Primordial Perturbation in Horava-Lifshitz Cosmology

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Recently, Ho\v{r}ava has proposed a renormalizable theory of gravity with critical exponent \( z = 3 \) in the UV. This proposal might imply the scale invariant primordial perturbation can be generated in any expansion of early universe with \( a \sim t^n \) and \( n > 1/3 \), which, in this note, will be validated by solving the motion equation of perturbation mode on super sound horizon scale for any background evolution of early universe. However, it is found that it seems that if we require the e folding number of primordial perturbation is suitable for observable universe, \( n \geq 1 \) still need to be satisfied, unless the scale of UV regime is quite low.

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Recently, Ho\v{r}ava has proposed a renormalizable theory of gravity at a Lifshitz point \(^1\), which can be a UV complete candidate of general relativity. In the UV, this theory has the critical exponent \( z = 3 \), at which the space and time scale differently, which describes the interacting nonrelativistic gravitons at short distances and is renormalizable. In the IR, this theory flows naturally to the relativistic value \( z = 1 \), and the general relativity is recovered.

Recently, the Ho\v{r}ava-Lifshitz (HL) gravitation theory has been studied intensively in Refs. \[2,3,4,5,6,7,8,9,10,11\]. In Refs. \[6,11\], the black hole solutions were studied. In Refs. \[4,6,9\], the cosmological solutions was explored. It was found that the early universe in HL cosmology may be able to escape singularities and has a nonsingular bounce. This might give an alternative to inflation, as has been discussed in \[8\] based on matter bounce. In Ref. \[7\], it was pointed out that, in UV regime of HL gravity, the spectrum of primordial perturbation induced by a scalar field may be scale invariant for any expansion with \( a \sim t^n \) and \( n > 1/3 \). This result is interesting. However, it might be required, and also significant to show it by solving the motion equation of perturbation mode on super sound horizon scale for any background evolution of early universe, which in some sense helps to understand how the perturbation generated in UV regime is matched to the observations on large IR scale. This will be done in this note.

In the UV of HL gravity, the action of a scalar mode, e.g. \( \Phi \), should have the critical exponent \( z = 3 \), which likes, see Refs. \[4,5,7\] for more discussions,

\[ I_{UV} \sim \int dt dx^3 \left( \Phi^2 + \Phi \frac{\Delta^3 \Phi}{a^6 M^4} \right), \]

where \( \Delta \equiv \partial_t \partial_t \) is the spacial Laplacian, \( a \) is the scale factor and \( M \) is the mass scale. The sign before \( \Phi \Delta^3 \Phi/a^6 M^4 \) is positive, which is required by the stability in the UV. In general, the term \( \Phi \Delta^3 \Phi/a^6 M^4 \) is important only when \( k/a \gtrsim M \), which means the physical wavelengths of the perturbation mode is quite short. This is consistent with the case of a sufficiently early period of expanding universe. When \( k/a \ll M \), which occurs after the universe expands some time, the term \( \Phi \Delta^3 \Phi/a^6 M^4 \) will be replaced with \( \Phi \Delta \Phi \), which means the field theory flows to the relativistic value \( z = 1 \), where the space and time will scale samely and the usual relativistic field theory will be acquired. The motion equation of perturbation of \( \Phi \) in the UV regime is given by, in the momentum space,

\[ u''_k + \left( \omega^2 - \frac{a''}{a} \right) u_k = 0, \]

where \( u_k \) is related to the perturbation \( \delta \Phi \) of \( \Phi \) by \( u_k \equiv a \delta \Phi a \) and the prime denotes the derivative with respect to the conformal time \( \eta \), and \( \omega = \frac{k^3}{a^2 M^2} \). In principle, we may generally take \( \omega = \frac{k^3}{(Ma)^2} \) for the calculation of Eq. \[2\], by which the perturbation spectrum can be obtained for different values of \( z \), which will be used in Eqs. \[4\] and \[5\]. We, no loosing generality, will take \( a \sim t^n \) for calculations, where \( n \) is a positive constant, thus for the conformal time \( a \sim \eta^{1-n} \).

The emergence of primordial perturbation in HL cosmology can be explained as follows. The universe is initially in the UV regime of HL gravity, \( \omega = \frac{k^3}{a^2 M^2} \). In this regime, since \( a \) is quite small, \( \omega \eta \gg 1 \), the perturbation modes can be regarded as adiabatic. The reason is that since \( \omega^2/\omega \sim 1/\eta \), thus the adiabatic condition \( \omega^2/\omega < \eta^2 \ll 1 \) is equivalent to \( \omega \eta \gg 1 \). Noting \( \eta \sim 1/(ah) \), we have \( \omega \eta \sim \omega/(ah) \gg 1 \), and thus obtain \( a/\omega \ll 1/h \), which corresponds to the case that the effective physical wavelength is quite deep into the horizon. Thus in this case, i.e. \( \omega \eta \gg 1 \),

\[ u_k \simeq \frac{1}{\sqrt{2\omega(\kappa, \eta)}} \exp (-i \int_\eta^\infty \omega(k, \eta) d\eta) \]

can be regarded as an approximate solution of Eq. \[2\]. \( \omega \eta \) will decrease with the expansion of \( a \). Thus at late time, we can expect \( \omega \eta \ll 1 \), i.e. \( a/\omega \gg 1/h \), which means that the effective wavelength will evolve faster than that of \( 1/h \), and thus will leave the horizon after some time, see the red dashed lines in Fig.1. This condition that \( \omega \eta \) decreases with the time equals that \( a^3 \) increase with the time, since \( \omega \sim 1/a^2 \) and \( \eta \sim 1/(ah) \). Thus when considering \( a \sim t^n \), \( n > 1/3 \) is required \[7\].
modes with wave number $\omega$. The red solid lines are the perturbation $c_s = \frac{\omega}{k}$, and the blue dashed line is that of $\ln(a)$. The left side of $a_e$ is the UV regime in which $z = 3$ and its right side is the IR regime in which $z = 1$. $a_e$ denotes the time when the UV regime ends. In the UV regime, initially $\omega a \gg 1$, we have $a/k \ll c_s/h$. Thus though $a/k \gg 1/h$, i.e. the physical wave length of perturbation mode is larger than the horizon, it is actually smaller than the “effective” sound horizon $c_s/h$, since $a^2 a^2$ is large. Thus a causal relation can be established on superhorizon scale. When the universe expands, $\frac{k^2}{a^2 M^2} \sim 1/a^2$ is decreased, thus the corresponding mode will leave this “effective” sound horizon and can be able to be responsible for seeds on large IR scale.

which corresponds to the state equation $w < 1$. While it is clear that if the universe is in contraction, which corresponds to $a \sim (-t)^n$ and in which the following Eqs. (4) and (5) can be also applied, the condition that the perturbation is able to leave the horizon is $n < 1/3$, i.e. $w > 1$.

This result can be also explained in another perspective, see Fig.1. When $\omega a \gg 1$, we have $a/k \ll (\frac{k^2}{a^2 M^2})/h$. Thus though $a/k \gg 1/h$, i.e. the physical wave length of perturbation mode is larger than the horizon, it actually smaller than the “effective” sound horizon $c_s/h$, where the effective sound speed is defined as $c_s = \frac{\omega}{a^2 M^2}$ since $\frac{k^2}{a^2 M^2} \sim 1/a^2$ is decreased, thus the corresponding mode will leave the “effective” sound horizon and can be able to be responsible for seeds in observable universe. The decrease of sound horizon $c_s/h$ with the time requires $a^2 h$ is increased with the time, thus $n > 1/3$ is obtained. In this sense, the generation of primordial perturbation in HL cosmology is similar to that in the scenario with the decaying speed of sound $\frac{\omega}{a}$, see also Ref. 17 for further illustration.

Eq. (2) is a deformed Bessel equation, which is slightly different from that used in usual calculations for perturbation, since here $\frac{k^2}{a^2 M^2}$ before $k^2$ is rapidly changed with the time. However, for primordial perturbation, such equation has been solved in Ref. 13 in detail. The general solutions of Eq. (2), which can be matched to Eq. (3) when the mode $u_k$ is quite deep inside the sound horizon, i.e. $\omega a \gg 1$, are the Hankel functions with the order $v$ and the variable $\omega a$. This solution on super sound horizon scale, i.e. $\omega a \ll 1$, is

$$u_k \simeq \frac{1}{\sqrt{2\omega}} (\omega a)^{0.5 - v} \simeq \frac{a M}{\sqrt{2k^3}} \left( \frac{k^2}{a^2 M^2} \frac{k a}{\omega} \right)^{0.5 - v},$$

(4)

which corresponds to the expansion of Hankel functions to the leading term of $\omega a$, where the prefactor of order one has been neglected, the upper equation is that for general $\omega$, while the lower equation is that for $z = 3$. In principle $v$ in this deformed Bessel equation is determined not only by $a^2/a$, but also by the dependence of $\omega$ on time. $v$ has been calculated in Ref. 13, which is

$$v = 0.5 \left| \frac{3n - 1}{n z - 1} \right|$$

(5)

for any $z$. It can be noticed that if $z = 1$, $v$ will be reduced to the usual result, in which only when $n > 1$ or $n = 2/3$, which correspond to that of the inflation and the contraction dominated by matter 16, 17, see also earlier 18, respectively, $v = 1.5$ and thus $P_{\Phi}^{1/2} \simeq k^{-3/2} \left| \frac{\omega a}{c_s} \right|$ is scale invariant, which is familiar result. While if $z = 2$, which corresponds to $\omega = \frac{a^2}{a\pi}$, in term of Eq. (5) and considering $P_{\Phi}^{1/2} \simeq k^{-3/2} \left| \frac{\omega a}{c_s} \right|$, it can be found that the scale invariance of spectrum requires $n > 1$ or $n = 5/12$. $n = 1$ apparently corresponds to that of inflation. When $z = 2$, for an expanding universe, the condition that the perturbation is able to leave the horizon requires that $a^2 h$ increases with the time. This means $n > 1/2$. Thus the case with $n = 5/12$ actually corresponds to the contraction dominated by the component with $w = 3/5$. In principle, in term of Eqs. (4) and (5), we can deduce in what background evolution of early universe, i.e. what value $n$ is, the perturbation spectrum generated is scale invariant for some special value of $z$.

Here $z = 3$, thus it is found that $v = 0.5$ and thus $u_k \simeq 1/k^{3/2}$ for any value of $n$. The spectrum of primordial perturbation induced by $\varphi$ is given by

$$P_{\Phi}^{1/2} \simeq k^{3/2} \left| \frac{u_k}{a} \right| \simeq M.$$

(6)

Thus on super sound horizon scale, the spectrum is scale invariant for any case. This means that if the universe is contracting, $n < 1/3$ is required for the emergence of

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1 The spectral index is actually $n_s - 1 = 3 - z \left| \frac{3n - 1}{n z - 1} \right|$. 

FIG. 1: The figure of $\ln(\frac{a}{\omega})$ with respect to $\ln(a)$, see the green solid lines. The red solid lines are the perturbation modes with wave number $k$. The “effective” sound speed $c_s = \frac{\omega}{k}$. The blue dashed line is that of $\ln(a)$. The left side of $a_e$ is the UV regime in which $z = 3$ and its right side is the IR regime in which $z = 1$. $a_e$ denotes the time when the UV regime ends. In the UV regime, initially $\omega a \gg 1$, we have $a/k \ll c_s/h$. Thus though $a/k \gg 1/h$, i.e. the physical wave length of perturbation mode is larger than the horizon, it is actually smaller than the “effective” sound horizon $c_s/h$, since $\frac{k^2}{a^2 M^2}$ is large. Thus a causal relation can be established on superhorizon scale. When the universe expands, $\frac{k^2}{a^2 M^2} \sim 1/a^2$ is decreased, thus the corresponding mode will leave this “effective” sound horizon and can be able to be responsible for seeds on large IR scale.
primordial perturbation, while if the universe is in expansion, \( n > 1/3 \) is required, which corresponds that the early universe is dominated by the component with \( w < 1 \). Thus we reproduce the result of Ref. [7], however, by solving the motion equation (2) of perturbation mode on super sound horizon for any background evolution of early universe. The effect of background evolution on spectrum is reflected in the term \((\omega \eta)^{0.5 - v} \) in Eq. (3), which is just 1 for \( z \) being exactly 3. In term of (3) and (5), the dependence of tilt of spectrum on \( z \) and \( n \) can be seen clearly.

The end of UV regime means the epoch at which the term \( \frac{k^2}{M^2} \) is replaced with \( \Phi \Delta \Phi \) in (1), which means that the universe is entering into the IR regime of HL gravity. This requires \( \frac{k^2}{M^2} \simeq 1 \), thus \( h_e \simeq M \), where the subscript ‘e’ denotes the end epoch of UV regime, since \( k^2 = a^2 h M^2 \) for the perturbation mode just leaving the horizon. This spectrum of \( \Phi \) field can be inherited by that of curvature perturbation in IR regime, which thus leads to the scale invariant curvature perturbation. The eefolding number for primordial perturbation is defined as

\[
\mathcal{N} = \ln \left( \frac{k_e}{k} \right),
\]

where \( k \) is the comoving wave number, which is equal to the value at the time when the corresponding perturbation mode leaves the sound horizon. This definition actually corresponds to the ratio of the physical wavelength of perturbation mode corresponding to the present observable scale to that at the end epoch of UV regime, which is generally not equal to the eefolding number of scale factor. We substitute the comoving wave number \( k = a h^{1/3} M^{2/3} \) and \( h \sim 1/a^{1/n} \) into (7), and have \( \mathcal{N} = (n - 1) \ln \left( \frac{h_e}{h} \right) \), which is consistent with the requirement of \( n > 1/3 \) discussed. This result indicates that, for fixed \( n \), the resulting \( \mathcal{N} \) depends on the ratio \( h \) to \( h_e \), which must be large enough to match the requirement of observable cosmology.

The eefolding number \( \mathcal{N} \) required is generally determined by the evolution of standard cosmology after the UV regime ends. In general, for simplicity, we assume that after the UV regime ends the energy density of background field can rapidly transferred into that of radiation, which will bring the universe to an evolution of standard cosmology. We regard \( M_e \) as the end scale of UV regime, which approximately equals to the reheating scale. In this case, the observation requires \( \mathcal{N} \simeq 68.5 + \ln(M_e/M_P) \) [20], which is actually consistent with that given by Ref. [19]. It can be noticed that \( M_e \simeq \sqrt{h} M_P \), since \( h_e \simeq M \).

We plot the figure of the \( \mathcal{N} \) with respect to \( \log(\frac{M_e}{M_P}) \) in Fig. 2, where that the UV regime begins at \( M_P \) has been set. We can see that for \( 1/3 < n < 1 \), it seems difficult to obtain the enough eefolding number, unless the scale \( M_e \) at which the UV regime ends is quite low. For example, for \( n = 2/3 \), we have \( \mathcal{N} \simeq 3 \) for \( M_e \simeq 10^{45} \text{Gev} \), while only when \( M_e \simeq 100 \text{Gev} \), can the enough eefolding number be acquired. It is clear that if \( n > 1 \), it is easier to have enough eefolding number suitable for observable universe, for example, for \( n = 3 \), we have \( \mathcal{N} \simeq 45 \) for \( M_e \sim 10^{45} \text{Gev} \), while when \( M_e \sim 10^{43} \text{Gev} \), \( N > 70 \).

The period of \( n > 1 \) corresponds to that of an accelerated expansion, i.e. inflation. Thus in HL cosmology, though we can obtain a scale invariant spectrum of primordial perturbation for any expansion with \( n > 1/3 \), but it seems that we still need a period of inflation to obtain enough eefolding number of primordial perturbation. However, it is significant that, compared with inflation with nearly exponential expansion, here \( n \gg 1 \) is not required, for example, \( n \simeq 3 \) is enough, which helps to relax the bounds for inflation model building. In principle, for \( 1/3 < n < 1 \), we can also consider some methods to obtain enough eefolding number, e.g. [13]. In addition, it is also interesting to explore above case in the bounce cosmology [8].

In conclusion, it is showed by solving the motion equation of perturbation mode for any background evolution of early universe that the primordial perturbation can be generated naturally in UV regime in HL cosmology for any expanding period of early universe with \( n > 1/3 \), which is scale invariant on large IR scale. However, it seems that if we require the eefolding number of primordial perturbation suitable for observable universe, \( n \gtrsim 1 \) still need to be satisfied, unless the scale of UV regime is quite low. The motion equation of tensor perturbation in UV regime is similar to that of scalar perturbation. Thus the similar discussions can be applied. This means that in principle we can have a detailed compare of results with recent observations [19], which will be considered.
work might be interesting for motivating further studies for HL cosmology.

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