A NEW APPROXIMATION THEORY WHICH UNIFIES SPHERICAL AND COHEN-MACAUAY APPROXIMATIONS

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Abstract. This paper gives a new approximation theory for finitely generated modules over commutative Noetherian rings, which unifies two famous approximation theorems; one is due to Auslander and Bridger and the other is due to Auslander and Buchweitz. Modules admitting such approximations shall be studied.

1. Introduction

In the late 1960s Auslander and Bridger [2] introduced a notion of approximation which they used to prove that every module whose $n$th syzygy is $n$-torsionfree can be described as the quotient of an $n$-spherical module by a submodule of projective dimension less than $n$. About two decades later Auslander and Buchweitz [3] introduced the notion of Cohen-Macaulay approximation which they used to show that the category of finitely generated modules over a Cohen-Macaulay local ring with the canonical module is obtained by gluing together the subcategory of maximal Cohen-Macaulay modules and the subcategory of modules of finite injective dimension. Our purpose is to give a new approximation theorem, which unifies these two notions of [2] and [3]. Before stating our own result, let us briefly summarize the theorems of [2] and [3].

Throughout let $R$ be a commutative Noetherian ring. Let $M$ be a finitely generated $R$-module and $n \geq 1$ an integer. Then we say that $M$ is $n$-spherical if $\text{Ext}^i_R(M, R) = 0$ for $1 \leq i \leq n$. We say that $M$ is $n$-torsionfree if the transpose $\text{Tr}M$ of $M$ is $n$-spherical. Let $\Omega^n M$ denote the $n$th syzygy of $M$. With this notation the approximation theorem of Auslander and Bridger is stated as follows.

Theorem 1.1. [2] The following are equivalent for a finitely generated $R$-module $M$.

1. $\Omega^n M$ is $n$-torsionfree.
2. There exists an exact sequence $0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0$ of $R$-modules such that $X$ is $n$-spherical and $\text{pd}_R Y < n$.

Let us call an exact sequence as above an $n$-spherical approximation of $M$.

The Cohen-Macaulay Approximation Theorem of Auslander and Buchweitz is the following.

Theorem 1.2. [3] Let $R$ be a Cohen-Macaulay local ring with the canonical module. Then for every finitely generated $R$-module $M$ there exists an exact sequence $0 \rightarrow$...
We call an exact sequence as above a Cohen-Macaulay approximation of \( M \).

For a moment suppose that \( R \) is a Gorenstein local ring of dimension \( d \). Then a finitely generated \( R \)-module has finite projective dimension if and only if it has finite injective dimension. The \( d \)-th syzygy of any finitely generated \( R \)-module is \( d \)-torsionfree because it is a maximal Cohen-Macaulay \( R \)-module. The local duality theorem guarantees that a finitely generated \( R \)-module is a maximal Cohen-Macaulay \( R \)-module if and only if it is \( d \)-spherical. Thus, although in general Theorems 1.1 and 1.2 have no implications to each other, Theorem 1.1 implies Theorem 1.2 in the case where \( R \) is a Gorenstein local ring, so that they yield the common consequence which asserts the existence of Cohen-Macaulay approximations over Gorenstein local rings. Added to it, there is a strong similarity between the two exact sequences in Theorems 1.1 and 1.2. It seems natural to guess that behind the above two theorems there is hidden a common source unifying them, which we are going to report in this paper.

Let us now state our own results, explaining how this paper is organized. In Section 2 we will make some definitions and state basic properties, which we need throughout this paper.

In Section 3 we will prove our main result. To state the result precisely we need new notation. Let \( M \) and \( C \) be finitely generated \( R \)-modules and let \( n \geq 1 \) be an integer. Let \( \lambda_M \) be the natural homomorphism \( M \to \text{Hom}_R(\text{Hom}_R(M, C), C) \). Then we say that \( M \) is 1-\( C \)-torsionfree if \( \lambda_M \) is a monomorphism, and that \( M \) is \( n \)-\( C \)-torsionfree if \( n \geq 2 \), \( \lambda_M \) is an isomorphism, and \( \text{Ext}^i_R(\text{Hom}_R(M, C), C) = 0 \) for \( 1 \leq i \leq n - 2 \). The \( R \)-module \( C \) is called 1-semidualizing if \( \lambda_R \) is a monomorphism and \( \text{Ext}^i_R(C, C) = 0 \), and \( C \) is called \( n \)-semidualizing if \( n \geq 2 \), \( \lambda_R \) is an isomorphism, and \( \text{Ext}^i_R(C, C) = 0 \) for \( 1 \leq i \leq n \). The \( R \)-module \( M \) is said to be \( n \)-\( C \)-spherical if \( \text{Ext}^i_R(M, C) = 0 \) for \( 1 \leq i \leq n \). We denote by \( C\dim R \) the infimum of nonnegative integers \( m \) such that there exists an exact sequence \( 0 \to C_m \to C_{m-1} \to \cdots \to C_0 \to M \to 0 \) of \( R \)-modules, where each \( C_i \) is a direct summand of a finite direct sum of copies of \( C \). The main result of this paper, which we shall prove in Section 3 (Theorem 3.5), is stated as follows.

**Theorem A.** Let \( M \) and \( C \) be finitely generated \( R \)-modules and \( n \geq 1 \) an integer. Assume that \( C \) is \( n \)-semidualizing. Then the following two conditions on \( M \) are equivalent to each other.

1. \( \Omega^n M \) is \( n \)-\( C \)-torsionfree.
2. There exists an exact sequence \( 0 \to Y \to X \xrightarrow{f} M \to 0 \) of \( R \)-modules such that \( X \) is \( n \)-\( C \)-spherical and \( C\dim R Y < n \).

The above map \( f \) is a right approximation over the full subcategory of \( \text{mod} R \) consisting of all \( n \)-\( C \)-spherical \( R \)-modules, where \( \text{mod} R \) denotes the category of finitely generated \( R \)-modules. We call an exact sequence as above an \( n \)-\( C \)-spherical approximation of \( M \).

Here let us briefly explain how Theorem A implies both Theorems 1.1 and 1.2. We notice that \( R \) is a \( n \)-semidualizing \( R \)-module and that \( R\dim R M = \text{pd}_R M \) for a finitely generated \( R \)-module \( M \). Our Theorem A thus implies Theorem 1.1. Let \( R \) be a \( d \)-dimensional Cohen-Macaulay local ring with the canonical module \( \omega \) and \( M \) a finitely generated \( R \)-module. Then \( \omega \) is \( d \)-semidualizing, and \( \Omega^d M \) is...
such that id\_R^n M be a projective resolution of \_R^n. We will give some results on \_C then the following two conditions on \_C are equivalent to each other.

1. \( \Omega^n M \) is \( n \)-C-torsionfree for any \_R-module \_M.
2. \( \text{id}_{p_C(p)} C_p < \infty \) for any \_p \in Spec \_R with depth \_R_p \leq n - 2.

Combining Theorems A and B, we see that if \_C is an \_n-semidualizing \_R-module such that \( \text{id}_{p_C(p)} C_p < \infty \) for any \_p \in Spec \_R with depth \_R_p \leq n - 2, then the \( n \)-C-spherical \_R-modules form a contravariantly finite subcategory of mod \_R.

In what follows, let \_R denote a commutative Noetherian ring. All \_R-modules considered are finitely generated.

2. Basic definitions

Let \_M be an \_R-module and let

\[
\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0
\]

be a projective resolution of \_M. Let \( \Omega^n M \) be the image of \( d_n \), which we call the \( n \)-th syzygy of \_M. This is uniquely determined up to projective summand. An \_R-module \_N is said to be \( n \)-syzygy if \_N is isomorphic to the \( n \)-th syzygy of some \_R-module \_M. Let \( (-)^* = \text{Hom}_{\_R}(-, \_R) \) denote the \_R-dual. We put \( \text{Tr}_M = \text{Coker}(d^n_1 : P^n_0 \to P^n_1) \) and call it the transpose of \_M. This is also uniquely determined up to projective summand. See [2] and [11] for more details on transposes.

**Definition 2.1.** [2] Let \( n \geq 1 \) be an integer and \_M an \_R-module. Then \_M is said to be \( n \)-torsionfree if \( \text{Ext}^i_{\_R}(\text{Tr}_M, \_R) = 0 \) for all \( 1 \leq i \leq n \).

Let \_M be an \_R-module. Then there exists an exact sequence

\[
0 \to \text{Ext}^1_{\_R}(\text{Tr}_M, \_R) \to M \overset{\rho_M}{\to} M^{**} \to \text{Ext}^2_{\_R}(\text{Tr}_M, \_R) \to 0
\]

of \_R-modules with \( \rho_M \) the natural homomorphism ([2] Proposition (2.6))). Therefore, since \( M^* \) is isomorphic to \( \Omega^2(\text{Tr}_M) \) up to projective summand, \_M is 1-torsionfree if and only if \( \rho_M \) is a monomorphism. When \( n \geq 2 \), \_M is \( n \)-torsionfree if and only if \( \rho_M \) is an isomorphism and \( \text{Ext}^i_{\_R}(M^*, \_R) = 0 \) for all \( 1 \leq i \leq n - 2 \).

In what follows, unless otherwise specified, let \( n \geq 1 \) be an integer, \_C an \_R-module, \( (-)^! = \text{Hom}_R(-, \_C) \) the \_C-dual, and \( \lambda_M : \_M \to \_M^{**!} \) the natural homomorphism (\_M an \_R-module). We then generalize the notion of an \( n \)-torsionfree module.

**Definition 2.2.** Let \_M be an \_R-module.

1. We say that \_M is 1-\_C-torsionfree if \( \lambda_M \) is a monomorphism.
Every \( n \)-C-torsionfree \( R \)-module is \( i \)-C-torsionfree for all \( 1 \leq i \leq n \). The \( n \)-\( R \)-torsionfree property is the same as the \( n \)-torsionfree property. If \( R \) is a Cohen-Macaulay local ring with the canonical module \( \omega \), then a maximal Cohen-Macaulay \( R \)-module is \( n \)-\( \omega \)-torsionfree for every \( n \geq 1 \).

**Definition 2.3.** (1) We say that \( C \) is 1-semidualizing if \( \lambda_R : R \rightarrow \text{Hom}_R(C,C) \) is a monomorphism and \( \text{Ext}^1_R(C,C) = 0 \).

(2) Suppose that \( n \geq 2 \). Then we say that \( C \) is \( n \)-semidualizing if \( \lambda_R : R \rightarrow \text{Hom}_R(C,C) \) is an isomorphism and \( \text{Ext}^i_R(C,C) = 0 \) for all \( 1 \leq i \leq n \).

An \( n \)-semidualizing \( R \)-module is \( i \)-semidualizing for all \( 1 \leq i \leq n \). If \( C \) is \( n \)-semidualizing, then \( R \) is \( n \)-C-torsionfree. The \( R \)-module \( R \) is an \( n \)-semidualizing \( R \)-module for every \( n \geq 1 \). Recall that an \( R \)-module \( C \) is called semidualizing if \( \lambda_R \) is an isomorphism and \( \text{Ext}^i_R(C,C) = 0 \) for any \( i \geq 1 \). A semidualizing \( R \)-module is \( n \)-semidualizing for every \( n \geq 1 \). In particular, the canonical module of a Cohen-Macaulay local ring is \( n \)-semidualizing for every \( n \geq 1 \).

The following proposition says that there are a lot of \( n \)-semidualizing modules. Recall that a local ring \( (R, \mathfrak{m}) \) has an isolated singularity if \( R_p \) is a regular local ring for any \( p \in \text{Spec} \, R - \{ \mathfrak{m} \} \).

**Proposition 2.4.** Let \( R \) be a Cohen-Macaulay local ring of dimension \( d \geq 2 \) with an isolated singularity. Let \( I \) be an ideal of \( R \) which is a maximal Cohen-Macaulay \( R \)-module. Then \( \lambda_R : R \rightarrow \text{End}_R(I) \) is an isomorphism and \( \text{Ext}^i_R(I,I) = 0 \) for every \( 1 \leq i \leq d - 2 \). Hence \( R \) is \( d \)-I-torsionfree, and \( I \) is \( (d-2) \)-semidualizing.

**Proof.** Note that \( R \) is a normal domain and that \( I \) is a nonzero ideal of \( R \). Hence it follows by [9, pp. 220-221] or [3, Theorem 2.1] that \( \lambda_R \) is an isomorphism. Let us prove that \( \text{Ext}^i_R(I,I) = 0 \) for \( 1 \leq i \leq d - 2 \). This statement can be shown by using the proof of [10, Proposition 2.5.1]. Set \( r = \sup \{ n \mid \text{Ext}^i_R(I,I) = 0 \text{ for } 1 \leq i \leq n \} \). We want to show that \( r \geq d - 2 \). Suppose that \( r < d - 2 \). Take a free resolution \( F_\bullet \) of the \( R \)-module \( I \). Dualizing \( F_\bullet \) by \( I \), we obtain an exact sequence

\[
0 \rightarrow R \rightarrow \text{Hom}_R(F_0, I) \rightarrow \cdots \rightarrow \text{Hom}_R(F_{r-1}, I) \rightarrow \text{Hom}_R(F_r, I) \rightarrow \text{Ext}^r_R(I,I) \rightarrow 0
\]

Put \( N_i = \text{Im} \delta_i \) for \( 0 \leq i \leq r \). The definition of \( r \) implies that \( \text{Ext}^r_R(I,I) \neq 0 \). Since \( R \) has an isolated singularity and \( I \) is maximal Cohen-Macaulay, the \( R \)-module \( \text{Ext}^r_R(I,I) \) has a finite length. Hence we have \( \text{depth}_R \text{Ext}^r_R(I,I) = 0 \). As \( \text{depth}_R \text{Hom}_R(\Omega^{r+1} I, I) \geq \min \{ 2, \text{depth}_R I \} = 2 > 0 \) (cf. [5, Exercise 1.4.19]), we obtain \( \text{depth}_R N_0 = 1 \) by the depth lemma. Noting that each \( \text{Hom}_R(F_i, I) \) is a maximal Cohen-Macaulay \( R \)-module, by the depth lemma, we get \( \text{depth}_R N_i = i + 1 \) for \( 0 \leq i \leq r \), and \( d = \text{depth} R = r + 2 < d \). This is a contradiction, which shows that \( r \geq d - 2 \) and the proof is completed.

We denote by \( \text{mod} \, R \) the category of finitely generated \( R \)-modules. Let \( \mathcal{X} \) be a full subcategory of \( \text{mod} \, R \). An \( R \)-homomorphism \( f : X \rightarrow M \) is called a right \( \mathcal{X} \)-approximation of \( M \) if \( X \) belongs to \( \mathcal{X} \) and the sequence \( \text{Hom}_R(-, X) (\sim f) \); \( \text{Hom}_R(-, M) \rightarrow 0 \), where \( (-, f) = \text{Hom}_R(-, f) \), is exact on \( \mathcal{X} \). We say that
\( \mathcal{X} \) is **contravariantly finite** if any \( X \in \mathcal{X} \) has a right \( \mathcal{X} \)-approximation. An \( R \)-complex \( \cdots \rightarrow X_i \rightarrow X_{i-1} \rightarrow \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \) is called a right \( \mathcal{X} \)-resolution of \( M \) if each \( X_i \) belongs to \( \mathcal{X} \) and the sequence \( \cdots \rightarrow (\text{Hom}_R(-, X_1) \rightarrow \cdots \rightarrow (\text{Hom}_R(-, X_0) \rightarrow \cdots \rightarrow \text{Hom}_R(-, M) \rightarrow 0 \) is exact on \( \mathcal{X} \). A left \( \mathcal{X} \)-approximation, a covariantly finite subcategory and a left \( \mathcal{X} \)-resolution are defined dually.

For an \( R \)-module \( X \), we denote by \( \text{add} X \) the full subcategory of \( \text{mod} R \) consisting of all direct summands of finite direct sums of copies of \( X \). For an \( R \)-module \( M \), we define \( \text{Cdim}_R M \), the add \( C \)-resolution dimension of \( M \), to be the infimum of nonnegative integers \( n \) such that there exists an exact sequence \( 0 \rightarrow C_n \rightarrow C_{n-1} \rightarrow \cdots \rightarrow C_0 \rightarrow M \rightarrow 0 \) with each \( C_i \) being in \( \text{add} C \). Note that add \( R \)-resolution dimension is the same as projective dimension. We make the following definition.

**Definition 2.5.** Let \( M \) be an \( R \)-module.

1. We say that \( M \) is \( n \)-spherical if \( \text{Ext}^i_R(M, R) = 0 \) for all \( 1 \leq i \leq n \). We call an exact sequence \( 0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0 \) of \( R \)-modules an \( n \)-spherical approximation if \( X \) is \( n \)-spherical and \( \text{pd}_R Y < n \).
2. We say that \( M \) is \( n \)-\( C \)-spherical if \( \text{Ext}^i_R(M, C) = 0 \) for all \( 1 \leq i \leq n \). We call an exact sequence \( 0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0 \) of \( R \)-modules an \( n \)-\( C \)-spherical approximation if \( X \) is \( n \)-\( C \)-spherical and \( \text{Cdim}_R Y < n \).

The \( n \)-\( R \)-spherical property is the same as the \( n \)-spherical property. By virtue of the local duality theorem, if \( R \) is a Cohen-Macaulay local ring with the canonical module \( \omega \), then an \( R \)-module is maximal Cohen-Macaulay if and only if it is \( d \)-\( \omega \)-spherical.

### 3. The approximation theorem

In this section, we will discuss when a given module has an \( n \)-\( C \)-spherical approximation. We shall actually give an equivalent condition for a module to have an \( n \)-\( C \)-spherical approximation in the case where \( C \) is an \( n \)-semidualizing module. Using this equivalent condition, we will prove two well-known approximation theorems: one is due to Auslander and Bridger, and the other is due to Auslander and Buchweitz.

First of all, we make a remark, and characterize \( n \)-\( C \)-torsionfree modules by using left add \( C \)-resolutions.

**Remark 3.1.** The map \( \lambda_M : M^\dagger \rightarrow M^{\dagger\dagger} \) is a split monomorphism for any \( R \)-module \( M \). Indeed, it is easy to check that the composite map \( (\lambda_M)^\dagger \cdot \lambda_M^\dagger \) is the identity map of \( M^\dagger \).

**Proposition 3.2.** Let \( M \) be an \( R \)-module. Then the following statements hold.

1. \( M \) is \( 1 \)-\( C \)-torsionfree if and only if \( M \) has an injective left add \( C \)-approximation.
2. (i) If \( M \) is \( 2 \)-\( C \)-torsionfree, then \( M \) has an exact left add \( C \)-resolution \( 0 \rightarrow M \rightarrow C_0 \rightarrow C_1 \).  
   (ii) The converse holds if \( \lambda_C \) is an isomorphism.
3. Let \( n \geq 3 \). Suppose that \( \lambda_C \) is an isomorphism and \( \text{Ext}^i_R(C^\dagger, C) = 0 \) for all \( 1 \leq i \leq n-3 \).
   (i) If \( M \) is \( n \)-\( C \)-torsionfree, then \( M \) has an exact left add \( C \)-resolution \( 0 \rightarrow M \rightarrow C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_{n-1} \).
   (ii) The converse holds if \( \text{Ext}^{n-2}_R(C^\dagger, C) = 0 \).
Proof. (1) Suppose that $M$ is 1-C-torsionfree. Dualizing a free cover $R^r \to M^\dagger$ of $M^\dagger$ by $C$, we have an injection $\alpha : M^{\dagger\dagger} \to C^r$. Hence we get an injection $\beta = \alpha \cdot \lambda_M : M \to C^r$. Let $f_1, \ldots, f_l$ be a system of generators of the $R$-module $\text{Hom}_R(M, C)$. Taking the direct sum of $f_1, \ldots, f_l$, we construct a homomorphism $f : M \to C^l$. It is easily seen that $f$ is a left add-$C$-approximation of $M$. In particular, the $C$-dual homomorphism $f^\dagger$ is surjective. Since $\beta$ factors through $f$, the homomorphism $f$ is injective. Thus we obtain an exact sequence $0 \to M \overset{f}{\to} C^l$ such that $(C^l)^{\dagger\dagger} \overset{f^\dagger}{\to} M^\dagger \to 0$ is also exact. Conversely, if there is an exact sequence $0 \to M \to C_0$ with $C_0 \in \text{add } C$ such that the $C$-dual sequence $C_0^{\dagger} \to M^\dagger \to 0$ is also exact, then dualizing the latter sequence by $C$, we get a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
\lambda_M & \downarrow & \lambda_{C_0} & \downarrow & \lambda_N & \downarrow & \\
0 & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & C_0^{\dagger\dagger} & \longrightarrow & N^{\dagger\dagger} & & \\
\end{array}
\]

with exact rows. Noting that $C_0$ is a direct summand of $C^l = (R^r)^{\dagger\dagger}$ for some $l > 0$ and that $\lambda_{C_0}$ is injective by Remark 3.1 we see that $\lambda_{C_0}$ is also injective. Therefore $\lambda_M$ is also injective, that is, $M$ is 1-C-torsionfree.

(2) According to (1), to show the assertion, we may assume that there is an exact sequence $0 \to M \to C_0 \to N \to 0$ with $C_0 \in \text{add } C$ whose $C$-dual sequence $0 \to N^{\dagger} \to C_0^{\dagger} \to M^\dagger \to 0$ is also exact. Dualizing the latter sequence by $C$ and using Remark 3.1 we have a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
\lambda_M & \downarrow & \lambda_{C_0} & \downarrow & \lambda_N & \downarrow & \\
0 & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & C_0^{\dagger\dagger} & \longrightarrow & N^{\dagger\dagger} & & \\
\end{array}
\]

with exact rows and columns. It follows from this diagram that if $M$ is 2-C-torsionfree then $N$ is 1-C-torsionfree, and that the converse holds when $\lambda_C$ is an isomorphism. By (1), $N$ is 1-C-torsionfree if and only if $N$ has an injective left add-$C$-approximation $N \to C_1$. If this is the case, then the spliced exact sequence $0 \to M \to C_0 \to C_1$ is a left add-$C$-resolution.

(3) As we did in the proof of (2), we may assume that there is an exact sequence $0 \to M \to C_0 \to N \to 0$ with $C_0 \in \text{add } C$ such that the $C$-dual sequence $0 \to N^{\dagger} \to C_0^{\dagger} \to M^\dagger \to 0$ is also exact. We obtain a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & C_0 & \longrightarrow & N & \longrightarrow & 0 \\
\lambda_M & \downarrow & \lambda_{C_0} & \downarrow & \lambda_N & \downarrow & \\
0 & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & C_0^{\dagger\dagger} & \longrightarrow & N^{\dagger\dagger} & & \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
\longrightarrow & \text{Ext}_R^1(M^\dagger, C) & \longrightarrow & \text{Ext}_R^1(C_0^{\dagger}, C) & \longrightarrow & \text{Ext}_R^1(N^{\dagger}, C) & \\
\longrightarrow & \text{Ext}_R^{n-2}(M^\dagger, C) & \longrightarrow & \text{Ext}_R^{n-2}(C_0^{\dagger}, C) & & & \\
\end{array}
\]
with exact rows. From this diagram and the assumptions we see that if $M$ is $n$-$C$-torsionfree then $N$ is $(n - 1)$-$C$-torsionfree, and that the converse holds when \( \text{Ext}^{n-2}_R(C^!, C) = 0 \). By induction on $n$, $N$ is $(n - 1)$-$C$-torsionfree if and only if it has an exact left add $C$-resolution $0 \to N \to C_1 \to C_2 \to \cdots \to C_{n-1}$. If this is the case, then the spliced exact sequence $0 \to M \to C_0 \to C_1 \to C_2 \to \cdots \to C_{n-1}$ is a left add $C$-resolution of $M$. \( \square \)

All the assumptions on $C$ in Proposition 3.2 are satisfied when $\text{Hom}_R(C, C)$ is isomorphic to $R$:

**Corollary 3.3.** Suppose that $\lambda_R$ is an isomorphism. The following are equivalent for an $R$-module $M$.

1. $M$ is $n$-$C$-torsionfree.
2. $M$ has an exact left add $C$-resolution $0 \to M \to C_0 \to C_1 \to \cdots \to C_{n-1}$.

*Proof.* Since $C^! \cong R$, the map $\lambda_C$ is an isomorphism and $\text{Ext}^i_R(C^!, C) = 0$ for any $i > 0$. The assertion follows from Proposition 3.2. \( \square \)

We give here an application of Proposition 3.2(1), which will be used later as a lemma.

**Corollary 3.4.** Suppose that $\text{Ext}^1_R(C, C) = 0$. An $R$-module $M$ is $1$-$C$-torsionfree if and only if there is an exact sequence $0 \to M \to C_0 \to N \to 0$ such that $C_0 \in \text{add} C$ and $\text{Ext}^1_R(N, C) = 0$.

*Proof.* Suppose that there is an exact sequence $0 \to M \to C_0 \to N \to 0$ such that $C_0 \in \text{add} C$ and $\text{Ext}^1_R(N, C) = 0$. Dualizing this sequence by $C$ gives an exact sequence $C_0^! \to M^! \to \text{Ext}^1_R(N, C) = 0$. It follows from Proposition 3.2(1) that $M$ is $1$-$C$-torsionfree. Conversely, if this is the case, then we have an injective left add $C$-approximation $f : M \to C_0$ by Proposition 3.2(1). Setting $N = \text{Coker} f$, we get an exact sequence $0 \to M \xrightarrow{f} C_0 \to N \to 0$ such that $0 \to N^! \to C_0^! \xrightarrow{f^!} M^! \to 0$ is an exact sequence. Since $\text{Ext}^1_R(C_0, C) = 0$, we have $\text{Ext}^1_R(N, C) = 0$. \( \square \)

Now, we can state and prove the following theorem, which is the main result of this paper.

**Theorem 3.5.** Let $C$ be an $n$-semidualizing $R$-module. The following are equivalent for an $R$-module $M$.

1. $\Omega^n M$ is $n$-$C$-torsionfree.
2. $M$ admits an $n$-$C$-spherical approximation.

*Proof.* Let $P_\bullet$ be a projective resolution of $M$.

1. $\Rightarrow$ 2: We have an exact sequence $0 \to \Omega^{i+1} M \to P_i \to \Omega^i M \to 0$ for each $i$. Set $X_0 = \Omega^n M$. Note that $X_0$ is $n$-$C$-torsionfree. Corollary 3.3 implies that there exists an exact sequence $0 \to X_0 \to C_0 \to Z_1 \to 0$ such that $C_0 \in \text{add} C$ and
Ext\textsuperscript{1}_R(Z_1, C) = 0. We construct the pushout diagram:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
0 & X_0 & \rightarrow & C_0 & \rightarrow & Z_1 & \rightarrow & 0 \\
\downarrow & & & & & & & \\
0 & P_{n-1} & \rightarrow & X_1 & \rightarrow & Z_1 & \rightarrow & 0 \\
\downarrow & & & & & & & \\
\Omega^{n-1}M & \rightarrow & \Omega^{n-1}M & & & \\
\downarrow & & & & & & & \\
0 & 0 & & & \\
\end{array}
\]

Since Ext\textsuperscript{1}_R(Z_1, C) = 0 = Ext\textsuperscript{1}_R(P_{n-1}, C), we have Ext\textsuperscript{1}_R(X_1, C) = 0. If \( n = 1 \), then the middle column is a desired exact sequence.

Let \( n \geq 2 \). We establish the following two claims:

**Claim 1.** \( Z_1 \) is \((n-1)\)-\( C \)-torsionfree.

**Proof of Claim.** We have an exact sequence \( 0 \rightarrow Z_1^\dagger \rightarrow C_0^\dagger \rightarrow X_0^\dagger \rightarrow 0 \). Dualize this again, and we have a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & X_0 & \rightarrow & C_0 & \rightarrow & Z_1 & \rightarrow & 0 \\
\downarrow & \lambda x_0 & \rightarrow & \lambda c_0 & \rightarrow & \lambda x_1 & \\
0 & \rightarrow & X_0^\dagger & \rightarrow & C_0^\dagger & \rightarrow & Z_1^\dagger & \\
\end{array}
\]

with exact rows. Noting that \( \lambda_R : R \rightarrow R^{\dagger \dagger} \) is an isomorphism, we see that so is \( \lambda_C \), hence so is \( \lambda_{C_0} \). Thus the snake lemma says that \( \lambda_{Z_1} \) is an injection. Therefore, \( Z_1 \) is \((n-1)\)-\( C \)-torsionfree when \( n = 2 \). When \( n \geq 3 \), noting that the map \( \eta \) in the above diagram is a surjection because \( \text{Ext}^1_R(X_0^\dagger, C) = 0 \), we see from the five lemma that \( \lambda_{Z_1} \) is an isomorphism. Since \( C_0 \) belongs to \( \text{add } C \) and \( \lambda_R : R \rightarrow C^{\dagger} \) is an isomorphism, we easily see that the \( R \)-module \( C_0^{\dagger} \) is projective, and hence \( \text{Ext}^i_R(C_0^{\dagger}, C) = 0 \) for \( i > 0 \). Since \( \text{Ext}^i_R(X_0^\dagger, C) = 0 \) for \( 1 \leq i \leq n-2 \), we get \( \text{Ext}^i_R(Z_1^\dagger, C) = 0 \) for \( 1 \leq i \leq n-3 \). It follows that \( Z_1 \) is \((n-1)\)-\( C \)-torsionfree. \( \Box \)

**Claim 2.** \( X_1 \) is \((n-1)\)-\( C \)-torsionfree.

**Proof of Claim.** We have an exact sequence \( 0 \rightarrow Z_1^\dagger \rightarrow X_1^\dagger \rightarrow P_{n-1}^\dagger \rightarrow 0 \), and dualizing this again yields a commutative diagram

\[
\begin{array}{cccc}
0 & \rightarrow & P_{n-1} & \rightarrow & X_1 & \rightarrow & Z_1 & \rightarrow & 0 \\
\downarrow & \lambda p_{n-1} & \rightarrow & \lambda x_1 & \rightarrow & \lambda x_1 & \\
0 & \rightarrow & P_{n-1}^{\dagger \dagger} & \rightarrow & X_1^{\dagger \dagger} & \rightarrow & Z_1^{\dagger \dagger} & \rightarrow & 0 \\
\end{array}
\]

with exact rows. If \( n = 2 \), then \( \lambda_{Z_1} \) is a monomorphism by Claim 1 and so is \( \lambda_{X_1} \). Hence \( X_1 \) is \((n-1)\)-\( C \)-torsionfree. If \( n \geq 3 \), then \( \lambda_{Z_1} \) is an isomorphism by Claim 1 and so is \( \lambda_{X_1} \). Since \( \text{Ext}^i_R(P_{n-1}^{\dagger \dagger}, C) = 0 \) for \( 1 \leq i \leq n \) and \( \text{Ext}^i_R(Z_1^{\dagger \dagger}, C) = 0 \) for
1 \leq i \leq n - 3$ by Claim 1, we obtain $\text{Ext}_R^i(X_1^1, C) = 0$ for $1 \leq i \leq n - 3$. Thus $X_1$ is $(n - 1)$-$C$-torsionfree.

According to Corollary 3.4, there is an exact sequence $0 \to X_1 \to C_1 \to Z_2 \to 0$ with $C_1 \in \text{add } C$ and $\text{Ext}_R^1(Z_2, C) = 0$. We construct the pushout diagram:

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & \downarrow \\
C_0 & C_0 \\
\downarrow & \downarrow \\
0 \longrightarrow X_1 & \longrightarrow C_1 & \longrightarrow Z_2 & \longrightarrow 0 \\
\downarrow & \downarrow & \vert & \\
0 \longrightarrow \Omega^{n-1}M & \longrightarrow Y_2 & \longrightarrow Z_2 & \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

Using the bottom row of the above diagram, we construct the pushout diagram:

\[
\begin{array}{ccccccccc}
0 & 0 \\
\downarrow & \downarrow \\
0 \longrightarrow \Omega^{n-1}M & \longrightarrow Y_2 & \longrightarrow Z_2 & \longrightarrow 0 \\
\downarrow & \downarrow & \vert & \\
0 \longrightarrow P_{n-2} & \longrightarrow X_2 & \longrightarrow Z_2 & \longrightarrow 0 \\
\downarrow & \downarrow & \downarrow \\
\Omega^{n-2}M & \longrightarrow \Omega^{n-2}M \\
\downarrow & \downarrow & \downarrow \\
0 & 0 \\
\end{array}
\]

From the first diagram, we immediately get $\text{Cdim } Y_2 < 2$, and $\text{Ext}_R^2(Z_2, C) = 0$ because $\text{Ext}_R^1(X_1, C) = 0 = \text{Ext}_R^2(C_1, C)$. Hence $\text{Ext}_R^i(Z_2, C) = 0$ for $i = 1, 2$, and we see from the middle row of the second diagram that $\text{Ext}_R^i(X_2, C) = 0$ for $i = 1, 2$. Thus, if $n = 2$, then the middle column of the second diagram is a desired exact sequence.

Let $n \geq 3$. Then similar arguments to the above claims show that both $Z_2$ and $X_2$ are $(n-2)$-$C$-torsionfree, and Corollary 3.4 yields an exact sequence $0 \to X_2 \to C_2 \to Z_3 \to 0$ such that $\text{Ext}_R^1(Z_3, C) = 0$. Similarly to the above, we construct two
pushout diagrams:

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
Y_2 & Y_2
\end{array}
\]

\[
\begin{array}{cccc}
0 & \rightarrow & X_2 & \rightarrow & C_2 & \rightarrow & Z_3 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Omega^{n-2}M & \rightarrow & Y_3 & \rightarrow & Z_3 & \rightarrow & 0
\end{array}
\]

and

\[
\begin{array}{cccc}
0 & 0 \\
\downarrow & \downarrow \\
0 & \rightarrow & \Omega^{n-2}M & \rightarrow & Y_3 & \rightarrow & Z_3 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & P_{n-3} & \rightarrow & X_3 & \rightarrow & Z_3 & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Omega^{n-3}M & \longrightarrow & \Omega^{n-3}M \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & 0
\end{array}
\]

If \( n = 3 \), then the middle column of the second diagram is a desired exact sequence.
If \( n \geq 4 \), then iterating this procedure, we eventually obtain an exact sequence \( 0 \rightarrow Y_n \rightarrow X_n \rightarrow M \rightarrow 0 \) such that \( \text{Ext}^i_R(X_n, C) = 0 \) for \( 1 \leq i \leq n \) and \( \text{Cdim} Y_n < n \).

(2) \( \Rightarrow \) (1): Let \( 0 \rightarrow Y \rightarrow X \rightarrow M \rightarrow 0 \) be an \( n \)-C-spherical approximation of \( M \). Since \( \text{Cdim} Y < n \), there exists an exact sequence \( 0 \rightarrow C_{n-1} \overset{d_{n-1}}{\rightarrow} C_{n-2} \overset{d_{n-2}}{\rightarrow} \cdots \overset{d_1}{\rightarrow} C_0 \overset{d_0}{\rightarrow} Y \rightarrow 0 \). Put \( Y_i = \text{Im} d_i \) for each \( i \). We have exact sequences \( 0 \rightarrow Y_{i+1} \rightarrow C_i \rightarrow Y_i \rightarrow 0 \) and \( 0 \rightarrow \Omega^{i+1}M \rightarrow P_i \rightarrow \Omega^i M \rightarrow 0 \), where \( P_i \) is projective.
$R$-module. The following pullback diagram is obtained:

\[
\begin{array}{cccccccc}
0 & 0 & \downarrow & \downarrow & \Omega M & \Omega M & \downarrow & \downarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \longrightarrow & Y & \longrightarrow & L & \longrightarrow & P_0 & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \longrightarrow & Y & \longrightarrow & X & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & 0 & \downarrow & \downarrow & \Omega M & \Omega M & \downarrow & \downarrow & 0 \\
\end{array}
\]

The projectivity of $P_0$ shows that the middle row splits; we have an isomorphism $L \cong Y \oplus P_0$. Adding $P_0$ to the exact sequence $0 \rightarrow Y \rightarrow C_0 \rightarrow Y \rightarrow 0$, we get an exact sequence $0 \rightarrow Y \rightarrow C_0 \oplus P_0 \rightarrow Y \oplus P_0 \rightarrow 0$. Thus the following pullback diagram is obtained:

\[
\begin{array}{cccccccc}
0 & 0 & \downarrow & \downarrow & Y_1 & \longrightarrow & Y_1 & \downarrow & \downarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \longrightarrow & X_1 & \longrightarrow & C_0 \oplus P_0 & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & \longrightarrow & \Omega M & \longrightarrow & Y \oplus P_0 & \longrightarrow & X & \longrightarrow & 0 \\
\downarrow & & & & \downarrow & & & & \downarrow \\
0 & 0 & \downarrow & \downarrow & \Omega M & \Omega M & \downarrow & \downarrow & 0 \\
\end{array}
\]

Applying a similar argument to the left column of the above diagram, we get exact sequences $0 \rightarrow X_{i+1} \rightarrow C_i \oplus P_i \rightarrow X_i \rightarrow 0$ for $0 \leq i \leq n - 1$, where $X_0 = X$ and $X_n = \Omega^n M$. The assumption yields $\text{Ext}^i_R(X_0, C) = 0 = \text{Ext}^i_R(C_0 \oplus P_0, C)$ for $1 \leq i \leq n$, hence we have an exact sequence $0 \rightarrow X_0^\dagger \rightarrow (C_0 \oplus P_0)^\dagger \rightarrow X_1^\dagger \rightarrow 0$ and $\text{Ext}^i_R(X_1, C) = 0$ for $1 \leq i \leq n - 1$. Inductively, for each $0 \leq i \leq n - 1$ an exact sequence $0 \rightarrow X_i^\dagger \rightarrow (C_i \oplus P_i)^\dagger \rightarrow X_{i+1}^\dagger \rightarrow 0$ is obtained and $\text{Ext}^j_R(X_i, C) = 0$ for $1 \leq j \leq n - i$. We have a commutative diagram

\[
\begin{array}{cccccccc}
0 & \longrightarrow & X_1 & \longrightarrow & C_0 \oplus P_0 & \downarrow & \lambda_{X_1} & \downarrow & \lambda_{C_0 \oplus P_0} & 0 \\
\downarrow & & \lambda_{C_0} & \downarrow & \lambda_{P_0} & & & & \downarrow \\
0 & \longrightarrow & X_1^\dagger & \longrightarrow & (C_0 \oplus P_0)^\dagger & & & & 0 \\
\end{array}
\]

with exact rows. The assumption says that $\lambda_R$ is a monomorphism, and we see from Remark 3.1 that so is $\lambda_C = \lambda_R^Y$. Hence so is the map $\lambda_{C_0 \oplus P_0}$, and so is $\lambda_{X_1}$. 
Therefore $X_1$ is 1-$C$-torsionfree. If $n \geq 2$, then $\lambda_R$ is an isomorphism, and so is $\lambda_C$.

There is a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & X_2 & \longrightarrow & C_1 \oplus P_1 & \longrightarrow & X_1 & \longrightarrow & 0 \\
\downarrow{\lambda_{x_2}} & & \downarrow{\lambda_{C_1 \oplus P_1}} & & \downarrow{\lambda_{x_1}} & & & & \\
0 & \longrightarrow & X_2^{\dagger} & \longrightarrow & (C_1 \oplus P_1)^{\dagger} & \longrightarrow & X_1^{\dagger} & & \\
\end{array}
$$

with exact rows. Since $\lambda_{C_1 \oplus P_1}$ is an isomorphism and $\lambda_{x_1}$ is a monomorphism, $\lambda_{x_2}$ is an isomorphism by the snake lemma. Hence $X_2$ is 2-$C$-torsionfree. If $n \geq 3$, then we have a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & X_3 & \longrightarrow & C_2 \oplus P_2 & \longrightarrow & X_2 & \longrightarrow & 0 \\
\downarrow{\lambda_{x_3}} & & \downarrow{\lambda_{C_2 \oplus P_2}} & & \downarrow{\lambda_{x_2}} & & & & \\
0 & \longrightarrow & X_3^{\dagger} & \longrightarrow & (C_2 \oplus P_2)^{\dagger} & \longrightarrow & X_2^{\dagger} & \longrightarrow & \text{Ext}_R^1(X_3^{\dagger}, C) & \longrightarrow & 0 \\
\end{array}
$$

with exact rows. From this diagram it follows that $\lambda_{x_3}$ is an isomorphism and $\text{Ext}_R^1(X_3^{\dagger}, C) = 0$, which means that $X_3$ is 3-$C$-torsionfree. Repeating a similar argument, we see that $X_i$ is $i$-$C$-torsionfree for every $1 \leq i \leq n$. Therefore $\Omega^n M = X_n$ is $n$-$C$-torsionfree, and the proof of the theorem is completed.

**Remark 3.6.** The proof of Theorem 3.5 actually shows a little stronger statements. Let $M$ be an $R$-module.

1. Assume that $\text{Ext}_R^1(C, C) = 0$. If $\Omega M$ is 1-$C$-torsionfree, then $M$ has a 1-$C$-spherical approximation.
2. Assume that $\lambda_R$ is injective and $\text{Ext}_R^i(C, C) = 0$ for $i = 1, 2$. If $\Omega^2 M$ is 2-$C$-torsionfree, then $M$ has a 2-$C$-spherical approximation.
3. Assume that $\lambda_R$ is injective. If $M$ has a 1-$C$-spherical approximation, then $\Omega M$ is 1-$C$-torsionfree. (In fact, the injectivity of $\lambda_R$ implies the conclusion that $\Omega M$ is 1-$C$-torsionfree by itself; see Lemma 4.1 below.)
4. Assume that $\lambda_R$ is bijective and $\text{Ext}_R^i(C, C) = 0$ for $1 \leq i \leq n - 1$. If $M$ has an $n$-$C$-spherical approximation, then $\Omega^n M$ is $n$-$C$-torsionfree.

**Remark 3.7.** Let $C$ be a semidualizing $R$-module (for the definition, see the following part of Definition 2.3). Recall the following:

1. We say that an $R$-module $M$ is totally $C$-reflexive if $\lambda_M : M \to M^{\dagger\dagger}$ is an isomorphism and $\text{Ext}_R^i(M, C) = \text{Ext}_R^i(M^{\dagger\dagger}, C) = 0$ for any $i > 0$.
2. The $G_C$-dimension of an $R$-module $M$, which is denoted by $G_C \dim_R M$, is defined as the infimum of integers $r$ such that there exists an exact sequence $0 \to X_r \to X_{r-1} \to \cdots \to X_0 \to M \to 0$ where each $X_i$ is totally $C$-reflexive.

Araya, Takahashi and Yoshino proved that every $R$-module of finite $G_C$-dimension admits an exact sequence which should be called an “$\infty$-$C$-spherical approximation” in our terminology; they proved that for every $R$-module $M$ with $G_C \dim M < \infty$ there exists an exact sequence $0 \to Y \to X \to M \to 0$ of $R$-modules such that $X$ is totally $C$-reflexive and $C \dim Y < \infty$.

As a direct corollary of Theorem 3.5, we have an approximation theorem of Auslander and Bridger:

**Corollary 3.8.** Let $M$ be an $R$-module. Then $\Omega^n M$ is $n$-torsionfree if and only if $M$ admits an $n$-spherical approximation.
Corollary 3.9. Let \((R, \mathfrak{m})\) be a \(d\)-dimensional Cohen-Macaulay local ring with the canonical module \(\omega\). Then every \(R\)-module \(M\) admits a Cohen-Macaulay approximation, namely, there exists an exact sequence \(0 \to Y \to X \to M \to 0\) such that \(X\) is a maximal Cohen-Macaulay \(R\)-module and \(Y\) is an \(R\)-module of finite injective dimension.

Proof. If \(d = 0\), then \(0 \to 0 \to M \to 0\) is a desired exact sequence. Let \(d \geq 1\). Then \(\omega\) is \(d\)-semidualizing, and \(\Omega^d M\) is \(d\)-\(\omega\)-torsionfree. Hence Theorem \(3.5\) guarantees the existence of an exact sequence \(0 \to Y \to X \to M \to 0\) such that \(X\) is \(d\)-\(\omega\)-spherical and \(\operatorname{wdim} Y < d\). Therefore \(X\) is maximal Cohen-Macaulay. On the other hand, noting that \(\omega\) is an indecomposable \(R\)-module, we have an exact sequence \(0 \to \omega^{d-1} \to \omega^{d-2} \to \cdots \to \omega^0 \to Y \to 0\). Decomposing this into short exact sequences and noting that \(\omega\) has finite injective dimension, one sees that \(Y\) also has finite injective dimension. \(\square\)

4. Modules admitting approximations

In the previous section, we proved that if \(C\) is \(n\)-semidualizing, then any module whose \(n\)-th syzygy is \(n\)-\(C\)-torsionfree admits an \(n\)-\(C\)-spherical approximation. In this section, we will consider modules whose \(n\)-th syzygies are \(n\)-\(C\)-torsionfree, and give several sufficient conditions for a given module to be such a module.

We begin by proving the following lemma.

Lemma 4.1. Let \(M\) be an \(R\)-module.

1. If \(R\) is \(1\)-\(C\)-torsionfree, then so is \(\Omega M\).

2. If \(R\) is \(2\)-\(C\)-torsionfree, then for each \(n \geq 2\) the map \(\lambda_{\Omega^n M}\) is a split monomorphism and the cokernel of \(\lambda_{\Omega^n M}\) is isomorphic to \(\operatorname{Ext}^2_R(M, C)\).

Proof. (1) There is an exact sequence \(0 \to \Omega M \xrightarrow{\theta} P \to M \to 0\) with \(P\) projective. One has the following commutative diagram:

\[
\begin{array}{ccc}
\Omega M & \xrightarrow{\theta} & P \\
\lambda_{\Omega M} \downarrow & & \lambda_P \downarrow \\
(\Omega M)^\dagger & \xrightarrow{\theta^\dagger} & P^\dagger
\end{array}
\]

The map \(\theta\) is injective, and the assumption implies that \(\lambda_P\) is injective. Hence \(\lambda_{\Omega M}\) is also injective.

(2) Putting \(X = \Omega^{n-2} M\), one has \(\Omega^n M = \Omega^2 X\) and \(\operatorname{Ext}^2_R(M, C) = \operatorname{Ext}^2_R(X, C)\). Hence, replacing \(M\) with \(X\), we have only to show the lemma when \(n = 2\). There is an exact sequence \(0 \to \Omega^2 M \to Q \to \Omega M \to 0\), where \(Q\) is a projective \(R\)-module. Applying the \(C\)-dual functor \((-)^\dagger\) yields an exact sequence \(0 \to (\Omega M)^\dagger \to Q^\dagger \xrightarrow{\sigma} (\Omega^2 M)^\dagger \to \operatorname{Ext}^2_R(M, C) \to 0\). Set \(N = \operatorname{Im} \sigma\) and decompose this sequence into two short exact sequences:

\[
\begin{align}
(4.1.1) & \quad 0 \to (\Omega M)^\dagger \to Q^\dagger \to N \to 0, \\
& \quad 0 \to N \xrightarrow{\tau} (\Omega^2 M)^\dagger \to \operatorname{Ext}^2_R(M, C) \to 0.
\end{align}
\]
Applying $(-)\dagger$ to the first sequence, one obtains a commutative diagram
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^2 M & \longrightarrow & Q & \longrightarrow & \Omega M & \longrightarrow & 0 \\
\eta \downarrow & & \lambda_Q \downarrow & & \lambda_{QM} \downarrow & & & & \\
0 & \longrightarrow & N^\dagger & \longrightarrow & Q^\dagger & \longrightarrow & (\Omega M)^\dagger & & \\
\end{array}
\]
with exact rows. The map $\lambda_Q$ is an isomorphism by the assumption, and $\lambda_{QM}$ is a monomorphism by (1). Therefore we see from the snake lemma that the map $\eta$ is an isomorphism. The diagram
\[
\begin{array}{ccc}
\Omega^2 M & \longrightarrow & \Omega^2 M \\
\lambda_{QM} \downarrow & & \eta \downarrow \cong \\
0 & \longrightarrow & \text{Ext}^2_R(M, C)^\dagger & \longrightarrow & (\Omega^2 M)^\dagger \longrightarrow & N^\dagger \\
\end{array}
\]
commutes, where the bottom row is the $C$-dual of the second sequence in (4.1.1). This commutative diagram shows that $\lambda_{QM}$ is a split monomorphism, that the sequence
\[(4.1.2) \quad 0 \to \text{Ext}^2_R(M, C)^\dagger \to (\Omega^2 M)^\dagger \to N^\dagger \to 0\]
is split exact and that $\text{Coker}(\lambda_{QM}) \cong \text{Ext}^2_R(M, C)^\dagger$. \hfill $\square$

For $R$-modules $M, N$, we define $\text{grade}(M, N)$ by the infimum of integers $i$ such that $\text{Ext}^i_R(M, N) \neq 0$. We state a criterion for $\Omega^2 M$ to be $i$-$C$-torsionfree for $1 \leq i \leq n$ in terms of grade.

**Proposition 4.2.** Let $C$ be an $R$-module such that $R$ is $(n-1)$-$C$-torsionfree.

1. If $\Omega^i M$ is $i$-$C$-torsionfree for every $1 \leq i \leq n$, then $\text{grade}(\text{Ext}^i_R(M, C), C) \geq i - 1$ for every $1 \leq i \leq n$.

2. The converse also holds if $R$ is $n$-$C$-torsionfree.

**Proof.** We prove the proposition by induction on $n$. When $n = 1$, the statement (1) is trivial and the statement (2) follows from Lemma 1.1.1.

Let $n = 2$. There is an exact sequence $0 \to \Omega^2 M \to P \to \Omega M \to 0$, where $P$ is projective. Applying the functor $(-)\dagger$ gives another exact sequence $0 \to (\Omega^2 M)^\dagger \to P^\dagger \to (\Omega^2 M)^\dagger \to \text{Ext}^2_R(M, C) \to 0$. Applying $(-)\dagger$ again, we get the following commutative diagram with exact rows:
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^2 M & \longrightarrow & P \\
\lambda_{\Omega^2 M} \downarrow & & \lambda_P \downarrow & & & \\
0 & \longrightarrow & \text{Ext}^2_R(M, C)^\dagger & \longrightarrow & (\Omega^2 M)^\dagger & \longrightarrow & P^\dagger \\
\end{array}
\]
Since $R$ is $1$-$C$-torsionfree by the assumption, $\lambda_P$ is injective, hence so is $\lambda_P$. Thus, if $\Omega^2 M$ is $2$-$C$-torsionfree, then $\lambda_{\Omega^2 M}$ is bijective and diagram chasing shows $\text{Ext}^2_R(M, C)^\dagger = 0$. Therefore the statement (1) holds. As for the statement (2), suppose that $R$ is $2$-$C$-torsionfree and $\text{Ext}^2_R(M, C)^\dagger = 0$. The induction hypothesis shows that $\Omega^2 M$ is $1$-$C$-torsionfree, and Lemma 1.1.2 implies that $\lambda_{\Omega^2 M}$ is bijective, i.e., $\Omega^2 M$ is $2$-$C$-torsionfree. Therefore the statement (2) holds.

Let $n \geq 3$. The induction hypothesis implies that we may assume that $\Omega^i M$ is $i$-$C$-torsionfree and $\text{grade}(\text{Ext}^i_R(M, C), C) \geq i - 1$ for $1 \leq i \leq n - 1$. There is an
exact sequence

\[0 \to \Omega^n M \to P \to \Omega^{n-1} M \to 0\]

with \(P\) projective, and dualizing this by \(C\) yields an exact sequence \(0 \to (\Omega^{n-1} M)^! \to P^! \to (\Omega^n M)^! \to \text{Ext}^n_R(M, C) \to 0\). Decompose this sequence into two short exact sequences as follows:

\[
\begin{align*}
0 & \to (\Omega^{n-1} M)^! \to P^! \to N \to 0, \\
0 & \to N \to (\Omega^n M)^! \to \text{Ext}^n_R(M, C) \to 0.
\end{align*}
\]

Applying \((-)^!\) to the first exact sequence and using the assumption that \(R\) is \((n-1)\)-\(C\)-torsionfree, we get \(\text{Ext}^i_R(N, C) = 0\) for \(1 \leq i \leq n - 3\) and a monomorphism \(\text{Ext}^{n-2}_R(N, C) \hookrightarrow \text{Ext}^{n-2}_R(P^!, C)\). Apply \((-)^!\) to the second exact sequence. Note that the proof of the existence of 14.2 in the proof of Lemma 4.1 shows that the induced sequence \(0 \to \text{Ext}^n_R(M, C)^! \to (\Omega^n M)^! \to N^! \to 0\) is exact. Hence we obtain isomorphisms \(\text{Ext}^i_R((\Omega^n M)^!, C) \cong \text{Ext}^i_R((\Omega^n M)^!, C)\) for \(1 \leq i \leq n - 3\) and an exact sequence \(0 \to \text{Ext}^{n-2}_R((\Omega^n M)^!, C) \to \text{Ext}^{n-2}_R((\Omega^n M)^!, C) \to \text{Ext}^{n-2}_R(N, C)\). Moreover, Lemma 4.1(2) guarantees that \(\text{Ext}^n_R(M, C)^! = 0\) if and only if \(\lambda_M\) is an isomorphism. Now it is easy to observe that if \(\Omega^n M\) is \(n\)-\(C\)-torsionfree then \(\text{grade}(\text{Ext}^n_R(M, C), C) \geq n - 1\), and that the converse holds when \(R\) is \(n\)-\(C\)-torsionfree. This completes the proof of the proposition.

Next we want to consider the difference between the condition that \(\Omega^n M\) is \(n\)-\(C\)-torsionfree and the condition that \(\Omega^i M\) is \(i\)-\(C\)-torsionfree for \(1 \leq i \leq n\). For this, let us study modules of finite add \(C\)-resolution dimension.

Lemma 4.3. Let \((R, m, k)\) be a local ring and \(C\) an \(R\)-module such that \(\lambda_R\) is an isomorphism. If \(M\) is an \(R\)-module with \(\text{Cdim } M < \infty\), then \(\text{Cdim } M \leq \text{depth } C\).

Proof. First of all, let us show that the \(R\)-module \(C\) is indecomposable: suppose that there is a direct sum decomposition \(C \cong X \oplus Y\) for some nonzero \(R\)-modules \(X, Y\). Since \(\lambda_R\) is an isomorphism, one has isomorphisms \(R \cong \text{Hom}_R(C, C) \cong \text{Hom}_R(X, X) \oplus \text{Hom}_R(Y, X) \oplus \text{Hom}_R(Y, Y)\). Both of the modules \(\text{Hom}_R(X, X)\) and \(\text{Hom}_R(Y, Y)\) are nonzero, because they contain the identity maps. Hence \(R\) is decomposable as an \(R\)-module, which contradicts the assumption that \(R\) is a local ring. Therefore \(C\) is an indecomposable \(R\)-module, and all objects of add \(C\) are finite direct sums of copies of \(C\).

Put \(\text{Cdim } M = s\). There exists an exact sequence \(0 \to C^{l_0} \xrightarrow{\phi_1} C^{l_{i-1}} \xrightarrow{\phi_{i-1}} \cdots \xrightarrow{\phi_1} C^{l_0} \to M \to 0\). Note that all \(l_i\) are nonzero. Since the homothety map \(\lambda_R : R \to \text{Hom}_R(C, C)\) is an isomorphism, one can identify each \(\phi_i\) with an \(l_{i-1} \times l_i\) matrix over \(R\). With respect to the suitable bases of \(C^{l_i}\) and \(C^{l_{i-1}}\), the matrix \(\phi_s\) has the form \((E \ 0)\), where \(E\) is an identity matrix and \(A\) is a matrix whose components are in the maximal ideal \(m\). Removing \(E\) from \(\phi_s\) and iterating this procedure, we may assume that all components of the matrix \(\phi_i\) belong to the maximal ideal \(m\) for each \(1 \leq i \leq s\).

Set \(t = \text{depth } C\). We want to prove that \(s\) is not bigger than \(t\). If \(s = 0\), then the inequality obviously holds. Let \(s > 0\). From the exact sequence \(0 \to C^{l_i} \xrightarrow{\phi_i} C^{l_{i-1}}\) we get an exact sequence \(0 \to \text{Hom}_R(k, C^{l_i}) \xrightarrow{f} \text{Hom}_R(k, C^{l_{i-1}})\), where \(f = \text{Hom}_R(k, \phi_s)\). Since all the components of \(\phi_s\) belong to \(m\), we have \(f = 0\). Hence \(\text{Hom}_R(k, C^{l_i}) = 0\), which implies that \(t = \text{depth } C > 0\). Putting \(M_i = \text{Im } \phi_i\), we have an exact sequence \(0 \to M_{i+1} \to C^{l_i} \to M_i \to 0\) for \(1 \leq
There are isomorphisms $Ext^i_R(M, C) \cong Ext^{i-1}_R(M_{s-1}, C) \cong \cdots \cong Ext^{i-s+1}_R(M_{s-1}, C) \cong Ext^{i-s}_R(C_s, C)$, where the last Ext module vanishes because $1 \leq r - s \leq r$.

The following is a well-known result concerning grade; see [5, Proposition 1.2.10(a),(e)], for example.

**Lemma 4.5.** For $R$-modules $M$ and $N$, one has $\text{grade}(M, N) = \inf \{ \text{depth } N_p | p \in \text{Supp } M \}$.

Combining the above three lemmas, we obtain the following.

**Lemma 4.6.** Let $C$ be an $R$-module such that $\lambda_R$ is an isomorphism and $Ext^i_R(M, C) = 0$ for $1 \leq i \leq n$. If $M$ is an $R$-module with $\text{Cdim } M < \infty$, then $\text{grade}(\text{Ext}^i_R(M, C), C) \geq i$ for all $1 \leq i \leq n$.

**Proof.** Fix an integer $i$ with $1 \leq i \leq n$. Let $p$ be an prime ideal of $R$ satisfying $\text{depth}_{R_p} C_p < i$. Then one has $\text{Cdim}_{R_p} M_p < i$ by Lemma 4.3 and hence $\text{Ext}^i_{R_p}(M_p, C_p) = 0$ by Lemma 4.3. Therefore $p \not\in \text{Supp } \text{Ext}^i_R(M, C)$, and the inequality $\text{grade}(\text{Ext}^i_R(M, C), C) \geq i$ follows from Lemma 4.3.

Now we can consider the relationship between the $n$-$C$-torsionfreeness of $\Omega^n M$ and the $i$-$C$-torsionfreeness of $\Omega^i M$ for $1 \leq i \leq n$. Under the assumption that $C$ is $n$-semidualizing, these properties are equivalent to each other. We should remark that an $n$-$C$-spherical approximation of $M$ plays an essential role in the proof.

**Proposition 4.7.** Let $C$ be an $n$-semidualizing $R$-module. The following are equivalent for an $R$-module $M$.

1. $\Omega^n M$ is $n$-$C$-torsionfree.
2. $\Omega^i M$ is $i$-$C$-torsionfree for every $1 \leq i \leq n$.

**Proof.** (2) $\Rightarrow$ (1): This implication is trivial.

(1) $\Rightarrow$ (2): Proposition 4.2 says that it is enough to prove that the inequality $\text{grade}(\text{Ext}^i_R(M, C), C) \geq i - 1$ holds for $1 \leq i \leq n$. This inequality automatically holds when $n = 1$, hence let $n \geq 2$. Theorem 3.5 yields an $n$-$C$-spherical approximation $0 \to Y \to X \to M \to 0$ of $M$. Hence we have an isomorphism...
Thus the assertion in the case where \( r \) only if \( \text{Hom} \) \( R \) has \( \text{Hom} \) \( x \) \( m \) where \( \text{Hom} \) \( \rightarrow \) is an exact sequence 0

Proof

Lemma 4.9. Let \( R \) be a local ring and \( r \) a positive integer. Suppose that \( \lambda_R \) is an isomorphism and \( \text{Ext}^i_R(C, C) = 0 \) for all \( 1 \leq i < r \). Then the following hold.

1. \( \text{depth} R \geq r \) if and only if \( \text{depth} C \geq r \).

2. Let \( R \) be a Cohen-Macaulay local ring with \( \dim R < r \). Then \( C \) is a maximal Cohen-Macaulay \( R \)-module.

Proof. (1) Let us show the assertion by induction on \( r \). Since \( R \cong \text{Hom}_R(C, C) \), one has \( \text{Hom}_R(k, R) \cong \text{Hom}_R(k, \text{Hom}_R(C, C)) \cong \text{Hom}_R(k \otimes_R C, C) \cong \text{Hom}_R(k, C)^m \), where \( m \) is the minimal number of generators of \( C \). Hence \( \text{Hom}_R(k, R) = 0 \) if and only if \( \text{Hom}_R(k, C) = 0 \). In other words, \( \text{depth} R \geq 1 \) if and only if \( \text{depth} C \geq 1 \). Thus the assertion in the case where \( r = 1 \) is proved.

Let \( r \geq 2 \). The induction hypothesis guarantees the existence of an element \( x \in R \) which is both \( R \)-regular and \( C \)-regular. Set \( (-)^\sim = (-) \otimes_R R/(x) \). There is an exact sequence \( 0 \to C \xrightarrow{\tau} C \to \overline{C} \to 0 \). Apply the functor \((-)^\sim \) to this, and make a long exact sequence; we see that the map \( R \to \text{Ext}^i_R(C, C) \) induced by the connecting homomorphism \( R \cong C^\sim \to \text{Ext}^i_R(C, C) \) is an isomorphism, and that \( \text{Ext}^i_R(C, C) = 0 \) for \( 2 \leq i < r \). Since there is a natural isomorphism \( \text{Ext}^i_R(C, C) \cong \text{Ext}^{i+1}_R(C, C) \) for any integer \( i \), the natural map \( R \to \text{Hom}_R(C, C) \) is an isomorphism and \( \text{Ext}^i_R(C, C) = 0 \) for \( 1 \leq i < r-1 \). It follows from the induction hypothesis that \( \text{depth} \overline{R} \geq r-1 \) if and only if \( \text{depth} \overline{C} \geq r-1 \). Thus \( \text{depth} R \geq r \) if and only if \( \text{depth} C \geq r \).

(2) Set \( d = \dim R \). According to the assumptions, the map \( \lambda_R \) is an isomorphism, \( \text{Ext}^i_R(C, C) = 0 \) for \( 1 \leq i \leq d \) and \( \text{depth} R = d \geq d \). Hence the assertion (1) yields \( \text{depth} C \geq d \), which means that \( C \) is maximal Cohen-Macaulay.

Lemma 4.9. Let \( R \) be a local ring and \( C \) a semidualizing \( R \)-module. Then the following hold for an \( R \)-module \( M \):

1. If \( G_C \dim_R M < \infty \), then \( G_C \dim_R M = \text{depth} R - \text{depth} C = \sup \{ i | \text{Ext}^i_R(M, C) \neq 0 \} \).

2. For a nonnegative integer \( r \), one has \( G_C \dim_R(\Omega^r M) = \sup \{ G_C \dim_R M - r, 0 \} \).

Let us give sufficient conditions for an \( R \)-module \( M \) to be such that \( \Omega^r M \) is \( n \)-\( C \)-torsionfree:

Proposition 4.10. Let \( M \) be an \( R \)-module, and let \( C \) be an \( R \)-module such that \( R \) is \( n \)-\( C \)-torsionfree. Suppose that either of the following holds:

1. \( \text{pd}_{R_p} M_p < \infty \) for any \( p \in \text{Spec} R \) with \( \text{depth} R_p \leq n-2 \).

2. \( C_p \) is a semidualizing \( R_p \)-module and \( G_C \dim_{R_p} M_p < \infty \) for any \( p \in \text{Spec} R \) with \( \text{depth} R_p \leq n-2 \).
Then $\Omega^n M$ is n-C-torsionfree.

**Proof.** The proposition can be proved by induction on $n$. When $n = 1$, by Lemma 4.9(1), the conclusion automatically holds. Hence let $n \geq 2$. It follows from the induction hypothesis that $\Omega^i M$ is $i$-C-torsionfree for $1 \leq i \leq n - 1$. Proposition 4.2 says that $\text{grade}(\text{Ext}^n_R(M, C), C) \geq i - 1$ for $1 \leq i \leq n - 1$, and that we have only to prove the inequality

$$\text{grade}(\text{Ext}^n_R(M, C), C) \geq n - 1.$$  

Let $p$ be a prime ideal of $R$ satisfying depth $C_p \leq n - 2$. Then depth $R_p \leq n - 2$ by Lemma 4.8(1).

Firstly, suppose that the condition (1) holds. Then $\text{pd}_{R_p} M_p < \infty$, and one has $\text{pd}_{R_p} M_p \leq \text{depth} R_p \leq n - 2$. Hence $\text{Ext}^i_{R_p}(M_p, X) = 0$ for any $R_p$-module $X$ and any $i > n - 2$. In particular, we get $\text{Ext}^n_{R_p}(M, C)_p \cong \text{Ext}^n_{R_p}(M_p, C_p) = 0$, and Lemma 4.5 yields the inequality $\text{grade}(\text{Ext}^n_R(M, C), C) \geq n - 1$.

Secondly, suppose that the condition (2) holds. Then $\text{Gdim}_{R_p} M_p < \infty$, and Lemma 4.9(1) implies that $\sup \{ i \mid \text{Ext}^i_{R_p}(M_p, C_p) \neq 0 \} = \text{Gdim}_{R_p} M_p \leq \text{depth} R_p \leq n - 2$. Therefore one has $\text{Ext}^n_{R_p}(M_p, C_p) = 0$. It follows from Lemma 4.5 that the inequality $\text{grade}(\text{Ext}^n_R(M, C), C) \geq n - 1$ holds. □

Recall that the G-dimension of an $R$-module $M$, which is denoted by Gdim$_R M$, is defined as the G$_R$-dimension of $M$. As a corollary of the above proposition, we get a theorem of Mašek.

**Corollary 4.11.** [11] Theorem 43] Let $X$ be an $R$-module such that Gdim$_R X_p < \infty$ for any $p \in \text{Spec} R$ with depth $R_p \leq n - 2$. Then $X$ is n-torsionfree if (and only if) $X$ is n-syzygy.

**Proof.** Suppose that $X$ is n-syzygy, i.e., $X \cong \Omega^n M$ for some $R$-module $M$. Then note by Lemma 4.8(2) that the $R_p$-module $M_p$ is also of finite G-dimension for $p \in \text{Spec} R$ with depth $R_p \leq n - 2$. Proposition 4.10 shows that $\Omega^n M$ is n-torsionfree. Hence $X$ is n-torsionfree. □

5. CONTRAVARIANT FINITENESS

In the previous section, we investigated those $R$-modules whose $n^\text{th}$ syzygies are n-C-torsionfree; we obtained several conditions for a given $R$-module to be such a module. In this section, we shall consider conditions for all $R$-modules to be such modules. After that, we will study the contravariant finiteness of the full subcategory of mod $R$ consisting of all n-C-spherical modules.

First of all, we prove the following theorem, which is the second main result of this paper.

**Theorem 5.1.** Suppose that $R$ is n-C-torsionfree. Then the following are equivalent.

1. $\text{id}_{R_p} C_p < \infty$ for any $p \in \text{Spec} R$ with depth $R_p \leq n - 2$.
2. $\Omega^n M$ is n-C-torsionfree for any $R$-module $M$.

**Proof.** We use induction on $n$ to prove the theorem. When $n = 1$, the assertion (1) holds because there is no prime ideal $p$ of $R$ satisfying depth $R_p \leq n - 2$, and the assertion (2) holds by Lemma 4.11(1). In the following, we consider the case where $n \geq 2$.
(1) ⇒ (2): Fix an \( R \)-module \( M \). The induction hypothesis shows that \( \Omega^i M \) is \( i \)-C-torsionfree for \( 1 \leq i \leq n - 1 \). By Proposition 4.2 we have \( \text{grade}(\text{Ext}^i_R(M, C)) \geq i - 1 \) for \( 1 \leq i \leq n - 1 \), and it suffices to prove that the inequality \( \text{grade}(\text{Ext}^n_R(M, C)) \geq n - 1 \) holds. Let \( p \in \text{Spec} R \). If depth \( C_p \leq n - 2 \), then depth \( R_p \leq n - 2 \) by Lemma 4.8(1). The assumption says that \( \text{id}_{R_p} C_p < \infty \), and \( \text{id}_{R_p} C_p = \text{depth} R_p \leq n - 2 \). Therefore \( \text{Ext}^n_R(M, C)_p \cong \text{Ext}^n_R(M_p, C_p) = 0 \). Thus we see from Lemma 4.5 that \( \text{grade}(\text{Ext}^n_R(M, C)) \geq n - 1 \), as desired.

(2) ⇒ (1): When \( n = 2 \), Lemma 4.1(2) implies that \( \text{Ext}^2_{R_p}(M, C_p) = 0 \) for all \( R \)-modules \( M \) and \( p \in \text{Ass} C \), because \( \text{Ass}(\text{Ext}^2_R(M, C)) = \text{Supp} \text{Ext}^2_R(M, C) \cap \text{Ass} C \). The isomorphism \( \lambda_R : R \rightarrow \text{Hom}_R(C, C) \) shows that \( C \) coincides with \( \text{Ass} C \). Hence, setting \( M = \Omega^i_R(R/p) \), one has \( \text{Ext}^{i+2}_R(\kappa(p), C_p) \cong \text{Ext}^2_{R_p}((\Omega^i_R(R/p))_p, C_p) = 0 \) for any \( p \in \text{Spec} R \) and any \( i > 0 \). Therefore \( \text{id}_{R_p} C_p < \infty \) for \( p \in \text{Spec} R \) with depth \( R_p = 0 \).

Let \( n \geq 3 \). Fix an \( R \)-module \( M \). We have an exact sequence \( 0 \rightarrow \Omega^{n+1}M \rightarrow P \rightarrow \Omega^n M \rightarrow 0 \) such that \( P \) is a projective \( R \)-module. From this we get another exact sequence \( 0 \rightarrow (\Omega^n M)^{\dagger} \rightarrow P^{\dagger} \rightarrow (\Omega^{n+1} M)^{\dagger} \rightarrow \text{Ext}^{n+1}_R(M, C) \rightarrow 0 \). Decompose this into short exact sequences:

\[
\begin{align*}
0 & \rightarrow (\Omega^n M)^{\dagger} \rightarrow P^{\dagger} \rightarrow N \rightarrow 0, \\
0 & \rightarrow N \rightarrow (\Omega^n M)^{\dagger} \rightarrow \text{Ext}^{n+1}_R(M, C) \rightarrow 0.
\end{align*}
\]

Note from the assumption that both \( \Omega^n M \) and \( \Omega^{n+1}M = \Omega^n(\Omega M) \) are \( n \)-C-torsionfree. Since \( R \) is \( n \)-C-torsionfree, we see from the first sequence in (5.1.1) that there is a commutative diagram

\[
\begin{array}{ccccccc}
0 & \rightarrow & (\Omega^n M)^{\dagger} & \rightarrow & P^{\dagger} & \rightarrow & N \\
\alpha & \downarrow & \lambda^{\dagger} & \cong & \lambda_{\Omega^n M} & \cong & \\
0 & \rightarrow & N^{\dagger} & \rightarrow & P^{\dagger} & \rightarrow & (\Omega^n M)^{\dagger} & \rightarrow & \text{Ext}^{n+1}_R(N, C) & \rightarrow & 0
\end{array}
\]

with exact rows, and \( \text{Ext}^p_R(N, C) = 0 \) for \( 2 \leq i \leq n - 2 \). This diagram shows that \( \alpha \) is an isomorphism and \( \text{Ext}^1_R(N, C) = 0 \). The second sequence in (5.1.1) gives an exact sequence \( 0 \rightarrow \text{Ext}^{n+1}_R(M, C)^{\dagger} \rightarrow (\Omega^{n+1} M)^{\dagger} \rightarrow N^\dagger \rightarrow \text{Ext}^1_R(\text{Ext}^{n+1}_R(M, C), C) \rightarrow 0 \) and \( \text{Ext}^p_R(\text{Ext}^{n+1}_R(M, C), C) = 0 \) for \( 2 \leq i \leq n - 2 \). Since the diagram

\[
\begin{array}{ccc}
\Omega^{n+1} M & \rightarrow & \Omega^{n+1} M \\
\cong & \alpha & \cong \\
(\Omega^{n+1} M)^{\dagger} & \rightarrow & N^\dagger \\
\beta & \cong & \\
\end{array}
\]

commutes, the map \( \beta \) is an isomorphism, and \( \text{Ext}^{n+1}_R(M, C)^{\dagger} = 0 = \text{Ext}^1_R(\text{Ext}^{n+1}_R(M, C), C) \). Thus we have \( \text{Ext}^i_R(\text{Ext}^{n+1}_R(M, C), C) = 0 \) for every \( i \leq n - 2 \), which means that the inequality \( \text{grade}(\text{Ext}^{n+1}_R(M, C), C) \geq n - 1 \) holds. Therefore, if \( p \) is a prime ideal of \( R \) with depth \( R_p \leq n - 2 \), then depth \( C_p \leq n - 2 \) by Lemma 4.8(1), and it follows from Lemma 4.5 that \( p \) does not belong to \( \text{Supp} \text{Ext}^{n+1}_R(M, C) \), i.e., \( \text{Ext}^{n+1}_R(M_p, C_p) = 0 \). Putting \( M = \Omega^i_R(R/p) \), we obtain \( \text{Ext}^{n+1}_R(k(p), C_p) \cong \text{Ext}^{n+1}_R((\Omega^i_R(R/p))_p, C_p) = 0 \) for any \( i > 0 \). This implies that \( \text{id}_{R_p} C_p < \infty \), and the proof is completed. \( \square \)
Remark 5.2. In the case where \( C \) is \( n \)-semidualizing, one can prove the above theorem more easily, as follows. Fix an \( R \)-module \( M \). The \( n \)-\( C \)-torsionfree property of \( \Omega^n M \) is equivalent to the \( i \)-\( C \)-torsionfree property of \( \Omega^i M \) for \( 1 \leq i \leq n \) by Proposition 4.7 which is equivalent to the inequality \( \text{grade}(\text{Ext}^i_R(M, C), C) \geq i - 1 \) for \( 1 \leq i \leq n \) by Proposition 4.2 which is equivalent to the condition that \( \text{Ext}^i_R(M_p, C_p) = 0 \) for any \( p \in \text{Spec} R \) with depth \( R_p \leq i - 2 \) by Lemmas 4.3 and 4.2(1). Hence, setting \( M = \Omega^j(R/p) \) for \( j \gg 0 \), we easily see that the two conditions in the theorem are equivalent.

Remark 5.3. It is easy to see that Theorem 5.1(1) holds in each of the following cases:

(1) \( R \) satisfies Serre’s condition \( (S_{n-1}) \) and \( C \) is locally of finite injective dimension in codimension \( n - 2 \), namely, \( \text{id}_{R_p} C_p < \infty \) for \( p \in \text{Spec} R \) with \( \text{ht} p \leq n - 2 \).

(2) \((R, m)\) is local, \( n \) is at most depth \( R + 1 \) and \( C \) is locally of finite injective dimension on the punctured spectrum of \( R \), namely, \( \text{id}_{R_p} C_p < \infty \) for \( p \in \text{Spec} R - \{m\} \).

Corollary 5.4. Let \( C \) be an \( n \)-semidualizing \( R \)-module. Suppose that \( R \) satisfies Serre’s conditions \( (R_{n-2}) \) and \( (S_{n-1}) \). Then every \( R \)-module \( M \) has an \( n \)-\( C \)-spherical approximation.

Proof. Take a prime ideal \( p \) of \( R \) with \( \text{ht} p \leq n - 2 \). Then \( R_p \) is a regular local ring, hence \( \text{id}_{R_p} C_p < \infty \). Thus the corollary follows from Remark 5.3(1) and Theorems 5.1 and 5.5. \( \square \)

The lemma below says that over a Gorenstein local ring of dimension \( d \geq 2 \), any \( n \)-semidualizing module is free for \( n \geq d \).

Lemma 5.5. Let \((R, m, k)\) be a \( d \)-dimensional Gorenstein local ring. If \( \lambda_R \) is an isomorphism and \( \text{Ext}^i_R(C, C) = 0 \) for \( 1 \leq i \leq d \), then \( C \cong R \).

Proof. Lemma 4.2(2) says that the \( R \)-module \( C \) is maximal Cohen-Macaulay. Hence there exists a sequence \( x = x_1, \ldots, x_d \) in \( m \) which is both \( R \)-regular and \( C \)-regular. Repeating a similar argument to the proof of Lemma 4.3(1), we obtain

\[
\text{Ext}^i_R(R/(x_i), C/x_i C, C/x_i C) \cong \begin{cases} R/(x_i) & \text{if } j = 0, \\ 0 & \text{if } 1 \leq j \leq d - i, \end{cases}
\]

where \( x_i = x_1, \ldots, x_i \), for each \( 1 \leq i \leq d \). It follows that there is an isomorphism \( \text{Hom}_R(C, C) \cong \overline{R} \), where \(-\) = \(-\) \( \otimes_R R/(x) \). Hence \( \text{Hom}_R(k, \overline{R}) \cong \text{Hom}_R(k, \text{Hom}_R(C, C)) \cong \text{Hom}_R(k \otimes_R C, C) \cong \text{Hom}_R(k, \overline{C})^m \), where \( m \) is the minimal number of generators of the \( R \)-module \( C \). Since \( R \) is artinian Gorenstein, we have \( \text{Hom}_R(k, \overline{R}) \cong k \). Thus one must have \( m = 1 \), hence \( C \) is a cyclic \( \overline{R} \)-module, and \( \overline{C} \cong \overline{R}/I \) for some ideal \( I \) of \( R \). Therefore \( \overline{R} \cong \text{Hom}_R(C, C) \cong \text{Hom}_R(\overline{R}/I, \overline{R}/I) \cong \overline{R}/I \), and we get \( I = 0 \), i.e., \( \overline{C} \cong \overline{R} \). This gives an isomorphism \( C \cong R_1 \).

Applying the above lemma, we can get a sufficient condition for \( R \) and \( C \) to satisfy Theorem 5.1(1).

Proposition 5.6. Suppose that \( R \) is \( n \)-\( C \)-torsionfree and that \( R_p \) is Gorenstein for any \( p \in \text{Spec} R \) with depth \( R_p \leq n - 2 \). Then \( \text{id}_{R_p} C_p < \infty \) for any \( p \in \text{Spec} R \) with depth \( R_p \leq n - 2 \). (Hence \( \Omega^n M \) is \( n \)-\( C \)-torsionfree for any \( R \)-module \( M \).)
Proposition 5.7. Define two full subcategories of $\mathcal{X}$ as follows:

$$\mathcal{X} = \{ X \in \text{mod} R \mid X \text{ is } n\text{-C-spherical} \},$$

$$\mathcal{Y} = \{ Y \in \text{mod} R \mid C\dim Y < n \}.$$

Let $0 \to Y \to X \xrightarrow{f} M \to 0$ be an exact sequence of $R$-modules with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then the homomorphism $f$ is a right $\mathcal{X}$-approximation of $M$.

Proof. We may assume $n \geq 2$. Take a prime ideal $p$ of $R$ with depth $R_p \leq n - 2$. Then $R_p$ is Gorenstein, and $\dim R_p = \text{depth } R_p \leq n - 2$. Since $R$ is $n$-$C$-torsionfree, one has

$$\text{Ext}^i_{R_p}(C_p, C_p) \cong \begin{cases} R_p & \text{if } i = 0, \\ 0 & \text{if } 1 \leq i \leq \dim R_p. \end{cases}$$

Hence Lemma 5.5 yields an isomorphism $C_p \cong R_p$. Therefore $\text{id}_{R_p} C_p = \text{id}_{R_p} R_p < \infty$. \hfill \□

Here we notice that an $n$-$C$-spherical approximation gives a right approximation:

Corollary 5.8. Let $C$ be an $n$-semidualizing $R$-module such that $\text{id}_{R_p} C_p < \infty$ for any $p \in \text{Spec } R$ with depth $R_p \leq n - 2$. Then the full subcategory

$$\mathcal{X} = \{ X \in \text{mod } R \mid X \text{ is } n\text{-C-spherical} \}$$

of $\text{mod } R$ is contravariantly finite.

Acknowledgments. The author would like to express his deep gratitude to Shiro Goto. He gave the author a lot of valuable comments and careful suggestions on both the mathematics and the construction of this paper. The author is convinced that this paper could not be in such a well-organized form without his great efforts. The author wants to give his hearty thanks to Futoshi Hayasaka and Osamu Iyama, who gave the author helpful comments. The author also thanks the referee for useful suggestions.

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