Research of Data Model under Exponential Distribution Based on Type-II Hybrid Censored Sample

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Abstract. The exponential distribution is one of the popular distributions in reliability and life testing experiments. In this paper, we discuss the exact inference of exponential distribution by considering the Tyoe-II hybrid censoring. We derive the exact conditional distribution of the maximum likelihood estimator (MLE) as well as the exact confidence intervals for the scale parameter of the exponential distribution.

Introduction

The exponential distribution is one of the popular distributions in reliability and life testing experiments. In this paper, we discuss the exact inference of the scale parameter of exponential distribution with consideration of Type-I hybrid censoring scheme. Using the Weibull-to-exponential transformation, we obtain the exact distribution of the scale parameter of exponential distribution. Based on the stochastic monotonicity, we construct the exact confidence interval for the scale parameter. For the complexity of exact conditional survival function, we provide approximate methods and bootstrap methods for constructing confidence interval. Epstein first introduced the Type-I hybrid censoring scheme and proposed two-sided confidence intervals for the failure rate of exponential distribution without any formal proof. Some modifications of Epstein¹ proposition were suggested by Fairbanks et al.² Balakrishnan and Iliopoulos³ formally proved that these conjectures are indeed true thus validating the exact inferential procedures developed by all these authors. Since then, more references about hybrid censoring can refer to Ebrahimi⁴, Chandrasekar, et al.⁵, Balakrishnan and Xie⁶, Kundu et al.⁷ Balakrishnan et al.⁸ Cheng Conghua, Chen Jinyuan⁹.

In a life testing experiment, m items are placed on the test. The failure times observed from such a life-test, Z₁ ≤ Z₂ ≤ ... ≤ Zₘ are the order statistics from a random sample of size m. However, one may not continue the experiment until the last failure since the waiting time for the final failure is unbounded or the experimental cost constrain. For this reason, the experiment may be terminated when the s th failure Zₘ is observed (1 ≤ s ≤ m), which is referred to as a Type-II censoring model. Childs et al.¹⁰ discussed a Type-II hybrid censoring scheme in order to avoid the problem of having too few failures. Type-II hybrid censoring scheme in which the life testing experiment is terminated at a random time T = max(Zₘ, T), Where 1 ≤ s ≤ m and T ∈ (0, +∞) are fixed in advance. Therefore, under this censoring scheme we have one of the following types of observations, Case1: Z₁ ≤ Z₂ ≤ ... ≤ Zₘ denote the observed failure times if 0 ≤ C < s and Case 2: Z₁ ≤ Z₂ ≤ ... ≤ Zₘ denote the observed failure times if s ≤ C ≤ m.

Life test plan (LTP) are developed such that the producer and consumer risks are satisfied. A rich literature exists on discussing LTP. Epstein¹¹ considered life testing situations where the lifetime follows the exponential distribution. Jeong et al.¹² developed a (s, T) hybrid sampling plans for the exponential distribution. CHEN et al.¹³. Presented Bayesian LTP under hybrid censoring of Weibull
distribution. Kim and Yum\textsuperscript{[14]} considered the LTP for the Weibull distribution under accelerated hybrid censoring. Tsai and LIN\textsuperscript{[15]} proposed a method for LTP under progressive interval censoring with likelihood ratio.

**Maximum Likelihood Estimators**

A random variable $Z$ is said to have exponential distribution if its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$\varphi(z; \lambda) = \frac{1}{\lambda} e^{-z/\lambda},$$
$$\Phi(z; \lambda) = 1 - e^{-z/\lambda}, z, \lambda > 0.$$

Here $\lambda$ is the scale parameters, respectively. Under the Type-II hybrid censoring scheme, the life test experiment is terminated at a random time $T^* = \max(Z_{x,m}, T)$, where $1 \leq s \leq m$ and $T$ are fixed in advance. Let $D$ denote the number of observed failures that occur before time $T^*$.

Case 1 If we get the observed data $Z_{x,m} \leq Z_{2,m} \leq \cdots \leq Z_{x,m}$. Based on the observed data, the likelihood function is

$$L(\lambda) = \frac{m!}{(m-s)!} \prod_{i=1}^{s} \varphi(Z_{x,m}^{i})(1 - \Phi(Z_{x,m}^{i}))^{m-s}$$

$$= \frac{m!}{(m-s)!} \lambda^{-s} \prod_{i=1}^{s} Z_{x,m}^{i} e^{-\lambda \sum_{i=1}^{s} Z_{x,m}^{i}} e^{-(m-s)Z_{x,m}^{i}/\lambda}.$$

Taking the derivatives with respect to parameters $\lambda$ and equating to zero, we have the log likelihood

$$0 \leq C \leq s - 1,$$

$$\frac{d}{d \lambda} \ln L(\lambda) = \frac{s}{\lambda} + \frac{1}{\lambda^2} (m-s) \sum_{i=1}^{s} Z_{x,m}^{i} \ln Z_{x,m}^{i} + \frac{1}{\lambda^2} \sum_{i=1}^{s} Z_{x,m}^{i} = 0. \quad (1)$$

From (1), we obtain

$$\lambda = \frac{(m-s) Z_{x,m}^{1} + \sum_{i=1}^{s} Z_{x,m}^{i}}{s} \quad (2)$$

Using a simple iterative procedure, we can obtain the approximate MLE of $\lambda$ from (3), denoted as

$$\hat{\lambda} = \frac{(m-s) Z_{x,m}^{1} + \sum_{i=1}^{s} Z_{x,m}^{i}}{s}$$

Case 2 If $s \leq C \leq m$, we get the observed data $Z_{x,m} \leq Z_{2,m} \leq \cdots \leq Z_{C,m}$. Based on the observed data, the likelihood function is

$$L(\lambda) = \frac{m!}{(m-C)!} \prod_{i=1}^{C} \varphi(Z_{x,m}^{i})(1 - \Phi(T))^{m-C}$$

$$= \frac{m!}{(m-C)!} \lambda^{-C} \prod_{i=1}^{C} Z_{x,m}^{i} e^{-(1/\lambda)(m-C)T^{i}}.$$

Using the same procedure, we get the MLE of $\lambda$

$$\hat{\lambda} = \frac{(m-C)T^{1} + \sum_{i=1}^{C} Z_{x,m}^{i}}{C}.$$

Form case 1 and case 2, the MLE of $\beta$ is
\[ \hat{\lambda} = \begin{cases} \frac{(m-s)Z_{m}^1 + \sum_{i=1}^{s} Z_{im}^i}{s} & \text{for } 0 \leq C \leq s-1, \\ \frac{(m-C)T^1 + \sum_{i=1}^{c} Z_{im}^i}{C} & \text{for } s \leq C \leq m, \end{cases} \]

**Exact Inference of Scale Parameter**

From the above discussion, we see that the MLE of \( \lambda \) have the explicit expression (3) depending on \( \hat{\lambda} \). Then the random variable \( Y = Z \) follows an exponential distribution \( E(1/\lambda) \) with density function \( \phi(z) = \lambda e^{-\lambda z}, z > 0 \).

Then, based on the observed failure data, pre-set observed number \( s \) and specifying Time \( T \). The converted data: \( Z_{1m} \leq Z_{2m} \leq \cdots \leq Z_{vm} \)., for \( 0 \leq C \leq s-1 \) and \( Z_{1m} \leq Z_{2m} \leq \cdots \leq Z_{m} \). for \( s \leq C \leq m \). are from an exponential distribution with failure rate \( 1/\lambda \). The type-II hybrid censoring time is \( T' = \max(Z_{1m}^1, T^1) \), because \( y = z \) is an increasing function with respect to \( z \). CHEN and Bhattacharyya [13] found the MLE of mean life \( \lambda \) as

\[ \hat{\lambda} = \begin{cases} \frac{(m-s)Z_{m}^1 + \sum_{i=1}^{s} Z_{im}^i}{s} & \text{for } 0 \leq C \leq s-1, \\ \frac{(m-C)T^1 + \sum_{i=1}^{c} Z_{im}^i}{C} & \text{for } s \leq C \leq m, \end{cases} \]

Since the converted data follows an exponential distribution with failure rate \( 1/\hat{\lambda} \), the exact distribution of the MLE of the scale parameter can be obtained. We conclude the results as the follows.

**Lemma 1** the moment generating function of \( \hat{\lambda} \) is given by:

\[ M_{\hat{\lambda}}(t) = (1 - \frac{\lambda t}{s})^{-c} \sum_{k=0}^{c-1} \frac{m^k}{k!} q^{(m-c)(1-\lambda t)/s} (1 - q^{(1-\lambda t)/c})^{c-k} \left( 1 - \frac{\lambda t}{c} \right)^{c-k}, \]

Where, \( q = e^{-t/\lambda} \).

Proof The proof of Lemma 1 is similar to Childs et al [10], one can refer to it.

**Theorem 1** The pdf of \( \hat{\lambda} \) is given by:

\[ \varphi_{\hat{\lambda}}(z) = \sum_{c=0}^{\infty} \sum_{k=0}^{c} Q_{k,c} g \left( c \left( \frac{z - b_{k,c}}{\lambda} \right), s \right) + \sum_{c=0}^{\infty} \sum_{k=0}^{c} Q_{k,c} \left( x - b_{k,c}^{2} ; \frac{c}{\lambda} \right), \]

Where

\[ Q_{k,c} = (-1)^{c} \frac{m^c}{c!} q^{m-c+k}, b_{k,c}^{1} = \frac{m-c+k}{s} T^1, b_{k,c}^{2} = \frac{m-c+k}{c} T^1, \]

and \( g(z; \lambda) = \frac{1}{T(\lambda)} z^{c-1} e^{-z}, z > 0 \) is the pdf of common gamma distribution.

Proof Using the moment generating function of \( \hat{\lambda} \) in Lemma 1, by expanding \( (1 - q^{(1-\lambda t)/c})^{c} \) and \( (1 - q^{(1-\lambda t)/c})^{c} \) binomially, we have

\[ M_{\hat{\lambda}}(t) = \left( 1 - \frac{\lambda t}{s} \right)^{-c} \sum_{k=0}^{c-1} \frac{m^k}{k!} q^{(m-c)(1-\lambda t)/s} + \sum_{c=0}^{\infty} \sum_{k=0}^{c} (-1)^{c} \frac{m^c}{c!} q^{(m-c+k)(1-\lambda t)/c} \left( \frac{\lambda t}{c} \right)^{-c} \]

\[ = \sum_{c=0}^{\infty} \sum_{k=0}^{c} Q_{k,c} e^{i(m-c+k)T^1/s} \left( 1 - \frac{\lambda t}{s} \right)^{-c} + \sum_{c=0}^{\infty} \sum_{k=0}^{c} Q_{k,c} e^{i(m-c+k)T^1/c} \left( 1 - \frac{\lambda t}{c} \right)^{-c}. \]

The proof is completed upon noting that \( e^{it(1-t)^{c}} \) is the moment generating function of
The random variable of $Y + d$, where $Y$ is a gamma random variable with density function $g(z; p)$, and $d$ is a real number.

**Theorem 2** The survival function of $\hat{\lambda}$ is given by

$$P_{\hat{\lambda}}(\hat{\lambda} \geq z) = \sum_{c=0}^{\infty} \sum_{k=0}^{\infty} \frac{Q_{k,c} f_{k,c}}{(s-1)!} \Gamma(s, G_{s}(b_{k,c}^1)) + \sum_{c=1}^{m} \sum_{k=0}^{\infty} \frac{Q_{k,c} f_{k,c}}{(c-1)!} \Gamma(c, G_{c}(b_{k,c}^2)),$$

where $G_{k} = k\lambda^{-1}(z - 1)$, $\langle y \rangle = \max(y, 0)$ and $\Gamma(a, b) = \int_{a}^{b} t^{(a-1)} e^{-t} dt$ is the incomplete gamma function.

**Proof** By integrating of the density function in Theorem 1, we can obtain the result.

**Confidence Intervals**

A standard method for constructing exact confidence intervals for a real parameter $\hat{\lambda}$ based on a statistic $\hat{\lambda}$ is “pivoting the cdf”, or, equivalently, the survival function; see, Casella and Berger. The method is applicable as long as $\hat{\lambda}$ is stochastically monotone with respect to $\lambda$, that is $P_{\hat{\lambda}}(\hat{\lambda} > z)$ is a monotone function of $\lambda$ for all $z$. Assuming without loss of generality that it is increasing, the method then proceeds as follows: Choose $\alpha_1$ and $\alpha_2$ such that $\alpha_1 + \alpha_2 = \alpha$ (for example, $\alpha_1 = \alpha_2 = \alpha/2$) and solve the equations $P_{\hat{\lambda}}(\hat{\lambda} > \hat{\lambda}_{obs}) = \alpha_1$, $P_{\hat{\lambda}}(\hat{\lambda} > \hat{\lambda}_{min}) = 1 - \alpha_2$ for $\hat{\lambda}$. Here, $\hat{\lambda}_{obs}$ is the observed value of $\hat{\lambda}$ determined from the given sample. The existence and uniqueness of the solutions of these equations are then guaranteed by the monotonicity of $P_{\hat{\lambda}}(\hat{\lambda} > \hat{\lambda}_{obs})$ with respect to $\hat{\lambda}$. Denote by $\hat{\lambda}_L < \hat{\lambda}_U$ these solutions. Then, $\hat{\lambda}_L$, $\hat{\lambda}_U$ is the realization of an exact 100$(1 - \alpha)$% confidence interval of $\lambda$ is given by solving the following equations $P_{\hat{\lambda}}(\hat{\lambda} \geq \hat{\lambda}_{obs}) = \alpha/2$, $P_{\hat{\lambda}}(\hat{\lambda} \geq \hat{\lambda}_{obs}) = 1 - \alpha/2$. Namely,

$$\sum_{c=0}^{\infty} \sum_{k=0}^{\infty} \frac{Q_{k,c}^1 f_{k,c}}{(s-1)!} \Gamma(s, G_{s}(b_{k,c}^1)) + \sum_{c=1}^{m} \sum_{k=0}^{\infty} \frac{Q_{k,c}^1 f_{k,c}}{(c-1)!} \Gamma(c, G_{c}(b_{k,c}^2)) = \frac{\alpha}{2},$$

$$\sum_{c=0}^{\infty} \sum_{k=0}^{\infty} \frac{Q_{k,c}^2 f_{k,c}}{(s-1)!} \Gamma(s, G_{s}(b_{k,c}^1)) + \sum_{c=1}^{m} \sum_{k=0}^{\infty} \frac{Q_{k,c}^2 f_{k,c}}{(c-1)!} \Gamma(c, G_{c}(b_{k,c}^2)) = 1 - \frac{\alpha}{2}.$$

Where $\hat{\lambda}_{obs}$ is the observed value of $\hat{\lambda}$ from (3),

$$Q_{k,c}^1 = (-1)^k \binom{m}{c} \binom{c}{k} \cdot e^{-(m-c-k+1)\lambda_k} / \lambda_k, Q_{k,c}^2 = (-1)^k \binom{m}{c} \binom{c}{k} \cdot e^{-(m-c-k+1)\lambda_k} / \lambda_k,$$

$$G_{s}(b_{k,c}^1) = \frac{s}{\hat{\lambda}_L} \langle \hat{\lambda}_{obs} - b_{k,c}^1 \rangle, G_{c}(b_{k,c}^2) = \frac{s}{\hat{\lambda}_L} \langle \hat{\lambda}_{obs} - b_{k,c}^2 \rangle,$$

$$G_{s}(b_{k,c}^1) = \frac{s}{\hat{\lambda}_U} \langle \hat{\lambda}_{obs} - b_{k,c}^1 \rangle, G_{c}(b_{k,c}^2) = \frac{s}{\hat{\lambda}_U} \langle \hat{\lambda}_{obs} - b_{k,c}^2 \rangle.$$

We can some numerical method to solve (7) and (8), such as Newton-Raphson method and bisection method to find the $\hat{\lambda}_L$ and $\hat{\lambda}_U$. If the $\hat{\lambda}_L$ and $\hat{\lambda}_U$ are obtained, the 100$(1 - \alpha)$% confidence interval of $(\hat{\lambda}_L, \hat{\lambda}_U)$.

Based on the derived Type-II hybrid censoring data from exponential distribution, we compute the fisher information $I(\lambda)$ with respect to $\lambda$, i.e., $I(\lambda) = \frac{2}{\hat{\lambda}^2} + \frac{2(\sum_{i=1}^{m} E(Z_{i,m})) + (m - J) E(W^1)}{\hat{\lambda}^2}$, where $J = s, W = Z_{s,m}$, for $0 \leq C \leq s - 1$ and $J = C, W = T$, for $s \leq C \leq m$. 173
Through computation, we have $E(Z_{1m}^1) = \lambda / m$. The approximate variance $A_i$ of $\hat{\lambda}$ is

$$A_i = I(\hat{\lambda})^{-1}$$  \hfill (8)

Since the exponential family satisfies all the regularity conditions, the MLE of the scale parameter $\lambda$ follows the asymptotic normal distribution with mean $\lambda$ and asymptotic variance $A_i$. Then the $100(1 - \alpha)\%$ approximate confidence interval of $\lambda$ is $(\hat{\lambda} - Z_{a/2}A_i^{1/2},\hat{\lambda} + Z_{a/2}A_i^{1/2})$. Meeker and Escobar \cite{17} reported that the confidence interval in based on the asymptotic theory of $\ln(\hat{\lambda})$ is superior to that of $\hat{\lambda}$. The approximate $100(1 - \alpha)\%$ confidence interval for $\ln(\hat{\lambda})$ is superior to that of $\hat{\lambda}$. The approximate $100(1 - \alpha)\%$ confidence interval for $\ln(\hat{\lambda})$ is

$$(\ln(\hat{\lambda}) - Z_{a/2}A_{\ln}^{1/2},\ln(\hat{\lambda}) + Z_{a/2}A_{\ln}^{1/2}).$$  \hfill (9)

Therefore, the approximate $100(1 - \alpha)\%$ confidence interval for $\hat{\lambda}$ becomes

$$(\hat{\lambda} e^{-Z_{a/2}A_{\ln}^{1/2}},\hat{\lambda} e^{Z_{a/2}A_{\ln}^{1/2}}).$$  \hfill (10)

Here $A_2 = \hat{\lambda}[2(\sum_{i=1}^J E(Z_{1m}^i) + (m - J)E(W^1)) - \hat{\lambda}J]^{-1}$, and $Z_{a/2}$ is the percentile of the standard normal distribution with right-tail probability $\alpha / 2$.

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