Stationary distributions for two-dimensional sticky Brownian motions: Exact tail asymptotics and extreme value distributions

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Abstract Sticky Brownian motions can be viewed as time-changed semimartingale reflecting Brownian motions, which find applications in many areas including queueing theory and mathematical finance. In this paper, we focus on stationary distributions for sticky Brownian motions. Main results obtained here include tail asymptotic properties in the marginal distributions and joint distributions. The kernel method, copula concept and extreme value theory are the main tools used in our analysis.

Keywords sticky Brownian motion, queueing model, stationary distribution, exact tail asymptotic, kernel method, extreme value distribution

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1 Introduction

Since the work of Feller [9–11], sticky Brownian motions have been explored in the literature extensively (see, for example, Billingsley [1], Itô and McKean [19, 20] and Ikeda and Watanabe [18]). Recall that a sticky Brownian motion on the half-line is a diffusion process that behaves like a Brownian motion everywhere except the origin, off which the process reflects after spending a positive amount of time there. According to Itô and McKean [19], sticky Brownian motions can be viewed as a time change of a semimartingale reflecting Brownian motion (SRBM). Recently, Rácz and Shkolnikov [32] introduced multidimensional sticky Brownian motions, which are a natural multidimensional extension of sticky Brownian motions on the half-line. This kind of processes has many applications in queueing theory. In the setting for single server queues, Welch [36] introduced an exceptional service for the first customer in each busy period and showed that a sticky Brownian motion on the half-line can be a heavy traffic limit. Later, with different exceptional service mechanisms, the same heavy traffic limit or the sticky Brownian motion was confirmed for other single server queueing models by Lemoine [29], Harrison and Lemoine [16],

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Yamada [39] and Yeo [40]. Similar to the one-dimensional setting case, we expect applications of multi-dimensional sticky Brownian motions to multi-server queueing networks. For other applications or potential applications, please refer to, for example, [32, Subsection 1.3].

For a stable process, it is interesting and important to study its stationary probabilities. However, except for very limited special cases, we cannot get a closed-form solution for the stationary probability distribution. This is one of motivations for us to study tail asymptotic properties in stationary probabilities, since performance bounds and approximations can often be developed from tail asymptotic properties. The stationarity of SRBMs has been studied in the literature. For example, Harrison and Hasenbein [15] and Harrison and Williams [17] studied the existence and the uniqueness of the stationary distribution, and Franceschi and Raschel [14] obtained Laplace transforms for the defined boundary measures. On the other hand, exact tail asymptotics for the SRBM have been obtained recently. The stationary tail behavior of the marginal distributions of the SRBM has been studied in terms of a geometric method in [6], through boundary value problems in [14], and using an analytic method in [13].

In this paper, we extend the above research to study exact tail asymptotic properties for a time-changed SRBM. We note that all the studies mentioned above are devoted to the asymptotic analysis for one-dimensional distributions (the boundary measures and the joint distribution along a given direction). In this article, we solve a harder one, to study exact tail asymptotics for the joint stationary distribution. This research was in part inspired by the recent work on time-changed SRBMs by Rácz and Shkolnikov [32]. In [32], Rácz and Shkolnikov studied the existence and the uniqueness of the stationary distributions of multidimensional sticky Brownian motions. Furthermore, under a rather strict condition, they presented an explicit expression of the stationary distribution.

The main contributions made in this paper include, under a stability condition, for a two-dimensional time-changed SRBM,

1) the extreme value distribution and exact tail asymptotics for the joint stationary distribution;
2) a study of exact tail asymptotics for the joint stationary distribution using a different method, a combination of the extreme value theory and copula. We expect that it can be extended to study exact tail asymptotic properties for many other stochastic models.

The rest of this paper is organized as follows. In Section 2, we first recall some preliminaries related to sticky Brownian motions. Section 3 is devoted to studying basic properties of stationary distributions of the sticky Brownian motion. In Section 4, we provide tail asymptotic results for the boundary stationary distributions and for the marginal stationary distributions, which are needed in Section 5. These results are available in the literature, but we derive them using a kernel method. In Section 5, we use the copula concept and extreme value theory to study the tail behavior of the joint stationary distribution.

2 Preliminaries

In this section, we introduce some preliminaries related to multidimensional sticky Brownian motions. We first recall the definition of the SRBM. SRBM models arise as an approximation for queueing networks of various kinds (see, for example, Williams [37,38]). A $d$-dimensional SRBM $\tilde{Z} = \{\tilde{Z}(t), t \geq 0\}$ is defined by the following:

$$\tilde{Z}(t) = X(t) + RL(t) \quad \text{for} \quad t \geq 0,$$

where $\tilde{Z}(0) = X(0) \in \mathbb{R}_+^d$ with

$$\mathbb{R}_+^d = \{(x_1, \ldots, x_d)' : x_i \geq 0, \ i = 1, \ldots, d\},$$

$X$ is an unconstrained Brownian motion with the drift vector $\mu = (\mu_1, \ldots, \mu_d)'$ and the covariance matrix $\Sigma = (\Sigma_{ij})_{d \times d}$, $R = (r_{ij})_{d \times d}$ is a $d \times d$ matrix specifying the reflection behavior at the boundaries, and $L = \{L(t)\}$ is a $d$-dimensional process with components $L_1, \ldots, L_d$ such that

(i) $L_j$ is continuous and non-decreasing with $L_j(0) = 0$;
(ii) $L_j$ only increases at times $t$ for which $\tilde{Z}_j(t) = 0$;
(iii) \( \tilde{Z}(t) \in \mathbb{R}^d_+, t \geq 0. \)

The existence of the SRBM has been studied extensively (see, for example, Taylor and Williams [35] and Reiman and Williams [33]). Recall that a \( d \times d \) matrix \( R \) is called an \( S \)-matrix, if there exists a \( d \)-vector \( \omega \geq 0 \) such that \( R\omega > 0 \), or equivalently, if there exists \( \omega > 0 \) such that \( R\omega > 0 \). Furthermore, \( R \) is called completely-\( S \) if each of its principal sub-matrices is an \( S \)-matrix. It was proved in [33,35] that for a given set of data \( (\Sigma, \mu, R) \) with \( \Sigma \) being positive definite, there exists an SRBM for each initial distribution of \( \tilde{Z}(0) \) if and only if \( R \) is completely-\( S \). Furthermore, when \( R \) is completely-\( S \), the SRBM is unique in distribution for each given initial distribution. Hereafter we assume that \( R \) is completely-\( S \). It is well known that a necessary condition (see, for example, Bramson et al. [2] and Kharroubi et al. [23]) for the existence of the stationary distribution for \( \tilde{Z} \) is that

\[ R \text{ is non-singular and } R^{-1}\mu < 0. \] (2.2)

Recall that if each principal submatrix of a square matrix has a positive determinant, then it is said to be a \( P \)-matrix. For a two-dimensional SRBM, Harrison and Hasenbein [15] showed that the condition (2.2) and \( R \) being a \( P \)-matrix are necessary and sufficient for the existence of a stationary distribution.

**Remark 2.1.** When \( d = 2 \), the equation (2.2) with \( R \) being a \( P \)-matrix is equivalent to

\[ r_{11} > 0, \quad r_{22} > 0, \quad r_{11}r_{22} - r_{12}r_{21} > 0 \] (2.3)

and

\[ r_{22}\mu_1 - r_{12}\mu_2 < 0, \quad r_{11}\mu_2 - r_{21}\mu_1 < 0. \] (2.4)

We note that the SRBM does not spend time on the boundary. As opposed to it, a sticky Brownian motion would spend a duration of time on the boundary. For the one-dimensional case, Feller [9–11] first observed the sticky boundary behavior for diffusion processes, and studied the problem that describes domains of the infinitesimal generators associated with a strong Markov process \( \tilde{X} \) in \([0, \infty)\). Moreover, \( \tilde{X} \) behaves like a standard Brownian motion in \((0, \infty)\), while at 0, a possible boundary behavior is described by

\[ f'(0+) = \frac{1}{2u} f''(0+), \] (2.5)

where \( u \in (0, \infty) \) is a given and fixed constant, and \( f \) are functions belonging to the domain of the infinitesimal generator of \( \tilde{X} \). The second order derivative \( f''(0+) \) measures the “stickiness” of \( \tilde{X} \) at 0. For this reason, the process \( \tilde{X} \) is called a sticky Brownian motion (sometimes, it is also referred to as a sticky reflecting Brownian motion in the literature). Itô and Mckean [19] first constructed the sample paths of \( \tilde{X} \). They showed that \( \tilde{X} \) can be obtained from the one-dimensional SRBM \( \tilde{Z} \) by the time change \( t \to T(t) \), where \( T(t) \) is the inverse of

\[ S(s) = s + \frac{1}{u} L_s \]

or \( T(t) = s \) is determined by the equation

\[ t = s + \frac{1}{u} L_s. \]

For more information about sticky Brownian motions on the half-line, please refer to Engelbert and Peskir [8] and the references therein.

Rácz and Shkolnikov [32] introduced multidimensional sticky Brownian motions, which is a natural extension of the sticky Brownian motion on the half-line, and proved the existence and the uniqueness of the multidimensional sticky Brownian motion. Similar to a sticky Brownian motion on the half-line, let

\[ S(t) = t + \sum_{i=1}^{d} u_i L_i(t), \] (2.6)

where \( u_i \in (0, \infty), i = 1, \ldots, d \) are fixed constants, and let \( T \) be the inverse of \( S(t) \) in (2.6). In the rest of this paper, we only consider bivariate sticky Brownian motions, i.e., \( d = 2 \).
Remark 2.2. (i) It follows from Kobayashi [24, Lemma 2.7] and (2.6) that $T$ has continuous paths.
(ii) It is obvious that $0 < T(1) \leq 1$ almost surely, since $L_i(0) = 0$ for $i = 1, \ldots, d$.
(iii) Compared with (2.5), in (2.6), we use $u_i$ to replace $\frac{1}{n}$ for convenience.

Ráce and Shkolnikov [32] pointed out that a multidimensional sticky Brownian motion
$Z = \{Z(t)\}$ can be written as

$$Z(t) = \tilde{Z}(T(t)).$$

For details about the representation (2.7), please refer to [32, Subsection 2.2]. It has been recently illus-
trated in [32] that the multidimensional sticky Brownian motion can serve as a diffusion approximation
of a particle movement system. It is also our expectation that it finds applications in the fields of queue-
ing theory and finance as a natural extension of the one-dimensional case. In the queueing field, it is
well known that the SRBM is a heavy traffic limit for many queueing networks such as open queueing
networks. As discussed in Section 1, in the setting for single server queues, a sticky Brownian motion on
the half-line can serve as a heavy traffic limit of a queueing system with exceptional service mechanisms.
It is reasonable to expect that a multidimensional sticky Brownian motion serves as a heavy traffic limit
for such multidimensional queueing networks with appropriately defined exceptional service mechanisms.

Remark 2.3. From (2.6), it is clear that the sticky Brownian motion degenerates to the SRBM, when
the parameters $u_i, i = 1, 2$ equal 0. Main results obtained in this paper also hold true for this special
case. At the same time, we note that as $u_i$ goes to infinity, the process $Z$ becomes an absorbed process,
which can be considered similarly.

3 Basic adjoint relation

Establishment of the basic adjoint relation (BAR) is the starting point for the analysis of our work in
this paper. Based on this equation, we can extend the kernel method (see, for example, Li and Zhao [30],
Dai et al. [4] and the references therein) to study exact tail asymptotics for stationary distributions of a
sticky Brownian motion. This is the focus of this section.

In the rest of this paper, we assume that $Z(0)$ follows the stationary distribution $\pi$ of $Z(t)$. Let $\mathcal{B}(\mathbb{R}_+^2)$
be the family of all Borel sets of $\mathbb{R}_+^2$. Recall that we can define the moment generating function (MGF)
for any finite measure on $\mathcal{B}(\mathbb{R}_+^2)$. For the stationary measure $\pi$, the MGF $\Phi(x, y)$ is defined as follows:

$$\Phi(x, y) = \int_{\mathbb{R}_+^2} \exp\{\langle \hat{w}, \hat{z} \rangle \} \pi(d\hat{z}),$$

where $\langle \cdot, \cdot \rangle$ is the inner product and $w = (x, y)' \in \mathbb{R}^2$. Similar to the SRBM, $\Phi(x, y)$ is closely related
to the MGFs of two boundary measures, which are defined below. For any set $B \in \mathcal{B}(\mathbb{R}_+^2)$, define for
$i = 0, 1, 2$,

$$V_i(B) = \mathbb{E}_\pi \left[ \int_0^{T(1)} 1_{\{\hat{Z}(s) \in B\}} dL_i(s) \right].$$

In addition, for any Borel set $B \in \mathcal{B}(\mathbb{R}_+^2)$, we define the joint measure for the time change

$$V_0(B) = \mathbb{E}_\pi \left[ \int_0^1 1_{\{Z(s) \in B\}} dT(s) \right].$$

According to Lemma 3.1 below, for $i = 0, 1, 2$, $V_i$ are finite measures on $\mathbb{R}_+^2$. Let $\Phi_i$ be the MGFs for
$V_i, i = 0, 1, 2$, i.e.,

$$\Phi_1(y) = \int_{\mathbb{R}_+^2} \exp\{\hat{z}_2 y\} V_1(d\hat{z}), \quad \Phi_2(x) = \int_{\mathbb{R}_+^2} \exp\{\hat{z}_1 x\} V_2(d\hat{z})$$
For these measures, we have the following BAR.

**Lemma 3.1.**  
(1) The boundary measures $V_i$, $i = 1, 2$, and the joint measure $V_0$ are all finite.  
(2) The MGFs of $V_i$, $i = 0, 1, 2$ have the following BAR: for any $w \in \mathbb{R}^2_-$, 
\[-\Psi_X(x, y) = \Phi_1(y)+\Phi_2(x)+ \Phi_0(x, y)\]  
(3.3)

where $R_i$ is the $i$-th column of the reflection matrix $R$, and $\Psi_X(x, y)$ is the Lévy exponent of the multi-dimensional Brownian vector $X(1)$, namely, 
\[\Psi_X(x, y) = \langle w, \mu \rangle + \frac{1}{2} \langle w, \Sigma w \rangle.\]

**Proof.** Since $Z(0)$ follows the stationary distribution $\pi$, for any $t \in \mathbb{R}_+$ and $\hat{z} = (\hat{z_1}, \hat{z_2})' \in \mathbb{R}^2$, 
\[P\{Z(t) < Z(0)\} = P\{Z < \hat{z}\}. \]

Now, we prove the first part of this lemma. From (3.1), we get that for all $i = 1, 2$, 
\[V_i(\mathbb{R}^2_+) = E_\pi[L_i(T(1))] \]  
(3.4)

and 
\[V_0(\mathbb{R}^2_+) = E_\pi[T(1)]. \]  
(3.5)

Hence, it suffices to prove that for any $i = 1, 2$, 
\[E_\pi[L_i(T(1))] < \infty \]  
(3.6)

and 
\[E_\pi[T(1)] < \infty. \]  
(3.7)

Here, we only prove (3.6) for $i = 1$. The other case can be proved in the same fashion. In fact, with probability 1, 
\[L_1(T(1)) = \int_0^1 dL_1(T(s)) = \int_0^{T(1)} dL_1(s) \leq \int_0^1 dL_1(s), \]  
(3.8)

since $L_1(s) \geq 0$ for all $s \geq 0$. It follows from [6] and (3.8) that 
\[E_\pi[L_1(T(1))] < \infty. \]  
(3.9)

On the other hand, it follows from Remark 2.2 that (3.7) holds naturally.

Below, we prove the second part of this lemma. We write $C^2(\mathbb{R}^2)$ to define the set of all the second order differentiable functions from $\mathbb{R}^2$ to $\mathbb{R}$, all of whose derivatives are continuous. We note that $\{Z(t)\}$ is a semimartingale. Since $T(t)$ is continuous and $S(t)$ is strictly increasing, it follows from [24, Corollary 3.4] that if $f : \mathbb{R}^2 \to \mathbb{R}$ is a $C^2(\mathbb{R}^2)$ function, then with probability 1, 
\[f(Z(t)) - f(Z(0)) = \sum_{i=1}^2 \mu_i \int_0^{T(t)} \frac{\partial f}{\partial \hat{z}_i}(\hat{Z}(u))du + \sum_{i,j=1}^2 \int_0^{T(t)} r_{ij} \frac{\partial f}{\partial \hat{z}_j}(\hat{Z}(u))dL_i(u) \]


$$+ \sum_{i=1}^{2} \int_{0}^{T(t)} \frac{\partial f}{\partial \tilde{z}_{i}} (\tilde{Z}(u)) dX_{i}(u) + \frac{1}{2} \sum_{i,j=1}^{2} \Sigma_{ij} \int_{0}^{T(t)} \frac{\partial^{2} f}{\partial \tilde{z}_{i} \partial \tilde{z}_{j}} (\tilde{Z}(u)) du. \quad (3.10)$$

Let

$$f(\hat{z}_{1}, \hat{z}_{2}) = \exp\{\hat{z}_{1}x + \hat{z}_{2}y\}$$

with $x, y \leq 0$ in (3.10). Then we have

$$\sum_{i=1}^{2} \mu_{r} E_{\pi} \left[ \int_{0}^{T(1)} \frac{\partial f}{\partial \tilde{z}_{i}} (\tilde{Z}(u)) du \right] + \sum_{i,j=1}^{2} E_{\pi} \left[ \int_{0}^{T(1)} \frac{\partial f}{\partial \tilde{z}_{i}} (\tilde{Z}(u)) dL_{i}(u) \right]$$

$$+ \frac{1}{2} \sum_{i,j=1}^{2} \Sigma_{ij} E_{\pi} \left[ \int_{0}^{T(1)} \frac{\partial^{2} f}{\partial \tilde{z}_{i} \partial \tilde{z}_{j}} (\tilde{Z}(u)) du \right] = 0. \quad (3.11)$$

By (3.11), (3.3) holds. The proof of this lemma is completed.

The following corollary immediately follows from Lemma 3.1.

**Corollary 3.2.** It holds that

$$E_{\pi}[T(1)] = 1 - \sum_{i=1}^{2} u_{i} E_{\pi}[L_{i}(T(1))],$$

where

$$E_{\pi}[L_{1}(T(1))] = \frac{\mu_{1}r_{22} - \mu_{2}r_{12}}{(r_{21} - \mu_{2}u_{1})(r_{12} - \mu_{1}u_{2}) - (r_{11} - \mu_{1}u_{1})(r_{22} - \mu_{2}u_{2})} \quad (3.12)$$

and

$$E_{\pi}[L_{2}(T(1))] = \frac{\mu_{1}r_{21} - \mu_{2}r_{11}}{(r_{22} - \mu_{2}u_{2})(r_{11} - \mu_{1}u_{1}) - (r_{12} - \mu_{1}u_{2})(r_{22} - \mu_{2}u_{1})}. \quad (3.13)$$

**Proof.** Let

$$f(\hat{z}_{1}, \hat{z}_{2}) = \exp\{\hat{z}_{1}x\}$$

with $x < 0$ and $\hat{z}_{1} \geq 0$ in (3.11). Then we have

$$f'(\hat{z}_{1}, \hat{z}_{2}) = x \exp\{\hat{z}_{1}x\} \quad \text{and} \quad f''(\hat{z}_{1}, \hat{z}_{2}) = x^{2} \exp\{\hat{z}_{1}x\}. \quad (3.14)$$

Hence, combining (3.11) and (3.14) gives

$$-\Psi_{X}(x,0)\Phi_{0}(x,0) = E_{\pi}[L_{1}(T(1))]r_{11}x + \Phi_{2}(x)r_{12}x. \quad (3.15)$$

Dividing $x < 0$ at both sides of (3.15) and letting $x \to 0$, we get

$$-\mu_{1} E_{\pi}[T(1)] = E_{\pi}[L_{1}(T(1))]r_{11} + E_{\pi}[L_{2}(T(1))]r_{12}. \quad (3.16)$$

Symmetrically, let

$$f(\hat{z}_{1}, \hat{z}_{2}) = \exp\{\hat{z}_{2}y\}$$

with $y < 0$ and $\hat{z}_{2} \geq 0$ in (3.11). Similar to (3.16), we can get

$$-\mu_{2} E_{\pi}[T(1)] = E_{\pi}[L_{1}(T(1))]r_{21} + E_{\pi}[L_{2}(T(1))]r_{22}. \quad (3.17)$$

At the same time, from the relation between $T$ and $S$, we get

$$T(t) = t - \sum_{i=1}^{2} u_{i} L_{i}(T(t)). \quad (3.18)$$
Hence, from (3.18), we have
\[ E_\pi[T(1)] = 1 - \sum_{i=1}^{2} u_i E_\pi[L_i(T(1))]. \tag{3.19} \]

Combining (3.16), (3.17) and (3.19) leads to
\[ \mu_1(r_{21} - \mu_2 u_1) - \mu_2(r_{11} - \mu_1 u_1) \\
= ((r_{22} - \mu_2 u_2)(r_{11} - \mu_1 u_1) - (r_{12} - \mu_2 u_1)(r_{21} - \mu_2 u_1))E_\pi[L_2(T(1))]. \tag{3.20} \]

If
\[ (r_{22} - \mu_2 u_2)(r_{11} - \mu_1 u_1) - (r_{12} - \mu_1 u_2)(r_{21} - \mu_2 u_1) \neq 0, \tag{3.21} \]
then (3.20) leads to (3.12) and (3.13) naturally. Hence, to prove this corollary, we only need to show (3.21).

If (3.21) does not hold, then we could get
\[ (r_{22} r_{11} - r_{12} r_{21}) = (r_{22} \mu_1 - r_{12} \mu_2) u_1 + (r_{11} \mu_2 - r_{21} \mu_1) u_2. \tag{3.22} \]

At the same time, it follows from (2.3) and (2.4) that
\[ (r_{22} r_{11} - r_{12} r_{21}) > 0 \tag{3.23} \]
and
\[ (r_{22} \mu_1 - r_{12} \mu_2) u_1 + (r_{11} \mu_2 - r_{21} \mu_1) u_2 < 0, \tag{3.24} \]
which contradict (3.22). From the above arguments, the proof of this corollary is completed. \qed

From Lemma 3.1, we can easily obtain the following result.

**Corollary 3.3.** (1) \( V_0(\cdot)/E_\pi[T(1)] \) and \( V_i(\cdot)/E_\pi[L_i(T(1))] \), \( i = 1, 2 \) are the stationary distribution and the boundary distributions of the corresponding reflecting Brownian motion \( \tilde{Z} \), respectively.

(2) For the stationary distribution \( \pi \) of the sticky Brownian motion \( Z \),
\[ \pi(B) = V_0(B) + \sum_{i=1}^{2} u_i V_i(B), \quad B \in \mathcal{B}(\mathbb{R}^+_2). \tag{3.25} \]

**Proof.** We first prove the first part of this corollary. To prove it, we apply [3, Lemma 2.1] (or [5, Theorem 1.1]), which presents a characterization of the stationary distribution and the associated boundary measures of the SRBM. Let \( C^2_b(\mathbb{R}^+_2) \) be the set of the functions \( f \) on \( \mathbb{R}^+_2 \) such that its first order derivatives and its second order derivatives are bounded and continuous. For any \( f \in C^2_b(\mathbb{R}^+_2) \), it follows from (3.10) that
\[ \sum_{i=1}^{2} \mu_i E_\pi \left[ \int_0^{T(1)} \frac{\partial f}{\partial z_i}(\tilde{Z}(u))du \right] + \sum_{i,j=1}^{2} E_\pi \left[ \int_0^{T(1)} r_{ij} \frac{\partial^2 f}{\partial z_i \partial z_j}(\tilde{Z}(u))dL_i(u) \right] \]
\[ + \frac{1}{2} \sum_{i,j=1}^{2} \Sigma_{ij} \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial z_i \partial z_j}(\tilde{Z}(u))du = 0. \tag{3.26} \]

Furthermore, from (3.1) and (3.2), we can rewrite (3.26) as
\[ \int_{\mathbb{R}^+_2} \mathcal{L} f(\tilde{z}) V_0(d\tilde{z}) + \sum_{i=1}^{2} \int_{\mathbb{R}^+_2} \langle \nabla f(\tilde{z}), R_i \rangle V_i(d\tilde{z}) = 0, \tag{3.27} \]
where \( \tilde{z} = (\tilde{z}_1, \tilde{z}_2)' \),
\[ \mathcal{L} f(\tilde{z}) = \frac{1}{2} \sum_{i,j=1}^{2} \Sigma_{ij} \frac{\partial^2 f}{\partial z_i \partial z_j}(\tilde{z}) + \sum_{j=1}^{2} \mu_j \frac{\partial f}{\partial z_j}(\tilde{z}), \]
and \( \Sigma_{ij} \) is the coefficient of the SRBM.
and $\nabla f(\hat{z})$ is the gradient of $f$. Noting (3.6) and (3.7), from [3, Lemma 2.1], we can get that

$$V_0(\cdot)/E_\pi[T(1)] \quad \text{and} \quad V_i(\cdot)/E_\pi[L_i(T(1))], \quad i = 1, 2$$

are the stationary distribution and the boundary distributions of the corresponding reflecting Brownian motion $\tilde{Z}$, respectively.

Below, we prove the second part of this corollary. In fact, for any Borel set $B \in \mathcal{B}(\mathbb{R}_+^2)$, we have

$$\pi(B) = E_\pi \left[ \int_0^{T(1)} 1_{\{Z(s) \in B\}} ds \right] = E_\pi \left[ \int_0^{T(1)} 1_{\{\tilde{Z}(s) \in B\}} dS(s) \right]. \quad (3.28)$$

By (2.6), (3.28) is equivalent to

$$E_\pi \left[ \int_0^{T(1)} 1_{\{\tilde{Z}(s) \in B\}} dt \right] + 2 \sum_{i=1}^2 u_i E_\pi \left[ \int_0^{T(1)} 1_{\{\tilde{Z}(s) \in B\}} dL_i(s) \right] = V_0(B) + 2 \sum_{i=1}^2 u_i V_i(B). \quad (3.29)$$

The proof of this corollary is now completed.

**Remark 3.4.** Combined with Corollary 3.3 and asymptotic results for the stationary distribution and boundary measures of the SRBM $\tilde{Z}$ in [6,7], (3.25) can be applied to analyze tail properties of the stationary distribution and boundary measures of $Z$.

**Theorem 3.5.** For each $w = (x, y)'$ with $\Phi(x, y) < \infty$, $\Phi_1(y) < \infty$ and $\Phi_2(x) < \infty$, we have

$$-\Psi_X(x, y) \Phi(x, y) = \Phi_1(y)(\langle w, R_1 \rangle - u_1 \Psi_X(x, y)) + \Phi_2(x)(\langle w, R_2 \rangle - u_2 \Psi_X(x, y)). \quad (3.30)$$

**Proof.** Similar to (3.3), from (3.10) and (3.25), we can get (3.30). Therefore, the proof of this theorem is finished.

### 4 Tail behavior of marginal distributions

In this paper, our main focus is on the study of the stationary tail behavior of the joint stationary distribution of the sticky Brownian motion $Z$, which is open, even for the SRBM $\tilde{Z}$. For this purpose, we need the asymptotic behavior of the marginal stationary distributions of $Z$. By Corollary 3.3, one can combine asymptotics of the stationary distribution along a given direction and the boundary measures of the SRBM $\tilde{Z}$, which can be found in [6, Theorems 2.2 and 2.3] and [7, Theorem 6.1] to obtain tail asymptotics of the marginal stationary distributions for $Z$. To make the paper self-contained, and for our convenience, we state these necessary results in this section, based on a kernel method (see, for example, [4]), which is an alternative method to other available literature methods (see, for example, Dai and Miyazawa [6,7] and Franceschi and Kurkova [13]).

For exact tail asymptotics for the marginal stationary distributions, we need tail asymptotic results for the boundary measures $V_i$, $i = 1, 2$. For this purpose, we study the kernel equation

$$\Psi_X(x, y) = 0. \quad (4.1)$$

For $(x, y)'$ satisfying (4.1), if the MGF $\Phi(x, y) < \infty$, then from (3.30), we have

$$\gamma_1(x, y) \Phi_1(y) + \gamma_2(x, y) \Phi_2(x) = 0, \quad (4.2)$$

where

$$\gamma_1(x, y) = xr_{11} + yr_{21}. \quad (4.3)$$
and

\[ \gamma_2(x, y) = x r_{12} + y r_{22}. \]  

(4.4)

We then rewrite the kernel equation in (4.1) as a quadratic form in \( y \) with coefficients that are polynomials in \( x \):

\[ \Psi_X(x, y) = ay^2 + b(x)y + c(x) = 0, \]

(4.5)

where

\[ a = \frac{1}{2} \Sigma_{22}, \quad b(x) = \mu_2 + \Sigma_{12}x \quad \text{and} \quad c(x) = x \mu_1 + \frac{1}{2} \Sigma_{11}x^2. \]

Let

\[ D_1(x) = 4[ (\Sigma_{12}^2 - \Sigma_{11} \Sigma_{22})x^2 + 2(\Sigma_{12} \mu_1 - \Sigma_{22} \mu_2)x + \mu_2^2 ] \]

(4.6)

be the discriminant of the quadratic form in (4.5). Therefore, in the complex plane \( \mathbb{C}_x \), for every \( x \), two solutions to (4.5) are given by

\[ Y_\pm(x) = -\frac{b(x) \pm \sqrt{b(x)^2 - 4ac(x)}}{2a}, \]

(4.7)

unless \( D_1(x) = 0 \), for which \( x \) is called a branch point of \( Y \). We emphasize that in using the kernel method, all the functions and variables are treated as complex ones unless specified.

Symmetrically, when \( x \) and \( y \) are interchanged, we have

\[ \Psi_X(x, y) = \tilde{a} x^2 + \tilde{b}(y)x + \tilde{c}(y) = 0, \]

(4.8)

where

\[ \tilde{a} = \frac{1}{2} \Sigma_{11}, \quad \tilde{b}(y) = \Sigma_{12}y + \mu_1 \]

and

\[ \tilde{c}(y) = \frac{1}{2} \Sigma_{22}y^2 + y \mu_2. \]

Let

\[ D_2(y) = \tilde{b}(y)^2 - 4\tilde{a}\tilde{c}(y). \]

(4.9)

For each fixed \( y \), two solutions to (4.8) are given by

\[ X_\pm(y) = -\frac{\tilde{b}(y) \pm \sqrt{\tilde{b}(y)^2 - 4\tilde{a}\tilde{c}(y)}}{2\tilde{a}}, \]

(4.10)

unless \( D_2(y) = 0 \), for which \( y \) is called a branch point of \( X \).

**Remark 4.1.** By some calculations, \( D_1(x) \) has two zeros satisfying \( x_1 \leq 0 < x_2 \) with \( x_i, i = 1, 2 \), being real numbers. Similarly, \( D_2(y) \) has two zeros satisfying \( y_1 \leq 0 < y_2 \) with \( y_i, i = 1, 2 \), being real numbers.

Now, let us introduce \( \tilde{x} \) and \( x^* \), which are candidates for the (rough) decay of the tail distributions for the marginal distribution along the \( x \)-direction, and their properties.

**About \( \tilde{x} \).**

(i) If \( \gamma_1(X, y) \) has a zero in \( (0, y_2) \), then it is unique, denoted by \( \tilde{y} \). Let \( \tilde{x} = X_+ (\tilde{y}) \). If \( Y_-(\tilde{x}) = \tilde{y} \), then let \( \tilde{x} = \tilde{x} \); otherwise, for convenience of comparison (with other candidates for the decay rate), let \( \tilde{x} = \infty \) (or any number greater than \( x_2 \)).

(ii) If \( \gamma_1(X, y) \) has no solution in \( (0, y_2) \), then for convenience of comparison, let \( \tilde{x} = \infty \).

**About \( x^* \).**

(i) If \( \gamma_2(x, Y_- (x)) = 0 \) has a solution in \( (0, x_2) \), then the solution is also unique, denoted by \( x^* \).
(ii) If $\gamma_2(x, Y_-(x))$ does not have a zero in $(0, x_2]$, then for convenience of comparison, let $x^* = \infty$.

An illustration of $x^* < \tilde{x} < x_2$ is provided in Figure 1. Here, we should point out that there is a total of seven possibilities from a comparison between the three candidates $x^*$, $\tilde{x}$ and $x_2$ for the dominant singularity, which results in a total of four cases with different tail asymptotics (see, for example, Theorem 4.2 below).

Let $\tau_2 = \min\{x^*, \tilde{x}, x_2\}$. Theorem 4.2 states the asymptotic results for the boundary measure $V_2$. By symmetry, asymptotic results for the boundary measure $V_1$ can also be easily stated.

**Theorem 4.2.** For the boundary measure $V_2$, we have the following tail asymptotic properties for large $x$:

**Case 1.** If $\tau_2 = x^* < \min\{\tilde{x}, x_2\}$ or $\tau_2 = \tilde{x} < \min\{x^*, x_2\}$ or $\tau_2 = \tilde{x} = x^* = x_2$, then

$$V_2(x, \infty) \sim K_1(\tau_2)e^{-\tau_2 x}.$$ (4.11)

**Case 2.** If $\tau_2 = x^* = x_2 < \tilde{x}$ or $\tau_2 = \tilde{x} = x_2 < x^*$, then

$$V_2(x, \infty) \sim K_2(\tau_2)e^{-\tau_2 x}x^{-\frac{1}{2}}.$$ 

**Case 3.** If $\tau_2 = x_2 < \min\{\tilde{x}, x^*\}$, then

$$V_2(x, \infty) \sim K_3(\tau_2)e^{-\tau_2 x}x^{-\frac{3}{2}}.$$ 

**Case 4.** If $\tau_2 = x^* = \tilde{x} < x_2$, then

$$V_2(x, \infty) \sim K_4(\tau_2)e^{-\tau_2 x}.$$ 

Here, $K_i(\tau_2)$, $i = 1, 2, 3, 4$ are finite constants depending on $\tau_2$.

**Remark 4.3.** As we mentioned earlier, the respective boundary measures of the SRBM and the sticky Brownian motion are equal up to a constant. Therefore, they have the same asymptotic results (up to a constant). We further note that [7, Theorem 6.1] illustrated exact tail asymptotics for the boundary measures of the SRBM based on Categories I–III defined in [6]. Hence, up to a constant, Theorem 4.2 is [7, Theorem 6.1] expressed in terms of our notations.

![Figure 1](image.png)  
**Figure 1** Illustration for $\tilde{x}$, $x^*$ and $x_2$
Remark 4.4. From a practical point of view, it is interesting and important to determine the value of $\tau_2$. Noting (4.1)–(4.4), based on the kernel method, one can follow the steps presented in [4, Section 7] to determine the value of $\tau_2$, and then the specific type of the asymptotics.

Next, we present the tail asymptotic behavior of the joint stationary distribution of $Z$. We only provide the result for $P\{Z_1 \geq x\}$, since the result for $P\{Z_2 \geq y\}$ can be obtained by symmetry.

From Lemma 3.1 and (3.30), we get that
\[
\Phi(x,0) = \frac{(x_{r1} - u_1\gamma_3(x))\Phi_1(0) + (x_{r2} - u_2\gamma_3(x))\Phi_2(x)}{\gamma_3(x)},
\]
where
\[
\gamma_3(x) = x\left(\mu_1 + \frac{1}{2}\Sigma_{11}x\right).
\]
It is obvious that the only non-zero solution to $\gamma_3(x) = 0$ is
\[
x_{\gamma_3} = -\frac{2\mu_1}{\Sigma_{11}}.
\]
Since $\Sigma_{11} > 0$, the zero of $\gamma_3(x)$ is not a pole of $\Phi(x,0)$ if $\mu_1 \geq 0$. In this case, up to a constant, $P\{Z_1 \geq x\}$ has the same tail asymptotics as the boundary measure $V_2$.

Theorem 4.5. Assume that $\mu_1 < 0$. Let $z = \min\{x^*, \bar{x}\}$ and $\alpha_1 = \min\{z, x_1, x_{\gamma_3}\}$. There are four types of exact tail asymptotics for the marginal survival distribution $P\{Z_1 \geq x\}$.

1. If $\min\{z, x_{\gamma_3}\} < x_2$, $x_{\gamma_3} \neq z$ and $x^* \neq \bar{x}$, or $\min\{z, x_{\gamma_3}\} < x_2$ and $x_{\gamma_3} = z$ with $Y_0(x^*) = 0$, or $z = x_{\gamma_3} = x_2$, or $z > x_{\gamma_3} = x_2$, or $x_{\gamma_3} > z = x_2$ with $\bar{x} = x^*$, then
   \[
P\{Z_1 \geq x\} \sim K_1(\alpha_1)\exp\{-\alpha_1 x\}.
   \]

2. If $\min\{z, x_{\gamma_3}\} < x_2$, $x_{\gamma_3} \neq z$ and $\bar{x} = x^*$, or $\min\{z, x_{\gamma_3}\} < x_2$ and $x_{\gamma_3} = z$ with $Y_1(x^*) = 0$, then
   \[
P\{Z_1 \geq x\} \sim K_2(\alpha_1)x\exp\{-\alpha_1 x\}.
   \]

3. If $x_{\gamma_3} > z = x_2$ with $\bar{x} \neq x^*$, then
   \[
P\{Z_1 \geq x\} \sim K_3(\alpha_1)x^{-\frac{1}{2}}\exp\{-\alpha_1 x\}.
   \]

4. If $x_2 < \min\{z, x_{\gamma_3}\}$, then
   \[
P\{Z_1 \geq x\} \sim K_4(\alpha_1)x^{-\frac{3}{2}}\exp\{-\alpha_1 x\}.
   \]

Here, $K_i(\alpha_1)$, $i = 1, 2, 3, 4$ are finite constants depending on $\alpha_1$.

Remark 4.6. Noting Corollary 3.3, up to a constant, we can also get Theorem 4.5 by combining asymptotic results for the stationary distribution of the SRBM along the direction $\vec{c} = (1, 0)'$ (see, for example, [6, Theorems 2.2 and 2.3]) and those for the boundary measures of the SRBM (see, for example, [7, Theorem 6.1]).

5 Tail behavior of the joint distribution

In this section, we focus on the tail behavior of the joint stationary distribution of $Z$. It is well known that the multivariate Gaussian vector with the correlation coefficients being strictly less than 1 is asymptotically independent. At the same time, we note that in the interior of the first quadrant, the sticky Brownian motion $Z$ behaves like the Brownian motion $X = (X_1, X_2)'$. Hence, it is expected that $Z$ is also asymptotically independent. In this section, we first prove this fact and study the extreme value distribution of the stationary distribution of $Z$. On the other hand, the extreme value distribution is very
useful since from a sample of vectors of maximum values, one can make inferences about the upper tail of the stationary distribution using the multivariate extreme value theory and copula. Much effort has been made to estimate tail probabilities based on the multivariate extreme value distributions. For more information, readers may refer to Fougères [12], Ledford and Tawn [28] and the references therein. Here, we apply the extreme value theory and copula to study exact tail asymptotics of the joint stationary distribution of the sticky Brownian motion $Z$.

In the rest of this paper, we assume that the correlation coefficient $\rho_{X_1, X_2} < 1$. For convenience, we let $F$ denote the stationary distribution function of $Z$, and $K$ denote an unspecified constant, whose value might be different from one case to another.

Now, we state the main result of this section.

**Theorem 5.1.** For the bivariate sticky Brownian motion $Z = (Z_1, Z_2)'$, we have

$$P\{Z_1 \geq x, Z_2 \geq y\}/(Kx^{\alpha_1}y^{\alpha_2} \exp\{-\alpha_1 x + \alpha_2 y\}) \to 1, \quad (x, y)' \to (\infty, \infty)' ,$$

(5.1)

where $\alpha_i$ is the decay parameter associated with $Z_i$ and

$$\bar{\mu}_i \in \left\{0, 1, -\frac{1}{2}, -\frac{3}{2}\right\}$$

is the exponent corresponding to $\alpha_i$ given in Theorem 4.5.

**Remark 5.2.** Since we cannot get the explicit expression for the stationary distribution $F(x, y)$, asymptotic properties in Theorems 4.5 and 5.1 are of more interest. They could be used to provide approximations or bounds for system performance, and approaches to computing the probability distribution $F(x, y)$. In fact, for a large enough threshold $\bar{T}$, an approximation for the joint probability is

$$F(x + \bar{T}, y + \bar{T}) \approx 1 - K((x + \bar{T})^{\mu_1} \exp\{-\alpha_1 (x + \bar{T})\} + (y + \bar{T})^{\mu_2} \exp\{-\alpha_2 (y + \bar{T})\}).$$

To prove Theorem 5.1, we first need to consider the extreme value distribution of the joint stationary distribution $F$. To achieve our goal, we first note that in the previous section, we obtained the tail equivalence of the marginal distributions. These results contain much information for studying the tail dependence of the stationary distribution. Once their dependence is specified, we can study the bivariate extreme value distribution of the stationary distribution. For this purpose, we first consider the extreme value distribution functions of the marginal functions $F_i(x)$, $i = 1, 2$. Before we state these results, we first introduce the notation of the domain of attraction for an extreme value distribution function $G$ and asymptotic independence.

**Definition 5.3 (Domain of attraction).** Assume that

$$\{X^{(n)} = (X_1^{(n)}, \ldots, X_d^{(n)})'\}$$

are independent and identically distributed (i.i.d.) multivariate random vectors with the common distribution $\tilde{F}$ and the marginal distributions $\tilde{F}_i$, $i = 1, \ldots, d$. If there exist normalizing constants $a_i^{(n)} > 0$ and $b_i^{(n)} \in \mathbb{R}$, $1 \leq i \leq d$, $n \geq 1$ such that as $n \to \infty$,

$$\mathbf{P}\left\{\frac{M_i^{(n)} - b_i^{(n)}}{a_i^{(n)}} \leq x_i, 1 \leq i \leq d\right\} \to \tilde{F}^n(a_1^{(n)}x_1 + b_1^{(n)}, \ldots, a_d^{(n)}x_d + b_d^{(n)})$$

$$\to G(x_1, \ldots, x_d),$$

where

$$M_i^{(n)} = \sqrt[n]{\sum_{k=1}^{n} X_i^{(k)}}$$

is the component-wise maxima, then we call the distribution function $G$ a multivariate extreme value distribution function, and $\tilde{F}$ is in the domain of attraction of $G$. We denote this by $\tilde{F} \in D(G)$. 
Definition 5.4 (Asymptotic independence). Assume that the extreme value distribution function $G$ has the marginal distributions $G_i$, $i = 1, \ldots, d$. If

$$
\tilde{F}_n^{(a_1^{(n)}x_1 + b_1^{(n)}), \ldots, a_d^{(n)}x_d + b_d^{(n)}} \to G(x_1, \ldots, x_d) = \prod_{i=1}^{d} G_i(x_i), \quad \text{as } n \to \infty,
$$

then we say that $\tilde{F}$ is asymptotically independent.

For the extreme value distribution functions of the marginal functions $F_i$, $i = 1, 2$, we have the following lemma.

Lemma 5.5. For the marginal stationary distribution $F_i(x) = \mathbb{P}\{Z_i \leq x\}$, $i \in \{1, 2\}$, of the $i$-th component of the stationary vector of the sticky Brownian motion $Z(t)$, we have

$$
F_i(x) \in D(G_1),
$$

i.e., there exist constants $a_n(\alpha_i, \bar{\mu}_i)$ and $b_n(\alpha_i, \bar{\mu}_i)$, which are functions of $\alpha_i$ and $\bar{\mu}_i$, such that as $n \to \infty$,

$$
F_i^n(a_n(\alpha_i, \bar{\mu}_i)x + b_n(\alpha_i, \bar{\mu}_i)) \to G_1(x),
$$

where

$$
G_1(x) = \exp\{-e^{-x}\}. \tag{5.3}
$$

Moreover, the normalizing constants $a_n(\alpha_i, \bar{\mu}_i)$ and $b_n(\alpha_i, \bar{\mu}_i)$ are given by (5.9) and (5.17) below, respectively.

Remark 5.6. It follows from Resnick [34, Proposition 0.3] that if a univariate function $\tilde{F} \in D(G)$, where $G$ is non-degenerate, then $G$ is of the type of one of the following three classes:

(i) Gumbel distribution:

$$
G_1(x) = \exp\{-e^{-x}\}, \quad -\infty < x < \infty.
$$

(ii) Fréchet distribution:

$$
G_2(x) = \exp\{-x^{-\theta}\}, \quad x > 0, \quad \theta > 0.
$$

(iii) Weibull distribution:

$$
G_3(x) = \exp\{-(x)^{\theta}\}, \quad x < 0, \quad \theta > 0.
$$

To prove Lemma 5.5, we require the following technical lemma.

Lemma 5.7. For any univariate continuous distribution function $\tilde{F}$, if

$$
1 - \tilde{F}(x) \sim x^\alpha \exp\{-\alpha x\}
$$

with $\alpha \in \mathbb{R}_+$ and $\bar{\mu} \in \mathbb{R}$, as $x \to \infty$, then we have

$$
\tilde{F}'(x) \sim \alpha x^\alpha \exp\{-\alpha x\}, \quad \text{as } x \to \infty
$$

and

$$
\tilde{F}''(x) \sim -\alpha^2 x^\alpha \exp\{-\alpha x\}, \quad \text{as } x \to \infty. \tag{5.5}
$$

Since both $\tilde{F}(x)$ and $1 - x^\alpha \exp\{-\alpha x\}$ are differentiable, we can apply L'Hôpital's rule to get the desired result. Here, we skip the proof of this lemma.

Remark 5.8. It can be shown that the inverse of Lemma 5.7 also holds.

Now, we prove Lemma 5.5.
Proof of Lemma 5.5. We only prove the case \(i = 1\). The other case can be considered in the same fashion.

It follows from Theorem 4.5 that we have

\[
1 - F_1(x) \sim K x^{\bar{\mu}_1} \exp\{-\alpha_1 x\}, \quad \text{as } x \to \infty.
\] (5.6)

It follows from the asymptotic equivalence (5.6) and Lemma 5.7 that

\[
\lim_{x \to \infty} \frac{F''_1(x)(1 - F_1(x))}{(F'_1(x))^2} = -1.
\] (5.7)

Then it follows from [34, p.40, Proposition 1.1] that \(F_1 \in D(G_1)\) with \(G_1(x)\) given by (5.3).

In the following, we identify suitable normalizing constants \(a_n(\alpha_1, \bar{\mu}_1)\) and \(b_n(\alpha_1, \bar{\mu}_1)\). Since we do not know the explicit expression of \(F_1\), we apply the tail equivalence to reach our goal.

First, since

\[
\lim_{x \to \infty} \frac{1 - F_1(x)}{F'_1(x)} = \frac{1}{\alpha_1},
\] (5.8)

according to [34, p.40, Proposition 1.1], we can choose

\[
a_n(\alpha_1, \bar{\mu}_1) = \frac{1}{\alpha_1}.
\] (5.9)

Next, we find a suitable \(b_n(\alpha_1, \bar{\mu}_1)\). Due to [34, p.40, Proposition 1.1], one choice of \(b_n\) is

\[
1 - F_1(b_n) = \frac{1}{n},
\] (5.10)

based on which and the tail asymptotic equivalence, we can choose \(b_n\) such that

\[-K(b_n)^{-\bar{\mu}_1} \exp\{-\alpha_1 b_n\} = \frac{1}{n}, \quad \text{for some constant } K,\] (5.11)

i.e.,

\[
\alpha_1 b_n + \bar{\mu}_1 \log(b_n) + \log(K) = \log(n).
\] (5.12)

To identify a solution to \(b_n\) in (5.12), without loss of generality, we assume that \(\bar{\mu}_1 \neq 0\) below. Then we have

\[
\log(n)/b_n \to \alpha_1, \quad \text{as } b_n \to \infty,
\]

i.e.,

\[
\alpha_1 b_n = \log(n) + r_n,
\] (5.13)

where \(r_n = o(\log(n))\).

Combining (5.12) and (5.13), we have

\[
\log(n) + r_n + \bar{\mu}_1 \log\left(\frac{1}{\alpha_1} \log n + \frac{1}{\alpha_1} r_n\right) + \log(K) = \log n.
\] (5.14)

By some calculations, (5.14) can be rewritten as

\[
r_n = -\bar{\mu}_1 \log\left(\frac{1}{\alpha_1} \left(1 + \frac{r_n}{\log n}\right)\right) - \bar{\mu}_1 \log \log(n) - \log K.
\] (5.15)

Due to (5.15), we have

\[
r_n = -\bar{\mu}_1 \log \log(n) + o(1) - \log K.
\] (5.16)
Combining (5.13) and (5.16), we have
\[
\frac{\alpha_1 b_n + \bar{\mu}_1 \log \log(n) - \log(n) - \log K}{a_n} = \frac{o(1)}{a_n} \to 0.
\]
Hence, we can choose
\[
b_n = \frac{1}{\alpha_1} (\log(n) - \bar{\mu}_1 \log \log(n) - \log(K)).
\]
Finally, it follows from [34, p.72, Proposition 1.19] and the convergence to types theorem (see [34, Propositions 0.2 and 0.3]) that we can set \( a_n(\alpha_1, \bar{\mu}_1) = a_n \) and \( b_n(\alpha_1, \bar{\mu}_1) = b_n \).

**Remark 5.9.** Here, we point out that the normalizing constants \( a_n(\alpha_i, \bar{\mu}_i) \) and \( b_n(\alpha_i, \bar{\mu}_i) \) are not unique. From the proof of Lemma 5.5, we get that \( F_i \) is a Von Mises function (see, for example, [34, p.40] for the definition). Hence, from [34, Proposition 1.1], we choose the normalizing constants as
\[
b_n(\alpha_i, \bar{\mu}_i) = \left( \frac{1}{1 - F_i} \right)^{(n)} \text{ and } a_n(\alpha_i, \bar{\mu}_i) = f_i(b_n(\alpha_i, \bar{\mu}_i)),
\]
where
\[
f_i = \frac{1 - F_i}{F_i'},
\]
and \( H^{-1}(x) \) denotes the (left continuous) inverse of a function \( H \) defined by
\[
H^{-1}(y) = \inf \{ s : H(s) \geq y \}.
\]
For the bivariate extreme value distribution of \( F(x, y) \), in fact, we have the following theorem.

**Theorem 5.10.** For the sticky Brownian motion \( Z = (Z_1, Z_2)' \) with the stationary distribution function \( F \), we have
\[
F^n(a_n(\alpha_i, \bar{\mu}_i)x_i + b_n(\alpha_i, \bar{\mu}_i), i = 1, 2) \to G_1(x_1)G_1(x_2), \quad \text{as } n \to \infty,
\]
where \( a_n(\alpha_i, \bar{\mu}_i) \) and \( b_n(\alpha_i, \bar{\mu}_i) \) are given by (5.9) and (5.17), respectively, and \( G_1(x) \) is given by (5.3).

**Remark 5.11.** From Theorem 5.10, we get that the maximum
\[
M^{(n)} = (M_1^{(n)}, M_2^{(n)})',
\]
of \( n \) i.i.d. observations from the stationary distribution \( F \) under suitable centering \( b_n(\alpha_i, \bar{\mu}_i) \) and scaling \( a_n(\alpha_i, \bar{\mu}_i) \) has a non-degenerate limit
\[
G(x_1, x_2) = G_1(x_1)G_1(x_2),
\]
which is a product measure, as \( n \) goes to infinity, i.e.,
\[
P \left\{ \frac{M_i^{(n)} - b_n(\alpha_i, \bar{\mu}_i)}{a_n(\alpha_i, \bar{\mu}_i)} \leq x_i, i = 1, 2 \right\} \to \exp\{-e^{-x_1}\} \exp\{-e^{-x_2}\}, \quad \text{as } n \to \infty.
\]
Hence, \( F \in D(G) \), and moreover \( F \) is asymptotically independent.

**Remark 5.12.** From the proofs of Theorems 5.1 and 5.10 below, we can see that similar results for the SRBM also hold.

**Remark 5.13.** Ráce and Shkolnikov [32, Theorem 5] studied the skew symmetric case of the multidimensional sticky Brownian motion. If the sticky Brownian motion \( Z \) is the skew symmetric case, then we have \( \bar{\mu}_i = 0, i = 1, 2 \) in (5.1).
Remark 5.14. As usual in the literature, for example Kotz and Nadarajah [26], (5.18) can be used to
study the stochastic behavior of the maximum and the minimum of i.i.d. random variables, which have
the common distribution $F$. At the same time, the tail of the distribution $F$ thereof may be evaluated by
means of statistical procedures based on extreme and intermediate order statistics or exceedances over
high thresholds.

To prove Theorem 5.10, we need the following result, which is a modified version of Proposition 5.27
in [34, p. 296].

Lemma 5.15. Suppose that
\[
\{X^{(n)} = (X_1^{(n)}, X_2^{(n)})', n \in \mathbb{N}\}
\]
are i.i.d. random vectors in $\mathbb{R}^2$ with the common joint continuous distribution $\tilde{F}$, and the marginal
distributions $\tilde{F}_i$, $i = 1, 2$. Moreover, we assume that $\tilde{F}_i$, $i = 1, 2$ are both in the domain of attraction
of some univariate extreme value distribution $\hat{G}_1$, i.e., there exist constants $a_i^{(n)}$ and $b_i^{(n)}$ such that as
$n \to \infty$,
\[
\tilde{F}_i^n(a_i^{(n)}x + b_i^{(n)}) \to \hat{G}_1(x).
\]

Then the following are equivalent:

1. $\tilde{F}$ is in the domain of attraction of a product measure, i.e.,
   \[
   \tilde{F}_i^n(a_i^{(n)}x_1 + b_i^{(n)}_1, a_2^{(n)}x_2 + b_2^{(n)}) \to \hat{G}_1(x_1)\hat{G}_1(x_2), \quad \text{as } n \to \infty.
   \]

2. \[
P\left\{\bigwedge_{i=1}^n X_1^{(i)} \leq a_1^{(n)}x_1 + b_1^{(n)}_1, \bigwedge_{i=1}^n X_2^{(i)} \leq a_2^{(n)}x_2 + b_2^{(n)}\right\} \to \hat{G}_1(x_1)\hat{G}_1(x_2), \quad \text{as } n \to \infty.
\]

3. For large enough $n \in \mathbb{N}$, such that
   \[
a_1^{(n)}x_1 + b_1^{(n)} > 0
   \]
   with $\hat{G}_1(x_i) > 0$, $i = 1, 2$,
   \[
   \lim_{n \to \infty} nP\{X_1^{(1)} > a_1^{(n)}x_i + b_1^{(n)}, i = 1, 2\} = 0.
   \]

(4) With $\lim_{x \to \infty} \tilde{F}_i(x) = 1$,
   \[
   \lim_{t \to \infty} P\{X_1^{(1)} > t, X_2^{(1)} > t\}/(1 - \tilde{F}_i(t)) \to 0.
   \]

By a slight modification of the proof of Proposition 5.27 in [34, p. 296], we can prove the above lemma,
details of which are omitted here.

Now, we are ready to prove Theorem 5.10. In order to prove it, we need to study the upper dependence
structure of the joint stationary distribution $F$. Before studying it, inspired by the study on upper tail
dependence, we introduce the following notation:
\[
\tilde{C}(u, u) = P\{Z_1 \geq F_1^{+}(u), Z_2 \geq F_2^{+}(u)\}, \quad u \in [0, 1].
\]

In fact, we have the following lemma.

Lemma 5.16. It holds that
\[
\lim_{u \to 1} \frac{\tilde{C}(u, u)}{1 - u} = 0.
\]

(5.19)
Proof. In the following proof, the reflection matrix $R$, the regulator process $L$ and the two-dimensional Brownian motion

$$X = (X_1, X_2)'$$

are the components in the definition of the SRBM given in (2.1), the time-changed process $T$ is defined through (2.6), and the sticky Brownian motion $\tilde{Z}$ is defined in (2.7). On the other hand, from Corollary 3.3, we get that the stationary distribution of the sticky Brownian motion $\hat{Z}$ and that of the SRBM $\tilde{Z}$ have the same upper tail dependence. Hence we turn to look at the dependence structure of the SRBM $\hat{Z}$.

For convenience, for any $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)' \in \mathbb{R}^2$, let

$$F_{\text{SRBM}}(\tilde{z}) = P\{\tilde{Z}_1 \geq \tilde{z}_1, \tilde{Z}_2 \geq \tilde{z}_2\}.$$ 

Furthermore, without loss of generality, we assume that $\tilde{Z}(0)$ follows the stationary distribution $F_{\text{SRBM}}$ of the SRBM $\tilde{Z}$. Hence, we have that for any $t \in \mathbb{R}_+$,

$$F_{\text{SRBM}}(\tilde{z}) := P\{\tilde{Z}_1 \leq \tilde{z}_1, \tilde{Z}_2 \leq \tilde{z}_2\} = P\{\tilde{Z}_1(t) \leq \tilde{z}_1, \tilde{Z}_2(t) \leq \tilde{z}_2\}. \quad (5.20)$$

To simplify the notations, for $i = 1, 2$, we further define

$$F_{(i, \text{SRBM})}(x) := P\{\tilde{Z}_i \leq x\}$$

and

$$\hat{C}(u, u) = P\{\tilde{Z}_1 \geq (F_{(1, \text{SRBM})})^{-}(u), \tilde{Z}_2 \geq (F_{(2, \text{SRBM})})^{-}(u)\}, \quad u \in [0, 1]. \quad (5.21)$$

Therefore, to prove (5.19), we only need to show

$$\lim_{u \to 1} \frac{\hat{C}(u, u)}{1 - u} = 0. \quad (5.22)$$

Next, we consider the upper tail dependence of $\hat{Z}$. We first define the last exit time $\tau_0(t)$ before $t$, out of the two boundaries

$$\tau_0(t) = \inf\{s : \hat{Z}(u) > 0 \text{ for all } s \leq u \leq t\}, \quad (5.23)$$

where $0 = (0, 0)' \in \mathbb{R}^2$. As a convention, let $\tau_0(t) = t$ if $\hat{Z}_i(t) = 0$ for some $i \in \{1, 2\}$. Therefore

$$0 \leq \tau_0(t) \leq t, \quad \text{a.s.} \quad (5.24)$$

On the other hand, noting that $X$ is a two-dimensional Brownian motion, we have that for large enough $t \in \mathbb{R}_+$,

$$\tau_0 > 0, \quad \text{a.s.} \quad (5.25)$$

From (2.1), it is obvious that

$$\hat{Z}(t) = \hat{Z}(\tau_0(t)) + X(t) - X(\tau_0(t)) + R(L(t) - L(\tau_0(t))). \quad (5.26)$$

Furthermore, due to the fact that $\hat{Z}$ has continuous sample paths,

$$\hat{Z}(t) > 0 \iff \tau_0(t) < t. \quad (5.27)$$

From [25, Proposition 1], (5.26) and (5.27), we get that if $\hat{Z}(t) > 0$, then

$$\hat{Z}(t) = \hat{Z}(\tau_0(t)) + X(t) - X(\tau_0(t)), \quad (5.28)$$

since $\hat{Z}$ has continuous sample paths. For convenience, we assume that the distribution function of $\tau_0(t)$ is $F_{\tau_0}^{(t)}$. 

To simplify the notation, let
\[ \tilde{u}_i = (F_{i,SRBM})^{-1}(u), \ i = 1, 2 \]
and
\[ \tilde{u} = (\tilde{u}_1, \tilde{u}_2)'. \] (5.29)

From Theorem 4.5, it is obvious that as \( u \to 1^- \),
\[ \tilde{u}_i \to +\infty. \] (5.30)

At the same time, from (5.20), (5.21) and (5.29), we have that for any normal random variable \( \tilde{Z}_i(\tau_0(t)) \) is a random variable for fixed \( t \in \mathbb{R}_+ \), to overcome this problem, we apply the truncation argument. Hence, from (5.26), we have
\[
P\{\tilde{Z}(t) \geq \bar{u}\} = \lim_{n \to \infty} P\{\tilde{Z}(\tau_0(t)) + X(t) - X(\tau_0(t)) + R(L(t) - L(\tau_0(t))) \geq \bar{u}, \tilde{Z}(\tau_0(t)) \leq n\}, \] (5.32)
where \( n = (n, n)' \) with \( n \in \mathbb{N} \). Hence, by (5.28) and (5.32),
\[
\lim_{u \to 1^-} \frac{P\{\tilde{Z}(t) \geq \bar{u}\}}{1 - u} = \lim_{n \to \infty} \lim_{u \to 1^-} \frac{P\{\tilde{Z}(\tau_0(t)) + X(t) - X(\tau_0(t)) + R(L(t) - L(\tau_0(t))) \geq \bar{u}, \tilde{Z}(\tau_0(t)) \leq n\}}{1 - u} = \lim_{n \to \infty} \lim_{u \to 1^-} \int_{\mathbb{R}_+} P\{|X(t) - X(u)| \geq \bar{u}_i - \tilde{Z}_i(s), \tilde{Z}_i(s) \leq n, i = 1, 2 | \tau_0(t) = s\} dF_{\tau_0}^{(t)}(s), \] (5.33)
where \([\tilde{u}]_i\) is the \( i \)-th entry of a vector \( \tilde{u} \in \mathbb{R}^2 \).

On the other hand, it is obvious that
\[
P\{|X(t) - X(u)| \geq \bar{u}_i - \tilde{Z}_i(s), \tilde{Z}_i(s) \leq n, i = 1, 2 | \tau_0(t) = s\} \leq P\{|X(t) - X(u)| \geq \bar{u}_i - n, i = 1, 2 | \tau_0(t) = s\}. \] (5.34)

Noting that the Brownian motion \( X \) is unconstrained, from (5.24), (5.33) and (5.34), we get
\[
\lim_{u \to 1^-} \int_{\mathbb{R}_+} P\{|X(t) - X(u)| \geq \bar{u}_i - \tilde{Z}_i(s), \tilde{Z}_i(s) \leq n, i = 1, 2 | \tau_0(t) = s\} dF_{\tau_0}^{(t)}(s) \leq \int_{\mathbb{R}_+} \lim_{u \to 1^-} \frac{P\{|X(t) - X(u)| \geq \bar{u}_i - n, i = 1, 2\}}{1 - u} dF_{\tau_0}^{(t)}(s). \] (5.35)

At the same time, it is known that the dependence structure is independent of marginal distributions. Hence
\[
\lim_{u \to 1^-} \frac{P\{|X(t) - X(u)| \geq \bar{u}_i - n, i = 1, 2\}}{1 - u} = \lim_{u \to 1^-} \frac{P\{|X(t) - X(u)| \geq \bar{u}_i - n, i = 1, 2\}}{P\{|X(t) - X(u)| \geq (F_2^{(t,s,n)})^{-1}(u) - n\}}, \] (5.36)
where \( F_2^{(t,s,n)} \) is the distribution function of the normal random variable \( |X(t) - X(u)| \). On the other hand, we know that for any normal random variable \( Y \), as \( y \to \infty \),
\[
P\{Y \geq y\} \sim K \exp\{-\beta y^2\}, \] (5.37)
where $\tilde{\beta}$ is a finite constant. Conversely, from Theorem 4.5 (or [6]), we have that as $z \to \infty$,\[P\{\tilde{Z}_i \geq z\} \sim K z^{\tilde{\beta}_i} \exp\{-\alpha_i z\}.\] (5.38)

Moreover, as $u \to 1^-$,
\[\lim_{u \to 1^-} \tilde{C}(u, u) = \lim_{n \to \infty} \int_{\mathbb{R}_+} \lim_{u \to 1^-} \frac{P\{[X(t) - X(s)]_i \geq \tilde{u}_i - n, i = 1, 2\}}{1 - u} dF_{\gamma_0}^{t}(s).\] (5.44)

Finally, it follows from (5.37) and (5.38) that for $u$ close to $1$,
\[\tilde{F}^{i(t,s,n)}_{\tilde{n}}(u) \leq \tilde{u}_i, \quad i = 1, 2.\] (5.40)

Combining (5.36) and (5.40) yields
\[\frac{P\{[X(t) - X(s)]_i \geq \tilde{u}_i - n, i = 1, 2\}}{P\{[X(t) - X(s)]_2 \geq (\tilde{F}^{i(t,s,n)}_{\tilde{n}})^{-1}(u) - n\}} \leq \frac{P\{([X(t) - X(s)]_i \geq \tilde{u}_i - n, i = 1, 2\}}{P\{[X(t) - X(s)]_2 \geq \tilde{u}_2 - n\}.\] (5.41)

At the same time, we know that for a Gaussian vector $\tilde{X} = (\tilde{X}_1, \tilde{X}_2)'$ with the correlation coefficient being less than 1, for any $j \in \{1, 2\},$
\[\lim_{x \to \infty} \frac{P\{\tilde{X}_j \geq x, i = 1, 2\}}{P\{\tilde{X}_j \geq x\}} = 0.\] (5.42)

From (5.36), (5.41) and (5.42), we get
\[\lim_{u \to 1^-} \frac{P\{[X(t) - X(s)]_i \geq \tilde{u}_i - n, i = 1, 2\}}{1 - u} = 0.\] (5.43)

On the other hand, from (5.31)–(5.35), we have
\[\lim_{u \to 1^-} \tilde{C}(u, u) \leq \lim_{n \to \infty} \int_{\mathbb{R}_+} \frac{P\{[X(t) - X(s)]_i \geq \tilde{u}_i - n, i = 1, 2\}}{1 - u} dF_{\gamma_0}^{t}(s).\] (5.44)

It follows from (5.43) and (5.44) that (5.22) holds. Hence
\[\lim_{u \to 1^-} \frac{\tilde{C}(u, u)}{1 - u} = 0.\] (5.45)

The proof of this lemma is finished. \[\square\]

Next, we use equivalent conditions in Lemma 5.15 to prove Theorem 5.10.

**Proof of Theorem 5.10.** The proof of the theorem follows now from (5.45) and Lemmas 5.5, 5.15 and 5.16. In fact, from Lemma 5.16, we can get that the equivalent condition in (4) of Lemma 5.15 holds. Then the equivalent condition in (1) of Lemma 5.15 implies that the joint stationary distribution $F$ is asymptotically independent. Finally, from the extreme value distributions of the marginal distributions $F_i, i = 1, 2$, which are given by (5.31), we get that Theorem 5.10 holds. \[\square\]

Now, we stand at a point to prove Theorem 5.1. In order to prove Theorem 5.1, here we use the copula. It is well known that for multivariate distributions, the univariate margins and the multivariate or dependence structure can be separated, and the multivariate structure can be represented by a copula. Before verifying Theorem 5.1, we first recall some definitions related to copulas, which come from Nelsen [31].

Let $\mathbb{R}$ denote the extended real line $[-\infty, +\infty]$, and $\bar{\mathbb{R}}^2$ denote the extended real plane $\mathbb{R} \times \mathbb{R}$.

**Definition 5.17.** Assume that $H(x, y)$ is a function whose domain, Dom$(H)$, is a nonempty subset of $\mathbb{R}^2$. Let
\[B = [x_1, x_2] \times [y_1, y_2]\]
be a rectangle all of whose vertices are in Dom$(H)$. Then we define the $H$-volume of $B$, $V_H(B)$, by
\[V_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1).\]

Moreover, the function $H$ is called 2-increasing, if $V_H(B) \geq 0$ for all the rectangles $B$ whose vertices lie in Dom$(H)$.
Definition 5.18 (Copula). A two-dimensional copula is a function \( \tilde{C} \) with the following properties:

(i) \( \text{Dom}(\tilde{C}) = [0, 1] \times [0, 1] \).

(ii) \( \tilde{C} \) is 2-increasing.

(iii) For any \( u, v \in [0, 1] \),

\[
\tilde{C}(u, 1) = u \quad \text{and} \quad \tilde{C}(1, v) = v
\]

and

\[
\tilde{C}(u, 0) = 0 \quad \text{and} \quad \tilde{C}(0, v) = 0.
\]

We refer readers to Nelsen [31] and Joe [21,22] for copulas.

Finally, we prove Theorem 5.1.

Proof of Theorem 5.1. To prove this theorem, we first introduce a transformation. Let

\[ X = (X_1, X_2)' \]

be a random vector with the joint distribution \( \tilde{F}(x,y) \) and the marginal distributions \( \tilde{F}_i(x) \), \( i = 1,2 \). Then we make the following transformation:

\[
X_i^* = \frac{-1}{\log(\tilde{F}_i(X_i))} \quad \text{for} \quad i = 1,2. \tag{5.46}
\]

By the transformation in (5.46), we transform each marginal \( \tilde{X}_i \) of a random vector \( \tilde{X} \) to a unit Fréchet variable \( X_i^* \), i.e.,

\[
P\{X_i^* < x\} = \exp\left\{ -\frac{1}{x} \right\} \quad \text{for} \quad x \in \mathbb{R}_+.
\]

Hence, for the bivariate extreme value distribution \( G(x,y) \) of \( Z \), we define

\[
G^*(x,y) = G\left( \frac{-1}{\log(G_1)} (x), \frac{-1}{\log(G_1)} (y) \right), \tag{5.47}
\]

where \( G_1 \) is given by (5.3). So by (5.46) and (5.47), one can see that \( G^* \) is the joint distribution function with the common marginal Fréchet distribution

\[
\tilde{G}_2(x) = \exp\{-x^{-1}\}.
\]

Furthermore, define

\[
Y_i = \frac{1}{1 - \tilde{F}_i(Z_i)}, \quad i = 1,2. \tag{5.48}
\]

Let \( F^*(y_1, y_2) \) be the joint distribution function of \( Y = (Y_1, Y_2)' \). Then it follows from [34, Proposition 5.10] and Theorem 5.10 that

\[
F^*(x,y) \in D(G^*(x,y)). \tag{5.49}
\]

By (5.49), we have that for any \( \tilde{y} = (y_1,y_2)' \in \mathbb{R}_+^2 \), as \( n \to \infty \),

\[
(F^*(n\tilde{y}))^n \to G^*(\tilde{y}). \tag{5.50}
\]

It follows from (5.50) that as \( n \to \infty \),

\[
F^*(n\tilde{y}) \sim (G^*(\tilde{y}))^{\frac{1}{n}}.
\]

By a simple monotonicity argument, we can replace \( n \) in the above equation by \( t \). Then we have that as \( t \to \infty \),

\[
F^*(t\tilde{y}) \sim (G^*(\tilde{y}))^{\frac{1}{t}}. \tag{5.51}
\]
At the same time, by Lemma 5.5, similar to (5.51), for any \( y \in \mathbb{R}_+ \),
\[
F_i^*(ty) \sim (G_1^*(y))^i, \quad i = 1, 2.
\]
(5.52)
Combining (5.51) and (5.52), we get that as \( t \to \infty \),
\[
F^*(ty) \sim F_1^*(ty_1) \cdot F_2^*(ty_2).
\]
(5.53)
It is obvious that for any \( x \in \mathbb{R}_+ \),
\[
\bar{F}_i^*(tx) := 1 - F_i^*(tx) \to 0, \quad \text{as } t \to \infty.
\]
(5.54)
From (5.53) and (5.54), we have that for any \((x,y) \in \mathbb{R}_+^2\),
\[
F^*(tx,ty) + \bar{F}_1^*(tx) + \bar{F}_2^*(ty) \sim F_1^*(tx) \cdot F_2^*(ty) + \bar{F}_1^*(tx) + \bar{F}_2^*(ty),
\]
(5.55)
since
\[
F^*(tx,ty) \to 1, \quad F_1^*(tx) \to 1 \quad \text{and} \quad F_2^*(ty) \to 1, \quad \text{as } t \to \infty.
\]
Note that (5.55) is equivalent to
\[
P\{Y_1 \geq tx, Y_2 \geq ty\} \sim \bar{F}_1^*(tx) \cdot \bar{F}_2^*(ty).
\]
(5.56)
Hence, we have that for any \((x,y) \in \mathbb{R}_+^2\),
\[
\lim_{t \to \infty} \frac{\bar{F}_i^*(tx,ty)}{F_i^*(tx) \cdot F_2^*(ty)} = 1,
\]
(5.57)
where \( \bar{F} := 1 - F \). To prove our theorem, we need to show
\[
\lim_{(x,y) \to (\infty, \infty)} \frac{\bar{F}_i^*(x,y)}{F_i^*(x) \cdot F_2^*(y)} = 1.
\]
(5.58)
Note that
\[
\bar{F}(x,y) = P\{\bar{F}_1^*(Y_1) \geq \bar{F}_1^*(x), \bar{F}_2^*(Y_2) \geq \bar{F}_2^*(y)\}.
\]
(5.59)
Hence, to prove (5.58), we only need to show
\[
\lim_{(u,v) \to (0,0) \text{ or } (u,v) \in I^2} \frac{\hat{C}(u,v)}{uv} = 1,
\]
(5.60)
where \( \hat{C}(\cdot, \cdot) \) is the survival copula of \( Y = (Y_1, Y_2)' \) defined by (5.48), and \( I = [0, 1] \). At the same time, note that
\[
\lim_{z \to 0} \frac{\log(1 - z)}{z} = 1.
\]
(5.61)
Hence, from (5.57), we get that for any \((u,v) \in I^2\),
\[
\lim_{t \to 0^+} \frac{\hat{C}(tu, tv)}{tu^2} = 1.
\]
(5.62)
We point out that the limit (5.60) has the form of \( \hat{C} \). Here, we apply the multivariate L’Hospital’s rule (see [27, Theorem 2.1]) to prove (5.60). Without much effort, we can construct a multivariate differentiable function \( \hat{C}(u,v) \) such that
\[
\hat{C}(u,v) = \hat{C}(u,v) \quad \text{for all } (u,v) \in I^2.
\]
and
\[ \tilde{C}(tu, tv) \sim t^2 uv, \quad \text{as } t \to 0. \]

Hence, to prove (5.60), it suffices to prove
\[ \lim_{(u, v) \to (0, 0)} \frac{\tilde{C}(u, v)}{uv} = 1. \] 

(5.63)

Near the origin \((0, 0)\)', the zero sets of both \(\tilde{C}(u, v)\) and \(uv\) consist of the lines \(u = 0\) and \(v = 0\). By the multivariate L'Hôpital's rule (see [27, Theorem 2.1]), to prove (5.63), we need to show that for each component \(E_i\) of \(\mathbb{R}^2 \setminus \mathcal{L}\), where
\[ \mathcal{L} = \{u = 0\} \cup \{v = 0\}, \]
we can find a vector \(\vec{z}\), not tangent to \((0, 0)\)' such that \(D_2(uv) \neq 0\) on \(E_i\) and
\[ \lim_{(u, v) \to (0, 0)'} \left( \frac{D_2 \tilde{C}(u, v)}{D_2(uv)} \right) = 1. \] 

(5.64)

For the component \(E_1\) bounded by the line \(\{u : u \geq 0\}\) and \(\{v : v \geq 0\}\), choose a vector, say \(\vec{z} = (1, 1)'\), and then \(\vec{z}\) is not tangent to the line \(v = 0\) at the origin \((0, 0)\)' . Next, we take the directional derivative along the direction \(\vec{z} = (1, 1)'\). It follows from (5.61) and (5.62) that
\[ \lim_{(u, v) \to (0, 0)'} \left( \frac{D_2 \tilde{C}(u, v)}{D_2(uv)} \right) = 1. \] 

(5.65)

From (5.64), (5.65) and [27, Theorem 2.1], we get
\[ \lim_{(u, v) \to (0, 0)'} \frac{\tilde{C}(u, v)}{uv} = 1. \] 

(5.66)

Hence, (5.58) holds.

Finally, it follows from (5.48) that for any \((x, y)' \in \mathbb{R}_+^2\),
\[ P[Z_1 \geq x, Z_2 \geq y] = P[F_1(Z_1) \geq F_1(x), F_2(Z_2) \geq F_2(y)] \]
\[ = P\left(Y_1 \geq \frac{1}{1 - F_1(x)}, Y_2 \geq \frac{1}{1 - F_2(y)} \right) \]
\[ = F^*(\frac{1}{F_1(x)}, \frac{1}{F_2(y)}), \] 

(5.67)

where \(F^*_i := 1 - F_i\). Combining (5.58) and (5.67), we get
\[ P[Z_1 \geq x, Z_2 \geq y] \left( \frac{1}{F_1(x)} \cdot \frac{1}{F_2(y)} \right) \to 1, \quad \text{as } (x, y)' \to (\infty, \infty)'. \] 

(5.68)

Conversely, we get
\[ \lim_{x \to 0} \frac{1 - \exp\{-x\}}{x} = 1. \] 

(5.69)

By (5.68) and (5.69), we get
\[ P[Z_1 \geq x, Z_2 \geq y] / (\hat{F}_1(x) \cdot \hat{F}_2(y)) \to 1, \quad \text{as } (x, y)' \to (\infty, \infty)'. \] 

(5.70)
Finally, it follows from Theorem 4.5 that as $x \to \infty$,

$$\hat{F}_i(x) \sim Kx^{\beta_i} \exp\{-\alpha_i x\}, \quad i = 1, 2. \quad (5.71)$$

From (5.70) and (5.71), the theorem is proved.

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