ESTIMATING THE DEEP SOLAR MERIDIONAL CIRCULATION USING MAGNETIC OBSERVATIONS AND A DYNAMO MODEL: A VARIATIONAL APPROACH

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ABSTRACT

We show how magnetic observations of the Sun can be used in conjunction with an axisymmetric flux-transport solar dynamo model in order to estimate the large-scale meridional circulation throughout the convection zone. Our innovative approach rests on variational data assimilation, whereby the distance between predictions and observations (measured by an objective function) is iteratively minimized by means of an optimization algorithm seeking the meridional flow that best accounts for the data. The minimization is performed using a quasi-Newton technique, which requires knowledge of the sensitivity of the objective function to the meridional flow. That sensitivity is efficiently computed via the integration of the adjoint flux-transport dynamo model. Closed-loop (also known as twin) experiments using synthetic data demonstrate the validity and accuracy of this technique for a variety of meridional flow configurations, ranging from unicellular and equatorially symmetric to multicellular and equatorially asymmetric. In this well-controlled synthetic context, we perform a systematic study of the behavior of our variational approach under different observational configurations by varying their spatial density, temporal density, and noise level, as well as the width of the assimilation window. We find that the method is remarkably robust, leading in most cases to a recovery of the true meridional flow to within better than 1%. These encouraging results are a first step toward using this technique to (i) better constrain the physical processes occurring inside the Sun and (ii) better predict solar activity on decadal timescales.

Key words: dynamo – magnetohydrodynamics (MHD) – methods: data analysis – Sun: activity – Sun: interior – Sun: magnetic fields

1. INTRODUCTION

The magnetic activity of our Sun has been monitored for centuries now. Modern systematic observations of sunspots started in the early 1600s. In addition, an analysis of the concentration of cosmogenic 10Be and 14C (found in ice cores and tree rings, respectively) makes it possible to trace back solar activity over the past 10,000 years (Beer et al. 1998 and Usoskin 2013). This monitoring has revealed a cyclic magnetic activity in our Sun: sunspots emerge at mid-latitudes during the rising phase of the cycle; they reach a maximum number 3–5 years later when the polar field flips polarity, while emerging gradually closer to the equator as the cycle progresses. If plotted against time, the location in latitude of sunspots then produces the so-called butterfly diagram. A quantitative estimate of the cycle strength was proposed by Wolf in 1859. He introduced the so-called Wolf number as \( R = k(10g + s) \), with \( g \) the number of sunspot groups, \( s \) the total number of individual sunspots in all groups, and \( k \) a variable scaling factor that accounts for instruments or observation conditions. The time series of the Wolf number starts in 1749, making the solar cycle that peaked in 1761 June Cycle number 1. At the time of writing, we just passed the maximum of Cycle 24. The monthly smoothed Wolf number reached a moderate maximum peak of about 82 in 2014 April, which will probably become the maximum of Cycle 24. This makes Cycle 24 the weakest cycle since Cycle 14, which peaked in 1906. This relatively low value is likely connected with the expected end of the current Gleissberg cycle (Abreu et al. 2008). Predicting solar activity has become very important for our technological society in which strong solar flares, coronal mass ejections (CMEs), or any violent event linked with solar activity can cause significant damage to satellites, air traffic, and telecommunication networks (Brun 2007). Consequently, a solar cycle panel, whose role is to produce predictions of forthcoming solar activity, was created in 1997. This panel provided estimates of the sunspot number for Cycles 23 and 24 (Joselyn et al. 1997; Biesecker 2007). In the ensemble of the 75 predictions for the Cycle 24 maximum sunspot number listed by Pesnell (2012), only 20 had anticipated such a moderate value of 82, taking into account the uncertainties provided for each prediction. This poor performance reveals the (expected) difficulties of producing reliable forecasts of magnetic activity for such a turbulent chaotic astrophysical system, especially if the prediction ignores the dynamics of the system and is entirely data-driven, as was the case for most predictions of Cycle 24, which relied on geomagnetic precursors or other statistical estimates (see Hathaway 2009; Pesnell 2012, for two reviews on the subject).

Recently, however, the use of numerical models in conjunction with observations (i.e., data assimilation) has started to emerge in the solar physics community. What is data assimilation? Let us assume that some observations of the Sun are available over a finite time interval \( [t_a, t_f] \) and that a numerical model governing the temporal evolution of the Sun over this interval is available. In a deterministic setting, the dynamic trajectory of the Sun is then entirely controlled by a set of initial conditions (at \( t = t_a \)) and, possibly, by a set of static control parameters.

In a first attempt of combining data and model, one can simply use the observations made at the Sun surface as
boundary conditions to impose on the numerical model between \( t_i \) and \( t_f \). This strategy was followed by Dikpati & Gilman (2006), Choudhuri et al. (2007), and Upton & Hathaway (2004) for the predictions of the amplitude and timing of the solar cycle, and by Cheung & DeRosa (2012) for eruptive events originating from active regions. However, this strategy is not optimal, in particular because it does not combine the uncertainties affecting the observations and the physical model.

More advanced techniques exist to remediate this problem, which can be classified into two categories: variational and sequential. Both share the same goal, which is to provide an optimal fit to the unknown reality (in a generalized least-squares sense) over the time window \([t_i, t_f]\) over which observations are available (hereafter, we will refer to this indifferently as the observation or sampling window). Variational assimilation provides a globally optimal fit over the whole time window. Sequential assimilation provides an optimal fit at the end of the window. The variational approach is rooted in the mathematical theory of control and aims at correcting the initial conditions (and possibly the set of static control parameters) by making use of all the data available over the entire \([t_i, t_f]\) interval. In contrast, the sequential approach rests on estimation (or filtering) theory. In that case, the stream of observations is assimilated sequentially, each time a new observation becomes available at, say, \( t = t_n \in [t_i, t_f] \). The sequential and variational approaches lead to the same results at the end of the assimilation window if the dynamics of the system is linear. More generally, and regardless of their respective merits, both approaches illustrate the same philosophy of combining data with numerical models. Both can lead to the production of a forecast for \( t > t_f \) and are used on a daily basis for the best-known problem of weather forecast, which requires several tens of millions of data to be assimilated every day into physical models of the atmosphere (and ocean), to first initialize a state (or ensemble of states) of the atmosphere (and ocean) and subsequently generate weather forecasts (see, e.g., Kalnay 2003).

The problem at hand may involve some nonlinearities, for instance, in the dynamics or in the relationships linking the state of the system to the available observations. If so, both sequential and variational approaches need be adapted. In practice, this amounts to performing a linearization at some stage in the analysis. In the case of the sequential approach, this leads to a class of methods known as the extended Kalman filter (EKF) and the ensemble Kalman filter, with the latter commonly known as the EnKF (Evensen 2009). In the variational framework, the most popular approach is the so-called 4D-Var approach, whose efficiency rests on the implementation of the so-called adjoint model (see Fournier et al. 2010; Talagrand 2010, for a recent review).

Regardless of the assimilation approach followed, the first question one may ask when seeking to apply data assimilation to the prediction of the solar cycle is: what type of numerical model of the origin of solar magnetism should be used? Indeed, in spite of some recent encouraging progress made by three-dimensional (3D) magnetohydrodynamic (MHD) simulations to self-consistently produce “solar-like” magnetic features (Ghizaru et al. 2010; Käpylä et al. 2012; Augustson et al. 2014; Warnecke et al. 2014), some of the ingredients needed to account for all the properties of the solar cycle remain to be understood. In particular, full 3D MHD simulations are not able to self-consistently produce, through an internal dynamo mechanism, sunspots emerging at the solar surface (refer to Nelson et al. 2013; Fan & Fang 2014 for a first step in that direction). The first attempts to apply data assimilation to solar physics have instead used simplified mean-field dynamo models, where strong simplifying assumptions are used and various physical processes are parametrized. These models are solar-like in the sense that they produce reversals of the large-scale magnetic field and butterfly diagrams resembling the observations. Sequential data assimilation was implemented for the first time by Kitiashvili & Kosovichev (2008) in such a mean-field dynamo model evolving jointly (in one spatial dimension) the three components of the magnetic field and a measure of magnetic helicity. The assimilated observations were the annually smoothed Wolf sunspot number for the period 1857–2007 in a sequential EnKF framework. The one-dimensional (1D) model used in that study was a standard \( \alpha \Omega \) dynamo model in which the toroidal field owes its origin to the differential rotation (the \( \Omega \)-effect) and the poloidal field is created by helical turbulence within the solar convection zone (the \( \alpha \)-effect). The prediction for the maximum sunspot number of Cycle 24 was 80 in 2013. This is remarkably close to the observed value of 82 for the amplitude of the cycle, and too early by one year for the timing of cycle maximum. Further, Cameron & Schüssler (2007) advocate that such predictions done within three years of the minimum are easier, as simple correlations reach up to 80% or so of success when applied to past cycles. More recently, data assimilation was performed with a more complete two-dimensional (2D) mean-field \( \alpha \Omega \) Cartesian model using the 4D-Var variational approach (Jouve et al. 2011); it was shown that the latitude-dependent profile of the \( \alpha \)-effect could be reconstructed from the assimilation of synthetic magnetic data and that the method was versatile and robust, which encouraged us to improve our model and method.

Such an improvement in the modeling can take the form of a flux-transport Babcock–Leighton model, in which the poloidal field is generated by the decay of active regions emerging at the solar surface (Babcock 1961; Leighton 1969), and where a large-scale meridional flow \( v_p \) acts to advect the magnetic field in the whole convection zone. The main strength of such models is that they incorporate physical processes that have observable counterparts, namely the active region evolution at the solar surface, and the meridional circulation amplitude and pattern. For the latter however, the observational constraints are limited. Most of our knowledge of \( v_p \) is provided by local helioseismology techniques that produce reliable measurements down to about 20 to 40 Mm below the surface (Haber et al. 2002; Zhao et al. 2004). We now know that the surface meridional flow is poleward and that the horizontal velocity amplitudes are between 10 and 20 m s\(^{-1}\). Inferences from the advection of super granules down to a depth of 70 Mm (Hathaway 2012), from \( p \)-mode frequencies (Mitra-Kraev & Thompson 2007) and improved time-distance analysis (Zhao et al. 2013; Jackiewicz et al. 2015) suggest a complex structure in the convection zone, organized in multiple cells, as also confirmed by recent global helioseismic methods (Schad et al. 2013).

Given the strong dependence of the magnetic cycle on the meridional flow amplitude and profile in flux-transport Babcock–Leighton dynamo models (Jouve & Brun 2007), the idea we pursue is to use data assimilation to better constrain this
flow. More specifically, our goal is to use time-dependent observations of the magnetic field to find the optimal meridional flow $v_p$ (in structure and amplitude) that minimizes the misfit between the observations and the values predicted by the model. To this end, preliminary studies have been performed to first characterize the sensitivity of the magnetic field evolution to changes in $v_p$ (Nandy et al. 2011; Dikpati & Anderson 2012) and to assess the so-called forecast horizon (i.e., the time interval for $t > t_p$) over which reliable predictions can be achieved; Sanchez et al. 2014). These studies concluded that a modification of the amplitude of $v_p$ would show large changes in the evolution of the magnetic field in a time much shorter than the typical circulation time. Sanchez et al. (2014) provided an estimate of the exponential growth rate of an initial perturbation on the model trajectory (the error growth rate), and found a corresponding $e$-folding time of 2.76 solar cycles, which is likely an overestimate of the practical horizon of predictability for the Sun. These results are promising for the possible use of magnetic field data to infer the meridional flow profile, and quite encouraging for possible future predictions of solar activity. Now that some key properties of this dynamical system (associated with this particular Babcock–Leighton dynamo model) are known, we can further consider applying data assimilation techniques to it. In a recent paper, Dikpati et al. (2014) used the EnKF to reconstruct the meridional flow speed at the solar surface (having fixed the meridional flow profile to one large circulation cell per hemisphere) from synthetic observations of the magnetic field. Dikpati et al. (2014) found that the best reconstruction of this unique, time-dependent parameter could be obtained, provided at least 10 observations were used with a time between two analyses of 15 days and an observational error of less than 10%. Here we propose working along the same philosophical line, aiming to estimate not only the amplitude of $v_p$ at a particular location, but also its structure throughout the whole convection zone. To do so, we use a variational data assimilation (4D-Var) technique akin to that developed by Jouve et al. (2011). We intend to demonstrate the successful application of 4D-Var to fully characterize the otherwise poorly constrained solar internal meridional flow $v_p$, by using proof-of-concept experiments based on a particular flux-transport dynamo model. Proof-of-concept experiments rely on data generated by a free run of the model (a run unconstrained by real data), and those synthetic data are subsequently used to verify and test the efficacy of any given assimilation scheme.

After presenting the details of the model in Section 2, we introduce our implementation of 4D-Var in Section 3. The results of proof-of-concept assimilation experiments with perfect data (i.e., without noise) are presented in Section 4, while results for noised data are given in Section 5. A discussion of the consequences of this study is given as a conclusion in Section 6.

2. ONE MAJOR INGREDIENT: THE MERIDIONAL CIRCULATION

In this work, we test our data assimilation technique with a spherical, axisymmetric mean-field dynamo model. We adopt the widely used flux-transport Babcock–Leighton model for which the meridional flow $v_p$ profile and amplitude is a major ingredient (Dikpati & Charbonneau 1999; Jouve & Brun 2007). The model equations are presented in Appendix A for reference. In this section, the meridional circulation, which should be optimized by the assimilation, is described in detail. Hereafter, the meridional circulation is expressed as the curl of a stream function, to ensure a divergence-free velocity field:

$$v_p = \nabla \times (\psi \phi),$$  \hspace{1cm} (1)

and the stream function is expanded as

$$\psi(r, \theta) = \frac{-2(r - r_{mc})^2}{\pi(1 - r_{mc})} \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij} \sin \left[ \frac{j\pi(r - r_{mc})}{1 - r_{mc}} \right] \times P_j^1(-\cos \theta)$$

$$+ \begin{cases} 0 & \text{if } r_{mc} \leq r \leq 1 \\ \eta_0 & \text{if } r_{bot} \leq r \leq r_{mc}, \end{cases}$$ \hspace{1cm} (2)

where $P_j^1$ are the associated Legendre polynomials, and the length is normalized with the solar radius $R_s$, the normalized radius $r$ varies between $r_{bot} = 0.6$ and $r_{top} = 1$, and the polar angle $\theta \in [0, \pi]$. The meridional flow is allowed to penetrate to a radius $r_{mc} = 0.65$ (i.e., slightly below the base of the convection zone located at $r_c = 0.7$). There is no flow between $r_{bot}$ and $r_{mc}$, which models a stationary layer of transition from the convection zone to the radiative zone. The magnetic diffusion time $R_s^2/\eta_0$ is chosen as the characteristic timescale, where $\eta_0$ is a typical value of the turbulent magnetic diffusivity (of the order of $10^{12} \text{ cm}^2 \text{ s}^{-1}$) in the convective envelope.

Three different profiles of the meridional flow are constructed using three different sets of expansion coefficients $d_{ij} \in \{0, 1, 2, \ldots, m, n, m, n \}$ in Equation (2). The size of the parameter space used in the following sections is $m = 2, n = 4$, which is an 8D parameter space. Those coefficients are given in Table 1. The non-listed coefficients are set to zero. We note that one can rewrite the expansion of the flow field (e.g., Equation (2)) using separable radial and latitudinal parts. The control vector would then be the expansion coefficients of the radial part and latitudinal part, resulting in a parameter space of lower dimension ($m + n$ versus $mn$). However, we find that this results in a larger misfit in the assimilation procedure.

Case 1 is a unicellular stream function (in each hemisphere), case 2 is a four-cell stream function with two cells in radius and two in latitude (in each hemisphere), and case 3 is a more general model with four cells in the northern hemisphere and only two cells in radius in the southern hemisphere. Figure 1 shows the contour plots of the stream functions of the meridional circulation of Cases 1–3. The other physical parameters of the three cases are given in Appendix A. From surface observations of the horizontal flows done by Ulrich (2010) and after projecting the data on associated Legendre polynomials $P_j^1(\theta)$, we note that the dominant mode is $\ell = 1$, and that $\ell$ modes beyond $\ell = 4$ are at least three orders of magnitudes smaller. We thus consider that stopping our latitudinal expansion of the stream function to $n = 4$ provides a fair reconstruction of the flow field. Figure 2 shows the resulting butterfly diagrams for the toroidal field at the base of the convection zone and the surface radial field. Those aspects are further documented in Appendix E.
3. SETTING UP THE ASSIMILATION PROCEDURE

The control vector, which is adjusted to fit the observational data, was chosen to be the expansion of the meridional flow profile on particular radial and latitudinal functions (Equation (2)). The idea of this work is to apply a variational data assimilation technique (or 4D-VAR, see Talagrand & Courtier 1987 for details) to this model, in which one seeks to minimize the misfit between the observations and the outputs of the model (characterized by an objective function $\mathcal{J}$) within a certain time interval. As a first step, we wish to proceed with twin (closed-loop) experiments, where the magnetic data are produced by a free run of the model, as described in Jouve et al. (2011). The assimilation will be considered successful when the true state (i.e., the value of the control vector that was used to produce the observations) is recovered, to a certain accuracy, as a result of the minimization of the objective function. In this section, the setup of these twin experiments is described: the generation of magnetic data, the choice of the objective function and the minimization algorithm, the choice of initial guess, and the diagnostics to assess the quality of the assimilation technique.

3.1. Generating Synthetic Observational Data

In our twin experiments, the synthetic observations are generated by the direct Babcock–Leighton dynamo model governed by Equations (9) and (10), with the expansion coefficients of the meridional circulation Equation (2) given by Table 1. The other parameters and grid size are given in Table 3 of Appendix A. In each case, the synthetic observations are the toroidal field at the tachocline and the vector potential of the surface poloidal field. These are taken from the reference trajectories of the magnetic field of the three cases described in Section 2, which are magnetic fields as a function of space and time, with the dipole field as the initial conditions, recorded when the periodic regime has been reached. The cycle period is $\sim 22$ years for all three cases (see Table 3). Our choice of synthetic data is motivated by the future application to real solar observations: the toroidal field at the tachocline is thought to be a proxy of the sunspot distribution, and the vector potential of the surface poloidal field should be a good proxy for the observed surface radial magnetic field, because the radial field is the spatial derivative of the vector potential. Our synthetic magnetic data are thus chosen to mimic what real solar observations provide (e.g., sunspot number, polar cap field, butterfly diagrams, etc.).

For the purpose of our twin experiments, we first introduce our synthetic observations into our assimilation code and analyze the converging behavior and performance of the code at various temporal observation windows, as well as various spatial sampling in latitudes. Second, we add random noise to the synthetic observations and study the effects on the performance and accuracy of the optimization. The noise added is centered, normally distributed, with standard deviation being a fraction of the root mean square (rms) of the magnetic field components produced by the dynamo model. Examples of synthetic observations for case 3 (error-free and noised with a noise level of 30% of rms) are shown in Figure 3; the counterparts of cases 1 and 2 are similar and thus not shown here.

### 3.2. Choice of the Objective Function and Minimization Algorithm

As stated above, the idea of the variational method is to minimize a well-defined objective function that measures the misfit between the observed quantities and corresponding outputs from the model. The objective functions being studied are the misfit of the toroidal field at the tachocline ($r_c = 0.7 R_s$),

$$\mathcal{J}_B = \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} \left[ B_{ai}(r_c, \theta_j, t_i) - B_{ai}^o(r_c, \theta_j, t_i) \right]^2 \sigma_{Bai}^2(r_c, \theta_j),$$

and, similarly, the misfit of the poloidal field vector potential at the solar surface,

$$\mathcal{J}_A = \sum_{i=1}^{N_r} \sum_{j=1}^{N_t} \left[ A_{ai}(R_s, \theta_j, t_i) - A_{ai}^o(R_s, \theta_j, t_i) \right]^2 \sigma_{Aai}^2(R_s, \theta_j),$$

and their sum, $\mathcal{J}_A + \mathcal{J}_B$.

Here $B_{ai}$ is the toroidal field predicted by our mean-field dynamo model and $B_{ai}^o$ is the synthetic observation. The weight $\sigma_{Bai}$ is the rms of the error-free toroidal field from the

### Table 1

| Case | $d_{1,1}$       | $d_{1,2}$       | $d_{2,2}$       | $d_{2,3}$       | $d_{2,4}$       |
|------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1    | $3.33 \times 10^{-1}$ | 0.00            | 0.00            | 0.00            | 0.00            |
| 2    | 0.00            | 0.00            | 0.00            | 0.00            | $9.47 \times 10^{-2}$ |
| 3    | 0.00            | $-5.74 \times 10^{-2}$ | $-8.75 \times 10^{-2}$ | $-3.83 \times 10^{-2}$ | $5.47 \times 10^{-2}$ |

Note. Coefficients that do not appear in this table are set to zero.
observations, and similarly for $A_{\phi}$, $A_{\phi}$, and $\sigma_{A}$. In real observations, $\sigma$ can also be time as well as space dependent, and can be adjusted to give more weight to some observations that are more reliable than others. For example, introduction of more advanced instruments would produce higher accuracy in more recent observations. The expression is summed over all the spatial observations in latitude ($N_0^o$) and temporal observations ($N^t_0$), so that the total number of observations is $N_0 = N_0^o N^t_0$.

The objective function is minimized with a quasi-Newton method, which requires the evaluation of the objective function $J$ and its gradient $\nabla J$ with respect to the control vector, but does not require the exact computation of the Hessian, which is instead approximated iteratively by the Broyden–Fletcher–Goldfarb–Shanno formula. In our calculations, we use the minimization routine m1qn3 developed by J. Gilbert and C. Lemaréchal (Gilbert et al. 2009) based on this algorithm. $J$ and $\nabla J$ are computed following the integration of the forward and adjoint models, respectively, details of which can be found in Appendices A and B, respectively. At the optimum, $\nabla J$
should be zero. Therefore, in the assimilation procedure, the
stopping criterion is defined by \(|\nabla J|/|\nabla J_0|\), which is the ratio
between the magnitude of the gradient of the objective function
(with respect to the control vector) after each iteration and the
magnitude of the gradient at the initial guess \((\nabla J_0)\). We call
this ratio the convergence criterion. In our twin experiments,
the assimilation will terminate when the criterion \(|\nabla J|/|\nabla J_0|\)
drops below \(10^{-6}\).

Unless otherwise stated, we present the results and analysis
adopting the objective function as the sum of the misfit on
toroidal field at the tachocline and on the surface poloidal
potential, \(J = J_B + J_A\). Most of the tests conducted with this
choice give a reasonable estimate of the meridional flow, so
that we can discuss any possible trends based on these results.
The effect of choosing only \(J_A\) or \(J_B\) as our objective function
has also been investigated and is discussed in Section 4.5. In
general, the performance is less satisfactory in terms of both
the misfit and the recovered flow pattern, if only \(A_v^0\) or \(B_v^0\) are
chosen for assimilation. Note that we are here using the toroidal
field at the tachocline, which in a realistic situation is not
directly available unless one develops an operator that relates
surface field observations to the field located at the base of
the convective envelope. Such a relationship will be the focus
of the future work; suffice it to say here that one could resort to
the three-halves law proposed by Bracewell (1988).

### 3.3. Choice of Initial Guess for \(v_p\) and Initial Condition for \(B\)

The assimilation code requires an initial guess for both the
meridional flow \(v_p\) and the magnetic field \(B\) as a starting point
of the minimization. Our initial guess for \(v_p\) is always that of
a unicellular flow, because we anticipate that this is the guess
we will most likely make in an operational setting (when dealing
with real data). For case 1 (i.e., the unicellular case, recall
Figure 1(a)), our initial guess for \(v_p\) was a unicellular flow, but
one that yields a magnetic cycle of 44 years (we thereby avoid
considering an initial \(v_p\) too close to the true \(v_p\)). For cases 2 and
3, we picked a unicellular \(v_p\) producing a 22-year cycle. We
stress that the performance of the assimilation method in all
three cases is stable with respect to the period of the cycle
determined by the choice of initial unicellular \(v_p\), up to a certain
margin. This margin shrinks as the complexity of the true \(v_p\)
increases. This statement is further illustrated by some examples
in Appendix D.

How do we set the initial condition (at \(t = t_0\), say) for \(B\)?
This is crucial because this choice determines the phase
difference between the modeled field and the observed one,
onece the modeled dynamo has entered its periodic regime. A
phase lag that is too large can be detrimental to the
optimization, to the point where it can simply fail, in particular
if the true \(v_p\) is substantially different from the initial guess we
just described (which is what happens for cases 2 and 3). A
possibility is to add \(B_0 \equiv B(t = t_0)\) to the control vector, and
perform an optimization of \(J\) by adjusting both \(v_p\) and \(B_0\). That
amounts to modifying the adjoint model (described in its
current form in Appendix B) in order to take the extra
sensitivity to \(B_0\) into account. The strategy that we choose for
this study is slightly different, and takes advantage of the
periodic nature of the system we are interested in. We take
different trials for \(B_0\) by regularly sampling a magnetic cycle
produced by the integration of the dynamo model based on the
initial guess of \(v_p\). In this framework, the best \(B_0\) is the one
leading to the most successful optimization of \(v_p\), following the
diagnostics described in the next subsection. In practice, this
involves multiple trials with an ensemble of initial conditions
for assimilating a single set of observations, and requires
considerable computer time. As we will demonstrate in the
upcoming sections, using this approach we always successfully
found an initial \(B_0\) that leads to a good final estimation of the
meridional circulation \(v_p\). We encourage the interested reader to
read Appendix C, where we give more details on the strategy
used to initialize the data assimilation algorithm. One should
also note that when trying to do a forecast, the initial condition
is usually taken from the previous assimilation cycle, and is
consequently much closer to the sought one than in the worst-
case scenario that we consider here.

### 3.4. Diagnostics to Assess the Quality of the Assimilation

The success of the assimilation depends on whether the
estimated expansion coefficients \(d_{ij}\) converge satisfactorily to
their true value. If convergence is achieved, we assess the
quality of the assimilation procedure by studying the number of
iterations required to achieve a given accuracy and the value of
the misfit at the end of minimization. In general, for a given set
of observations, the assimilation halts when the magnitude of
the gradient of the objective function decreases by a preset
factor. This indicates that the misfit is minimized, but from this
alone there is no information about the accuracy and
uniqueness of the optimal solution. However, in twin
experiments, as observations are artificially generated by a
known true state, the accuracy of the assimilation can be
studied quantitatively by defining the following (relative)
discrepancy:

\[
\frac{\Delta \rho}{\rho} = \sqrt{\frac{\sum_{i=1}^{m} \sum_{j=1}^{n} (d_{ij} - d_{ij}^{\text{true}})^2}{\sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^{\text{true}}}}
\]

where the coefficients with subscript “true” are those used to
generate the synthetic observations (see Table 1). This
discrepancy is the Euclidean metric between the estimated
control vector and its true counterpart, which is normalized by
the norm of the true vector.

In Section 5, when our synthetic observations are noised, we
can measure the performance of the assimilation technique by
the normalized misfit defined by

\[
\mathcal{J}_{\text{norm}} = \frac{1}{\epsilon} \left(\frac{\mathcal{J}}{N}\right)^{1/2}
\]

where \(\epsilon\) is the level of noise introduced (as a fraction of rms
of the field trajectory) and \(N\) is the total number of observations.
The fraction \(\epsilon\) is used instead of the absolute standard deviation
because from Equations (3) and (4), the objective functions are
already normalized with the rms value. In our case, the
observations are (unless otherwise noted) the tachocline
toroidal field and the surface potential vector, implying that
\(N = 2N^P\). Statistically, an assimilation trial with ideal fitting
of data will give \(\mathcal{J}_{\text{norm}} \sim 1\), while \(\mathcal{J}_{\text{norm}} \gg 1\) implies under fitting
where the misfit is considered to be too large. On the other
hand, \(\mathcal{J}_{\text{norm}} \ll 1\) implies over fitting, given what is expected
from the statistics.
4. RESULTS USING DATA WITHOUT NOISE

In this section, we investigate the quality of the convergence of the assimilation for cases 1–3 when the observational data are perfect (i.e., when they are realizations of the direct model and are not perturbed). The results of this section will be used for a systematic analysis and testing of the assimilation method, and will serve as a reference for the following section where noise is added. We first show the efficiency of our assimilation technique on an ideal case, with a reference distribution of observations. We then study the behavior of our technique when we vary the sampling of observations both in time and latitude. We focus on the effect of the width of temporal windows over which observations are available, the temporal sampling, and the latitudinal sampling. We investigate each of the above for case 1 and, for comparison, illustrate some typical examples for the two other cases.

4.1. Recovery of True Flow and Observed Magnetic Field in an Ideal Case

An illustrative example is considered here for the three cases. The distribution of observations is uniform in latitude ($\Delta \theta = 2.83^\circ$) and time ($\Delta t = 33$ days) and the observational window has a width of $t_e - t_s = 1.5$ cycles. We show how the stream functions evolve from the unicellular initial guess to the final structure after assimilation. Figures 4(a)–(c) show the evolution for cases 1–3, respectively. We can see that in cases 2 and 3, new cells appear at the expense of shrinking existing cells, and that it takes more iterations for the multicellular asymmetric structure to develop in case 3 compared with the relatively simpler case 2. Note that in cases 1 and 2, the intermediate states of meridional circulation as the minimization proceeds are always antisymmetric with respect to the equator, because the symmetry of predicted values is preserved throughout the algorithm. For case 3, the synthetic data are asymmetric, and the meridional flow slowly shifts from the symmetric prior to a more adequate asymmetric structure, as shown in Figure 4(c). In all cases, the basic structure of the meridional circulation is rapidly and nicely recovered. In the last 10 iterations, it comes to fine adjustments to further decrease the data misfit. This confirms that a unicellular prior is an appropriate initial guess, and it also demonstrates the potential of our assimilation method that can recover meridional circulation flow with multicellular and asymmetric profiles with respect to the equator and its deeper inner structure.

We now focus on the associated magnetic fields we get from the recovered meridional flow at the end of the assimilation
increases. In this study, the accuracy of the estimate $\Delta p/p$ increases as $N^o$ increases. In this study, the iteration count increases as $J_0/J_0$ and $\Delta p/p$ with the corresponding iteration count, respectively ($J_0$ is the value of objective function at the initial guess of meridional circulation). The results are shown in Figure 6. In all cases the normalized objective functions $J_i/J_0$ (panel (a)) and discrepancies $dp/p$ (panel (b)) diminish as the iteration count increases, showing that the optimization method is giving an improved estimate of meridional circulation after each iteration, thereby reducing the misfit between the observations and the field predicted by the dynamo model. Note that the objective function and discrepancy can get extremely small values ($\Delta p/p < 10^{-5}$) in case 1 compared with cases 2 and 3. This is because the initial condition of the assimilation is always the dynamo field from the unicellular flow model, and the synthetic observations in case 1 are also from the same unicellular profile (but the strength of the flow is not the same, so the assimilation is not trivial), thus the model prediction is essentially the same as the synthetic observations when the true meridional circulation is recovered. In cases 2 and 3, assimilation is initialized using a 1-cell configuration which is different from the 4-cell and asymmetric profiles used to generate the synthetic observations, resulting in a more challenging minimization, making it more difficult to reach the absolute minimum. As shown, the latter is of the order $>10^{-2}$, even when the synthetic observations are noise-free. Nonetheless, we are still able to reconstruct the true flow and the trajectory of the field (which will be shown in the following sections). The number of iterations required (for a fixed convergence criterion) increases as the complexity of the model increases: case 1 requires the least iterations and case 3 requires most. In case 3, only the trial with a sampling window of width 1.5 cycles can give a discrepancy of $10^{-2}$ (others are higher), because the objective function for such a complex meridional flow is also complicated and hard to optimize. In some situations, the assimilation algorithm terminates at a local minimum of the objective function in the parameter space, like the trials in case 3 with temporal windows of 1.0 and 2.0 cycles. In those cases, the performance in terms of minimization of misfit are lower than that of the 1.5 cycles trial.

For all cases, a short sampling window of 0.5 cycle lowers the accuracy of the results (see the blue curves in Figure 6), as the short observational window may not provide enough constraints on the meridional circulation.

Another feature of the assimilation method is that the accuracy increases slowly at the beginning of the minimization and speeds up eventually as the improved forecast gets closer to the true state, where the objective function is close to quadratic. This is a characteristic of the assimilation algorithm. As the meridional circulation becomes more complicated, it takes more iterations for the estimated control vector to reach the region of quadratic convergence.

In this particular study of the impact of the sampling window width on the quality of the assimilation, we find that a window width of 1.5 cycles gives on average the best a posteriori fit to the observations.

4.3. Convergence Behavior at Different Sampling Frequencies

In the previous section, we identified that the optimal observational window width $t_c - t_s$ for almost all cases was 1.5...
cycles. We now fix this temporal window width and investigate the effect of changing the sampling frequency. Spatial sampling is held constant and evenly distributed ($\Delta t = 2.83^\circ$). The results are shown in Figure 7. We see that the convergence behavior is relatively insensitive to the change in sampling frequencies, with only one exception for case 3. The highest sampling frequency, shown at $\Delta t = 33$ days, corresponds to $N_o^\theta = 181$, while $N_o^\theta = 16$ for the sparsest sampling, $\Delta t = 396$ days. The sparsest sampling consists of a total of $N_o^\theta = 1008$ observations, which is sufficient to estimate the eight expansion coefficients of the meridional flow, provided that the sampling window is wide enough to characterize the flow. We thus find that in all these cases where $N_o^\theta \geq 8$, having more temporal observations within a fixed sampling window does not introduce any new characteristics to the objective function during the assimilation process. However, the trial with the sparsest sampling in case 3 is an exception, because the true meridional flow is the most complex. This probably requires more frequent observations to correctly estimate the flow structure. Overall, a sampling frequency of one month to a trimester seems adequate when considering the perspective of using real solar data.

4.4. Convergence Behavior at Different Latitudinal Samplings

In this section, we fix the temporal window to 1.5 cycles and sampling frequency to one month. We now investigate the results of the assimilation procedure when the distribution of observations in latitude is varied. We show the results with (I) uniform sampling in latitude and (II) nonuniform sampling (i.e., sampling in one hemisphere only and sampling in the activity band only; latitude $-45^\circ$ to $+45^\circ$). The results for (I) and (II) are shown in Figure 8. For (I) (panel (a)), there is no systematic trend relating the sampling density and convergence behavior. Unlike the situation of changing sampling frequencies, the objective function is more sensitive to a change of sampling density in latitudes, and a denser spatial sampling does not necessarily result in faster convergence. A latitudinal sampling of $\Delta \theta = 4^\circ$ requires the least iterations for convergence for cases 1 and 2, but the $\Delta \theta = 2.83^\circ$ sampling results in higher accuracy. For case 3, the sparsest spatial sampling fails to converge and assimilation stops eventually without significant optimization, showing that for a flux-transport dynamo with a complex flow, more spatial observations are needed to estimate the flow structure.

In (II) (panels (b) and (c)), a uniform sampling always gives better convergence than other nonuniform patterns for the same number of observations. Again, case 1 gives the smallest misfit and discrepancy. For sampling in one hemisphere only, notice the following features.

i. In case 1, the convergence paths shown for sampling in the northern or southern hemisphere only coincide. Indeed, because the data and prior are both antisymmetric about the equator, the resulting normalized objective
functions $J/J_0$ (panel b)) are exactly the same. Note, however, that for case 2, which is also antisymmetric, the results for the northern and southern hemispheres are close but not exactly the same. This is due to the fact that the initial condition for the assimilation is unicellular and the system thus has to undergo a transient, resulting in not exactly the same steady states for both trials. This is not true for case 3, where the model is asymmetric.

ii. Although the discrepancy from the true state is significantly higher than in the case with uniform sampling, the meridional flow in the whole domain can be recovered by sampling in one hemisphere only, even for the asymmetric case 3. This can be explained by the fact that we do not try to reconstruct the point-wise meridional flow, but look for the coefficients of a particular expansion (Equation (2)), which implies certain symmetries. Also, this shows that the magnetic observations in one hemisphere give information for the meridional circulation in the other hemisphere by various physical processes like hemispheric coupling. It thus demonstrates the global structure of the magnetic field produced in these models.

iii. Let us now focus on case 3, with observations in the northern hemisphere only (black dashed line in Figure 8 (c)). As there is no constraint on the hemisphere where the field is not sampled, the optimization algorithm produces a state that is slightly different from the true flow. This is illustrated in Figure 9(a), where the result for the meridional flow is shown at the end of the assimilation. Compared to Figure 1(c), we see that the northern hemisphere where observations are available is much better recovered than the southern hemisphere (as stated before, the initial guess was indeed a unicellular flow). To have more quantitative estimates of the quality of the recovery of the solution in this case, we plot the deviation between the forecast and the true trajectory for the magnetic poloidal potential at some selected latitudes in Figure 10. We find that during the observational window, the misfit on $A_0$ in the northern hemisphere is of the order of half a percent when it is about 10 times higher for the southern hemisphere (which is still a low value given the fact we have no observations in this hemisphere). This figure also
From the above systematic study of the sampling patterns, a temporal observational window of 1.5 solar cycles with monthly sampling and uniform latitude (with $\Delta \theta = 2.83^\circ$) gives relatively robust convergence among all the trials (although not necessarily giving the fastest convergence in all three cases). Therefore, for illustration purposes, we use this reference sampling in most of the analysis and demonstrations below.

4.5. Other Characteristics of the Convergence

We study some additional properties of the minimization procedure. We first focus on the choice of the objective function. We can assimilate the tachocline toroidal field or surface potential vector only (i.e., using the objective functions (3) or (4) for optimization). For case 1, data assimilation is effective in terms of minimizing the misfit and recovering the true meridional circulation, even if we use (3) or (4) only. For the more complex cases 2 and 3, using only one of the objective functions lowers the performance of optimization (i.e., more iterations are required to recover the meridional circulation to the same accuracy, compared with the case where the sum of both objective functions is used). For example, for case 3, it takes 94 iterations to assimilate both components, but 141 iterations are needed if only the toroidal field is given, to reach a discrepancy of $\sim 2\%$. Also, in most cases, the misfit at the end of the assimilation is higher than when both fields are considered. We also showed that when only one component of the field is observed, the decline of performance cannot be compensated by a longer sampling time, for example, 2 solar cycles (one complete magnetic cycle) instead of 1.5. Overall, both $A^u(R, \theta, t)$ and $B^u(r, \theta, t)$ are necessary for the assimilation procedures to capture a meridional circulation that can optimize the misfit, especially when the true meridional circulation is far more complex than one cell per hemisphere.

For the next section, we thus use the synthetic observations both on $A_s$ and $B_o$ to perform the assimilation. As noted above, in a real situation we will not have direct access to the toroidal field at the base of the convection zone, but instead to surface field (via for instance sunspot observations) that we will have to relate to the deep toroidal field via an adequately defined operator. We have tested the influence on the convergence of shifting the location of the toroidal sampling, and found that in the unicellular case it has little influence. For the multicellular
cases 2 and 3, as long as the sampled depth probes the deeper secondary cell, the data assimilation procedure behaves the same. Nevertheless in the Babcock–Leighton framework it is natural to relate the sunspot number to the field strength at the base of the convection zone, because it mimics the rise of toroidal flux tubes to the surface. Defining the corresponding operator will be the subject of our next study.

We also investigate the relationship between the accuracy of the estimate parameters, defined in Equation (5) and the convergence criterion. Plots of $\Delta p/p$ against $|\nabla J|/|\nabla J_0|$ are shown in Figure 11. In all cases, $\Delta p/p$ decreases as the preset criteria is lowered, although in cases 2 and 3, the discrepancy cannot get lower than $\sim 1\%$. Indeed, as mentioned above, the true flow structure in cases 2 and 3 is more complicated than in case 1, making the minimization more difficult. The number of iterations required for various criteria are shown in Figure 11(b). It shows that when $|\nabla J|/|\nabla J_0| < 10^{-3}$, convergence becomes quadratic and a few further iterations can decrease the gradient by two to three orders of magnitude.

5. RESULTS USING DATA WITH NOISE

In this section, we move to a more realistic situation where we carry out the same twin experiments but with noised data. The synthetic observations are perturbed by a normally distributed random noise, with standard deviation being a factor of the rms of the magnetic field/vector potential at steady state, denoted by $\epsilon$ in the preceding sections.

5.1. Dependence of Convergence Behavior on the Noise Level

For our reference sampling, we carry out assimilation with synthetic observations noised with different values of $\epsilon$ in cases 1–3. The evolution of the objective function and the discrepancy as the minimization proceeds are shown in Figure 12. The normalized objective function $J/J_0$ (panel (a)) at the optimal parameters is no longer exactly zero, but is positive, increasing as the noise level increases. Similarly, $\Delta p/p$ at the optimum also increases with the noise level. The number of iterations required is slightly more than most of the
corresponding perfect (error-free) situations, i.e., it takes \(\sim 20\) iterations for the simplest unicellular case to converge, about \(\sim 60\) for case 2 and \(\sim 110\) for case 3, where the true meridional flow becomes more complex.

Note that for case 1, as the noise level \(\epsilon\) increases, the optimization remains successful: the normalized objective function reaches a value proportional to \(\epsilon^2\), as it should in a pure linear situation. For cases 2 and 3, the effect of noise on assimilation is less straightforward, for the reasons outlined above in the perfect data case (see Section 4.2). This contributes to the misfit of data at the end of the assimilation, in addition to the artificially added noise.

Another feature is the relationship between the residual discrepancy \(\Delta p/p\) and the noise level \(\epsilon\). For case 1, it is visible in Figure 12(b) that \(\Delta p/p\) increases linearly with the noise level (i.e., the accuracy decreases linearly as the noise level is increased). The linear relationship implies that the change in magnetic field with respect to that of the norm of the control vector \(\{d,\alpha\}\) is linear. Moreover, the discrepancy when no noise is added is nonzero. This is because the optimization terminates when the preset convergence criterion is reached, and because the criteria is not obvious, and the discrepancy increases very slowly before, the minimization for more complicated situations, i.e., it takes \(\sim 10\) to 44 years, while for cases 2 and 3 the prior is also a unicellular flow but producing a 22-year cycle. Again, the reference sampling (1.5 solar cycles, sampling every month) is used. The distribution of the deviations \(A_\phi - A_\phi^0\) for the most complicated case 3 is shown in Figure 13. The noise added to the data is \(\epsilon = 10\%\) for this example. Initially, the innovations are broadly distributed, showing the large discrepancy between the true trajectory and the initial state. After assimilation, the residuals show a peak that resembles a Gaussian (verified with a least square fit shown in black line in the figure), with a kurtosis of 3.05 (a value of 3 is expected for a perfect Gaussian) and a skewness of 1.04 \(\times\) \(10^{-2}\) (zero is expected for unbiased distribution).

The corresponding average and standard deviation are consistent with the settings of our twin experiment (i.e., zero mean and \(\sigma = 10\%\) rms; Table 2). The normalized misfit is 1.03. In this example, we illustrate that the assimilation algorithm not only minimizes the misfit and gives an estimate of the meridional circulation close to the true one (Section 5.1), but also correctly recovers the normal distribution of the synthetic noise, with a normalized misfit being statistically consistent (i.e., \(\sim 1\)).

We now consider a more realistic situation, where the field in the activity band only is observed (keeping the temporal sampling the same as the reference), for cases 2 and 3. We plot the distribution in Figure 14 for case 3 (case 2 is similar and thus not shown), with a noise level of \(\epsilon = 1\%\). Starting from a broadly distributed innovation as in the previous case, the residuals crowd around zero. However, the standard deviations are now larger than that of the synthetic noise: 2.86 for case 2 and 3.23 for case 3 (where it should be 1 for an ideal situation). Moreover, the distributions depart somewhat from pure Gaussians and are biased, with a kurtosis \(\sim 8.09\) indicative of extended wings and a skewness \(\sim 1.79\) for the case 3 shown in Figure 14. The corresponding normalized misfits are 3.01 and 2.71, respectively, which is an indication of under fitting. The estimated meridional circulation (shown in Figure 9(c)) is nevertheless still close to the truth. Therefore, it is still possible to estimate a complex multicellular flow in this more realistic example of nonuniform sampling and noised data, even if the statistics of the residuals clearly indicate that the recovery is not perfect.

Finally, we show in Figure 15 the recovered field of the assimilation for case 3, when the data files are noised with \(\epsilon = 30\%\). The trajectory obtained from the initial guess is shown in green, the final forecast after assimilation in blue, and the true trajectory in orange. Note that the forecast trajectory only deviates by a small amount from the true trajectory as time evolves.

6. DISCUSSION AND CONCLUSIONS

This study presented a first step toward predicting future solar activity using data assimilation. A variational data assimilation technique was applied to a mean-field axisymmetric flux-transport Babcock–Leighton dynamo model, which is widely used in the community to reproduce key properties of the real solar cycle. As a proof of concept, we focused our study on the estimation of the meridional circulation by assimilating magnetic proxies into our data assimilation procedure. We successfully adjusted a control vector representing the expansion coefficients of the meridional flow onto radial and latitudinal functions to minimize the deviations...
are normalized by the root mean square of the model itself, but where noise can be added to perturb the observations. We show that with adapted sampling of the different components of the magnetic field, and starting from a classical unicellular meridional circulation, we are able to minimize the misfit to the observations and recover a complex flow profile, with multiple cells and asymmetry with respect to the equator.

By performing a systematic study of the effects of the spatial and temporal sampling patterns on the efficiency of the assimilation method, we find an optimal sampling with an observational window of width 1.5 cycles, uniform sampling in latitude with $\Delta \theta \sim 3^\circ$, and monthly observations for almost all cases considered. An interesting aspect of this systematic study, however, is that observing in one hemisphere only or in the activity belt only can produce flow reconstructions of reasonably good agreement with the true state, even for a complex flow structure. When noise is added to the observational data, a normalized misfit close to one is found in the optimal case, showing that the true state is very well recovered. When an even more realistic case is considered (i.e., complex flow and 30% noise or activity band sampling), minimization remains effective and the estimated flow is still reasonable, although the residuals deviate slightly from Gaussian distribution with a higher normalized misfit. Overall, proof is made that the assimilation method is successful throughout our systematic studies.

The variational technique can be very useful for testing the sensitivity of various outputs of the models to poorly constrained input parameters, such as the meridional flow profile. Indeed, below $\sim 40 \text{ Mm}$, inversions based on helioseismic techniques, such as ring diagram or time-distance analysis, do not seem to give consistent results (Zhao et al. 2013; Jackiewicz et al. 2015). We showed that the high sensitivity of the components of the magnetic field to the flow structure and amplitude in flux-transport Babcock–Leighton models makes it possible to estimate the meridional circulation profile by assimilating magnetic field observations. If such a model is a good representation of what actually happens in the Sun, this technique is extremely promising to allow for much better constraint of this internal flow.

As far as predictions of future solar activity are concerned, our present studies are still limited at a stage of proof of concept based on twin experiments where the assimilated data are not real solar observations. However, we used magnetic outputs that are believed to be good proxies for real magnetic measurements at the solar surface, such as sunspot distributions, polar fields, and the structure of the butterfly diagram. Note that we did use a direct measure of the toroidal field at the base of the convective envelope, and that in reality one would have to construct an operator that adequately relates the observed surface field to the deeply anchored and hidden

| Case | Mean Innovation | Std. Dev. of Innovations (%) | Norm. Misfit Before Assim. | Mean Residual | Std. Dev. of Residuals (%) | Norm. Misfit After Assim. |
|------|----------------|-------------------------------|--------------------------|---------------|---------------------------|--------------------------|
| 1    | 0.72           | 152                           | 13.6                     | $-1.89 \times 10^{-4}$ | 9.86                     | 0.995                    |
| 2    | -1.61          | 164                           | 23.2                     | $2.65 \times 10^{-2}$ | 11.2                     | 1.11                     |
| 3    | -0.99          | 524                           | 39.3                     | $9.27 \times 10^{-3}$ | 10.4                     | 1.03                     |

Note. Shown are the statistics of the innovations (prior to assimilation) and those of the residuals (after completion of assimilation). Averages and standard deviations are normalized by the root mean square of $A_\ell$ of the reference solution. The values for the residuals are consistent with the noise added to the data (i.e., zero average, $\sigma = 10\%$ rms) and the normalized misfits are close to 1.

**Figure 14.** Same as Figure 13, for a latitudinal sampling restricted to the activity band only and a noise level of $\epsilon = 1\%$. Note that the distribution of residuals slightly deviates from a Gaussian distribution.

**Figure 15.** Same as Figure 5, save that synthetic data are noised with a noise level of 30% relative to the rms. Note that the deviation grows slowly as time evolves.
magnetic field. Defining such an operator will be the subject of our next study.

At this stage, the meridional circulation in our model is expressed as an expansion on basis functions, which defines the flow globally in the whole convection zone. This basis was limited to two sinusoidal radial functions and four Legendre polynomials in latitude in the present analysis for practical purposes, as nothing prevents us to go beyond. This global definition obviously implies some symmetries and possibly artificial coupling between hemispheres, which would probably not be the case with a point-wise definition. The latter would involve many more coefficients to recover and require us to impose a constraint that the recovered flow is divergence-free. When projecting surface meridional circulation onto the associated Legendre polynomials, as we have done using data from Ulrich (2010), we find that small velocity structures contain little power compared with the global low order modes. Hence, even though our strong formulation to recover the meridional circulation has its own limitations, we do capture the dominant components, which encourages us to apply it to real solar data. Note also that in the present work we have deliberately chosen to initialize our assimilation procedure with a magnetic field that was only weakly related to the solution we were looking for (recall Section 3.3 and Appendix C). In an operational forecasting procedure, one would use instead the magnetic field obtained from the previous assimilation cycle, which would presumably give rise to a faster convergence to the optimum.

In this work, only synthetic observations with a constant cycle period were used. This deviates from the activity of the real Sun, which presents strong modulation of its magnetic cycle, both in duration and amplitude. If we want to be able to take into account the temporal modulation of the solar cycle (which is necessary to make predictions), the next step is to produce a model with such modulations. This is possible in the mean-field dynamo framework if stochastic fluctuations of the dynamo coefficients or on the meridional flow are introduced (see for example Charbonneau & Dikpati 2000; Ossendrijver et al. 2002), or if the back-reaction of the magnetic field on the flow is considered through the Malkus–Proctor effect (e.g., Malkus & Proctor 1975; Tobias 1997; Moss & Brooke 2000; Bushby 2006; Rempel 2006). The idea would then be to assimilate synthetic data based on a model with a time-dependent meridional flow, such that the auto-correlation of the modeled flow is similar to that of the observed flow in the real Sun. The predictive skills, predictability limit, and sensitivity of these models to perturbations of the input parameters have to be studied, to gain some insights on the reliability of predictions that could be provided. Incorporating real solar data coming from instruments on board satellites like Hinode, Stereo, SDO, and the future Solar Orbiter mission into a well-suited model is then the ultimate goal of our work. If flux-transport models are valid to explain the large-scale solar magnetic activity, they should enable us to produce a quantitative estimate of the meridional flow in the solar interior. Based on such an estimate, predictions of the timing, amplitude, and shape of the next solar cycle will be made and compared with that provided by other existing methods relying on geomagnetic precursors or other statistical estimates (see for instance, Hathaway 2010; Petrovay 2010).

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**APPENDIX A**

**THE BABCOCK–LEIGHTON MEAN-FIELD DYNAMO MODEL**

In this section, we present the Babcock–Leighton dynamo model and the corresponding physical ingredients we adopted for the assimilation for reader’s reference.

We start from the induction equation to model the solar dynamo, describing the evolution of large-scale magnetic field $B$,

$$\partial_t B = \nabla \times (v \times B) - \nabla \times (\eta \nabla \times B),$$

(7)

where $\eta$ is the effective magnetic diffusivity. We adopt a kinematic formulation (i.e., the velocity field $v$ is prescribed instead of being a dynamical variable). By introducing the spherical coordinate system and assuming axisymmetry, we rewrite the magnetic field and velocity field as a sum of poloidal and toroidal components:

$$B = B_\phi e_\phi + \nabla \times \left( A_\phi e_\phi \right)$$

and $v(r, \theta) = v_p(r, \theta) + r \sin \theta \Omega(r, \theta) e_\phi$, (8)

where $B_\phi$ is the toroidal field, $A_\phi$ is the poloidal potential, and $\Omega(r, \theta)$ is the differential rotation.

Then the induction Equation (7) can be rewritten in poloidal and toroidal components as

$$\partial_t A_\phi = \frac{\eta}{\eta_t} \left( \nabla^2 - \frac{1}{\alpha^2} \right) A_\phi - \text{Re} \frac{v_p}{\alpha} \cdot \nabla \left( \alpha A_\phi \right)$$

$$+ C_s S (r, \theta, B_\phi),$$

(9)

$$\partial_t B_\phi = \frac{\eta}{\eta_t} \left( \nabla^2 - \frac{1}{\alpha^2} \right) B_\phi + \frac{\partial (\nabla B_\phi)}{\partial r} \frac{\partial (\eta/\eta_t)}{\partial r}$$

$$- \text{Re} \alpha v_p \cdot \nabla \left( \frac{B_\phi}{\alpha} \right) - \nabla B_\phi \cdot v_p$$

$$+ C_\Omega \left[ \nabla \times \left( A_\phi e_\phi \right) \right] \cdot \nabla \Omega,$$

(10)

where $\alpha = r \sin \theta$. The domain is $r \in [0.6, 1]$ and $\theta \in [0, \pi]$ as specified in Section 2. The toroidal field $B_\phi = 0$ at the boundary of the domain, and for $A_\phi$ we impose the pure radial field approximation at the surface, that is, $\partial_r (r A_\phi) = 0$ at $r = 1$, and $A_\phi = 0$ on all other boundaries. Here and in the following, the length is normalized with solar radius $R_s$ and time is normalized with the diffusive timescale $R_s^2/\eta$, where $\eta$ is the envelope diffusivity. Using this normalization, we introduce three-dimensionless parameters, namely the Reynolds number based on the meridional flow speed $Re = R_s u_s/\eta$, the strength of the Babcock–Leighton source $C_s = R_s s_0/\eta$, and the strength of the $\Omega$-effect $C_\Omega = \Omega_s R_s^2/\eta$, with $u_s$ and $s_0$ given in Table 3 and $\Omega_s = 2\pi \times 456$ nHz.

The physical ingredients for the model include a differential rotation that generates the toroidal magnetic field from the
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Table 3
Parameters of the Three Models Being Studied

| Case | Resolution $n_x \times n_y$ | Time Step $t_0.7$ | $u_0$ (cm s$^{-1}$) | $\eta$ (cm$^2$ s$^{-1}$) | $\lambda_0$ (cm s$^{-1}$) | Cycle Period (yrs) |
|------|-----------------------------|-------------------|------------------|----------------|----------------|------------------|
| 1    | $128^2$                    | $10^{-6}$         | 690              | $10^{11}$      | 50             | 22.0             |
| 2    | $128^2$                    | $10^{-6}$         | 1379             | $2 \times 10^{11}$ | 201           | 21.6             |
| 3    | $128^2$                    | $10^{-6}$         | 1034             | $2.4 \times 10^{11}$ | 17.2          | 21.7             |

Note. Unicellular (case 1), four cells (case 2), asymmetric (case 3).

Poloidal field:

$$\Omega(r, \theta) = \Omega_c + \frac{1}{2} (1 - \Omega_c - c_2 \cos^2 \theta) \times \left[ 1 + \tanh \left( \frac{r - r_c}{d_1} \right) \right].$$

(11)

with $d_1 = 0.016$, $r_c = 0.7$ (base of the convection zone), $\Omega_c = 0.92$ and $c_2 = 0.2$, the Babcock–Leighton source of poloidal field, with a quenching term to prevent the magnetic energy from growing exponentially:

$$S(r, \theta, B_\phi) = \frac{1}{2} \left[ 1 + \tanh \left( \frac{r - r_c}{d_2} \right) \right] \left[ 1 - \tanh \left( \frac{r - 1}{d_2} \right) \right]^{-1} \times \cos \theta \sin \theta \left\{ 1 + \left[ \frac{B_\phi(r_c, \theta, t)}{B_0} \right]^2 \right\}^{-1} \times B_\phi(r_c, \theta, t),$$

(12)

with $d_2 = 0.008$, $r_2 = 0.95$, and $B_0 = 10^4$. Note the dependence of the toroidal field at the base of the convection zone results in a nonlocal source term, and the magnetic diffusivity which is given by

$$\frac{\eta}{\eta_t} = \frac{n_t}{2 \eta_t} \left[ 1 + \tanh \left( \frac{r - r_c}{d_1} \right) \right] + \frac{1}{2} \left[ 1 + \tanh \left( \frac{r - r_c}{d_1} \right) \right],$$

(13)

where $\eta_t = 10^9$ cm$^2$ s$^{-1}$.

Another important ingredient is the meridional flow $v_p$, which advects the magnetic field in the meridian plane. Because this ingredient is at the center of this present study, it is specified in the main body of the text in Section 2.

APPENDIX B
DERIVATION OF THE ADJOINT
BABCOCK–LEIGHTON MODEL.

In this appendix, we follow and adapt the procedure described by Talagrand (2003) in order to derive the adjoint dynamo model needed to express efficiently the sensitivity of the objective function to its control vector. The novelty here with respect to the previous derivation by Jouve et al. (2011) stands in the fact that we are operating in spherical geometry with a Babcock–Leighton flux-transport dynamo model, as opposed to Cartesian geometry with a simpler $\alpha$–$\Omega$ dynamo model. Let us consider the coupled induction Equations (9) and (10) for the fields $A_\phi$ and $B_\phi$. We look for solutions over the domain $D = [r_{bot}, r_{top}] \times [0, \pi] \times [0, 2\pi] \times [t_0, t_1]$ in the $(r, \theta, \phi, t)$-space. These equations are first order in time and second order in space.

Consider $A_\phi(r, t)$ and $B_\phi(r, t)$ as our observations over the domain $D$. Because we assimilate data on the toroidal and poloidal fields, our objective function

$$\mathcal{J} = \frac{1}{2} \int d^3r \int dt \left[ (B_\phi - B_\phi^o)^2 + (A_\phi - A_\phi^o)^2 \right].$$

(14)

where the spatial integration is over the domain of the dynamo model described in Section A. The system considered is axisymmetric, so that integration with respect to $\phi$ is equivalent to multiplication by a factor of $2\pi$.

We aim to express the variations of the objective function $\mathcal{J}$ subject to variations of $A_\phi$ and $B_\phi$ for all points in space at the initial time $t = t_o$, as well as to variations in the meridional flow $v_p$. Such variations are constrained by the dynamo equations. Let us linearize and differentiate Equations (9) and (10) with respect to $A_\phi$, $B_\phi$, and $v_p$. The corresponding equations write

$$\partial_t \delta A_\phi = \frac{\eta}{\eta_t} (\nabla^2 - \frac{1}{\omega^2}) \delta A_\phi + Re \frac{\omega v_p}{\omega} \cdot \nabla (\omega \delta A_\phi)$$

$$- Re \frac{\omega v_p}{\omega} \cdot \nabla (\omega \delta B_\phi) + C_f(r) \delta (\omega \delta A_\phi)$$

$$\times \left[ 1 - B_\phi^o(r_c, \theta, t)/B_0^2 \right] \delta B_\phi(r_c, \theta, t),$$

(15)

$$\partial_t \delta B_\phi = \frac{\eta}{\eta_t} (\nabla^2 - \frac{1}{\omega^2}) \delta B_\phi + \frac{1}{\omega} \frac{\partial (\omega \delta B_\phi)}{\partial r} \frac{\partial (\eta/\eta_t)}{\partial r}$$

$$- Re \omega \delta v_p \cdot \nabla (\omega \delta B_\phi) - Re \delta B_\phi \nabla \cdot v_p$$

$$- Re B_\phi \nabla \cdot \delta v_p + \frac{C_{1\alpha}}{\omega} \left[ \nabla \times (\delta A_\phi \epsilon_\phi) \right] \cdot \nabla \Omega,$$

(16)

where we use the explicit expression of the Babcock–Leighton source (12), and the variation is up to first order in $\delta B_\phi$.

The variations of $\mathcal{J}$ are subject to the constraints defined by these last two equations. Consequently, we introduce the Lagrange multipliers $-A_\phi^a(r, t)$ and $-B_\phi^a(r, t)$, respectively, for Equations (15) and (16). The notations $A_\phi^a$ and $B_\phi^a$ are used so that when the derivation proceeds, we can identify them as defining the adjoint magnetic field ($A_\phi^a$ and $B_\phi^a$ being the adjoint poloidal potential and toroidal field, respectively). We
get

\[
\delta J = \int d^3r \int dt \left\{ \left( B_0 + B_0^2 \right) \delta B_0 + \left( A_\phi - A_\phi^2 \right) \delta A_\phi \right. \\
- A_\phi \left[ \partial_t \delta A_\phi - \frac{\eta}{\eta_t} \left( \nabla^2 - \frac{1}{c_0^2} \right) \delta A_\phi \right] \\
+ Re \frac{\delta p}{\omega} \cdot \nabla \left( \omega \delta A_\phi \right) + Re \frac{\delta p}{\omega} \cdot \nabla \left( \omega \delta A_\phi \right) \\
- C_f (r) g (\theta) \left[ \frac{1 - B_0^2 (r, \theta, t)}{1 + B_0^2 (r, \theta, t)} \delta B_0 (r, \theta, t) \right. \\
- B_0 \left[ \partial_t \delta B_0 - \frac{\eta}{\eta_t} \left( \nabla^2 - \frac{1}{c_0^2} \right) \delta B_0 - \frac{1}{c_0} \right] \\
\left. \left. \times \frac{\partial (\omega \delta B_0)}{\partial r} \frac{\partial (\eta/\eta_t)}{\partial r} \right] + Re \omega \delta p \cdot \nabla \left( B_0 \right) \omega \\
+ Re \omega \delta p \cdot \nabla \left( B_0 \right) \omega + Re \delta B_0 \nabla \delta p \\
+ Re B_0 \nabla \delta p - C_{\Omega} \omega \left[ \nabla \times \left( \delta A_\phi e_\phi \right) \right] \cdot \nabla \Omega \right\}.
\]

The differential operators acting on the variations of \( A_\phi, B_\phi \), and \( v_p \) can be removed via integration by parts, at the expense of introducing boundary integrals, either over the surface \( \partial V \) of the spatial domain (we remove the notation \( \partial V \) for clarity after its first introduction) or at the end-points in time \( t_s \) and \( t_e \).

For example, for the time derivative and diffusion of \( A_\phi \) one gets

\[
- \int d^3r \int dt A_\phi \partial_t \delta A_\phi = \int d^3r \int dt \delta A_\phi \partial_t A_\phi - \int d^3r A_\phi \delta A_\phi \bigg|_{t_e},
\]

and

\[
\int d^3r \int dt \frac{A_\phi \eta}{\eta_t} \left( \nabla^2 - \frac{1}{c_0^2} \right) \delta A_\phi \\
= \int d^3r \int dt \delta A_\phi \left( \frac{\nabla^2}{\eta_t} - \frac{1}{c_0^2} \right) A_\phi \\
+ \int dt \int_{\partial V} da \cdot \left( \frac{\nabla^2}{\eta_t} \delta A_\phi - \delta A_\phi \nabla \frac{\nabla^2}{\eta_t} \delta A_\phi \bigg|_{\partial V} \right)
\]

respectively. In addition, the rearrangements for the nonlocal Babcock–Leighton source term (a novelty of this study) write

\[
\int d^3r \int dt A_\phi \delta A_\phi \bigg|_{t_e} - \int d^3r B_\phi \delta B_\phi \bigg|_{t_e} \\
+ \int dt \int da \cdot \left( \frac{A_\phi \eta}{\eta_t} \nabla \delta A_\phi - \delta A_\phi \nabla \frac{A_\phi \eta}{\eta_t} \right) \\
- v_p A_\phi Re \delta A_\phi + C_{\Omega} e_\phi \times \nabla \Omega \delta A_\phi \bigg|_{\partial V} \\
+ \int dt \int da \cdot \left( \frac{B_\phi \eta}{\eta_t} \nabla \delta B_\phi - \delta B_\phi \nabla \frac{B_\phi \eta}{\eta_t} \right)
\]

Note the introduction of the \( \delta \)-function \( \delta (r' - r_c) \) in the first right-hand side of Equation (20), which should be distinguished from the \( \delta \) symbol used to represent variations of field variables.

By grouping the terms by variations, we get the following equation for \( \delta J \):

\[
\delta J = \int d^3r \int dt \left[ \left( \frac{\partial A_\phi}{\partial t} \right) \delta A_\phi \right. \\
+ Re \omega \nabla \cdot \frac{A_\phi}{\omega} v_p + \nabla \times \left( A_\phi e_\phi \right) \cdot \nabla \left( \omega A_\phi \right) \\
\left. \times \nabla \Omega + \left( A_\phi - A_\phi^2 \right) \delta A_\phi \right] \\
+ \left[ \partial_t B_\phi + \left( \frac{\nabla^2}{\eta_t} - \frac{1}{c_0^2} \right) B_\phi + Re \frac{1}{c_0} \nabla \cdot \left( \omega B_\phi \right) v_p \\
- \frac{1}{r} \left. \partial_r \left( r B_\phi \frac{\partial}{\partial r} \frac{\eta}{\eta_t} \right) \right] - Re B_\phi \nabla \delta p \\
+ C_{\Omega} g (\theta) \left[ \frac{1 - B_0^2 (r, \theta, t)}{1 + B_0^2 (r, \theta, t)} \delta B_0 (r, \theta, t) \right] \\
\times \left[ \int d^3r \int dt A_\phi C_f (r) g (\theta) \int dr' \delta (r' - r_c) \right. \\
\left. \times \frac{1 - B_0^2 (r, \theta, t)}{1 + B_0^2 (r, \theta, t)} \delta B_0 (r, \theta, t) \right]
\]

(20)
\[
\begin{aligned}
- v_p B^*_p R e \delta B_\phi + e_r B^*_p \delta B_\phi \frac{\partial \eta / \eta_\ell}{\partial r} \\
- R e \left. \delta v_p B^*_p B_\phi \right|_{\partial V}.
\end{aligned}
\] (21)

The expression is valid for any \( A^*_\phi (r, t) \) and \( B^*_\phi (r, t) \). The first three lines of the above equations vanish if we require \( A^*_\phi \) and \( B^*_\phi \) to satisfy the following partial differential equations:

\[
- \partial_r A^*_\phi = \left( \nabla^2 - \frac{1}{\omega^2} \right) \eta A^*_\phi + Re \frac{1}{\omega} \nabla \cdot \left( \nabla B^*_\phi \right) v_p + C_\ell e_\phi \cdot \left( n B^*_\phi \right) \nabla \Omega + \left( A_\phi - A_\phi^0 \right), \quad (22)
\]

and

\[
- \partial_r B^*_\phi = \left( \nabla^2 - \frac{1}{\omega^2} \right) \eta B^*_\phi + Re \frac{1}{\omega} \nabla \cdot \left( \nabla B^*_\phi \right) v_p - \frac{1}{r} \partial _r \left( r B^*_\phi \eta / \eta_\ell \right) - Re B^*_\phi \nabla \cdot v_p + C_\ell g(\theta) \frac{1 - B^2_\phi / B^2_0}{r^2} \frac{\delta (r - r_c)}{2} \times \int dr' r'^2 A^*_\phi (r', \theta, t) f (r') + \left( B_\phi - B^*_\phi \right). \quad (23)
\]

Now we can identify Equations (22) and (23) as the adjoint equations of the forward dynamo model (9) and (10), respectively, with the adjoint field variables \( A^*_\phi \) and \( B^*_\phi \). Note that the nonlocality of the (forward) Babcock-Leighton effect results in a nonlocality of its adjoint and in the introduction of the \( \delta \)-function. Interestingly, in these adjoint equations, differential rotation now acts upon \( A^*_\phi \), while the Babcock-Leighton effect has an imprint on \( B^*_\phi \). This is contrary to the “forward” situation and illustrates nicely the general mechanism of the “transposition” that forms the backbone of any adjoint-based variational approach (Talagrand 2003).

Let us now inspect the boundary terms (in space). The boundary conditions of the forward dynamo model are that

\[
\partial_r \left( r A_\phi \right) = 0 \quad \text{at} \quad r = r_{\text{top}},
\]

\[
A_\phi = 0 \quad \text{at} \quad r = r_{\text{bot}}, \quad \text{and}
\]

\[
B_\phi = 0 \quad \text{at} \quad r = r_{\text{bot}} \quad \text{and} \quad r = r_{\text{top}}.
\]

The respective variations of these quantities must consequently vanish. We also have that \( \partial_r \cdot v_p = 0 \) on \( \partial V \), which suppresses some of the components of the boundary terms appearing in the last two lines of the above equation. Now, if we further require that

\[
\partial_r \left( A^*_\phi \eta / \eta_\ell \right) = 0 \quad \text{at} \quad r = r_{\text{top}},
\]

\[
A^*_\phi = 0 \quad \text{at} \quad r = r_{\text{bot}}, \quad \text{and}
\]

\[
B^*_\phi = 0 \quad \text{at} \quad r = r_{\text{bot}} \quad \text{and} \quad r = r_{\text{top}}.
\]

at all times, the surface integrals in (21) identically vanish. These conditions are the adjoint boundary conditions that are naturally associated with the adjoint problem. At this stage, the variations of \( J \) reduce to

\[
\delta J = \int d^3 r \ Re \delta v_p \cdot \int dt \left[ \nabla \left( B^*_\phi B_\phi \right) \right]
\]

\[
- A^*_\phi \frac{1}{\omega} \nabla \left( \omega A_\phi \right) - \omega B^*_\phi \nabla B_\phi \frac{\delta}{\omega}
\]

\[
- \int d^3 r A^*_\phi \delta A_\phi \left|_t \right. - \left. \int d^3 r \nabla B^*_\phi \delta B_\phi \left|_t \right. \right. \quad (30)
\]

The adjoint fields are auxiliary fields whose task is to help us compute the sensitivity of \( J \) to its control parameters, and we can conveniently ask them to satisfy the following terminal conditions

\[
A^*_\phi = B^*_\phi = 0 \quad \text{at} \quad t = t_r.
\]

This leaves us with the following variation of \( J \)

\[
\delta J = \int d^3 r \ Re \delta v_p \cdot \int dt \left[ \nabla \left( B^*_\phi B_\phi \right) \right]
\]

\[
- A^*_\phi \frac{1}{\omega} \nabla \left( \omega A_\phi \right) - \omega B^*_\phi \nabla B_\phi \frac{\delta}{\omega}
\]

\[
+ \left[ \int d^3 r \left( A^*_\phi \delta A_\phi + B^*_\phi \delta B_\phi \right) \right] \left. \right|_{t=t_r}. \quad (32)
\]

This expression shows that the partial derivatives of \( J \) with respect to \( A_\phi (r, t), B_\phi (r, t) \), and \( v_p (r) \) write, respectively,

\[
\frac{\partial J}{\partial A_\phi} (r, t) = A^*_\phi (r, t),
\]

\[
\frac{\partial J}{\partial B_\phi} (r, t) = B^*_\phi (r, t),
\]

\[
\frac{\partial J}{\partial v_p} (r) = Re \int dt \left[ \nabla \left( B^*_\phi B_\phi \right) \right]
\]

\[
- A^*_\phi \frac{1}{\omega} \nabla \left( \omega A_\phi \right) - \omega B^*_\phi \nabla B_\phi \frac{\delta}{\omega}. \quad (33)
\]

The actual calculations of these derivatives demand, in particular, the knowledge of \( A^*_\phi (r, t) \) and \( B^*_\phi (r, t) \). Those are obtained from the integration of the adjoint Equations (22) and (23), subject to the boundary and terminal conditions we just discussed. The integration is carried out backwards from \( t_c \) to \( t_r \), which is what makes it stable: the partial time derivative is preceded by a minus sign, so the Laplacian on the righthand sign does not lead to instabilities (Talagrand & Courtier 1987).

In practice, instead of discretizing and numerically integrating the adjoint Equations (22) and (23), we develop the adjoint model directly from the discretized forward model (Talagrand 1991). This ensures that the computation of the gradient is consistent between the forward and adjoint models. Furthermore, as we observe the magnetic proxies at discrete times and positions, and with uncertainties, the driving terms \( A_\phi - A^*_\phi \) and \( B_\phi - B^*_\phi \) in the adjoint model are collections of delta functions in space and time, divided by the appropriate variances. This is what we effectively implemented in our optimization routine for the twin experiments.

In this study, as discussed in Section 3.3, \( A_\phi (r, t) \) and \( B_\phi (r, t) \) are not included in the control vector. We are therefore left with the sensitivity of \( J \) to the sole \( v_p (r) \). This steady flow is mathematically represented by a stream function, recall Equation (2). It is thus divergence-free, that is, the term
Again, the calculation of this derivative requires the integration of Equation (33) need not be considered. We can then rewrite the variation of the meridional flow in terms of the stream function,

\[
\int d\tau \int dt \left( -\text{Re} \int d\tau \int dt \left[ A_0^* \frac{1}{\omega} \nabla (\phi A_0) \right. \right.
\]

\[+ \left. \phi B_0^* \nabla B_0 \cdot \nabla \phi \right) \right) \cdot \Delta \phi = \text{Re} \int d\tau \int dt \left[ \frac{A_0^*}{\omega} \frac{1}{\omega} \nabla (\phi A_0) \right.
\]

\[+ \nabla (\phi B_0^*) \times \nabla B_0 \cdot e_\phi \right)
\]

\[+ \text{Re} \int dt \int da \cdot A_0^* \frac{1}{\omega} \nabla (\phi A_0) \right)
\]

\[+ \phi B_0^* \nabla B_0 \cdot e_\phi \right) \right) \cdot \Delta \phi \right). \tag{34}
\]

The surface term again vanishes, and by substituting the variation of

\[\psi(r, \theta) = \frac{-2(r - r_{mc})^2}{\pi (1 - r_{mc})}
\]

\[\sin \frac{i \pi (r - r_{mc})}{1 - r_{mc}} \times P_l^m (\cos \theta) \quad \text{if} \quad r_{mc} \leq r \leq 1
\]

\[0 \quad \text{if} \quad r_{bot} \leq r < r_{mc},
\]

into Equation (34), the partial derivative of the objective function with respect to each expansion coefficient \(d_{ij}\) writes

\[
\frac{\partial J}{\partial d_{ij}} = \text{Re} \int d\tau \left\{ P_l^m (\cos \theta) \int dt \left[ \frac{A_0^*}{\omega} \nabla (\phi A_0) \right.
\]

\[+ \nabla (\phi B_0^*) \times \nabla B_0 \cdot e_\phi \right) \right) \cdot \Delta \phi \right) \right).
\]

Again, the calculation of this derivative requires the integration of the adjoint equations for \(A_0^*\) and \(B_0^*\) subject to the boundary and terminal conditions already discussed. Note also that the radial part of the volumetric integration is now restricted to the domain \(r_{mc} < r < r_{bot} = 1\).

APPENDIX C

ASSIMILATION MODEL STARTED FROM AN ENSEMBLE OF INITIAL CONDITIONS TAKEN FROM A UNICELLULAR DYNAMO MODEL

As discussed in Section 3.3, the assimilation is carried out without the knowledge of the true meridional circulation \(\psi\) and the true initial condition for \(B\). In practice, the assimilation tests different initial conditions for \(A_0\) and \(B_0\), based on a collection of snapshots from a 22-year periodic reference dynamo model whose variability is controlled by a unicellular meridional flow. Under favorable circumstances, there is potentially a time for which the predicted magnetic field can be almost in phase with the synthetic data, opening the way to a successful recovery of the meridional circulation; otherwise, too large a phase difference leads to a misfit remaining suboptimal, and an unsuccessful recovery. For each trial, we let the forward model iterate through the transient regime. When the periodic regime is reached, the the misfit between the synthetic observations and the predicted trajectory (i.e., the objective function) is evaluated. Those multiple trials of assimilation for the same set of observations are performed in an embarrassingly parallel framework.

Let us illustrate this further by considering a twin experiment for case 3. The synthetic observations are obtained using an asymmetric flow associated with a 22-year magnetic cycle period and the other parameters given in Tables 1 and 3. The observations are noised with \(\epsilon = 10\%\). The sampling window has a width of 1.5 solar cycles. The field is sampled monthly and the observations are uniformly distributed in latitude. We take as an initial guess an ensemble of initial conditions from a 22-year reference dynamo model with a unicellular flow, evenly distributed within this period of 22 years. To label these snapshots in the reference model, we plot the time evolution of the toroidal field at the tachocline at the latitude +35° over a period. We define the instant when the field is zero with the positive time derivative to be year zero. We assimilate the same set of data for each initial condition and the objective function used is the sum \(J_A + J_B\). The convergence criteria is \(10^{-6}\). The result for an ensemble of 10 such snapshots is shown in Table 4.

The iteration converges to a minimum discrepancy of \(\sim 1.55\%\) when the snapshot of year 14.0 is chosen as the initial condition for assimilation, and the corresponding normalized misfit is close to one. This shows that our assimilation is robust with respect to the choice of initial conditions, provided that the choice is physical and the forward model is allowed to iterate through the transient regime.

APPENDIX D

SENSITIVITY OF ASSIMILATION RESULTS TO THE PERIOD OF UNICELLULAR INITIAL GUESSES

In this appendix we show some examples demonstrating the stability of the performance of the assimilation method with respect to the choice of period of the dynamo model based on unicellular meridional circulations of various amplitude, in the vicinity of a period of 44 years for case 1 and 22 years for cases 2 and 3, as mentioned in Section 3.2. The reference sampling is used (i.e., 1.5 cycles, monthly in time and uniform in space at \(\Delta \theta = 2.83^\circ\)). We start with a unicellular stream function and record the performance of convergence. The data is noised at \(\epsilon = 10\%\). We tabulate the tested trials in Table 5. The convergence criterion is fixed to \(|\nabla J|/|\nabla J_0| < 10^{-6}\), and we indicate a successful converging trial with the corresponding iterations required and a divergent trial with an “x”. The results show that the convergence is stable with respect to the change of period of the dynamo model with a unicellular initial guess within a finite margin (in the vicinity of 44 years for case 1 and 22 years for cases 2 and 3). This also justifies our choice of initial guesses in the previous analysis. However, this margin of stability shrinks as the meridional flow gets more complex from the unicellular case to the most complicated asymmetric case.
APPENDIX E
PARAMETER SPACE USING A SEPARABLE STREAM FUNCTION

In this appendix, we discuss the differences for the data assimilation procedure between using a separable stream function in the dynamo model or using the general linear combination defined in (2). We also illustrate with a few examples how well we can recover the separable \( \{a_i, b_j\} \), \( i = 1, \ldots, m \), \( j = 1, \ldots, n \) coefficients.

The key difference between the two mathematical structures of the stream function is that the general expansion \( \sum_{i,j} d_{i,j} \sin \frac{\pi x}{a} \cos \frac{\pi y}{b} \) constitutes a complete set in the 2D physical space, as \( m, n \) approaches infinity, whereas expanding the radial and polar dependencies separately \( \sum_{i=1}^m a_i \sin \frac{\pi x}{a} \sum_{j=1}^n b_j P_j^i (-\cos \theta) \) is only a subset of the general expansion in the 2D space. Therefore, theoretically, there is a trade-off between the two situations. The separable stream function model is neater and uses fewer parameters \( (m+n) \) to fit the observations compared with the general 2D expansion \( (mn) \) for constant \( m, n \). However, the misfit of using the separable model of the stream function in assimilation will always be essentially equal to or greater than the general expansion, as there are more degrees of freedom to control for the latter case during optimization. Also, having more parameters to adjust in the general structure makes it easier for the assimilation algorithm to reach a region with a lower gradient in the parameter space, compared with the separable expansion.

In an operational sense, the dependencies of the stream function (and so does the objective function and its gradient in the parameter space) on the parameters for the separable model are more complicated than those of the general expansion. For the general 2D expansion, the parameters to be estimated are \( d_i, s \), which appear as a linear combination in the expansion of the stream function in (2). While expanding the radial and polar dependencies separately, the parameters being estimated would be \( a_5 \) and \( b_5 \), which are nonlinearly coupled in the stream function, and the expression must first be linearized (in \( a_5 \) and \( b_5 \)) to evaluate the adjoint during operation.

In the following examples, we still limit the parameter space to \( m = 2, n = 4 \), so that \( m + n = 6 \) versus \( mn = 8 \) in the general case. Because it is the product, \( a_5 b_5 \) characterizes the stream function, and two different pairs of \( a_5, b_5 \) can describe the same meridional flow if the product \( a_5 b_5 \) is the same. In the evaluation of \( \Delta p/p \) using the separable model, we replace \( d_{i,j} \) with \( a_5 b_5 \) (the forecast) in (5).

We compare the performance of one assimilation trial for cases 1–3 using the reference sampling. We use the sum of both objective functions \( J_g + J_B \) and a one-cell flow as initial guess. The convergence criterion for both the separable and general expansion model are \( |\nabla J_g|/|\nabla J| < 10^{-6} \) (note that as the parameter spaces for two-stream function structures are different, the criteria may correspond to different accuracies as the gradients are taken with respect to different sets of variables). The synthetic observations are unnoised to rule out the possible effects of noise on the comparison.

The performance in terms of the convergence behavior as iteration evolves is plotted in Figure 16. The efficiency of assimilation using the separable model is slightly higher than that of the general expansion in case 1, but significantly lower than the latter in cases 2 and 3 when the flow becomes more

---

**Table 4**

Assimilation Results for Case 3, Showing the Need to Resort to an Ensemble of Initial Magnetic Conditions in Order to Eventually Achieve a Good Recovery of the Meridional Circulation

| Snapshot (year) | 1.07 | 3.22 | 5.37 | 7.52 | 9.67 | 11.8 | 14.0 | 16.1 | 18.3 | 20.4 |
|----------------|------|------|------|------|------|------|------|------|------|------|
| \( n_{\text{iter}} \) | 9 | 8 | 142 | 9 | 75 | 93 | 106 | 172 | 91 | 87 |
| \( \Delta p/p \) | \( x \) | \( x \) | 3.28 | 13.4 | 0.236 | 0.544 | 1.55 \( \times 10^{-2} \) | 1.48 | 0.326 | 8.65 \( \times 10^{-2} \) |
| \( J_{\text{norm}} \) | \( x \) | 9.90 | 10.2 | 2.34 | 2.57 | 1.03 | 3.22 | 2.24 | 1.18 |

**Note.** The 10 equally spaced snapshots defining the initial conditions are extracted from a 22-year cycle of a dynamo model driven by our initial guess of \( r_p \). The sampling window for assimilation has a width of 1.5 solar cycles, with observations made every month and uniformly distributed in latitude. A divergent iteration is marked with an “x” for the final discrepancy and normalized misfit. The assimilation is successful with minimum discrepancy of 1.55% if the snapshot of year 14.0 is used as the initial condition for the assimilation.

**Table 5**

Assimilation Performance with Respect to Initial Guesses of Unicellular Stream Function with Different Strength, Which Makes the Period of the Dynamo Vary

| Case | guessed \( d_{i,j} \) | 0.13 | 0.145 | 0.16 | 0.19 | 0.22 |
|------|-----------------|------|------|------|------|------|
| corr. period (yrs) | 54.2 | 48.3 | 43.5 | 36.6 | 31.6 |
| \( n_{\text{iter}} \) | 19 | 20 | 19 | 16 | 16 |
| \( J_{\text{norm}} \) | 1.00 | 1.00 | 1.00 | 1.00 | 1.00 |
| Case 2, Guessed \( d_{i,j} \) | 0.130 | 0.150 | 0.163 | 0.175 | 0.180 |
| Corr. Period (yrs) | 27.1 | 23.6 | 21.6 | 20.0 | 19.4 |
| \( n_{\text{iter}} \) | 49 | 46 | 50 | 45 | 32 |
| \( J_{\text{norm}} \) | 1.11 | 1.11 | 1.11 | 1.11 | 1.13 |
| Case 3, Guessed \( d_{i,j} \) | 0.180 | 0.185 | 0.190 | 0.195 | 0.200 |
| Corr. Period (yrs) | 23.1 | 22.6 | 22.2 | 21.8 | 21.4 |
| \( n_{\text{iter}} \) | 76 | 83 | 106 | 95 | 94 |
| \( J_{\text{norm}} \) | 3.04 | 1.03 | 1.03 | 1.03 | 6.36 |

**Note.** In case 1, the 22 years’ unicellular meridional flow case is the same model as the data file, so no essential assimilation is done, as a result, it is not included in the test. A successful trial shows a normalized misfit close to one, whereas a trial that is underfitted has a misfit much greater than one. The convergence is stable in the vicinity of the periods chosen to be the initial guesses for the analysis above, but the margin of the variation of the period shrinks as the model gets more complicated.
Figure 16. $\Delta p/p$ as the iteration evolves for the three cases, sampling monthly for 1.5 solar cycles, uniformly in latitudes. Note that the general expansion model is more efficient than the separable stream function expansion in cases 2 and 3.

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