Triviality and Landau poles are often greeted as harbingers of new physics at 1 TeV. After briefly reviewing the ideas behind this, a model of singular quantum mechanics is introduced. Its ultraviolet structure, as well as some features of its vacuum, related to triviality, very much parallel $\lambda \phi^4$. The model is solvable, exactly and perturbatively, in any dimension. From its analysis we learn that Landau poles do not appear in any exactly computed observable, but only in truncated perturbation theory, when perturbation theory is performed with the wrong sign coupling. If these findings apply to the standard model no new physics at 1 TeV should be expected but only challenges for theorists.
I. Introduction

Our first model of weak interactions was given by the four fermion Fermi interaction [1] with a dimensionful coupling $G_F$. As it was not perturbatively renormalizable, one could not make much quantum field-theoretical sense out of it. This was in sharp contrast with the situation of electromagnetic lepton interactions, for which quantum electrodynamics (QED) was so successful. The search of a perturbatively renormalizable theory of weak interactions, together with symmetry considerations, led to the Weinberg-Salam (WS) theory of electroweak interactions [2], which subsumed QED. The Fermi model was at once understood (modulus confinement) as part of the low energy effective theory which remains once one integrates out the massive gauge fields, $W^\pm$ and $Z^0$ which got their mass form the Higgs mechanism. This leads to $G_F = \frac{1}{\sqrt{2} v^2} \sim M_W^{-2}$, $v$ being the vacuum expectation value of the scalar field, and a range of validity of the Fermi model given by $E < M_W$.

The WS theory works remarkably well up to the maximum present day energies, but it contains the seeds of difficulties at energies approaching the TeV if the Higgs particle is not found by then. These difficulties go under the heading of Landau poles [3], of triviality [4], or of growing coupling, or equivalently, breakdown of perturbation theory [5]. These are of course related phenomena [6] due to the scalar or Higgs sector of the WS theory. The central question is: are they telling us something about new physics at $E \sim 1$ TeV or are they indicating that perturbation theory and some of its hidden remnants in non-perturbative studies [7], like the relation between the Higgs coupling and mass and the vacuum expectation value

$$\lambda = \frac{M_H^2}{2v^2}$$

should be given up or more properly understood? Certainly the first scenario is much more exciting, and probably theoretically easier, once the new physics is known. The second scenario only promises theoretical difficulties in explaining smooth experimental data at 1 TeV. It is understandably less popular, and considered by some somewhat heterodox. Contributing to its understanding is the aim of this work.

Let us go a bit more into these two alternatives. Recall that new physics has, up to now, never come at the hand of Landau poles or triviality. Electron QED, obtained from lepton QED by integrating out muons and taus is an effective theory which has a perturbatively renormalizable truncation, valid for $E < m_\mu$. The new physics comes at $E \approx m_\mu$, zillions of orders of magnitude below its Landau pole, which is beyond the Planck mass. Five flavour quantum chromodynamics (QCD), obtained by integrating out the top quark, is an effective theory with a perturbatively renormalizable truncation, valid for $E < m_t$. The new physics comes at $E \approx m_t$, and QCD has no Landau pole. New physics often does not mean a new particle, but a new vacuum. Again QCD provides examples at the chiral symmetry breaking scale, $\Lambda_\chi$, and at the confinement scale, $\Lambda_{QCD}$. These thresholds, with their effective theories below them, for $E < \Lambda_{QCD}$ [8] and for $\Lambda_{QCD} < E < \Lambda_\chi$ [9], are totally unrelated to any Landau pole. Thus, these effective...
theories, effective à la Weinberg [10], do not provide justification for expecting Landau pole suggested new physics at \(E \sim 1\) TeV. In fact the very WS theory is expected to change its vacuum at these energies which correspond to the electroweak symmetry breaking scale, \(\Lambda_{EW}\). But this is unrelated to Landau poles too.

What about the effective theories à la Wilson [11], the ones the renormalization group produces? They are related to the previous ones, but different [12]. They are valid for \(E < a^{-1}\), \(a\) being the distance up to which one has integrated out the short distance degrees of freedom. The distance \(a\) is arbitrary. There is no qualitative new physics at \(E \sim a^{-1}\), just more of the old stuff. They often have truncations which are perturbatively renormalizable, as when one only keeps relevant and marginal terms around a gaussian fixed point. The couplings depend on \(a\). What happens if the theory is trivial? Could then triviality lead to new physics?

Let us shortly recall in the setting of renormalization group improved perturbation theory the standard logic in favour of new physics. Recalling that there does not exist a strong coupling lattice Higgs model which can be regarded an effective continuum theory at low energies [7] our setting should be qualitatively sound. For \(\lambda \phi^4\) the bare coupling is given to one loop by

\[
\lambda_o(a) = \frac{\lambda(M)}{1 + \frac{3\lambda(M)}{16\pi^2} \ln(Ma)}
\]

so that at

\[
a_L = M^{-1} \exp\left[-\frac{16\pi^2}{3\lambda(M)}\right]
\]

the bare coupling diverges. This is the Landau pole, which, for \(\lambda(M) > 0\), lies on the way towards the continuum limit, \(a \to 0\). One could insist in keeping \(\lambda_o(a) > 0\) bounded as \(a \to 0\), say \(\lambda_o(a) < 1\), so that perturbation theory makes sense. To see how this works, invert (2),

\[
\lambda(M) = \frac{\lambda_o(a)}{1 - \frac{3\lambda_o(a)}{16\pi^2} \ln(Ma)}
\]

Now, \(\lambda_o(a)\) is an adimensional function of \(a\Lambda\), \(\Lambda\) being the physical scale which quantifies the strength of the interaction, as \(\Lambda_{QCD}\) does for QCD. Then one can take the \(a \to 0\) limit keeping \(a\Lambda\) fixed, which thus keeps \(\lambda_o(a)\) fixed, and bounded. The result is

\[
\lim_{a \to 0, a\Lambda \text{ fixed}} \lambda(M) = 0
\]

We have avoided the Landau pole, but no interaction is left. This is known as the Moscow zero. The theory is trivial.

In the Dashen - Neuberger approximation [13], where the gauge and Yukawa couplings are considered a small perturbation, the above results for O(4) scalar fields apply to the WS theory. Then (1) and (5) are only compatible if one keeps \(a \neq 0\), and the WS
theory is an effective theory à la Wilson without non-trivial continuum limit as necessarily \( a > a\bar{H}, \) \( a\bar{H} \) being given by (4) (slightly modified for \( \text{O}(4) \)) and (1) with \( \lambda(M) = \lambda, \) \( M_H = M \) and \( \lambda_o \) fixed, and bounded. The minimal spacing \( a_H \) depends on \( M_H. \) However, as long as

\[
M_H^{-1} > a_H
\]

it has continuum physics. As \( M_H \) increases the left hand side of (6) decreases and approaches the right hand side. At \( M_H \sim 1 \text{ TeV} \) (6) ceases to be valid and continuum physics vanishes. Something must happen if the Higgs is not found by then: new physics. But even if the Higgs is found before, by the same argument new physics is expected for energies \( E \sim a_H^{-1}. \)

This seems a tight scenario, but never before new physics came this way. So alternatives should not be dismissed beforehand. There are alternatives. They are based on giving up (1), which is certainly incompatible with the continuum limit. Recall that triviality does not mean to have a symmetric vacuum, one can have triviality and spontaneous symmetry breaking at the same time. One point functions can be non-vanishing, only three and more point proper Green functions have to vanish [14]. This allows to take the continuum limit, \( a \rightarrow 0, \) with \( \lambda = 0 \) but still \( v \neq 0 \) and \( M_W \neq 0. \) Concrete ways of how to implement this have been worked out [15]. Even some lattice computations hinting in this direction have been published [16]. The Higgs sector would be very weakly interacting and no new physics would be expected. The existence of the Landau pole would be unrelated to new physics.

This alternative scenario leads to important new questions: Is there any physics then in the Landau pole? What is the meaning of perturbation theory if the theory is trivial? It is these questions we want to address here.

It is basically impossible to solve these issues for our \( 3 + 1 \) dimensional quantum field theories (QFT), where we cannot do much more than to compute a few orders of perturbation theory. We will thus study a model of singular quantum mechanics [17], where everything can be worked out exactly, and which has an ultraviolet (UV) structure which matches very precisely the one of \( \lambda\phi^4. \) The picture which emerges is the following: the Landau pole is a feature which only appears in non-observable magnitudes, or in observable ones computed approximately, as one always does in perturbation theory. In the exact expressions for observables there are no Landau poles. The only meaning of the Landau pole is that one started with the wrong sign coupling in doing perturbation theory, so that in converging to the exact result the coupling has to change sign and does so at the Landau pole. The Landau pole would then only be a feature of truncated perturbation theory and a nightmare for theorists, but experimentalists would see nothing of it.

In carrying over our findings, which are exact and rigorous, to the WS theory there are two steps to be taken. First to go from singular quantum mechanics to \( \lambda\phi^4. \) This means going from a theory with only coupling constant renormalization to one with several independent renormalizations. It also means going from a perturbative expansion with a
finite radius of convergence to one which is asymptotic. It finally means going from a theory without non-perturbative contributions to one which will have non-perturbative contributions, like the ones dues to instantons \[18\] and maybe renormalons \[19\]. The other differences are more likely to be irrelevant to the problem at hand. Second to go from $\lambda \phi^4$ to WS. If our findings should be relevant to the WS theory we have to assume that also in presence of gauge symmetries triviality of the scalar sector is compatible with the Higgs mechanism in the continuum limit.

Our model are contact interactions in $d = 2$ quantum mechanics (QM), $V = \lambda \delta^{(2)}(\vec{r})$. We will first give six reasons, some of them related, of why we believe it to be a faithful model for the UV structure and high energy behaviour, and even for the vacuum, of $\lambda \phi^4$ in $D = 3+1$.

1. Both interactions are classically scale invariant.

2. Both require regularization (or alternatively selfadjoint extensions and differential renormalization \[20\] respectively) and show dimensional transmutation \[21\]. The role played by the infinite number of degrees of freedom of $\lambda \phi^4$ is played by the singular character of $\lambda \delta^{(2)}(\vec{r})$ in quantum mechanics. Regularization puts them on a somewhat equivalent footing.

3. Both are perturbatively renormalizable.

4. Both are very likely trivial for $\lambda > 0$.

5. Both very likely collapse for $\lambda < 0$, but are asymptotically free \[22\].

6. Both are given at their critical dimensions in the random walk picture. In the random walk representation of $\lambda \phi^4$ the interaction is given by the crossing of two random paths \[23\]. As the Hausdorff dimension of the random path is $d_H = 2$, $D = 3 + 1 = 2d_H$ is the critical dimension. Thus for $D > 2d_H$ the theory is trivial and for $D < 2d_H$ it is interacting. For $\lambda \delta^{(2)}(\vec{r})$ the interaction is given by the crossing of the origin by the path in the path integral. Again $d = d_H$ is the critical dimension.

The triviality issue of $\lambda \phi^4$ has been studied by taking its non-relativistic limit, which leads to $V = \frac{\lambda}{m^3} \delta^{(3)}(\vec{r})$ \[24\]. We feel that the non-relativistic limit is not adequate for studying an UV issue as triviality. Most of the above reasons do not hold. On the other hand $\lambda \delta^{(2)}(\vec{r}), \lambda < 0$, has been used as a model for asymptotic freedom \[25\]. Here we are interested mainly in $\lambda > 0$, where Landau poles appear, because $\lambda \phi^4$ is just a sick theory for $\lambda < 0$ when taken as it stands \[14\].

To buttress the analogy of $\lambda \delta^{(2)}(\vec{r})$ and $\lambda \phi^4_{3+1}$ we will study $\lambda \delta^{(d)}(\vec{r})$ for $d = 1, 2$ and 3 \[25, 26\] and see how its UV structure matches the one of $\lambda \phi^4_D$ with $D < 3 + 1$, $D = 3+1$ and $D > 3 + 1$ respectively. Our units are $\hbar = 1, 2m = 1$. 

\[5\]
II. \( d = 1 \)

The potential \( V(x) = \lambda \delta(x) \) is solved in many textbooks [27]. Although singular it does not require regularization. For \( \lambda < 0 \) there is one bound state of energy

\[
E_o = -\frac{\lambda^2}{4}
\]  
(7)

For \( E = k^2 \geq 0 \) the scattering amplitude \( f(k) \) is defined as

\[
\Psi_k(x)|_{x \to \infty} \sim e^{ikx} + e^{i(k|x|+\frac{\pi}{2})} (f_+(k)\theta(x) + f_-(k)\theta(-x))
\]  
(8)

For the Dirac delta potential \( f_+ = f_- \equiv f \) reads, for any \( \lambda \),

\[
f(k) \equiv \frac{e^{2i\delta_0} - 1}{2i} = -\frac{\lambda}{2k + i\lambda}
\]  
(9)

which leads to the even phase shift (the odd phase shift vanishes)

\[
tg\delta_0(k) = -\frac{\lambda}{2k}
\]  
(10)

At high energies there is no scattering,

\[
\lim_{k >> \lambda} \delta_0(k) = 0
\]  
(11)

The connection between high and low energies is correctly given by Levinson’s theorem

\[
\frac{1}{2} + \frac{1}{\pi}[\delta_0(0) - \delta_0(\infty)] = 1, \quad \lambda < 0
\]  
(12)

and counts the number of bound states.

Perturbation theory is best built upon the Lippmann - Schwinger equation, which for any \( d \) reads

\[
\Psi(\vec{r}) = \Psi_o(\vec{r}) - \int d^d r' G_{k^+}(\vec{r} - \vec{r}') V(\vec{r}') \Psi(\vec{r}')
\]  
(13)

where \( \Psi_o \) is a solution of the free Schrödinger equation and \( G_{k^+} \) the free propagator. For \( d = 1 \)

\[
G_{k^+}(x) = i^e^{ik|x|}
\]  
(14)

is the Green’s function with the adequate large \( |x| \) behaviour. From (8), (13) and (14) one obtains in perturbation theory

\[
f(k) = -\frac{\lambda}{2k} + i\frac{\lambda^2}{4k^2} + \frac{\lambda^3}{(2k)^3} + ...
\]  
(15)
This is a geometric series which can be summed at high energies, $2k > |\lambda|$. The result is the exact result (9), which is thus reproduced by analytic continuation.

The theory is effectively asymptotically free as the expansion (15) is actually in the dimensionless function

$$\hat{\lambda}(k) \equiv \frac{\lambda}{2k}$$

which vanishes for high energies. One can always introduce a dimensionless constant, the renormalized coupling

$$\hat{\lambda}_r(\mu) \equiv \hat{\lambda}(k = \mu)$$

where $\mu$ is an arbitrary scale. Then (9) reads

$$f(k) = \frac{-\hat{\lambda}_r(\mu)}{k + i\hat{\lambda}_r(\mu)} = \frac{-\hat{\lambda}(k)}{1 + i\hat{\lambda}(k)}$$

The $\mu$-dependence in $f(k)$ is of course fictitious.

At low energies $\hat{\lambda}(k)$ blows up. Notice however that none of the observables, i.e. the scattering amplitude, in any sense reflects this singularity. This is not so for the perturbative result (15), which does not hold at low energies, where it diverges.

So far the two body problem. What happens with the N body problem when $N$ becomes large with fixed density? It is known [28] that $N$ bosons in a space $0 \leq x_i \leq L$ interacting via attractive two body contact interactions ($\lambda < 0$) have a ground state energy per particle which goes as

$$\lim_{N \to \infty} N \frac{E_0(N)}{N} \sim -N$$

This implies that no vacuum exists for $\lambda < 0$. We call vacuum in QM the state of lowest energy per particle for any $N$ at zero particle density or for any particle density as $N \to \infty$. It is this analysis in QM which corresponds to studying the existence of the vacuum in QFT for $\lambda < 0$.

$\lambda\phi^4_{2+1}$ is a very similar QFT. It is perturbatively superrenormalizable and only the vacuum and the mass require renormalization up to a finite order. As the coupling constant is not renormalized this corresponds to no renormalization in QM. The coupling has dimensions and is asymptotically free in very much the same sense as above. The theory is not trivial, it interacts for $\lambda > 0$. For $\lambda < 0$ no ground state exists, but perturbation theory goes through.

One might think that at one point the analogy breaks down for $\lambda < 0$, as QM allows a bound state but QFT has no ground state, and therefore does not make sense. This is because in QM, being a theory which does not allow creation and annihilation of...
particles, a collapse for $N \to \infty$ does not affect the study of the two body problem, and the existence of its possible bound states. In QFT, not having a ground state spoils any sensible physical content of the theory. But it is precisely this feature of QM, and its solvability, which allows the study we are performing, and leads to the understanding we will gain along this work.

III. $d=2$ [29]

We know that $V(\vec{r}) = \lambda \delta^{(2)}(\vec{r})$ requires regularization. We will do so by writing

$$V_R(r) = \frac{\lambda}{2\pi R} \delta(r - R), \quad R > 0$$

which for $R \to 0$ reproduces $V(\vec{r})$. Finite results for physical magnitudes as $R \to 0$ are obtained only if

$$\lambda(R) = \frac{2\pi}{\ln \frac{R}{R_o}} < 0, \quad R < R_o$$

where subleading terms as $R \to 0$ are irrelevant and have been dropped and where $R_o$ is a length which characterizes the strength of the interaction (as $\Lambda_{QCD}$ does for strong interactions). Notice that $\lambda(R)$ is negative. It is called the bare coupling. It leads to a bound state of energy

$$E_o = -\frac{4}{R_o^2} e^{-2\gamma}$$

$\gamma$ being Euler’s constant. $\lambda(R)$ has been traded for $R_o$; this is dimensional transmutation.

One might want to reintroduce a coupling, as a renormalized coupling,

$$\lambda_r(\mu) \equiv \frac{\lambda(R)}{1 - \frac{\lambda(R)}{2\pi} [\ln \frac{\mu R}{2} + \gamma]} = \frac{-2\pi}{\ln \frac{\mu R}{2} + \gamma}$$

in terms of which (22) reads.

$$E_o = -\mu^2 \exp \frac{4\pi}{\lambda_r(\mu)}$$

Notice that $E_o$ does not depend on $\mu$, which is arbitrary and physically irrelevant, but on $\lambda_r(\mu)$. The only relevance of $\mu$ comes from it having dimensions. As

$$\lambda_r(\mu) = \frac{2}{R} e^{-\gamma} = \lambda(R)$$

bare and renormalized couplings are basically the same, with one caveat: the regulator $R$ is introduced for small distances, $R < R_o$, while $\mu$ can take any value.
For scattering, $E = k^2 > 0$, again only $\lambda(R)$ as given by (21) leads to finite results as $R \to 0$. One obtains for the scattering amplitude, defined as

$$\Psi(\vec{r})_{r \to \infty} \sim e^{i\vec{k} \cdot \vec{r}} + \frac{1}{\sqrt{r}} e^{i(kr + \frac{\pi}{4})} f(k, \theta)$$

(26)

the following result

$$f(k, \theta) = \sqrt{\frac{\pi}{2k \ln \frac{kR_o}{2} + \gamma - \frac{i}{2} \pi}} = -\frac{1}{2\sqrt{k\pi}} \frac{\lambda_r(\mu)}{1 + \lambda_r(\mu) \left[ \frac{i}{4} - \frac{1}{2\pi} \ln \frac{k}{\mu} \right]}$$

(27)

It is convenient to introduce a running coupling constant, a function in fact,

$$\lambda_r(k) = \lambda_r(\mu = k)$$

(28)

so that (27) reads

$$f(k, \theta) = \frac{-1}{2\sqrt{k\pi}} \frac{\lambda_r(k)}{1 + \frac{i}{4} \lambda_r(k)}$$

(29)

which displays the $\mu$-Independence of the scattering amplitude as well as the absence of logarithms of $k$ when the running coupling is used. Only $s$-waves scatter.

From the partial wave expansion

$$f(k, \theta) = \frac{-i}{\sqrt{2\pi k}} \sum_{l=-\infty}^{\infty} (e^{zi\delta_l} - 1)e^{il\theta}$$

(30)

one obtains

$$\tan \delta_0(k) = \frac{\pi}{2(ln \frac{kR_o}{2} + \gamma)} = \frac{\pi}{2(ln \frac{k}{\mu} - \frac{2\pi}{\lambda_r(\mu)}} = -\frac{\lambda_r(k)}{4}$$

(31)

There is no scattering at high energies,

$$\lim_{kR_o \to \infty} \delta_0(k) = 0$$

(32)

and Levinson’s theorem holds,

$$\frac{1}{\pi} [\delta_0(0) - \delta_0(\infty)] = 1$$

(33)

The result (32) reflects asymptotic freedom,

$$\lim_{R << R_o} \lambda(R) = 0$$

$$\lim_{kR_o \to \infty} \lambda_r(k) = 0$$

(34)
Notice the existence of a harmless Landau pole,

\[ \lambda(R_L = R_o) = \infty \]
\[ \lambda_r(\mu_L = k_L = \frac{2}{R_o}e^{-\gamma}) = \infty \] (35)

Since \( R \) can always be taken smaller than \( R_o \) and \( \mu \) does not show up in observables, the Landau pole is innocuous. One can indeed go all the way to \( k = 0 \) in (29) or (31) without encountering singularities in physical magnitudes. The coupling \( \lambda_r(k) \) however changes sign and becomes positive. Notice that

\[ k_L^2 = -E_o \] (36)

The existence of the Landau pole is therefore linked to the existence of a bound state, which of course only exists for attractive interactions.

To work out perturbations we only need the propagator

\[ G_{k^+}(\vec{r}) = \frac{i}{4}H_o^{(1)}(kr) \] (37)

where \( H_o^{(1)} \) is the first Hankel function of zero order. To first order one obtains

\[ f^{(1)}(k) = -\frac{\lambda_p}{2\sqrt{2\pi}k} + O(R) \] (38)

where the coupling of (20) has been rebaptized \( \lambda_p \). At second order it diverges as \( Rk \to 0 \),

\[ f^{(2)}(k) = \frac{\lambda_p^2}{2\sqrt{2\pi}k}(\frac{i}{4} - \frac{1}{2\pi}(\ln \frac{kR}{2} + \gamma)) + O(R) \] (39)

In order to get a finite result one has to add a counterterm to the interaction:

\[ V_{R}(r) = V_{R}^{(1)}(r) + V_{R}^{(2)}(r) = \frac{\lambda_p}{2\pi R}(1 - \frac{\lambda_p}{2\pi}(\ln \frac{R\nu}{2} + \gamma))\delta(r - R) \] (40)

where \( \nu \) is an arbitrary scale. As \( V_R \) should not depend on it, \( \lambda_p \) will have to, \( \lambda_p(\nu) \). The first and second order result obtained from (40), which is finite by construction and called renormalized, is

\[ f_r(k) = -\frac{\lambda_p(\nu)}{2\sqrt{2\pi}k}[1 - \frac{\lambda_p(\nu)}{2\pi}(\ln \frac{k}{\nu})] + O(\lambda_p^3) \] (41)

One can proceed to next order. Geometric series appear both for \( V_{R}(r) \) and \( f_r(k) \). The results of the summations are

\[ V_{R}(r) = \frac{1}{2\pi R} \frac{\lambda_p(\nu)}{1 + \frac{\lambda_p(\nu)}{2\pi}(\ln \frac{R\nu}{2} + \gamma)} \delta(r - R) \] (42)
and
\[ f_r(k) = \frac{-1}{2\sqrt{\pi k}} \frac{\lambda_p(\nu)}{1 + \lambda_p(\nu)\left(\frac{i}{4} - \frac{1}{2\pi}\ln \frac{k}{\nu}\right)} \]  

Both reproduce the exact results with the identification
\[ \lambda_p(\nu = \mu) = \lambda_r(\mu) \]  

As for d=1 the exact result is an analytic continuation of the summed perturbative one.

Notice that perturbation theory can be performed both for \( \lambda_p > 0 \) and \( \lambda_p < 0 \). And yet the interaction is eventually always attractive. This makes the Landau pole more significant in perturbation theory. From eq. (42) and for \( \lambda_p > 0 \) but \( \lambda_p << 1 \) one encounters a singularity as \( R\nu \to 0 \). And yet for the scattering amplitude, eq. (43), which is the only predicted observable (\( E_o \) is used to fix \( R_o \)), no singularity appears anywhere. The Landau pole is irrelevant to physics as given by the exact expressions. What it tells us is that if \( \lambda_p > 0 \) but the interaction is actually attractive, the coupling will have to change sign and it does so at the Landau pole. The Landau pole thus only has to be crossed in going from perturbation theory to the exact theory if one started with a coupling of the wrong sign. As seen from eq. (36), when it exists it implies that the true interaction is attractive. On the contrary, if one starts from \( \lambda_p < 0 \) one does not have to cross the Landau pole in taking the continuum limit \( R\nu \to 0 \). But one always sees the Landau pole in \( \lambda_p(k) \) at \( k_L \) given by eq. (36). We have learned, however, that the observables do not notice the Landau pole; the exactly computed observables, of course. Any perturbative truncation will make the observables notice the Landau pole.

What happens for the N-body problem? It is only known that for finite N the system does not collapse, as a lower bound exists [30]. There are strong reasons, however, to believe that as \( N \to \infty \) with fixed density, the ground state energy per particle is unbounded from below. These reasons, based on continuity in going from d=1 to d=3, will become clearer when we study next d=3. If we accept these reasons, then no vacuum exists for \( \lambda < 0 \) in QM.

\( \lambda \phi^{4}_{3+1} \) is a QFT of analogous features. It is perturbatively renormalizable both for \( \lambda > 0 \) and \( \lambda < 0 \). It requires vacuum, mass, field and coupling constant renormalization. In QM this means coupling constant renormalization. For \( \lambda > 0 \) there is a worrisome Landau pole and the theory is believed to be trivial. For \( \lambda < 0 \) the exact theory will not exist as such. QM gives us a model of what could happen: the Landau pole is actually not a problem, but it signals that although \( \lambda_p > 0 \) eventually \( \lambda < 0 \). This is fine in finite N QM, but does not allow to define the QFT, which is thus trivial, \( \lambda = 0 \).

**IV. d=3**

We will use as regulator of \( V(\vec{r}) = \lambda \delta^{(3)}(\vec{r}) \)
\[ V_R(\vec{r}) = \frac{\lambda}{4\pi R^2} \delta(r - R), \quad R > 0 \]  

\[ 11 \]
Finiteness as \( R \to 0 \) requires (omitting irrelevant subleading terms)
\[
\lambda(R) = -4\pi R(1 + \frac{R}{R_o}) < 0, \quad R < |R_o|
\]  
(46)

For \( R_o > 0 \) there is a bound state of energy
\[
E_o = -\frac{1}{R_o^2}
\]  
(47)

but the interaction is always attractive, even when \( R_o < 0 \).

The scattering amplitude, defined as
\[
\Psi(r)_{r \to \infty} \sim e^{i\vec{k} \cdot \vec{r}} + \frac{1}{r} e^{ikr} f(k, \theta, \varphi)
\]  
(48)

comes out to be
\[
f(k) = \frac{-R_o}{1 + iR_o k}
\]  
(49)

Again only s-waves scatter. From the partial wave expansion for rotationally invariant potentials
\[
f(k, \theta) = \frac{1}{2ik} \sum_{l=0}^{\infty} (2l + 1)(e^{2i\delta_l} - 1)P_l(\cos\theta)
\]  
(50)

one obtains
\[
tg\delta_0(k) = -R_o k
\]  
(51)

Notice that there is scattering at high energies,
\[
\lim_{k R_o \to \infty} \delta_0(k) \neq 0
\]  
(52)

This is a surprising result, as the theory looks asymptotically free,
\[
\lim_{R \to 0} \lambda(R) = 0
\]  
(53)

As \( \lambda(R) \) has dimensions this result is not very meaningful. Let us introduce a dimensionless coupling which weighs the UV behaviour of \( V_R(r) \),
\[
\hat{\lambda}(R) \equiv \frac{\lambda(R)}{R}
\]  
(54)

Then
\[
\lim_{R < < R_o} \hat{\lambda}(R) = -4\pi \neq 0
\]  
(55)

and there is no asymptotic freedom. There is interaction, even at high energies, and (52) follows. Levinson’s theorem does not hold either.
Had we taken the limit $k \to \infty$ before $R \to 0$ the result would have been

$$\lim_{kR_o \to \infty} \delta_o(k, R) = 0$$

(56)

The limits $k \to \infty$ and $R \to 0$ do not commute. This is a feature which has catastrophic consequences for three particle systems, known as Thomas effect [31] (see ref. 32 for a more modern presentation): in $d = 3$ the binding energy of the three-particle system increases beyond limit as the range of the two-body short range interaction goes to zero. In other words, non-trivial contact interactions are just too strong in $d = 3$. Eq. (52) hints at it for two particles, the Thomas effect shows it bluntly for three particles. No vacuum exists. Only in the two body channel some, somewhat weird, physics remains.

Furthermore, there is no Landau pole. This is an ominous result. Recall that perturbation theory is blind to the sign of $\lambda_p$, and that it was the Landau pole where the interaction becomes attractive. Since there is no Landau pole, and the exact interaction is attractive, something must go wrong with perturbations. Indeed, it does.

The propagator is

$$G_{k+}(\vec{r}) = \frac{1}{4\pi r} e^{ikr}$$

(57)

First order perturbation theory gives

$$f^{(1)}(k) = -\frac{\lambda_p}{4\pi} + 0(R^2)$$

(58)

while at second order the result is UV divergent,

$$f^{(2)}(k) = \left(\frac{\lambda_p}{4\pi}\right)^2 \left(\frac{1}{R} + ik\right) + O(R)$$

(59)

Adding a counterterm to the interaction,

$$V_R(r) = \frac{\lambda_p}{4\pi R^2} \left(1 + \frac{\lambda_p}{4\pi R}\right)\delta(r - R)$$

(60)

gives

$$f_r(k) = -\frac{\lambda_p}{4\pi} + i\left(\frac{\lambda_p}{4\pi}\right)^2 k + 0(\lambda^3_p)$$

(61)

To third order one obtains

$$f^{(3)}(k, \theta) = \left(\frac{\lambda_p}{4\pi}\right)^2 \left(\frac{1}{R^2} + \frac{2}{3}k^2 + \frac{5}{27}k^2\cos\theta\right)$$

(62)

Notice the $\theta$-dependence! A new counterterm is added to (60)

$$V_R(r) = \frac{\lambda_p}{4\pi R^2} \left(1 + \frac{\lambda_p}{4\pi R} + \left(\frac{\lambda_p}{4\pi R}\right)^2\right)\delta(r - R)$$

(63)
This gives

\[
f_r(k, \theta) = -\frac{\lambda_p}{4\pi} + i\left(\frac{\lambda_p}{4\pi}\right)^2 k + \frac{1}{3}\left(\frac{\lambda_p}{4\pi}\right)^3 k^2 (3 - \frac{4}{9} \cos \theta) + 0(\lambda_p^4) \tag{64}\]

which is \(\theta\)-dependent. For a pointlike, though regularized, potential this is a surprising result. It hints at an interaction which is too strong.

Next, fourth order. The result for the UV-divergent contribution is

\[
f^{(4)}(k, \theta) = \left(\frac{\lambda_p}{4\pi}\right)^4 \left[ \frac{1}{R^3} + \frac{19}{81} \frac{k^2 \cos \theta - \frac{1}{3} \frac{k^2}{R}}{R} \right] + \text{finite terms} \tag{65}\]

There do not exist counterterms functions of only \(R\) and eventually \(\theta\) which make (65) finite. Perturbation theory is not renormalizable. \(\lambda_p\) has the wrong dimensions.

One could have thought of doing perturbations in \(\tilde{\lambda}_p\) as defined in eq. (54). Then all perturbative contributions vanish. One could try intermediate choices, like \(\lambda'_p \equiv R^{-1/2}\lambda_p(R)\). This only shifts the UV problem to higher orders. There does not exist a non-trivial renormalizable perturbation theory. There is no intermediate situation for the coupling between being too weak and being too strong. A further insight into the UV structure of this theory has been gained recently by studying the perturbative \(N > 2\)-body problem [33].

\(\lambda \phi_{4+1}^4\) is a very similar QFT. It is perturbatively non-renormalizable. The coupling has the wrong dimensions. It is trivial, in a much more straightforward sense than \(\lambda \phi_{3+1}^4\), exactly as contact interactions in \(d=3\) are trivial in a more straightforward sense, due to the Thomas effect, than in \(d=2\).

For \(d > 3\) \(\lambda \delta^{(d)}(\vec{r})\) is trivial for any \(\lambda\), and perturbatively non-renormalizable. So is \(\lambda \phi_D^4\) for \(D > 4 + 1\).

**V. Conclusions**

If the WS theory follows the pattern of singular QM no new physics is expected at 1 TeV on the grounds of the existence of a Landau pole in the Higgs sector. That weak interactions become strong only reflects the inadequacy of truncated perturbation theory near a Landau pole. That weak interactions are an effective theory without reasonable continuum limit only follows from sticking to (1). Weak interactions become as difficult at high energies as strong interactions do at low energies. Both have ranges where perturbations work, and ranges where they do not. Neither triviality nor Landau poles announce new physics, they are only a nuisance to theorists. So is infrared slavery.

If the WS theory does not follow the pattern presented here, we are still left with a, both exactly and perturbatively solvable model, with a rich UV structure, and which gives a clear meaning to Landau poles: they signal attractive perturbatively renormalizable interactions, they are not seen in physical magnitudes, they are only a problem when
one truncates perturbation theory which uses the wrong sign coupling and then takes the continuum limit, or when one truncates perturbation theory which uses the right sign coupling, and takes the low energy limit.

Our results also hint at one direction where theoretical progress could be made in QFT when a Landau pole is nearby. The Landau pole is seen in (2) or, equivalently, (42), but it is not seen in the amplitude (43) because it has the coupling also in the denominator. One needs beyond the running coupling a (even if partial) summation of the very observable one is computing in perturbation theory, as e.g. done with K-matrix methods and Padé-approximants [34].

On the other hand the link between Landau poles and triviality is not clear. Landau poles appear in the N=2 channel, triviality for \( N \to \infty \) (for \( d=2 \)). Remember that in our framework, basically equivalent to a non-relativistic QFT but not the one corresponding to the non-relativistic limit of the relativistic QFT we are interested in, the Hilbert spaces of different N are not connected to each other. In other words, is there a model of singular quantum mechanics, perturbatively renormalizable, with Landau pole, and with a non-trivial vacuum?

Our model also lacks non perturbative contributions, say instantons, as perturbation theory reproduces the exact result. Another relevant question thus is: is there a model of singular quantum mechanics, perturbatively renormalizable, with Landau poles and non-perturbative contributions? What do Landau poles mean then?

Let us finish with a final comment on singular QM: it is so predictive, because it is very constrained by the combined requirement of exact and perturbative renormalizability. So much so that e.g. no model with two bound states (two Landau poles) is known. Satisfying both requirements allows to go beyond what we can actually accomplish for QFT. This is why it might add to our understanding of QFT.

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