There is no APTAS for 2-dimensional vector bin packing: Revisited

Arka Ray
Indian Institute of Science, Bengaluru, India

Abstract
We study vector bin packing and vector bin covering problems, multidimensional generalizations of the classical bin packing and bin covering problems, respectively. In Vector Bin Packing we are given a set of $d$-dimensional vectors from $(0, 1]^d$, and the aim is to partition the set into the minimum number of bins such that for each bin $B$, we have $\|\sum_{v \in B} v\|_\infty \leq 1$. Woeginger [17] claimed that the problem has no APTAS for dimensions greater than or equal to 2. We note that there was a slight oversight in the original proof. Hence, we give a revised proof using some additional ideas from [3, 8]. In fact, we show that it is NP-hard to get an asymptotic approximation ratio better than $\frac{600}{599}$.

An instance of vector bin packing is called $\delta$-skewed if every item has at most one dimension greater than $\delta$. As a natural extension of our general $d$-dimensional vector bin packing result we show that for $\epsilon \in (0, \frac{1}{2500})$ it is NP-hard to obtain a $(1 + \epsilon)$-approximation for $\delta$-skewed vector bin packing if $\delta > 20\sqrt{\epsilon}$.

The Vector Bin Covering problem given a set of $d$-dimensional vectors from $(0, 1]^d$, the aim is to obtain a family of disjoint subsets (called bins) with the maximum cardinality such that for each bin $B$, we have $\sum_{v \in B} v \geq 1$. Using ideas similar to our vector bin packing result, we show that for vector bin covering there is no APTAS for dimensions greater than or equal to 2. In fact, we show that it is NP-hard to get an asymptotic approximation ratio better than $\frac{998}{997}$.

2012 ACM Subject Classification Theory of computation → Problems, reductions and completeness; Theory of computation → Packing and covering problems

Keywords and phrases vector bin packing, hardness of approximation, vector bin covering, skewed items

Funding Supported by IISc CSA M.Tech Program

Acknowledgements I would like to thank Arindam Khan for introducing me to multidimensional bin packing problems (especially the then open problem of finding a $\Omega(\ln d)$ lower bound for VBP) and all the discussions we had on them. I would also like to thank KV N Sreenivas, Rameesh Paul and Eklavya Sharma for their valuable comments.

1 Introduction

In the $d$-dimensional vector bin packing problem we are given a set of $d$-dimensional vectors (say $S$) each of whose components belong to $(0, 1]$, i.e., $S \subseteq (0, 1]^d$. The aim is to find the minimum cardinality partition of $S$ into bins $\{B_i | i \in [m]\}$ such that $\|\sum_{v \in B_i} v\|_\infty \leq 1$, for any $i \in [m]$. Whenever the above condition holds for a set of vectors $B$ we say that the vectors in $B$ fit in a bin. This is a natural generalization of the classical bin packing problem, which can be obtained by setting $d = 1$.

We also studied related problem called the vector bin covering problem. This is a generalization of another classical problem called the bin covering problem. In the $d$-dimensional vector bin covering problem we are again given a set of $d$-dimensional vectors (say $S$) each of whose components belong to $(0, 1]$. The aim is to obtain a family of disjoint subsets (also called unit cover or bin) with the maximum cardinality such that for each bin $B$, we have $\sum_{v \in B} v \geq 1$.

There is a well-known reduction from the partition problem to the classical bin packing showing that it is NP-hard to approximate bin packing within a factor of $3/2$. This guarantee
There is no APTAS for 2-dimensional vector bin packing: Revisited

holds even for vector bin packing. However, note that this does not even rule out extremely strong guarantees like $\text{OPT} + 1$. Nevertheless, if we take a closer look at the reduction, we can conclude that the above lower bound holds only for instances with solutions with small values (e.g., a small number of bins in bin packing), which is not typical of most instances. Hence, it is instructive to look at the asymptotic approximation ratio ($R_\infty$) for a minimization problem is defined by,

$$R_\infty = \limsup_{n \to \infty} R_n$$

$$R_n = \max_i \left\{ \frac{A(I)}{\text{OPT}(I)} \middle| \text{OPT}(I) = n \right\}$$

where $I$, $A(I)$, $\text{OPT}(I)$ denote the instances of the problem, the value of the output of an algorithm $A$, and the optimal value for the instance $I$, respectively. An asymptotic polynomial time approximation scheme (APTAS) for a problem is a class of algorithms for the problem in which there is an $1 + \epsilon$ approximation algorithm for each $\epsilon > 0$. For a maximization problem, we have $R_n = \max \left\{ \frac{\text{OPT}(I)}{A(I)} \middle| \text{OPT}(I) = n \right\}$ and $R_\infty = \limsup R_n$ while the definition of APTAS remains the same. In fact, even in case of bin covering there is another well-known reduction from the partition problem to bin covering showing it is NP-hard to get an absolute approximation ratio of $2 - \epsilon$. Hence, we will be looking at the asymptotic approximation ratio for both these problems. For the remainder of this paper, we will refer to the asymptotic approximation ratio simply as the approximation ratio.

1.1 Related Works

For vector bin packing, in the case where $d$ has been supplied as part of the input, de la Vega and Lueker gave $(d + \epsilon)$-approximate algorithm in [11]. This algorithm is almost optimal as there is a reduction from the vertex coloring problem known, which shows a $d^{1-\epsilon}$ hardness (see [3]).

If $d$ is kept fixed, i.e., it is not supplied as part of the input, then the above lower bound does not hold, and in fact, much better approximation factors are known for this case. The barrier of $d$ was broken by Chekuri and Khanna [6] by obtaining a $1 + \epsilon d + H_{\epsilon-1}$ approximation where $H_k = \sum_{m=1}^{k} \frac{1}{m}$. Notice that taking $\epsilon = \frac{1}{d}$, this implies a $O(\ln d)$ approximation (in fact $\ln d + 2 + \gamma$). This was further improved to $\ln d + 1$ by Bansal, Caprara, and Sviridenko [2] and then to $\ln(d + 1) + 0.807$ by Bansal, Eliáš and Khan [4]. Recently Sandeep [16] showed that the best approximation ratio any vector bin packing algorithm, for high enough dimensions, can have is $\Omega(\ln d)$. In the $d = 2$ case, Bansal, Caprara, and Sviridenko [2] gave a 1.693 approximation algorithm which was later improved to 1.406 by Bansal, Eliáš, and Khan [4].

Bansal, Eliáš, and Khan [4] also note that skewed instance constitute the hard instances for rounding based algorithms for 2-dimensional vector bin packing. An item is called $\delta$-large if at least two dimensions are larger than $\delta$, for some constant $\delta > 0$, otherwise it is called $\delta$-skewed. In fact, the case where all items are skewed forms an important sub-case for many packing problems. Gálvez et al. [12] study the strip packing problem in this context and give a $(\frac{3}{2} + \epsilon)$-approximation and also show an (almost) matching $(\frac{3}{2} - \epsilon)$ lower bound. Recently, Khan and Sharma [14] gave an APTAS for 2-dimensional geometric bin packing with skewed items. They also note that it is possible to solve Maximum Independent Set of Rectangle (MISR)

\footnote{\(\gamma \approx 0.57721\) is the Euler-Mascheroni constant}
and 2-dimensional geometric knapsack (2GK) exactly in polynomial time if all items are δ-large. Finally, not only are the skewed instances of packing problems of theoretical interest but they occur in practice as well, e.g., in scheduling, it captures scenarios where no job can consume a significant amount of a shared resource (energy, memory space, etc.) for a significant amount of time.

For vector bin covering, when \( d \) is supplied as part of the input the best known algorithm by Alon et al. [1] gives an approximation ratio of \( O(\ln d) \). In the same paper, they also give another algorithm which gives \( d \)-approximation which is better than their \( O(\log d) \) algorithm for small values of \( d \). Finally, Sandeep [16] also gave a lower bound of \( \Omega(\frac{\log d}{\log \log d}) \) for arbitrary value of dimension \( d \).

For further information on approximation and online algorithms for multidimensional variants of the bin packing and bin covering problems, we refer the reader to the survey [10] by Christensen et al.

### 1.2 Our Results

It was believed that, [17] shows there is no APTAS for vector bin packing with \( d \geq 2 \). However, as we show in Section 3, there was a minor oversight in the original proof. Specifically, an essential claim made in the proof fails to hold for a few special cases. We also examine some natural modifications to the claim, which exclude those special cases. We conclude that it is impossible to obtain the final result if we try to use the same arguments. Unfortunately, this oversight is also present in the \( \frac{391}{390} \) lower bound for vector bin packing by Chlebík and Chlebíková [9]. Hence, we present a revision to the original proof in Section 2. Although our proof uses essentially the same construction as the original proof, the final analysis is slightly different, and the main ideas for the analysis are borrowed from [3] [8]. Specifically, we have obtained a gap reduction instead of an approximation ratio preserving reduction. The APX-hardness of vector bin packing, though not stated explicitly, was considered to be a simple corollary of the original result (see [3]). Although the revised proof gives us a constant lower bound on the approximation ratio, we cannot conclude that the vector bin packing problem is APX-hard. We note that Sandeep’s lower bound of \( \Omega(\ln d) \) does not hold for low dimensions, and hence, it does not even rule out the possibility of APTAS in the 2-dimensional case.

Our second result is concerning vector bin packing with skewed items. Extending the proof of our non-existence of APTAS for vector bin packing we show that for \( \epsilon \in (0, \frac{1}{2500}) \) we need \( \delta \leq 20\sqrt{\epsilon} \) to obtain a \((1 + \epsilon)\)-approximation for \( \delta \)-skewed \( d \)-dimensional vector bin packing. This rules out a APTAS for a fixed \( \delta \) for \( d \)-dimensional vector bin packing (for \( d \geq 2 \)). Finally, we adapt our proof for vector bin covering to show that there is no APTAS for vector bin covering with dimension \( d \geq 2 \).

### 1.3 Preliminaries

As is the case in [17] and [3], we start with maximum 3-dimensional matching (denoted by MAX-3-DM) and reduce it to an instance of 4-Partition and finally reduce it to a vector bin packing instance. In a 3-dimensional matching instance we have three sets \( X = \{x_1,x_2,\ldots,x_q\}, Y = \{y_1,y_2,\ldots,y_q\} \), and \( Z = \{z_1,z_2,\ldots,z_q\} \) and we are given a set of tuples \( T \subseteq X \times Y \times Z \). The aim is to find the maximum cardinality subset \( T' \subseteq T \) such that no element from \( X, Y, \) or \( Z \) occurs in more than one tuple. In the bounded variant of MAX-3-DM (denoted by MAX-3-DM-B), it is assured that any element which belongs to either \( X, Y, \) or \( Z \) will appear in at most \( B \) tuples. Kann [13] showed that bounded maximum
There is no APTAS for 2-dimensional vector bin packing: Revisited

matching with bound 3 is MAX SNP-hard which in turn implies it is APX-hard. Later, it was shown by Petrank \cite{15} that it is NP-hard to distinguish between instances where there is a solution $T'$ with $|T'| = q$ and from instances for which every solution $T'$, $|T'| \leq (1-\epsilon)q$ for a constant $\epsilon$. There is also a more restricted variant of the problem, which is frequently studied where there are exactly $B$ tuples for each element of the sets $X, Y$, and $Z$ called the exact maximum 3-dimensional matching (denoted by MAX-3-DM-E$B$). In case of MAX-3-DM-E$2$ it was shown by Berman and Karpinski \cite{5} that it is NP-hard to approximate with ratio better than $\frac{98}{97}$, which Chlebík and Chlebíková \cite{7} improved to $\frac{94}{93}$. Finally, Chlebík and Chlebíková \cite{8} also note an useful corollary of their $\frac{94}{93}$ bound, which is the following result for the promise variant of MAX-3-DM-E$2$.

**Theorem 1.** \cite{8} Let $I_M$ be a MAX-3-DM-E$2$ instance comprising of sets $X, Y, Z$ and tuples $T \subseteq X \times Y \times Z$ with $|X| = |Y| = |Z| = q$. Then, it is NP-hard to distinguish between the case with $\text{OPT}(I_M) \geq \lceil \beta_0 q \rceil$ and $\text{OPT}(I_M) \leq \lfloor \alpha_0 q \rfloor$, where $\alpha_0 = 0.9690082645$ and $\beta_0 = 0.979338843$.

For our result for the skewed item case we note that the size of the items in the reduction from MAX-3-DM to 2-dimensional vector bin packing can be reduced by going through $m$-Partition instead of 4-Partition. Finally, for our vector bin covering result we use the same reduction to 4-Partition to finally obtain a reduction to a vector bin covering instance.

## 2 The Main Result

In this section, we prove our main result, i.e., there is no APTAS for vector bin packing. We do so by modifying the construction in the original proof given in \cite{17} by adding a set of dummy vectors\cite{2}. The final analysis is based on the analysis in \cite{3} for the geometric bin packing lower bound.

We start by defining a few integers based on the given MAX-3-DM instance $I_M$. Let $r = 64q$, where $q = |X| = |Y| = |Z|$ and $b = r^4 + 15$. Define integers $x'_i, y'_i, z'_i$ corresponding to $x_i \in X, y_j \in Y, z_k \in Z$ to be,

\[
x'_i = ir + 1, \quad y'_i = ir^2 + 2, \quad z'_i = ir^3 + 4,
\]

and for $t_{(i,j,k)} = (x_i, y_j, z_k) \in T$ define $t'_{(i,j,k)}$ as,

\[
t'_{(i,j,k)} = r^4 - kr^3 - jr^2 - ir + 8.
\]

Let $U'$ be the set of integers constructed as above. Also, note that for any integer $a' \in U'$ constructed as above we have $0 < a' < b$. These integers were constructed so that the following statement holds (cf. Observation 2 in \cite{17}).

**Lemma 2.** A set of four integers from $U'$ add up to $b$ if and only if they correspond to some elements $x_i \in X, y_j \in Y, z_k \in Z$ and tuple $t_{(i,j,k)} \in T$ where $t_{(i,j,k)} = (x_i, y_j, z_k)$.

\footnote{We have tried to stay as close to the original proof as possible in terms of notations and arbitrary choices. Yet we have made two notable changes, (i) using $r = 64q$, and (ii) using $t_{(i,j,k)}$ to denote a tuple.}
Proof. (If) Suppose \( x_i \in X, y_j \in Y, z_k \in Z \) and \( t_{(i,j,k)} \in T \) where \( t_{(i,j,k)} = (x_i, y_j, z_k) \) then it is easy to verify that indeed
\[
t'_{(i,j,k)} + x'_i + y'_j + z'_k = b
\]
(Only if) Conversely, suppose that four integers \( a'_1, a'_2, a'_3, a'_4 \) sum to \( b \). Considering the equation modulo \( r \) and using the fact \( 1 + 2 + 4 + 8 \) is the only possible way of obtaining 15 as a sum of four elements (possibly with repetition) from the set \( \{1, 2, 4, 8\} \) and therefore we conclude the integers must correspond one element each from \( X, Y, Z, T \). This means we can write \( a'_1, a'_2, a'_3, a'_4 \) as \( x'_i, y'_j, z'_k, t'_{(i,j,k)} \). Now, considering the equation modulo \( r^2, r^3, r^4 \) gives us \( i = i_0, j = j_0, k = k_0 \). □

To obtain a vector bin packing instance for each integer \( a' \) constructed above construct the following vector,
\[
a = \left( \frac{1}{5} + \frac{a'}{5b}, \frac{3}{10} - \frac{a'}{5b} \right)
\]
We also construct additional \( |T| + 3q - 4\beta(I_M) \) dummy vectors as follows,
\[
d = \left( \frac{3}{5}, \frac{3}{5} \right)
\]
where \( \beta() \) is an arbitrary function which will be fixed later. We now note a few properties of the vectors. First of these pertains to how many vectors can fit in a bin (cf. Observation 4, Lemma 2.5 from [17] and [3]).

▶ Lemma 3. A bin can contain at most 4 vectors. If a bin contains a dummy vector it can contain at most one more vector. Furthermore, two dummy vectors will not fit in a bin while any other set of two vectors fit in a bin.

Proof. The first part follows from the fact that the first component of any vector is strictly greater than \( \frac{1}{5} \). The second part of the claim follows from the fact any vector in the instance has first component greater than \( \frac{1}{5} \) and the dummy vector has first component equal to \( \frac{2}{5} \). For the third part, observe if there are two dummy vectors then both the components would add up to \( 2 \cdot \frac{2}{5} > 1 \). However, if one of them is a non-dummy vector, then notice that both components of a non-dummy vector are less than \( \frac{2}{5} \), and both components of any vector are less than \( \frac{1}{5} \). Hence, both components of the sum are less than 1. □

The following lemma shows that a configuration corresponding to a tuple is an optimal configuration (cf. Observation 3 from [17]).

▶ Lemma 4. A set of four vectors fits in a bin if and only if it corresponds to a tuple.

Proof. (If) For a tuple \( t_{(i,j,k)} = (x_i, y_j, z_k) \), we have \( t'_{(i,j,k)} + x'_i + y'_j + z'_k = b \) by Lemma 2. So, we have
\[
t_{(i,j,k)} + x_i + y_j + z_k = \left( \frac{4}{5} + \frac{t'_{(i,j,k)}}{5b}, \frac{6}{5} - \frac{t'_{(i,j,k)}}{5b} \right) = (1, 1).
\]
(Only if) Suppose there are four vectors \( a_1, a_2, a_3, a_4 \) which fit in a bin. By Lemma 3 all the vectors are non-dummy vectors. Hence, each vector can be written as:
\[
a_i = \left( \frac{1}{5} + \frac{a'_i}{5b}, \frac{3}{10} - \frac{a'_i}{5b} \right).
\]
As they fit in a bin we get the following two conditions,

\[
\frac{4}{5} \sum_{i=1}^{4} a_i' + \frac{5b}{5b} \leq 1,
\]

\[
\frac{6}{5} - \frac{4}{5b} \sum_{i=1}^{4} a_i' \leq 1,
\]

which simplify to \( \sum_{i=1}^{4} a_i' \leq b \) and \( \sum_{i=1}^{4} a_i' \geq b \). Combining the inequalities we get \( \sum_{i=1}^{4} a_i' = b \).

Therefore, by Lemma 2 the vectors correspond to a tuple. \( \blacklozenge \)

Now, we show that the above construction is a gap reduction from MAX-3-DM to 2-dimensional vector packing (cf. Theorem 2.1 from [3]),

\textbf{Lemma 5.} If a MAX-3-DM instance \( I_M \) has a solution with \( \beta(I_M) \) tuples the constructed vector bin packing instance has a solution with \( |T| + 3q - 3\beta(I_M) \) bins. Otherwise, if all the solutions of the MAX-3-DM instance have at most \( \alpha(I_M) \) tuples then the constructed instance needs at least \( |T| + 3q - \frac{3\beta(I_M)}{3} \) bins where \( \alpha(\cdot) \) is an arbitrary function.

\textbf{Proof.} First, we show that if a MAX-3-DM instance has a matching consisting of \( \beta(I_M) \) tuples, then the vector bin packing instance has a solution of \( |T| + 3q - 3\beta(I_M) \) bins. Using Lemma 4, the \( 4\beta(I_M) \) vectors corresponding to the \( \beta(I_M) \) tuples and their elements can be packed into \( \beta(I_M) \) bins. Each of the \( |T| + 3q - 4\beta(I_M) \) non-dummy vectors can be packed along with a dummy vector into \( |T| + 3q - 4\beta(I_M) \) bins by Lemma 3.

Now, suppose that for a given instance, all the solutions have at most \( \alpha(I_M) \) tuples. Let \( n_g \) be the number of bins with 4 vectors, \( n_d \) be the number of bins with dummy vectors, and \( n_r \) be the rest of the bins. Now, since any solution to the bin packing instance must cover all the non-dummy vectors,

(a) any bin containing four vectors consists of only non-dummy vectors by Lemma 4

(b) any bin containing a dummy vector contains at most one non-dummy vector, by Lemma 3

and

(c) any other bin can contain at most 3 vectors by Lemma 3

Therefore, we have

\[
4n_g + 3n_r + n_d \geq 3q + |T|.
\]

Now, by Lemma 3 we have \( n_d = |T| + 3q - 4\beta(I_M) \). Hence, the above inequality simplifies to,

\[
4n_g + 3n_r \geq 4\beta(I_M) \\
\Rightarrow n_g + n_r \geq \frac{4}{3} \beta(I_M) - \frac{n_g}{3} \\
\Rightarrow n_g + n_r + n_d \geq |T| + 3q - \frac{n_g}{3} - \frac{8}{3} \beta(I_M) \quad \text{[Since, } n_d = |T| + 3q - 4\beta(I_M)]
\]

Since there are at most \( \alpha(I_M) \) triples in the MAX-3-DM instance, by Lemma 4 we have \( n_g \leq \alpha(I_M) \). Therefore, the number of bins is at least

\[
|T| + 3q - \frac{\alpha(I_M)}{3} - \frac{8\beta(I_M)}{3}.
\]

\( \blacklozenge \)
The following inapproximability for vector bin packing directly follows from Lemma 5.

**Theorem 6.** There is no APTAS for the d-dimensional vector bin packing with \( d \geq 2 \) unless \( P=NP \). Furthermore, for the 2-dimensional case there is no algorithm with asymptotic approximation ratio better than \( \frac{600}{599} \).

**Proof.** Suppose that there is an algorithm with approximation ratio

\[
1 + \frac{\beta_0 - \alpha_0}{15 - 9}\beta_0.
\]

Then we can distinguish between MAX-3-DM-E2 instances (i) having a solution of \( \lceil \alpha_0 q \rceil \) triples and (ii) having no solutions with more than \( \lfloor \alpha_0 q \rfloor \) tuples using Lemma 5 with \( \alpha(I_M) = \lfloor \alpha_0 q \rfloor \) and \( \beta(I_M) = \lceil \beta_0 q \rceil \). By Theorem 1, we know it is NP-hard to distinguish between these two types of MAX-3-DM-E2 instances with \( \beta_0 = 0.979338843 \) and \( \alpha_0 = 0.9690082645 \). Hence, we obtain the bound of

\[
1 + \frac{\beta_0 - \alpha_0}{15 - 9}\beta_0.
\]

Simple calculations will show this is at least \( 1 + \frac{1}{599} \).

3 The Original Proof

The original proof uses essentially the same reduction as ours, i.e., there we had 

\[
r = 32q, \quad b = r^4 + 15
\]

and then for each \( x_i \in X, y_i \in Y, z_i \in Z \) we had,

\[
\begin{align*}
  x_i' &= ir + 1, \\
  y_i' &= ir^2 + 2, \\
  z_i' &= ir^3 + 4,
\end{align*}
\]

and for \( t_l \in T \) was \( t_l' \) defined by,

\[
t_l' = r^4 - kr^3 - jr^2 - ir + 8.
\]

And finally, to obtain a vector bin packing instance for each integer \( a' \) constructed above construct the following vector,

\[
a = \left( \frac{1}{5} + \frac{a'}{5b}, \frac{3}{10} - \frac{a'}{5b} \right)
\]

The above set of vectors forms a 2-dimensional vector bin packing instance \( U \). A noticeable difference from our reduction being the absence of dummy vectors. In [17], Woeginger claimed that,

▷ Claim (Observation 4 in [17]). Any set of 3 vectors in \( U \) can be packed in a unit-bin. No set of 5 vectors in \( U \) can be packed into a unit-bin.

We show that this claim does not hold in general. In particular, all sets of 3 vectors can not be packed into a unit-bin.

Consider the tuple vectors for the tuples \( t_1 = (x_1, y_1, z_1), t_2 = (x_2, y_1, z_1), \) and \( t_3 = (x_3, y_1, z_1) \). According to the claim, the vectors \( t_1, t_2, t_3 \) corresponding to the above tuples can be packed in a bin. Suppose \( t_1, t_2, t_3 \) can indeed be packed in a bin. This implies that the first components of the vectors do not exceed 1, i.e.,

\[
\frac{3}{5} + \frac{t_1' + t_2' + t_3'}{5b} \leq 1,
\]

which simplifies to,

\[
t_1' + t_2' + t_3' \leq 2b.
\]

\[3\] Also notice that \( r = 32q \) and tuples are denoted by \( t_l \).
Finally, using
\[ t_1' = r^4 - r^3 - r^2 - r + 8, \]
\[ t_2' = r^4 - r^3 - r^2 - 2r + 8, \]
\[ t_3' = r^4 - r^3 - r^2 - 3r + 8, \]
and
\[ b = r^4 + 15, \]
along with further simplification we get,
\[ r^4 \leq 3r^3 + 3r^2 + 6r + 6. \]
But this inequality does not even hold for \( r \geq 32 \) whereas 32 is the smallest value for \( r = 32q \). Thus the claim is incorrect. This implies his main lemma (Lemma 5 in [17]) fails to hold.

\[ \triangleright \text{Claim (Lemma 5 in [17]).} \]
Let \( \alpha > 0 \) be an integer such that \( |T| - \alpha \) is divisible by 3. Then there exists a feasible solution for the instance \( I_M \) for MAX-3-DM that contains at least \( \alpha \) triples if and only if there exists a feasible packing for the instance \( U \) of the 2-dimensional vector packing problem that uses at most \( |T| + \frac{1}{3}(q - \alpha) \) unit bins.

We now consider, and rule out, a few natural attempts at fixing the proof. As a first attempt, we modify the claim of Observation 4 from [17] to exclude the case of 3 tuples. In this case, we see that it is not possible prove the original claim of Lemma 5 from [17]. Another attempt inspired by the failures of the above attempt is to consider modifying the claim of Observation 4 to apply to any set of 2 vectors. In this case, it is clear that the original claim of Lemma 5 can not be proven and even the natural modification to the claim of Lemma 5 to consider at most \( \frac{1}{2}(3q + |T|) - \alpha \) bins instead of \( q + \frac{1}{3}(q - \alpha) \) bins can not be proven. It seems that it may not be possible to show that the construction given in [17] is an approximation preserving reduction. Finally, the analysis in [19] also uses the relation in Lemma 5 of Woeginger’s proof to obtain their \( \frac{391}{390} \) inapproximability which unfortunately means that their analysis also suffers from the same oversight. Hence, the need for the revision.

\section{Vector Bin Packing with skewed items}

In this section, we adapt\[ \footnote{Proofs of statements in this section are similar to ones in Section 2 hence, omitted (see Appendix A).} \] the reduction presented in Section 2 to show that any algorithm for \( \delta \)-skewed \( d \)-dimensional vector bin packing cannot have an approximation ratio better than \( 1 + \epsilon \) if \( \delta > 20\sqrt{\epsilon} \) for small values of \( \epsilon \).

Again, we start by defining a few integers based on the given MAX-3-DM instance \( I_M \). Let \( m = \lceil \frac{\delta}{2} \rceil - 1 \), for some \( \delta \in (0, \frac{2}{5}) \). Choose \( n > m2^m \) and set \( r = nq \) and \( b = r^m + 2^{m+1} - 1 \). Define integers \( x'_i, y'_i, z'_i \) corresponding to \( x_i \in X, y_i \in Y, z_i \in Z \) to be,
\[ x'_i = ir + 1, \]
\[ y'_i = ir^2 + 2, \]
\[ z'_i = ir^3 + 4. \]
and for \( t_{(i,j,k)} = (x_i, y_j, z_k) \in T \) define \( t'_{(i,j,k)} \) as,

\[
t'_{(i,j,k)} = r^m - \sum_{l=4}^{m-1} r^l - kr^3 - jr^2 - ir + 2^m.
\]

Finally, we add additional \(|T|\) integers for each \( l \in \{4, \ldots, m - 1\} \),

\[
c'_l = r^l + 2^l.
\]

Let \( U' \) be the set of integers constructed as above. Also, note that for any integer \( a' \in U' \) constructed as above we have \( 0 < a' < b \). These integers were constructed so that the following statement holds.

**Lemma 7.** A subset of integers \( S \subseteq U' \) with \(|S| = m\) adds up to \( b \) if and only if there are \( x_{i}', y_{j}', z_{k}' \in S \) corresponding to some elements \( x_i \in X, y_j \in Y, z_k \in Z \) and tuple \( t_{(i,j,k)} \in T \) where \( t_{(i,j,k)} = (x_i, y_j, z_k) \) and \( c'_l \in S \) for each \( l \in \{4, \ldots, m - 1\} \).

To obtain a vector bin packing instance for each integer \( a' \) constructed above construct the following vector,

\[
a = \left( \frac{1}{m + 1} + \frac{a'}{(m + 1)b}, m + 2 = \frac{m + 2}{m(m + 1)} - \frac{a'}{(m + 1)b} \right).
\]

We also construct additional \((m - 3)|T| + 3q - m \beta(I_M)\) dummy vectors as follows,

\[
d = \left( \frac{m - 1}{m + 1}, 0 \right),
\]

where \( \beta() \) is an arbitrary function which will be fixed later. Notice that each of these vectors has a dimension whose size is less than \( \frac{2}{m+1} \leq \delta \). Again, we note a few properties of the vectors. First of these pertains to how many vectors can fit in a bin.

**Lemma 8.** A bin can contain at most \( m \) vectors. If a bin contains a dummy vector it can contain at most one more vector. Furthermore, two dummy vectors will not fit in a bin while any other set of two vectors fit in a bin.

The following lemma shows that a configuration corresponding to a tuple is an optimal configuration.

**Lemma 9.** A set \( S \) of \( m \) vectors fits in a bin if and only if there are \( x_i, y_j, z_k, t_{(i,j,k)} \in S \) corresponding to some elements \( x_i \in X, y_j \in Y, z_k \in Z \) and tuple \( t_{(i,j,k)} \in T \) where \( t_{(i,j,k)} = (x_i, y_j, z_k) \) and \( c'_l \in S \) for each \( l \in \{4, \ldots, m - 1\} \).

Now, we show that the above construction is a gap reduction from MAX-3-DM to 2-dimensional vector packing.

**Lemma 10.** If a MAX-3-DM instance \( I_M \) has a solution with \( \beta(I_M) \) tuples the constructed vector bin packing instance has a solution with \((m - 3)|T| + 3q - (m - 1)\beta(I_M)\) bins. Otherwise, if all the solutions of the MAX-3-DM instance have at most \( \alpha(I_M) \) tuples then the constructed instance needs at least \((m - 3)|T| + 3q - \frac{\alpha(I_M)}{m-1} - \frac{m(m-2)}{m-1} \beta(I_M)\) bins where \( \alpha() \) is an arbitrary function.

**Proof.** First, we show that if a MAX-3-DM instance has a matching consisting of \( \beta(I_M) \) tuples, then the vector bin packing instance has a solution of \((m - 3)|T| + 3q - (m - 1)\beta(I_M)\) bins.
There is no APTAS for 2-dimensional vector bin packing: Revisited

bans. Using Lemma 4, the \( m\beta(I_M) \) vectors corresponding to the \( \beta(I_M) \) tuples and their elements and one vector \( c_l \) for each \( l \in \{4, \ldots, m-1\} \) can be packed into \( \beta(I_M) \) bins. Each of the \((m-3)|T| + 3q - m\beta(I_M)\) remaining non-dummy vectors can be packed along with a dummy vector into \((m-3)|T| + 3q - m\beta(I_M)\) bins by Lemma 8.

Now, suppose that for a given instance, all the solutions have at most \( \alpha(I_M) \) tuples. Let \( n_g \) be the number of bins with \( m \) vectors, \( n_d \) be the number of bins with dummy vectors, and \( n_r \) be the rest of the bins. Now, since any solution to the bin packing instance must cover all the non-dummy vectors,

(a) any bin containing \( m \) vectors consists of only non-dummy vectors by Lemma 9,
(b) any bin containing a dummy vector contains at most one non-dummy vector, by Lemma 8
and
(c) any other bin can contain at most \( m - 1 \) vectors by Lemma 8.

Therefore, we have

\[
 mn_g + (m-1)n_r + n_d \geq 3q + (m-3)|T|.
\]

Now, by Lemma 8 we have \( n_d = (m-3)|T| + 3q - m\beta(I_M) \). Hence, the above inequality simplifies to,

\[
 mn_g + (m-1)n_r \geq m\beta(I_M)
\]
\[
 \Rightarrow n_g + n_r \geq \frac{m}{m-1} \beta(I_M) - \frac{n_g}{m-1}
\]
\[
 \Rightarrow n_g + n_r + n_d \geq (m-3)|T| + 3q - \frac{n_g}{m-1} - \frac{m(m-2)}{m-1} \beta(I_M).
\]

Since there are at most \( \alpha(I_M) \) triples in the MAX-3-DM instance, by Lemma 9 we have \( n_g \leq \alpha(I_M) \). Therefore, the number of bins is at least

\[
 (m-3)|T| + 3q - \frac{\alpha(I_M)}{m-1} - \frac{m(m-2)}{m-1} \beta(I_M).
\]

The following inapproximability for vector bin packing directly follows from Lemma 10.

**Theorem 11.** For \( \epsilon \in (0, \frac{4 \sqrt{\epsilon}}{25(\sqrt{\epsilon})}) \) there is no \( 1 + \epsilon \) for the \( d \)-dimensional vector bin packing with \( d \geq 2 \) if there are items with at least two dimensions greater than \( 20\sqrt{\epsilon} \) unless \( P=NP \).

**Proof.** Suppose that there is an algorithm with approximation ratio \( 1 + \frac{\beta_0-\alpha_0}{m(2m-3)-(m-1)^2\beta_0} \). Then we can distinguish between MAX-3-DM-E2 instances (i) having a solution of \( \lfloor \beta_0q \rfloor \) triples and (ii) having no solutions with more than \( \lfloor \alpha_0q \rfloor \) tuples using Lemma 10 with \( \alpha(I_M) = \lfloor \alpha_0q \rfloor \) and \( \beta(I_M) = \lceil \beta_0q \rceil \). By Theorem 1, we know it is NP-hard to distinguish between these two types of MAX-3-DM-E2 instances with \( \beta_0 = 0.979338843 \) and \( \alpha_0 = 0.9690082645 \). Hence, we obtain the bound of \( 1 + \frac{\delta_0-\alpha_0}{m(2m-3)-(m-1)^2\delta_0} \). Simple calculations will show that this is strictly greater than \( 1 + \frac{\epsilon^2}{445} \). Using \( \delta = 20\sqrt{\epsilon} \) we get the desired result.

## 5 Vector Bin Covering has no APTAS

In this section, we prove that vector bin covering has no APTAS unless \( P=NP \) by adapting the proof presented in Section 2. The analysis is slightly more complicated and bears some resemblance to the analysis of the reduction to geometric bin covering problem presented in 8. To show this we obtain a gap preserving reduction from MAX-3-DM to 2-dimensional
vector bin covering. We start with the same set of integers $U'$ we had in Section 2. To obtain a vector bin covering instance for each integer $a'$ in $U'$ construct the following vector,

$$a = \left( \frac{1}{5} + \frac{a'}{55}, \frac{3}{10} - \frac{a'}{55} \right)$$

we also construct additional $|T| + 3q - 4\beta(I_M)$ dummy vectors as follows,

$$d = \left( \frac{9}{10}, \frac{9}{10} \right)$$

where $\beta(\cdot)$ is an arbitrary function which will be fixed later. If a unit cover (or bin) has at least one dummy vector then we call it a D-bin. Otherwise, if a unit cover has no dummy vectors the we call it a non-D-bin.

**Observation 12.** Any set of 5 vectors can cover a bin. Any vector along with a dummy vector can cover a bin. At least 2 vectors are needed to form a unit cover. Though, as we observe, any bin can be covered using a dummy vector along with another vector. We now show that for a non-D-bin the optimal cover has cardinality 4.

**Lemma 13.** A set of four vectors covers a non-D-bin if and only if it corresponds to a tuple.

Now, we are ready to prove our main lemma showing our reduction is indeed a gap preserving reduction.

**Lemma 14.** If a MAX-3-DM instance $I_M$ has a solution with $\beta(I_M)$ tuples then there is a solution to the vector bin covering instance with $|T| + 3q - 3\beta(I_M)$ tuples. Otherwise, if all the solutions of $I_M$ have at most $\alpha(I_M)$ tuples then the constructed instance can cover at most $|T| + 3q - 4\beta(I_M)$ bins, where $\alpha(\cdot)$ is an arbitrary function.

**Proof.** Suppose that $\text{OPT}(I_M) \geq \beta(I_M)$. Then we can cover $\beta(I_M)$ bins using $4\beta(I_M)$ vectors corresponding to the $\beta(I_M)$ tuples from the solution of $I_M$ using Lemma 13. Now, we have $|T| + 3q - 4\beta(I_M)$ non-dummy vectors left along with exactly $|T| + 3q - 4\beta(I_M)$ dummy vectors. These vectors, by Observation 12, can cover $|T| + 3q - 4\beta(I_M)$ bins.

Now, suppose that every solution of the MAX-3-DM instance has value at most $\alpha(I_M)$. Consider an optimal solution to the constructed vector bin covering instance. We can normalize such a solution without any loss in the number of bins covered as follows,

(a) Each bin has at most one dummy element. Clearly, more than $|T| + 3q - 4\beta(I_M)$ bins are covered in an optimal solution. So, if there are $n$ bins with two dummy vectors then there must be at least $n$ non-D-bins. Therefore, we can pick one non-D-bin for each bin with 2 dummy vectors, which must contain 2 non-dummy vectors by Observation 12. Again by Observation 12 we can obtain two unit cover each with one dummy vector and one non-dummy vector. Similar arguments can be used for bins having $k$ dummy vectors, i.e., there are at least $(k - 1)n$ non-D-bins and then similar rearrangements can be done to obtain $kn$ unit covers with one dummy vector.

(b) No subset of a unit cover is an unit cover. To that end, some vectors can left out, i.e., they may be designated as not part of any unit cover. Now, using arguments similar to (a) we see that every dummy vector must be part of a unit cover. Also, note that number of such vectors can be at most four as five vectors always form a unit cover.

---

5 Note the difference in size of dummy vectors from Section 2.
There is no APTAS for 2-dimensional vector bin packing: Revisited

Let \( n_d \) be the number of D-bins, \( n_g \) be the number of non-D-bins covered by 4 vectors and \( n_r \) be the number of non-D-bin covered by 5 vectors. By our normalization every dummy vector is part of unit cover with exactly one dummy vector, i.e., \( n_d = |T| + 3q - 4\beta(I_M) \). Since there are \( 3q + |T| \) non-dummy vectors,

\[
\begin{align*}
    n_d + 4n_g + 5n_r & \leq 3q + |T| \\
\Rightarrow n_g + 5n_r & \leq 4\beta(I_M) \quad \text{[Since, } n_d = |T| + 3q - 4\beta(I_M) \text{]} \\
\Rightarrow n_g + n_r & \leq \frac{4}{5}\beta(I_M) + \frac{n_d}{5}
\end{align*}
\]

By Lemma 13, we have \( n_g \leq \alpha(I_M) \). Therefore,

\[
    n_g + n_r \leq \frac{4}{5}\beta(I_M) + \frac{\alpha(I_M)}{5}
\]

\[
    \Rightarrow n_d + n_g + n_r \leq |T| + 3q - \frac{16}{5}\beta(I_M) + \frac{\alpha(I_M)}{5}
\]

In other words, the number of bins covered is at most

\[
    |T| + 3q - \frac{16}{5}\beta(I_M) + \frac{\alpha(I_M)}{5}
\]

\[\blacktriangleright\]

**Theorem 15.** There is no APTAS for \( d \)-dimensional vector bin covering with \( d \geq 2 \) unless \( P=NP \). Furthermore, for the 2-dimensional vector covering there is no algorithm with asymptotic approximation ratio better than \( \frac{998}{997} \).

**Proof.** Suppose there is an algorithm with approximation ratio \( 1 + \frac{\beta_0 - \alpha_0}{25 - \alpha_0} \). Then we can distinguish between MAX-3-DM-E2 instances (i) having a solution of \( \lceil \beta_0 q \rceil \) tuples and (ii) with all solution less than \( \lfloor \alpha_0 q \rfloor \) using Lemma 14 with \( \alpha(I_M) = \lfloor \alpha_0 q \rfloor \) and \( \beta(I_M) = \lceil \beta_0 q \rceil \). By Theorem 1, we know it is NP-hard to distinguish between these two types of MAX-3-DM-E2 instances with \( \beta_0 = 0.979339943 \), and \( \alpha_0 = 0.9690082645 \). Hence, we obtain the bound of \( 1 + \frac{\beta_0 - \alpha_0}{25 - \alpha_0} \). Simple calculations will show this is at least \( 1 + \frac{1}{997} \).

\[\blacktriangleright\]

**Concluding Remarks**

In the original proof of the non-existence of an APTAS for vector bin packing reduction used claimed to be an approximation preserving reduction, and hence, the APX-hardness of the vector bin packing problem was a simple corollary. This is no longer the case now as we could only show that our reduction is a gap preserving reduction, and hence, it is not known if 2-dimensional vector bin packing is indeed APX-hard. More importantly, there is still a considerable gap between the best-known approximation ratio for \( d = 2 \) case (1.406) and our lower bound (1.00167). We also make similar observation in case of the vector bin covering problem, i.e., we have not showed that it is APX-hard, and the gap between the best-known algorithm for the two-dimensional case (2) and our lower bound (1.001). As Sandeep observed, his reduction (in [16]) does not give the exact lower bounds on the approximation ratio. Hence, the problem of finding the optimal lower bound on the approximation ratio in the exact sense is still open for vector bin packing with fixed dimension. Also, note that in case of vector bin covering there is a gap of factor \( c \log \log d \) between the lower bound and upper bound on the approximation ratio. Finally, we observe that Lemma 2 implies that the intermediate reduction to 4-Partition is still approximation preserving, and hence, 4-Partition is indeed APX-hard.
References

1. Noga Alon, Yossi Azar, János Csirik, Leah Epstein, Sergey V. Sevastianov, Arjen P. A. Vestjens, and Gerhard J. Woeginger. On-line and off-line approximation algorithms for vector covering problems. *Algorithmica*, 21(1):104–118, 1998. doi:10.1007/PL00009203.

2. Nikhil Bansal, Alberto Caprara, and Maxim Sviridenko. A new approximation method for set covering problems, with applications to multidimensional bin packing. *SIAM J. Comput.*, 39(4):1256–1278, 2009. doi:10.1137/080736831.

3. Nikhil Bansal, José R. Correa, Claire Kenyon, and Maxim Sviridenko. Bin packing in multiple dimensions: Inapproximability results and approximation schemes. *Math. Oper. Res.*, 31(1):31–49, 2006. doi:10.1287/moor.1050.0168.

4. Nikhil Bansal, Marek Eliás, and Arindam Khan. Improved approximation for vector bin packing. In Robert Krauthgamer, editor, *Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms, SODA 2016, Arlington, VA, USA, January 10-12, 2016*, pages 1561–1579. SIAM, 2016. doi:10.1137/1.9781611974331.ch106.

5. Piotr Berman and Marek Karpinski. Improved approximation lower bounds on small occurrence optimization. *Electron. Colloquium Comput. Complex.*, 10(008), 2003. URL: http://eccc.hpi-web.de/eccc-reports/2003/TR03-008/index.html.

6. Chandra Chekuri and Sanjeev Khanna. On multidimensional packing problems. *SIAM J. Comput.*, 33(4):837–851, 2004. doi:10.1137/S0097539799356265.

7. Miroslav Chlebík and Janka Chlebíková. Complexity of approximating bounded variants of optimization problems. *Theor. Comput. Sci.*, 354(3):320–338, 2006. doi:10.1016/j.tcs.2005.11.029.

8. Miroslav Chlebík and Janka Chlebíková. Hardness of approximation for orthogonal rectangle packing and covering problems. *J. Discrete Algorithms*, 7(3):291–305, 2009. doi:10.1016/j.jda.2009.02.002.

9. Janka Chlebíková and Miroslav Chlebík. Hardness of asymptotic approximation for orthogonal rectangle packing and covering problems. *Electron. Colloquium Comput. Complex.*, 019, 2006. URL: http://eccc.hpi-web.de/eccc-reports/2006/TR06-019/index.html.

10. Henrik I. Christensen, Arindam Khan, Sebastian Pokutta, and Prasad Tetali. Approximation and online algorithms for multidimensional bin packing: A survey. *Comput. Sci. Rev.*, 24:63–79, 2017. doi:10.1016/j.cosrev.2016.12.001.

11. Weneceslas Fernandez de la Vega and George S. Lueker. Bin packing can be solved within 1+ε in linear time. *Comb.*, 1(4):349–355, 1981. doi:10.1007/BF02579458.

12. Waldo Gálvez, Fabrizio Grandoni, Afrouz Jabal Ameli, Klaus Jansen, Arindam Khan, and Malin Rau. A tight (3/2+ε) approximation for skewed strip packing. In Jaroslav Byrka and Raghu Meka, editors, *Approximation, Randomization, and Combinatorial Optimization. Algorithms and Techniques, APPROX/RANDOM 2020, August 17-19, 2020, Virtual Conference*, volume 176 of *LIPIcs*, pages 44:1–44:18. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2020. doi:10.4230/LIPIcs.APPROX/RANDOM.2020.44.

13. Viggo Kann. Maximum bounded 3-dimensional matching is MAX snp-complete. *Inf. Process. Lett.*, 37(1):27–35, 1991. doi:10.1016/0020-0190(91)90246-E.

14. Arindam Khan and Eklavya Sharma. Tight approximation algorithms for geometric bin packing with skewed items. *CoRR*, abs/2105.02827, 2021. URL: https://arxiv.org/abs/2105.02827.

15. Erez Petrank. The hardness of approximation: Gap location. *Comput. Complex.*, 4:133–157, 1994. doi:10.1007/BF01202866.

16. Sai Sandeep. Almost optimal inapproximability of multidimensional packing problems. *Electron. Colloquium Comput. Complex.*, 28:7, 2021. URL: http://eccc.weizmann.ac.il/report/2021/007.

17. Gerhard J. Woeginger. There is no asymptotic PTAS for two-dimensional vector packing. *Inf. Process. Lett.*, 64(6):293–297, 1997. doi:10.1016/S0020-0190(97)00179-8.
A Omitted Proofs

Proof of Lemma 7 (If) Suppose \( x_i \in X, y_j \in Y, z_k \in Z \) and \( t_{(i,j,k)} \in T \) where \( t_{(i,j,k)} = (x_i, y_j, z_k) \) then it is easy to verify that indeed

\[
 t'_{(i,j,k)} + x'_i + y'_j + z'_k + \sum_{i=4}^{m-1} c'_i = b
\]

(Only if) Conversely, suppose that there is a subset \( S \subseteq U' \) with \( |S| = m \) which sums to \( b \). Let \( S = \{a'_1, \ldots, a'_m\} \). Considering the equation modulo \( r \) and using the fact \( \sum_{i=1}^{m} 2^i \) is the only possible way of obtaining \( 2^{m+1} - 1 \) as a sum of \( m \) elements (possibly with repetition) from the set \( \{1, 2, \ldots, 2^m\} \). Therefore, we conclude there are some elements \( x_i \in X, y_j \in Y, z_k \in Z \) and a tuple \( t_{(i_0,j_0,k_0)} \in T \) such that \( x'_i, y'_j, z'_k, t'_{(i_0,j_0,k_0)} \in S \). Now, considering the equation modulo \( r^2, r^3, r^4 \) gives us \( i = i_0, j = j_0 \), and \( k = k_0 \).

Proof of Lemma 8 The first part follows from the fact that the first component of any vector is strictly greater than \( \frac{1}{m+1} \). The second part of the claim follows from the fact the sum are less than \( 1 \).

Proof of Lemma 9 (If) For a tuple \( t_{(i,j,k)} = (x_i, y_j, z_k) \), we have \( t'_{(i,j,k)} + x'_i + y'_j + z'_k + \sum_{i=4}^{m-1} c'_i = b \) by Lemma 7. So, we have

\[
 t_{(i,j,k)} + x_i + y_j + z_k + \sum_{i=4}^{m-1} c'_i
 = \left( \frac{m}{m+1} + \frac{t'_{(i,j,k)} + x'_i + y'_j + z'_k + \sum_{i=4}^{m-1} c'_i}{(m+1)b} \right)m + 2 \left\lfloor \frac{m+2}{m+1} - \frac{t'_{(i,j,k)} + x'_i + y'_j + z'_k + \sum_{i=4}^{m-1} c'_i}{(m+1)b} \right\rfloor
 = (1, 1).
\]

(Only if) Suppose there is a set \( S = \{a_i\} \) of \( m \) vectors which fit in a bin. By Lemma 8 all the vectors are non-dummy vectors. Hence, each vector can be written as:

\[
a_i = \left( \frac{1}{m+1} + \frac{a'_i}{(m+1)b} \right)m + \frac{2}{(m+1)b} - \frac{a'_i}{(m+1)b}.
\]

As they fit in a bin we get the following two conditions,

\[
\frac{m - 1}{m + 1} + \frac{\sum_{i=1}^{m} a'_i}{(m+1)b} \leq 1,
\]

\[
\frac{m + 2}{m + 1} - \frac{\sum_{i=1}^{m} a'_i}{(m+1)b} \leq 1,
\]

which simplify to \( \sum_{i=1}^{m} a'_i \leq b \) and \( \sum_{i=1}^{m} a'_i \geq b \). Combining the inequalities we get \( \sum_{i=1}^{m} a'_i = b \). Therefore, by Lemma 7 we have shown what we set out to show.
Proof of Lemma 13. (If) For a tuple \( t_{(i,j,k)} = (x_i, y_j, z_k) \), we have \( t'_{(i,j,k)} + x'_i + y'_j + z'_k = b \) by Lemma 2. So, we have

\[
 t_{(i,j,k)} + x_i + y_j + z_k = \left( \frac{4}{5} + \frac{t'_{(i,j,k)} + x'_i + y'_j + z'_k}{5b}, \frac{6}{5} - \frac{t'_{(i,j,k)} + x'_i + y'_j + z'_k}{5b} \right) = (1, 1).
\]

(Only if) Suppose there are four vectors \( a_1, a_2, a_3, a_4 \) which cover a non-D-bin. By our assumption all the vectors are non-dummy vectors. Hence each vector can be written as,

\[
a_i = \left( \frac{1}{5} + \frac{a'_i}{5b}, \frac{3}{10} - \frac{a'_i}{5b} \right).
\]

As they fit in a bin we get the following two conditions,

\[
\begin{align*}
\frac{4}{5} + \frac{\sum_{i=1}^{4} a'_i}{5b} & \geq 1, \\
\frac{6}{5} - \frac{\sum_{i=1}^{4} a'_i}{5b} & \geq 1,
\end{align*}
\]

which simplify to \( \sum_{i=1}^{4} a'_i \geq b \) and \( \sum_{i=1}^{4} a'_i \leq b \). Combining the inequalities, we get \( \sum_{i=1}^{4} a'_i = b \).

Therefore, by Lemma 2 the vectors correspond to a tuple. \( \square \)