Small Gamma Products with Simple Values

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Introduction. Central as the gamma function is, it is surprising that its only known simple specific values are $\Gamma(m)$ and $\Gamma(m + \frac{1}{2})$ for integral $m$. There are, however, numerous formulas that relate specific values of $\Gamma$ to other functions, such as elliptic or hypergeometric functions - or to other specific values of $\Gamma$ itself. Among the latter there are products that have very simple values. An example is

$$\Gamma\left(\frac{1}{14}\right) \Gamma\left(\frac{9}{14}\right) \Gamma\left(\frac{11}{14}\right) = 4\pi^{3/2},$$

which recently occurred in the Problems section of the “Monthly” [1], see also [2]. There is also, of course, the classical multiplier formula

$$\prod_{k=1}^{m-1} \Gamma\left(\frac{k}{m}\right) = (2\pi)^{(m-1)/2} m^{-1/2},$$

but it would be more interesting to find simple values for products of fewer factors. In this note we do just that, for a large class of products, and with little computational effort.

Consider any odd integer $n > 1$ and the set $\Phi(2n)$ of numbers in $[0, 2n]$ that are relatively prime to $2n$. (Its cardinality is $\varphi(2n)$, Euler’s totient function.) $\Phi(2n)$ is a group with respect to multiplication modulo $2n$. Let $\nu(n)$ be the order of the subgroup generated by $n + 2$, and let $A$ be this subgroup or any one of its cosets. Let $b(A)$ count the $x \in A$ that are larger than $n$. Our main result is

Theorem.

$$\prod_{x \in A} \Gamma\left(\frac{x}{2n}\right) = 2^{b(A)} \pi^{\nu(n)/2}.$$
The Formula. Throughout this paper, \( n > 1 \) will denote a “fixed” odd integer, and let \( \Phi(n) \) be the set (group) of all integers in the interval \([0, n]\) relatively prime to \( n \). Then \( \phi(n) = \phi(2n) \), and the map \( \alpha : \Phi(n) \to \Phi(2n) \) given by

\[
\alpha : \Phi(n) \to \Phi(2n), \quad \alpha(y) = y \quad \text{if} \quad y \quad \text{odd}, \quad \text{else} \quad \alpha(y) = y + n
\]

is a group isomorphism. (The proof is a simple exercise, distinguishing 3 cases.) The inverse is

\[
\alpha^{-1} : \Phi(2n) \to \Phi(n), \quad \alpha^{-1}(x) = x \quad \text{if} \quad x < n, \quad \text{else} \quad \alpha^{-1}(x) = x - n.
\]

We also need a map \( \beta \) which “halves” the elements of \( \Phi(n) \) (modulo \( n \)):

\[
\beta : \Phi(n) \to \Phi(n), \quad \beta(y) = y/2 \quad \text{if} \quad y \quad \text{even}, \quad \text{else} \quad \beta(y) = (y + n)/2.
\]

The doubling formula for \( \Gamma \) is needed in the following form

\[
\Gamma(t) = c_t \Gamma(2t) / \Gamma \left( t + \frac{1}{2} \right),
\]

where \( c_t = (2\sqrt{\pi})2^{-2t} \). In (7) set \( t = x/2n \), where \( x \in \Phi(2n) \),

\[
\Gamma \left( \frac{x}{2n} \right) = c_{x/2n} \Gamma \left( \frac{x}{n} \right) / \Gamma \left( \frac{x + n}{2n} \right).
\]

This equation is of the form

\[
\Gamma \left( \frac{x}{2n} \right) = \varepsilon(x)c_{x/2n} \Gamma \left( \frac{y}{n} \right) / \Gamma \left( \frac{z}{n} \right)
\]

with \( y = x, z = (x + n)/2 \in \Phi(n) \) and \( \varepsilon(x) = 1 \) when \( x < n \). When \( x > n \), we apply the reduction formula \( \Gamma(t) = (t - 1)\Gamma(t - 1) \) to both \( \Gamma \)'s on the right in (8), and cancel factors \((x - n)/n\). The result is

\[
\Gamma \left( \frac{x}{2n} \right) = 2c_{x/2n} \Gamma \left( \frac{x - n}{n} \right) / \Gamma \left( \frac{x - n}{2n} \right),
\]
which is of the form (9) with \( y = x - n, z = (x - n)/2 \in \Phi(n) \) and \( \varepsilon(x) = 2 \).

**Lemma.** Let \( n \) be an odd integer, \( n > 1 \), and \( x \in \Phi(2n) \). Then (9) holds, where 

\[
y = \alpha^{-1}(x), \ z = \beta(y) \in \Phi(n).
\]

Further, \( x - n = 2y - 2z \) and \( y \equiv 2z \pmod{n} \).

**Proof.** Distinguish the two cases \( x < n \) and \( x > n \). (Note that \( x \in \Phi(2n) \) is odd.)

If \( x < n \), then \( \alpha^{-1}(x) = x = y \) (odd), so \( \beta(y) = (y + n)/2 = (x + n)/2 = z \).

Also, \( 2y - 2z = 2x - (x + n) = x - n \) and \( 2z = y + n \equiv y \pmod{n} \).

If \( x > n \), then \( \alpha^{-1}(x) = x - n \) (even), so \( \beta(y) = y/2 = (x - n)/2 = z \).

Also, \( 2y - 2z = 2(x - n) - (x - n) = x - n \), and \( 2z = y \).

**Proof of (3).** The members of \( \Phi(n) \) are taken as vertices of a directed labeled graph. The edges are the pairs \((y, z) = (y, \beta(y))\); such an edge is labeled \( x = \alpha(y) \in \Phi(2n) \).

The vertices have in- and outdegree 1, so the connected components are cycles. In fact, the component of 1 is the cyclic group generated by 2; denote its order by \( \nu(n) \). The other components are the cosets. Let \( B \) be any one of these cycles. Form the product \( P = \prod \Gamma(x/2n) \) of the left sides of (9), where the product extends over \( x \in \alpha B \), i.e., the labels of the edges of \( B \). Similarly, take the product of the right sides of (9), and note that the product telescopes as all the \( \Gamma \)'s cancel, leaving only

\[
P = \prod_{\alpha B} \varepsilon(x) e_{-x/2n} = (2\sqrt{\pi})^{\nu(n)} 2^{-2(\Sigma x)/2n} \prod_{\alpha B} \varepsilon(x).
\]

Similarly, we have the telescoping sum \( \sum(x - n) = \sum(2y - 2z) = 0 \), so \( \sum x = n\nu(n) \).

Therefore,

\[
P = (2\sqrt{\pi})^{\nu(n)} 2^{-2n\nu(n)/2n} \prod_{x \in \alpha B} \varepsilon(x) = \pi^{\nu(n)/2} 2^{b(\alpha B)},
\]

where \( b(\alpha B) \) is the number of \( x \in \alpha B \) that are bigger than \( n \). In summary,

\[
\prod_{x \in \alpha B} \Gamma \left( \frac{x}{2n} \right) = 2^{b(\alpha B)} \pi^{\nu(n)/2}.
\]

Since \( \alpha \) is an isomorphism, \( A = \alpha B \) is (a coset of) the subgroup of \( \Phi(2n) \) generated by \( \alpha(2) = n + 2 \). That yields (3).
Corollaries.

1. In Zucker [2] we find, for \( n = 2^m - 1, \ m > 1 \), that

\[
\Gamma \left( \frac{1}{2n} \right) \prod_{k=1}^{m-1} \Gamma \left( \frac{2k + n}{2n} \right) = 2^{m-1} \pi^{m/2}.
\]

Proof: This is a special case of (3). The left side equals

\[
\prod_{k=0}^{m-1} \Gamma \left( \frac{\alpha(2^k)}{2n} \right),
\]

while \( \beta(1) = 2^{m-1}, \beta(2^k) = 2^{k-1} \) \((k > 0)\). All numerators except for one are \( > n \).

2. Complementation. Let \( A \) be as in (3), and \( A^* = \{ 2n - x | x \in A \} \), then (3) holds with \( A^* \) replacing \( A \), and \( b(A^*) = \nu(n) - b(A) \).

Proof: Verify that if \( x' \equiv x(n + 2) \pmod{2n} \) then \( 2n - x' \equiv (2n - x)(n + 2) \pmod{2n} \).

3. Take the product in (3) over all of \( \Phi(2^n) \); that is, multiply both sides of (3) as \( A \) ranges over the subgroup and its cosets. There are \( \varphi(n)/\nu(n) \) choices of \( A \). Each \( x \in \Phi(2^n) \) greater than \( n \) will occur exactly once.

\[
\prod_{x \in \Phi(2^n)} \Gamma \left( \frac{x}{2n} \right) = (2\pi)^{\varphi(n)/2}.
\]

4. Some numerical examples. We determined, for which odd \( n < 100 \), the number of sets \( A \) exceeds 2. There are 9 such values. Among these, only \( n = 43 \) has 3 \( A \)'s, and all of these are self-complementary [2]. In all other cases the number is even, and can be as big as 8. At the other end, there are 16 values of \( n \) for which \( \nu(n) = \phi(n) \); that can only happen when \( n \) is a prime or a prime power.

We list the six sets \( A \) for \( n = 31 \) because only two of them are usually mentioned [2].

\[
\begin{align*}
n = 31 : (1, 33, 35, 39, 47), & \quad (3, 17, 37, 43, 55), \quad (5, 9, 41, 49, 51), \\
(7, 19, 25, 45, 59), & \quad (11, 13, 21, 53, 57), \quad (15, 23, 27, 29, 61).
\end{align*}
\]

The first one, written out in full, is

\[
\Gamma \left( \frac{1}{62} \right) \Gamma \left( \frac{33}{62} \right) \Gamma \left( \frac{35}{62} \right) \Gamma \left( \frac{39}{62} \right) \Gamma \left( \frac{47}{62} \right) = 2^4 \pi^{5/2}.
\]
Here, $\nu = 5$, the length of the product, and $b = 4$, the number of numerators bigger than 31. Each numerator, multiplied by $33 \pmod{62}$ yields the next one, in circular order.

In a personal note, H. Chen states the problem of finding minimum sizes of gamma products that have simple values. This paper may be a step in that direction, but any definitive answer will depend on a suitable definition of “simple value”.

References

[1] Glasser, M. L., Problem 11426, Amer. Math. Monthly, 116, p. 365, (2009)

[2] Zucker, L.J., Personal notes (1994)

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