Singularities of plane gravitational waves and their memory effects

Tongzheng Wang1,2, Jared Fier3, Bowen Li3, Guoliang Li2, Zhaojun Wang3, Yumei Wu1,3, and Anzhong Wang1,3

1 Institute for Advanced Physics & Mathematics, Zhejiang University of Technology, Hangzhou, 310032, China
2 School of Physical Science and Technology, Xinjiang University, Urumqi, 830046, China
3 GCAP-CASPER, Department of Physics, Baylor University, Waco, TX, 76798-7316, USA

(Dated: April 2, 2019)

Abstract

Similar to the Schwarzschild coordinates for spherical black holes, the Baldwin, Jeffery and Rosen coordinates for plane gravitational waves are often singular, and extensions beyond such singularities are necessary, before studying asymptotic properties of such spacetimes. In this paper, we point out that all the spacetimes are actually singular physically at the focused point \( u = u_s \), except for only two particular cases with \( \alpha = 1/2 \) or 1, where the constant \( \alpha \) characterizes the strength of the singularities and appears in the expression \( \chi \equiv \left| \det (g_{ab}) \right|^{1/4} = (u - u_s)^{\beta} \sum_{n=0}^{\infty} \chi_n (u - u_s)^n \) with \( \chi_0 \neq 0 \), where \( g_{ab} \) denotes the reduced metric on the two-dimensional plane orthogonal to the propagation direction of the wave. As a result, all the spacetimes, except for the ones with \( \alpha = 1/2, 1 \), cannot be used to study memory effects and soft graviton theorems, as the latter are closely connected with the asymptotical behaviors of the spacetimes at null infinities, which are not parts of the manifolds of the spacetimes when \( \alpha \neq 1/2, 1 \).

I. INTRODUCTION

The memory effects of gravitational waves (GWs) have been attracted lots of attention (see, for example, [1–4] and references therein), especially after the recent observations of several GWs emitted from remote binary systems of either black holes [5–8] or neutron stars [9]. (The detections of several more GWs were announced lately [10]). Such effects might be possibly detected by LISA [11] or even by current generation of detectors, such as aLIGO and aVIRGO [12]. Recently, such investigations gained new momenta due to the close relations between asymptotically symmetric theorems of soft gravitons and GW memory effects [13, 14].

The characteristic feature of these effects is the permanent displacement of a test particle after a burst of a GW passes [15–20]. In addition, the passage of the GW affects not only the position of the test particle, but also its velocity. In fact, the change of the velocity of the particle is also permanent [21–25].

When far from the sources, the emitted GWs can be well approximated by plane GWs, a subject that has been extensively studied, including their nonlinear interactions [20–24]. The spacetimes for plane GWs can be cast in various forms, depending on the choice of the coordinates and gauge-fixing. One of them was originally due to Baldwin, Jeffery and Rosen (BJR) [28, 29]. Despite its several attractive features, the system of the BJR coordinates is often singular within a finite width of a wave, and when studying the asymptotic behavior of the spacetime, extension beyond this singular surface is needed. In this paper, we point out that there exist actually two kind of singularities in plane gravitational wave spacetimes, one represents coordinate singularities, which can be removed by proper coordinate transformations, and the other represents really spacetime singularities, and physical quantities, such as distortions of test particles, become infinitely large when such singularities are approaching. Therefore, in the latter these singularities already represent the boundaries of the spacetimes and extensions beyond them are not only impossible but also not needed. Since gravitational memory effects and soft graviton theorems are closely related to the asymptotical behaviors of plane GW spacetimes, in the latter the spacetimes cannot be used to study such properties.

In general relativity (GR), there are powerful Hawking-Penrose theorems [30], from which one can see that spacetimes with quite “physically reasonable” conditions are singular. However, the theorems did not tell the nature of the singularities, and Ellis and Schmidt classified them into two different kinds, *spacetime curvature singularities* and *coordinate singularities* [31]. The former is real and cannot be removed by any coordinate transformations,

\[
x^\mu \rightarrow x'^\mu = \xi^\mu (x^n), \quad (\mu, \nu = 0, 1, 2, 3),
\]

while the latter is coordinate-dependent, and can be removed by proper coordinate transformations [1]. One typical example is the coordinate singularity of the

\[1\] It is interesting to note that in [32] this classification was generalized to Horava theory [33], a theory that has only foliation-preserving diffeomorphism, \( t' = f(t), \quad x'^\mu = \xi^\mu (t, x^i) \), \( (i, j = 1, 2, 3) \). For a recent review of Horava theory, see, for example, [34].
Therefore, all the plane GW spacetimes are singular physically at the focused point \( \alpha = 1 \), only the ones with ties appearing in the BJR coordinates, and then study II we shall first give a brief review over the singularities and soft graviton theorems be studied.

In the spacetimes of plane GWs, all the 14 independent scalars vanish identically \([23, 24]\), so in such spacetimes the singularities can be either non-scalar (but real spacetime) singularities or coordinate singularities. In this paper, we shall clarify this important point, by studying tidal forces and distortions of freely falling observers. In particular, we find that the singularities can be in general characterized by

\[
\chi(u) \equiv e^{-U(u)/2} = (u - u_s)^\alpha \hat{\chi}(u),
\]

(1.2)

where the plane GWs are moving along the null direction of \( u = \text{Constant} \), \( U(u) \) is defined in Eq.(2.1), \( \alpha > 0 \), and \( \hat{\chi}(u) \) is given by Eq.(2.20) with \( \hat{\chi}(u_s) \neq 0 \). But, the Einstein vacuum field equations require \( 0 < \alpha \leq 1 \) (See the discussions given in the next section). Then, we find that the tidal forces and distortions are finite across the singular surface \( u = u_s \) only in two particular cases, in which we have

\[
(i) \ \alpha = \frac{1}{2} \quad \text{or} \quad (ii) \ \alpha = 1.
\]

(1.3)

Therefore, all the plane GW spacetimes are singular physically at the focused point \( u = u_s \), exceptions are only the ones with \( \alpha = 1/2 \) or 1. As a result, all the plane GW spacetimes cannot be used to study memory effects and soft graviton theorems, except the ones with \( \alpha = 1/2, 1 \), as only these spacetimes that can be possibly extended to null infinity, whereby can memory effects and soft graviton theorems be studied.

Specifically, the paper is organized as follows. In Sec. II we shall first give a brief review over the singularities appearing in the BJR coordinates, and then study II. SINGULARITIES IN SPACETIMES OF PLANE GRAVITATIONAL WAVES

The spacetimes for plane GWs in the BJR coordinates can be cast in the form \([20, 27]\),

\[
ds^2 = -2e^{-U}dv + e^{-U} \left[ e^V \cosh Wdy^2 - 2 \sinh W dydz + e^{-V} \cosh W dz^2 \right],
\]

(2.1)

where \( M, U, V \) and \( W \) are functions of \( u \) only. The spacetime in general represents a plane GW moving along the null surfaces \( u = \text{constant} \) with two polarizations, one is along the \( y \)-axis, often referred to as the “+” polarization, and the other is along an axis which is at a 45° degree with respect to the \( y \)-axis, often referred to as the “×” polarization (Cf. Fig.1). According to the Petrov classifications, the corresponding spacetimes belong to Petrov Type N \([20, 27]\).

When the metric coefficients are functions of both \( u \) and \( v \), an interesting phenomenon raises, the gravita-
tional Faraday rotation, but now, the medium is pro-

vided by the nonlinear interaction of the oppositely mov-

ing gravitational wave. 39 40.

A. Linearly Polarized Plane Gravitational Waves

Note that by rescaling the null coordinate \( u \rightarrow u' = \int e^{-M(u)} du \), without loss of the generality, one can always set

\[
M = 0, \tag{2.2}
\]

a gauge that will be adopted in this paper. In addition, for our current purpose, it is sufficient to consider only the linearly polarized case in which we have \( W = 0 \), so the metric takes the simple form,

\[
ds^2 = -2dudv + e^{-U(u)} \left( e^{V(u)}dy^2 + e^{-V(u)}dz^2 \right). \tag{2.3}
\]

It can be shown that the corresponding Riemann tensor has only two independent components, given, respectively, by

\[
R_{uyuy} = \frac{1}{4} e^{-U-V} \left[ 2(U'' - V'') - (U' - V')^2 \right],
\]

\[
R_{uzuz} = \frac{1}{4} e^{-U+V} \left[ 2(U'' + V'') - (U' + V')^2 \right], \tag{2.4}
\]

where \( U' \equiv dU/du \), etc. All the fourteen independent scalars 35, made of the Riemann tensor and its derivatives, vanish identically 26 27, so there are no scalar singularities in the spacetimes of plane GWs.

Decomposing it into the Weyl and Ricci tensor 26 27, we find that each of them has only one independent component. In particular, the independent component of the Ricci tensor is given by,

\[
R_{uu} = U'' - \frac{1}{2} (U'^2 + V'^2), \tag{2.5}
\]

while the independent component of the Weyl tensor is given by,

\[
\Psi_4 = -C_{\mu\nu\alpha\beta} n^\mu \tilde{m}^\nu n^\alpha \tilde{m}^\beta = -\frac{1}{2} A^2 (V'' - U'V'), \tag{2.6}
\]

which represents the plane GWs propagating along the hypersurfaces \( u = \text{const.} \), as illustrated in Fig.1 where

\[
l^\mu = A^{-1} \delta_u^\mu, \quad n^\mu = A \delta_{u}^\mu, \quad m^\mu = \zeta^2 \delta_u^\mu + \zeta^3 \delta_3^\mu, \quad \tilde{m}^\mu = \bar{\zeta}^2 \delta_u^\mu + \bar{\zeta}^3 \delta_3^\mu,
\]

form a null tetrad, with \( A \) being an arbitrary function of \( u \) only, and

\[
\zeta^2 \equiv e^{(U-V)/2} / \sqrt{2}, \quad \zeta^3 \equiv i e^{(U+V)/2} / \sqrt{2}. \tag{2.7}
\]

An over bar denotes the complex conjugate. As noticed in various occasions, the BJR coordinates are not harmonic, typically not global, and contain coordinate singularities, see, for example, 2 41 42 and references therein.

To overcome these problems, the Brinkmann coordinates \((\hat{u}, \hat{v}, \hat{y}, \hat{z})\) are often used, defined by,

\[
\hat{v} \equiv v + \frac{1}{4} y^2 e^{V-U} (V' - U'), \quad \hat{u} \equiv u, \quad \hat{y} \equiv e^{(V-U)/2} y, \quad \hat{z} \equiv e^{-V-U/2} z, \tag{2.9}
\]

in terms of which, the metric \eqref{2.3} takes the form 38,

\[
ds^2 = -2d\hat{u}d\hat{v} + d\hat{y}^2 + d\hat{z}^2 + \frac{1}{2} \mathcal{A}(\hat{u}) (\hat{y}^2 - \hat{z}^2) d\hat{u}^2, \tag{2.10}
\]

where

\[
\mathcal{A}(\hat{u}) \equiv \frac{1}{2} \left[ 2(V'' - U'') + (V' - U')^2 \right]. \tag{2.11}
\]

As we mentioned previously, in this paper we would like to point out that these singularities are not always coordinate ones. In fact, all singularities are really spacetime singularities at the focused point \( u = u_s \), except only the ones that asymptotically behave as that given by Eqs. \eqref{2.7} and \eqref{2.8} at the neighborhood of the focused point. To show our claim, we find that it is easier to work in the BJR coordinates. Since the nature of singularities does not depend on the choice of coordinates, they must be singular in any coordinate system, including the Brinkmann system of coordinates.

B. Spacetime Singularities

In the vacuum case, the Einstein field equations \( R_{\mu\nu} = 0 \) have only one independent component, given by \( R_{uu} = 0 \), and from Eq.\eqref{2.5} we find that it can be written as,

\[
\chi'' + \omega^2 \chi = 0, \tag{2.12}
\]

where

\[
\chi \equiv e^{-U/2}, \quad \omega \equiv \frac{1}{2} V'. \tag{2.13}
\]

Then, from Eq.\eqref{2.12} we can see that, for any given initial value, \( \chi_0 > 0 \), there always exists a moment, say, \( u = u_s \) at which \( \chi \) vanishes 2,

\[
\chi(u_s) = 0, \quad \text{or} \quad U(u_s) = +\infty, \tag{2.14}
\]

that is, a singularity of the metric \eqref{2.3} appears at \( u = u_s \), which is surely not a scalar singularity, since, as mentioned above, all the fourteen independent scalars made of the Riemann tensor in such spacetimes vanish identically. Does this mean that the singularity must be a coordinate one? The answer is not always affirmative. This is because spacetime singularities can be not only scalar ones but also non-scalar ones 31. The non-scalar spacetime singularities can be indicated by, for example, the divergence of distortions of a freely falling observer, which are the twice integrations of the tidal force with respect to the proper time of the observer 39.
To calculate distortions of a freely falling observer, let us first consider her trajectory, which follows the timelike geodesics. In the present case, we just consider the ones laid in the \((u,v)\)-plane, that is, \((u,v,y,z) = (u(\xi), v(\xi), y_0, z_0)\), where \(\xi\) denotes the proper time of the observer, and \(y_0\) and \(z_0\) are constants. Then, the timelike geodesics are simply given by

\[
u = \gamma_0 \lambda, \quad v = \frac{\lambda}{2\gamma_0}, \quad y = y_0, \quad z = z_0, \quad (2.15)
\]

where \(\gamma_0\) is an integration constant. Define \(e_0^{(u)} = dx^\mu/d\lambda\), we can construct a tetrad, \(e_{(u)}^\mu\) \((u = 0, 1, 2, 3)\), by

\[
e_{(0)}^\mu = \gamma_0 \delta_0^\mu, \quad e_{(1)}^\mu = \gamma_0 \delta_0^\mu - \frac{1}{2\gamma_0} \delta_v^\mu, \quad e_{(2)}^\mu = e^{\frac{u-v}{\mu}} \delta_b^\mu, \quad e_{(3)}^\mu = e^{\frac{u+v}{\mu}} \delta_z^\mu, \quad (2.16)
\]

which satisfies the relations

\[
e_{(a)}^\mu e_{(\beta)}^\nu g_{\mu\nu} = \eta_{\alpha\beta}, \quad e_{(a)}^\mu e_{(\alpha)}^\nu e_{(0)}^\mu = 0, \quad (2.17)
\]

that is, they are unit orthogonal vectors and parallelly transported along the timelike geodesics, so that they form a freely falling frame \([20]\). Then, the projection of the Riemann tensor onto this frame, \(R_{(a)(b)(c)(d)} = R_{\mu
u\rho\lambda} e_{(a)}^\mu e_{(b)}^\nu e_{(c)}^\rho e_{(d)}^\lambda\), yields two independent components,

\[
R_{(0)(2)(0)(2)} = \frac{2}{\gamma_0} e^{U-V} R_{uyuy}, \quad R_{(0)(3)(0)(3)} = \frac{2}{\gamma_0} e^{U-V} R_{uzuz}, \quad (2.18)
\]

where \(R_{\mu\nu\rho\lambda}\)'s are given by Eq.\((2.4)\).

To study the nature of the singularities at \(u = u_s\), we assume that in the neighborhood of \(u = u_s\), the function \(\chi\) takes the form,

\[
\chi(u) = (u - u_s)^\alpha \tilde{\chi}(u), \quad (2.19)
\]

where \(\alpha > 0\) and \(\tilde{\chi}(u_s) \neq 0\) but finite. Thus, expanding it as

\[
\tilde{\chi}(u) = \sum_{n=0}^{\infty} \chi_n (u - u_s)^n, \quad (2.20)
\]

we must assume that \(\chi_0 \neq 0\), since \(\tilde{\chi}(u_s) \neq 0\). Then, from Eqs.\((2.12)\) and \((2.13)\) we find that

\[
V' = \left( -\frac{4\chi''}{\chi} \right)^{1/2} = \frac{2}{u - u_s} \left[ \alpha(1 - \alpha) - 2\alpha(u - u_s) \frac{\chi'}{\chi} - (u - u_s)^2 \frac{\chi''}{\chi} \right]^{1/2},
\]

\[
U = -2 \ln \chi = -2 \ln(u - u_s) - 2 \ln \tilde{\chi}(u). \quad (2.21)
\]

Note that, in writing the above expression for \(V'\) we had chosen the plus sign, without loss of generality. To study the singular behavior of the solutions at the focused point further, it is found convenient to consider the cases with and without \(\alpha = 1\), separately.

### 1. \(\alpha = 1\)

In this case, inserting Eq.\((2.20)\) into Eq.\((2.21)\), we obtain

\[
V' = \frac{2\sqrt{-2\chi_1/\chi_0}}{(u - u_s)^{1/2}} \sum_{n=0}^{\infty} v_n (u - u_s)^n, \quad (2.22)
\]

where

\[
v_0 = 1, \quad v_1 = -\frac{\chi_0^2 - 3\chi_0 \chi_2}{2\chi_0 \chi_1}, \quad v_2 = \frac{3\chi_1^4 - 10\chi_0 \chi_1^2 + 9\chi_0^2 \chi_2 + 24\chi_0 \chi_3}{8\chi_0^2 \chi_1^2}, \quad (2.23)
\]

and \(\chi_n\) are coefficients appearing in Eq.\((2.20)\). Hence, from Eqs.\((2.4)\) and \((2.18)\) we find

\[
R_{(0)(2)(0)(2)} = -R_{(0)(3)(0)(3)} = \frac{1}{2} \gamma_0^2 (U'V' - V''),
\]

\[
= -\frac{3\sqrt{-\chi_1/\chi_0}}{(u - u_s)^{3/2}} \chi_0^2 + \frac{3(\chi_1^2 + 5\chi_0 \chi_2) \gamma_0^2}{2\chi_0^2 \sqrt{-2\chi_1/\chi_0 (u - u_s)^{1/2}}} + O\left((u - u_s)^{1/2}\right), \quad (2.24)
\]

and

\[
\int d\lambda \int d\lambda R_{(0)(2)(0)(2)}(\lambda) = 6 \sqrt{\frac{2\chi_1}{\chi_0}} \gamma_0^{1/2} (\lambda - \lambda_s)^{1/2} + O\left((\lambda - \lambda_s)^{1/2}\right), \quad (2.25)
\]

which is finite as \(\lambda \to \lambda_s\), where \(\lambda_s \equiv u_s/\gamma_0\).

### 2. \(\alpha \neq 1\)

In this case, we find that

\[
V' = \frac{2}{u - u_s} \sum_{n=0}^{\infty} v_n (u - u_s)^n, \quad (2.22)
\]

\[
V'' = \frac{2}{(u - u_s)^2} \sum_{n=0}^{\infty} (n - 1) v_n (u - u_s)^n, \quad (2.22)
\]

\[
U' = \frac{2}{u - u_s} \left[ \alpha + \frac{\chi_1}{\chi_0} (u - u_s) - \frac{\chi_1^2 - 2\chi_0 \chi_2}{\chi_0^2} (u - u_s)^2 + ... \right], \quad (2.26)
\]
but now with
\[ v_0 = \sqrt{(1 - \alpha)^2}, \quad v_1 = -\frac{\alpha\chi_1}{\chi_0 \sqrt{(1 - \alpha)^2}}, \]
\[ v_2 = -\frac{\alpha[\chi_2(1 - 2\alpha) + 2\chi_0\chi_2(1 + \alpha - 2\alpha^2)]}{2\chi_0^2(1 - \alpha)^{3/2}}. \]  

(2.27)

Clearly, to have the metric coefficient \( V \) be real, we must assume that \( 0 < \alpha < 1 \). Then, we find that
\[ R_{(0)(2)(0)(2)} = -\frac{1}{2} R_{(0)(3)(0)(3)} = (U'V' - V'') \]
\[ = \frac{\alpha^2}{\chi_0} \left( (1 - 2\alpha) \sqrt{(1 - \alpha)} \alpha \right) \]
\[ + \frac{2\chi_1\alpha(2\alpha - 1) \chi_0^2}{\chi_0 \sqrt{(1 - \alpha)^2}} \alpha \chi_0^2 (1 - 2\alpha) \sqrt{(1 - \alpha)} \alpha \]
\[ + \frac{2\chi_1\alpha(2\alpha - 1) \chi_0^2}{\chi_0 \sqrt{(1 - \alpha)^2}} \alpha \chi_0^2 (1 - 2\alpha) \sqrt{(1 - \alpha)} \alpha \]
\[ = \chi_0^2 (1 - 2\alpha) \sqrt{(1 - \alpha)} \alpha \]
\[ + \frac{2\chi_0\chi_2}{\chi_0 \sqrt{(1 - \alpha)^2}} \left( -1 + \alpha - 8\alpha^2 + 8\alpha^3 \right) \]
\[ + \mathcal{O}(u - u_s). \]  

(2.28)

Note that only the first term leads to divergence in the distortions. In fact, we have
\[ \int d\lambda_1 \int d\lambda_2 \]  
\[ R_{(0)(2)(0)(2)} = (2\alpha - 1) \sqrt{\alpha(1 - \alpha)} \]
\[ \times \ln(\lambda - \lambda_s) + \mathcal{O}(\lambda - \lambda_s), \]  

(2.29)

for \( 0 < \alpha < 1 \). Clearly, they are always singular, unless \( \alpha = 1/2 \).

Combining the above with the case \( \alpha = 1 \), we conclude that, unless
\[ (i) \; \alpha = \frac{1}{2}, \quad \text{or} \quad (ii) \; \alpha = 1, \]  

(2.30)

the singularities located at the focused point \( u = u_s \) are always really spacetime singularities.

III. SINGULARITIES IN BRINKMANN COORDINATES

As mentioned previously, gravitational memory effects are frequently studied in the Brinkmann coordinates. So, it would be very interesting to see how the metric behaves in the neighborhood of \( u = u_s \) in the Brinkmann coordinates.

From Eqs. (2.13) and (2.19), we find that
\[ U = -2\alpha \ln(\lambda - \lambda_s) - 2\ln \tilde{\chi}(u), \]
\[ V'^2 = \frac{4\alpha(1 - \alpha)}{(u - u_s)^2} - \frac{4}{\tilde{\chi}} \left( \tilde{\chi}'' + \frac{2\chi_1}{u - u_s} \tilde{\chi}' \right), \]  

(3.1)

where we can expand \( \tilde{\chi}(u) \) in the neighborhood of \( u = u_s \) as that given by Eq. (2.20).

In the vacuum, Eq. (2.12) holds, from which we find that
\[ 2U'' - U'^2 = V'^2, \quad (R_{\mu\nu} = 0). \]  

(3.2)

Then, Eq. (2.11) reduces to,
\[ \mathcal{A}(u) = V'' - V'U', \quad (R_{\mu\nu} = 0). \]  

(3.3)

Note that in writing the above expression we used the coordinate transformations (2.9), from which we simply find \( u = \bar{u} \). Inserting Eqs. (3.1) and (2.20) into Eq. (3.3), we can find the behavior of \( \mathcal{A}(u) \) in the neighborhood of \( u = u_s \). We find that it is convenient to consider the cases, (i) \( 0 < \alpha < 1, \; \alpha \neq 1/2 \); (ii) \( \alpha = 1/2 \); and (iii) \( \alpha = 1 \), separately.

A. \( 0 < \alpha < 1, \; \alpha \neq \frac{1}{2} \)

In this case, inserting Eqs. (3.1) and (2.20) into Eq. (2.11), we find that
\[ \mathcal{A}(u) = \sum_{n=-2}^{\infty} \mathcal{A}_n (u - u_s)^n, \]  

(3.4)

where the first three coefficients that show the singular behavior of \( \mathcal{A}(u) \) are given by,
\[ \mathcal{A}_{-2} = -2(1 - 2\alpha) \sqrt{(1 - \alpha)} \alpha, \]
\[ \mathcal{A}_{-1} = 4\chi_1 \alpha(1 - 2\alpha) \sqrt{(1 - \alpha)} \alpha, \]
\[ \mathcal{A}_0 = -\frac{\alpha}{[(1 - \alpha)^2]^{3/2}} \left[ \chi_0^2 (7 - 12\alpha + 8\alpha^2) \right. \]
\[ + \left. 2\chi_0 \chi_2 (1 - \alpha + 8\alpha^2 - 8\alpha^3) \right]. \]  

(3.5)

Since \( 0 < \alpha < 1 \) and \( \alpha \neq 1/2 \), we have \( \mathcal{A}_{-2} \neq 0 \), so the leading divergent term now is \( (u - u_s)^{-2} \), and \( \mathcal{A}(u) \) behaves as
\[ \mathcal{A}(u) = \frac{\mathcal{A}_{-2}(u)}{(u - u_s)^2} + \frac{\mathcal{A}_{-1}(u)}{(u - u_s)} + \mathcal{A}_0 (\chi_0, \chi_1, \chi_2) \]
\[ + \mathcal{O}(u - u_s), \]  

(3.6)

in the neighborhood of \( u = u_s \), where \( \mathcal{A}_{-1}(\alpha) \) is a function of \( \alpha \) only, which is also non-zero for \( 0 < \alpha < 1 \) and \( \alpha \neq 1/2 \), as it can be seen from Eq. (3.5). As mentioned in the last section, the spacetime now is singular, and no extension beyond this surface is possible, so \( u = u_s \) represents a real boundary of the spacetime.

From Eq. (3.1) we find that
\[ U(u) = -2\alpha \ln(u - u_s) + \tilde{U}(u), \]
\[ V(u) = 2\sqrt{(1 - \alpha)} \ln(u - u_s) + \tilde{V}(u), \]  

(3.7)

where \( \tilde{U} \) and \( \tilde{V} \) are regular and finite functions of \( u \) across the hypersurface \( u = u_s \). Note that in writing down the above expressions, we took the positive sign of \( V' \), without loss of generality, as we did previously. In addition, \( \tilde{U} \) and \( \tilde{V} \) are not independent, as they must satisfy the field equation (3.2).
B. $\alpha = \frac{1}{2}$

In this case, the singularity at $u = u_s$ is a coordinate singularity, which can be removed by the coordinate transformations of Eq. (2.9), and the resulted metric is the Brinkmann metric (2.10) with

\[ \mathcal{A}(u) = \sum_{n=0}^{\infty} s_n (u-u_s)^n, \]  

where the first term $s_0$ is given by,

\[ s_0 = -\frac{6(\chi_1^2 + 2\chi_0\chi_2)}{\chi_0^2}. \]  

Clearly, in this case $\mathcal{A}(u)$ is well-behaved in the neighborhood of $u = u_s$, and the Brinkmann metric (2.10) can be considered as its extension beyond the hypersurface $u = u_s$. If such obtained $\mathcal{A}(u)$ is analytical, then the extension is unique.

On the other hand, from Eq. (3.1) we find that

\[ U(u) = -\ln (u-u_s) + \hat{U}(u), \]
\[ V(u) = \ln (u-u_s) + \hat{V}(u), \quad (\alpha = 1/2), \]  

where $\hat{U}$ and $\hat{V}$ are regular and finite functions of $u$ across the hypersurface $u = u_s$, and are related each other through Eq. (3.2).

C. $\alpha = 1$

In this case, from Eq. (3.1) we find that

\[ U = -2\ln (u-u_s) + \hat{U}(u), \]
\[ V = -\left(\frac{\chi'}{\chi}\right) \frac{8}{u-u_s} - \frac{4\chi''}{\chi}, \]  

where $\hat{U}$ takes the form of Eq. (2.20) with $\chi_0 \neq 0$. Thus, depending on values of $\chi_1$, $\chi_2$ and $\chi_3$, the function $\mathcal{A}(u)$ can have different singular behaviors. Therefore, in the following, let us consider them separately.

1. $\chi_1 \neq 0$

In this case, we find that

\[ U = -2\ln (u-u_s) + \hat{U}(u), \]
\[ V = 4\sqrt{2}\mathcal{D}_1 (u-u_s)^{1/2} + \mathcal{O}(u-u_s)^{3/2}), \]
\[ A = \frac{1}{(u-u_s)^{3/2}} \sum_{n=0}^{\infty} s_n (u-u_s)^n, \]  

where $\mathcal{D}_1 \equiv \sqrt{-\chi_1/\chi_0}$, $\hat{U}$ is regular and finite functions of $u$ across the hypersurface $u = u_s$, and the leading terms of $s_n$ that show clearly the singular behavior of $A$ are given by,

\[ s_0 = 3\sqrt{2} \mathcal{D}_1, \]
\[ s_1 = -\frac{3}{\sqrt{2} \chi_0^2 \mathcal{D}_1} (\chi_1^2 + 5\chi_0\chi_2), \]
\[ s_2 = -\frac{3}{4\sqrt{2} \chi_0^4 \mathcal{D}_1^2} (9\chi_1^4 - 14\chi_0\chi_1^2\chi_2 + 21\chi_0^2\chi_1^2 \]
\[ - 56\chi_0^3 \chi_1 \chi_3). \]  

It is interesting to note that in the current case the Brinkmann metric is still singular at $u = u_s$, although the distortions felt by the freely falling observers defined by Eq. (2.15) are all finite. Hence, now there are two possibilities: (i) Distortions felt by other freely falling observers diverge at $u = u_s$, so the singularity is a real spacetime singularity, and the spacetime cannot be extended beyond this surface. (ii) Distortions felt by any of freely falling observers are finite, and the singularity is a coordinate one. Note that proving the latter is not an easy task, and it might be more effective to find coordinate transformations that bring the BJR metric to a non-singular one, if the singularity is indeed a coordinate one. Clearly, the ones given by Eq. (2.9) fail to do so, and we need to find other one(s), that bring the singular BJR metric to a non-singular one(s). Unfortunately, we have not been successful in this direction, and intend to believe that it might belong to the first possibility. In the next section, we shall come back to this issue again.

2. $\chi_1 = 0$, $\chi_2 \neq 0$

When $\chi_1 = 0$ and $\chi_2 \neq 0$, we find that

\[ U = -2\ln (u-u_s) + \hat{U}(u), \]
\[ V = 2\sqrt{6}\mathcal{D}_2 (u-u_s) + \mathcal{O}(u-u_s)^2), \]
\[ A = \frac{1}{u-u_s} \sum_{n=0}^{\infty} s_n (u-u_s)^n, \]  

where $\mathcal{D}_2 \equiv \sqrt{-\chi_2/\chi_0}$, $\hat{U}$ is regular and finite functions of $u$ across the hypersurface $u = u_s$, and the leading terms of $s_n$ are given by,

\[ s_0 = 4\sqrt{6}\mathcal{D}_2, \]
\[ s_1 = \frac{6\sqrt{6}\mathcal{D}_2 \chi_3}{\chi_2}, \]
\[ s_2 = \frac{4\sqrt{2/3}\mathcal{D}_2^2}{\chi_2^2} (3\chi_3^2 - 3\chi_0\chi_3^2) \]
\[ + 10\chi_0\chi_2\chi_4). \]  

Thus, now the Brinkmann metric is also singular near the hypersurface $u = u_s$. Similar to the last case, it is difficult to see the nature of the singularity, and further investigations are needed.
3. \( \chi_1 = \chi_2 = 0, \chi_3 \neq 0 \)

In this case, we find that
\[
U = -2 \ln (u - u_s) + \hat{U}(u), \\
V = \frac{8\sqrt{3}}{3} D_3 (u - u_s)^{3/2} + \mathcal{O} \left( (u - u_s)^{5/2} \right), \\
\mathcal{A} = \frac{1}{(u - u_s)^{2}} \sum_{n=0}^{\infty} \epsilon_n (u - u_s)^n, \tag{3.16}
\]
where \( D_3 = \sqrt{-\chi_3/x_0} \) and
\[
\epsilon_0 = 10\sqrt{3} D_3, \quad \epsilon_1 = \frac{35 D_3 \chi_4}{\sqrt{3} \chi_3}. \tag{3.17}
\]
Again, in this case the Brinkmann metric is also singular, and it is not clear if the singularity is a coordinate one or not. To clarify the nature of this singularity, further investigations are needed.

4. \( \chi_1 = 0 = \chi_2 = \chi_3 = 0, \chi_4 \neq 0 \)

In this case, we find that
\[
U = -2 \ln (u - u_s) + \hat{U}(u), \\
V = 2\sqrt{5} D_4 (u - u_s)^2 + \mathcal{O} \left( (u - u_s)^3 \right), \\
\mathcal{A} = \sum_{n=0}^{\infty} \mathcal{A}_n (u - u_s)^n, \tag{3.18}
\]
where \( D_4 = \sqrt{-\chi_4/x_0} \) and \( \mathcal{A}_0 = 12\sqrt{5} D_4 \). In this case, it is clear that the Brinkmann metric becomes non-singular, and Eq. (2.9) represents an extension of the singular BJR metric beyond the hypersurface \( u = u_s \). So, in this case it is sure that the singularity encountering in the BJR metric is a coordinate one, and the Brinkmann metric is one of its extensions. Note that the extension will be unique, if such obtained \( \mathcal{A}(u) \) is analytical across \( u = u_s \).

D. Examples of \( \mathcal{A}(u) \)

In the studies of gravitational wave memory effects, several interesting cases have been considered. For example, in [2] [43], the function \( \mathcal{A}(u) \) was chosen as
\[
\mathcal{A}(u) = \frac{1}{2} (u - u_s)^2 \left( 3 - 2u^2 \right) e^{-u^2}. \tag{3.19}
\]
Once \( \mathcal{A}(u) \) is given, we can solve Eqs. (2.11) and (3.2),
\[
2 (V'' - U'') + (V' - U')^2 = 2 \mathcal{A}(u), \tag{3.20}
2U'' - U'^2 = V'^2, \tag{3.21}
\]
to find \( U \) and \( V \). However, due to the nonlinearity of these equations, usually it is difficult to find analytical solutions. In [2] it was found numerically that the singularity in the BJR coordinates happen at \( u_s \approx 0.593342 \).

From Eq. (3.22) we can see that \( \mathcal{A}(u) \) is finite and well-behaved in the neighborhood of this point. So, it must belong to either the case with \( \alpha = 1/2 \), or the case with \( \alpha = 1 \) and \( \chi_i = 0 \) \((i = 1, 2, 3)\). Some modified versions of the above example were considered in [3] [25] [44].

Another example with
\[
\mathcal{A}(u) = \frac{2}{\pi} \frac{\varepsilon^3}{(u^2 + \varepsilon^2)^{3/2}}, \tag{3.22}
\]
was considered in [4], where \( \varepsilon \) is a constant. When \( \varepsilon \) is very small, the above expression gives rise to an impulse gravitational waves, recently studied in [41]. Clearly, in all of these models, \( \mathcal{A}(u) \) is always finite and well-behaved across the singularity located at \( u = u_s \) in the BJR coordinates. So, they all belong to the non-singular cases (either \( \alpha = 1/2 \) or \( \alpha = 1 \), \( \chi_i = 0 \) \((i = 1, 2, 3)\)), presented in the current paper.

IV. CONCLUSIONS AND DISCUSSING REMARKS

The memory effects of gravitational waves are tightly related to the asymptotical properties of the spacetime at the future null infinity (see Ref. [11] [25] and references therein), and so are the soft gravitons and black holes [13] [43]. However, it is well-known that in the BJR coordinates, \( \mathcal{A}(u) \), the metric coefficients often become singular, and extensions beyond the singularities are needed before studying these important issues.

In this paper, we first pointed out that such extensions are not always possible, as some of these singularities are physically real singularities. In particular, distortions experienced by freely falling observers in the \((u, v)\)-plane can be divergent, and any objects trying across the singular surface will be destroyed by these distortions [cf. Eq. (2.9)]. As a result, in these cases the singularities actually represent the boundaries of the spacetimes. In particular, if the metric coefficient \( e^{-U} \) vanishes at the singularity \( u = u_s \) as,
\[
\chi = e^{-U/2} = (u - u_s)^{\alpha} \hat{\chi}(u), \tag{4.1}
\]
where \( \alpha > 0 \) and \( \hat{\chi}(u_s) \neq 0 \), we found that distortions experienced by such freely falling observers always diverge, unless \( \alpha = 1/2 \) or \( \alpha = 1 \). Therefore, only in the cases where \( \alpha = 1/2 \) or \( \alpha = 1 \), the spacetimes at \( u = u_s \) are possibly non-singular, and extensions of the spacetimes beyond this surface is needed, whereby we are able to study the memory effects of gravitational waves and soft gravitons and black holes.

Coordinate transformations from the BJR coordinates to the Brinkmann ones are carried out by Eq. (2.9). It is interesting to note that in the Brinkmann coordinates there is only one unknown function \( \mathcal{A} \), while in the BJR coordinates there are two, \( \hat{U} \) and \( \hat{V} \). However, the vacuum Einstein field equation [3.2] relates \( \hat{U} \) to \( \hat{V} \), so finally
there is only one independent component, too. In fact, for any given \( V \), from Eq.(3.2) one can find \( U \), and then the function \( \mathcal{A} \) is uniquely determined by Eq.(3.3). It is also interesting to note that the inverse is not unique, that is, for any given \( \mathcal{A}(u) \), Eqs. (3.20) and (3.21) will have a family of solutions of the form, \( U(u, u_1, u_2) \) and \( V(u, v_1, v_2) \), where \( u_i's \) and \( v_i's \) are the integration constants.

With the above in mind, we find that \( \mathcal{A} \) is finite and well-behaved across \( u = u_\alpha \) for \( \alpha = 1/2 \) [cf. Eq.(3.8)]. However, in the case \( \alpha = 1 \), we found that \( \mathcal{A} \) is finite and well-behaved across \( u = u_s \) only when \( \chi_1 = \chi_2 = \chi_3 = 0 \), where \( \chi_n \) are the expansion coefficients of \( \chi(u) \), given in Eq.(2.20). If any of these three coefficients is not zero, \( \mathcal{A}(u) \) will be singular across \( u = u_s \), although the distortions of the freely falling observers considered in this paper are finite. There are two possibilities for these cases: (i) The corresponding spacetimes are indeed singular, and distortions become unbounded across \( u = u_s \) for other kinds of observers. (ii) The corresponding singularities are coordinate ones, but the proper coordinate transformations are not given by Eq.(2.9), and instead they are given by something else. Then, it would be very interesting to find them, although it is clearly not an easy task.

Finally we note that our results are expected to be valid when both of the two polarizations exist, that is, \( W \neq 0 \) in Eq.(2.21), although in the current paper we only considered the case \( W = 0 \).

**ACKNOWLEDGEMENTS**

We would like to thank J. Oost for his earlier collaboration and valuable comments and suggestions. We would like also to express our gratitude to P. A. Horváthy for valuable suggestions and comments. This work was supported in part by the National Natural Science Foundation of China under Nos. 11473024, 11363005, 11763007, 11563008, 11365022, 11375153 and 11675145, and the XinJiang Science Fund for Distinguished Young Scholars under No. QN2016YX0049.

---

[1] L. Bieri, D. Garfinkle, N. Yunes, Gravitational Waves and Their Mathematics, AMS Notices, 64, 07 (2017); Gravitational wave memory in \( \Lambda \)CDM cosmology, Class. Quantum Grav. 34, 215002 (2017).

[2] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horváthy, The memory effect for plane gravitational waves, Phys. Lett. B772, 743 (2017).

[3] J. W. Maluf, J. F. da Rocha-Neto, S. C. Ulhoa, and F. L. Carneiro, Plane Gravitational Waves, the Kinetic Energy of Free Particles and the Memory Effect, arXiv:1707.06874.

[4] K. Andrzejewski, and S. Prencel, Memory effect, conformal symmetry and gravitational plane waves, Phys. Lett. B782, 421 (2018).

[5] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 061102 (2016).

[6] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 116, 241103 (2016).

[7] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 118, 221101 (2017).

[8] B. P. Abbott et al. (LIGO Scientific Collaboration and Virgo Collaboration), Phys. Rev. Lett. 119, 141101 (2017).

[9] B. P. Abbott et al. (Virgo, LIGO Scientific Collaboration), Phys. Rev. Lett. 119, 161101 (2017).

[10] Abbott B.P. et al (The LIGO Scientific and Virgo Collaborations) GWTC-1: A gravitational-wave transient catalog of compact binary mergers observed by LIGO and Virgo during the first and second observing runs, arXiv:1811.12907.

[11] M. Favata, Class. Quantum Grav. 27, 084036 (2010).

[12] P. D. Lasky, E. Thrane, Y. Levin, J. Blackman, and Y. Chen, Detecting gravitational wave memory with LIGO: GW150914, Phys. Rev. Lett. 117, 061102 (2016).

[13] S. W. Hawking, M. J. Perry, A. Strominger, Soft hair on Black Holes, Phys. Rev. Lett. 116, 231301 (2016); Superrotation charge and supertranslation hair on Black Holes, JHEP 05, 161 (2017).

[14] A. Strominger, Lectures on the Infrared Structure of Gravity and Gauge Theory, (2017) arXiv:1703.05448.

[15] Ya. B. Zeldovitch and A. G. Polnarev, Astron. Zh. 51, 30 (1974); Sov. Astron. 18, 17 (1974).

[16] V. P. Braginsky and L. P. Grishchuk, Zh. Eksp. Teor. Fiz. 89, 744 (1985).

[17] D. Christodoulou, Nonlinear nature of gravitation and gravitational wave experiments, Phys. Rev. Lett. 67, 1486 (1991).

[18] L. Blanchet and T. Damour, Hereditary effects in gravitational radiation, Phys. Rev. D46, 4304 (1992).

[19] K. S. Thorne, Gravitational-wave bursts with memory: the Christodoulou effect, Phys. Rev. D45, 520 (1992).

[20] A. I. Harte, Strong lensing, plane gravitational waves and transient flashes, Class. Quantum Grav. 30, 075011 (2013).

[21] J.-M. Souriau, Ondes et radiations gravitationnelles, in: Colloques Internationaux du CNRS, Paris, 220, 243 (1973).

[22] V. B. Braginsky, K. S. Thorne, Gravitational-wave bursts with memory and experimental prospects, Nature 327, 123 (1987).

[23] H. Bondi, Energy conversion by gravitational waves, Nature 332, 212 (1988); H. Bondi, F. A. E. Pirani, Gravitational waves in general relativity. 13: Caustic property of plane waves, Proc. R. Soc. Lond. A 421, 395 (1989).
[24] L. P. Grishchuk, A. G. Polnarev, Gravitational wave pulses with velocity memory, Zh. Eksp. Teor. Fiz. 96, 1153 (1989).
[25] P.-M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, Velocity Memory Effect for polarized gravitational waves, JCAP 05, 030 (2018).
[26] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, Exact Solutions of Einstein’s Field Equations, Cambridge Monographs on Mathematical Physics, Second Edition (Cambridge University Press, Cambridge, 2009), Chapters 24 & 25.
[27] J. B. Griffiths, Colliding Plane Waves in General Relativity (Dover Publications, Inc. Mineola, New York, 2016).
[28] O. R. Baldwin and G. B. Jeffery, The relativity theory of plane waves, Proc. R. Soc. A 111, 95 (1926).
[29] N. Rosen, Plane polarized waves in the general theory of relativity, Phys. Z. Sowjetunion, 12, 366 (1937).
[30] S. W. Hawking and G. F. R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, 1972).
[31] G. F. R. Ellis and B. G. Schmidt, Gen. Relativ. Grav. 8, 915 (1977).
[32] R.-G. Cai and A. Ori, Singularities in Horava-Lifshitz theory, Phys. Lett. B 686, 166 (2010).
[33] P. Horava, JHEP 0903, 020 (2009); Phys. Rev. D 79, 084008 (2009); Phys. Rev. Lett. 102, 161301 (2009).
[34] A. Wang, Int. J. Mod. Phys. D 26, 1730014 (2017).
[35] S. J. Campbell and J. Wainwright, Gen. Relativ. Grav. 8, 987 (1977).
[36] A. Ori, Phys. Rev. D 61, 064016 (2000); B.C. Nolan, Phys. Rev. D 62, 044015 (2000); E. W. Hirschmann, A. Wang, and Y. Wu, Class. Quantum Grav. 21, 1791 (2004); P. Sharma, A. Tziolas, A. Wang, and Z. C. Wu, Int. J. Mod. Phys. A 26, 273 (2011).
[37] P. A. Horváthy, Extended Feynman formula for harmonic oscillator, Int. Jour. Theor. Phys. 18, 245 (1979); The Maslov correction in the semiclassical Feynman integral, Cent. Eur. J. Phys. 9, 1 (2011).
[38] M. W. Brinkmann, Einstein spaces which are mapped conformally on each other, Math Ann. 94, 119 (1925).
[39] A. Wang, Interacting Gravitational, Electromagnetic, Neutrino and other waves in the context of Einstein’s General Theory of Relativity (A dissertation submitted to Physics Department in partial fulfillment of the requirements for the degree of Doctor of Philosophy, the University of Ioannina, Greece, 1991).
[40] A. Wang, Gravitational Faraday rotation induced from interacting gravitational plane waves, Phys. Rev. D 44, 1120 (1991).
[41] N. Rosen, Plane polarized waves in the general theory of relativity, Phys. Z. Sowjetunion, 12, 366 (1937).
[42] H. Bondi, F. A. E. Priani and I. Robinson, Gravitational waves in general relativity. 3. Exact plane waves, Proc. Roy. Soc. Lond. A 251, 519 (1959).
[43] G. W. Gibbons and S. W. Hawking, Theory of the Detection of Short Bursts of Gravitational Radiation, Phys. Rev. D 4, 2191 (1971).
[44] P.-M. Zhang, C. Duval, and P. A. Horvathy, Memory effect for impulsive gravitational waves, Class. and Quantum Grav. 35, 065011 (2018).