Instanton representation of Plebanski gravity: XVIII. Quantization and proposed resolution of the Kodama state

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Abstract

In this paper we have constructed a Hilbert space of states solving the initial value constraints of GR in the instanton representation of Plebanski gravity. The states are labelled by two free functions of position constructed from the eigenvalues of the CDJ matrix. This comprises the physical degrees of freedom of GR with a semiclassical limit corresponding to spacetimes of Petrov Type I, D and O. The Hamiltonian constraint in this representation is a hypergeometric differential equation on the states, for which we have provided a closed form solution. Additionally, we have clarified the role of the Kodama state within this Hilbert space structure, which provides a resolution to the issue of its normalizability raised by various authors.
1 Introduction

One of the main outstanding issues in quantum gravity has been the construction of solutions to the quantum Hamiltonian constraint, in the full theory, with a well-defined semiclassical limit. In the Ashtekar formalism of general relativity there is one known special solution which satisfies this requirement, known as the Kodama state $\psi_{\text{Kod}} (\mathcal{2}, \mathcal{3})$. It has been argued by various authors the dangers inherent in attempting to associate the Kodama state with a wavefunction of the universe for gravity (See e.g. [4] by analogy to the pathologies of the Chern–Simons functional for Yang–Mills theory). Counterarguments by Smolin indicate that not all of the properties of Yang–Mills theory extend to gravity, particularly in view of the fact that the latter has additional constraints which must be satisfied. In [5] it is concluded that the Kodama state cannot be considered a normalizable state of Lorentzian gravity, though in the Euclidean case it is delta-function normalizable in minisuperspace.

In the present paper we will provide a proposal for the resolution of these issues, by constructing a Hilbert space of solutions to the constraints of the full theory for Lorentzian signature spacetime using the instanton representation of Plebanski gravity. In this description of gravity the states will be labelled by the eigenvalues of the antiself-dual part of the Weyl curvature tensor, which as shown in Paper XIII encode the Petrov classification of the corresponding spacetime. Within this description the role of the Kodama state is clear, as is the resolution to the aforementioned questions and the existence of a Hilbert structure for gravity. In this paper we construct wavefunctions annihilated by the Hamiltonian constraint which satisfy the requirements of a Hilbert space of states, restricted to quantizable configurations of the instanton representation. These configurations correspond to spacetimes of Petrov Type I, D and O where the CDJ matrix has three linearly independent eigenvectors.

The organization of this paper is as follows. In section 2 we derive the instanton representation of Plebanski gravity starting from the Ashtekar variables. The rationale is to provide a link from the Ashtekar variables to a new space of solutions which like the Kodama state have a well-defined semiclassical limit while at the same time forming a genuine Hilbert space for gravity. Specifically, we implement the kinematic constraints at the level of the action, leaving three degrees of freedom per point to be constrained by the Hamiltonian constraint upon quantization. Section 3 puts in place the canonical structure, delineating the canonically conjugate variables for

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1 This is also known as the Wheeler–DeWitt equation [1], which has posed difficulties due to the existence of singular operator products.
quantization. Sections 4 and 5 systematically deal with the issues plaguing the Wheeler–Dewitt equation, solving the Hamiltonian constraint first for a discretization of 3-space and then passing to the continuum limit. We first treat the case of vanishing cosmological constant $\Lambda = 0$, demonstrating the regularization-independence of the Hilbert space. Section 6 attempts to extend the results to $\Lambda \neq 0$, extending the Lippmann–Schwinger perturbative approach of quantum mechanics to field theory. It is found that the condition of finiteness of the wavefunction imposes a restriction on the allowable states linked to an expansion in powers of $\Lambda$. Sections 7 and 8 lift this restriction by carrying out the expansion in inverse powers of $\Lambda$. The $\Lambda \neq 0$ states are in three to one correspondence with points in $C_2$, two copies of the complex plane per spatial point. The solution reduces to the Kodama state in the limit of Type O spacetimes, which corresponds to the origin of $C_2$. It is shown in the general case that the states are solutions to a certain hypergeometric differential equation, which is the Hamiltonian constraint in the instanton representation. In section 9 we provide a discussion and conclusion, showing that the Kodama state can be regarded as a time variable on the configuration space of the instanton representation, which the gravitational degrees of freedom stationary with respect to this time.\(^2\) We then provide the proposal for resolving the issue of its normalizability.

\(^2\)The idea of the Chern–Simons functional as a time variable has been first suggested in [6], for which the results of the present paper provide additional support.
2 Transformation into the instanton representation

Our starting point for the transformation into the instanton representation will be from the Ashtekar variables \((A^a_i, \text{\tilde{\sigma}}^i_a)\) where \(A^a_i\) is the left-handed \(SU(2)_-\) Ashtekar connection with densitized triad \(\text{\tilde{\sigma}}^i_a\). The 3 + 1 decomposition of the action for vacuum general relativity in the Ashtekar variables is given by the following totally constrained system [7],[8],[9]

\[
I_{\text{Ash}} = \int_0^T dt \int_{\Sigma} d^3x \text{\tilde{\sigma}}^i_a \dot{A}^a_i - \mathcal{N} H - N^i H_i + A^a_0 G_a, \tag{1}
\]

where \(\mathcal{N}\) is the lapse function with lapse density \(\mathcal{N} = N/\sqrt{\det\sigma}\), and \(N^i\) and \(A^a_0\) are respectively the shift vector and \(SU(2)_-\) rotation angles. Here \(B^i_a\) is the Ashtekar magnetic field given by

\[
B^i_a = \epsilon^{ijk} \partial_j A^k_a + \frac{1}{2} \epsilon^{ijk} f_{abc} A^b_j A^c_k \tag{2}
\]

with structure constants \(f_{abc}\). The Gauss’ law and diffeomorphism constraints, the kinematic constraints, are given by

\[
H_i = \epsilon_{ijk} \text{\tilde{\sigma}}^j_a B^k_a, \quad G_a = D_i \text{\tilde{\sigma}}^i_a \tag{3}
\]

with \(SU(2)_-\) covariant derivative \(D_i = (D_i)^{ab} = \delta^{ab} \partial_i + f^{abc} A^c_i\). Then the Hamiltonian constraint is given by

\[
H = \epsilon_{ijk} \epsilon^{abc} \left( \frac{\Lambda}{3} \text{\tilde{\sigma}}^i_a \text{\tilde{\sigma}}^j_b \text{\tilde{\sigma}}^k_c + \text{\tilde{\sigma}}^i_a B^i_c \right) \tag{4}
\]

where \(\Lambda\) is the cosmological constant.

We will now perform a change of variables using the CDJ Ansatz,

\[
\text{\tilde{\sigma}}^i_a = \Psi_{ae} B^i_e. \tag{5}
\]

Here \(\Psi_{ae}\) is a \(SO(3, C) \times SO(3, C)\)-valued matrix known as the CDJ matrix, named after Capovilla, Dell and Jacobson [10]. The (inverse) CDJ matrix was used as a Lagrange multiplier designed to enforce the equivalence of the nonmetric formulation [10] to Einstein’s general relativity. The instanton

\text{\footnotesize 3By convention, lowercase Latin indices from the beginning of the alphabet \(a, b, c, \ldots\) denote internal \(SU(2)_-\) indices, while those from the middle \(i, j, k, \ldots\) denote spatial indices in three space \(\Sigma\).}
representation of Plebanski gravity treats the eigenvalues of the symmetric part of $\Psi_{ae}$ as dynamical variables which will be quantized. First, we substitute (5) into (1). The kinematic constraints (3) then transform into

$$H_i = \epsilon_{ijk}\tilde{\sigma}^j_i B^k_a = \epsilon_{ijk} B^j_i B^k_e \Psi_{ae}; \quad G_a = D_i \tilde{\sigma}^i_a = B^i_a D_i \Psi_{ae},$$

(6)

where we have used the Bianchi identity $D_i B^i_a = 0$ in the second equation of (6). The Hamiltonian constraint (4) under (5) transforms into

$$H = (\det B)(\Lambda \det \Psi + \frac{1}{2} \text{Var} \Psi),$$

(7)

where we have defined $\text{Var} \Psi = (\text{tr} \Psi)^2 - \text{tr} \Psi^2$. Substitution of (5) into the canonical one form (1) yields

$$\tilde{\sigma}^i_a \dot{A}^a_i = \Psi_{ae} B^i_e \dot{A}^a_i.$$  

(8)

If we could define a variable $X^{ae}$ such that

$$\Psi_{ae} B^i_e \dot{A}^a_i = \dot{X}^{ae},$$

(9)

then the canonical structure of (8) would suggest that $X^{ae}$ is the configuration space coordinate canonically conjugate to $\Psi_{ae}$, seen as a momentum space variable. But $X^{ae}$ in general does not exist since $\delta X^{ae} = B^i_e \delta A^a_i$ is in general not an exact functional one form, a point which we will return to shortly.  

Let us nevertheless proceed to designate $\Psi_{ae}$ as a fully dynamical variable, and no longer part of an Ansatz. Then (5) should rather be read from right to left, wherein $B^i_e$ is freely specifiable with $\tilde{\sigma}^i_a$ derived from $\Psi_{ae}$, which is the fundamental object. Then (1) becomes

$$L_{Inst} = \int_0^T dt \int d^3x \left[ \Psi_{ae} B^i_e \dot{A}^a_i + N^i \epsilon_{ijk} B^j_i B^k_e \Psi_{ae} + A^a_0 \mathbf{w}_e(\Psi_{ae}) - N(\det B)^{1/2} \frac{1}{\sqrt{\det \Psi}} (\frac{1}{2} \text{Var} \Psi + \Lambda \det \Psi) \right],$$

(10)

where $\mathbf{w}_e \equiv B^i_e D_i$ is the Gauss’ law differential operator which acts on the CDJ matrix $\Psi_{ae}$. We define the action (10) as the instanton representation for Plebanski gravity, which implies (1) for nondegenerate $B^i_e$ and nondegenerate $\Psi_{ae}$ upon substitution of (5). Note, as shown in Paper II, that (10)  

\footnote{Note that the trace of $X^{ae}$ does exist, and is the antiderivative of $\tilde{l}_{CS}[A] = B^i_e \dot{A}^a_i$, where $\tilde{l}_{CS}[A]$ is the Chern–Simons Lagrangian for the connection $A^a_i$.}
can also be derived directly from the Plebanski action and is in a sense dual to the Ashtekar action.

We will perform a quantization of the physical degrees of freedom of (10), which entails the implementation first of the kinematic constraints at the level of the action. First, the diffeomorphism constraint

\[ H_i = \epsilon_{ijk} B_j^a B_k^a \Psi_{ae} = 0, \]  

implies that \( \Psi_{[ae]} = 0 \), namely that the CDJ matrix is symmetric \( \Psi_{ae} = \Psi_{(ae)} \). Next, moving on to the Hamiltonian constraint, note that when diagonalizable, \( \Psi_{(ae)} \) can be written as a polar decomposition\(^5\)

\[ \Psi_{(ae)} = (e^{\theta \cdot T})_{af} \lambda_f (e^{-\theta \cdot T})_{fe}, \]  

where \( \vec{\theta} = (\theta^1, \theta^2, \theta^3) \) are three complex angles with \( SO(3) \) generators \( T_a \), and \( \lambda_f = (\lambda_1, \lambda_2, \lambda_3) \) are the eigenvalues, which will serve as momentum space variables. Upon substitution of (12) into (7), the angles \( \vec{\theta} \) drop out on account of the cyclic property of the trace and we obtain

\[ h = (\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1) + \Lambda \lambda_1 \lambda_2 \lambda_3 = 0. \]  

Using (12), the Gauss’ law constraint for \( \Psi_{[ae]} = 0 \) is given by

\[ B_i^a D_i \{ \lambda_f (e^{-\theta \cdot T})_{fa} (e^{-\theta \cdot T})_{fe} \} = 0, \]  

where the gauge covariant derivative \( D_i \) acts in the tensor representation of \( SO(3, C) \). From (13) it is clear that the eigenvalues \( \lambda_f \) should be regarded as the physical degrees of freedom and not the angles \( \vec{\theta} \). Therefore we will regard (14) as a condition which for each triple of \( \lambda_f \) satisfying (13), fixes an equivalence class of \( SO(3, C) \) frames \( \vec{\theta} = \vec{\theta} [\vec{\lambda}; A] \) associated with the set of configurations of the connection \( A_i^a \).\(^6\) Therefore it is appropriate to apply the quantization procedure to (13) rather than (14).

The last point prior to proceeding with quantization regards the canonical structure of the theory, which brings us back the issue of whether a variable \( X^{ae} \) as suggested by (9) actually exists. Since we have designated the three eigenvalues \( \lambda_f = (\lambda_1, \lambda_2, \lambda_3) \) as momentum space D.O.F., then we must identify three configuration space D.O.F. which are canonically conjugate to \( \lambda_f \). Let us choose a configuration where \( A_i^a = \delta_{ai} A_i^a \) is diagonal

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\(^5\)This requires the existence of three linearly independent eigenvectors [11].

\(^6\)The procedure for solving the differential equations (14) is treated in papers VI, VII and VIII and therefore will not be covered here.
\[
A^a_i = \begin{pmatrix}
A_1^1 & 0 & 0 \\
0 & A_2^2 & 0 \\
0 & 0 & A_3^3
\end{pmatrix};\quad
B^i_a = \begin{pmatrix}
A_2^2 & -\partial_3 A_3^3 & \partial_2 A_3^3 \\
\partial_3 A_1^1 & A_3^3 A_1^1 & -\partial_1 A_3^3 \\
-\partial_2 A_1^1 & \partial_1 A_2^2 & A_1^1 A_2^2
\end{pmatrix}
\]

and \(A^\ell_f \neq 0\). For this configuration \(A^a_i\) is nondegenerate as a three by three matrix, and as shown in Paper XIII to be one of the six quantizable configurations in the full theory for the instanton representation. We can now form the velocity \(\dot{X}_a^a\) in (9)

\[
B^i_a \dot{A}^a_i = \begin{pmatrix}
A_2^2 A_3^3 A_1^1 & -(\partial_3 A_3^3) A_2^2 & (\partial_2 A_3^3) A_3^3 \\
(\partial_3 A_1^1) A_2^2 & A_3^3 A_1^1 A_2^2 & -(\partial_1 A_3^3) A_3^3 \\
-(\partial_2 A_1^1) A_2^2 & (\partial_1 A_2^2) A_3^3 & A_1^1 A_2^2 A_3^3
\end{pmatrix}
\]

Upon contraction with a diagonal CDJ matrix \(\Psi_{ae} = \delta_{ae} \lambda_e\) this leads to the canonical one-form

\[
\Psi_{ae} B^i_a \delta A^a_i = \lambda_1 A_2^2 A_3^3 \delta A_1^1 + \lambda_2 A_3^3 A_1^1 \delta A_2^2 + \lambda_3 A_1^1 A_2^2 \delta A_3^3
\]

\[
= (A_1^1 A_2^2 A_3^3) \left[\lambda_1 \left(\frac{\delta A_1^1}{A_1^1}\right) + \lambda_2 \left(\frac{\delta A_2^2}{A_2^2}\right) + \lambda_3 \left(\frac{\delta A_3^3}{A_3^3}\right)\right],
\tag{15}
\]

where \(\det A = A_1^1 A_2^2 A_3^3\). Note in (15) that due to the choice of configuration, all terms containing spatial gradients \(A^a_i\) have vanished from the canonical one form. But nowhere have we imposed any spatial homogeneity on the variables. The variables \(A_f^\ell = A_f^\ell(x)\) constitute three independent dynamical degrees of freedom per point, therefore this is not minisuperspace. It is a configuration of the full theory which shares the advantages of the simplicity of minisuperspace.
3 Canonical structure of the kinematic phase space

From the initial value constraints we have identified three configuration space degrees of freedom corresponding to the three eigenvalues of $\Psi_{(ae)}$. We will now put in place the canonical structure of the kinematic phase space $\Omega_{Kin}$ in preparation for a quantization.\footnote{We define $\Omega_{Kin}$ as the phase space at the level subsequent to implementation of the kinematic constraints (11) and (14), and prior to implementation of the Hamiltonian constraint (13). For notational purposes we denote $\Gamma_{Kin}$ as the configuration space at this level, and $P_{Kin}$ the momentum space.}

The canonical one form $\theta_{Kin}$ is given by

$$\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \Psi_{ae} B_i^a \delta A_i^a$$

where we have taken the CDJ matrix without loss of generality to be already in diagonal form. Equation (16) in present form does not have a globally holonomic coordinate on the kinematic configuration space $\Gamma_{Kin}$. We remedy this by defining densitized momentum variables $\tilde{\Psi}_{ae} = \Psi_{ae} (\det A)$, where $(\det A) \neq 0$. For (16) this is given by

$$\tilde{\Psi}_{11} = \Psi_{11} (A_1 A_2 A_3); \quad \tilde{\Psi}_{22} = \Psi_{22} (A_1 A_2 A_3); \quad \tilde{\Psi}_{33} = \Psi_{33} (A_1 A_2 A_3). \quad (17)$$

In the densitized variables (16), then (17) is given by

$$\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \left( \tilde{\Psi}_{11} \left( \frac{\delta A_1^1}{A_1^1} \right) + \tilde{\Psi}_{22} \left( \frac{\delta A_2^2}{A_2^2} \right) + \tilde{\Psi}_{33} \left( \frac{\delta A_3^3}{A_3^3} \right) \right). \quad (18)$$

Next, rewrite (16) in the form

$$\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \left( (\tilde{\Psi}_{11} - \tilde{\Psi}_{33}) \frac{\delta A_1^1}{A_1^1} + (\tilde{\Psi}_{22} - \tilde{\Psi}_{33}) \frac{\delta A_2^2}{A_2^2} \\
+ \tilde{\Psi}_{33} \left( \frac{\delta A_1^1}{A_1^1} + \frac{\delta A_2^2}{A_2^2} + \frac{\delta A_3^3}{A_3^3} \right) \right) \quad (19)$$

and make the following definitions

$$\tilde{\Psi}_{11} - \tilde{\Psi}_{33} = \Pi_1; \quad \tilde{\Psi}_{22} - \tilde{\Psi}_{33} = \Pi_2; \quad \tilde{\Psi}_{33} = \Pi \quad (20)$$

for the momentum space variables. For the configuration space variables define

$$\Pi_1, \Pi_2, \Pi$$
\[
\delta A_1^1 = \delta X; \quad \delta A_2^2 = \delta Y; \quad \frac{\delta A_1^1}{A_1^1} + \frac{\delta A_2^2}{A_2^2} + \frac{\delta A_3^3}{A_3^3} = \delta T. \tag{21}
\]
Equation (21) provides holonomic coordinates \((X, Y, T) \in \Gamma_{Kin}\), given by
\[
X = \ln\left(\frac{A_1^1}{a_0}\right); \quad Y = \ln\left(\frac{A_2^2}{a_0}\right); \quad T = \ln\left(\frac{A_1^1 A_2^2 A_3^3}{a_3^0}\right), \tag{22}
\]
where \(a_0\) is a numerical constant of mass dimension \([a_0] = 1\). The ranges of the configuration space variables are \(-\infty < (|X|, |Y|, |T|) < \infty\) corresponding to \(0 < |A_f^f| < \infty\), and the mass dimensions of all dynamical variables are
\[
[\Pi_1] = [\Pi_2] = [\Pi] = 1; \quad [X] = [Y] = [T] = 0. \tag{23}
\]
In the densitized momentum space variables (20), the canonical one form \(\theta_{Kin}\) is given by
\[
\theta_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x (\Pi \delta T + \Pi_1 \delta X + \Pi_2 \delta Y). \tag{24}
\]
Equation (24) provides canonical pairs, which upon promotion to quantum operators satisfy the equal time canonical commutation relations
\[
[\hat{T}(x, t), \hat{\Pi}(y, t)] = [\hat{X}(x, t), \hat{\Pi}_1(y, t)] = [\hat{Y}(x, t), \hat{\Pi}_2(y, t)] = (\hbar G) \delta^{(3)}(x, y), \tag{25}
\]
with respect to a quantum state \(\ket{\psi}\) with all other relations vanishing. Note, with the fundamental phase space variables as defined, that (24) implies the symplectic two form
\[
\Omega_{Kin} = \frac{1}{G} \int_{\Sigma} d^3x \left( \delta \Pi_1 \wedge \delta X + \delta \Pi_2 \wedge \delta Y + \delta \Pi \wedge \delta T \right) \tag{26}
\]
such that \(\Omega_{Kin} = \delta \theta_{Kin}\) and \(\delta \Omega_{Kin} = 0\).

We must now express the Hamiltonian constraint in terms of the phase space variables of (24). For a diagonal \(\Psi_{ae}\) the Hamiltonian constraint (13) in the original undensitized variables is given by\(^8\)
\[
H = (\Psi_{11}\Psi_{22} + \Psi_{22}\Psi_{33} + \Psi_{33}\Psi_{11}) + \Lambda \Psi_{11}\Psi_{22}\Psi_{33} = 0. \tag{27}
\]
\(^8\)There is no loss in choosing a diagonal CDJ matrix, since one can simply identify the diagonal elements with the eigenvalues.
Using (17) in (27), we have

\[ (\text{det} A)^{-2} \left( \bar{\Psi}_{11} \bar{\Psi}_{22} + \bar{\Psi}_{22} \bar{\Psi}_{33} + \bar{\Psi}_{33} \bar{\Psi}_{11} + \Lambda (\text{det} A)^{-1} \bar{\Psi}_{11} \bar{\Psi}_{22} \bar{\Psi}_{33} \right) = 0. \]  (28)

Since \((\text{det} A) \neq 0\), then we can omit the pre-factor of \((\text{det} A)^{-2}\). Then upon using (20) and (22) we have the following Hamiltonian constraint

\[ H = \Pi (\Pi + \Pi_1) + (\Pi + \Pi_1)(\Pi + \Pi_2) + (\Pi + \Pi_2)\Pi + \left( \frac{\Lambda}{a_0^3} \right) e^{-T} \Pi (\Pi + \Pi_1)(\Pi + \Pi_2) = 0. \]  (29)

In this paper we will solve the quantum version of (29) both for vanishing and for nonvanishing cosmological constant. For \(\Lambda = 0\), (29) reduces to

\[ H = \Pi^2 + \frac{2}{3} (\Pi_1 + \Pi_2)\Pi + \frac{1}{3} \Pi_1 \Pi_2 = 0, \]  (30)

which is invariant under a rescaling of the momenta. Equation (30) can be seen as the flow of the momentum vector \((\Pi_1, \Pi_2, \Pi)\) along null geodesics of the superspace metric

\[ G_{ae} = \begin{pmatrix} 1 & 1 & 1 \\ \frac{1}{3} & 0 & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} & 0 \end{pmatrix}. \]

The characteristic equation

\[ \det (G_{ae} - r\delta_{ae}) = r^3 - 3r^3 + \frac{3}{4} r + \frac{3}{4} = 0 \]  (31)

has one negative and two positive roots, which suggests that there exists a \(SO(3, C)\) frame in which one of the momenta is timelike with the other two momenta spacelike with respect to this metric.
4 Hilbert space on a discretization of 3-space

Define by $\Delta_N(\Sigma)$, a discretization of 3-space $\Sigma$ into a lattice of $N$ points, where $\nu$ is the size of an elementary cell given by

$$\nu = \frac{L^3}{N}.$$  

(32)

where $[\nu] = -3$ and where $L$ is the characteristic linear dimension of $\Sigma$. Define at each $x \in \Delta_N(\Sigma)$ a kinematic Hilbert space $H_{Kin}(x)$ of entire analytic functions $f(X_x)$ in the holomorphic representation, based upon the resolution of the identity

$$I = \int \delta \mu_x |T_x, X_x, Y_x\rangle\langle T_x, X_x, Y_x|,$$  

(33)

where the measure $\delta \mu_x$ at point $x$ is given by

$$\delta \mu_x = \prod_x \delta X_x \delta X_x \delta Y_x \delta Y_x \exp \left[ -\left( |X_x|^2 + |Y_x|^2 \right) \right].$$  

(34)

Then $f$ belongs to $H_{Kin}$ if it is square integrable with respect to the measure (34). The inner product of two functionals $f[X_x]$ and $f'[X_x]$ is given by

$$\langle f x | f'_x \rangle = \int \Gamma f(X_x)f'(X_x)\delta \mu_x.$$  

(35)

Using (33), an arbitrary state $|\psi_x\rangle$ can be expanded in the basis states $\langle X_x, Y_x, T_x |$ to produce a wavefunctional $\psi_x \equiv \psi[X_x, Y_x, T_x] = \langle X_x, Y_x, T_x | \psi_x \rangle$. The configuration and momentum space operators act respectively on $\psi$ by multiplication

$$\hat{T}_x \psi(X_y) = \delta_{xy} T_x \psi(X_y); \quad \hat{X}_x \psi(X_y) = \delta_{xy} X_x \psi; \quad \hat{Y}_x \psi(X_y) = \delta_{xy} Y_x \psi(X_y).$$  

(36)

and by differentiation

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9 This is not to say the physical volume but rather, a region of space whose characteristic linear dimension is of size $\nu^{1/3}$. Hence $\nu$ is a numerical constant, while the physical volume is a dynamical variable which may be promoted to an operator upon quantization.

10 We will use the subscript $x$ to signify that the quantity in question is defined with respect to the elementary cell containing the point $x$. So we may view the cell as one copy of a minisuperspace, where all $x$ in the cell are equivalent.

11 Note that $\langle \psi_x | \psi_y \rangle = \delta_{xy} |\psi_x|^2 \forall x, y \in \Delta_x(\Sigma)$, where $\delta_{xy}$ is the Kronecker delta of $x$ and $y$ (not the Dirac Delta). This signifies that the Hilbert spaces at each separate point are independent of each other.

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\[ \hat{\Pi}_x \psi(X_y) = \delta_{xy} (hG)^{-1} \frac{\partial}{\partial T_x} \psi(X_y); \]
\[ (\hat{\Pi}_1)_x \psi(X_y) = \delta_{xy} (hG)^{-1} \frac{\partial}{\partial X_x} \psi(X_y); \quad (\hat{\Pi}_2)_x \psi(X_y) = \delta_{xy} (hG)^{-1} \frac{\delta}{\delta Y_x} \psi(X_y). \] (37)

Starting from a set of states
\[ |\lambda, \alpha, \beta\rangle_x = |\alpha_x\rangle \otimes |\beta_x\rangle \otimes |\lambda_x\rangle \] (38)
construct a family of plane wave-type states in the holomorphic representation of the kinematic configuration space at \( x \) \( (\Gamma_{Kin})_x \) using
\[ \langle X, Y, T | \lambda_x, \alpha_x, \beta_x \rangle = N(\alpha_x, \beta_x) e^{\nu(hG)^{-1}(\alpha_x X + \beta_x Y + \gamma_x T)}, \] (39)
where \( N(\alpha_x, \beta_x) \) is a normalization constant which depends on \( \alpha_x \) and \( \beta_x \).

The states (39) are eigenstates of the momentum operators
\[ \hat{\Pi}_x |\lambda_x\rangle = \lambda_x |\lambda_x\rangle; \quad (\hat{\Pi}_1)_x |\alpha_x\rangle = \alpha_x |\alpha_x\rangle; \quad (\hat{\Pi}_2)_x |\beta_x\rangle = \beta_x |\beta_x\rangle. \] (40)

Upon quantization the Hamiltonian constraint (30) for \( \Lambda = 0 \) at the point \( x \) becomes promoted to an operator \( \hat{H}_x \), given by
\[ \hat{H}_x = (\hat{\Pi} \hat{\Pi} + \frac{2}{3} (\hat{\Pi}_1 + \hat{\Pi}_2) \hat{\Pi} + \frac{1}{3} \hat{\Pi}_1 \hat{\Pi}_2)_x. \] (41)

Note that the states (39) are also eigenstates of \( \hat{H} \), with eigenvalue
\[ \nu^{-2} (hG)^2 \left[ \frac{\partial^2}{\partial T_x^2} + \frac{2}{3} \left( \frac{\partial}{\partial X_x} + \frac{1}{3} \frac{\partial}{\partial T_x} + \frac{\partial^2}{\partial X_x \partial Y_x} \right) \frac{\partial}{\partial T_x} + \frac{\partial^2}{\partial X_x \partial Y_x} \right] \psi_x \]
\[ = \left( \lambda_x^2 + \frac{2}{3} (\alpha_x + \beta_x) \lambda_x + \frac{1}{3} \alpha_x \beta_x \right) \psi_x. \] (42)

We now search for states \( \psi_x \in Ker\{\hat{H}_x\} \) solving the Hamiltonian constraint, which requires that (42) vanish. This leads to the dispersion relation
\[ \lambda \equiv \gamma_x^\pm = \frac{1}{3} (\alpha_x + \beta_x \pm \sqrt{\alpha_x^2 - \alpha_x \beta_x + \beta_x^2}) \quad \forall x \in \Delta_N(\Sigma). \] (43)

Defining \( \lambda_x \equiv (\lambda_{\alpha, \beta})_x \), the wavefunctions \( \psi_x \in Ker\{\hat{H}_x\} \) are given by
\[ |\lambda_{\alpha, \beta}, \alpha, \beta\rangle_x = N(\alpha, \beta) e^{\nu(hG)^{-1}(\alpha_x X + \beta_x Y + \gamma_x^\pm T)}, \] (44)
which are labelled by two free parameters $\alpha_x$ and $\beta_x$ per point $x$.

The measure (34) guarantees square integrability of the wavefunctions. Hence for the norm we have that

$$
\left| \langle \lambda_{\alpha,\beta}, \alpha, \beta \rangle_x \right|^2 = |N|^2 \int \delta \mu_x e^{i\nu(hG)^{-1}(\alpha_x X_x + \beta_x Y_x + \lambda_x T_x)} e^{i\nu(hG)^{-1}(\alpha_x X_x + \beta_x Y_x + \lambda_x T_x)} = |N|^2 e^{\nu^2(hG)^{-2}(|\alpha_x|^2 + |\beta_x|^2)} \exp \left[ 2\nu(hG)^{-1}Re\{\lambda_x T_x\} \right] = 1. \tag{45}
$$

We have not performed an integration over the variable $T_x$ since we will use $T_x$ as a clock variable on configuration space $(\Gamma_{Kin})_x$. For $\Lambda = 0$ the state at $x$ is labelled by two arbitrary complex numbers $\alpha_x$ and $\beta_x$. The normalization factor is given by

$$
N \equiv N(\alpha, \beta) = \exp \left[ \nu(hG)^{-1}Im\{\lambda_x T_x\} \right] \exp \left[ \frac{1}{2} \nu^2(hG)^{-2} \left( |\alpha_x|^2 + |\beta_x|^2 \right) \right], \tag{46}
$$

which leads to the following normalized state

$$
|\lambda_{\alpha,\beta}, \alpha, \beta \rangle = e^{i\nu(hG)^{-1}Im\{\lambda_x T_x\}} e^{-\frac{1}{2} \nu^2(hG)^{-2} \left( |\alpha_x|^2 + |\beta_x|^2 \right)} e^{(hG)^{-1}(\alpha_x X_x + \beta_x Y_x)}. \tag{47}
$$

Note that the dependence of the normalized state (47) on the variable $T_x$ designated as the configuration space time variable, is just a phase factor. Hence the overlap of two normalized $\Lambda = 0$ states is given by

$$
\left| \langle \lambda_{\alpha,\beta}, \alpha, \beta | \lambda_{\zeta,\sigma}, \zeta, \sigma \rangle_x \right|^2 = e^{-\nu^2(hG)^{-2}|\alpha_x - \zeta_x|^2} e^{-\nu^2(hG)^{-2}|\beta_x - \sigma_x|^2}, \tag{48}
$$

whence the phase factor cancels out leaving an overlap characterized completely by the degrees of freedom excluding $T_x$. Hence for each pair of complex numbers $\alpha_x$ and $\beta_x$, there are two states corresponding to $\Lambda = 0$. The labels $(\alpha_x, \beta_x)$ define a point on $C_2$, a two dimensional complex Euclidean manifold, for which these states are in two to one correspondence. If one uses the flat metric to measure distance on $C_2$ as in

$$
d(\alpha_x, \beta_x; \zeta_x, \sigma_x) = |\alpha_x - \zeta_x|^2 + |\beta_x - \sigma_x|^2, \tag{49}
$$

then it is clear that there is always a nontrivial overlap between any two states is of the form $e^{-d}$. Lastly, note that probability density is conserved for all normalized states in the sense that

\[12\]This is motivated by the observation in [12] and [6] that the Kodama state $\psi_{Kod}$ can play the role of a time variable on configuration space $\Gamma$. If this were to be the case, then an integration over $T$ would be tantamount to normalization of a wavefunction in time.
since all dependence on $T_x$ has dropped out.

4.1 The continuum limit

We have constructed a Hilbert space $H_x$ of states satisfying the Hamiltonian constraint for $\Lambda = 0$ at one point $x$, with a quantum mechanical interpretation, such that $\text{Supp}(H_x) = x \forall x \in \Delta_N(\Sigma)$. We have also circumvented the necessity to perform a regularization of $H_x$, since the functional space is finite dimensional for each $N < \infty$. The un-normalized solution at $x$ is

$$\psi_x = e^{\nu(hG)^{-1}\alpha_x X_x} e^{\nu(hG)^{-1}\beta_x Y_x} e^{\nu(hG)^{-1}\gamma_x T_x}. \quad (51)$$

But we would like for our wavefunctions to have support on all of 3-space $\Sigma$ in the continuum limit $\lim_{N \to \infty} \Delta_N(\Sigma)$. We will pass to the continuum limit in two stages. First, we will associate a wavefunctional $\Psi = \Psi(\Delta_N(\Sigma))$ to the full discretization $\Delta_N$ by taking the direct product of $N$ copies of (51) over the entire lattice

$$\Psi(\Delta_N(\Sigma)) = \bigotimes_x \psi_x = \prod_{k=1}^N \psi(x_k). \quad (52)$$

Then we pass to the continuum limit by taking the limit as $\nu$ approaches zero and as $N$ approaches infinity in (32). In the continuum limit, the direct product of the wavefunctions should become a wavefunctional. For example we have

$$\prod_x e^{\nu(hG)^{-1}\alpha_x X_x} = \exp \left[ (hG)^{-1} \sum_{k=1}^N \nu \alpha(x_k) X(x_k) \right], \quad (53)$$

which in the continuum limit becomes

$$\lim_{N \to \infty} \prod_x e^{\nu(hG)^{-1}\alpha_x X_x} = \exp \left[ (hG)^{-1} \int_{\Sigma} d^3 x \alpha(x) X(x) \right]. \quad (54)$$

We see that in the continuum limit, the argument of the exponential approaches a Riemannian integral.

We must also understand the manner in which partial derivatives in the functional space at $x$ become promoted to functional derivatives in the continuum limit. For the functional space at $x$ we have
\[ \nu^{-1} \frac{\partial}{\partial T_x} F(T_y) = \delta_{xy} F'(T) \longrightarrow \frac{\delta}{\delta T(x)} F(T(y)) = \delta(x, y) F'(T) \] (55)

where \( F'(T) = \partial F / \partial T \). Observe that the inverse size of the elementary cell of the discretization enters as part of the definition of the derivative. Though \( T \) is dimensionless, this implies that the functional derivative with respect to \( T \) is of mass dimension 3, hence the factor of \( \nu^{-1} \) in the discretization of this functional derivative. This is consistent with the definition of the delta function of the continuum limit since

\[ \int_{\Sigma} d^3 x \delta(x, y) = 1, \] (56)

implying that the mass dimension of the three dimensional Dirac delta function is \([\delta^{(3)}(x, y)] = 3\). Hence, the adaptation of the definition of the functional derivative in terms of its action on (52) is given by

\[ (hG) \frac{\delta}{\delta T(x)} \Psi[T] \longrightarrow (hG) \nu^{-1} \frac{\partial}{\partial T_x} \Psi = \lambda_x \Psi. \] (57)

### 4.2 The volume operator

To construct the volume operator we will need \( \sqrt{h} = \sqrt{\det h_{ij}} \), where \( h_{ij} \) is the 3-metric of \( \Sigma \). This is given in the original Ashtekar variables by

\[ hh^{ij} = \tilde{\sigma}^i \tilde{\sigma}^j. \] (58)

The determinant of (58), upon use of the CDJ Ansatz (5), implies that

\[ \sqrt{h} = \sqrt{\det B} \sqrt{\det \Psi}. \] (59)

In terms of the variables we have quantized this is given by

\[ \sqrt{h} = (\det B)^{1/2} (\det A)^{-3/2} \sqrt{\Psi_{11} \Psi_{22} \Psi_{33}}, \] (60)

and the volume of 3-space \( \Sigma \) can be computed from

\[ V = Vol(\Sigma) = \int_{\Sigma} d^3 x \sqrt{h} = \int_{\Sigma} d^3 x (\det B)^{1/2} (\det A)^{-3/2} \sqrt{\Pi(\Pi + \Pi_1)(\Pi + \Pi_2)}. \] (61)
We will now quantize (61), which will enable us to calculate the volume corresponding to our quantum states. For an ordering of the momenta to the right of the coordinates, defining \( \det B \equiv (U \det A)^2 \) for some \( U \), and using \( (\det A) = a_0^3 e^T \), the quantum version of (61) is given by

\[
\hat{V} = a_0^{-3/2} \int_\Sigma d^3x U e^{-T/2} \sqrt{\Pi_1(\Pi + \Pi_1)(\Pi + \Pi_2)}.
\]  
(62)

The volume operator (62) has the following action on the states (70)

\[
\hat{V} |\lambda, \alpha, \beta\rangle = a_0^{-3/2} \left( \int_\Sigma d^3x U e^{-T/2} \sqrt{\lambda(\lambda + \alpha)(\lambda + \beta)} \right) |\lambda, \alpha, \beta\rangle.
\]  
(63)

For solutions to the Hamiltonian constraint one substitutes \( \lambda_{\alpha, \beta} \) for \( \lambda \) in (63).
5 Continuum Hilbert space structure for $\Lambda = 0$

We will now proceed to construct a quantum theory and Hilbert space corresponding to $\Lambda = 0$, by extending the previous steps to the infinite dimensional spaces of field theory. Since the dynamical variables are complex, we apply the construction of [13] to infinite dimensional spaces. Define a kinematic Hilbert space $H_{Kin}$ of entire analytic functionals $f[X]$ in the holomorphic representation, based upon the resolution of the identity

$$I = \int D\mu[T,X,Y]\langle T,X,Y|,$$  \hspace{1cm} (64)

where the measure $D\mu$ is given by

$$D\mu = \prod_x \delta X \delta X \delta Y \delta Y \exp\left[-\nu'^{-1} \int_{\Sigma} d^3x \left(|X(x)|^2 + |Y(x)|^2\right)\right],$$ \hspace{1cm} (65)

with $\nu'$ a numerical constant of mass dimension $[\nu'] = -3$ necessary to make the argument of the exponential dimensionless. Then $f$ belongs to $H_{Kin}$ if it is square integrable with respect to the measure (65). The inner product of two functionals $f[X]$ and $f'[X]$ is given by

$$\langle f|f'\rangle = \int_{\Gamma} f[X] f'[X] D\mu$$ \hspace{1cm} (66)

which is an infinite product of integrals in the functional space $\Gamma$, one integral for each spatial point $x \in \Sigma$.

Using (64), an arbitrary state $|\psi\rangle$ can be expanded in the basis states $\langle X,Y,T|$ to produce a wavefunctional $\psi \equiv \psi[X,Y,T] = \langle X,Y,T|\psi\rangle$. The configuration and momentum space operators act respectively on $\psi$ by multiplication

$$\hat{T}(x)\psi[X] = T(x)\psi[X]; \hspace{0.5cm} \hat{X}(x)\psi[X] = X(x)\psi; \hspace{0.5cm} \hat{Y}(x)\psi[X] = Y(x)\psi[X],$$ \hspace{1cm} (67)

and by functional differentiation

$$\hat{\Pi}(x)\psi[X] = (\hbar G) \frac{\delta}{\delta T(x)} \psi[X]; \hspace{0.5cm} \hat{\Pi}_1(x)\psi[X] = (\hbar G) \frac{\delta}{\delta X(x)} \psi[X]; \hspace{0.5cm} \hat{\Pi}_2(x)\psi[X] = (\hbar G) \frac{\delta}{\delta Y(x)} \psi[X].$$ \hspace{1cm} (68)

Starting from a set of states
construct a family of plane wave-type states in the holomorphic representation of $\Gamma_{K_{in}}$ using

$$\langle X, Y, T | \lambda, \alpha, \beta \rangle = N(\alpha, \beta) e^{(\hbar G)^{-1}(\alpha \cdot X + \beta \cdot Y + \lambda \cdot T)}, \quad \text{(70)}$$

where $N(\alpha, \beta)$ is a normalization constant which depends on $\alpha$ and $\beta$. The dot in (70) signifies a Riemannian integration over 3-space $\Sigma$, as in

$$\alpha \cdot X = \lim_{\nu \to 0; N \to \infty} \sum_{n=1}^{N} \nu \alpha(x_n) X(x) = \int_{\Sigma} d^3 x \alpha(x) X(x), \quad \text{(71)}$$

where $\nu$ is the volume of an elementary cell in the discretization $\Delta_N(\Sigma)$. In (70) $\alpha$, $\beta$ and $\lambda$ are at this stage time independent arbitrary functions of position, with no functional dependence on $(X, Y, T)$. The states (70) are eigenstates of the momentum operators

$$\Pi(x) | \lambda \rangle = \lambda(x) | \lambda \rangle; \quad \Pi_1(x) | \alpha \rangle = \alpha(x) | \alpha \rangle; \quad \Pi_2(x) | \beta \rangle = \beta(x) | \beta \rangle. \quad \text{(72)}$$

Upon quantization the Hamiltonian constraint becomes promoted to an operator $\hat{H}$, given by

$$\hat{H} = \hat{\Pi} \hat{\Pi} + \frac{2}{3} (\hat{\Pi}_1 + \hat{\Pi}_2) \hat{\Pi} + \frac{1}{3} \hat{\Pi}_1 \hat{\Pi}_2. \quad \text{(73)}$$

Note that the states (70) are also eigenstates of $\hat{H}$, with eigenvalue

$$(\hbar G)^2 \left[ \frac{\delta^2}{\delta T(x) \delta T(x)} + \frac{2}{3} \left( \frac{\delta}{\delta X(x)} + \frac{1}{3} \frac{\delta}{\delta Y(x)} \right) \frac{\delta}{\delta T(x)} + \frac{\delta^2}{\delta X(x) \delta Y(x)} \right] \psi = (\lambda^2 + \frac{2}{3}(\alpha + \beta) \lambda + \frac{1}{3} \alpha \beta) \psi = (\lambda + \gamma^-)(\lambda + \gamma^+) \psi. \quad \text{(74)}$$

Note that the action of the quantum Hamiltonian constraint $\psi$ is free of ultraviolet singularities in spite of the multiple functional derivatives acting at the same point, since the momentum labels $(\alpha, \beta, \lambda)$ are functionally independent of the configuration variables $(X, Y, T)$. Therefore a regularization of (74) is not necessary. However, we will perform a regularization in order to make the link to the discretization formalism presented earlier.

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\textsuperscript{13}Hence, $[\alpha] = [\beta] = [\gamma] = 1$ so that the exponential is dimensionless on account of the volume factor from integration over $\Sigma$, which is of mass dimension $-3$. 

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5.1 Regularization of the Hamiltonian constraint

Let us now examine the effect of using a regularization of the Hamiltonian constraint. Let us start from a wavefunction of the form

$$\Psi_{\alpha,\beta} = e^{(\hbar G)^{-1}\alpha \cdot X} e^{(\hbar G)^{-1}\beta \cdot Y} \psi[T],$$  \hspace{1cm} (75)

where the part dependent on $T$ is given by the following semiclassical Ansatz

$$\psi[T] = \exp \left[ (\hbar G)^{-1} \int_{\Sigma} d^3 x I(T) \right],$$  \hspace{1cm} (76)

where for each $x \in \Sigma$,

$$I(T(x)) = \int_{\Gamma} \lambda(T(x)) \delta T(x).$$  \hspace{1cm} (77)

Equation (77) is the antiderivative (in the functional sense) of the exact one form $\lambda \delta T \in \bigwedge^1 (\Gamma_{Kin})$, which is defined at each spatial point. The Hamiltonian constraint is given by

$$\hat{H} \psi = \left( (\hbar G) \frac{\delta}{\delta T(x)} + \gamma^-(x) \right) \left( (\hbar G) \frac{\delta}{\delta T(x)} + \gamma^+(x) \right) \psi = 0,$$  \hspace{1cm} (78)

where we have made the definition

$$\gamma^+ = -\frac{1}{3} \left( \alpha + \beta + \sqrt{\alpha^2 - \alpha \beta + \beta^2} \right);$$

$$\gamma^- = -\frac{1}{3} \left( \alpha + \beta - \sqrt{\alpha^2 - \alpha \beta + \beta^2} \right)$$  \hspace{1cm} (79)

so that the dispersion relation is given by $(\lambda + \gamma^-)(\lambda + \gamma^+) = 0$. To deal with the double functional derivatives at the same point in (78) let us introduce a regulating function $f_\epsilon(x, y)$, such that

$$\int_{\Sigma} d^3 x f_\epsilon(x, y) \phi(y) = \phi(x)$$  \hspace{1cm} (80)

for all $\phi(x) \in C^\infty(\Sigma)$, where $\epsilon$ is a continuous parameter. Next, perform a point splitting regularization of (78) in accordance with [14] and [15], which requires that the factors appearing in an operator product be smeared individually with smearing functions. Hence the regularized Hamiltonian constraint is given by\(^{14}\)

\(^{14}\)Note, since there are only two functional derivatives, that it is necessary only to smear one factor in the operator product.
\[ \hat{H}_\epsilon(x)\psi = \int_\Sigma d^3y f_\epsilon(x, y) \left( (hG) \frac{\delta}{\delta T(y)} + \gamma^-(y) \right) \left( (hG) \frac{\delta}{\delta T(x)} + \gamma^+(x) \right) \psi. \quad (81) \]

Using (76) and (77), equation (81) is given by

\[ \hat{H}_\epsilon(x)\psi = \int_\Sigma d^3y f_\epsilon(x, y) \left( (hG) \frac{\delta}{\delta T(y)} + \gamma^-(y) \right) \left( (hG) \frac{\delta}{\delta T(x)} + \gamma^+(x) \right) \psi \]

\[ = \int_\Sigma d^3y f_\epsilon(x, y) \left[ (\lambda(T(x)) + \gamma^+(x))(\lambda(T(y)) + \gamma^-(y)) + (hG) \left( \frac{\partial \lambda(T)}{\partial T} \right)_x \delta^{(3)}(y, x) \right] \psi. \quad (82) \]

Performing the integration over the delta function, we obtain

\[ \hat{H}_\epsilon(x)\psi = \left[ (\lambda(T(x)) + \gamma^+(x))(\lambda_\epsilon(T(x)) + \gamma^-(x)) + (hG f_\epsilon(0)) \left( \frac{\partial \lambda(T)}{\partial T} \right)_x \right] \psi = 0, \quad (83) \]

where we have defined \( f_\epsilon(0) = f_\epsilon(x, x) \).

We must now remove the regulator by taking the limit \( \epsilon \to 0 \). Application of (80) to the semiclassical term of (83), namely the term of zeroth order in \( hG \), yields

\[ \lim_{\epsilon \to 0} (\lambda(T(x)) + \gamma^+(x))(\lambda_\epsilon(T(x)) + \gamma^-(x)) = (\lambda(T) + \gamma^+(x))(\lambda(T) + \gamma^-(x)) \quad (84) \]

for each \( x \in \Sigma \), which is finite. However, application of (80) to the term of order \( hG \) leads to a \( f_\epsilon(0) \) singularity which blows up as \( \epsilon \to 0 \). A necessary condition for the Hamiltonian constraint to be satisfied with no singularities is that the coefficient of this singularity be zero, namely that \( \partial \lambda / \partial T = 0 \), which implies that \( \lambda \) is functionally independent of \( T \). Additionally, the semiclassical term of (83) must be required to vanish which imposes the condition \( \lambda + \gamma_\pm = 0 \). This is simply the condition that the Hamiltonian constraint be satisfied at the classical level.

### 5.2 Construction of the solution space

We now search for states \( \psi \in Ker\{\hat{H}\} \) solving the constraints, which requires that (83) vanish in the limit of removal of the regulator. This leads to the dispersion relation

\[ \lambda \equiv \lambda_{\alpha, \beta} = -\frac{1}{3} \left( \alpha + \beta \pm \sqrt{\alpha^2 - \alpha \beta + \beta^2} \right) \forall x. \quad (85) \]

The wavefunctions \( \psi \in Ker\{\hat{H}\} \) are given by
\[ \left| \lambda_{\alpha, \beta}, \alpha, \beta \right\rangle = N(\alpha, \beta)e^{(\hbar G)^{-1}(\alpha \cdot X + \beta \cdot Y + \lambda_{\alpha, \beta} \cdot T)}, \]  

(86)

which are labelled by two free functions of position \( \alpha \) and \( \beta \), which are directly related to the densitized eigenvalues of \( \Psi_{(ae)} \). Additionally, there is a choice of two Hilbert spaces, corresponding to either of the two roots (85).

Since the variables are complex, as is the case generally for a spacetime of Lorentzian signature, we require a Gaussian measure in order to have square integrable wavefunctions. Hence for the norm we have that

\[
\left| \lambda_{\alpha, \beta}, \alpha, \beta \right\rangle \left\langle \lambda_{\alpha, \beta}, \alpha, \beta \right| = |N|^2 \int D\mu(X, Y) e^{(\hbar G)^{-1}(\alpha^* \cdot X + \beta^* \cdot Y + \lambda^* T)} e^{(\hbar G)^{-1}(\alpha \cdot X + \beta \cdot Y + \lambda T)} = |N|^2 e^{\nu'(\hbar G)^{-2}(|\alpha^2| + |\beta^2|)} \exp \left[ 2(\hbar G)^{-1} \int_\Sigma d^3 x \text{Re}\{\lambda T\} \right] = 1. \tag{87}
\]

(87)

In direct analogy to the discretized version, we have not performed an integration over the variable \( T \) since we will use \( T \) as a clock variable on configuration space \( \Gamma_{\text{Kin}} \). For \( \Lambda = 0 \) the state is labelled by two arbitrary functions \( (\alpha(x), \beta(x)) \in C^0(\Sigma) \), and the normalization factor is given by

\[ N \equiv N(\alpha, \beta) = \exp \left[ (\hbar G)^{-1} \int_\Sigma d^3 x \text{Re}\{\lambda T\} \right] \exp \left[ -\nu'(\hbar G)^{-2} \int_\Sigma d^3 x \left( |\alpha|^2 + |\beta|^2 \right) \right]. \tag{88}
\]

The overlap of two normalized \( \Lambda = 0 \) states is given by

\[
\left| \langle \lambda_{\alpha, \beta}, \alpha, \beta | \lambda_{\zeta, \sigma}, \zeta, \sigma \right\rangle \right|^2 = e^{-\nu'(\hbar G)^{-2}|\alpha - \zeta|^2} e^{-\nu'(\hbar G)^{-2}|\beta - \sigma|^2}, \tag{89}
\]

whence the \( \lambda_{\alpha, \beta} \) part of the label becomes superfluous. For \( \Lambda = 0 \) there is a two to one correspondence between states and points in \( C_2 \otimes C_2 \otimes C_2 \ldots \), one copy of \( C_2 \) per point \( x \in \Sigma \), and the overlap is of the form \( e^{-d} \), where \( d \) is given by

\[ d(\alpha, \beta; \zeta, \sigma) = \int_\Sigma d^3 x \left( |\alpha(x) - \zeta(x)|^2 + |\beta(x) - \sigma(x)|^2 \right). \tag{90}
\]

In analogy to (50), probability density in conserved also in the continuum limit \( \forall x \), since (89) is functionally independent of \( T \).

Hence the transition from the discrete into the continuum can be described as follows. Starting from a discretization \( \Delta N(\Sigma) \) of 3-space \( \Sigma \), construct \( \Psi(\Delta N(\Sigma)) \in \text{Ker}\{\hat{H}\} \) as in (52). Note that this forms a Cauchy sequence as \( N \) increases, such that

\[ \lim_{N \to \infty} \Psi(\Delta N(\Sigma)) = \Psi(\Delta_\infty(\Sigma)) \in \text{Ker}\{\hat{H}\}. \tag{91} \]
The result is that for vanishing cosmological constant, the solution for the continuum limit is an element of the same Hilbert space of any discretization satisfying the Hamiltonian constraint. Therefore for $\Lambda = 0$ the Hilbert space of solutions $H$ is in this sense Cauchy complete.
6 Incorporation of a nonzero cosmological constant

Having constructed a complete Hilbert space of normalizable states for $\Lambda = 0$, we will now generalize the construction to incorporate a non-vanishing $\Lambda$. The effect of a nonzero $\Lambda$ will be to introduce a length scale $l \sim \sqrt{\frac{1}{\Lambda}}$ into the theory, which destroys the invariance of the Hamiltonian constraint under rescaling of momenta enjoyed in the $\Lambda = 0$ case. The Hamiltonian constraint at the classical level for $\Lambda \neq 0$ is given by

$$O = -re^{-T}Q,$$  \hspace{1cm} (92)

where we have defined the numerically constant length scale $r$, given by

$$r = \left( \frac{\Lambda}{3a_0^3} \right)$$ \hspace{1cm} (93)

and we have defined

$$O = \Pi^2 + \frac{2}{3}(\Pi_1 + \Pi_2)\Pi + \frac{1}{3}\Pi_1\Pi_2 \equiv \Pi_-\Pi_+; \quad Q = \Pi(\Pi + \Pi_1)(\Pi + \Pi_2).$$ \hspace{1cm} (94)

We will now utilize the previous construction for quantization, whereupon (94) becomes promoted to the quantum operators

$$\hat{O} = \hat{\Pi}_-\hat{\Pi}_+; \quad \hat{Q} = \hat{\Pi}(\hat{\Pi} + \hat{\Pi}_1)(\hat{\Pi} + \hat{\Pi}_2).$$ \hspace{1cm} (95)

The operators $\hat{O}$ and $\hat{Q}$ in (95) have the following action on the states (70)

$$\hat{O}|\lambda,\alpha,\beta\rangle = (\lambda + \gamma^-)(\lambda + \gamma^+)|\lambda,\alpha,\beta\rangle;$$

$$\hat{Q}|\lambda,\alpha,\beta\rangle = \lambda(\lambda + \alpha)(\lambda + \beta)|\lambda,\alpha,\beta\rangle,$$ \hspace{1cm} (96)

with $\gamma^\pm$ as given in (79). We will now quantize the Hamiltonian constraint (92) for an operator ordering with $e^{-T}$ sandwiched between $O$ and $Q$ for illustrative purposes. The quantum Hamiltonian constraint is given by

$$\hat{O}|\psi\rangle = -re^{-T}\hat{Q}|\psi\rangle.$$ \hspace{1cm} (97)

Recall from the previous section that $\psi \in Ker\{\hat{O}\}$ solve the Hamiltonian constraint for $\Lambda = 0$. These states are given by

$$|\lambda_\pm\rangle_{\alpha,\beta,\alpha,\beta} = e^{(hG)^{-1}\alpha \cdot X}e^{(hG)^{-1}\beta \cdot Y}e^{(hG)^{-1}\gamma_+ \cdot T};$$

$$|\lambda_-\rangle_{\alpha,\beta,\alpha,\beta} = e^{(hG)^{-1}\alpha \cdot X}e^{(hG)^{-1}\beta \cdot Y}e^{(hG)^{-1}\gamma_- \cdot T},$$ \hspace{1cm} (98)
with $\gamma_-$ and $\gamma_+$ as in (79). We will solve (97) by expansion about the states (98). First, assuming that $\hat{O}$ is invertible, we act on both sides of (97) with $\hat{O}^{-1}$ to obtain

$$|\psi\rangle = |\lambda_{\alpha,\beta}, \alpha, \beta\rangle - r\hat{O}^{-1}e^{-T\hat{Q}}|\psi\rangle.$$  \hspace{1cm} (99)

Then we re-arrange (99) into the form

$$(1 + r\hat{O}^{-1}e^{-T\hat{Q}})|\psi\rangle = |\lambda_{\alpha,\beta}, \alpha, \beta\rangle,$$  \hspace{1cm} (100)

where $|\lambda_{\alpha,\beta}\rangle \in Ker\{\hat{O}\}$ are elements of the Hilbert space corresponding to $\Lambda = 0$. From (100) we can now perform the inversion

$$|\psi\rangle = \left(\frac{1}{1 + r\hat{O}^{-1}e^{-T\hat{Q}}}\right)|\lambda_{\alpha,\beta}\rangle \equiv \left(\frac{1}{1 + \hat{q}}\right)|\lambda_{\alpha,\beta}, \alpha, \beta\rangle.$$  \hspace{1cm} (101)

Equation (101) on the surface appears formal, but it will be justified by the fact that the operator $\hat{q}$ has a well-defined action on the $\Lambda = 0$ Hilbert space. We will in fact use the following operator expansion in powers of $r$

$$(1 + \hat{q})^{-1} = \sum_{n=1}^{\infty}(-r)^n(\hat{O}^{-1}e^{-T\hat{Q}})^n,$$  \hspace{1cm} (102)

to solve the constraint. Note that the zeroth order term of (101) is simply given by $|\lambda_{\alpha,\beta}, \alpha, \beta\rangle$. This approach bears an analogy to the Lippman–Schwinger method of quantum mechanics applied to perturbation theory, where $\hat{O}$ plays the role of a kinetic operator with propagator $\hat{O}^{-1}$, and $\hat{Q}$ plays the role of an interaction term.

### 6.1 Action of the constituent operators

We will encounter an issue commonly encountered in the continuum limit of quantum field theory, namely that quantum operators acting at the same spatial point produce ultraviolet singularities. The action of the conjugate momentum $\Pi(x)$ on the state $|\lambda\rangle$ is finite without regularization, since

\footnote{Since the Hamiltonian constraint must be satisfied point by point, we apply this method independently at each point $x \in \Sigma$, and then to reconstruct the full wavefunction we take the direct product of the Hilbert spaces at each point. A regularization can be adopted in which the volume of an elementary cell of a lattice is given by $\nu$. We should obtain the continuum limit by taking $\nu \to 0$.}
\[ \hat{\Pi}(x)\{e^{(hG)^{-1}\lambda \cdot T}\} = (hG) \frac{\delta}{\delta T(x)} \exp\left[(hG)^{-1} \int_{\Sigma} d^3y \lambda(y) T(y)\right] \]

\[ = \left[ \int_{\Sigma} d^3y \delta^{(3)}(x, y) \lambda(y) \right] e^{(hG)^{-1}\lambda \cdot T} = \lambda(x) e^{(hG)^{-1}\lambda \cdot T} \]  

(103)
on account of the integration of the delta function over 3-space \(\Sigma\). However the action of \(\hat{\Pi}(x)\) on \(e^{-T(x)}\), which is evaluated at a single point \(x\), would yield a \(\delta^{(3)}(0)\) singularity. To deal with this we will use (80). The regularized action of the functional derivative on \(e^{-T}\) is given by

\[ \hat{\Pi}_{\epsilon}(x)\{e^{-T}\} = (hG) \int_{\Sigma} d^3y f_{\epsilon}(x, y) \frac{\delta}{\delta T(y)} \exp\left[-T(x)\right] \]

\[ = -(hG) \int_{\Sigma} d^3y f_{\epsilon}(x, y) \delta^{(3)}(x, y) \exp\left[-T(x)\right] = -(hG)f_{\epsilon}(0)e^{-T}, \]  

(104)
where we have defined \(f_{\epsilon}(0) \equiv f_{\epsilon}(x, x)\). Define a new constant \(\mu'\) by

\[ \mu' = (hG)f_{\epsilon}(0). \]  

(105)
Since \([f_{\epsilon}] = 3\), then eigenvalue of (104) has mass dimension of \([\mu'] = 1\), the same as \([\lambda]\). Hence we have the following relation

\[ \hat{\Pi}(x)\{e^{(hG)^{-1}\lambda \cdot T} e^{-T}\} = (\lambda(x) - \mu') e^{(hG)^{-1}\lambda \cdot T} e^{-T}, \]  

(106)
which suggests the identification of \(e^{-T}\) with a state

\[ |\mu'\rangle \equiv \exp\left[-(hG)^{-1} \int_{\Sigma} d^3y \left(\frac{(hG)}{Vol_{\epsilon}(\Sigma)}\right) T(x)\right] = e^{-T(x)}, \]  

(107)
whereupon the volume factor cancels upon integration. Defining

\[ |\lambda_{\alpha,\beta}\rangle \equiv |\alpha\rangle \otimes |\beta\rangle e^{(hG)^{-1}\lambda_{\alpha,\beta} \cdot T} \]  

(108)
then we have

\[ e^{-T} |\lambda_{\alpha,\beta}\rangle = |\lambda_{\alpha,\beta} - \mu'\rangle. \]  

(109)
For \(Re\{\lambda\} < 0\) \(\hat{q}\) acts as a raising operator on \(\Lambda = 0\) states, and for \(Re\{\lambda\} > 0\) it acts as a lowering operator.
One may at first balk that in the limit of removal of the regulator, 
\[ \lim_{\varepsilon \to 0} f_{\varepsilon}(0) = \infty, \] 
since the increments of \( \mu' \) in relation to the densitized 
eigenvalues \( \lambda, \alpha \) and \( \beta \) would be infinite. However, recall that the undensitized 
eigenvalues \( \lambda_f \) are given by

\[ \lambda_f \sim \lambda (\det A)^{-1} = \lambda a_0^{-3} e^{-T}. \] (110)

Hence equation (109), which corresponds to a decrement of \( \lambda \) in steps of size \( \mu' \), actually corresponds to a decrement in \( \lambda_f \) of size

\[ \Delta \lambda_f = \left( \frac{\hbar G}{a_0^3} \right) f_{\varepsilon}(0). \] (111)

The mass scale \( a_0 \) of the connection \( A_i^a \) has thus far remained unspecified. 
A choice \( a_0 = (f_{\varepsilon}(0))^{1/3} \) sets the scale of incrementation of \( \lambda_f \) in steps of 
\( l_{Pl}^2 = \hbar G \), where \( l_{Pl} \) is the Planck length. Hence the action of \( \hat{q} \) on the 
states would provide very small, though still discrete, increments of the 
(undensitized) CDJ matrix \( \Psi_{ae} \) in comparison.

With this interpretation, we now continue from (97), obtaining

\[ \hat{q} \lambda, \alpha, \beta \rangle = r \hat{O}^{-1} e^{-T} \hat{O} \lambda, \alpha, \beta \rangle = r \lambda (\lambda + \alpha)(\lambda + \beta) \hat{O}^{-1} e^{-T} \lambda, \alpha, \beta \rangle = r \lambda \left( \frac{\lambda + \alpha}{\lambda + \gamma - \mu'} - \frac{\lambda + \beta}{\lambda + \gamma + \mu'} \right) |\lambda - \mu', \alpha, \beta \rangle. \] (112)

Repeating this \( n \) times, we have

\[ \hat{q}^n \lambda, \alpha, \beta \rangle = r^n \prod_{k=0}^{n-1} \left( \frac{\lambda - k\mu'}{\lambda + \gamma - k\mu'} \right) \left( \frac{\lambda + \alpha}{\lambda + \gamma - \mu'} - \frac{\lambda + \beta}{\lambda + \gamma + \mu'} \right) |\lambda - n\mu', \alpha, \beta \rangle. \] (113)

Then the full solution using (102) is given by

\[ |\psi_{\alpha, \beta} \rangle = \sum_n (-\mu')^n \left( \prod_{k=0}^{n-1} \left( \frac{\lambda - k\mu'}{\lambda + \gamma - k\mu'} \left( \frac{\lambda + \alpha}{\lambda + \gamma - \mu'} - \frac{\lambda + \beta}{\lambda + \gamma + \mu'} \right) \right) \right) |\lambda - n\mu', \alpha, \beta \rangle \]

\[ = \sum_n (-\mu' e^{-T})^n \left( \prod_{k=0}^{n-1} \left( \frac{\lambda - k\mu'}{\lambda + \gamma - k\mu'} \left( \frac{\lambda + \alpha}{\lambda + \gamma - \mu'} - \frac{\lambda + \beta}{\lambda + \gamma + \mu'} \right) \right) \right) |\lambda, \alpha, \beta \rangle \] (114)

which has acquired the label of the \( \Lambda = 0 \) basis states.
6.2 Sufficient condition for convergence of the state

Let us define the dimensionless quantities

\[ a \equiv \frac{\alpha}{\mu'}; \quad b \equiv \frac{\beta}{\mu'}; \quad c \equiv \frac{\lambda}{\mu'}, \]  

which expresses the densitized eigenvalues of the CDJ matrix in units of the regulating factor \( \mu' \). Then using the Pochammer symbols \((p)_k\), defined by

\[ (p)_k = \frac{\Gamma(p+k)}{\Gamma(p)} = p(p+1)\ldots(p+k-1), \]

the solution can be written

\[ |\psi_{\alpha,\beta}\rangle = P_{\alpha,\beta}(T)|\lambda_{\alpha,\beta}\rangle \]  

where we have defined the hypergeometric series

\[ P_{a,b}(T) = \sum_{n=0}^{\infty} \frac{(c)_n(a)_n(b)_n}{(c_-+1)n(c_+1)n}(-\mu' re^{-T})^n. \]

In order to obtain a sensible wavefunction, we must require (117) to converge. However, the numerator \( Q \) of each term exceeds the denominator \( O \) and for large \( n \) this goes roughly as \( n \rightarrow \infty \), yielding a zero radius of convergence. In order to guarantee convergent wavefunctions, a sufficient condition is that the series (114) be required to terminate at finite order by setting \( Q = 0 \). This leads to three possibilities, namely \( \lambda = N\mu' \), \( \lambda = N\mu' - \alpha \) or \( \lambda = N\mu' - \beta \) for some integer \( N \), so that the series becomes truncated at order \( N \). This amounts to a restriction of the allowable states, which can be seen from the dispersion relation

\[ 3\lambda^2 + 2(\alpha + \beta)\lambda + \alpha\beta = 0, \]

which determines the \( \Lambda = 0 \) states that are being used for determine the \( \Lambda \neq 0 \) counterparts. The solution to (119) is given by

\[ \beta = -\lambda \left( \frac{3\lambda + 2\alpha}{2\lambda + \alpha} \right); \quad \alpha = -\lambda \left( \frac{3\lambda + 2\beta}{2\lambda + \beta} \right). \]

\[ ^{16} \text{This } N \text{ is not to be confused with the } N \text{ which we used previously to denote the number of lattice sites in a discretization of 3-space } \Sigma, \text{ nor should it be confused with the lapse function.} \]
There are three possibilities for each state. For $\lambda = N\mu'$ we have

$$\frac{\beta}{\mu'} = -N\left(\frac{3N + 2\alpha}{2N + \mu'}\right),$$

(121)

for $\lambda = N\mu' - \alpha$ we have

$$\frac{\beta}{\mu'} = -\left(N - \frac{\alpha}{\mu'}\right)\left(\frac{3N - \alpha}{2N - \mu'}\right),$$

(122)

and for $\lambda = N\mu' - \beta$ we have

$$\frac{\alpha}{\mu'} = -\left(N - \frac{\beta}{\mu'}\right)\left(\frac{3N - \beta}{2N - \mu'}\right).$$

(123)

The result is that the $\Lambda \neq 0$ states are labelled by one continuous index $\alpha = \alpha(x)$ and one discrete index $n \in \mathbb{Z}$ at each point, which are arbitrary. Recall for $\Lambda = 0$ that the state labels define a two-dimensional complex manifold $(\alpha, \beta) \in \mathbb{C}^2$ per point. The effect of a nonzero $\Lambda$ is to cause a reduction $\mathbb{C}^2 \to \mathbb{C} \otimes T^1$, where $\mathbb{C}$ is the complex plane and $T^1$ is the one-dimensional torus with spacing $l_{Pl}^2$, thus implying a quantization according to the three cases analyzed above. One may relabel the states using the index $n$ as $|\psi_{n,\alpha}\rangle$, which corresponds to an infinite tower of states

$$\psi = P_{n,\alpha}[T(x)]e^{(hG)^{-1} \frac{\alpha}{\mu'} X(x)}e^{(hG)^{-1} \frac{\beta}{\mu'} Y(x)}e^{(hG)^{-1} \lambda_{\alpha,\beta} T(x)},$$

(124)

which produces a Hilbert space of normalizable states at each point $x$. To form a Hilbert space with support on 3-space we must take the direct product of (124) over all points $x \in \Sigma$,

$$\Psi = P_{n,\alpha}[T]e^{(hG)^{-1} \frac{\alpha}{\mu'} X}e^{(hG)^{-1} \frac{\beta}{\mu'} Y}.$$

(125)

However, since the argument of the exponentials in (125) is directly proportional to $\mu'$, which blows up in the continuum limit, then such a wavefunction can be used only for discretized 3-space.\(^\text{17}\) This brings us to the improved momentum sequence of the next section.

While the reduction of the state manifold $\mathbb{C}^2 \to \mathbb{C} \otimes T^1$ has produced convergent quantum states, it would be unsatisfactory if the presence of a

\(^{17}\)Hence, while we obtain a convergent hypergeometric solution, the state is not finite in the sense of [16] on account of the field-theoretical infinities induced upon removal of the regulator.
nonzero $\Lambda$ were to cause a reduction in the available states. This means that there must be additional states which the above procedure has missed. Nevertheless it still admits a physical interpretation. Recalling the definition of the volume operator in equation (63), one sees that terms of the hypergeometric series (114) enhance states of large volume, while suppressing states of small volume. The physical interpretation of the convergent states is that the series terminates on states of zero volume, corresponding to each $N$. Hence the state (124) is a superposition of states of decreasing volume labelled by the integers, where the length scale occurs in increments of $\frac{1}{\mu}$ which is the Planck length for $a_0 = (f(0))^{1/3}$ in (111). Hence for each choice of the continuous label $\alpha$, the 3-volume for the state is quantized according to $N$.

The overlap of two un-normalized states is given by

$$
\langle \Psi_{\alpha;n} | \Psi_{\alpha';m} \rangle = P_{\alpha;n}[T] P_{\alpha';m}[T] e^{-\nu f_0(0)|m-n|^2} e^{-\nu'(\hbar G)^{-2}|\alpha-\alpha'|^2}, 
$$

where the bold quantities signify the direct product of the unbolded counterparts over $\Sigma$, as the discretization becomes finer.
7 Expansion in inverse $\Lambda$

Given that the expansion in powers of $\Lambda$ has led to restrictions required for convergence of solutions to the Hamiltonian constraint, let us instead try an expansion in inverse powers of $\Lambda$. Redefine the operators

$$\hat{O} = \hat{\Pi}(\hat{\Pi} + \hat{\Pi}_1)(\hat{\Pi} + \hat{\Pi}_2),$$

(127)

and

$$\hat{Q} = \hat{\Pi}_+ \hat{\Pi}_- = \hat{\Pi}^2 + \frac{2}{3}(\hat{\Pi}_1 + \hat{\Pi}_2)\hat{\Pi} + \frac{1}{3}\hat{\Pi}_1\hat{\Pi}_2.$$  

(128)

Hence, the operators in (96) have switched roles. Also redefine the constant $r$ such that

$$r = \left(\frac{3\alpha^3}{\Lambda}\right).$$

(129)

The quantum Hamiltonian constraint is now given by

$$\hat{O}|\psi\rangle = -r e^T \hat{Q}|\psi\rangle.$$  

(130)

The action of (127) and (128) on the $\Lambda = 0$ basis states is given by

$$\hat{O}|\lambda,\alpha,\beta\rangle = \lambda(\lambda + \alpha)(\lambda + \beta)|\lambda,\alpha,\beta\rangle;$$

$$\hat{Q}|\lambda,\alpha,\beta\rangle = \lambda_- \lambda_+ |\lambda,\alpha,\beta\rangle = (\lambda^2 + \frac{2}{3}(\alpha + \beta) + \frac{1}{3}\alpha\beta)|\lambda,\alpha,\beta\rangle,$$

(131)

where $\lambda_- = \lambda + \gamma_-$ and $\lambda_+ = \lambda_+ + \gamma_+$ with $\gamma_-$ and $\gamma_+$ given by (79). Next, we must find $|\lambda_{\alpha,\beta}\rangle \in Ker\{\hat{O}\}$. From (127) one sees that $\hat{O}$ annihilates states with $\lambda = 0$, $\lambda = -\alpha$ and $\lambda = -\beta$, which are states of zero volume. Therefore

$$|\lambda_{\alpha,\beta},\alpha,\beta\rangle = \{0,\alpha,\beta\},|\alpha,\alpha,\beta\rangle,|\beta,\alpha,\beta\rangle \in Ker\{\hat{O}\}$$

(132)

are the desired states about which we will perform the Lippman–Schwinger type expansion. These states are given by

$$|0,\alpha,\beta\rangle = e^{(hG)^{-1}\alpha \cdot X} e^{(hG)^{-1}\beta \cdot Y};$$

$$|\alpha,\alpha,\beta\rangle = e^{(hG)^{-1}\alpha \cdot (X-T)} e^{(hG)^{-1}\beta \cdot Y};$$

$$|\beta,\alpha,\beta\rangle = e^{(hG)^{-1}\alpha \cdot X} e^{(hG)^{-1}\beta \cdot (Y-T)};$$

(133)
The physical interpretations are as follows: If we view $T$ as a time variable on configuration space $\Gamma$, then $|0, \alpha, \beta\rangle$ is a timeless state, and $|-\alpha, \alpha, \beta\rangle$ and $|-\beta, \alpha, \beta\rangle$ correspond to plane waves travelling at unit speed in respectively the $X$ and in the $Y$ directions for each $x \in \Sigma$. In other words, we have chosen to perform the expansion about states of zero volume which mimic the motion of a free particle on a two dimensional configuration space per point. We will now compute the Hamiltonian constraint for expansion about $(133)$. This is given by

$$|\psi_{\alpha, \beta}\rangle = \left(\frac{1}{1+\hat{q}}\right)|\lambda_{\alpha, \beta}\rangle = \left(1 - \hat{q} + \hat{q}^2 - \hat{q}^3 + \ldots\right)|\lambda_{\alpha, \beta}\rangle, \quad (134)$$

where

$$\hat{q} = r\hat{O}^{-1}e^T\hat{Q}. \quad (135)$$

The action of $\hat{\Pi}$ is given, recalling the results of (103) and (104), by

$$\hat{\Pi}(x)|\lambda\rangle = \lambda(x)|\lambda\rangle; \quad \hat{\Pi}_\epsilon(x)e^T(x) = \mu'e^T(x), \quad (136)$$

where $\mu' = hGf_\epsilon(0)$, so that

$$\hat{\Pi}(x)e^T(x)|\lambda\rangle = (\lambda(x) + \mu')e^T(x)|\lambda\rangle \sim (\lambda + \mu')|\lambda + \mu'\rangle. \quad (137)$$

Hence $e^T(x)$ is a lowering operator for $Re\{\lambda\} < 0$ and a raising operator for $Re\{\lambda\} > 0$. Whatever the case, the point is that we will obtain a hypergeometric function that is well-defined.

Let us now evaluate the action of $\hat{q}$ on an arbitrary state

$$\hat{q}|\lambda_{\alpha, \beta}\rangle = r\hat{O}^{-1}e^T\hat{Q}|\lambda\rangle = r(\lambda + \gamma^-)(\lambda + \gamma_+)^{-1}e^T|\lambda, \alpha, \beta\rangle = r(\lambda + \gamma^-)(\lambda + \gamma_+)^{-1}|\lambda + \mu'\rangle$$

$$= r\left(\frac{1}{\lambda + \mu'}\right)\left(\frac{\lambda + \gamma^-}{\lambda + \alpha + \mu'}\right)\left(\frac{\lambda + \gamma_+}{\lambda + \beta + \mu'}\right)|\lambda + \mu'\rangle. \quad (138)$$

Repeating this $n$ times, we have

$$\hat{q}^n|\lambda, \alpha, \beta\rangle = r^n\mu'^n\prod_{k=0}^{n-1}\left(\frac{\lambda + \gamma^-}{\mu'} + k\right)\left(\frac{\lambda + \gamma_+}{\mu'} + k\right)\left(\frac{\lambda + \alpha + \mu'}{\mu'} + k\right)\left(\frac{\lambda + \beta + \mu'}{\mu'} + k\right)|\lambda + n\mu', \alpha, \beta\rangle. \quad (139)$$

\[^{18}\text{In what follows we will occasionally omit the } \alpha, \beta \text{ part of the state labels } |\lambda, \alpha, \beta\rangle \to |\lambda\rangle \text{ in order to avoid cluttering up the notation. It should hopefully be clear from the context that these labels are implicit.}\]
where we have divided the numerator and the denominator of each term by a common factor of $\mu'$. Equation (139) can be written using the Pochammer symbols (116), combined with bringing out the exponential factor of $eT$ from the state

$$\hat{q}^n |\lambda,\alpha,\beta\rangle = \frac{1}{n!} \left(\frac{re^T}{\mu'}\right)^n \frac{(1_n)(\frac{\lambda+\gamma^-}{\mu'})_n (\frac{\lambda+\gamma^+}{\mu'})_n}{(\frac{\lambda}{\mu'}+1)_n (\frac{\lambda+\alpha}{\mu'}+1)_n (\frac{\lambda+\beta}{\mu'}+1)_n} |\lambda,\alpha,\beta\rangle \quad (140)$$

whence $\mu'$ now appears in the denominator in contrast to (114). Defining the dimensionless variable $z$, given by

$$z \equiv \frac{re^T}{\mu'} = \frac{3a_0^3 e^T}{hG\Lambda f_e(0)}, \quad (141)$$

Then the full solution is given by

$$|\psi_{\alpha,\beta,\alpha,\beta}\rangle = \sum_n (-\hat{q})^n |\lambda,\alpha,\beta\rangle = \sum_{n=0}^{\infty} (-z)^n \frac{1}{n!} \left(\frac{\lambda+\gamma^-}{\mu'}+1\right)_n \left(\frac{\lambda+\alpha}{\mu'}+1\right)_n \left(\frac{\lambda+\beta}{\mu'}+1\right)_n |\lambda,\alpha,\beta\rangle \quad (142)$$

Equation (142) can be written as a hypergeometric function

$$3F_3\left(1, \frac{\lambda + \gamma^-}{\mu'}, \frac{\lambda + \gamma^+}{\mu'}; \frac{\lambda}{\mu'} + 1, \frac{\lambda + \alpha}{\mu'} + 1, \frac{\lambda + \beta}{\mu'} + 1; z\right)|\lambda,\alpha,\beta\rangle, \quad (143)$$

which is a solution to the hypergeometric differential equation

$$z \frac{d}{dz} \left( z \frac{d}{dz} + \frac{\lambda}{\mu'} \right) \left( z \frac{d}{dz} + \frac{\lambda + \alpha}{\mu'} \right) \left( z \frac{d}{dz} + \frac{\lambda + \beta}{\mu'} \right) \psi(z) = z \left( z \frac{d}{dz} + 1 \right) \left( z \frac{d}{dz} + \frac{\lambda + \gamma^-}{\mu'} \right) \left( z \frac{d}{dz} + \frac{\lambda + \gamma^+}{\mu'} \right) \psi(z). \quad (144)$$

One requirement of our quantization procedure is that for $\alpha(x) = \beta(x) = 0$, we should obtain the Kodama state $\psi_{Kod}$, since the Hamiltonian constraint in this case is given by

$$\hat{H} |\psi\rangle = (\hat{\Pi} \hat{\Pi} \hat{\Pi} + 3a_0^3(hG\Lambda)^{-1}e^T \hat{\Pi}) |\psi\rangle = 0. \quad (145)$$

It is clear (142) does not satisfy this boundary condition, which implies that the operator ordering for $\hat{q}$ has been chosen incorrectly.
Let us attempt an alternate operator ordering, this time with the momenta to the left to the coordinates upon quantization. Returning to the level of (134), the solution is given by

\[ |\psi_{\alpha,\beta}\rangle = \left(\frac{1}{1 + \hat{q}}\right)|\lambda_{\alpha,\beta}\rangle = \left(1 - \hat{q} + \hat{q}^2 - \hat{q}^3 + \ldots\right)|\lambda_{\alpha,\beta}\rangle, \]  

(146)

where

\[ \hat{q} = r\hat{O}^{-1}\hat{Q}e^T. \]  

(147)

The action of \( \hat{q} \) on the state (70) is given by

\[ \hat{q}|\alpha,\beta,\lambda\rangle = \left(r\hat{O}^{-1}\hat{Q}e^T\right)|\lambda,\alpha,\beta\rangle = r\hat{O}^{-1}\hat{Q}|\lambda + \mu',\alpha,\beta\rangle = r \left(\frac{\lambda + \gamma^- + \mu'}{\lambda + \alpha + \mu'}(\lambda + \beta + \mu')\right)|\lambda + \mu',\alpha,\beta\rangle. \]  

(148)

To first order in \( \hat{q} \), both the numerator and the denominator of (148) have been augmented by \( \mu' \), unlike in (138). The \( n^{th} \) repeated action is given by

\[ \hat{q}^n|\lambda,\alpha,\beta\rangle = \left(r\right)^n\frac{n!}{\mu'} \prod_{k=1}^{n} \left(\frac{1}{\mu'} + k\right) \left(\frac{1}{\mu'} + \beta + k\right) \left(\frac{1}{\mu'} + \alpha + k\right) |\lambda + n\mu',\alpha,\beta\rangle. \]  

(149)

where we have divided the numerator and the denominator of each term by a common factor of \( \mu' \). Bringing out the exponential factor of \( e^T \) from the state, we have

\[ \hat{q}^n|\lambda,\alpha,\beta\rangle = \frac{1}{n!} \left(r\right)^n \langle e^T \right|^n \left(\frac{1}{\mu'} + 1\right) \left(\frac{1}{\mu'} + \beta + 1\right) \left(\frac{1}{\mu'} + \alpha + 1\right) |\lambda,\alpha,\beta\rangle. \]  

(150)

Defining the dimensionless variable \( z \), given by

\[ z \equiv \frac{r e^T}{\mu'} = \frac{3a_0^3 e^T}{hG\Lambda f_e(0)} = \frac{3(\text{det}A)}{hG\Lambda f_e(0)}. \]  

(151)

then the full solution is given by
\[
\psi_{\alpha,\beta,\lambda} = \sum_n (-\hat{q})^n |\lambda, \alpha, \beta\rangle
\]
\[
= \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \frac{(\lambda+\gamma^- + 1)(\lambda+\gamma^+ + 1)}{(\mu' + 1)^n (\frac{\lambda+\alpha}{\mu'} + 1)^n (\frac{\lambda+\beta}{\mu'} + 1)^n} |\lambda, \alpha, \beta\rangle
\]  \hfill (152)

Equation (152) can be written as a hypergeometric function

\[
3F_3\left(1, \frac{\lambda + \gamma^-}{\mu'} + 1, \frac{\lambda + \gamma^+}{\mu'} + 1; \frac{\lambda + \alpha}{\mu'} + 1, \frac{\lambda + \beta}{\mu'} + 1; z\right) |\lambda, \alpha, \beta\rangle, \hfill (153)
\]

which solves the hypergeometric differential equation

\[
z \frac{d}{dz} \left( z \frac{d}{dz} + \frac{\lambda}{\mu'} \right) \psi(z) = z \left( z \frac{d}{dz} + 1 \right) \left( z \frac{d}{dz} + \frac{\lambda + \gamma^-}{\mu'} + 1 \right) \left( z \frac{d}{dz} + \frac{\lambda + \gamma^+}{\mu'} + 1 \right) \psi(z). \hfill (154)
\]

For \( \alpha(x) = \beta(x) = 0 \) \( \forall x \), (153) reduces to

\[
3F_3\left(1, 1, 1; 1, 1; z\right) = e^z = e^{z(x)} \hfill (155)
\]

for each \( x \). To obtain the Hilbert space we must form the direct product of the solution \( \forall x \in \Delta_N(\Sigma) \), and then take the continuum limit

\[
\Psi_{0,0} = \bigotimes_x e^{z(x)} = \lim_{\epsilon \to 0} \prod_x \exp \left[ -3(h\Lambda f_\epsilon(0))^{-1} a_0^3 e^{T(x)} \right]. \hfill (156)
\]

We recognize the reciprocal of the regulating function \( f_\epsilon(0) \) as \( \nu \), the size of an elementary lattice cell in the discretization \( \Delta_N(\Sigma) \). In this sense the argument of the exponential in (156) in the limit of removal of the regulator approaches the Riemannian integral

\[
\exp \left[ -3(h\Lambda)^{-1} \lim_{\epsilon \to 0} \sum \nu \text{det} A(x_n) \right] = \exp \left[ -3(h\Lambda)^{-1} \int_{\Sigma} d^3 x \mathcal{L}_{CS} \right] = \psi_{Kod} \hfill (157)
\]

where we have used \( \text{det} A = a_0^3 e^T \). We have obtained the proper limit for \( \alpha = \beta = 0 \), namely the Kodama state evaluated on the diagonal connection used for quantization.\footnote{This corresponds to spacetimes of Petrov Type O, where all eigenvalues of the CDJ matrix are equal.}

33
8.1 Verification of the Hamiltonian constraint

The previous exercise has demonstrated two things. First, the correct operator ordering must have the momenta to the left of the coordinates, in order to produce the Kodama state $\psi_{Kod}$ which is a known solution to the Hamiltonian constraint for $\alpha = \beta = 0$. Secondly, we have proven that

$$\lim_{N \to \infty} \psi_{Kod}(\Delta_N(\Sigma)) = \psi_{Kod}(\Delta_\infty(\Sigma)) \in \operatorname{Ker}\{\hat{H}\}. \quad (158)$$

This is another way of saying that the solution space is Cauchy complete with respect to $\psi_{Kod}$, since its continuum limit is part of the same solution space each discretized version identically annihilated by the same Hamiltonian constraint. Having obtained the $\psi_{Kod}$ in the proper limit, we may now attempt to construct the solution in the general case $(\alpha, \beta) \neq (0, 0)$. But first, note that the operator ordering of (154) has $z$ to the left on the right hand side, whereas the ordering which has produced $\psi_{Kod}$ must have $z$ to the right. So we must verify the consistency with (154) with the correct operator ordering. Using the identity

$$z\left(z \frac{d}{dz} + 1\right)F = z \frac{d}{dz}(zF), \quad (159)$$

we can commute the factor of $z$ to the right, subtracting 1 for each differential operator traversed. The result is that (154) is the same as

$$z \frac{d}{dz}\left(z \frac{d}{dz} + \frac{\lambda}{\mu'}\right)\left(z \frac{d}{dz} + \frac{\lambda + \alpha}{\mu'}\right)\left(z \frac{d}{dz} + \frac{\lambda + \beta}{\mu'}\right)\psi(z) = z \frac{d}{dz}\left(z \frac{d}{dz} + \frac{\lambda + \gamma}{\mu'}\right)\left(z \frac{d}{dz} + \frac{\lambda + \gamma}{\mu'}\right)z\psi(z). \quad (160)$$

The common operator $z \frac{d}{dz}$ in front can be dropped, yielding

$$\left(z \frac{d}{dz} + \frac{\lambda}{\mu'}\right)\left(z \frac{d}{dz} + \frac{\lambda + \alpha}{\mu'}\right)\left(z \frac{d}{dz} + \frac{\lambda + \beta}{\mu'}\right)\psi(z) = \left(z \frac{d}{dz} + \frac{\lambda + \gamma}{\mu'}\right)\left(z \frac{d}{dz} + \frac{\lambda + \gamma}{\mu'}\right)z\psi(z). \quad (161)$$

The quantum Hamiltonian constraint for an operator ordering of momenta to the left of the coordinates is given in the Schrödinger representation by

$$\mu' \frac{\delta}{\delta T} \left(\mu' \frac{\delta}{\delta T} + \alpha\right)\left(\mu' \frac{\delta}{\delta T} + \beta\right)\psi = -\left(3\alpha^3_i\frac{3}{\Lambda}\right)\left(\mu' \frac{\delta}{\delta T} + \gamma_{-}\right)\left(\mu' \frac{\delta}{\delta T} + \gamma_{+}\right)e^T\psi \psi(162)$$
where $\mu'$ will be fixed by consistency condition. Dividing (162) by $\mu'^3$, we obtain

$$\frac{\delta}{\delta T} \left( \frac{\delta}{\delta T} + \frac{\alpha}{\mu'} \right) \left( \frac{\delta}{\delta T} + \frac{\beta}{\mu'} \right) \psi = - \left( \frac{3\alpha^3_0}{\mu'\Lambda} \left( \frac{\delta}{\delta T} + \frac{\gamma_-}{\mu'} \right) \left( \frac{\delta}{\delta T} + \frac{\gamma_+}{\mu'} \right) e^T \psi \right).$$

(163)

Upon comparison of (163) with (161) we can make the identification $\mu' = \hbar G f_\epsilon(0)$, since this is precisely the regularization term induced by the action of the functional derivative on $e^T$. The general solution is given by

$$\psi_{\alpha,\beta}(z) = {_2F_2}(\gamma_-/\mu' + 1, \gamma_+/\mu' + 1; \alpha/\mu', \beta/\mu'; z(x)) \Phi_{\alpha,\beta}(X, Y)$$

(164)

where we have identified $\Phi_{\alpha,\beta}$ with the $\Lambda = 0$ basis states

$$\Phi_{\alpha,\beta} = e^{(hG)^{-1} \alpha \cdot X} e^{(hG)^{-1} \beta \cdot Y},$$

(165)

with the $T$ dependence given by $z = e^T$.

### 8.2 Hypergeometric functional formalism

We will put the Hamiltonian constraint into standard notation, for ease of identification with known functions. Define the dimensionless quantities

$$a = \frac{\alpha}{\mu'}; \quad b = \frac{\beta}{\mu'}; \quad c_{\pm} = \frac{\gamma_{\pm}}{\mu'}.$$

(166)

The Hamiltonian constraint for the appropriate operator ordering necessary to produce $\psi_{Kod}$ in the correct limit is given by\(^{20}\)

$$z \frac{d}{dz} \left( z \frac{d}{dz} + a \right) \left( z \frac{d}{dz} + b \right) \psi(z) = \left( z \frac{d}{dz} + c_+ \right) \left( z \frac{d}{dz} + c_- \right) z \psi(z).$$

(167)

To put (167) into the form of the hypergeometric differential equation, we commute $z$ to the left on the right hand side of (167), yielding

$$z \frac{d}{dz} \left( z \frac{d}{dz} + a \right) \left( z \frac{d}{dz} + b \right) \psi(z) = z \left( z \frac{d}{dz} + c_+ + 1 \right) \left( z \frac{d}{dz} + c_- + 1 \right) \psi(z).$$

(168)

The quantum wavefunction satisfying the constraint is given by

\(^{20}\)We have replaced the action of the operators on the part of the wavefunctional that depends on $X$ and $Y$ with their eigenvalues.
\[ |\psi_{a,b}(x)\rangle = P_{a,b}(z(x)) |a, b\rangle_x, \]  
\[ (169) \]
where the subscript \(x\) labels the point at which the solution is evaluated. The pre-factor \(P_{a,b}(z)\) is also evaluated at the same point \(x\) and is the solution to (168), given by

\[ P_{a,b}(z) = 2F_2\left(c_- + 1, c_+ + 1; a, b; z\right). \]  
\[ (170) \]

The full state is then a direct product over all points in \(\Sigma\)

\[ |\Psi_{a,b}\rangle = \bigotimes_x |\psi_{a,b}(x)\rangle \].  
\[ (171) \]

The Kodama state \(\psi_{Kod}\) corresponds to the choice \(a = b = 0\), whence the infinite product of hypergeometric functionals is measurable.

Using the hypergeometric formalism, we can also write a general solution for the states at the opposite extreme which in an earlier section we required to terminate at finite order in the series. Starting from (168), which is the hypergeometric form of the Hamiltonian constraint, divide by \(z\) to obtain

\[ \frac{1}{z} \left( z \frac{d}{dz} \right) \left( z \frac{d}{dz} + a \right) \left( z \frac{d}{dz} + b \right) \psi(z) = \left( z \frac{d}{dz} + c_+ + 1 \right) \left( z \frac{d}{dz} + c_- + 1 \right) \psi(z). \]  
\[ (172) \]

Now make the following transformation

\[ u = \frac{1}{z}; \quad z \frac{d}{dz} = -u \frac{d}{du}. \]  
\[ (173) \]

Inserting (173) into (172), we obtain

\[ -u \left( u \frac{d}{du} \right) \left( u \frac{d}{du} - a \right) \left( u \frac{d}{du} - b \right) \Phi(u) = \left( u \frac{d}{du} - c_- - 1 \right) \left( u \frac{d}{du} - c_+ - 1 \right) \Phi(u). \]  
\[ (174) \]

where \(\Phi(u) = \psi(1/z)\). Now act on (166) with \(u(d/du)\)

\[ u \frac{d}{du} \left( u \frac{d}{du} - c_- - 1 \right) \left( u \frac{d}{du} - c_+ - 1 \right) \Phi(u) \]

\[ = -\left( u \frac{d}{du} \right) u \left( u \frac{d}{du} - a \right) \left( u \frac{d}{du} - b \right) \Phi(u), \]  
\[ (175) \]

then commute \(u\) to the left to put into the standard form
\[ u \frac{d}{du} \left( u \frac{d}{du} - c_- - 1 \right) \left( u \frac{d}{du} - c_+ - 1 \right) \Phi(u) = -u \left( u \frac{d}{du} \right) u \left( u \frac{d}{du} + 1 \right) \left( u \frac{d}{du} - a \right) \left( u \frac{d}{du} - b \right) \Phi(u). \]  

(176)

The solution to (176) is given by

\[ \Phi_{a,b}(u) = _4F_2(0, 1, -a, -b; -c_-, -c_+; u). \]  

(177)

This converges only when \( a \) or \( b \) is an integer whence the series terminates as in (125). This yields an infinite tower of states obtained by replacing \( \Phi_{a,b} \) with \( \psi_{a,b} \) in (171).

8.3 States for \( \alpha = \beta \neq 0 \)

It is an easy matter to verify the case where two eigenvalues are equal and nonvanishing, which corresponds to one degree of freedom. For \( \Lambda = 0 \) the dispersion relation (85) still holds, quoted here for completeness

\[ \lambda \equiv \lambda_{\alpha,\beta} = -\frac{1}{3} \left( \alpha + \beta \pm \sqrt{\alpha^2 - \alpha\beta + \beta^2} \right) \forall x. \]  

(178)

But we must now restrict (178) to the case \( \alpha = \beta \), which yields the solution \( \lambda_{\alpha,\beta} \equiv \lambda_{\alpha} \), given by

\[ \lambda_{\alpha,\alpha} = (-\alpha, -\frac{1}{2}\alpha). \]  

(179)

in the case \( \Lambda = 0 \). This corresponds to states of the form

\[ \Phi_{\alpha,\beta} = \Phi_{\alpha,\alpha} = e^{(hG)^{-1} \alpha \cdot (X - T)}; \quad e^{(hG)^{-1} \alpha \cdot (X - \frac{1}{2}T)} e^{\lambda_{\alpha,\alpha} \cdot T}, \]  

(180)

which correspond to plane waves travelling at speeds 1 and \( \frac{1}{3} \) in the \( X \) direction of a one-dimensional configuration space per point. To obtain the \( \Lambda \neq 0 \) case, we may perform an expansion about (180) using the improved momentum ordering to the left. The classical Hamiltonian constraint for \( \Lambda \neq 0 \) and \( \alpha = \beta \neq 0 \) is given by

\[ (3\Pi + \alpha)(\Pi + \alpha)re^T = \Pi(\Pi + \alpha)^2. \]  

(181)

We can cancel the common factor \( \Pi + \alpha \) to reduce the order of the equation
\[(3\Pi + \alpha)re^T = \Pi(\Pi + \alpha).\]  

(182)

Upon making the identification \( z = re^T \), with \( r = -\frac{3}{\alpha_3} \), this yields a quantum version of

\[
(z \frac{d}{dz} + \frac{a}{3})z\psi(z) = z \frac{d}{dz} \left( z \frac{d}{dz} + \frac{a}{3} \right) \psi(z).
\]  

(183)

Commuting the factor of \( z \) into the standard form of a hypergeometric equation

\[
z \left( z \frac{d}{dz} + \frac{a}{3} + 1 \right) \psi(z) = z \frac{d}{dz} \left( z \frac{d}{dz} + \frac{a}{3} \right) \psi(z),
\]  

(184)

we see that the solution is given by

\[
\psi = \hypergeom{1}{2}{\frac{a}{3} + 1; \frac{a}{3}; z}.
\]  

(185)

This should correspond to Type D spacetimes, with two equal eigenvalues of the CDJ matrix.


9 Normalizability of the Kodama state

We have constructed a Hilbert space of quantum gravitational states solving the constraints of GR, which provides a possible resolution to the issue of normalizability of the Kodama state raised in [4] and [5]. The Kodama state is given by

$$\psi_{Kod}[A] = e^{-3(hG\Lambda)^{-1}I_{CS}[A]}, \quad (186)$$

where $I_{CS}[A]$ is the Chern–Simons functional of the Ashtekar connection, given in two form notation by

$$I_{CS} = \int_{\Sigma} A \wedge dA + \frac{2}{3} A \wedge A \wedge A. \quad (187)$$

For DeSitter spacetime the Petrov classification is type O, which corresponds to three equal (undensitized) eigenvalues of $\Psi_{ae}$ given by

$$\lambda_1 = \lambda_2 = \lambda_3 = -\frac{3}{\Lambda}. \quad (188)$$

In this case $\alpha = \beta = 0$ and one is reduced to a single degree of freedom on per point configuration space $\Gamma$, namely $T(x)$.\(^{21}\) Hence, (186) is given by

$$(\psi_{Kod})_{\text{Inst}} = \exp\left[-3\epsilon_0^3(hG\Lambda)^{-1}\int_{\Sigma} d^3xe^{T(x)}\right] = \psi_{Kod}[T]. \quad (189)$$

The Chern–Simons functional depends completely on $T$, which plays the role of a time variable on configuration space $\Gamma_{Kin}$. The proposed resolution to [4] then simply is that one does not normalize a wavefunction in time. However, one does normalize the wavefunction with respect to the physical degrees of freedom which are orthogonal to the time direction, namely $(X,Y)$, and we have done so using a Gaussian measure for the states in the holomorphic representation. This is consistent with the results of [12], and moreover corresponds to the full theory restricted to quantizable configurations in the instanton representation.

\(^{21}\)The cosmological constant $\Lambda$ fixes the characteristic length scale of the universe at $l \sim \Lambda^{-1/2}$, which is large compared to the discreted Planck length sized scale of quantized increments of the undensitized $\Psi_{ae}$. 

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9.1 Doublecheck on the procedure

We will now doublecheck the consistency of our procedure for passing from the discretized quantum theory to the continuum limit, starting with the Kodama state. Start from the functional differential equation defining the Hamiltonian constraint in the case $\alpha = \beta = 0$

$$
\frac{\Lambda}{3a_0^3}(hG)^3 \frac{\delta^3}{\delta T^3} \psi = -(hG)\frac{\delta^2}{\delta T^2} e^T \psi.
$$

(190)

Factoring out a pair of functional derivatives we have

$$
(hG)^2 \frac{\delta^2}{\delta T^2} \left( \left( \frac{hG}{3a_0^3} \frac{\delta}{\delta T} + e^T \right) \psi \right) = 0.
$$

(191)

We require the argument of the wavefunctional $\psi$ to have support on 3-space, therefore it must be expressible as an integral over 3-space $\Sigma$. Hence

$$
\psi[T] = e^{I[T]},
$$

(192)

where the integral is defined by the limit of a Riemann sum for an discretization of lattice size $\nu$

$$
I[T] = \lim_{\nu \to 0} \sum \nu L(x_n) = \int_{\Sigma} d^3x L(x).
$$

(193)

Equation (191) then reduces to the term in brackets, which is given by

$$
\left( \frac{hG\Lambda}{3a_0^3} \right) \frac{\delta I}{\delta T(x)} + e^T(x) = 0.
$$

(194)

The usual field-theoretical method to integrate (194) would be to perform a contraction over all of 3-space

$$
\left( \frac{hG\Lambda}{3a_0^3} \right) \int \Sigma d^3x \frac{\delta I}{\delta T(x)} \delta T(x) = -\int \Sigma d^3x e^T(x) \delta T(x).
$$

(195)

Since the left hand side is just the functional variation of $I$, this leads to

$$
\delta I = -3a_0^3(hG\Lambda)^{-1} \int \Sigma d^3x \delta(e^T(x)).
$$

(196)
Since both sides of (196) are exact variations in the functional space of fields, we may use the usual rules of antidifferentiation to obtain

\[ I = -3a_0^3(hG\Lambda)^{-1} \int_\Sigma d^3x e^{T(x)}. \]  \hspace{1cm} (197)

We will now derive this result as the continuum limit of discretization without recourse to field theory, starting with the discretized version of (196)

\[ \delta I_x = -3a_0^3(hG\Lambda)^{-1} e^{T_x} \delta T_x \forall x, \]  \hspace{1cm} (198)

as follows. Since both sides of (198) are exact functional variations, we should be able to integrate it with respect to \( T \) at each point \( x \) of the discretization

\[ I_x = \int_\Gamma \delta I_x = -3a_0^3(hG\Lambda)^{-1} \int_\Gamma e^{T_x} \delta T_x, \]  \hspace{1cm} (199)

which brings us to the question of how to perform \( \int \delta T_x \) at a fixed spatial point. The functional derivative in the continuum limit of field theory involves the following action at a single point upon point-splitting regularization

\[ \left( \frac{\delta}{\delta T(x)} e^{T(x)} \right)_\epsilon = \int_\Sigma d^3x f_\epsilon(x, y) \frac{\delta}{\delta T(y)} e^{T(x)} = f_\epsilon(0) e^{T(x)}. \]  \hspace{1cm} (200)

The analogue for the discretized case is exemplified by (55), (56) and (57)

\[ \frac{\delta}{\delta T_x} e^{T_x} \equiv \nu^{-1} \frac{\partial}{\partial T_x} e^{T_x} = \frac{1}{\nu} e^{T_x}, \]  \hspace{1cm} (201)

whence one identifies the regularization function \( f_\epsilon(0) = \frac{1}{\nu} \) with the inverse of the size of the elementary cell of the discretization \( \Delta_N(\Sigma) \). Since the inverse operation of differentiation is antidifferentiation, then the functional integral for the discretized case should be given by

\[ \int \delta T_x e^{T_x} = \nu e^{T_x}. \]  \hspace{1cm} (202)

Therefore the regularized functional integral, which plays the role of an antiderivative on functional space, is given in the continuum limit by

---

\(^{22}\)This is in the functional sense, where the antidifferentiation is carried out independently at each spatial point \( x \in \Sigma \). Note from [16] that functional variation in \( \Gamma \) must commute with spatial variation in \( \Sigma \).
\[
\int_\Gamma e^{T(x)\delta T(x)} = \frac{1}{f_\nu(0)} e^{T(x)} \equiv \nu e^{T(x)}
\]

whence the volume \( \nu \) of the elementary lattice cell comes into play. The prescription for obtaining the wavefunctional is to take the direct product of the exponential of (203) over all points, which produces the Kodama state.

### 9.2 Continuum limit in the general case \( \alpha, \beta \neq 0 \)

We will now attempt to obtain the continuum limit of the \( T \)-dependent part of the wavefunctional for algebraically general spacetimes. Utilizing the previous discretization, assign a value to each point of \( z(x_n) = \nu T(x_n) \) for each \( n \). The hypergeometric part of the solution is of the form

\[
\Phi(z) = 1 + A z + B z^2 + \cdots = e^{\ln \Phi(z)}.
\]

It will suffice to demonstrate this result to second order in \( \nu \), and the remaining orders automatically follow. Now expand the logarithm

\[
\ln(1 + A z + B z^2 + \cdots) = A z + (B - \frac{A^2}{2}) z^2 + \cdots.
\]

Inserting (205) into the right hand side of (204), and taking a product over all \( n \), we have

\[
\prod_{n=1}^{N} e^{\nu A(x_n)T(x_n)} e^{\nu^2(B(x_n) - A^2(x_n)/2)T^2(x)} \ldots
\]

which is the exponential of the sum

\[
\exp\left[\sum_{n=1}^{N} \nu A(x_n)T(x_n)\right] \exp\left[\sum_{n=1}^{N} \nu^2 \left( B(x_n) - \frac{1}{2} A^2(x_n) \right) \right] \cdots = P_1 P_2 \ldots \quad (207)
\]

Recalling that \( \nu \) is the fundamental volume per lattice site of the discretization, we see that the first term of (206) approaches a Riemannian integral

\[
\lim_{\nu \to 0, N \to \infty} P_1 = \lim_{N \to \infty, \nu \to 0} \sum_{n=1}^{N} \nu A(x_n)T(x_n) = \int_{\Sigma} d^3x A(x)T(x). \quad (208)
\]

We have assumed that the space of lattice points is measurable in writing (208), since it has been shown to be measurable in the case of the Kodama state \( \psi_{Kod} \). For the second term of (207) we have
\lim_{\nu \to 0; N \to \infty} P_2 = \exp \left[ \nu \int_{\Sigma} d^3 x \left( B(x) - \frac{1}{2} A^2(x) \right) \right] \to 1. \quad (209)

Assuming that the integral is convergent, then \( \nu \) can be set to zero, which causes this term to vanish. The same effect occurs for higher orders of \( \nu \). In the continuum limit, the \( T \) dependent part of the state would be given by

\[
\Psi[T] = \exp \left[ -6(hG) \Lambda^{-1} \int_{\Sigma} d^3 x a_0^2 e^T \frac{(c + c^- + 1)(c + c^+ + 1)}{(c + 1)(c + a + 1)(c + b + 1)} \right]. \quad (210)
\]

We will assume that (210) is not annihilated by the Hamiltonian constraint in the continuum limit, since the exact solution for all discretizations requires all the higher order terms of (207). Hence, while \( \psi_{\alpha,\beta}(\Delta N(\Sigma)) \in Ker\{\hat{H}\} \), we have that

\[
\lim_{N \to \infty} \psi_{\alpha,\beta}(\Delta N(\Sigma)) = \Psi_{\alpha,\beta}(\Delta \infty(\Sigma)) \not\subset Ker\{\hat{H}\} \quad (211)
\]

unless \( \alpha = \beta = 0 \). The result is that for \( \Lambda = 0 \) the solution space to the Hamiltonian constraint is Cauchy complete, but for \( \Lambda \neq 0 \) it is not Cauchy complete except for the Kodama state. One obtains an exact solution by hypergeometric series for any discretization \( \Delta N(\Sigma) \ \forall N < \infty \). Equation (210) is not a solution in the continuum limit, but all solutions in the discretized case get arbitrarily close to (210) as \( N \to \infty \). This is analogous to approximating a real number using rational numbers, whence the latter set is dense in the former. We may complete the Hilbert space by enlarging it to include the states (210), which with the exception of \( \psi_{\text{Kod}} \) is excluded from the space of solutions.\(^{23}\)

\(^{23}\)Hence while not a solution to the Hamiltonian constraint, (210) can be used as a good approximation for the solution provided that \( \Sigma \) remains discrete. Then the only question is the appropriate length scale for the discretization, which can be chosen to be the Planck length.
10 Conclusion

In this paper we have quantized the full theory of gravity for spacetimes of Petrov Type I, D and O. These spacetimes correspond to quantizable configurations of the kinematic phase space of the instanton representation. The momentum space variables of the instanton representation are chosen to be the densitized eigenvalues of the CDJ matrix, which are directly related to the algebraic classification of spacetime. We have demonstrated the existence of a natural Hilbert space structure of physical states labelled by these eigenvalues. For vanishing cosmological constant the construction of the states is straightforward due to scale invariance of the Hamiltonian constraint, and the continuum limit lies within the same Hilbert space as the discretized version. The $\Lambda \neq 0$ case introduces a length scale into the theory, which admits an expansion of the state in powers of this length scale. We have utilized a Lippman–Schwinger like approach to perform the expansion in the length scale and its inverse. In the former case criterion of convergence requires that the series be terminated at finite order, resulting in an infinite tower of states labelled by one free function and the integers. In the latter case the automatic convergence of the series lifts this restriction, whence the states revert to the continuous labels of the $\Lambda = 0$ states. We have expressed the general solution for the states in a compact notation in terms of hypergeometric functions. The continuum limit of the $\Lambda \neq 0$ states do not solve the Hamiltonian constraint, while the discretized versions do.

The last main area regards the address of the normalizability of the Kodama state $\psi_{Kod}$. The term ‘state’ is a misnomer in that $\psi_{Kod}$ is dependent entirely upon a variable $T$, which plays the role of a clock variable on the configuration space of the instanton representation. The salient characteristics of the state are encapsulated in the aforementioned Hilbert space structure, which is labelled by two functions $\alpha$ and $\beta$ and depend on configuration space variables $(X,Y)$ which are orthogonal to the $T$ direction. $\psi_{Kod}$ is a state in the sense that it corresponds to Type O spacetimes, where $\alpha = \beta = 0$. But this is the direct analogue of minisuperspace on functional configuration space $\Gamma_{Inst}$, since it depends only on ‘time’.\textsuperscript{24} The resolution to the issue normalizability raised by [4]) is that $\psi_{Kod}$ is a time variable, and one should not normalize a wavefunction in time. For $\alpha = \beta = 0$ the normalizable degrees of freedom $X$ and $Y$ become eliminated from the state and there is nothing to normalize. But when nonzero, namely for spacetimes not of Petrov Type O, there is $(X,Y)$ dependence in the state and one carries out a normalization of the state with respect to $X$ and $Y$, while leaving the $T$ dependence intact. The degrees of freedom $(X,Y)$ are orthogonal to the $T$ direction is the same manner that space is orthogonal to time in a spacetime manifold. In the case of gravity, the time dependence

\textsuperscript{24}Note that it is still the full theory with respect to 3-space $\Sigma$. 

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of the state is fixed a hypergeometric function of $\psi_{K\alpha\delta}$ labelled by $\alpha$ and $\beta$, which is the solution to a hypergeometric differential equation in the time $T$. Hence the instanton representation provides a new approach which can be applied to the quantization of the full theory of GR. In this approach the gravitational labels $(\alpha, \beta)$ are stationary with respect to the time $T$. We have not implemented reality conditions on the instanton representation or on the Ashtekar variables. This is because, as shown in Paper XIII, the complex nature of the CDJ matrix already has a physical significance of its own with respect to the determination of principal null directions of space-time. The application of reality conditions, specifically in the sense of the original Ashtekar variables, will be carried out in a separate paper.
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