NONLINEAR PERTURBATIONS OF A MAGNETIC NONLINEAR CHOQUARD EQUATION WITH HARDY-LITTLEWOOD-SOBOLEV CRITICAL EXPONENT

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ABSTRACT. In this paper, we consider the following magnetic nonlinear Choquard equation

\[-(\nabla + iA(x))^2 u + V(x)u = \lambda \left( \frac{1}{|x|^{\alpha}} * |u|^p \right) |u|^{p-2} u + \left( \frac{1}{|x|^{\alpha}} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2} u, \]

where \(2^*_\alpha = \frac{2N-\alpha}{N-2}\) is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality, \(\lambda > 0\), \(N \geq 3\), \(\frac{2N-\alpha}{N} < p < 2^*_\alpha\) for \(0 < \alpha < N\), \(A : \mathbb{R}^N \to \mathbb{R}^N\) is a \(C^1\), \(\mathbb{Z}^N\)-periodic vector potential and \(V\) is a continuous scalar potential given as a perturbation of a periodic potential. Using variational methods, we prove the existence of a ground state solution for this problem if \(p\) belongs to some intervals depending on \(N\) and \(\lambda\).

Keywords: Variational methods, magnetic Choquard equation, Hardy-Littlewood-Sobolev critical exponent

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1. Introduction

In this article we consider, the problem

\[-(\nabla + iA(x))^2 u + V(x)u = \lambda \left( \frac{1}{|x|^{\alpha}} * |u|^p \right) |u|^{p-2} u + \left( \frac{1}{|x|^{\alpha}} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2} u, \]

where \(\nabla + iA(x)\) is the covariant derivative with respect to the \(C^1\), \(\mathbb{Z}^N\)-periodic vector potential \(A : \mathbb{R}^N \to \mathbb{R}^N\), i.e.,

\[A(x + y) = A(x), \quad \forall x \in \mathbb{R}^N, \quad \forall y \in \mathbb{Z}^N.\]

The exponent \(2^*_\alpha = \frac{2N-\alpha}{N-2}\) is critical, in the sense of the Hardy-Littlewood-Sobolev inequality, \(\lambda > 0\), \(N \geq 3\), \(\frac{2N-\alpha}{N} < p < 2^*_\alpha\), \(0 < \alpha < N\) and \(V : \mathbb{R}^N \to \mathbb{R}\) is a continuous scalar potential. Inspired by the papers \([2, 24]\), we assume that there is a continuous potential \(V_P : \mathbb{R}^N \to \mathbb{R}\), also \(\mathbb{Z}^N\)-periodic, constants \(V_0, W_0 > 0\) and \(W \in L^\infty(\mathbb{R}^N)\) with \(W(x) \geq 0\) such that

- (V1) \(V_P(x) \geq V_0, \quad \forall x \in \mathbb{R}^N\);
- (V2) \(V(x) - V_P(x) - W(x) \geq W_0, \quad \forall x \in \mathbb{R}^N\),

where the last inequality is strict on a subset of positive measure in \(\mathbb{R}^N\).

Under these assumptions, we will show the existence of a ground state solution to problem \((1)\). Besides being considered in the whole \(\mathbb{R}^N\), which leads to the loss of compactness of the Sobolev immersion, problem \((1)\) has a critical nonlinearity in the Hardy-Littlewood-Sobolev sense, which increases the difficulty in verifying a compactness condition.

Our paper is motivated by Gao and Yang in \([19]\), where a classical Choquard equation is considered in a bounded domain, i.e., the case \(A \equiv 0\) and \(V \equiv 0\). There is a huge literature about the Choquard equation and we cite only Moroz and Van Schaftingen \([25]\) for a good review of results on this important equation. In \([19]\), Gao and Yang proved the existence of a ground state solution under restriction on \(N\) and \(\lambda\). (Variations on the right-hand side of the equation are also handled in that paper.) Other recent advances in the study of the Choquard equation can be found, e.g., in \([11, 15, 16, 17, 21, 25, 29]\).

In Mukherjee and Sreenadh \([27]\), the magnetic problem

\[-(\nabla + iA(x))^2 u + \mu g(x)u = \lambda u + \left( \frac{1}{|x|^{\alpha}} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2} u\]
was examined. In this equation $\mu > 0$ is also a parameter that interacts with the linear term in the right-hand side of the equation. Under suitable hypotheses on $g$, they prove the existence of a ground state solution.

Changing the right-hand side of (1) to
\begin{equation}
(3)
\left(\frac{1}{|x|^\alpha} + |u|^p\right) |u|^{p-2}u,
\end{equation}
the problem was studied by Cingolani, Clapp and Secchi in [13]. In that paper the authors prove existence and multiplicity of solutions. In [12], that right-hand side [3] was generalized, but the multiplicity result depend on more restrictive hypotheses than in [13].

Recent years have witnessed a growth of interest in the study of magnetic equations. The progress in this research can be found in a series of articles, e.g., [3, 7, 8, 9, 14, 15].

Initially, we are going to prove the existence of a ground state solution for the problem (1) considering the potential $V = V_P$, that is, we consider the problem,
\begin{equation}
(4)
-(\nabla + iA(x))^2 u + V_P(x) u = \lambda \left(\frac{1}{|x|^\alpha} + |u|^p\right) |u|^{p-2}u + \left(\frac{1}{|x|^\alpha} + |u|^{2^*_\alpha}\right) |u|^{2^*_\alpha - 2}u
\end{equation}
where we maintain the notation introduced before and suppose that $(V_1)$ is valid.

As in Gao and Yang in [19], the key step to proof the existence of a ground state solution of problem (1) is the use of cut-off techniques on the extreme function that attains the best constant $S_{H,L}$ naturally attached to the problem. This allows us to estimate the mountain pass value $c_\lambda$ associated to the energy functional $J_{A,V_P}$ related with (1) in terms of the Sobolev constant $S_{H,L}$. In a demanding proof, this lead us to consider different cases for $p$, if it belongs to some intervals depending on $N$ and $\lambda$, as in the seminal work of Brézis and Nirenberg [11]. After that, the proof is completed by showing the mountain pass geometry, introducing the Nehari manifold associated with (1) and applying concentration-compactness arguments.

In the sequel, we consider the general case and prove that (1) has one nontrivial solution.

Our main result is the following.

\textbf{Theorem 1.} Under the hypotheses already stated on $A$, $V$ and $\alpha$, problem (1) has at least one ground state solution if either
\begin{enumerate}[(i)]
\item $\frac{N+2-\alpha}{N-2} < p < 2^*_\alpha$, $N = 3, 4$ and $\lambda > 0$;
\item $\frac{2N}{N-2} - \frac{\alpha}{2} < p \leq \frac{2N}{N-2} - \frac{\alpha}{2}$, $N = 3, 4$ and $\lambda$ sufficiently large;
\item $\frac{2N}{N-2} - \frac{\alpha}{2} < p < 2^*_\alpha$, $N \geq 5$ and $\lambda > 0$;
\item $\frac{2N}{N-2} - \frac{\alpha}{2} < p \leq \frac{2N}{N-2} - \frac{\alpha}{2}$, $N \geq 5$ and $\lambda$ sufficiently large.
\end{enumerate}

Problems (1) and (4) are then related by showing that the minimax value $d_\lambda$ of the latter satisfies $d_\lambda < c_\lambda$. Once more, concentration-compactness arguments are applied to show the existence of a ground state solution.

Also as in [19], we have a existence result for the magnetic Choquard equation with Sobolev critical exponent and subcritical nonlocal term, that is,

\textbf{Theorem 2.} Under the hypotheses already stated on $A$, $V$ and $\alpha$ the problem
\[-(\nabla + iA(x))^2 u + V(x) u = \lambda \left(\frac{1}{|x|^\alpha} + |u|^p\right) |u|^{p-2}u + |u|^{2^*_\alpha - 2}u, \text{ in } \mathbb{R}^N,\]
has at least one ground state solution in the same intervals described in Theorem 1.

This paper is organized as follows. In Section 2 some preliminary results will be established. In Section 3 we deal with the periodic problem. Finally, in Section 4 we deal with the general case of problem (1) and also prove Theorem 2.

\section{Preliminary results}

We define
\[
\nabla_A u = \nabla u + iA(x)u
\]
and consider the space
\[
H_{A,V_P}^1(\mathbb{R}^N, \mathbb{C}) = \{ u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla_A u \in L^2(\mathbb{R}^N, \mathbb{C}) \}.\]
endowed with scalar product 

\[ (u, v)_{A,V_p} = 9i\pi \int_{\mathbb{R}^N} (\nabla_A u \cdot \nabla_A v + V_p(x) uv) \]

and, therefore 

\[ \|u\|^2_{A,V_p} = \int_{\mathbb{R}^N} |\nabla_A u|^2 + V_p |u|^2. \]

Observe that the norm generated by this scalar product is equivalent to the norm obtained by considering \( V \equiv 1 \), see [23, Definition 7.20].

If \( u \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \), then \( |u| \in H^1(\mathbb{R}^N) \) and the diamagnetic inequality is valid (see [23, Theorem 7.21], [13])

\[ |\nabla |u(x)\| \leq |\nabla u(x) + iA(x)u(x)|, \quad \text{a.e. } x \in \mathbb{R}^N. \]

As a consequence of the diamagnetic inequality, we have the continuous immersion

\[ H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \hookrightarrow L^q(\mathbb{R}^N, \mathbb{C}) \]

for any \( q \in [2, \frac{2N}{N-2}] \). We denote \( 2^* = \frac{2N}{N-2} \).

It is well-known that \( C^\infty(\mathbb{R}^N, \mathbb{C}) \) is dense in \( H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \), see [23, Theorem 7.22].

Following Gao and Yang [20], we denote by \( S_{H,L} \)

\[ S_{H,L} : = \inf_{u \in D^{1,2}(\mathbb{R}^N, \mathbb{R}) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{\left( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^\alpha} \, dx \, dy \right)^{\frac{2 \alpha}{N-2}}}, \]

where the last equality was proved in [27]. We remark that \( S_A \) is attained if and only if \( \text{rot} A = 0 \), see [10, Theorem 1.1] and [27, Theorem 4.1].

We state a result proved in [20].

**Proposition 3.** The constant \( S_{H,L} \) defined in (6) is achieved if and only if

\[ u = C \left( \frac{b}{b^2 + |x-a|^2} \right)^{\frac{N-2}{2}}, \]

where \( C > 0 \) is a fixed constant, \( a \in \mathbb{R}^N \) and \( b \in (0, \infty) \) are parameters. Furthermore,

\[ S_{H,L} = \frac{S}{C^{\frac{N-2}{N-\alpha}}}, \]

where \( S \) is the best Sobolev constant of the immersion \( D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*} (\mathbb{R}^N) \) and \( C_{N,\alpha} \) depends on \( N \) and \( \alpha \).

If we consider the minimizer for \( S \) given by \( \bar{U}(x) := \frac{[N(N-2)]^{\frac{N-2}{2}}}{(N+\alpha)(N+2-\alpha)} (1+|x|^2)^{\frac{N-2}{2}} \) (see [30, Theorem 1.42]), then

\[ \bar{U}(x) = S^{\frac{(N-\alpha)(2-\alpha)}{(N+\alpha)(N+2-\alpha)}} C(N, \alpha)^{\frac{2-N}{2(N+2-\alpha)}} \left( \frac{|N(N-2)|^{\frac{N-2}{2}}}{(1+|x|^2)^{\frac{N-2}{2}}} \right) \]

is the unique minimizer for \( S_{H,L} \) that satisfies

\[ -\Delta u = \left( \int_{\mathbb{R}^N} \frac{|u|^2}{|x-y|^\alpha} \, dy \right) |u|^2 \] \( \text{em } \mathbb{R}^N \)

with

\[ \int_{\mathbb{R}^N} |\nabla \bar{U}|^2 \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\bar{U}(x)|^2 |\bar{U}(y)|^2}{|x-y|^\alpha} \, dx \, dy = S_{H,L}^{\frac{2N-\alpha}{N-\alpha}}. \]
Proposition 4 (Hardy-Littlewood-Sobolev inequality). Let $t, r > 1$ $e 0 < \alpha < N$ with $\frac{1}{t} + \frac{\alpha}{N} + \frac{1}{r} = 2$, $f \in L^t(\mathbb{R}^N)$ and $h \in L^r(\mathbb{R}^N)$. There exists a sharp constant $C(t, N, \alpha, r)$, independent of $f, h$, such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)h(y)}{|x-y|^\mu} \, dx \, dy \leq C(t, N, \mu, r)|f|_{L^t}|h|_{L^r},$$

where $| \cdot |_q$ denote the norm of $L^q(\mathbb{R}^N)$ for $q \in [1, \infty]$. If $t = r = \frac{2N}{2N-\mu}$, then

$$C(t, N, \mu, r) = C(N, \mu) = \pi^{\frac{\mu}{2}} \frac{\Gamma\left(\frac{N+\mu-2}{2}\right)}{\Gamma\left(\frac{N-\mu}{2}\right)} \left\{ \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma(N)} \right\}^{-1+\frac{\mu}{2}}.$$

In this case there is equality if and only if $f$ is constant and $h(x) = A(\gamma^2 + |x-a|^2)^{-(2N-\mu)/2}$ for some $A \in \mathbb{C}$, $0 < \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^N$.

Remark 2.1. Let us consider the case $F(t) = |t|^q$. By applying the Hardy-Littlewood-Sobolev inequality for $t = r = p$ we have that

$$\int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * F(|u|) \right) F(|u|)$$

is well-defined if $F(|u|) \in L^p(\mathbb{R}^N)$ for $p > 1$ defined by

$$\frac{2}{p} + \frac{\alpha}{N} = 2 \Rightarrow \frac{1}{p} = \frac{1}{2} \left( 2 - \frac{\alpha}{N} \right).$$

Consequently, in order to apply the immersion (7), we should have

$$pr \in [2, 2^*) \Rightarrow \frac{2N-\alpha}{N} \leq r \leq \frac{N}{N-2} \left( 2 - \frac{\alpha}{N} \right) = \frac{2N-\alpha}{N-2}.$$

3. The periodic problem

In this section we deal with problem (4). We observe that the energy functional $J_{A, V_p}$ on $H^1_{A, V_p}(\mathbb{R}^N, \mathbb{C})$ is given by

$$J_{A, V_p}(u) := \frac{1}{2} \|u\|^2_{H^1_{A, V_p}} - \frac{\lambda}{2p} B(u) - \frac{1}{2} \cdot 2^\alpha D(u),$$

where

$$B(u) = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * |u|^p \right) |u|^p = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^p |u(y)|^p}{|x-y|^\alpha} \, dx \, dy$$

and

$$D(u) = \int_{\mathbb{R}^N} \left( \frac{1}{|x|^\alpha} * |u|^{2^\alpha} \right) |u|^{2^\alpha} = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)|^{2^\alpha} |u(y)|^{2^\alpha}}{|x-y|^\alpha} \, dx \, dy.$$

Observe that

$$S_{H, L} = \inf_{u \in D^{1, 2}(\mathbb{R}^N) \setminus \{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx}{D(u)^{\frac{N}{N-2}}}.$$

The functional $J_{A, V_p}$ is well-defined as a consequence of the Hardy-Littlewood-Sobolev inequality (see [23], Theorem 4.3).

Proposition (4) implies that

$$|B(u)| \leq C_1(N, \alpha) \|u\|_{2^p}^{2^p}$$

and

$$|D(u)| \leq C_2(N, \alpha) \|u\|_{2^\alpha}^{2^\alpha},$$

where $C_1(N, \alpha)$ and $C_2(N, \alpha)$ are as given in Proposition (4). By (3), we know that $B$ and $D$ are well defined for $u \in H^1_{A, V_p}(\mathbb{R}^N, \mathbb{C})$. Consequently, $J_{A, V_p}$ is well defined.
Definition 3.1. A function \( u \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \) is a weak solution of (4) if
\[
\langle u, \varphi \rangle_{A,V_p} - \lambda \Re \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * |u|^p \right) |u|^{p-2} u \varphi - \Re \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2} u \varphi = 0
\]
for all \( \psi \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \).

Since the derivative of the energy functional \( J_{A,V_p} \) is given by
\[
J'_{A,V_p}(u) \cdot \varphi = \langle u, \varphi \rangle_{A,V_p} - \lambda \Re \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * |u|^p \right) |u|^{p-2} u \varphi - \Re \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} * |u|^{2^*_\alpha} \right) |u|^{2^*_\alpha - 2} u \varphi,
\]
we see that critical points of \( J_{A,V_p} \) are weak solutions of (4).

Note that, if \( \varphi = u \) we obtain
\[
J'_{A,V_p}(u) \cdot u := ||u||^2_{A,V_p} - \lambda B(u) - D(u).
\]

Lemma 5. The functional \( J_{A,V_p} \) satisfies the mountain pass geometry. Precisely,

(i) there exist \( \rho, \delta > 0 \) such that \( J_{A,V_p} \big|_S \geq \delta > 0 \) for any \( u \in S \), where
\[
S = \{ u \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) : ||u||_{A,V_p} = \rho \};
\]

(ii) for any \( u_0 \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \setminus \{ 0 \} \) there exists \( \tau \in (0, \infty) \) such that \( ||\tau u_0||_{A,V_p} > \rho \) \( e \) \( J_{A,V_p}(\tau u_0) < 0 \).

Proof. Inequalities (9) and (10) yields
\[
J_{A,V_p}(u) \geq \frac{1}{2} ||u||^2_{A,V_p} \frac{\lambda C_1(\alpha,N)}{2^p} ||u||^2_{A,V_p} - \frac{\lambda \alpha C_2(\alpha,N)}{2^p \cdot 2^*_\alpha} ||u||^{2^*_\alpha}_{A,V_p},
\]
thus implying (i) when we take \( ||u||_{A,V_p} = \rho \) sufficiently small.

In order to prove (ii), fix \( u_0 \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \setminus \{ 0 \} \) and consider the function \( g_{u_0} : (0, \infty) \rightarrow \mathbb{R} \) given by
\[
g_{u_0}(t) := J_{A,V_p}(tu_0) = \frac{1}{2} ||tu_0||^2_{A,V_p} - \frac{\lambda}{2^p} B(tu_0) - \frac{1}{2} \frac{\alpha}{2^*_\alpha} D(tu_0).
\]

We have
\[
B(tu_0) = t^{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^p |u_0(y)|^p}{|x-y|^\alpha} \, dx \, dy = t^{2p} B(u_0).
\]

and
\[
D(tu_0) = t^{2^*_\alpha} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_0(x)|^{2^*_\alpha} |u_0(y)|^{2^*_\alpha}}{|x-y|^\alpha} \, dx \, dy = t^{2^*_\alpha} D(u_0).
\]

Thus,
\[
g_{u_0}(t) = \frac{1}{2} t^{2\alpha} ||u_0||^2_{A,V_p} - \frac{\lambda}{2^p} t^{2p} B(u_0) - \frac{1}{2} \frac{\alpha}{2^*_\alpha} t^{2^*_\alpha} D(u_0)
\]
\[
= \frac{1}{2} t^{2^*_\alpha} \left( t^{(2^*-2^*_\alpha)} ||u_0||^2_{A,V_p} - \frac{\lambda}{p} t^{(2^*-2^*_\alpha)} B(u_0) - \frac{1}{2^*_\alpha} D(u_0) \right)
\]

Consequently,
\[
g_{u_0}(t) = \frac{1}{2} t^{2^*_\alpha} \left( \frac{||u_0||^2_{A,V_p}}{t^{(2(2^*_\alpha-1))}} \frac{\lambda}{p} \frac{B(u_0)}{t^{(2(2^*_\alpha-\rho))}} - \frac{1}{2^*_\alpha} D(u_0) \right).
\]

Since \( 1 < p < 2^*_\alpha \), we have
\[
\lim_{t \to +\infty} J_{A,V_p}(tu_0) = -\infty
\]
proving that there exists \( \tau \in (0, +\infty) \) sufficiently large such that
\[
||\tau u_0||_{A,V_p} > \rho \ \text{and} \ J_{A,V_p}(\tau u_0) < 0
\]
for all \( u_0 \in H^1_{A,V_p}(\mathbb{R}^N, \mathbb{C}) \setminus \{ 0 \} \).
The mountain pass Theorem without the PS condition (see [30, Theorem 1.15]) yields a Palais-Smale sequence \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) such that
\[
(12) \quad J'_{A,V}(u_n) \to 0 \quad \text{and} \quad J_{A,V}(u_n) \to c_\lambda,
\]
where \(c_\lambda = \inf_{\alpha \in \Gamma} \max_{t \in [0,1]} J_{A,V}(\gamma(t))\), and \(\Gamma = \{ \gamma \in C^1([0,1], H^1_{A,V}(\mathbb{R}^N, \mathbb{C})) : \gamma(0) = 0, J_{A,V}(\gamma(1)) < 0 \}\).

Lemma 6. There exists a unique \(t_u = t_u(u) > 0\) such that \(t_u u \in \mathcal{M}_{A,V}\) for all \(u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) \(\setminus \{0\}\) and \(J_{A,V}(t_u u) = \max_{t \geq 0} J_{A,V}(t u)\). Moreover \(c_\lambda = c_\lambda^* = c_\lambda^{**}\), where
\[
c_\lambda = \inf_{u \in \mathcal{M}_{A,V}} J_{A,V}(u),
\]
and
\[
c_\lambda^{**} = \inf_{u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} J_{A,V}(t u).
\]

**Proof.** Let \(u \in H^1_{A,V}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}\) and \(g_u\) defined on \((0, +\infty)\) given by
\[
g_u(t) = J_{A,V}(t u).
\]
By Lemma 5 there exists \(t_u > 0\) such that
\[
g_u(t_u) = \max_{t \geq 0} g_u(t) = \max_{t \geq 0} J_{A,V}(t u).
\]
Hence
\[
0 = g_u'(t_u) = J'_{A,V}(t_u u) \cdot u = J'_{A,V}(u_n u) \cdot u_n,
\]
implying that \(t_u u \in \mathcal{M}_{A,V}\). In the sequel, will show that \(t_u\) is unique. To this end, we suppose that there exists \(s_u > 0\) such that \(s_u u \in \mathcal{M}_{A,V}\). This way,
\[
(14) \quad \|u\|_{A,V}^2 = 0 = \inf_{\alpha \in \Gamma} \max_{t \in [0,1]} J_{A,V}(\gamma(t)) = \inf_{u \in \mathcal{M}_{A,V}} J_{A,V}(u)
\]
and
\[
(15) \quad \|u\|_{A,V}^2 = \inf_{u \in \mathcal{M}_{A,V}} J_{A,V}(u).
\]
Hence
\[
0 = \lambda \left( t^{2(p-1)} - s^{2(p-1)} \right) B(u) + \left( t^{2*_{2,1}} - s^{2*_{2,1}} \right) D(u).
\]
From \(\lambda > 0, 1 < p < 2_{*a}, B(u) > 0\) and \(D(u) > 0\), it follows that \(t_u = s_u\), since both terms in parentheses have the same sign if \(t_u \neq s_u\).

Now, the proof follows by using similar arguments found in [1, 13, 28] and [30]. □

Standard arguments prove the next affirmative:

Lemma 7. If \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\) is a \((PS)\) sequence for \(J_{A,V}\), then \((u_n)\) is bounded.

Lemma 8 (P.L. Lions). Let \(R > 0\) and \(2 \leq q < 2^*\) be. If \((u_n)\) is bounded in \(H^1(\mathbb{R}^N)\) and
\[
\sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^q \, dx \to 0, \quad n \to \infty,
\]
then \(u_n \to 0\) in \(L^p(\mathbb{R}^N)\) para \(2 < p < 2^*\).

**Proof.** See [30, Lemma. 1.21].

The following result presents an interesting property involving the \((PS)_{c_\lambda}\) sequences of \(J_{A,V}\).
Lemma 9. If \((u_n) \subset H^1_{A,V_F}(\mathbb{R}^N, \mathbb{C})\) is a sequence \((PS)_{c_\lambda}\) for \(J_{A,V_F}\) such that
\[
(16) \quad u_n \rightharpoonup 0 \quad \text{weakly in } H^1_{A,V_F}(\mathbb{R}^N, \mathbb{C}) \quad \text{as } n \to \infty,
\]
with
\[
(17) \quad c_\lambda < \frac{N + 2 - \alpha}{4N - 2\alpha} S^{rac{2N-\alpha}{H,L}},
\]
then there exists a sequence \((y_n) \in \mathbb{R}^N\) and constants \(R, \theta > 0\) such that
\[
\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^2 \, dx \geq \theta > 0
\]
where \(B_R(y)\) denotes the ball in \(\mathbb{R}^N\) of center \(y\) and radius \(R > 0\).

Proof. Suppose that the lemma does not hold, i.e., for all sequence \((y_n) \subset \mathbb{R}^N\) and for all \(R, \theta > 0\)
\[
\liminf_{n \to \infty} \int_{B_R(y_n)} |u_n|^2 \, dx < \eta.
\]
Then, Lemma 8 gives
\[
(18) \quad \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^2 \, dx \to 0, \quad \text{as } n \to \infty.
\]
for some \(y \in \mathbb{R}^N\).

From [13] we conclude \(u_n \rightharpoonup 0 \) as \(n \to \infty\) in \(L^q(\mathbb{R}^N)\), for all \(q \in (2, 2^*)\).

We have
\[
0 \leq B(u_n) \leq C_1(N, \alpha) \|u_n\|^{2p}_{L^{p}(\mathbb{R}^N)}.
\]
Since \(2p \in (2, 2^*)\), it follows
\[
(19) \quad B(u_n) \to 0 \quad \text{quando } n \to \infty.
\]
From (11) we have
\[
(20) \quad \|u_n\|^2_{A,V_F} = J'_{A,V_F}(u_n) \cdot u_n + \lambda B(u_n) + D(u_n).
\]
Since \(J'_{A,V_F}(u_n)u_n = O(1)\) as \(n \to \infty\), then, by (19), we have
\[
(21) \quad \|u_n\|^2_{A,V} = O_n(1) + D(u_n)
\]
Suppose that
\[
(22) \quad \|u_n\|^2_{A,V} \to l, \quad l > 0 \quad \text{as } n \to \infty.
\]
From (21) and (22) we also have \(D(u_n) \to l\).

Since \(J_{A,V_F}(u_n) \to c_\lambda\), as \(n \to \infty\), it follows that
\[
c_\lambda = l \left( \frac{N + 2 - \alpha}{4N - 2\alpha} \right).
\]
On the other hand, it follows from
\[
\|u_n\|^2_{A,V_F} \geq \int_{\mathbb{R}^N} |\nabla u_n|^2 \, dx \geq S_{H,L}(D(u_n))^{rac{N-2}{2N-\alpha}}, \quad \forall u \in D^{1,2}(\mathbb{R}^N),
\]
that \(l \geq S_{H,L}^\frac{N-2}{2N-\alpha}\).

Since \(c_\lambda = \left( \frac{4N-2\alpha}{N+2-\alpha} \cdot c_\lambda \right) l\), using the inequality above, we have \(c_\lambda \geq \frac{N+2-\alpha}{4N-2\alpha} S_{H,L}^\frac{2N-\alpha}{H,L}\), which is a contradiction. Then, the lemma holds. \(\square\)

Lemma 10. Let \(U \subseteq \mathbb{R}^N\) be any open set. For \(1 < p < \infty\), let \((f_n)\) be a bounded sequence in \(L^p(U, \mathbb{C})\) such that \(f_n(x) \to f(x)\) a.e. Then \(f_n \to f\) in \(L^p(U, \mathbb{C})\).
The proof of Lemma 11 follows by adapting the arguments given for the real case, as in [22, Lemmte 4.8, Chapitre 1].

**Lemma 11.** Suppose that \( u_n \to u_0 \) in \( H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C}) \) and \( u_n(x) \to u_0(x) \) a.e \( x \in \mathbb{R}^N \). Then
\[
\frac{1}{|x|^{\alpha}} |u_n|^p \to \frac{1}{|x|^{\alpha}} |u_0|^p \text{ in } L^{\frac{2N-\alpha}{N}}(\mathbb{R}^N),
\]
for all \( \frac{2N-\alpha}{N} < p \leq 2^*_\alpha \).

**Proof.** In this proof we adapt some ideas of [12].

From (5) we have that \( |u_n|^p \) is bounded in \( L^{\frac{2N}{N+2}}(\mathbb{R}^N) \). Since we can suppose that \( u_n(x) \to u_0(x) \) a.e. in \( \mathbb{R}^N \), it follows that \( |u_n(x)|^p \to |u_0(x)|^p \). From Lemma 11 it follows
\[
|u_n(x)|^p \to |u_0(x)|^p \text{ in } L^{\frac{2N}{N+2}}(\mathbb{R}^N, \mathbb{C}).
\]

As a consequence of the Hardy-Littlewood-Sobolev inequality, we have that
\[
\frac{1}{|x|^{\alpha}} * w(x) \in L^{\frac{2N}{N+2}}(\mathbb{R}^N)
\]
for all \( w \in L^{\frac{2N}{N+2}}(\mathbb{R}^N) \). Thus, the Riesz potential defines a linear continuous \( L^{\frac{2N}{N+2}}(\mathbb{R}^N) \) to \( L^{\frac{2N}{N+2}}(\mathbb{R}^N) \). A new application of Lemma 11 yields (23).

**Corollary 12.** Suppose that \( u_n \to u_0 \) and consider
\[
B'(u_n) \cdot \psi = \mathcal{R} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} |u_n|^p \right) |u_n|^{p-2} u_n \psi dx
\]
and
\[
D'(u_n) \cdot \psi = \mathcal{R} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} |u_n|^2 \right) |u_n|^{2-2} u_n \psi dx,
\]
for \( \psi \in C_c^\infty(\mathbb{R}^N, \mathbb{C}) \). Then \( B'(u_n) \cdot \psi \to B'(u_0) \cdot \psi \) and \( D'(u_n) \cdot \psi \to D'(u_0) \cdot \psi \).

Consequently, \( u_0 \) is a weak solution of problem (11).

**Proof.** It follows from (5) that \( |u_n|^p u_n \) is bounded in \( L^{\frac{2N}{N+2}}(\mathbb{R}^N, \mathbb{C}) \). Since we can suppose that \( u_n(x) \to u_0(x) \) a.e. in \( \mathbb{R}^N \), it follows that \( |u_n(x)|^p \to |u_0(x)|^p \). By applying Lemma 11 we conclude, for all \( \frac{2N-\alpha}{N} < p \leq 2^*_\alpha \),
\[
|u_n|^p u_n \to |u_0|^p u_0 \text{ in } L^{\frac{2N}{N+2}}(\mathbb{R}^N, \mathbb{C}).
\]
as \( n \to +\infty \).

Combining (22) with (23) yields
\[
\left( \frac{1}{|x|^{\alpha}} |u_n|^p \right) |u_n|^{p-2} u_n \to \left( \frac{1}{|x|^{\alpha}} |u_0|^p \right) |u_0|^{p-2} u_0 \text{ in } L^{\frac{2N}{N+2}}(\mathbb{R}^N)
\]
if \( n \to +\infty \), for all \( \frac{2N-\alpha}{N} < p \leq 2^*_\alpha \). Consequently, for \( \psi \in C_c^\infty(\mathbb{R}^N, \mathbb{C}) \), it follows that
\[
\mathcal{R} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} |u_n|^p \right) |u_n|^{p-2} u_n \psi dx \to \mathcal{R} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} |u_0|^p \right) |u_0|^{p-2} u_0 \psi dx
\]
and
\[
\mathcal{R} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} |u_n|^2 \right) |u_n|^{2-2} u_n \psi dx \to \mathcal{R} \int_{\mathbb{R}^N} \left( \frac{1}{|x|^{\alpha}} |u_0|^2 \right) |u_0|^{2-2} u_0 \psi dx,
\]
that is
\[
B'(u_n) \cdot \psi \to B'(u_0) \cdot \psi \quad \text{and} \quad D'(u_n) \cdot \psi \to D'(u_0) \cdot \psi.
\]
Since for \( \psi \in C_c^\infty(\mathbb{R}^N, \mathbb{C}) \),
\[
J'_{A,V}(u_n) \cdot \psi = \langle u_n, \psi \rangle_{A,V,p} - \lambda B'(u_n) \cdot \psi - D'(u_n) \cdot \psi \to 0,
\]
passing to the limit when $n \to +\infty$, we obtain
\[
\langle u_0, \psi \rangle_{A,V,p} = \lambda B'(u_0) \cdot \psi + D'(u_0) \cdot \psi
\]
for any $\psi \in C_{c}^{\infty}(\mathbb{R}^N, \mathbb{C})$, which means $u_0$ is a weak solution of $\mathfrak{M}$. \hfill \Box

Lemma 13. There exists $u_\varepsilon$ such that
\[
\sup_{t \geq 0} J_{A,V,p}(tu_\varepsilon) < \frac{N + 2 - \alpha}{4N - 2\alpha} (S_{H,L})^{\frac{2N-\alpha}{N-2}}.
\]
provided that either
(i) $\frac{N+2-\alpha}{N-2} < p < 2^{*}_\alpha$, $N = 3,4$ and $\lambda > 0$;
(ii) $\frac{2N-\alpha}{N-2} < p \leq \frac{N+2-\alpha}{N-2}$, $N = 3,4$ and $\lambda$ sufficiently large;
(iii) $\frac{2N-2-\alpha}{N-2} < p < 2^{*}_\alpha$, $N \geq 5$ and $\lambda > 0$;
(iv) $\frac{2N-\alpha}{N-2} < p \leq \frac{2N-2-\alpha}{N-2}$, $N \geq 5$ and $\lambda$ sufficiently large.

The arguments of this proof were adapted from the articles \cite{[19]} \cite{[24]}.

Proof. From Theorem 1.42 in \cite{[30]}, we know that $U(x) = \frac{N(N-2)}{(1+|x|^2)^{\frac{N-2}{2}}}$ is a minimizer for $S$, the best Sobolev constant. By Proposition \cite{[3]} we know that $U(x)$ is also a minimizer for $S_{H,L}$. If $B_r$ denotes the ball in $\mathbb{R}^N$ of center at origin and radius $r$, consider the balls $B_{\delta}$ and $B_{2\delta}$ and take $\psi \in C_{c}^{\infty}(\mathbb{R}^N)$ such that, for a constant $C$,
\[
\psi(x) = \begin{cases} 
1, & \text{if } x \in B_{\delta}, \\
0, & \text{if } x \in \mathbb{R}^N \setminus B_{2\delta},
\end{cases} \quad 0 < |\psi(x)| \leq 1, \quad \forall \ x \in \mathbb{R}^N, \quad |D\psi(x)| \leq C \quad \forall \ x \in \mathbb{R}^N.
\]

We define, for $\varepsilon > 0$,
\[
U_\varepsilon(x) := \varepsilon^{(2-N)/2}U \left( \frac{x}{\varepsilon} \right) \quad \text{and} \quad u_\varepsilon(x) := \psi(x)U_\varepsilon(x)
\]

From \cite{[20]}, we have that
\[
\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \, dx = C(N,\alpha) \frac{N-2}{2} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})
\]
and
\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u_\varepsilon(x)|^2 |u_\varepsilon(y)|^{2\alpha}}{|x-y|^\alpha} \, dx \, dy \geq C(N,\alpha) \frac{N}{2} S_{H,L}^{\frac{N-\alpha}{N-2}} - O(\varepsilon^{N-\frac{N}{2}}).
\]

Consider the function $f : [0, +\infty) \to \mathbb{R}$ defined by
\[
f(t) = J_{A,V,p}(tu_\varepsilon) = \frac{t^2}{2} \|u_\varepsilon\|_{A,V,p}^2 - \frac{\lambda t^{2p}}{2p} B(u_\varepsilon) - \frac{t^{2-2\alpha}}{2 \cdot 2^\alpha} D(u_\varepsilon).
\]

We have $f(0) = 0$, $f(t) > 0$ for all $t > 0$ sufficiently small and $\lim_{t \to +\infty} f(t) = -\infty$. Thus, there exists $t_\varepsilon > 0$ such that $f(t) = f(t_\varepsilon)$, that is, $\sup_{t \geq 0} J_{A,V,p}(tu_\varepsilon) = J_{A,V,p}(t_\varepsilon u_\varepsilon)$.

Since $t_\varepsilon > 0$, $B(u_\varepsilon) > 0$ and $f'(t_\varepsilon) = 0$, we obtain
\[
0 < t_\varepsilon < \left( \frac{\|u_\varepsilon\|_{A,V,p}^2}{D(u_\varepsilon)} \right)^{\frac{2\alpha}{N-\alpha}} := S_{H,L}(\varepsilon) \quad \Rightarrow \quad \|u_\varepsilon\|_{A,V,p}^2 = D(u_\varepsilon) (S_{H,L}(\varepsilon))^{2(2_\varepsilon^*-1)}.
\]

Define $g : [0, S_{H,L}(\varepsilon)] \to \mathbb{R}$ by
\[
g(t) = \frac{t^2}{2} \|u_\varepsilon\|_{A,V,p}^2 - \frac{t^{2-2\alpha}}{2 \cdot 2^\alpha} D(u_\varepsilon) = D(u_\varepsilon) \left( \frac{t^2}{2} (S_{H,L}(\varepsilon))^{2(2_\varepsilon^*-1)} - \frac{t^{2-2\alpha}}{2 \cdot 2^\alpha} \right).
\]

Since $t > 0$ and $D(u_\varepsilon) > 0$, it follows that $g'(t) > 0$, and, consequently, $g$ is increasing in this interval. Thus,
\[
0 < g(t_\varepsilon) < \frac{N + 2 - \alpha}{2(2N - \alpha)} D(u_\varepsilon)(S_{H,L}(\varepsilon))^{2-2_\varepsilon^*}.
\]
From (29) follows
\[
D(u_\varepsilon)(S_{H,L}(\varepsilon))^{2\alpha - \frac{2N}{N+2}} = \left(\frac{\|u_\varepsilon\|_{A,V_P}^2}{D(u_\varepsilon)}\right)^{2N-\alpha}.
\]

Thus,
\[
0 < g(t_\varepsilon) < \frac{N + 2 - \alpha}{2(2N - \alpha)} \left(\frac{\|u_\varepsilon\|_{A,V_P}^2}{D(u_\varepsilon)}\right)^{\frac{2N-\alpha}{N+2}}.
\]

Since \(J_{A,V_P}(t) = g(t) - \frac{\lambda}{2p} t^2 B(u_\varepsilon)\), we obtain
\[
J_{A,V_P}(t_\varepsilon) < N + 2 - \alpha \left(\frac{\|u_\varepsilon\|_{A,V_P}^2}{D(u_\varepsilon)}\right)^{\frac{2N-\alpha}{N+2}} - \frac{\lambda}{2p} t_\varepsilon^2 B(u_\varepsilon).
\]

But \(\|u_\varepsilon\|_{A,V_P}^2 = \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx + \int_{\mathbb{R}^N} (|A(x)|^2 + V_P(x)|u_\varepsilon|^2) dx\) implies that
\[
\frac{\|u_\varepsilon\|_{A,V_P}^2}{D(u_\varepsilon)^{\frac{2N-\alpha}{N+2}}} = \frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx
\]
\[
+ \frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_P(x)|u_\varepsilon|^2) dx.
\]

Therefore, (31) and (32) allow us to conclude that
\[
J_{A,V_P}(t_\varepsilon) < \frac{N + 2 - \alpha}{2(2N - \alpha)} \left(\frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \right)^{\frac{2N-\alpha}{N+2}} - \frac{\lambda}{2p} t_\varepsilon^2 B(u_\varepsilon).
\]

The Mean Value Theorem implies that, for all \(\beta \geq 1\) and for any \(a, b > 0\), we have
\[
(a + b)^\beta \leq a^\beta + \beta(a + b)^{\beta-1} b.
\]

Thus, considering
\[
a = \frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx, \quad b = \frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_P(x)|u_\varepsilon|^2) dx \quad \text{and} \quad \beta = \frac{N - 2}{2N - \alpha},
\]
it follows from (33) that
\[
J_{A,V_P}(t_\varepsilon) < \frac{N + 2 - \alpha}{2(2N - \alpha)} \left[ \left(\frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \right)^{\frac{2N-\alpha}{N+2}} \right]
\]
\[
+ \frac{2N - \alpha}{N + 2 - \alpha} \left(\frac{1}{(D(u_\varepsilon))^{\frac{N-2}{N+2}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx + \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{N+2}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_P(x)|u_\varepsilon|^2) dx \right)^{\frac{N-2}{N+2}}
\]
\[
\cdot \left[ \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{N+2}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_P(x)|u_\varepsilon|^2) dx \right] - \frac{\lambda}{2p} t_\varepsilon^2 B(u_\varepsilon).
\]

Taking into account (27) and (28), we conclude that
\[
\frac{1}{(D(u_\varepsilon))^{\frac{2N-\alpha}{N+2}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 dx \leq \frac{(C(N, \alpha))^{\frac{2N-\alpha}{2N}} \cdot \frac{N}{2} S_{H,L} \cdot O(\varepsilon^{N-2})}{(C(N, \alpha)^{\frac{2N-\alpha}{2N}} S_{H,L} \cdot O(\varepsilon^{\frac{2N-\alpha}{2N}}))^{\frac{2N-\alpha}{N+2}}},
\]
an inequality that implies

\[
\left( \frac{1}{(D(u_\varepsilon))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \, dx \right)^{\frac{2N-\alpha}{N+2-\alpha}} \leq \left( \frac{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} S_{H,L}^{\frac{N}{2}} - O(\varepsilon^{2N-\alpha})} \right)^{\frac{2N-\alpha}{N+2-\alpha}}.
\]

Since

\[
\frac{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} S_{H,L}^{\frac{N}{2}} - O(\varepsilon^{2N-\alpha})} = \frac{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} (S_{H,L})^{\frac{N}{2}} + O(\varepsilon^{N-2})}{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} (S_{H,L})^{\frac{N}{2}} - O(\varepsilon^{2N-\alpha})}
\]

we conclude that

\[
\left( \frac{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} S_{H,L}^{\frac{N}{2}} + O(\varepsilon^{N-2})}{(C(N, \alpha))^{\frac{N-2}{2N-\alpha}} S_{H,L}^{\frac{N}{2}} - O(\varepsilon^{2N-\alpha})} \right)^{\frac{2N-\alpha}{N+2-\alpha}} = (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}} \cdot \frac{1 + O(\varepsilon^{N-2})}{\left(1 - O\left(\varepsilon^{2N-\alpha}\right)\right)^{\frac{2N-\alpha}{N+2-\alpha}}}.
\]

On the other hand,

\[
\frac{1 + O(\varepsilon^{N-2})}{\left(1 - O\left(\varepsilon^{2N-\alpha}\right)\right)^{\frac{2N-\alpha}{N+2-\alpha}}} = \frac{(1 - O(\varepsilon^{2N-\alpha}))(1 - O(\varepsilon^{2N-\alpha}))}{\left(1 - O\left(\varepsilon^{2N-\alpha}\right)\right)^{\frac{2N-\alpha}{N+2-\alpha}}}
\]

\[
= (1 - O(\varepsilon^{2N-\alpha}))^{1 - \frac{N-2}{2N-\alpha}} + O(\varepsilon^{N-2}) - O(\varepsilon^{2N-\alpha})
\]

\[
= (1 - O(\varepsilon^{2N-\alpha}))^{\frac{N-2}{2N-\alpha}} + O(\varepsilon^{N-2}) - O(\varepsilon^{2N-\alpha})
\]

From this it follows that

\[
\left( \frac{1 + O(\varepsilon^{N-2})}{\left(1 - O\left(\varepsilon^{2N-\alpha}\right)\right)^{\frac{2N-\alpha}{N+2-\alpha}}} \right)^{\frac{2N-\alpha}{N+2-\alpha}}
\]
where in the penultimate inequality we use the Mean Value Theorem considering

\[ a = \left(1 - O\left(\varepsilon^{2N-\alpha}\right)\right)^{\frac{N+2-\alpha}{N-\alpha}}, \quad b = \frac{O(\varepsilon^{N-2}) + O\left(\varepsilon^{2N-\alpha}\right)}{\left(1 - O\left(\varepsilon^{2N-\alpha}\right)\right)^{\frac{N-\alpha}{N+2-\alpha}}} \]

and \( \beta = \frac{2N - \alpha}{N + 2\alpha} \).

For \( \varepsilon > 0 \) sufficiently small, we have

\[ (1 - O(\varepsilon^{N-2,2N-\alpha})) \geq \frac{1}{2}, \]

which implies that

\[ \left(1 + O(\varepsilon^{N-2})\right)^{\frac{2N-\alpha}{N+2-\alpha}} \leq 1 + 2C(N, \alpha) \left(O\left(\varepsilon^{N-2}\right) + O\left(\varepsilon^{2N-\alpha}\right)\right) < 1 + O\left(\varepsilon^{\min(N-2,2N-\alpha)}\right). \]

Thus, it follows from (35), (36) and (37) that, for any \( \varepsilon > 0 \) sufficiently small, we have

\[ \left(S_{H,L}\right)^{\frac{2N-\alpha}{N+2-\alpha}} \leq (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}} \cdot \left(1 + O(\varepsilon^{N-2})\right)^{\frac{2N-\alpha}{N+2-\alpha}} < (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}} + O\left(\varepsilon^{\min(N-2,2N-\alpha)}\right). \]

Combining (38) with (36), for \( \varepsilon \) sufficiently small, we have

\[ J_{A,p}(t_{\varepsilon}u_{\varepsilon}) < \frac{N + 2 - \alpha}{2(2N - \alpha)} (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}} + O\left(\varepsilon^{\min(N-2,2N-\alpha)}\right) \]

\[ \frac{1}{2} \left(\frac{1}{(D(u_{\varepsilon}))^{\frac{2N-\alpha}{2N-\alpha}}} \int_{\mathbb{R}^N} |\nabla u_{\varepsilon}|^2 \, dx + \frac{1}{(D(u_{\varepsilon}))^{\frac{2N-\alpha}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x))|u_{\varepsilon}|^2 \, dx \right) \]

\[ + \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{1}{(D(u_{\varepsilon}))^{\frac{N-2}{2N-\alpha}}} \int_{\mathbb{R}^N} (|A(x)|^2 + V_{\mathcal{P}}(x))|u_{\varepsilon}|^2 \, dx \right) - \frac{\lambda}{2p} t_{\varepsilon}^{2p} B(u_{\varepsilon}). \]

We claim that there is a positive constant \( C_0 \) such that, for all \( \varepsilon > 0 \)

\[ t_{\varepsilon}^{2p} \geq C_0. \]
In fact, suppose that there is a sequence \((\varepsilon_n) \subset \mathbb{R}, \varepsilon_n \to 0\) as \(n \to \infty\), such that \(t_{\varepsilon_n} \to 0\) as \(n \to \infty\). Thus,
\[
0 < C_\Lambda \leq \sup_{t \geq 0} J_{A,V}(tu_{\varepsilon_n}) = J_{A,V}(t_{\varepsilon_n} u_{\varepsilon_n}).
\]
Since \(u_{\varepsilon_n} \in H^1_{A,\nu_p}(\mathbb{R}^N, \mathbb{C})\) is bounded and \(t_{\varepsilon_n} \to 0\), as \(n \to \infty\), we have \(t_{\varepsilon_n} u_{\varepsilon_n} \to 0\) as \(n \to \infty\), em \(H^1_{A,\nu_p}(\mathbb{R}^N, \mathbb{C})\).

From the continuity of \(J_{A,V}\) it follows that \(J_{A,V}(t_{\varepsilon_n} u_{\varepsilon_n}) \to J_{A,V}(0) = 0\).

Therefore
\[
0 < C_\Lambda \leq \lim_{n \to \infty} J_{A,V}(t_{\varepsilon_n} u_{\varepsilon_n}) = 0,
\]
which is a contradiction that proves the claim.

From (39) e (40) we conclude that, for some constant \(C_0 > 0\) and \(\varepsilon > 0\) sufficiently small
\[
J_{A,V}(t_{\varepsilon} u_{\varepsilon}) < \frac{N + 2 - \alpha}{2(2N - \alpha)} (S_{H,L})^{\frac{2N}{N + 2 - \alpha}} + O(\varepsilon^\eta) + C_1 \int_{\mathbb{R}^N} a(x)|u_{\varepsilon}|^2 dx - C_0 B(u_{\varepsilon}),
\]
where \(a(x) = |A(x)|^2 + V_p(x)\) and \(\eta = \min\{N - 2, \frac{2N - \alpha}{2}\}\).

By direct computation we know that, for \(\varepsilon < 1\),
\[
B(u_{\varepsilon}) = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varepsilon^{(2N-2)p} |N(N-2)|^{\frac{2N-2}{4}}}{(1 + |x|^2)^\frac{N-2}{2}} |x - y|^\alpha (1 + |y|^2)^\frac{N-2}{2} dx dy
\]
\[
= [N(N-2)]^{\frac{2N-2}{4}} \varepsilon^{2N-\alpha-(N-2)p} \int_{B_{\frac{\varepsilon}{2}}} \int_{B_{\frac{\varepsilon}{2}}} \frac{1}{(1 + |x|^2)^\frac{N-2}{2}} |x - y|^\alpha (1 + |y|^2)^\frac{N-2}{2} dx dy
\]
\[
\geq [N(N-2)]^{\frac{2N-2}{4}} \varepsilon^{2N-\alpha-(N-2)p} \int_{B_{\frac{\varepsilon}{2}}} \int_{B_{\frac{\varepsilon}{2}}} \frac{1}{(1 + |x|^2)^\frac{N-2}{2}} |x - y|^\alpha (1 + |y|^2)^\frac{N-2}{2} dx dy.
\]
Thus,
\[
B(u_{\varepsilon}) \geq [N(N-2)]^{\frac{2N-2}{4}} \varepsilon^{2N-\alpha-(N-2)p} \int_{B_{\frac{\varepsilon}{2}}} \int_{B_{\frac{\varepsilon}{2}}} \frac{1}{(1 + |x|^2)^\frac{N-2}{2}} |x - y|^\alpha (1 + |y|^2)^\frac{N-2}{2} dx dy.
\]

From (41) and (42) we obtain
\[
J_{A,V}(t_{\varepsilon} u_{\varepsilon}) < \frac{N + 2 - \alpha}{4N - 2\alpha} (S_{H,L})^{\frac{2N}{N + 2 - \alpha}} + O(\varepsilon^\eta) + C_2 \int_{\mathbb{R}^N} |u_{\varepsilon}(x)|^2 dx - C_3 \varepsilon^{2N-\alpha-(N-2)p}.
\]

We are going to show that
\[
\lim_{\varepsilon \to 0} \varepsilon^{-\eta} \left( C_2 \int_{\mathbb{R}^N} |u_{\varepsilon}(x)|^2 dx - C_3 \varepsilon^{2N-\alpha-(N-2)p} \right) = -\infty.
\]

In order to do that, it suffices to show that
\[
\lim_{\varepsilon \to 0} \varepsilon^{-\eta} \left( C_2 \int_{B_{\frac{\varepsilon}{2}}} |u_{\varepsilon}(x)|^2 dx - C_3 \varepsilon^{2N-\alpha-(N-2)p} \right) = -\infty
\]
and
\[
C_2 \int_{B_{\frac{\varepsilon}{2}} \setminus B_{\varepsilon}} |u_{\varepsilon}(x)|^2 dx - C_3 \varepsilon^{2N-\alpha-(N-2)p} = O(\varepsilon^\eta).
\]

Assuming (45) and (46), let us proceed with our proof.
Since
\[ O(\varepsilon^N) + C_2 \int_{\mathbb{R}^N} |u_\varepsilon|^2 \, dx - C_3 \varepsilon^{2(N-\alpha - (N-2)p} = \varepsilon^{\eta} \left[ \frac{O(\varepsilon^N)}{\varepsilon^0} + \varepsilon^{-\eta} \left( C_2 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2(N-\alpha - (N-2)p} \right) \right], \]
from (44) we conclude that
\[ (47) \quad O(\varepsilon^N) + C_2 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2(N-\alpha - (N-2)p} < 0 \]
for \( \varepsilon > 0 \) sufficiently small.

Thus, from (43) and (47) we have
\[ \sup_{t \geq 0} J_{A,V_p}(tu_\varepsilon) < \frac{N+2-\alpha}{4N-2\alpha} \left( S_{H,L} \right)^{\frac{2N-\alpha}{N+2-\alpha}}, \]
for \( \varepsilon > 0 \) sufficiently small and fixed.

Now, we are going to consider the following cases:

1. \( \frac{N+2-\alpha}{N+2-\alpha} < p < \frac{N+2-\alpha}{N+2-\alpha} \) para \( N = 3, 4 \) and \( \lambda \) sufficiently large;
2. \( \frac{2N-\alpha}{N+2-\alpha} < p < \frac{2N-\alpha}{N+2-\alpha} \) para \( N \geq 5 \) and \( \lambda \) large enough.

Let \( \varepsilon > 0 \) fixed in (20) and \( g_\lambda : [0, +\infty) \to \mathbb{R} \) given by
\[ g_\lambda(t) := J_{A,V_p}(tu_\varepsilon) = \frac{t^2}{2} \int_{\mathbb{R}^N} \left[ |\nabla u_\varepsilon|^2 + (|A(x)|^2 + V_p(x)) |u_\varepsilon|^2 \right] \, dx - \frac{\lambda}{2p} t^{2p} B(u_\varepsilon) - \frac{1}{2} t^{2\alpha} D(u_\varepsilon). \]

From
\[ \lim_{t \to +\infty} g_\lambda(t) = -\infty \]
when \( t \to +\infty \), we have that \( \max g_\lambda(t) \) is attained at some \( t_\lambda > 0 \) and \( t_\lambda \) satisfies
\[ t_\lambda \int_{\mathbb{R}^N} \left[ |\nabla u_\varepsilon|^2 + (|A(x)|^2 + V_p(x)) |u_\varepsilon|^2 \right] \, dx = \frac{\lambda t_\lambda^{2p-1} B(u_\varepsilon)}{2p} + t_\lambda^{2\alpha-1} D(u_\varepsilon) \]
that is
\[ \int_{\mathbb{R}^N} \left[ |\nabla u_\varepsilon|^2 + (|A(x)|^2 + V_p(x)) |u_\varepsilon|^2 \right] \, dx = \lambda t_\lambda^{2(p-1)} B(u_\varepsilon) + t_\lambda^{2(\alpha-1)} D(u_\varepsilon), \]
since \( g_\lambda'(t_\lambda) = 0 \). Thus \( t_\lambda \to 0 \) as \( \lambda \to +\infty \) and
\[ \max_{t \geq 0} J_{A,V_p}(tu_\varepsilon) = J_{A,V_p}(t_\lambda u_\varepsilon) \]
\[ = \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} \left[ |\nabla u_\varepsilon(x)|^2 + (|A(x)|^2 + V_p(x)) |u_\varepsilon(x)|^2 \right] \, dx - \frac{\lambda}{2p} t_\lambda^{2p} B(u_\varepsilon) - \frac{1}{2} t_\lambda^{2\alpha} D(u_\varepsilon) \]
\[ < \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} \left[ |\nabla u_\varepsilon|^2 + (|A(x)|^2 + V_p(x)) |u_\varepsilon(x)|^2 \right] \, dx \]

Since \( t_\lambda \to 0 \) as \( \lambda \to +\infty \) and \( \frac{N+2-\alpha}{4N-2\alpha} (S_{H,L})^{\frac{2N-\alpha}{N+2-\alpha}} > 0 \), we conclude that
\[ \frac{t_\lambda^2}{2} \int_{\mathbb{R}^N} \left[ |\nabla u_\varepsilon|^2 + (|A(x)|^2 + V_p(x)) |u_\varepsilon(x)|^2 \right] \, dx < \frac{N+2-\alpha}{2(2N-\alpha)} \left( S_{H,L} \right)^{\frac{2N-\alpha}{N+2-\alpha}}, \]
for \( \lambda > 0 \), sufficiently small.

Therefore
\[ \sup_{t \geq 0} J_{A,V_p}(tu_\varepsilon) < \frac{N+2-\alpha}{4N-2\alpha} \left( S_{H,L} \right)^{\frac{2N-\alpha}{N+2-\alpha}}, \]
for \( \lambda > 0 \) sufficiently small. \( \square \)

We now prove (65).

**Lemma 14.** It holds
\[ \lim_{\varepsilon \to 0} \varepsilon^{-\eta} \left( C_2 \int_{B_\delta} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2(N-\alpha - (N-2)p} \right) = -\infty. \]
Proof. We initially observe that direct computation allows us to conclude that
\begin{equation}
\int_{B_\delta} |u_\varepsilon(x)|^2 dx = N \omega_N [N(N - 2)]^{\frac{N-2}{2}} \varepsilon^2 \int_0^\frac{\delta}{\varepsilon} \frac{r^{N-1}}{(1 + r^2)^{N-2}} dr,
\end{equation}
where $\omega_N$ denotes the volume of the unit ball in $\mathbb{R}^N$.

Now, we define
\begin{align*}
I_\varepsilon &= \varepsilon^{-n} \left( C_2 \int_{B_\delta} |u_\varepsilon(x)|^2 dx - C_3 \varepsilon^{2N - \alpha - (N - 2)p} \right) \\
&= \varepsilon^{-n} \left( C_4 \varepsilon^2 \int_0^\frac{\delta}{\varepsilon} \frac{r^{N-1}}{(1 + r^2)^{N-2}} dr - C_3 \varepsilon^{2N - \alpha - (N - 2)p} \right),
\end{align*}
taking into account (48).

It is easy to show that
\begin{equation}
\varepsilon^2 \int_0^\frac{\delta}{\varepsilon} \frac{r^2}{1 + r^2} dr = \varepsilon \left( \delta - \varepsilon \arctan \left( \frac{\delta}{\varepsilon} \right) \right).
\end{equation}

We now consider different cases.

- **The case $N = 3$.** In this case, we have $0 < \alpha < 3$ and $\frac{6 - \alpha}{3} < p < 6 - \alpha$.

Since $0 < \alpha < 3$, it follows that
\begin{align*}
I_\varepsilon &= \varepsilon^{-1} \left[ C_4 \varepsilon \left( \delta - \varepsilon \arctan \left( \frac{\delta}{\varepsilon} \right) \right) - C_3 \varepsilon^{6 - \alpha - p} \right] \\
&= C_4 \left( \delta - \varepsilon \arctan \left( \frac{\delta}{\varepsilon} \right) \right) - C_3 \varepsilon^{5 - p - \alpha}.
\end{align*}

Considering $5 - \alpha < p < 6 - \alpha$ and using again $0 < \alpha < 3$ we ensure that $\frac{6 - \alpha}{3} < 5 - \alpha$ and $5 - p - \alpha < 0$. Therefore, if $N = 3$ and $5 - \alpha < p < 6 - \alpha$, we deduce that $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

- **The case $N = 4$.** In this case, we have $0 < \alpha < 4$ and $\frac{8 - \alpha}{4} < p < \frac{8 - \alpha}{2}$.

Changing variables, we obtain
\begin{equation}
\varepsilon^2 \int_0^\frac{\delta}{\varepsilon} \frac{r^3}{(1 + r^2)^2} dr = \frac{\varepsilon^2}{2} \left[ \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right) + \frac{\varepsilon^2}{\varepsilon^2 + \delta^2} - 1 \right]
\end{equation}

Since $0 < \alpha < 4$, it follows
\begin{align*}
I_\varepsilon &= \varepsilon^{-\min\left(2, \frac{8 - \alpha}{2}\right)} \left[ C_4 \varepsilon^2 \left( \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right) + \frac{\varepsilon^2}{\varepsilon^2 + \delta^2} - 1 \right) - C_3 \varepsilon^{8 - \alpha - 2p} \right] \\
&= \varepsilon^{-2} \left[ \frac{C_4}{2} \varepsilon^{2} \left( \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right) + \frac{\varepsilon^2}{\varepsilon^2 + \delta^2} - 1 \right) - C_3 \varepsilon^{-2} \varepsilon^{8 - \alpha - 2p} \right] \\
&= \frac{C_4}{2} \left( \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right) + \frac{\varepsilon^2}{\varepsilon^2 + \delta^2} - 1 \right) - C_3 \varepsilon^{5 - \alpha - 2p} \\
&= \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right) \left[ \frac{C_4}{2} + \frac{C_4}{2 \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right) \varepsilon^2}{\varepsilon^2 + \delta^2} - \frac{C_4}{2 \ln \left( 1 + \frac{\delta^2}{\varepsilon^2} \right)} - C_3 \varepsilon^{6 - \alpha - 2p} \right]
\end{align*}

Considering $\frac{6 - \alpha}{2} < p < \frac{8 - \alpha}{2}$, we obtain $\frac{8 - \alpha}{4} < p < \frac{6 - \alpha}{2}$ and $6 - 2p - \alpha < 0$, since $0 < \alpha < 4$. Therefore, if $N = 4$ and $\frac{6 - \alpha}{2} < p < \frac{8 - \alpha}{2}$ we deduce that $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$.

**Remark 3.1.** Since $0 < \alpha < N$, it is easy to see that $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$ for $N = 3, 4$ and $\frac{N + 2 - \alpha}{N - 2} < p < \frac{2N - \alpha}{N - 2}$. Moreover, if $N = 3, 4$ then $N > \frac{2(p + 1 - \alpha)}{p - 1}$.

- **The case $N \geq 5$.** There are two cases to be considered: a) $0 < \alpha < 4$ and $N \geq 5$ and b) $\alpha \geq 4$ and $N \geq 5$.

Let us consider initially the case $0 < \alpha < 4$ and $N \geq 5$. 


Since $0 < \alpha < 4$ we see that

$$N - 2 < \frac{2N - \alpha}{2},$$

consequently,

$$2 - \eta = 2 - \min\{N - 2, \frac{2N - \alpha}{2}\} = -N + 4 < 0$$

where the last inequality is justified by the hypothesis $N \geq 5$.

Considering $\frac{2N - \alpha - 2}{N - 2} < p < \frac{2N - \alpha}{N - 2}$, we ensure that $2N - \alpha - (N - 2)p - 2 < 0$; moreover, using the hypothesis $0 < \alpha < N$, we have $\frac{2N - \alpha}{N - 2} < p < \frac{2N - \alpha - 2}{N - 2}$.

It is easy to show that the integral

$$\lim_{\varepsilon \to 0} \int_0^\frac{\varepsilon}{\delta} \frac{r^{N-1}}{(1 + r^2)^{N-2}} dr$$

converges, if $N \geq 5$.

Therefore, we obtain $I_\varepsilon \to -\infty$ as $\varepsilon \to 0$, if $N \geq 5$, $0 < \alpha < 4$ and $\frac{2N - \alpha - 2}{N - 2} < p < \frac{2N - \alpha}{N - 2}$ and we are done.

Now we consider the case $\alpha \geq 4$ and $N \geq 5$.

In this case, the hypothesis $\alpha \geq 4$ implies that

$$N - 2 \geq \frac{2N - \alpha}{2}$$

and, consequently,

$$2 - \eta = 2 - \min\{N - 2, \frac{2N - \alpha}{2}\} = 2 - N + \frac{\alpha}{2} < 0,$$

where the last inequality is justified by the hypothesis $0 < \alpha < N$ and by the following implication

$$\alpha \geq 4 \Rightarrow \frac{\alpha}{2} \geq 2.$$

Therefore, in this case,

$$I_\varepsilon = \varepsilon^{2-N+\frac{\alpha}{2}} \left[ C_4 \int_0^{\frac{\varepsilon}{\delta}} \frac{r^{N-1}}{(1 + r^2)^{N-2}} dr - C_3 \varepsilon^{2N - \alpha - (N-2)p - 2} \right]$$

and, thus, we have that $I_\varepsilon \to -\infty$ if $N \geq 5$, $0 < \alpha < N$ and $\frac{2N - \alpha - 2}{N - 2} < p < \frac{2N - \alpha}{N - 2}$. Summarizing, we have $I_\varepsilon \to -\infty$ if $N \geq 5$, $0 < \alpha < N$ and $\frac{2N - \alpha - 2}{N - 2} < p < \frac{2N - \alpha}{N - 2}$. We are done.

We now prove (16).

**Lemma 15.** We have

$$C_2 \int_{B_{2\delta} \setminus B_\delta} |u_\varepsilon(x)|^2 dx - C_3 \varepsilon^{2N - \alpha - (N-2)p} = O(\varepsilon^\eta).$$

**Proof.** First of all, fix $\delta > 0$ sufficiently large such that $(U_\varepsilon(x))^2 \leq \varepsilon^{1+\eta}$, for all $|x| \geq \delta$.

It is easy to see that

$$\frac{1}{\varepsilon^\eta} \left[ C_2 \int_{B_{2\delta} \setminus B_\delta} |u_\varepsilon(x)|^2 dx - C_3 \varepsilon^{2N - \alpha - (N-2)p} \right] < \frac{C_2}{\varepsilon^\eta} \int_{B_{2\delta} \setminus B_\delta} |u_\varepsilon(x)|^2 dx$$

$$= \frac{C_2}{\varepsilon^\eta} \int_{B_{2\delta} \setminus B_\delta} (\psi(x))^2 (U_\varepsilon(x))^2 dx.$$

Thus,

$$\frac{1}{\varepsilon^\eta} \left[ C_2 \int_{B_{2\delta} \setminus B_\delta} |u_\varepsilon(x)|^2 dx - C_3 \varepsilon^{2N - \alpha - (N-2)p} \right] < C_2 \varepsilon \int_{B_{2\delta} \setminus B_\delta} (\Psi(x))^2 dx \leq C_2 \varepsilon \int_{R^N} (\Psi(x))^2 dx$$

$$\leq C_2 \varepsilon \|\psi\|_{L^p_{(R^N)}}$$

$$\leq C_1 \varepsilon \|\psi\|_{L^p_{A,V}} \leq C,$$

which shows that (16) is verified.

We now state our result about the periodic problem.
Theorem 16. Under the hypotheses already stated on $A$ and $\alpha$, suppose that $(V_1)$ is valid. Then problem (4) has at least one ground state solution if either

(i) $\frac{N+2-2\alpha}{N-2} < p < 2\alpha$, $N = 3, 4$ and $\lambda > 0$;
(ii) $\frac{2N}{N-2} < p < \frac{N+2-2\alpha}{N-2}$, $N = 3, 4$ and $\lambda$ sufficiently large;
(iii) $\frac{2N}{N-2} < p < 2^*_\alpha$, $N \geq 5$ and $\lambda > 0$;
(iv) $\frac{2N}{N-2} < p \leq \frac{2N-2\alpha}{N-2}$, $N \geq 5$ and $\lambda$ sufficiently large.

Observe that the conditions stated in this result are exactly the same of Lemma [13] and Theorem [11].

Proof. Since the energy functional satisfy the mountain pass geometry, there exists a $(PS)_{c_\lambda}$ sequence of $J_{A,V,p}$, that is, $(u_n) \subset H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C})$ such that

$$J'_{A,V,p}(u_n) \to 0 \quad \text{and} \quad J_{A,V,p}(u_n) \to c_\lambda,$$

where

$$c_\lambda = \inf_{u \in H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} J_{A,V,p}(tu).$$

and

$$\Gamma = \{ \gamma \in C^1([0,1], H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C})) : \gamma(0) = 0, J_{A,V,p}(\gamma(1) < 0) \}$$

with $c_\lambda < \frac{N+2-2\alpha}{N-2\alpha}(S_{H,L})^{\frac{2N}{N-2}}$, under the conditions stated in the theorem.

Lemma [12] guarantees that $(u_n)$ is bounded. So, passing to a subsequence if necessary, there is $u \in H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C})$ such that

$$u_n \rightharpoonup u \quad \text{in} \quad H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C}), \quad u_n \to u \quad \text{in} \quad H^1_{A,V,p}(\mathbb{R}^N, \mathbb{C}) \quad \text{and} \quad u_n \to u \quad \text{a.e.} \quad x \in \mathbb{R}^N.

If $u \neq 0$ we are done. If $u = 0$, it follows from Lemma [9] the existence of $\theta > 0$ and $(y_n) \subset \mathbb{Z}^N$ such that

$$\lim_{n \to \infty} \inf_{r \in \mathbb{R}^N} \int_{B_r(y_n)} |u_n|^2 \, dx \geq \theta > 0.$$ 

A direct computation shows that we can assume that $(y_n) \subset \mathbb{Z}^N$. Let

$$v_n(x) := u_n(x + y_n).$$

Since that $V_p$ and $A$ are $\mathbb{Z}^N$- periodic, we have

$$\|v_n\|_{A,V,p} = \|u_n\|_{A,V,p} = J_{A,V,p}(u_n) = J_{A,V,p}(u_n) \quad \text{and} \quad J'_{A,V,p}(v_n) \to 0, \quad \text{when} \quad n \to \infty.$$ 

Therefore there exists $v \in H^1_{A,V,p}$ such that $v_n \rightharpoonup v$ weakly in $H^1_{A,V,p}$ (as $n \to \infty$).

We claim that $v \neq 0$. In fact, from (19) it follows

$$0 < \theta \leq \|v_n\|_{L^2(B_R(0))} \leq \|v_n - v\|_{L^2(B_R(0))} + \|v\|_{L^2(B_R(0))}.$$ 

Since $v_n \rightharpoonup v$ in $L^2_{loc}(\mathbb{R}^N)$, we have that

$$\|v_n - v\|_{L^2(B_R(0))} \to 0 \quad \text{as} \quad n \to \infty,$$

and we conclude that $v \neq 0$.

But Corollary [12] guarantees that $J'_{A,V,p}(v_n) \cdot \phi \to J'_{A,V,p}(v) \cdot \phi$ and it follows that $J'_{A,V,p}(v) \cdot \phi = 0$. Consequently $v$ is a ground state solution of problem (4) \( \Box \)

4. Proof of Theorem [11]

Some arguments of this proof were adapted from the articles [2,24].

Maintaining the notation introduced in Section [3], consider the functional $I_{A,V} : E \to \mathbb{R}$ associated to problem [1], defined by

$$I_{A,V}(u) := \frac{1}{2} \|u\|^2_{A,V} - \frac{\lambda}{2p} B(u) - \frac{1}{2 \cdot 2^\alpha} D(u).$$

where

$$E = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) : \nabla A u \in L^2(\mathbb{R}^N, \mathbb{C}), \quad \int_{\mathbb{R}^N} V(x)|u(x)|^2 \, dx < \infty \right\}$$
is endowed with the norm
\[ \|u\|_{A,V} = \left( \int_{\mathbb{R}^N} (|\nabla_A u|^2 + V(x)|u|^2) \, dx \right)^{\frac{1}{2}}. \]

We denote by \( \mathcal{N}_{A,V} \) the Nehari Manifold related to \( I_{A,V} \), that is,
\[ \mathcal{N}_{A,V} = \left\{ u \in E \setminus \{0\} : \|u\|_{A,V} = \lambda B(u) + D(u) \right\}, \]

which is non-empty as a consequence of Theorem [16]. As before, the functional \( I_{A,V} \) satisfies the mountain pass geometry. Thus, there exists a \((PS)_d\) sequence \((u_n) \subset H^1_{A,V}(\mathbb{R}^N, \mathbb{C})\), that is, a sequence satisfying
\[ I_{A,V}(u_n) \to 0 \quad \text{and} \quad I_{A,V}(u_n) \to d_\lambda, \]

where \( d_\lambda \) is the minimax level, also characterized by
\[ d_\lambda = \inf_{u \in E \setminus \{0\}} \max_{t \geq 0} I_{A,V}(tu) = \inf_{\mathcal{N}_{A,V}} I_{A,V}(u) > 0. \]

We stress that, as a consequence of \((V_2)\), we have \( I_{A,V}(u) < J_{A,V_p}(u) \) for all \( u \in E \).

The next lemma shows as important inequality involving the levels \( d_\lambda \) and \( c_\lambda \).

**Lemma 17.** The levels \( d_\lambda \) and \( c_\lambda \) verify the inequality
\[ d_\lambda < c_\lambda < \left( \frac{N + 2 - \alpha}{4N - 2\alpha} (S_{H,L})^{\frac{2N+2\alpha}{N-2}} \right) \]
for all \( \lambda > 0 \).

**Proof.** Let \( u \) be the ground state solution of problem \([1]\). There is \( \bar{t}_u > 0 \) such that \( \bar{t}_u u \in \mathcal{N}_{A,V} \), that is
\[ 0 < d_\lambda \leq \sup_{t \geq 0} I_{A,V}(tu) = I_{A,V}(\bar{t}_u u). \]

It follows from \((V_2)\) that
\[ 0 < d_\lambda \leq I_{A,V}(\bar{t}_u u) < J_{A,V_p}(\bar{t}_u u) \leq \sup_{t \geq 0} J_{A,V_p}(tu) = J_{A,V_p}(u) = c_\lambda. \]

Therefore,
\[ d_\lambda < c_\lambda. \]

Let \((u_n)\) be a \((PS)_{d_\lambda}\) sequence for \( I_{A,V} \). As before, \((u_n)\) is bounded in \( E \). Thus, there exists \( u \in E \) such that
\[ u_n \rightharpoonup u \quad \text{in} \quad E. \]

By the same arguments given in the proof of Theorem [16] if \( u \neq 0 \), then \( u \) is a ground state solution of problem \([1]\).

Now, as in [2], we will show that \( u = 0 \) cannot occur. Indeed, if \( u = 0 \), then \( u_n \rightharpoonup 0 \) in \( E \). On one hand, since \( W \in L_+^\infty(\mathbb{R}^N, \mathbb{C}) \), Lemma [16] yields
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} W|u_n|^2 \, dx = 0 \]
and so,
\[ |J_{A,V_p}(u_n) - I_{A,V}(u_n)| = o_n(1) \]
showing that
\[ J_{A,V_p}(u_n) \rightharpoonup d_\lambda. \]

On the other hand, taking \( \varphi \in E \), with \( \|\varphi\|_{A,V} \leq 1 \), we obtain
\[ \|J'_{A,V_p}(u_n) - I'_{A,V}(u_n)) \cdot \varphi\| \leq \left( \int_{\mathbb{R}^N} W|u_n|^2 \, dx \right)^{\frac{1}{2}} = o_n(1). \]

Thus,
\[ J_{A,V_p}(u_n) = o_n(1) \]
Let \( t_n > 0 \) such that \( t_n u_n \in M_{A,V_p} \). Mimicking the argument found in [1] [18] [28] [30], it follows that \( t_n \rightharpoonup 1 \) as \( n \to \infty \). Therefore,
\[ c_\lambda \leq J_{A,V_p}(t_n u_n) = J_{A,V_p}(u_n) + o_n(1) = d_\lambda + o_n(1). \]
Letting $n \to +\infty$, we get

$$c_\lambda \leq d_\lambda$$

obtaining a contradiction with Lemma 17. This completes the proof of Theorem 1. \qed

As observed by Gao and Yang [BrezisN1, the proof of the next result is analogous to the proof of Theorem 1. The principal distinction is that the $(PS)_{C_\lambda}$ condition holds true below the level $\frac{1}{N}S^{\frac{N}{2}}$.

**Proof of Theorem 2.** It follows from [30, Lemma 1.46] that

$$\int_{\mathbb{R}^N} |\nabla u_\varepsilon|^2 \, dx = S^{\frac{N}{2}} + O(\varepsilon^{N-2})$$

and

$$\int_{\mathbb{R}^N} |u_\varepsilon|^2 \, dx = S^{\frac{N}{2}} + O(\varepsilon^N).$$

So, we have

$$\sup_{t \geq 0} J_{A,V_p}(t \varepsilon u_\varepsilon) < \frac{1}{N}S^{\frac{N}{2}} + O(\varepsilon^{N-2}) + C_2 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2N-\alpha-(N-2)p}$$

since

$$\lim_{\varepsilon \to 0} \varepsilon^{-(N-2)} \left( C_2 \int_{\mathbb{R}^N} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2N-\alpha-(N-2)p} \right) = -\infty.$$ 

Observe that (52) is a consequence of

$$\lim_{\varepsilon \to 0} \varepsilon^{-(N-2)} \left( C_2 \int_{B_\varepsilon} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2N-\alpha-(N-2)p} \right) = -\infty$$

and

$$C_2 \int_{B_{2\varepsilon} \setminus B_\varepsilon} |u_\varepsilon(x)|^2 \, dx - C_3 \varepsilon^{2N-\alpha-(N-2)p} = O(\varepsilon^{N-2}).$$ \qed

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