Paths and stability number in digraphs

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Abstract

The Gallai-Milgram theorem says that the vertex set of any digraph with stability number \( k \) can be partitioned into \( k \) directed paths. In 1990, Hahn and Jackson conjectured that this theorem is best possible in the following strong sense. For each positive integer \( k \), there is a digraph \( D \) with stability number \( k \) such that deleting the vertices of any \( k - 1 \) directed paths in \( D \) leaves a digraph with stability number \( k \). In this note, we prove this conjecture.

1 Introduction

The Gallai-Milgram theorem [7] states that the vertex set of any digraph with stability number \( k \) can be partitioned into \( k \) directed paths. It generalizes Dilworth’s theorem [4] that the size of a maximum antichain in a partially ordered set is equal to the minimum number of chains needed to cover it. In 1990, Hahn and Jackson [8] conjectured that this theorem is best possible in the following strong sense. For each positive integer \( k \), there is a digraph \( D \) with stability number \( k \) such that deleting the vertices of any \( k - 1 \) directed paths in \( D \) leaves a digraph with stability number \( k \). Hahn and Jackson used known bounds on Ramsey numbers to verify their conjecture for \( k \leq 3 \). Recently, Bondy, Buchwalder, and Mercier [3] used lexicographic products of graphs to show that the conjecture holds if \( k = 2^a 3^b \) with \( a \) and \( b \) nonnegative integers. In this short note we prove the conjecture of Hahn and Jackson for all \( k \).

Theorem 1 For each positive integer \( k \), there is a digraph \( D \) with stability number \( k \) such that deleting the vertices of any \( k - 1 \) directed paths leaves a digraph with stability number \( k \).

To prove this theorem we will need some properties of random graphs. As usual, the random graph \( G(n, p) \) is a graph on \( n \) labeled vertices in which each pair of vertices forms an edge randomly and independently with probability \( p = p(n) \).

Lemma 1 For \( k \geq 3 \), the random graph \( G = G(n, p) \) with \( p = 20n^{-2/k} \) and \( n \geq 2^{15k^2} \) a multiple of \( 2k \) has the following properties.

(a) The expected number of cliques of size \( k + 1 \) in \( G \) is at most \( 20^{k+1} \).

(b) With probability more than \( 2/3 \), every induced subgraph of \( G \) with \( n^{2k} \) vertices has a clique of size \( k \).

Proof: (a) Each subset of \( k + 1 \) vertices has probability \( p^{(k+1)/2} \) of being a clique. By linearity of expectation, the expected number of cliques of size \( k + 1 \) is

\[
\binom{n}{k+1} p^{(k+1)/2} = \binom{n}{k+1} 20^{(k+1)/2} n^{-k-1} \leq 20^{(k+1)/2}.
\]

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(b) Let $U$ be a set of $\frac{n^2}{2k}$ vertices of $G$. We first give an upper bound on the probability that $U$ has no clique of size $k$. For each subset $S \subset U$ with $|S| = k$, let $B_S$ be the event that $S$ forms a clique, and $X_S$ be the indicator random variable for $B_S$. Since $k \geq 3$, by linearity of expectation, the expected number $\mu$ of cliques in $U$ of size $k$ is

$$
\mu = \mathbb{E}\left[\sum_{S} X_S\right] = \binom{n}{k} p^{(k)} \geq \frac{n^k}{2^{(2k)k!}} 20^{(k)} n^{1-k} \geq 2n.
$$

Let $\Delta = \sum \Pr[B_S \cap B_T]$, where the sum is over all ordered pairs $S, T$ with $|S \cap T| \geq 2$. We have

$$
\Delta = \sum_{i=2}^{k-1} \sum_{|S \cap T|=i} \Pr[B_S \cap B_T] = \sum_{i=2}^{k-1} \sum_{|S \cap T|=i} p^{2i}(\frac{i}{k}) \geq \sum_{i=2}^{k-1} \binom{n}{i} \left(\frac{n-i}{k-1}\right) \left(\frac{n-k}{k-i}\right) p^{2i}(\frac{i}{k}) \leq 20^{k^2} \sum_{i=2}^{k-1} n^{2i-1} / (k-1) \leq k 20^{k^2} n^{2/k}.
$$

Here we used the fact that $i(i-1)/k - i$ for $2 \leq i \leq k-1$ clearly achieves its maximum when $i = 2$ or $i = k-1$.

Using that $k \geq 3$ and $n \geq 2^{15k^2}$, it is easy to check that $\Delta \leq n$. Hence, by Janson’s inequality (see, e.g., Theorem 8.11 of [2]) we can bound the probability that $U$ does not contain a clique of size $k$ by $\Pr[\land_S B_S] \leq e^{-\mu + \Delta/2} \leq e^{-n}$. By the union bound, the probability that there is a set of $\frac{n}{2k}$ vertices of $G(n, p)$ which does not contain a clique of size $k$ is at most $\left(\frac{n}{2k}\right) e^{-n} \leq 2^n e^{-n} < 1/3$. □

The proof of Theorem 1 combines the idea of Hahn and Jackson of partitioning a graph into maximum stable sets and orienting the graph accordingly with Lemma 1 on properties of random graphs.

**Proof of Theorem 1.** Let $k \geq 3$ and $n \geq 2^{15k^2}$. By Markov’s inequality and Lemma 1(a), the probability that $G(n, p)$ with $p = 20n^{-2/k}$ has at most $2 \cdot 20^{(k+1)}$ cliques of size $k+1$ is at least $1/2$. Also, by Lemma 1(b), we have that with probability at least $2/3$ every set of $\frac{n^2}{2k}$ vertices of this random graph contains a clique of size $k$. Hence, with positive probability (at least $1/6$) the random graph $G(n, p)$ has both properties. This implies that there is a graph $G$ on $n$ vertices which contains at most $2 \cdot 20^{(k+1)}$ cliques of size $k+1$ and every set of $\frac{n^2}{2k}$ vertices of $G$ contains a clique of size $k$. Delete one vertex from each clique of size $k+1$ in $G$. The resulting graph $G'$ has at least $n - 2 \cdot 20^{(k+1)} \geq 3n/4$ vertices and no cliques of size $k+1$. Next pull out vertex disjoint cliques of size $k$ from $G'$ until the remaining subgraph has no clique of size $k$, and let $V_1, \ldots, V_t$ be the vertex sets of these disjoint cliques of size $k$. Since every induced subgraph of $G$ of size at least $\frac{n^2}{2k}$ contains a clique of size $k$, then $|V_1 \cup \ldots \cup V_t| \geq \frac{3n}{4} - \frac{n}{2k} \geq \frac{n}{2}$. Define the digraph $D$ on the vertex set $V_1 \cup \ldots \cup V_t$ as follows. The edges of $D$ are the nonedges of $G$. In particular, all sets $V_i$ are stable sets in $D$. Moreover, all edges of $D$ between $V_i$ and $V_j$ with $i < j$ are oriented from $V_i$ to $V_j$. By construction, the stability number of $D$ is equal to the clique number of $G'$, namely $k$. Also any set of $\frac{n}{2k}$ vertices of $D$ contains a stable set of size $k$. Note that every directed path in $D$ has at most one vertex in each $V_i$. Hence, deleting any $k-1$ directed paths in $D$ leaves at least $|D|/k \geq \frac{n}{2k}$ remaining vertices. These remaining vertices contain a stable set of size $k$, completing the proof. □

**Remark.** Note that in order to prove Theorem 1 we only needed to find a graph $G$ on $n$ vertices with no clique of size $k+1$ such that every set of $\frac{n^2}{2k}$ vertices of $G$ contains a clique of size $k$. The existence of such graphs were first proved by Erdős and Rogers [6], who more generally asked to estimate the minimum $t$ for which there is a graph $G$ on $n$ vertices with no clique of size $s$ such that every set of
$t$ vertices of $G$ contains a clique of size $r$. Since then a lot of work has been done on this question, see, e.g., [9] [1] [10] [5]. Although most result for this problem used probabilistic arguments, Alon and Krivelevich [1] give an explicit construction of an $n$-vertex graph $G$ with no clique of size $k + 1$, such that every subset of $G$ of size $n^{1-\epsilon}$ contains a $k$-clique. Since we only need a much weaker result to prove the conjecture of Hahn and Jackson, we decided to include its very short and simple proof to keep this note self-contained.

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