Abstract

We prove a generalized Albanese property for the Picard stack of a smooth projective curve, which in particular implies Deligne’s unramified geometric class field theory. This Albanese property also specializes to the Cartier self-duality of the Picard stack and is in fact equivalent to it. We give a novel proof of the self-duality, by showing that any family of multiplicative line bundles on the space of invertible rational functions is trivial. This argument contains a new extension result for families of line bundles on projective space, communicated to the author by A. Beilinson.

1 Introduction

1.1 Fix a ground field \( k \) and a smooth, projective, and geometrically connected curve \( X \) over \( k \). Write \( \text{Pic} \) for the Picard stack of \( X \), which parameterizes line bundles on \( X \). The Abel-Jacobi morphism \( X \to \text{Pic} \) which sends \( x \mapsto \mathcal{O}_X(x) \) has the well-known Albanese property: any map \( X \to G \) into an abelian variety factors uniquely through a morphism \( \text{Pic} \to G \) of commutative group stacks.

In this paper we formulate and prove a generalization of this Albanese property for \( G \) belonging to a large class of commutative group stacks. Taking \( G = BA \) the classifying stack of a finite abelian group \( A \), we recover Deligne’s geometric formulation of global unramified Artin reciprocity. If \( k \) is a finite field, then the \( \text{fonctions-faisceaux} \) correspondence can be used to deduce unramified Artin reciprocity for the function field of \( X \).

The proof of our Albanese property reduces to the case \( G = B\mathbb{G}_m \), where it becomes the canonical self-duality of the Picard stack. The original content of this paper is mostly contained in a new proof of this self-duality inspired by Gaitsgory’s contractibility theorem in [3].

1.2 We call a symmetric monoidal category in which every object is invertible a Picard groupoid. All Picard groupoids will be assumed to be strictly commutative, meaning that the commutativity constraint on the square of any object is the identity. A commutative group stack is a sheaf of Picard groupoids on the fppf site of affine schemes over \( k \).

The Cartier 1-dual of a commutative group stack \( \mathcal{G} \) is the commutative group stack defined by

\[
\mathbb{D}(\mathcal{G}) := \text{Hom}(\mathcal{G}, B\mathbb{G}_m).
\]

Here \( \text{Hom} \) is the internal Hom functor given by

\[
\text{Map}(S, \text{Hom}(\mathcal{G}_1, \mathcal{G}_2)) := \text{Hom}_S(\mathcal{G}_1 \times S, \mathcal{G}_2 \times S)
\]

for any commutative group stacks \( \mathcal{G}_1, \mathcal{G}_2 \) and any affine scheme \( S \). Similarly, for any prestack \( \mathcal{Y} \) (i.e. a contravariant functor from affine schemes to groupoids), there is a commutative group stack \( \text{Map}(\mathcal{Y}, \mathcal{G}) \) defined by the formula

\[
\text{Map}(S, \text{Map}(\mathcal{Y}, \mathcal{G})) := \text{Map}_S(\mathcal{Y} \times S, \mathcal{G} \times S).
\]

Definition 1.2.1. We call a commutative group stack \( \mathcal{G} \) reflexive if the canonical morphism

\[
\mathcal{G} \to \mathbb{D}(\mathbb{D}(\mathcal{G}))
\]

is an isomorphism.
Evidently, Cartier 1-duality defines an involutive autoequivalence on the category of reflexive commutative

group stacks. See [2] for a recent general treatment of Cartier 1-duality.

The following is the main theorem of this paper.

**Theorem 1.2.2.** If \( \mathcal{G} \) is reflexive, then the morphism of commutative group stacks

\[
\text{Hom}(\text{Pic}, \mathcal{G}) \longrightarrow \text{Map}(X, \mathcal{G})
\]

given by restriction along the Abel-Jacobi map is an isomorphism.

As stated above, the case \( \mathcal{G} = BA \) is essentially Deligne’s version of unramified geometric class field

theory. Since a local system on \( X \) with finite coefficients does not admit any continuous deformation, this

case of Theorem 1.2.2 is not interesting “in families.” However, for other \( \mathcal{G} \) there may be nontrivial families

\( X \times S \rightarrow \mathcal{G} \), and we make essential use of them.

1.3 We will show that Theorem 1.2.2 reduces easily to the case \( \mathcal{G} = B \mathbb{G}_m \), which expresses the self-duality

of the Picard stack.

**Theorem 1.3.1.** Restriction along the the Abel-Jacobi map induces an isomorphism

\[
\text{D}(\text{Pic}) = \text{Hom}(\text{Pic}, B \mathbb{G}_m) \sim \text{Map}(X, B \mathbb{G}_m) = \text{Pic}.
\]

We will give a self-contained and more or less elementary proof of this isomorphism.

Below we will define the moduli spaces \( \text{Div} \) and \( \text{Rat} \) of divisors and invertible rational functions on \( X \),

respectively. We view them as discrete commutative group stacks, i.e. sheaves of abelian groups on the big

fppf site.

**Proposition 1.3.2.** The natural sequence

\[
\text{Rat} \longrightarrow \text{Div} \longrightarrow \text{Pic}
\]

is a cofiber sequence of commutative group stacks.

On the other hand, we have the following result about the moduli space of divisors.

**Lemma 1.3.3.** The canonical morphism \( X \rightarrow \text{Div} \) induces an equivalence of commutative group stacks

\[
\text{D}(\text{Div}) = \text{Hom}(\text{Div}, B \mathbb{G}_m) \sim \text{Map}(X, B \mathbb{G}_m) = \text{Pic}.
\]

Combining Proposition 1.3.2 and Lemma 1.3.3, the proof of Theorem 1.3.1 reduces to the following

vanishing statement, which is the main technical result of this work.

**Lemma 1.3.4.** The commutative group stack \( \text{D}(\text{Rat}) \) is trivial.

This lemma bears a resemblance to the main theorem of [4], in that it asserts a certain cohomological

triviality of spaces of rational maps. It is also similar to Proposition 4.9.1 in [1]. Our argument is based on

the proof sketched in *loc. cit.*, but we carefully prove the following extension lemma, which is implicit there.

**Lemma 1.3.5.** Let \( V \) be a vector space and \( U \subset \mathbb{P}(V) \) an open subset whose complement has codimension

at least three. Then restriction induces an isomorphism

\[
\text{Map}(\mathbb{P}(V), B \mathbb{G}_m) \sim \text{Map}(U, B \mathbb{G}_m).
\]

If \( S \) is smooth, then a line bundle on \( U \times S \) extends uniquely to \( \mathbb{P}(V) \times S \) for general reasons provided
the complement of \( U \) has codimension at least two. We do not know if Lemma 1.3.5 in which \( S \) is arbitrary,
holds under this relaxed hypothesis.
1.4 Theorem 1.3.1 defines an equivalence

\[ \mathbb{D}(\text{Pic}) \rightarrow \text{Pic}, \]

but in fact we can give an explicit formula for the inverse as well. This amounts to writing down the corresponding pairing \( \text{Pic} \times \text{Pic} \rightarrow B\mathbb{G}_m \). It is a well-known formula due to Deligne.

Given a line bundle \( \mathcal{L} \) on \( X \times S \), the pushforward \( R_p \mathcal{L} \) is a perfect complex because the projection \( p : X \times S \rightarrow S \) is proper and flat. Thus we may consider its determinant line bundle \( \det R_p \mathcal{L} \) on \( S \).

**Proposition 1.4.1.** The pairing on \( \text{Pic} \) induced by Theorem 1.3.1 is given by the formula

\[ \langle \mathcal{L}_1, \mathcal{L}_2 \rangle = \det R_p (\mathcal{L}_1 \otimes \mathcal{O}_{X \times S} \mathcal{L}_2) \otimes \mathcal{O}_{S} \left( \det R_p \mathcal{L}_1 \right)^{-1} \otimes \mathcal{O}_{S} \left( \det R_p \mathcal{L}_2 \right)^{-1} \otimes \mathcal{O}_S \det R_p \mathcal{O}_{X \times S}. \]

1.5 In Section 2 we review definitions and prove some basic results about moduli spaces of divisors, including Lemma 1.3.3. Section 3 contains the proofs of the remaining results stated in this introduction. In Section 4 we explain how to deduce unramified Artin reciprocity for function fields from Theorem 1.2.2.

Although the applications of Theorem 1.2.2 given here are known theorems, Lemmas 1.3.4 and 1.3.5 appear to be new, and may be of independent interest. In future work we hope to prove a generalization of Theorem 1.2.2 to higher stacks. The case \( \mathcal{G} = B^2 \mathbb{G}_m \) is especially interesting, and can be used to prove that the the Brauer group of \( X \) vanishes for \( k \) finite. We also hope to extend the formulation and proofs of these results to allow \( X \) to be an open curve or a punctured formal disk, which would in particular yield new proofs of global ramified and local class field theory, respectively, when \( k \) is finite.

A related problem is to study the Fourier-Mukai transform between quasicoherent sheaves on \( \mathcal{G} \) and on \( \mathbb{D}(\mathcal{G}) \). In the appendix to [3] it is stated as an informal principle that for \( \mathcal{G} \) reflexive, the universal line bundle on \( \mathcal{G} \times \mathbb{D}(\mathcal{G}) \) defines such a transform, and that it is an equivalence. Many interesting applications of this principle, such as the categorical geometric Langlands correspondence for a torus (also known as the Fourier-Laumon transform), require derived algebraic geometry. Even if \( \mathcal{G} \) is classical, its (appropriately defined) dual \( \mathbb{D}(\mathcal{G}) \) may not be. It also seems likely that the scope of Theorem 1.2.2 could be widened by reformulating it in the setting of derived algebraic geometry.

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## 2 Divisors and the affine Grassmannian

2.1 The following definition is standard, but we include it for completeness and to fix notations. We continue to denote by \( S \) an arbitrary affine scheme.

**Definition 2.1.1.** An \( S \)-family of effective Cartier divisors on \( X \) is a closed subscheme \( Z \subset X \times S \), finite flat over \( S \), whose ideal sheaf \( \mathcal{I}_Z \) is a line bundle on \( X \times S \).

Equivalently, an \( S \)-family of effective Cartier divisors on \( X \) is a line bundle on \( X \times S \) equipped with a monomorphism into \( \mathcal{O}_{X \times S} \). We use the standard notation

\[ \mathcal{O}_{X \times S}(Z) := \mathcal{I}_Z^{-1}. \]

The degree of \( Z \) is the degree of the map \( Z \rightarrow S \), which is a locally constant function on \( S \).

As \( S \) varies, effective divisors form an fpfp sheaf \( \text{Div}^+ \) valued in commutative monoids, with the binary operation defined by tensor product of ideal sheaves. The moduli space \( \text{Div} \) of Cartier divisors on \( X \) is by definition the Zariski sheafified group completion of \( \text{Div}^+ \) (we will see below that \( \text{Div} \) is an fpfp sheaf). The degree extends to a homomorphism \( \text{Div} \rightarrow \mathbb{Z} \) of sheaves of abelian groups, where \( \mathbb{Z} \) is viewed as a discrete group scheme.
For any \( d \geq 0 \) let \( X^{(d)} \) denote the \( d \)th symmetric power of \( X \), i.e. the quotient of \( X^d \) by the action of the symmetric group \( \Sigma_d \) in the sense of geometric invariant theory. Using the maps \( X^{(d)} \times X^{(e)} \to X^{(d+e)} \) for all \( d, e \geq 0 \) we give \( \coprod_{d \geq 0} X^{(d)} \) the structure of a commutative monoid in schemes.

Note that there is a canonical morphism \( X \to \text{Div}_+ \) which sends a point of \( X \) to its graph. The following proposition is well-known.

**Proposition 2.2.1.** There is a unique isomorphism \( \coprod_{d \geq 0} X^{(d)} \xrightarrow{\sim} \text{Div}_+ \) of commutative monoids which extends \( X \to \text{Div}_+ \).

Now we can prove one of the lemmas from which Theorem 1.3.1 will follow.

**Proof of Lemma 1.3.3.** Since \( \mathbb{B} \text{G}_m \) is a group stack, it suffices to prove that

\[
\text{Hom}(\text{Div}_+, \mathbb{B} \text{G}_m) \xrightarrow{\sim} \text{Map}(X, \mathbb{B} \text{G}_m) = \text{Pic}
\]

is an equivalence. We explicitly describe the inverse as a pairing

\[
\text{Pic} \times \text{Div}_+ \to \mathbb{B} \text{G}_m.
\]

Namely, it sends \((\mathcal{L}, Z) \mapsto \langle \mathcal{L}, \mathcal{O}_{X \times S}(Z) \rangle\), using the pairing introduced in Proposition 1.4.1. A standard argument using the theorem of the square shows that the pairing is bilinear.

To see that \( \text{Pic} \to \text{Hom}(\text{Div}_+, \mathbb{B} \text{G}_m) \to \text{Pic} \) is isomorphic to the identity, we produce an isomorphism

\[
\langle \mathcal{L}, \mathcal{O}_{X \times S}(\Gamma_x) \rangle \xrightarrow{\sim} x^* \mathcal{L}
\]

for any point \( x : S \to X \). For this, use the exact sequences

\[
0 \to \mathcal{O}_{X \times S}(-\Gamma_x) \to \mathcal{O}_{X \times S} \to \mathcal{O}_{\Gamma_x} \to 0
\]

and

\[
0 \to \mathcal{L}(-\Gamma_x) \to \mathcal{L} \to x^* \mathcal{L} \to 0.
\]

Pushing forward to \( S \) and taking determinants, we obtain the desired isomorphism.

Conversely, we must show that the composition

\[
\text{Hom}(\text{Div}_+, \mathbb{B} \text{G}_m) \to \text{Pic} \to \text{Hom}(\text{Div}_+, \mathbb{B} \text{G}_m)
\]

is isomorphic to the identity. Let \( \mathcal{M} \) be a multiplicative line bundle on \( \text{Div}_+ \times S \), and \( \mathcal{\hat{M}} \) the result of applying this composition to \( \mathcal{M} \). The multiplicativity implies that the restriction of \( \mathcal{M} \) to \( X^d \times S \) is

\[
\langle \mathcal{M}|_X \rangle \boxtimes \cdots \boxtimes \langle \mathcal{M}|_X \rangle,
\]

with the obvious \( \Sigma_d \)-equivariant structure. The previous paragraph supplies an isomorphism \( \mathcal{M}|_X \to \mathcal{\hat{M}}|_X \), and since \( \mathcal{\hat{M}} \) is multiplicative, it may therefore be identified with \( \mathcal{M} \) over \( X^d \times S \). By Proposition 2.2.1 it remains to show that the resulting isomorphism is \( \Sigma_d \)-equivariant. This follows from the fact that \( \mathcal{M} \), viewed as a functor

\[
\text{Div}_+ \times S \to \mathbb{B} \text{G}_m \times S
\]

between commutative group stacks over \( S \), is symmetric monoidal (by its construction in terms of \( \langle \ , \ \rangle \)).
2.3 We now introduce the affine Grassmannian $\operatorname{Gr}$ for $\mathbb{G}_m$ and identify it with $\operatorname{Div}$. First we define the effective locus: a map $S \to \operatorname{Gr}_+$ is an isomorphism class of pairs $(\mathcal{L}, \ell)$ made up of a line bundle $\mathcal{L}$ on $X \times S$ and a global section $\ell \in \Gamma(X \times S, \mathcal{L})$ such that

$$0 \to \mathcal{O}_{X \times S} \xrightarrow{\ell} \mathcal{L} \to \mathcal{L}/\mathcal{O}_{X \times S} \to 0$$

is exact, the support of $\mathcal{L}/\mathcal{O}_{X \times S}$ is proper (equivalently, finite) over $S$, and $\mathcal{L}/\mathcal{O}_{X \times S}$ is flat over $S$ (equivalently, $\ell$ does not vanish identically on the fiber over any closed point of $S$). Tensor product and pullback of line bundles extend to product and pullback operations on such pairs, making $\operatorname{Gr}_+$ a sheaf of commutative monoids. The following proposition is more or less tautological.

**Proposition 2.3.1.** The map $Z \mapsto (\mathcal{O}(Z), 1)$ is an isomorphism $\operatorname{Div}_+ \to \operatorname{Gr}_+$.

When working with moduli spaces of generic data, we use the following notion.

**Definition 2.3.2.** We call an open set $U \subset X \times S$ a domain relative to $S$ if the projection $U \to S$ is surjective.

Then we define the full affine Grassmannian $\operatorname{Gr}$ as follows: a map $S \to \operatorname{Gr}$ consists of a line bundle on $X \times S$ equipped with a rational trivialization, i.e. a trivialization defined over a domain relative to $S$, up to isomorphism preserving the trivialization. We view $\operatorname{Gr}$ as a sheaf of abelian groups on the big fppf site.

Evidently there is a canonical isomorphism $\operatorname{Gr}_+ \to \operatorname{Gr}$. By Proposition 2.3.1 we obtain a canonical homomorphism $\operatorname{Div} \to \operatorname{Gr}$.

**Theorem 2.3.3.** The morphism $\operatorname{Div} \to \operatorname{Gr}$ is an isomorphism.

Although this theorem follows from some fairly standard geometric arguments, the author could not find it stated in this form in the literature.

2.4 In order to prove Theorem 2.3.3, we need a couple of lemmas.

**Lemma 2.4.1.** Let $U \subset X \times S$ be a domain. Then, Zariski locally on $S$, there is an effective relative Cartier divisor $Z \subset X \times S$ such that $X \times S \setminus Z \subset U$, and fppf locally on $S$ we can take $Z$ to be a union of graphs of points $S \to X$.

**Proof.** The second statement follows from the first in view of Proposition 2.2.1, and we may assume that $S$ is affine. Fix a closed point $s \in S$: we will produce a Zariski neighborhood of $s$ where the first claim holds. Also choose a closed point $x \in X$ and write $X = X \setminus \{x\}$. Now let $Y = X \times S \setminus U$ and choose a function $f$ on $X \times S$ which vanishes on $Y \cap X \times S$ but does not vanish identically on the fiber $X_s$. We can think of $f$ as a global section of $\mathcal{O}_{X \times S}(nx)$ for some $n \geq 0$, and by Serre’s theorem for large enough $n$ we have $R^i\pi_*\mathcal{O}_{X \times S}(nx) = 0$ for $i \neq 0$ (here $\pi : X \times S \to S$ is the projection). Since $\pi$ is proper and flat $R\pi_*\mathcal{O}_{X \times S}(nx)$ is a perfect complex, and being concentrated in degree zero it must be a vector bundle. Now $f$ can be viewed as a global section of this vector bundle, and the locus in $S$ where this section does not vanish is the desired neighborhood of $s$.

**Lemma 2.4.2.** The canonical morphism $\operatorname{Gr}_+ \to \operatorname{Gr}$ identifies $\operatorname{Gr}$ with the Zariski sheafification of the group completion of $\operatorname{Gr}_+$.

**Proof.** It is straightforward to verify that the group completion of $\operatorname{Gr}_+$ injects into $\operatorname{Gr}$, so we show surjectivity. Fix an $S$-point $(\mathcal{L}, \ell)$ of $\operatorname{Gr}$, where $\ell$ has domain $U \subset X \times S$. By Proposition 2.4.1 after Zariski localization on $S$ we may assume that $U$ is the complement of an effective Cartier divisor $Z \subset X \times S$. Thus

$$\ell \in \Gamma(X \times S, \mathcal{L}(nZ))$$

for sufficiently large $n$, and it suffices to prove that $(\mathcal{L}(nZ), \ell)$ defines a map $S \to \operatorname{Gr}_+$. Injectivity of $\ell : \mathcal{O}_{X_s} \to \mathcal{L}(nZ)$ follows from the fact that $\ell$ is nonzero over the schematically dense open set $U$. The support of $\mathcal{L}/\mathcal{O}_{X_s}$ is automatically proper over $S$ because $X$ is proper, and flat over $S$ because $\ell$ does not vanish identically on any fiber $X_s$ for $s \in S$.

**Proof of Theorem 2.3.3.** The theorem follows immediately from Lemma 2.4.2 and the definition of $\operatorname{Div}$. 

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5
3 Self-duality and Albanese property of the Picard stack

3.1 From now on $X$ is projective. First we show that the equivalence between line bundles and divisors modulo rational equivalence, well-known at the level of $k$-points, is valid in families.

Define the moduli space $\text{Rat}$ of invertible rational functions on $X$ as follows: a point $S \to \text{Rat}$ is a map $U \to \mathbb{G}_m$ where $U \subset X \times S$ is a domain, and we identify maps which agree on a domain. This is the datum of a rational trivialization of the trivial line bundle on $X \times S$, so there is a canonical morphism

$$\text{Rat} \to \text{Gr} \hookrightarrow \text{Div}.$$ 

There is also a canonical morphism $\text{Gr} \to \text{Pic}$ which forgets the rational trivialization, and the composition $\text{Rat} \to \text{Gr} \to \text{Pic}$ is canonically trivialized.

**Proof of Proposition 1.3.2.** It is immediate from the definitions that the morphism

$$\text{Map}(S, \text{cofib}(\text{Rat} \to \text{Gr})) \to \text{Map}(S, \text{Pic})$$

is fully faithful for any $S$. To see essential surjectivity we prove that, Zariski locally on $S$, any line bundle on $X \times S$ admits a rational trivialization. Fix a closed point $s \in S$ and trivialize the restriction of $\mathcal{L}$ to $X_s$ over an open set $U \subset X_s$. An argument similar to the one in Lemma 2.4.1 shows that this trivialization extends to the product of $U$ and an open neighborhood of $s$.

3.2 Before proving Lemma 1.3.4 we need the following extension result.

**Proof of Lemma 1.3.3.** Fix a test scheme $S$. Extending scalars if necessary, we may assume that $U$ has a rational point $p \in U(k)$. Let $U_0 \subset U \setminus \{p\}$ be the open subset consisting of points $x$ such that the line through $x$ and $p$ is contained in $U$. The complement of $U_0$ is the union of lines through $p$ and points in $\mathbb{P}(V) \setminus U$, hence has codimension at least two. It follows that functions on $U_0 \times S$ extend to $\mathbb{P}(V) \times S$, so the restriction functors

$$\text{Map}(\mathbb{P}(V) \times S, B\mathbb{G}_m) \to \text{Map}(U \times S, B\mathbb{G}_m) \to \text{Map}(U_0 \times S, B\mathbb{G}_m)$$

are fully faithful. Therefore it suffices to show that for any line bundle $\mathcal{L}$ on $U \times S$, the restriction $\mathcal{L}|_{U_0 \times S}$ extends to $\mathbb{P}(V) \times S$.

Let $E \subset U \times U_0$ be the set of pairs $(x, y)$ such that $x$, $y$, and $p$ are collinear. Write $\pi : E \to U$ and $\rho : E \to U_0$ for the projections. Since $\rho$ is a $\mathbb{P}^1$-bundle, the degree of $(\pi \times \text{id}_S)^*\mathcal{L}$ is a locally constant function on $U_0 \times S$. Thus, by twisting appropriately we can assume that $(\pi \times \text{id}_S)^*\mathcal{L} \cong (\rho \times \text{id}_S)^*\mathcal{L}'$ for some line bundle $\mathcal{L}'$ on $U_0 \times S$. Now we are trying to prove that $\mathcal{L}|_{U_0 \times S}$ is pulled back from $S$.

Restricting the chosen isomorphism along $\Delta_{U_0} \times \text{id}_S$, we obtain $\mathcal{L}|_{U_0 \times S} \cong \mathcal{L}'$. On the other hand, we can restrict the same isomorphism along the section of $\rho \times \text{id}_S$ with image $\{p\} \times U_0 \times S \subset E \times S$, which gives an identification of $\mathcal{L}'$ with the pullback to $U_0 \times S$ of $(\rho \times \text{id}_S)^*\mathcal{L}'$.

Let $K$ denote the $k$-vector space of rational functions on $X$. Any vector space can be viewed as an ind-scheme, but $K$ has the canonical presentation

$$\colim Z \Gamma(X, \mathcal{O}_X(Z)) = K$$

where $Z$ runs through all effective divisors on $X$.

Observe that the ind-scheme $K^\times := K \setminus \{0\}$ is a commutative monoid under multiplication, although not a group. There is an injective monoid homomorphism $K^\times \to \text{Rat}$, which can be seen as follows. A morphism $S \to K^\times$ lands in some $\Gamma(X, \mathcal{O}_X(Z)) \setminus \{0\}$, hence determines a function $f : (X \setminus Z) \times S \to \mathbb{A}^1$ which does not vanish identically on any fiber over a point of $S$. This means that the complement of the vanishing locus of $f$ is a domain.
Lemma 3.2.1. The morphism \( K^\times \to \text{Rat} \) realizes \( \text{Rat} \) as the Zariski sheafified group completion of the monoid \( K^\times \).

Proof. The group completion of \( K^\times \) injects into \( \text{Rat} \) because the multiplication in \( K^\times \) is cancellative, i.e. \( K^\times \) injects into its group completion. Thus it remains to prove surjectivity.

Fix a domain \( U \subset X \times S \) and an invertible function \( f : U \to \mathbb{G}_m \). Fix a point \( x \in X \) and resume the notation from the proof Lemma 2.4.3. Shrink \( U \) if necessary so that \( U \subset X \times S \), and fix a point \( s \in S \). Then there is a function \( g \) on \( X \times S \) which vanishes on the complement of \( U \) but does not vanish identically on the fiber \( X_s \). Replacing \( S \) by a Zariski neighborhood of \( s \), we may assume \( g \) does not vanish identically on any fiber over a point of \( S \). Now there exists \( m \) so that \( fg^m \) is regular on \( X \times S \), which means for sufficiently large \( n \) we have
\[
fg^n \in \Gamma(X \times S, \mathcal{O}_X(nx) \boxtimes \mathcal{O}_S)
\]
and likewise for \( g^m \). Thus \( f \) can be written as the ratio of two \( S \)-points of \( \Gamma(X, \mathcal{O}_X(nx)) \setminus \{0\} \).

The previous lemma implies that \( K^\times \times K^\times \to \text{Rat} \) given by \( (f,g) \mapsto f/g \) is a Zariski epimorphism. Moreover, it is evidently \( \mathbb{G}_m \)-equivariant with respect to trivial \( \mathbb{G}_m \)-action on \( \text{Rat} \), so this map factors through
\[
(K^\times \times K^\times)/\mathbb{G}_m \to \text{Rat}.
\]
Observe that the domain is the ind-projective space \( \mathbb{P}(K \oplus K) \) with the subspaces \( \mathbb{P}(K \oplus 0) \) and \( \mathbb{P}(0 \oplus K) \) removed. Moreover, the fiber over \( 1 \in \text{Rat} \) is the diagonally embedded \( \mathbb{P}(K) \). Since (3.2.1) is a monoid homomorphism and \( \text{Rat} \) is a group, it follows that (3.2.1) is a bundle with fiber \( \mathbb{P}(K) \).

Proof of Lemma 1.3.3. Suppose we are given a function \( (K^\times \times K^\times)/\mathbb{G}_m \times S \to \mathbb{A}^1 \). We claim it extends to \( \mathbb{P}(K \oplus K) \times S \), which will imply it is pulled back from \( S \). Since (3.2.1) is a projective space bundle, it will follow that any function on \( \text{Rat} \times S \) is pulled back from \( S \), and hence \( \text{Hom}(\text{Rat}, \mathbb{G}_m) = 1 \). Namely, if \( V \subset K \) is a finite-dimensional subspace with \( \dim V > 1 \), then we obtain a function on the complement in \( \mathbb{P}(V \oplus V) \times S \) of \( (\mathbb{P}(V \oplus \{0\}) \cup \mathbb{P}((0 \oplus V)) \times S \). Since \( \mathbb{P}(V \oplus \{0\}) \) and \( \mathbb{P}((0 \oplus V)) \) are subspaces of codimension at least two, the function extends to \( \mathbb{P}(V \oplus V) \times S \) as desired.

Given a multiplicative line bundle on \( \text{Rat} \times S \), pull back along the map (3.2.1) to obtain a line bundle \( \mathcal{L} \) on \( (K^\times \times K^\times)/\mathbb{G}_m \times S \). It is enough to show that \( \mathcal{L} \) is pulled back from \( S \), because the fibers of (3.2.1) are isomorphic to \( \mathbb{P}(K) \), and hence pullback of line bundles along (3.2.1) is fully faithful. Lemma 1.3.3 implies that \( \mathcal{L} \) extends to \( \mathbb{P}(K \oplus K) \times S \). Thus, in order to prove that \( \mathcal{L} \) is pulled back from \( S \), we need only show that its degree is zero. For this, note that \( \mathcal{L} \) is trivial when restricted to the diagonal \( \mathbb{P}(K) \) because (3.2.1) is trivial on \( \mathbb{P}(K) \).

Proof of Theorem 1.3.1. The Abel-Jacobi map factors as \( X \to \text{Div} \to \text{Pic} \). By Proposition 1.3.3 pullback along \( X \to \text{Div} \) induces an equivalence \( \text{Hom}(\text{Div}, \mathbb{G}_m) \to \text{Pic} \). Together Theorem 1.3.4 and Proposition 1.3.2 imply that
\[
\text{Hom}(\text{Pic}, \mathbb{G}_m) \to \text{Hom}(\text{Div}, \mathbb{G}_m)
\]
is an equivalence.

3.3 We now deduce Proposition 1.4.1 and Theorem 1.2.2 from Theorem 1.3.1.

Proof of Proposition 1.4.1 The calculation from the proof of Lemma 1.3.3 shows that
\[
\text{Pic} \to \text{Hom}(\text{Pic}, \mathbb{G}_m) \to \text{Pic}
\]
is isomorphic to the identity, where the first morphism is the pairing \( (\cdot, \cdot) \) and the second is restriction along the Abel-Jacobi morphism. The claim now follows from Theorem 1.3.1.
Proof of Theorem 1.2.2. Since Pic and \( \mathcal{G} \) are reflexive, we have

\[
\text{Hom}(\text{Pic}, \mathcal{G}) \cong \text{Hom}(\text{Hom}(\mathcal{G}, B\mathbb{G}_m), \text{Hom}(\text{Pic}, B\mathbb{G}_m)),
\]

which in turn identifies with \( \text{Hom}(\text{Hom}(\mathcal{G}, B\mathbb{G}_m), \text{Pic}) \) by Theorem 1.3.1. The bilinear pairing

\[
\text{Map}(X, \mathcal{G}) \times \text{Hom}(\mathcal{G}, B\mathbb{G}_m) \longrightarrow \text{Pic}
\]

induces a morphism

\[
\text{Map}(X, \mathcal{G}) \longrightarrow \text{Hom}(\text{Hom}(\mathcal{G}, B\mathbb{G}_m), \text{Pic}).
\]

A routine calculation shows that this is the desired inverse.

\[\square\]

4 Class field theory

4.1 In this section we explain how Theorem 1.2.2 implies unramified global geometric class field theory, and then deduce unramified Artin reciprocity for \( k \) finite. Let \( A \) be a finite abelian group.

For a Picard groupoid \( \mathcal{C} \), we write \( \pi_0 \mathcal{C} \) for the abelian group of isomorphism classes in \( \mathcal{C} \). In the case of a commutative group stack \( G \), we use \( \pi_0 G \) to denote the fppf sheafification of the functor \( S \mapsto \pi_0 \text{Map}(S, G) \).

Theorem 4.1.1. Pullback along \( X \to \text{Pic} \) induces an equivalence

\[
\text{Hom}(\text{Pic}, BA) \cong \text{Map}(X, BA).
\]

Proof. By Theorem 1.2.2 it suffices to prove that if we consider \( A \) as a discrete group scheme over \( k \), then \( BA \) is reflexive. Note that \( (BA)^\vee \) is the 0-Cartier dual \( A^* \) of \( A \), and

\[
\pi_0 ((A^*)^\vee) = \text{Ext}^1(A^*, \mathbb{G}_m) = 1
\]

by Proposition 3.3.1 in [5]. Therefore the canonical morphism \( BA \to (A^*)^\vee \) is an isomorphism.

\[\square\]

Maps \( X \to BA \) are equivalent to \( \acute{\text{e}} \text{tale} \) local systems on \( X \) with coefficients in \( A \), or equivalently continous homomorphisms \( \pi_1(X, \eta) \to A \), where \( \eta \) is a geometric generic point of \( X \). A homomorphism \( \text{Pic} \to BA \) is a multiplicative local system on \( \text{Pic} \) with coefficients in \( A \).

4.2 From now on we assume that \( k = \mathbb{F}_q \) is finite. The following instance of the fonctions-faisceaux correspondence is well-known, but we include a proof for completeness.

Proposition 4.2.1. Taking the trace of Frobenius at the \( \mathbb{F}_q \)-points of \( \text{Pic} \) induces an isomorphism

\[
\pi_0 \text{Hom}(\text{Pic}, BA) \cong \text{Hom}(\pi_0 \text{Pic}(\mathbb{F}_q), A).
\]

Proof. There is a splitting \( \text{Pic} \cong \text{Pic}^0 \times \mathbb{Z} \), and the claim is obvious for \( \mathbb{Z} \) so we are reduced to proving it for \( \text{Pic}^0 \). Since

\[
\text{Hom}(B\mathbb{G}_m, BA) = \text{Hom}(\mathbb{G}_m, A) = 1,
\]

we obtain an equivalence

\[
\text{Hom}(\text{Pic}^0, BA) \cong \text{Hom}(\pi_0 \text{Pic}^0, BA).
\]

Moreover \( \pi_0 B\mathbb{G}_m = 1 \), so we have reduced to the case of the classical Jacobian \( J = \pi_0 \text{Pic}^0 \), an abelian variety. Lang’s isogeny \( J \to J \), defined by \( g \mapsto g^{-1} \text{Fr}_q(g) \), is a Galois covering with group \( J(\mathbb{F}_q) \). Thus any homomorphism \( J(\mathbb{F}_q) \to A \) induces a multiplicative \( A \)-local system on \( J \), which supplies the inverse map.

\[\square\]

So we have constructed functorial isomorphisms

\[
\text{Hom}(\pi_0 \text{Pic}(\mathbb{F}_q), A) \cong \text{Hom}(\pi_1^{\acute{\text{e}} \text{t}}(X, \eta), A).
\]

(4.2.1)
Theorem 4.2.2 (Unramified Artin reciprocity). There is a unique injection with dense image

\[ \pi_0 \text{Pic}(F_q) \longrightarrow \pi_1^{\text{ét}}(X, \overline{\eta})^{\text{ab}} \]

which, for any closed point \( x \in X \), sends \( \mathcal{O}_X(x) \) to the Frobenius element \( \text{Fr}_x \).

Proof. The existence of such an injection follows from the isomorphism of functors \([4.2.1]\) and the Yoneda lemma; it has dense image because it induces an isomorphism after passing to the profinite completion of \( \pi_0 \text{Pic}(F_q) \). Uniqueness is immediate, because the line bundles \( \mathcal{O}_X(x) \) generate \( \pi_0 \text{Pic}(F_q) \).

Recall that \( \text{Fr}_x \) is the image of 1 under the homomorphism

\[ \hat{\mathbb{Z}} = \pi_1(F_q(x)) \xrightarrow{x} \pi_1^{\text{ét}}(X, \overline{\eta})^{\text{ab}}. \]

To show that \( \mathcal{O}_X(x) \) maps to \( \text{Fr}_x \), we must prove that for any abelian group \( A \) and any homomorphism \( \varphi : \pi_0 \text{Pic}(F_q) \rightarrow A \), the homomorphism \( \psi : \pi_1^{\text{ét}}(X, \overline{\eta})^{\text{ab}} \rightarrow A \) induced by \([4.2.1]\) sends \( \text{Fr}_x \) to \( \varphi(\mathcal{O}_X(x)) \). By Proposition \([4.2.1]\) there is a unique multiplicative \( A \)-local system \( E_\varphi \) on \( \text{Pic} \) which gives rise to \( \varphi \) by taking trace of Frobenius. By construction, the pullback of \( E_\varphi \) along the Abel-Jacobi map corresponds to \( \psi \). Thus \( \psi(\text{Fr}_x) \) equals the trace of Frobenius acting on the fiber of \( E_\varphi \) at \( \mathcal{O}_X(x) \), which is just \( \varphi(\mathcal{O}_X(x)) \). \( \square \)

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