LIMIT CYCLE BIFURCATIONS OF PIECEWISE SMOOTH NEAR-HAMILTONIAN SYSTEMS WITH A SWITCHING CURVE

HUANHUAN TIAN
School of Mathematics and Statistics
Anhui Normal University
Wuhu, Anhui, 241000, China

MAOAN HAN\textsuperscript{a,b,}\,
\textsuperscript{a} Department of Mathematics
Zhejiang Normal University
Jinhua, Zhejiang, 321004, China
\textsuperscript{b} Department of Mathematics
Shanghai Normal University
Shanghai, 200234, China

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Abstract. This paper deals with the number of limit cycles for planar piecewise smooth near-Hamiltonian or near-integrable systems with a switching curve. The main task is to establish a so-called first order Melnikov function which plays a crucial role in the study of the number of limit cycles bifurcated from a periodic annulus. We use the function to study Hopf bifurcation when the periodic annulus has an elementary center as its boundary. As applications, using the first order Melnikov function, we consider the number of limit cycles bifurcated from the periodic annulus of a linear center under piecewise linear polynomial perturbations with three kinds of quadratic switching curves. And we obtain three limit cycles for each case.

1. Introduction. In the real world, non-smooth phenomena exist in large numbers because of the influence of natural laws and many factors. Many models established in the fields of biology \cite{5,11}, electrical engineering \cite{1} and system control \cite{2} and so on have the following form

\[
\begin{align*}
(\dot{x}, \dot{y}) &= \begin{cases} (P_1(x,y), Q_1(x,y)), & \text{if } S(x,y) > 0, \\
(P_2(x,y), Q_2(x,y)), & \text{if } S(x,y) < 0, \end{cases}
\end{align*}
\]

which is a planar piecewise smooth differential system. The set

\[
\Sigma = \{(x,y)|S(x,y) = 0\}
\]

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* Corresponding author: M. Han.
is known as the switching manifold which divides the plane into two regions, where the systems are smooth in each region. For system (1), one of the main topics is to determine the number of limit cycles which cross the switching manifold. Usually, $\Sigma$ is a straight line. In this case, for piecewise linear systems, Huan and Yang [9], Llibre and Ponce [17] presented examples with three limit cycles. For piecewise quadratic systems, Tian and Yu [18] provided an example with ten limit cycles. Consider

$$
\begin{align*}
\dot{x} &= H_y(x,y) + \epsilon P(x,y), \\
\dot{y} &= -H_x(x,y) + \epsilon Q(x,y)
\end{align*}
$$

where

$$
H(x,y) = \begin{cases} 
H^+(x,y), & x \geq 0, \\
H^-(x,y), & x < 0,
\end{cases}
$$

$$(P(x,y), Q(x,y)) = \begin{cases} 
(P^+(x,y), Q^+(x,y)), & x \geq 0, \\
(P^-(x,y), Q^-(x,y)), & x < 0.
\end{cases}
$$

Recently, piecewise smooth near-Hamiltonian system (2) has gained considerable attention (see [8, 12, 13, 15, 16, 19]) and general methods have been obtained for finding the number of limit cycles. When system (2) $|\epsilon|=0$ has a family of periodic orbits, Liu and Han [15] introduced the definition of the first order Melnikov function and derived an expression of it which can be used to study the number of limit cycles bifurcated from the periodic orbits.

When $\Sigma$ is not a straight line, there are some valuable results. Braga and Mello [3] constructed a piecewise linear system which has a non-smooth polygonal line as switching manifold and has $n$ limit cycles for any given positive integer $n$. Zou and Yang [20] showed the existence of piecewise linear system whose switching manifold is an analytic curve. For any given positive integer $n$, the system has exactly $n$ limit cycles. Cardin and Torregrosa [4] and Liang et al. [14] studied limit cycle bifurcations in two classes of piecewise smooth near-Hamiltonian systems. The switching manifold of each system is formed by two rays starting at the origin.

Motivated by the above references, we aim to study limit cycle bifurcations for piecewise smooth near-Hamiltonian or near-integrable systems with a switching curve. As an application of our general results, we particularly study the number of limit cycles of system

$$
\begin{align*}
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} &= \begin{cases} 
\begin{pmatrix} y + \epsilon P^+(x,y) \\
-x + \epsilon Q^+(x,y)
\end{pmatrix}, & S(x,y) \geq 0, \\
\begin{pmatrix} y + \epsilon P^-(x,y) \\
-x + \epsilon Q^-(x,y)
\end{pmatrix}, & S(x,y) < 0,
\end{cases}
\end{align*}
$$

where $0 < \epsilon \ll 1$, $P^\pm$ and $Q^\pm$ are polynomials of degree one. For the case $S(x,y) = x$, Liu and Han [15] proved that system (3) has at most one limit cycle and this bound can be reached if the first order Melnikov function is not zero identically. If $S(x,y)$ is a quadratic curve, in this paper, we show that system (3) can have three limit cycles bifurcated from the periodic annulus of the origin of system (3) $|\epsilon|=0$ by using the first order Melnikov function. It indicates that the type of switching manifold makes a big difference on the number of limit cycles of piecewise smooth system.

In the following three sections, we present our results in details.
2. The formula of the first order Melnikov function. In this section, we establish the Melnikov function method for piecewise smooth near-Hamiltonian system with a switching curve defined by $S(x, y) = x - \varphi(y) = 0$ or a closed curve by following the idea of [15]. In addition, we study Hopf bifurcation for the case where the switching curve is defined by $S(x, y) = x - \varphi(y) = 0$.

2.1. Case 1: The switching curve $x = \varphi(y)$. Consider

$$
\begin{align*}
\left(\begin{array}{c}
\dot{x} \\
\dot{y}
\end{array}\right) &= \begin{cases}
H^+_y(x, y) + \epsilon f^+(x, y) \\
-H^+_x(x, y) + \epsilon g^+(x, y) \\
H^-_y(x, y) + \epsilon f^-(x, y) \\
-H^-_x(x, y) + \epsilon g^-(x, y)
\end{cases}, \\
x &\geq \varphi(y), \\
x &< \varphi(y),
\end{align*}
$$

(4)

where $H^\pm, f^\pm, g^\pm$ and $\varphi$ are all $C^\infty$ functions satisfying $\varphi(0) = 0, \epsilon > 0$ is a small parameter. For system $(4)|_{\epsilon = 0}$, we make the following assumptions:

(A1): There exists an open interval $J$ such that for each $h \in J$, there are two points $A(h)$ and $B(h)$ on the curve $x = \varphi(y)$ with $A(h) = (\varphi(a(h)), a(h))$, $B(h) = (\varphi(b(h)), b(h))$ and satisfying

$$
H^+(A(h)) = H^+(B(h)) = h, \\
H^-(A(h)) = H^-(B(h)),
$$

(5)

and $b(h) < 0 < a(h)$.

(A2): There is a family of periodic orbits surrounding the origin with clockwise orientation and denoted by $L_h = L^+_h \cup L^-_h$, $h \in J$ where $L^+_h$ is defined by $H^+(x, y) = h, x \geq \varphi(y)$ and starting from $A(h)$, ending at $B(h)$, $L^-_h$ is defined by $H^-(x, y) = H^-(A(h)), x \leq \varphi(y)$ and starting from $B(h)$, ending at $A(h)$.

(A3): Curves $L^\pm_h$, $h \in J$ are not tangent to curve $x = \varphi(y)$ at points $A(h)$ and $B(h)$. In other words, for each $h \in J$,

$$
H^\pm_x(x, y)\varphi'(y) + H^\pm_y(x, y) \neq 0
$$

at points $A(h)$ and $B(h)$.

Consider the orbit of $(4)$ starting from $A(h)$. Denote its first intersection point with curve $x = \varphi(y)$ by $B_r(h) = (\varphi(b_r(h)), b_r(h))$. For the orbit of system $(4)$ starting from $B_r(h)$, we denote its first intersection point with curve $x = \varphi(y)$ by $A_r(h) = (\varphi(a_r(h)), a_r(h))$. See Figure 1.

![Figure 1. The orbit $\hat{AA}_r$ of system (4)](image-url)
Because of the dependence of solutions on parameters, \( A_\epsilon(h) \) and \( B_\epsilon(h) \) are \( C^\infty \) smooth with respect to \( \epsilon \) and satisfy
\[
A_\epsilon(h)|_{\epsilon=0} = A(h), \quad B_\epsilon(h)|_{\epsilon=0} = B(h).
\]
For the property of \( A(h), B(h) \) with respect to \( h (h \in J) \), we have

**Lemma 2.1.** Consider system (4). Under assumptions (A1) and (A3), it holds that \( A(h), B(h) \in C^\infty(J) \).

**Proof.** Let
\[
\tilde{H}(y, h) = H^+(\varphi(y), y) - h.
\]
Clearly, \( \tilde{H} \) is \( C^\infty \). Take \( h = h_0 \in J \). We obtain \( \tilde{H}(a(h_0), h_0) = H^+(A(h_0)) - h_0 = 0 \) from assumption (A1). Further by (A3), we get
\[
\tilde{H}_y(a(h_0), h_0) = [H_+^+(\varphi(y), y)\varphi'(y) + H_+^y(\varphi(y), y)]|_{y=a(h_0)} \neq 0.
\]
Hence, by the implicit function theorem, \( a(h) \) is the unique \( C^\infty \) function such that \( \tilde{H}(a(h), h) \equiv 0 \) for \( h \) near \( h_0 \). So, \( a(h) \) is \( C^\infty \) at \( h = h_0 \). Moreover, since \( h_0 \) is arbitrary, we can conclude that \( a(h) \in C^\infty(J) \). Consequently, \( A(h) \in C^\infty(J) \). By using the same argument, we can show \( B(h) \in C^\infty(J) \). This ends the proof. \( \square \)

Similar to the proof of Lemma 2.1 we can show that \( A_\epsilon(h) \) and \( B_\epsilon(h) \) are \( C^\infty \) functions. Let
\[
H^+(A_\epsilon(h)) - H^+(A_\epsilon(h)) = \epsilon F(h, \epsilon), \quad M(h) = F(h, 0).
\]
We define \( F(h, \epsilon) \) as the bifurcation function and \( M(h) \) as the first order Melnikov function of system (4). For the property of \( F(h, \epsilon) \) and \( M(h) \), we have the following lemma from the above analysis, similar to [7, 15].

**Lemma 2.2.** Let \( h_0 \in J \). Then
(i): For sufficiently small \( |\epsilon| + |h - h_0| \) and \( h \in J \), \( F(h, \epsilon) \) is a \( C^\infty \) function with respect to \( (h, \epsilon) \). In particular, \( M(h) \in C^\infty(J) \).
(ii): System (4) has a periodic solution (resp., a limit cycle) near \( L_{h_0} \) if and only if for sufficiently small \( \epsilon > 0 \), \( F(h, \epsilon) \) has a zero (resp., an isolated zero) on \( h \) near \( h_0 \).
(iii): If \( M(h_0) = 0 \) and \( M'(h_0) \neq 0 \), then for sufficiently small \( |\epsilon| \), system (4) has a unique limit cycle near \( L_{h_0} \).

Hence, Theorem 1.1 of [7] holds for system (4), that is

**Lemma 2.3.** Under assumptions (A1)-(A3), we have
(i): If \( M(h) \) has \( k \) zeros in \( h \) on the interval \( J \) with each having an odd multiplicity, then system (4) has at least \( k \) limit cycles bifurcated from the periodic annulus for sufficiently small \( \epsilon \);
(ii): If the function \( M(h) \) has at most \( k \) zeros in \( h \) on the interval \( J \), taking multiplicities into account, then there exist at most \( k \) limit cycles of (4) bifurcated from the periodic annulus.

Generalizing Theorem 1.1 of [15], we can obtain a formula of the Melnikov function \( M(h) \) of system (4) as follows.

**Theorem 2.4.** Under assumptions (A1)-(A3), for the first order Melnikov function of system (4), we have
\[
M(h) = \int_{L^+_{h_0}} g^+ dx - f^+ dy + \frac{H^+_x(A)\varphi'(a(h)) + H^+_y(A)}{H^+_x(A)\varphi'(a(h)) + H^+_y(A)} \int_{L^-_h} g^- dx - f^- dy.
\]
Proof. We will divide the left side of (7) into four parts and study each part respectively. More precisely, let
\[
H^+(A_\epsilon) - H^+(A) = \left[ H^+(A_\epsilon) - H^- (A_\epsilon) \right] + \left[ H^- (A_\epsilon) - H^- (B_\epsilon) \right] + \left[ H^- (B_\epsilon) - H^+(B_\epsilon) \right] + \left[ H^+(B_\epsilon) - H^+(A) \right]
\]
defined as \( l_1 + l_2 + l_3 + l_4 \).

Then
\[
M(h) = \frac{\partial l_1}{\partial \epsilon}|_{\epsilon=0} + \frac{\partial l_2}{\partial \epsilon}|_{\epsilon=0} + \frac{\partial l_3}{\partial \epsilon}|_{\epsilon=0} + \frac{\partial l_4}{\partial \epsilon}|_{\epsilon=0}.
\]
By (6) and the proof of Theorem 1.1 of [15], we derive
\[
l_4 = \int_{AB} g^+ dx - f^+ dy.
\]
On the other hand, since \( l_4 = H^+(B_\epsilon) - H^+(A) \), we have
\[
\frac{\partial l_4}{\partial \epsilon}|_{\epsilon=0} = \frac{\partial H^+(b_\epsilon(h), b_\epsilon(h))}{\partial \epsilon}|_{\epsilon=0} = H_+^+(B) \frac{\partial \phi(b_\epsilon(h))}{\partial \epsilon}|_{\epsilon=0} + H_+^+(B) \frac{\partial b_\epsilon(h)}{\partial \epsilon}|_{\epsilon=0}.
\]
Under (A3), combing the preceding equality with (10), we derive
\[
\frac{\partial b_\epsilon(h)}{\partial \epsilon}|_{\epsilon=0} = \int_{AB} g^+ dx - f^+ dy.
\]
By using the same argument, we have
\[
\frac{\partial l_2}{\partial \epsilon}|_{\epsilon=0} = \int_{BA} g^- dx - f^- dy.
\]
Furthermore, we obtain from (6) and \( l_2 = H^- (A_\epsilon) - H^- (B_\epsilon) \) that
\[
\frac{\partial l_2}{\partial \epsilon}|_{\epsilon=0} = \left[ H^-_-(A) \phi'(a(h)) + H^-_-(A) \right] \frac{\partial a_\epsilon(h)}{\partial \epsilon}|_{\epsilon=0} - \left[ H^-_-(B) \phi'(b(h)) + H^-_-(B) \right] \frac{\partial b_\epsilon(h)}{\partial \epsilon}|_{\epsilon=0}.
\]
Hence Remark 1.

If Combing (13) together with (A3), (11), (12), it follows that

\[
\frac{\partial \alpha_\epsilon(h)}{\partial \epsilon} \bigg|_{\epsilon=0} = \frac{\int_{\hat{A}} g^- dx - f^- dy}{H_x^{-}(A) \phi'(a(h)) + H_y^{-}(A)} + \frac{\left[H_x^{+}(B) \phi'(b(h)) + H_y^{+}(B)\right] \int_{\hat{A}} g^+ dx - f^+ dy}{\left[H_x^{+}(A) \phi'(a(h)) + H_y^{+}(A)\right] \left[H_x^{+}(B) \phi'(b(h)) + H_y^{+}(B)\right]}.
\]

(14)

Similarly, it holds that

\[
\frac{\partial \beta_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} = \left[H_x^{-}(B) \phi'(b(h)) + H_y^{-}(B) - \left(H_x^{+}(B) \phi'(b(h)) + H_y^{+}(B)\right)\right] \frac{\partial b_\epsilon(h)}{\partial \epsilon} \bigg|_{\epsilon=0}.
\]

and

\[
\frac{\partial \gamma_\epsilon}{\partial \epsilon} \bigg|_{\epsilon=0} = \left[H_x^{+}(A) \phi'(a(h)) + H_y^{+}(A) - \left(H_x^{-}(A) \phi'(a(h)) + H_y^{-}(A)\right)\right] \frac{\partial \alpha_\epsilon(h)}{\partial \epsilon} \bigg|_{\epsilon=0}.
\]

Consequently, by (9), (10), (11), (12), (14), (15) and (16), we obtain

\[
M(h) = \frac{H_x^{+}(A) \phi'(a(h)) + H_y^{+}(A)}{H_x^{-}(A) \phi'(a(h)) + H_y^{-}(A)} \left[H_x^{+}(B) \phi'(b(h)) + H_y^{+}(B)\right] \int_{\hat{A}} g^+ dx - f^+ dy
\]

\[
+ \int_{\hat{B}} g^- dx - f^- dy.
\]

(17)

Taking derivative on both sides of the two equalities in (5) with respect to \( h \) gives

\[
(H_x^{+}(A) \phi'(a(h)) + H_y^{+}(A)) \frac{da(h)}{dh} = (H_x^{+}(B) \phi'(b(h)) + H_y^{+}(B)) \frac{db(h)}{dh} = 1,
\]

\[
(H_x^{-}(A) \phi'(a(h)) + H_y^{-}(A)) \frac{da(h)}{dh} = (H_x^{-}(B) \phi'(b(h)) + H_y^{-}(B)) \frac{db(h)}{dh}.
\]

Hence

\[
\frac{H_x^{+}(A) \phi'(a(h)) + H_y^{+}(A) H_x^{-}(B) \phi'(b(h)) + H_y^{-}(B)}{H_x^{+}(A) \phi'(a(h)) + H_y^{+}(A)} = 1.
\]

(18)

Substituting (18) into (17) gives (8). This ends the proof.

Remark 1. If \( \phi(y) \equiv 0 \), the Melnikov function in Theorem 2.4 reduces to

\[
M(h) = \int_{L_k^+} g^+ dx - f^+ dy + \frac{H_y^{+}(A)}{H_y^{-}(A)} \int_{L_k^-} g^- dx - f^- dy
\]

which is the same as that in [12] and [15].

2.2. Hopf bifurcation of case 1. Suppose

\[
H_x^{+}(0, 0) = H_y^{+}(0, 0) = 0, \quad \det \frac{\partial (H_y^{+}, -H_x^{+})}{\partial (x, y)}(0, 0) > 0.
\]

(19)

Clearly, under (A1)-(A3) the origin is an elementary center of system (4) at \( \epsilon=0 \).

Consider

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{cases}
H_x^{+}(x, y) + \epsilon f^+(x, y, \delta), & x \geq \phi(y), \\
-H_x^{-}(x, y) + \epsilon g^+(x, y, \delta), & x < \phi(y),
\end{cases}
\]

(20)

where \( H^\pm, \varphi \) are the same as those in Section 2.1, \( f^\pm, g^\pm \) are \( C^\infty \) functions, \( 0 < \epsilon \ll 1 \) and \( \delta = (\delta_1, \delta_2, \ldots, \delta_m) \in D \subset \mathbb{R}^m \) with \( D \) compact. For (20), the first order Melnikov function \( M \) depends on both \( h \) and \( \delta \), denoted by \( M(h, \delta) \). Now, suppose that assumptions \( (A1)-(A3) \) and (19) hold for system (20) with \( J = (0, \beta) \), \( \beta > 0 \) and

\[
\begin{align*}
f^\pm(0, 0, \delta) &= g^\pm(0, 0, \delta) = 0. 
\end{align*}
\]

Note that (21) implies that the origin is always a singular point under perturbation.

Let

\[
\begin{align*}
u &= x - \varphi(y), \quad \hat{H}^\pm(u, y) = H^\pm(u + \varphi(y), y).
\end{align*}
\]

Hence,

\[
\begin{align*}
\hat{H}^+_u(u, y) &= H^+_u(u + \varphi(y), y), \\
\hat{H}^+_y(u, y) &= H^+_x(u + \varphi(y), y) + H^+_y(u + \varphi(y), y).
\end{align*}
\]

Noticing that \( \varphi(0) = 0 \), we obtain from (19) that

\[
\begin{align*}
\hat{H}^+_u(0, 0) &= H^+_x(0, 0) = 0, \\
\hat{H}^+_y(0, 0) &= 0,
\end{align*}
\]

\[
\det \left( \frac{\partial (\hat{H}^+_y, -\hat{H}^+_u)}{\partial (u, y)} \right)(0, 0) = \det \left( \frac{\partial (H^+_y, -H^+_x)}{\partial (x, y)} \right)(0, 0) > 0.
\]

Making the change of variables

\[
\begin{align*}
u &= x - \varphi(y), \\
y &= y,
\end{align*}
\]

system (20) becomes the following \( C^\infty \) system

\[
\begin{align*}
\left( \begin{array}{c}
\dot{u} \\
\dot{y}
\end{array} \right) &= \left\{ \begin{array}{ll}
\hat{H}^+_y(u, y) + cp^+(u, y, \delta) & \text{, } u \geq 0, \\
-H^+_x(u, y) + cp^+(u, y, \delta) & \text{, } u < 0,
\end{array} \right.
\end{align*}
\]

where \( \hat{H}^\pm(u, y) \) satisfy (22) and

\[
\begin{align*}
p^\pm(u, y, \delta) &= f^\pm(u + \varphi(y), y, \delta) - \varphi'(y)g^\pm(u + \varphi(y), y, \delta), \\
q^\pm(u, y, \delta) &= g^\pm(u + \varphi(y), y, \delta).
\end{align*}
\]

By (21), we obtain

\[
\begin{align*}
p^\pm(0, 0, \delta) &= q^\pm(0, 0, \delta) = 0.
\end{align*}
\]

Denote the images of \( A_\epsilon(\varphi(a(h)), a(h)) \), \( B(\varphi(b(h)), b(h)) \), \( A_\epsilon(\varphi(a_\epsilon(h, \delta)), a_\epsilon(h, \delta)) \) and \( B_\epsilon(\varphi(b_\epsilon(h, \delta)), b_\epsilon(h, \delta)) \) under (23) by \( \hat{A}(0, a(h)) \), \( \hat{B}(0, b(h)) \), \( \hat{A}_\epsilon(0, a_\epsilon(h, \delta)) \) and \( \hat{B}_\epsilon(0, b_\epsilon(h, \delta)) \) respectively.

Clearly, the bifurcation function of system (20) satisfies

\[
\begin{align*}
\epsilon F(h, \epsilon, \delta) &= H^+(A_\epsilon) - H^+(A) \\
&= H^+(\varphi(a_\epsilon(h, \delta)), a_\epsilon(h, \delta)) - H^+(\varphi(a(h)), a(h)) \\
&= H^+(0, a_\epsilon(h, \delta)) - H^+(0, a(h)) \\
&= \hat{H}^+(\hat{A}_\epsilon) - \hat{H}^+(\hat{A}).
\end{align*}
\]

Hence, by Theorem 2.1 of [7] which studied Hopf bifurcation of system (24), we can obtain the following theorem directly.
Theorem 2.5. Consider system (20). Let assumptions \((A1)-(A3), (19)\) and \((21)\) hold with \(J = (0, \beta)\), \(\beta > 0\). Then

1: The function \(M(h, \delta)\) has an expansion of the form

\[
M(h, \delta) = \sum_{j \geq 2} b_{j-1}(\delta) h^j;
\]

2: If there exist \(k\) and \(\delta_0 \in \mathcal{D}\) such that

\[
b_j(\delta_0) = 0, \quad j = 1, \cdots, k, \quad b_{k+1}(\delta_0) \neq 0,
\]

and

\[
\text{rank} \frac{\partial (b_1, b_2, \cdots, b_k)}{\partial (\delta_1, \delta_2, \cdots, \delta_m)}(\delta_0) = k,
\]

then system (20) has at most \(k\) limit cycles in a neighborhood of the origin for all \((\epsilon, \delta)\) near \((0, \delta_0)\) and \(k\) limit cycles appear for some \((\epsilon, \delta)\) near \((0, \delta_0)\). If condition (21) does not hold, the following theorem can be proved similarly to Theorem 2.3 of [7].

Theorem 2.6. Consider system (20). Let assumptions \((A1)-(A3)\) and \((19)\) hold with \(J = (0, \beta)\), \(\beta > 0\). Then

1: We have formally

\[
M(h, \delta) = \sum_{i \geq 1} b_i(\delta) h^i;
\]

2: If there exist \(k\) and \(\delta_0 \in \mathcal{D}\) such that

\[
b_1(\delta_0) = b_2(\delta_0) = \cdots = b_k(\delta_0) = 0, b_{k+1}(\delta_0) \neq 0\]

and

\[
\text{rank} \frac{\partial (b_1, b_2, \cdots, b_k)}{\partial (\delta_1, \delta_2, \cdots, \delta_m)}(\delta_0) = k,
\]

then system (20) can have at least \(k\) limit cycles in an arbitrary neighborhood of the origin.

If a piecewise smooth near-Hamiltonian system has a closed curve as its switching curve, we study its Melnikov function in the following part.

2.3. Case 2: The closed switching curve \(S(x, y) = 0\). Assume that \(S(x, y) = 0\) defines a closed curve \(L\) passing through the origin with \(S(x, y) \in C^\infty\) and \(D \overset{\text{def}}{=} \text{Int}.(L)\) is a convex set. Hence, the closed curve \(L\) has a highest point and a lowest point which divide curve \(L\) into two curve segments. We denote them by \(L_1:\ x = \varphi_1(y)\) and \(L_2:\ x = \varphi_2(y)\) for \(y \in I\) respectively, where \(\varphi_1(y) \leq \varphi_2(y)\) and \(I\) is a closed interval. Now, consider the following system which has \(L\) as its switching curve

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} =
\begin{cases}
\begin{pmatrix}
H^+(x, y) + \epsilon f^+(x, y) \\
H^+_x(x, y) + \epsilon g^+(x, y)
\end{pmatrix}, & (x, y) \in D, \\
\begin{pmatrix}
H^-(x, y) + \epsilon f^-(x, y) \\
H^-_x(x, y) + \epsilon g^-(x, y)
\end{pmatrix}, & (x, y) \notin D,
\end{cases}
\]

where \(H^\pm, f^\pm\) and \(g^\pm\) are all \(C^\infty\) functions, \(0 < \epsilon \ll 1\). Similar to (4), we make the following assumptions for system (31)\(_{\epsilon=0}\):
(B1): There exists an open interval \( J = J_1 \cup J_2 \) with \( J_1 \) and \( J_2 \) being two intervals having no points in common such that for each \( h \in J \), there exist two points \( A(h) \) and \( B(h) \) on curve \( L \) satisfying

\[
H^+(A(h)) = H^+(B(h)) = h, \quad H^-(A(h)) = H^-(B(h)),
\]

where \( A(h) = (\varphi_1(a(h)), a(h)) \in L_1 \) (resp., \( A(h) = (\varphi_2(a(h)), a(h)) \in L_2 \) if \( h \in J_1 \) (resp., \( h \in J_2 \) and \( B(h) \in L_1 \) or \( L_2 \).

(B2): There exists a family of periodic orbits with clockwise orientation surrounding the origin, denoted by \( L_h = L_h^+ \cup L_h^- \), \( h \in J \) where \( L_h^+ \) is defined by \( H^+(x, y) = h \), \( (x, y) \in D \), starting from \( A(h) \) and ending at \( B(h) \), \( L_h^- \) is defined by \( H^-(x, y) = H^-(A(h)) \), \( (x, y) \notin D \), starting from \( B(h) \) and ending at \( A(h) \).

(B3): Curves \( L_h^\pm, h \in J \) are not tangent to \( L \) at the intersection points \( A(h) \) and \( B(h) \).

Consider the orbit of system (31) starting from \( A(h) \). We denote its first two intersection points with curve \( L \) by \( B_{\epsilon}(h) \) and \( A_{\epsilon}(h) \). See Figure 2. Let \( H^+(A_{\epsilon}(h)) - 

\[
\text{Figure 2. The orbit } \hat{AA}_{\epsilon} \text{ of system (31)}
\]

\( H^+(A(h)) = \epsilon F(h, \epsilon) \) and \( M(h) = F(h, 0) \). Similar to system (4), we define \( M(h) \) as the first order Melnikov function of (31). By the proof of Theorem 2.4, we obtain the following conclusion.

**Theorem 2.7.** Consider system (31). Under assumptions (B1)-(B3), we have

(1): If \( h \in J_1 \), then

\[
M(h) = \int_{L_h^+} g^+ dx - f^+ dy + \frac{H_x^+(A)\varphi'_1(a(h)) + H_y^+(A)}{H_x^+(A)\varphi'_1(a(h)) + H_y^+(A)} \int_{L_h^-} g^- dx - f^- dy.
\]

(2): If \( h \in J_2 \), then

\[
M(h) = \int_{L_h^+} g^+ dx - f^+ dy + \frac{H_x^-(A)\varphi'_2(a(h)) + H_y^-(A)}{H_x^-(A)\varphi'_2(a(h)) + H_y^-(A)} \int_{L_h^-} g^- dx - f^- dy.
\]

In the next section, we will generalize the conclusions of piecewise smooth near-Hamiltonian systems to piecewise smooth near-integrable systems.
3. The Melnikov function of piecewise smooth near-integrable systems.

Now, consider the following piecewise smooth near-integrable system (32)

\[
\begin{align*}
\dot{x} &= \frac{H^+_x(x,y)}{R^+(x,y)} + \varepsilon f^+(x,y), \quad x \geq \varphi(y), \\
\dot{y} &= -\frac{H^+_y(x,y)}{R^+(x,y)} + \varepsilon g^+(x,y),
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= \frac{H^-_x(x,y)}{R^-(x,y)} + \varepsilon f^-(x,y), \quad x < \varphi(y), \\
\dot{y} &= -\frac{H^-_y(x,y)}{R^-(x,y)} + \varepsilon g^-(x,y),
\end{align*}
\]

where \(H^\pm, f^\pm, g^\pm, \varphi\) and \(\varepsilon\) are all the same as those given in (4), \(R^+(x,y)\) and \(R^-(x,y)\) are the integrating factors of systems \((32a)\)|_{\(\varepsilon=0\)} and \((32b)\)|_{\(\varepsilon=0\)} respectively. We assume that system \((32)\)|_{\(\varepsilon=0\)} satisfies conditions (A1)-(A3), that is, system \((32)\)|_{\(\varepsilon=0\)} has a periodic annulus with clockwise orientation around the origin.

Assume that the orbit of system (32) starting from point \(A(h)\) intersects with curve \(x = \varphi(y)\) firstly at \(B_s(h)\) and secondly at \(A_r(h)\). Let \(H^+(A_r(h)) - H^+(A(h)) = \varepsilon F(h, \epsilon)\) and \(M(h) = F(h, 0)\). We also define \(M(h)\) as the first order Melnikov function of system (32). Clearly, Lemmas 2.2 and 2.3 still hold. Note that (32) is equivalent to

\[
\begin{pmatrix}
\dot{x} \\
\dot{y}
\end{pmatrix} = \begin{cases}
\begin{pmatrix}
H^+_x(x,y) + \varepsilon R^+(x,y) f^+(x,y) \\
-\frac{H^+_y(x,y)}{R^+(x,y)} + \varepsilon g^+(x,y)
\end{pmatrix}, & x \geq \varphi(y), \\
\begin{pmatrix}
H^-_x(x,y) + \varepsilon R^-(x,y) f^-(x,y) \\
-\frac{H^-_y(x,y)}{R^-(x,y)} + \varepsilon g^-(x,y)
\end{pmatrix}, & x < \varphi(y).
\end{cases}
\]

Hence, by Theorem 2.4, the following theorem holds directly.

**Theorem 3.1.** Consider system (32). Under assumptions (A1)-(A3), the first order Melnikov function \(M(h)\) has the following expression

\[
M(h) = \int_{L^+_h} R^+ g^+ dx - R^+ f^+ dy + \frac{H^+_x(A) \varphi'(a(h)) + H^+_y(A)}{H^+_x(A) \varphi'(a(h)) + H^+_y(A)} \int_{L^-_h} R^- g^- dx - R^- f^- dy.
\]

Consider system (33)

\[
\begin{align*}
\dot{x} &= \frac{H^+_x(x,y)}{R^+(x,y)} + \varepsilon f^+(x,y), \quad (x,y) \in D, \\
\dot{y} &= -\frac{H^+_y(x,y)}{R^+(x,y)} + \varepsilon g^+(x,y),
\end{align*}
\]

and

\[
\begin{align*}
\dot{x} &= \frac{H^-_x(x,y)}{R^-(x,y)} + \varepsilon f^-(x,y), \quad (x,y) \notin D, \\
\dot{y} &= -\frac{H^-_y(x,y)}{R^-(x,y)} + \varepsilon g^-(x,y),
\end{align*}
\]

where \(D, H^\pm, f^\pm, g^\pm\) and \(\epsilon\) are all the same as those in system (31), \(R^+(x,y)\) and \(R^-(x,y)\) are the integrating factors of systems \((33a)\)|_{\(\varepsilon=0\)} and \((33b)\)|_{\(\varepsilon=0\)} respectively. Assume that conditions (B1)-(B3) hold for system (33)|_{\(\varepsilon=0\)}. Then there exists a periodic annulus surrounding the origin with clockwise orientation.

Assume that the orbit of system (33) starting from \(A(h)\) intersects with closed curve \(S(x,y) = 0\) at \(B_s(h)\) and \(A_r(h)\) in turn. Define the first order Melnikov function of system (33) as that of system (32). Hence, Lemmas 2.2 and 2.3 still hold. By Theorem 2.7 and the proof of Theorem 3.1, we obtain the expression of \(M(h)\) as follows.

**Theorem 3.2.** Consider system (33). Under assumptions (B1)-(B3), we have
(1): If $h \in J_1$, then
\[
M(h) = \int_{J_1} R^+ g^+ dx - R^+ f^+ dy + \frac{H^+_x(A)\varphi'_1(a(h)) + H^+_y(A)}{H^+_x(A)\varphi'_1(a(h)) + H^+_y(A)} \int_{L_1} R^- g^- dx - R^- f^- dy.
\]

(2): If $h \in J_2$, then
\[
M(h) = \int_{J_2} R^+ g^+ dx - R^+ f^+ dy + \frac{H^+_x(A)\varphi'_2(a(h)) + H^+_y(A)}{H^+_x(A)\varphi'_2(a(h)) + H^+_y(A)} \int_{L_2} R^- g^- dx - R^- f^- dy.
\]

In the following section, we present some applications.

4. The number of limit cycles of piecewise linear system with a quadratic switching curve. At first, we present the definition and an equivalent condition for extended complete Chebyshev system (for short, ECT-system).

**Definition 4.1.** (See [6] or [19]) Let $p_0(x), p_1(x), \cdots, p_{n-1}(x)$ be analytic functions on an open interval $J \subset \mathbb{R}$. The ordered set $(p_0(x), p_1(x), \cdots, p_{n-1}(x))$ is said to be an ECT-system on $J$ if, for all $k = 1, 2, \cdots, n$, any nontrivial linear combination
\[
\alpha_0 p_0(x) + \alpha_1 p_1(x) + \cdots + \alpha_{k-1} p_{k-1}(x)
\]
has at most $k - 1$ isolated zeros on $J$ counted with multiplicities.

**Lemma 4.2.** (See [10] or [19]) The ordered set $(p_0(x), p_1(x), \cdots, p_{n-1}(x))$ is an ECT-system on $J$ if and only if, for each $k = 1, 2, \cdots, n$,
\[
W(p_0, p_1, \cdots, p_{k-1}) \neq 0, \quad \text{for all } x \in J,
\]
where $W(p_0, p_1, \cdots, p_{k-1})$ is the Wronskian of functions $p_0(x), p_1(x), \cdots, p_{k-1}(x)$.

Now, consider system
\[
\begin{align*}
\dot{x} &= y + \epsilon f(x, y), \\
\dot{y} &= -x + \epsilon g(x, y),
\end{align*}
\]
where
\[
f(x, y) = \begin{cases} f^+(x, y), & x \geq \varphi(y), \\
f^-(x, y), & x < \varphi(y), \end{cases} \quad g(x, y) = \begin{cases} g^+(x, y), & x \geq \varphi(y), \\
g^-(x, y), & x < \varphi(y), \end{cases}
\]
\[
f^\pm(x, y) = a^\pm_{10} + a^\pm_{10} x + a^\pm_{01} y, \quad g^\pm(x, y) = b^\pm_{00} + b^\pm_{10} x + b^\pm_{01} y, \quad 0 < \epsilon \ll 1.
\]

Denote
\[
\begin{align*}
c_0 &= \pi(a^+_{10} - b^+_{01}), & c_1 &= 2(a^+_{00} - a^-_{00}), \\
c_2 &= -2(b^+_{10} - b^-_{10}), & c_3 &= a^+_{10} + b^+_{01} - a^-_{10} - b^-_{01}.
\end{align*}
\]

For the number of limit cycles of system (34), the following theorem holds.

**Theorem 4.3.** Consider system (34).

(i): Let $\varphi(y) = y^2$ which means that the switching curve of (34) is a parabola. If the first order Melnikov function of system (34) is not zero identically, system (34) has at most 3 limit cycles bifurcated from the period annulus $\{x^2 + y^2 = h \mid h \in (0, \frac{3}{2})\}$ and this upper bound can be reached for sufficiently small $\epsilon$ and some $(c_0, c_1, c_2, c_3)$ near $(-\frac{\pi}{2}, 0, 0, 0)$ with $c_3 \neq 0$. 
(ii): Let \( \varphi(y) = -1 + \sqrt{1 + y^2} \) which means that the switching curve of (34) is the branch of hyperbola \((x + 1)^2 - y^2 = 1\) satisfying \(x \geq 0\). If the first order Melnikov function of system (34) is not zero identically, system (34) has at most 3 limit cycles bifurcated from the period annulus \( \{x^2 + y^2 = h \mid h \in (0, 4)\} \) and this upper bound can be reached for sufficiently small \( \epsilon \) and some \((c_0, c_1, c_2, c_3)\) near \((-\frac{\pi}{2}, 0, 0, \bar{c}_3)\) with \(\bar{c}_3 \neq 0\).

**Proof.** Evidently, the Hamiltonian function of system (34)|\(\epsilon = 0\) is

\[
H(x, y) = \frac{x^2 + y^2}{2}.
\]

For each \(h > 0\), \(H(x, y) = \frac{h}{2}\) defines a periodic orbit with clockwise orientation.

First, we prove conclusion (i). For the periodic orbit \(H(x, y) = \frac{h}{2}\), we denote its intersection points with \(x = y^2\) by \(A(h) = (a(h), a(h))\) and \(B(h) = (a^2(h), -a(h))\), and denote its intersections points with \(y\)-axis by \(C(h) = (0, -\sqrt{h})\) and \(D(h) = (0, \sqrt{h})\), where

\[
a(h) = \sqrt{-1 + \sqrt{1 + 4h}}.
\]

See Figure 3.

![Figure 3. Periodic orbits and switching curve of system (34)|\(\epsilon = 0\)](image)

Define \(L_h^\pm\) as follows

\[
\begin{align*}
L_h^+ : \overrightarrow{AB} &= \{(x, y)|x = \sqrt{h - y^2}, \ y \in (-a(h), a(h))\}, \\
L_h^- : \overrightarrow{BA} &= \overrightarrow{BC} \cup \overrightarrow{CD} \cup \overrightarrow{DA},
\end{align*}
\]

where

\[
\begin{align*}
\overrightarrow{BC} &= \{(x, y)|x = \sqrt{h - y^2}, \ y \in (-\sqrt{h}, -a(h))\}, \\
\overrightarrow{CD} &= \{(x, y)|x = -\sqrt{h - y^2}, \ y \in (-\sqrt{h}, \sqrt{h})\}, \\
\overrightarrow{DA} &= \{(x, y)|x = \sqrt{h - y^2}, \ y \in (a(h), \sqrt{h})\}.
\end{align*}
\]

By Theorem 2.4, Melnikov function of system (34) is

\[
M(h) = \int_{L_h^+} g^+ dx - f^+ dy + \int_{L_h^-} g^- dx - f^- dy.
\]
Let

\[ M^\pm (h) = \int_{L^\pm} g^+ dx - f^+ dy. \]

Then

\[ M(h) = M^+(h) + M^-(h). \]  \hfill (38)

In the following, we calculate \( M^\pm (h) \) respectively. Clearly,

\[ M^+(h) = \int_{AB \cup BA} g^+(x, y)dx - f^+(x, y)dy + \int_{BA} f^+(x, y)dy. \]  \hfill (39)

Denote \( x_0 = a^2(h) \) and

\[ \bar{q}^\pm (x, y) = f^\pm (x, y) - f^\pm (x_0, y) + \int_{x_0}^x g^\pm_y (u, y)du. \]

Hence, \( \bar{q}_x^\pm (x, y) = f_x^\pm (x, y) + g_y^\pm (x, y) \). By applying Green formula to the closed curve integral in (39), we obtain

\[
M^+(h) = - \int_{AB \cup BA} \bar{q}^+(x, y)dy + \int_{BA} f^+(x_0, y)dy \\
= - \int_{AB \cup BA} \left[ f^+(x, y) - f^+(x_0, y) + \int_{x_0}^x g^+_y (u, y)du \right] dy \\
+ \int_{BA} f^+(x_0, y)dy \\
= - \int_{AB} \left[ f^+(x, y) + \int_{x_0}^x g^+_y (u, y)du \right] dy. 
\]

Substituting the equation of curve \( AB \) (see (37)) into (40), it follows that

\[
M^+(h) = \int_{-a(h)}^{a(h)} \left[ f^+(\sqrt{h-y^2}, y) + \int_{x_0}^{\sqrt{h-y^2}} g^+_y (u, y)du \right] dy. 
\]

By (35), we have

\[
f^+(\sqrt{h-y^2}, y) + \int_{x_0}^{\sqrt{h-y^2}} g^+_y (u, y)du = (a_{00}^+ - b_{10}^- x_0) + (a_{10}^+ + b_{01}^-) \sqrt{h-y^2} + a_{01}^+ y. 
\]

Hence, combing this with the fact \( x_0 = a^2(h) \), we obtain

\[
M^+(h) = 2 \int_{-a(h)}^{a(h)} \left[ (a_{00}^+ - b_{10}^- x_0) + (a_{10}^+ + b_{01}^-) \sqrt{h-y^2} \right] dy \\
= 2(a_{00}^+ - b_{10}^- x_0) a(h) + 2(a_{10}^+ + b_{01}^-) T(a(h), h) \\
= 2a_{00}^+ a(h) - 2b_{10}^- a^3(h) + 2(a_{10}^+ + b_{01}^-) T(a(h), h). 
\]

where

\[
T(y, h) = \frac{1}{2} \left( y \sqrt{h-y^2} + h \arcsin \frac{y}{\sqrt{h}} \right). 
\]

Clearly, \( T(y, h) \) is an odd function with respect to \( y \). Similarly, we derive

\[
M^-(h) = \int_{BCDA \cup AB} g^-(x, y)dx - f^-(x, y)dy + \int_{BA} f^-(x, y)dy \\
= - \int_{BCDA} \left[ f^-(x, y) + \int_{x_0}^x g_y^-(u, y)du \right] dy.
\]
where most three limit cycles bifurcated from the period annulus.

By substituting (44) into (43), it follows that

Further, by (37) and (42), one derives

By Lemma 4.2, we compute that for

By substituting (44) into (43), it follows that

Substituting (41) and (45) into (38) and combining with (36), we obtain the Melnikov function of system (34)

By Lemma 4.2, we compute that for $h > 0$,

where

Note that $f(h)$ has zeros $h = 0, -\frac{1}{4}$ only. Hence, $f(h) \neq 0$ for $h > 0$. This means that the ordered set $(h, a(h), a^3(h), 2T(a(h), h))$ is an ECT-system in $h \in (0, \frac{4}{9})$. Thus, if $M(h) \neq 0$, it has at most three isolated zeros in $(0, \frac{4}{9})$. By Lemma 2.3, there exist at most three limit cycles bifurcated from the period annulus \( \{x^2 + y^2 = h \mid h \in (0, \frac{4}{9})\} \).
Now, we verify this bound can be reached. Easy calculations give for $h$ near 0

$$a(h) = h^2 - \frac{1}{2} h^2 + \frac{7}{8} h^3 + O(h^4),$$

$$a^3(h) = h^2 - \frac{3}{2} h^2 + O(h^3),$$

$$2T(a(h), h) = \frac{\pi}{2} h - \frac{2}{3} h^2 + O(h^3).$$

Therefore, we derive the expansion of Melnikov function (46) at $h = 0$ below

$$M(h) = c_1 h^2 + (c_0 + \frac{\pi}{2} c_3) h + (-\frac{1}{2} c_1 + c_2) h^2 + \frac{7}{8} c_1 - \frac{3}{2} c_2 - \frac{2}{3} c_3) h^3 + O(h^4).$$

Denote

$$\delta = (c_0, c_1, c_2, c_3), \quad b_1(\delta) = c_1, \quad b_2(\delta) = c_0 + \frac{\pi}{2} c_3,$$

$$b_3(\delta) = -\frac{1}{2} c_1 + c_2, \quad b_4(\delta) = \frac{7}{8} c_1 - \frac{3}{2} c_2 - \frac{2}{3} c_3.$$

Hence

$$M(h) = b_1(\delta) h^2 + b_2(\delta) h + b_3(\delta) h^2 + b_4(\delta) h^3 + O(h^4).$$

For any given $\bar{c}_3 \neq 0$ and $\delta_0 = (-\frac{\pi}{2} \bar{c}_3, 0, 0, \bar{c}_3)$, we have

$$b_1(\delta_0) = b_2(\delta_0) = b_3(\delta_0) = 0, \quad b_4(\delta_0) = -\frac{2}{3} \bar{c}_3 \neq 0 \quad \text{and} \quad \operatorname{rank} \frac{\partial (b_1, b_2, b_3)}{\partial (c_0, c_1, c_2)}(\delta_0) = 3.$$

By Theorem 2.6, system (34) can have 3 limit cycles in an arbitrary neighborhood of the origin. Hence, we obtain conclusion (i).

Now we prove conclusion (ii). Similar to the proof of conclusion (i), we obtain the Melnikov function of system (34)

$$M(h) = c_0 h + c_1 a(h) + c_2 \bar{a}(h) a(h) + 2c_3 T(a(h), h),$$

(47)

where

$$\bar{a}(h) = \frac{-1 + \sqrt{1 + 2h}}{2}, \quad a(h) = \sqrt{\frac{h - 1 + \sqrt{1 + 2h}}{2}},$$

and $T(y, h)$ is presented in (42).

By Lemma 4.2, we compute that for $h > 0$,

$$W(h) = h \neq 0,$$

$$W(h, a(h)) = -2 + 3h + h\sqrt{1 + 2h} - 2\sqrt{1 + 2h} \neq 0,$$

$$W(h, a(h), \bar{a}(h) a(h)) = -\frac{2 - h + 2\sqrt{1 + 2h}}{4(h - 1 + \sqrt{1 + 2h})(1 + 2h)\frac{\pi}{2}} \neq 0, \text{ if } h \neq 4,$$

$$W(h, a(h), \bar{a}(h) a(h), 2T(a(h), h)) = \frac{f^*(h)}{g^*(h)} \neq 0,$$
where
\[
\begin{align*}
\overline{f}(h) &= h^9 + h^8 \sqrt{1 + 2h} - 9h^8 + 40h^7 - 40h^6 \sqrt{1 + 2h} - 40h^6 \\
&
+ 16h^5 \sqrt{1 + 2h} - 240h^5 - 80h^4 \sqrt{1 + 2h} + 528h^4 \\
&
- 512h^3 \sqrt{1 + 2h} + 448h^3 + 320h^2 \sqrt{1 + 2h} - 960h^2 \\
&
+ 768h \sqrt{1 + 2h} - 1024h + 256 \sqrt{1 + 2h} - 256, \\
g^*(h) &= 4h^2 (h + 1 - \sqrt{1 + 2h})^2 (1 + 2h)^3 (h - 1 + \sqrt{1 + 2h})^3.
\end{align*}
\]

Note that \( f^*(h) \) has zero \( h = 0 \) only. This means that the ordered set
\[(h, a(h), \bar{a}(h) a(h), 2T(a(h), h))
\]
is an ECT-system in \( h \in (0, 4) \). Thus, \( M(h) \) in (47) has at most three isolated zeros in \((0, 4)\) if \( M(h) \neq 0 \). Hence, by Lemma 2.3, there exist at most three limit cycles bifurcated from the period annulus \( \{x^2 + y^2 = h \mid h \in (0, 4)\} \).

Further, by the following expansions for \( 0 < h \ll 1 \)
\[
\begin{align*}
a(h) &= h^\frac{1}{2} - \frac{1}{8} h^\frac{3}{2} + \frac{15}{128} h^2 + O(h^\frac{7}{2}), \\
\bar{a}(h) a(h) &= \frac{1}{2} h^\frac{3}{2} - \frac{5}{16} h^2 + O(h^\frac{7}{2}), \\
2T(a(h), h) &= \frac{\pi}{2} h^2 - \frac{1}{12} h^\frac{7}{2} + O(h^2),
\end{align*}
\]
we obtain the expansion of \( M(h) \) at \( h = 0 \)
\[
M(h) = c_1 h^\frac{1}{2} + (c_0 + \frac{\pi}{2} c_3) h + (-\frac{1}{8} c_1 + \frac{1}{2} c_2) h^\frac{3}{2} \\
+ (\frac{15}{128} c_1 - \frac{5}{16} c_2 - \frac{1}{12} c_3) h^2 + O(h^\frac{7}{2}).
\]

Similar to the analysis of case \( \varphi(y) = y^2 \), for any given \( c_3 \neq 0 \) and \( \delta_0 = (-\frac{\pi}{2} c_3, 0, 0, \bar{c}_3) \), by Theorem 2.6, it follows that system (34) can have 3 limit cycles in an arbitrary neighborhood of the origin. This ends the proof. \( \square \)

**Remark 2.** By making the change of variables
\[
u = x - \varphi(y), \quad y = y,
\]

system (34) becomes
\[
\begin{cases}
\dot{u} = y + \varphi'(y) [u + \varphi(y)] + \epsilon p(u, y), \\
\dot{y} = -[u + \varphi(y)] + \epsilon q(u, y),
\end{cases}
\]
where
\[
\begin{align*}
p(u, y) &= \begin{cases}
f^+(u + \varphi(y), y) - \varphi'(y) g^+(u + \varphi(y), y), & u \geq 0, \\
f^-(u + \varphi(y), y) - \varphi'(y) g^-(u + \varphi(y), y), & u < 0,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
q(u, y) &= \begin{cases}
g^+(u + \varphi(y), y), & u \geq 0, \\
g^-(u + \varphi(y), y), & u < 0.
\end{cases}
\end{align*}
\]

Clearly, if \( \varphi(y) = y^2 \) then system (48) is a piecewise smooth system with a straight switching line whose unperturbed system contains nonlinear items. This leads that system (34) with \( \varphi(y) = y^2 \) can generate more limit cycles than that with \( \varphi(y) \equiv 0 \).
Now, consider the following system which has circle $L$: $(x - 1)^2 + y^2 = 1$ as its switching curve

\[
\begin{align*}
\dot{x} &= y + \epsilon f(x,y), \\
\dot{y} &= -x + \epsilon g(x,y),
\end{align*}
\]  

(49)

where

\[
f(x,y) = \begin{cases} 
    f^+(x,y), & (x,y) \in \text{Int.}(L), \\
    f^-(x,y), & (x,y) \notin \text{Int.}(L),
\end{cases}
\]

\[
g(x,y) = \begin{cases} 
    g^+(x,y), & (x,y) \in \text{Int.}(L), \\
    g^-(x,y), & (x,y) \notin \text{Int.}(L),
\end{cases}
\]

\[
f^\pm(x,y) = a^\pm_{00} + a^\pm_{10}x + a^\pm_{01}y, \quad g^\pm(x,y) = b^\pm_{00} + b^\pm_{10}x + b^\pm_{01}y, \quad 0 < \epsilon \ll 1.
\]

For the number of limit cycles of system (49), the following theorem holds.

**Theorem 4.4.** Consider system (49) and assume (36) holds. Then,

(i): If the first order Melnikov function of system (49) is not zero identically, there exist at most 3 limit cycles bifurcated from the period annulus $\{x^2 + y^2 = h \mid h \in (0, 1)\}$ and this upper bound can be reached for some $(\epsilon, c_0, c_1, c_2, c_3)$ near $(0, -\frac{\epsilon}{2}c_3, 0, 0, c_3)$ with $c_3 \neq 0$.

(ii): If the first order Melnikov function of system (49) is not zero identically, there exist at most 3 limit cycles bifurcated from the period annulus $\{x^2 + y^2 = h \mid h \in (1, 4)\}$ and this upper bound is reached for some $(\epsilon, c_0, c_1, c_2, c_3)$ near $(0, -\frac{3\sqrt{3}}{8}c_2, \frac{1}{4}c_2, c_2, 0)$ with $c_2 \neq 0$.

**Proof.** By Theorem 2.7, we know that no matter point $A(h)$ on which section of $L$, the Melnikov function of system (49) is unique. By the similar proof of Theorem 4.3, we obtain

\[
M(h) = c_0 h + c_1 a(h) + \frac{1}{2} c_2 h a(h) + 2 c_3 T(a(h), h), \quad 0 < h < 4,
\]

(50)

where

\[
a(h) = \sqrt{\frac{h(4-h)}{2}}
\]

and $T(y,h)$ is presented in (42). Let

\[
p_0(h) = h, \quad p_1(h) = 2a(h), \quad p_2(h) = \frac{1}{2} h a(h), \quad p_3(h) = 2T(a(h), h).
\]

For $0 < h < 4$, the following is calculated

\[
W(p_0(h)) = h \neq 0,
\]

\[
W(p_0(h), p_1(h)) = -2 \sqrt{\frac{h}{4-h}} \neq 0,
\]

\[
W(p_0(h), p_1(h), p_2(h)) = \frac{h-1}{4-h} \neq 0, \quad \text{if } h \neq 1,
\]

\[
W(p_0(h), p_1(h), p_2(h), p_3(h)) = \frac{1}{2(4-h)^{3/2}} \sqrt{h} \neq 0.
\]

By Lemma 4.2, for $h \in (0, 1)$ or $h \in (1, 4)$, $(p_0, p_1, p_2, p_3)$ is an ECT-system. Hence, if $(c_0, c_1, c_2, c_3) \neq 0$, that is $M(h) \neq 0$, by Definition 4.1, $M(h)$ has at most 3 isolated zeros on $(0, 1)$ or $(1, 4)$. Further, by Lemma 2.3, there exist at most 3
limit cycles of (49) bifurcated from the period annulus \( \{x^2 + y^2 = h \mid h \in (0, 1) \} \) or \( \{x^2 + y^2 = h \mid h \in (1, 4) \} \).

On the other hand, for \( 0 < h \ll 1 \), \( p_i \) can be expanded as

\[
\begin{align*}
p_1(h) &= 2h^\frac{3}{2} - \frac{1}{4}h^2 - \frac{1}{64}h^\frac{5}{2} + O(h^\frac{7}{2}), \\
p_2(h) &= \frac{1}{2}h^\frac{3}{2} - \frac{1}{16}h^2 + O(h^\frac{5}{2}), \\
p_3(h) &= \frac{\pi}{2} - \frac{1}{12}h^\frac{3}{2} + O(h^\frac{5}{2}).
\end{align*}
\]

Thus, we have

\[
M(h) = c_1h^\frac{3}{2} + (c_0 + \frac{\pi}{2}c_3)h + (-\frac{1}{8}c_1 + \frac{1}{2}c_2)h^\frac{5}{2} - (\frac{1}{128}c_1 + \frac{1}{16}c_2 + \frac{1}{12}c_3)h^\frac{7}{2} + O(h^\frac{9}{2}).
\]

For any given \( \bar{c}_3 \neq 0 \) and \( \bar{\delta}_0 = (-\frac{3\sqrt{3}}{8}\bar{c}_2, \frac{1}{2}\bar{c}_2, \bar{c}_2, 0) \), the conditions of Theorem 2.6 hold for \( k = 3 \). Hence, system (49) can have 3 limit cycles in an arbitrary neighborhood of the origin.

If \( h \in (1, 4) \) with \( h - 1 \) small, \( p_i \) can be expanded as

\[
\begin{align*}
p_0(h) &= 1 + (h - 1), \\
p_1(h) &= \sqrt{3} + \frac{\sqrt{3}}{3}(h - 1) - \frac{2\sqrt{3}}{9}(h - 1)^2 + \frac{2\sqrt{3}}{27}(h - 1)^3 + O((h - 1)^4), \\
p_2(h) &= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{3}(h - 1) + \frac{\sqrt{3}}{36}(h - 1)^2 - \frac{\sqrt{3}}{27}(h - 1)^3 + O((h - 1)^4), \\
p_3(h) &= (\frac{\pi}{3} + \frac{\sqrt{3}}{4}) + (\frac{\pi}{3} + \frac{\sqrt{3}}{6})(h - 1) - \frac{\sqrt{3}}{9}(h - 1)^2 - \frac{\sqrt{3}}{36}(h - 1)^3 + O((h - 1)^4).
\end{align*}
\]

So, we can get an expansion of Melnikov function in (50) at \( h = 1 \) below

\[
M(h) = b_0(\delta) + b_1(\delta)(h - 1) + b_2(\delta)(h - 1)^2 + b_3(\delta)(h - 1)^3 + O((h - 1)^4),
\]

where

\[
\begin{align*}
b_0(\delta) &= c_0 + \frac{\sqrt{3}}{2}c_1 - \frac{\sqrt{3}}{4}c_2 + (\frac{\pi}{3} + \frac{\sqrt{3}}{4})c_3, \\
b_1(\delta) &= c_0 + \frac{\sqrt{3}}{6}c_1 + \frac{\sqrt{3}}{3}c_2 + (\frac{\pi}{3} + \frac{\sqrt{3}}{6})c_3, \\
b_2(\delta) &= -\frac{\sqrt{3}}{9}c_1 + \frac{\sqrt{3}}{36}c_2 - \frac{\sqrt{3}}{9}c_3, \\
b_3(\delta) &= \frac{\sqrt{3}}{27}c_1 - \frac{\sqrt{3}}{27}c_2 - \frac{\sqrt{3}}{36}c_3, \\
\delta &= (c_0, c_1, c_2, c_3).
\end{align*}
\]

For any given \( \bar{c}_2 \neq 0 \) and \( \bar{\delta}_0 = (-\frac{3\sqrt{3}}{8}\bar{c}_2, \frac{1}{2}\bar{c}_2, \bar{c}_2, 0) \), we obtain

\[
b_0(\bar{\delta}_0) = b_1(\bar{\delta}_0) = b_2(\bar{\delta}_0) = 0, \quad b_3(\bar{\delta}_0) = -\frac{\sqrt{3}}{36}\bar{c}_2 \neq 0 \quad \text{and} \quad \text{rank} \frac{\partial(b_0, b_1, b_2)}{\partial(c_0, c_1, c_3)}(\bar{\delta}_0) = 3.
\]

Obviously, \( b_0, b_1, b_2 \) can be taken as free parameters. For definiteness, let \( \bar{c}_2 < 0 \). It follows that \( b_3(\bar{\delta}_0) > 0 \). Then, we can choose \( b_0, b_1, b_2 \) satisfying

\[
0 < -b_0 \ll b_1 \ll -b_2 \ll 1
\]

which implies that \( M(h) \) can have three simple zeros on \((1, 4)\) arbitrarily near \( h = 1 \). By using Lemma 2.3, the proof is completed. \( \square \)
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E-mail address: tianhuanhuan8880163.com
E-mail address: mahan@shnu.edu.cn