Estimates and properties of certain $q$-Mellin transform on generalized $q$-calculus theory

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Abstract

This paper deals with the generalized $q$-theory of the $q$-Mellin transform and its certain properties in a set of $q$-generalized functions. Some related $q$-equivalence relations, $q$-quotients of sequences, $q$-convergence definitions, and $q$-delta sequences are represented. Along with that, a new $q$-convolution theorem of the estimated operator is obtained on the generalized context of $q$-Boehmians. On top of that, several results and $q$-Mellin spaces of $q$-Boehmians are discussed. Furthermore, certain continuous $q$-embeddings and an inversion formula are also discussed.

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1 Introduction and preliminaries

The quantum calculus or the $q$-calculus theory has been given a noticeable importance and popularity due to its wide application in various fields of mathematics, statistics, and physics [1]. The $q$-calculus theory has appeared as a connection between mathematics and physics. Recently, this topic has attracted the attention of several researchers, and a variety of results have been derived in various areas of research including number theory, hypergeometric functions, orthogonal polynomials, quantum theory, combinatorics, and electronics as well. The $q$-calculus begins with the definition of the $q$-analogue $d_qg$ of the differential

$$d_qg(t) = g(qt) - g(t)$$

of the function $g$, where $q$ is a fixed real number such that $0 < q < 1$ (see [1–3]). Having said this, we immediately get the $q$-analogue of the derivative of $g$ as

$$D_qg(t) := \frac{d_qg(t)}{d_qt} := \frac{g(t) - g(qt)}{(1 - q)t} \quad \text{for } t \neq 0$$

and $D_qg(0) = \lim_{t \to 0} D_qg(t) = g'(0)$ provided $g'(0)$ exists. Also, when $g$ is differentiable, the $q$-derivative $D_qg$ tends to $g'(0)$ as $q$ tends to 1. It also satisfies the $q$-analogue of the Leibniz
rule

\[ D_q(g_1(t)g(t)) = g(t)D_qg_1(t) + g_1(qt)D_qg(t). \]

The Jackson \( q \)-integrals from 0 to \( x \) and respectively from 0 to \( \infty \) are defined by \[1, 4\]

\[
\int_0^x g(t) \, dq(t) = (1-q)x \sum_{k=0}^{\infty} g(q^k)q^k, \tag{1}
\]

\[
\int_0^\infty g(t) \, dq(t) = (1-q) \sum_{k=\infty}^{0} g(q^k)q^k, \tag{2}
\]

when the sums converge absolutely. The Jackson \( q \)-integral on the generic interval \([a, b]\) is, therefore, given by \[1, 5\]

\[
\int_a^b g(t) \, dq(t) = \int_a^b g(t) \, dq(t) - \int_a^b g(t) \, dq(t).
\]

The \( q \)-integration by parts for two functions \( f \) and \( g \) is defined by

\[
\int_a^b g_2(t)D_qg_1(t) \, dq(t) = g_1(b)g_2(b) - g_1(a)g_2(a) - \int_a^b g_1(qt)D_qg_2(t) \, dq(t).
\]

Arising from the notion of regular operators \[6\], the notion of a Boehmian was firstly introduced by Mikusinski and Mikusinski \[7\] to generalize distributions and regular operators \[8\]. Boehmians are equivalence classes of quotients of sequences constructed from an integral domain when the operations are interpreted as addition and convolution, see, e.g., \[9–20\]. In terms of the \( q \)-calculus concept, we introduce the concept of \( q \)-Boehmians to popularize the concept of \( q \)-calculus theory as follows:

For a complex linear space \( V \) and a subspace \((W, \ast_q)\) of \( V \), let \( \bullet : V \times W \to V \) be a binary operation such that the undermentioned axioms (1)–(5) hold:

1. \((g_1 + g_2) \bullet \psi = g_1 \bullet \psi + g_2 \bullet \psi, \forall g_1, g_2 \in V \text{ and } \psi \in W.\)
2. \((\alpha g) \bullet \psi = \alpha (g \bullet \psi), \forall \alpha \in \mathbb{C}, \forall g \in V \text{ and } \psi \in W.\)
3. \(g_1 \bullet (g_2 \ast_q \psi_2) = (g_1 \bullet g_2) \ast_q \psi_2, \forall g \in V \text{ and } \psi_1, \psi_2 \in W.\)
4. \(g_n \rightarrow g \text{ in } V \text{ as } n \rightarrow \infty \text{ and } \psi \in W, \text{ then } g_n \bullet \psi \rightarrow g \bullet \psi \text{ as } n \rightarrow \infty \text{ in } V.\)

5. A collection \( \Delta_q \) of sequences from \( W \) such that, for all \((e_n), (\phi_n) \in \Delta_q \) and \((g_n) \in W,\)

we have \( e_n \bullet \phi_n \in \Delta_q \) and

\[
\text{if } g_n \rightarrow g \text{ in } V \text{ as } n \rightarrow \infty, \text{ then } g_n \bullet e_n \rightarrow g \text{ as } n \rightarrow \infty.\]

Once the preceding axioms are applied, the name of a \( q \)-Boehmian is set to mean the equivalence class \( \frac{g_n}{e_n} \) that arises from the equivalence relation

\[ g_n \bullet e_m = g_m \bullet e_n, \quad \forall m, n \in \mathbb{N}, \tag{4} \]
where \((g_n) \in V\) and \((\varepsilon_n) \in \Delta_q\). The collection of all \(q\)-Boehmians is denoted by \(B_q\) which is the so-called Boehmian space. The classical linear space \(V\) is identified as a subset of the space \(B_q\) which can be recognized from the relation

\[ g \longrightarrow g \bullet \varepsilon_n, \]

where \((\varepsilon_n) \in \Delta_q\) is arbitrary. Two \(q\)-Boehmians \(g_n \bullet \varepsilon_n\) and \(\varphi_n \bullet \varepsilon_n\) are said to be equal in \(B_q\) if

\[ g_n \bullet \varepsilon_n = \varphi_m \bullet \varepsilon_n, \quad \forall m, n \in \mathbb{N}. \]

Addition in the space \(B_q\) is defined as

\[ g_n \bullet \varepsilon_n + \varphi_n \bullet \varepsilon_n = g_n \bullet \varepsilon_n + \varphi_n \bullet \varepsilon_n. \]

The scalar multiplication in the space \(B_q\) is defined as

\[ \alpha \cdot g_n \bullet \varepsilon_n = \alpha g_n \bullet \varepsilon_n, \quad \alpha \in \mathbb{C}. \]

The \(q\)-convergence of type \(\delta\), \(\beta_n \xrightarrow{\Delta_q} \beta\), is defined in the space \(B_q\) when for \((\psi_n) \in \Delta_q\) and each \(k \in \mathbb{N}\) such that

\[ \beta_n \bullet \varepsilon_k \in V, \quad \forall k, n \in \mathbb{N}, \beta \bullet \varepsilon_k \in V, \]

we have \(\beta_n \bullet \varepsilon_k \to \beta \bullet \varepsilon_k\) as \(n \to \infty\) in \(V\). The \(q\)-convergence \(\beta_n \xrightarrow{\Delta_q} \beta\) of type \(\Delta_q\) is defined when for some \((\varepsilon_n) \in \Delta_q\) we have

\[ (\beta_n - \beta) \bullet \varepsilon_n \in V, \quad \forall n \in \mathbb{N} \quad \text{and} \quad (\beta_n - \beta) \bullet \varepsilon_n \to 0 \quad \text{as} \quad n \to \infty \quad \text{in} \quad V. \]

The space of \(q\)-Boehmians emerging from the \(q\)-convergence assigns a complete quasi-normed space.

In recent work, several remarkable integral transforms were given different \(q\)-analogues in a \(q\)-calculus context [4, 21–24]. In the sequence of such \(q\)-integral transforms, we recall the \(q\)-Laplace integral transform [25–29], the \(q\)-Sumudu integral transform [2, 30–32], the \(q\)-Weyl fractional integral transform [33], the \(q\)-wavelet integral transform [34], the \(q\)-Mellin type integral transform [35], the Mangontarum integral transform [36, 37], the \(E_{2;1}\) integral transform [38, 39], the natural integral transform [3], and many others, to mention but a few. In this paper, we discuss the generalized \(q\)-theory of the \(q\)-Mellin transform and obtain several results.

Let \(g\) be a function defined on \(\mathbb{R}_{q_1}, \mathbb{R}_{q_2} = \{q^n : n \in \mathbb{Z}\}\), then the \(q\)-Mellin transform was defined by [40], p. 521 as

\[ M_q(g(t))(\xi) = \int_0^\infty t^{\xi-1} g(t) dq_t, \]

provided the \(q\)-integral converges. The integral (9) is analytic on the fundamental strip \((a_{q_2}, b_{q_1})\) and periodic with period \(2i\pi \log(q)\). The inversion formula for the \(q\)-analogue
(9) is given by
\[ g(t) = \frac{\log(q)}{2\pi i(1-q)} \int_{c-i\pi}^{c+i\pi} M_q(g)(\zeta) t^{\zeta} d\zeta, \quad t \in \mathbb{R}_{q,+}, \]
where \( \alpha_{qg} < c < \beta_{qg} \). The asymptotic properties as well as the asymptotic singularities of the \( q \)-Mellin transform into asymptotic expansions of the original function for \( x \to 0 \) and \( x \to \infty \) are given in [40]. Additionally, the asymptotic behavior at 0 or \( \infty \) is studied using the \( q \)-Mellin transform.

**Definition 1** The function \( g \) is said to be \( q \)-integrable on an interval \([0, \infty]\) provided the infinite series
\[ \sum_{n \in \mathbb{Z}} q^n g(q^n) \]
converges absolutely. The space of all \( q \)-integrable functions on \([0, \infty]\) is denoted by \( L^1_q(\mathbb{R}_{q,+}) \). In a better recognition, the space \( L^1_q(\mathbb{R}_{q,+}) \) is defined to be the space of all \( q \)-integrable functions \( g \) on \( \mathbb{R}_{q,+} \) such that
\[ L^1_q(g)(t) = \frac{1}{1-q} \int_0^\infty |g(t)| d_q t < \infty. \]
(10)

We denote by \( \mathbb{D}_q \) the \( q \)-space of test functions of compact supports on \( \mathbb{R}_{q,+} \), i.e., \( \mathbb{D}_q \) is the \( q \)-space of all smooth functions \( \kappa \in C^\infty(\mathbb{R}_{q,+}) \) such that
\[ \mathbb{D}_q = \left\{ \kappa \in C^\infty(\mathbb{R}_{q,+}) : \sup_{0<a<\infty} |D_q \kappa(t)| < \infty \right\}. \]
(11)

However, this theory is new and might be developing a new area of research. It investigates a generalization to the \( q \)-theory of calculus [40] and hence all results can be popularized. Every element in the space \( L^1_q(\mathbb{R}_{q,+}) \) is identified as a member in the generalized theory. To this aim, we spread our results into five sections. In Sect. 1, we recall some definitions and preliminaries from the \( q \)-calculus theory. In Sect. 2, we derive \( q \)-delta sequences, \( q \)-convolution theorems and establish a space of \( q \)-Boehmians. In Sect. 3, we establish a space of \( q \)-ultraBoehmians. In Sect. 4, we generalize definitions and obtain several properties of the \( q \)-Mellin transform. In Sect. 5 we include several results.

**2 The space \( \mathbb{B} \)**

In this section, we strive to establish the space \( \mathbb{B} \) of \( q \)-Boehmians. Henceforth, we denote by \( \Delta_q \) the set of all sequences from \( \mathbb{D}_q \) such that the undermentioned identities \( \Delta_q^1 - \Delta_q^3 \) hold, where
\[ \Delta_q^1 : \int_0^\infty |\epsilon_n(t)| d_q t = 1, \quad \forall n \in \mathbb{N}, \]
\[ \Delta_q^2 : |\epsilon_n(t)| < M, \quad M > 0, M \in \mathbb{R}_+, \]
\[ \Delta_q^3 : \text{supp}(\epsilon_n) \subseteq (0, b_n), \quad b_n \to 0 \text{ as } n \to \infty, 0 < b_n, \forall n \in \mathbb{N}. \]
(12)
On the other hand, we denote by $\hat{\cdot}$ the Mellin type $q$-convolution product defined on $L^1_q(\mathbb{R}_{q,+})$ by

$$ (g_1 \hat{\cdot} g_2)(x) = \int_0^\infty t^{-1} g_1(t^{-1}x) g_2(t) d_q t, \quad (13) $$

provided the integral part exists for every $x > 0$. It is clear from the context that $g_1 \hat{\cdot} g_2 \in L^1_q(\mathbb{R}_{q,+})$ for all $g_1$ and $g_2$ in $L^1_q(\mathbb{R}_{q,+})$. On that account, the $q$-convolution theorem of the $q$-Mellin transform of the product $g_1 \hat{\cdot} g_2$ can be easily established as follows.

**Theorem 2** Let $L^1_q(\mathbb{R}_{q,+})$ be the space of all $q$-integrable functions on $\mathbb{R}_{q,+}$. Then the $q$-convolution theorem of the transform $M_q$ is given by

$$ M_q(g_1 \hat{\cdot} g_2)(\zeta) = M_q(g_1)(\zeta) M_q(g_2)(\zeta) \quad \text{for} \ g_1 \text{ and } g_2 \in L^1_q(\mathbb{R}_{q,+}). $$

**Proof** By applying the definition of the $M_q$ transform to the product $g_1 \hat{\cdot} g_2$, we get

$$ M_q(g_1 \hat{\cdot} g_2)(\zeta) = \int_0^\infty (g_1 \hat{\cdot} g_2)(x) x^{-\zeta-1} d_q x $$

$$ = \int_0^\infty \left( \int_0^\infty g_1(t) g_2(t^{-1}x) x^{-1} d_q t \right) x^{-\zeta-1} d_q x. $$

Therefore, employing the substitution $z = t^{-1}x$ and, hence, $d_q z = t^{-1} d_q x$, in collaboration with simple computations, reveals

$$ M_q(g_1 \hat{\cdot} g_2)(\zeta) = M_q(g_1)(\zeta) M_q(g_2)(\zeta). $$

Hence, the proof of this theorem is completed. $\Box$

The following is an imperative result for initiating the $q$-delta sequence concept.

**Lemma 3** Let $(\epsilon_n)$ and $(\eta_n)$ be sequences in $\Delta_q$. Then $(\epsilon_n \hat{\cdot} \eta_n)$ is a sequence in $\Delta_q$.

**Proof** To establish this lemma, we examine the performance of the sequence $(\epsilon_n \hat{\cdot} \eta_n)$. To inspect the correctness of the property $\Delta_q^1$, we use the integral equation (3) to get

$$ \int_0^\infty (\epsilon_n \hat{\cdot} \eta_n)(x) d_q x = \int_0^\infty t^{-1} \epsilon_n(t) \left( \int_0^\infty \eta_n(t^{-1}x) d_q x \right) d_q t. \quad (14) $$

Therefore, by using the change of variables $t^{-1}x = y$ and, hence, $d_q x = t d_q y$, (14) we indicate

$$ \int_0^\infty (\epsilon_n \hat{\cdot} \eta_n)(x) d_q x = \left( \int_0^\infty \epsilon_n(t) d_q t \right) \left( \int_0^\infty \eta_n(y) d_q y \right) = 1. $$

This proves the $\Delta_q^1$ part. The proof of the $\Delta_q^2$ part follows from similar techniques, whereas the $\Delta_q^3$ part is clearly valid, by conducting the fact

$$ \text{supp}(\epsilon_n \hat{\cdot} \eta_n) \subset \text{supp}(\epsilon_n) + \text{supp}(\eta_n) \quad \text{for} \ (\epsilon_n), (\eta_n) \in \Delta_q. $$

This ends the proof of the lemma. $\Box$
Lemma 3, hence, displays that every sequence in $\Delta_q$ forms, to a great extent, the $q$-delta sequence concept.

Lemma 4 Let $g_1, g_2 \in L^1_q(\mathbb{R}_q, +)$, $\kappa_1, \kappa_2 \in \mathbb{D}_q$, and $\alpha \in \mathbb{C}$. Then the following assertions are valid:

- (i) $\kappa_1 \diamond_q \kappa_2 = \kappa_2 \diamond_q \kappa_1$,
- (ii) $(g_1 + g_2) \diamond_q \kappa_1 = g_1 \diamond_q \kappa_1 + g_2 \diamond_q \kappa_1$,
- (iii) $(\alpha g_1) \diamond_q \kappa_1 = \alpha (g_1 \diamond_q \kappa_1)$,
- (iv) $g_1 \diamond_q (\kappa_1 \diamond_q \kappa_2) = (g_1 \diamond_q \kappa_1) \diamond_q \kappa_2$.

Proof (i) As the convolution product of the functions $\kappa_1$ and $\kappa_2$ in $\mathbb{D}_q$ is exceptionally given by

$$
(k_1 \diamond_q k_2)(x) = \int_0^\infty t^{-1} \kappa_1(t^{-1}x)k_2(t) d_q t,
$$

the change of variables $t^{-1}x = y$ reveals us to write (15) into the form

$$
(k_1 \diamond_q k_2)(x) = \int_0^\infty y^{-1} \kappa_2(x^{-1}y)k_1(y) d_q y.
$$

Hence (i) follows. To prove (ii) and (iii), we merely follow simple integral calculus. To prove (iv), we employ the definition of the product $\bullet$ to get

$$
(g_1 \bullet (k_1 \bullet k_2))(x) = \int_0^\infty t^{-1} g_1(t^{-1}x)(k_1 \bullet k_2)(t) d_q t
$$

$$
= \int_0^\infty t^{-1} g_1(t^{-1}x) \left(\int_0^\infty y^{-1} \kappa_1(y^{-1}t)k_2(y) d_q y\right) d_q t.
$$

That is,

$$
(g_1 \bullet (k_1 \bullet k_2))(x) = \int_0^\infty y^{-1} \left(\int_0^\infty t^{-1} g_1(t^{-1}x)k_1(y^{-1}t) d_q t\right)k_2(y) d_q y.
$$

(16)

Now, by employing the change of variables $y^{-1}t = z$, we write down equation (16) into the form

$$
(g_1 \bullet (k_1 \bullet k_2))(x) = \int_0^\infty y^{-1} \left(\int_0^\infty z^{-1} g_1(z^{-1}y^{-1}x)k_1(z) d_q z\right)k_2(y) d_q y
$$

$$
= \int_0^\infty y^{-1} (g_1 \diamond_q k_1)(y^{-1}x)k_2(y) d_q y.
$$

This ends the proof of the lemma. □

To proceed in our construction, we establish the following lemma.

Lemma 5 (i) Let $g_1$ and $g_2$ be integrable functions in $L^1_q(\mathbb{R}_q, +)$ and $(\varepsilon_n)$ be a delta sequence in the set $\Delta_q$ such that $g_1 \diamond_q \varepsilon_n = g_2 \diamond_q \varepsilon_n$. Then $g_1 \equiv g_2$ in $L^1_q(\mathbb{R}_q, +)$ for every $n \in \mathbb{N}$. 
(ii) Let \( g \) and \((g_n)\) be integrable functions in \( L^1_q(\mathbb{R}_{q,+}) \) such that \( g_n \to g \) as \( n \to \infty \) in \( L^1_q(\mathbb{R}_{q,+}) \). Then

\[
g_n \bullet^q \psi \to g \bullet^q \psi \quad \text{for every} \quad \psi \in D_q \quad \text{as} \quad n \to \infty.
\]

**Proof** To prove (i), we merely need to show that \( g_1 \bullet^q \varepsilon_n = g_1 \in L^1_q(\mathbb{R}_{q,+}) \). By using \( \Delta^1_q \) and \( \Delta^3_q \), we obtain

\[
\int_0^\infty \left| (g_1 \bullet^q \varepsilon_n)(x) - g_1(x) \right| \, dq_x \leq \int_0^\infty \int_0^\infty \left| t^{-1} g_1(t^{-1} x) - g_1(x) \right| \, dq_t \, dq_x
\]

\[
= \int_0^\infty \int_{a_n}^{b_n} \left| t^{-1} g_1(t^{-1} x) - g_1(x) \right| \, dq_t \, dq_x.
\]

Therefore,

\[
\int_0^\infty \left| (g_1 \bullet^q \varepsilon_n)(x) - g_1(x) \right| \, dq_x \leq \int_0^\infty \int_{a_n}^{b_n} \left| t^{-1} \right| \, dq_t \, dq_x
\]

\[
+ \int_0^\infty \int_{a_n}^{b_n} \left| g_1(x) \right| \, dq_t \, dq_x.
\]

(17)

Hence, for \( g_1 \in L^1_q(\mathbb{R}_{q,+}) \), by using (17) we turn to write

\[
\int_0^\infty \left| (g_1 \bullet^q \varepsilon_n)(x) - g_1(x) \right| \, dq_x \leq A \int_0^{b_n} \left| t^{-1} \right| \, dq_t + A \int_0^{b_n} \left| \varepsilon_n(t) \right| \, dq_t.
\]

Therefore, by the properties of the delta sequences \( \Delta^2_q \) and \( \Delta^3_q \), we conclude that

\[
\int_0^\infty \left| (g_1 \bullet^q \varepsilon_n)(x) - g_1(x) \right| \, dq_x \leq AM \ln(b_n) + AM(b_n) \to 0
\]

as \( n \to \infty \).

Proof of (ii) follows from simple integration. We therefore omit the details. Hence the proof of this lemma is ended. \( \square \)

**Lemma 6** Let \( g_1 \) be an integrable function in the space \( L^1_q(\mathbb{R}_{q,+}) \). Then \( g_1 \bullet^q \varepsilon_n \to g_1 \) as \( n \to \infty \) for every \( (\varepsilon_n) \in \Delta_q \).

The proof of this lemma is a straightforward conclusion from the proof of Lemma 4. Hence, we delete the details.

Thus, the space \( B \) with \( (L^1_q(\mathbb{R}_{q,+}), \bullet^q), (\mathbb{D}_q, \bullet^q) \), and \( \Delta_q \) is defined. The canonical embedding of \( L^1_q(\mathbb{R}_{q,+}) \) in \( B \) is given by

\[
g \to \frac{g \bullet \varepsilon_n}{\varepsilon_n}.
\]

(18)
That is, every element in the space $L^1_q(\mathbb{R}_q)$ can be identified as a member of the space $B$. Addition, scalar multiplication, differentiation, $\Delta_q$ and $\delta_q$ convergence are defined in a natural way as follows:

If $(\varphi_n) \in L^1_q(\mathbb{R}_q)$ and $(\epsilon_n) \in \Delta_q$, then the pair $(\varphi_n, \epsilon_n)$ (or $\frac{\varphi_n}{\epsilon_n}$) is said to be a $q$-quotient of sequences if $\frac{\varphi_n}{\epsilon_n} \cdot m = \frac{\varphi_m}{\epsilon_m}, \forall n, m \in \mathbb{N}$. Therefore, if $\frac{\varphi_n}{\epsilon_n}$ and $\frac{\varphi_m}{\epsilon_m}$ are $q$-quotients of sequences and $g \in L^1_q(\mathbb{R}_q)$, then it is easy to see that

$$\frac{g}{\epsilon_n}$$

and

$$\frac{\varphi_n}{\epsilon_n} + \frac{g}{\epsilon_n}$$

are $q$-quotients of sequences. Two $q$-quotients of sequences $\frac{\varphi_n}{\epsilon_n}$ and $\frac{\varphi_m}{\epsilon_m}$ are said to be equivalent if

$$\frac{\varphi_n}{\epsilon_n} \cdot m = \frac{\varphi_m}{\epsilon_m}, \forall n, m \in \mathbb{N}.$$

We can easily check the following equivalence relations:

$$\frac{\varphi_n}{\epsilon_n} \sim \frac{\varphi_n}{\epsilon_n} \cdot g$$

and

$$\frac{\varphi_n}{\epsilon_n} \sim \frac{\varphi_n}{\epsilon_n} \cdot g.$$

The equivalent class $\tilde{w} = (\frac{\varphi_n}{\epsilon_n})$ of $q$-quotients of sequences containing $\frac{\varphi_n}{\epsilon_n}$ is said to be a $q$-Boehmian. The space of such $q$-Boehmians is denoted by $B$.

**Remark 7** For two $q$-Boehmians $\tilde{w} = (\frac{\varphi_n}{\epsilon_n})$ and $\tilde{z} = (\frac{\varphi_n}{\epsilon_n})$ in $B$, we have the following identities:

(i) $\tilde{w} + \tilde{z} = \left(\frac{\varphi_n}{\epsilon_n} \cdot g_n \cdot \epsilon_n\right)$,

(ii) $\beta \tilde{w} = \left(\frac{\beta \varphi_n}{\epsilon_n}\right)$,

(iii) $\tilde{w} \cdot \tilde{z} = \left(\frac{\varphi_n}{\epsilon_n} \cdot g_n \cdot \epsilon_n\right)$,

(iv) $D^k \tilde{w} = \left(\frac{D^k \varphi_n}{\epsilon_n}\right)$,

(v) $\tilde{w} \cdot g = \left(\frac{\varphi_n}{\epsilon_n} \cdot g\right)$,

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $D^k \tilde{w}$ is the $k$th derivative of $\tilde{w}$, and $\psi \in L^1_q(\mathbb{R}_q)$.

**Definition 8** (i) For $n = 1, 2, 3, \ldots$ and $\tilde{w}_n, \tilde{w} \in B$, the sequence $(\tilde{w}_n)$ is $\delta_q$-convergent to $\tilde{w}$, denoted by $\delta_q - \lim_{n \to \infty} \tilde{w}_n = \tilde{w}$, provided there can be found a $q$-delta sequence $(\epsilon_n)$ such that

$$(\tilde{w}_n \cdot \epsilon_k), (\tilde{w} \cdot \epsilon_k) \in L^1_q(\mathbb{R}_q) \quad \text{and} \quad \lim_{n \to \infty} \tilde{w}_n \cdot \epsilon_k = \tilde{w} \cdot \epsilon_k \in L^1_q(\mathbb{R}_q) \quad (\forall k \in \mathbb{N}).$$
(ii) For $n = 1, 2, 3, \ldots$ and $\tilde{w}_n, \tilde{w} \in \mathbb{B}$, the sequence $(\tilde{w}_n)$ is said to be $\Delta_q$-convergent to $\tilde{w}$, denoted by $\Delta_q\lim_{n \to \infty} \tilde{w}_n = \tilde{w}$, provided there can be found a $q$-delta sequence $(\varepsilon_n)$ such that
\[
(\tilde{w}_n - \tilde{w}) \hat{\varepsilon}_n \in L^1_q(\mathbb{R}_q, \mathbb{R}) \quad (\forall n \in \mathbb{N}) \quad \text{and} \quad \lim_{n \to \infty} (\tilde{w}_n - \tilde{w}) \hat{\varepsilon}_n = 0 \quad \text{in} \quad L^1_q(\mathbb{R}_q, \mathbb{R}).
\]

Now we have the following few corollaries.

**Corollary 9** (i) Let $g \in L^1_q(\mathbb{R}_q, \mathbb{R})$ and $(\varepsilon_n) \in \Delta_q$ be fixed. Then the mapping
\[
g \mapsto \tilde{w},
\]
where $\tilde{w} = \frac{q \varepsilon_n}{\varepsilon_0}$ is an injective mapping from $L^1_q(\mathbb{R}_q, \mathbb{R})$ into $\mathbb{B}$.

(ii) Let $(\varepsilon_n) \in \Delta_q$. Then, if $g_n \rightarrow g$ in $L^1_q(\mathbb{R}_q, \mathbb{R})$ as $n \rightarrow \infty$, then for all $k \in \mathbb{N}$,
\[g_n \hat{\varepsilon}_k \rightarrow g \hat{\varepsilon}_k \quad \text{and} \quad \tilde{w}_n \rightarrow \tilde{w} \quad \text{in} \quad \mathbb{B} \quad \text{as} \quad n \rightarrow \infty.
\]

Therefore, it can be easily checked that $L^1_q(\mathbb{R}_q, \mathbb{R})$ can be mathematically identified as a subspace of $\mathbb{B}$.

The above corollary leads to the following corollary.

**Corollary 10** The $q$-embedding, $g \mapsto \tilde{w}$, $\tilde{w} = \frac{q \varepsilon_n}{\varepsilon_0}$, of the space $L^1_q(\mathbb{R}_q, \mathbb{R})$ into the space $\mathbb{B}$ is continuous.

### 3 The $q$-ultraBoehmian space $\mathbb{B}_M$

In this section, we provide sufficient axioms to define the $q$-ultraBoehmian space $\mathbb{B}_M$ with the set $(L_M, \circ)$, the subset $(\hat{D}_M, \circ)$, the set $(\Delta_q M, \circ)$ of $q$-delta sequences, and the product $\circ$, where $L_M, \hat{D}_M$, and $\Delta_q M$ are the $q$-Mellin transforms of the sets $L^1_q(\mathbb{R})$, $D_q$, and $\Delta_q$ respectively. To this end, we introduce the following convolution operation.

**Definition 11** Let $\omega_1$ and $\omega_2$ be in $\mathbb{B}_M$. For $\omega_1$ and $\omega_2$, we define a product $\circ$ as
\[
(\omega_1 \circ \omega_2)(t) = \omega_1(t)\omega_2(t). \quad (19)
\]

The following assertion holds in the space $L_M$.

**Theorem 12** Let $\omega_1$ be in $L_M$. Then $\omega_1 \circ \eta \in L_M$ for all $\eta \in \hat{D}_M$.

**Proof** Let $\omega_1 \in L_M$. Then, by the definition of the space $L_M$ and the definition of the product $\circ$, we write
\[
(\omega_1 \circ \omega_2)(t) = \omega_1(t)\omega_2(t) = M_q(g_1)M_q(g_2) \quad (20)
\]
for some $g_1, g_2 \in L^1_q(\mathbb{R}_q, \mathbb{R})$. Hence, by virtue of Def. 11, (20) can be written in the form
\[
(\omega_1 \circ \omega_2)(t) = M_q(g_1 \hat{\varepsilon} g_2). \quad (21)
\]
Therefore, as \( g_1 \circ g_2 \in L_q^1(\mathbb{R}_q, \mathbb{R}) \), it follows from (21) that \( \omega_1 \circ \eta \in L_d \). This ends the proof of the theorem. 

\[ \square \]

**Theorem 13** Let \( \omega \) be an integrable function in \( L_d \). Then \( \omega \circ (\eta_1 \circ \eta_2) = (\omega \circ \eta_1) \circ \eta_2 \) for all \( \eta_1, \eta_2 \in \mathbb{D}_q \).

**Proof** By the concept of the convolution \( \circ \), we get

\[
(\omega \circ (\eta_1 \circ \eta_2))(t) = \omega(t)(\eta_1 \circ \eta_2)(t) = \omega(t)\eta_1(t)\eta_2(t).
\]

By using Def. 11 twice, we write the preceding equation as

\[
(\omega \circ (\eta_1 \circ \eta_2))(t) = (\omega \circ \eta_1)(t)\eta_2(t) = ((\omega \circ \eta_1) \circ \eta_2)(t).
\]

This ends the proof of the theorem. 

\[ \square \]

The following axioms are straightforward.

**Theorem 14** (i) Let \( \omega_1 \) and \( \omega_2 \) be in \( L_d \). Then \( (\omega_1 + \omega_2) \circ \eta = \omega_1 \circ \eta + \omega_2 \circ \eta \) for all \( \eta \in \mathbb{D}_q \).

(ii) Let \( \omega_1 \) be in \( L_d \). Then \( (\alpha \omega_1 \circ \eta) = \alpha(\omega_1 \circ \eta) \) for all \( \eta \in \mathbb{D}_q \) and \( \alpha \in \mathbb{C} \).

**Proof** (i) Let \( \omega_1 \) and \( \omega_2 \) be in \( L_d \). Then, by Def. 11, we write

\[
((\omega_1 + \omega_2) \circ \eta)(t) = (\omega_1 + \omega_2)(t)\eta(t) = \omega_1(t)\eta(t) + \omega_2(t)\eta(t) = (\omega_1 \circ \eta)(t) + (\omega_2 \circ \eta)(t).
\]

The proof of the first part is finished. The proof of the second part is trivial. This completes the proof of the theorem. 

\[ \square \]

**Theorem 15** (i) Let \( \omega_1 \) and \( (\omega_n) \) be members of the space \( L_d \) and \( \eta \in \mathbb{D}_M \). If \( \omega_n \to \omega_1 \) in \( L_d \) as \( n \to \infty \), then \( \omega_n \circ \eta \to \omega_1 \circ \eta \) as \( n \to \infty \).

(ii) Let \( \omega_1 \) and \( \omega_2 \) be in \( L_d \) and \( (\upsilon_n) \in \Delta_{q,M} \). If \( \omega_1 \circ \upsilon_n = \omega_2 \circ \upsilon_n \), then \( \omega_1 = \omega_2 \) in \( L_d \).

(iii) Let \( \omega_1 \) be an integrable function in \( L_d \) and \( (\upsilon_n) \in \Delta_{q,M} \). If \( \upsilon_n(t) \neq 0 \) for all \( t \in \mathbb{R}_{q,*} \). Then \( \omega_1 \circ \upsilon_n \to 0 \) in \( L_d \) as \( n \to \infty \).

**Proof** To prove (i), let \( \omega_1 \) and \( (\omega_n) \) be members of \( L_d \) and \( \eta \in \mathbb{D}_M \). If \( \omega_n \to \omega_1 \) in \( L_d \) as \( n \to \infty \), then by Def. 11 and Theo. 14, we have

\[
(\omega_n \circ \eta - \omega_1 \circ \eta)(t) = (\omega_n - \omega_1)(t)(\eta(t) = \omega_n(t)\eta(t) - \omega_1(t)\eta(t).
\]

Hence, by the hypothesis of the theorem, we obtain

\[
\omega_n \circ \eta - \omega_1 \circ \eta \to 0\quad \text{as } n \to \infty.
\]

Hence, the first part of the theorem is completely proved. To prove (ii), let \( \omega_1 \) and \( \omega_2 \) be in \( L_d \) and \( (\upsilon_n) \in \Delta_{q,M} \). If \( \omega_1 \circ \upsilon_n = \omega_2 \circ \upsilon_n \), then \( \omega_1(t)\upsilon_n(t) = \omega_2(t)\upsilon_n(t) \). Hence,

\[
(\omega_1 - \omega_2)(t)\upsilon_n(t) = 0\quad \text{for all } t \in \mathbb{R}_{q,*}.
\]
Hence, the theorem is completely proved.

Two $\omega$-quotients of sequences are said to be equivalent if

$$
\omega_n \circ \omega_m = \omega_m \circ \omega_n, \quad \forall n, m \in \mathbb{N}.
$$

Therefore, if $\omega_n$ and $\omega_m$ are $q$-quotients of sequences and $\omega \in L_M$, then it is easy to see that

$$
\frac{\omega \circ \epsilon_n}{\epsilon_n} \quad \text{and} \quad \frac{\omega_n \circ \epsilon_n + g_n \circ \epsilon_n}{\epsilon_n \circ \psi_n}
$$

are $q$-quotients of sequences. Furthermore, it is easy to see the following equivalence relations:

$$
\frac{\omega_n}{\epsilon_n} \sim \frac{\omega_n \circ \omega}{\epsilon_n} \quad \text{and} \quad \frac{\omega_n \circ \epsilon_n + g_n \circ \epsilon_n}{\epsilon_n \circ \psi_n} \sim \frac{\omega_n \circ g_n}{\epsilon_n}.
$$

Two $q$-quotients of sequences $\frac{\omega_n}{\epsilon_n}$ and $\frac{\omega_m}{\epsilon_m}$ are said to be equivalent if $\omega_n \circ \psi_m = g_m \circ \epsilon_n, \forall n, m \in \mathbb{N}$. The equivalent class $\tilde{\omega} = (\frac{\omega_n}{\epsilon_n})$ of $q$-quotients of sequences containing $\frac{\omega_n}{\epsilon_n}$ is said to be a $q$-Boehmian. The space of such $q$-Boehmians is denoted by $B_M$.

**Remark 16** For two $q$-Boehmians $\tilde{\omega} = (\frac{\omega_n}{\epsilon_n})$ and $\tilde{\omega} = (\frac{\omega_m}{\epsilon_m})$ in $B_M$, the following are well defined on $B_M$:

(i) $\tilde{\omega} + \tilde{\omega} = \left( \frac{\omega_n \circ \epsilon_n + g_n \circ \epsilon_n}{\epsilon_n \circ \psi_n} \right)$,

(ii) $\beta \tilde{\omega} = \left( \frac{\beta \omega_n}{\epsilon_n} \right)$,

(iii) $\tilde{\omega} \circ \tilde{\omega} = \left( \frac{\omega_n \circ g_n}{\epsilon_n \circ \psi_n} \right)$,

(iv) $D^k \tilde{\omega} = \left( \frac{D^k \omega_n}{\epsilon_n} \right)$,

(v) $\tilde{\omega} \circ \omega = \left( \frac{\omega_n \circ \omega}{\epsilon_n} \right)$,

where $k \in \mathbb{R}$, $\beta \in \mathbb{C}$ and $D^k \tilde{\omega}$ is the $k$th derivative of $\tilde{\omega}$, and $\psi \in L_M$.

**Definition 17** (i) For $n = 1, 2, 3, \ldots$ and $\tilde{\omega}_n, \tilde{\omega} \in B_M$, the sequence $(\tilde{\omega_n})$ is said to be $\delta_q$-convergent to $\tilde{\omega}$, denoted by $\delta_q - \lim_{n \to \infty} \tilde{\omega}_n = \tilde{\omega}$, provided there can be found a $q$-delta sequence $(\psi_n)$ such that

$$
(\tilde{\omega}_n \circ \psi_k, \tilde{\omega} \circ \psi_k) \in L_M \quad (\forall n, k \in \mathbb{N}) \quad \text{and} \quad \lim_{n \to \infty} \tilde{\omega}_n \circ \psi_k = \tilde{\omega} \circ \psi_k \quad \text{in} \quad L_M \quad (\forall k \in \mathbb{N}).
$$

(ii) For $n = 1, 2, 3, \ldots$ and $\tilde{\omega}_n, \tilde{\omega} \in B_M$, the sequence $(\tilde{\omega}_n)$ is said to be $\Delta_q$-convergent to $\tilde{\omega}$, denoted by $\Delta_q - \lim_{n \to \infty} \tilde{\omega}_n = \tilde{\omega}$, provided there can be found a $q$-delta $q$-sequence $(\psi_n)$ such that

$$
(\tilde{\omega}_n - \tilde{\omega}) \circ \psi_n \in L_M \quad (\forall n \in \mathbb{N}) \quad \text{and} \quad \lim_{n \to \infty} (\tilde{\omega}_n - \tilde{\omega}) \circ \psi_n = 0 \quad \text{in} \quad L_M.
$$
Corollary 18 (i) Let $\omega \in L_M$ and $(\upsilon_n) \in \Delta_q$ be fixed. Then the mapping

$$\omega \rightarrow \tilde{w},$$

where $\tilde{w} = \frac{\omega \circ \upsilon_n}{\upsilon_n}$ is an injective mapping from $L_M$ into $B_M$.

(ii) Let $(\upsilon_n) \in \Delta_q$. Then, if $\omega_n \rightarrow \omega$ in $L_M$ as $n \rightarrow \infty$, then for all $k \in \mathbb{N}$,

$$\omega_n \circ \upsilon_k \rightarrow \omega \circ \upsilon_k \quad \text{and} \quad \tilde{w}_n \rightarrow \tilde{w} \quad \text{in} \quad B_M \quad \text{as} \quad n \rightarrow \infty.$$ (22)

Therefore, it can be easily checked that $L_M$ may be identified as a subspace of $B_M$.

The above corollary can bestated as follows.

Corollary 19 The $q$-embedding $\psi \rightarrow \tilde{w}, \quad \tilde{w} = \frac{\omega \circ \upsilon_n}{\upsilon_n}$, of the space $L_M$ into the space $B_M$ is continuous.

4 The $q$-Mellin transform of the generalized $q$-theory

This section aims to discuss a definition and some basic properties of the generalized $q$-Mellin transform in a context of the new $q$-theory. All results are brief and concise, and may give the reader a general overview of the generalized $q$-theory of the Mellin operator. However, by virtue of the preceding analysis, we introduce the following definition.

Definition 20 Let $\frac{g_n}{\varepsilon_n} \in B$, then we define the $q$-Mellin transform of the $q$-Boehmian $\frac{g_n}{\varepsilon_n}$ as

$$M_q \frac{g_n}{\varepsilon_n} = \tilde{\omega}_n,$$ (23)

where $\tilde{\omega}_n = \frac{\omega_n}{\upsilon_n}, \omega_n = M_q g_n$, and $\upsilon_n = M_q \varepsilon_n$. Indeed $\tilde{\omega}_n$ belongs to $B_M$.

Theorem 21 The operator $M_q : B \rightarrow B_M$ is sequentially continuous, i.e., if $\Delta_q - \lim_{k \rightarrow \infty} \tilde{\omega}_{n,k} = \tilde{\omega}_n$ in $B$, then $\Delta_q - \lim_{n \rightarrow \infty} M_q \tilde{\omega}_{n,k} = M_q \tilde{\omega}_n$ in $B_M$.

Proof Let $\Delta_q - \lim_{k \rightarrow \infty} \tilde{\omega}_{n,k} = \tilde{\omega}_n$ in $B$, then there is $(e_n) \in \Delta_q$ such that

$$\Delta_q - \lim_{n \rightarrow \infty} (\tilde{\omega}_{n,k} - \tilde{\omega}_n)^{q} e_n = 0 \quad \text{in} \quad B.$$

The continuity of the integral operator gives

$$\Delta_{q,M} - \lim_{n \rightarrow \infty} M_q ((\tilde{\omega}_{n,k} - \tilde{\omega}_n)^{q} e_n) = \Delta - \lim_{n \rightarrow \infty} (M_q(\tilde{\omega}_{n,k} - \tilde{\omega}_n) \circ \upsilon_n) = 0,$$

where $M_p e_n = \upsilon_n$. Thus, we have $\Delta_{q,M} - \lim_{n \rightarrow \infty} M_q \tilde{\omega}_{n,k} = M_q \tilde{\omega}_n$ in $B_M$.

This finishes the proof of the theorem. \hfill \Box

Theorem 22 (i) $M_q$ is a linear isomorphism from the space $B$ onto the space $B_M$.

(ii) $M_q$ is continuous with respect to $\varepsilon_q$ and $\Delta_q$-convergence.

(iii) The operator $M_q$ coincides with the operator $M_q$. 


Proof We prove Part (iii) since similar proofs for Part (i)–Part (ii) are available in literature. Let \( g \in L^1_{\mathbb{R}_q} \) and \( \bar{g} = g \ast \varepsilon_n \) be its representative in \( \mathcal{B} \), where \( (\varepsilon_n) \in \Delta_q \ (\forall n \in \mathbb{N}) \). Clearly, for all \( n \in \mathbb{N} \), \( (\varepsilon_n) \) is independent from the representative. Let \( M_q \varepsilon_n = \upsilon_n \), then, by the \( q \)-convolution theorem, we get

\[
M_q g = M_q g \ast \varepsilon_n = M_q \frac{M_q g \circ M_q \varepsilon_n}{M_q \varepsilon_n} = M_q \frac{M_q g}{\varepsilon_n} \circ \frac{\upsilon_n}{\upsilon_n}.
\]

Hence, the \( q \)-Boehmian \( \frac{\omega_n}{\upsilon_n} \) is the representative of \( M_q \) in the space \( L_M \), where \( \omega = M_q g \).

The proof is, therefore, ended. \( \square \)

We introduce the inverse transform of \( M_q \) as follows.

Definition 23 We define the inverse integral operator of \( M_q \) of a \( q \)-Boehmian \( \frac{\omega_n}{\upsilon_n} \) in \( \mathcal{B}_M \) as follows:

\[
N_q \left( \frac{\omega_n}{\upsilon_n} \circ \omega \right) = \frac{g_n}{\varepsilon_n} \circ g \quad \text{and} \quad M_q \left( \frac{g_n}{\varepsilon_n} \circ g \right) = \frac{\omega_n}{\upsilon_n} \circ \omega.
\]

Theorem 24 Let \( \frac{\omega_n}{\upsilon_n} \in \mathcal{B}_M \) and \( \omega \in L_M \). Then we have

\[
N_q \left( \frac{\omega_n}{\upsilon_n} \circ \omega \right) = \frac{g_n}{\varepsilon_n} \circ g \quad \text{and} \quad M_q \left( \frac{g_n}{\varepsilon_n} \circ g \right) = \frac{\omega_n}{\upsilon_n} \circ \omega.
\]

Proof Assume \( \frac{\omega_n}{\upsilon_n} \in \mathcal{B}_M \) where \( \omega_n = M_q g_n \). Then, for every \( \omega = M_q g \in L_M \) and \( \upsilon_n = M_q \varepsilon_n \), we have

\[
N_q \left( \frac{\omega_n}{\upsilon_n} \circ M_q g \right) = N_q \left( \frac{\omega_n}{\upsilon_n} \circ \omega \right) = N_q \left( \frac{M_q g_n \circ \omega}{\upsilon_n} \right) = \frac{g_n}{\varepsilon_n} \circ g = \frac{g_n}{\varepsilon_n} \circ g.
\]

The proof of the first part is finished. The proof of the second part is almost similar. Hence, we omit the details.

This completely ends the proof of the theorem. \( \square \)

5 Conclusion

This paper has given an extension of the quantum theory of the \( q \)-Mellin transform operator \([40]\) to sets of \( q \)-generalized functions named \( q \)-Boehmians and \( q \)-ultraBoehmians. Every element \( g \) in the function space \( L^1_{\mathbb{R}_q} \) is identified as a member in the generalized space \( \mathcal{B} \) by the identification formula

\[
g \rightarrow g \ast \varepsilon_n,
\]

where \( (\varepsilon_n) \) is an arbitrary delta sequence. It also shows that the \( q \)-embedding

\[
g \rightarrow \bar{g}, \quad \bar{g} = g \ast \varepsilon_n.
\]
of the space $L^1(q)(\mathbb{R}_q,+)$ into the space $\mathbb{B}$ is continuous, $(\varepsilon_n)$ being an arbitrary $q$-delta sequence. The $q$-Mellin transform operator is extended to the generalized $q$-calculus theory, and many properties are discussed. Further, the inversion of the $q$-Mellin transform operator is also discussed.

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References
1. Jackson, F.H.: On $q$-definite integrals. Q. J. Pure Appl. Math. 41, 193–203 (1910)
2. Albayrak, D., Purohit, S.D., Uçar, F.: On $q$-analogues of Sumudu transform. An. Științ. Univ. ‘Ovidius’ Constanța 21(1), 239–260 (2013)
3. Al-Omari, S.K.: On $q$-analogues of the natural transform of certain $q$-Bessel functions and some application. Filomat 31(9), 2587–2598 (2017)
4. Yasmin, G., Muhyi, A.: Certain results of 2-variable $q$-generalized tangent-Apostol type polynomials. J. Math. Comput. Sci. 22(3), 238–251 (2021)
5. Araci, S., Duran, U., Acikgoz, M.: On weighted $q$-Dahee polynomials with their applications. Indag. Math. 30, 365–374 (2019)
6. Boehme, T.K.: The support of Mikusinski operators. Trans. Am. Math. Soc. 176, 319–334 (1973)
7. Mikusinski, J., Mikusinski, P.: Quotients of suites et. Comptes Rendus 293, 463–464 (1981)
8. Schwartz, L.: Théorie des Distributions, I. Hermann, Paris (1950)
9. Al-Omari, S.K.: $q$-analogues and properties of the Laplace-type integral operator in the quantum calculus theory. J. Inequal. Appl. 203, 1–14 (2020)
10. Mikusinski, P.: Convergence of Boehmians. Jpn. J. Math. 9, 159–179 (1983)
11. Mikusinski, P.: On flexibility of Boehmians. Integral Transforms Spec. Funct. 4, 141–146 (1996)
12. Al-Omari, S.K.: On a class of generalized Meijer–Laplace transforms of Fox function type kernels and their extension to a class of Boehmians. Georgian Math. J. 25(1), 1–8 (2018)
13. Al-Omari, S.K., Agarwal, P.: Some general properties of a fractional Sumudu transform in the class of Boehmians. Kuwait J. Sci. 43(2), 16–30 (2016)
14. Karunakaran, V., Vembu, R.: On point values of Boehmians. Rocky Mt. J. Math. 35, 181–193 (2005)
15. Kananthai, A.: The distribution solutions of ordinary differential equation with polynomial coefficients. Southeast Asian Bull. Math. 25, 129–134 (2001)
16. Al-Omari, S.K.: An extension of certain integral transform to a space of Boehmians. J. Assoc. Arab Univ. Basic Appl. Sci. 17, 36–42 (2015)
17. Al-Omari, S.K.: The $q$-Sumudu transform and its certain properties in a generalized $q$-calculus theory. Adv. Differ. Equ. 10, 1–14 (2021)
18. Mikusinski, P.: Boehmians and generalized functions. Acta Math. Hung. 51, 271–281 (1988)
19. Nemzer, D.: Periodic Boehmians. Int. J. Math. Math. Sci. 12, 685–692 (1989)
20. Looiker, D., Banerji, P.K.: Solution of integral equations by generalized wavelet transform. Bol. Soc. Parana. Mat. 33(2), 89–94 (2015)
21. Selvakumaran, K.A., Choi, J., Purohit, S.D.: Certain subclasses of analytic functions defined by fractional $q$-calculus operators. Appl. Math. E-Notes 21, 72–80 (2021)
22. Muhyi, A., Araci, S.: A note on $q$-Fubini–Appell polynomials and related properties. J. Funct. Spaces 2021, 1–9 (2021)
23. Purohit, S., Raina, R.: Certain subclasses of analytic functions defined by fractional $q$-calculus operators. Appl. Math. E-Notes 21, 72–80 (2021)
24. Acikgoz, M., Araci, S., Duran, U.: New extensions of some known special polynomials under the theory of multiple $q$-calculus. Turk. J. Anal. Number Theory 3(5), 128–139 (2015)
25. Albayrak, D., Purohit, S., Ucar, F.: On $q$-Laplace and $q$-Sumudu transforms of a product of generalized $q$-Bessel functions. Math. Eng. Sci. Aerosp. MESA 11(2), 355–369 (2020)
26. Vyas, V., Al-Jarrah, A., Purohit, S.D., Araci, S., Nisar, K.: $q$-Laplace transform for product of general class of $q$-polynomials and $q$-analogue of $f$-function. J. Inequal. Spec. Funct. 11(3), 21–28 (2020)
27. Abdi, W.H.: On $q$-Laplace transforms. Proc. Natl. Acad. Sci. 29, 389–408 (1961)
28. Purohit, S.D., Kalla, S.L.: On $q$-Laplace transforms of the $q$-Bessel functions. Fract. Calc. Appl. Anal. 10(2), 189–196 (2007)
29. Uçar, F., Albayrak, D.: On $q$-Laplace type integral operators and their applications. J. Differ. Equ. Appl. 18(6), 1001–1014 (2012)
30. Purohit, S.D., Uçar, F.: An application of $q$-Sumudu transform for fractional $q$-kinetic equation. Turk. J. Math. 42, 726–734 (2018)
31. Uçar, F.: $q$-Sumudu transforms of $q$-analogues of Bessel functions. Sci. World J. 2014, 1–12 (2014)
32. Albayrak, D., Purohit, S.D., Uçar, F.: On $q$-Sumudu transforms of certain $q$-polynomials. Filomat 27(2), 413–429 (2013)
33. Yadav, R.K., Purohit, S.D., Kalla, S.L.: On generalized Weyl fractional $q$-integral operator involving generalized basic hypergeometric functions. Fract. Calc. Appl. Anal. 11(2), 129–142 (2008)
34. Fitouhi, A., Bettaibi, N.: Wavelet transforms in quantum calculus. J. Nonlinear Math. Phys. 13(3), 492–506 (2006)
35. Fitouhi, A., Bettaibi, N.: Applications of the Mellin transform in quantum calculus. J. Math. Anal. Appl. 328, 518–534 (2007)
36. Al-Omari, S.K.: On $q$-analogues of the Mangontarum transform for certain $q$-Bessel functions and some application. J. King Saud Univ., Eng. Sci. 28(4), 375–379 (2016)
37. Al-Omari, S.K.: On $q$-analogues of Mangontarum transform of some polynomials and certain class of $H$-functions. Nonlinear Stud. 23(1), 51–61 (2016)
38. Salem, A., Uçar, F.: The $q$-analogue of the $E_{2,1}q$-transform and its applications. Turk. J. Math. 40(1), 98–107 (2016)
39. Al-Omari, S.K., Baleanu, D., Purohit, S.D.: Some results for Laplace-type integral operator in quantum calculus. Adv. Differ. Equ. 124, 1–10 (2018)
40. Fitouhi, A., Bettaibi, N.: Applications of the Mellin transform in quantum calculus. J. Math. Anal. Appl. 328, 518–534 (2007)