Curvature perturbations and non-Gaussianities from the waterfall phase transition during inflation

Ali Akbar Abolhasani\textsuperscript{1,2}, Hassan Firouzjahi\textsuperscript{2} and Mohammad Hossein Namjoo\textsuperscript{3}

\textsuperscript{1} Department of Physics, Sharif University of Technology, Tehran, Iran
\textsuperscript{2} School of Physics, Institute for Research in Fundamental Sciences (IPM), PO Box 19395-5531, Tehran, Iran
\textsuperscript{3} E-mail: abolhasani@ipm.ir, firouz@ipm.ir and mh.namjoo@ipm.ir

Received 3 November 2010, in final form 13 February 2011
Published 7 March 2011
Online at stacks.iop.org/CQG/28/075009

Abstract

We consider a variant of hybrid inflation where the waterfall phase transition occurs during inflation. By adjusting the parameters associated with the mass of the waterfall field, we can achieve a phase transition that is not sharp, thus inflation can proceed for about 50–60 e-folds after the waterfall phase transition. We show that one can work within the limit where the quantum back-reactions are subdominant compared to the classical back-reactions. It is shown that a significant amount of large scale curvature perturbations are induced from the entropy perturbations. The curvature perturbation spectral index is either blue or red depending on whether the mode of interest leaves the horizon before the phase transition or after the phase transition. This can have interesting observational consequences on CMB. The non-Gaussianity parameter $f_{NL}$ is calculated to be $\lesssim 1$ but much bigger than the slow-roll parameters.

PACS number: 98.80.Cq

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Recent observations [1] strongly support inflation as a correct theory of early universe and structure formation [2]. Thanks to the precision data, different inflationary models can be distinguished based on their predictions for the curvature perturbation spectral index, the amplitude of the primordial gravitational waves and the level of non-Gaussianity.

It proved very difficult to obtain an appreciable amount of non-Gaussianity in simple models of inflation. One has to have either multiple field inflationary scenarios or non-trivial sound speed, for a review see [3–9] and the references therein. Specifically, in models of multiple field inflation when the slow-roll conditions are violated temporarily on the
field space, one may naively expect that an appreciable amount of non-Gaussianities can be produced. Careful examinations in the context of double inflation indicate that this may not be the case [10, 11]. Therefore, it would be interesting to extend this analysis to similar models where non-trivial dynamics such as a sudden change in sound speed during inflation, fields annihilations [12, 13], particle creations [14–17] and phase transition [18–20] occur either during inflation or at the end of inflation [21–27].

In this work we consider a variant of hybrid inflation [28, 29] where the waterfall phase transition occurs during the early stage of inflation. By tuning the effective mass of the waterfall field, we arrange that the waterfall phase transition is mild enough so that inflation continues for a long period, such as 55 e-foldings [30, 31]. To bring the effects of phase transition into the cosmic microwave background (CMB) observational window, we assume that the phase transition occurs around the first few e-foldings, say the first five e-folds. We follow the dynamics of both fields so that our treatment is a two-field inflationary mechanism throughout. We would like to see, first, whether the entropy perturbations can induce a significant amount of large-scale curvature perturbations, and second, whether a significant amount of non-Gaussianity can be produced. These questions [32, 33] attracted new interest in the literature for the model of standard hybrid inflation where the waterfall phase transition occurs very efficiently at the end of inflation. As demonstrated in [34–37] the quantum back-reactions from very small-scale inhomogeneities produced during the waterfall phase transition uplift the tachyonic instability and shut off inflation very efficiently. As a consequence, the large scale curvature perturbations are exponentially suppressed. By the same reasoning, it seems natural to ask whether this conclusion can be averted if one relaxes the model parameters such that the waterfall phase transition, occurring during the early stage of inflation, is not very sharp. This is one of our main goals in this work which we elaborate in the subsequent sections.

As a remark, somewhat related to our work here, there have been many works in the literature concerning curvature perturbations and obtaining non-Gaussianities in models where there are local features during inflation. In [38, 39] this was translated into a sudden violation of the slow-roll condition in a single-field model. In [40, 41] the starting model is a multiple-field scenario, where the power spectrum features are induced from slow-roll violation or phase transition due to other non-inflationary fields. However, the analysis of curvature perturbations and non-Gaussianities in these models is as in single-field inflationary models.

The rest of the paper is organized as follows. In section 2 we present our background and the classical field equations. In section 3 we study the quantum excitations of the waterfall field in detail and calculate the power spectrum of the entropy perturbations. In section 4 we compare the quantum back-reactions and the classical back-reactions and specify the limit where the former can be safely ignored compared to the latter one. In section 5 we study the curvature perturbations induced from the entropy perturbations in detail. In section 6 we use the complementary $\delta N$ formalism to calculate the level of non-Gaussianity in our model. Conclusion and discussions are given in section 7 followed by an appendix describing limiting behaviors of the Bessel functions used extensively in our analysis.

2. Waterfall phase transition during inflation

As explained above we consider a variant of the hybrid inflation model where the waterfall phase transition takes place during inflation. The potential is

$$V(\phi, \psi) = \frac{\lambda}{4} \left( \psi^2 - \frac{M^2}{\lambda} \right)^2 + \frac{1}{2} m^2 \phi^2 + \frac{1}{2} g^2 \phi^2 \psi^2,$$  \hspace{1cm} (1)
where $\phi$ is the conventional inflaton field, $\psi$ is the waterfall field and $\lambda$ and $g$ are two dimensionless couplings. The system has a global minimum given by $\phi = 0$ and $\psi = M/\sqrt{\lambda}$. Our assumption is that the first period of inflation takes place during $\phi_c < \phi < \phi_i$ where $\phi_i$ is the initial value of the inflaton field and $\phi_c = M/g$ is the critical value of $\phi$ where the waterfall field becomes instantaneously massless. Unlike conventional hybrid inflation, we assume that the waterfall field is not heavy so that it can also slowly roll down during the first stage of inflation. For $\phi < \phi_c$ the waterfall becomes tachyonic triggering an instability in the system. Again, unlike standard hybrid inflation, we assume that this phase transition is mild enough so that it will take a long time for the waterfall field to settle down to its global minimum. This provides the second stage of inflation. Our parameters should be such that the second stage of inflation is long enough, 55 e-foldings or so.

It is well known that the potential (1) with the discrete $Z_2$ symmetry is plagued with the domain-wall formations at the end of inflation [42] which are dangerous cosmologically. To get rid of this problem one has to consider $\psi$ to be a complex scalar field so that at the end of inflation cosmic strings are produced which can be safe cosmologically. However, the process of topological defect formation is beyond the scope of this work and we proceed with $\psi$ to be real as is customary in many models of hybrid inflation in the literature.

2.1. The background field dynamics

Here we study the classical evolutions of background fields $\phi$ and $\psi$ before, during and after the phase transition. The background spacetime metric, as usual, is

$$ds^2 = -dt^2 + a(t)^2 dx^2,$$

(2)

where $a(t)$ is the scale factor.

It is more convenient to use the number of e-foldings, $N$, as the clock $dN = H \, dt$. The background field equations are written as

$$\phi'' + (3 - \epsilon)\phi' + \left(\alpha + g^2 \frac{\psi^2}{H^2}\right)\phi = 0$$

(3)

$$\psi'' + (3 - \epsilon)\psi' + \left(-\beta + g^2 \frac{\phi^2}{H^2} + \lambda \frac{\psi^2}{H^2}\right)\psi = 0,$$

(4)

where the prime denotes the differentiation with respect to the number of e-foldings and $\epsilon \equiv -H/H^2$ is the slow-roll parameter which is assumed to be very small by construction. Also the dimensionless parameters $\alpha$ and $\beta$ are defined as

$$\alpha = \frac{m^2}{H^2}, \quad \alpha_0 = \frac{m^2}{H_0^2} = \frac{3\lambda m_p^2 m_p^2}{2\pi M^4},$$

(5)

and

$$\beta = \frac{M^2}{H^2}, \quad \beta_0 = \frac{M^2}{H_0^2} = \frac{3\lambda m_p^2}{2\pi M^2},$$

(6)

where $m_p = 1/G$ for $G$ being the Newton constant and $H_0 \equiv \sqrt{2\pi/3\lambda} M^2/m_p$. Here the quantities $\alpha_0$ and $\beta_0$ respectively represent the values of the parameters $\alpha$ and $\beta$ in the limit when we neglect the variation of $H$ during inflation so that $H \simeq H_0$, $\alpha \simeq \alpha_0$ and $\beta \simeq \beta_0$. However, in order to obtain accurate enough solutions it is important to consider the running of $H$ in our analytical treatments. We work in the limit where $\alpha \ll 1$ corresponds to a light $\phi$ field. Also since we are interested in a mild phase transition, we take $\beta \lesssim 1$. This
in contrast to the standard hybrid inflation model with a sharp phase transition at the end of inflation where $\beta \approx \beta_0 \gg 1$.

To simplify the notation, we take the critical point as the reference point and define $n \equiv N - N_c$. We use the convention that at the start of inflation for $\phi = \phi_i$, $N = 0$, at the time of phase transition $N = N_c$, and at the end of inflation $N = N_e$. With this convention $n < 0$ before the phase transition whereas $n > 0$ afterward.

Let us for the moment assume that $\alpha$ and $\beta$ are constants. At the end of the analysis, we will include the running of $\alpha$ and $\beta$ effectively in the analysis. Also we are in the limit where $g^2 \psi^2 / H^2 \ll \alpha$, i.e. the back-reaction of the waterfall field on the inflaton field is small during inflation. With these assumptions one can easily solve equation (3) to obtain

$$\phi(n) \simeq \phi_c \exp(-r_0 n)$$

with

$$r_0 = \left( \frac{3}{2} - \frac{9}{4 - \alpha_0} \right) \simeq \frac{\alpha_0}{3}.$$  

(8)

Equivalently, one also has

$$N_c \simeq \frac{1}{r_0} \ln \left( \frac{\phi_i}{\phi_c} \right).$$

(9)

As explained before, one cannot neglect the running of $\alpha$ and $\beta$ which results in significant errors in the results above. Here we take into account the running of $\alpha$ and $\beta$. For this purpose, we consider the next leading term in $H$ from the Friedmann equation

$$H^2 \simeq \frac{8\pi}{3 m_p^2} \left( \frac{M_4^4}{4\lambda} + \frac{1}{2} m_p^2 \phi_0^2 \right).$$

(10)

It emerges that the correction to $\alpha$ is less crucial as compared to $\beta$ and an overall averaging would suffice. We define the averaged $H^2$ via

$$\overline{H^2} = \frac{1}{N} \int_0^N H^2(n) \, dn.$$  

(11)

Plugging the result from equation (10) and performing the integral, one obtains

$$\overline{H^2}(N) \simeq H_0^2 \left( 1 + \frac{2\pi \phi_c^2}{N m_p^2} \right).$$  

(12)

Using this averaged value of $H^2$ into the definition of $\alpha$ results in

$$\alpha \simeq \alpha_0 \left( 1 - \frac{2\pi \phi_c^2}{N_c m_p^2} \right),$$

(13)

in which $N_c$ is the total number of e-foldings. This equation modifies $r_0$ to an effective value of $r = \alpha / 3$. Now plugging equations (7) and (12) in the definition of $\beta$ (with $r_0 \to r$) one obtains the first-order correction to $\beta$:

$$\beta \simeq \beta_0 \left( 1 - \Gamma e^{-2\Gamma n} \right),$$

(14)

where $\Gamma \equiv \frac{4\pi}{3} \alpha_0 \left( \frac{\phi_0}{m_p} \right)^2 \ll 1$.

Let us now turn to the dynamics of the $\psi$ field. As it can be confirmed from our full numerical results, we are in the limit where $\psi^2 / H_0^2 \ll \beta / \lambda$ so that the equation for $\psi$ simplifies to

$$\psi'' + 3\psi' + \beta (e^{-2\Gamma n} - 1) \psi = 0.$$  

(15)
Using the modified form of $\beta$ from equation (14), the equation of motion for $\psi$ modifies to

$$\psi'' + 3\psi' - \beta_0[1 - (1 + \Gamma) e^{-2n} + \Gamma e^{-b_\nu n}]\psi = 0.$$  

(16)

The solutions of the above equation are given in terms of the Whittaker functions $e^{(r-3/2)\nu}M_{\nu,\mu}(\frac{\sqrt{r\nu}}{r} e^{-2n})$ and $e^{(r-3/2)\nu}W_{\nu,\mu}(\frac{\sqrt{r\nu}}{r} e^{-2n})$, in which $\kappa \equiv \frac{\sqrt{r\nu}}{4r\sqrt{r\nu}}$, $\nu \equiv \frac{1}{2}\sqrt{\nu_0 + 9/4}$ and $\mu \equiv \nu/2$. In the solution above one can check that $\kappa \gg 1$ and we can approximate the Whittaker functions with the Bessel functions and

$$\psi \simeq e^{-3n/2} \left[ cJ_n \left( \frac{\sqrt{\beta}}{r} e^{-rn} \right) + c'Y_n \left( \frac{\sqrt{\beta}}{r} e^{-rn} \right) \right],$$  

(17)

where $\tilde{\beta} \equiv \beta_0(1 + \Gamma)$.

For our range of parameters one can easily show that $\nu > \frac{\sqrt{\beta}}{r} e^{-rn} \gg 1$ and in this limit $Y_n(x)$ is much larger than $J_n(x)$. After imposing the initial conditions the contributions from $J_n$ in equation (17) become negligible and the classical trajectory can be approximated by

$$\psi \simeq \psi i e^{-3N/2} \left[ Y_n \left( \frac{\sqrt{\beta}}{r} e^{-rn} \right) \right].$$  

(18)

We have checked that this analytic formula for $\psi$ is in good agreement with the results obtained from the full numerical analysis.

3. Dynamics of quantum fluctuations

In this section we study the dynamics of waterfall field quantum fluctuations which play the role of entropy perturbations. The goal is to calculate the curvature perturbations induced from these entropy perturbations which will be the subject of the studies in section 5. As we see, the adiabatic curvature perturbation from the inflaton field is subdominant compared to the curvature perturbation induced from the entropy field.

The equation governing the dynamics of waterfall quantum fluctuations, $\delta \psi_k$, in the momentum space $k$ is

$$\delta \psi_k'' + 3\delta \psi_k' + \left( \frac{k^2}{a^2H^2} - \beta + \frac{\dot{\phi}^2}{H^2} \right) \delta \psi_k = 0.$$  

(19)

By substituting the equation of the $\phi$ field from equation (7) one has

$$\delta \psi_k'' + 3\delta \psi_k' + \left( \frac{k^2}{a^2H^2} + \beta(e^{-2n} - 1) \right) \delta \psi_k = 0.$$  

(20)

Since the effective mass of $\delta \psi_k$ is at the same order as $H$, that is $\beta \lesssim 1$, one can neglect the term containing $\beta$ in equation (20) for the sub-horizon perturbations, and the solution of the $\delta \psi_k$ excitations inside the horizon is given in terms of the Hankel functions $H_{3/2}(k e^{-n}/k_c)$ and $H_{3/2}^{(2)}(k e^{-n}/k_c)$ where $k_c = H e^{N_c}$ is the critical mode which exits the horizon at the moment of phase transition. We require that deep inside the horizon the solutions start from the Bunch–Davis vacuum

$$\delta \psi_k \to \frac{e^{-ik\tau}}{a\sqrt{2k}} \quad \text{as} \quad -k\tau \to \infty,$$  

(21)

where $\tau$ is the conformal time, $d\tau = -a d\tau$. With this initial condition the incoming solution for the modes inside the horizon is obtained to be

$$\delta \psi_k^{(-)}(n) = -\sqrt{\frac{\pi}{4k_c}} e^{-N_c} e^{-3n/2} H_{3/2}^{(1)} \left( \frac{k}{k_c} e^{-n} \right).$$  

(22)
As can be seen from this expression, the amplitude of the quantum fluctuations at the time of horizon crossing \( n_s \) when \( k = e^{c_{1}} k_c \) is given by

\[
|\delta \psi_{k}| \simeq \frac{H}{\sqrt{2k^3}}.
\]  

(23)

Note that here and below an asterisk represents the values of the corresponding quantities at the time of horizon crossing.

After horizon crossing one can neglect the term containing \( k^2 \) in equation (20) and the equation for \( \psi_k \) becomes identical to the background \( \psi \) equation, equation (15), with the answer similar to equation (18):

\[
\delta \psi_k^* \simeq e^{-3n/2} \left[ c_1 J_0 \left( \frac{\sqrt{2} \beta}{r} e^{-n_\alpha} \right) + c_2 Y_0 \left( \frac{\sqrt{2} \beta}{r} e^{-n_\alpha} \right) \right].
\]  

(24)

We need to fix the constants of integrations \( c_1 \) and \( c_2 \) by imposing the matching conditions connecting the outgoing solution \( \delta \psi_k \) to the incoming solution \( \delta \psi_k^* \). The matching condition is performed at \( n = n_m \) when the term containing \( \beta \) in equation (20) becomes comparable to the term containing \( k^2 \) which results in the following equation for \( n_m \):

\[
\left( \frac{k}{k_c} \right)^2 e^{-2n_\alpha} = \beta | e^{-2r_{n_\alpha}} - 1 | \simeq 2 \beta_0 | n_m |.
\]  

(25)

From the above equation one observes that \( n_m \) can be either positive or negative for the physically relevant modes. This means that the time of matching can occur either before or after the waterfall phase transition. Furthermore, comparing equation (25) with the equation of \( n_s \), that is \( e^{n_s} = k/k_c \), one concludes that \( n_m > n_s \) and for a given mode the time of performing the matching condition is always after the time when that mode leaves the horizon.

To further simplify the analysis of matching conditions we can neglect the running of \( n_m \) compared to \( e^{n_\alpha} \) and replace equation (25) with the following simpler equation:

\[
\left( \frac{k}{k_c} \right)^2 e^{-n_\alpha} = 2 \beta_0 r.
\]  

(26)

Imposing the conditions \( \delta \psi_k^*(n_m) = \delta \psi_k^* \) and \( \delta \psi_k'(n_m) = \delta \psi_k'^* \) we can fix \( c_1 \) and \( c_2 \):

\[
c_1 = -\frac{\sqrt{\pi} \pi e^{-N_c}}{4 r} \left[ \beta e^{-r_{n_\alpha}} H_{1/2}^{(1)} Y_0 - \frac{k}{k_c} e^{-n_\alpha} H_{3/2}^{(1)} Y_0 \right],
\]

\[
c_2 = +\frac{\sqrt{\pi} \pi e^{-N_c}}{4 r} \left[ \beta e^{-r_{n_\alpha}} H_{1/2}^{(1)} J_0 - \frac{k}{k_c} e^{-n_\alpha} H_{3/2}^{(1)} J_0 \right].
\]  

(27)

Here primes denote derivatives with respect to the argument of the Bessel and the Hankel functions. Also the arguments of the Hankel and the Bessel functions, respectively, are the same as those in (22) and (24) with \( n = n_m \). We have checked that with these values of \( c_1 \) and \( c_2 \), our analytical solutions (22) and (24) are in very good agreement with the results obtained from the full numerical analysis.

In the appendix we presented approximate formulae for \( c_1 \) and \( c_2 \). Considering the fact that we are in the limit where \( v \gtrsim \sqrt{\beta_0 (1 + T)} e^{-r_{n_\alpha}}/r \gg 1 \) and using the approximate expressions for \( c_1 \) and \( c_2 \) one can check that the term containing \( J_0 \) in (24) becomes subdominant and

\[
\delta \psi_k^* \simeq c_2 e^{-3n/2} Y_0 \left( \frac{\sqrt{2} \beta}{r} e^{-n_\alpha} \right).
\]  

(28)

Now it is time to compute the power spectrum of entropy perturbations, \( S \). Following the prescription of [43] we can perform a local rotation from \( \phi - \psi \) field space to \( \sigma - s \)
field space where \( d\sigma \equiv \cos\theta \, d\phi + \sin\theta \, d\psi \) represents the adiabatic field tangential to the classical trajectory while \( ds \equiv \cos\theta \, d\psi - \sin\theta \, d\phi \) represents the entropy field orthogonal to the classical trajectory. Here \( \theta = \tan^{-1}(\psi'/\phi') \) is the angle between \( \phi \) and \( \psi \) in the field space [43]. As can be verified from our numerical analysis we are in the limit where the trajectory in \( \psi - \phi \) space is very flat, corresponding to \( \theta, \theta' \ll 1 \). We will further elaborate on this point in section 5. This implies that \( \delta s_k \simeq \delta \psi_k \) and \( S_k \equiv \dot{H} \delta s_k \simeq \frac{H}{\psi} \delta \psi_k \) which, using (28), results in

\[
\mathcal{P}_S \simeq \left( \frac{H}{\phi} \right)^3 \frac{4\pi k^3}{(2\pi)^3} |c_2|^2 Y_0^2 \left( \frac{\sqrt{\beta}}{r} e^{-r n} \right) e^{-3n}.
\]

The \( k \)-dependence of the entropy power spectrum is only due to the pre-factor \( k^3 \) and the constant of integration \( c_2 \). Using (26) one has \( d\ln k = dn_m \), so the spectral index of entropy perturbation, \( n_s \), in terms of \( n_m \) is

\[
n_s - 1 = 3 + \frac{\ln |c_2|^2}{dn_m}.
\]

In order to compute the spectral index analytically we use the approximate expression for \( c_2 \) given by equation (A.9) and the limiting behavior of the Bessel function given by equation (A.7) to obtain

\[
n_s - 1 \simeq \frac{4\beta_0}{3} \left( \frac{\beta_0}{9} - r n_m \right).
\]

We have checked that this gives a qualitatively good approximation for \( n_s \) when compared to the full numerical analysis. As explained before, \( n_m \) can be either positive or negative. For modes which leave the horizon before the phase transition \( n_m < 0 \), whereas for modes leaving the horizon after the phase transition \( n_m > 0 \). This means that \( n_s \) can change from a blue spectrum to a red spectrum, depending on whether the mode of interest leaves the horizon before the phase transition or after the phase transition. This conclusion has been verified numerically.

4. Back-reactions: quantum or classical?

Before we proceed to calculate the curvature perturbation and its power spectrum, we have to determine whether or not the quantum back-reactions are small compared to the classical back-reactions. In the standard hybrid inflation model with \( \beta \gg 1 \), corresponding to a very sharp phase transition, it was shown in [34–37] that the quantum back-reactions from very small-scale inhomogeneities produced during the waterfall phase transition dominate exponentially over the classical back-reactions. The back-reactions of quantum fluctuations uplift the tachyonic instability during the waterfall phase transition and shut off inflation very efficiently. In our case at hand with \( \beta \ll 1 \), corresponding to a mild phase transition, one may expect that depending on model parameters the quantum back-reactions are sub-leading and one can only use the classical back-reactions induced from \( \lambda \psi^4 \) and \( g^2 \psi^2 \phi^2 \) interactions to terminate inflation. As shown in [35], the latter becomes important slightly sooner than the former.

The expectation value of the quantum fluctuations sometime after the phase transition is

\[
\langle \delta \psi^2 \rangle = \int \frac{d^3k}{2\pi^3} \delta \psi_k^2.
\]

As mentioned before, \( \theta \ll 1 \) during most of the inflationary period and \( \delta s_k \simeq \delta \psi_k \). We already calculated \( \delta \psi_k \) for super-horizon modes, given by equation (28). We also note that for
super-horizon modes $\delta \psi_k$ evolve as the background $\psi$ field given by equation (18) such that $\delta \psi_k(N) = \Omega(k) \psi(N)$ where

$$\Omega(k) \equiv \frac{c_2(k)}{\psi_i} Y_\nu \left( \frac{\sqrt{\beta} r e^{r N_k}}{2} \right) e^{3 N_k/2}.$$  

(33)

Here we provide an approximation for $\Omega(k)$ which helps us to evaluate the integral in equation (32). Using equation (A.9) one has

$$|c_2| \equiv \gamma(x) \left( \frac{2 \pi e^{-N_i}}{k_c} \frac{e^{-3N_i}}{8r^2} \right) J_\nu(x),$$  

(34)

in which

$$x \equiv \frac{\sqrt{\beta_r (\Gamma^j + 1)}}{r} e^{-r n_m} \gg 1, \quad z \equiv \frac{k}{k_c} e^{-r n_m} \simeq \sqrt{2b_0 r} < 1,$$

(35)

and

$$\gamma(x) \equiv \left[ r(i + z) \left( \frac{2v^2 - x^2}{v} \right) + 3z + 3i \right] \simeq 6i + O(\beta^2).$$  

(36)

One can check that the main $k$-dependence of $c_2$ comes from the Bessel function and in our approximations

$$\Omega(k) \simeq \frac{H_0}{\sqrt{2 \pi} \psi_i} \frac{6}{4v \psi_i} \simeq \frac{H_0}{\sqrt{2 \pi} \psi_i}.$$  

(37)

As a measure of the strength of the quantum back-reactions, we calculate the ratio $\frac{\delta \psi^2(\psi)}{\psi^2}$ and see under what conditions this ratio is small so that one can safely neglect the quantum back-reactions. Using the above approximations one has

$$\frac{\langle \delta \psi^2 \rangle}{\psi^2} \simeq \frac{H_0^2 \psi_i}{4r \psi_i^2} \int_{k_i}^{k_f} \frac{d^3k}{(2\pi)^3} \frac{1}{2k^3} = \frac{H_0^2}{4\pi^2 \psi_i^2} \ln \frac{k_f}{k_i}.$$  

(38)

Here $k_i$ and $k_f$, respectively, correspond to the largest and smallest modes which become tachyonic during inflation. For the smallest scale which becomes tachyonic during inflation we have $k_f \simeq \sqrt{2b_0 r} \exp(N_f) H$ and for the largest mode we can set $k_i = H$. Plugging these into equation (38) and noting that $\sqrt{2b_0 r} \exp(N_f) H \lesssim 1$ results in

$$\frac{\langle \delta \psi^2 \rangle}{\psi^2} \simeq \frac{H_0^2 N_f}{\psi_i^2 4\pi^2}.$$  

(39)

This equation shows that the quantum back-reactions can be safely ignored if $H_0 < \psi_i$, that is if one starts with large enough classical waterfall field values at the start of inflation. It would be more instructive to express the ratio $H_0^2 / \psi_i^2$ in terms of the number of e-foldings and the mass parameters. Using equation (18) one obtains

$$\frac{H_0^2}{\psi_i^2} \simeq \frac{g^2}{\alpha} e^{-3N_f} \left[ \frac{Y_\nu \left( \frac{\sqrt{\beta} r e^{-r N_i}}{2} \right)}{Y_\nu \left( \frac{\sqrt{\beta} r e^{r N_i}}{2} \right)} \right]^2.$$  

(40)

To get this relation, it was assumed that the end of inflation is determined by the back-reactions of the waterfall field on the inflaton field [35], so $g^2 \psi_i^2 \sim m^2 \phi^2$. Using the approximations for the Bessel functions given in equation (A.7), the ratio above is simplified to

$$\frac{H_0^2}{\psi_i^2} \simeq \frac{g^2}{\alpha} \exp \left[ (2r v - r - 3) N_f - \frac{\beta}{2r^2} e^{2r N_i} \right] \simeq \frac{g^2}{\alpha} \exp \left[ \frac{2\beta}{3} \left( N_f - \frac{1}{2r} \right) \right].$$  

(41)
Combined with equation (39), the condition under which one can safely neglect the quantum back-reactions until the end of inflation is translated into

$$\frac{g^2}{\alpha} \ll \exp\left[\frac{-2\beta}{3} \left( N_f - 1 \right) \right] \sim \exp\left[-30\beta\right], \quad (42)$$

where the final approximation is for typical values of $r$ and $N_f$ used in our numerical analysis. For our numerical example with $\beta = 0.7$, $\alpha \simeq 0.04$ and $g^2 = 2 \times 10^{-12}$ this condition can be met easily.

Equation (42) indicates that the strength of the quantum back-reactions is exponentially sensitive to the parameter $\beta$. For fixed values of the inflaton mass and coupling $g$, one has to start with a small enough parameter $\beta$, such that the quantum back-reactions can be safely ignored. This conclusion is consistent with our starting intuition that if the phase transition is mild enough one can neglect the quantum back-reactions. This is also consistent with the conclusion drawn in the model of standard hybrid inflation with $\beta \gg 1$ that the quantum back-reactions dominate exponentially over the classical back-reactions [35].

5. Curvature perturbations

We now have all the materials to calculate the final power spectrum of curvature perturbations. For this purpose we need to know the amplitude of adiabatic curvature perturbations at the time of phase transition as the initial condition, and integrate the evolutions of curvature perturbation from the time of phase transition to the end of inflation. Therefore, the final amplitude of curvature perturbation is

$$R_f = R_0 + \int_{n_f}^{n_i} \mathcal{R}' \, dn, \quad (43)$$

where $R_0$ represents the adiabatic curvature perturbations in the absence of entropy perturbations.

As demonstrated in [43] the evolution of curvature perturbations for the super-horizon modes, induced by the entropy perturbations, can be written as

$$\mathcal{R}' = \frac{2}{\sigma'} \delta s. \quad (44)$$

As one can see from the above equation, both $\theta'$ and $\delta s$ can source the curvature perturbations. We also note that $\theta'$ represents the acceleration of $\psi$, especially during the phase transition. As can be seen from our full numerical analysis, the classical background is such that during inflation and phase transition, $\theta, \theta' \ll 1$. Inflation ends when the classical back-reactions from $g^2 \phi^2\psi^2$ and $\lambda \psi^4$ interactions induce large masses for $\phi$ and $\psi$ such that they roll rapidly to the global minimum. Therefore, in the below analysis, we work in the limit where $\theta, \theta' \ll 1$ and consider the end of inflation when $\theta = \theta_f \simeq 1$.

To calculate the evolution of curvature perturbation from equation (44) we need to estimate $\theta'$ and $\delta s$. The derivative of $\theta$ in field space is

$$\theta' \simeq \tan \theta' = \frac{\psi''}{\phi'} - \frac{\psi' \phi''}{\phi^2}. \quad (45)$$

Since $r \simeq 1/N_e \ll 1$, the first term is much larger than the second term by a factor of $1/r$ and $\theta' \simeq \psi''/\phi'$. Furthermore, as mentioned before, $\delta s = \Omega(k) \delta\psi_k$ for $\theta, \theta' \ll 1$ where $\Omega(k)$ is given by equation (33). Combining the above expressions for $\theta'$ and $\delta s$, the final curvature perturbation is given by

$$R(n_f) \simeq R_0 - 2 \int_{n_f}^{n_i} \frac{\Omega(k)}{\phi^2} \psi'' \psi \, dn. \quad (46)$$
There are some comments in order before we move forward. First, one can check that the integrand is above scales such as $e^{2 \beta n / 3 r}$, which is fast growing, so that one can safely ignore the contribution from the lower limit of the integral. Second, almost all $k$-dependence in the above expression comes from $\Omega_1(k)$. This means that the main contribution to the spectral index is induced from the entropy perturbations. In other words, the spectral index of the curvature perturbations and the entropy perturbations is more or less the same.

Now we proceed to approximately evaluate the integral in equation (46). For this purpose note that $\phi'$ scales like $\exp(r n)$ which is nearly constant so that it can be taken out of the integral. Also we can use the background $\psi$ equation, equation (15), to replace $\psi''$ in favor of $\psi$ and $\psi'$. After these simplifications, one obtains

$$R(n_f) \simeq R_0 + \frac{2 \Omega(k)}{\phi'^2} \left[ \int_0^{n_f} 3 \psi \psi' \, d\psi + \beta(e^{-2\nu n_f} - 1) \int_0^{n_f} \psi^2 \, d\psi \right],$$

in which the function $\beta(e^{-2\nu n} - 1)$ is taken out of the integral by the same reasoning as for $\phi'$. The first integral is a total derivative which can be calculated easily. To calculate the second integral, note that from equations (18) and (A.8) one has

$$\psi \simeq \psi' \left( -\frac{3}{2} + r \nu - \frac{\beta e^{-2\nu n}}{2r \nu} \right)^{-1}.$$

This can be used to transform the second integral above into an approximate total derivative containing $\psi \psi'$. With these simplifications employed, one obtains

$$\beta(e^{-2\nu n} - 1) \int_0^{n} \psi^2 \, d\psi \simeq - \left( \frac{3}{2} + \frac{\beta}{3}(1 - e^{-2\nu n}) \right) \frac{\psi^2(n)}{2}.$$

Plugging this into equation (47) yields our analytic formula for the curvature perturbation

$$R(n) = R_0 + \frac{\beta}{3} (1 - e^{-2\nu n}) \Omega(k) \frac{\psi^2}{\phi'^2},$$

where $\Omega(k)$ is given in equation (33).

In figure 1 we have plotted the predictions of the curvature perturbations from equation (50) and compared them with the full numerical results. As can be seen, they are in good agreement. Furthermore, the induced curvature perturbations from the entropy perturbations dominate by about two orders of magnitude over the initial adiabatic perturbations. Below we find an analytic expression for this enhancement factor.

Now the important question is ‘what is the final amplitude of the curvature perturbation?’. As long as we are not concerned about the curvature perturbations’ $k$-dependence, we can find a simple answer to this question. For this purpose we also need to determine when inflation ends and the curvature perturbations saturate. So far we have been working in the limit where the classical back-reactions are subdominant and $\theta \ll 1$. Once the back-reactions become important we expect $\theta$ to increase significantly. Specifically, from equation (45) one observes that

$$\theta' \simeq \frac{\psi''}{\phi' \left( 1 + \tan^2(\theta) \right)},$$

so once $\theta$ increases significantly $\theta'$ vanishes quickly, indicating that both fields approach their minima. This suggests that the time when the curvature perturbations saturate, which is nearly the time of the end of inflation, is when

$$\theta(n_f) \simeq \frac{\psi(n_f)}{\phi(n_f)} \simeq 1.$$
Here we plot $\ln |R_k|$ for different modes. The blue solid curves are obtained from the full numerical analysis whereas the red dashed curves are obtained from our analytical formula, equation (50), setting $\theta_f = .85$. The induced curvature perturbations from the entropy perturbations dominate over the initial adiabatic curvature perturbations by about two orders of magnitude. Note that the apparent singularity at $N \simeq 30$ is due to the fact that $R_k$ vanishes at this point so that $\ln |R_k|$ diverges, otherwise it has no physically significant meaning. From top to bottom, the curves correspond to modes which leave the horizon at $N = 3$, $N = 6$ and $N = 9$ e-foldings respectively. The waterfall phase transition occurs at $N_c = 7$. The parameters are $M = 7.8 \times 10^{-7} m_p$, $m = 2.5 \times 10^{-7} m_p$ and $\lambda = 2 \times 10^{-12}$.

Imposing this criteria in equation (50) and using the approximations $\phi'(n_f) \simeq e^{-r n_f} \phi'(n_\ast)$ and $\psi' \simeq \frac{3}{4}(1 - e^{-2 r n}) \psi$ (derived from equation (48)) yields the following result for the amplitude of the curvature perturbations at the end of inflation

$$R_f \simeq R_0 \left[ 1 + \frac{\psi(n_f)}{\psi_i} e^{r n_f} \right].$$

This is an interesting result. This indicates that the induced curvature perturbations from the entropy perturbations dominate over the initial adiabatic curvature perturbations $R_0$ by the factor $\frac{\psi(n_f)}{\psi_i} e^{r n_f} \gg 1$. This enhancement can be seen in figure 1. Note however that this enhancing factor cannot be arbitrarily large. As can be seen from equation (39), the quantum back-reactions can become important if we start with an arbitrarily small $\psi_i$. It is worth mentioning that equation (53) has no precise $k$-dependence and if one is interested in the $k$-dependence of the curvature perturbation one should use the original formula (equation (50)) with the $k$-dependence dictated by $\Omega(k)$.

In figure 2 we have plotted the spectral index of curvature perturbations, $n_R - 1$, obtained from our analytical formula, equation (50), compared with the full numerical analysis. As can be seen they are in good agreement. Both curves indicate the running from a blue spectrum to a red spectrum, depending on whether the mode of interest leaves the horizon before the phase transition or after the phase transition. This is in light of discussions below (equation (31)).

6. $\delta N$ formalism and non-Gaussianity

In the previous sections we have employed the perturbative approaches in detail to calculate the curvature perturbations induced from the entropy perturbations. In this section we use the complementary $\delta N$ formalism to verify the previous results for the curvature perturbations. This also enables us to calculate the non-Gaussianity parameter $f_{NL}$ in our model directly.

In $\delta N$ formalism [44–50] curvature perturbation on super-horizon scales can be determined by the variation of the number of e-folds with respect to the field values at
Figure 2. The spectral index of curvature perturbations, $n_R - 1$, for modes which leave the horizon at e-folding $N$. The blue dotted curve is obtained from the full numerical analysis whereas the dashed green curve and the thin solid red curve, respectively, are obtained from our perturbative analysis, equation (50), and the $\delta N$ formalism, equation (62). The parameters are as in figure 1. As explained in the text, modes which leave the horizon approximately before the phase transition ($N_c = 7$) have a blue spectrum, whereas modes which leave the horizon approximately after the phase transition become red-tilted.

the time of horizon crossing

$$\zeta \simeq R \simeq \delta N,$$

in which

$$\delta N \equiv N(\dot{\phi} + \delta \phi, \dot{\psi} + \delta \psi) - \bar{N}(\bar{\phi}, \bar{\psi}).$$

Here $\bar{N}$, $\dot{\phi}$ and $\dot{\psi}$, respectively, are the background number of e-foldings and the background fields starting from the initial flat hyper-surface to the final constant energy density hyper-surface. One should note that in the $\delta N$ formalism the variation should be performed with respect to the field values at the horizon crossing. It is also worth noting that in our case the hyper-surface of the end of inflation is nearly the same as the surface of constant energy density. This is because the former hyper-surface is given by the relation $\psi^2 = \alpha H^2 g^2$, where the back-reaction of the waterfall field on the inflaton field terminates the slow-roll condition [35]. From our previous analysis we know that the main part of the energy at the end of inflation is due to the $\psi$ field so that a constant $\psi$ hyper-surface nearly coincides with a constant energy density hyper-surface.

To employ the $\delta N$ formalism we have to use the background classical trajectory. Using equation (18) one can obtain an implicit function of $n$ in terms of fields at the time of horizon crossing

$$\psi(n) \simeq \psi_* e^{-3(n - n_*)/2} \frac{Y_i \left(\sqrt{\beta} e^{-rn}\right)}{Y_e \left(\sqrt{\beta} e^{-rn_*}\right)},$$

in which $\psi_*$ is the value of $\psi$ at horizon crossing $n_*$. The variation of the above equation results in

$$n_\psi \equiv \frac{dn}{d\psi_*} = \frac{1}{\psi_* f(n)},$$

where

$$f(n) \equiv \frac{3}{2} + \sqrt{\beta} e^{-rn} Y'(\sqrt{\beta} e^{-rn}) \frac{Y_e(\sqrt{\beta} e^{-rn})}{Y_e'(\sqrt{\beta} e^{-rn})}.\quad (58)$$
Here the prime denotes the derivative with respect to the argument of the Bessel function. At leading order the curvature perturbation at the end of inflation is given by
\[ R \simeq \frac{\partial n_e}{\partial \psi^*} \delta \psi^*, \tag{59} \]
where \( \delta \psi^* \) is the value of quantum fluctuations at horizon crossing:
\[ \delta \psi^* = \left( \frac{H}{\sqrt{2\pi k}} \right)_*. \tag*{} \]

The power spectrum of curvature perturbation can be obtained by the above equations with
\[ P_R = \frac{\mathcal{P}}{\pi^2} (\delta \psi_e \delta \psi_e) \]
which results in
\[ P_R = \left( \frac{H}{2\pi \psi} \right)_*^2 \frac{1}{f(n_e)^2}. \tag{60} \]

where \( n_e \equiv N_e - N_c \). In our case \( n_e \) is about 55. This expression for the curvature perturbation should be compared with equation (50) for the curvature perturbations obtained from the perturbative approach. We have checked that they are in good agreement with themselves and with the full numerical solution. From the above expression one can observe that the leading \( k \)-dependence of curvature perturbation comes from the \( 1/\psi^* \) factor, so the spectral index of curvature perturbation can be obtained as
\[ n_R - 1 \simeq -2 \frac{d \ln \psi^*}{d \ln k} = -2 \frac{d \ln \psi^*}{d n_e}. \tag{61} \]

Using equation (57) the final result for the spectral index is as follows:
\[ n_R \simeq 1 - 2 f(n_e) = \frac{5}{2} + \sqrt{\beta} e^{-n_e} \frac{Y_1' \left( \sqrt{\beta} \right) e^{-n_e}}{Y_1 \left( \sqrt{\beta} \right) e^{-n_e}}. \tag{62} \]

This expression for the spectral index of curvature perturbations should be compared with equation (31), the spectral index of the entropy perturbations obtained from the perturbative approach. As demonstrated in the previous section, the main source of curvature perturbations are the entropy perturbations, so the spectral index of curvature perturbation has the same form as that of the entropy perturbations. Furthermore, from equation (62) one observes that for modes which leave the horizon before the phase transition, that is \( n_e < 0 \), the spectral index is blue-tilted. However, for modes which leave the horizon (slightly) after the phase transition the spectral index is red-tilted. In figure 2 we have plotted the spectral index obtained from equation (62) and compared it with the spectral index obtained from the full numerical analysis and from the perturbative analysis, equation (50). The running from a blue spectrum to a red spectrum is common in all three curves.

Finally we can easily obtain the non-Gaussianity parameter in our model by the following formula:
\[ \frac{6}{5} f_{NL} = \frac{n_{\psi \psi}}{n_{\psi}^2} \simeq -f(n_e). \tag{63} \]

Using the approximate behavior of Bessel functions given in the appendix, equations (A.7) and (A.8), to simplify \( f(n_e) \) we obtain
\[ f_{NL} \simeq \frac{5}{6} \left( rv - \frac{3}{2} \right) \simeq 5 \beta \frac{18}{18}. \tag{64} \]

This is a very interesting formula, indicating that the level of non-Gaussianity in our model is controlled by the parameter \( \beta \) measuring the sharpness of the phase transition during inflation. For our numerical example with \( \beta = 0.7 \), we obtain \( f_{NL} \simeq 0.2 \). In figure 3 we have plotted \( f_{NL} \) obtained from our full numerical analysis for different modes. As we see, \( f_{NL} \) measured
Here we present \( \ln |f_{NL}| \) for different modes obtained from the full numerical analysis. The solid blue curve, the dashed red curve and the dashed-dotted green curve, respectively, correspond to modes which leave the horizon at \( N = 3 \), \( N = 6 \) and \( N = 9 \) e-foldings. As can be seen, the final value of \( f_{NL} \) saturates at \( f_{NL} \approx 0.2 \) which is very well approximated by our analytical formula (equation (64)). The bump at \( N \approx 30 \) is because \( R_k = 0 \) at this point and since \( f_{NL} \sim R^{-1} k \) it peaks at this point. Otherwise, it has no physically significant meaning. Similarly, the apparent singularity at \( N \approx 40 \) appears because \( f_{NL} \) changes sign at the final stage of inflation and \( \ln |f_{NL}| \) diverges at this point. As before, it has no significant meaning because we measure \( f_{NL} \) and \( R_k \) at the end of inflation. The parameters are the same as in figure 1.

at the end of inflation is very well approximated by our analytical estimation (equation (64)) and saturates at the expected value \( f_{NL} = 0.2 \). Also, as can be seen from the plot, \( f_{NL} \) has no significant \( k \)-dependence.

We see that generically \( f_{NL} \lesssim 1 \) in our model because of the assumption that the phase transition is mild during inflation. However, this level of non-Gaussianity is much bigger than the level of non-Gaussianity predicted in simple inflationary models which is at the order of the slow-roll parameters. By increasing \( \beta \) by one or two orders of magnitude one can obtain significant non-Gaussianities [37]. However, with \( \beta \gg 1 \) we approach the standard hybrid inflation model where the waterfall phase transition is very sharp and the quantum back-reactions dominate exponentially over the classical back-reactions and one should take their effects into account. This conclusion is also consistent with the bound obtained in equation (42) in order to neglect the quantum back-reactions compared to the classical back-reactions.

### 7. Conclusion and discussions

In this work we considered a variant of hybrid inflation where the waterfall phase transition is mild so that a long period of inflation can be obtained after the phase transition. We found the model parameters where the quantum back-reactions are sub-dominant compared to the classical back-reactions. As can be seen from equation (42), the strength of quantum back-reactions is exponentially sensitive to the parameter \( \beta \). For a fixed value of inflaton mass and the coupling \( g \), one has to start with a small enough \( \beta \), such that the quantum back-reactions are negligible.

We have calculated the curvature perturbations and the spectral index perturbatively as well as using the complementary \( \delta N \) formalism. We have shown that the curvature perturbations induced from the entropy perturbations dominate over the adiabatic perturbations by the enhancing factor \( \frac{\psi_r}{\psi_i} e^{n_{ns}} \gg 1 \). Consequently, the spectral index of curvature perturbations has the same form as the spectral index of the entropy perturbations. We have shown that \( n_S \) runs from a blue spectrum to a red spectrum depending on the time when the mode of
interest leaves the horizon. For the modes which leave the horizon approximately before the phase transition $n_\mathcal{R} > 1$, whereas for the modes which leave the horizon approximately after the phase transition $n_\mathcal{R} < 1$. This may have interesting consequences when compared to the WMAP data [1] to fit the CMB observations. We would like to return to this question in future.

Using the $\delta N$ formalism we have calculated the level of non-Gaussianity in our model. We found the interesting result that $f_{NL} \simeq \beta^{18} \ll 1$. This indicates that the parameter $\beta$, which is a measure of the sharpness of the phase transition, controls not only the strength of the quantum back-reactions but also the level of non-Gaussianity. Although this level of non-Gaussianity may not be detectable observationally in near future, it is one order of magnitude larger than the level of non-Gaussianity predicted in simple models of inflation.

As studied in [30], in our model with $\beta \ll 1$, primordial black holes can be formed copiously which may be dangerous cosmologically. The analysis of black hole formation and their cosmological consequences are beyond the scope of this work. We would like to come back to this question in future.

Acknowledgments

We would like to thank P Creminelli, J Gong, M Sasaki, T Tanaka and J Yokoyama for useful discussions and correspondence. HF would like to thank the Yukawa Institute for Theoretical Physics (YITP) for their hospitality during the events ‘Gravity and Cosmology 2010’ and ‘YKIS2010 Symposium: Cosmology—The Next Generation’ where this work was in progress. AAA also would like to thank YITP for their hospitality when this work was in its final stage.

Appendix. Approximate behaviors of Bessel functions

In this appendix we present some asymptotic behaviors of Bessel functions for the range of parameters relevant to our analysis. These approximate relations are very useful in evaluating $c_2$ in equation (27) which is also used to calculate the curvature perturbation power spectrum and the spectral index.

The Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ satisfy the following equation of motion:

$$f''(x) + \frac{1}{x} f'(x) + \left(1 - \frac{\nu^2}{x^2}\right) f(x) = 0. \tag{A.1}$$

We are in the limit where $\nu \gtrsim x \gg 1$. In this limit the above equation simplifies to approximately

$$f''(x) - \frac{\nu^2}{x^2} f(x) \simeq 0, \tag{A.2}$$

which has a simple solution of

$$f(x) \simeq c x^{\pm(\nu^2 + 1)/2} \sim c x^{\pm \nu}. \tag{A.3}$$

Here $c$ is a constant of integration and throughout this appendix the upper and lower signs belong to the Bessel function of the first and second kind, respectively. To improve our approximation, note that the main error in the analysis above is in ignoring the factor 1 in comparison with $\nu^2/x^2$ in (A.1). Therefore, to the next leading order, the differential equation is

$$f''(x) + \left(1 - \frac{\nu^2}{x^2}\right) f(x) = 0. \tag{A.4}$$
This equation modifies the solution (A.3) by allowing \( c \) to be \( x \)-dependent, whereas previously it was a constant. Since this is the second-order approximation we can assume that \( c(x) \) is a slowly varying function of \( x \) and ignore its second derivative. Using this assumption and substituting (A.3) into (A.4) results in the following equation for \( c \):

\[
(\pm \sqrt{4\nu^2 + 1} + 1)c'(x) + xc(x) \simeq 0,
\]

(A.5)

which has a simple solution

\[
c(x) \simeq \hat{c} \exp \left( -\frac{x^2}{\pm 2\sqrt{4\nu^2 + 1} + 2} \right) \simeq \hat{c} \exp \left( -\frac{x^2}{4\nu} \right).
\]

(A.6)

Finally, we should determine the constant \( \hat{c} \). Since the overall behavior of our approximate solution looks like the standard small argument limit of the Bessel function \( (\nu \sim x \ll 1) \), this can be used to fix \( \hat{c} \) so that our approximate solution conforms to the small argument expansion of the Bessel function. This leads to our desired formulae

\[
J_{\nu}(x) \simeq \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{\nu} \right)^{\nu} \exp \left( -\frac{x^2}{4\nu} \right)
\]

\[
Y_{\nu}(x) \simeq -\frac{\Gamma(\nu)}{\pi} \left( \frac{x}{\nu} \right)^{\nu} \exp \left( \frac{x^2}{4\nu} \right).
\]

(A.7)

One can check numerically that the above equations are a good approximation of Bessel functions.

Now we obtain an approximate expression for \( c_1 \) and \( c_2 \) in equation (27) which we need for the calculation of spectral index and quantum fluctuations. From (A.7) one has

\[
J'_{\nu}(x) \simeq J_{\nu}(x) \left( \frac{\nu}{x} - \frac{x}{2\nu} \right)
\]

\[
Y'_{\nu}(x) \simeq Y_{\nu}(x) \left( -\frac{\nu}{x} + \frac{x}{2\nu} \right),
\]

(A.8)

where the prime denotes the derivative with respect to the argument \( x \). Using these expressions and also the explicit form of Hankel functions one obtains the following approximations for \( c_1 \) and \( c_2 \):

\[
c_1 \simeq \sqrt{\frac{2}{k_c}} \pi e^{-N_c} \left( \frac{x}{8r} \right)^{\nu} \left[ 1 + \frac{z}{2r} \right] \left( -r(1 + z) + 3z + 3i \right)
\]

\[
c_2 \simeq -\sqrt{\frac{2}{k_c}} \pi e^{-N_c} \left( \frac{x}{8r} \right)^{\nu} \left[ 1 + \frac{z}{2r} \right] \left( r(1 + z) + 3z + 3i \right).
\]

(A.9)

Here \( x \) and \( z \) are the arguments of Bessel and Hankel functions in equation (27), respectively, defined by

\[
x \equiv \sqrt{\frac{\rho_0(\Gamma + 1)}{r}} e^{-\rho_n} \simeq \sqrt{2\rho_0/r} \ll 1.
\]

(A.10)

References

[1] Komatsu E et al (WMAP Collaboration) 2011 Seven-year Wilkinson microwave anisotropy probe (WMAP) observations: cosmological interpretation Astrophys. J. Suppl. 192 18 (arXiv:1001.4538 [astro-ph.CO])

[2] Guth A H 1981 The inflationary universe: a possible solution to the horizon and flatness problems Phys. Rev. D 23 347
Linde A D 1982 A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primordial monopole problems Phys. Lett. B 108 389
Albrecht A and Steinhardt P J 1982 Cosmology for grand unified theories with radiatively induced symmetry breaking Phys. Rev. Lett. 48 1220
[3] Chen X, Huang M X, Kachru S and Shiu G 2007 Observational signatures and non-Gaussianities of general single field inflation J. Cosmol. Astropart. Phys. JCAP01(2007)002 (arXiv:hep-th/0605045)
[4] Bartolo N, Matarrese S and Riotto A 2010 Non-Gaussianity and the cosmic microwave background anisotropies Adv. Astron. 2010 157079 (arXiv:1001.3957 [astro-ph.CO])
[5] Wands D 2010 Local non-Gaussianity from inflation Class. Quantum Grav. 27 124002 (arXiv:1004.0818 [astro-ph.CO])
[6] Koyama K 2010 Non-Gaussianity of quantum fields during inflation Class. Quantum Grav. 27 124001 (arXiv:1002.0600 [hep-th])
[7] Byrnes C T and Choi K Y 2010 Review of local non-Gaussianity from multi-field inflation Adv. Astron. 2010 724525 (arXiv:1002.3110 [astro-ph.CO])
[8] Suyama T, Takahashi T, Yamaguchi M and Yokoyama S 2010 On classification of models of large local-type non-Gaussianity arXiv:1009.1979 [astro-ph.CO]
[9] Komatsu E 2010 Hunting for primordial non-Gaussianity in the cosmic microwave background Class. Quantum Grav. 27 124010 (arXiv:1003.6097 [astroph-ph])
[10] Barnaby N, Huang Z, Kofman L and Pogosyan D 2009 Curvature perturbation spectrum from false vacuum inflation J. Cosmol. Astropart. Phys. JCAP01(2009)001 (arXiv:0804.4488 [astro-ph])
[11] Nambu Y and Sasaki M 1990 Purely quantum derivation of density fluctuations in the inflationary universe Prog. Theor. Phys. 83 37
[12] Li M and Wang Y 2009 Multi-stream inflation J. Cosmol. Astropart. Phys. JCAP07(2009)033 (arXiv:0903.2123 [hep-th])
[13] Li M and Wang Y 2009 Multi-stream inflation J. Cosmol. Astropart. Phys. JCAP07(2009)033 (arXiv:0903.2123 [hep-th])
[14] Byrnes C T, Choi K Y and Hall L M H 2009 Large non-Gaussianity from two-component hybrid inflation J. Cosmol. Astropart. Phys. JCAP02(2009)017 (arXiv:0812.0807 [astro-ph])
[15] Byrnes C T, Choi K Y and Hall L M H 2008 Conditions for large non-Gaussianity in two-field slow-roll inflation J. Cosmol. Astropart. Phys. JCAP10(2008)008 (arXiv:0807.1101 [astro-ph])
[24] Alabidi L 2006 Non-Gaussianity for a two component hybrid model of inflation J. Cosmol. Astropart. Phys. JCAP10(2006)015 (arXiv:astro-ph/0604611)
[25] Alabidi L and Lyth D 2006 Curvature perturbation from symmetry breaking the end of inflation J. Cosmol. Astropart. Phys. JCAP08(2006)006 (arXiv:astro-ph/0604569)
[26] Yokoyama S and Soda J 2008 Primordial statistical anisotropy generated at the end of inflation J. Cosmol. Astropart. Phys. JCAP08(2008)005 (arXiv:0805.4265 [astro-ph])
[27] Cogollo H R S, Rodriguez Y and Valenzuela Toledo C A 2008 On the issue of the zeta series convergence and loop corrections in the generation of observable primordial non-Gaussianity in slow-roll inflation: 1. The bispectrum J. Cosmol. Astropart. Phys. JCAP08(2008)029 (arXiv:0806.1546 [astro-ph])
Rodriguez Y and Valenzuela Toledo C A 2010 On the issue of the zeta series convergence and loop corrections in the generation of observable primordial non-Gaussianity in slow-roll inflation: 2. The trispectrum Phys. Rev. D 81 023531 (arXiv:0811.4092 [astro-ph])
[28] Linde A D 1994 Hybrid inflation Phys. Rev. D 49 748 (arXiv:astro-ph/9307002)
[29] Copeland E J, Liddle A R, Lyth D H, Stewart E D and Wands D 1994 False vacuum inflation with Einstein gravity Phys. Rev. D 49 6410 (arXiv:astro-ph/9401011)
[30] Garcia-Bellido J, Linde A D and Wands D 1996 Density perturbations and black hole formation in hybrid inflation Phys. Rev. D 54 6040–58 (arXiv:astro-ph/9605094)
[31] Clesse S 2010 Hybrid inflation along waterfall trajectories arXiv:1006.4522 [gr-qc]
[32] Taruya A and Nambu Y 1998 Cosmological perturbation with two scalar fields in reheating after inflation Phys. Lett. B 28 37 (arXiv:gr-qc/9709035)
Bassett B A, Kaiser D I and Maartens R 1999 General relativistic preheating after inflation Phys. Lett. B 455 84 (arXiv:hep-ph/9808404)
Finelli F and Brandenberger R H 1999 Parametric amplification of gravitational fluctuations during reheating Phys. Rev. Lett. 82 1362 (arXiv:hep-ph/9809490)
Finelli F and Brandenberger R H 2000 Parametric amplification of metric fluctuations during reheating in two field models Phys. Rev. D 62 083502 (arXiv:hep-ph/0003172)
Zibin J P, Brandenberger R H and Scott D 2001 Backreaction and the parametric resonance of cosmological fluctuations Phys. Rev. D 63 043511 (arXiv:hep-ph/0007219)
Bassett B A and Viniegra F 2000 Massless metric preheating Phys. Rev. D 62 043507 (arXiv:hep-ph/9909353)
Bassett B A, Gordon C, Maartens R and Kaiser D I 2000 Restoring the sting to metric preheating Phys. Rev. D 61 061302 (arXiv:hep-ph/9909482)
Jedamzik K and Sigl G 2000 On metric preheating Phys. Rev. D 61 023519 (arXiv:hep-ph/9906287)
Liddle A R, Lyth D H, Malik K A and Wands D 2000 Super-horizon perturbations and preheating Phys. Rev. D 61 103509 (arXiv:hep-ph/9912473)
Ivanov P 2000 On generation of metric perturbations during preheating Phys. Rev. D 61 023505 (arXiv:astro-ph/9906415)
Tanaka T and Bassett B 2003 Application of the separate universe approach to preheating arXiv:astro-ph/0302544
Tsujikawa S and Bassett B A 2002 When can preheating affect the CMB? Phys. Lett. B 536 9 (arXiv:astro-ph/0204031)
Tsujikawa S, Parkinson D and Bassett B A 2003 Correlation-consistency cartography of the double inflation landscape Phys. Rev. D 67 083516 (arXiv:astro-ph/0210322)
Levasseur L P, Laporte G and Brandenberger R 2010 Analytical study of mode coupling in hybrid inflation Phys. Rev. D 82 123524 (arXiv:1004.1425 [hep-th])
[33] Barnaby N and Cline J M 2007 Non-Gaussianity from tachyonic preheating in hybrid inflation Phys. Rev. D 75 086004 (arXiv:astro-ph/0611750)
Barnaby N and Cline J M 2006 Non-Gaussian and nonscale-invariant perturbations from tachyonic preheating in hybrid inflation Phys. Rev. D 73 106012 (arXiv:astro-ph/0601481)
Mazumdar A and Rocher J 2011 Particle physics models of inflation and curvaton scenarios Phys. Rep. 497 85–215 (arXiv:1001.0993 [hep-ph])
Enqvist K, Jokinen A, Mazumdar A, Multamaki T and Vaihkonen A 2005 Non-Gaussianity from preheating Phys. Rev. Lett. 94 161301 (arXiv:astro-ph/0411394)
Enqvist K, Jokinen A, Mazumdar A, Multamaki T and Vaihkonen A 2005 Non-Gaussianity from instant and tachyonic preheating J. Cosmol. Astropart. Phys. JCAP03(2005)010 (arXiv:hep-ph/0501076)
Kohri K, Lyth D H and Valenzuela Toledo C A 2010 Preheating and the non-Gaussianity of the curvature perturbation J. Cosmol. Astropart. Phys. JCAP02(2010)023 (arXiv:0904.0793 [hep-ph])
Lyth D H 2010 Issues concerning the waterfall of hybrid inflation arXiv:1005.2461 [astro-ph.CO]
[35] Abolhasani A A and Firouzjahi H 2010 No large scale curvature perturbations during waterfall of hybrid inflation arXiv:1005.2934 [hep-th]

[36] Fonseca J, Sasaki M and Wands D 2010 Large-scale perturbations from the waterfall field in hybrid inflation J. Cosmol. Astropart. Phys. JCAP09(2010)012 (arXiv:1005.4053 [astro-ph.CO])

[37] Gong J O and Sasaki M 2010 Waterfall field in hybrid inflation and curvature perturbation arXiv:1010.3405 [astro-ph.CO]

[38] Adams J A, Cresswell B and Easther R 2001 Inflationary perturbations from a potential with a step Phys. Rev. D 64 123514 (arXiv:astro-ph/0102236)

[39] Chen X, Easther R and Lim E A 2007 Large non-Gaussianities in single field inflation J. Cosmol. Astropart. Phys. JCAP06(2007)023 (arXiv:astro-ph/0611645)

[40] Hotchkiss S and Sarkar S 2010 Non-Gaussianity from violation of slow-roll in multiple inflation J. Cosmol. Astropart. Phys. JCAP05(2010)024 (arXiv:0910.3373 [astro-ph.CO])

[41] Jaladat, Sahni V and Starobinsky A A 2008 A new universal local feature in the inflationary perturbation spectrum Phys. Rev. D 77 023514 (arXiv:0711.1585 [astro-ph])

[42] Felder G, Garcia-Bellido J and Greene B et al 2001 Dynamics of symmetry breaking and tachyonic preheating Phys. Rev. Lett. 87 011601 (arXiv:hep-ph/0012142)

[43] Gordon C, Wands D, Bassett B A and Maartens R 2001 Adiabatic and entropy perturbations from inflation Phys. Rev. D 63 023506 (arXiv:astro-ph/0009131)

[44] Starobinsky A A 1985 Multicomponent de Sitter (inflationary) stages and the generation of perturbations JETP Lett. 42 152

[45] Salopek D S and Bond J R 1990 Nonlinear evolution of long wavelength metric fluctuations in inflationary models Phys. Rev. D 42 3936

[46] Sasaki M and Stewart E D 1996 A general analytic formula for the spectral index of the density perturbations produced during inflation Prog. Theor. Phys. 95 71 (arXiv:astro-ph/9507001)

[47] Sasaki M and Tanaka T 1998 Super-horizon scale dynamics of multi-scalar inflation Prog. Theor. Phys. 99 763 (arXiv:gr-qc/9801017)

[48] Wands D, Malik K A, Lyth D H and Liddle A R 2000 A new approach to the evolution of cosmological perturbations on large scales Phys. Rev. D 62 043527 (arXiv:astro-ph/0003278)

[49] Lyth D H, Malik K A and Sasaki M 2005 A general proof of the conservation of the curvature perturbation J. Cosmol. Astropart. Phys. JCAP05(2005)004 (arXiv:astro-ph/0411220)

[50] Lyth D H and Rodriguez Y 2005 The inflationary prediction for primordial non-Gaussianity Phys. Rev. Lett. 95 121302 (arXiv:astro-ph/0504045)