Well-posedness for the Navier–Stokes equations in critical mixed-norm Lebesgue spaces

TUOC PHAN

Abstract. We study the Cauchy problem in $n$-dimensional space for the system of Navier–Stokes equations in critical mixed-norm Lebesgue spaces. Local well-posedness and global well-posedness of solutions are established in the class of critical mixed-norm Lebesgue spaces. Being in the mixed-norm Lebesgue spaces, both of the initial data and the solutions could be singular at certain points or decaying to zero at infinity with different rates in different spatial variable directions. Some of these singular rates could be very strong, and some of the decaying rates could be significantly slow. Besides other interests, the results of the paper demonstrate the persistence of the anisotropic behavior of the initial data under the evolution. To achieve the goals, fundamental analysis theory such as Young’s inequality, time decaying of solutions for heat equations, the boundedness of the Helmholtz–Leray projection, and the boundedness of the Riesz transform are developed in mixed-norm Lebesgue spaces. These analysis results are topics of independent interests, and they are potentially useful in other problems.

1. Introduction and main results

This paper establishes local and global well-posedness of the Cauchy problem for Navier–Stokes equations in critical mixed-norm Lebesgue spaces. We consider the following initial value problem for the system of Navier–Stokes equations of incompressible fluid in $n$-dimensional space

$$
\begin{align*}
  u_t - \Delta u + (u \cdot \nabla)u + \nabla P &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T),
  \\
  \text{div}(u) &= 0 \quad \text{in} \quad \mathbb{R}^n \times (0, T),
  \\
  u|_{t=0} &= a_0 \quad \text{on} \quad \mathbb{R}^n,
\end{align*}
$$

where $u = (u_1, u_2, \ldots, u_n) : \mathbb{R}^n \times (0, T) \rightarrow \mathbb{R}^n$ is the unknown velocity of the considered fluid with some $T > 0$ and $n \geq 2$. Moreover, $P : \mathbb{R}^n \times (0, T) \rightarrow \mathbb{R}$ is the unknown fluid pressure, and $a_0$ is a given vector field initial data function which is assumed to be divergence-free. Global well-posedness of small solutions in critical mixed-norm Lebesgue spaces and local well-posedness for large solutions in critical mixed-norm Lebesgue spaces are established. Being in the mixed-norm Lebesgue spaces, both of the initial data and the solutions could be singular at certain points or decaying to zero at infinity with different rates in different spatial variable directions. Some of these singular rates could be very strong, and some of the decaying rates could be significantly slow.
spaces, both of the initial data and the solutions obtained in the paper could possibly decay to zero with different rates as $|x| \to \infty$ in different directions. Similarly, they could also be singular at certain points in $\mathbb{R}^n$ with different rates in different directions of the spatial $x$-variable. As a result, this paper demonstrates the persistence of the anisotropic properties of the initial data under the evolution of the Navier–Stokes equations.

To explain the ideas, motivation and to put our results in perspective, let us review and discuss known results concerning the Cauchy problem for the system of the Navier–Stokes equations (1.1) with possibly irregular initial data in critical spaces. In 1984, in the well-known work, Kato [17] initiated the study of (1.1) with initial data belonging to the space $L_n(\mathbb{R}^n)$ and he proved the global existence and uniqueness of solutions of (1.1) in a subspace of $C([0, \infty), L_n(\mathbb{R}^n))$ provided the norm $\|a_0\|_{L_n(\mathbb{R}^n)}$ is sufficiently small. Similarly, local existence and uniqueness of solutions were also obtained in [17] with initial data $a_0 \in L_n(\mathbb{R}^n)$. As found in [15,18,20,21,31], in [5,6,28] and [1, Theorem 5.40, p. 234], this kind of global and local existence and uniqueness of solutions continues to hold with initial data in homogeneous Morrey spaces $\mathcal{M}^{q,q}(\mathbb{R}^n)$ for $1 \leq q \leq n$, and, respectively, in homogeneous Besov spaces $\dot{B}^{-1+\frac{n}{p_0}}_{p_0,\infty}(\mathbb{R}^n)$ for $n \leq p_0 < \infty$. Here, for $1 \leq q < \infty$ and $0 < \lambda \leq n$, we say that a $L_q$-locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$ belongs to the Morrey space $\mathcal{M}^{q,\lambda}(\mathbb{R}^n)$ provided that its norm

$$
\|f\|_{\mathcal{M}^{q,\lambda}(\mathbb{R}^n)} = \sup_{B_\rho(x_0) \subset \mathbb{R}^n} \left\{ \rho^{\lambda-n} \int_{B_\rho(x_0)} |f(x)|^q \, dx \right\}^{\frac{1}{q}} < \infty,
$$

where $B_\rho(x_0)$ denotes the ball in $\mathbb{R}^n$ of radius $\rho > 0$ and centered at $x_0 \in \mathbb{R}^n$. Also, for $q \in [1, \infty]$ and $\alpha > 0$, $\dot{B}^{-\alpha,q}_{q,\infty}(\mathbb{R}^n)$ denotes the homogeneous Besov space consisting of distributions $f$ whose norm can be equivalently defined by

$$
\|f\|_{\dot{B}^{-\alpha,q}_{q,\infty}(\mathbb{R}^n)} \approx \sup_{t > 0} t^{\frac{\alpha}{q}} \|e^{\Delta t} f(\cdot)\|_{L_q(\mathbb{R}^n)} < \infty. \tag{1.2}
$$

The significant breakthrough is due to the work by Koch and Tataru [19]. In this work, the authors established the global well-posedness of the Cauchy problem (1.1) for small initial data in the borderline BMO$^{-1}(\mathbb{R}^n)$ space. Here, the space BMO$^{-1}(\mathbb{R}^n)$ can be defined as the space of all distributional divergences of BMO($\mathbb{R}^n$) vector fields. On the other hand, it should be also noted that it has been shown recently by Bourgain and Pavlović [2] that the Cauchy problem (1.1) is ill-posedness in a space even smaller than $\dot{B}^{-1,\infty}_{\infty,\infty}(\mathbb{R}^n)$.

We note that all of the spaces that appear in the mentioned papers are invariant with respect to the scaling

$$
f(\cdot) \to \lambda f(\lambda \cdot), \quad \lambda > 0 \tag{1.3}
$$
in the sense that for every $f$ in some space $E$ that we just mentioned, then

$$
\|f(\cdot)\|_E = \|\lambda f(\lambda \cdot)\|_E, \quad \forall \lambda > 0.
$$
In other words, up to now, \( \text{BMO}^{-1}(\mathbb{R}^n) \) is the largest known space that is invariant under the scaling (1.3) on which the Cauchy problem for the system of the Navier–Stokes equations (1.1) is globally well-posed for small initial data. Interested readers may find in [14,16] for related results in bounded domains, and in [1, Chapter 5], [24, Chapters 7–9] and [32, Chapter 5] for further results, discussion, and more related references.

Motivated by the mentioned work, this paper continues the study of the well-posedness of the Cauchy problem (1.1) in critical spaces. We plan to revisit [17] and extend it to a different point of view. In particular, we investigate the class of initial data and solutions for the Cauchy problem (1.1) that possibly decay to zero with different rates as \( |x| \to \infty \) in different directions. Some of these rates could be extremely slow. Similarly, the class of initial data and solutions investigated in this paper could also be singular at certain points in \( \mathbb{R}^n \) with different singularity rates in different spatial directions, some of which could be very strong. As the initial data and the solutions are in the same class of such functions, the results of this paper particularly imply the persistence of the anisotropic properties of the initial data under the evolution of the Navier–Stokes equations. To the best of our knowledge, this phenomenon is not very well documented even for the heat equations. To achieve the goals, we follow the spirit of Krylov in the work [23] and use mixed-norm Lebesgue spaces to capture the features of those kinds of functions. Several important analysis inequalities and estimates in mixed-norm Lebesgue spaces will be also developed in this paper. See also [8–10,30] for some other related work and [22] for a survey paper on some interesting features regarding mixed-norm Lebesgue spaces.

For \( p_1, p_2, \ldots, p_n \in [1, \infty) \), and for a given measurable function \( f : \mathbb{R}^n \to \mathbb{R} \), we say that \( f \) belongs to the mixed-norm Lebesgue space \( L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n) \) if its norm

\[
\| f \|_{L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} = \left( \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \left( \cdots \left( \int_{\mathbb{R}} |f(x_1, x_2, \ldots, x_n)|^{p_1} \, dx_1 \right)^{p_2/p_1} \, dx_2 \ldots \right)^{p_{n-1}/p_{n-2}} \, dx_{n-1} \right)^{p_n/p_{n-1}} \, dx_n \right)^{1/p_n} < \infty.
\]

Similar definitions can be also formulated if some of the indices in \( \{p_1, p_2, \ldots, p_n\} \) are equal to \( \infty \). Note that it follows directly from the definition that if \( p = p_1 = p_2 = \cdots = p_n \), then \( L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n) \) is the same as the usual Lebesgue space \( L_p(\mathbb{R}^n) \).

To clearly explain our ideas as well as to understand the importance of the mixed-norm Lebesgue spaces, let us consider the following example about a function that is decaying to zero at different rates as \( |x| \to \infty \). Similar examples can be easily produced about different rates of singularity of functions at some certain points. We consider a bounded measurable function \( f : \mathbb{R}^3 \to \mathbb{R} \) satisfying

\[
|f(x)| \leq \frac{N}{|x_1|^{10^{-2}}|x'|^{10^k}}, \quad x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^2, \quad |x| > 1,
\] (1.4)
with some given constant $N > 0$ and $k \in \mathbb{N}$ which could be very large. It can be seen that $f \in L_{p_1, p_2, p_3}(\mathbb{R}^3)$ with $p_1 > 10^k$ and $p_2 = p_3 > \frac{2}{10^k}$. However, if we consider the usual Lebesgue space, then $f \in L_p(\mathbb{R}^3)$ only if $p > 10^k$, which can be very large when we choose $k$ sufficiently large. In other words, the very fast decaying directions in $(x_2, x_3)$-variable of the function $f$ is completely invisible in the usual unmixed Lebesgue spaces. As a consequence, in the unmixed spaces, the class of functions $f$ and $p^N$ with some given constant $N$ and $k$ when we choose $g$ and moreover $p$, holds, we have $\|p^N\|_{L^p} < \infty$. Therefore, in some certain sense, this paper can be considered as a natural but completely non-trivial extension of the work [17].

Before stating our results, let us introduce some notations used in the paper. For given numbers $p_1, p_2, \ldots, p_n \in [1, \infty]$, the mixed-norm space $L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)$ is invariant under the scaling (1.3) if and only if

$$\frac{1}{p_1} + \frac{1}{p_2} + \cdots + \frac{1}{p_n} = 1. \quad (1.5)$$

The class of critical mixed-norm spaces $L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)$ such that (1.5) holds is the one we will establish the well-posedness for solutions of the Navier–Stokes equations (1.1) in this paper. Note that in the special case when $p_1 = p_2 = \cdots = p_n$ and (1.5) holds, we have $p_1 = p_2 = \cdots = p_n = n$. On the other hand, we also note, as an example, that for the class of functions as in (1.4), it is possible to choose (1.1) in this paper. Note that in the special case when $p_1 = p_2 = \cdots = p_n$ and (1.5) holds, we have $p_1 = p_2 = \cdots = p_n = n$. On the other hand, we also note, as an example, that for the class of functions as in (1.4), it is possible to choose $p_1 > 10^k$ and $p_2 = p_3 > 2$ but sufficiently close to 2 so that the triple $(p_1, p_2, p_3)$ satisfies the condition (1.5). Therefore, in some certain sense, this paper can be considered as a natural but completely non-trivial extension of the work [17].

Before stating our results, let us introduce some notations used in the paper. For given $p_k \in [1, \infty)$, we write $PL_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)$ the space of all vector fields $f \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)$ such that

$$\text{div}(f) = 0 \quad \text{in} \quad \mathbb{R}^n \quad \text{in the sense of distributions.}$$

Also, for given $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$ such that $p_k \in (1, \infty)$ and $q_k \in [p_k, \infty)$ for all $k = 1, 2, \ldots, n$. Assume that (1.5) holds and

$$\frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} = \delta \in (0, 1). \quad (1.6)$$

Then, with given $T \in (0, \infty]$, we denote $X_{p,q,T}$ the space consisting of all measurable vector field functions $f : \mathbb{R}^n \times [0, T) \rightarrow \mathbb{R}^n$ such that for

$$g(x, t) := t^\frac{1}{\delta} f(x, t) \quad \text{and} \quad \tilde{g}(x, t) := t^\frac{1}{\delta} D_x f(x, t), \quad \text{with} \quad (x, t) \in \mathbb{R}^n \times (0, T)$$

then

$$g \in C([0, T), PL_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)), \quad \tilde{g} \in C([0, T), PL_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n))$$

and moreover $g(x, 0) = 0$, $\tilde{g}(x, 0) = 0$ and the norm

$$\|f\|_{X_{p,q,T}} = \sup_{t \in (0,T)} \left[ \|g(\cdot, t)\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} + \|\tilde{g}\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \right] < \infty. \quad (1.7)$$
We also denote $\mathcal{Y}_{p,T}$ the space consisting of all vector field functions $f \in C([0, T), PL_{p_1,p_2,...,p_n}(\mathbb{R}^n))$ such that $t^{\frac{4}{n}}D_x f \in C([0, T), PL_{p_1,p_2,...,p_n}(\mathbb{R}^n))$ and

$$\|f\|_{\mathcal{Y}_{p,T}} = \sup_{t \in (0, T)} \left[ \|f(t)\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)} + t^{\frac{1}{2}}\|D_x f(t)\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)} \right] < \infty.$$ (1.8)

The following theorem on local and global well-posedness of the Cauchy problem (1.1) in the critical mixed-norm Lebesgue spaces $L_{p_1,p_2,...,p_n}(\mathbb{R}^n)$ is the main result of the paper.

**Theorem 1.9.** Let $p = (p_1, p_2, \ldots, p_n)$ and $q = (q_1, q_2, \ldots, q_n)$. Assume that $p_k \in [2, \infty)$, $q_k \in [p_k, \infty)$ for all $k = 1, 2, \ldots, n$, and the conditions (1.5) and (1.6) hold. Moreover, if for some $k = 1, 2, \ldots, n$ so that $p_k = 2$, we assume $q_k > 2$. Then, there exist a sufficiently small constant $\lambda_0 > 0$ and a large number $N_0 > 0$ depending only on $n, p,$ and $q$ such that the following assertions hold.

(i) For every $a_0 \in L_{p_1,p_2,...,p_n}(\mathbb{R}^n)$ with $\nabla \cdot a_0 = 0$, if $\|a_0\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)} \leq \lambda_0$, then the Cauchy problem (1.1) has unique global time solution $u \in \mathcal{X}_{p,q,\infty} \cap \mathcal{Y}_{p,q,\infty}$ with

$$\|u\|_{\mathcal{X}_{p,q,\infty}} \leq N_0\|a_0\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)} \quad \text{and} \quad \|u\|_{\mathcal{Y}_{p,q,\infty}} \leq N_0\|a_0\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)}.$$ 

(ii) For every $a_0 \in L_{p_1,p_2,...,p_n}(\mathbb{R}^n)$ with $\nabla \cdot a_0 = 0$, there exists $T_0 > 0$ sufficiently small depending on $n, p, q$, and $a_0$ such that the Cauchy problem (1.1) has unique local time solution $u \in \mathcal{X}_{p,q,T_0} \cap \mathcal{Y}_{p,q,T_0}$ with

$$\|u\|_{\mathcal{X}_{p,q,T_0}} \leq N_0\|a_0\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)} \quad \text{and} \quad \|u\|_{\mathcal{Y}_{p,q,T_0}} \leq N_0\left[\|a_0\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)} + \|a_0\|_{L_{p_1,p_2,...,p_n}(\mathbb{R}^n)}^2\right].$$

To the best of our knowledge, this is the first time that the kinds of solutions of Navier–Stokes equations in critical mixed-norm Lebesgue spaces are discovered. As demonstrated in the example in (1.4) and the discussion after (1.5), it is possible to choose some of $p_1, p_2, \ldots, p_n$ to be very large numbers so that the given numbers $(p_1, p_2, \ldots, p_n)$ still satisfy the condition (1.5). Due to this reason, in some directions, the class of the initial data and the solutions in Theorem 1.9 could decay significantly slow. Similarly, some of the singularity rates in some spatial directions could be very strong. Hence, our class of solutions may not belong to $L_n(\mathbb{R}^n)$ nor $L_2(\mathbb{R}^n)$, and the solutions obtained in Theorem 1.9 may not belong to the classes of solutions found in the papers [17,25,26]. Observe also that if $p_1 = p_2 = \cdots = p_n$ and (1.5) holds, then $p_1 = p_2 = \cdots = p_n = n$. In this sense, this paper can be considered as a natural, but completely non-trivial extension of the work [17].

Now, we summarize the above discussion with the following remarks regarding Theorem 1.9.

**Remark 1.10.** The following interesting points are worth highlighting.
(i) Under the condition (1.5), the initial data and the solutions obtained in Theorem 1.9 may decay to zero very slow as \( |x| \to \infty \). Similarly, they could also be strongly singular in some spatial directions. Therefore, the solutions obtained in Theorem 1.9 may not be in \( L_n(\mathbb{R}^n) \) nor \( L_2(\mathbb{R}^n) \). Consequently, these solutions may not be the same as the ones obtained in [17, 25, 26].

(ii) Let \( p_0 \in (\max\{p_1, p_2, \ldots, p_n, n\}, \infty) \), where \( p_1, p_2, \ldots, p_n \) are as in Theorem 1.9. Then, if \( a_0 \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n) \), it follows from the characterization of Besov spaces with negative regularity (see Remark 2.21 below) that
\[
a_0 \in \dot{B}^{-1+\frac{n}{p_0}}_{p_0, \infty}(\mathbb{R}^n) \subset \text{BMO}^{-1}(\mathbb{R}^n).
\] In view of this and the results obtained in [5, 6, 19, 28] and [1, Theorem 5.40, p. 234], Theorem 1.9 can be seen as a refinement of these results regarding the persistence of the anisotropic properties of the initial data under the evolution of the Navier–Stokes equations. See also [3, Section 3.3] for some different but related results.

To prove Theorem 1.9, we follow the approach developed in [11, 12, 17] and in [14, 16, 19, 27]. To implement the method, several important and fundamental analysis estimates in mixed-norm Lebesgue spaces are developed. In Sect. 2, we develop and prove a version of Young’s inequality in mixed-norm Lebesgue spaces. We then use Young’s inequality in mixed-norm Lebesgue spaces to establish the time decaying estimates for solutions of heat equations in mixed norm Lebesgue spaces. The boundedness of the Riesz transform and the boundedness of the Helmholtz–Leray projection in mixed-norm Lebesgue spaces are also established and proved in Sect. 2. Clearly, these analysis inequalities and estimates are topics of independent interests and they can be useful in many other problems. Also, the mentioned mixed-norm analysis estimates seem to be documented for the first time in this paper. As such, besides the study of the Navier–Stokes equations, this paper also provides some contribution in real and harmonic analysis theory. The paper concludes with Sect. 3 which provides the proof of Theorem 1.9.

2. Preliminaries on analysis inequalities in mixed-norm Lebesgue spaces

This section gives some main ingredients for the proof of the main theorem. In particular, we develop Young’s inequality in mixed-norm Lebesgue spaces, time decaying rate estimates for solutions of the Cauchy problem for the heat equation in mixed-norm Lebesgue spaces, and Helmholtz–Leray projection in mixed-norm Lebesgue spaces.

2.1. Young’s inequality in mixed-norm Lebesgue spaces

This section establishes a result on Young’s inequality in mixed norm Lebesgue spaces. The result will be useful in the study of heat equations in mixed-norm Lebesgue spaces. Our theorem can be stated as in the following.
**Theorem 2.1.** (Young’s inequality in mixed norm) Let $p_k$, $r_k$ and $q_k$ be given numbers in $[1, \infty]$ that satisfy

$$\frac{1}{p_k} + 1 = \frac{1}{q_k} + \frac{1}{r_k}, \quad k = 1, 2, \ldots, n.$$ 

Then

$$\|f * g\|_{L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \|f\|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \|g\|_{L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} \tag{2.2}$$

for every $f \in L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)$ and $g \in L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)$.

**Proof.** We use induction on $n$. Observe that when $n = 1$, the inequality (2.2) is the classical Young’s inequality. We now assume that the inequality holds true in $(n-1)$-dimension and prove it for $n$-dimension with $n \geq 2$. Let us denote $p_k', q_k', r_k'$ the Hölder’s conjugates of $p_k, q_k, r_k$, respectively. By the assumption, we see that

$$\frac{1}{r_k'} + \frac{1}{p_k} + \frac{1}{q_k} = 1, \quad \frac{r_k'}{p_k} + \frac{r_k}{q_k} = 1, \quad \text{and} \quad \frac{q_k}{r_k'} + \frac{q_k}{p_k} = 1, \quad k = 1, 2, \ldots, n. \tag{2.3}$$

We split the proof into three different cases.

**Case I** We assume that $p_1 < \infty$ and $q_1 < \infty$. In this case, we also see that $r_1 < \infty$. For $x, y \in \mathbb{R}^n$, we write $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and $y = (y_1, y') \in \mathbb{R} \times \mathbb{R}^{n-1}$. As $q_1 < \infty$, by using the last two identities in (2.3), we have

$$|(f * g)(x)| \leq \int_{\mathbb{R}^n} |f(y)||g(x - y)|dy$$

$$= \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}} |f(y_1, y')||g(x_1 - y_1, x' - y')|dy_1 \right] dy'$$

$$= \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}} |f(y_1, y')|^{\frac{q_1}{r_1}} \left( |f(y_1, y')|^{\frac{q_1}{p_1}} |g(x_1 - y_1, x' - y')|^{\frac{r_1}{p_1}} \right) \right]^{\frac{r_1}{q_1}} dy_1 dy'.$$

Note that in the above inequality also holds when $r_1' = \infty$ with $\frac{1}{r_1'} = 0$. Now, by using the first identity in (2.3) and the Hölder’s inequality with respect to the integration in $y_1$-variable, we obtain

$$|(f * g)(x)|$$

$$\leq \int_{\mathbb{R}^{n-1}} \left[ \left( \frac{\int_{\mathbb{R}} |f(y_1, y')|^{\frac{q_1}{r_1}} dy_1 \right)^{\frac{r_1}{q_1}} \left( \int_{\mathbb{R}} |g(y_1, y')|^{\frac{r_1}{q_1}} |g(x_1 - y_1, x' - y')|^{\frac{r_1}{q_1}} dy_1 \right)^{\frac{1}{r_1}} \right] dy',$$ \hspace{1cm} (2.4)

where we denote

$$f_1(y') = \left( \int_{\mathbb{R}} |f(y_1, y')|^{\frac{q_1}{r_1}} dy_1 \right)^{\frac{1}{q_1}} \quad \text{and} \quad$$

$$g_1(y') = \left( \int_{\mathbb{R}} |g(y_1, y')|^{\frac{r_1}{q_1}} dy_1 \right)^{\frac{1}{r_1}}, \quad \text{for a.e. } y' \in \mathbb{R}^{n-1}. \tag{2.5}$$
As $p_1 < \infty$, it follows from (2.4) that

$$G(x') := \left( \int_\mathbb{R} |(f \ast g)(x_1, x')|^{p_1} \, dx_1 \right)^{\frac{1}{p_1}}$$

$$\leq \left\{ \left( \int_{\mathbb{R}^{n-1}} \left| f_1(y') \right|^{\frac{q_1}{r_1}} |g_1(x' - y')|^{\frac{r_1}{q_1}} \right) \left( \int_{\mathbb{R}} |f(y_1, y')|^{q_1} |g(x_1 - y_1, x' - y')|^{r_1} \, dy_1 \right)^{\frac{1}{p_1}} \, dx_1 \right\}^{\frac{1}{p_1}}.$$

From this, and by using the Minkowski’s inequality, we see that

$$\left( \int_{\mathbb{R}} |(f \ast g)(x_1, x')|^{p_1} \, dx_1 \right)^{\frac{1}{p_1}} \leq \int_{\mathbb{R}^{n-1}} \left( |f_1(y')|^{\frac{q_1}{r_1}} |g_1(x' - y')|^{\frac{r_1}{q_1}} I(x', y') \right) \, dy',$$

where

$$I(x', y') = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y_1, y')|^{q_1} |g(x_1 - y_1, x' - y')|^{r_1} \, dy_1 \right) \right)^{\frac{1}{p_1}}.$$

By the Fubini’s theorem, we see that

$$I(x', y') = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f(y_1, y')|^{q_1} |g(x_1 - y_1, x' - y')|^{r_1} \, dx_1 \right) \right)^{\frac{1}{p_1}}$$

$$\quad = \|g(\cdot, x' - y')\|_{L_1(\mathbb{R})}^{\frac{r_1}{p_1}} \|f(\cdot, y')\|_{L_{q_1}(\mathbb{R})}^{\frac{q_1}{p_1}}$$

$$\quad = |g_1(x' - y')|^{\frac{r_1}{p_1}} |f_1(y')|^{\frac{q_1}{p_1}}.$$ 

Therefore,

$$G(x') \leq \int_{\mathbb{R}^{n-1}} \left( |f_1(y')|^{\frac{q_1}{r_1}} |g_1(x' - y')|^{\frac{r_1}{q_1}}, \frac{r_1}{p_1} \right) \, dy'.$$

From this, and by using the last two identities in (2.3), we obtain

$$G(x') = (f_1 \ast g_1)(x').$$

Then, by induction hypothesis, we see that

$$\|f \ast g\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} = \|G\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^{n-1})} \leq \|g_1\|_{L_{r_1, r_2, \ldots, r_n}(\mathbb{R}^{n-1})} \|f_1\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^{n-1})}$$

$$\leq \|g\|_{L_{r_1, r_2, \ldots, r_n}(\mathbb{R}^{n-1})} \|f\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^{n-1})}.$$ 

This proves the desired estimate for the case $q_1 < \infty$ and $p_1 < \infty$. 
Case II  We assume that $p_1 = \infty$ and $q_1 < \infty$. In this case, we observe that $r_1' = q_1 \in [1, \infty)$. In this case, we write
\[ |f \ast g(x)| \leq \int_{\mathbb{R}^{n-1}} \left[ \int_{\mathbb{R}} |f(y_1, y')||g(x_1 - y_1, x' - y')|dy_1 \right]dy'. \]
If $r_1 < \infty$, as $\frac{1}{q_1} + \frac{1}{r_1} = 1$, we apply the Hölder's inequality for the integration with respect to $y_1$ to obtain
\[ |(f \ast g)(x)| \leq \int_{\mathbb{R}^{n-1}} f_1(y')g(x' - y')dy' = (f_1 \ast g_1)(x'), \]
for a.e. $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$,

where $f_1, g_1$ are defined as in (2.5). Observe also that the similar estimate can be also done when $r_1 = \infty$. From this, the desired inequality follows by the induction hypothesis as in Case I. The proof for this case is therefore completed.

Case III  We are left to consider the case that $q_1 = \infty$. In this case, it follows that $p_1 = \infty$ and $r_1 = 1$. By defining
\[ G(x') = \sup_{x_1 \in \mathbb{R}} |(f \ast g)(x_1, x')|, \quad f_1(x') = \sup_{x_1 \in \mathbb{R}} |f(x_1, x')|, \quad g_1(x') \]
\[ = \int_{\mathbb{R}} |g(x_1, x')|dx_1 \]
we see that
\[ G(x') \leq (f_1 \ast g_1)(x'). \]

Then, we also obtain the same desired estimate. The proof is then completed.

Remark 2.6. Theorem 2.1 gives the classical unmixed-norm Young's inequality when $p_1 = p_2 = \cdots = p_n$ and $q_1 = q_2 = \cdots = q_n$.

2.2. Heat equations in mixed-norm Lebesgue spaces

This section develops estimates of time decaying rates for solutions of heat equations in mixed-norm Lebesgue spaces. We consider the Cauchy problem for the heat equation
\[ \left\{ \begin{array}{ll}
  u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\
  u|_{t=0} = u_0 & \text{on } \mathbb{R}^n.
\end{array} \right. \tag{2.7} \]

Under some suitable conditions on the initial data $u_0$, it is well known that
\[ u(x, t) = e^{\Delta t}u_0(x) = (G_t \ast u_0)(x), \quad (x, t) \in \mathbb{R}^n \times (0, \infty), \tag{2.8} \]
is a solution of (2.7), where
\[ G_t(x) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}, \quad (x, t) \in \mathbb{R}^n \times (0, \infty). \]

The following result on the time decaying rates of the solutions (2.8) of the heat equation (2.7) in mixed norm Lebesgue spaces is the main result of this section.
Theorem 2.9. (Time decaying of solutions for heat equation in mixed-norm) Let $1 \leq q_k \leq p_k \leq \infty$. There exists a positive constant $N$ depending only on $p_1, p_2, \ldots, p_n$, $q_1, q_2, \ldots, q_n$ such that for the solution $u(x, t) = e^{\Delta t}u_0(x)$ defined in (2.8) of the Cauchy problem (2.7) with $u_0 \in L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)$, it holds that

$$\|u(\cdot, t)\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq N t^{-\frac{1}{2}} \sum_{k=1}^{n} \left(\frac{1}{q_k} - \frac{1}{p_k}\right) \|u_0\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \quad \text{for } t > 0. \quad (2.10)$$

Moreover, for every $l = 1, 2, \ldots$ and for $t > 0$

$$\|D^l_x u(\cdot, t)\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq N t^{-\frac{1}{2} - \frac{l}{2}} \sum_{k=1}^{n} \left(\frac{1}{q_k} - \frac{1}{p_k}\right) \|u_0\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)}, \quad (2.11)$$

where $D^l_x$ denotes the $l$th-derivative in $x$-variable.

Proof. We begin with the proof of (2.10). For each $k = 1, 2, \ldots, n$, by the assumption that $q_k \leq p_k$, we can find $r_k \in [1, \infty)$ such that

$$\frac{1}{p_k} + 1 = \frac{1}{r_k} + \frac{1}{q_k}. \quad (2.12)$$

Then, because $u(x, t) = (G_t * u_0)(x)$, we can use the mixed-norm Young’s inequality in Theorem 2.1 to see that

$$\|u(\cdot, t)\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \|G_t(\cdot)\|_{L_{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} \|u_0\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)}. \quad (2.13)$$

We now note that we can write $G_t$ as

$$G_t(x) = g_t(x_1) g_t(x_2) \cdots g_t(x_n) \quad \text{for } x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad t > 0$$

where $g_t$ is the heat kernel in $\mathbb{R}$:

$$g_t(s) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{s^2}{4t}}, \quad s \in \mathbb{R}, \quad t > 0. \quad (2.14)$$

Note also that for $r \in [1, \infty)$ we have

$$\|g_t(\cdot)\|_{L_r(\mathbb{R})} = \frac{1}{\sqrt{4\pi t}} \left(\int_{\mathbb{R}} e^{-\frac{s^2}{4t}} \right)^{\frac{1}{r}} = \frac{1}{\sqrt{4\pi t}} \left(\int_{\mathbb{R}} \frac{1}{\sqrt{t}} \left(\int_{\mathbb{R}} e^{-\frac{z^2}{2}} dz\right)^{\frac{1}{r}} \right) \quad (2.15)$$

Moreover, we also see that

$$\|g_t(\cdot)\|_{L_{\infty}(\mathbb{R})} \leq N t^{-\frac{1}{2}}, \quad t > 0.$$
From this, we infer that
\[
\|G_t(\cdot)\|_{L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} = \|g_t(\cdot)\|_{L^{r_1}(\mathbb{R})} \|g_t(\cdot)\|_{L^{r_2}(\mathbb{R})} \cdots \|g_t(\cdot)\|_{L^{r_n}(\mathbb{R})}
\]
\[
= N(r_1, r_2, \ldots, r_n) t^{-\frac{1}{2}(n-\sum_{i=1}^{n} \frac{1}{r_i})}
\]
\[
= N(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n) t^{-\frac{1}{2} \sum_{i=1}^{n} (\frac{1}{q_i} - \frac{1}{p_i})}, \quad t > 0,
\]
where we have used (2.12) in the last estimate. This last estimate together with (2.13) implies (2.10).

Next, we prove (2.11). We only demonstrate the proof of (2.11) with \(l = 1\) as the general case can be done in a similar way. We observe that for each \(i = 1, 2, \ldots, n\)
\[
D_{x_i} u(x, t) = ([D_{x_i} G_t] \ast u_0)(x), \quad (x, t) \in \mathbb{R}^n \times (0, \infty).
\]
Then, by the mixed-norm Young’s inequality in Theorem 2.1, we have
\[
\|D_{x_i} u(\cdot, t)\|_{L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \|D_{x_i} G_t(\cdot)\|_{L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} \|u_0(\cdot)\|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)}. \tag{2.16}
\]
It remains to estimate the mixed norm \(\|D_{x_i} G_t(\cdot)\|_{L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)}\). Note that
\[
D_{x_i} G_t(x) = -\frac{1}{(4\pi t)^{\frac{n}{2}}} \gamma \frac{x_i}{2t} e^{-\frac{|x|^2}{4t}}, \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \quad t > 0.
\]
Consequently,
\[
D_{x_i} G_t(x) = h_t(x_i) \left( \prod_{k \neq i} g_t(x_k) \right), \quad x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \quad t > 0,
\]
where \(g_t\) is defined as in (2.14) and
\[
h_t(s) = -\frac{1}{\sqrt{4\pi t}} \frac{s}{2t} e^{-\frac{s^2}{4t}}, \quad s \in \mathbb{R} \quad \text{and} \quad t > 0.
\]
We observe that
\[
|h_t(s)| \leq \frac{N}{t} \frac{|s|}{\sqrt{4t}} e^{-\frac{s^2}{4t}} = \frac{N}{t} |z| e^{-|z|^2} \quad \text{where} \quad z = \frac{s}{\sqrt{4t}}.
\]
As \(|z| e^{-z^2}\) is a bounded function for \(z \in \mathbb{R}\), we conclude that
\[
\|h_t\|_{L^{\infty}(\mathbb{R})} \leq \frac{N}{t}, \quad t > 0.
\]
On the other hand, if \(r_i \in [1, \infty)\), we see that
\[
\|h_t\|_{L^{r_i}(\mathbb{R})} = \frac{N(r_i)}{t} \left[ \int_{\mathbb{R}} \left( \frac{\sqrt{r_i} |s|}{\sqrt{4t}} \right)^{r_i - \frac{n}{2}} e^{-\frac{s^2}{4t}} ds \right]^{\frac{1}{r_i}}
\]
\[
= N(r_i) t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{r_i})} \left[ \int_{\mathbb{R}} \frac{dz}{|z|^n} e^{-z^2} \right]^{\frac{1}{r_i}}
\]
\[
= N(r_i) t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{r_i})}.
\]
Therefore, for \( r_i \in [1, \infty] \), we have

\[
\|h_t\|_{L_{r_i}(\mathbb{R})} = N(r_i)t^{-\frac{1}{2} - \frac{1}{2}(1 - \frac{1}{r_i})}, \quad t > 0.
\]

From this estimate and (2.15), we see that

\[
\|Dx_i G_t\|_{L_{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} = \|h_t\|_{L_{r_i}(\mathbb{R})} \prod_{k \neq i} \|g_t\|_{L_{r_k}(\mathbb{R})} \leq N(r_1, r_2, \ldots, r_n)t^{-\frac{1}{2} - \frac{1}{2}(n - \sum_{k=1}^n \frac{1}{r_k})}.
\]

From this, and by using (2.12), we infer that

\[
\|Dx_i G_t\|_{L_{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} \leq N(r_1, r_2, \ldots, r_n)t^{-\frac{1}{2} - \frac{1}{2}\sum_{k=1}^n (\frac{1}{q_k} - \frac{1}{p_k})}.
\]

This last estimate and (2.16) imply (2.11) with \( l = 1 \). The proof of the lemma is complete. \( \square \)

Next, we introduce and prove the following simple lemma on the continuity property of the solutions of the heat equation (2.7) in mixed-norm spaces. The result will be useful in the paper.

**Lemma 2.17.** For each \( k = 1, 2, \ldots, n \), let \( p_k \in [1, \infty) \). Assume that \( u_0 \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n) \). Let \( u(x, t) = e^{\Delta t}u_0 \) be the solution of the heat equation (2.7) defined in (2.8). Then, we have \( u \in C([0, \infty), L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)) \) and

\[
\lim_{t \to 0^+} \|u(\cdot, t) - u_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} = 0. \tag{2.18}
\]

**Proof.** We only need to prove (2.18), as the proof of the continuity of \( u \) at \( t_0 > 0 \) can be done similarly. Let \( \epsilon > 0 \), by using the truncation and a multiplication by a suitable cut-off function, we can find a bounded compactly support function \( \tilde{u}_0 \) such that

\[
\|u_0 - \tilde{u}_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \frac{\epsilon}{4N_0},
\]

where \( N_0 = N_0(n, p_1, p_2, \ldots, p_n) > 1 \) is the number defined in Theorem 2.9. In particular, by Theorem 2.9, we have

\[
\|e^{\Delta t}(u_0 - \tilde{u}_0)\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq N_0\|u_0 - \tilde{u}_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \frac{\epsilon}{4}.
\]

From the previous two estimates, we see that

\[
\|e^{\Delta t}(u_0 - \tilde{u}_0) - (u_0 - \tilde{u}_0)\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \frac{\epsilon}{4} + \frac{\epsilon}{4N_0} \leq \frac{\epsilon}{2}. \tag{2.19}
\]

Our next goal is to show that

\[
\lim_{t \to 0^+} \|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} = 0.
\]
Take $p > \max\{p_1, p_2, \ldots, p_n\}$ and choose the numbers $q_k \in (p_k, \infty)$ such that
\[
\frac{1}{q_k} = \frac{1}{p_k} - \frac{1}{p}, \quad k = 1, 2, \ldots, n.
\]

Then, by applying the Hölder’s inequality repeatedly for each integration with respect to each variable $x_k$, we see that
\[
\|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} 
\leq \|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_p(\mathbb{R}^n)} \|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} 
\leq N \|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_p(\mathbb{R}^n)} \|\tilde{u}_0\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)}.
\]

Observe that as $\tilde{u}_0$ is bounded and compactly supported, $\|\tilde{u}_0\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} < \infty$. Therefore,
\[
\|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \tilde{N} \|e^{\Delta t} \tilde{u}_0 - \tilde{u}_0\|_{L_p(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow 0^+,
\]
where in the last assertion, we used the classical result of the continuity of the heat flow in $L_p(\mathbb{R}^n)$ and the fact that $\tilde{u}_0 \in L_p(\mathbb{R}^n)$. From this and (2.19), we conclude that there is $\delta = \delta(\epsilon) > 0$ such that
\[
\|e^{\Delta t} u_0 - u_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \epsilon, \quad \forall t \in (0, \delta_0).
\]

This proves (2.18) as desired.

\begin{remark}
Note that Theorem 2.9 shows the persistence of the anisotropic properties of the initial data under the evolution of the heat equations. Theorem 2.9 and Lemma 2.17 recover the classical results when $p_1 = p_2 = \cdots = p_n$ and $q_1 = q_2 = \cdots = q_n$.
\end{remark}

\begin{remark}
For given numbers $p_1, p_2, \ldots, p_n \in (1, \infty)$ that satisfy (1.5), if $u_0 \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)$, by Theorem 2.9, we see that
\[
\|e^{\Delta t} u_0 \|^\frac{1}{2}(1-\frac{n}{p_0}) \|u_0\|_{L_{p_0}(\mathbb{R}^n)} \leq N \|u_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)},
\]
for $p_0 \in (\max\{p_1, p_2, \ldots, p_n, n\}, \infty)$ and for $N = N(n, p_0, p_1, p_2, \ldots, p_n)$. Then, it follows from the characterization of Besov spaces with negative regularity (see [1, Theorem 2.34, p. 72], or [24, eqn (8.6), p. 177], and also [4, 5]) that $u_0 \in B_{p_0, \infty}^{-1+\frac{n}{p_0}}(\mathbb{R}^n)$ with its norm is defined as in (1.2)
\[
\|u_0\|_{B_{p_0, \infty}^{-1+\frac{n}{p_0}}(\mathbb{R}^n)} \approx \sup_{t > 0} \|t^{\frac{1}{2}(1-\frac{n}{p_0})} e^{\Delta t} u_0\|_{L_{p_0}(\mathbb{R}^n)} < \infty.
\]
In particular, it follows from this and [19, eqn (23)] that $u_0 \in BMO^{-1}(\mathbb{R}^n)$.
\end{remark}
2.3. Helmholtz–Leray projection in mixed-norm Lebesgue spaces

Let $\mathbb{P} = \text{Id} - \nabla \Delta^{-1} \nabla \cdot$ be the Helmholtz–Leray projection onto the divergence-free vector fields. This section proves that

$$\|\mathbb{P}(f)\|_{L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})} \leq N \|f\|_{L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})},$$

for every $f \in L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})$ and for $p_{1}, p_{2}, \ldots, p_{n} \in (1, \infty)$. This estimate is an important ingredient in our paper. To achieve it, we need to recall the following definition of Muckenhoupt $A_{q}(\mathbb{R}^{n})$-class of weights, which is needed for the proof of Theorem 2.23 below. For each $q \in (1, \infty)$, a nonnegative, locally integrable function $\omega : \mathbb{R}^{n} \rightarrow \mathbb{R}$ is said to be in the Muckenhoupt $A_{q}(\mathbb{R}^{n})$-class of weights if

$$[\omega]_{A_{q}} := \sup_{R > 0, x_{0} \in \mathbb{R}^{n}} \left( \frac{1}{|B_{R}(x_{0})|} \int_{B_{R}(x_{0})} \omega(x) \, dx \right) \left( \frac{1}{B_{R}(x_{0})} \int_{B_{R}(x_{0})} \omega(x)^{-\frac{1}{q-1}} \, dx \right)^{q-1} < \infty,$$

where $B_{R}(x_{0})$ denotes the ball in $\mathbb{R}^{n}$ of radius $R$ centered at $x_{0} \in \mathbb{R}^{n}$. In the following, for each given $p \in [1, \infty)$ and each given weight $\omega : \mathbb{R}^{n} \rightarrow \mathbb{R}$, a measurable function $f : \mathbb{R}^{n} \rightarrow \mathbb{R}$ is said to be in the weighted Lebesgue space $L_{p}(\mathbb{R}^{n}, \omega)$ if its norm

$$\|f\|_{L_{p}(\mathbb{R}^{n}, \omega)} = \left( \int_{\mathbb{R}^{n}} |f(x)|^{p} \omega(x) \, dx \right)^{\frac{1}{p}} < \infty.$$

We also recall the following amazing result from [22, Theorem 6.2], which is a beautiful application of the Rubio De Francia extrapolation theory (see [7] for instance).

**Theorem 2.22.** Let $p_{k} \in (1, \infty)$ for all $k = 1, 2, \ldots, n$. Then, there exists a constant $K_{0} = K_{0}(n, p_{1}, p_{2}, \ldots, p_{n}) \geq 1$ such that the following holds true. For a pair of given measurable functions $f, g : \mathbb{R}^{n} \rightarrow \mathbb{R}$ such that if

$$\|f\|_{L_{p_{1}}(\mathbb{R}^{n}, \omega)} \leq \|g\|_{L_{p_{1}}(\mathbb{R}^{n}, \omega)}$$

for every $\omega \in A_{p_{1}}$ with $[\omega]_{A_{p_{1}}} \leq K_{0}$, then we have

$$\|f\|_{L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})} \leq 4^{n} \|g\|_{L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})}.$$

Now, we begin with the following important result on the boundedness of the Riesz transform in mixed-norm Lebesgue spaces. Interested readers may find [8, Corollary 2.7] and [30, Lemma 2.1] for other interesting related results in mixed-norm spaces.

**Theorem 2.23.** For any $j = 1, 2, \ldots, n$ and any $p_{1}, p_{2}, \ldots, p_{n} \in (1, \infty)$, there exists a positive constant $N = N(p_{1}, p_{2}, \ldots, p_{n}, n)$ such that

$$\|\mathcal{R}_{j}(f)\|_{L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})} \leq N \|f\|_{L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})}$$

for every $f \in L_{p_{1},p_{2},\ldots,p_{n}}(\mathbb{R}^{n})$, where $\mathcal{R}_{j}$ is the $j$th-Riesz transform defined by $\mathcal{R}_{j}(f) = \partial_{x_{j}}(-\Delta)^{-\frac{1}{2}} f$. 
Proof. We plan to apply Theorem 2.22. For given \( p_1, p_2, \ldots, p_n \in (1, \infty) \), let \( K_0 \) be as in Theorem 2.22. By using the truncation and a multiplication with suitable cut-off functions, we can approximate \( f \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n) \) by a sequence of bounded compactly supported functions. Therefore, we may assume that \( f \) is bounded and compactly supported in \( \mathbb{R}^n \). Without loss of generality, we can also assume that \( p_1 = \min \{ p_1, p_2, \ldots, p_n \} \). Under these assumptions, we see that \( f \in L_{p_1(\mathbb{R}^n, \omega)} \) for every weight \( \omega \in A_{p_1} \). Then, since \( p_1 \in (1, \infty) \), by the classical Calderón–Zygmund theory (see [7,13] for instance), there exists a constant \( N = N(p_1, n, K_0) \) such that

\[
\| \mathcal{R}_j(f) \|_{L_{p_1(\mathbb{R}^n, \omega)}} \leq N \| f \|_{L_{p_1(\mathbb{R}^n, \omega)}},
\]

for every \( \omega \in A_{p_1} \) with \([\omega]_{A_{p_1}} \leq K_0\). From (2.24) and Theorem 2.22, we infer that

\[
\| \mathcal{R}_j(f) \|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq 4^n N \| f \|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)}.
\]

This is the desired estimate and the proof is therefore completed. \( \square \)

The following consequence of Theorem 2.23 gives the boundedness of the Helmholtz–Leray projection in mixed-norm Lebesgue spaces, which is an important ingredient in the paper.

Corollary 2.25. Let \( \mathbb{P} = \text{Id} - \nabla \Delta^{-1} \nabla \cdot \) be the Helmholtz–Leray projection onto the divergence-free vector fields. Let \( p_1, p_2, \ldots, p_n \in (1, \infty) \). Then, one has

\[
\| \mathbb{P}(f) \|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq N \| f \|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)},
\]

for every \( f \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n) \), where \( N = N(p_1, p_2, \ldots, p_n, n) \) is a positive constant.

Proof. Note that with \( f = (f_1, f_2, \ldots, f_n) \in L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)^n \), we have \( \mathbb{P}(f) = (\mathbb{P}(f)_1, \mathbb{P}(f)_2, \ldots, \mathbb{P}(f)_n) \) with

\[
\mathbb{P}(f)_k = f_k + \mathcal{R} \sum_{j=1}^{n} \mathcal{R}_j f_j, \quad k = 1, 2, \ldots, n,
\]

where \( \mathcal{R}_j \) is the \( j \)th-Riesz transform. Therefore, it follows from Theorem 2.23 that

\[
\| \mathbb{P}(f) \|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq N \| f \|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)}
\]

which is our desired estimate. \( \square \)

3. Navier–Stokes equations in critical mixed-norm Lebesgue spaces

This section provides the proof of Theorem 1.9. We follow the approach introduced in [11,12,17] and in [19,27]. Recall that \( \mathbb{P} \) denotes the Helmholtz–Leray projection.
which is defined in Corollary 2.25. By applying $\mathbb{P}$ on the system (1.1), we see that the system (1.1) is recast in the following abstract form

$$\begin{cases} u_t + Au + F(u, u) = 0 & \text{in } \mathbb{R}^n \times (0, \infty), \\ u(\cdot, 0) = a_0(\cdot) & \text{on } \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $A = -\mathbb{P}\Delta = -\Delta\mathbb{P}$ and

$$F(u, v) = \mathbb{P}((u \cdot \nabla)v). \tag{3.2}$$

By the Duhamel’s principle, the system (3.1) is then converted to the following integral equation

$$u = u_0 + G(u, u), \tag{3.3}$$

where

$$u_0(t) = e^{-At}a_0, \text{ and } G(u, v)(t) = -\int_0^t e^{-(t-s)A}F(u(s), v(s))ds. \tag{3.4}$$

To proceed, we need several estimates. We begin with the following lemma on the time decaying properties for the semi-group $e^{-At}$ in mixed-norm Lebesgue spaces.

**Lemma 3.5.** For each $k = 1, 2, \ldots, n$, let $1 < p_k \leq q_k < \infty$ be given numbers. Also, let $\sigma \geq 0$ be defined by

$$\sigma = \sum_{k=1}^n \left[ \frac{1}{p_k} - \frac{1}{q_k} \right].$$

(i) There exists a number $N$ depending only on $n, p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_n$ such that

$$\| e^{-At}\mathbb{P}f \|_{L_{q_1,q_2,\ldots,q_n}(\mathbb{R}^n)} \leq N t^{-\sigma} \| f \|_{L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)},$$

$$\| D_x e^{-At}\mathbb{P}f \|_{L_{q_1,q_2,\ldots,q_n}(\mathbb{R}^n)} \leq N t^{-\frac{1}{2}(1+\sigma)} \| f \|_{L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}, \tag{3.6}$$

for every $f \in L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)^n$.

(ii) For each $f \in L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)^n$, the following assertions hold

$$\lim_{t \to 0^+} t^{\frac{\sigma}{2}} \| e^{-At}\mathbb{P}f \|_{L_{q_1,q_2,\ldots,q_n}(\mathbb{R}^n)} = 0 \text{ if } \sigma > 0 \text{ and also}$$

$$\lim_{t \to 0^+} \| [e^{-At}\mathbb{P}f] - \mathbb{P}f \|_{L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} = 0,$$

and

$$\lim_{t \to 0^+} t^{-\frac{1}{2}(1+\sigma)} \| D_x e^{-At}\mathbb{P}f \|_{L_{q_1,q_2,\ldots,q_n}(\mathbb{R}^n)} = 0. \tag{3.7}$$

**Proof.** We begin with the proof of (i). As $A = -\mathbb{P}\Delta = -\Delta\mathbb{P}$, we see that $A = -\Delta$ when acting on the class of divergence-free vector fields. Therefore, $e^{-At}\mathbb{P} = e^{\Delta t}\mathbb{P}$. Then, by using the decay estimate for the heat equation in mixed norm developed in Theorem 2.9, we see that

$$\| e^{-At}\mathbb{P}f \|_{L_{q_1,q_2,\ldots,q_n}(\mathbb{R}^n)} \leq N t^{-\frac{\sigma}{2}} \| \mathbb{P}(f) \|_{L_{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}.$$

On the other hand, from Corollary 2.25, we see that the Helmholtz–Leray projection
\[ P : L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)^n \rightarrow L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)^n \]
is bounded. From this fact and the previous estimate, we obtain the first estimate in (3.6). The second estimate in (3.6) can be proved in the same way.

Next, we prove (ii). We assume that \( \sigma > 0 \), and we will prove the first assertion in (3.7). We may assume that \( f \) is bounded and compactly supported if needed. Let \( \epsilon > 0 \).

Then, by using approximation, we can find \( g \in L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)^n \cap L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)^n \) such that
\[
\| f - g \|_{L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \frac{\epsilon}{2N}
\]
where \( N > 0 \) is defined in (i). Now, using the first assertion in (i), we see that
\[
t^\alpha \| e^{-At}P(f - g) \|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \leq N \| f - g \|_{L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \frac{\epsilon}{2}.
\]

On the other hand, using the first assertion in (i) again, we also obtain
\[
t^\alpha \| e^{-At}Pg \|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \leq Nt^\alpha \| g \|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \rightarrow 0 \text{ as } t \rightarrow 0^+.
\]

Now, combine the last two estimates, we infer that there is small number \( \delta_0 = \delta_0(\epsilon) > 0 \) such that
\[
t^\alpha \| e^{-At}Pf \|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \leq \epsilon, \quad \forall t \in (0, \delta_0).
\]

This implies that
\[
\lim_{t \rightarrow 0^+} t^\alpha \| e^{-At}Pf \|_{L^{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} = 0,
\]

and the first assertion in (3.7) is proved. Observe also that the last assertion in (ii) can be done in a similar way. Meanwhile, the second assertion of (3.7) is due to the continuity of the heat semi-group in Lemma 2.17 and the continuity of the Helmholtz–Leray in the mixed norm \( L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)^n \) as from Corollary 2.25. The proof of the lemma is therefore completed.

Our next lemma gives some important estimates in mixed norm for the bilinear term \( G(u, v) \) defined in (3.4).

**Lemma 3.8.** Let \( p_k \in (1, \infty) \) and \( \alpha_k, \beta_k, \gamma_k \in (0, 1] \) be given numbers satisfying
\[
\gamma_k \leq \alpha_k + \beta_k < p_k, \quad k = 1, 2, \ldots, n.
\]

Let
\[
\alpha = \sum_{k=1}^{n} \frac{\alpha_k}{p_k}, \quad \beta = \sum_{k=1}^{n} \frac{\beta_k}{p_k}, \quad \text{and} \quad \gamma = \sum_{k=1}^{n} \frac{\gamma_k}{p_k}.
\]
Then,
\[
\|G(u, v)(t)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \\
\leq N \int_0^t (t - s)^{-\frac{\alpha_1 + p_1 + \beta_1 + \gamma_1}{2}} \|u(s)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \|D_x v(s)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} ds,
\]
\[
\|D_x G(u, v)(t)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \\
\leq N \int_0^t (t - s)^{-\frac{\alpha_1 + p_1 + \beta_1 + \gamma_1}{2}} \|u(s)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \|D_x v(s)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} ds,
\]
where \(N\) is a positive number depending only on \(n, p_k, \alpha_k, \beta_k, \gamma_k\) for \(k = 1, 2, \ldots, n\).

**Proof.** We only prove the first assertion in the lemma as the proof of the second one can be done similarly. By applying the first estimate in (3.6), we see that
\[
\|G(u, v)(t)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \\
\leq N \int_0^t (t - s)^{-\frac{\alpha_1 + p_1 + \beta_1 + \gamma_1}{2}} \|F(u(s), v(s))\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} ds,
\]
where the bilinear function \(F\) is defined in (3.2). From this and the boundedness of the Helmholtz–Leray projection \(P\) as stated in Corollary 2.25, we see that
\[
\|G(u, v)(t)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \\
\leq N \int_0^t (t - s)^{-\frac{\alpha_1 + p_1 + \beta_1 + \gamma_1}{2}} \|(u \cdot \nabla) v\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} ds.
\]
Then, as
\[
\frac{\alpha_k + \beta_k}{p_k} = \frac{\alpha_k}{p_k} + \frac{\beta_k}{p_k}, \quad \text{for all } k = 1, 2, \ldots, n
\]
we can repeatedly apply the Hölder’s inequality for each integration with respect to each variable \(x_k\) to find that
\[
\|(u \cdot \nabla) v\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \leq \|u(s)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)} \|D_x v(s)\|_{L^{\frac{p_1}{\alpha_1 + p_1} \cdot \frac{p_2}{\alpha_2 + p_2} \cdots \frac{p_n}{\alpha_n + p_n}}(\mathbb{R}^n)}.
\]
The desired estimate then follows and the proof is complete. \(\square\)

To prove Theorem 1.9, our goal is to show that the abstract equation (3.3) has unique fixed point in suitable spaces. For this purpose, let us recall the following abstract lemma which is useful in the study of initial value problem for Navier–Stokes equations, see [29, Lemma 3.1] and also [27].

**Lemma 3.9.** Let \(X\) be a Banach space with norm \(\|\cdot\|_X\). Let \(G : X \times X \to X\) be a bilinear map such that there is \(N_0 > 0\) so that
\[
\|G(u, v)\|_X \leq N_0 \|u\|_X \|v\|_X, \quad \forall u, v \in X.
\]
Then, for every \( u_0 \in X \) with \( 4N_0 \| u_0 \|_X < 1 \), the equation

\[
u = u_0 + G(u, u)\]

has unique solution \( u \in X \) with

\[\|u\|_X \leq 2\|u_0\|_X.\]

We are now ready to prove Theorem 1.9.

**Proof of Theorem 1.9.** Let \( p = (p_1, p_2, \ldots, p_n) \), \( q = (q_1, q_2, \ldots, q_n) \) with \( p_k \in [2, \infty), q_k \in [p_k, \infty) \) for \( k = 1, 2, \ldots, n \). Assume that (1.5) and (1.6) hold, and assume that with some \( k \in \{1, 2, \ldots, n\} \) if \( p_k = 2 \), then \( q_k > 2 \). Let \( a_0 \in L_{p_1, p_2, \ldots, p_n} (\mathbb{R}^n)^n \) with \( \nabla \cdot a_0 = 0 \) and recall that

\[
\delta = \frac{1}{q_1} + \frac{1}{q_2} + \cdots + \frac{1}{q_n} \in (0, 1).
\]

(3.10)

We now prove (i). Recall the definitions of \( \mathcal{X}_{p,q,\infty} \) and \( \mathcal{Y}_{p,q,\infty} \) in (1.7) and (1.8). We plan to prove the existence of solution \( u \in \mathcal{X}_{p,q,\infty} \) of (3.3), and then prove that the solution \( u \in \mathcal{Y}_{p,q,\infty} \). Our goal is to apply Lemma 3.9 to obtain the existence and uniqueness of solution of (3.3) in \( \mathcal{X}_{p,q,\infty} \). To this end, we begin with the proof that \( u_0 \in \mathcal{X}_{p,q,\infty} \). From (i) of Lemma 3.5 and the definition of \( u_0 \) in (3.4), we have

\[
\|u_0(t)\|_{L_{q_1,q_2,\ldots,q_n} (\mathbb{R}^n)} \leq N_1 t^{-\frac{1}{q_k}} \|a_0\|_{L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)} \quad \text{and} \quad \|D_x u_0(t)\|_{L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)} \leq N_1 t^{-\frac{1}{2}} \|a_0\|_{L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)}, \quad \forall \ t > 0,
\]

where \( N_1 > 0 \) is a universal constant depending only on \( n, p \) and \( q \). Moreover, it follows from (i)–(ii) of Lemma 3.5 that \( t^{\frac{1}{q_k}} e^{-\beta t} P \) is uniformly bounded from \( L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)^n \) to \( PL_{q_1,q_2,\ldots,q_n} (\mathbb{R}^n) \) and tends to zero as \( t \to 0^+ \), we see that \( t^{\frac{1}{q_k}} u_0 \) vanishes as \( t = 0 \). Similarly, as \( t^{\frac{1}{2}} D_x e^{-\beta t} P \) is uniformly bounded from \( L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)^n \) to \( PL_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)^n \) and tends to zero as \( t \to 0^+ \), we also have \( t^{\frac{1}{2}} D_x u_0 \) equals to zero as \( t \to 0^+ \). In conclusion, we have shown that \( u_0 \in \mathcal{X}_{p,q,\infty} \) and

\[
\|u_0\|_{\mathcal{X}_{p,q,\infty}} \leq N_1 \|a_0\|_{L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)}.
\]

(3.11)

It now remains to prove that the bilinear form \( G : \mathcal{X}_{p,q,\infty} \times \mathcal{X}_{p,q,\infty} \to \mathcal{X}_{p,q,\infty} \) is bounded. By (1.5), (3.10), and the assumption that \( 2 < q_k \) whenever \( p_k = 2 \), we are able to apply the first assertion in Lemma 3.8 with \( \beta_k = 1 \) and \( \gamma_k = \alpha_k = \frac{p_k}{q_k} \in (0, 1] \) to find that

\[
\|G(u, v)(t)\|_{L_{q_1,q_2,\ldots,q_n} (\mathbb{R}^n)} \leq N_1 \int_0^t (t - s)^{-\frac{1}{2}} \|u(s)\|_{L_{q_1,q_2,\ldots,q_n} (\mathbb{R}^n)} \|D_x v(s)\|_{L_{p_1,p_2,\ldots,p_n} (\mathbb{R}^n)} ds \leq N_1 \|u\|_{\mathcal{X}_{p,q,\infty}} \|v\|_{\mathcal{X}_{p,q,\infty}} \int_0^t (t - s)^{-\frac{1}{2}} s^{-\frac{1}{2}} + \frac{1}{2} ds.
\]
To control the integration in the last estimate, we split it into two time intervals $(0, t/2)$ and $(t/2, t)$. We then obtain

$$\|G(u, v)(t)\|_{L_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)} \leq N\|u\|_{\mathcal{X}_{p, q, \infty}} \|v\|_{\mathcal{X}_{p, q, \infty}} \left[ \int_{0}^{t/2} (t - s)^{-\frac{1}{2}} s^{-1+\frac{1}{2}} ds + \int_{t/2}^{t} (t - s)^{-\frac{1}{2}} s^{-1+\frac{1}{2}} ds \right]$$

$$\leq N\|u\|_{\mathcal{X}_{p, q, \infty}} \|v\|_{\mathcal{X}_{p, q, \infty}} \left[ t^{-\frac{1}{2}} \int_{0}^{t/2} s^{-1+\frac{1}{2}} ds + t^{-1+\frac{1}{2}} \int_{t/2}^{t} (t - s)^{-\frac{1}{2}} ds \right]$$

$$\leq N t^{-\frac{1}{2}} \|u\|_{\mathcal{X}_{p, q, \infty}} \|v\|_{\mathcal{X}_{p, q, \infty}}. \quad (3.12)$$

From the last two estimates and the definition of $G(u, v)$ and Lemma 3.5, it follows that $t^{1/2} G(u, v) : [0, \infty) \to PL_{q_1, q_2, \ldots, q_n}(\mathbb{R}^n)$ is continuous and vanishes at $t = 0$. Similarly, we can also prove that $t^{1/2} D_x G(u, v) : [0, \infty) \to PL_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)$ is continuous and vanishes at $t = 0$. Therefore, we conclude that $G(u, v) \in \mathcal{X}_{p, q, \infty}$ and

$$\|G(u, v)\|_{\mathcal{X}_{p, q, \infty}} \leq N_2 \|u\|_{\mathcal{X}_{p, q, \infty}} \|v\|_{\mathcal{X}_{p, q, \infty}}, \quad \forall u, v \in \mathcal{X}_{p, q, \infty}, \quad (3.13)$$

where $N_2$ is a constant depending only on $n$, $p$ and $q$. In other words, the bilinear form $G : \mathcal{X}_{p, q, \infty} \times \mathcal{X}_{p, q, \infty} \to \mathcal{X}_{p, q, \infty}$ is bounded.

Next, let us choose $\lambda_0 > 0$ and sufficiently small so that

$$4N_1 N_2 \lambda_0 < 1, \quad (3.14)$$

where $N_1$ is defined in (3.11), and $N_2$ is defined in (3.13). Note that both of these numbers depend only on $p$, $q$ and $n$. Now, if $\|a_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq \lambda_0$, then it follows from (3.11) that

$$4N_2 \|u_0\|_{\mathcal{X}_{p, q, \infty}} \leq 4N_1 N_2 \|a_0\|_{L_{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)} \leq 4N_1 N_2 \lambda_0 < 1.$$
From this and by applying Lemma 3.9, we can find a unique solution $u \in \mathcal{X}_{p,q,\infty}$ of the equation (3.3) such that
\begin{equation}
\|u\|_{\mathcal{X}_{p,q,\infty}} \leq 2\|u_0\|_{\mathcal{X}_{\infty}} \leq 2N_1 \|a_0\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}.
\end{equation}
(3.15)

Now, to complete the proof (i), we need to show that $u \in \mathcal{Y}_{p,q,\infty}$. We recall that the definition of $\mathcal{Y}_{p,q,\infty}$ is given in (1.8). Since
\begin{equation}
\begin{aligned}
\|u(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} &\leq \|u_0(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} + \|G(u,u)(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}, \\
\|D_x u(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} &\leq \|D_x u_0(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} + \|D_x G(u,u)(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}.
\end{aligned}
\end{equation}
(3.16)

Then, by applying Lemma 3.5, we see that
\begin{equation}
\begin{aligned}
\|u_0(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} &\leq N\|a_0\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}, \\
\|D_x u_0(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} &\leq Nt^{-\frac{1}{2}}\|a_0\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}.
\end{aligned}
\end{equation}
(3.17)
(3.18)

On the other hand, by (1.5), (3.10), and the fact that $2 < q_k$ if $p_k = 2$, we are able to apply the first assertion in Lemma 3.8 with $\gamma_k = 1$, $\alpha_k = \frac{p_k}{q_k} \in (0, 1]$ and $\beta_k = 1$ to infer that
\begin{equation}
\begin{aligned}
\|G(u,u)(t)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} &\leq N \int_0^t (t-s)^{-\frac{1}{2}}\|u(s)\|_{L^{q_1,q_2,\ldots,q_n}(\mathbb{R}^n)} \|D_x u(s)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} ds \\
&\leq N\|u\|^2_{\mathcal{X}_{p,q,\infty}} \int_0^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}-(1-\frac{1}{2})} ds \\
&= N\|u\|^2_{\mathcal{X}_{p,q,\infty}} \left[ \int_0^{t/2} (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds + \int_{t/2}^t (t-s)^{-\frac{1}{2}} s^{-\frac{1}{2}} ds \right] \\
&= N\|u\|^2_{\mathcal{X}_{p,q,\infty}} \left[ t^{-\frac{1}{2}} \int_0^{t/2} s^{-\frac{1}{2}} ds + t^{-\frac{1}{2}} \int_{t/2}^t (t-s)^{-\frac{1}{2}} ds \right] \\
&\leq N\|a_0\|^2_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)},
\end{aligned}
\end{equation}
(3.19)

where in the last estimate, we used (3.15). Also, by (3.12) and (3.15), it follows that
\begin{equation}
\|D_x G(u,u)\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)} \leq Nt^{-\frac{1}{2}}\|a_0\|^2_{\mathcal{X}_{p,q,\infty}} \leq Nt^{-\frac{1}{2}}\|a_0\|^2_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}.
\end{equation}
(3.20)

Then, from the estimates (3.16), (3.17), (3.19), (3.20) and the fact that $\|a_0\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}$ is sufficiently small that, we see that
\begin{equation}
\|u\|_{\mathcal{Y}_{p,q,\infty}} \leq N_0\|a_0\|_{L^{p_1,p_2,\ldots,p_n}(\mathbb{R}^n)}.
\end{equation}
The proof of (i) is therefore complete.

Now, we turn to prove (ii). As in the proof of (3.11), we see that $u_0 \in \mathcal{X}_{p,q,\infty}$. From the definition of the norm of the space $\mathcal{X}_{p,q,\infty}$ in (1.7), the continuity and the vanishes of $t^{1+\frac{\delta}{2}}u_0$ and of $t^\frac{\delta}{2}D_xu_0$ at $t = 0$, we can choose a sufficiently small number $T_0 > 0$ depending on $n, p, q$ and $a_0$ so that

$$\|u_0\|_{\mathcal{X}_{p,q,T_0}} \leq \lambda_0,$$

where $\lambda_0$ is defined as in (3.14). Moreover, by following the proof of (3.13), we can also see that the bilinear form $G : \mathcal{X}_{p,q,T_0} \times \mathcal{X}_{p,q,T_0} \to \mathcal{X}_{p,q,T_0}$ is bounded with

$$\|G(u, v)\|_{\mathcal{X}_{p,q,T_0}} \leq N_2 \|u\|_{\mathcal{X}_{p,q,T_0}} \|v\|_{\mathcal{X}_{p,q,T_0}}, \quad \forall u, v \in \mathcal{X}_{p,q,T_0}.$$

Then, applying Lemma 3.9 again, we can find a unique local time solution $u \in \mathcal{X}_{p,q,T_0}$ of (3.3) satisfying

$$\|u\|_{\mathcal{X}_{p,q,T_0}} \leq 2N_1 \|a_0\|_{L^{p_1, p_2, \ldots, p_n}(\mathbb{R}^n)}.$$

Now, we only need to prove that the solution $u$ that we found is indeed in $\mathcal{Y}_{p,q,T_0}$. However, this can be done exactly the same as in the proof that $u \in \mathcal{Y}_{p,q,\infty}$ in (i), and we skip it. The proof of the theorem is then complete. \qed

**Remark 3.20.** The pressure $P$ in (1.1) can be solved from the solution $u = (u_1, u_2, \ldots, u_n)$ as

$$P = \sum_{i, j=1}^n \mathcal{R}_i \mathcal{R}_j (u_i u_j),$$

where $\mathcal{R}_j$ is the $j$th Riesz transform, which is defined in Theorem 2.23. Then, with given $r_k \in (2, \infty)$ for all $k = 1, 2, \ldots, n$, it follows from Theorem 2.23 that

$$\|P(\cdot, t)\|_{L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)} \leq N \|u(\cdot, t)\|_{L^{r_1, r_2, \ldots, r_n}(\mathbb{R}^n)}, \quad \text{for } t > 0.$$

**Acknowledgements**

The author wishes to thank anonymous referees for their important remarks and suggestions that significantly improve the manuscript. The author also would like to thank professor Lorenzo Brandolese (Institut Camille Jordan, Université Lyon 1) and professor Nam Le (Indiana University) for their interests and valuable comments.

**Publisher’s Note** Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.
REFERENCES

[1] H. Bahouri, J.-Y. Chemin, R. Danchin, *Fourier analysis and nonlinear partial differential equations*. Grundlehren der Mathematischen Wissenschaften, 343. Springer, Heidelberg, 2011.

[2] J. Bourgain, N. Pavlović, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*, J. Funct. Anal. 255 (2008), 2233–2247.

[3] L. Brandolese, F. Vigneron, *New asymptotic profiles of nonstationary solutions of the Navier-Stokes system*. J. Math. Pures Appl. (9) 88 (2007), no. 1, 64–86.

[4] M. Cannone, *Ondelettes, paraproducts et Navier-Stokes*. Diderot Editeur, Paris, 1995.

[5] M. Cannone, *A generalization of a theorem by Kato on Navier-Stokes equations*, Rev. Mat. Iberoam. 13 (1997), 515–541.

[6] M. Cannone, F. Planchon, *On the non-stationary Navier-Stokes equations with an external force*. Adv. Differential Equations 4 (1999), no. 5, 697–730.

[7] D. V. Cruz-Uribe, J. M. Martell, and C. Pérez. *Weights, extrapolation and the theory of Rubio de Francia*, volume 215 of Operator Theory: Advances and Applications. Birkhäuser/Springer Basel AG, Basel, 2011.

[8] H. Dong, D. Kim, *On L_p-estimates for elliptic and parabolic equations with A_p weights*. Trans. Amer. Math. Soc. 370 (2018), no. 7, 5081–5130.

[9] H. Dong, N.V. Krylov, *Fully nonlinear elliptic and parabolic equations in weighted and mixed-norm Sobolev spaces*, arXiv:1806.00077.

[10] H. Dong, T. Phan, *Mixed norm L_p-estimates for non-stationary Stokes systems with singular VMO coefficients and applications*, arXiv:1805.04143.

[11] T. Kato, H. Fujita, *On the nonstationary Navier-Stokes system*. Rend. Sem. Mat. Univ. Padova 32 1962 243–260.

[12] H. Fujita, T. Kato, *On the Navier-Stokes initial value problem. I*. Arch. Rational Mech. Anal. 16, 1964, 269–315.

[13] J. García-Cuerva, J. L. Rubio de Francia, *Weighted norm inequalities and related topics*. North-Holland Mathematics Studies, 116. Notas de Matemática, 104. North-Holland Publishing Co., Amsterdam, 1985.

[14] Y. Giga, T. Miyakawa, *Solutions in L' of the Navier-Stokes initial value problem*. Arch. Rational Mech. Anal. 89 (1985), no. 3, 267–281.

[15] Y. Giga and T. Miyakawa, *Navier-Stokes flow in R^3 with measures as initial vorticity and Morrey spaces*, Comm. Partial Differential Equations 14 (1989), 577–618.

[16] Y. Giga, *Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system*. J. Differential Equations 62 (1986), no. 2, 186–212.

[17] T. Kato, *Strong L^p-solutions of the Navier-Stokes equation in R^n*, with applications to weak solutions. Math. Z. 187 (1984), no. 4, 471–480.

[18] T. Kato, *Strong solutions of the Navier-Stokes equation in Morrey spaces*, Bol. Soc. Brasil. Mat. (N.S.) 22 (1992), 127–155.

[19] H. Koch, D. Tataru, *Well-posedness for the Navier-Stokes equations*. Adv. Math. 157 (2001), no. 1, 22–35.

[20] H. Kozono, M. Yamazaki, *Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data*, Comm. Partial Differential Equations 19 (1994), 959–1014.

[21] H. Kozono, M. Yamazaki, *The stability of small stationary solutions in Morrey spaces of the Navier-Stokes equation*, Indiana Univ. Math. J. 44 (1995), 1307–1335.

[22] N.V. Krylov, *Rubio de Francia extrapolation theorem and related topics in the theory of elliptic and parabolic equations. A survey*, arXiv:1901.00549.

[23] N. V. Krylov, *Parabolic equations with VMO coefficients in Sobolev spaces with mixed norms*. J. Funct. Anal. 250 (2007), no. 2, 521–558.

[24] P. R. Lemarié-Rieusset, *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016.

[25] J. Leray, *Étude de diverses équations intégrales non linéaires et de quelques problèmes que pose l’Hydrodynamique*, J. Math. Pures Appl. 9 (1933), 1–82.
[26] J. Leray, *Sur le mouvement d’un liquide visqueux emplissant l’espace*, Acta Math. 63 (1934), no. 1, 193–248.

[27] Y. Meyer, *Wavelets, Paraproducts and Navier-Stokes Equations*. Current Developments in Mathematics, 1996 (Cambridge, MA), Int. Press, Boston, MA, 1997, pp. 105–212.

[28] F. Planchon, *Global strong solutions in Sobolev or Lebesgue spaces to the incompressible Navier-Stokes equations in $\mathbb{R}^3$*. Ann. Inst. Henri Poincare, Anal. Non Lineaire 13 (1996), 319–336.

[29] T. V. Phan, N. C. Phuc, *Stationary Navier-Stokes equations with critically singular external forces: existence and stability results*. Adv. Math. 241 (2013), 137–161.

[30] T. Phan, *Liouville type theorems for 3D stationary Navier-Stokes equations in weighted mixed-norm Lebesgue spaces*, arXiv:1812.10135.

[31] M.E. Taylor, *Analysis of Morrey spaces and applications to Navier-Stokes and other evolution equations*, Comm. Partial Differential Equations 17 (1992), 1407–1456.

[32] T.-P. Tsai, *Lectures on Navier-Stokes equations*. Graduate Studies in Mathematics, 192. American Mathematical Society, Providence, RI, 2018.

---

Tuoc Phan  
Department of Mathematics  
University of Tennessee  
227 Ayres Hall, 1403 Circle Drive  
Knoxville TN 37996-1320  
USA  
E-mail: phan@math.utk.edu