Characteristic foliation on the discriminant hypersurface of a holomorphic Lagrangian fibration

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CHARACTERISTIC FOLIATION ON THE DISCRIMINANT HYPERSURFACE OF A HOLOMORPHIC LAGRANGIAN FIBRATION

By JUN-MUK HWANG and KEIJI OGISO

Abstract. We give a Kodaira-type classification of general singular fibers of a holomorphic Lagrangian fibration in Fujiki’s class \( C \). Our approach is based on the study of the characteristic vector field of the discriminant hypersurface, which naturally arises from the defining equation of the hypersurface via the symplectic form. As an application, we show that the characteristic foliation of the discriminant hypersurface has algebraic leaves which are either rational curves or smooth elliptic curves.

1. Introduction. We work in the category of complex analytic sets. Let \( (M, \omega) \) be a holomorphic symplectic manifold and \( Y \subset M \) be a reduced hypersurface. We will not assume that \( M \) is compact. The restriction of \( \omega \) on the smooth locus \( Y_{\text{reg}} \) of \( Y \) has a kernel of rank 1 at every point, defining a foliation of rank 1 on \( Y_{\text{reg}} \). We will call it the characteristic foliation on \( Y \). (See Section 2 for details.) A leaf of this foliation is said to be \textit{algebraic} if its closure in \( M \) is an algebraic curve, i.e., a compact complex variety of dimension 1. A natural question is the following.

\textit{Question 1.1. Are the leaves of the characteristic foliation algebraic?}

The answer is no in general. Note that if \( Y \) is a smooth projective variety with Picard number 1, it does not admit a foliation with algebraic leaves. For example, if \( M \) is a projective holomorphic symplectic manifold with Picard number 1 of dimension \( \geq 4 \), which certainly exists (see e.g., [BD], [Og]), then for any smooth hypersurface \( Y \) in \( M \), the general leaves of the characteristic foliation are not algebraic. The essential point of Question 1.1 is to \textit{find} a suitable class of hypersurfaces in which the leaves of the characteristic foliations are closed.

The motivation for the current paper comes from [HR], where the Hitchin system \( f: M \rightarrow B \) was studied. Here, \( M \) is the moduli space of Higgs bundles over a curve and \( B \) is a certain affine space. In [HR], the discriminant hypersurface \( Y \subset M \), i.e., the locus of singular fibers of \( f \), was studied and it was noticed ([HR, Remark 4.9]) that the closures of the leaves of the characteristic foliation...
on $Y$ are rational curves called Hecke curves. In other words, the discriminant hypersurface of the Hitchin system is one example where we have a positive answer for Question 1.1. One of our main results is a generalization of this fact in the following way. See Sections 2 and 3 for the precise definitions of a holomorphic Lagrangian fibration and a discriminant hypersurface.

**Theorem 1.2.** Let $M$ be a holomorphic symplectic manifold and let $f : M \to B$ be a holomorphic Lagrangian fibration over a complex manifold $B$. Assume that each irreducible component of any general singular fiber of $f$ belongs to class $C$, i.e., is bimeromorphic to a compact Kähler manifold. Let $Y$ be the discriminant hypersurface of $f$. Then the characteristic foliation on $Y$ has algebraic leaves and the closures of the leaves are either rational curves or elliptic curves.

In the course of proving Theorem 1.2, we are naturally led to study the structure of general singular fibers of $f$, which will be of its own interest. The results can be summarized as the following two theorems (Theorems 1.3 and 1.4). Both theorems are directly motivated by the study of the characteristic foliation.

**Theorem 1.3.** Let $M$ be a holomorphic symplectic manifold of dimension $2n$ and let $f : M \to B$ be a holomorphic Lagrangian fibration over a complex manifold $B$ of dimension $n$. Assume that each irreducible component of any general singular fiber of $f$ belongs to class $C$. Let $\Delta \subset B$ be the hypersurface consisting of critical values of $f$. For a general point $b$ of an irreducible component $D$ of $\Delta$, let $X$ be the underlying variety of an irreducible component of the singular fiber $M_b = f^{-1}(b)$ and let $\hat{X}$ be the normalization of $X$. Then $\hat{X}$ is a compact complex manifold of class $C$ such that the Albanese variety $\text{Alb}(\hat{X})$ has dimension $n-1$ and the Albanese map $\alpha : \hat{X} \to \text{Alb}(\hat{X})$ is either a $\mathbb{P}_1$-bundle or an elliptic fiber bundle. The latter case occurs only when $M_b$ is irreducible and nonreduced, in which case, the reduction $Y_b := (M_b)_{\text{red}}$ of $M_b$ is smooth.

Theorem 1.3 describes the structure of each irreducible component of a general singular fiber. The next theorem describes the structure of the whole general singular fiber.

For the statement, we need a few more notions. In the same notation as in Theorem 1.3, let $Y_b$ be the reduction of any general singular fiber $M_b$. We call an irreducible curve $\Theta$ on $Y_b$ a characteristic curve if it is the image of some fiber of the Albanese map $\alpha : \hat{X} \to \text{Alb}(\hat{X})$, where $\hat{X}$ is the normalization of some irreducible component $X$ of $Y_b$. We say that two points $y_1$ and $y_2$ on $Y_b$ are equivalent if there exist finitely many characteristic curves $\Theta_1, \ldots, \Theta_N$, such that $y_1 \in \Theta_1, y_2 \in \Theta_N$ and $\Theta_i \cap \Theta_{i+1} \neq \emptyset$ for each $1 \leq i \leq N-1$. Then, each equivalence class is of the form $\bigcup_{s \in \Lambda} \Theta_s$, where $\Theta_s$ are characteristic curves. Here the index set $\Lambda$ is possibly an infinite set. For each characteristic curve $\Theta_s$, we define the multiplicity $r_s$ to be the multiplicity of the unique irreducible
component $H$ of the discriminant hypersurface $f^{-1}(D)$ such that $\Theta_s \subset H$. We call the 1-cycles

$$\sum_{s \in \Lambda} r_s \Theta_s, \sum_{s \in \Lambda} \Theta_s,$$

possibly with infinitely many irreducible components, a \textit{characteristic 1-cycle} and a \textit{reduced characteristic 1-cycle} respectively. Note that $Y_b$ is a disjoint union of the reduced characteristic 1-cycles on $Y_b$.

**Theorem 1.4.** In the same notation as in Theorem 1.3, let $Y_b$ be the reduction of a general singular fiber $M_b$. Then the natural $\mathbb{C}^n - 1$-action on $Y_b$ (cf. Proposition 2.2) induces a transitive action on the set of characteristic 1-cycles on $Y_b$. In particular, the characteristic 1-cycles on $Y_b$ are all isomorphic. Moreover, modulo common divisors of $r_s$, the characteristic 1-cycle is of the form of:

1. one of the singular fibers of a relatively minimal elliptic fibration listed by Kodaira [Kd, Theorem 6.2];
2. a 1-cycle of Type $A_\infty$, i.e., a 1-cycle $\sum_{i \in \mathbb{Z}} C_i$ consisting of infinitely many $\mathbb{P}_1$’s such that $C_i \cap C_{i+1} = \{ P_i \}$ (the intersections are quasi-transversal and $P_i \neq P_j$ if $i \neq j$), and such that $C_i \cap C_j = \emptyset$ if $|i - j| \geq 2$; or
3. a 1-cycle of Type $D_\infty$, i.e., a 1-cycle $C_0 + C_1 + \sum_{i \geq 2} 2C_i$ consisting of infinitely many $\mathbb{P}_1$’s such that $C_i \cap C_{i+1} = \{ P_i \}$ for each $i \geq 1$, $C_0 \cap C_2 = \{ P_0 \}$ (all the intersections are quasi-transversal and $P_i \neq P_j$ if $i \neq j$) and such that $C_i \cap C_j = \emptyset$ for other pairs $i \neq j$.

For a more precise formulation, see Propositions 4.10, 4.11 and 4.12. Here, we say that the intersection of two nonsingular curves is quasi-transversal, if their tangent spaces at the intersection point are distinct. Theorem 1.4 says that a general singular fiber is a disjoint union of a family of 1-cycles, each of which is either one of Kodaira’s singular fibers in [Kd, Theorem 6.2] or an infinite cycle described in (2) or (3). As Matsushita pointed out to us, the case (2) actually occurs. His example will be given in Proposition 4.13 in Section 4. Unfortunately, we do not have a concrete example of the case (3).

When $f$ is a projective morphism, Theorem 1.3 and some part of Theorem 1.4 follow from the works of Matsushita ([M1], [M2]). In particular, Matsushita gave a more or less complete classification when $n = 2$ and $f$ is projective in [M1], which contains more refined information than ours, especially about the multiplicities and monodromy. However, the view-point of the characteristic 1-cycles in Theorem 1.4 did not appear in his classification. We believe that our view-point gives a new perspective even in the situation of [M1].

The argument used by Matsushita requires the projectivity of $f$, because it depends on the classification theory of degeneration of abelian varieties. No analog of this theory is known for nonalgebraic complex tori. Our proof of Theorems 1.3 and 1.4 uses a completely different approach and uses the properties of La-
grangian fibration more directly. A key tool of our approach is twisted vector fields on the singular fibers. By examining the Chern numbers of the leaves and the degree of the twisting, we can control the degeneration of the fibers. This idea goes back to Siu’s work [Si] where he used certain twisted vector fields to control the degeneration of complex structures. Another key ingredient in the control of the singularity of a general singular fiber is the theory of dualizing sheaves for singular curves, in particular, the classical result of Rosenlicht’s in [Se, Chapter IV].

In Section 2, we will introduce the basic geometric objects associated to a proper holomorphic Lagrangian fibrations, called the characteristic foliation of a vertical hypersurface, and show that Theorem 1.2 follows from Theorem 1.3. In Section 3, we study the characteristic foliation arising from the determinantal hypersurface more closely and prove Theorems 1.3. In Section 4, we will prove Theorem 1.4, using the characteristic vector fields and the theory of dualizing sheaves for singular curves.

While writing this paper, we learned about a preprint of J. Sawon [Sa] where Question 1.1 was studied from a different view-point. In particular, he found many interesting examples for which the characteristic foliations are algebraic. But there is no overlap with our result. Also after we finished a preliminary version of this paper, we received Matsushita’s preprint [M3] which gives a classification of the general singular fibers when $f$ is a projective morphism, refining his previous works [M1] and [M2]. His result is more refined than ours when $f$ is projective. His approach generalizes that of [M1] and is completely different from ours.

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2. Characteristic foliation of a vertical hypersurface. Let $M$ be a connected (not necessarily compact) complex manifold of dimension $2n$. Assume that there exists a symplectic form $\omega$ on $M$. This means that $\omega$ is a $d$-closed holomorphic 2-form on $M$ which is nondegenerate at every point of $M$. The pair $(M, \omega)$ is called a holomorphic symplectic manifold of dimension $2n$. The symplectic form $\omega$ defines a natural $O_M$-linear isomorphism $\iota_\omega: T^*(M) \longrightarrow T(M)$, which is the inverse of the contraction map $T(M) \longrightarrow T^*(M)$ defined by $v \mapsto \omega(v, *)$.

Let $Y \subset M$ be a reduced (not necessarily irreducible) hypersurface in $M$ and let $N^*_Y$ be the conormal sheaf of $Y \subset M$. This is an invertible sheaf on $Y$; in fact,

$$N^*_Y \cong O(-Y)|_Y.$$

Via the natural map $N^*_Y \longrightarrow T^*(M)|_Y$, we get a canonical section of $T^*(M) \otimes$
\( \mathcal{O}(Y)|_Y \) uniquely defined up to a nonzero multiplicative function. Via the isomorphism \( \iota_\omega: T^*(M) \rightarrow T(M) \), this gives rise to a nonzero section

\[
\lambda_Y \in H^0(Y, T(M) \otimes \mathcal{O}(Y)).
\]

Locally, \( \lambda_Y \) is defined as follows. Let \( h, \) or more precisely \( h = 0, \) be a local defining equation of the hypersurface \( Y \) in an open subset \( U \subset M. \) Then on \( U \cap Y, \)

\[
\lambda_Y = \iota_\omega (dh), \text{ i.e., } \omega(\lambda_Y, *) = dh(*).
\]

From this local description, it is clear that the leaves of the twisted vector field \( \lambda_Y \) are tangent to \( Y, \) defining a foliation on the smooth part of \( Y_{\text{reg}} \) of \( Y. \) This foliation of \( Y_{\text{reg}} \) is exactly the characteristic foliation of \( Y \) defined in the introduction. The twisted vector field \( \lambda_Y \) will be called the characteristic vector field on \( Y. \)

Now let \( B \) be a complex manifold of dimension \( n \) and let \( f: M \rightarrow B \) be a holomorphic Lagrangian fibration, i.e., \( f \) is a proper morphism with connected fibers such that the underlying variety of each fiber of \( f \) is a Lagrangian subvariety of \( M. \) Recall that a reduced (not necessarily irreducible) subvariety \( L \) of \( (M, \omega) \) is called a Lagrangian subvariety if \( L \) is of pure dimension \( n = \dim M/2 \) and \( \omega|_{L_{\text{reg}}} = 0. \)

Given a holomorphic function \( g \) on \( B, \) the holomorphic vector field \( \iota_\omega(f^* dg) \) on \( M \) is called the Hamiltonian vector field associated to \( g. \) It is well known that the Hamiltonian vector field is complete, because \( f \) is proper, and it is tangent to the fibers of \( f, \) in the sense that it maps the defining ideal of the reduction of each fiber to itself. For two functions \( g_1 \) and \( g_2, \) their Hamiltonian vector fields

\[
\iota_\omega(f^* dg_1) \text{ and } \iota_\omega(f^* dg_2)
\]

commute, i.e., \( [\iota_\omega(f^* dg_1), \iota_\omega(f^* dg_2)] = 0 \) under the Lie bracket (see e.g., [Ar]).

We will say that a reduced hypersurface \( Y \subset M \) is vertical with respect to \( f \) if the set-theoretic image \( f(Y) \) is an irreducible hypersurface in \( B. \) The requirement of the irreducibility of \( f(Y) \) is somewhat artificial, but will be very convenient when stating our results below. Note that \( Y \) itself is not necessarily irreducible. We will denote the restriction of \( f \) to \( Y \) by \( f|_Y: Y \rightarrow f(Y). \) Here and hereafter, we regard the image \( f(Y) \) as an analytic subset with its reduced structure. But we regard the preimage \( f^{-1}(Z) \) of an analytic subset \( Z \) as an analytic subspace of \( Y \) in the scheme-theoretic sense.

**Proposition 2.1.** Let \( Y \subset M \) be a vertical hypersurface with respect to a Lagrangian fibration \( f: M \rightarrow B. \) Then the characteristic foliation on \( Y_{\text{reg}} \) is tangent to fibers of \( f|_Y. \)

**Proof:** The statement is local on \( Y. \) So we may assume that \( Y \) is irreducible and replace \( B \) by a coordinate neighborhood of a smooth point of the hypersurface \( f(Y). \) Let \( y \in Y \) be a smooth point of \( Y \) such that \( f(y) \) is a smooth point of \( f(Y). \)
Let \( h \in \mathcal{O}_{M,y} \) be a local defining equation of \( Y \) at \( y \) and \( g \in \mathcal{O}_{B,f(y)} \) be a local defining equation of \( f(Y) \) at \( f(y) \). Then

\[
f^* g = \xi \cdot h^k \quad \text{for some } \xi \in \mathcal{O}_{M,y}^* \text{ and an integer } k > 0.
\]

The characteristic vector field of \( Y \) is defined locally by the 1-form \( dh \). On the other hand,

\[
f^* dg = h^k \, d\xi + k\xi h^{k-1} \, dh.
\]

The Hamiltonian vector field \( \iota_\omega (f^* dg) \) is tangent to the fibers of \( f \) and vanishes on \( Y \) with multiplicity \( k - 1 \). Thus the local vector field

\[
\frac{1}{h^{k-1}} \iota_\omega (f^* dg)
\]

defined in a neighborhood of \( y \) in \( M \) is also tangent to the fibers of \( f \). Therefore, its restriction to \( Y \) will be tangent to fibers of \( f|_Y \) as well. But the restriction is just

\[
\iota_\omega \left( \frac{1}{h^{k-1}} f^* dg \right)|_Y = \iota_\omega (k \xi \, dh)|_Y.
\]

The right-hand side is exactly the characteristic vector field on \( Y \) multiplied by the nonvanishing function \( k \xi \). It follows that the characteristic vector field of \( Y \) is tangent to the fibers of \( f|_Y \).

**Proposition 2.2.** Let \( Y \subset M \) be a vertical hypersurface with respect to a Lagrangian fibration \( f: M \rightarrow B \). Then there exist \( n - 1 \) commuting holomorphic vector fields \( v_1, \ldots, v_{n-1} \) in a neighborhood of a general fiber \( Y_b \) of \( f|_Y \), which are linearly independent at every point of \( Y_b \). These vector fields generate an effective holomorphic action of the commutative additive complex Lie group \( \mathbb{C}^{n-1} \) on a general fiber of \( f|_Y \) such that each orbit has dimension \( n - 1 \). In particular, the normalization \( \hat{Y}_b \) of a general fiber \( Y_b \) of \( f|_Y \) is smooth and the connected component of the singular locus of \( Y_b \) is a variety of dimension \( n - 1 \) homogeneous under the action of \( \mathbb{C}^{n-1} \), so that \( \text{Sing}(Y_b) \) is a disjoint union of \( (n-1) \)-dimensional tori.

**Proof.** Choose a general point \( b \) of \( f(Y) \). Let \( z_1, \ldots, z_{n-1} \) be local holomorphic functions in a neighborhood \( U \) of \( b \) in \( B \) whose restriction to \( f(Y) \) give holomorphic coordinates in a neighborhood of \( b \) in \( f(Y) \). Consider the holomorphic map \( \Psi: U \rightarrow \mathbb{C}^{n-1} \) defined by \( \Psi := (z_1, \ldots, z_{n-1}). \) By shrinking \( U \), we may assume that \( \Psi|_{f(Y) \cap U}: f(Y) \cap U \rightarrow \Psi(U) \) is biholomorphic. A general fiber of \( \Psi \circ f \) is smooth by Sard’s theorem. From the assumption that \( b \) is a general point of \( f(Y) \), we may assume that the fiber of \( \Psi \circ f \) over \( \Psi(b) \) is smooth. In other words, the 1-forms \( f^* dz_1, \ldots, f^* dz_{n-1} \), defined in a neighborhood of the
fiber $f^{-1}(b)$ are linearly independent at every point of $Y_b$. Via the isomorphism $\iota_\omega: T^*(M) \rightarrow T(M)$, these 1-forms define Hamiltonian vector fields

$$v_i := \iota_\omega(f^*dz_i)$$

in a neighborhood of $Y_b$. Recall that these Hamiltonian vector fields are tangent to the fibers of $f$. Moreover, they are linearly independent at every point of $Y_b$ and they commute:

$$[v_i, v_j] = 0, \text{ for each } 1 \leq i, j \leq n - 1.$$ 

Since $f$ is proper, these Hamiltonian vector fields are complete. Thus, they generate a holomorphic action of the commutative complex Lie group $\mathbb{C}^{n-1}$ on $Y_b$. Each point of $Y_b$ has an $(n - 1)$-dimensional (not necessarily closed) orbit under this action, as $v_1, \ldots, v_{n-1}$ are linearly independent at each point of $Y_b$. This $\mathbb{C}^{n-1}$-action can be lifted to the normalization $\hat{Y}_b$ of $Y_b$, because any local analytic automorphism of a complex analytic variety can be lifted to its normalization. Let $\text{Sing} (\hat{Y}_b)$ be the singular locus of $\hat{Y}_b$. By the normality, $\text{Sing} (\hat{Y}_b)$ has dimension $\leq n - 2$. But each orbit of the $\mathbb{C}^{n-1}$-action on $\hat{Y}_b$ has dimension $n - 1$, as the natural lifts of $v_1, \ldots, v_{n-1}$ under the natural map $\hat{Y}_b \rightarrow M$, say, $\hat{v}_1, \ldots, \hat{v}_{n-1}$, are linearly independent at each point of $\hat{Y}_b$. It follows that $\text{Sing} (\hat{Y}_b)$ must be empty. 

**Proposition 2.3.** Let $Y \subset M$ be a vertical hypersurface with respect to a Lagrangian fibration $f: M \rightarrow B$. Let $Y_b$ be a general fiber of $f|_Y$ and $\hat{Y}_b$ be its normalization, which is smooth by Proposition 2.2. Let $\hat{v}_i$ be the vector field on $\hat{Y}_b$ which is the natural lift of the vector field $v_i$ defined in Proposition 2.2. Then there exist $n - 1$ holomorphic 1-forms $\varphi_1, \ldots, \varphi_{n-1}$ on $\hat{Y}_b$ with the following properties:

(i) $\varphi_i(\hat{v}_j) = \delta_{ij}$ for each $1 \leq i, j \leq n - 1$,

(ii) $\varphi_1, \ldots, \varphi_{n-1}$ are linearly independent at every point of $\hat{Y}_b$, and

(iii) each $\varphi_i$ annihilates the lift of the characteristic vector field $\lambda_Y$ to $\hat{Y}_b$.

**Proof.** Let $\nu: \hat{Y} \rightarrow Y$ be the normalization of $Y$ and let $\mu: \hat{Y} \rightarrow f(Y)$ be the composition $\mu = f \circ \nu$. Let $b \in f(Y)$ be a general smooth point of $f(Y)$. Then by Proposition 2.2, $\hat{Y}_b = \mu^{-1}(b)$ is smooth and therefore $\mu$ is a smooth morphism around $\hat{Y}_b$. In particular, $\hat{Y}$ is smooth in a neighborhood of $\hat{Y}_b$. Let $\frac{\partial}{\partial z_i} \in T_b(f(Y))$ be the coordinate vector with respect to the coordinates $z_1, \ldots, z_{n-1}$ on $f(Y)$ used in the proof of Proposition 2.2. For each $i = 1, \ldots, n - 1$, we will define a nonzero 1-form $\varphi_i$ on $\hat{Y}_b$ as follows. For each $z \in \hat{Y}_b$, pick an element $u_i \in T_z(\hat{Y})$ such that $\mu_*(u_i) = \frac{\partial}{\partial z_i}$, where $\mu_*: T_z(\hat{Y}) \rightarrow T_b(f(Y))$ is the differential of $\mu$, which is surjective. Then one can define the holomorphic 1-form $\varphi_i$ on $\hat{Y}_b$ by setting for each tangent vector $v \in T_z(\hat{Y}_b)$,

$$\varphi_i(v) := \omega(\nu_*(u_i), \nu_*(v)),$$
where $\nu_\ast: T_z(\hat{Y}) \rightarrow T_{\nu(z)}(M)$ is the differential of $\nu: \hat{Y} \rightarrow M$. This definition does not depend on the choice of $u_i$ because $f$ is a Lagrangian fibration. From this and the definitions of $v_i$ and $\iota_\omega$, (i) is immediate. (ii) is then a direct consequence of (i), as $\partial_1, \ldots, \partial_{n-1}$ are linearly independent at each point of $\hat{Y}_b$.

To check (iii), it suffices to check it at a general point of $\hat{Y}_b$. Choose a point $z \in \hat{Y}_b$ over the smooth locus of $Y$ and let $\nu \in T_z(\hat{Y}_b)$ be a local lift of the characteristic vector field $\lambda_Y$. By the definition of $\varphi_i$,

$$\varphi_i(\nu) = \omega(\nu_\ast(u_i), \lambda_Y)$$

which is zero by the definition of the characteristic foliation.

**Proposition 2.4.** Let $Y \subset M$ be a vertical hypersurface with respect to a Lagrangian fibration $f: M \rightarrow B$. Let $Y_b$ be a general fiber of $f|_Y$ and let $\hat{Y}_b$ be its normalization, which is smooth by Proposition 2.2. Let $\hat{X}$ be an irreducible component of $\hat{Y}_b$. Assume that $\hat{X}$ is of class $C$, so that the Hodge decomposition holds for $\hat{X}$. Suppose that $\dim H^0(\hat{X}, T^*(\hat{X})) \leq n - 1$, where $2n = \dim M$. Then, the Albanese map $\alpha: \hat{X} \rightarrow \text{Alb}(\hat{X})$ is surjective, has connected fibers, and the fibers of $\alpha$ correspond to the leaves of the characteristic foliation on $Y$.

**Proof.** Recall that the holomorphic 1-forms $\varphi_1, \ldots, \varphi_{n-1}$ on $\hat{X}$ (in Proposition 2.3) are linearly independent at each point of $\hat{X}$. Therefore, the dimension of the image of the Albanese map is at least $n - 1$. Thus $\alpha$ is surjective by our assumption. We can say more. Let $\beta: \hat{X} \rightarrow A$ be the Stein factorization of $\alpha$. By Proposition 2.3 (ii), the induced morphism $A \rightarrow \text{Alb}(\hat{X})$ is étale and surjective. It follows that $A$ is also an $(n-1)$-dimensional complex torus, and that $H^0(\hat{X}, T^*(\hat{X})) = \beta^*(H^0(A, T^*(A)))$. Thus, $\alpha$ coincides with $\beta$. Hence the general fibers of Albanese map are connected algebraic curves on $\hat{X}$. They correspond to the leaves of the characteristic foliation by Proposition 2.3(iii).

Note that by Proposition 2.4, Theorem 1.2 follows from Theorem 1.3.

**3. Characteristic foliation of a discriminant hypersurface.** Let $f: M \rightarrow B$ be a holomorphic Lagrangian fibration satisfying the assumption of Theorem 1.3. Consider the set of the critical values of $f$, i.e.,

$$\Delta = \{b \in B, f^{-1}(b) \text{ is singular}\}.$$

In this section, we study the characteristic foliation of the special vertical hypersurface called the discriminant hypersurface more closely. First of all, we notice the following:

**Proposition 3.1.** (1) $M_s = f^{-1}(s)$ is an $n$-dimensional complex torus if $s \notin \Delta$.

(2) The critical set $\Delta$ is a hypersurface of $B$ unless $\Delta = \emptyset$. 


The second statement is nontrivial. In fact, there is a flat morphism from a 
4-dimensional manifold to a 2-dimensional manifold whose critical set consists
of just one point [Ra].

**Proof.** (1) is well-known as Liouville’s theorem for a Lagrangian fibration
(e.g., [Ar]).

Let us show (2). It suffices to show that \( f: M \rightarrow B \) is a smooth morphism
if \( f \) is smooth over \( B \setminus Z \) for some analytic subset \( Z \) of codimension \( \geq 2 \). Let
\( b \in Z \). Since the statement is local at each \( b \), we may (and will) assume that \( B \)
is a germ at \( b \) and shrink it whenever it is more convenient. Choose a smooth
point \( P \) of \( Y_b \). Then, there is a smooth \( n \)-dimensional manifold \( U \subset M \) such that
\( U \) meets \( Y_b \) at \( P \) transversally. The induced map \( f|_U: U \rightarrow B \) is a finite map
which is unramified over \( B \setminus Z \). Since \( Z \) is of codimension \( \geq 2 \), by the purity of
branched loci for a finite map, \( f|_U \) is unramified over \( B \). Thus \( U \) is a section of
\( f \). In particular, \( f^{-1}(b) \) is reduced at \( P \). We denote \( u_t = (f|_U)^{-1}(t) \) \( (t \in B) \).

Let \( z_1, \ldots, z_n \) be a local coordinate system of \( B \) at \( b \). Using these coordinates,
we obtain \( n \) Hamiltonian vector fields, \( \iota_\omega(f^*dz_1), \ldots, \iota_\omega(f^*dz_n) \), which
are commutative, tangent to each fiber and linearly independent at each point of
\( M \setminus f^{-1}(Z) \). Thus, as in the proof of Proposition 2.2, they generate a holomorphic
action of the commutative Lie group \( C^n \) on \( M \) over \( B \) and define a holomorphic map
\[
\rho: C^n \times B \rightarrow M; \quad (g, t) \mapsto g(u_t).
\]

For each \( t \in B \), we denote the stabilizer subgroup \( \{ g \in C^n; g(u_t) = u_t \} \) by \( \Lambda_t \). If
\( t \in B \setminus Z \), then \( M_t = C^n / \Lambda_t \), as \( M_t \) is an \( n \)-dimensional complex torus by (1). Thus,
we can naturally regard \( \Lambda_t \) as \( H_1(M_t, \mathbb{Z}) \), which is canonically dual to \( H^1(M_t, \mathbb{Z}) \).

Since \( B \setminus Z \) is simply-connected, the local system \( R^1(f|_M)^{-1}(Z) \) is actually constant;
\( R^1(f|_M)^{-1}(Z) \) is \( \mathbb{Z}^{2n} \) over \( B \setminus Z \). The canonical basis \( e_1, \ldots, e_{2n} \)
of the constant sheaf \( \mathbb{Z}^{2n} \) gives the generators of \( H^1(M_t, \mathbb{Z}) \) and therefore the
generators of \( \Lambda_t \) simultaneously for \( t \in B \setminus Z \). In this way, we obtain \( 2n \) \( C^n \)-valued holomorphic functions \( f_1(t), \ldots, f_{2n}(t) \) on \( B \setminus Z \) such that
\[
\Lambda_t = \langle f_1(t), \ldots, f_{2n}(t) \rangle
\]
for each \( t \in B \setminus Z \).

Let \( \Gamma := \pi_1(M \setminus f^{-1}(Z)) \). Since \( C^n \times (B \setminus Z) \) is simply connected, \( \Gamma \cong \mathbb{Z}^{2n} \) and \( \Gamma \) acts by the holomorphic deck transformations on the universal cover \( C^n \times (B \setminus Z) \)
over \( B \setminus Z \). The deck transformations on each fiber \( C^n \times \{ t \} \) \( (t \in B \setminus Z) \) are just
translations by the elements of the lattice \( \Lambda_t \). Since \( C^n \times Z \) is of codimension
\( \geq 2 \) in \( C^n \times B \), the action of \( \Gamma \) uniquely extends to an action on \( C^n \times B \) over \( B \).
The induced action on \( C^n \times \{ t \} \) \( (t \in B) \) is by translations. (At the moment, we do
not know if it is faithful and discontinuous or not.)
Let \( \hat{M} \) be the universal covering space of \( M \). As \( f^{-1}(Z) \) is of codimension \( \geq 2 \), we have \( \pi_1(M) \cong \Gamma \) and \( \Gamma \) acts on the complex manifold \( \hat{M} \) by holomorphic deck transformations. Since \( \mathbb{C}^n \times (B \setminus Z) \) is simply connected, the natural inclusion \( M \setminus f^{-1}(Z) \subset M \) lifts to a natural inclusion \( \mathbb{C}^n \times (B \setminus Z) \subset \hat{M} \), which is compatible with \( \Gamma \). Let \( \hat{f} : \hat{M} \to B \) be the composition of \( f \) and the universal covering map. By the shape of the action of \( \Gamma \), \( \hat{f} \) coincides with the second projection \( \mathbb{C}^n \times (B \setminus Z) \to B \setminus Z \) on \( \mathbb{C}^n \times (B \setminus Z) \).

Since \( \mathbb{C}^n \times B \) is also simply-connected, the morphism \( \rho : \mathbb{C}^n \times B \to M \) lifts to the morphism \( \hat{\rho} : \mathbb{C}^n \times B \to \hat{M} \). \( \hat{\rho} \) is equivariant under the action of \( \Gamma \) and commutes with the morphisms to \( B \), as it is so on the common open set \( \mathbb{C}^n \times (B \setminus Z) \). Therefore the action of \( \Gamma \) on \( \mathbb{C}^n \times B \) is free and discontinuous, as it is so on \( \hat{M} \).

Thus, we obtain a smooth complex torus fibration \( f' : (\mathbb{C}^n \times B)/\Gamma \to B \) and a bimeromorphic morphism \( \tau : (\mathbb{C}^n \times B)/\Gamma \to M = \hat{M}/\Gamma \) over \( B \). This morphism is also finite, as both \( (\mathbb{C}^n \times B)/\Gamma \) and \( M \) are proper over \( B \) of equidimensional fibers. Thus, \( \tau \) is an isomorphism, as both \( (\mathbb{C}^n \times B)/\Gamma \) and \( M \) are normal (actually smooth). Hence, \( f \) is a smooth complex torus fibration as well. \( \square \)

In what follows, we always assume that \( \Delta \neq \emptyset \), and regard \( \Delta \) as a reduced hypersurface of \( B \). We call \( \Delta \) the critical hypersurface of \( f \). We will fix one irreducible component \( D \) of \( \Delta \). The scheme-theoretic preimage \( f^{-1}(D) \) is a divisor on \( M \). We call \( f^{-1}(D) \) the discriminant hypersurface of \( f \). Let \( Y \) be the underlying reduced hypersurface of \( f^{-1}(D) \), i.e., \( Y = f^{-1}(D)_{\text{red}} \). Then \( Y \) is a vertical hypersurface in the sense of Section 2. Let \( Y_b \) be a fiber of \( f|_Y \) over a general point \( b \in D \). Note that \( Y_b \) is reduced (and of pure dimension \( n \)), as \( b \) is general.

**Proposition 3.2.** \( Y_b \) cannot be a complex torus (See Prop. 3.5 for a stronger statement).

**Proof.** It is well-known that deformation of a Lagrangian complex torus in a holomorphic symplectic manifold is unobstructed, locally forming a fibration over an \( n \)-dimensional base space (e.g., [DM, Theorem 8.7]). Thus if \( Y_b \) is complex torus, the scheme theoretic fiber \( f^{-1}(b) \) must coincide with (its reduction) \( Y_b \), a contradiction to \( b \in \Delta \). \( \square \)

Let \( \nu : \hat{Y}_b \to Y_b \) be the normalization of \( Y_b \). Then \( \hat{Y}_b \) is a compact complex manifold by Proposition 2.2. Let \( L \) be a line bundle on \( M \). Let \( w \in H^0(Y_b, T(M) \otimes L) \) be a twisted vector field which is tangent to \( Y_b \), in the sense that it preserves the defining ideal of \( Y_b \). The 1-parameter subgroup of local analytic automorphisms of \( M \) generated by \( w \) in the neighborhood of any point \( y \in Y_b \) preserves \( Y_b \). Since any local analytic automorphism of a complex analytic variety can be lifted to its normalization (and \( \hat{Y}_b \) is smooth by Proposition 2.2), \( w \) can
be lifted to a twisted vector field

\[ \hat{w} \in H^0(\hat{Y}_b, T(\hat{Y}_b) \otimes \nu^* L). \]

Here, the existence of the smooth ambient manifold \( M \) and the fact that \( L \) is a line bundle defined (not only on \( Y_b \) but also) on \( M \) are important for the existence of \( \hat{w} \) on \( \hat{Y}_b \).

The next proposition is very crucial in the sequel.

**Proposition 3.3.** Let \( L \) be a line bundle on \( M \) and \( w \in H^0(Y_b, T(Y_b) \otimes L) \). Assume that \( w \) is tangent to \( Y_b \) and that the lift \( \hat{w} \) is annihilated by the 1-forms \( \varphi_1, \ldots, \varphi_{n-1} \) of Proposition 2.3. Then \( \hat{w} \) must vanish on the points of \( \hat{Y}_b \) lying over the singular locus of \( Y_b \).

**Proof.** Since \( w \) is locally defined in a neighborhood of \( Y_b \) and is tangent to \( Y_b \), \( w \) must be tangent to the singular locus of \( Y_b \). By Proposition 2.2, each component \( E \) of the singular locus of \( Y_b \) is a homogeneous variety under the group action generated by commuting Hamiltonian vector fields \( v_1, \ldots, v_{n-1} \). Thus the set \( \nu^{-1}(E) \) is a smooth hypersurface in \( \hat{Y}_b \) and the lifted vector fields \( \hat{v}_1, \ldots, \hat{v}_{n-1} \) generate the tangent spaces of \( \nu^{-1}(E) \) at every point. Suppose \( \hat{w} \) does not vanish at some point \( z \in \nu^{-1}(E) \). Then

\[ \hat{w}_z = c_1 \hat{v}_1 + \cdots + c_{n-1} \hat{v}_{n-1} \]

for some complex numbers \( c_1, \ldots, c_{n-1} \), one of which, say \( c_1 \) is nonzero. This means that \( \varphi_1(\hat{w}) \) is not identically zero, and \( \hat{w} \) is not annihilated by \( \varphi_1 \), a contradiction. \( \square \)

Let \( X \) be an irreducible component of \( Y_b \). Let \( k \) be the multiplicity of \( f^{-1}(b) \) at a general point of \( X \). In other words, the divisor \( f^{-1}(D) \) has multiplicity \( k \) along the irreducible component \( H \) of \( Y \) such that \( X \subset H \). Let \( g \) be a local defining equation of \( f(Y) \) at \( b \). Then \( f^*dg \) vanishes on \( H \) with multiplicity \( k - 1 \). The vector field \( v_\omega(f^*dg) \) on \( M \) vanishes on \( H \) with multiplicity \( k - 1 \) as well. We can then consider the \( \mathcal{O}(-(k-1)H) \)-valued vector field \( \gamma \) defined locally by

\[ \gamma := \frac{1}{h^{k-1}} v_\omega(f^*dg), \]

where \( h \) is a local defining equation of the reduced hypersurface \( H \) in \( M \). Then, globally,

\[ \gamma \in H^0(Y_b, T(M) \otimes \mathcal{O}(-(k-1)H)). \]
PROPOSITION 3.4. The twisted vector field $\gamma$ defined on $Y_b$ as above is proportional to the characteristic vector field $\lambda_Y$ on the components of $Y_b$ where $\gamma$ is not identically zero.

Proof. It is clear that $\gamma$ is proportional to $\lambda_Y$ on a component of $Y_b$ not lying on $H$. Let $X$ be a component of $Y_b$, which lies on $H$. At a general point $y \in X$, we have

$$f^*g = \xi \cdot h^k$$

for some $\xi \in \mathcal{O}_{M,Y}$. As in the proof of Proposition 2.1. The local expression of $\gamma$ is proportional to $\lambda_Y$, as we have seen it in the proof of Proposition 2.1.

PROPOSITION 3.5. Let $X$ be an irreducible component of $Y_b$ and let $\hat{X}$ be its normalization. Then $\dim H^0(\hat{X}, T^*(\hat{X})) = n - 1$. In particular, the Albanese map $\alpha: \hat{X} \to \text{Alb}(\hat{X})$ is surjective and has connected fibers. Consequently, $X$ cannot be a complex torus and the characteristic foliation of $Y$ has algebraic leaves by Proposition 2.4.

Proof. Let $\nu: \hat{X} \to X$ be the normalization map. Since $\gamma \in H^0(Y_b, T(M) \otimes \mathcal{O}(-(k-1)H))$ and $\gamma$ is proportional to $\lambda_Y$ from Proposition 3.4, we can apply Proposition 3.3 for $\gamma$, as the lift $\hat{\lambda}_Y$ of $\lambda_Y$ is annihilated by 1-forms $\varphi_1, \ldots, \varphi_{n-1}$ constructed in Proposition 2.3. Thus the lifted vector field $\hat{\gamma} \in H^0(\hat{X}, T(\hat{X}) \otimes \nu^*\mathcal{O}(-(k-1)H))$ on $\hat{X}$ vanishes at the points lying over the singular locus of $Y_b$. We have also the characteristic vector field $\lambda_H \in H^0(X, T(M) \otimes \mathcal{O}(H))$ of the vertical hypersurface $H$, which can be lifted to $\hat{\lambda}_H \in H^0(\hat{X}, T(\hat{X}) \otimes \nu^*\mathcal{O}(H))$. We will use these two twisted vector fields $\hat{\gamma}$ and $\hat{\lambda}_H$ on $\hat{X}$ to prove Proposition 3.5.

Let $\varphi$ be a nonzero 1-form on $\hat{X}$. We need to show that $\varphi$ is linearly dependent on $\varphi_1, \ldots, \varphi_{n-1}$.

Claim. At every point $z \in \hat{X}$, $\varphi_z$ is linearly dependent on $\varphi_1, \ldots, \varphi_{n-1}$.

Proof of Claim. Suppose to the contrary that at a general point $z \in \hat{X}$, $\varphi_z$ is linearly independent from $\varphi_1, \ldots, \varphi_{n-1}$. Then both $\varphi_z(\hat{\lambda}_H)$ and $\varphi_z(\hat{\gamma})$ are nonzero. Thus we have nonzero sections

$$\varphi(\hat{\lambda}_H) \in H^0(\hat{X}, \nu^*\mathcal{O}(H)), \quad \varphi(\hat{\gamma}) \in H^0(\hat{X}, \nu^*\mathcal{O}(-(k-1)H)).$$

Then $\varphi(\hat{\lambda}_H)^{\otimes (k-1)} \cdot \varphi(\hat{\gamma})$ is a nonzero holomorphic function on $\hat{X}$. But $\hat{\gamma}$ vanishes at a point of $\hat{X}$ over singular locus of $Y_b$. This leads to a contradiction unless $X$ does not contain any singular point of $Y_b$.

From now, we assume furthermore that $X$ does not contain any singular point of $Y_b$. Note then that $\hat{X} = X = Y_b$ and $Y_b$ is smooth. If $k > 1$, then both $\mathcal{O}(H)$
and \(O(-(k-1)H)\) have nonzero sections on \(Y_b\). Thus \(O(H)\) is a trivial bundle on \(Y_b\). This implies that \(\gamma \in H^0(Y_b, T(Y_b))\). Then \(\gamma, \nu_1, \ldots, \nu_{n-1}\) are vector fields on \(Y_b\) which are linearly independent at \(z\). Here \(\nu_1, \ldots, \nu_{n-1}\) are vector fields in Proposition 2.2. If a linear combination of these \(n\) vector fields vanishes at some point of \(\hat{X} = Y_b\), the holomorphic function \(\varphi(\gamma) = \varphi(\xi)\) is nonconstant, giving a contradiction. Thus, the tangent bundle of \(Y_b\) is trivial. In particular, there are global 1-forms \(\eta_1, \ldots, \eta_n\) which are linearly independent at every point of \(\hat{X}\). Then the Albanese map of \(\alpha: Y_b \rightarrow \text{Alb}(Y_b)\) is étale and surjective; It is in fact an isomorphism as in the proof of Proposition 2.4. Thus, \(Y_b\) is a complex torus, a contradiction to Proposition 3.2. This completes the proof of Claim.

By this claim, at every point \(z \in \hat{X}\), \(\varphi_z\) is linearly dependent on \(\varphi_1, \ldots, \varphi_{n-1}\). Since \(\varphi_1, \ldots, \varphi_{n-1}\) are linearly independent at every point of \(\hat{X}\), for each \(y \in \hat{X}\), we can write

\[
\varphi_y = c_{1,y}\varphi_1 + \cdots + c_{n-1,y}\varphi_{n-1}
\]

for uniquely determined complex numbers \(c_{1,y}, \ldots, c_{n-1,y}\). Then \(c_{i,y}\) is a holomorphic function on \(\hat{X}\) and must be independent of \(y\). It follows that \(\varphi\) is linearly dependent on \(\varphi_1, \ldots, \varphi_{n-1}\) as 1-forms on \(\hat{X}\).

Now let us study the fibers of the Albanese map of \(\hat{X}\).

**Proposition 3.6.** Let \(C\) be a general fiber of the Albanese map for \(\hat{X}\). Then the \(C^{n-1}\)-action on \(\hat{X}\) generated by \(\hat{v}_1, \ldots, \hat{v}_{n-1}\) (cf. Proposition 2.2) induces a transitive action by translations on \(\text{Alb}(\hat{X})\) and \(\hat{X}\) is a holomorphic fiber bundle over \(\text{Alb}(\hat{X})\) with \(C\) as the fiber.

**Proof.** By the universality of the Albanese map, the action of \(C^{n-1}\) on \(\hat{X}\) descends to an action on \(\text{Alb}(\hat{X})\). The action of \(C^{n-1}\) on \(\hat{X}\) is the one corresponding to the global vector fields \(\hat{v}_1, \ldots, \hat{v}_{n-1}\), which are pointwise dual to the basis \(\varphi_1, \ldots, \varphi_{n-1}\) of \(H^0(\hat{X}, T^*(\hat{X}))\) (Proposition 2.3). Thus, by the construction of the Albanese map, the action of \(C^{n-1}\) is the natural translation on the \((n-1)\)-dimensional complex torus \(\text{Alb}(\hat{X})\). This implies the result.

The next proposition describes the structure of a smooth fiber \(Y_b\). It implies, in particular, that a smooth fiber is nothing but a hyperelliptic surface when \(\dim M = 4\) (cf. [M1], Table 4 Type I_0).

**Proposition 3.7.** Suppose \(Y_b\) is smooth. Then the Albanese map of \(Y_b(=\hat{X})\) is an elliptic fiber bundle. Moreover, there exists an action of an \((n-1)\)-dimensional torus on \(Y_b\) inducing a transitive action on \(\text{Alb}(Y_b)\) and providing \(Y_b\) with a structure of a Seifert fibration over \(\mathbb{P}_1\).
See e.g. [Ho] for the definition of a Seifert fibration.

Proof. If $Y_b$ is smooth, then it is irreducible and $\tilde{X} = X = Y_b$. We have $Y = H$, $k > 1$ and $\mathcal{O}(kH)$ is a trivial bundle on $Y_b$. Thus $C \cdot H = 0$ for the fiber $C$ of the Albanese map. On the other hand, the characteristic vector field $\lambda_H$ gives rise to a nonzero element of $H^0(C, T(C) \otimes \mathcal{O}(H))$. Thus $\deg(T(C) \otimes \mathcal{O}(H)) \geq 0$. Therefore $\deg T(C) \geq 0$ by $C \cdot H = 0$. It follows that $C$ is either a smooth elliptic curve or $\mathbb{P}_1$. Suppose $C = \mathbb{P}_1$. The hypersurface $Y$ on $M$ is smooth in a neighborhood of $C$ and the normal bundle of $C$ in $Y$ is trivial. Moreover, the normal bundle of $Y$ in $M$ is just $\mathcal{O}(H)$. This implies that the normal bundle of $C$ in $M$ is trivial by $C \cdot H = 0$. But then $K_M \cdot C = \deg K_C = -2$ by the adjunction formula. This contradicts the triviality of $K_M$, which follows from the existence of the symplectic structure on $M$. Thus $C$ is an elliptic curve. This means that $Y_b$ contains no rational curve, as the Albanese map is a fiber bundle by Proposition 3.6. Applying [Fu, Theorem 5.5], we see that $\text{Aut}_{\nu}(Y_b)$ is an $(n - 1)$-dimensional torus isogenous to $\text{Alb}(Y_b)$. From the general result about torus actions in [Ho], we have a Seifert fibration $\sigma: Y_b \to C'$ over a smooth curve $C'$ such that fibers of $\sigma$ are $\text{Aut}_{\nu}(Y_b)$-orbits. If $C'$ has a nontrivial 1-form $\psi$, then $\sigma^*\psi$ is a nontrivial 1-form on $Y_b$ which is nontrivial on $C$ as well. This is a contradiction to the fact that $C$ is a fiber of the Albanese map for $Y_b$. Thus $C' \cong \mathbb{P}_1$. \hfill $\square$

Proposition 3.8. Suppose $Y_b$ is singular. Then for each component $X$ of $Y_b$, its normalization $\tilde{X}$ is a $\mathbb{P}_1$-bundle over the complex torus $\text{Alb}(\tilde{X})$. In particular, each component of $Y_b$ is covered by rational curves and each rational curve on $Y_b$ is the image of the Albanese fiber of $\tilde{X}$ for some component $X$.

Proof. We will use the notation in the proof of Proposition 3.5. The fiber $C$ of the Albanese map of $\tilde{X}$ is a leaf of the two twisted vector fields

$$\tilde{\lambda}_H \in H^0(\tilde{X}, T(\tilde{X}) \otimes \nu^* \mathcal{O}(H)), \quad \tilde{\gamma} \in H^0(\tilde{X}, T(\tilde{X}) \otimes \nu^* \mathcal{O}(-(k - 1)H)).$$

Moreover, by Propositions 3.3, both $\tilde{\lambda}_H$ and $\tilde{\gamma}$ vanish on the divisor $E$ lying over $\text{Sing} Y_b$. The divisor $E$ certainly meets $C$ properly by Propositions 2.2 and 3.6. Therefore, both twisted vector fields

$$0 \neq \tilde{\lambda}_H|_C \in H^0(C, T(C) \otimes \nu^* \mathcal{O}(H)), \quad 0 \neq \tilde{\gamma}|_C \in H^0(C, T(C) \otimes \nu^* \mathcal{O}(-(k - 1)H))$$

have nonempty zeros. Hence

$$\deg T(C) + \nu^* H \cdot C > 0, \quad \deg T(C) - (k - 1)\nu^* H \cdot C > 0.$$ 

Therefore, $\deg T(C) > 0$. Hence $C = \mathbb{P}_1$. A rational curve on $Y_b$ cannot lie on $\text{Sing}(Y_b)$ by Proposition 2.2. Thus it must come from a rational curve on the
normalization, that is, it must be the image of the Albanese fiber of $\hat{X}$ for some component $X$.

Propositions 3.5, 3.7 and 3.8 complete the proof of Theorem 1.3 and consequently the proof of Theorem 1.2 via Proposition 2.4.

4. Structure of general singular fibers. In this section, we shall give a Kodaira-type classification of general singular fibers of a holomorphic Lagrangian fibration $f: M \to B$ satisfying the assumption in Theorem 1.3. Theorem 1.4 will follow from Propositions 3.7, 3.8, 4.10, 4.11 and 4.12.

In what follows, unless stated otherwise, we use the same notation as in Section 3. For instance, $2n = \dim M = \dim B$; $b$ is a general point of an irreducible component $D$ of the critical hypersurface $\Delta \subset B$; $Y$ is the underlying reduced hypersurface of $f^{-1}(D)$; $Y_b$ is the fiber $(f|_Y)^{-1}(b)$; $X$ is an irreducible component of $Y_b$; $\nu: \hat{X} \to X$ is the normalization of $X$; $\alpha: \hat{X} \to \text{Alb}(\hat{X})$ is the Albanese map (recall that $\hat{X}$ is smooth by Proposition 2.2 and that $\alpha$ is surjective with connected fibers by Proposition 3.5); $H$ is the irreducible component of $f^{-1}(D)$ containing $X$, and $k$ is the multiplicity of $H$ in $f^{-1}(D)$. Besides these notations, we denote by $C$ a fiber of the Albanese map $\alpha: \hat{X} \to \text{Alb}(\hat{X})$. Note that $\alpha$ is a holomorphic fiber bundle with typical fiber $C$ (Proposition 3.6) and that $C$ is a smooth elliptic curve when $Y_b$ is smooth and $C = \mathbb{P}_1$ when $Y_b$ is singular (Propositions 3.7 and 3.8).

Since Theorem 1.4 is local at $b \in D \subset B$ in the classical topology, we will freely shrink $B$ around $b$, whenever it is more convenient. For example, we may (and will) assume that $X = H \cap Y_b$ for an irreducible component $X$ of $Y_b$.

In Propositions 4.1–4.7 below, we study the structure of a germ of $Y_b$ at its singular point. In Proposition 4.1, we consider the case where the germ is not irreducible. We consider the case where the germ is irreducible in Lemma 4.2–Proposition 4.7.

To state Proposition 4.1, it is convenient to define the following notions. Let $x$ be a singular point of $Y_b$ and $\mathcal{X}_i, i = 0, \ldots, \ell$ be the irreducible components of the germ of $Y_b$ at $x$. By Proposition 3.8, there exists a unique germ $\mathcal{C}_i \subset X_i$ of a rational curve on $Y_b$ through $x$ for each $i$. Suppose that for two components $\mathcal{X}_i$ and $\mathcal{X}_j, i \neq j$, the scheme-theoretic intersection $\mathcal{X}_i \cap \mathcal{X}_j$, which is of dimension $n - 1$ by Proposition 2.2, defines a Cartier divisor $\mathcal{D}_i$ on $\mathcal{X}_i$ and a Cartier divisor $\mathcal{D}_j$ on $\mathcal{X}_j$. For example, this is the case if both $\mathcal{X}_i$ and $\mathcal{X}_j$ are smooth. We will say that $\mathcal{X}_i$ and $\mathcal{X}_j$ intersect transversely if the local intersection numbers at $x$ are

$$\mathcal{D}_i \cdot \mathcal{C}_i = \mathcal{D}_j \cdot \mathcal{C}_j = 1.$$

We will say that the two components intersect with multiplicity 2 if

$$\mathcal{D}_i \cdot \mathcal{C}_i = \mathcal{D}_j \cdot \mathcal{C}_j = 2.$$
Proposition 4.1. If the germ of $Y_b$ at a point is not irreducible, then one of the following holds.

(Case 1) The germ has three irreducible components $X_0, X_1, X_2$. Each component is smooth. Each pair of them intersect transversely along the common intersection $X_0 \cap X_1 = X_1 \cap X_2 = X_0 \cap X_2$.

(Case 2) The germ has two irreducible components. Each component is smooth and the two components intersect with multiplicity 2.

(Case 3) The germ has two irreducible components. Each component is smooth and the two components intersect transversely.

Proof. Let $x \in Y_b$ be such a point and let $h$ be the local defining equation of $D$ at $b$. Let

$$f^*h = h_0^{a_0} \cdots h_\ell^{a_\ell}, \quad 0 < a_0 \leq \cdots \leq a_\ell$$

be a unique factorization into irreducible factors modulo units in $O_M$. Put $H_i = \text{div } h_i$. These $H_i$ form local analytic irreducible components of $Y = f^{-1}(D)_{\text{red}}$ in a neighborhood $U$ of $x$ in $M$. We choose as $H$ the global irreducible component of $Y$ such that $H_0 \subset H$. We put $X_i = H_i \cap Y_b$. Let $X$ be a global irreducible component of $Y_b$ such that

$$x \in X_0 \subset X \subset H.$$

We also choose an Albanese fiber $C$ of $\hat{X}$ such that $x \in \nu(C)$. Since $f^{-1}(D)$ is a principal divisor on $Y_b$, it is trivial on $\hat{X}$ as a line bundle. Thus, so is it on $C$ and $C \cdot \nu^*(f^{-1}(D)) = 0$. Recall that $Y = f^{-1}(D)_{\text{red}}$. Then $C \cdot \nu^*Y \leq 0$, as $\nu(C) \subset X$ and $a_0 \leq \cdots \leq a_\ell$. Since $C = \mathbb{P}_1$, one can then regard the restriction $\hat{\lambda}_Y|_C$ of the lift of the characteristic vector field $\lambda_Y$ as a nonzero vector field on $C$ (and we shall do so). Again, since $C = \mathbb{P}_1$, the vector field has at most two zeros over $x$ counted with multiplicities. $C$ meets each divisor of $\hat{X}$ lying over $\text{Sing } X_0$ (if it is not empty) and also meets the divisors lying over $X_i \cap X_0$ ($1 \leq i \leq \ell$). There, $\hat{\lambda}_Y|_C$ is zero with multiplicities counted by the local expression $\iota_\omega(h_1 \cdots h_\ell dh_0)$. Hence $\ell \leq 2$ and the Case (1) occurs when $\ell = 2$.

Consider the case where $\ell = 1$. If $X_0$ is singular at $x$, then $\nu^*h_1$ has two zeros counted with multiplicities on $C$ and $\nu^*(dh_0)$ has additional zeros on $C$, a contradiction. Thus $X_0$ is smooth and $X_0$ intersects the other component with multiplicities at most 2 at $x$. If $a_0 = a_1$, then, by changing the role of $h_0$ and $h_1$, we see that the other local irreducible component is smooth as well.

It remains to consider the case where $\ell = 1$ and $a_0 < a_1$. We use the twisted vector field $\lambda_H$ instead of $\lambda_Y$ above. By $a_0 \neq a_1$, the global irreducible component $H'$ of $Y$ such that $H_i \subset H'$ is different from $H$. Therefore, the global irreducible decomposition of $f^{-1}(D)$ is of the form $a_0H + a_1H' + \cdots$. Since $C \cdot \nu^*H' > 0$ and...
$0 < a_0 < a_1$, one has then

$$0 = C \cdot f^{-1}(D) \geq a_0(C \cdot \nu^*H) + a_1(C \cdot \nu^*H') > a_0(C \cdot \nu^*H + C \cdot \nu^*H').$$

Thus, if $C \cdot \nu^*H' \geq 2$, then $C \cdot \nu^*H < -3$. However, this contradicts the fact that

$$0 \neq \hat{\lambda}_H|_C \in H^0(C, T(C) \otimes \nu^*\mathcal{O}(H)).$$

Thus $C \cdot \nu^*H' = 1$. In particular, the two local irreducible components are smooth and meet transversely at $x$. This completes the proof.

To study the irreducible germ of $Y_b$, we need some preliminary results regarding dualizing sheaves. Let $V'$ be a complex variety with the property that its normalization is smooth and the dualizing sheaf $\omega_{V'}$ is invertible. Denoting by $\nu': V \rightarrow V'$ the normalization map, we have a natural injective sheaf map $\omega_V \rightarrow \nu^*\omega_{V'}$. Since this is an inclusion of invertible sheaves on the smooth variety $V$, we can naturally identify $\omega_V = \nu^*\omega_{V'} \otimes \mathcal{O}_V(-E)$ for some effective divisor $E$ on $V$. In Lemma 4.2 and Proposition 4.5 below, we will study this effective divisor $E$ in some cases.

**Lemma 4.2.** Let $R'$ be an irreducible germ of an analytic curve with an isolated singularity $Q$ and let $\nu: R \rightarrow R'$ be the normalization morphism. Denote by $\omega_R$ and $\omega_{R'}$ the dualizing sheaves of the curves. Assume that $\nu^{-1}(Q) = P \in R$ (as a set) and that $\omega_{R'}$ is locally free. Then

$$\omega_R = \nu^*\omega_{R'} \otimes \mathcal{O}_R(-2\delta P)$$

where $\delta$ is the codimension of $\mathcal{O}_{R', Q}$ in $\mathcal{O}_{R, P}$.

**Proof.** Since $\omega_{R'}$ is locally free and $\nu^{-1}(Q) = P$, it follows from [Se, Page 72] that

$$\nu^*\omega_{R'} = \mathcal{O}_R \cdot \frac{\eta}{\eta^2}, \quad \omega_R = \mathcal{O}_R \cdot \eta.$$

Here $\eta$ is a local generator of $\omega_R$ and $t$ is a generator of the maximal ideal $\mathfrak{m}_{R, P}$. Thus,

$$\nu^*\omega_{R'} = \omega_R(2\delta P) = \omega_R \otimes \mathcal{O}_R(2\delta P).$$

Since $\omega_R$ is invertible, this gives the desired equality.

**Lemma 4.3.** Let $V'$ be an irreducible locally complete intersection variety in a smooth manifold $V$ and let $\rho: V' \rightarrow Z$ be a surjective morphism to an irreducible
variety \( Z \). Then for a general fiber \( R' \) of \( \rho \), the dualizing sheaf \( \omega_{R'} \) is the restriction of the dualizing sheaf \( \omega_{Y'} \) to \( R' \).

This is an immediate consequence of the adjunction formula.

**Proposition 4.4.**

1. Each irreducible component of \( Y_b \), as well as \( Y_b \) itself, is a locally complete intersection variety. Consequently, it has invertible dualizing sheaf.
2. Let \( x \) be a point of \( Y_b \). Then there exists a germ of an analytic curve \( R' \) in \( \mathbb{C}^2 \) and a biholomorphic map from the germ of \( Y_b \) at \( x \) to the product \( R' \times M \) where \( M \) denotes the germ of an \((n - 1)\)-dimensional complex manifold. Moreover, this biholomorphic map can be chosen in such a way that the orbits of \( \mathbb{C}^{n-1} \) in \( Y_b \) are sent to the slices \( \{ r \} \times M (r \in \mathbb{R}) \).
3. Let \( x \) be a singular point of \( Y_b \). Then there exist an open neighborhood \( U \) of \( x \) in \( M \) and a (not necessarily integrable) distribution \( D \) of rank 2 on \( U \) such that it is normal to \( \text{Sing}(Y_b) \) at \( x \) and such that the germ of any rational curve \( C' \) of \( Y_b \) through \( x \) is tangent to \( D \) at smooth points of \( C' \).

**Proof.** Let \( z_1, \ldots, z_{n-1} \) be the same as in the proof of Proposition 2.2. From the nonvanishing of \( f^*dz_1, \ldots, f^*dz_{n-1} \) in a neighborhood of \( Y_b \), we see that \( Z_b := f^{-1}(z_1 = b_1, \ldots, z_{n-1} = b_{n-1}) \) is smooth in a neighborhood of \( Y_b \) for general \( b = (b_1, \ldots, b_{n-1}) \in D \) and \( Y_b \) is just a reduced hypersurface in \( Z_b \). This implies (1).

The commuting vector fields \( \iota_1, \ldots, \iota_{n-1} \) of Proposition 2.2 are defined in a neighborhood of \( Y_b \). In a neighborhood \( U \) of \( x \) in \( Z_b \), we can define coordinates \( w_1, \ldots, w_{n+1} \) such that \( \iota_i = \frac{\partial}{\partial w_i} \) for \( 1 \leq i \leq n - 1 \). Since the hypersurface \( Y_b \) in \( Z_b \) is invariant under \( \iota_1, \ldots, \iota_{n-1} \), we can choose a local defining equation of \( Y_b \) on \( U \) as a holomorphic function depending only on the two variables \( w_n \) and \( w_{n+1} \). This implies (2).

To see (3), it suffices to construct such a distribution on the germ of \( Z_b \) at \( x \). We use the construction in the proof of Proposition 2.3. Let \( \frac{\partial}{\partial z_i}, 1 \leq i \leq n - 1 \), be the vector fields on \( B \) transversal to \( D \) near \( b \) and let \( u_i \) be a vector field defined in a neighborhood of \( x \) in \( M \) satisfying \( f_*(u_i) = \frac{\partial}{\partial z_i} \) for each \( 1 \leq i \leq n - 1 \). Let \( \phi_i \) be the 1-form in this neighborhood defined by \( u_i = i_{\omega}(\phi_i) \). Note that our \( \varphi_i \) in Proposition 2.3 is the pull-back of \( \phi_i \) to the normalization of \( Y_b \). Unlike \( \varphi_j \), \( \phi_i \) is defined only in a neighborhood of \( x \) because its definition depends on the choice of \( u_i \). These \( n - 1 \) differential forms restricted to the \((n + 1)\)-dimensional manifold \( Z_b \) define a distribution \( D \) of rank 2 on \( Z_b \). From the obvious relation \( \phi_i(t_j) = \delta_{ij}, \) this distribution is normal to the \( \mathbb{C}^{n-1} \)-orbits and consequently normal to \( \text{Sing}(Y_b) \). Since the rational curve \( C' \) is the fiber of the Albanese map defined by \( \varphi_1, \ldots, \varphi_{n-1}, C' \) is tangent to \( D \) at its smooth point. \( \square \)

**Proposition 4.5.** Let \( X \) be a component of \( Y_b \). Assume that there exists a codimension-1 irreducible component \( S' \) of the singular locus of \( X \) such that the
germ of $X$ at a general point of $S'$ is irreducible so that the support of $\nu^{-1}(S')$ is an irreducible reduced divisor $S$ on $\hat{X}$ and $\nu|_S$ is bimeromorphic. Then

$$\omega_{\hat{X}} = \nu^* \omega_X \otimes \mathcal{O}(-2\delta S) \otimes \mathcal{O}(-E')$$

for some positive integer $\delta$ and an effective divisor $E'$ on $\hat{X}$.

**Proof.** Since the statement is local in a neighborhood of a general point of $S'$, it suffices to prove

$$\omega_{\hat{X}} = \nu^* \omega_X \otimes \mathcal{O}(-2\delta S)$$

for the germ of $X$ at a general point of $S'$. By the description of the germ in Proposition 4.4, this follows immediately from Lemma 4.2 and Lemma 4.3. 

**Proposition 4.6.** Let $X$ be a component of $Y_b$ and let $C$ be a fiber of the Albanese map of $\hat{X}$. Then:

1. $C \cdot (\nu \circ f)^{-1}(D) = 0$ and $C \cdot \nu^* \omega_X = C \cdot \nu^* H \leq 0$.
2. Assume that $Y_b$ is not irreducible. Then $C \cdot \nu^* \omega_X = C \cdot \nu^* H \leq -1$.

**Proof.** By $\omega_M \simeq \mathcal{O}_M$, we have $\mathcal{O}(H)|_H \simeq \omega_H$ by the adjunction formula. Since we have assumed, by shrinking $B$, that $X$ is a fiber of $f|_H$: $H \rightarrow f(H)$, we have $\omega_H|_X = \omega_X$ as in Lemma 4.3. Thus, $\mathcal{O}(H)|_X \simeq \omega_X$. This implies the second equality in (1). We have $C \cdot \nu^*(f^{-1}(D)) = 0$ as $f^{-1}(D)$ is a principal divisor on $Y_b$. We have also that $C \cdot \nu^* H' \geq 0$ for each irreducible component $H'$ of $Y = f^{-1}(D)_{\text{red}}$ such that $H' \neq H$. Hence $C \cdot \nu^* H \leq 0$. This implies (1).

If $Y_b$ is not irreducible, then there is at least one irreducible component $H'$ of $Y = f^{-1}(D)_{\text{red}}$ such that $H' \neq H$. Moreover, $C \cdot \nu^* H' > 0$ at least one of such $H'$, as $Y_b$ is connected. Therefore $C \cdot \nu^* H \leq -1$.

**Proposition 4.7.** Suppose a component $X$ of $Y_b$ is locally irreducible but singular at a point $x$. Then the following hold:

1. $Y_b$ is irreducible and the singular locus of $X$ is irreducible.
2. The germ of $X$ at $x$ is biholomorphic to the product of the singularity of the rational cuspidal cubic curve in the plane and a complex manifold.
3. Each rational curve on $Y_b$ is isomorphic to the rational cuspidal cubic curve in the plane.

**Proof.** Let $C$ be a fiber of the Albanese map of $\hat{X}$ such that $x \in \nu(C)$. In the notation of Proposition 4.5

$$-2 = \omega_{\hat{X}} \cdot C = \nu^* \omega_X \cdot C - 2\delta S \cdot C - E' \cdot C,$$

which implies that $\nu^* \omega_X \cdot C = \nu^* \mathcal{O}(H) \cdot C \geq 0$. By Proposition 4.6 (1), we
conclude that
\[ \nu^* \mathcal{O}(H) \cdot C = 0, \quad \delta = 1, \quad S \cdot C = 1, \quad E' \cdot C = 0. \]

This implies that \( Y_b \) is irreducible by Proposition 4.6 (2). Also, it implies that \( \nu(S) \) is the only component of \( \text{Sing}(X) \) where \( X \) is locally irreducible, because otherwise \( E' \cdot C > 0 \). In particular, if there exists another component of \( \text{Sing}(X) \), there are two distinct points \( x_1, x_2 \) on \( C \) disjoint from \( S \) such that \( \nu(x_1), \nu(x_2) \in \text{Sing}(X) \). From \( \nu^* \mathcal{O}(H) \cdot C \geq 0 \), \( \nu^* \mathcal{O}(H)|_{C} \) is a trivial line bundle on \( C = \mathbb{P}_1 \). The twisted vector field \( \tilde{\lambda}_H|_C \neq 0 \) is then a vector field on \( C \) vanishing at the points \( x_1, x_2 \) and \( \nu^{-1}(x) \), a contradiction. Thus \( \text{Sing}(X) \) is irreducible. This completes the proof of (1).

By Proposition 4.4 (2), we already know that the germ of \( X \) at \( x \) is of the form \( R' \times M \). Let us use the notation of Lemma 4.2. From [Se, p. 59, equation (1)], we have the inclusion
\[ C + c_x \subset \mathcal{O}_{R',x} \subset \mathcal{O}_{R,\nu^{-1}(x)}, \]
where \( c_x \) is the conductor of \( R' \) at \( x \). Since \( \delta = 1 \), \( c_x \) is the square of the maximal ideal of \( R \) at \( \nu^{-1}(x) \) by [Se, p.71, Section 11]. Then \( C + c_x = \mathcal{O}_{R',x} \) and \( R' \) is the germ of the rational cuspidal cubic plane curve at the cusp. This completes the proof of (2).

For (3), it suffices to show that the germ of \( C' = \nu(C) \) at \( x \) is isomorphic to that of \( R' \). Since \( C \) is transversal to \( S \), we see that under the projection \( R' \times M \longrightarrow R' \), the germ of \( C' \) is projected to \( R' \) bijectively. From the property of the cusp of \( R' \), either the germ of \( C' \) is isomorphic to \( R' \) or \( C' \) is nonsingular. In the latter case, \( C' \) corresponds to a smooth curve on \( R' \times M \), and must be tangent to \( \text{Sing}(Y_b) \) at \( x \). However, from Proposition 4.4 (3), if \( C' \) is smooth it must be tangent to the distribution \( \mathcal{D} \) normal to \( \text{Sing}(Y_b) \), a contradiction. This finishes the proof of (3).

**Proposition 4.8.** Suppose \( Y_b \) is not irreducible. Then each rational curve \( C' \) on \( Y_b \) is smooth. Moreover, for each point \( x' \in C' \), the irreducible component of the germ of \( Y_b \) at \( x' \) containing the germ of \( C' \) at \( x' \) is smooth.

**Proof.** A rational curve \( C' \) on \( Y_b \) must be the image of the Albanese fiber \( C \) of some component \( \hat{X} \) by Proposition 3.8. Thus, the statement follows if \( \nu_C : C \longrightarrow C' \) is an isomorphism. Note that \( \nu_C \) is of degree 1, as \( C \) is not contained in \( \nu^{-1}(\text{Sing}(X)) \) by Proposition 3.6.

Suppose that \( \nu_C \) is not an isomorphism for some \( C \). Since \( \nu_C \) is of degree 1, the image \( C' = \nu(C) \) must be singular at some point, say at \( x' \in C' \). If \( X \) is locally irreducible at \( x' \), we get contradiction from Proposition 4.7. Thus \( X \) is not locally irreducible at \( x' \) and each component of the germ at \( x' \) is smooth by Proposition 4.1.
Hence the germ of $C'$ at $x$ has at least two irreducible components. Therefore, $\nu_C^{-1}(x')$ contains at least two different points, say $x_1$ and $x_2$.

The twisted vector field $0 \neq \hat{\nu}_H \in H^0(C, T(C) \otimes \nu^*\mathcal{O}(H))$ vanishes at $x_1$ and $x_2$ by Proposition 3.3. On the other hand, since $Y_b$ is not irreducible, we have $\deg T(C) \otimes \nu^*\mathcal{O}(H) \leq 1$ by Proposition 4.6, a contradiction.

For the statement of the next proposition, it is convenient to introduce the notion of the irreducible germ of $Y_b$ along a rational curve. Suppose $Y_b$ is not irreducible. Let $C'$ be a rational curve on $Y_b$, which is necessarily smooth by Proposition 4.8. Since $C'$ is smooth, at each point $x' \in C'$, there is a unique irreducible component, say $\mathcal{V}_{x'}$, of the germ of $Y_b$ at $x'$ containing the germ of $C'$ at $x'$. By Proposition 4.8, $\mathcal{V}_{x'}$ is smooth. Other irreducible components of the germ of $Y_b$ at $x'$ intersect properly with $C'$. Since $C'$ is compact and irreducible, we can choose $\{\mathcal{V}_{x'}, x' \in C'\}$ so that $\cup_{x' \in C'} \mathcal{V}_{x'}$ forms a germ of a complex submanifold in $M$ containing $C'$. We call $\cup_{x' \in C'} \mathcal{V}_{x'}$ the irreducible germ of $Y_b$ along $C'$.

**Proposition 4.9.** Suppose $Y_b$ is not irreducible. Let $C'$ be a (necessarily smooth) rational curve on $Y_b$. Let $\mathcal{V}_0$ be the irreducible germ of $Y_b$ along $C'$. Let $\mathcal{X}_1, \ldots, \mathcal{X}_\ell$ be the other components of the germs of $Y_b$ at all the points of $C'$ where the germ of $Y_b$ is reducible, and let $\mathcal{H}_i$ be the unique irreducible component of the germ $Y = f^{-1}(D)_{\text{red}}$ at $C' \cap \mathcal{X}_i$ such that $\mathcal{X}_i \subset \mathcal{H}_i$. Denote by $a_i$ the multiplicity of $\mathcal{H}_i$ in $f^{-1}(D)$. Then,

$$2a_0 = a_1(C' \cdot \mathcal{H}_1) + \cdots + a_\ell(C' \cdot \mathcal{H}_\ell).$$

Note that $a_i$ coincides with the multiplicity of the global irreducible component $H$ of $f^{-1}(D)$ such that $\mathcal{H}_i \subset H$. The intersection number $(C' \cdot \mathcal{H}_i)$ is the one counted with multiplicities at the intersection point. The intersection number $(C' \cdot \mathcal{H}_0)$ also makes a sense, as $\mathcal{O}(\mathcal{H}_0)$ can be naturally regarded as a line bundle on the complete curve $C'$.

**Proof.** Let $h$ be the defining equation of the critical divisor $D$ at $b$. Let $x' \in C'$ and let $h_i$ be the local equation of $\mathcal{H}_i$ at $x'$. Then $f^*h = h_0^{a_0} \cdot h_1^{a_1} \cdots h_\ell^{a_\ell}$ in $\mathcal{O}_{M, x'}$. Note that $\mathcal{H}_0$ is a globally defined divisor around $C'$. Thus, we can consider the $\mathcal{O}(-(a_0 - 1)\mathcal{H}_0)$-valued vector field $\gamma$ in a neighborhood of $C'$ in $M$, which is locally defined by

$$\gamma = \frac{1}{h_0^{a_0-1}} t_\omega(df^*h) = a_0 h_1^{a_1} \cdots h_\ell^{a_\ell} t_\omega dh_0 + \gamma_1,$$

where $\gamma_1$ is a vector field vanishing on $\mathcal{H}_0$. Note that the adjunction formula
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gives

\[ H_0 \cdot C' = K_{H_0} \cdot C' = \deg K_{C'} = -2 \]

because \( K_M \) and the normal bundle of \( C' \) in \( H_0 \) are both trivial. By the same argument as in the proof of Proposition 2.1, \( \gamma \) is tangent to \( C' \) and it gives rise to a nonzero global section \( \gamma|_{C'} \) of the line bundle \( T(C') \otimes O(-(a_0 - 1)H_0) \) on \( C' \). Since \( C' = P_1 \), this line bundle is of degree

\[-(a_0 - 1)(C' \cdot H_0) + 2 = 2(a_0 - 1) + 2 = 2a_0.\]

On the other hand, from the local expression of \( \gamma \) above and the fact that \( H_0 \) is smooth along \( C' \), we see that \( \gamma|_{C'} \) has exactly

\[ a_1(C' \cdot H_1) + \cdots + a_\ell(C' \cdot H_\ell) \]

zeros counted with multiplicities. It follows that

\[ \sum_{i=1}^{\ell} a_i(C' \cdot H_i) = 2a_0. \]

Now we are ready to prove Theorem 1.4 when \( Y_b \) is singular. By construction, the \( C^{n-1} \)-action in Proposition 2.2 descends to the action on the set of characteristic 1-cycles on \( Y_b \). In particular, they are all isomorphic. There are the following three possibilities:

(Case i) Each singular germ of \( Y_b \) falls into the (Case 3) in Proposition 4.1 and all rational curves on \( Y_b \) are smooth.

(Case ii) Some singular germ of \( Y_b \) falls into (Case 1) or (Case 2) in Proposition 4.1 and all rational curves on \( Y_b \) are smooth.

(Case iii) \( Y_b \) is irreducible and all rational curves on \( Y_b \) are singular.

In fact, if \( Y_b \) is not irreducible, its germ at any singular point is reducible, thus falls into (Case i) or (Case ii) by Proposition 4.8. If \( Y_b \) is irreducible and a rational curve on \( Y_b \) is smooth, it belongs to (Case i) or (Case ii) by Proposition 4.7.

**Proposition 4.10.** Suppose \( Y_b \) is in (Case i). Then the characteristic 1-cycle

\[ \sum_s \frac{1}{r_s} \Theta_s \]

divided by \( r = \gcd\{r_s\} \) is of the form of one of singular fibers of a relatively minimal elliptic fibration listed by Kodaira [Kd, Theorem 6.2], 1-cycle of Type A\(_{\infty}\) or 1-cycle of Type D\(_{\infty}\), defined in Theorem 1.4.

Here (and also in Propositions 4.11 and 4.12 below), we say that a 1-cycle in a variety is of the form of a 1-cycle in another variety, if the reductions of the two 1-cycles are isomorphic as reduced varieties and under that isomorphism the corresponding multiplicities coincide.

**Proof.** Let \( X_1, \ldots, X_N \) be the (global) irreducible components of \( Y_b \) and let \( H_i \) be the (global) irreducible component of \( Y = f^{-1}(D)_{\text{red}} \) such that \( X_i \subset H_i \). We denote by \( C'_i \) any smooth rational curve in \( X_i \). Each \( \Theta_s \) is \( P_1 \) by our assumption,
and it belongs to one of $X_1, \ldots, X_N$. Let us denote by $\{\Theta_j^i, 1 \leq j \leq \tau(i)\}$, the collection of the components contained in $X_i$ the cardinality $\tau(i)$ of which may be infinite. Let us define the intersection number $\Theta_j^i \cdot \Theta_p^m$ as the cardinality of the set-theoretic intersection $\Theta_j^i \cap \Theta_p^m$, except when $i = m$ and $j = p$ in which case we decree it to be $-2$. Set $a_{im} := C_i^j \cdot H_m$. This number does not depend on the choice of $C_i^j$ in $X_i$ by Proposition 3.8. Note that for each $1 \leq i \neq m \leq N$ and for each $j$, there are only finitely many $p$ such that $\Theta_j^i \cap \Theta_p^m \neq \emptyset$. Indeed, in (Case i), such curves $\Theta_p^m$ bijectively correspond to the set $\Theta_j^i \cap \text{Sing}(Y_b)$, which consists of at most finitely many points. Thus, for each $1 \leq i \neq m \leq N$ and $1 \leq i \leq \tau(i)$, we have a well-defined equality

$$a_{im} = \Theta_j^i \cdot H_m = \sum_p \Theta_j^i \cdot \Theta_p^m.$$ 

This equation is also true when $i = m$, as

$$a_{ii} = \Theta_j^i \cdot H_i = \Theta_j^i \cdot \mathcal{H}_{i,0} + \Theta_j^i \cdot \mathcal{H}_{i,1} + \cdots + \Theta_j^i \cdot \mathcal{H}_{i,q} \text{ and } \Theta_j^i \cdot \mathcal{H}_{i,0} = -2 = \Theta_j^i \cdot \Theta_j^i.$$ 

Here $\mathcal{H}_{i,0}$ is a germ of $H_i$ along $\Theta_j^i$ and $\mathcal{H}_{ij}^{j'}$ ($j' \geq 1$) are other germs of $H_i$ which meets $\Theta_j^i$ at some points.

Then, by $\Theta_j^i \cdot f^{-1}(D) = 0$, we have

$$\sum_{1 \leq m \leq N, 1 \leq p \leq \tau(m)} r_m \Theta_j^i \cdot \Theta_p^m = \sum_{1 \leq m \leq N} r_m a_{im} = 0.$$ 

This means the intersection numbers $\Theta_t \cdot \Theta_s$ and the coefficient $r_s$ of the cycle $\sum_s r_s \Theta_s$ satisfy

$$2r_s = \sum_{t \neq s} r_t \Theta_t \cdot \Theta_s.$$ 

This formula, which we will call the key formula later, is exactly the formula in [Kd, p. 567, equation (6.5)]. The only difference is that the cardinality of indices here is possibly infinite. Now all the arguments in [Kd, pp. 567–571] for the classification of $\sum_s r_s \Theta_s$, starting from a component with minimal multiplicity, say $r_0$, goes through modulo the following three points:

(a) the great common divisor $r$ of $r_s$ may not be 1 (even if it is not of Type $I_n$);

(b) the process of [Kd, pp 568, lines 23–25, the case $(\beta)$] does not terminate if $\sum_s r_s \Theta_s$ is an infinite 1-cycle, so that we can not conclude that $\sum_s r_s \Theta_s$ contains a (finite) cyclic chain even if $\Theta_0$ meets at least two components;

(c) the process of [Kd, pp 569, the case $(\beta_2)$] does not terminate if $\sum_s r_s \Theta_s$ is an infinite 1-cycle.

For (a), we may just divide the cycle by $r$. From the key formula above, the cycle is of Type $A_\infty$ when the process (b) does not terminate. Again, from
the key formula above, the cycle is of Type $D_{\infty}$ when the process (c) does not terminate. This completes the proof.

**Proposition 4.11.** When $Y_b$ is in (Case ii), one of the following holds.

1. If a local germ of $Y_b$ contains two components, there are at most two components in $Y_b$ and the singular locus of $Y_b$ is irreducible. The multiplicities of $f^{-1}(D)$ along all components of $Y_b$ are equal. Moreover through each point of the singular locus there are exactly two smooth rational curves, meeting each other with contact order 2. In this case, the characteristic 1-cycles are of the form of the Kodaira fiber of Type III in [Kd] up to a common integer factor. (Both an example of an irreducible $Y_b$ and an example of $Y_b$ with two components exist; see e.g. [M1].)

2. If a local germ of $Y_b$ contains three components, there are at most three components in $Y_b$ and the singular locus of $Y_b$ is irreducible. The multiplicities of $f^{-1}(D)$ along all components of $Y_b$ are equal. Moreover through each point of the singular locus there are exactly three smooth rational curves, meeting each other transversely. In this case, the characteristic 1-cycles are of the form of the Kodaira fiber of Type IV in [Kd] up to a common integer factor. (There are examples of this type where the number of components of $Y_b$ can be one, two or three; see e.g. [M1].)

**Proof.** Let $\sum s \gamma_r \Theta_s$ be a characteristic 1-cycle. By our assumption, $\Theta_s = P_1$. Define the intersection number $\Theta_j^i \cdot \Theta_m^l$ as the cardinality of the set-theoretic intersection, weighted by the order of contact. The order of contact can be 1 or 2 by Proposition 4.1. We also put $\Theta_j \cdot \Theta_j = -2$ as in the proof of Proposition 4.10. Then the same argument as in the proof of Proposition 4.10, following [Kd, proof of Theorem 6.2], goes through. By the assumption in (Case ii) and by the shape of singular fibers in Kodaira’s list [Kd], our characteristic 1-cycle is a finite 1-cycle and, as 1-cycles, it would be of Type III if $Y_b$ had a singular germ in (Case 2) of Proposition 4.1 and of Type IV if $Y_b$ had a singular germ in (Case 1) of Proposition 4.1.

Since Kodaira fibers of type III and IV are not semi-normal (see [GPR, Chapter I, Section 15] for a definition), it is not immediate that our cycle $\sum s \gamma_r \Theta_s$ is actually biholomorphic to the Kodaira fiber. In other words, the equality of the coefficients and the intersection numbers only guarantees that the semi-normalization of the characteristic cycle is isomorphic to the semi-normalization of the corresponding Kodaira fiber. The biholomorphicity can be seen as follows.

In the case of Type III, since the germ of two smooth analytic curves having contact of order 2 is always biholomorphic to the germ of two such curves on a smooth surface, it is immediate that the characteristic cycle is biholomorphic to the Kodaira fiber.

In the case of Type IV, it is easy to see that the germ of three smooth curves intersecting at one point, with transversal pairwise intersection, is isomorphic to the germ of the Kodaira fiber if and only if the three tangent vectors at the intersecting point are linearly dependent. But this is true for the characteristic
cycle by Proposition 4.4 (3). Thus the characteristic cycle is biholomorphic to the Kodaira fiber.

All the other statements in Proposition 4.11 follow immediately from the properties of the corresponding Kodaira fibers.

**Proposition 4.12.** When $Y_b$ is in (Case iii), one of the following holds.

1. Each rational curve on $Y_b$ is isomorphic to the cuspidal cubic curve in the plane. The singular locus of $Y_b$ is irreducible and consists of the cusps of the rational curves. In this case, the characteristic 1-cycles are irreducible and of the form of the fiber of Type II in $[Kd]$ up to multiplicity.

2. Each rational curve on $Y_b$ is an immersed $\mathbb{P}_1$ with one node. In this case, $Y_b$ is semi-normal and the characteristic 1-cycles are irreducible and of the form of the fiber of Type I in $[Kd]$, up to multiplicity.

**Proof.** If an irreducible component of the germ of $Y_b$ is singular, then we are in the situation of Proposition 4.7 and (1) follows. If each component of the germ of $Y_b$ is smooth, then a rational curve on $Y_b$ is an immersed $\mathbb{P}_1$ with self-intersection. Since $Y_b$ is irreducible, $\lambda_H$ is a vector field along $C$ vanishing at the points over the self-intersection points as in the proof of Proposition 4.7. The germ of $Y_b$ at the singular point can have transversal intersection or contact of order 2. If it has contact of order 2, we see that the vector field $\hat{\lambda}_H|_C$ has double zeroes at the two points of $C$ over the self-intersection, a contradiction. Thus there is only one nodal point and (2) follows.

As Matsushita pointed out to us, the case when the characteristic 1-cycle is an infinite cycle of Type $A_{\infty}$ actually occurs. The following proposition and its proof are due to Matsushita:

**Proposition 4.13.** There is a 4-dimensional Lagrangian fibration $f : M \longrightarrow \Delta^2$ such that $f$ is projective and such that the characteristic 1-cycles of a general singular fiber of $f$ are of Type $A_{\infty}$.

**Proof.** Let $R_k = \text{SpecAn} \mathbb{C}[u^{k+1}v^{-1}, u^{-k}v] \ (k \in \mathbb{Z})$. There is a natural morphism $g_k : R_k \longrightarrow \text{SpecAn} \mathbb{C}[u]$. Let $E$ be an elliptic curve. Using the morphisms $g_k$, which are compatible with the natural gluing of the spaces $R_k$, we obtain a morphism

$$(\bigcup_{k \in \mathbb{Z}} R_k) \times E \times \text{SpecAn} \mathbb{C}[y] \longrightarrow \text{SpecAn} \mathbb{C}[u, y].$$

Restricting this morphism over a sufficiently small 2-dimensional disk $\Delta^2$ (centered at $(u, y) = (0, 0)$), we obtain a fibration

$$\tilde{f} : \tilde{M} \longrightarrow \Delta^2_{(u, y)}.$$
The fiber over \( u = 0 \) is an infinite chain of \( \mathbb{P}_1 \times E \), while the fiber over \( u \neq 0 \) is \( \mathbb{C}^* \times E \). Let \( \alpha \) be a non-torsion point of \( E \). Then \( \mathbb{Z} \) acts on \( \tilde{f} : \tilde{M} \to \Delta^2 \) if we define the action of \( m \in \mathbb{Z} \) by

\[
(u^{k+1}v^{-1}, u^{-k}v, x, y) \mapsto (u^{k+1+m}v^{-1}, u^{-k-m}v, x + m\alpha, y).
\]

By [Na, Theorem 2.6], this action is properly discontinuous (and is free). Moreover, by [Na, Section 5], the induced morphism

\[
f : M = \tilde{M}/\mathbb{Z} \to \Delta^2
\]

is projective. As the symplectic 2-form \( du \wedge dv/v + dx \wedge dy \) on \( \tilde{M} \) is \( \mathbb{Z} \)-invariant, it descends to a symplectic 2-form on \( M \). With respect to this form, \( f \) is a Lagrangian fibration. By construction, the normalization of each singular fiber is \( \mathbb{P}_1 \times E \). Characteristic curves are exactly the images of the fibers \( \mathbb{P}_1 \) of \( \mathbb{P}_1 \times E \). Since \( \alpha \) is of infinite order, the characteristic 1-cycles of \( f \) are then of Type \( A_{\infty} \). \( \square \)

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