From global to local dynamics: effects of the expansion on astrophysical structures

A Balaguera-Antolín and M Nowakowski

1 Max Planck Institut für Extraterrestrische Physik, Garching, Glienkenbachstrasse 1 D-85748, Garching, Germany
2 Departamento de Física, Universidad de los Andes, AA 4976, Bogotá, DC, USA

E-mail: a-balagu@uniandes.edu.co and mnowakos@uniandes.edu.co

Received 8 February 2007, in final form 5 April 2007
Published 30 April 2007
Online at stacks.iop.org/CQG/24/2677

Abstract
We explore the effects of background cosmology on large-scale structures with non-spherical symmetry by using the concept of quasi-equilibrium which allows certain internal properties (e.g. angular velocity) of the bodies to change with time. In accordance with the discovery of the accelerated phase of the universe, we model the cosmological background by two representative models: the \( \Lambda \)CDM model and the Chaplygin gas model. We compare the effects of the two models on various properties of large astrophysical objects. Different equations of state are also invoked in the investigation.

PACS numbers: 95.30.Sf, 98.62.Dm, 98.80.Jk, 98.52.Eh, 98.56.Ew

(Some figures in this article are in colour only in the electronic version)

1. Introduction
The interest in the impact of the cosmological expansion on bound astrophysical systems can be traced back to the paper by Einstein and Straus [1]. Over the subsequent years different approaches have been used to gain some insight into the interplay between cosmology and astrophysical structures [2–6]. With the advent of the discovery of the accelerated universe [7], it seems timely to pick up the topic once again and contrast the effects of different models which can explain the current accelerated phase. Indeed, in [8], such an examination was performed for phantom [9] and quintessence cosmologies [10]. In this paper, we will use the same approach as [2, 8] and apply it to models with a positive cosmological constant and Chaplygin gas models. These two models will serve us as representatives for cosmological models which can explain the current stage of acceleration. In addition, we will also invoke a general model with variable equation of state of the form \( \omega_x = p_x/\rho_x \) (DECDM). The approach mentioned above consists of generalizing the Newtonian limit by allowing terms
which are specific to the background cosmology. This is not limited to the inclusion of the cosmological constant. Note that the effects of the background cosmology on bound systems are now generally time dependent. This is, however, not in contradiction with the fact that we rightly consider most of the astrophysical structures to be in gravitational equilibrium. The time-dependent background cosmology will not tear the system apart unless its density is diluted, but will change only, over cosmological time, the internal properties of the system (such as inner velocities or angular velocity, etc). As a mathematical tool, we will make use of the virial theorem ([11] for different methods) by allowing certain internal properties to be time dependent. The virial theorem can be easily derived even in the presence of the cosmological background as the latter enters only the Newtonian limit (i.e. the Poisson equation for the gravitational potential which now includes a time-dependent part).

As shown in [12], dark energy can affect certain astrophysical static properties which have to do with either the virialization of the system [13] or the motion of test bodies [2]. For instance, the cosmological constant sets itself scales of distance \( r_{\Lambda} = \Lambda^{-1/2} \),\(^3\) time and mass, which are of the same order of magnitude as the radius of the observed universe, the age of the universe and the total mass of the universe (the so-called coincidence problem). At the first glance it looks hopeless to expect any effects from \( \Lambda \) at astrophysical scales. This is indeed so as long as no other scale enters the theory. If the latter appears in the theory a combination of the large cosmological and the small internal scale can yield values of astrophysical relevance (e.g. \( r_{\Lambda} r_{s} \))\(^{1/3} \), where \( r_{s} \) is the Schwarzschild radius) [14]. Another enhancement mechanism of background cosmology emerges when we are dealing with non-spherically symmetric objects. In such a case, the ratio of two length scales to some positive power (bigger than one) often goes hand in hand with \( \Lambda \). For this reason, in examining time-dependent effects, we will mostly employ ellipsoidal configurations. For general consideration of equilibrium in the spherical case, see [15, 16], the quasi-spherical case has been discussed in [17].

In this paper, we will entirely concentrate on the effects of dark energy followed over cosmological times, past and future. We do not expect these effects to be large; however, we think that it is of some importance to probe into these matters, especially in view of the accelerated universe. In [18], we restricted ourselves to effects over small time scales. In this sense, these two articles complement each other.

2. Local dynamics with background cosmology

In standard cosmology, the universe is described as an ideal fluid which implies the properties of homogeneity and isotropy. Such properties are mathematically represented in the Friedmann–Robertson–Walker (FRW) line element, which is written for a flat universe as 
\[
d s^2 = -dt^2 + R(t)^2 \left( dr^2 + r^2 d\Omega^2 \right).
\]

Einstein field equations and the FRW line element yields the acceleration equation and the well-known Friedmann equation, written, respectively, as
\[
\left[ \frac{R(t)}{R(t)} \right]^2 = H(t)^2 = \frac{8}{3} \pi \rho_b(t), \quad \frac{R(t)}{R(t)} = -\frac{4}{3} \pi \left[ \rho_b(t) + 3 p_b(t) \right]. \tag{1}
\]

In conventional cosmology, the total energy density \( \rho_b \) is a contribution of radiation, cold dark matter and dark energy. By neglecting interaction among these components, each one evolves through the conservation equation \( \dot{\rho} = -3H(\rho + p) \), so that radiation scales as \( \rho_{\text{rad}} = \rho_{\text{rad}}(R = 1) R^{-4} \), cold dark matter scales as \( \rho_{\text{cdm}} = \rho_{\text{cdm}}(R = 1) R^{-3} \), while the dark energy component associated with the cosmological constant is \( \rho_{\text{vac}} = \text{constant} \). The major contribution in connection to the cosmological constant is approximately 70% of total energy density [7].

\(^3\) We work in the so-called geometrized units, i.e., \( G_N = c = 1 \).
2.1. Dark energy

As pointed out before, almost 70% of the content of energy density in the universe is ruled by a dark energy component, and astronomical observations imply that at the present time this dark energy component might be represented by the cosmological constant. That is, at the present time, the equation of state that displays the best fit with astronomical observation is simply $p_{\text{vac}} = -\rho_{\text{vac}}$, i.e., $\omega = -1$. Nevertheless, several models have been proposed in order to reproduce the present value of dark energy but maintaining a time-dependent energy density. One generalizes the equation of state for the dark energy by $p_x(R) = \omega_x(R)\rho_x(R)$.

Dark energy then scales as $\rho_x(R) = \rho_x(R = 1)a^{-f(R)}$, where the function $f(R)$ is defined as

$$f(R) \equiv \frac{3}{\ln R} \int_{1}^{R} \frac{\omega_x(R')}{R'} dR'.$$

The model is constrained by requiring that at the present time, the dark energy density reproduces the vacuum energy density associated with the cosmological constant, that is, $\rho_\chi(R = 1) = \rho_{\text{vac}}$. As pointed out before, the case $\omega_\chi = -1$ corresponds to the cosmological constant $\rho_x = \rho_{\text{vac}} = \Lambda / 8\pi$, while for $0 > \omega_\chi > -1$ one stands in the dark energy realms. Models with $\omega_\chi < -1$ lead to future singularities in the so-called Phantom regime [9]. The description of a dynamical dark energy is compatible with the description of inflationary models.

In the DECDM model, the Friedmann equation and the acceleration equation are written as

$$\left(\frac{\ddot{R}(t)}{R(t)}\right)^2 = H_0^2\Omega_{\text{vac}}h_1(R) - \frac{\dot{R}(t)}{R(t)} = -H_0^2\Omega_{\text{vac}}h_2(R) = -\frac{8}{3}\pi\rho_{\text{vac}}h_2(R).$$

The functions $h_{1,2}(R)$ are given as

$$h_1(R) = \frac{\Omega_{\text{cdm}}}{\Omega_{\text{vac}}} R^{-3} + R^{-f(R)}, \quad h_2(R) = \frac{1}{2} \left[ \frac{\Omega_{\text{cdm}}}{\Omega_{\text{vac}}} R^{-3} + R^{-f(R)} (1 + 3\omega_\chi(R)) \right].$$

2.2. Chaplygin gas

The main stages of the universe that have passed through in the standard cosmology scenario can be reproduced by the introduction of the equation of state of an exotic ideal relativistic gas written as

$$p_{\text{ch}} = \kappa \rho_{\text{ch}} - \kappa_2 \rho_{\text{ch}}^\gamma.$$

The relevance of this equation of state lies in three different facts: (a) it violates the strong energy condition, which is necessary to obtain an accelerated phase at the present time, (b) it generates a well-defined speed of sound, which is relevant to the process of structure formation and (c) it unifies the early radiation or dark-matter behaviour with the late dark energy dominance. The so-called pure Chaplygin gas is obtained with $\kappa_1 = 0$ and $\gamma = 1$.

The time evolution of the Chaplygin gas energy density is given from the integration of the mass–energy conservation equation as

$$\rho_{\text{ch}}(R) = [A + B R^{-\beta}]^\gamma,$$

with $n = 3(1 + \kappa_1)$, $\beta = \gamma + 1$ and where $A$ and $B$ are integration constants given as

$$B = \rho_{\text{ch}}(R = 1)^\beta - A, \quad A = \frac{3}{n} \kappa_2 \equiv -\alpha_\beta H_0^2.$$
with $\rho_{\text{ch}}(R = 1) = \rho_{\text{crit}}$ as the Chaplygin gas energy density at the present time (only for a flat universe). In order to write the integration constants as we have done, we have re-parametrized the Chaplygin gas equation of state with $\kappa = \alpha n H_0^{2/3}$ such that a finite age and a present accelerated phase for the Chaplygin gas universe is reached for

$$\alpha_n < \frac{n}{3} \left(\frac{3}{8\pi}\right)^{1/\beta}.$$  

(8)

This bound can be derived after integration of the Friedmann equation

$$H(R)^2 = \frac{\dot{R}^2}{R^2} = \frac{8}{3\pi} [A + B R^{-n\beta}]^{1/\beta},$$

(9)

in order to determine the age of the universe. The age of the universe is given in terms of hypergeometric function as

$$T = \frac{1}{q H_0} \left[ \left(\frac{3}{8\pi}\right)^{1/\beta} \frac{3}{n\alpha_n} \right]^{1/\beta} \sqrt{\frac{3}{2\pi}}$$

$$\times _2 F_1 \left[ \frac{q}{2n\beta}, \frac{1}{2\beta} \frac{1}{2n\beta} + \frac{n + 4}{2n\beta} \right],$$

(10)

with $q \equiv 3\omega + 7$. For the pure Chaplygin gas, one then obtains a finite age and the observed acceleration at the present time only if the parameter $\alpha$ is such that $\alpha \frac{3}{n} < 10^{-2}$, which in turn implies a bound on the Chaplygin parameter $\kappa < \rho_{\text{crit}}^{\frac{2}{3}}$. This warrants that the equation of state is $\omega_{\text{ch}}(\text{today}) > -1$. The solution given in equation (6) allows us to describe the behaviour of Chaplygin gas at different ages. For early times, the $\kappa_1$-term dominates and Chaplygin gas behaves as $\rho_{\text{ch}} = B^{1/\beta} R^{-n}$. This scales as radiation for $n \equiv 4$ ($\kappa_1 = 1/3$). Hence we could, in principle, relate the integration constant $B$ with cosmological parameters of the concordance model as $B = \rho_{\text{mat}}^0 (R = 1)$. For $\kappa_1 = 0$, the model displays a matter-dominated epoch at early times with $\rho_{\text{ch}} \approx B^{1/\beta} R^{-3}$. In this case, one can write the integration constant $B = \rho_{\text{mat}}^0 (R = 1)$ and also the constant $A$ as $A = \rho_{\text{crit}}^0 (1 - \Omega_{\text{mat}}^\beta)$. Finally, for large times, Chaplygin gas acquires a vacuum-like behaviour with

$$\rho_{\text{ch}}(t \to \infty) = A^{1/\beta} = \left(\frac{3\alpha_n}{n}\right)^{1/\beta} H_0^2.$$  

(11)

This would lead to an effective vacuum energy density $\rho_{\text{vac}}^{\text{eff}}$ which can be associated with the current vacuum energy density $\rho_{\text{vac}}$ via

$$\rho_{\text{vac}}^{\text{eff}} = \rho_{\text{vac}} \left[ \frac{1 + B}{A} \right]^{1/\beta}.$$  

(12)

With $\rho_{\text{vac}}^{\text{eff}} > \rho_{\text{vac}}$, the effects of this effective vacuum energy density may be relevant in the Newtonian limit when we explore the equilibrium conditions of large-scale structures. Let us concentrate on the case $\kappa_1 = 0$, which does not reproduce an early-radiation-dominated era but still unifies an early dark matter dominance and a vacuum-dominated era at large-scale factors. The acceleration equation for this model reads

$$\frac{\ddot{R}(t)}{R(t)} = -\frac{4}{3\pi} [\rho_{\text{ch}}(R) \varrho_{\text{ch}}(R)]$$

$$= -\frac{4}{3\pi} [A + B R(t)^{-n\beta}]^{1/\beta} \left[ 1 - \frac{3\kappa_2}{A + B R(t)^{-n\beta}} \right].$$  

(13)
Figure 1. Left: equation of state for Chaplygin gas \( \omega_{ch} = p_{ch}/\rho_{ch} \) for different Chaplygin parameters. Right: acceleration equation as a function of the (future) scale factor for different Chaplygin parameters.

where \( \eta_{ch} \equiv 1 + 3\omega_{ch} \) and

\[
\omega_{ch} = \omega_{ch}(R) \equiv \frac{p_{ch}}{\rho_{ch}} = -\kappa^2 \rho_{ch}(R)^{-\beta}
\]

is the time-dependent equation of state. In figure 1, we show the behaviour of the equation of state \( \omega_{ch} \) and the acceleration equation for different values of Chaplygin parameters.

2.3. Newtonian description

The effects of the background can be explored through the Newtonian limit of field equations [19], from which one can derive a modified Poisson’s equation [2]

\[
\nabla^2 \Phi = 4\pi \delta \rho - 3 \left( \frac{\ddot{R}}{R} \right),
\]

where \( \delta \rho \) is the overdensity that gives rise to the ‘gravitational contribution’ of the potential. The total energy density within the clustered configuration \( \rho \) is a contribution of the background \( \rho_b \) and the collapsed fraction \( \delta \rho \). The background energy density depends on the model we are interested in:

\[
\rho_b = \rho(R) - \delta \rho = \begin{cases} 
\rho_{cdm}(R) & \text{DECDM} \\
\rho_{ch}(R) & \text{Chaplygin}. 
\end{cases}
\]

Note that the last expression implies that in the DECDM model, one has assumed that only the cold dark matter component collapses and forms vitalized structures, while in Chaplygin gas the perturbation is done on the total energy density. The solution for the potential can be simplified as

\[
\Phi(r, R) = \Phi_{grav}(r) = \left( \frac{\dot{R}}{R} \right) r^2, \quad \Phi_{grav}(r) = -\int \frac{\delta \rho(r')}{|r - r'|} d^3 r'.
\]
where one has neglected the terms associated with boundary conditions (the potential cannot be zero at infinity) \[19\]. The last term of equation (15) reduces to \(\rho_{\text{vac}}\) for \(\omega_\Lambda = -1\) and negligible contribution from the cold dark matter component, that is, for low redshifts. This limit represents the Newton–Hooke spacetime, which aside from the gravitational interaction has an external force of the Hooke form \(\sim \Lambda r^2\).

Gravitational equilibrium is represented through the Euler equation \(\rho \ddot{u} = -\nabla p - \rho \nabla \Phi\), where \(\rho\) is given by equation (16). The next generation of equilibrium equations comes from taking moments on the Euler equation and deriving the second-order virial equation \[12, 19, 20\]

\[
\frac{d^2 I_{ik}}{dt^2} = 2T_{ik} + \mathcal{W}_{ik}^{\text{grav}} + \left(\frac{\dot{R}}{R}\right) I_{ik},
\]

(18)

where \(\mathcal{W}_{ik}^{\text{grav}}\) is the gravitational potential energy tensor, whose trace corresponds to the gravitational potential energy, \(K_{ik}\) is the kinetic energy tensor, \(I_{ik}\) is the moment of inertia tensor and \(\Pi_{ik}\) is the dispersion tensor. The trace of these quantities leads to the well-known forms for gravitational potential energy:

\[
\mathcal{W}_{grav} = -\int_V \rho r_i \partial_i \Phi_{grav} d^3r = \frac{1}{2} \int_V \rho \Phi_{grav} d^3r,
\]

(19)

together with \(I \equiv \int_V \rho r^2 d^3r\) and \(\Pi \equiv \int_V p d^3r\). As usual, the set of equilibrium equations is closed with the equation for mass conservation, energy conservation and an equation of state \(p = p(\delta \rho, s)\).

The tensor virial equation can be thought of as a differential equation for the moment of inertia, which in turn is converted to a differential equation for the parameters determining the geometrical properties of the configuration. In order to explore the contribution from the background, let us write the virial equation (18) as

\[
\frac{d^2 I_{ik}}{dR^2} = \frac{2}{R^2 H^2(R)} \left[ 2T_{ik} + \mathcal{W}_{ik}^{\text{grav}} + \left(\frac{\dot{R}}{R}\right) \left( I_{ik} - \frac{1}{2} \frac{dI_{ik}}{dt} \right) + \Pi_{ik} \right],
\]

(20)

3. Dynamical equilibrium

If we assume equilibrium via \(\ddot{I} \approx 0\), we obtain the virial theorem

\[
|\mathcal{W}_{grav}| = 2K + \left(\frac{\dot{R}}{R}\right) I, \quad K = T + \frac{3}{2} \Pi.
\]

(21)

Strictly speaking, a formal equilibrium configuration is never reached since the energy-like terms in (18) are time dependent. Nevertheless, we can use equation (21) by assuming that as long as \(R\) evolves in time, the configuration evolves through successive states of equilibrium.

The expressions derived in the last section, specially (15) and (21), can be used for testing dark energy models on configurations in equilibrium. In this section, we will concentrate on possible effects of the two cosmological models described above on cosmological structures with non-spherical symmetry, such as low density galaxies and galactic clusters.

For ellipsoidal homogeneous configurations, the gravitational potential energy tensor and the moment of inertia tensor are written as

\[
\mathcal{W}_{ik}^{\text{grav}} = -\frac{8}{15}\pi^2 \rho (\rho - \rho_b) a_1 a_2 a_3^2 A_i \delta_{ik} = -2\pi \delta \rho A_i I_{ik},
\]

\[
I_{ik} = \frac{4}{15} \pi \rho a_1 a_2 a_3^2 \delta_{ik},
\]

(22)

where the quantities \(A_i\) are functions of the eccentricities of each case: for an oblate configuration we have \(a_1 = a_2 > a_3\), \(e_{\text{oblate}}^2 = 1 - q_2^2\), while for prolate \(a_1 = a_2 < a_3\).
In order to determine the angular velocity, one may consider a non-zero isotropic dispersion
\[ W_{\Omega_1} = \frac{2}{3} \left( \sqrt{1-e^2} \right) \left[ \frac{\arcsin e}{e} - \sqrt{1-e^2} \right] \]
\[ \omega_x = 1 - \frac{2q_3}{\Omega_1} \] with \( q_i \equiv a_i/a_1 \). The functions \( A_i \) are given as [21]
\[ A_1 = A_2 = \left\{ \begin{array}{ll}
\sqrt{1-e^2} \left[ \frac{\arcsin e}{e} - \sqrt{1-e^2} \right] & \text{oblate} \\
\frac{1}{2e} \left[ \frac{1 + \ln \left( \frac{1+e}{1-e} \right)}{1-e} - 1 \right] & \text{prolate}
\end{array} \right. \] (23)
and
\[ A_3 = \left\{ \begin{array}{ll}
2 \sqrt{1-e^2} \left[ \frac{1}{\sqrt{1-e^2}} - \frac{\arcsin e}{e} \right] & \text{oblate} \\
2 \frac{1}{2e} \left[ \ln \left( \frac{1+e}{1-e} \right) - 1 \right] & \text{prolate}.
\end{array} \right. \] (24)
In the limit \( e \to 0 \) one has \( A_i \to 2/3 \) for the prolate and oblate configurations. Using equations (22), the potential energies are then written for the models of interest as
\[ \mathcal{V}^{\text{grav}} = \mathcal{V}^{\text{grav}}_{\text{tot}} + \frac{\mathcal{R}}{R} \mathcal{I}_{\text{ik}} = \left\{ \begin{array}{ll}
\frac{8}{15} \pi^2 \rho^2 a_1 a_2 a_3 a_4^2 A_1 \delta_{ik} - \frac{4}{3} \pi \eta_x \rho_x \mathcal{I}_{\text{ik}} + 2 \pi \rho_{\text{cdm}} \left( A_i - \frac{2}{3} \right) \mathcal{I}_{\text{ik}} & \text{DECDM} \\
\frac{8}{15} \pi^2 \rho^2 a_1 a_2 a_3 a_4^2 A_3 \delta_{ik} + 2 \pi \rho_{\text{ch}} \left( A_i - \frac{2}{3} \eta_{\text{ch}} \right) \mathcal{I}_{\text{ik}} & \text{Chaplygin}
\end{array} \right. \] (25)
for DECDM and Chaplygin gas, respectively, with \( \eta_x = 1 + 3 \omega_x \) and \( \rho_x = \rho_{\text{vac}} (1 + \omega_x)^{f(z)} \). Note that the last term on the DECDM model vanishes in two different situations: the first is in the case of spherical symmetry and when we neglect the contribution from the cold dark matter component with respect to the proper density of the system and the dark energy contribution. This approximation is only valid for very low redshifts. At high redshifts we approach a matter-dominated universe and hence the term proportional to \( \rho_{\text{cdm}} \) is relevant.

3.1. Oblate systems

When considering oblate configurations \( (a_1 = a_2 > a_3) \), we assume that the kinetic energy is due to constant rotation along the minor axis. We may then write the kinetic energy tensor as
\[ T_{ik} = \frac{1}{2} \left( \Omega_{\text{tot}}^2 \mathcal{I}_{ik} - \Omega_{\text{rot}} \mathcal{I}_{ik} \Omega_{\text{rot}} \right). \] (26)
In order to determine the angular velocity, one may consider a non-zero isotropic dispersion tensor \( \Pi_{ik} = \delta_{ik} \Pi \). We then have
\[ \Omega_{\text{rot}}^2 = \frac{\mathcal{V}^{\text{grav}}_{\text{tot}} - \mathcal{V}^{\text{grav}}_{\text{rot}}}{\mathcal{I}_{xx}} = \left( \frac{\mathcal{R}}{R} \right) \left( 1 - \frac{\mathcal{I}_{zz}}{\mathcal{I}_{xx}} \right), \] (27)
which follows from eliminating the trace of the dispersion tensor from the tensor virial equations. Using equation (25), we can explicitly write for the angular velocity
\[ \frac{\Omega_{\text{rot}}^2}{2 \pi \delta \rho} = \left( A_1 - A_3 q_3^2 \right) \left( 1 + \frac{1 - q_3^2}{2 \pi \delta \rho (A_1 - A_3 q_3^2)} \times \frac{H_0^2 \rho_{\text{vac}} h_2(R)}{4 \sqrt{3} \pi \rho_{\text{ch}} (R) \eta_{\text{ch}} (R) \text{DECDM}} \right), \] (28)
In the Newton–Hooke spacetime with \( \omega_x = -1 \) \( (h_2 \to -1) \) this reduces to
\[ \frac{\Omega_{\text{rot}}^2}{2 \pi \delta \rho} \approx A_1 - A_3 q_3^2 - \frac{4}{3} \left( \frac{\rho_{\text{vac}}}{\delta \rho} \right) (1 - q_3^2), \] (29)
which in turn reproduces the Maclaurin formula for \( \rho_{\text{vac}} = 0 \).
3.2. Prolate systems

For prolate configurations \(a_1 = a_2 < a_3\), we solve for the velocity dispersion of main components rather than for an angular velocity (rotation of prolate configurations, although rare, has been observed and some properties of this situation have been explored in \[18\]). Hence we use the virial theorem (21) with the kinetic energy given by

\[
\mathcal{K} = \frac{1}{2} \int \rho \langle v^2 \rangle \, d^3r = \frac{2}{3} \pi \rho a_1^2 a_3 \langle v^2 \rangle. \tag{30}
\]

In analogy with equation (28), the virial theorem then implies for the velocity dispersion

\[
\frac{\langle v^2 \rangle}{2 \pi \delta \rho} = \frac{1}{5} a_1^2 (2 A_1 + q_3^2 A_3) \left( 1 + \frac{2 + q_3^2}{2 \pi \delta \rho (2 A_1 + q_3^2 A_3)} \times \frac{H_0^2 \Omega_{\text{vac}} h_2(R)}{4 \pi \delta \rho h(R)} \right) \tag{31}
\]

for the DECDM and Chaplygin gas model, respectively. As in (28), one obtains a scale factor (or redshift) dependence of the velocity dispersion through the contribution of the cosmological background. In the Newton–Hooke spacetime one reduces to

\[
\frac{\langle v^2 \rangle}{2 \pi \delta \rho} \approx \frac{1}{5} a_1^2 \left[ 1 - \frac{4 \rho_{\text{vac}}}{3 \delta \rho} \left( \frac{2 + q_3^2}{2 A_1 + q_3^2 A_3} \right) \right]. \tag{32}
\]

which in turn reduces for spherical symmetry to the known expression \[12, 22\]

\[
\frac{\langle v^2 \rangle}{2 \pi \delta \rho} \approx \frac{1}{5} a_1^2 \left[ 1 - 2 \frac{\rho_{\text{vac}}}{\delta \rho} \right]. \tag{33}
\]

The structure of the resulting expressions for the angular velocity and the velocity dispersion can be written in a similar way. Let \(\Delta\) measure the ratio of each velocities with respect to the velocities when there is no background contribution. We may then write

\[
\Delta_{\Omega, v} = \frac{\Omega_{\text{tot}}^2 (\rho_b = 0)}{\langle v^2 \rangle (\rho_b = 0)} = 1 - G_{\Omega, v} \left( \frac{1}{2 \pi \delta \rho} \right) \frac{R}{R}, \tag{34}
\]

where the geometrical factor \(G\) is written for each case as

\[
G = \begin{cases} 
G_\Omega = \frac{1 - q_3^2}{A_1 - q_3^2 A_3} & \text{oblate, rotational velocity} \\
G_v = \frac{2 + q_3^2}{2 A_1 + q_3^2 A_3} & \text{prolate, velocity dispersion}.
\end{cases} \tag{35}
\]

Returning to the oblate case, in a generalized Newton–Hooke spacetime (i.e. with \(-1 < \omega < 0\), one observes that the dark energy component decreases the angular velocity (or the velocity dispersion in the prolate case) in order for the system to maintain equilibrium (since \(1 + 3 \omega < 0\)). At the present time \((R = 1, z = 0)\) we get \(\Delta_\Omega \sim 1 - 0.8 g(e)(\rho_{\text{vac}}/\delta \rho)\), where

\[
g(e) \equiv \frac{4}{\pi} e^5 \left[ (1 - e^2)^{1/2} (3 - 2 e^2) \arcsin e - 3 e (1 - e^2) \right]^{-1}. \tag{36}
\]

An extreme situation is reached in the limit \(\Delta_\Omega \to 0\), which implies \(\Omega_\text{vac} \to 0\). For \(e \sim 0.8\) this would require \(\xi \equiv 2 \rho_{\text{vac}}/\delta \rho \sim 0.5\), which is a very diluted configuration. For higher eccentricities (say \(e \sim 0.97\)) one would need a system with \(\xi \sim 0.2\) which is
still a rather low value. In the ΛCDM model Ω = 0 is reached at a redshift given by [23]

\[ z_c = \left( \frac{2\Omega_{\text{vac}}(\xi g(e) - 2))}{(\xi g(e)\Omega_{\text{cdm}})} \right)^{1/3} - 1. \]

For realistic examples as \( \delta \rho \sim 200\rho_{\text{cdm}} \) the value \( z_c \) could be reached for \( e \) very close to 1. That is, the value \( z_c \) requires extreme flat objects.

In figures 2 and 3, we show the behaviour of the ratio \( \Delta_v \) for different values of eccentricities and proper densities in the DECDM case with \( \omega_x = -1 \) and in the Chaplygin gas model, respectively. The larger effects appear for low densities and large eccentricities, as expected. However, the fractional change is very low. For \( \delta \rho = 10^2\rho_{\text{vac}} \), the larger effect occurs for scale factors \( \geq 2.6 \) with a change of 1%.

In figure 4, we show the behaviour of \( \Delta_\Omega - 1 \) as a function of the redshift for different densities and \( e = 0.9, 0.95 \) for three different values of the equation of state for dark energy. It is clear that the effects associated with a cosmological constant are stronger than those associated with other dark energy models, but in general those effects are small for realistic values of \( \xi \), as used in the figure. We would have to measure the angular velocity at different redshifts very exactly to see an effect over a range of \( z \). However, the difference between the models is more significant.

4. Dynamical evolution

In this section, we want to explore the virial equation with the cosmological background without insisting on the equilibrium condition (dynamical equilibrium). Even though we will not make extensive use of these new equations, we want to demonstrate that the equilibrium condition is not a necessary ingredient for studying the effects of different background cosmologies. The results are differential equations for the geometrical parameters of a given configuration. Our starting point is equation (20). In the first approach, we will concentrate...
on pressureless systems with homogeneous density. The problem is then reduced to determine $a_1 = a_1(R)$ and $a_3 = a_3(R)$. As pointed out before, equation (20) represents a set of two
nonlinear coupled second-order equations for \( i = x, z \)

\[
\frac{d^2 I_{ii}}{dR^2} = \frac{2}{R^2 H^2(R)} \left[ 2T_{ii} + \mathcal{W}^{\text{grav}}_{ii} + \left( \frac{\ddot{R}}{R} \right) \left( I_{ii} - \frac{1}{2} \frac{dI_{ii}}{dt} \right) \right],
\]

(37)

where we have used equation (25). The derivatives of the moment of inertia tensor are given as follows:

\[
\frac{dI_{ii}}{dR} = \frac{8}{15} \pi \rho a_i^2 a_j \frac{da_i}{dR} = 2a_i^{-1} \frac{da_i}{dR} T_{ii},
\]

\[
\frac{d^2 I_{ii}}{dR^2} = \frac{8}{15} \pi \rho a_i^2 a_j a_i \left[ \left( \frac{da_i}{dR} \right)^2 + a_i \frac{da_i}{dR} \right] = 2a_i^{-1} \left[ \left( \frac{da_i}{dR} \right)^2 + a_i \frac{d^2 a_i}{dR^2} \right] T_{ii}.
\]

(38)

These expressions enclose our approximation: the volume of the system is constant even if the semi-axes may change, so that one can write the moment of inertia tensor as

\[
I_{ik} = \frac{1}{5} M a_i^2 \delta_{ik},
\]

(39)

where the mass \( M \) is

\[
M = M_b + \delta M = \frac{4}{3} \pi a_1^2 a_3 \rho_b(R) + \delta M, \quad \delta M = \int_V \delta \rho \, d^3r.
\]

(40)

In order to deal with the kinetic energy tensor, we follow [24] and transform from proper coordinates to fixed co-moving coordinates via \( r_i = A_{i\alpha} x_\alpha \), where \( A_{i\alpha} \) is such that

\[
A_{i\alpha} = a_i \delta_{i\alpha}
\]

and hence the ellipsoid is characterized by the constraint \( \delta_{\alpha\beta} x_\alpha x_\beta = 1 \). We have

\[
T_{i\alpha} = \frac{1}{2} \rho \int_V \left( \frac{dA_{i\alpha}}{dt} \right) \left( \frac{dA_{i\beta}}{dt} \right) x_\alpha x_\beta A \, d^3x,
\]

(41)

where \( A = \det(A_{i\alpha}) = a_1 a_2 a_3 \). Using the definition of the moment of inertia tensor, one has

\[
I_{ik} = \rho A_{i\alpha} A_{i\beta} A \int x_\alpha x_\beta d^3x,
\]

(42)

and hence

\[
T_{i\alpha} = \frac{1}{2} A_{i\alpha}^{-1} A_{i\beta}^{-1} \left( \frac{dA_{i\alpha}}{dt} \right) \left( \frac{dA_{i\beta}}{dt} \right) I_{ik}.
\]

(43)

We may then write

\[
T_{ii} = \frac{1}{2} a_i^{-2} \left( \frac{da_i}{dt} \right)^2 I_{ii} = a_i^{-2} R^2 \left( \frac{da_i}{dR} \right)^2 I_{ii} \times \left\{ \begin{array}{ll}
\frac{1}{2} H_b^2 \Omega_{vac} h_1(R) \\
n \frac{4}{3} \pi \rho_b(R)
\end{array} \right.
\]

(44)

for the DECDM and Chaplygin gas, respectively. Equation (37) together with (38) is the differential equation to be solved. The functions \( A_i \) depend on the eccentricity of the system and hence they are also functions of the semi-axis \( A_i(z) = A_i(a_1(R), a_3(R)) \). The differential equation can then be cast into the following form:

\[
\frac{d^2 a_i}{dR^2} = \frac{1}{R^2 H^2(R)} \left[ 2T_{ii} + \mathcal{W}^{\text{grav}}_{ii} + \left( \frac{\ddot{R}}{R} \right) \left( \frac{1}{a_i R} \frac{da_i}{dR} \right) - a_i \left( \frac{da_i}{dR} \right)^2 \right].
\]

(45)
5. Conclusion

In this paper, we have examined the effects of the expansion of the universe on certain quasi-static properties of large astrophysical bodies. We have done this by invoking a dynamical equilibrium, i.e., the time-dependent response of the astrophysical objects to the time-dependent cosmological background is taken into account by allowing some internal properties to become epoch dependent. In spite of the fact that the expansion of the universe is accelerated, the effects are rather small, at most a few percents over a large cosmological time stretch in the case of the Chaplygin gas model. However, a qualitative investigation, done in this paper, seems mandatory to establish the size of the effects. Different equations of state can be clearly distinguished theoretically; however, in practice this would require a good knowledge of the angular velocities at different redshifts. In spite of the smallness of the effects, the results clearly demonstrate that background cosmology affects, in principle, local properties of astrophysical bodies and furthermore the details of these effects are model dependent. In this paper, we focused on the dynamical equilibrium. One can enlarge the concept of the influence of background cosmology by dropping the equilibrium condition and going over to a fully dynamical scenario. A first step in this direction was done in section 4. Supplementing these equations with the Euler equation for the angular velocity, one has a full set of differential equations determining the time evolution of the parameters during (and after) virialization. We will come back to this point in future publications.

References

[1] Einstein A and Straus E G 1945 Rev. Mod. Phys. 17 2–3
[2] Noerdlinger P and Petrosian V 1971 Astrophys. J. 168 1
[3] Bona C and Stela J 1987 Phys. Rev. D 36 2915
[4] Baker G A 2001 Preprint astro-ph/0112320
[5] MacVittie C G 1933 Mon. Not. R. Astron. Soc. 93 325
[6] Nunes N J and Mota D F 2006 Mon. Not. R. Astron. Soc. 368 751 (Preprint astro-ph/0409481)
[7] Perlmutter S et al 1998 Nature 391 51
Perlmutter S et al 1999 Astrophys. J. 517 565
[8] Nesseris S and Perivolaropoulos L 2004 Phys. Rev. D 70 123129
[9] Nojiri S and Odinstov S D 2005 Preprint hep-th/0505215
Godlowski W and Szydłowski M 2005 Preprint astro-ph/0507322
Polarski D and Ramiéquet A 2005 Preprint astro-ph/0507299
[10] Hu W 2005 Phys. Rev. D 71 047301 (Preprint astro-ph/0410680)
Barreiro T, Copeland E and Nunes N J 2000 Phys. Rev. D 61 127301
Steinhardt P J, Wang L and Zlatev I 1999 Proc. Rev. D 59 123504
Albrecht L R and Skordis K 2000 Phys. Rev. Lett. 84 2076
Wang L, Caldwell R R, Ostriker J P and Steinhardt P J 2000 Astrophys. J. 530 17–35 (Preprint astro-ph/9901388)
[11] Cardoso V and Gualtieri L 2006 Class. Quantum Grav. 23 7198
[12] Balaguera-Antolínez A and Nowakowski M 2005 Astron. Astrophys. 441 23
[13] Mota D F and van de Bruck C 2004 Astron. Astrophys. 421 71
[14] Balaguera-Antolínez A, Böhmer C and Nowakowski M 2006 Class. Quantum Grav. 23 485–96
[15] Boehmer C G 2004 Gen. Rel. Grav. 36 1039
[16] Boehmer C G and Harko T 2005 Phys. Rev. D 71 084026
[17] Debnath U, Nath S and Chakraborty S 2006 Mon. Not. Astron. Soc. 369 1961
[18] Balaguera-Antolínez A, Mota D F and Nowakowski M 2006 Class. Quantum Grav. 23 4497–510
[19] Nowakowski M 2001 Int. J. Mod. Phys. D 10 649
[20] Nowakowski M, Sanabria J-C and García A 2002 Phys. Rev. D 66 023003
[21] Binney J and Tremaine S 1987 Galactic Dynamics (Princeton, NJ: Princeton University Press)
[22] Wang L and Steinhardt P J 1998 Astrophys. J. 508 483
[23] Balaguera-Antolínez A and Nowakowski M 2006 Preprint astro-ph/0603624
[24] Peebles P J E 1980 The Large-Scale Structure of the Universe (Princeton, NJ: Princeton University Press)