Asymmetries in the tunneling probability of Bose-Einstein condensate in an accelerating optical lattice

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Abstract

We derive a two-band finite-dimensional model for description of the condensate tunneling in an accelerating optical lattice, taking into account the fine Bloch band structure. The model reveals a very strong dependence of the final band populations on the initial populations and phases. Most importantly, additionally to the known asymmetric dependence on the nonlinearity, there is also a notable asymmetry in the sensitivity of the tunneling probability to the nonlinearity-induced initial population of the Bloch band to which the tunneling takes place. This fact can explain the experimentally observed unexpected independence of the upper-to-lower tunneling probability on the nonlinearity. Finally, we compare the predictions of the two-band model with that of the well-known nonlinear Landau-Zener model and find disagreement when the two bands are initially populated. The disagreement can be qualitative and reveals itself even for a negligible nonlinearity. However, the two models agree remarkably well if just one band is populated initially.

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I. INTRODUCTION

Since the successful realization of Bose-Einstein condensation (BEC) in an optical lattice \[1, 2\] the experimental efforts are directed to explore and understand the behavior of BEC in a periodic potential. Many of the BEC phenomena in an optical lattice have an analog in the solid state physics. For instance, Bloch oscillations and Landau-Zener tunneling of BEC were experimentally observed \[3, 4, 5, 6\] in an accelerating lattice. The atomic interactions in BEC, however, give rise to the essentially nonlinear effects such as the asymmetry of the tunneling probability between the two adjacent bands \[5, 6\] and the Landau and dynamical instabilities \[6, 7, 8, 9, 10\]. Accordingly, two essentially different regimes of BEC dynamics in an optical lattice can be identified \[6, 11\]: the “instability regime”, which realizes for a small to intermediate values of the lattice acceleration and the “Landau-Zener regime”, which realizes for larger accelerations. When the acceleration is small, the condensate slowly crosses the edge of the Brillouin zone allowing the unstable modes to dominate the order parameter \[11, 12\]. For instance, in one of the bands the modulational instability \[13\] will result in the bright soliton formation even in a repulsive BEC \[14\].

For the “Landau-Zener regime” a reduced two-level model (the nonlinear Landau-Zener model) was proposed in Refs. \[15, 16\] to account for the effect of the atomic interactions on the tunneling probability of BEC (see also Refs. \[12, 17\] for further development). This model is derived for a weak optical lattice by an analog of the method of free electrons in the solid state theory (see for instance, Ref. \[18\]) and is essentially the Landau-Zener model \[19\] modified by the nonlinear terms due to the atomic interactions in BEC.

The predictions of the nonlinear Landau-Zener model were experimentally tested \[5, 6\]. In accordance with the theoretical prediction the atomic interactions in BEC make the tunneling probability asymmetric, but still a significant disagreement was discovered. Hence, a more accurate model is in order for description of the condensate tunneling.

To account for the modulational instability present in one of the Bloch bands a spatio-temporal model was proposed recently \[11\]. Essentially, it consists of two linearly coupled nonlinear Schrödinger equations for the slowly varying amplitudes of the two Bloch waves bordering the Brillouin zone edge (i.e. at \( k = k_B \)). The key role of the modulational instability of Bloch wave amplitudes in the transition from the reversible tunneling to the asymmetry in the band populations is revealed. Therefore, this model is suitable for descrip-
tion of the “instability regime”, when the Brillouin zone edge crossing time is comparable
to the characteristic time of development of the modulational instability. However, its ap-
plicability is restricted to the case of a very small lattice acceleration and thus it cannot be
applied to the “Landau-Zener regime” (as defined above).

The aim of this paper is to propose an improved finite-dimensional system (the two-band
model) for BEC tunneling in an accelerating optical lattice. We take into account the fine
structure of the Bloch bands, which is of crucial importance if the initial conditions are
arbitrary. Only the “Landau-Zener regime” is considered. Hence, we can safely neglect the
spatial inhomogeneity of the Bloch wave amplitudes. We expand the BEC order parameter
over the Bloch waves with a time-dependent wave index. Thus our method is similar to
Houston’s approach \[20\] to the accelerating electrons in the solid state theory. The resulting
finite-dimensional model can also have application in the study of ultracold atoms in a
periodic potential submitted to a constant external force \[21, 22\].

There are two main results. First, the two-band model reveals a dramatic dependence of
the final Bloch band populations on the initial populations and the phase difference. Most
importantly, additionally to the asymmetry in the dependence of the tunneling probability
on the nonlinearity strength, we reveal also the asymmetry in the sensitivity of the tunneling
probability to the population of the adjacent Bloch band (i.e. to which the tunneling takes
place). This fact, combined with the nonlinearity-induced population of the adjacent band,
can explain the independence of the upper-to-lower tunneling probability on the nonlinearity
suggested by the experimental results of Ref. \[5\]. Second, there is a significant disagreement
between the predictions of the two-band model and the Landau-Zener model if the two
bands are initially populated. The disagreement can be qualitative and reveals itself even
for a negligible nonlinearity. The two models agree in the predictions if only one Bloch band
is populated initially.

The paper is organized as follows. In the next section a derivation of the two-band
model is presented. Some details of the derivation are relegated to appendix A. Then, in
section III the case of a weak lattice is considered. The predictions of the two-band model
are compared to those of the nonlinear Landau-Zener model in section IV. The two-band
model in the diabatic basis is given in appendix B. Section V contains discussion of the
asymmetric BEC tunneling. Finally, a general discussion and perspectives are contained in
the concluding section VI.
II. DERIVATION OF THE TWO-BAND MODEL

We consider a quasi one-dimensional BEC, i.e. the transverse width is much smaller than the length of the condensate. For not too large number of atoms the transverse dynamics of BEC is that of a quantum particle confined to the lower energy level of the parabolic trap [24]. In terms of the $s$-wave scattering length $a_s$, the transverse oscillator length $\ell_\perp = \sqrt{\hbar/m\omega_\perp}$ and the linear condensate density $n_{1D} = N_d/d$, where $N_d$ is the number of atoms per lattice site, the condensate can be considered as quasi one-dimensional under the condition $n_{1D}a_s \ll 1$ [25]. In the experiments the usual values are: $d \sim 1\mu m$ and $a_s \sim 1$nm. Hence, to satisfy the quasi one-dimensionality condition one has to have $N_d \ll 1000$. The BEC order parameter $\Phi(\vec{r}_\perp, x, t)$ then can be approximated as a product of the wave-function for the ground state of the transverse parabolic trap and a function describing the longitudinal motion of the condensate:

$$\Phi(\vec{r}_\perp, x, t) = \frac{1}{\sqrt{\pi\ell_\perp}} \exp\left\{-\frac{r_{\perp}^2}{2\ell_\perp^2}\right\} \psi(x, t).$$

(1)

For the reduced order parameter $\psi(x, t)$ the Gross-Pitaevskii equation describing the condensate in an accelerated optical lattice reads

$$ih\partial_t \psi = -\frac{\hbar^2}{2m} \partial_x^2 \psi + \left(V_0 \cos(2k_L[x - x_0(t)]) + \frac{m\omega_\parallel^2}{2} x^2 \right) \psi + \frac{gn_{1D}}{2\pi\ell_\perp^2} |\psi|^2 \psi,$$

(2)

where $g = 4\pi\hbar^2a_s/m$, $k_L$ is the optical lattice index and $x_0(t)$ describes the acceleration of the lattice. Here the order parameter is normalized as follows $d^{-1} \int_0^d d x |\psi|^2 = 1$ with $d = \pi/k_L$ being the lattice spacing. The following dimensionless variables will be used below:

$$\tilde{t} = \frac{8E_R}{\hbar} t, \quad \tilde{x} = 2k_L[x - x_0(t)], \quad v_0 = \frac{V_0}{8E_R}, \quad c = \frac{a_s n_{1D}}{2\ell_\perp^2 k_L^2}, \quad \lambda^2 = (2\ell_\parallel k_L)^{-2},$$

(3)

where $E_R = \hbar^2 k_L^2/(2m)$ is the recoil energy and $\ell_\parallel = \sqrt{\hbar/m\omega_\parallel}$. We have

$$i\partial_{\tilde{t}} \tilde{\psi} = \frac{1}{2} \left(-i\partial_{\tilde{x}} - \frac{d\tilde{x}_0}{dt} \right)^2 \tilde{\psi} + \left(v_0 \cos(\tilde{x}) + \frac{\lambda^2}{2} [\tilde{x} + \tilde{x}_0]^2 \right) \tilde{\psi} + c|\tilde{\psi}|^2 \tilde{\psi},$$

(4)

where $\tilde{x}_0 = 2k_Lx_0$ and

$$\tilde{\psi} = \exp\left\{-\frac{i}{2} \int_0^{\tilde{t}} d\tau \left(\frac{d\tilde{x}_0}{d\tau} \right)^2 \right\} \psi.$$

The normalization now reads $(2\pi)^{-1} \int_0^{2\pi} d\tilde{x} |\tilde{\psi}|^2 = 1$. As we will use only the dimensionless variables of equation (4), we drop all the tildes from $x$, $t$, and $\psi$, for simplicity.
We assume that the quasi-momentum spread of the condensate in the optical lattice is negligible compared to the width of the Brillouin zone. One necessary condition is that the optical lattice be long enough, i.e. the parabolic trap contains many lattice periods: $\lambda \ll 1$. Then, the effect of the parabolic trap is negligible. We set $\lambda = 0$.

An additional condition is that the time of the Brillouin zone edge crossing is much less than the characteristic time of the modulational instability development. This sets a lower limit on the acceleration $\alpha_0$. The latter time can be estimated as follows. In the slowly varying envelope approximation \[11, 13\] the spatial Bloch amplitude will be governed by a nonlinear Schrödinger equation with an effective interaction coefficient $c\chi_{nn}$. The characteristic time of the instability development is the inverse of the maximal growth rate $t_{MI} \sim |c|^{-1}$.

Formation of gap solitons is also possible for a nonlinearity of the order of the gap between the Bloch bands \[26\]. In the following, however, we consider only the weak nonlinearity limit $c \ll v_0$, used in the experiment \[5\].

Under the conditions that the condensate has a narrow initial momentum spread and the time of the Brillouin zone edge crossing is much less than the characteristic time of the modulational instability development the solution to equation (4) can be expanded in the basis of the Bloch waves with a definite (time-dependent) index $q(t)$ and an overall linear phase due to the acceleration:

$$
\psi = e^{iv(t)x} \sum_{n=1}^{\infty} A_n(t) \Psi_{n,q}(x),
$$

where $v(t) \equiv \dot{x}_0$ and $\Psi_{n,q}(x) \equiv e^{i\pi x} u_{n,q}(x)$ satisfies

$$
\left\{ -\frac{1}{2} \frac{\partial^2}{\partial x^2} + v_0 \cos(x) \right\} \Psi_{n,q}(x) = E_n(q) \Psi_{n,q}(x).
$$

Due to the normalization and orthogonality of the Bloch functions, \( (2\pi)^{-1} \int_{-\pi}^{\pi} dx \Psi_{n,k}^*(x) \Psi_{m,k}(x) = \delta_{nm} \), the amplitudes satisfy the condition \( \sum_{n=1}^{\infty} |A_n|^2 = 1 \).

It is shown below that, in a vicinity of the Brillouin zone edge $q = k_B$, for a small acceleration and under the condition $E_3(k_B) - E_2(k_B) \gg E_2(k_B) - E_1(k_B)$ one can neglect the effect of the upper bands on the tunneling between the first two Bloch bands. The nonlinear term with $c \ll v_0$ can be taken into account by using a perturbation theory.

The expansion (5) is in the spirit of Houston’s approach in the theory of electrons accelerating in a crystal lattice \[20\]. It takes into account, for instance, that in the experiments \[5, 6\] the condensate was in a Bloch state before the acceleration was turned on.
A. The linear limit \( c = 0 \)

For the moment let us drop the nonlinear term and consider the linear limit of equation (4). The contribution from the nonlinear term is accounted later on. Substituting the expression (5) into equation (4) and dropping the overall imaginary exponent gives

\[
- \left[ \frac{dq}{dt} + \frac{dv}{dt} \right] x \sum_n A_n u_{n,q}(x) + \sum_n i \frac{dA_n}{dt} u_{n,q}(x) + i \frac{dq}{dt} \sum_n A_n \partial_q u_{n,q}(x)
\]

\[
= \sum_n E_n(q) A_n u_{n,q}(x).
\]

Setting \( q(t) = q_0 - v(t) \) to cancel the non-periodic in \( x \) term and projecting the resulting equation onto the basis function \( u_{n,q}(x) \) we get an equation for the amplitude

\[
i \frac{dA_n}{dt} = E_n(q) A_n + i \alpha(t) \sum_m A_m \langle u_{n,q} | \partial_q u_{m,q} \rangle,
\]

where \( \alpha = \bar{x}_0 \). Here the inner product over the lattice period is used: \( \langle f_1 | f_2 \rangle \equiv (2\pi)^{-1} \int_{-\pi}^{\pi} dx f_1^*(x) f_2(x) \) with \( \langle u_{n,q} | u_{m,q} \rangle = \delta_{n,m} \).

In appendix A it is shown that for any periodic potential \( v = v(x) \), with \( v(x) \) being an even function, \( u_{n,q}^*(x) = u_{n,q}(-x) \) and, hence, the diagonal inner product \( \langle u_{n,q} | \partial_q u_{m,q} \rangle = 0 \), while the non-diagonal is given as

\[
\kappa_{nm}(q) \equiv \langle u_{n,q} | \partial_q u_{m,q} \rangle = -i \frac{\langle u_{n,q} | \partial_x u_{m,q} \rangle}{E_m(q) - E_n(q)} = -i \frac{\langle u_{n,q} | \partial_x u_{m,q} \rangle}{[E_m(q) - E_n(q)]^2}, \quad n \neq m.
\]

The coupling coefficient \( \kappa_{nm} \) is real and satisfies \( \kappa_{nm}(q) = -\kappa_{mn}(q) \).

Note that the coupling coefficient between the Bloch bands is inversely proportional to the band level spacing \( E_m(k) - E_n(k) \), since \( \kappa_{nm}(k) = \mathcal{O}(v_0/[E_m(k) - E_n(k)]^2) \). Thus, for a small acceleration and an optical lattice satisfying \( E_3(k_B) - E_2(k_B) \gg E_2(k_B) - E_1(k_B) \), the influence of the upper bands on BEC tunneling between the first two Bloch bands at the Brillouin zone edge is negligible. For instance, the above condition is satisfied by a weak optical lattice (see section III).

Dropping the terms corresponding to the upper bands from equation (8) and making the transformation \( A_1 = \exp \left\{ -i \int_0^t d\tau \bar{E}(q(\tau)) \right\} a_1 \) and \( A_2 = \exp \left\{ -i \int_0^t d\tau \bar{E}(q(\tau)) \right\} (-i)a_2 \), with \( \bar{E} = (E_1 + E_2)/2 \), we obtain the two-band model for the linear case:

\[
i \frac{da_1}{dt} = -\varepsilon(q)a_1 + \alpha\kappa_{12}(q)a_2,
\]

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\[ i \frac{da_2}{dt} = \varepsilon(q)a_2 + \alpha \kappa_{12}(q)a_1. \]  

(11)

Here we have used that \( \kappa_{21}(q) = -\kappa_{12}(q) \) and set \( \varepsilon(q) = (E_2(q) - E_1(q))/2 \). Recall that \( \alpha(t) = \ddot{x}_0 \) is the lattice acceleration and \( q(t) = q_0 - \dot{x}_0 \) is a time-dependent parameter. The system of equations (10)-(11) is consistent with the two-band approximation, since it conserves the total number of atoms in the two bands: \( |a_1(t)|^2 + |a_2(t)|^2 = 1 \).

The linear two-band model (10)-(11) is important in its own right, since it can have direct application in the study of ultracold atoms in a periodic potential submitted to a constant external force [21, 22].

**B. Weak nonlinearity: \( c \ll v_0 \)**

In the experiments on BEC tunneling in an optical lattice [5, 6] the nonlinearity satisfies the condition \( c \ll v_0 \). In this case, the nonlinear term can be treated as a perturbation of the optical lattice potential. Here it is pertinent to mention that the nonlinear term was also treated by introduction of an effective optical lattice potential [6, 27]. This treatment is of limited validity.

Under the condition \( c \ll v_0 \) the nonlinear term preserves the qualitative Bloch band structure, but modifies the Bloch functions and the band energies. The nonlinear term serves as an additional lattice potential (in general, time-dependent), the same for all eigenfunctions. In fact, if we define the nonlinear modes \( \hat{u}_{n,k}(x,t) \) as follows

\[
\left\{ \frac{1}{2}(-i\partial_x - k)^2 + v_0 \cos(x) + c \left| \sum_{m=1}^{\infty} A_m(t)\hat{u}_{m,k}(x,t) \right|^2 \right\} \hat{u}_{n,k}(x,t) = \hat{E}_n(k,t)\hat{u}_{n,k}(x,t), \quad (12)
\]

then the effect of the nonlinear term can be treated by using the usual perturbation theory for the eigenvalues \( \hat{E}_n(k,t) \) of a perturbed ((\( k, t \))-dependent) linear operator. Notice that the nonlinear modes \( \hat{u}_{n,q} \) are orthogonal, \( \langle \hat{u}_{m,k}|\hat{u}_{n,k}\rangle = \delta_{n,m} \), and for \( c = 0 \) coincide with the respective Bloch functions. Thus, for small \( c \), they constitute a complete orthogonal basis (in this respect we generalize the approach of Refs. [28, 29]). In the first order of the perturbation theory we get

\[
\hat{E}_n(k,t) = E_n(k) + c\langle u_{n,q}^*(x) \left| \sum_{m=1}^{\infty} A_m(t)u_{m,q}(x) \right|^2 u_{n,q}(x) \rangle + \mathcal{O}(c^2),
\]
with \( \langle ... \rangle = (2\pi)^{-1} \int_{-\pi}^{\pi} dx \langle ... \rangle \), while the eigenfunctions \( \hat{u}_{n,q}(x) = u_{n,q}(x) + \mathcal{O}(c) \) which is a sufficient approximation for the following. In the two-band approximation we get the following expressions for the energies of the first two Bloch bands (the variable \( t \) is omitted for simplicity):

\[
\hat{E}_1(k) = E_1(k) + c \left\{ \chi_{12}(k)|A_1|^2 + \chi_{12}(k)|A_2|^2 + \sigma_{12}(k)A_1^2 + \sigma_{12}^*(k)A_1^*A_2 \right\},
\]

\[
\hat{E}_2(k) = E_2(k) + c \left\{ \chi_{22}(k)|A_2|^2 + \chi_{12}(k)|A_1|^2 + \sigma_{21}(k)A_2A_1^* + \sigma_{21}^*(k)A_2^*A_1 \right\}.
\]

Here

\[
\chi_{nn} = \langle |u_{n,k}|^4 \rangle, \quad \chi_{12} = \langle |u_{1,k}|^2|u_{2,k}|^2 \rangle, \quad \sigma_{12} = \langle |u_{1,k}|^2u_{1,k}^*u_{2,k}^* \rangle, \quad \sigma_{21} = \langle |u_{2,k}|^2u_{2,k}u_{1,k}^* \rangle.
\]

The terms with \( \sigma_{nm} \) can be neglected, since at the Brillouin zone edge these coefficients are exactly zero due the properties of the Bloch waves. Numerics shows that they grow linearly in \( k - k_B \).

The nonlinear generalization of the system (10)-(11) is derived by using the nonlinear Bloch waves \( \hat{\Psi}_{n,k} = e^{ikx}\hat{u}_{n,k} = \Psi_{n,k} + \mathcal{O}(c) \) in the expansion (5). For a small acceleration the term proportional to \( \alpha c \) can be neglected, thus the inter-band coupling coefficient \( \kappa_{12} \) in the nonlinear system is given by the same formula as in the linear case. Setting now

\[
A_1 = \exp \left\{ -i \int_0^t d\tau F(q(\tau)) \right\} a_1 \quad \text{and} \quad A_2 = \exp \left\{ -i \int_0^t d\tau F(q(\tau)) \right\} (-ia_2),
\]

with \( F = (E_1 + E_2)/2 + c\chi_{12} \), we obtain the nonlinear two-band model:

\[
i \frac{da_1}{dt} = \{-\epsilon(q) + c\gamma_{11}(q)|a_1|^2\} a_1 + \alpha \kappa_{12}(q)a_2,
\]

\[
i \frac{da_2}{dt} = \{\epsilon(q) + c\gamma_{22}(q)|a_2|^2\} a_2 + \alpha \kappa_{12}(q)a_1.
\]

Here we have used the modified Bloch energies from equation (13) and defined

\[
\gamma_{nn} = \chi_{nn} - \chi_{12} = \langle |u_{n,k}|^4 - |u_{1,k}|^2|u_{2,k}|^2 \rangle, \quad n = 1, 2.
\]

The system (14)-(15) depends on the fine \( k \)-structure of the Bloch bands, since \( q(t) \) scans through the edge of the Brillouin zone. Hence, the expressions for the Bloch waves in a vicinity of the Brillouin zone edge have to be available. For a general optical lattice the coefficients \( \epsilon(q) \), \( \kappa_{12}(q) \), and \( \gamma_{nn}(q) \) can be computed numerically. However, such expressions are available in the analytical form (see below) for a weak optical lattice used in the experiments [5, 6].

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III. WEAK LATTICE APPROXIMATION

Similar to the free electrons approach in the solid state theory \cite{18}, in the vicinity of the Brillouin zone edge the Bloch waves from the two lowest bands of a weak optical lattice can be sought in the form

\[ \Psi = ae^{i qx} + b e^{i (q - 2k_B)x}. \]  

For the cosine lattice \( v(x) = v_0 \cos(x) \) we get

\[ E_{1,2}(q) = \frac{q^2}{2} + k_B(k_B - q)^2 \mp \varepsilon(q), \quad \varepsilon(q) \equiv \left( k_B^2 (k_B - q)^2 + \frac{v_0^2}{4} \right)^{\frac{1}{2}}. \]  

The corresponding Bloch waves read

\[ \Psi_{1,q} = \frac{\nu(q)e^{i qx} - e^{i(q-2k_B)x}}{\sqrt{1 + \nu^2(q)}}, \quad \Psi_{2,q} = \frac{e^{iqx} + \nu(q)e^{i(q-2k_B)x}}{\sqrt{1 + \nu^2(q)}}, \]  

where \( \nu(q) = 2[k_B(k_B - q) + \varepsilon(q)]/v_0. \) Then the remaining two coefficients of the two-band model are

\[ \kappa_{12}(q) = \frac{1}{2\varepsilon(q)} \frac{\nu(q)}{1 + \nu^2(q)}, \quad \gamma(q) \equiv \gamma_{nn}(q) = \frac{4\nu^2(q)}{[1 + \nu^2(q)]^2}. \]  

\[ \text{FIG. 1: The Bloch energy of the first two bands, panel (a), and the inter-band coupling coefficient } \kappa_{12}, \text{ panel (b), vs. the band index } q. \text{ Numerically found values are given by the solid lines and the analytical weak-lattice approximations are given by the dashed lines. Here } v_0 = 0.1375. \]

For the linear case, the analytical formulae in equations (18) and (20) are in good agreement with the exact numerical results for the experimentally used values \[5, 6\] (we use
$v_0 = 0.1375$). In fig. 1 the analytical energy levels $E_{1,2}(q)$ (left panel) and the inter-band coupling coefficient $\kappa_{12}(q)$ (right panel) vs. their numerical values are given. The coupling coefficients to the higher Bloch bands are small as compared to $\kappa_{12}$, in accordance with the theory of section II A. For instance, we have for the maximum value: $\kappa_{n3}(k_B) < 0.2$, $n = 1, 2$, for $v_0 = 0.1375$.

![Graph of nonlinear coefficients $\gamma_{11}(q)$ and $\gamma_{22}(q)$. The numerically computed coefficients are given by the solid lines and the analytical weak-lattice approximation is given by the dotted line. The dashed lines are the scaled analytical coefficients as indicated in the text. Here $v_0 = 0.1375$.](image)

**FIG. 2:** The nonlinear coefficients $\gamma_{11}(q)$ and $\gamma_{22}(q)$. The numerically computed coefficients are given by the solid lines and the analytical weak-lattice approximation is given by the dotted line. The dashed lines are the scaled analytical coefficients as indicated in the text. Here $v_0 = 0.1375$.

The weak lattice approximation cannot, however, capture the weak asymmetry in the two nonlinear coefficients $\gamma_{11}(q)$ and $\gamma_{22}(q)$, since their analytical values always coincide. In fact, in the vicinity of the Brillouin zone edge, the analytical approximation (dotted line in fig. 2) is close to the algebraic average of the exact numerical coefficients. To have a better approximation the analytical coefficients must be scaled by the maximum value (at $q = k_B$). For instance, $\gamma_{11}(q) = 1.132\gamma(q)$ and $\gamma_{22}(q) = 0.842\gamma(q)$ for $v_0 = 0.1375$ (the dashed lines in fig. 2).

**IV. COMPARISON WITH THE LANDAU-ZENER MODEL**

To make a quantitative comparison with the nonlinear Landau-Zener model

$$i\frac{da}{dt} = \left\{ \frac{\alpha t}{2} + \frac{c}{2}(|b|^2 - |a|^2) \right\} a + \frac{v_0}{2} b,$$

(21)
introduced in Refs. [15, 16], one has to relate the amplitudes $a$ and $b$ of the incident and Bragg-scattered waves with the amplitudes of the Bloch waves from the first two bands.

There are two approaches. The first one is based on the fact that the adiabatic energy levels of the quantum-mechanical Hamiltonian

$$H = \begin{pmatrix} \frac{\alpha t}{2} & \frac{v_0}{2} \\ \frac{v_0}{2} & -\frac{\alpha t}{2} \end{pmatrix}$$

associated with the system (21)-(22), mimic the structure of the first two Bloch bands. Therefore, the Bloch amplitudes are obtained by projecting the vector $(a, b)$ onto the instantaneous adiabatic eigenvectors of $H$. We obtain the following relation

$$a_1 = \frac{\alpha a - b}{\sqrt{1 + \alpha^2}}, \quad a_2 = \frac{a + \alpha b}{\sqrt{1 + \alpha^2}},$$

where $\alpha = (E - \alpha t)/v_0$ with $E = \sqrt{\alpha^2 t^2 + v_0^2}$. The amplitudes $a_1$ and $a_2$ in equation (24) correspond to the instantaneous adiabatic energy levels $E_{1,2} = \mp E/2$, thus they can be taken as the occupation amplitudes of the first two Bloch bands according to the nonlinear Landau-Zener model (however, one can get only the absolute values of the amplitudes $a_{1,2}$, their phases are left unknown due to arbitrary phases of the eigenvectors).

The second approach is to use the analytical results for a weak lattice and equate directly the order parameters corresponding to the two models. The two-band model (15)-(16) corresponds to the order parameter

$$\psi = e^{i\theta(t) + iv(t)x} \left[ a_1(t)\Psi_{1,q(t)}(x) - ia_2(t)\Psi_{2,q(t)}(x) \right],$$

with $\theta(t)$ being an unimportant phase and $q(t) = q_0 - v(t)$. The nonlinear Landau-Zener model (21)-(22) corresponds to the following order parameter [15]

$$\psi = e^{-3ict/2} \left[ a(t)e^{ikx} + b(t)e^{i(k-2k_B)x} \right].$$

It is straightforward to relate the band amplitudes using the expression for the Bloch waves [19] and associating $k = q_0$. The common $x$-independent phase can be discarded. We have

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{\nu}{\sqrt{1 + \nu^2}} & -\frac{i}{\sqrt{1 + \nu^2}} \\ -\frac{1}{\sqrt{1 + \nu^2}} & \frac{-i\nu}{\sqrt{1 + \nu^2}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$
The transformation given by equation (27) is linear and unitary. Asymptotically it trivializes. Indeed, \( \nu \to \infty \) as \( t \to -\infty \), while \( \nu \to 0 \) as \( t \to \infty \) (here \( \alpha > 0 \)). We get

\[
a(-\infty) = a_1(-\infty), \quad b(-\infty) = -ia_2(-\infty), \quad a(\infty) = -ia_2(\infty), \quad b(\infty) = -a_1(\infty).
\]  

(28)

FIG. 3: The tunneling probability vs. the lattice acceleration according to the Landau-Zener formula (solid line) and in the two-band system (points) in the case of negligible nonlinearity and trivial initial condition (i.e. just one band being populated initially). Here \( v_0 = 0.134 \) and \( c = 0 \).

The overall conclusion to be argued below is that, whatever the way one relates the two models, the predictions of the nonlinear Landau-Zener model (21)-(22) and the two-band model (15)-(16) disagree significantly in general, if the initial condition is non-trivial, i.e. both amplitudes \( a \) and \( b \) and, respectively, \( a_1 \) and \( a_2 \) are non-zero initially. The disagreement is apparent even for a negligible nonlinearity, which case is treated below in detail. A remarkable agreement between the two models if just one band is populated initially (i.e. either \( a_1 \) or \( a_2 \) is zero initially) must also be stressed.

When initially the atoms populated just one band the two models give the same results. For instance, for \( c = 0 \) the probability of tunneling, defined as \( P = |a_n(\infty)|^2 \) for \( a_n(-\infty) = 0 \), is given by the Landau-Zener formula \( P = e^{-\pi v_0^2/(2\alpha)} \) [19], which is illustrated in fig. 3.

When both bands are populated initially, the two models do not agree in the prediction of the final populations. This is illustrated in fig. 4 where we plot the time-dependence of the the second band population for different initial conditions but with the majority of atoms populating initially the first band. Note that in panel (a) the two models give the
FIG. 4: The population of the second Bloch band vs. time in the two-band model (solid lines) and in the nonlinear Landau-Zener model (dashed lines). We have used $v_0 = 0.1375$, $\alpha = 0.036$, and $c = 0$ and the following initial conditions: (a) $(a_1, a_2) = (1, 0)$, (b) $(a_1, a_2) = (\sqrt{0.99}, \sqrt{0.01})$, (c) $(a_1, a_2) = (\sqrt{0.95}, \sqrt{0.05})$, and (d) $(a_1, a_2) = (\sqrt{0.95}, e^{-i\pi/5}\sqrt{0.05})$. The initial amplitudes $a$ and $b$ for the Landau-Zener model and the corresponding functions $a_1(t)$ and $a_2(t)$ were computed using equation (27). The initial time was $t_0 = -200$.

same results if related by equation (27). However, the same equation leads to significant disagreement in panels (b) and (c). Similar conclusion can be drawn using the first relation scheme for the two models, i.e. equation (24).

Two more conclusions are apparent from fig. 4. When the first band is initially dominantly populated, a weak population of the second band makes a strong effect on its final population, (compare panels (a) and (c)). Moreover, at the fixed initial populations, the
initial phase difference can have a significant effect on the final populations (compare panels (c) and (d)). The same conclusions are true for the first band.

FIG. 5: The populations of the Bloch bands vs. time in the two-band model (thick lines) and in the Landau-Zener model (thin lines). The solid lines give $|a_1|^2$. The initial time was $t_0 = -200$. Here $v_0 = 0.1375$, $\alpha = 0.036$, and $c = 0$. We have used the following initial condition $(a_1, a_2) = (i\sqrt{0.15}, \sqrt{0.85})$. The initial amplitudes $a$ and $b$ for the Landau-Zener model and the corresponding functions $a_1(t)$ and $a_2(t)$ were computed using equation (27).

The significant disagreement between the two models can be manifested by using a larger initial population of the less populated Bloch band. In this case the disagreement is qualitative, see fig. 5.

The transformation (27) can be used to derive the two-band model in the diabatic basis, i.e. in terms of the amplitudes of the incident ($a$) and Bragg scattered ($b$) waves. The resulting system is given in appendix B. The main feature of the two-band model in the diabatic basis is that both the sweep function and the coupling coefficient are complicated functions of the band index $q$ with the coupling coefficient being complex valued. This latter fact can explain the disagreement in the predictions of the two-band model and the nonlinear Landau-Zener model when the two bands are initially populated.
V. ASYMMETRIC TUNNELING FOR $c \neq 0$

The tunneling probability of BEC at the edge of the Brillouin zone in an accelerated optical lattice was found to have an asymmetric dependence on the nonlinearity coefficient $c$. The asymmetry in the tunneling probability was first theoretically predicted in Ref. [16] within the nonlinear Landau-Zener model. The two-band model (15)-(16) confirms the existence of the asymmetric tunneling, see fig. 6.

![Graph showing asymmetric tunneling probabilities.]

FIG. 6: The tunneling probabilities in the two-band model (solid lines) when just one band is initially populated. The upper line gives the lower-to-upper tunneling probability and the lower one gives the upper-to-lower tunneling probability. For comparison, the tunneling probabilities in the nonlinear Landau-Zener model are also given (dashed lines). Here $v_0 = 0.1375$ and $\alpha = 0.036$.

The experimental results (see fig. 2 in Ref. [5]) also suggest that the upper-to-lower tunneling probability, in contrast to the lower-to-upper one, is unaffected by the nonlinearity (there is, however, a large error due to imperfections in the experiment). Below we argue that, indeed, there is an additional asymmetry in the sensitivity of the tunneling probability to the initial nonlinearity-induced population of the adjacent Bloch band, which, together with the asymmetric dependence on the nonlinearity coefficient, can give a complete explanation of the experimental results.

In section IV (see fig. 4) we have found that the final populations of the Bloch bands are affected by the initial band populations and phases. Due to imperfections in the experimental preparation of the initial state, there is always a small population of BEC atoms in the
adjacent Bloch band. The condensate is initially prepared at the center of the Brillouin zone ($k = 0$) with a large fraction of atoms populating one of the two Bloch bands. Besides the experimental imperfections, nonlinearity (i.e. the atomic interactions in BEC) also contributes to a small number of BEC atoms in the adjacent band, since the nonlinearity couples the Bloch bands in the higher orders of the perturbation theory discarded in section II B.

To find out the effect of the nonlinearity-induced population of the Bloch band on the tunneling probability to this band, we choose to use the following initial condition $a_j = \sqrt{r c}$, where $j = 2$ for the lower-to-upper tunneling and $j = 1$ for the upper-to-lower tunneling (we do not know the experimental initial share $|a_j|^2$ of the number of atoms in the adjacent Bloch band, however, this number grows with the nonlinearity coefficient $c$ [31]). Since the Bloch waves are real for $k = 0$ the initial phase difference between the amplitudes $a_1$ and $a_2$ of the two-band model is close to zero and does not affect the general picture. For the coefficient $r$ we have used the following set of values $\{0.01; 0.05; 0.075; 0.1; 0.15; 0.2\}$. The results of the numerical simulations of the two-band model are presented in fig. 7. The probability of tunneling (defined here as the final population of the Bloch band to which the tunneling takes place, neglecting a very small initial population of this band) is given vs. the nonlinearity coefficient $c$. The asymmetry in the spread of the numerical curves with different $r$ in the two cases of tunneling is apparent. Here we should mention that fig. 7 can provide only the qualitative account of the asymmetry in the sensitivity of tunneling to the initial Bloch band populations, since the weak lattice approximation developed in section III applies only in the vicinity of the Brillouin zone edge.

It is easy to explain the tunneling probability change if the Bloch band to which the tunneling takes place is populated with a small number of atoms. Indeed, the nonlinear terms in the two-band system (12)-(15) effectively raise the band energy levels. Therefore, in the case of the lower-to-upper tunneling a small population of the adjacent band increases the energy gap, while in the case of the upper-to-lower tunneling it decreases the energy gap, thus the tunneling probability is decreased in the first case and increased in the second one. However, this argument cannot explain the fact that the strength of the sensitivity to the population is quite different in the two cases.

In conclusion, a very small initial population of the Bloch band to which the tunneling takes place (in fact for $r = 0.2$ we get $|a_j|^2 = 0.12$) has a dramatic effect on its final
population in the case of the upper-to-lower tunneling as compared to that in the case of
the lower-to-upper tunneling (if one consider the relative change in the tunneling proba-
bility when increasing $r$). This fact, together with the known asymmetry of the tunneling
probability, provides a complete explanation of the experimental results presented in Ref.

\[ P \]

![Graph](image)

**FIG. 7:** The lower-to-upper (a) and upper-to-lower tunneling (b) in the two-band model with a
weak nonlinearity-induced initial population of the adjacent Bloch band (the upper band in case of 
(a) and the lower one in case of (b)). The probability of tunneling (the final population of the Bloch
band to which the tunneling takes place) is given vs. the nonlinearity coefficient $c$. The initial
conditions are (a) $[a_1, a_2] = [\sqrt{1-r}c, \sqrt{r}c]$ and (b) $[a_1, a_2] = [\sqrt{r}c, \sqrt{1-r}c]$. The dashed lines give
the tunneling probability for $r = 0$. The solid lines have $r = \{0.01; 0.05; 0.075; 0.1; 0.15; 0.2\}$ (the
solid lines deviate from the dashed one as $r$ increases). Here $v_0 = 0.134$ and $\alpha = 0.036$. The time
interval is $[-60, 60]$.

VI. CONCLUSION

In the present work a finite-dimensional model has been derived for the condensate tun-
neling at the edge of the Brillouin zone in an accelerating optical lattice. The fine Bloch
band structure was taken into account. A special attention was paid to the case of a weak
lattice, which could have seemed a redundant task due to the existence of the nonlinear
Landau-Zener model proposed for this case previously [15, 16]. However, it turns out that the two-band model derived in this paper and the Landau-Zener model disagree significantly even in the case of a negligible nonlinearity if both adjacent bands are populated initially. Only in the special case of just one Bloch band being initially populated the two models agree in the prediction of the final populations. In particular, the Landau-Zener result [19] is reproduced by our model.

The two-band model confirms the asymmetry in the tunneling probability dependence on the nonlinearity, predicted theoretically in Ref. [16] and discovered experimentally in Ref. [5]. However, besides the qualitative agreement between the theory and the experiment, a quantitative disagreement was also revealed. For instance, the experimental upper-to-lower tunneling seems to be unaffected by the nonlinearity [5]. We have proposed an explanation for this fact based on the discovered strong dependence of the final Bloch band populations on the initial populations. Indeed, additionally to the asymmetry in the dependence of the tunneling probability on the nonlinearity strength, we find also the asymmetry in the sensitivity of the tunneling probability to the initial populations: the upper-to-lower tunneling probability is stronger affected by a variation of the initial Bloch band populations than the lower-to-upper tunneling probability. For instance, about one per cent of BEC atoms populating the lower Bloch band due to nonlinearity-induced Bloch band coupling significantly modifies the upper-to-lower tunneling probability, whereas the lower-to-upper tunneling probability is only weakly affected by the same initial population of the upper Bloch band.

Though it is in the dependence of the final Bloch band populations on the initial ones where our model disagrees with the nonlinear Landau-Zener model, nevertheless, both models show strong dependence of the final Bloch band populations on the initial populations. Thus an experimental study and/or numerical simulations of the full GPE equation with various initial Bloch band populations is in order to confirm the strong dependence of the final populations on the initial conditions and compare with the two models.

So far we have discussed the so-called “Landau-Zener regime” of BEC tunneling neglecting the spatial modulation of the Bloch wave amplitudes. Recently the modulational instability of a Bloch wave amplitude was found to result in the irreversibility of tunneling and the asymmetry in the final band populations [11]. A unified approach, capable to cover the two identified regimes, the “Landau-Zener regime” and the “instability regime” can also be
developed. The solution of this problem is left for a future publication.

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APPENDIX A: THE COUPLING COEFFICIENT BETWEEN THE BLOCH BANDS

The inner product \( \langle u_{n,q} | \partial_q u_{m,q} \rangle \) can be easily evaluated by using the properties of the Bloch functions \( u_{n,q}(x) \). For generality, consider an arbitrary periodic potential \( v = v(x) \). Define the Hermitian operator \( L_q \equiv \frac{1}{2}(-i\partial_x + q)^2 + v(x) \). We have then \( L_q u_{n,q} = E_n(q) u_{n,q} \) and, hence

\[
L_q \partial_q u_{n,q} = E_n(q) \partial_q u_{n,q} + \frac{dE_n(q)}{dq} u_{n,q} + i\partial_x u_{n,q} - qu_{n,q}.
\]

(A1)

Using this result and orthogonality of the functions \( u_{n,q} \) we get

\[
E_n(q) \langle u_{n,q} | \partial_q u_{m,q} \rangle = \langle u_{n,q} | L_q | \partial_q u_{m,q} \rangle = E_m(q) \langle u_{n,q} | \partial_q u_{m,q} \rangle + i \langle u_{n,q} | \partial_x u_{m,q} \rangle.
\]

Therefore, for \( m \neq n \) we obtain

\[
\langle u_{n,q} | \partial_q u_{m,q} \rangle = -i \frac{\langle u_{n,q} | \partial_x u_{m,q} \rangle}{E_m(q) - E_n(q)}.
\]

(A2)

Now, let us evaluate the product on the r.h.s. of equation (A2). We have

\[
L_q \partial_x u_{m,q} = E_m(q) \partial_x u_{m,q} - \frac{dv(x)}{dx} u_{m,q}.
\]

Hence,

\[
E_n(q) \langle u_{n,q} | \partial_x u_{m,q} \rangle = \langle u_{n,q} | L_q | \partial_x u_{m,q} \rangle = E_m(q) \langle u_{n,q} | \partial_x u_{m,q} \rangle - \langle u_{n,q} | \frac{dv(x)}{dx} | u_{m,q} \rangle
\]

with the result

\[
\kappa_{nm} \equiv \langle u_{n,q} | \partial_q u_{m,q} \rangle = -i \frac{\langle u_{n,q} | \partial_x u_{m,q} \rangle}{E_m(q) - E_n(q)} = -i \frac{\langle u_{n,q} | \frac{dv(x)}{dx} | u_{m,q} \rangle}{(E_m(q) - E_n(q))^2}, \quad n \neq m.
\]

(A3)
What is left is to consider the diagonal inner product \( \langle u_{n,q} | \partial_q u_{n,q} \rangle \). Here we use that the potential is even function: \( v(-x) = v(x) \). Then the functions \( u_{n,q}(x) \) can be selected to satisfy \( u_{n,q}^*(x) = u_{n,q}(-x) \). We get

\[
\langle u_{n,q} | \partial_q u_{n,q} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \: u_{n,q}(-x) \partial_q u_{n,q}(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \: (-\partial_q u_{n,q}(-x)) u_{n,q}(x)
\]

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} dx \: (-\partial_q u_{n,q}(x)) u_{n,q}(-x) = -\langle u_{n,q} | \partial_q u_{n,q} \rangle,
\]

where to replace the \( q \)-derivative we have used the independence of the inner product on \( q \):

\( \langle u_{n,q} | u_{n,q} \rangle = 1 \). Hence \( \langle u_{n,q} | \partial_q u_{n,q} \rangle = 0 \). A similar reasoning for \( \langle u_{n,q} | (-i)\partial_x u_{m,q} \rangle \) can be used also to establish that \( \kappa_{nm}(q) \) is real and that \( \kappa_{nm}(q) = -\kappa_{mn}(q) \).

### APPENDIX B: THE TWO-BAND MODEL IN THE DIABATIC BASIS

Unitary transformation (27) for the two-band model can be interpreted as a relation between the adiabatic basis \((a_1, a_2)\), where the amplitudes are the band populations, and the diabatic basis \((a, b)\), where the variables have the physical meaning of the amplitudes of the incident and Bragg scattered waves. For simplicity, we consider below only the linear case, \( c = 0 \) (the nonlinearity does not pose any problems, but complicates the presentation). After simple calculations one arrives at the following system in the diabatic basis:

\[
i \frac{d a}{d t} = E(q) a + K(q) b,
\]

\[
\frac{d b}{d t} = -E(q) b + K^*(q) a,
\]

where the coefficients are as follows

\[
E(q) = \frac{1 - \nu^2(q)}{1 + \nu^2(q)} \varepsilon(q), \quad K(q) = \frac{2\nu(q)\varepsilon(q)}{1 + \nu^2(q)} + \frac{i\alpha}{\varepsilon(q)} \frac{\nu(q)}{1 + \nu^2(q)}.
\]

Here the functions \( \varepsilon(q) \) and \( \nu(q) \) are taken from section III. (The system (B1)-(B3) was also verified numerically against the two-band model.)

It is easy to see that \( E(q) \) is an odd function of \( k_B - q \), while \( K(q) \) is an even function of this variable. The following asymptotic properties of the coefficients can be easily established.
Recalling that \( \dot{q} = -\alpha \) we get as \( t \to \pm \infty \):

\[
E(q) \to -\frac{\alpha t}{2}, \quad K(q) \to \frac{v_0}{2}.
\] (B4)

At the Brillouin zone edge we get \( E(k_B) = 0 \) and \( K(k_B) = v_0/2 + i\alpha/v_0 \).

However, an important fact is observed about the system \( (B1)-(B2) \): the coupling coefficient \( K(q) \) is a complex valued function. Therefore, to obtain an equivalent Landau-Zener model (with a real coupling coefficient), one has to perform the following time-dependent phase transformation

\[
a = a^{(LZ)} e^{i\Theta(q)}, \quad b = b^{(LZ)} e^{-i\Theta(q)}, \quad \Theta(q) \equiv \arg\{K(q)\}/2.
\] (B5)

For the phase-transformed amplitudes we obtain:

\[
i \frac{da^{(LZ)}}{dt} = E^{(LZ)}(q)a^{(LZ)} + |K(q)|b^{(LZ)},
\] (B6)

\[
i \frac{db^{(LZ)}}{dt} = -E^{(LZ)}(q)b^{(LZ)} + |K(q)|a^{(LZ)},
\] (B7)

where \( E^{(LZ)}(q) = E(q) + \alpha \Theta'(q) \) (\( \Theta' \equiv d\Theta/dq \)).

It is easy to see that \( E^{(LZ)} \to -\alpha t/2 \) as \( t \to \pm \infty \) and \( E^{(LZ)}(k_B) = 0 \), i.e. we have all the necessary prerequisites for application of the arguments due to Landau and Zener \[19\] to the system \( (B6)-(B7) \). This explains the agreement between the Landau-Zener model and the two-band model when just one band is populated initially.

[1] B. P. Anderson and M. A. Kasevich, Science 282, 1686 (1998).
[2] F. S. Cataliotti, S. Burger, C. Fort, P. Maddaloni, F. Minardi, A. Trombettoni, A. Smerzi, and M. Inguscio, Science 293, 843 (2001).
[3] O. Morsch, J. H. Müller, M. Cristiani, D. Ciampini, and E. Arimondo, Phys. Rev. Lett. 87, 140402 (2001).
[4] M. Cristiani, O. Morsch, J. H. Müller, D. Ciampini, and E. Arimondo, Phys. Rev. A 65, 063612 (2002).
[5] M. Jona-Lasinio, O. Morsch, M. Cristiani, N. Malossi, J. H. Müller, E. Courtade, M. Anderlini, and E. Arimondo, Phys. Rev. Lett. 91, 230406 (2003).
[6] M. Jona-Lasinio, O. Morsch, M. Cristiani, E. Arimondo, and C. Menotti, e-print cond-mat/0501572.

[7] S. Burger, F. S. Cataliotti, C. Fort, F. Minardi, and M. Inguscio, M. L. Chiofalo and M. P. Tosi, Phys. Rev. Lett. 86, 4447 (2001).

[8] L. Fallani, L. De Sarlo, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. Lett. 93 140406 (2004).

[9] F. Cataliotti, L. Fallani, P. Ferlaino, C. Fort, P. Maddaloni, and M. Inguscio, New. J. Phys. 5, 71 (2003).

[10] L. De Sarlo, L. Fallani, J. E. Lye, M. Modugno, R. Saers, C. Fort, and M. Inguscio, Phys. Rev. A 72, 013603 (2005).

[11] V. V. Konotop, P. G. Kevrekidis, and M. Salerno, Phys. Rev. A 72, 023611 (2005).

[12] B. Wu and Q. Niu, New J. Phys. 5, 104 (2003) and references therein.

[13] V. V. Konotop and M. Salerno, Phys. Rev. A 65, 021602 (2002).

[14] B. Eiermann, Th. Anker, M. Albiez, M. Taglieber, P. Treutlein, K.-P. Marzlin, and M. K. Oberthaler, Phys. Rev. Lett. 92, 230401 (2004).

[15] B. Wu and Q. Niu, Phys. Rev. A 61, 023402 (2000).

[16] O. Zobay and B. M. Garry, Phys. Rev. A 61, 033603 (2000).

[17] J. Liu, L. Fu, B.-Y. Ou, S.-G. Chen, D.-I. Choi, B. Wu, and Q. Niu, Phys. Rev. A 66, 023404 (2002).

[18] J. M. Ziman, The principles of the theory of solids, (Cambridge University Press, 1972).

[19] L.D. Landau, Phys. Z. Sowjetunion 2 (1932) 46; C. Zener, Proc. R. Soc. London A 137 (1932) 696.

[20] W. V. Houston, Phys. Rev. 57, 184 (1940).

[21] M. B. Dahan, E. Peik, J. Reichel, Y. Castin, and C. Salomon, Phys. Rev. Lett. 76, 4508 (1996).

[22] E. Peik, M. B. Dahan, I. Bouchoule, Y. Castin, and C. Salomon, Phys. Rev. A 55, 2989 (1997).

[23] D. A. Garanin, and R. Schilling, Phys. Rev. B 66, 174438 (2002).

[24] L. Pitaevskii and S. Stringari, Bose-Einstein Condensation (Clarendon Press, Oxford, 2003).

[25] The condition $a_s n_{1D} \ll 1$ means that the transverse kinetic term is much larger than the nonlinear term, which is a requisite for an effective factorization of the order parameter.
Indeed, “adding back” the transverse terms to equation (2) we arrive at the condition: \[ \frac{\hbar^2}{2m\ell_\perp} \gg \frac{m\nu|\psi|^2}{2\pi\ell_\perp}. \]
Using \(|\psi|^2 \sim 1\), due to the normalization as indicated in the text, and the expression for \(g\) one arrives at \(a_s n_{1D} \ll 1\).

[26] V. V. Konotop and G. P. Tsironis, Phys. Rev. A 53, 5393 (1996).

[27] D.-I. Choi and Q. Niu, Phys. Rev. Lett. 82, 2022 (1999).

[28] V. N. Serkin, M. Matsumo, and T. L. Belyaeva, JETP Letters 73, 59 (2001); V. N. Serkin and T. L. Belyaeva, Quantum Electronics, 31, 1006 (2001).

[29] P. J. Louis, E. A. Ostrovskaya, C. M. Savage, and Yu. S. Kivshar, Phys. Rev. A 67, 013602 (2003).

[30] B. Wu, private communication.

[31] The particular choice of the dependence of the initial population in the adjacent band on the nonlinearity coefficient \(c\) is not crucial as long as the population grows with \(c\). The point is that the asymmetry in the dependence of the tunneling probability on the initial band populations would be qualitatively the same. On the other hand, since we use the analytical formulae of the weak lattice approximation valid only in the vicinity of the Brillouin zone edge, we can claim only the qualitative correspondence with the full GPE equation.