c}{1 - X_c} \right) = X_c + \frac{1}{3} X_c^3 + \frac{1}{5} X_c^5 + \cdots, \tag{31}
\]

it is easy to check the decoupling of the c-quark effect in the \(m_c \to \infty\) limit in Eq. (31).

We show the behavior of the effective coupling for \(\xi_{\text{bg}} = 0\) and \(\xi_{\text{bg}} = 1\) in Figure 2. Since we want to calculate them at NLO level, we adopt the two-loop \(\overline{\text{MS}}\) coupling in Eq. (27) and Eq. (30). We also assumed \(\mu = \sqrt{-k^2}\) in the plot. We note that the effective coupling \(\alpha_s^\text{eff}\) is significantly larger than the \(\overline{\text{MS}}\) one.

![Graph showing the running of the MS (one- and two-loop) and effective (\(\xi_{\text{bg}} = 0\) and \(\xi_{\text{bg}} = 1\)) couplings in QCD. The boundary condition of RGE is taken to be \(\alpha_s^\overline{\text{MS}}(m_b) = 0.2197\), which corresponds to \(\alpha_s^\overline{\text{MS}}(M_Z) = 0.1172\) at the scale of \(M_Z\).

![Graph showing typical behavior of the IR regularized effective coupling.](image)
the SD equation.) In this paper, we adopt an IR regularization scheme $\overline{\text{MS}}$ coupling. We note that the correction to the wave function parameters. We consider so called Higashijima-Miransky \[1\] formed in an analytical manner. In this section, we adopt the same IR regularization for the two IR coupling parameters $t_0$, $C$ being determined by

$$\alpha_s^{\text{eff}}(z = \Lambda^2 e^{t_1}) = \alpha_1,$$

$$t_0 = t_1 + \frac{2(\alpha_0 - \alpha_1)}{\alpha_s^{\text{eff}}(z = \Lambda^2 e^{t_1})},$$

$$C = \frac{1}{2(\alpha_0 - \alpha_1)} \left[ \left. \frac{d}{dz} \alpha_s^{\text{eff}} \right|_{z = \Lambda^2 e^{t_1}} \right]^2$$

We also adopt the same IR regularization for the $\overline{\text{MS}}$ coupling. The typical behavior of the IR regularized coupling $\alpha_s$ is depicted in Figure 3. Note here that our IR regularization Eq. \[2\] contains two IR coupling parameters $\alpha_0$ and $\alpha_1$. We regard the result of the SD equation reliable only when they are not too sensitive to these IR parameters.

V. CALCULATING $f_\pi$ WITH HIGASHIJIMA-MIRANSKY APPROXIMATION

The angular integral Eq. \[3\] cannot be performed in an analytical manner. In this section, we consider so called Higashijima-Miransky \[2\] approximation, in which $\alpha_s(z)$ in Eq. \[2\] is replaced as

$$\alpha_s(z) \rightarrow \alpha_s(\max(x, y)).$$

We can then analytically perform the angular integral. We obtain

$$K_A(x, y) = \frac{\pi}{4} \left[ \alpha_s(x) \frac{\theta(x - y) + (x \leftrightarrow y)}{x} \right],$$

$$K_B(x, y) = \frac{3 + \pi}{4} \left[ \alpha_s(x) \frac{\theta(x - y) + (x \leftrightarrow y)}{x} \right].$$

It is known that the existence of non-vanishing correction to the wave function $A(x)$ makes the ladder approximation inconsistent with the gauge symmetry. We thus take Landau gauge $\xi = 0$ here in our numerical calculation of the improved ladder SD equation.

The quark mass function $B(x)$ can be regarded as an order parameter of the chiral phase transition in QCD. Non-zero solution of the SD equation $B(x)$ thus indicates dynamical chiral symmetry breaking and implies the appearance of the Nambu-Goldstone (NG) boson, the pion. The decay constant of pion $f_\pi$ can be related with the mass function $B(x)$.

In the following numerical analysis, we use the Pagels-Stokar formula for the pion decay constant $\overline{\text{MS}}$,

$$f_{\text{PS}}^2 = \frac{N_c}{4\pi^2} \int_0^\Lambda \frac{B^2(x) - \frac{x}{4} \frac{d}{dx} B^2(x)}{(x + B^2(x))^2}.$$  \[36\]

The Pagels-Stokar formula Eq. \[36\], however, does not fully take account of the pion wavefunction of the ladder approximation. The formula is therefore not ladder-exact one. We thus relate it with the ladder-exact $f_\pi$ by

$$f_\pi = N_{\text{PS}} f_{\text{PS}}.$$  \[37\]

Fortunately, the Pagels-Stokar formula is known to work extremely well within the Higashijima-Miransky approximation of the ladder SD equation. \[2\] We thus assume

$$N_{\text{PS}} \simeq 1,$$  \[38\]

in this section.

We are now ready to perform numerical analyses of the improved ladder SD equation. The UV cutoff $\Lambda$ is taken to be $\Lambda = m_b$. We adopt an algorithm similar to the one described in Ref. \[18\] in our numerical analysis.

Let us first confirm previous results of Ref. \[1, 2\] where the momentum dependence of $\alpha_s(z)$ is given by the $\overline{\text{MS}}$ coupling with the renormalization scale $\mu$ being naively identified with $\sqrt{s}$.

Figure 4 shows the result of $f_\pi$ for the one- and two-loop $\overline{\text{MS}}$ coupling. We note that the calculated $f_\pi$ is extremely stable over wide range of the IR coupling regularization parameters $\alpha_0$ and $\alpha_1$. The result is also consistent with previous one that the calculated $f_\pi$ is significantly smaller than its experimentally observed value $f_\pi = 92.4\text{MeV}$.

As we expect from the running behavior of $\alpha_s$ (Figure 2), the $f_\pi$ calculated from two-loop $\overline{\text{MS}}$
α₀ is larger than the value from the one-loop \( \overline{\text{MS}} \) coupling. Although three- or higher-loop running effects may further enhance the value of \( f_\pi \), we expect that they are not enough to reproduce the value \( f_\pi = 92.4 \text{MeV} \), since the higher-loop \( \overline{\text{MS}} \) coupling is still well below the effective coupling at the scale \( \sim 700 \text{MeV} \), where the dynamical chiral symmetry breaking takes place in the analysis of the ladder SD equation.

We also emphasize that the mass function at the zero momentum \( B(0) \) depends significantly on the choice of the IR parameters \( \alpha_0 \) and \( \alpha_1 \). We find, however, that the “constituent quark mass” defined as

\[
    m_{\text{const}} = B(4m_{\text{const}}^2),
\]

is rather insensitive to \( \alpha_0 \) and \( \alpha_1 \). (See Figure 4.) Our result on \( m_{\text{const}} \) is also consistent with the previous analysis.

We next consider the improved ladder SD equation combined with the effective coupling Eq. (27) and Eq. (30). We adopt two-loop \( \overline{\text{MS}} \) coupling as \( \alpha_{s}^{\overline{\text{MS}}} (\mu) \) in Eq. (27) and Eq. (30). The leading-log dependence on \( \mu \) cancels in the definition of \( \alpha_{s}^{\text{eff}} \) as we noted before. We take \( \alpha_{s}^{\text{eff}} (-k^2 = z^2, \mu = \sqrt{z}) \) in order to take account of the next-to-leading-log effects.

We show our results on \( f_\pi \) and \( m_{\text{const}} \) in Figure 4. Again, the results are quite insensitive to the variation of IR regularization parameters \( \alpha_0 \) and \( \alpha_1 \). We also note that the calculated \( f_\pi \) from \( \alpha_{s}^{\text{eff}} \) with \( \xi_{bg} = 0 \) agrees very well with its experimentally observed value \( f_\pi = 92.4 \text{MeV} \). The result suggests the validity of our method to settle the scale ambiguity in the improved ladder SD equation. It may also imply importance of the ladder diagrams in the mechanism of the dynamical chiral symmetry breaking of QCD.

If we adopt \( \xi_{bg} = 1 \) in the effective coupling, on the other hand, the calculated \( f_\pi \) becomes sizably larger than 92.4MeV (of order 120MeV). This deviation of \( f_\pi \) in \( \xi_{bg} = 1 \) may be understood as an indication of the double counting of the ladder diagrams as we described before. This point should be investigated further in future.

In order to figure out the effects of the two-loop RGE in the effective coupling, we next perform analyses using one-loop \( \alpha_{s}^{\overline{\text{MS}}} (\mu) \) in Eq. (27) and Eq. (30). In this case, the \( \mu \)-dependence is completely canceled in the effective coupling. Our results are shown in Figure 5. These results are sizably smaller than the results of Figure 4. The effect of two-loop RGE is therefore sizable in the effective coupling. Although the \( f_\pi \) at \( \xi_{bg} = 1 \) agrees well with its experimental value 92.4MeV, the agreement should be regarded as an accidental coincidence. Actually, the results shown in Figure 5 sizably decrease if we adopt Eq. (24) directly, instead of Eq. (27), as the boundary condition of the one-loop RGE.

We have so far used the central value of Eq. (27), \( \alpha_{s}^{\overline{\text{MS}}} (m_0) = 0.2197 \) as an input parameter of the QCD coupling. Larger the \( \alpha_{s}^{\overline{\text{MS}}} (m_0) \), larger \( f_\pi \) is obtained in our framework, however. The uncertainty of the high energy QCD coupling thus leads to an uncertainty of our results on \( f_\pi \). There also exists uncertainty coming from the choice of the IR regularization parameters \( \alpha_0 \), \( \alpha_1 \), and \( \alpha_2 \). This suggests the necessity of further improvement of our method.
The Higashijima-Miransky approximation is adopted in the kernel of the ladder SD equation. The boundary condition of RGE is assumed to be the effective couplings neglecting their two-loop RGE effects. The cases with $\xi_{bg} = 0$ and $\xi_{bg} = 1$ are shown. The Higashijima-Miransky approximation is adopted in the kernel of the ladder SD equation. The boundary condition of RGE is assumed to be $\alpha_0^\text{MS}(m_0) = 0.2197$, which corresponds to $\alpha_0^\text{MS}(M_Z) = 0.1172$ at the scale of $M_Z$. $N_{PS}$ is taken to be 1.

\[ \begin{align*}
\text{FIG. 5: Results of (a) the pion decay constant } f_\pi \text{ and (b) the "constituent quark mass" } m_{\text{const}} \text{ with use of the effective couplings (two-loop RGE + finite part). The cases with } \xi_{bg} = 0 \text{ and } \xi_{bg} = 1 \text{ are shown.} \\
\text{The Higashijima-Miransky approximation is adopted in the kernel of the ladder SD equation. The boundary condition of RGE is assumed to be } \alpha_0^\text{MS}(m_0) = 0.2197, \text{ which corresponds to } \alpha_0^\text{MS}(M_Z) = 0.1172 \text{ at the scale of } M_Z. \text{ } N_{PS} \text{ is taken to be 1.}
\end{align*} \]

\[ \begin{align*}
\text{FIG. 6: Results of (a) the pion decay constant } f_\pi \text{ and (b) the "constituent quark mass" } m_{\text{const}} \text{ with use of the effective couplings neglecting their two-loop RGE effects. The cases with } \xi_{bg} = 0 \text{ and } \xi_{bg} = 1 \text{ are shown.} \\
\text{The Higashijima-Miransky approximation is adopted in the kernel of the ladder SD equation. The boundary condition of RGE is assumed to be } \alpha_0^\text{MS}(m_0) = 0.2197, \text{ which corresponds to } \alpha_0^\text{MS}(M_Z) = 0.1172 \text{ at the scale of } M_Z. \text{ } N_{PS} \text{ is taken to be 1.}
\end{align*} \]

and $\alpha_1$. Since our results (Figures 5, 6) are rather insensitive to $\alpha_0$ and $\alpha_1$, however, the bulk of uncertainty comes from the high energy QCD coupling here within the Higashijima-Miransky approximation.

We show the uncertainty of our results in Figure 7. The high energy QCD coupling is taken in range of Eq. (23). The IR regularization parameters are also varied in range of $2\pi < \alpha_0 < 4\pi$, $0.4 < \alpha_1/\alpha_0 < 0.8$ in the plot. The experimentally observed value of $f_\pi = 92.4\text{MeV}$ and the decay constant in the chiral limit $f_\pi = 86\text{MeV}$ are also plotted in Figure 7. It is very interesting that the calculated value of $f_\pi$ agrees well with its experimental value for the effective coupling with $\xi_{bg} = 0$.

VI. CALCULATING $f_\pi$ WITH NON-LOCAL GAUGE

We next try to perform the angular integral of the kernel Eq. (4) numerically without using the approximation Eq. (23). Unfortunately, the function $A(x)$ receives non-vanishing correction even if we take $\xi = 0$ in this case. In order to keep $A \equiv 1$, we need to use so called “non-local gauge” fixing in which the gauge parameter $\xi$ is regarded as a function of the gluon momentum. [30, 37]

It is known that $A \equiv 1$ can be achieved when
we take a particular form \[ \xi(z) = \frac{3}{z^2 \alpha_s(z)} \int_0^z dz z^2 \frac{d}{dz} \alpha_s(z). \] \tag{40}

We then obtain

\begin{align}
K_A(x, y) &= 0, \tag{41a} \\
K_B(x, y) &= \frac{1}{2\pi^2} \int_0^\pi d\theta \sin^2 \theta \frac{(3 + \xi(z))\alpha_s(z)}{z}, \tag{41b}
\end{align}

with \( z = x + y - 2\sqrt{xy} \cos \theta \).

We first start with naive improved ladder case, in which \( \alpha_s^{\overline{\text{MS}}} (\mu = \sqrt{z}) \) is used as the running coupling of the SD equation. The naive improved ladder SD equation with one-loop \( \alpha_s^{\overline{\text{MS}}} \) has been investigated in full detail by Ref. [6]. Unlike the case with Higashijima-Miransky approximation, it was found that the Pagels-Stokar formula Eq.\([6]\) gives sizably larger \( f_\pi \) than the ladder-exact value of \( f_\pi \) in the non-local gauge fixing method. The factor \( N_{\text{PS}} \) in Eq.\([7]\) is then smaller than unity. We extract here the value of \( N_{\text{PS}} \) from Ref.\([6]\) for various IR coupling parameters \( \alpha_0, \alpha_1 \). The results are summarized in Table I. We find \( N_{\text{PS}} \) is of order 0.8 and it is a decreasing function of \( \alpha_0, \alpha_1 \). We assume here \( N_{\text{PS}} = 0.8 \) irrespective to \( \alpha_0, \alpha_1 \) throughout in this section.

![Diagram](image)

**Figure 7:** Uncertainties of our results on (a) the pion decay constant \( f_\pi \) and (b) the “constituent quark mass” \( m_{\text{const}} \). The strong coupling constant \( \alpha_s \) is taken in range of \( \alpha_s^{\overline{\text{MS}}} (m_0) = 0.2197 \pm 0.0075 \), which corresponds to \( \alpha_s^{\overline{\text{MS}}} (M_Z) = 0.1172 \pm 0.0020 \) at the scale of \( M_Z \). The IR regularization \( \alpha_0 \) and \( \alpha_1 \) are \( 2\pi < \alpha_0 < 4\pi \) and \( 0.4 < \alpha_1/\alpha_0 < 0.8 \). \( N_{\text{PS}} \) is taken to be 1. The vertical lines in (a) indicate \( f_\pi = 92.4 \text{MeV} \) (the experimental value of \( f_\pi \)) and \( f_\pi = 86 \text{MeV} \) (the decay constant in the chiral limit).

| \( \alpha_0/\pi \) | \( \alpha_1/\pi \) | \( N_{\text{PS}} \) |
|-----------------|-----------------|-----------|
| 2.3             | 0.74            | 0.85      |
| 3.1             | 0.89            | 0.83      |
| 4.4             | 1.1             | 0.80      |
| 7.2             | 1.5             | 0.76      |

**Table I:** The value of \( N_{\text{PS}} \) for various \( \alpha_0 \) and \( \alpha_1 \) with one-loop \( \overline{\text{MS}} \) RGE.

The MS coupling is consistent with the analysis of Ref.\([6]\). Although both \( f_\pi \) and \( m_{\text{const}} \) are relatively stable over the variation of \( \alpha_1/\alpha_0 \), their dependence on \( \alpha_0 \) is of non-negligible order. We are thus not able to calculate \( f_\pi \) in a reliable manner within the non-local gauge fixing method. The size of the calculated \( f_\pi \) is, however, significantly smaller than its experimental value 92.4MeV, even if we use two-loop \( \alpha_s^{\overline{\text{MS}}} \) and assume very strong coupling in the IR region \( \alpha_0 = 4\pi \). This fact again suggests the importance of the scale ambiguity in the improved ladder SD equation.

It should be emphasized that \( N_{\text{PS}} = 0.8 \) is assumed irrespective to the value of \( \alpha_0 \) in Figure 8. Since the factor \( N_{\text{PS}} \) is a decreasing function of \( \alpha_0 \), the ladder-exact value of \( f_\pi \) is considered more stable than the result of the Pagels-Stokar formula. In order to make the analysis more reliable, we therefore need to evaluate the ladder-exact value of \( f_\pi \) using the non-local gauge fixing method. Such a calculation is technologically difficult to be performed, however. We thus leave the subject as a problem to be examined in future.
FIG. 8: Results of (a) the pion decay constant $f_\pi$ and (b) the “constituent quark mass” $m_{\text{const}}$ with use of the one- and two-loop MS couplings. The non-local gauge fixing method is adopted in the ladder SD equation. The boundary condition of RGE is assumed to be $\alpha_s(\mu_0) = 0.2197$, which corresponds to $\alpha_s(M_Z) = 0.1172$ at the scale of $M_Z$. $N_{\text{PS}}$ is taken to be 0.8.

FIG. 9: Results of (a) the pion decay constant $f_\pi$ and (b) the “constituent quark mass” $m_{\text{const}}$ with use of the effective coupling (two-loop RGE + finite part) with $\xi_{\text{bg}} = 0$. The non-local gauge fixing method is adopted in the ladder SD equation. The boundary condition of RGE is assumed to be $\alpha_s(\mu_0) = 0.2197$, which corresponds to $\alpha_s(M_Z) = 0.1172$ at the scale of $M_Z$. $N_{\text{PS}}$ is taken to be 0.8.

FIG. 10: Results of (a) the pion decay constant $f_\pi$ and (b) the “constituent quark mass” $m_{\text{const}}$ with use of the effective coupling (two-loop RGE + finite part) with $\xi_{\text{bg}} = 1$. The non-local gauge fixing method is adopted in the ladder SD equation. The boundary condition of RGE is assumed to be $\alpha_s(\mu_0) = 0.2197$, which corresponds to $\alpha_s(M_Z) = 0.1172$ at the scale of $M_Z$. $N_{\text{PS}}$ is taken to be 0.8.
We next turn to the case with the effective coupling, in which the scale ambiguity is expected to be resolved at the leading-order. Since we use the non-local gauge parameter $\xi(z)$ Eq.(40) in our analysis of the SD equation, there does not exist a priori choice of the gauge parameter $\xi_{bg}$ in the effective coupling. We thus adopt both $\xi_{bg} = 0$ (Figure 1) and $\xi_{bg} = 1$ (Figure 2) and compare the results. Again, the $\alpha_0$-dependence of the results is of non-negligible order. The calculated $f_\pi$ is relatively close to the observed value $f_\pi = 92.4\text{MeV}$ for $\alpha_0 = 2-4\pi$ with $\xi_{bg} = 0$, while $\xi_{bg} = 1$ leads to a little bit larger predictions for the same range of $\alpha_0$. Considering that the non-local gauge parameter method, we were not able to obtain stable results. The results, however, were shown to be consistent with the results of the Higashijima-Miransky approximation in order of magnitude.

VII. SUMMARY AND DISCUSSIONS

In this paper, we have calculated the pion decay constant $f_\pi$ from the high energy QCD coupling strength $\alpha_s^{MS}(M_Z)$ by using the improved ladder Schwinger-Dyson (SD) equation. The SD equation was analyzed both with and without the Higashijima-Miransky approximation for its angular integral. The non-local gauge parameter method was adopted in the analysis without the Higashijima-Miransky approximation in order to keep the wave-function factor trivial $A \equiv 1$. The effective coupling was calculated in the background gauge fixing method with arbitrary covariant gauge parameter $\xi_{bg}$. Analyzing the Landau gauge improved ladder SD equation combined with the $\xi_{bg} = 0$ next-to-leading-order effective coupling, we obtained $f_\pi = 85-106\text{MeV}$ depending on the value of $\alpha_s^{MS}(M_Z) = 0.1172 \pm 0.0020$ within the Higashijima-Miransky approximation. Our result impressively agrees with its experimental value $f_\pi = 92.4\text{MeV}$ and suggests quantitative validity of the improved ladder SD equation. It is interesting to compare our result on $f_\pi$ with the previous analyses 3, 4 where the value of the calculated $f_\pi$ is less than a half of its experimental value $f_\pi = 92.4\text{MeV}$. In the previous analyses, the renormalization scale $\mu$ was naively identified with the gluon momentum $\sqrt{|k^2|}$ in the ladder SD equation. Such an identification has the problem of the scale ambiguity, however, as we pointed out in this paper. The improvement achieved in our analysis comes mainly from our use of the effective coupling, instead of the naive use of the one-loop MS coupling. The leading-order $\ln \mu$ dependence of $\alpha_s^{MS}(\mu)$ cancels with the $\mu$-dependence of the finite correction in our effective coupling.

We found that the effective coupling with $\xi_{bg} = 1$ (the effective coupling derived in the pinch technique (PT)) leads to a larger value of $f_\pi$. This deviation of $f_\pi$ in $\xi_{bg} = 1$ may be understood as an indication of the double counting of the ladder diagrams in the analysis of the ladder SD equation in $\xi = 0$ combined with the PT effective coupling ($\xi_{bg} = 1$). In the analysis of the non-local gauge parameter method, we were able to obtain stable results. The results, however, were shown to be consistent with the results of the Higashijima-Miransky approximation in order of magnitude.

We next comment on a different approach to the scale ambiguity, in which $\alpha_s^{MS}(\mu)$ is used with $\mu$ tuned so as to minimize the finite correction. In the case of QED (or QCD in $N_F \rightarrow \infty$ limit), such a scale is given by $\mu = \sqrt{|k^2|}\exp(-5/6)$ with $k$ being the momentum of photon (gluon) propagator, while it is $\mu = \sqrt{|k^2|}\exp(-205/264)$ in the case of the gluon propagator ($\xi_{bg} = 0$) of $N_F = 0$ QCD. Since these scales are close to each other, it may be possible to use universally the scale $\mu = \sqrt{|k^2|}\exp(-5/6)$ in the analysis of the improved ladder SD equation. It is straightforward to evaluate the $f_\pi$ in such a simple prescription. Thanks to the scale invariant property of the SD equation, we just need to multiply the factor $\exp(5/6)$ to the result of $f_\pi$ in the naive ladder SD equation with the MS coupling. Figure 3 thus reads $f_\pi = 64-82\text{MeV}$ for the one-loop MS and $f_\pi = 122-155\text{MeV}$ for the two-loop MS with this prescription. The difference between this method and the effective coupling method comes from the rest of the finite corrections and the treatment of the two-loop RGE.

Many issues remain unsolved and need further investigation. In order to calculate the value of $f_\pi$ more precisely with the non-local gauge parameter method, we need to evaluate the ladder-exact $f_\pi$ by solving the Bethe-Salpeter equation of the pion. The problem of the $\xi_{bg}$ dependence of the effective coupling should also be studied. It is rather non-trivial task to find adequate relation between $\xi$ (the gauge parameter in the SD equation) and $\xi_{bg}$ (the gauge parameter in the effective coupling), especially in the non-local gauge parameter method. It is also interesting to investigate the critical behavior of the dense
and/or hot QCD using the effective coupling described in this paper. Since the method of the ladder SD equation is now smoothly connected with the high energy QCD, the high-density and high-temperature behaviors of the chiral phase transition can now be studied in a more trustful manner than before.

Finally, the success of the ladder QCD implies that a bulk of driving force of the dynamical chiral symmetry breaking comes from the ladder-type diagrams. The result presented in this paper therefore provides deeper understanding of the low energy QCD dynamics.

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