Intervals of Permutations with a Fixed Number of Descents are Shellable

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Abstract The set of all permutations, ordered by pattern containment, is a poset. We present an order isomorphism from the poset of permutations with a fixed number of descents to a certain poset of words with subword order. We use this bijection to show that intervals of permutations with a fixed number of descents are shellable and present a formula for the Möbius function of these intervals. We present an alternative proof for a result on the Möbius function of intervals $[1, \pi]$ where $\pi$ has exactly one descent. We also prove that if $\pi$ has exactly one descent and avoids 456123 and 356124 then the intervals $[1, \pi]$ have no non-trivial disconnected subintervals and conjecture that these intervals are shellable.

1 Introduction and Preliminaries

Let $\sigma$ and $\pi$ be permutations of positive integers. We define an occurrence of $\sigma$ as a pattern in $\pi$ to be a subsequence of $\pi$ with the same relative order of size as the letters in $\sigma$. For example, if $\sigma = 213$ and $\pi = 23514$ then there are two occurrences of $\sigma$ in $\pi$, as the subsequences 214 and 314. The set of all permutations forms a poset $\mathcal{P}$, with a partial ordering defined as $\sigma \leq \pi$ if $\sigma$ occurs as a pattern in $\pi$. An interval $[\sigma, \pi]$ in $\mathcal{P}$ is a subposet consisting of all permutations $z \in \mathcal{P}$ with $\sigma \leq z \leq \pi$. A chain in a poset $\mathcal{P}$ is a totally ordered subset $C = \{c_1 < \cdots < c_t\}$. For example, $21 < 2341 < 24513$ is a chain in $[1, 24513]$. A descent occurs at $i$ in a permutation $\pi = \pi_1 \cdots \pi_n$ if $\pi_i > \pi_{i+1}$. An example, 23154 has descents at 2 and 4. It is straightforward to show that if the permutations $\sigma$ and $\pi$ both have exactly $k$ descents then any permutation $\tau \in [\sigma, \pi]$ also has exactly $k$ descents. We denote the induced subposet of all permutations with exactly $k$ descents as $\mathcal{P}_k$. The Möbius function for a poset is defined recursively as follows: $\mu(a, b) = 0$ if $a \not\leq b$, $\mu(a, a) = 1$ for all $a$ and
\[
\mu(a,b) = -\sum_{a \leq z < b} \mu(a,z).
\]

Given an interval \( I = [\sigma, \pi] \) we construct a simplicial complex \( \Delta(I) \), called the order complex of \( I \), whose vertices are the elements of the interior of \( I \), that is, elements of \( (\sigma, \pi) := [\sigma, \pi] \setminus \{\sigma, \pi\} \), and whose faces are the chains of \((\sigma, \pi)\). When we attribute a topological property to an interval we mean the corresponding property of its order complex. We refer the reader to [14] for extensive background on the subject of order complexes.

A simplicial complex is pure if all its maximal faces, which are called facets, have the same dimension. The order complex of an interval of permutations is always pure. A pure simplicial complex \( \Delta \) is shellable if its facets can be arranged in linear order \( F_1, F_2, \ldots, F_t \) in such a way that the subcomplex \( \bigcup_{k=1}^{t-1} F_k \cap F_k \) is pure and \( (\dim \Delta - 1) \)-dimensional for all \( k = 2, \ldots, t \), where \( F \) is the subcomplex generated by \( F \). Again we refer the reader to [14] for extensive background on the subject of shellability.

Let \( \mathcal{A} \) be the poset of words on the alphabet of positive integers, with the partial order called subword order where if two words \( v \) and \( w = w_1 \ldots w_n \) belong to \( \mathcal{A} \) then \( v \leq w \) if there is a subsequence \( w_{i_1} \ldots w_{i_m} \) in \( w \) such that \( v = w_{i_1} \ldots w_{i_m} \). For example, \( 2132 \leq 212312 \) but \( 2132 \not\leq 21233 \). In [2] a formula is given for computing the M"obius function on intervals of \( \mathcal{A} \) in polynomial time and it is shown that all intervals in \( \mathcal{A} \) are shellable. In this paper we present an order isomorphism, that is, an order preserving bijection, between each interval in the permutation posets \( P_k \) and a corresponding interval in \( \mathcal{A} \). This allows us to easily compute the M"obius function of intervals from the posets \( P_k \) and to show they are shellable.

The Betti number \( \beta_k(X) \) of a simplicial complex \( X \) is the rank of the \( k \)-th homology group of \( X \) (for background on the homology of simplicial complexes we refer the reader to [14]). The Philip Hall Theorem and the Euler-Poincaré formula, which appear as Proposition 1.2.6 and Theorem 1.2.8 in [14], combined state:

\[
\mu(I) = \tilde{\chi}(\Delta(I)) = \sum_{i=-1}^{\dim \Delta(I)} (-1)^i \beta_i(\Delta(I)), \quad (1.1)
\]

where \( I \) is an interval and \( \tilde{\chi}(\Delta(I)) \) is the reduced Euler characteristic of the order complex of \( I \).

An important property of simplicial complexes is Cohen-Macaulayness, which has its origins in commutative algebra. A simplicial complex \( \Delta \) is said to be Cohen-Macaulay if \( H_i(\ell k_{\Delta} F) = 0 \) for all \( F \in \Delta \) and \( i < \dim \ell k_{\Delta} F \), where \( \ell k_{\Delta} F \) denotes the link of \( F \) and \( H_i \) denotes the \( i \)-th homology group. For a full explanation of this definition see [14, Section 4]. A shellable simplicial complex is Cohen-Macaulay as pointed out in [11]. We use this property to compute the homology of intervals from the posets \( P_k \) for any \( k \geq 0 \).
There is a generalised subword order, defined in [9], where we take a poset $P$ and let $P^*$ denote the poset of finite words whose letters are elements of $P$. If $u, w \in P^*$ then $u \leq_{P^*} w$ if there is a subword $w_{i_1} \ldots w_{i_\ell}$ having length $\ell = |u|$ such that $u_j \leq w_{i_j}$, for all $1 \leq j \leq \ell$. If $P$ is an antichain then generalised subword order is equal to subword order. In [9] a formula is presented for the Möbius function of words with generalised subword order when $P$ is a chain. It is also shown there is an order isomorphism between posets of these words and posets of layered permutations, that is permutations that can be expressed as a direct sum of decreasing permutations, such as $1 \oplus 21 \oplus 321 \oplus 1 \oplus 1 = 13265478$.

In [7] a formula is presented for the Möbius function of words with generalised subword order for any poset $P$, which covers both the words considered in the present paper and in [9]. In [1] it is shown that if an interval $I$ contains a non-trivial disconnected subinterval, that is, a disconnected subinterval of rank at least three, then $I$ is not shellable. The first major result on the topology of intervals from the poset $P$ is given in [6], where it is shown that if $P$ is a rooted forest then any interval $[u, v]$ in $P^*$ that does not contain a non-trivial disconnected subinterval is shellable. This result is then used to show that intervals of layered permutations that do not contain a non-trivial disconnected subinterval are shellable and we conjecture that the same applies to the more general class of separable permutations.

In Section 2 we present a bijection between $\mathcal{P}$ and a subposet of $\mathcal{A}$. We show that when we restrict this bijection to $\mathcal{P}_k$ it is an order isomorphism. This allows us to draw on many useful results that have been proven for subword order, such as the shellability of intervals, and apply these results to permutations. In Section 3 we use this order isomorphism to present a formula for the Möbius function of intervals from the posets $\mathcal{P}_k$. We use this formula to prove a conjecture made in [10] and to present an alternative, simpler proof of [10, Theorem 5] on the Möbius function of intervals $[1, \pi]$ where $\pi$ has one descent. In Section 4 we show that if $\pi$ has exactly one descent and avoids 456123 and 356124 then $[1, \pi]$ has no non-trivial disconnected subintervals and we conjecture that these intervals are shellable.

2 Bijection From Permutations to Words

In this section we present an order isomorphism between the poset $P_k$ of permutations with exactly $k$ descents and a subposet of $\mathcal{A}$. Let $\max(w)$ be the value of the largest letter in the word $w$. We now define the poset of words we consider:

**Definition 2.1** Let $\hat{\mathcal{A}}$ denote the poset of words with subword order on the alphabet of all positive integers with the additional conditions that for any $w \in \hat{\mathcal{A}}$:

AC1: There is at least one occurrence of each letter $i \in \{1, \ldots, \max(w)\}$.
AC2: The rightmost occurrence of each letter $i \in \{1, \ldots, \max(w) - 1\}$ is preceded by an occurrence of $i + 1$. 


Let \( \hat{A}_k \) denote the subposet of \( \hat{A} \) of words \( w \) where \( \max(w) = k \).

Note that the additional conditions in Definition 2.1 are very similar to the definition of a restricted growth function, which can be used to encode set partitions, see [3]. To see the similarity we use the definition of a restricted growth function which appears in Question 106 in [1]. A restricted growth function is a sequence of the positive integers \( 1, \ldots, k \) with each letter occurring at least once and the first occurrence of \( i \) appearing before the first occurrence of \( i+1 \), for all \( 1 \leq i \leq k-1 \). If we consider AC2 reworded as beginning at the right end of the word, and travelling left, we get: the first occurrence of \( i \) must appear before the last occurrence of \( i+1 \). The key difference is that AC2 requires at least one occurrence of \( i+1 \) after the first \( i \) whereas a restricted growth function requires that all occurrences of \( i+1 \) are after the first \( i \). As such it is easy to see that \( \hat{A}_k \) is a larger class than the class of restricted growth functions.

We know that the number of permutations of length \( n \) in \( P_k \) is the Eulerian number \( A(n,k) \), see [12]. We show that there is a bijection between \( \hat{A}_k \) and \( P_k-1 \), which implies the number of permutations of length \( n \) in \( \hat{A}_k \) is given by the Eulerian number \( A(n,k-1) \).

Before proceeding we define a term used for both permutations and words:

**Definition 2.2** Given two elements \( a \leq b \) of a poset of either words or permutations, an embedding of \( a \) in \( b \) is a sequence \( \eta \) of length \( |b| \) such that the nonzero positions in \( \eta \) are the positions of an occurrence of \( a \) in \( b \) and removal of all the zeroes from \( \eta \) results in \( a \).
Lemma 2.5

The map \( \text{id} \) implies there are \( k \) the sections of \( g \) \( \left( f \circ g \right) \) \( \subseteq \). Proof We prove this by showing that \( t + 1 \) because \( i \) is the letter at the location of the \( i \)-th descent will map to \( i \). All that remains to be shown is that \( w \) satisfies AC2 in Definition 2.7.

Let \( w_i = i \) be the rightmost occurrence of the letter \( i \). This implies the letter \( t = \pi_t \) in \( \pi \) is the rightmost letter that is preceded by exactly \( i \) descents, which implies a descent occurs directly after \( \pi_t \). This means the letter \( \pi_{j+1} \) is mapped to \( i+1 \). However as \( \pi_{j+1} \leq \pi_j \) then \( \pi_{j+1} \) is mapped to an earlier location in \( w \) than \( \pi_j \). Therefore \( w_i = i \) is preceded by an occurrence of \( i+1 \) and as this is true for any \( i \) this concludes the first case.

For the second case we know there are \( k \) descents because the largest letter in \( \{ i : w_i = t \} \) must have a greater value than the smallest letter in \( \{ i : w_i = t + 1 \} \), by AC2 in Definition 2.7. Therefore, there is a descent between each of the sections of \( g(w) \) built from the sets \( \{ i : w_i = t \} \), of which there are \( k + 1 \), which implies there are \( k \) descents.

Lemma 2.4

Let \( f \) and \( g \) be defined as above.

1. If \( \pi \in \mathcal{P}_k \) then \( f(\pi) \in \mathcal{A}_{k+1} \).
2. If \( w \in \mathcal{A}_{k+1} \) then \( g(w) \in \mathcal{P}_k \).

Proof For the first case consider \( \pi \in \mathcal{P}_k \) and let \( w = f(\pi) \). It is clear that \( w \) is a word and that \( d_f(\pi_\ell) = k + 1 \), also there must be an occurrence of all the letters \( 1, \ldots, k \) because for each \( i \) the letter at the location of the \( i \)-th descent will map to \( i \). All that remains to be shown is that \( w \) satisfies AC2 in Definition 2.7.

Let \( w_i = i \) be the rightmost occurrence of the letter \( i \). This implies the letter \( t = \pi_t \) in \( \pi \) is the rightmost letter that is preceded by exactly \( i \) descents, which implies a descent occurs directly after \( \pi_t \). This means the letter \( \pi_{j+1} \) is mapped to \( i+1 \). However as \( \pi_{j+1} \leq \pi_j \) then \( \pi_{j+1} \) is mapped to an earlier location in \( w \) than \( \pi_j \). Therefore \( w_i = i \) is preceded by an occurrence of \( i+1 \) and as this is true for any \( i \) this concludes the first case.

For the second case we know there are \( k \) descents because the largest letter in \( \{ i : w_i = t \} \) must have a greater value than the smallest letter in \( \{ i : w_i = t + 1 \} \), by AC2 in Definition 2.7. Therefore, there is a descent between each of the sections of \( g(w) \) built from the sets \( \{ i : w_i = t \} \), of which there are \( k + 1 \), which implies there are \( k \) descents.

Lemma 2.5

The map \( f \) is a bijection with inverse \( g \).

Proof We prove this by showing that \( f \circ g = \text{id}_\mathcal{A} \) and \( g \circ f = \text{id}_\mathcal{P} \).

First consider \( w \in \mathcal{A} \) and \( v = f(g(w)) \). If \( w_i = t \) then \( d_g(w)(i) = t \), because \( i \) would be put in the \( [ i : w_i = t ] \) part of \( g(w) \) and this means it would be preceded by \( t \) descents. So we know that \( v_i = d_g(w)(i) = t = w_i \) and as this true for any \( i \) it implies \( w = v \) for any \( w \in \mathcal{A} \).

Now consider \( g(f(\pi)) \) where \( \pi \in \mathcal{P}_k \) and let \( \pi_1 \ldots \pi_{\ell+\lambda} \) be the subpermutation of \( \pi \) where all the letters are preceded by exactly \( \ell \) descents for some \( 1 \leq \ell \leq k + 1 \). Each \( \pi_t \), where \( t \leq \ell \leq t + \lambda \), is mapped to the letter \( j \) in \( f(\pi) \) and these are the only letters to map to \( j \). In turn only those letters are mapped into the \( [ i : w_i = j ] \) part of \( f(f(\pi)) \). So the letters preceded by exactly \( j \) descents are mapped to the letters that are preceded by exactly \( j \) descents and both are listed in increasing order. As this is true for all \( j \) it implies \( g(f(\pi)) = \pi \) for any \( \pi \in \mathcal{P} \).

So \( f \) is a bijection between \( \mathcal{P} \) and \( \mathcal{A} \). Finally we need to see if this bijection is order-preserving. This is not true in the general case. For example, consider the permutations \( 132 \leq 2143 \): If we apply \( f \) we get \( f(132) = 121 \not\leq 2132 = f(2143) \).
However, if we consider $f_k := f|_{\mathcal{P}_k}$ then we can show $f_k$ is order preserving and we know by Lemma 2.4 that the image of $f_k$ is $\hat{\mathcal{A}}_{k+1}$ and that the image of $g_{k+1} := g|_{\hat{\mathcal{A}}_{k+1}}$ is $\mathcal{P}_k$, so by this and Lemma 2.5 we know $f_k$ is a bijection.

**Theorem 2.6** The bijection $f_k$ is an order isomorphism.

**Proof** Consider two permutations $\sigma, \pi \in \mathcal{P}_k$ with $\sigma \leq \pi$. As $\sigma$ and $\pi$ have the same number of descents this implies that for any occurrence of $\sigma$ in $\pi$ the part of $\pi$ that is preceded by exactly $t$ descents must occur in the part of $\sigma$ that is preceded by exactly $t$ descents. So let $\pi_{k_1} \ldots \pi_{k_m}$ be a subsequence of $\pi$ that is an occurrence of $\sigma$. Then $d_\sigma(\pi_{k_1}) \ldots d_\sigma(\pi_{k_m}) = d_\pi(1) \ldots d_\pi(m)$ and we let $\pi_{k_1} \ldots \pi_{k_m}$ be the reordering of $\pi_{k_1} \ldots \pi_{k_m}$ in increasing order. Then $d_\pi(1) \ldots d_\pi(m)$ occurs in $f_k(\pi)$ and is equal to $f_k(\sigma)$, hence $f_k(\sigma) = f_k(\pi).

Now consider two words $v, w \in \hat{\mathcal{A}}_k$ with $v \leq w$. Let $\eta$ be an embedding of $v$ in $w$ and let $g_k(\eta) = [i : \eta_i = 1][i : \eta_i = 2] \ldots [i : \eta_i = k + 1]$. It is easy to see that $\{i : \eta_i = t\} \subseteq \{i : \eta_i = t\}$, which implies $g_k(\eta) \leq g_k(w)$ and $g_k(\eta)$ is an occurrence of $g_k(v)$, therefore $g_k(v) \leq g_k(w)$.

Hence we have an order isomorphism between $\mathcal{P}_k$ and $\hat{\mathcal{A}}_{k+1}$. The following important corollary follows directly from [2, Theorem 3] and Theorem 2.6.

**Corollary 2.7** Any interval $[\sigma, \pi]$, where $\sigma$ and $\pi$ are permutations with the same number of descents, is dual CL-shellable.

Note that (dual) CL-shellability implies (dual) shellability and that dual shellability implies shellability. For a good survey of the implications of different types of shellability we refer the reader to [14, Section 4.1].

We can also consider $f$ as a map to the poset of words on generalised subword order where the underlying poset is the chain of positive integers. In this case $f$ is order-preserving, but $g$ is not. For example, $211 \leq 212$ but $g(211) = 231 \nleq 213 = g(212)$.

We can use Corollary 2.7 to analyse the homotopy type of these intervals:

**Proposition 2.8** If $\sigma$ and $\pi$ are permutations with the same number of descents then $\Delta([\sigma, \pi])$ is homotopy equivalent to a wedge of $|\mu(\sigma, \pi)|$ spheres of dimension $n = \dim(\Delta([\sigma, \pi]))$.

**Proof** A shellable pure simplicial complex is homotopically equivalent to a wedge of top dimensional spheres, see [3, Theorem 12.3]. So all that remains is computing the number of these spheres, which is equivalent to $\beta_n(\Delta([\sigma, \pi]))$. These intervals are shellable hence they are Cohen-Macaulay which implies the Betti numbers $\beta_k(\Delta([\sigma, \pi]))$ are zero except when $k = n$. We can combine this with the Philip Hall Theorem and the Euler-Poincaré formula, that is, Equation (1.7), to get:

$$\beta_n(\Delta([\sigma, \pi])) = |\mu(\sigma, \pi)|.$$

$\square$
3 Computing the Möbius function

We can use Theorem 2.6 along with [2, Theorem 1], which also appears as [9, Theorem 2.1], to compute the Möbius function of any interval between permutations with the same number of descents. To do this we first need to define what a normal embedding is in the case of permutations. The definition we use is induced by the definition of a normal embedding in [9] after applying the bijection from Theorem 2.6:

Definition 3.1 An adjacency in a permutation is a sequence of consecutively valued letters in increasing consecutive order. The tail of an adjacency is all but the first letter of the adjacency. An embedding η of σ in π is normal if η_i is nonzero for each letter π_i in the tail of an adjacency. We use the notation from [2] and denote the number of normal embeddings of σ in π as \((π \circ σ)_n\).

Note there is an analogous decreasing adjacency but we are only interested in increasing adjacencies.

Example 3.2 As in Example 2.3 consider 213 and 142356. The adjacencies in 142356 are 23 and 56 so the tails of the adjacencies are 3 and 6. Hence the only normal embedding is 020103 and therefore \((142356 \circ 213)_n = 1\).

We use this definition to state the following result:

Proposition 3.3 If σ and π are permutations with the same number of descents, then

\[ \mu(σ, π) = (-1)^{|π| - |σ|} \binom{|π|}{|σ|}_n. \]

Proof This follows directly from Theorem 2.6 and [2, Theorem 1].

In [2] it is shown that \((π \circ σ)_n\) can be computed in polynomial time. In Section 3.1 we use Proposition 3.3 to give a simpler proof of a result which appears in [10] and prove a conjecture from the same paper, but first we derive from it two corollaries:

Corollary 3.4 Consider σ, π ∈ P_k. Let t be the total number of letters in all the tails of all the adjacencies in π. If t > |σ| then \(μ(σ, π) = 0\).

This result doesn’t hold if we remove the restriction on the number of descents. For example, consider σ = 213 and π = 569341278, which have one and two descents, respectively. The total number of letters in all the tails of 569341278 is t = 4 and |σ| = 3, but \(μ(312, 6745123) = 1 ≠ 0\).

Corollary 3.4 is another part of the answer to a question posed in [3] asking when is \(μ(σ, π) = 0\). Whilst we cannot yet give a simple definitive answer to this question (which may not exist) there are results which present several classes of intervals with a zero Möbius function, such as results in [3, 10] and [3, 13].

A result in [3] shows that if σ and π are separable permutations then \(|μ(σ, π)|\) is at most the number of occurrences of σ in π.
this is also the case if we fix the number of descents, since an embedding corresponds to a unique occurrence.

**Corollary 3.5** If \( \sigma \) and \( \pi \) have the same number of descents, then \( |\mu(\sigma, \pi)| \) is at most the number of occurrences of \( \sigma \) in \( \pi \).

### 3.1 Möbius Function of Permutations With at Most One Descent

Proposition 3.3 allows us to compute the Möbius function of an interval between two permutations with the same number of descents but says nothing about intervals between permutations with different number of descents. Now we consider the intervals \([1, \pi]\), where \( \pi \in P_1 \) and 1 denotes the permutation 1. In particular we present an alternative proof, which is both shorter and simpler than the original, of [10, Theorem 5]. We begin with a useful lemma which gives a formula for \( \mu(1, \pi) \) for every permutation \( \pi \) with one descent.

**Lemma 3.6** If \( \pi \) has exactly one descent then \( \mu(1, \pi) = -\mu(21, \pi) \).

Lemma 3.6 can be proved directly by considering the effect the removal of the increasing permutations has on the Möbius function. However, it also follows from Theorem 4.2 so we omit the proof here.

We now present the alternative proof of [10, Theorem 5]. As in [10], we use the notation \( \mu(\pi) := \mu(1, \pi) \). A triple adjacency indicates an adjacency of three letters, for example 234 in 52341. Unless otherwise stated an adjacency has two letters. The value and position of an adjacency are given by the value and position of the first letter of the adjacency. We denote the two permutations of length \( n \) without adjacencies as \( M_n = 246 \ldots 135 \) and \( W_n = 135 \ldots 246 \). For example, \( M_6 = 246135 \) and \( W_5 = 13524 \).

As pointed out in [10], in Theorem 3.7 any overlap of cases agree in value. For example, if \( \pi \) contains the triple adjacency 234 then equivalently \( \pi \) contains the two adjacencies 23 and 34 the first of which has lower value, both of which imply \( \mu(\pi) = 0 \).

**Theorem 3.7** Given a permutation \( \pi \) of length \( n > 2 \), with exactly one descent, the value of \( \mu(\pi) \) can be computed from the number and positions of adjacencies in \( \pi \), as follows:

1. If \( \pi \) begins with 12 or ends in \((n - 1)n\) then \( \mu(\pi) = 0 \).
2. If \( \pi \) has a triple adjacency then \( \mu(\pi) = 0 \).
3. If \( \pi \) has more than two adjacencies then \( \mu(\pi) = 0 \).
4. If \( \pi \) has exactly two adjacencies then:
   - (a) If the first adjacency has greater value than the second then \( \mu(\pi) = \pm 1 \),
   - (b) If the first adjacency has lower value than the second then \( \mu(\pi) = 0 \).
5. If \( \pi \) has exactly one adjacency, at position \( i \in \{1, \ldots, n-1\} \), and the descent is at position \( d \), then (see item 7 for calculating the sign)
   - (a) If \( i < d \) and \( \pi_1 \neq 1 \) then \( \mu(\pi) = \pm i \),
   - (b) If \( i < d \) and \( \pi_1 = 1 \) then \( \mu(\pi) = \pm (i - 1) \),
(c) If \( i > d \) and \( \pi_n \neq n \) then \( \mu(\pi) = \pm (n - i) \),
(d) If \( i > d \) and \( \pi_n = n \) then \( \mu(\pi) = \pm (n - i - 1) \).

6. If \( \pi \) has no adjacencies then:
(a) If \( n \) is even and \( \pi_1 = 1 \), so \( \pi = W_n \), then \( \mu(\pi) = -\left(\frac{n}{2}\right) \),
(b) If \( n \) is even and \( \pi_1 = 2 \), so \( \pi = M_n \), then \( \mu(\pi) = -\left(\frac{n+1}{2}\right) \),
(c) If \( n \) is odd then \( \mu(\pi) = \left(\frac{n+1}{2}\right) \).

7. If \( \mu(\pi) \neq 0 \) then \( \mu(\pi) \) is positive if and only if \( n \) is odd.

Proof By Lemma 3.6 we know that \( \mu(\pi) = -\mu(21, \pi) \). We can use Proposition 3.3 to compute \( \mu(21, \pi) \) which implies the sign of \( \mu(21, \pi) \) is given by \( (-1)^{|\pi|-2} \). Therefore \( \mu(21, \pi) \) is positive if and only if \( n \) is even, combining this with \( \mu(\pi) = -\mu(21, \pi) \) gives part 7.

We need to show that the absolute value of \( \mu(21, \pi) \), which equals the number of normal embeddings, agrees with each of the cases in the theorem. We refer to the permutation 21 as \( \sigma \) to avoid confusion between letters and permutations.

Case 3: If \( \pi \) begins with 12 we must embed the 2 of \( \sigma \) as the 2 in \( \pi \) and there is no letter after the descent of value less than 2 so we cannot embed the 1 of \( \sigma \) anywhere. Similarly if \( \pi \) ends in \( n-1 \) we must embed the 1 of \( \sigma \) as \( n \) in \( \pi \) and therefore we cannot embed the 2 of \( \sigma \) anywhere. So there are no normal embeddings of \( \sigma \) in \( \pi \).

Case 4: If \( \pi \) has a triple adjacency at \( \pi_i \pi_{i+1} \pi_{i+2} \) a normal embedding of \( \sigma \) in \( \pi \) must be non-zero for \( \pi_{i+1} \pi_{i+2} \) so \( \sigma \) must contain 12, which 21 does not. So there are no normal embeddings of \( \sigma \) in \( \pi \).

Case 5 follows directly from Corollary 3.4.

Case 6: When there are two adjacencies, at locations \( k \) and \( j \), there is only one embedding that might be normal, namely \( \eta = \ldots \eta_k 0 \ldots \eta_j 0 \ldots \). If \( \pi_k > \pi_j \), then we can set \( \eta_k = 2 \) and \( \eta_j = 1 \) so we can get one normal embedding of \( \sigma \) in \( \pi \). If \( \pi_k < \pi_j \), there is no way to make \( \eta \) an embedding of \( \sigma \) so there are no normal embeddings of \( \sigma \) in \( \pi \).

Case 7: In these cases we must embed one of the letters of 21 in the adjacency and can choose an appropriate place for the other letter. Denote the locations of the descent and adjacency as \( d \) and \( i \), respectively. If \( i < d \) then an embedding \( \eta \) of \( \sigma \) in \( \pi \) must have \( \eta_{i+1} = 2 \) and we can then embed the 1 from \( \sigma \) in any of the letters after the descent that have value less than \( \pi_i \). As the rest of \( \pi \) follows the same alternating pattern, since there are no more adjacencies, it is easy to see that this gives the desired results. The argument is analogous if \( i > d \).

Case 8: As there are no adjacencies in \( \pi \) any embedding is normal, so we need only count the number of embeddings. First consider the case when \( n \) is even and \( \pi_1 = 1 \). If we embed the letter 2 of \( \sigma \) in locations 1, 2, \ldots, \( \frac{n}{2} \) and then count where we can embed the letter 1 we get the following sequence
0, 1, 2, ..., $\frac{n}{2} - 1$ hence $\binom{n}{2n} = \binom{\frac{n}{2}}{2}$. Repeating this for each case gives the desired results. \qed

We can also use Proposition 3.3 to prove one of the conjectures presented in [10]. In Proposition 3.9 we count the number of size two adjacencies, so a triple adjacency counts as two adjacencies and a length $k$ adjacency counts as $k - 1$ adjacencies. We say that two permutations with exactly one descent are related if they have the letter 1 on the same side of the descent. Let $\lfloor x \rfloor$ denote the floor of $x$, that is, the largest integer not greater than $x$.

**Lemma 3.8** Let $\sigma$ be a permutation of length $m$ with exactly one descent and $i$ adjacencies. In $\sigma$ the letter $m$ occurs on the same side of the descent as the letter 1 if and only if $m - i$ is odd.

**Proof** If $\sigma$ begins with the letter 1 let $\tau = W_m$ otherwise let $\tau = M_m$. We can build $\sigma$ from $\tau$ by going through each letter $k = 2, \ldots, m$ in $\tau$ and if $k$ is not on the same side of the descent as $k$ is in $\sigma$ move it to the opposing side of the descent, in the unique way that will not create a new descent.

When we move a letter $k$ with $1 < k < n$ there are three possibilities: if $k$ is not part of an adjacency it will create two new adjacencies $(k - 1)k$ and $k(k + 1)$, if $k$ is part of one adjacency moving it will destroy one adjacency but create another and if $k$ is part of two adjacencies moving it will destroy both adjacencies. If $k = n$ then moving it will either create or destroy the adjacency $(n - 1)n$. Therefore, each move of a letter $k$ will change the number of adjacencies by $-2, 0$ or $2$ for all $1 < k < n$ and by $1$ or $-1$ if $k = n$.

If $m$ is odd then 1 and $m$ are on the same side of the descent in $\tau$. If $m$ is not moved whilst building $\sigma$ from $\tau$ then $m - i$ must be odd and $m$ must be on the same side of the descent as 1 in $\sigma$. If $m$ is moved it will be on the opposite side of the descent and $m - i$ is even. The argument is analogous if $m$ is even.

**Proposition 3.9** Given a permutation $\sigma \in \mathcal{P}_1$ of length $m$, let $i$ be the number of size two adjacencies in $\sigma$. If $\sigma \preceq \pi$ where $\pi \in \{M_m, W_m\}$, then:

$$\mu(\sigma, \pi) = (-1)^{n-m} \binom{\lfloor \frac{n+m-i-a}{2} \rfloor}{m}$$

where $a = \begin{cases} 0, & \text{if } \sigma \text{ and } \pi \text{ are related} \\ 1, & \text{otherwise} \end{cases}$.

**Proof** As both $\sigma$ and $\pi$ have exactly one descent we can apply Proposition 3.3. The sign part of the result follows immediately. Because $\pi$ has no adjacencies any embedding of $\sigma$ in $\pi$ is normal, hence we need only count the number of embeddings. To do this we find it simpler to consider $f(\sigma)$ and $f(\pi)$ which are binary strings. Note that we consider an occurrence of a substring to occur in consecutive positions, for example 101 has an occurrence of 10 but no occurrence of 11. We will consider the different cases depending on whether $n$ is odd or even and whether $\sigma$ and $\pi$ are related.
First consider the case when $\sigma$ and $\pi$ are related and $n$ is even. Suppose $\pi = M_n$, then $f(\pi) = 1010 \ldots$ and can be split into $n/2$ blocks, each consisting of a single 10. We can choose to embed a 10 from $f(\sigma)$ in either a single block of $f(\pi)$ or two separate blocks. For any other letter of $f(\sigma)$ we choose a single block of $f(\pi)$ in which to embed it. Thus, once we decide which 10s of $f(\sigma)$ to embed in single blocks of $f(\pi)$, all we need to do to determine an embedding is to pick a subset of blocks of $f(\pi)$.

Suppose we embed none of the 10s of $f(\sigma)$ in a single block of $f(\pi)$. Then we need to pick $m$ of the $n/2$ blocks of $f(\pi)$, to embed one letter of $f(\sigma)$ in each of the selected blocks, which can be done in $\binom{\frac{n}{2}}{m}$ ways. If we select $r$ of the 10s in $f(\sigma)$ to embed in a single block, then we need to choose $m - r$ of the 10s in $f(\pi)$ in which to embed the parts of $f(\sigma)$. Thus, we need to pick a total of $m$ objects, some of them blocks of $f(\pi)$ to embed in and some of them 10s in $f(\sigma)$ to embed in a single block of $f(\pi)$. An occurrence of 10 in $f(\sigma)$ corresponds to a letter in $\sigma$ that is after the descent and not the start of an adjacency and there are $\lfloor \frac{m}{2} \rfloor$ such letters. Therefore, we have $\binom{\frac{n}{2} + \lfloor \frac{m}{2} \rfloor}{m}$ embeddings and because $n$ is even this gives the desired result. If $\pi = W_n$, $n$ is even and $\sigma$ and $\pi$ are related, the proof is analogous to when $\pi = M_n$ but considering substrings $01$ instead of 10.

Now consider the case when $n$ is odd and $\sigma$ and $\pi$ are related. By Lemma 3.8 we know that the largest letters in $\sigma$ and $\pi$ will be on same sides of the descent if and only if $m - i$ is odd. So if $m - i$ is even we cannot embed anything in the final letter of $\pi$, thus this case is equivalent to when $n$ is even and $\sigma$ and $\pi$ are related. If $m - i$ is odd then we can embed a letter of $\sigma$ in the largest letter of $\pi$, and thus we have $\frac{m - 1}{2}$ blocks of $f(\pi)$ to embed in. The remaining argument is analogous to when $n$ is even, using the fact that as $n$ and $m - i$ are odd $\lfloor \frac{n - 1}{2} \rfloor + \lfloor \frac{m - 1}{2} \rfloor = \lfloor \frac{n + m - 1}{2} \rfloor$.

Finally consider the cases when $\sigma$ and $\pi$ are not related. In these cases we cannot embed anything in the first letter of $\pi$. Therefore, we cannot remove the first letter from $\pi$ without changing the number of embeddings. So these cases are equivalent to when $\sigma$ and $\pi$ are related and $\pi$ is of length $n - 1$, the latter point accounting for the $-a$ in the equation. \hfill \Box

4 Intervals of $[1, \pi]$ Where $\pi$ Has One Descent

We have shown that intervals between two permutations with the same number of descents are shellable. Now we consider intervals of the form $[1, \pi]$ where $\pi \in P_1$. First we present a useful tool called the Quillen Lemma, which can be found as Theorem 15.28 in [1], in which we refer to the upper ideal $Q_{\geq x} := \{y \in Q : y \geq x\}$:

**Proposition 4.1 (Quillen Lemma)** Let $\phi : P \to Q$ be an order-preserving map between posets such that for any $x \in Q$ the complex $\Delta(\phi^{-1}(Q_{\geq x}))$ is contractible. Then the induced map between simplicial complexes $\Delta(\phi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.
Note that the order complex of an upper ideal \( Q \geq x \) is always contractible to the vertex \( x \). Now we consider the homology of the order complexes of intervals \([1, \pi]\) where \( \pi \in \mathcal{P}_1 \).

**Theorem 4.2** If \( \pi \in \mathcal{P}_1 \) then the order complex \( \Delta([1, \pi]) \), is homotopy equivalent to a suspension of \( \Delta([21, \pi]) \). Therefore, the Betti numbers of \( \Delta([1, \pi]) \) are given as \( \beta_n(\Delta([1, \pi])) = \beta_{n-1}(\Delta([21, \pi])) \), for \( n > 0 \), and \( \beta_0(\Delta([1, \pi])) = 0 \).

**Proof** Let \( X = (1, \pi) \) and \( A = X \setminus [123, k] \) where \( k = 1 \ldots k \) is the largest increasing permutation that occurs in \( \pi \). The only permutations in \( A \) not in \( (21, \pi) \) are 21 and 12. The permutations 21 and 12 occur as a pattern in every permutation in \( (21, \pi) \). Therefore in the order complex of \( A \), each of the vertices associated to 12 and 21 is the apex of a cone over \( \Delta([21, \pi]) \), so \( \Delta(A) \) is a suspension of \( \Delta([21, \pi]) \). We use the Quillen Lemma to show that \( \Delta(X) \) is homotopically equivalent to \( \Delta(A) \), and to this end we define a map \( f : X \to A \) as:

\[
f(\sigma) = \begin{cases} 
12, & \text{if } \sigma \in \mathcal{P}_0 \\
\sigma, & \text{if } \sigma \in \mathcal{P}_1
\end{cases}
\]

This map is order-preserving and \( f^{-1}(A \geq a) = X \geq a \) which is an upper ideal thus \( \Delta(f^{-1}(A \geq a)) \) is contractible. Therefore \( f \) induces a homotopy equivalence between \( \Delta(X) \) and \( \Delta(A) \) by the Quillen Lemma. So \( \Delta(X) \) is homotopically equivalent to a suspension of \( \Delta([21, \pi]) \) and the result on the Betti numbers then follows directly from the property of the suspension that \( H_{n+1}(\text{susp } X) = H_n(X) \).

It is not true that all intervals \([1, \pi]\), \( \pi \in \mathcal{P}_1 \), are shellable, as can be seen by the following example:

**Example 4.3** Consider the permutations 456123 and 356124. In the interval \([1, 456123]\) the subinterval \([123, 456123]\) is disconnected and of rank 3 which implies \([1, 456123]\) is not shellable. Similarly in \([1, 356124]\) the subinterval \([123, 356124]\) is disconnected and of rank 3. As a consequence, if a permutation \( \pi \in \mathcal{P}_1 \) contains 456123 or 356124 the interval \([1, \pi]\) is not shellable.

Whilst it is not true that the intervals \([1, \pi]\) are all shellable we conjecture that containing 456123 or 356124 are the only obstructions to shellability for the intervals \([1, \pi]\) when \( \pi \in \mathcal{P}_1 \).

**Conjecture 4.4** If \( \pi \in \mathcal{P}_1 \) and \( \pi \) avoids 456123 and 356124 the interval \([1, \pi]\) is shellable.

We have been unable to prove this conjecture, but we show that these intervals have no non-trivial disconnected subintervals. We prove this below, but in order to do so we need a result from [6] and the following definition:

**Definition 4.5** Let \( \eta \) be an embedding of \( \sigma \) in \( \pi \). The zero set of \( \eta \), which we denote \( Z_\eta \), is the set \( \{i : \eta_i = 0\} \). The zero set \( Z_E \) of a set of embeddings \( E \) is the union of the zero sets of all the embeddings in the set \( E \).
Example 4.6 Let $\sigma = 213$ and $\pi = 245136$ and consider the following embeddings of $\sigma$ in $\pi$: $\eta_1 = 200130$, $\eta_2 = 200103$ and $\eta_3 = 020103$. These embeddings have zero sets $Z_{\eta_1} = \{2, 3, 6\}$, $Z_{\eta_2} = \{2, 3, 5\}$ and $Z_{\eta_3} = \{1, 3, 5\}$, so the set $\{\eta_1, \eta_2, \eta_3\}$ has zero set $\{1, 2, 3, 5, 6\}$.

Lemma 4.7 (see [6, Proposition 5.3]) Suppose permutations $\sigma < \pi$ satisfy $|\pi| - |\sigma| \geq 3$. Then the interval $[\sigma, \pi]$ is not disconnected if the embeddings of $\sigma$ in $\pi$ cannot be partitioned into two non-empty sets $E_1$ and $E_2$ such that $Z_{E_1} \cap Z_{E_2} = \emptyset$.

Proposition 4.8 If $\pi \in P_1$ and $\pi$ avoids 456123 and 356124 then the interval $[1, \pi]$ has no disconnected subintervals of rank 3 or more.

Proof By Corollary 2.7 we know that intervals between two permutations in $P_1$ are shellable hence have no disconnected subintervals. So all that remains is subintervals of the form $[\alpha, \beta]$, of rank 3 or more, with $\alpha \in P_0$ (so $\alpha$ is an increasing permutation) and $\beta \in P_1$. We show there is no way to split the embeddings of $\alpha$ in $\beta$ into two sets with disjoint zero sets. To do this we separate the embeddings into three disjoint sets:

1. Embeddings with all of $\alpha$ embedded before the descent in $\beta$ constitute the set $E_1$.
2. Embeddings with all of $\alpha$ embedded after the descent in $\beta$ constitute the set $E_2$.
3. Embeddings with part of $\alpha$ embedded before the descent in $\beta$, and part after, constitute the set $E_3$.

Note that each embedding in $E_1$ has zeros in all positions after the descent and similarly all embeddings in $E_2$ have zeros in all positions before the descent. So it is not possible to split $E_1$ or $E_2$ into smaller sets that have disjoint zero sets. Also $E_3$ cannot be split into smaller sets with disjoint zero sets, since it is always possible to swap a nonzero letter with a zero letter directly to the right if after the descent, or directly to the left if before the descent. We can use this to build a sequence of embeddings between any two embeddings where the elements in each adjacent pair in the sequence have only one letter differing in their zero sets. If the zero sets differ by only one element they cannot be disjoint. As we can build such a sequence between any two embeddings it is not possible to split $E_3$ into two sets with disjoint zero sets.

Suppose that all three sets are non-empty. As both $E_1$ and $E_2$ are non-empty this means that it is not possible to make an embedding that uses all letters from one side of the descent and some letters from the other. This means that each embedding in $E_3$ must have a zero on both sides of the descent. So all embeddings in $E_1$ must be placed in the same set, all embeddings in $E_3$ must be placed in the same set as the embeddings in $E_1$ and all embeddings in $E_2$ must be placed in the same set as the embeddings in $E_3$. So we cannot split the embeddings into two sets with disjoint zero sets.

We now analyse three cases, depending on which of the three sets are empty.

First suppose $E_3$ is empty and that $E_2$ and $E_3$ are non-empty. Consider the embeddings in $E_3$. Unless an embedding embeds all its letters before the
descent (and then some after) it will have a zero before the descent so must be put into the same set as $E_2$. And as $E_3$ cannot be split into two sets with disjoint zero sets, the only way for $E_2$ and $E_3$ to have disjoint zero sets is if all the embeddings in $E_3$ have no zeros before the descent. We show that the only way such an embedding can exist is if $\beta = \beta_1 \beta_2 ... \beta_d ... \beta_n$ with $\beta_3 > \beta_d$ and any letter strictly between $\beta_d$ and $\beta_4$ is less than $\beta_d$. Also the number of letters not between $\beta_d$ and $\beta_4$ must be exactly $|\alpha|$ and thus we can embed $\alpha$ as

$$\eta = \alpha_1 ... \alpha_d 0 ... 0 \alpha_{d+1} ... \alpha_n,$$

where $\alpha_{d+1}$ is embedded in position $i$. To see this is the only possible embedding suppose there is another embedding $\tilde{\eta} \neq \eta$, because there cannot be a zero before the descent this implies there must be a zero after $\tilde{\eta}_i$. This implies it would also be possible to embed the sequence $\alpha_d ... \alpha_n$ after the descent, leaving a zero before the descent, contradicting our requirement for $E_3$.

If $\eta$ is a valid embedding, then $\beta_{d-2} \beta_{d-1} \beta_d$ must be of one of two forms, either $c(c+1)(c+2)$ or $c(c+2)(c+3)$. If this were not the case we could build valid embeddings of the form

$$\alpha_1 ... \alpha_{d-2} 00 ... 0 \alpha_{d-1} \alpha_d \alpha_{d+1} ... \alpha_n,$$

which has a zero before the descent, contradicting our requirement for $E_3$. We also know that there are $|\beta| - |\alpha|$ letters smaller than $\beta_d$ that occur after $\beta_d$ which we know to be at least 3. These two things imply that the embedding $\eta$ can only exist if there is an occurrence of either 456123 or 356124 in $\beta$. As $\beta$ avoids both these permutations $\eta$ cannot be a valid embedding. So if $E_1$ is empty the embeddings cannot be split into disjoint zero sets.

An analogous argument shows that if $E_2$ is empty the embeddings cannot be split into disjoint zero sets.

Now suppose $E_3$ is empty but $E_1$ and $E_2$ are not. As $E_3$ is empty there can be no increasing sequence of the same length as $\alpha$ spread across both sides of the descent. Using this we can repeat the same argument as above showing that $\beta_{d-2} \beta_{d-1} \beta_d$ must be of one of the forms $c(c+1)(c+2)$ or $c(c+2)(c+3)$. Therefore, if $\beta$ avoids 456123 and 356124 this case cannot arise.

So we have shown that if $\pi \in P_3$ and $\pi$ avoids 456123 and 356124, then for any $1 \leq \alpha \leq \pi$ the embeddings of $\alpha$ in $\beta$ cannot be split into two sets with disjoint zero sets. Thus, by Lemma 4.7, the interval $[\alpha, \beta]$ cannot be disconnected. Therefore $[I, \pi]$ has no disconnected subintervals of rank 3 or more.

\[ \square \]

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