SPECTRAL STRUCTURE OF QUANTUM LINE WITH A DEFECT

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We study the spectral properties of one-dimensional quantum wire with a single defect. We reveal the existence of the non-trivial topological structures in the spectral space of the system, which are behind the exotic quantum phenomena that have lately been found in the system.

With the progress of the nanotechnology, it has become possible to manufacture quantum systems with desired specification.\textsuperscript{1} The theoretical study of simple quantum system with nontrivial properties is now a legitimate and relevant subject in wider context outside of mathematical physics. It has been lately pointed out that one of such simple model systems of the idealized quantum wire with a single defect\textsuperscript{2,3} possesses the properties such as strong vs weak coupling duality and spiral spectral anholonomy\textsuperscript{4,5}, the features usually associated with the non-Abelian gauge field theories. Despite its simplicity, the model is a very generic one in the sense that it represents the long wave-length limit of arbitrary one-dimensional potential with finite spatial support. As such, probing those phenomena in its precise working is worthwhile, if only for its mathematical feasibility. That is exactly what we attempt in this paper.

We consider a quantum particle in one-dimensional line with a single defect placed at $x = 0$. In formal language, the system is described by the Hamiltonian

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2},$$

defined on proper domains in the Hilbert space $\mathcal{H} = L^2(\mathbb{R} \setminus \{0\})$. We ask what the most general condition at $x = 0$ is. We define the two-component vectors,

$$\Phi = \begin{pmatrix} \varphi(0_+) \\ \varphi(0_-) \end{pmatrix}, \quad \Phi' = \begin{pmatrix} \varphi'(0_+) \\ -\varphi'(0_-) \end{pmatrix},$$

from the values and derivatives of a wave function $\varphi(x)$ at the left $x = 0_-$ and the right $x = 0_+$ of the missing point. The requirement of self-adjointness of the Hamiltonian operator is satisfied if probability current $j(x) = -i\hbar((\varphi^*)'\varphi - \varphi^*\varphi')/(2m)$ is continuous at $x = 0$. In terms of $\Phi$ and $\Phi'$, this requirement is expressed as

$$\Phi'^\dagger \Phi - \Phi^\dagger \Phi' = 0,$$

which is equivalent to $|\Phi - iL_0\Phi'| = |\Phi + iL_0\Phi'|$ with $L_0$ being an arbitrary constant in the unit of length. This means that, with a two-by-two unitary matrix $U \in U(2)$,
we have the relation,

\[(U - I)\Phi + iL_0(U + I)\Phi' = 0.\]  \hspace{1cm} (4)

This shows that the entire family \(\Omega\) of contact interactions admitted in quantum mechanics is given by the group \(U(2)\). In mathematical term, the domain in which the Hamiltonian \(H\) becomes self-adjoint is parametrized by \(U(2)\) — there is a one-to-one correspondence between a physically distinct contact interaction and a self-adjoint Hamiltonian. We use the notation \(H_U\) for the Hamiltonian with the contact interaction specified by \(U \in \Omega \simeq U(2)\).

We now consider following generalized parity transformations:

\[P_1 : \varphi(x) \rightarrow (P_1\varphi)(x) := \varphi(-x),\]  \hspace{1cm} (5)

\[P_2 : \varphi(x) \rightarrow (P_2\varphi)(x) := i(\Theta(-x) - \Theta(x))\varphi(-x).\]  \hspace{1cm} (6)

\[P_3 : \varphi(x) \rightarrow (P_3\varphi)(x) := |\Theta(x) - \Theta(-x)|\varphi(x).\]  \hspace{1cm} (7)

These transformations satisfy the anti-commutation relation

\[P_i P_j = \delta_{ij} + i\epsilon_{ijk} P_k.\]  \hspace{1cm} (8)

Since the effect of \(P_i\) on the boundary vectors \(\Phi\) and \(\Phi'\) are given by \(\Phi \xrightarrow{P_i} \sigma_i \Phi\), \(\Phi' \xrightarrow{P_i} \sigma_i \Phi'\), where \(\{\sigma_i\}\) are the Pauli matrices, the transformation \(P_i\) on an element \(H_U \in \Omega\) induces the unitary transformation

\[U \xrightarrow{P_i} \sigma_i U \sigma_i\]  \hspace{1cm} (9)

on an element \(U \in U(2)\). The crucial fact is that the transformation \(P_i\) turns one system belonging to \(\Omega\) into another one with same spectrum. In fact, with any \(P\) defined by \(P := \sum_{j=1}^3 c_j P_j\) with real \(c_j\) with constraint \(\sum_{j=1}^3 c_j^2 = 1\), one has a transformation

\[PH_P = H_{U_P}\]  \hspace{1cm} (10)

where \(U_P\) is given by

\[U_P := \sigma U \sigma\]  \hspace{1cm} (11)

with \(\sigma := \sum_{j=1}^3 c_j \sigma_j\). One sees, from (11), that the system described by the Hamiltonians \(H_U\) has a family of systems \(H_{U_P}\) which share the same spectrum with \(H_U\).

Let us suppose that the matrix \(U\) is diagonalized with appropriate \(V \in SU(2)\) as

\[U = V^{-1}DV.\]  \hspace{1cm} (12)

With the explicit representations

\[D = e^{i\xi} e^{i\sigma_3} = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix}, \quad \theta_{\pm} := \xi \pm \rho, \quad \text{and} \quad V = e^{i\frac{\pi}{4}\sigma_2} e^{i\frac{\pi}{4}\sigma_3},\]  \hspace{1cm} (13)

one can show easily that with \(\sigma_V := e^{-i\frac{\pi}{4}\sigma_3} e^{-i\frac{\pi}{4}\sigma_2} e^{i\frac{\pi}{4}\sigma_3} \sigma_3 = \sigma_V^{-1}\), one has

\[U = \sigma_V D \sigma_V\]  \hspace{1cm} (14)
Figure 1. (a) [left]: The parameter space \( \{ (\theta_+, \theta_-, \mu, \nu) \} \) is a product of the spectral torus \( T^2 \) specified by the angles \( (\theta_+, \theta_-) \) and the isospectral sphere \( S^2 \) specified by the angles \( (\mu, \nu) \). (b) [right]: In the top figure, the distinct spectral space \( \Sigma \) is the triangle surrounded by edges \( A_1 + A_2, B \) and \( B' \). Since a subtriangle is spectrally identical to its isospectral image \( B - C' - A_2, \Sigma \) can be represented by the square \( A_1 - C' - A_2 - C \) in the middle figure. When the two spectrally identical edges \( C \) and \( C' \) are stitched together with the right orientation, we obtain the Möbius strip with boundary \( A_1 - A_2 \) (the bottom figure).

which is of the type \([\text{1}]\). One can therefore conclude that [A] the spectrum of the system described by \( H_U \) is uniquely determined by the eigenvalue of \( U \), and [B] a point interaction characterized by \( U \) possesses the isospectral subfamily

\[ \Omega_{iso} := \{ H_{V^{-1}DV} | V \in SU(2) \} \],

which is homeomorphic to the 2-sphere specified by the polar angles \((\mu, \nu)\).

\[ \Omega_{iso} = \{ (\mu, \nu) | \mu \in [0, \pi], \nu \in [0, 2\pi) \} \simeq S^2. \] (16)

There is of course an obvious exception to this for the case of \( D \propto I \), in which case, \( \Omega_{iso} \) consists only of \( D \) itself.

To see the structure of the spectral space, i.e. the part of parameter space \( U(2) \) that determines the distinct spectrum of the system, it is convenient to make the
spectrum of the system discrete. Here, for simplicity, we consider the line \( x \in [-l, l] \) with Dirichlet boundary, \( \varphi(-l) = \varphi(l) = 0 \). One then has

\[
V \left( \frac{\varphi(0_+)}{\varphi(0_-)} \right) = \sin kl \Phi_0, \quad V \left( \frac{\varphi'(0_+)}{-\varphi'(0_-)} \right) = k \cos kl \Phi_0,
\]

with some common constant vector \( \Phi_0 \). From (4), we obtain

\[
1 + kL_0 \cot kl \cot \theta = 0, \quad 1 + kL_0 \cot kl \cot \theta = 0.
\]

This means that the spectrum of the system is effectively split into that of two separate systems of same structure, each characterized by the parameters \( \theta_+ \) and \( \theta_- \). So the spectra of the system is uniquely determined by two angular parameters \( \{\theta_+, \theta_-\} \).

The entire parameter space \( \Omega = \{\theta_+, \theta_-, \mu, \nu\} \) is a product of spectral space \( \Omega_{sp} = \{\theta_+, \theta_-\} \) which is homeomorphic to the 2-torus \( T^2 = S^1 \times S^1 \) and the isospectral space \( \Omega_{iso} = \{\mu, \nu\} \approx S^2 \) (See Fig. 1a). Note, however, that this parameter space provides a double covering for the family of point interactions \( \Omega \approx U(2) \) due to the arbitrariness in the interchange \( \theta_+ \leftrightarrow \theta_- \). Accordingly, two systems with interchanged values for \( \theta_+ \) and \( \theta_- \) are isospectral. So the space of distinct spectra \( \Sigma \) is the torus \( T^2 = \{(\theta_+, \theta_-)|\theta_\pm \in [0, 2\pi)\} \) subject to the identification \( (\theta_+, \theta_-) \equiv (\theta_-, \theta_+) \). Thus we have

\[
\Sigma := \{\text{Spec}(H_U)|U \in \Omega\} = T^2/\mathbb{Z}_2,
\]

which is homeomorphic to a Möbius strip with boundary (Fig. 1b).

To relate the non-trivial topological structure found here and the exotic quantum phenomena we have alluded to in the introduction, the readers are referred to other publications. Here we simply observe that the homotopy \( \pi_1(T^2) = \mathbb{Z} \times \mathbb{Z} \) is behind the double spiral ahology, and the isospectral family \( S^2 \) is the generalization of the duality found earlier.

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References

1. See, for example, J.H. Thywissen et. al., Euro. Phys. J. D7, 361 (1999).
2. P. Šeba, Czech. J. Phys. B36, 667 (1986).
3. S. Albeverio, F. Gesztesy, R. Høegh-Krohn and H. Holden, Solvable Models in Quantum Mechanics (Springer, Heidelberg, 1988).
4. T. Cheon, Phys. Lett. A248, 285 (1998).
5. T. Cheon and T. Shigehara, Phys. Rev. Lett. 82, 2539 (1999).
6. T. Fülöp and I. Tsutsui Phys. Lett. A264, 366 (2000).
7. I. Tsutsui, T. Fülöp and T. Cheon, J. Phys. Soc. Jpn, 69, 3473 (2000).
8. T. Cheon, T. Fülöp and I. Tsutsui, to be published in Ann. of Phys. (NY) (2001). LANL preprint quant-ph/0008123
9. I. Tsutsui, T. Fülöp and T. Cheon, to be published in Journ. Math. Phys. (2001). LANL preprint quant-ph/0105066