Abstract

For every variety $\Theta$ of universal algebras we can consider the category $\Theta^0$ of the finite generated free algebras of this variety. The quotient group $A/Y$, where $A$ is a group of the all automorphisms of the category $\Theta^0$ and $Y$ is a subgroup of the all inner automorphisms of this category measures difference between the geometric equivalence and automorphic equivalence of algebras from the variety $\Theta$.

In [8] the simple and strong method of the verbal operations was elaborated on for the calculation of the group $A/Y$ in the case when the $\Theta$ is a variety of one-sorted algebras. In the first part of our paper (Sections 1, 2 and 3) we prove that this method can be used in the case of many-sorted algebras.

In the second part of our paper (Section 4) we apply the results of the first part to the universal algebraic geometry of many-sorted algebras and refine and reprove results of [5] and [10] for these algebras. For example we prove in the Theorem 4.3 that the automorphic equivalence of algebras can be reduced to the geometric equivalence if we change the operations in the one of these algebras.

In the third part of this paper (Section 5) we consider some varieties of many-sorted algebras. We prove that automorphic equivalence coincide with geometric equivalence in the variety of the all actions of semigroups over sets and in the variety of the all automats, because the group $A/Y$ is trivial for this varieties. We also consider the variety of the all representations of groups and the all representations of Lie algebras. The group $A/Y$ is not trivial for these varieties and for both these varieties we give an examples of the representations which are automorphically equivalent but not geometrically equivalent.
1 Introduction.

In this paper we consider many-sorted algebras. We suppose that there is a finite set of names of sorts $\Gamma$. Many-sorted algebra, first of all, is a set $A$ with the "sorting": mapping $\eta_A : A \rightarrow \Gamma$. We call the set $\eta^{-1}_A(i)$ for $i \in \Gamma$ - set of elements of the sort $i$ of the algebra $A$. We denote $\eta^{-1}_A(i) = A^{(i)}$. If $a \in A^{(i)}$, then we will many time denote $a = a^{(i)}$, with a view to emphasize that $a$ is an element of the sort $i$. Contrary to the common approach we allow that $A^{(i)} = \emptyset$.

We denote $\text{im} \eta_A = \{ i \in \Gamma \mid A^{(i)} \neq \emptyset \} = \Gamma_A$.

Also we suppose that there is a set of operations (signature) $\Omega$. Every operation $\omega \in \Omega$ has a type $\tau_\omega = (i_1, \ldots, i_n; j)$, where $n \in \mathbb{N}$, $i_1, \ldots, i_n, j \in \Gamma$. Operation $\omega \in \Omega$ of the type $(i_1, \ldots, i_n; j)$ is a partially defined mapping $\omega : A^n \rightarrow A$. This mapping is defined only for tuples $(a_1, \ldots, a_n) \in A^n$ such that $a_k \in A^{(i_k)}$, $1 \leq k \leq n$. The images of these tuples are elements of the sort $j$: $\omega(a_1, \ldots, a_n) \in A^{(j)}$. We suppose that all operations $\omega \in \Omega$ are closed. It means that for all $a_k \in A^{(i_k)}$, $1 \leq k \leq n$, there exists $\omega(a_1, \ldots, a_n) \in A^{(j)}$.

If there is at least one $k \in \{1, \ldots, n\}$ such that $A^{(i_k)} = \emptyset$, then the operation $\omega \in \Omega$ with the type $\tau_\omega = (i_1, \ldots, i_n; j)$ defined only on empty set. But we still consider algebra $A$ as algebra with operation $\omega$. It is possible that $n = 0$. In this case the operation $\omega \in \Omega$ with the type $\tau_\omega = (i_1, \ldots, i_n; j)$ is the operation of the taking a constant $\omega = c^{(j)}$ of the sort $j$.

Now we will define the notion of the homomorphism of the two many-sorted algebras. We assume that two many-sorted algebras $A$ and $B$ have the same set of names of sorts $\Gamma$. We denote the set of operations in the algebra $A$ by $\Omega^A = \{ \omega^A_\alpha \mid \alpha \in I \}$ and the set of operations in the algebra $B$ by $\Omega^B = \{ \omega^B_\alpha \mid \alpha \in I \}$. We assume that between these sets there is a one-to-one and onto correspondence such that operations $\omega^A_\alpha$ and $\omega^B_\alpha$ have the same type $\tau_\alpha = (i_1, \ldots, i_n; j)$. Homomorphism from $A$ to $B$ is the mapping $\varphi : A \rightarrow B$, which conforms with the "sorting" $\eta_A$ and $\eta_B$ and conforms with operations $\omega^A_\alpha$ and $\omega^B_\alpha$. "Conforms with the sorting" it means that

$$\eta_A = \eta_B \varphi$$

(1.1)

(the diagram

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\eta_A & \downarrow & \eta_B \\
\Gamma & & \\
\end{array}$$

is commutative). "Conforms with the operations" it means that for every $\alpha \in I$ and every $a^{(i_k)} \in A^{(i_k)}$, $1 \leq k \leq n$, fulfills

$$\varphi \left( \omega^A_\alpha \left( a^{(i_1)}, \ldots, a^{(i_n)} \right) \right) = \omega^B_\alpha \left( \varphi \left( a^{(i_1)} \right), \ldots, \varphi \left( a^{(i_n)} \right) \right).$$

(1.2)

If for any $k$ such that $1 \leq k \leq n$ the $A^{(i_k)} = \emptyset$ holds then this equality fulfills by the principle of the empty set. From (1.1) we can conclude that if $i \in \Gamma$, $B^{(i)} = \emptyset$, $A^{(i)} \neq \emptyset$ then homomorphisms from $A$ to $B$ are not defined: $\text{Hom} (A, B) = \emptyset$. In other words, if $\Gamma_A \not\subseteq \Gamma_B$ then $\text{Hom} (A, B) = \emptyset$. 

2
The notions of a congruence and a quotient algebra we define by natural way. The congruence must be conform with the ”sorting”, so for every algebra $A$ and every congruence $T \subseteq A^2$ the $T \subseteq \bigcup_{i \in \Gamma_A} (A^{(i)})^2$ holds. We denote by $\Delta_A$ the minimal congruence in the algebra $A$: $\Delta_A = \{(a, a) \mid a \in A\}$.

Now we will define the notion of the varieties of the many-sorted algebras. We fix the set of names of sorts $\Gamma$ and the signature $\Omega$. We take a set $X$, which we will call an alphabet. We suppose that this set has a ”sorting”: a mapping $\chi : X \to \Gamma$. After this we define an algebra of terms over the alphabet $X$.

The notion of the term over the alphabet $X$ we define by the induction by the construction: if $x = x^{(i)} \in \chi^{-1}(i) = X^{(i)}$, where $i \in \Gamma$, then $x^{(i)}$ is a term of the sort $i$. If $\omega \in \Omega$ has the type $\tau_\omega = (i_1, \ldots, i_n; j)$ and $t_k$ is a term of the sort $i_k$, $1 \leq k \leq n$, then $\omega(t_1, \ldots, t_n)$ is a term of the sort $j$. We denote the algebra of terms over the alphabet $X$ by $F = \tilde{F}(X)$. It is clear that $\chi_{\mid X} = (\eta_{\tilde{F}})_{\mid X}$.

Pairs $(w_1, w_2) \in \left(\left(\tilde{F}(X)\right)^{(i)}\right)^2$, $i \in \Gamma$, which we denote as $w_1 = w_2$, will be identities. We denote by $X'$ the finite subset $X' \subset X$ of the letters from alphabet $X$ which are really included in the term $w_1$ or in the term $w_2$. The $w_1, w_2 \in \tilde{F}(X')$ holds. Now we consider an arbitrary algebra $A$ with the set of names of sorts $\Gamma$ and the signature $\Omega$. We say that the identity $w_1 = w_2$ fulfills in the algebra $A$ if for every $\varphi \in \text{Hom}(\tilde{F}(X'), A)$ the $\varphi(w_1) = \varphi(w_2)$ holds.

If there is a $i \in \Gamma$ such that $(X')^{(i)} = X' \cap X^{(i)} \neq \emptyset$ and $A^{(i)} = \emptyset$ then $w_1 = w_2$ fulfills in the algebra $A$ by the principle of the empty set.

If $\mathcal{J} \subset \bigcup_{i \in \Gamma} \left(\left(\tilde{F}(X)\right)^{(i)}\right)^2$ then the family of algebras in which fulfill all identities from $\mathcal{J}$ called the variety of algebras defined by the identities $\mathcal{J}$. We denote this variety by $\Theta(\mathcal{J}) = \Theta$.

We say that the algebra $F \in \Theta$ is a free algebra of this variety with the set of free generators $X \subset F$ if for every algebra $A \in \Theta$, such that $\Gamma_A \supseteq \eta_F(X)$, and every mapping $f : X \to A$, such that $\eta_A f(x) = \eta_F(x)$ for every $x \in X$, there exists only one homomorphism $\varphi : F \to A$, such that $\varphi(x) = f(x)$ for every $x \in X$. This algebra we denote by $F = F(X)$. It is clear that the algebra $\tilde{F}(X)$ of terms over the alphabet $X$ is a free algebra with the free generators $X$ in the variety $\Theta(\emptyset)$ defined by the empty set of identities. When we will construct the free algebras $F(X)$ in arbitrary variety $\Theta$, we must consider the set $\mathcal{J}_\Theta(X)$ of the all identities from $\bigcup_{i \in \Gamma_A} \left(\left(\tilde{F}(X)\right)^{(i)}\right)^2$ which fulfill in the all algebras $A \in \Theta$. The set $\mathcal{J}_\Theta(X)$ is a congruence and $F(X) = \tilde{F}(X) / \mathcal{J}_\Theta(X)$.

In the end of this section we will say that the first and the second theorems of homomorphisms, the projective propriety of free algebras fulfill according to our approach to the notions of many-sorted algebras, their homomorphisms and their varieties. If $A \in \Theta$ and there exists $i \in \Gamma$ such that $A^{(i)} = \emptyset$ then there are free algebras $F(X)$ of the variety $\Theta$, such that $\text{Hom}(F(X), A) = \emptyset$. But
for every $A \in \Theta$ there exists free algebra $F(X) \in \Theta$, such that $A \cong F(X)/T$, where $T$ is a congruence. The Birkhoff theorem about varieties can be proved according to our approach.

Also we will say that our approach to the notions of many-sorted algebras, their homomorphisms and their varieties coincide with the common approach, for example, when our signature $\Omega$ has operation of the taking a constant $e^{(j)}$ for the every sort $j \in \Gamma$. It means that our approach coincide with the common one in the cases of representations of groups, representations of linear algebras, actions of monoids over sets with stationary points and many other cases.

2 Category $\Theta^0$ and its automorphisms. Decomposition theorem.

Now we consider an arbitrary variety $\Theta = \Theta(\mathcal{J})$ of the many-sorted algebras with the set of names of sorts $\Gamma$, the set of operations $\Omega$ and defined by the identities $\mathcal{J}$. We take $X_0$ and $\chi : X_0 \rightarrow \Gamma$ such that $\chi^{-1}(i) = X_0^{(i)}$ is an infinite countable set for every $i \in \Gamma$. Free algebras $F(X)$ such that $X \subseteq X_0$, $|X| < \infty$ will be objects of the category $\Theta^0$, the homomorphisms of these algebras will be morphisms of this category.

From now on we assume that the following condition holds in our variety $\Theta$:

**Condition 2.1** $\Phi(F(x^{(i)})) \cong F(x^{(i)})$ for every automorphism $\Phi$ of the category $\Theta^0$, every sort $i \in \Gamma$ and every element of this sort $x^{(i)} \in X_0^{(i)} \subseteq X_0$.

The following theorem is a generalization of the first part of the Theorem 1 from [3].

**Theorem 2.1** If $\Phi$ is an automorphism of the category $\Theta^0$ then for every $A \in \text{Ob}\Theta^0$ there exists a bijection $s_A : A \rightarrow \Phi(A)$ such that $\eta_A = \eta_{\Phi(A)}s_A$ and for every $\mu \in \text{Mor}_{\Theta^0}(A,B)$ the

$$\Phi(\mu) = s_B\mu s_A^{-1}$$

(2.1)

holds.

**Proof.** We take $a^{(i)} \in A^{(i)} \subseteq A$, $x^{(i)} \in X_0^{(i)} \subseteq X_0$ and $F(x^{(i)}) \in \text{Ob}\Theta^0$. There exists one homomorphism $\alpha : F(x^{(i)}) \rightarrow A$ such that $\alpha(x^{(i)}) = a^{(i)}$. $\Phi(\alpha) : \Phi(F(x^{(i)})) \rightarrow \Phi(A)$. By Condition 2.1 there exists isomorphism $\sigma : F(x^{(i)}) \rightarrow \Phi(F(x^{(i)}))$. We define $s_A(a^{(i)}) = \Phi(\alpha)\sigma(x^{(i)})$. $\Phi(\alpha)\sigma : F(x^{(i)}) \rightarrow \Phi(A)$ is a homomorphism, so by (2.1) $s_A(a^{(i)}) \in (\Phi(A))^{(i)}$. Therefore $\eta_A = \eta_{\Phi(A)}s_A$.

We take $b^{(i)} \in (\Phi(A))^{(i)} \subseteq \Phi(A)$. There exists homomorphism $\beta : F(x^{(i)}) \rightarrow \Phi(A)$ such that $\beta(x^{(i)}) = b^{(i)}$ and homomorphism $\beta\sigma^{-1} : \Phi(F(x^{(i)})) \rightarrow \Phi(A)$. Therefore exists a homomorphism $\Phi^{-1}(\beta\sigma^{-1}) : F(x^{(i)}) \rightarrow A$. $\Phi^{-1}(\beta\sigma^{-1})(x^{(i)}) = a^{(i)} \in A^{(i)}$, so $\Phi^{-1}(\beta\sigma^{-1}) = \alpha$, such that $\alpha(x^{(i)}) = a^{(i)}$. Hence $\beta = \Phi(\alpha)\sigma$, $\Phi(\alpha)\sigma(x^{(i)}) = b^{(i)}$ and $b^{(i)} = s_A(a^{(i)})$. So $s_A$ is a surjection.
We assume that $a_1^{(i)}, a_2^{(i)} \in A^{(i)}$ and $s_A (a_2^{(i)}) = s_A (a_2^{(i)}) = (\Phi (A))^{(i)}$. We consider the homomorphisms $\alpha_1, \alpha_2 : F (x^{(i)}) \to A$ such that $a_j (x^{(i)}) = a_j^{(i)}$, $j = 1, 2$. We have that $\Phi (\alpha_1) \sigma (x^{(i)}) = \Phi (\alpha_2) \sigma (x^{(i)})$. $\Phi (\alpha_1) \sigma, \Phi (\alpha_2) \sigma : F (x^{(i)}) \to \Phi (A)$, so $\Phi (\alpha_1) \sigma = \Phi (\alpha_2) \sigma$. Therefore $\alpha_1 = \alpha_2$ and $a_1^{(i)} = a_2^{(i)}$. So $s_A$ is a injection.

We consider $\mu \in \text{Mor}_{\Theta} (A, B)$ and $a^{(i)} \in A^{(i)} \subset A$. $s_B \mu (a^{(i)}) = \Phi (\beta) \sigma (x^{(i)})$, where $\sigma$ is an isomorphism $\sigma : F (x^{(i)}) \to \Phi (F (x^{(i)}))$, $\beta$ is a homomorphism $\beta : F (x^{(i)}) \to B$ such that $\beta (x^{(i)}) = \mu (a^{(i)})$. $\Phi (\mu) s_A (a^{(i)}) = \Phi (\mu) \Phi (\alpha) \sigma (x^{(i)})$, where $\alpha$ is a homomorphism $\alpha : F (x^{(i)}) \to A$ such that $\alpha (x^{(i)}) = a^{(i)}$. $\mu \alpha (x^{(i)}) = \mu (a^{(i)})$, so $\mu \alpha = \beta$ and $\Phi (\beta) = \Phi (\mu) \Phi (\alpha)$. Therefore $s_B \mu (a^{(i)}) = \Phi (\mu) s_A (a^{(i)})$ and $\Phi (\mu) = s_B \mu s_A^{\text{-1}}$.

From this theorem we conclude that for every automorphism $\Phi$ of the category $\Theta_0$ and every $\mu \in \text{Mor}_{\Theta_0} (A, B)$ the diagram

\[
\begin{array}{ccc}
A & \to & \Phi (A) \\
\downarrow \mu & & \downarrow \\
B & \to & \Phi (B)
\end{array}
\]

commutes.

The following theorem is a generalization of the second part of the Theorem 1 from [5].

**Theorem 2.2** If $\Phi$ is an automorphism of the category $\Theta_0$ then for every $A \in \text{Ob} \Theta_0$ the $\Phi (A) \cong A$.

**Proof.** We denote $\Phi (A) = B$, $\Phi^{-1} (A) = C$. $A = F (X)$ such that $X \subset X_0$, $|X| < \infty$. By Theorem 2.1 there exist bijections $s_C : C \to A$, $s_A : A \to B$, so $\Gamma_A = \Gamma_B = \Gamma_C$. Hence there exists one homomorphism $\sigma : A \to B$, such that $\sigma (x) = s_A (x)$ and there exists one homomorphism $\tau : A \to C$, such that $\tau (x) = s_C^{\text{-1}} (x)$ for every $x \in X$. $\Phi (\tau) : \Phi (A) = B \to \Phi (C) = A$. $\Phi (\tau) \sigma (x) = s_C^{\text{-1}} s_A^{\text{-1}} \sigma (x) = s_C \tau (x) = x$ holds for every $x \in X$. Therefore $\Phi (\tau) \sigma = \text{id}_A$.

$\Phi^{-1} (\sigma) : \Phi^{-1} (A) = C \to \Phi^{-1} (B) = A$. $\Phi^{-1} (\sigma) \tau (x) = s_A^{\text{-1}} s_C \tau (x) = s_A^{\text{-1}} (x) = x$ holds for every $x \in X$. Therefore $\Phi^{-1} (\sigma) \tau = \text{id}_A$. Automorphism $\Phi$ provides an isomorphisms of monoids $\text{End} A \to \text{End} \Phi (A)$, $\text{id}_A$ is an unit of $\text{End} A$, $\text{id}_{\Phi (A)}$ is an unit of $\text{End} \Phi (A)$, so $\Phi (\Phi^{-1} (\sigma) \tau) = \sigma \Phi (\tau) = \text{id}_B$.

**Definition 2.1** An automorphism $\Upsilon$ of an arbitrary category $\mathcal{K}$ is **inner**, if it is isomorphic as a functor to the identity automorphism of the category $\mathcal{K}$.

This means that for every $A \in \text{Ob} \mathcal{K}$ there exists an isomorphism $s_A^{\Upsilon} : A \to \Upsilon (A)$ such that for every $\mu \in \text{Mor}_{\mathcal{K}} (A, B)$ the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{s_A^{\Upsilon}} & \Upsilon (A) \\
\downarrow \mu & & \downarrow \\
B & \xrightarrow{s_B^{\Upsilon}} & \Upsilon (B)
\end{array}
\]
commutes. It is clear that the set of all inner automorphisms of an arbitrary category $\mathcal{A}$ is a normal subgroup of the group of all automorphisms of this category.

**Definition 2.2** An automorphism $\Phi$ of the category $\Theta^0$ is called strongly stable if it satisfies the conditions:

1. $\Phi$ preserves all objects of $\Theta^0$,
2. there exists a system of bijections $S = \{s_F : F \rightarrow F \mid F \in \text{Ob} \Theta^0\}$ such that all these bijections conform with the sorting:
   \[ \eta_F = \eta_{FSF} \]
3. $\Phi$ acts on the morphisms $\mu \in \text{Mor}_{\Theta^0}(A, B)$ of $\Theta^0$ by this way:
   \[ \Phi(\mu) = s_B \mu s_A^{-1}, \quad (2.2) \]
4. $s_F |_X = id_X$, for every $F(X) \in \text{Ob} \Theta^0$.

It is clear that the set of all strongly stable automorphisms of the category $\Theta^0$ is a subgroup of the group of all automorphisms of this category. We denote by $\mathfrak{A}$ the group of the all automorphisms of the category $\Theta^0$, by $\mathfrak{A}$ the group of the all inner automorphisms of this category, by $\mathfrak{S}$ the group of the all strongly stable automorphisms of this category.

The following decomposition theorem is a is a generalization of the Theorem 2 from [8].

**Theorem 2.3** $\mathfrak{A} = \Psi \mathfrak{S} = \Psi \mathfrak{S}$, where $\mathfrak{A}$ is a group of the all automorphisms of the category $\Theta^0$.

**Proof.** We consider $\Phi \in \mathfrak{A}$. In the Theorem 2.2 we prove that there is a system of bijections $\{s_A \mid A \in \text{Ob} \Theta^0\}$ such that $s_A : A \rightarrow \Phi(A)$ and $\Phi(\mu) = s_B \mu s_A^{-1}$ if $\mu \in \text{Mor}_{\Theta^0}(A, B)$. In the Theorem 2.2 we prove that for every $A = F(X) \in \text{Ob} \Theta^0$ there is an isomorphism $\sigma_A : A \rightarrow \Phi(A)$, such that $\sigma_A(x) = s_A(x)$ for every $x \in X$.

Now we will define two automorphisms $\Upsilon$ and $\Psi$. $\Upsilon$ we define by this way: $\Upsilon(A) = \Phi(A)$ for every $A \in \text{Ob} \Theta^0$, $\Upsilon(\mu) = \sigma_B \mu \sigma_A^{-1}$ for every $\mu \in \text{Mor}_{\Theta^0}(A, B)$. $\Psi$ we define by this way: $\Psi(A) = A$ for every $A \in \text{Ob} \Theta^0$, $\Psi(\mu) = \sigma_B^{-1} \mu \sigma_A^{-1} \sigma_A$ for every $\mu \in \text{Mor}_{\Theta^0}(A, B)$. It is easy to check that $\Upsilon$ and $\Psi$ are functors. Also we can remark that $\Upsilon$ and $\Psi$ have inverted functors $\Upsilon^{-1}$ and $\Psi^{-1}$: if we define $\Upsilon^{-1}(A) = \Psi^{-1}(A)$ for every $A \in \text{Ob} \Theta^0$, $\Upsilon^{-1}(\mu) = \Psi^{-1}(\mu^{-1}(B)) \mu \Psi^{-1}(A)$ for every $\mu \in \text{Mor}_{\Theta^0}(A, B)$ and $\Psi^{-1}(A) = A$ for every $A \in \text{Ob} \Theta^0$, $\Psi^{-1}(\mu) = s_B^{-1} \sigma_B \mu \sigma_A^{-1} s_A$ for every $\mu \in \text{Mor}_{\Theta^0}(A, B)$ - then $\Upsilon^{-1} \Upsilon = \Psi^{-1} \Psi = \Psi^{-1} \Phi^{-1} = \text{id}_{\Theta^0}$. Therefore $\Upsilon$ and $\Psi$ are automorphisms.

If $A = F(X) \in \text{Ob} \Theta^0$ then $s_A^{-1} \sigma_A(x) = x$ for every $x \in X$, so $\Psi$ is a strongly stable automorphism.
We can conclude from $\Phi(\mu) = s_B(\mu s_A^{-1}) = \sigma_B^{-1}(s_B(\mu s_A^{-1}))\sigma_A^{-1}$ for every $\mu \in \text{Mor}_{\Theta^0}(A,B)$ that $\Phi = \Upsilon \Psi$. It is well known that the group $\Upsilon$ is a normal subgroup of the group $\Psi$. This completes the proof. □

3 Strongly stable automorphisms and systems of verbal operations.

3.1 Strongly stable automorphisms and systems of bijections.

If we have a strongly stable automorphism $\Phi$ of the category $\Theta^0$, then by Definition 2.2 there exists a system of bijections $S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0\}$, such that

B1) for every $F \in \text{Ob}\Theta^0$ the $\eta_F = \eta_F s_F$ holds,

B2) for every $A,B \in \text{Ob}\Theta^0$ and every $\mu \in \text{Mor}_{\Theta^0}(A,B)$ the mappings $s_B(\mu s_A^{-1}), s_B^{-1}(\mu s_A)$: $A \to B$ are homomorphisms,

B3) for every $F(X) \in \text{Ob}\Theta^0$ the $s_F \mid_X = id_X$ holds.

Proposition 3.1. For strongly stable automorphism $\Phi$ of the category $\Theta^0$ there is only one system of bijections $S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0\}$ such that $\Phi$ acts on homomorphisms by these bijections and the system $S$ fulfills conditions B1) - B3).

Proof. By Definition 2.2 there exists a system of bijections $S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0\}$ such that for every $F \in \text{Ob}\Theta^0$ the $\eta_F = \eta_F s_F$ holds, for every $A,B \in \text{Ob}\Theta^0$ and every $\mu \in \text{Mor}_{\Theta^0}(A,B)$ the $\Phi(\mu) = s_B(\mu s_A^{-1})$ holds and $s_F \mid_X = id_X$, for every $F(X) \in \text{Ob}\Theta^0$. $\Phi^{-1}(\mu) = s_B^{-1}(\mu s_A)$ is also homomorphism, so $S$ fulfills conditions B1) - B3). For every $F \in \text{Ob}\Theta^0$ and every $f^{(i)} \in F^{(i)}$, $i \in \Gamma$, we take $A = F(x^{(i)})$ where $x^{(i)} \in X^{(i)}_0$. There exist homomorphism $\alpha : A \ni x^{(i)} \mapsto f^{(i)} \in F^{(i)}$. The $s_F f^{(i)} = s_F(\alpha(x^{(i)})) = s_F(a(x^{(i)})) = \Phi(a)(x^{(i)})$ holds, so all bijections of $S$ uniquely defined by automorphism $\Phi$. □

The systems of bijections $S$ which is subject of this Proposition we denote by $S^\Phi$.

Proposition 3.2. The mapping $\Phi \to S^\Phi$ is one to one and onto correspondence between the family of the all strongly stable automorphisms of the category $\Theta^0$ and the family of the all systems of bijections $S = \{s_F : F \to F \mid F \in \text{Ob}\Theta^0\}$ which fulfills conditions B1) - B3).

Proof. If we have a system of bijections $S$ which fulfills conditions B1) - B3) we can define a functor $\Phi : \Theta^0 \to \Theta^0$, such that preserves all objects of $\Theta^0$ and for every $A,B \in \text{Ob}\Theta^0$ and every $\mu \in \text{Mor}_{\Theta^0}(A,B)$ the $\Phi(\mu) = s_B(\mu s_A^{-1})$ holds. There is an inverse functor $\Phi : \Theta^0 \to \Theta^0$, such that the $\Phi^{-1}(\mu) = s_B^{-1}(\mu s_A)$ holds.
For every operation \( w \) that fulfills conditions B1) - B3), every algebra \( w \) called the verbal operation defined by the word \( f \), denote by \( W \) the verbal operation \( \) defined. So, if \( \) is a homomorphism then \( \phi \) is a homomorphism too.

Hence our correspondence is onto.

If \( \Phi \) and \( \Psi \) are strongly stable automorphisms of the category \( \Theta^0 \) and \( S^\Phi = S^\Psi = S = \{ s_F : F \to F \mid F \in \text{Ob} \Theta^0 \} \), then \( \Phi = \Psi \), because, for \( F \in \text{Ob} \Theta^0 \) the \( \Phi(F) = \Psi(F) = F \) holds and for every \( A, B \in \text{Ob} \Theta^0 \) and every \( \mu \in \text{Mor}_{\Theta^0}(A, B) \) the \( \Phi(\mu) = s_B s_A^{-1} = \Psi(\mu) \) holds. Hence our correspondence is one to one.

3.2 Systems of bijections and systems of verbal operations.

We take the word \( w = w(x_1, \ldots, x_n) \in F(x_1, \ldots, x_n) = F \in \text{Ob} \Theta^0 \). For every algebra \( H \in \Theta \) we can define an operation \( w^*_H \): if \( h_1, \ldots, h_n \in H \) such that \( \eta_H(h_i) = \eta_F(x_i) \), where \( 1 \leq i \leq n \), then \( w^*_H(h_1, \ldots, h_n) = \alpha(w) \), where \( \alpha : F \to H \) homomorphism such that \( \alpha(x_i) = h_i \). If \( \eta_F\{x_1, \ldots, x_n\} \not\subseteq \Gamma_H \) then the operation \( w^*_H \) is defined on the empty subset of \( H^n \). The operation \( w^*_H \) is called the verbal operation defined by the word \( w \). This operation we consider as the operation of the type \( \eta_F(x_1), \ldots, \eta_F(x_n) ; \eta_F(w) \) even if not all free generators \( x_1, \ldots, x_n \) really enter to the word \( w \).

If we have a system of words \( W = \{ w_i \mid i \in I \} \) then for every \( H \in \Theta \) we denote by \( H^*_W \) the universal algebra which coincide with \( H \) as a set with the "sorting", but has only verbal operations defined by the words from \( W \). It is easy to prove that if \( H_1, H_2 \in \Theta \) and \( \varphi : H_1 \to H_2 \) is a homomorphism then \( \varphi(w^*_H(h_1, \ldots, h_n)) = w^*_H(\varphi(h_1), \ldots, \varphi(h_n)) \) if both sides of this equality are defined. So, if \( W \) is a system of words and \( \varphi : H_1 \to H_2 \) is a homomorphism then \( \varphi : (H_1)_W \to (H_2)_W \) is a homomorphism too.

Now we assume that there exists a system of bijections \( S = \{ s_A : A \to A \mid A \in \text{Ob} \Theta^0 \} \) which fulfills conditions B1) - B3).

For the operation \( \omega \in \Omega \) which has a type \( \tau_\omega = (i_1, \ldots, i_n; j) \), we take \( A_\omega = F(X_\omega) \in \text{Ob} \Theta^0 \) such that \( X_\omega = \{ x^{(i_1)}, \ldots, x^{(i_n)} \} \), \( \eta_{A_\omega}(x^{(i_k)}) = i_k \), \( 1 \leq k \leq n \).

\[
\alpha_{A_\omega}(\omega(x^{(i_1)}, \ldots, x^{(i_n)})) = w_\omega(x^{(i_1)}, \ldots, x^{(i_n)}) \in A_\omega. \tag{3.1}
\]

For every \( H \in \Theta \) we define by the word \( w_\omega \) the verbal operation \( \omega^*_H \). We denote \( W = \{ w_\omega \mid \omega \in \Omega \} \). The types of the operations \( \omega \) and \( \omega^* \) coincides.

Proposition 3.3 For every \( F \in \text{Ob} \Theta^0 \) the bijection \( s_F \) is an isomorphism \( s_F : F \to F_W^* \).

Proof. We consider the operation \( \omega \in \Omega \) which has a type \( \tau_\omega = (i_1, \ldots, i_n; j) \).

We need to prove that for every \( f^{(i_1)}, \ldots, f^{(i_n)} \in F \) such that \( \eta_F(f^{(i_r)}) = i_r \), \( 1 \leq r \leq n \), the \( s_F \omega(f^{(i_1)}, \ldots, f^{(i_n)}) = \omega^*(s_F(f^{(i_1)}), \ldots, s_F(f^{(i_n)})) \) holds. We consider the homomorphisms \( \alpha, \beta : A_\omega \to F \) such that \( \alpha(x^{(i_r)}) = f^{(i_r)} \), \( \beta(x^{(i_r)}) = s_F(f^{(i_r)}) \), \( 1 \leq r \leq n \). By our assumption \( s_F \alpha A_\omega^{-1} \) is also homomorphism from \( A \) to \( F \). \( s_F \alpha A_\omega^{-1}(x^{(i_r)}) = s_F \alpha(x^{(i_r)}) = s_F(f^{(i_r)}) \), therefore...
\[ \beta = s_F \alpha s_A^{-1}. \] So
\[ s_F \omega \left( f^{(i_1)}, \ldots, f^{(i_n)} \right) = s_F \alpha s_A^{-1} s_A \omega \left( f^{(i_1)}, \ldots, f^{(i_n)} \right) = \]
\[ \beta w_\omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) = \omega^* \left( s_F \left( f^{(i_1)} \right), \ldots, s_F \left( f^{(i_n)} \right) \right). \]

\[ \text{Op1) } W = \{ w_\omega \in A_\omega \mid \omega \in \Omega \}, \text{ where } A_\omega \text{ is defined as above, and} \]

\[ \text{Op2) for every } F(X) \in \text{Ob} \Theta^0 \text{ there exists a bijection } s_F : F \to F \text{ such that } \]
\[ (s_F)_X = id_X \text{ and } s_F : F \to F^*_W \text{ is an isomorphism.} \]

The verbal operations defined by the word \( w_\omega \) we denote by \( \omega^* \). We have a new signature \( \Omega^* = \{ \omega^* \mid \omega \in \Omega \} \).

Because \( s_F : F \to F^*_W \) is an isomorphism, \( F^*_W \) such that \( F \in \text{Ob} \Theta^0 \) is also a free algebra in the variety \( \Theta \).

By our assumption there is a symmetry between the signatures \( \Omega \) and \( \Omega^* \).

**Proposition 3.4** Every \( \omega \in \Omega \) is a verbal operations defined by the some word written in the signature \( \Omega^* \).

**Proof.** \( s_{A_\omega} \) is a bijection, so there is \( u_\omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \in A_\omega \) such that \( s_{A_\omega} \left( u_\omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \right) = \omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \). \( s_{A_\omega} \) is an isomorphism and \( (s_F)_X = id_X \), so \( \omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) = u_\omega^* \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \) where \( u_\omega^* \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \) we achieve from the word \( u_\omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \) when we change the all operations from \( \Omega \) by corresponding operation from \( \Omega^* \). So for every \( H \in \Theta \) and every \( h^{(i_k)} \in H^{(i_k)}, 1 \leq k \leq n, \omega_H \left( h^{(i_1)}, \ldots, h^{(i_n)} \right) = \alpha \omega \left( x^{(i_1)}, \ldots, x^{(i_n)} \right) \) we have a homomorphism. But also \( \alpha \) is a homomorphism from \( (A_\omega)_W \) to \( H^*_W \). \( \blacksquare \)

**Corollary 1** If \( H_1, H_2 \in \Theta \) and \( \mu : (H_1)_W^* \to (H_2)_W^* \) is a homomorphism then \( \mu : H_1 \to H_2 \) is also a homomorphism.

**Corollary 2** For every \( A, B \in \text{Ob} \Theta^0 \) and every \( \mu \in \text{Mor} \Theta^0 (A, B) \) the mappings \( s_B \mu s_A^{-1}, s_B^{-1} \mu s_A : A \to B \) are also homomorphisms.

**Proposition 3.5** If \( H \in \Theta \) then \( H^*_W \in \Theta \).

**Proof.** There is \( F \in \text{Ob} \Theta^0 \) such that exists epimorphism \( \varphi : F \to H \). \( \varphi \) is also an epimorphism from \( F^*_W \) to \( H^*_W \). Therefore \( \varphi s_F \) is also an epimorphism from \( F \) to \( H^*_W \). \( \blacksquare \)

If we have a system of words \( W \) which fulfills the conditions Op1) and Op2) then by Proposition 3.5 the bijections \( \{ s_F : F \to F \mid F \in \text{Ob} \Theta^0 \} \) which are subjects of the condition Op2) uniquely defined by the system of words \( W \). This system of bijections we denote by \( S^W \).
Proposition 3.6 The mapping $W \rightarrow S^W$ is one to one and onto correspondence between the family of the all systems of words which fulfill the conditions Op1) and Op2) and the family of the all systems of bijections $S = \{s_F : F \rightarrow F \mid F \in \text{Ob} \Theta^0\}$ which fulfills conditions B1) - B3).

Proof. We consider a system of words $W$ which fulfills the conditions Op1) and Op2). By condition Op2) for every $F(X) \in \text{Ob} \Theta^0$ the $s_F \mid X = \text{id}_X$ holds and all the bijections $s_F : F \rightarrow F$ conform with the sorting of the algebras $F$. So the system of bijections $S^W$ fulfills the conditions B1) and B2). By Corollary 2 from the Proposition 3.4 (the system of bijections $S^W$ fulfills the condition B2).

If we have the systems of bijections $S$ which fulfills conditions B1) - B3) we can define the system of words $W = \{w_\omega \in A_\omega \mid \omega \in \Omega\}$ by formula (3.1). This system of words fulfills the condition Op1) and by Proposition 3.3 - the condition Op2). The $S^W = S$ holds, so our correspondence is onto.

We assume that we have two systems of words $W = \{w^W_\omega \in A_\omega \mid \omega \in \Omega\}$ and $V = \{v^V_\omega \in A_\omega \mid \omega \in \Omega\}$ which fulfill the conditions Op1) and Op2) and $S^W = S = S^V$. We take $\omega \in \Omega$ such that $\tau_\omega = (i_1, \ldots, i_n, j)$, $i_1, \ldots, i_n, j \in \Gamma$, $A_\omega = F(X_\omega)$ as above and $x^{(i_j)}$ such that $\eta_{A_\omega} (x^{(i_j)}) = i_k$, $1 \leq k \leq n$. By condition Op2) $s^W_{A_\omega} (\omega (x^{(i_1)}, \ldots, x^{(i_n)})) = s^W_{A_\omega} (\omega (x^{(i_1)}, \ldots, x^{(i_n)})) = s^V_{A_\omega} (\omega (x^{(i_1)}, \ldots, x^{(i_n)})) = w^V_{\omega} (x^{(i_1)}, \ldots, x^{(i_n)}).$ Therefore $W = V$ and our correspondence is one to one.

From Propositions 3.2 and 3.6 we conclude the

Theorem 3.1 There is one to one and onto correspondence $\Phi \rightarrow W^\Phi$ between the family of the all strongly stable automorphisms of the category $\Theta^0$ and the family of the all systems of words which fulfill the conditions Op1) and Op2). The systems of words $W^\Phi = W$ defined by formula (3.7) where bijections $s_{A_\omega}$ are subjects of the items A2) - A4) of the Definition 2.2.

We will denote by $W \rightarrow \Phi^W$ the correspondence which inverse to the correspondence $\Phi \rightarrow W^\Phi$.

3.3 Strongly stable automorphisms and inner automorphisms.

Proposition 3.7 The strongly stable automorphism $\Phi$ which corresponds to the system of words $W^\Phi = W$ is inner if and only if there is a system of isomorphisms $\{\tau_F : F \rightarrow F^\Phi \mid F \in \text{Ob} \Theta^0\}$ such that for every $A, B \in \text{Ob} \Theta^0$ and every $\mu \in \text{Mor}_{\Theta^0}(A, B)$ the

$$\tau_B \mu = \mu \tau_A$$

holds.

Proof. If automorphism $\Phi$ is inner then by Definition 2.1 there is a system of isomorphisms $\{\sigma_F : F \rightarrow F \mid F \in \text{Ob} \Theta^0\}$ such that for every $A, B \in \text{Ob} \Theta^0$ and every $\mu \in \text{Mor}_{\Theta^0}(A, B)$ the $\Phi(\mu) = \sigma_B \mu \sigma_A^{-1} = s_B \mu s_A^{-1}$ holds, where
bijections $s_A, s_B$ are subjects of the items A2) - A4) of the Definition 2.2. So the $\mu\sigma^{-1}_A s_A = \sigma^{-1}_B s_B \mu$ holds. By Proposition 3.3 $\tau_A = \sigma^{-1}_A s_A : A \to A_W$ is an isomorphism.

Now we assume that $\Phi$ is a strongly stable automorphism and holds $s_A, s_B$ are subjects of the items A2) - A4) of the Definition 2.2. From (2.2) we conclude $\Phi(\mu) = s_B \mu s_A^{-1} = s_B \tau_B^{-1} \mu \tau_A s_A^{-1}$. By Proposition 3.3 $\tau_A s_A^{-1} : A_W \to A_W$ is an isomorphism. Hence, by Corollary 1 from Proposition 3.4 $\Phi(\tau_A s_A^{-1}) = \tau_A s_A^{-1} : A \to A$ is an isomorphism. Hence $\sigma_A = (\tau_A s_A^{-1})^{-1} : A \to A$ is an isomorphism and $\Phi(\mu) = \sigma_B \mu \sigma_A^{-1}$.

4 Application to the universal algebraic geometry of many-sorted algebras.

The basic notion of the universal algebraic geometry defined in [3], [4] and [5]. These notion can be defined for many-sorted algebras. The universal algebraic geometry of many classes of many-sorted algebras was considered in [6] and [7]. We can consider the universal algebraic geometry over the arbitrary variety of algebras $\Theta$. Our equations will be pairs of the elements of the free algebras of the variety $\Theta$. It is natural that in the case of many-sorted algebras we compare elements of the same sort, so for the equation $(t_1, t_2) \in F^2$, where $F \in \text{Ob} \Theta$, the $(t_1, t_2) \in (F(i))^2$ holds, where $i \in \Gamma$. Therefore, $T \subseteq \bigcup_{i \in \Gamma} (F(i))^2$ holds for every system of equations $T$. Solutions of this system of equations we can find in arbitrary algebra $H \in \Theta$. It will be a homomorphisms $\varphi \in \text{Hom}(F, H)$, such that $\varphi(t_1) = \varphi(t_2)$ holds for every $(t_1, t_2) \in T$. We denote by $T'_H$ the set of all solutions of the system of equations $T$ in the algebra $H$. The $T'_H = \{ \varphi \in \text{Hom}(F, H) \mid T \subseteq \ker \varphi \}$ holds. Also we can consider the algebraic closure of the system $T$ over the algebra $H$: $T''_H = \bigcap_{\varphi \in T'_H} \ker \varphi = \bigcap_{T \subseteq \ker \varphi} \ker \varphi$.

$T''_H$ is a set of all equations which we can conclude from the system in the algebra $H$ or the maximal system of equations which has in the algebra $H$ same solutions as the system $T$. It is clear that algebraic closure of arbitrary system $T \subseteq \bigcup_{i \in \Gamma} (F(i))^2$ is a congruence and also the $T''_H \subseteq \bigcup_{i \in \Gamma} (F(i))^2$ must fulfill.

If $\text{Hom}(F, H) = \emptyset$, then for every $T \subseteq \bigcup_{i \in \Gamma} (F(i))^2$ the $T'_H = \emptyset$ holds and $T''_H = \bigcap_{\varphi \in T'_H} \ker \varphi = \bigcup_{i \in \Gamma} (F(i))^2$ fulfills by the principle of the empty set.

Definition 4.1 [3] The set $T \subseteq \bigcup_{i \in \Gamma} (F(i))^2$ is called algebraically closed over the algebra $H$ if $T''_H = T$. 

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If \( F \in \text{Ob} \Theta^0 \) then the family of the subsets of \( \bigcup_{i \in \Gamma} \left( (F)^{(i)} \right)^2 \) which are algebraically closed over the algebra \( H \in \Theta \) we denote by \( Cl_H(F) \). The minimal subset in the \( Cl_H(F) \) is \((\Delta_F)''_H = \bigcap_{\varphi \in \text{Hom}(F,H)} \ker \varphi(.)\).

**Definition 4.2** We say that algebras \( H_1, H_2 \in \Theta \) are **geometrically equivalent** if for every \( F \in \text{Ob} \Theta^0 \) and every \( T \subseteq \bigcup_{i \in \Gamma} \left( (F)^{(i)} \right)^2 \) the \( T''_{H_1} = T''_{H_2} \).

It is clear that algebras \( H_1, H_2 \) are geometrically equivalent if and only if the \( Cl_{H_1}(F) = Cl_{H_2}(F) \) holds for every \( F \in \text{Ob} \Theta^0 \).

### 4.1 Automorphic equivalence of many-sorted algebras.

**Definition 4.3** We say that algebras \( H_1, H_2 \in \Theta \) are **automorphically equivalent** if there exist an automorphism \( \Phi : \Theta^0 \rightarrow \Theta^0 \) and the bijections

\[
\alpha(\Phi)_A : Cl_{H_1}(A) \rightarrow Cl_{H_2}(\Phi(A))
\]

for every \( A \in \text{Ob} \Theta^0 \), coordinated in the following sense: if \( A_1, A_2 \in \text{Ob} \Theta^0 \), \( \mu_1, \mu_2 \in \text{Hom}(A_1, A_2) \), \( T \in Cl_{H_1}(A_2) \) then

\[
\tau \mu_1 = \tau \mu_2,
\]

if and only if

\[
\tau \Phi (\mu_1) = \tau \Phi (\mu_2),
\]

where \( \tau : A_2 \rightarrow A_2/T, \tau : \Phi(A_2) \rightarrow \Phi(A_2)/\alpha(\Phi)_{A_2}(T) \) are the natural epimorphisms.

If \( A, B \in \text{Ob} \Theta^0 \), \( f : A \rightarrow B \) is a mapping, \( T \subseteq A^2 \) then we denote \( f(T) = \{(f(t_1), f(t_2)) \mid (t_1, t_2) \in T\} \).

This result is a refinement of [5, Proposition 8].

**Theorem 4.1** If an automorphism \( \Phi : \Theta^0 \rightarrow \Theta^0 \) provides an automorphic equivalence of algebras \( H_1, H_2 \in \Theta \) and \( \Phi \) acts on the morphisms of \( \Theta^0 \) by formula (2.7) with the system of bijections \( \{s_A : A \rightarrow \Phi(A) \mid A \in \text{Ob} \Theta^0 \} \), then \( \alpha(\Phi)(T) = s_A(T) \) holds for every \( T \in Cl_{H_1}(A) \).

Vice versa, if an automorphism \( \Phi : \Theta^0 \rightarrow \Theta^0 \) acts on the morphisms of \( \Theta^0 \) by formula (2.7) and for every \( A \in \text{Ob} \Theta^0 \) the \( s_A : Cl_{H_1}(A) \rightarrow Cl_{H_2}(\Phi(A)) \) is a bijection then the automorphism \( \Phi : \Theta^0 \rightarrow \Theta^0 \) provides an automorphic equivalence of algebras \( H_1, H_2 \in \Theta \) with \( \alpha(\Phi)_A = s_A \).

**Proof.** We assume that \( T \in Cl_{H_1}(A) \). We suppose that \( (t_1, t_2) \in (A^{(i)})^2 \cap T \), where \( i \in \Gamma \). We consider \( F = F(x^{(i)}) \in \text{Ob} \Theta^0 \) and two homomorphisms \( \mu_1, \mu_2 : F \rightarrow A \), such that \( \mu_j(x^{(i)}) = t_j, j = 1, 2 \). \( T \) is a congruence, so we can prove by induction by the length of the terms in the algebra \( F \), that
\( (\mu_1 (f), \mu_2 (f)) \in T \) holds for every \( f \in F \). Therefore \( \tau \mu_1 = \tau \mu_2 \) and \( \tilde{\sigma} \Phi (\mu_1) = \tau \mu_2 \), where \( \tau : A \to A/T, \tilde{\tau} : \Phi (A) \to \Phi (A) / \alpha (\Phi) (T) \) are the natural epimorphisms. So \((s_A \mu_1 s_F^{-1} (u), s_A \mu_2 s_F^{-1} (u)) \in \alpha (\Phi) (T) \) holds for every \( u \in \Phi (F) \). We take \( u = s_F (x^{(i)}) \in \Phi (F) \) and conclude that \((s_A (t_1), s_A (t_2)) \in \alpha (\Phi) (T) \). So \( s_A (T) \subseteq \alpha (\Phi) (T) \).

Now we suppose that \((u_1, u_2) \in \left((\Phi (A))^{(i)} \right)^2 \cap \alpha (\Phi) (T), i \in \Gamma \). We consider \( F = F (x^{(i)}) \in \text{Ob} \Theta^0 \) and two homomorphisms \( \varphi_1, \varphi_2 : F \to \Phi (A) \), such that \( \varphi_j (x^{(i)}) = u_j, j = 1, 2 \). By proof of the Theorem 2.2 there is an isomorphism \( \sigma : F \to \Phi (F) \), such that \( \sigma (x^{(i)}) = s_F (x^{(i)}) \). We denote \( \nu_j = \varphi_j \sigma^{-1}, j = 1, 2 \). \( \nu_j \) are homomorphisms from \( \Phi (F) \) to \( \Phi (A) \). \( \nu_j (\sigma (x^{(i)})) = u_j, j = 1, 2 \). \( \alpha (\Phi) (T) \) is a congruence, \( \sigma (x^{(i)}) \) is a free generator of \( \Phi (F) \), so we can prove by induction by the length of the terms in the algebra \( F \) that \( \tilde{\tau} \nu_1 = \tilde{\tau} \nu_2 \). \( \nu_j = \Phi (\mu_j), j = 1, 2 \), \( \mu_j \) are homomorphisms from \( F \) to \( A \). Hence, by 2.1, \( s_A \mu_j (x^{(i)}) = \nu_j s_F (x^{(i)}) = \varphi_j \sigma^{-1} s_F (x^{(i)}) = \varphi_j (x^{(i)}) = u_j, j = 1, 2 \). We denote \( t_j = \mu_j (x^{(i)}), j = 1, 2 \). By Definition 4.3 we have that \( \tau \mu_1 = \tau \mu_2, \) so \((t_1, t_2) \in T \) and \((u_1, u_2) \in s_A (T) \). Hence, \( \alpha (\Phi) (T) \subseteq s_A (T) \).

Now we prove the second statement of the Theorem. We assume that \( A_1, A_2 \in \text{Ob} \Theta^0, \mu_1, \mu_2 \in \text{Hom} (A_1, A_2), T \in \text{Cl} (A_1, A_2) \) and \( \tau : A_2 \to A_2 / T, \tilde{\tau} : \Phi (A_2) \to \Phi (A_2) / s_{A_2} (T) \) are the natural epimorphisms. If \( \tau \mu_1 = \tau \mu_2 \) holds, then for every \( u \in A_1 \) that \( ((\Phi (\mu_1)) (u), (\Phi (\mu_2)) (u)) = (s_{A_2} \mu_1 s_{A_1}^{-1} (u), s_{A_2} \mu_2 s_{A_1}^{-1} (u)) \in s_{A_2} (T) \) holds. So \( \tilde{\tau} (\mu_1) = \tilde{\tau} (\mu_2) \).

If \( \tilde{\tau} (\mu_1) = \tilde{\tau} (\mu_2) \), then for every \( u \in A_1 \) the \( ((\Phi (\mu_1)) (u), (\Phi (\mu_2)) (u)) = (s_{A_2} \mu_1 s_{A_1}^{-1} (u), s_{A_2} \mu_2 s_{A_1}^{-1} (u)) \) in \( s_{A_2} (T) \). Because \( s_{A_1} \) is a bijection, we have that for every \( a \in A_1 \) the \( (\mu_1 (a), \mu_2 (a)) \) in \( T \) holds and \( \tau \mu_1 = \tau \mu_2 \).

**Corollary 1** Automorphic equivalence of algebras in \( \Theta \) is equivalence.

**Proof.** We need to prove that automorphic equivalence is reflexive, symmetric and reflexive relation. The identity automorphism provides automorphic equivalence of the algebra \( H \in \Theta \) to itself, so this relation is reflexive.

We assume that the automorphism \( \Phi \in \mathfrak{A} \) provides automorphic equivalence of the algebra \( H_1 \) to the algebra \( H_2 \), where \( H_1, H_2 \in \Theta \), and acts on the morphisms of \( \Theta^0 \) by formula 2.4 with the system of bijections \( \{s_{A}^{\Phi} : A \to \Phi (A) \mid A \in \text{Ob} \Theta^0 \} \). Then by Theorem 4.1 \( s_{A}^{\Phi} : \text{Cl} (H_1, \Phi^{-1} (A)) \to \text{Cl} (H_2, A) \) is a bijection. So \( \left( s_{A}^{\Phi^{-1} (A)} \right)^{-1} : \text{Cl} (H_2, A) \to \text{Cl} (H_1, \Phi^{-1} (A)) \) is also a bijection. But we can conclude from consideration of the formula 2.4 that automorphism \( \Phi^{-1} \) acts on the morphisms of \( \Theta^0 \) by formula 2.1 with the system of bijections \( \{s_{A}^{\Phi^{-1} (A)} = \left( s_{A}^{\Phi^{-1} (A)} \right)^{-1} : A \to \Phi^{-1} (A) \mid A \in \text{Ob} \Theta^0 \} \). Therefore the automorphism \( \Phi^{-1} \) provides automorphic equivalence of the algebra \( H_2 \) to the algebra \( H_1 \).

Now we assume that the automorphism \( \Phi \in \mathfrak{A} \) provides automorphic equivalence of the algebra \( H_1 \) to the algebra \( H_2 \) and the automorphism \( \Psi \in \mathfrak{A} \).
provides automorphic equivalence of the algebra $H_2$ to the algebra $H_3$, where $H_1, H_2, H_3 \in \Theta$. We by $\{s^\Phi_A\}^\Gamma$ and $\{s^\Psi_A\}^\Gamma$ the systems of bijections which present by formula (2.1) the acting of automorphisms $\Phi$ and $\Psi$ correspondingly on the morphisms of $\Theta^0$. By Theorem 4.1 $s^\Phi_A : Cl_{H_1}A \to Cl_{H_1}\Phi(A)$ and $s^\Psi_{\Phi(A)} : Cl_{H_2}\Phi(A) \to Cl_{H_2}\Psi\Phi(A)$ are bijections. So $s^\Psi_{\Phi(A)}s^\Phi_A : Cl_{H_1}A \to Cl_{H_2}\Psi\Phi(A)$ is also a bijection. But we can conclude from consideration of the formula (2.1) that automorphism $\Psi\Phi$ acts on the morphisms of $\Theta^0$ by formula (2.1) with the system of bijections $\{s^\Psi_{\Phi(A)}s^\Phi_A : A \to \Psi\Phi(A) \mid A \in Ob\Theta^0\}$. Therefore the automorphism $\Psi\Phi$ provides automorphic equivalence of the algebra $H_1$ to the algebra $H_3$. ■

**Definition 4.4** We say that the system of bijections $\{c_F : F \to F \mid F \in Ob\Theta^0\}$ is a **central function** if $c_{B\mu} = \mu c_A$ holds for every $\mu \in Mor_{\Theta^0}(A, B)$.

**Proposition 4.1** If $A \in Ob\Theta^0$, $T \subseteq \bigcup_{i \in \Gamma} (\{A\}^{(i)})^2$ is a congruence and $\{c_F : F \to F \mid F \in Ob\Theta^0\}$ is a central function then $c_A(T) = T$.

**Proof.** We assume that $(a_1, a_2) \in T \cap (\{A\}^{(i)})^2$, where $i \in \Gamma$. We consider $F = F(x^{(i)}) \in Ob\Theta^0$ and two homomorphisms $\mu_j : F \to A$, $j = 1, 2$, such that $\mu_j(x^{(i)}) = a_j$. We can prove by induction by the length of the terms in the algebra $F$, that $(\mu_1(f), \mu_2(f)) \in T$ holds for every $f \in F$. The $c_A(a_1) = c_A\mu_j(x^{(i)}) = \mu_jc_F(x^{(i)})$ holds for $j = 1, 2$. Therefore $(c_A(a_1), c_A(a_2)) \in T$ and $c_A(T) \subseteq T$.

If $\{c_F : F \to F \mid F \in Ob\Theta^0\}$ is a central function then $\{c^{-1}_F : F \to F \mid F \in Ob\Theta^0\}$ is also a central function, so $c^{-1}_A(T) \subseteq T$. ■

**Corollary 1** The bijections $\{\alpha(\Phi)_A \mid A \in Ob\Theta^0\}$ (see Definition 4.3) are depend only from the automorphism $\Phi$ and not from the presentation of the action of it automorphism over morphisms by bijections.

**Theorem 4.2** (see [5, Proposition 9]) If an inner automorphism $\Upsilon$ provides the automorphic equivalence of the algebras $H_1$ and $H_2$, where $H_1, H_2 \in \Theta$, then $H_1$ and $H_2$ are geometrically equivalent.

**Proof.** By Definition 2.1 the automorphism $\Upsilon$ acts on the morphisms of $\Theta^0$ by system of isomorphisms $\{\sigma_A : A \to \Phi(A) \mid A \in Ob\Theta^0\}$. By Theorem 4.1 $\sigma_A : Cl_{H_1}A \to Cl_{H_2}\Phi(A)$ is a bijection for $A \in Ob\Theta^0$. We take the set $T \in Cl_{H_1}A$. $\sigma_A(T) \in Cl_{H_2}\Phi(A)$, so $\sigma_A(T) = \bigcap_{\varphi \in Hom(\Phi(A), H_2)} \sigma^{-1}_A(\ker \varphi).$ Therefore

$$T = \sigma^{-1}_A \sigma_A(T) = \bigcap_{\varphi \in Hom(\Phi(A), H_2), \sigma_A(T) \subseteq \ker \varphi} \sigma^{-1}_A \ker \varphi = \bigcap_{\varphi \in Hom(\Phi(A), H_2), \sigma_A(T) \subseteq \ker \varphi} \ker \varphi \sigma_A.$$
If \( \sigma_A(T) \subseteq \ker \phi \) then \( T \subseteq \ker \phi \sigma_A \), so \( T \supseteq \bigcap_{\psi \in \text{Hom}(A, H_2)} \ker \psi = T_{H_2}'' \) and \( T \in \text{Cl}_{H_2} A \). Therefore \( \text{Cl}_{H_1} A \subseteq \text{Cl}_{H_2} A \).

Hence we must remark that if \( \{ \varphi \in \text{Hom}(\Phi(A), H_2) \mid \sigma_A(T) \subseteq \ker \varphi \} = \emptyset \), in particular, because \( \Gamma_{\Phi(A)} \varphi \sigma_A(T) \subseteq \Gamma_{H_2} \), and \( \sigma_A(T) = \bigcup_{i \in F} \left( (\Phi(A))^{(i)} \right)^2 \in \text{Cl}_{H_2} \Phi(A) \) then \( \{ \psi \in \text{Hom}(A, H_2) \mid T \subseteq \ker \psi \} = \emptyset \), in particular, because \( \Gamma_A = \Gamma_{\Phi(A)} \varphi \sigma_A(T) \subseteq \Gamma_{H_2} \), and \( T = \bigcup_{i \in F} (\Phi(A))^{(i)} \in \text{Cl}_{H_2} A \).

By Corollary 4.1 from the Theorem 4.1 the automorphism \( \Upsilon^{-1} \) provides the automorphic equivalence of the algebras \( H_2 \) and \( H_1 \) and by similar consideration we conclude that \( \text{Cl}_{H_2} A \subseteq \text{Cl}_{H_1} A \). ■

From this Theorem we conclude that the possible difference between the automorphic equivalence and geometric equivalence is measured by quotient group \( \mathfrak{A}/\mathfrak{Q} \cong \mathfrak{S}/\mathfrak{S} \cap \mathfrak{Q} \). The elements of the group \( \mathfrak{S} \) we can find by using of the Theorem 3.3. The elements of the group \( \mathfrak{S} \cap \mathfrak{Q} \) we can find by using of the Proposition 3.3.

From here to the end of the Subsection we assume the system of words \( W = \{ \omega \mid \omega \in \Omega \} \) is a subject of conditions \( \text{Op1} \) and \( \text{Op2} \), \( \{ s_F \mid F \in \text{Ob} \Theta^0 \} \) is a system of bijection subject of condition \( \text{Op2} \), \( \Phi \in \mathfrak{S} \) corresponds to the system of words \( W \).

**Proposition 4.2** For every \( H \in \Theta \) and every \( F \in \text{Ob} \Theta^0 \) the mapping \( s_F^{-1} : \text{Cl}_H(F) \to \text{Cl}_{H^*_W}(F) \) is a bijection.

**Proof.** Algebras \( H \) and \( H^*_W \) have different operations but same sets of elements, so \( \Gamma_H = \Gamma_{H^*_W} \). If \( \Gamma_H \not\subseteq \Gamma_F \) then \( \text{Cl}_H(F) = \text{Cl}_{H^*_W}(F) = \left\{ \bigcup_{i \in \Gamma_F} (F^{(i)})^2 \right\} \) and

\[
s_F^{-1} \left( \bigcup_{i \in \Gamma_F} (F^{(i)})^2 \right) = \bigcup_{i \in \Gamma_F} (F^{(i)})^2,
\]

because \( s_F^{-1} \) is a bijection which conforms with the \( \eta_F \). So in this case the Proposition is proved.

Now we consider the case when \( \Gamma_H \supseteq \Gamma_F \) and \( \text{Hom}(F, H) \neq \emptyset \). In this case also \( \text{Hom}(F, H^*_W) \neq \emptyset \). We consider the diagram

\[
\begin{array}{ccc}
F & \xrightarrow{s_F} & F \\
\downarrow \psi & & \downarrow \psi \\
H^*_W & & H \\
\end{array}
\]

\( s_F : F \to F^*_W \) is an isomorphism. If \( \varphi \in \text{Hom}(F, H) \) then by Corollary 4.1 from Proposition 3.4 \( \varphi \in \text{Hom}(F^*_W, H^*_W) \) and \( s_F \varphi \in \text{Hom}(F, H^*_W) \). If \( \psi \in \text{Hom}(F, H^*_W) \) then \( \psi s_F^{-1} \in \text{Hom}(F^*_W, H^*_W) \) and also by Corollary 4.1 from Proposition 3.3 \( \psi s_F^{-1} \in \text{Hom}(F, H) \).

If \( T \in \text{Cl}_H(F) \), then \( T = \bigcap_{\varphi \in T_H} \ker \varphi \). If \( \varphi \in T^*_H \) then \( \varphi \in \text{Hom}(F, H) \), \( \ker \varphi \supseteq T^*_H \). So \( s_F \varphi \in \text{Hom}(F, H^*_W) \) and \( \ker \varphi s_F = s_F^{-1} \ker \varphi \supseteq s_F^{-1} T^*_H \). Hence
\[\varphi s_F \in (s_F^{-1}T)_H^{\psi}. \text{ Therefore } \bigcap_{\psi \in (s_F^{-1}T)_H^{\psi}} \ker \psi \subseteq \bigcap_{\varphi \in T^\psi_H} \ker \varphi = s_F^{-1} \left( \bigcap_{\varphi \in T^\psi_H} \ker \varphi \right) = s_F^{-1}T. \]  
So \( s_F^{-1}T \in Cl_{H^*_W}(F) \). Therefore \( s_F^{-1} \) is a mapping from \( Cl_H(F) \) to \( Cl_{H^*_W}(F) \).

We need to prove that \( s_F^{-1} \) is a bijection. For this purpose we will prove that \( s_F \) is a mapping from \( Cl_{H^*_W}(F) \) to \( Cl_H(F) \). If \( T \in Cl_{H^*_W}(F) \) then \( T = \bigcap_{\psi} \ker \psi \). If \( \psi \in T^\psi_H \) then \( \psi \in \text{Hom}(F,H^*_W) \), \( \ker \psi \supseteq T \). So \( \psi s_F^{-1} \in \bigcap_{\psi \in T^\psi_H} \ker \psi = T \).

\[ \text{Hom}(F,H) \text{ and } \ker \psi s_F^{-1} = s_F \ker \psi \supseteq s_F T. \text{ Hence } \psi s_F^{-1} \in (s_F T)_H^{\psi}. \]  
Therefore \( \bigcap_{\psi \in (s_F T)_H^{\psi}} \ker \psi \subseteq \bigcap_{\varphi \in T^\psi_H} \ker \varphi = s_F T. \) So \( s_F T \in Cl_H(F) \). Therefore \( s_F \) is a mapping from \( Cl_{H^*_W}(F) \) to \( Cl_H(F) \). \( s_F \) is an inverse mapping for the \( s_F^{-1} \), so both of them are bijections.

**Corollary 1** For every \( H \in \Theta \) the automorphism \( \Phi^{-1} \) provides an automorphic equivalence of algebras \( H \) and \( H^*_W \).

**Proof.** By Theorem 4.1.

**Theorem 4.3** Algebras \( H_1, H_2 \in \Theta \) are automorphically equivalent if and only if there is a system of words \( W = \{w_\omega \mid \omega \in \Omega \} \) subject of conditions Op1) and Op2), such that algebras \( H_1 \) and \( (H_2)_W \) are geometrically equivalent.

**Proof.** We assume that the automorphism \( \Psi \in \mathfrak{A} \) provides an automorphic equivalence of algebras \( H_1 \) and \( H_2 \). By Theorem 2.3 we can present the automorphism \( \Psi \) in the form \( \Psi = \Phi \Upsilon \), where \( \Phi \in \mathfrak{S}, \Upsilon \in \mathfrak{I} \). Therefore \( \Upsilon = \Phi^{-1} \Psi \).

By Corollary 4 from the Proposition 4.2 the automorphism \( \Phi^{-1} \) provides an automorphic equivalence of algebras \( H_2 \) and \( (H_2)_W^* \), where \( W = \{w_\omega \mid \omega \in \Omega \} \) is the system of the words, which is a subject of conditions Op1) and Op2) and corresponds to the automorphism \( \Phi \) by the Theorem 3.1.

By the Proof of the Corollary from the Theorem 4.1 we have that the inner automorphism \( \Upsilon = \Phi^{-1} \Psi \) provides an automorphic equivalence of algebras \( H_1 \) and \( (H_2)_W^* \).

By the Theorem 4.2 we conclude that algebras \( H_1 \) and \( (H_2)_W^* \) are geometrically equivalent.

Now we assume that there is a system of words \( W = \{w_\omega \mid \omega \in \Omega \} \) subject of conditions Op1) and Op2), such that algebras \( H_1 \) and \( (H_2)_W^* \) are geometrically equivalent. By definition for every \( F \in \text{Ob} \Theta^0 \) there is a system a bijection \( id_F : Cl_{H_1}(F) \to Cl_{(H_2)_W^*}(F) \). The identity automorphism of the category \( \Theta^0 \) acts on the morphisms of \( \Theta^0 \) by formula (2.1) with bijections \( \{id_F : F \to F \mid F \in \text{Ob} \Theta^0 \} \). So, by Theorem 4.1 the identity automorphism of the category \( \Theta^0 \) provides an automorphic equivalence of algebras \( H_1 \) and \( (H_2)_W^* \). We denote by \( \Phi \) the strongly stable automorphism which corresponds to the system of words \( W \) by the Theorem 3.1 From Corollary 4.1 from the Proposition 4.2 and from the Proof of the Corollary 5 from the Theorem 4.1 we
conclude that automorphism $\Phi$ provides an automorphic equivalence of algebras $(H_2)^*_W$ and $H_2$ and same automorphism provides an automorphic equivalence of algebras $H_1$ and $H_2$. ■

4.2 Automorphic equivalence and coordinate algebras.

Categories $C_\Theta (H)$ of the coordinate algebras were defined in [3]. Here $\Theta$ is an arbitrary variety of algebras, $H \in \Theta$. The objects of the category $C_\Theta (H)$ are algebras $F/T$ where $F \in \text{Ob}\Theta$, $T \in Cl_H(F)$. Morphisms of the category $C_\Theta (H)$ are homomorphisms of these algebras. Now we will formulate the notion of the automorphic equivalence of algebras in the language of the coordinate algebras.

Theorem 4.4 Automorphism $\Phi : \Theta^0 \to \Theta^0$ provides an automorphic equivalence of algebras $H_1, H_2 \in \Theta$ if and only if there is an isomorphism $\Psi : C_\Theta (H_1) \to C_\Theta (H_2)$ subject of conditions:

1. for every $F \in \text{Ob}\Theta$ and every $T \in Cl_{H_1}(F)$ the $\Psi(F/T) = \Phi(F)/\tilde{T}$ holds, where $\tilde{T} \in Cl_{H_2}(\Phi(F))$;

2. for every $F \in \text{Ob}\Theta$ the $\Psi(F/(\Delta F)''_{H_1}) = \Phi(F)/(\Delta \Phi(F))''_{H_2}$ holds,

3. for every $F \in \text{Ob}\Theta$ and every $T \in Cl_{H_1}(F)$ the isomorphism $\Psi$ transforms the natural epimorphism $\tau : F/(\Delta F)''_{H_1} \to F(X)/T$ to the natural epimorphism $\Psi(\tau) : \Phi(F)/(\Delta \Phi(F))''_{H_2} \to \Psi(F/T)$;

4. for every $F_1, F_2 \in \text{Ob}\Theta$ and every $\nu \in \text{Mor}_{C_\Theta(H_1)}(F_1/(\Delta F_1)''_{H_1}, F_2/(\Delta F_2)''_{H_1})$ if the diagram

\[
\begin{array}{ccc}
F_1 & \overset{\delta_1}{\to} & F_1/(\Delta F_1)''_{H_1} \\
\downarrow \mu & & \downarrow \nu \\
F_2 & \overset{\delta_2}{\to} & F_2/(\Delta F_2)''_{H_1}
\end{array}
\]

is commutative then the diagram

\[
\begin{array}{ccc}
\Phi(F_1) & \overset{\delta_1}{\to} & \Phi(F_1)/(\Delta \Phi(F_1))''_{H_2} \\
\downarrow \Phi(\mu) & & \downarrow \Phi(\nu) \\
\Phi(F_2) & \overset{\delta_2}{\to} & \Phi(F_2)/(\Delta \Phi(F_2))''_{H_2}
\end{array}
\]

is also commutative, where $\delta_i$ and $\tilde{\delta}_i$ are the natural epimorphisms, $i = 1, 2$, and $\mu \in \text{Mor}_{\Theta}(F_1, F_2)$. The isomorphism $\Psi$ is uniquely determined by automorphism $\Phi : \Theta^0 \to \Theta^0$.

Proof. We suppose that there are an automorphism $\Phi : \Theta^0 \to \Theta^0$ and an isomorphism $\Psi : C_\Theta (H_1) \to C_\Theta (H_2)$ subject of conditions 1.-4. We will prove that the automorphism $\Phi : \Theta^0 \to \Theta^0$ provide an automorphic equivalence of
algebras $H_1, H_2 \in \Theta$. If $F/T \in \text{Ob}C_\Theta (H_1)$ then by condition 1. there exists $\overline{T} \in \text{Cl}_{H_2}(\Phi (F))$ such that $\Psi (F/T) = \Phi (F)/\overline{T}$. We denote by $\alpha (\Phi)_{\overline{T}} (T) = \overline{T}$. $\alpha (\Phi)_T$ is a mapping $\text{Cl}_{H_2}(\Phi (F)) \rightarrow \text{Cl}_{H_2}(\Phi (F))$. We consider $A_1, A_2 \in \text{Ob}\Theta^0$. If $\text{Hom} (A_1, A_2) = \emptyset$, in particular, because $\Gamma_{A_1} \not\subseteq \Gamma_{A_2}$, then condition: for every $\mu_1, \mu_2 \in \text{Hom} (A_1, A_2)$ and every $T \in \text{Cl}_{H_1}(A_2)$ the $\tau \mu_1 = \tau \mu_2$ holds if and only if when the $\overline{\tau} \Phi (\mu_1) = \overline{\tau} \Phi (\mu_2)$ holds, where $\tau : A_2 \rightarrow A_2/T$, $\overline{\tau} : \Phi (A_2) \rightarrow \Phi (A_2)/\alpha (\Phi)_{\overline{T}} (T)$ are the natural epimorphisms - fulfills. Now we consider the case when $\Gamma_{A_1} \subseteq \Gamma_{A_2}$. We take $\mu_1, \mu_2 \in \text{Hom} (A_1, A_2)$, $T \in \text{Cl}_{H_1}(A_2)$ and suppose that

$$\tau \mu_1 = \tau \mu_2,$$

(4.3)

where $\tau : A_2 \rightarrow A_2/T$ is the natural epimorphism. We will consider the diagrams

$$\begin{align*}
A_1 \xrightarrow{\delta_i} & A_1/(\Delta_{A_1})''_{H_1}, \\
\downarrow \mu_i & \quad \downarrow \overline{\mu_i}, \\
A_2 \xrightarrow{\delta_3} & A_2/(\Delta_{A_2})''_{H_1},
\end{align*}$$

(4.4)

where $\delta_i$ are the natural epimorphisms, $i = 1, 2$. If $(a_1, a_2) \in (\Delta_{A_1})''_{H_1}$ then for every $\varphi \in \text{Hom} (A_2, H_1)$ the $\varphi \mu_i (a_1) = \varphi \mu_i (a_2)$ holds, because $\varphi \mu_i \in \text{Hom} (A_1, H_1)$. So $\mu_i (\Delta_{A_1})''_{H_1} \subseteq (\Delta_{A_2})''_{H_1}$. In the degenerate cases, if $\Gamma_{A_1} \not\subseteq \Gamma_{H_1}$, then $(\Delta_{A_1})''_{H_1} = \bigcup_{i \in \Gamma} \left( A_1^{(i)} \right)^2$. If in this case $\Gamma_{A_2} \subseteq \Gamma_{H_1}$ then $\Gamma_{A_1} \subseteq \Gamma_{A_2} \subseteq \Gamma_{H_1}$, and we have a contradiction. So $\Gamma_{A_2} \not\subseteq \Gamma_{H_1}$ and $(\Delta_{A_2})''_{H_1} = \bigcup_{i \in \Gamma} \left( A_2^{(i)} \right)^2$. From this equality the $\mu_i (\Delta_{A_1})''_{H_1} \subseteq (\Delta_{A_2})''_{H_1}$ is concluded in all cases. Therefore in all cases the homomorphism $\overline{\mu_i}$, which closes the diagram (4.4), commutative, exists.

We will consider the natural epimorphisms $\tau : A_2/(\Delta_{A_2})''_{H_1} \rightarrow A_2/T$. The $\tau = \overline{\tau} \delta_2$ holds. From commutativity of the diagrams (4.4) and from (4.3) we conclude that $\overline{\tau} \delta_2 \mu_1 = \overline{\tau} \delta_2 \mu_2 = \overline{\tau} \overline{\mu_1} \delta_1 = \overline{\tau} \overline{\mu_2} \delta_1$. So $\overline{\tau} \overline{\mu_1} = \overline{\tau} \overline{\mu_2}$ and $\Psi (\tau) \Psi (\overline{\tau}) = \Psi (\tau) \Psi (\overline{\tau})$, because $\overline{\tau}, \overline{\mu_1}, \overline{\mu_2} \in \text{Mor}C_\Theta (H_1)$.

By condition 4. and 2. we have that the diagrams

$$\begin{align*}
\Phi (A_1) \xrightarrow{\delta_i} & \Phi (A_1)/(\Delta_{\Phi (A_1)})''_{H_2}, \\
\downarrow \Phi (\mu_i) & \quad \downarrow \Psi (\overline{\tau}), \\
\Phi (A_2) \xrightarrow{\delta_3} & \Phi (A_2)/(\Delta_{\Phi (A_2)})''_{H_2},
\end{align*}$$

(4.5)

where $\delta_i$ are the natural epimorphisms, $i = 1, 2$, is also commutative. From commutativity of these diagrams we have that $\Psi (\tau) \Psi (\overline{\tau}) \delta_1 = \Psi (\tau) \Psi (\overline{\tau}) \delta_2 = \Psi (\tau) \delta_3 \Phi (\mu_1) = \Psi (\tau) \delta_3 \Phi (\mu_2)$. By condition 3. $\Psi (\tau) \delta_2 = \overline{\tau}$, where $\overline{\tau} : \Phi (A_2) \rightarrow \Phi (A_2)/\alpha (\Phi)_{\overline{T}} (T)$ is the natural epimorphism. So $\overline{\tau} \Phi (\mu_1) = \overline{\tau} \Phi (\mu_2)$.

Vice versa, if the $\overline{\tau} \Phi (\mu_1) = \overline{\tau} \Phi (\mu_2)$ holds, then from commutativity of the diagrams (4.5) we, as above, conclude that $\Psi (\tau) \Psi (\overline{\tau}) = \Psi (\tau) \Psi (\overline{\tau})$. 

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So $\tau \mu_1 = \tau \mu_2$. And from commutativity of the diagrams (4.4) we have that $\tau \mu_1 = \tau \mu_2$.

We also must prove that $\alpha (\Phi)_F$ is a bijection for every $F \in \text{Ob} \Theta^0$, if $R \in Cl_{H_2}(\Phi (F))$, then $\Phi (F)/R \in \text{Ob} \Theta^0 (H_2)$. There exists $F_1/T \in \text{Ob} \Theta^0 (H_1)$ such that $\Phi (F)/R = \Psi (F_1/T) = \Phi (F_1)/\alpha (\Phi)_{F_1}(T)$. So $\Phi (F) = \Phi (F_1)$, $F = F_1$ and $R = \alpha (\Phi)_{F_1}(T)$. So $\alpha (\Phi)_F$ is a surjection. If $T_1, T_2 \in Cl_{H_2}(F)$ and $\alpha (\Phi)_F(T_1) = \alpha (\Phi)_F(T_2) = T$. Then $\Psi(F/T_1) = \Psi(F/T_2) = \Phi(F)/T$, so $F/T_1 = F/T_2$ and $T_1 = T_2$. So $\alpha (\Phi)_F$ is an injection.

Now suppose that the automorphism $\Phi : \Theta^0 \to \Theta^0$ provides an automorphic equivalence of algebras $H_1, H_2 \in \Theta$. By Theorem 2.11 there exists a system of bijections $\{ s_F : F \to \Phi (F) \mid F \in \text{Ob} \Theta^0 \}$, such that $\Phi$ acts on the morphisms of $\Theta^0$ according the formula (2.1). By Theorem 4.11 $s_F : Cl_{H_2}(F) \to Cl_{H_2}(\Phi(F))$ is the bijection subject of the Definition 4.3. We will construct the functor $\Psi : H_1 \to H_2$.

We consider $\nu \in \text{Mor}_{Cl_{H_1}}(F_1/T_1, F_2/T_2)$. By projective propriety of the free algebras there exists homomorphism $\mu$ such that the diagram

$$
\begin{array}{ccc}
F_1 & \xrightarrow{\tau_1} & F_1/T_1 \\
\downarrow \mu & & \downarrow \nu \\
F_2 & \xrightarrow{\tau_2} & F_2/T_2 \\
\end{array}
$$

is commutative. Here $\tau_i$ are the natural epimorphisms, $i = 1, 2$. If $(f_1, f_2) \in T_1$ then $\tau_2 \mu (f_1) = \nu \tau_1 (f_1) = \nu \tau_1 (f_2) = \tau_2 \mu (f_2)$. Therefore $\mu(T_1) \subseteq T_2$ and $\Phi(\mu)(s_{F_1}(T_1)) = s_{F_2} \mu s_{F_1}(T_1) \subseteq s_{F_2}(T_2)$. Hence there exists a homomorphism $\overline{\varphi} : \Phi(F_1)/s_{F_1}(T_1) \to \Phi(F_2)/s_{F_2}(T_2)$ such that the diagram

$$
\begin{array}{ccc}
\Phi(F_1) & \xrightarrow{\tau_1} & \Psi(F_1/T_1) = \Phi(F_1)/s_{F_1}(T_1) \\
\downarrow \Phi(\mu) & & \downarrow \overline{\varphi} \\
\Phi(F_2) & \xrightarrow{\tau_2} & \Psi(F_2/T_2) = \Phi(F_2)/s_{F_2}(T_2) \\
\end{array}
$$

where $\tau_i$ are the natural epimorphisms, $i = 1, 2$, is commutative.

The homomorphism $\overline{\varphi}$ is not depend on the choice of the homomorphism $\mu$, which make the diagram (4.6) commutative. Indeed, if two homomorphisms $\mu_1, \mu_2 : F_1 \to F_2$ make the diagram (4.6) commutative, then $\tau_2 \mu_1 = \nu \tau_1 = \tau_2 \mu_2$. So by Definition 4.3 and Theorem 4.11 $\overline{\varphi}_2 \Phi (\mu_1) = \overline{\varphi}_2 \Phi (\mu_2)$. So if we have two commutative diagrams

$$
\begin{array}{ccc}
\Phi(F_1) & \xrightarrow{\tau_1} & \Phi(F_1)/s_{F_1}(T_1) \\
\downarrow \Phi(\mu_1) & & \downarrow \overline{\varphi}_1 \\
\Phi(F_2) & \xrightarrow{\tau_2} & \Phi(F_2)/s_{F_2}(T_2) \\
\end{array} \quad \text{and} \quad
\begin{array}{ccc}
\Phi(F_1) & \xrightarrow{\tau_1} & \Phi(F_1)/s_{F_1}(T_1) \\
\downarrow \Phi(\mu_2) & & \downarrow \overline{\varphi}_2 \\
\Phi(F_2) & \xrightarrow{\tau_2} & \Phi(F_2)/s_{F_2}(T_2) \\
\end{array}
$$

then $\overline{\varphi}_1 = \overline{\varphi}_2 \Phi (\mu_1) = \overline{\varphi}_2 \Phi (\mu_2) = \overline{\varphi}_2 \tau_1$ and $\overline{\varphi}_1 = \overline{\varphi}_2$. We denote $\overline{\varphi} = \Psi (\nu)$, where $\overline{\varphi}$ is the homomorphism from the diagram (4.7).
Now we will check that $\Psi$ is a functor. If $\nu_1 \in \text{Mor}_{C_\Theta(H_1)}(F_1/T_1, F_2/T_2)$, $\nu_2 \in \text{Mor}_{C_\Theta(H_1)}(F_2/T_2, F_3/T_3)$ then, by the consideration of the two diagrams which are similar to the diagram (4.6), we conclude that $\Psi (\tau_1 \nu_1)$ commutative. And now, by the consideration of the big squares of the diagrams which are similar to the diagram (4.6), we conclude that the diagram

\[
\begin{array}{ccc}
F_1 & \rightarrow & F_1/T_1 \\
\downarrow \mu_1 & & \downarrow \nu_1 \\
F_2 & \rightarrow & F_2/T_2 \\
\downarrow \mu_2 & & \downarrow \nu_2 \\
F_3 & \rightarrow & F_3/T_3
\end{array}
\]

(4.8)

is commutative. After this, by the consideration of the two diagrams which are similar to the diagram (4.7), we conclude that the diagram

\[
\begin{array}{ccc}
\Phi (F_1) & \rightarrow & \Psi (F_1/T_1) \\
\downarrow \Phi (\mu_1) & & \downarrow \Psi (\nu_1) \\
\Phi (F_2) & \rightarrow & \Psi (F_2/T_2) \\
\downarrow \Phi (\mu_2) & & \downarrow \Psi (\nu_2) \\
\Phi (F_3) & \rightarrow & \Psi (F_3/T_3)
\end{array}
\]

(4.9)

is commutative. And now, by the consideration of the big squares of the diagrams (4.8) and (4.9), we conclude that $\Psi (\nu_1 \nu_2) = \Psi (\nu_1) \Psi (\nu_2)$. So $\Psi$ is a functor.

Now we will prove that $\Psi$ is an isomorphism. By Corollary 1 from the Theorem 4.11 the automorphism $\Phi^{-1}$ provides an automorphic equivalence of algebras $H_2$ and $H_1$. By using of the automorphism $\Phi^{-1}$ and the system of bijections \( \{ s_F^{-1} = (s_{\Phi^{-1}(F)})^{-1} : F \rightarrow \Phi^{-1}(F) \mid F \in \text{Ob}\Theta \} \) we construct the functor $\Psi$. If $F/T \in \text{Ob}_{C_\Theta}(H_1)$ then $\Psi(F/T) = \Phi(F)/s_F(T) \in \text{Ob}_{C_\Theta}(H_2)$, $s_F(T) \in C_{H_2}(\Phi(F))$ and $\Psi\Psi(F/T) = \Phi(F)/s_F(T) = \Phi^{-1}(F)/s_{\Phi^{-1}(F)}s_F(T)$. But $s_{\Phi^{-1}(F)}s_F(T) = (s_{\Phi^{-1}(F)})^{-1} s_F(T) = T$, so $\Psi\Psi(F/T) = F/T$. If $F/T \in \text{Ob}_{C_\Theta}(H_2)$ then $\Phi(F/T) = \Phi^{-1}(F)/s_{\Phi^{-1}(F)}^{-1}(T) \in \text{Ob}_{C_\Theta}(H_1)$, $(s_{\Phi^{-1}(F)})^{-1}(T) \in C_{H_1}(\Phi^{-1}(F))$ and $\Psi\Psi(F/T) = \Phi\Phi^{-1}(F)/s_{\Phi^{-1}(F)}(s_{\Phi^{-1}(F)})^{-1}(T) = F/T$. Now we consider $\nu \in \text{Mor}_{C_\Theta(H_1)}(F_1/T_1, F_2/T_2)$. There exists homomorphism $\mu : F_1 \rightarrow F_2$ such that the diagram (4.6) and the diagram (4.7) with $\overline{\mu} = \Psi(\nu)$ are commutative. So, as above, the diagram

\[
\begin{array}{ccc}
\Phi^{-1}(F_1) & \rightarrow & \tilde{\Psi}\Psi(F_1/T_1) \\
\downarrow \Phi^{-1}(\mu) & & \downarrow \tilde{\Psi}\Psi(\nu) \\
\Phi^{-1}(F_2) & \rightarrow & \tilde{\Psi}\Psi(F_2/T_2)
\end{array}
\]

(4.10)

where $\tilde{\tau}_i$ are the natural epimorphisms, $i = 1, 2$, is also commutative. But $\Phi^{-1}(F_i) = F_i$, $\Phi^{-1}(\mu) = \mu$, $\tilde{\Psi}\Psi(F_i/T_i) = F_i/T_i$, $i = 1, 2$. So $\tilde{\tau}_i = \tau_i$, $i = 1, 2$, and diagram (4.10) coincide with the diagram (4.6), only instead the
homomorphism $\nu$ we have a homomorphism $\tilde{\Psi} \Psi (\nu)$. From $\nu \tau_1 = \tilde{\Psi} \Psi (\nu) \tau_1$ we conclude that $\nu = \tilde{\Psi} \Psi (\nu)$. Now we consider $\nu \in \text{Mor}_{C_o(H_2)} (F_1/T_1, F_2/T_2)$. We denote by $\mu$ the homomorphism which makes the diagram (4.6) commutative. As above, the diagram

$$
\begin{align*}
\Phi^{-1} (F_1) & \xrightarrow{\tau_i} \tilde{\Psi} (F_1/T_1) \\
\downarrow \Phi^{-1} (\mu) & \downarrow \tilde{\Psi} (\nu) \\
\Phi^{-1} (F_2) & \xrightarrow{\tau_i} \tilde{\Psi} (F_2/T_2)
\end{align*}
$$

is commutative. So the diagram

$$
\begin{align*}
\Phi \Phi^{-1} (F_1) & \xrightarrow{\tilde{\tau}_i} \tilde{\Psi} \Psi (F_1/T_1) \\
\downarrow \Phi \Phi^{-1} (\mu) & \downarrow \Psi (\nu) \\
\Phi \Phi^{-1} (F_2) & \xrightarrow{\tilde{\tau}_i} \tilde{\Psi} \Psi (F_2/T_2)
\end{align*}
$$

(4.11)

where $\tilde{\tau}_i$ are the natural epimorphisms, $i = 1, 2$, is commutative. $\Phi \Phi^{-1} (F_i) = F_i$, $\tilde{\Psi} \Psi (F_i/T_i) = F_i/T_i$, $i = 1, 2$, $\Phi^{-1} \Phi (\mu) = \mu$. So $\tilde{\tau}_i = \tau_i$, $i = 1, 2$, and diagram (4.11) coincide with the diagram (4.6), only instead the homomorphism $\nu$ we have a homomorphism $\tilde{\Psi} \Psi (\nu)$. And as above we have that $\tilde{\Psi} \Psi (\nu) = \nu$. So the functor $\Psi$ is an isomorphism, because it has an inverse functor $\tilde{\Psi}$.

Now we must prove that the isomorphism $\Psi$ is a subject of the conditions 1.-4. Condition 1. fulfills by the definition of $\Psi$. Condition 2. follows from the monotony of the bijection $s_F : Cl_{H_1} (F) \to Cl_{H_2} (\Phi (F))$. We consider the commutative diagram

$$
\begin{array}{ccc}
F & \xrightarrow{\delta} & F / (\Delta F)'_{H_1} \\
\downarrow id_F & & \downarrow \tau \\
F & \xrightarrow{\tau} & F / T
\end{array}
$$

where $F \in \text{Ob} \Theta^0$, $T \in Cl_{H_1} (F)$, $\delta, \tau, \tau'$ are the natural epimorphisms. As above, the diagram

$$
\begin{array}{ccc}
\Phi (F) & \xrightarrow{\tilde{\delta}} & \Phi (F) / (\Delta \Phi (F))'_{H_2} \\
\downarrow id_{\Phi (F)} & & \downarrow \Psi (\tau) \\
\Phi (F) & \xrightarrow{\tilde{\tau}} & \Psi (F / T)
\end{array}
$$

where $\tilde{\delta}$ and $\tilde{\tau}$ are the natural epimorphisms, is commutative. From $\Psi (\tau) \tilde{\delta} = \tilde{\tau}$ we conclude that $\Psi (\tau)$ is the natural epimorphism. So, condition 3. holds. Condition 4. we can conclude from condition 2. and the definition of $\Psi$.

**Proposition 4.3** If there is a pair: an automorphism $\Phi : \Theta^0 \to \Theta^0$ and an isomorphism $\Psi : C_o (H_1) \to C_o (H_2)$ subject of conditions 1. - 4., where $H_1, H_2 \in \Theta$, then the isomorphism $\Psi$ is uniquely determined by the automorphism $\Phi$. 

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we denote by \( F/T \). The morphisms of the category \( \Theta \) are acted on by automorphism \( \Phi \). So by condition 1. we have that for every \( F/T \in \Theta \), the \( \Psi (F/T) \) uniquely determined.

We just have to prove that the acting of the isomorphism \( \Psi \) on morphisms of the category \( C_{\Theta}(H_1) \) also uniquely determined. We consider \( \nu \in \text{Mor}_{\Theta}(H_1) \). As it was proved in the previous Theorem, there exists homomorphism \( \mu : F_1 \rightarrow F_2 \) such that diagram (4.6) is commutative. Also, as it was proved in the previous Theorem when diagrams (4.4) were considered, there exists homomorphism \( \tau_i : F_i \rightarrow F_i/T_i \) such that

\[
\delta_2 \mu = \overline{\tau}_1 \delta_1 , \quad (4.12)
\]

where \( \delta_i : F_i \rightarrow F_i / (\Delta F_i)_{H_1} \) are the natural epimorphisms, \( i = 1, 2 \). As above we denote by \( \overline{\tau}_i : F_i / (\Delta F_i)_{H_1} \rightarrow F_i / T_i \) the natural epimorphisms, \( i = 1, 2 \). The \( \tau_i = \overline{\tau}_i \delta_i \) holds, where \( \tau_i : F_i \rightarrow F_i / T_i \) the natural epimorphisms. So

\[
\nu \overline{\tau}_i \delta_i = \nu \tau_i = \tau_2 \mu = \overline{\tau}_2 \delta_1 = \overline{\tau}_2 \mu , \quad (4.13)
\]

We suppose that there are two isomorphisms \( \Psi_1, \Psi_2 : C_{\Theta}(H_1) \rightarrow C_{\Theta}(H_2) \) subjects of conditions 1. - 4, and will prove that \( \Psi_1(\nu) = \Psi_2(\nu) \). All homomorphisms in the \( \{ 1, 2 \} \) are morphisms of the category \( C_{\Theta}(H_1) \). We apply to (4.13) both isomorphisms: as \( \Psi_1 \), as \( \Psi_2 \) - achieve two commutative diagrams

\[
\Psi_i \left( F_i / (\Delta F_i)_{H_1}'' \right) \xrightarrow{\Psi_i(\overline{\tau}_i)} \Psi_i \left( F_i / T_i \right) \quad \Psi_i \left( F_i / T_i \right) \xrightarrow{\Psi_i(\nu)} \Psi_i \left( F_i / T_i \right)
\]

where \( i = 1, 2 \). As it was proved above, the corresponding objects of these diagrams are coincide. The morphisms in the rows of these diagrams are also coincide, because by condition 3. they are natural epimorphisms which exists between the corresponding objects. That is, in fact, we have these two commutative diagrams

\[
\Phi \left( F_1 / (\Delta \Phi(F_1))_{H_2}'' \right) \xrightarrow{\tau_1} \Phi \left( F_1 / s_{F_1}^\circ (T_1) \right) \quad \Phi \left( F_1 / s_{F_1}^\circ (T_1) \right) \xrightarrow{\nu \tau_1} \Phi \left( F_2 / s_{F_2}^\circ (T_2) \right)
\]

where \( i = 1, 2 \). Automorphism \( \Phi \) acts on the morphisms of the category \( \Theta^0 \) by formula (2.1) with the system of bijections \( \{ s_F^0 : F \rightarrow \Phi(F) \mid F \in \text{Ob}\Theta^0 \} \), \( \tau_1 \),
the natural epimorphisms. Now we consider the diagrams

\[
\Phi(F_1) \xrightarrow{\delta_1} \Phi(F_1)/\left(\Delta\Phi(F_1)\right)''_{H_2} \xrightarrow{\tau_1} \Phi(F_1)/s_{F_1}(T_1)
\]
\[
\downarrow \Phi(\mu) \quad \Psi_1(\tau) \downarrow \quad \Psi_1(\nu) \downarrow, 
\]
\[
\Phi(F_2) \xrightarrow{\delta_2} \Phi(F_2)/\left(\Delta\Phi(F_2)\right)''_{H_2} \xrightarrow{\tau_2} \Phi(F_2)/s_{F_2}(T_2)
\]

where \(i = 1, 2\), \(\delta_1\), \(\delta_2\) the natural epimorphisms. From (1.12) and condition 4, we can conclude that the left small squares of these diagrams are commutative. Therefore both these diagrams are commutative and \(\tau_2\delta_2\Phi(\mu) = \Psi_1(\nu)\tau_1\delta_1 = \Psi_2(\nu)\tau_1\delta_1\) - is an epimorphism so \(\Psi_1(\nu) = \Psi_2(\nu)\).

### 5 Examples.

In this Section we will consider some varieties of many-sorted algebras. We will calculate the group \(\mathcal{A}(\mathcal{V})\) for these varieties and if this group is not trivial we will give examples of algebras which are automorphically equivalent but are not geometrically equivalent. We must prove in all these considerations that in our varieties Condition 2.4 fulfills.

We use for this aim the algebraic proprieties of the automorphisms of the category \(\Theta^0\). First of all we must say that every automorphism \(\Phi\) of the category \(\Theta^0\) transforms every identity morphism \(id_F \in \text{Mor}_{\Theta^0}(F, F)\), where \(F \in \text{Ob}\Theta^0\), to the identity morphism \(id_{\Phi(F)} \in \text{Mor}_{\Theta^0}(\Phi(F), \Phi(F))\), because \(id_F\) is an unit of the monoid \(\text{Mor}_{\Theta^0}(F, F)\). Therefore every automorphism \(\Phi\) of the category \(\Theta^0\) transforms the isomorphism \(\alpha : F_1 \rightarrow F_2\), where \(F_1, F_2 \in \text{Ob}\Theta^0\), to the to the isomorphism \(\Phi(\alpha) : \Phi(F_1) \rightarrow \Phi(F_2)\), because isomorphisms are the invertible morphisms. Also every automorphism \(\Phi\) of the category \(\Theta^0\) preserves the coproducts: \(\Phi\left(\bigsqcup_{i \in I} F_i\right) \cong \bigsqcup_{i \in I} \Phi(F_i)\), where \(F_i \in \text{Ob}\Theta^0\) - because the coproduct is defined by algebraic conditions on the morphisms.

**Definition 5.1** We say that the variety \(\Theta\) has an **IBN propriety** if for every \(F(X), F(Y) \in \text{Ob}\Theta^0\) the \(F(X) \cong F(Y)\) holds if and only if the \(|X(i)| = |Y(i)|\) holds for every \(i \in \Gamma\).

It is clear that if \(F(X), F(Y) \in \text{Ob}\Theta^0\), then \(F(X) \sqcup F(Y) \cong F(Z)\), where \(|X(i)| + |Y(i)| = |Z(i)|\), \(i \in \Gamma\). Therefore, as it was explained in [3, Proposition 5.2], if the variety \(\Theta\) has an IBN propriety every automorphism \(\Phi\) of the category \(\Theta^0\) induces the automorphism \(\varphi\) of the additive monoid \(\mathbb{N}^{[\Gamma]}\), such that \(\varphi\left((m_i)_{i \in \Gamma}\right) = (m_i)_{i \in \Gamma}\) if and only if \(\Phi(F(X)) \cong F(Y)\), where \(|X(i)| = n_i\), \(|Y(i)| = m_i\), \(i \in \Gamma\). The automorphism \(\varphi\) transforms the minimal set of the generators of the \(\mathbb{N}^{[\Gamma]}\): \(\{e_i \mid i \in \Gamma\}\), where \(e_i = (0, \ldots, 0, 1, 0, \ldots, 0)\), 1 is located on the \(i\) place - to itself. Its enough to prove Condition 2.4 if \(|\Gamma| = 1\). In the case \(|\Gamma| > 1\), we must make additional efforts.

**Definition 5.2** We say that homomorphism \(\iota : A \rightarrow B\), where \(A, B \in \Theta\), is an **embedding** if \(\ker \iota = \Delta_A\).
We denote \( \iota : A \rightarrow B \).

**Proposition 5.1** We consider \( F_1, F_2 \in \text{Ob}\Theta^0 \). A homomorphism \( \iota : F_1 \rightarrow F_2 \) is an embedding if and only if for every \( F \in \text{Ob}\Theta^0 \) and every \( \alpha, \beta \in \text{Hom}(F, F_1) \) from \( \alpha \iota = \iota \beta \) we can conclude that \( \alpha = \beta \) holds.

**Proof.** We assume that homomorphism \( \iota : F_1 \rightarrow F_2 \) is not an embedding, so there are \( f_1^{(i)}, f_2^{(i)} \in F_1^{(i)} \subseteq F_1 \), where \( i \in \Gamma \), such that \( f_1^{(i)} \neq f_2^{(i)} \) but \( \iota \left(f_1^{(i)}\right) = \iota \left(f_2^{(i)}\right) \). We consider \( F = F \left(x^{(i)}\right) \in \text{Ob}\Theta^0 \) and homomorphisms \( \alpha, \beta \in \text{Hom}(F, F_1) \) such that \( \alpha \left(x^{(i)}\right) = f_1^{(i)}, \beta \left(x^{(i)}\right) = f_2^{(i)} \). \( \alpha \neq \beta \), but \( \alpha \iota = \iota \beta \).

The proof of the converse is obvious. ■

Therefore embeddings in the category \( \Theta^0 \) can be defined by algebraic propriety of morphisms. So every automorphism of the category \( \Theta^0 \) transforms an embedding to the embedding. We will use this fact in the prove of Condition 2.1

### 5.1 Actions of semigroups over sets.

In this Subsection \( \Theta \) is a variety of the all actions of semigroups over sets. \( \Gamma = \{1, 2\} \): the first sort is a sort of elements of semigroups, the second sort is a sort of elements of sets. \( \Omega = \{, \cdot\} \): \( \cdot \) is a multiplication in the semigroup, \( \circ \) is an action of the elements of the semigroup over the elements of the set. \( \tau_\cdot = (1, 1; 1), \tau_\circ = (1, 2; 2) \). A free algebras \( F \left(X\right) \) in our variety have this form: \( \left(F \left(X\right)\right)^{(1)} = S \left(X^{(1)}\right), \left(F \left(X\right)\right)^{(2)} = \left(S \left(X^{(1)}\right) \circ X^{(2)}\right) \cup X^{(2)} \), where \( S \left(X^{(1)}\right) \circ X^{(2)} = \{ s \circ x^{(2)} \mid s \in S \left(X^{(1)}\right), x^{(2)} \in X^{(2)} \} \), \( S \left(X^{(1)}\right) \) is a free semigroup generated by the set of free generators \( X^{(1)} \).

**Proposition 5.2** The variety of the all actions of semigroups over sets has an IBN propriety.

**Proof.** We consider \( F \left(X\right) \in \text{Ob}\Theta^0 \). We denote \( \left(F \left(X\right)\right)^{(1)} = S \). The quotient semigroup \( S/S^2 \) has \( |X^{(1)}| + 1 \) elements.

We consider in \( \left(F \left(X\right)\right)^{(2)} \) the relation \( R = \{ (s \circ y, y) \mid s \in S, y \in \left(F \left(X\right)\right)^{(2)} \} \) and the relation \( Q \) - the minimal equivalence in \( \left(F \left(X\right)\right)^{(2)} \) which contain \( R \). Quotient set \( \left(F \left(X\right)\right)^{(2)}/Q \) has \( |X^{(2)}| \) elements. ■

**Proposition 5.3** Condition 2.1 fulfills in the variety of the all actions of semigroups over sets.

**Proof.** By previous Proposition every automorphism \( \Phi \) of the category \( \Theta^0 \) induces the automorphism \( \varphi \) of the additive monoid \( \mathbb{N}^2 \), such that \( \varphi \left((n_1, n_2)_{i \in \Gamma}\right) = (m_1, m_2)_{i \in \Gamma} \) if and only if \( \Phi \left(F \left(X\right)\right) \cong F \left(Y\right) \), where \( |X^{(i)}| = n_i, |Y^{(i)}| = m_i, i \in \{1, 2\} \). This automorphism transforms the minimal set of the generators of the \( \mathbb{N}^2 \): \( \{(1, 0), (0, 1)\} \) to itself. We will prove that it is not possible that \( \varphi \left(1, 0\right) = (0, 1), \varphi \left(0, 1\right) = (1, 0) \). Indeed, if \( \varphi \left(1, 0\right) = (0, 1) \), then
\( \Phi \left( F \left( x^{(1)} \right) \right) \cong F \left( x^{(2)} \right) \) and \( \Phi \left( F \left( x_1^{(1)}, \ldots, x_n^{(1)} \right) \right) \cong F \left( x_1^{(2)}, \ldots, x_n^{(2)} \right), n \geq 1. \)

\( F \left( x_1^{(1)}, \ldots, x_n^{(1)} \right) = S \left( x_1^{(1)}, \ldots, x_n^{(1)} \right) \) is a free semigroup with \( n \) free generators. \( F \left( x_1^{(2)}, \ldots, x_n^{(2)} \right) = \left\{ x_1^{(2)}, \ldots, x_n^{(2)} \right\} \) is a set with \( n \) elements. There is an embedding \( \iota : F \left( x_1^{(1)}, \ldots, x_n^{(1)} \right) \hookrightarrow F \left( x_1^{(2)}, x_2^{(2)} \right) \), because in the semigroup \( S \left( x_1^{(1)}, x_2^{(1)} \right) \), for example, elements \( x_1^{(1)} x_2^{(1)}, x_1^{(1)} x_2^{(2)} \) are free. By Proposition 5.1 automorphism \( \Phi \) transforms this embedding to the embedding \( \Phi \left( \iota \right) : \Phi \left( F \left( x_1^{(1)}, \ldots, x_n^{(1)} \right) \right) \hookrightarrow \Phi \left( F \left( x_1^{(1)}, x_2^{(2)} \right) \right) \), where \( \Phi \left( F \left( x_1^{(1)}, \ldots, x_n^{(1)} \right) \right) \cong F \left( x_1^{(2)}, \ldots, x_n^{(2)} \right), \Phi \left( F \left( x_1^{(1)}, x_2^{(1)} \right) \right) \cong F \left( x_1^{(2)}, x_2^{(2)} \right). \)

But it is not possible, because when \( n > 2 \) we can not embed the set with \( n \) elements to the set with 2 elements. Therefore \( \varphi \left( 1, 0 \right) = \left( 1, 0 \right), \varphi \left( 0, 1 \right) = \left( 0, 1 \right) \) must fulfills. So \( \Phi \left( F \left( x^{(1)} \right) \right) \cong F \left( x^{(1)} \right), \Phi \left( F \left( x^{(2)} \right) \right) \cong F \left( x^{(2)} \right) \).

**Theorem 5.1**
The automorphic equivalence in the variety of the all actions of semigroups over sets coincides with the geometric equivalence.

**Proof.** By previous Proposition we can use the method of verbal operations for calculating group \( \mathfrak{A} \left( \mathfrak{G} \right) \) for the variety \( \Theta \). We will find the systems of words \( W = \left\{ w, w_0 \right\} \) subjects of conditions Op1 and Op2). Any such sequence corresponds to the strongly stable automorphism of the category \( \Theta^0 \). By Op1 \( w, w_0 \in F \left( x_1^{(1)}, x_2^{(1)} \right) = S \left( x_1^{(1)}, x_2^{(1)} \right) \). We have only two possibility: \( w, x_1^{(1)} x_2^{(1)} = x_1^{(2)} x_2^{(1)} \) or \( w, x_1^{(1)} x_2^{(1)} = x_1^{(1)} x_2^{(2)} \). \( w, x_1^{(1)} x_2^{(1)} \) can not be shorter or longer than these words, because, otherwise, the mapping \( s_F \left( x_1^{(1)} x_2^{(1)} \right) : F \left( x_1^{(1)}, x_2^{(1)} \right) \rightarrow F \left( x_1^{(1)}, x_2^{(1)} \right)_W \) which preserves elements \( x_1^{(1)} \) and \( x_2^{(1)} \) and will be a homomorphism can not be an isomorphism.

\( w_0 \in F \left( x^{(1)}, x^{(2)} \right) \). Here we have only one possibility: \( w_0 \left( x_1^{(1)}, x_2^{(2)} \right) = x_1^{(1)} \circ x_2^{(2)} \). Also \( w_0 \left( x_1^{(1)}, x_2^{(2)} \right) \) can not be shorter or longer, because, otherwise, we can not construct a bijection \( s_F \left( x^{(1)}, x^{(2)} \right) : F \left( x^{(1)}, x^{(2)} \right) \rightarrow F \left( x^{(1)}, x^{(2)} \right) \) subject of condition Op2.

But the system of words \( \left\{ w, x_1^{(1)}, x_2^{(1)} \right\} = x_2^{(1)} x_1^{(1)}, w_0 \left( x^{(1)}, x^{(2)} \right) = x_1^{(1)} \circ x_2^{(2)} \left( x^{(1)} \right) \circ x_2^{(2)} \right) \) also is not a subject of conditions Op1 and Op2). Indeed, if we have an isomorphism \( s_F \left( x_1^{(1)}, x_2^{(1)} \right) : F \left( x_1^{(1)}, x_2^{(1)} \right) \rightarrow F \left( x_1^{(1)}, x_2^{(1)} \right)_W^* \), then, because \( x_1^{(2)} x_2^{(1)} \circ x_2^{(2)} = x_1^{(1)} \circ x_2^{(2)} \), the \( w_0 \left( x_1^{(1)}, x_2^{(1)} \right), x_2^{(1)} = w_0 \left( x_1^{(1)}, x_2^{(1)} \right) \) must hold. But

\[
\begin{align*}
\left( w_0 \left( x_1^{(1)}, x_2^{(1)} \right), x_2^{(2)} \right) &= w_0 \left( x_2^{(2)} x_1^{(1)}, x_2^{(2)} \right) = \left( x_2^{(1)} x_1^{(1)} \right) \circ x_2^{(2)} \\
\left( w_0 \left( x_1^{(1)}, x_2^{(1)} \right), x_2^{(2)} \right) &= w_0 \left( x_1^{(1)} x_2^{(1)} \circ x_2^{(2)} \right) = x_1^{(1)} \circ \left( x_2^{(1)} \circ x_2^{(2)} \right) = \left( x_1^{(1)} x_2^{(1)} \right) \circ x_2^{(2)}.
\end{align*}
\]
This contradiction proves that there is only one possibility for the system $W$:

$$W = \left\{ w, (x_1^{(1)}, x_2^{(1)}) = x_1^{(1)} x_2^{(1)}, w_0 (x_1^{(1)}, x_2^{(2)}) = x_1^{(1)} \circ x_2^{(2)} \right\}.$$  

So $\mathcal{S} = \{1\}$ in our variety $\Theta$ and by Theorem 2.3 $\mathfrak{A} / \mathfrak{Q} = \{1\}$. By Theorem 4.2 the proof is complete.  

**5.2 Automatons.**

In this Subsection $\Theta$ is a variety of the all automatons. $\Gamma = \{1, 2, 3\}$: the first sort is a sort of input signals, the second sort is a sort of statements of automatons, the third sort is a sort of output signals. $\Omega = \{\ast, \circ\}$. Operation $\ast$ gives as a new statement of automaton according an input signal and a previous statement of automaton. $\tau_\ast = (1, 2; 2)$. Operation $\circ$ gives as an output signal according an input signal and a statement of automaton. $\tau_\circ = (1, 2; 3)$. A free algebras $F(X)$ in our variety has this form: $(F(X))^{(1)} = X^{(1)}, (F(X))^{(2)} = (X^{(1)})^\infty \ast X^{(2)} = \left\{ x_i^{(1)} \ast (\ldots (x_i^{(1)} \ast x_j^{(2)})) \mid x_j^{(2)} \in X^{(2)}, x_i^{(1)}, \ldots, x_i^{(1)} \in X^{(1)}, n \in \mathbb{N} \right\}$. Here, when $n = 0$, we must understand $x_i^{(1)} \ast (\ldots (x_i^{(1)} \ast x_j^{(2)})) = x_j^{(2)}$.  

$\langle F(X) \rangle^{(3)} = \left( \left( (X^{(1)})^\infty \circ (F(X))^{(2)} \right) \cup X^{(3)} \right)$, where $(X^{(1)})^\infty \circ (F(X))^{(2)} = \left\{ x_i^{(1)} \circ (\ldots (x_1^{(1)} \circ y^{(2)})) \mid y^{(2)} \in (F(X))^{(2)}, x_i^{(1)}, \ldots, x_i^{(1)} \in X^{(1)}, n \geq 1 \right\}$.  

**Proposition 5.4** The variety of the all automatons has an IBN propriety.

**Proof.** We consider $F(X) \in \text{Ob}\Theta^0$. $|X^{(1)}| = \left|(F(X))^{(1)} \right|$. As in Proposition 5.2 we consider in $(F(X))^{(2)}$ the relation $R = \left\{ (s \ast y, y) \mid s \in (F(X))^{(1)}, y \in (F(X))^{(2)} \right\}$ and the relation $Q$ - the minimal equivalence in $(F(X))^{(2)}$ which contain $R$. Quotient set $(F(X))^{(2)} / Q$ has $|X^{(2)}|$ elements.

$|X^{(3)}| = \left|(F(X))^{(3)} \setminus \langle (F(X))^{(1)} \cup (F(X))^{(2)} \rangle^{(3)} \right|$, where $\langle (F(X))^{(1)} \cup (F(X))^{(2)} \rangle$ is a subalgebra of $F(X)$ generated by set $(F(X))^{(1)} \cup (F(X))^{(2)}$.  

**Proposition 5.5** Condition 2.4 fulfills in the variety of the all automatons.

**Proof.** The automorphism $\varphi$ of the additive monoid $\mathbb{N}^3$ induced by automorphism $\Phi$ of the category $\Theta^0$ permutes the minimal set of generators of this monoid $E = \{e_1, e_2, e_3\}$. We will prove that only the identity permutation is possible. For this aim we will construct the embedding $\iota : A = F \left( x_1^{(1)}, x_2^{(1)}, x_1^{(2)}, \ldots, x_n^{(2)} \right) \rightarrow B = F \left( x_1^{(1)}, x_2^{(1)}, x_1^{(2)} \right)$ for every $n > 1$. There exists homomorphism $\iota : A \rightarrow B$ such that $\iota \left( x_i^{(1)} \right) = x_i^{(1)}$, $i = 1, 2$, $\iota \left( x_j^{(2)} \right) = x_2^{(1)} \ast (\ldots (x_1^{(1)} \ast (x_1^{(1)} \ast (x_2^{(2)}))) = y_j^{(2)}$, where $x_1^{(1)}$ appear $j$ times in the sequence $x_2^{(1)} \ast (\ldots (x_1^{(1)} \ast (x_1^{(1)} \ast (x_2^{(2)}))))$ and $1 \leq j \leq n$. This homomorphism
is embedding, because for every element of the subalgebra \( \langle x_1^{(1)}, x_2^{(1)}, y_1^{(2)}, \ldots, y_n^{(2)} \rangle \subset F \left( x_1^{(1)}, x_2^{(1)}, x_1^{(2)} \right) \) we can uniquely calculate its coinage according \( \iota \).

If automorphism \( \varphi \) of the additive monoid \( \mathbb{N}^3 \) induced by automorphism \( \Phi \) of the category \( \Theta^0 \) permutes set \( E \) by permutation \( (1, 2) \), i.e., \( \Phi \left( F \left( x_1^{(1)} \right) \right) = F \left( x_1^{(2)} \right), \Phi \left( F \left( x_2^{(1)} \right) \right) = F \left( x_2^{(1)} \right), \Phi \left( F \left( x_3^{(1)} \right) \right) = F \left( x_3^{(2)} \right) \), then \( \Phi \left( \iota \right) \) must be an embedding \( \Phi \left( A \right) = F \left( x_1^{(1)}, \ldots, x_n^{(2)} \right) \hookrightarrow \Phi \left( B \right) = F \left( x_1^{(1)}, x_2^{(1)}, x_2^{(2)} \right) \).

But this embedding can not exist, because \( \left| \left( \Phi \left( A \right) \right)^{(1)} \right| = n, \left| \left( \Phi \left( B \right) \right)^{(1)} \right| = 1 \).

If \( \varphi \) permutes \( E \) by permutation \( (1, 3) \), i.e., \( \Phi \left( F \left( x_1^{(1)} \right) \right) = F \left( x_3^{(3)} \right), \Phi \left( F \left( x_2^{(1)} \right) \right) = F \left( x_2^{(3)} \right), \Phi \left( F \left( x_3^{(1)} \right) \right) = F \left( x_1^{(2)} \right), \) then \( \Phi \left( \iota \right) \) must be an embedding \( \Phi \left( A \right) = F \left( x_1^{(3)}, x_1^{(3)}, x_2^{(3)} \right) \hookrightarrow \Phi \left( B \right) = F \left( x_1^{(1)}, x_2^{(1)}, x_2^{(3)} \right) \).

But this embedding can not exist, because \( \left| \left( \Phi \left( A \right) \right)^{(3)} \right| = n, \left| \left( \Phi \left( B \right) \right)^{(3)} \right| = 1 \).

If \( \varphi \) permutes \( E \) by permutation \( (1, 2, 3) \), then \( \Phi \left( \iota \right) \) must be an embedding \( \Phi \left( A \right) = F \left( x_1^{(2)}, x_2^{(2)}, x_3^{(3)} \right) \hookrightarrow \Phi \left( B \right) = F \left( x_1^{(2)}, x_2^{(2)}, x_1^{(3)} \right) \). But this embedding can not exist, because \( \left| \left( \Phi \left( A \right) \right)^{(3)} \right| = n, \left| \left( \Phi \left( B \right) \right)^{(3)} \right| = 1 \). If \( \varphi \) permutes \( E \) by permutation \( (1, 3, 2) \), then \( \Phi \left( \iota \right) \) must be an embedding \( \Phi \left( A \right) = F \left( x_1^{(1)}, \ldots, x_n^{(1)}, x_2^{(3)} \right) \hookrightarrow \Phi \left( B \right) = F \left( x_1^{(1)}, x_2^{(3)}, x_2^{(3)} \right) \). But this embedding can not exist, because \( \left| \left( \Phi \left( A \right) \right)^{(1)} \right| = n, \left| \left( \Phi \left( B \right) \right)^{(1)} \right| = 1 \). So only the identity permutation is possible. Therefore \( \Phi \left( F \left( x_1^{(i)} \right) \right) = F \left( x_1^{(i)} \right), i = 1, 2, 3. \)

**Theorem 5.2** The automorphic equivalence in the variety of the all automata coincides with the geometric equivalence.

**Proof.** We will find systems of words \( W = \{w_*, w_0\} \) subjects of conditions Op1 and Op2. But we have only one possibility: \( w_* \left( x_1^{(1)}, x_2^{(2)} \right) = x_1^{(1)} + x_2^{(2)} \), \( w_0 \left( x_1^{(1)}, x_2^{(2)} \right) = x_1^{(1)} \circ x_2^{(2)} \). As in the proof of Theorem 5.1 these words can not be shorter or longer, because, otherwise, we can not construct a bijection \( s_{F \left( x_1^{(1)}, x_2^{(2)} \right)} : F \left( x_1^{(1)}, x_2^{(2)} \right) \to F \left( x_1^{(1)}, x_2^{(2)} \right) \) subject of condition Op2.

5.3 Representations of groups.

In this Subsection we reprove one result of [5]. \( \Theta \) will be a variety of the all representations of groups over linear spaces over field \( k \). We assume that \( k \) has a characteristic 0. \( \Gamma = \{1, 2\} \): the first sort is a sort of elements of groups, the second sort is a sort of vectors of linear spaces. \( \Omega = \{1, -1, 0, \lambda (\lambda \in k), +, \circ\} \). 1 is a 0-ary operation of the taking a unit in the group, \( \tau_1 = (1) \). -1 is an
unary operation of the taking an inverse element in the group, \( \tau_{-1} = (1; 1) \).

0 is 0-ary operation of the taking a zero vector in the linear space, \( \tau_0 = (2) \).

\(-\) is an unary operation of the taking a negative vector in the linear space, \( \tau_- = (2; 2) \). For every \( \lambda \in k \) we have an unary operation: multiplication of vectors from the linear space by scalar \( \lambda \). We denote this operation by \( \lambda \) and \( \tau_\lambda = (2; 2) \). \(+\) is an operation of the addition of vectors of the linear space, \( \tau_+ = (2; 2) \). \( \circ \) is an operation of the action of elements of the group on vectors from the linear space, \( \tau_o = (1, 2; 2) \).

A free algebras \( F(X) \) in our variety has this form: \( (F(X))^{(1)} = G(X^{(1)}) \), where \( G(X^{(1)}) \) is a free group with the set of free generators \( X^{(1)} \). \( (F(X))^{(2)} = kG(X^{(1)}) \circ X^{(2)} = \bigoplus_{i \in I} (kG(X^{(1)}) \circ x_i^{(2)}) \), where \( kG(X^{(1)}) \) is a group \( k \)-algebra of the group \( G(X^{(1)}) \) and \( kG(X^{(1)}) \circ X^{(2)} \) is a free \( kG(X^{(1)}) \)-module generated by the set \( X^{(2)} = \{x_i^{(2)} | \ i \in I\} \) of the free generators. We understand

\[
\left( \sum_{g \in I} \lambda_g g \right) \circ x_i^{(2)},
\]

where \( g \in G(X^{(1)}) \), \( \lambda_g \in k \), \( I \subset G(X^{(1)}) \), \( |I| < \infty \), as

\[
\sum_{g \in I} \lambda_g \left( g \circ x_i^{(2)} \right).
\]

**Proposition 5.6** The variety of the all representations of groups over linear spaces over field \( k \) has an IBN property.

**Proof.** \(|X^{(1)}|\) is equal to the rang of the free abelian group \( G(X^{(1)}) / [G(X^{(1)}), G(X^{(1)})] \).

We consider in \( kG(X^{(1)}) \) the ideal of augmentation \( \Delta \). We denote \( kG(X^{(1)}) \circ X^{(2)} = M \), \( G(X^{(1)}) = G \). \( \Delta \circ M \) will be a submodule of the module \( M \) and quotient module \( M/\Delta \circ M \) will be a \( kG(X^{(1)})/\Delta \)-module, i.e., a \( k \)-linear space. We will prove that \( \text{Sp}_k \left( x_i^{(2)} + \Delta \circ M \mid i \in I \right) = M/\Delta \circ M \). Indeed, for every \( m \in M \) the \( m = \sum f_i \circ x_i^{(2)} \) holds, where \( f_i \in kG \), \( f_i = \delta (f_i) \) + \( h_i \), where \( \delta : kG \rightarrow k \) is a homomorphism of augmentation, \( h_i \in \Delta, 1 \in G \). So \( m \equiv \sum \delta (f_i) x_i^{(2)} \) (mod \( \Delta \circ M \)). We will prove that \( \left\{ x_i^{(2)} + \Delta \circ M \mid i \in I \right\} \) is a linear independent set. Indeed, if \( \sum_{i \in I} \alpha_i x_i^{(2)} \in \Delta \circ M \), where \( \{\alpha_i | i \in I\} \), then

\[
\sum_{i \in I} \alpha_i x_i^{(2)} = \sum_{i \in I} \alpha_i 1 \circ x_i^{(2)} = \sum_{i \in I} h_i \circ x_i^{(2)}, \text{ where } h_i \in \Delta. \left\{ x_i^{(2)} \mid i \in I \right\} \text{ is a set of the free generators of the module } M, \text{ so } \alpha_i 1 \in \Delta \text{ and } \alpha_i = 0 \text{ for every } i \in I.
\]

Therefore \( |X^{(2)}| = \text{dim}_k (M/\Delta \circ M) \). \( \blacksquare \)

**Proposition 5.7** Condition \( (2.1) \) fulfills in the variety of the all representations of groups.

**Proof.** The proof of this Proposition is similar to the proof of the Proposition \[5.3\] we use the fact that we can embed the free group generated by \( n \) free
we have only one possibility: $w = 1$. By [2] we have two possibilities: $w \cdot (x_1^{(1)}, x_2^{(1)}) = x_1^{(1)} \cdot x_2^{(1)}$ or $w \cdot (x_1^{(1)}, x_2^{(1)}) = x_2^{(1)} \cdot x_1^{(1)}$. After this by [11, Proposition 2.1] we have that for $w^{-1}(x^{(1)}) \in F(x^{(1)})$ we have only one possibility: $w^{-1}(x^{(1)}) = (x^{(1)})^{-1}$.

By consideration which was used in [8] and in [12], for $w_{\lambda}(x^{(2)}) \in F(x^{(2)})$ we have only one possibility: $w_{\lambda}(x^{(2)}) = \varphi(\lambda)x^{(2)}$ holds, where $\varphi \in \text{Aut}_k$.

Therefore $w_\phi(x^{(1)}, x^{(2)}) = w_\phi(x^{(1)} \circ x^{(2)})$, where $w(x^{(1)}) \in kG(x^{(1)})$. Since $F(x^{(1)}, x^{(2)}) \to (F(x^{(1)}, x^{(2)}))^*$ is an isomorphism, so for every $g_1, g_2 \in G(x^{(1)})$ the $w_\phi(g_1, w_\phi(g_2, x^{(2)})) = w_\phi(\phi(g_1, g_2), x^{(2)})$ must hold. $G(x^{(1)})$ is a commutative group, so $w(g_1, g_2) = g_2g_2$. Therefore $w_\phi(g_1, w_\phi(g_2, x^{(2)})) = w_\phi(g_1, g_2, x^{(2)})\phi$. $w_\phi(g_1, g_2, x^{(2)}) = w_\phi(g_1, w_\phi(g_2, x^{(2)})) = w_\phi(g_1, g_2 \circ x^{(2)}) = w_\phi(g_1, g_2, x^{(2)}).$ Hence $w(g_1)w(g_2) = w(g_1, g_2) and w: G(x^{(1)}) \to kG(x^{(1)})$ is a multiplicative homomorphism. If we define $w\left(\sum_{i=n}^{m} \alpha_i(x^{(1)})^i\right) = \sum_{i=n}^{m} \alpha_i w(x^{(1)})^i$, where $\alpha_i \in k, n, m \in \mathbb{Z}$, we achieve that $w: kG(x^{(1)}) \to kG(x^{(1)})$ is a homomorphism. If we define $w\left(g(x^{(1)}) = \frac{f(x^{(1)})}{h(x^{(1)})}\right)$, where $f(x^{(1)}, h(x^{(1)}) \in k[x^{(1)}]$, we can extend this $w$ to the endomorphism of the field $k(x^{(1)})$. The $w_\phi(1, x^{(2)}) = w(1) \circ x^{(2)} = 1 \circ x^{(2)}$ must hold, so $w(1) = 1$ and $\ker w = 0$. im$\phi(x^{(1)}, x^{(2)}) = (F(x^{(1)}, x^{(2)}))^2 = kG(x^{(1)}) \circ x^{(2)}$, so for every $f(x^{(1)}) \in kG(x^{(1)}) \in kG(x^{(1)})$ there exists $h(x^{(1)}) \in kG(x^{(1)})$ such that $f(x^{(1)}) \circ x^{(2)} = \phi(x^{(1)}, x^{(2)}) \circ x^{(2)} = h(x^{(1)}) \circ x^{(2)} = w_\phi(h(x^{(1)}) \circ x^{(2)}) = w_\phi(h(x^{(1)}) \circ x^{(2)}) = w(h(x^{(1)})) \circ x^{(2)}$ and $f(x^{(1)}) = w(h(x^{(1)}))$. So im$w \supset$
\( kG(x^{(1)}) \) and \( \text{im} w = k(x^{(1)}) \). Therefore \( w \) is an automorphism of the \( k(x^{(1)}) \).

So \( w(x^{(1)}) = \frac{ax^{(1)} + b}{cx^{(1)} + d} \), where \( a, b, c, d \in k, \ ad - bc \neq 0 \).

But \( \text{im} w = k(x^{(1)}) \Rightarrow w(x^{(1)}) = \sum_{i=0}^{m} \alpha_i (x^{(1)})^i \in kG(x^{(1)}), \ \alpha_i \in k, \ \alpha_n, \alpha_m \neq 0, \ n, m \in \mathbb{Z} \).

So \( ax^{(1)} + b = w(x^{(1)}) \left( cx^{(1)} + d \right) \). If \( a, b, c, d \neq 0 \) then \( m + 1 = 1, \ n = 0 \), so \( w(x^{(1)}) \in k \). This is a contradiction with \( ad - bc \neq 0 \). If \( a = 0 \) then \( b \neq 0, \ c \neq 0 \), \( w(x^{(1)}) = \frac{1}{\alpha x^{(1)} + \beta} \) and \( w(x^{(1)}) \left( \alpha x^{(1)} + \beta \right) = 1 \), where \( \alpha, \beta \in k, \ \alpha \neq 0 \). If in this case \( \beta \neq 0 \), then \( m + 1 = 0, \ n = 0 \). It is impossible. So \( \beta = 0 \) and \( w(x^{(1)}) = \alpha^{-1} (x^{(1)})^{-1} \). \( \left( \alpha x^{(1)} + \beta \right)^2 = \left( x^{(1)} \right)^2 \) must holds, but \( w \left( \left( x^{(1)} \right)^2 \right) = \alpha^{-1} \left( x^{(1)} \right)^{-2}, \left( w(x^{(1)}) \right)^2 = \alpha^{-2} \left( x^{(1)} \right)^{-2} \). Hence \( \alpha = 1 \) and \( w(x^{(1)}) = (x^{(1)})^{-1} \). If \( b = 0 \) then \( d \neq 0 \) and \( w(x^{(1)}) = \frac{ax^{(1)}}{cx^{(1)} + d} \).

So \( w(x^{(1)}) (cx^{(1)} + d) = ax^{(1)} \) and \( m + 1 = 1, \ n = 1 \). It is impossible. If \( c = 0 \) then \( d \neq 0, \ a \neq 0 \) and \( w(x^{(1)}) = \alpha x^{(1)} + \beta \), where \( \alpha, \beta \in k, \ \alpha \neq 0 \). From \( w \left( \left( x^{(1)} \right)^2 \right) = \left( w(x^{(1)}) \right)^2 \) we conclude that \( \alpha (x^{(1)})^2 + \beta = \alpha^2 (x^{(1)})^2 + 2 \alpha \beta x^{(1)} + \beta^2 \) and \( \beta = 0, \ \alpha = 1 \). So \( w(x^{(1)}) = x^{(1)} \). If \( d = 0 \) then then \( c \neq 0, \ b \neq 0 \) and \( w(x^{(1)}) = \alpha (x^{(1)})^{-1} + \beta \), where \( \alpha, \beta \in k, \ \alpha \neq 0 \). As above from \( w \left( \left( x^{(1)} \right)^2 \right) = \left( w(x^{(1)}) \right)^2 \) we conclude that \( w(x^{(1)}) = (x^{(1)})^{-1} \). Therefore we have two possibilities: \( w_0 \left( x^{(1)}, x^{(2)} \right) = x^{(1)} \circ x^{(2)} \) or \( w_0 \left( x^{(1)}, x^{(2)} \right) = (x^{(1)})^{-1} \circ x^{(2)} \).

We will prove that the situation when \( w \left( x^{(1)}, x^{(2)} \right) = x^{(1)} \circ x^{(2)} \) is impossible. Indeed, there is an isomorphism \( F \left( x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2 \right) \rightarrow \left( F \left( x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2 \right) \right)^* \), so the \( w_0 \left( x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2 \right) \) must hold. But in our situation \( w_0 \left( x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2 \right) = \left( x^{(1)}_1 \right)^{-1} \circ \left( x^{(1)}_2 \right)^{-1} \circ x^{(2)}_1 \circ x^{(2)}_2 \) and

\[
\begin{align*}
w_0 \left( w \left( x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2 \right) \right) &= \left( x^{(1)}_1 \right)^{-1} \circ \left( x^{(1)}_2 \right)^{-1} \circ x^{(2)} = \left( \left( x^{(1)}_1 \right)^{-1} \cdot \left( x^{(1)}_2 \right)^{-1} \right) \circ x^{(2)} \end{align*}
\]

Also the situation when \( w \left( x^{(1)}_1, x^{(1)}_2 \right) = x^{(1)}_2 \cdot x^{(1)}_1, \ w_0 \left( x^{(1)}_1, x^{(2)}_1 \right) = x^{(1)} \circ x^{(2)} \) is impossible, because in this situation \( w_0 \left( x^{(1)}_1, x^{(1)}_2, x^{(2)}_1, x^{(2)}_2 \right) = x^{(1)} \circ x^{(2)} \) and \( w_0 \left( w \left( x^{(1)}_1, x^{(1)}_2 \right), x^{(2)} \right) = \left( x^{(1)}_1 \right)^{-1} \cdot \left( x^{(1)}_2 \right)^{-1} \circ x^{(2)} \).

So for systems of the words \( \{1, -1\} \) we have these possibilities:

\[
W = \left\{ w_1 = 1^{(1)}, w_{-1} \left( x^{(1)} \right) = \left( x^{(1)} \right)^{-1}, w \left( x^{(1)}_1, x^{(1)}_2 \right) = x^{(1)}_1 \cdot x^{(1)}_2 \right\}
\]
\[ w_0 = 0^{(2)}, w_+ (x^{(2)}) = \varphi (\lambda) x^{(2)} (\lambda \in k), \quad (5.2) \]
\[ w_0 = 0^{(2)}, w_+ (x^{(2)}) = -x^{(2)}, w_\lambda (x^{(2)}) = \varphi (\lambda) x^{(2)} (\lambda \in k), \]

or
\[ W = \{ w_1 = 1^{(1)}, w_+ (x^{(1)}) = -x^{(1)} - 1, w_+ (x^{(1)}) = x^{(1)} \cdot x^{(1)}, \]
\[ w_0 = 0^{(2)}, w_+ (x^{(2)}) = -x^{(2)}, w_\lambda (x^{(2)}) = \varphi (\lambda) x^{(2)} (\lambda \in k), \quad (5.3) \]

where \( \varphi \in \text{Aut} k \).

Now we will prove that conditions for Op1) and Op2) fulfills for all these systems of the words. By direct computations we can prove that if \( H \in \Theta \) then, for all these systems of the words \( W \), in the algebra \( H_W^* \) fulfill all identities (axioms) which define our variety \( \Theta \). So \( H_W^* \in \Theta \). Therefore for every \( F = F(X) \in \Theta^0 \) there exists homomorphism \( s_F : F \rightarrow F_W^* \) such that \( (s_F)_X = \text{id}_X \).

We will prove that \( s_F \) is an an isomorphism. We assume that \( W \) has form \( G. \) First of all we prove that \( s_F \) is a monomorphism. We consider \( g = (x^{(1)})^{\varepsilon_1} \cdot \ldots \cdot (x^{(1)})^{\varepsilon_n} \in G(X^{(1)}) \), where \( x_i^{(1)}, \ldots, x_i^{(1)} \in X^{(1)} \), \( \varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\} \), \( n \in \mathbb{N} \) and in the word we can not make cancellations. Also we can not make cancellations in the word \( s_F (g) = (x^{(1)})^{\varepsilon_n} \cdot \ldots \cdot (x^{(1)})^{\varepsilon_1} \). So if \( s_F (g) = 1^{(1)} \) then \( n = 0 \) and \( g = 1 \). Now we consider \( \sum_{i \in I} f_i \circ x_i^{(2)} \in kG(X^{(1)}) \circ X^{(2)} \), such that \( s_F \left( \sum_{i \in I} f_i \circ x_i^{(2)} \right) = \sum_{i \in I} s_F \left( f_i \circ x_i^{(2)} \right) = 0 \). If \( f_i = \sum_{g \in I} \lambda_{i,g} g \in kG(X^{(1)}) \) and \( x_i^{(2)} \in X^{(2)} \), then \( s_F \left( f_i \circ x_i^{(2)} \right) = s_F \left( \sum_{g \in I} \lambda_{i,g} \left( g \circ x_i^{(2)} \right) \right) \)

\[ \sum_{g \in I} \varphi (\lambda_{i,g}) \left( g^{-1} \circ x_i^{(2)} \right) = \left( \sum_{g \in I} \varphi (\lambda_{i,g}) g^{-1} \right) \circ x_i^{(2)} \in kG(X^{(1)}) \circ x_i^{(2)} \]

where \( \varphi \in \text{Aut} k \). So, if \( \sum_{i \in I} s_F \left( f_i \circ x_i^{(2)} \right) = 0 \) then the \( s_F \left( f_i \circ x_i^{(2)} \right) = 0 \) holds for every \( x_i^{(2)} \in X^{(2)} \) and \( \sum_{g \in I} \varphi (\lambda_{i,g}) g^{-1} = 0 \). Hence \( \varphi (\lambda_{i,g}) = 0 \) and \( \lambda_{i,g} = 0 \) holds for every \( g \in I \). Therefore \( f_i \circ x_i^{(2)} = 0 \) and \( \sum_{i \in I} f_i \circ x_i^{(2)} = 0 \). So \( s_F \) is a monomorphism.

For every \( g = (x^{(1)})^{\varepsilon_1} \cdot \ldots \cdot (x^{(1)})^{\varepsilon_n} \in G(X^{(1)}) \) the \( s_F \left( \left( (x^{(1)})^{\varepsilon_n} \cdot \ldots \cdot (x^{(1)})^{\varepsilon_1} \right) \right) = g \) holds. And for every \( v = \sum_{i \in I} f_i \circ x_i^{(2)} \in kG(X^{(1)}) \circ X^{(2)} \), where \( f_i = \sum_{g \in I} \lambda_{i,g} g \), the \( s_F \left( \sum_{g \in I} \varphi^{-1} (\lambda_{i,g}) \left( g^{-1} \circ x_i^{(2)} \right) \right) = f_i \circ x_i^{(2)} \), holds. Therefore \( v \in \text{im} s_F \).
Hence \( s_F \) is an an isomorphism. If \( W \) has form \([5.2]\) we can even prove easier that \( s_F \) is an isomorphism.

Now we know all elements of the group \( \mathfrak{S} \): by Theorem \([3.1]\) they are all strongly stable automorphisms \( \Phi^W \), which correspond to the systems of words \([5.2]\) and \([5.3]\). Now we will study the multiplication in this group. The automorphisms \( \Phi^W \), which correspond to the system of words \([5.2]\) we will denote by \( \Phi(\varphi, 1) \) and the automorphisms \( \Phi^W \), which correspond to the system of words \([5.3]\) we will denote by \( \Phi(\varphi, \sigma) \). We consider \( \Phi_1, \Phi_2 \in \mathfrak{S} \). We denote, as in the Section \([3]\) \( S^{\Phi_1} = \{ s_{i, F} : F \to F_{W_i} \mid F \in \text{Ob}\Theta^0 \} \), where \( W_i = W^{\Phi_i}, i = 1, 2 \). The automorphism \( \Phi_2\Phi_1 \) acts on morphisms of the category \( \Theta^0 \) according the formula \([2.2]\) by bijections \( S^{\Phi_2\Phi_1} = \{ s_{2, Fs_{1, F}} \mid F \in \text{Ob}\Theta^0 \} \). The system of words \( W = W^{\Phi_2\Phi_1} \) can be calculated by the formula \([3.1]\). If \( \Phi_i = \Phi(\varphi_i, \sigma), i = 1, 2 \), then \( W = W^{\Phi_2\Phi_1} \) has form \([5.1]\) with \( w_i(x^{(1)}_1, x^{(1)}_2) = s_{2, Fs_{1, F}}(x^{(1)}_1 \cdot x^{(1)}_2) = s_{2, F}(x^{(1)}_1 \cdot x^{(1)}_2) = x^{(1)}_1 \cdot x^{(1)}_2, w_\lambda(x^{(1)}) = s_{2, Fs_{1, F}}(\lambda x^{(1)}) = s_{2, F}(\varphi_1(\lambda)x^{(1)}) = \varphi_2\varphi_1(\lambda)x^{(1)} \) and \( w_\sigma(x^{(1)}, x^{(2)}) = s_{2, Fs_{1, F}}(x^{(1)} \circ x^{(2)}) = s_{2, F}(x^{(1)} \circ x^{(2)}) = x^{(1)} \circ x^{(2)} \). Therefore \( \Phi(\varphi_2, \sigma)\Phi(\varphi_1, \sigma) = \Phi(\varphi_2\varphi_1, 1) \). By similar calculation we prove that \( \Phi(\varphi_2, \sigma)\Phi(\varphi_1, 1) = \Phi(\varphi_2, 1)\Phi(\varphi_1, \sigma) = \Phi(\varphi_2\varphi_1, \sigma) \) and \( \Phi(\varphi_2, 1)\Phi(\varphi_1, 1) = \Phi(\varphi_2\varphi_1, 1) \). Therefore \( \mathfrak{S} \cong \text{Aut}_k \times \mathbb{Z}_2 \).

Now we must calculate the group \( \mathfrak{S} \cap \mathfrak{A} \). If \( \Phi = \Phi(\varphi, \sigma) \) or \( \Phi = \Phi(\varphi, 1) \) and \( \varphi \neq id_k \) then \( \Phi \) is not inner automorphism. The proof of this fact is even easier than proof of the similar fact from \([12]\) Proposition 4.1. If \( \Phi = \Phi(id_k, \sigma) \) than for every \( F \in \text{Ob}\Theta^0 \) we define the mapping \( \tau_F : F \to F \) by this way: if \( g \in F^{(1)} \) then \( \tau_F(g) = g^{-1} \), if \( v \in F^{(2)} \) then \( \tau_F(v) = v \). We will that prove \( \tau_F : F \to (F)_W^* \), where \( W = W^\Phi \), is an isomorphism. For every \( g \in F^{(1)} \) the \( \tau_F(g)^{-1} = (g^{-1})^{-1} = g \) holds. Also \( g_1, g_2 \in F^{(1)} \) the \( \tau_F(g_1 \cdot g_2) = (g_1 \cdot g_2)^{-1} = g_2^{-1} \cdot g_1^{-1} = w(\tau_F(g_1), \tau_F(g_2)) \) holds for every \( g_1, g_2 \in F^{(1)} \). And the \( w_\circ(\tau_F(g) \circ \tau_F(v)) = w_\circ(g^{-1} \circ v) = g \circ v = \tau_F(g \circ v) \) holds for every \( g \in F^{(1)}, v \in F^{(2)} \). Also it is clear that \( \tau_F \) is a bijection. So \( \tau_F : F \to (F)_W^* \) is an isomorphism. For every \( F_1, F_2 \in \text{Ob}\Theta^0 \) and every \( \mu \in \text{Mor}_{\Theta^0}(F_1, F_2) \) we have that the \( \tau_{F_2}\mu(g) = \mu(g)^{-1} = \mu\tau_{F_1}(g) \) holds for every \( g \in F_1^{(1)} \) and the \( \tau_{F_2}\mu(v) = \mu(v) = \mu\tau_{F_1}(v) \) holds for every \( v \in F_2^{(2)} \). So, by Proposition \([5.1]\) \( \Phi \) is an inner automorphism. Therefore \( \mathfrak{A}(\mathfrak{J}) \cong \mathfrak{S} \cap \mathfrak{A} \cong \text{Aut}_k \).

Now we will give an example of two representations of groups which are automorphically equivalent but not geometrically equivalent. In \([8]\) this matter was not discussed. We take \( k = \mathbb{Q}(\theta_1, \theta_2) \) - the transcendental extension of degree 2 of the field \( \mathbb{Q} \). We consider a free representation \( F = F(x^{(1)}, x^{(2)}) \). \( F^{(1)} = G(x^{(1)}), F^{(2)} = kG(x^{(1)}) \circ x^{(2)} \). In the representation \( F \) we will consider the congruence \( T \) generated by pair \( (x^{(1)} \circ x^{(2)}, \theta_1 x^{(2)}) \). The pair of the normal submodule of \( F^{(1)} \) and \( kG(x^{(1)}) \)-submodule of \( F^{(2)} \) which corresponds to this congruence (see \([7]\)) is \( \{ (1^{(1)}), (x^{(1)} - \theta_1) kG(x^{(1)}) \circ x^{(2))} \). We will denote the quotient representation \( F/T = H \). We will denote the natural epimorphism
\( \tau : F \rightarrow F/T \). The equalities \( \tau (x^{(1)})^m \circ \tau (x^{(2)}) = \theta_1^m \tau (x^{(2)}) \) hold in the \( H \) for every \( m \in \mathbb{Z} \). We will consider the automorphism \( \varphi \) of the field \( k \) such that \( \varphi (\theta_1) = \theta_2, \varphi (\theta_2) = \theta_1 \). \( W \) will be the system of words which has form \([5,2]\) with the mentioned \( \varphi \). By Corollary \([4,2]\) from Proposition \([4,2]\) the representations \( H \) and \( H^*_W \) are automorphically equivalent.

**Proposition 5.8** The representations \( H \) and \( H^*_W \) are not geometrically equivalent.

**Proof.** \( (T)_H'' = \bigcap_{\psi \in \text{Hom}(F,H)} \ker \psi, \ker \tau = T, \) so \( (T)_H'' = T \) and \( T \in \text{Cl}_H(F) \).

If \( H \) and \( H^*_W \) are geometrically equivalent then \( T \in \text{Cl}_{H^*_W}(F) \) so, by Theorem \([4,4]\) and Corollary \([4,2]\) from Proposition \([4,2]\), \( s_F(T) \in \text{Cl}_H(F) \). The pair of the normal subgroup of \( F^{(1)} \) and \( kG \langle x^{(1)} \rangle \)-submodule of \( F^{(2)} \) which corresponds to the congruence \( s_F(T) \) is \( \{(1^{(1)}), (x^{(1)} - \theta_2) kG \langle x^{(1)} \rangle \circ x^{(2)} \}. \) \( (s_F(T))_H'' = \bigcap_{\psi \in \text{Hom}(F,H)} \ker \psi. \) If \( \psi \in \text{Hom}(F,H) \) then by projective propriety of the free \( \psi \langle (T) \rangle \subseteq \ker \psi \)

algebras there exists \( \alpha \in \text{End}(F) \) such that \( \tau \alpha = \psi. \) \((x^{(1)} \circ x^{(2)}, \theta_2 x^{(2)}) \in s_F(T), \) so if \( s_F(T) \subseteq \ker \psi \) then \( \tau \alpha (x^{(1)}) \circ \alpha (x^{(2)}) = \tau (\theta_2 \alpha (x^{(2)})) \) and \( \alpha (x^{(1)}) \circ \alpha (x^{(2)}) \in T. \) \( \alpha (x^{(1)}) = (x^{(1)})^m, \alpha (x^{(2)}) = f (x^{(1)}) \circ x^{(2)}, \) where \( m \in \mathbb{Z}, f (x^{(1)}) \in kG \langle x^{(1)} \rangle. \) So \( (x^{(1)})^m f (x^{(1)} \circ x^{(2)}, \theta_2 f (x^{(1)} \circ x^{(2)}) \in T. \) It means that \( (x^{(1)})^m - \theta_2 \) f \( (x^{(1)}) \circ x^{(2)} \in (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle \circ x^{(2)} \) or \( (x^{(1)})^m - \theta_2 \) f \( (x^{(1)}) \in (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle. \) The ideal \( (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle \) is a prime ideal in the \( kG \langle x^{(1)} \rangle. \) Therefore or \( (x^{(1)})^m - \theta_2 \in (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle, \) or \( f (x^{(1)}) \in (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle. \) If \( (x^{(1)})^m - \theta_2 \in (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle \) then \( (x^{(1)})^m - \theta_2 = 0, \) but it is not possible. If \( f (x^{(1)}) \in (x^{(1)} - \theta_1) kG \langle x^{(1)} \rangle \) then \( \psi (x^{(2)}) = 0. \) So \( (s_F(T))_H'' = \bigcap_{\psi \in \text{Hom}(F,H)} \ker \psi \supseteq \Delta_{F^{(1)}} \cup (F^{(2)})^2 \supseteq s_F(T). \)

Therefore \( s_F(T) \notin \text{Cl}_H(F). \) This contradiction finishes the prove. \( \blacksquare \)

### 5.4 Representations of Lie algebras.

In this Subsection we reprove result of \([9]\). \( \Theta \) will be a variety of the all representations of Lie algebras over linear spaces over field \( k. \) We assume that \( k \) has a characteristic 0. \( \Gamma = \{1,2\}: \) the first sort is a sort of elements Lie algebras, the second sort is a sort of vectors of linear spaces.

\[ \Omega = \{0^{(1)}, -^{(1)}, \lambda^{(1)} (\lambda \in k), +^{(1)}, [,], 0^{(2)}, -^{(2)}, \lambda^{(2)} (\lambda \in k), +^{(2)} \}. \]

\( 0^{(2)}, -^{(2)}, \lambda^{(2)} (\lambda \in k), +^{(2)} \) are the operations in the linear space, \( 0^{(1)}, -^{(1)}, \lambda^{(1)} (\lambda \in k), +^{(1)} \) are the similar operations in the Lie algebra. \([,]\) is the Lie
brackets; this operation has type \( \tau_{(1]} = (1, 1; 1) \). \( \circ \) is an operation of the action of elements of the Lie algebra on vectors from the linear space, \( \tau_{\circ} = (1, 2; 2) \).

A free algebras \( F(X) \) in our variety has this form: \( (F(X))^{(1)} = L(X^{(1)}) \), where \( L(X^{(1)}) \) is a free Lie algebra with the set of free generators \( X^{(1)} \). \( (F(X))^{(2)} = A(X^{(1)}) \circ X^{(2)} = \bigoplus_{i \in I} (A(X^{(1)}) \circ x_i^{(2)}) \), where \( A(X^{(1)}) \) is a free associative algebra with unit generated by the set of free generators \( X^{(1)} \) and \( (x_1^{(1)} \ldots x_n^{(1)}) \circ \circ \circ \circ \circ \circ \circ \circ \) we understand as \( x_1^{(1)} \circ (\ldots \circ (x_n^{(1)} \circ v^{(2)}) \ldots ) \) and so on by linearity, \( x_1^{(1)}, \ldots, x_n^{(1)} \in X^{(1)} \), \( v^{(2)} \in (F(X))^{(2)} \).

By [9, Theorem 5.1] the variety \( \Theta \) has an IBN propriety.

**Proposition 5.9** Condition (2.1) fulfills in the variety of the all representations of Lie algebras.

**Proof.** The proof of this Proposition is similar to the proof of the Proposition 5.3. We use the following two facts. The first is: we can embed the free Lie algebra generated by \( n \) free generators to the free Lie algebra generated by 2 free generators, for example, by [1, 2.4.2] the elements \( [1, 2.4.2] \) the elements \( \{x_1, x_2\} \) and so on are free in the algebra \( L(x_1, x_2) \). And the second is: we can not embed the \( n \)-dimension linear space to the 2-dimension linear space when \( n > 2 \).

**Theorem 5.4** \( \mathcal{A}/\mathcal{Q} \cong \text{Autk for the variety of the all representations of Lie algebras.} \)

**Proof.** We will find the systems of words

\[
W = \{w_{01}, w_{11}, w_{12}, \lambda \in k, w_{11}, w_{01}, w_{02}, w_{12}, w_{22}, w_{00}\}
\]

subjects of conditions Op1) and Op2).

As in the proof of the Theorem 5.3 we have that \( w_{01} = 0^{(1)} \), \( w_{11} \left( x^{(1)} \right) = -x^{(1)} \), \( w_{12} \left( x^{(1)} \right) = \varphi \left( \lambda \right) x^{(1)} \), \( w_{01} \left( x^{(1)} \right) = x^{(1)} + x^{(2)} \), \( w_{02} = 0^{(2)} \), \( w_{12} \left( x^{(2)} \right) = -x^{(2)} \), \( w_{22} \left( x^{(2)} \right) = \psi \left( \lambda \right) x^{(2)} \), \( w_{11} \left( x^{(2)} \right) = x^{(2)} + x^{(2)} \), where \( \varphi, \psi \in \text{Autk} \). By [10, 2.5] \( W \left[ x^{(1)}, x^{(2)} \right] = \left[ x^{(1)}, x^{(2)} \right] \), where \( a \in k \setminus \{0\} \).

\[
w_{00} \left( x^{(1)}, x^{(2)} \right) = f \left( x^{(1)} \right) \circ x^{(2)}, \text{ where } f \left( x^{(1)} \right) \in k \left[ x^{(1)} \right].
\]

There is an isomorphism \( s_{F(x^{(1)}, x^{(2)})} : F \left( x^{(1)}, x^{(2)} \right) \rightarrow \left( F \left( x^{(1)}, x^{(2)} \right) \right)^{\text{Autk}} \), so the \( w_{12} \left( w_{00} \left( x^{(1)}, x^{(2)} \right) \right) = w_{00} \left( w_{12} \left( x^{(1)}, x^{(2)} \right) \right) \) must hold for every \( \lambda \in k \). \( k \) has a characteristic 0, so \( 2 \in \mathbb{Q} \subseteq k \), \( 2^2 = 2 \) if and only if \( i = 1 \), and the \( \varphi \left( 2 \right) = 2 \) holds for every \( \varphi \in \text{Autk} \).

We denote \( f \left( x^{(1)} \right) = \sum_{i=0}^{n} \alpha_i \left( x^{(1)} \right)^i, \text{ where } \alpha_i \in k \). Then \( w_{12} \left( w_{00} \left( x^{(1)}, x^{(2)} \right) \right) = \psi \left( 2 \right) f \left( x^{(1)} \right) \circ x^{(2)} = \sum_{i=0}^{n} \alpha_i \left( x^{(1)} \right)^i \circ x^{(2)} \), \( w_{00} \left( w_{12} \left( x^{(1)}, x^{(2)} \right) \right) = \sum_{i=0}^{n} \alpha_i \left( \varphi \left( 2 \right) x^{(1)} \right)^i \circ \)

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\[ x^{(2)} = \sum_{i=0}^{n} 2^{i} \alpha_{i} (x^{(1)})^{i} \circ x^{(2)}. \] So \( \alpha_{i} = 0 \) if \( i \neq 1 \) and \( w_{o} (x^{(1)}, x^{(2)}) = b x^{(1)} \circ x^{(2)}, \) where \( b \in k \setminus \{0\}. \)

Also the

\[ w_{o} \left( w_{i} \right) \left( x_{1}^{(1)}, x_{2}^{(1)}, x^{(2)} \right) = w_{o} \left( x_{1}^{(1)}, w_{o} \left( x_{2}^{(1)}, x^{(2)} \right) \right) - w_{o} \left( x_{2}^{(1)}, w_{o} \left( x_{1}^{(1)}, x^{(2)} \right) \right) \]

must hold in the \( F \left( x_{1}^{(1)}, x_{2}^{(1)}, x^{(2)} \right). \)

\[ w_{o} \left( w_{i} \right) \left( x_{1}^{(1)}, x_{2}^{(1)}, x^{(2)} \right) = ab \left( \left[ x_{1}^{(1)}, x_{2}^{(1)} \right] \circ x^{(2)} \right), \]

\[ w_{o} \left( x_{1}^{(1)}, w_{o} \left( x_{2}^{(1)}, x^{(2)} \right) \right) - w_{o} \left( x_{2}^{(1)}, w_{o} \left( x_{1}^{(1)}, x^{(2)} \right) \right) = b^{2} \left( x_{1}^{(1)} \circ x_{2}^{(1)} \circ x^{(2)} - x_{2}^{(1)} \circ x_{1}^{(1)} \circ x^{(2)} \right) = b^{2} \left( \left[ x_{1}^{(1)}, x_{2}^{(1)} \right] \circ x^{(2)} \right), \]

so \( a = b. \)

Also the \( w_{o} \left( w_{\lambda(1)} \left( x^{(1)} \right), x^{(2)} \right) = w_{o} \left( x^{(1)}, w_{\lambda(2)} \left( x^{(2)} \right) \right) \) must hold for every \( \lambda \in k. \)

\[ w_{o} \left( w_{\lambda(1)} \left( x^{(1)} \right), x^{(2)} \right) - w_{o} \left( x^{(1)}, w_{\lambda(2)} \left( x^{(2)} \right) \right) = a \phi (\lambda) x^{(1)} \circ x^{(2)}, \]

\( \text{so } \phi = \psi. \) So for systems of the words \( 5.4 \) we have these possibilities:

\[ W = \{ w_{0(1)} = 0^{(1)}, \ w_{\lambda(1)} \left( x^{(1)} \right) = -x^{(1)}, \ w_{\lambda(1)} \left( x^{(1)} \right) = \phi (\lambda) x^{(1)} (\lambda \in k), \] \[ w_{0(2)} = 0^{(2)}, \ w_{\lambda(2)} \left( x^{(2)} \right) = -x^{(2)}, \ w_{\lambda(2)} \left( x^{(2)} \right) = \phi (\lambda) x^{(2)} (\lambda \in k), \]

where \( \phi \in \text{Aut}k, \ a \in k^{*}. \)

Now we will prove that conditions for Op1 and Op2) fulfills for all these systems of the words. By direct computations we can prove that if \( H \in \Theta \) then, for all these systems of the words \( W, \) in the algebra \( H_{W}^{*} \) fulfill all identities (axioms) which define our variety \( \Theta. \) So \( H_{W}^{*} \in \Theta. \)

Therefore for every \( F = F(X) \in \text{Ob}\Theta^{\Theta} \) there exists homomorphism \( s_{F} : F \rightarrow F_{W}^{*} \) such that \( (s_{F})_{|X} = id_{X}. \)

We will prove that \( s_{F} \) is an an isomorphism. First of all we prove that \( s_{F} \) is a monomorphism. We consider a basis \( E \) of the free Lie algebra \( L \left( X^{(1)} \right) \) such that the elements of \( E \) are monomials of \( L \left( X^{(1)} \right). \) If \( l = \sum_{e \in E^{\prime}} \alpha_{e} e \in L, \) where \( \alpha_{e} \in k, \ E^{\prime} \subset E, \ |E^{\prime}| < \infty, \) and \( s_{F} \left( l \right) = \sum_{e \in E^{\prime}} \phi (\alpha_{e}) a^{i(e)} e = 0, \) where \( l(e) \) is a length of the monomial \( e, \) then the \( \phi (\alpha_{e}) a^{i(e)} = 0 \) holds for every \( e \in E^{\prime}. \)
Now we consider $v \in (F(X))^{(2)} = A (X^{(1)}) \circ X^{(2)}$. $v = \sum_{i \in I} f_i \circ x_i^{(2)}$, where $\{x_i^{(2)} | i \in I\} = X^{(2)}$, $f_i \in A (X^{(1)})$. $s_F (v) = \sum_{i \in I} s_F (f_i \circ x_i^{(2)})$. We denote $f_i = \sum_{j \in J_i} \beta_{i,j} m_j$, where $\beta_{i,j} \in k$, $m_j$ are monomials of $A (X^{(1)})$, $|J_i| < \infty$. $s_F (f_i \circ x_i^{(2)}) = \sum_j \varphi (\beta_{i,j}) a_{\deg(m_j)} m_j \circ x_i^{(2)} \in A (X^{(1)}) \circ x_i^{(2)}$. Hence if $s_F (v) = 0$ then the $s_F (f_i \circ x_i^{(2)}) = 0$ holds for every $i \in I$. Therefore the $\varphi (\beta_{i,j}) = 0$ and $\beta_{i,j} = 0$ holds for every $i \in I$ and every $j \in J_i$. So $v = 0$ and $s_F$ is a monomorphism.

By previous calculation we can see that for every $f \in F(X)$ there exists $h \in F(X)$, such that $s_F (v) = f$. So $s_F$ is an isomorphism.

Now we know all elements of the group $\mathcal{G}$: they are all strongly stable automorphisms $\Phi^W$, which correspond to the systems of words $[\text{[3],5}]$. Now we will study the multiplication in this group. The automorphism $\Phi^W$, which correspond to the system of words $[\text{[3],5}]$ we will denote by $\Phi (\varphi, a)$. We consider $\Phi_1 = \Phi (\varphi_1, a_1), i = 1, 2$. The system of bijections corresponding to the automorphism $\Phi$, we denote $S^\Phi = \{s_{i,F} : F \rightarrow F^*_W, F \in \text{Ob}(\Theta^0)\}$. The system of words $W = W^\Phi_2 \Phi_1$ can be calculated by the formula $[\text{[3],1}]$: the $W = W^\Phi_2 \Phi_1$ has form $[\text{[5],4}]$ with $w_{\lambda(1)} (x^{(1)}) = s_{2,F} s_{1,F} (\lambda x^{(1)}) = s_{2,F} (\varphi_1 (\lambda) x^{(1)}) = \varphi_2 \varphi_1 (\lambda) x^{(1)}$, similarly $w_{\lambda(2)} (x^{(2)}) = \varphi_2 \varphi_1 (\lambda) x^{(2)}$, $w_{|I} (x^{(1)}, x^{(2)}) = s_{2,F} s_{1,F} \left( x^{(1)}, x^{(2)} \right) = a_2 \varphi_2 (a_1) \left[ x^{(1)}, x^{(2)} \right]$, similarly $w_{w_0} (x^{(1)}, x^{(2)}) = a_2 \varphi_2 (a_1) x^{(1)} \circ x^{(2)}$. So $\Phi (\varphi_2, a_2) \Phi (\varphi_1, a_1) = \Phi (\varphi_2 \varphi_1, a_2 \varphi_2 (a_1))$.

If $\Phi = \Phi (\varphi, a)$ and $\varphi \neq \text{id}_k$ then $\Phi$ is not an inner automorphism. The proof of this fact is even easier than proof of the similar fact from $[\text{[12], Proposition 4.1}]$. Now we consider the automorphism $\Phi = \Phi (\text{id}_k, a)$. We can define for every $F \in \text{Ob}(\Theta^0)$ a mapping $\tau_F : F \rightarrow F^*_W$, such that $\tau_F (f) = a^{-1} f$ for every $f \in F$. This mapping is a homomorphism: for example

$$
\tau_F \left[ f_1^{(1)}, f_2^{(1)} \right] = a^{-1} \left[ f_1^{(1)}, f_2^{(1)} \right], \\
\tau_F \left[ f_1^{(1)} \right], \tau_F \left[ f_2^{(1)} \right] = a \left[ \tau_F \left( f_1^{(1)} \right), \tau_F \left( f_2^{(1)} \right) \right] = \\
a \left[ a^{-1} f_1^{(1)}, a^{-1} f_2^{(1)} \right] = a^{-1} \left[ f_1^{(1)}, f_2^{(1)} \right]
$$

and

$$
\tau_F \left( f^{(1)} \circ f^{(2)} \right) = a^{-1} \left( f^{(1)} \circ f^{(2)} \right), \\
w_{w_0} \left[ \tau_F \left( f^{(1)} \right), \tau_F \left( f^{(2)} \right) \right] = a \left( a^{-1} f^{(1)} \circ a^{-1} f^{(2)} \right) = a^{-1} \left( f^{(1)} \circ f^{(2)} \right)
$$

and so on. Obviously, that $\tau_F$ is an invertible mapping. So $\tau_F$ is an isomorphism. For every $F_1, F_2 \in \text{Ob}(\Theta^0)$ and every $\mu \in \text{Mor}_0 (F_1, F_2)$ the $\mu \tau_{F_1} (f) = \mu (a^{-1} f) = a^{-1} \mu (f) = \tau_{F_2} \mu (f)$.
holds for every \( f \in F_1 \). Therefore, by Proposition 3.7, \( \Phi \) is an inner automorphism. So \( \mathfrak{A}/\mathfrak{g} \cong \mathfrak{S}/\mathfrak{S} \cong \mathfrak{S}/\mathfrak{g} \cong \text{Aut}_k \).

Now we will give an example of two representations of Lie algebras which are automorphically equivalent but not geometrically equivalent. This example will be similar to the Example 3. Congruences are the kernels of homomorphisms. So it is easy to see that for every congruence in the free presentation \( \Phi \) is automorphically equivalent but not geometrically equivalent. This example holds for every \( A \varepsilon \).

We will consider the two sided ideal \( T \) and \( \Phi (\varphi, 1) \varphi \neq \text{id}_k \). So there is \( \lambda \in k \) such \( \varphi(\lambda) \neq \lambda \). We will consider the free representation \( F = F (x_1^{(1)}, x_2^{(1)}, x_2^{(2)}) \). In \( L = L (x_1^{(1)}, x_2^{(1)}) \) there are linear independent elements

\[
[x_1^{(1)}, [x_1^{(1)}, x_1^{(1)}, x_2^{(1)}]]] = e_1, [x_1^{(1)}, [x_1^{(1)}, [x_1^{(1)}, x_1^{(1)}, x_2^{(1)}]]] = e_2, \\
[x_1^{(1)}, [x_1^{(1)}, x_1^{(1)}, x_2^{(1)}], x_2^{(1)}] = e_3, [x_1^{(1)}, [x_1^{(1)}, x_1^{(1)}, x_2^{(1)}], [x_1^{(1)}, x_2^{(1)}]] = e_4, \\
[x_1^{(1)}, x_2^{(1)}, [x_1^{(1)}, x_1^{(1)}, x_2^{(1)}], x_2^{(1)}] = e_5, [x_1^{(1)}, x_1^{(1)}, x_2^{(1)}, [x_1^{(1)}, x_2^{(1)}], x_2^{(1)}] = e_6.
\]

These elements will be linear independent also in \( A = A (x_1^{(1)}, x_2^{(1)}) \). In \( A \) we will consider the two sided ideal \( (x_1^{(1)}, x_2^{(1)}) = N \) and the two sided ideal \( I = \langle t, N^6 \rangle \), where \( t = \lambda e_2 + e_4 \). In \( F \) we will consider the congruence \( T \) corresponds to the subrepresentation \( Q \), such that \( Q^{(1)} = \langle 0 \rangle, Q^{(2)} = I \circ x^{(2)} \). We will denote \( H = F/T \) and \( W = W^F \). As above \( H \) and \( H_W^p \) are automorphically equivalent.

**Proposition 5.10** The representations \( H \) and \( H_W^p \) are not geometrically equivalent.

**Proof.** As above \( T \in Cl_H (F) \). If \( H \) and \( H_W^p \) are geometrically equivalent then \( T \in Cl_{H_W^p} (F) \) and \( s_F (T) \in Cl_H (F) \), where \( s_F : F \to F_W^p \) is an isomorphism subject of condition Op2). We will calculate \( (s_F (T))^p_H \) with a view to prove that \( (s_F (T))^p_H \neq s_F (T) \). This contradiction will finish the prove.

If \( s_F (T) \) is a congruence then it corresponds to the subrepresentation \( s_F (Q) \), such that \( (s_F (Q))^{(1)} = \langle 0 \rangle, (s_F (Q))^{(2)} = s_F (I) \circ x^{(2)} \), where \( s_F (I) = \langle s_F (t), N^6 \rangle \). We denote the natural epimorphism \( \tau : F \to F/T \). As above for every \( \psi \in \text{Hom} (F, H) \) there exists \( \alpha \in \text{End} (F) \) such that \( \tau \alpha = \psi \). If \( s_F (T) \subseteq \ker \psi \), then

\[
\alpha (s_F (t \circ x^{(2)})) = \alpha s_F (t) \circ \alpha (x^{(2)}) \in I \circ x^{(2)} , \alpha (x^{(2)}) = \left( \mu + n \left( x_1^{(1)}, x_2^{(1)} \right) \right) \circ x^{(2)}, \text{where} \ \mu \in k, n \left( x_1^{(1)}, x_2^{(1)} \right) \in N.
\]

\[
\alpha s_F (t) \circ \alpha (x^{(2)}) = \alpha s_F (t) \left( \mu + n \left( x_1^{(1)}, x_2^{(1)} \right) \right) \circ x^{(2)} \in I \circ x^{(2)}
\]

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if and only if

\[ \alpha s_F(t) \left( \mu + n \left( x^{(1)}_1, x^{(1)}_2 \right) \right) \in I = \text{sp}_k \{ t \} + N^6. \]  \hspace{1cm} (5.6)

\( s_F(t) = \varphi(\lambda) e_2 + e_4 \in L^5 \subset N^5 \), so \( \alpha s_F(t) \in L^5 \subset N^5 \). If \( \mu = 0 \), then 
\[ \alpha s_F(t) \left( \mu + n \left( x^{(1)}_1, x^{(1)}_2 \right) \right) \in N^6 \subset I. \]
But, if \( \mu = 0 \), the \( \alpha \left( l \left( \mu + n \left( x^{(1)}_1, x^{(1)}_2 \right) \right) \right) \in N^6 \subset I \) holds for every \( l \in N^5 \). In this case \( \left( \text{sp}_k \{ e_1, \ldots, e_6 \} \right) \circ x^{(2)} \subseteq \ker \psi \), where \( \ker \psi \) we understand as subrepresentation corresponding to this congruence. If \( \mu \neq 0 \), then from (5.6) we conclude that \( \alpha s_F(t) \in I = \text{sp}_k \{ t \} + N^6 \). It fulfills, as we can see from the calculation of the \([13\text{ Example 3]}\), if and only if \( \alpha \left( x^{(1)}_1 \right) \) and \( \alpha \left( x^{(1)}_2 \right) \) are linear depend modulo \( L^2 \). In this case \( \alpha \left[ x^{(1)}_1, x^{(1)}_2 \right] \in L^4 \), \( \alpha \left( \text{sp}_k \{ e_1, \ldots, e_6 \} \right) \subseteq L^5 \subset N^6 \subset I \) and also \( \left( \text{sp}_k \{ e_1, \ldots, e_6 \} \right) \circ x^{(2)} \subseteq \ker \psi \). Therefore \( (s_F(T))''_H = \bigcap_{\psi \in \text{Hom}(F,H)} \ker \psi \supseteq \left( \text{sp}_k \{ e_1, \ldots, e_6 \} \right) \circ x^{(2)} \) and 
\( (s_F(T))''_H \neq s_F(T). \) □

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References

[1] Ju. A. Bahturin, Identities in the Lie algebras, Moscow, Nauka, 1985. (In Russian.)

[2] A. Morimoto, A lemma on a free group. Nagoya Math. Journal, 7(1954), pp. 149-150.

[3] B. Plotkin, Varieties of algebras and algebraic varieties. Categories of algebraic varieties. Siberian Advanced Mathematics, Allerton Press, 7:2 (1997), pp. 64 – 97.

[4] B. Plotkin, Some notions of algebraic geometry in universal algebra, Algebra and Analysis, 9:4 (1997), pp. 224 – 248, St. Peterburg Math. J., 9:4, (1998) pp. 859 – 879.

[5] B. Plotkin, Algebras with the same (algebraic) geometry, Proceedings of the International Conference on Mathematical Logic, Algebra and Set Theory, dedicated to 100 anniversary of P.S. Novikov, Proceedings of the Steklov Institute of Mathematics, MIAN, 242 (2003), pp. 127 – 207.
[6] B. Plotkin, A. Tsurkov, Action type geometrical equivalence of representations of groups. *Algebra and Discrete Mathematics, 4* (2005), pp. 48 - 79.

[7] B.I. Plotkin, S.M. Vovsi, Varieties of group representation, *Riga, Zinatne, 1983.* (In Russian.)

[8] B. Plotkin, G. Zhitomirski. On automorphisms of categories of free algebras of some varieties, *Journal of Algebra, 306:2* (2006), pp. 344 – 367.

[9] I. Shestakov, A. Tsurkov. Automorphic equivalence of the representations of Lie algebras. *Algebra and Discrete Mathematics, 15* (2013), pp. 96 – 126.

[10] Tsurkov A., Automorphic equivalence of algebras, *International Journal of Algebra and Computation. 17:5/6,* (2007), pp. 1263 – 1271.

[11] Tsurkov A., Automorphisms of the category of the free nilpotent groups of the fixed class of nilpotency. *International Journal of Algebra and Computation. 17:5/6,* (2007), pp. 1273 – 1281.

[12] Tsurkov A., Automorphic Equivalence of Linear Algebras, [http://arxiv.org/abs/1106.4853](http://arxiv.org/abs/1106.4853)

[13] Tsurkov A., Automorphic Equivalence in the Classical Varieties of Linear Algebras, [http://arxiv.org](http://arxiv.org)