Electromagnetic interactions of three-body systems in the covariant spectator theory

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We derive a complete Feynman diagram expansion for the elastic form factor and the three-body photo and electrodisintegration of the three-body bound state using the covariant spectator theory. We show that the equations obtained are fully consistent with bound-state equations and the normalization condition previously derived for the covariant three-body bound state, and that the results conserve current.

I. INTRODUCTION

The covariant spectator formalism\(^1\) has been applied successfully to the description of \(NN\) scattering and the deuteron bound state\(^2\), the deuteron form factors\(^4\),\(^5\)\(^6\),\(^7\),\(^8\)\(^9\) electrodisintegration of the deuteron\(^2\),\(^6\) and the three-body system\(^8\), with emphasis on the numerical solution of the three-body bound state equations\(^8\),\(^9\). While most of the theory has been developed using diagrammatic methods, all of the two-body theory\(^10\) and the covariant normalization condition for three-body spectator wave functions\(^11\) have also been derived algebraically. Kvinikhidze and Blankleider have also studied the current for both the two and three-body spectator equations\(^12\),\(^13\)\(^14\) using the gauging of equations method.

In this paper we use the same diagrammatic method that is the basis of the results given in Refs.\(^1\),\(^2\),\(^3\),\(^4\),\(^5\),\(^6\),\(^7\),\(^8\),\(^9\) to obtain the complete set of gauge invariant Feynman diagrams needed to evaluate the three-body form factors and three-body breakup amplitudes for three-body photo or electrodisintegration. Our results agree with those recently derived algebraically by Adam and Van Orden\(^14\), and, although they have a different form, can also be shown to agree with the results of Kvinikhidze and Blankleider\(^13\). These results, together with the relativistic wave functions solutions found in Ref.\(^8\), are currently being used\(^15\) to study the three-body electrodisintegration process\(^3\)He\((e, e'pp)n\) recently measured at the Jefferson Laboratory\(^16\).

This paper is divided into five sections and a long Appendix. Following this short introduction, we first review the definitions and some properties of the covariant three-body spectator subamplitudes and spectator equations. In Sec. III, the basic results for both the elastic and inelastic three-body currents are derived from an analysis of the infinite series of Feynman diagrams that consistently defines both the scattering equation and the currents. The derivation is based on the gauging method of Kvinikhidze and Blankleider\(^13\) (referred to as KB throughout this paper), with a detailed diagrammatic proof of gauge invariance and a full discussion of double counting. Then, in Sec. IV, explicit algebraic forms of the final results for the currents are presented and in Sec. V conclusions, and a detailed comparison with KB are given. An algebraic proof that the elastic current conserves the charge of the three-body bound state is given in the Appendix.

II. WORKING WITH SPECTATOR THREE-BODY AMPLITUDES

A. Covariant Faddeev subamplitudes

In this paper \(\{i, j, k\}\) denote any permutation of particles \(1, 2, 3\), so that \(j \neq i, k \neq i\) and \(j \neq k\), and, for example, \(i\) can represent any of the three particles. Then, in the nonrelativistic Faddeev theory, the full three-body vertex function, which we denote \(|\Gamma\rangle\), is decomposed into three subamplitudes \(|\Gamma^i\rangle\) which denote that part of the vertex in which particle \(i\) is the spectator and the other two particles \((j\text{ and }k)\) were the last to interact. The full vertex is a sum of the three subvertices

\[|\Gamma\rangle = \sum_i |\Gamma^i\rangle = |\Gamma^1\rangle + |\Gamma^2\rangle + |\Gamma^3\rangle. \]  

(2.1)

In the covariant spectator theory\(^12\),\(^13\), the spectator is on-shell, and one of the two interacting particles is...
satisfy a different constraint, and care must be taken in Feynman diagrams (as we will discuss below).

Also on-shell. Hence there are now 6 possible Faddeev subvertex functions, denoted by \( |\Gamma_i^j\rangle \), where \( i \) is the (on-shell) spectator and \( j \) is also on-shell, so that only one of the three particles, \( k \) in this example, is off-shell. The diagrammatic representation of this amplitude is shown in Fig. 1(a). If the three particles are identical (as we assume in this paper) it can be shown \( \ref{fig:diagram} \), under the permutation operator \( P_{ij} \) that interchanges particle \( i \) and \( j \), that

\[
\begin{align*}
P_{ij} |\Gamma_i^j\rangle &= \zeta |\Gamma_i^j\rangle \\
P_{jk} |\Gamma_j^k\rangle &= \zeta |\Gamma_j^k\rangle \\
P_{ik} |\Gamma_k^j\rangle &= \zeta |\Gamma_k^j\rangle
\end{align*}
\]

where \( \zeta = \pm \) depending on whether or not the particles are bosons or fermions [a diagrammatic representation of the first two of these equations is given in Fig. 1(b)]. In practice, this means that the particles may be freely relabeled in Feynman diagrams (as we will discuss below).

The subvertex functions in the spectator theory each satisfy a different constraint, and care must be taken to match this constraint to the physics. For example, the energy of particle 3 in the subvertex function \( |\Gamma_2^1\rangle \) is \( k_{30} = M_B - E_1 - E_2 \), where throughout this paper \( E_i = \sqrt{m^2 + k_i^2} \) will always represent a physical energy, \( M_B \) is the bound state mass, and \( k_i \) are the three-momenta of the three particles in the three-body c.m. However, the energy of particle 3 in the subvertex functions \( |\Gamma_3^1\rangle \) and \( |\Gamma_1^3\rangle \) is \( E_3 \). The energy domains of each of the subamplitudes is illustrated diagrammatically in Fig. 2. Note that, if \( M_B < 3m \), there are three distinct domains that do not overlap.

We now describe briefly how to derive the bound state and scattering equations for three identical particles. To make the discussion simple and intuitive, we use a diagrammatic approach (similar to that used in the original Ref. \( \ref{fig:diagram} \)).

![FIG. 1: (Color online) Figure (a) shows the notation for the vertex subamplitude \( \Gamma_2^1 \). Lines marked by an \( \times \) are on mass-shell, and the spectator particle is the one that connects to the small dot inside the oval. Note that \( P_{12} \Gamma_2^1 = \zeta \Gamma_3^1 \), as shown in (b), and \( P_{32} \Gamma_2^1 = \Gamma_3^1 \), as shown in (c).](image1)

![FIG. 2: The three shaded areas are kinematically allowed regions of the energies of the three particles, subject to the constraint that their total is \( M_B < 3m \), and that two are on-shell. To read the figure, note that each point on the plane defines three unique energies. The energy of each particle is the perpendicular distance from one of the sides of the large central triangle (positive if “above” the side and negative if “below”, where the positive direction is shown by the arrows). The geometry of this Dalitz-like plot insures that the sum of the three energies at each point equals the altitude of the triangle (chosen to be \( M_B \)). The on-shell condition \( k_{30} > m \) requires that \( k_{30} \) lie beyond the line labeled with the number \( i \) at each end, and the shaded areas labeled by the number \( i \) (equal to 1, 2, or 3) are regions where particle \( i \) is off-shell, and \( j \) and \( k \) have energies \( E > m \). Note that these three areas do not overlap.](image2)

![FIG. 3: (Color online) Diagrammatic representation of the construction of the symmetrized three-body scattering subamplitude \( \Gamma_{23}^1 \) (represented by the oval) from unsymmetrized subamplitudes \( \Gamma_{ij}^{ij'} \) (represented by rounded rectangles). Only the symmetrization of the final state is shown. The 6 figures correspond to the 6 terms in Eq. \( \ref{eq:three-body} \) as follows: (a) \( \rightarrow 1 \), (b) \( \rightarrow \zeta P_{23} \), (c) \( \rightarrow \zeta P_{12} \), (d) \( \rightarrow P_{23} P_{12} \), (e) \( \rightarrow \zeta P_{12} \), and (d) \( \rightarrow P_{12} P_{23} \).](image3)

### B. The three-body scattering equation

For simplicity, this paper addresses only cases in which the three-body scattering is a succession of two-body scatterings. This means that three-body forces of relativistic origin, as defined in Ref. \( \ref{fig:diagram} \), will be neglected.

We begin with a review of some of the results of Ref. \( \ref{fig:diagram} \). The quantity of primary interest is the symmetrized three-body subamplitude \( \Gamma_{ij}^{ij'} \), which can be obtained from the unsymmetrized subamplitudes \( \Gamma_{ij}^{ij'} \). Here the superscripts \(obile \( i, i' \) denote the on-shell spectators in the
initial and final state, respectively, and the subscripts $j', j$ the on-shell interacting particle in the initial and final state, respectively. The symmetrized subamplitude can be obtained from the unsymmetrized subamplitude by action of the three-particle antisymmetrization projection operator

$$A_3 = \frac{1}{6} \left\{ 1 + \zeta \mathcal{P}_{12} + \zeta \mathcal{P}_{13} + \zeta \mathcal{P}_{23} + \mathcal{P}_{12} \mathcal{P}_{23} + \mathcal{P}_{23} \mathcal{P}_{12} \right\} \, (2.3)$$

normalized to $(A_3)^2 = A_3$. The fully symmetrized amplitude is

$$T_{jj'}^{jj'} = A_3 T_{jj'}^{jj'} A_3 \, (2.4)$$

Expanding out the final state gives, for example,

$$T_{jj'}^{11} = A_3 T_{jj'}^{11} A_3 + \zeta T_{jj'}^{21} A_3 + \zeta T_{jj'}^{31} A_3 + \zeta T_{jj'}^{12} A_3 + \zeta T_{jj'}^{13} A_3 + \zeta T_{jj'}^{22} A_3 + \zeta T_{jj'}^{23} A_3 + \zeta T_{jj'}^{32} A_3 + \zeta T_{jj'}^{33} A_3 \, (2.5)$$

corresponding to the six diagrams shown in Fig. 8. Each of these 6 diagrams generates 6 more terms when the initial state is symmetrized, for a total of $6 \times 6 = 36$ terms in all. Equation (2.6) holds for both the initial and final states, and because of this there is really only one distinct subamplitude (all others are related to it by a phase), which we choose by convention to be $T_{jj'}^{11}$. The symmetrization process introduces various weight factors into the power series expansion of the amplitude $T_{jj'}^{11}$, and hence into the equation for this amplitude. These were derived algebraically in Ref. 11; here they are obtained diagrammatically from a study of Figs. 4–6.

Figure 6 shows how each of the unsymmetrized subamplitudes appearing in Fig. 8 is expanded up to third order (in the two-body scattering). Considering the symmetrization of the final state only, there are a total of two first-order terms, four second-order terms, and 8 third-order terms. The contributions from the first-order terms are illustrated in Fig. 6; the end result is that the

symmetrized first-order scattering diagram enters with a weight factor of $\frac{1}{6}$. Applying the same argument to the other terms gives the series shown in Fig. 6 which is unchanged by the symmetrization of the initial state. This series results from the iteration of the scattering equation shown diagrammatically in Fig. 7, and hence this is the correct scattering equation. This is the result obtained previously in Ref. 11.

In algebraic form, this scattering equation is

$$T_{jj'}^{11} = \frac{1}{3} M_{jj'}^{11} - 2 \zeta M_{jj'}^{11} G_{12}^1 \mathcal{P}_{12} T_{jj'}^{11} \, (2.6)$$

deidentical to Eq. (3.55) of Ref. 11. Here $M_{jj'}^{11}$ is the symmetrized amplitude for the two-body scattering of particles 2 and 3 [with particle 1 a spectator], and $G_{12}^1$ is the propagator for particle 3 off-shell. The amplitude $M_{jj'}^{11}$ satisfies the integral equation

$$M_{jj'}^{11} = V_{jj'}^{11} - V_{jj'}^{11} G_{12}^1 M_{jj'}^{11} = V_{jj'}^{11} - M_{jj'}^{11} G_{12}^1 V_{jj'}^{11} \, (2.7)$$

FIG. 7: (Color online) Diagrammatic representation of the equation for the spectator three-body scattering subamplitude $T_{jj'}^{11}$. This equation is equivalent to the series shown in Fig. 6.
where $V_{22}^{1}$ is the kernel, or driving terms, of the $NN$ interaction. This equation is illustrated in Fig. 9. In diagrams drawn in this paper, the minus sign in the second term of (2.6) and (2.7) will be associated with the propagator $G_{1}^{2}$, so in the figures the propagator is $-G_{1}^{2}$. However, the factors $\frac{1}{2}$ and $2\zeta$ will be shown explicitly.

The three-body bound state produces a pole in the $s = P^{2}$ channel (with $P^{\mu}$ the total momentum four-vector),

$$T_{22}^{1} = -\frac{|\Gamma_{1}^{1} \rangle \langle \Gamma_{1}^{2} |}{M_{1}^{2} - P^{2}} + \mathcal{R}, \quad (2.8)$$

where the remainder term, $\mathcal{R}$, is regular at the pole, and the spin structure of the propagating bound state is included in the vertex function $|\Gamma_{2}^{1} \rangle$. Substituting (2.8) into (2.6), approaching the pole, and equating residues gives the three-body bound state equation

$$|\Gamma_{2}^{1} \rangle = -2\zeta M_{22} G_{2}^{1} P_{12} |\Gamma_{1}^{1} \rangle. \quad (2.9)$$

Note that the inhomogeneous term in Eq. (2.8) has no three-body bound state pole, and hence does not contribute to the bound state equation. The bound state equation is therefore diagrammatically identical to Fig. 7 without the inhomogeneous term.

We conclude this section with a discussion of how to define the spectator amplitudes when one of the spectators is off shell. The definitions needed in the subsequent discussion are shown in Fig. 9. Here the principle is to expose the two-body interaction which connects to the off-shell particle, because the two-body amplitude can be extended off-shell by pulling out the last two-body interaction (which is always defined with both particles off shell). Note that, when the spectator is off-shell [Fig. 9(b, c)], the equation must be iterated twice to get the desired result.

We now turn to the main subject of this paper, the diagrammatic derivation of the three-body current operator.

### III. SPECTATOR THREE-BODY CURRENTS

#### A. The problem of double counting

To expose one of the central issues in the construction of three-body currents, we begin by looking at what appears to be the lowest order result in the Bethe-Salpeter formalism, and show that this expected result leads to over counting.

The full BS three-body vertex function is the sum of three subamplitudes, as shown in Fig. 10(A). Guided by nonrelativistic theory, we might expect the impulse approximation to the current to be related to the square of the wave function, as illustrated in Fig. 10(B). However, if this proposed current is expanded using the wave equations, it leads to two terms of type Fig. 10(d), while direct examination of the ladder sum (for example) shows that there should be only one such term. Unless an interaction term of type (d) is explicitly subtracted from the “impulse” approximation, it will be double counted. The same problem does not arise in nonrelativistic theory because there the diagrams represent a sequence of operators which, in general, do not commute. The interaction of (a)×(b) gives a different contribution from that of (b)×(a), and both must be present.
this problem in the context of the Bethe-Salpeter theory is discussed in Ref. [17].

It turns out that the spectator theory, like nonrelativistic theory, also does not suffer from double counting. Furthermore, the topology of the terms shown in Fig. 10 is discussed in Ref. [17].

The construction of the current will be carried out in three steps. First the coupling to internal lines and vertices will be constructed. Then the wave equations will be used to rearrange the result into a more usable form. Finally, the extension of the result to inelastic processes requires coupling to the final state nucleons, and the correct way to do this will be developed last.

The coupling to internal lines and vertices is very nicely obtained using the gauging of equations method of Kvinikhidze and Blankleider [12]. The first step in this construction is to note that the coupling of the photon satisfies the distributive rule of differential calculus. Starting from the scattering equation (2.6), the photon coupling must satisfy the equation

\[ (T_{22}^{T_1})^{\mu} = \frac{1}{3} (M_{22}^{T_1})^{\mu} - 2 \zeta (M_{22}^{T_1})^{\mu} G_{2}^{T_1} P_{12} T_{22}^{T_1} \quad \text{(3.1)} \]

where \(X^{\mu}\) denotes the coupling of the photon to all internal lines and vertices in the series of Feynman diagrams that make up \(X\). The gauging of the spectator propagator generates three terms with the operators connecting each of these terms having different arguments. Denoting these operators by \(A\) and \(B\), the gauged \(G_{2}^{T_1}\) is

\[ A \left( G_{2}^{T_1} \right)^{\mu} B = A(p_2, p_3^{\pm}) G(p_3^{\pm}) j_2^{\mu}(p_3^{\pm}, p_3) G(p_3) (m + p_2) B(p_2, p_3) \]

\[ + A(p_2, p_3) (m + p_2) j_2^{\mu}(p_2, p_2) G(p_2) G(p_3) B(p_2, p_3) \]

\[ + A(p_2^{\pm}, p_3) G(p_2^{\pm}) j_2^{\mu}(p_2^{\pm}, p_2) G(p_3)(m + p_2) B(p_2, p_3) \quad \text{(3.2)} \]

In each term the particle with momentum \(p_2\) is on shell, \(p_2^{\pm} = m^2\), and \(p_i^{\pm} = p_i \pm q\), and

\[ G(p) = (m - \not{p})^{-1} . \quad \text{(3.3)} \]

Each term is a direct product of Dirac operators on the space of particle 2 and 3 (with the space on which the operators act implied by the momentum labels, so, for example, \(p_2\) operates on the space of particle 2). This equation is illustrated diagrammatically in Fig. 11

Using Fig. 11, Eq. (3.1) is represented in Fig. 12. This is an equation for \((T_{22}^{T_1})^{\mu}\), and can be solved by iteration, using the series representation for the three-body scattering amplitude given in Fig. 4. The solution is shown in Fig. 13. To obtain this solution diagrammatically is straightforward, if not familiar to many. Fig. 11 demonstrates diagrammatically how Fig. 13(c) is obtained by iterating the inhomogenous term in Fig. 12(c).

In subsequent applications the discussion will be limited to those cases in which the initial state is bound. These diagrams are extracted from the general result shown in Fig. 13 by approaching the bound state pole in the initial scattering and retaining the residue. Diagrams (a) and (a1) do not have such a pole, and therefore do not contribute. The result for the bound state internal current is shown in Fig. 15.

The parts of this figure involving rescattering in the final state are identical to Fig. 4 of KB [12]. To show this, first compare our two-body scattering Eq. (2.7) with the KB two-body scattering equation (Eq. (26) of Ref. [12]). Note that \(G_{2}^{T_1} = -\delta_{2d_3}\), so that the equations are identical if \(2M_{22}^{1} = \lambda_1\) and \(2V_{22}^{1} = \epsilon_1\), corresponding to a different normalization of the two-body amplitudes. Next, note...
that our three-body scattering Eq. (2.6) is identical to the similar Eq. (15) of KB if we choose $\zeta = -1$ and set $6T_{22}^{11} = X t_1$, which corresponds to a different normalization of three-body scattering. With these replacements, our diagrams (b), (c), and (d) are identical to Fig. [4] in KB [recalling that second figure in the KB Fig. [4] is the same as the sum of our (d) and (d)]. Since the KB result shown in Fig. [4] applies only to transitions to connected final states, diagrams like our Fig. (b) will be discussed after we have finished our discussion of the breakup process.

Note that Fig. (d) includes a contribution in which the spectator (particle 1) is off-shell. While recognizing that further reductions are possible, KB elect to leave their answer in this form. In Step II of our derivation, we use the wave equations (2.6) and (2.8) to replace this amplitude by an equivalent one in which the spectator is on-shell. This replacement does not change the total result, but is still very useful for numerical applications, and leads to a nice demonstration of how the spectator equations avoid the double counting problem in a natural way.

C. Step II: Removal of off-shell spectator contributions

The diagram (d) of Fig. (c) can be transformed as shown in Fig. (c). The first step is to replace the initial bound state amplitude (which has its spectator off-shell) using the definition shown in Fig. (c). Then we recognize that, if one of the two NN scattering amplitudes introduced by this substitution is identified with the final state, and one with the initial state (as outlined by
FIG. 16: (Color online) Diagrammatic representation of the transformation of diagram (d₁) in Fig. 15. The first step uses Fig. 9(c) to replace the initial three body amplitude [denoted by (a) in the figure]. Then the two parts labeled (b) and (c) are isolated, and Fig. 9(a) is used to replace (b) and Fig. 9(b) to replace (c). The final result is the two terms shown in the lower panel. Note that the diagram (b₂)(c) cancels an identical diagram that comes from the first iteration in the final state scattering amplitude in (b₁)(c) [c.f. diagram (a₁) of Fig. 9].

FIG. 17: (Color online) Diagrammatic representation of contributions from internal photon couplings to the three-body breakup process. Diagram (a) is constructed from the core diagrams shown in Fig. 18 (simply add the two external half ellipses to each diagram in Fig. 18 to complete the diagram). Diagram (b) is the same diagram (b₂)(c) that appeared in Fig. 16 when the off-shell spectator was replaced by an on-shell spectator. It cancels the first iteration of the final state interaction in Fig. 19(e₂) so that the sum of all of these contributions contains no couplings to external nucleons.

FIG. 18: (Color online) Diagrammatic representation of the gauged (M₂₂)μ where, in our application, the on-shell particle is #2 and the off-shell particle #3. Figures (b₁) and (b₂) arise from the gauging of the propagator (G₁)μ, and (c)-(e) include contributions from the two-body interaction current, represented by the photon coupling inside a shaded rectangle. There are no three-body forces, and hence no three-body interaction currents.

FIG. 19: (Color online) This figure shows “core” diagrams common to all interactions. The half circles in the initial and final state can be either the bound state, or part of a three-body scattering amplitude. Substituting into Fig. 18 gives the complete internal current. (A) the core equals the sum of (b) and (d) photon coupling to off shell particle, either with or without exchange of the 2 on-shell particles, (c) and (e) the coupling of the photon to the on-shell interacting particle (arranged so that the spectator is always on-shell), and (f) two-body interaction current diagrams.

FIG. 20: (Color online) The three-body bound state form factor is constructed only from the core diagrams of Fig. 19.
$+M_{22} G_2^I (V_{22}^I) \gamma^a G_2^I M_{22} + M_{22}^I (G_2^I) \gamma^a M_{22}, \quad (3.4)$

illustrated in Fig. 17. This replacement exposes the two-body interaction current, $(V_{22}^I)\gamma^a$ which includes the photon coupling to all exchanged mesons and meson-nucleon vertices (when present). Using this expansion, and the wave equations, gives the diagrams shown in Fig. 18 with the “core” diagrams defined in Fig. 19. The core diagrams also define the three-body form factor. To extract this form factor from the result shown in Fig. 18, go to the bound-state pole in the final scattering amplitude, and note that the diagrams (b), (c) and (d) will not contribute. The result is summarized in Fig. 20.

It remains now to (i) demonstrate that the form factor conserves current, (ii) find the additional diagrams (describing eolys coupling to external nucleons) that must be added to Fig. 18 to compete the description of the three-body breakup process, and (iii) show that the three-body breakup also conserves current.

It turns out that the best way to proceed with these remaining tasks is to first derive the WT identity for the core diagrams, Fig. 19.

**D. Step III: The WT identity for the core diagrams**

Our study of current conservation is based on a generalization of the arguments of Gross and Riska [18]. There it was shown that the current will be conserved if (i) it is constructed from elementary nucleon and interaction currents that satisfy the appropriate WT identities, and (ii) contributions from all possible couplings of the photon to nucleons and interactions are included in a consistent manner. The core diagrams derived in the previous subsection provide a consistent scheme for coupling photons to all two-body interactions and internal nucleons, and hence provide the solution to condition (ii) when there are no external free nucleons (true for the form factor, but not for the three-body breakup). It remains now to show explicitly how the WT identities for the elementary nucleon and interaction currents insure that this is so.

The nucleon current is constructed to satisfy the WT identity

$$q_\mu J_\mu (p', p) = \left[ G^{-1}(p) - G^{-1}(p') \right], \quad (3.5)$$

where the particle charge is excluded from the definition of the current, and conservation of four-momentum at every vertex implies that $q = p' - p$. From this it follows that the divergence of Eq. (3.4) is

$$q_\mu \mathcal{A} (G_2^I) \gamma^a \mathcal{B} =$$

$$\mathcal{A}(p_2, p_2^+) (m + p_2) G(p_2^+) \mathcal{B}(p_2, p_3)$$

$$- \mathcal{A}(p_2, p_2^+) (m + p_2) G(p_3) \mathcal{B}(p_2, p_3)$$

$$+ \mathcal{A}(p_2, p_3) (m + p_2) G(p_3) \mathcal{B}(p_2, p_3)$$

$$- \mathcal{A}(p_2, p_3) (m + p_2) G(p_3) \mathcal{B}(p_2, p_3). \quad (3.6)$$

This relation is illustrated diagrammatically in Fig. 21. Note that Fig. 21(d) is a shorthand notation for the 1st and 3rd terms on the r.h.s. of Eq. (3.6) and Fig. 21(e) for the 2nd and 4th terms. Similarly, the WT identity satisfied by the interaction current (for further discussion, see Eq. (3.3) of [18]) is

$$q_\mu \left( V_{22}^I \right) = \left[ V_{22}^I (p_2, p_3) - V_{22}^I (p_2, p_3) \right]$$

$$+ \left[ V_{22}^I (p_2^+, p_3) - V_{22}^I (p_2, p_3^+) \right], \quad (3.7)$$

with $p_i^\pm$ defined above. In this equation, the photon momentum is inserted wherever there is a $p_i^\pm$, as in Eq. (3.6). Using the notation of Fig. 21, this equation is illustrated in Fig. 22.

The derivation of the WT identity for the core diagrams depends on the observation that the three-body bound state and scattering equations can be used to express the amplitudes in terms of the two-body kernel $V_{22}$, as illustrated in Fig. 23. Using this relation, and the WT identities (3.6) and (3.7), the WT identity for the core diagrams can be found. The steps are outlined in Fig. 24. Cancellations occur when the identities of Fig. 24 are used to reexpress the diagrams (24)(b), (24)(e). In detail, (24)(a) cancels (24)(b) and (24)(c), (24)(b) cancels (24)(f) and
The top line shows the wave equation with the amplitude (A) equal to the inhomogeneous part (a) and the rescattering part (b). In the second line, $M_{22}^{22}$ in (b) is replaced by its scattering equation, Fig 8 and in the third line one of these terms, (c), is replaced by the three-body scattering equation. Because of the cancellation of diagrams (a) and (f), the final result (last line) is the sum of only three diagrams, (g), (e) and (d). The diagrams (a), (f), and (g) do not contribute to bound states, leaving only diagrams (e) and (d).

This result implies that the bound-state form factor conserves current, because in this case neither of the diagrams $24(m)$ or $24(n)$ are present, and the WT identity gives zero. Discussion of the three-body breakup process requires additional diagrams, which will be discussed now.

E. Step IV: Photon coupling to external nucleons

Using the results of Fig. 24 we obtain the WT identity for the internal photon couplings to the three-body breakup process [Fig. 13]. The result is shown in Fig. 25.

The coupling to external nucleons will produce the terms needed to cancel the diagrams in Fig. 25(a), 25(b), 25(c), and 25(e). Since 25(a) results only from Fig. 18(b), 18(b) will be removed from the final three-body breakup current. [Removal of this diagram means that the core contributions will contain a diagram like Fig. 18(b) that can be interpreted as an interaction with a free final state particle, and care must be taken not to overlook this term. We will discuss this further in the conclusions.] The final result for three-body breakup, including photon coupling to external nucleons, is shown in Fig. 26.

The three-body breakup diagrams (a) and (a2) will be referred to as the relativistic impulse approximation (RIA), diagrams (c) and (d) as interaction currents (denoted by I, but not to be confused with isobar currents which, if present, are included in the interaction currents), and the core contribution (b), which includes final state interactions driven by both the RIA and the I (and
FIG. 26: (Color online) Diagrams that describe the three-body breakup process. Diagrams (a) are the RIA, (b) the FSI, and (c) and (d) are interaction currents (I).

\[
q^\mu \begin{pmatrix}
\text{(a)} \\
\text{(b)} \\
\text{(c)} \\
\text{(d)}
\end{pmatrix}
\equiv \begin{pmatrix}
\text{(a)}_1 \\
\text{(a)}_2 \\
\text{(b)} \\
\text{(c)} \\
\text{(d)}
\end{pmatrix} + 2\zeta \begin{pmatrix}
\text{(c)} \\
\text{(d)}
\end{pmatrix} = 0
\]

FIG. 27: (Color online) Proof that the breakup diagrams conserve current. Diagram (a) results from application of the WT identity on the RIA terms (with the final state on shell), (b) and (c) are from Figs. 24 and (d) and (e) result from the the application of Fig. 24 on the I diagrams 26(c) and 26(d), respectively. After the cancellations shown, the application of Fig. 24 to diagram (a) cancels (d) and (e), giving zero.

denoted FSI). A proof that this set of diagrams conserves current is given in Fig. 27.

We have completed our derivation of the current, and return now to the discussion of double counting first introduced in the last section.

F. Conclusion: Removal of double counting in the Spectator theory

We now demonstrate that the problem of double counting referred to in subsection A above is solved by the current operator given in Fig. 26.

Guided by Fig. 10, we examine the “exchange” terms \( (e_1) \) and \( (e_2) \) of Fig. 19 that contribute to the core process. These diagrams are reproduced in Figs. 28(a) and (b) for the case when the initial state is bound and the final state is three-body breakup. Using the bound state wave equation with Fig. 28(a) and the scattering equation with Fig. 28(b) gives the three contributions \((c), (d), \text{ and (e)}\). Figures (d) and (e) are untangled in the last line of the figure, so that they may be more easily compared with diagrams 10(d) of the Bethe Salpeter theory. Note that in the spectator theory, both diagrams (d) and (e) must occur, because they describe different processes, with the spectator on shell either “before” or “after” the interaction (note that here “before” or “after” refer to a topological ordering and not a time ordering). Our insistence that the spectator always be on shell has eliminated the double counting problem.

Finally, look at diagram 28(c), which arises from final state interactions. In Fig. 29 this diagram is rearranged to look like an RIA contribution with the spectator off shell. This would lead to double counting if we had included such processes in the RIA, but these contributions are explicitly excluded from the RIA contributions \((a_1) \) and \( (a_2) \) shown in Fig. 26.

We now record, for future use, the algebraic expressions corresponding to our major results.
IV. ALGEBRAIC EXPRESSIONS FOR THE WAVE FUNCTIONS AND CURRENTS

In this section we record the algebraic form of the three-body vertex functions, wave functions, form factors, and three-body breakup. The Appendix shows in detail how the covariant three-body normalization condition leads to the conservation of charge.

A. The three-body vertex function and wave function

The three-body vertex function defined in Ref. [9] [Eq. (3.14)] is denoted

$$\Gamma_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) \equiv \langle k_1 \lambda_1 | k_2 \lambda_2, k_3 \alpha \rangle |\Gamma_2^1\rangle,$$  (4.1)

where $\langle k_1 \lambda_1 | k_2 \lambda_2, k_3 \alpha \rangle |\Gamma_2^1\rangle$ is the three-body vertex function describing the coupling of the $^3$He nucleus to the three-nucleon system, with the first pair of arguments $[k_1, \lambda_1]$ the four-momentum and helicity of the on-shell spectator, the second pair $[k_2, \lambda_2]$ the four-momentum and helicity of the on-shell particle in the interacting pair, and the last pair $[k_3, \alpha]$ the four-momenta and Dirac index of the off-shell particle in the interacting pair. Different notations will sometimes be used for these momenta; it is their location in the argument list that identifies them as spectator, on-shell interacting particle, or off-shell particle. In this notation the momenta and spin indices of all three particles are defined in the same system, with

$$P = k_1 + k_2 + k_3,$$  (4.2)

the total four momentum of the bound state. In the rest frame of three-body bound state, the energy of the off-shell particle is

$$k_3^0 = M_B - E_1 - E_2 < m.$$  (4.3)

In this representation, the symmetry of the amplitude is simply

$$\Gamma_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) = \zeta \Gamma_{12} \Gamma_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) = \zeta \Gamma_{\lambda_2\lambda_1\alpha}(k_1, k_2, k_3).$$  (4.4)

When solving the three-body bound state equations it is convenient to work in the rest frame of the bound state, but to boost the interacting pair to its own rest frame, where the partial wave decomposition of the two-body amplitude that drives the equation, $M_{22}$, is defined. The numerical solutions reported in Ref. [8] were carried out in this mixed frame, i.e., with $k_1$ and $\lambda_1$ defined in the three-body rest system and the remaining variables defined in the rest system of the interacting $23$ pair. The connection between these two representations will be discussed in a subsequent paper, where we will calculate the electrodisintegration of $^3$He [15]. In this section we will use the representation [14].

The wave function, when needed, is defined by

$$\Psi_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) = G_{\alpha,\alpha'}(k_3) \Gamma_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3),$$  (4.5)

where the nucleon propagator, $G$, was given in Eq. (3.3).

B. The three-body form factors

The diagrams needed to calculate the form factors were displayed in Figs. [19] and [20]. Following work on the deuteron form factors [2], the diagrams Figs. [19] (b)–(e) are referred to as the complete impulse approximation (CIA). Diagrams Figs. [19] (f) are the interaction currents, denoted by I.

Some of the CIA diagrams require knowledge of the vertex function with the two interacting nucleons off-shell. This vertex function was defined in Fig. [19] (b). A more convenient expression for this vertex function can be found using the equation for the two-body scattering amplitude with both particles in the final state off-shell [this generalization of Fig. [5] is shown in Fig. [20]. Substituting Fig. [20] into Fig. [19] (b), and using Fig. [20] (b) a second time, gives the result shown in Fig. [21] for the bound-state amplitude with particles 2 and 3 off-shell. [This is a generalization of a result previously shown in Fig. [23]. The algebraic form of the equation shown in Fig. [23] is

$$\Gamma_{\lambda_1\beta\alpha}(k_1, k_2, k_3) = - \int \frac{m \, d^3 k_3'}{(2\pi)^3} \sum_{\lambda_2} V_{\beta\alpha, \lambda_2\alpha'}(k_2, k_3; k_2', k_3') \left[ 1 + 2 \zeta \right] \Psi_{\lambda_1\lambda_2\alpha'}(k_1, k_2', k_3'),$$  (4.6)
where $V$ is the same two-body kernel used in Eq. (2.7), and summation over repeated Dirac indicies is implied. The phase of each spectator amplitude is computed from $i^{(i-n)}$, where $n$ is the total number of off-shell propagators plus interactions (either vertices or kernels) in the integrand of the amplitude. Equation (4.6) has one off-shell propagator (contained in $\Psi$), and two interactions ($V$ and $\Gamma$), for a phase of $i^{(i-1)} = -1$. For convenience we have adopted the notation

$$V_{\beta\alpha,\beta\alpha'}(k_2, k_3; k'_2, k'_3) = V_{\beta\alpha,\lambda_1\alpha'}(k_2, k_3; k'_2, k'_3),$$

(4.7)

so care must be taken to distinguish Dirac indicies from helicity indicies. Whenever a Dirac index is replaced by a helicity index, a contraction with an on-shell, positive energy spinor, such is shown in Eq. (1.7), is implied. The on-shell Dirac spinors are normalized to $\bar{u}u = 1$.

The algebraic result for the six diagrams that make up the CIA can now be written

$$J_{\text{CIA}}^\mu = 3e \int \frac{m^3 d^3k_1 d^3k_2}{E_1 E_2 (2\pi)^6} \sum_{\lambda_1,\lambda_2} \left\{ \bar{\Psi}_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) [1 + 2 \zeta \mathcal{P}_{12}] j_{\gamma'\alpha}(k_1, k_3) \Psi_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) 
+ \bar{\Gamma}_{\lambda_1\beta\alpha}(k_1, k'_2, k_3) G_{\beta\gamma'}(k'_2, k_2) [1 + 2 \zeta \mathcal{P}_{12}] u_\gamma(k_2, \lambda_2) \Psi_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3) 
+ \bar{\Psi}_{\lambda_1\lambda_2\alpha}(k_1, k_2, k'_3) \bar{u}_\gamma(k_2, \lambda_2) [1 + 2 \zeta \mathcal{P}_{12}] j_{\gamma'\beta}(k_2, k'_3) G_{\gamma'\gamma}(k'_3, k_3) \Gamma_{\lambda_1\beta\alpha}(k_1, k'_3, k_3) \right\},$$

(4.8)

where the doubly off-shell vertex functions are evaluated using (4.6), $j_{\gamma'\alpha}(k', k)$ is the single nucleon current for off-shell nucleons with incoming (outgoing) four-momenta $k'^\mu (k''\nu), k'^\mu = k_1 \pm q$, and in every term $k^2_1 = k^2_2 = m^2$ and $k_1 + k_2 + k_3 = P$, where $P$ is the four-momenta of the incoming deuteron. Each diagram has two off-shell propagators and three interactions, for a phase of $i^{(i-1)} = 1$. The I diagrams are

$$J_{\text{I}}^\mu = 3e \int \frac{m^3 d^3k_1 d^3k_2 d^3k_3}{E_1 E_2 E_3 (2\pi)^6} \sum_{\lambda_1,\lambda_2,\lambda_3} \bar{\Psi}_{\lambda_1\lambda_2\alpha}(k_1, k'_2, k_3) [1 + 2 \zeta \mathcal{P}_{12}] \times V_{\lambda_1\lambda_2\alpha,\lambda_3\alpha'}(k_2' k_3, k_2, k_3) [1 + 2 \zeta \mathcal{P}_{12}] \Psi_{\lambda_1\lambda_2\alpha}(k_1, k_2, k_3),$$

(4.9)

where $k_1 + k_2 + k_3 = P, k_1 + k'_2 + k'_3 = P' = P + q, k^2_1 = k^2_2 = m^2$, and $V_{\lambda_1\lambda_2\alpha,\lambda_3\alpha'}(k_2' k_3, k_2, k_3)$ is the symmetrized two-body interaction current for nucleons with incoming four-momenta $k_2, k_3$ and outgoing four-momenta $k'_2, k'_3$. In $V_{\lambda_1\lambda_2\alpha,\lambda_3\alpha'}(k_2' k_3, k_2, k_3)$ the first four-momentum listed in each pair describes an on-shell nucleon with incoming helicity $\lambda_2$ and outgoing helicity $\lambda_3$. The second nucleon is off-shell, with incoming Dirac indicies $\alpha$ and outgoing Dirac indicies $\alpha'$. The equations (4.8) and (4.9), with the two particle off-shell vertex function defined by Eq. (4.6), are convenient for numerical calculations of the form factor at non-zero $q^2$. However, when $q \rightarrow 0$, the nucleon propagators in the second and third terms of the CIA result (4.8) develop singularities that cancel, leading to terms involving the derivatives of the two-body kernel. The Appendix evaluates these diagrams in the $q \rightarrow 0$ limit, and gives an algebraic demonstration that the singularities cancel. The work is somewhat lengthy, and already contained implicitly in the proof of gauge invariance. A similar (but algebraically different) demonstration of charge conservation in the two-body case was already given in Ref. [10].

We now turn to the expressions for the breakup current.

C. The three-body breakup current

The three-body breakup current was shown in Fig. 26. The final state in all the diagrams is antisymmetric. As discussed in Sec. 11.3 this implies that the each diagram is multiplied by the projection operator $A_3$ [defined in Eq. (2.3)]. Hence the diagrams all have the form given explicitly in Eq. (4.23).

The symmetrized RIA diagrams Fig. 26 (a) are
Similarly, the interaction current diagrams, Fig. 26 (c) and (d) are

\[ J_1^\mu = -A_3 \bar{u}_3(p_3, \lambda_3) \int \frac{m \, d^3k_2}{E_2(2\pi)^3} V_{12,\gamma,\gamma'}(p_2, p_3; k_2, k_3) \left[ 1 + 2\zeta P_{12} \right] \Psi_{12,\gamma,\gamma'}(p_1, k_2, k_3) \]  

(4.11)

and the final state interaction (FSI) terms, Fig. 26 (b), arise from the core diagrams, and parallel those already given for the form factor in Eqs. (4.8) and (4.9)

\[ J_{FSI}^{\mu} = 3e \int \frac{m^2 \, d^3k_1 \, d^3k_2 \, d^3k_3}{E_1 E_2 E_3 (2\pi)^9} \sum_{\mu_1 \mu_2} \frac{m^2 \, d^3k_1 \, d^3k_2 \, d^3k_3}{E_1 E_2 E_3 (2\pi)^9} \sum_{\lambda_1 \lambda_2 \lambda_3} \frac{m^2 \, d^3k_1 \, d^3k_2 \, d^3k_3}{E_1 E_2 E_3 (2\pi)^9} \sum_{\mu_1 \mu_2} \left\{ T_{12,\lambda_1 \lambda_2,\mu_1 \mu_2}^\mu(p_1, p_2, p_3; k_1, k_2, k_3) \right\} \left[ 1 + 2\zeta P_{12} \right] \Psi_{12,\gamma,\gamma'}(p_1, k_2, k_3) \]  

(4.12)

where \( T_{12,\lambda_1 \lambda_2,\mu_1 \mu_2}^\mu(p_1, p_2, p_3; k_1, k_2, k_3) \) is the symmetrized three-body scattering amplitude with the final state on-shell. The off-shell unsymmetrized three-body scattering amplitude satisfies the equation shown in Fig. 26. This is

\[ T_{22,\lambda_1 \lambda_2,\mu_1 \mu_2}^{11}(p_1, p_2, p_3; k_1, k_2, k_3) = \frac{1}{3} E_3 \int \frac{m^2 \, d^3k_1 \, d^3k_2 \, d^3k_3}{E_1 E_2 E_3 (2\pi)^9} \sum_{\lambda_1 \lambda_2 \lambda_3} \frac{m^2 \, d^3k_1 \, d^3k_2 \, d^3k_3}{E_1 E_2 E_3 (2\pi)^9} \sum_{\mu_1 \mu_2} \left\{ T_{12,\lambda_1 \lambda_2,\mu_1 \mu_2}^\mu(p_1, p_2, p_3; k_1, k_2, k_3) \right\} \left[ 1 + 2\zeta P_{12} \right] \Psi_{12,\gamma,\gamma'}(p_1, k_2, k_3) \]  

(4.13)

The antisymmetrized amplitude is obtained by applying \( A_3 \) to both the initial and final state, and illustrated in Eq. (2.4).

V. CONCLUSIONS

Using diagrammatic techniques, we have derived a three-nucleon current consistent with the three-body spectator equations, and have shown explicitly that it is conserved. We obtain the current for elastic scattering, shown in Fig. 26 and Eqs. (4.8) and (4.9), and for the three-body breakup reaction, shown Fig. 26 and Eqs. (4.10), (4.11), and (4.12). The appendix also shows explicitly that this current conserves the charge of the bound state, even with an arbitrary choice of electromagnetic form factors and in the presence of energy-dependent interactions.

Our results show that the spectator current will be free of any double counting if we always choose to keep the spectator on-shell. This simple rule leads to an organizational principal that also resolves some ambiguities in the choice of diagrams that might otherwise be present.

Our derivation and discussion relies heavily on the beautiful method of gauging equations, developed by Kvinikhidze and Blankleider. As already noted in Sec. (1113) the normalization of the two-body and three-body scattering amplitude used by KB is different from ours, giving different weights to various diagrams, but the total results are the same.

Another difference, also mentioned in Sec. (1113) is that KB do not rearrange their amplitudes so that the spectator is always on-shell. This rearrangement has some advantages; it not only makes the equations more tractable, but it also displays how the spectator theory avoids the double counting problem (recall Fig. 28), the importance of which has been emphasized by Kvinikhidze and Blankleider.

This rearrangement also leads to a different interpretation of the diagrams in the theory. For example, Eq. (60) of KB includes RIA terms in which the photon couples to all three of the final nucleons, while our RIA terms, given in Fig. 26 (a1) and (a2), include only couplings to the particles in the final state interacting pair, with no term describing coupling to the final state spectator (both approaches require the results be antisymmetrized).

At first glance it might seem that our result cannot be correct – surely the photon must couple to all of the outgoing nucleons. But our total result does include coupling to the final state spectator as part of the FSI term discussed in Fig. 26 (since this term arose from the rearrangement, it is not present in KB). In Fig. 26 the off-shell equations of Fig. 26 are used to show that the
We plan to use the results of this paper to calculate the high energy breakup processes recently measured at JLab.

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APPENDIX A: DEMONSTRATION OF CHARGE CONSERVATION

The explicit demonstration that charge is conserved begins by expanding the diagrams (c) and (c′) using Fig. 31. The result is shown in Fig. 32. The 8 diagrams are organized into 4 pairs, with an integral over a three momentum $k_2$. All the contributions from these 8 diagrams can be written

$$J_{c_1+c_2}^\mu = 3e \int \int \frac{m^3 d^3 k_1 d^3 k_2' d^3 k_2''}{E_1 E_2' E_2'' (2\pi)^3} \bar{\Psi}_{\lambda_1 \lambda_2 \alpha'}(k_1, k_2', k_2'') \left[ 1 + 2 \xi P_{12} \right] \times j_{c_1+c_2}^{\mu}(k_1, k_2', k_2''; k_3') \left[ 1 + 2 \xi P_{12} \right] \Psi_{\lambda_1 \lambda_2 \alpha}(k_1, k_2', k_3'),$$

where the integrand $j_{c_1+c_2}^\mu$ is the sum of two contributions that each become singular as $q \to 0$

$$j_{c_1+c_2}^\mu \equiv j_{c_1+c_2}^{\mu, \lambda_1 \lambda_2 \alpha}(k_1, k_2', k_2''; k_3') = j_{c_1}^{\mu}(k_1, k_2', P_{23}', k_2'' P_{23}')$$

$$= \int \frac{m^3 d^3 k_2}{E_2 (2\pi)^3} \left\{ V_{2, \alpha', \beta', \gamma}(k_2', k_2 + q; P_{23}) G_{\gamma', \gamma}(P_{23}' - k_2) \left[ \frac{-1}{m - k_2 + q} \right] j_{N}^\mu(k_2 + q, k_2) \Lambda(k_2) \right\} V_{\beta', \gamma; \lambda_2, \alpha}(k_2, k_2''; P_{23}')$$

$$+ V_{2, \alpha', \beta', \gamma}(k_2', k_2; P_{23}') G_{\gamma', \gamma}(P_{23}' - k_2) \left[ \Lambda(k_2) j_{N}^\mu(k_2, k_2 - q) \right] \left[ \frac{1}{m - k_2 + q} \right] V_{\beta, \gamma; \lambda_2, \alpha}(k_2 - q, k_2''; P_{23}) \right\},$$

with $j_{N}^\mu$ the single nucleon current and $\Lambda(k_2) = (m + k_2)/(2m)$ the positive energy projection operator for a particle with four-momentum $k_2$. The integrals in (A2) are both given in terms of the on-shell four momentum $k_2 = \{E_2, k_2\}$. The two-body kernels, $V$, conserve four-momentum, and are expressed in terms of independent variables, which we choose to be the initial and final momenta of particle 2 (usually on-shell) and the total momentum of the two-body system, so that generically

$$V \equiv V(k_2', k_3'; k_2, k_3) = V(k_2', k_2; P_{23})$$
with $P_{23} = P - k_1 = k_2 + k_3 = k'_2 + k'_3$.

To calculate the $q \to 0$ limit of Eq. (A2), we follow the method developed in Ref. [11]. Since the full current is gauge invariant, we know that the singular terms must vanish as $q \to 0$, so we can find the $q \to 0$ limit of Eq. (A2) by contracting $q_\mu$ into both sides, expanding about $q_0$, and extracting the coefficient of the linear term. Using the Ward identity for the nucleon current

$$q_\mu j^\mu_N(k', k) = (m' - k') - (m - k')$$  \hspace{1cm} (A4)

and, for $k_2$ on-shell,

$$(m - k_2)\Lambda(k_2) = 0,$$ \hspace{1cm} (A5)

Eq. (A2) reduces to

$$q_\mu j^\mu_c = -\int \frac{m^3 k_2}{E_2(2\pi)^3} \left\{ V_{\lambda'_2,\alpha',\beta',\gamma'}(k''_2, k_2 + q; P''_{23}) G_{\gamma'}(P''_{23} - k_2) \Lambda(k_2) \beta' \lambda_2, \alpha(k_2, k'_2; P'_{23})
- V_{\lambda'_2,\alpha',\beta',\gamma'}(k''_2, k_2 + q; P''_{23}) G_{\gamma'}(P''_{23} - k_2) \Lambda(k_2) \beta' \lambda_2, \alpha(k_2, k'_2; P'_{23})
+ V_{\lambda'_2,\alpha',\beta',\gamma'}(k''_2, k_2; P''_{23}) G_{\gamma'}(P''_{23} - k_2) \Lambda(k_2) V_{\lambda'_2,\gamma'}(k_2, k'_2; P'_{23})
- V_{\lambda'_2,\alpha',\beta',\gamma'}(k''_2, k_2; P''_{23}) \frac{\partial}{\partial P''_{23}} G_{\gamma'}(P''_{23} - k_2) \Lambda(k_2, k'_2; P'_{23}) \right\} \equiv \mathcal{O}(q^2).$$  \hspace{1cm} (A6)

Use the shorthand notation

$$\frac{\partial}{\partial k_2} V(k''_2, k_2; P''_{23}) \equiv \delta_{k''}^k V; \quad \frac{\partial}{\partial k_2} V(k_2, k'_2; P'_{23}) \equiv \delta_{k'}^k V; \quad \frac{\partial G(k_3)}{\partial k_3} \equiv G^{\mu}(k_3)$$  \hspace{1cm} (A7)
second two rows use Fig. 34 (to replace the amplitudes enclosed in the dashed boxes). The resulting 8 diagrams are labeled by both the parent diagram (a–d in the first two rows) and the term (1 or 2) in Fig. 34 from which they originate. They are organized into 4 pairs, with members of each pair arranged above and below each other on lines 3 and 4 of the figure [diagrams (a2) and (c2) are an example].

FIG. 34: (Color online) The second two-body scattering amplitude in Fig. 34(c) can be replaced by its integral equation, and further simplified using the on-shell version of 9(b). This form of the vertex function that is useful in the reduction of diagrams 9(e1) and (e2).

Inserting this back into the original expression (A11), and using the wave equation (4.6) with the second particle on-shell, gives

\[
\lim_{q \to 0} j_{\mu}^{\nu} = \int \frac{m^2 d^3 k_2}{E_2 (2\pi)^3} \sum_{\lambda_2, \lambda_1} \left\{ V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_3, k_2, P_2) G_{\gamma, \gamma}^{\nu \mu} (k_3) V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_3, k_2, P_2) G_{\gamma, \gamma}^{\nu \mu} (k_3) - \delta_\mu^\nu V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_2, k_2, P_2) G_{\gamma, \gamma}^{\nu \mu} (k_3) V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_3, k_2, P_2) G_{\gamma, \gamma}^{\nu \mu} (k_3) \right. \\
\left. - V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_2, k_2, P_2) G_{\gamma, \gamma}^{\nu \mu} (k_3) \delta_\mu^\nu V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_3, k_2, P_2) G_{\gamma, \gamma}^{\nu \mu} (k_3) \right\} .
\]

(A8)

Inserting this back into the original expression (A11), and using the wave equation (4.6) with the second particle on-shell, gives

\[
\lim_{q \to 0} j_{\mu}^{\nu} = 3 e \int \frac{m^2 d^3 k_2}{E_2 (2\pi)^3} \sum_{\lambda_2, \lambda_1} \left\{ \Gamma_{\lambda_1 \lambda_2 \gamma, \gamma, \gamma (k_1, k_2, k_3) G_{\gamma, \gamma}^{\nu \mu} (k_3) \Gamma_{\lambda_1 \lambda_2 \gamma, \gamma, \gamma (k_1, k_2, k_3) G_{\gamma, \gamma}^{\nu \mu} (k_3) \right. \\
\left. + \int \frac{m d^3 k_2}{E_2 (2\pi)^3} \sum_{\lambda_2} \Psi_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_1, k_2, k_3) \left[ 1 + 2 \xi P_{12} \right] \delta_\mu^\nu V_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_2, k_2, P_2) \Psi_{\lambda_1 \lambda_2 \alpha (k_1, k_2, k_3) \right] + \int \frac{m d^3 k_2}{E_2 (2\pi)^3} \sum_{\lambda_2} \Psi_{\lambda_1 \lambda_2 \alpha; \gamma, \gamma, \gamma (k_1, k_2, k_3) \left[ 1 + 2 \xi P_{12} \right] \Psi_{\lambda_1 \lambda_2 \alpha (k_1, k_2, k_3) \right} .
\]

(A9)

where, in general, \( \delta_\mu^\nu \) refers to the derivative with respect to the initial \( k_2 \) momentum, and \( \delta_\mu^\nu \) with respect to the final \( k_2 \) momentum. With this notation

Note that the relative signs of the terms change because of the sign in Eq. (4.6).

We now turn to diagrams Fig. 34(e1) and (e2). The object here is to isolate the electromagnetic coupling in a loop as we did above, but because the loop now involves a spectator, we need to pull out two interactions. In preparation,
we note that the Fig. 9 (c) can be written in an alternative form that involves the kernel, as shown in Fig. 34. The algebraic form of this equation is

$$
\Gamma_{\lambda_1 \lambda_2 \omega}(k_1, k_2, k_3) = 2 \zeta P_{12} \int \frac{m^2 \delta^4 k_1' \delta^3 k_2' \delta^3 k_3'}{E_1' E_2' E_3'} \sum_{\lambda_1' \lambda_2' \lambda_2} M_{\lambda_1, \lambda_1', \gamma}(k_1', k_1; P_{13} - k_1') \times V_{\lambda_2, \gamma, \alpha'}(k_2, k_2'; P_{23}') [1 + 2 \zeta P_{12}] \Psi_{\lambda_1', \lambda_2', \alpha'}(k_1', k_2', k_2'), \tag{A10}
$$

where $P_{13} = P - k_2$ and $P_{23}' = P - k_1'$.

Using this substitution, the diagrams Fig. 10 (e1) and (e2) can be written as shown in Fig. 35. These 8 diagrams collect together into

$$
J_{e_1 + e_2}^\mu = -12 e \int \int \int \frac{m^4 \delta^4 k_1' \delta^3 k_2' \delta^3 k_3'}{E_1' E_2' E_3'} \sum_{\lambda_1' \lambda_2' \lambda_2} M_{\lambda_1, \lambda_1', \gamma}(k_1', k_1; P_{13}) \times V_{\lambda_2, \gamma, \alpha'}(k_2', k_2'; P_{23}') [1 + 2 \zeta P_{12}] \Psi_{\lambda_1', \lambda_2', \alpha'}(k_1', k_2', k_2'), \tag{A11}
$$

where the common internal loop is again the sum of two (canceling) singular terms

$$
\begin{align*}
J_{e_1 + e_2}^\mu &= j_{\lambda_1', \lambda_2', \gamma, \alpha'}(k_1', k_2', k_3; P_{23}') = j_{\lambda_1', \lambda_2', \gamma, \alpha'}(k_1', k_2', P_{23}'; k_1, k_2', P_{23}') = \int \frac{m^3 \delta^3 k_2}{E_2 (2\pi)^3} \\
\left\{ V_{\lambda_2, \gamma, \alpha'}(k_2', k_2; q; P_{23}') \left[ \frac{1}{m - k_2} \right] j_{\lambda_1'}(k_2 + q, k_2) \Lambda(k_2) \right\}_{\beta' \beta} \left( V_{\gamma, \alpha'}(k_2', k_2; P_{23}) \left[ \frac{1}{2} \right] \right)_{\beta' \beta} \\
&+ V_{\lambda_2, \gamma, \alpha'}(k_2', k_2; P_{23}') \left[ \Lambda(k_2) \right] j_{\lambda_1'}(k_2, k_2 - q) \left[ \frac{1}{m - k_2} \right]_{\beta' \beta} \right\}, \tag{A12}
\end{align*}
$$

where

$$
O_{\lambda_1', \gamma, \alpha'}(k_1', k_1; P_{13}) = G_{\gamma, \alpha'}(P_{13} - k_1') M_{\lambda_1, \alpha', \lambda_1}(k_1, k_1; P_{13}) G_{\gamma, \alpha'}(P_{13} - k_1') \tag{A13}
$$

and $P_{23}' = P - k_1', P_{23}' = P' - k_1$, $P' = P + q$, and $P_{13} = P' - k_2$. Following the same arguments that lead from Eq. (A2) to (A8), we obtain

$$
\lim_{q \to 0} J_\mu = \int \frac{m^3 \delta^3 k_2}{E_2 (2\pi)^3} \sum_{\lambda_2} \left\{ V_{\lambda_2, \gamma, \alpha'}(k_2', k_2; P_{23}') \Delta \mu \delta_{\lambda_1', \lambda_1}(k_1', k_1; P_{13}) V_{\lambda_2', \gamma, \alpha\lambda_2, \alpha}(k_2, k_2'; P_{23}') \\
+ \delta_{\lambda_1', \lambda_1}(k_1', k_1; P_{13}) \delta_{\lambda_2, \lambda_2}(k_2, k_2'; P_{23}') \right\}, \tag{A14}
$$

where $\delta_{\lambda_1', \lambda_1}(k_1', k_1; P_{13})$ and $\delta_{\lambda_2, \lambda_2}(k_2, k_2'; P_{23}')$ have been previously defined, and

$$
\Delta \mu \delta_{\lambda_1', \lambda_1}(k_1', k_1; P_{13}) = G_{\gamma, \alpha'}(P_{13} - k_1') M_{\lambda_1, \alpha', \lambda_1}(k_1, k_1; P_{13}) G_{\gamma, \alpha'}(P_{13} - k_1') \\
+ \Gamma_{\lambda_1, \lambda_2, \gamma}(k_1, k_2, k_3) G_{\gamma, \alpha'}(k_3) 4 \zeta P_{12} \Gamma_{\lambda_1, \lambda_2, \gamma}(k_1, k_2, k_3) \\
+ \Gamma_{\lambda_1, \lambda_2, \gamma}(k_1, k_2, k_3) G_{\gamma, \alpha'}(k_3) \delta_{\lambda_2, \lambda_1}(k_1, k_1; P_{13}) G_{\gamma, \alpha'}(P_{13} - k_1') \tag{A15}
$$

with

$$
\delta_{\lambda_1', \lambda_1}(k_1', k_1; P_{13}) = \left( \frac{\partial}{\partial P_{13}} \right) M_{\lambda_1, \lambda_1}(k_1', k_1; P_{13}) \tag{A16}
$$

the derivative with respect to the total two-body momentum. Inserting (A14) into (A11), and using the replacement (A15) and the wave equations (4.6) and (4.10), gives an intermediate result

$$
\lim_{q \to 0} J_\mu_{e_1 + e_2}^\mu = 3 e \int \frac{m^3 \delta^3 k_1 \delta^3 k_2}{E_1 E_2 (2\pi)^6} \sum_{\lambda_1 \lambda_2} \left\{ \tilde{\Gamma}_{\lambda_1, \lambda_2, \gamma}(k_1, k_2, k_3) G_{\gamma, \alpha'}(k_3) 4 \zeta P_{12} \Gamma_{\lambda_1, \lambda_2, \gamma}(k_1, k_2, k_3) \\
+ \frac{m^3 \delta^3 k_2}{E_2 (2\pi)^3} \sum_{\lambda_2} \Psi_{\lambda_1, \lambda_2, \alpha}(k_2, k_2', k_3) \lambda_{\alpha'} \lambda_2, \alpha(k_2, k_2', k_2) \Psi_{\lambda_1, \lambda_2, \alpha}(k_1, k_2, k_3) \right\}. \tag{A17}
$$
Therefore, the $\delta$ and recalling that $k$ rewriting the WT identity for the interaction current, Eq. (3.7), in terms of the independent variables gives

$$\text{Collecting all terms from Eqs. (A9), (A17), and (A22) gives}$$

Therefore, the $\delta^\mu_p M$ term in (A18) becomes

Using this expression and the original form of the wave equation, (A20), the term involving the derivative of $M$ reduces to

Collecting all terms from Eqs. (A19), (A21), and (A22) gives

where

The evaluation of the derivative of $M$ can be carried out using the fact that $M$ is an infinite series of interactions:

$$M = V - VGV + VGVGV - \cdots,$$

$$\text{and recalling that } k_3 = P_{23} - k_2, \text{ so that } \partial G(k_3)/\partial (P_{23}) = \partial G(k_3)/\partial (k_3) = G^\mu$$

$$\delta^\mu_p M = \delta^\mu_p V - (\delta^\mu_p V)GV - VGV^\mu V - VG(\delta^\mu_p V) + (\delta^\mu_p V)GVGV + VGVGV + VG(\delta^\mu_p V)GV + \cdots$$

$$= \delta^\mu_p V - (\delta^\mu_p V)GM - MG(\delta^\mu_p V) - MG^\mu M + MG(\delta^\mu_p V)GM.$$
Substituting this into (4.19) gives a result with the same form as the second term in (A.23), and combining this with (A.24) gives a total contribution of $-3\delta_\mu^\nu V$. Combining this with the contributions from diagrams (b) and (d), and the other terms from (A.23), gives the total result

$$J^\mu_{\text{total}}(0) = 9e \int \int \frac{m^2 d^3 k_1 d^3 k_2}{E_1 E_2 (2\pi)^6} \Gamma_{\lambda_1 \lambda_2 \gamma}(k_1, k_2, k_3) \frac{G^\mu \gamma(k_3)}{[1 + 2\zeta P_{12}] \Gamma_{\lambda_1 \lambda_2 \gamma}(k_1, k_2, k_3)}$$

where the last line uses the fact that the total charge of a bound state of three identical particles of charge $e$ is $3e$ (because isospin has been ignored). Hence charge is conserved if the normalization of the wave function is

$$1 = 3 \int \int \langle \Gamma \frac{P^\mu}{M_B} \frac{\partial G}{\partial P^\mu} [1 + 2\zeta P_{12}] \Gamma \rangle - 3 \int \int \langle \Psi [1 + 2\zeta P_{12}] \frac{P^\mu}{M_B} \frac{\partial V}{\partial P^\mu} [1 + 2\zeta P_{12}] \Psi \rangle .$$  \tag{A28}

Noting that

$$\frac{P^\mu}{M_B} \frac{\partial G}{\partial P^\mu} = 2M_B \frac{\partial G}{\partial p^2} ,$$  \tag{A29}

and similarly for $V$, we recover a normalization condition equal to $2M_B$ times that originally derived in Ref. [11]. This difference is due to the fact that the all spinors in Ref. [11] were normalized to $2m$ (or $2M_B$ for the bound state), so our result agrees with Ref. [11], completing our demonstration.

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