A renormalization group study of a class of reaction-diffusion model, with particles input

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Abstract. We study a class of reaction-diffusion model extrapolating continuously between the pure coagulation-diffusion case ($A + A \rightarrow A$) and the pure annihilation-diffusion one ($A + A \rightarrow \emptyset$) with particles input ($\emptyset \rightarrow A$) at a rate $J$. For dimension $d \leq 2$, the dynamics strongly depends on the fluctuations while, for $d > 2$, the behaviour is mean-field like. The models are mapped onto a field theory which properties are studied in a renormalization group approach. Simple relations are found between the time-dependent correlation functions of the different models of the class. For the pure coagulation-diffusion model the time-dependent density is found to be of the form $c(t, J, D) = (\frac{J}{D})^{1/\delta} F[(J/D)^{\Delta}Dt]$, where $D$ is the diffusion constant. The critical exponent $\delta$ and $\Delta$ are computed to all orders in $\epsilon = 2 - d$, where $d$ is the dimension of the system, while the scaling function $F$ is computed to second order in $\epsilon$. For the one-dimensional case an exact analytical solution is provided which predictions are compared with the results of the renormalization group approach for $\epsilon = 1$.

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1. Introduction

It is by now well established that reaction-diffusion models can have a rich dynamics governed in low dimensions by the fluctuations. Several physical quantities behave as power law and the associated critical exponents have some universal properties. The renormalization group (RG) method developed in the framework of equilibrium statistical physics [1], provides a suitable tool to study such dynamics. To be able to approach the problem of reaction-diffusion in term of RG, one has first to go from the initial microscopic master equation to a coarsed-grained description. The standard way to do this consists in using a Fock space formalism (see the works of Doi [2], Grassberger and Scheunert [3] and Peliti [4]). One ends up with a model which dynamics is defined by the action of a continuous field theory. Among several applications, this method has been used to study the two species annihilation reaction problem \( A + B \rightarrow \emptyset \); for homogeneous initial state with equal densities [5], or unequal densities [6], rigorous predictions of Bramson and Lebowitz [7] have been reproduced and novel results obtained. For the case, in which the two species are initially spatially separated, Howard and Cardy [8], have confirmed scaling arguments developed by Cornell and Droz [9].

Another interesting family of reaction-diffusion processes is formed by the one-species diffusion-annihilation and the diffusion-coagulation models. In 1986, Peliti [10] showed that the coagulation-diffusion model \( A + A \rightarrow A \) belongs to the same universality class than the annihilation-diffusion one \( A + A \rightarrow \emptyset \). He also showed that the associated field theory is super-renormalizable. Thus, only the coupling constant needs to be renormalized. Moreover, this renormalization can be done to all orders in \( \epsilon = d_u - d \), where \( d_u \), the upper critical dimension is 2 in this case. Peliti showed that the concentration \( c(t) \) of the reactant in the long time regime behaves as:

\[
    c(t) \propto t^{-\alpha}, \quad \alpha = \frac{d}{2}, \quad (t \rightarrow \infty).
\]

However, he made no predictions concerning the amplitude, neglecting the initial conditions in its approach. It turns out that the initial conditions may play a very important role. This aspect has been taken into account within this formalism only in 1991 by Ohtsuki [11] and in 1992 by Friedman et al [12]. In 1994, Lee [13] gave the first complete RG analysis of the annihilation-diffusion model. It turns out that the initial conditions show up as a local source into the action. Lee was able to treat this term to all orders in perturbation theory. It was shown later (see [8] for the two-species annihilation and [14] for the one-species case) that such infinite resummation is equivalent to a shift of the fields in the action by their classical values.

Meanwhile, in 1993, Droz and Sasvári [15] addressed the problem of both annihilation-diffusion and coagulation-diffusion in presence of a source \( J \) of particles:

\[
    A + A \xrightarrow{k} \emptyset, \quad A + A \xrightarrow{g} A, \quad \emptyset \xrightarrow{J} A.
\]
Performing a renormalization procedure, they found that the density of particles obeys the scaling law

\[ c(t, J, D) = \left( \frac{J}{D} \right)^{d/(d+2)} \mathcal{F}\left[ (J/D)^{2/(d+2)} Dt \right], \]

for sufficiently small value of \( J \). However, the scaling function \( \mathcal{F} \) was not computed. Making ad hoc assumptions on the asymptotic behavior of \( \mathcal{F} \) in the limits \( t \to \infty \), they showed that the stationary particle density

\[ c \sim J^{1/\delta}, \quad \delta = 1 + \frac{2}{d}, \]

was approached with a characteristic relaxation time \( \tau \) given by

\[ \tau \sim J^{-\Delta}, \quad \Delta = \frac{2}{d+2}. \]

Moreover, considerations on the \( J \to 0 \) limit, allow them to reproduce the scaling laws postulated phenomenologically by Rácz [16], namely:

\[ \alpha \delta \Delta = 1, \quad \Delta + \frac{1}{\delta} = 1. \]

The goal of this paper is two folds. First to provide a complete renormalization group analysis of such models by computing not only the exponents but also the scaling function \( \mathcal{F} \) defined by (1) in the framework of an \( \epsilon = 2 - d \) expansion. This is done in section 2. Second, to give an exact analytical solution of the one-dimensional coagulation-diffusion model with infinite reaction rate and source, by extending to time dependent regime an approach developed by Doering and ben-Avraham [17]. These exact results are compared with the RG predictions in the limit \( \epsilon = 1 \) in section 3. Remarks and conclusions are given in section 4.

2. Field theoretical approach and renormalization group analysis

2.1. The model and the associated field theory

We shall not derive here in details how one obtains the field theoretical model. The interested reader is referred to the original papers of Doi [2], Grassberger and Scheunert [3] and Peliti [4, 10] or to the short reviews presented in [13] and in [18]. As the number of particles is not conserved, the basic idea is to introduce a Fock space representation. The time evolution operator of the problem can then be cast in a path integral form which, in the continuous (coarsed grained) limit is characterized by the action:

\[ S_\gamma [a, \bar{a}, J] = \int d^d x \int_0^t d\bar{t} \left[ \bar{a} \left( \frac{\partial}{\partial \bar{t}} - D \nabla^2 \right) a + \gamma \lambda \bar{a} a^2 + \lambda \bar{a}^2 a^2 - J \bar{a} \right], \]

(2)
A whole class of models indexed by the parameter $\gamma \in [1, 2]$ is thus defined. For $\gamma = 1$ one has the pure coagulation-diffusion model and for $\gamma = 2$ the pure annihilation one. For $1 < \gamma < 2$, both reactions are possible with a given probability depending on $\gamma$ (see [18] for more details). The coupling constant $\lambda$ is related to the reaction rates $g$ and $k$ via $\lambda = (\tilde{\gamma} + 1)k$ and $\gamma = (\tilde{\gamma} + 2)(\tilde{\gamma} + 1)^{-1}$, with $\tilde{\gamma} = k/g$. The particles diffuse in an infinite $d$-dimensional space with a diffusion constant $D$. The above action could models two different types of colliding particles in some appropriate limit $s$. First point-like particles living on a $d$-dimensional hypercubic lattice [13], second extended particles living in a $d$-dimensional continuous space [18].

The time and position dependent fields $a$ and $\bar{a}$ obey bosonic like commutation relations. The field $a(x, t)$ is related to the local particle density, while the auxiliary field $\bar{a}(x, t)$ has no particular physical meaning.

Within this formalism, correlation functions are expressed by functional integrals.

$$G_{\gamma}^{N, \bar{N}}(\{x_i, t_i\}_{i=1}^{N+\bar{N}}) = \int \mathcal{D}a \mathcal{D}\bar{a} a(x_1, t_1) \ldots a(x_N, t_N)$$

$$\times \bar{a}(x_{N+1}, t_{N+1}) \ldots \bar{a}(x_{N+\bar{N}}, t_{N+\bar{N}}) \exp(-S_\gamma[a, \bar{a}, J]).$$

In particular, the particle density $c_\gamma(x, t)$ at point $x$ and time $t$ reads:

$$c_\gamma(x, t) = \int \mathcal{D}a \mathcal{D}\bar{a} a(x, t) \exp(-S_\gamma[a, \bar{a}, J])$$

As we have shown in a previous work [18], the correlation functions of different $\gamma$ models are closely related. In particular, one has

$$S_\gamma[a, \bar{a}, J] = S_1[\gamma a, \gamma^{-1} \bar{a}, J],$$

which implies for the concentration

$$c_\gamma(x, t; J) = \gamma^{-1} c_1(x, t; \gamma J).$$

Accordingly, it suffices to study one particular model belonging to the class to know the behaviour of the other members. From now on, we shall study the pure coagulation-diffusion model ($\gamma = 1$), with an initial state empty, which action is:

$$S[a, \bar{a}, J] = \int d^dx \int_0^t dt \left[ a \left( \frac{\partial}{\partial t} - D \nabla^2 \right) a + \lambda \bar{a} a^2 + \lambda \bar{a}^2 a^2 - J \bar{a} \right].$$

For the sake of simplicity, we do not write the index $\gamma = 1$ in the following. When $J = \lambda = 0$, one has a free theory (pure diffusion) and the spatial Fourier transform of the free propagator is simply $G_0(p, t) = \theta(t) \exp(-Dp^2t)$, where $\theta(t)$ is the usual Heaviside function. Simple power counting shows that the upper critical dimension of action (6) is $d_u = 2$. For $d > d_u$, the quadrivertex $\lambda \bar{a} a^2$ is irrelevant and can be discarded. Below $d_u$, the quadrivertex $\lambda \bar{a}^2 a^2$ is relevant and leads to singularities that have to be renormalized. At $d = d_u$, this vertex is marginal, and one expects logarithmic corrections to the mean-field behaviour.
2.2. Mean-field solution

We first consider the case $d > d_u = 2$, where the behavior is mean-field like. At the mean-field level, the equations of motion for $a$ and $\bar{a}$ are obtained from the action (6), by the usual saddle point argument and read:

$$\frac{\delta S}{\delta \bar{a}} = \left( \frac{\partial}{\partial t} - D \nabla^2 \right)a + \lambda a^2 + 2\lambda \bar{a}a^2 - J = 0$$

(7)

and

$$\frac{\delta S}{\delta a} = -\left( \frac{\partial}{\partial t} + D \nabla^2 \right)\bar{a} + 2\lambda \bar{a}a + 2\lambda \bar{a}^2 a = 0.$$  

(8)

Assuming that $a$ and $\bar{a}$ are homogeneous, it follows that as expected $\bar{a} = 0$ is a solution and (7) becomes

$$\frac{\partial a}{\partial t} = -\lambda a^2 + J$$

with the initial condition $a|_{t=0} = 0$. Thus, the mean-field or classical solution is:

$$a_{cl}(t) = \sqrt{\frac{J}{\lambda}} \left( 1 - 2 \frac{\exp(-2\sqrt{J/\lambda} t)}{1 + \exp(-2\sqrt{J/\lambda} t)} \right),$$

(9)

where the subscript ”cl” stands for classical.

2.3. Renormalization

Let us now consider the problem below 2 dimensions. A brute force computation of the particle density from (4) leads to divergences. Thus the coupling constant $\lambda$ needs to be renormalized. Neither the fields $a$ and $\bar{a}$ nor the diffusion constant $D$ require a renormalization [10, 13], and as a consequence, the particles input rate $J$ does not either.

We define the temporally extended vertex function $\lambda(p, t)$ to be the sum of the diagrams shown in figure 1. These diagrams can be summed to all orders and the Laplace-transformed vertex function reads (see [13]):

$$\tilde{\lambda}(p, s) = \frac{\lambda}{1 + 2\left[\lambda/(8\pi D)^{d/2}\right] \Gamma(1/2) \left( s + \frac{Dp^2}{2} \right)^{-\epsilon/2}}.$$

(10)

We define $g_R$, the dimensionless renormalized coupling (or running coupling constant), using the minimal subtraction scheme. That is we define

$$g_R = Z_g \kappa^{-\epsilon} \frac{\lambda}{2\pi D}$$

(11)

where $\kappa$ is a normalization point and $Z_g = 1 + \sum_{i=1}^{\infty} a_i g_R^i$. The $a_i$ are chosen such as to exactly cancel the poles of order $1/\epsilon$ appearing in $\tilde{\lambda}(p, s)$. From (10), we have

$$\tilde{\lambda}(p, s) = \frac{\lambda}{1 + \kappa^{-\epsilon} \lambda[1 + \epsilon A(\epsilon)]/(2\pi D \epsilon)}.$$
Figure 1. Diagrammatic representation of the temporally extended vertex function \( \lambda(p, t_2 - t_1) \). The propagator is represented by a full line. Here is shown the diagrammatic expansion for the trivertex \( \lambda \bar{a} a^2 \). Similar diagrams can be drawn for the quadrivertex \( \lambda \bar{a}^2 a^2 \).

where \( A(\epsilon) = O(1) \). With (11), one finds

\[
\tilde{\lambda}(p, s) = \frac{k^\epsilon g_R}{2\pi D} \left( Z_g + g_R/\epsilon + g_R A(\epsilon) \right).
\]

Accordingly to our prescription, we have to choose \( a_1 = -1/\epsilon \) and \( a_i = 0 \) for \( i > 1 \), giving \( Z_g = 1 - g_R/\epsilon \) (exact to all orders). The \( \beta \) function is defined by

\[
\beta(g_R) \equiv \kappa \frac{\partial g_R}{\partial \kappa} = -\epsilon g_R + g_R^2.
\]

It is exactly quadratic in \( g_R \) and has a fixed point given by \( \beta(g_R^*) = 0 \) at \( g_R^* = \epsilon \).

The bare coupling may thus be expressed in terms of the renormalized one:

\[
\kappa^{-\epsilon} \frac{\lambda}{2\pi D} = \frac{g_R}{1 - g_R/g_R^*} = g_R + \frac{g_R^2}{g_R} + \ldots
\]

(12)

The perturbation theory can then be written as an expansion in powers of \( g_R \).

### 2.4. Renormalization group equations

An arbitrary renormalized correlation function \( G_{N, \bar{N}}^R(x_i, t_i) \) (where the subscript \( R \) stands for renormalized) is related to its bare expression (3) through

\[
G_{N, \bar{N}}^R(x_i, t_i) \equiv G_{N, \bar{N}}^R(x_i, t_i)_{1}^{N+\bar{N}}; g_R, D, J, \kappa = G_{N, \bar{N}}^{N, \bar{N}}(x_i, t_i)_{1}^{N+\bar{N}}; g_R, D, J, \kappa).
\]

The independence of the bare functions on the normalization scale can be expressed via the condition:

\[
\left( \kappa \frac{\partial}{\partial \kappa} + \beta(g_R) \frac{\partial}{\partial g_R} \right) G_{N, \bar{N}}^{N, \bar{N}}(x_i, t_i)_{1}^{N+\bar{N}}; g_R, D, J, \kappa = 0.
\]

The formal solution (obtained by the method of characteristic) is

\[
G_{N, \bar{N}}^{N, \bar{N}}(x_i, t_i)_{1}^{N+\bar{N}}; g_R, D, J, \kappa = G_{N, \bar{N}}^{N, \bar{N}}(x_i, t_i)_{1}^{N+\bar{N}}; g_R(\rho), D, J, \rho \kappa)
\]

(13)

with

\[
g_R(\rho) = g_R^\epsilon \left( 1 + \frac{g_R^* - g_R}{g_R} \right)^{-1}.
\]

(14)
Note that in the small $\rho$ limit, $g_R(\rho) \to g_R^*$. We can implement (13) with a dimensional analysis. The dimensions of the different quantities, expressed in term of momentum $\kappa$ and energy $E$ are

$$
\begin{align*}
[t] &= E^{-1}, & [D] &= E\kappa^{-2}, & [J] &= E\kappa^d \\
[a] &= \kappa^d, & [\bar{a}] &= 1, & [G^N,\bar{N}_R(\{x_i, t_i\}^N_{1+N}; g_R, D, J, \kappa)] &= \kappa^{Nd},
\end{align*}
$$

Thus,

$$
G^N,\bar{N}_R(\{x_i, t_i\}^N_{1+N}; g_R, D, J, \kappa) = \kappa^{Nd}g^N,\bar{N}_R(\{\rho\kappa x_i, (\rho\kappa)^2Dt_i\}^N_{1+N}; g_R, 1, \kappa^{-d-2}J/D, 1). \tag{15}
$$

The combination of (13) and (15) leads to

$$
G^N,\bar{N}_R(\{x_i, t_i\}^N_{1+N}; g_R, D, J, \kappa)
= (\rho\kappa)^{Nd}G^N,\bar{N}_R(\{\rho\kappa x_i, (\rho\kappa)^2Dt_1\}^N_{1+N}; g_R(\rho), 1, (\rho\kappa)^{-d-2}J/D, 1). \tag{16}
$$

We can then use the following strategy to compute the correlation functions: first an expansion in power of $\lambda$ is established; then it is converted into an expansion in power of $g_R$ through (12). The singularities in $\epsilon$ are eliminated using the renormalization scheme. Now, for a correctly renormalized theory, we can rewrite the $g_R$ expansion into an $\epsilon$ expansion using (16) and (14). Indeed, introducing the $\rho$ dependence through (16) and letting $\rho \to 0$, $g_R(\rho) \to g_R^*$ one obtains $G^N,\bar{N}_R$ as an expansion in power of $\epsilon$.

Up to now $\rho$ is an arbitrary parameter, and several choices are possible. For example, if we choose $\rho$ such that

$$
(\rho\kappa)^{-d-2}\frac{J}{D} = 1. \tag{17}
$$

the limit $\rho \to 0$ becomes equivalent to $J \to 0$: when the source rate is vanishing small, the running coupling approaches its fixed point value.

Another choice is

$$
(\rho\kappa)^2Dt_1 = 1
$$

and the limit $\rho \to 0$ may be exchanged with $t_1 \to \infty$. However, from (16), we see that $(\rho\kappa)^{-d-2}J/D$ diverges. Thus, with this choice, one should know the behaviour of $G^N,\bar{N}_R$ for arbitrary large value of $J$. However, for large values of $J$, the $\epsilon$-expansion breaks down (see Appendix A). Accordingly, we shall choose $\rho$ according to condition (17) in what follows.

2.5. Density calculation

The density is first calculated using (4), in the framework of a perturbation expansion in power of $\lambda$. At the tree-level, we find out the mean-field result and we can directly use the RG equation (no renormalization is needed). The first correction to the classical
behaviour is given by the one-loop contribution. The corresponding diagram may be calculated using the action obtained from (6) by shifting the field \(a\) by its classical value:

\[
S[\eta, \bar{\eta}, J] = \int d^d x \int_0^t dt \left[ \bar{\eta} \left( \frac{\partial}{\partial t} - D \nabla^2 + 2\lambda a_{cl} \right) \eta + \lambda \bar{\eta} \eta^2 + \lambda \eta^3 (a_{cl}^2 + 2a_{cl} \eta + \eta^2) \right],
\]

(18)

where \(\eta = a - a_{cl}\) and \(\bar{\eta} = \bar{a}\), with \(a_{cl}\) given by (9). We have

\[
G_{\eta \bar{\eta}}(p, t, t') = \theta(t-t') \exp \left[ -Dp^2(t-t') \right] \left( \frac{\cosh(\sqrt{J\lambda} t')}{\cosh(\sqrt{J\lambda} t)} \right)^2.
\]

Note that because of the initial condition, \(G_{\eta \bar{\eta}}\) is not invariant under time-translation.

Obtaining divergent expressions for the one-loop corrections, we shall renormalize them using the renormalization scheme developed above. We give below a summary of these results.

2.5.1. Tree level  Applying the RG formalism as developed above on the mean-field equation, we find for \(J\) sufficiently small,

\[
c_R(t; g_R, D, J, \kappa) = \frac{1}{\sqrt{2\pi\epsilon}} \left( \frac{J}{D} \right)^{d/(d+2)} \times \left( 1 - \frac{2 \exp[-2\sqrt{2\pi\epsilon} (J/D)^{2/(d+2)} D t]}{1 + 2 \exp[-2\sqrt{2\pi\epsilon} (J/D)^{2/(d+2)} D t]} \right) [1 + \mathcal{O}(\epsilon)],
\]

which is universal (independent of \(g_R\)). This result is valid for any time \(t\), because we only need to tune \(J\) to be in the critical domain.

For small \(t\) we find the expected result

\[
c_R(t) = Jt + \mathcal{O}(t^2)
\]

and for long time (\(t \to \infty\))

\[
c_R(t) = \frac{1}{\sqrt{2\pi\epsilon}} \left( \frac{J}{D} \right)^{d/(d+2)} \{ 1 - 2 \exp[-2\sqrt{2\pi\epsilon} (J/D)^{2/(d+2)} D t] \} [1 + \mathcal{O}(\epsilon)].
\]

The steady state value is thus given by

\[
c_R(\infty) = \frac{1}{\sqrt{2\pi\epsilon}} \left( \frac{J}{D} \right)^{d/(d+2)} [1 + \mathcal{O}(\epsilon)]
\]

(21)

and it is approached exponentially in time as:

\[
\delta c_R(t) \equiv c_R(t) - c_R(\infty) \propto \exp[-\gamma_{\epsilon} (J/D)^{2/(d+2)} D t] \quad (t \to \infty)
\]

(22)

with \(\gamma_{\epsilon} = 2\sqrt{2\pi\epsilon} [1 + \mathcal{O}(\epsilon)]\).
2.5.2. One-loop corrections  The diagram corresponding to the one-loop correction is given in figure 2; its analytic expression is

\[c^{(1)}(t) = \frac{2J\lambda}{\cosh^2(\sqrt{J\lambda} t)} \int_0^t dt_2 \int_0^{t_2} dt_1 \int d^d p \left[ -2Dp^2(t_2 - t_1) \right] \times \frac{\cosh^2(\sqrt{J\lambda} t_1) \sinh^2(\sqrt{J\lambda} t_1)}{\cosh^2(\sqrt{J\lambda} t_2)}.\]

The integral over the momentum gives the factor \([8\pi D(t_2 - t_1)]^{-d/2}\). We thus obtain

\[c^{(1)}(t) = \frac{2J\lambda}{(8\pi D)^{-d/2} \cosh^2(\sqrt{J\lambda} t)} \frac{t^{1+\epsilon/2}}{\cosh^2(\sqrt{J\lambda} t)} \int_0^1 dx_2 \int_0^1 dx_1 x_1^{-1+\epsilon/2} \times \frac{\cosh^2[\sqrt{J\lambda} x_2(1 - x_1)] \sinh^2[\sqrt{J\lambda} x_2(1 - x_1)]}{\cosh^2(\sqrt{J\lambda} x_2)}.\]

In the framework of an \(\epsilon\)-expansion, we eventually find (treating \(x_1^{-1+\epsilon/2}\) as a generalized function [20], see Appendix B):

\[c^{(1)}(t) = \frac{2J\lambda}{(8\pi D)^{-d/2} \cosh^2(\sqrt{J\lambda} t)} \left( \frac{2\sinh(2\sqrt{J\lambda} t)}{2\sqrt{J\lambda} t} + \frac{\phi(\sqrt{J\lambda} t)}{\sqrt{J\lambda} t} + O(\epsilon) \right),\]

where

\[\phi(\xi) = \xi \int_0^1 dx_2 \ln x_2 \sinh^2(\xi x_2) + \xi \int_0^1 \frac{dx_1}{x_1} \int_0^1 dx_2 \left( \frac{\cosh^2[\xi x_2(1 - x_1)] \sinh^2[\xi x_2(1 - x_1)]}{\cosh^2(\xi x_2)} - \sinh^2(\xi x_2) \right).\]

Putting together the mean-field result and the one-loop correction, we verify that the density is indeed divergence free, and using the RG equation (16) we find, for small values of \(J\)

\[c_R(t; g_R, D, J, \kappa) = \frac{1}{\sqrt{2\pi \epsilon}} \left( \frac{J}{D} \right)^{d/(d+2)} \left\{ \tanh[\sqrt{2\pi \epsilon}(J/D)^{2/(d+2)} Dt] \right. \right. \]

\[+ \epsilon \left. \left. \left[ \frac{\phi[\sqrt{2\pi \epsilon}(J/D)^{2/(d+2)} Dt]}{\cosh^2[\sqrt{2\pi \epsilon}(J/D)^{2/(d+2)} Dt]} \right] + \frac{1}{2} \ln[8\pi \epsilon(J/D)^{2/(d+2)} Dt] \right. \right. \]

\[\times \left. \left( \tanh[\sqrt{2\pi \epsilon}(J/D)^{2/(d+2)} Dt] - \frac{\sqrt{2\pi \epsilon}(J/D)^{2/(d+2)} Dt}{\cosh^2[\sqrt{2\pi \epsilon}(J/D)^{2/(d+2)} Dt]} \right) \right] + O(\epsilon^2).\]
\[
= \left( \frac{J}{D} \right)^{d/(d+2)} \mathcal{F}[(J/D)^{2/(d+2)}] dt .
\]  

(23) 

We immediately identify the scaling function \( \mathcal{F} \) defined in (1). Taking the limit \( t \to \infty \) we find
\[
c_R(\infty; g_R, D, J, \kappa) = \frac{1}{\sqrt{2 \pi \epsilon}} \left( \frac{J}{D} \right)^{d/(d+2)} \left\{ 1 - \frac{\epsilon}{4} \left[ \gamma_E - \ln \left( \frac{2 \pi}{\epsilon} \right) \right] + O(\epsilon^2) \right\}.
\]  

(24) 

where \( \gamma_E \) is the Euler constant (\( \gamma_E \approx 0.5772 \)). Unfortunately, due to the complicated form of \( \phi \), we are unable to give a more compact form for the asymptotics of \( \mathcal{F} \). Notice that the empty initial condition implies that \( \lim_{J \to 0} c_R(t) = 0 \).

Let us consider two particular cases of interest. The first one is the case \( \epsilon = 1 \) (\( d = 1 \)). Then the steady-state density is
\[
c_R(\infty) = \frac{1}{\sqrt{2 \pi}} \left( \frac{J}{D} \right)^{1/3} \left[ 1 - \frac{1}{2} (\gamma_E - \ln 2\pi) + \cdots \right]
\]  

(25) 

and it is asymptotically approached (at the tree-level) as:
\[
\delta c_R(t) \propto \exp \left[ -\sqrt{8\pi} \left( \frac{J}{D} \right)^{1/2} Dt \right].
\]  

(26) 

We shall compare the accuracy of these expressions with exact results in the next section. Note however that, in principle, nothing ensures us that the terms we neglected are small.

The second case is \( \epsilon = 0 \) (\( d = 2 \)). The running coupling is given by
\[
g_R(\rho) = \frac{g_R}{1 - g_R \ln \rho},
\]  

which goes to \(- (\ln \rho)^{-1} \) when \( \rho \to 0 \). By replacing this expression in our previous formula, we find for the steady-state density
\[
c_R(\infty) = \left( \frac{2J}{\pi D} \right)^{1/2} \left[ (\ln(J/\kappa^4D))^{-1/2} + O(\ln(J/\kappa^4D))^{-1}) \right]
\]  

(27) 

and
\[
\delta c_R(t) \propto \exp \left[ -\sqrt{8\pi} (J/D)^{1/2} \ln(J/\kappa^4D) Dt \right].
\]  

(28) 

as anticipated, logarithmic corrections to the mean-field result are obtained.

3. Exact results in one dimension

A large amount of work has been done to solve exactly one-species diffusion reaction models in one dimension (see for example [21]). In particular, the diffusion-annihilation and the diffusion-coagulation reactions have been considered with an input of particles. In the diffusion-annihilation case, Rácz [22] obtained the steady-state concentration by mapping its model to the kinetic Glauber-Ising model [23]. In 1988, Doering and ben-Avraham [24] calculated exactly the time dependent concentration for a simple diffusion-coagulation model, using the interparticle distribution function. Since, their method
has been generalized to other diffusion-coagulation processes (see for example [25]) and in particular the steady-state concentration has been obtained [17] for the diffusion-coagulation with an input source of particle.

In this section we aim at testing the validity of the RG predictions for $d = 1$. For this purpose we shall extend Doering’s and ben-Avraham’s results and compute the time-dependent concentration.

We consider an infinite chain (our one-dimensional space) initially empty, and we allow particles to appear randomly at rate $J$ (per unit time and per unit length). Thus initially

$$\frac{dc(t)}{dt}\bigg|_{t=0} = J$$

where $c(t)$ is as before the particle concentration. The particles diffuse on the line (with a diffusion constant $D$) and when two particles meet, they instantaneously coagulate ($A + A \rightarrow A$). Note that this model is the same as the pure diffusion-coagulation process of section 2, but with an infinite reaction rate $\lambda$. Of course, for such a reaction rate, the perturbation expansion in power of $\lambda$ is meaningless and one may argue that the two models are not equivalent. However by examining the relation between the renormalized coupling $g_R$ and $\lambda$, we see (from (12)) that:

$$g_R = g_R^\ast(1 + 2\pi g_R^\ast \kappa' D / \lambda)^{-1};$$

and thus, when $\lambda$ is infinite, $g_R = g_R^\ast$. We argue that the infinite reaction rate limit may be obtained by taking the fixed point coupling limit ($g_R \rightarrow g_R^\ast$), that is to say by taking the $\lambda \rightarrow \infty$ limit after having performed the path integrals. This will be confirmed in the following, at least in one dimension.

In a one-dimensional space, the particles concentration can be related to the probability $E(x, t)$ that an interval of length $x \geq 0$ is empty at time $t$, via

$$c(t) = -\frac{\partial E(x, t)}{\partial x}\bigg|_{x=0}.$$ 

As shown by Doering and ben-Avraham [17], $E(x, t)$ has the advantage to obey a closed equation of evolution, namely:

$$\frac{\partial E}{\partial t} = 2D \frac{\partial^2 E}{\partial x^2} - Jx E,$$

with the two conditions

$$E(0, t) = 1, \quad E(\infty, t) = 0. \quad (30)$$

From this equation, one can immediately obtain the steady-state, by setting the left hand side to zero. One then recognizes the Airy equation, whose solution is (taking into account conditions (30))

$$E(x, \infty) = \frac{\text{Ai}\left((J/2D)^{1/3}x\right)}{\text{Ai}(0)},$$
where $\text{Ai}(z)$ is the Airy’s function (see [26]). As a consequence, the asymptotic concentration reads

$$ c(\infty) = -\frac{\text{Ai}'(0)}{\text{Ai}(0)} \left( \frac{J}{2D} \right)^{1/3} $$  \hspace{1cm} (31) $$

$\text{Ai}'(z)$ is the first derivative of $\text{Ai}(z)$. Note that $\text{Ai}'(0) < 0$.

Before comparing (31) with the RG results, we shall compute the time dependent part of the concentration. For this purpose, we shall solve (29) using the Laplace transform $\tilde{E}(x, s)$ defined by $\tilde{E}(x, s) = \int_0^\infty e^{-st} E(x, t) \, dt$. Equation (29) becomes

$$ 2D \frac{\partial^2 \tilde{E}(x, s)}{\partial x^2} - (Jx + s) \tilde{E}(x, s) + E(x, 0) = 0, $$  \hspace{1cm} (32) $$

where $E(x, 0)$ is the initial condition (for an empty system, $E(x, 0) = 1$). Equation (32) is an inhomogeneous second order ordinary differential equation. Its general solution is the sum of the homogeneous solution $\tilde{E}_h(x, s)$ and a particular solution. The homogeneous solution is

$$ \tilde{E}_h(x, s) = \left( \frac{J}{2D} \right)^{-2/3} [\alpha_1(s) \text{Ai}(z) + \alpha_2(s) \text{Bi}(z)], $$

where $z = (J/2D)^{1/3}x = (J/2D)^{-2/3}Jx + s/2D - 1/2D.$

Writing $\tilde{E}(x, s) = (J/2D)^{-2/3}A(z, s)$, (32) becomes

$$ \frac{\partial^2 A(z, s)}{\partial z^2} - zA(z, s) = -\frac{1}{2D}, $$

for which a well known solution is

$$ A(z, s) = -\frac{\pi}{2D} \int_0^z dv [\text{Ai}(v) \text{Bi}(z) - \text{Ai}(z) \text{Bi}(v)]. $$

The two boundary conditions (30) permit us to determine the two unknown functions $\alpha_1(s)$ and $\alpha_2(s)$. We eventually find for the general solution of (32)

$$ \tilde{E}(x, s) = \frac{\pi}{2D} \left( \frac{J}{2D} \right)^{-2/3} \left\{ \text{Ai}(\xi + \sigma) \int_0^\xi dv \text{Bi}(v) + \text{Bi}(\xi + \sigma) \left[ \frac{1}{3} - \int_0^\xi dv \text{Ai}(v) \right] \right\} $$

$$ + \frac{\text{Ai}(\xi + \sigma)}{\text{Ai}(\sigma)} \left[ \frac{1}{\pi \sigma} + \text{Bi}(\sigma) \left( \int_0^\sigma dv \text{Ai}(v) - \frac{1}{3} \right) \right], $$  \hspace{1cm} (33) $$

where $\xi = (J/2D)^{1/3}x$ and $\sigma = (J/2D)^{-2/3}s/2D$.

The probability $E(x, t)$ is then obtained by Laplace-inverting (33). For $t > 0$, we only have to care for the poles of $\tilde{E}(x, s)$. They are located at $\sigma = 0$ and at $\sigma = a_n$, $n = 1, 2, 3, \ldots$, where $a_n$ is the $n$-th zero of $\text{Ai}(x)$ ($a_n < 0$). We finally obtain

$$ E(x, t) = \frac{\text{Ai}(\xi)}{\text{Ai}(0)} + \sum_{n=1}^\infty \frac{\text{Ai}(\xi + a_n)}{\text{Ai}'(a_n)} \left[ \frac{1}{a_n} + \pi \text{Bi}(a_n) \left( \int_0^{a_n} dv \text{Ai}(v) - \frac{1}{3} \right) \right] e^{-|a_n|t}, $$
with \( \tau = 2Dt(J/2D)^{2/3} \) and
\[
c(t) = \left( \frac{J}{2D} \right)^{1/3} \left\{ -\frac{Ai'(0)}{Ai(0)} - \sum_{n=1}^{\infty} \left[ \frac{1}{a_n} + \pi Bi(a_n) \left( \int_0^{a_n} dv \, Ai(v) - \frac{1}{3} \right) \right] e^{-|a_n|\tau} \right\}.
\] (34)

We are now in position to compare these results with the RG results for \( \epsilon = 1 \). For the steady-state, the RG gives (up to the one-loop corrections) (26)
\[
c_R(\infty) \simeq 0.53 \left( \frac{J}{D} \right)^{1/3}
\] (35)
(for small \( J \)) while (putting \( t = \infty \) inside (34)) the exact solution gives:
\[
c(\infty) = -\frac{Ai'(0)}{Ai(0)} \left( \frac{J}{2D} \right)^{1/3} \simeq 0.58 \left( \frac{J}{D} \right)^{1/3}
\] (36)
(for arbitrary \( J \)). Surprisingly, the difference is only of the order of 10%.

The comparison for the approach to the steady-state is less convincing, mainly due to the fact that we do not know the one-loop corrections. The RG gives, from equation (26)
\[
\delta c_R(t) = -\left[ 1 - \frac{1}{4}(\gamma_E - \ln 2\pi) \right] \left( \frac{2}{\pi} \right)^{1/2} \left( \frac{J}{D} \right)^{1/3} \exp\left[ -\sqrt{8\pi(J/D)^{2/3}} Dt \right]
\] (37)
whereas the exact result is
\[
\delta c(t) = \left[ \frac{1}{a_1} + \pi Bi(a_1) \left( \int_0^{a_1} dv \, Ai(v) - \frac{1}{3} \right) \right] \left( \frac{J}{2D} \right)^{1/3} e^{-|a_1|\tau}
\] (with \( a_1 \simeq -2.33 \)). Both amplitudes in front of the exponential are in good agreement (because we used the one-loop result of the steady-state). However, the tree-level amplitude into the exponential is quite different (of almost 70%) from the one given by the exact theory. The inclusion of the one-loop correction should lead to a better agreement.

Note that the exact results are valid without any restriction on \( J \), on the contrary to the RG results, which apply only for small \( J \). This restriction was introduced to ensure that \( g_R \) to be inside the critical domain (\( i.e. \ g_R \) near \( g^*_R \)). In view of the predictions of the exact theory, it turns out that this restriction over \( J \) is unnecessary. The coupling stays inside the critical domain for any value of \( J \). This justifies why when \( \lambda \to \infty \), \( g_R \to g^*_R \).

4. Discussion and conclusive remarks

We have shown that the RG is a suitable formalism to compute the density in a wide class of diffusion-reaction model, with an input of particle. In particular, we easily
calculated the critical exponents in arbitrary dimension. However, the computation of the universal scaling function is generally much more difficult (see for example (23)) and can only be achieved within a power expansion in $\epsilon = 2 - d$. However, the comparison of the first order results for $\epsilon = 1$ are not too far from the results obtained exactly in one dimension by a different approach.

One of the main interest of the RG method is that, contrary to most unidimensional exact approach, the RG approach is not restricted to the computation of the particle density but higher correlation functions can also be calculated (although the computation can become quite involved, see [13]).

Another advantage of RG approach, is the fact that the properties of a whole class of model can be simply related. This allows us using (5), and knowing the particle density for the diffusion-coagulation model, to obtain the density of any mixed annihilation-coagulation process. In particular, for the pure diffusion-annihilation model ($\alpha = 2$) in one dimension, one recovers the steady-state density previously calculated by Rácz [22].

The present work can be generalized in several directions. Generalization to $mA \rightarrow \emptyset$ ($m > 2$) reactions with a source of particles is also straightforward. Indeed, following Lee, it turns out that apart the lowering of the critical dimension to $d_u = 2/(m-1)$, only minor changes occur (for example the amplitudes become $m$-dependent). This is due to the fact that in our renormalization scheme only the fixed point $g^*_R$ of the coupling is modified. The structure of the equations remain unchanged and in particular, the RG equation (16) still holds. As a consequence, the critical exponents below $d_u$ are the same than for $m = 2$. The steady-state concentration can be written (for $d < d_u$) as:

$$c_{m,R}(\infty) = A_m \left( \frac{J}{D} \right)^{d/(d+2)},$$

and the universal amplitude $A_m$ can be computed within an $\epsilon$-expansion. The approach to the steady-state is still exponential with $k$-dependent amplitudes. Extension to the reactions $mA \rightarrow lA$ with $l < m$ ($m > 2$) is also possible (see [13]).

Another natural extension of this work is to consider the possibility of reversible reaction (for example $A + A \rightleftharpoons A$). For such system a completely different physics is expected. Indeed, a quick investigation of the associated action shows that the upper critical dimension is no longer 2 but 4. In addition, it appears that a wavefunction renormalization is needed, giving rise to anomalous dimensions. The computation of scaling functions for this problem is currently under investigation.

Appendix A.

We aim to show that in the limit $J \rightarrow \infty$, the $\epsilon$-expansion breaks down. For this purpose, let us consider the action (18) obtained by the shift of the field $a$ by its classical value. The shift permits us to suppress the source term in the action; in other words, shifting...
the field is equivalent to perform the infinite sum generated by the source term. One could then think that any quantity can be calculated for arbitrary \( J \). This is however not true for the following reason. To calculate a given quantity, we expand the action with respect to the \( \lambda \bar{\eta}^2 (a_{c1}^2 + 2a_{c1}\eta + \eta^2) \) term. We shall then obtain a power expansion in term of \( \lambda \) with coefficient proportional to \( a_{c1}^2 \). But for large \( J \), \( a_{c1} \) tends to \( (J/\lambda)^{1/2} \).

The expansion in \( \lambda \) is partly replaced by an expansion in \( J \), which does not lead to an \( \epsilon \)-expansion. The only possibility to revert this would be to treat non-perturbatively the action, which is out of question.

Appendix B.

The problem is to compute the \( \epsilon \)-expansion of

\[
\int_0^1 dx \ x^{-1+\epsilon} f(x).
\]

Putting naively \( x^{-1+\epsilon} = x^{-1}[1+\epsilon \ln x + O(\epsilon^2)] \) in (B1) clearly fails, because if \( f(0) \neq 0 \), \( \int_0^1 dx \ x^{-1} f(x) \) diverges. One way to avoid this problem, is to treat \( x^{-1+\epsilon} \) as a generalized function. We shall not give here a detailed discussion of the generalized functions (see [20] for an introduction). We only quote the result, namely:

\[
\int_0^1 dx \ x^{-1+\epsilon} f(x) = \frac{1}{\epsilon} f(0) + \int_0^1 dx \ x^{-1} [f(x) - f(0)] + \epsilon \int_0^1 dx \ x^{-1} \ln x [f(x) - f(0)] + O(\epsilon^2).
\]

Remark that there is a pole of order \( 1/\epsilon \).

References

[1] Wilson K G and Kogut J 1974 Phys. Rep. 12C 75
[2] Doi M 1976 J. Phys. A: Math. Gen. 9 1465
    Doi M 1976 J. Phys. A: Math. Gen. 9 1479
[3] Grassberger P and Scheunert M 1980 Fortschr. Phys. 28 547
[4] Peliti L 1985 J. Physique 46 1469
[5] Lee B P and Cardy J 1995 J. Stat. Phys. 80 971
[6] Rey P-A and Droz M, unpublished
[7] Bramson M and Lebowitz J 1991 J. Stat. Phys. 62 297
[8] Howard M and Cardy J 1995 J. Phys. A: Math. Gen. 28 3599
[9] Cornell S and Droz M 1993 Phys. Rev. Lett. 70 3824
[10] Peliti L 1986 J. Phys. A: Math. Gen. 19 L365
[11] Ohtsuki T 1991 Phys. Rev. A 43 6917
[12] Friedman B, Levine G and O'Shaughnessy B 1992 Phys. Rev. A 46 R7343
[13] Lee B P 1994 J. Phys. A: Math. Gen. 27 2633
[14] Rey P-A, unpublished
[15] Droz M and Sasvári L 1993 *Phys. Rev. E* 48 R2343
[16] Rácz Z 1985 *Phys. Rev. A* 32 1129
[17] Doering C R and ben-Avraham D 1989 *Phys. Rev. Lett.* 62 2563
[18] Balboni D, Rey P-A and Droz M 1995 *Phys. Rev. E* 52 6220
[19] Amit D J 1984 *Field Theory, the Renormalization Group, and Critical Phenomena* (Singapore: World Scientific)
[20] Gel’fand I M and Shilov G E 1964 *Generalized functions* vol 1 (New York and London: Academic Press) p 84ff
[21] Alcaraz F C, Droz M, Henkel M and Rittenberg V 1994 *Ann. Phys., NY* 230 250
  Lushnikov A A 1987 *Phys. Lett.* A120 135
  Balding D, Clifford P and Green N J B 1988 *Phys. Lett.* A126 481
  Spouge J L 1988 *Phys. Rev. Lett.* 60 871
  Privman V, Cadilhe A M R and Glasser M L 1995 *J. Stat. Phys.* 81 881
  Grynberg M D and Stinchcombe R B 1995 *Phys. Rev. Lett.* 74 1242
[22] Rácz Z 1985 *Phys. Rev. Lett.* 55 1707
[23] Glauber R J 1963 *J. Math. Phys.* 4 294
[24] Doering C R and ben-Avraham D 1988 *Phys. Rev. A* 38 3035
[25] ben-Avraham D, Burschka M A and Doering C R 1990 *J. Stat. Phys.* 60 695
  Doering C R, Burschka M A and Horsthemke W 1991 *J. Stat. Phys.* 65 953
[26] Antosiewicz H A 1972 *Handbook of mathematical functions* edited by Abramowitz M and Stegun I A (National Bureau of Standards) p 446ff