On the boundary components of arbitrary central streams

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Abstract

The foliation on the space of $p$-divisible groups are studied by Oort in 2004. In his theory, special leaves, which are called central stream are important. There is a lot of problems on the boundaries of central streams. In this paper, we classify the boundary components of all central streams. This classification will be a key step to determine the generic Newton polygons of boundary components.

1 Introduction

Oort introduced the notion of minimal $p$-divisible groups in [10, p.1023]. Here, $p$-divisible groups are often called Barsotti-Tate group. Oort showed in [10, 1.2] that the property: Let $X$ be a minimal $p$-divisible groups over an algebraically closed field $k$ of characteristic $p$, and let $Y$ be a $p$-divisible group over $k$. If $X[p] \simeq Y[p]$, then $X \simeq Y$, where, $X[p]$ is the $p$-kernel of $p$-multiplication. See Section 2.1 for the definition of minimal $p$-divisible groups.

Let $k$ be an algebraically closed field of characteristic $p$. Fix a $p$-divisible group $X_0$ over $k$. Let $\text{Def}(X_0) = \text{Spf}(\Gamma)$ be the deformation space of $X_0$. The deformation space is the formal scheme pro-representing the functor $\text{Art}_k \to \text{Sets}$ which sends $R$ to the isomorphism classes of $p$-divisible groups $X$ over $R$ such that $X_k \simeq X_0$, where, $\text{Art}_k$ is the category of local artinian rings with residue field $k$. The given $p$-divisible group over $\text{Def}(X)$ comes from a $p$-divisible group $X$ over $\Delta = \text{Spec}(\Gamma)$. In [5, 2.4.4] de Jong proved that the category of $p$-divisible groups over $\text{Spf}(\Gamma)$ is equivalent to the category the $p$-divisible groups over $\Delta$. Let $X' \to \text{Spf}(\Gamma)$ and we denote by $X$ the $p$-divisible groups over $\Delta$ obtained from $X'$ by the above equivalence. He called this leaf $C_Y(S)$ the leaf associated to $Y$ in $S$; see (1) in section 2.1.

For a $p$-divisible group $Y$ over $k$, Oort introduced a leaf $C_Y(S)$ for a $p$-divisible group $Y$ over $S$ characterized by $s$ belongs to $C_Y(S)$ if and only if $Y_s$ is isomorphic to $Y$ over an algebraically closed field containing $k(s)$ and $k$; see [9, 2.1] for details. We are interested in the case $Y = X$ and $S = \Delta$. In particular, for the case $Y$ is minimal, he called the leaf the central stream. This notion is a “central” tool in the theory of foliations.

Let $X$ and $Y$ be $p$-divisible groups over $k$. We say $X$ is a specialization of $Y$ if there exists a family of $p$-divisible group $X \to \text{Spec}(R)$ with discrete valuation ring $R$ in characteristic of $p$ such that $X$ is isomorphic to $Y$ over an algebraically closed field containing $L$ and $k$, and $X_k$ is isomorphic to $X$ over an algebraically closed field containing $K$ and $k$, where, $L$ is the field of fractions of $R$, and $K = R/m$ is the residue field of $R$. For a $p$-divisible group $X$, we define the length $\ell(X[p])$ of the $p$-kernel by the length of the element of the Weyl group corresponding to $X[p]$. We say a specialization $X$ of $Y$ is generic if $\ell(X[p]) = \ell(Y[p]) - 1$ holds.

Let $\xi$ be a Newton polygon. We have the minimal $p$-divisible group $H(\xi)$; see [2] and [3] for details. For Newton polygons $\xi$ and $\zeta$, we say that $\zeta \prec \xi$ if each point of $\zeta$ is above or on
\(\xi\). In particular, we say \(\zeta \prec \xi\) is saturated if there exists no Newton polygon \(\eta\) satisfying that \(\xi \nless \eta \nless \xi\).

In \([4]\), we determine the Newton polygon of each generic specialization of \(H(\xi)\), where \(\xi = (m_1, n_1) + (m_2, n_2)\) consists of two slopes satisfying that \(n_2/(m_2 + n_2) < 1/2 < n_1/(m_1 + n_1)\).

The main result of this paper is a generalization of this result:

**Theorem 1.1.** Let \(\xi\) be an arbitrary Newton polygon. Let \(X\) be a generic specialization of \(H(\xi)\). Then there exists a Newton polygon \(\zeta\) such that \(\zeta \prec \xi\) is saturated, and \(H(\zeta)\) appears as a specialization of \(X\).

This paper is organized as follows. In section \([2]\) we recall notions of \(p\)-divisible groups, Newton polygons and truncated Dieudonné modules. Moreover, we introduce the definition of arrowed binary sequences which is the main tool to show the main result. In section \([3]\) we show that the problem of classification of boundary components come down to the case Newton polygon consists of two segments. In section \([4]\), we treat central streams corresponding to Newton polygons consisting of two slopes, and give a criterion of boundary component in Theorem \([4.1]\).

## 2 Preliminary

### 2.1 Definitions

In this section we fix a prime number \(p\). Let \(h\) be a non-negative integer. Let \(S\) be a scheme in characteristic \(p\). A \(p\)-divisible group (Barsotti-Tate group) of height \(h\) over \(S\) is an inductive system \((G_v, i_v)_{v \geq 1}\), where \(G_v\) is a finite locally free commutative group scheme over \(S\) of order \(p^v\), and for every \(v\), there exists the exact sequence of commutative group schemes

\[
0 \longrightarrow G_v \xrightarrow{i_v} G_{v+1} \xrightarrow{p^v} G_{v+1},
\]

with canonical inclusion \(i_v\). Let \(X = (G_v, i_v)\) be a \(p\)-divisible group over \(S\). Let \(T\) be a scheme over \(S\). Then we have the \(p\)-divisible group \(X_T\) over \(T\), which is defined as \((G_v \times_S T, i_v \times \text{id})\).

For the case \(T\) is a closed point \(s = \text{Spec}(k)\) over \(S\), we call the \(p\)-divisible group \(X_s\) fiber of \(X\) over \(s\). In \([4, 2.1]\) Oort defined a leaf by

\[
\mathcal{C}_Y(S) = \{s \in S \mid Y_s \text{ is isomorphic to } Y \text{ over an algebraically closed field}\},
\]

and he showed this leaf is a locally closed subscheme of \(S\) endowed with the induced reduced structure.

Fix a perfect field \(K\) of characteristic \(p\). Let \(W(K)\) denote the ring of Witt-vectors with coefficients in \(K\). Let \(\sigma\) be the frobenius over \(K\). We denote by the same symbol \(\sigma\) the frobenius over \(W(K)\) if no confusion can occur. A **Dieudonné module over** \(K\) is a finite \(W(K)\)-module \(M\) equipped with \(\sigma\)-homomorphism \(F : M \to M\) and \(\sigma^{-1}\)-homomorphism \(V : M \to M\) satisfying that \(F \circ V = V \circ F\) and \(V \circ F\) equal the multiplication by \(p\). For each \(p\)-divisible group \(X\), we have the Dieudonné module \(\mathbb{D}(X)\) using the Dieudonné functor.

Let \(\{(m_i, n_i)\}_i\) be finite number of pairs of coprime non-negative integers satisfying that if \(i < j\) then \(\lambda_j < \lambda_i\), where \(\lambda_i = n_i/(m_i + n_i)\) for each \(i\). For this pairs, a Newton polygon is a lower convex polygon in \(\mathbb{R}^2\), which breaks on integral coordinates and consists of slopes \(\lambda_i\). We denote by

\[
\xi = \sum_i (m_i, n_i)
\]
the Newton polygon. For a Newton polygon \( \xi = \sum (m_i, n_i) \), we define the \( p \)-divisible group \( H(\xi) \) by

\[
H(\xi) = \bigoplus_i H_{m_i, n_i},
\]

(3)

where \( H_{m,n} \) is the \( p \)-divisible group over \( \mathbb{F}_p \) which is of dimension \( n \), and its Serre-dual is of dimension \( m \). Moreover the Dieudonné module \( \mathcal{D}(H_{m,n}) \) satisfies that

\[
\mathcal{D}(H_{m,n}) = \bigoplus_{i=1}^{m+n} W(\mathbb{F}_p)e_i,
\]

(4)

with \( W(\mathbb{F}_p) \) is the ring of Witt-vectors over \( \mathbb{F}_p \), and \( e_i \) is a basis. In this case \( W(\mathbb{F}_p) \) is equal to the ring of \( p \)-adic integers \( \mathbb{Z}_p \). For the basis \( e_i \), operations \( F \) and \( V \) satisfy that \( Fe_i = e_{i-m} \), \( Ve_i = e_{i-n} \) and \( e_{i-(m+n)} = pe_i \).

We say a \( p \)-divisible group \( X \) is minimal if \( X \) is isomorphic to \( H(\xi) \) over an algebraically closed field for a Newton polygon \( \xi \). For a \( p \)-divisible group \( X \), the \( p \)-kernel \( X[p] \) is obtained by the kernel of the multiplication by \( p \). It is known that a truncated Dieudonné module of level one is obtained by the Dieudonné module of \( H(\xi)[p] \). In other words, a \( DM_1 \) appears as a Dieudonné module of a truncated Barsotti-Tate group of level one. We define truncated Dieudonné module of level one by

**Definition 2.1.** A truncated Dieudonné module of level one (abbreviated as \( DM_1 \)) over \( K \) of height \( h \) is the triple \((N, F, V)\) consisting of a \( K \)-vector space \( N \) of height \( h \), a \( \sigma^{-1} \)-linear map and a \( \sigma \)-linear map from \( N \) to itself satisfying that \( \text{ker } F = \text{im } V \) and \( \text{im } F = \text{ker } V \).

Let \( \xi = \sum (m_i, n_i) \) be a Newton polygon. We denote by \( N_\xi \) the \( DM_1 \) associated to the \( p \)-kernel of \( H(\xi) \). Then \( N_\xi \) is described as

\[
N_\xi = \bigoplus_{i} N_{m_i, n_i},
\]

(5)

where \( N_{m,n} \) is the \( DM_1 \) corresponding to the \( p \)-kernel of \( H_{m,n} \).

In the case \( k \) is algebraically closed, we know the following; see Kraft [6], Oort [8] and Moonen-Wedhorn [7].

**Theorem 2.2.** There exists a one-to-one correspondence:

\[
\{0, 1\}^h \leftrightarrow \{DM_1 \text{ over } k \text{ of height } h\}/\cong.
\]

By the above we have a map \( D : \{0, 1\}^h \to \{DM_1 \text{ of height } h\} \). We write \( s(i) \) for \( i \)-th coordinate of \( s \in \{0, 1\}^h \). From this one-to-one, we give a construction of the arrowed binary sequence from a \( DM_1 \). Let \( s \in \{0, 1\}^h \). We obtain the \( DM_1 \) \( D(s) \) by the following. Put \( N = k e_1 \oplus \cdots \oplus k e_h \). We define maps \( F \) and \( V \) as follows:

\[
F e_i = \begin{cases} e_j, & j = \# \{x \mid s(x) = 0, \ x \leq i \} \quad \text{for } s(i) = 0, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( j_1, \ldots, j_m \), with \( j_1 < \cdots < j_m \), be natural numbers satisfying \( s(j_x) = 1 \). Put \( n = h - m \). Then a map \( V \) is defined by

\[
V e_i = \begin{cases} e_{j_x}, & x = i - n \quad \text{for } i > n, \\ 0 & \text{otherwise}. \end{cases}
\]

Therefore \( D(s) \) is given by \( D(s) = (N, F, V) \).

Thus we can identify \( DM_1 \)'s with sequences consisting of 0 and 1. Using this bijection we will introduce the notion of arrowed binary sequences.
2.2 Arrowed binary sequences

In this section we introduce the notion of arrowed binary sequences, which is used for constructing specializations of DM$_1$’s.

**Definition 2.3.** An *arrowed binary sequence* (we often abbreviate as ABS) $S$ of height $h$ is the triple $(\Lambda, \delta, \pi)$ consisting of an ordered symbol set $\Lambda = \{t_1 < t_2 < \cdots < t_h\}$, a map $\delta : \Lambda \to \{0, 1\}$ and a bijection $\pi : \Lambda \to \Lambda$. For an ABS $S$, let $T(S)$ denote the ordered symbol set of $S$. Similarly, we denote by $\Delta(S)$ (resp. $\Pi(S)$) the map from $T(S)$ to $\{0, 1\}$ (resp. from $T(S)$ to itself).

Let $s \in \{0, 1\}^h$. For the DM$_1$ $D(s) = (N, F, V)$, we construct the arrowed binary sequence $(\Lambda, \delta, \pi)$ as follows. For an ordered symbol set $\Lambda = \{t_1, \ldots, t_h\}$, let $\delta : \Lambda \to \{0, 1\}$ be the map which sends $t_i$ to the $i$-th coordinate of $s$. We define a map $\pi : \Lambda \to \Lambda$ by

$$
\pi(t_i) = \begin{cases} 
  t_j, & \text{for } Fe_i = e_j \text{ if } \delta(t_i) = 0, \\
  t_j, & \text{for } V e_i = e_i \text{ otherwise.}
\end{cases}
$$

(6)

Note that this map $\pi$ is bijective, and $\pi$ is uniquely obtained by $s$. As an example, for the DM$_1$ $N_{m,n}$ corresponding to the $p$-divisible group $H_{m,n}$, we get the ABS $S$ as follows. We see $T(S) = \{t_1, \ldots, t_{m+n}\}$. The map $\Delta(S)$ is defined by $\Delta(S)(t_i) = 1$ if $i \leq m$, and $\Delta(S)(t_i) = 0$ otherwise. The map $\Pi(S)$ is defined by $\Pi(S)(t_i) = t_i - m \mod (m+n)$.

**Definition 2.4.** Let $S$ be an ABS. We define the *length* $\ell(S)$ of $S$ by

$$
\ell(S) = \# \{(t, t') \in T(S) \times T(S) \mid t < t' \text{ with } \delta(t) = 0 \text{ and } \delta(t') = 1\}.
$$

**Definition 2.5.** Let $S$ be an ABS. Put $\delta = \Delta(S)$ and $\pi = \Pi(S)$. Let $t \in T(S)$. The *binary expansion* $b(t)$ of $t$ is the real number $b(t) = 0.b_1b_2\ldots,$ where $b_i = \delta(\pi^{-i}(t))$.

Let $S$ be the ABS associated to a minimal DM$_1$. The elements of $T(S)$ are ordered by binary expansions determined by $\Pi(S)$, i.e., for symbols $t_i$ and $t_j$ of $T(S) = \{t_1, t_2, \ldots, t_h\}$, if $i < j$, then $b(t_i) < b(t_j)$ holds.

**Definition 2.6.** Let $S$ be an ABS. Let $t'$ and $t''$ be elements of $T(S)$. We define the *small modification* $\pi$ by $t'$ and $t''$ as follows. Let $u'$ and $u''$ be the inverse image of $t'$ and $t''$ by $\Pi(S)$ respectively. We define the map $\pi : T(S) \to T(S)$ to be $\pi$ maps $u'$ (resp. $u''$) to $t''$ (resp. $t'$), and the others $t$ to $\Pi(S)(t)$.

**Definition 2.7.** Let $S$ be an ABS. Let $\pi$ be the small modification by $t'$ and $t''$. We define the *specialization* by $t'$ and $t''$. A specialization $S'$ of $S$ is an ABS satisfying the following properties: We define $T(S')$ to be the set $T(S)$ endowed with the order determined by binary expansions by $\pi$. Set $\Delta(S') = \Delta(S)$, and put $\Pi(S') = \pi$. We say a specialization $S'$ by $t'$ and $t''$ of $S$ is *good* if $\ell(S') = \ell(S) - 1$ holds.

**Definition 2.8.** Let $S_1$ and $S_2$ be ABS’s. We define the *direct sum* $S = S_1 \oplus S_2$ of $S_1$ and $S_2$ as follows. For the set $T(S_1) \cup T(S_2)$, we define the map $\delta : T(S_1) \cup T(S_2) \to \{0, 1\}$ to be $\delta|_{T(S_i)} = \Delta(S_i)$ for $i = 1, 2$. Let $\pi$ be the map from $T(S_1) \cup T(S_2)$ to itself satisfying that $\pi|_{T(S_i)} = \Pi(S_i)$ for $i = 1, 2$. Let $T(S)$ be the set $T(S_1) \cup T(S_2)$ equipped with the order determined by binary expansions by $\delta$ and $\pi$. Put $\Delta(S) = \delta$ and $\Delta(S) = \pi$.

**Notation 2.9.** Let $N_\xi$ be the minimal DM$_1$ of a Newton polygon $\xi = \sum_{i=1}^\infty(m_i, n_i)$. Put $\lambda_i = n_i/(m_i + n_i)$. Let $S$ be the ABS associated to $N_\xi$. Then $S$ is described as $S = \bigoplus_{i=1}^\infty S_i$, where $S_i$ is the ABS associated to the DM$_1$ $N_{m_i,n_i}$. If an element $t$ of $T(S)$ belongs to $T(S_i)$, then we denote by $t'$ or $t''$ this element $t$ with $\tau = \Delta(S)(t)$. If we want to say that the element $t'$ is the $i$-th element of $T(S_i)$, we write $t_i'$ for the element $t_i'$. Furthermore, we often write $\tau_i'$ for the element $t_i'$ of $T(S)$ with $\tau = \Delta(S)(t_i')$. 

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In this section, we will prove Theorem 1.1 to give a criterion of boundary components for a minimal DM. The following lemma is useful.

Lemma 2.10. Let $N_\xi$ be the minimal DM of $\xi = (m_1, n_1) + (m_2, n_2)$ with $\lambda_2 < 1/2 < \lambda_1$. For the above notation, the sequence $S$ associated to $N_\xi$ is obtained by the following:

\[
1^1_{m_1} \cdots 1^1_{m_1} 0^1_{n_1+1} \cdots 0^1_{n_1} 1^2_{n_2} \cdots 2^1_{n_2} 0^1_{n_1+1} \cdots 0^1_h 1^2_{m_2-n_2} \cdots 1^2_{m_2-n_2} 0^2_{n_2+m_2+1} \cdots 0^2_h.
\]

Proof. See [1], Proposition 4.20. \qed

In Construction 2.11 and Construction 2.12, we introduce a method to construct specializations $S'$ of $S$. This method is useful for classifying the specializations satisfying that $\ell(S') = \ell(S) - 1$.

Construction 2.11. Let $S$ be the ABS of a minimal DM $N_\xi$ with $\xi = \sum_{i=1}^z (m_i, n_i)$. We write $\delta$ for $\Delta(S)$. Let $t^\prime_i$ and $t^\prime_j$ be elements of $T(S)$ satisfying that $\delta(t^\prime_i) = 0$ and $\delta(t^\prime_j) = 1$ with $r < q$. Let $\pi$ be the small modification by $t^\prime_i$ and $t^\prime_j$. Put $\alpha_n = \pi^n(t^\prime_i)$ for non-negative integers $n$. We define the ABS $S(0)$ to be

\[
T(S(0)) = \{ t^\prime_1 < \cdots < t^\prime_{r-1} < t^\prime_j < t^\prime_{r+1} < \cdots < t^\prime_{q-1} < t^\prime_i < t^\prime_{q+1} < \cdots < t^\prime_h \},
\]

with $h_z = |T(S_z)|$, $\Delta(S(0)) = \delta$ and $\Pi(S(0)) = \pi$. We define a set $A_0$ to be

\[
A_0 = \{ t \in T(S(0)) \mid t < \alpha_0 \text{ and } \alpha_1 < \pi(t) \text{ in } T(S(0)), \text{ with } \delta(t) = 0 \}
\]

endowed with the order induced from $T(S(0))$. We construct an ABS $S(n)$ and a set $A_n$ by ABS’s $S(0), \ldots, S(n-1)$ and sets $A_0, \ldots, A_{n-1}$ inductively. We define an ABS $S(n)$ to be the triple $(\Lambda, \delta, \pi)$, where $\Lambda$ is the set consisting of elements of $T(S)$ endowed with the order

(1) $\pi(t_{\text{max}}) < \alpha_n$, and there is no element $t$ satisfying $\pi(t_{\text{max}}) < t < \alpha_n$, where $t_{\text{max}}$ is the maximum element of $A_{n-1}$;

(2) the others are ordered by the same way as $T(S(n-1))$.

Moreover, we define the set $A_n$ by

\[
A_n = \{ t \in T(S(n)) - T(S_q) \mid t < \alpha_n \text{ and } \alpha_{n+1} < \pi(t) \text{ in } T(S(n)) \text{ with } \delta(t) = \delta(\alpha_n) \}
\]

endowed with the order induced from $S(n)$. Thus we construct ABS’s $S(n)$ and sets $A_n$ for non-negative integers $n$.

3 Proof

In this section, we will prove Theorem 1.1 to give a criterion of boundary components for arbitrary DM’s.

We use the notation of Notation 2.3. Furthermore, we fix the following notation. Let $S$ be the ABS associated to $N_\xi$. Let $\pi$ be the small modification by $t^\prime_i$ and $t^\prime_j$ satisfying that $\delta(t^\prime_i) = 0$ and $\delta(t^\prime_j) = 1$ with $r < q$. Then we obtain arrowed binary sequences $S(0), S(1), \ldots$ and sets $A_0, A_1, \ldots$ by Construction 2.11. Let $\delta$ and $\pi$ denote $\Delta(S(n))$ and $\Pi(S(n))$ respectively for non-negative integers $n$. We set $\alpha_n = \pi^n(t^\prime_i)$ and $\beta_n = \pi^n(t^\prime_j)$ for non-negative integers $n$.

Proposition 3.1. Let $n$ be a natural number. The set $A_n$ is given by

\[
A_n = \{ \pi(t) \mid t \in A_{n-1}, \pi(t) \not\in T(S_q) \text{ and } \delta(\pi(t)) = \delta(\alpha_n) \}.
\]
Proof. First, fix an element \( t \) of \( A_{n-1} \). Let us show that if \( \pi(t) \) satisfies \( t \notin T(S_q) \) and \( \delta(t) = \delta(\alpha_t) \), then \( \pi(t) \) belongs to \( A_n \). We have \( \alpha_{n+1} < \pi(\pi(t)) \) in \( T(S^{(n-1)}) \) and \( T(S^{(n)}) \). Furthermore, by construction, \( \pi(t) < \alpha_n \) holds in \( S^{(n)} \). Hence \( \pi(t) \) belongs to \( A_n \). Conversely, let \( \pi(t) \) be an element of \( A_n \). Note that for an element \( t’ \) of \( T(S^{(n-1)}) \), if \( t’ < \alpha_n \) and \( \delta(t’) = \delta(\alpha_t) \), then \( \pi(t’) < \alpha_{n+1} \). It suffices to show that \( t \) belongs to \( A_{n-1} \). In \( T(S^{(n-1)}) \) and \( T(S^{(n)}) \), we have \( t < \alpha_{n+1} \). Moreover, by construction, in \( T(S^{(n-1)}) \), we have \( \alpha_n < \pi(t) \). Hence we see that \( t \) belongs to \( A_{n-1} \).

**Proposition 3.2.** Every set \( A_n \) does not contain elements \( \alpha_m \) with \( m \leq n \).

**Proof.** Note that for all non-negative integers \( n \), sets \( A_n \) do not contain the inverse image of \( \alpha_0 \), which is an element of \( T(S_q) \). Let us show the assertion by induction on \( n \). The case of \( n = 0 \) is obvious. For a natural number \( n \), suppose that \( A_n \) contains \( \alpha_m \) for a non-negative integer \( m \) with \( m \leq n \). By Proposition 3.1, \( A_{n-1} \) contains \( \alpha_{m-1} \). It contradicts with the hypothesis of induction.

**Proposition 3.3.** If there exists no non-negative integer \( a \) such that \( A_a = \emptyset \), then the specialization \( S’ \) is not good.

**Proof.** Let \( a’ \) be the minimum number satisfying \( \alpha_{a’} = \beta_0 \). We define the set \( B_{-1} \) by

\[
B_{-1} = \{ t \in T(S^{(a’-1)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1, \text{ with } \delta(t) = 1 \}.
\]

If the set \( A_{a’} \) is not empty, then elements \( t \) of \( A_{a’} \) satisfy that \( t’ < t \) for all elements \( t’ \) of \( B_{-1} \). By construction, the set \( \{ t \in T(S^{(a’)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1, \text{ with } \delta(t) = 1 \} \) is empty. If there exists no non-negative integer \( a \) such that \( A_a = \emptyset \), then we have a non-negative integer \( m \) satisfying that \( |A_m| = |A_{m+1}| = \ldots \). The elements of \( T(S^{(m)}) \) are ordered by binary expansions determined by \( \pi \). Put \( A_m = \{ u_1, u_2, \ldots, u_n \} \). Consider the subset \( \{ u_1, \ldots, u_n, u_{n+1} \} \) of \( T(S^{(m)}) \), with \( u_{n+1} = \alpha_m \). We define the map \( \pi’ \) on \( T(S^{(m)}) \) to be \( \pi’ \) maps \( u_x \) to \( \pi(u_{x-1}) \) if \( x > 1 \) and \( u_1 \) to \( \pi(u_{n+1}) \), and the others \( t \) to \( \pi(t) \). Thus we obtain \( S’ \) by \( S’ = (T(S^{(m)}), \delta, \pi’). \) We see that \( \ell(S^{(m)}) = \ell(S’). \)

Here, let us compare lengths of \( S \) and \( S’ \). Put

\[
B’_0 = \{ t \in T(S^{(0)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1, \text{ with } \delta(t) = 1 \}.
\]

We have \( \ell(S) - \ell(S^{(0)}) = |A_0| + |B’_0| + 1. \) Since \( \ell(S^{(m)}) - \ell(S^{(0)}) \leq |A_0| - |A_m|, \) we see \( \ell(S’) < \ell(S) - 1. \)

By Proposition 3.3, we may assume that there exists a non-negative integer \( a \) such that \( A_a \) is an empty set to classify good specializations of arrowed binary sequences.

**Construction 3.4.** Let \( S \) be the ABS of a minimal \( DM_1 N_\xi \) with \( \xi = \sum_{i=1}^z (m_i, n_i) \). We write \( \delta \) for \( \Delta(S) \). Let \( t_i^r \) and \( t_i^q \) be elements of \( T(S) \) satisfying that \( \delta(t_i^r) = 0 \) and \( \delta(t_i^q) = 1 \) with \( r < q \). Let \( \pi \) be the small modification by \( t_i^r \) and \( t_i^q \). Put \( \beta_n = \pi^n(t_i^q) \) for non-negative integers \( n \). For the minimum non-negative integer \( a \) such that \( A_a \) is empty, we define a set \( B_0 \) by

\[
B_0 = \{ t \in T(S^{(a)}) \mid \beta_0 < t \text{ and } \pi(t) < \beta_1, \text{ with } \delta(t) = 1 \}
\]

endowed with the order induced from \( T(S^{(a)}) \). For ABS’s \( S^{(a)}, S^{(a+1)}, \ldots, S^{(a+n-1)} \) and sets \( B_0, B_1, \ldots, B_{n-1} \), we define an ABS \( S^{(a+n)} \) to be the triple \( (\Lambda, \delta, \pi) \), where \( \Lambda \) consists of elements of \( T(S) \) equipped with the order

(1) \( \beta_n < \pi(t_{\text{min}}) \), and there exists no element \( t \) such that \( \beta_n < t < \pi(t_{\text{min}}) \), where \( t_{\text{min}} \) is the minimum element of \( B_{n-1} \).
We define the set \( B_n \) as
\[
B_n = \{ t \in T(S^{(a+n)}) \mid \beta_n < t \text{ and } \pi(t) < \beta_{n+1} \text{ in } T(S^{(a+n)}) \text{ with } \delta(t) = \delta(\beta_n) \}.
\]
Thus we obtain \( \text{ABS} \)'s \( S^{(a+n)} \) and sets \( B_n \) for non-negative integers \( n \).

We will show that to classify specializations \( S' \) satisfying that \( \ell(S') = \ell(S) - 1 \), it suffices to consider the case there exists a non-negative integer \( b \) such that \( B_b = \emptyset \). Let us see some properties of sets \( B_n \).

**Proposition 3.5.** Let \( n \) be a natural number. The set \( B_n \) is obtained by
\[
B_n = \{ \pi(t) \mid t \in B_{n-1} \text{ and } \delta(\pi(t)) = \delta(\beta_n) \}.
\]

**Proof.** A proof is given by the same way as Proposition 3.1.

**Proposition 3.6.** If there exists no non-negative integer \( b \) such that \( B_b = \emptyset \), then the specialization is not good.

**Proof.** In this hypothesis, there exists a non-negative integer \( m \) such that \( |B_m| = |B_{m+1}| = \cdots \). For this \( m \), we set \( B_m = \{ u_1, \ldots, u_n \} \), and consider the subset \( \{ u_0, u_1, \ldots, u_n \} \) of \( T(S^{(a+m)}) \), where \( u_0 = \beta_m \). Let \( \pi' \) be the map on \( T(S^{(a+m)}) \) which maps \( u_x \) to \( \pi(u_{x+1}) \) if \( x < n \), and \( u_n \) to \( \pi(u_0) \), and the others to \( \pi(t) \). Thus we obtain the specialization \( S' = (T(S^{(a+m)}), \delta, \pi') \).

Let us compare lengths of \( S \) and \( S' \). We have \( \ell(S) - \ell(S^{(a)}) \geq |B_0| + 1 \). Since \( \ell(S^{(a+m)}) - \ell(S^{(a)}) = |B_0| - |B_m| \) with \( |B_m| > 0 \), we have
\[
\ell(S') < \ell(S) - 1 - |B_m|.
\]
Thus we see that \( \ell(S') < \ell(S) - 1 \).

For the above \( b \) and the \( \text{ABS} \) \( S^{(a+b)} \), if elements \( t \) and \( t' \) of \( T(S^{(a+b)}) \) satisfy that \( t < t' \) and \( \delta(t) = \delta(t') \), then \( \pi(t) < \pi(t') \) holds. By Proposition 3.3 and Proposition 3.6, to classify specializations \( S' \) satisfying that \( \ell(S') = \ell(S) - 1 \), we may assume that there exists non-negative integers \( a \) and \( b \) such that \( A_a \) and \( B_b \) are empty, and we have then \( S' = S^{(a+b)} \). We call this \( \text{ABS} \) \( S^{(a+b)} \) the full modification by \( t_i^a \) and \( t_j^b \).

Here, we introduce some properties of Newton polygons. These properties are useful for investigating the construction of the \( \text{ABS} \)'s of minimal DM1's. Let \( \xi = \sum_{i=1}^{z} (m_i, n_i) \) be a Newton polygon. We define the Newton polygon \( f(\xi) \) by
\[
f(\xi) = \sum_{i=1}^{z} (m_{z-i+1}, n_{z-i+1}).
\]
We call this \( f(\xi) \) the dual of \( \xi \). If \( \xi \) satisfies \( m_i < n_i \) for all \( i \), we define the Newton polygon \( g(\xi) \) by
\[
g(\xi) = \sum_{i=1}^{z} (m_i, n_i - m_i),
\]
and we call this \( g(\xi) \) the curtailment of \( \xi \). For the Newton polygon \( \xi \), we denote by \( g^{-1}(\xi) \) the Newton polygon \( \sum_{i=1}^{z} (m_i, h_i) \) with \( h_i = m_i + n_i \).

**Lemma 3.7.** Let \( \xi = \sum_{x=1}^{z} (m_x, n_x) \) be a Newton polygon. By repeating to construct the dual and curtailment, we obtain the Newton polygon \( \xi' = \sum_{x=1}^{z} (c_x, d_x) \) such that \( \mu_x < 1/2 < \mu_1 \), where \( \mu_x = d_x/(c_x + d_x) \).
Proposition 2.10. If \( \xi \) satisfies that \( \lambda_2 < \lambda_1 \) implies that \( m_1m_2(q_2 - q_1) < m_2r_1 - m_1r_2 \). Suppose \( q_1 < q_2 \). Then \( m_1m_2 < m_2r_1 - m_1r_2 \) holds. It induces that \( m_2(r_1 - m_1) > m_1r_2 \), and we get \( m_1 < 0 \) or \( r_2 < 0 \). It is a contradiction.

Let us construct the Newton polygon \( \xi' \). If \( q_2 < q_1 \), then constructing curtailments, we obtain the desired Newton polygon \( \xi' \) by \( \xi' = (m_1, n_1 - q_2m_1) + (m_2, r_2) \). Assume \( q_1 = q_2 \). Constructing curtailments, we obtain the Newton polygon \( (m_1, r_1) + (m_2, r_2) \). Take the dual \( (r_2, m_2) + (r_1, m_1) \), and we can repeat constructing curtailments. We may assume \( r_1 = 1 \) or \( r_2 = 1 \) since \( \gcd(m_x, n_x) = 1 \). We divide the proof into the three cases:

1. \( \xi = (1, n_1) + (1, n_2) \);
2. \( \xi = (m_1, n_1) + (1, n_2) \);
3. \( \xi = (1, n_1) + (m_2, n_2) \).

In the case (1), if \( n_1 > n_2 + 1 \), then we obtain \( \xi' = (1, n_2 - n_1) + (1, 0) \). If \( n_2 = n_1 + 1 \), then by the Newton polygon \( (1, 1) + (1, 0) \) obtained by curtailments, we take the dual, and we get \( \xi' = (0, 1) + (1, 0) \).

In the case (2), we have \( q_2 = n_2 \). If \( q_1 = q_2 \), then by the Newton polygon \( (m_1, r_1) + (1, 0) \) obtained by curtailments, constructing the dual and curtailments we get \( \xi' = (0, 1) + (r_1, r) \), where \( r \) is the reminder obtained by dividing \( m_1 \) by \( r_1 \).

In the case (3), if \( q_1 = q_2 = n_1 \), then we have \( n_2 = n_1m_2 + r_2 \). On the other hand, it follows that \( n_1m_2 - n_2 > 0 \) by the condition of the Newton polygon \( \xi \). It implies \( r_2 < 0 \), and we have a contradiction. If \( q_1 > q_2 + 1 \), then we get \( \xi' = (1, n_1 - q_2) + (m_2, r_2) \). If \( q_2 = q_1 + 1 \), then by curtailments and the dual we have the Newton polygon \( (r_2, m_2) + (1, 1) \). By the case (1) and (2), we can get the desired Newton polygon \( \xi' \) of this Newton polygon.

Finally, for the case \( z > 2 \), we consider the Newton polygon \( \eta' = (c_1, d_1) + (c_2, d_2) \) of \( \eta = (m_1, n_1) + (m_z, n_z) \), which is obtained by constructing duals and curtailments satisfying \( d_z/\max(c_2, d_2) < 1/2 < d_1/\max(c_1, d_1) \). Apply to \( \xi \) the same operation with the operation \( f \) and \( g \) to obtain \( \eta' \) from \( \eta \), and we get the desired \( \xi' = \sum_{i=1}^{z} (c_x, d_x) \).

**Lemma 3.8.** Let \( S \) be the ABS of the minimal DM \( N_\xi \) with \( \xi = \sum_{i=1}^{z} (m_i, n_i) \). For natural numbers \( r \) and \( q \) with \( r < q \), we have

1. \( 1^r_1 < 1^q_1 \),
2. \( 0^r_{h_i} < 0^q_{h_q} \),
3. \( 0^r_{m_r+1} < 0^q_{m_q+1} \)

in the set \( T(S) \), where \( h_i = m_i + n_i \).

**Proof.** If (i) holds, then (iii) immediately holds since \( 0^r_{m_r+1} \) and \( 0^q_{m_q+1} \) are the inverse image of \( 1^r_1 \) and \( 1^q_1 \) respectively. Moreover, considering the dual \( f(\xi) \) of \( \xi \), if (i) is true, then (ii) holds.

To show the lemma, it suffices to deal with the case \( z = 2 \). For a Newton polygon \( \xi \), let \( P(\xi) \) denote the proposition: The ABS associated to the minimal DM \( N_\xi \) satisfies (i). By Proposition 2.10 if \( \xi \) satisfies that \( \lambda_2 < 1/2 < \lambda_1 \), then \( P(\xi) \) is true. To show that \( P(\xi) \) is true for arbitrary Newton polygons \( \xi \), we claim

(A) If \( P(\xi) \) holds, then \( P(f(\xi)) \) also holds;

(B) If \( P(\xi) \) holds, then \( P(g^{-1}(\xi)) \) also holds.
The claim (A) is obvious by the duality. Moreover, by the construction of $N_{ξ}$ and $N_{g^{-1}(ξ)}$, we see that (B) holds. The assertion of the lemma follows from (A), (B) and Lemma 3.7.  

In the following Propositions, we use symbols of Notation 2.9.

**Proposition 3.9.** Let $S$ be the ABS associated to $N_{ξ}$ with $ξ = \sum_{i=1}^{7} (m_i, n_i)$. Let $0^r$ and $1^q$ be elements of $T(S)$ with $r + 1 < q$. If $0^r < 1^q$ holds in $T(S)$, then there exists an element $t^x$ of $T(S)$ such that $r < x < q$ and $0^r < t^x < 1^q$.

**Proof.** For a Newton polygon $ξ$, we write $Q(ξ)$ for the proposition: For elements $0^r$ and $1^q$ of the ABS associated to $N_{ξ}$ satisfying that $0^r < 1^q$ and $r + 1 < q$, there exists an element $t^x$ of $S$ such that $r < x < q$ and $0^r < t^x < 1^q$. It suffices to treat the case $z = 3$, $r = 1$ and $q = 3$. If $λ_1 = λ_2$ (resp. $λ_2 = λ_3$) holds, then we immediately have the desired term $t^x$ since for elements $0^1_1 < 1^3_j$, the element $0^2_1$ (resp. $0^2_j$) satisfies $0^1_1 < 0^2_1 < 1^3_j$ (resp. $0^1_1 < 1^2_j < 1^3_j$). After this, we assume that the slopes are different from each other.

Now we treat Newton polygons satisfying one of the following:

(i) $λ_3 < 1/2 ≤ λ_2 < λ_1$,  

(ii) $λ_3 < λ_2 ≤ 1/2 < λ_1$.

By the duality, if $Q(ξ)$ is true for all $ξ$ satisfying (i), then $Q(ξ)$ holds for all $ξ$ satisfying (ii). Suppose that $ξ$ satisfies (i). Put $h_x = m_x + n_x$ for all $x$. By Lemma 2.10 in the ABS of $N_{(m_1, n_1)+(m_3, n_3)}$, there exists no element $t$ of the ABS satisfying that $0^1_{n_1} < t < 1^3_{n_3+1}$. Hence it is enough to show that there exists an element $t^x_z$ such that $0^1_{n_1} < t^x_z < 1^3_{n_3+1}$. If $λ_2 > 1/2$, then these elements are obtained by $0^2_{n_2}$ and $0^0_{n_2}$ respectively. In fact, by Lemma 3.8 (ii), we have $0^1_{n_1} < 0^2_{n_2}$ and $0^0_{n_1} < 0^2_{n_2}$. Moreover, by the construction of the ABS corresponding to the DM1 $N_{(m_2, n_2)+(m_3, n_3)}$ we have $0^2_{n_2} < 1^3_{n_3+1}$ and $0^2_{n_2} < 1^3_{n_2}$. If $λ_2 = 1/2$, then we can show that the desired elements $t^x_z$ and $t^y_y$ are obtained by $1^2_1$ and $0^2_2$.

To treat the remaining case, we claim that

(A) If $Q(ξ)$ holds, then $Q(g^{-1}(ξ))$ also holds;  

(B) If $Q(ξ)$ holds, then $Q(g^{-1}(ξ))$ also holds.

If the claim (A) and (B) are true, then by Lemma 3.7 the proposition arrives at the case (i) or (ii), and we complete the proof. The claim (A) is obvious. Let us show (B). Put $ξ' = g^{-1}(ξ)$, and let $S$ (resp. $R$) denote the ABS associated to $N_{ξ}$ (resp. $N_{ξ'}$). We can regard $T(S)$ as a subset of $T(R)$. Let $U$ (resp. $V$) be the subset of $T(S) \times T(S)$ (resp. $T(R) \times T(R)$) consisting of the pair $(0^1_1, 1^3_j)$ of elements of $T(S)$ (resp. $T(R)$) satisfying $0^1_1 < 1^3_j$. By construction of $S$ and $R$, we have $U = V$, hence clearly (B) holds. □

**Proposition 3.10.** For the small modification of $0^r$ and $1^q$, if there exists a non-negative integer $n$ such that one of the cases:

(i) the set $A_0$ contains an element $0^r$ with $r < x$,  

(ii) the set $B_0$ contains an element $1^r$ with $x < q$,  

then the specialization is not good.

**Proof.** For the elements $0^r$ and $1^q$ of the assertion, set $α_n = π^n(0^r)$ and $β_n = π^n(1^q)$. Proposition 3.3 and Proposition 3.6 imply that we may assume that there exists the full modification $S^{(a+b)}$ by $0^r$ and $1^q$. Put

$$B'_0 = \{ t \in T(S^{(0)}) \mid β_0 < t \text{ and } π(t) < β_1 \text{ with } δ(t) = 1 \}.$$
For this set, \( \ell(S) - \ell(S^{(0)}) = |A_0| + |B_0'| + 1 \) holds. In general, we have \( \ell(S^{(n+1)}) - \ell(S^{(n)}) \leq d(n) \), where

\[
d(n) = \begin{cases} 
|A_n| - |A_{n+1}| & \text{if } n < a, \\
|B_n| - |B_{n+1}| & \text{if } a \leq n.
\end{cases}
\]

Clearly \( \ell(S') - \ell(S^{(0)}) \leq |A_0| + |B_0'| \) holds. First, we show that \( \ell(S') - \ell(S^{(0)}) \leq |A_0| + |B_0'| \) holds. Let \( I \) be the subset of \( B_0 \) consisting of elements which are of the form \( \alpha_m \). We have then \( |B_0| \leq |B_0'| + |I| \). Let \( m \) be a non-negative integer such that \( A_m \) contains the inverse image of \( \beta_0 \). Since \( \delta(\alpha_{m+1}) = \delta(\beta_0) = 1 \), we have \( \ell(S^{(m+1)}) - \ell(S^{(m)}) = d(m) - 1 \). Moreover, \( \alpha_{m+1} \) belongs to \( I \). Hence we see \( \ell(S^{(a)}) - \ell(S^{(0)}) \leq |A_0| - |I| \) and the desired inequality.

Let us see that in the case (i) the specialization is not good. Let \( m \) be the minimum number such that the set \( A_m \) contains no element \( t^x \) with \( r < x \). Fix an element \( t^x \) of \( A_{m-1} \). Put \( t = \pi(t^x) \). If \( \delta(t) = 0 \) and \( \delta(\alpha_m) = 1 \) is true, then there exists an element \( 1^x < t \) in \( T(S) \). In fact, if \( t' < \alpha_m \) holds in \( T(S) \) for all \( t' \in T(S_x) \) with \( \delta(t') = 1 \), then we have \( 1^x_m < 1^{x'}_m \) with \( r < x \). It contradict with Lemma 3.3. Thus we see that the set \( A_m \) contains the element \( 1^x_m \) and it is a contradiction. Hence we have \( \delta(t) = 1 \) and \( \delta(\alpha_m) = 0 \), and it implies that \( \ell(S^{(m)}) - \ell(S^{(m-1)}) \leq d(m) \).

Similarly, if \( B_0 \) contains an element \( t^x \) with \( x < q \), then there exists a non-negative integer \( m \) such that \( \ell(S^{(m)}) - \ell(S^{(m-1)}) < d(m) \). In fact, for the minimum number \( m \) such that \( B_m \) contains no element \( t^x \) with \( x < q \), fix an element \( t^x \) of \( B_{m-1} \). Then for \( t = \pi(t^x) \), we have \( \delta(t) = 0 \) and \( \delta(\beta_m) = 1 \) since if \( \delta(t) = 1 \) and \( \delta(\beta_m) = 0 \) is true, then there exists an element \( 0^x \) of \( T(S) \) satisfying that \( t < 0^x < \beta_m \). It implies that \( B_m \) contains an element \( 0^x \), and it is a contradiction.

By the above, in the case (i) and (ii), we have \( \ell(S') - \ell(S^{(0)}) < |A_0| + |B_0'| \), and it follows that \( \ell(S') < \ell(S) - 1 \).

**Corollary 3.11.** If \( r + 1 < q \), then the specialization is not good.

**Proof.** For a small modification of \( 0^r \) and \( 1^r \), by Proposition 3.9, there exists an element \( t^x \) of \( T(S) \) such that \( 0^r < t^x < 1^r \) and \( r < x < q \). If \( \delta(t^x) = 0 \), then the element \( t^x \) belongs to \( A_0 \), and the desired claim follows from Proposition 3.10. Let us see the case \( \delta(t^x) = 1 \). If the set \( B_0 \) contains \( t^x \), then by Proposition 3.10 we complete the proof. If \( B_0 \) does not contain \( t^x \), then we have \( a \geq a' \) with \( a' \) is the minimum number satisfying that \( \alpha_{a'} = \beta_0 \). Then \( |B_0| < |B_0'| + |I| \) holds, and we see \( \ell(S') < \ell(S) - 1 \).

Let \( \xi = \sum_{i=1}^n (m_i, n_i) \) be a Newton polygon. Let \( S \) be the ABS of the DM \( N_\xi \). Recall that the ABS \( S \) is described as \( S = \bigoplus S_i \) for ABS’s \( S_i \) corresponding to the DM \( N_{m_i, n_i} \). We say a full modification \( S' \) is good if \( \ell(S') = \ell(S) - 1 \). By the above propositions, all good specializations are given by full modifications. The following proposition implies that to classify specializations \( S' \) satisfying that \( \ell(S') = \ell(S) - 1 \), it suffices to deal with the case the Newton polygon consists of two slopes.

**Proposition 3.12.** Let \( r \) be a natural number with \( r < z \). The full modification by \( 0^r \) and \( 1^{r+1} \) is good if and only if the full modification of \( S_r \oplus S_{r+1} \) by \( 0^r \) and \( 1^{r+1} \) is good.

**Proof.** If the set \( A_0 \) contains an element \( 0^x \) with \( x \neq r \), then \( x < r \) holds. In fact, if \( r < x \) is true, then since \( 0^r < \beta_0 \) with \( \beta_0 \in T(S_{r+1}) \), we have \( r + 1 < x \) and \( 1^r < 1^{r+1} \). It contradict with Lemma 3.3. Similarly, if \( 1^y \) with \( y \neq r + 1 \) belongs to the set \( B_0 \), we have then \( r + 1 < y \).

Let \( m \) be the minimum number such that \( \alpha_m = 1^r_m \). We claim that the set \( A_m \) contains no element \( t^x \) with \( x \neq r \). Suppose that the set \( A_{m-1} \) contains an element \( t^x \). Since \( 1^r_m < 1^r_m \), we see \( \delta(\pi(t^x)) = 0 \), and \( \pi(t^x) \) does not belong to \( A_m \).
Fix an element \(0^x\) of \(A_0\), and let \(n\) be the maximum number satisfying that \(A_n\) contains \(\pi^n(0^x)\). Put \(t = \pi^n(0^x)\). If \(\delta(\alpha_{n+1}) = 0\) and \(\delta(t) = 1\), we have then \(1^2_1 < 1^2_1\) with \(x < r\). It is a contradiction. Thus we see \(\delta(\alpha_{n+1}) = 1\) and \(\delta(t) = 0\).

Put \(R = S_r \oplus S_{r+1}\). Let \(C_0, C_1, \ldots\) be sets obtained by Construction 2.11 for the small modification by \(0^x\) and \(1^y\), and let \(D_0, D_1, \ldots\) be sets obtained by Construction 3.4. We have \(A_0\) (resp. \(B_0\)) consisting of elements \(0^x\) (resp. \(1^y\)) with \(x \neq r\) (resp. \(y \neq r+1\)). Then we have \(\ell(S) - \ell(S^{(0)}) = \ell(R) - \ell(R^{(0)}) + |E| + |F|\). Moreover, by the above, we have \(\ell(S') - \ell(S^{(0)}) = \ell(R') - \ell(R^{(0)}) + |E| + |F|\). This completes the proof. \(\square\)

4 The case of Newton polygons of two slopes

In Section 3 we have seen that it suffices to deal with Newton polygons consisting of two segments to classify boundary components of central streams. In this section, we give a criterion of boundary components of central streams using arrowed binary sequences.

We use notation of Section 2 and Section 3. Let \(S = (m_1, n_1) + (m_2, n_2)\) be a Newton polygon consisting of two segments. Let \(N_\xi\) be the minimal DM\(_1\), and let \(S\) be the ABS associated to \(N_\xi\). The main result of this section is

**Theorem 4.1.** An specialization obtained by \(0^1_1\) and \(1^2_2\) is good if and only if sets \(A_n\) and \(B_n\) do not contain the inverse images of \(1^2_1\) and \(0^1_1\) respectively for all non-negative integers \(n\).

**Proof.** First, we suppose that for all non-negative integer \(n\) the sets \(A_n\) and \(B_n\) do not contain the inverse images of \(1^2_1\) and \(0^1_1\) respectively. In this hypothesis, we have \(\Delta \ell(n) = d(n)\) for all \(n\), and the non-negative integer \(a\) and \(b\) are obtained by the natural number \(m\) and \(n\) satisfying that \(\pi^m(0^1_1) = 1^1_m\) and \(\pi^n(1^2_1) = 0^2_m+1\) respectively. Hence we have \(\sum_n \pi^m(0^1_1) = |A_0| + |B_0|\) and \(\ell(S) - \ell(S^{(0)}) = |A_0| + |B_0| + 1\). It induces that \(\ell(S') = \ell(S) - 1\).

Next, we assume that for a non-negative integer \(n\) the set \(A_n\) contains the inverse image of \(1^2_1\). For this non-negative integer \(n\), if \(\delta(\alpha_n) = 1\), then \(\alpha_n\) belongs to \(I\). Proposition 3.10 induces the desired claim. After this, we assume \(I = \emptyset\). If \(\delta(\alpha_n) = 0\), then we have \(\Delta \ell(n) < 0\) and \(\ell(S^{(0)}) - \ell(S^{(0)}) < |A_0|\).

Next, let us see the case that the set \(B_n\) contains the inverse image of \(0^1_1\) for a non-negative integer \(n\). If \(\delta(\alpha_n) = 1\), then we have \(\Delta \ell(a + n) < 0\) and \(\ell(S') - \ell(S^{(0)}) < |B_0|\). Moreover, if \(\delta(\alpha_n) = 0\), then the set \(B_{n+1}\) contains \(0^1_1\). By Proposition 3.10 we have the desired claim. \(\square\)

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