Abstract

We study the index of $\mathcal{N} = 4$ Yang-Mills theory on $S^3 \times \mathbb{R}$ at large angular momenta. A generalized Cardy limit exhibits macroscopic entropy at large $N$. Our result is derived using free QFT analysis, and also a background field method on $S^3$. The index sets a lower bound on the entropy. It saturates the Bekenstein-Hawking entropy of known supersymmetric AdS$_5$ black holes, thus accounting for their microstates. We further analyze the so-called Macdonald index, exploring small black holes and possibly new black holes reminiscent of hairy black holes. Finally, we study aspects of large supersymmetric AdS$_7$ black holes, using background field method on $S^5$ and 't Hooft anomalies.
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### 1 Introduction and summary

Microscopic understanding of black holes is a major achievement of string theory. Many successes are made using 2d QFT approaches, starting from the pioneering work of [1]. These are mostly related to the AdS$_3$/CFT$_2$ duality. In fact, AdS/CFT [2] is an ideal setting to study black holes using quantum field theory. In higher dimensional AdS$_d$, at $d > 3$, there are interesting physics of black holes to be better understood: see e.g. [3, 4] and references thereof. Especially, supersymmetric black holes in $d > 3$ suggest quantitative challenges to CFTs.\footnote{Magnetic/dyonic black holes in AdS$_4$, with fluxes on $S^2$ boundary, were recently studied microscopically from topologically twisted 3d QFTs [5, 6]. Our interest in this paper will be the electric black holes, whose microstates consist of excitations from the unique vacuum of the radially quantized CFT.}

It has been believed that such BPS black holes in AdS$_5$ defied quantitative understandings from indices of SCFTs on $S^3 \times \mathbb{R}$ [7, 8]. There have been many speculations on why the index fails to capture black holes. A possible reason is that bosonic/fermionic states undergo big cancelation. For instance, the index cannot see the deconfinement phase transition at an order 1 temperature in the unit of AdS$_5$ radius [7], which is the QFT dual of the Hawking-Page transition of AdS black holes [9]. So the index cannot capture all the physics of generic supersymmetric AdS$_5$ black holes. Direct studies of BPS operators at weak coupling did not discover enough microstates for such black holes either [10, 11, 12, 13], at least so far.

In this paper, we show that the index of 4d $\mathcal{N} = 4$ Yang-Mills theory does capture large supersymmetric AdS$_5$ black holes [14, 15, 16] in an asymptotic Cardy-like limit. Our Cardy
limit is more refined than [17], in that the imaginary parts of chemical potentials are tuned to optimally obstruct boson/fermion cancelations. The entropy of our asymptotic index is macroscopic, meaning that it is proportional to $N^2$ when all the charges are at this order. This sets a lower bound on the true microscopic entropy of BPS states, assuring the existence of BPS black holes in $AdS_5 \times S^5$. In particular, when a charge relation is met, our asymptotic free energy agrees with the Bekenstein-Hawking entropy of known supersymmetric $AdS_5$ black holes [14], thereby microscopically counting them. The asymptotic free energy of our index is the recently suggested entropy function for supersymmetric $AdS_5$ black holes [18], in our large black hole limit. At general values of charges, perhaps our findings may have implications to possible supersymmetric hairy black holes in $AdS_5 \times S^5$ [19, 20]. The last suggestion is indirectly supported by studying the asymptotic free energy of the so-called Macdonald index [21]. Here, depending on charge regime, the Cardy-like free energy differs from the entropy function of [18], showing properties reminiscent of hairy black holes in $AdS_5 \times S^5$.

Our derivation is based on two methods. One is the free QFT. Another is a background field method on $S^3$, in which the Chern-Simons terms of these background fields yield the asymptotic free energy. The relevant Chern-Simons terms are determined by ’t Hooft anomalies. The latter method can be useful for non-Lagrangian QFTs. We apply it to the 6d $(2,0)$ theory and study aspects of large supersymmetric black holes in $AdS_7 \times S^4$ [22, 23].

The rest of this paper is organized as follows. In section 2, we derive the asymptotic free energy of the index of 4d $\mathcal{N} = 4$ Yang-Mills theory, in a generalized Cardy-like limit. This free energy counts known supersymmetric $AdS_5$ black holes. In section 3, we study similar asymptotic free energy of the index in the Macdonald limit, suggesting rich structures such as small black holes and new saddle points reminiscent of hairy black holes. In section 4, we apply the background field method to study supersymmetric $AdS_7$ black holes. In section 5, we summarize with comments on future directions.

As we were finalizing our draft, we received [24] whose subject overlaps with our section 2. However, their claims appear to be different from ours.

## 2 Large supersymmetric $AdS_5$ black holes

We study the the partition function of $\mathcal{N} = 4$ Yang-Mills theory on $S^3 \times \mathbb{R}$, focussing on the index limit [7]. The partition function counts states carrying six charges. The first one is the energy $E$, made dimensionless by multiplying the $S^3$ radius. Three charges $Q_1, Q_2, Q_3$ are for the Cartans of $SO(6)$ R-symmetry, defined to be the angular momenta on three orthogonal 2-planes on $\mathbb{R}^6$, being $\pm \frac{1}{2}$ for spinors. The final two are the angular momenta $J_1, J_2$ on $S^3$, being $\pm \frac{1}{2}$ for spinors. The BPS states of our interest saturate the bound $E \geq Q_1 + Q_2 + Q_3 + J_1 + J_2$,.
in section 2.3, we study the physics of the derived log terms of background fields, determined either from a weakly-coupled 4d QFT or using 't Hooft action is a series expansion in small $\beta, \omega$. Subtle than it may naively sound, we shall argue that the derivative expansion of the effective action is a series expansion in small $\beta, \omega_1, \omega_2$. The leading log $Z$ is given by the Chern-Simons terms of background fields, determined either from a weakly-coupled 4d QFT or using 't Hooft anomalies. Evaluating these Chern-Simons terms, we obtain same asymptotic log $Z$. Then in section 2.3, we study the physics of the derived log $Z$ and discuss the dual black holes.

The complex chemical potentials $\Delta_I, \omega_i$ satisfy five periodicity conditions $\Delta_I \sim \Delta_I + 4\pi i$, $\omega_i \sim \omega_i + 4\pi i$. The 16 supercharges are $Q^{Q_1,Q_2,Q_3}$. 16 possible values of $Q_I, J_i$ carried by $Q$ are $\pm \frac{1}{2}$, where the product of all 5 $\pm$ signs is +. The conformal supercharges are $S^{Q_1,Q_2,Q_3}$ with five charges being $\pm \frac{1}{2}$, where the product of signs is -. Taking the trace without $(-1)^F$, the fermionic fields are anti-periodic along temporal circle, twisted by $\Delta_I, \omega_i$. So the SUSY connecting periodic bosons and anti-periodic fermions are generally broken. In a sense, the supercharges are anti-periodic which has no zero modes on temporal $S^1$. However, if

$$
\sum_{I=1}^{3} s_I \Delta_I - \sum_{i=1}^{2} t_i \omega_i = 2\pi i \quad \text{(mod } 4\pi i\text{)} , \quad s_I, t_i = \pm 1 \quad \text{satisfying } s_1s_2s_3t_1t_2 = +1 ,
$$

becomes an index if one takes $\beta \to 0^+$. This is because

$$
e^{-\Delta Q - \omega J} e^{Q_{\pm 1,-1,-2}} = e^{-\frac{s_1s_2s_3}{2} - \Delta Q - \omega J} e^{-\Delta Q - \omega J} = -Q_{-1,-1,-2} e^{-\Delta Q - \omega J} ,
$$

so that translating $Q_{s_1,s_2,s_3}^{-1}$ along the trace will cause extra $-1$ sign, creating a zero mode of this supercharge. So restricting $Z$ to this hypersurface of $\Delta_I, \omega_i$, it becomes an index which counts $\frac{1}{16}$-BPS states annihilated by $Q \equiv Q_{s_1,s_2,s_3}^{-1}$ and $S \equiv S_{t_1,t_2}$. From the algebra

$$
\{Q, S\} = E - \sum_{I=1}^{3} s_I \Delta_I - \sum_{i=1}^{2} t_i J_i ,
$$

one finds $E = s_I Q_I + t_i J_i$. Therefore, having in mind that we shall eventually live on one of the hyperspaces (2.2), we study $Z$ in the ‘formal high temperature limit’ $\beta \to 0^+$. We shall analyze log $Z$ in an asymptotic Cardy-like limit $|\omega_i| \ll 1$. In our limit, $\Delta_I$ is kept complex, $O(1)$, and generic. The computation will be made using two complementary approaches. One is the free QFT analysis, which is reliable because $Z$ will be independent of the coupling constant at the hyperspace (2.2). This will be presented in section 2.1. Another method, explored in section 2.2, is a background field approach on $S^3$. To understand this, note that $Z$ has a path integral representation on $S^3 \times S^1$, where the size of the temporal circle is given by vanishing $\beta$. $\beta, \Delta_I, \omega_i$ are realized as background fields on this space. At small $\beta$, we reduce the system on small circle, integrating out the KK modes on $S^1$. The partition function will then acquire contribution from dynamical zero modes on $S^3$, while the KK modes will yield an effective action of 3d background fields. Although the KK reduction is a bit subtle than it may naively sound, we shall argue that the derivative expansion of the effective action is a series expansion in small $\beta, \omega_1, \omega_2$. The leading log $Z$ is given by the Chern-Simons terms of background fields, determined either from a weakly-coupled 4d QFT or using 't Hooft anomalies. Evaluating these Chern-Simons terms, we obtain same asymptotic log $Z$. Then in section 2.3, we study the physics of the derived log $Z$ and discuss the dual black holes.
2.1 Free QFT analysis

The partition function \((2.1)\) of weakly-coupled \(\mathcal{N} = 4\) Yang-Mills theory is given by \[ Z = \frac{1}{N!} \int \frac{d\alpha}{2\pi} \prod_{a=1}^{N} \prod_{a<b} \left(2 \sin \frac{\alpha_{ab}}{2}\right)^2 \exp \left[ \sum_{a,b=1}^{N} \sum_{n=1}^{\infty} \frac{1}{n} \left( f^v_B(n\beta, n\omega_i) + (-1)^{n-1} f^v_F(n\beta, n\omega_i) \right) \right] \]

\[ + \chi_3(n\Delta_I)(f^a_B(n\beta, n\omega_i) + (-1)^{n-1} f^a_F(n\beta, n\omega_i)) + \chi_3(n\Delta_I)(f^a_B(n\beta, n\omega_i) + (-1)^{n-1} f^a_F(n\beta, n\omega_i)) e^{i\alpha_{ab}} \]

where \(\alpha_{ab} \equiv \alpha_a - \alpha_b\), \(\chi_3 = \sum_{i=1}^3 e^{-\Delta_i}\), \(\chi_3 = \sum_{i=1}^3 e^{-\Delta_i}\), and

\[ f^v_B = \frac{e^{-\beta}(1 - e^{-2\beta})(e^{\omega_1} + e^{\omega_2} + e^{-\omega_1} + e^{-\omega_2}) - 1 + e^{-4\beta}}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})} + 1 \]

\[ f^v_F = \frac{e^{-\frac{3}{2}\beta}(e^{\Delta} - e^{-\Delta}e^{-\beta})(e^{\omega_+} + e^{-\omega_+} + e^{-\frac{3}{2}\beta}(e^{-\Delta} - e^{-\Delta}e^{-\beta})(e^{-\omega_+} + e^{-\omega_-})}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})} \]

\[ f^a_B = f^a_B = \frac{e^{-\beta}(1 - e^{-2\beta})}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})} \]

\[ f^c_B = \frac{e^{-\frac{3}{2}\beta - \Delta}}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})} \]

\[ f^a_F = \frac{e^{-\frac{3}{2}\beta + \Delta}}{(1 - e^{-\beta+\omega_1})(1 - e^{-\beta+\omega_2})(1 - e^{-\beta-\omega_1})(1 - e^{-\beta-\omega_2})} \]

with \(\Delta \equiv \frac{\Delta_1 + \Delta_2 + \Delta_3}{2}\), \(\omega_{\pm} \equiv \frac{\omega_1 \pm \omega_2}{2}\). The superscripts \(v, c, a\) refer to \(\mathcal{N} = 1\) vector, chiral, anti-chiral multiplets, respectively, with the chiral supercharges \(Q_{\alpha} \equiv Q_{\alpha}^{+,+,+} (at (t_1, t_2) = (+, +), (-, -))\).

With the understanding that one of the BPS index conditions \((2.2)\) will be taken, we study the \(\beta \to 0^+\) limit of this partition function. One might worry that, before reaching \(\beta \to 0\), the factors \(1 - e^{-\beta+\omega_{1,2}}\) in the denominators will hit zeros or make the sum divergent if \(\text{Re}(\omega_{1,2}) > 0\) (for BPS states with \(t_1 = t_2 = +1\)). These are divergences caused by two non-BPS derivatives, losing fugacity factors smaller than 1. In general partition function, going beyond this point will probably have no meaning, analogous to going beyond infinite temperature. However, having in mind imposing \((2.2)\) at \(t_1 = t_2 = 1\), these poles are canceled between bosons/fermions, so that one can reduce \(\beta\) below \(\omega_{1,2}\). Anyway, later in this subsection, we shall present a complementary derivation manifestly within the index. (However, we think the analysis presented now has a conceptual advantage.) In this limit, one finds \(f^a_B \to 0\) due to the vanishing of the equation of motion factor \(1 - e^{-2\beta} \to 0\) on the numerators. Also, one finds \(f^c_B \to 1\) for the same reason. The fermionic letter partition functions reduce to

\[ f^v_F \to \frac{(e^{\Delta} - e^{-\Delta})(e^{\omega_+} + e^{-\omega_+} - e^{\omega_-} - e^{-\omega_-})}{(1 - e^{\omega_1})(1 - e^{\omega_2})(1 - e^{-\omega_1})(1 - e^{-\omega_2})} = \frac{e^{\Delta} - e^{-\Delta}}{2 \sinh \frac{\omega_1}{2} \cdot 2 \sinh \frac{\omega_2}{2}} \]

\[ f^c_F \to \pm \frac{e^{\pm \Delta}}{2 \sinh \frac{\omega_1}{2} \cdot 2 \sinh \frac{\omega_2}{2}}. \]
Z then becomes
\[
Z \to \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \prod_{a<b} (2 \sin \frac{\alpha_{ab}}{2})^2 \exp \left[ \sum_{a,b=1}^{N} \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 + \sum_{s_1,s_2,s_3 = \pm 1} \frac{s_1 s_2 s_3 (-1)^{n-1} e^{\frac{n s_1 \Delta_1 + n s_2 \Delta_2 + n s_3 \Delta_3}{2}}}{2 \sinh \frac{n \omega_1}{2} \cdot 2 \sinh \frac{n \omega_2}{2}} \right) e^{i n \alpha_{ab}} \right].
\] (2.8)

Note that the sum over \( n \) in the exponent is convergent with nonzero real parts of \( \omega_{1,2} \). For instance, let us have in mind imposing \( \sum_i \Delta_I = \sum_i \omega_i + 2 \pi i \) for an index, with all chemical potentials having positive real part. For the terms with given \( s_1, s_2, s_3 \), the sum over \( n \) is separately convergent if \( (s_1, s_2, s_3) \neq (+, +, +) \). This is because, for large \( n \), one finds
\[
\sim \sum_n \frac{(-1)^{n-1}}{n} e^{-\frac{n}{2} (\omega_1 + \omega_2)} e^{i n \alpha_{ab} e^{\frac{n s_1 \Delta_1 + n s_2 \Delta_2 + n s_3 \Delta_3}{2}}} = - \sum_n \frac{1}{n} e^{-\frac{n}{2} (1-s_1) \Delta_I} e^{i n \alpha_{ab}}.
\] (2.9)

If some \( s_I \) is \(-1\), this sum is convergent at large \( n \), due to an exponential damping. On the other hand, the remaining terms in the exponent are the first term ‘1’ and the term with \( (s_1, s_2, s_3) = (+, +, +) \). The sum over each term over \( n \) may be divergent, for instance at \( \alpha_{ab} = 0 \). For \( a = b \), divergence at \( \alpha_{ab} = 0 \) is fine because there is a suppression factor given by the Haar measure \((2 \sinh \frac{\Delta_{ab}}{2})^2\). For the Cartans, \( a = b \), one has to study the possible convergence of the sum of these two terms without resorting to the phase factor \( e^{i \alpha_{ab}} \) or the Haar measure. The sum of these two terms at large \( n \) behaves as
\[
\sum_n \frac{1}{n} \left( 1 + \frac{(-1)^{n-1} e^{\frac{n \Delta_1 + n \Delta_2 + n \Delta_3}{2}}}{2 \sinh \frac{n \omega_1}{2} \cdot 2 \sinh \frac{n \omega_2}{2}} \right) e^{i n \alpha_{ab}} = \sum_n \frac{1}{n} \left( 1 + \frac{(-1)^{n-1} e^{\frac{n \omega_1 + n \omega_2 + 2 \pi i}{2}}}{2 \sinh \frac{n \omega_1}{2} \cdot 2 \sinh \frac{n \omega_2}{2}} \right) e^{i n \alpha_{ab}}
\sim \sum_n \frac{1}{n} \left[ 1 - (1 - O(e^{-n \omega})) \right] e^{i n \alpha_{ab}}.
\] (2.10)

So even if \( \alpha_{ab} = 0 \), or \( a = b \), the sum over \( n \) converges.

Having realized that the sum converges at \( \sum_I \Delta_I = \omega_1 + \omega_2 + 2 \pi i \), we also note here that it will be useful later to consider this sum slightly away from this surface. Namely, we shall consider the approximation of the index in the ‘Cardy limit’ \( |\omega_i| \ll 1 \). Imposing the relation \( \sum_I \Delta_I = \omega_1 + \omega_2 + 2 \pi i \), \( \Delta_I \)'s will share the \( O(1) \) imaginary part \( 2 \pi i \), and furthermore will have small real parts to match \( \text{Re}(\omega_1 + \omega_2) \). However, for convenient intermediate manipulations, we shall take \( \Delta_I \)'s slightly away from this surface by temporarily demanding them to be of order 1 and purely imaginary. This parameter deformation clearly does not affect the convergence analysis of (2.9) for \( (s_I) \neq (+, +, +) \). So as for this part, the function is well defined even after slight deformations. However, for (2.10), the convergence issue becomes tricky after the deformation. Just working with the left hand side of (2.10) with \( \Delta_1 + \Delta_2 + \Delta_3 \) being imaginary, the second term containing \( \Delta_I, \omega_i \) will be convergent by itself, for any \( a, b \), while the first term ‘1’ will remain divergent at \( a = b \). Therefore, in the analysis below, we shall separate the Cartan parts at \( a = b \) and the off-diagonal parts at \( a \neq b \). The former has an exponential proportional to \( N \), and it can be taken out of the \( \alpha_a \) integral. The latter part has \( N^2 - N \) terms, and only for these terms we shall make a deformation to purely imaginary \( \Delta_I \)'s. Ignoring the former
contribution to the free energy \( \sim \mathcal{O}(N^1) \) will be justified if one obtains a free energy and entropy of order \( N^2 \) from the latter part only. So with this understanding, we shall often ignore the exponents at \( a = b \) in the discussions below. Note also that, for the off-diagonal parts, the term ‘1’ in the exponent completely cancels the Haar measure part, so we can ignore this term together with the Haar measure.\(^2\)

Now we consider the Cardy limit \( |\omega_i| \ll 1 \), keeping \( \Delta_I \) order 1 and purely imaginary. The sum over \( n \) can be divided into two parts: the ‘dominant part’ till \( n \ll |\omega_i| \), and the ‘suppressed part’ from \( n \gtrsim |\omega_i| \). As for the ‘dominant’ part, we can approximate \( 2\sinh \frac{n\omega}{2} \approx n\omega_i \). The terms in the exponent of (2.8) from these \( n \)'s is given by

\[
\frac{s_1s_2s_3}{\omega_1\omega_2} \sum_{n < n_0} \frac{(-1)^{n-1}}{n^3} e^{\frac{n}{2} + i\alpha_{ab}}
\]

where \( n_0 \ll |\omega|^{-1} \) is a ‘cut-off’ which defines the ‘dominant part.’ (Note again that we consider the terms at \( a \neq b \) only, and we ignored the term 1 which cancels the Haar measure.) The summation over \( n \) is now independent of the cut-off value \( n_0 \), as the summand is independent of \( \omega_i \) and converges when \( e^{\frac{n}{2} + i\alpha_{ab}} \) is a pure phase. So one obtains the dominant part given by

\[
\frac{s_1s_2s_3}{\omega_1\omega_2} \sum_{n < n_0} \frac{(-1)^{n-1}}{n^3} e^{n\frac{n}{2} + i\alpha_{ab}} \approx \frac{s_1s_2s_3}{\omega_1\omega_2} \text{Li}_3 \left( -e^{\frac{n}{2} + i\alpha_{ab}} \right) \tag{2.11}
\]

Before proceeding, we note that if one wishes, one can take the cut-off \( n_0 \) to be as big as \( |\omega|^{-1} \). This is because at \( n \sim |\omega|^{-1} \), both summands \( e^{\frac{n}{2} + i\alpha_{ab}} \) and \( \frac{1}{2n} \sinh \frac{n\omega}{2} \cdot 2 \sinh \frac{n\omega_i}{2} \) are very small, much smaller than the final asserted result (2.12) which is \( \mathcal{O}(1) \). So we proceed with assuming \( n_0 \sim |\omega|^{-1} \) below. Now we discuss the ‘suppressed’ part. It is easy to see that it is indeed suppressed at \( |\omega_i| \ll 1 \). This is because

\[
\left| \sum_{n \gtrsim |\omega|^{-1}} \frac{s_1s_2s_3}{\omega_1\omega_2} \frac{(-1)^{n-1}}{n} e^{n\frac{n}{2} + i\alpha_{ab}} \right| \lesssim \sum_{n \gtrsim |\omega|^{-1}} \frac{1}{n} \cdot 2 \sinh \frac{n\omega_1}{2} \cdot 2 \sinh \frac{n\omega}{2} \lesssim |\omega| \sum_{n \gtrsim |\omega|^{-1}} \left( 2 \sinh \frac{n\omega}{2} \right)^{-2}
\]

which is indeed much smaller than \( \frac{1}{\omega^2} \). With these approximations, one then obtains

\[
Z \sim \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \exp \left[ -\frac{1}{\omega_1\omega_2} \sum_{a \neq b} \sum_{s_1s_2s_3=\pm 1} s_1s_2s_3 \text{Li}_3 \left( -e^{\frac{i\alpha_{ab}}{2}} \right) \right]
\]

where we used the series definition \( \text{Li}_3(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^3} \) when \( |x| \leq 1 \). The summations over \( a \neq b, (s_1, s_2, s_3) \) can be arranged so that

\[
Z \sim \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \exp \left[ -\frac{1}{\omega_1\omega_2} \sum_{a \neq b} \sum_{s_1s_2s_3=+1} \left( \text{Li}_3 \left( -e^{\frac{i\alpha_{ab}}{2}} \right) - \text{Li}_3 \left( -e^{-\frac{i\alpha_{ab}}{2}} \right) \right) \right] \tag{2.15}
\]

\(^2\) Probably, using asymptotic properties of special functions in the integrand carefully, one can do the approximation below without using our small deformations of \( \Delta_I \). We just regard it as a short-cut derivation, similar to familiar ‘i.e’ prescriptions which often makes many calculus more straightforward.
Here, we note an identity
\[
\text{Li}_3(-e^x) - \text{Li}_3(-e^{-x}) = -\frac{x^3}{6} - \frac{\pi^2 x}{6},
\]
valid for \(-\pi < \text{Im}(x) < \pi\), taking \(-e^x = e^{x+\pi i}, -e^{-x} = e^{-(x+\pi i)}\), respectively. When \((2p-1)\pi < \text{Im}(x) < (2p+1)\pi\) for \(p \in \mathbb{Z}\), similar identity holds with \(x \to x - 2\pi ip\) on the right hand side. This identity can be continued to include either positive or negative real parts of \(x\).

Now we treat the integrals over \(\alpha_a\)’s by a saddle point approximation at \(|\omega_1\omega_2| \ll 1\). Considering a pair of terms \(\text{Li}_3(-e^{\frac{\Delta I}{2}+i\alpha_{ab}}) + \text{Li}_3(-e^{\frac{-\Delta I}{2}-i\alpha_{ab}})\) at given \(s_I\), one finds that \(\alpha_a\) derivative of these are all zero at \(\alpha_1 = \alpha_2 = \cdots = \alpha_N\). We assume the dominance of this \(U(N)\) saddle point in our generalized Cardy limit. The dominance of such a saddle point was assumed in the Cardy limit of \([17]\). But it may fail to be dominant in certain models, e.g. for other gauge groups than \(U(N)\), with fields in certain representations \([25]\). Here and later, we shall basically assume the dominance of our saddle point. In particular, it will reproduce the physics of known large black holes. As a very basic check, we confirmed at \(N = 2\) that \(\alpha_1 = \alpha_2\) is the global maximum of \(\log Z\), making its real part maximal and imaginary part stationary, along the line of \([25]\). However, since our free energy will depend on various complex parameters \(\Delta_I, \omega_i\), we have tested it self-consistently at the extremal values of \(\Delta_I, \omega_i\) found in section 2.3, only at \(Q_1 = Q_2 = Q_3, J_1 = J_2\). More conceptually, \([25]\) discussed the relation between other possible saddle points and the behaviors of the \(S^3\) partition function of 4d QFT reduced on small \(S^1\). Depending on how bad the IR divergence of this partition function is \([25, 17]\), one may either expect more nontrivial saddle points to be dominant, or otherwise zero modes like \(\alpha_a\) to cause subleading \(N^1\log \omega\) corrections. As we shall discuss further in section 2.2, our reduced QFT on \(S^3\) is maximal SYM, belonging to the latter class \([25, 17]\). The expected log correction at \(N^1\) order should come from the Cartan terms that we have ignored. So mostly in this paper, we shall proceed assuming that the above ‘maximally deconfining’ saddle point is dominant. (In only section 3, we discuss a different saddle point in a non-Cardy scaling limit.)

Perhaps as a related issue, one may worry from the Haar measure factor \(\sim (2\sin\frac{\alpha_a-\alpha_b}{2})^2\) that there is a net factor of 0 when all \(\alpha_a\) are the same, making this saddle point suppressed. Indeed, in the usual large \(N\) saddle point analysis (see e.g. \([4]\)), the Haar measure provides relative repulsions between pairs of \(\alpha_a\)’s, forbidding them to be on top of each other. However, in our Cardy saddle point, log of Haar measure is sub-dominant \(O(\omega^0)\). So \(\alpha_1 = \cdots = \alpha_N\) should make sense only as the asymptotic Cardy saddle point at \(\omega \ll 1\).

So assuming this saddle point, one finds
\[
\log Z \sim -\frac{N^2}{\omega_1\omega_2} \sum_{s_1s_2s_3=+1} \left[ \text{Li}_3\left(-e^{\frac{\Delta_I}{2}}\right) - \text{Li}_3\left(-e^{-\frac{\Delta_I}{2}}\right) \right]
\]
(2.17)
where we used \( N^2 - N \sim N^2 \). Now using the identity (2.16), one obtains

\[
\log Z \sim \frac{N^2}{6\omega_1 \omega_2} \sum_{s_1 s_2 s_3 = +1} \left[ \left( \frac{s \cdot \Delta}{2} - 2\pi p_s \right)^3 + \pi^2 \left( \frac{s \cdot \Delta}{2} - 2\pi p_s \right) \right]
\]

(2.18)
in the chamber defined by

\[
(2p_s - 1) < \pi \sum_{I=1}^{3} \text{Im} \left( \frac{s_I \Delta_I}{2} \right) < (2p_s + 1)\pi , \quad p_s \in \mathbb{Z} , \quad s_1 s_2 s_3 = +1 .
\]

(2.19)

Let us consider the ‘canonical chamber,’ with all four \( p_s = 0 \). This chamber is an octahedron in the space of \( \text{Im}(\Delta_I) \). In this chamber, summing over 4 values of \( s \), one obtains

\[
\log Z \sim \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2\omega_1 \omega_2} .
\]

(2.20)

This is the final form of our free energy in the generalized Cardy-like limit. Now we can continue \( \Delta_I \)'s to have (small) real parts, to go back to one of the surfaces (2.2). This formula is reliable at strong coupling on any hypersurface (2.2). Note that in our notation, it appears that [17] restricted their interest to \( \omega_1 = \omega_2 \equiv \omega \ll 1 \), one of \( \Delta_I \)'s \( 2\pi i + \mathcal{O}(\omega) \), and the remaining two of \( \Delta_I \)'s at \( \mathcal{O}(\omega) \). The partition function is trivial in this setting. However, as we shall explain in section 2.3, complex \( \Delta_I \sim \mathcal{O}(1) \) are required for all \( I = 1,2,3 \) to see the black hole saddle points, with minimally obstructed boson/fermion cancelation by the phases of fugacities.

We discussed the asymptotic free energy in the octahedral ‘canonical chamber,’ defined by

\[
-2\pi < \text{Im}(\Delta_1 + \Delta_2 + \Delta_3) < 2\pi , \quad -2\pi < \text{Im}(\Delta_1 - \Delta_2 - \Delta_3) < 2\pi , \quad -2\pi < \text{Im}(\Delta_1 + \Delta_2 - \Delta_3) < 2\pi , \quad -2\pi < \text{Im}(\Delta_1 - \Delta_2 + \Delta_3) < 2\pi .
\]

(2.21)

Here, note that we should seek for an expression on one of the surfaces (2.2). For instance, let us consider \( \Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi i \). Since \( \omega_{1,2} \) are very small in our scaling limit, our hypersurface is very close to the right boundary of the first inequality, \( \text{Im}(\Delta_1 + \Delta_2 + \Delta_3) = 2\pi \). Whether one is within the octahedral chamber or not will depend on the small imaginary parts of \( \omega_i \)'s. So one may wonder if the expression (2.20) can be used or not. This issue does not matter, as (2.18) is continuous across \( \Delta_1 + \Delta_2 + \Delta_3 = 2\pi i \). To see this, note that one uses

\[
\text{Li}_3(-e^x) - \text{Li}_3(-e^{-x}) = -\frac{(x - 2\pi i)^3}{6} - \frac{\pi^2 (x - 2\pi i)}{6} ,
\]

(2.22)

outside the boundary, instead of (2.16), where \( x = \frac{\Delta_1 + \Delta_2 + \Delta_3}{2} \). However, the differences between the right hand sides of (2.22) and (2.16) is \( \pi i (x - \pi i)^2 \), being continuous and differentiable at \( x = \pi i \). We shall therefore use (2.20) at the surface (2.2).

Note that we used large \( N \) limit very trivially so far, just to ignore the Cartans. We basically relied on \( |\omega_i| \ll 1 \) to approximate the calculations. This is similar to the Cardy limit of 2d
QFT’s describing black holes or strings. There, central charge $c$ is kept fixed while the chemical potential $\tau$ conjugate to the left Hamiltonian is taken small. However, the entropy in the Cardy limit is useful to study black holes with large $c$, sometimes beyond the Cardy regime. To derive the true large $N$ free energy in the non-Cardy regime, one should consider the large $N$ saddle point approximation of $\alpha_a$ integrals, at finite $\Delta_I, \omega_i$. As we explained above, we expect a more complicated saddle point. Also, we are not sure how the graviton phase will get converted to the black hole phase as we change chemical potentials. In section 3, in the Macdonald limit, we find that (2.20) may not be true in general. However, still there might be other regime in which (2.20) is true, which we shall partly probe in the Macdonald limit. With this in mind, in section 2.3, we shall also explore the ‘thermodynamics phenomenology’ of (2.20) beyond Cardy limit, especially pointing out the existence of a Hawking-Page transition of this free energy.

So far, we took the limit $\beta \to 0$ first, having in mind imposing the index condition (2.2) later. We think this is completely fine, but some people might think that this way of thinking is dangerous. Appreciating possible worries, we start from the index given by [7] and rederive (2.20) at (2.2). A direct consideration of the index will also give interesting lessons beyond the Cardy limit, in the Macdonald limit [21]. Let us insert the following shifted values to the chemical potentials in (2.1),

$$\Delta_I \to \Delta_I - \beta, \quad \omega_i \to \omega_i - \beta,$$

(2.23)

after which the partition function is given by

$$Z(\beta, \Delta_I, \omega_i) = \text{Tr} \left[ e^{-\beta(E-\sum_I Q_I-\sum_i J_i)} e^{-\Delta_I Q_I-\omega_i J_i} \right].$$

(2.24)

Now imposing the condition $\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi i$, the measure in the trace commutes with the supercharge $Q^{++}, S_{++}$, at any value of $\beta$. We take $\beta \to \infty$ to suppress the contributions from all non-BPS letters. Let us redefine one of the chemical potentials, say $\Delta_1$, so that the index condition becomes

$$\Delta_1 + \Delta_2 + \Delta_3 = \omega_1 + \omega_2.$$

(2.25)

Then, the shift by $2\pi i$ generates extra $e^{-2\pi i Q_1} = (-1)^F$ in the trace formula (2.24), making it a manifest index. (This redefinition can be made with any one of the five chemical potentials.) After this redefinition, and taking $\beta \to \infty$ in (2.5), one obtains [7]

$$Z = \frac{1}{N!} \int \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \prod_{a<b} \left( 2 \sin \frac{\alpha_{ab}}{2} \right)^2 PE \left[ \left( 1 - \frac{\prod_{I=1}^3 (1 - t^3 y)}{(1 - t^3 y) (1 - t^3 y)} \right) \sum_{a,b=1}^N e^{i\alpha_{ab}} \right],$$

(2.26)

where $v_i$’s satisfying $v_1 v_2 v_3 = 1$ are the fugacities for $SU(3) \subset SO(6)$ part of R-symmetry. The parameters $t, v_i, y$ are related to our parameters in (2.5) by $(e^{-\omega_1}, e^{-\omega_2}, e^{-\Delta_1}) = (t^3 y, t^3 y, t^3 y)$, manifestly satisfying (2.25). This is rewritten as

$$Z = \frac{1}{N!} \int \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \prod_{a<b} \left( 2 \sin \frac{\alpha_{ab}}{2} \right)^2 \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \frac{\prod_{I=1}^3 2 \sinh \frac{n\Delta_I}{2} \cdot 2 \sinh \frac{n\omega_i}{2}}{2 \sinh \frac{n\omega_i}{2}} \right) \sum_{a,b=1}^N e^{i\alpha_{ab}} \right].$$

(2.27)
We again take $|\omega_i| \ll 1$, keeping them complex with $\text{Re}(\omega_{1,2}) > 0$. Had we been taking this limit with real positive $\Delta_i$’s, which is the canonical range for the chemical potentials, $\Delta_i$’s should also vanish at order $\mathcal{O}(\omega_{1,2})$ due to the relation (2.25). This will make the free energy to be small, $\sim \frac{\Delta^3}{\omega^2} \ll 1$, making the index uninteresting. However, we keep finite imaginary parts of $\Delta_i$’s while taking the limit $\omega_{1,2} \to 0$. Physically, we take advantage of the possibility of tuning the phases of bosonic/fermionic terms to maximally obstruct their cancelations. The asymptotic limit of (2.25) is $\Delta_1 + \Delta_2 + \Delta_3 \approx 0$, so we take all $\Delta_i$’s to be purely imaginary whose sum is zero, and continue back to complex numbers later. The details of the approximation is the same as we presented above. Following very similar procedures, again taking out the Cartan parts and ignoring them, one obtains

$$Z \sim \frac{1}{N!} \int \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \exp \left[ -\frac{1}{\omega_1\omega_2} \sum_{s_1,s_2,s_3=+1} \sum_{a \neq b} \left( \text{Li}_3 \left( e^{\frac{s_3}{2} + i\alpha_{ab}} \right) - \text{Li}_3 \left( e^{-\frac{s_3}{2} - i\alpha_{ab}} \right) \right) \right].$$  \tag{2.28}

Here, note that $\text{Li}_3(e^x) - \text{Li}_3(e^{-x}) = -\left(\frac{2\pi i}{3}\right) B_3 \left( \frac{1}{2\pi i} \right)$ for $\text{Re}(x) \geq 0$ and $0 < \text{Im}(x) < 2\pi$, with $B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x$. For $2\pi p < \text{Im}(x) < 2\pi(p+1)$ with an integer $p$, one finds

$$\text{Li}_3(e^x) - \text{Li}_3(e^{-x}) = -\left(\frac{2\pi i}{3}\right) B_3 \left( \frac{x}{2\pi i} - p \right) = -\frac{(x-2\pi ip)^3}{6} + \frac{\pi i(x-2\pi ip)^2}{2} + \frac{\pi^2(x-2\pi ip)}{3}. \tag{2.29}$$

When the arguments are pure phase, one finds

$$\text{Li}_3(e^{ix}) - \text{Li}_3(e^{-ix}) = i \left[ \frac{(x-2\pi p)^3}{6} - \frac{\pi(x-2\pi p)^2}{2} + \frac{\pi^2(x-2\pi p)}{3} \right] \equiv if(x). \tag{2.30}$$

in the interval $x \in (2\pi p, 2\pi(p+1))$. $f$, defined piecewise as above, is an odd function. Inserting $\Delta_I = i T_I$ with real $T_I$’s, one obtains

$$Z = \frac{1}{N!} \int \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \exp \left[ -\frac{i}{\omega_1\omega_2} \sum_{s_1,s_2,s_3=+1} \sum_{a \neq b} f \left( \frac{s \cdot T}{2} + \alpha_{ab} \right) \right]. \tag{2.31}

Using $\sum_I T_I = 0$, one can easily notice that

$$\mathcal{F}(\alpha_a) \equiv \sum_{s_1,s_2,s_3=+1} \sum_{a \neq b} f \left( \frac{s \cdot T}{2} + \alpha_{ab} \right) = \sum_{a \neq b} \left( f(\alpha_{ab}) + \sum_{I=1}^3 f(T_I + \alpha_{ab}) \right) = \sum_{a \neq b} \sum_{I=1}^3 f(T_I + \alpha_{ab}). \tag{2.32}

At the last step, we used the fact that $f$ is an odd function to set $f(\alpha_{ab}) + f(\alpha_{ba}) = 0$. Since $\mathcal{F}(\alpha_a)$ is an even function in all $\alpha_{ab}$’s, its derivatives with respect to all $\alpha_a$’s vanish at the ‘completely deconfining configuration’ $\alpha_1 = \alpha_2 = \cdots = \alpha_N$. This can be easily seen as follows:

$$\mathcal{F}(\alpha_{\neq a} = k, \alpha_a = k + \epsilon) = \sum_{I=1}^3 \left( (N-1) \left( f(T_I + \epsilon) + f(T_I - \epsilon) \right) + (N^2 - 3N + 2) f(T_I) \right)$$

$$\Rightarrow \frac{\partial}{\partial \alpha_a} \mathcal{F}(\alpha_1, \cdots, \alpha_N) \bigg|_{\alpha_1 = \cdots = \alpha_N = k} = 0. \tag{2.33}$$
Hence, the ‘completely deconfining configuration’ \( \alpha_1 = \alpha_2 = \cdots = \alpha_N \) is a saddle point of the function \( \mathcal{F}(\alpha_i) \) at large \( \frac{1}{\omega_1 \omega_2} \). Again assuming the dominance of this saddle point, one can asymptotically evaluate the integral \( (2.31) \) by the saddle point method as

\[
\log Z \sim -\frac{i}{\omega_1 \omega_2} \mathcal{F}(\alpha_1 = \alpha_2 = \cdots = \alpha_N) = -\frac{iN^2}{\omega_1 \omega_2} \sum_{l=1}^{3} f(T_l). \tag{2.34}
\]

Without loss of generality, we now assume that

\[ 2\pi p_I < T_I < 2\pi(p_I + 1), \quad p_I \in \mathbb{Z}. \tag{2.35} \]

Since \( \sum_I T_I = 0 \) for the index, we find a constraint on \( \sum_I p_I \) as

\[
2\pi \sum_{l=1}^{3} p_I < \sum_{l=1}^{3} T_I = 0 < 2\pi(\sum_{l=1}^{3} p_I + 3) \Rightarrow -3 < \sum_{l=1}^{3} p_I < 0 \Rightarrow \sum_{l=1}^{3} p_I = -1, -2. \tag{2.36}
\]

When \( \sum_I p_I = -1 \), one can show that

\[
\sum_{l=1}^{3} f(T_l) = \sum_{l=1}^{3} f(T_l - 2\pi p_I) = \frac{1}{2}(T_1 - 2\pi p_1)(T_2 - 2\pi p_2)(T_3 - 2\pi p_3). \tag{2.37}
\]

On the other hand, when \( \sum_I p_I = -2 \), one will find that

\[
\sum_{l=1}^{3} f(T_l) = \sum_{l=1}^{3} f(T_l - 2\pi p_I) = \frac{1}{2}(T_1 + 2\pi(1 + p_2 + p_3))(T_2 + 2\pi(1 + p_3 + p_1))(T_3 + 2\pi(1 + p_1 + p_2)). \tag{2.38}
\]

These two results can be all expressed as the following equation,

\[
\sum_{l=1}^{3} f(T_l) = \frac{1}{2}(T_1 + 2\pi(1 + p_2 + p_3))(T_2 + 2\pi(1 + p_3 + p_1))(T_3 + 2\pi(1 + p_1 + p_2)). \tag{2.39}
\]

Therefore, one obtains

\[
\log Z \sim -\frac{iN^2}{2\omega_1 \omega_2} (T_1 + 2\pi(1 + p_2 + p_3))(T_2 + 2\pi(1 + p_3 + p_1))(T_3 + 2\pi(1 + p_1 + p_2)). \tag{2.40}
\]

Converting back to \( \Delta_I = iT_I \), one obtains

\[
\log Z \sim \frac{N^2}{2\omega_1 \omega_2} \prod_{l=1}^{3} (\Delta_I + 2\pi in_I). \tag{2.41}
\]

where \( n_1 \equiv 1 + p_2 + p_3, n_2 \equiv 1 + p_3 + p_1, n_3 \equiv 1 + p_1 + p_2 \), satisfying \( \sum_{l=1}^{3} n_I = \pm 1 \). This agrees with the previous analysis, supposing that \( \Delta_1 + \Delta_2 + \Delta_3 \) there and here are related by a shift of \( 2\pi i \) (mod \( 4\pi i \)).
2.2 Background field analysis on $S^3$

We consider an alternative approach to compute the asymptotic free energy of the index. The chemical potentials $\beta, \omega_i$ are reflected in the background metric of $S^3 \times S^1$ as

$$ds^2 = r^2 \left[ d\theta^2 + \sum_{i=1}^2 n_i^2 \left( d\phi_i - \frac{i\omega_i}{\beta} d\tau \right)^2 \right] + d\tau^2, \quad (2.42)$$

where $(n_1, n_2) = (\cos \theta, \sin \theta)$, $0 \leq \theta \leq \frac{\pi}{2}$. The Euclidean time $\tau$ has period $\tau \sim \tau + \beta$, and we restored the radius $r$ of $S^3$. $\Delta_I$ are encoded in the background $U(1)^3 \subset SO(6)$ gauge fields

$$A^I = - \frac{i\Delta_I}{\beta} d\tau. \quad (2.43)$$

The partition function is given by a path integral over the $\mathcal{N} = 4$ Yang-Mills fields at coupling constant $g_{YM}$, coupled to these background fields in a canonical manner. Again having in mind imposing (2.2) to get the index, we take $\beta \to 0^+$. Very naively, one might think that a Kaluza-Klein reduction to $S^3$ would be possible, integrating out heavy KK fields, because the circle size $\beta$ is small. If one can integrate out the heavy fields, they will contribute to an effective action of the background fields, arranged in the derivative expansion which is a series in small $\beta$. This will turn out to be a much subtler issue, because $\beta^{-1}$ appears in other background fields. Indeed, naively doing the KK reduction, one would see shortly that the 3d metric, dilaton and $U(1)^3$ fields all see inverse powers of $\beta$. Still, when $\omega_{1,2} \ll 1$, we will show that the KK fields can be integrated out, whose effect will be arranged in a derivative expansion. The expansion will be a series in small $\beta, \omega_{1,2}$, whose leading terms will be given by Chern-Simons terms. The effect of 3d zero modes is also expected to be subleading in our model. The analysis is similar to [17], except that our setting is subtler with new aspects.

Having these in mind, we arrange the 4d background fields as 3d background fields. To this end, we rewrite (2.42) in terms of 3d metric, gravi-photon $a$, and the dilaton $\Phi$ as

$$ds_4^2 = r^2 \left[ d\theta^2 + \sum_i n_i^2 d\phi_i^2 + \frac{r^2 (\sum_i \omega_i n_i^2 d\phi_i)^2}{\beta^2 (1 - r^2 \sum_i \frac{n_i^2 \omega_i^2}{\beta^2})} \right] + e^{-2\Phi} (d\tau + a)^2 \equiv ds_3^2 + e^{-2\Phi} (d\tau + a)^2$$

$$e^{-2\Phi} = 1 - r^2 \sum_i \frac{n_i^2 \omega_i^2}{\beta^2}, \quad a = - i \frac{r^2 \sum_i \omega_i n_i^2 d\phi_i}{\beta (1 - r^2 \sum_i \frac{n_i^2 \omega_i^2}{\beta^2})}. \quad (2.44)$$

The 4d $U(1)^3$ background fields $A^I$ are arranged to 3d gauge field $A^I$ and the scalar $A_4^I$ as $A^I = A_4^I (d\tau + a) + A^I$, where

$$A_4^I = - \frac{i\Delta_I}{\beta} \equiv \frac{\alpha^I}{\beta}, \quad A^I = - A_4^I a. \quad (2.45)$$

We take $\beta$ to be the smallest variable, eventually intending to take the limit $\beta \to 0^+$. $\omega_i \ll 1$ are also small, but still satisfying $\frac{\beta}{r\omega_i} \ll 1$. One might worry that some background fields may
behave badly due to the factor $1 - r^2 \sum_i \frac{n_i^2 \omega_i^2}{\beta^2}$ in denominators. We temporarily circumvent this issue by taking $\omega_i$ to be complex and generic, evading the poles. Physically, this has to do with the fact that non-BPS derivatives’ effect is present before imposing (2.2).

We first consider the limiting behaviors of the 3d background fields for $\frac{\beta}{r} \ll |\omega_i| \ll 1$:

$$ds_3^2 \sim r^2 \left[ ds^2(S^3_{\text{round}}) - \frac{\left( \sum_i \omega_i n_i^2 d\phi_i \right)^2}{\sum_i n_i^2 \omega_i^2} + \cdots \right]$$

$$\beta^2 e^{-2\Phi} \sim -r^2 \sum_i n_i^2 \omega_i^2 + \cdots, \quad \frac{a}{\beta} \sim \frac{i \left( \sum_i \omega_i n_i^2 d\phi_i \right)}{2 \sum_i n_i^2 \omega_i^2} + \cdots . \quad (2.46)$$

The omitted terms $\cdots$ are suppressed by positive powers of $\frac{\beta}{r\omega_i} \ll 1$. Note that in the 3d metric, one has a canonical round sphere metric, accompanied by the second term which is an $O(1)$ negative length element along one direction. For instance, if $\omega_1 = \omega_2 \equiv \omega$, this direction is the Hopf fiber of $S^3$. Along this direction, leading $O(1)$ length elements cancel and its $[\text{length}]^2$ becomes smaller, at a positive power in $\frac{\beta}{\omega}$. This is one reason why a naive KK reduction becomes subtle in our case. The dilaton field $\beta^2 e^{-2\Phi}$ for the $[\text{circumference}]^2$ of temporal circle is suppressed to be small $|\omega_i| \ll 1$, which is an intuitive reason why we should also keep $\omega_i$ small to trust the derivative expansion. The 3d background fields are highly singular (e.g. $\omega, \beta$ dependence), presumably having short wavelength components on $S^3$, so that one might wonder if the whole spirit of using derivative expansion is relevant or not. In general, using these fields will be highly problematic in the general effective field theory. For instance, if one wishes to make variation of this effective action in background fields to generate correlation functions, this probably might be tricky. However, our strategy here is very practical, having in mind using this EFT just for our particular background. In other words, we use it just as a way of expressing the series expansion of a particular observable $\log Z$ in $\beta, \omega_1, \omega_2$. So no matter how singular the fields may look, we just care about whether the actual values of terms after spatial integrals are sequentially suppressed as an infinite series. We will show (more precisely, strongly illustrate) that this is indeed true.

In this background, we consider the path integral of 4d $\mathcal{N} = 4$ Yang-Mills theory. We formally decompose the 4d dynamical fields into 3d ‘zero modes’ and ‘KK fields,’ depending on the momentum mode on $S^1$. We schematically call the zero modes $\Phi_L$ and KK modes $\Phi_H$, where $L/H$ stand for ‘light/heavy.’ $\Phi_H$ couples to the background field $a$, while $\Phi_L$ does not. The path integral is done by integrating over $\Phi_H$ at fixed $\Phi_L$, and then integrating over $\Phi_L$.

We discuss the structure of the path integral over $\Phi_H$, at fixed $\Phi_L$. In our scaling limit of small $S^1$ radius, the path integral over $\Phi_H$ gives an effective action that depends only on the 3d background fields, but not on $\Phi_L$ which are held fixed for the moment. To see this, consider the schematic structure of the 3d action for $\Phi_H$. It takes the form of

$$\mathcal{L} \sim \Phi_H (\partial^2 + M_{KK}^2) \Phi_H + g_{5\text{d}}^2 V(\Phi_L, \Phi_H) \quad (2.47)$$
where $V$ denotes a potential quartic in $\Phi_H, \Phi_L$, with order 1 coefficients. Here we consider the case in which $\Phi_H, \Phi_L$ are bosonic, for simplicity. Both $M_{KK}$ and $g_{3d}^2$ have dimension of mass, proportional to the inverse-radius of the temporal circle $\sim \frac{1}{r\omega}$ (where $\omega \sim \omega_{1,2}$.) The solution to $\Phi_H$ at given $\Phi_L$ is schematically given by $\Phi_H \sim \frac{g_{3d}^2}{\partial^2 + M_{KK}^2} \partial \Phi_H V$. The propagator factor scales like $g_{3d}^2 \sim r\omega$, which suppresses the $\Phi_H$ tadpole and fluctuations depending on $\Phi_L$. $\Phi_H$’s path integral is effectively Gaussian, depending on background fields only. So after integrating out $\Phi_H$, $Z$ consists of two factors: one given by the 3d background fields, and another given by the path integral of ‘zero modes’ $\Phi_L$ canonically coupled to 3d background fields, obtained by classical dimensional reduction of 4d $\mathcal{N} = 4$ Yang-Mills theory. In the latter sector, the dilaton appears as the 3d coupling constant (which may depend on spatial coordinate if $\omega_1 \neq \omega_2$), while the gravi-photon $\beta^{-1} a$ does not couple to the classical 3d Yang-Mills.

We first consider the factor coming from the path integral over $\Phi_L$. It consists of the fields of 3d maximal super-Yang-Mills, whose action is deformed to be less supersymmetric by various parameters. Here, we simply discuss how its contribution to $\log Z$ will depend on various parameters. The 3d effective coupling is given by $g_{3d}^2 \sim \frac{1}{r\omega}$. The 3d metric consists of 2d base whose length scale is $r$, and a fiber whose length scale is $\frac{\beta}{\omega} \ll r$. As we shall see below from background effective actions (which is also obvious from BPS kinematics), the leading free energy will be of order $\sim \frac{\beta^0}{\omega^2}$ at $\frac{1}{r} \ll \omega \ll 1$. We can argue that the path integral of $\Phi_L$ will yield much smaller terms than this. Suppose otherwise, and the $\Phi_L$’s path integral contributes a term at this order. Then, the divergent $\omega^{-2}$ part would come either from positive power in the 3d gauge coupling $g_{3d}^2 \sim \frac{g_{YM}^2}{r\omega}$, or positive power in the Hopf fiber radius $\sim (\beta \omega^{-1})^\#$. But acquiring this factor from the Hopf fiber radius is accompanied by a positive power in $\beta$, which is subleading. So $\beta^0 \omega^{-2}$ dependence would come from the divergent 3d coupling, $g_{3d}^2 \sim \omega^{-1}$.

However, it is also hard to imagine (probably inconsistent) that a 3d QFT partition function diverges as the coupling grows, as the 3d QFT seems to be perfectly well defined. The only way in which we can imagine a divergent dependence on large $g_{3d}$ is when the observable suffers from infrared divergence, since $g_{3d} \rightarrow \infty$ is a sort of IR limit in 3d. More concretely, the partition function of 3d maximal SYM on $S^3$ is well known to have an IR divergence [26]. As studied in [17, 25], this is due to the $N$ gauge holonomies of $U(N)$ on $S^3$ being non-compact in the small circle limit. At small but finite circle radius, $\sim r\omega$, the holonomies have period given by $\sim \frac{1}{r\omega}$, thus providing an IR cutoff. This would yield a factor of $\sim \omega^{-N}$ to $Z$, contributing at a subleading order $\sim N \log \omega$ to the free energy. Thus, we expect the divergent leading part $\propto \beta^0 \omega^{-2}$ of the net free energy to be unaffected by the 3d dynamical fields.

So it suffices to consider the effect of integrating out the ‘KK fields’ $\Phi_H$, yielding an effective

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3We expect a caveat when $\Phi_L$ has zero modes held at large value without a potential cost, making $\partial \Phi_H V$ large. There are two types of such modes, again depending on the IR divergent behaviors of $Z_{S^3}$ for $\Phi_L$ [25]. In our 4d $U(N)$ theory, or 6d (2,0) theory for $N$ M5-branes, we assume the absence of such dangerous modes. See the next two paragraphs for more discussions.
action of $g_{\mu\nu}$, $a_\mu$, $\Phi$, $A_I^\mu$, $A_I^4$. There are infinitely many terms in this effective action, arranged in a derivative expansion, whose coefficients are mostly unknown. At generic points of the background fields, before imposing the BPS index constraint (2.2), all fermions of the 4d theory will go to $\Phi_H$, due to the anti-periodic boundary conditions. At (2.2), some fermion modes may be massless. Across this surface, as we shall see, these transiently massless fermions at (2.2) will simply change some Chern-Simons coefficients, without further effects on the effective action. Below, we will show that: (1) the derivative expansion is arranged in a series of $\beta, \omega_1, \omega_2$; (2) the leading terms are at order $\beta \omega_1 \omega_2$; (3) the Chern-Simons coefficients can be determined either from the free 4d QFT, or by an anomaly consideration. We shall discuss these issues in the order of (3) $\rightarrow$ (2) $\rightarrow$ (1).

We first discuss possible Chern-Simons terms of $A_I$, $a$. (One might also think of the gravitational Chern-Simons term $\sim \omega \wedge R$. We think its coefficient is zero, but anyway it will be subleading in our scaling limit, as illustrated below.) There can be standard gauge-invariant Chern-Simons terms of the forms [27, 17]

$$\beta^{-2} \int a \wedge da, \quad \beta^{-1} \int A^I \wedge da, \quad \int A^I \wedge dA^I,$$

whose coefficients are dimensionless and quantized. There can also be gauge non-invariant Chern-Simons terms which are needed for anomaly matching [27, 17]. Since their coefficients are all quantized, either from gauge invariance or anomaly matching, one can determine them by integrating out KK fermions of the 4d QFT at weak coupling.

We follow [17] to compute these coefficients for $U(1)^3 \subset SO(6)$ times the gravi-photon $U(1)$. There are four Weyl fermions $\Psi_\alpha^{Q_1,Q_2,Q_3}$, where $\alpha = \pm \frac{1}{2}$, and with $(Q_1, Q_2, Q_3) = (-, +, +), (+, -+, (+++, -), (-, --, -)$. $\pm$’s for $Q_I$’s denote $\pm \frac{1}{2}$. The fermions with anti-periodic boundary conditions are labeled by the Kaluza-Klein level $n \in \mathbb{Z} + \frac{1}{2}$. The contributions to the Chern-Simons terms from the $n$’th KK modes are given by [17]

$$S^{(n)}_{CS} = \frac{iN^2}{8\pi} \sum_{(Q_1,Q_2,Q_3)} \text{sgn} \left( n - \frac{\beta}{2\pi} A_I^4 Q_I \right) \int_{S^3} \left( Q_1 Q_J A^I \wedge dA^J + 2Q_I \frac{2\pi n}{\beta} A^I \wedge da + \frac{(2\pi n)^2}{\beta^2} a \wedge da \right).$$

(2.49)

There are infinitely many contributions from the tower of KK modes, which should be regularized. Following [17], we sum over all $n \in \mathbb{Z} + \frac{1}{2}$ using the zeta function regularization.\footnote{The overall sign is chosen to be consistent with our chirality/convention.} To

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\footnote{There are various proposals for regularizing $Z[S^3 \times S^1]$ [17, 28, 29, 30], concerning the supersymmetric Casimir energy [31, 32, 33]. Employing the regularization of [17], we obtain a free energy unspoiled by the formal Casimir energy factor of [32]. Although we have no clear reasoning for this, note that Casimir energy is very sensitive to regularization, while the integral spectrum part should be more robust. Especially, our setup respects all the periodicities of holonomies, which is a property of the spectral part of log $Z$ but not of the Casimir energy [32]. So our regularization appears to disallow a room for vacuum energy factor like [32].}
start with, when \(-\frac{1}{2} < \mu \equiv \frac{\beta}{2\pi} Q_I A_I^4 < \frac{1}{2}\) for a fermion mode with given \(Q_I\), one obtains

\[
\sum_n \text{sgn}(n - \mu) \sim 2\mu, \quad \sum_n \text{sgn}(n - \mu)n \sim \mu^2 + \frac{1}{12}, \quad \sum_n \text{sgn}(n - \mu)n^2 \sim \frac{2}{3}\mu^3.
\] (2.50)

If \(A_I^4\)’s are chosen so that \(\frac{\beta}{2\pi} Q_I A_I^4\) is in the range \((-\frac{1}{2}, \frac{1}{2})\) for all possible \(Q_I\)’s, one obtains

\[
S_{CS} = \frac{iN^2}{4\pi} \sum_{(Q_1,Q_2,Q_3)} \int \left[ \frac{\beta}{2\pi} Q_I Q_J Q_K A_I^4 A_J^4 \wedge dA^K + \frac{2\pi}{\beta} Q_I \left( Q_J Q_K \frac{\beta^2}{(2\pi)^2} A_I^4 A_J^4 + \frac{1}{12} \right) A^I \wedge da \\
+ \frac{\beta}{3} \cdot \frac{2\pi}{\beta} Q_I Q_J Q_K A_I^4 A_J^4 A_K^4 a \wedge da \right] .
\] (2.51)

Here, note that

\[
\sum_{(Q_1,Q_2,Q_3)} Q_I Q_J Q_K = -\frac{1}{2} C_{IJK}, \quad \sum_{(Q_1,Q_2,Q_3)} Q_I = 0 ,
\] (2.52)

where \(C_{IJK}\) is symmetric in \(I,J,K\), \(C_{123} = 1\), and \(C_{IJK} = 0\) if any two of \(I,J,K\) are same. (These are the anomaly coefficients of \(U(1)^3\).) Using these facts, one obtains

\[
S_{CS} = -\frac{iN^2}{8\pi} \cdot \frac{\beta}{2\pi} \int_{S^3} C_{IJK} \left( A_I^4 A_J^4 \wedge dA^K + A_I^4 A_J^4 A_K^4 \wedge da + \frac{1}{3} A_I^4 A_J^4 A_K^4 a \wedge da \right) .
\] (2.53)

Note that the gauge invariant Chern-Simons terms (2.48) are all zero in this chamber, with \(-\frac{1}{2} \leq \frac{\beta}{4\pi}(\pm A_1^4 \pm A_2^4 \pm A_3^4) \leq \frac{1}{2}\) for all four possible sign choices satisfying \(± \cdot ± \cdot ± = -1\).

In general chambers of \(A_I^4\), one takes

\[
-\frac{1}{2} + pQ \leq \frac{\beta}{2\pi} Q_I A_I^4 \leq \frac{1}{2} + pQ ,
\] (2.54)

where \(Q\) runs over 4 possible cases, with integral \(pQ\)’s. In this chamber, the regularized sums are now given by

\[
\sum_n \text{sgn}(n - \mu) = \sum_{n'} \text{sgn}(n' - \mu') \sim 2(\mu - p)
\] (2.55)

\[
\sum_n \text{sgn}(n - \mu)n = \sum_{n'} \text{sgn}(n' - \mu')(n' + p) \sim (\mu - p)^2 + \frac{1}{12} + 2p(\mu - p)
\]

\[
\sum_n \text{sgn}(n - \mu)n^2 = \sum_{n'} \text{sgn}(n' - \mu'((n')^2 + 2pn' + p^2) \sim \frac{2}{3}(\mu - p)^3 + 2p(\mu - p)^2 + \frac{p}{6} + 2p^2(\mu - p) ,
\]

where \(n' = n - p, \mu' = \mu - p\). In this chamber, one obtains

\[
S_{CS} = \frac{iN^2}{4\pi} \cdot \frac{\beta}{2\pi} \sum_{(Q_1,Q_2,Q_3)} \int \left[ (Q_I A_I^4 - \frac{2\pi pQ}{\beta}) Q_J Q_K A_I^4 \wedge dA^K \\
+ Q_I \left( Q \cdot A_4 - \frac{2\pi pQ}{\beta} \right)^2 + \frac{1}{12} \cdot \frac{(2\pi)^2}{\beta^2} + 2pQ \cdot \frac{2\pi}{\beta} \left( Q \cdot A_4 - \frac{2\pi pQ}{\beta} \right) \right) A^I \wedge da
\] (2.56)
We shall mostly work with the result \( (2.53) \) in the canonical chamber.

One can also determine \( (2.53) \) by just knowing 't Hooft anomalies and discrete symmetries. Firstly, the gauge non-invariant terms \( (2.53) \) are completely fixed in \([27, 17]\), by demanding that its gauge variation yields the expected 't Hooft anomaly of the 4d \( U(1)^3 \subset SO(6)_R \) symmetry. (More precisely, \( (2.53) \) matches the covariant anomalies.) To complete the argument, we discuss why gauge invariant CS terms \( (2.48) \) should vanish. Firstly, \( a \wedge da \) is forbidden by the 3d parity after \( S^1 \) reduction, which is a symmetry of the mother 4d theory if an object is blind to \( SO(6)_R \), such as \( a \wedge da \). Similarly, \( \mathcal{A}^I \wedge d\mathcal{A}^I \) with a given \( I \) is forbidden since the mother 4d \( \mathcal{N} = 4 \) theory is invariant under parity with sign flip of odd number of \( \mathcal{A}^I \) fields. The latter flip is charge conjugation, flipping \( 4 \leftrightarrow \bar{4} \). The remaining gauge invariant CS terms are forbidden simply from the Weyl symmetry of \( SO(6) \). We consider the Weyl reflections which reflects two of the three \( \mathcal{A}^I \)'s, leaving one invariant. This reflection also acts on \( A^I_4 \). But they cannot affect the gauge invariant CS terms, so in the canonical chamber which is left invariant under these reflections, the gauge invariant CS terms should respect this symmetry. For \( \mathcal{A}^I \wedge da \) with any given \( I \), a reflection which flips \( I \) and another \( J(\neq I) \) flips sign of this term, forbidding its generation. Similarly, for \( \mathcal{A}^I \wedge d\mathcal{A}^J \) at given pair \( I \neq J \), reflection of \( I \) and \( K(\neq I, J) \) forbids its generation. This completes a symmetry-based argument for \( (2.53) \). Such an approach may be useful for some non-Lagrangian theories, if there are enough discrete symmetries. In section 4, we shall make similar studies with 6d \((2, 0)\) theory, although it appears that such intrinsic arguments are less predictive there.

We now evaluate these CS terms for our backgroud fields, in the canonical chamber. We first consider the background R-symmetry fields \( (2.45) \) with real \( \alpha' = -i\Delta' \), and later continue to complex \( \Delta' \). Also, we keep \( \epsilon_i \equiv -i\omega_i \) real for a moment, and later continue to complex \( \omega_i \). \( (2.53) \) is given by

\[
S_{CS} = -\frac{iN^2}{48\pi^2\beta^2}C_{IJK}\alpha^I\alpha^J\alpha^K \int_{S^3} a \wedge da . \tag{2.57}
\]

Inserting \( a \) in \( (2.44) \), one finds

\[
\int a \wedge da = \frac{r^4}{\beta^2} \int \frac{\epsilon_i n_i^2 d\phi_i \wedge \epsilon_j d(n_j^2) \wedge d\phi_j}{\left(1 + \frac{r^2 n_i^2 \epsilon_i^2}{\beta^2}\right)^2} = \frac{(2\pi)^2 r^4 \epsilon_1 \epsilon_2}{\beta^2} \int \frac{y dx - x dy}{\left(1 + \frac{r^2(\epsilon_1^2 + \epsilon_2^2)}{\beta^2}\right)} \tag{2.58}
\]

\[
= \frac{(2\pi)^2 r^4 \epsilon_1 \epsilon_2}{\beta^2} \int_0^1 dx \left( \frac{x}{1 + \frac{r^2}{\beta^2}(1 + 2\epsilon_1 \epsilon_2 - 1)} \right) = \frac{(2\pi)^2 r^4 \epsilon_1 \epsilon_2}{\beta^2(1 + \frac{r^2}{\beta^2})(1 + \frac{r^2}{\beta^2})},
\]

where \( x \equiv n_1^2, y \equiv n_2^2 = 1 - x \). So one finds

\[
S_{CS} = -\frac{iN^2 r^4 \epsilon_1 \epsilon_2}{12\beta^4(1 + \frac{r^2}{\beta^2})(1 + \frac{r^2}{\beta^2})}C_{IJK}\alpha^I\alpha^J\alpha^K \tag{2.59}
\]
in the canonical chamber. Inserting \( \alpha^I = -i \Delta^I \), \( \epsilon_i = -i \omega_i \) and taking \( \beta \to 0^+ \), one obtains
\[
S_{CS} \to -\frac{N^2 C_{IJK} \Delta^I \Delta^J \Delta^K}{12 \omega_1 \omega_2} = -\frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2 \omega_1 \omega_2}
\] (2.60)
in the canonical chamber. If \( S_{CS} \) is the dominant term in the effective action (which we will show shortly), this yields the asymptotic free energy by the relation \( Z \sim e^{-S_{CS}} \). So \( \log Z \sim -S_{CS} \) completely agrees with the free QFT analysis in the previous subsection. The extension of this result to different chambers also agrees with the result from free QFT.

Now to complete the analysis of the free energy, we show that all the other terms in the effective action are subleading in our scaling limit, suppressed by small \( \beta, \omega_{1,2} \). The background fields are the 3d metric \( g_{\mu\nu} \), dilaton \( \Phi \), graviphoton \( a_\mu \), gauge boson \( A^I_\mu \), and scalar \( A^I_\mu \). Greek indices run over the coordinates \( \{ \phi_1, \phi_2, \theta \} \), and small Latin indices used below will run over the locally flat coordinates \( \{1,2,3\} \). There are rich possibilities in constructing the effective action. However, many possible terms are eliminated by taking into account the actual background value (2.44) and (2.45). First, the Riemann curvature \( R_{\mu\nu\rho\sigma} \) has non-zero components only at \( \{\mu, \nu\} = \{\rho, \sigma\} \) or \( \{\mu, \nu\} \cap \{\rho, \sigma\} = \emptyset \). Second, the background value (2.44) and (2.45) depends only on the \( \theta \) coordinate, so that the field strengths \( F^0_{\mu\nu} \equiv \frac{1}{23}(\partial_\mu a_\nu - \partial_\nu a_\mu) \) and \( F^I_{\mu\nu} \equiv \frac{1}{2}(\partial_\mu A^I_\nu - \partial_\nu A^I_\mu) \) of the graviphoton \( a_\mu \) and gauge field \( A^I_\mu \) have non-zero components only at \( \{\mu, \nu\} \supset \emptyset \). For the same reason, the derivative of any scalar function of the background fields \( \partial_\mu f(\omega_{\rho}, \Phi, a_\mu, A^I_\mu, A^I_\mu) \) can have non-zero components only at \( \{\mu\} = \emptyset \). Third, the graviphoton \( a_\mu \) and gauge field \( A^I_\mu \) have non-zero components only at \( \{\mu\} \supset \emptyset \}. \) We will further assume that \( \omega_1 = \omega_2 = \omega \) for simplification, so that the dilaton \( \Phi \) becomes a constant.

Let us first examine the possible terms that involve the volume integral \( \int d^3x \sqrt{g} \) of gauge-invariant Lagrangian densities, formed by contracting tensors without \( \epsilon^{\mu\nu\rho} \). When we consider the scalar contraction between the curvature \( R_{\mu\nu\rho\sigma} \) and the field strength \( F^0_{\mu\nu} \) or \( F^I_{\mu\nu} \), only an even number of \( F^0_{\mu\nu} \) or \( F^I_{\mu\nu} \) can appear in the non-vanishing Lagrangian densities. It can be shown as follows: the scalar contraction of \( R_{\mu\nu\rho\sigma}, F^0_{\mu\nu}, F^I_{\mu\nu} \) can be encoded in the circular sequence of antisymmetric pairs of tensor indices \( [\alpha\beta][\gamma\delta] \cdots [\zeta\alpha] \), where adjacent indices in adjoining pairs are contracted to each other. We distinguish the curvature tensor indices by using capital letters. Then the contraction to a Lorentz scalar can be generally written as
\[
[a_{1,1}, b_{1,1}] \cdots [a_{1,n_1}, b_{1,n_1}] A_1 B_1 [a_{2,1}, b_{2,1}] \cdots [a_{2,n_2}, b_{2,n_2}] A_2 B_2 \cdots [a_{2j,1}, b_{2j,1}] \cdots [a_{2j,n_2}, b_{2j,n_2}] \quad \text{with} \quad \sum_{i=1}^{2j} n_i \in 2\mathbb{Z} + 1.
\]
The set of the field strength indices \( \{a_{k,1}, b_{k,n_k}\} \) in \( [a_{k,1}, b_{k,1}] \cdots [a_{k,n_k}, b_{k,n_k}] \) can only be either
\[
\{a_{k,1}, b_{k,n_k}\} = \begin{cases} \{\phi_1, \theta\} \text{ or } \{\phi_2, \theta\} & \text{if } n_k \in 2\mathbb{Z} + 1 \\ \{\theta\} \text{ or } \{\phi_1, \phi_2\} \text{ or } \{\phi_1\} \text{ or } \{\phi_2\} & \text{if } n_k \in 2\mathbb{Z}. \end{cases}
\] (2.61)
Collecting the sets of the curvature indices \( \{A_k, B_k\} \) for \( k = 1, \cdots, 2j \), there are always an odd number of \( \{\phi_1, \theta\} \) or \( \{\phi_2, \theta\} \) and an odd number of \( \{\phi_1, \phi_2\} \). Any complete pairings in this
collection have at least one pair between \( \{\phi_1, \phi_2\} \) and \( \{\phi_{1,2}, \theta\} \), so each term in the contraction refers to \( R_{\phi_1 \phi_2 \phi_3 \theta} = 0 \). This exhausts many possible terms in the effective action. Here we evaluate and list all non-vanishing terms which involve up to 4 derivatives: (Below we assume \( I, J, K, L \) run over \( 0, 1, 2, 3 \), and \( \Delta^0 \equiv -i \)).

\[
1 \frac{1}{(2\pi)^2} \int \beta^{-3} e^{3\Phi} \sqrt{g} = \frac{\beta r^3}{2(\beta^2 - r^2 \omega^2)^2} = \frac{\beta}{2r \omega^4} + O \left( \frac{\beta^3}{r^3 \omega^4} \right)
\]

\[
1 \frac{1}{(2\pi)^2} \int \beta^{-1} e^\Phi \sqrt{g} R_{ab}^{ab} = \frac{r(3\beta^3 - 4\beta r^2 \omega^2)}{(\beta^2 - r^2 \omega^2)^2} = -\frac{4\beta}{r \omega^2} + O \left( \frac{\beta^3}{r^3 \omega^4} \right)
\]

\[
1 \frac{1}{(2\pi)^2} \int \beta^{-1} e^{-\Phi} \sqrt{g} \mathcal{F}_{ab}^{Iab} \mathcal{F}_{ab}^{Jab} = \beta \Delta^I \Delta^J \frac{r^3 \omega^2}{(\beta^2 - r^2 \omega^2)^2} = \frac{\beta \Delta^I \Delta^J}{r \omega^2} + O \left( \frac{\beta^5}{r^5 \omega^4} \right)
\]

\[
1 \frac{1}{(2\pi)^2} \int \beta^{-1} e^{-3\Phi} \sqrt{g} (\nabla_c \mathcal{F}_{ab}^I)(\nabla^c \mathcal{F}_{ab}^{Jab}) = 2 \beta \Delta^I \Delta^J \frac{r^3 \omega^2}{(\beta^2 - r^2 \omega^2)^2} = \frac{2 \beta \Delta^I \Delta^J}{r \omega^2} + O \left( \frac{\beta^5}{r^5 \omega^4} \right)
\]

These terms are all much smaller than \( \text{[2.59]} \) in the scaling limit \( \beta/r \ll \omega \ll 1 \). Extrapolating a pattern from the above terms, an action made of \( n_1 \) curvature strengths, \( n_3 \) graviphoton field strengths, \( n_3 \) background \( U(1) \subset SO(6) \) field strengths, and \( n_4 \) derivatives should behave as

\[
\frac{\beta^{1+n_4} \Delta^{n_3}}{r^{1+n_4} \omega^{4-2n_1-n_2-n_3}} + O \left( \frac{\beta^{3+n_4}}{r^{3+n_4} \omega^{6-2n_1-n_2-n_3}} \right)
\]

which would be suppressed in the limit \( \beta/r \ll \omega \ll 1 \).

As a next step, we turn to the effective action that contains a totally antisymmetric tensor \( e^{\mu \nu \rho} \). This consists of Chern-Simons terms and those terms associated with a gauge invariant Lagrangian density. We can further distinguish the gauge non-invariant Chern-Simons terms from the gauge invariant ones. The gauge non-invariant Chern-Simons terms are entirely dictated by the chiral anomaly, so that no other terms than \( \text{[2.51]} \) can arise \( \text{[27, 17]} \). And also, the gauge invariant Chern-Simons terms displayed in \( \text{[2.48]} \) are already shown to be absent in
the canonical chamber. The gravitational Chern-Simons term \(\text{tr} (\omega \wedge R + \frac{2}{3} \omega \wedge \omega \wedge \omega)\), even if present, makes only a sub-dominant contribution in the limit \(\beta/r \ll \omega \ll 1\):

\[
\frac{1}{3!} \frac{1}{(2\pi)^2} \int e^{\mu \nu \rho} \left( \omega_{\mu}^{ab} R_{\nu \rho}^{ab} + \frac{2}{3} \omega_{\mu}^{ab} \omega_{\nu}^{bc} \omega_{\rho}^{ca} \right) = -\frac{4\beta^2}{r^2\omega^2} + O \left( \frac{\beta^4}{r^4\omega^4} \right). \tag{2.64}
\]

Other gauge invariant Lagrangian densities containing \(e^{\mu \nu \rho}\) are constrained by the symmetry-based argument, which was used to argue the gauge invariant CS terms \((2.48)\) are absent. Each allowed term should have odd numbers of three different \(U(1) \subset SO(6)\) field strengths \(F_{1,2,3}\). So even a minimal term of this sort has \(3\ U(1) \subset SO(6)\) field strengths coupled to one another. Some non-vanishing sample terms are evaluated and displayed below:

\[
\int \frac{\beta^6 e^{-6\Phi}}{3! (2\pi)^2} e^{\mu \nu \rho} \left( \nabla_\alpha F_{\rho \beta}^2 \right) F_{3 \alpha \sigma} F_{\sigma \beta} = -\frac{i\beta^2 r^2 \omega^4 \Delta_1 \Delta_2 \Delta_3}{3(\beta^2 - r^2 \omega^2)^2} = -\frac{i\beta^2 \Delta_1 \Delta_2 \Delta_3}{3r^2} + O \left( \frac{\beta^4}{r^4\omega^2} \right)
\]

\[
\int \frac{\beta^{10} e^{-10\Phi}}{3! (2\pi)^2} e^{\mu \nu \rho} \frac{F_{1 \mu \nu}}{\rho \beta} \frac{F_{1 \alpha \beta}}{\rho \sigma} \left( \nabla_\lambda F_{\rho \sigma}^2 \right) F_{3 \lambda \delta} F_{\delta \sigma} = -\frac{i\beta^2 r^2 \omega^6 \Delta_1 \Delta_2 \Delta_3}{3(\beta^2 - r^2 \omega^2)^2} = -\frac{i\beta^2 \omega^2 \Delta_1 \Delta_2 \Delta_3}{3r^2} + O \left( \frac{\beta^4}{r^4} \right) \tag{2.65}
\]

Notice that these leading corrections exhibit the same scaling behavior as \((2.63)\). In any case, all these terms become sub-dominant in the limit \(\beta/r \ll \omega \ll 1\). One can probably make a systematic proof of this statement, but we content ourselves here by illustrating the suppressions. This establishes our claimed result \((2.60)\), rederived from an effective action approach.

### 2.3 AdS\(_5\) black holes

In this subsection, we make a Legendre transformation of the free energy \((2.20)\) to the microcanonical ensemble, as the macroscopic saddle point approximation of the inverse Laplace transformation. One should extremize the following entropy function

\[
S(\Delta_I, \omega_i; Q_I, J_i) = \frac{N^2}{2} \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2} + \sum_{I=1}^{3} Q_I \Delta_I + \sum_{i=1}^{2} J_i \omega_i. \tag{2.66}
\]

Since this free energy is reliable only at one of the surfaces \((2.2)\), we make variation with 4 independent variables, which couples to four combinations of 5 charges. This is our ignorance due to restricting considerations to the index. We consider the surface

\[
\Delta_1 + \Delta_2 + \Delta_3 - \omega_1 - \omega_2 = 2\pi i, \quad J_1 + J_2.
\]

Let us first make a basic consideration on what this extremization does. Although the entropy function \((2.66)\) has real coefficients only, it should have complex solutions for \(\Delta_I, \omega_i\) due to the constraint \((2.67)\). During the extremization, we will be led to distribute \(2\pi i\) on
the right hand side suitably to the 5 chemical potentials. We assert that one should pay attention to nontrivial distribution of this phase to the fugacities. Allowing nontrivial imaginary parts of $\Delta_I, \omega_i \pmod{2\pi i}$ satisfying (2.67), one can hope to reduce unnecessary boson/fermion cancelations in the index. Namely, we insert $(-1)^F$ in the index because we want pairs of bosonic/fermionic states related by $Q, S$ to cancel. If the index does not acquire contributions from such states, it can be computed at any coupling constant. However, inserting $-1$ factor to all fermions, it may cause unnecessary cancelations between bosonic/fermionic states which are not superpartners of each other. So as long as it is allowed by (2.67), we attempt to insert extra phase factor $e^{-i\phi}$ for each state, defined by $e^{-i\text{Im}(\Delta_I Q_I + \omega_i J_i)} \equiv (-1)^F e^{-i\phi}$, trying to maximally obstruct cancelations. Converting to microscopic ensemble at definite charges, the ‘entropy’ is counted with such phase factor inserted for each state:

$$e^{S(Q, J)} \sim \sum_B e^{-i\phi_B} - \sum_F e^{-i\phi_F} = \sum_B e^{-i\phi_B} + \sum_F e^{-i(\phi_F + \pi)}.$$  

(2.68)

Morally, the real parts of chemical potentials are extremized to tune the system to definite charges in the microscopic ensemble, while imaginary parts are tuned to make (2.68) maximally unobstructed. However, the two extremizations are intertwined, so that both real and imaginary parts participate in both processes. If one is lucky so that all phases $\varphi_B, \varphi_F + \pi$ at a saddle point are same (mod $2\pi$) for all microstates, then $\text{Re}(S)$ of the index would be the true BPS entropy. See the red arrows of the left figure of Fig. 1. In the figure, each arrow denotes a phase like $e^{-i\phi_B}$ or $e^{-i(\phi_F + \pi)}$. In the unlucky case that one cannot make all these phases collinear, $\text{Re}(S)$ would be smaller than the entropy: see the blue arrows of the left figure of Fig. 1. In any case, $\text{Re}(S)$ computed from our index sets a lower bound on the true entropy, and there is no a priori way of knowing when this bound saturates the true entropy. In particular, there seems to be no a priori reason to care about $\text{Im}(S)$, as the saturation may happen or not irrespective of whether $\text{Im}(S)$ assumes a specific value. E.g., see the red arrows on the left figure, and the blue arrows on the right figure. With this in mind, we consider the extremization of (2.66).
This extremization problem was considered in [18]. Below, we shall be essentially reviewing the calculations of [18], however employing our viewpoints stated above, and hopefully making some calculus more explicit and transparent.

One first solves the constraint \( \sum_I \Delta_I - \sum_i \omega_i = 2\pi i \) by the following parametrization:

\[
\Delta_I = \frac{2\pi iz_I}{1 + z_1 + z_2 + z_3 + z_4} \quad , \quad \omega_1 = -\frac{2\pi iz_4}{1 + z_1 + z_2 + z_3 + z_4} \quad , \quad \omega_2 = -\frac{2\pi i}{1 + z_1 + z_2 + z_3 + z_4}.
\]

Now \( z_{1,2,3,4} \) are unconstrained variables. Extremization in \( z_1 \) yields

\[
\frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2 \omega_1 \omega_2} + Q_I \Delta_I + J_i \omega_i = \pi i N^2 \frac{\Delta_2 \Delta_3}{\omega_1 \omega_2} + 2\pi i Q_1 \ ,
\]

while one obtains similar equations with cyclic permutations of three \( (Q_I, \Delta_I) \), to get the extremization conditions for \( z_2, z_3 \). Extremization in \( z_4 \) yields

\[
\frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2 \omega_1 \omega_2} + Q_I \Delta_I + J_i \omega_i = \pi i N^2 \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2} - 2\pi i J_1 \quad (2.71)
\]

Inserting \( \omega_1 = \sum_I \Delta_I - \omega_2 - 2\pi i \) to the term \( J_1 \omega_1 \) on the left hand side of (2.71), one obtains

\[
(Q_I + J_1) \Delta_I + (J_2 - J_1) \omega_2 = \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2 \omega_1 \omega_2} \left( \frac{2\pi i}{\omega_1} - 1 \right) \ ,
\]

which can be rewritten as

\[
(Q_I + J_1) z_I - (J_2 - J_1) = -\frac{N^2}{2} \frac{z_1 z_2 z_3}{z_4} \left( \frac{1 + z_1 + z_2 + z_3 + z_4}{z_4} + 1 \right) \ .
\]

On the other hand, subtracting (2.70) and (2.71), and doing similar subtractions for \( I = 2, 3 \), one obtains

\[
N^2 \Delta_1 \Delta_2 \Delta_3 \left( \frac{1}{\Delta_I} - \frac{1}{\omega_1} \right) = -2(Q_I + J_1) \rightarrow N^2 \frac{z_1 z_2 z_3}{z_4} \left( \frac{1}{z_4} + \frac{1}{z_I} \right) = -2(Q_I + J_1) .
\]

Viewing these four equations (2.73), (2.74) as equations for \( Q_I + J_1, J_2 - J_1 \), one can ‘solve’ them for these charges and obtain

\[
Q_I + J_1 = -\frac{N^2}{2} \frac{z_1 z_2 z_3}{z_4} \left( \frac{1}{z_I} + \frac{1}{z_4} \right) \ , \quad J_2 - J_1 = \frac{N^2}{2} \frac{z_1 z_2 z_3}{z_4} \left( \frac{1}{z_4} - 1 \right) .
\]

To get further useful arrangements, we view this equation as those for \( \frac{1}{z_{1,2,3,4}} \) with given overall \( \frac{z_1 z_2 z_3}{z_4} \) factor. Namely, with \( f \equiv \frac{N^2 z_1 z_2 z_3}{z_4} \), one obtains

\[
\frac{1}{z_4} = \frac{J_2 - J_1}{f} + 1 \ , \quad \frac{1}{z_I} = -\frac{Q_I + J_2}{f} - 1 .
\]

From the definition of \( f \), one obtains the following equation for \( f \):

\[
f = -\frac{N^2}{2} \frac{f^2 (J_2 - J_1 + f)}{(Q_1 + J_2 + f)(Q_2 + J_2 + f)(Q_3 + J_2 + f)} .
\]
This is a cubic equation of $f$,

$$(f + Q_1 + J_2)(f + Q_2 + J_2)(f + Q_3 + J_2) + \frac{N^2}{2} f(f + J_2 - J_1) = 0 . \tag{2.78}$$

After suitably applying the saddle point equations, the entropy $S$ can be expressed as

$$S = -2\pi i(f + J_2) . \tag{2.79}$$

Then (2.78) yields the following a cubic equation for $S$,

$$(S - 2\pi iQ_1)(S - 2\pi iQ_2)(S - 2\pi iQ_3) - \pi iN^2(S + 2\pi iJ_1)(S + 2\pi iJ_2) = 0 . \tag{2.80}$$

If we solve this equation, we will get three solutions. At general charges, all solutions for $S$ will be complex. If $\text{Re}(S) \leq 0$, that solution will not represent black holes. Furthermore, if the solution for any of $\Delta_I, \omega_i$ has negative real part, that solution will not describe a good saddle point. Finally, if the constructed solution has negative $\text{Re}(\log Z)$, that saddle point will lose against thermal gravitons whose free energy is of subleading order $N^0$, meaning that the system is below the Hawking-Page transition [9]. If there are multiple physical solutions, one should compare their $\text{Re}(\log Z)$ to see which one is thermodynamically dominant. All these issues will be gradually addressed, below in this subsection and also in section 3.

Before analyzing the solutions of (2.80) in detail, we first seek for a special situation, in which case $\text{Im}(S) = 0$. As emphasized, we see no a priori physical reason to expect this to be a special locus, but it will just turn out that the saddle points for known black hole solutions stay there. Demanding a real solution for $S$ will force the 5 charges $Q_I, J_i$ to stay on a codimension 1 locus, to be determined below. In this setting, one can demand that the real and imaginary parts of (2.80) are separately zero, which are of the form $S^3 + \alpha S = 0, \beta S^2 + \gamma = 0$ with real $\alpha, \beta, \gamma$. These equations determine $S$ twice, leading to

$$S (= \sqrt{-\alpha}) = 2\pi \sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2} (J_1 + J_2)}$$

$$S (= \sqrt{-\gamma/\beta}) = 2\pi \sqrt{\frac{N^2}{2} J_1J_2 - \frac{N^2}{2} Q_1 + Q_2 + Q_3} . \tag{2.81}$$

Compatibility of the two expressions yields the charge relation for $\text{Im}(S) = 0$.

Here note that, all known BPS black hole solutions of [14, 16] satisfy a charge relation (whose physical reason is unclear, at least to us, if any). The 4 parameter solutions in the last reference of [14] have the following charges:

$$Q_1 = \frac{N^2}{2\ell^2} \left[ \mu_1 + \frac{1}{2\ell^2} (\mu_1\mu_2 + \mu_1\mu_3 - \mu_2\mu_3) \right] \tag{2.82}$$

$$Q_2, Q_3 = \text{(obtained from } Q_1 \text{ by cyclic permutations of } \mu_1, \mu_2, \mu_3)$$

$$J_1 = \frac{N^2}{2\ell^4} \left[ \frac{1}{2} (\mu_1\mu_2 + \mu_2\mu_3 + \mu_3\mu_1) + \frac{\mu_1\mu_2\mu_3}{\ell^2} + \ell^4 \left( \sqrt{\Xi_b/\Xi_a} - 1 \right) \mathcal{J} \right]$$

23
\[ J_2 = (\text{obtained from } J_1 \text{ by } a \leftrightarrow b) \ . \]

\[ \ell = g^{-1} \text{ is the radius of AdS}_5 \text{ (and also } S^5) \), \[ N^2 = \frac{\pi^3}{2\ell^2} \text{ where } G \text{ is the 5d Newton constant on AdS}_5 \), and \]

\[ \Xi_a = 1 - \frac{a^2}{\ell^2} \ , \ \Xi_b = 1 - \frac{b^2}{\ell^2} \ , \ \mathcal{J} = \prod_{i=1}^{3} \left( 1 + \frac{\mu_i}{\ell^2} \right) . \quad (2.83) \]

(We multiplied \( \ell \) to the expressions of \( Q_I \)'s presented in [14] to get charges in our convention.)

The four independent parameters are \( \mu_I \)'s and \( a, b \) constrained by

\[ \mu_1 + \mu_2 + \mu_3 = \frac{1}{\sqrt{\Xi_a \Xi_b}} \left[ 2\ell(a + b) + 2ab + \frac{3}{\ell^2} \left( 1 - \sqrt{\Xi_a \Xi_b} \right) \right] . \quad (2.84) \]

Inserting these expressions, One can show

\[ Q_1Q_2Q_3 + \frac{N^2}{2} J_1J_2 = \left( \frac{N^2}{2} + Q_1 + Q_2 + Q_3 \right) \left( Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1 + J_2) \right) . \quad (2.85) \]

The charge relation of these known solutions is precisely the equation obtained by equating the two right hand sides of (2.81). This means that, somehow, the technically chosen surface \( \text{Im}(S) = 0 \) is where the known BPS black holes sit. Furthermore, assuming this charge relation, the first expression in (2.81) was shown to be equal to the Bekenstein-Hawking entropy of these black holes [34]. Since the lower bound of entropy given by our index saturates the black hole entropy, we have microscopically accounted for all their microstates. We do not have a good understanding on why/whether the locus \( \text{Im}(S(Q,J)) = 0 \) is physically special.

So far, the analysis was general, without assuming the Cardy limit, as first discovered in [18]. So even though we managed to derive it only in our Cardy limit, the free energy (2.66) could be the correct one describing the known black holes. However, beyond the Cardy regime \(|\omega_i| \ll 1\), it is not guaranteed that there are no more black hole saddle points, so that the true free energy of large \( N \mathcal{N} = 4 \) Yang-Mills may be more complicated. Indeed, in section 3, we find that the true free energy may be more complicated than (2.66), by studying another special limit. Now focussing on our Cardy limit, it demands \( z_4 \) to be order 1 while \( z_{1,2,3} \) to be much larger than 1. From (2.75), this implies that the four combinations of charges \( Q_I + J_1, J_2 - J_1 \) are much larger than \( N^2 \) (unless \( z_4 = 1 \), so one considers microstates with equal rotations \( J_1 = J_2 \)). This is all one can say intrinsically from the index. However, we can discuss the implication of the Cardy limit on the known black hole solutions that we have just counted, on the surface (2.85). From the expressions (2.82), \( Q_I + J_1, J_2 - J_1 \) can be taken to be much larger than \( N^2 \) by taking \( \mu_I \gg \ell^2 \), assuming that \( a, b \) are further tune to meet (2.84). So generically, \( Q_I \propto N^2 \mu^2 \), \( J_i \propto N^2 \mu^3 \). One can then approximate the right hand side of (2.85) by dropping \( \frac{N^2}{2} \) term on the first factor, and \(-\frac{N^2}{2}(J_1 + J_2)\) term on the second factor, yielding the asymptotic relation

\[ (Q_1 + Q_2 + Q_3)(Q_1Q_2 + Q_2Q_3 + Q_3Q_1) - Q_1Q_2Q_3 \approx \frac{N^2}{2} J_1J_2 . \quad (2.86) \]
When all charges are equal, \( \equiv Q \), and also when all angular momenta are equal \( \equiv J \), it becomes \((J/N^2)^2 \approx 16(Q/N^2)^3\). So our Cardy limit on known solutions demands \( J/N^2 \gg Q/N^2 \gg 1 \).

To finalize our discussion on the saddle points on the surface \( \text{Im}(S) = 0 \), we should confirm that all \( \text{Re}(\Delta_I) \), \( \text{Re}(\omega_i) \) agree with the BPS chemical potentials of the black hole solutions. \( z_{1,2,3,4} \), can be determined from \( S \) that we just expressed in terms of \( Q_I, J_i \). From (2.75), and from the relation \( S = -2\pi i(f + J_2) \) with \( f \equiv N^2/N^2, \frac{z_1 z_3}{z_2 z_4} \), one obtains

\[
Q_I + J_1 = \left( \frac{S}{2\pi i} + J_2 \right) \left( \frac{1}{z_1} + \frac{1}{z_4} \right), \quad J_1 - J_2 = \left( \frac{S}{2\pi i} + J_2 \right) \left( \frac{1}{z_4} - 1 \right) .
\]  

(2.87)

This determines \( z_{1,2,3,4} \) as

\[
z_I = -\frac{S + 2\pi i J_2}{S - 2\pi i Q_I} , \quad z_4 = \frac{S + 2\pi i J_2}{S + 2\pi i J_1} .
\]  

(2.88)

Inserting these into (2.69), one obtains \( \Delta_I, \omega_i \), whose real parts \( \text{Re}(\Delta_I) \equiv \xi_I, \text{Re}(\omega_i) \equiv \zeta_i \) are the chemical potentials coupling to \( Q_I, J_i \). We have shown that \( \xi_I \) and \( \zeta_i \) agree with those of the dual black holes. Since this involved computerized calculations of complicated functions, we simply outline the procedures.

To compute these chemical potentials from gravity, one has to start from the non-BPS solutions and take the zero temperature BPS limit to find \( \xi_I, \zeta_i \). The general non-BPS solutions with unequal \( Q_I, J_i \) is known in \[35\]. Here, we study the solutions of \[16\] with independent charges \( Q_I \), but only at equal angular momenta. So we made comparisons only at \( J \equiv J_1 = J_2, \zeta \equiv \zeta_1 = \zeta_2 \). The non-BPS black holes of \[16\] at \( J_{1,2} = J \) has 5 parameters \( m, a \) and \( \delta_{1,2,3} \). Its energy, charges and entropy are given by \[6\]

\[
E = \frac{1}{gG} \cdot \frac{1}{4} m \pi \left( 3 + a^2 g^2 + 2 s_1^2 + 2 s_2^2 + 2 s_3^2 \right) , \quad Q_I = \frac{1}{gG} \cdot \frac{1}{2} m \pi s_1 c_I
\]

\[
J = \frac{1}{G} \cdot \frac{1}{2} m a \pi \left( c_1 c_2 c_3 - s_1 s_2 s_3 \right) , \quad S = \frac{1}{G} \cdot \frac{\pi^2}{2} \sqrt{f_1(r_+)} ,
\]  

(2.89)

where \( G \) is the Newton’s constant and

\[
s_I = \sinh \delta_I , \quad c_I = \cosh \delta_I , \quad H_I = 1 + \frac{2m}{r^2} s_I^2
\]

\[
f_1 = r^6 H_1 H_2 H_3 + 2ma^2 r^2 + 4ma^2 \left( 2(c_1 c_2 c_3 - s_1 s_2 s_3) s_1 s_2 s_3 - s_1^2 s_2^2 - s_2^2 s_3^2 - s_3^2 s_1^2 \right)
\]

\[
f_2 = 2ma(c_1 c_2 c_3 - s_1 s_2 s_3) r^2 + 4m^2 a s_1 s_2 s_3
\]

\[
f_3 = 2ma^2 (1 + g^2 r^2) + 4g^2 m^2 a^2 \left( 2(c_1 c_2 c_3 - s_1 s_2 s_3) s_1 s_2 s_3 - s_1^2 s_2^2 - s_2^2 s_3^2 - s_3^2 s_1^2 \right)
\]

\[
Y = f_3 + g^2 r^6 H_1 H_2 H_3 + r^4 - 2mr^2 .
\]  

(2.90)

\( r = r_+ \) is the largest positive root of \( Y(r) = 0 \). The BPS condition and smooth horizon condition yield

\[
a = \frac{1}{g} e^{-\delta_1 - \delta_2 - \delta_3}, \quad m = \frac{4e^{2\delta_1 + 2\delta_2 + 2\delta_3}}{g^2 (e^{2\delta_1 + 2\delta_2} - 1)(e^{2\delta_2 + 2\delta_3} - 1)(e^{2\delta_3 + 2\delta_1} - 1)} .
\]

(2.91)

\[6\] Compared to (3.10), (3.11) of \[16\], we multiplied factors containing \( g \) and/or \( G \), to meet our convention.
For BPS black holes, the outer horizon is located at
\[ r_+ = \sqrt{\frac{4e^{2\delta_1 + 2\delta_2 + 2\delta_3} - 2e^{2\delta_1 + 2\delta_2} - 2e^{2\delta_2 + 2\delta_3} - 2e^{2\delta_3 + 2\delta_1} + 2}{g^2(e^{2\delta_1} - 1)(e^{2\delta_2} - 1)(e^{2\delta_3} - 1)}}. \] (2.92)

Inserting (2.91) and (2.92) to (2.89), one can check \[ E \equiv 2J + Q_1 + Q_2 + Q_3 \] and (2.81). Chemical potentials are given by
\[ T = \frac{1}{g} \cdot \frac{1}{4\pi \sqrt{f_1}} \frac{\partial Y}{\partial r}, \quad \Omega = \frac{1}{2g} \cdot \frac{2f_2}{f_1}, \quad \Phi_I = \frac{2m}{r^2 H_I} \left( s_I c_I + \frac{2a f_2}{f_1} (c_I s_J s_K - s_I c_J c_K) \right). \] (2.93)

where functions are evaluated at \( r = r_+ \). They satisfy the following first law of thermodynamics:
\[ dE = T dS + 2\Omega dJ + \sum_{I=1}^{3} \Phi_I dQ_I. \] (2.94)

The free energy \( F \) in the canonical ensemble for all \( \Omega, \Phi_I \) is given by
\[ F = E - TS - 2\Omega J - \Phi_I Q_I. \] (2.95)

Defining \( \Delta E = E - \sum I Q_I - 2J \), the energy beyond BPS bound, one finds
\[ \frac{F}{T} = \frac{\Delta E}{T} - S + \sum I \frac{1}{T} \Phi_I Q_I + 2 \frac{1 - \Omega}{T} J. \] (2.96)

Taking the BPS limit (2.91), the black hole chemical potentials approach \( \Phi_I \to 1, \Omega \to 1 \) in our normalization. Since \( T \to 0 \) is associated with taking the BPS limit, one finds that
\[ \xi_I \equiv \lim_{T \to 0} \frac{1 - \Phi_I}{T}, \quad \zeta \equiv \lim_{T \to 0} \frac{1 - \Omega}{T}. \] (2.97)

is finite. Since \( S \) is finite in the limit, one finds that the BPS limit of \( \frac{F - \Delta E}{T} \equiv F_{\text{BPS}}(\mu_I, \nu) \) should exist. Therefore, one finds
\[ -F_{\text{BPS}} = S - \sum I \xi_I Q_I - 2\zeta J. \] (2.98)

\(-F_{\text{BPS}}\) is nothing but \( \log Z \), where \( Z \) is the partition function in the BPS limit. \( \xi_I, \zeta \) defined by (2.97) are functions of \( Q_I, J \) subject to a charge relation. This can be directly compared to our result from the entropy function. We find that the two results agree.

Having found that both entropy and chemical potentials derived from (2.66) agree those of known BPS black holes, even away from the Cardy limit \( |\omega_i| \ll 1 \), we can understand \(-\text{Re} (\log Z) = \text{Re} \left( \frac{N^2 \Delta_1 \Delta_2 \Delta_3}{2\omega_1 \omega_2 \omega_3} \right) \) more generally as the free energy of the known black holes. For instance, \( F_{\text{BPS}} \) defined by the right hand side of (2.98) will automatically be the same as the

\[ \text{Here, a typo of (3.75) in [16] is corrected.} \]
\[ \text{(r_+)}_{\text{here}} = -(r_0)_{\text{there}} \]
\[ \text{We changed normalization by multiplying } \frac{1}{g} \text{ to } T, \text{ and } \frac{1}{2g} \text{ to } \Omega. \text{ We also corrected a typo in (3.10) of [16]: the \text{+} sign in front of the second term of } \Phi_I \text{ in our (2.93) was } - \text{ there.} \]
extremal value of it. Of course an unclear part is whether this saddle point is thermodynamically dominant or not, against other possible black holes. However, there is a point in discussing (2.20) beyond the Cardy limit, as a tool to study known black holes better. We shall sometimes assume this attitude below.

Now we work more intrinsically within the index without imposing any charge relation by hand (e.g. \( \text{Im}(S) = 0 \)), and study \( \text{Re}(S) \). This will depend only on 4 combinations of the 5 charges \( Q_I, J_i \). For simplicity, let us consider the case with equal electric charges, \( Q_1 = Q_2 = Q_3 \equiv Q \), and equal angular momenta \( J_1 = J_2 \equiv J \). Then we also set the corresponding chemical potentials to be equal, \( \Delta_1 = \Delta_2 = \Delta_3 \equiv \Delta \), \( \omega_1 = \omega_2 \equiv \omega \). The constraint on chemical potentials is \( 3\Delta = 2\pi i + 2\omega \). Inserting this, the entropy function is given by

\[
S = \frac{N^2}{2} \left( \frac{(2\pi i + 2\omega)^3}{\omega^2} \right) + 2\omega(J + Q) + 2\pi iQ. \tag{2.99}
\]

We ignore the last constant term \( 2\pi iQ \), as this will not contribute to \( \text{Re}(S) \). (In fact, \( e^{2\pi iQ} = \pm 1 \) from charge quantization.) The saddle point equation \( \frac{\partial S}{\partial \omega} = 0 \) yields

\[
J + Q = \frac{N^2}{54} \left( \frac{(2\pi i + 2\omega)^3}{\omega^3} - 3 \frac{(2\pi i + 2\omega)^2}{\omega^2} \right). \tag{2.100}
\]

As mentioned in the previous paragraph, we allow general \( \omega \), not necessarily small. \( \omega \) will be complex, but since the left hand side of (2.100) is real, it is helpful to write \( \omega = \omega_R + i\omega_I \) with real \( \omega_{I,R} \). Then (2.100) can be separated to real and imaginary parts. Setting the imaginary part to zero, one obtains three solutions for \( \omega_R \) at given \( \omega_I \):

\[
\omega_R = \begin{cases} 
0 & \text{for } \omega_I \in (-\infty, \infty) \\
\pm \omega_I \sqrt{\frac{3\pi + 3\omega_I}{\pi - 3\omega_I}} & \text{for } \omega_I \in (-\pi, \frac{\pi}{3})
\end{cases}. \tag{2.101}
\]

If one inserts (2.101) to (2.100), the real part of this equation becomes

\[
J + Q = \begin{cases} 
\frac{2N^2(2\pi - \omega_I)(\pi + \omega_I)^2}{\omega_I^4} & \text{if } \omega_R = 0 \\
- \frac{N^2(\pi - 2\omega_I)^2(\pi + \omega_I)}{\omega_I^7} & \text{if } \omega_R = \pm \omega_I \sqrt{\frac{3\pi + 3\omega_I}{\pi - 3\omega_I}}
\end{cases}. \tag{2.102}
\]

Also, the ‘free energy’ \( \log Z = \frac{N^2}{2} \Delta^3/\omega^3 \) becomes

\[
\log Z = \begin{cases} 
\frac{i 8N^2(\pi + \omega_I)^3}{27\omega_I^3} & \text{if } \omega_R = 0 \\
+ \frac{N^2}{9} \frac{\pi^3 - 9\pi \omega_I^2 - 8\omega_I}{\omega_I^9} \sqrt{\frac{\pi + \omega_I}{3\pi - 9\omega_I}} - i \frac{N^2(\pi - 8\omega_I)(\pi + \omega_I)^2}{27\omega_I^4} & \text{if } \omega_R = \pm \omega_I \sqrt{\frac{3\pi + 3\omega_I}{\pi - 3\omega_I}}
\end{cases}. \tag{2.103}
\]

The solution with \( \omega_R = 0 \) will yield imaginary \( \log Z \) and therefore \( \text{Re}(S) \) = 0, making it an irrelevant solution. In the remaining two solution, the free parameter \( \omega_I \) is related to the unique charge combination \( J + Q \) captured by the index, which can be used to express \( \log Z \) and \( S \).

We further discuss which of the remaining solutions corresponds to black holes. Since \( \omega_R \) should be positive, one should choose the upper sign for \( 0 < \omega_I < \frac{\pi}{3} \), and lower sign for...
$-\pi < \omega_I < 0$. Also, since $J + Q$ has to be positive, one obtains $\omega_I < 0$ from the second line of (2.102). Therefore the physical solution is $\omega_R = -\omega_I \sqrt{\frac{3\pi + 3\omega_I}{\pi - 3\omega_I}}$ for $-\pi < \omega_I < 0$. Various quantities labeled by $\omega_I$ are summarized as

$$
\omega = -\omega_I \sqrt{\frac{3\pi + 3\omega_I}{\pi - 3\omega_I}} + i\omega_I, \quad -\pi < \omega_I < 0
$$

$$
J + Q = -\frac{N^2}{54} \frac{(\pi - 2\omega_I)^2(\pi + \omega_I)}{\omega_I^3}
$$

$$
\log Z = \frac{N^2}{9} \frac{\pi^3 - 9\pi \omega_I^2 - 8\omega_I^3}{\omega_I^2} \sqrt{\frac{\pi + \omega_I}{3\pi - 9\omega_I}} - \frac{i}{27} \frac{N^2}{\omega_I^2} \frac{(\pi - 8\omega_I)(\pi + \omega_I)^2}{\omega_I^2}.
$$

In the Cardy limit we derived, $\omega_I$ should be a small negative number.

As emphasized, this is the free energy of known black hole saddle points. Our microscopic analysis assures that this is the dominant one for large black holes in the Cardy limit. But for not-so-large or small black holes, the situation is unclear. In particular, numerical studies are made recently on hairy BPS black holes [19], predicting more general black holes as one approaches the zero temperature BPS limit. In particular, as far as we see from the reported charge regimes in [19], evidences for new black holes are found for small angular momenta, at around $\frac{J}{N\pi} \lesssim 0.05$. If we take these results seriously, the true free energy may deviate from (2.20) for small black holes. Of course, there could be a possibility that the intrinsic prediction from the index has its own ambiguity, in that the physical charges $Q, J$ cannot be separately be specified. In any case, we find it worthwhile to investigate the ‘phenomenology’ of the thermodynamics shown by this entropy function, all the way from large to small black holes. For instance, one may ask if this saddle point is more dominant or not compared to the thermal graviton phase in AdS5. This can be answered by comparing the free energy of black holes and thermal gravitons. At given temperature, gravitons do not see $N$ so that their free
energy is $\mathcal{O}(N^0)$. This is much smaller than the free energy of our entropy function. Therefore, we should compare the extremal value of $-F_{\text{BPS}} = \text{Re}(\log Z)$ in (2.104) with 0. The plot of the free energy as a function of $Q + J$ is shown in Fig. 2. One finds that $\log Z = 0$ at $\omega_I = -\pi$ or $\omega_I = -\frac{1+\sqrt{33}}{16}\pi$, for $J + Q = 0$ and $J + Q = \frac{3+\sqrt{33}}{18}N^2 \simeq 0.486N^2$, respectively. So taking this entropy function (2.20) down to order 1 values of $\frac{Q+J}{N^2}$, one finds that the Hawking-Page transition would happen at

$$\omega_R = \pi \sqrt{414 - 66\sqrt{33}} \simeq 1.16 , \quad J + Q = \frac{3 + \sqrt{33}}{18}N^2 \simeq 0.486N^2 , \quad (2.105)$$

if the phase structure does not get interrupted by other factors, like yet unknown saddle points.

### 3 The $\frac{1}{8}$-BPS Macdonald sector

In this section, we investigate the Cardy-like and non-Cardy-like free energy of the index in the so-called Macdonald limit [21]. We first explain the Macdonald index in the context of $\mathcal{N} = 4$ Yang-Mills theory. Consider the index

$$Z = \text{Tr} \left[ (-1)^F e^{-\Delta I Q_I - \omega_I J_I} \right] \quad (3.1)$$

at $\Delta_1 + \Delta_2 + \Delta_3 = \omega_1 + \omega_2$, which is obtained from (2.21) by shifting a chemical potential by $2\pi i$, and by sending $\beta \to \infty$. This is an index counting $\frac{1}{16}$-BPS states preserving $Q^{++}$ and $S^{++}$. Eliminating $\omega_2 = \Delta_1 + \Delta_2 + \Delta_3 - \omega_1$, one obtains

$$Z = \text{Tr} \left[ (-1)^F e^{-\Delta_1 (Q_1 + J_2) - \Delta_2 (Q_2 + J_2) - \Delta_3 (Q_3 + J_2) - \omega_1 (J_1 - J_2)} \right] \quad (3.2)$$

in terms of four independent variables $\Delta_I, \omega_I$ with positive real parts. Now we take the limit $\Delta_3 \to \infty$, projecting to states satisfying $Q_3 + J_2 = 0$. One can show that this projection demands the BPS states to be annihilated by an extra pair of supercharges, $Q^{++}$, $S^{++}$. A quick way to see this is that the new pair demands the BPS energy relation $E = Q_1 + Q_2 - Q_3 + J_1 - J_2$, which is satisfied by imposing the original BPS bound $E = Q_1 + Q_2 + Q_3 + J_1 + J_2$ and the new projection condition $Q_3 + J_2 = 0$. This is a limit which takes $\Delta_3, \omega_2 \to \infty$, with $\frac{\Delta_3}{\omega_2} \to 1$. One also has to keep $\Delta_3 - \omega_2 = (\omega_1 - \Delta_1 - \Delta_2)$ finite. This way, one obtains the Macdonald index for $\frac{1}{8}$-BPS states depending on $\Delta_1, \Delta_2, \omega_1$.

In the weakly interacting theory, $\frac{1}{16}$-BPS operators are made of: 3 anti-chiral scalars $\Phi^{Q_I}$ with $(Q_I) = (1, 0, 0), (0, 1, 0), (0, 0, 1)$; three chiralinos $\Psi^{Q_I}_{\pm \frac{1}{4}, \pm \frac{1}{2}}$ with $(Q_I) = (-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}), \ (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}), \ (\frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$; two gauginos $\Psi^{\pm \frac{1}{4}, \pm \frac{1}{2}}$; one self-dual component of field strength $f_{+,+1}$; two covariant derivatives $D_{1,0}, D_{0,1}$. In the Macdonald limit, $\frac{1}{8}$-BPS operators are made of: two complex scalars $\Phi^{1,0,0}, \Phi^{0,1,0}$; two fermions $\Psi^{\pm \frac{1}{4}, \pm \frac{1}{2}}, \pm \frac{1}{2}$; one derivative $D_{1,0}$. Despite preserving enhanced SUSY, the full spectrum of this sector is not completely solved yet even at weak
coupling, to the best of our knowledge. This is in contrast to other $\frac{1}{8}$-BPS sectors of $\mathcal{N} = 4$ Yang-Mills theory. There are two more inequivalent $\frac{1}{8}$-BPS subsectors of the above canonical $\frac{1}{16}$-BPS sector, specified by either $J_1 + J_2 = 0$ or $Q_1 + Q_2 = 0$. The former is the well-known chiral ring sector, completely solved in, e.g. [7]. The solution in the second sector can be found, e.g. in [12]. It might be surprising that the last $\frac{1}{8}$-BPS sector given by the Macdonald limit is still unsolved. As we shall see below, perhaps the reason is that this sector is too rich to admit a simple exact solution.

We shall study a new Cardy-like limit and a non-Cardy-like limit of the Macdonald index at $|\omega_1| \ll 1$. Although we also call the former a Cardy limit, it is different from the one in section 2 in that $\omega_2$ is sent large. In a way, the previous one is a 4d Cardy limit, acquiring large contributions from two BPS derivatives. Here, it is more like a 2d Cardy limit.

In the Macdonald limit $\Delta_3, \omega_2 \to \infty$, $\Delta_3/\omega_2 \to 1$, the index (2.27) reduces to

$$Z = \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \prod_{a<b} \left( 2 \sin \frac{\alpha_{ab}}{2} \right)^2 \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \left( 1 - \frac{(1 - e^{-n\Delta_1})(1 - e^{-n\Delta_2})}{1 - e^{-n\omega_1}} \right) \right] \sum_{a,b=1}^{N} e^{i\alpha_{ab}} \right].$$

(3.3)

As before, we ignore the exponents for the Cartans, $a = b$, which will give $O(N^1)$ contribution to the free energy. Then, for $a \neq b$, the term ‘1’ in the exponent will cancel with the Haar measure. Taking $\omega_1 \ll 1$ with the remaining non-Abelian terms, with $\Delta_{1,2}$ kept fixed, and again assuming the maximally deconfining saddle point $\alpha_1 \approx \cdots \approx \alpha_N$, one obtains

$$\log Z \sim -\frac{N^2}{\omega_1} \left[ \text{Li}_2(1) - \text{Li}_2(e^{-\Delta_1}) - \text{Li}_2(e^{-\Delta_2}) + \text{Li}_2(e^{-\Delta_1 - \Delta_2}) \right]$$

(3.4)

with unconstrained $\Delta_1, \Delta_2, \omega_1$. This is the Macdonald-Cardy limit of the index.

On the other hand, had (2.20) or the result of [18] been exact for general $\omega_{1,2}$, one would have obtained a very different result from (3.4). Namely, taking the Macdonald limit of (2.20) assuming its validity at general $\omega_{1,2}$, $\Delta_3, \omega_2 \to +\infty$ with $\Delta_3/\omega_2 \to 1$, one would have obtained

$$\log Z \sim \frac{N^2 \Delta_1 \Delta_2}{2\omega_1}$$

(3.5)

without any constraint on $\Delta_1, \Delta_2, \omega_1$. But keeping $\omega_1 \ll 1$ and $\Delta_{1,2}$ finite, we derive (3.4) instead of (3.5) (assuming maximally deconfining saddle points). So the true phase structure of black holes may be richer than simply the known black holes, or [18], even in the $\frac{1}{8}$-BPS Macdonald sector.

However, before proceeding, we explain that there appears to be a scaling limit of the Macdonald index which yields (3.5). To see this, let us scale $\omega_1 \ll 1$, but also take $\Delta_1, \Delta_2 \ll 1$

9However, [36] solved the Schur index problem, which is an unrefined version of the Macdonald index. The Schur limit of the general $\frac{1}{16}$-BPS index (2.27) is defined as $\Delta_3 = \omega_2$. In the Macdonald index, to be studied shortly, one further unrefines as $\Delta_1 + \Delta_2 = \omega_1$ to get the Schur index.
keeping $\frac{\Delta_1 \Delta_2}{\omega_1}$ finite. In this case, we take large $N$ and disregard the integrand factors for the Cartans, $a = b$, assuming that this $O(N^1)$ term will not affect our scaling free energy at $O(N^2)$. In fact, as we shall see later, the last assumption will fail, with an interesting implication: however, let us proceed for now to derive (3.5) first. With the summation in the exponent restricted to $a \neq b$, (3.3) can be written as

$$Z \sim \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \exp \left[ -\frac{\Delta_1 \Delta_2}{\omega_1} \sum_{n=1}^{\infty} \sum_{a \neq b} e^{i\alpha_{ab}} \right]. \quad (3.6)$$

Since

$$\sum_{n=1}^{\infty} \sum_{a \neq b} e^{i\alpha_{ab}} = \sum_{n \neq 0} \sum_{a < b} e^{i\alpha_{ab}} = \sum_{a < b} (2\pi \delta(\alpha_a - \alpha_b) - 1), \quad (3.7)$$

one obtains

$$Z \sim \frac{1}{N!} \int \prod_{a=1}^{N} \frac{d\alpha_a}{2\pi} \exp \left[ -\sum_{a < b} V_{\text{eff}}(\alpha_a - \alpha_b) \right], \quad V_{\text{eff}}(\theta) \equiv \frac{\Delta_1 \Delta_2}{\omega_1} \left[2\pi \delta(\theta) - 1\right], \quad (3.8)$$

where $\delta(\theta)$ is the delta function on a circle, with $\theta \sim \theta + 2\pi$. Therefore, by keeping $\text{Re}(\frac{\Delta_1 \Delta_2}{\omega_1}) > 0$, one finds an effective potential with very small repulsive core. Whether this is satisfied or not will be controversial at the end, for a reason to be explained shortly. In any case, let us assume this and proceed. In this case, if $\alpha_a$’s are not equal, the potential is at its flat minimum, with constant negative energy. Since the repulsive core is scaling to zero size in our scaling limit, one can take $V_{\text{eff}} = -\frac{\Delta_1 \Delta_2}{\omega_1}$ for most values of $\alpha_a$. It makes real part of $\log Z$ maximal, and imaginary part stationary. Therefore, one approximates

$$\log Z \sim \left( N^2 - N \right) \frac{\Delta_1 \Delta_2}{2\omega_1} \approx \frac{N^2 \Delta_1 \Delta_2}{2\omega_1}. \quad (3.9)$$

In fact, as we will show below, the assumption that $O(N^1)$ terms are ignorable will fail, by the free energy (3.3) failing to have nontrivial large $N$ saddle point with $\log Z \sim N^2$. But we shall use this free energy as a probe of small black holes.

We shall now discuss the thermodynamic aspects of two free energies (3.4) and (3.5).

It is first illustrative to see what is the consequence of (3.5). As we emphasized in section 2.3, we can regard (2.20) as describing known black holes, even beyond the Cardy limit. Firstly, from the known black hole solutions, one can show that the horizon area vanishes as one takes limit $Q_3 + J_2 \to 0^+$. To see this, we start from the charge relation (2.85). Plugging in $J_2 = -Q_3$ on both sides, and rearranging, one obtains

$$0 = \left( Q_1 + Q_2 + \frac{N^2}{2} \right) \left( Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{N^2}{2} (J_1 + J_2) + Q_3^2 \right), \quad (3.10)$$

where we suitably inserted back $Q_3 \to -J_2$ on the second factor. The first factor is positive since $Q_1 + Q_2 \geq 0$ in the BPS sector. On the second factor, $Q_3^2 \geq 0$ for the last term. The
remaining terms in the second factor are simply square of the black hole entropy \( (\frac{S}{2\pi})^2 \), from the first line of (2.81). So the solution becomes meaningless if this is negative. So from the vanishing of (3.10) on the solutions without naked singularities, one finds

\[
Q_3 \to 0, \quad Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2} (J_1 + J_2) = \left(\frac{S}{2\pi}\right)^2 \to 0.
\]

(3.11)

We conclude that the known black solutions become ‘small black holes’ in the Macdonald limit. Here ‘small’ and ‘large’ is an entropic notion, different from those used in the other part of this paper: the above configuration has small entropy at large charges. Collecting all the conditions, the charges carried by these small black holes satisfy

\[
Q_1Q_2 = \frac{N^2}{2} J_1, \quad Q_3 = J_2 = 0,
\]

(3.12)

where the first relation is the vanishing condition of the horizon area when \( Q_3 = J_2 = 0 \).

Similar conclusion can be obtained from (3.5), in a rather curious manner. Note that

\[
\frac{N^2 \Delta_1 \Delta_2}{2\omega_1} + (Q_1 + J_2) \Delta_1 + (Q_2 + J_2) \Delta_2 + (J_1 - J_2) \omega_1
\]

is homogeneous degree 1 in three independent variables \( \Delta_1, \Delta_2, \omega_1 \). Therefore, the overall scaling mode of them plays the role of Lagrange multiplier, making the extremized entropy to vanish. Since the remaining two ratios of the chemical potentials determine three charges \( Q_1 + J_2, \quad Q_2 + J_2, \quad J_1 - J_2 \), the charges satisfy a relation. The relation is

\[
(Q_1 + J_2)(Q_2 + J_2) = \frac{N^2}{2} (J_1 - J_2).
\]

(3.14)

We find it as closest as one can get to (3.12) from the index, without extra input on the charges that the index cannot see (such as \( Q_3 = J_2 = 0 \)). However, we emphasize that both approaches predict small black holes \( S \to 0 \) in the \( \frac{1}{8} \)-BPS Macdonald limit. And coming back to the derivation of (3.5) ignoring \( O(N^4) \) terms, we simply arrive at the conclusion that we may have to include them to obtain the leading entropy. In any case, both known black hole solutions and the QFT analysis in the non-Cardy scaling limit predicts small black holes. As an additional comment, we cannot determine in this framework whether \( \Re \left( \frac{\Delta_1 \Delta_2}{\omega_1} \right) \) is positive or not, because an overall scaling mode is a Lagrange multiplier which cannot be determined. The sign of this quantity was important above, when we want to regard (3.13) as derived from the Macdonald index in a scaling limit. Perhaps it is related to the degenerate nature of this saddle point, which one may resolve clearly by going slightly beyond the Macdonald limit and doing a more careful calculation. We leave a more detailed study to the future.

Now we study the free energy (3.4). We study the associated entropy function:

\[
S = \log Z + Q_1 \Delta_1 + Q_2 \Delta_2 + Q_3 \Delta_3 + J_1 \omega_1 + J_2 \omega_2 + \frac{N^2}{\omega_1} \left[ \Li_2(1) - \Li_2(e^{-\Delta_1}) - \Li_2(e^{-\Delta_2}) + \Li_2(e^{-\Delta_1-\Delta_2}) \right]
\]

\[
+ (Q_1 + J_2) \Delta_1 + (Q_2 + J_2) \Delta_2 + (J_1 - J_2) \omega_1.
\]

(3.15)
Extremizing, one obtains

\[ Q_1 + J_2 = \frac{N^2}{\omega_1} \left[ -\log (1 - e^{-\Delta_1}) + \log (1 - e^{-\Delta_1 - \Delta_2}) \right], \]

\[ Q_2 + J_2 = \frac{N^2}{\omega_1} \left[ -\log (1 - e^{-\Delta_2}) + \log (1 - e^{-\Delta_1 - \Delta_2}) \right], \]

(3.16)

\[ J_1 - J_2 = -\frac{N^2}{\omega_1^2} \left[ \text{Li}_2 (1) - \text{Li}_2 (e^{-\Delta_1}) - \text{Li}_2 (e^{-\Delta_2}) + \text{Li}_2 (e^{-\Delta_1 - \Delta_2}) \right]. \]

From now on, we shall use some identities of \( \text{Li}_2 \) to make a semi-analytic study. However, all solutions below are cross-checked numerically against (3.16).

Using the following identity (W. Schaeffer, 1846)

\[ \text{Li}_2 (xy) - \text{Li}_2 (x) - \text{Li}_2 (y) + \text{Li}_2 (1) = \text{Li}_2 \left( \frac{1 - x}{1 - xy} \right) - \text{Li}_2 \left( y \frac{1 - x}{1 - xy} \right) + \log(x) \log \left( \frac{1 - x}{1 - xy} \right), \]

(3.17)

the extremized entropy becomes

\[ S = \frac{N^2}{\omega_1} \left[ - \text{Li}_2 \left( \frac{1 - e^{-\Delta_1}}{1 - e^{-\Delta_1 - \Delta_2}} \right) - \text{Li}_2 \left( \frac{1 - e^{-\Delta_2}}{1 - e^{-\Delta_1 - \Delta_2}} \right) + \text{Li}_2 \left( e^{-\Delta_2} \frac{1 - e^{-\Delta_1}}{1 - e^{-\Delta_1 - \Delta_2}} \right) + \text{Li}_2 \left( e^{-\Delta_1} \frac{1 - e^{-\Delta_2}}{1 - e^{-\Delta_1 - \Delta_2}} \right) \right]. \]

(3.18)

From this formula, one finds \( S < 0 \) if \( \Delta_1, \Delta_2, \omega_1 \) are strictly real and positive. This is because \( \text{Li}_2 (x) \) is an increasing function of \( x > 0 \), so that first plus third terms are negative, and second plus fourth terms are also negative. Hence, in order to get black holes with \( \text{Re}(S) > 0 \) at positive chemical potential, we should turn on the imaginary part of chemical potentials. Physically, this again implies that one should turn on phases of fugacities to obstruct boson/femrion cancelation in the index to see black holes.

Now, for simplicity, we consider the case with equal charge: \( Q_1 = Q_2 \). Below, we will frequently use (3.17) at \( x = y \) and the Euler’s reflection formula:

\[ \text{Li}_2 (x^2) - 2\text{Li}_2 (x) + \text{Li}_2 (1) = \text{Li}_2 \left( \frac{1}{1 + x} \right) - \text{Li}_2 \left( \frac{x}{1 + x} \right) - \log(x) \log (1 + x), \]

(3.19)

\[ \text{Li}_2(x) + \text{Li}_2(1 - x) = \text{Li}_2(1) - \log(x) \log(1 - x). \]

Then, setting \( \Delta \equiv \Delta_1 = \Delta_2 \), one obtains

\[ q \equiv \frac{Q_1 + J_2}{N^2} = \frac{Q_2 + J_2}{N^2} = \frac{1}{\omega_1} \log \left( 1 + e^{-\Delta} \right), \]

(3.20)

\[ j \equiv \frac{J_1 - J_2}{N^2} = -\frac{1}{\omega_1^2} \left[ \text{Li}_2 (1) - 2\text{Li}_2 (e^{-\Delta}) + \text{Li}_2 (e^{-2\Delta}) \right] \]

\[ = -\frac{1}{\omega_1^2} \left[ \text{Li}_2 (1) - 2\text{Li}_2 \left( \frac{1}{1 + e^\Delta} \right) - (\log(1 + e^{-\Delta}))^2 \right]. \]
(a) Re[f(r)]: Green line denotes real parts of both yellow and green lines in Fig. 3(b).

(b) (Re[f(r)], Im[f(r)]): Arrows denote an increase of r. Yellow and green lines are complex conjugate to each other.

Figure 3: Various solutions $f(r)$ of (3.21)

\[ f(r)^2 r = 2 \text{Li}_2(1-e^{-f(r)}) - \text{Li}_2(1). \]  

Note that $\text{Li}_2(1) = \frac{\pi^2}{6}$. We expect macroscopic physical solutions only when $q > 0$ and $j > 0$. Indeed, with some efforts, one can check this fact explicitly from the above formulae.

Due to the complexity of these equations, we numerically/graphically solve this problem. For $r = \frac{j}{q^2} - 1 > 0$, one finds that $f(r)$ is a double-valued, while for $-1 < r < 0$, it is single-valued. See Fig. 3. We find that only when $r > r_0 \equiv 0.2003559478...$, Im($f(r)$) $\neq 0$. If $r$ is smaller than this critical value $r_0$, $f(r)$ is strictly real. Then, one finds that $\omega_1, \Delta$ are also real, from the definition of $f$ and the first equation of (3.20), since $f = q\omega_1 > 0$. Namely, only when $j > (1+r_0)q^2$, Im($\omega_1$), Im($\Delta$) $\neq 0$, and we may expect a solution with macroscopic entropy and positive chemical potentials. One can see that we have two distinct solutions $f(r) = x(r) \pm iy(r)$ when $r > r_0$. In fact, one can analytically show that if $f(r) = x(r) + iy(r)$ is one solution of its defining equation (3.21) at certain $r$, then $(f(r))^* = x(r) - iy(r)$ becomes another solution. Correspondingly, for given $j, q$, one will find the following form of two distinct solutions for
the chemical potentials and entropy: \( \omega_1 = \omega_1^R \pm i \omega_1^I \), \( \Delta = \Delta^R \pm i \Delta^I \), and \( S = S^R \pm i S^I \). So the directly observable physical quantities, given by the real parts \( \omega_1^R, \Delta^R, S^R \), are uniquely determined in terms of \( j, q \). As commented below (3.18), the region \( r < r_0 \) does not yield sensible saddle points.

For \( r > r_0 \), we study whether \( \text{Re}(\Delta), \text{Re}(\omega_1) \) are actually positive. In Fig. 4, \( \omega_1, \Delta \) are plotted with respect to \( q \), at fixed \( j \). Note that among two solutions of \( f(r) \), we chose the blue one and the yellow one in Fig. 3. From Fig. 4(b), \( \text{Re}(\Delta) \) decreases to zero as \( q \) increase to a finite quantity, \( q_{\text{max}}(j) \). We find that only for \( r > r_c \approx 1.9488532 \ldots \), i.e. \( j > (1+r_c)q^2 \approx 2.9488532q^2 \), \( \text{Re}(\Delta) > 0 \). So at given angular momentum \( j \), a sensible saddle point at \( \text{Re}(\Delta) > 0 \) exists only when the electric charge \( q \) is smaller than a maximal value \( q_{\text{max}}(j) = \sqrt{\frac{j}{1+r_c}} \approx 0.582336j^{\frac{1}{2}} \). If \( r \) is smaller than this critical value \( r_c \), \( \text{Re}(\Delta) < 0 \). Note that in the BPS partition function, \( \text{Re}(\Delta) \rightarrow 0^+ \) is analogous to infinite temperature limit, since its dual charge is positive. It is curious to find such an ‘infinite temperature limit’ at finite \( q_{\text{max}}(j) \). See a related comment below. In Fig. 5, \( s \) is plotted with respect to \( j, q \). As before, we chose the blue and yellow
One can see that Re(s) > 0 for arbitrary j, q > 0. Also, when \( j > (1 + r_c)q^2 \), the entropy \( S \) increases as the charges \( j, q \) increases, as expected.

One may want to find explicit forms of chemical potentials and entropy, in terms of charges, at least in certain asymptotic regime. This amounts to knowing the function \( f(r) \). An explicit asymptotic form of \( f(r) \) can be deduced at very large \( r \). When \( r \gg 1, f(r) \to 0 \). Hence, we can approximate the equation (3.21) as

\[
(f(r))^2 r \sim 2\text{Li}_2(f(r)) - \text{Li}_2(1) \sim 2f(r) - \text{Li}_2(1) \to f(r) \sim \frac{1}{r} \left(1 \pm i\pi \sqrt{\frac{r}{6}}\right). \tag{3.22}
\]

So when \( r \gg 1 \), i.e. \( j \gg q^2 \), one obtains the asymptotic formula of the chemical potentials and the entropy in terms of \( j, q \) as follows:

\[
\omega_1 = \frac{f(r)}{q} \sim \frac{1}{qr} \left(1 \pm i\pi \sqrt{\frac{r}{6}}\right) \sim \frac{1}{j} \left(q \pm i\pi \sqrt{\frac{j}{6}}\right),
\]

\[
\Delta = -\log(e^{q\omega_1} - 1) = -\log(e^{f(r)} - 1) \sim -\log f(r) \sim \log r - \log \left(1 \pm i\pi \sqrt{\frac{r}{6}}\right)
\sim \frac{1}{2} \log r - \frac{1}{2} \log \frac{\pi^2}{6} \mp \log i \sim \frac{1}{2} \log \frac{j}{q^2} - \frac{1}{2} \log \frac{\pi^2}{6} \mp \log i,
\]

\[
s = 2(q\Delta + j\omega_1) \sim q \log \frac{j}{q^2} + \left(2 - \log \frac{\pi^2}{6} \mp 2 \log i\right) q \pm i\pi \sqrt{\frac{2j}{3}}.
\tag{3.23}
\]

One finds that the Cardy-like condition \(|\omega_1| \ll 1\) is met in this regime, since \(\text{Re}(\omega_1) \sim \frac{j}{q} \ll 1\) and \(\text{Im}(\omega_1) \sim j^{-\frac{1}{2}} \ll 1\). In fact, just as a side comment, the above approximate entropy formula is very well-fitted even from \( r > r_c \). At, \( r = r_c, \left|\frac{S - S_{\text{approx}}}{S}\right| \sim 0.07\).

We study the validity of our Cardy approximation \(|\omega_1| \ll 1\) for more general \(q, j\)'s, at \( r > r_c \). This can be easily seen in Fig. 6 where we showed the lines with constant \(|\omega_1|\) on the \(q-j\) space. We can highly trust our approximation when \(|\omega_1| \ll 1\). When \( r > r_c \), one can see that if \( j \gtrsim 200 \), then \(|\omega_1| < 0.1\). Therefore, we can say that when \( r > r_c \) and \( j \gtrsim 200 \), our results are within the Cardy regime.

In summary, only when \( j > (1 + r_c)q^2 \), or \( q < q_{\text{max}}(j) = \sqrt{\frac{j}{1 + r_c}} \approx 0.582532 j^{\frac{1}{2}} \), all chemical potentials \(\omega_1, \Delta\) and the macroscopic entropy \(S\) have positive chemical potentials. Otherwise, we find solutions with Re(\(\Delta\)) < 0, which we disregard.

So far, we presented a semi-analytical analysis, using some identities of \(\text{Li}_2\) functions to simplify the structures. However, to be absolutely sure, we plugged in our numerical saddle points back to the original extremization conditions (3.13) without any analytic treatment, to numerically reconfirm the correctness of our results, at least when Re(\(\Delta\)) > 0 in which case \(\text{Li}_2(e^{-\Delta})\), \(\text{Li}_2(e^{-2\Delta})\) are very safely well defined.
|\omega_1| > 0.1 in the region encircled by the red dashed line.

We also note that, in the regime \( q < q_{\text{max}}(j) \), we numerically analyzed the Hessian

\[
H_{ij} \equiv -\frac{\partial^2 \text{Re}(S(Q))}{\partial Q_i \partial Q_j}, \quad (Q_1 = q, \ Q_2 = j)
\]

for \( S \) at the saddle point, to study the local thermodynamic stability. At least for \( q < q_{\text{max}}(j) \), we find that both eigenvalues of \( H_{ij} \) are positive, implying that all susceptibility parameters are positive. Also, we find that \( \log Z \) at the saddle point is always positive in our Cardy regime with large charges, making it more dominant than the gravitons.

Now we turn to discuss some aspects of our results. First of all, it is interesting to see where the small black holes satisfying \( Q^2 = \frac{N^2}{2} J_1 \) are located. Since \( J_2 = 0 \) on the known solutions, this charge condition translates to \( q = J \frac{1}{\sqrt{2}} \approx 0.707 j^{\frac{1}{2}} \). This is the charge region where our new predicted saddle points cannot exist, since its \( q \) is larger than \( q_{\text{max}}(j) \). So to conclude, our free energy predicted new \( \frac{1}{8} \)-BPS black hole-like saddle points with macroscopic entropy, when \( q < q_{\text{max}}(j) = \sqrt{\frac{j}{1+r_c}} \approx 0.582532 j^{\frac{1}{2}} \), in the Cardy regime. Since no such black holes are known so far in this sector, including the small black hole limits of \([14]\), one may ask where to seek for such objects in the gravity dual.

Here we note that there has been some endeavors to construct black holes beyond those known in the literature, based on allowing condensations of matters outside the event horizon. These black holes are called hairy black holes. In the context of global \( AdS_5 \times S^5 \), \([37, 38]\) made studies of hairy black holes with one electric charge \( Q \equiv Q_1 = Q_2 = Q_3 \) at \( J_1 = J_2 = 0 \). At zero angular momentum, one finds that the hairy black hole horizon disappears as one reduces the energy to its BPS bound \( E \lesssim 3Q \), with fixed \( Q \). The end point is either a smooth AdS
soliton when $Q$ is smaller than a critical value $Q_c$, or a singular horizonless solution if $Q > Q_c$. Studying the temperature as $E \searrow 3Q$, the subcritical solutions have zero temperature $T = 0$, while the supercritical solutions have $T = \infty$. As for hairy black holes with nonzero angular momenta, [19, 20] studied those at nonzero $Q \equiv Q_1 = Q_2 = Q_3$ and $J \equiv J_1 = J_2$. In this case, as $E$ is reduced to its BPS bound $M \searrow 3Q + 2J$ at fixed $Q, J$, one still finds black holes with nonzero entropy. Again here, one finds a signal of two different types of endpoints. In the subcritical region $Q < Q_{\text{max}}(J)$, the temperature of the limiting hairy black hole goes to 0. In the supercritical region, $Q > Q_{\text{max}}(J)$, the temperature blows up to $\infty$. The critical charge depends on $J$. It seems that due to numerical limitations, the precise value of $Q_{\text{max}}(J)$ could not be determined [19].

Even if the hairy black holes explained above are in a different charge sector, we find some qualitative similarities with the new saddle points that we find in the Macdonald-Cardy limit. This is because our new saddle points also exist only in a subcritical region $q < q_{\text{max}}(j) \approx 0.582532 j^{\frac{2}{3}}$. The reason why this gets spoiled at $q = q_{\text{max}}(j)$ is because the chemical potential $\text{Re}(\Delta)$ approaches zero, which is analogous to the high temperature limit in the BPS sector. It will be interesting to see if this more than just an analogy.

### 4 Large supersymmetric AdS$_7$ black holes

In this section, we apply the method of section 2.2 to the 6d $\mathcal{N} = (2,0)$ SCFT living on $N$ M5-branes. We shall again rely on a background field method on $S^5$, reducing the system on small temporal $S^1$ in a Cardy-like limit. The results in this section are by no means a ‘derivation’ or ‘full microscopic account’ of AdS$_7$ black holes, even in our highly progressive standard. Technically, in the setup of section 2.2, this is mainly due to the fact that we do not have arguments on why we can ignore finite number of gauge invariant Chern-Simons terms of background fields. We shall assume this, probably appealing to a $\frac{1}{N}$ suppression. Other than this drawback, we show that gauge non-invariant Chern-Simons terms determined by ’t Hooft anomalies derive the free energy suggested in [43] in the Cardy limit, which completely captures the large supersymmetric AdS$_7$ black holes. And then we explain that other higher derivative terms are suppressed in our BPS Cardy limit. So in a sense, our studies reduce the problem of large BPS black holes to studies of finitely many gauge-invariant CS terms on $S^5$. Note that the absence or $\frac{1}{N}$ suppression of some terms are already partly addressed in the literature [17, 31], as we shall explain below.

The SCFT is put on $S^5 \times \mathbb{R}$. The 6d partition function is given by

$$Z = \text{Tr} \left[ e^{-\beta E} e^{-\Delta_1 Q_1 - \Delta_2 Q_2} e^{-\sum_{i=1}^{2} \omega_i J_i} \right],$$

where $Q_1, Q_2$ are two charges for $U(1)^2 \subset SO(5)_5$, and $J_{1,2,3}$ are three $U(1)^3 \subset SO(6)$ angular
momenta on $S^5$. The 6d theory has 16 Poincare supercharges $Q_{J_1,J_2,J_3}$ where $(Q_1, Q_2) = (\pm \frac{1}{2}, \pm \frac{1}{2})$, and $J_i = \pm \frac{1}{2}$ with the product of three $\pm$ signs of $J_i$’s being $-1$. We choose $Q \equiv Q^{++}$ and its conjugate $S$, and constrain $\Delta_I, \omega_i, \beta$ to make $Z$ an index. One should constrain
\[
\Delta_1 + \Delta_2 - \omega_1 - \omega_2 - \omega_3 = 2\pi i \pmod{4\pi i}
\]
and take $\beta \to 0^+$. We will study $\log Z$ at $|\omega_i| \ll 1$, again keeping finite imaginary parts of $\Delta_I$ to admit saddle points in which boson/fermion cancelations are obstructed.

We consider the 6d QFT on $S^5 \times S^1$ coupled to the following background fields:
\[
ds^2 = r^2 \sum_{i=1}^3 \left[ dn_i^2 + n_i^2 \left( d\phi_i - \frac{i\omega_i}{\beta} d\tau \right)^2 \right] + d\tau^2
\]
where $n_i$ label two of the coordinates of $S^5$, constrained as $n_1^2 + n_2^2 + n_3^2 = 1$. The other angles satisfy $\phi_i \sim \phi_i + 2\pi$. $\tau$ has period $\beta$. The $U(1)^2 \subset SO(5)_R$ gauge fields are given by
\[
A^I = -\frac{i\Delta_I}{\beta} d\tau .
\]

In the absence of any 6d Lagrangian description, we find it awkward to concretely discuss the KK modes and follow all the discussions presented in section 2.2. However, the structure of zero modes are well known, given by 5d maximal SYM (deformed by various parameters) on $S^5$. If the $S^1$ radius for KK reduction is small, the 5d zero modes are weakly coupled. Also, we simply assume here that nontrivial holonomy issues of [25] are absent, at least for the $A_{N-1}$ type theory which is of our main concern. The contribution from 5d zero modes’ perturbative partition function on $S^5$ can surely be ignored. This can be seen either by relying on arguments similar to section 2.2, or simply by a $\frac{1}{N}$ suppression since this part will be proportional to $N^2$.

So we study the structure of the effective action of our background fields, which encodes the effects of 6d KK modes along $S^1$. We organize the background fields to the following 5d fields after the KK reduction:
\[
ds_6^2 = ds_5^2 + e^{-2\Phi} (d\tau + a)^2
\]
\[
ds_5^2 = r^2 \left[ d\theta_1^2 + \sin^2 \theta_1 d\theta_2^2 + n_i^2 d\phi_i^2 + \frac{r^2 (\omega_i n_i^2 d\phi_i)^2}{\beta^2 (1 - r^2 n_i^2 \omega_i^2 / \beta^2)} \right]
\]
where the dilaton field $\Phi$ and the gravi-photon field $a$ are given by
\[
e^{-2\Phi} = 1 - r^2 \frac{n_i^2 \omega_i^2}{\beta^2}, \quad a = -i \frac{r^2 \omega_i n_i^2 d\phi_i}{\beta (1 - r^2 n_i^2 \omega_i^2 / \beta^2)}.
\]

\[\text{It will be interesting if one can address whether there are nontrivial issues with outer automorphism twists}\]

[39], whose zero modes are 5d Yang-Mills theories with non-ADE gauge groups. [10] studied such partition functions on $\mathbb{R}^4 \times T^2$ from 5d instanton calculus, which may provide microscopic clues to this question.
The 6d background fields $A^I$ are rewritten as 5d gauge fields $\mathcal{A}^I$ and scalars $A_6^I$ as $A^I = A_6^I (d\tau + a) + \mathcal{A}^I$, where

$$A_6^I = -\frac{i\Delta_I}{\beta}, \quad \mathcal{A}^I = -A_6^I a. \quad (4.7)$$

In our scaling limit ($\beta \ll |\omega_i| \ll 1$), the leading terms will turn out to come from Chern-Simons terms, at order $\frac{\beta^0}{|\omega_i| |\omega_j|}$. So it is crucial to know all their coefficients to get the free energy in our Cardy limit. The gauge non-invariant CS terms are again dictated by the 't Hooft anomalies of $SO(5)_R$, which will be presented below. The gauge invariant Chern-Simons terms of $\mathcal{A}^I$ and $a$ take the forms of [17]

$$\beta^{-3} a \wedge da \wedge da, \quad \beta^{-2} \mathcal{A}^I \wedge da \wedge da, \quad \beta^{-1} \mathcal{A}^I \wedge d\mathcal{A}^J \wedge da, \quad \mathcal{A}^I \wedge d\mathcal{A}^J \wedge d\mathcal{A}^K. \quad (4.8)$$

Here, just like in section 2.2, we do not discuss Chern-Simons terms involving gravitational fields since they will be absent or subleading in our scaling limit. (See below in this section.) Now, unlike the 3d CS terms for 4d $N = 4$ theory, we are not given enough discrete symmetries of 6d (2,0) theory to forbid them all. In fact, some of them are believed to be nonzero.

Trying to see if one can use abstract symmetry-based arguments to forbid CS terms, one can only partly achieve the goal. Firstly, $\mathcal{A}^I \wedge d\mathcal{A}^J \wedge d\mathcal{A}^K$ at $I, J, K = 1, 2, \mathcal{A}^I \wedge d\mathcal{A}^J \wedge da$ at $I \neq J$ and $\mathcal{A}^I \wedge da \wedge da$ can be forbidden from the Weyl symmetry of $SO(5)_R$, just like we excluded $\mathcal{A}^I \wedge da$ or $\mathcal{A}^I \wedge d\mathcal{A}^J$ at $I \neq J$ in section 2.2. In section 2.2, one used parity (suitably blind to $SO(6)_R$) to forbid other terms. However, in 6d (2,0) theory, the system is intrinsically chiral, so that we have no simple argument to forbid

$$\beta^{-3} a \wedge da \wedge da, \quad \beta^{-1} \sum_{I=1}^2 \mathcal{A}^I \wedge d\mathcal{A}^I \wedge da. \quad (4.9)$$

A proposal made in [17] had a consequence that the coefficient of $a \wedge da \wedge da$ is zero for the (2,0) theory. This is partly supported from a SUSY calculus of the index on $S^5 \times S^1$ at high temperature [31], by not exhibiting a free energy at order $\beta^{-3}$ (although the calculus was carried out after turning off many chemical potentials). Also, the $\beta^{-1}$ term of the free energy studied in [31] was at order $N^1$. This may be related to an argument that the second term of (4.9) is $\frac{1}{N}$ suppressed. Anyway, in the remaining part of this section, we shall assume the vanishing or suppression of (4.9). Perhaps carefully studying the microscopically computed partition functions of the (2,0) theory at high temperature (e.g. see [11]), one may be able to determine these coefficients.

The gauge non-invariant Chern-Simons terms for $\mathcal{A}^I, A_6^I$ can be determined from the 't Hooft anomaly of $SO(5)_R$. Note that the anomaly 8-form of 6d (2,0) $A_{N-1}$ theory is

$$I_8 = \frac{N^3 - N}{24} p_2(N) + \frac{N}{48} \left[ p_2(N) - p_2(T) + \frac{1}{4} (p_1(T) - p_1(N))^2 \right] \quad (4.10)$$
with

\[p_1(N) = -\frac{1}{2(2\pi)^2} \text{tr} F^2, \quad p_2(N) = \frac{1}{(2\pi)^4} \left( -\frac{1}{4} \text{tr} F^4 + \frac{1}{8} (\text{tr} F^2)^2 \right).\]  

(4.11)

\([11]\) discussed the gauge non-invariant Chern-Simons term for \(A_0^1 + A_0^2 = 0, A^1 + A^2 = 0,\) to study certain asymptotic aspects of the free energy of (2, 0) theory on \(\mathbb{R}^4 \times T^2.\) Generalizing the calculus of \([11]\) for \(U(1)^2,\) one obtains\(^{11}\)

\[S_{CS} = \frac{i(N^3 - N)}{192\pi^3} \beta \int_{S^5} \left[ 2 \left( A_0^1 A^1 \wedge dA^2 \wedge dA^2 + A_0^2 A^2 \wedge dA^1 \wedge dA^1 \right) \\
+ \left( 4A_0^1 A_0^2 A_1 \wedge dA^2 \wedge da + (A_0^2)^2 A_0^1 \wedge dA^2 \wedge da + (A_0^2)^2 A_0^1 \wedge dA^1 \wedge da \right) \\
+ 2 \left( (A_0^2)^2 A_0^1 A^1 \wedge da \wedge da + (A_0^1)^2 A_0^2 A^2 \wedge da \wedge da \right) + (A_0^2)^2 (A_0^1)^2 a \wedge da \wedge da \right] \\
+ iN\beta \sum_{I=1}^{2} \int_{S^5} \left[ 4A_0^1 A^I \wedge dA^1 \wedge dA^I + 6(A_0^1)^2 A^I \wedge dA^I \wedge da \\
+ 4(A_0^1)^3 A^I \wedge da \wedge da + (A_0^1)^4 a \wedge da \wedge da \right].\]

(4.12)

Inserting (4.6), (4.7) to (4.12), one obtains

\[S_{CS} = -\frac{iN^3}{192\pi^3} \frac{\Delta_1^2 \Delta_2^2}{\beta^3} \int_{S^5} a \wedge da \wedge da + \mathcal{O}(N^1).\]

(4.13)

Evaluating \(\int a \wedge da \wedge da\) with (4.6), one obtains

\[\int_{S^5} a \wedge da \wedge da = -\frac{(2\pi)^3(-i)^3 r^6 \omega_1 \omega_2 \omega_3}{\beta^3} \frac{1}{\left(1 - \frac{r^2 \omega_3^2}{\ell^2}\right) \left(1 - \frac{r^2 \omega_2^2}{\ell^2}\right) \left(1 - \frac{r^2 \omega_1^2}{\ell^2}\right)}.\]

(4.14)

Taking the \(\beta \to 0^+\) limit, one obtains

\[S_{CS} = \frac{N^3}{24} \frac{\Delta_1^2 \Delta_2^2}{\omega_1 \omega_2 \omega_3}.\]

(4.15)

Therefore, the asymptotic free energy one obtains from \(S_{CS}\) is

\[\log Z \sim -S_{CS} = -\frac{N^3}{24} \frac{\Delta_1^2 \Delta_2^2}{\omega_1 \omega_2 \omega_3},\]

supposing that other higher derivative terms are suppressed.

We now examine other background terms in the \(S^5\) effective action, assuming the absences or large \(N\) suppressions of particular low-order terms (4.13), as discussed above. All other terms arranged in an infinite tower of derivative expansion will turn out to be suppressed in the scaling limit \(\beta/r \ll \omega \ll 1,\) as we shall illustrate with sample terms below. We shall study the case

\(^{11}\)We flipped the overall sign of \(S_{CS}\) compared with \([11],\) due to opposite 6d chirality conventions. E.g., in \([11],\) supercharges contain (anti-chiral)\(_{\mathbb{R}^4}\times\) (right chiral)\(_{T^2},\) which is in (0, 2) spinors in our convention here.
without $\epsilon^{\mu\nu\rho\sigma\lambda}$ first and then the other case. The analysis on the $S^5$ background action will be parallel to that on the $S^3$ action done in section 2.2. So we shall keep our discussion more concise, inspecting a few sample terms rather than attempting an exhaustive list of corrections to certain order, as in (2.62). Below we assume $\omega_1 = \omega_2 = \omega_3 \equiv \omega$ for simplification, so that the dilaton $\Phi$ becomes a constant.

We first consider the background action built from the scalar contraction of tensors without $\epsilon^{\mu\nu\rho\sigma\lambda}$. Evaluating a few terms which involve 0, 2, and 4 derivatives, we find

$$\frac{1}{(2\pi)^3} \int \beta \omega e^{5\Phi} \sqrt{g} = \frac{\beta r^5}{8(\beta^2 - r^2\omega^2)^3} = -\frac{\beta}{8\omega^6} + O \left( \frac{\beta^3}{r^3\omega^8} \right) \tag{4.17}$$

$$\frac{1}{(2\pi)^3} \int \beta^{-3} e^{3\Phi} \sqrt{g} R_{\mu\nu} R^{\mu\nu} = \frac{5\beta^3 r^3 - 6\beta r^5 \omega^2}{2(\beta^2 - r^2\omega^2)^3} = \frac{3\beta}{r\omega^4} + O \left( \frac{\beta^3}{r^3\omega^6} \right)$$

$$\frac{1}{(2\pi)^3} \int \beta^{-1} e^{\Phi} \sqrt{g} F_{ab} F^{ab} = \frac{\beta r^5 \omega^2 \Delta^I \Delta^J}{2(\beta^2 - r^2\omega^2)^3} = \frac{\beta \Delta^I \Delta^J}{2r\omega^4} + O \left( \frac{\beta^3}{r^5\omega^8} \right)$$

$$\frac{1}{(2\pi)^3} \int \beta e^{-\Phi} \sqrt{g} (\nabla_c F^I) (\nabla_c F^{ab}) = \frac{\beta^3 \Delta^I \Delta^J r^3 \omega^2}{(\beta^2 - r^2\omega^2)^3} = -\frac{\beta^3 \Delta^I \Delta^J}{r^3\omega^4} + O \left( \frac{\beta^3}{r^5\omega^6} \right)$$

$$\frac{1}{(2\pi)^3} \int \beta e^{-\Phi} \sqrt{g} R_{abcd} R^{abcd} = \frac{24\beta r^5 \omega^4 - 12\beta^3 r^3 \omega^2 + 5\beta^5 r}{(\beta^2 - r^2\omega^2)^3} = \frac{24\beta}{r\omega^2} + O \left( \frac{\beta^3}{r^3\omega^4} \right)$$

$$\frac{1}{(2\pi)^3} \int \beta e^{-\Phi} \sqrt{g} F_{ab} F^{cd} R_{abcd} = -\frac{\Delta^I \Delta^J (6\beta r^5 \omega^4 - \beta^3 r^3 \omega^2)}{(\beta^2 - r^2\omega^2)^3} = \frac{6\beta \Delta^I \Delta^J}{r^2\omega^2} + O \left( \frac{\beta^3}{r^3\omega^4} \right)$$

$$\frac{1}{(2\pi)^3} \int \beta e^{-5\Phi} \sqrt{g} F_{ab} F^{cd} F_{cd} = \frac{2\beta r^5 \omega^4 - \Delta^I \Delta^J \Delta^K \Delta^L}{(\beta^2 - r^2\omega^2)^3} = -\frac{2\beta \Delta^I \Delta^J \Delta^K \Delta^L}{r^2\omega^2} + O \left( \frac{\beta^3}{r^3\omega^4} \right)$$

where the indices $I, J, K, L$ run over $0, 1, 2, 3$ and $\Delta^0 \equiv -i$. These terms are all much smaller than (4.15) in the scaling limit $\beta/r \ll \omega \ll 1$. Moreover, their leading behavior is consistent with the following speculation: An action made of $n_1$ curvature tensors, $n_2$ graviphoton field strengths, $n_3$ background $U(1)^2 \subset SO(5)_R$ field strengths, $n_4$ derivatives scales as

$$\frac{\beta^{1+n_4} \Delta^{n_3}}{r^{1+n_4} \omega^{6-2n_1-2n_2-n_3}} + O \left( \frac{\beta^{3+n_4}}{r^{3+n_4} \omega^{8-2n_1-2n_2-n_3}} \right), \tag{4.18}$$

Notice that it differs from (2.63) due to the additional factor $r^2 \cdot (\beta e^{-\Phi})^{-2} \sim \omega^{-2}$. All these terms would be suppressed by taking the scaling limit $\beta/r \ll \omega \ll 1$.

Now we turn to the background action associated to a pseudo-scalar Lagrangian density which has $\epsilon^{\mu\nu\rho\sigma\lambda}$. It can be either a Chern-Simons action or the action coming from a gauge invariant Lagrangian density. Gauge non-invariant CS terms have been determined to be (4.12) from 6d ’t Hooft anomaly. The analogue of the gravitational CS term (2.60) that involves the spin connection $\omega^{ab}_\mu$ cannot exist in 5 dimensions, but only in 3, 7, 11 dimensions [42]. The Weyl symmetry of $SO(5)_R$ restricts the other gauge invariant CS terms to be invariant under the simultaneous sign flip of $\mathcal{A}^{I=1}$ and $\mathcal{A}^{I=2}$. Displaying all possible CS terms,

$$\frac{\beta^{-3}}{5(2\pi)^3} \int \epsilon^{\mu\nu\rho\sigma\lambda} a_\mu (da)_\nu (da)_\sigma = \frac{\beta r^6 \omega^3}{120 (\beta^2 - r^2\omega^2)^3} = -\frac{i}{120\omega^3} + O \left( \frac{\beta^2}{r^2\omega^5} \right) \tag{4.19}$$
\[
\frac{\beta^{-1}}{5!(2\pi)^3} \int \epsilon^{\mu \nu \rho \lambda} a_\mu R_{\nu \rho} \alpha^\beta R_{\sigma \alpha \lambda \beta} = -\frac{ir_5^6 \omega^5}{5 (\beta^2 - r^2 \omega^2)^3} = i \frac{5 \omega}{\beta^2} + \mathcal{O} \left( \frac{\beta^2}{r^2 \omega^3} \right) \quad (4.20)
\]

\[
\frac{\beta^{-1}}{5!(2\pi)^3} \int \epsilon^{\mu \nu \rho \lambda} \beta J_{\mu} \beta F_{\nu \rho} (da)_{\sigma \lambda} = -\frac{i \Delta^i \Delta^j \omega^3}{120 (\beta^2 - r^2 \omega^2)^3} = i \frac{\beta^2}{120 \omega^3} (\beta^2 - r^2 \omega^2) \quad (4.21)
\]

In fact, as asserted earlier, CS terms containing gravitational terms are suppressed, while other gauge invariant CS terms are not. As noted above, we assume (partly relying on assertions/observations made in the literature) that their coefficients are either exactly zero or 1 suppressed. Then we move to study the action associated to the gauge invariant Lagrangian density containing \( \epsilon^{\mu \nu \rho \lambda} \). We compute some non-vanishing terms of this kind, e.g.,

\[
\frac{1}{5!(2\pi)^3} \int e^{-6 \Phi} \epsilon^{\mu \nu \rho \lambda} F^I_{\mu \alpha} F^J_{\nu \beta} F^0_{\rho \sigma} \left( \nabla_\alpha F^I_{J \beta} \right) F^I_{\alpha \delta} F^0_{\beta} = \frac{i \beta^2 r^4 \omega^2}{30 (\beta^2 - r^2 \omega^2)^3} \quad (4.22)
\]

\[
\frac{1}{5!(2\pi)^3} \int e^{-14 \Phi} \epsilon^{\mu \nu \rho \lambda} F^I_{\mu \alpha} F^J_{\nu \beta} F^0_{\rho \sigma} \left( \nabla_\alpha F^I_{J \beta} \right) F^I_{\alpha \delta} F^J_{\beta \gamma} = \frac{i \beta^2 r^4 \omega^4}{30 (\beta^2 - r^2 \omega^2)^3} \quad (4.23)
\]

We observe that their scaling behavior in the limit \( \beta/r \ll \omega \ll 1 \) follows (4.18). All these terms would be subleading corrections to the free energy.

Now, we perform Legendre transformation of (4.16) to the microcanonical ensemble. One should extremize the following entropy function:

\[
S(\Delta_I, \omega_i; Q_I, J_I) = -\frac{N^3}{24} \frac{\Delta^2 \omega^2}{\omega \omega \omega} + \sum_{I=1}^2 Q_I \Delta_I + \sum_{i=1}^3 J_i \omega_i \quad (4.24)
\]

This problem was studied in [43], reproducing the entropy of known BPS \( \text{AdS}_7 \) black holes of [22, 23]. We will review the calculation in [43]. As in section 2.3, we also extend the studies of [43] by checking the agreements of chemical potentials. This allows us to regard the real part of (4.16) as the free energy of known BPS black holes, even away from the Cardy limit.

Since we consider the index, the extremization should be performed on the specific surface of the chemical potential space where

\[
\Delta_1 + \Delta_2 - \omega_1 - \omega_2 - \omega_3 = 2\pi i \quad (4.23)
\]

This also reflects the ignorance of the index on one of the five charges. The relevant BPS states saturate the bound \( E \geq 2Q_1 + 2Q_2 + J_1 + J_2 + J_3 \). On the surface (4.23), one can reparameterize the chemical potentials with four unconstrained complex variables \( z_{1,2,3,4} \).

\[
\Delta_I = \frac{2\pi i z_I}{1 + z_1 + z_2 + z_3 + z_4}, \quad I = 1, 2
\]

\[
\omega_1 = \frac{-2\pi i z_3}{1 + z_1 + z_2 + z_3 + z_4}, \quad \omega_2 = \frac{-2\pi i z_4}{1 + z_1 + z_2 + z_3 + z_4}, \quad \omega_3 = \frac{-2\pi i}{1 + z_1 + z_2 + z_3 + z_4} \quad (4.24)
\]
With this reparametrization, the entropy function (4.22) becomes

$$S = \frac{2\pi i}{1 + z_1 + z_2 + z_3 + z_4} \left(\frac{N^3(z_1 z_2)^2}{z_1 z_2} + Q_1 z_1 + Q_2 z_2 - J_1 z_3 - J_2 z_4 - J_3\right)$$  \hspace{1cm} (4.25)$$

Extremization in \(z_i\) yields four saddle point equations, which can be reorganized as follows:

$$Q_I + J_3 = -\frac{N^3 (z_1 z_2)^2}{z_3 z_4} \left(1 + \frac{2}{z_I}\right),$$
$$J_1 - J_3 = -\frac{N^3 (z_1 z_2)^2}{z_3 z_4} \left(-1 + \frac{1}{z_3}\right),$$
$$J_2 - J_3 = -\frac{N^3 (z_1 z_2)^2}{z_3 z_4} \left(-1 + \frac{1}{z_4}\right).$$  \hspace{1cm} (4.26)$

At the saddle point, the black hole entropy becomes

$$S = 2\pi i \left(-\frac{N^3 (z_1 z_2)^2}{z_3 z_4} - J_3\right).$$  \hspace{1cm} (4.27)$$

Using the last expression, one can replace the common factor \(-\frac{N^3 (z_1 z_2)^2}{z_3 z_4}\) in (4.20) into \(S = \frac{S}{2\pi i} + J_3\). Then the saddle point values of \(z_i\) can be expressed in terms of the charges and the entropy as follows:

$$z_I = -\frac{2S + 2\pi i J_3}{S - 2\pi i Q_I},$$
$$z_3 = \frac{S + 2\pi i J_1}{S + 2\pi i J_1},$$
$$z_4 = \frac{S + 2\pi i J_2}{S + 2\pi i J_2}.$$  \hspace{1cm} (4.28)$$

Plugging in these values for \(z_{1,2,3,4}\) to (4.27), one obtains a simple quartic equation for \(S\) in terms of charges:

$$\left(S - 2\pi i Q_1\right)^2 \left(S - 2\pi i Q_2\right)^2 + \frac{4\pi i N^3}{3} (S + 2\pi i J_1) \left(S + 2\pi i J_2\right) \left(S + 2\pi i J_3\right) = 0.$$  \hspace{1cm} (4.29)$$

The equation (4.29) has four complex solutions \(S\), at given five real charges. Again, our general attitude on \(\text{Im}(S)\) is that it is the phase factors that one may end up with, by allowing imaginary parts of chemical potentials to ideally obstruct boson/fermion cancellations. However, just as in the case of section 2.2, special solutions are somehow known at the surface \(\text{Im}(S) = 0\). So among the four solutions of (4.29), we study the special sets of charges which allow a real and positive solution for \(S\). Note that (4.29) has the form of \((a_4 S^4 + a_2 S^2 + a_0) + i(a_3 S^3 + a_1 S^1) = 0\) with real coefficients \(a_i\). Demanding a real solution requires \(a_4 S^4 + a_2 S^2 + a_0\) and \(a_3 S^3 + a_1 S^1\) to separately vanish. This leads to the two alternative expressions for the entropy:

$$\left(S\right)^2 = \frac{3(Q_1^2 Q_2 + Q_1 Q_2^2) - N^3(J_1 J_2 + J_2 J_3 + J_3 J_1)}{3(Q_1 + Q_2) - N^3},$$
$$\left(S\right)^2 = \left(\frac{N^3}{3} \left(J_1 + J_2 + J_3\right) + \frac{Q_1^2 + Q_2^2}{2} + 2Q_1 Q_2\right) \times \left(1 - \frac{2\pi i N^3 J_1 J_2 J_3 + Q_1^2 Q_2^2}{\left(N^3 \left(J_1 + J_2 + J_3\right) + \frac{Q_1^2 + Q_2^2}{2} + 2Q_1 Q_2\right)^2}\right).$$  \hspace{1cm} (4.30)$$

The compatibility of two expressions require a charge relation for \(\text{Im}(S) = 0\).
Here, note that the known BPS black hole solutions also satisfy a charge relation. Unfortunately, black hole solutions with all unequal $Q_1, Q_2, J_1, J_2, J_3$ are yet unknown. This is most probably just a technical limitation. A class of non-extremal solutions studied in [22] has unequal $Q_1, Q_2$, but equal angular momentum $J_1 = J_2 = J_3 \equiv J$. Together with energy $E$, there are 4 parameter solutions for independent $E, Q_1, Q_2, J$. However, imposing a BPS limit for $E = 2Q_1 + 2Q_2 + 3J$, one also has to impose a separate condition that the smooth horizon is not spoiled. So one ends up with a 2 parameter solution with nonzero $Q_1, Q_2, J$, where the last three charges meet a relation. A different slice of black hole solutions was found in [23]. The solutions here satisfy $Q_1 = Q_2 \equiv Q$, with independent $J_1, J_2, J_3$ and energy $E$. Again imposing the BPS condition for $E = 4Q + J_1 + J_2 + J_3$ and smooth horizon condition, one obtains a 3 parameter solution with $Q, J_1, J_2, J_3$, so that the charges again meet a relation. In both cases, one finds that the charge relation is precisely the two right hand sides of (4.30) being equal. So the known BPS black holes happen to live on the surface $\text{Im}(S)$, for which we again do not have a good physical insight. On this surface, one can again show that the Bekenstein-Hawking entropy of these black holes precisely agree with our (4.30). The results summarized in this paragraph have all been reported in [43] already. Now we have a sort of ‘derivation’ of (1.22) in the Cardy regime $|\omega_i| \ll 1$, with certain assumptions stated earlier in this section. We hope the discussions presented so far in this section to shed good lights on AdS$_7$ black holes, and also to 6d (2,0) theory (especially about the CS coefficients in the high temperature expansion).

In the remaining part of this section, we supplement [43] by showing that the chemical potentials of black holes agree with the real parts of $\Delta_\mathcal{I}, \omega_i$. We only do so for the case with general $Q_1, Q_2$ and equal angular momenta $J_1 = J_2 = J_3 \equiv J$. As in section 2.3, we should start from non-BPS solutions and take $T \to 0$ BPS limit to read off BPS chemical potentials.

The energy, charges and entropy for non-extremal black holes of [22, 10] are given in terms of four parameters $\delta_{1,2}$, $m$ and $a$\textsuperscript{12}.

$$E = \frac{1}{g_N} \cdot \frac{m \pi^2}{16 \Xi^4} \left[ 12 \Xi^2_+(\Xi^2_+ - 2) - 2c_1c_2a^2g^2(21\Xi^4_+ - 20\Xi^3_+ - 15\Xi^2_+ - 10\Xi_+ - 6) + (c_1^2 + c_2^2)(21\Xi^6_+ - 62\Xi^5_+ + 40\Xi^4_+ + 13\Xi^2_+ - 2\Xi_+ + 6) \right]$$

$$J = -\frac{1}{g_N} \cdot \frac{ma \pi^2}{16 \Xi^4} \left[ 4ag\Xi^2_+ - 2c_1c_2(2\Xi^5_+ - 3\Xi^4_+ - 1) + ag(c_1^2 + c_2^2)(\Xi_+ + 1)(2\Xi^3_+ - 3\Xi^2_+ - 1) \right]$$

$$Q_1 = \frac{1}{2g_N} \cdot \frac{m \pi^2 s_1}{4 \Xi^3} \left[ a^2g^2c_2(2\Xi^2_+ + 1) - c_1(2\Xi^3_+ - 3\Xi_+ - 1) \right]$$

$$Q_2 = \frac{1}{2g_N} \cdot \frac{m \pi^2 s_2}{4 \Xi^3} \left[ a^2g^2c_1(2\Xi^2_+ + 1) - c_2(2\Xi^3_+ - 3\Xi_+ - 1) \right]$$

$$S = \frac{1}{4g_N} \cdot \frac{\pi^3 (r^2 + a^2)}{\Xi^3} \sqrt{f_1(r_+)} . \quad (4.31)$$

\textsuperscript{12}We change the normalization of (4.7) and (4.9) in [10], by multiplying the first factors put before $\cdot$ on all right hand sides of our (4.31). This is mostly to convert to our convention.
Here, the parameters and functions are defined by:

\[ s_i = \sinh \delta_i, \quad c_i = \cosh \delta_i, \quad \Xi_{\pm} = 1 \pm ag, \quad \Xi = 1 - a^2 g^2, \quad \rho = \sqrt{\Xi} r, \quad H_i = 1 + \frac{2 ms_i^2}{\rho^3} \]

\[ \alpha_1 = c_1 - \frac{1}{2} (1 - \Xi_+^2)(c_1 - c_2), \quad \alpha_2 = c_2 + \frac{1}{2} (1 - \Xi_+^2)(c_1 - c_2), \quad \beta_1 = -a \alpha_2, \quad \beta_2 = -a \alpha_1 \]

\[ f_1(r) = \Xi \rho^6 H_1 H_2 - \frac{4 \Xi_+^2 m^2 a^2 s_1^2 s_2^2}{\rho^4} + \frac{1}{2} m a^2 \left( 4 \Xi_+^2 + 2 c_1 c_2 (1 - \Xi_+^4) + (1 - \Xi_+^2)^2 (c_1^2 + c_2^2) \right) \]

\[ f_2(r) = -\frac{1}{2} g \Xi_+ \rho^6 H_1 H_2 + \frac{1}{4} m a \left( 2 (1 + \Xi_+^4) c_1 c_2 + (1 - \Xi_+^4) (c_1^2 + c_2^2) \right) \]

\[ Y(r) = g^2 \rho^6 H_1 H_2 + \Xi \rho^6 + \frac{1}{2} m a^2 \left( 4 \Xi_+^2 + 2 (1 - \Xi_+^4) c_1 c_2 + (1 - \Xi_+^2)^2 (c_1^2 + c_2^2) \right) \]

\[ - \frac{1}{2} m \rho^2 \left( 4 \Xi + a^2 g^2 (6 + 8a g + 3a^2 g^2) c_1 c_2 - a^2 g^2 (2 + a g) (2 + 3ag) (c_1^2 + c_2^2) \right). \] (4.32)

\[ r = r_+ \] is the largest positive root of \( Y(r) = 0 \). The BPS limit is achieved by setting:

\[ a = \frac{2}{3g} \frac{1}{1 - e^{\delta_1 + \delta_2}}, \quad m = \frac{128 e^{\delta_1 + \delta_2} (3 e^{\delta_1 + \delta_2} - 1)^3}{729 g^4 (e^{2 \delta_1} - 1) (e^{2 \delta_2} - 1) (e^{\delta_1 + \delta_2} + 1) (e^{\delta_1 + \delta_2} - 1)^4}. \] (4.33)

Then the outer horizon is located at:

\[ r_+ = \sqrt{\frac{16}{3g^2 (e^{\delta_1 + \delta_2} + 1) (e^{\delta_1 + \delta_2} - 5)}}. \] (4.34)

Inserting (4.33) and (4.34) to (4.31), one can obtain BPS relation \( E = 3J + 2Q_1 + 2Q_2 \). Here, the seven dimensional Newton’s constant is given by \( G_N = \frac{3 g^2}{16 \pi N \pi} \) for \( AdS_7 \times S^4 \) for \( N \) M5-branes.

g is the inverse-radius of \( AdS_7 \).

The first law of black hole thermodynamics is given by:

\[ dE = T dS + 3 \Omega dJ + \Phi_1 dQ_1 + \Phi_2 dQ_2, \] (4.35)

with the chemical potentials are:

\[ T = \frac{1}{4 \pi g \rho^3 \Xi f_1} \frac{\partial Y}{\partial r}, \quad \Omega = \frac{1}{g} \left( g + \frac{2 f_2}{f_1} \Xi_- \right), \quad \Phi_i = \frac{4 m s_i}{\rho^4 H_i} \left( \alpha_i \Xi_- + \beta_i \frac{2 f_2 \Xi_-}{f_1} \right). \] (4.36)

All functions are evaluated at \( r = r_+ \). The free energy \( F \) in the canonical ensemble is given by:

\[ F = E - TS - 3 \Omega J - \Phi_I Q_I \] (4.37)

Defining \( \Delta E = E - 2 \sum_i Q_i - 3J \), one finds:

\[ \frac{F}{T} = \frac{\Delta E}{T} - S + \sum_i \frac{2 - \Phi_i}{T} Q_i + 3 \frac{1 - \Omega}{T} J \] (4.38)

---

13 We corrected a typo in (4.5) of [10], where we correct \( \rho_{\text{ours}} = \sqrt{r^2 + a^2} \) by \( \rho_{\text{ours}} = \sqrt{\Xi} r \).

14 We corrected a typo in (4.46) of [10]: \((3e^{\delta_1 + \delta_2} - 1)^2 \rightarrow (3e^{\delta_1 + \delta_2} - 1)^3 \) in the numerator of \( m \).

15 We changed normalization and corrected typo in (4.7) of [10], by all the factors shown with red colors. The correct temperature and chemical potentials can be derived from the metric (2.5) of [13].
Taking the BPS limit (4.33), the black hole chemical potentials approach $\Phi_I \to 2$ and $\Omega \to 1$. Therefore, we can define BPS chemical potentials as

$$\xi_I = \lim_{T \to 0} \frac{2 - \Phi_I}{T}, \quad \zeta = \lim_{T \to 0} \frac{1 - \Omega}{T}.$$  \hfill (4.39)

Since the entropy $S$ is finite in BPS limit, $F_{\text{BPS}} \equiv \frac{F - \Delta E}{T}$ should remain finite. Therefore,

$$S = -F_{\text{BPS}} + \sum_I \xi_I Q_I + 3\zeta J.$$  \hfill (4.40)

We checked that $\xi_I, \zeta$ computed from (4.39) agree with Re($\Delta_I$), Re($\omega$), computed from (4.22).

5 Discussions and future directions

We first discuss possible subtleties of our results. We also try to suggest conservative interpretations of our results, in case some readers might be worrying about subtleties.

- Throughout this paper, we mostly took (with one exception) Cardy-like limits which suppress the fluctuations relying on large $J$. However, general black holes are semi-classical saddle points at large $N$, rather than large charges. So we are assuming an interpolation, which connects large $N$ saddle points given by black holes and large $J$ saddle points of our QFT. This often turned out to provide the correct quantitative results, starting from the seminal work [1]. The fact that our Cardy free energy successfully captures known black holes of [14, 15] makes us to hope that a similar situation is happening here.

- In our Cardy limit, we took the $U(N)$ gauge holonomies $\alpha_a$ to be at the maximally deconfining point. One cannot imagine such saddle points at finite charges (or finite $\omega$), because the Haar measure repulsion forbids $\alpha_a$’s to be on top of another [4, 7]. We expect our maximally deconfining saddle point to actually mean that the distances of $\alpha_a$’s are suppressed by small $\omega$. It is easy to check that this is the local saddle point in the Cardy limit, but one may ask if this is the global minimum of free energy. There are examples of $4d \mathcal{N} = 1$ QFTs in which this fails to be true [25]. Considering the empirical relation between more nontrivial saddle points and the behaviors of $Z[S^3]$ [25], it seems that our model should be safe of this issue. We checked explicitly that our $U(2)$ saddle point is the global minimum, but only in a self-consistent way at the specific value of $\Delta_I, \omega_i$ for equal charge black holes. Studies at higher $N$ and more general values of $\Delta, \omega$ appear to be cumbersome. However, at the very least, we have identified their dual black hole saddle points, no matter stable or metastable. So our maximally deconfining saddle points should have substantial physical implications to the large $N$ gravity dual.
• The fact that BPS black holes exist only with a charge relation might be somewhat puzzling from the QFT dual side, especially after we claimed that we have counted them (at large charges). We have little to comment on it, especially in our Cardy regime in which other solutions seem to be unknown so far [19, 20]. Especially, intertwined with the ignorance of the index on one of the 5 charges, the possibility of more general black holes seems not easy to address within our results. However, technically from the gravity side, such charge relations of BPS black holes are ubiquitous. Familiar examples are single-centered 4d black holes [44] at zero angular momentum, or 5d BMPV black holes [45] with self-dual angular momenta. By now we know much richer families of BPS black solutions, such as 4d multi-centered black holes [46] or 5d black rings [47], which violate such charge relations. In AdS, one can naturally seek for hairy black holes. The BPS version of such black holes were recently reported [19, 20], even though it appears not in our large rotation regime (at least from the data presented there).

• We studied Cardy-like and non-Cardy-like scaling limits of the $1^\frac{1}{8}$-BPS Macdonald index. In the latter, we have identified the small black hole limit of the known BPS solutions (third reference of [14]). In the former, our Cardy free energy is quite nontrivial, and exhibits rich saddle points. These saddle points exhibit properties very reminiscent of hairy black holes [37, 19]. If one can again trust the smooth interpolation between our Cardy saddle point and the large $N$ saddle point, we can claim that we have predicted new (hairy) black holes in the Macdonald sector. Since no solutions are actually constructed yet, we are much less confident about the issues raised above in this section. Perhaps actual constructions of such gravity solutions can clear the uncertainty.

• There were extra assumptions in our discussions of large AdS$_7$ black holes. One issue is the unknown coefficients of gauge invariant CS terms on $S^5$, in the high temperature expansion. With gradually accumulating studies on this 6d SCFT, we may hopefully be able to answer this question conclusively.

We think there are many interesting future directions to pursue. We finish this paper by briefly mentioning some of them.

• Having seen macroscopic entropies from the index, one should expect an explicit construction of such operators at weak-coupling. At 1-loop level, the BPS states are mapped to cohomologies of the supercharge $Q$. [7, 10, 11, 12, 13]. Considering the free QFT analysis of section 2.1, (2.7) and comments above it, fermionic fields may be responsible for our asymptotic free energy. [10, 11] considered a class of such operators called ‘Fermi liquid operators.’ Unfortunately, the operators discussed there were shown to be (weakly) renormalized, even at weak coupling. As already mentioned in [11] as a possible scenario, dressing these operators with other fermion fields might yield large number of new BPS
states. Perhaps a clever ‘ansatz’ for such operators using all four fermions should be discovered, generalizing [10]. [13] performed a systematic analysis of this cohomology at $N = 2, 3$, up to certain energy order, without using an ansatz. However, it is not completely clear to us whether the energy orders covered in [13] are definitely well above $N^2$. For instance, our Cardy limit demands $\omega$ to be small. Its conjugate $J$ is given by $J \sim \frac{1}{\omega^3}$. So even if one generously accepts $\omega \sim 0.1$ to be small, the associated charge will be $J \sim 10^3$, definitely out of reach in [13].

- On the other hand, the roles of fermions seen around (2.7) might be an ‘emergent’ one. This is because, if we study the Cardy limit honestly from the index, (2.28) is obtained by both bosons and fermions. Here, note that there is a known toy model in which a fermion picture emerges. This is the half-BPS sector of 4d $\mathcal{N} = 4$ Yang-Mills theory, exhibiting a Fermi droplet picture [48, 49]. It may be interesting to clarify the true nature of the ‘fermion picture’ we think we see around (2.7).

- As also commented at various places earlier, it will be interesting to see what one obtains by going beyond the Cardy limit, seeking for large $N$ saddle points of $N$ integral variables, again carefully tuning the imaginary parts of the chemical potentials. The analysis of [7] already seems to set some limitation of this approach, but it would be interesting (if possible) to see how their results at order 1 chemical potentials get connected to our results in the Cardy-like limit. However, at least at the moment, this appears to be a very challenging calculus.

- In the $\frac{1}{8}$-BPS Macdonald sector, our studies ‘predict’ that there should be black holes, in case one believes that our Cardy saddle points will transmute to large $N$ saddle points. Known black holes reduce to small black holes with vanishing entropy in this limit. Considering some qualitative aspects similar to the recently explored hairy black holes, we speculate that they might be hairy $\frac{1}{8}$-BPS black holes. Since one is now equipped with 4 real Killing spinors, perhaps combining the general SUSY analysis with a clever ansatz may shed lights on such solutions.

- It may be straightforward to generalize the background field methods of sections 2.2 and 4 to 4d $\mathcal{N} = 1$ or 6d $\mathcal{N} = (1, 0)$ SCFTs, with or without gravity duals. For those with gravity duals, [18, 43] already suggest expressions in terms of the anomaly polynomials of the SCFTs. It will also be interesting to find possible caveats of our discussions in various models, coming from zero mode structures, as explored in [25].

- It may be useful to employ the background field approach at small $S^1$, to explore large non-BPS AdS black holes. Of course in this case, we expect that additional dynamical information has to be put in, unlike BPS black holes. Maybe not too surprisingly, we find similar structures as the hydrodynamic approach to the large AdS black holes [50].
• One is naturally led to the question of BPS black holes in AdS$_4$ \cite{16} and AdS$_6$ \cite{52}. Although the free QFT or anomaly-based studies are not available, some microscopic/macroscopic studies are to appear \cite{53}.

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