Faddeev eigenfunctions for two-dimensional Schrödinger operators via the Moutard transformation

I.A. Taimanov *   S.P. Tsarev †

Abstract

We demonstrate how the Moutard transformation of two-dimensional Schrödinger operators acts on the Faddeev eigenfunctions on the zero energy level and present some explicitly computed examples of such eigenfunctions for smooth fast decaying potentials of operators with non-trivial kernel and for deformed potentials which correspond to blowing up solutions of the Novikov–Veselov equation.

1 Introduction

In [1] we used the Moutard transformation for constructing examples of two-dimensional Schrödinger operators with fast decaying smooth potentials and nontrivial kernels and of blowing up solutions of the Novikov–Veselov equation.

Here we show how the Moutard transformation acts on the Faddeev eigenfunctions corresponding to the zero energy level. In particular, we construct some explicit examples which correspond to operators constructed in [1]. It appears that such Faddeev eigenfunctions have very interesting analytical properties and dynamics under the Novikov–Veselov flow.

The presented procedure also enables us to extend in an essentially new way a list of two-dimensional Schrödinger operators for which the Faddeev eigenfunctions are explicitly computed at least for one energy level.

* Sobolev Institute of Mathematics, 630090 Novosibirsk, Russia; e-mail: taimanov@math.nsc.ru
† Siberian Federal University, 79 Svobodny Prospect, Krasnoyarsk 660041 Russia; e-mail: sptsarev@mail.ru.
For multi-dimensional Schrödinger operator
\[ L = -\Delta + u \]
the Faddeev eigenfunctions \( \psi(x, k), k \in \mathbb{C}^n, x \in \mathbb{R}^n \), are formal solutions to the equation
\[ L\psi = E\psi, \quad E = \langle k, k \rangle, \]
meeting the condition
\[ \psi = e^{i\langle k, x \rangle}(1 + o(1)) \text{ as } |x| \to \infty. \quad (1) \]
Here \( \langle x, y \rangle = \sum_{j=1}^{n} x_j y_j \) is the standard inner product in \( \mathbb{C}^n \).

These functions were defined in [2] and play the crucial role in the multi-dimensional inverse scattering problem [3, 4].

In the sequel we consider two-dimensional Schrödinger operators:
\[ L = -\Delta + u = -\left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) + u = -4\partial\bar{\partial} + u, \]
where in the right-hand side we have \( \partial = \frac{\partial}{\partial x} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \bar{\partial} = \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), (x_1, x_2) = (x, y) \in \mathbb{R}^2, z = x + iy \in \mathbb{C}. \)

In this case, as well as for higher dimensions, there are not many examples for which these eigenfunctions were computed explicitly. In fact, they are as follows:

1) for the point potential
\[ u(x) = \varepsilon \delta(x), \]
the Faddeev eigenfunctions are computed for all energy levels in [5];

2) for the Grinevich–Zakharov potentials introduced by Grinevich [6] and Zakharov (these potentials are reflectionless on this energy level \( E \) and decay as \( |x|^{-2} \)) explicit formulas for these Faddeev eigenfunctions on the energy level \( E \) may be extracted from computations in [6].

In [7] it is shown that for negative energy levels \( E \) the functions \( \chi(x, k) = e^{-i\langle k, x \rangle} \psi(x, k) = \exp \left( -\frac{i}{2}\sqrt{-E} \left( \lambda z - \frac{1}{\lambda} \right) \right) \psi(x, k) \), where \( k_1 = \sqrt{-E} \left( \lambda + \frac{1}{2} \right), k_2 = -\sqrt{-E} \left( \lambda - \frac{1}{2} \right) \), satisfy the generalized Cauchy–Riemann equation
\[ \frac{\partial \chi}{\partial \lambda} = T\bar{\chi}, \]
i.e. they are generalized analytical functions, and if \( T \) is nonsingular, then this level is below the ground state. Moreover in the latter case the inverse scattering problem of reconstructing potentials (without assuming that these potentials are sufficiently small) is solved by methods of generalized analytic functions theory [9].
In this article we consider the Faddeev eigenfunctions on the zero energy level $E = 0$. They meet the following condition

$$\psi(z, \bar{z}, \lambda) = e^{\lambda z} \left(1 + \frac{A(\lambda, \bar{\lambda})}{z} + B(\lambda, \bar{\lambda})\frac{1}{\bar{z}} + O \left(\frac{1}{|z|^2}\right)\right) \text{ as } |z| \to \infty.$$  \hfill (2)

In fact, this is one branch of these functions and another branch is given by solutions to the equation $(-\Delta + u)\psi = 0$ with asymptotics $e^{\mu z}(1 + o(1))$ as $|z| \to \infty$. For non-zero energy levels the analogous asymptotic expansions were given in [10] and for the zero energy level it is derived in a similar way from the analytical properties of the Green–Faddeev function of the Laplace operator. The detailed analysis of the latter function, in particular, was given in [8].

2 The Moutard transformation

Given $\omega$ satisfying the equation

$$L \omega = (-\Delta + u)\omega = 0,$$

the Moutard transformation of $L$ is defined as

$$L \to \tilde{L} = -\Delta + u - 2\Delta \log \omega = -\Delta - u + 2\frac{\omega_x^2 + \omega_y^2}{\omega^2}.$$

It is easy to check that

- if $\varphi$ satisfies the equation $L \varphi = 0$, then the function $\theta$ defined up to term $C\omega$, $C = \text{const}$, from the system

$$\left(\omega \theta\right)_x = -\omega^2 \left(\frac{\varphi}{\omega}\right)_y, \quad \left(\omega \theta\right)_y = \omega^2 \left(\frac{\varphi}{\omega}\right)_x$$ \hfill (3)

satisfies $\tilde{L} \theta = 0$.

If the potential $u$ depends only on $x$, then a certain reduction of the Moutard transformation gives the famous Darboux transformation of one-dimensional Schrödinger operators:

$$L = -\frac{d^2}{dx^2} + u = \left(-\frac{d}{dx} + v\right) \left(-\frac{d}{dx} + v\right) \to \tilde{L} = \left(-\frac{d}{dx} + v\right) \left(-\frac{d}{dx} + v\right).$$

This is the case when $\omega = f(x)e^{\sqrt{Ey}}$ and $\left(-\frac{d^2}{dx^2} + u\right)f = Ef$.

In [11] we used a double iteration of the Moutard transformation starting from the Laplace operator $L_0 = -\Delta$. Given two harmonic functions $\omega_1$ and $\omega_2$, $\Delta \omega_1 = \Delta \omega_2 = 0$, we construct a pair of Moutard transformations corresponding to $\omega_1$ and $\omega_2$:

$$L_1 = -\Delta + u_1 \xleftarrow{\omega_1} L_0 = -\Delta \xrightarrow{\omega_2} L_2 = -\Delta + u_2.$$
Both potentials \( u_1 = -2\Delta \log \omega_1 \) and \( u_2 = -2\Delta \log \omega_2 \) have singularities at zeroes of functions \( \omega_1 \) and \( \omega_2 \), respectively, and we have to iterate the Moutard transformation to achieve smooth (i.e. bounded and differentiable) potentials.

Let us denote by \( \theta_1 \) and \( \theta_2 \) the transforms of \( \omega_2 \) and \( \omega_1 \) via (3) for \( \omega = \omega_1 \) and \( \omega = \omega_2 \), respectively. Such transformations are defined up to \( C_\omega \) and, for \( \theta_1 \) fixed, we put \( \theta_2 = -\frac{\omega_1}{\omega_2} \theta_1 \) (cf. Fig. 1). We have

- the functions \( \theta_1 \) and \( \theta_2 \) define the Moutard transformations of \( L_2 \) and \( L_1 \), respectively, which give the same operator

\[
L_1 = -\Delta + u_1 \xrightarrow{\theta_1} L = L_{12} = -\Delta + u_{12} \xleftarrow{\theta_2} L_2 = -\Delta + u_2.
\]

- the functions \( \varphi_1 = \frac{1}{\theta_1} \) and \( \varphi_2 = \frac{1}{\theta_2} \) meet the equation \( L\psi = 0 \).

The general formula for \( u = u_{12} \) takes the form

\[
u = -2\Delta \log i \left( (p_1 p_2 - p_2 p_1) + \int ((p_1' p_2 - p_1 p_2') dz + (\bar{p}_1 \bar{p}_2' - \bar{p}_1' \bar{p}_2) d\bar{z}) \right),
\]

where \( \omega_1 = p_1(z) + p_1(\bar{z}) \), \( \omega_2 = p_2(z) + \bar{p}_2(\bar{z}) \), \( p_1(z) \) and \( p_2(z) \) are holomorphic functions of \( z \), and the free scalar parameter which we mentioned above appears as the integration constant in (4).

For \( L_0 = -\Delta \) the Faddeev eigenfunctions (2) are \( \psi_0(z, \lambda) = e^{\lambda z} \) and they are transformed via (3) to \( \psi_1 \) and \( \psi_2 \):

\[
\psi_1 \xleftarrow{\omega_1} \psi_0 = e^{\lambda z} \xrightarrow{\omega_2} \psi_2.
\]

As we mentioned above these transformations are not uniquely defined however at least for functions \( \omega_1 \) and \( \omega_2 \) which are polynomial in \( z \) and \( \bar{z} \) it is possible to choose the branches of \( \psi_1 \) and \( \psi_2 \) such that they would meet the condition (1). Then “the cubic superposition formula” (see [1] and Fig. 2) implies that

\[
\psi(z, \bar{z}, \lambda) = e^{\lambda z} + \frac{\omega_2}{\theta_1}(\psi_2 - \psi_1)
\]
satisfies the equation $L\psi = 0$. It seems that for smooth fast decaying potentials $u = u_{12}$ the function $\psi$ is the Faddeev eigenfunction on the zero energy level. We do not state that here as a rigorous mathematical result however for the potentials found in [1] this is the case.

Example. Let $p_1(z) = (1 - \frac{1}{4})z^2 + \frac{1}{2}z$. $p_2(z) = \frac{1}{4}(3 - 5i)z^2 + \frac{1-i}{2}z$. For some appropriate constant $C$ in $\theta_1$ we obtain

$$u = -\frac{5120(4 - i)z + 1}{(160 + |z|^2)(4 - i)z + 2|i|^2}.$$  

After simplifying multiplications by some constants $\varphi_1$ and $\varphi_2$ take the form

$$\varphi_1 = \frac{2(z + \bar{z}) + (4 - i)z^2 + (4 + i)\bar{z}^2}{160 + |z|^2|4 - i)z + 2|^2},$$

$$\varphi_2 = \frac{2(1 - i)z + 2(1 + i)\bar{z} + (3 - 5i)z^2 + (3 + 5i)\bar{z}^2}{160 + |z|^2|4 - i)z + 2|^2}.$$

The Faddeev eigenfunction (2) for $L = -\Delta + u$ is as follows:

$$\psi(z, \bar{z}, \lambda) = e^{\lambda z} \left( 1 + \frac{1}{\lambda} \frac{(8i - 32)z\bar{z} - 8\bar{z} - (16 + 4i)\bar{z}^2 - 68z\bar{z}^2}{160 + |z|^2|4 - i)z + 2|^2} + \frac{1}{\lambda^2} \frac{32 - 8i)\bar{z} + 68\bar{z}^2}{160 + |z|^2|4 - i)z + 2|^2} \right) =$$

$$= e^{\lambda z} \left( 1 - \frac{4}{\lambda z} + O \left( \frac{1}{|z|^2} \right) \right) \text{ as } |z| \to \infty.$$  

The functions $u$, $\varphi_1$, and $\varphi_2$ are smooth and real-valued, and the function $\psi$ is also smooth in $z, \bar{z}$. The potential $u$ decays like $1/|z|^6$ as
$|z| \to \infty$, and $\varphi_1$ and $\varphi_2$ decay like $1/|z|^2$ as $|z| \to \infty$. Hence $\varphi_1$ and $\varphi_2$ span a two-dimensional subspace in the kernel of $L$ and, since $u \leq 0$ and $u$ decays sufficiently fast, there are also negative eigenvalues of $L$.

We also see that “the scattering data” takes a very simple form:

$$A(\lambda, \bar{\lambda}) = -\frac{4}{\lambda}, \quad B(\lambda, \bar{\lambda}) = 0.$$  

Similar formulas for the Faddeev eigenfunctions may be easily derived for another smooth and fast decaying potential with non-trivial kernel which was found in [1] and corresponds to $p_1(z) = (i-1)z^3 + \left(\frac{1}{10} + \frac{3}{20}i\right) z^2 + \frac{1}{2}z$. In this case “the scattering data” are $A(\lambda, \bar{\lambda}) = -\frac{6}{\lambda}$, $B(\lambda, \bar{\lambda}) = 0$.

3 The extended Moutard transformation and the Novikov–Veselov equation

In order to comply with the traditional notation related to the Novikov–Veselov equation we renormalize the Schrödinger operator as follows

$$H = \partial \bar{\partial} + U = \frac{1}{4} \Delta - \frac{u}{4}.$$  

In these terms the Moutard transformation of the potential and eigenfunctions is given by the formulas

$$U \to U + 2\partial \bar{\partial} \log \omega, \quad V \to V + 2\partial^2 \log \omega.$$  

Let us assume that $\varphi$ depends also on the temporal variable $t$ and satisfies the equations

$$H \varphi = 0, \quad \partial_t \varphi = \left(\partial^3 + \bar{\partial}^3 + 3V\partial + 3\bar{V}\bar{\partial}\right)\varphi, \quad \bar{\partial}V = \partial U$$  

where $U = \bar{U}$. It is checked by straightforward computations that

- given real-valued functions $\omega$ and $U$, the system (5) is invariant under the extended Moutard transformation [1, 12].

$$\varphi \to \theta = \frac{i}{\omega} \int \left(\varphi \partial \omega - \omega \partial \varphi\right) dz - \left(\varphi \bar{\partial} \omega - \omega \bar{\partial} \varphi\right) d\bar{z} + [\varphi \partial^3 \omega - \omega \partial^3 \varphi + \omega \partial^3 \varphi - \varphi \bar{\partial}^3 \omega + 2(\partial^2 \varphi \partial \omega - \partial \varphi \partial^2 \omega) - 2(\partial^2 \varphi \bar{\partial} \omega - \bar{\partial} \varphi \partial^2 \omega) + 3V(\varphi \partial \omega - \omega \partial \varphi) + 3\bar{V}(\omega \partial \varphi - \varphi \partial \omega)] dt,$$

$$U \to U + 2\partial \bar{\partial} \log \omega, \quad V \to V + 2\partial^2 \log \omega.$$  

(6)
The compatibility condition for (5) is the well-known Novikov–Veselov equation \[11\]

\[ U_t = \partial^3 U + \bar{\partial}^3 U + 3 \partial (VU) + 3 \bar{\partial} (\bar{V}U) = 0, \quad \bar{\partial} V = \partial U. \] \hspace{1cm} (7)

It is a two-dimensional generalization of the Korteweg–de Vries equation to which it reduces in the case \( U = U(x), U = V \).

We may obtain analogs of all constructions of §2 by replacing the Moutard transformation by its extended version (6) and construct by this mean non-trivial real-valued solutions of the Novikov–Veselov equation (7) (cf. \[1\]):

- if \( p_1(z, t) \) and \( p_2(z, t) \) are holomorphic functions which satisfy

\[ \frac{\partial p}{\partial t} = \frac{\partial^3 p}{\partial z^3}, \]

then \( \omega = p_1 + \bar{p}_1 \) and \( \varphi = p_2 + \bar{p}_2 \) satisfy (8) with \( U = V = 0 \).

Let us consider the extended Moutard transformation of \( H_0 = \partial \bar{\partial} \) defined by \( \omega_1 \), and let \( \theta_1 \) be the image of \( \omega_2 \) under this transformation. Let \( H = \partial \bar{\partial} + U \) is obtained by the iteration of the Moutard transformation defined by \( \theta_1 \). Then a real-valued function

\[ U(z, \bar{z}, t) = 2 \partial \bar{\partial} \log i \left( (p_1 \bar{p}_2 - p_2 \bar{p}_1) + \int ((p_1' \bar{p}_2 - p_2' \bar{p}_1)dz + \right. \]

\[ + (p_1 \bar{p}_2' - p_1' \bar{p}_2)dz) + \int (p_1''' \bar{p}_2 - p_1''' \bar{p}_2 + 2(p_1'' \bar{p}_2' - p_1'' \bar{p}_2') + 
\]

\[ \left. \right) \right) \]

satisfies the Novikov–Veselov equation

\[ U_t = \partial^3 U + \bar{\partial}^3 U + 3 \partial (VU) + 3 \bar{\partial} (\bar{V}U) = 0, \quad \bar{\partial} V = \partial U. \]

**Example.** Let \( p_1(z) = iz^2 \) and \( p_2 = (1 + i)z + z^2 \). Although these functions are independent on \( t \) the formula (8) gives a non-stationary solution of the Novikov–Veselov equation. The Faddeev eigenfunction is obtained by applying iterations of (6) to \( \psi_0(z, \bar{z}, t) = e^{\lambda z + \lambda^* t} \).

These data lead to the following functions:

\[ U(z, \bar{z}, t) = \frac{F_U(z, \bar{z}, t)}{Q^2(z, \bar{z}, t)}, \quad V(z, \bar{z}, t) = \frac{F_V(z, \bar{z}, t)}{Q^2(z, \bar{z}, t)} \]

where

\[ F_U(z, \bar{z}, T) = 6i(24t(i\bar{z} - iz + z + \bar{z}) + 2iz^4 \bar{z} + 4iz^3 \bar{z} - 2iz \bar{z}^4 - 4iz \bar{z}^3 + \]

\[ + 4iz^2 \bar{z}^2 + 2iz \bar{z}^3 - 2iz^2 \bar{z} - 4iz \bar{z}^2 + 2iz^3 \bar{z} - 4iz^2 \bar{z}) \]
+60iz - 60i\bar{z} + 96tz\bar{z} + 2z^4\bar{z} + z^4 + 6z^2\bar{z}^2 + 2z\bar{z}^4 - 240z\bar{z} - 60z + 5z^4 - 60\bar{z}),

F_V(z, \bar{z}, t) = 6i(24t(z - i\bar{z} + z + \bar{z}) + iz^4 + 4iz^3\bar{z}^2 - 12iz^2\bar{z}^3 - 6iz^2\bar{z}^2 + +6iz^4 - 60iz + 2i\bar{z}^5 + 5iz^4 + 60iz + 48tz^2 + 4z^3\bar{z}^2 + 4z^3\bar{z} + 12z^2\bar{z}^4 + 12z^2\bar{z}^3 + +6z\bar{z}^4 + 4z\bar{z}^3 - 60z - 2\bar{z}^2 - 60\bar{z}),

and

Q(z, \bar{z}, t) = (12 - 12i)t - z^3 - (3 - 3i)z^2\bar{z}^2 + 3i\bar{z}^2 - 3z\bar{z}^2 + i\bar{z}^3 - (30 - 30i) = 

= \frac{1}{1 + t} (24t - [6(x^2 + y^2) - 2(6x^2 + 8(x^2 + y^3) + 60)].

The Faddeev eigenfunctions for the potentials $U(z, \bar{z}, t)$ take the form

$$
\psi(z, \bar{z}, t, \lambda) = e^{\lambda z} \left( 1 + \frac{\mu_1(z, \bar{z}, t)}{\lambda} + \frac{\mu_2(z, \bar{z}, t)}{\lambda^2} \right) = 

= e^{\lambda z} \left( 1 + \frac{1}{\lambda} \cdot \frac{6(-2iz\bar{z}^2 - 2iz\bar{z} + z^2 + 2z\bar{z}^2 + \bar{z}^2)}{Q(z, \bar{z}, t)} + +\frac{1}{\lambda^2} \cdot \frac{12(i\bar{z}^2 + i\bar{z} - z - \bar{z})}{Q(z, \bar{z}, t)} \right).

From the latter formula it is easy to derive “the scattering data” for $U(z, \bar{z}, t)$:

$$
\psi(z, \bar{z}, t, \lambda) = e^{\lambda z} \left( 1 - \frac{4}{\lambda z} + O \left( \frac{1}{|z|^2} \right) \right) \quad \text{as} \quad |z| \to \infty,

i.e. we have

$$
A(\lambda, \bar{\lambda}, t) = -\frac{4}{\lambda}, \quad B(\lambda, \bar{\lambda}, t) = 0.

The function $U(z, \bar{z}, t)$ gives an example of a solution to the Novikov–Veselov equation which has smooth and fast decaying (like $1/|z|^3$ as $|z| \to \infty$) initial data $U(z, \bar{z}, 0)$ at $t = 0$ and which blows up at finite time (in this case it is $T_s = \frac{2\pi}{72}$, the time at which $Q(z, \bar{z}, t)$ vanishes at some point). The scenario of this blow up is described in [1].

We remark that

1) “the scattering data” $A(\lambda, \bar{\lambda}, t)$ and $B(\lambda, \bar{\lambda}, t)$ are conserved and do not feel the blow-up;

2) the real and imaginary parts of $\mu_2(z, \bar{z}, t)$ give solutions of the equation $H_\varphi = 0$ which for $t = 0$ are non-singular and decay like $O \left( \frac{1}{|z|^2} \right)$ as $|z| \to \infty$, i.e. they are eigenfunctions of $H$ at the zero energy level. Hence the formula for $\mu_2(z, \bar{z}, t)$ describes a deformation of some functions from the discrete spectrum via the Novikov–Veselov
equation: these eigenfunctions stay nonsingular and quadratically integrable until the critical time $T^*$ when they do obtain the same singularities as $U$.

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