Geometrical Construction of Supertwistor Theory

Kazuki Hasebe
Department of General Education,
Takuma National College of Technology,
Takuma-cho, Mitoyo-city, Kagawa 769-1192, Japan
Email: hasebe@dg.takuma-ct.ac.jp

Supertwistor theory is geometrically constructed based on the SUSY Hopf map. We derive a new incidence relation for the geometrical supertwistor theory. The present supertwistor exhibits remarkable properties: Minkowski space need not be complexified to introduce spin degrees of freedom, and even number SUSY is automatically incorporated by the geometrical set-up. We also develop a theory for massless free particle in Minkowski superspace, which physically corresponds to the geometrical supertwistor theory. The spin degrees of freedom are originated from fermionic momenta as well as fermionic coordinates. The geometrical supertwistor is quantized to reproduce same physical contents as in the original supertwistor theory. Relationships to superspin formalism and SUSY quantum Hall effect are also discussed.

I. INTRODUCTION

As is well known, twistor theory is an approach towards a geometrical quantization of space-time, originally proposed by Penrose [1]. In the twistor program, the twistor space is regarded more fundamental than space-time. A light ray (massless particle) has special importance in twistor theory, and the twistor space is naturally introduced as parameter space of massless particle. A time slice of light-cone is given by a celestial sphere, and the mathematical foundation of the twistor theory is intimately related to the Hopf map:

$$S^3 \rightarrow S^2.$$  

(1.1)

This particular notion of the nontrivial homotopy from sphere to sphere in different dimensions plays a crucial role in constructing the twistor theory [2]. It is known that 2-dimensional sphere is a special manifold that accommodates complex structure, and mathematical progress initiated by the twistor formalism has exclusively indebted to analytic properties of the twistor space $\mathbb{R}^2$.

In this paper, we construct a supersymmetric extension of twistor theory (supertwistor theory) based on a purely geometrical set-up: the supersymmetric extension of the Hopf map (SUSY Hopf map) $\mathbb{R}^4 \rightarrow S^2$: [3, 4, 5]:

$$S^{3|2} \rightarrow S^{2|2}.  \quad (1.2)$$

The fermionic components are geometrically introduced by the SUSY Hopf map, and bring spin degrees of freedom. In the conventional twistor theory, the complexified Minkowski space is postulated to introduce spin degrees of freedom, and the hermiticity of Minkowski space is sacrificed. In the supertwistor theory first introduced by Ferber [5], imaginary coordinates of the complexified Minkowski space are replaced by fermion bispinor forms, but the complexified Minkowski space is still postulated. In the present geometrical approach, Minkowski space need not be complexified, and the hermiticity of Minkowski space is promoted to the super-hermiticity in superspace. The incidence relation is also naturally promoted to a SUSY framework, and provides a new nonlocal relation between Minkowski superspace and supertwistor space. The supersymmetry has a geometrical meaning given by the SUSY Hopf map, and the number of supersymmetry always takes even number. It is known that the number of supersymmetry has to be even to provide integer or half integer helicity multiplets in quantized supertwistor theory [4], and the number of supersymmetry has been conventionally fixed by hands. In the present approach, even number of supersymmetry is automatically incorporated by the geometrical set-up.

This paper is organized as follows. In Sect. II we review the Hopf map and its relation to the twistor theory. In Sect. III replacing the Hopf map with the SUSY Hopf map, we develop a geometrical supertwistor theory. Properties of the super incidence relation with emphasis on differences to Ferber’s approach are discussed in Sect. IV. In Sect. V we explore a massless particle model in Minkowski superspace, which corresponds to the geometrical supertwistor formalism. In Sect. VI the geometrical supertwistor is quantized to yield same physical contents obtained in the original supertwistor theory. Relations to Bloch supersphere and SUSY quantum Hall effect are discussed in Sect. VII. Sect. VIII is devoted to summary and discussion. In Appen.A several definitions used in super Lie group are briefly explained.

II. REVIEW OF HOPF MAP AND INCIDENCE RELATION

First, we introduce Hopf map and discuss its relation to the twistor theory. The Hopf map $S^3 \rightarrow S^2$ is explicitly given by

$$\phi \rightarrow x^a = \phi^b \sigma^a \phi,$$  

(2.1)

where $\phi$ is a normalized two-component complex (Hopf) spinor: $\phi^b \phi = 1$, and $\sigma^a$ ($a = 1, 2, 3$) denote the Pauli matrices. By the normalization constraint, $\phi$ is regarded
as the coordinates on $S^3$, and $x^a$ defined by (2.1) satisfy the relation $x^a x^a = 1$ that represents two-sphere with unit radius. The Hopf map is the template for more complicated twistor theory, and as a preparation, we exploit its basic features here. By reversing the Hopf map (2.1), the Hopf spinor is given by

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{2(1 + x^3)}} \begin{pmatrix} 1 + x^3 \\ x^1 + ix^2 \end{pmatrix}, \quad e^{i\chi},$$  

(2.2)

where $e^{i\chi}$ is the $U(1)$ phase factor canceled in the mapping (2.1), and the projective Hopf spinor space is defined as $S^3/S^1 \approx S^2 \approx \mathbb{CP}^1$. The Hopf map (2.1) suggests that the Hopf spinor is a zero-mode of the "space-matrix" $r$:

$$r = -1 + x^a \sigma^a = \begin{pmatrix} -1 + x^3 \\ x^1 - ix^2 \end{pmatrix}.$$  

(2.3)

With the stereographic coordinates $x = x^1/(1 + x^3)$ and $y = x^2/(1 + x^3)$, the Hopf spinor is rewritten as

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \frac{1}{\sqrt{1 + x^2 + y^2}} \begin{pmatrix} 1 \\ x + iy \end{pmatrix}, \quad e^{i\chi},$$  

(2.4)

and the upper component and lower component in the Hopf spinor is simply related as

$$\phi_1 = (x + iy)\phi_2.$$  

(2.5)

This is the simplest incident relation that specifies one-to-one correspondence between points on the projective Hopf spinor space $S^2$ and points on the stereographic space $R^2$ (except for the infinite distance). The incidence relation is gauge independent in the sense that the $U(1)$ phase factor does not appear in (2.5). It is straightforward to generalize the above set-up for two-sphere with arbitrary radius $t$:

$$t^2 = x^a x^a.$$  

(2.6)

The Hopf mapping is rephrased as

$$t = \phi^1 \phi, \quad x^a = \psi^1 \sigma^a \phi,$$  

(2.7)

and the space matrix is naturally promoted to the "space-time" matrix $x$:

$$x = -t + x^a \sigma^a = \begin{pmatrix} -t + x^3 \\ x^1 - ix^2 \end{pmatrix}, \quad \begin{pmatrix} x^1 + ix^2 \end{pmatrix}.$$  

(2.8)

It is important to notice if we identify $t$ as time, the present two-sphere is regarded as time-slice of a light cone, i.e. celestial sphere made of light rays passing through the origin of Minkowski space. With this identification, the sphere condition (2.6) becomes null vector condition for $x^a = (t, x^1, x^2, x^3)$:

$$\eta_{\mu\nu} x^\mu x^\nu = \det(x) = 0,$$  

(2.9)

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The coordinates $x^\mu$ can be inversely obtained from $x$ as

$$x^\mu = \eta_{\mu\nu} tr(x \sigma^\nu),$$  

(2.10)

where $\sigma^\mu = (1, \sigma^a)$. With the space-time matrix (2.8), the incidence relation in the twistor theory is given by (1)

$$\begin{pmatrix} Z^1 \\ Z^2 \end{pmatrix} = i \begin{pmatrix} -t + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & -t - x^3 \end{pmatrix} \begin{pmatrix} Z^3 \\ Z^4 \end{pmatrix},$$  

(2.11)

where $Z^a = (Z^1, Z^2, Z^3, Z^4)$ represent twistor variables, and $x^\mu$ are real coordinates in Minkowski space (and are not necessarily a null vector). As in the simplest incidence relation (2.20), (2.21) specifies relations between points in the twistor space and space-time events in Minkowski space. Conventionally, two spinor components of the twistor are introduced $Z^a = (\omega^\alpha, \pi^\beta)$ ($\alpha, \beta = 1, 2$), and the incidence relation is written as

$$\omega^\alpha = i x^a \pi^b \pi^a.$$  

(2.12)

The space-time matrix is not affected by any complex scaling of the twistor variables, and the projective twistor space is defined as $\mathbb{CP}^3 = S^7/S^1$ where $S^1$ represents the overall $U(1)$ phase freedom. Unlike the simplest version (2.8), the incidence relation (2.22) connects the space-time events and the twistor points nonlocally. When a point in twistor space is given, the corresponding space-time point is determined up to the gauge transformation

$$x^\alpha \beta \rightarrow x^\alpha \beta + ax^\alpha \pi^\beta,$$  

(2.13)

where $a$ is an arbitrary real parameter to keep the hermiticity of $x^\alpha \beta$, and $\pi^\alpha$ is defined as $\pi^\alpha = (-\pi_2, \pi_1)$. Such gauge degree of freedom corresponds to a null direction in Minkowski space, since the null vector $p^\mu$ is constructed by the gauge part as

$$p^\mu = -2 \eta_{\mu\nu} (x^\nu) \pi^\alpha \pi^\alpha \pi^\beta.$$  

(2.14)

Thus, a point in the twistor space is nonlocally transformed to a light ray in Minkowski space. The inverse transformation from Minkowski space to the twistor space is explained as follows. Here, the coordinates in Minkowski space are supposed to be real, then the twistors satisfy the null condition:

$$Z^a \pi^a = 0,$$  

(2.15)

where $Z^a = (\pi^a, \omega^a)$ represents the dual twistor. Thus, the corresponding (projective) twistor space is given by the real five dimensional manifold called the projective null twistor space $\mathbb{PN}$. Provided the lower spinor component $\pi^a$ given, the entire twistor point is uniquely determined by the incidence relation. Since the lower component $\pi^a$ geometrically represents $S^2$, a point in Minkowski space corresponds to a two-sphere in the projective twistor space. Such nonlocal transformations are the most particular feature in the twistor theory.

III. SUSY HOPF MAP AND SUPER SPACE-TIME MATRIX

It has been reported the existence of the SUSY extension of the Hopf map [3, 4, 7], that is the SUSY
Hopf map: $S^{3|2} \rightarrow S^{2|2}$. The 3-component super (Hopf) spinor $\psi = (\psi_1, \psi_2, \psi_0)^t$, in which $\psi_1$ and $\psi_2$ are Grassmann even components and $\psi_0$ is a Grassmann odd component, plays a crucial role in constructing the SUSY Hopf map explicitly. The super Hopf spinor is normalized as $\psi^\dagger \psi = 1$ with $\psi^\dagger = (\psi_1^*, \psi_2^*, -\psi_0^*)$. Here, * is not the conventional complex conjugation but the super-conjugation. (For the definition of the super-conjugation, see Appen. A) The SUSY Hopf map is given by

$$2\psi^\dagger l^a \psi = x^a, \quad 2\psi^\dagger l^a \psi = \theta^a, \quad (3.1)$$

where $l^a$ and $l^a$ are

$$l^a = \frac{1}{2} \begin{pmatrix} \sigma^a & 0 \\ 0 & 0 \end{pmatrix}, \quad l^a = \frac{1}{2} \begin{pmatrix} 0 & \tau^a \\ -(C\tau^a)^t & 0 \end{pmatrix}, \quad (3.2)$$

with $\tau^1 = (1,0)^t$, $\tau^2 = (0,1)^t$, and $C$ is the charge conjugation matrix:

$$C = C_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad C^t = C_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.3)$$

The spinor index is raised or lowered as $\phi^a = C_{\alpha\beta} \phi_\beta$ and $\phi_\alpha = C_{\alpha\beta} \phi^\beta$. $l^a$ and $l^a$ satisfy the $OSp(1|2)$ algebra

$$[l^a, l^b] = i\epsilon^{abc} l^c, \quad [l^a, l^b] = \frac{1}{2} (\sigma^a)_{\beta}^\alpha l^\beta, \quad \{l^a, l^b\} = \frac{1}{2} (C\sigma^a)_{\alpha\beta} l^a, \quad (3.4)$$

with $\epsilon^{123} = 1$. Under the definition of the super-conjugation, $x^a$ and $\theta^a$ become (pseudo-)real in the sense:

$$x^{a^*} = x^a, \quad \theta^{a^*} = \theta^a, \quad (3.5)$$

where $\theta^a = C_{\alpha\beta} \theta^a$. Besides, from the normalized super spinor $\psi$, $x_a$ and $\theta_a$ satisfy the condition

$$x^a x^a + C_{\alpha\beta} \theta^a \theta^\beta = 1, \quad (3.6)$$

which defines the supersphere with unit radius. Reversing (3.1), the super Hopf spinor is expressed as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_0 \end{pmatrix} = \frac{1}{\sqrt{2(1+x^2)}} \begin{pmatrix} (1+x^3)(1-\frac{1}{4(1+x^2)} C\theta) \\ x^1 + ix^2 + 1 \quad x^3 \\ 1 + x^3 \theta^{1} + (x^1 + ix^2)^2 \end{pmatrix} \psi^x, \quad (3.7)$$

Following the discussion in Sect. III the “super space” matrix is similarly introduced as

$$R = -2l^0 + 2x^a l^a + 2C_{\alpha\beta} \theta^\alpha \theta^\beta = \begin{pmatrix} -1 + x^3 & x^1 - ix^2 & -\theta_2 \\ x^1 + ix^2 & -1 - x^3 & \theta_1 \\ -\theta_1 & -\theta_2 & 1 \end{pmatrix}, \quad (3.8)$$

where $l^0$ is

$$l^0 = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (3.9)$$

The SUSY Hopf map (3.1) suggests that the super Hopf spinor is a zero-mode of the super space matrix: $R \psi = 0$. The super stereographic coordinates are introduced as

$$z = \psi_2 \psi_1 = \frac{x^1 + ix^2}{1 + x^3} \left(1 + \frac{1}{2(1+x^3)} C\theta\right), \quad \theta = \frac{\psi_0}{\psi_1} = \theta^1 + z\theta^2, \quad (3.10)$$

and the SUSY Hopf spinor is represented as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_0 \end{pmatrix} = \frac{1}{\sqrt{1 + zz^* + \theta^\alpha \theta^\beta}} \begin{pmatrix} 1 \\ z \\ \theta \end{pmatrix} \cdot \psi^x. \quad (3.11)$$

The super incidence relations are

$$\psi_2 = z \psi_1, \quad \psi_0 = \theta \psi_1. \quad (3.12)$$

It is easy to generalize the above discussion for the supersphere with arbitrary radius $t$:

$$l^2 = x^a x^a + C_{\alpha\beta} \theta^a \theta^\beta. \quad (3.13)$$

The corresponding super Hopf map is given by

$$2\psi^\dagger t \psi = t, \quad 2\psi^\dagger l^a \psi = x^a, \quad 2\psi^\dagger l^a \psi = \theta^a. \quad (3.14)$$

Identifying $t$ as time, the present supersphere is regarded as a celestial supersphere that is equal to a time slice of super light-cone passing through the origin of Minkowski superspace. Here, Minkowski superspace is referred to $M_{4|2}$ which has 6 (pseudo-)real coordinates, 4 of which are bosonic $x^\mu (x^{\mu*} = x^\mu)$, 2 are fermionic $\theta^a (\theta^{a*} = \theta_a)$. The supersphere condition (3.13) is rephrased as the null super vector condition of $x^\mu$ and $\theta^a$:

$$\eta_{\mu\nu} x^\mu x^\nu + C_{\alpha\beta} \theta^a \theta^\beta = -t \cdot sdet X = 0, \quad (3.15)$$

where the “super space-time” matrix $X$ is defined as

$$X = 2\eta_{\mu\nu} x^\mu l^\nu + 2C_{\alpha\beta} \theta^a \theta^\beta = \begin{pmatrix} -t + x^3 & x^1 - ix^2 & -\theta^2 \\ x^1 + ix^2 & t^2 - x^3 & \theta^1 \\ -\theta^1 & -\theta^2 & t \end{pmatrix}. \quad (3.16)$$

The coordinates in Minkowski superspace are inversely obtained as

$$x^0 = -\text{str}(X l^0), \quad x^a = \frac{1}{2} \eta_{\mu\nu} \text{str}(X l^\mu), \quad \theta^a = \frac{1}{2} \text{str}(X l^a). \quad (3.17)$$

It should be noted that the super space-time matrix is super-hermitian under the definition of the super-adjoint $\dagger$ in Appen. A

$$X^\dagger = X. \quad (3.18)$$

IV. SUPER INCIDENCE RELATION

Based on the analogy to the original incidence relation (2.11), we introduce the super incidence relation as

$$\begin{pmatrix} Z^1 \\ Z^2 \\ \zeta^1 \end{pmatrix} = i \begin{pmatrix} -t + x^3 & x^1 - ix^2 & -\theta^2 \\ x^1 + ix^2 & t^2 - x^3 & \theta^1 \\ -\theta^1 & -\theta^2 & t \end{pmatrix} \begin{pmatrix} Z^4 \\ Z^5 \\ \zeta^2 \end{pmatrix}. \quad (4.1)$$
where $x^\mu$ and $\theta^i$ need not be a super null vector [6,15]. Since the super space-time matrix is given by the 3×3 supermatrix, the corresponding superwistor has 6 components: $Z_A = (Z^1, Z^2, Z^3, Z^4, \xi^1, \xi^2)$ where $Z^1, Z^2, Z^3$ and $Z^4$ are Grassmann even while $\xi^1$ and $\xi^2$ are Grassmann odd quantities. It should be noted in the present approach, the number of the Grassmann odd components is fixed to 2 by the geometrical set-up. In [6,11], the super space-time matrix is invariant under the arbitrary complex scaling of the superwistors, and the projective superwistor space is defined by the projection of the complex scaling, and hence has the (pseudo-)real dimension 6/4. Introducing two super spinors $\pi_A$ and $\omega^A$:

$$\omega^A = (\omega^\alpha, \omega) = (Z^1, Z^2, \xi^1),$$

$$\pi_A = (\pi_\alpha, \pi) = (Z^3, Z^4, \xi^2),$$ (4.2)

the super incidence relation (4.1) is written as

$$\omega^\alpha = i\varepsilon^{\alpha\beta} a_{\beta\alpha} - i \theta_{\alpha i} \pi_i,$$

$$\omega = -i \theta^\alpha a_{\alpha\beta} + it \pi_i.$$ (4.3)

The super incidence relation specifies nonlocal relations between superwistor space and Minkowski superspace. With given a point in twistor space, the corresponding point in Minkowski superspace cannot be determined uniquely due to the existence of the gauge degree of freedom in [13]:

$$x^{\alpha \beta} \rightarrow x^{\alpha \beta} + a(2 \pi^{\alpha \star} \pi^{\beta} - \delta^{\alpha \beta} \pi^{\star} \pi),$$

$$\theta^\alpha \rightarrow \theta^\alpha - a(\pi^{\alpha \star} \pi + \pi^{\star} \pi^{\alpha}),$$ (4.4)

where $a$ is an arbitrary real parameter. The transformation of $t = x^{33}$ follows from that of $x^{\alpha \beta}$:

$$t \rightarrow t - a(\pi^{1 \star} \pi + \pi^{2 \star} \pi - \pi^{\star} \pi),$$ (4.5)

and, similarly, $\theta_\alpha$ follows from $\theta^\alpha$:

$$\theta_\alpha \rightarrow \theta_\alpha + a(\pi^{\alpha \star} \pi - \pi^{\star} \pi^{\alpha}).$$ (4.6)

Such gauge degrees of freedom represent a direction of a super light ray (this will be discussed in detail in Sect.VI), and a point in superwistor space is nonlocally transformed to a super light ray in Minkowski superspace. Since the space-time matrix is super-hermitian, the superwistor variables satisfy the super null condition

$$Z_A Z^A = 0,$$ (4.7)

where $Z^A$ denote the dual superwistor defined by $Z^A = (Z^*_\alpha, \xi^*_i) = (\pi^{\alpha \star}, \omega^{1 \star}, \pi, \omega^*).$ Thus, the present projective superwistor is null, and carries (pseudo-)real 5/4 degrees of freedom. With given a super space-time point, the corresponding point in the superwistor space is uniquely determined provided the lower components $\pi_A = (\pi_\alpha, \pi)$ given. This indicates that a point in Minkowski superspace is nonlocally transformed to a supersphere $S^{2|3}$ in the projective superwistor space. The super incidence relation is easily generalized to include N flavor Grassmann odd coordinates:

$$\omega^\alpha = i x^{\alpha \beta} \pi_{\beta} - i \theta_{\alpha i} \pi_i,$$

$$\omega^i = -i \theta^\alpha a_{\alpha \beta} + it \pi_i.$$ (4.8)

where $i$ is the flavor index for Grassmann odd coordinates, $i = 1, 2, \cdots, N$. The corresponding superwistor is $Z^A = (\omega^\alpha, \pi_{\alpha}, \omega^i, \pi_i)$, and its dual is $Z_A = (\pi^{\alpha \star}, \omega^{\alpha \star}, \pi^{i \star}, \omega^i)$. One may notice that the number of the fermion components in $Z^A$ is necessarily even, 2N, due to the appearance of pairs of $\theta^i$ and $\pi_i$.

Here, we comment differences between the present incidence relation (4.8) and Ferber’s original relation [8]:

$$\omega^\alpha = i (x^{\alpha \beta} + \frac{i}{2} \theta^{\alpha \star} \theta^\beta) \pi_{\beta},$$

$$\omega^i = i \theta^\alpha a_{\alpha \beta}.$$ (4.9)

where Grassmann coordinate index $i$ runs to arbitrary integer $N$, and $\ast$ represents the conventional complex conjugation. First of all, in Ferber’s superwistor, the superwistors consist of $(\omega^\alpha, \pi_{\alpha}, \omega^i)$ and fermionic counterparts of $\pi_{\alpha}$, namely $\pi_i$, do not exist. The fermion components in the present superwistors are double compared to the original Ferber’s set-up, and this discrepancy becomes important in discussing the spin degrees of freedom in quantum superwistor theory (See Sect.VI). Next, in Ferber’s incidence relation (4.9), the space-time matrix is given by $x^{\alpha \beta} + \frac{i}{2} \theta^{\alpha \star} \theta^\beta$ and is not hermitian due to the imaginary factor in front of fermionic bilinears, while, in the present, the space-time matrix is promoted to a super-hermitian matrix (4.8). Besides, in the present, the gauge freedom is the bosonic one (4.3) only, while in Ferber’s, fermionic gauge freedoms exist as well as bosonic one:

$$x^{\alpha \beta} \rightarrow x^{\alpha \beta} + a \pi^{\alpha \star} \pi^{\beta} - i (\beta_i \pi^{i \star} \pi^{\beta} + \beta_i^* \pi^{\alpha \star} \pi_i),$$

$$\theta^\alpha_i \rightarrow \theta^\alpha_i + \beta_i \pi^{i \star}.$$ (4.10)

where $a$ is a Grassmann even real parameter and $\beta_i$ are Grassmann odd complex parameters. The geometrical meaning of the bosonic gauge transformation is apparent as in the bosonic twistor: a direction of a light ray in Minkowski space, while the geometrical meaning of the fermionic gauge transformation is not clear. Similarly, a space-time point in Minkowski is transformed to two-sphere (not supersphere) in the superwistor space in Ferber’s incidence relation.

V. MASSLESS PARTICLE IN MINKOWSKI SUPER SPACE-TIME

The massless particle set-up provides a complementary physical approach to the purely mathematical construction [6], and here, such massless particle model for the
geometrical supertwistor theory is explored. Hereafter, we consider Minkowski superspace $\mathcal{M}_{4|2}$ with the metric
\[ dt^2 = \eta_{\mu\nu} dx^\mu dx^\nu + C_{\alpha\beta} d\theta^\alpha d\theta^\beta. \] (5.1)
The free particle action in $\mathcal{M}_{4|2}$ is given by
\[ S = \frac{\mu}{2} \int dt (\eta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + C_{\alpha\beta} \dot{\theta}^\alpha \dot{\theta}^\beta), \] (5.2)
where $\mu$ denotes the mass of the particle and $\cdot$ the derivative about the invariant length $\tau$. Introducing the auxiliary variable $p_\mu$ and $p_\alpha$, (5.2) is rewritten as
\[ S = \int d\tau (\dot{p}_\mu + \dot{\theta}^\alpha p_\alpha - \frac{1}{2\mu} p^\mu p_\mu - \frac{1}{2\mu} \theta^\alpha p_\alpha), \] (5.3)
where $x^\mu$, $\theta^\alpha$, $p^\mu$, and $p^\alpha$ are treated as independent variables. We are interested in the case of the massless particle in which $p^\mu$ and $p^\alpha$ satisfy the super null condition:
\[ \eta_{\mu\nu} p^\mu p^\nu + C_{\alpha\beta} p^\alpha p^\beta = 0. \] (5.4)
The super momenta, $p^\mu$ and $p^\alpha$, subject to the condition, can be simply expressed as the bilinear forms of 3-component superspinor $\pi_A = (\pi_1, \pi_2, \tau^-)^t$:
\[ p^0 = \pi^+ \pi, \quad p^1 = 2\pi^1 \tau^-, \quad p^2 = 2\tau^+ \tau^-. \] (5.5)
$\pi_A$ are the “square root” of the super null momenta, and regarded as more fundamental variables than super momenta. With use of the superspinor, the massless free action becomes
\[ S_0 = \int dt \pi_A^* \dot{x}^{AB} \pi_B = \int dt (\dot{x}^{\alpha\beta} \pi^\alpha \pi^\beta + \dot{\theta}^{\alpha} (\pi^\alpha \pi_\alpha + C_{\alpha\beta} \pi^\beta \pi^\alpha) + \dot{\pi}_A^* \pi^A), \] (5.6)
where $x^{AB}$ denotes the components of $\pi_A$. $S_0$ is invariant under the global translation in the supertwistor space,
\[ x^{AB} \rightarrow x^{AB} + c^{AB}, \quad \pi_A \rightarrow \pi_A. \] (5.7)
In detail,
\[ x^{\alpha\beta} \rightarrow x^{\alpha\beta} + c^{\alpha\beta}, \quad \theta^\alpha \rightarrow \theta^\alpha + \gamma^\alpha, \] (5.8)
where $c^{\alpha\beta}$ denote Grassmann even constants and $\gamma^\alpha$ Grassmann odd constants. From (5.6), the equations of motion for $x^\mu$ and $\theta_\alpha$ are derived as
\[ \frac{d}{d\tau} (\pi^\alpha \dot{\pi}_\beta - \frac{1}{2} \delta_{\alpha\beta} \pi^\alpha \pi^\alpha) = 0, \]
\[ \frac{d}{d\tau} (\pi^\alpha \pi_\beta + C_{\alpha\beta} \pi^\alpha \pi^\beta) = 0. \] (5.9)
These provide 6 independent real equations, and suggest
\[ \dot{\pi}_A = 0, \] (5.10)
which is consistent with the assumption that the particle is free and hence carries conserved momenta. Similarly, the equations of motion of $\pi_A$ are derived as
\[ \dot{x}^{\alpha\beta} \pi_\beta - C_{\alpha\beta} \dot{\theta}^\beta \pi = 0, \quad \dot{\pi}_A \pi = 0, \] (5.11)
or
\[ \dot{x}^{\alpha\beta} = \pi^{\alpha\beta} \pi_\beta - \frac{1}{2} \delta_{\alpha\beta} \pi^\alpha \pi^\alpha, \]
\[ \dot{\theta}^\alpha = -\frac{1}{2} \pi^\alpha \pi - \frac{1}{2} \pi^\alpha \pi^\alpha, \]
\[ \dot{\pi}_3 = -\frac{1}{2} (\pi^\alpha \pi^\alpha + \pi^\alpha \pi^\alpha - \pi^\alpha \pi^\alpha). \] (5.12)
The right-hand-sides of (5.12) are concisely represented by the super momentum matrix $p$:
\[ p = 2\eta_{\mu\nu} p^\mu p^\nu + 2C_{\alpha\beta} p^\alpha p^\beta = \begin{pmatrix} -p_0 + p_3 - p_1 - ip_2 & p_1 + ip_2 & -p_0 \\ -p_0 & -p_3 & p_1 \\ -p_1 & -p_2 & p_0 \end{pmatrix}. \] (5.13)
From (5.5), the components of $p$ are given by
\[ p^{\alpha\beta} = -2\pi^{\alpha\beta} \pi_\beta + \delta^{\alpha\beta} \pi^\alpha \pi, \]
\[ p^{\alpha 3} = \pi^{\alpha -} \pi^+ \pi^-, \]
\[ p^{3\alpha} = -\pi^{3} \pi^+ \pi^-, \]
\[ p^{33} = \pi^{3} \pi^+ \pi^+. \] (5.14)
Then, (5.12) is simply expressed as
\[ \dot{x}^{AB} = -\dot{a}^A \pi^{AB}, \] (5.15)
and the solution is obtained as
\[ x^{AB} = x_0^{AB} - a(\tau) p_0^{AB}, \] (5.16)
where we have used (5.10), and $p_0^{AB}$ represent a constant super momentum matrix. Substituting (5.14) to (5.16), one may find that the gauge transformation in the super incidence relation (1.3) is reproduced. Thus, the massless particle formulation in Minkowski superspace is a physical set-up for the geometrical supertwistor theory.

VI. SUPERTWISTOR ACTION AND QUANTIZATION

Generally, the gauge degree of freedom of the solution is a consequence of that of the action. Indeed, the massless superparticle action (5.6) is invariant under the gauge transformation
\[ x^{AB} \rightarrow x^{AB} - a(\tau) p_0^{AB}. \] (6.1)
($\pi_A$ is a zero-mode of $p_0^{AB}$, $p_0^{AB} \pi_B = 0$.) Then, the super space-time matrix $x^{AB}$ is a gauge dependent quantity, and the gauge invariant quantity is introduced as
\[ \omega^A = ix^{AB} \pi_B. \] (6.2)
This is nothing but the super incidence relation \((4.1)\). Its (pseudo-)complex conjugation is given by
\[
\omega^{A*} = -i\pi_B^{*}e^{BA}.
\] (6.3)

Now, the super massless particle action \((5.6)\) is concisely expressed as
\[
S_0 = -i \int d\tau (\pi^{A*} \omega_A + \omega^{B*} \pi_B),
\] (6.4)
and, with the supertwistor variables \(Z^A = (\omega^A, \pi_A, \omega, \pi)\), further simplified as
\[
S_0 = -i \int d\tau \frac{d}{d\tau} Z^A_{\Lambda},
\] (6.5)
where \(Z^A_{\Lambda}\) and \(Z^A_{\Lambda}^*\) represent the twistor and dual twistor variables subject to the constraint \((4.7)\). Up to total derivatives, the action \((6.5)\) is invariant under the global translation in supertwistor space:
\[
Z^A \rightarrow Z^A + D^A
\] (6.6)
with constant supertwistor \(D^A\). The supertwistor action \((6.5)\) and the constraint \((4.7)\) are “diagonalized” by recombination of the supertwistor variables:
\[
Z_D = \frac{1}{\sqrt{2}} \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix} Z.
\] (6.7)

With use of \(Z_D\), the supertwistor norm is represented as
\[
Z_D Z^A = Z_D^1 Z_1^{*} + Z_D^2 Z_2^{*} - Z_D^3 Z_3^{*} + Z_D^4 Z_4^{*} + \xi_1^{*} \xi_1^j - \xi_2^{*} \xi_2^j - \xi_3^{*} \xi_3^j,
\] (6.8)
and thus the supertwistor space has the metric: \(\text{diag}(+,-,-,-,+,+,-,-)\). The action and the null constraint are invariant under the \(SU(2, 2|1, 1)\) global transformation of the supertwistor variables. Generally, with \(N\)-flavor fermionic coordinates, the number of SUSY is \(N' = 2N\), and the global symmetry becomes \(SU(2, 2|N, N)\).

Next, we discuss the quantization of the supertwistor. For simplicity, we consider one-flavor fermion case \((N = 1\) then \(N' = 2)\). With use of the bosonic and fermionic components of supertwistors, the action \((6.5)\) is rewritten as
\[
S_0 = -i \int d\tau (Z_\Lambda^* \frac{d}{d\tau} Z^\Lambda + \xi^i_d \frac{d}{d\tau} \xi^i_d).
\] (6.9)
and the super null condition \((4.7)\) becomes
\[
Z_\Lambda^* Z^\Lambda + \xi^i_d \xi^i_d = 0.
\] (6.10)

Apparently, \((6.9)\) and \((6.10)\) are equal to what used in the original supertwistor theory \((5)\), so the quantization reproduces same physical contents as in the original supertwistor. We briefly explain the quantization procedure and results. From \((6.9)\), the canonical conjugation of \(Z_\Lambda\) is obtained as \(-iZ_\Lambda^*,\) and that of \(\xi^i_d\) is \(i\xi^i_d\). Applying the canonical quantization condition to these variables
\[
[Z_\Lambda^a, Z_\Lambda^b] = -\delta^a_b, \quad \{\xi^i_d, \xi^j_d\} = \delta^i_j, \quad \{\xi^i_d, \xi^j_d^*\} = 0,
\] (6.11)

and the constraint \((4.7)\) are invariant under the \((pseudo-)\)complex conjugation is given by
\[
\xi^i_d = \frac{\partial}{\partial \xi^i_d}.
\] (6.12)
The super null condition \((6.10)\) is expressed as \([Z^a, Z^*_a] + [\xi^i_d, \xi^*_i_d] = 0\) and imposed to the Hilbert space:
\[
\{[Z^a, Z^*_a] + [\xi^i_d, \xi^*_i_d]\} \Psi = 0.
\] (6.13)
In the coordinate representation, \((6.13)\) is rewritten as
\[
(Z^a \frac{\partial}{\partial Z^a} + \xi^i_d \frac{\partial}{\partial \xi^i_d} + 1) \Psi = 0,
\] (6.14)
where \(Z^a \frac{\partial}{\partial Z^a}\) is known as the Euler homogeneity operator. Then, \(\Psi\) should be a homogeneous function of \(Z^a\) and \(\xi^i_d\), and the sum of the powers of \(Z^a\) and \(\xi^i_d\) should be \(-1\). Thus, in general, \(\Psi\) is expressed as
\[
\Psi = t_{1/2}(Z^a) + t_0(Z^a)\xi^1_d + t_{-1/2}(Z^a)\xi^2_d, \quad \text{for simplicity, we consider one-flavor fermion case (5)}.
\] (6.15)

Thus, the number of SUSY (charges) is equal to that of the fermionic components of supertwistor \(\xi^i_d\). It is straightforward to introduce \(N\) fermionic components in supertwistors \(\xi^i_d\) \((i = 1, 2, \cdots, N)\). In such \(N\)-SUSY case, \((6.14)\) is generalized as
\[
(s + \frac{1}{2} \xi^i_d \frac{\partial}{\partial \xi^i_d} - \frac{N}{4}) \Psi = 0,
\] (6.21)
where $\frac{\partial}{\partial x^i} \xi_i = -\xi_i \frac{\partial}{\partial x^i} + \mathcal{N}$ was used. Since the operator $\xi^i \frac{\partial}{\partial x^i}$ can take the eigenvalues $0, 1, 2, \cdots, \mathcal{N}$, the eigenvalues of the helicity operator are distributed as

$$s = -\frac{\mathcal{N}}{4} - \frac{\mathcal{N}}{4} + 1, \cdots, \frac{\mathcal{N}}{4} - \frac{\mathcal{N}}{2}, \frac{\mathcal{N}}{2}. \quad (6.22)$$

Although the resultant quantum supertwistor is superficially equal to that of the original supertwistor, there are important differences. In the original supertwistor, the basic quantities are given by $x^\mu, p^\mu$ and $\theta^\alpha_i$ that amount to the complex coordinates: $y^\mu = \frac{1}{2} \sigma_{\alpha\beta}^\mu \theta^\alpha \theta^\beta$, while in the present, the basic quantities are $x^\mu, p^\mu, \theta^\alpha_i$ and $p_i^\alpha (i = 1, 2, \cdots, \mathcal{N})$, and complex space-time is not introduced. In both approaches, the spin degrees of freedom are originated from the existence of the fermionic variables, since, from the null supertwistor condition \( (4.7) \), the helicity $s$ \( (6.17) \) is restated as

$$s = -\frac{1}{2} \xi^i \xi_i. \quad (6.23)$$

However, in the present geometrical formalism, the momentum space and the space-time are treated equivalently, and there always exist pairs of fermionic variables: \( (p_i^\alpha, \theta^\alpha_i) \) or \( (\pi_i, \omega^i) \). Then, the helicity $s$ is expressed as

$$s = -\frac{1}{2} (\pi^i \omega^i + \omega^i \pi^i) \quad (6.24)$$

with $\omega^i$ given by \( (4.3) \), and such fermion sets amounts to even number SUSY $\mathcal{N} = 2N$. Meanwhile in the Ferber’s original supertwistor, the helicity is given by

$$s = -\frac{1}{2} \omega^i \omega_i \quad (6.25)$$

with $\omega^i$ given by \( (4.9) \), and the number of SUSY is $\mathcal{N} = N$. Even number of SUSY is physically required to bring integer of half-integer helicities (See \( (6.22) \)), and it has been fixed by hand in the original supertwistor. Meanwhile in the geometrical construction, such condition is automatically satisfied because $\mathcal{N} = 2N$. Thus, even number of SUSY is necessarily incorporated in the geometrical supertwistor. Besides, in Ferber’s approach the signatures of the fermionic space are not uniquely determined, while in the present they are unique: $N$ for $+$ and $N$ for $-$.\]

VII. RELATIONS TO SUPERSPIN AND SUSY QUANTUM HALL EFFECT

It is known that the (bosonic) Hopf map is a mathematical background of quantum mechanics of spin \( 10 \) and quantum Hall effect on two-sphere \( 11 \). Here, we discuss how their structures are generalized and related to the geometrical supertwistor when the SUSY Hopf map is adopted.

A. Relation to Superspin on Bloch supersphere

In the context of spin quantum mechanics, the Hopf spinor is used to construct a spin coherent state

$$|\phi >= \phi_1 |> + \phi_2 |>, \quad (7.1)$$

where $\phi_1, \phi_2$ is the Hopf spinor \( (2.2) \) that specifies a point on Bloch sphere by the Hopf map \( (2.1) \). It is well known that the $SU(2)$ spin mechanics is reformulated by introducing Schwinger bosons, $a$ and $b$, $|> = a^0|0 >$, $|> = b^0|0 >$. In other words, the Hopf spinor is a coherent state representation of the Schwinger boson:

$$<\phi|a >= \phi_1, \quad <\phi|b >= \phi_2. \quad (7.2)$$

The spin magnitude corresponds to half of the total number of Schwinger bosons. For instance, to represent spin $1/2$, the Schwinger boson operators satisfy the constraint

$$a^a b + b^b a = 1. \quad (7.3)$$

Meanwhile in the present, we have used the super Hopf spinor which contains two Grassmann even and one Grassmann odd components. Then, the corresponding operators may be given by two bosonic operators $a$ and $b$, and one fermionic operator $f$:

$$(\psi_1, \psi_2, \psi_0) \rightarrow (a, b, f). \quad (7.4)$$

The normalization condition for the SUSY Hopf spinor is transformed to the constraint of the operators:

$$1 = a^a b + b^b f + f^b a, \quad (7.5)$$

which represents the superspin $1/2$. Such formalism is known as the slave fermion formalism in condensed matter physics, where the fermionic operator is introduced to deal with the inequivalent condition

$$a^a b + b^b a \leq 1. \quad (7.6)$$

Thus, in the slave fermion formalism, spin $1/2$ and $0$ are treated simultaneously. With the super Hopf spinor \( (7.7) \), the supersymmetric extension of the spin coherent state is constructed as

$$|\psi >= |\psi_1 |> + \psi_2 |> + \psi_0 |f >= \psi_1 |a > + \psi_2 |b > + \psi_0 |f >, \quad (7.7)$$

which is also known as the spin-hole state \( 12 \). Thus, Bloch supersphere is the hidden geometry behind the slave fermion formalism \( 21 \), and based on this observation, a supersymmetric antiferromagnetic valence bond solid model was constructed recently \( 13 \).

B. Relation to SUSY Quantum Hall Effect

Based on the SUSY Hopf map, a supersymmetric extension of the quantum Hall effect is constructed in
where the fermionic variables are interpreted as spin degrees of freedom. In the present supertwistor model, the number of (minimal) SUSY is $\mathcal{N} = 2$, while in the SUSY quantum Hall effect $\mathcal{N} = 1$. This two-fold difference suggests the geometrical supertwistor may consist of two copies of the SUSY quantum Hall effect. Here, we pursue this heuristic observation. The supertwistor action (6.9) is rewritten as

$$S = -i \int d\tau Z^1_+ \frac{d}{d\tau} Z^1_+ + i \int d\tau Z^1_- \frac{d}{d\tau} Z^1_-,$$  
(7.8)

where $Z^1_+$ and $Z^1_-$ denote the diagonal supertwistors: $Z^1_+ = (Z^1_D, Z^2_D, Z^3_D, Z^4_D)^t$ and $Z^1_- = (Z^1_D, Z^2_D, Z^3_D, Z^4_D)^t$, which satisfy the super null condition: $Z^1_+ Z^1_+ - Z^1_- Z^1_- = 0$. We focus on a “slice” of the null supertwistor space

$$Z^1_+ Z^1_- = Z^1_- Z^1_+ = R^2,$$  
(7.9)

with some constant $R$. From (6.10), the coordinates on supersphere are naturally defined as

$$x^a = 2Z^1_+ l^a Z^1_+,$$

$$y^a = 2Z^1_- l^a Z^1_-,$$  
(7.10)

that satisfy the relation: $x^a x^a + C_{\alpha\beta} \theta^\alpha \theta^\beta = y^a y^a + C_{\alpha\beta} \theta^\alpha \theta^\beta = R^2$, and the SUSY monopole gauge fields are induced as

$$-i Z^1_+ dZ^1_+ = dx^a A_\alpha + d\theta^\alpha A_\alpha,$$

$$-i Z^1_- dZ^1_- = dy^a A_\alpha + d\theta^\alpha A_\alpha.$$  
(7.11)

Then, the action (7.8) becomes

$$S = \int d\tau \frac{dx^a}{d\tau} A_\alpha + \int d\tau \frac{d\theta^\alpha}{d\tau} A_\alpha - \int d\tau \frac{dy^a}{d\tau} A_\alpha - \int d\tau \frac{d\theta^\alpha}{d\tau} A_\alpha,$$  
(7.12)

which is formally equivalent to two copies of one-particle action used in the SUSY quantum Hall effect by replacing invariant time $\tau$ with time $t$. The opposite signs in front of the two copies suggest that the magnetic fields are inversely aligned in such two copies. This is something similar to the spin Hall effect [13], where up-spin and down-spin feel opposite effective magnetic fields. Indeed, the bosonic part of the action (7.12) is equal to what was used in the context of quantum spin Hall effect [14]. Thus, in the slice of the supertwistor space, SUSY spin Hall analogous system is supposed to be realized.

VIII. SUMMARY AND DISCUSSION

We have geometrically constructed a supertwistor theory based on the SUSY Hopf map. The basic variables are different from those of Ferber’s original supertwistor; fermionic momenta are newly introduced by geometrical reasoning. The new super incidence relation is naturally derived based on the arguments of the celestial supersphere. The super space-time matrix becomes super-hermitian and relates the Minkowski superspace and the supertwistor space nonlocally in the sense: a point in Minkowski superspace is transformed to a supersphere in the supertwistor space, and a point in the supertwistor space is transformed to a super light ray in the Minkowski superspace. The quantum theory of the geometrical supertwistor reproduces same physical contents as in the original supertwistor theory, and besides, the present formalism has following remarkable properties. First of all, the space-time is not complexified to introduce the spin degrees of freedom. The space-time is promoted to a super-hermitian superspace and the (pseudo)real fermion variables yield the origin of spin degrees of freedom. Pairs of fermionic momenta and fermionic coordinates are introduced, which necessarily amount to even number of SUSY to bring half integer or integer helicity states. We have also discussed relations to super-spin quantum mechanics and the SUSY quantum Hall effect. Bloch supersphere is the template geometry of the present model, and provides the hidden geometry of the slave fermion formalism. With an appropriate choice of the slice of supertwistor space, the SUSY spin Hall analogous system is supposed to be realized. Twistor theory shares many analogous properties with quantum Hall effect, such as holomorphicity of twistor functions and lowest Landau level functions, fuzzy geometry in space(-time) [14] [18]. We would like report detail analyses of their relations elsewhere. The higher dimensional SUSY Hopf maps are proposed in [19], and it is also interesting to see what geometrical supertwistor models come out based on such higher dimensional SUSY set-up.

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APPENDIX A: SEVERAL DEFINITIONS FOR SUPERMATRIX

We briefly summarize several definitions used in super Lie group. (For more detail, see [20].) The superconjugation acts to Grassmann odd quantities $\eta$ and $\xi$ as

$$(\eta\xi)^* = \eta^*\xi^*, \quad (\eta^*)^* = -\eta.$$  
(A1)

With the supermatrix taking the form of

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix},$$  
(A2)

where $A$ and $D$ are Grassmann even component matrices, $B$ and $C$ are Grassmann odd component matrices, the super-adjoint $\dagger$ is defined as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^\dagger = \begin{pmatrix} A^\dagger & C^\dagger \\ -B^\dagger & D^\dagger \end{pmatrix}.$$  
(A3)
The supertrace is given by

\[
\text{str} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \text{tr}A - \text{tr}D, \quad (A4)
\]

and the superdeterminant is

\[
\text{sdet} \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{\det(A - BD^{-1}C)}{\det D} = \frac{\det A}{\det(D - CA^{-1}B)}. \quad (A5)
\]