Exit densities of Super-Brownian motion
as extreme $X$-harmonic functions

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Abstract: Let $X$ be a super-Brownian motion (SBM) defined on a domain $E \subset \mathbb{R}^d$ and $(X_D)$ be its exit measures indexed by sub-domains of $E$. The relationship between the equation $1/2 \Delta u = 2u^2$ and Super-Brownian motion (SBM) is analogous to the relationship between Brownian motion and the Laplace's equation, and substantial progress has been made on the study of the solutions of this semi-linear p.d.e. exploring this analogy. An area that remains to be explored is Martin boundary theory. Martin boundary in the semi-linear case is defined as the convex set of extreme $X$-harmonic functions which are functions on the space of finite measures supported in a domain $E$ of $\mathbb{R}^d$ and characterized by a mean value property with respect to the Super-Brownian law. So far no probabilistic construction of Martin boundary is known.

In this paper, we consider a bounded smooth domain $D$, and we investigate exit densities of SBM, a certain family of $X$ harmonic functions, $H^\nu$, indexed by finite measures $\nu$ on $\partial D$. These densities were first introduced by E.B. Dynkin and also identified by T.Salisbury and D. Sezer as the extended $X$-harmonic functions corresponding to conditioning SBM on its exit measure $X_D$ being equal to $\nu$. $H^\nu(\mu)$ can be thought as the analogue of the Poisson kernel for Brownian motion. It is well known that Poisson kernel for a smooth domain $D$ is equivalent to the so called Martin kernel, the class of extreme harmonic functions for $D$. We show that a similar result is true for Super-Brownian motion as well, that is $H^\nu$ is extreme for almost all $\nu$ with respect to a certain measure.

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1. Introduction

The classical Martin boundary is concerned with harmonic functions: the non-negative solutions of Laplace’s equation $\Delta u = 0$ in a given domain $D$ of $\mathbb{R}^n$. Harmonic functions are extensively studied and well understood. A major milestone in this domain is the integral representation of harmonic functions in terms of the Martin kernel, (see e.g. Doob (1984)).

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A key feature of the p.d.e. \( \frac{1}{2} \Delta u = 0 \) is that it renders itself to probabilistic analysis, and as a result many important results on this p.d.e can be reformulated probabilistically in terms of Brownian motion. This not only brings new insights into the theory, but also inspires similar ideas to be used for other p.d.e.’s. A striking example of this is the formulation of the solutions of the semi-linear equation \( \frac{1}{2} \Delta u = 2u^2 \) in terms of Super-Brownian motion (SBM). Indeed most of the recent progress on understanding the solutions of this p.d.e has been made using probabilistic methods. The two books of E. B. Dynkin, Dynkin (2002) and Dynkin (2004b), give a systematic account of the theory of super-processes and its applications to semi-linear elliptic p.d.e.s, culminating in the result that all solutions of \( \frac{1}{2} \Delta u = 2u^2 \) are \( \sigma \) moderate i.e. can be approximated by solutions that are bounded by harmonic functions. This was first proved by B. Mselati Mselati (2004) for the p.d.e. \( \frac{1}{2} \Delta u = 2u^2 \), and then extended by Dynkin to a more general class of semi-linear elliptic p.d.e.’s.

The connection between SBM and the p.d.e. \( \frac{1}{2} \Delta u = 2u^2 \) might allow us to build a Martin boundary theory for this non-linear p.d.e as well. As one of the initiators of this idea, Dynkin in a series of papers began to develop and explore some of the key constructs, in particular X-harmonic functions. Consider a super-Brownian motion \( X = (X_{D'}, P_{\mu}) \), a family of random measures (exit measures) and their associated probability laws where \( D' \) is an open subset of a given domain \( D \) in \( \mathbb{R}^d \) and \( \mu \) is a finite measure on \( D \). We write \( D' \subset D \) if \( D' \) is open and its closure is a compact subset of \( D \). A non-negative function \( H \) is X-harmonic if for any \( D' \subset D \) and any finite measure \( \mu \) with support in \( D' \),

\[
P_{\mu}(H(X_D)) = H(\mu).
\]

(1)

Given an X-harmonic function one can define an \( h \)-transform of the SBM law \( P_{\mu} \) by setting \( P_{\mu}^H(Y) = \frac{P_{\mu}(YH(X_{D'}))}{P_{\mu}(Y)} \) for any \( Y \) measurable with respect to \( \mathcal{F}_{D'} = \sigma(X_{D'}, \tilde{D} \subset D') \), for \( D' \subset D \) and then extending this measure to \( \mathcal{F}_{D} = \sigma(X_{D'}, D' \subset D) \) using the X-harmonicity of \( H \).

Let us fix \( a \in D \) and let \( \mathcal{H}^a \) be the set of all X-harmonic functions s.t. \( P_{\mu}(H(X_D)) = 1 \). Note that \( \mathcal{H}^a \) is convex. Martin boundary is defined as the set of extreme elements of \( \mathcal{H}^a \) and this set is independent of the reference point \( a \) (see e.g. Dynkin (2004a)). Dynkin (2006b) aimed to obtained the extreme X-harmonic functions by a limiting procedure from the Radon-Nikodyn densities \( H_{D}'(\mu) = \frac{dP_{\mu}^H}{dP_{\mu}}(\nu) \) of \( P_{\mu,X_{D'}}(d\nu) = P_{\mu}(X_{D'} \in d\nu) \) with respect to \( P_{\nu,X_{D'}}(d\nu) = P_{\nu}(X_{D'} \in d\nu) \) using classical exit theory for general Markov chains. More precisely, Dynkin showed that if \( H \) is an extreme X harmonic function then \( H(\mu) = \lim_{n \to \infty} H_{D_n}^{X_{D_n}}(\mu) \), \( P_{\mu}^H \) almost surely. In a further paper Dynkin (2006a) derived a formula for \( H_{D}'(\mu) \) using diagram description of moments.

X-harmonic functions are used to characterize conditional distributions of SBM. Salisbury and Sezer (2012a) identified the densities \( H_{D}'(\mu) \) as the extended X-harmonic functions (that is, they satisfy the mean value property 1 but may fail to be everywhere finite) corresponding to conditioning SBM on its exit measure \( X_D \). In other words, \( P_{\mu}^{H_{D}'(\mu)} \) is the conditional law of SBM given \( X_D = \)
They also gave an infinite fragmentation system description for \( P_{\mu}^{H_{\nu}} \). In a subsequent paper, Salisbury and Sezer (2012b) proved that \( H_{\nu}(\mu) < \infty \) for all \( \mu \) compactly supported in \( D \) establishing that the exit densities \( H_{\nu}^{D} \) are \( X \)-harmonic when \( D \) is a regular domain.

In this paper we will assume that \( D \) is a smooth domain, and show that the \( X \)-harmonic functions \( H_{\nu} := H_{\nu}^{D} \) are extreme in \( D \) for \( R \)-almost all \( \nu \) with respect to a certain measure \( R \). Our starting point is the paper Dynkin (2002), which gives a proof of a result which is originally due to Evans (1993), stating that the \( X \)-harmonic function \( \langle \mu, 1 \rangle \) is extreme in \( \mathbb{R}^d \). The conditioning corresponding to this \( X \)-harmonic function is Evans and Perkins’s conditioning, i.e. conditioning on survival. The idea in this proof is to formulate an equivalent problem in terms of the tail \( \sigma \)-field of the immortal particle. In his reformulation of Evans’s proof, Dynkin shows us how to make a similar idea work for the \( X \)-harmonic functions \( H_{\nu} \). Indeed, one can reduce the problem to showing that the tail \( \sigma \)-field of the branching backbone in the fragmentation description of Salisbury and Sezer (2012a) is trivial. This idea was also used by Verzani (2008) who showed that another class of \( X \)-harmonic functions are extreme. The resulting h-transform for Verzani’s class has a branching backbone representation with finitely many leaves.

There is a major difference between \( H_{\nu} \) and Verzani’s class; that is, the branching backbone representation of \( H_{\nu} \) has infinitely many particles moving according to an implicit \( \gamma_{\tilde{\nu}} \)-transform where \( \gamma_{\tilde{\nu}} \) are shown to be superharmonic in fact a potential. Hence the particles die before they reach the boundary. Consequently the tail behavior of the system can not be studied by looking at the tail behavior of a single particle, unlike in Verzani’s case. We will overcome this difficulty by looking at a slightly different system, that is, the branching backbone system corresponding to conditioning SBM on a Poisson random measure \( Y_n \) with intensity \( nX_{D} \). This type of conditioning was studied in Salisbury and Sezer (2012a) as a specialized case of conditioning SBM on its so called boundary statistics, random variables defined on an auxiliary probability space generated by sampling from the exit measure \( X_{D} \). A Poisson random measure \( Y_n \) with intensity \( nX_{D} \) is an example of a boundary statistic. In Salisbury and Sezer (2012a) a certain class of \( X \)-harmonic functions is associated with this conditioning. These \( X \)-harmonic functions belong to the class of \( X \)-harmonic functions considered first by Salisbury and Verzani (1999) and the resulting H-transform of the SBM laws also have branching backbone representations, but this time with finitely many particles involved hence much simpler. It is useful to consider \( H_{n,\nu_n} \) because we can show

\[
\lim_{n \to \infty} P_{\mu}^{n, Y_n}(A) = P_{\mu}^{X_D}(A), P_{\mu} \text{almost surely}
\]

(2)

via a martingale convergence argument. This tells us that for any \( A \) in the tail \( \sigma \)-field,

\[
\lim_{n \to \infty} P_{\mu}^{n, \nu_n}(A) = P_{\mu}^{\nu}(A)
\]

(3)

for a certain approximating sequence of finite atomic measures \( \nu_n \), \( R \)-almost surely.
all $\nu$. Although the $H_\nu$ are not extreme, we can show that as $n$ goes to $\infty$, $P^{n,\nu_n}_\mu(A)$ converges to a limit that can only be 0 or 1. This does not immediately give us the zero-one law because of the dependence of the $R$-null set on $A$ in the equation 3, as the tail $\sigma$-algebra may not be countably generated. However, using Dynkin’s concept of sufficient statistics, we will be able to find a countably generated sub-$\sigma$-algebra which differs from the tail sigma algebra only up to “null sets” a notion we will make precise later. This will give us the zero-one law.

2. Is $H_\nu$ extreme?

We will first fix the version of the exit densities $H_\nu$ that we are working with. Let $V^*$ be the space of nonzero and finite measures on $\partial D$. Define

$$P_{\mu,X_D}(A) := P_{\mu}(X_D \in A, X_D \neq 0).$$

Theorem 5.3.2 of Dynkin (2004b) shows that $P_{\mu_1,X_D}$ and $P_{\mu_2,X_D}$ are mutually absolutely continuous. We fix $x_0 \in D$, and take $R = P_{\delta_{x_0},X_D}$ as a reference measure. For each $\mu$ we could define $H_\nu(\mu)$ as any version of the Radon-Nikodym density $\frac{dP_{\mu,X_D}}{dR}(\nu)$, however this would not guarantee sufficient regularity for our purposes. There are several theorems on the existence of regular versions (Dynkin (2006b), Salisbury and Sezer (2012a)); however, we will use the most refined one given in Theorem 8 of Salisbury and Sezer (2012a). This version involves the excursion law $N_x$ of SBM. As is well known, $N_x$ is a $\sigma$ finite measure on the same probability space where the SBM law $P_{\mu}$ is defined and $P_{\mu}$ can be recovered from $N_x$ through a Poisson random measure

$$\Pi(dx) = \sum_i \delta_{\chi_i},$$

with intensity $R_\mu(A) = \int N_x(X_D \in A, X_D \neq 0)\mu(dx)$. More precisely $P_{\mu}$ law of $X_D$ is the same as the law of $\sum \chi_i = \int \chi \Pi(dx)$.

Let $\mu(x) = N_x(X_D \neq 0)$. For a bounded regular domain, Salisbury and Sezer (2012a) construct a family of functions $\gamma_\nu : D \rightarrow (0, \infty), \nu \in V^*$ such that the mapping $(\nu, y) \mapsto \gamma_\nu(y)$ is measurable, each $\gamma_\nu$ is superharmonic, and for all $y \in D$

$$N_{y,X_D}(dv) := N_y(X_D \in dv, X_D \neq 0) = \gamma_\nu(y)R(dv).$$

In addition, Salisbury and Sezer (2012a) construct a measurable strictly positive kernel $K_\nu(\nu; dv_1, dv_2, \ldots, dv_n)$ from $V^*$ to $(V^*)^n$, concentrated on $\{(\nu_1, \ldots, \nu_n) : \nu_1 + \cdots + \nu_n = \nu\}$, and an $R$-null set $V_0$, such that

$$H_\nu(\mu) = \begin{cases} e^{-(\mu,u) + u(x_0)}, & \text{if } \nu = 0 \\ \sum_{n=1}^{\infty} \frac{(-\mu,u)}{n!}K_\nu(\nu; dv_1, \ldots, dv_n)\langle \mu, \gamma_{\nu_1} \rangle \cdots \langle \mu, \gamma_{\nu_n} \rangle, & \text{if } \nu \neq 0 \end{cases}$$

(4)

is extended $X$-harmonic for each $\nu \notin V_0$, is a version of $\frac{dP_{\mu,X_D}}{dR}(\nu)$ for each $\mu$, and also satisfies

$$\gamma_\nu(y) = N_y(H_\nu(X_D)).$$
for every $y$, every $D' \subseteq D$ such that $y \in D'$, and every $\nu \notin V_0$.

Salisbury and Sezer (2012b) established that $0 < H^\nu(\mu) < \infty$ for all $\mu \in \mathcal{M}_D$, $R$-almost all $\nu$ in $V_0$. Let $V_1$ be the subset of $V_0$ such that $0 < H^\nu(\mu) < \infty$. In the rest of the paper we consider the class of $X$-harmonic functions $\{H^\nu, \nu \in V_1\}$, and refer to this class as the “exit densities”.

In this section we are going to investigate the question whether the $X$-harmonic functions $H^\nu, \nu \in V_1$ are extreme. Our main result is the following:

**Theorem 1.** Let $D$ be a smooth domain. Then the $X$-harmonic functions $\{H^\nu, \nu \in V_1\}$ are extreme for $R$-almost all $\nu \in V_1$.

We defer the proof of Theorem 1 to the end of this section. It will follow from Theorem 2 and Theorem 3. Theorem 2 is based on the characterization of Evans and Perkins (1990) of extreme $X$-harmonic functions in terms of tail event probabilities or SBM. We will follow Dynkin’s reformulation of Evans and Perkins’s proof using exit laws of an associated Markov Chain as in Dynkin (2002). Let $D_n \subseteq D$ be a sequence of domains exhausting $D$. Consider the sequence $X_n := X_{D_n}$. Let $\mathcal{M}_n$ be the space of finite measures on $\partial D_n$. Because of the Markov property of SBM, $X_n$ is a Markov chain with transition function

$$p(r, \mu, n, B) = P^\mu_n(X_{D_n} \in B)$$

for $r \leq n$, $\mu \in \mathcal{M}_r$, and $B$ a measurable subset of $\mathcal{M}_n$. To every $\mu \in \mathcal{M}_r$ there corresponds a probability measure $\mathbb{P}^\mu_r$ on $\mathcal{F}_\geq r := \sigma(X_{D_r}, X_{D_{r+1}}, \ldots)$ whose finite dimensional distributions are defined in terms of the above transition function as it is standard in literature. Let $H$ be an $X$-harmonic function, and $F^\mu(\mu)$ be the restriction of $H$ to $\mathcal{M}_n$, which is an exit law since $H$ is $X$-harmonic.

Remark 3.1 of Dynkin (2002) establishes that $H$ is extreme if and only if the exit law $F = (F^\mu)_{n \geq 1}$ is extreme for the Markov chain $X_n$. Let $\mathbb{P}^F_r,\mu$ denote the $F$ transform of $\mathbb{P}^\mu_r$. It is further established that an exit law is extreme if and only if $\mathbb{P}^F_{r,\mu}(C) = 0$ or $1$ for any $C$ in the tail $\sigma$-field $\mathcal{F}_r$ of $X_n$, that is, $\mathcal{F}_T = \cap_r \mathcal{F}_{\geq r}$.

Let $F^\nu$ be the exit law associated to the $X$-harmonic function $H^\nu$, and let $\mathbb{P}^F_{r,\mu} := \mathbb{P}^{F^\nu}_{r,\mu}$.

**Theorem 2.** Let $C \in \mathcal{F}_T$. For $R$-almost all $\nu$,

$$\mathbb{P}^{F^\nu}_{r,\mu}(C) = 0 \text{ or } 1$$

(5)

for all $\mu \in \mathcal{M}_r$, $r \geq 1$.

We will prove Theorem 2 in Section 3. Note that Theorem 2 is weaker than Theorem 1 because the null set where equation 5 does not hold depends on $C$. The following theorem will enable us to get Theorem 1 from Theorem 2.

**Theorem 3.** There exists a countably generated sub-$\sigma$-algebra $\mathcal{F}_0$ in $\mathcal{F}_T$ such that for any $A$ in $\mathcal{F}_T$ there exists an $A_0 \in \mathcal{F}_0$ such that $1_A = 1_{A_0} \mathbb{P}^F_{r,\mu}$ almost surely for any exit law $F$. 


Consider the product space $\prod_{\geq 1} Q$ that is generated by coordinate maps, denoted by $X_n$, (with respect to the $\sigma$-algebra $\mathcal{B}_n$ on $\mathcal{M}_n$, generated by sets of the form $\{\mu \in \mathcal{M}_n : \mu(A) \leq l\}$, where $A$ is a Borel subset of $\partial D_n$, and $l \in \mathbb{R}^+$). Let $\mathcal{E}_{\geq n} = \sigma(X_n, X_{n+1}, \ldots)$, and $\mathcal{E}_T = \cap_{n \geq 0} \mathcal{E}_{\geq n}$.

Each $M_n$ is a Lusin space, and $\mathcal{B}_n$ coincides with the Borel sigma algebra on $\mathcal{M}_n$ (see for example Kallenberg (1977)). Since $E$ is the product of $M_n$, it follows that $E$ is a Lusin space as well and similarly $\mathcal{E}$ coincides with the Borel subsets of $\mathcal{E}$.

For the rest of the discussion let us fix a reference point $c$ on $\partial D_1$ and set $P_{1,c}$ as the induced law on $(E, \mathcal{E})$ by the Markov chain $X_{D_n}$, with initial value $\delta_c$, the Dirac measure at point $c$. Similarly define $P_{1,c}^F$ as the induced law with respect to $\mathcal{E}_T$, where $F$ is an exit law.

We will use the theory of sufficient statistics of Dynkin (1978) to show that there exists a common transition kernel $Q^\omega$ such that for any $A \in \mathcal{E}$

$$P_{1,c}^F(A|\mathcal{E}_T)(\omega) = Q^\omega, \quad \text{almost surely all } \omega$$

for all exit laws $F$.

$Q^\omega$ is constructed asymptotically from a family of conditional distributions $\Pi_{\geq n}(\omega)$ defined as follows. Let $A_1 \in A \in \mathcal{E}_{\geq n}$ and $A_2 \in \mathcal{E}_{<n}$. For each $\nu$, $\Pi_{\geq n}(\omega)$ is the unique probability measure on $E, \mathcal{E}$ such that $\Pi_{\geq n}(A_1 \cap A_2)(\omega) = 1_{A_1}(\omega)g_\nu(X_n(\omega), A_2)$ where

$$g(\nu, A_2) = \frac{P_{1,c}(A_2 H_{D_n}^\nu(X_{n-1})}{H_{D_n}(\delta_c)}$$

Here $H_{D_n}^\nu, \nu \in \mathcal{M}_n$ are the exit densities for the domain $D_n$.

The family $(\mathcal{E}_{\geq n}, \Pi_{\geq n})$ satisfies the properties of a so called specification system in $E, \mathcal{E}$, and furthermore, each $P_{1,c}^F$ is a Gibbs state specified by this system. We only give a proof of the last assertion: Let $F$ be any exit law and let $f$ be $\mathcal{E}_{<n}$-measurable and $g$ be $\mathcal{E}_{\geq n} \cap \mathcal{E}_{<n}$ measurable for some $m > n$.

$$P_{1,c}^F(\Pi_{\geq n} f) = \int dP_{1,c}^F(\omega) \left[ g(\omega) \frac{P_{1,c}(f H_{D_n}^\nu(X_{n-1})}{H_{D_n}(\delta_c)} \right]$$

$$= \int dP_{1,c}(\omega) \frac{F^m(X_m(\omega))}{F^1(\delta_c)} \left[ g(\omega) \frac{P_{1,c}(f H_{D_n}^\nu(X_{n-1}))}{H_{D_n}(\delta_c)} \right]$$

$$= P_{1,c} \left( f \int dP_{1,c}(\omega) \frac{F^m(X_m(\omega))}{F^1(\delta_c)} \frac{H_{D_n}(\delta_c)}{H_{D_n}(\delta_c)} g(\omega) \right)$$

$$= P_{1,c} \left( f \int dP_n(X_{n-1}) \frac{F^m(X_m(\omega))}{F^1(\delta_c)} g(\omega) \right)$$

$$= P_{1,c} f(g)$$
Above we used Fubini theorem, the fact that 

\[
H_{\nu} D_n(X_{D_n} - 1) H_{\nu} D_n(\delta c)
\]

is the Radon Nikodym density of \(P_{n-1, X_{D_n-1}}(X_{D_n} \in d\nu)\) with respect to \(P_{n-1, X_{D_n-1}}(X_{D_n} \in d\nu)\) and the Markov property of \(X_n\). By a monotone class argument we get that

\[
P^F_{1,c}(f|\mathcal{E}_{\geq n}) = \prod_{\geq n} f
\]

\(P^F_{1,c}\) almost surely for all \(F\), proving that \(P^F_{1,c}\) is a Gibbs state for \((\mathcal{E}_{\geq n}, \Pi_{\geq n})\).

Existence of \(Q^\omega\) follows from Theorem 5.1 of Dynkin (1978). For fixed \(\omega\), it is identified as the limit of \(\Pi_{\geq n}(\omega)\), (whenever it exists).

One can view \(Q^\omega\) as a measurable map \(Q^\omega : (\mathcal{E}, \mathcal{E}) \to (\mathbb{P}(E), \mathcal{P})\) the space of probability measures on \(E\) endowed with the \(\sigma\)-algebra \(\mathcal{P}\) generated by the sets of the form \(\{P : P(A) \leq l\}\) where \(A\) is in \(E\) and \(l \in \mathbb{R}^+\). Indeed,

\[
Q^{-1}\{\{P : P(A) \leq l\}\} = \{\omega : Q^\omega(A) \leq l\}
\]

which is in \(\mathcal{E}\) since for fixed \(A\), \(Q^\omega\) is \(\mathcal{E}\), in fact \(\mathcal{E}_T\) measurable.

Let \(E_0 = \sigma(Q)\). Because \(\mathcal{P}\) is countably generated, so is \(E_0\). Moreover, for any \(A\) in \(\mathcal{E}_T\)

\[
1_A = P^F_{1,c}(A|\mathcal{E}_T) = Q^\omega A, P^F_{1,c,a.s.}
\]

So letting \(\hat{A} = \{\omega \in \{Q^\omega(A) = 1\}\}\), we get that \(\hat{A} \in \mathcal{E}_0\) and \(1_A = 1_{\hat{A}} P^F\) almost surely. Taking \(\mathcal{F}_0\) as the \(\sigma\)-algebra generated by the pre-images of sets in \(\mathcal{E}_0\) under the map \(X : (X_{D_n})_{n \geq 1}\) gives the theorem.

\[\square\]

**Proof.** (of Theorem 1)

Let \(A_1, A_2 \ldots\) be a countable collection of sets generating \(\mathcal{F}_0\). From Theorem 2 there exists a set \(V_i \subset V\) such that \(R(V_i^c \cap V) = 0\), and for all \(\nu \in V_i\)

\[
\mathbb{P}^\nu(A_i) = 0 \text{ or } 1
\]

Let \(V_1 \cap V_i\). It follows that \(\mathbb{P}^\nu(A) = 0\) or \(1\) for all \(A \in \mathcal{F}_0\). For \(A \in \mathcal{F}_T\), there exists \(\hat{A}\) such that \(\mathbb{P}^\nu(A) = \mathbb{P}^\nu(\hat{A})\) hence the theorem follows.

\[\square\]

### 3. Proof of Theorem 2

Let \(G\) be the space of finite atomic measures on \(\partial D\). Let \((\mathcal{G}_n)_{n \geq 1}\) be the filtration generated by the finite projection maps \(W_n\) on \(G^\infty\), (i.e. \(W_n(g) = (g_1, \ldots, g_n), g \in G^\infty\)) and

\[
\mathcal{G}_\infty = \bigvee_{n=1}^{\infty} \mathcal{G}_n.
\]

For any finite measure \(\nu\) on \(\partial D\), let \(P^\nu\) be the law on \(G^\infty\), corresponding to the law of independent and identically distributed Poisson random measures with characteristic measure \(\nu\). We define the following probability law on \(\Omega \times G^\infty:\n
\[
P_\mu(Y Z) = P_\mu(YP^{X_\nu}(Z))
\]
We extend both $X_D$, $D' \in D$ and $W_n$, $n \geq 1$, to $\Omega \times G^\infty$ by letting $X_D'(\omega, g) = X_D'(\omega)$ and $W_n(\omega, g) = W_n(g)$. Clearly, the law under $\hat{P}_\mu$ of $X = (X_D')_{D' \in D}$ is the same as its law under $P_\mu$. We also extend $F_{\subset D-}$, $\sigma(X_D)$ and $\mathcal{G}_n$ to $\sigma$-algebras on $\Omega \times G^\infty$ in the obvious way. If $W_n = V_1, \ldots, V_n$, let $U_n = V_1 + \ldots + V_n$.

We note that $W_n$ is a “boundary statistic” as defined in Salisbury and Sezer (2012a). A boundary statistic is any random variable that is conditionally independent of $X$ given $X_D$. Theorem 2 of Salisbury and Sezer (2012a) asserts that for any boundary statistic $S$ with a state space $\Sigma_S$ there exists a family of extended $X$-harmonic functions $H^s$, $s \in \Sigma_S$ such that if one defines an $H$-transform of $P_\mu$ by

$$P^H_\mu(Y) = P_\mu(Y H^s(X_D))$$

then $P^H_\mu$ is the conditional law of $X$ given $S$.

Hence there exists a family of extended $X$-harmonic functions $H^w_{n,W}$ and probability laws $P^H_{\mu,w}$, $w \in \Sigma_{W_n}$ such that the conditional law of $(X_D')_{D' \in D}$ given $\mathcal{G}_n$ is given by $P^{H^w_{n,W}}_\mu$. That is, we have

$$\bar{P}_\mu(A|\mathcal{G}_n) = P^{H^w_{n,W}}_\mu(A).$$

Similarly, $U_n$ is a boundary statistic, and for which there is also a family of extended $X$-harmonic functions $H^w_{n,U}$, $u \in \Sigma_{U_n}$. We note that

$$P^{H^w_{n,U}}_\mu(A) = P^{H^w_{n,U}}_\mu(A),$$

as any information in $\mathcal{G}_n$ relevant for $X$ is already carried by $U_n$. In what follows we will denote $P^{H^w_{n,U}}_\mu$ by $P^{n,u}_\mu$.

Recall $P^H_\mu$ is the $H^\nu$ transform of the SBM law $P_\mu$ and $H^\nu$ is the $X$-harmonic family which we are trying to show to be extreme.

**Theorem 4.** Let $A \in F_{\subset D-}$. $\bar{P}_\mu$ almost surely the following holds:

i) $U_n$ is a unit atomic measure for every $n$.

ii) $\frac{1}{n} U_n \Rightarrow X_D$

iii) $P^{X_D}_\mu(A) = \lim_{n \to \infty} P^{n,U_n}_\mu(A)$

**Proof.** Conditional on $X_D = \nu$, $U_n$ is a Poisson random measure with characteristic measure $n \nu$. Conditioning on a Poisson random measure with an intensity $\beta X_D$ is studied in Section 3.1 of Salisbury and Sezer (2012a). Their results assert that $U_n$ is a unit atomic measure $\bar{P}_\mu$ almost surely, clearly this implies (i).

By conditional independence of $G_\infty$ and $F_{\subset D-}$ given $X_D$, for any $A \in F_{\subset D}$ we have $\bar{P}_\mu$ a.s.

$$\bar{P}_\mu(A|G_\infty \vee \sigma(X_D)) = \bar{P}_\mu[A|\sigma(X_D)]$$

(9)

Note that $\bar{P}_\mu$ a.s.

$$\lim_{n \to \infty} \frac{1}{n} U_n = X_D,$$
where the limit is in the weak convergence sense. If for some \((\omega, g)\) the above limit does not hold, we exclude those pairs from \(\Omega \times G^\infty\). This way we have that \(X_D \) is measurable with respect to \(G^\infty\), therefore \(P_\mu[A| \sigma(X_D)]\) is measurable with respect to \(G^\infty\) as well. This and Equation 9 imply 
\[
P_\mu[A| \sigma(X_D)] = \bar{P}_\mu[A| G^\infty].
\]
By the martingale convergence theorem we get that 
\[
\bar{P}_\mu[A| \sigma(X_D)] = \lim_{n \to \infty} \bar{P}_\mu[A| G_n].
\]

By Theorem 2 of Salisbury and Sezer (2012a), the left side is equal to \(P^{X_D}_\mu(A)\), and the left side is equal to \(\lim_{n \to \infty} P^{n,U_n}_\mu(A)\). So, except on a \(\bar{P}_\mu\) null set, for all \((\omega, g)\), we have 
\[
P^{X_D}_\mu(A) = \lim_{n \to \infty} P^{n,U_n}_\mu(A).
\]

We will also need the following technical lemma in the proof of Theorem 1.
Let \((D_k)_{k \geq 0}\) be an exhausting sequence of domains for \(D\). Let \(m_k\) be a sequence s.t.
\[
\sup_{D_k} u_{m_k}(x) - u(x) < \frac{1}{k}
\]
where \(u_{m_k}\) is the solution of \(\Delta u = \frac{1}{2} u^2\) which blows up on \(\partial D_{m_k}\) and \(u\) is the solution which blows up on \(\partial D\). Note that we can do this because \(u(x) = \lim_{k \to \infty} u_{m_k}\). This is true because \(D\) is a smooth domain and therefore \(\Delta u = \frac{1}{2} u^2\) has a unique large solution (see Bandle and Marcus (1992)). Since both \(u(x)\) and \(\lim_{k \to \infty} u_{m_k}\) are large solutions they must be equal. Let \(u^{(n)}\) be the solution of \(\Delta u = \frac{1}{2} u^2\) equal to \(n\) on \(\partial D\). That is, \(u^{(n)} = V_D(n)\)

**Lemma 5.** For any \(\mu\), there exist a sequence \(n_l\) such that \(\bar{P}_\mu\) almost surely,
\[
\lim_{l \to \infty} P^{n_l,U_{n_l}}_\mu(\liminf_k e^{-\langle X_{D_{k}} u_{D_{m_k}} - u^{(n_l)} \rangle}) = 1.
\]

**Proof.** Since 
\[
P^{n,U_n}_\mu \liminf_k e^{-\langle X_{D_{k}} u_{D_{m_k}} - u^{(n)} \rangle} = \bar{P}_\mu(\liminf_k e^{-\langle X_{D_{k}} u_{D_{m_k}} - u^{(n)} \rangle}|U_n)
\]
we have 
\[
\bar{P}_\mu[P^{n,U_n}_\mu \liminf_k e^{-\langle X_{k} u_{D_{m_k}} - u^{(n)} \rangle}] = \bar{P}_\mu(\liminf_k e^{-\langle X_{k} u_{D_{m_k}} - u^{(n)} \rangle})
\]
\(P_\mu\) almost surely \(\lim_{k \to \infty} e^{-\langle X_{k} u_{D_{m_k}} - u^{(n)} \rangle}\) exists. Because, first \(e^{-\langle X_{k} u_{D_{m_k}} \rangle}\) is a bounded \(P_\mu\) sub-martingale:
\[
P_\mu(e^{-\langle X_{k+1} u_{D_{m_{k+1}}} \rangle}|F_k) = e^{-\langle X_{k} u_{D_{m_{k+1}}} \rangle} \geq e^{-\langle X_{k} u_{D_{m_{k}}} \rangle} \quad \text{since } u_{D_{m_{k}}} \geq u_{D_{m_{k+1}}}.
\]
Second, $e^{-\langle X_k, u_n \rangle}$ is a $P_\mu$-martingale with a non-zero limit. By the dominated convergence theorem we get that

$$P_\mu[P_{\mu,U_n} \lim_k e^{-\langle X_k, u_{Dm_k} - u(n) \rangle}] = P_\mu(P_{\mu,U_n} e^{-\langle X_k, u_{Dm_k} - u(n) \rangle}) = \lim_k e^{-\langle \mu, VDR(u_{Dm_k} - u_n) \rangle} \geq \lim_k e^{-\langle \mu, (u - u_n) \rangle} e^{-\langle \mu, \frac{1}{2}%n \rangle} = e^{-\langle \mu, u - u_n \rangle}.$$ 

Note

$$\lim_n e^{-\langle \mu, u - u(n) \rangle} = 1$$

which implies that $P_{\mu,U_n} [\lim_k e^{-\langle X_k, u_{Dm_k} - u(n) \rangle}]$ converges to 1 in probability, therefore there exists a subsequence $n_t$ s.t. $P_{\mu,U_{n_t}} [\lim_k e^{-\langle X_k, u_{Dm_k} - u(n_t) \rangle}]$ converges to $1 \bar{P}_\mu$ a.s.

Our strategy to prove Theorem 1 is to get a formula for $P_{\mu,\nu_n}(A)$ when $A$ is in the tail $\sigma$-field and find a subsequence $n_t$ and $\nu_{n_t}$ so that $\lim_{k \to 0} P_{\mu,\nu_n}(A)$ is 0 or 1.

Our proof is analogous to Dynkin’s proof. We will consider two Markov Chains. The first one is $X_k = X_{Dk}$, $k \geq 1$, associated to the family $P_{\mu,U_n}$, indexed by $(r, \mu)$ such that $\mu$ is a finite measure supported on $\partial D_r$ and $r = 1, 2, \ldots$. $P_{\mu,U_n}$ is the restriction of the law $P_{\mu,U_n}$ to $\sigma(X_{D_{r+1}} \times X_{D_{r+2}} \times \ldots)$.

We construct the second Markov chain using the branching forest construction of $P_{\mu,U_n}$, which we obtain from the results of Salisbury and Verzani (1999).

Let $\Theta_D$ be the space of finite point measures $\eta$ on the pairs $(x, m)$ such that $x \in D$, and each $m$ is a finite measure and if the support of $\eta$ consists of $p$ points, $(x_1, m_1), \ldots, (x_p, m_p)$, then $m_1 + \cdots + m_p = \nu_n$. Recall, $\mathcal{M}_D$ is the space of finite measures compactly supported in $D$.

Let $\mu \in \mathcal{M}_D$ and $\eta \in \Theta_D$. We will consider a probability measure $Q_{\mu,\nu_n}^\eta$ on an auxiliary probability space $\Omega_D$ where a branching diffusion $Y_t$ and conditional on this branching diffusion a Poisson random measure is generated on $\mathcal{M}_{D_1} \times \mathcal{M}_{D_2} \times \ldots$ endowed with the $\sigma$-algebra $\mathcal{F}$ generated by coordinate maps $\mathcal{F}_{D_t}$. The system gives rise to the exit measures $(Y_{D_t}, \mathcal{F}_{D_t}), t \geq 1$ as follows:

Let $z_1, \ldots, z_k$ be the points in the support of $\nu_n$. $\eta$ gives us a finite partition $\gamma = \{C_1, \ldots, C_p\}$ of $\{1, \ldots, k\}$ and a cluster of points $x_1, \ldots, x_p$ in $D$. We start a branching backbone system labeled with $z_{C_i}$ from each $x_i$. Here $z_C = (z_i, i \in C)$. The evolution of each branching backbone is as follows. Recall $u(n)(x) = -\log P_x(e^{-\langle X_{Dn}, u(n) \rangle})$. A particle starts from $x_i$ following a $\rho_{C_i}$ transform of a $L_u(n) \colon \frac{1}{2}\Delta - u(n)$ diffusion where for $C = \{i\}$

$$\rho_C = \begin{cases} \nu(\cdot, z_i), & \text{for } C = \{i\} \\ \frac{1}{2} \sum_{A \subseteq C, \emptyset \neq A \neq C} G_D^{\nu}(\cdot, z) \beta_A \beta_{C \setminus A}, & \text{for } |C| > 1. \end{cases}$$

(10)
where $k^n(x,z_i)$ is the Poisson kernel for $\mathcal{L}_u^{(n)}$ and $G_D^{(n)}$ is the Green operator for $D$.

As $\rho_{C_i}$ is a potential, the particle dies somewhere in $D$, say at $y$. Two new particles start at $y$, each assigned a new label $z_{C_i}$ and $z_{C_i \setminus C'}$ respectively where $C'$ is a proper subset of $C_i$ and chosen randomly according to a density proportional to $\rho_{C_i}(y)\rho_{C_i \setminus C'}(y)$. We repeat the same procedure for each particle unless the particle is assigned the label $z_i$ for some $i$, in which case we let it evolve as a $k^n(\cdot, z_i)$ transform of an $\mathcal{L}_u^{(n)}$ diffusion.

Let $\Upsilon_t$ denote the measure-valued process putting a unit point mass at the historical paths $z$ of each particle alive at time $t$, (A historical path of a given particle at time $t$ is the path $z : [0,t] \to D$ describing for any $s < t$ the location of the particle or whichever ancestor that is alive at time $s$). Next we are going to generate a Poisson random measure on $\mathcal{M}_{D_1} \times \mathcal{M}_{D_2} \times \ldots$ with uniform intensity along the backbone to create mass to give rise to exit measures $Y_D$. That is, for each $t$ and historical path $z$ we define the following map $X^{t,z} : \Omega \to \mathcal{M}_{D_1} \times \mathcal{M}_{D_2} \times \ldots$

$$X^{t,z}(\omega) = \begin{cases} X_D(\omega) & \text{if } \tau_{D'}(z) > t \\ 0 & \text{otherwise} \end{cases}$$

and generate a Poisson random measure on $\mathcal{M}_{D_1} \times \mathcal{M}_{D_2} \times \ldots$ with intensity

$$\lambda(A) = \int_0^\infty 4\Upsilon_t(dz)\mathbb{N}_{z(t)}((X^{t,z})^{-1}(A)) \, dt$$

Here $\mathbb{N}_{z(t)}$ is the excursion measure of a super-diffusion whose spatial motion is a Brownian motion killed at rate $u_n$. Adding up the resulting measure-valued processes gives us $Y_D := Y_{D_1}, Y_{D_2}, \ldots$.

Let $\Upsilon_{D'}$ be the point measure which puts mass on the pair of points $(y,m)$ where the $y$ are the first exit points from $D'$ of the particles in the branching forest system whose ancestors were all born inside $D'$, and the $m$ are the point measures associated to their labels. (That is $m = \sum_{i \in C} \delta_{z_i}(dz)$ if $C$ is the label of the particle exited at $y$). Let $\Upsilon_D := \Upsilon_{D_1}, \Upsilon_{D_2}, \ldots$.

On an another auxiliary probability space $\Omega_2$ we consider a probability law $P_{n}^{(n)}$, under which the exit measures $W_{D_1}, W_{D_2}, \ldots$ evolve according to the law of a super-Brownian motion whose spatial motion is killed at rate $u^{(n)}$.

Now, we define a process $(Z_{D_k}, \Upsilon_{D_k})$ on $\Omega_1 \times \Omega_2$ such that

$$(Z_{D_k}(\omega_1, \omega_2), \Upsilon_{D_k}(\omega_1, \omega_2)) = (W_{D_k}(\omega_2) + Y_{D_k}(\omega_1, \Upsilon_{D_k}(\omega_1)))$$

By construction $(Z_k, \Upsilon_k)_{k \geq 1}$ is a Markov chain with respect to the family of probability laws $\{P_{r,\mu,\eta}^{\mu,\eta}, r \geq 1, \mu \in \mathcal{M}_{D_1}^r, \eta \in \Theta_{D_r}\}$, where $P_{r,\mu,\eta}^{\mu,\eta}$ is the restriction of $Q_{\eta}^{\mu,\eta} \times P_{\mu}^{(n)}$ to $\sigma(Z_{r+1}, \Upsilon_{r+1}), (Z_{r+2}, \Upsilon_{r+2}), \ldots$. The original branching forest construction is due to Salisbury and Verzani (1999) where they assumed a general $g$ as the killing rate function and a recursive family $v_i$ obtained from initial $C^g$ harmonic functions $v_i$ through equation (10). See also Verzani (2008) for a similar formulation of this system as a Markov chain using labeled trees.
Proof. (of Theorem 1)  
The tail σ-field of \((X_k)_{k \geq 1}\) coincides with the intersection of all \(\mathcal{F}_{\geq D'}\), \(D' \subset D\) Dynkin (2004a). If \(C\) is in the tail the σ-field of \((X_k)_{k \geq 1}\), and \(P_{\mu}^\nu(C) = 1\) for some \(\mu\) supported on \(\partial D\), this implies \(P_{\mu}^\nu(C) = 1\). By the absolute continuity of the measures \(P_{\mu}\), this implies \(P_{\mu}^\nu(C) = 1\) for all \(\mu\). Therefore without loss of generality in what follows we fix \(\mu\), a finite measure supported on \(\partial D_1\).

\textbf{Step 0}  
Let \(\mu\) be a measure supported on \(\partial D_1\). We fix \(C\) in the tail σ-field of \(X_k = X_{\partial D}, k \geq 1\). Let \(\Omega^\nu\) be the set of \((\omega, g)\) such that \(\frac{1}{L}U_n(g) \Rightarrow X_D\), \(P_{\mu}^{X_D(\omega)}(C) = \lim P_{\mu, U_n(g)}^\nu(C), U_n(g)\) is a unit atomic measure for every \(n\), and for a certain subsequence \(n_l\), \(\lim_{l \to \infty} P_{\mu, U_n(g)}^\nu(\lim inf_k e^{-(X_k · D - u_n)}) = 1\). (Existence of the subsequence \(n_l\) follows from Lemma 5). From Theorem 4 and Lemma 5 we know that \(P_{\mu}(\Omega^\nu) = 1\). Let \(\nu = X_D(\omega, g)\) and \(\nu_{n_l} = U_n(\omega, g)\).

Assume \(P_{1,\mu}^\nu(C) > 0\). Our goal is to show that \(P_{1,\mu}^\nu(C) = 1\). Note that this is enough to prove Theorem 1, since \(R(\{\nu : \nu = X_D(\omega, g)\}, \{\nu, (\nu_{n_l})\}) = 0\). Let \(\epsilon \equiv P_{1,\mu}^\nu(C)\). Then there exists a number \(K\), s.t. for all \(l \geq K\)

(a) \(P_{1,\mu}^{\nu_{n_l}}(\lim inf_k P_{k, X_k}^\nu(X_{m_k} = 0)) > 1 - \frac{\epsilon}{2}\),
(b) \(P_{1,\mu}^{\nu_{n_l}}(C) > \frac{\epsilon}{2}\).

Note that the choice of \(\nu\) and \(\nu_{n_l}\) depends on \(C\) and \(\mu\). Note also that \(K\) depends on \(\nu\), \((\nu_{n_l})_{l \geq 0}\), \(C\), \(\mu\).

\textbf{Step 1}  
We show that there exists an event \(\tilde{C}\) in the tail σ-field of \((Z_k, \Upsilon_k)_{k \geq 1}\) and probability measure \(\tilde{\mu}\) on \(\Theta\) s.t.

\[P_{1,\mu}^{\nu_{n_l}}(C) = \int P_{1,\mu}^{\nu_{n_l}}(C) \tilde{\mu}(d\nu)\]

To see this, let \(\tilde{\mu}\) be a probability measure on \(\Theta\) constructed as follows. We pick a random partition \(\gamma\) of \(\{1, \ldots, k\}\) with probability proportional to \(\Pi_{\mathcal{A} \in \gamma} (\mu, \psi_A)\). For each \(A \in \gamma\), we then independently choose a starting point \(x\) with law \(\frac{1}{\mu} \psi_A(x) \mu(dx)\). \(\tilde{\mu}\) is the law of the point measure which puts unit mass on each \((x_i, m_i)\), where if \(\gamma = (A_1, \ldots, A_p)\) then \(m_i = \sum_{j \in A_i} \delta_{z_j}(dz)\).

Let \(\tilde{P}_{1,\mu}^{\nu_{n_l}}\) be the law defined by

\[\tilde{P}_{1,\mu}^{\nu_{n_l}}(d\omega) = \int \tilde{\mu}(d\nu) P_{1,\mu}^{\nu_{n_l}}(d\nu)\]

From Salisbury and Verzani (1999) we know that \(\tilde{P}_{1,\mu}^{\nu_{n_l}}\)-distribution of \(Z_k, k \geq 1\) is the same as the distribution of \((X_k)_{k \geq 1}\) w.r.t. \(P_{1,\mu}^{\nu_{n_l}}\), so for any measurable non-negative \(f_1, \ldots, f_m\) we have

\[P_{1,\mu}^{\nu_{n_l}}[f_1(X_k) \ldots f_m(X_{k+m})] = \tilde{P}_{1,\mu}^{\nu_{n_l}}(f_1(Z_k), \ldots, f_m(Z_{k+m}))\]

hence by a straightforward application of monotone class theorem we get the result.

\textbf{Step 2}:
We show that for any \( \tilde{C} \) in the tail \( \sigma \) field of the Markov Chain \((Z_k, \Upsilon_k)\), and \( \tilde{\mu} \) and \( \eta \) supported on \( \partial D_r \), we have
\[
P^{n,\nu}_r(\tilde{\mu},\eta)(C) = P^{n,\nu}_r(W_m = 0)P^{n,\nu}_r(\tilde{C}) + J^{n,\nu}_r(\tilde{C})
\]
where
\[
0 \leq J^{n,\nu}_r(\tilde{C}) \leq (1 - P^{n,\nu}_r(W_m = 0)).
\]
To see this, we use the Markov property of \((Z_k, \Upsilon_k)\), to get
\[
P^{n,\nu}_r(\tilde{\mu},\eta)(C) = P^{n,\nu}_r(g_m(Z_m, \Upsilon_m))
\]
where
\[
g_m(\tilde{\mu}, \tilde{\eta}) = P^{n,\nu}_r(W_m = 0)
\]
Therefore,
\[
P^{n,\nu}_r(\tilde{C}) = I_m + J_m
\]
where
\[
I_m = \int P^{n,\nu}_r(d\omega_1)Q_\eta(d\omega_2)1\{W_m(\omega_1) = 0\}g_m[Z_m(\omega_1, \omega_2), \Upsilon_m(\omega_1, \omega_2)]
\]
\[
= P^{n,\nu}_r(W_m = 0)P^{n,\nu}_r(\tilde{C})
\]
and
\[
J_m = \int P^{n,\nu}_r(d\omega_1)Q_\eta(d\omega_2)1\{W_m(\omega_1) \neq 0\}g_m[Z_m(\omega_1, \omega_2), \Upsilon_m(\omega_1, \omega_2)]
\]
\[
\leq P^{n,\nu}_r(W_m \neq 0).
\]

**Step 3:**
Following Dynkin, we show for any \( \tilde{C} \) in the tail \( \sigma \)-field of \((Z_k, \Upsilon_k)\) for any \( k \geq r \), \( \tilde{P}^{n,\nu}_1(\tilde{C}) \) a.s.
\[
1_C = \lim_{k \to \infty} P^{n,\nu}_1(Z_k, \Upsilon_k)(\tilde{C}).
\]
This is simply a consequence of the Markov property and the martingale convergence theorem.

**Step 4:**
Step 3 and Step 1 imply that \( \tilde{P}^{n,\nu}_1(\tilde{C}) \) a.s.
\[
1_C = \lim_{k \to \infty} \left( P^{n,\nu}_k(W_m = 0)P^{n,\nu}_k(\tilde{C}) + J^{n,\nu}_k(\tilde{C}) \right)
\]

**Step 5:**
We argue that \( \liminf_k P^{n,\nu}_k(\tilde{C}) \) is constant, \( \tilde{P}^{n,\nu}_1(\tilde{C}) \) a.s. This will follow once we show that the tail \( \sigma \)-field of \((\Upsilon_k)\) is trivial. Moreover, it is enough if we show that the tail \( \sigma \)-field of the process \((\Upsilon_t)_{t \geq 0}\) is trivial. \((\Upsilon_t)\) is defined on \( \Omega_1 \).

Let \( \tau \) be the following stopping time:
\[
\tau := \inf\{t \geq 0 : |\Upsilon_t| = k\}.
\]
For \( \eta \) such that \(|\eta| = k\), the \( Q_\eta \)-distribution of \( \Upsilon_t \) is the same as the distribution of
\[
\sum_{i=1}^{k} \delta_{\xi_i(t), \delta_x_i}
\]
where \( \xi_i \) are independent and each \( \xi_i \) is a \( k_{un}((\cdot, z_i)) \) transform of an \( \mathcal{L}_{n} \) diffusion with starting point \( y_i \in D \) determined by \( \eta \). If \( A \) is in the tail \( \sigma \)-field of \( \Upsilon_t \), then for such \( \eta \), \( Q_\eta(A) \) is 0 or 1 since each \( \xi_i \) has a trivial \( \sigma \)-field with respect to their laws. Now, by the Markov property of \( \Upsilon_t \),
\[
\tilde{F}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(A) = Q_{\Upsilon_t}(A)
\]
which is 0 or 1 as we have just argued.

**Step 6:** Note
\[
P^{(n)}_{k, Z_k}(X_{m_k} = 0) = e^{-(Z_k, u_{D_{m_k}} - u_{n})},
\]
and \( \tilde{F}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} \) distribution of \((Z_k)_{k \geq 1}\) is the same as the distribution of \((X_k)_{k \geq 1}\) under \( P^{n_{i_1}, \nu_{n_{i_1}}}_{\mu} \). Hence
\[
\tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} \lim_{k} \inf_{k} P^{n}_{k, Z_k}(W_{m_k} = 0) = \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} \lim_{k} \inf_{k} e^{-(X_k, u_{D_{m_k}} - u_{n})}.
\]

Let \( K \) be the constant we set in Step 0. From Step 0, and from the equality above we get that \( \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} (\lim_{k} \inf_{k} P^{n}_{k, Z_k}(X_{m_k} = 0) = 0) \) is always less than \( \epsilon/2 \) for \( l \geq K \) and converges to 0 as \( l \to \infty \). Since \( \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(C) = \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(C)_{\Upsilon_t} \), from Step 0, we also have that \( \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\tilde{C}) > \epsilon/2 \), for \( l \geq K \).

Let \( l \geq K \). Step 5 implies that either (1) \( \liminf_{k} \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\tilde{C}) < 1 \), \( \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} \) a.s.; or (2) \( \liminf_{k} \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\tilde{C}) = 1 \), \( \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} \) a.s. If (1) is correct, then this would imply, if \( \omega \in \tilde{C} \) is such that the equality in step (5) holds, then \( \liminf_{k} P_{k, Z_k}(X_{m_k} = 0) \) = 0. Hence if (1) were correct we would get
\[
\tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\liminf_{k} P_{k, Z_k}(X_{m_k} = 0)) = 0 \geq \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\tilde{C}).
\]
which is a contradiction, since according to what we argued in the previous paragraph, the left side is less than \( \epsilon/2 \), where the right side is strictly greater than \( \epsilon/2 \).

Hence we conclude that \( \liminf_{k} P^{n_{i_1}, \nu_{n_{i_1}}}_{k, 0, T_k}(\tilde{C}) = 1 \), \( \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu} \) a.s. for \( l \geq K \).

**Step 7:**

Step 6 and Step 5 imply that
\[
\tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\tilde{C}^c) \leq \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\liminf_{k} P_{k, Z_k}(X_{m_k} = 0) = 0)).
\]
The limit of the right side is 0 as \( l \to \infty \), hence,
\[
P^{(n)}_{1, \mu}(C) = \lim_{l \to \infty} P^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(C) = \lim_{l \to \infty} \tilde{P}^{n_{i_1}, \nu_{n_{i_1}}}_{1, \mu}(\tilde{C}) = 1.
\]
\(\square\)
4. Concluding remarks

We have shown that when $D$ is a smooth domain the family $H^\nu$ are extreme for all $\nu$ in a certain subset $V_2$ of the space of finite measures on $\partial D$. We know that this set is non-empty since $R(V_2^+) = 0$. At this time the following questions remain open and we hope to resolve them in future work:

(a) Are there other elements of the Martin boundary besides $H^\nu$ when $D$ is a smooth domain? There is at least one: the $X$-harmonic function $H^0$ corresponding to conditioning $SBM$ on $X_D = 0$. A $SBM$ conditioned on $X_D = 0$ is simply a $SBM$ whose spatial motion is a Brownian motion killed at rate $u = \log P_x(X_D = 0)$. Let $P^\mu_\nu$ denote this law. If $C$ is in the tail $\sigma$-algebra the Markov property gives us that

$$P^\mu_\nu(C) = P^\mu_\nu(P^n_{X_D}(C))$$
$$= P^\mu_\nu(X_{D_n} = 0)P^n_0(C) + P^\mu_\nu(1_{X_D\neq 0}P^n_{X_D}(C))$$

$P^n_\mu(X_{D_n} = 0)$ converges to 1 as $n \to \infty$, $P^\nu_0(C)$ is either 0 or 1, hence $P^\mu_\nu(C)$ can only be 0 or 1. Of course, this is a trivial example, we would like to know if there are any nontrivial $X$-harmonic functions aside from $H^\nu$ and $H^0$.

(b) We do not know which measures $\nu$ are in $V_2$. To make progress in this one may seek for continuous versions of $H^\nu$. Such a version may be constructed through the family $H^{n,\nu}$ through a limiting procedure. We hope to be able to characterize $V_2$ as the closure of the space of unit atomic measures on $\partial D$ with respect to a certain metric.

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