INTEGRAL APOLLONIAN CIRCLE
PACKINGS AND PRIME CURVATURES

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Summary: It is shown that any primitive integral Apollonian circle packing captures a fraction of the prime numbers. Basically the method consists in applying the circle method, considering the curvatures produced by a well-chosen family of binary quadratic forms.

Introduction

In this paper, we pursue a line of research initiated in [GLMWY] and [S] on the arithmetical properties of integral Apollonian circle packings (ACP for short) in the plane. The reader is also referred to [B-F1] for certain background material.

Throughout the paper, we consider bounded ACP’s which are primitive, meaning that all curvatures of the circles in the packing do not share a factor greater than one. Let us recall that the set of curvatures in a given packing $P$ is obtained by action of the Apollonian group $A$ on the root quadruple $(a, b, c, d)$ of co-prime integers $a < 0 \leq b \leq c \leq d$, $a + b + c \geq d$. The group $A$ is a subgroup of the orthogonal group associated to the Descartes quadratic form

$$Q(x_1, x_2, x_3, x_4) = 2(x_1^2 + x_2^2 + x_3^2 + x_4^2) - (x_1 + x_2 + x_3 + x_4)^2$$

whose vanishing is tantamount with $x_1, x_2, x_3, x_4$ being curvatures of mutually tangent circles. The group $A$ is generated by the matrices

$$S_1 = \begin{pmatrix}
-1 & 2 & 2 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad \quad S_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
2 & -1 & 2 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}$$

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$$S_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 2 & 2 & -1 \end{pmatrix}$$

The basic problem is to describe the set of curvatures appearing in a given packing $P$; the ultimate hope is to establish a local to global principle cf. [F-S]. More modestly, [GLMWY] put forward the ‘positive density’ conjecture, according to which the set of curvatures in an ACP form a subset of $\mathbb{Z}$ of positive density. Following up on a technique proposed by P. Sarnak, this problem was solved affirmatively in [B-F1] (a slightly stronger result is obtained in this paper; see Theorem 1 and the Remark following its proof). Using a result due to Iwaniec on representing shifted primes by binary quadratic forms, Sarnak also pointed out that any primitive ACP produces at least $c \frac{X}{(\log X)^{3/2}}$ distinct prime curvatures at most $X$, for $X \to \infty$. Based on new results on the representation of integers by binary quadratic forms of large discriminant, previous lower bound for the number of prime curvatures is improved further in [B-F2] to at least $cX^{\frac{3}{2}} \frac{\log 2}{\log X} + \epsilon$.

The main result in this paper gives the correct order of magnitude.

**Theorem 2.** Given an integral primitive ACP, there is a positive $c$, such that for $X$ large the number of prime numbers less than $X$ which are curvatures of circles in the ACP, is at least $c \frac{X}{\log X}$ (with $c > 0$ an absolute constant).

Compared with the arguments due to Sarnak and refined in [B-F2], that are based on Iwaniec’ theorem and representations by individual quadratic forms, the strategy used here is different. Our approach consists in introducing a generating function by considering the collected contribution of suitable families of binary quadratic forms (constructed in §0, §1 of the paper). These generating functions can then be analyzed using the circle method (in a rather standard way), to the extent of providing a main (arithmetical) contribution with an error term. In particular, we are able to establish Theorem 2 (relying also on the so-called ‘majorant property’ for the set of the prime numbers, established in [B], [G]). The technique applied here may be organized better as to allow a treatment of the major-arcs contribution by spectral methods (using the spectral analysis for the full Apollonian group), in the spirit of [B-K]. This leads to better error terms and statements that come close to a local to global principle. That program is pursued in the forthcoming paper [B-K2]. Let us also mention the paper [F-S] that gives evidence for the only congruence obstructions to appear (mod 24).

In this discussion, we should cite the paper [K-O], where counting results for the curvatures, with multiplicity, are obtained based on spectral techniques (see also [BGS]).
In particular, it is shown in [K-O] that in any ACP $P$ the number of curvatures at most $X$ is of the order

$$X^\delta \text{ for } X \to \infty$$

with $\delta = 1, 30068..$ is independent of the packing. This amounts also to the number of quadruples bounded by $X$ in the orbit of the root quadruple under the Apollonian group $A$.

Let us briefly recall how binary quadratic forms enter the analysis (see [S]). While $A$ is a ‘thin’ (non-arithmetic) group, its subgroup $A_1 = \langle S_2, S_3, S_4 \rangle$ (stabilizer of $x_1$) and similarly $A_2, A_3, A_4$ are arithmetic. More precisely, considering the map

$$y = (y_2, y_3, y_4) = (x_2, x_3, x_4) + (a, a, a)$$

the affine action of $A_1$ on $(x_2, x_3, x_4)$ is conjugated to the action of a finite index subgroup $\Gamma$ of $O_g(\mathbb{Z})$, $g$ denoting the quadratic form

$$g(y) = y_2^2 + y_3^2 + y_4^2 - 2y_2y_3 - 2y_2y_4 - 2y_3y_4.$$ 

By a further coordinate change

$$A = y_2, B = \frac{1}{2}(y_1 - y_3 + y_4), C = y_4$$

transforming $g(y)$ in the quadratic form $\Delta(A, B, C) = B^2 - AC$, $\Gamma$ is conjugated to the subgroup of $O_\Delta(\mathbb{Z})$ generated by the reflections

$$\begin{bmatrix} 1 & -4 & 4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 4 & -4 & 1 \end{bmatrix}.$$ 

Consider the spin double cover of $SO_\Delta(\mathbb{Z})$ realized as image of $GL_2(\mathbb{Z})$ under the homomorphism

$$\rho: \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \frac{1}{\alpha\delta - \beta\gamma} \begin{bmatrix} \alpha^2 & 2\alpha\gamma \\ \alpha\beta & \alpha\delta + \beta\gamma \\ \beta^2 & 2\beta\delta \\ \gamma^2 & \delta^2 \end{bmatrix}$$

with kernel $\pm I$. Then $\rho^{-1}(SO_\Delta(\mathbb{Z}) \cap \tilde{\Gamma})$ contains $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ and hence the principal congruence subgroup $\Lambda(2)$ of $SL_2(\mathbb{Z})$.

It turns out that the set of values of $y_2 = A$, $y_3 = A + C - 2B, y_4 = C$ contains at least those of $A$ with $(A, B, C)$ ranging in an orbit $\rho(SL_2(\mathbb{Z}))(A_0, B_0, C_0)^t$, i.e. the integers represented primitively by the binary quadratic form

$$A_0\alpha^2 + 2B_0\alpha\gamma + C_0\gamma^2 \text{ with } (\alpha, \gamma) = 1.$$
The preceding provides an explicit recipe to produce curvatures in a given packing \(P\). Assume \((a_0, b_0, c_0, d_0) \in S = S(P) = A(a, b, c, d)\) and set

\[
A_0 = a_0 + b_0, \quad 2B_0 = a_0 + b_0 - c_0 + d_0, \quad C_0 = a_0 + d_0.
\]

Then all integers represented by the quadratic form

\[
A_0 x^2 + 2B_0 xy + C_0 y^2 - a_0 \quad \text{with} \quad x, y \in \mathbb{Z}, \quad (x, y) = 1
\]

appears as curvatures of circles in the packing \(P\).

This observation made in [S] plays a key role in [B-F1] and also in the construction of an appropriate family of binary quadratic forms described in §0, §1 of this paper.

(0). Preliminary construction of a set of curvature quadruples

Let \(R_1\) be a large integer and denote \(S_{R_1}\) the set of quadruples \((a, b, c, d) \in S = S(P)\) of the Apollonian packing \(P\) satisfying

\[
\max(|a|, |b|, |c|, |d|) \sim R_1.
\]

Thus

\[
|S_{R_1}| > R_1^\delta \quad \text{with} \quad \delta > \frac{13}{10}.
\]

Let

\[
R_2 = \frac{R_1}{10}.
\]

Given \((a, b, c, d) \in S_{R_1}\), let \(A = a + b, C = a + d, 2B = a + b - c + d\) and consider the set of integers

\[
S_{a,b,c,d} = \{Ax^2 + 2Bxy + Cy^2 - a; x, y \in \mathbb{Z}, 0 \leq x, y < R_2 \quad \text{and} \quad (x, y) = 1\}.
\]

Recall that \((A, B, C) = 1\) and \(a^2 = AC - B^2\) (by Descartes’ equation).

As explained in [B-F1] the set \(S_{a,b,c,d}\) is contained in the set of curvatures produced in the orbit of \((a, b, c, d)\) under group elements of \(A_1 = \langle S_2, S_3, S_4 \rangle\) of norm bounded by \(R_2^2\). Denote \(S(a, b, c, d) \subset S_{R_1, R_2} \cap \langle S_2, S_3, S_4 \rangle \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right)\) a set of quadruples \((a, b', c', d')\) in one-to-one correspondence with \(S_{a,b,c,d}\) by projection on the \(b'\)-coordinate.

Thus for each \(0 \leq x, y < R_2\), \((x, y) = 1\) there is some \(g_{x,y} \in \langle S_2, S_3, S_4 \rangle\) such that

\[
S(a, b, c, d) \subset \left\{ g_{x,y} \left( \begin{array}{c} a \\ b \\ c \\ d \end{array} \right); \quad 0 \leq x, y < R_2 \quad \text{and} \quad (x, y) = 1 \right\}.
\]
Obviously, if we fix \( x, y \), all quadruples \( g_{xy} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \) are distinct and hence

\[
\sum_{(a,b,c,d) \in S_{R_1}} 1_{S(a,b,c,d)} \leq R_2^2. \tag{0.6}
\]

Given \( b' \), it follows from Descartes’ equation that

\[
\pi_2^{-1}(b') \cap S_{R_1 R_2^2} \ll R_1^{1+\varepsilon} R_2^2 \tag{0.7}
\]

and (0.6), (0.7) imply that

\[
\sum_{(a,b,c,d) \in S_{R_1}} 1_{S_{a,b,c,d}} \ll R_1^{1+\varepsilon} R_2^4. \tag{0.8}
\]

To each \((a, b, c, d) \in S_{R_1}\), associate the distribution \( \lambda_{a,b,c,d} \) on \( \mathbb{Z} \) obtained as image measure of

\[
[0 \leq x, y < R_2; \ (x, y) = 1, f_a(x, y) \sim R_1 R_2^2 \text{ and } (f_a(x, y), \prod_{p < R_2^2} p) = 1]
\]

under the map

\[
(x, y) \mapsto f_a(x, y) = Ax^2 + 2Bxy + Cy^2 - a.
\]

Hence, \( \text{supp} \lambda_{a,b,c,d} \subseteq S_{a,b,c,d}; \ |\lambda_{a,b,c,d}|_{\infty} \ll R_1^{\varepsilon} \) and elementary sieving implies that certainly

\[
\frac{R_2^2}{(\log R_2)^2} \lesssim |\lambda_{a,b,c,d}|_{1} \lesssim R_2^2. \tag{0.9}
\]

Also, from sieving, we obtain that for all \( q \in \mathbb{Z}_+ \)

\[
\sum_{z \equiv u \ (\text{mod} \ q)} \lambda_{a,b,c,d}(z) \lesssim \left( \frac{1}{q} + \frac{1}{R_2} \right)^{1/\varphi} |\lambda_{a,b,c,d}|_{1} \tag{0.10}
\]

(in the argument, we distinguish the cases \( q > (\log R_2)^{100} \) and \( q < (\log R_2)^{100}; \) we only need a crude estimate for our purpose).

Define

\[
\lambda = \sum_{(a,b,c,d) \in S_{R_1}} \lambda_{a,b,c,d} \tag{0.11}
\]
which is a distribution on \([b' \in \mathbb{Z}; b' \sim R_1R_2^2]\). From (0.8), (0.9)
\[
|S_{R_1}| \frac{R_2^2}{(\log R_2)^2} < \|\lambda\|_1 \leq |S_{R_1}|.R_2^2 \quad \text{and} \quad \|\lambda\|_\infty \ll R_1^{1+\varepsilon}R_2^4. \quad (0.12)
\]

By construction, \(z \in \text{supp} \lambda\) has no prime factors less than \(R_2^{\frac{1}{Q}}\) and from (0.10) obviously
\[
\sum_{z \equiv u \pmod q} \lambda(z) \lesssim \left( \frac{1}{q} + \frac{1}{R_2} \right)^{\frac{1}{10^5}} \|\lambda\|_1. \quad (0.13)
\]

Let \(\eta_{a,b,c,d}\) be a distribution on \(S(a, b, c, d)\) which image measure under projection on the \(b'\)-coordinate equals \(\lambda_{a,b,c,d}\) and set
\[
\eta = \sum_{(a, b, c, d) \in S_{R_1}} \eta_{a,b,c,d}. \quad (0.14)
\]

Hence
\[
\|\eta\|_1 = \|\lambda\|_1. \quad (0.15)
\]

Since clearly \(\|\eta_{a,b,c,d}\|_\infty \leq \|\lambda_{a,b,c,d}\|_\infty \ll R_1^\varepsilon\), it follows from (0.6) that
\[
\|\eta\|_\infty \ll R_2^2R_1^\varepsilon \quad (0.16)
\]

(0.13) may be rephrased as
\[
\sum_{b' \equiv u \pmod q} \eta(a', b', c', d') \lesssim \left( \frac{1}{q} + \frac{1}{R_2} \right)^{\frac{1}{10^5}} \|\eta\|_1 \quad \text{for all} \quad q \in \mathbb{Z}. \quad (0.17)
\]

Next, we replace the distribution \(\eta\) by a subset \(C \subset S_{R_1R_2^2}\) which we construct probabilistically by selecting \((a', b', c', d') \in C\) with probability

\[
\delta \eta(a', b', c', d') < 1 \quad \text{where} \quad \delta = R_2^{-3} \quad \text{(cf. (0.16))}. \]

By (0.12), (0.15), we obtain
\[
|C| \approx \delta.\|\eta\|_1 > R_2^{-2}|S_{R_1}| \quad (0.18)
\]

and also
\[
\sum_{b'} |C_{b'}|^2 \lesssim \delta \|\eta\|_1 + \delta^2\|\eta\|_1 \|\lambda\|_\infty < |C|.R_1R_2^2 \quad (0.19)
\]
with $C_{b'}$ denoting the fibers of $C$.

From (0.17) and standard large deviation inequalities, we deduce that for $q \in \mathbb{Z}_+$, $u \in \mathbb{Z}$

$$|\{(a', b', c', d') \in C; b' \equiv u \pmod{q}\}| \leq$$

$$\delta \sum_{b' \equiv u \pmod{q}} \eta(a', b', c', d') + c \sqrt{\log R_1(\delta \|\eta\|_1)}^{1/2} \lesssim$$

$$\left(\frac{1}{q} + \frac{1}{R_2}\right)^{\frac{3}{10}} |C|.$$  \hspace{1cm} (0.20)

Relabeling the quadruples, we obtain a subset $C \subset S_R, R = R_1 R_2^2$ with the following properties

(0.21) $|C| > R^{5-\frac{1}{10}} > R^{6/5}$

(0.22) $a \sim R$ and $|b|, |c|, |d| \lesssim R$ for $(a, b, c, d) \in C$

(0.23) $\sum_a |C_a|^2 < R^{-1/5}|C|^2$

(0.24) For $(a, b, c, d) \in C$, $a$ has no prime factors less than $R^{3/5}$

(0.25) $|\{(a, b, c, d) \in C; a \equiv u \pmod{q}\}| \lesssim (q^{-1} + R^{-1})^{\frac{3}{10}} |C|$ for all $q \in \mathbb{Z}_+$

(1). Introducing a family of quadratic forms

Let $C \subset S_R$ be the set constructed in §0. Let $A = \pi_a(C) \subset \mathbb{Z}_+$.

To each $(a, b, c, d) \in C$, we associate again the binary form

$$f(x, y) = Ax^2 + 2Bxy + Cy^2$$  \hspace{1cm} (1.1)

with

$$A = a + b, C = a + d, 2B = a + b - c + d, \text{ disc } f = -4a^2, a^2 = AC - B^2.$$  \hspace{1cm} (1.2)

Thus $(A, B, C) = 1$. Since $|A|, |B|, |C| \lesssim R$ and $|A.C| \gtrsim a^2 \sim R^2$, it follows that $|A|, |C| \sim R$.

Denote $\mathcal{F}$ the family of quadratic forms (1.1) obtained from $C$ and by $\mathcal{F}_a \subset \mathcal{F}$ those obtained from $C_a$. We show that if we fix the discriminant, the number of equivalent forms in $\mathcal{F}$ is $O(1)$.
Thus if $Ax^2 + 2Bxy + Cy^2$ and $A_1x^2 + 2B_1xy + C_1y^2$ are equivalent, then

$$\begin{align*}
A_1 &= \alpha^2 A + 2\alpha\gamma B + \gamma^2 C \\
B_1 &= \alpha\beta A + (\alpha\delta + \beta\gamma)B + \gamma\delta C \\
C_1 &= \beta^2 A + 2\beta\delta B + \delta^2 C
\end{align*}$$

for some $\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$.

Thus

$$A_1 = A\left(\alpha + \frac{B}{A}\gamma\right)^2 + \frac{a^2}{A}\gamma^2.$$ 

Hence

$$\gamma^2 < \frac{AA_1}{a^2} < O(1)$$

and since $|A|, |A_1| \sim R, \left|\alpha + \frac{B}{A}\gamma\right| < O(1), |\alpha| < O(1) + O(1)\left|\frac{B}{A}\right| < O(1)$.

Similarly, we see that, $|\beta|, |\delta| < O(1)$. This shows that at most $O(1)$ quadratic forms obtained from $C_a$ are equivalent. This proves our claim.

As a consequence, we obtain that for fixed $a$ and $M$

$$\#\{(f, f_1, x, y, x_1, y_1) \in \mathcal{F}_a \times \mathcal{F}_a \times [1, M]^4; f(x, y) = f_1(x_1, y_1)\} \ll (RM)^{\varepsilon}M^2|\mathcal{F}_a|.$$ 

(1.3)

Indeed, fix $f \in \mathcal{F}_a$ and $x, y$. The integer $z = f(x, y)$ is at most $RM^2$ and is represented by at most $2^{\omega(z)}$ classes with discriminant $-4a^2$. From the preceding, there are at most $O(1)2^{\omega(z)}$ possibilities for $f_1 \in \mathcal{F}_a$ and for each $f_1, f_1(x_1, y_1) = z$ holds for at most $(RM)^{\varepsilon}$ values of $(x_1, y_1)$. This establishes (1.3).

2. Application of the circle method

Let $R$ and $\mathcal{F}$ be as in §1. Let $P$ be large and assume $R < (\log P)^C$.

For $f \in \mathcal{F}$, let $\omega_f$ be the image measure on $\mathbb{Z}$ of $[1, P]^2$ under the map

$$(x, y) \mapsto f_a(x, y) \text{ with } f_a = f - a, -4a^2 = \text{disc } f$$

under restriction $(x, y) = 1$.

Actually it is technically more convenient to consider the image measure of

$$1_{(x, y) = 1}\gamma\left(\frac{x}{P}\right) \otimes \gamma\left(\frac{y}{P}\right),$$

where $0 \leq \gamma \leq 1$ is a smooth bumpfunction supported on $[0, 1]$. 

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Note that \( \text{supp} \omega_f \subset [-RP^2, RP^2] \) (by construction of \( \mathcal{F} \)) and we may assume \( \text{supp} \omega_f \subset [0, RP^2] \) (by assuming \( A > 0 \)). Define
\[
\omega = \sum_{f \in \mathcal{F}} \omega_f \quad \text{where} \quad \sum_{f \in \mathcal{F}} = \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}}.
\]

The co-primality condition \((x, y) = 1\) leads to technical complications (with no effect on the basic scheme of the argument).

Fix some integer \( B = (\log P)^{10} \) and replace the restriction \( 1 \equiv (x, y) = 1 \) by
\[
\sum_{d \mid (x, y), d < B} \mu(d).
\]
This expression equals 1 if \((x, y) = 1\), vanishes if \( 1 < (x, y) < B \) and is bounded by \( \tau((x, y)) \). Hence this replacement introduces an error at most \( \sum_{x, y < P} \tau((x, y)) \ll \frac{P^2}{B}(\log P)^2 \) for the counting functions \( \omega_f \) and \( \omega \) (in \( \ell^1(\mathbb{Z}) \)-norm), which is harmless.

With the above modification, we obtain
\[
\hat{\omega}(\theta) = \sum_{d < B} \mu(d) \left\{ \sum_{\mathcal{F}} \sum_{d \mid (x, y)} \gamma\left(\frac{x}{P}\right) \gamma\left(\frac{y}{P}\right) e\left(f_a(x, y)\theta\right) \right\}
\]
and
\[
|\hat{\omega}(\theta)|^2 \ll \sum_{d < B} d^{1+\epsilon} \left| \sum_{\mathcal{F}} \sum_{d \mid (x, y)} \gamma\left(\frac{x}{P}\right) \gamma\left(\frac{y}{P}\right) e\left(f_a(x, y)\theta\right) \right|^2.
\]

For \( q < RP \), denote for \((q, b) = 1\)
\[
\mathcal{M}(q, b) = \left[ \left| \theta - \frac{b}{q} \right| < \frac{1}{qRP} \right] \subset \mathbb{T} = \mathbb{R}/\mathbb{Z}.
\]

Our main concern is to obtain suitable bounds on
\[
\sum_{q \sim Q} \sum_{(b, q) = 1} \int_{\mathcal{M}(q, b)} |S_\omega(\theta)|^2 d\theta.
\]
Let $\theta = \frac{b}{q} + \varphi \in \mathcal{M}(q, b), |\varphi| < \frac{1}{d_1q}$. Then

$$f_a(x, y)\theta = f_a(x, y)\frac{b}{q} + f_a(x, y)\varphi$$

and

$$S_{\omega_f}(\theta) = \sum_{d|(x,y)} \gamma \left( \frac{x}{P} \right) \gamma \left( \frac{y}{P} \right) e(f_a(x, y)\theta) =$$

$$\sum_{0 \leq k, \ell < q \atop d_0|(k, \ell)} e(f_a(k, \ell)\frac{b}{q}) \left[ \sum_{x \equiv k, y \equiv \ell \pmod{q}} e(f_a(x, y)\varphi) \gamma \left( \frac{x}{P} \right) \gamma \left( \frac{y}{P} \right) \right]$$  \hspace{1cm} (2.4)

where $d_0 = (d, q), d = d_0d_1$.

We distinguish 2 cases.

**Case I.** $q < P$.

Rewrite the second factor in (2.4) as

$$\sum_{r, s \leq \frac{q}{d_1q} \atop rq+k \equiv sq+\ell \equiv 0 \pmod{d_1}} \gamma \left( \frac{rq+k}{P} \right) \gamma \left( \frac{sq+k}{P} \right) e(f(qk, sk + \ell)\varphi)$$

(dropping a multiplicative factor)

$$= \sum_{r', s' \leq \frac{q}{d_1q}} \gamma \left( \frac{r'd_1q+k'}{P} \right) \gamma \left( \frac{s'd_1q+k'}{P} \right) e(f(qd_1k', sd_1q+k')\varphi)$$

with $0 \leq k', \ell' < d_1q, k' \equiv k \pmod{q}, \ell' \equiv \ell \pmod{q}, k' \equiv \ell' \equiv 0 \pmod{d_1}$.

From the Poisson summation formula, we obtain

$$\frac{1}{q^2d_1^2} \sum_{m, n \in \mathbb{Z}} \int \int \gamma \left( \frac{y+k'}{P} \right) \gamma \left( \frac{z+\ell'}{P} \right) e(f(y+k', z+\ell')\varphi) e\left( m \frac{y}{qd_1} + n \frac{z}{qd_1} \right) dy dz$$

$$= \frac{1}{q^2d_1^2} \sum_{m, n \in \mathbb{Z}} J_f(qd_1, m, n, \varphi) e_q(-md_1k - nd_1\ell)$$  \hspace{1cm} (2.5)

where $d_1d_1 \equiv 1 \pmod{q}$ and

$$J_f(q, m, n, \varphi) = \int \int \gamma \left( \frac{y}{P} \right) \gamma \left( \frac{z}{P} \right) e(f(y, z)\varphi) e\left( m \frac{y}{q} + n \frac{z}{q} \right) dy dz.$$  \hspace{1cm} (2.6)
Note that, by stationary phase
\[ |J_f(q, m, n, \varphi)| \lesssim \frac{1}{|\varphi||\text{discr}(f)|^{1/2}} \lesssim \frac{1}{R|\varphi|}. \] (2.7)

Also, in (2.5) the significant contributions come from values \( m, n \) satisfying
\[ \left| \frac{m}{qd_1}, \frac{n}{qd_1} \right| \lesssim |\nabla f|, |\varphi| \lesssim R.P. \frac{1}{QRP} \] (2.8)

hence \( |m|, |n| = 0(d_1) \).

From (2.7), there is an obvious bound \( 0\left(\frac{1}{q^2 R|\varphi|}\right) \) on (2.5) and \( 0\left(\frac{1}{R|\varphi|}\right) \) on (2.4), (2.2). Substituting in (2.3), we see that the contribution of \( |\varphi| > \varphi_* \) is at most
\[ \frac{B^2 Q^2}{R^2 \varphi_*} < \frac{P^2}{R} (\log P)^{-C} \text{ provided } \varphi_* > (\log P)^C Q^2 \frac{P}{P^2}. \]

If \( Q < P^{1/2} \), we may therefore restrict \( |\varphi| < P^{-\frac{3}{2}} \) and obtain \( m = n = 0 \) in (2.8). Thus we distinguish the ranges
\( Q < P^{1/2} \). Then (2.4) is replaced by
\[ \frac{1}{q^2 d_1^2} \left[ \sum_{0 \leq k, \ell < q} e\left(f_a(k, \ell) \frac{b}{q}\right) \right] J_f(\varphi) \] (2.9)

contributing in (2.3) for
\[ \frac{1}{Q^2} \sum_{q \sim Q} \frac{1}{d_1^2} \sum_{(b, q) = 1} \left| \sum_{\mathcal{F}} c_f \sum_{0 \leq k, \ell < q} e\left(f_a(k, \ell) \frac{b}{q}\right) \right|^2 \int_{|\varphi| < \frac{P}{P^2}} \left(\min \left(P^2, \frac{1}{R|\varphi|}\right)\right)^2 d\varphi \] (2.10)

(where \( |c_f| \leq 1 \)).

Summing over \( d \), we obtain
\[ \frac{P^2}{R} \sum_{d_0} d_0^{1+\varepsilon} \left\{ \sum_{q \sim Q} \sum_{(b, q) = 1} \left| \sum_{\mathcal{F}} c_f S_{f, d_0}(q, b, 0, 0) \right|^2 \right\} \] (2.11)

denoting
\[ S_f(q, b, m, n) = \frac{1}{q^2} \sum_{0 \leq k, \ell < q} e_q(bf_a(k, \ell) - mk - n\ell) \] (2.12)
\[ S_{f, d_0}(q, b, m, n) = \frac{1}{q^2} \sum_{0 \leq k, \ell < q} e_q(bf_a(k, \ell) - mk - n\ell) \] (2.12')
and where \(d_0 < B\) is a square-free divisor of \(q\).

\[
\frac{P^{16}}{\psi} \leq Q \leq P
\]

By (2.5), (2.4) becomes

\[
\frac{1}{d_1^2} \sum_{\left| m \right|, \left| n \right| < 0(d_1)} S_{f,d_0}(q, b, m d_1, n d_1) J_f(qd_1, m, n, \varphi) \tag{2.13}
\]

and the contribution in (2.3) is at most

\[
B^{2+\epsilon} \frac{P^2}{R} \max_{d_0, d_1, m, n} \sum_{q \sim Q} \left\{ \sum_{d_0 \mid q} \left| c_{f,q} S_{f,d_0}(q, b, m d_1, n d_1) \right|^2 \right\} \tag{2.14}
\]

with \(|c_{f,q}| \leq 1, d, d_1\) square free, \((d_0, d_1) = 1, d_0 d_1 < B\) and \(m, n = 0(d_1)\).

**Case II.** \(P < Q < PR\)

Since for \(\theta = \frac{b}{q} + \varphi \in M(q, b)\)

\[
f_a(x, y) \theta = f_a(x, y) \frac{b}{q} + f_a(x, y) \varphi \quad \text{with} \quad |f_a(x, y)| \varphi < R, P^2 \frac{1}{RPQ} < O(1)
\]

we may replace \(\theta\) by \(\frac{b}{q}\) (dropping \(\varphi\)) and \(S_\omega(\theta)\) becomes

\[
\sum_{\mathcal{F}} \sum_{1 \leq k, \ell \leq P} e_q(b f_a(k, \ell)) \gamma \left( \frac{k}{P} \right) \gamma \left( \frac{\ell}{P} \right). \tag{2.15}
\]

Thus the inner sum in (2.15) equals

\[
\sum_{d_0 \mid (k, \ell)} e_q(b f_a(d_1 k, d_1 \ell)) \gamma \left( \frac{d_1 k}{P} \right) \gamma \left( \frac{d_1 \ell}{P} \right)
\]

and completing the sum, we obtain

\[
e_q(-ab) \sum_{\left| u \right|, \left| v \right| < \frac{Q}{d_1} \ b f(x, y) + ux + vy) \]

\[
= q^2 \sum_{\left| u \right|, \left| v \right| < \frac{Q}{d_1}} S_{f,d_0}(q, b, d_1 u, d_1 v). \tag{2.16}
\]
The contribution to (2.3) may be bounded by
\[
\frac{B^{2+\epsilon}Q^3}{RP} \sum_{q \sim Q} \sum_{(b,q) = 1} \left| \sum_{u,v < \frac{Q}{d_1}} \sum_{d \in \mathcal{D}} S_{f,d_0}(q,b,d_1u,d_1v) \right|^2
\]
with \(d_0, d_1\) square free, \((d_0,d_1) = 1\) and \(d_0d_1 < B\).

(3). Evaluation of the Gauss sum

Analyzing further (2.11), (2.14), (2.17), we evaluate expressions of the form
\[
\sum_{(b,q) = 1} S_f(q,b,u,v) S_{f_1}(q,b,u_1,v_1)
\]
and also
\[
\sum_{(b,q) = 1} S_{f,d_0}(q,b,u,v) S_{f_1,d_0}(q,b,u_1,v_1).
\]

Consider first (3.1)

Factoring \(q\) as a product of prime power \(p^r\), (3.1) factors correspondingly. Recall that
\[
S_f(q,b,u,v) = \frac{1}{q^2} \sum_{0 \leq x,y < q} e_q(bf(x,y) + ux + vy - ba)
\]
with \(f(x,y) = Ax^2 + 2Bxy + Cy^2, (A,B,C) = 1\).

Let \(q = p^r\) be a prime power. We may assume \((A,p) = 1\).

Write (assuming \(p \neq 2\); for \(p = 2\) there are some extra technicalities that we omit here and to which we will return in §7) and using the notation \(\tau\) for the multiplicative inverse \((\text{mod } q)\)
\[
bf(x,y) + ux + vy - ab = b(Ax^2 + 2Bxy + Cy^2) + ux + vy - ab \\
= bA(x + B\bar{A}y)^2 + ba^2\bar{A}y^2 + ux + vy - ab \\
= bA(x + B\bar{A}y + \frac{2bA}{u}u)^2 + ba^2\bar{A}y^2 + (v - B\bar{A}u)y - \frac{4bA}{u^2}u^2 - ab \quad (\text{mod } q).
\]
Hence, by Gauss sum evaluation, we obtain (cf. [BEW])
\[
S_f(q,b,u,v) \sim \frac{1}{q^{3/2}} \left( \frac{bA}{q} \right) e_q(-4Abu^2 - ab) \left[ \sum_{0 \leq y < q} e_q(ba^2\bar{A}y^2 + (v - B\bar{A}u)y) \right]. \quad (3.3)
\]
Let $a^2 = \tilde{a}$ with $(\tilde{a}, q) = 1$. Thus writing $y = \tilde{y} + zq$, $0 \leq \tilde{y} < \tilde{q}$, $0 \leq z \leq \frac{q}{q} = (q, a^2)$

$$\sum_{0 \leq y < q} \cdots = \sum_{0 < \tilde{y} < \tilde{q}} \sum_{0 \leq z < (a^2, q)} e\left(bA \frac{\tilde{a}(\tilde{y})^2}{q} + (v - B\tilde{A}u)\frac{\tilde{y}}{q} + (v - B\tilde{A}u)\frac{z}{(a^2, q)}\right)$$

(3.4)

and (3.4) = 0 unless $(a^2, q)|v - B\tilde{A}u$, in which case we set

$$v - B\tilde{A}u = (v - B\tilde{A}u)^\sim (a^2, q)$$

Thus (3.4) becomes

$$(a^2, q) \sum_{0 \leq y < \tilde{q}} e\tilde{q} (b\tilde{A}\tilde{a}y^2 + (v - B\tilde{A}u)^\sim y)$$

$$= (a^2, q) \sum_{0 \leq y < \tilde{q}} e\tilde{q} (b\tilde{A}\tilde{a}(y + (Av - Bu)^\sim 2\bar{b}\tilde{a}^2) - 4\bar{b}\tilde{a} A((Av - Bu)^\sim)^2)$$

$$\sim (a^2, q)(\tilde{q})^{1/2} \left(\frac{b\tilde{A}\tilde{a}}{\tilde{q}}\right) e\tilde{q} (-4\bar{b}\tilde{a} A((Av - Bu)^\sim)^2).$$

(3.5)

Hence, from (3.3), (3.5)

$$S_f(q, b, u, v) = \frac{(a^2, q)^{1/2}}{q} \left(\frac{bA}{q}\right) \left(\frac{b\tilde{A}\tilde{a}}{\tilde{q}}\right) e\tilde{q}(-4\bar{A}b u^2 - ab) e\tilde{q}((-4\bar{b}\tilde{a} A((Av - Bu)^\sim)^2)$$

$$= \begin{cases} \frac{(a^2, q)^{1/2}}{q} e\tilde{q}(-4\bar{A}b u^2 - ab) e\tilde{q}((-4\bar{b}\tilde{a} A((Av - Bu)^\sim)^2) & \text{if } \tilde{q} \neq 1 \\ \frac{(a^2, q)^{1/2}}{q} \left(\frac{bA}{q}\right) e\tilde{q}(-4\bar{A}b u^2 - ab) & \text{if } q|a^2. \end{cases}$$

(3.6)

It follows that

$$(3.1) = \frac{(a^2, q)^{1/2}(a_1, q)^{1/2}}{q^2} \sum_{(b, q) = 1} e\tilde{q} \left(\frac{4\bar{A}u_1^2 - 4\overline{A}u^2}{\tilde{b}} + (a_1 - a)b\right) E(b) E_1(b)$$

(3.7)

where $E(b)$ is either

$$\left(\frac{bA}{q}\right) \text{ or } e\tilde{q}((-4\bar{A}a((Av - Bu)^\sim)^2) \tilde{b} \text{ with } \tilde{q} = \frac{q}{(q, a^2)}$$

(3.8)
where we assume \((q, a^2)|Av - Bu\) and similarly for \(E_1(b)\).

Thus the sum in (3.7) is of Kloosterman or Salié-type.

There is the following elementary estimate (which suffices for our needs)

\[
\left| \sum_{(b,p)=1}^{0<b<p} e_{p^r}(cb + db) \right| < (p^r)^{3/4} (p^r, c,d)^{1/4}
\]  
(3.9)

and also

\[
\left| \sum_{(b,p)=1}^{0<b<p} \left( \frac{b}{p} \right) e_{p^r}(cb + db) \right| < (p^r)^{3/4} (p^r, c,d)^{1/4}.
\]  
(3.10)

Hence we may state the following bound for \(q = p^r\)

\[
(3.7) < (a^2, q)^{1/2} (a_1^2, q)^{1/2} (q, a - a_1)^{1/4} q^{-5/4}
\]  
(3.11)

and also, for \(a = a_1\)

\[
(3.7) < q^{-5/4} (a^2, q) \left( q, A_1 u_1 - 4A_1 u_2 + (q, a^2) \right) A_1 \left( u_1 - B_1 u_1 \right)^2 - (q, a^2) \left( Av - Bu \right)^2 \right)^{1/4}
\]  
(3.12)

where \((a^2, q)|Av - Bu, (a^2, q)|A_1 v_1 - B_1 u_1\) and \(\bar{a} = \frac{a^2}{(a^2, q)}, (Av - Bu)^{\sim} = \frac{Av - Bu}{(a^2, q)}\).

Consider next (3.1’) for which there is again factorization.

If \(q = p^r\) and \(p|d_0\), then

\[
S_{f,p}(p^r, b, u, v) = p^{-2r} e_{p^r}(-ba) \sum_{0 \leq x, y < p^{r-1}} e_{p^{r-1}}(bf(x, y)p + ux + vy).
\]  
(3.13)

Note that by assumption (0.24) and since \(p < (\log P)^{10}, (a, p) = 1\).

We distinguish several cases

\(3.14 \ r = 1\)

Then \(|(3.13)| = p^{-2}\) and \(|(3.1’)\| = (p - 1)p^{-4}\), which is certainly bounded by

\[
p^{-\frac{7}{2}} \ \text{(3.11)}
\]  
(3.15)
and
\[ p^{-\frac{3}{4}} (3.12). \] (3.16)

(3.17) \( r \geq 2 \)

Clearly (3.13) vanishes unless \( p|(u, v) \). Writing \( u = pu_1, v = pv_1 \),
\[ (3.13) = e_{pr} (-ba) p^{-2} p^{-2(r-2)} \left[ \sum_{0 \leq x, y < p^{r-2}} e_{p^{r-2}} (bf(x, y) + u_1x + v_1y) \right]. \] (3.18)

If \( r > 2 \), repeating the analysis of the exponential sum with \( q \) replaced by \( q_1 = p^{-2}q \),
we obtain instead of (3.6)
\[ \frac{1}{q} e_{q_1} (-4Abu^2 - 4ba^2 A((Av - Bu)^2) e_q(-ab). \] (3.19)

This gives for (3.1) the bound
\[ (q, a - a_1)^{1/4} q^{-5/4}. \] (3.20)

Factor
\[ q = \prod_{(p, d_0) = 1} p^r \cdot \prod_{p|d_0, r=1} p \cdot \prod_{p|d_0, r>1} p^r \]
and consider the corresponding factorization of (3.1'). Apply (3.11), (3.12) if \( (d, p) = 1 \),
(3.15), (3.16) if \( p|d_0, r = 1 \) and (3.20) if \( p|d_0, r > 1 \).

(4). Estimation of (2.11), (2.14)

Expressing the square of the inner sum and carrying out the summation over \( b \), we evaluate (3.1), (3.1') by (3.11), (3.15), (3.20).

Hence for \( \sum_{(b, q) = 1} | \cdots |^2 \) we obtain the bound
\[ Q^{-5/4} \sum_{f, f_1 \in F} (a^2, q)^{1/2} (a_1^2, q)^{1/2} (q, a - a_1)^{1/4} \left( \prod_{p|d_0} p^{-7/4} \right). \] (4.1)

Substitution in (2.11) gives
\[ \frac{P^2}{RQ^{5/4}} \sum_{f, f_1 \in F} \sum_{d_0} \sum_{q' \sim Q} (a^2, q)^{1/4} (a_1^2, q)^{1/4} (a - a_1, q)^{1/4} \left( \prod_{p|d_0} p^{-\frac{3}{4} + \epsilon} \right) \left( \prod_{p|d_0} p^{1+\epsilon} \right) \]
\[ \leq \frac{P^2}{RQ^{5/4}} \sum_{f, f_1 \in F} \sum_{q' \sim Q} (a^2, q_1)^{1/4} (a_1^2, q_1)^{1/4} (a - a_1, q_1)^{1/4} q_0^{\frac{5}{2} + \epsilon} \] (4.2)
where \( q = q_1d_{0,2} \) and \( d_{0,2} = \prod_{p|d_0} p \). Note that \( d_0|q_1 \) and hence for given \( q_1 \), there are at most \( \min(q_1^\varepsilon, B^{1+\varepsilon}) \) possibilities for \( d_0,1, d_0,2 \) and \( q \).

Specify \( q_1 \sim Q_1 \) where \( Q_1 < Q < \min(BQ_1, Q_2^2) \) and replace (4.2) by

\[
\frac{P^2}{RQ_1^{5/4}} \min(B^2, Q_1^\varepsilon) \sum_{f, f_1 \in F, q_1 \sim Q_1} \sum (a^2, q_1)^{1/2}(a_1^2, q_1)^{1/2}(a - a_1, q_1)^{1/2}. \tag{4.3}
\]

Next, we evaluate

\[
\frac{P^2}{RQ_1^{5/4}} \sum_{f, f_1 \in F} \sum_{q \sim Q} (a^2, q)^{1/2}(a_1^2, q)^{1/2}(a - a_1, q)^{1/4}. \tag{4.4}
\]

The contribution for \( a = a_1 \) may be bounded by

\[
\frac{P^2}{RQ} \frac{1}{|F|^2} \sum_{a \in A} |F_a|^2 \sum_{q \sim Q} (a^2, q) \leq \frac{P^2}{R Q_1^{1-\varepsilon}} \frac{1}{|F|^2} \sum_a |F_a|^2 \leq \frac{P^2}{R} R^{-\frac{1}{4}+\varepsilon}. \tag{4.5}
\]

For \( a \neq a_1 \), we obtain

\[
\frac{P^2}{RQ_1^{5/4}} \frac{1}{|F|^2} \sum_{a, a_1 \in A, a \neq a_1} |F_a| \sum_{q \sim Q} (a^2, q)^{1/2}(a_1^2, q)^{1/2}(a - a_1, q)^{1/4}. \tag{4.6}
\]

Clearly, for fixed \( a \neq a_1 \)

\[
\sum_{q \sim Q} (a^2, q)^{1/2}(a_1^2, q)^{1/2}(a - a_1, q)^{1/4} \leq \sum_{d|a^2, d_1|a_1^2} d^{1/2}d_1^{1/2} \frac{Q_1^{5/4}}{[d, d_1]} \leq Q_1^{5/4} \tag{4.7}
\]

since \( a, a_1 \) are pseudo-prime by (0.24).

Also

\[
\sum_{q \sim Q} (a^2, q)^{1/2}(a_1^2, q)^{1/2}(a - a_1, q)^{1/4} \leq \sum_{q \sim Q} (a, q)(a_1, q)(a - a_1, q)^{1/4}
\]

\[
\leq \sum_{d|a, d_1|a_1, d' | a-a_1} d^{1/4} \frac{Q}{[d, d_1, d']} \tag{4.8}
\]

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Denote \( f = (d, d_1) \). Thus \( f \mid d' \) and since \( f^{-1}d, f^{-1}d_1, f^{-1}d' \) are pairwise coprime, 
\([d, d_1, d'] \geq f^{-3}dd_1d' \) and therefore

\[
(4.8) \lesssim Q(a, a_1)^3 \sum_{d'|a-a_1 \atop d' \leq Q} (d')^{-3/4} . \tag{4.9}
\]

From (4.7), (4.9)

\[
(4.6) \lesssim \frac{P^2}{R} \frac{1}{|F|^2} \sum_{a, a_1 \in A \atop a \neq a_1} |F_a| |F_{a_1}| \min \left(1, Q^{-\frac{3}{4}}(a, a_1)^3 \sum_{d'|a-a_1 \atop d' < Q} \left(\frac{1}{d'}\right)^{\frac{3}{4}}\right) . \tag{4.10}
\]

We distinguish two cases.

Assume \((a, a_1) = \Delta > Q^{10^{-4}}\). Estimate

\[
(4.10) \lesssim \frac{P^2}{R} \frac{1}{|F|^2} \sum_{a \in A} |F_a| \sum_{\Delta | a, \Delta > Q^{10^{-4}}} \sum_{a_1 \equiv a \pmod{\Delta}} |F_{a_1}| . \tag{4.11}
\]

Again, since \( a \) is pseudo-prime, \( \Delta \) is restricted to \( 0(1) \) values, once \( a \) fixed. From (0.25),

\[
|\{f \in F; a \equiv u \pmod{\Delta}\}| \lesssim \Delta^{-\frac{1}{320}} |F|
\]

and hence

\[
(4.11) < \frac{P^2}{R} Q^{-10^{-7}} . \tag{4.12}
\]

Assume \((a, a_1) \leq Q^{10^{-4}}\). Then

\[
(4.10) \lesssim \frac{P^2}{R} Q^{-\frac{4}{7} + 3.10^{-4}} \frac{1}{|F|^2} \sum_{a, a_1 \in A \atop a \neq a_1} |F_a| |F_{a_1}| \sum_{d'|a-a_1 \atop d' < Q} \left(\frac{1}{d'}\right)^{3/4}
\]

\[
\lesssim \frac{P^2}{R} Q^{-\frac{4}{7} + 3.10^{-4}} \sum_{d' < Q} \left(\frac{1}{d'}\right)^{3/4} \left(\frac{1}{d'}\right)^{\frac{3}{320}}
\]

\[
< \frac{P^2}{R} Q^{-\frac{1}{100}} . \tag{4.13}
\]

From (4.5), (4.12), (4.13), we obtain

\[
(4.4) < \frac{P^2}{R} (R^{-\frac{4}{7}} + Q^{-10^{-7}}) . \tag{4.14}
\]
We assume $R \sim (\log P)^C$ with $C$ a sufficiently large constant.

Since $B \sim (\log P)^{10}$, it follows from (4.14) that

$$ (4.3) < \frac{P^2}{R} \left( R^{-\frac{1}{2}} + Q^{-10-s} \right). \quad (4.15) $$

The same bound also holds for (2.14).

(5). Estimation of (2.17)

Expressing the square of the inner sums in (2.17) gives

$$ \frac{B^{2+\varepsilon}Q^3}{RP} \sum_{q \sim Q} \sum_{f, f_1 \in F} \sum_{u, v, u_1, v_1 < \frac{Q}{d_1} (b, q) = 1} S_{f, d_0}(q, b, d_1 u, d_1 v) \overline{S}_{f_1, d_0}(q, b, d_1 u_1, d_1 v_1) \quad (5.1) $$

where the inner sum is of type (3.1’).

Note that here $d_0, d_1$ are fixed.

To estimate the contribution for $a \neq a_1$, use again the bound (3.11) on (3.1) (which from previous discussion is always valid). We obtain

$$ B^{2+\varepsilon}Q^\frac{7}{4} \frac{1}{RP} \sum_{f, f_1 \in F} \sum_{q \sim Q} (a^2, q) \frac{1}{2} (a_1^2, q) \frac{1}{2} (a - a_1, q)^{1/4} \ll \frac{1}{P} Q^{\frac{11}{4}} R^\frac{3}{2} B^{2+\varepsilon} < P^{7/4} R^5 < \frac{P^2}{R} R^{-1}. \quad (5.2) $$

Next the $a = a_1$ contribution

Let $q = q_1 q_2$ with $q_2 = \prod_{p|d_0, r \geq 1} p^r$ and factor (3.1’) according to $q = \prod_{r} p^r$. If $p|q_1$, the bound (3.12) applies to (3.1’) with $q = p^r$.

For $p|q_2$, apply (3.11) which gives the bound $q_2^{-1}$ on the $q_2$-factor. Let $q_1 \sim Q_1, q_2 \sim Q_2$ (noting that, by construction, the number of $q_2$-values is at most $Q_2^{\frac{5}{2}}$). Thus the contribution to (5.1) may be bounded by

$$ \frac{B^{2+\varepsilon}Q^3 Q_2^{5/2}}{RP} \sum_{q_1 \sim Q_1} \frac{1}{|F|^2} \sum_{a \in A} \sum_{f, f_1 \in F_a} \sum_{u, v, u_1, v_1 < \frac{Q}{d_1} (b, q) = 1} \quad (5.3) \quad (5.4) $$
with
\[
(5.3) = \prod_{p|q_1} \sum_{(b,p)=1} S_{f,d_0,1}(p^r, b, d_1 u, d_1 v)S_{f_1,d_0,1}(p^r, b, d_1 u_1, d_1 v_1)
\]
and \(d_{0,1} = \prod_{p|d_0, r=1} p\).

Because \((d_0, d_1) = 1\), the factor \(d_1\) in (5.3) turns out to be irrelevant and we drop it for simplicity.

Applying (3.12) for each prime \(p|q\), we obtain
\[
|\langle 5.3 \rangle| \leq (a^2, q_1)Q_1^{-5/4} \prod_{p|q_1} (p^r, 4A_1 u_1^2 - 4Au^2 + (p^r, a^2)4A_1 a((A_1 v_1 - B_1 u_1)^2 - (p^r, a^2)4A_1 a((Av - Bu)^2)^{1/4}).
\]

(5.5)

We distinguish several cases for the factors in (5.5).

Assume
\[
(p^r, 4A_1 u_1^2 - 4Au^2 + (p^r, a^2)4A_1 a((A_1 v_1 - B_1 u_1)^2 - (p^r, a^2)4A_1 a((Av - Bu)^2) > p^{\lceil \frac{5}{2} \rceil}.
\]

(5.6)

Multiplying with \(a^2\) gives then
\[
4Au^2a^2 + \bar{a}4Aa(Av - Bu)^2 \equiv 4A_1 u_1^2a^2 + \bar{a}4A_1 a(A_1 v_1 - B_1 u_1)^2 \pmod{p^{\lceil \frac{5}{2} \rceil}+1}
\]
and since \(\bar{a}\bar{a} \equiv 1\pmod{\frac{p^r}{(a^2, p^r)}}\) and \(Av - Bu \equiv A_1 v_1 - B_1 u_1 \equiv 0 \pmod{(a^2, p^r)}\)
\[
\bar{A}u^2a^2 + \bar{A}(Av - Bu)^2 \equiv \bar{A}_1 u_1^2a^2 + \bar{A}_1 (A_1 v_1 - B_1 u_1)^2 \pmod{p^{\lceil \frac{5}{2} \rceil}+1}.
\]

(5.7)

Hence, since \(a^2 = AC - B^2 = A_1 C_1 - B_1^2\),
\[
f(v, -u) \equiv f_1(v_1, -u_1) \pmod{p^{\lceil \frac{5}{2} \rceil}+1}.
\]

(5.8)

Since \(|A|, |B|, |C|, |u|, |v| \lesssim R\), it follows that either
\[
f(v, -u) = f_1(v_1, -u)
\]

(5.9)
or (5.8) can only hold for a set \(\sigma\) of primes \(p|q\) such that
\[
\prod_{p \in \sigma} p^{\lceil \frac{5}{2} \rceil}+1 < R^4.
\]

(5.10)
Hence, if (5.9) fails,

\[(5.5) < (a^2, q_1) Q_1^{-5/4} Q_1^{1/8} R^2. \quad (5.11)\]

The contribution to (5.4) is at most

\[
\frac{B^{2+\varepsilon}}{PR} Q_1^2 Q_2^{5/2} \frac{1}{|F|^2} \sum_{a \in A} |F_a|^2 \sum_{q \sim Q_1} (a^2, q) Q_1^{-9/8} R^2 < \frac{B^{2+\varepsilon} R^3}{P} Q_1^{23} < \frac{P^{15}}{R^7} < \frac{P^2}{R^2}. \quad (5.12)
\]

Finally, the contribution of \(a = a_1\) and \((u, v, u_1, v_1)\) satisfying (5.9). We obtain

\[
\frac{B^{2+\varepsilon} Q_1^2 Q_2^{5/2}}{RP} \frac{1}{|F|^2} \left( \frac{P}{Q d_1} \right)^4 \sum_{q \sim Q_1} \sum_{a \in A} (a^2, q). \quad (5.13)
\]

with

\[
(5.13) = \left| \left\{ (f, f_1, u, v, u_1, v_1) \in F_a^2 \times \left[ 1, \frac{Q d_1}{P} \right]^4 : f(u, v) = f(u_1, v_1) \right\} \right|.
\]

Recalling (1.3), it follows that

\[(5.13) \ll R^\varepsilon Q_1^2 B^2 |F_a|\]

and

\[
(5.14) \ll R^\varepsilon B^2 \frac{Q_1^2 Q_2^{5/2}}{RP} \frac{1}{|F|^2} \frac{P^2}{Q^2} \sum_{q \sim Q_1} \sum_{a \in A} (a^2, q) |F_a| < B^2 \frac{QP}{R^{1-\varepsilon}|F|} \ll \frac{P^2}{R^{11/10}}, \quad (5.15)
\]

(under proper assumption on \(R\)).

Hence from (5.2) and (5.15)

\[(2.17) < \frac{P^2}{R} R^{-\frac{1}{10}}. \quad (5.16)\]

(6). Minor arcs estimate

From (4.15), (5.16), we obtain the following bound on (2.3)

\[(2.3) < \frac{P^2}{R} \left( R^{-\frac{1}{10}} + Q^{-10^{-8}} \right). \quad (6.1)\]
Take $R = (\log P)^{10^{10}}$ and $1 \ll Q_0 \leq R$.

Summing (6.1) over $Q > Q_0$ gives

$$\int \bigcup_{q > Q_0} \bigcup_{(b,a)=1} M(q,b) |S_\omega(\theta)|^2 d\theta < \frac{P^2}{R} Q_0^{-10^{-8}}. \quad (6.3)$$

Returning to the definition of $M(q,b)$ in §2, we may further reduce the arcs $M(q,b)$, defining for $q < Q_0$, $(b,q) = 1$

$$M_0(q,b) = \left[ \left| \theta - \frac{b}{q} \right| < \frac{Q_0^2}{RP^2} \right]. \quad (6.4)$$

It follows from (2.4) and (2.7) that for $\theta \in M(q,b), \theta = \frac{b}{q} + \varphi$

$$|S_\omega(\theta)| \lesssim \frac{1}{\sqrt{qR}|\varphi|} \quad (6.5)$$

and therefore

$$\int_{M(q,b) \setminus M_0(q,b)} |S_\omega(\theta)|^2 \lesssim \frac{1}{qR^2} \int_{|\varphi| > \frac{Q_0^2}{RP^2}} \frac{1}{|\varphi|^2} < \frac{P^2}{RQ_0^2q} \quad (6.6)$$

which collected contribution is at most $\frac{P^2}{RQ_0^2}$. Therefore

$$\int_{\mathbb{T} \setminus \bigcup_{q \leq Q_0} \bigcup_{(b,a)=1} M_0(q,b)} |S_\omega(\theta)|^2 d\theta < \frac{P^2}{R} Q_0^{-10^{-8}}. \quad (6.7)$$

where $1 \ll Q_0 < (\log P)^{10^{10}}$ is a parameter.

(7). Contribution of the major arcs

Let

$$1 \ll Q_{00} < (\log P)^{1\frac{1}{2}} \quad (7.1)$$

and $Q_1 = \prod_{q \mid Q_{00}} q$. Let

$$M = \frac{P^2}{Q_1} \quad (7.2)$$
and $\lambda$ a smooth even function on $\mathbb{R}$, supp $\lambda \subset [-1, 1], 0 \leq \lambda \lesssim \frac{1}{M}$ s.t.
\[
\sum_m \lambda \left( \frac{m}{M} \right) = 1. \tag{7.3}
\]

Let $\nu$ be the distribution on $\mathbb{Z}$ defined by
\[
\nu(n) = \sum_{m \in \mathbb{Z}} \lambda \left( \frac{m}{M} \right) \omega(n - mQ_1).
\] \tag{7.4}

Hence
\[
\hat{\nu}(\theta) = \left[ \sum_m \lambda \left( \frac{m}{M} \right) e(-mQ_1 \theta) \right] S_\omega(\theta). \tag{7.4'}
\]

Estimate
\[
\int_T |\hat{\nu}(\theta) - S_\omega(\theta)|^2 d\theta = \int \left| 1 - \sum_m \lambda \left( \frac{m}{M} \right) e(-mQ_1 \theta) \right|^2 |S_\omega(\theta)|^2 d\theta
\]
\[
< \frac{P^2}{R} Q_0^{-10^{-s}} + \sum_{q < Q_0} \sum_{(b,q) = 1} \int_{\mathcal{M}_{00}(q,b)} \left| 1 - \sum_m \lambda \left( \frac{m}{M} \right) e(-mQ_1 \theta) \right|^2 |S_\omega(\theta)|^2 d\theta
\]
where we applied (6.7) with $Q_0$ replaced by $Q_{00}$ and denote
\[
\mathcal{M}_{00}(q,b) = \left[ \left| \theta - \frac{b}{q} \right| < \frac{Q_{00}^2}{RP^2} \right].
\tag{7.5}
\]

Since $q|Q_1$ and $\lambda$ is even
\[
\int_{\mathcal{M}_{00}(q,b)} \left| 1 - \sum_m \lambda \left( \frac{m}{M} \right) e(-mQ_1 \theta) \right|^2 |S_\omega(\theta)|^2 d\theta
\]
\[
\overset{(6.5)}{\leq} \int_{|\varphi| < \frac{Q_{00}^2}{qR^2}} \max_{|m| < M} |1 - \cos(mQ_1 \varphi)|^2 \frac{1}{qR^2 \varphi^2} d\varphi
\]
\[
< \frac{M^4 Q_1^4}{qR^2} \left( \frac{Q_{00}^2}{RP^2} \right)^3 < \frac{P^2 Q_{00}^6}{qR^5}
\] \tag{7.2}
and we obtain from the choice of $R, Q_{00}$
\[
\|\hat{\nu} - S_\omega\|_2^2 < \frac{P^2}{R} Q_{00}^{-10^{-s}}. \tag{7.6}
\]
Hence, by Parseval
\[ \sum |\nu(n) - \omega(n)|^2 < \frac{P^2}{R} Q_{00}^{-10^{-8}}. \] (7.7)

Next, from definition of \( \nu \) and \( \omega \), clearly
\[ \nu(n) \sim \frac{1}{M} \sum_{f} \left| \{(x, y) \in [1, P]^2; (x, y) = 1, f_a(x, y) \equiv n \pmod{Q_1} \mbox{ and } |f_a(x, y) - n| < P^2\} \right|. \] (7.8)

Replacing the condition \((x, y) = 1\) by the weaker condition \((x, y, p) = 1\) for \(p < B\) introduces an error at most \(\frac{P^2}{B}\) with respect to the \(\ell_1(\mathbb{Z})\)-norm.

With this replacement, we obtain the following lower bound on (7.8) (since \(Q_1, B \ll P^\epsilon\)).
\[ \frac{1}{M} \sum_{f} \left| \{0 \leq x, y < Q_1; (x, y, Q_{00}) = 1 \mbox{ and } f_a(x, y) \equiv n \pmod{Q_1}\} \right|. \] (7.9)

where (7.9) equals
\[ \text{mes} \left[ |s|, |t| < \frac{P}{Q_1}; \left| f(s, t) - \frac{n}{Q_1^2} \right| < \frac{P^2}{Q_1^2} \right] - \sum_{Q_{00} < p < B} \text{mes} \left[ |s|, |t| < \frac{P}{pQ_1}; \left| f(s, t) - \frac{n}{p^2Q_1^2} \right| < \frac{P^2}{p^2Q_1^2} \right]. \] (7.10')

From the assumption on \(f \in \mathcal{F}\), it follows that taking \(n \sim RP^2\)
\[ \text{mes} \left[ |s|, |t| < \frac{P}{Q_1}; \left| f(s, t) - \frac{n}{Q_1^2} \right| < \frac{P^2}{Q_1^2} \right] \sim \frac{P^2}{Q_1^2 R} \] (7.11)
and similarly for the terms in (7.10'). Hence \(\nu(n)\) may be substituted by
\[ \nu(n) \sim \frac{1}{Q_1 R} \sum_{f \in \mathcal{F}} \left| \{0 \leq x, y < Q_1; f(x, y) \equiv n + a \pmod{Q_1} \mbox{ and } (x, y, Q_{00}) = 1\} \right| \]
\[ = \frac{1}{R} \sum_{f \in \mathcal{F}} \prod_{p < Q_{00}} \left( p^{-r} \left| \{0 \leq x, y < p^r; (x, y, p) = 1 \mbox{ and } f(x, y) \equiv n + a \pmod{p^r}\} \right| \right) \] (7.12)
where \(Q_1 = \prod_{p < Q_{00}} p^r\).

Recalling (0.24), \(a \in \mathcal{A}\) has no prime factors less than \(R^{3/4}\) and hence \((a, Q_1) = 1\).

Let
\[ f(x, y) = Ax^2 + 2Bxy + Cy^2 \in \mathcal{F} \mbox{ where } a^2 = AC - B^2. \]

Fixing $p < Q_{00}$, we may, since $(A, B, C) = 1$, assume $(A, p) = 1$ (the other cases are similar, except for $p = 2$, $A, C$ even and $B$ odd; we leave the adjustment to the reader).

The equation

$$f_a(x, y) \equiv n + a \pmod{p^r}$$

becomes

$$(Ax + By)^2 + a^2y^2 \equiv A(n + a) \pmod{p^r}$$

or equivalently

$$x^2 + y^2 \equiv A(n + a) \pmod{p^r}. \quad (7.13)$$

We seek for a lower bound on the number of solutions of (7.13) with $(x, y, p) = 1$.

Assume $p > 2$ (the local factor at $p = 2$ requires an additional congruence assumption on $n$ at the place $p = 2$).

Clearly, the number of solutions of (7.13) is at least

$$p^{r-1} | \{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^* \times \mathbb{Z}/p\mathbb{Z}; x^2 + y^2 \equiv A(n + a)(\pmod{p})\}|. \quad (7.14)$$

Using Gauss sums, we obtain

$$| \{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^* \times \mathbb{Z}/p\mathbb{Z}; x^2 + y^2 \equiv A(n + a)(\pmod{p})\}|$$

$$= \frac{1}{p} \sum_{b=0}^{p-1} \left[ \sum_{x=1}^{p-1} e_p(bx^2) \right] \left[ \sum_{y=0}^{p-1} e_p(by^2) \right] e_p(- bA(n + a))$$

$$= p - 1 + \frac{1}{p} \sum_{b=1}^{p-1} \left( \zeta \sqrt{p} - \left( \frac{b}{p} \right) \right) \zeta \sqrt{p} e_p(- bA(n + a)) \quad (7.15)$$

with $\zeta = 1$ (resp. $\zeta = i$) if $p \equiv 1(\pmod{4})$ (resp. $p \equiv 3(\pmod{4})$).

Assume

$$(n + a, p) = 1. \quad (7.16)$$

For $p \equiv 1(\pmod{4})$,

$$(7.15) = p - 2 - \frac{1}{\sqrt{p}} \sum_{b=1}^{p-1} \left( \frac{b}{p} \right) e_p(- bA(n + a)) = p - 2 + \alpha, |\alpha| = 1.$$

For $p \equiv 3(\pmod{4})$,

$$(7.15) = p - \frac{i}{\sqrt{p}} \sum_{b=1}^{p-1} \left( \frac{b}{p} \right) e_p(- bA(n + a)) = p + \alpha', |\alpha'| = 1.$$
Hence, if (7.16),

\[ (7.14) \geq \begin{cases} p^r \left( 1 - \frac{3}{p} \right) & \text{if } p \equiv 1 \pmod{4} \\ p^r \left( 1 - \frac{1}{p} \right) & \text{if } p \equiv 3 \pmod{4} \end{cases} \]  

(7.17)

Consequently

\[ (7.12) \gtrsim \frac{1}{R} \sum_{\mathcal{F}} \ast \nu(n) = 1_{[(n+a, p) = 1 \text{ for } 2 < p < Q_00]} \sum_{3 < p < Q_00} \left( 1 - \frac{3}{p} \right) \]  

and

\[ \nu(n) > (\log Q_00)^{-C} R^{-1} \sum_{\mathcal{F}} \ast \nu(n) = 1_{[(n+a, p) = 1 \text{ for } 2 < p < Q_00]} \]  

(7.18)

where \( \sum \ast \) refers to an additional congruence condition on \( n, f \pmod{8} \).

As an immediate consequence of (7.7) and (7.18), we obtain an alternative proof of the ‘positive density’ conjecture, first established in [B-F1].

**Theorem 1.** Any integral Apollonian circle packing produces a set of curvatures of positive density in \( \mathbb{Z} \).

**Proof.**

With previous notation, let \( S \subset [1, P^2R] \cap \mathbb{Z} \) satisfy \( |S| > (1 - \tau)P^2R \) with \( \tau > 0 \) a sufficiently small constant. We show that

\[ \sum_{n \in S} \omega(n) > 0. \]  

(7.19)

Since \( S \) is arbitrary, this will imply that \([1, P^2R]\) contains at least \( \tau P^2R \) curvatures and hence Theorem 1.

Let \( Q_{00} \) be a large constant and \( K = \prod_{2 < p < Q_{00}} p \). From (7.7), (7.18)

\[ \sum_{n \in S} \omega(n) \geq \sum_{n \in S} \nu(n) - |S|^{1/2} \left( \sum_{n \in S} |\nu(n) - \omega(n)| \right)^{1/2} \]

\[ > (\log Q_{00})^{-C} R^{-1} \sum_{\mathcal{F}} \sum_{n \in S} \ast \nu(n) = 1_{[(n+a, K) = 1]} - P^2Q_{00}^{-\frac{1}{2}} 10^{-8} \]  

(7.20)

(\( \ast \) refers to a congruence condition \( \pmod{8} \) on \( n + a \)).
Next
\[
\sum_{n \in S}^* 1_{[(n+a, K) = 1]} > \sum_{n < P^2R}^* 1_{[(n+a, K) = 1]} - \tau P^2 R \\
\geq \frac{1}{8} \prod_{2 < p < Q_{00}} \left(1 - \frac{1}{p}\right) P^2 R - \tau P^2 R \\
\geq (\log Q_{00})^{-1} P^2 R
\]  
(7.21)
for \(\tau = \tau(Q_{00})\) small enough. Substituting in (7.20) gives
\[
\sum_{n \in S} \omega(n) > P^2 [c (\log Q_{00})^{-c} - Q_{00}^{-\frac{1}{10} - 8}] > 0
\]
for \(Q_{00}\) large enough.

**Remark.** It should be noted that the previous argument establishes a stronger statement in fact. It follows indeed that the set of curvatures in the ACP contains a coset of the integers, up to a ‘small’ exceptional set (where ‘small’ refers to small density).
We do not attempt here to make quantitatively stronger statements, as better results will be obtained in the forthcoming paper [B-K2].

(8). **Prime Curvatures**

Another application of our analysis is an analogue of Theorem 1 for primes. Thus

**Theorem 2.** The set of curvatures produced by any primitive integral Apollonian circle packing contains a subset of the primes of positive density.

Denote \(\mathcal{P}\) the set of primes and assume \(S \subset [1, P^2R] \cap \mathcal{P}\) satisfies
\[
\sum_{n \in S} \Lambda(n) > (1 - \tau) P^2 R. \tag{8.1}
\]
A straightforward adjustment of the proof of Theorem 1 shows that it will suffice to establish a bound
\[
\left| \sum_{n \in S} \Lambda(n)[\omega(n) - \nu(n)] \right| < Q_{00}^{-c} P^2 \tag{8.2}
\]
\((c > 0\) some fixed constant, \(Q_{00}\) is a sufficiently large constant and \(\tau = \tau(Q_{00}) > 0\)).

Defining
\[
T(\theta) = \sum_{n \in S} \Lambda(n)e(n\theta) \tag{8.3}
\]
we have to bound
\[ \left| \int_T [S_\omega - \hat{\nu}] T \right| \leq \int_T |S_\omega - \hat{\nu}| |T|. \] (8.4)

Invoking an additional ingredient, we will rely on certain distributional properties of exponential sums of the form (8.3), assuming \(|S| \sim P^2 R\).

As a consequence of the majorant property for the set of primes (see [B], [Gr]), there is the distributional inequality
\[ \text{mes} \{ \theta \in \mathbb{T}; |T(\theta)| > \delta P^2 R \} \ll \delta^{-2-\varepsilon} (P^2 R)^{-1} \text{ for } \delta > (\log P)^{-A} \] (8.5)
(and hence for all \(\delta > 0\)).

Hence, if \(\Omega \subset \mathbb{T}\) such that \(P^2 R |\Omega| > 1\), we have
\[ \int_\Omega |T(\theta)|^2 d\theta \ll (|\Omega| P^2 R)^\varepsilon P^2 R. \] (8.6)

Take \(Q_0 = (\log P)^{10^{10}}\) and let \(M_0(q, b), M_{00}(q, b)\) be defined by (6.4), (7.5).

Decompose (8.4) as follows
\[ |(8.4)| \leq \int_{\mathbb{T}} \bigcup_{q \leq Q_0 \atop \text{dyadic}} M_0(q, b) \left| S_\omega - \hat{\nu} \right| |T| \] (8.7)
\[ + \sum_{Q_0 < Q < Q_0 \atop \text{dyadic}} \int_{q \sim Q \atop \text{dyadic}} M_0(q, b) \left| S_\omega - \hat{\nu} \right| |T| \] (8.8)
\[ + \sum_{q \leq Q_{00} \atop \text{dyadic}} \int_{M_0(q, b)} \left| S_\omega - \hat{\nu} \right| |T| \] (8.9)

Recall that \(|\hat{\nu}| \leq |S_\omega|\) by (7.4').

Since certainly \(\|T\|_2 < (\log P)^{\frac{1}{2}} \sqrt{RP}\), we have
\[ (8.7) \lesssim (\log P)^{\frac{1}{2}} \sqrt{RP} \left[ \int_{\mathbb{T}} \bigcup_{q \leq Q_0 \atop \text{dyadic}} M_0(q, b) \right]^{1/2} \lesssim (\log P)^{\frac{1}{2}} \sqrt{R P} \frac{P}{\sqrt{R}} Q_{00}^{-10^{-7}} < P^2 (\log P)^{-4}. \] (8.10)
We estimate (8.8). Fix \( Q_{00} < Q < Q_0 \) (Q dyadic) and define
\[
\mathcal{M}'(q, b) = \left[ \left| \theta - \frac{b}{q} \right| < \frac{q^2}{RP^2} \right].
\] (8.11)

Then
\[
\int \left| S_\omega - \hat{\nu} \right| |T| \leq \int_{q \sim Q} \mathcal{M}_0(q, b) + \sum_{q \sim Q} \mathcal{M}'(q, b) = (8.12) + (8.13).
\]

By (6.7) and (8.6)
\[
(8.12) \leq \frac{P}{\sqrt{R}} Q^{-\frac{3}{2}} 10^{-7} \left\| \mathcal{M}(q, b) \right\|_2 \ll \frac{P}{\sqrt{R}} Q^{-\frac{3}{2}} 10^{-7} Q^3 \epsilon P \sqrt{R} < Q^{-\frac{3}{2}} 10^{-7} Q^2 \quad (8.14)
\]
and, using (6.5) and (8.6)
\[
(8.13) \leq \sum_{q \sim Q} \frac{1}{RQ^{1/2}} \int_{Q^2/RP^2 < |\varphi| < \frac{Q_0^2}{RP^2}} |\varphi|^{-1} |T(\frac{b}{q} + \varphi)| d\varphi
\]
\[
< \frac{1}{RQ^{1/2}} \sum_{s, Q^2 < 2^s < Q_0^2} P^2 R 2^{-s} \sum_{q \sim Q} \int_{|\varphi|^{-1} < \frac{2^s}{RP^2}} |T(\frac{b}{q} + \varphi)| d\varphi
\]
\[
\ll \frac{P^2}{Q^{1/2}} \sum_{2^s > Q^2} 2^{-s} \left( \frac{2^s}{RP^2 Q^2} \right) \left( 2^s Q^2 \right)^{3/2} P \sqrt{R} < P^2 Q^{-1/3}. \quad (8.15)
\]

and from (8.14), (8.15)
\[
(8.8) < Q_{00}^{-\frac{3}{2}} 10^{-7} P^2. \quad (8.16)
\]

Finally,
\[
(8.9) \leq \int \left| S_\omega - \hat{\nu} \right| |T| \leq \sum_{q \leq Q_{00}} \int_{\left| \theta - \frac{b}{q} \right| < \frac{Q_{00}^2}{RP^2}} |S_\omega| |T| = (8.17) + (8.18)
\]
From (7.6), (8.6),

\[ (8.17) \ll \|S_\omega - \hat{\nu}\|_2 Q_0^{3\varepsilon} (P^2 R)^{1/2} < \frac{P}{\sqrt{R}} Q_0^{-\frac{1}{2}10^{-7}} Q_4^{3\varepsilon} P \sqrt{R} < Q_0^{-\frac{1}{4}10^{-7}} P^2 \]  \hspace{1cm} (8.19)

and (8.18) is bounded as (8.13)

\[ (8.18) < P^2 Q_0^{-1/3}. \]  \hspace{1cm} (8.20)

Hence

\[ (8.9) < Q_0^{-\frac{1}{4}10^{-7}} P^2. \]  \hspace{1cm} (8.21)

From (8.10), (8.16), (8.21),

\[ (8.4) < Q_0^{-\frac{1}{4}10^{-7}} P^2. \]  \hspace{1cm} (8.22)

which is the desired inequality (8.2). This proves Theorem 2.

References

[B]. J. Bourgain, On \( \Lambda(p) \)-subsets of the squares, Israel J. Math. 67 (1989), 291–311.

[BEW]. B. Berndt, R. Evans, K. Williams, Gauss and Jacobi sums, Canadian Math. Soc., Vol. 21 (1998).

[B-F1]. J. Bourgain, E. Fuchs, A proof of the positive density conjecture for integer Apollonian circle packings, to appear in JAMS.

[B-F2]. J. Bourgain, E. Fuchs, On representation of integers by binary quadratic forms, submitted to IMRN.

[BGS]. J. Bourgain, A. Gamburd, P. Sarnak, Generalization of Selberg’s 3/16 theorem and affine sieve, to appear in Acta Math.

[B-K]. J. Bourgain, A. Kontorovich, On representations of integers in thin subgroups of \( SL_2(\mathbb{Z}) \), GAFA, Vol. 20, N5, 2010, p. 1144–1174.

[B-K2]. J. Bourgain, A. Kontorovich, On the strong density conjecture for integral Apollonian circle packings, in preparation.

[F-S]. E. Fuchs, K. Sanden, Prime numbers and the local to global principle in Apollonian circle packings, preprint.

[G]. B. Green, Roth’s theorem in the primes, Annals of Math. (2), 161 (2005), no 3, 1609–1636.

[G-S]. V. Guillemin, S. Sternberg, Geometric asymptotics, Math. Surveys 14, AMS 1977.

[GLMWY]. R. Graham, L. Lagarias, C. Mallows, A. Wilks, C. Yan, Apollonian circle packings: number theory, J. of Number Theory, 100 (1), pp 1–45 (2003).

[K-O]. A. Kontorovich, H. Oh, Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds, to appear in JAMS.

[S]. P. Sarnak, Letter to Lagarias on integral Apollonian packings, (2007).