NEW COMPLEMENTARY SETS OF LENGTH $2^m$ AND SIZE 4

GAOFEI WU
State Key Laboratory of Integrated Service Networks
Xidian University, Xi’an, 710071, China

YUQING ZHANG
National Computer Network Intrusion Protection Center
UCAS, Beijing 100043, China

XUEFENG LIU*
State Key Laboratory of Integrated Service Networks
Xidian University, Xi’an, 710071, China

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Abstract. We construct new complementary sequence sets of size 4, using a graphical description. We explain how the construction can be seen as a special case of a less explicit array construction by Parker and Riera and, at the same time, a generalization of another construction by the same authors. Some generalizations of the construction are also given, which are not in the construction of Parker and Riera. Lower bounds and upper bounds of the number of sequences in the constructions are analyzed.

1. Introduction

Orthogonal frequency-division multiplexing (OFDM) is a communication technique used in several wireless communication standards such as IEEE 802.16 Mobile WiMAX. A major problem with OFDM is the large peak-to-mean envelope power ratio (PMEPR) of uncoded OFDM signals. Please refer to Litsyn’s book [7] for a general source on PMEPR control.

Golay complementary sequences [6] have PMEPR less than or equal to 2.0, which is very low, and Davis and Jedwab [3] showed how to construct ‘standard’ $2^h$-ary Golay complementary sequences of length $2^m$, comprising some second-order cosets of the generalized first-order Reed-Muller codes $RM_{2h}(1,m)$. Paterson [11] extended the theory, and Chen et al. [2] gave a tighter upper bound on the PMEPR of all the cosets of the generalized first Reed-Muller codes $RM_q(1,m)$. In [19], Yu and Gong proposed a method to construct near-complementary sequences, and gave several classes of near-complementary sequences with PMEPR $\leq 4$ by using shortened and...
extended Golay pairs as the seed pairs. In [16, 18], the PMEPR bound of these near-complementary sequences were improved to asymptotically equivalent to 2, and new near-complementary sequences constructed by other seed pairs were also proposed.

Fiedler, Jedwab, and Parker [4] proposed a matrix framework for constructions of Golay sequences. [5,8–10] show that the complementary set construction is primarily an array construction, where sequence sets are obtained by considering suitable projections of the arrays. It is desirable to propose constructions that significantly improve code rate without greatly compromising the upper bound on PMEPR or pairwise distance. See [1,17] for some recent results on such constructions.

In this paper, we propose constructions for complementary 4-sets instead of complementary pairs, having a very simple graphical description and with pairwise Hamming distance $\geq 2^{n-2}$. We show how this is a special case of the general array construction given in [8], and clarify aspects of section 5 of [10]. Our construction also generalizes a more explicit construction in [8]. Lower and upper bounds on the number of sequences generated are analyzed. Some generalizations of the construction are also given, which are not in the construction given in [8].

2. Preliminaries

Let $\xi = e^{2\pi \sqrt{-1}/H}$, where $H$ is an even positive integer. In an OFDM system with $n$ subcarriers and $H$-PSK modulation, the transmitted signal for an $H$-ary sequence $a = (a_0, a_1, \cdots, a_{n-1})$ can be modeled as the real part of the complex envelope, and can be written as

$$s_a(t) = \sum_{i=0}^{n-1} \xi^{a_i}e^{i2\pi(f_0+i\Delta f)t}, t \in [0, 1/\Delta f),$$

where $\Delta f$ is the frequency separation between adjacent subcarriers, $f_0$ is the carrier frequency, and $j = \sqrt{-1}$. Let $A = (A_0, A_1, \cdots, A_{n-1})$, where $A_i = \xi^{a_i}$, and let $A(z) = \sum_{i=0}^{n-1} \xi^{a_i}z^i$, where $z \in \{e^{i2\pi t}|0 \leq t < 1\}$. Then

$$\text{PMEPR}(a) = \frac{1}{n} \sup_{|z|=1} |A(z)|^2,$$

where

$$|A(z)|^2 = n + \sum_{\tau=1}^{n-1} C_a(\tau)z^{-\tau} + \sum_{\tau=1}^{n-1} \overline{C_a(\tau)}z^\tau,$$

and

$$C_a(\tau) = \sum_{i=0}^{n-1-\tau} \xi^{a_i-a_{i+\tau}}, 0 \leq \tau < n,$$

is the aperiodic autocorrelation of $a$. $(a,b)$ is a Golay complementary pair of length $n$ if $C_a(\tau) + C_b(\tau) = 0$ for $0 < \tau < n$. Each sequence of a pair is called a complementary sequence. The associated polynomials, $A(z), B(z)$, satisfy $|A(z)|^2 + |B(z)|^2 = 2n$, and PMEPR($a$) $\leq 2$ since $|A(z)|^2 \leq 2n$ for any $z$ with $|z| = 1$.

**Definition 2.1.** A set of $N$ length $n$ sequences $a^0, a^1, \cdots, a^{N-1}$ is a complementary set if

$$C_{a^0}(\tau) + C_{a^1}(\tau) + \cdots + C_{a^{N-1}}(\tau) = 0, \text{ for all } \tau \neq 0$$

and the associated polynomials, $A^0(z), A^1(z), \cdots, A^{N-1}(z)$, satisfy

$$|A^0(z)|^2 + |A^1(z)|^2 + \cdots + |A^{N-1}(z)|^2 = Nn.$$
Let $f : \mathbb{Z}_2^n \rightarrow \mathbb{Z}_H$ be a generalized Boolean function, and $u = (u_0, u_1, \ldots, u_{m-1})$ be the binary expansion of $u$, where $u_{m-1}$ is the most significant bit. $f(x)$ can be represented by its algebraic normal form (ANF):

$$ f(x) = \sum_{u=0}^{2^m-1} \lambda_u \prod_{i=0}^{m-1} x_i^{u_i}, $$

where $\lambda_u \in \mathbb{Z}_H$. The sequence of length $2^m$ over $\mathbb{Z}_H$ associated with $f(x)$ is given by

$$ f = [f(0), f(1), \ldots, f(2^m - 1)], $$

where $f(i) = f(i_0, \ldots, i_{m-1}), i = \sum_{l=0}^{m-1} i_l 2^l$ and $i_l \in \mathbb{Z}_2$. The generalized $r$th order Reed-Muller code $RM_H(r, m)$ of length $2^m$ is generated by the monomials of degree at most $r$. Davis and Jedwab [3] showed that $2^k$-ary standard complementary sequences of length $2^m$ can be obtained from an explicit ANF, and Paterson [11] generalized the alphabet to an even positive integer $H$.

**Theorem 2.2** ([3,11]). Let

$$ a = \frac{H}{2} \sum_{i=0}^{m-2} x_{\pi(i)}x_{\pi(i+1)} + \sum_{i=0}^{m-1} u_i x_i + e, \quad b = a + \frac{H}{2} x_{\pi(0)}, $$

where $u_i, e \in \mathbb{Z}_H$, and $\pi$ is a permutation of $\{0, \ldots, m-1\}$. Then $(a, b)$ is a Golay complementary pair, i.e. a standard pair.

**Definition 2.3** ([12]). Let $m$ and $k$ be positive integers such that $m \geq k$. Let $n = 2^k$. Consider a sequence $a = [a_0, a_1, \ldots, a_{n-1}]$ of length $n$, its generalized $k$-variable Boolean function is $f(x_0, x_1, \ldots, x_{k-1})$, and its complex-valued sequence of length $n$ is $A = [\xi f(0), \xi f(1), \ldots, \xi f(n-1)] = [A_0, A_1, \ldots, A_{n-1}]$. Let

$$ 0 \leq i_0 < i_1 < \cdots < i_{k-1} < m $$

and write

$$ 0 \leq j_0 < j_1 < \cdots < j_{m-k-1} < m $$

for the remaining indices. Let $x = (x_{j_0}, \ldots, x_{j_{m-k-1}})$. Let $d = (d_0 d_1 \cdots d_{m-k-1})$ be a binary word of length $m - k$. We define an extended sequence $A_{[x=d]}$ of length $2^m$ as follows. As $(u_0 u_1 \cdots u_{k-1})$ ranges over $\mathbb{Z}_2^k$, at position

$$ \sum_{\alpha=0}^{k-1} u_\alpha 2^i + \sum_{\alpha=0}^{m-k-1} d_\alpha 2^j $$

the sequence $A_{[x=d]}$ is equal to $A_i$, where $i = \sum_{\alpha=0}^{k-1} u_\alpha 2^\alpha$, and equal to zero otherwise.

**Example 1.** Let $a = [1011]$, $k = 2$, and $A = [- + -]$ , where ‘+’ and ‘−’ represent $1$, $−1$, respectively. If $x = (x_0 x_1)$ and $d = (10)$, then $A_{[x=d]} = (0\ 000+000-000-000)$ and, if $x = (x_2)$ and $d = (0)$, then $A_{[x=d]} = (- + -0000)$.

In Definition 2.3, take $(i_0, i_1, \ldots, i_{k-1}) = (0, 1, \ldots, k-1)$ and $m = k + 1$, we have $x = (x_k)$, $d = (c), c \in \mathbb{Z}_2$. If $c = 0$, $A_{[x=d]} = (A0)$. If $c = 1$, $A_{[x=d]} = (0A)$, where $0$ is a $0$-valued sequence of length $2^k$. 

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Definition 2.4 ([11, 14]). Let $\mathbf{A}$ and $\mathbf{a}$ be defined as in Definition 2.3. Let $0 < j_0 < j_1 < \cdots < j_r < k$, and $\mathbf{x} = (x_{j_0}, \cdots, x_{j_r-1}), \mathbf{d} = (d_0, d_1, \cdots, d_{r-1}) \in \mathbb{Z}_2^r$. Define the restricted vector $\mathbf{A}|_{x=d}$ to be the complex-valued vector with component $i = \sum_{j=1}^{k-r} j$ equal to $A_i$ if $j_i = d_i$ for each $0 \leq i < r$, and equal to 0 otherwise. Obtain $\hat{\mathbf{A}}|_{x=d}$ of length $2^{k-r}$ from $\mathbf{A}|_{x=d}$ by deleting the zeros in $\mathbf{A}|_{x=d}$. We call $\hat{\mathbf{A}}|_{x=d}$ the compressed sequence of $\mathbf{A}|_{x=d}$.

Example 2. Let $\mathbf{A} = (A_0, A_1, A_2, A_3)$. If $\mathbf{x} = (x_0)$ and $\mathbf{d} = (1)$ then $\mathbf{A}|_{x=d} = (0, A_1, 0, A_3)$, and $\hat{\mathbf{A}}|_{x=d} = (A_1, A_3)$. If $\mathbf{x} = (x_1)$ and $\mathbf{d} = (1)$ then $\mathbf{A}|_{x=d} = (0, 0, A_2, A_3)$, and $\hat{\mathbf{A}}|_{x=d} = (A_2, A_3)$.

Definition 2.5 ([11]). Define $Q : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_H$ by

$$Q(\mathbf{x}) = \sum_{0 \leq i < j < m} q_{ij} x_i x_j,$$

where $q_{ij} \in \mathbb{Z}_H$. Associate $m$-vertex labeled graph $G(Q)$ with $Q$, where the edge between vertex $i$ and $j$ is labeled $q_{ij}$.

For $m \geq 2$, $Q(\mathbf{x}) = \frac{m}{2} \sum_{\alpha=0}^{m-1} x_{\pi(\alpha)} x_{\pi(\alpha+1)}$ corresponds to a path graph.

3. Construction of Complementary Sets of Size 4

Theorem 3.1. Define $A, B : \mathbb{Z}_2^m \rightarrow \mathbb{Z}_H$ by $A = \frac{m}{2} \sum_{j=0}^{k-2} x_{\pi(j)} x_{\pi(j+1)}$ and $B = \frac{m}{2} \sum_{j=k}^{m-2} x_{\pi(j)} x_{\pi(j+1)}$, respectively, where $\pi$ is a permutation of $\{0, 1, \cdots, m-1\}$ and $1 \leq k \leq m-1$. Let $(r_1, r_0, r_1, r_2, \cdots, r_k)$ satisfy $r_{-1} = 0 \leq r_0 \leq r_1 \leq r_2 \leq \cdots \leq r_{k-1} \leq m - k - 1$, and

$$Q_i = \sum_{j_i=k+r_{i-1}}^{k+r_i} a_{j_i} x_{\pi(i)} x_{\pi(j_i)} + e_i x_{\pi(i)} x_{\pi(k+r_i)},$$

where $0 \leq i \leq k-1$, and $a_{j_i} \in \{0, H/2\}, e_i = H/2, \forall i$. Let

$$Q(x_0, x_1, \cdots, x_{m-1}) = A + B + \sum_{i=0}^{k-1} q_i Q_i, \quad q_i \in \{0, 1\}.$$

Then for any $a_i, a \in \mathbb{Z}_H$,

$$\{Q + \sum_{i=0}^{m-1} a_i x_i + a + \frac{H}{2} (d_0 x_{\pi(k-1)} + d_1 x_{\pi(m-1)}) : d_0, d_1 \in \{0, 1\}\}$$

is a complementary set of size 4.

From now on, we call the set of all the quadratic terms in $Q_i, 0 \leq i \leq k-1$ the set of intermediate terms. Note that there are $m - 1$ intermediate terms for a given $Q$. Let $S_m$ be the set of all $m$-variable functions constructed by Theorem 3.1.

Example 3. Let $m = 10, k = 5, H = 4$, and

$$Q(x_0, x_1, \cdots, x_9) = 2 \sum_{j=0}^{8} x_j x_{j+1} + 2 \sum_{j=5}^{8} x_j x_{j+1} + 4 \sum_{i=0}^{4} Q_i,$$
Lemma 3.3. Let \( Q_0 = 2x_0(x_5 + x_6), Q_1 = 2x_1(x_6 + x_7 + x_8), Q_2 = 2x_2x_8, Q_3 = 2x_3(x_8 + x_9), \) and \( Q_4 = 2x_4x_9. \) The graph is shown in Fig. 1. From Theorem 3.1,
\[
\{Q + \sum_{i=0}^{9} a_i x_i + a + 2(d_0 x_4 + d_1 x_9) : d_0, d_1 \in \{0, 1\}\}
\]
is a complementary set of size 4, for any choice of \( a, a \in \mathbb{Z}_4. \) In this example, the \( m - 1 = 9 \) intermediate terms are
\[
\{x_0x_5, x_0x_6, x_1x_6, x_1x_7, x_1x_8, x_2x_8, x_2x_9, x_3x_9, x_4x_9\}.
\]

To prove Theorem 3.1, we need the following lemmas.

Lemma 3.2 ([11, Th. 9]). Let \( \mathbf{a} \) be a Golay sequence of length \( n \) over \( \mathbb{Z}_4 \) given by (2), where \( n = 2^m, \) and \( \mathbf{A} \) be the complex-valued sequence of \( \mathbf{a}. \) Then both \( \mathbf{A}\{x_\pi(m-1)=0, A|_{x_\pi(m-1)=1}\} \) and \( \mathbf{A}\{x_\pi(m-1)=0, \hat{A}|_{x_\pi(m-1)=1}\} \) are Golay pairs. Moreover, both
\[
(A|x_{\pi(m-1)=0}, A|_{x_\pi(m-1)=1}) \text{ and } (\hat{A}|_{x_\pi(m-1)=0}, \hat{A}|_{x_\pi(m-1)=1})
\]
and their compressed sequences
\[
(A|x_{\pi(m-1)=0}, A|_{x_\pi(m-1)=1}) \text{ and } (\hat{A}|_{x_\pi(m-1)=0}, \hat{A}|_{x_\pi(m-1)=1})
\]
are Golay pairs.

Lemma 3.3. Let \( (\mathbf{a}, \mathbf{b}) \) and \( (\mathbf{c}, \mathbf{d}) \) be Golay pairs of lengths \( n \) and \( m, \) respectively, over \( \mathbb{Z}_4. \) Then \( \{a + c_0, a + c_1, \cdots, a + c_{m-1}, b + d_0, b + d_1, \cdots, b + d_{m-1}\} \) is a complementary set of size \( 4, \) where each sequence has length \( mn. \)

![Figure 1. The graph \( G(Q) \) of the quadratic function \( Q \) in Example 3](image)

**Proof.** Let \( A(z), B(z), C(z), D(z) \) be the associated polynomials of \( \mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}, \) respectively. The associated polynomial of \( (a + c_0, a + c_1, \cdots, a + c_{m-1}) \) is
\[
\xi^{c_0}A(z) + \xi^{c_1}A(z)z^n + \cdots + \xi^{c_{m-1}}A(z)z^{(m-1)n} = A(z)C(z^n).
\]
Similarly, we obtain \( B(z)D(z^n), A(z)D(z^n), B(z)C(z^n). \) Then
\[
|A(z)C(z^n)|^2 + |A(z)D(z^n)|^2 + |B(z)C(z^n)|^2 + |B(z)D(z^n)|^2
\]
\[
= |A(z)|^2 \cdot 2m + |B(z)|^2 \cdot 2m
\]
(4)
\[
= 4mn.
\]
This completes the proof.

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Lemma 3.4. Let \( c = \frac{H}{T} \sum_{i=0}^{t-2} x_\pi(i) x_\pi(i+1) + \sum_{i=0}^{t-1} u_i x_\pi(i) + u \) and \( d = c + \frac{H}{T} x_\pi(t-1) \) be a standard Golay pair of length \( m = 2^t \). Let \( C \) and \( D \) be their complex-valued sequences, respectively. Then for any \( 0 \leq i < t \) and \( r_i \in \mathbb{Z}_2 \),
\[
(C_{x_i=r_0}, D_{x_i=r_1})
\]
is also a Golay pair, where each sequence has length \( 2^{t+1} \).

Proof. Let \( c' = \frac{H}{T} \sum_{j=0}^{t-1} x_\pi(j) x_\pi'(j+1) + \sum_{j=0}^{t-1} u_j x_\pi(j) + u \) be a Golay sequence of length \( 2^{t+1} \), where \( \pi'(t) = i \). For \( 0 \leq j \leq t-1 \), \( \pi'(j) = \pi(j) + 1 \) if \( \pi(j) \geq i \), and \( \pi'(j) = \pi(j) \) otherwise. Let \( C' \) be its associated complex-valued sequence. Then \( C'|_{x_i=0} = C|_{x_i=0} \) and \( C'|_{x_i=1} = D|_{x_i=1} \). From Lemma 3.2, \( (C'|_{x_i=0}, C'|_{x_i=1}) \) and \( (C|_{x_i=0}, D|_{x_i=1}) \) are both Golay pairs. Thus the desired conclusion follows from \( |C|_{x_i=0}(z)|^2 = |C|_{x_i=1}(z)|^2 \).

Using Lemma 3.4 several times, we have the following corollary.

Corollary 1. Let \( c = \frac{H}{T} \sum_{i=0}^{t-2} x_\pi(i) x_\pi(i+1) + \sum_{i=0}^{t-1} u_i x_\pi(i) + u \) and \( d = c + \frac{H}{T} x_\pi(t-1) \) be a standard Golay pair of length \( m = 2^t \). Let \( C \) and \( D \) be their complex-valued sequences, respectively. For any \( 0 \leq l < t \) and \( r \in \mathbb{Z}_2 \), let \( x = (x_{i_1}, x_{i_2}, \cdots, x_{i_l}) \), where \( 0 \leq i_j < t \). Then
\[
(C|_{x=r}, D|_{x=r})
\]
is also a Golay pair, each sequence has length \( 2^{t+l} \).

Example 4. Let \( c = \sum_{i=0}^{l} x_i x_{i+1} \) and \( d = \sum_{i=0}^{r} x_i x_{i+1} + x_5 \). Then \( C|_{x_i=0} = 0 \) and \( D|_{x_i=0} = 0 \) is a Golay pair. To see this, let \( c' = x_0 x_1 + x_1 x_3 + x_3 x_5 + x_5 x_6 + x_6 x_7 + x_7 x_2 + x_2 x_4 \). Then \( C|_{x_i=0} = C'|_{x_i=0} \) and \( D|_{x_i=0} = C'|_{x_i=0} \). From Lemma 3.2, we know that \( C'|_{x_i=0} \) and \( C'|_{x_i=0} \) is a golay pair, so is \( C|_{x_i=0} \) and \( D|_{x_i=0} \). Or we can let \( c' = x_0 x_1 + x_1 x_3 + x_3 x_5 + x_5 x_6 + x_6 x_7 + x_7 x_2 + x_2 x_4 \). Then \( C|_{x_i=0} = C'|_{x_i=0} \) and \( D|_{x_i=0} = C'|_{x_i=0} \). Thus, \( C|_{x_i=0} \) and \( D|_{x_i=0} \) is a Golay pair according to Lemma 3.2.

Lemma 3.5. Let \( (A, B, C, D) \) be a complementary set of size 4, where each sequence has length \( n = 2^k \). Then \( (A|_{x_0=r_0}, B|_{x_0=r_1}, C|_{x_0=r_0}, D|_{x_0=r_1}) \) is a complementary set of size 4, and each sequence has length \( 2^{k+1} \).

Proof. For \( A = (A_0, A_1, \cdots, A_{n-1}) \), we have \( A|_{x_0=0} = (A_0, 0, \cdots, A_{n-1}, 0) \), so \( A|_{x_0=0}(z) = A(z^2) \). The identity \( |A(z)|^2 + |B(z)|^2 + |C(z)|^2 + |D(z)|^2 = 4n \) follows from \( |A(z)|^2 + |B(z)|^2 + |C(z)|^2 + |D(z)|^2 = 4n \).

Using Lemma 3.5 several times, we have the following lemma.

Lemma 3.6. Let \( (A, B, C, D) \) be a complementary set of size 4, each complex-valued sequence has length \( n = 2^k \). Then for any \( r > 0 \) and any \( d \in \mathbb{Z}_2^r \), the complex-valued sequences sets
\[
(A|_{x_0 x_1 \cdots x_{r-1} = d}, B|_{x_0 x_1 \cdots x_{r-1} = d}, C|_{x_0 x_1 \cdots x_{r-1} = d}, D|_{x_0 x_1 \cdots x_{r-1} = d})
\]
and
\[
(A|_{x_k x_{k+1} \cdots x_{k+r-1} = d}, B|_{x_k x_{k+1} \cdots x_{k+r-1} = d}, C|_{x_k x_{k+1} \cdots x_{k+r-1} = d}, D|_{x_k x_{k+1} \cdots x_{k+r-1} = d})
\]
of length \( 2^{k+r} \) are both complementary sets of size 4.
Proof. (of Theorem 3.1 by induction.)

There are two initial cases: $\frac{H}{2}(x_0x_1, (x_0x_1 + x_0), (x_0x_1 + x_1), (x_0x_1 + x_0 + x_1))$ and $\frac{H}{2}(0, x_0, x_1, (x_0 + x_1))$.

The first initial case $\frac{H}{2}(x_0x_1, (x_0x_1 + x_0), (x_0x_1 + x_1), (x_0x_1 + x_0 + x_1))$ is a complementary set of size 4, because $\frac{H}{2}(x_0x_1, (x_0x_1 + x_0))$ and $\frac{H}{2}((x_0x_1 + x_1), (x_0x_1 + x_0 + x_1))$ are both Golay pairs. Similarly, $\frac{H}{2}(0, x_0, x_1, (x_0 + x_1))$ is also a complementary set of size 4. We only give the proof of the theorem for the first initial case $\frac{H}{2}(x_0x_1, (x_0x_1 + x_0), (x_0x_1 + x_1), (x_0x_1 + x_0 + x_1))$, one can prove the theorem for the second initial case in a similar way.

From Corollary 1, we have $(A_0, B_0, C_0, D_0) = \frac{H}{2}((x_0x_1x_0), x_0x_1 + x_0, x_0x_1 + x_1, x_0x_1 + x_0 + x_1)$ is a complementary set of size 4, each sequence has 2 nonzero elements, where $\pi = (x_0, x_1, \ldots, x_{m-1}) \setminus \pi$. Then $|A_0(z)|^2 + |B_0(z)|^2 + |C_0(z)|^2 + |D_0(z)|^2 = 4 \cdot 2^3$. Note that the end vertexes of the two paths (in this case, each path have only one vertex) are $x_0(z)$ and $x_1(z)$, respectively. The vertices next to $x_0(z)$ and $x_1(z)$ are $x_1(z)$ and $x_1(z)$, respectively.

- Case 1: Suppose that $x_{k+1}(z)$ is connected with both $x_k(z)$ and $x_{k+1}(z)$. Let

$$A_1(z) = A_0(z) + z^{2^{k+1}}D_0(z), D_1(z) = A_0(z) - z^{2^{k+1}}D_0(z),$$

$$B_1(z) = B_0(z) + z^{2^{k+1}}C_0(z), C_1(z) = B_0(z) - z^{2^{k+1}}C_0(z).$$

Then we have

$$|A_1(z)|^2 + |B_1(z)|^2 + |C_1(z)|^2 + |D_1(z)|^2 = 2(|A_0(z)|^2 + |B_0(z)|^2 + |C_0(z)|^2 + |D_0(z)|^2) = 4 \cdot 2^3,$$

which implies that $(A_1(z), B_1(z), C_1(z), D_1(z))$ is a complementary set of size 4, each sequence has 2 nonzero elements. One can see that $A_1(z)$ is the associated polynomial of the sequence $\frac{H}{2}Q_{[x'=0]}$, where $x' = x \setminus x_{k+1}(z)$, and $Q = x_0(z) + x_{k+1}(z)$. Similarly, $B_1(z)$, $C_1(z)$, and $D_1(z)$ are the associated polynomials of the sequences $\frac{H}{2}Q_{[x'=0]}$, $\frac{H}{2}Q_{[x'=0]}$, $\frac{H}{2}Q_{[x'=0]}$, respectively. Now the end vertexes of $Q$ are $x_0(z)$ and $x_{k+1}(z)$. The vertices next to $x_0(z)$ and $x_{k+1}(z)$ are $x_{k+1}(z)$ and $x_{k+1}(z)$, respectively.

- Case 2: Suppose that $x_{k+1}(z)$ is connected with both $x_k(z)$ and $x_{k+1}(z)$. Let

$$A_1(z) = A_0(z) + z^{2^{k+1}}D_0(z), D_1(z) = A_0(z) - z^{2^{k+1}}D_0(z),$$

$$B_1(z) = C_0(z) + z^{2^{k+1}}B_0(z), C_1(z) = C_0(z) - z^{2^{k+1}}B_0(z).$$

Then we have

$$|A_1(z)|^2 + |B_1(z)|^2 + |C_1(z)|^2 + |D_1(z)|^2 = 2(|A_0(z)|^2 + |B_0(z)|^2 + |C_0(z)|^2 + |D_0(z)|^2) = 4 \cdot 2^3,$$

which implies that $(A_1(z), B_1(z), C_1(z), D_1(z))$ is a complementary set of size 4, each sequence has 2 nonzero elements. One can see that $A_1(z)$ is the associated polynomial of the sequence $\frac{H}{2}Q_{[x'=0]}$, where $x' = x \setminus x_{k+1}(z)$, and $Q = x_0(z) + x_{k+1}(z)$. 

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Similarly, $B_1(z), C_1(z), D_1(z)$ are the associated polynomials of the sequences $\frac{H}{\pi} (Q + x_{\pi(k)})|_{x' = 0}$, $\frac{H}{\pi} (Q + x_{\pi(k)})|_{x' = 0}$, $\frac{H}{\pi} (Q + x_{\pi(1)})|_{x' = 0}$, respectively. Now the end vertexes of $Q$ are $x_{\pi(k)}$ and $x_{\pi(1)}$. The vertexes next to $x_{\pi(1)}$ and $x_{\pi(k)}$ are $x_{\pi(2)}$ and $x_{\pi(k+1)}$, respectively.

- Case 3: Suppose that $x_{\pi(k+1)}$ is connected with $x_{\pi(k)}$, but not connected with $x_{\pi(0)}$. Let

$$A_1(z) = A_0(z) + z^{2^{\pi(k+1)}} C_0(z), \quad C_1(z) = A_0(z) - z^{2^{\pi(k+1)}} C_0(z),$$

$$B_1(z) = B_0(z) + z^{2^{\pi(k+1)}} D_0(z), \quad D_1(z) = B_0(z) - z^{2^{\pi(k+1)}} D_0(z).$$

Then we have

$$|A_1(z)|^2 + |B_1(z)|^2 + |C_1(z)|^2 + |D_1(z)|^2 = 2(|A_0(z)|^2 + |B_0(z)|^2 + |C_0(z)|^2 + |D_0(z)|^2),$$

which implies that $(A_1(z), B_1(z), C_1(z), D_1(z))$ is a complementary set of size 4, each sequence has $2^3$ nonzero elements. One can see that $A_1(z)$ is the associated polynomial of the sequence $\frac{H}{\pi} Q|_{x' = 0}$, where $x' = x \setminus x_{\pi(k+1)}$, and $Q = x_{\pi(0)}x_{\pi(k)} + x_{\pi(k)}x_{\pi(k+1)}.$

- Case 4: Suppose that $x_{\pi(1)}$ is connected with $x_{\pi(0)}$, but not connected with $x_{\pi(k)}$. This case can be proved in a similar way as in Case 3.

Suppose now that $(A_1, B_1, C_1, D_1) = (\frac{H}{\pi}Q|_{x'' = 0}, \frac{H}{\pi} (Q + x_{\pi(k+1)})|_{x'' = 0}, \frac{H}{\pi} (Q + x_{\pi(k+1)})|_{x'' = 0}, e)$ is a complementary set of size 4, each sequence has $2^{i+2}$ nonzero elements, where $e \in \mathbb{Z} = \mathbb{Z}$. Note that the two end vertexes are $x_{\pi(k_1)}$ and $x_{\pi(k_2)}$, where $x''$ is a list of $m - (i + 2)$ variables. The vertexes next to $x_{\pi(k_1)}$ and $x_{\pi(k_2)}$ are $x_{\pi(k_1+1)}$ and $x_{\pi(k_2+1)}$, respectively. Similarly, there are four cases. We only show the proof for the first case, one can prove the other three cases similarly.

Assume that $x_{\pi(k_2+1)}$ is connected with both $x_{\pi(k_1)}$ and $x_{\pi(k_2)}$. Let

$$A_{i+1}(z) = A_i(z) + z^{2^{\pi(k_2+1)}} D_i(z), \quad D_{i+1}(z) = A_i(z) - z^{2^{\pi(k_2+1)}} D_i(z),$$

$$B_{i+1}(z) = B_i(z) + z^{2^{\pi(k_2+1)}} C_i(z), \quad C_{i+1}(z) = B_i(z) - z^{2^{\pi(k_2+1)}} C_i(z).$$

Then we have

$$|A_{i+1}(z)|^2 + |B_{i+1}(z)|^2 + |C_{i+1}(z)|^2 + |D_{i+1}(z)|^2 = 2(|A_i(z)|^2 + |B_i(z)|^2 + |C_i(z)|^2 + |D_i(z)|^2),$$

which implies that $(A_{i+1}(z), B_{i+1}(z), C_{i+1}(z), D_{i+1}(z))$ is a complementary set of size 4. Note that $A_i(z)$ and $z^{2^{\pi(k_2+1)}} D_i(z)$ have no common support. One can see that $A_{i+1}(z)$ is the associated polynomial of the sequence $\frac{H}{\pi} Q|_{x'' = 0} + e x_{\pi(k_2+1)}$, where $x'' = x'' \setminus x_{\pi(k_2+1)}$, and

$$Q' = Q + (x_{\pi(k_1)} + x_{\pi(k_2)})x_{\pi(k_2+1)}.$$
respectively. The vertexes next to \(x_{\pi(k_1)}\) and \(x_{\pi(k_2+1)}\) are \(x_{\pi(k_1+1)}\) and \(x_{\pi(k_2+2)}\), respectively.

Using the iteration \(m-2\) times, we get \((A_{m-2}, B_{m-2}, C_{m-2}, D_{m-2}) = (\frac{H}{2}Q + L, \frac{H}{2}(Q + x_{\pi(k-1)}) + L, \frac{H}{2}(Q + x_{\pi(m-1)}) + L)\) is a complementary set of size 4, where \(L\) is a linear function of \(m\) variables. Each sequence has \(2^m\) nonzero elements, which means that they are sequences over \(\mathbb{Z}_H\), where \(Q\) is defined in Theorem 3.1. This completes the proof. 

4. Generalizations of the construction

Corollary 2. Let \(A, B, Q, \pi\) be defined as in Theorem 3.1. The coefficients \(a_i\) and \(e_i\) of quadratic terms in \(Q\pi\) need not to be constrained to \(\{0, H/2\}\), but can satisfy \(a_i, e_i \in \mathbb{Z}_H\) and \(e_i \in \mathbb{Z}_H\setminus\{0\}\).

Proof. From Theorem 3.1, we know that

\[
\{Q + \sum_{i=0}^{m-1} a_i x_i + a + \frac{H}{2}(d_0 x_{\pi(k-1)} + d_1 x_{\pi(m-1)}) : d_0, d_1 \in \{0, 1\}\}
\]

is a complementary set of size 4, where \(x_{\pi(k-1)}\) and \(x_{\pi(m-1)}\) are end variables of the two paths in \(Q\), respectively. Let \(Q' = Q + x_{\pi(k-1)}x_{\pi(m-1)}\). We need to prove that

\[
\{Q' + \sum_{i=0}^{m-1} a_i x_i + a + \frac{H}{2}(d_0 x_{\pi(k-1)} + d_1 x_{\pi(m-1)}) : d_0, d_1 \in \{0, 1\}\}
\]

is a complementary set of size 4. Let \(Q(z), Q'(z)\) be the associated polynomials of \(Q, Q'\), respectively. Divide \(Q(z)\) into four parts: \(Q(z) = Q_0(z) + Q_1(z) + Q_2(z) + Q_3(z)\), where \(Q_0(z), Q_1(z), Q_2(z), Q_3(z)\) are the associated polynomials of \(Q|(x_{\pi(k-1)}x_{\pi(m-1)})=(00); Q|(x_{\pi(k-1)}x_{\pi(m-1)})=(10); Q|(x_{\pi(k-1)}x_{\pi(m-1)})=(01);\) and \(Q|(x_{\pi(k-1)}x_{\pi(m-1)})=(11);\) respectively. Then

\[
\begin{align*}
Q'(z) &= Q_0(z) + Q_1(z) + Q_2(z) + Q_3(z) \\
(Q' + \frac{H}{2}x_{k-1})(z) &= Q_0(z) + Q_1(z) + Q_2(z) + Q_3(z) \\
(Q' + \frac{H}{2}x_{m-1})(z) &= Q_0(z) + Q_1(z) + Q_2(z) + Q_3(z) \\
(Q' + \frac{H}{2}(x_{k-1} + x_{m-1}))(z) &= Q_0(z) + Q_1(z) + Q_2(z) + Q_3(z)
\end{align*}
\]

From (5),

\[
\begin{align*}
|Q'(z)|^2 + |(Q' + \frac{H}{2}x_{k-1})(z)|^2 + |(Q' + \frac{H}{2}x_{m-1})(z)|^2 \\
&\quad + |(Q' + \frac{H}{2}(x_{k-1} + x_{m-1}))(z)|^2 \\
&= 4(|Q_0(z)|^2 + |Q_1(z)|^2 + |Q_2(z)|^2 + |Q_3(z)|^2) \\
&= |Q(z)|^2 + |(Q + \frac{H}{2}x_{k-1})(z)|^2 + |(Q + \frac{H}{2}x_{m-1})(z)|^2 \\
&\quad + |(Q + \frac{H}{2}(x_{k-1} + x_{m-1}))(z)|^2 \\
&= 4 \cdot 2^m.
\end{align*}
\]

This completes the proof. 

\[
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\]
Remark 1. Let $\hat{S}_m$ be the set of all the sequences of length $2^m$ constructed by Corollary 2. Define the Lee weight of a length $n$ sequence $a = (a_0, a_1, \cdots, a_{n-1})$ to be $wt_L(a) = \sum_{i=0}^{n-1} \min\{a_i, H - a_i\}$. The Lee distance between $a$ and $b$ is defined by $d_L(a, b) = wt_L(a - b \mod H)$. Define the minimum Lee distance of a code $C$ over $\mathbb{Z}_H$ to be $d_L(C) = \min\{d_L(a, b) : a, b \in C, a \neq b\}$.

Then from [11, Th. 4], we have $d_L(\hat{S}_m) = 2^{m-2}$ for $H \geq 2$, $d_L(S_m) = 2^{m-1}$ for $H \geq 4$, where $S_m$ is the set of all $m$-variable functions constructed by Theorem 3.1.

Inspired by [11, Th. 12] and [13, Th. 5], we have

Theorem 4.1. Suppose $m > k$. Let $0 \leq j_0 < j_1 < \cdots < j_{k-1} < m$ be a list of $k$ indices, and write $x = (x_{j_0}, x_{j_1}, \cdots, x_{j_{k-1}})$. Let $f$ be a Boolean function ($f$ is not necessary quadratic) of $m$ variables such that for each $d \in \{0, 1\}^k$, the restricted function $f|_{x=d}$ is quadratic and the graph associated with $f|_{x=d}$ is of the form $Q$ in $m-k$ vertices given in Corollary 2. Then

$$\{f + \frac{H}{2} \left( \sum_{i=0}^{k-1} c_i x_{j_i} + c_1 e_1 + c_2 e_2 \right) : c_i, c_1, c_2 \in \{0, 1\} \}$$

is a Golay complementary set of size $2^{k+2}$, where

$$e_1 = \sum_{d \in \{0, 1\}^k} x_{a_d} \prod_{i=0}^{k-1} x_{j_i}^{d_i} (1 - x_{j_i})^{1-d_i},$$

$$e_2 = \sum_{d \in \{0, 1\}^k} x_{b_d} \prod_{i=0}^{k-1} x_{j_i}^{d_i} (1 - x_{j_i})^{1-d_i},$$

and $a_d, b_d$ are the end vertexes of the two paths in $Q$.

Example 5. Let $m = 11$, $k = 5$ and $H = 4$. Define the quadratic form $Q'$ of a sequence $f$ as

$$Q'(x_0, x_1, \cdots, x_9, x_{10}) = Q + x_{10}(x_0 + x_1 + x_2 + x_6 + x_8 + x_9).$$
New complementary sets of length $2^m$ and size 4

Figure 3. Extra symmetries induced by different partitions that lead to the same quadratic function: $x_0(x_1 + x_2 + x_3) + x_2x_3$

Since deleting the vertex 10 will obtain the quadratic form $Q$, as shown in Fig. 2.

According to Theorem 4.1,

\[ \{ f + \frac{H}{2}(d_0x_4 + d_1x_9 + d_2x_{10}) : d_0, d_1, d_2 \in \{0, 1\} \} \]

is a Golay complementary set of size 8.

5. Enumerations

It remains open to exactly enumerate and uniquely generate all the sequences constructed by Corollary 2, due to extra symmetries, induced by permutation and partition of the variables into two (or more) paths. For example, partitions $(0, 1, 2), (3)$, and $(1, 0), (3, 2)$ both lead to $x_0(x_1 + x_2 + x_3) + x_2x_3$ in Fig. 3.

Theorem 5.1. Let $m \geq 4$. A lower bound on the number of cosets with PMEPR \( \leq 4 \) given in Corollary 2 is $N$, where

\[
N = \frac{m!}{2} + N_1 + \left( \frac{m}{2} \right) - 1 \frac{m!}{4}(H - 1)^{m-3}(1 + 2(H - 2)) + \frac{(m)!}{2} + \frac{m!}{2}(m - 2) + \frac{m!}{2}(m - 1) - 1 + \frac{(m - 1)!}{2} + \frac{m!}{2}(m - 3) - 1 - \frac{m!}{2} \left( \frac{m}{2} \right) + \frac{m!}{2} \left( \frac{m}{3} \right) - 1 - m + \frac{m!}{2} [H^{m-1} - (Hm - H - 2m + 4)2^{m-2}],
\]

where

\[
N_1 = \frac{m!}{2}(H - 2)^2 \left( \frac{m}{2} \right) (H - 1)^{m-3} + \left( \frac{m}{2} \right) - 1 \right) H^{m-3}.
\]

When $m = 3$, the number of cosets with PMEPR \( \leq 4 \) given in Corollary 2 is $H^3 - (H - 1)^3$.

An upper bound on the number of cosets with PMEPR \( \leq 4 \) given in Corollary 2 is

\[
B = 2^{m-3} \frac{m!}{2} H^{m-1}.
\]
Table 1. Lower and upper bounds on the number of cosets given in Theorem 5.1

| $H$ | $m$ | Lower bound $N$ | Low bound $L_1 + L_2$ in [11] | Exact number $N$ in Table 1 in [10] | Upper bound $I$ of Construction 14 in [13] |
|-----|-----|-----------------|-------------------------------|---------------------------------|----------------------------------------|
| 2   | 3   | 7               | 6                             | 7                               | 4                                      |
| 2   | 4   | 67              | 30                            | 36                              | 192                                    |
| 2   | 5   | 762             | 210                           | 3840                           | 192                                    |
| 2   | 6   | 10320           | 3240                          | 9240                           | 1920                                   |
| 2   | 7   | 152400          | 63000                         | 2580480                        | 23040                                  |
| 2   | 8   | 2522520         | 1360800                       | 82575360                       | 322560                                 |
| 4   | 3   | 37              | 18                            | 37                             | 16                                     |
| 4   | 4   | 727             | 414                           | 1536                           | 576                                    |
| 4   | 5   | 20802           | 10710                         | 61440                          | 36864                                  |
| 4   | 6   | 736800          | 300600                        | 2949120                        | 3686400                                |
| 4   | 7   | 26138640        | 9200520                       | 165150720                      | 530841600                              |
| 4   | 8   | 1.1453 ⋅ 10^9   | 309728160                     | 1.0570 ⋅ 10^11                 | 1.0404 ⋅ 10^11                         |

The proof of Theorem 5.1 is given in the Appendix.

Table 1 shows lower and upper bounds of cosets given in Theorem 5.1 for $H = 2, 4$, and $3 \leq m \leq 8$. In Table 1,

$$L_1 = \frac{m!}{2} [2^{m-1} - \binom{m-1}{3} - \binom{m-1}{2} - m + 1]$$

if $H = 2$, and

$$L_1 = \frac{m!}{2} [H^{m-1} - (Hm - H - 2m + 4)2^{m-2} + 1]$$

if $H > 2$, is the low bound of cosets given in [11] and

$$L_2 = \sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{i! (m-i)!} \frac{1}{2} = \frac{m!}{4} (\lfloor \frac{m}{2} \rfloor + 1)$$

is the number of cosets with PMEPR $\leq 4$ obtained by two standard Golay pairs. For $H = 4$, the number $I$ ($I = \frac{(m-1)!}{2^{m-1}}$ if $H = 2$, and $I = \frac{(m-1)!}{4} H^{m-1}$ if $H = 2^h, h \geq 2$) of cosets with PMEPR $\leq 4$ in [13, Construction 14] is larger than the number of cosets in Corollary 2. This is because that the sequences in [13, Construction 14] have algebraic degree 3 and effective degree 2 (see [13]), but our sequences have both algebraic degree and effective degree 2. It should be noted that lots of sequences in Corollary 2 are not in the sequences family given in [13], moreover, Corollary 2 can be applied to any even $H$, while in [13], $H$ should be a power of 2, i.e., $H = 2^h$, where $h \geq 1$.

If we set $H = 2^h$, then by using Theorem 4.1 and the method given in [13], we can obtain a large family of sequences with PMEPR $\leq 8$ and Lee weight $2^{m-2}$. First we introduce some definitions in [13]. Let $f : \{0, 1\}^m \rightarrow \mathbb{Z}_{2h}$ be a generalized Boolean function. Define the effective degree of $f$ to be

$$\max_{0 \leq i < h} \deg (f \text{ mod } 2^{i+1} - i).$$
Define $\mathcal{F}(r, m, h)$ to be the set of all generalized Boolean functions of $m$ variables and effective degree at most $r$. It is shown in [13] that
\[
\log_2|\mathcal{F}(r, m, h)| = \sum_{i=0}^{r} \binom{m}{i} + \sum_{i=1}^{h-1} (m - i) \binom{m}{r+i}.
\]
Thus, we have $\log_2|\mathcal{F}(0,1, h)| = 2h - 1$ and $\log_2|\mathcal{F}(1,1, h)| = 2h$. For $0 \leq k < m$, $0 \leq r \leq k + 1$, and $h \geq 1$, define the code $\mathcal{A}(k, r, m, h)$ to be the set of words corresponding to a word in $A$ as defined in Corollary 2. Thus according to Theorem 4.1, the PMEPR of the coset $D$ is at most $2^{1+2} = 8$. Now we are ready to give the following construction.

**Construction 1.** Let $k = 1$. Let $2 \leq r \leq k + 2 = 3$ for $h = 1$, and $1 \leq r \leq k + 1 = 2$ for $h > 1$. Let $r' = \min\{r, 2\}$. Now take the union of $N_{2^{\min\{1, r, h-3\}}}$ distinct cosets of $\mathcal{A}(1, r', m, h)$, each containing a word in $R(m, h)$ with effective degree at most $r$. The PMEPR in this code is at most $8$, and from [13, Th. 9], one can show that its minimum Lee weight is at least $2^{m-r}$. The number of words in this code is $N_{2^2}^{2^{\min\{1, r, h-3\}}} 2^s$.

**Remark 2.** Note that we restrict $r' = \min\{r, 2\}$ in Construction 1. This is because that $r' - 1 \leq k = 1$, and to make sure that any word in $\mathcal{A}(1, r', m, h)$ has effective algebraic degree at most $r$ (as a consequence, it has minimum Lee weight at least $2^{m-r}$), thus we have $r' \leq r$.

In Table 2, we list the lower bound and upper bound of the number of the cosets in Construction 1. It can be seen that for $H = 2$, $r = 2, 3$ or $H = 4$, $r = 1$, the lower bound of our construction is larger than that of [13]. For $H = 4$, $r = 2$, our upper bound is much larger than the number of cosets in [13], and our lower bound is a bit small than the number of cosets in [13]. We hope to tighten the bounds in the future work.
Table 2. Lower and upper bounds on the number of cosets with PMEPR ≤ 8 given by Construction 1

| H  | r | m | Lower bound | Coset number I in [13, Construction 14] | Upper bound |
|----|---|---|-------------|----------------------------------------|-------------|
| 2  | 2 | 5 | 880         | 384                                    | 3072        |
| 2  | 2 | 6 | 20544       | 6144                                   | 122880      |
| 2  | 2 | 7 | 568320      | 122880                                 | 5898240     |
| 2  | 2 | 8 | 17571840    | 2949120                                | 330301440   |
| 2  | 3 | 5 | 48400       | 9216                                   | 589824      |
| 2  | 3 | 6 | 13189248    | 1179648                                | 471859200   |
| 2  | 3 | 7 | 5.0467 · 10^9 | 235929600                              | 5.4358 · 10^{14} |
| 2  | 3 | 8 | 2.4123 · 10^{12} | 6.7948 · 10^{19} | 8.5234 · 10^{14} |
| 4  | 1 | 5 | 880         | 384                                    | 3072        |
| 4  | 1 | 6 | 20544       | 6144                                   | 122880      |
| 4  | 1 | 7 | 568320      | 122880                                 | 5898240     |
| 4  | 1 | 8 | 17571840    | 2949120                                | 330301440   |
| 4  | 2 | 5 | 774400      | 1179648                                | 9437184     |
| 4  | 2 | 6 | 422055936   | 603979776                              | 1.5099 · 10^{11} |
| 4  | 2 | 7 | 3.2299 · 10^{11} | 4.8318 · 10^{14} | 3.4789 · 10^{11} |
| 4  | 2 | 8 | 3.0877 · 10^{14} | 5.5663 · 10^{14} | 1.0910 · 10^{14} |

6. Relationship to Previous Constructions

Theorem 3.1 is a special case of construction (17) in [8] (see also section 5 of [10]), and, when \( H = 2 \), is a generalization of construction (18) in [8] for \( t = 2 \), which generates a vector of multivariate polynomials using the recursion:

\[
\begin{pmatrix}
F_{j,0}(z_j) \\
F_{j,1}(z_j) \\
\vdots \\
F_{j,2^t-1}(z_j)
\end{pmatrix} = \mathcal{H}^\otimes t P_{\gamma,j} \bigotimes_{k=0}^{t-1} \begin{pmatrix} 1 & 0 \\ 0 & z_{t+j+k} \end{pmatrix} \bigotimes_{k=0}^{t-1} \begin{pmatrix} 1 & 0 \\ 0 & z_{t+j+k} \end{pmatrix}
\begin{pmatrix}
F_{j-1,0}(z_j) \\
F_{j-1,1}(z_j) \\
F_{j-1,2^t-1}(z_j)
\end{pmatrix},
\]

where \( P_{\theta,j} \) and \( P_{\gamma,j} \) are \( 2^t \times 2^t \) permutation matrices, \( O_j \) is a \( 2^t \times 2^t \) diagonal matrix with diagonal entries in \( \{1, -1\} \), \( \bigotimes_{k=0}^{t-1} \begin{pmatrix} 1 & 0 \\ 0 & z_{t+j+k} \end{pmatrix} \) is a \( 2^t \times 2^t \) matrix in \( t \) variables, \( z_{t+j}, z_{t+j+1}, \ldots, z_{t+j+t-1} \), and where \( \{F_{j,r}(z_j), 0 \leq r \leq 2^t\} \) is a complementary set of size \( 2^t \) comprising \( 2^t \) multivariate polynomials in \( (j + 1)t \) variables, representing \( (j + 1)t \)-dimensional, \( 2 \times 2 \times \ldots \times 2 \) bipolar arrays. \( \mathcal{H}^\otimes t \) is the \( 2^t \times 2^t \) Walsh-Hadamard transform, where \( \mathcal{H} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \).

(6) can be generalised as follows:

\[
\begin{pmatrix}
F_{j,0}(z_j) \\
F_{j,1}(z_j) \\
\vdots \\
F_{j,2^t-1}(z_j)
\end{pmatrix} = \mathcal{H}_a P_{\gamma,j} Q_{\theta,j} O_j P_{\theta,j} \bigotimes_{k=0}^{t-1} \begin{pmatrix} 1 & 0 \\ 0 & z_{t+j+k} \end{pmatrix}
\begin{pmatrix}
F_{j-1,0}(z_j) \\
F_{j-1,1}(z_j) \\
F_{j-1,2^t-1}(z_j)
\end{pmatrix},
\]

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where \( a_j = (a_{0,j}, a_{1,j}, \ldots, a_{t-1,j}) \in \mathbb{Z}_2^t, \)

\[
Q_j.a_j = \bigotimes_{k=0}^{t-1} \begin{pmatrix} 1 & 0 & a_{k,j} \\ 0 & z_{tj+k} \end{pmatrix},
\]

and

\[
\mathcal{H}_{a_j} = \bigotimes_{k=0}^{t-1} \mathcal{H}^{a_{k,j}},
\]

where \( z_{tj+k}^0 = 1, z_{tj+k}^1 = z_{tj+k} \) and, similarly, \( \mathcal{H}^0 = I, \mathcal{H}^1 = \mathcal{H}. \)

In (7), let \( t = 2, F_{-1,i}(x_{-1}) = 1, \) where \( 0 \leq i \leq 3. \) Let \( a_0 = (a_{0,0}, a_{1,0}) = (1,1). \) Then

\[
Q_{0,a_0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_0 & 0 & 0 \\ 0 & 0 & z_1 & 0 \\ 0 & 0 & 0 & z_0z_1 \end{pmatrix}, \quad \text{and} \quad \mathcal{H}_{a_0} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.
\]

Let \( f_{j,r} \) be the associated \( \sum_{e=0}^{t-1} wt(a_e) \)-variable Boolean function of the sequence \( F_{j,r}, \) for any \( j \) and \( r \leq 2^t, \) where \( F_{j,r} \) is the coefficient sequence obtained, by projection, from the polynomial \( F_{j,r}(z_j), \) by equating \( z_{i+1} = \pi_i^2, \forall i. \) Let

\[
F_{0,0}(z_0) = (-1)^{f_{0,0}}, \quad F_{0,1}(z_0) = (-1)^{f_{0,1}},
\]

\[
F_{0,2}(z_0) = (-1)^{f_{0,2}}, \quad F_{0,3}(z_0) = (-1)^{f_{0,3}}.
\]

If \( O_0 = I, \) where \( I \) is the identity matrix then, by (7),

\[
f_{0,\pi(0)} = 0 + c_0, \quad f_{0,\pi(1)} = x_0 + c_1, \quad f_{0,\pi(2)} = x_1 + c_2, \quad f_{0,\pi(3)} = x_0 + x_1 + c_3,
\]

where \( \pi \) is a permutation of \( \{0,1,2,3\}, c_i \in \mathbb{Z}_2, \) and \( \pi, c_i \) are determined by \( P_{\gamma,0}. \) Similarly, for any other \( O_0 \) which have two or four \(-1\) elements, we have,

\[
(8) \quad \{0 + c_0, x_0 + c_1, x_1 + c_2, x_0 + x_1 + c_3\}, \quad \forall c_i \in \mathbb{Z}_2,
\]

is a complementary set of size 4. If \( O_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \) then, by (7),

\[
\begin{align*}
& f_{0,\pi(0)} = x_0x_1 + c_0, \\
& f_{0,\pi(1)} = x_0x_1 + x_0 + c_1, \\
& f_{0,\pi(2)} = x_0x_1 + x_1 + c_2, \\
& f_{0,\pi(3)} = x_0x_1 + x_0 + x_1 + c_3,
\end{align*}
\]

and \( \pi, c_i \) are determined by \( P_{\gamma,0}. \) Similarly, for any other \( O_0 \) which have one or three \(-1\) elements, we have,

\[
(9) \quad \{x_0x_1 + c_0, x_0x_1 + x_0 + c_1, x_0x_1 + x_1 + c_2, x_0x_1 + x_0 + x_1 + c_3\}, \quad \forall c_i \in \mathbb{Z}_2,
\]

is a complementary set of size 4. (8) and (9) are the complementary sets of size 4 of Theorem 3.1 for \( m = 2. \)
Let $a_j = (a_{0,j}, a_{1,j}) = (1, 0)$ for $j \geq 1$. Then we have

$$Q_{j,a_j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_{tj} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z_{tj} \end{pmatrix},$$

and $H_{a_j} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

Let $P_j, j = I, \forall j \geq 1$. Let $O_j = \begin{pmatrix} c_0 & 0 & 0 & 0 \\ 0 & c_1 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 \end{pmatrix}, \forall j \geq 1, c_i \in \mathbb{Z}_2$. Suppose that

$$(f_{j,0}(x), f_{j,1}(x), f_{j,2}(x), f_{j,3}(x)) = (P, P + \frac{H}{2}x_a, P + \frac{H}{2}x_b, P + \frac{H}{2}(x_a + x_b))$$

is a complementary set of size 4, with length $2^{j+2}$, where $P \in S_{j+2}$, and $x_a$ and $x_b$ are the end vertices of the two paths. If $P_{0,j} = I$ then, by (7),

$$f_{j+1,0}(x, x_{j+2}) = (f_{j,0} + c_0)(x)(x_{j+2} + 1) + (f_{j,1} + c_1)(x)(x_{j+2}) = \frac{H}{2}x_{j+2}(x_a + c_0 + c_1) + P + c_0,$$

$$f_{j+1,1}(x, x_{j+2}) = f_{j+1,0}(x, x_{j+2}) + \frac{H}{2}x_{j+2},$$

$$f_{j+1,2}(x, x_{j+2}) = (f_{j,2} + c_2)(x)(x_{j+2} + 1) + (f_{j,3} + c_3)(x)(x_{j+2}) = \frac{H}{2}x_{j+2}(x_a + c_2 + c_3) + P + \frac{H}{2}x_b + c_2,$$

$$f_{j+1,3}(x, x_{j+2}) = f_{j+1,2}(x, x_{j+2}) + \frac{H}{2}x_{j+2}.$$

Note that $P' = P + \frac{H}{2}x_{j+2}x_a$ is the quadratic function of $j + 3$ variables in case 4 of the proof of Theorem 3.1. If $P_{0,j} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ then, by (7),

$$f_{j+1,0}(x, x_{j+2}) = (f_{j,0} + c_0)(x)(x_{j+2} + 1) + (f_{j,3} + c_3)(x)(x_{j+2}) = \frac{H}{2}x_{j+2}(x_a + x_b + c_0 + c_3) + P + c_0,$$

$$f_{j+1,1}(x, x_{j+2}) = f_{j+1,0}(x, x_{j+2}) + \frac{H}{2}x_{j+2},$$

$$f_{j+1,2}(x, x_{j+2}) = (f_{j,2} + c_2)(x)(x_{j+2} + 1) + (f_{j,1} + c_1)(x)(x_{j+2}) = \frac{H}{2}x_{j+2}(x_a + x_b + c_1 + c_2) + P + \frac{H}{2}x_b + c_2,$$

$$f_{j+1,3}(x, x_{j+2}) = f_{j+1,2}(x, x_{j+2}) + \frac{H}{2}x_{j+2}.$$

Note that $P' = P + \frac{H}{2}x_{j+2}(x_a + x_b)$ is the quadratic function of $j + 3$ variables in case 2 of the proof of Theorem 3.1.

Finally, by applying $j + 2$ successive projection mappings to the $j+3$ dimensional arrays $f_{j+1,r}(x, x_{j+2}), r \in \{0, 1, 2, 3\}$, one obtains sequences

$$\{Q + \sum_{i=0}^{m} a_i x_i + a + \frac{H}{2}(d_0 x_a + d_1 x_b) : d_0, d_1 \in \{0, 1\}\}.$$
where $Q$ is the quadratic function of $j + 3$-variables defined in Theorem 3.1, and $x_a$ and $x_b$ are the two end vertices of the two paths, respectively.

At the end of this section, we point out that some sequences constructed in Theorem 3.1 and Corollary 2 cannot be obtained by Theorem 1 in [2]. For example, the sequence represented by Fig. 1 in Example 3 cannot be obtained by [2, Theorem 1].

7. Conclusion

We have given an explicit construction for complementary 4-sets, being a special case of a more general, but less explicit, construction by Parker and Riera [8]. Our construction generalizes, for $H = 2$, construction (18) in [8]. Weak lower and upper bounds on the number of sequences generated by our construction are given, and we hope to tighten these bounds in the future work. Some generalizations of the construction are also given, which are not in the construction given in [2, 8].

APPENDIX

Proof of Theorem 5.1. Let $m$ be a positive integer and $\pi$ be a permutation of $\{0, 1, \cdots , m - 1\}$. Let $A, B, Q_i, Q$ be defined as in Theorem 3.1. Let $a_{ij}$ be the coefficient of $x_{\pi(i)}x_{\pi(j)}$. For $0 \leq i \leq k - 1$, let

$$m_i = |\{a_{ij} \neq 0 | k \leq j \leq m - 1\}|,$$

where $|A|$ is the cardinality of the set $A$.

Note that the maximum number of quadratic terms in $Q$ is $m - 2 + m - 1 = 2m - 3$, where there are $m - 2$ terms (i.e. the two paths) must have coefficient $\frac{H}{2}$, and the other $m - 1$ intermediate terms can have any coefficient in $\mathbb{Z}_H$.

Proof of the lower bound: We obtain the low bound by restraining the coefficients of the $m - 1$ intermediate terms.

- **Case 1a:** In this case, let the coefficients of $x_{\pi(0)}x_{\pi(k)}$ and $x_{\pi(k-1)}x_{\pi(m-1)}$ be restrained in $\mathbb{Z}_H \setminus \{0, \frac{H}{2}\}$, and the other $m - 3$ coefficients of intermediate terms be restrained in $\mathbb{Z}_H \setminus \{0\}$. We have $m = \sum_{i=1}^{k-1} m_i$, where $m_i \geq 1$. We call $(m_0, m_1, m_2, \cdots, m_{k-1})$ a $k$ decomposition of $m$. Let $G_m^k$ be the number of different ordered $k$ decompositions of $m$. For example, let $m = 4$ and $k = 2$. Then $(1, 3), (2, 2), (3, 1)$ are all the three different ordered decompositions of 4, so $G_4^2 = 3$. It is easy to see that $G_m^1 = 1$, $G_m^2 = m - 1 = \binom{m-1}{1}$, and

$$G_m^3 = G_m^2 + G_m^2 + G_m^2 + \cdots + G_m^2 = \binom{m - 2}{1} + \binom{m - 3}{1} + \cdots + \binom{2}{2} = \binom{m - 1}{2}.$$

Similarly, we have $G_m^k = \binom{m-1}{k-1}$. Suppose that the lengths of the two paths $A, B$ are $k$ and $m - k$, respectively, where $k \leq \lfloor m/2 \rfloor$. For each $2 \leq k \leq \lfloor m/2 \rfloor$, the number of different cosets in case 1a is

$$\binom{m}{k} \frac{(m - k)!}{2} k! (H - 2)^2 (H - 1)^{m - 3} G_{m-k}^k.$$

---

1Our construction can also be seen as a special case and clarification of section 5 of [10].
For \( m \geq 4 \), summing up all the \( k \)'s satisfy \( 2 \leq k \leq \lfloor m/2 \rfloor \), one can get the number \( N_{1a} \) of different cosets in case 1a is

\[
N_{1a} = \begin{cases} 
\frac{m!}{2} (H - 2)^2 (H - 1)^{m-3} \sum_{k=2}^{(m-1)/2} \binom{m-2}{k-1}, & \text{m odd}, \\
\frac{m!}{2} (H - 2)^2 (H - 1)^{m-3} \left( \sum_{k=2}^{m/2-1} \binom{m-2}{k-1} + \frac{1}{2} \binom{m-2}{m/2-1} \right), & \text{m even}, 
\end{cases}
\]

(11) \( = \frac{2^{m-3} - 1}{2} \frac{m!}{2} (H - 2)^2 (H - 1)^{m-3} \),

Note that in this case there are \( m - 2 + m - 1 \) coefficients are nonzero, of which, there are at least \( m - 2 \) coefficients are \( \frac{H}{2} \), and at least 2 coefficients of intermediate terms belong to \( \mathbb{Z}_H \setminus \{0, \frac{H}{2}\} \).

**Case 1b:** In this case, let the coefficients of \( x_{\pi(0)}x_{\pi(k)} \) and \( x_{\pi(k-1)}x_{\pi(m-1)} \) be restrained in \( \mathbb{Z}_H \setminus \{0, \frac{H}{2}\} \). Note that we need to exclude the case 1a, in which there are \( m - 2 + m - 1 \) nonzero coefficients. Then for \( m \geq 4 \), the number of different cosets in case 1b is

\[
N_{1b} = \frac{1}{2} (H - 2)^2 (H^{m-3} - (H - 1)^{m-3}) \sum_{k=2}^{m/2} \binom{m}{k} k!(m - k)!
\]

(12) \( = \left( \lfloor \frac{m}{2} \rfloor - 1 \right) \frac{m!}{2} (H - 2)^2 (H^{m-3} - (H - 1)^{m-3}) \).

(12) is obtained by first changing \( (H - 1)^{m-3}G_{m-1} \) to \( H^{m-3} - (H - 1)^{m-3} \) in (10), and then summing up all the \( k \)'s satisfy \( 2 \leq k \leq \lfloor m/2 \rfloor \).

Note that in Case 1b, there are at most \( m - 2 + m - 2 \) coefficients in \( Q \) are nonzero, of which, there are at least \( m - 2 \) coefficients of quadratic terms are \( \frac{H}{2} \), and at least 2 coefficients of intermediate terms belong to \( \mathbb{Z}_H \setminus \{0, \frac{H}{2}\} \).

**Case 2:** In this case, let the coefficients of \( x_{\pi(0)}x_{\pi(k)} \) or \( x_{\pi(k-1)}x_{\pi(m-1)} \) be 0, and the other coefficients of intermediate terms be restrained in \( \mathbb{Z}_H \setminus \{\frac{H}{2}\} \). Then one can see that for \( m \geq 4 \), the number of cosets in case 2 is

\[
N_2 = \sum_{k=2}^{\lfloor m/2 \rfloor} \binom{m}{k} k! \left( \frac{m-k}{2} \right)! (H - 1)^{m-3}(1 + 2(H - 2)) + \binom{m}{1} \frac{(m-1)!}{2}
\]

(13) \( = \left( \lfloor \frac{m}{2} \rfloor - 1 \right) \frac{m!}{4} (H - 1)^{m-3}(1 + 2(H - 2)) + \frac{m!}{2} \).

Note that in this case there are \( m - 2 \) coefficients of quadratic terms in \( Q \) are \( \frac{H}{2} \).

**Case 3:** In this case, let \( a_0 = \frac{H}{2} \), for some \( j \) satisfies \( k + 1 \leq j \leq m - 2 \), and the other \( m - 2 \) coefficients of intermediate terms be 0. Note that in this case there are \( m - 2 + 1 \) coefficients are \( \frac{H}{2} \), and the other coefficients of quadratic terms in \( Q \) are 0. For \( m \geq 4 \), the number of different cosets in case 3 is

\[
N_3 = \frac{m!}{3!} \cdot S_3 + \frac{m!}{2!} \cdot S_2 + m! \cdot S_0 = \frac{m!}{6} \binom{m-2}{2},
\]

where \( S_0, S_2, S_3 \) are defined as follows. Let

\[
A_0 = \{(a, b, c) \mid m - 1 = a + b + c, 1 \leq a < b < c \leq m - 2, a, b, c \text{ are integers} \},
\]
\[ A_2 = \{(a,b,c)|m - 1 = a + b + c, 1 \leq a = b < c \leq m - 2 \text{ or } 1 \leq a < b = c \leq m - 2, a, b, c \text{ are integers}\}, \]

and

\[ A_3 = \{(a,b,c)|m - 1 = a + b + c, 1 \leq a = b = c \leq m - 2, a, b, c \text{ are integers}\}. \]

Let \( S_0, S_2, S_3 \) be the cardinalities of \( A_0, A_2, A_3 \), respectively. One can see that
\[
\frac{m!}{2} \cdot S_3 + \frac{m!}{2} \cdot S_2 + m! \cdot S_0 = \frac{m!}{6} \times G_{m-1} = \frac{m!}{6} (m-2)^2.
\]

- **Case 4:** In this case, let \( k = 1 \), and there are 2 intermediate terms have coefficient \( \frac{H}{4} \), while the other coefficients of intermediate terms are 0. Note that in this case, there are \( m - 2 + 2 \) quadratic terms in \( Q \) have coefficient \( \frac{H}{4} \), and the other coefficients of quadratic terms in \( Q \) are 0. If \( a_{01} = a_{0,m-1} = \frac{H}{2} \), then there are \( \frac{(m-1)!}{6} \) different permutations (since in this case the graph \( G(Q) \) of \( Q \) is a circle), otherwise there are \( \frac{m!}{4} \) different permutations. In the later case, we choose two coefficients of intermediate terms to be \( \frac{H}{4} \), the number of choice is \( \binom{m-1}{2} - 1 \). So that, for \( m \geq 4 \),
\[
N_4 = \frac{m!}{2} \left( \binom{m-1}{2} - 1 \right) + \frac{(m-1)!}{2}.
\]

- **Case 5:** In this case, there are 3 intermediate terms have coefficient \( \frac{H}{4} \), and the other coefficients of intermediate terms are 0. Note that in this case, there are \( m - 2 + 3 \) coefficients of quadratic terms in \( Q \) are \( \frac{H}{4} \), and the other coefficients are 0. For any \( 3 \leq i \leq m - 1 \), we first choose \( i \) vertexes from \( m \) vertexes to form a circle, then add an edge in the circle, and then choose a vertex in the circle to link the remaining \( m - i \) vertexes. This gives
\[
\binom{m}{i} \frac{(i-1)!}{2} \binom{i}{2} (m-i)! (\frac{1}{2}) - i = \frac{m!}{2} (i) - i \text{ different cosets for } i \leq m - 1 \text{, and}
\]
\[
\binom{m}{i} \frac{(i-1)!}{2} \binom{i}{2} (m-i)! (\frac{1}{2}) - i = \frac{m!}{2} (i) - i \text{ different cosets for } i = m \text{. Thus, for } m \geq 4,
\]
\[
N_5 \geq \frac{(m-1)!}{2} \left( \binom{m}{2} - m \right) + \frac{m!}{2} \left( \binom{m-1}{2} - (m-1) + \binom{m-2}{2} - (m-2) + \cdots + \binom{4}{2} - 4 \right)
\]
\[
= \frac{m!}{2} \left( \binom{m}{3} - 1 \right) - \frac{m!}{4} (m+1)(m-3).
\]

- **Case 6:** In this case, let \( k = 1 \), and suppose that there are \( i \) \( (4 \leq i \leq m - 1) \) intermediate terms have coefficient \( \frac{H}{4} \), and the other coefficients of intermediate terms are 0. Note that in this case, there are \( m - 2 + i \) \( (4 \leq i \leq m - 1) \) coefficients of quadratic terms in \( Q \) are \( \frac{H}{2} \), and the other coefficients are 0. From [11, Lemma 17], for \( m \geq 4 \),
\[
N_6 = \frac{m!}{2} \left[ 2^{m-1} - \binom{m-1}{3} - \binom{m-1}{2} - m \right].
\]

- **Case 7:** In this case, let \( k = 1 \). There exist \( 1 \leq i \neq j \leq m - 1 \) such that \( a_{0i}, a_{0j} \in \mathbb{Z}_H \setminus \{0, \frac{H}{2}\} \). From [11, Lemma 18], for \( m \geq 4 \), the number of cosets in case 7 is
\[
N_7 = \frac{m!}{2} \left[H^{m-1} - (Hm - H - 2m + 4)2^{m-2}\right].
\]
Let $N_1 = N_{1a} + N_{1b}$. Note that all the cases do not overlap with each other, so a lower bound of the number of sequences with PMEPR $\leq 4$ in corollary 2 is

$$\frac{m!}{2} + N_1 + N_2 + N_3 + N_4 + N_5 + N_6 + N_7,$$

where $\frac{m!}{2}$ is the number of cosets of standard Golay sequences.

Proof of the case $m = 3$: For $m = 3$, we know that there are 3 quadratic terms, and at least one of them has coefficient $\frac{H}{2}$. Thus the number of cosets is $H^3 - (H - 1)^3$.

Proof of the upper bound: For each $1 \leq k \leq \lfloor m/2 \rfloor$, the number of cosets in Corollary 2 is less than

$$\binom{m}{k} \frac{(m-k)!}{2} k! H^{m-1} G_{m-1}^k.$$

Summing up all the $k$’s satisfy $1 \leq k \leq \lfloor m/2 \rfloor$, one can get the upper bound

$$B = \begin{cases} 
\frac{1}{2} H^{m-1} m! \sum_{k=1}^{(m-1)/2} \binom{m-2}{k-1}, m \text{ odd}, \\
\frac{1}{2} H^{m-1} m! \sum_{k=1}^{m/2-1} \binom{m-2}{k-1} + \frac{1}{2} \binom{m-2}{m/2-1}, m \text{ even}, 
\end{cases}$$

$$= 2^{m-3} \frac{m!}{2} H^{m-1}.$$

This completes the proof. □

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E-mail address: wugf@nipc.org.cn
E-mail address: zhangyq@nipc.org.cn
E-mail address: liuxf@nipc.org.cn