Comparisons of $H_\infty$ Loop Shaping Method and Fixed-structure $H_\infty$ Method for Attitude Controllers of a Quadrotor

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Abstract. The $H_\infty$ loop shaping method and fixed-structure $H_\infty$ method are employed to design controllers of a quadrotor. The validity of these two controllers is verified by numerical simulations, and it is observed that both the methods can achieve the design requirements and have a well tracking performance and restraining noise performance. Compared to $H_\infty$ loop shaping controller, the fixed-structure $H_\infty$ controller has not only the general nature based on h-infinity theory, but also takes into account easy realization.

Introduction

The quadrotor has been attracted more and more attention from military and civilian application domains and have been employed in fields of mapping 3-D environments, transporting etc.[1]. The realization of all the activities are based on a designed flight control system with well attitude control performances. However, the dynamics model of a quadrotor is coupled and nonlinear. In addition, the parameters of the model are uncertain because of the working environment is very complicated. Based on the above mentioned factors, it is necessary to design an advanced controller to improve the performance and maneuverability of the quadrotor. At present, $H_\infty$ theory with direct and efficient control performance has been applied in many fields [2, 3]. $H_\infty$ loop shaping and fixed-structure $H_\infty$ control are two common methods of designing controller based on $H_\infty$ theory. In this paper, vertical height, pitch angle, yaw angle and roll angle controllers of a quadrotor are designed with these two methods, respectively. The characteristics of these two methods are compared and analyzed through the design procedure and control performance.

Since the $H_\infty$ loop shaping design procedure based on robust stabilization combined with classical loop-shaping was proposed by McFarlane and Glover [4], it has been applied in fields of aerospace, mechanical control [5, 6]. The design method of fixed-structure $H_\infty$ control was first proposed by Apkarian and Gahinet in 2012 [7], and the algorithmic approach is a non-smooth technique which gets the locally optimal solution of the practical problem. At present, the fixed-structure $H_\infty$ control method has been employed to solve various problems[8].

Dynamics Model

Some assumptions has been adopted before modeling [9], and the inputs are defined as,

$$u = [u_1, u_2, u_3, u_4]^T = [F_1 + F_2 + F_3 + F_4, F_2 - F_4, F_3 - F_1, F_2 + F_4 - F_1 - F_3]^T$$ (1)
where \( \mathbf{F}_i (i=1\sim 4) \) are the forces acting on the motors, then the dynamic equations are described as follows [10],

\[
\begin{align*}
\dot{\mathbf{X}} &= \left( \theta \cos \psi + \gamma \sin \psi \right) \frac{u_1}{m}, \\
\dot{\mathbf{Y}} &= \left( \theta \sin \psi + \gamma \cos \psi \right) \frac{u_2}{m}, \\
\dot{\mathbf{Z}} &= -g + \frac{u_3}{m} \\
\dot{\gamma} &= \frac{u_4}{J_X}, \\
\dot{\theta} &= \frac{u_5}{J_Y}, \\
\dot{\psi} &= \frac{u_6}{J_Z}
\end{align*}
\]

where \( \gamma, \theta, \psi \) are angles of roll, pitch and yaw, respectively; \( J_X, J_Y, J_Z \) are moments of inertia in \( x, y, z \) direction; \( l \) is the length from the gravity center of the quadrotor to each rotor; \( g \) is the gravitational acceleration. Then, the state space realization of the above mentioned dynamic model is,

\[
\dot{\mathbf{X}} = A\mathbf{X} + B\mathbf{U}, \quad \mathbf{Y} = C\mathbf{X} + D\mathbf{U}
\]

where \( \mathbf{X} = [\dot{x} \quad \dot{y} \quad \dot{z} \quad \dot{\theta} \quad \dot{\psi} \quad \gamma \quad \theta \quad \psi]^T \), \( \mathbf{Y} = [z \quad \gamma \quad \theta \quad \psi]^T \), \( \mathbf{U} = [u_1 \quad u_2 \quad u_3 \quad u_4]^T \).

The transfer function matrix \( G_i \) is obtained,

\[
G_i = \text{diag}[1/m^2, \quad l/J_x s^2, \quad l/J_y s^2, \quad l/J_z s^2]
\]

and the parameters of the quadrotor are shown in table 1.

| Parameter | \( m \) | \( l \) | \( J_X \) | \( J_Y \) | \( J_Z \) |
|-----------|-------|-------|-------|-------|-------|
| Unit      | Kg    | m     | N*m*s^2 | N*m*s^2 | N*m*s^2 |
| Value     | 1.24  | 0.225 | 12.674e-3 | 12.674e-3 | 24.368e-3 |

The motor module is approximated as \( G_2 \) (\( a, b \) are parameters of the motor, \( a=0.78, b=0.1 \)),

\[
G_2 = \text{diag}[a / (1+bs), a / (1+bs), a / (1+bs), a / (1+bs)]
\]

Then, the completed transfer function is obtained as,

\[
G = G_1G_2 = \text{diag}\left[\begin{array}{cccc}
\frac{3.46}{s^2(1+0.1s)} & \frac{13.847}{s^2(1+0.1s)} & \frac{13.847}{s^2(1+0.1s)} & \frac{7.202}{s^2(1+0.1s)}
\end{array}\right]
\]

\( H_\infty \) Control Design

Controller Design Using \( H_\infty \) Loop Shaping

\( H_\infty \) loop shaping design is a well-known robust control design method combined classical loop-shaping concepts with \( H_\infty \) synthesis [11]. It is used to solve the problem of robust stabilization with coprime factor when transform the left coprime factorization of a
perturbation system to the general structure [12]. In order to stabilize $G_\Delta$ with $K$ upon the acceptance, $G$ is internally stable under the action of $K$ and $\|\Delta_M, \Delta_N\| < \varepsilon$ [13], where $G_\Delta = (\tilde{M} + \Delta_M)^{-1}(\tilde{N} + \Delta_N)$ represents unstructured uncertainty in a plant via coprime factor perturbations when the nominal plant is written as $G = \tilde{M}^{-1}\tilde{N}$; $\tilde{M}, \tilde{N}$ is a left coprime factorization of $G$; $K$ is the controller, a requirement $\|T_{w \to z}\|_\infty \leq 1/\varepsilon$ must be met [14], where $\varepsilon$ is a robust stability margin; $w$ and $z$ represent the disturbances of input and output, respectively; $T_{w \to z}$ is the transfer function matrix from $w$ to $z$.

$W_1$ and $W_2$ are the pre- and post-compensators,

$$
W_1 = \text{diag} \left[ \begin{array}{cccc}
12(14s+4) & 28(11s+6.5) & 28(11s+6.5) & 12(14s+4) \\
0.19s+19 & 1.8s+93 & 1.8s+93 & 0.19s+19
\end{array} \right],
W_2 = \text{diag} \left[ \begin{array}{cccc}
12 / (3.9s+40) & 21 / (10s+75) & 21 / (10s+75) & 9 / (4.2s+32)
\end{array} \right]
$$

(7)

The controller is designed to meet the following requirements,

1)For each channel, the overshoot is less than 20%, the steady-state error is less than 2% and the settling time is as short as possible.

2) The stability margin is between 0.3 to 0.4.

The final solution of the controller is a $4 \times 4$ matrix with the absolute values of the elements are less than $1 \times 10^{-7}$ in addition to the diagonal elements, hence the controller is,

$$
K_{11} = \frac{-95227(s+0.2644)(s^2+21.02s+113.44)}{(s+99.94)(s+26.08)(s+10.26)(s^2+20.08s+325.8)},
K_{22} = \frac{-859.28(s+0.5289)(s^2+18.65s+89.79)}{(s+53.31)(s+7.5)(s^2+29.47s+436.7)}
$$

$$
K_{33} = \frac{-859.28(s+0.5289)(s^2+18.65s+89.79)}{(s+53.31)(s+7.5)(s^2+29.47s+436.7)},
K_{44} = \frac{-4531.1(s+0.2739)(s^2+19.31s+98.34)}{(s+100.5)(s+7.619)(s^2+33.82s+536.3)}
$$

(8)

at the same time, the stability margin is 0.386 that meet the design requirement.

In Fig.1, The singular values of the sensitivity function is under the 0dB line after the response of the controller. It means that the disturbance is restrained well and the input is tracked well. As depicted in Fig.2, noises are filtered effectively under the condition that the singular values of the complementary sensitivity function roll off rapidly at high-frequency.

![Figure 1. Singular Value of Sensitivity Function.](image1)

![Figure 2. Singular Value of Complementary Sensitivity Function.](image2)
**Controller Design Using Fixed-structure $H_{\infty}$**

The method firstly fixes the order of the controller, then adjusts some parameters of the control system according to the algorithm. Multiple requirements are described as,

$$\| T_{w_1 \to z_1} (C(s, p)) \| \leq 1, \ldots, \| T_{w_N \to z_N} (C(s, p)) \| \leq 1$$

where $(\omega_i, z_i)$ are design requirements of I/O; $C(s, p)$ is the controller which meets the constraints; $T_{w \to z}$ is the closed-loop map from $w$ to $z$; The symbol $\parallel \parallel$ refers to either the $H_{\infty}$ or the $H_2$ norms; The parameter $p \in R^n$ regroups all tunable parameters in the controller.

After both soft objectives and hard constraints are defined, Eq. 9 is formalized as,

$$\min_p \max_{i=1, \ldots, n_c} \{ \| T_{w_i \to z_i} (C(s, p)) \| \} \quad \text{subject to} \quad \| T_{w_j \to z_j} (C(s, p)) \| \leq 1, j = 1, \ldots, n_c$$

(10)

A specialized non-smooth technique is employed to solve the problem with searching local solutions after extending the generalized plant originates to multiple model instances and enriching supplement constraints [15].

Rate feedback loop is added to improve the performance of the system as depicted in Fig. 3, $k_c$ is equal to 13, 9, 9, 6 in the channels, respectively. Then the whole system will be placed in the framework of fixed-structure $H_{\infty}$ to design controllers. The design requirements which are concerned in the algorithms are described as follows,

1) The response of controlled plant closely matches the reference model, the closed-loop response from the noise to output of the controller should roll off $8$ rad/s with a slope of at least $40$ dB/dec.
2) The stability margins at the plant input should be at least $7$ dB and $45$ degrees.

The simulation model of fixed-structure $H_{\infty}$ is shown in Fig. 4.

![Figure 3. Rate Feedback.](image)

![Figure 4. Simulation Model.](image)

where $K_f, K_i, K_p$ are tuned blocks; $F_{ro}(s)$ is a second-order filter; reference model $G_{ref} (s) = \frac{w_n^2}{(s^2 + 2\zeta w_n s + w_n^2)}$ is a second-order system, where $w_n = 3$, $\zeta = 0.7$. The parameters of the controller is listed in table 2.
Table 2. Parameters of the Controller.

|                | $K_i$  | $K_p$  | $K_f$  | $\omega_n$ | $\zeta$ |
|----------------|--------|--------|--------|-------------|---------|
| Vertical height| 0.001807 | 1.807  | -0.02501 | 7.84       | 0.85    |
| Pitch/roll     | 0.001789 | 1.789  | 0.002568 | 7.45       | 0.865   |
| Yaw            | 0.00181  | 1.81   | 0.03397  | 7.86       | 0.855   |

Analysis and Simulation Results

Analysis

It is known from the above design procedures, the design requirements can’t be unified because of the design method is different. $H_\infty$ loop shaping control can’t make the requirements of time domain indexes directly. And the procedure of choosing weight functions depends heavily on the experiences of designers. In addition, it is necessary to make a trade-off between robust performance and shaping performance when choose the weight functions. Although there are some optimization algorithms which can be used to search weight function [16], the cost of obtaining results doesn’t match its performances. As for fixed-structure $H_\infty$, requirements of time domain indexes can be reflected by the reference model directly. The second-order filter can filter out high frequency noises of the motor input.

The controller of $H_\infty$ loop shaping is complex and can’t ensure the specific impact of a parameter on the system, this problem leads to troubles for the practical application of quadrotors. However, the structure of fixed-structure $H_\infty$ controller is simple, and it is provably effective both in terms of speed of execution and quality of the solutions.

Simulation Results

The simulation results are shown from Fig 5 to Fig 6, and the input is the unit step signal.

As shown in simulation results, the dynamic performances indicate that $H_\infty$ loop shaping controller has faster response, longer settling time and larger overshoot. For a quadrotor, it is dangerous when there is a large overshoot. Therefore, we prefer slow response and even small steady-state error rather than large overshoot.
Table 3. Dynamic Performances.

| Channel    | Dynamic performances | $H_\infty$ loop shaping | Fixed-structure $H_\infty$ |
|------------|----------------------|--------------------------|-----------------------------|
| Vertical   |                      |                          |                             |
| height     | Rising time (s)      | 0.227                    | 0.9                         |
|            | Overshoot (%)        | 12                       | 0                           |
|            | Adjusting time (s)   | 4.57                     | 1.1                         |
| Pitch/roll |                      |                          |                             |
|            | Rising time (s)      | 0.178                    | 0.9                         |
|            | Overshoot (%)        | 20                       | 1                           |
|            | Adjusting time (s)   | 2.89                     | 1.05                        |
| Yaw        |                      |                          |                             |
|            | Rising time (s)      | 0.155                    | 0.9                         |
|            | Overshoot (%)        | 19                       | -0.8                        |
|            | Adjusting time (s)   | 2.25                     | 1.2                         |

Conclusions
Quadrotor attitude controllers are designed using two methods and the design procedures and simulation results are compared. The results show that both of them can approach the design requirements and have noise restraining performance and tracking performance. The fixed-structure $H_\infty$ control is recommended to choose as the practical applications of quadrotors as.

1) Fixed-structure $H_\infty$ controller is realizable because of the structure is simple, and the order can be set artificially.

2) Due to the design procedure depends on the experiences of designers heavily, there are too many human factors in $H_\infty$ loop shaping designing. In addition, the adjustment of the controller parameters is unintuitive in practical applications.

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