Multi-hop Cooperative Wireless Networks: Diversity Multiplexing Tradeoff and Optimal Code Design

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Abstract

We consider single-source single-sink (ss-ss) multi-hop relay networks, with slow-fading links and single-antenna half-duplex relay nodes. While two-hop cooperative relay networks have been studied in great detail in terms of the diversity-multiplexing tradeoff (DMT), few results are available for more general networks. In this paper, we identify two families of networks that are multi-hop generalizations of the two-hop network: $K$-Parallel-Path (KPP) networks and layered networks.

KPP networks, can be viewed as the union of $K$ node-disjoint parallel relaying paths, each of length greater than one. KPP networks are then generalized to KPP(I) networks, which permit interference between paths and to KPP(D) networks, which possess a direct link from source to sink. We characterize the DMT of these families of networks completely for $K > 3$. Layered networks are networks comprising of layers of relays with edges existing only between adjacent layers, with more than one relay in each layer. We prove that a linear DMT between the maximum diversity $d_{\text{max}}$ and the maximum multiplexing gain of 1 is achievable for single-antenna fully-connected layered networks. This is shown to be equal to the optimal DMT if the number of relaying layers is less than 4. For multiple-antenna KPP and layered networks, we provide an achievable DMT, which is significantly better than known lower bounds for half duplex networks.

For arbitrary multi-terminal wireless networks with multiple source-sink pairs, the maximum achievable diversity is shown to be equal to the min-cut between the corresponding source and the sink, irrespective of whether the network has half-duplex or full-duplex relays. For arbitrary ss-ss single-antenna directed acyclic networks with full-duplex relays, we prove that a linear tradeoff between maximum diversity and maximum multiplexing gain is achievable.

Along the way, we derive the optimal DMT of a generalized parallel channel and derive lower bounds for the DMT of triangular channel matrices, which are useful in DMT computation of various protocols. We also give alternative and often simpler proofs of several existing results and show that codes achieving full diversity on a MIMO Rayleigh fading channel achieve full diversity on arbitrary fading channels. All protocols in this paper are explicit and use only amplify-and-forward (AF) relaying. We also construct codes with short block-lengths based on cyclic division algebras that achieve the optimal DMT for all the proposed schemes.

Two key implications of the results in the paper are that the half-duplex constraint does not entail any rate loss for a large class of cooperative networks and that simple AF protocols are often sufficient to attain the optimal DMT.

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I. INTRODUCTION

A. Prior Work

The concept of user cooperative diversity was introduced in [1]. Cooperative diversity protocols were first discussed in [2] for the two-hop relay network (Fig. 1), where the authors develop and analyze the Orthogonal Amplify and Forward (OAF) protocol and the Selection Decode and Forward (SDF) protocol for the case of a single relay network.

Zheng and Tse [3] proposed the Diversity-Multiplexing gain Tradeoff (DMT) as a tool to evaluate point-to-point multiple-antenna schemes in the context of slow fading channels. The DMT was used as a tool to compare various protocols for half duplex two-hop cooperative networks in [4], [5]. As noted in [8], the DMT is a valuable tool in the study of cooperative relay networks, because it is simple enough to be analytically tractable and powerful enough to compare different protocols.

In [4], the SDF protocol is analyzed for an arbitrary number of relays, where the authors give upper and lower bounds on the DMT of the protocol. In these protocols, the relays and the source node participate for equal time instants and the maximum multiplexing gain $r$ that could be achieved was 0.5.

For any network, an upper bound on the achievable DMT has been given by the cut-set bound [8], [33]. A fundamental question in this area is whether the two-hop cooperative wireless system in Fig. 1 can mimic a Multiple Input Single Output (MISO) system with $N + 1$ transmit antennas and 1 receive antenna and achieve the DMT corresponding to the MISO system. This question still remains open, see [9], [10] for a detailed comparison of existing achievable regions.

Fig. 1. Two Hop Cooperative Relay Network

In [5], Azarian et al. analyze the class of Non Orthogonal amplify and Forward (NAF) protocols, introduced earlier by Nabar et al. in [6]. In [5], the authors establish the improved DMT of the NAF protocol in comparison to the class of OAF protocols considered in [4]. However it has been shown in [9] that the DMT of the NAF protocol can be obtained for the OAF protocols as well using appropriate unequal slot lengths for source and relay transmissions.

The authors of [5] also introduce the Dynamic Decode and Forward (DDF) protocol wherein the time for which the relays listen to the source depends on the source-relay channel gain. They show that for the single relay case, the DMT of the DDF protocol achieves the transmit diversity bound for $r \leq 0.5$, beyond which the DMT falls below the transmit diversity bound.

Jing and Hassibi [7] consider cooperative communication protocols where the relay nodes apply a linear transformation to the received signal. The network model that they consider is the same as the one shown in Fig. 1, except that there is no direct link between source and sink in their model. The authors consider the case when both the source and the relays transmit for an equal number of channel uses and the linear transformation applied by the relays are restricted to the class of unitary matrices. Rao and Hassibi [23] consider two-hop half-duplex multi-antenna cooperative networks without direct link and analyze the DMT performance.

Yang and Belfiore consider a class of protocols called Slotted Amplify And Forward (SAF) protocols in [17], and show that these improve upon the performance of the NAF protocol [5] for the case of two relays. The authors
also provide an upper bound on the DMT of the SAF protocol with any number of slots, and show that this upper bound tends towards the transmit diversity bound as the number of slots increases. Under the assumption of relay isolation and relay ordering, the naive SAF scheme proposed in [17] is shown to achieve the SAF protocol upper bound.

Yuksel and Erkip in [8] have considered the DMT of the DF and compress-and-forward (CF) protocols. They show that the CF protocol achieves the transmit diversity bound for the case of a single relay. We note however, that in the CF protocol, the relays are assumed to know all the fading coefficients in the system. The authors also translate cut-set upper bounds in [33] for mutual information into the DMT framework for a general multi-terminal network.

Yang and Belfiore in [16] consider AF protocols on a family of MIMO multihop networks (termed as multi-antenna layered networks in the current paper). They derive the optimal DMT for the Rayleigh-product channel which they prove is equal to the DMT of the AF protocol applied to this channel. They also propose AF protocols to achieve the optimal diversity of these multi-antenna layered networks.

Oggier and Hassibi [27] have proposed distributed space time codes for multi-antenna layered networks that achieve a diversity equal to the minimum number of relay nodes among the hops. Recently, Vaze and Heath [28] have constructed distributed space time codes based on orthogonal designs that achieve the optimal diversity of the multi-antenna layered network.

Borade, Zheng and Gallager in [22] consider AF schemes on a class of multi-hop layered networks where each layer has the same number of relays (termed as Regular networks in the current paper). They show that AF strategies are optimal in terms of multiplexing gain. They also compute lower bounds on the DMT of the product Rayleigh channel.

From a capacity perspective as well, there have been some investigations into single-source single-sink wireless networks. Recently, Avestimehr, Diggavi and Tse [26] have evaluated the capacity of deterministic wireless networks with broadcast and interference constraints. They have also shown that schemes from these deterministic networks can be lifted to gaussian networks, to give achievable regions that are within a constant away from outer-bounds. However, it must be noted that they consider only full-duplex networks. The degrees of freedom of arbitrary full-duplex ss-ss and multicast wireless networks is established in [21] using a connection with deterministic wireless networks.

From the point of code design for multiple antenna systems, Space-Time codes from Cyclic Division Algebra (CDA) was introduced in [18]. Certain codes constructed from CDAs were proved to be DMT optimal (in fact approximately universal - see [11]) for the general MIMO channel in [12]. These codes were tailored to suit the structure of various static protocols for two-hop cooperation and proved to be DMT optimal in [9]. For the Dynamic Decode and Forward protocol, DMT optimal codes were constructed for arbitrary number of relays with multiple antennas in [13]. Recently, in [14], codes for the single relay single antenna DDF channel were constructed, which are not only DMT optimal, but also have probability of error close to the outage probability. In this paper, we present a DMT optimal code design for all proposed protocols based on the approximately universal codes in [12].

Cooperative networks with asynchronous transmissions have also been studied in the literature [39], [40], [41]. However, we consider networks in which relays are synchronized. Codes for two-hop cooperative networks having low decoding complexity and full diversity are studied in [42], [41] and [43]. While decoding complexity is not the primary focus of the present paper, we do provide a successive-interference-cancellation technique to reduce the code length and therefore the complexity.

B. Classification of Networks

In this section, we define the classes of networks under consideration here. Unless otherwise stated, all networks considered possess a single source and a single sink and we will apply the abbreviation ss-ss to these networks.

A cooperative wireless network can be built out of a collection of spatially distributed nodes in many ways. For instance, we can identify paths connecting source to the sink through a series of nodes in such a manner that any two adjacent nodes fall in the Rayleigh zone [8]. This process can be continued barring those nodes which are already chosen. Such a construction will result in a set of paths from the source to the sink. In the simplest model, we can further impose the constraint that these paths do not interfere each other, see Fig.1 thus motivating the study of a class of multi-hop network which we shall refer to as the set of K-Parallel Path (KPP) networks.
Alternatively, a layered network model can be identified from a collection of nodes between the source and the sink. This will result in a layered network model, which is described in [22].

1) Representation by a graph: Any wireless network can be associated with a directed graph, with vertices representing nodes in the network and edges representing connectivity between nodes. If an edge is bidirectional, we will represent it by two edges one pointing in either direction. An edge in a directed graph is said to be live at a particular time instant if the node at the head of the edge is transmitting at that instant. An edge in a directed graph is said to be active at a particular time instant if the node at the head of the edge is transmitting and the tail of the edge is receiving at that instant.

Remark 1: Since most networks considered in this paper will have bidirectional links, we will represent a bidirectional link by an un-directed edge. Therefore, un-directed edges must be interpreted as two directed edges, with one edge pointing in either direction.

A wireless network is characterized by broadcast and interference constraints. Under the broadcast constraint, all edges connected to a transmitting node are simultaneously live and transmit the same information. Under the interference constraint, the symbol received by a receiving end is equal to the sum of the symbols transmitted on all incoming live edges. We say a protocol avoids interference if only one incoming edge is live for all receiving nodes.

In wireless networks, the relay nodes operate in either half or full-duplex mode. In case of half-duplex operation, a node cannot simultaneously listen and transmit, i.e., an incoming edge and an outgoing edge of a node cannot be simultaneously active.

2) K-Parallel-Path Networks: One way of generalizing the two-hop relay network is to consider this network as a collection of K parallel, relaying paths from the source to sink, each of length greater than 1. This immediately leads to a more general network that is comprised of K parallel paths of varying length, linking source and sink. More formally:

Definition 1: A set of edges \((v_1, v_2), (v_2, v_3), \ldots, (v_{n-1}, v_n)\) connecting the vertices \(v_1\) to \(v_n\) is called a path. The length of a path is the number of edges in the path. The K-parallel path (KPP) network is defined as a ss-ss network that can be expressed as the union of K vertex-disjoint paths, each of length greater than one, connecting the source to the sink. Each of the node-disjoint paths is called a relaying path. All edges in a KPP network are bidirectional (see Fig. 3).

The communication between the source and the sink takes place in K parallel paths, labeled with the indices \(P_1, P_2, \ldots, P_K\). Along path \(P_i\), the information is transmitted from source to sink through multiple hops with the aid of \(n_i - 1\) intermediate relay nodes \(\{R_{ij}\}_{j=1}^{n_i-1}\).

Remark 2: A network similar to the KPP network in Definition 1 is considered in [37], albeit from a symbol error probability perspective.

Definition 1 of KPP networks precludes the possibility of either having a direct link between the source and the sink, or of the existence of links connecting nodes lying on distinct node-disjoint paths. We now expand the definition of KPP networks to include both possibilities.
Definition 2: If a given network is a union of a KPP network and a direct link between the source and sink, then the network is called a KPP network with direct link, denoted by KPP(D). If a given network is a union of a KPP network and links interconnecting relays in various paths, then the network is called a KPP network with interference, denoted by KPP(I). If a given network is a union of a KPP network, a direct link and links interconnecting relays in various paths, then the network is called a KPP network with interference and direct path, denoted by KPP(I, D).

Remark 3: We adopt following terminology: For a KPP(D), KPP(I) or a KPP(I, D) network, we consider the union of the $K$ node disjoint paths as the backbone KPP network (When there are several choices for the $K$ node-disjoint paths, we are free to choose any one set of $K$ node-disjoint paths and refer to this collection of $K$ paths as the backbone KPP network). The $K$ relaying paths in these networks are referred to as the $K$ backbone paths. A start node and end node of a backbone path are the first and the last relays respectively in the path.

Fig. 3 below provides examples of all four variants of KPP networks.

![Diagram of KPP networks](image)

**Fig. 3.** The KPP network

**Fig. 4.** Examples of KPP networks with $K = 2$

For a KPP(D), KPP(I) or a KPP(I, D) network, we consider the union of the $K$ node disjoint paths as the backbone KPP network. While there may be many choices for the $K$ node disjoint paths, we can choose any one such choice and call that the backbone KPP network. These $K$ relaying paths in these networks are referred to as the $K$ backbone paths. A start node and end node of a backbone path are the first and the last relays respectively in the path.

In a general KPP network, let $P_i, i = 1, 2, ..., K$ be the $K$ backbone paths. Let $P_i$ have $n_i$ edges. The $j$-th edge on the $i$-th path $P_i$ will be denoted by $e_{ij}$ and the associated fading coefficient by $g_{ij}$.

3) Layered Network: A second way of generalizing a two-hop relay network is to view the two-hop network as a network comprising of a single layer of relays. The immediate generalization is to allow for more layers of...
relays between source and sink, with the proviso that all links are either inside a layer or between adjacent layers. We label this class of multi-hop relaying networks as \textit{layered networks}:

\textbf{Definition 3}: Consider a ss-ss single-antenna bidirectional network. A network is said to be a layered network if there exists a a partition of the vertex set $V$ into subsets $V_0, V_1, ..., V_L, V_{L+1}$, such that

- $V_0, V_{L+1}$ denote the singleton sets corresponding to the source and sink respectively.
- If there is an edge between a node in vertex set $V_i$ and a node in $V_j$, then $|i - j| \leq 1$. We assume $|V_i| > 1$, $i = 1, 2, \ldots, L$.

We call $V_1, ..., V_L$ as the relaying layers of the network. A layered network is said to be fully connected if for any $i$, $v_1 \in V_i$ and $v_2 \in V_{i+1}$, then the $(v_1, v_2)$ is an edge in the network.

It must be noted that a fully connected layered network may or may not have links inside of a layer. However, whenever we say fully connected layered network, it applies to both networks that have intra-layer links and those that do not have such links. Examples of both these types of networks are shown in Fig. 5(c) and Fig. 5(d).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig5.png}
\caption{Examples of Layered and Regular networks}
\end{figure}

Every layered network will have a layer containing only the source, and another layer containing only the sink. In Fig 5 examples of layered networks are given. Layered networks were also considered in [16] and [22]. In particular, [22] considered layered networks with equal number of relays on all layers. We refer to such layered networks as regular networks.

\textbf{Remark 4}: In this remark, we characterize the intersection of KPP(I) networks and layered networks. First we observe that one is not contained in the other. Consider the subgraph of a given KPP(I) network graph, consisting of all the nodes of the original network except for the source and the sink. This subgraph will have the property that the number of node-disjoint and edge-disjoint paths is equal to the number of relay nodes immediately adjacent to the source. This is a key property of KPP(I) networks, which in general, does not hold for layered networks. On the other hand, there can be cross links between the parallel paths in a KPP(I) network in such a way that the network cannot be viewed as being layered. However, these two classes of networks are not mutually exclusive and in fact, we term networks that lie in the intersection of the two classes as regular networks.

\textbf{Definition 4}: The $(K, L)$ Regular network is defined as a KPP(I) network which is also a layered network [16] with $L$ layers of relays (see Fig. 5(b)).

\textbf{Remark 5}: The two-hop relay network [Fig 1] is a KPP(I,D) network with $K = M$, $M$ being the number of relays. If we assume relay isolation, then it is a KPP(D) network with $K = M$. If we exclude the direct link, then we have a $(M, 1)$ regular network.

\textbf{C. Setting and Channel Model}

Between any two adjacent nodes $v_x, v_y$ of a wireless network, we assume the following channel model.
\[ y = Hx + w , \]  

where \( y \) corresponds to the received signal at node \( v_y \), \( w \) is the noise vector, \( H \) is a matrix and \( x \) is the vector transmitted by the node \( v_x \).

We follow the literature in making the assumptions listed below. Our description is in terms of the equivalent complex-baseband, discrete-time channel.

1) All channels are assumed to be quasi-static and to experience Rayleigh fading and hence all fade coefficients are i.i.d., circularly-symmetric complex Gaussian \( \mathbb{C} \mathcal{N}(0, 1) \) random variables.

2) The additive noise at each receiver is also modeled as possessing an i.i.d., circularly-symmetric complex Gaussian \( \mathbb{C} \mathcal{N}(0, 1) \) distribution.

3) Each receiver (but none of the transmitters) is assumed to have perfect channel state information of all the upstream channels in the network.

An AF protocol \( \wp \), i.e., a protocol \( \wp \) in which each node in the network operates in an amplify-and-forward fashion, induces the following linear channel model between source and sink:

\[ y = H(\wp)x + w , \]

where \( y \in \mathbb{C}^m \) denotes the signal received at the sink, \( w \) is the noise vector, \( H(\wp) \) is the \((m \times n)\) induced channel matrix and \( x \in \mathbb{C}^n \) is the vector transmitted by the source. The components of the \( n \)-tuple \( x \) are the \( n \) symbols transmitted by the source and similarly, the components of the \( m \)-tuple \( y \) represent the symbols received at the sink. Typically \( m \) equals \( n \). We impose the following energy constraint on the transmitted vector \( x \)

\[ \text{Tr}(\Sigma_x) := \text{Tr}(E\{xx^\dagger\}) \leq n\rho \]

where \( \text{Tr} \) denote the trace operator, and we will regard \( \rho \) as representing the SNR on the network. We will assume a symmetric power constraint on the relays and the source. However it will turn out that given our high SNR perspective here, the exact power constraint is not of significant importance. We consider both half and full-duplex operation at the relay nodes.

1) Diversity-Multiplexing Gain Tradeoff: Let \( R \) denote the rate of communication across the network in bits per network use. Let \( \wp \) denote the protocol used across the network, not necessarily an AF protocol. Let \( r \) denote the multiplexing gain associated to rate \( R \) defined by

\[ R = r \log(\rho). \]

The probability of outage for the network operating under protocol \( \wp \), i.e., the probability of the induced channel in (2) is then given by

\[ P_{\text{out}}(\wp, R) = \inf_{\Sigma_x \geq 0, \text{Tr}(\Sigma_x) \leq n\rho} \Pr(I(x; y) \leq nR|H(\wp) = H(\wp)). \]

Let the outage exponent \( d_{\text{out}}(\wp, r) \) be defined by

\[ d_{\text{out}}(\wp, r) = -\lim_{\rho \to \infty} \frac{P_{\text{out}}(\wp, R)}{\log(\rho)} \]

and we will indicate this by writing

\[ \rho^{-d_{\text{out}}(\wp, r)} \leq P_{\text{out}}(\wp, R). \]

The symbols \( \geq, \leq \) are similarly defined.

The outage \( d_{\text{out}}(r) \) of the network associated to multiplexing gain \( r \) is then defined as the supremum of the outages taken over all possible protocols, i.e.,

\[ d_{\text{out}}(r) = \sup_{\wp} d_{\text{out}}(\wp, r). \]

\(^1\)However, for the protocols proposed in this paper, the CSIR is utilized only at the sink, since all the relay nodes are required to simply amplify and forward the received signal.
A distributed space-time code (more simply a code) operating under a protocol $\varphi$ is said to achieve a diversity gain $d(\varphi, r)$ if

$$P_e(\varphi, \rho) \leq \rho^{-d(\varphi, r)},$$

where $P_e(\rho)$ is the average error probability of the code $C(\rho)$ under maximum likelihood decoding. Using Fano’s inequality, it can be shown (see [3]) that for a given protocol,

$$d(\varphi, r) \leq d_{\text{out}}(\varphi, r).$$

We will refer to the outage exponent $d_{\text{out}}(r)$ as the DMT $d(r)$ of the corresponding channel since for every protocol discussed in this paper we shall identify a corresponding coding strategy in Section IX-A whose diversity gain $d(\varphi, r)$ equals $d_{\text{out}}(r)$.

For each of the networks described in this paper, we can get an upper bound on the DMT, based on the cut-set upper bound on mutual information [33]. This was formalized in [8] as follows:

**Lemma 1.1:** Given a cut $C_i, i = 1, 2, ..., M$ between any source and sink, let $r(C_i) \log(\rho)$ be the rate of information flow across the cut. Given a cut, there is a $H$ matrix connecting the input terminals of the cut to the output terminals. Let us call the DMT corresponding to this $H$ matrix as the DMT of the cut, $d_{\text{out}}(r(C_i))$. Then the DMT between the source and the sink is upper bounded by

$$d(r) \leq \min\{d_{\text{out}}(r(C_i))\}.$$ 

**Definition 5:** Given a random matrix $H$ of size $m \times n$, we define the DMT of the matrix $H$ as the DMT of the associated channel $y = Hx + w$ where $y$ is a $m$ length received column vector, $x$ is a $n$ length transmitted column vector and $w$ is a $CN(0, I)$ column vector. We denote the DMT by $d_H(.)$

**D. Results**

The principal results of this paper are tabulated in Table I. Some of these results were presented in conference versions of this paper [19], [20]. We have characterized achievable DMT/diversity for many classes of networks as given in the table. When compared against the cut-set upper bound, in many cases, the optimal DMT is achieved. In other cases, we prove that a linear DMT between the maximum multiplexing gain and maximum diversity is achievable, while the cut-set upper bound can be concave in general. Explicit schemes and code design is established for all the achievable DMT. In the table, $M$ refers to the min-cut of the network of interest.

For arbitrary co-operative networks with multiple sources and sinks, each potentially equipped with multiple antennas, we characterize the maximum achievable diversity gain and give a scheme that achieves this maximum diversity using an amplify-and-forward protocol in Section VII-A. For arbitrary ss-ss networks with full duplex operation, we prove that a linear tradeoff between maximum diversity and maximum multiplexing gain is achievable using an amplify and forward protocol in Section IV.

For both KPP and layered networks, we propose an explicit protocol that achieves a diversity multiplexing tradeoff that is linear between the maximum diversity and maximum multiplexing gain points in Section VII. For KPP networks, this coincides with the upper-bound on the DMT as given by the cut-set bound, thus characterizing the DMT of this entire family of networks completely. For layered networks, the cut-set bound turns out to be concave in the general case and does not coincide with the achievable region. For general layered networks, we give a sufficient condition for the achievability of a linear DMT between the maximum diversity and the maximum multiplexing gain in Lemma IX-A.

Along the way, we derive the optimal DMT of parallel channel in Lemma IX-A provide alternative and often simpler proofs of several existing results and in Section IX-B prove that codes achieving full diversity on a MIMO Rayleigh fading channel achieve full diversity on arbitrary fading channels.

In Section IX-A we give explicit codes with short block-lengths based on cyclic division algebras that achieve the best possible DMT for all the schemes proposed above. We also prove (Section IX-B) that full diversity codes for all networks in this paper can be obtained by using codes that give full diversity on a Rayleigh fading MIMO channel.

For KPP and layered networks with multiple antenna nodes, we examine certain protocols and establish achievable DMT for these protocols in Section VII.
## Table I

### Principal Results Summary

| Network | No of sources/sinks | No of antennas in nodes | FD/HD | Direct Link | Upper bound on Diversity/DMT $d_{\text{bound}}(r)$ | Achievable Diversity/DMT $d_{\text{achieved}}(r)$ | Is upper bound achieved? | Reference |
|---------|---------------------|-------------------------|-------|-------------|-----------------------------------------------|-----------------------------------------------|---------------------------|-----------|
| Arbitrary | Multiple | Multiple | FD/HD | ✓ | $d(0) = M$ | $d(0) = M$ | ✓ | Theorem 4.1 |
| Arbitrary | Multiple | Multiple | FD/HD | ✗ | $d(0) = M$ | $d(0) = M$ | ✓ | Theorem 4.1 |
| Arbitrary Directed Acyclic Networks | Single | Single | FD | ✓ | Concave in general | $M(1-r)^+$ | A linear DMT between $d_{\text{max}}$ and $r_{\text{max}}$ is achieved | Theorem 4.2 |
| KPP($K \geq 3$) | Single | Single | HD | ✗ | $(K-1)(1-r)^+$ | $K(1-r)^+$ | ✓ | Theorem 5.10 |
| KPP(D)($K \geq 3$) | Single | Single | HD | ✓ | $(K+1)(1-r)^+$ | $(K+1)(1-r)^+$ | ✓ | Theorem 5.11 |
| KPP(I)($K \geq 3$) | Single | Single | HD | ✗ | $K(1-r)^+$ | $K(1-r)^+$ | ✓ | Theorem 6.7 |
| Fully Connected Layered | Single | Single | HD | ✗ | Concave in general | $M(1-r)^+$ | A linear DMT between $d_{\text{max}}$ and $r_{\text{max}}$ is achieved. ✓ for $L < 4$ | Theorem 7.5 |
| General Layered (satisfying Lemma 7.3) | Single | Single | HD | ✗ | Concave in general | $M(1-r)^+$ | A linear DMT between $d_{\text{max}}$ and $r_{\text{max}}$ is achieved | Lemma 7.3 |
| $(K,L)$ Regular | Single | Single | HD | ✗ | $K(1-r)^+$ | $K(1-r)^+$ | ✓ | Theorem 6.3 |

### II. Relation to Existing Literature

In this section, we present how the results in this paper relate to other in this area. Certain results in this paper can be used to recover existing results on cooperative communication in a simpler, concise and more intuitive manner.

1) **Proof of Conjecture 1 in the paper by Rao and Hassibi [23] and [24]:**
   The general NAF protocol considered in Example 3 in Section III-E of the present paper is the same as that considered by Rao and Hassibi. The results here proves Conjecture 1 given in [23] and [24].

2) **The lower bound on the DMT of various AF Protocols:** We prove lower bounds on the DMT of various AF protocols. While most are previously known, the new method employed here presents a simpler derivation. As it turns out, all lower bounds for single antenna systems provided here are tight.
   - **NAF Protocol:** The DMT of the NAF protocol was computed in [5]. We prove a lower bound on the DMT which turns out to be tight.
   - **SAF Protocol:** The Slotted Amplify and Forward protocol is proposed in [17] and upper and lower bounds on its DMT under relay isolation is evaluated and shown to be equal. For doing so, matrix theoretic techniques
are employed in [17]. In the current paper, in Example 2 of Section III-E, the lower bound for the same is developed using information theoretic techniques, which lends insight into the form of the DMT.

N-Relay MIMO NAF Channel given in [15]:
In [15], the authors consider a two-hop relay network with a direct link and \( N \) relays. We prove an improved lower bound on the DMT for the MIMO NAF protocol considered in that paper (See Example 4 of Section III-E).

3) The diversity of arbitrary cooperative networks.
We characterize completely the maximum diversity order attainable for arbitrary cooperative networks and it is shown that an amplify and forward scheme is sufficient to achieve this. Special cases of these were derived for the MIMO two-hop relay channel in [15], under a certain condition on the number of antennas (See Corollary 1 in that paper). Also, the diversity order of layered networks using amplify and forward networks is characterized in [16]. In [38], upper bounds on the diversity order of an arbitrary single-source single-sink network under the two cases of common and independent code-books was derived. However, no achievability results are given there.

4) The optimal DMT of the two-hop cooperative channel without direct link.
The optimal DMT of a \((K,L)\) regular network is derived in Theorem 6.3 in Section VI of this paper. In an independent (parallel) work by Gharan, Bayesteh and Khandani [25], the optimal DMT of a two-hop network, which is a special case of a regular network (in particular it is a \((K,1)\) Network), is derived to be \( d(r) = L(1-r) \). The protocol they propose is the same as the protocol employed in the present paper. In fact, both these protocols are simply the SAF (Slotted Amplify and Forward) protocol [17] applied in the situation when there is no direct link between source and sink. It must be noted however, that the proof techniques used in this paper are entirely different from those used in [25].

5) The DMT of the parallel channel in closed form is obtained in Lemma 3.5. A special case of this result is derived in [16] where the authors characterize the parallel channel DMT when all the individual channels have the same DMT.

6) For an arbitrary full-duplex networks, it is shown in the present paper, that a linear DMT between the maximum diversity and the maximum multiplexing gain is achievable. A special case of this result is proved for the case of layered networks in [16].

A. Outline
In Section III, we present techniques and general results which will of use in later sections. In this section, we introduce the Information Flow diagram (i-f diagram), and prove the result that min-cut equals diversity. In Section IV we consider the case with full duplex relays. We present schemes achieving optimal DMT for KPP(I,D) networks. In Section V we focus on half-duplex KPP networks and present protocols achieving optimal DMT for \( K \geq 3 \). In Section VI KPP(I) networks with half-duplex relays are considered, and schemes achieving optimal DMT are presented for KPP(I) networks allowing certain types of interference. In Section VII we consider layered networks and show that a linear DMT between max multiplexing of 1 and diversity of \( d_{\text{max}} \) is obtained, which is indeed optimal if the number of layers is lesser than 4. In Section VIII-A we consider multi-antenna layered and KPP networks and give an achievable DMT, which improves significantly on known bounds. Finally, in Section IX-A we give explicit CDA based codes of low complexity for all the DMT optimal protocols.

III. TECHNIQUES AND GENERAL RESULTS FOR COOPERATIVE NETWORKS
A. Amplify and Forward Protocols
We consider only amplify-and-forward (AF) protocols in this paper by which we mean that relays are allowed to perform only linear processing on their received signals prior to transmission. In particular, they are not permitted to decode and then re-encode.

In all of our protocols, we assume that the relays perform the simplest form of linear processing; transmission upon scaling the incoming by an appropriate constant to meet a transmit-power constraint. Furthermore, it is

\(^2\) More sophisticated linear processing techniques would include matrix transformations of the incoming signal.
known [5], that this constant does not matter in the scale of interest. Therefore, without loss of accuracy, we will assume that this constant is indeed 1.

It follows that, for any given network, we only need specify the schedule to completely specify the protocol. Once the schedule is specified, each node transmits the last received signal in the next time instant in accordance with the schedule. This will create a transfer matrix between the signal transmitted from the source and the sink, with the noise being no longer white. To compute the DMT offered by the protocol, we need to compute the DMT of the equivalent channel $y = Hx + w$, where $H$ is the effective transfer matrix and $w$ is the noise vector, which is potentially colored.

In this section, we will develop techniques to handle non-white noise and a general method to compute lower bounds on the DMT of matrices with certain structure.

**B. The Information Flow Diagram**

We begin by introducing the notion of an information-flow (i-f) diagram as a means of characterizing the mutual information between the source and the sink in a ss-ss relay network. A ss-ss relay network will have many paths between the source and the sink, including a direct link. Protocols employed in a wireless network need to take into account the half-duplex, interference and broadcast constraints at each of the nodes. Due to the complexity of the network graph, it is in general difficult to characterize the network information-theoretically under the wireless constraints. The i-f diagram, that we propose, is an attempt to abstract out the details of network graph, and to focus our attention only on the mutual information between source and sink, given a protocol.

As will be seen, the i-f diagram is well suited to studying amplify and forward relay networks.

![Fig. 6. Single Relay Channel](image-url)

**Example 1** Consider a ss-ss, single-relay scenario, operating under the Non-orthogonal Amplify and Forward (NAF) protocol of [5] (Fig 6). This is a two slot protocol, wherein during the first slot, the source transmits to both relay and sink. During the second slot, the relay re-transmits the information that it received during the first time slot, while the source transmits new information at this time. Let us represent the random vectors associated to source transmissions at time slot one and two by $x_1, x_2$ and the corresponding data received by sink in the two time slots by $y_1, y_2$.

Then the input-output relation takes on the following form

$$y = Hx + n,$$  \hspace{1cm} (3)

where

$$n = \begin{bmatrix} w_1 \\ h_2v + w_2 \end{bmatrix},$$

$$H = \begin{bmatrix} g_1 & 0 \\ g_2h_2 & g_1 \end{bmatrix},$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}. $$
Given that $H$ is known, the covariance matrices of the noise and signal vector are denoted by

$$
\Sigma_n := \mathbb{E}(nn^\dagger) = \begin{bmatrix}
\sigma_w^2 & 0 \\
0 & \sigma_w^2 + |h_2|^2 \sigma_v^2 \\
\end{bmatrix}
$$

and

$$
\Sigma_x := \mathbb{E}(xx^\dagger),
$$

where $\sigma_w^2$, $\sigma_v^2$ denote the variances of the corresponding noise vectors. We will assume $\sigma_v^2 = \sigma_w^2 = 1$ without loss of generality since the exact value does not matter in the scale of interest.

We represent the induced channel by the i-f diagram in Fig. 7.

Fig. 7. Information flow diagram of single relay channel

In the i-f diagram in Fig. 7 we have used the subscript $s$ denoting straight coupling, and subscript $c$ denoting cross coupling. So the following equivalence holds.

$$
H_d := g_1 \\
H_c := g_2 h_2 \\
\Sigma_c := 1 + |h_2|^2
$$

The interpretation of the arrows in the i-f diagram is illustrated in Fig. 8 and Fig. 9.

Fig. 8. Equivalent channel model of a single link in i-f diagram.

Fig. 9. Equivalent channel model of multiple-access links in i-f diagram.
The two tuple notation \((H, \Sigma)\) for each link is used to specify the channel matrix for the signal and the noise covariance matrix. \(H\) is the channel matrix and therefore the transmitted signal \(x\) is multiplied by \(H\) to give \(Hx\) at the receiver. The noise is potentially correlated because it is accumulated over multiple links, in such a way that the noise added on one link gets multiplied by the channel matrix of the next link.

The input output relation for the single link in the i-f diagram Fig.8 is explained as follows:

\[ y_1 = Hx_1 + z \]

where \(z\) is a complex gaussian random variable, with \(\Sigma_z = \mathbb{E}(zz^\dagger)\).

The input output relation for the multiple links terminating in a given node in the i-f diagram Fig.9 is explained as follows:

\[ y = \sum_{i=1}^{N} H_i x_i + \sum_{i=1}^{N} z_i + z_0 \]

where \(z_i\) and \(z_0\) are independent complex gaussian random variables, \(\Sigma_k = \mathbb{E}((z_k + z_0)(z_k + z_0)^\dagger)\).

C. White in the scale of interest

In this section, we provide two lemmas that will be extensively used in all future sections: Lemma 3.1 which states that noise, even though correlated can be treated as white in the scale of interest and Lemma 3.2 which proves that i.i.d. gaussian inputs are sufficient to attain the outage exponent of any channel of the form \(n x + w\).

**Lemma 3.1:** Consider a channel of the form \(y = Hx + z\). Let \(H, F_j, j = 1, 2, .., L\) be \(n \times n\) independent random matrices, with entries in each of the matrices being i.i.d. random variables with complex Gaussian \(\mathbb{C}N(0,1)\) distribution. Let \(G_i, i = 1, 2, .., M\) comprise of finite products of various matrices from the set of \(F_j\). Let \(z = z_0 + \sum_{i=1}^{M} G_i z_i\). Let \(\{z_i\}\) be i.i.d. circularly symmetric \(n\)-dimensional complex Gaussian \(\mathbb{C}N(0, I)\) random vectors.

Then \(z\) is white in the scale of interest, i.e.,

1) \(\lambda_i = \rho^0 \ \forall i\) with probability one, where \(\lambda_i\) are eigenvalues of the noise covariance matrix \(\Sigma\).
2) \(\log \det(I + \rho HH^\dagger \Sigma^{-1}) = \log \det(I + \rho HH^\dagger)\) with probability one.
3) \(Pr(\log \det(I + \rho HH^\dagger \Sigma^{-1}) \leq r \log \rho) = Pr(\log \det(I + \rho HH^\dagger) \leq r \log \rho)\)

**Proof:** For a fixed set of values of \(F_i\) and \(H\), the noise covariance matrix is given by,

\[ \Sigma = \mathbb{E}[zz^\dagger] = I + \sum_{i=1}^{M} G_i G_i^\dagger \]

(4)

Let \(\lambda_i(A), \lambda_{max}(A)\) and \(\lambda_{min}(A)\) denote the \(i\)th, maximum and minimum eigenvalues of the positive semi-definite matrix \(A\). If the context is clear, we may avoid specifying the matrix, and just use \(\lambda_i\), \(\lambda_{max}\) and \(\lambda_{min}\) respectively.

By Theorem 6.1.1 in [34] due to Gersgorin, each eigenvalue of \(\Sigma\), when properly ordered, is bounded within the interval

\[ \Sigma_{ii} - R_i(\Sigma) \leq \lambda_i(\Sigma) \leq \Sigma_{ii} - R_i(\Sigma) \]

(5)

where,

\[ R_i(\Sigma) := \sum_{j=1, j\neq i}^{n} |\Sigma_{ij}| \]

For \(i = 1, 2, \ldots, M\), let \(G_i\) be a product of \(n_i\) matrices from the set \(\{F_j : j = 1, 2, \ldots, L\}\), and let them be labeled as \(F_{ij}, j = 1, 2, \ldots, n_i\). Let \(F_{ij}(k, l)\) denote the \((k, l)\)th entry of the matrix \(F_{ij}\). Note that each of \(F_{ij}(k, l)\) ~ \(\mathbb{C}N(0,1)\). Also, let \(G_{ij}(k, l)\) denote the \((k, l)\)th entry of \(G_i\). Then,
\[ \Sigma_{ii} = 1 + \sum_{\ell} \sum_{k} |G_\ell(i, k)|^2 \]  \hspace{1cm} (6)

\[ |\Sigma_{ij}| = |\sum_{\ell} \sum_{k} G_\ell(i, k)G_\ell^*(j, k)| \]
\[ = |\sum_{\ell} \sum_{k} G_\ell(i, k)(G_\ell(j, k))^*| \]
\[ \leq \sum_{\ell} \sum_{k} |G_\ell(i, k)||G_\ell(j, k)| \]  \hspace{1cm} (7)

Now every \( G_\ell(i, j) \) is a polynomial function of \( \mathbb{C}N(0, I) \) entries of \( F_{\ell m} \), \( m = 1, 2, \ldots, n_\ell \). Define a random variable \( v \) such that \( |G_\ell(i, j)|^2 \approx \rho^{-v} \). Now we will prove that \( v \geq 0 \) with probability one, for every \( \ell, i \) and \( j \). Let \( v \) denote a realization of the random variable \( v \). It can be proved that polynomial functions of independent random variables that have finite mean and variance have finite mean and variance. Therefore \( \mathbb{E}(|G_\ell(i, j)|^2) \) is finite.

Let the pdf of \( v \) be \( p_v(v) \). We have to prove that \( P(v < 0) = 0 \). Suppose we have proved that \( P(v < -\frac{1}{n}) = 0 \), for all \( n \in \mathbb{N} \), then we have:

\[ P(v < 0) = P(\bigcup_{n=1}^{\infty} \{v < -\frac{1}{n}\}) \]
\[ \leq \sum_{n=1}^{\infty} P(v < -\frac{1}{n}) \]
\[ = \sum_{n=1}^{\infty} 0 \]
\[ = 0 \]

Now we will prove that indeed \( P(v < -\frac{1}{n}) = 0 \), \( \forall n \). Now, for any given \( n \) and \( \rho \),

\[ \infty > \mathbb{E}(|G_\ell(i, j)|^2) \]
\[ = \mathbb{E}(\rho^{-v}) \]
\[ = \int_{-\infty}^{+\infty} \rho^{-v}p_v(v)dv \]
\[ \geq \int_{-\infty}^{-\frac{1}{n}} \rho^{-v}p_v(v)dv \]
\[ \geq \rho^\frac{1}{n} \int_{-\infty}^{-\frac{1}{n}} p_v(v)dv \]
\[ = \rho^\frac{1}{n}P(v < -\frac{1}{n}) \]

Taking limit as \( \rho \) tends to infinity on both sides

\[ \infty > \lim_{\rho \to \infty} \rho^\frac{1}{n}P(v < -\frac{1}{n}) \]

This can only imply that \( P(v < -\frac{1}{n}) = 0 \) since otherwise, the RHS will grow to infinity as \( \rho \) tends to infinity. Hence with probability 1,

\[ |G_\ell(i, j)|^2 \approx \rho^{-v} \quad \text{with} \quad vv > 0. \]  \hspace{1cm} (8)

By equations (5), (8), (7) and (6), it follows that with probability one, the following equations are true:

\[ \lambda_i(\Sigma) \leq 1 + \rho^{-v} \]
\[ \approx \rho^0 \quad \forall \, i \]
\[ \Rightarrow \lambda_{\max} \leq \rho^0 \]  \hspace{1cm} (9)
We now provide a lower bound for each $\lambda_i(\Sigma)$. Let $e_i$ be the eigen vector corresponding to $\lambda_i(\Sigma)$. Then,

$$
\lambda_i \| e_i \|^2 = e_i^\dagger \Sigma e_i \\
= e_i^\dagger (I + \sum_{i=1}^{M} G_i G_i^\dagger) e_i \\
= \| e_i \|^2 + e_i^\dagger \left( \sum_{i=1}^{M} G_i G_i^\dagger \right) e_i \\
\geq \| e_i \|^2 \\
\Rightarrow \lambda_i \geq 1 \quad \forall i \\
\Rightarrow \lambda_{\min} \geq \rho^0
$$

(10)

By (9), (10), we have that with probability one:

$$
\lambda_i = \rho^0 \quad \forall i
$$

(11)

To prove the second assertion of the lemma, we use the Amir-Moez bound on the eigen values of the product of Hermitian, positive-definite matrices [36]. By this bound, for any two positive definite $n \times n$ Hermitian matrices $A, B$:

$$
\lambda_i(A)\lambda_{\min}(B) \leq \lambda_i(AB) \leq \lambda_i(A)\lambda_{\max}(B)
$$

So we get,

$$
\det(I + \rho AB) = \prod_i (1 + \rho \lambda_i(AB)) \\
\leq \prod_i (1 + \rho \lambda_i(A)\lambda_{\max}(B)) \\
= \det(I + \rho \lambda_{\max}(B)A)
$$

Similarly,

$$
\det(I + \rho AB) \geq \det(I + \rho \lambda_{\min}(B)A)
$$

Therefore,

$$
\det(I + \rho \lambda_{\min}(B)A) \leq \det(I + \rho AB) \leq \det(I + \rho \lambda_{\max}(B)A)
$$

(12)

Applying (12) to $A = HH^\dagger$ and $B = \Sigma^{-1}$, we get

$$
\Rightarrow \det(I + \rho HH^\dagger \lambda_{\min}(\Sigma^{-1})) \leq \det(I + \rho HH^\dagger \Sigma^{-1}) \leq \det(I + \rho HH^\dagger \lambda_{\max}(\Sigma^{-1}))
$$

(13)

(14)

Since the eigenvalue of $\Sigma$ and of $\Sigma^{-1}$ are reciprocals, it follows that $\lambda_{\max}(\Sigma^{-1}) = \lambda_{\min}(\Sigma) \doteq \rho^0$, and $\lambda_{\min}(\Sigma^{-1}) = \lambda_{\max}(\Sigma^{-1}) \doteq \rho^0$ with probability one. Hence, we have with probability one,

$$
\det(I + \rho HH^\dagger \Sigma^{-1}) = \det(I + \rho HH^\dagger)
$$

(15)

This proves the second assertion of the lemma.

Continuing from (13) and (14), we have

$$
Pr\{\log(\det(I + \rho HH^\dagger \lambda_{\min}(\Sigma^{-1}))) < r \log \rho\} \geq Pr\{\log \det(I + \rho HH^\dagger \Sigma^{-1}) < r \log \rho\} \geq Pr\{\log(\det(I + \rho HH^\dagger \lambda_{\max}(\Sigma^{-1}))) < r \log \rho\}
$$

(16)
In the following, we will prove that both the bounds coincide as $\rho \to \infty$. We begin with the bounds on $\lambda_{\min}(\Sigma)$ and $\lambda_{\max}(\Sigma)$. By (4), we know that

$$\lambda_{\min}(\Sigma) \geq 1$$

$$\lambda_{\max}(\Sigma^{-1}) \leq 1$$

Hence,

$$Pr\{\log \det(I + \rho H H^\dagger \Sigma^{-1}) < r \log \rho\} \geq Pr\{\log \det(I + \rho H H^\dagger \lambda_{\max}(\Sigma^{-1})) < r \log \rho\} \geq Pr\{\log \det(I + \rho H H^\dagger) < r \log \rho\}$$

Now bounding $\lambda_{\max}(\Sigma)$,

$$\lambda_{\max}(\Sigma) = \lambda_{\max}(I + \sum_{i=1}^{M} G_i G_i^\dagger)$$

$$= 1 + \lambda_{\max}(\sum_{i=1}^{M} (G_i G_i^\dagger))$$

$$\leq 1 + Tr(\sum_{i=1}^{M} G_i G_i^\dagger)$$

$$= 1 + \sum_{i=1}^{M} Tr(G_i G_i^\dagger)$$

$$= 1 + \sum_{i=1}^{M} ||G_i||^2_F$$

$$\leq 1 + \sum_{i=1}^{M} \prod_{j=1}^{n_i} ||F_{ij}||^2_F$$

$$\leq f(u_1, u_2, \ldots, u_S)$$

Now, it follows that RHS of (18) is a multinomial in random variables $u_1, u_2, \ldots, u_S$ with constant term 1 and non-negative integer coefficients. Here, each $u_i$ is the squared norm of a $\mathbb{C}N(0, 1)$ random variable, and therefore has a exponentially distribution.

$$f(u_1, u_2, \ldots, u_S) = \sum_{\varepsilon \in E} c_{\varepsilon} u_1^{\varepsilon_1} u_2^{\varepsilon_2} \cdots u_S^{\varepsilon_S}$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_S) \in E \subset \mathbb{Z}_+^S$, $|E| < \infty$

$$c_{\varepsilon} \in \mathbb{Z}_+$$

Clearly,

$$f(u_1, u_2, \ldots, u_S) > \rho^\varepsilon \Rightarrow \exists \varepsilon \text{ s.t. } u_1^{\varepsilon_1} u_2^{\varepsilon_2} \cdots u_S^{\varepsilon_S} > \frac{\rho^\varepsilon}{T_\varepsilon}, \text{ where}$$

$$T_\varepsilon \text{ is the number of terms in the multinomial.}$$

$$\Rightarrow Pr\{f(u_1, u_2, \ldots, u_S) > \rho^\varepsilon\} \leq Pr\left\{\bigcup_{\varepsilon} \left( u_1^{\varepsilon_1} u_2^{\varepsilon_2} \cdots u_S^{\varepsilon_S} > \frac{\rho^\varepsilon}{T_\varepsilon} \right) \right\}$$

Now we evaluate a single term in the RHS of (20). Define $T := \max_{\varepsilon} T_\varepsilon$. 

$$f(u_1, u_2, \ldots, u_S)$$
\[
Pr \left( \frac{\rho^\ell}{T_\rho} > \frac{\rho^i}{S T_\rho} \right) \leq S Pr \left( u_i > \frac{\rho^i}{S T_\rho} \right) \\
 \leq Pr \left( u_i > \frac{\rho^i}{S T_\rho} \right) = Pr \left( u_i > a \rho^f \right) = \exp(-a \rho^f) \\
\Rightarrow Pr \left( \frac{\rho^\ell}{T_\rho} > \frac{\rho^i}{S T_\rho} \right) \leq \exp(-a \rho^f)
\]

where \( G \) is the maximum degree of \( f \) in any variable and \( a \) is a constant. 
Continuing with (20),

\[
Pr\{f(u_1, u_2, \ldots, u_S) > \rho^f\} \leq \sum_{\lambda} \exp(-a \rho^f) = |E| \exp(-a \rho^f) = \exp(-a \rho^f)
\]

So we have,

\[
Pr\{f(u_1, u_2, \ldots, u_S) > \rho^f\} \leq \exp(-a \rho^f) \\
\Rightarrow Pr\{\lambda_{max}(\Sigma) > \rho^f\} \leq \exp(-a \rho^f)
\]

Let \( \mathcal{H} \) denote the support of all the fading coefficients in the network, and let \( h \in \mathcal{H} \) denote a realization of the fading coefficients. Clearly, once a \( h \) is given, the values of the matrices \( H, G, \) and \( F_t \) are all well defined. 
Let \( A = \{ \rho \in \mathcal{H} \mid \log \det(I + \rho H H^\dagger \Sigma^{-1}) < \rho^f \} \) and \( B = \{ \rho \in \mathcal{H} \mid \lambda_{max}(\Sigma) > \rho^f \} \). Then,

\[
Pr(A) = Pr(A \cap B^c) + Pr(A \cap B) \\
\leq Pr(A \cap B^c) + Pr(B) \\
\leq Pr(A \cap B^c) + \exp(-a \rho^f)
\] (21)

Now, \( A \subset \{ \rho \in \mathcal{H} \mid \log \det(I + \rho H H^\dagger \lambda_{min}(\Sigma^{-1})) < \rho^f \} \)
\[
= \{ \rho \in \mathcal{H} \mid \log \det(I + \rho H H^\dagger (\lambda_{max}(\Sigma)^{-1}) < \rho^f \}
\]
\[
A \cap B^c \subset \{ \rho \in \mathcal{H} \mid \log \det(I + \rho^{1-\epsilon} H H^\dagger) < \rho^f \}
\] (22)

\[
\log \frac{Pr(A)}{\log \rho} \leq \log \frac{Pr(A \cap B^c) + Pr(B)}{\log \rho} \\
\leq \log \frac{Pr(h \in \mathcal{H} \mid \log \det(I + \rho^{1-\epsilon} H H^\dagger) < \rho^f) + \exp(-\rho^f)}{\log \rho}
\]

\[
\lim_{\rho \to \infty} \log \frac{Pr(A)}{\log \rho} \leq \lim_{\rho \to \infty} \log \frac{Pr(h \in \mathcal{H} \mid \log \det(I + \rho^{1-\epsilon} H H^\dagger) < \rho^f)}{\log \rho}
\] (23)

The last equation follows since the first term in the RHS is polynomial in \( \rho \) whereas the second term is exponential and therefore the sum is dominated by the first term.

After doing the variable change, \( \rho^f = \rho^{1-\epsilon} \) and using the variable \( \rho \) itself in place of \( \rho^f \),

\[
\lim_{\rho \to \infty} \log \frac{Pr(A)}{\log \rho} \leq (1-\epsilon) \lim_{\rho \to \infty} \log \frac{Pr(h \in \mathcal{H} \mid \log \det(I + \rho H H^\dagger) < \rho^{1-\epsilon})}{\log \rho}
\] (24)

In (24), \( \epsilon \) is arbitrary, and we tend it to zero. Hence, by (24) and (17), the exponents for both the bounds in (16) coincide and hence we get,

\[
Pr\{\log \det(I + \rho H H^\dagger \Sigma^{-1}) < r \log \rho\} \leq Pr\{\log \det(I + \rho H H^\dagger) < r \log \rho\}
\]
This proves the third assertion of the lemma. □

Lemma 3.2: [3] For any channel that is of the form \( y = Hx + w \) with \( w \) being white gaussian noise, i.i.d. gaussian inputs are sufficient to attain the best possible outage exponent of the channel.

Proof: Proof is available in [3]. We sketch the outline of the same proof for completeness. The outage probability is given by,

\[
P_{\text{out}}(R) = \inf_{\Sigma_x: \text{Tr}(\Sigma_x) \leq P} \Pr\{I(x; y \mid H = H) \leq R\}
\]

If \( x, y \in \mathbb{C}^m \), then the outage probability can be bounded below and above as,

\[
\Pr\{\log \det(I + \frac{\rho}{m}HH^\dagger) \leq R\} \geq P_{\text{out}}(R) \geq \Pr\{\log \det(I + \rho HH^\dagger) \leq R\}
\]

As \( \rho \to \infty \), it can be shown that the bounds are tight and hence we get (Equation (9) in [3]),

\[
P_{\text{out}}(R) \equiv \Pr(\log \det(I + \rho HH^\dagger) < R) \quad (25)
\]

Remark 6: Because of Lemma 3.2, it is sufficient to consider i.i.d. gaussian input distribution for characterizing the outage exponent. Also, for characterizing outage exponent, we are allowed to assume that the noise is white in the scale of interest (see Lemma 3.1). It can be verified that noise that we deal with in this paper is always satisfies the conditions in Lemma 3.1. Hence we will make these two assumptions throughout the paper

- Signal is distributed as i.i.d. gaussian.
- Noise is white in the scale of interest.

D. A DMT Lower Bound

Definition 6: Consider a set of \( N_i \times N_j \) matrices \( A_{ij}, j = 1, 2, \ldots, N_i, i \geq j \). Let \( A \) be a matrix comprised of the block matrices \( A_{ij} \) in the \((i, j)\)th position, i.e.,

\[
A = \begin{bmatrix}
A_{11} & 0 & \ldots & 0 \\
A_{21} & A_{22} & \ldots & 0 \\
& \ddots & \ddots & \vdots \\
A_{N1} & A_{N2} & \ldots & A_{NN}
\end{bmatrix}.
\]

We will call \( A \) as a block lower-triangular matrix. Define the \( l \)-th sub-diagonal matrix, \( A_l \) of a block lower triangular matrix \( A \) as the block lower triangular matrix comprising of entries \( A_{(l+1)1}, A_{(l+1)2}, \ldots, A_{(l+N-1)N} \) and zeros everywhere else i.e.,

\[
(A_l)_{ij} = A_{ij} \text{ if } i - j = l - 1, \text{ else } (A_l)_{ij} = 0_{N_i \times N_j}.
\]  

The last sub-diagonal matrix of \( A \) is defined as the sub-diagonal matrix \( A_{\ell} \) of \( A \), with the maximum \( l \) such that \( A_{\ell} \) is a non-zero matrix.

Theorem 3.3: Consider a block lower triangular random matrix \( H \) made of matrices \( H_{ij} \) of size \( N_i \times N_j \). Let \( M := \sum_{i=1}^{N} N_i \) be the size of the square matrix \( H \). Consider a channel of the form \( y = Hx + w \), where \( H \) is the \( M \times M \) block lower triangular random matrix, \( x, y, w \) are \( M \times 1 \) vectors. Let \( w \) be a noise vector, which is white in the scale of interest. Let \( x, y, w \) be vectors of length \( N_i \) such that \( x = [x_1, x_2, \ldots, x_N]^T, y = [y_1, y_2, \ldots, y_N]^T \) and \( w = [w_1, w_2, \ldots, w_N]^T \).

Let \( H_d \) be the block-diagonal part of the matrix \( H \) and \( H_\ell \) denote the last sub-diagonal matrix of \( H \), as per Definition 6. Then
1. \( d_H(r) \geq d_{H_2}(r) \).
2. \( d_H(r) \geq d_{H_1}(r) \).
3. In addition, if the entries of \( H_1 \) are independent of the entries in \( H_d \), then \( d_H(r) \geq d_{H_1}(r) + d_{H_2}(r) \).

Proof: The channel is given by \( y = Hx + w \). Since the noise is white in the scale of interest, by Lemma 3.1, the DMT of this channel is the same as that of a channel with the noise distributed as \( \mathbb{C} \mathbb{N}(0, I) \). Therefore, without loss of generality, we assume that \( w \) is distributed as \( \mathbb{C} \mathbb{N}(0, I) \).

We have the block-diagonal part of \( H \), \( H_d = \text{diag}\{(H_{11}, H_{22}, \ldots, H_{NN})\} \) and the last sub-diagonal matrix \( H_1 \) contains \( N - l + 1 \) non-zero entries \( \{H_{11}, H_{(l+1)2}, \ldots, H_{N(N-l+1)}\} \) in the \( l \)-th sub-diagonal.

The outage probability exponent [3] is given by

\[
\rho^{-d(r)} = \inf_{\Sigma_x : \text{Tr} \Sigma_x \leq \rho} \Pr\{ I(x; y : H = H) \leq r \log \rho \}
\]

In order to evaluate this exponent, we first evaluate the mutual information. Let us assume that the input \( x \) is distributed as \( \mathbb{C} \mathbb{N}(0, I) \). By Lemma 3.2, this input distribution is indeed DMT optimal. We will compute the mutual information terms under this assumption that the inputs are iid gaussian.

See Fig. 10 for the i-f diagram. Now, we proceed to find a lower bound on the DMT of the protocol.

Consider the following series of inequalities for all \( i = 1, \ldots, N \).

\[
I(x_i; y|H = H, x_{i-1}^{-1}) \geq I(x_i; y|H = H, x_{i-1}^{-1}) = I(x_1; H_1x_1 + H_{(1l)}x_{1-1} + \ldots + H_{i(i-l)}x_{i-l} + w_1|H = H, x_{i-1}^{-1}) = I(x_i; H_{ii}x_i + H_{i(i-1)}x_{i-1} + \ldots + H_{i(i-l)}x_{i-l} + w_1|H = H, x_{i-1}^{-1}) = I(x_i; H_{ii}x_i + w_i|x_{i-1}^{-1}) = I(x_i; H_{ii}x_i + w_i)
\]

The last step follows since \( \{x_i\} \) are independent.

\[
\Rightarrow I(x; y|H = H) = \sum_{i=1}^{M} I(x_i; y|H = H, x_{i-1}^{-1}) \geq \sum_{i=1}^{M} I(x_i; H_{ii}x_i + w_i) \geq I(x; H_{dd}x + w|H_d = H_d)
\]

In the above, whenever the index of a variable is not positive, we assume that the variable is not present in the conditioning, in order to simplify the notation.

Now by equation (27),

\[
\rho^{-d_H(r)} = \Pr\{ I(x; y|H = H) \leq r \log \rho \} \leq \Pr\{ I(x; H_{dd}x + w|H_d = H_d) \leq r \log \rho \} = \rho^{-d_{H_d}(r)}
\]

\[
d_H(r) \geq d_{H_d}(r)
\]

We have another series of inequalities for all \( i = 1, \ldots, M - 1 \).
Fig. 10. The i-f diagram for the block lower triangular channel matrix.

\[
I(x_{i-\ell}; y | H = H, x_{i-\ell+1}^N) \geq I(x_{i-\ell}; y | H = H, x_{i-\ell+1}^N) = I(x_{i-\ell}; H_i x_i + H_i(\ell-1)x_{i-1} + \ldots + H_{i(i-\ell)}x_{i-\ell} + w_i | H = H, x_{i-\ell+1}^N) = I(x_{i-\ell}; H_{i(\ell)}x_i + H_{i(i-\ell)}x_{i-1} + \ldots + H_{i(i-\ell)}x_{i-\ell} + w_i | x_{i-\ell+1}^N) = I(x_{i-\ell}; H_{i(\ell)}x_i + w_i | x_{i-\ell+1}^N) = I(x_{i-\ell}; H_{i(\ell)}x_i + w_i) \\
\Rightarrow I(x; y | H = H) = \sum_{i=1}^{l+1} I(x_{i}; y | H = H, x_{i+1}^N) \geq \sum_{i=1}^{l+1} I(x_{i-\ell}; y | H = H, x_{i-\ell+1}^N) \geq \sum_{i=1}^{l+1} I(x_{i-\ell}; H_{i(i-\ell)}x_{i-\ell} + w_i) = I(x; H_{\ell}x + w | H_{\ell} = H_{\ell})
\]

(30)

Now by equation (27),

\[
\rho^{-d_H(r)} = Pr \{ I(x; y | H = H) \leq r \log \rho \} \leq Pr \{ I(x; H_{\ell}x + w | H_{\ell} = H_{\ell}) \leq r \log \rho \} = \rho^{-d_{H_{\ell}}(r)}
\]

(31)

Therefore,

\[
I(x; y | H = H) \geq \max(I(x; H_{\ell}d x + w | H_{\ell} = H_{\ell}), I(x; H_{\ell}x + w | H_{\ell} = H_{\ell}))
\]

(33)

The outage probability exponent [3] is given by

\[
\rho^{-d(r)} = \inf_{\Sigma_2: Tr\Sigma_2 \leq P} Pr \{ I(x; y | H = H) \leq r \log \rho \}
\]

Now by equation (33),
\( \rho^{-d_H(r)} = \Pr \{ I(x; y : H = H) \leq r \log \rho \} \) 
\leq \Pr \{ \max(I(x; H_d x + w|H_d = H_d), I(x; H_\ell x + w|H_\ell = H_\ell)) \leq r \log \rho \} 
= \Pr \{ I(x; H_d x + w|H_d = H_d) \leq r \log \rho, I(x; H_\ell x + w|H_\ell = H_\ell) \leq r \log \rho \} 
= \Pr \{ I(x; H_d x + w|H_d = H_d) \leq r \log \rho \} \times \Pr \{ I(x; H_\ell x + w|H_\ell = H_\ell) \leq r \log \rho \} 
= \rho^{-d_{H_d}(r)} \rho^{-d_{H_\ell}(r)} 
= \rho^{-d_{H_d}(r) + d_{H_\ell}(r)} 
\geq d_{H_d}(r) + d_{H_\ell}(r) 
\tag{35} 
\]

where the first step comes about because of the independence of the entries in \( H_d \) and \( H_\ell \), which is indeed the case because of the assumption that all the fading coefficients in the system are independent. The second step is because iid complex gaussian inputs are optimal in the scale of interest.

**Corollary 3.4:** Theorem 3.3 holds even for the case when the matrix \( H \) is block upper-triangular instead of block lower-triangular.

**Proof:** Follows from the proof of Theorem 3.3 since the DMT of a matrix \( H \) and its transpose \( H^T \) are the same.

**Remark 7:** The following two matrix inequalities can be deduced from the proof of Theorem 3.3 with \( H_d \) and \( H_\ell \) defined as in the theorem:

\[ \det(I + \rho H H^\dagger) \geq \det(I + \rho H_d H_d^\dagger) \]
and 
\[ \det(I + \rho H H^\dagger) \geq \det(I + \rho H_\ell H_\ell^\dagger) \]

**Remark 8:** The DMT of a matrix \( H \) is greater than or equal to the DMT of the block diagonal matrix \( H_d \). This bound will be most frequently used whenever we recall Theorem 3.3.

### E. Example Applications of the Main Theorem

In this section, we recover lower bounds on DMT of various existing amplify and forward protocols. While these are already known, the derivations presented here are surprisingly simple and they lead to intuitive explanation of how these protocols achieve the DMT.

**Example 1: Single Source, Single Sink, Single relay, NAF protocol**

Consider the relay network in Fig.\( \ref{fig:naf_network} \) considered in Section \( \ref{sec:example} \). The i-f diagram is given in Fig.\( \ref{fig:naf_if_dia} \)

\[ y = H x + n, \tag{36} \]

where 

\[
H = \begin{bmatrix}
g_1 & 0 \\
g_2 h_2 & g_1 \\
\end{bmatrix}
\]

\[
n = \begin{bmatrix}
w_1 \\
w_2 + h_2 v
\end{bmatrix}
\]

Since two time instants are used in order to obtain the equivalent channel matrix, we have a rate loss by a factor of 2, and hence \( d(r) = d_H(2r) \). It can be checked that the noise vector \( n \) satisfies the conditions in Lemma 3.1 and therefore is white in the scale of interest. Now it is sufficient to study the DMT of the matrix \( H \). Let \( H_d = H \otimes I \),
where $\otimes$ denotes the Hadamard product (entry-wise product) of matrices. Let $H_\ell$ denote the matrix with only the lower triangular entry and set all other entries to zero, i.e.,

$$H_d := \begin{bmatrix} g_1 & 0 \\ 0 & g_1 \end{bmatrix} \quad H_\ell := \begin{bmatrix} 0 & 0 \\ g_2 h_2 & 0 \end{bmatrix}$$

The fading coefficients $g_1, g_2, h_2$ are independent and therefore $H_d$ is independent of $H_\ell$. We use Theorem 3.3 and we get that:

$$d_H(r) \geq d_{H_d}(r) + d_{H_\ell}(r)$$

It is easy to evaluate $d_{H_d}(r)$ and $d_{H_\ell}(r)$:

$$d_{H_d}(r) = (1 - \frac{r}{2})^+$$
$$d_{H_\ell}(r) = (1 - r)^+$$

$$\Rightarrow d_H(r) \geq (1 - \frac{r}{2})^+ + (1 - r)^+$$

We can get the DMT of the protocol as

$$d(r) = d_H(2r)$$
$$\Rightarrow d(r) \geq (1 - r)^+ + (1 - 2r)^+$$

From [5] we know that this bound is indeed tight. However, we will not proceed to find an upper-bound here.

**Example 2: Single source, Single sink, Multiple relays, SAF**

Consider the network in Fig[1] with $N$ relays. We employ an $M$-slot amplify-and-forward protocol termed Slotted Amplify-and-Forward (SAF) introduced in [17]. Each of symbols transmitted by the source reach the sink through the direct link, and through a relayed path. For the case when relays are isolated from each other (see [17] for a description), the induced channel matrix for a $M$ slot protocol is given by a $M \times M$ channel matrix, with $g_d$, the fading coefficient of the direct link, along the diagonal and $g_1, \ldots, g_N$, the product coefficients on relay paths, repeating cyclically along the second sub-diagonal. Let $M = kN + 1$ be the slot length, with $k$ a positive integer.

For example, for $M = 5$, $N = 2$, $k = 2$ case, the induced channel matrix is given by:

$$H := \begin{bmatrix} g_d & 0 & 0 & 0 & 0 \\ g_1 & g_d & 0 & 0 & 0 \\ 0 & g_2 & g_d & 0 & 0 \\ 0 & 0 & g_1 & g_d & 0 \\ 0 & 0 & 0 & g_2 & g_d \end{bmatrix}$$

See Fig[1] for the i-f diagram, where $H_d := g_d, H_\ell := g_{(i-1 \mod 2)+1}, \Sigma_i = 1 + |f_i|^2$. Since the channel is used for $M$ time slots, we have the relation $d(r) = d_H(Mr)$ between the DMT of the protocol, $d(r)$, and the DMT of the matrix $d_H(r)$. Now, we proceed to find a lower bound on the DMT of the matrix.

Let $H_d = g_d I$ be the diagonal matrix corresponding to $H$. Let $H_\ell$ be the second sub-diagonal matrix corresponding to $H$. It contains $g_1, \ldots, g_M$ each for $k$ times in the second sub-diagonal. From Theorem 3.3 the DMT of $H$ can be lower bounded as:

$$d_H(r) \geq d_{H_d}(r) + d_{H_\ell}(r) \quad (37)$$

We already have

$$d(r) = d_H(Mr) \quad (38)$$
$$\Rightarrow d(r) \geq d_{H_d}(Mr) + d_{H_\ell}(Mr) \quad (39)$$

Now the DMT of the matrices $H_d$ and $H_\ell$ can be easily derived as: $d_{H_d}(r) = (1 - \frac{r}{M})^+$ and $d_{H_\ell}(r) = N(1 - \frac{r}{M-1})^+$.
The right hand side is in fact shown to be equal to the DMT of the SAF protocol in [17].

**Example 3: Single Source, Single Sink, Multiple Antenna, Single relay, NAF protocol**

Let us first consider a single relay network with the source, the relay and sink equipped with multiple antennas $n_s, n_r, n_d$. Let us use the NAF protocol [5] in this scenario, as is done in [15]. The channel matrix turns out to be

$$H := \begin{bmatrix} H_d & 0 \\ H_\ell & H_d \end{bmatrix}$$  \hspace{1cm} (41)

where $H_d$ is the $n_d \times n_s$ fading matrix between source and the sink, $H_\ell$ is the product fading matrix of an $n_r \times n_s$ matrix between the source and the relay and an $n_d \times n_r$ matrix between relay and sink. Proceeding in the same manner as in Example 1, we can get that $d(r) \geq d_{H_d}(r) + d_{H_\ell}(2r)$, where $d_{H_d}(r)$ is the DMT of the direct link matrix $H_d$, and $d_{H_\ell}(r)$ is the DMT of the product matrix $H_\ell$. This lower bound was derived as Theorem 1 of [15].

Let us now consider a generalized NAF protocol (see [23]) where, for the first $T$ time instants, the source transmits to the relays and then the relays transmit a linear transformation of the received vector over the $T$ time instants. Even in this case, the input output transformation can be represented using an equation of the form (41). However $H$ is now a $2Tn_d \times 2Tn_s$ matrix, $H_d$ is a $Tn_s \times Tn_d$ block diagonal matrix with the direct link fading matrix repeated $T$ times and $H_\ell$ is any $Tn_d \times Tn_s$ matrix (which depends on the linear transformations used at the relays) relating the inputs to the output at the sink due to the relaying path. Let $d_{C}(r) := d_{H_\ell}(Tr)$ denote the DMT of the same scheme used without the direct link and with full duplex relays. Let $d_{D}(r) := d_{H_d}(Tr)$ denote the DMT of the direct path fading matrix.

Then Theorem 3.3 can be used to get the following inequality for the DMT of this generalized NAF scheme:

$$d(r) \geq d_{D}(r) + d_{C}(2r)$$

This proves Conjecture 1 of [23].

**Example 4: Single Source, Single Sink, Multiple Antenna, Multiple relays, NAF protocol**

In [15], the authors consider a two-hop relay network with a direct link and $N$ relays. Consider the NAF protocol for the $N$ relay case suggested in [15] in which each path is used for equal duration. Here we consider a general version of the NAF Protocol, where different relaying paths are activated for different fractions of time. Let the relaying path through relay $i$ be used for $f_i$ fraction of the time. For this protocol, let us derive the DMT. The matrix connecting the input and the output is a block lower-triangular matrix with the direct-link fading matrix $H_d$.
repeated on the block-diagonal. The second sub-diagonal contains entries matrices $R_1, R_2, \ldots, R_N$, where $R_i$ is the product matrix along the $i$th relay. We can bound the DMT of resulting matrix using Theorem 3.8

$$d(r) \geq d_{H_d}(r) + d_C(2r) \quad \quad (42)$$

where $d_C(r)$ is the DMT of a parallel channel with entries $R_i$ occurring for a fraction $f_i$ of the time. We can evaluate $d_C(r)$ explicitly from the DMT $d_i(r)$ of the product channel $R_i$.

The DMT of this channel can be computed using the parallel channel formula given in equation Equation (55) in Lemma 3.8 and it is given by,

$$d_C(r) = \sup_{(f_1, f_2, \ldots, f_K) \in J} \inf_{(r_1, r_2, \ldots, r_K) : \sum_{i=1}^{K} f_i r_i = r} \sum_{i=1}^{K} d_i(r_i) \quad \quad (43)$$

where $d_i(r)$ is the DMT of the product channel in the $i$th channel and corresponds to the DMT of the product matrix $G_i H_i$.

Therefore the overall DMT is given by

$$d(r) \geq d_{H_d}(r) + \sup_{(f_1, f_2, \ldots, f_K) \in J} \inf_{(r_1, r_2, \ldots, r_K) : \sum_{i=1}^{K} f_i r_i = r} \sum_{i=1}^{K} d_i(r_i) \quad \quad (44)$$

As a particular choice, if $f_i = 1/N$ for all $i$, then

$$d_C(r) = \inf_{(r_1, r_2, \ldots, r_K) : \sum_{i=1}^{K} r_i = N r} \sum_{i=1}^{K} d_i(r_i) \quad \quad (45)$$

Let $\theta_i := \frac{r_i}{N r}$. Then we have

$$d_C(r) = \inf_{(\theta_1, \theta_2, \ldots, \theta_K) : \sum_{i=1}^{K} \theta_i = 1} \sum_{i=1}^{K} d_i(N \theta_i r) \quad \quad (46)$$

We plug this equation into (42) and get

$$d(r) \geq d_{H_d}(r) + \inf_{(\theta_1, \theta_2, \ldots, \theta_K) : \sum_{i=1}^{K} \theta_i = 1} \sum_{i=1}^{K} d_i(2N \theta_i r) \quad \quad (47)$$

which is indeed the formula in Theorem 2 of [15]. However the lower bound on DMT that we have in Equation (44) is better than the lower bound in Theorem 2 of [15] since we allow for arbitrary periods of activation which is a more general approach.

Remark 9: In the notation of [15], $d_{H_d}(r) = d_F(r)$ since $F$ is the matrix of transformation between source and sink through the direct link. Also $G_i$ is the matrix between source to relay $i$ and $H_i$ matrix between relay $i$ to sink. According to notation of [15], $d_{G_i H_i}(r)$ is the DMT corresponding to the product matrix $G_i H_i$.

F. DMT of elementary network connections

1) Parallel Network:

Lemma 3.5: Consider a parallel channel with $M$ links, the each link being represented by $y_i = H_i x_i + w_i$, and let the optimal DMT of the $i$th link be $d_i(.)$. Then the optimal DMT of the parallel channel is given by

$$d(r) = \inf_{(r_1, r_2, \ldots, r_M) : \sum_{i=1}^{M} r_i = r} \sum_{i=1}^{M} d_i(r_i) \quad \quad (48)$$

Proof: The input-output relation of the parallel channel is given by
The equality in the last equation occurs if all the \( x_i \) are independent. So we will choose the \( x_i \) to independent, for the rest of the discussion, since this maximizes the mutual information and hence minimizes the error probability.

Define \( Z_i := I(x_i; y_i | H_i = H_i) \). Now \( Z_i \) is a random variable which depends on the realization of the channel. Since \( \{H_i\} \) are independent, \( \{Z_i\} \) are also independent. Let \( R_i = r_i \log(\rho) \) and \( R = r \log(\rho) \) for \( i = 1, 2 \).

Now our goal is to evaluate \( P \{ \sum_{i=1}^{M} Z_i \leq r \log(\rho) \} \). To do this, first we consider the case when \( M = 2 \) and
we evaluate \( P\{ Z_1 + Z_2 \leq r \log(\rho) \} \). Then we extend this to general \( M \) by induction.

\[
\begin{align*}
F_{Z_i}(R_i) & := P\{ Z_i < R_i \} \\
 f_{Z_i}(R_i) & := \frac{d}{dR_i} F_{Z_i}(R_i)
\end{align*}
\]

Let \( F_{Z_i}(R_i) \leq \rho^{-d_i(r_i)} \)

Then \( f_{Z_i}(R_i) \leq \frac{d}{dr_i} \rho^{-d_i(r_i)} \)

\[
\begin{align*}
F_{Z_i}(R_i) & := \int_0^\infty f_{Z_i}(R_i) F_{Z_2}(R-R_1) dR_1 \\
 & = \int_0^\infty \rho^{-d_i(r_i)} \rho^{-d_2(r-r_1)} \ln(\rho) d(r_1)
\end{align*}
\]

By Varadhan’s Lemma [30], the SNR exponent integral can be evaluated in the scale of interest as:

\[
d(r) = \inf_{r_i \geq 0} d_1(r_1) + d_2(r - r_1)
\]

\[
= \inf_{(r_1, r_2): r_1 + r_2 = r} \frac{2}{i=1} d_i(r_i)
\]

Now, consider the general case with \( M \) parallel channels

\[
\rho^{-d(r)} \leq P\{ \sum_{i=1}^M Z_i \leq r \log(\rho) \}
\]

Proceeding by induction, we get:

\[
d(r) = \inf_{(r_1, r_2, \ldots, r_M): \sum_{i=1}^M r_i = r} \sum_{i=1}^M d_i(r_i)
\]

**Remark 10:** The following lower and upper bounds on the outage exponent are immediate from Equation (48):

\[
d(r) \leq \sum_{i=1}^M d_i(\frac{r}{K}) \tag{51}
\]

\[
d(r) \geq \sum_{i=1}^M d_i(r) \tag{52}
\]

We recall the following Lemma from the theory of majorization [32]:

**Lemma 3.6:** [32] If \( f(.) \) is a symmetric function in variables \( r_1, r_2, \ldots, r_N \) and is convex in each of the variables \( r_i, i = 1, 2, \ldots, N \), then,

\[
\inf_{(r_1, r_2, \ldots, r_N): \sum_{i=1}^N r_i = r} f(r_1, r_2, \ldots, r_N) = f\left(\frac{r}{N}, \frac{r}{N}, \ldots, \frac{r}{N}\right) \tag{53}
\]

**Lemma 3.7:** The DMT of a parallel channel with all the individual channels being identical and having a convex DMT is given by:

\[
d(r) = Md_1\left(\frac{r}{M}\right) \tag{54}
\]
Proof: Consider \( \sum_{i=1}^{M} d_i(r_i) \) as a function of the variables \( r_i \). Then the function satisfies the conditions of Lemma 3.6. Therefore,

\[
d(r) = \inf_{(r_1, r_2, \ldots, r_M): \sum_{i=1}^{M} r_i = r} \sum_{i=1}^{M} d_i(r_i)
\]

\[
= \sum_{i=1}^{M} d_i\left(\frac{r}{M}\right)
\]

\[
= Md_i\left(\frac{r}{M}\right)
\]

\[\blacksquare\]

2) Parallel Channel with Repeated Coefficients:

Lemma 3.8: Consider a parallel channel with \( M \) links with repeated channel matrices. Let there be \( N \) distinct channel matrices \( H^{(1)}, H^{(2)}, \ldots, H^{(N)} \), with \( H^{(i)} \) repeating in \( n_i \) sub-channels, such that \( \sum_{i=1}^{N} n_i = M \). Let \( f_i = \frac{n_i}{M} \). Then the DMT of the parallel channel is given by,

\[
d(r) = \inf_{(r_1, r_2, \ldots, r_M): \sum_{i=1}^{M} f_i r_i = \sum_{i=1}^{N} d_i(r_i)}
\]

Fig. 13. The Parallel Network with repeated coefficients

Proof: Following the same line of arguments in the proof of Lemma 3.5, choose \( x_i \) to be independent. For computing the DMT, we know from Lemma 3.2 that the inputs can in fact be independent and identically distributed with a \( \mathbb{C} \mathbb{N}(0, I) \) distribution. So we have

\[
I(x; y|H = H) = \sum_{i=1}^{M} I(x_i; y_i|H_i = H_i)
\]

\[
P\{I(x; y|H = H) \leq r \log p\} = P\{\sum_{i=1}^{M} I(x_i; y_i|H_i = H_i) \leq r \log p\}
\]

\[
= P\{\sum_{i=1}^{N} n_i I(x_i; y_i|H_i = H_i) \leq r \log p\}
\]

\[
= P\{\sum_{i=1}^{N} n_i I(x_i; y_i|H_i = H_i) \leq r \log p\}
\]
Now, define $Z_i := n_i I(x_i; y_i | H_i = H_i)$. Also let
\[
\rho^{-d_i(r)} \doteq P\{Z_i < r \log(\rho)\} \\
= P\{I(x_i; y_i | H_i = H_i) < \left(\frac{r}{n_i}\right) \log(\rho)\} \\
= \rho^{-d_i(i)}
\]
where, $\rho^{-d_i(r)} \doteq P\{\mu(H(i), p_x) < r \log(\rho)\}$.

Using the same convolution argument in the proof of Lemma 3.5,
\[
d(r) = \inf_{(r_1, r_2, \ldots, r_N): \sum_{i=1}^{N} r_i = r} \sum_{i=1}^{N} d_i(r_i)
\]
\[
= \left(\frac{\rho}{n} \right) \inf_{(r_1, r_2, \ldots, r_N): \sum_{i=1}^{N} r_i = r} \sum_{i=1}^{N} d_i\left(\frac{r_i}{n_i}\right)
\]
\[
= \inf_{(r_1, r_2, \ldots, r_N): \sum_{i=1}^{N} r_i = r} \sum_{i=1}^{N} d_i(r_i)
\]

### G. Achievability of outage exponent

In all the above derivations, it was assumed that the outage exponent was equal to the DMT. It needs to be shown that the outage exponent can indeed be achieved. We first give a simple compound channel argument for the achievability, similar to the argument in [11]. Consider a compound channel, where a channel, $s$ is chosen from a set of possible channels $S$ and the channel remains fixed. Then the capacity of the compound channel is given by

\[
C = \sup_{p_X} \inf_{s \in (S)} I(X; Y | S = s)
\]

If the maximizing input distribution $p_X^*(x)$ is the same for all possible channels $s \in S$, then

\[
C = \inf_{s \in S} C_s, \text{ where} \\
C_s := I(X; Y | S = s)
\]
evaluated for $p_X^*(x)$, which is indeed the capacity of the channel $s$.

Consider the set of all channels not in outage, $\mathcal{H}$. Then $\mathcal{H}$ is defined as

\[
\mathcal{H} = \{H : I(X; Y | H = H) > r \log(\rho)\}
\]

If the optimizing distribution is independent of $H$ in $\mathcal{H}$, then the capacity of the compound channel $\mathcal{H}$ is given by $C = r \log \rho$.

This means that there exists a code for this compound channel, whose probability of error is less than $\epsilon$ for any given $\epsilon > 0$. The probability of error of this code when used on the slow fading channel is given by

\[
P_e = P_{\text{out}} P_{\text{out}} + P_{\text{out}}^c P_{\text{e/out}}
\]
\[
\leq P_{\text{out}} + P_{\text{e/out}}^c
\]
\[
\leq P_{\text{out}} + \epsilon
\]
\[
\leq P_{\text{out}}
\]

where $P_{\text{out}}$ is the probability of the channel being in outage and $P_{\text{out}}^c$ is the probability of the channel not being in outage. $P_{\text{e/out}}$ is the probability of error of the code given the channel is in outage and $P_{\text{e/out}}^c$ is the probability of error of the code given the channel is not in outage. Thus the outage probability is achievable if the optimizing distribution is independent of $H$. 
Since the outage exponent optimizing distribution is iid gaussian, which is independent of $H$, as shown in Lemma 3.2, we can show that outage exponent is achievable using universal codes. It should be pointed out here that short approximately universal codes for the MIMO parallel channel were given recently in [13]. These codes indeed achieve the outage exponent of the parallel channels considered in Section 3.5.

IV. FULL DUPLEX RELAY NETWORKS

In this section, we consider networks equipped with full duplex (FD) relay nodes. First, we draw a general result on the optimum diversity of a multi-terminal network. We also provide an achievable DMT region for an ss-ss network with single antenna nodes.

A. Mincut equals Diversity

**Theorem 4.1:** Consider a multi-terminal fading network with nodes having multiple antennas with each edge having iid Rayleigh-fading coefficients. The maximum diversity achievable for any flow is equal to the min-cut between the source and the sink corresponding to the flow. Each flow can achieve its maximum diversity simultaneously.

**Proof:** First we consider the case where there is only a single source-sink pair. We will prove the theorem in two cases: the single antenna antenna case and the multiple antenna case. We shall assume that all the fade coefficients are independent.

**Case I: Network with single antenna nodes**

Let the source be $S_i$ and sink be $D_j$. Let $C_{ij}$ denote the set of all cuts between $S_i$ and $D_j$. From cutset bound [8],

\[
d(r) \leq \min_{C \in C_{ij}} d_C(r) \Rightarrow d(0) \leq \min_{C \in C_{ij}} d_C(0) =: m
\]

where $m$ is the number of edges in the min-cut between $S_i$ and $D_j$.

Sufficient to prove that diversity order of $m$ is achievable. We know that the number of edges in the min-cut is the maximum number of edge disjoint paths between source and the sink. Schedule the network in such a way that each edge in a given edge disjoint path is activated one by one. Same is repeated for all the edge disjoint paths. Thus, the same data symbol is transmitted through all the edge disjoint paths from $S_i$ to $D_j$.

Let the number of edges in the $i$th edge disjoint path be $n_i$. The $j$th edge in the $i$th edge disjoint path is denoted by $e_{ij}$ and the associated fading coefficient be $h_{ij}$. So the activation schedule will be as follows: $e_{11}, e_{12}, \ldots, e_{1(n_1)}, e_{21}, \ldots, e_{2(n_2)}, \ldots, e_{m1}, e_{m2}, \ldots, e_{m(n_m)}$. Now define $h_i := \prod_{j=1}^{n_i} h_{ij}$. Let the total number of time slots required be $N = \sum_{i=1}^{m} n_i$.

With this protocol in place, the equivalent channel seen by a symbol is

\[
H = \begin{bmatrix}
h_1 & 0 & \cdots & 0 \\
0 & h_2 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & h_m
\end{bmatrix}
\]

If $d_e(r)$ is the outage exponent for this channel,
\[ \rho^{-d_e(r)} = \Pr \left\{ \sum_{i=1}^{m} \log(1 + |h_i|^2) \leq r \log \rho \right\} \]
\[ = \Pr \left\{ \sum_{i=1}^{m} \log \left( 1 + \prod_{j=1}^{n_i} |h_{ij}|^2 \right) \leq r \log \rho \right\} \]
\[ = \Pr \left\{ \sum_{i=1}^{m} \log \left( 1 + \rho^{1 - \sum_{j=1}^{n_i} u_{ij}} \right) \leq r \log \rho \right\} \]
\[ \leq \Pr \left\{ \prod_{i=1}^{m} \left( \rho^{1 - \sum_{j=1}^{n_i} u_{ij}} \right)^+ \leq \rho^r \right\} \]

Following the same lines of arguments as in [3],
\[ d(r) = \inf_{A} \sum_{i=1}^{m} \sum_{j=1}^{n_i} u_{ij} \] (62)
where
\[ A = \{ u_{ij} : \sum_{i=1}^{m} (1 - \sum_{j=1}^{n_i} u_{ij})^+ \leq r \} \] (63)

Let \( \sum_{j=1}^{n_i} u_{ij} = u_i \). Then,
\[ d(r) = \inf_{A'} \sum_{i=1}^{m} u_i \]
where \( A' = \{ u_i : \sum_{i=1}^{m} (1 - u_i)^+ \leq r \} \)
\[ \Rightarrow d_e(r) = m - r \]

Since we use \( N \) channel uses, the effective outage exponent is given by,
\[ d(r) = d_e(Nr) = m - Nr \] (64)

Hence the maximum achievable diversity is \( m \).

**Case II: Network with multiples antenna nodes**

In the multiple antenna case, we regard any link between a \( n_t \) transmit and \( n_r \) receive antenna as being composed of \( n_t n_r \) links, with one link between each transmit and each receive antenna. Note that it is possible to selectively activate precisely one of the \( n_t n_r \) Tx-antenna-Rx-antenna pairs by appropriately transmitting from just one antenna and listening at just one Rx antenna. The same strategy as in the single antenna case can then be applied to achieve this diversity in the network.

Fig. [14] illustrates this conversion for the case of a single source \( S \), two relays \( R_1 \) and \( R_2 \) and a sink \( D \). Having converted the multiple antenna network into one with single antenna nodes, **Case II** follows from **Case I**.

Thus the proof is complete for the single flow from \( S_i \) to \( D_j \).

When there are multiple flows in the network, we simply schedule the data of all the flows in a time-division manner. This will entail a rate loss - however, since we are interested only in the diversity, we can still achieve each flow's maximum diversity simultaneously.

**Definition 7:** Consider a network \( N \) and a path \( P \) from source to sink. This path \( P \) is said to have an intermediate direct path if there is a direct link in \( N \) connecting two non-consecutive nodes in \( P \).

**Theorem 4.2:** Consider a ss-ss full-duplex network with single antenna nodes. Let the min-cut of the network be \( M = d_{\max} \). Let the network satisfy **either** of the two conditions:

1) *None* of the \( M \) edge disjoint paths between source and sink have intermediate direct paths, or
2) The directed graph representing the network has no directed cycles.
Then, a linear DMT $d(r) = M(1 - r)^+$ between the maximum multiplexing gain of 1 and maximum diversity is achievable.

Proof: Given that the network has min-cut $M$, it means that there are $M$ edge disjoint paths from source to sink. By the hypothesis of the lemma, we have that these edge disjoint paths do not have any intermediate direct paths. Let us call the edge disjoint paths $e_1, e_2, \ldots, e_M$. Let the product of the fading coefficients along the path $e_i$ be $g_i$. Let $D_i$ be the delay of each path. Let $D = \max D_i$. Add delays $D - D_i$ to the path $e_i$ such that now all paths have equal delay. We follow the following steps in order to activate the edges:

1) a) Activate edge disjoint path $e_1$ for a period $T$, where $T > D$: activating all edges of the edge disjoint path simultaneously. This will create a transfer matrix from the source symbols to sink symbols as a diagonal matrix with zeros on the first $D$ rows, and only one non-zero thread in the matrix comprised of coefficients equal to $g_1$ which is the product coefficient on path $e_1$. After this is done, the various nodes in the network store the data that have not yet been passed to the sink for future use.

   b) Repeat Step 1.a for all edge disjoint paths $e_1, \ldots, e_M$. The net transfer matrix will comprise $MD$ zero rows and one non-zero thread which contains each $g_i$ for $T - D$ durations.

2) Activate all the edge disjoint paths each for time $T$. This time, the net transfer matrix will comprise of a single non-zero thread which contains each product coefficient $g_i$ for $T$ durations. There will be no zero rows since all nodes always have information to transmit.

3) Repeat Step 2 for $L - 2$ more times, thereby all edge disjoint paths have been activated for $L$ times.

Now the induced channel matrix from source to sink will comprise of $MD$ zeros initially and on removing these rows we get a transfer matrix, $H$. $d(r) = d_H(LMTT)$. For $L$ large, we will have $d(r) = d_H(LMTT)$.

This matrix $H$ will have each $g_i$ for $LT - D$ times along the diagonal. This matrix will be lower triangular if none of the $M$ edge disjoint paths between source and sink have intermediate direct paths. This matrix will be upper triangular if the directed graph representing the network has no directed cycles. In either case, we can use Theorem 3.3 and Corollary 3.4 we get that $d_H(r) \geq d_H^+(r)$, where $H_d$ is the diagonal matrix corresponding to the matrix $H$. But $H_d$ contains $LT - D$ entries each of $g_i$, therefore this matrix DMT is given by $d_H(r) = d_H(LMTT)$ where $H_1 = \text{diag}(g_1, \ldots, g_M)$. $d(r) = d_H(LMTT) \geq d_H^+(LMTT) = d^+(LT - D)$. For $LT$ tending to $\infty$, we get $d(r) \geq d_H^+(M)$. Now $d(H^+(r) = (M - r)^+$. Since $M = d_{\max}$, we get

$$d(r) \geq d_{\max}(1 - r)^+$$

(65)

Corollary 4.3: For the full duplex KPP networks without direct link (i.e. KPP(I) networks) and full duplex layered networks, a DMT of $M(1 - r)^+$ which is a linear DMT between the maximum diversity and maximum multiplexing gain can be achieved.

Proof: It can be easily shown that the $M$ edge disjoint paths between source and sink for KPP(I) and layered networks do not have any intermediate direct path. Therefore it satisfies condition (1) of Theorem 3.2 and hence proved.
V. HALF DUPLEX NETWORKS WITH ISOLATED PATHS - KPP NETWORKS

In this section, we consider single-source single-sink (ss-ss) half duplex networks in which relaying paths are isolated (i.e., interference between the paths is absent). Every node is equipped with a single antenna. In general, it is assumed that half duplex networks incur a loss in multiplexing gain by a factor of 2. But we will establish that we can achieve the same performance in DMT with half duplex relays as that of full duplex ones, in most of the cases. We will show systematic ways of constructing protocols for multi-hop networks with half duplex relays. We will show that we can achieve the same optimal DMT of KPP networks with/without direct link.

We first consider KPP networks in the absence of a direct link. At the end of this section we extend the results to KPP(D) networks.

A. Protocols for K-Parallel Path Networks

We consider amplify-and-forward (AF) protocols in this paper. In the class of AF protocols considered in this paper, the communication takes place in a block of \( N \) time instants, during which the channel fading coefficients remain fixed. We assume that the edge activations are periodic, and we refer to \( N \) as the cycle length of the protocol. We shall describe all our protocols in a simple manner, as an edge coloring scheme. Let \( C = \{c_1, c_2, \ldots, c_N\} \) be the set of \( N \) colors used in the scheme. All the edges in the network are assigned a subset of colors from the set \( C \). The subset of colors assigned to the edge \( e_{ij} \) will be denoted by \( A_{ij} \). Each color in \( A_{ij} \) represents the time instants during which the edge \( e_{ij} \) is active. However, due to the broadcast nature, a node will experience interference if there is any other node connected to this one is transmitting, apart from its intended transmitting node. A protocol which avoids this interference is said to be an interference free protocol, which will be of interest to us. Also, in the class of AF protocols that we consider, we assume that neither the source broadcasts simultaneously to different nodes nor does the sink listen to simultaneous transmission by different nodes. We will see later that imposing such a constraint on the protocol is not restrictive, since we are able to achieve the best possible DMT performance with such a protocol.

The upper bound on DMT for the class of KPP networks using the cutset bound (Lemma 1.1) is given by:

\[
d(r) \leq K(1 - r).
\]

Hence, for each of the KPP networks, we shall try to approach this bound. Since this bound corresponds to a MISO channel, we refer to this as the MISO bound. We shall prove, by constructing protocols and computing their DMT, that this bound can be achieved for all \( K \geq 3 \).

B. Protocols achieving MISO bound

In this section we propose protocols for the \( K \)-parallel path network and compute their DMT. For the case when \( K \geq 3 \) the DMT of proposed protocols achieve the MISO bound. Also, for the case \( K = 2 \) we find the maximum multiplexing gain that a protocol can achieve among the class of AF protocols considered in this paper.

Definition 8: A half duplex protocol is said to be an orthogonal protocol if at any node, at a given time instant, only one of the incoming or outgoing edges is active and none of the nodes perform any processing of the symbols, but just forwards the incoming packets. We put a further condition that an orthogonal protocol for a KPP network has all edges on a given parallel path activated equal number of times.

Remark 11: In networking literature [29], a network is said to have orthogonal channels if interference is avoided at all nodes and each node can communicate with at most one other node at any given time. While Definition 8 is similar to this, the notion of orthogonal protocols will be generalized to networks with interference as well in Section VI.

Proposition 1: Let \( C = \{c_1, c_2, \ldots, c_N\} \) be the set of colors. An edge coloring is a map \( \psi : E \to \mathcal{P}_C \) which takes \( e_{ij} \) to \( A_{ij} \).

Every orthogonal protocol can be described as an edge coloring of the network satisfying the following constraints. Similarly, every edge coloring satisfying the following constraints describes an orthogonal protocol.

\(^3\)We assume that the network is in operation for sufficient amount of time, so that if an edge is active, the node at beginning of the edge always has a symbol to transmit.
A_I \cap A_J = \phi, i \neq j. \quad (66)

A_{in} \cap A_{jn} = \phi, i \neq j. \quad (67)

A_{ij} \cap A_{ij+1} = \phi, j = 1, 2, ..., n_i - 1. \quad (68)

|A_{ij}| = m_j, j = 1, 2, ..., n_i. \quad (69)

Each color in \( C \) represents a time slot and so the length of the cycle for the protocol is \( N \). Each color in \( A_{ij} \) represents the time slots during which the edge \( e_{ij} \) is active.

The first constraint corresponds to the fact that for an orthogonal protocol, only one outgoing edge is active at the source. Similarly the second constraint corresponds to the fact that for an orthogonal protocol, only one incoming edge is active at the sink. The third constraint captures the half duplex nature of the protocol. The last constraint indicates that all the edges in a given path are active for equal duration of time so that all the symbols transmitted by the source are forwarded to the sink.

**Definition 9:** The rate, \( R \) of an orthogonal protocol is defined as the ratio of the number of symbols transmitted by the source to the total number of time slots. In the notation above, we have

\[
R = \frac{\sum_{i=1}^{K} m_i}{N}
\]

**Definition 10:** Consider a KPP network. Let \( v_1, v_2, v_3, v_4 \) be four consecutive vertices lying on one of the \( K \) paths leading from source to sink. Let \( v_1 \) and \( v_3 \) transmit, thereby causing the edges \( (v_1, v_2) \) and \( (v_3, v_4) \) to be active. Due to the broadcast and interference constraints, transmission from \( v_3 \) interferes with the reception at \( v_2 \). This is termed as back-flow, and is illustrated in Fig. 15.

![Fig. 15. Back-flow on a path](image)

Back-flow can be avoided if we make sure that there is at least two inactive edges between any two active edges. We formalize this in the following remark:

**Remark 12:** An orthogonal protocol avoids back-flow if the corresponding coloring satisfies the following condition:

\[
A_{ij} \cap A_{ij+2} = \phi, j = 1, 2, ..., n_i - 2.
\]

By Remark 12, it is evident that any three adjacent edges \( e_{ij}, e_{i(j+1)}, \) and \( e_{i(j+2)} \) will map to disjoint sets of colors when the coloring scheme corresponds to an orthogonal protocol avoiding back-flow. Moreover, it remains consistent with the constraints to repeat the same set of colors in every third edge. This suggests an easy way of describing the edge coloring. For a given path in the network, we will have three sets of colors in order and they are cyclically associated to edges starting from source to sink. For reasons that will become apparent later, the last edge (edge connected to the sink) in the given path may get associated to a different set of colors. So, to describe an orthogonal protocol, we define a tuple of sets \( G_i = [G_{i0}, G_{i1}, G_{i2}] \) and a set \( F_i \) for all \( i \) such that

\[
A_{ij} = \begin{cases} 
G_{i(j \mod 3)}, & j \neq n_i \\
F_i, & j = n_i 
\end{cases} \quad (70)
\]

Hereafter, we will use \( G^i \) and \( F^i \) for \( i = 1, 2, ..., K \) to completely describe an orthogonal protocol. Here, \( G^i \) specifies the colors that are repeated cyclically on the edges of the path \( P_i \) and \( F^i \) specifies the color on the last edge \( e_{in} \) of path \( P_i \).

**Lemma 5.1:** Consider a KPP network. If an orthogonal protocol satisfies the following constraints:
1) The rate of the protocol is equal to one.
2) In every cycle, the sink receives equal number of symbols from each one of the $K$ parallel paths.
3) The protocol avoids back-flow.

Then the protocol achieves the MISO bound\[d(r) = K(1 - r)^+\]

**Proof:** The induced channel matrix for any orthogonal protocol for a $K$-parallel path network can be split into block diagonal matrices. This is by virtue of the fact that we are dealing with $K$ parallel paths and at any time instant, the sink receives a symbol from only one of the $K$ paths. Further, the input symbols can be reordered such that the matrices $H_i$ on the block-diagonal contain fading coefficients corresponding to the $i$-th path.

So, the induced channel matrix $H$ between the source and sink, considering $mK$ time instants of transmission, can be written in terms of the channel matrices $H_i$, $i = 1, 2, \cdots, K$, where $H_i$ is the $m \times m$ channel matrix for path $P_i$.

\[
H = \begin{bmatrix}
H_1 & 0 & \cdots & 0 \\
0 & H_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & H_K
\end{bmatrix}.
\] (71)

\[
\Rightarrow \det(I + \rho HH^\dagger) = \prod_{i=1}^{K} \det(I + \rho H_i H_i^\dagger)
\] (72)

For protocols which avoid back-flow and use all paths equally, the channel matrix for path $P_i$ is given by

\[
H_i = g_i I_m, \quad i = 1, 2, \cdots, K.
\] (73)

where $g_i = \prod_{j=1}^{n_i} g_{ij}$

Consider one cooperation frame of the protocol satisfying the above constraints. Let $x_i$ be the column vector of $m$ symbols transmitted by the source to path $P_i$ and $y_i$ be the column vector of $m$ symbols received by the sink from the path $P_i, 1 \leq i \leq K$. Since $x_i$ passes through all the edges $e_{ij}, 1 \leq j \leq n_i$, before reaching the sink, the channel model for one cooperation frame can be written as

\[
\begin{bmatrix}
y_1 \\
y_2 \\
\vdots \\
y_K
\end{bmatrix} = \begin{bmatrix}
g_1 I_m \\
g_2 I_m \\
\vdots \\
g_K I_m
\end{bmatrix} + \begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_K
\end{bmatrix} + n
\] (74)

\[
y = Hx + n
\] (75)

where $n$ is the equivalent colored noise seen at the sink and $H$ is the equivalent parallel channel. It can be easily shown that the noise becomes white, in the scale of interest [9]. The DMT of the above channel, $H$, can be shown to be,

\[
d(r) = K(1 - r)^+,
\]

which is the MISO bound. Here, the notation $(1 - r)^+$ indicates that we must choose the maximum of $0$ and $1 - r$.

**Corollary 5.2:** If any orthogonal protocol has a channel matrix $H$, with $H_i$ as the channel matrix for the path $P_i$, such that $\det(I_m + \rho H_i H_i^\dagger) = \det(I_m + \rho H_i H_i^\dagger)$, where $H_i = g_i I_m$ for $i = 1, 2, \cdots, K$, then that protocol achieves the MISO bound.

**Proof:** The DMT depends only upon $\det(I + \rho HH^\dagger)$ which remains the same as that in (72). Therefore the DMT remains same.

\[\text{Throughout the paper, keeping in mind that the number } N \text{ of symbols transmitted can be made large, we ignore a rate-loss factor of } \frac{N}{N+D} \text{ arising from the presence of } D \text{ units of delay in the network.}\]
Theorem 5.3: When $K \geq 4$, there exists a protocol achieving MISO bound for KPP networks.

Proof: We now establish an orthogonal protocol for the case when $K \geq 4$. By Prop [1], it is sufficient to establish a coloring of the edges. We will give the map $\psi$ explicitly for the given network by specifying $A_{ij}$, $\forall i,j$.

We will be using the set of colors $C = \{c_1, c_2, ..., c_K\}$. In the following, whenever we refer to color $c_i$ assume $c_0 = c_K$ and for $i > K$, $c_i = c(i \mod K)$.

We will specify the coloring scheme by giving a tuple of sets $G_i = [G_{i0}, G_{i1}, G_{i2}]$ and a set $F_i$ for all $i$.

$$G_i = \{c_i\}, \{c_{i+1}\}, \{c_{i+2}\}$$
$$F_i = \{c_{i+3}\}$$

It is easy to verify that the scheme described satisfies all the constraints of Lemma 5.1, and therefore will achieve the MISO bound.

C. Back-flow does not impair the DMT

Lemma 5.4: Consider a network running an orthogonal protocol, which, in the absence of back-flow creates a block-diagonal matrix as the transfer matrix between the input and the output. For such a network, the DMT when back-flow is present, is lower bounded by the DMT in the absence of back-flow.

Proof: The presence of back-flow creates entries in the strictly lower-triangular portion of the transfer matrix. Since the DMT of a lower triangular matrix is lower bounded by the DMT of the corresponding diagonal matrix (by Theorem 3.3), we have that the system with back-flow will yield a better DMT than the one without back-flow.

1) Back Flow does not alter DMT in the Single Antenna Case: Since we already have a lower bound on the DMT of the networks with back-flow, it is sufficient to get an upper bound on the DMT, which is the same as the lower bound.

Lemma 5.5: Consider a KPP network running an orthogonal protocol with single antenna nodes, which in the absence of back-flow creates a diagonal matrix as the transfer matrix between the input and the output. For such a network, the DMT when back-flow is present, is the same as the DMT in the absence of back-flow.

Proof: If the network has back-flow, then the channel matrix would be

$$H = \begin{bmatrix}
    h_1 & 0 & \cdots & 0 \\
    h_2(g_{22}) & h_2 & 0 & 0 \\
    h_3(g_{31}) & h_3(g_{32}) & h_3 & \cdots \\
    \vdots & \vdots & \ddots & \ddots \\
    h_n(g_{n1}) & \cdots & \cdots & h_n
    \end{bmatrix},$$

If the network did not have back-flow, then the channel matrix would be

$$H_d = \begin{bmatrix}
    h_1 & 0 & \cdots & 0 \\
    0 & h_2 & 0 & 0 \\
    0 & 0 & h_3 & 0 \\
    \vdots & \vdots & \ddots & \ddots \\
    0 & \cdots & \cdots & h_n
    \end{bmatrix}.$$

$(I + \rho HH^\dagger)$ is a positive definite Hermitian matrix and by invoking Theorem 16.8.2 of [33], we have that the determinant is upper bounded by the product of row-norms:

$$\det(I + \rho HH^\dagger) \leq (1 + \rho|h_1|^2)(1 + \rho|h_2|^2 + \rho|g_{21}|^2|h_2|^2)\cdots$$
$$\cdots (1 + \rho|h_n|^2 + \rho|g_{n(n-1)}|^2|h_n|^2 + \cdots + \rho|g_{n1}|^2|h_n|^2)$$
$$= \prod_{i=1}^{n}(1 + \rho|h_i|^2)$$
$$= \det(I + \rho H_d H_d^\dagger)$$

(76)
The dot equivalence \((76)\) follows from equation \((8)\) in the proof of Lemma \(5.4\). Already we have from Lemma \(5.4\)

\[
\det(I + \rho_{HH}^\dagger) \geq \det(I + \rho_{Hd}H_d^\dagger)
\]

Therefore we get

\[
\det(I + \rho_{HH}^\dagger) = \det(I + \rho_{Hd}H_d^\dagger).
\]

Therefore, the DMT with back-flow is the same as without back-flow.

**Theorem 5.6:** When \(K = 3\), there exists a protocol achieving MISO bound for KPP networks.

**Proof:** By Prop \(1\) it is sufficient to establish a coloring of the edges. We will give the map \(\psi\) explicitly for the given network by specifying \(A_{ij}, \forall i, j\). Define

\[
a_i = \begin{cases} 
1, & n_i = 1 \text{ mod } 3 \\
0, & n_i \neq 1 \text{ mod } 3
\end{cases}
\]

\[(77)\]

Without loss of generality we assume that the paths are ordered such that for the first \(l\) paths, \(a_i = 1\) followed by the paths for which \(a_i = 0\). We give a protocol for various possibilities of \(l\).

- **Case 1:** \((l = 0, 1, \text{ or } 3)\)

  We will give a coloring scheme such that the corresponding protocol avoids back-flow, uses all paths equally, and achieves rate 1. By Lemma \(5.4\) this protocol will achieve the transmit diversity bound.

  We will specify the coloring scheme by giving the tuple of sets \(G_i = [G_{i0}, G_{i1}, G_{i2}]\) for all \(i\). \(G_i\) is defined exactly the same way how it is in the proof of Theorem \(5.3\).

  The set of colors used is \(C = \{c_1, c_2, c_3\}\). In the following, whenever we refer to color \(c_i\), assume \(c_0 = c_3\) and for \(i > 3\), \(c_i = c(i \text{ mod } 3)\).

  For \(l = 0\),

  \[
  G_i = \begin{cases} 
  \{c_1\}, \{c_1\}, \{c_1\}, & n_i = 0 \text{ mod } 3 \\
  \{c_1\}, \{c_1\}, \{c_1\}, & n_i = 2 \text{ mod } 3
  \end{cases}
  \]

  For \(l = 1\),

  \[
  G_1 = \{c_1\}, \{c_2\}, \{c_3\},
  G_2 = \begin{cases} 
  \{c_2\}, \{c_1\}, \{c_3\}, & n_2 = 0 \text{ mod } 3 \\
  \{c_2\}, \{c_1\}, \{c_3\}, & n_2 = 2 \text{ mod } 3
  \end{cases}
  \]

  \[
  G_3 = \begin{cases} 
  \{c_3\}, \{c_1\}, \{c_2\}, & n_3 = 0 \text{ mod } 3 \\
  \{c_3\}, \{c_2\}, \{c_1\}, & n_3 = 2 \text{ mod } 3
  \end{cases}
  \]

  For \(l = 3\),

  \[
  G_i = \{c_1\}, \{c_1\}, \{c_1\}, \{c_1\}, \{c_1\}, \{c_1\}, \{c_1\}
  \]

- **Case 2:** \((l = 2)\)

  For \(l = 2\), we shall now come up with a protocol such that only one node in the third path encounters back-flow. Then, we show that the DMT for this protocol is equal to the MISO bound. We describe the coloring scheme for the protocol as follows.

  \[
  G_1 = \{c_1\}, \{c_2\}, \{c_3\},
  G_2 = \{c_2\}, \{c_3\}, \{c_1\},
  G_3 = \{c_3\}, \{c_1\}, \{c_2\}
  \]

  After this assignment, we make the following modifications to \(A_{ij}\):

  \[
  A_{3(n_3)} = \{c_3\},
  A_{3(n_3-1)} = \{c_1\}, \text{ if } n_3 = 2 \text{ mod } 3
  \]

  One can check that this will lead to back-flow at only one node, say \(R_{ij}\), in the third path, whose position will depend on whether \(n_3 = 0 \text{ (mod } 3)\) or \(n_3 = 2 \text{ (mod } 3)\).
For the given protocol, there is no back-flow in paths \( P_1 \) and \( P_2 \), and therefore,

\[
H_i = g_i I_m \text{ for } i = 1, 2. \tag{78}
\]

For path \( P_3 \), the channel matrix is no longer diagonal because there is back-flow, rather the matrix is lower triangular. But according to Lemma 5.5, 
\[
\det (I_m + \rho H_3 H_3^\dagger) = \det (I_m + \rho H'_3 H'_3^\dagger), \text{ where } H'_3 = g_3 I_m.
\]

Therefore, the DMT of the proposed protocol is the same as the case when \( H_3 \) is a diagonal matrix, and hence, would achieve the MISO bound by Corollary 5.2.

**Theorem 5.7:** For \( K = 2 \) and \( n_i > 1 \), the maximum achievable rate for any orthogonal protocol is given by

\[
R_{\text{max}} \leq \left\{ \begin{array}{ll}
1, & n_1 + n_2 = 0 \mod 2 \\
\frac{2n_2 - 1}{2n_2}, & n_1 + n_2 = 1 \mod 2
\end{array} \right. \tag{79}
\]

where \( n_1 \leq n_2 \).

**Proof:** By Prop 1, any orthogonal protocol corresponds to a coloring of the edges, described by the map \( \psi \).

For \( K = 2 \), we consider the network as a cycle with edges \( l_1, l_2, \ldots, l_{n_1+n_2} \) with associated sets of colors \( D_1, D_2, \ldots, D_{n_1+n_2} \) respectively. Here,

\[
l_j = \begin{cases} 
e_{1j}, & j \leq n_1 \\ e_{2(n_2+n_i+1-j)}, & n_1 < j \leq n_1 + n_2 \end{cases} \\
D_j = \begin{cases} 
A_{1j}, & j \leq n_1 \\ A_{2(n_2+n_i+1-j)}, & n_1 < j \leq n_1 + n_2 \end{cases}
\]

with a single constraint,

\[
D_j \cap D_{(j+1) \mod (n_1+n_2)} = \phi \tag{80}
\]

Now suppose we have a coloring scheme with \( N \) colors. Then each color can be an element of the sets of colors corresponding to at most \( \left\lfloor \frac{n_1+n_2}{2} \right\rfloor \) edges. This is because, if there are more colors, then the half duplex constraint must be violated. So we have,

\[
\sum_{i=1}^{2} \sum_{j=1}^{n_i} |A_{ij}| \leq \left\lfloor \frac{n_1+n_2}{2} \right\rfloor N \\
\text{i.e., } n_1 m_1 + n_2 m_2 \leq \left\lfloor \frac{n_1+n_2}{2} \right\rfloor N \tag{81}
\]

Since \( n_i \geq 2 \) in each of the paths, the constraint (81) also implies that,

\[
2m_1 \leq N \tag{82} \\
2m_2 \leq N \tag{83}
\]

To find the maximum rate, we pose the maximization problem:

Maximise \( \left( \frac{m_1}{N} + \frac{m_2}{N} \right) \) subject to (81), (82), and (83).

This is easily solved to be,

\[
\frac{m_1}{N} = 0.5 \\
\frac{m_2}{N} = \frac{1}{n_2} \left( \left\lfloor \frac{n_1+n_2}{2} \right\rfloor - \frac{n_1}{2n_2} \right).
\]
So the maximum rate of the protocol is given by,

\[
R_{\text{max}} \leq \begin{cases} 
1, & n_1 + n_2 = 0 \mod 2 \\
\frac{2n_2 - 1}{2n_2}, & n_1 + n_2 = 1 \mod 2
\end{cases}
\]  

(84)

where \( n_1 \leq n_2 \).

**Construction 5.8:** This construction establishes an orthogonal protocol for \( K = 2 \) which achieves maximum rate. By Prop\[1\] it is sufficient to establish a coloring of the edges. We will give the map \( \psi \) explicitly for the given network by specifying \( A_{ij} \forall i, j \).

We consider the network as a cycle with edges \( l_1, l_2, ..., l_{n_1+n_2} \) with associated sets of colors \( D_1, D_2, ..., D_{n_1+n_2} \), as in the proof of Theorem 5.7.

For \( K = 2 \), respectively. Here,

\[
l_j = \begin{cases} 
e_{1j}, & j \leq n_1 \\
e_{2(n_2+n_1+1-j)}, & n_1 < j \leq n_1 + n_2
\end{cases}
\]

\[
D_j = \begin{cases} 
A_{1j}, & j \leq n_1 \\
A_{2(n_2+n_1+1-j)}, & n_1 < j \leq n_1 + n_2
\end{cases}
\]

with a single constraint, \( D_j \cap D_{(j+1) \mod (n_1+n_2)} = \phi \).

**Case 1:** \( (n_1 + n_2) = 0 \mod 2 \)

We will have \( C = \{c_1, c_2\} \). Define \( \psi \) to be such that

\[
D_j = \begin{cases} 
\{c_1\}, & j = 1, 3, ..., n_1 + n_2 - 1 \\
\{c_2\}, & j = 2, 4, ..., n_1 + n_2
\end{cases}
\]

**Case 2:** \( (n_1 + n_2) = 1 \mod 2 \)

We have the set of colors \( C = \{c_1, c_2, ..., c_N\} \), where \( N = 2n_2 \). We will add colors to \( D_j \) using the following algorithm.

1) Step 1: \( D_j \leftarrow \phi \forall j \in \{1, 2, ..., n_1+n_2\} \).
2) Step 2: Now we will add colors to each of the set \( D_j \) using the following algorithm. In the algorithm, whenever we refer to \( D_j \), with \( j > n_1 + n_2 \), we mean \( D_j \mod (n_1+n_2) \) and with \( j = 0 \), we mean \( D_j = D_{n_1+n_2} \).

\[
t \leftarrow 1;
\]

\[
\text{For } k = 1 \text{ to } n_2 \text{ in steps of 1:}
\]

\[
\{ 
\text{For } i = 1 \text{ to } n_1 + n_2 - 2 \text{ in steps of 2:}
\}
\]

\[
D_{i-k+1} \leftarrow D_{i-k+1} \cup \{c_t\};
\]

\[
t \leftarrow t + 1;
\]

\[
\}.
\]

\[
\text{For } k = 1 \text{ to } n_2 \text{ in steps of 1:}
\]

\[
\{ 
\text{For } i = 1 \text{ to } n_1 + n_2 - 2 \text{ in steps of 2:}
\}
\]

\[
D_{(n_1+n_2)-(k-1)-i} \leftarrow D_{(n_1+n_2)-(k-1)-i} \cup \{c_t\};
\]

\[
t \leftarrow t + 1;
\]

\[
\}.
\]

**Remark 13:** The orthogonal protocol shown in construction (5.8) achieves maximum rate given in Theorem 5.7. In case 1, it is clear that rate achieved is 1. In case 2, the number of colors used are \( 2n_2 \). In the first loop of the
construction, out of the $n_2$ colors used, $n_2 - 1$ colors are added to either $D_{n_1}$ or $D_{n_1 + 1}$ and all the $n_2$ colors are added to either $D_1$ or $D_{n_1 + n_2}$. In the second loop of the construction, out of the $n_2$ colors used, $n_2 - 1$ colors are added to either $D_1$ or $D_{n_1 + n_2}$ and all the $n_2$ colors are added to either $D_{n_1}$ or $D_{n_1 + 1}$. So the rate of the protocol would be $\frac{2n_2 - 1}{2n_2}$.

![Diagram](image)

Fig. 16. Protocol Illustration: $(n_1, n_2) = (3, 4)$ [contd...]

2) Geometric Interpretation: In this subsection, we interpret the protocol constructed by Construction (5.3) in a geometric manner. We assume $n_1 < n_2$ as in the previous section. As explained earlier, at any given time instant a maximum of $\lfloor \frac{n_1 + n_2}{2} \rfloor$ edges can be active. Now $n_1 + n_2$ is odd, and due to the half duplex constraint, only alternate edges can be active. This means that, if we consider the entire network at any time instant, every alternate edge will be colored except for one place, where there will be two consecutive edges that are not active. We will give the protocol by specifying at which two consecutive places the edges will not be active, at every time slot.

Consider the longer path and fix our pointer on the first edge $e_{21}$ of the longer path $P_2$. Start a cycle from this edge (consider the whole network as a cycle now), and activate alternate edges beginning from the next edge following the pointer in the clockwise direction, for the first time slot. This defines the set of edges, which are active for the first time slot. Hereafter, a set of edges which are simultaneously active at a time slot will be referred to as the activation set for that time slot. Now, move the pointer to the next edge $e_{22}$ of the longer path $P_2$ and repeat the same procedure. Now the activation set for the second time slot is defined. Continue the procedure, moving the pointer to all of the edges $e_{2i}$, $i = 1, 2, ..., n_2$. Thus the activation sets for the first $n_2$ time slots of the protocol is specified. For the next $n_2$ time slots of the protocol, the same procedure is followed, except that an anti-clockwise cycle is used instead of clockwise cycle.

Thus the cycle length of the protocol equals $2n_2$. By using this procedure, the edges in the shorter path $P_1$ always gets activated every alternate time instant. So, each edge in the shorter path gets $n_2$ colors. On the other hand, the edges on the longer path $P_2$ also get activated alternately except that they give up their transmission opportunity twice during the whole duration of $2n_2$ time slots. So each edge in the longer path $P_2$ gets $n_2 - 1$ colors.

This illustrated with an example, $(n_1, n_2) = (3, 4)$. In Fig. 16 activation sets for first $n_2$ time slots of the protocol
are defined. Here, we can observe that the pointer moves in the clockwise direction. In Fig. 17 activation sets for the next $n_2$ time slots of the protocol are defined. Pointer is moved in the clockwise direction in Fig. 16; in contrast, it is moved anti-clockwise in Fig. 17.

**Theorem 5.9:** For a 2-PP network, if the two path lengths are equal modulo 2, then the DMT achieved by the orthogonal protocol of Construction 5.8 is equal to the MISO bound, i.e., $d(r) = 2(1 - r)^+$.  

**Proof:** The proof follows from Lemma 5.1, Lemma 5.5 and Theorem 5.7.

**Theorem 5.10:** For a KPP network, there exists an orthogonal protocol achieving the MISO bound as long as $K \geq 3$ or $K = 2$ and $n_1 = n_2 \mod 2$.

**Proof:** Clear by combining Theorem 5.3, Theorem 5.6 and Theorem 5.9.

**D. KPP Networks with Direct Link**

**Theorem 5.11:** For KPP(D) networks with half duplex relays, single antenna nodes and with a direct link, the MISO bound on DMT is achievable whenever there is an orthogonal protocol avoiding back-flow that achieves the MISO bound in the absence of direct link.

**Proof:** By hypothesis, the given KPP network with half duplex relays and single antenna nodes, achieves optimal DMT in the absence of direct link. We know by Theorem 5.3 all KPP networks with $K > 3$ achieve optimal DMT.

Consider any KPP network with $K \geq 4$. We have also established that there exists a protocol, $P$, with cycle length $K$, achieving optimal DMT, in which the source sends one symbol each through every path during one cycle. Now assume that a direct link added between the source and the sink.

Define a protocol $P'$ as $P$ with a modification such that nodes preceding the sink do not forward the symbols, but buffer them. (Each node is assumed to have enough buffer length for this). The protocol $P'$ is run for $D$ time slots on the network till all the nodes preceding the sink have at least one symbol in their buffer. Now switch back to the protocol $P$. 

Fig. 17. Protocol Illustration: $(n_1, n_2) = (3,4)$ [..contd.]
Up to and including $D$ time slots, the sink receives $D$ symbols through the direct link. After $D$ time slots, the sink receives one symbol through the direct link, and another through a relayed path. By the definition of the protocol, each symbol transmitted by the source reaches the sink node through the direct link, and through exactly one relayed path. Note that each symbol arrives at the sink through the direct link, and a relayed path with a delay characteristic of the path. This is the same setting as in Theorem 3.3, and we invoke the results from there.

Let the total time slots elapsed be $M = mK + D$ for some positive integer $m$. Then the lower bound for DMT, $d(\cdot)$ is given by,

$$
\begin{align*}
    d(r) & \geq d_D(r) + d_C\left(\frac{M}{M - D}\right) \\
    \text{where, } d_D(r) &= (1 - r)^+ \\
    d_C(r) &= K(1 - r)^+
\end{align*}
$$

As $m$ tends to infinity, the DMT lower bound coincides with the cut-set bound, and thus the optimal DMT is achieved.

VI. HALF DUPLEX KPP(I) NETWORKS

In this section, we consider KPP networks in the presence of interference links between paths, i.e., KPP(I) networks. There is no direct link is KPP(I) networks as per the definition. We prove that the MISO bound is achievable even in KPP(I) networks.

The basic idea here is to consider the backbone KPP network for the given KPP(I) network. An orthogonal protocol is designed for the backbone network. This protocol is run on the KPP(I) network. It is obvious that there are now interference terms in the transfer matrix. However, if the transfer matrix can be written as a lower triangular matrix with the $K$ product coefficients on the diagonal, then we can use Theorem 3.3 and prove that the MISO bound is achievable.

A. Interference does not impair DMT

Next, we consider the case of causal interference, which we define first.

**Definition 11:** Consider a KPP(I) network with single antenna nodes. Let us operate the backbone KPP network using an orthogonal protocol which induces an AF protocol on the KPP(I) network. Let $H$ denote the channel matrix induced by the AF protocol in the KPP(I) network and $H_1$ denote the diagonal channel matrix induced by the orthogonal protocol in the backbone KPP network. If the protocol is such that

- $H$ is lower triangular,
- Diagonal entries of $H$ are same as that of $H_1$,

then the KPP(I) network is said to admit causal interference under that protocol.

Now we prove a Lemma which asserts that the DMT of a KPP(I) network with causal interference is same as that of the backbone KPP network under the same protocol.

**Lemma 6.1:** Consider a KPP(I) network with single antenna nodes, running on an AF protocol which admits causal interference. Let the induced channel matrix be $H$, and the diagonal part of $H$ be $H_d$. Then the DMT of $H$ is same as that of $H_d$.

**Proof:** The presence of causal interference creates entries in the strictly lower-triangular portion of the transfer matrix. Since the DMT of a lower triangular matrix is lower bounded by the DMT of the corresponding diagonal matrix, by Theorem 3.3 $d_H(r) \geq d_{H_d}(r)$.

Now shall prove that $d_H(r) \leq d_{H_d}(r)$, which will complete the proof of the lemma. Since an orthogonal protocol is employed, all the entries in a row of the matrix $H$ will have a common term $h_i$ corresponding to the fading
coefficient of the last link connecting to the sink. So with causal interference, then the channel matrix would be

\[
H = \begin{bmatrix}
  h_1(g_{11}) & 0 & \cdots & 0 \\
  h_2(g_{21}) & h_2(g_{22}) & 0 & 0 \\
  h_3(g_{31}) & h_3(g_{32}) & h_3(g_{33}) & \ddots \\
  \vdots & \vdots & \ddots & \ddots \\
  h_n(g_{n1}) & \cdots & \cdots & h_n(g_{nn}) \\
\end{bmatrix},
\]

where every \( g_{ij} \) is a polynomial function of Rayleigh fading coefficients. Since the interference is causal, if the network does not have interference links (i.e., in the backbone KPP network), the same protocol would yield a channel matrix,

\[
H_d = \begin{bmatrix}
  h_1(g_{11}) & 0 & \cdots & 0 \\
  0 & h_2(g_{22}) & 0 & 0 \\
  0 & 0 & h_3(g_{33}) & 0 \\
  \vdots & \vdots & \ddots & \ddots \\
  0 & \cdots & \cdots & h_n(g_{nn}) \\
\end{bmatrix},
\]

\[
(I + \rho HH^\dagger) \text{ is a positive definite Hermitian matrix and by invoking Theorem 16.8.2 of [33], we have that the determinant is upper bounded by the product of row-norms:}
\]

\[
\det(I + \rho HH^\dagger) < (1 + \rho|h_1|^2|g_{11}|^2)(1 + \rho|h_2|^2|g_{22}|^2 + \rho|g_{21}|^2|h_2|^2) \cdots \]
\[
(1 + \rho|h_n|^2|g_{nn}|^2 + \rho|g_{n(n-1)}|^2|h_n|^2 + \cdots + \rho|g_{n1}|^2|h_n|^2)
\]
\[
= \prod_{i=1}^{n}(1 + \rho|h_i|^2(|g_{ii}|^2 + |g_{i(i-1)}|^2 + \cdots + |g_{i1}|^2))
\]
\[
= \prod_{i=1}^{n}(1 + \rho|h_i|^2)
\]
\[
\det(I + \rho H_1 H_1^\dagger) \quad \text{(85)}
\]

The dot equivalence (85) follows from equation (8) in the proof of Lemma 3.1.

Now, \( \det(I + \rho HH^\dagger) \leq \det(I + \rho H_1 H_1^\dagger) \)
\[
\approx \det(I + \rho H_d H_d^\dagger)
\]
\[
\Rightarrow d_H(r) \leq d_{H_d}(r) \quad \text{(86)}
\]

Equation (86) follows from the fact that product of absolute value of Rayleigh random variables is equivalent to a single Rayleigh random variable in the scale of interest, as long as all the variables involved in the two matrices \( H_1 \) and \( H_d \) are independent.

\[ \blacksquare \]

B. Causal Interference

By Lemma 6.1 it is clear that the cut-set bound for a KPP(I) network can be attained if there is a protocol that yields a lower triangular matrix with \( K \) independent coefficients along its diagonal repeated periodically (except maybe the first \( D \) time instants). Specifically if the input-output relation can be written in the following form, then a DMT of \( d(r) = K(1-r)^+ \) is achievable.
Since the network satisfies \( g \) in \( K \) be lower triangular. This is ensured by from the first node on the actual path to the sink. That we are interested in. We want to get a lower triangular matrix with the which this data can reach the sink, since these contribute to the entries other than the diagonal entries in the matrix the diagonal. This means that in the representation given by Equation (88), a given column corresponding to the input \( x_{m_i} \) will look like: \( \text{Column}_i = [0 0 \ldots g_i \ast \ast \ast \ast]^T \), where \( \ast \) denotes some entry (zero or non-zero).
This clearly means that the matrix representation is lower triangular with \( g_i \) on the diagonal repeating periodically. i.e., it is of the form \( (87) \) and therefore, by Theorem \( 3.3 \) the upper bound on DMT is achievable: \( d(r) = K(1 - r) \). 

**Remark 14:** The conditions in this proposition depend on the actual delays experienced by the data travelling through various paths. However, the actual delays depend on the protocol used. To simplify the criterion in terms of characteristics of network topology, we define a class of protocols with “almost continuous activation” in the next section. This modified criterion can be computed by a simple examination of the network.

**C. Protocols with Almost Continuous Activation**

In this section, we define a class of protocols with “almost continuous activation” where in conditions in Proposition \( 2 \) can be reduced to conditions on the path lengths of the network.

**Definition 12:** An orthogonal protocol for a KPP network is said to have continuous activation at a relay node if the node transmits whatever it receives from the incoming edge in the last instant in the immediately next time instant.

**Definition 13:** An orthogonal protocol for a KPP network is said to have continuous activation at all relay nodes.

**Definition 14:** An orthogonal protocol for a KPP network is said to have almost continuous activation if the protocol has continuous activation at all relay nodes except possibly the first hop node on each parallel path.

Protocols with almost continuous activation will be used in the future sections to establish a sufficient condition for achievability of DMT upper bound. Protocols with almost continuous activation have the property that the data passes continuously through the edges of the backbone paths of the KPP network in successive instants after the first hop.

**Theorem 6.2:** For a KPP network without interference, there exists a protocol with almost continuous activation whenever \( K \geq 3 \).

**Proof:** Let us assume without loss of generality that the paths are ordered in ascending order of their sizes ordered modulo \( K \). Let us consider a given path \( P_i \). Let us fix the color on the first edge to be \( c_i \), i.e., \( A_{i1} = c_i \).

The next edge can be anything other than \( c_i \) in order to satisfy the half duplex constraint. Once the color on the next edge is fixed, the colors on the rest of the edges are known because the protocol must have almost continuous activation. Let the next edge have color \( c_m \). \( A_{i2} = c_m \) and we know that \( m \neq i \). So we must color the remaining edges consecutively: \( A_{ij} = c_{m+j-2}, j \geq 2 \).

We have \( K - 1 \) choices for \( m \) and therefore these will lead to \( K - 1 \) different colors for the last edge \( e_{i_{in}} \). These are all possible colors \( c_1, c_2, ..., c_K \) except the one color that will appear on the last edge if \( A_{i2} = c_i \). Let us try to determine the one color that can not appear on the last edge, because if it does, then the half duplex constraint will be violated.

Let \( n_i = a \mod K \). Then if \( A_{i2} = c_i \), then \( A_{in_i} = c_{(i+a-2) \mod K} \).

This means that if the starting color is \( c_i \), then there are \( K - 1 \) colors allowed except the one stated here: \( A_{in_i} \neq c_{(i+n_i-2) \mod K} \). Let \( S_i = C \setminus \{ c_{(i+n_i-2) \mod K} \} \). Therefore, \( S_i \) is the set of all allowed colors on the last edge in path \( P_i \). We represent this symbolically by \( c_i \leftarrow \leftrightarrow c_j, \forall c_j \in S_i \), where \( \leftarrow \leftrightarrow \) denotes the terminal edge compatibility relation.

Now we have a set of starting colors \( A = \{ c_i, i = 1, 2, ..., K \} \). The set of ending colors (i.e., the colors on the ending edges) should also be the set \( B = \{ c_i, i = 1, 2, ..., K \} \) since we want a rate one protocol. Now visualize a bipartite graph \( G \) between the sets \( A \) and \( B \). Where \( c_i \) in \( A \) is connected to \( c_j \) in \( B \) if \( c_j \in S_i \).

**Definition 15:** A complete matching on this bipartite graph \( G \) is a subgraph of \( G \) where every node in \( A \) is connected to exactly one node in \( B \) and these nodes in \( B \) are distinct.

Any complete matching on \( G \) specifies a protocol with almost continuous activation and vice versa, since a protocol with almost continuous activation is specified by just the starting and the ending colors. From the theory of bipartite matching [31], we have the following proposition:

**Proposition 3:** Let \( G \) be a bi-partite graph from set \( A \) to set \( B \). Let \( X \subset A \) be any subset of \( A \). A complete matching from \( A \) to \( B \) exists iff

\[
|\Gamma(X)| \geq |X|, \forall X \subset A \tag{89}
\]
where $\Gamma(X)$ denotes the set of all nodes that are adjacent to any node in $X$ on the graph $G$.

**Proposition 4:** The bipartite graph $G$ has a complete matching whenever $K \geq 3$

**Proof:** The bipartite graph $G$ has a complete matching iff $|\Gamma(X)| \geq |X|$, $\forall X \subset A$.

Since each element in $A$ is connected to $K-1$ nodes in the set $B$, we have that $|\Gamma(X)| \geq K-1$, $\forall X \subset A$, $X \neq \Phi$.

This means that the condition is satisfied automatically for the sets for which $0 < |X| \leq K-1$.

Now the only condition to check is when $|X| = K$. In this case the condition (89) reduces to

$$\bigcup_{i=1}^{K} S_i = C \quad (90)$$

This condition is violated $\iff$ all the $S_i$ are equal.

$\iff$ All the $c_{(i+n_i-2) \mod K}$ are equal.

$\iff$ All the $(i+n_i-2) \mod K$ are equal.

$\iff$ All the $(i+n_i) \mod K$ are equal to $M$ (say).

$\iff$ All the $n_i$ are distinct modulo $K$ and $(i+n_i) \mod K$ are equal to $M$, for $i = 1, 2$.

Now since, all the $n_i$ are distinct modulo $K$ and the paths are ordered in ascending order of their sizes ordered modulo $K$, we have $n_i = i-1 \mod K$.

$\Rightarrow$ All the $n_i$ are distinct modulo $K$ and $(1+0) \mod K = (2+1) \mod K$.

$\iff$ all the $n_i$ are distinct modulo $K$ and $0 = 2 \mod K$.

$\Rightarrow K \leq 2$.

Therefore there is no complete matching on the bipartite graph $\Rightarrow K \leq 2$. The contra-positive of this statement is that, $K > 2 \Rightarrow$ There is a complete matching on the bipartite graph.

Therefore a complete matching exists whenever $K \geq 3$. This proves the proposition.

Since a protocol with almost continuous activation exists whenever a complete matching on the corresponding bipartite graph exists, we have that protocols with almost continuous activation exist whenever $K \geq 3$. Hence the theorem.

Now, we can translate conditions on the delay in Proposition 2 into conditions on path lengths while using protocols with almost continuous activation. This is formalized in the following proposition:

**Proposition 5:** If the interference in a KPP network, running a protocol with almost continuous activation, has the following two properties, then the matrix connecting the output and a permuted version of the input will be lower triangular with $K$ independent coefficients along its diagonal repeated periodically (except maybe the first $D$ time instants). For each backbone path,

- **Condition 1:** The length of any other path from the first node should be no lesser than the delay on the backbone path from the first node to the sink.

- **Condition 2:** The unique shortest path from the first node on the given path to the last node on that path is through the backbone path from that node to the sink.

**Proof:** This follows directly from Proposition 2 and Remark 14.

1) **Optimal DMT for regular networks:** Now we show that the MISO bound is achievable for regular networks.

**Theorem 6.3:** The optimal DMT $d(r) = L(1-r)^+$ of (K,L) Regular networks is achievable.

**Proof:** Consider a (K,L) regular network. It can be treated as a KPP(I) network and therefore the back-bone KPP network can be run using an orthogonal protocol with almost continuous activation. Consider the following protocol with almost continuous activation. Let the colors be $c_1, c_2, \ldots, c_K$, and assume $c_0 = c_K$ and $c_\ell = c_\ell \mod K$.

$A_{ij} = c_{i+(j-1)}, i = 1, 2, \ldots, K, j = 1, 2, \ldots, L+1$.

With this protocol it can be seen that interference is causal, i.e., interference satisfies the conditions of Prop. 2.

Therefore, the optimal DMT of $L(1-r)^+$ is achievable for these networks.

**Corollary 6.4:** For a (2L) layered network, a lower triangular transfer matrix which contains the two product coefficients corresponding to the two parallel paths alternately on the diagonal can be obtained using the protocol with almost continuous activation.

**Corollary 6.5:** For the two-hop relay network without direct link, the optimal DMT is achieved.

**Proof:** The two-hop relay network without the direct link is a (K,1) regular network, where $K$ denotes the number of relays in the network. Thus Theorem 6.3 implies this corollary.

**Remark 15:** The result in Corollary 6.5 was also proved in an independent work [25]. The protocol used in this paper and in [25] are essentially the same as the SAF protocol [17], except that it is used in a network without direct link. However, the proof techniques used here and in [25] are very different.
2) Optimal DMT for KPP(I) networks: In this section, we prove that the MISO bound can be achieved on all KPP(I) networks, with \( K \geq 3 \).

In Prop. [2] we gave a sufficient condition to establish when a network can be used along with a given protocol in order to achieve the optimal DMT. Later in Prop. [5] we gave a sufficient condition on paths lengths in a network such that the network can be used along with a protocol with almost continuous activation to get the optimal DMT. Suppose the network does not meet the sufficient condition given in Prop. [5]. It is possible that the protocol can be modified to make the network meet the sufficient condition of Prop. [2]. We do so here by adding delays to internal nodes of the network such that, even though the path lengths do not satisfy the constraints, the delays do. By appropriately choosing a protocol and adding delays, we can make the network and the protocol jointly satisfy the conditions of Prop. [2]. This leads us to the following Theorem:

**Theorem 6.6:** Consider a KPP(I) network with \( K = 3 \). There exists a set of delays which when added appropriately to various nodes in the networks, and when used along with the protocol with almost continuous activation, satisfies the conditions of Prop. [2].

**Proof:** The proof is omitted here for brevity. The proof makes use of decomposing the given network into various layers, each of which can be balanced individually and the layers can put together to give a solution for the entire network.

**Theorem 6.7:** Consider a KPP(I) network with \( K \geq 3 \). The cut-set bound on the DMT \( d(r) = K(1 - r)^+ \) is achievable.

**Proof:** For \( K = 3 \), it follows from Theorem [6.6].

Now, we will consider the case when \( K > 3 \). Consider a 3 parallel path sub-network of the original network. By Theorem [6.6], we can get a matrix with these three product coefficients along the diagonal. There are now \( K^3 \) possible 3PP subnetworks. If each of these subnetworks is activated in succession, it would yield a lower triangular matrix with all the \( K \) product coefficient \( g_i \) repeated thrice \( K \) choose 3 times on the diagonal. By Theorem [3.3], the DMT of this matrix is better than that of the diagonal matrix alone. The diagonal matrix has a DMT equal to \( K(1 - r)^+ \). Therefore a DMT of \( d(r) \geq K(1 - r)^+ \) can be obtained. However, since \( d(r) \leq K(1 - r)^+ \) by cutset bound, we have \( d(r) = K(1 - r)^+ \).

VII. Layered Networks

**Lemma 7.1:** Let \( \mathcal{H} \subset \{ h_{11}, h_{12}, ..., h_{1M_1} \} \times \{ h_{21}, h_{22}, ..., h_{2M_2} \} \times ... \times \{ h_{K1}, h_{K2}, ..., h_{KM_K} \} \). Let \( |\mathcal{H}| = N \). Let each \( h_{ij} \) appear in \( N_i \) of the terms in \( \mathcal{H} \) irrespective of \( j \). Then \( N_i M_i = N \). Let \( N_{\text{max}} := \max_{i=1}^{N} N_i \) and \( M_{\text{min}} := \min_{i=1}^{K} M_i \).

Let \( h_{i} \), \( i = 1, 2, ..., N \) be the elements of \( \mathcal{H} \).

Let \( \psi : \mathcal{H} \rightarrow G \) be a map such that \( \psi((a_1, a_2, ..., a_K)) = \prod_{j=1}^{K} a_i \). Now let \( g_i = \psi(H_i) \), \( i = 1, 2, ..., N \). Then each \( g_i \) is of the form \( \prod_{l=1}^{K} h_{kl}(i,k) \), where \( l(i,k) \) is a map from \([N] \rightarrow [M_k] \) for a fixed \( k \in [K] \).

Let \( H \) be a \( N \times N \) diagonal matrix with the diagonal elements given by \( H_{ii} = g_i \).

The DMT of the parallel channel \( H \) is a linear DMT between a diversity of \( \frac{N}{N_{\text{max}}} \) and a multiplexing gain of \( N \):

\[
d(r) = \frac{(N - r)^+}{N_{\text{max}}}
\]

**Proof:**

Let us assume without loss of generality that \( N_1 \geq N_2 \geq ... N_K \).

\[
H = \text{diag} (H_{ii}). H_{ii} = \prod_{k=1}^{K} h_{kl}(i,k).
\]

Consider a variable transformation where \( \alpha_{kj} \) is defined such that \( \rho^{-\alpha_{kj}} = |h_{kj}|^2 \).
Now the DMT $d(r)$ is given by the following defining equation:

$$
\rho^{d(r)} = Pr\{\log \det(I + \rho H H^\dagger) \leq r \log \rho\}
$$

$$
= Pr\{\det(I + \rho H H^\dagger) \leq \rho^r\}
$$

$$
= Pr\{\prod_{i=1}^{N} (1 + \rho |h_{ii}|^2) \leq \rho^r\}
$$

$$
= Pr\{\prod_{i=1}^{N} (1 + \rho \sum_{k=1}^{K} |h_{kl(i,k)}|^2) \leq \rho^r\}
$$

$$
= Pr\{\prod_{i=1}^{N} (1 + \rho \sum_{k=1}^{K} \alpha_{kl(i,k)} - \rho^r\}
$$

$$
= Pr\{\prod_{i=1}^{N} (1 + \rho^{1 - \sum_{k=1}^{K} \alpha_{kl(i,k)}}) \leq \rho^r\}
$$

$$
= Pr\{\prod_{i=1}^{N} \rho^{1 - \sum_{k=1}^{K} \alpha_{kl(i,k)}} \leq \rho^r\}
$$

$$
= Pr\{\sum_{i=1}^{K} (1 - \sum_{k=1}^{K} \alpha_{kl(i,k)})^+ \leq r\} 
$$

$$
\leq Pr\{\sum_{i=1}^{K} (1 - \sum_{k=1}^{K} \alpha_{kl(i,k)}) \leq r\}
$$

$$
= Pr\{N - \sum_{k=1}^{K} M_k \sum_{j=1}^{N_k} \alpha_{kj} \leq r\}
$$

The last equality follows since each $|h_{ij}|^2$ appear in $N_i$ of the terms in $H$ irrespective of $j$ and so do the corresponding $\alpha_{ij}$. Let $d_1(r)$ be defined as the SNR exponent of the RHS in the last equation above, i.e.,

$$
Pr\{N - \sum_{k=1}^{K} M_k \sum_{j=1}^{N_k} \alpha_{kj} \leq r\} = \rho^{-d_1(r)}
$$

(94)

Now,

$$
\rho^{d(r)} \geq \rho^{d_1(r)}
$$

(95)

$$
= \inf_{\{N - \sum_{k=1}^{K} M_k \sum_{j=1}^{N_k} \alpha_{kj} \leq r, \ \alpha_{kj} \geq 0\}} \sum_{k=1}^{K} \sum_{j=1}^{M_k} \alpha_{kj}
$$

(96)

Define

$$
\alpha_k := \sum_{j=1}^{M_k} \alpha_{kj}
$$

(97)

$$
= \inf_{\{N - \sum_{k=1}^{K} N_k \alpha_k \leq r, \ \alpha_k \geq 0\}} \sum_{k=1}^{K} \alpha_k
$$

(98)

$$
= \inf_{\{\sum_{k=1}^{K} N_k \alpha_k \geq N - r, \ \alpha_k \geq 0\}} \sum_{k=1}^{K} \alpha_k
$$

(99)

Claim: The infimum of $\sum_{k=1}^{K} \alpha_k$ under the constraint $\{\sum_{k=1}^{K} N_k \alpha_k \geq N - r\}$, $\alpha_k \geq 0$ is attained by $\alpha_1 = \frac{N - r}{N_1}$, $\alpha_i = 0$, $\forall i = 2, ..., N$ and the value of the infimum is $\frac{N - r}{N_1}$.

Proof: The proof is simple and is skipped here.

This claim implies that $d(r) \geq d_1(r) = \frac{N - r}{N_1}$.

Now we will check that this lower bound is infact equal to the DMT of the channel. Let us consider an assignment of $\alpha_{kj}$ suggested by the claim above: $\alpha_{1j} = \frac{N - r}{N_1 M_1} = \frac{N - r}{N}$, $j = 1, 2, ..., M_1$.

From (92), we know that

$$
d(r) = \inf_{\{\sum_{i=1}^{N} (1 - \sum_{k=1}^{K} \alpha_{kl(i,k)})^+ \leq r, \ \alpha_{kj} \geq 0\}} \sum_{k=1}^{K} \sum_{j=1}^{M_k} \alpha_{kj}
$$

(100)
We have to verify that this assignment yields the infimum under the constraint stated here.

**Claim:** The infimum of \( \sum_{k=1}^{K} \sum_{j=1}^{M_k} \alpha_{kj} \) under the constraint \( \{ \sum_{i=1}^{N} (1 - \sum_{k=1}^{K} \alpha_{k(i,k)})^+ \leq r \; , \; \alpha_{kj} \geq 0 \} \) is attained by \( \alpha_{1j} = \frac{N-r}{N} \), \( j = 1, 2, \ldots, M_1 \), \( \alpha_{kj} = 0 \), \( \forall k > 1 \) and the value of the infimum is \( \frac{N-r}{N} \).

**Proof:** Since the objective function is convex, local minimum is the same as global minimum. It is sufficient to prove that the stated \( \{ \alpha_{kl} \} \) is a local minimum. To prove that, we show that the objective function does not decrease in a neighbourhood of the claimed optimal point. Let us assume that \( \alpha'_{kl} = \delta_{kl} \geq 0, i = 2, \ldots, K \).

Since \( \alpha_{1j} = \frac{N-r}{N} \leq 1 \), we have that all terms in the summation \( \sum_{i=1}^{N} (1 - \sum_{k=1}^{K} \alpha'_{k(i,k)})^+ \) are non-zero. By choosing \( \delta_{kl} \) small enough, we can ensure that all terms in the summation are non-zero.

\[
\sum_{i=1}^{N} (1 - \sum_{k=1}^{K} \alpha'_{kl(i,k)})^+ \leq r
\]

\[
\sum_{i=1}^{N} (1 - \sum_{k=1}^{K} \alpha'_{kl(i,k)}) \leq r
\]

\[
N - \sum_{k=1}^{K} \sum_{j=1}^{M_k} \alpha'_{kj} \leq r
\]

\[
\sum_{k=1}^{K} \sum_{j=1}^{M_k} \alpha'_{kj} \geq N - r
\]

The last equation follows since \( N_1 - N_k \geq 0, k \geq 2 \) and \( \delta_{kj} \geq 0 \).

Therefore \( \alpha_{1j} = \frac{N-r}{N} \), \( j = 1, 2, \ldots, M_1 \), \( \alpha_{kj} = 0 \), \( \forall k > 1 \) is a local minimum, and thereby a global minimum. This yields a DMT of

\[
d(r) = \frac{M_k}{N} \sum_{k=1}^{K} \sum_{j=1}^{M_k} \alpha_{kj} = \frac{M_1}{N} \sum_{j=1}^{M_1} \alpha_{1j}
\]

\[
d(r) = \frac{M_1}{N} \sum_{j=1}^{M_1} (N-r) = \frac{N-r}{N_1}
\]

Thus \( d(r) = d_1(r) = \frac{N-r}{N_1} \) is indeed the DMT of the channel described.

**Definition 16:** Given a set of paths \( P \) in a layered network, the bipartite graph corresponding to the path set \( P \) is defined as follows:

- Construct a bi-partite graph with vertices \( P \) on the left and vertices \( P \) again on the right.
- Connect an element \( P_i \) on the left to \( P_j \) on the right if the two paths are edge disjoint.

**Lemma 7.2:** Consider a set of paths \( A := \{ a_i, i = 1, 2, \ldots, N \} \) in a given layered network. Let the product of the fading coefficient on the \( i \)-th edge disjoint path \( a_i \) be \( g_i \). Construct the bi-partite graph corresponding to \( A \) according to Definition. If there exists a complete matching in this bi-partite graph, then these edges can be
activated in such a way that the DMT of this protocol is greater than or equal to the DMT of a parallel channel with fading coefficients $g_i, i = 1, 2, ..., N$ with the rate reduced by a factor of $N$, i.e., $d(r) \geq d_{H_d}(N r)$, where $H_d = \text{diag}(g_1, g_2, ..., g_N)$

Proof:

Suppose there is a complete matching $\pi$ on the graph constructed as above. The complete matching specifies for every edge disjoint path on the left $a_i$, a partner on the right $a_{\pi_i}$. The length of each path and therefore the delay is equal to $D := L + 1$.

Step - 1: Activate path $a_1$ along with path $a_{n_1}$ for a period $2T$, where $T > D$: treating these two paths as a $2 - PP$ Network, since these two paths are node disjoint. This network potentially has interference, but no direct link. Since this network is a subnetwork of a layered network, this 2-PP network has both the edges to be of the same length and causal interference and therefore rate-1 can be achieved on this network by Corollary 6.4. So the technique used in Section VI-C.1 can be used on this network to get a matrix, with zeros on the first $D$ rows. After deleting these $D$ rows, the matrix will be lower triangular due to causal interference and the diagonal in the matrix comprised of coefficients equal to $g_1$ and $g_{n_1}$, alternately for $T - D$ durations each. After this is done, the various nodes in the network store the data that have not yet been passed to the sink. This data will be used in the future when this path is activated again.

Step - 2: Repeat Step - 1 for all the paths $a_1, ..., a_N$. The net transfer matrix will comprise $ND$ zero rows, which effectively signifies a rate loss.

On removing these zero rows we get a transfer matrix, $H$. The DMT of the protocol is $d(r) = d_{H}(2NTr)$. By using Theorem 6.3, we get that $d_H(r) \geq d_{H_1}(r)$, where $H_1$ is the diagonal matrix corresponding to the matrix $H$. But $H_1$ contains $2T - D$ entries each of $g_i$, therefore this matrix DMT is given by $d_{H_1}(r) = d_{H_d}(\frac{1}{2T - D}r)$ where $H_d = \text{diag}(g_1, ..., g_N)$. If $d(r) = d_{H_1}(2NTr) \geq d_{H_1}(2NTr) = d_{H_d}(N \frac{2T}{2T - D}r)$. For $T$ tending to infinity, we get $d(r) \geq d_{H_d}(N r)$.

Remark 16: This activation can also be done in a cyclic way in order to reduce the delay of data transfer. In the modified scheme, the method used above can be repeated for $L$ cycles. Now, instead of letting $T$ going to infinity, we can tend $L$ to infinity to get the same DMT as above.

A sufficient condition that guarantees that a linear DMT between the maximum diversity and multiplexing gain on a general layered network is given in Lemma 7.3.

Lemma 7.3: For a general layered network, a linear diversity multiplexing tradeoff of $d(r) = d_{\max}(1 - r)^+$ between the maximum diversity gain $d_{\max}$ and the maximum multiplexing gain 1 is achievable whenever the bipartite graph corresponding to the set of edge disjoint paths $e_i, i = 1, 2, ..., d_{\max}$ from the source to the sink has a complete matching.

Proof:

By using Lemma 7.2, we will be able to get a DMT of $d(r) = d_{H_d}(d_{\max} r)$. But since the paths are edge disjoint, the fading coefficients are independent, we get $d_{H_d}(r) = (d_{\max} - r)^+$. Therefore, we get $d(r) = d_{\max}(1 - r)^+$.

Definition 17: A path from a source to sink in a layered network is said to be forward-directed if all the edges in the path are directed from one layer to the next layer towards the sink (i.e., no edge in the path goes from one layer to the previous layer and there is no edge which starts and ends in the same layer.)

Lemma 7.4: Let $P_1, ..., P_N$ be the set of all forward directed paths in a fully connected layered network. Then the bipartite graph of the path set $P$ has a complete matching.

Proof: We will prove this by producing an explicit complete matching on the bipartite graph. Let the layered network have $L$ layers. Let there be $R_i$ relays in the $i$-th layer. Let us fix an (arbitrary) ordering on the relays in each hop. Let the relays in the $j$-th hop be indexed $0, 1, ..., R_j - 1$. The number of paths is given to be equal to $N$.

A forward-directed path $P_i$ is specified completely if all the relays through which the path passes is denoted by the $L$ tuple $B_i = (b_{i1}, ..., b_{iL})$, where $b_{ij}$ denotes the index of the relay in the $j$-th hop through which path $P_i$ passes. Each $L$-tuple specifies a path from source to sink, since the layered network is fully connected. Now in this notation, two forward-directed paths $P_i$ and $P_j$ are node-disjoint if the tuples $B_i$ and $B_j$ are distinct in all the $L$ positions.

Consider a map $\alpha : P \rightarrow P$, where

$$\alpha(P_i) = \alpha(B_i) = \alpha(b_{i1}, b_{i2}, ..., b_{iL}) = (b_{i1} + 1 \mod R_1, b_{i2} + 1 \mod R_2, ..., b_{iL} \mod R_L).$$
It can be checked that this map is a bijection from $P$ to $P$. Since $R_i > 1 \forall i$, $B_i$ and $\alpha(B_i)$ are point-wise distinct, and thereby the paths $P_i$ and $\alpha(P_i)$ are node disjoint. Therefore the map $\alpha$ defines a complete matching on the graph.

*Theorem 7.5:* For a fully-connected layered network, a linear DMT between maximum diversity and maximum multiplexing gain of $1$ is achievable.

*Proof:* Consider a fully connected layered network with $L$ layers. Let there be $R_i$ relays in the $i$-th layer for $i = 0, 1, ..., L + 1$. Let $R_0 = R_{L+1} = 1$ since there is one source and one sink and $M_i := R_{i-1} R_i, i = 1, 2, ..., L + 1$ be the number of fading coefficients in the $i$-th hop. Let $h_{ij}, j = 1, 2, ..., M_i$ be the fading coefficients on the $i$-th hop for $i = 1, 2, ..., L + 1$. Let $N$ be the total number of forward-directed paths from source to sink, and $P_i, i \in [N]$ be the various forward-directed paths. Let $P$ denote the set of all these forward-directed paths. Then $|P| = N = \Pi_{i=1}^{L} R_i$. Let $g_{ij}$ be the product fading coefficient on path $P_i$.

Let $M_{\min} = \min_{i=1}^{L+1} M_i$. Then $d_{\max} = M_{\min}$ by Theorem 4.1.

By Lemma 7.4, the bipartite graph corresponding to $P$ has a complete matching. $P$ satisfies the criterion of Lemma 7.2, and therefore, we can obtain a DMT of $d(r) \geq d_{H_d}(N r)$. Now, we need to compute $d_{H_d}(r)$. To that effect, we make the following observations, which will enable us utilize Lemma 7.1.

A given path $P_i$ can be alternately represented as the set $G_i = \{h_{11(i,1)}, h_{21(i,2)}, ..., h_{(L+1)l(i,L+1)}\}$ of fading coefficients on that path. Consider the set of all $G_i$, i.e., $G = \{G_i, i \in [N]\}$.

Now let $g_{ik}, k \in [N]$ be the product fading coefficient on path $G$. Now clearly

$$G \subset \{h_{11}, h_{12}, ..., h_{1M_1}\} \times \{h_{21}, h_{22}, ..., h_{2M_2}\} \times ... \times \{h_{(L+1)1}, h_{(L+1)2}, ..., h_{(L+1)M_{L+1}}\}$$

Now each $h_{ij}$ appears in the same number $N_i$ of terms in $G$ irrespective of $j$, where $N_i = \frac{N}{M_i}$ and $N_{\max} = \max_{i=1}^{L+1} N_i$.

If $\psi$ is defined as in Lemma 7.1 then $g_i = \psi(G_i)$. Now we have satisfied all the conditions of Lemma 7.1 and therefore, $d_{H_d}(r) = \frac{N - r}{N_{\max}}$.

Now

$$d(r) \geq d_{H_d}(N r) = \frac{(N - N r)^+}{N_{\max}} = M_{\min}(1 - r)^+$$

$$\Rightarrow d(r) \geq d_{\max}(1 - r)^+$$

For fully connected layered networks with $L < 4$, the min-cut is either at the source side or at the sink side, and hence we have the following corollary:

*Corollary 7.6:* For a fully connected layered network with $L < 4$, the optimal DMT is achievable.

*Proof:* Consider a layered network with $L = 1$, i.e., there is only one layer. Let there be $n_1$ relay antennas in the relaying layer. The DMT upper bound is $n_1(1 - r)^+$ from the cut-set bound, which is achieved.

Let $L = 2$ and there be $n_1$ and $n_2$ relays in layers 1 and 2. Then the cutset bound on DMT is $\min\{n_1, n_2\} (1 - r)^+$, which is achieved.

Let $L = 3$ and there be $n_1, n_2, n_3$ relay antennas in the corresponding layer. It can be seen that $d_{\max} = \min\{n_1, n_2\}$ and that the DMT upper bound is $\min\{n_1, n_2\} (1 - r)^+$, which is indeed achieved.

**VIII. Networks with Multiple Antenna Nodes**

In this section we consider families of single source single sink networks with potentially all nodes having multiple antennas. We consider KPP networks with interference and Layered networks under both half duplex and full duplex constraint.
A. Achievable DMT for Certain Networks with Multiple antenna nodes

1) Full Duplex Layered Networks: We consider layered networks with multiple antennas at the source and the sink. Multiple antennas at relays can be handled by replacing the relay with multiple single-antenna relays in the same layer. We do not assume directed antennas and consider undirected edges. However this creates a back-flow, which induces a lower triangular matrix, that we handle using Theorem 3.3.

Definition 18: A single source single sink layered network with multiple antennas at the source and the sink is referred to as an \((n_0, n_1, \ldots, n_L, n_{L+1})\) network if the network has \(L\) layers, with the source having \(n_0\) antennas, the sink having \(n_{L+1}\) antennas, and the \(i\)-th layer of relays having \(n_i\) nodes with single antennas.

In [16], parallel AF and flip-and-forward (FF) protocols have been proposed for the \((n_0, n_1, \ldots, n_{L+1})\) network with full duplex operation and directed antennas, so that back-flow is avoided. The parallel AF protocol aims to achieve the full diversity for the network, whereas FF achieves the extreme points of full multiplexing gain and the full diversity gain. In [16], it has been proved that FF achieves a better DMT than AF. However, the DMT curves of both these protocols lie far away from the cut-set DMT bound. We propose a protocol with achievable DMT better than the existing protocols for a \((n_0, n_1, \ldots, n_{L+1})\) network under the full-duplex constraint.

In parallel AF and FF, the key idea is to partition the relay nodes in each layer into subsets of nodes called super nodes. A sequence of consecutive super nodes from source to sink form an AF path, and a set of AF paths is defined as a parallel partition in [16]. An independent parallel partition is defined as a parallel partition where any two different AF paths do not share common edges [16].

We propose a protocol which uses different partitioning depending upon the multiplexing gain \(r\) (we will refer to \(r\) as the rate by abuse of notation). The basic intuition is that, at lower rates, we can exploit the diversity of the network by creating more parallel AF paths. At higher rates, super nodes are to be chosen such that each AF path has enough degrees of freedom.

We propose a protocol which uses different partitioning depending upon the multiplexing gain \(r\) (we will refer to \(r\) as the rate by abuse of notation). The basic intuition is that, at lower rates, we can exploit the diversity of the network by creating more parallel AF paths. At higher rates, super nodes are to be chosen such that each AF path has enough degrees of freedom.

Let \(P_i\) be the number of partitions in layer \(i\). Let \(P\) denote a particular partitioning which is specified by the vector of \((P_0, P_1, P_2, \ldots, P_{L+1})\) and let \(\mathcal{P}\) denote all possible partitionings.

Given that the layer \(i\) has \(P_i\) partitions, the number of independent AF paths is

\[
N = \min_{\{i=0,1,2,\ldots,L\}} P_i P_{i+1}
\]

The protocol is as follows: Activate all the \(N\) parallel paths successively so that each path is activated for \(T\) time instants. During the activation of \(i\)th path, we will get a transfer matrix that is block lower-triangular with \(H_i\), the product matrix for the \(i\)-th path on the diagonal. Since the matrix is lower triangular, the DMT of this matrix is better than the DMT of \(H_i\). Let \(d_i(r)\) be the DMT of this matrix, which can be computed using the techniques.

\[5\] The idea of varying the protocol parameters depending on \(r\) was used in [9] for the NSDF protocol.
for computing the DMT of product Rayleigh matrices in [16]. Now the DMT of this induced channel can be given using Theorem 3.3 and the parallel channel DMT in Lemma 3.5:

\[ d_H(r) \geq \sup_{\{P \in \mathcal{P}\}} \inf_{\{(r_1, r_2, \cdots, r_N) : \sum_{i=1}^N r_i = r\}} \sum_{i=1}^N d_i(r_i) \]

The DMT of the protocol can be given as \( d(r) = d_H(Nr) \).

Since the optimization is over the set of all possible partitions, it might be difficult to compute the DMT in general. So we consider a restricted case when the source and sink are unpartitioned, and all the relay layers are partitioned into the same size, \( P \). Under this assumption, we have that \( 1 \leq P \leq n_{\text{min}} \). Let \( d_{(n_0, n_1, \ldots, n_L+1)}(r) \) denote the DMT of a product channel \((n_0, n_1, \ldots, n_L+1)\), which we can compute using the technique given in [16]. Let \( n_i^P := \left\lfloor \frac{n_i}{P} \right\rfloor \), \( i = 1, 2, \ldots, L \). When the relay layer \( i \) is partitioned into \( P_i \) partitions, each partition contains at-least \( n_i^P \) relays. If it contains more, the remaining relays are requested to be silent. This is done for simplicity of computing the DMT.

The strategy of Theorem 4.2 can be used to obtain a DMT of \( d_{\max}(1 - r)^+ \) for a layered network (see Corollary 4.3). By combining this strategy with the aforementioned strategy and choosing the one with the better DMT based on \( r \), we get a DMT of

\[ d(r) \geq \max\{d_{\max}(1 - r)^+, \sup_{\{P \in \mathcal{P}\}} P d_{(n_0, n_1^P, \ldots, n_L^P, n_L+1)}(r)\} \]  

(101)

The proposed protocol is essentially the same as [16] except for the following differences:

- We consider un-directed graph which gives rise to back-flow. We are able to handle back-flow by using Theorem 3.3.
- We consider partitions of arbitrary size. Evaluating the DMT with arbitrary sized partitions is made possible because of the parallel channel DMT in Lemma 3.5.
- The size of the partition is made variable with respect to the rate.
- We will show that this result can be extended to half-duplex networks under the assumption that all partitions are of equal size with \( P_i > 1 \).

It can be shown that the DMT of the RHS in (101) is strictly better than that of the FF protocol.

Example 1: Consider a \((2, 4, 2)\) layered network. The achievable DMT curve using the FF protocol, the proposed protocol and the cut-set bound are plotted in the Figure 18.

2) Half-Duplex Layered Networks: We consider multi-antenna Layered networks with the additional constraint of half-duplex relay nodes. We prove that the methods provided above for full duplex networks can be generalized for the half duplex network with bidirectional links.

Consider the partitioning method stated for full-duplex layered networks, with \( P_i = P, \forall i = 1, 2, \ldots, L \), i.e., the relaying layers are partitioned into equal number of partitions. Let the source and sink be un-partitioned. When the relay layer \( i \) is partitioned into \( P_i \) partitions, each partition contains at-least \( n_i^P \) relays. If it contains more, the remaining relays are requested to be silent, as in the full duplex case.

The following observations are in place: Once we replace the nodes corresponding to the same partition by a super-node, this virtual network forms a regular network. This is because each relaying layer has the same number of partitions and therefore the same number of super-nodes. Therefore, this network can be treated as a KPP networks with paths having equal lengths if \( P > 1 \). We use a protocol with continuous activation on this regular network. Since the paths are of equal length, the interference is causal making the induced channel matrix lower triangular. This has better DMT than the corresponding diagonal matrix by Theorem 3.3. This yields the same lower bound on DMT as in the full duplex case. Thus the DMT of the half duplex network with the protocol is better than using the network with a full duplex protocol and using the same partitioning. So we get:

\[ d(r) \geq \max\{d_{\max}(1 - r)^+, \sup_{\{P \in \{2, 3, \ldots, n_{\text{min}}\}\}} P d_{(n_0, n_1^P, \ldots, n_L^P, n_L+1)}(r)\} \]  

(102)

\(^6\)However, the fact that FF protocol does not depend on \( r \) can make practical implementation simpler.
Example 2: For the case of $(2, 4, 2)$ network with half-duplex constraint, the proposed protocol achieves the same DMT as the full duplex case of Example 1. However, the FF protocol used naively for a half-duplex system will entail multiplexing gain loss by a factor of $\frac{1}{2}$.

B. KPP(I) Networks

Consider KPP(I) networks with multiple antennas at the source and sink and potentially at all intermediate nodes.

1) Full duplex KPP(I) Networks: We consider full-duplex KPP(I) networks with multiple antenna nodes. Given an underlying path $P_i$, we activate all edges in the $P_i$ simultaneously. Let us call this process as activating the path $P_i$ and the fading matrix thus obtained as $G_i$. So $G_i = \prod_{j=1}^{K} H_{ij}$. Let the DMT corresponding to this product matrix be $d_i(r)$, which depends only on the number of the antennas on the path $P_i$ and can be computed according to formulae given in [16].

Since activating different paths can potentially have different DMTs, it is not optimal in general to use all paths equally.

When one is operating at a higher multiplexing gain, one might want to use a path with higher multiplexing gain more frequently in order to get greater average rate. While operating at a low rate, all the paths must be used in order to get maximum diversity. We consider a generic case where path $i$ is activated for a fraction $f_i$ of the duration. These fractions can be chosen depending on $r$ in order to maximize $d(r)$.

By so doing, we will get a parallel channel with repeated coefficients. The DMT of such a channel was evaluated in Lemma 3.8. The conversion however entails a loss factor, which is equal to the total number of time instants for which the channel was used. After making this rate correction, we get the following formula by modifying equation (55). So the achievable DMT is given by,

$$d(r) \geq \sup_{(f_1, f_2, \ldots, f_K)} \inf_{(r_1, r_2, \ldots, r_K): \sum_{i=1}^{K} f_i r_i = r} \sum_{i=1}^{K} d_i(r_i)$$

(103)

2) Half Duplex KPP(I) Networks: From Section VI, we know that under the half duplex constraint, there exists a protocol activating the $K$ paths equally for KPP(I) networks with $K \geq 3$ causing only causal interference. We can use the same protocol notwithstanding the fact that the relays contain multiple antennas. By doing so, we will get a transfer matrix which will be lower triangular. Also, the diagonal entries of this channel matrix would remain the same as though the relay nodes operate under full-duplex mode. By Theorem 3.3, this gives a lower bound on the DMT, and it is equal to DMT lower bound of the full duplex network in (103). Therefore even when there is half duplex constraint, we can achieve the same DMT given by the (103) with $f_i = \frac{1}{K}$ instead of the supremum.

If we want to achieve different fractions of activation for different parallel paths, then we can follow a different trick for $K \geq 4$. In this case, we can use the $K C_3$ 3-parallel path networks, but activate each 3-parallel-path network for a different fraction of time. Using this strategy, we can show that, for $K \geq 4$, all time fractions $f_i$ for the parallel path $P_i$ can be obtained as long as $(f_1, f_2, \ldots, f_K) \in \mathcal{F}$ where

$$\mathcal{F} := \{(f_1, f_2, \ldots, f_K) : \sum_{i=1}^{K} f_i = 1, \ 0 \leq f_i \leq \frac{1}{3}\}$$

For $K \geq 4$, this yields a DMT of

$$d(r) \geq \sup_{(f_1, f_2, \ldots, f_K) \in \mathcal{F}} \inf_{(r_1, r_2, \ldots, r_K): \sum_{i=1}^{K} f_i r_i = r} \sum_{i=1}^{K} d_i(r_i)$$

(104)

This is the same as the lower bound on the DMT for the full duplex case, except that we are constrained to have all activation fractions $f_i$ to be lesser than one-third.

IX. CODE DESIGN

A. Design of DMT achieving codes

Consider any network and protocol described above, and let us say the network is operated for $M$ slots. Let $L$ be the period of the protocol and let us assume $M = mL + D$ for simplicity. We will assume that after $D$ time
instants the KPP network comes to steady state, and we will neglect the first $D$ time instants. Even though there is a rate loss of $\frac{M}{M+D}$ associated with that, we can make this loss arbitrarily small by making $M$ large enough.

The induced channel is given by $Y = HX + W$ where $X,Y,W$ is a $M \times 1$ vector and $H$ is a $M \times M$ matrix. However, to design an optimal code for this channel, we need to use a space time code matrix $X$. In order to obtain an induced channel with $X$ being a $M \times T$ matrix, we do the following. Instead of transmitting a single symbol, each node transmits a row vector comprising of $T$ symbols during each activation. Then the induced channel matrix takes the form: $Y = HX + W$, with $X,Y,W$ being $M \times T$ matrices and $H$ the same $M \times M$ matrix as earlier.

So there are totally $MT$ symbols transmitted. In the matrix $X$, let us call the row vector of $T$ symbols in slot $i$ as $x_i$. To address a specific symbol: the $j$-th symbol in slot $i$, we use the notation $x_{ij}$. Let us use similar notation for the output: $y_{ij}$ denotes the $j$-th symbol received in the $i$-th time slot, and $y_i$ denotes the row vector of $T$ symbols received in the $i$-th time slot.

Now from [11], we know that if we use an approximately universal code for $X$, then it will achieve the optimal DMT of the channel matrix $H$ irrespective of the statistics of the channel. Explicit minimal delay approximately universal codes for the case when $T = M$ are given in [12], constructed based on appropriate cyclic division algebras [18]. These codes can be used here to achieve the optimal DMT of the induced channel matrix.

1) Short DMT Optimal Code Design: The code construction provided above affords a code length of $TM = M^2$. Also we need $M$ very large for the initial delay overhead to be minimal. This entails a very large block length, and indeed very high decoding complexity. Now a natural question is whether optimal DMT performance can be achieved with shorter block lengths. We answer this question for KPP networks by constructing DMT optimal codes that have $T = L$ and a block length of $L^2$, where $L$ is the period of the protocol used. We also provide a DMT optimal decoding strategy that also requires only decoding a $L \times L$ matrix at a time. This is a constant which does not depend on $M$ and therefore, even if we make $M$ large, the delay and decoding complexity are unaffected. This code construction can be easily extended to other networks considered in this paper as well.

After $D$ time instants, the KPP network attains steady state. Consider the first $L$ inputs after attaining steady state $x_{D+1}, x_{D+2}, \ldots, x_{D+L}$. If the channel matrix is restricted to these $L$ time slots alone, then channel matrix would be a lower triangular matrix with the $L$ independent coefficients $g_i, i = 1, 2, \ldots, K$ repeated periodically. The DMT of this matrix, after adjusting for rate, is $d_K(r) = K(1-r)^+$. So if we use a $L \times L$ DMT optimal matrix as the input (this can be done by setting $T = L$ and using a $L \times L$ approximately universal CDA based code for the input), we will be able to obtain a DMT of $d_K(r)$ for this subset of the data. This means that the probability of error for this vector comprising of $T$ input symbols will be of exponential order $P_e = \rho^{-d_K(r)}$ if an ML decoder is used to decode the $L \times L$ matrix.

Let us assume that the first $L$ symbols has been decoded independently. Let us now focus on the next $L$ received symbols $y_{D+L+1}, y_{D+L+2}, \ldots, y_{D+L+L}$. These symbols potentially depend on the previous block of $L$ symbols and it is optimal to decode all of these together. However we show that a Successive Interference Cancellation (SIC) based method is DMT optimal as well. After the first block of $L$ symbols are decoded, its effect will be subtracted out from the remaining symbols, and then the next block of $L$ symbols decoded independently. For the third block, the effect of the first two blocks each of length $L$ will be subtracted out and the third block decoded independently and so on.

Let us evaluate the probability of error when this SIC based method is used. Let us find the probability of error for $B$ blocks after the initial $D$ instants of silence. Let $E_i$ denote the event that there is an error in any of the first $i$ blocks, $F_i$ denote the event that there is an error in decoding the $i$-th block. Proceeding by induction on the $i$-th statement $P(E_i) = \rho^{-d_K(r)}$, we get

\[
\begin{align*}
P(F_i) &= P(F_i/E_{i-1})P(E_{i-1}) + P(F_i/E_{i-1})P(E_{i-1}) \\ &\leq P(E_{i-1}) + P(F_i/E_{i-1}) \\ &\leq \rho^{-d_K(r)} + \rho^{-d_K(r)} \\ &= \rho^{-d_K(r)}
\end{align*}
\]
$\Rightarrow P(E_i) = P\left(\bigcup_{j=1}^{i} F_j\right) \leq \sum_{j=1}^{i} P(F_j) \leq \sum_{j=1}^{i} \rho^{-d_K(r)} \equiv \rho^{-d_K(r)}$

Therefore, we have that the entire probability of error is of the exponential order of $\rho^{-d_K(r)}$ and the scheme achieves the optimal DMT of the $H$ matrix.

B. Universal Full-Diversity Codes

Consider a input output equation of the form $Y = HX + W$ where $X, Y, H, W$ are $M \times M$ matrices.

Usually the code design criterion given for a input matrix to have full diversity for rayleigh fading is that the difference of any two possible input matrices be full rank. In this section we show that such a criterion is sufficient to get full diversity on any channel matrix distribution. By full diversity here, we mean that the code will attain a diversity equal to $d(0)$ for the channel.

We quote the following theorem from the theory of approximately universal codes (Theorem 3.1 in [11]):

Theorem 9.1: [11] A sequence of codes of rate $R(\rho) := r \log \rho$ bits/symbol is approximately universal over the MIMO channel if and only if, for every pair of codewords,

$$\lambda_1^2 \lambda_2^2 \cdots \lambda_{n_{\text{min}}}^2 \geq \frac{1}{2^{R(\rho)+o(\log \rho)}} = \frac{1}{\rho^r 2^{o(\log \rho)}},$$

where $\lambda_1, \ldots, \lambda_{n_{\text{min}}}$ are the smallest $n_{\text{min}}$ singular values of the normalized (by $\frac{1}{\sqrt{\rho}}$) codeword difference matrix. A sequence of codes achieves the DMT of any channel matrix if and only if it is approximately universal.

Substituting $r = 0$ corresponding to a multiplexing gain of 0 in Theorem 9.1 we get that the criterion is

$$\lambda_1^2 \lambda_2^2 \cdots \lambda_{n_{\text{min}}}^2 \geq \frac{1}{2^{o(\log \rho)}},$$

In particular, if a code satisfies, for all pairs of codewords, the difference determinant is non-zero, i.e.,

$$\lambda_1^2 \lambda_2^2 \cdots \lambda_{n_{\text{min}}}^2 \geq L > 0,$$

then the code is approximately universal for a rate of $r = 0$, and therefore achieves, the $d(0)$ of any given channel matrix.

This criterion is the same as the criterion for full diversity on a rayleigh channel. This means that all codes with full diversity designed for the rayleigh fading MIMO channel are indeed full diversity for a MIMO channel with any fading distribution. Therefore we can use a full-diversity code designed for a rayleigh fading MIMO channel to get full-diversity for any KPP or Layered network, when used along with the corresponding protocol for these networks.

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