An asymptotically superlinearly convergent semismooth Newton augmented Lagrangian method for Linear Programming

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Abstract

Powerful interior-point methods (IPM) based commercial solvers such as Gurobi and Mosek have been hugely successful in solving large-scale linear programming (LP) problems. The high efficiency of these solvers depends critically on the sparsity of the problem data and advanced matrix factorization techniques. For a large scale LP problem with data matrix $A$ that is dense (possibly structured) or whose corresponding normal matrix $AA^T$ has a dense Cholesky factor (even with re-ordering), these solvers may require excessive computational cost and/or extremely heavy memory usage in each interior-point iteration. Unfortunately, the natural remedy, i.e., the use of iterative methods based IPM solvers, although can avoid the explicit computation of the coefficient matrix and its factorization, is not practically viable due to the inherent extreme ill-conditioning of the large scale normal equation arising in each interior-point iteration. To provide a better alternative choice for solving large scale LPs with dense data or requiring expensive factorization of its normal equation, we propose a semismooth Newton based inexact proximal augmented Lagrangian (SNIPAL) method. Different from classic IPMs, in each iteration of SNIPAL, iterative methods can efficiently be used to solve simpler yet better conditioned semismooth Newton linear systems. Moreover, SNIPAL not only enjoys a fast asymptotic superlinear convergence but is also proven to enjoy a finite termination property. Numerical comparisons with Gurobi have demonstrated encouraging potential of SNIPAL for handling large-scale LP problems where the constraint matrix $A$ has a dense representation or $AA^T$ has a dense factorization even with an appropriate re-ordering. For a few large LP instances arising from the correlation clustering, our algorithm can be up to $25 - 140$ times faster than Gurobi in solving the problems to the high accuracy of $10^{-8}$ in the relative KKT residual.

Keywords: Linear programming, semismooth Newton method, augmented Lagrangian method

AMS subject classifications: 90C05, 90C06, 90C25, 65F10

1 Introduction

It is well known that primal-dual interior-point methods (IPMs) as implemented in highly optimized commercial solvers such as Gurobi and Mosek are powerful methods for solving large scale

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linear programming (LP) problems with conducive sparsity. However, the large scale normal equation arising in each interior-point iteration is generally highly ill-conditioned (especially when the barrier parameter tends to zero) and typically it is necessary to employ a direct method such as the sparse Cholesky factorization to solve the equation stably and accurately. Various attempts, for examples in [6, 11, 18, 28], have been made in using an iterative solver such as the preconditioned conjugate-gradient (PCG) method to solve the normal equation when it is too expensive to compute the coefficient matrix or the sparse Cholesky factorization because of excessive computing time or memory usage due to fill-ins. For more details on the numerical performance of iterative methods based IPMs for solving large scale LP, we refer the readers to [11] and the references therein. However, the extreme ill-conditioning of the normal equation (and also of the augmented equation) makes it extremely costly for an iterative method to solve the equation either because it takes excessive number of steps to converge or because constructing an effective preconditioner is prohibitively expensive. Thus far, iterative methods based IPMs have not been proven convincingly to be more efficient in general than the highly powerful solvers such as Gurobi and Mosek on various large scale LP test instances.

The goal of this paper is to design a semismooth Newton inexact proximal augmented Lagrangian (Snipal) method for solving large scale LP problems, which has the following key properties: (a) The Snipal method can achieve fast local linear convergence; (b) The semismooth Newton equation arising in each iteration can fully exploit the solution sparsity in addition to data sparsity; (c) The semismooth Newton equation is typically much better conditioned than its counterparts in IPMs, even when the iterates approach optimality. The latter two properties thus make it cost effective for one to use an iterative method such as the PCG method to solve the aforementioned linear system when it is large. It is these three key properties that give the competitive advantage of our Snipal method over the highly developed IPMs for solving certain classes of large scale LP problems which we will describe shortly.

Consider the following primal and dual LP problems:

\[(P) \quad \min \left\{ c^T x + \delta_K(x) \mid Ax = b, \ x \in \mathbb{R}^n \right\} \]

\[(D) \quad \max \left\{ -\delta_K(A^*y - c) + b^T y \mid y \in \mathbb{R}^m \right\} \]

where \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n \) are given data. The set \( K = \{ x \in \mathbb{R}^n \mid l \leq x \leq u \} \) is a simple polyhedral set, where \( l, u \) are given vectors. We allow the components of \( l \) and \( u \) to be \(-\infty\) and \( \infty \), respectively. In particular, \( K \) can model the nonnegative orthant \( \mathbb{R}^n_+ \). In the above, \( \delta_K(\cdot) \) denotes the indicator function over the set \( K \) such that \( \delta_K(x) = 0 \) if \( x \in K \) and \( \delta_K(x) = \infty \) otherwise. The Fenchel conjugate of \( \delta_K \) is denoted by \( \delta_K^*(\cdot) \). We note that while we focus on the indicator function \( \delta_K(\cdot) \) in (P), the algorithm and theoretical results we have developed in this paper are also applicable when \( \delta_K \) is replaced by a closed convex polyhedral function \( p : \mathbb{R}^n \to (-\infty, \infty] \). We made the following assumption on the problems (P) and (D).

**Assumption 1.** The solution set of (P) and (D) is nonempty and \( A \) has full row rank (hence \( m \leq n \)).

Our Snipal method is designed for the dual LP but the primal variable is also generated in each iteration. In order for the fast local convergence property to kick-in early, we warm-start the Snipal method by an alternating direction method of multipliers (ADMM), which is also applied to the dual LP. We should mention that our goal is not to use Snipal as a general purpose solver for LP but to complement the excellent general solvers (Gurobi and Mosek) when the latter are
too expensive or have difficulties in solving very large scale problems due to memory limitation. In particular, we are interested in solving large scale LP problems having one of the following characteristics.

1. The number of variables \( n \) in (P) is significantly larger than the number of linear constraints \( m \). We note that such a property is not restrictive since for a primal problem with a huge number of inequality constraints \( Ax \leq b \) and \( m \gg n \), we can treat the dual problem (D) as the primal LP, and the required property is satisfied.

2. The constraint matrix \( A \) is large and dense but it has an economical representation such as being the Kronecker product of two matrices, or \( A \) is sparse but \( AA^T \) has a dense factorization even with an appropriate re-ordering. For such an LP problem, it may not be possible to solve it by using the standard interior-point methods as implemented in Gurobi or Mosek since \( A \) cannot be stored explicitly. Instead, one would need to use a Krylov subspace iterative method to solve the underlying large and dense linear system of equations arising in each iteration of an IPM or SNIPAL.

In \[39\], Wright proposed an algorithm for solving the primal problem (P) for the special case where \( K = \mathbb{R}^n_+ \). The proposed method is in fact the proximal method of multipliers applied to (P) while keeping the nonnegative constraint in the quadratic programming (QP) subproblem. More specifically, suppose that the iterate at the \( k \)th iteration is \((x^k, y^k)\) and the penalty parameter is \( \gamma_k = \sigma_k^{-1} \). Then the QP subproblem is given by \( \min \left\{ \frac{1}{2}((\sigma_k A^* A + \sigma_k^{-1} I_n)x, x) + \langle x, c - A^* y^k - \sigma_k^{-1} x^k - \sigma_k A^* b \rangle \mid x \geq 0 \right\} \). In \[39\], an SOR (successive over-relaxation) method is used to solve the QP subproblem. But it is unclear how this subproblem can be solved efficiently when \( n \) is large. In contrast, in this paper, we propose a semismooth Newton based inexact proximal augmented Lagrangian (SNIPAL) method that is applied to the dual problem (D) and the associated subproblems are solved efficiently by a semismooth Newton method with guaranteed quadratic convergence.

In the pioneering work of De Leone and Mangasarian \[25\], an augmented Lagrangian method is applied to an equivalent reformulation of (D), and the QP subproblem of the form \( \min \left\{ -b^T y + \frac{1}{2} \| A^* y + z - c + \sigma^{-1} x^k \|_2^2 \mid y \in \mathbb{R}^m, z \geq 0 \right\} \) in each iteration is solved by a projected SOR method. Interestingly, in a later paper \[27\], based on the results obtained in \[26\], Mangasarian designed a generalized Newton method to first solve a penalty problem of the form \( \min \left\{ -eb^T y + \frac{1}{2} \| \Pi_{\mathbb{R}^n_+} (A^* y - c) \|_2^2 \right\} \) and then use its solution to indirectly solve (P), for \( K = \mathbb{R}^n_+ \), under the condition that the positive parameter \( \epsilon \) must be below a certain unknown threshold and a strong uniqueness condition holds. Soon after, \[14\] observed that the restriction on the parameter in \[27\] can be avoided by modifying the procedure in \[27\] via the augmented Lagrangian method but the corresponding subproblem in each iteration must be solved exactly. As the generalized Newton system is likely to be singular, in both \[27\] and \[14\], the system is modified by adding a scalar multiple of the identity matrix to the generalized Hessian. Such a perturbation, however, would destroy the fast local convergence property of the generalized Newton method. We also note that to obtain the minimum norm solution of primal problem (P), \[20\] proposed a generalized Newton method for solving \( \min \left\{ \frac{1}{2} \| \Pi_{\mathbb{R}^n_+} (A^* y - rc) \|_2^2 - \langle b, y \rangle \right\} \) with the positive parameter \( r \) being sufficiently large. Although \[20\] contains no computational results, the authors obtained the global convergence and finite termination properties of the proposed method under the assumption that the involved Newton linear systems are solved exactly and certain regularity condition on the nonsingularity of generalized Jacobians holds. More recently, \[41\] designed an ALM for the primal problem (P) for which a bound-constrained convex QP subproblem must be solved in each iteration. In the paper, this
subproblem is solved by a randomized coordinate descent (RCD) method with an active set implementation. There are several drawbacks to this approach. First, solving the QP subproblem can be very time consuming. Second, the RCD approach can hardly exploit any specific structure of the matrix $A$ to speed up the computation of the QP subproblem. Finally, it also does not exploit the sparsity structure presented in the Hessian of the underlying QP subproblem to speed up the computation.

Here, we employ the an inexact proximal augmented Lagrangian (PAL) method to (D) to simultaneously solve (P) and (D). Our entire algorithmic design is dictated by the focus on computational efficiency and generality. From this perspective, now we elaborate on the key differences between our paper and \cite{14}. First, our algorithm is applicable to problems with a more general set $K$ instead of just $\mathbb{R}^n$ as in \cite{27} and \cite{14}. Second, we use the inexact PAL framework which ensures that in each iteration, an unconstrained minimization subproblem involving the variable $y$ is strongly convex and hence the semismooth Newton method we employ to solve this subproblem will have local quadratic convergence. Third, the flexibility of allowing the PAL subproblems to be solved inexactly can lead to substantial computing savings, especially during the initial phase of the algorithm. Fourth, for computational efficiency, we warm-start our inexact PAL method by using a first-order method. Finally, as solving the semismooth Newton linear systems is the most critical component of the entire algorithm, we have devoted a substantial part of the paper on proposing novel numerical strategies to solve the linear systems efficiently.

**Notation.** We use $\mathcal{X}$ and $\mathcal{Y}$ to denote finite dimensional real Euclidean spaces each endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. For any self-adjoint positive semidefinite linear operator $\mathcal{M} : \mathcal{X} \to \mathcal{X}$, we define $\langle x, x' \rangle_\mathcal{M} := \langle x, \mathcal{M}x' \rangle$ and $\|x\|_\mathcal{M} := \sqrt{\langle x, \mathcal{M}x \rangle}$ for all $x, x' \in \mathcal{X}$. The largest eigenvalue of $\mathcal{M}$ is denoted by $\lambda_{\max}(\mathcal{M})$. A similar notation is used when $\mathcal{M}$ is replaced by a matrix $M$. Let $D$ be a given subset of $\mathcal{X}$. We write the weighted distance of $x \in \mathcal{X}$ to $D$ by $\text{dist}_\mathcal{M}(x, D) := \inf_{x' \in D} \|x - x'\|_\mathcal{M}$. If $\mathcal{M}$ is the identity operator, we just omit it from the notation so that $\text{dist}(\cdot, D)$ is the Euclidean distance function. If $D$ is closed, the Euclidean projector over $D$ is defined by $\Pi_D(x) := \text{argmin}\{\|x - d\| \mid d \in D\}$. Let $F : \mathcal{X} \rightrightarrows \mathcal{Y}$ be a multivalued mapping. We define the graph of $F$ to be the set $\text{gph}F := \{(x, y) \in \mathcal{X} \times \mathcal{Y} \mid y \in F(x)\}$. The range of a multifunction is defined by $\text{Range}(F) := \{y \mid \exists x \text{ with } y \in F(x)\}$.\n
**2 A preconditioned proximal point algorithm**

In this section, we present a preconditioned proximal point algorithm (PPA) and study its convergence properties. In particular, following the classic framework developed in \cite{34, 35}, we prove the global convergence of the preconditioned PPA. Under a mild error bound condition, global linear rate convergence is also derived. In fact, by choosing the parameter $c_k$ in the algorithm to be sufficiently large, the linear rate can be as fast as we please. We further show in Section 3.1 that our main Algorithm SNIPAL is in fact an application of the preconditioned proximal point algorithm. Hence, SNIPAL’s convergence properties can be obtained as a direct application of the general theory developed here.

Let $\mathcal{X}$ and $\mathcal{Y}$ be finite dimensional Hilbert spaces and $T : \mathcal{X} \to \mathcal{X}$ be a maximal monotone operator. Throughout this section, we assume that $\Omega := T^{-1}(0)$ is nonempty. We further note from \cite{36} Exercice 12.8 that $\Omega$ is a closed set. The preconditioned proximal point algorithm generates for any start point $z^0 \in \mathcal{X}$ a sequence $\{z^k\} \subseteq \mathcal{X}$ by the following approximate rule:

$$z^{k+1} \approx P_k(z^k), \quad \text{where} \quad P_k = (M_k + c_k T)^{-1} M_k. \quad (1)$$
Here \( \{c_k\} \) and \( \{M_k\} \) are some sequences of positive real numbers and self-adjoint positive definite linear operators over \( \mathcal{X} \). If \( M_k = \mathcal{I} \) for all \( k \geq 0 \), the updating scheme (1) recovers the classical proximal point algorithm considered in [34]. Since \( M_k + c_k \mathcal{T} \) is a strongly monotone operator, we know from [36, Proposition 12.54] that \( P_k \) is single-valued and is Lipschitz continuous globally.

Here, we further assume that \( \{c_k\} \) bounded away from zero and

\[
M_k \geq M_{k+1}, \quad M_k \geq \lambda_{\min} \mathcal{I} \quad \forall k \geq 0
\]

with some constant \( \lambda_{\min} > 0 \). Therefore, we know that for all \( k \geq 0 \), \( \lambda_{\max}(M_k) \geq \lambda_{\max}(M_{k+1}) \) and there exists a positive number \( \lambda_\infty \) such that

\[
\lim_{k \to \infty} \lambda_{\max}(M_k) = \lambda_\infty \geq \lambda_{\min} > 0.
\]  

(2)

Note that if \( \mathcal{T} \) is a linear operator, one may rewrite \( P_k \) as \( P_k = (\mathcal{I} + c_k M_k^{-1} \mathcal{T})^{-1} \). We show in the next lemma that this expression in fact holds even for a general maximal monotone operator \( \mathcal{T} \). Therefore, we can regard the self-adjoint positive definite linear operator \( M_k \) as a preconditioner for the maximal monotone operator \( \mathcal{T} \). Based on this observation, we name the algorithm described in (1) as the preconditioned proximal point algorithm.

**Lemma 1.** Given a constant \( \alpha > 0 \), a self-adjoint positive definite linear operator \( M \) and a maximal monotone operator \( \mathcal{T} \) on \( \mathcal{X} \), it holds that \( \text{Range}(\mathcal{I} + \alpha M^{-1} \mathcal{T}) = \mathcal{X} \) and \( (\mathcal{I} + \alpha M^{-1} \mathcal{T})^{-1} \) is a single-valued mapping. In addition,

\[
(\mathcal{M} + \alpha \mathcal{T})^{-1} \mathcal{M} = (\mathcal{I} + \alpha M^{-1} \mathcal{T})^{-1}.
\]

**Proof.** We first argue that \( \text{Range}(\mathcal{I} + \alpha M^{-1} \mathcal{T}) = \mathcal{X} \). For any \( z \in \mathcal{X} \), let \( x = (\mathcal{M} + \alpha \mathcal{T})^{-1} \mathcal{M} z \in \mathcal{X} \). We know that \( \mathcal{M} z \in (\mathcal{M} + \alpha \mathcal{T}) x \), and consequently, \( z \in M^{-1}(\mathcal{M} + \alpha \mathcal{T}) x \), i.e., \( z \in (\mathcal{I} + \alpha M^{-1} \mathcal{T}) x \). Therefore, \( \text{Range}(\mathcal{I} + \alpha M^{-1} \mathcal{T}) = \mathcal{X} \).

Next, we show that \( (\mathcal{I} + \alpha M^{-1} \mathcal{T})^{-1} \) is a single-valued mapping from all of \( \mathcal{X} \) to itself. For any given \( z \in \mathcal{X} \), suppose that \( z_1, z_2 \in (\mathcal{I} + \alpha M^{-1} \mathcal{T})^{-1}(z) \). Then, it holds that

\[
z \in (\mathcal{I} + \alpha M^{-1} \mathcal{T}) z_1 \quad \text{and} \quad z \in (\mathcal{I} + \alpha M^{-1} \mathcal{T}) z_2,
\]

and consequently,

\[
\mathcal{M} z \in (\mathcal{M} + \alpha \mathcal{T}) z_1 \quad \text{and} \quad \mathcal{M} z \in (\mathcal{M} + \alpha \mathcal{T}) z_2.
\]

Since \( (\mathcal{M} + \alpha \mathcal{T})^{-1} \) is a single-valued operator [36, Proposition 12.54], we know that

\[
z_1 = (\mathcal{M} + \alpha \mathcal{T})^{-1} \mathcal{M} z = z_2,
\]

i.e., \( (\mathcal{I} + \alpha M^{-1} \mathcal{T})^{-1} \) is single-valued and \( (\mathcal{I} + \alpha M^{-1} \mathcal{T})^{-1} z = (\mathcal{M} + \alpha \mathcal{T})^{-1} \mathcal{M} z \) for all \( z \in \mathcal{X} \). Therefore, we have the desired equation and thus complete the proof of this lemma. \( \square \)

In the literature, the updating scheme (1) is closely related to the so-called “variable metric proximal point algorithms”; see for examples [2] [30] [9] [4] [3] [5] [29]. Among these papers, [2] [30] [9] focus only on the case of optimization, i.e., the maximal monotone operator \( \mathcal{T} \) is the subdifferential mapping of a convex function. In addition, they emphasize more on the combination of the proximal point algorithm with quasi Newton method. Meanwhile, [4] and the subsequent papers [3] [5], are concerned about the general maximal monotone operator \( \mathcal{T} \) and study the following scheme in the exact setting:

\[
z^{k+1} = z^k + M_k ((\mathcal{I} + c_k \mathcal{T})^{-1} - \mathcal{I}) z^k.
\]

(3)
The global convergence of the scheme (3) requires a rather restrictive assumption about $M_k$ \cite{Snipal}, although $M_k$ is not required to be self-adjoint. In fact, the authors essentially assumed that the deviation of $M_k$ from the identity operator should be small, and the verification of the assumption can be quite difficult. As far as we are aware of, \cite{29} may be the most related work to ours. In \cite{29}, the authors consider a variable metric hybrid inexact proximal point method whose updating rule consists of an inexact proximal step and a projection step. Moreover, some specially designed stopping criteria for the inexact solution of the proximal subproblem are also used. However, due to the extra projection step, the connection between their algorithm and the proximal method of multipliers \cite{4} is no longer available. Therefore, the results derived in \cite{29} cannot be directly used to analyze the convergence properties of Snipal proposed in this paper, which is a variant of the proximal method of multipliers. Since the scheme (1) under the classical setting of \cite{34, 35} fits our context best, we conduct a comprehensive examination of its convergence properties which, to our best knowledge, are currently not available in the literature.

For all $k \geq 0$, define mapping $Q_k := I - P_k$. Clearly, if $0 \in T(z)$, we have that $P_k(z) = z$ and $Q_k(z) = 0$ for all $k \geq 0$. Similar to \cite{34} Proposition 1], we summarize the properties of $P_k$ and $Q_k$ in the following proposition:

**Proposition 1.** It holds for all $k \geq 0$ that:

(a). $z = P_k(z) + Q_k(z)$ and $c_k^{-1}M_kQ_k(z) \in T(P_k(z))$ for all $z \in X$;

(b). $(P_k(z) - P_k(z'), Q_k(z) - Q_k(z'))M_k \geq 0$ for all $z, z' \in X$;

(c). $\|P_k(z) - P_k(z')\|^2_{M_k} + \|Q_k(z) - Q_k(z')\|^2_{M_k} \leq \|z - z'\|^2_{M_k}$ for all $z, z' \in X$.

**Proof.** The proof can be obtained via simple calculations and is similar to the proof of \cite{34} Proposition 1]. We omit the details here. \hfill $\square$

We list the following two general criteria for the approximate calculation of $P_k(z^k)$ which are analogous to those proposed in \cite{34}:

\begin{align*}
\text{A) } & \|z^{k+1} - P_k(z^k)\|_{M_k} \leq \epsilon_k, \quad 0 \leq \epsilon_k, \quad \sum_{k=0}^{\infty} \epsilon_k < \infty, \\
\text{B) } & \|z^{k+1} - P_k(z^k)\|_{M_k} \leq \delta_k\|z^{k+1} - z^k\|_{M_k}, \quad 0 \leq \delta_k < 1, \quad \sum_{k=0}^{\infty} \delta_k < \infty.
\end{align*}

**Theorem 1.** Suppose that $\Omega = T^{-1}(0) \neq \emptyset$. Let $\{z^k\}$ be any sequence generated by the mPPA (1) under criterion (A). Then $\{z^k\}$ is bounded and

$$\text{dist}_{M_k+1}(z^{k+1}, \Omega) \leq \text{dist}_{M_k}(z^k, \Omega) + \epsilon_k \quad \forall k \geq 0. \tag{4}$$

In addition, $\{z^k\}$ converges to a point $z^\infty$ satisfy $0 \in T(z^\infty)$.

**Proof.** Let $\bar{z} \in X$ be a point satisfying $0 \in T(\bar{z})$. We have

$$\|z^{k+1} - \bar{z}\|_{M_k} - \epsilon_k \leq \|P_k(z^k) - \bar{z}\|_{M_k} = \|P_k(z^k) - P_k(\bar{z})\|_{M_k} \leq \|z^k - \bar{z}\|_{M_k}.$$

Since $M_k \succeq M_{k+1}$, we know that

$$\|z^{k+1} - \bar{z}\|_{M_{k+1}} - \epsilon_k \leq \|z^{k+1} - \bar{z}\|_{M_k} - \epsilon_k \leq \|z^k - \bar{z}\|_{M_k}. \tag{5}$$
Let $\Pi_\Omega(z)$ denote the projection of $z$ onto $\Omega$. By noting that $0 \in \mathcal{T}(\Pi_\Omega(z^k))$, we get from the above inequality (by setting $\bar{z} = \Pi_\Omega(z^k)$) that
\[
\text{dist}_{\mathcal{M}_{k+1}}(z^{k+1}, \Omega) - \epsilon_k \leq \|z^{k+1} - \Pi_\Omega(z^k)\|_{\mathcal{M}_{k+1}} - \epsilon_k
\]
\[
\leq \|z^k - \Pi_\Omega(z^k)\|_{\mathcal{M}_k} = \text{dist}_{\mathcal{M}_k}(z^k, \Omega).
\]
Since $\sum_{k=0}^{\infty} \epsilon_k < \infty$, (5) further implies
\[
\lim_{k \to \infty} \|z^k - \bar{z}\|_{\mathcal{M}_k} = \lim_{k \to \infty} \|z^{k+1} - \bar{z}\|_{\mathcal{M}_k} = \mu < \infty,
\]
and consequently, $\lim_{k \to \infty} \|\mathcal{P}(z^k) - \bar{z}\|_{\mathcal{M}_k} = \mu$. The boundedness of $\{z^k\}$ thus follows directly from the fact that $\mathcal{M}_k \geq \lambda_{\text{min}}\mathcal{I}$ for all $k \geq 0$. Therefore, $\{z^k\}$ has at least one cluster point $z^\infty$.

From Proposition 1, we know that for all $k \geq 0$
\[
0 \leq \|\mathcal{Q}_k(z^k)\|_{\mathcal{M}_k}^2 \leq \|z^k - \bar{z}\|_{\mathcal{M}_k}^2 - \|\mathcal{P}_k(z^k) - \bar{z}\|_{\mathcal{M}_k}^2.
\]
Therefore, $\lim_{k \to \infty} \|\mathcal{Q}_k(z^k)\|_{\mathcal{M}_k}^2 = 0$. It follows that
\[
\lim_{k \to \infty} c_k^{-1} \mathcal{M}_k \mathcal{Q}_k(z^k) = \lim_{k \to \infty} \mathcal{Q}_k(z^k) = 0,
\]
because the number $c_k$ is bounded away from zero and $\mathcal{M}_k \geq \lambda_{\text{min}}\mathcal{I}$ for all $k \geq 0$. Since
\[
\|\mathcal{Q}_k(z^k)\|_{\mathcal{M}_k} = \|(z^k - z^{k+1}) + (z^{k+1} - \mathcal{P}_k(z^k))\|_{\mathcal{M}_k} \geq \|z^k - z^{k+1}\|_{\mathcal{M}_k} - \epsilon_k,
\]
we further have $\lim_{k \to \infty} \|z^k - z^{k+1}\| = 0$.

Since $z^\infty$ is a cluster point of $z^k$ and
\[
\lim_{k \to \infty} \|\mathcal{P}_k(z^k) - z^{k+1}\| = \lim_{k \to \infty} \|z^{k+1} - z^k\| = 0,
\]
$z^\infty$ is also a cluster point of $\mathcal{P}_k(z^k)$. From Proposition 1 (a), we have that for any $w \in \mathcal{T}(z)$
\[
0 \leq \langle z - \mathcal{P}_k(z^k), w - c_k^{-1} \mathcal{M}_k \mathcal{Q}_k(z^k) \rangle \quad \forall k \geq 0,
\]
which, together with (6), implies
\[
0 \leq \langle z - z^\infty, w \rangle \quad \forall z, w \text{ satisfying } w \in \mathcal{T}(z).
\]
From the maximality of $\mathcal{T}$, we know that $0 \in \mathcal{T}(z^\infty)$. Hence, we can replace $\bar{z}$ in (6) by $z^\infty$. Therefore,
\[
\lim_{k \to \infty} \|z^k - z^\infty\|_{\mathcal{M}_k} = 0.
\]
That is $\lim_{k \to \infty} z^k = z^\infty$. \hfill \Box

Next, we study the convergence rate of the preconditioned proximal point algorithm. The following error bound assumption associated with $\mathcal{T}$ is critical to the study of the convergence rate of the preconditioned PPA.

**Assumption 2.** For any $r > 0$, there exists $\kappa > 0$ such that
\[
\text{dist}(x, \mathcal{T}^{-1}(0)) \leq \kappa \text{dist}(0, \mathcal{T}(x)) \quad \forall x \in \mathcal{X} \text{ satisfying } \text{dist}(x, \mathcal{T}^{-1}(0)) \leq r.
\]
In Rockafellar’s classic work [34], the asymptotic Q-superlinear convergence of PPA is established under the assumption that \( \mathcal{T}^{-1} \) is Lipschitz continuous at zero. Note that the Lipschitz continuity assumption on \( \mathcal{T}^{-1} \) is rather restrictive, since it implicitly implies that \( \mathcal{T}^{-1}(0) \) is a singleton. In [24], Luque extended Rockafellar’s work by considering the following relaxed condition over \( \mathcal{T} \): there exist \( \gamma > 0 \) and \( \epsilon > 0 \) such that
\[
\text{dist}(x, \mathcal{T}^{-1}(0)) \leq \gamma \text{dist}(0, \mathcal{T}(x)) \quad \forall \; x \in \{ x \in \mathcal{X} \mid \text{dist}(0, \mathcal{T}(x)) < \epsilon \}.
\]
We show in the following lemma that this condition in fact implies Assumption 2. Thus, our Assumption 2 is quite mild and weaker than condition (10).

**Lemma 2.** Let \( F \) be a multifunction from \( \mathcal{X} \) to \( \mathcal{Y} \) with \( F^{-1}(0) \neq \emptyset \). If \( F \) satisfies condition (10), then Assumption 2 holds for \( F \), i.e., for any \( r > 0 \), there exists \( \kappa > 0 \) such that
\[
\text{dist}(x, F^{-1}(0)) \leq \kappa \text{dist}(0, F(x)) \quad \forall \; x \in \mathcal{X} \text{ satisfying } \text{dist}(x, F^{-1}(0)) \leq r.
\]

**Proof.** Since \( F \) satisfies condition (10), there exist \( \epsilon > 0 \) and \( \kappa_0 \geq 0 \) such that if \( x \in \mathcal{X} \) satisfies \( \text{dist}(0, F(x)) < \epsilon \), then
\[
\text{dist}(x, F^{-1}(0)) \leq \kappa_0 \text{dist}(0, F(x)).
\]
For any \( r > 0 \) and \( x \) satisfying \( \text{dist}(x, F^{-1}(0)) \leq r \), if \( \text{dist}(0, F(x)) < \epsilon \), then \( \text{dist}(x, F^{-1}(0)) \leq \kappa_0 \text{dist}(0, F(x)) \); otherwise if \( \text{dist}(0, F(x)) \geq \epsilon \), then
\[
\text{dist}(0, F(x)) \geq \epsilon \geq \frac{\epsilon}{r \kappa_0} \text{dist}(x, F^{-1}(0)),
\]
i.e., \( \text{dist}(x, F^{-1}(0)) \leq \frac{r}{\epsilon} \text{dist}(0, F(x)) \). Therefore, the desired inequality holds for \( \kappa = \max\{\kappa_0, \frac{r}{\epsilon}\} \). \( \square \)

**Remark 1.** In fact, condition (10) is exactly the local upper Lipschitz continuity of \( \mathcal{T}^{-1} \) at the origin which was introduced by Robinson in [31]. Later, Robinson established in [32] the celebrated result that every polyhedral multifunction is locally upper Lipschitz continuous, i.e., satisfies condition (10). Thus from Lemma 2, we know that any polyhedral multifunction \( F \) with \( F^{-1}(0) \neq \emptyset \) satisfies Assumption 2.

Since the nonnegative sequence \( \{\epsilon_k\} \) in the stopping criterion (A) is summable, we can choose \( r \) to be a positive number satisfying \( r > \sum_{i=0}^{\infty} \epsilon_k \). Assume that \( \mathcal{T} \) satisfies Assumption 2 then associated with \( r \), there exists a positive constant \( \kappa \) such that (9) holds. With these preparations, we prove in the following theorem the asymptotic Q-superlinear (R-superlinear) convergence of the weighted (unweighted) distance between the sequence generated by the preconditioned PPA and \( \Omega \).

**Theorem 2.** Suppose that \( \Omega \neq \emptyset \) and the initial point \( z^0 \) satisfies \( \text{dist}_{\mathcal{M}_0}(z^0, \Omega) \leq r - \sum_{i=0}^{\infty} \epsilon_k \). Let \( \{z^k\} \) be the infinite sequence generated by the preconditioned PPA under criteria (A) and (B) with \( \{c_k\} \) nondecreasing (\( c_k \uparrow c_{\infty} \leq \infty \)). Then for all \( k \geq 0 \), it holds that
\[
\text{dist}_{\mathcal{M}_{k+1}}(z^{k+1}, \Omega) \leq \mu_k \text{dist}_{\mathcal{M}_k}(z^k, \Omega),
\]
where \( \mu_k = (1 - \delta_k)^{-1}(\delta_k + (1 + \delta_k)\kappa \lambda_{\max}(\mathcal{M}_k)/\sqrt{c_k^2 + \kappa^2 \lambda_{\max}^2(\mathcal{M}_k)}) \) and
\[
\lim_{k \to \infty} \mu_k = \mu_{\infty} = \frac{\kappa \lambda_{\infty}}{\sqrt{c_{\infty}^2 + \kappa \lambda_{\infty}^2}} < 1 \quad (\mu_{\infty} = 0 \text{ if } c_{\infty} = \infty)
\]
with \( \lambda_{\infty} \) given in (2). In addition, one has that for all \( k \geq 0 \),
\[
\text{dist}(z^{k+1}, \Omega) \leq \frac{\mu_k}{\lambda_{\min}(\mathcal{M}_{k+1})} \text{dist}_{\mathcal{M}_k}(z^k, \Omega).
\]
Proof. From (4) in Theorem 1, we know that for all \( k \geq 0 \), \( \text{dist}_{M_k}(z^k, \Omega) \leq \text{dist}_{M_0}(z^0, \Omega) + \sum_{i=0}^{\infty} \epsilon_k \leq r \), and consequently,

\[
\text{dist}_{M_k}(\mathcal{P}_k(z^k), \Omega) \leq \|\mathcal{P}_k(z^k) - \Pi_\Omega(z^k)\|_{M_k} \leq \text{dist}_{M_k}(z^k, \Omega) \leq r \quad \forall k \geq 0.
\]

From Proposition \( \Box \) (a), we have

\[
c_k^{-1}M_k Q_k(z^k) \in \mathcal{T}(\mathcal{P}_k(z^k)),
\]

which, together with Assumption \( \Box \) implies that for all \( k \geq 0 \)

\[
\text{dist}(\mathcal{P}_k(z^k), \Omega) \leq \kappa c_k^{-1} \|M_k Q_k(z^k)\|.
\]

It further implies that for all \( k \geq 0 \),

\[
\frac{1}{\sqrt{\lambda_{\text{max}}(M_k)}} \text{dist}_{M_k}(\mathcal{P}_k(z^k), \Omega) \leq \text{dist}(\mathcal{P}_k(z^k), \Omega) \leq \sqrt{\lambda_{\text{max}}(M_k) \kappa c_k^{-1} \|Q_k(z^k)\|}. \tag{14}
\]

Now taking \( \bar{z} = \Pi_\Omega(z^k) \), we deduce from (7) that for all \( k \geq 0 \),

\[
\begin{align*}
\|Q_k(z^k)\|_{\text{dist}_{M_k}}^2 & \leq \|z^k - \Pi_\Omega(z^k)\|_{\text{dist}_{M_k}}^2 - \|\mathcal{P}_k(z^k) - \Pi_\Omega(z^k)\|_{\text{dist}_{M_k}}^2 \\
& \leq \text{dist}_{M_k}^2(z^k, \Omega) - \text{dist}_{M_k}^2(\mathcal{P}_k(z^k), \Omega).
\end{align*}
\]

Therefore, it holds that

\[
\text{dist}_{M_k}(\mathcal{P}_k(z^k), \Omega) \leq \frac{\kappa \lambda_{\text{max}}(M_k)}{\sqrt{c_k^2 + \kappa^2 \lambda_{\text{max}}^2(M_k)}} \text{dist}_{M_k}(z^k, \Omega) \quad \forall k \geq 0. \tag{15}
\]

Under stopping criterion (B), we further have for all \( k \geq 0 \),

\[
\begin{align*}
\|z^{k+1} - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} & \leq \|z^{k+1} - \mathcal{P}_k(z^k)\|_{M_k} + \|\mathcal{P}_k(z^k) - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} \\
& \leq \delta_k \|z^{k+1} - z^k\|_{M_k} + \|\mathcal{P}_k(z^k) - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} \\
& \leq \delta_k (\|z^{k+1} - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} + \|z^k - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k}) + \|\mathcal{P}_k(z^k) - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k}.
\end{align*}
\]

Thus,

\[
(1 - \delta_k)\|z^{k+1} - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} \leq \delta_k \|z^k - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} + \|\mathcal{P}_k(z^k) - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k}.
\]

Now

\[
\delta_k \|z^k - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} \leq \delta_k \|\mathcal{P}_k(z^k) - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} + \delta_k \|Q_k(z^k)\|_{M_k} \\
\leq \delta_k \|\mathcal{P}_k(z^k) - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} + \delta_k \text{dist}_{M_k}(z^k, \Omega),
\]

where the last inequality follows from (13). By using the above inequality in the previous one, we get

\[
(1 - \delta_k)\|z^{k+1} - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} \leq \delta_k \text{dist}_{M_k}(z^k, \Omega) + (1 + \delta_k) \text{dist}_{M_k}(\mathcal{P}_k(z^k), \Omega).
\]

Therefore, from the last inequality and (15), it holds that for all \( k \geq 0 \),

\[
\text{dist}_{M_{k+1}}(z^{k+1}, \Omega) \leq \text{dist}_{M_k}(z^{k+1}, \Omega) \leq \|z^{k+1} - \Pi_\Omega(\mathcal{P}_k(z^k))\|_{M_k} \leq \mu_k \text{dist}_{M_k}(z^k, \Omega),
\]

where \( \mu_k = (1 - \delta_k)^{-1} \left( \delta_k + (1 + \delta_k) \kappa \lambda_{\text{max}}(M_k) / \sqrt{c_k^2 + \kappa^2 \lambda_{\text{max}}^2(M_k)} \right) \). That is, (11) holds for all \( k \geq 0 \). Since for all \( k \geq 0 \), \( M_k \geq \lambda_{\text{min}} \mathcal{I} \), (12) and (13) can be obtained through simple calculations. \( \square \)
Remark 2. Suppose that $\{\delta_k\}$ in criterion (B) is nonincreasing. Since $\{c_k\}$ is nondecreasing and $\lambda_{\text{max}}(M_k)$ is nonincreasing, we know that $\{\mu_k\}$ is nonincreasing. Therefore, if one chooses $c_0$ large enough such that $\mu_0 < 1$, then we have $\mu_k \leq \mu_0 < 1$ for all $k \geq 0$. The inequality (\ref{ineq:mu_k}) thus implies the global Q-linear convergence of $\{\text{dist}_{M_k}(z^k, \Omega)\}$. In addition, (\ref{ineq:dist_k}) implies that for all $k \geq 0$,

$$\text{dist}(z^{k+1}, \Omega) \leq (\text{dist}_{M_0}(z^0, \Omega)/\sqrt{\lambda_{\text{min}}}) \prod_{i=0}^{k} \mu_i \leq (\mu_0)^{k+1}(\text{dist}_{M_0}(z^0, \Omega)/\sqrt{\lambda_{\text{min}}})$$

i.e., $\{\text{dist}(z^k, \Omega)\}$ converges globally R-linearly.

3 A semismooth Newton augmented Lagrangian method

Note that we can equivalently rewrite problem (D) in the following minimization form:

$$(D) \quad \min \left\{ g(y) := \delta_K(A^*y - c) - b^Ty \right\}.$$  

Associated with this unconstrained formulation, we write the augmented Lagrangian function following the framework developed in [36, Examples 11.46 and 11.57]. To do so, we first identify (D) with the problem of minimizing $g(y) = \tilde{g}(y,0)$ over $\mathbb{R}^m$ for

$$\tilde{g}(y, \xi) = -b^Ty + \delta_K(A^*y - c + \xi) \quad \forall (y, \xi) \in \mathbb{R}^m \times \mathbb{R}^n.$$  

Obviously, $\tilde{g}$ is jointly convex in $(y, \xi)$. Now, we are able to write down the Lagrangian function $l: \mathbb{R}^m \times \mathbb{R}^n$ through partial dualization as follows:

$$l(y;x) := \inf_{\xi} \{ \tilde{g}(y, \xi) - \langle x, \xi \rangle \} = -b^Ty - \langle x, c - A^*y \rangle - \delta_K(x).$$

Thus, the KKT condition associated with (P) and (D) is given by

$$-b + Ax = 0, \quad A^*y - c \in \partial \delta_K(x), \quad (x, y) \in \mathbb{R}^n \times \mathbb{R}^m.$$  

Given $\sigma > 0$, the augmented Lagrangian function corresponding to (D) can be obtained by

$$L_\sigma(y;x) := \sup_{s \in \mathbb{R}^n} \left\{ l(y,s) - \frac{1}{2\sigma} \|s - x\|^2 \right\}$$

$$= -b^Ty - \inf_{s \in \mathbb{R}^n} \left\{ \delta_K(s) + \langle s, c - A^*y \rangle + \frac{1}{2\sigma} \|s - x\|^2 \right\}$$

$$= -b^Ty - \langle \Pi_K(x - \sigma(c - A^*y)), c - A^*y \rangle - \frac{1}{2\sigma} \|\Pi_K(x - \sigma(c - A^*y)) - x\|^2.$$  

We propose to solve (D) via an inexact proximal augmented Lagrangian method. Our algorithm is named as the semi-smooth Newton inexact proximal augmented Lagrangian (SNIPAL) method because we will design a semi-smooth Newton method to solve the underlying augmented Lagrangian subproblems. Its template is given as follows.
Algorithm 1: SNIPAL: Semi-smooth Newton inexact proximal augmented Lagrangian

Let $\sigma_0, \sigma_\infty > 0$ be given parameters, $\{\tau_k\}_{k=0}^\infty$ be a given nonincreasing sequence such that $\tau_k > 0$ for all $k \geq 0$. Choose $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^m$. For $k = 1, \ldots$, perform the following steps in each iteration.

**Step 1.** Compute

$$y^{k+1} \approx \arg\min_{y \in \mathbb{R}^m} \left\{ L_{\sigma_k}(y; x^k) + \frac{\tau_k}{2\sigma_k} \|y - y^k\|^2 \right\}$$

via the semismooth Newton method.

**Step 2.** Compute $x^{k+1} = \Pi_{K}(x^k - \sigma_k(c - A^*y^{k+1}))$.

**Step 3.** Update $\sigma_{k+1} \uparrow \sigma_\infty \leq \infty$.

Note that different from the classic proximal method of multipliers in [35] with $\tau_k \equiv 1$ for all $k$, we allow an adaptive choice of the parameter $\tau_k$ in the proximal term $\frac{\tau_k}{2\sigma_k} \|y - y^k\|^2$ in the inner subproblem (17) of Algorithm SNIPAL. Here, the proximal term is added to guarantee the existence of the optimal solution to the inner subproblem (17), and to ensure the positive definiteness of the coefficient matrix of the underlying semi-smooth Newton linear system. Moreover, our numerical experience with SNIPAL indicates that having the additional flexibility of choosing the parameter $\tau_k$ can help to improve the practical performance of the algorithm. We shall emphasize here that comparing with [35], our modification emphasizes more on the computational and implementational aspects.

While the introduction of the parameters $\{\tau_k\}$ brings us more flexibility and some promising numerical advantages, it also makes the convergence analysis of the algorithm more challenging. Fortunately, we are able to rigorously characterize the connection between our Algorithm SNIPAL and the preconditioned PPA studied in Section 2. As one will see in the subsequent text, this connection allows us to conduct a comprehensive convergence analysis for Algorithm SNIPAL. From the convergence analysis, we also note that $\frac{\tau_k}{2\sigma_k} \|y - y^k\|^2$ can be replaced by a more general proximal term, i.e., $\frac{1}{2\sigma_k} \|y - y^k\|^2$ with a symmetric positive definite matrix $T_k$.

### 3.1 Global convergence properties of SNIPAL

In this section, we present a comprehensive analysis for the convergence properties of SNIPAL. The global convergence and global linear-rate convergence of SNIPAL are presented as an application of the theory of the preconditioned PPA.

To establish the connection between SNIPAL and the preconditioned PPA, we first introduce some notation. To this end, for $k = 0, 1, \ldots$, and any given $(\bar{y}, \bar{x}) \in \mathbb{R}^m \times \mathbb{R}^n$, define the function

$$P_k(\bar{y}, \bar{x}) := \arg\min_{y,x} \left\{ l(y, x) + \frac{\tau_k}{2\sigma_k} \|y - \bar{y}\|^2 - \frac{1}{2\sigma_k} \|x - \bar{x}\|^2 \right\}.$$  

Corresponding to the closed proper convex-concave function $l$, we can define the maximal monotone operator $T_l$ [33 Corollary 37.5.2], by

$$T_l(y, x) := \{(y', x') \mid (y', -x') \in \partial l(y, x)\}$$

$$= \{(y', x') \mid y' = -b + Ax, \ x' \in c - A^*y + \partial \delta_K(x)\},$$
whose corresponding inverse operator is given by

$$\mathcal{T}_l^{-1}(y', x') := \arg \min_{y, x} \{ I(y, x) - \langle y', x \rangle + \langle y, x' \rangle \}. \quad (19)$$

Since $K$ is a polyhedral set, $\partial \delta_K$ is known to be a polyhedral multifunction (see, e.g., [21, p. 108]). As the sum of two polyhedral multifunctions is also polyhedral, $\mathcal{T}_l$ is also polyhedral. Define, for $k = 0, 1, \ldots,$

$$\Lambda_k = \text{Diag} (\tau_k I_m, I_n) \succ 0. \quad (20)$$

The optimal solution of problem (18), i.e., $P_k(\bar{v}, \bar{x})$, can be obtained via the following lemma.

**Lemma 3.** For all $k \geq 0$, it holds that

$$P_k(\bar{y}, \bar{x}) = (\Lambda_k + \sigma_k \mathcal{T}_l)^{-1} \Lambda_k (\bar{y}, \bar{x}) \quad \forall (\bar{y}, \bar{x}) \in \mathbb{R}^m \times \mathbb{R}^n. \quad (21)$$

If $(y^*, x^*) \in \mathcal{T}_l^{-1}(0)$, then $P_k(y^*, x^*) = (y^*, x^*)$.

In SNIPAL, at $k$-th iteration, denote

$$\psi_k(y) := L_{\Lambda_k}(y; x^k) + \frac{\tau_k}{2\sigma_k} \| y - y^k \|^2. \quad (22)$$

From the property of the proximal mapping, we know that $\psi_k$ is continuously differentiable and

$$\nabla \psi_k(y) = -b + A \Pi_K(x^k + \sigma_k(A^*y - c)) + \tau_k \sigma_k^{-1}(y - y^k).$$

As a generalization of Proposition 8 in [35], the following proposition about the weighted distance between $(y^{k+1}, x^{k+1})$ generated by SNIPAL and $P_k(y^k, x^k)$ is important for designing the stopping criteria for the subproblem [17] and establishing the connection between SNIPAL and the preconditioned PPA.

**Proposition 2.** Let $P_k, \Lambda_k$ and $\psi_k$ be defined in (18), (20) and (22), respectively. Let $(y^{k+1}, x^{k+1})$ be generated by Algorithm SNIPAL at iteration $k + 1$. It holds that

$$\|(y^{k+1}, x^{k+1}) - P_k(y^k, x^k)\|_{\Lambda_k} \leq \frac{\sigma_k}{\min(\sqrt{\tau_k}, 1)} \| \nabla \psi_k(y^{k+1}) \|. \quad (23)$$

**Proof.** Since $\nabla \psi_k(y^{k+1}) = \nabla_y L_{\Lambda_k}(y^{k+1}, x^k) + \tau_k \sigma_k^{-1}(y^{k+1} - y^k)$, we have

$$\nabla \psi_k(y^{k+1}) + \sigma_k^{-1} \tau_k (y^k - y^{k+1}) \in \nabla_y L_{\Lambda_k}(y^{k+1}, x^k),$$

which, by [35] Proposition 7, implies $(\nabla \psi_k(y^{k+1}) + \sigma_k^{-1} \tau_k (y^k - y^{k+1}), \sigma_k^{-1}(x^k - x^{k+1})) \in \mathcal{T}_l(y^{k+1}, x^{k+1})$. Thus,

$$\sigma_k(\nabla \psi_k(y^{k+1}, 0) + \Lambda_k((y^k, x^k) - (y^{k+1}, x^{k+1})) \in \sigma_k \mathcal{T}_l(y^{k+1}, x^{k+1})$$

and $\sigma_k(\nabla \psi_k(y^{k+1}, 0) + \Lambda_k(y^k, x^k) \in (\Lambda_k + \sigma_k \mathcal{T}_l)(y^{k+1}, x^{k+1})$, or equivalently,

$$(y^{k+1}, x^{k+1}) = (\Lambda_k + \sigma_k \mathcal{T}_l)^{-1} \Lambda_k(\sigma_k \nabla \psi_k(y^{k+1}, 0) + (y^k, x^k)).$$

Then, by Lemma 3 and Proposition 1 we know that

$$\|(y^{k+1}, x^{k+1}) - P_k(y^k, x^k)\|_{\Lambda_k}$$

$$= \|(\Lambda_k + \sigma_k \mathcal{T}_l)^{-1} \Lambda_k(\sigma_k \nabla \psi_k(y^{k+1}, 0) + (y^k, x^k)) - (\Lambda_k + \sigma_k \mathcal{T}_l)^{-1} \Lambda_k((y^k, x^k))\|_{\Lambda_k}$$

$$\leq \|\Lambda_k^{-1}(\sigma_k \nabla \psi_k(y^{k+1}, 0), 0)\|_{\Lambda_k} \leq \frac{\sigma_k}{\min(\sqrt{\tau_k}, 1)} \| \nabla \psi_k(y^{k+1}) \|. \quad \square$$

This completes the proof for the proposition.

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Based on Proposition 2 we propose the following stopping criteria for the approximate computation of $y^{k+1}$ in Step 1 of SNIPAL:

\begin{align*}
(A') \quad & \|\nabla \psi_k(y^{k+1})\| \leq \frac{\min(\sqrt{\tau_k}, 1)}{\sigma_k} \epsilon_k, \quad 0 \leq \epsilon_k, \quad \sum_{k=0}^{\infty} \epsilon_k < \infty, \\
(B') \quad & \|\nabla \psi_k(y^{k+1})\| \leq \frac{\delta_k}{\sigma_k} \min(\sqrt{\tau_k}, 1) \|\psi(y^{k+1}) - \psi(y^k)\|_{A_k}, \quad 0 \leq \delta_k < 1, \quad \sum_{k=0}^{\infty} \delta_k < \infty.
\end{align*}

For the convergence of SNIPAL, we also need the following assumption on $\tau_k$:

**Assumption 3.** The positive sequence \(\{\tau_k\}\) is non-increasing and bounded away from zero, i.e., \(\tau_k \downarrow \tau_{\infty} > 0\) for some positive constant \(\tau_{\infty}\).

Under Assumption 3 we have, for all \(k \geq 0\),

\[ \Lambda_k \geq \Lambda_{k+1} \text{ and } \Lambda_k \geq \min(1, \tau_{\infty}) I_{m+n}. \]

We now present the global convergence results for SNIPAL in the following Theorem. Similar to the case in [35], it is in fact a direct application of Theorem 1.

**Theorem 3 (Global convergence of SNIPAL).** Suppose that Assumptions 1 and 2 hold. Let \(\{y^k, x^k\}\) be the sequence generated by Algorithm SNIPAL with the stopping criterion \((A')\). Then \(\{y^k, x^k\}\) is bounded. In addition, \(\{x^k\}\) converges to an optimal solution of \((P)\) and \(\{y^k\}\) converges to an optimal solution of \((D)\), respectively.

Since \(\mathcal{T}_l\) is a polyhedral multifunction, we know from Lemma 2 and Remark 1 that \(\mathcal{T}_l\) satisfies Assumption 2. Let \(r\) be a positive number satisfying \(r > \sum_{i=0}^{\infty} \epsilon_k\) with \(\epsilon_k\) being the summable sequence in \((A')\). Then, there exists \(\kappa > 0\) associated with \(r\) such that for any \((y, x) \in \mathbb{R}^m \times \mathbb{R}^n\) satisfying \(\text{dist}((y, x), \mathcal{T}_l^{-1}(0)) \leq r\),

\[ \text{dist}((y, x), \mathcal{T}_l^{-1}(0)) \leq \kappa \text{dist}(0, \mathcal{T}(y, x)). \quad (24) \]

As an application of Theorem 2 we are now ready to show the asymptotic superlinear convergence of SNIPAL in the following theorem.

**Theorem 4 (Asymptotic superlinear convergence of SNIPAL).** Suppose that Assumptions 1 and 2 hold and the initial \(z^0 := (y^0, x^0)\) satisfies \(\text{dist}_{\Lambda_0}(z^0, \mathcal{T}_l^{-1}(0)) \leq r - \sum_{i=0}^{\infty} \epsilon_k\). Let \(\kappa\) be the modulus given in (24) and \(\{z^k := (y^k, x^k)\}\) be the infinite sequence generated by the preconditioned PPA under criteria \((A')\) and \((B')\). Then, for all \(k \geq 0\), it holds that

\[
\begin{align*}
\text{dist}_{\Lambda_{k+1}}(z^{k+1}, \mathcal{T}_l^{-1}(0)) & \leq \mu_k \text{dist}_{\Lambda_k}(z^{k+1}, \mathcal{T}_l^{-1}(0)), \\
\text{dist}(z^{k+1}, \mathcal{T}_l^{-1}(0)) & \leq \frac{\mu_k}{\sqrt{\min(1, \tau_{k+1})}} \text{dist}_{\Lambda_k}(z^k, \mathcal{T}_l^{-1}(0)),
\end{align*}
\]

where \(\mu_k = (1 - \delta_k)^{-1} \left(\delta_k + (1 + \delta_k) \kappa \gamma_k / \sqrt{\sigma_k^2 + \kappa^2 \gamma_k^2}\right)\) with \(\gamma_k := \max(\gamma_k, 1)\) and

\[
\lim_{k \to \infty} \mu_k = \mu_\infty = \frac{\kappa \gamma_\infty}{\sqrt{\sigma_\infty^2 + \kappa^2 \gamma_\infty^2}} < 1 \quad (\mu_\infty = 0 \text{ if } \sigma_\infty = \infty)
\]

with \(\gamma_\infty = \max(\gamma_k, 1)\).

**Remark 3.** Suppose that \(\{\delta_k\}\) in criterion \((B')\) is nonincreasing. We know from 2 that if one chooses \(\sigma_0\) large enough such that \(\mu_0 < 1\), then \(\mu_k \leq \mu_0 < 1\) for all \(k \geq 0\). Thus, from (25), we have the global linear convergence of \(\{\text{dist}_{\Lambda_k}(z^k, \mathcal{T}_l^{-1}(0))\}\) and \(\{\text{dist}(z^k, \mathcal{T}_l^{-1}(0))\}\).
3.2 Semismooth Newton method for subproblems

In this subsection, we discuss how the subproblem (17) in SNIPAL can be solved efficiently. As is mentioned in the name of SNIPAL, we propose to solve (17) via an inexact semismooth Newton method which converges at least superlinearly. In fact, the convergence rate can even be quadratic. This inner solver is thus the cornerstone for the high performance of SNIPAL.

For given \((\tilde{x}, \tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m\) and \(\tau, \sigma > 0\), define the function \(\psi : \mathbb{R}^m \rightarrow \mathbb{R}\) as

\[
\psi(y) := L(\sigma; \tilde{x}) + \frac{\sigma}{2\sigma}||y - \tilde{y}||^2 \quad \forall y \in \mathbb{R}^m,
\]

and we aim to solve

\[
\min_{y \in \mathbb{R}^m} \psi(y). \tag{26}
\]

Note that \(\psi\) is strongly convex and continuously differentiable over \(\mathbb{R}^m\) with

\[
\nabla \psi(y) = -b + A \Pi_K(\tilde{x} + \sigma(A^*y - c)) + \tau\sigma^{-1}(y - \tilde{y}).
\]

Hence, we know that for any given \(\alpha \in \mathbb{R}\), the level set \(L_\alpha := \{y \in \mathbb{R}^m \mid \psi(y) \leq \alpha\}\) is a nonempty closed and bounded convex set. In addition, problem (26) has a unique optimal solution which we denote as \(\bar{y}\).

As an unconstrained optimization problem, the optimality condition for (26) reads

\[
\nabla \psi(y) = 0, \quad y \in \mathbb{R}^m, \tag{27}
\]

and \(\bar{y}\) is the unique solution to this nonsmooth equation. Since \(\Pi_K\) is a Lipschitz continuous piecewise affine function, we have that \(\nabla \psi\) is strongly semismooth. Hence, we propose to solve the nonsmooth equation (27) via a semismooth Newton method. For this purpose, we define the following operator:

\[
\hat{\partial}^2 \psi(y) := \tau\sigma^{-1}I_m + \sigma A \partial \Pi_K(\tilde{x} + \sigma(A^*y - c))A^* \quad \forall y \in \mathbb{R}^m,
\]

where \(\partial \Pi_K(\tilde{x} + \sigma(A^*y - c))\) is the Clarke subdifferential [10] of the Lipschitz continuous mapping \(\Pi_K(\cdot)\) at \(\tilde{x} + \sigma(A^*y - c)\). Note that from [19] Example 2.5, we have that

\[
\hat{\partial}^2 \psi(y)d = \partial^2 \psi(y)d \quad \forall d \in \mathbb{R}^m,
\]

where \(\partial^2 \psi(y)\) denotes the generalized Hessian of \(\psi\) at \(y\). However, we caution the reader that it is unclear whether \(\hat{\partial}^2 \psi(y) = \partial^2 \psi(y)\). Given any \(y \in \mathbb{R}^m\), define

\[
H := \tau\sigma^{-1}I_m + \sigma AU A^* \tag{28}
\]

with \(U \in \partial \Pi_K(\tilde{x} + \sigma(A^*y - c))\). Then, we know that \(H \in \hat{\partial}^2 \psi(y)\) and \(H\) is symmetric positive definite.

After these preparations, we are ready to present the following semismooth Newton method for solving the nonsmooth equation (27) and we can expect a fast superlinear convergence.
Algorithm 2 Ssn: A Semi-smooth Newton method for solving (27) (Ssn(\(\tilde{x}, \tilde{y}, \sigma, \tau\)))

Given \(\tau > 0, \sigma > 0\), choose parameters \(\mu \in (0, 1/2), \eta \in (0, 1), \gamma \in (0, 1)\) and \(\delta \in (0, 1)\) and set \(y^0 = \tilde{y}\). Iterate the following steps for \(j = 0, 1, \ldots\).

**Step 1.** Choose \(U_j \in \partial \Pi_{K}(\tilde{x} + \sigma(A^*y^j - c))\). Set \(H_j := \tau \sigma^{-1}I_m + \sigma A U_j A^*\). Solve the linear system

\[
H_j d = -\nabla \psi(y^j)
\]

exactly or by the conjugate gradient (CG) algorithm to find \(d\) such that \(\|H_j d + \nabla \psi(y^j)\| \leq \min(\bar{\eta}, \|\nabla \psi(y^j)\|^{1+\gamma})\).

**Step 2.** (Line search) Set \(\alpha_j = \delta^{m_j}\), where \(m_j\) is the first nonnegative integer \(m\) for which

\[
\psi(y^j + \delta^m d^j) \leq \psi(y^j) + \mu \delta^m \langle \nabla \psi(y^j), d^j \rangle.
\]

**Step 3.** Set \(y^{j+1} = y^j + \alpha_j d^j\).

The convergence results of Algorithm Ssn are stated in the following theorem.

**Theorem 5.** Let \(\{y^j\}\) be the infinite sequence generated by Algorithm Ssn. It holds that \(\{y^j\}\) converges to the unique optimal solution \(\bar{y}\) of (26) and \(\|y^{j+1} - \bar{y}\| = O(\|y^j - \bar{y}\|^{1+\gamma})\).

**Proof.** We know from [42, Proposition 3.3] that \(d^j\) is always a descent direction. Then, the strong convexity of \(\psi\) and [42, Theorem 3.4] imply that \(\{y^j\}\) converges to the unique optimal solution \(\bar{y}\) of (26). By (28), we have that the symmetric positive definite matrix \(H_j \in \hat{\partial}^2 \psi(y^j)\) satisfies the property that \(H_j \succeq \tau \sigma^{-1}I_m\) for all \(j\). The desired results thus can be obtained by following the proof of [42, Theorem 3.5]. We omit the details here.

### 3.3 Finite termination property of Snipal

In our extensive numerical experience with Snipal, we observe that Snipal nearly possesses a certain finite convergence property for solving (P) and (D) when \(\sigma_k\) and \(1/\tau_k\) are sufficiently large. We note that most available theoretical results corresponding to the finite termination property of proximal point algorithms require each subproblem involved to be solved exactly, e.g., see [34], [35] and [24]. Hence, all these results cannot be directly adopted to support our numerical findings. In this section, we aim to investigate the finite termination property of Snipal by showing that it is possible to obtain a solution pair of (P) and (D) without requiring the exact solutions of each and every subproblem involved in Snipal.

Our arguments build upon an interesting property called “staircase property” associated with subdifferential mappings of convex closed polyhedral functions. Let

\[
f(x) := c^T x + \delta_K(x) + \delta_{\{x|Ax=b\}}(x).
\]

Clearly, \(f\) is a convex closed polyhedral function. From [13, Sec. 6] and earlier works in [12, 24], we know that its subdifferential mapping enjoys the following staircase property, i.e., there exists \(\delta > 0\) such that

\[
w \in \partial f(x), \|w\| \leq \delta \Rightarrow 0 \in \partial f(x).
\]

Based on the staircase property of \(\partial f\), we present the finite convergence property of Snipal in the following theorem.
**Theorem 6.** Suppose that Assumptions [4] and [5] hold and let \( \{ (y^l, x^l) \} \) be the infinite sequence generated by Snipal with the stopping criterion (A'). For any given \( k \geq 0 \), suppose that \( \tilde{y}^{k+1} \) is an exact solution to the following optimization problem:

\[
\tilde{y}^{k+1} = \arg\min_{y} L_{\sigma_k}(y; x^k).
\]  

Then, the following results hold.

(a) The point \( \bar{x}^{k+1} := \Pi_K(x^k - \sigma_k(c - A^*\tilde{y}^{k+1})) \) is the unique solution to the following proximal problem:

\[
\min \left\{ c^T x + \frac{1}{2\sigma_k} \| x - x^k \|^2 \mid Ax = b, x \in K \right\}.
\]  

(b) There exists a positive scalar \( \bar{\sigma} \) independent of \( k \) such that for all \( \sigma_k \geq \bar{\sigma} \), \( \bar{x}^{k+1} \) also solves the problem (P).

(c) If \( x^k \) is a solution of (P), then \( \tilde{y}^{k+1} \) also solves (D).

**Proof.** (a) Observe that the dual of (31) is exactly (32), and the KKT conditions associated with (31) and (32) are given as follows:

\[
x = \Pi_K(x^k - \sigma_k(c - A^*y)), \quad Ax = b = 0.
\]  

Since \( \tilde{y}^{k+1} \) is a solution of the problem (31), it holds from the optimality condition associated with (31) that \( \Pi_K(x^k - \sigma_k(c - A^*\tilde{y}^{k+1})) = b \). Thus, \( (\bar{x}^{k+1}, \bar{y}^{k+1}) \) satisfy (33). Therefore, \( \bar{x}^{k+1} \) solves (32). The uniqueness of \( \bar{x}^{k+1} \) follows directly from the strong convexity of (32).

(b) By Theorem 3, we know that \( x^l \to x^* \) as \( l \to \infty \) for some \( x^* \in \partial f^{-1}(0) \). Therefore, there exists a constant \( M > 0 \) (independent of \( k \)) such that

\[
\|x^l - x^*\| \leq M \quad \forall \, l \geq 0.
\]  

From the optimality of \( \tilde{x}^{k+1} \) and the definition of \( f \), we have that

\[
\frac{1}{\sigma_k}(x^k - \tilde{x}^{k+1}) \in \partial f(\tilde{x}^{k+1}).
\]

It also holds from the nonexpansive property of the proximal mapping that \( \|\tilde{x}^{k+1} - x^*\| \leq \|x^k - x^*\| \), which, together with (31), further implies that

\[
\|\tilde{x}^{k+1} - x^k\| \leq \frac{2M}{\sigma} \leq \delta,
\]

where \( \delta > 0 \) is the constant given in (30). Thus, by using the “staircase” property (30), we know that

\[
0 \in \partial f(\tilde{x}^{k+1}).
\]

That is, \( \tilde{x}^{k+1} \) solves the problem (P).

(c) Next, consider the case when \( x^k \) is a solution of (P). From the minimization property of \( x^k \), it is clear that the unique solution of (32) must be \( \bar{x}^{k+1} = x^k \). Thus, \( x^k = \Pi_K(x^k - \sigma_k(c - A^*\tilde{y}^{k+1})) \) and \( Ax^k = b \). Note that it can be equivalently rewritten as:

\[
A^*\tilde{y}^{k+1} - c \in \partial \delta_K(x^k), \quad A x^k = b,
\]

i.e., \( (x^k, \tilde{y}^{k+1}) \) satisfy the KKT conditions for (P) and (D) in (16). Thus, \( \tilde{y}^{k+1} \) solves (D). 

\[ \square \]
Remark 4. We now remark on the significance of the above theorem. Essentially, it says that when \( \sigma_k \) is sufficiently large with \( \sigma_k \geq \sigma \), then \( \hat{x}^{k+1} \) solves \((P)\), and it holds that \( \hat{y}^{k+2} = \text{arg} \min L_{\sigma_{k+1}}(y; \hat{x}^{k+1}) \) solves \((D)\).

From the fact that the SSN method used to solve \([27]\) has the finite termination property \([16, 37]\), we know that \( y^{k+1} \) computed in Step 1 of SNIPAL is in fact the exact solution of the subproblem \( \min \psi_k(y) \) when the corresponding linear system is solved exactly. In addition, when \( \sigma_k \) is sufficiently large and \( \tau_k \) is small enough, we have that

\[
0 = \nabla L_{\sigma_k}(y^{k+1}; x^k) + \tau_k \sigma_k^{-1}(y^{k+1} - \hat{y}^k) \approx \nabla L_{\sigma_k}(y^{k+1}; x^k),
\]

and consequently, \( y^{k+1} \) can be regarded as a highly accurate solution to the problem \( \min L_{\sigma_k}(y; x^k) \). In this sense, Theorem 6 explains the finite termination phenomenon in the practical performance of SNIPAL.

Remark 5. In fact, one could also obtain certain finite termination results different from those stated in Theorem 6. Indeed, in the proof of Theorem 6\((b)\), since \( x^i \to x^* \), we know that there exists \( K > 0 \) such that for all \( k \geq K \), \( \| x^k - x^* \| \leq \frac{\delta_0}{2} \), and consequently, \( \| x^{k+1} - x^k \| \leq 2 \| x^k - x^* \| \leq \delta_0 \) with the constant \( \delta > 0 \) given in \((30)\). Hence, we have

\[
\frac{1}{\sigma_k} \| \bar{x}^{k+1} - x^k \| \leq \frac{1}{\sigma_0} \| x^{k+1} - x^k \| \leq \delta \quad \forall k \geq K.
\]

From the staircase property of \( \partial f \), we know that for all \( k \geq K \), \( \bar{x}^{k+1} \) solves the problem \((P)\).

4 Solving the linear systems arising from the semismooth Newton method

Note that the most expensive operation in Algorithm SSN is the computation of the search direction \( d \in \mathbb{R}^m \) through solving the linear system \((29)\). To ensure the efficiency of SSN and consequently the efficiency of SNIPAL, in this section, we shall discuss efficient approaches for handling \((29)\) in Algorithm SSN. Given \( c, \bar{x} \in \mathbb{R}^n, \bar{y} \in \mathbb{R}^m \), the parameters \( \tau, \sigma > 0 \) and the current iterate of SSN \( \bar{y} \in \mathbb{R}^m \), let

\[
g := -\nabla \phi(\bar{y}) = R_p - \tau \sigma^{-1}(\bar{y} - \bar{y}),
\]

where \( R_p = b - A\Pi_K(w(\bar{y})) \) with \( w(\bar{y}) := \bar{x} + \sigma(A^* \bar{y} - c) \). At each SSN iteration, we need to solve a linear system of the form:

\[
H \Delta y = g,
\]

where \( H = \tau \sigma^{-1}I_m + \sigma AU^* \) with \( U \in \partial \Pi_K(w(\bar{y})) \). Define the index set \( \mathcal{J} = \{ i \mid l_i < |w(\bar{y})|, u_i, i = 1, \ldots, n \} \) and \( p = |\mathcal{J}| \), i.e., the cardinality of \( \mathcal{J} \). In the implementation, we always construct the generalized Jacobian matrix \( U \in \partial \Pi_K(w(\bar{y})) \) as a diagonal matrix in the following manner:

\[
U = \text{Diag}(u) \quad \text{with} \quad u_i = \begin{cases} 1 & \text{if} \quad i \in \mathcal{J}, \\ 0 & \text{otherwise,} \end{cases} \quad i = 1, \ldots, n.
\]

Without the loss of generality, we can partition \( A \equiv [A_{\mathcal{J}}, A_{\mathcal{N}}] \) with \( A_{\mathcal{J}} \in \mathbb{R}^{m \times p}, A_{\mathcal{N}} \in \mathbb{R}^{m \times (n-p)} \), and hence

\[
H = \sigma A_{\mathcal{J}} A_{\mathcal{J}}^* + \tau \sigma^{-1}I_m = \sigma(A_{\mathcal{J}} A_{\mathcal{J}}^* + \rho I_m)
\]
where $\rho := \tau \sigma^{-2}$. To solve the linear system \((35)\) efficiently, we need to consider various scenarios. In the discussion below, we use \text{nnzden}(M)\) to denote the density of the nonzero elements of a given matrix \(M\).

(a) First, we consider the case where \(p \geq m\) and the sparse Cholesky factorization of \(A_J A_J^*\) can be computed at a moderate cost. In this case, the main cost of solving the linear system is in forming the matrix \(A_J A_J^* + \rho I_m\) at the cost of \(O(m^2 p \text{nnzden}(A_J))\) and computing the sparse Cholesky factorization of \(A_J A_J^* + \rho I_m\).

Observe that the index set \(J\) generally changes from one SSN iteration to the next. However, when the SSN method is converging, the index set \(J\) may only change slightly from the current iteration to the next. In this case, one can update the inverse of \(H\) via a low-rank update by using the Sherman-Morrison-Woodbury formula.

When it is expensive to compute and factorize \(H\), one would naturally use a preconditioned conjugate gradient (PCG) method to solve \((35)\). Observe that the condition number of \(H\) is given by

$$\kappa(H) = \frac{\omega_{\text{max}}^2 + \rho}{\omega_{\text{min}}^2 + \rho}$$

if \(p \geq m\), where \(\omega_{\text{max}}, \omega_{\text{min}}\) are the largest and smallest singular value of \(A_J\), respectively.

(b) Next we consider the case where \(p < m\). In this case, it is more economical to solve \((35)\) by using the Sherman-Morrison-Woodbury formula to get

$$\Delta y = H^{-1} g = \tau^{-1} \sigma (I_m - P_J) g,$$

where \(P_J = A_J G^{-1} A_J^*\), \(G = \rho I_p + A_J^* A_J \in \mathbb{R}^{p \times p}\). Thus to compute \(\Delta y\), one needs only to solve a smaller \(p \times p\) linear system of equations \(Gv = A_J^* g\). Observe that when \(\rho \ll 1\), \(\tau \sigma^{-1} \Delta y\) is approximately the orthogonal projection of \(g\) onto the null space of \(A_J\).

To solve \((36)\), one can compute the sparse Cholesky factorization of the symmetric positive definite matrix \(G \in \mathbb{R}^{p \times p}\) if the task can be done at a reasonable cost. In this case, the main cost involved in \((36)\) is in computing \(A_J^* A_J\) at the cost of \(O(p^2 m \text{nnzden}(A_J))\) operations and the sparse Cholesky factorization of \(G = \rho I_p + A_J^* A_J\).

When it is too expensive to compute and factorize \(G\), one can use a Krylov iterative method to solve the \(p \times p\) linear system of equations:

$$Gv = (\rho I_p + A_J^* A_J) v = A_J^* g.$$  \hfill (37)

To estimate the convergence rate of the Krylov iterative method, it is important for us to analyse the conditioning of the above linear system, as is done in the next theorem.

**Theorem 7.** Let \(B \in \mathbb{R}^{m \times p}\) with \(p < m\). Consider linear system \(Gv = B^* g\), where \(G = B^* B + \rho I_p\) and \(g \in \mathbb{R}^m\). Then the effective condition number for solving the system by the MINRES (minimum residual) method with zero initial point is given by

$$\kappa = \frac{\omega_{\text{max}}^2 + \rho}{\omega_{\text{min}}^2 + \rho},$$

where \(\omega_{\text{max}}\) is the largest singular value and \(\omega_{\text{min}} > 0\) is the smallest positive singular value of \(B\), respectively.

**Proof.** Consider the following full SVD of \(B\):

$$B = U \Sigma V^T = [U_1, U_2] \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix},$$

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where $\hat{\Sigma}$ is the diagonal matrix consisting of the positive singular values of $B$. Let $P_k^0$ be the set of polynomials $p_k$ with degree at most $k$ and $p_k(0) = 1$. Then for $p_k \in P_k^0$, we have that

$$p_k(G)B^* g = Vp_k(\Sigma^T \Sigma + \rho I)\Sigma^T U^T g = [V_1, V_2] \begin{bmatrix} p_k(\hat{\Sigma}^2 + \rho I)\hat{\Sigma} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1^T g \\ U_2^T g \end{bmatrix}$$

$$= V_1 p_k(\hat{\Sigma}^2 + \rho I)\hat{\Sigma} U_1^T g.$$ 

Since the $k$-th iteration of the MINRES method computes an approximate solution $x_k$ such that its residual $\xi = p_k(G)B^* g$ satisfies the condition that

$$\|\xi\| = \|p_k(G)B^* g\| = \min_{p_k \in P_k^0} \|p_k(G)B^* g\| \leq \|\hat{\Sigma} U_1^T g\| \min_{p_k \in P_k^0} \|p_k(z)\| |\omega_{\text{min}}^2 + \rho, \omega_{\text{max}}^2 + \rho|,$$

thus we see the convergence rate of the MINRES method is determined by the best approximation of the zero function by the polynomials in $P_k^0$ over the interval $[\omega_{\text{min}}^2 + \rho, \omega_{\text{max}}^2 + \rho]$. Hence the convergence rate of the MINRES method is determined by $\kappa$. □

After (37) is solved via the MINRES method, one can compute the residual vector of associated with system (36) without much difficulty. Indeed, let the computed solution of (36) be given as follows:

$$\hat{\Delta} y = \tau^{-1} \sigma (g - A_J v)$$

where $Gv = A_J^* g - \xi$ with $\xi$ being the residual vector obtained from the MINRES iteration. Now the residual vector associated with (36) is given by

$$\eta := g - \mathcal{H}\hat{\Delta} y = g - \tau^{-1} \sigma \mathcal{H} g + \tau^{-1} \sigma \mathcal{H} A_J G^{-1} (A_J^* g - \xi)$$

$$= g - \tau^{-1} \sigma \mathcal{H} (g - P_J g) - \tau^{-1} \sigma \mathcal{H} A_J G^{-1} \xi$$

$$= -\tau^{-1} \sigma \mathcal{H} A_J G^{-1} \xi = -\rho^{-1} A_J \xi,$$

where the last equation follows directly from $\mathcal{H} A_J = \sigma A_J G$. Based on the computed $\eta$, one can check the termination condition for solving the linear system in (29).

Now, we are ready to bound the condition numbers of the Newton linear systems involved in Snipal. As can be observed from the above discussions, for both cases (a) and (b), the effective condition number of the linear system involved is upper bounded by

$$\kappa \leq \frac{\omega_{\text{max}}^2 + \rho}{\rho},$$

where $\omega_{\text{max}}$ is the largest singular value of $A_J$ and $\rho = \tau \sigma^{-2}$. Since $A_J$ is always a sub-matrix of $A$, it holds that $\omega_{\text{max}} \leq \|A\|_2$. Hence, for any linear systems involved in the $k$-th iteration of Snipal, we can provide an upper bound for the condition number as follows:

$$\kappa \leq 1 + \frac{\|A\|_2^2 \sigma_k^2}{\tau_k}.$$  \hfill (38)

From our assumptions on Snipal, we note that $\sigma_k \leq \sigma_\infty$ and $\tau_k \geq \tau_\infty > 0$ for all $k \geq 0$. Hence, for all the linear systems involved in Snipal, there exists an uniform upper bound for the corresponding condition number:

$$\kappa \leq 1 + \frac{\|A\|_2^2 \sigma_\infty^2}{\tau_\infty}.$$
As long as $\sigma_\infty < +\infty$, we have shown that all these linear systems have bounded condition numbers. This differs significantly from the setting in interior-point based algorithms where the condition numbers of the corresponding normal equations are asymptotically unbounded. The competitive advantage of Snipal can be partially explained from the above observation. Meanwhile, in the $k$-th iteration of Snipal, to maintain a small condition number based on (38), one should choose small $\sigma_k$ but large $\tau_k$. However, the convergence rate of Snipal developed in Theorem 2 requires the opposite choice, i.e., large $\sigma_k$ and $\tau_k$ should be moderate. The preceding discussion thus reveals the trade-off between the convergence rate of the ALM and the condition numbers of the Newton linear systems. Clearly, in the implementation of Snipal, the parameters $\{\sigma_k\}$ and $\{\tau_k\}$ should be chosen to balance the progress of the outer and inner algorithms, i.e., the ALM and the semismooth Newton method.

5 Warm-start algorithm for Snipal

As is mentioned in the introduction, to achieve high performance, it is desirable to use a simple first-order algorithm to warm start Snipal. For this purpose, we present an ADMM algorithm for solving (D). We note that a similar strategy has also been employed for solving large scale semidefinite programming and quadratic semidefinite programming problems [40, 22].

We begin by rewriting (D) into the following equivalent form:

$$\min \{\delta^*_K(-z) - b^T y \mid z + A^* y = c\}. \quad (39)$$

Given $\sigma > 0$, the augmented Lagrangian function associated with (39) can be written as

$$L_\sigma(z, y; x) = \delta^*_K(-z) - b^T y + \langle x, z + A^* y - c \rangle + \frac{\sigma}{2} \|z + A^* y - c\|^2$$

for all $(x, y, z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^n$. After these preparations, we are able to present the template of the classic ADMM for solving (39) as follows:

**Algorithm 3 ADMM: An ADMM method for solving (39)**

Select the initial point $(x^0, y^0) \in \mathbb{R}^n \times \mathbb{R}^m$. For $k = 1, \ldots$, perform the following steps in each iteration.

**Step 1.** Compute

$$z^{k+1} = \text{argmin} L_\sigma(z, y^k; x^k). \quad (40)$$

**Step 3.** Compute

$$y^{k+1} = \text{argmin} L_\sigma(z^{k+1}, y; x^k). \quad (41)$$

**Step 3.** Compute

$$x^{k+1} = x^k + \gamma \sigma (z^{k+1} + A^* y^{k+1} - c).$$

After some calculations, we can simplify (40) and (41) in the following manner:

$$z^{k+1} = \frac{1}{\sigma} \left( \Pi_K(x^k + \sigma (A^* y^k - c)) - (x^k + \sigma (A^* y^k - c)) \right),$$

$$y^{k+1} = (AA^*)^{-1} \left( b/\sigma - A(x^k/\sigma + z^{k+1} - c) \right).$$

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The convergence of the above classic ADMM for solving the two-block optimization problem (39) with the steplength $\tau \in (0, (1 + \sqrt{5})/2)$ can be readily obtained from the vast literature about the ADMM. Here, we adopt a newly developed result from [8] stating that the above ADMM is in fact an inexact proximal ALM. This interesting interpretation allows us to choose the steplength $\gamma$ in the large interval $(0, 2)$ which usually leads a better numerical performance. We summarize the convergence results in the following theorem. The detailed proof can be found in [8].

**Theorem 8.** Suppose that Assumption 1 holds and $\tau \in (0, 2)$. Let $\{(x^k, y^k, z^k)\}$ be the sequence generated by Algorithm ADMM. Then, $\{x^k\}$ converges to an optimal solution of $(P)$ and $\{(y^k, z^k)\}$ converges to an optimal solution of (39), respectively.

**Remark 6.** In the above algorithm, one can also handle (41) by adding an appropriate proximal term or by using an iterative method to solve the corresponding linear system. The convergence of the resulting proximal or inexact ADMM with large steplengh $\gamma$ has also been discussed in [8]. For simplicity, we only discussed the exact version here.

### 6 Numerical experiments

In this section, we evaluate the performance of SNIPAL against the powerful commercial solver Gurobi on various LP data sets. Our goal is to compare the performance of our solver against Gurobi in terms of its speed and ability to solve the tested instances to the relatively high accuracy of $10^{-8}$ in the relative KKT residual, which, for a given computed solution $(x, y, z)$ is defined as follows:

$$
\eta = \max \left\{ \frac{\|b - Ax\|}{1 + \|b\|}, \frac{\|A^Ty + z - c\|}{1 + \|c\|}, \frac{\|x - \Pi_K(x - z)\|}{1 + \|x\| + \|z\|} \right\}. \quad (42)
$$

In our comparison, the default setting is used for Gurobi unless stated otherwise. We should note that sometimes the presolve phrase in Gurobi is too time consuming and does not lead to any reduction in the problem size. In that case, we turn off the presolve phase in Gurobi to give it a competitive advantage.

All the numerical experiments in this paper are run in MATLAB on a Dell Laptop with Intel(R) Core i7-6820HQ CPU @2.70GHz and 16GB of RAM.

#### 6.1 Randomly generated sparse LP in [27]

Here we test very large synthetic LP problems generated as in [27]. In particular, the matrix $A$ is generated as follows:

$$
A = sprand(m, n, d); \quad A = 100*(A - 0.5*spones(A));
$$

In this case, we turn off the presolve phase in Gurobi as this phase is too time consuming for these randomly generated problems. As we can observe from Table 1, SNIPAL is able to outperform Gurobi by a factor of about 2 – 3.5 times in computational time in most cases.

Note that for the column “it (itsub)” in Table 1 we report the number of SNIPAL iterations and the total number of semismooth Newton linear systems solved in Algorithm 2. For the columns “time (RAM)” and “Gurobi time (RAM)”, we report the wall-clock time and the memory consumed by SNIPAL and Gurobi, respectively.
Table 1: Numerical results for random sparse LPs. The reported times are in seconds.

| m   | n   | d   | it (itsub) | time (RAM)   | Gurobi time (RAM) |
|-----|-----|-----|-----------|--------------|-------------------|
| 2e3 | 1e5 | 0.025 | 4 (20)   | 3.7 (0.8GB) | 9.8 (1.0GB)      |
|     |     | 0.050 | 5 (20)   | 7.2 (0.8GB) | 18.7 (1.3GB)     |
| 5e3 | 1e5 | 0.025 | 4 (15)   | 14.8 (1.5GB) | 36.2 (1.7GB)    |
|     |     | 0.050 | 4 (16)   | 30.4 (1.8GB) | 70.0 (2.6GB)    |
| 10e3| 1e5 | 0.025 | 5 (24)   | 50.4 (3.2GB) | 154.9 (3.2GB)  |
|     |     | 0.050 | 5 (24)   | 98.8 (5.3GB) | 317.6 (6.0GB)  |
| 1e3 | 1e6 | 0.025 | 6 (34)   | 11.1 (2.0GB) | 41.2 (4.1GB)   |
|     |     | 0.050 | 7 (38)   | 20.7 (3.2GB) | 62.0 (6.0GB)   |
| 2e3 | 1e6 | 0.025 | 6 (34)   | 26.2 (3.2GB) | 86.4 (6.1GB)   |
|     |     | 0.050 | 6 (34)   | 53.1 (5.2GB) | 158.0 (9.6GB)  |
| 5e3 | 1e6 | 0.025 | 5 (27)   | 88.1 (4.5GB) | 277.6 (10.0GB) |
| 10e3| 1e6 | 0.010 | 8 (44)   | 87.1 (4.3GB) | 321.1 (8.5GB)  |

6.2 Transportation problem

In this problem, $s$ supplies $a_i$ ($i = 1, \ldots, s$) of goods must be transported to meet $t$ demands $b_j$ ($j = 1, \ldots, t$) of customers and the cost of transporting one unit of $i$th good to $j$th customer is $c_{ij}$. The objective is to find a transportation plan denoted by $x_{ij}$ to solve the following LP:

$$\begin{align*}
\min & \sum_{i=1}^{s} \sum_{j=1}^{t} c_{ij} x_{ij} \\
\text{s.t.} & \sum_{j=1}^{t} x_{ij} = a_i, \quad i \in [s] \\
& \sum_{i=1}^{s} x_{ij} = b_j, \quad j \in [t] \\
& x_{ij} \geq 0, \quad \forall i \in [s], j \in [t].
\end{align*}$$

In the above problem, we assume that $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j$. We can write the transportation LP compactly as follows:

$$\min \left\{ \langle C, X \rangle \mid A(X) = [a;b], X \geq 0 \right\} \quad (43)$$

where

$$A(X) = \begin{bmatrix} \hat{e}^T \otimes I_s \\ I_t \otimes e^T \end{bmatrix} \vec(X),$$

$e \in \mathbb{R}^s$ and $\hat{e} \in \mathbb{R}^t$ are vector of all ones, and $\vec(X)$ is the $st$-dimensional column vector obtained from $X$ by concatenating its columns sequentially.

In Table 2, we report the results for some randomly generated transportation instances. For each pair of given $s,t$, we generate a random transportation instance as follows:

```plaintext
M=abs(rand(s,t)); a=sum(M,2); b=sum(M,1)'; C=ceil(100*rand(s,t));
```

Note that we give Gurobi the most favorable parameter selection. Firstly, we turn off the presolve phase in Gurobi as this phase is too time consuming (about 20–30% of the total time) and there is no benefit in cutting down the computation time per iteration. Secondly, we turn on its crossover capability which typically helps to cut down the number of expensive interior-point iterations.

We can observe that for this class of problems, SNIPAL is able to outperform the highly powerful solver Gurobi by a factor of about 3–5 times in terms of computation times. Moreover, our solver SNIPAL consumed less peak memory than Gurobi. For the largest instance where the primal LP has 12,000 linear constraints and 27 millions variables, our solver is at least five times faster than Guorbi, and it only needs 5.4GB of RAM whereas Gurobi required 12.8GB.
Table 2: Numerical results for transportation LPs.

| s  | t   | it (itsub) | time (RAM) | Gurobi time (RAM) | Gurobi optimal  |
|----|-----|-----------|------------|-------------------|-----------------|
| 2000 | 3000 | 5 (17)    | 14.3 (1.8GB) | 40.1 (4.8GB)     | 4.78715022e+06  |
| 2000 | 4000 | 5 (18)    | 19.3 (2.1GB) | 55.2 (6.5GB)     | 6.38301484e+06  |
| 2000 | 6000 | 5 (18)    | 28.8 (3.4GB) | 117.5 (8.9GB)    | 9.57359762e+06  |
| 3000 | 4500 | 5 (17)    | 34.6 (3.5GB) | 103.8 (9.2GB)    | 1.07707226e+07  |
| 3000 | 6000 | 5 (17)    | 45.4 (4.0GB) | 148.2 (10.3GB)   | 1.43597750e+07  |
| 3000 | 9000 | 5 (20)    | 55.1 (5.4GB) | 281.7 (12.8GB)   | 2.15412784e+07  |

6.3 Generalized transportation problem

The Generalized Transportation Problem (GTP) was introduced by Ferguson and Dantzig [15] in their study of an aircraft routing problem. Eisemann and Lourie [17] applied it to the machine loading problem. In that problem, there are \( m \) types of machines which can produce \( n \) types of products such that machine \( i \) would take \( h_{ij} \) hours at the cost of \( c_{ij} \) to produce one unit of product \( j \). It is assumed that machine \( i \) is available for at most \( a_i \) hours, and the demand for product \( j \) is \( b_j \). The problem is to determine \( x_{ij} \), the amount of product \( j \) to be produced on machine \( i \) during the planning period so that the total cost is minimized, namely,

\[
\begin{align*}
\min & \quad \sum_{i=1}^{s} \sum_{j=1}^{t} c_{ij} x_{ij} \\
\text{s.t.} & \quad \sum_{j=1}^{t} h_{ij} x_{ij} = a_i, \quad i \in [s] \\
& \quad \sum_{i=1}^{s} x_{ij} = b_j, \quad j \in [t] \\
& \quad x_{ij} \geq 0, \quad \forall \ i \in [s], \ j \in [t].
\end{align*}
\]

In addition to assuming that \( \sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j \), we also apply the normalization \( \sum_{i=1}^{s} \sum_{j=1}^{t} h_{ij} = st \).

Table 3 presents the results for randomly generated generalized transportation LPs where \( a, b, c \) are generated as in the last subsection. The weight matrix \( H = (h_{ij}) \) is generated by setting \( H = \text{rand}(s,t); \quad H = (s*t/\text{sum(\text{sum}(H))))*H \). We can observe that SNIPAL can be up to 5 times faster than Gurobi when the problems are large.

Table 3: Numerical results for generalized transportation LPs.

| s  | t   | it (itsub) | time (RAM) | Gurobi time (RAM) | Gurobi optimal  |
|----|-----|-----------|------------|-------------------|-----------------|
| 2000 | 2000 | 6 (22)    | 20.2 (1.6GB) | 23.6 (2.9GB)     | 3.19127056e+06  |
| 2000 | 4000 | 5 (18)    | 23.7 (2.5GB) | 52.5 (5.5GB)     | 6.38302875e+06  |
| 2000 | 6000 | 5 (19)    | 36.5 (3.6GB) | 101.4 (8.0GB)    | 9.57359762e+06  |
| 3000 | 3000 | 5 (18)    | 43.8 (3.4GB) | 61.7 (6.0GB)     | 7.18025216e+06  |
| 3000 | 6000 | 5 (17)    | 57.0 (4.0GB) | 150.8 (12.9GB)   | 1.43598011e+07  |
| 3000 | 9000 | 5 (19)    | 70.6 (5.8GB) | 347.7 (13.1GB)   | 2.15412784e+07  |
6.4 Covering and packing LPs

Given a nonnegative matrix \( A \in \mathbb{R}^{m \times n} \) and cost vector \( c \in \mathbb{R}_+^n \), the covering and packing LPs are defined by

(Covering) \[ \min \left\{ \langle c, x \rangle \mid Ax \geq e, x \geq 0 \right\} \]

(Packing) \[ \min \left\{ \langle -c, x \rangle \mid Ax \leq e, x \geq 0 \right\} \]

It is easy to see that by adding a slack variable, the above problems can be converted into the standard form expressed in (P).

In our numerical experiments in Table 4, we generate \( A \) and \( c \) randomly as follows:

\[
\begin{align*}
    c &= \text{rand}(n,1); \\
    A &= \text{sprand}(m,n,\text{den}); \\
    A &= \text{round}(A);
\end{align*}
\]

Table 4 presents the numerical performance of Snipal versus Gurobi on some randomly generated large scale covering and packing LPs. As we can observe, Snipal is competitive against Gurobi in solving these large scale LPs, and the former can be up to three times faster than Gurobi.

Table 4: Numerical results for covering and packing LPs.

| Type | m  | n  | den | it (itsub) | time | Gurobi time | Gurobi optimal |
|------|----|----|-----|------------|------|-------------|----------------|
| C    | 1e3| 5e5| 0.2 | 22 (148)   | 47.6 | 113.6       | 1.79109338e-02 |
| C    | 2e3| 5e5| 0.1 | 25 (151)   | 92.2 | 139.9       | 6.32319966e-02 |
| C    | 2e3| 1e6| 0.05| 24 (160)   | 83.8 | 191.2       | 6.69164798e-02 |
| C    | 3e3| 5e6| 0.01| 22 (140)   | 155.2| 461.6       | 9.49310633e-02 |
| P    | 1e3| 5e5| 0.2 | 28 (160)   | 45.6 | 94.9        | -1.37831164e+01 |
| P    | 2e3| 5e5| 0.1 | 29 (160)   | 91.0 | 132.2       | -2.54519423e+01 |
| P    | 2e3| 1e6| 0.05| 30 (173)   | 68.9 | 148.2       | -5.84484904e+01 |
| P    | 3e3| 5e6| 0.01| 34 (231)   | 154.8| 460.3       | -6.12178098e+02 |

6.5 LP arising from correlation clustering

A correlation clustering problem \[ \] is defined over an undirected graph \( G = (V,E) \) with \( p \) nodes and edge weights \( c_e \in \mathbb{R} \) (for each \( e \in E \)) that is interpreted as a confidence measure of the similarity or dissimilarity of the edge’s end nodes. In general, for \( e = (u,v) \in E \), \( c_e \) is given a negative value if \( u,v \) are dissimilar, and a positive value if \( u,v \) are similar. For the goal of finding a clustering that minimizes the disagreements, the problem can be formulated as an integer programming problem as follows. Suppose that we are given a clustering \( S = \{S_1, \ldots, S_N\} \) where each \( S_t \subset V \) for some \( t = 1, \ldots, N \), denotes a cluster. For each edge \( e = (u,v) \in E \), set \( y_e = 0 \) if \( u,v \in S_t \) for some \( t \), and set \( y_e = 1 \) otherwise. Observe that \( 1 - y_e \) is 1 if \( u,v \) are in the same cluster, and 0 if \( u,v \) are in different clusters. Now define the constants

\[
m_e = \min\{0,c_e\}, \quad p_e = \max\{0,c_e\}.
\]

Then the cost of disagreements for the clustering \( S \) is given by \( \sum_{e \in E} m_e (1 - y_e) + \sum_{e \in E} p_e y_e \).

A version of the correlation clustering problem is to find a valid assignment (i.e., it satisfies the triangle inequalities) of \( y_e \) for all \( e \in E \) to minimize the disagreements’ cost. We consider the
relaxation of this integer program to get the following LP:

\[
\begin{align*}
\min & \quad \sum_{(i,j) \in E} m_{ij} (1 - y_{ij}) + \sum_{(i,j) \in E} P_{ij} y_{ij} \\
\text{s.t.} & \quad -y_{ij} \leq 0, \quad y_{ij} \leq 1 \quad \forall (i,j) \in E \\
& \quad -y_{ij} - y_{jk} + y_{ik} \leq 0 \quad \forall 1 \leq i < j < k \leq n, \text{ such that } (i,j),(j,k),(i,k) \in E.
\end{align*}
\]

In the above formulation, we assumed that the edge set \( E \) is a subset of \( \{(i,j) \mid 1 \leq i < j \leq p\} \). Let \( M \) be the number of all possible triangles in \( E \). Define \( T : \mathbb{R}^{|E|} \rightarrow \mathbb{R}^M \) to be the linear map that maps \( y \) to all the \( M \) terms \(-y_{ij} - y_{jk} + y_{ik}\) in the triangle inequalities. We can express the above LP in the dual form as follows:

\[
\langle m, 1 \rangle - \max \left\{ \langle m - p, y \rangle \mid \begin{bmatrix} -I & I & T \\
- & & \\
\end{bmatrix} y \leq \begin{bmatrix} 0 \\
1 \\
0 \\
\end{bmatrix} \right\},
\]

and the corresponding primal LP is given by

\[
\langle m, 1 \rangle - \min \left\{ \langle 0; 1; 0 \rangle, x \mid [-I, I, T^*] x = m - p, x \in \mathbb{R}^{|2|E|+M} \right\}. \tag{44}
\]

Observe that the primal LP has \(|E|\) equality constraints and a large number of \(2|E| + M\) variables.

In Table 5 we evaluate the performance of our algorithm on correlation clustering LPs on data that were used in [38]. One can observe that for the LP problem (44), our solver Snipal is much more efficient than Gurobi, and the former can be up to 140 times faster for the largest problem.

Table 5: Numerical results for correlation clustering LPs.

| Data      | \( p \) | \(|E|\)   | \(2|E| + M\) | it (itsub) | time | Gurobi time (RAM) | Gurobi optimal |
|-----------|--------|---------|--------------|-----------|------|-------------------|---------------|
| planted(5)| 200    | 19900   | 1353200      | 6 (75)    | 40.3 | 1159.6 (2.6GB)    | 2.42924179e+03|
| planted(10)| 200   | 19900   | 1353200      | 7 (97)    | 35.7 | 1459.5 (2.6GB)    | 1.48767133e+03|
| planted(5)| 300    | 44850   | 4544800      | 7 (84)    | 162.9| 15112.7 (8.9GB)   | 5.55522547e+03|
| planted(10)| 300   | 44850   | 4544800      | 9 (135)   | 165.2| 23803.1 (8.9GB)   | 3.1496312e+03 |
| stocks    | 200    | 19900   | 1353200      | 6 (72)    | 63.2 | 1639.9 (2.6GB)    | 1.5309430e+03 |
| stocks    | 300    | 44850   | 4544800      | 7 (80)    | 291.9| 27137.4 (9.0GB)   | 3.24858698e+03|

7 Conclusion

In this paper, we proposed a method called Snipal for solving the primal and dual LP problems. Snipal is an inexact proximal augmented Lagrangian method where the inner subproblems are solved via an efficient semismooth Newton method. By connecting the inexact proximal augmented Lagrangian method with the preconditioned proximal point algorithm, we are able to show the global and local asymptotic superlinear convergence of Snipal. Our novel analysis also reveals that Snipal can enjoy a certain finite termination property. To achieve high performance, we further conducted a comprehensive study on efficient approaches for tackling the large linear systems in the semismooth Newton method. Our findings indicate that the linear systems solved in Snipal have uniformly bounded condition numbers and are much better conditioned than those involved in the classic interior point algorithms. Building upon all the aforementioned desirable properties, our algorithm Snipal has demonstrated a clear computational advantage in solving various large-scale LP problems in the numerical experiments when tested against one of the best commercial LP solvers Gurobi.
Appendix

Here we show that the dual of (17) with $\tau = 0$ is given by (32). Consider the augmented Lagrangian function

$$\inf_y L_{\sigma_k}(y; x^k) = \inf_y \max_x \left\{ l(y; x) - \frac{1}{2\sigma} \|x - x^k\|^2 \right\} = \max_x \left\{ -\frac{1}{2\sigma} \|x - x^k\|^2 + \inf_y l(y; x) \right\}$$

$$= \max_x \left\{ -\delta_K(x) - \langle c, x \rangle - \frac{1}{2\sigma} \|x - x^k\|^2 \mid Ax = b \right\},$$

where $l(y; x) = -b^T y - \langle x, c - A^* y \rangle - \delta_K(x)$ for any $(y, x) \in \mathbb{R}^m \times \mathbb{R}^n$. The interchange of $\inf_y$ and $\max_x$ follows from the growth properties in $x$ of the “minimaximand” in question [33, Theorem 37.3]. See also the proof of [35, Proposition 6].

References

[1] N. Bansal, A. Blum, and S. Chawla, Correlation clustering, IEEE Symp. on Foundations of Computer Science, 2002.

[2] J. F. Bonnans, J. Ch. Gilbert, C. Lemaréchal, and C. A. Sagastizábal, A family of variable metric proximal methods, Mathematical Programming, 68 (1995), pp. 15–47.

[3] J. V. Burke and M. Qian, A variable metric proximal point algorithm for monotone operators, SIAM Journal on Control and Optimization, 37 (1999), pp. 353–375.

[4] J. V. Burke and M. Qian, On the local super-linear convergence of a matrix secant implementation of the variable metric proximal point algorithm for monotone operators, in Reformulation - Nonsmooth, Piecewise Smooth, Semismooth and Smoothing Methods, M. Fukushima and L. Qi, eds., Kluwer Academic Publishers, Norwell, MA, 1999, pp. 317–334.

[5] J. V. Burke and M. Qian, On the superlinear convergence of the variable metric proximal point algorithm using Broyden and BFGS matrix secant updating, Mathematical Programming, 88 (2000), pp. 157–181.

[6] J. S. Chai and K.-C. Toh, Preconditioning and iterative solution of symmetric indefinite linear systems arising from interior point methods for linear programming, Computational Optimization and Applications, 36 (2007), pp. 221–247.

[7] J. Chen and S. Burer, A first-order smoothing technique for a class of large-scale linear programs, SIAM Journal on Optimization, 24 (2014), pp. 598–620.

[8] L. Chen, X. D. Li, D. F. Sun and K.-C. Toh, On the equivalence of inexact proximal ALM and ADMM for a class of convex composite programming, arXiv:1803.10803, 2018.

[9] X. Chen and M. Fukushima, Proximal quasi-Newton methods for nondifferentiable convex optimization, Math. Program., 85 (1999), pp. 313–334.

[10] F. Clarke, Optimization and Nonsmooth Analysis, John Wiley and Sons, New York, 1983.
[11] Y. Cui, K. Morikuni, T. Tsuchiya, and K. Hayami, Implementation of interior-point methods for LP based on Krylov subspace iterative solvers with inner-iteration preconditioning, preprint available at https://arxiv.org/abs/1604.07491

[12] R. Durier, On locally polyhedral convex functions, in Trends in Mathematical Optimization (Irsee, 1986), Internat. Schriftenreihe Numer. Math., 84, Birkhäuser, Basel, 1988, pp. 55–66.

[13] J. Eckstein and D. P. Bertsekas, On the Douglas-Rachford splitting method and the proximal point algorithm for maximal monotone operators, Mathematical Programming, 55 (1992), pp. 293–318.

[14] Yu. Evtushenko, A.I. Golikov, and N. Mollaverdy, Augmented Lagrangian method for large-scale linear programming problems, Optimization Methods and Software, 20 (2005), pp. 515–524.

[15] A.R. Ferguson and G.B. Dantzig, The allocation of aircrafts to routes - an example of linear programming under uncertain demand, Management Science, 3 (1956).

[16] A. Fischer and C. Kanzow, On finite termination of an iterative method for linear complementarity problems, Mathematical Programming, 74 (1996), pp. 279–292.

[17] K. Eisemann and J.R. Lourie, The machine loading problem, IBM 704 Program, BML-1, IBM Application Library, New York, 1959.

[18] G. Al-Jeiroudi, J. Gondzio and J.A.J. Hall, Preconditioning indefinite systems in interior point methods for large scale linear optimization, Optimization Methods and Software, 23 (2008), pp. 345–363.

[19] J.-B. Hiriart-Urruty, J.-J. Strodiot, and V. H. Nguyen, Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data, Applied Mathematics and Optimization, 11 (1984), pp. 43–56.

[20] C. Kanzow, H. Qi, and L. Qi, On the minimum norm solution of linear programs, Journal of Optimization Theory and Applications, 116 (2003), pp. 333–345.

[21] D. Klatte and B. Kummer, Nonsmooth Equations in Optimization, Regularity, Calculus, Methods and Applications, Kluwer Academic Publishers, Dordrecht, the Netherlands, 2002.

[22] X.D. Li, D.F. Sun, and K.C. Toh, QSDPNAL: A two-phase augmented Lagrangian method for convex quadratic semidefinite programming, Mathematical Programming Computation, 10 (2018), pp. 703–743.

[23] X.D. Li, D.F. Sun, and K.C. Toh, On the efficient computation of a generalized Jacobian of the projector over the Birkhoff polytope, Mathematical Programming, in print, arXiv:1702.05934, 2018.

[24] F. J. Luque, Asymptotic convergence analysis of the proximal point algorithm, SIAM Journal on Control and Optimization, 22 (1984), pp. 277–293.
[25] R. De Leone and O. L. Mangasarian, *Serial and parallel solution of large scale linear programs by augmented Lagrangian successive overrelaxation*, in A. Kurzhanski, K. Neumann, and D. Pallaschke, editors, Optimization, Parallel Processing and Applications, pp. 103–124, Berlin, 1988. Springer-Verlag.

[26] O. L. Mangasarian and R. R. Meyer, *Nonlinear perturbation of linear programs*, SIAM J. Control and Optimization, 17 (1979), pp. 745–752.

[27] O. L. Mangasarian, *A Newton method for linear programming*, Journal of Optimization Theory and Applications, 121 (2004), pp. 1–18.

[28] A. R. L. Oliveira and D. C. Sorensen, *A new class of preconditioners for large-scale linear systems from interior point methods for linear programming*, Linear Algebra and its Applications, 394, 1-24, 2005.

[29] L. A. Parente, P. A. Lotito, and M. V. Solodov, *A class of inexact variable metric proximal point algorithms*, SIAM Journal on Optimization, 19 (2008), pp. 240–260.

[30] L. Qi and X. Chen, *A preconditioning proximal Newtons method for nondifferentiable convex optimization*, Mathematical Programming, 76 (1995), pp. 411–430.

[31] S. M. Robinson, *An implicit-function theorem for generalized variational inequalities*, Technical summary report no. 1672, Mathematics Research Center, University of Wisconsin-Madison, (1976); available from National Technical Information Service under Accession No. ADA031952.

[32] S. M. Robinson, *Some continuity properties of polyhedral multifunctions*, in Mathematical Programming at Oberwolfach, Math. Program. Stud., Springer, Berlin, Heidelberg, 1981, pp. 206–214.

[33] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, N.J., 1970.

[34] R. T. Rockafellar, *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization, 14 (1976), pp. 877–898.

[35] R. T. Rockafellar, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Mathematics of operations research, 1 (1976), pp. 97–116.

[36] R. T. Rockafellar and R. J.-B. Wets, *Variational Analysis*, Springer, New York, 2009.

[37] D. F. Sun, J. Y. Han, and Y. Zhao, *On the finite termination of the damped-newton algorithm for the linear complementarity problem*, Acta Mathematica Applicatae Sinica, 21 (1998), pp. 148–154.

[38] N. Veldt, A. Wirth, and D. Gleich, *Correlation clustering with low-rank matrices*, Proceedings of the 26th International Conference on World Wide Web, 2017, pp. 1025–1034.

[39] S. J. Wright, *Implementing proximal point methods for linear programming*, Journal of Optimization Theory and Applications, 65 (1990), pp. 531–554.
[40] L.Q. Yang, D.F. Sun, and K.C. Toh, *SDPNAL+: a majorized semismooth Newton-CG augmented Lagrangian method for semidefinite programming with nonnegative constraints*, Mathematical Programming Computation, 7 (2015), pp. 331–366.

[41] E.-H. Yen, K. Zhong, C.-J. Hsieh, P. K. Ravikumar, and I. S. Dhillon, *Sparse linear programming via primal and dual augmented coordinate descent*, Advances in Neural Information Processing Systems, 2015, pp. 2368–2376.

[42] X.-Y. Zhao, D. F. Sun, and K.-C. Toh, *A Newton-CG augmented Lagrangian method for semidefinite programming*, SIAM Journal on Optimization, 20 (2010), pp. 1737–1765.