Explicit Bivariate Rate Functions for Large Deviations in AR(1) and MA(1) Processes with Gaussian Innovations

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Abstract

We investigate large deviations properties for centered stationary AR(1) and MA(1) processes with independent Gaussian innovations, by giving the explicit bivariate rate functions for the sequence of random vectors \((S_n)_{n \in \mathbb{N}} = (n^{-1}(\sum_{k=1}^{n} X_k, \sum_{k=1}^{n} X_k^2))_{n \in \mathbb{N}}\). In the AR(1) case, we also give the explicit rate function for the bivariate random sequence \((W_n)_{n \geq 2} = (n^{-1}(\sum_{k=1}^{n} X_k^2, \sum_{k=2}^{n} X_k X_{k+1}))_{n \geq 2}\). Via Contraction Principle, we provide explicit rate functions for the sequences \((n^{-1} \sum_{k=1}^{n} X_k)_{n \in \mathbb{N}}, (n^{-1} \sum_{k=1}^{n} X_k^2)_{n \geq 2}\) and \((n^{-1} \sum_{k=2}^{n} X_k X_{k+1})_{n \geq 2}\), as well. In the AR(1) case, we present a new proof for an already known result on the explicit deviation function for the Yule-Walker estimator.

Keywords: Large Deviations; Empirical Autocovariance; Quadratic and Sample Means; Autoregressive Processes; Moving Average Processes; Yule-Walker Estimator

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1 Introduction

Since the first establishments on the Large Deviations theory, there has been a great expansion of the number of surveys on Large Deviations Principles (LDP). Nowadays, we can find a variety of examples applied to the time series analysis and stochastic processes in general; for instance, LDPs for Stable laws (see, e.g. Heyde [22], Rozovskii [31], Rozovskii [32] and Zaigraev [36]), stationary Gaussian processes (see, e.g. Bercu et al. [3], Bercu et al. [4], Bryc and Dembo [9], Donsker and Varadhan [17] and Zani [37]), autoregressive and moving average processes (see, e.g. Bercu [2], Bryc and Smolenski [10], Burton and Dehling [12], Djellout and Guillin [14], Macci and Trapani [25], Mas and Menetenteau [27], Miao [29] and Wu [35]) and continuous processes (see, e.g. Bercu and Richou [5] and Bercu and Richou [6]).

When considering the Empirical Autocovariance function

\[ \hat{\gamma}_n(h) = \frac{1}{n} \sum_{k=1}^{n-h} X_k X_{k+h}, \quad \text{for } 0 \leq h \leq d \quad \text{and} \quad d \in \mathbb{N}, \]

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of a process \((X_n)_{n \in \mathbb{N}}\), few results on LDP are known. Regarding Gaussian distributions, one of the first studies in the literature is the one from Bryc and Smolenski [10], concerning the LDP for the Quadratic Mean

\[ \tilde{\gamma}_n(0) = \frac{1}{n} \sum_{k=1}^{n} X_k^2. \]

Bryc and Dembo [9] showed that an LDP for the vector \((\tilde{\gamma}_n(h))_{h=0}^{d} \) is available when \((X_n)_{n \in \mathbb{N}}\) is an independent and identically distributed (i.i.d.) process, with \(X_n \sim \mathcal{N}(0, 1)\). It is well known that most of the relevant stochastic processes are not independent and, as the authors have claimed, their approach needs some adjustments when trying to show that a similar LDP works, for instance, when dealing with the classical centered stationary Gaussian AR(1) process (see example 1 in Bryc and Dembo [9]). On the other hand, Bercu et al. [4] proved an LDP for Toeplitz quadratic forms of centered stationary Gaussian processes in an univariate setting. Their survey eliminated the need for the variables of \((X_n)_{n \in \mathbb{N}}\) to be independent, extending the result in Bryc and Dembo [9] by including the AR(1) process. However, it is not clear if the LDP was available even for the bivariate random vector \((\tilde{\gamma}_n(0), \tilde{\gamma}_n(1))\), once the LDP has only been proved for each one of the components separately. More precisely, the results in Bercu et al. [4] only cover the LDP of the random variable

\[ W_n = \frac{1}{n} X^{(n)T} M_n X^{(n)}, \]

where \(X^{(n)} = (X_1, \cdots, X_n)\), with \(X^{(n)T}\) denoting the transpose of \(X^{(n)}\), and where \((M_n)_{n \in \mathbb{N}}\) is a sequence of \(n \times n\) Hermitian matrices.

In a more general setting, Carmona et al. [13] present a level-1 LDP for the empirical autocovariance function of order \(h\) for any innovation processes, that encompasses the AR(\(d\)) process with Gaussian innovations. In this paper, the authors used the level-2 LDP together with the Contraction Principle. The process itself is obtained from iterations of a continuous uniquely ergodic transformation, preserving the Lebesgue measure on the circle. In Carmona and Lopes [14], the authors considered a similar problem where the dynamics are given by an expanding transformation on the circle. In the same line of research, Wu [35] proved an LDP for \((\tilde{\gamma}_n(h))_{h=0}^{d} \) under the assumption that \(\mathbb{E}(\exp(\lambda \varepsilon_n^2))\) is finite, for \(\lambda > 0\), where \((\varepsilon_n)_{n \in \mathbb{N}}\) is the white noise of an AR(\(d\)) process, excluding in turn the Gaussian case.

In the present manuscript, we take into account the studies from Bercu et al. [4] and Bryc and Dembo [9] to give a proof that the sequence \((W_n)_{n \geq 2}\), given by

\[ W_n = \frac{1}{n} (\tilde{\gamma}_n(0), \tilde{\gamma}_n(1)) = \frac{1}{n} \left( \sum_{k=1}^{n} X_k^2, \sum_{k=2}^{n} X_k X_{k-1} \right), \text{ for } n \geq 2, \]

does, in fact, satisfy an LDP when \((X_n)_{n \in \mathbb{N}}\) is a centered stationary Gaussian AR(1) process and we present its explicit bivariate rate function. The asymptotical behavior of the sequence \((W_n)_{n \geq 2}\) is well known (see Brockwell and Davis [8]), that is

\[ W_n \xrightarrow{n \to \infty} \left( \frac{1}{1-\theta^2}, \frac{\theta}{1-\theta^2} \right), \text{ almost surely.} \]

By definition of almost sure convergence, as \(n \to \infty\), the sequence of probabilities

\[ P \left( \left\| W_n - \left( \frac{1}{1-\theta^2}, \frac{\theta}{1-\theta^2} \right) \right\| > \delta \right), \quad (1.1) \]
converges to zero, for all \( \delta > 0 \). However, if the convergence of these probabilities is very slow, even for large \( n \), we have a certain reasonable chance of choosing a bad sample \( X_1, \ldots, X_n \) from \( (X_n)_{n \in \mathbb{N}} \), such that \( W_n \) is distant from the true value \( (\frac{1}{2}\theta_1, \frac{1}{2}\theta_2) \).

The Large Deviations theory considers the asymptotic behavior of the probabilities presented in (1.1), ensuring that they converge to zero approximately in exponential rate (see chapter 1 in Bucklew [11]). Its usual definition is given as follows (see Dembo and Zeitouni [15]).

**Definition 1.1.** A sequence of random vectors \( (V_n)_{n \in \mathbb{N}} \) of \( \mathbb{R}^d \), for \( d \in \mathbb{N} \), satisfies a Large Deviation Principle (LDP) with speed \( n \) and rate function \( J(\cdot) \), if \( J(\cdot) : \mathbb{R}^d \to [0, \infty] \) is a lower semi-continuous function such that,

- **Upper bound:** for any closed set \( F \subset \mathbb{R}^d \),
  \[
  \limsup_{n \to \infty} \frac{1}{n} \log P(V_n \in F) \leq -\inf_{x \in F} J(x);
  \]

- **Lower bound:** for any open set \( G \subset \mathbb{R}^d \),
  \[
  -\inf_{x \in G} J(x) \leq \liminf_{n \to \infty} \frac{1}{n} \log P(V_n \in F).
  \]

Moreover, \( J(\cdot) \) is said to be a good rate function if its level sets \( J^{-1}([0, b]) \) are compact, for all \( b \in \mathbb{R} \).

**Remark 1.** In this work, we only deal with good rate functions, but for short, we sometimes write rate function instead.

In general, it is not easy to prove that an arbitrary sequence of random vectors satisfies an LDP (see, e.g. Bercu and Richou [6], Bryc and Dembo [9], Dembo and Zeitouni [15], Ellis [18], Macci and Trapani [25] and Mas and Menneteau [27]). An elegant way of proving such property is to verify the validity of the Gärtner-Ellis’ theorem conditions (see theorem 2.3.6 in Dembo and Zeitouni [15]), which is a counterpart to the very well known Cramér-Chernoff’s theorem (see theorem 2.2.30 in Dembo and Zeitouni [15]). It is worth mentioning that, within the conditions of the Gärtner-Ellis’ theorem, little use of the dependency structure is made and the focus mainly rests in the behavior of the limiting cumulant generating function, defined by

\[
L(\lambda) = \lim_{n \to \infty} L_n(\lambda), \text{ for all } \lambda \in \mathbb{R}^2,
\]

where \( L_n(\cdot) : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\} \) denotes the normalized cumulant generating function of \( W_n \),

\[
L_n(\lambda) = \frac{1}{n} \log \mathbb{E} [\exp (n \langle \lambda, W_n \rangle)].
\]

We shall present an explicit expression for \( L(\cdot) \) in the case \( \lambda = (\lambda_1, \lambda_2) \) depends on two variables. As a result, we obtain the explicit rate function through the Fenchel-Legendre transform of \( L(\cdot) \).

In the second part of our study, we shall analyze the LDP of the sequence of bivariate random vectors \( (S_n)_{n \in \mathbb{N}} \), where

\[
S_n = \frac{1}{n} \left( \sum_{k=1}^{n} X_k, \sum_{k=1}^{n} X_k^2 \right).
\]
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We shall call $S_n$ as the bivariate SQ-Mean, for short, since its first and second components are, respectively, the Sample Mean and the Quadratic Mean. We dedicate our study to the particular cases when $(X_n)_{n \in \mathbb{N}}$ follows an AR(1) or an MA(1) process. This study is based on a particular result presented in Bryc and Dembo [9] and which has a very interesting application when the Contraction Principle can be applied.

Our study is organized as follows. Section 2 is dedicated to the proof of the LDP and computations of the explicit rate function for the random sequence $(W_n)_{n \geq 2}$, under the assumption that $(X_n)_{n \in \mathbb{N}}$ follows an AR(1) process. In Section 3, we obtain the LDP for some particular cases, namely, the Quadratic Mean and the first order Empirical Autocovariance of a random sample $X_1, \cdots, X_n$ from the AR(1) process. Moreover, the LDP for the Yule-Walker estimator is provided likewise. As a direct application of the studies in Section 2, we dedicate Section 4 to show that the LDP for the SQ-Mean $(S_n)_{n \in \mathbb{N}}$ of an AR(1) process is available. Next, we give the details of the LDP for the Quadratic Mean of an MA(1) process and, as a consequence, the LDP for the SQ-Mean. Section 5 gives insights on future work and concludes the manuscript.

2 LDP and the centered stationary Gaussian AR(1) process

Consider the autoregressive process $(X_n)_{n \in \mathbb{N}}$ defined by the equation

$$X_{n+1} = \theta X_n + \varepsilon_{n+1}, \quad \text{for } |\theta| < 1 \text{ and } n \in \mathbb{N}, \tag{2.1}$$

where $(\varepsilon_n)_{n \geq 2}$ is a sequence of i.i.d. random variables, with $\varepsilon_n \sim \mathcal{N}(0,1)$, for all $n \geq 2$. Assume that $X_1$ is independent of $(\varepsilon_n)_{n \geq 2}$, with $\mathcal{N}(0,1/(1-\theta^2))$ distribution. Then $(X_n)_{n \in \mathbb{N}}$ is a centered stationary Gaussian AR(1) process with (positive) spectral density function defined as

$$g_{\theta}(\omega) = \frac{1}{1 + \theta^2 - 2\theta \cos(\omega)}, \quad \omega \in T = [-\pi, \pi). \tag{2.2}$$

Throughout this section, we shall study the existence of an LDP for the random vector

$$W_n = \frac{1}{n} \left( \sum_{k=1}^{n} X_k^2, \sum_{k=2}^{n} X_k X_{k-1} \right). \tag{2.3}$$

Consider $\lambda = (\lambda_1, \lambda_2) \in \mathbb{R}^2$. Let $L_n(\cdot, \cdot) : \mathbb{R}^2 \to \mathbb{R}$ represent the normalized cumulant generating function associated to the sequence $(W_n)_{n \geq 2}$, defined by

$$L_n(\lambda_1, \lambda_2) = \frac{1}{n} \log \mathbb{E} \left( e^{n \langle (\lambda_1, \lambda_2), W_n \rangle} \right), \quad \text{for } n \geq 2, \tag{2.4}$$

where $\langle (x_1, y_1), (x_2, y_2) \rangle := x_1 x_2 + y_1 y_2$ denotes the usual inner product in $\mathbb{R}^2$. We want to apply the Gärtner-Ellis’ theorem, which requires the convergence of $L_n(\cdot, \cdot)$, as $n \to \infty$. 

2.1 Analysis of the normalized cumulant generating function

We shall present below, the expression for the limiting function $L(\cdot, \cdot)$, when $n \to \infty$, of the sequence of functions $(L_n(\cdot, \cdot))_{n \geq 2}$. In particular, we use the function $L(\cdot, \cdot)$ by applying the Gärtner Ellis' theorem in order to obtain the rate function of the sequence $(\mathcal{W}_n)_{n \geq 2}$.

With $X^{(n)} = (X_1, \cdots, X_n)$ and $X^{(n)T}$ denoting the transpose of $X^{(n)}$, note that, one can rewrite (2.3) as

$$\mathcal{W}_n = \frac{1}{n} \left( X^{(n)T} T_n(\varphi_1) X^{(n)}, X^{(n)T} T_n(\varphi_2) X^{(n)} \right),$$

where $\varphi_1 : T \to \{1\}$ and $\varphi_2 : T \to [-1,1]$ are real valued functions, given respectively by

$$\varphi_1(\omega) = 1 \quad \text{and} \quad \varphi_2(\omega) = \cos(\omega).$$

The matrix $T_n(f)$ represents the Toeplitz matrix associated to the function $f : T \to \mathbb{R}$, which is defined by

$$T_n(f) = \left[ \frac{1}{2\pi} \int_\pi e^{i(j-k)\omega} f(\omega) \, d\omega \right]_{1 \leq j,k \leq n}.$$  

Remark 2. A vast literature comprehending Toeplitz matrices has emerged in the last century and one of the most famous and referenced works is given in Grenander and Szegö [21]. A modern treatment about this subject may be found in Gray [20] and in Nikolski [30].

Inserting (2.5) into (2.4), we obtain

$$L_n(\lambda_1, \lambda_2) = \frac{1}{n} \log \mathbb{E} \left( e^{X^{(n)T} (\lambda_1 T_n(\varphi_1) + \lambda_2 T_n(\varphi_2)) \cdot X^{(n)}} \right)$$

and, by linearity of Toeplitz matrices, we get

$$L_n(\lambda_1, \lambda_2) = \frac{1}{n} \log \mathbb{E} \left( e^{X^{(n)T} T_n(\varphi_\lambda) X^{(n)}} \right),$$

where $\varphi_\lambda : T \to \mathbb{R}$ is defined by

$$\varphi_\lambda(\omega) = \lambda_1 \varphi_1(\omega) + \lambda_2 \varphi_2(\omega) = \lambda_1 + \lambda_2 \cos(\omega).$$

Observe that $\varphi_\lambda(\cdot)$ depends on the choice of $\lambda = (\lambda_1, \lambda_2)$ and that

$$T_n(\varphi_\lambda) = \frac{1}{2} \begin{bmatrix} 2\lambda_1 & \lambda_2 & 0 & 0 & \cdots & 0 \\ \lambda_2 & 2\lambda_1 & \lambda_2 & \cdots & \vdots \\ 0 & \lambda_2 & 2\lambda_1 & \lambda_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_2 & 2\lambda_1 & 0 \\ 0 & \cdots & 0 & 0 & \lambda_2 & 2\lambda_1 \end{bmatrix}.$$  

The fact that $X^{(n)}$ has multivariate Gaussian distribution gives us some advantage here. A standard result from Probability theory (see section B.6 in Bickel and Doksum [7]) shows that there is always a multivariate Gaussian vector $Y^{(n)} = (Y_{n,1}, \cdots, Y_{n,n})$ with independent components, such that

$$X^{(n)} = T_n(g_n)^{1/2} Y^{(n)},$$
where \( g_\theta(\cdot) \) is given in (2.2) and \( T_n(g_\theta)^{1/2} \) is the square root matrix of \( T_n(g_\theta) \). We also note that \( (T_n(g_\theta))_{n \in \mathbb{N}} \) is the sequence of autocovariance matrices associated to the process \( (X_n)_{n \in \mathbb{N}} \). Therefore, since \( T_n(g_\theta) \) is a positive definite matrix, the sequence of matrices \( (T_n(g_\theta)^{1/2})_{n \in \mathbb{N}} \) is well defined.

From (2.8) we obtain
\[
X^{(n)T} T_n(\varphi_\lambda) X^{(n)} = Y^{(n)T} T_n(g_\theta)^{1/2} T_n(\varphi_\lambda) T_n(g_\theta)^{1/2} Y^{(n)}. \tag{2.9}
\]

Since \( T_n(g_\theta)^{1/2} T_n(\varphi_\lambda) T_n(g_\theta)^{1/2} \) is a real symmetric matrix, there exists a sequence of orthogonal matrices \( (P_n)_{n \in \mathbb{N}} \) such that
\[
T_n(g_\theta)^{1/2} T_n(\varphi_\lambda) T_n(g_\theta)^{1/2} = P_n \Lambda_n P_n^T, \tag{2.10}
\]
with \( \Lambda_n = \text{Diag}(\alpha_{n1}^k, \ldots, \alpha_{nn}^k) \) a diagonal \( n \times n \) matrix, where \( (\alpha_{nk}^k)_{k=1}^n \) are the eigenvalues of \( T_n^{1/2}(g_\theta) T_n(\varphi_\lambda) T_n^{1/2}(g_\theta) \).

**Remark 3.** It is interesting to note that \( (\alpha_{nk}^k)_{k=1}^n \) are also the eigenvalues of \( T_n(\varphi_\lambda) T_n(g_\theta) \).

From (2.9) and (2.10) we obtain
\[
Y^{(n)T} T_n(g_\theta)^{1/2} T_n(\varphi_\lambda) T_n(g_\theta)^{1/2} Y^{(n)} = Y^{(n)T} P_n \Lambda_n P_n^T Y^{(n)}. \tag{2.11}
\]

As \( P_n \) is orthogonal, the product \( P_n^T Y^{(n)} \) has a multivariate Gaussian distribution with independent components. From (2.9) and (2.11), it is easy to conclude that
\[
X^{(n)T} T_n(\varphi_\lambda) X^{(n)} = \sum_{k=1}^n \alpha_{nk}^k Z_{n,k}, \tag{2.12}
\]
where \( Z_{1,n}, \ldots, Z_{n,n} \) are i.i.d. random variables, each one having a \( \chi_1^2 \) distribution with moment generating function given by
\[
M_{Z_{n,k}}(t) = \mathbb{E}(e^{t Z_{n,k}}) = \begin{cases} 1, & t < \frac{1}{2}, \\ \infty, & t \geq \frac{1}{2}, \end{cases} \tag{2.13}
\]
for \( k = 1, \ldots, n \).

Returning to the analysis of (2.6) and considering (2.12), as \( Z_{1,n}, \ldots, Z_{n,n} \) are mutually independent, we conclude that
\[
L_n(\lambda_1, \lambda_2) = \frac{1}{n} \log \mathbb{E} \left( e^{\sum_{k=1}^n \alpha_{nk}^k Z_{n,k}} \right) = \frac{1}{n} \log \left( \prod_{k=1}^n \mathbb{E} \left( e^{\alpha_{nk}^k Z_{n,k}} \right) \right). \tag{2.14}
\]

From (2.13), we observe that \( \mathbb{E} \left( e^{\alpha_{nk}^k Z_{n,k}} \right) \) is only defined if each one of the \( \alpha_{nk}^k < 1/2 \). In other words, (2.13) is finite if
\[
0 < 1 - 2 \alpha_{nk}^k, \quad \text{for all } k \text{ such that } 1 \leq k \leq n. \tag{2.15}
\]

The condition in (2.15) is equivalent to requiring that \( I_n - 2 T_n(\varphi_\lambda) T_n(g_\theta) \) must be positive definite (see Bercu et al. [4]). Since \( T_n(g_\theta) \) is a positive definite matrix and
\[
I_n - 2 T_n(\varphi_\lambda) T_n(g_\theta) = (T_n^{-1}(g_\theta) - 2 T_n(\varphi_\lambda)) T_n(g_\theta),
\]
it is sufficient to show that

\[ D_{n, \lambda} = T_n^{-1}(g_\theta) - 2T_n(\varphi_\lambda) = \begin{pmatrix} r_1 & q & 0 & \cdots & 0 \\ q & p & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & p \\ 0 & \cdots & 0 & q & r_1 \end{pmatrix} \]  

(2.16)

is positive definite, where \( r_1 = 1 - 2\lambda_1, \ p = 1 + \theta^2 - 2\lambda_1 \) and \( q = -\theta - \lambda_2 \). The domain \( \mathcal{D} \subseteq \mathbb{R}^2 \), where \( D_{n, \lambda} \) (and so \( I_n = 2T_n(\varphi_\lambda)T_n(g_\theta) \)) is positive definite, is given by the following lemma.

**Lemma 2.1.** If the pair \((\lambda_1, \lambda_2)\) belongs to the domain \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \), where

\[
\mathcal{D}_1 = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \lambda_1 \leq \frac{1-\theta^2}{2}, \ 4(\theta + \lambda_2)^2 < (1 + \theta^2 - 2\lambda_1)^2 \right\}, \\
\mathcal{D}_2 = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \frac{1-\theta^2}{2} < \lambda_1 < \frac{1}{2}, \ (\theta + \lambda_2)^2 < \theta^2(1 - 2\lambda_1) \right\},
\]

(2.17)

then, for \( n \) large enough, the tridiagonal matrix \( D_{n, \lambda} \), given in (2.16), is positive definite.

**Proof.** The proof is given in Appendix A. \( \square \)

To illustrate the domains presented in Lemma 2.1 Figures 2.1 and 2.2 show the graphs of \( \mathcal{D} \) when \( \theta = 0.9 \). In particular, Figure 2.1 shows the sets \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) separately, while Figure 2.2 shows the union \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \).

Figure 2.1: Regions \( \mathcal{D}_1 \) (in the left) and \( \mathcal{D}_2 \) (in the right) defined in (2.17) in the particular case when \( \theta = 0.9 \).

The knowledge of the domain where the matrix \( D_{n, \lambda} \) is positive definite, allows one to give continuity to the computations of \( L_n(\cdot, \cdot) \) and its limiting function when \( n \to \infty \). It is shown in Bryc and Dembo \( \diamondsuit \) (see page 330), for the special case \( \theta = 0 \), that

\[
\lim_{n \to \infty} L_n(\lambda_1, \lambda_2) = \begin{cases} -\frac{1}{2} \log \left( \frac{1 - 2\lambda_1 + \sqrt{(1 - 2\lambda_1)^2 - 4\lambda_2^2}}{2} \right), & \text{if } (\lambda_1, \lambda_2) \in \mathcal{D}_\varphi, \\
\infty, & \text{otherwise}, \end{cases}
\]
Figure 2.2: Region \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \), when \( \theta = 0.9 \).

where

\[
\mathcal{D}_\varphi = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \mid \sup_{\omega \in \mathcal{T}} \varphi_\lambda(\omega) < 1/2\}.
\]

Even though representing a particular degenerate case, it is important to note such result. If \( \theta = 0 \), the process \( (X_n)_{n \in \mathbb{N}} \) in (2.1) reduces itself to an i.i.d. sequence of random variables with standard Gaussian distribution. We shall generalize the result in Bryc and Dembo [9] on a bivariate setting, for the case when \( \theta \neq 0 \).

**Lemma 2.2.** Let \( L_n(\cdot, \cdot) \) denote the normalized cumulant generating function of \( (\mathcal{W}_n)_{n \geq 2} \), then

\[
\lim_{n \to \infty} L_n(\lambda_1, \lambda_2) = L(\lambda_1, \lambda_2),
\]

where \( L : \mathbb{R}^2 \to \mathbb{R} \cup \{\infty\} \) is defined by

\[
L(\lambda_1, \lambda_2) = \begin{cases} 
\frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\lambda_1 + \sqrt{(1 + \theta^2 - 2\lambda_1)^2 - 4(\theta + \lambda_2)^2}}{2} \right), & \text{for } (\lambda_1, \lambda_2) \in \mathcal{D}, \\
\infty, & \text{otherwise},
\end{cases}
\]

with the domain \( \mathcal{D} \) given in Lemma 2.1.

**Proof.** Let \( (\alpha_{n,k})_{k=1}^n \) represent the sequence of eigenvalues of \( T_n(\varphi_\lambda)T_n(g_\theta) \), with \( g_\theta(\cdot) \) denoting the spectral density function, defined in (2.2), and \( \varphi_\lambda(\cdot) \) the function given in (2.7). If \( (\lambda_1, \lambda_2) \in \mathcal{D} \), Lemma 2.1 guarantees that \( \alpha_{n,k}^\lambda < 1/2 \), for all \( 1 \leq k \leq n \) and \( n \) large enough. Then, from (2.13) and (2.14) it follows that

\[
L_n(\lambda_1, \lambda_2) = -\frac{1}{2n} \sum_{k=1}^n \log(1 - 2\alpha_{n,k}^\lambda), \quad \text{for } (\lambda_1, \lambda_2) \in \mathcal{D} \text{ and } n \text{ large enough}.
\]
Nonetheless, if \((\lambda_1, \lambda_2) \notin \mathcal{D}\) and \((\alpha_{n,\lambda}^n)_{n \in \mathbb{N}}\) represents the sequence of maximum eigenvalues of \(T_n(\varphi) T_n(g_0)\), we can always find a subsequence \((\alpha_{n_j,\lambda}^n)_{j \in \mathbb{N}}\) of \((\alpha_{n,\lambda}^n)_{n \in \mathbb{N}}\) such that \(\alpha_{n_j,\lambda}^n \geq 1/2\), for all \(j \in \mathbb{N}\). In that case, we have \(M_{Z_{n_j,\lambda}}(\alpha_{n_j,\lambda}^n) = \infty\), for all \(j \in \mathbb{N}\), implying that \(\lim_{n \to \infty} L_n(\lambda_1, \lambda_2) = \infty\). Henceforth, we only need to take care when \((\lambda_1, \lambda_2) \notin \mathcal{D}\), because in this case, \(L_n(\lambda_1, \lambda_2)\) is finite for \(n\) large enough and it is given by (2.20).

Consider in what follows the measure space \(L^\infty(\mathbb{T}) := L^\infty(\mathbb{T}, \mathcal{B}(\mathbb{T}), \nu)\), were \(\nu(\cdot)\) is the Lebesgue measure acting on \(\mathcal{B}(\mathbb{T})\), the Borel \(\sigma\)-algebra over \(\mathbb{T}\). If \(h \in L^\infty(\mathbb{T})\), the usual norm \(||h||_\infty = \inf\{h^\ast S(\mathbb{N}) \mid N \in \mathcal{B}(\mathbb{T}), \nu(N) = 0\}\), with \(h^\ast S(\mathbb{N}) = \sup\{|h(x)| : x \notin N\}\), shall be considered. Since \(\varphi_\lambda, g_0 \in L^\infty(\mathbb{T})\), it is straightforward to show that (see Avram [1])

\[
||\alpha_{n,k}^\lambda|| \leq ||\varphi_\lambda||_\infty ||g_0||_\infty, \quad \text{for all } 1 \leq k \leq n \text{ and } n \in \mathbb{N}. \tag{2.21}
\]

Let \(m_{\varphi_\lambda g_0}\) and \(M_{\varphi_\lambda g_0}\) denote, respectively, the essential infimum and essential supremum (see Grenander and Szegö [21]) of the continuous mapping \(\varphi_\lambda g_0 : \mathbb{T} \to \mathbb{R}\), belonging to \(L^\infty(\mathbb{T})\) and defined by

\[
(\varphi_\lambda g_0)(\omega) = \varphi_\lambda(\omega) g_0(\omega) = \frac{\lambda_1 + \lambda_2 \cos(\omega)}{1 + \theta^2 - 2\theta \cos(\omega)}. \tag{2.22}
\]

The function \((\varphi_\lambda g_0)(\cdot)\) is continuous and bounded in \([-\pi, \pi]\), hence it attains a maximum and a minimum in that interval. It follows that

\[
m_{\varphi_\lambda g_0} = \min_{\omega \in \mathbb{T}}\{(\varphi_\lambda g_0)(\omega)\} \quad \text{and} \quad M_{\varphi_\lambda g_0} = \max_{\omega \in \mathbb{T}}\{(\varphi_\lambda g_0)(\omega)\}.
\]

Since

\[
\frac{d}{d\omega}(\varphi_\lambda g_0)(\omega) = \frac{\lambda_2 \sin(\omega)}{1 + \theta^2 - 2\theta \cos(\omega)} - \frac{2\theta(\lambda_1 + \lambda_2 \cos(\omega)) \sin(\omega)}{(1 + \theta^2 - 2\theta \cos(\omega))^2},
\]

we notice that \((\varphi_\lambda g_0)(\omega)\) has two critical points at \(\omega = -\pi\) and \(\omega = 0\). Moreover,

- if \(\lambda_2 < -2\theta\lambda_1/(1 + \theta^2)\), then \(\frac{d^2}{d\omega^2}(\varphi_\lambda g_0)(-\pi) < 0\) and \(\frac{d^2}{d\omega^2}(\varphi_\lambda g_0)(0) > 0\);
- if \(\lambda_2 > -2\theta\lambda_1/(1 + \theta^2)\), then \(\frac{d^2}{d\omega^2}(\varphi_\lambda g_0)(-\pi) > 0\) and \(\frac{d^2}{d\omega^2}(\varphi_\lambda g_0)(0) < 0\);
- if \(\lambda_2 = -2\theta\lambda_1/(1 + \theta^2)\), then \((\varphi_\lambda g_0)(\omega) = \lambda_1\) is constant.

Therefore, since

\[
(\varphi_\lambda g_0)(-\pi) = \frac{\lambda_1 - \lambda_2}{1 + \theta^2 + 2\theta} \quad \text{and} \quad (\varphi_\lambda g_0)(0) = \frac{\lambda_1 + \lambda_2}{1 + \theta^2 - 2\theta},
\]

we conclude that

\[
m_{\varphi_\lambda g_0} = \begin{cases} 
\frac{\lambda_1 + \lambda_2}{1 + \theta^2 - 2\theta}, & \text{if } \lambda_2 < -\frac{2\theta\lambda_1}{1 + \theta^2}, \\
\frac{\lambda_1 - \lambda_2}{1 + \theta^2 + 2\theta}, & \text{if } \lambda_2 \geq -\frac{2\theta\lambda_1}{1 + \theta^2}, 
\end{cases}
\quad \text{and} \quad
M_{\varphi_\lambda g_0} = \begin{cases} 
\frac{\lambda_1 - \lambda_2}{1 + \theta^2 + 2\theta}, & \text{if } \lambda_2 < -\frac{2\theta\lambda_1}{1 + \theta^2}, \\
\frac{\lambda_1 + \lambda_2}{1 + \theta^2 - 2\theta}, & \text{if } \lambda_2 \geq -\frac{2\theta\lambda_1}{1 + \theta^2}.
\end{cases}
\]

Considering the case in which \((\lambda_1, \lambda_2) \in \mathcal{D}\), it follows that

\[
2|\theta + \lambda_2| < 1 + \theta^2 - 2\lambda_1,
\]
Explicit Rate Functions for LDP

whence

\[-\left(\frac{1 + \theta^2 - 2\lambda_1}{2}\right) < \theta + \lambda_2 < \frac{1 + \theta^2 - 2\lambda_1}{2} \tag{2.23}\]

From the left-hand side of (2.23), we get

\[
\frac{\lambda_1 - \lambda_2}{1 + \theta^2 + 2\theta} < \frac{1}{2},
\]

while from the right-hand side of (2.23), we obtain

\[
\frac{\lambda_1 + \lambda_2}{1 + \theta^2 - 2\theta} < \frac{1}{2}.
\]

Hence, we conclude that \(M_{\varphi, g_0} < 1/2\). On the other hand, from

\[
||\varphi||_\infty ||g_0||_\infty \geq ||\varphi_A g_0||_\infty = \max\{|M_{\varphi, g_0}|, |m_{\varphi, g_0}|\} \geq -m_{\varphi, g_0},
\]

we conclude that \(m_{\varphi, g_0} \geq -||\varphi||_\infty ||g_0||_\infty\). Therefore, we just proved that

\[
[m_{\varphi, g_0}, M_{\varphi, g_0}] \subseteq [-||\varphi||_\infty ||g_0||_\infty, 1/2). \tag{2.24}
\]

The denominator in the left-hand side of (2.22) satisfies

\[
\inf_{\omega \in \mathbb{T}} |1 + \theta^2 - 2\theta \cos(\omega)| = \min\{1 + \theta^2 - 2\theta, 1 + \theta^2 + 2\theta\} > 0, \text{ for all } \theta \in (-1, 1).
\]

Then, it follows from theorem 5.1 in Tyrtyshnikov [34] that, if \(F\) is any arbitrary continuous function with bounded support (i.e., the set of those \(x \in \mathbb{R}\) for which \(F(x) \neq 0\) is bounded), we get

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} F(\alpha_{n,k}) = \frac{1}{2\pi} \int_{\mathbb{T}} (F \circ (\varphi_A g_0))(\omega) \, d\omega. \tag{2.25}
\]

In particular, the latter convergence applies itself when considering the continuous function \(F: [-||\varphi||_\infty ||g_0||_\infty, 1/2) \to \mathbb{R}\) defined by

\[
F(x) = \frac{\log(1 - 2x)}{2}.
\]

Indeed, from (2.24) and (2.24), combined with the result of Lemma 2.1, we conclude that \(F(\cdot)\) has bounded support and that \(F(\alpha_{n,k})\) are finite, for every \(1 \leq k \leq n\) and \(n\) large enough. Besides that, \((F \circ (\varphi_A g_0))(\omega) = \log(1 - 2(\varphi_A g_0)(\omega))\) is finite, for every \(\omega \in \mathbb{T}\), due to (2.24). Therefore, the two sides of (2.25) are well defined and such convergence holds, giving

\[
\lim_{n \to \infty} L_n(\lambda_1, \lambda_2) = \lim_{n \to \infty} \frac{1}{2n} \sum_{k=1}^{n} \log(1 - 2\alpha_{n,k}) = \lim_{n \to \infty} \frac{1}{2n} \sum_{k=1}^{n} F(\alpha_{n,k})
\]

\[
= -\frac{1}{2\pi} \int_{\mathbb{T}} (F \circ (\varphi_A g_0))(\omega) \, d\omega = -\frac{1}{4\pi} \int_{\mathbb{T}} \log(1 - 2(\varphi_A g_0)(\omega)) \, d\omega
\]

\[
= -\frac{1}{2} \log \left(1 + \theta^2 - 2\lambda_1 + \sqrt{(1 + \theta^2 - 2\lambda_1)^2 - 4(\theta + \lambda_2)^2}\right),
\]

where the last equality was achieved using equation 4.224(9) in Gradshteyn and Ryzhik [19].
2.2 LDP of the random sequence \((W_n)_{n \geq 2}\)

Here we use the lemmas of Subsection 2.1, combined with the Gärtner-Ellis’ theorem, to prove that the sequence \((W_n)_{n \geq 2}\) in (2.3) satisfies an LDP. There are two conditions that must be satisfied in order to apply the Gärtner-Ellis’ theorem (see pages 43-44 in Dembo and Zeitouni [15]).

- **Condition A:** for each \((\lambda_1, \lambda_2) \in \mathbb{R}^2\), the limiting cumulant generating function \(L(\cdot, \cdot)\), defined as the limit in (2.18) and explicitly given by (2.19), exists as an extended real number. Moreover, if
  \[
  D_L = \{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 | L(\lambda_1, \lambda_2) < \infty \}
  \]  
  (2.26)
  denotes the effective domain of \(L(\cdot, \cdot)\), the origin must belong to \(D_L^o\) (the interior of \(D_L\)).

- **Condition B:** \(L(\cdot, \cdot)\) is an essentially smooth function, that is,
  1. \(D_L^o\) is non-empty;
  2. \(L(\cdot, \cdot)\) is differentiable throughout \(D_L^o\);
  3. \(L(\cdot, \cdot)\) is steep, i.e., we get \(\lim_{n \to \infty} ||\nabla L(\lambda_1, n, \lambda_2, n)|| = \infty\), in the case \((\lambda_1, n, \lambda_2, n)_{n \in \mathbb{N}}\) is a sequence in \(D_L^o\) converging to a boundary point of \(D_L^o\), where \(||(x, y)|| = \sqrt{x^2 + y^2}\) denotes the usual Euclidean norm in \(\mathbb{R}^2\).

Note that, if **Condition A** above is satisfied, then **Condition B.1** is redundant. In the following proposition, we verify that both **Conditions A** and **B** are satisfied when considering the LDP for \((W_n)_{n \geq 2}\). The cornerstone of our proof stands on the observation that the effective domain \(D_L\), defined in (2.26), contains the domain \(D\), given in Lemma 2.1.

**Proposition 2.1.** The sequence of random vectors \((W_n)_{n \geq 2}\), defined in (2.3), satisfies an LDP with good rate function

\[
J(x, y) = \begin{cases} 
\frac{1}{2} \left( x(1 + \theta^2) - 1 - 2y\theta + \log\left(\frac{x}{x^2 + y^2}\right) \right), & \text{for } 0 < x \text{ and } |y| < x, \\
\infty, & \text{otherwise}.
\end{cases}
\]  
(2.27)

**Proof.** Let \(L(\cdot, \cdot)\) denote the function in (2.18) and \(D\) the domain defined in Lemma 2.1. The effective domain of \(L(\cdot, \cdot)\) is given by

\[
D_L = \left\{ (\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1 < \frac{1}{2}, 4(\theta + \lambda_2)^2 < (1 + \theta^2 - 2\lambda_1)^2 \right\}.
\]

Notice that \(D\) is a proper subset of \(D_L\). Furthermore, if \((\lambda_1, \lambda_2) = (0, 0)\), then

\[
0 < (1 - \theta^2)^2 \Rightarrow 0 < 1 - 2\theta^2 + \theta^4 \Rightarrow 4\theta^2 < (1 + \theta^2)^2.
\]

Whence, the origin \((0, 0) \in \mathbb{R}^2\) belongs to the interior of \(D_L\), for any \(\theta \in (-1, 1)\), proving that **Condition A** above is fulfilled. The proof that \(L(\cdot, \cdot)\) is an essentially smooth function follows the same steps as the proof given in section 3.6 of Bryc and Dembo [9], so that **Condition B** is also verified.
Let \( J : \mathbb{R}^2 \to \mathbb{R} \) denote the Fenchel-Legendre dual of \( L(\cdot, \cdot) \), defined by the supremum
\[
J(x, y) = \sup_{(\lambda_1, \lambda_2) \in \mathbb{R}^2} \{x\lambda_1 + y\lambda_2 - L(\lambda_1, \lambda_2)\} = \\
\sup_{(\lambda_1, \lambda_2) \in \mathcal{D}} \left\{ x\lambda_1 + y\lambda_2 + \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\lambda_1 + \sqrt{(1 + \theta^2 - 2\lambda_1)^2 - 4(\theta + \lambda_2)^2}}{2} \right) \right\}. \tag{2.28}
\]
From the Gärtner-Ellis’ theorem, \((\mathcal{W}_n)_{n \geq 2}\) satisfies an LDP with good rate function \( J(\cdot, \cdot) \). To explicitly compute \( J(\cdot, \cdot) \), consider the auxiliary function \( K : \mathcal{D} \to \mathbb{R} \), defined by
\[
K(\lambda_1, \lambda_2) = x\lambda_1 + y\lambda_2 - L(\lambda_1, \lambda_2), \quad \text{with } (x, y) \in \mathbb{R}^2.
\]
The partial derivatives of \( K(\cdot, \cdot) \) are
\[
K_{\lambda_1}(\lambda_1, \lambda_2) = x - \frac{1}{\sqrt{1 - 4\lambda_1 + 4\lambda_1^2 - 4\lambda_2^2 - 8\theta\lambda_2 - 2\theta^2(2\lambda_1 + 1) + \theta^4}}
\]
and
\[
K_{\lambda_2}(\lambda_1, \lambda_2) = y + \frac{-2(\theta + \lambda_2)}{\left(1 + \theta^2 - 2\lambda_1 + \sqrt{(1 + \theta^2 - 2\lambda_1)^2 - 4(\theta + \lambda_2)^2}\right) \sqrt{(1 + \theta^2 - 2\lambda_1)^2 - 4(\theta + \lambda_2)^2}}.
\]
Provided that \( x > 0 \) and \( x > |y| \), the solution to the system of equations
\[
\begin{cases}
K_{\lambda_1}(\lambda_1, \lambda_2) = 0, \\
K_{\lambda_2}(\lambda_1, \lambda_2) = 0,
\end{cases}
\]
is given by
\[
\lambda_1^* = \frac{1 + \theta^2}{2} - \frac{x^2 + y^2}{2x(x^2 - y^2)} \quad \text{and} \quad \lambda_2^* = \frac{y}{x^2 - y^2} - \theta.
\]
It is not difficult to prove that \((\lambda_1^*, \lambda_2^*)\) is the point where the supremum in (2.28) is attained. Hence, it follows that
\[
J(x, y) = K(\lambda_1^*, \lambda_2^*) = \frac{1}{2} \left[ x(1 + \theta^2) - 1 - 2y\theta + \log \left( \frac{x}{x^2 - y^2} \right) \right],
\]
where \( 0 < x \) and \( |y| < x \).

Note that, the restrictions \( 0 < x \) and \( |y| < x \) are related to the inequalities \( 0 < \sum_{k=2}^n X_k^2 \) (see McLeod and Jiménez [28]) and \( |\sum_{k=2}^n X_kX_{k-1}| < \sum_{k=1}^n X_k^2 \).

If \( x \leq 0 \) or \( |y| \geq x \), we may define \( J(x, y) = \infty \), since \( K(\lambda_1, \lambda_2) \) is unbounded. Indeed, if \( x < 0 \), then
\[
\lim_{\lambda_1 \to -\infty} K(\lambda_1, \lambda_2) \approx \lim_{\lambda_1 \to -\infty} x\lambda_1 = \infty,
\]
because the linear part \( x\lambda_1 \) rules over the logarithmic part of \( K(\lambda_1, \lambda_2) \), while if \( x = 0 \), then
\[
\lim_{\lambda_1 \to -\infty} K(\lambda_1, \lambda_2) = \lim_{\lambda_1 \to -\infty} y\lambda_2 - L(\lambda_1, \lambda_2) = \infty.
\]
If \( x > 0 \), but \( |y| \geq x \), then we have two cases to consider: the first one is when \( y \leq -x \), whereby

\[
\lim_{\lambda_2 \to -\infty} K(\lambda_1, \lambda_2) \approx \lim_{\lambda_2 \to -\infty} y\lambda_2 = \infty;
\]

the second case is when \( y \geq x \), for which it follows that

\[
\lim_{\lambda_2 \to \infty} K(\lambda_1, \lambda_2) \approx \lim_{\lambda_2 \to \infty} y\lambda_2 = \infty.
\]

A graph of the function \( J(\cdot, \cdot) \), in (2.27), is shown in Figure 2.3 when \( \theta = 0.3 \). Since \( L(\cdot, \cdot) \) is a convex function, \( J(\cdot, \cdot) \) must also be a convex function (see section VI.5 in Ellis [18]).

![Graph of the function](image)

Figure 2.3: Graph of the function \( J(x, y) \) given in (2.27) when \( \theta = 0.3 \), \( x \in (0, 3) \) and \( y \in (-3, 3) \).

### 3 Particular cases

We dedicate this section to show three particular examples where the reasoning of the last section can be used, via Contraction Principle, to get explicit rate functions for univariate random sequences. Two of these examples were already known from Bercu et al. [4] and Bryc and Smoleski [10]. We shall obtain them as a continuous transform of the random vector \( W_n \), defined in (2.3). In Subsection 3.2, we present a result which we believe is new in the literature.

The Contraction Principle will be of great importance for the computations of the rate functions.

**Theorem 3.1** (Contraction Principle). If a sequence of random vectors \((V_n)_{n \in \mathbb{N}}\) with values in \( E \subseteq \mathbb{R}^d \) satisfies an LDP with good rate function \( J(\cdot) : \mathbb{R}^d \to [0, \infty] \) and \( U_n = f(V_n) \), where \( f(\cdot) : E \to \mathbb{R} \) is a continuous function, then the random sequence \((U_n)_{n \in \mathbb{N}}\) also satisfies an LDP with good rate function \( I(\cdot) : \mathbb{R} \to [0, \infty] \) given by

\[
I(c) = \inf_{x \in E} \{ J(x) \mid \text{with } x \text{ such that } f(x) = c \}, \quad \text{for all } c \in \mathbb{R}.
\]

**Proof.** See section 4.2.1 in Dembo and Zeitouni [15].
Explicit Rate Functions for LDP

Since the sequence of random vectors \((W_n)_{n \geq 2}\) satisfies an LDP with rate function \(J(\cdot, \cdot)\), given in (2.27), the Contraction Principle ensures that any sequence of vectors \((f(W_n))_{n \geq 2}\), for \(f: \mathbb{R}^2 \to \mathbb{R}\) continuous, satisfies an LDP with good rate function

\[
I(c) = \inf_{(x,y) \in \mathbb{R}^2} \{J(x,y) \mid \text{with } (x,y) \text{ such that } f(x,y) = c\}, \quad \text{for all } c \in \mathbb{R}. \tag{3.1}
\]

There is a standard procedure involving Calculus techniques for computing the infimum in (3.1), namely, checking for the critical points of the derivatives from \(J(\cdot, \cdot)\). In the examples considered below, the Wolfram Mathematica software (version 11.2.0.0) was used in the calculations.

Note that, \(f(W_n) = f\left(\frac{1}{n} \left(\sum_{k=1}^{n} X_k^2, \sum_{k=2}^{n} X_k X_{k-1}\right)\right)\) is a continuous function involving only the components \(\frac{1}{n} \sum_{k=1}^{n} X_k^2\) and \(\frac{1}{n} \sum_{k=2}^{n} X_k X_{k-1}\). Any statistic that can be written in terms of these components, as a continuous transform of \(W_n\), is suitable for our method. In particular, in Sections 3.1-3.3 we shall consider \(f: \mathbb{R}^2 \to \mathbb{R}\) as being respectively defined by

1. \(f(x, y) = x;\)
2. \(f(x, y) = y;\)
3. \(f(x, y) = \frac{y}{x}, \text{ for } x > 0.\)

Other continuous functions \(f: \mathbb{R}^2 \to \mathbb{R}\) could also be considered, however, in the present work, we shall restrict our attention to these three cases.

### 3.1 LDP for the quadratic mean

Consider the Quadratic Mean of a random sample \(X_1, \cdots, X_n\) which satisfies (2.1), given by

\[
\tilde{\gamma}_n(0) = \frac{1}{n} \sum_{k=1}^{n} X_k^2.
\]

Bryc and Smolenski [10] proved that the sequence \((\tilde{\gamma}_n(0))_{n \in \mathbb{N}}\) satisfies an LDP with rate function given by

\[
I(c) = \begin{cases} 
\frac{1}{2} \left[ c \left(1 + \theta^2\right) - \sqrt{1 + 4 \theta^2 c^2} - \log \left(\frac{2c}{1+\sqrt{1+4\theta^2c^2}}\right)\right], & \text{if } c > 0, \\
\infty, & \text{if } c \leq 0.
\end{cases} \tag{3.2}
\]

Here we obtain the result from Bryc and Smolenski [10] as a particular case, by using Proposition 2.1 and the Contraction Principle.

Note that \(\tilde{\gamma}_n(0)\) may be obtained as the projection on the first coordinate of the vector \(W_n\), given in (2.3). Consider \(f_1: \mathbb{R}^2 \to \mathbb{R}\) the continuous function given by \(f_1(x,y) = x\). Then \(f_1(W_n) = \tilde{\gamma}_n(0)\) and, since \((W_n)_{n \geq 2}\) satisfies an LDP with rate function \(J(\cdot, \cdot)\), given in (2.27), the Contraction Principle ensures that \((\tilde{\gamma}_n(0))_{n \in \mathbb{N}}\) satisfies an LDP with rate function, which we shall denote by \(I_1: \mathbb{R} \to [0, \infty]\). Then \(I_1(\cdot)\) can be computed from (3.1) in the following way.
• By the Contraction Principle, if \( c > 0 \), then

\[
I_1(c) = \inf_{\{0 < x, \ |y| < x\}} \{ J(x, y) \mid f_1(x, y) = c \} = \inf_{\ |y| < c} \left\{ \frac{1}{2} \left[ c(1 + \theta^2) - 1 - 2y\theta + \log \left( \frac{c}{c^2 - y^2} \right) \right] \right\} ;
\]

(3.3)

• The infimum in (3.3) is attained at

\[
y_c = \begin{cases} 
-1 + \frac{\sqrt{1 + 4c^2\theta^2}}{2\theta}, & \text{if } 0 < |\theta| < 1, \\
0, & \text{if } \theta = 0; 
\end{cases}
\]

• If \( \theta = 0 \), then it immediately follows that

\[
I_1(c) = \frac{c - 1 - \log(c)}{2};
\]

• If \( 0 < |\theta| < 1 \), then, after some algebraic computations, we obtain

\[
I_1(c) = \frac{1}{2} \left[ c(1 + \theta^2) - 1 - 2y\theta + \log \left( \frac{c}{c^2 - y^2} \right) \right] = \frac{1}{2} \left[ c(1 + \theta^2) - \sqrt{1 + 4c^2\theta^2} + \log \left( \frac{2c\theta^2}{\sqrt{1 + 4c^2\theta^2} - 1} \right) \right],
\]

and since

\[
\log \left( \frac{2c\theta^2}{\sqrt{1 + 4c^2\theta^2} - 1} \right) = \log \left( \frac{2c\theta^2(\sqrt{1 + 4c^2\theta^2} + 1)}{(\sqrt{1 + 4c^2\theta^2} - 1)(\sqrt{1 + 4c^2\theta^2} + 1)} \right) = \log \left( \frac{\sqrt{1 + 4c^2\theta^2} + 1}{2c} \right) = -\log \left( \frac{2c}{\sqrt{1 + 4c^2\theta^2} + 1} \right),
\]

we obtain

\[
I_1(c) = \frac{1}{2} \left[ c(1 + \theta^2) - \sqrt{1 + 4c^2\theta^2} - \log \left( \frac{2c}{\sqrt{1 + 4c^2\theta^2} + 1} \right) \right];
\]

• Considering that \( I_1(c) = \infty \), for \( c \leq 0 \), we conclude that \( I_1(c) = I(c) \), for all \( c \in \mathbb{R} \), with \( I(\cdot) \) defined in (3.2).

Therefore, we get the same result as in expression (1.2) in Bryc and Smolenski [10]. The graphs of \( I_1(\cdot) \) are illustrated in Figure 3.1 for four different values of \( \theta \). Notice that \( I_1(\cdot) \) is symmetric with respect to the values of \( \theta \), i.e., \( I_1(\cdot) \) is the same function for \( \theta \) and \(-\theta\), given that \( \theta \in (0, 1) \).

### 3.2 LDP for the first order empirical autocovariance

Consider now the first order Empirical Autocovariance of \( X_1, \ldots, X_n \), defined below as

\[
\tilde{\gamma}_n(1) = \frac{1}{n} \sum_{k=2}^{n} X_k X_{k+1}.
\]
Explicit Rate Functions for LDP

By the same reasoning as for the Quadratic Mean, we show here that the sequence \((\tilde{\gamma}_n(1))_{n \geq 2}\) satisfies an LDP, under the assumption that \((X_n)_{n \in \mathbb{N}}\) follows an AR(1) process, as defined in (2.1). We present the explicit expression for the deviation function, a result which we believe has not yet been shown in the literature.

Consider the continuous function \(f_2 : \mathbb{R}^2 \to \mathbb{R}\), with law \(f_2(x, y) = y\). Since \(f_2(W_n) = \tilde{\gamma}_n(1)\), it follows from Proposition 2.1 and the Contraction Principle that \((\tilde{\gamma}_n(1))_{n \geq 2}\) satisfies an LDP with rate function \(I_2 : \mathbb{R} \to [0, \infty)\). To give an explicit expression for \(I_2(\cdot)\), we proceed as follows.

- By the Contraction Principle,
  \[
  I_2(c) = \inf_{0 < x, |c| < x} \{ J(x, y) | f_2(x, y) = c \} = \inf_{0 < x, |c| < x} \{ J(x, c) \}
  = \inf_{0 < x, |c| < x} \left\{ \frac{1}{2} \left[ x(1 + \theta^2) - 2c\theta + \log \left( \frac{x}{x^2 - c^2} \right) \right] \right\};
  \tag{3.4}
  \]
- Denote by \(A(c, \theta) = \sqrt[3]{1 + 18c^2 (1 + \theta^2)^2 + 3\sqrt{3}} \sqrt{-c^2 (1 + \theta^2)^2 \left( c^4 (1 + \theta^2)^4 - 11c^2 (1 + \theta^2)^2 - 1 \right)}\).

Then, the infimum in \((3.3)\) is attained at

\[
  x_c = \frac{1 + 3c^2 (1 + \theta^2)^2 + A(c, \theta) + A(c, \theta)^2}{3 (1 + \theta^2) A(c, \theta)},
  \tag{3.5}
\]

provided that

\[
  c^2 < \frac{11 + 5\sqrt{5}}{2(1 + \theta^2)^2}.
  \tag{3.6}
\]

Figure 3.1: Graphs of \(I_1(\cdot)\) for \(\theta \in \{0, 0.3, 0.6, 0.9\}\) and \(c \in (0, 10]\).
The condition in (3.6) guarantees that $A(c, \theta)$ is real valued when $c \in \left[ -\sqrt{\frac{11+5\sqrt{5}}{2(1+\theta^2)}}, \sqrt{\frac{11+5\sqrt{5}}{2(1+\theta^2)}} \right]$.

- Inserting (3.5) into (3.4), we obtain

$$I_2(c) = \frac{1}{2} \left[ x_n(1 - 2c + \theta^2) - 1 + \log \left( \frac{1}{x_n(1 - c^2)} \right) \right]$$

$$= 1 + 3c^2(1 + \theta^2)^2 + \left( -2 - 6c + 3 \log \left( \frac{1 + 3c^2(1 + \theta^2)^2 + A(c, \theta) + A(c, \theta)^2}{3(1 + \theta^2)A(c, \theta)} \right) \right) \frac{A(c, \theta) + A(c, \theta)^2}{6A(c, \theta)}$$

for $c^2 < \frac{11 + 5\sqrt{5}}{2(1+\theta^2)}$.

- By setting $I_2(c) = +\infty$, if $c^2 \geq \frac{11 + 5\sqrt{5}}{2(1+\theta^2)}$, we concluded that $(\tilde{\gamma}_n(1))_{n \geq 2}$ satisfies an LDP with rate function $I_2(\cdot)$.

The graph of $I_2(\cdot)$ is illustrated in Figure 3.2 for five different values of $\theta$.

![Figure 3.2: Graphs of $I_2(\cdot)$ for $\theta \in \{-0.99, -0.6, 0, 0.6, 0.99\}$. The vertical lines $c^2 = (11 + 5\sqrt{5})/2$ (in blue), $c^2 = (11 + 5\sqrt{5})/3.699$ (in red) and $c^2 = (11 + 5\sqrt{5})/7.841$ (in green) represent the values of $c$ where $I_2(\cdot)$ changes to $\infty$, when $\theta = 0$, $|\theta| = 0.6$ and $|\theta| = 0.99$, respectively.]

3.3 LDP for the Yule-Walker estimates

Consider the Yule-Walker estimator

$$\hat{\theta}_n = \frac{\sum_{k=2}^{n} X_k X_{k-1}}{\sum_{k=1}^{n} X_k^2}$$

(3.7)
of the parameter $\theta$, for the AR(1) processes given in (2.1). The asymptotical behavior of such estimator is well known (see Brockwell and Davis [8]), so that

$$\sqrt{n}(\hat{\theta}_n - \theta) \Rightarrow N(0, 1 - \theta^2)$$

and that (see Mann and Wald [26])

$$\hat{\theta}_n \xrightarrow{n \to \infty} \theta, \text{ almost surely.}$$

In Bercu et al. [4] it was proved that the Yule-Walker estimator satisfies an LDP with rate function given by

$$S(c) = \begin{cases} \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right), & \text{if } |c| < 1, \\ \infty, & \text{otherwise.} \end{cases}$$

Later on, Bercu et al. [3] provided a Sharp Large Deviation Principle (SLDP) for Hermitian quadratic forms of stationary Gaussian processes, obtaining the Yule-Walker’s SLDP as a particular case. In Bercu [2], the study on LDP of the Yule-Walker estimator in AR(1) processes was extended to the unstable ($|\theta| = 1$) and explosive ($|\theta| > 1$) cases.

Here we obtain the result from Bercu et al. [4] by using Proposition 2.1 and the Contraction Principle. Since the rate function can be related to the sequence of probabilities $P(\hat{\theta}_n \geq c)$, for $|\hat{\theta}_n| < 1$ and $n \geq 2$, it actually makes sense to get $S(c)$ finite, for $|c| < 1$, and infinite when $|c| \geq 1$.

From (3.7), note that

$$\hat{\theta}_n = \frac{\sum_{k=2}^{n} X_kX_{k-1}}{\sum_{k=1}^{n} X_k^2} = f(W_n),$$

where $W_n$ is the random vector given in (2.3) and $f : \mathbb{R}^2 \to \mathbb{R}$ is the continuous function defined by

$$f(x, y) = \frac{y}{x}, \quad \text{for } 0 < x \text{ and } |y| < x.$$  \hspace{1cm} (3.8)

Since $(W_n)_{n \geq 2}$ satisfies an LDP with rate function $J(.\cdot)$, given in (2.27), the Contraction Principle is applicable and $(\hat{\theta}_n)_{n \geq 2}$ must satisfy an LDP with rate function, given by $I_\theta(\cdot) : \mathbb{R} \to [0, \infty]$. Then $I_\theta(\cdot)$ can be computed from (3.1) and (3.8) as follows.

- By the Contraction Principle,

$$I_\theta(c) = \inf_{\{0 < x, |y| < x\}} \{J(x, y) \mid f(x, y) = c\} = \inf_{\{0 < x, |y| < x\}} \{J(x, y) \mid y - cx = 0\}$$

$$= \inf_{0 < x} \left\{ \frac{1}{2} \left( x(1 - 2c\theta + \theta^2) - 1 + \log \left( \frac{1}{x(1 - c^2)} \right) \right) \right\}, \quad \text{for } |c| < 1; \hspace{1cm} (3.9)$$

- The infimum in (3.9) is attained at

$$x_c = \frac{1}{1 - 2c\theta + \theta^2}; \hspace{1cm} (3.10)$$

- Inserting (3.10) into (3.9), $I_\theta(c)$, for $|c| < 1$, reduces itself to

$$I_\theta(c) = \frac{1}{2} \left[ x_c(1 - 2c\theta + \theta^2) - 1 + \log \left( \frac{1}{x_c(1 - c^2)} \right) \right] = \frac{1}{2} \log \left( \frac{1 + \theta^2 - 2\theta c}{1 - c^2} \right);$$
• Considering that \( I_\theta(c) = \infty \) for \(|c| \geq 1\), we obtain \( I_\theta(c) = S(c), \forall c \in \mathbb{R} \).

Therefore, we get the same result as in expression (4.6) in Bercu et al. [4]. The graph of \( I_\theta(\cdot) \) is illustrated in Figure 3.3 for three different values of \( \theta \).

4 Large deviations for the bivariate SQ-Mean

After finding the rate function for the random sequence \((n^{-1} \sum_{k=1}^{n} X_k^2)_{n \geq 2}\) in Section 3.1, there exists a simple variation of that approach leading to the LDP for the sequence of bivariate SQ-Mean \((S_n)_{n \in \mathbb{N}}\), where

\[
S_n = \frac{1}{n} \left( \sum_{k=1}^{n} X_k, \sum_{k=1}^{n} X_k^2 \right).
\]

(4.1)

We shall use a result proved in Bryc and Dembo [9], which we enunciate below for completeness.

**Proposition 4.1.** Let \((X_n)_{n \in \mathbb{N}}\) be a real-valued centered stationary Gaussian process whose spectral density \(f(\cdot)\) is differentiable. Then, \((S_n)_{n \in \mathbb{N}}\), for \(S_n\) given in (4.1), satisfies an LDP (in \(\mathbb{R}^2\)) with good rate function

\[
K(x, y) = I(y - x^2) + \frac{x^2}{2f(0)}.
\]

(4.2)

where \(0/0 := 0\) in (4.2) and \(I(\cdot)\) is the rate function associated to \((n^{-1} \sum_{k=1}^{n} X_k^2)\).

**Proof.** See section 3.5 in Bryc and Dembo [9].
Explicit Rate Functions for LDP

We dedicate the next two subsections to the particular study of the LDP of the bivariate SQ-Mean when \((X_n)_{n \in \mathbb{N}}\) is an AR(1) process (Subsection 4.1) and is an MA(1) process (Subsection 4.2). Since the LDP for the Quadratic Mean is already available for the AR(1) process, it is easy to show such property in this case. For the MA(1) process, however, we must first derive the LDP of the Quadratic Mean in order to apply Proposition 4.1 and to provide the LDP for the bivariate SQ-Mean, likewise.

4.1 AR(1) process

Since the AR(1) process \((X_n)_{n \in \mathbb{N}}\) in (2.1) is a real-valued centered stationary Gaussian process, it follows from Proposition 4.1 that \((S_n)_{n \in \mathbb{N}}\) satisfies an LDP with rate function

\[
J_S(x, y) = I(y - x^2) + \frac{x^2}{2g_\theta(0)},
\]

where \(I(\cdot)\) is defined by (3.2) and \(g_\theta(\cdot)\) denotes the spectral density function given in (2.2). Note that \(g_\theta(\cdot)\) is differentiable. The explicit rate function is given by

\[
J_S(x, y) = \begin{cases} 
\frac{1}{2} \left[ y(1 + \theta^2) - 2x^2\theta - \sqrt{1 + 4\theta^2(y - x^2)^2} - \log \left( \frac{2(y - x^2)}{1 + \sqrt{1 + 4\theta^2(y - x^2)^2}} \right) \right], & \text{if } y > x^2, \\
\infty, & \text{if } y \leq x^2.
\end{cases}
\]

As a consequence, by an application of the Contraction Principle with the auxiliary continuous function \(f_1(x, y) = x\), we are able to obtain the rate function for the AR(1) Sample Mean \(\bar{X}_n = \sum_{k=1}^{n} X_k\). Following the same steps from Section 3.3, notice that the infimum

\[
I_{\bar{X}}(c) = \inf_{y > c^2} \{ J_S(x, y) \mid f_1(x, y) = c \} = \inf_{y > c^2} J_S(c, y)
\]

is attained at

\[
y_c = \frac{1 + c^2(1 - \theta^2)}{1 - \theta^2}.
\]

Hence, the sequence \((\bar{X}_n)_{n \in \mathbb{N}}\) satisfies an LDP with rate function

\[
I_{\bar{X}}(c) = J_S(c, y_c) = \frac{c^2(1 - \theta)^2}{2}, \quad \text{for } c \in \mathbb{R}.
\]

The graphs of \(I_{\bar{X}}(\cdot)\) are depicted in Figure 4.1 for three different values of \(\theta\). Notice that, \(I_{\bar{X}}(\cdot)\) has the shape of a parabola.

4.2 MA(1) process

Consider the MA(1) process, defined by the equation

\[
Y_n = \varepsilon_n + \phi \varepsilon_{n-1}, \quad \text{with } |\phi| < 1 \text{ and } n \in \mathbb{N}.
\]

(4.3)
Figure 4.1: Graph of $I_X(\cdot)$ for $\theta \in \{-0.5, 0, 0.5\}$ and $c \in [-10, 10]$.  

Here, we assume that the innovations $(\varepsilon_n)_{n \geq 0}$ are i.i.d., with $\varepsilon_n \sim \mathcal{N}(0,1)$. Then, $Y_n \sim \mathcal{N}(0,1+\phi^2)$, for each $n \in \mathbb{N}$, and the spectral density function associated to $(Y_n)_{n \in \mathbb{N}}$ is given by

$$h_\phi(\omega) = 1 + \phi^2 + 2\phi \cos(\omega), \quad \text{for } \omega \in \mathbb{T} = [-\pi, \pi).$$

The process $(Y_n)_{n \in \mathbb{N}}$ is stationary for any $\phi \in \mathbb{R}$ (see definition 3.4 in Shumway and Stoffer [33]). Nevertheless, the assumption $|\phi| < 1$ in (4.3) ensures that the process is also invertible and that $h_\phi(\cdot)$ is positive for all $\omega \in \mathbb{T}$.

Let us denote by

$$\tilde{\gamma}_n(1) = \frac{1}{n} \sum_{k=1}^{n} Y_k^2$$

the Quadratic Mean of a random sample $Y_1, \ldots, Y_n$, following the MA(1) process described in (4.3). Since the autocovariance function of $(Y_n)_{n \in \mathbb{N}}$ is equal to

$$\gamma_Y(k) = \begin{cases} 1 + \phi^2, & \text{if } k = 0, \\ \phi, & \text{if } |k| = 1, \\ 0, & \text{if } |k| > 1, \end{cases}$$

it is known (see section 7.3 in Brockwell and Davis [8]) that

$$\tilde{\gamma}_n(0) \xrightarrow{n \to \infty} \gamma_Y(0) = 1 + \phi^2,$$

almost surely.

We shall prove that the sequence $(\tilde{\gamma}_n(0))_{n \in \mathbb{N}}$ satisfies an LDP. For this reason, consider the normalized cumulant generating function

$$L_n(\lambda) = \frac{1}{n} \log E(e^{\lambda \tilde{\gamma}_n(0)}).$$
Explicit Rate Functions for LDP

In this case, the asymptotic distribution of \( L_n(\cdot) \) is known (see Grenander and Szeg"o [21]) and we immediately obtain the convergence

\[
\lim_{n \to \infty} L_n(\lambda) = L(\lambda) = \begin{cases} 
-\frac{1}{4\pi} \int_T \log[1 - 2\lambda h(\omega)] \, d\omega, & \text{if } \lambda \in \left(-\infty, \frac{1}{2M_{h\phi}}\right), \\
\infty, & \text{otherwise,}
\end{cases}
\]

where \( M_{h\phi} \) denotes the essential supremum of \( h(\cdot) \), given by

\[
M_{h\phi} = \begin{cases} 
\frac{1}{2(1+\phi)^2}, & \text{if } \phi \geq 0, \\
\frac{1}{2(1-\phi)^2}, & \text{if } \phi < 0.
\end{cases}
\]

As presented in Bercu et al. [4] and corollary 1 in Bryc and Dembo [9], \((\tilde{\gamma}_n(0))_{n\in\mathbb{N}}\) satisfies an LDP whose good rate function is the Fenchel-Legendre dual of \( L(\cdot) \), given by

\[
K_{\phi}(x) = \begin{cases} 
\sup_{\lambda < \lambda_{\phi}} \left\{ x\lambda + \frac{1}{4\pi} \int_T \log[1 - 2\lambda h(\omega)] \, d\omega \right\}, & \text{for } x > 0, \\
\infty, & \text{for } x \leq 0.
\end{cases}
\]

(4.4)

Since

\[
\int_T \log[1 - 2\lambda h(\omega)] \, d\omega = \int_T \log[1 - 2\lambda(1 + \phi^2) - 4\lambda \phi \cos(\omega)] \, d\omega = \log \left[ \frac{1 - 2\lambda(1 + \phi^2) + \sqrt{(1 - 2\lambda(1 + \phi^2))^2 - 16\lambda^2\phi^2}}{2} \right],
\]

the supremum in (4.4) is attained at

\[
\lambda_{\phi}(x) = \frac{A_{\phi}(x) + B_{\phi}(x)}{12x^2(\phi^2 - 1)^2} + C_{\phi}(x),
\]

(4.5)

where

\[
A_{\phi}(x) = 4x \left( x(\phi^2 + 1) - (\phi^2 - 1)^2 \right),
\]

\[
B_{\phi}(x) = 4x^2 \left( x^2(\phi^4 + 14\phi^2 + 1) + 4x(\phi^2 + 1)(\phi^2 - 1)^2 + (\phi^2 - 1)^4 \right),
\]

and

\[
C_{\phi}(x) = -\left( 1 + i\sqrt{3} \right) \left[ -x^6(\phi^6 - 33\phi^4 - 33\phi^2 + 1) - 6x^5(\phi^2 - 1)^2(\phi^4 - 10\phi^2 + 1) \\
+ 6x^4(\phi^2 - 1)^4(\phi^2 + 1) + x^3(\phi^2 - 1)^6 + 3\sqrt{3}c_{\phi}(x) \right]^{1/3},
\]

with

\[
c_{\phi}(x) = -x^8(\phi^2 - 1)^4 \left( 4x^4\phi^2 + 32x^3(\phi^4 + \phi^2) + x^2(\phi^4 + 46\phi^2 + 1)(\phi^2 - 1)^2 \\
+ 6x(\phi^2 + 1)(\phi^2 - 1)^4 + (\phi^2 - 1)^6 \right).
\]
Remark 4. Although \( C_\phi(\cdot) \) appears in a complex form, it can be proved that \( B_\phi(x)/C_\phi(x) + C_\phi(x) \in \mathbb{R} \), for any \( x > 0 \). In fact, \( \lambda_\phi(x) \) in (4.5) is one of the solutions from the polynomial equation

\[
\lambda^3 \left( 4x^2 \phi^4 - 8x^2 \phi^2 + 4x^2 \right) + \lambda^2 \left( -4x^2 \phi^2 - 4x^2 + 4x\phi^4 - 8x\phi^2 + 4x \right) + \lambda \left( x^2 - 4x\phi^2 - 4x + \phi^2 - 2\phi^2 + 1 \right) + x - \phi^2 - 1 = 0,
\]

which has three real roots if \( x > 0 \). Moreover, we have \( \lambda_\phi(x) < \frac{1}{2M_{\phi}} \).

We conclude that \((\tilde{S}_n(0))_{n \in \mathbb{N}}\) satisfies an LDP with rate function given by

\[
K_\phi(x) = x\lambda_\phi(x) + \frac{1}{2} \log \left[ \frac{1 - 2\lambda_\phi(x)(1 + \phi^2) + \sqrt{(1 - 2\lambda_\phi(x)(1 + \phi^2))^2 - 16\lambda_\phi(x)^2\phi^2}}{2} \right]
= \frac{f_\phi(x)}{12x(\phi^2 - 1)^2} + \frac{1}{2} \log \left( \frac{1}{2} - \frac{(\phi^2 + 1)f_\phi(x)}{12x^2(\phi^2 - 1)^2} + \frac{\left( \frac{1}{2} - \frac{(\phi^2 + 1)f_\phi(x)}{12x^2(\phi^2 - 1)^2} \right)^2 - \frac{\phi^2f_\phi(x)^2}{36x^4(\phi^2 - 1)^4}}{2} \right),
\]

for all \( x > 0 \) and \( K_\phi(x) = \infty \), for \( x \leq 0 \), with

\[
f_\phi(x) = A_\phi(x) + \frac{B_\phi(x)}{C_\phi(x)} + C_\phi(x).
\]

The graph of \( K_\phi(\cdot) \) is illustrated in Figure 4.2 for four different values of \( \phi \) and \( x \in (0, 5] \).

![Figure 4.2: Graphs of \( K_\phi(\cdot) \) for \( \phi \in \{0.2, 0.4, 0.6, 0.8\} \) and \( x \in (0, 5] \).](image-url)

By Proposition 4.1 we may now conclude that \((S_n)_{n \in \mathbb{N}}\) satisfies an LDP with rate function

\[
K_S(x, y) = \begin{cases} 
K_\phi(y - x^2) + \frac{x^2}{2(1+y^2)}, & \text{if } y > x^2, \\
\infty, & \text{if } y \leq x^2.
\end{cases}
\]
where $K_{\phi}(\cdot)$ is given in (4.6).

Note that, by the Contraction Principle, the sequence $(f_1(S_n))_{n \in \mathbb{N}} = (n^{-1} \sum_{k=1}^{n} Y_k)_{n \in \mathbb{N}}$, where $f_1(x, y) = x$ and \( S_n = n^{-1} (\sum_{k=1}^{n} Y_k, \sum_{k=1}^{n} Y_k^2) \), must satisfy an LDP with rate function

\[
I_{Y}(c) = \inf_{y > c^2} \left\{ K_S(x, y) \mid f_1(x, y) = c \right\} = \inf_{y > c^2} K_S(c, y) = \inf_{y > c^2} \left\{ K_{\phi}(y - x^2) + \frac{x^2}{2(1 + \phi)^2} \right\}.
\] (4.7)

However, when trying to compute the infimum in (4.7), we face a non-trivial problem.

Fortunately, an LDP for the Sample Mean of moving average processes has already been given in Burton and Dehling [12], where the authors considered the sequence

\[ X_n = \sum_{k \in \mathbb{Z}} a_{k+n} \varepsilon_k, \quad \text{for } n \in \mathbb{Z}, \]

with $(\varepsilon_n)_{n \in \mathbb{Z}}$ a sequence of i.i.d. random variables. They proved the LDP under the hypotheses that $(a_n)_{n \in \mathbb{Z}}$ is an absolutely summable sequence and that the moment generating function $E(e^{t \varepsilon_1})$ is finite, for all $t \in \mathbb{R}$. In Djellout and Guillin [16], a similar approach has been given. In this paper, the authors proved an analogous result under the hypotheses that the sequence $(\varepsilon_n)_{n \in \mathbb{Z}}$ is bounded and that $\sum_{k \in \mathbb{Z}} a_k^2 < \infty$.

If we set $a_{2n} = 1$, $a_{2n-1} = \phi$ and $a_k = 0$ for $k \in \mathbb{Z} \setminus \{2n, 2n-1\}$, then $X_n = \varepsilon_n + \phi \varepsilon_{n-1}$ is the MA(1) process given in (4.3), as long as the same hypotheses for the distribution of $(\varepsilon_n)_{n \geq 0}$ are considered. Then by theorem 2.1 in Burton and Dehling [12], the Sample Mean $(\bar{Y}_n)_{n \in \mathbb{N}} = (n^{-1} \sum_{k=1}^{n} Y_k)_{n \in \mathbb{N}}$ satisfies an LDP with rate function

\[
I_{\bar{Y}}(c) = \sup_{\lambda \in \mathbb{R}} \left\{ \frac{c \lambda}{1 + \phi} - \frac{\lambda^2}{2} \right\} = \frac{c^2}{2(1 + \phi)^2}, \quad \text{for } c \in \mathbb{R}.
\]

5 Conclusion

In this work, we showed that an LDP is available for the sequence $(W_n)_{n \geq 2}$, given in (2.3). The same technique to find such LDP is not restricted to the AR(1) process. There may exist other classes of processes that can be explored as well. If we take another process $(Z_n)_{n \in \mathbb{N}}$ which still has a multivariate Gaussian distribution, equipped with another spectral density function, other than the one given in (2.2), the proposed technique may remain valid. The LDP is, however, not always guaranteed and in most cases, the rate function is hard to compute. This difficulty mainly arises when trying to compute a closed form for the Fenchel-Legendre transform. Besides that, to obtain a similar convergence result as given in Lemma 2.2 for another class of Gaussian processes remains an intriguing problem. A remarkable class of processes that requires a more sophisticated approach, is the class of MA(1) process, which was not covered in this work when evaluating the LDP for the random vectors $(W_n)_{n \geq 2}$.

In Section 3, we presented three important particular examples using the previous reasoning from Section 2, together with the Contraction Principle. Two of these examples were already known from Bercu et al. [4] and Bryc and Smolenski [10] for univariate sequences. Here we obtained them as a
continuous transformation of the random vector $W_n$, given in (2.3). In Subsection 3.2, we presented a result which we believe is new in the literature. In Subsection 3.3, the LDP for the Yule-Walker estimator was obtained, via the Contraction Principle, getting the same result as in Bercu et al. [4]. The approach used here, first proving an LDP for bivariate random vectors and then particularizing to univariate random sequences via Contraction Principle has recently been used with continuous stochastic processes by Bercu and Richou [5], where the authors investigated the LDP of the maximum likelihood estimates for the Ornstein-Uhlenbeck process with shift. A similar approach was subsequently used by the same authors in Bercu and Richou [6], allowing them to circumvent the classical difficulty of non-steepness.

In Section 4, we provided an LDP for the sequence of bivariate SQ-Mean, for both AR(1) and MA(1) processes. For the AR(1) process, the computations were simple and the previous technique of proving an LDP for the bivariate random vector $W_n$ was extremely helpful. Nevertheless, when dealing with the MA(1) process, we found some issues due to the complexity of the computations involved. The same technique explored here may perhaps be available for general AR($d$) processes with Gaussian innovations. This is an important issue to be explored in the future.

A Proof of Lemma 2.1

In this appendix, we give the details for the proof of Lemma 2.1 which was based on the techniques given in page 270 in Jensen [24]. In summary, we use Sylvester’s Criterion (see theorem 7.2.5 in Horn [23]) to check for the positive definiteness of each principal minor of $D_{n,\lambda}$ and resort to the use of an auxiliary function with its corresponding iterates.

By Sylvester’s Criterion, $D_{n,\lambda}$ is positive definite, if and only if, the principal minors of $D_{n,\lambda}$ are positive. Hence, we analyze each one of the principal minors of $D_{n,\lambda}$ as follows:

1-st Step: since the first principal minor of $D_{n,\lambda}$ is $r_1 = 1 - 2\lambda_1$, we require that $r_1 > 0$. As a consequence, since $p = r_1 + \theta^2$, we obtain

$$0 < r_1 < p \Rightarrow 0 < p.$$  

2-nd Step: the second principal minor of $D_{n,\lambda}$ is defined as the determinant

$$\begin{vmatrix} r_1 & q \\ q & p \end{vmatrix} = p r_1 - q^2 = \left( p - \frac{q^2}{r_1} \right) r_1.$$  

(A.1)

Since we already restricted our analysis for $r_1 > 0$, (A.1) requires in addition that $r_2 := p - \frac{q^2}{r_1} > 0$.

3-rd Step: the third principal minor of $D_{n,\lambda}$ is the determinant

$$\begin{vmatrix} r_1 & q & 0 \\ q & p & q \\ 0 & q & p \end{vmatrix} = p^2 r_1 - q^2 r_1 - q^2 p = \left( p - \frac{q^2}{p} \right) \left( p - \frac{q^2}{r_1} \right) r_1.$$  

(A.2)
Explicit Rate Functions for LDP

Since we already restricted our analysis for $r_1 > 0$ and $p - \frac{q^2}{r_1} > 0$, (A.2) requires that $r_3 := \left(p - \frac{q^2}{r_1}\right) > 0$.

- **k-th Step**: by induction, the $k$-th principal minor of $D_{n,\lambda}$, for $1 \leq k \leq n - 1$, is the determinant
  \[
  \begin{vmatrix}
  r_1 & q & 0 & \cdots & 0 \\
  q & p & \ddots & \ddots & \vdots \\
  0 & q & \ddots & q & 0 \\
  \vdots & \ddots & \ddots & p & q \\
  0 & \cdots & 0 & q & r_1
  \end{vmatrix}
  \]

  for $r_2 = p - \frac{q^2}{r_1}$, $r_3 = \left(p - \frac{q^2}{p - \frac{q^2}{r_1}}\right)$ and $r_k = G^{k-1}(r_1)$, where $G^k$ denotes the $k$-th iterate of $G : (0, \infty) \rightarrow (0, \infty)$, given by
  \[
  G(a) = p - \frac{q^2}{a}.
  \]

Since $n \in \mathbb{N}$ is arbitrary, we must require that $G^k(r_1) > 0$, for all $k \in \mathbb{N}$. Without loss of generality, we may assume that $q \neq 0$ (if $q = 0$, then $D_{n,\lambda}$ is a diagonal matrix; this happens if and only if $\lambda_2 = -\theta$).

Notice that $G(\cdot)$ has the following two fixed points
  \[
  R = \frac{1}{2} \left(p - \sqrt{p^2 - 4q^2}\right) \quad \text{and} \quad Q = \frac{1}{2} \left(p + \sqrt{p^2 - 4q^2}\right).
  \]

The point named $Q$ is an attractor point and the point named $R$ is a repulsor point, provided that $p^2 > 4q^2$. If $p^2 = 4q^2$, then $P = Q = p$ is neither an attractor, neither a repulsor point. Let us consider henceforth $p^2 > 4q^2$.

Observe that $G(\cdot)$ is an increasing concave function. Therefore, the problem of knowing when $G^k(r_1) > 0$, for all $k \in \mathbb{N}$, reduces to knowing where $r_1 > R$. In one hand, every point greater than $R$ converges towards $Q$ and since $R > 0$, it follows that
  \[
  G^k(r_1) > R > 0, \quad \text{for all } k \in \mathbb{N}.
  \]

On the other hand,
  \[
  \forall x < R, \quad \exists n_0 \in \mathbb{N}; \quad G^{n_0}(x) < 0.
  \]

Since $r_1 = 1 - 2\lambda_1 = p - \theta^2$, we get
  \[
  r_1 > R \iff r_1 > \frac{p - \sqrt{p^2 - 4q^2}}{2} \iff \sqrt{p^2 - 4q^2} > p - 2r_1 = p - 2(p - \theta^2) = 2\theta^2 - p. \tag{A.3}
  \]

If $p > 2\theta^2$, then obviously $r_1 > R$, since the right-hand side of (A.3) is non-positive. But if $p < 2\theta^2$, then
  \[
  r_1 > R \iff p^2 - 4q^2 > (2\theta^2 - p)^2 \iff p^2 - 4q^2 > 4\theta^4 - 4\theta^2 p + p^2 \iff \theta^2(p - \theta^2) > q^2.
  \]
Therefore, we obtain the domain \( D = D_1 \cup D_2 \), where
\[
D_1 = \{ r_1 > 0, \; p^2 > 4q^2, \; p \geq 2\theta^2 \} \quad \text{and} \quad D_2 = \{ r_1 > 0, \; p^2 > 4q^2, \; p < 2\theta^2, q^2 < \theta^2(p - \theta^2) \}.
\]
Notice that \( r_1 > 0 \) is equivalent to \( p > \theta^2 \). Moreover, from
\[
0 > -\left( \frac{\theta^2 - \frac{p^2}{2}}{2} \right)^2 = -\theta^4 + 2\theta^2 \frac{p^2}{2} - \frac{p^2}{4} = -\theta^4 + \theta^2 p - \frac{p^2}{4} = \theta^2(p - \theta^2) - \frac{p^2}{4},
\]
we conclude that \( \theta^2(p - \theta^2) < \frac{p^2}{4} \). Hence, if \( q^2 < \theta^2(p - \theta^2) \), it follows that \( 4q^2 < p^2 \). Therefore, if \( p, q \) belong to
\[
D_1 = \{ p \geq 2\theta^2, \; p^2 > 4q^2 \} \quad \text{or} \quad D_2 = \{ \theta^2 < p < 2\theta^2, \; q^2 < \theta^2(p - \theta^2) \},
\]
then \( G^k(r_1) > 0 \), for all \( k \in \mathbb{N} \).

- **n-th Step**: last but not least, the \( n \)-th principal minor (or determinant) of \( D_{n,\lambda} \) is
\[
|D_{n,\lambda}| = \begin{vmatrix} r_1 & q & 0 & \cdots & 0 \\ q & p & & \cdots & \\ 0 & q & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & p & q \\ 0 & \cdots & 0 & q & r_1 \end{vmatrix} = r_1(r_{n-1} r_{n-2} \cdots r_2 r_1) - q^2(r_{n-2} \cdots r_2 r_1)
\]
\[
= \left( r_1 - \frac{q^2}{r_{n-1}} \right) (r_{n-1} r_{n-2} \cdots r_2 r_1) = (G^{n-1}(r_1) - \theta^2)(r_{n-1} r_{n-2} \cdots r_2 r_1).
\]
It is not difficult to see that, for \( n \) large enough, we eventually obtain \( G^{n-1}(r_1) > \theta^2 \). Indeed
\[
p > \theta^2 \Rightarrow Q > \theta^2 \Rightarrow \lim_{n \to \infty} G^n(r_1) = Q > \theta^2,
\]
so that
\[
\exists n_0 \in \mathbb{N}; n \geq n_0 \Rightarrow G^n(r_1) > \theta^2.
\]

The set \( D_1 \cup D_2 \) is therefore, the closed domain where all principal minors of \( D_{n,\lambda} \) are positive, and consequently, where the matrix \( D_{n,\lambda} \) is positive definite. Converting the domains \( D_1 \) and \( D_2 \) to the \((\lambda_1, \lambda_2)\) notation, we obtain the desired expressions given by (2.17).

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