Subharmonic Dynamics of Wave Trains in Reaction Diffusion Systems

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Abstract

We investigate the stability and nonlinear local dynamics of wave trains in reaction-diffusion systems. For each $N \in \mathbb{N}$, such $T$-periodic traveling waves are easily seen to be nonlinearly asymptotically stable (with asymptotic phase) with exponential rates of decay when subject to $NT$-periodic perturbations. However, both the allowable size of perturbations and the exponential rates of decay depend on $N$, and, in particular, they tend to zero as $N \to \infty$, leading to a lack of uniformity in such subharmonic stability results. In this work, we build on recent work by the authors and introduce a methodology by which a stability result for subharmonic perturbations which is uniform in $N$ may be achieved. Our work is motivated by the dynamics of such waves when subject to perturbations which are localized (i.e. integrable on the line), which has recently received considerable attention by many authors.

1 Introduction

In this work, we consider the local dynamics of periodic traveling wave solutions, i.e. wave trains, in reaction diffusion systems of the form

$$(1.1) \quad u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \quad t \geq 0, \quad u \in \mathbb{R}^n$$

where $n \in \mathbb{N}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $C^K$-smooth nonlinearity for some $K \geq 3$. Such systems arise naturally in many areas of applied mathematics, and the behavior of such wave train solutions when subject to a variety of classes of perturbations has been studied intensively over the last decade. Most commonly in the literature, one studies the stability and instability of such periodic traveling waves to perturbations which are localized, i.e. are integrable on the line, or which are nonlocalized, accounting for asymptotic phase differences at infinity. See, for example, [2, 6, 7, 9, 12, 13] and references therein.

Here, we consider the stability and long-time dynamics of $T$-periodic traveling wave solutions of (1.1) when subjected to $NT$-periodic, i.e. subharmonic, perturbations for some $N \in \mathbb{N}$. More precisely, suppose that $u(x,t) = \phi(k(x - ct))$ is a periodic traveling wave solution of (1.1) with

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period $T = 1/k$, where we choose $k \in \mathbb{R}$ so that the profile $\phi \in H_{loc}^1(\mathbb{R})$ is a 1-periodic stationary solution of
\begin{equation}
k u_t - k cu_x = k^2 u_{xx} + f(u),
\end{equation}
i.e. it satisfies the profile equation
\begin{equation}
k^2 \phi'' + k c \phi' + f(\phi) = 0.
\end{equation}
Given such a solution, note that a function of the form $u(x, t) = \phi(x) + v(x, t)$ is a solution of (1.2) provided it satisfies a system of the form
\begin{equation}
k v_t = k \mathcal{L}[\phi]v + \mathcal{N}(v),
\end{equation}
where here $\mathcal{N}(v)$ is at least quadratic in $v$ and $\mathcal{L}[\phi]$ is the linear differential operator
\begin{equation}
k \mathcal{L}[\phi] := k^2 \partial_x^2 + kc \partial_x + Df(\phi).
\end{equation}
Naturally, the domain of the operator $\mathcal{L}[\phi]$ is determined by the chosen class of perturbations $v$ of the underlying standing wave $\phi$ and, as mentioned above, several choices are available in the literature. As we are interested in subharmonic perturbations, i.e. perturbations with period $N \in \mathbb{N}$, we consider $\mathcal{L}[\phi]$ as a closed, densely defined linear operator acting on $L^2_{per}(0, N)$ with 1-periodic coefficients.

The stability analysis of periodic waves to such subharmonic perturbations naturally relies on a detailed understanding of the spectrum of $\mathcal{L}[\phi]$ acting on $L^2_{per}(0, N)$. Since the domain of $\mathcal{L}[\phi]$ is compactly embedded in $L^2_{per}(0, N)$, the spectrum of $\mathcal{L}[\phi]$ will consist of an at most countable set of eigenvalues with finite algebraic multiplicity which will have no finite accumulation point: see [10], for example. Further, since $\mathcal{L}[\phi]$ is clearly a sectorial operator acting on $L^2_{per}(0, N)$, in the case where $\phi$ is stable it is natural to assume that as many of the eigenvalues as possible lies in the open left-half plane. Observe, however, that differentiating the profile equation (1.3) immediately yields
\begin{equation}
\mathcal{L}[\phi]\phi' = 0,
\end{equation}
which, since $\phi$ is certainly $N$ periodic for every $N \in \mathbb{N}$, implies that the $N$-periodic generalized kernel of $\mathcal{L}[\phi]$ is always at least one-dimensional. Thus, $\mathcal{L}[\phi]$ always has at least one eigenvalue at the origin, which is associated to the spatial translation invariance of the governing equation (1.1). Assuming $\lambda = 0$ is indeed a simple eigenvalue, and that all the remaining eigenvalues of $\mathcal{L}[\phi]$ are uniformly bounded away from the imaginary axis, it is easy to show that there exist constants $C_N$, $\delta(N) > 0$ such that
\begin{equation}
\|e^{\mathcal{L}[\phi]t} (1 - \mathcal{P}_1) \|_{L^2_{per}(0, N)} \leq C_N e^{-\delta(N)t} \|f\|_{L^2_{per}(0, N)}.
\end{equation}
for all $f \in L^2_{per}(0, N)$, where here $\mathcal{P}_1$ denotes the projection of $L^2_{per}(0, N)$ onto the $N$-periodic kernel of $\mathcal{L}[\phi]$ spanned by $\phi'$. Equipped with this linear estimate, one can establish the following nonlinear stability result.

**Proposition 1.1.** Let $\phi \in H_{loc}^1$ be a 1-periodic stationary solution of (1.2) and fix $N \in \mathbb{N}$. Assume that $\phi$ is diffusively spectrally stable, in the sense of Definition 2.3 below and, for each $N \in \mathbb{N}$, take $\delta_N > 0$ such that
\begin{equation}
\max Re \left( \sigma_{L^2_{per}(0, N)} (\mathcal{L}[\phi]) \setminus \{0\} \right) = -\delta_N
\end{equation}

holds. Then for each \( N \in \mathbb{N} \), \( \phi \) is asymptotically stable to subharmonic \( N \)-periodic perturbations. More precisely, for every \( \delta \in (0, \delta_N) \) there exists an \( \varepsilon = \varepsilon_\delta > 0 \) and a constant \( C = C_\delta > 0 \) such that whenever \( u_0 \in H^1_{\text{per}}(0, N) \) and \( \|u_0 - \phi\|_{H^1(0,N)} < \varepsilon \), then the solution \( u \) of (1.2) with initial data \( u(0) = u_0 \) exists globally in time and satisfies

\[
\|u(\cdot, t) - \phi(\cdot - \gamma_\infty)\|_{H^1(0,N)} \leq C e^{-\delta t} \|u_0 - \phi\|_{H^1(0,N)}
\]

for all \( t > 0 \), where here \( \gamma_\infty = \gamma_\infty(N) \) is some constant.

The proof of Proposition 1.1 is standard, and can be completed by following appropriate texts: see, for example, [10, Chapter 5]. The main idea is that the linear estimate (1.5) suggests that if \( u(x, t) \) is a solution of (1.2) which is initially close to \( \phi \) in \( L^2_{\text{per}}(0, N) \), then there exists a (small) time-dependent modulation function \( \gamma(t) \) such that \( u(x, t) \) essentially behaves for large time as

\[
u(x, t) \approx \phi(x) + \gamma(t)\phi'(x) \approx \phi(x + \gamma(t)),
\]

corresponding to standard asymptotic (orbital) stability of \( \phi \). With this insight gained from (1.5), a straightforward nonlinear iteration scheme completes the proof of Proposition 1.1.

While Proposition 1.1 establishes nonlinear stability of \( \phi \) in \( L^2_{\text{per}}(0, N) \) for each fixed \( N \in \mathbb{N} \), it lacks uniformity in \( N \) in two important (and related) aspects. Indeed, note that the exponential rate of decay \( \delta \) and the allowable size of initial perturbations \( \varepsilon = \varepsilon_\delta \) are both controlled completely in terms of the size of the spectral gap \( \delta_N > 0 \). Since \( \delta_N \to 0 \) as \( N \to \infty \), it follows that both \( \delta \) and \( \varepsilon \) chosen in Proposition 1.1 necessarily tend to zero\(^1\) as \( N \to \infty \). With this observation in mind, it is natural to ask if one can obtain a stability result to \( N \)-periodic perturbations which is uniform in \( N \). In such a result, one should naturally require that both the rate of decay and and the size of initial perturbations be independent of \( N \), thus depending only on the background wave \( \phi \). This is precisely achieved in our main result.

**Theorem 1.2** (Uniform Subharmonic Asymptotic Stability). Fix \( K \geq 3 \). Suppose \( \phi \in H^1_{\text{loc}}(\mathbb{R}) \) is a 1-periodic stationary solution of (1.2) that is diffusively spectrally stable, in the sense of Definition 2.3. There exists an \( \varepsilon > 0 \) and a constant \( C > 0 \) such that, for each \( N \in \mathbb{N} \), whenever \( u_0 \in H^1_{\text{per}}(0, N) \) and

\[
E_0 := \|u_0 - \phi\|_{L^1_{\text{per}}(0,N) \cap H^K_{\text{per}}(0,N)} < \varepsilon,
\]

there exists a function \( \tilde{\psi}(x,t) \) satisfying \( \tilde{\psi}(\cdot, 0) \equiv 0 \) such that the solution of (1.2) with initial data \( u(0) = u_0 \) exists globally in time and satisfies

\[
\|u(\cdot, t) - \psi(\cdot, t)\|_{L^2_{\text{per}}(0,N)}, \quad \|\nabla_{x,t} \tilde{\psi}(\cdot, t)\|_{L^2_{\text{per}}(0,N)} \leq C E_0 (1 + t)^{-3/4}.
\]

and

\[
\|
\tilde{\psi}(\cdot, t)\|_{L^2_{\text{per}}(0,N)} \leq C E_0 (1 + t)^{-1/4}
\]

for all \( t \geq 0 \).

Moreover, we note that since the above decay rates are sufficiently fast, we can obtain the following result accounting for only time-dependent modulations yet offering slower uniform decay rates.

\(^1\)Additionally, this degeneracy can be seen in the linear estimate (1.5) since both \( \delta_N \to 0^+ \) and \( C_N \to \infty \) as \( N \to \infty \).
Corollary 1.3. Under the hypotheses of Theorem 1.2, there exists an \( \varepsilon > 0 \) and a constant \( C > 0 \) such that, for each \( N \in \mathbb{N} \), whenever \( u_0 \in H^1_{\text{per}}(0,N) \) and \( E_0 < \varepsilon \), there exists a function \( \gamma(t) \) satisfying \( \gamma(0) = 0 \) the solution \( u \) of (1.2) with initial data \( u(0) = u_0 \) exists globally in time and satisfies

\[
\left\| u \left( \cdot - \frac{1}{N} \gamma(t), t \right) - \phi \right\|_{L^2_{\text{per}}(0,N)} \leq CE_0(1 + t)^{-1/4}.
\]

for all \( t > 0 \). Further, the time-dependent modulation function \( \gamma(t) \) satisfies

\[
|\gamma'(t)| \leq CE_0(1 + t)^{-3/2}
\]

and hence, in particular, there exists a \( \gamma_\infty \in \mathbb{R} \)

\[
|\gamma(t) - \gamma_\infty| \leq CE_0(1 + t)^{-1/2}
\]

for all \( t > 0 \).

The key point in the above results is that both the admissible size of initial perturbations \( \varepsilon \) and the rate of decay of perturbations are both uniform in \( N \). Even more, the decay rates guaranteed in Theorem 1.2 are precisely those predicted by considering the dynamics of such periodic wave trains to localized perturbations: see \([8, 6, 7, 9]\), for example. Formally, this should not be too surprising since, up to appropriate translations, a sequence of \( N \)-periodic functions may converge (locally) as \( N \to \infty \) to functions in \( L^2_{\text{per}}(\mathbb{R}) \).

Remark 1.4. Using the methods in \([6, 9]\), the results in Theorem 1.2 can easily be extended to establish uniform (in \( N \)) decay rates of perturbations in \( L^p_{\text{per}}(0,N) \) for any \( 2 \leq p \leq \infty \) provided the initial perturbations are again sufficiently small in \( L^1_{\text{per}}(0,N) \cap H^K_{\text{per}}(0,N) \). For simplicity, however, and to establish proof of concept, in this work we concentrate on the \( L^2_{\text{per}} \)-based theory only.

In the proof of Theorem 1.2, we make the above intuition precise by first providing a delicate decomposition of the semigroup \( e^{\mathcal{L}[\phi]t} \) acting on the the space \( L^2_{\text{per}}(0,N) \) with \( N \in \mathbb{N} \). This decomposition will not only recover, at the linear level, the exponential decay rates exhibited in Proposition 1.1, but they will also provide the uniform (in \( N \)) rates of decay in Theorem 1.2. As we will see, our linear analysis predicts that if \( u(x,t) \) is a solution of (1.2) which is initially close to \( \phi \) in \( L^2_{\text{per}}(0,N) \) then there exists a (small) space-time dependent, \( N \)-periodic (in \( x \)) modulation function \( \tilde{\psi}(x,t) \) such that \( u(x,t) \) essentially behaves for large time like

\[
u(x,t) \approx \phi(x) + \tilde{\psi}(x,t)\phi'(x) \approx \phi \left( x + \tilde{\psi}(x,t) \right)
\]

giving a refined insight into the long-time local dynamics near \( \phi \) beyond the more standard asymptotic stability (with asymptotic phase) as in Proposition 1.1. Motivated by this initial linear analysis, we then will build a nonlinear iteration scheme to complete the proof of Theorem 1.2.

The outline of the paper is as follows. In Section 2 we review several preliminary results, including a review in Section 2.1 of Floquet-Bloch theory in the context of \( N \)-periodic function spaces. This will provide us with a characterization of \( N \)-periodic eigenvalues of the 1-periodic coefficient differential operator \( \mathcal{L}[\phi] \) in terms of the associated Bloch operators. We further collect several properties of the Bloch operators and their associated semigroups. In Section 2.2 we introduce our specific notion of \textit{diffusive spectral stability} and record some immediate consequences regarding the
decay properties of the corresponding Bloch semigroups. In Section 3 we establish our key linear estimates by providing a delicate decomposition of the semigroup $e^{L[\phi]t}$ acting on $L^2_{\text{per}}(0,N)$, which allows us to identify polynomial decay rates on the linear evolution which are uniform in $N$: see Proposition 3.1. These linear estimates form the backbone for our nonlinear analysis, which is detailed in Section 4. In Section 4.1 we use intuition gained from the linear estimates of Section 3 to introduce an appropriate nonlinear decomposition of a small $L^2_{\text{per}}(0,N)$ neighborhood of the underlying diffusively stable 1-periodic wave $\phi$, and we develop appropriate perturbation equations satisfied by the corresponding perturbation and modulation functions. In Section 4.2, we apply a nonlinear iteration scheme to the system of perturbation equations obtained in Section 4.1 and complete the proof of Theorem 1.2. Finally, a proof of some technical results from Section 3 are provided in an Appendix.

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2 Preliminaries

In this section, we review several preliminary results. First, to aid in our description of the spectrum of the linearization $L[\phi]$ we review general results from Floquet-Bloch theory as applied to subharmonic perturbations. From this, we then provide our main spectral stability assumption and elementary semigroup estimates for the associated Bloch operators. Throughout the remainder of the paper, for notational convenience we set for each $N \in \mathbb{N}$ and $p \geq 1$

$$L^p_N := L^p_{\text{per}}(0,N).$$

2.1 Floquet Bloch Theory for Subharmonic Perturbations

Motivated by Floquet-Bloch theory for linear differential operators with periodic coefficients acting on $L^2(\mathbb{R})$ (see [3, 8, 11], for example), we review a modification of this theory (restricted to the present reaction-diffusion context) for the study of subharmonic perturbations.

Suppose that $\phi$ is a 1-periodic stationary solution of (1.2), and consider the linearized operator $L[\phi]$. Since the coefficients of $L[\phi]$ are 1-periodic, Floquet theory implies that for each $\lambda \in \mathbb{C}$ any non-trivial solution of the ordinary differential equation $L[\phi]v = \lambda v$

cannot be integrable on $\mathbb{R}$ and that, at best, they can be bounded functions of the form

$$v(x) = e^{i\xi x}w(x)$$

(2.1)

for some $\xi \in [-\pi, \pi)$ and non-trivial function $w \in L^2_{\text{per}}(0,1)$. For a given $N \in \mathbb{N}$, setting

$$\Omega_N := \{\xi \in [-\pi, \pi) : e^{i\xi N} = 1\}$$

\footnote{See also [4] for more information regarding this subharmonic extension.}
we see from (2.1) that the perturbation $v$ satisfies $N$-periodic boundary conditions if and only if $\xi \in \Omega_N$. In particular, it can be shown that $\lambda \in \mathbb{C}$ belongs to the $L^2_N$-spectrum of $L[\phi]$ if and only if there exists a $\xi \in \Omega_N$ and a non-trivial $w \in L^2_{\text{per}}(0, 1)$ such that

$$\lambda w = e^{-i\xi x} L[\phi] e^{i\xi x} w =: L_{\xi}[\phi] w.$$  

The operators $L_{\xi}[\phi]$ are known as the Bloch operators associated to $L[\phi]$, and the parameter $\xi$ is referred to as the Bloch frequency. Note that each $L_{\xi}[\phi]$ acts on $L^2_{\text{per}}(0, 1)$ with densely defined and compactly embedded domain $H^1_{\text{per}}(0, 1)$, and hence their spectrum consists entirely of isolated eigenvalues with finite algebraic multiplicities which, furthermore, depend on continuously on $\xi$. In fact, we have the spectral decomposition

$$\sigma_{L^2_N}(L[\phi]) = \bigcup_{\xi \in \Omega_N} \sigma_{L^2_{\text{per}}(0, 1)}(L_{\xi}[\phi]).$$

This characterizes the $N$-periodic spectrum of $L[\phi]$ in terms of union of 1-periodic eigenvalues for the Bloch operators $\{L_{\xi}[\phi]\}_{\xi \in \Omega_N}$.

**Remark 2.1.** For definiteness, we note that the set $\Omega_N$ may be written explicitly when $N$ is even by

$$\Omega_N = \left\{\xi_j = \frac{2\pi j}{N} : j = -\frac{N}{2}, -\frac{N}{2} + 1, \ldots, \frac{N}{2} - 1\right\},$$

and when $N$ is odd by

$$\Omega_N = \left\{\xi_j = \frac{2\pi j}{N} : j = -\frac{N - 1}{2}, \ldots, \frac{N - 1}{2}, \ldots, \frac{N - 1}{2} - 1\right\}.$$ 

In particular, observe that we have $0 \in \Omega_N$ and $|\Omega_N| = N$ for all $N \in \mathbb{N}$ and that, furthermore, $\Delta \xi_j := \xi_j - \xi_{j-1} = \frac{2\pi}{N}$ for each appropriate $j$.

From the above, it is clearly desirable to have the ability to decompose arbitrary functions in $L^2_N$ into superpositions of functions of the form $e^{i\xi x} w(x)$ with $\xi \in \Omega_N$ and $w \in L^2_{\text{per}}(0, 1)$. This is achieved by noting that a given $g \in L^2_N$ admits a Fourier series representation

$$g(x) = \frac{1}{N} \sum_{m \in \mathbb{Z}} e^{2\pi i mx/N} \tilde{g}(2\pi m/N)$$

where here $\tilde{g}$ denotes the Fourier transform of $g$ on the torus given by

$$(2.2) \quad \tilde{g}(z) := \int_{-N/2}^{N/2} e^{-izy} g(y) dy.$$

Together with the identity (valid for any $f$ for which the sum converges)

$$\sum_{m \in \mathbb{Z}} f(2\pi m/N) = \sum_{\xi \in \Omega_N} \sum_{\ell \in \mathbb{Z}} f(\xi + 2\pi \ell),$$

it follows that $g$ may be represented as

$$g(x) = \frac{1}{N} \sum_{\xi \in \Omega_N} \sum_{\ell \in \mathbb{Z}} e^{i(\xi + 2\pi \ell)} \tilde{g}(\xi + 2\pi \ell).$$
In particular, defining for $\xi \in \Omega_N$ the 1-periodic Bloch transform of a function $g \in L^2_N$ as
\[ B_1(g)(\xi,x) := \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell x} \hat{g}(\xi + 2\pi \ell), \]
the above yields the inverse Bloch representation formula
\[ g(x) = \frac{1}{N} \sum_{\xi \in \Omega_N} e^{i \xi x} B_1(g)(\xi,x), \]
which is valid for all $g \in L^2_N$. Note that the function $B_1(g)(\xi, \cdot)$ is clearly 1-periodic for each $\xi \in \Omega_N$, and hence the above representation formula decomposes arbitrary $N$-periodic functions in the desired fashion.

Before proceeding, we note that, in fact, the 1-periodic Bloch transform
\[ B_1 : L^2_N \to \ell^2 \left( \Omega_N : L^2_{\text{per}}(0,1) \right) \]
as defined above satisfies the subharmonic Parseval identity
\[ \langle f, g \rangle_{L^2_N} = \frac{1}{N} \sum_{\xi \in \Omega_N} \langle B_1(f)(\xi, \cdot), B_1(g)(\xi, \cdot) \rangle_{L^2(0,1)} \]
valid for all $f, g \in L^2_N$. In particular, this yields the useful identity
\[ \|g\|^2_{L^2_N} = \frac{1}{N} \sum_{\xi \in \Omega_N} \|B_1(g)(\xi, \cdot)\|^2_{L^2(0,1)} \]
valid for all $g \in L^2_N$, establishing that (up to normalization) $B_1$ is an isometry. Furthermore, we note that
\[ B_1 \left( \mathcal{L}[\phi] v \right)(\xi,x) = \mathcal{L}[\phi] B_1(v)(\xi,x) \quad \text{and} \quad \mathcal{L}[\phi] v(x) = \frac{1}{N} \sum_{\xi \in \Omega_N} e^{i \xi x} \mathcal{L}[\phi] B_1(v)(\xi,x). \]
and hence we may view the Bloch operators $\mathcal{L}[\phi]$ as operator valued symbols associated to $\mathcal{L}[\phi]$ under the action of the 1-periodic Bloch transform $B_1$. Since the operator $\mathcal{L}[\phi]$ and its corresponding Bloch operators $\mathcal{L}[\phi]$ are clearly sectorial on $L^2_N$ and $L^2_{\text{per}}(0,1)$, respectively, they clearly generate analytic semigroups on their respective function spaces and, further, it is now straightforward to check that the associated semigroups satisfy
\[ B_1 \left( e^{\mathcal{L}[\phi] t} v \right)(\xi,x) = \left( e^{\mathcal{L}[\phi] t} B_1(v)(\xi, \cdot) \right)(x) \quad \text{and} \quad e^{\mathcal{L}[\phi] t} v(x) = \frac{1}{N} \sum_{\xi \in \Omega_N} e^{i \xi x} e^{\mathcal{L}[\phi] t} B_1(v)(\xi,x). \]
Combined with (2.3), this latter identity allows us to conclude information about the semigroup $e^{\mathcal{L}[\phi] t}$ acting on $L^2_N$, by synthesizing (over $\xi \in \Omega_N$) information about the Bloch semigroups $e^{\mathcal{L}[\phi] t}$ acting on $L^2_{\text{per}}(0,1)$. This decomposition is key to our forthcoming linear analysis.

Finally, we end by recalling the following useful identity.

**Lemma 2.2.** Let $N \in \mathbb{N}$ If $f \in L^2_{\text{per}}(0,1)$ and $g \in L^2_N$, then
\[ B_1(fg)(\xi,x) = f(x) B_1(g)(\xi,x). \]
In particular, for such $f$ and $g$ we have the identity
\[ \langle f, g \rangle_{L^2_N} = \langle f, B_1(g)(0, \cdot) \rangle_{L^2(0,1)}. \]

The proof of Lemma 2.2 is straightforward and can be found in [4].
2.2 Diffusive Spectral Stability & Properties of Semigroups

With the above characterization of the $L^2_N$-spectrum of the linearized operator $L[\phi]$ about a 1-periodic stationary solution $\phi$ of (1.2), we now introduce an appropriate notion of spectral stability.

**Definition 2.3.** A 1-periodic stationary solution $\phi \in H^1_{\text{loc}}(\mathbb{R})$ of (1.2) is said to be **diffusively spectrally stable** provided the following conditions hold:

(i) The spectrum of the linear operator $L[\phi]$ acting on $L^2(\mathbb{R})$ satisfies

$$\sigma_{L^2(\mathbb{R})}(L[\phi]) \subset \{\lambda \in \mathbb{C} : \text{Re}(\lambda) < 0\} \cup \{0\};$$

(ii) There exists a $\theta > 0$ such that for any $\xi \in [-\pi, \pi)$ the real part of the spectrum of the Bloch operator $L_\xi[\phi]$ acting on $L^2_{\text{per}}(0,1)$ satisfies

$$\text{Re}\left(\sigma_{L^2_{\text{per}}(0,1)}(L_\xi[\phi])\right) \leq -\theta \xi^2;$$

(iii) $\lambda = 0$ is a simple eigenvalue of $L_0[\phi]$ with associated eigenfunction $\phi'$.

Since the pioneering work of Schneider [14, 15], the above notion of spectral stability has been taken as the standard spectral assumption in nonlinear stability results for periodic traveling/standing waves in reaction diffusion systems. Specifically, the above notion of spectral stability is sufficiently strong to allow one to immediately conclude important details regarding the nonlinear dynamics of $\phi$ under localized perturbations, including long-time asymptotics of the associated modulation functions. For more information, see [2, 6, 7, 12, 13] and references therein.

**Remark 2.4.** Note the assumption on simplicity of the eigenvalue $\lambda = 0$ is natural since such periodic standing waves typically appear as one-parameter families parametrized only by translational invariance. Indeed, solutions of (1.3) are readily seen to rely (up to translation invariance) on the $n + 2$ parameters $(\phi(0), k, c)$, while periodicity requires the enforcement of $n$ constraints, leaving in general a two-parameter family of 1-periodic solutions

$$u(x - ct - x_0; c, x_0)$$

which satisfy (1.3) with $k = k(c)$. Due to the secular dependence of the frequency $k$ on the wave speed $c$, variations in $c$ do not preserve periodicity and hence, generically, it follows one should expect the kernel of $L[\phi]$ to be one-dimensional, which leads to (iii) in Definition (2.3) above.

In our present subharmonic context, we note that if $\phi$ is a diffusively spectrally stable standing solution of (1.2), then for each $N \in \mathbb{N}$ there exists a $\delta_N > 0$ such that

$$\text{Re}\left(\sigma_{L^2_N}(L[\phi]) \setminus \{0\}\right) \leq -\delta_N,$$

i.e. the non-zero $N$-periodic eigenvalues of $L[\phi]$ are uniformly bounded away from the imaginary axis. In particular, by standard spectral perturbation theory, we immediately have that the following spectral properties hold.

**Lemma 2.5** (Spectral Preparation). Suppose that $\phi$ is a 1-periodic stationary solution of (1.2) which is diffusively spectrally stable. Then the following properties hold.
(i) For any fixed \( \xi_0 \in (0, \pi) \), there exists a constant \( \delta_0 > 0 \) such that
\[
\text{Re}(L_\xi[\phi]) < -\delta_0
\]
for all \( \xi \in [-\pi, \pi] \) with \( |\xi| > \xi_0 \).

(ii) There exists positive constants \( \xi_1 \) and \( \delta_1 \) such that for any \( |\xi| < \xi_1 \) the spectrum of \( L[\phi] \) decomposes into two disjoint subsets
\[
\sigma(L_\xi[\phi]) = \sigma_-(L_\xi[\phi]) \cup \sigma_0(L_\xi[\phi])
\]
with the following properties:

(a) \( \text{Re} \sigma_-(L_\xi[\phi]) < -\delta_1 \) and \( \text{Re} \sigma_0(L_\xi[\phi]) > -\delta_1 \);

(b) the set \( \sigma_0(L_\xi[\phi]) \) consists of a single eigenvalue \( \lambda_c(\xi) \) which is analytic in \( \xi \) and expands as
\[
\lambda_c(\xi) = ia\xi - d\xi^2 + O(\xi^3)
\]
for \( |\xi| \ll 1 \) and some constants \( a \in \mathbb{R} \) and \( d > 0 \);

(c) the eigenfunction associated to \( \lambda_c(\xi) \) is analytic near \( \xi = 0 \) and expands as
\[
\Phi_\xi(x) = \phi'(x) + O(\xi)
\]
for \( |\xi| \ll 1 \).

The proof of (i) follows immediately from the properties (i) and (ii) in Definition 2.3, while the second part follows since \( \lambda = 0 \) is a simple eigenvalue of the co-periodic operator \( L_0[\phi] \) and that the coefficients of \( L_\xi[\phi] \) clearly vary analytically on \( \xi \).

With the above spectral preparation result in hand, we now record some key induced features of the associated semigroups. These estimates are immediate consequences of Lemma 2.5 and the fact that the Bloch operators are clearly sectorial when acting on \( L^2_{\text{per}}(0, 1) \).

**Proposition 2.6.** Suppose that \( \phi \) is a 1-periodic stationary solution of (1.2) which is diffusively spectrally stable. Then the following properties hold.

(i) For any fixed \( \xi_0 \in (0, \pi) \) there exists positive constants \( C_0 \) and \( d_0 \) such that
\[
\left\| e^{L_\xi[\phi]t} f \right\|_{B(L^2_{\text{per}}(0, 1))} \leq C_0 e^{-d_0 t}
\]
valid for all \( t \geq 0 \) and all \( \xi \in [-\pi, \pi] \) with \( |\xi| > \xi_0 \).

(ii) With \( \xi_1 \) chosen as in Lemma 2.5, there exists positive constants \( C_1 \) and \( d_1 \) such that for any \( |\xi| < \xi_1 \), if \( \Pi(\xi) \) denotes the (rank-one) spectral projection onto the eigenspace associated to \( \lambda_c(\xi) \) given by Lemma 2.5(ii), then
\[
\left\| e^{L_\xi[\phi]t} (1 - \Pi(\xi)) \right\|_{B(L^2_{\text{per}}(0, 1))} \leq C e^{-d_1 t}
\]
for all \( t \geq 0 \).

Coupled with an appropriate decomposition of \( e^{L[\phi]t} \), the above linear estimates form the core of our forthcoming linear analysis (which, in turn, forms the backbone of our nonlinear iteration scheme).
3 Uniform Subharmonic Linear Estimates

We begin our analysis by obtaining decay rates on the semigroup $e^{\mathcal{L}_\phi t}$ acting on classes of subharmonic perturbations in $L^2_N$ which are uniform in $N$. This analysis is based on a delicate decomposition of the semigroup. In particular, we use (2.4) to study the action of $e^{\mathcal{L}_\phi t}$ on $L^2_N$ in terms of associated Bloch operators, which is accomplished by separating the semigroup into appropriate critical-frequency and non-critical frequency components. Note that, due to Lemma 2.5 we expect the “critical frequency” component to be dominated by the translational mode $\phi'$. This decomposition was recently carried out in detail (in a related context) in [4], and for completeness we review it here. Note the decomposition is heavily motivated by the corresponding decomposition used in the case of localized perturbations: see [8, 6].

To begin, let $\xi_1 \in (0, \pi)$ be defined as in Lemma 2.5 and let $\rho$ be a smooth cutoff function satisfying $\rho(\xi) = 1$ for $|\xi| < \frac{\xi_1}{2}$ and $\rho(\xi) = 0$ for $|\xi| > \xi_1$. For a given $v \in L^2_N$, we use (2.4) to decompose $e^{\mathcal{L}_\phi t}$ into low-frequency and high-frequency components as

$$
e^{\mathcal{L}_\phi t}v(x) = \frac{1}{N} \sum_{\xi \in \Omega_N} \rho(\xi) e^{i\xi x} e^{\mathcal{L}_\phi t} B_1(\xi, x) + \frac{1}{N} \sum_{\xi \in \Omega_N} (1 - \rho(\xi)) e^{i\xi x} e^{\mathcal{L}_\phi t} B_1(\xi, x)$$

$$= S_{lf,N}(t)v(x) + S_{hf,N}(t)v(x).$$

Using Proposition 2.6 and the subharmonic Parseval identity 2.3 it follows that there exists a constants $C, \eta > 0$, both independent of $N$, such that

$$\|S_{h,N}(t)v\|_{L^2_N}^2 = \frac{1}{N} \sum_{\xi \in \Omega_N} \left\|(1 - \rho(\xi)) e^{\mathcal{L}_\phi t} B_1(\xi, \cdot)\right\|^2_{L^2(0,1)}$$

$$\leq \frac{1}{N} \sum_{\xi \in \Omega_N} (1 - \rho(\xi))^2 \left\|e^{\mathcal{L}_\phi t}\right\|^2_{B(L^2(0,1))} \|B_1(\xi, \cdot)\|^2_{L^2(0,1)}$$

$$\leq Ce^{-2\eta t} \left(\frac{1}{N} \sum_{\xi \in \Omega_N} \|B_1(\xi, \cdot)\|^2_{L^2(0,1)}\right),$$

which, again using Parseval’s identity (2.3), yields the exponential decay estimate

$$\|S_{h,N}(t)v\|_{L^2_N} \leq Ce^{-\eta t} \|v\|_{L^2_N}. \quad (3.2)$$

For the low-frequency component, for each $|\xi| < \xi_1$ define the rank-one spectral projection onto the critical mode of $\mathcal{L}_\phi$ by

$$\Pi(\xi) : L^2_{\text{per}}(0, 1) \to \ker (\mathcal{L}_\phi - \lambda_\phi(\xi) I)$$

$$\Pi(\xi)g(x) = \left\langle \tilde{\Phi}_\xi, g \right\rangle_{L^2(0,1)} \Phi_\xi(x) \quad (3.3)$$

where here $\tilde{\Phi}_\xi$ denotes the element of the kernel of the adjoint $\mathcal{L}_\phi^+ - \lambda_\phi(\xi) I$ satisfying the normalization condition $\left\langle \Phi_\xi, \Phi_\xi \right\rangle_{L^2(0,1)} = 1$. The low-frequency operator $S_{lf,N}$ can thus be further decomposed into the contribution from the critical mode and the contribution from low-frequency
spectrum bounded away from $\lambda = 0$ via
\begin{equation}
S_{f,N}(t)v(x) = \frac{1}{N} \sum_{\xi \in \Omega_N} \rho(\xi)e^{i\xi x}e^{C[\phi]t} \Pi(\xi)B_1(v)(\xi, x) + \frac{1}{N} \sum_{\xi \in \Omega_N} \rho(\xi)e^{i\xi x}e^{C[\phi]t} (1 - \Pi(\xi)) B_1(v)(\xi, x)
\end{equation}

$$=: S_{c,N}v(x) + \tilde{S}_{f,N}(t)v(x).$$

As with the exponential estimate (3.2), Proposition 2.6 implies, by possibly choosing $\eta > 0$ smaller, that there exists a constant $C > 0$ independent of $N$ such that
\begin{equation}
\|\tilde{S}_{f,N}(t)v\|_{L^2_N} \leq Ce^{-\eta t}\|v\|_{L^2_N}.
\end{equation}

For the critical component $S_{c,N}$, note by Lemma 2.5(ii) that we can write

$$S_{c,N}(t)v(x) = \frac{1}{N} e^{C[\phi]t} \Pi(0)B_1(v)(0, x) + \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \rho(\xi)e^{i\xi x}e^{C[\phi]t} \Pi(\xi)B_1(v)(\xi, x)$$

$$= \frac{1}{N} \phi'(x) \left\langle \tilde{\Phi}_0, B_1(v)(0, \cdot) \right\rangle_{L^2(0,1)} + \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \rho(\xi)e^{i\xi x}e^{C[\phi]t} \left\langle \tilde{\Phi}_\xi, B_1(v)(\xi, x) \right\rangle_{L^2(0,1)}.$$

and hence, recalling Lemma 2.2 and expanding $\Phi_\xi$,

$$S_{c,N}(t)v(x) = \frac{1}{N} \phi'(x) \left\langle \tilde{\Phi}_0, v \right\rangle_{L^2_N} + \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \rho(\xi)e^{i\xi x}e^{C[\phi]t} \left\langle \tilde{\Phi}_\xi, B_1(v)(\xi, x) \right\rangle_{L^2(0,1)}$$

$$+ \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \rho(\xi)e^{i\xi x}e^{C[\phi]t} \left( \frac{\tilde{\Phi}(x) - \phi'(x)}{i\xi} \right) \left\langle \tilde{\Phi}_\xi, B_1(v)(\xi, x) \right\rangle_{L^2(0,1)}$$

$$=: \frac{1}{N} \phi'(x) \left\langle \tilde{\Phi}_0, v \right\rangle_{L^2_N} + \phi'(x)s_{p,N}(t)v(x) + \tilde{S}_{c,N}(t)v(x).$$

Taken together, it follows that the linear solution operator $e^{C[\phi]t}$ can be decomposed as

$$e^{C[\phi]t}v(x) = \frac{1}{N} \phi'(x) \left\langle \tilde{\Phi}_0, v \right\rangle_{L^2_N} + \phi'(x)s_{p,N}(t)v(x) + \tilde{S}_{N}(t)v(x)$$

where
\begin{equation}
s_{p,N}(t)v(x) = \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \rho(\xi)e^{i\xi x}e^{C[\phi]t} \left\langle \tilde{\Phi}_\xi, B_1(v)(\xi, x) \right\rangle_{L^2(0,1)}
\end{equation}

and

$$\tilde{S}_{N}(t)v(x) = S_{h,f,N}(t)v(x) + \tilde{S}_{f,N}(t)v(x) + \tilde{S}_{c,N}(t)v(x).$$

Equipped with the above, we can establish our main set of linear estimates.

**Proposition 3.1 (Linear Estimates).** Suppose that $\phi$ is a 1-periodic stationary solution of (1.2) which is diffusively spectrally stable. There exists a constant $C > 0$ such that for all $t \geq 0$, $N \in \mathbb{N}$ and all $l, m \geq 0$ we have

$$\left\| \partial_x^l \partial_t^m s_{p,N}(t)v \right\|_{L^2_N} \leq C(1 + t)^{-1/4-l+4/m}/2\|v\|_{L^2_N}.$$
Furthermore, there exists constants $C, \eta > 0$ such that for all $t \geq 0$ and $N \in \mathbb{N}$ we have

$$\left\| \tilde{S}_N(t) \right\|_{L^2_N} \leq C \left( (1 + t)^{-3/4} \|v\|_{L^1_N} + e^{-\eta t} \|v\|_{L^1_N} \right).$$

Remark 3.2. While the bounds above on the derivatives of $s_{p,N}(t)$ are largely unmotivated by our linear analysis, they will be essential in our forthcoming nonlinear theory.

Proof. First observe that, by definition of $B_1$, we have

$$\left\| \Phi_{\xi} B_1(v)(\xi, \cdot) \right\|_{L^2(0,1)}^2 = \int_0^1 \Phi_{\xi}(x) \sum_{\ell \in \mathbb{Z}} e^{2\pi i \ell x} \hat{\varphi}(\xi + 2\pi \ell) dx = \sum_{\ell \in \mathbb{Z}} \hat{v}(\xi + 2\pi \ell) \int_0^1 \Phi_{\xi}(x) e^{2\pi i \ell x} dx = \sum_{\ell \in \mathbb{Z}} \hat{v}(\xi + 2\pi \ell) \Phi_{\xi}(x)(2\pi \ell)$$

and hence, using the fact that (2.2) implies $\|\hat{v}\|_{L^\infty(\mathbb{R})} \leq \|v\|_{L^1_N}$ along with Cauchy-Schwartz, it follows that

$$\rho(\xi) \left\| \Phi_{\xi} B_1(v)(\xi, \cdot) \right\|_{L^2(0,1)}^2 \leq \rho(\xi) \left\| v \right\|_{L^1_N}^2 \left( \sum_{\ell \in \mathbb{Z}} (1 + |\ell|^2)^{1/2} \left| \Phi_{\xi}(x)(2\pi \ell) \right| (1 + |\ell|^2)^{-1/2} \right)^2 \leq C \|v\|_{L^1_N}^2 \sup_{\xi \in [-\pi, \pi]} \left( \rho(\xi) \left\| \Phi_{\xi} \right\|_{H^1_{p,\alpha}(0,1)}^2 \right),$$

valid for all $\xi \in \Omega_N$. Using Lemma 2.5, it follows by Parseval’s identity (2.3) that there exists constants $C, d > 0$, independent of $N$, such that

$$\left\| \partial_t^l \partial_{\xi}^m s_{p,N}(t)v \right\|_{L^2_N}^2 = \frac{1}{N} \sum_{\xi \in \Omega_N} \left[ \rho(\xi)(i\xi)^l (\lambda_c(\xi))^m e^{\lambda_{-c}(\xi)t} \left\langle \Phi_{\xi}, B_1(v)(\xi, \cdot) \right\rangle_{L^2(0,1)}^2 \right] \leq C \|v\|_{L^1_N} \left( \frac{1}{N} \sum_{\xi \in \Omega_N} |\xi|^{2(l+m)} e^{-2d|\xi|^2 t} \right).$$

By similar considerations, we find that

$$\left\| \tilde{S}_N(t)v \right\|_{L^2_N}^2 \leq Ce^{-2\eta t} \left\| v \right\|_{L^2_N}^2 + C \|v\|_{L^1_N}^2 \left( \frac{1}{N} \sum_{\xi \in \Omega_N} |\xi|^{2(l+m)} e^{-2d|\xi|^2 t} \right).$$

It remains to provide uniform in $N$ decay rates on the finite sums

$$\frac{1}{N} \sum_{\xi \in \Omega_N} |\xi|^{2(l+m)} e^{-2d|\xi|^2 t} \quad \text{and} \quad \frac{1}{N} \sum_{\xi \in \Omega_N} |\xi|^2 e^{-2d|\xi|^2 t}.$$

To gain some intuition on how to uniformly bound these sums, notice that they can be interpreted as Riemann sum approximations (up to a harmless rescaling) of the integrals

$$\int_{-\pi}^{\pi} \xi^{2(l+m)} e^{-2d|\xi|^2 t} d\xi, \quad \int_{-\pi}^{\pi} \xi^2 e^{-2d|\xi|^2 t} d\xi,$$
which, through an elementary scaling argument, exhibit \((1 + t)^{-1/2-(\ell+m)}\) and \((1 + t)^{-3/2}\) decay for large time, respectively. The proof that the Riemann sums are uniformly controlled by these decay rates is provided in Lemma A.1 in the Appendix, which completes the proof.

Before continuing to our nonlinear analysis, we pause to interpret the above results. Suppose that \(\phi\) is a 1-periodic diffusively spectrally stable stationary solution of \((1.2)\), and let \(u(x,t)\) be a solution of \((1.2)\) with initial data \(u(x,0) = \phi(x) + \varepsilon v(x)\) with \(\varepsilon \ll 1\) and \(v \in L_N^1 \cap L_N^2\). From Proposition 3.1, it follows that one may expect that the solution \(u\) behaves for large time like

\[
\lim_{t \to \infty} u(x,t) \approx \phi(x) + e^{L[\phi]t}v(x)
\]

\[
\approx \phi(x) + \phi'(x) \left( \frac{1}{N} \left\langle \Phi_0, v \right\rangle_{L_N^2} + s_{p,N}(t)v(x) \right)
\]

\[
\approx \phi \left(x + \frac{1}{N} \left\langle \Phi_0, v \right\rangle_{L_N^2} + s_{p,N}(t)v(x) \right).
\]

In the next section, we use this intuition to develop a nonlinear iteration scheme and complete the proof of Theorem 1.2 and Corollary 1.3.

### 4 Uniform Nonlinear Asymptotic Stability

In this section, we use the decomposition of the linearized solution operator \(e^{L[\phi]t}\) and the associated linear estimates in Proposition 3.1 to develop a nonlinear iteration scheme to complete the proof of Theorem 1.2. As discussed at the end of Section 3, the linear estimates in Proposition 3.1 suggest that if \(\phi\) is a 1-periodic diffusively spectrally stable stationary solution of \((1.2)\) then \(N\)-periodic perturbations of \(\phi\) should, for large time, behave essentially like space-time modulated version of \(\phi\). This suggests a nonlinear decomposition of \(N\)-periodic perturbations of \(\phi\), which we develop in Section 4.1 below. With this decomposition in hand, the proof of Theorem 1.2 will be completed in Section 4.2 through an appropriate nonlinear iteration scheme.

#### 4.1 Nonlinear Decomposition and Perturbation Equations

Suppose \(\phi\) is a 1-periodic diffusively spectrally stable stationary solution of \((1.2)\). Motivated by the work in the previous section, we introduce a decomposition of nonlinear perturbations of the background wave \(\phi\) which accounts for the critical phase-shift contribution \(s_{p,N}(t)\) of the linear operator.

To begin, let \(\tilde{u}(x,t)\) be a solution of \((1.2)\) and define a spatially modulated function

\[
(4.1) \quad u(x,t) := \tilde{u} \left(x - \frac{1}{N} \gamma(t) - \psi(x,t), t\right)
\]

where both \(\gamma : \mathbb{R}_+ \to \mathbb{R}\) and \(\psi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}\) are functions to be determined later. Taking \(\tilde{u}\) to be initially close to \(\phi\) in some sense, we attempt to decompose \(u\) as

\[
(4.2) \quad u(x,t) = \phi(x) + v(x,t),
\]

where here \(v\) denotes a nonlinear perturbation. As a preliminary step, we derive equations that must be satisfied by the perturbation \(v\) and the modulation functions \(\gamma\) and \(\psi\). To this end, we
note that in [6, 9] it is shown through elementary, but tedious, manipulations that if \( u(x,t) \) is as above then the triple \((v,γ,ψ)\) satisfies
\[
(4.3) \quad (k\partial_t - k\mathcal{L}[ϕ]) \left( v + \frac{1}{N} φ'γ + φ'ψ \right) = k\tilde{N}, \quad \text{where } k\tilde{N} := \tilde{Q} + k\tilde{R}_x + k\tilde{S}_t + \tilde{T},
\]
with
\[
\tilde{Q} := f(ϕ + v) - f(ϕ) - Df(ϕ)v, \quad \tilde{R} := -ψ_tv - \frac{1}{N} ϕ_t v + k \left( \frac{ψ_x}{1 - ψ_x} v_x \right) + k \left( \frac{ψ_x^2}{1 - ψ_x} ϕ' \right),
\]
and
\[
\tilde{S} := ψ_x v, \quad \tilde{T} := -ψ_x [f(ϕ + v) - f(ϕ)].
\]
Rearranging slightly as in [1] to remove temporal derivatives of the perturbation \( v \) in present in \( \tilde{N} \) in (4.3) yields the following.

**Lemma 4.1.** The nonlinear residual \( v \) defined in (4.2) and modulation functions \( γ \) and \( ψ \) in (4.1) satisfy
\[
(4.4) \quad (k\partial_t - k\mathcal{L}[ϕ]) \left( (1 - ψ_x)v + \frac{1}{N} φ'γ + φ'ψ \right) = k\mathcal{N}, \quad \text{where } k\mathcal{N} = Q + kR_x,
\]
where here
\[
(4.5) \quad Q = (1 - ψ_x) \left[ f(ϕ + v) - f(ϕ) - Df(ϕ)v \right],
\]
and
\[
(4.6) \quad R = -ψ_tv - \frac{1}{N} ϕ_t v + cψ_x v + k(ψ_x v)_x + k \left( \frac{ψ_x}{1 - ψ_x} v_x \right) + k \left( \frac{ψ_x^2}{1 - ψ_x} ϕ' \right).
\]

Our goal is to now obtain a closed nonlinear iteration scheme by integrating (4.4) and exploiting the decomposition of the linear solution operator \( e^{\mathcal{L}[ϕ]t} \) provided in Proposition 3.1. To motivate this, we first provide an informal description of how to determine the modulation functions \( γ \) and \( ψ \) to separate out the principle nonlinear behavior. Using Duhamel’s formula, we can write (4.4) as the implicit integral equation
\[
(1 - ψ_x(x,t))v(x,t) + \frac{1}{N} φ'(x)γ(t) + ϕ'(x)ψ(x,t) = e^{\mathcal{L}[ϕ]t}v(x,0) + \int_0^t e^{\mathcal{L}[ϕ](t-s)}\mathcal{N}(x,s)ds,
\]
with initial data \( γ(0) = 0, ψ(\cdot,0) = 0 \) and \( v(\cdot,0) = \tilde{u}(\cdot,0) - φ(\cdot) \). Recalling that Proposition 3.1 implies the linear solution operator can be decomposed as
\[
e^{\mathcal{L}[ϕ]t}f(x) = φ'(x) \left[ \frac{1}{N} \left( \Phi_0, f \right)_{L_2^N} + s_p,N(t)f(x) \right] + \tilde{S}(t)f(x)
\]
with phase modulation faster decaying residual
\[
(4.7)
\]
it follows that we can remove the principle (i.e. slowest decaying) part of the nonlinear perturbation by implicitly defining
\[
\gamma(t) \sim \left( \Phi_0, v(0) \right)_{L_2^N} + \int_0^t \left( \Phi_0, \mathcal{N}(s) \right)_{L_2^N} ds
\]
and \( \psi(x,t) \sim s_p,N(t)v(0) + \int_0^t s_p,N(t-s)\mathcal{N}(s)ds, \)
\[
(4.8)
\]
where here \( \sim \) indicates equality for \( t \geq 1 \). This choice then yields the implicit description

\[
v(x,t) \sim \psi(x,t)v(x,t) + \tilde{S}(t)v(0) + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds
\]

involving only the faster decaying residual component of the linear solution operator.

Note the above choices clearly cannot extend all the way to \( t = 0 \) due to an incompatibility of these choices with the initial data on \((v, \gamma, \psi)\). Here, we choose to keep the above choices for all \( t \geq 1 \) while interpolating between the initial data and the right hand sides of (4.8)-(4.9) on the initial layer \( 0 \leq t \leq 1 \). Specifically, we let \( \chi(t) \) be a smooth cutoff function that is zero for \( t \leq 1/2 \) and one for \( t \geq 1 \), and define the modulation functions \( \gamma \) and \( \psi \) implicitly for all \( t \geq 0 \) as

\[
\begin{aligned}
\gamma(t) &= \chi(t)
\left[ \left< \Phi_0, v(0) \right>_{L^2_N} + \int_0^t \left< \Phi_0, \mathcal{N}(s) \right>_{L^2_N} ds \right], \\
\psi(x,t) &= \chi(t)
\left[ s_{p,N}(t)v(0) + \int_0^t s_{p,N}(t-s)\mathcal{N}(s)ds \right],
\end{aligned}
\]

leaving the system

\[
v(x,t) = (1 - \chi(t))
\left[ e^{\mathcal{C}_p(t-s)}v(x,0) + \int_0^t e^{\mathcal{C}_p(0-s)}\mathcal{N}(s)ds \right]
\]

\[
+ \chi(t) \left( \psi(x,t)v(x,t) + \tilde{S}(t)v(0) + \int_0^t \tilde{S}(t-s)\mathcal{N}(s)ds \right).
\]

We note that from the differential equation (4.4), along with the system of integral equations (4.10)-(4.11), we readily obtain short-time existence and continuity with respect to \( t \) of a solution \((v, \gamma, \psi) \in H^1_N \) and \( \gamma \in W^{1,\infty}(0, \infty) \) by a standard contraction mapping argument, treating (4.4) as a forced heat equation: see, for example, [5]. Associated with this solution, we now aim to obtain \( L^2 \) estimates on \((v, \gamma, \psi)\) and some of their derivatives.

Noting that the nonlinear residual \( \mathcal{N} \) in (4.4) involves only derivatives of the modulation functions \( \gamma \) and \( \psi \), we may then expect to extract a closed system in \((v, \gamma, \psi)\), and some of their derivatives, and then recover \( \gamma \) and \( \psi \) through the slaved system (4.8). In particular, observe that using (4.11) we see that control of \( v \) in, say, \( L^2_N \) requires (in part) control \( v \) in \( H^1_N \). This loss of derivatives is compensated by the following result, established by energy estimates in [6, 9], which uses the dissipative nature of the governing evolution equation to control higher derivatives of \( v \) by lower ones, enabling us to close our nonlinear iteration.

**Proposition 4.2 (Nonlinear Damping).** Suppose the nonlinear perturbation \( v \) defined in (4.2) satisfies \( v(\cdot,0) \in H^1_N \), and suppose that for some \( T > 0 \) the \( H^1_N \) norm of \( v \) and \( \psi_t \), the \( H^{1+1}_N \) norm of \( \psi_x \), and the \( L^\infty_N \) norms of \( \gamma \) and \( \gamma_t \) remain bounded by a sufficiently small constant for all \( 0 \leq t \leq T \). Then there exists a positive constants \( \theta, C > 0 \), both independent of \( N \) and \( T \), such that

\[
\|v(t)\|^2_{H^1_N} \leq e^{-\theta t}\|v(0)\|^2_{H^1_N} + \int_0^t e^{-\theta(t-s)} \left( \|v(s)\|^2_{L^2_N} + \|\psi_x(s)\|^2_{H^1_N} + \|\psi_t(s)\|^2_{H^1_N} + |\gamma_t(s)|^2 \right) ds
\]

for all \( 0 \leq t \leq T \).
Proof. The proof strategy is by now standard, and can be found, or example, in [6, 9]. For completeness, here we simply outline the main details. First, one rewrites (4.4) as the forced heat equation

\[(1 - \psi_x) (kv_t - k^2 v_{xx}) = -k \left( \psi_t + \frac{1}{N} \nabla_t \right) \phi' + k^2 \left( \frac{\psi_x}{1 - \psi_x} \phi' \right)_x - \psi_x f(\phi + v) + f(\phi + v) - f(\phi) + kv_x \left( c - \psi_t - \frac{\gamma_t}{N} \right) + k^2 \left[ \left( \frac{1}{1 - \psi_x} + 1 \right) \psi_x v_x \right]_x.\]

Multiplying by \(\sum_{j=0}^{K} (-1)^j \frac{\partial^j v}{\partial^j \psi_x}\), integrating over \([0, N]\), using integration by parts and rearranging yields a bound of the form\(^3\)

\[
\partial_t \|v\|^2_{H^N} + 2k \|v\|^2_{H^{K+1}} + \frac{1}{\varepsilon} \left\| \frac{\psi_t}{1 - \psi_x} \phi' \right\|^2_{H^{K+1}} + \frac{1}{\varepsilon} \left\| \frac{1}{1 - \psi_x} \partial_x \left( \frac{\psi_x}{1 - \psi_x} \phi' \right) \right\|^2_{H^{K+1}} \]
\[
+ \frac{1}{\varepsilon} \left\| \frac{\psi_x}{1 - \psi_x} f(\phi + v) \right\|^2_{H^{K+1}} + \frac{1}{\varepsilon} \left\| \frac{1}{1 - \psi_x} (f(\phi + v) - f(\phi)) \right\|^2_{H^{K+1}} \]
\[
+ \frac{1}{\varepsilon} \left\| v_x \right\|^2_{H^{K-1}} + \frac{1}{\varepsilon} \left\| \psi_t v_x \right\|^2_{H^{K-1}} + \frac{1}{\varepsilon} \left\| \frac{1}{1 - \psi_x} \partial_x \left[ \left( \frac{1}{1 - \psi_x} + 1 \right) \psi_x v_x \right] \right\|^2_{H^{K-1}}. \]

Using the Sobolev interpolation

\[\|g\|^2_{H^N} \leq \tilde{C}^{-1} \|\partial_x^{K+1} g\|^2_{L^2_N} + \tilde{C}' \|g\|^2_{L^2_N},\]

valid for some constant \(\tilde{C} > 0\) independent of \(N\), now gives

\[
\frac{d}{dt} \|v\|^2_{H^N}(t) \leq -\theta \|v(t)\|^2_{H^N} + C \left( \|v(t)\|^2_{L^2_N} + \|\psi_x\|^2_{H^N} + \|\psi_t\|^2_{H^N} + |\gamma_t(t)|^2 \right) .
\]

The proof is now complete by an application of Gronwall’s inequality. \(\square\)

4.2 Nonlinear Iteration

To complete the proof of Theorem 1.2, associated to the solution \((v, \gamma_t, \gamma_t, \gamma_x)\) of of (4.10)-(4.11) we define, so long as it is finite, the function

\[\zeta(t) := \sup_{0 \leq s \leq t} \left( \|v(s)\|^2_{H^N} + \|\psi_x(s)\|^2_{H^N} + \|\psi_t(s)\|^2_{H^N} + |\gamma_t(s)| \right)^{1/2} (1 + s)^{3/4}.\]

Combining the linear estimates in Proposition 3.1 with the damping estimate in Proposition 4.2, we now establish a key inequality for \(\zeta\) which will yield global existence and stability of our solutions.

\(^3\)Below, the symbol \(A \lesssim B\) implies there exists a constant \(C > 0\), independent of \(N\), such that \(A \leq CB\).
Proposition 4.3. Under the assumptions of Theorem 1.2, there exists positive constants $C, \varepsilon > 0$, both independent of $N$, such that if $v(\cdot, 0)$ is such that

$$E_0 := \|v(\cdot, 0)\|_{L_N^1 \cap H_N^k} \leq \varepsilon \quad \text{and} \quad \zeta(T) \leq \varepsilon$$

for some $T > 0$, then we have

$$\zeta(t) \leq C \left( E_0 + \zeta^2(t) \right)$$

valid for all $0 \leq t \leq T$.

Proof. Recalling Lemma 4.1 we readily see that there exists a constant $C > 0$, independent of $N$, such that

$$\|Q(t)\|_{L_N^1 \cap H_N^k} \leq C \left( 1 + \|\psi_x(t)\|_{H_N^k} \right) \|v(t)\|_{H_N^k}^2$$

and

$$\|R(t)\|_{L_N^1 \cap H_N^k} \leq C \left( \|v, v_x, \psi_x, \psi_xx, \psi_t\|_{H_N^k}^2 + |\gamma(t)|^2 \right)$$

so that, using the linear estimates in Proposition 3.1, we have for so long as $\zeta(t)$ remains small that

$$\|Q(t)\|_{L_N^1 \cap H_N^k}, \|R(t)\|_{L_N^1 \cap H_N^k} \leq C\zeta^2(t)(1 + t)^{-3/2}$$

for some constant $C > 0$ which is independent of $N$. Since $kN = Q + kR_x$, it follows there exists a constant $C > 0$ independent of $N$ such that

$$\|N(t)\|_{L_N^1 \cap H_N^k} \leq C \|v, v_x, v_xx, \psi_x, \psi_xx, \psi_t, \psi_tx\|_{H_N^k}^2 + |\gamma(t)|^2 \leq C\zeta^2(t)(1 + t)^{-3/2}.$$  

for so long as $\zeta(t)$ remains small. Applying the bounds in Proposition 3.1 to the implicit equation (4.11), it immediately follows that

$$\|v(t)\|_{L_N^2} \leq \|v(\cdot, t)\psi_x(t)\|_{L_N^2} + CE_0(1 + t)^{-3/4} + C \int_0^t (1 + t - s)^{-3/4} \|N(s)\|_{L_N^1 \cap L_N^2} ds$$

$$\leq \zeta(t)^2(1 + t)^{-3/2} + CE_0(1 + t)^{-3/4} + C\zeta(t)^2 \int_0^t (1 + t - s)^{-3/4}(1 + s)^{-3/2} ds$$

$$\leq C \left( E_0 + \zeta(t)^2 \right) (1 + t)^{-3/4}$$

for some constant $C > 0$ independent of $N$. In particular, observe the loss of derivatives in the above estimate: control of the $L_N^2$ norm of $v(t)$ requires control of the $H_N^k$ norm of $v(t)$. This loss of derivatives may be compensated by the nonlinear damping estimate in Proposition 4.2, assuming we can obtain appropriate estimates on the modulation functions and their derivatives.

To this end, we observe that by using (4.10) for $0 \leq \ell \leq K + 1$ we have that

$$\partial_x^\ell \psi_x(x, t) = \partial_x^{\ell+1} s_{p,N}(t)v(0) + \int_0^t \partial_s^{\ell+1} s_{p,N}(t-s) N(s) ds,$$

and for $0 \leq \ell \leq K$

$$\partial_x^\ell \psi_t(x, t) = \partial_x^{\ell} \partial_t [s_{p,N}](t)v(0) + \partial_x^{\ell} s_{p,N}(0) N(t) + \int_0^t \partial_s^{\ell} \partial_t [s_{p,N}](t-s) N(s) ds,$$

and hence that

$$\|\psi_x\|_{H_N^{K+1}}, \|\psi_t\|_{H_N^K} \leq C \left( E_0 + \zeta(t)^2 \right) (1 + t)^{-3/4}.$$
Similarly, using (4.10)(i) we find\footnote{Note here we use an $L^\infty$-$L^1$ bound to control the inner product. This is opposed to using Cauchy-Schwartz, which would contribute the growing factor $\|f_0\|_{L^2_N} = O(N)$.}

$$|\gamma(t)| = \left| \left\langle \Phi_0, \mathcal{N}(t) \right\rangle_{L^2_N} \right| \leq C\|\mathcal{N}(t)\|_{L^1_N} \leq C \left( E_0 + \zeta^2(t) \right) (1 + t)^{-3/2}. $$

Using the damping result in Proposition 4.2, we conclude that

$$\|v(t)\|_{H^2_N}^2 \leq CE_0^2 e^{-\theta t} + C \left( E_0 + \zeta^2(t) \right)^2 \int_0^t e^{-\theta(t-s)} (1 + s)^{-3/2} ds \leq CE_0^2 e^{-\theta t} + C \left( E_0 + \zeta^2(t) \right)^2 (1 + t)^{-3/2} \leq C \left( E_0 + \zeta^2(t) \right)^2 (1 + t)^{-3/2}. $$

(4.14)

Since $\zeta(t)$ is a non-decreasing function, it follows that for a given $t \in (0, T)$ we have

$$\left( \|v(s)\|_{H^2_N}^2 + \|\psi_x(s)\|_{H^2_N}^2 + \|\psi_t(s)\|_{H^2_N}^2 + |\gamma_t(s)|^2 \right)^{1/2} (1 + s)^{3/4} \leq C \left( E_0 + \zeta^2(t) \right)^2$$

valid for all $s \in (0, t)$. Taking the supremum over $s \in (0, t)$ completes the proof.

The proof of Theorem 1.2 now follows by continuous induction. Indeed, $\zeta(t)$ is continuous so long as it remains small, Proposition 4.3 implies that if $E_0 < \frac{1}{2\delta}$ then $\zeta(t) \leq 2CE_0$ for all $t \geq 0$. Noting that $C > 0$ is independent of $N$, this completes the proof of Theorem 1.2 by taking

$$\bar{\psi}(x, t) := \frac{1}{N} \gamma(t) + \psi(x, t). $$

Further, Corollary 1.3 follows by (4.14) and the triangle inequality since

$$\left\| u \left( \cdot - \frac{1}{N} \gamma(t), t \right) - \phi \right\|_{L^2_N} \leq \|u_x\|_{L^\infty} \|\psi(x, t)\|_{L^2_N} + CE_0(1 + t)^{-3/4} \leq CE_0 (1 + t)^{-1/4},$$

as claimed. Further, note that since for $0 < t < s$ we have

$$|\gamma(t) - \gamma(s)| \leq \int_t^s |\gamma'(z)|dz \leq CE_0(1 + t)^{-1/2}$$

it follows that $\gamma(t)$ converges to some\footnote{Note since the modulation function $\gamma$ depends on $N$, so does the limiting phase shift $\gamma_\infty$.} $\gamma_\infty \in \mathbb{R}$ as $t \to \infty$ with rate

$$|\gamma(t) - \gamma_\infty| \leq \int_t^\infty |\gamma'(z)|dz \leq CE_0(1 + t)^{-1/2},$$

which completes the proof of Corollary 1.3. Note, in fact, that from (4.10) and (4.13), we can identify $\gamma_\infty = \left\langle \Phi_0, v(0) \right\rangle_{L^2_N}$, emphasizing the dependence of the asymptotic phase shift on the initial data $v(0)$ and on $N$. Further, from (4.7) we see that $\gamma_\infty$ agrees exactly with the asymptotic phase shift suggested by the linear theory in Section 3.
A Bounds on Discrete Sums

In order to establish the uniform linear bounds in Proposition 3.1, we need to establish uniform-in-\(N\) bounds on finite sums of the form

\[
\frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \xi^{2r} e^{-2d\xi^2 t}
\]

where \(N \in \mathbb{N}\). Following the ideas in [4], we note that the above finite sum is, up to a simple rescaling, a Riemann sum approximation for the integral

\[
\int_{-\pi}^{\pi} \xi^{2r} e^{-2d\xi^2 t} \, d\xi
\]

which, through an elementary scaling argument, exhibits \((1 + t)^{-r-1/2}\) decay for large time. Using this as motivation, we now establish the following key estimate.

**Lemma A.1.** Let \(d > 0\) and \(r \in \mathbb{N} \cup \{0\}\) be given. Then there exists a constant \(C > 0\), independent of \(N\), such that for every \(N \in \mathbb{N}\) we have

\[
\frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \xi^{2r} e^{-2d\xi^2 t} \leq C (1 + t)^{-r-1/2},
\]

valid for all \(t \geq 0\).

**Proof.** First, consider the case when \(r = 0\) and note that, for each \(t > 0\), the function \(\xi \mapsto e^{-2d\xi^2 t}\) is even and monotonically decreasing for \(\xi > 0\). Together with the equality \(\xi_j - \xi_{j-1} = 2\pi/N\), monotonicity allows us to treat the sum over \(\xi \in \Omega_N\), \(\xi > 0\) as a right-endpoint Riemann sum (i.e. an under-approximation). Parity then tells us the sum over \(\xi \in \Omega_N\), \(\xi < 0\) is also an under-approximation, yielding

\[
\frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} e^{-2d\xi^2 t} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-2d\xi^2 t} \, d\xi \lesssim (1 + t)^{-1/2}.
\]

For \(r \geq 1\), the analysis is complicated by the fact that the function

\[
f(\xi, t) := \xi^{2r} e^{-2d\xi^2 t},
\]

defined for \(\xi \in \mathbb{R}\) and \(t > 0\), is not monotonically decreasing for \(\xi > 0\). However, we may use similar analysis via the following procedure.

First, observe that, for fixed \(t > 0\), \(f(\cdot, t)\) has a global minimum at 0 and global maxima at

\[
\pm R := \pm \left( \frac{r}{2d} \right)^{1/2} t^{-1/2}, \quad \text{with} \quad f(\pm R, t) = \left( \frac{r}{2de} \right)^{r} t^{-r}.
\]

If \(0 < t \leq r/(2d\pi^2)\), then \(R \geq \pi\) so that \(\pm R \notin (-\pi, \pi)\). We can then easily estimate the sum

\[
(A.1) \quad \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \xi^{2r} e^{-2d\xi^2 t} \leq \frac{1}{N} \sum_{\xi \in \Omega_N \setminus \{0\}} \pi^{2r} \leq \pi^{2r}.
\]
For \( t > r/(2d\pi^2) \), we define the auxiliary function

\[
G(\xi, t) := \begin{cases} 
  f(R, t), & |\xi| \leq R \\
  f(\xi, t), & |\xi| > R 
\end{cases}
\]

Notice that \( G(\cdot, t) \) is even and monotonically decreasing for \( \xi > 0 \). Furthermore, notice that

\[
\int_{-\pi}^{\pi} G(\xi, t) \, d\xi \leq 2e^{1/2} \left( \frac{r}{2de} \right)^{r+1/2} t^{-r-1/2} + \int_{-\pi}^{\pi} f(\xi, t) \, d\xi \lesssim (1 + t)^{-r-1/2},
\]

where the last inequality follows from (A.1). Consequently, we may modify the monotonicity trick from the \( r = 0 \) case to obtain

\[
\frac{1}{N} \sum_{\xi \in \Omega \setminus \{0\}} \xi^2 e^{-2d\xi^2 t} \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} G(\xi, t) \, d\xi \lesssim (1 + t)^{-r-1/2}.
\]

In [4], the authors further established that, in the cases \( r = 0 \) and \( r = 1 \), the decay rate in Lemma A.1 is indeed sharp, providing also a uniform lower bound for the corresponding finite sums. A similar analysis applied to the present situation establishes the sharpness of these bounds for all \( r \geq 0 \). While not necessary in the present analysis, it provides yet a deeper connection between the current uniform analysis of subharmonic perturbations and the “limiting” localized theory.

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