ON THE DENSITY OF COPRIME \( m \)-TUPLES OVER HOLOMORPHY RINGS

GIACOMO MICHELI AND RETO SCHNYDER

ABSTRACT. Let \( \mathbb{F}_q \) be a finite field, \( \mathbb{F}/\mathbb{F}_q \) be a function field of genus \( g \) having full constant field \( \mathbb{F}_q \), \( S \) a set of places of \( \mathbb{F} \) and \( H \) the holomorphy ring of \( S \). In this paper we compute the density of coprime \( m \)-tuples of elements of \( H \). As a side result we obtain that whenever the complement of \( S \) is finite, the density only depends on the number of \( \mathbb{F}_q \)-rational points of the curve associated to the function field, for \( i \) that runs from 1 to \( g \). In the genus zero case, classical results for the density of coprime \( m \)-tuples of polynomials are obtained as corollaries.

Keywords: Function fields, Density, Polynomials, Riemann-Roch spaces, Zeta function.
MSC: 11R58, 11M38, 11T06

1. Introduction

Let \( \mathbb{F}_q \) be a finite field with \( q \) elements and let \( \mathbb{F} \) be an algebraic function field\(^1\) having full constant field \( \mathbb{F}_q \). Let \( C \) be the set of places of \( \mathbb{F} \) and \( S \subseteq C \) be a non-empty proper subset. The holomorphy ring of \( S \) is \( H = \bigcap_{P \in S} \mathcal{O}_P \), where \( \mathcal{O}_P \) is the valuation ring of the place \( P \).

In what follows we will say that an \( m \)-tuple of elements of \( H \) is coprime if its components generate the unit ideal in \( H \) (in analogy to the case of the ring of integers in [1]). In this paper we define a notion of density for subsets of \( H^m \), using Moore-Smith convergence for nets [3, Chapter 2]. We then wish to study the density of the set of coprime \( m \)-tuples in \( H \), considered as a subset of \( H^m \).

The special case \( F = \mathbb{F}_q(x) \) and \( H = \bigcap_{P \neq P_{\infty}} \mathcal{O}_P = \mathbb{F}_q[x] \) has been studied for \( m = 2 \) in [8] and more generally in [2]. We will explain how to interpret the densities presented in these papers as particular cases of our general framework. In fact, using the Riemann-Roch Theorem and the absolute convergence of the Zeta function of \( F \), we are able to show that the density of coprime \( m \)-tuples of elements of a holomorphy ring exists and is equal to \( \frac{1}{Z_H(q^{-m})} \), where \( Z_H \) is the Zeta function of the holomorphy ring, which will be defined in Section 2. Moreover, since our definition of density is independent of the choice of a graded basis of \( H \), as a side effect of Theorem 1, we find an additional symmetry for the set of coprime \( m \)-tuples that is explained in Remark 7.

Finally, we provide an example in the case of the affine ring of coordinates of an elliptic curve to show a concrete application of the main result.

The results in this paper provide a function field version of a classical result for the ring of integers, where the natural density of the set of coprime \( m \)-tuples of \( \mathbb{Z} \) is proven

\(^1\)In this note we will mostly use the language and notation of [7].
to be equal to $\frac{1}{\zeta(m)}$, $\zeta$ being the classical Riemann zeta function (see for example [5]). Similar results also hold in the rings of integers of arbitrary number fields (see [1] and [6]).

1.1. **Notation.** Let $F/\mathbb{F}_q$ be an algebraic function field with full constant field $\mathbb{F}_q$, let $g$ be the genus of $F$, and let $\mathcal{C}$ be the set of its places. Let $H$ be the holomorphy ring of a nonempty set of places $\mathcal{S} \subseteq \mathcal{C}$. For a fixed positive integer $m$, we wish to study the set of coprime $m$-tuples of elements of the ring $H$. Let us denote this set by $U$:

$$U := \{ f = (f_1, \ldots, f_m) \in H^m \mid I_f = H \},$$

where $I_f$ denotes the ideal of $H$ generated by the set $\{f_1, \ldots, f_m\}$.

Define furthermore $\mathcal{D} := \{ D \in \text{Div}(F) \mid D \geq 0 \land \text{supp}(D) \subseteq \mathcal{C} \setminus \mathcal{S} \}$, the set of positive divisors supported away from $\mathcal{S}$. It follows that

$$H = \bigcup_{D \in \mathcal{D}} \mathcal{L}(D),$$

where $\mathcal{L}(D)$ denotes the Riemann-Roch space\(^2\) associated to a divisor $D$. Recall that we have a bijection between the set of places $\mathcal{S}$ and the maximal ideals of $H$ given by $P \mapsto P \cap H =: P_H$ (see for example [7, Proposition 3.2.9]). In analogy to the natural density of integers, we define the *superior density* of a subset $L \subseteq H^m$ as

$$\mathbb{D}(L) := \limsup_{D \in \mathcal{D}} \frac{|L \cap \mathcal{L}(D)^m|}{|\mathcal{L}(D)^m|}.$$  

This limit can be defined via Moore-Smith convergence [3, Chapter 2]. To be precise, the set of divisors $\mathcal{D}$ with the usual partial order $\leq$ is a directed set, so the map from $\mathcal{D}$ to the topological space $\mathbb{R}$ defined as

$$D \mapsto \frac{|L \cap \mathcal{L}(D)^m|}{|\mathcal{L}(D)^m|}$$

is a net. Now, since $\mathbb{R}$ is Hausdorff, the definition in (1) is well posed. Analogously one can define the *inferior density* as

$$\underline{\mathbb{D}}(L) := \liminf_{D \in \mathcal{D}} \frac{|L \cap \mathcal{L}(D)^m|}{|\mathcal{L}(D)^m|}.$$  

Moreover, whenever $\mathbb{D}(L) = \underline{\mathbb{D}}(L)$, we call this value the *density* of $L$ and denote it by $\mathbb{D}(L)$.

2. **The density of $U$**

Recall that the Zeta function of the function field $F$ is given by

$$Z_F(T) := \prod_{P \in \mathcal{C}} \left(1 - T^{\deg(P)}\right)^{-1}$$

\(^2\)The reader should refer to the definition [7, Def. 1.4.4]
for $0 < T < q^{-1}$. Analogously, we define the Zeta function of the holomorphy ring $H$ corresponding to the set of places $S$ as

$$Z_H(T) := \prod_{P \in S} \left(1 - T^{\deg(P)}\right)^{-1}.$$  

We will now state our main result.

**Theorem 1.** The density of the set of coprime tuples of length $m \geq 2$ of the holomorphy ring $H$ is $\frac{1}{Z_H(q^m)}$.

**Proof.** We first enumerate the set of places of $S = \{Q_1, Q_2, \ldots, Q_t, \ldots\}$. Let us define

$$U_t := \{f = (f_1, \ldots, f_m) \in H^m \mid I_f \not\subseteq (Q_i)_H \quad \forall i \in \{1, \ldots, t\}\}$$

and notice that $U_t \supseteq U$. Observe that the condition $I_f \not\subseteq (Q_i)_H$ is equivalent to the fact that for each $i$ there exists at least one $f_j$ that does not belong to $Q_i$. Consider now the projection $\pi: H \rightarrow H/((Q_1)_H \cdots (Q_t)_H)$ and observe that

$$H/((Q_1)_H \cdots (Q_t)_H) \cong \prod_{i=1}^{t} H/(Q_i)_H \cong \prod_{i=1}^{t} \mathbb{F}_{q^{\deg Q_i}},$$

by the Chinese remainder theorem over the ideals $(Q_i)_H$. This gives us a homomorphism

$$\phi: H \rightarrow \prod_{i=1}^{t} \mathbb{F}_{q^{\deg Q_i}},$$

which we can extend to $m$-tuples by

$$\hat{\phi}: H^m \rightarrow \prod_{i=1}^{t} \mathbb{F}_{q^{\deg Q_i}}.$$  

By construction, this homomorphism satisfies

$$U_t = \hat{\phi}^{-1}\left(\prod_{i=1}^{t} (\mathbb{F}_{q^{\deg Q_i}} \setminus \{0\})\right).$$

Consider now a divisor $D \in D$. We wish to count the number of elements in $U_t \cap \mathcal{L}(D)^m$. First, we will show that $\phi$ maps $\mathcal{L}(D)$ surjectively onto $\prod_{i=1}^{t} \mathbb{F}_{q^{\deg Q_i}}$ if $\deg D$ is large enough.

For this, note that the image of $\mathcal{L}(D)$ under $\pi$ is $\mathcal{L}(D)/(\mathcal{L}(D) \cap ((Q_1)_H \cdots (Q_t)_H))$. The space

$$\mathcal{L}(D) \cap ((Q_1)_H \cdots (Q_n)_H) = \mathcal{L}(D) \cap Q_1 \cap \cdots \cap Q_t$$

consists of all elements in $\mathcal{L}(D)$ with at least a root at each $Q_i$, so it is equal to $\mathcal{L}(D - \sum_{i=1}^{t} Q_i)$ (note that the $Q_i$ cannot be in the support of $D$). Hence, its dimension as an $\mathbb{F}_q$-vector space is $\ell(D - \sum_{i=1}^{t} Q_i)$, which is equal to $\deg D - \sum_{i=1}^{t} \deg Q_i + 1 - g$ if $\deg D$ is large enough by Riemann-Roch Theorem. On the other hand, the dimension of $\mathcal{L}(D)$ is then $\ell(D) = \deg D + 1 - g$, and so the image has dimension $\sum_{i=1}^{t} \deg Q_i$, the same as $\prod_{i=1}^{t} H/(Q_i)_H$. Therefore $\pi$ and $\phi$ restricted to $\mathcal{L}(D)$ are surjective.

We can now count the elements of $U_t \cap \mathcal{L}(D)^m$ using (2). As we have just seen, the dimension of the kernel of $\phi$ restricted to $\mathcal{L}(D)$ is $\ell(D - \sum_{i=1}^{t} Q_i)$, so each element of
\[ \prod_{i=1}^{t} (\mathbb{F}_{q^m}^{m \cdot \deg Q_i} \setminus \{0\}) \] is the image under \( \hat{\phi} \) of exactly \( q^m(t - \sum_{i=1}^{t} \deg Q_i) \) elements of \( U_t \cap \mathcal{L}(D)^m \). Hence we get
\[ \frac{|U_t \cap \mathcal{L}(D)^m|}{|\mathcal{L}(D)^m|} = q^m(t - \sum_{i=1}^{t} \deg Q_i) \cdot \prod_{i=1}^{t} (q^m \deg Q_i - 1) = \prod_{i=1}^{t} (1 - q^{-m \deg Q_i}) \]
if \( \deg D \) is large enough. It follows that the density of \( U_t \) is well-defined and equals
\[ \mathbb{D}(U_t) = \lim_{D \to \infty} \frac{|U_t \cap \mathcal{L}(D)^m|}{|\mathcal{L}(D)^m|} = \prod_{i=1}^{t} (1 - q^{-m \deg Q_i}). \]

Since \( U \subseteq U_t \), it follows that \( \mathbb{D}(U) \leq \mathbb{D}(U_t) \).

To get an estimate in the other direction, let us write \( (\mathcal{L}(D) \cap U) \cup (\mathcal{L}(D) \cap (U \setminus U)) = \mathcal{L}(D) \cap U \). We have
\[ \mathbb{D}(U) = \liminf_{D \to \infty} \frac{|U \cap \mathcal{L}(D)^m|}{q^m(t)} \geq \lim_{D \to \infty} \frac{|U_t \cap \mathcal{L}(D)^m|}{q^m(t)} - \limsup_{D \to \infty} \frac{|(U_t \setminus U) \cap \mathcal{L}(D)^m|}{q^m(t)} , \]
hence we have the inequalities
\[ (3) \quad \mathbb{D}(U_t) \geq \mathbb{D}(U) \geq \mathbb{D}(U_t) - \limsup_{D \to \infty} \frac{|(U_t \setminus U) \cap \mathcal{L}(D)^m|}{q^m(t)} . \]

Now, passing to the limit in \( t \), we get that \( \lim_{t \to \infty} \mathbb{D}(U_t) = 1/\mathcal{Z}_H(q^{-m}) \). Therefore it remains to prove that
\[ \lim_{t \to \infty} \limsup_{D \to \infty} \frac{|(U_t \setminus U) \cap \mathcal{L}(D)^m|}{q^m(t)} = 0. \]

In order to prove the last claim, let us denote by \( Q_t \) the set \( \{ Q_1, \ldots, Q_t \} \), and notice
\[ U_t \setminus U \subseteq \bigcup_{P \in S \setminus Q_t} \{ A \in H^m \mid I_A \subseteq P_H \} = \bigcup_{P \in S \setminus Q_t} P_m^m, \]
where by \( P_m^m \) we mean the Cartesian product of \( m \) copies of the ideal \( P_H \). Fix now a divisor \( D \in \mathcal{D} \). It follows that
\[ (U_t \setminus U) \cap \mathcal{L}(D)^m \subseteq \bigcup_{P \in S \setminus Q_t} (P_m \cap \mathcal{L}(D))^m = \bigcup_{P \in S \setminus Q_t} \mathcal{L}(D - P)^m = \bigcup_{P \in S \setminus Q_t \text{ \ deg } P \leq \deg D} \mathcal{L}(D - P)^m. \]

The last equality holds because \( \mathcal{L}(D - P) = 0 \) if \( \deg D - \deg P < 0 \). With this containment, we can now estimate the last term of (3):
\[ \limsup_{D \to \infty} \frac{|\mathcal{L}(D) \cap (U_t \setminus U)|}{q^m(t)} \leq \limsup_{D \to \infty} \left| \bigcup_{P \in S \setminus Q_t \text{ \ deg } P \leq \deg D} \mathcal{L}(D - P)^m \right| \cdot q^{-m \ell(D)} \]
\[ \leq \limsup_{D \to \infty} \sum_{P \in S \setminus Q_t \text{ \ deg } P \leq \deg D} |\mathcal{L}(D - P)^m| \cdot q^{-m \ell(D)} \]
\[ = \limsup_{D \to \infty} \sum_{P \in S \setminus Q_t \text{ \ deg } P \leq \deg D} q^{m(\ell(D) - \ell(P))} \]
Now observe that we have \( \ell(D) \geq \deg(D) + 1 - g \) and \( \ell(D - P) \leq \deg(D - P) + 1 \), since \( \deg(D - P) \geq 0 \) [7, Eq. 1.21 and Theorem 1.4.17]. It follows that the above is less or
equal to
\[
\limsup_{D \in \mathcal{D}} \sum_{P \in \mathcal{S} \setminus \mathcal{Q}, \deg P \leq \deg D} q^m(q^{-\deg P}) = \limsup_{d \to \infty} \sum_{P \in \mathcal{S} \setminus \mathcal{Q}, \deg P \leq d} (q^{-m})^{\deg P} = q^m \sum_{P \in \mathcal{S} \setminus \mathcal{Q}, \deg P \leq d} (q^{-m})^{\deg P}.
\]
Observe that \( \sum_{P \in \mathcal{S} \setminus \mathcal{Q}, \deg P \leq d} (q^{-m})^{\deg P} \) is the tail of a subseries of the Zeta function of \( F \) evaluated at \( q^{-m} < q^{-1} \), which is absolutely convergent (see for example [4, Chapter 3]). As \( t \) goes to infinity, it converges to 0, from which our claim follows. \( \square \)

3. Consequences

The reader should observe that in Theorem 1 both \( \mathcal{S} \) and \( \mathcal{C} \setminus \mathcal{S} \) could possibly be infinite and the result will still hold. Nevertheless, the density depends on the Zeta function of the holomorphy ring, which may be hard to compute. First of all notice that this is not the case when \( \mathcal{S} \) is finite since under this condition \( \mathcal{Z}_H \) is a finite product. The following immediate corollary covers the case in which \( \mathcal{C} \setminus \mathcal{S} \) is finite.

**Corollary 2.** Let \( F \) be a function field, \( \mathcal{S} \) a set of places of \( F \) and \( H \) the holomorphy ring of \( \mathcal{S} \). Let \( L_F(T) \) be the \( L \)-polynomial of \( F \). Then
\[
\mathcal{Z}_H(q^{-m}) = \frac{L_F(q^{-m})}{(1 - q^{-m})(1 - q^{-m+1})} \prod_{P \in \mathcal{C} \setminus \mathcal{S}} \left(1 - \frac{1}{q^{\deg(P) - m}}\right).
\]

**Proof.** The corollary follows from Theorem 1, the definition of \( \mathcal{Z}_H \) and the expression of the Zeta function of \( F \) in terms of the \( L \)-polynomial. \( \square \)

**Remark 3.** Observe now that in the case where \( \mathcal{C} \setminus \mathcal{S} \) is finite, the density of coprime \( m \)-tuples of \( H \) essentially only depends on the degrees of the places in \( \mathcal{C} \setminus \mathcal{S} \) and the \( L \)-polynomial of the function field, which again only depends on the \( \mathbb{F}_q \)-rational points of the curve associated to the function field for \( i \in \{1, \ldots, g\} \) (see for example [7, Corollary 5.1.17]).

**3.1. An example.** Let \( \text{char}(\mathbb{F}_q) \neq 2, 3 \) for simplicity. Let \( a, b \in \mathbb{F}_q \) and \( p(x, y) = y^2 - x^3 - ax - b \) be a polynomial defining an elliptic curve \( E \) over \( \mathbb{F}_q \). Let us define
\[
A(E) := \mathbb{F}_q[x, y]/(p(x, y)).
\]
Let \( E(\mathbb{F}_q) \) denote the set of (projective) \( \mathbb{F}_q \)-rational points of \( E \) (i.e. the places of degree one of the function field of \( E \)).

**Corollary 4.** The density of \( m \)-tuples of coprime elements of \( A(E) \) is
\[
\mathbb{D}(U) = \frac{1 - q^{-m+1}}{1 + a_q q^{-m} + q^{-2m+1}}
\]
where \( a_q = q + 1 - |E(\mathbb{F}_q)| \).

**Proof.** Observe that the Zeta function of an elliptic curve is
\[
\mathcal{Z}_E(T) = \frac{1 + a_q T + q T}{(1 - q T)(1 - T)}.
\]
The result follows from Theorem 1 applied to the holomorphy ring $A(E) = \bigcap_{P \neq P_\infty} \mathcal{O}_P$ where $P_\infty$ is the place at infinity of $E$ with respect to $p(x, y)$.

\[ \text{Remark 5.} \] The reader should notice again that (4) depends only on the number of $\mathbb{F}_q$-rational points of $E$, since the genus of $E$ equals one. The probabilistic interpretation of Corollary 4 is the following: select uniformly at random $m$ elements of $A(E)$ of degree at most $N$, then the probability that they generate the unit ideal in $A(E)$ approaches

\[ \frac{1 - q^{-m+1}}{1 + a_qq^{-m} + q^{-2m+1}} \]

as $N \to \infty$.

3.2. \textbf{The case $F = \mathbb{F}_q(x)$}. In the remaining part of this section we show how the result of [2] about $\mathbb{F}_q[x]$ fits in our framework. We denote by $P_\infty$ the place at infinity of the function field $\mathbb{F}_q(x)$.

The reader should observe that the definition of density for $\mathbb{F}_q[x]$ given in [2, 8] agrees with ours for $H = \mathbb{F}_q[x] = \bigcap_{P \neq P_\infty} \mathcal{O}_P$, so we have [8, Theorem 1] as a corollary with $m = 2$. More generally, we obtain the result in [2] for unimodular rows over $\mathbb{F}_q[x]$:

\[ \text{Corollary 6.} \] Let $m > 1$ be an integer. The density of unimodular rows of length $m$ over $\mathbb{F}_q[x]$ is

\[ \mathbb{D}(U) = 1 - \frac{1}{q^{m-1}} \]

\[ \text{Proof.} \] It is enough to notice that the Zeta function of the function field $\mathbb{F}_q(x)$ (i.e. the Zeta function of the projective line) is

\[ Z_{\mathbb{F}_q(x)}(T) = \frac{1}{(1-T)(1-qT)} \]

and then the Zeta function of the holomorphy ring $\mathbb{F}_q[x] = \bigcap_{P \neq P_\infty} \mathcal{O}_P$ is

\[ Z_{\mathbb{F}_q[x]}(T) = \frac{1}{1 - qT} \]

The claim follows by inverting the expression above and evaluating at $q^{-m}$. \hfill \Box

\[ \text{Remark 7.} \] Another interesting implication of Theorem 1 is the fact that the result in Corollary 6 holds for any graded basis of $\mathbb{F}_q[x]$ in the following sense: the definition of density in [2] and [8] depends on the particular choice of the basis $\{1, x, \ldots, x^n, \ldots\}$ in the limit [2, Eq. 6]. Our result shows something more: for any covering of $\mathbb{F}_q[x]$ via a basis $\{p_0, \ldots, p_i, \ldots\}$ $\deg(p_i) = i$ for $\mathbb{F}_q[x]$ (as an $\mathbb{F}_q$-vector space) we get the same density. This essentially depends on the fact that the choice $\{\mathcal{L}(nP_\infty)\}_{n \in \mathbb{N}}$ as a covering of $\mathbb{F}_q[x]$ is “canonical”. 

\textbf{Acknowledgements}

The authors want to thank Andrea Ferraguti for useful discussions and suggestions. The authors are thankful to Swiss National Science Foundation grant number 149716 and Armasuisse.
ON THE DENSITY OF COPRIME $m$-TUPLES OVER HOLOMORPHY RINGS

References

[1] Andrea Ferraguti and Giacomo Micheli. On Cesaro Theorem for number fields. 2014. URL http://arxiv.org/abs/1409.6527.

[2] Xiangqian Guo and Guangyu Yang. The probability of rectangular unimodular matrices over $\mathbb{F}_q[x]$. Linear Algebra and its Applications, 438(6):2675–2682, 2013.

[3] John L. Kelley. General topology. New York: Van Nostrand, 1955.

[4] Carlos Moreno. Algebraic Curves over Finite Fields. Cambridge University Press, 1991. ISBN 9780511608766. Cambridge Books Online.

[5] J. E. Nymann. On the probability that $k$ positive integers are relatively prime. Journal of Number Theory, 4(5):469–473, 1972.

[6] B. D. Sittinger. The probability that random algebraic integers are relatively r-prime. Journal of Number Theory, 130(1):164 – 171, 2010. doi: http://dx.doi.org/10.1016/j.jnt.2009.06.008.

[7] Henning Stichtenoth. Algebraic function fields and codes, volume 254. Springer, 2009.

[8] Hiroshi Sugita and Satoshi Takanobu. The probability of two $\mathbb{F}_q[x]$-polynomials to be coprime. Probability and number theory, Advanced Studies in Pure Mathematics, 49:455–478, 2007.

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland.

E-mail address: giacomo.micheli@math.uzh.ch

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057 Zurich, Switzerland.

E-mail address: reto.schnyder@math.uzh.ch