Exact analytical soliton solutions of \( N \)-component coupled nonlinear Schrödinger equations with arbitrary nonlinear parameters

Ning Mao and Li-Chen Zhao*

School of Physics, Northwest University, Xi'an, 710127, China
Peng Huannan Center for Fundamental Theory, Xi'an 710127, China and Shaanxi Key Laboratory for Theoretical Physics Frontiers, Xi'an, 710127, China

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Exact analytical soliton solutions play an important role in soliton fields. Soliton solutions were obtained with some special constraints on the nonlinear parameters in nonlinear coupled systems, but they usually do not hold in real physical systems. We successfully release all usual constrain conditions on nonlinear parameters for exact analytical vector soliton solutions in \( N \)-component coupled nonlinear Schrödinger equations. The exact soliton solutions and their existence condition are given explicitly. Applications of these results are discussed in several present experimental parameters regimes. The results would motivate experiments to observe more novel vector solitons in nonlinear optical fibers, Bose-Einstein condensates, and other nonlinear coupled systems.

Introduction—Analytical soliton solutions can describe the dynamics of localized waves in many nonlinear systems well, and some of them have even been used to direct experimental observations [1–7]. Analytical solutions usually contain exact solutions and approximate solutions. Exact analytical soliton solutions have been paid much more attention due to their beauty and convenience for uncovering underlying physics [8–13]. Unfortunately, they are usually given with some special constrain conditions [14–18]. For example, most soliton solutions of multi-component coupled nonlinear systems were obtained with some special constraints on the nonlinear parameters, e.g., the Manakov model [14]. However, those special constrain conditions usually do not holds in real nonlinear coupled systems [3–6, 19, 20]. It highly demands us to release the constrain conditions on nonlinear parameters to derive exact soliton solutions.

In this letter, we release all usual constrain conditions on nonlinear parameters for exact analytical vector soliton solutions in \( N \)-component coupled nonlinear Schrödinger equations, which have played important roles in soliton fields due to their simplicity and wide roles in nonlinear systems [19–21]. Our attempt could motivate more efforts to derive exact soliton solutions of other nonlinear models [22–24]. Based on our proper trial functions and the modified Lagrangian variational method, we give the exact soliton solutions and their existence condition explicitly, which covers all previously known exact soliton solutions and predicts some new physical properties for vector solitons. As an example, we show two- and three-component vector soliton solutions and their existence region in detail. Applications of these results are discussed in Bose-Einstein condensates with present experimental parameters regimes.

The exact vector soliton solutions and Lagrangian

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method—Coupled nonlinear Schrödinger equations have been used to describe nonlinear wave dynamics in many different physical systems [21, 25, 26], such as Bose–Einstein condensates [26] and nonlinear optical fibers [19, 27]. We consider a general \( N \)-component coupled nonlinear Schrödinger equations with arbitrary nonlinear parameters, which can be written as

\[
i\partial_t \Psi = \left(-\frac{1}{2} \partial_{xx} + \sum_{j=1}^{N} g_{ij} |\psi_j|^2 \right) \Psi, \tag{1}
\]

where \( \Psi = (\psi_1, \cdots, \psi_i, \cdots, \psi_N)^T \) and where \( \psi_j \) stands for the wave function of the \( j \)-th component. The nonlinear coefficient \( g_{ij} \) is the intra-interaction in one component (inter-action between two components) for \( i = j \) (\( i \neq j \)), and \( g_{ij} = g_{ji} \). They admit scalar solitons (\( N = 1 \)) and vector solitons (\( N > 1 \)). The exact scalar bright and dark soliton solutions were derived by the Bäcklund transformation [15, 16], inverse scattering method [17], and Hirota bilinear method [18] (noting that \( N = 1 \) cases are always integrable). But there are many nonlinear parameters when \( N > 1 \), which makes the coupled systems usually no longer integrable. When all these nonlinear parameters are equal, the model becomes the well-known integrable Manakov model [14], whose vector soliton solutions have been derived by many different methods [15–18]. The soliton solutions of the Manakov model have motivated many experiments to observe dark-dark (DD) solitons [28, 29], dark-bright (DB) solitons [3], etc.

However, the constrain conditions on nonlinear parameters usually do not holds in real nonlinear coupled systems [3, 4, 20, 30–33]. Many scientists tried to address the cases for which the special constrain conditions are violated, and some approximate analytic solutions were obtained [34–39]. More approximate analytic solutions could be given by the perturbation method [40], multiscale expansion method [41, 42], and Lagrangian varia-
tional method \[43–48\]. We wish to find exact analytic solutions for more general conditions. Notably, some exact soliton solutions can still be derived by the Lagrangian variational methods \[49, 50\]. This hints that it is possible to derive exact soliton solutions for nonintegrable cases and even more general cases, by developing the methods.

For the Lagrangian variational methods, the trial functions are critical for precision. The Gaussian profile is usually used for localized wave packets due to its simplicity and easy calculation. However, this form usually fails to obtain exact solutions in nonlinear systems. It was further suggested that the sech- (tanh-)type ansatz could be more accurate than the Gaussian ansatz \[34\]. For bright or dark solitons in the \(i\)th component, we introduce the trial wave function as

\[
\psi_{iD} = (i\sqrt{a_i^2 - f_i^2(t)} + f_i(t) \tanh\{w_i(t)[x - b(t)]\})e^{i\theta_i(t)},
\]

\[
\psi_{iB} = f_i(t) \sech\{w_i(t)[x - b(t)]\}e^{i\phi_i(t)},
\]

where \(\psi_{iD} (\psi_{iB})\) denotes the wave function of the dark (bright) soliton. \(f_i\) and \(w_i\) describe the amplitude and width of the dark (bright) soliton, respectively, \(a_i\) is the background of the dark component, the central position of the soliton is \(b(t)\), \(\theta_i\) and \(\xi_i\) are the time-dependent phases of the dark and bright components, respectively, and \(\phi_i\) is related to the velocity of the bright soliton.

We introduce the Lagrangian density as \(\mathcal{L} = \sum_{i=1}^{N} \left\{ \frac{1}{2} (\psi_i^* \partial_t \psi_i - \psi_i^* \partial_i \psi_i^*) - a_i^2 \right\} \left( \partial_{\psi_i}^2 \psi_i^2(t)\right) - \frac{1}{2} b_i^2 |\partial_x \psi_i|^2 - \sum_{j=1}^{N} \frac{g_{ij}}{2} (|\psi_j|^2 - a_j^2 \delta_{D,j})(|\psi_i|^2 - a_i^2 \delta_{D,i}) \right\} \), where

\[
\delta_{D,i} = \begin{cases} 0, & \tilde{\psi}_i \text{ denotes bright soliton,} \\ 1, & \tilde{\psi}_i \text{ denotes dark soliton,} \end{cases}
\]

to obtain Euler-Lagrangian equations. In particular, the terms \(1 - a_i^2/|\psi_i|^2 \delta_{D,i}\) and \(|\psi_i|^2 - a_i^2 \delta_{D,i}\) were introduced in the Lagrangian density for dark solitons \[50\]. Notably, the modified \(\mathcal{L}\) corresponds to dynamical equation \(\partial_t \psi_i = -\frac{i}{2} \partial_{xx} \psi_i + \left[ \sum_{j=1}^{N} g_{ij} (|\psi_j|^2 - a_j^2 \delta_{D,j}) \right] \psi_i \) \((i = 1, \ldots, N)\). This is different from Eq. \(1\), but the solutions of the two dynamical equations can be transformed to each other by the phase factor \(\psi_i/\psi_i = e^{i\theta_i(t)}\), where \(\theta_i(t) = -\sum_{j=1}^{N} \sum_{i=1}^{N} g_{ij} a_j^2 \delta_{D,j}\). We obtain the Lagrangian \((\mathcal{L})\) by substituting Eq. \((2)\) into \(\mathcal{L}\) and integrating over space from \(-\infty\) to \(+\infty\), which can be simplified as

\[
L = \sum_{i=1}^{N} \left\{ \left[ 2 f_i^2(t) \phi_i(t) \frac{\partial^2}{\partial x^2} \psi_i(t) - \frac{1}{3} f_i^2(t) w_i(t) - f_i^2(t) w_i(t) \right] - \frac{2}{3} f_i^2(t) \frac{\partial^2}{\partial x^2} w_i(t) \right\} \delta_{D,i} + \sum_{j=1}^{N} \sum_{i=1}^{N} g_{ij} f_i^2(t) f_j^2(t) G_{ij}, \quad (\delta_{D,i} = |\delta_{D,i} - 1|).
\]

Specifically, the soliton width parameter \(w_i\) is set to be different for different components, which is in contrast to the previous trial functions for vector solitons \[34–39\]. In most previous studies, the trial functions were chosen as the Gaussian (sech- or tanh-type) ansatz for the unequal (equal) width parameter setting \[34–39\], partly because the Gaussian ansatz with different widths is much easier to calculate than the sech (tanh) type. This makes the Lagrangian variational results only give an approximate solution for the vector model \[34–39\]. However, one encounters the problem that it is difficult to accommodate the factor \(G_{ij} = \int_{-\infty}^{\infty} (\pm \sech^2\{w_{ij}(t)[x - b(t)]\}) (\pm \sech^2\{w_{ji}(t)[x - b(t)]\}) dx\), and the sign \(-\) \((+)\) corresponds to the dark (bright) component. We take these width parameters independently when deriving the Euler-Lagrangian equations. After obtaining the equations of motion by the Euler-Lagrangian formula, we finally calculate \(G_{ij}, \frac{\partial G_{ij}}{\partial w_{ij}},\) and \(\frac{\partial G_{ij}}{\partial w_{ij}}\) by setting \(w_i = w_j = w\) (see details in Supplemental Material \[51\]). These operations bring us more constraint conditions on soliton parameters and finally enable us to obtain exact analytical soliton solutions.

Based on all simplified Euler-Lagrangian equations, we have the essential constraint equations of \(f_i\) and \(w\) as

\[
\begin{pmatrix}
g_{11} & \cdots & g_{1j} & \cdots & g_{1N} & 1 \\
\vdots & & \vdots & & \vdots & \vdots \\
g_{N1} & \cdots & g_{Nj} & \cdots & g_{NN} & 1
\end{pmatrix}
\begin{pmatrix}
\pm f_1^2 \\
\vdots \\
\pm f_N^2 \\
\pm f_j^2
\end{pmatrix} = 0.
\]

The sign \(-\) \((+)\) corresponds to the dark (bright) component. The other parameters can be obtained as \(b(t) = vt\), \(\phi_i = v\), \(\xi_i(t) = \frac{1}{2}(w^2 + v^2) t + \theta_i(t)\), and the soliton velocity \(v = w\sqrt{a_i^2 - f_i^2}/f_i\). The constraint equation on backgrounds of dark components is \(a_i/a_j = f_i/f_j\) (for
details, see [51]). In this way, we obtain many exact analytic vector soliton solutions when the nonlinear parameters can be arbitrary. Different types of vector solitons usually exist in different regions in nonlinear coefficient space. Their existence regions can also be clarified by the constraint conditions on soliton parameters. The above solution can cover all previously reported vector solitons for integrable cases (g_{1,2} are all equal) [13, 52–62]. We choose two- and three-component systems as examples to show the variational result explicitly based on vector soliton experiments in Ref. [3–6].

Two-component coupled systems—For two-component systems \( (N = 2) \), the condition for vector soliton solutions can be derived from Eq. (4), which is simplified as

\[
\alpha_2 f_2^2 = \frac{g_{11} - g_{12}}{g_{22} - g_{12}} \alpha_1 f_1^2, \quad \alpha_3 f_3^2 = \frac{g_{12}^2 - g_{11} g_{22}}{g_{22} - g_{12}} \alpha_1 f_1^2, \quad w^2 = \frac{g_{12}^2 - g_{11} g_{22}}{g_{22} - g_{12}} \alpha_1 f_1^2, \tag{5}
\]

where \( \alpha_i = \pm 1 \) and where the sign \( - (+) \) is chosen for the dark (bright) soliton. Then, we clarify the existence regimes for different vector solitons by analyzing the soliton parameters of Eq. (5). The phase diagram for different vector solitons is summarized in Fig. 1 (\( g_{12} \neq 0 \)), in which (a) and (b) show the results for the \( g_{12} < 0 \) and \( g_{12} > 0 \) cases, respectively. When \( g_{12} = 0 \), the model decouples into two scalar models. Our variational method extends the existence region of exact vector soliton solutions from the “orange dot” (Manakov model) to the whole parameter plane. The vector solitons can still be classified into four families, similar to the integrable model [13, 52–55]. However, there are many additional constraints on the soliton parameters and nonlinear parameters, which are absent for integrable models [13, 52–55].

For bright-bright (BB) solitons, Eq. (5) gives the conditions \( \alpha_1 f_1^2/a_1 = f_2^2/f_1 \) in nonintegrable cases. This is in sharp contrast to the Manakov model, for which there is no constraint on the ratio of background amplitudes for DD solitons [53, 54].

For dark-bright (DB) or bright-dark (BD) solitons, the regions for them are shown in Fig. 1. The DB and BD solitons are defined by the total density of the two components, which admits a dip (for DB) or hump (for BD). In particular, the total density can be uniform when \( 2g_{12} = g_{11} + g_{22} \), and the vector soliton in this case becomes the spin soliton [63] (denoted by green lines in Fig. 1). Note that the amplitude of the bright soliton can be larger than the dark soliton’s background amplitude; i.e., BD soliton can exist even though the nonlinear interactions are all repulsive (all nonlinear parameters are positive). This characteristic is absent for the Manakov case [13], for which only DB soliton exists for repulsive interactions. Similarly, DB soliton can still exist with the nonlinear interactions all attractive (all nonlinear parameters are negative) for nonintegrable cases. Our DB soliton solution with zero velocity can be reduced to the ones given in Ref. [64]. The exact BB and BD (DD and DB) soliton solutions cannot coexist in the given nonlinear parameters for the nonintegrable cases (see Fig. 1), in contrast to the integrable case [13, 52–55].

Our variational results can be reduced to a previously known exact soliton solution in the two-component model [13, 52–55, 63]. The general exact vector soliton solutions of two-component cases are provided explicitly in [51], which partly overlap with the ones given by the periodic wave expansion method [65, 66]. We emphasize that our variational method here can be extended more conveniently to arbitrary \( N \)-component cases. Experiments have been performed on three-component systems and even five-component Bose–Einstein condensates [4, 20, 30–33], but the exact analytic soliton solutions for them are still absent. These results motivate us to derive soliton solutions explicitly for more components cases. Next, we apply our variational result to three-component systems.

Three-component coupled systems—The exact soliton solution of three-component systems has been widely studied for integrable models [56–60]. Our variational result greatly widens the existence region of vector soliton solutions, which is very meaningful for soliton experiments [4]. The essential conditions for soliton solutions can be given as

\[
\alpha_2 f_2^2 = \frac{P_{23}}{P_{213}} \alpha_1 f_1^2, \quad \alpha_3 f_3^2 = \frac{P_{32}}{P_{213}} \alpha_1 f_1^2, \quad w^2 = \frac{Q}{P_{213}} \alpha_1 f_1^2, \tag{6}
\]

where \( Q = 2g_{12} g_{13} g_{23} - g_{12}^2 g_{33} - g_{13}^2 g_{22} - g_{11} (g_{23} - g_{22} g_{33}) \), \( P_{lmn} = g_{ln} - g_{lm} g_{mn} + g_{lm} (g_{ln} - g_{mn}) + g_{mn} (g_{ln} - g_{lm}) \), and \( \alpha_i = \pm 1 \) (\( i = 1, 2, 3 \)). The exact
vector solitons based on our variational results. With the nonlinear parameters \[3\]. This is partly because ex-

ton transport for more general cases \[63\]. Our attempt

ution become nonintegrable. The theoretical analysis of

The dynamical equations in the mean-field approxima-

ing lengths are \[a\].

TABLE I: Parameter conditions of different soliton solutions

| Soliton type | Parameter region |
|--------------|------------------|
| | \(P_{123}/P_{213}\) | \(P_{132}/P_{213}\) | \(Q/P\) |
| BBB | + | + | + |
| DDB | – | – | – |
| DDB | + | – | – |
| DDD | + | + | – |

which is smaller than the sound velocity 1 mm/s given by

The nonlinear parameters in our rescaling model satisfy

The constraint on amplitude and width is de-

We cannot give explicit conditions of existence for dif-

applications in experiments—We first discuss the ap-

Conclusion—Our work successfully releases all usual

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