PICARD AND CHAZY SOLUTIONS TO THE PAINLEVE’ VI EQUATION

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Abstract. I study the solutions of a particular family of Painlevé VI equations with the parameters $\beta = \gamma = 0$, $\delta = \frac{1}{2}$ and $2\alpha = (2\mu - 1)^2$, for $2\mu \in \mathbb{Z}$. I show that the case of half-integer $\mu$ is integrable and that the solutions are of two types: the so-called Picard solutions and the so-called Chazy solutions. I give explicit formulae for them and completely determine their asymptotic behaviour near the singular points $0$, $1$, $\infty$ and their nonlinear monodromy. I study the structure of analytic continuation of the solutions to the PVI$\mu$ equation for any $\mu$ such that $2\mu \in \mathbb{Z}$. As an application, I classify all the algebraic solutions. For $\mu$ half-integer, I show that they are in one to one correspondence with regular polygons or star-polygons in the plane. For $\mu$ integer, I show that all algebraic solutions belong to a one-parameter family of rational solutions.

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1. Introduction.

In this paper I study the following particular case of Painlevé VI equation (see [Pain], [Gamb]):

$$y_{xx} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-x} \right) y_x^2 - \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{y-x} \right) y_x + \frac{1}{2} \frac{y(y-1)(y-x)}{x^2(x-1)^2} \left( (2\mu - 1)^2 + \frac{x(x-1)}{(y-x)^2} \right),$$

in the complex variable $x$, for resonant values of the parameter $\mu$, i.e. $2\mu \in \mathbb{Z}$. In the first part (see Sections 2, 3, 4) I show that, for any half-integer $\mu$, the PVI$\mu$ equation is integrable and compute the solutions in terms of known special functions. In particular, I completely describe the asymptotic behaviour around the critical points $0$, $1$, $\infty$ for every branch of all solutions, and their nonlinear monodromy. I show that for any half-integer $\mu$, PVI$\mu$ admits a countable set of algebraic solutions. In second part (see Sections 5, 6, 7, 8) I describe the structure of analytic continuation of the solutions of the PVI$\mu$ equation for any $2\mu \in \mathbb{Z}$ and show that the algebraic solutions of PVI$\mu$ with half-integer $\mu$ are in one to one correspondence with regular polygons or star-polygons in the plane. For $\mu$ integer, I show that there are no algebraic solutions except a one parameter family of rational solutions.

The fact that the PVI$\mu$ equation with $\mu = \frac{1}{2}$ is integrable and admits an infinite set of algebraic solutions, was already known to Picard, see [Pic]. The Picard solutions, of PVI$\mu$ with $\mu = \frac{1}{2}$ are described in Section 2. They have the form

$$y(x; \nu_1, \nu_2) = \wp(\nu_1 \omega_1 + \nu_2 \omega_2; \omega_1, \omega_2) + \frac{x + 1}{3}.$$
where \( \omega_{1,2}(x) \) are two linearly independent solutions of the following Hypergeometric equation
\[
x(1-x)\omega''(x) + (1-2x)\omega'(x) - \frac{1}{4}\omega(x) = 0,
\]
\(\nu_1, \nu_2\) are complex numbers such that \(0 \leq \text{Re}\nu_i < 2\) and \(\wp(u;\omega_1, \omega_2)\) is the Weierstrass elliptic function with the half-periods \(\omega_1, \omega_2\). I show that all the other PVI\(\mu\) equations with half-integer \(\mu \neq \frac{1}{2}\) have “more” solutions. Let me briefly explain what I mean. Let the solutions of Picard type be the solutions of PVI\(\mu\) with \(\mu + \frac{1}{2} \in \mathbb{Z}\backslash\{1\}\) which are images via birational canonical transformations of Picard solutions. I show that, while the Picard solutions exhaust all the possible solutions of PVI\(\mu\) with \(\mu = \frac{1}{2}\), the solutions of Picard type do not cover all the possible solutions of PVI\(\mu\) for all the other half integer values of \(\mu\), i.e. for \(\mu + \frac{1}{2} \in \mathbb{Z}\backslash\{1\}\). Indeed, there exists a one-parameter family of transcendental solutions of PVI\(\mu\) with \(\mu + \frac{1}{2} \in \mathbb{Z}\backslash\{1\}\), the so-called Chazy solutions, which are not of Picard type. I describe the Chazy solutions of PVI\(\mu\) with \(\mu = -\frac{1}{2}\) in Section 3. They are a one parameter family \(y(x; \nu)\) of the form
\[
y = \frac{1}{8} \left\{ [(\nu\omega_2 + \omega_1 + 2x(\nu\omega_2' + \omega_1')]^2 - 4x(\nu\omega_2' + \omega_1')]^2 \right\} \bigg/ (\nu\omega_2 + \omega_1)(\nu\omega_2' + \omega_1'][2(x-1)(\nu\omega_2' + \omega_1') + \nu\omega_2 + \omega_1][\nu\omega_2 + \omega_1 + 2x(\nu\omega_2' + \omega_1')],
\]
where \(\omega_{1,2}(x)\) are chosen as above, \(\omega_{1,2}'(x)\) are their derivatives with respect to \(x\) and \(\nu\) is a complex parameter. The set of Chazy and Picard type solutions covers all the possible solutions of PVI\(\mu\) with any half-integer \(\mu \neq \frac{1}{2}\). I compute explicitly the asymptotic behaviour of Picard and Chazy solutions for any choice of the parameters \((\nu_1, \nu_2)\) and \(\nu\) respectively (see Lemma 2 and Lemma 5). I show that structure of the nonlinear monodromy is given by the action of \(\Gamma(2)\) on \((\nu_1, \nu_2)\) and \(\nu\), i.e. given a branch \(y(x; \nu_1, \nu_2)\) (resp. \(y(x; \nu)\)) of a Picard (resp. Chazy) solution, all the other branches of the same solutions are of the form \(y(x; \tilde{\nu}_1, \tilde{\nu}_2)\) (resp. \(y(x; \tilde{\nu})\)) with
\[
\begin{pmatrix}
\tilde{\nu}_1 \\
\tilde{\nu}_2
\end{pmatrix} =
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix}, \quad \tilde{\nu} = \frac{a\nu + b}{c\nu + d}.
\]
Concerning the algebraic solutions, I show that, for any half-integer \(\mu\), they form a countable set. They coincide with all the Picard type solutions with rational \((\nu_1, \nu_2)\).

One of the main tools to prove the above results are the symmetry transformations between solutions of Painlevé equations with different values of the parameters (see [Ok]). In Section 4, I prove that all the solutions of PVI\(\mu\) equations with any half-integer \(\mu, \mu \neq -\frac{1}{2}\), are transformed via birational canonical transformations to solutions of the case \(\mu = -\frac{1}{2}\) and that the birational canonical transformations mapping the case \(\mu = 8\) to the case \(\mu = -\frac{1}{2}\) diverge when applied to the Chazy solutions. This is the reason why Chazy solutions are lost in the case of \(\mu = -\frac{1}{2}\).

Even if the nonlinear monodromy of the solutions of PVI\(\mu\) with half-integer \(\mu\) is completely described in the first part of this paper (see Theorems 1 and 3), it is interesting to study the structure of analytic continuation of the solutions by a geometric approach which allows us to parametrize the algebraic solutions in terms of regular polyhedra or star
polyhedra in the plane and to describe also the case of integer \( \mu \). The main tool to study the structure of analytic continuation of the solutions to PVI_{\mu} with any \( 2\mu \in \mathbb{Z} \) is the isomonodromy deformation method (see [FlN], [ItN]), i.e PVI_{\mu} is treated as isomonodromy deformation equation of the auxiliary Fuchsian system

\[
\frac{dY}{dz} = \left( \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_x}{z-x} \right) Y, \tag{1.1}
\]

where \( A_0, A_1, A_x \) are \( 2 \times 2 \) nilpotent matrices and \( A_0 + A_1 + A_x = -A_{\infty} \), with

\[
A_{\infty} = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix}, \quad \text{for } \mu \neq 0
\]

\[
A_{\infty} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \text{for } \mu = 0
\]

The technique is similar to the one developed in [DM], with some subtleties due to the resonance of the matrix \( A_{\infty} \), so I give a brief resume of it in Sections 5, 6. In Section 7, I give the explicit relation between algebraic solutions and monodromy data of the system (1.1). In Section 8, I show that the algebraic solutions of PVI_{\mu} with half-integer \( \mu \) are in one to one correspondence with regular polygons or star-polygons in the plane and thus, the non linear monodromy of the algebraic elliptic curves of Weierstrass is described by the regular polygons or star-polygons in the plane. For \( \mu \) integer, I show that there are no algebraic solutions except a one parameter family of rational solutions.

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2. Picard Solutions.

In this section, I describe the two parameter family of solutions of PVI_{\mu=\frac{1}{2}} introduced by Picard (see [Pic]), their asymptotic behaviour and their monodromy.

For the case \( \mu = 1/2 \), Picard (see [Pic]) produced the following family of elliptic solutions:

\[
y(x) = \wp(\nu_1 \omega_1 + \nu_2 \omega_2; \omega_1, \omega_2) + \frac{x + 1}{3} \tag{2.1}
\]

where \( \omega_1,2(x) \) are two linearly independent solutions of the Hypergeometric equation

\[
x(1-x)\omega''(x) + (1-2x)\omega'(x) - \frac{1}{4} \omega(x) = 0, \tag{2.2}
\]
and \( \nu_1, \nu_2 \) are two complex numbers chosen such that \( 0 \leq \Re \nu_1, \nu_2 < 2 \) (all the other values of \( \Re \nu_1, \nu_2 \) can be reduced to this case by the periodicity of the Weierstrass function). I choose the branch cuts in the \( x \)-plane on the real axis \( \pi_1 = [-\infty, 0] \) and \( \pi_2 = [1, +\infty] \).

**Remark 1.** Picard solutions obviously satisfy the Painlevé property. Indeed, they are regular functions of \( \omega_{1,2}(x) \), which are analytic on the universal covering of \( \mathbb{T} \setminus \{0,1,\infty\} \).

**Lemma 1.** All the solutions of \( \text{PVI}_{\mu=1/2} \) are of the form (2.1).

The proof is due to R. Fuchs (see [Fuchs] and [Man]).

**Remark 2.** The general solution obtained by Hitchin (see [Hit]) in terms of Jacobi theta-functions in the case of \( \text{PVI}_{\frac{1}{2},\pm \frac{1}{2},\pm \frac{1}{2}} \) can be obtained from (2.1) via the birational canonical transformation \( w = w_1 w_2 w_1 \) described in [Ok].

### 2.1 Asymptotic behaviour and monodromy of the Picard solutions

Here, I describe the monodromy of the Picard solutions and show that, for any \( \nu_1, \nu_2 \in \mathbb{C} \setminus \{(0,0)\} \), all the solutions (2.1) have asymptotic behaviour of algebraic type (see (2.5) below).

First of all, let me fix a branch of a particular Picard solution (2.1), i.e. a pair of values \( (\nu_1, \nu_2) \), and a branch of \( \omega_{1,2} \) with respect to the above branch cuts \( \pi_1, \pi_2 \). For example, in a neighborhood of 0, one can take:

\[
\omega_1^{(0)}(x) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}, 1, x\right), \quad \omega_2^{(0)}(x) = -\frac{i}{2} g\left(\frac{1}{2}, \frac{1}{2}, 1, x\right),
\]

where

\[
F(a,b,c,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!(c)_k} z^k,
\]

\[
g(a,b,z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{k!} z^k [\ln z + \psi(a + k) + \psi(b + k) - 2\psi(k + 1)].
\]

Now, I fix some paths \( \gamma_1 \) and \( \gamma_2 \) along which the above basis is analytically continued to 1 and to \( \infty \) as in figure 1.

![Fig.1](image-url) The paths \( \gamma_1 \) and \( \gamma_2 \) along which the basis \( \omega_{1,2}^{(0)} \) is analytically continued.
Along the paths $\gamma_1$ and $\gamma_2$, the basis $\omega_{1,2}^{(0)}$ has the following analytic continuation:

\[
\omega_{1}^{(0)} \rightarrow \begin{cases} 
\omega_{1}^{(1)} = -\frac{1}{2} g \left( \frac{1}{2}, \frac{1}{2}, 1, 1-x \right), & \text{as } x \to 0, \\
\omega_{1}^{(\infty)} = \frac{1}{\sqrt{x}} \left[ i g \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{x} \right) + \pi F \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{x} \right) \right], & \text{as } x \to \infty, 
\end{cases} \quad (2.4)
\]

\[
\omega_{2}^{(0)} \rightarrow \begin{cases} 
\omega_{2}^{(1)} = \frac{i \pi}{2} F \left( \frac{1}{2}, \frac{1}{2}, 1, 1-x \right), & \text{as } x \to 1, \\
\omega_{2}^{(\infty)} = -\frac{i}{2\sqrt{x}} \left[ g \left( \frac{1}{2}, \frac{1}{2}, 1, \frac{1}{x} \right) \right], & \text{as } x \to \infty. 
\end{cases}
\]

**Lemma 2.** The above chosen branch of the solution (2.1) of PVI$\mu$ with $\mu = \frac{1}{2}$, for $\nu_{1,2} \in \mathbb{C}^{2}\{(0, 0)\}$, $0 \leq \text{Re} \nu_{1,2} < 2$, has the following asymptotic behaviour:

\[
y(x) \sim \begin{cases} 
a_0 x^0 \left( 1 + \mathcal{O}(x^\varepsilon) \right), & \text{as } x \to 0, \\
1 - a_1 (1 - x)^{1l} \left( 1 + \mathcal{O}((1 - x)^\varepsilon) \right), & \text{as } x \to 1, \\
a_\infty x^{1l} \left( 1 + \mathcal{O}(x^{-\varepsilon}) \right), & \text{as } x \to \infty, 
\end{cases} \quad (2.5)
\]

where $l_0, l_1, l_\infty$ are given by

\[
l_0 = \begin{cases} 
\nu_2, & \text{if } \text{Re } \nu_2 \leq 1 \\
2 - \nu_2, & \text{if } \text{Re } \nu_2 > 1 
\end{cases} \quad l_1 = \begin{cases} 
\nu_1, & \text{if } \text{Re } \nu_1 \leq l \\
2 - \nu_1, & \text{if } \text{Re } \nu_1 > 1 
\end{cases} \quad (2.6)
\]

and

\[
l_\infty = \begin{cases} 
2 + \nu_2 - \nu_1, & \text{if } \text{Re } (\nu_2 - \nu_1) < 0 \\
\nu_2 - \nu_1, & \text{if } \text{Re } (\nu_2 - \nu_1) \leq 1 \\
2 - (\nu_2 - \nu_1), & \text{if } \text{Re } (\nu_2 - \nu_1) > 1 
\end{cases} \quad (2.7)
\]

$a_0, a_1, a_\infty$ are three non-zero complex numbers depending on $\nu_{1,2}$ and $\varepsilon > 0$ is small enough.

**Remark 3.** The solutions (2.1), for $(\nu_1, \nu_2) \in \mathbb{C}^{2}\{(0, 0)\}$, with $\text{Re } \nu_{1,2} > 2$ or $\text{Re } \nu_{1,2} < 0$, can be reduced to the previous case thanks to the periodicity of the Weiestrass $\wp$ function. In this way, I show that the solutions (2.1) have asymptotic behaviour of algebraic type for any $(\nu_1, \nu_2) \in \mathbb{C}^{2}\{(0, 0)\}$, and that the exponents $l_i$ have always real part in the interval $[0, 1]$.

**Proof of Lemma 2.** First, let me analyze the asymptotic behaviour of $y(x)$ as $x \to 0$. Observe that, as $x \to 0$ along any direction in the complex plane, the function $\tau(x)$, defined by $\tau = \frac{\omega_1}{\omega_2}$, has imaginary part that tends to infinity, while the real part remains limited. In fact, for $\omega_{1,2}(x)$ defined in (2.3):

\[
\tau = -i \frac{g \left( \frac{1}{2}, \frac{1}{2}, 1, 1-x \right)}{\pi F \left( \frac{1}{2}, \frac{1}{2}, 1, x \right)} \sim -i \pi \log |x| + \frac{\arg(x)}{\pi},
\]
where, as $x \to 0$, $\log |x| \to -\infty$, while $\arg (x)$ remains bounded for any fixed branch. This fact permits to use the formula of the Fourier expansion of the Weierstrass function $\wp (u, \omega_1, \omega_2)$ (see [SG]):

$$\wp (u, \omega_1, \omega_2) = -\frac{\pi^2}{12 \omega_1^3} + \frac{2 \pi^2}{\omega_1^2} \sum_{k=1}^{\infty} \frac{k q^{2k}}{1 - q^{2k}} \left( 1 - \cos \frac{k \pi u}{\omega_1} \right) + \frac{\pi^2}{4 \omega_1^2} \csc^2 \left( \frac{\pi u}{2 \omega_1} \right), \tag{2.8}$$

where $q = \exp \left( \frac{i \pi \omega_2}{\omega_1} \right)$ and $u$ is such that

$$-2 \Re \left( \frac{\omega_2}{i \omega_1} \right) < \Re \left( \frac{u}{i \omega_1} \right) < 2 \Re \left( \frac{\omega_2}{i \omega_1} \right). \tag{2.9}$$

For $u = \nu_1 \omega_1 + \nu_2 \omega_2$, (2.9) reads:

$$|\Im \nu_1 + \Re \nu_2 \Im \tau + \Im \nu_2 + \Re \tau| < 2 \Im \tau,$$

that is always verified for $x \to 0$ along any direction in the complex plane, and $|\Re \nu_2| < 2$, because, $\Im \tau \to \infty$ while $\Re \tau$ remains limited.

First, suppose that $\Re \nu_2 \neq 0$. In this case,

$$q = \exp \left( \frac{i \pi \omega_2}{\omega_1} \right) \quad \text{with} \quad \frac{i \pi \omega_2}{\omega_1} = g \left( \frac{1}{2}, \frac{1}{2}; 1, x \right) = \log (x) + 1 + O(x).$$

Then:

$$\exp \left( \frac{i \pi u}{\omega_1} \right) = \exp (i \pi \nu_1) \exp (\nu_2 \log (x) + \nu_2 (1 + O(x))) \sim \exp (i \pi \nu_1 + \nu_2) x^{\nu_2} (1 + O(x)),$$

and, for $\Re \nu_2 > 0$,

$$\csc^2 \left( \frac{\pi u}{2 \omega_1} \right) = -\frac{4}{\exp \left( \frac{i \pi u}{\omega_1} \right) + \exp \left( -\frac{i \pi u}{\omega_1} \right) - 2} \sim -4 \exp (i \pi \nu_1) x^{\nu_2} + O(x^{2 \nu_2}),$$

and

$$\frac{\pi^2}{12 \omega_1^3} \sim \frac{1}{3} + O(x),$$

and

$$\frac{k q^{2k}}{1 - q^{2k}} \left( 1 - \cos \frac{k \pi u}{\omega_1} \right) \sim (k x^{2k} + O(x^{2k+1}))(\frac{-\exp (-i \pi k \nu_1)}{2} x^{-k \nu_2} + O(1)).$$

As a consequence,

$$y (x) \sim -4 \exp (i \pi \nu_1) x^{\nu_2} - 4 \exp (-i \pi \nu_1) x^{2 - \nu_2} + O \left( x^{3 - \nu_2} \right) + O \left( x^{4 - 2 \nu_2} \right) + O \left( x^2 \right) + O \left( x^{2 \nu_2} \right).$$

This gives the required asymptotic behaviour around 0.
For $\Re \nu_2 = 0$, with $\Im \nu_2 \neq 0$,
\[
y(x) \sim \frac{x}{3} - \frac{4x^2(\alpha(x) - 1)^2 - \alpha^2}{\alpha(x)(\alpha(x) - 1)} + O(x^2),
\]
where $\alpha(x) = \exp(i\pi \nu_1 + \nu_2)x^{\nu_2}$ remains limited for $x \to 0$ along any direction in the complex plane and
\[
- \frac{4x^2(\alpha(x) - 1)^2 - \alpha^2}{\alpha(x)(\alpha(x) - 1)} \sim \frac{\exp(i\pi \nu_1 + \nu_2)x^{\nu_2}}{\exp(i\pi \nu_1 + \nu_2)x^{\nu_2} - 1},
\]
that gives the required asymptotic behaviour around 0 for $l_0 = \nu_2$. If $\nu_2 = 0$, then $\frac{\pi u}{\omega_1} = \pi \nu_1$, and
\[
y(x) \sim \csc \left( \frac{\pi \nu_1}{2} \right) + O(x),
\]
which is again the required asymptotic behaviour around 0 for $l_0 = 0$.

The asymptotic behaviour of $y(x)$ as $x \to 1$ can be obtained observing that, as $x \to 1$, the chosen branch of the basis $\omega_{1,2}$ is analytically continued to $\omega_{1,2}^{(1)}$ given by (2.4). As a consequence
\[
y(x) \sim \frac{x + 1}{3} + \wp \left( \nu_1 \omega_1^{(1)} + \nu_2 \omega_2^{(1)}, \omega_1^{(1)}, \omega_2^{(1)} \right)
\]
and, with the change of variable $x = 1 - z$, one obtains, for $z \to 0$,
\[
y(z) \sim \frac{2 - z}{3} + \wp \left( -\nu_1 \omega_2^{(0)} + \nu_2 \omega_1^{(0)}, -\omega_2^{(0)}, \omega_1^{(0)} \right) = \frac{2 - z}{3} + \wp \left( (\nu_2 - \nu_1)^{0} + \nu_1 \omega_2^{(0)}, \omega_1^{(0)}, \omega_2^{(0)} \right).
\]
Using the previously obtained asymptotic behaviour of $y(x)$ around zero, one immediately obtains the required asymptotic behaviour around 1 with $l_1$ given by (2.6).

Analogously, one can derive the asymptotic behaviour around $\infty$. The only delicate point is that $\nu_2 - \nu_1$ might be negative. In this case one takes $2 + \nu_2 - \nu_1$ which is positive because $\nu_2 - \nu_1 > -2$. This concludes the proof of the lemma. QED

**Theorem 1.** The monodromy of the Picard solutions (2.1) is described by the action of the group $\Gamma(2)$ on the parameters $(\nu_1, \nu_2)$, i.e. given a branch $y(x; \nu_1, \nu_2)$, all the other branches of the same solution are of the form $y(x; \tilde{\nu}_1, \tilde{\nu}_2)$ with all the $(\tilde{\nu}_1, \tilde{\nu}_2)$ such that
\[
\begin{pmatrix}
\tilde{\nu}_1 \\
\tilde{\nu}_2
\end{pmatrix} = \begin{pmatrix} a & b \\
c & d \end{pmatrix} \begin{pmatrix} \nu_1 \\
\nu_2 \end{pmatrix} \text{ for } \begin{pmatrix} a & b \\
c & d \end{pmatrix} \in \Gamma(2).
\]

**Proof.** Let us fix a particular Picard solution (2.1), i.e. a particular pair of values of $(\nu_1, \nu_2)$. A branch is given by the choice of a branch of the basis $\omega_{1,2}$ of solutions of the hypergeometric equation (2.2). As a consequence, the monodromy of the Picard solutions is described by the monodromy of the hypergeometric equation (2.2). This is given by the
action of the group $\Gamma(2)$ on $\omega_{1,2}$. In fact, let us fix a basis $\gamma_0, \gamma_1$ of loops in $\pi_1(C \setminus \{0, 1, \infty\})$ like in figure 2. Let us consider $\omega_{1,2}$ chosen as in (2.3), (2.4). The result of the analytic continuation of $\omega^{(0)}_{1,2}$ along $\gamma_0$ is given by:

$$\begin{pmatrix} \omega^{(0)}_1 \\ \omega^{(0)}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \omega^{(0)}_1 \\ 2\omega^{(0)}_1 + \omega^{(0)}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} \omega^{(0)}_1 \\ \omega^{(0)}_2 \end{pmatrix}$$

and the result of the analytic continuation of $\omega^{(1)}_{1,2}$ along $\gamma_1$ is given by:

$$\begin{pmatrix} \omega^{(1)}_1 \\ \omega^{(1)}_2 \end{pmatrix} \rightarrow \begin{pmatrix} \omega^{(1)}_1 - 2\omega^{(1)}_2 \\ \omega^{(1)}_2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^{(1)}_1 \\ \omega^{(1)}_2 \end{pmatrix}.$$
if \( \nu_i = \frac{p_i}{q_i} \) for some pairs of coprime integers \( (p_i, q_i) \), \( i = 1, 2 \), \( N \) is the smallest common multiple of \( q_1, q_2 \) and \( M \) is the largest common divisor of \( \frac{p_1 N}{q_1} \) and \( \frac{p_2 N}{q_2} \).

Proof. For any \( \nu_1, \nu_2 \in \mathbb{Q} \), by the use of the addition and bisection formulae for the Weierstrass function, it is easy to see that \( \wp(\nu_1 \omega_1 + \nu_2 \omega_2, \omega_1, \omega_2) \) is an algebraic expression of the invariants \( e_1, e_2, e_3 \), which are given by

\[
e_1 = 1 - \frac{x + 1}{3}, \quad e_2 = x - \frac{x + 1}{3}, \quad e_3 = \frac{1}{3}.
\]

This shows that for any \( \nu_1, \nu_2 \in \mathbb{Q} \), \( y(x) \) is an algebraic function of \( x \). It remains to show that two branches \( y(x, \nu_1, \nu_2) \) and \( y(x, \tilde{\nu}_1, \tilde{\nu}_2) \) are branches of the same solution (up to the transformations \( x \to 1 - x, y \to 1 - y \) and \( x \to \frac{1}{x}, y \to \frac{\omega}{x} \)) if and only if the correspondent integers \( (M, N) \) and \( (\tilde{M}, \tilde{N}) \), defined as in the statement of the theorem, coincide. Indeed, I show that the ratio \( \frac{M}{N} \) is preserved under the analytic continuation. Due to Theorem 1, the analytic continuation of a solution \( y(x) \) is described by the action of \( \Gamma(2) \) on \( \nu_1, \nu_2 \), that preserves the ratio \( \frac{M}{N} \). Indeed, we can write \( \nu_1 = m_1 \frac{M}{N} \) and \( \nu_2 = m_2 \frac{M}{N} \), where \( m_{1,2} \in \mathbb{Z} \) and \( (m_1, m_2) = 1 \), i.e. \( m_1 \) and \( m_2 \) are coprime integers. Consider a matrix

\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \Gamma(2).
\]

Then the new values of the parameters are given by

\[
\begin{pmatrix}
\tilde{\nu}_1 \\
\tilde{\nu}_2
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix} = \frac{M}{N} \begin{pmatrix}
am_1 + bm_2 \\
cm_1 + dm_2
\end{pmatrix} = \frac{M}{N} \begin{pmatrix}
\tilde{m}_1 \\
\tilde{m}_2
\end{pmatrix},
\]

where \( (\tilde{m}_1, \tilde{m}_2) = 1 \) because \( ab - cd = 1 \), and then \( (a, c) = (a, d) = (b, c) = (b, d) = 1 \).

Now, consider any two numbers \( \nu_1 = m_1 \frac{M}{N} \) and \( \nu_2 = m_2 \frac{M}{N} \), for some given integers \( (M, N, m_1, m_2) \) such that \( (M, N) = 1 \) and \( (m_1, m_2) = 1 \). There are three possibilities. i) \( m_1 \) and \( m_2 \) are odd integers. Then there exists a \( \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \Gamma_2 \) such that \( a + b = m_1, c + d = m_2 \). In fact, for any \( \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \Gamma_2 \), the numbers \( a + b \) and \( c + d \) are odd and coprime. As a consequence, the branch specified by \( \nu_1 = m_1 \frac{M}{N} \) and \( \nu_2 = m_2 \frac{M}{N} \), belongs to the same solution as the branch specified by \( \nu_1 = \frac{M}{N} = \nu_2 \). ii) \( m_1 \) is even and \( m_2 \) is odd. Then there exists a \( \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \Gamma_2 \) such that \( b = m_1 \) and \( d = m_2 \). As a consequence, the branch specified by \( \nu_1 = m_1 \frac{M}{N} \) and \( \nu_2 = m_2 \frac{M}{N} \), belongs to the same solution as the branch specified by \( \nu_1 = 0 \) and \( \nu_2 = \frac{M}{N} \). ii) \( m_1 \) is odd and \( m_2 \) is even. Then there exists a \( \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \Gamma_2 \) such that \( a = m_1 \) and \( c = m_2 \). As a consequence, the branch specified by \( \nu_1 = m_1 \frac{M}{N} \) and \( \nu_2 = m_2 \frac{M}{N} \), belongs to the same solution as the branch specified by \( \nu_1 = \frac{M}{N} \) and \( \nu_2 = 0 \). It easy to see that the above three cases are related one to the other by the transformations \( x \to 1 - x, y \to 1 - y \) and \( x \to \frac{1}{x}, y \to \frac{\omega}{x} \). This concludes the proof of the lemma. QED
3. Chazy Solutions.

In this section, I introduce a one-parameter family of transcendental solutions of \( PVI_{\mu} \), with \( \mu = -\frac{1}{2} \), compute their asymptotic behaviour and describe their monodromy.

**Theorem 2.** There exists a one-parameter family of solutions of \( PVI_{\mu} \) with \( \mu = -\frac{1}{2} \) of the form:

\[
y(x) = \frac{1}{8} \left\{ \left[ \nu \omega_2 + \omega_1 + 2x(\nu \omega'_2 + \omega'_1) \right]^2 - 4x(\nu \omega'_2 + \omega'_1)^2 \right\}^2 \frac{1}{(\nu \omega_2 + \omega_1)(\nu \omega'_2 + \omega'_1)[2(x - 1)(\nu \omega'_2 + \omega'_1) + \nu \omega_2 + \omega_1][\nu \omega_2 + \omega_1 + 2x(\nu \omega'_2 + \omega'_1)]}
\]

where \( \omega_{1,2}(x) \) are two linearly independent solutions of the Hypergeometric equation (2.2) and \( \nu \in \mathbb{C} \) is the parameter.

Proof. Substitute (3.1) in \( PVI_{\mu=-\frac{1}{2}} \). By straightforward computations, it is easy to verify that if \( \omega_{1,2} \) are solutions of (2.2), then (3.1) is a solution of \( PVI_{\mu=-\frac{1}{2}} \) for any \( \nu \in \mathbb{C} \). QED

Observe that the solutions (3.1) are regular functions of \( \omega_{1,2}(x) \), which are analytic on the universal covering of \( \mathbb{T} \backslash \{0,1,\infty\} \), so they obviously satisfy the Painlevé property. I call Chazy solutions the solutions (3.1) and all the ones obtained from them via the transformations \( x \to 1 - x \), \( y \to 1 - y \) and \( x \to \frac{1}{x} \), \( y \to \frac{1}{y} \), which preserve the \( PVI_{\mu} \) equation. The reason of this name is that they correspond to the following solution of WDVV equations in the variables \((t^1,t^2,t^3)\) (see [Dub]):

\[
F = \frac{(t^1)^2t^3}{2} + \frac{t^1(t^2)^2}{2} - \frac{(t^2)^4}{16} \gamma(t^3)
\]

where the function \( \gamma(t^3) \) is a solution of the equation of Chazy (see [Cha]):

\[
\gamma'' = 6\gamma' - 9\gamma^2.
\]

3.1. Derivation of the Chazy solutions. I briefly outline how to derive (3.1) from (3.2). Using the procedure explained in appendix E of [Dub], it is possible to show that the solution \( y(x) \) of \( PVI_{\mu=-\frac{1}{2}} \) correspondent to the solution (3.2) of WDVV equations is given by:

\[
y(\tau) = \frac{(w_2(\tau)w_3(\tau) - w_1(\tau)w_2(\tau) - w_1(\tau)w_3(\tau))^2}{4w_1(\tau)w_2(\tau)w_3(\tau)(w_1(\tau) - w_3(\tau))},
\]

\[
x(\tau) = \frac{w_2(\tau) - w_1(\tau)}{w_3(\tau) - w_1(\tau)}
\]

where \( \tau = t^3 \) and \( (w_1,w_2,w_3) \) are solutions of the Halphen system (see [Hal]):

\[
\frac{d}{d\tau} w_1 = - w_1(w_2 + w_3) + w_2w_3,
\]

\[
\frac{d}{d\tau} w_2 = - w_2(w_1 + w_3) + w_1w_3,
\]

\[
\frac{d}{d\tau} w_3 = - w_3(w_1 + w_2) + w_1w_2,
\]

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that is related to the Chazy equation. Indeed \((w_1, w_2, w_3)\) are the roots of the following cubic equation
\[
w^3 + \frac{3}{2} \gamma(\tau) w^2 + \frac{3}{2} \gamma'(\tau) w + \frac{1}{4} \gamma''(\tau) = 0.
\]
I want to derive (3.1) from (3.3). The following lemma will be useful:

**Lemma 4.** The transformation property
\[
\tilde{w}_i(\tau) = \frac{1}{ct+d} w_i \left( \frac{\alpha \tau + \beta}{ct + d} \right) + \frac{c}{ct+d}, \quad \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2, \mathbb{C}),
\]
and the formulae
\[
w_1 = -\frac{1}{2} \frac{d}{dt} \log \frac{\lambda'}{\lambda}, \quad w_2 = -\frac{1}{2} \frac{d}{dt} \log \frac{\lambda'}{\lambda-1}, \quad w_3 = -\frac{1}{2} \frac{d}{dt} \log \frac{\lambda'}{\lambda(\lambda-1)},
\]
where \(\lambda(\tau)\) is a solution of the Schwartzian ODE:
\[
\{\tau, \lambda\} = \frac{1}{2} \left[ \frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{\lambda(1-\lambda)} \right],
\]
with
\[
\{\tau, \lambda\} = -\left[ \frac{\lambda''}{\lambda'} - \frac{3}{2} \left( \frac{\lambda''}{\lambda'} \right)^2 \right] \frac{1}{\lambda^2},
\]
provide the general solution of (3.4).

The proof of this result can be found in [Tak].

The Schwartzian differential equation (3.6) can be reduced to the hypergeometric equation (2.2) via a standard procedure (see [Ince]). Let \(\tau(\lambda) = \frac{\omega_1(\lambda)}{\omega_2(\lambda)}\). Then \(\omega_{1,2}\) are two linearly independent solutions of the ODE:
\[
\omega'' + p(\lambda) \omega' + q(\lambda) \omega = 0,
\]
were \(p(\lambda)\) and \(q(\lambda)\) are two rational functions of \(\lambda\) such that:
\[
2q(\lambda) - \frac{1}{2} p(\lambda)^2 - p'(\lambda) = \frac{1}{2} \left[ \frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} + \frac{1}{\lambda(1-\lambda)} \right].
\]
Requiring that (3.7) is a hypergeometric equation, we obtain
\[
p(\lambda) = \frac{(1-2\lambda)}{(1-\lambda)\lambda}, \quad q(\lambda) = -\frac{1}{4(1-\lambda)^2}.
\]
Using the formula (3.3) for \(x(\tau)\), and the formulae for the Halphen functions (3.5), we see that \(x(\tau) \equiv \lambda(\tau)\). As a consequence, (3.6) is reduced to (3.7) that coincides with (2.2).

Putting \(\tau(x) = \frac{\omega_1(x)}{\omega_2(x)}\), we can compute all the derivatives \(x'(\tau)\) and \(x''(\tau)\) in terms of \(x\), and by (3.5), \(w_i(x)\). In this way we obtain (3.1) by substitution in (3.3).

### 3.2. Asymptotic behaviour and monodromy of the Chazy solutions.

Here I compute the asymptotic behaviour of the Chazy solutions and show that they are transcendental functions. Moreover I describe their nonlinear monodromy.
Lemma 5. The solutions (3.1), for any $\nu \in \mathbb{C}$, and with branch cuts in the $x$-plane $\pi_1, \pi_2$ on the real axis, $\pi_1 = [-\infty, 0]$, $\pi_2 = [1, -\infty]$, have the following asymptotic behaviour around the singular points $0, 1, \infty$:

$$y(x) \sim \begin{cases} 
- \log(x)^{-2} + b_0 \log(x)^{-3} + \mathcal{O}(\log(x)^{-4}) & \text{as } x \to 0, \\
1 + \log(1-x)^{-2} + b_1 \log(1-x)^{-3} + \mathcal{O}(\log(1-x)^{-4}), & \text{as } x \to 1 \\
-x \log\left(\frac{1}{x}\right)^{-2} + b_\infty x \log\left(\frac{1}{x}\right)^{-3} + \mathcal{O}\left(\log\left(\frac{1}{x}\right)^{-4}\right), & \text{as } x \to \infty 
\end{cases}, \quad (3.8)$$

where

$$b_0 = 1 + \frac{i\pi}{\nu} - 4 \log 2, \quad b_1 = 2[i\pi(\nu - 1) - 1 + 4 \log 2], \quad b_\infty = 2[(\nu - 1)(1 - 4 \log 2) + i\pi].$$

Proof. First of all one fixes a particular Chazy solution (3.1), i.e. a value $\nu$, and take a branch of it, i.e. a branch of $\omega_{1,2}$ for example (2.3) and (2.4). The correspondent branch $y(x)$ has the asymptotic behaviour (3.8) around the singular points $0, 1, \infty$. QED.

Notice that the leading term of the asymptotic behaviour does not depend on the chosen particular solution, i.e. it does not depend on $\nu$. The dependence on $\nu$ appears in the second term. To derive the asymptotic behaviour of any other branch of $y(x)$, one can use the following:

Theorem 3. The monodromy of the Chazy solutions (3.1) is described by the by the action of the group $\Gamma(2)$ on the parameter $\nu$, for a fixed basis $\omega_{1,2}$, i.e. given a branch $y(x; \nu)$, all the other branches of the same solutions are of the form $y(x; \tilde{\nu})$ with all $\tilde{\nu}$ such that

$$\tilde{\nu} = \frac{a\nu + b}{cv + d} \quad \text{for} \quad \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \Gamma(2).$$

The proof is analogous to the one of Theorem 1.

4. Relations between half-integer values of $\mu$.

We have seen in Section 2 that the PVI$_{\mu}$ for $\mu = \frac{1}{2}$ equation is integrable and its solutions are called Picard solutions. I call solutions of Picard type the solutions of PVI$_{\mu}$ with any half-integer $\mu$, which are images of Picard solutions via the birational canonical transformations described below. In this Section, I prove that, roughly speaking, while the Picard solutions exhaust all the possible solutions of PVI$_{\mu}$ for $\mu = \frac{1}{2}$, the Picard type ones do not exhaust all the possible solutions of the PVI$_{\mu}$ equation with half-integer $\mu$, $\mu \neq \frac{1}{2}$. All the PVI$_{\mu}$ equations with any half-integer $\mu$, $\mu \neq \frac{1}{2}$, are equivalent via birational canonical
Lemma 6. The proof is based on the following lemma: transformations to the case \( \mu = -\frac{1}{2} \), for which Picard type solutions and Chazy solutions are distinct and provide a complete set of solutions. I recall that the so-called singular solutions of the \( PVI_\mu \) equation are \( y(x) = 0, 1, \infty \). They are not solutions of the \( PVI_\mu \) equation because the \( PVI_\mu \) equation itself becomes singular on them.

Theorem 4. i) All the solutions \( y(x) \) of \( PVI_\mu \) equations with \( \mu + \frac{1}{2} \in \mathbb{Z}, \mu \neq \frac{1}{2} \), are mapped via birational canonical transformations to solutions of \( PVI_{\mu=-1/2} \). ii) Chazy solutions of \( PVI_\mu \) are not of Picard type and vice-versa. iii) Chazy solutions and Picard type solutions exhaust all the possible solutions of \( PVI_\mu \) with \( \mu + \frac{1}{2} \in \mathbb{Z}, \mu \neq \frac{1}{2} \).

The proof is based on the following lemma:

**Lemma 6.** The formula:

\[
\tilde{y} = y \frac{(p_0(y')^2 + p_1y' + p_2)^2}{q_0(y')^4 + q_1(y')^3 + q_2(y')^2 + q_3y' + q_4},
\]

where:

\[
p_0 = x^2(x - 1)^2, \\
p_1 = 2x(x - 1)(y - 1)(2\mu(y - x) - y), \\
p_2 = y(y - 1)(y(y - 1) - 4\mu(y - 1)(y - x) + 4\mu^2(y - x)(y - x - 1)], \\
q_0 = x^4(x - 1)^4, \\
q_1 = -4x^3(x - 1)^3y(y - 1), \\
q_2 = 2x^2(x - 1)^2y(y - 1)[3y(y - 1) + 4\mu^2(y - x)(1 + x - 3y)], \\
q_3 = 4x(x - 1)y^2(y - 1)^2[-y(y - 1) - 16\mu^3(y - x)^2 + 4\mu^2(y - x)(3y - x - 1)], \\
q_4 = y^2(y - 1)^2 \{y^2(y - 1)^2 + 64\mu^3y(y - 1)(y - x)^2, -8\mu^2y(y - 1)(y - x)(3y - x - 1) + 16\mu^4(y - x)^2[(x - 1)^2 + y(2 + 2x - 3y)]\},
\]

transforms solutions of \( PVI_\mu \) to solutions of \( PVI(-\mu) \), or equivalently of \( PVI(\mu + 1) \).

Proof. Suppose \( y(x) \) is a solution of \( PVI_\mu \). Take \( \tilde{y} \) as in (4.1) and compute \( \tilde{y}' \) and \( \tilde{y}'' \) in terms of \( (x, y, y') \) by derivating (4.1) and substituting to \( y'' \) the righthand side of \( PVI_\mu \). Substituting \( \tilde{y}, \tilde{y}' \) and \( \tilde{y}'' \) in \( PVI_{-\mu} \) it is easy to verify that \( PVI_{-\mu} \) is satisfied.

Proof of theorem 4. Let us consider a solution \( y(x) \) of \( PVI_\mu \) for any half-integer \( \mu \). We want to apply the transformation (4.1) to \( y(x) \). This is possible if the denominator \( Q(y_x, y, x, \mu) \) of the formula (4.1) does not vanish identically on \( y(x) \). In [DM], it is shown that \( Q \) can identically vanish on solutions of \( PVI_\mu \) only for \( \mu = -1/2 \) or \( \mu = 0 \). Let me consider the case \( PVI_{\mu=-1/2} \). This is the same as \( PVI_{\mu=3/2} \) (in fact \( \mu \) and \( 1 - \mu \) give the same value of the parameter \( \alpha \) in \( PVI \)). For \( \mu = 3/2 \) the denominator \( Q(y_x, y, x, \mu) \) never vanishes and we can apply the transformation (4.1). Moreover \( Q(y_x, y, x, \mu) \) never vanishes for any \( \mu = 3/2 + n, n \geq 0 \), so we can apply (4.1) iteratively. In this way we obtain all the \( PVI_{\mu=\pm(3/2+n)} \) for any \( n \geq 0 \). The above transformations are all invertible. In fact, starting
Fig. 3. The birational canonical transformations relating solutions of PVI\(\mu\) equations with half-integer values of \(\mu \neq \frac{1}{2}\).

from any PVI\(_{\mu=\pm(3/2+n)}\), for \(n \geq 0\), we arrive at PVI\(_{\mu=-1/2}\), via rational transformations of the form (4.1), the determinant of which never vanishes. The idea of what happens is shown in figure 2.

We have proved the claim i) of Theorem 4. Claims ii) and iii) follow from the following two lemmas.

**Lemma 7.** The one-parameter family of Chazy solutions exhaust all the possible solutions of the differential equation \(Q(y_x, y, x) \equiv 0\).

Proof. Let us consider the algebraic differential equation \(Q(y_x, y, x) \equiv 0\). It has the following roots

\[
y_x = \frac{y(y-1) - \sqrt{y(y-1)(y-x)} + \sqrt{y(y-1)(y-x)} \sqrt{2y-1 + 2\sqrt{y(y-1)}}}{x(x-1)},
\]

\[
y_x = \frac{y(y-1) - \sqrt{y(y-1)(y-x)} - \sqrt{y(y-1)(y-x)} \sqrt{2y-1 + 2\sqrt{y(y-1)}}}{x(x-1)},
\]

\[
y_x = \frac{y(y-1) + \sqrt{y(y-1)(y-x)} + \sqrt{y(y-1)(y-x)} \sqrt{2y-1 - 2\sqrt{y(y-1)}}}{x(x-1)},
\]

\[
y_x = \frac{y(y-1) + \sqrt{y(y-1)(y-x)} - \sqrt{y(y-1)(y-x)} \sqrt{2y-1 - 2\sqrt{y(y-1)}}}{x(x-1)}.
\]

All the above differential equations are equivalent: the first is mapped in the third by the transformation \(x \to 1-x, y \to 1-y\), to the forth by \(x \to \frac{1}{x}, y \to \frac{y}{x}\) and to the second by \(x \to \frac{1}{1-x}, y \to \frac{1-y}{1-x}\). All these transformations preserve the class of Chazy solutions of PVI\(_{\mu=-1/2}\). Thus it is enough to prove that the one-parameter family of Chazy solutions exhaust all the possible solutions of the first differential equation. Indeed it easy to verify that \(y(x, \nu)\) of the form (3.1) solves it for any value of the parameter \(\nu\). To conclude, we have to show that \(\forall (x_0, y_0) \in \mathbb{C} \times \mathbb{C}\), there exists a value of the parameter \(\nu\) such that

\[
y(x_0, \nu) = y_0.
\]

If \(y_0 = \infty\), we can take \(\nu = \nu_\infty\) such that \(w_1'(x_0) + \nu_\infty w_2'(x_0) = 0\). Let us suppose that \(y_0 \neq \infty\). Then \(w_1'(x_0) + \nu w_2'(x_0) \neq 0\) for every \(\nu \neq \nu_\infty\) and \(y(x)\) can be written in the
form:

\[ y(x) = \frac{[(W(x, \nu) + 2x)^2 - 4x]}{8W(x, \nu)(W(x, \nu) + 2x)[W(x, \nu) + 2(x - 1)]} \]

where \( W(x, \nu) = \frac{w_1(x) + \nu w_2(x)}{w_1(x) + \nu w_2(x)} \). If we show that given any \((x_0, W_0)\), there exists \(\nu_0\) such that \(W(x_0, \nu_0) = W_0\), we are done. Indeed, for

\[ \nu_0 = -\frac{w_1(x_0) - W_0 w_1'(x_0)}{w_2(x_0) - W_0 w_2'(x_0)} \]

\[ W(x_0, \nu_0) = W_0. \]

**Lemma 8.** The denominator \(Q(y_x, y, x)\) never vanishes on Picard type solutions.

Proof. Consider any Picard solution \(y(x)\) and its correspondent Picard type solution \(\tilde{y}\), obtained by the transformation (4.1). I want to show that \(\tilde{y}(x)\) is such that \(Q(\tilde{y}_x, \tilde{y}, x) \neq 0\).

By straightforward computations, one obtains:

\[ Q(\tilde{y}_x, \tilde{y}, x) = (x - 1)^2 x^2 \left(y - y^2 - xy_x^2 + x^2 y_x^2\right)^4 \left(y^2 - y - 2xy_x(y - 1) - xy_x^2 + x^2 y_x^2\right)^4 \times \]
\[ \times \left(y^2 - y - 2y y_x(x - 1) - xy_x^2 + x^2 y_x^2\right)^4 \left\{ y^2(y - 1)^2 - 4y^2(y - 1)^2 y_x + \right. \]
\[ \left. + 2(y - 1)y y_x^2(4xy - x - x^2 - 2y) - 4(x - 1)x(y - 1)y y_x^2 + (x - 1)^2 x^2 y_x^4 \right\}^{-1}. \]

The above quantity can not vanish on any Picard solution \(y\). In fact none of the polynomials

\[ Q_1(y_x, y, x) = y - y^2 - xy_x^2 + x^2 y_x^2, \]
\[ Q_2(y_x, y, x) = y^2 - y - 2xy_x(y - 1) - xy_x^2 + x^2 y_x^2, \]
\[ Q_3(y_x, y, x) = y^2 - y - 2y y_x(x - 1) - xy_x^2 + x^2 y_x^2, \]

can vanish on any Picard solution. Indeed, eliminating \(y_{xx}\) and \(y_x\), form the system

\[ y_{xx} = \frac{1}{2} \left( \frac{1}{y} + \frac{1}{y - 1} + \frac{1}{y - x} \right) y_x^2 - \left( \frac{1}{x} + \frac{1}{x - 1} + \frac{1}{y - x} \right) y_x + \frac{y (y - 1)}{2x(x - 1)(y - x)}, \]
\[ Q_i(y_x, y, x, \mu) = 0, \]
\[ \frac{d}{dx} Q_i(y_x, y, x, \mu) = 0, \]

for each \(i = 1, 2, 3\), we obtain the following resultant:

\[ (x - 1)x(x - y)^2(y - 1)^3 y^3, \quad (x - 1)^3 x(x - y)^4(y - 1)^4 y^2, \quad (x - 1)x^3(x - y)^4(y - 1)^2 y^4, \]

which never vanish for nosingular solutions of PVI\(\mu\). This concludes the proof of Lemma 8.

QED
Now, I conclude the proof of Theorem 4. Claim ii) follows from the fact that, thanks to Lemma 5, $Q$ does not vanish on Picard type solutions, while, thanks to Lemma 7, it vanishes on Chazy solutions. Claim iii) is due to the fact that any solution of $PVI_{\mu=\frac{1}{2}}$ such that $Q(y_x, y, x, \mu) \neq 0$ is necessarily of Picard type. In fact, being $Q(y_x, y, x, \mu) \neq 0$, the birational canonical transformation (4.1) can be applied to $y$ and it gives rise to a Picard solution.

QED

Remark 4. In the notations of the paper [Ok], the symmetry from $\mu = -1/2$ to $\mu = 1/2$ is given by the transformation $l_3^2$ applied on the canonical coordinates $(p, q)$ of $PVI_{\mu=\frac{-1}{2}}$. The condition $Q(y_x, y, x) \equiv 0$ is exactly the condition that the intermediate coordinates $l_3(p, q)$ are singular, i.e. the associated auxiliary Hamiltonian $h(l_3(p, q))$ is linear in $t$. In particular, this means that the transformation $l_3$ can map non-singular solutions into singular ones. This is exactly what happens when we apply (4.1) on the Chazy solutions.

3.1. Chazy solutions as limit of Picard type solutions. For $\nu_1 = \nu_2 = 0$ the Weierstrass $\wp$-function has a pole and the correspondent function $y(x)$ defined by (2.1) does not exist.

Lemma 9. Chazy solutions of $PVI_{\mu=-\frac{1}{2}}$ can be obtained as the limit for $\nu_1, \nu_2 \to 0$, with $\frac{\nu_2}{\nu_1} = \nu$, of the Picard type solution obtained applying the symmetry (4.1) to the solution (2.1) of $PVI_{\mu=\frac{-1}{2}}$.

The above result is not surprising, in fact, as observed above, Chazy solutions are transformed, via the symmetry (4.1), to singular solutions of $PVI_{\mu=\frac{-1}{2}}$ which are identically equal to $\infty$.

Proof of Lemma 9. Consider a solution $y(x)$ of $PVI_{\mu=-\frac{1}{2}}$, $y(x)$ given by the formula (2.1). Fix the ratio $\frac{\nu_2}{\nu_1} = \nu$ and let $\nu_1, \nu_2 \to \infty$. Since the Weierstrass function has a pole of order two in 0, one has

$$
\lim_{\nu_1 \to 0} \nu_1^2 y(x) = \frac{1}{(\omega_1 + \nu \omega_2)^2},
$$

and

$$
\lim_{\nu_1 \to 0} \nu_1^2 y'(x) = \frac{1}{(\omega_1 + \nu \omega_2)^2},
$$

and, applying the transformation (4.1) for $\mu = \frac{1}{2}$ to $y(x)$ given by (2.1), and taking the limit as $\nu_1, \nu_2 \to 0$ with fixed ratio $\frac{\nu_2}{\nu_1} = \nu$, one obtains the formula (3.1). This concludes the proof of the lemma. QED

4.2. Algebraic solutions. Here, I classify all the algebraic solutions of $PVI_{\mu}$, for any $\mu + \frac{1}{2} \in \mathbb{Z}$. As shown in Section 2.2, $PVI_{\mu}$ with $\mu = \frac{1}{2}$ admits a countable set of algebraic solutions. Now I show that for all the other half integer values of $\mu$, the algebraic solutions are all of Picard type, so they are equivalent via birational canonical transformations to the ones of Lemma 3.

Theorem 5. For any half-integer $\mu$, $PVI_{\mu}$ admits a countable family of algebraic solutions. All the algebraic solutions are of Picard type for some $\nu_1, \nu_2 \in \mathbb{Q}$, $0 \leq \nu_1, \nu_2 < 2$.
Proof. The algebraic solutions are preserved under the transformations (4.1). So we obtain a countable family of algebraic solutions $PVI_\mu$ for any half-integer $\mu$. Moreover Chazy solutions are transcendental, so the algebraic solutions can only be of Picard type. 

QED

For example, we can recover the solutions found in [Dub], (E.34a), (E.36) and (E.37). In fact they are mapped by the symmetry (4.1), respectively to

$$y = \frac{(s - 1)^2}{(s - 3)(1 + s)}, \quad x = \frac{(s - 1)^3(3 + s)}{(s - 3)(1 + s)^3},$$  \hspace{1cm} (4.3)

that is a Picard solution with $N = 3$ and $M = 2$, to

$$y = \frac{2 + s}{4}, \quad x = \frac{(s + 2)^2}{8s}$$  \hspace{1cm} (4.4)

that is a Picard solution with $N = 2$, i.e. $y(x) = x + \sqrt{(x-1)x}$ and to

$$y = \frac{3(3-t)(1+t)}{(3+t)^2}, \quad x = \frac{(3-t)^3(1+t)}{(1-t)(3+t)^3},$$  \hspace{1cm} (4.5)

that is a Picard solution with $N = 3, M = 1$. In section 8, I show how these solutions correspond to the affine Weyl groups $A_2$, $B_2$ and $G_2$ respectively and show that all the other algebraic solutions correspond to different presentations of the dihedral group.

5. Painlevé VI equation as isomonodromy deformation equation.

First of all, I briefly describe how $PVI_\mu$, for $2\mu \in \mathbb{Z}$, can be interpreted as the isomonodromy deformation equation of the following auxiliary Fuchsian system (see [JMU], [ItN], [FlN]) with four regular singularities at $z = u_1, u_2, u_3, \infty$:

$$\frac{d}{dz}Y = A(z)Y, \quad z \in \mathbb{C}\{u_1, u_2, u_3, \infty\}$$  \hspace{1cm} (5.1)

where

$$A(z) = \frac{A_1}{z - u_1} + \frac{A_2}{z - u_2} + \frac{A_3}{z - u_3},$$

$u_1, u_2, u_3$ being pairwise distinct complex numbers and $A_i$ being $2 \times 2$ matrices satisfying the following conditions:

$$A_i^2 = 0 \text{ and } -A_1 - A_2 - A_3 = A_\infty$$  \hspace{1cm} (5.2)

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where \( A_\infty = \begin{pmatrix} \mu & 0 \\ 0 & -\mu \end{pmatrix} \) for \( \mu \neq 0 \) and \( A_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) for \( \mu = 0 \).

The solution \( Y(z) \) of the system (5.1) is a multi-valued analytic function on the punctured Riemann sphere, \( \mathbb{C}\setminus\{u_1, u_2, u_3\} \), and its multivaluedness is described by the so-called \textit{monodromy matrices}. To define them, we fix the basis \( \gamma_1, \gamma_2, \gamma_3 \) of loops in the fundamental group

\[ \pi_1 \left( \mathbb{C}\setminus\{u_1, u_2, u_3, \infty\}, \infty \right), \]

as in figure 4 and a fundamental matrix.

![Fig.4: The cuts \( \pi_i \) between the singularities \( u_i \), ordered according to the order of the points \( u_1, u_2, u_3 \), and the oriented loops \( \gamma_i \), starting and finishing at infinity, going around \( u_i \) in positive direction (\( \gamma_i \) is oriented counter-clockwise, \( u_i \) lies inside, while the other singular points lie outside) and not crossing the cuts \( \pi_i \).](image)

**Proposition 1.** There exists a fundamental matrix of the system (5.1) of the form

\[
Y_\infty = \left( 1 + \mathcal{O}\left( \frac{1}{z} \right) \right) z^{-A_\infty} z^R, \quad \text{as} \quad z \to \infty,
\]

as in figure 4 and a fundamental matrix.

\[
Y_\infty = \begin{pmatrix} 1 + \mathcal{O}\left( \frac{1}{z} \right) \end{pmatrix} z^{-A_\infty} z^R, \quad \text{as} \quad z \to \infty,
\]

(5.3)

where the matrix \( R \) is defined as follows for \( 2\mu = n \in \mathbb{Z} \),

\[
\begin{align*}
\text{for } n > 0, & \quad R_{12} = \sum_{k=1}^{3} (A_k)_{12} u_k^n + \sum_{l=1}^{n-1} \left( G^{(n-l)} \sum_{k=1}^{3} A_k u_k^l \right)_{12}, \quad R_{11} = R_{21} = R_{22} = 0, \\
\text{for } n < 0, & \quad R_{21} = \sum_{k=1}^{3} (A_k)_{21} u_k^n + \sum_{l=1}^{n-1} \left( G^{(n-l)} \sum_{k=1}^{3} A_k u_k^l \right)_{21}, \quad R_{11} = R_{12} = R_{22} = 0, \\
\text{for } n = 0, & \quad R_{11} = R_{12} = R_{21} = R_{22} = 0,
\end{align*}
\]

where for \( l = 1, 2, \cdots |n| - 1 \), \( G^{(l)} \) are uniquely determined by

\[
G^{(l)}_{11} = -\sum_{k=1}^{3} (A_k)_{11} \frac{u_k^l}{l} - \sum_{i=1}^{l-1} \left( G^{(l-i)} \sum_{k=1}^{3} A_k u_k^i \right)_{11}, \quad (5.4)
\]

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\[
G_{12}^{(l)} = -\sum_{k=1}^{3} (A_k)_{12} \frac{u_k^l}{l-2\mu} - \sum_{i=1}^{l-1} \left( G^{(l-i)} \sum_{k=1}^{3} A_k u_k^i \right)_{12}, \quad (5.5)
\]

\[
G_{21}^{(l)} = -\sum_{k=1}^{3} (A_k)_{21} \frac{u_k^l}{l+2\mu} - \sum_{i=1}^{l-1} \left( G^{(l-i)} \sum_{k=1}^{3} A_k u_k^i \right)_{21}, \quad (5.6)
\]

\[
G_{22}^{(l)} = -\sum_{k=1}^{3} (A_k)_{22} \frac{u_k^l}{l} - \sum_{i=1}^{l-1} \left( G^{(l-i)} \sum_{k=1}^{3} A_k u_k^i \right)_{22}, \quad (5.7)
\]

and \( z^\mu \) is defined as \( e^{\mu \log z} \), with the choice of the principal branch of the logarithm with the branch-cut along the common direction of the cuts \( \pi_1, \pi_2, \pi_3 \) of figure 4.

Proof. Consider the Fuchsian system near infinity. Perform a change of variable \( z \to \frac{1}{z} \), so that the Fuchsian system becomes

\[
\frac{d}{dz} Y = -\frac{1}{z} \left( A_\infty + \sum_{l=1}^{\infty} A^{(l)} z^l \right) Y,
\]

where

\[
A^{(l)} = A_1 u_1^l + A_2 u_2^l + A_3 u_3^l.
\]

We look for a gauge

\[
\tilde{Y} = G(z) Y, \quad G(z) = 1 + \sum_{l=1}^{\infty} G^{(l)} z^l
\]

such that

\[
\frac{d}{dz} \tilde{Y} = -\frac{1}{z} \left( A_\infty + \sum_{l=1}^{N} R^{(l)} z^l \right) Y,
\]

where

\[
R^{(l)}_{ij} = A^{(l)}_{ij} + G^{(l)}_{ij} (l + (A_\infty)_{jj} - (A_\infty)_{ii}) + \sum_{i=1}^{l-1} \left( G^{(l-i)} A^{(i)} \right)_{ij}.
\]

One chooses \( R^{(l)}_{ij} \neq 0 \) if and only if \( (A_\infty)_{ii} - (A_\infty)_{jj} = l \). Since \( (A_\infty)_{11} = \mu \) and \( (A_\infty)_{22} = -\mu \), there exists a unique \( l \) such that the above condition is fullfilled, i.e. \( l = N = |2\mu| \). All the other \( R^{(l)} \) are zero and for every \( l < N \), \( G^{(l)} \) is uniquely determined by (5.4), (5.5), (5.6), (5.7). QED

The monodromy matrix \( M_\gamma \), by definition, is a constant invertible \( 2 \times 2 \) matrix such that

\[
\gamma[Y_\infty(z)] = Y_\infty(z) M_\gamma,
\]

where \( \gamma[Y_\infty(z)] \) is the analytic continuation of \( Y_\infty(z) \) along the loop \( \gamma \). Particularly, the matrix \( M_\infty := M_{\gamma_\infty} \) is given by:

\[
M_\infty = \exp 2\pi i (A_\infty + R), \quad (5.8)
\]
where $\gamma_\infty$ is a loop around infinity in the clock-wise direction. The matrices $M_i := M_{\gamma_i}$, where the $\gamma_i$ for $i = 1, 2, 3$ are the generators of the fundamental group, are given by:

$$M_i = C_i^{-1} \exp(2\pi i J_i) C_i, \quad i = 1, 2, 3,$$

where the matrices $C_1, C_2, C_3$ are the so-called connection matrices. The matrices $M_1$, $M_2$, $M_3$ generate the monodromy group of the system, and satisfy the following relations:

$$\det(M_i) = 1, \quad \text{Tr}(M_i) = 2, \quad \text{for} \quad i = 1, 2, 3,$$

with $M_i = 1$ if and only if $A_i = 0$. Since the loop $(\gamma_1 \gamma_2 \gamma_3)^{-1}$ is homo-topic to $\gamma_\infty$, the following relation holds true:

$$M_\infty M_3 M_2 M_1 = 1.$$  \hspace{1cm} (5.11)

**Lemma 10.** For $\mu \neq 0$, under the assumption that $R \neq 0$, two Fuchsian systems of the form (5.1) with the same poles $u_1, u_2$ and $u_3$, and the same resonant value of $\mu$, coincide if and only if they have the same monodromy matrices $M_1, M_2, M_3$, with respect to the same basis of the loops $\gamma_1, \gamma_2$ and $\gamma_3$ and the same value of $R$. For $\mu = 0$ two Fuchsian systems of the form (5.1) with the same poles $u_1, u_2$ and $u_3$ coincide if and only if they have the same monodromy matrices $M_1, M_2, M_3$, with respect to the same basis of the loops $\gamma_1, \gamma_2$ and $\gamma_3$.

Proof. Fix for example $\mu > 0$, i.e. $M_\infty$ upper-triangular. Suppose that there are two Fuchsian systems of the form (5.1) with the same poles $u_1, u_2$ and $u_3$, and the same value of $\mu$, the same monodromy matrices $M_1, M_2, M_3$, the same value of $R$. The fundamental matrices at $\infty$ of the form (5.3) exist. All the fundamental matrices of the form

$$Y_\infty = \left(1 + O\left(\frac{1}{z}\right)\right) z^{-A_\infty} z^R \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

with any $a \in \mathbb{C}$, give the same monodromy matrix at infinity. So, for a given $M_\infty$, the fundamental matrices at $\infty$ of the two Fuchsian system can be fixed as

$$Y_\infty^{(i)} = \left(1 + O\left(\frac{1}{z}\right)\right) z^{-A_\infty} z^R \begin{pmatrix} 1 & a^{(i)} \\ 0 & 1 \end{pmatrix}, \quad i = 1, 2$$

for some $a^{(1)}$ and $a^{(2)}$. For any choice of $a^{(1)}$ and $a^{(2)}$, the following matrix:

$$Y(z) := Y_\infty^{(2)}(z) Y_\infty^{(1)}(z)^{-1}.$$ 

$Y(z)$ is an analytic function around infinity:

$$Y(z) = 1 + O\left(\frac{1}{z}\right), \quad \text{as} \quad z \to \infty,$$

and, since the monodromy matrices coincide, $Y(z)$ is a single valued function on the punctured Riemann sphere $\mathbb{P} \setminus \{ u_1, u_2, u_3 \}$. As shown in [DM] Lemma 4.1, near the point $u_i$, we can choose the fundamental matrices $Y_i^{(1)}(z)$ and $Y_i^{(2)}(z)$ in such a way that

$$Y_\infty^{(1),(2)}(z) = Y_i^{(1),(2)}(z) C_i \quad i = 1, 2, 3.$$
with the same connection matrices $C_i$ and

$$Y_i = G_i \left(1 + \mathcal{O}(z - u_i)\right) (z - u_i)^J.$$ 

Then near the point $u_i$, $Y(z)$ is again analytic

$$Y(z) = G_i^{(2)} \left(1 + \mathcal{O}(z - u_i)\right) \left[G_i^{(1)} \left(1 + \mathcal{O}(z - u_i)\right)\right]^{-1}.$$ 

This proves that $Y(z)$ is an analytic function on all $\overline{\mathbb{C}}$ and then, by the Liouville theorem $Y(z) = 1$ and the two Fuchsian systems must coincide. The proof in the case of $\mu = 0$ is analogous. QED

**Remark 5.** The above argument fails for $R = 0$, $\mu \neq 0$. Indeed, the fundamental matrices at $\infty$ of the form (5.3), with $R = 0$, exist, but all the fundamental matrices of the form

$$Y_\infty = \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-A_\infty} B$$

with any constant matrix $B$, give the same monodromy matrix at infinity. Thus, chosen the fundamental matrices at $\infty$ of the two Fuchsian systems as

$$Y_\infty^{(i)} = \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{-A_\infty} B^{(i)}, \quad i = 1, 2,$$

for some constant matrices $B^{(1)}$ and $B^{(2)}$, the above defined matrix $Y(z)$ is no more an analytic function near infinity and thus the uniqueness is not assured.

The theory of the isomonodromy deformations is described by the following two results (see [Sch]):

**Theorem 6.** Let $M_1$, $M_2$, $M_3$ be the monodromy matrices of the Fuchsian system:

$$\frac{d}{dz} Y^0 = \left(\frac{A^0_1}{z - u^0_1} + \frac{A^0_2}{z - u^0_2} + \frac{A^0_3}{z - u^0_3}\right) Y^0,$$ 

(5.12)

of the above form (5.2), with $R \neq 0$, with pair-wise distinct poles, and with respect to some basis $\gamma_1, \gamma_2, \gamma_3$ of the loops in $\pi_1 \left(\overline{\mathbb{C} \setminus \{u^0_1, u^0_2, u^0_3, \infty\}}, \infty\right)$. Then there exists a neighborhood $U \subset \mathbb{C}^3$ of the point $u^0 = (u^0_1, u^0_2, u^0_3)$ such that, for any $u = (u_1, u_2, u_3) \in U$, there exists a unique triple $A_1(u), A_2(u), A_3(u)$ of analytic matrix valued functions such that:

$$A_i(u^0) = A_i^0, \quad i = 1, 2, 3,$$

and the monodromy matrices of the Fuchsian system

$$\frac{d}{dz} Y = A(z; u) Y = \left(\frac{A_1(u)}{z - u_1} + \frac{A_2(u)}{z - u_2} + \frac{A_3(u)}{z - u_3}\right) Y,$$ 

(5.13)
with respect to the same basis\(^1\) \(\gamma_1, \gamma_2, \gamma_3\) of the loops, coincide with the given \(M_1, M_2, M_3\). The matrices \(A_i(u)\) are the solutions of the Cauchy problem with the initial data \(A_i^0\) for the following Schlesinger equations:

\[
\frac{\partial}{\partial u_j} A_i = \frac{[A_i, A_j]}{u_i - u_j}, \quad \frac{\partial}{\partial u_i} A_i = -\sum_{j \neq i} \frac{[A_i, A_j]}{u_i - u_j}.
\] (5.14)

The solution \(Y_\infty^0(z)\) of (5.12), with the form (5.3), can be, uniquely continued, for \(z \notin U\), to an analytic function \(Y_\infty(z, u), u \in U\), such that

\[
Y_\infty(z, u^0) = Y_\infty^0(z).
\]

Moreover the functions \(A_i(u)\) and \(Y_\infty(z, u)\) can be continued analytically to global meromorphic functions on the universal coverings of

\[
\mathbb{C}^3 \backslash \{\text{diags}\} := \{(u_1, u_2, u_3) \in \mathbb{C}^3 \mid u_i \neq u_j \text{ for } i \neq j\},
\]

and

\[
\{(z, u_1, u_2, u_3) \in \mathbb{C}^4 \mid u_i \neq u_j \text{ for } i \neq j \text{ and } z \neq u_i, i = 1, 2, 3\},
\]

respectively.

The proof of this theorem can be found, for example, in [Mal], [Miwa], [Sib].

**Theorem 7.** Given three arbitrary non commuting matrices \(M_1, M_2, M_3\), satisfying (5.10) and (5.11), with \(M_\infty\) of the form (5.8), with \(R \neq 0\) and given a point \(u^0 = (u_1^0, u_2^0, u_3^0) \in \mathbb{C}^3 \backslash \{\text{diags}\}\), for any neighborhood \(U\) of \(u^0\), there exist \((u_1, u_2, u_3) \in U\) and a Fuchsian system of the form (5.1), with the given monodromy matrices, with the given \(R\), with poles in \(u_1, u_2, u_3\) and with a fixed value \(\mu\) such that \(\text{Tr}M_\infty = 2 \cos \pi \mu\).

Proof. First, observe that if the matrices \(M_1, M_2, M_3\), satisfying (5.10) and (5.11), with \(M_\infty\) of the form (5.8) with \(R \neq 0\) commute then they are all lower triangular or all upper triangular. Since their eigenvalues are all equal to one, \(M_\infty\) has eigenvalues equal to one, and thus \(\mu \in \mathbb{Z}\).

Now, consider three arbitrary matrices \(M_1, M_2, M_3\), satisfying (5.10) and (5.11), with \(M_\infty\) of the form (5.8), with \(R \neq 0\). In [Dek] it is proved that for any given point \(u^0 = (u_1^0, u_2^0, u_3^0) \in \mathbb{C}^3 \backslash \{\text{diags}\}\), and for any neighborhood \(U\) of \(u^0\), there exist \((u_1, u_2, u_3) \in U\) and a Fuchsian system

\[
\frac{d}{dz} Y = \left(\frac{A_1}{z - u_1} + \frac{A_2}{z - u_2} + \frac{A_3}{z - u_3}\right) Y, \quad z \in \overline{\mathbb{C}} \backslash \{u_1, u_2, u_3, \infty\},
\]

with the given monodromy matrices, and with \(\mu\) fixed up to \(\mu \to \mu + n, n \in \mathbb{Z}\).

\(^1\) Observe that the basis \(\gamma_1, \gamma_2, \gamma_3\) of \(\pi_1(\overline{\mathbb{C}} \backslash \{u_1, u_2, u_3, \infty\}, \infty)\) varies continuously with small variations of \(u_1, u_2, u_3\). This new basis is homotopic to the initial one, so we can identify them.
We want to build two gauge transformations which map the obtained Fuchsian system of the form (5.1), with some given non-commuting monodromy matrices and some value of $\mu$, to another Fuchsian system of the same form with the same monodromy matrices and with the value $-\mu$ and $\mu + 1$ respectively.

For $\mu \neq 0$ the constant gauge transformation
\[ G = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
is such that the new Fuchsian system with $\tilde{A}_i = G^{-1}A_i G$, has the same monodromy matrices $M_1, M_2, M_3$ and
\[ \tilde{A}_\infty = \begin{pmatrix} -\mu & 0 \\ 0 & \mu \end{pmatrix}. \]
So, the above gauge transformation maps the obtained Fuchsian system correspondent to the given monodromy matrices and some value of $\mu \neq 0$ to another Fuchsian system of the same form with the same monodromy matrices and with the value $-\mu$. Now, we want to build the analogous gauge transformation mapping $\mu$ to $\mu + 1$.

First, observe that the matrices $A_i$ can be parameterized as follows
\[ A_i = \begin{pmatrix} a_i b_i & -b_i^2 \\ a_i^2 & -a_i b_i \end{pmatrix}, \tag{5.15} \]
for some $a_i, b_i \in \mathbb{C}$, $i = 1, 2, 3$, with

for $\mu \neq 0$, \[ \sum_{i=1}^{3} a_i b_i = -\mu, \quad \sum_{i=1}^{3} a_i^2 = \sum_{i=1}^{3} b_i^2 = 0 \]
for $\mu = 0$, \[ \sum_{i=1}^{3} a_i b_i = 0, \quad \sum_{i=1}^{3} a_i^2 = 0, \quad \sum_{i=1}^{3} b_i^2 = 1. \]

If $a_i = 0$ (or $b_i = 0$) for every $i = 1, 2, 3$, then all the matrices $A_i$ are upper (resp. lower) triangular, then the matrices $M_1, M_2, M_3$ are upper (resp. lower) triangular, and thus commuting. So, for a triple of non-commuting monodromy matrices at least one of the $a_i$ and one of the $b_i$ must be different from zero. Moreover, for $R \neq 0$, and for every $\mu$, \[ \sum_{i=1}^{3} a_i^2 u_i = 0. \] In fact, if $\mu = -\frac{1}{2}$, \[ \sum_{i=1}^{3} a_i^2 u_i = R_{21} \neq 0. \] For $\mu \neq -\frac{1}{2}$, if $\sum_{i=1}^{3} a_i^2 u_i = 0$ then, being $a_2^2 = -a_2^3 - a_3^2$, one obtains $a_2^2 = -a_3^2 \frac{u_3 - u_1}{u_2 - u_1}$ and thus
\[ \frac{\partial}{\partial u_1} a_2^2 = \frac{u_3 - u_1}{u_2 - u_1} \frac{\partial}{\partial u_1} a_3^2 - a_3^2 \frac{u_3 - u_2}{(u_2 - u_1)^2}. \]

By the Schlesinger equations
\[ 2a_1 b_1 a_2^2 - 2a_2 b_2 a_1^2 = 2a_3 b_3 a_2^2 - 2a_1 b_1 a_2^2 - a_3^2 \frac{u_3 - u_2}{u_2 - u_1}, \]

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and imposing \( \sum a_i^2 = 0, \sum a_i b_i = 0 \), one obtains
\[
2a_1^2 \mu + a_3^2 \frac{u_3 - u_2}{u_2 - u_1} = 0
\]
that for \( \mu = 0 \) leads to \( a_3^2 = 0 \) and thus \( a_2 = 0 \) and \( a_1 = 0 \), for \( \mu \neq 0 \) leads to
\[
a_1^2 = -\frac{a_3^2}{2 \mu} \frac{u_3 - u_2}{u_2 - u_1}.
\]
Imposing \( \sum_{i=1}^3 a_i^2 = 0 \), one obtains for \( \mu \neq 0 \)
\[
a_3^2 \frac{(1 + 2 \mu)(u_3 - u_2)}{2 \mu(u_2 - u_1)} = 0,
\]
that for \( \mu \neq -\frac{1}{2} \) implies \( a_3^2 = 0 \) and thus \( a_2 = 0 \) and \( a_1 = 0 \). Analogously, one can show that for \( \mu \neq 0 \) and \( R \neq 0, \sum b_i^2 u_i \neq 0 \).

For \( \mu \neq -1, 0, -\frac{1}{2} \) the gauge transform \( Y = G(z) \tilde{Y} \) with
\[
G(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} \frac{a}{b} & b \\ -\frac{1}{b} & 0 \end{pmatrix}
\]
for \( b = \frac{2 \mu + 1}{\sum_{i=1}^3 a_i^2 u_i} \) and \( a = -\frac{b^2}{2(1+\mu)} \left( \frac{2}{b} \sum_{i=1}^3 a_i b_i u_i + \sum_{i=1}^3 a_i^2 u_i^2 \right) \) is well defined because as observed above \( \sum_{i=1}^3 a_i^2 u_i \neq 0 \) and it is such that the new Fuchsian system
\[
\frac{d}{dz} \tilde{Y} = \left( \frac{\tilde{A}_1}{z - u_1} + \frac{\tilde{A}_2}{z - u_2} + \frac{\tilde{A}_3}{z - u_3} \right) \tilde{Y},
\]
with \( \tilde{A}_i = G(u_i)^{-1} A_i G(u_i) \), has the same monodromy matrices \( M_1, M_2, M_3 \) and
\[
\tilde{A}_\infty = \begin{pmatrix} \mu + 1 & 0 \\ 0 & -\mu - 1 \end{pmatrix}.
\]
So, the above gauge transformation maps the obtained Fuchsian system corresponding to the given monodromy matrices and some value of \( \mu \neq 0, -1, -\frac{1}{2} \) to another Fuchsian system of the same form with the same monodromy matrices and with the value \( \mu + 1 \).

In this way, all the half-integer values and all the non-zero integer values of the index \( \mu \) are related via some gauge transformation. To conclude the proof, one has to consider the case of \( \mu = 0 \). For a triple of non-commuting monodromy matrices with \( \mu = 0 \), the gauge transformation \( Y = G(z) \tilde{Y} \) with
\[
G(z) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} z + \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & 0 \end{pmatrix}
\]
with \( g_{21} = -\sum a_i^2 u_i, g_{11} = \frac{1}{2} \left( g_{21} - 2 \sum_{i=0}^3 a_i b_i u_i + \frac{1}{g_{21}} \sum_{i=0}^3 a_i^2 u_i^2 \right) \), is well defined and it maps the Fuchsian system corresponding to the given triple of monodromy matrices to
a new Fuchsian system with \[ \tilde{A}_i = G(u_i)^{-1} \mathcal{A}_i G(u_i), \] with the same monodromy matrices \( M_1, M_2, M_3 \) and
\[ \tilde{A}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

In the same way, the gauge transformation \( Y = G(z) \tilde{Y} \) with
\[ G(z) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} z + \begin{pmatrix} g_{12} \\ g_{21} \\ g_{22} \end{pmatrix} \]
\[ g_{12} = -\sum b_i^2 u_i, \quad g_{21} = \sum_{i=0}^3 a_i b_i u_i - \frac{1}{g_{12}} \sum_{i=0}^3 b_i^2 u_i^2, \] and any \( g_{22} \neq 0 \), is well defined and it maps any Fuchsian system with
\[ \tilde{A}_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
corresponding to the given triple of non-commuting monodromy matrices to a new Fuchsian system with \( \tilde{A}_i = G(u_i)^{-1} \mathcal{A}_i G(u_i) \), with the same monodromy matrices \( M_1, M_2, M_3 \) and
\[ \tilde{A}_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

This concludes the proof of the theorem. QED

**Remark 6.** Existence statements of Theorems 6 and 7 can be proved also for triples of monodromy matrices such that \( R = 0 \), but as stressed in Remark 5, uniqueness is lost.

Let me now explain, following [JMU], how to rewrite the Schlesinger equations (5.2) in terms of the PVI\( \mu \) equation. Observe that, for \( \mu \neq 0 \), the Schlesinger equations (5.2) with fixed \( \mathcal{A}_\infty \) are invariant with respect to the gauge transformations of the form:
\[ \mathcal{A}_i \mapsto D^{-1} \mathcal{A}_i D, \quad i = 1, 2, 3, \text{ for any } D \text{ diagonal matrix.} \quad (5.16) \]

Such a diagonal conjugation changes the value of \( R \). So, we introduce two coordinates \((p, q)\) on the quotient of the space of the matrices satisfying (5.14) with respect to the equivalence relation (5.16) and a coordinate \( k \) that takes account of the changes of \( R \) due to the above diagonal conjugations. The coordinate \( q \) is the root of the following linear equation:
\[ [\mathcal{A}(q; u_1, u_2, u_3)]_{12} = 0, \]
and \( p \) and \( k \) are given by:
\[ p = [\mathcal{A}(q; u_1, u_2, u_3)]_{11}, \quad k = [\mathcal{A}(z; u_1, u_2, u_3)]_{12} \frac{P(z)}{\mu(q - z)}, \]
where \( \mathcal{A}(z; u_1, u_2, u_3) \) is given in (5.13) and \( P(z) = (z - u_1)(z - u_2)(z - u_3) \). The matrices \( \mathcal{A}_i \) are uniquely determined by the coordinates \((p, q)\), and \( k \) and expressed rationally in
terms of them:

\[
\begin{align*}
(A_i)_{11} &= -(A_i)_{22} = \frac{q - u_i}{2\mu P'(u_i)} \left[ P(q)p^2 + 2\mu \frac{P(q)}{q - u_i} p + \mu^2(q + 2u_i - \sum_j u_j) \right], \\

(A_i)_{12} &= -\mu k \frac{q - u_i}{P'(u_i)} , \\

(A_i)_{21} &= k^{-1} \frac{q - u_i}{4\mu^3 P'(u_i)} \left[ P(q)p^2 + 2\mu \frac{P(q)}{q - u_i} p + \mu^2(q + 2u_i - \sum_j u_j) \right]^2,
\end{align*}
\]

for \( i = 1, 2, 3 \), where \( P'(z) = \frac{dP}{dz} \).

The Schlesinger equations (5.14) in the \((p, q, k)\) variables are

\[
\begin{align*}
\frac{\partial q}{\partial u_i} &= \frac{P(q)}{P'(u_i)} \left[ 2p + \frac{1}{q - u_i} \right] , \\

\frac{\partial p}{\partial u_i} &= -\frac{P'(q)p^2 + (2q + u_i - \sum_j u_j)p + \mu(1 - \mu)}{P'(u_i)},
\end{align*}
\]

and

\[
\frac{\partial \log(k)}{\partial u_i} = (2\mu - 1) \frac{q - u_i}{P'(u_i)} ,
\]

for \( i = 1, 2, 3 \). The system of the reduced Schlesinger equations (5.18) is invariant under the transformations of the form

\[
u_i \mapsto a u_i + b, \quad q \mapsto aq + b, \quad p \mapsto \frac{p}{a}, \quad \forall a, b \in \mathbb{C}, \quad a \neq 0.\]

We introduce the following new invariant variables:

\[
x = \frac{u_2 - u_1}{u_3 - u_1}, \quad y = \frac{q - u_1}{u_3 - u_1};
\]

the system (5.18), expressed in these new variables, reduces to the PVI\(\mu\) equation for \( y(x) \).

The reduced Schlesinger equations admit the following singular solutions

\[
q = u_i \quad \text{for} \quad i = 1, 2, 3.
\]

In [DM] is shown that \( q = u_i \) if and only if the matrix \( A_i \) is identically equal to 0, or, equivalently, the monodromy matrix \( M_i \) is equal to the identity. The singular solutions do not give any solution of the PVI\(\mu\) equation, while all the other solutions of the reduced Schlesinger equations do.

For \( \mu = 0 \), the matrices \( A_i \) can be parameterized as in (5.15) for some \( a_i, b_i \in \mathbb{C} \). One can introduce the coordinates \((p, q)\) as above and obtain:

\[
a_1^2 = \frac{p^2 P(q)}{\Delta} (u_3 - u_2)(q - u_1), \quad b_1^2 = 0, \quad a_1 b_1 = 0,
\]
\[ a_2^2 = \frac{p^2 P(q)}{\Delta} (u_1 - u_3)(q - u_2), \quad b_2^2 = \frac{q - u_3 u_2 - u_1}{q - u_1 u_2 - u_3}, \quad a_2 b_2 = p \frac{(q - u_2)(q - u_3)}{u_2 - u_3}, \]

\[ a_3^2 = \frac{p^2 P(q)}{\Delta} (u_2 - u_1)(q - u_3), \quad b_3^2 = \frac{q - u_2 u_3 - u_1}{q - u_1 u_3 - u_2}, \quad a_3 b_3 = -p \frac{(q - u_2)(q - u_3)}{u_2 - u_3}, \]

where \( P(q) = (q - u_1)(q - u_2)(q - u_3) \) and \( \Delta = (u_3 - u_2)(u_3 - u_1)(u_2 - u_1) \). Introducing the variables \((y, x)\) as above, it is straightforward to verify that the Schlesinger equations for the matrices \((5.15)\) are satisfied iff \( \mu \) is zero.

Observe that the Schlesinger equations for the matrices \((5.15)\) admit the trivial solutions \( a_i = 0, \forall \, i = 1, 2, 3 \) or \( b_i = 0, \forall \, i = 1, 2, 3 \) corresponding respectively to the triple of commuting monodromy matrices

\[
M_i = \begin{pmatrix} 1 & -2\pi i b_i^2 \\ 0 & 1 \end{pmatrix}, \quad \text{or} \quad M_i = \begin{pmatrix} 1 & 0 \\ 2\pi i a_i^2 & 1 \end{pmatrix}.
\]

All the triples of commuting monodromy matrices can be realized by a trivial solution to the Schlesinger equations for \( \mu = 0 \). For such trivial solutions the coordinate \( q \) is not defined. By the way, one cannot say that the triples of commuting monodromy matrices do not correspond to any solution of PVI\(_{\mu=0}\). Indeed this equation coincides with PVI\(_{\mu=1}\) and it is not excluded that particular triples of commuting monodromy matrices are realized by non-trivial Fuchsian system with \( \mu = 1 \) (in Section 6, I show that there exists a unique triple of this kind and determine it).

Till now I supposed \( R \neq 0 \), because the uniqueness of the correspondence between triples of monodromy matrices and Fuchsian systems with a given set of poles is not assured for \( R = 0 \). Thus I treat the case of \( R = 0 \) separately.

**Lemma 11.** For half integer values of \( \mu \), the equation \( R = 0 \) is satisfied if and only if the reduced Schlesinger equations give rise to Chazy solutions for any \( \mu \neq \frac{1}{2} \) or to the singular solution \( q = \infty \) for \( \mu = \frac{1}{2} \).

Proof. Consider the case \( \mu = -\frac{1}{2} \) (as shown in Section 4, all the other cases with half integer \( \mu \neq \frac{1}{2} \) are equivalent to it). The equation \( R = 0 \) is satisfied iff

\[
(A_1 u_1 + A_2 u_2 + A_3 u_3)_{21} = 0.
\]

Writing the above equation in terms of \( y, y' \) and \( x \), one realizes that it coincides with the equation \( Q(y, y', x) = 0 \) which is satisfied only by the Chazy solutions.

In the case \( \mu = \frac{1}{2} \), the equation \( R = 0 \) is satisfied iff

\[
(A_1 u_1 + A_2 u_2 + A_3 u_3)_{12} = 0,
\]

that leads to the singular solution \( q = \infty \). In fact, in the equation for \( q \)

\[
(A_1 (u_2 + u_3) + A_2 (u_1 + u_3) + A_3 (u_1 + u_2))_{12} q = (A_1 u_2 u_3 + A_2 u_1 u_3 + A_3 u_1 u_2)_{12},
\]

the coefficient of \( q \) is zero and the right-hand side is non-zero because \((A_i)_{12} \neq 0, \forall \, i = 1, 2, 3 \). In fact if one of the \((A_i)_{12}\) is zero then, being \( \sum (A_i)_{12} = 0 \), for \( R = 0 \) all of them are 0. Requiring that the determinant of the matrices \( A_i \) is zero, one obtains that also the elements \((A_i)_{11}\) are zero, that is \( \mu = 0 \) that leads a contradiction. \( \text{QED} \)
Lemma 12. The equation $R = 0$ is not satisfied on any solution of the reduced Schlesinger equations for integer $\mu$.

Proof. Consider the cases of integer $\mu$. They can all be treated as the case $\mu = 1$. In fact the case $\mu = 0$ gives rise to the same PVI equation as the case $\mu = 1$. Moreover all the other integer values of $\mu$ are related to $\mu = 1$ via birational canonical transformations of the form (4.1), the denominator of which never vanishes. For $\mu = 1$ the equation $R = 0$ in the $(p, q)$ coordinates is

$$R_{12} = k \left[ p^2 (q - u_1)(q - u_2)(q - u_3) - 2q + 2u_1 + 2u_2 + 2u_3 \right], \quad (5.21)$$

which is never zero on the solutions of the reduced Schlesinger equations. In fact from (5.21), one obtains:

$$p^2 = \frac{2(q - u_1 - u_2 - u_3)}{(q - u_1)(q - u_2)(q - u_3)}.$$

Differentiating both sides with respect to $u_i$ for all $i = 1, 2, 3$, and substituting the reduced Schlesinger equation for $\frac{\partial p}{\partial u_i}$, one obtains

$$p = -\frac{q - u_1 - u_2}{(q - u_1)(q - u_2)} = -\frac{q - u_1 - u_3}{(q - u_1)(q - u_3)} = -\frac{q - u_3 - u_2}{(q - u_3)(q - u_2)},$$

which can be satisfied only for $u_1 = u_2 = u_3$. QED

As shown above, the case $R = 0$ can be realized only for half integer $\mu$. It gives rise to the singular solution $q = \infty$ in the case of $\mu = \frac{1}{2}$ and to the Chazy solutions in the case of half-integer $\mu \neq \frac{1}{2}$.

We resume the results of the section in the following:

Theorem 8. Given any triple of non-commuting monodromy matrices $M_1, M_2, M_3$ satisfying (5.10) and (5.11) with $M_\infty$ given by (5.8), for $R \neq 0$, none of them being equal to 1, considered modulo diagonal conjugations, there exists unique branch of a non-Chazy solution to the PVI$_\mu$ equation near a given point $x_0 \in \mathbb{C} \setminus \{0, 1, \infty\}$ which defines a Fuchsian system of the form (5.1) with the prescribed monodromy matrices $M_1, M_2, M_3$. Vice versa, given any branch of a non-Chazy solution to the PVI$_\mu$ equation near a given point $x_0 \in \mathbb{C} \setminus \{0, 1, \infty\}$, the correspondent triple of monodromy matrices $M_1, M_2, M_3$ satisfying (5.10) and (5.11) with $M_\infty$ given by (5.8), for $R \neq 0$, none of them being equal to 1 is unique modulo diagonal conjugations.

Observe that permutations of the poles $u_i$ induce transformations of $(y, x)$ of the type $x \rightarrow 1 - x$, $y \rightarrow 1 - y$ and $x \rightarrow \frac{1}{x}$ and $y \rightarrow \frac{y}{x}$ and their compositions. These transformations preserve the PVI$_\mu$ equation.
6. The structure of the analytic continuation.

We parameterized branches of the non-Chazy solutions of PVI$\mu$ by triples of monodromy matrices. Recall that, according to Theorem 6, the solutions of PVI$\mu$, defined in a neighborhood of a given point $x_0 \in \mathbb{C}\setminus\{0, 1, \infty\}$, can be analytically continued to a meromorphic function on the universal covering of $\mathbb{C}\setminus\{0, 1, \infty\}$. In [DM] it is shown that the procedure of the analytic continuation is described by the action of the pure braid group with three strings (see [Bir]), $P_3 = \pi_1(\mathbb{C}^3\setminus\{diags\}; u^0)$, on $\pi_1(\mathbb{C}\setminus\{0, 1, \infty\})$. To simplify the computations, the procedure of the analytic continuation is extended to the full braid group $B_3$ that admits a presentation with generators $\beta_1$ and $\beta_2$ shown in figure 2, and defining relations $\beta_1 \beta_2 \beta_1 = \beta_2 \beta_1 \beta_2$.

![Fig.5. The geometric representation of the generators of the braid group $B_3$.](image)

The action of $B_3$ on the generating loops of $\pi_1(\mathbb{C}\setminus\{0, 1, \infty\})$ can be expressed in terms of monodromy matrices:

**Lemma 13.** For the generators $\beta_1, \beta_2$ shown in the figure 2, the matrices $M_i^\beta$ have the following form:

\[
M_1^{\beta_1} = M_2, \quad M_2^{\beta_1} = M_2 M_1 M_2^{-1}, \quad M_3^{\beta_1} = M_3, \quad (6.1)
\]

\[
M_1^{\beta_2} = M_1, \quad M_2^{\beta_2} = M_3, \quad M_3^{\beta_2} = M_3 M_2 M_3^{-1}. \quad (6.2)
\]

The proof can be found in [DM].

The action (6.1), (6.2) of the braid group on the triples of monodromy matrices commutes with the diagonal conjugation of them; moreover the class of the singular solutions is closed under the analytic continuation. In fact if some of the matrices $M_i$ is equal to 1, then for any $\beta$ there is a $j$ such that $M_j^\beta = 1$. Moreover $R$ is preserved and the class of non-Chazy solutions is closed under the analytic continuation. As a consequence the structure of the analytic continuation of the non-Chazy solutions of the PVI$\mu$ equation is determined by the action (6.1), (6.2) of the braid group on the triples of monodromy matrices.

I want to introduce a parameterization of the monodromy matrices and write this action in terms of the parameters in the space of the monodromy data. I follow the same procedure of [DM], suitably modified due to the resonant value of $\mu$. The following lemmas can be proved as in [DM] (see lemma 1.4 and 1.5).

**Lemma 14.** Let $M_1, M_2$ and $M_3$ be three linear non commuting operators $M_i : \mathbb{C}^2 \to \mathbb{C}^2$ satisfying (5.10) and (5.11), with $M_\infty$ given by (5.8) with $R \neq 0$. If two of the following numbers

\[
\text{Tr}(M_1 M_2), \quad \text{Tr}(M_1 M_3), \quad \text{Tr}(M_3 M_2) \quad (6.3)
\]
are equal to 2, then one of the matrices of $M_i$ is equal to one.

**Lemma 15.** Let $M_1, M_2, M_3$ as in lemma 14.

i) If $\text{Tr}(M_1M_2) \neq 2$, then there exists a basis in $\mathbb{C}^2$ such that, in this basis, the matrices $M_1, M_2$ and $M_3$ have the form

$$
M_1 = \begin{pmatrix}
1 & -x_1 \\
0 & 1
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
1 & 0 \\
x_1 & 1
\end{pmatrix}, \quad M_3 = \begin{pmatrix}
1 + \frac{x_2 x_3}{x_1} & -\frac{x_1^2}{x_1} \\
\frac{x_3}{x_1} & 1 - \frac{x_2 x_3}{x_1}
\end{pmatrix},
$$

where

$$
\text{Tr}(M_1M_2) = 2 - x_1^2, \quad \text{Tr}(M_3M_2) = 2 - x_2^2, \quad \text{Tr}(M_1M_3) = 2 - x_3^2,
$$

and

$$
x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4\sin^2 \pi \mu. \quad (6.5)
$$

ii) If two triples of matrices $M_1, M_2, M_3$ and $M_1', M_2', M_3'$, satisfying (5.11), with none of them equal to 1, have the form (6.4) with parameters $(x_1, x_2, x_3)$ and $(x_1', x_2', x_3')$ respectively, then these triples are conjugated

$$
M_i = T^{-1}M'_iT
$$

with some invertible matrix $T$, if and only if the triple $(x_1', x_2', x_3')$ is equal to the triple $(x_1, x_2, x_3)$, up to the change of the sign of two of the coordinates.

Let me stress that for $\mu \in \mathbb{Z}$ the monodromy matrices can commute, i.e. they can be of the form (5.20). The action (6.1), (6.2) of the braid group simply permutes them. Thus the action (6.1), (6.2) of the braid group does not mix the commuting triples (5.20) with the ones admitting the parameterization (6.3).

**Lemma 16.** There exists only a one-parameter family of triples of commuting monodromy matrices which give rise to solutions to PVI$\mu$ equation. They are (up to diagonal conjugations)

$$
M_1 = \begin{pmatrix}
1 & i\pi a \\
0 & 1
\end{pmatrix}, \quad M_2 = \begin{pmatrix}
1 & i\pi(1-a) \\
0 & 1
\end{pmatrix}, \quad M_3 = \begin{pmatrix}
1 & i\pi \\
0 & 1
\end{pmatrix},
$$

where $a$ is a parameter. The correspondent solutions to PVI$_{\mu=1}$ consist of a one parameter family of rational solutions of the form:

$$
y(x) = \frac{ax}{1-(1-a)x}, \quad \text{for } a \neq 0. \quad (6.7)
$$

Proof. Consider a triple of commuting monodromy matrices. As shown above, they are necessarily of the form (5.20), i.e. either all upper triangular or all lower triangular. Then the corresponding Fuchsian system admits a single-valued solution $Y(z)$. For $\mu = 1$ (as
shown in the proof of Lemma 12, all the other cases with integer \( \mu \) are equivalent to this) such solution has only a pole of order one at infinity, i.e.

\[
Y(z) = \begin{pmatrix} az + b \\ cz + d \end{pmatrix}.
\]

for some \( a, b, c, d \in \mathbb{C} \). Substituting \( Y \) in the Fuchsian system, one obtains \( a = 0, b = ck, d = \frac{q}{2}(q-u_1-u_2-u_3) \) and \( c \neq 0 \) iff \( p \equiv 0 \). By direct substitution in the reduced Schlesinger equations, one can compute \( q \) and determine the explicit form of Fuchsian system. Thus it is straightforward to compute the monodromy matrices, which turn out to have the form (6.6). Their orbit under the action of the braid group (6.1), (6.2) consists of one point, up to permutations. Thanks to Theorem 8, the correspondent solution of PVI\( \mu, \; \mu \in \mathbb{Z} \) consists only of one branch, i.e. it is rational and it is easy to see that, being \( p \equiv 0 \) it has the form (6.7). QED

For the case when at most one of the numbers (6.3) is equal to 2, or equivalently the monodromy matrices do not commute, one gives the following:

**Definition:** a triple \((x_1, x_2, x_3)\) is called *admissible* if it has at most one coordinate equal to zero. Two triples are called *equivalent* if they are equal up to the change of two signs of the coordinates. The equivalence class of a triple \((x_1, x_2, x_3)\) is denoted by \([x_1, x_2, x_3]\)

Observe that for an admissible triple \((x_1, x_2, x_3)\), none of the matrices (6.4) is equal to the identity. So the admissible triples give rise to non-singular solutions of the reduced Schlesinger equations. Moreover two equivalent triples such that \( R \neq 0 \) generate the same solution.

**Theorem 9.** In the case of \( \mu \in \mathbb{Z} \) there exists a one parameter family of rational solutions of the form (6.7). The one-parameter family of Chazy solutions in the case of half-integer \( \mu, \; \mu \neq \frac{1}{2} \), corresponds to \([2, 2, 2]\). All the other solutions of PVI\( \mu, \; \mu \in \mathbb{Z} \), have branches which, near a given point \( x_0 \in \overline{\mathbb{C}}\setminus\{0, 1, \infty\} \), are in one-to-one correspondence with the equivalence classes of the admissible triples satisfying (6.5). The structure of the analytic continuation of them is determined by the action

\[
\beta_1 : (x_1, x_2, x_3) \mapsto (-x_1, x_3 - x_1 x_2, x_2),
\]

\[
\beta_2 : (x_1, x_2, x_3) \mapsto (x_3, -x_2, x_1 - x_2 x_3).
\] (6.8)

of the braid group on the triples \((x_1, x_2, x_3)\).

Proof. The first claim is proved in Lemma 16. The second claim follows from the fact that as proved in Lemma 11, Chazy solutions correspond to the case \( R = 0 \), i.e. to \( M \infty = -\mathbf{1} \). Since \( M \infty \) is invariant with respect to conjugations, it must be equal to \(-\mathbf{1}\) also in the canonical form (6.4). Solving the equations in \((x_1, x_2, x_3)\), one obtains that the triple of monodromy matrices is given by

\[
M_1 = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}
\] (6.9)

The third and fourth claim of this theorem were proved in [DM]. QED
7. Solutions of the PVI$_\mu$ equation having finite branching and their monodromy data.

In this section I classify all the monodromy data corresponding to solutions with a finite number of branches. The solutions corresponding to commuting triples of monodromy matrices were already found in Lemma 16. All the other triples admit the parameterization (6.4), and the strategy is essentially the same of [DM]. I recall here the ideas and results of [DM] omitting all the proofs.

Let $y(x)$ be a finite-branching solution. According to the Painlevé property, the ramification points of $y(x)$ are allowed to lie only at 0, 1, $\infty$ and the correspondent monodromy matrices, defined modulo diagonal conjugations, have a finite orbit under the action of the braid group (6.1), (6.2).

**Lemma 17.** An admissible triple $(x_1, x_2, x_3) \notin [2, 2, 2]$, specifies a finite-branching solution of PVI$_\mu$, for $\mu$ given by (6.5), if and only if its orbit, under the action (6.8) of the braid group, is finite.

**Remark 7.** Observe that even if the orbit of the triple $(2, 2, 2)$ consists only of one point, up to equivalence, it gives rise to a one parameter family of transcendental solutions (the Chazy solutions). This is not surprising because this triple corresponds to the case $R = 0$. As stressed in Lemma 10, the uniqueness of the Fuchsian system associated to triples of monodromy matrices in not assured in this case. Observe that the first term of the asymptotic behaviour of the Chazy solutions does not depend on the parameter $\nu$. This fact could be related to the fact that the monodromy matrices associated to the Chazy solutions do not depend on $\nu$.

Due to Lemma 17, the problem of the classification of the regular solutions of the PVI$_\mu$ reduces again to the problem of the classification of all the finite orbits of the action (6.8) under the braid group in the three dimensional space. The following simple necessary condition for a triple $(x_1, x_2, x_3)$ to belong to a finite orbit is proved in [DM].

**Lemma 18.** Let $(x_1, x_2, x_3)$ be a triple belonging to a finite orbit. Then:

$$x_i = -2 \cos \pi r_i, \quad r_i \in \mathbb{Q}, \quad 0 \leq r_i \leq 1, \quad i = 1, 2, 3,$$

(7.1)

here $\mathbb{Q}$ is the set of the rational numbers.

**Remark 8.** Thanks to the above lemma, for the finite orbits of the braid group, it is equivalent to deal with the triples $(x_1, x_2, x_3)$, or with the triangles of angles $(\pi r_1, \pi r_2, \pi r_3)$, with $x_i = -2 \cos \pi r_i$ and $0 \leq r_i \leq 1, \ r_i \in \mathbb{Q}$.

7.1. **Classification of the monodromy data corresponding to finite-branching solutions: case of $\mu \in \mathbb{Z}$.** In this case the classification theorem of [DM] (see Theorem 1.6) is still valid, so there are no triples $(x_1, x_2, x_3)$ having a finite orbit. As a consequence, the only finite-branching solutions are the ones corresponding to triples of monodromy matrices admitting the parameterization (5.20).
Lemma 19. The only finite-branching solutions of PVI$_\mu$, for $\mu = 1$, consist of a one-parameter family of rational solutions of the form (6.7).

All the other cases of PVI$_\mu$, for $\mu \in \mathbb{Z}\backslash\{0\}$ can be obtained from it via birational canonical transformations (4.1), the denominator of which never vanishes. The case $\mu = 0$ is the same as $\mu = 1$.

7.2. Classification of the triples $(x_1, x_2, x_3)$ corresponding to finite-branching solutions: case of half-integer $\mu$. For $\mu + \frac{1}{2} \in \mathbb{Z}$, the triples $(x_1, x_2, x_3)$ satisfy

$$x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3 = 4,$$

and the correspondent triangle is flat, $r_1 + r_2 + r_3 = 1$.

Lemma 20. For every admissible triple $(x_1, x_2, x_3)$, $x_i = -2\cos \pi r_i$, with $r_i \in \mathbb{Q}$, the correspondent solution of PVI$_\mu$, for $\mu$ half-integer, is finite-branching.

Proof. The action of the braid group on flat triangles can be written in the form

$$\beta_1 : (r_1, r_2, r_3) \mapsto (|1 - r_1|, |1 - r_2|, r_2),$$

$$\beta_2 : (r_1, r_2, r_3) \mapsto (r_3, |1 - r_2|, |r_3 - r_2|).$$

(7.2)

As a consequence, it maps triangles with rational angles to triangles with rational angles. Moreover all the orbits are finite. In fact, let $r_i = \frac{\tilde{p}_i}{q_i}$, for $p_i, q_i \in \mathbb{Z}$, $p_i < q_i$, $i = 1, 2, 3$ and $n$ be the smallest common factor of $q_1, q_2, q_3$. The action of the braid group (7.2) does not increase $n$, and all the images of $(r_1, r_2, r_3)$ have the form $\left(\frac{\tilde{p}_1}{n}, \frac{\tilde{p}_2}{n}, \frac{\tilde{p}_3}{n}\right)$, with $\tilde{p}_i < n$. The number of possible triples of this kind is trivially finite. QED

Recall that since the Chazy solutions are transcendental, the finite-branching solutions in the case of half-integer $\mu$, are necessarily of Picard type, and thus they have asymptotic behaviour of algebraic type, see (2.5). In the following Proposition, I relate the parameters $\nu_{1,2}$ of Picard solutions to the triples $(r_1, r_2, r_3)$.

Proposition 2. i) The monodromy matrices corresponding to a solution $y(x)$ of the form (2.1) are given by

$$M_1 = \begin{pmatrix} 1 & -2\cos\left(\frac{\pi \nu_2}{2}\right) \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & 0 \\ 2\cos\left(\frac{\pi \nu_2}{2}\right) & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 + \frac{2\cos\left(\frac{\pi \nu_1}{2}\right)\cos\left(\frac{\pi (\nu_1 - \nu_2)}{2}\right)}{\cos\left(\frac{\pi \nu_2}{2}\right)} & -2\cos\left(\frac{\pi \nu_2}{2}\right) \\ \frac{2\cos\left(\frac{\pi (\nu_1 - \nu_2)}{2}\right)}{\cos\left(\frac{\pi \nu_2}{2}\right)} & 1 - \frac{2\cos\left(\frac{\pi \nu_1}{2}\right)\cos\left(\frac{\pi (\nu_1 - \nu_2)}{2}\right)}{\cos\left(\frac{\pi \nu_2}{2}\right)} \end{pmatrix}.$$

That is the parameters $\nu_{1,2}$ and the triples $(x_1, x_2, x_3)$ are related as follows:

$$r_1 = \frac{\nu_2}{2}, \quad r_2 = 1 - \frac{\nu_1}{2}, \quad r_3 = \frac{\nu_1 - \nu_2}{2}, \quad \text{for} \quad \nu_1 > \nu_2,$$

$$r_1 = 1 - \frac{\nu_2}{2}, \quad r_2 = \frac{\nu_1}{2}, \quad r_3 = \frac{\nu_2 - \nu_1}{2}, \quad \text{for} \quad \nu_1 < \nu_2,$$

(7.3)
and viceversa

\[ \nu_1 = 2 - 2r_2, \quad \nu_2 = 2r_1, \quad (7.4) \]

where \( x_i = -2 \cos \pi r_i \). ii) All the finite-branching solutions are indeed algebraic. iii) The action of the braid group \( B_3 \) (pure braid group \( P_3 \)) on \( (x_1, x_2, x_3) \) corresponds to the action of \( \Gamma (\Gamma(2)) \) on \( (\nu_1, \nu_2) \).

Proof. i) The relation between the parameters \( \nu_1, \nu_2 \) and the exponents \( (l_0, l_1, l_\infty) \) of the asymptotic behaviour was derived in Lemma 2. It remains to find the relation between the exponents and the triangles. For \( l_i \neq 0 \) for any \( i = 0, 1, \infty \), this was already done in [DM] for PVI\( \mu, 2\mu \notin \mathbb{Z} \). This relation is

\[ l_i = \begin{cases} 2r_i & \text{for } 0 < r_i \leq \frac{1}{2}, \\ 2 - 2r_i & \text{for } \frac{1}{2} \geq r_i < 1. \end{cases} \quad (7.5) \]

This result can be extended to the case of \( \mu = \frac{1}{2} \). Indeed, the procedure of reduction of the Fuchsian system to the systems \( \tilde{\Sigma} \) and \( \tilde{\Sigma} \) of [DM] is the same. They are again reduced to the Gauss equation. The only difference appears in the computation of the connection matrices of the system \( \Sigma \). In fact, for \( \mu = \frac{1}{2} \), the fundamental matrix at infinity is a Jordan block and has logarithmic type behaviour. The computations of the analytic continuation can be performed following the formulae of [Nor] and the connection matrices are computed as in [DM]. Then, using (7.5), (2.6) and (2.7), one can show (7.3), for \( \nu_i \neq 0 \). Let us suppose \( \nu_i = 0 \) for some \( i \), for example \( \nu_2 = 0 \), i.e. \( l_0 = 0 \). Then \( \nu_1 \neq 0 \), and \( l_1 = l_\infty, l_0 = 0 \). I can take \( r_2 = 1 - r_3 = \frac{l_1}{2} \). Since \( r_1 + r_2 + r_3 = 1 \), \( r_1 \) must be 0, and the lemma is proved also for \( \nu_2 = 0 \).

ii) Follows from the fact that finite-braching solutions correspond to rational values of \( (r_1, r_2, r_3) \), i.e. to rational \( (\nu_1, \nu_2) \), thus they are algebraic (see Lemma 3).

iii) The fact that the action of the braid group \( B_3 \) (pure braid group \( P_3 \)) on \( (x_1, x_2, x_3) \) corresponds to the action of \( \Gamma (\Gamma(2)) \) on \( (\nu_1, \nu_2) \) is easily derived by the formulae (7.3), (7.4) relating \( (\nu_1, \nu_2) \) and the angles, and by the formula (7.2). QED

Remark 9. Observe that in the limit \( \nu_1, \nu_2 \to 0 \) the above formulae for the monodromy matrices give the matrices (6.9). Indeed, as shown in Section 4.1, Chazy solutions are limits of Picard type solutions for \( \nu_1, \nu_2 \to 0 \).

8. Algebraic solutions and finite irreducible reflection groups.

We reformulate here the above parameterization of the monodromy data, for the case of half-integer \( \mu \), by flat triangles \( (r_1, r_2, r_3) \), or equivalently, by couples of constants \( (\nu_1, \nu_2) \) in a more geometric way. Consider the Euclidean three-dimensional space \( E \) and three
planes \((p_1, p_2, p_3)\) all intersecting in one point, let \((r_1, r_2, r_3)\) be the angles between them and \((e_1, e_2, e_3)\) the vectors normal to them. Define three reflections \(R_1, R_2, R_3\) with respect to the three planes \((p_1, p_2, p_3)\):

\[
R_i : E \to E \\
x \mapsto x - (e_i, x)e_i \quad i = 1, 2, 3.
\]

Let us consider the group \(G\) of the linear transformations of \(E\), generated by the three reflections \(R_1, R_2, R_3\). Its Gram matrix is

\[
g := \begin{pmatrix} 2 & x_1 & x_3 \\
x_1 & 2 & x_2 \\
x_3 & x_2 & 2 \end{pmatrix}
\]  

\[\text{(8.1)}\]

where \(x_i = -2 \cos \pi r_i\). Observe that \(g\) is always singular:

\[
\det g = 8 - 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2x_3) = 8 \cos^2 \pi \mu = 0,
\]

then the normal vectors \((e_1, e_2, e_3)\) are linearly dependent.

Observe that the group \(G\) is invariant under analytic continuation. In fact, as shown in [DM], the action of the braid group on the triples \((x_1, x_2, x_3)\) can be interpreted as an action on the correspondent generating reflections

\[
\begin{align*}
\beta_1 : (R_1, R_2, R_3) &\mapsto (R_1, R_2, R_3)^{\beta_1} := (R_2, R_2R_1R_2, R_3), \\
\beta_2 : (R_1, R_2, R_3) &\mapsto (R_1, R_2, R_3)^{\beta_2} := (R_1, R_3R_2R_3, R_3),
\end{align*}
\]

\[\text{(8.2)}\]

where \(\beta_1, \beta_2\) are the standard generators of the braid group. The groups generated by the reflections \((R_1, R_2, R_3)\) and \((R_1, R_2, R_3)^{\beta}\) coincide for any \(\beta \in B_3\). In particular the following lemma holds true:

**Lemma 21.** For any braid \(\beta \in B_3\), the images \(\beta(R_1, R_2, R_3)\) are reflections with respect to some planes orthogonal to some new basic vectors \((e_1^\beta, e_2^\beta, e_3^\beta)\). The Gram matrix with respect to the basis \((e_1^\beta, e_2^\beta, e_3^\beta)\) has the form:

\[
\begin{align*}
(e_1^\beta, e_1^\beta) &= 2, \quad i = 1, 2, 3, \\
(e_1^\beta, e_2^\beta) &= x_1^\beta, \\
(e_1^\beta, e_3^\beta) &= x_2^\beta, \\
(e_2^\beta, e_3^\beta) &= x_3^\beta,
\end{align*}
\]

where \((x_1^\beta, x_2^\beta, x_3^\beta) = \beta(x_1, x_2, x_3)\).

Consider any algebraic solution of PVI\(_\mu\) with half-integer \(\mu\). According to Lemma 3, it is specified by a couple of coprime integers \(0 \leq M < N\), and, thanks to Proposition 2, the correspondent triangles belong to the orbit of \((0, \frac{M}{2N}, 1 - \frac{M}{2N})\). Thus two mirrors coincide and form an angle \(\frac{\pi M}{2N}\) with the third. This is called a *dihedral kaleidoscope* (see [Cox]). The generated group is the dihedral group \(D(\hat{N})\) realized as symmetry group of a regular star-polygon with \(\hat{N}\) edges and density \(\hat{M}\), where

\[
\hat{N} = \begin{cases} 
N & \text{if } M \text{ is even,} \\
2N & \text{if } M \text{ is odd,}
\end{cases}
\]

\[
\hat{M} = \begin{cases} 
\frac{M}{2} & \text{if } M \text{ is even,} \\
M & \text{if } M \text{ is odd.}
\end{cases}
\]

Resuming, I proved the following
**Theorem 10.** The algebraic solutions of PVI$\mu$ with any half-integer $\mu$, are in one to one correspondence with regular polygons and star-polygons in the plane.

**Remark 10.** The algebraic solutions (4.3), (4.4) and (4.5) correspondent to the values $N = 3$ and $M = 2$, $N = 2$ and $M = 1$, $N = 3$ and $M = 1$ respectively, correspond to $D(3)$, $D(4)$, $D(6)$ which coincide with $A_2$, $B_2$ and $G_2$ respectively.

In the Chazy case, all the mirrors coincide and the generated group is the cyclic group of order two, the symmetry group of a dighon.

For integer $\mu$ the only algebraic solutions are the ones correspondent to the triple $(0,0,0)$. The correspondent Gram matrix is

$$g = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$  

This means that the basic vectors are all orthogonal one to each other, i.e. they form a Cartesian frame, the correspondent reflection group is generated by the three inversions of the Cartesian coordinates and it is abelian.

**BIBLIOGRAPHY**

[Bir] J.S. Birman, *Braids, Links, and Mapping Class groups*, Ann. Math. Stud. Princeton University (1975).

[Cha] J. Chazy, Sur les equations differentielles dont l’integrale generale possede un coupure essentielle mobile, *C.R. Acad. Paris* 150 (1910), 456-458.

[Dek] W. Dekkers, The matrix of a connection having regular singularities on a vector bundle of rank 2 on $P^1(\mathbb{C})$, Lec. Notes Math. 712, p. 33-43 (1979).

[Dub] B. Dubrovin, *Geometry of 2D Topological Field Theories*, Lect. Notes Math. 1620, (1996).

[DM] B. Dubrovin and M. Mazzocco, Monodromy of certain Painlevé transcendents and reflection groups, SISSA preprint n.149/97/FM (1997).

[ItN] A.R. Its and V.Yu. Novokshenov, The isomonodromic deformation method in the theory of Painlevé equations, Lec. Not. Math. 1191 (1986) Springer.

[FlN] H. Flashka and A.C. Newell, Monodromy and spectrum preserving deformations, *Comm. Math. Phys.* 76 (1980).

[Fuchs] R. Fuchs, Über lineare homogene Differentialgleichungen zweiter Ordnung mit im drei im Endrichen gelegene wesentlich singulären Stellen, *Math. Ann.* 63 (1907) 301-321.

[Gamb] B. Gambier, Sur les Equations Differentielles du Second Ordre et du Primier Degré dont l’Integrale est a Points Critiques Fixes, *Acta Math.* 33, (1910) 1-55.

[Hal] G.H. Halphen, Sur un systeme d’équations differentielles, *C.R. Acad Sc. Paris* 92 (1881) 1001-1007.
[Hit] N.J. Hitchin, Twistor spaces, Einstein metrics and isomonodromic deformations, J. Differential Geometry 42, No.1 July 1995.

[Ince] E.L. Ince, Ordinary Differential Equations, Dover Publications, New York (1956).

[Man] Yu.I. Manin, Sixth Painlevé equation, universal elliptic curve and mirror of $\mathbb{P}^2$, preprint (1996).

[Mal] B. Malgrange, Sur les deformations isomonodromiques I, singularités régulières, Séminaire de l’Ecole Normale Supérieure 1979-1982, Progress mathematics 37, Birkhäuser Boston (1983) 401-426.

[Miwa] T. Miwa, Painlevé property of monodromy preserving deformation equations and the analyticity of $\tau$-function, Publ. RIMS, Kyoto Univ. 17 (1981) 703-721.

[Nor] N.E. Nörlund, The logarithmic solutions of the hypergeometric equation, Mat. Fys. Skr. Dan. Vid. Selsk. 2, no.5 (1963).

[Ok] K. Okamoto, Studies on the Painlevé equations I, sixth Painlevé equation, Ann. Mat. Pura Appl. 146 (1987) 337-381.

[Pain] P. Painlevé, Sur les Equations Differentielles du Second Ordre et d’Ordre Superieur, dont l’Interable Generale est Uniforme, Acta Math. 25 (1902) 1-86.

[Pic] E. Picard, Mémoire sur la théorie des functions algébriques de deux variables, Journal de Liouville 5 (1889), 135-319.

[Sch] L. Schlesinger, Über eine Klasse von Differentsial system beliebiger Orduung mit festen kritischer Punkten, J. fur Math. 141 (1912), 96-145.

[Sib] Y. Sibuya, Linear differential equations in the complex domain: problems of analytic continuation, AMS TMM 82 (1990).

[SG] G. Sansone and J. Gerretsen, Lectures on the theory of functions of a complex variable, P. Noordhoff editor, Groningen (1960).

[Tak] L.A. Takhtajan, Modular forms as tau-functions for certain integrable reductions of the Yang-Mills equations, PREPRINT (1992)