Coupled Tensor Decomposition for Hyperspectral and Multispectral Image Fusion with Variability

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Abstract—Coupled tensor approximation has recently emerged as a promising approach for the fusion of hyperspectral and multispectral images, reconciling state of the art performance with strong theoretical guarantees. However, tensor-based approaches previously proposed assume that the different observed images are acquired under exactly the same conditions. A recent work proposed to accommodate spectral variability in the image fusion problem using a matrix factorization-based formulation, but did not account for spatially-localized variations. Moreover, it lacks theoretical guarantees and has a high associated computational complexity. In this paper, we consider the image fusion problem while accounting for both spatially and spectrally localized changes in an additive model. We first study how the general identifiability of the model is impacted by the presence of such changes. Then, assuming that the high-resolution image and the variation factors admit a Tucker decomposition, two new algorithms are proposed — one purely algebraic, and another based on an optimization procedure. Theoretical guarantees for the exact recovery of the high-resolution image are provided for both algorithms. Experimental results show that the proposed method outperforms state-of-the-art methods in the presence of spectral and spatial variations between the images, at a smaller computational cost.

Index Terms—Hyperspectral data, multispectral data, spectral variability, super-resolution, image fusion, tensor decomposition.

I. INTRODUCTION

Hyperspectral (HS) cameras are able to acquire images with very high spectral resolution. However, the fundamental compromise between signal-to-noise ratio, spatial resolution, and spectral resolution means that their spatial resolution is usually low [1]. Multispectral (MS) devices, on the other hand, are able to achieve a much higher spatial resolution since they contain only a small number of spectral bands. An approach that attempts to circumvent the physical limitations of imaging sensors consists in combining HS and MS images (MSI) of the same scene to obtain images with high spatial and spectral resolution, in a multimodal image fusion problem [2], commonly referred to as hyperspectral super resolution.

Different algorithms have been proposed to solve this problem. Early approaches were based on component substitution or on multiresolution analysis, which attempt to extract high-frequency spatial details from the MSI and combine them with the HS image (HSI) [3], [4]. Subspace-based formulations have later received a significant amount of interest as they explore the natural representation of the pixels in an HSI as the linear combination of a small number of spectral signatures [2], [3], [6]. Different algorithms have been proposed following this approach using, e.g., Bayesian formulations [7] or sparse representations on learned dictionaries [8], and different kinds of matrix factorization formulations employing sparse and spatial regularizations [9], [10], or estimating both the basis vectors and their coefficients blindly/unsupervisedly from the images [11].

The natural representation of HSIs and MSIs as 3-dimensional tensors has been successfully exploited for hyperspectral unmixing, denoising [12]–[15] and super-resolution. Superior super-resolution performance and exact recovery guarantees have been obtained using this formulation [16], [17].

The image fusion problem was formulated in [16] as a coupled tensor approximation problem. Assuming that the high resolution (HR) image admits a low-rank canonical polyadic decomposition (CPD), the problem was solved using an alternating optimization strategy. The recovery of the correct HR image (HRI) was shown to be guaranteed provided that the CPD of the MSI is identifiable, and state of the art performance was achieved. A recent work extended this approach by assuming the high resolution images to follow a block term decomposition (BTD), which shows a closer connection to the physical mixing model when compared to the CPD [18]. A simpler approach was later proposed in [19] by requiring only the computation of one CPD of the MSI and a singular value decomposition (SVD) of the HSI.

A Tucker decomposition-based approach was later considered in [17], [20]. Closed form SVD-based algorithms were proposed for the image fusion problem, achieving results comparable to [16] at a very small computational complexity; exact recovery guarantees were also provided. A coupled Tucker approximation was also considered in [21] using an alternating optimization approach and employing a sparsity regularization on the elements of the core tensor. Another approach considered the CPD of non-local similar patch tensors to explore the non-local redundancy of the image [22].
Most existing algorithms, however, share a common limitation: they assume that the HSI and the MSI are acquired under the same conditions. However, despite the short revisit cycles provided by the increasing number of optical satellites orbiting the Earth (e.g., Sentinel, Orbview, Landsat and Quickbird missions), the number of platforms carrying both HS and MS sensors is still considerably limited [23], [24]. This makes combining HS and MS observations acquired on board of different satellites of great interest to obtain HRIs [25], [26]. Images acquired at different time instants can be impacted by, e.g., illumination, atmospheric or seasonal changes. This may result in significant variations between the HSI and the MSI [27], negatively impacting traditional image fusion algorithms.

Recently, a method was proposed to combine HSIs and MSIs accounting for seasonal spectral variability [28]. Using a low-rank matrix formulation, the set of spectral basis vectors of the HRIs underlying the HS and the MS observations are allowed to be different from each other, with variations introduced by a set of multiplicative scaling factors [29]. This algorithm led to significant performance improvements when the HSI and MSI are subject to spatially uniform seasonal or acquisition variations. However, it does not account for spatially localized changes commonly seen in practical scenes [27]. Moreover, the algorithm in [28] presented high computation times and does not offer any theoretical guarantees.

In this paper, we propose a tensor-based image fusion formulation that accounts for localized spatial and spectral changes between the HSI and MSI. A general observation model is considered, in which the HRI underlying the MSI admits an additive variability term to account for changes between the scenes. Studying the general identifiability of this model, we show that this variability term can only be identified in general up to its smooth structure (which is defined according to the degradation operators). To introduce additional a priori information and mitigate the ambiguity associated with the proposed model, both the HRI and the additive perturbations are assumed to have low multilinear rank (i.e., to admit a Tucker decomposition). Two algorithms are then proposed, one totally algebraic and another based on an optimization procedure. Theoretical guarantees for the exact recovery of the HRI are provided for both. Simulation results show that the proposed optimization-based algorithm yields superior performance at a considerably lower computational cost when compared to [28], especially when spatially localized variability is considered.

II. TENSORS – BACKGROUND

A. Notation and definitions

An order-3 tensor $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ is an $N_1 \times N_2 \times N_3$ array whose elements are indexed by $[\mathcal{T}]_{n_1,n_2,n_3}$. Each dimension of a tensor is called a mode. A mode-$k$ fiber of tensor $\mathcal{T}$ is the one-dimensional subset of $\mathcal{T}$ which is obtained by fixing all but the $k$-th dimension. Similarly, a slab or slice of a tensor $\mathcal{T}$ is a matrix whose elements are the two-dimensional subset of $\mathcal{T}$ obtained by fixing all but two of its modes. Operator vec(·) represents the standard matrix column-major vectorization, or tensor vectorization. The (left) pseudo-inverse of matrix $X$ is denoted by $X^\dagger$. We denote scalars by lowercase (x) or uppercase (X) plain font, vectors and matrices by lowercase (x) and uppercase (X) bold font, respectively, and order-3 tensors by calligraphic plain font ($\mathcal{T}$) or using the blackboard Greek alphabet ($\mathcal{T}$). In the following, we review some useful operations of multilinear (tensor) algebra that will be used in the rest of the manuscript (see, e.g., [30], [31] for more details).

**Definition 1.** The mode-$k$ product between a tensor $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ and a matrix $B \in \mathbb{R}^{M_k \times N_k}$ is denoted by $\mathcal{U} = \mathcal{T} \times_k B$, and is evaluated such that each mode-$k$ fiber of $\mathcal{T}$ is multiplied by $B$, yielding $\mathcal{U}_{...,n_k-1,m_k,n_k+1,...} = \sum_{i=1}^{N_k} \mathcal{T}_{...,n_k-1,i,n_k+1,...} B_{m_k,i}$.

**Definition 2.** The full multilinear product is denoted by $\mathcal{T} \odot (B^{(1)}, B^{(2)}, B^{(3)})$, and consists of the successive application of mode-$k$ products between a tensor $\mathcal{T}$ and matrices $B^{(k)}$, which is represented as $\mathcal{T} \times_1 B^{(1)} \times_2 B^{(2)} \times_3 B^{(3)}$.

**Definition 3.** The mode-$k$ matricization of an order-3 tensor $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ denoted by $\mathcal{T}^{(k)}$, arranges its mode-$k$ fibers to be the columns of the resulting matrix $\mathcal{T}^{(k)} \in \mathbb{R}^{N_k \times N_1 N_2}$, $k, \ell, m \in \{1, 2, 3\}$, $k \neq \ell \neq m$, where the $n_k$-th row of $\mathcal{T}^{(k)}$ consists of the vectorization of the slice of $\mathcal{T}$ obtained by fixing the index of the $k$-th mode of $\mathcal{T}$ as $n_k$.

**Definition 4.** We define by tSVD$_R(X)$ the operator which returns a matrix containing the $R$ leading left singular vectors of a matrix $X$.

B. Tensor decompositions

The canonical polyadic (CP) rank of an order-3 tensor $\mathcal{T}$ is defined as the smallest number of rank one tensors that must be added together to represent $\mathcal{T}$ [32]. The polyadic decomposition decomposes a given rank-$K$ tensor $\mathcal{T} \in \mathbb{R}^{N_1 \times N_2 \times N_3}$ as a sum of at least $K$ outer products of three vectors, and is called the canonical polyadic decomposition when this decomposition contains exactly $K$ terms. Mathematically, the CPD of a tensor $\mathcal{T}$ can be represented using the mode-$k$ or the full multilinear product as

$$\mathcal{T} = \mathcal{G}_{CP} \times_1 B^{(1)} \times_2 B^{(2)} \times_3 B^{(3)}$$

(1)

$$= \left[\mathcal{G}_{CP}; B^{(1)}, B^{(2)}, B^{(3)}\right],$$

(2)

where $B^{(i)} \in \mathbb{R}^{N_i \times K_i}$, for $i \in \{1, 2, 3\}$ and $\mathcal{G}_{CP}$ is a diagonal tensor satisfying $[\mathcal{G}_{CP}]_{n_1 n_2 n_3} = \lambda_i \in \mathbb{R}$ for $n_1 = n_2 = n_3$, and otherwise equal to zero.

One striking difference between CPD for tensors and the low-rank decomposition for the matrices lies in the identifiability of the resulting factors. While the matrix factorization is ambiguous, the CPD of higher order ($P \geq 3$) tensors is unique up to scaling and permutations of the estimated factors under mild conditions on the tensor rank [32].

The Tucker decomposition [31] adds more flexibility to the CPD by allowing the core tensor $\mathcal{G}$ to be full, instead of diagonal. This allows for interactions between all the columns of the different factor matrices. This increased amount of freedom makes the Tucker decomposition a good candidate for tensor approximation and compression. Considering the factor matrices given by $B^{(i)} \in \mathbb{R}^{N_i \times K_i}$, $i \in \{1, 2, 3\}$, where...
The BTD also suffers from the same non-uniqueness problems as the Tucker decomposition. Nevertheless, by controlling the value of $R$, its increased flexibility allows one to consider different ranks for each mode of the decomposition without necessarily increasing the size of the core tensors or limiting the representation capability. Moreover, the BTD also benefits from uniqueness results which, although not as strong as those of the CPD, are still interesting for many applications [33, Section 5].

III. PROPOSED MODEL AND ITS UNDETERMINACIES

A. The imaging model

Let an HSI with high spectral resolution and low spatial resolution be represented as an order-3 tensor $\mathbf{Y}_h \in \mathbb{R}^{N_1 \times N_2 \times L_h}$, where $N_1$ and $N_2$ are the spatial and $L_h$ the spectral dimensions. Similarly, an MSI with high spatial and low spectral resolution is denoted by an order-3 tensor $\mathbf{Y}_m \in \mathbb{R}^{M_1 \times M_2 \times L_m}$, where $M_1 > N_1$ and $M_2 > N_2$ are the spatial and $L_m < L_h$ the spectral dimensions. Both the HSI and the MSI are assumed to be degraded versions of a tensor $\mathbf{Z} \in \mathbb{R}^{M_1 \times M_2 \times L_h}$, with high spectral and spatial resolutions. This degradation process is commonly described as [16], [17], [20], [21]:

\[
\mathbf{Y}_h = \mathbf{Z} \times_1 \mathbf{P}_1 \times_2 \mathbf{P}_2 + \mathbf{E}_h ,
\]

\[
\mathbf{Y}_m = \mathbf{Z} \times_3 \mathbf{P}_3 + \mathbf{E}_m ,
\]

where tensors $\mathbf{E}_m \in \mathbb{R}^{M_1 \times M_2 \times L_m}$ and $\mathbf{E}_h \in \mathbb{R}^{N_1 \times N_2 \times L_h}$ represent additive noise, matrix $\mathbf{P}_3 \in \mathbb{R}^{L_m \times L_h}$ contains the spectral response functions (SRF) of each band of the multispectral sensor, and matrices $\mathbf{P}_1 \in \mathbb{R}^{N_1 \times M_1}$ and $\mathbf{P}_2 \in \mathbb{R}^{N_2 \times M_2}$ represent the spatial blurring and downsampling in the hyperspectral sensor, which we assume to be separable for each spatial dimension as previously done in, e.g., [16], [17], [20], [21].

To make notation more convenient, we also denote the (linear) spatial and spectral degradation operators more compactly as

\[
\mathcal{P}_{1,2}(\mathbf{T}) = \mathbf{T} \times_1 \mathbf{P}_1 \times_2 \mathbf{P}_2 ,
\]

\[
\mathcal{P}_3(\mathbf{T}) = \mathbf{T} \times_3 \mathbf{P}_3 .
\]

Most previous works consider that $\mathbf{Y}_h$ and $\mathbf{Y}_m$ are acquired under the same conditions, implicitly assuming that no variability occur between the images. However, significant changes may result from, e.g., variations in atmospheric, illumination or seasonal conditions [27], [34], [35], motivating the development more flexible models.

Recently, spatially uniform spectral variability has been considered in [28]. The image fusion problem was formulated as a matrix factorization problem, and the (multiplicative) spectral variability as well as the spatial coefficients were estimated from the observed images. However, this work still did not address two fundamental problems: 1) How to account for both spatial and spectral variability and 2) what theoretical guarantees can be offered for the recovery of the HRI and (possibly) of the variability factors under these more challenging conditions.

To address these issues, we adopt a more general approach by considering two different HRIs $\mathbf{Z}_h \in \mathbb{R}_{+}^{N_1 \times M_1 \times L_h}$ and $\mathbf{Z}_m \in \mathbb{R}_{+}^{N_2 \times M_2 \times L_h}$, both with high spectral and spatial resolutions, underlying the observed HSI and the MSI, respectively. This leads to the following extension of model (9)–(10):

\[
\mathbf{Y}_h = \mathcal{P}_{1,2}(\mathbf{Z}_h) + \mathbf{E}_h ,
\]

\[
\mathbf{Y}_m = \mathcal{P}_3(\mathbf{Z}_m) + \mathbf{E}_m .
\]

Both HR images $\mathbf{Z}_h$ and $\mathbf{Z}_m$ are related to each other as follows:

\[
\mathbf{Z}_h = \mathbf{Z}, \quad \mathbf{Z}_m = \mathbf{Z} + \mathbf{\Psi} ,
\]

where $\mathbf{Z}$ denotes the underlying HRI that is shared by both modalities, and $\mathbf{\Psi} \in \mathbb{R}_{+}^{M_1 \times M_2 \times L_h}$ is an additive variability tensor representing changes between the scenes.

Considering the variability model (15) along with (11)–(14), we obtain the following observation model for the acquired HSI and MSI:

\[
\mathbf{Y}_h = \mathcal{P}_{1,2}(\mathbf{Z}) + \mathbf{E}_h ,
\]

\[
\mathbf{Y}_m = \mathcal{P}_3(\mathbf{Z} + \mathbf{\Psi}) + \mathbf{E}_m .
\]

B. The image fusion problem and its undeterminacies

The image fusion problem in this case consists in recovering $\mathbf{\Psi}$ and $\mathbf{Z}$ from the observed images $\mathbf{Y}_h$ and $\mathbf{Y}_m$. 

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*Note:* Each tensor $\mathbf{B}_{(i)}$ can have a possibly different number of columns (i.e., $K_i \neq K_j$), the Tucker decomposition is given by

\[
\mathcal{T} = \left[ \mathcal{G}_{TK}; \mathbf{B}_{(1)}, \mathbf{B}_{(2)}, \mathbf{B}_{(3)} \right] ,
\]

where $\mathcal{G}_{TK} \in \mathbb{R}^{K_1 \times K_2 \times K_3}$ is the full core tensor. Note that giving up on the diagonality of $\mathcal{G}_{TK}$ allows the rank along each mode of the tensor to be unique. This property can be very useful since it allows one to set a higher rank to specific modes of the decomposition in order to adequately represent the data diversity while still keeping the model parsimonious. The tuple $(K_1, K_2, K_3)$ is called the multilinear rank of $\mathcal{T}$. Each value $K_i$ is also equal to the rank of the mode-$i$ unfolding of $\mathcal{T}$ [30]. The matricizations and vectorization of a tensor $\mathcal{T}$ following the Tucker decomposition (3) are given by [31]:

\[
\text{vec}(\mathcal{T}) = (\mathbf{B}_{(1)} \otimes \mathbf{B}_{(2)} \otimes \mathbf{B}_{(3)}) \text{vec}(\mathcal{G}_{TK} ) ,
\]

\[
\mathcal{T}_{(1)} = \mathbf{B}_{(1)} \mathcal{G}_{TK} \mathbf{B}_{(3)} \otimes \mathbf{B}_{(2)} ,
\]

\[
\mathcal{T}_{(2)} = \mathbf{B}_{(2)} \mathcal{G}_{TK} \mathbf{B}_{(3)} \otimes \mathbf{B}_{(1)} ,
\]

\[
\mathcal{T}_{(3)} = \mathbf{B}_{(3)} \mathcal{G}_{TK} \mathbf{B}_{(2)} \otimes \mathbf{B}_{(1)} .
\]
Proof. If the operator \( \mathcal{P}_3 \) has nontrivial nullspace, then we can find \( Z, Z' \), different from one another, such that

\[
\mathcal{P}_3(Z) = \mathcal{P}_3(Z'),
\]

implying that \( \mathcal{Y}_h = \mathcal{Y}_h' \). Now, we can always find \( \Psi > 0 \) and \( \Psi' > 0 \) satisfying

\[
Z + \Psi = Z' + \Psi',
\]

which implies that \( \mathcal{Y}_m = \mathcal{Y}_m' \) (i.e., the model is not identifiable). Similarly, if operator \( \mathcal{P}_3 \) has nontrivial nullspace, then suppose we select \( Z = Z' \). This makes \( \mathcal{Y}_h = \mathcal{Y}_h' \). Then, we can select \( \Psi > 0 \) and \( \Psi' > 0 \), distinct from one another, satisfying

\[
\mathcal{P}_3(\Psi) - \mathcal{P}_3(\Psi') = 0,
\]

where \( 0 \) is the tensor of zeros. Since \( Z = Z' \), this leads to:

\[
\mathcal{P}_3(Z) = \mathcal{P}_3(Z') = \mathcal{P}_3(Z') + \mathcal{P}_3(\Psi) - \mathcal{P}_3(\Psi'),
\]

which also implies that \( \mathcal{Y}_m = \mathcal{Y}_m' \) (i.e., the model is not identifiable).

Theorem 1. Suppose that the HSI and MSI are generated following the model in (16)–(17), that the observation noise is zero (i.e., \( \mathcal{E}_h = \mathcal{E}_m = 0 \)) and that \( \Omega_h = \mathbb{R}^{M_1 \times M_2 \times L_h} \) and \( \Omega_\Psi = \mathbb{R}^{M_1 \times M_2 \times L_\Psi} \). Then, if either the operator \( \mathcal{P}_3 \) or the operator \( \mathcal{P}_{1,2} \) has nontrivial nullspace (e.g., if \( L_m < L_h \) or if \( N_1 N_2 < M_1 M_2 \)), then the pair \( (\Psi, Z) \) cannot be uniquely identified from the observations \( (\mathcal{Y}_h, \mathcal{Y}_m) \).

Proof. If the operator \( \mathcal{P}_{1,2} \) has nontrivial nullspace, then we can find \( Z, Z' \), different from one another, such that

\[
\mathcal{P}_{1,2}(Z) = \mathcal{P}_{1,2}(Z'),
\]

implying that \( \mathcal{Y}_h = \mathcal{Y}_h' \). Now, we can always find \( \Psi > 0 \) and \( \Psi' > 0 \) satisfying

\[
Z + \Psi = Z' + \Psi',
\]

which implies that \( \mathcal{Y}_m = \mathcal{Y}_m' \) (i.e., the model is not identifiable). Similarly, if operator \( \mathcal{P}_3 \) has nontrivial nullspace, then suppose we select \( Z = Z' \). This makes \( \mathcal{Y}_h = \mathcal{Y}_h' \). Then, we can select \( \Psi > 0 \) and \( \Psi' > 0 \), distinct from one another, satisfying

\[
\mathcal{P}_3(\Psi) - \mathcal{P}_3(\Psi') = 0,
\]

where \( 0 \) is the tensor of zeros. Since \( Z = Z' \), this leads to:

\[
\mathcal{P}_3(Z) = \mathcal{P}_3(Z') = \mathcal{P}_3(Z') + \mathcal{P}_3(\Psi) - \mathcal{P}_3(\Psi'),
\]

which also implies that \( \mathcal{Y}_m = \mathcal{Y}_m' \) (i.e., the model is not identifiable).
factors of the sets \( \Omega \) obtained unless we provide stricter a priori characterizations of the variability factors have to satisfy
\[
\Psi - \Psi' = -Z - Z' + X'' \quad \text{for all } \Psi, \Psi', \Psi'' \in \ker(\mathcal{F}) \quad \text{and all } X'', X' \in \ker(\mathcal{F}).
\]
(29)
The set of all possible \( \Psi - \Psi' \) satisfying (29) is the sum of \( \ker(\mathcal{F}_3) \) and \( \ker(\mathcal{F}_1 \circ \mathcal{F}_2) \). We can readily see that \( \Psi \) cannot be recovered from the observations, only the spectrally degraded variability factors \( \mathcal{F}_3(\Psi) \) can be uniquely recovered (which comes “for free” with the recovery of \( Z \) since it can be computed as \( \mathcal{F}_3(\Psi) = Y_m - \mathcal{F}_3(Z) \)). This makes it sufficient to study the capability of an algorithm to recover \( Z \) in our model. Since the matrices \( P_i \), \( i \in \{1, 2, 3\} \) are essentially low-pass filtering and downsampling operators, their nullspaces intuitively encode high-frequency information along each tensor mode. Thus, only the smooth structure of \( \Psi \) can be identified uniquely from observations \( (Y_h, Y_m) \), since otherwise we cannot separate the effects of \( \Psi \) from \( Z \).

We also note that each EC in (25), which contains all factors \( \Psi \) whose difference lies in the nullspace of the combined operator \( \mathcal{F}_1 \circ \mathcal{F}_3 \), is strictly larger than if we considered changes that occur in the nullspace of each of these operators individually (i.e., \( \mathcal{F}_{1,2} \) and \( \mathcal{F}_3 \)).

Theorem 2 guarantees that tensors belonging to different ECs will result in different observations, which is the minimal requirement for having identifiable \( Z \) and \( \mathcal{F}_3(\Psi) \). However, the coresets inverse problem still remains ill-posed as the number of unknowns is greater than the number of available data. Thus, stronger identifiability conditions cannot be obtained unless we provide stricter a priori characterizations of the sets \( \Omega_Z \) and \( \Omega_\Psi \).

C. A Low-Multilinear-Rank Model

One possible condition that can be imposed on the structures of both \( \Omega_\Psi \) and \( \Omega_Z \) is the low-rank tensor model. This kind of structure makes it possible to obtain identifiability and exact recovery guarantees for problem (18), where spatial and spectral variabilities are present. Moreover, it also makes the problem well-posed and easier to solve since the number of unknowns becomes smaller than the amount of available data.

Suppose that \( Z \) and \( \Psi \) have multilinear ranks \( (K_{Z,1}, K_{Z,2}, K_{Z,3}) \) and \( (K_{\Psi,1}, K_{\Psi,2}, K_{\Psi,3}) \), respectively. This means that they can be represented as
\[
Z = \left[ g_Z; B(Z,1), B(Z,2), B(Z,3) \right],
\]
\[
\Psi = \left[ g_\Psi; B(\psi,1), B(\psi,2), B(\psi,3) \right],
\]
(30)
\( (31) \)

where \( B(Z,i) \in \mathbb{R}^{M \times K_{Z,i}}, B(\psi,i) \in \mathbb{R}^{L \times K_{\psi,i}}, i \in \{1, 2, 3\}, B(Z,1) \in \mathbb{R}^{Lz \times K_{Z,1}}, B(\psi,1) \in \mathbb{R}^{L_{\psi,1} \times K_{\psi,1}} \) are the factor matrices and \( g_Z \in \mathbb{R}^{L_{\psi,1} \times K_{Z,1} \times K_{Z,2} \times K_{Z,3}}, g_\Psi \in \mathbb{R}^{K_{_\psi,1} \times K_{\psi,2} \times K_{\psi,3}} \) are the core tensors.

Our objective is to study the identifiability and exact recovery of these variables given the observation model in (16)–(17). Using this model, the noiseless case of the degradation model (16)–(17) can be written as
\[
Y_h = \left[ g_Z; P_1 B(Z,1), P_2 B(Z,2), B(Z,3) \right],
\]
\[
Y_m = \left[ g_Z; B(Z,1), B(Z,2), P_3 B(Z,3) \right]
\]
\[
+ \left[ g_\Psi; B(\psi,1), B(\psi,2), P_3 B(\psi,3) \right].
\]
(32)
(33)

Note that we can represent the multiespectral image model in (33) equivalently using a standard Tucker model as:
\[
Y_m = \left[ C_m; C_{m,1}, C_{m,2}, C_{m,3} \right],
\]
where \( C_{m,i} \), \( i \in \{1, 2, 3\} \) and \( C_m \) are the factor matrices and the core tensor of the MSI, which satisfy:
\[
C_m = g_Z \oplus g_\Psi,
\]
\[
C_{m,i} = \left[ B(Z,i); B(\psi,i) \right], \quad i \in \{1, 2, 3\},
\]
and \( \oplus \) is a binary operator that returns the block-diagonal tensor whose blocks are the operands.

IV. AN ALGEBRAIC ALGORITHM

Considering the model in Section III-C, the image fusion problem consists in estimating the factors and core tensor \( g_Z, B(Z,i), i \in \{1, 2, 3\} \). However, if the values composing the multilinear rank of \( Z \) are sufficiently low, these variables can be computed by solving the following coupled system of equations:
\[
\begin{align*}
Y_h &= \left[ g_Z; C_{h,1}, C_{h,2}, B(Z,3) \right] \\
Y_m &= \left[ C_m; C_{m,1}, C_{m,2}, P_3 C_{m,3} \right], \\
C_{h,i} &= \left[ P_i B(Z,i) \right], \quad i \in \{1, 2\} \\
C_{m,i} &= \left[ B(Z,i); B(\psi,i) \right], \quad i \in \{1, 2, 3\}
\end{align*}
\]
(34)
(35)
(36)
(37)

where \( C_{h,i}, i \in \{1, 2\} \) denote the spatial factor matrices of the HSI, and the HRI is obtained from the solution of (37) as \( Z = \left[ g_Z; B(Z,1), B(Z,2), B(Z,3) \right] \).

If we suppose that \( K_{Z,i} + K_{\psi,i} \leq N_i, i \in \{1, 2\} \), (37) can be solved using an efficient, algebraic approach detailed in Algorithm 1, which we call CT-STAR (Coupled Tucker decompositions for hyperspectral Super-resolution via vARiability).

It is important to note that CT-STAR does not use the block diagonal structure of the core tensor of the MSI (described in (35)). The following theorem gives a constructive proof of exact recovery conditions from which Algorithm 1 is derived.

Theorem 3. Suppose that the HRI \( Z \) and the variability tensor \( \Psi \) have multilinear ranks \( (K_{Z,1}, K_{Z,2}, K_{Z,3}) \) and
Algorithm 1: Algebraic image fusion method (CT-STAR)

Input : Images \( Y_h, Y_m \) ranks \( K_{Z,i}, K_{\Psi,i}, i \in \{1,2,3\} \)
Output: HRI \( \hat{Z} \), spectrally degraded variability factors \( \Phi(\hat{Z}) \)

1. Check if \( K_{Z,i} + K_{\Psi,i} \leq N_i, i \in \{1,2\} \);
2. Compute \( \hat{C}_{h,3} = \text{tSVD}_{K_{Z,3}}(Y_{h,3}) \);
3. Compute \( \hat{C}_{m,i} = \text{tSVD}_{K_{Z,i}}(Y_{m,i}) \) for \( i \in \{1,2\} \);
4. Compute \( \bar{Q}_{h,i} \) for \( i \in \{1,2\} \), as \( \bar{Q}_{h,i} = (P_i \hat{C}_{m,i})^t \text{tSVD}_{K_{Z,3}}(Y_{h,3}) \);
5. Compute \( \hat{C}_{m,i} = \hat{C}_{m,i} \bar{Q}_{h,i} \) for \( i \in \{1,2\} \);
6. Compute \( \hat{Q}_Z \)
   by solving \( (C_{h,3} \otimes P_3 \hat{C}_{m,2} \otimes P_1 \hat{C}_{m,1}) \vec{\phi}(\hat{Q}_Z) = \vec{\phi}(Y_h) \);
7. Compute \( \hat{Z} = [\hat{Q}_Z; \hat{C}_{m,1}, \hat{C}_{m,2}, \hat{C}_{h,3}] \);
8. Compute \( \Phi(\hat{Z}) = Y_m - \hat{Z} \times P_3 \).

\( (K_{\Psi,1}, K_{\Psi,2}, K_{\Psi,3}) \), respectively, that \( Y_h \) and \( Y_m \) admit Tucker decompositions as denoted in (37), that the observation noise is zero (i.e. \( \hat{E}_h = 0, \hat{E}_m = 0 \)), and that

\[
\text{rank}(P_i B_{Z,i}) = K_{Z,i}, \ i \in \{1,2\} \tag{38}
\]

\[
\text{rank}(P_i B_{\Psi,i}) \leq K_{\Psi,i}, \ i \in \{1,2\} \tag{39}
\]

\[
\text{rank}(Y_{m,i}) = K_{Z,i}, \ i \in \{1,2,3\} \tag{40}
\]

\[
\text{rank}(Y_{m,ij}) = K_{Z,i} + K_{\Psi,i} \leq N_i, \ i \in \{1,2\} \tag{41}
\]

Then, if all columns in \( P_i B_{Z,i} \) are linearly independent from those in \( P_i B_{\Psi,i} \), for \( i \in \{1,2\} \), Algorithm 1 exactly recovers \( Z \) from the observations.

Proof. Let us compute matrices \( \hat{C}_{m,i}, i \in \{1,2,3\} \), as the left-singular vectors associated with the non-zero singular values of \( Y_{m,ij} \), \( i \in \{1,2\} \) and \( Y_{h,ij} \), \( i \in \{1,2,3\} \), respectively. Then, due to (40)–(41) and to the non-uniqueness of matrix decomposition, these matrices satisfy:

\[
\hat{C}_{h,i} = P_i B_{Z,i} Q_{h,i}, \ i \in \{1,2\} \tag{42}
\]

\[
\hat{C}_{h,3} = B_{Z,3} Q_{h,3} \tag{43}
\]

\[
\hat{C}_{m,i} = C_{m,i} Q_{m,i}, \ i \in \{1,2\} \tag{44}
\]

for invertible matrices \( Q_{h,i} \in \mathbb{R}^{K_{Z,i} \times K_{Z,i}}, i \in \{1,2,3\} \) and \( Q_{m,i} \in \mathbb{R}^{(K_{Z,i} + K_{\Psi,i}) \times (K_{Z,i} + K_{\Psi,i})}, i \in \{1,2\} \).

Now, the main problem caused by the presence of variability is that matrices \( Q_{m,1} \) and \( Q_{m,2} \) preclude us from distinguishing the factors \( B_{Z,1} \) and \( B_{Z,2} \), associated with \( Z \), from \( B_{\Psi,1} \) and \( B_{\Psi,2} \), associated with \( \Psi \), using only information available in the MSI. These two become mixed in the spatial factors \( \hat{C}_{m,i} \). Nonetheless, consider the relationship between spatial degradation of the factors estimated from the MSI and the spatial factors of the HSI:

\[
P_i \hat{C}_{m,i} = P_i C_{m,i} Q_{m,i} \tag{45}
\]

\[
= [P_i B_{Z,i}, P_i B_{\Psi,i}] Q_{m,i} \tag{46}
\]

Now, let us compute matrices \( \hat{Q}_{h,i} \in \mathbb{R}^{(K_{Z,i} + K_{\Psi,i}) \times K_{Z,i}}, i \in \{1,2\} \) such that

\[
\hat{C}_{h,i} = P_i \hat{C}_{m,i} \hat{Q}_{h,i} \tag{47}
\]

By partitioning the following matrix product as \( Q_{m,i} \hat{Q}_{h,i} = [Q_{Z,i}, Q_{\Psi,i}]^t, \) (47) can be represented as

\[
\hat{C}_{h,i} = P_i B_{Z,i} Q_{h,i} + P_i B_{\Psi,i} Q_{\Psi,i} \tag{48}
\]

Since all columns in \( P_i B_{Z,i} \) are linearly independent from those in \( P_i B_{\Psi,i} \), equality (47) will be satisfied if and only if the result of the product \( \hat{C}_{m,i} \hat{Q}_{h,i} \) (i.e., the right hand side of (48)) does not contain any nontrivial linear combination of the columns of \( P_i B_{\Psi,i} \). Thus, \( \hat{Q}_{\Psi,i} = 0 \) and \( Q_{Z,i} = Q_{h,i} \) due to (38) and (40). This allows us to “separate” the variability and image factors as

\[
\hat{C}_{m,i} = \hat{C}_{m,i} \hat{Q}_{h,i} = B_{(Z,i)} Q_{h,i}, \tag{49}
\]

for \( i \in \{1,2\} \). Now, consider the vectorization of the HSI as:

\[
(\hat{C}_{h,3} \otimes P_3 \hat{C}_{m,2} \otimes P_1 \hat{C}_{m,1}) \vec{\phi}(\hat{Q}_Z) = \vec{\phi}(Y_h) \tag{50}
\]

since the matrix in the l.h.s. has full column rank, \( \hat{Q}_Z \) can be uniquely recovered from this equation, and will satisfy

\[
\hat{Z} = [\hat{Q}_Z; \hat{C}_{m,1}, \hat{C}_{m,2}, \hat{C}_{h,3}] \tag{51}
\]

and

\[
\psi \times P_3 = Y_m - \hat{Z} \times P_3, \tag{52}
\]

which completes the proof.

\( \square \)

Note that although CT-STAR cannot forego knowledge of the spatial degradation operators \( P_1 \) and \( P_2 \), the spectral degradation operation \( P_3 \) is only used in (52), and is not necessary if we only want to recover \( \hat{Z} \). This makes it possible to obtain an algorithm “blind” in the spatial dimension, similarly to the spatially blind methods proposed in [16], [17].

The CT-STAR algorithm is fast, but only works for the cases where the ranks of the spatial modes of \( Z \) are smaller than the dimensions of the HSI, which is quite restrictive. Moreover, both Algorithm 1 and Theorem 3, in considering model (37), made no assumptions about the (block diagonal) structure of the core tensor of the MSI. Although this led to more freedom from a modeling perspective, the recoverability conditions turned out to be restrictive.

V. AN OPTIMIZATION-BASED ALGORITHM

In this section, we pursue a different approach. Assume that model (32)–(33) holds and that the values forming the multilinear ranks of both \( Z \) and \( \Psi \) are sufficiently low so that \( Y_m \) admits a block term decomposition (BTD) in the noiseless case [33]. We can then use uniqueness results thereof to guarantee the identifiability of \( Z \) under less restrictive conditions.

Let us consider the image fusion problem as the solution to the following optimization problem:

\[
\min_{\Theta} J(\Theta) = \left\| Y_h - [\hat{Q}_Z; P_1 B_{Z,1}, P_2 B_{Z,2}, B_{Z,3}] \right\|_F^2 + \left\| Y_m - \sum_{i \in \{Z,\Psi\}} [\hat{Q}_i; B_{(i,1)}, B_{(i,2)}, P_3 B_{(i,3)}] \right\|_F^2 \tag{53}
\]

where \( \Theta = \{ \hat{Q}_i, B_{(i,\ell)} : \ell \in \{Z,\Psi\}, i \in \{1,2,3\} \} \).

A. Optimization

In order to minimize the cost function in (53), we consider a block coordinate descent strategy (as in, e.g., [37]), which
successively minimizes $J$ with respect to each of the variables in $\Theta$ while keeping the remaining ones fixed. Let us denote by $J(X|\Theta(X))$ the cost function $J$ in which all variables but $X$ are fixed.

Note that we apply the QR factorization after computing each of the factor matrices $B_{i,j}$ to constrain them to be unitary at all the iterations, as performed in [37]. This normalization prevents convergence issues by avoiding under/over-flow and keeping these matrices well-conditioned.

The optimization procedure is detailed in Algorithm 2, which we call CB-STAR (Coupled Block term decompositions for hyperspectral Super-resolution with vARIability).

1) Optimizing w.r.t. $B_{(z,i)}$, $i \in \{1,2,3\}$: To save space, we present only the case where $i = 1$. The extension to $i \in \{2,3\}$ is straightforward. Note that the cost function $J(B_{(z,1)}|\Theta(B_{(z,1)}))$ can be equivalently reformulated using the mode-1 matricization as

$$J(B_{(z,1)}|\Theta(B_{(z,1)})) = \|Y_{m}^{(1)} - X_{2}B_{(z,1)}P_{1}^{T}\|_{F}^{2},$$

where matrices $X_{1}, X_{2}$ and $X_{3}$ are given by

$$X_{1} = (B_{(z,3)} \otimes P_{3}B_{(z,2)})G_{Z,1},$$

$$X_{2} = (P_{2}B_{(z,3)} \otimes B_{(z,2)})G_{Z,1},$$

$$X_{3} = (P_{3}B_{(z,3)} \otimes B_{(z,2)})G_{Z,1}. $$

Computing the derivative of (54) and setting it equal to zero results in the following expression:

$$X_{1}^{T}X_{1}B_{(z,1)}^{T} + X_{2}^{T}X_{2}B_{(z,1)}^{T}P_{1} = X_{1}^{T}(Y_{m}^{(1)} - X_{3}B_{(z,1)}P_{1}).$$

This is a Sylvester equation that can be directly solved using existing software with, e.g., the Hessenberg-Schur or the Bartels-Stewart algorithms (see [38] and references therein).

2) Optimizing w.r.t. $G_{Z}$: The cost function $J(G_{Z}|\Theta(G_{Z}))$ can be equivalently reformulated using the tensor vectorization as

$$J(G_{Z}|\Theta(G_{Z})) = \|\text{vec}(Y_{h}) - X_{2}g_{Z}\|_{F}^{2},$$

where $g_{Z} = \text{vec}(G_{Z})$ is the vectorization of the core tensor, and $X_{1}, X_{2}$ and $X_{3}$ are given by

$$X_{1} = (P_{3}B_{(z,3)} \otimes B_{(z,2)} \otimes B_{(z,1)}),$$

$$X_{2} = (B_{(z,3)} \otimes P_{2}B_{(z,2)} \otimes P_{1}B_{(z,1)}),$$

$$X_{3} = (P_{3}B_{(z,3)} \otimes B_{(z,2)} \otimes P_{1}B_{(z,1)})\text{vec}(G_{Z}).$$

The solution that minimizes (59) can be computed through the normal equations, which can be written as

$$(X_{1}^{T}X_{1} + X_{2}^{T}X_{2})g_{Z} = X_{1}^{T}(\text{vec}(Y_{m}) - x_{3}) + X_{2}^{T}\text{vec}(Y_{h}).$$

As shown in [17], this set of equations can be alternatively interpreted as a generalized Sylvester equation, for which efficient solvers can be used.

3) Optimizing w.r.t. $\Psi_{q}$ and $B_{(\Psi,q)}$, $i \in \{1,2,3\}$: This optimization problem can be written equivalently as

$$\min_{\Psi_{q},B_{(\Psi,q)}}\|X_{1} - \left[\mathcal{G}_{q} : B_{(\Psi,1)}, B_{(\Psi,2)}, B_{(\Psi,3)}\right]\|_{2}^{2},$$

where $X_{1} = \mathcal{Y}_{m} - \left[\mathcal{G}_{q} : B_{(\Psi,1)}, B_{(\Psi,2)}, B_{(\Psi,3)}\right]$ and $X_{2} = P_{3}B_{(\Psi,3)}$. This problem can be solved by computing the high-order SVD of $X_{1}$ with rank $(K_{\Psi,1}, K_{\Psi,2}, K_{\Psi,3})$ [39]. Note that problem (64) only returns $X_{2} = P_{3}B_{(\Psi,3)}$ instead of $B_{(\Psi,3)}$. This is not a problem since the variations of $B_{(\Psi,3)}$ in the nullspace of $P_{3}$ are not identifiable.

4) Initialization: Since this optimization problem is non-convex, the choice of initialization can have a significant impact on the performance of the algorithm. This can be particularly prominent in this algorithm since the model considered in (16)–(17) allows for a significant amount of ambiguity. Fortunately, for practical scenes, we can consider a simple strategy to provide a reasonably accurate initial guess.

In the noiseless case, the following relation is satisfied:

$$\mathcal{P}_{1,2}(\mathcal{Y}_{m}) = \mathcal{P}_{1,2}(\mathcal{P}_{3}(\mathcal{Z} + \Psi))$$

$$= \mathcal{P}_{3}(\mathcal{Y}_{h}) + \mathcal{P}_{1,2}(\mathcal{P}_{3}(\Psi)).$$

Thus, we can obtain a spatially and spectrally degraded version of $\Psi$ directly from the HSI and MSI simply as:

$$\tilde{\Psi} = \mathcal{P}_{1,2}(\mathcal{Y}_{m}) - \mathcal{P}_{3}(\mathcal{Y}_{h})$$

$$= \mathcal{P}_{1,2}(\mathcal{P}_{3}(\Psi)).$$

Then, if $\Psi$ is smooth, we can spatially upscale $\tilde{\Psi}$ using some form of interpolation (e.g., bicubic), leading to $\mathcal{P}_{3}(\Psi(0))$. Finally, we can use the Tucker decomposition of $\mathcal{P}_{3}(\Psi(0))$ to initialize $G_{q}, B_{(\Psi,i)}, i \in \{1,2,3\}$. We call this the interpolation initialization.

Another option is to try to invert (66) using the pseudoinverse of operator $\mathcal{P}_{1,2}$, which can be computed using properties of the tensor vectorization and Kronecker product:

$$(\mathcal{P}_{1,2})^{\dagger} = (\cdot)^{T} P_{1}^{T} \times_{2} P_{2}^{T},$$

where $X^{\dagger}$ denotes the pseudoinverse of $X$. The initialization can be then computed as $\mathcal{P}_{3}(\Psi(0)) = \tilde{\Psi} \times_{1} P_{1}^{T} \times_{2} P_{2}^{T}$. We call this the pseudoinverse initialization.

The initialization of $B_{(z,i)}, i \in \{1,2,3\}$ can then be performed as $B_{(z,i)} = \text{tSVD}_{K_{X,i}}(\mathcal{X}_{i}^{(0)})$ and $B_{(z,i)} = \text{tSVD}_{K_{X,i}}(\mathcal{X}_{i}^{(0)})$ for $i \in \{1,2\}$, where $X' = \mathcal{Y}_{m} - \mathcal{P}_{3}(\Psi(0))$. 


B. Exact Recovery

Suppose that $K_{Z,i} = K_{\Psi,i}$, $i \in \{1, 2, 3\}$, without loss of generality, so that the MSI follows a standard BTD as considered in [33]. Then we have the following result regarding the identifiability of the proposed algorithm.

**Theorem 4.** Suppose that $K_i \equiv K_{Z,i} = K_{\Psi,i}$, $i \in \{1, 2, 3\}$, that the observations are noise free (i.e., $E_i = 0$, $E_m = 0$), that $\{G_i, B_{(i,i)} : i \in \{Z, \Psi\}, i \in \{1, 2, 3\}\}$ are drawn from some joint absolutely continuous distribution, and that the following conditions on the dimensions hold:

\[
\begin{align*}
M_1 &\geq 2K_1 \text{ and } M_2 \geq 2K_2 \quad (68) \\
K_{Z,3} &\leq \min\{N_1N_2, K_{Z,1}K_{Z,2}\} \quad (69)
\end{align*}
\]

and either one of the following:

\[
\begin{align*}
\left\{ \begin{array}{ll}
K_3 > K_1 + K_2 - 2, \quad \text{or} \\
|K_1 - K_2| > K_3 - 2, \\
\text{and } L_m \geq 2K_3
\end{array} \right.
\end{align*}
\]

or

\[
K_1 = K_2, \quad K_3 \geq 3 \quad \text{and} \quad K_3 < L_m, \quad (71)
\]

is satisfied. Then, the solution to optimization problem (53) satisfies $\hat{Z} = Z$ almost surely.

**Proof.** Since there is no additive noise, the optimal solution to optimization problem (53) will necessarily make both terms in the cost function equal to zero. This implies that

\[
\begin{align*}
\mathcal{Y}_h &= \left[ \hat{G}_Z; \mathcal{P}_2 B_{(Z,1)}, \mathcal{P}_2 B_{(Z,2)}, \mathcal{P}_3 B_{(Z,3)} \right], \\
\mathcal{Y}_m &= \sum_{i \in \{Z, \Psi\}} \left[ \hat{G}_i; \mathcal{B}_{(i,1)}, \mathcal{B}_{(i,2)}, \mathcal{P}_3 \mathcal{B}_{(i,3)} \right],
\end{align*}
\]

where $\{G_i, \mathcal{B}_{(i,i)} : i \in \{Z, \Psi\}, i \in \{1, 2, 3\}\}$ denotes a solution to (53).

Since the dimension conditions (68) and either (70) or (71) are satisfied and the core tensor and factor matrices are drawn from joint absolutely continuous distributions, the BTD decomposition of the MSI in (73) is essentially unique according to Theorems 5.1 and 5.5 in [33]. This means that the following conditions are satisfied:

\[
\begin{align*}
\mathcal{B}_{(Z,i)} &= B_{(i,i)} Q_{i1,i}, \quad i \in \{1, 2\} \quad (74) \\
\mathcal{B}_{(\Psi,i)} &= B_{(i,i)} Q_{i2,i}, \quad i \in \{1, 2\} \quad (75) \\
\mathcal{P}_3 \mathcal{B}_{(Z,3)} &= \mathcal{P}_3 B_{(1,3)} Q_{i1,3} \quad (76) \\
\mathcal{P}_3 \mathcal{B}_{(\Psi,3)} &= \mathcal{P}_3 B_{(1,3)} G_{i2,3} \quad (77) \\
\hat{G}_Z &= G_{i1} x_1 Q_{i1,1}^{-1} x_2 Q_{i1,2}^{-1} x_3 Q_{i1,3}^{-1} \quad (78) \\
\hat{G}_\Psi &= G_{i2} x_1 Q_{i2,1}^{-1} x_2 Q_{i2,2}^{-1} x_3 Q_{i2,3}^{-1} \quad (79)
\end{align*}
\]

where the indexes $i_1$ and $i_2$ represent the possible permutations of the BTD terms, and thus can be either $(i_1, i_2) = (Z, \Psi)$ or $(i_1, i_2) = (\Psi, Z)$, and $Q_{Z,i}$, $Q_{\Psi,i}$, $i \in \{1, 2, 3\}$ are invertible matrices of appropriate size, which account for other ambiguities of the model.

Let us consider the degraded mode-3 unfolding of the HSI:

\[
\begin{align*}
P_3 \mathcal{Y}_h(3) &= P_3 \mathcal{B}_{(Z,3)} \hat{G}_Z(3) \left( P_2 \mathcal{B}_{(Z,2)} \otimes P_1 \mathcal{B}_{(Z,1)} \right)^T \\
&= P_3 \mathcal{B}_{(Z,3)} \hat{G}_Z(3) \left( P_2 B_{(Z,2)} \otimes P_1 B_{(Z,1)} \right)^T,
\end{align*}
\]

where $\hat{X}$ and $X$ are generically full row rank by the assumption (69) on the ranks and on the dimensions. Therefore, we have

\[
\text{span} \left( P_3 \mathcal{B}_{(Z,3)} \right) = \text{span} \left( P_3 B_{(Z,3)} \right). \quad (80)
\]

However, since $K_3 \leq L_m$ and due to the distributional assumptions on the factor matrices, we have that, generically, matrices $P_3 B_{(Z,3)}$ and $P_3 \mathcal{B}_{(Z,3)}$ both have rank $K_1$, and matrix $\left[ P_3 B_{(Z,3)} \mathcal{P}_3 B_{(\Psi,3)} \right]$ is full column rank (i.e., it has rank greater than $K_3$). Therefore, the subspaces spanned by $P_3 B_{(Z,3)}$ and $P_3 \mathcal{B}_{(Z,3)}$ are different, and it is not possible to have $P_3 B_{(Z,3)} = P_3 \mathcal{B}_{(\Psi,3)} S$, for any matrix $S$. Thus, (80) implies that the equation $\hat{G}_Z(3) = P_3 \mathcal{B}_{(Z,3)} Q_{\Psi,3}$ is not possible (i.e., it cannot be satisfied for any matrix $Q_{\Psi,3}$). This ensures, due to (76) (and to the essential uniqueness of the MSI BTD), that we must have $i_1 = Z$ in (74), (76) and (78), (i.e., the factor matrices related to $\Psi$ can not fit the HSI), which shows that the correct permutation of the BTD terms is selected.

Finally, since $\hat{X}$ is generically full row rank, (or since $\mathcal{Y}_h(3)$ has rank $K_3$), we have that $\hat{B}_{(Z,3)} = B_{(Z,3)} S$ for some $S$, and (76) ensures that $S = Q_{Z,3}$. This means that $\hat{B}_{(Z,3)} = B_{(Z,3)} Q_{Z,3}$ (almost surely), and, using (74) and (78) and $i_1 = Z$, the reconstructed image consequently satisfies

\[
\hat{Z} = \left[ \hat{G}_Z; \hat{B}_{(Z,1)}, \hat{B}_{(Z,2)}, \hat{B}_{(Z,3)} \right] = \left[ G_Z; B_{(Z,1)}, B_{(Z,2)}, B_{(Z,3)} \right] = Z, \quad (81)
\]

(almost surely), which concludes the proof.

By taking the block diagonal structure of $C_m$ into account, Theorem 4 obtains generally less restrictive recovery conditions. Comparing theorems 3 and 4, we can see that: 1) conditions (40) and (69) are equivalent; 2) the conditions for the spectral ranks are not directly comparable but are similarly restrictive for both theorems; and, most notably, 3) The constraint (41) on spatial ranks is much more restrictive when compared to the one in (68), required by Theorem 4.

VI. EXPERIMENTS

In this section, the performance of the proposed approach is illustrated through numerical experiments considering both synthetic and real data containing spatial and spectral variability. All simulations were coded in MATLAB and run on a desktop with a 4.2 GHz Intel Core i7 and 16GB RAM.

A. Experimental Setup

We compared CT-STAR and CB-STAR to both matrix and tensor factorization-based algorithms. Among the matrix factorization-based methods, we considered the HySure [10] and CNMF [11] methods, the FuVar [28] method, which accounts for spectral variability, and the multiresolution
analysis-based GLP-HS algorithm [40]. We also considered the STEREO [16] and SCOTT [17] algorithms, which are tensor factorization methods based on optimization and on algebraic methods, respectively.

The real HSI and MSI, which were acquired at different time instants but at the same spatial resolution, were pre-processed as described in [10]. It consists in the manual removal of water absorption and low SNR bands, followed by the normalization of all bands of the HSI and MSI such that the 0.999 intensity quantile corresponds to a value of 1. Afterwards, the HSI was denoised (as described in [41]) to yield the high-SNR reference image $Z$ [2]. The observed HSIs $Y_{ij}$ were then generated from $Z$ by applying a separable degradation operator, with $P_1 = P_2$ (a Gaussian filter with unity variance followed by a subsampling with a decimation factor of two\(^1\)). A Gaussian noise was also added to obtain an SNR of 30dB. The observed MSIs $Y_{m}$ were generated by adding noise to the reference MSI to obtain an SNR of 40dB. The spectral response function $P_3$ was obtained from calibration measurements and known a priori\(^2\).

The parameters of the algorithms were selected as follows. We selected the ranks and regularization parameters for HySure, CNMF and FuVar according to the original works [10], [11], [28]. For STEREO, we selected the rank in the interval $[5, 80]$ which led to the best reconstruction results. Similarly, for SCOTT and for the proposed algorithms, we selected the spatial ranks in the intervals $[10, 80]$ and the spectral ranks in the interval $[2, 30]$, which led to the best reconstruction results. The spatial and spectral degradation operators (or, equivalently, the blurring kernels for HySure, CNMF and FuVar) were assumed to be known a priori for all methods. The BCD procedure in Algorithm 2 was performed until the relative change in the objective function value was smaller than $10^{-3}$. Both the interpolation and the pseudoinverse initializations described in Section V-A4 were considered, but only the first one (which performed better) is shown in the visual results.

To evaluate the quality of the reconstructed images $\hat{Z}$, we considered four quantitative metrics, which were previously used in [2], [10], [28]. The first metric is the peak signal to noise ratio (PSNR), defined as

$$PSNR(Z, \hat{Z}) = \frac{1}{L} \sum_{\ell=1}^{L} 10 \log_{10} \left( \frac{M_1 M_2 E\{\max(Z_{\cdot \cdot \cdot \ell})\}}{\|Z_{\cdot \cdot \cdot \ell} - \hat{Z}_{\cdot \cdot \cdot \ell}\|^2_F} \right),$$

where $E\{\cdot\}$ denotes the expectation operator.

The second metric is the Spectral Angle Mapper (SAM):

$$SAM(Z, \hat{Z}) = \frac{1}{M_1 M_2} \sum_{m,n} \cos^{-1} \left( \frac{Z_{\cdot \cdot \cdot n,m}^\top \hat{Z}_{\cdot \cdot \cdot n,m}}{\|Z_{\cdot \cdot \cdot n,m}\|_2 \|\hat{Z}_{\cdot \cdot \cdot n,m}\|_2} \right).$$

The ERGAS [42] metric provides a global statistical measure of the quality of the fused data, and is defined as:

$$ERGAS(Z, \hat{Z}) = \frac{M_1 M_2}{N_1 N_2} \sqrt{\frac{1}{L_h} \sum_{\ell=1}^{L_h} \left( \frac{\|Z_{\cdot \cdot \cdot \ell} - \hat{Z}_{\cdot \cdot \cdot \ell}\|^2_F}{(1/\langle M_1 M_2 \rangle)^2} \right)}.$$

The last metric is the average of the bandwise UIQI [43], which evaluates image distortions caused by loss of correlation and by luminance and contrast distortion, with value approaching one as $\hat{Z}$ approaches $Z$.

We also evaluate the reconstructed images visually, by displaying true- and pseudo-color representations of the visual and infrared spectra of $\hat{Z}$ (corresponding to the wavelengths 0.45, 0.56 and 0.66 \(\mu m\), and 0.80, 1.50 and 2.20 \(\mu m\), respectively). Due to space limitations, we only display the results of FuVar, STEREO, SCOTT, CT-STAR and CB-STAR, since these are the methods which performed best, and the ones which were conceptually closest to our approach. The spectrally degraded additive factors $\mathcal{P}_3(\hat{\Psi})$ estimated by CT-STAR and CB-STAR are also evaluated visually, through pseudo-color representations of its visible and infrared spectra, and by the norm (over all bands) of each of its pixels.

### Table I

| Algorithm | SAM | ERGAS | PSNR | UIQI | time |
|-----------|-----|-------|------|------|------|
| HySure    | 0.67| 21.32 | 9.3  | 0.65 | 5.94 |
| CNMF      | 0.97| 7.59  | 22.84| 0.81 | 16.4 |
| GLPHS     | 0.82| 4.26  | 27.81| 0.94 | 7.19 |
| FuVar     | 0.98| 6.7   | 23.86| 0.89 | 98.4 |
| STEREO    | 2.65| 11.5  | 18.48| 0.78 | 0.59 |
| SCOTT     | 0.6 | 7.33  | 21.89| 0.94 | 0.05 |
| CT-STAR, rk:20, 10, 5, & 5, 3 | 0.56 | 0.65 | 44.69 | 1 | 0.09 |
| CT-STAR, rk:20, 20, 7, & 7, 3, 3 | 0.54 | 0.62 | 45.05 | 1 | 0.12 |
| CT-STAR, rk:20, 20, 7, & 7, 3, 3 | 1.12 | 1.22 | 39.5 | 1 | 0.24 |
| CB-STAR, rk:20, 5, 5, & 3, 3, 2 | 0.58 | 0.64 | 44.98 | 1 | 0.29 |
| CB-STAR, rk:10, 10, 5, & 5, 3, 3 | 0.51 | 0.57 | 45.99 | 1 | 0.44 |
| CB-STAR, rk:10, 10, 7, & 7, 3, 3 | 1.56 | 1.65 | 37.31 | 0.99 | 1.12 |

### B. Example – Synthetic data

To evaluate the algorithms in a controlled scenario, we first considered a simulation with synthetic data. The tensors $Z$ and $\Psi$, of dimensions $100 \times 100 \times 200$, were generated following the Tucker model, with uniformly distributed entries on the interval $[0, 1]$ and ranks $[10, 10, 5]$ and $[5, 5, 3]$, respectively. The spectral response function $P_3 \in \mathbb{R}^{100 \times 200}$ was constructed by uniformly averaging groups of 20 bands, and the rest of the simulation setup was the same as described in Section VI-A. For this experiment, we initialized CB-STAR with the results of CT-STAR. The ranks of STEREO and SCOTT were 50 and (60, 60, 5), respectively, and the proposed methods were run with three different ranks, indicated in Table I.

The results in Table I show that the proposed methods yielded significant improvements when compared to the other algorithms, which is expected since this dataset was generated according to the model (16)–(17). Moreover, the performance of both CT-STAR and CB-STAR as a function of the ranks was similar, with the best results when the rank was the same as the ground truth, but with similar performance when the rank was underestimated. When the selected rank was overestimated, the performance of the proposed methods degraded more sharply (with a more prominent decrease for CB-STAR), indicating that the ranks should not be much greater than the true values in order to obtain the best performance.

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1Details on how to construct $P_1$ and $P_2$ can be found in [17].

2Available for download [here](#).
Figure 1. Hyperspectral and multispectral images with a small acquisition time difference used in the experiments.

Figure 2. Hyperspectral and multispectral images with a large acquisition time difference used in the experiments.

C. Example – Real data

In this example, we evaluated the algorithms using real HS and MS images acquired at different time instants, thus presenting different acquisition and seasonal conditions. The reference hyperspectral and multispectral images, with a pixel size of 20 m, acquired by the AVIRIS and by the Sentinel-2A instruments, respectively, were originally considered in [28]. Four sets of image pairs were available. Two of which contained images acquired less than three months apart (thus containing moderate variability). The other two contained images acquired with a time difference of more than one year (thus containing more significant variability). The HSI and MSI contained \( L_h = 173 \) and \( L_m = 10 \) bands, respectively. The selected ranks for the tensor-based methods are shown in Table II.

| Algorithm | SAM | ERGAS | PSNR | UIQI | time |
|-----------|-----|-------|------|------|------|
| HySure    | 3.38| 7.79  | 23.65| 0.88 | 4.63 |
| CNMF      | 2.57| 5.64  | 27.6 | 0.89 | 8.83 |
| GLPHS     | 2.57| 5.32  | 28.39| 0.91 | 4.74 |
| FuVar     | 2.37| \textbf{4.29} | 30.59| 0.95 | 218  |
| STEREO    | 3.49| 5.51  | 28.72| 0.93 | 1.14 |
| SCOTT     | 2.52| 4.91  | 29.93| 0.95 | \textbf{0.18} |
| CT-STAR   | 2.96| 5.25  | 28.36| 0.92 | 1.82 |
| CB-STAR, init=interp | 2.19| 4.35  | \textbf{31.47} | \textbf{0.96} | 18.8 |
| CB-STAR, init=pseudoinv | 2.22| 4.34  | 31.41| \textbf{0.96} | 17.98 |

Table II

RANKS OF THE TENSOR-BASED ALGORITHMS USED IN THE EXPERIMENTS

| Algorithm | SAM | ERGAS | PSNR | UIQI | time |
|-----------|-----|-------|------|------|------|
| HySure    | 2.41| 8.97  | 21.19| 0.73 | 3.52 |
| CNMF      | 1.85| 6.02  | 26.36| 0.84 | 6.72 |
| GLPHS     | 2   | 4.69  | 29.33| 0.92 | 3.83 |
| FuVar     | 1.73| \textbf{3.65} | 32.28| 0.97 | 197.95 |
| STEREO    | 3.22| 4.85  | 30.86| 0.95 | 1.02 |
| SCOTT     | 2.36| 4.68  | 30.76| 0.96 | \textbf{0.16} |
| CT-STAR   | 1.83| 4.47  | 30.56| 0.94 | 1.83 |
| CB-STAR, init=interp | \textbf{1.38}| 3.72 | \textbf{34.42} | \textbf{0.98} | 3.25 |
| CB-STAR, init=pseudoinv | 1.51| 3.9   | 33.86| 0.97 | 3.25 |

1) Moderate variability: The first pair of images considered in this example contained \( 80 \times 80 \) pixels and were acquired over the region surrounding Lake Isabella, on 2018-06-27 and on 2018-08-27. The second pair of images contained \( 80 \times 100 \) pixels and was acquired near Lockwood, on 2018-08-20 and on 2018-10-19. A true color representation of the HSI and MSI for this example can be seen in Fig. 1. Due to the relatively small difference between the acquisition dates of both images, the HSI and MSI look similar. However, there are slight differences between them, as seen in the overall color hue of the images and in the upper right part of the Lake Isabella HSI. The quantitative performance metrics of all algorithms are shown in Tables III and IV, while the reconstructed images are presented in Figs. 3 and 4.

2) Significant variability: The remaining image pairs used in this example were acquired over the Ivanpah Playa and over Lake Tahoe area. The Ivanpah Playa image pair contained \( 80 \times 128 \) pixels and was acquired on 2015-10-04 and on 2017-12-17. For the Lake Tahoe region, we considered two different image pairs ("A" and "B"), both with \( 100 \times 80 \) pixels, the first one acquired on 2014-10-04 and on 2017-10-24, and the second...
one acquired on 2014-09-19 and on 2017-10-24. A true color representation of the HSI and MSI for this example can be seen in Fig. 2. Due to the considerable difference between the acquisition date/time of the HSI and MSI, significant differences can be found between them. For the Ivanpah Playa images, there are large variations between the sand colors in the central part of the image. For the Lake Tahoe region, significant differences are observed in both image pairs, with differences in the color hues of the ground and of the crop circles for the image pair A, and also a large change in the water level of the lake in the image pair B. The quantitative performance metrics of all algorithms are shown in Tables V, VI, and VII, while the reconstructed images are presented in Figs. 5, 6 and 7.

Table V

| Algorithm | SAM   | ERGAS | PSNR | UIQI | time |
|-----------|-------|-------|------|------|------|
| HySure    | 1.78  | 4.53  | 23.35| 0.57 | 6.19 |
| CNMF      | 1.24  | 3.22  | 26.65| 0.78 | 16.36|
| GLPHS     | 1.59  | 3.17  | 26.84| 0.82 | 5.97 |
| FuVar     | 1.06  | 2.04  | 30.6 | 0.96 | 254.97|
| STEREO    | 28.17 | 9.840 | 20.43| 0.61 | 0.74 |
| SCOTT     | 35.74 | 38.25 | 11.4 | 0.44 | 0.21 |
| CT-STAR   | 1.49  | 3.44  | 26.09| 0.71 | 0.18 |
| CB-STAR, init=interp | 1.22 | 1.84 | 31.56 | 0.95 | 71.47|
| CB-STAR, init=pseudoinv | 1.51 | 2.14 | 30.3 | 0.92 | 48.94|

Table VI

| Algorithm | SAM   | ERGAS | PSNR | UIQI | time |
|-----------|-------|-------|------|------|------|
| HySure    | 11.3  | 13.99 | 17.37| 0.71 | 4.5  |
| CNMF      | 8.79  | 14.59 | 18.37| 0.71 | 12.1 |
| GLPHS     | 5.65  | 7.45  | 24.08| 0.91 | 4.65 |
| FuVar     | 3.91  | 4.73  | 27.98| 0.97 | 270.91|
| STEREO    | 27.07 | 1.540 | 20.19| 0.68 | 0.92 |
| SCOTT     | 33.17 | 43.100| 11.21| 0.39 | 1.47 |
| CT-STAR   | 5.41  | 5.25  | 27.25| 0.96 | 2.88 |
| CB-STAR, init=interp | 4.25 | 3.78 | 30.1 | 0.98 | 63.71|
| CB-STAR, init=pseudoinv | 4.7  | 3.94 | 29.67| 0.98 | 31.6 |

Table VII

| Algorithm | SAM   | ERGAS | PSNR | UIQI | time |
|-----------|-------|-------|------|------|------|
| HySure    | 7.17  | 19.08 | 13.62| 0.35 | 4.36 |
| CNMF      | 8.08  | 14.7  | 16.16| 0.42 | 12.42|
| GLPHS     | 3.61  | 5.58  | 24.57| 0.86 | 4.53 |
| FuVar     | 2.58  | 3.38  | 28.86| 0.96 | 342.39|
| STEREO    | 28.18 | 6.220 | 19.99| 0.63 | 0.75 |
| SCOTT     | 38.45 | 2.960 | 10.87| 0.31 | 1.42 |
| CT-STAR   | 3.07  | 4.3   | 26.82| 0.92 | 2.88 |
| CB-STAR, init=interp | 2.17 | 2.64 | 31.19 | 0.97 | 46.46|
| CB-STAR, init=pseudoinv | 2.34 | 2.73 | 30.74 | 0.96 | 30.08|

a) Discussion: The quantitative results show that CB-STAR achieved again the overall best results for this example, outperforming the remaining algorithms in most metrics, except in the SAM and UIQI for the Ivanpah Playa HSI and in the SAM of the Lake Tahoe A HSI. Moreover, there was a stronger gap between the performance of the methods that consider variability and the remaining algorithms. CT-STAR, although better than STEREO and SCOTT, performed significantly worse than CB-STAR due to its stringent constraints on the image ranks. The visual inspection of the results again indicates that CB-STAR provides reconstructions closest to the ground truth when compared to the remaining methods. Although FuVar also provided good results, the reconstructions by CB-STAR were closer to the ground truth, as can be observed in the color shades of the upper part of the Ivanpah Playa image and of the crop circles of the Lake Tahoe A image, and especially in the overall colors in the more uniform regions containing soil and water and vegetation in the Lake Tahoe B image. STEREO and SCOTT, on the other hand, produced significant artifacts in all reconstructed images. CT-STAR also produced significant artifacts, which as in the previous example are due to the stringent rank constraints. The estimated factors $P_n(\hat{\psi})$ were in close agreement with the variability seen in the scenes, notably in the sand region of the central part of the Ivanpah Playa image (which lies at the bottom of a hill), and in the regions near the lake in the Lake Tahoe A and B images, which undergo variations in the water level. Moreover, the overall amplitude of the variables was significantly larger than in the previous example, in which the differences between the images were more moderate. The computation times of all algorithms were similar to those observed in the previous example, except for that of CB-STAR, which was higher since it underwent a larger number of iterations for the data in this example. Nonetheless, the computation times of CB-STAR were still considerably smaller than those of FuVar.

VII. CONCLUSIONS

In this paper, we proposed a novel framework for multimodal (hyperspectral and multispectral) image fusion accounting for spatially and spectrally localized changes. We first studied the general identifiability of the considered model, which becomes more ambiguous due to the presence of changes. Then, assuming that the high resolution image and the variation factors admit a Tucker decomposition, two new algorithms were proposed – one being purely algebraic (which was computationally more efficient), and another based on an optimization procedure (which allowed for more relaxed specification of the multilinear ranks). Theoretical guarantees for the exact recovery of the high resolution image were provided for both algorithms. The proposed optimization-based algorithm achieved superior experimental performance in the presence of spectral and spatial variations between the images, while also exhibiting a smaller computational cost.

REFERENCES

[1] G. A. Shaw and H.-h. K. Burke, “Spectral imaging for remote sensing,” Lincoln laboratory journal, vol. 14, no. 1, pp. 3–28, 2003.
[2] N. Yokoya, C. Grohnfeldt, and J. Chanussot, “Hyperspectral and multispectral data fusion: A comparative review of the recent literature,” IEEE Geoscience and Remote Sensing Magazine, vol. 5, no. 2, pp. 29–56, 2017.
[3] W. J. Carper, T. M. Lillesand, and R. W. Kiefer, “The use of intensity-hue-saturation transformations for merging SPOT panchromatic and multispectral image data,” Photogrammetric Engineering and Remote Sensing, vol. 56, no. 4, pp. 459–467, 1990.
[4] J. Liu, “Smoothing filter-based intensity modulation: A spectral preserve image fusion technique for improving spatial details,” International Journal of Remote Sensing, vol. 21, no. 18, pp. 3461–3472, 2000.
Figure 3. Left: Visible (top) and infrared (bottom) representation of the true and estimated versions of the Lockwood HSI. Right: Spectrally degraded additive scaling factors $\beta_k(\theta)$ estimated by CT-STAR and CB-STAR.

Figure 4. Left: Visible (top) and infrared (bottom) representation of the true and estimated versions of the Isabella Lake HSI. Right: Spectrally degraded additive scaling factors $\beta_k(\theta)$ estimated by CT-STAR and CB-STAR.

Figure 5. Left: Visible (top) and infrared (bottom) representation of the true and estimated versions of the Ivanpah Playa HSI. Right: Spectrally degraded additive scaling factors $\beta_k(\theta)$ estimated by CT-STAR and CB-STAR.

[5] N. Keshava and J. F. Mustard, “Spectral unmixing,” IEEE Signal Processing Magazine, vol. 19, no. 1, pp. 44–57, 2002.

[6] R. A. Borsoi, T. Imbiriba, J. C. M. Bermudez, and C. Richard, “A fast multiscale spatial regularization for sparse hyperspectral unmixing,” IEEE Geoscience and Remote Sensing Letters, vol. 16, no. 4, pp. 598–602, April 2019.

[7] R. C. Hardie, M. T. Eismann, and G. L. Wilson, “MAP estimation for hyperspectral image resolution enhancement using an auxiliary sensor,” IEEE Transactions on Image Processing, vol. 13, no. 9, pp. 1174–1184, 2004.

[8] Q. Wei, J. Bioucas-Dias, N. Dobigeon, and J.-Y. Tourneret, “Hyperspectral and multispectral image fusion based on a sparse representation,” IEEE Transactions on Geoscience and Remote Sensing, vol. 53, no. 7, pp. 3658–3668, 2015.

[9] R. Kawakami, Y. Matsushita, J. Wright, M. Ben-Ezra, Y.-W. Tai, and K. Ikeuchi, “High-resolution hyperspectral imaging via matrix factorization,” in Computer Vision and Pattern Recognition (CVPR), 2011 IEEE Conference on. IEEE, 2011, pp. 2329–2336.

[10] M. Simões, J. Bioucas-Dias, L. B. Almeida, and J. Chanussot, “A convex formulation for hyperspectral image super-resolution via subspace-based regularization,” IEEE Transactions on Geoscience and Remote Sensing, vol. 53, no. 6, pp. 3373–3388, 2015.

[11] N. Yokoya, T. Yairi, and A. Iwasaki, “Coupled nonnegative matrix factorization unmixing for hyperspectral and multispectral data fusion,” IEEE Transactions on Geoscience and Remote Sensing, vol. 50, no. 2, pp. 528–537, 2012.

[12] T. Imbiriba, R. A. Borsoi, and J. C. M. Bermudez, “Low-rank tensor modeling for hyperspectral unmixing accounting for spectral variability,” IEEE Transactions on Geoscience and Remote Sensing (accepted), 2019.

[13] M. A. Veganzones, J. E. Cohen, R. C. Farias, J. Chanussot, and P. Comon, “Nonnegative tensor CP decomposition of hyperspectral data,” IEEE Transactions on Geoscience and Remote Sensing, vol. 54, no. 5, pp. 2577–2588, 2015.

[14] T. Imbiriba, R. A. Borsoi, and J. C. M. Bermudez, “A low-rank tensor regularization strategy for hyperspectral unmixing,” in Proc. IEEE Statistical Signal Processing Workshop (SSP), 2018, pp. 373–377.

[15] X. Liu, S. Bourrennane, and C. Fossati, “Denoising of hyperspectral images using the PARAFAC model and statistical performance analysis,” IEEE Transactions on Geoscience and Remote Sensing, vol. 50, no. 10, pp. 3717–3724, 2012.

[16] C. I. Kanatsoulis, X. Fu, N. D. Sidiropoulos, and W.-K. Ma, “Hyperspectral super-resolution: A coupled tensor factorization approach,” IEEE Transactions on Signal Processing, vol. 66, no. 24, pp. 6503–6517, 2018.

[17] C. Prévost, K. Usevich, P. Comon, and D. Brie, “Hyperspectral super-resolution with coupled tucker approximation: Recoverability and SVD-based algorithms,” IEEE Transactions on Signal Processing, vol. 68, pp. 931–946, 2020.

[18] G. Zhang, X. Fu, K. Huang, and J. Wang, “Hyperspectral super-resolution: A coupled nonnegative block-term tensor decomposition approach,” in Proc. 8th International Workshop on Computational Advances in Multi-Sensor Adaptive Processing (CAMSAP). Le Gosier, Guadeloupe: IEEE, 2019.

[19] C. I. Kanatsoulis, X. Fu, N. D. Sidiropoulos, and W.-K. Ma, “Hyperspectral super-resolution: Combining low rank tensor and matrix structure,” in 2018 25th IEEE International Conference on Image Processing (ICIP). IEEE, 2018, pp. 3318–3322.

[20] C. Prévost, K. Usevich, P. Comon, and D. Brie, “Coupled tensor low-rank multilinear approximation for hyperspectral super-resolution,” in Proc. IEEE International Conference on Acoustics, Speech and Signal Processing (ICASSP). Brighton, U.K.: IEEE, 2019, pp. 5536–5540.
A. Cichocki, D. Mandic, L. De Lathauwer, G. Zhou, Q. Zhao, C. Caiafa, and H. A. Phan, “Tensor decompositions for signal processing applications: From two-way to multiway component analysis,” IEEE Signal Processing Magazine, vol. 32, no. 2, pp. 145–163, 2015.

T. G. Kolda and B. W. Bader, “Tensor decompositions and applications,” SIAM review, vol. 51, no. 3, pp. 455–500, 2009.

N. D. Sidiropoulos, L. De Lathauwer, X. Fu, K. Huang, E. E. Papalexakis, and C. Faloutsos, “Tensor decomposition for signal processing and machine learning,” IEEE Transactions on Signal Processing, vol. 65, no. 13, pp. 3531–3582, 2017.

L. De Lathauwer, “Decompositions of a higher-order tensor in block terms—Part II: Definitions and uniqueness,” SIAM Journal on Matrix Analysis and Applications, vol. 30, no. 3, pp. 1033–1066, 2008.

B. Somers, G. P. Asner, L. Tits, and P. Coppin, “Endmember variability in spectral mixture analysis: A review,” Remote Sensing of Environment, vol. 115, no. 7, pp. 1603–1616, 2011.

A. Zare and K. C. Ho, “Endmember variability in hyperspectral analysis: Addressing spectral variability during spectral unmixing,” IEEE Signal Processing Magazine, vol. 31, pp. 95–104, January 2014.

Y. Li, K. Lee, and Y. Bresler, “Identifiability in bilinear inverse problems with applications to subspace or sparsity-constrained blind gain and phase calibration,” IEEE Transactions on Information Theory, vol. 63, no. 2, pp. 822–842, 2016.

L. De Lathauwer and D. Nion, “ Decompositions of a higher-order tensor in block terms—Part III: Alternating least squares algorithms,” SIAM journal on Matrix Analysis and Applications, vol. 30, no. 3, pp. 1067–1083, 2008.

V. Simoncini, “Computational methods for linear matrix equations,” SIAM Review, vol. 58, no. 3, pp. 377–441, 2016.

L. De Lathauwer, B. De Moor, and J. Vandewalle, “A multilinear singular value decomposition,” SIAM journal on Matrix Analysis and Applications, vol. 21, no. 4, pp. 1253–1278, 2000.

B. Aiazzi, L. Alparone, S. Baronti, A. Garzelli, and M. Selva, “MTF-tailored multiscale fusion of high-resolution MS and pan imagery,” Photogrammetric Engineering & Remote Sensing, vol. 72, no. 5, pp. 591–596, 2006.

R. E. Roger and J. F. Arnold, “Reliably estimating the noise in AVIRIS hyperspectral images,” International Journal of Remote Sensing, vol. 17, no. 10, pp. 1951–1962, 1996.

L. Wald, “Quality of high resolution synthesised images: Is there a simple criterion?” in Third conference Fusion of Earth data: merging point measurements, raster maps and remotely sensed images”, SEE/URISA, 2000, pp. 99–103.

Z. Wang and A. C. Bovik, “A universal image quality index,” IEEE signal processing letters, vol. 9, no. 3, pp. 81–84, 2002.