Thin-shell wormholes in $d$-dimensional general relativity: Solutions, properties, and stability

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We construct thin-shell electrically charged wormholes in $d$-dimensional general relativity with a cosmological constant. The wormholes constructed can have different throat geometries, namely, spherical, planar and hyperbolic. Unlike the spherical geometry, the planar and hyperbolic geometries allow for different topologies and in addition can be interpreted as higher-dimensional domain walls or branes connecting two universes. In the construction we use the cut-and-paste procedure by joining together two identical vacuum spacetime solutions. Properties such as the null energy condition and geodesics are studied. A linear stability analysis around the static solutions is carried out. A general result for stability is obtained from which previous results are recovered.

PACS numbers: 04.20.Gz, 04.20.Jb, 04.40.-b

I. INTRODUCTION

A. Visser’s book

In physics, in many situations, the study of a system passes through three stages. First, one finds a solution that emulates the system itself. Second, in turn, through the solution one studies the peculiarities that the system might have. Third, one performs a stability analysis on the solution to gather whether the system can be realistic or not. Traversable wormhole physics is not different. Throughout its development, traversable wormhole physics has complied to this bookkeeping.

The field of traversable wormhole physics has been systematized by Visser in his book “Lorentzian wormholes: from Einstein to Hawking” [1]. The three stages of wormhole study are in one form or another in the book. For instance, not only it displays several solutions as it also shows how simple wormhole construction can be in the paper. In addition, the book emphasizes that one of the most important properties of traversable wormhole physics is that, within general relativity, its constitutive matter must possess one kind or another of exotic properties. Indeed, as the wormhole geometry, with a throat and two mouths, acts as tunnels from one region of spacetime to another, its constitutive matter possesses the peculiar property that its stress-energy tensor violates at least the null energy condition, i.e., its matter must be exotic. In general, the known classical fields obey the null energy condition but quantum fields in very special circumstances may not obey it. Since the circumstances in which the null energy condition is not obeyed are very restricted, the creation of these exotic fields for wormhole building is a difficult task. In this setting it is thus important to minimize the usage of exotic fields by finding conditions on the wormhole geometries which can contain arbitrarily small quantities of matter violating the averaged null energy condition. This has been clearly explained in Visser’s book [1] and pursued further later [2]. Finally, stability analysis, perhaps the most difficult stage of all, is also studied, though timidly, in the book [1].

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B. Wormhole solutions divided into three classes

The wormhole solutions that appeared previously have been put into a wider context through the book [1], which in turn helped to inspire developing the field still further as can be seen by the great many number of solutions dealing with wormhole physics and geometry that have appeared since. A methodical way of organizing now this field is by dividing wormhole solutions into three classes: I. Wormholes generated by continuous fundamental fields with exotic properties; II. Wormholes generated from matching an interior exotic solution to an exterior vacuum, at a junction surface, (just like one does with relativistic stars); and III. Wormholes generated from exotic thin shells. Class I could be thought of as nature given, of cosmological origin. Classes II and III, instead, should be built by cosmic engineers, first by providing the exotic matter, then building the wormhole itself and lastly joining it smoothly into the cosmic vacuum. Class III is the most simple to construct theoretically, and perhaps also practically.

In relation to class I (wormholes from continuous fundamental fields) it is striking that the first wormhole solutions are indeed of this class [3]. Ellis [3] discussed a wormhole solution, which he called a drainhole, as a model for a particle in general relativity in which an exotic continuous scalar field through spacetime provides the means to open up the throat and mouths of the wormhole. Bronnikov [3] found a general class of solutions within scalar-tensor theories, among which he managed to glimpse solutions with a neck, as he called it, i.e., wormhole solutions. In the same vein, Kodama [3] studied a wormhole solution, which he called a kink, with an exotic continuous scalar field being the necessary field to maintain an Einstein-Rosen bridge open. Many other wormholes, whose subsistence hinges on some or another form of exotic matter put into the model or on the introduction of interactions of a new type which in turn mimic the exotic matter itself, have been discussed with profusion. One can name a few of those solutions: wormholes in a cosmological constant scenario, wormhole solutions in semiclassical gravity, wormhole solutions in Brans-Dicke theory, wormholes on the brane, wormholes supported by matter with an exotic equation of state such as phantom energy and tachyon matter, wormholes in nonlinear electrodynamics, wormholes in nonminimal Einstein–Yang-Mills matters originating a Wu-Yang magnetic wormhole, wormholes with nonminimal and ghostlike scalar field and nonminimal matter [4]. In this setting, cylindrical wormholes, which permit two definitions of the throat (one of them allowing non-violation of the energy conditions), with several classes of fields such as scalar, spinor, Maxwell, and nonlinear electric fields have also been studied [4]. Stability is always a hard issue and only some of these systems have had their stability analyzed, see [5] as well as some of the works in [4] where for instance the stability of the Ellis wormhole is studied.

In relation to class II (wormholes generated from matching an interior exotic solution to an exterior vacuum at a junction surface) there are many interesting works. Morris and Thorne [6] created the idea that an arbitrarily advanced civilization could construct wormholes and traverse them to and fro, initiating thus the systematic study of traversable wormholes, later organized in a coherent whole in [1]. In Morris and Thorne’s work [6] a finite quantity of exotic matter was used in wormhole building. This matter was then to be carefully matched to the environment. In theory this matching is done through the junction conditions. In practice, it certainly can be tricky. Morris-Thorne type of wormholes were later generalized to include a positive or negative cosmological constant in [7], where a review of wormhole solutions was also taken, and it was suggested that perhaps one should call a civilization that constructs wormholes an absurdly advanced civilization rather than an arbitrarily advanced civilization. Other wormhole solutions generated from matching an interior exotic solution to a vacuum, as well as their exotic matter properties, have been studied. Namely, other wormholes in spacetimes with a cosmological constant, wormholes on the brane, wormholes made of phantom energy, wormholes with a generalized Chaplygin gas, Van der Waals quintessence wormholes [8]. For some study of stability of wormholes within this class see [9].

In relation to class III (wormholes generated from exotic thin shells) many solutions have also been studied. Since thin shells need less matter, the construction of wormholes from thin shells gives an elegant way of minimizing the usage of exotic matter. The exotic matter is concentrated at a shell at the wormhole throat alone, yielding a thin shell wormhole solution. In this limiting case of a thin shell, the throat and the two mouths are all at the same location, they are indistinguishable. This concentration of the entire content of exotic matter at the shell throat of the wormhole is provided by the cut-and-paste technique used for the first time in relation to wormholes in [10] (see also [1]), although Morris and Thorne [6] had used before heuristic methods in such a construction. Thin-shell wormholes in a cosmological constant background, wormhole with surface stresses on a thin shell, plane symmetric thin shell wormholes with a negative cosmological constant, cylindrical thin-shell wormholes, charged thin-shell Lovelace wormholes in a dilaton-gravity theory, five dimensional thin-shell wormholes in Einstein-Maxwell theory with a Gauss-Bonnet term, solutions in higher dimensional Einstein-Maxwell theory, thin-shell wormholes associated with global cosmic strings, wormhole solutions in heterotic string theory, a new type of thin-shell wormhole by matching two tidal charges black hole solutions localized on a three brane in the five dimensional Randall-Sundrum gravity scenario, thin shell and other wormholes within pure Gauss-Bonnet gravity which can be built without the need of using exotic matter (the gravity itself being already exotic), spherically symmetric thin-shell wormholes in a string cloud background spacetime, and many other solutions have been discussed in detail [11]. Now, in relation to stability,
these thin-shell wormholes are extremely useful as one may consider a linearized stability analysis around the static solution. In the end, the stability analysis will tell whether the throat can be kept open or not, i.e., whether the static solution holds under small perturbations. Within general relativity this has been done for several thin-shell wormhole systems, namely, four-dimensional spherical symmetric systems in vacuum [12], with a cosmological constant [13], and with electric charge [14], planar systems with a cosmological constant [15], and \( d \)-dimensional spherical symmetric systems with electric charge [16]. Also for stability analyses with dilaton, axion, phantom and other types of matter, and in cylindrical symmetry see [17]. This method does not require a specific equation of state, although it can be added to the analysis as was done in some works previously cited. One can also apply other techniques like a dynamic stability analysis to these thin-shell wormholes [18].

C. Aim and motivation for this work

In this work we want to extend further the study of class III wormholes, wormholes generated from exotic thin shells. We study here \( d \)-dimensional thin shell electrically charged wormhole spacetimes with spherical, planar, and hyperbolic geometry (each with possible different topologies), with a cosmological constant term within general relativity. The motivation for this general analysis comes from several fronts.

The study of objects in \( d \)-dimensions was already of interest when one invoked Kaluza-Klein small extra dimensions within fundamental theories, and it has become even more interesting and important since the proposal of the possible existence of a universe with extra large dimensions [19]. Assuming such a scenario of large extra dimensions is correct, one can possibly make tiny small distinct objects such as black holes and wormholes of sizes of the order of the new Planck scale.

The introduction of a cosmological constant in the study of new objects is advisable since from astronomical observations, it seems that we presently live in a world with a positive cosmological constant, \( \Lambda > 0 \), i.e., in an asymptotically de Sitter universe. In addition, a spacetime with \( \Lambda < 0 \), an anti-de Sitter spacetime, is also of significant relevance since it allows a consistent physical interpretation when one enlarges general relativity into a gauge extended supergravity theory in which the vacuum state has negative energy density, i.e., a negative cosmological constant. If these theories are correct, they imply that the anti-de Sitter spacetime should be considered as a symmetric phase of the theory, although it must have been broken as we do not presently live in a universe with \( \Lambda < 0 \). Moreover a \( \Lambda < 0 \) anti-de Sitter spacetime permits a consistent theory of strings in any dimension, and it has been conjectured that these spacetimes have a direct correspondence with certain conformal field theories on the boundary of that space, the AdS-CFT conjecture. Even with its preference to negative cosmological constant scenarios, string theory can, although in a contrived way, produce a landscape of positive cosmological constant universes, indicating perhaps that one can transit between both signs of the cosmological constant, see [20] for all these aspects related to the cosmological constant.

Once one starts to discuss extra dimensions and a cosmological constant term then the objects under study, black holes or wormholes, say, can have different geometries and topologies. Indeed, in the \( \Lambda < 0 \) case, besides spherical symmetric horizons, black holes can have planar and hyperbolic symmetric horizons, each one of these new geometries admitting different topologies [21]. Also, contrary to black holes with a spherical horizon, black holes with planar and hyperbolic symmetric horizons have the property that infinity carries the same topology as the throat. One can go beyond black hole solutions and build, upon addition of exotic matter, traversable wormholes with the same corresponding symmetries and topologies, by natural extension of the corresponding black hole solutions. In these new planar and hyperbolic geometries, infinity carries the same topology as the throat, whereas in the spherical geometry infinity is the usual corresponding asymptotic spacetime. In this sense, the construction of the planar and hyperbolic symmetric wormholes does not alter the topology of the background spacetime (i.e., spacetime is not multiply-connected), so that these solutions can be considered higher-dimensional domain walls or branes. Note that the \( \Lambda = 0 \) and \( \Lambda < 0 \) wormhole spacetimes allow valid definitions for the mass and charge, and when the wormholes have rotation angular momentum is also well defined [21]. So, this gives further reasons to study, besides wormhole structures in positive cosmological constant spacetimes, wormhole structures in zero and negative cosmological constant spacetimes.

Finally, electric charge is a fundamental property of matter, and the electromagnetic field is a fundamental interaction between electrically charged objects. Objects with a distribution of electric charge of the same sign suffer internal repulsion, which counteracts the effects of attractive gravitation. Moreover, the concept of wormholes was invented by Wheeler to provide a mechanism for having charge without charge, since the field lines seen in one part of the universe could thread the handles of a multiply connected spacetime and reappear in the other part. This idea was indeed corroborated by Wheeler and other authors when they showed the Schwarzschild and Reissner-Nordström solutions, when fully extended, really should be interpreted as wormholes, although non-traversable, the latter a wormhole with electric charge [22]. As such it is always of interest to see the effects of a net electric charge in wormholes.

So these are the fields and matter which we consider in our wormhole and domain wall building. Using the usual
cut-and-paste technique [1] we find, from the vacuum solutions, thin-shell electrically charged wormholes with different geometric-topologies in $d$-dimensional general relativity with a cosmological constant. We study the properties of the solutions and analyze perturbations in the linear regime. Our general solutions encompass previous solutions which can be reobtained as particular cases of our analysis. Indeed, previous results in $d=4$ and in other dimensions can be taken from our general expression, in particular the results in [12–16] are reobtained as particular cases.

D. Structure of the work

The paper is structured as follows. In Sec. II we use the cut-and-paste technique in order to build static wormholes from the gluing, above the gravitational radius (or the would-be horizon), of the $d$-dimensional electrically charged spacetimes, with a negative cosmological constant and with several geometries and topologies. In Sec. III we study the properties of these wormholes by providing the exotic conditions and analyzing the motion of test particles in the wormhole background. In Sec. IV a linearized stability analysis is performed around the static configurations. Our general result is then shown to recover previously calculated conditions for static wormholes in particular dimensions, in particular the results in [12–16] are reobtained as particular cases. Indeed, previous results in $d=4$ and in other dimensions can be taken from our general expression, in particular the results in [12–16] are reobtained as particular cases.

II. SPACETIME SURGERY AND STATIC WORMHOLES

A. Cut-and-paste techniques

We consider Einstein’s equations in the form

$$G_{\alpha\beta} + \frac{(d-1)(d-2)}{6} \Lambda g_{\alpha\beta} = 8\pi T_{\alpha\beta},$$

where $G_{\alpha\beta}$ is the Einstein tensor, $d$ is the dimension of the spacetime, $\Lambda$ is the cosmological constant, $g_{\alpha\beta}$ is the generic metric, and $T_{\alpha\beta}$ is the energy-momentum tensor. Greek indices are spacetime indices, latin indices are spacetime indices out of the shell. Equation (1) is supplemented by the Maxwell equation, but since it is rather trivial, it is not necessary to explicitly display it.

The general static metric solution for a $d$-dimensional vacuum spacetime, with electric charge, cosmological constant and different $(d-2)$ geometric-topologies is given in the form

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2d\Omega^2_{d-2},$$

with the metric function $f(r)$ given by

$$f(r) = k - \frac{\Lambda}{3} \frac{r^2}{r^{d-3}} - \frac{M}{r^{(d-3)}} + \frac{Q^2}{r^{2(d-3)}},$$

and where $\{t, r, \theta_1, \ldots, \theta_{d-2}\}$ are Schwarzschildian coordinates, $M$ and $Q$ are mass and charge parameters, respectively, and the cosmological constant $\Lambda$ being zero or of either sign, $\Lambda < 0$, $\Lambda = 0$, and $\Lambda > 0$. Here $k$ is the geometric-topological factor of the $(d-2)$-dimensional $t = \text{constant}$, $r = \text{constant}$ surfaces, with $k = 1$, $0$, $-1$ for spherical, planar, and hyperbolic geometries, respectively. The spherical geometry can only have $(d-2)$-dimensional surfaces with spherical topology and with infinity behaving normally. On the other hand, the $(d-2)$-dimensional hyperbolic and planar geometries can have different topologies, depending on whether they are infinite in extent or somehow compactified to produce different $(d-2)$-dimensional surfaces, and with infinity following the topology of those $(d-2)$-dimensional surfaces. For instance, in $d = 4$ the situation simplifies, the $t = \text{constant}$, $r = \text{constant}$ surfaces are 2-surfaces and can be classified. For spherical symmetry each 2-surface is compact (genus zero) spherical surface. In planar geometry each 2-surface can be an infinite plane, a cylindrical surface, or a compact (genus one) toroidal surface. In hyperbolic geometry each 2-surface can be an infinite hyperbolic plane, or a compact (genus two or greater) toroidal surface with several holes. For surfaces in higher dimensions the topologies can be even a lot more complicated [21]. Now, $(d\Omega^2_{d-2})$ is given by three different expressions according to the value of the parameter $k$,

$$d\Omega^2_{d-2} = d\theta^2_1 + \sin \theta_1^2 d\theta_2^2 + \ldots + \prod_{i=2}^{d-3} \sin \theta_i^2 d\theta_{d-2}^2,$$
\[ d\Omega^0_{d-2} = d\theta_1^2 + d\theta_2^2 + \ldots + d\theta_{d-2}^2, \]
\[ d\Omega^{d-1}_{d-2} = d\theta_1^2 + (\sinh \theta_1)^2 d\theta_2^2 + \ldots + (\sinh \theta_1)^2 \prod_{i=2}^{d-3} \sin \theta_i^2 d\theta_{i-2}^2. \]  

One should note that the mass and charge terms above in Eq. (3), that is \( M \) and \( Q \), are not the ADM mass and charge of the solutions, but rather \( M \) and \( Q \) are parameters proportional to the ADM mass and electric charge, respectively. For instance, in the spherical case and for zero cosmological constant one has \( m = \left( \frac{(d-2)\Sigma^{1}_{d-2}}{8\pi} \right) M \), and \( q^2 = \left( \frac{(d-2)(d-3)}{2} \right) Q^2 \), where \( m \) and \( q \) are the ADM mass and electric charge, respectively, and \( \Sigma^{1}_{d-2} \) is the area of the \((d-2)\)-dimensional unit sphere, \( \Sigma^{1}_{d-2} = 2\pi \frac{d-1}{d} / \Gamma \left( \frac{d-1}{d} \right) \). When \( \Lambda > 0 \) the ADM quantities are not well defined. The zeros of \( f(r) \) in Eq. (3) give the gravitational radius \( r_g \). The full metric vacuum solution given in Eqs. (2)-(3), when extended to all \( r \) \((0 \leq r < \infty)\), yield black hole solutions. Thus, in studying wormholes the matter is put outside the gravitational radius \( r_g \).

We now consider two copies of the vacuum solution, Eqs. (2)-(4), removing from each copy the spacetime region given by

\[ \Omega^\pm = \{ r^\pm \leq a | a > r_g \}, \]

where \( a \) is a constant and \( r_g \) is the gravitational radius given as the largest positive solution \( r = r_g \) to the equation \( f(r) = 0 \). The latter condition, \( a > r_g \), is important in order not to have an event horizon. With the removal of these regions of each spacetime, we are left with two geodesically incomplete manifolds, with the following timelike hypersurfaces as boundaries

\[ \partial \Omega^\pm = \{ r^\pm = a | a > r_g \}. \]

The identification of these two timelike hypersurfaces, \( \partial \Omega^+ = \partial \Omega^- = \partial \Omega \), results in a manifold, now geodesically complete, where two regions are connected by a wormhole. This wormhole has a throat at \( \partial \Omega \), which is the separating surface between the two regions \( \Omega^\pm \). In fact, this cut-and-paste technique treats the throat as a hypersurface between two regions of spacetime, where all the exotic matter is concentrated, making the wormhole solution a thin-shell wormhole solution. In this thin-shell case the location of the two mouths coincide with that of the throat. To continue the analysis we need to use the Darmois-Israel formalism. The intrinsic metric of this separating hypersurface \( \partial \Omega \) can now be written as

\[ ds_{\partial \Omega}^2 = -d\tau^2 + a^2(\tau)(d\Omega_{d-2}^2)^2, \]

with \( \tau \) being the proper time along the hypersurface \( \partial \Omega \), and \( a(\tau) \), a quantity that defines the radius at which the throat is located in each partial manifold \( \Omega^\pm \), now a function of the proper time of the throat. In the bulk spacetime, the coordinates of this hypersurface are given by \( x^\gamma(\tau, \theta_1, \theta_2, \ldots, \theta_{d-2}) \), with the respective 4-velocity written as

\[ u_+^\gamma = \frac{dx^\gamma}{d\tau}. \]

The intrinsic stress-energy tensor is defined through the Lanczos equation as

\[ S^i_j = -\frac{1}{8\pi}(\kappa^j_i - \delta^j_i \kappa^i), \]

where the indices written in the latin alphabet run as \( i = (\tau, \theta_1, \ldots, \theta_{d-2}) \). The quantity \( \kappa_{ij} \) represent the discontinuity in the extrinsic curvature \( K_{ij} \), and one has \( \kappa_{ij} = K^+_{ij} - K^-_{ij} \). Each of the extrinsic curvatures, on each of the original manifolds \( \Omega^\pm \), is defined through

\[ K^\pm_{ij} = \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \nabla^\alpha n_\beta, \]

\[ = -n_\gamma \left( \frac{\partial^2 x^\gamma}{\partial \xi^i \partial \xi^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial x^\alpha}{\partial \xi^i} \frac{\partial x^\beta}{\partial \xi^j} \right), \]

where \( n_\alpha \) is the unit normal to \( \partial \Omega \) in the bulk spacetime, the ± superscripts refer to the spacetime parcel \( \Omega^\pm \), and \( \Gamma^\gamma_{\alpha\beta} \) refers to the respective spacetime Christoffel symbols. The parametric equation for the hypersurface \( \partial \Omega \) can be written as \( f(x^\alpha(\xi^i)) = 0 \). Using this equation we arrive at the formula for the normal vector

\[ n_\alpha = \pm \left| g^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial f}{\partial x^\beta} \right|^{-\frac{1}{2}} \frac{\partial f}{\partial x^\alpha}. \]
It can be ascertained that $n^\alpha n_\alpha = +1$, which makes it spacelike, indeed $n^\alpha$ is normal vector to a timelike hypersurface. Applying Eqs. (8) and (11) to our case, we have $x^\alpha (\tau, \theta_1, \ldots, \theta_{d-2}) = (t(\tau), a(\tau), \theta_1, \ldots, \theta_{d-2})$, with the 4-velocity written as

$$u^\alpha_\pm = \left(\frac{\sqrt{f(r) + \dot{a}^2}}{f(r)}, \dot{a}, 0, \ldots, 0\right),$$

and the normal vector $n_\alpha\pm$ as

$$n_\alpha\pm = \left(-\dot{a}, \frac{\sqrt{f(r) + \dot{a}^2}}{f(r)}, 0, \ldots, 0\right),$$

where $\partial / \partial \tau$. This last result can be arrived at through the relations $u^\alpha n_\alpha = 0$ and $n^\alpha n_\alpha = +1$, as well as through the relation (11). Given the symmetry properties of the solutions we are working with, the discontinuity of the extrinsic curvatures can be written as $\kappa_j^\alpha = \text{diag}(\kappa_\ell^\tau, \kappa_\theta^\theta_1, \ldots, \kappa_\theta^\theta_{d-2})$. This allows us to write the surface intrinsic energy-momentum tensor as $S_j^\alpha = \text{diag}(-\sigma, \mathcal{P}, \ldots, \mathcal{P})$, where $\sigma$ is the surface energy density and $\mathcal{P}$ is the surface pressure. From the Lanczos equation, Eq. (9), we obtain $S_{00}^0 = -\frac{1}{4\pi} (\kappa_0^0 - \kappa_1^0) = \frac{1}{4\pi} (\kappa_1^1 + \kappa_2^2 + \ldots + \kappa_{d-2}^{d-2}) = \frac{8\pi}{(d-2)f} \sqrt{f + \dot{a}^2}$, and so,

$$\sigma = -\frac{(d-2)}{4\pi a} \sqrt{f + \dot{a}^2}.\tag{14}$$

Using $S_1^1$ we obtain

$$\mathcal{P} = \frac{1}{8\pi} \frac{2\dot{a} + f'}{\sqrt{f + \dot{a}^2}} - \frac{d-3}{d-2} \tau.\tag{15}$$

For the latter results we have used the following expressions for the extrinsic curvatures, defined in Eq. (10), $K_0^0\pm = \pm \frac{f' + \dot{a}^2}{\sqrt{f + \dot{a}^2}}$, $K_1^1\pm = \pm \frac{1}{a} \sqrt{f + \dot{a}^2}$, $K_2^2\pm = \pm \frac{1}{a} \sqrt{f + \dot{a}^2}$, and so on. One can substitute one of the two equations (14) and (16) by the conservation equation, which can be written as

$$\frac{d}{d\tau} (\sigma a^{d-2}) + \mathcal{P} \frac{d}{d\tau} (a^{d-2}) = 0.\tag{16}$$

In order to solve Eqs. (14), (15), or if we prefer Eqs. (14) and (16), for $\sigma(\tau)$ and $a(\tau)$, one would need to choose an equation of state, the most simple one would be a cold equation of state $\mathcal{P} = \mathcal{P}(\sigma)$.

### B. Static wormholes

Now, we resort to a static shell for which $\dot{a} = \ddot{a} = 0$. In this case Eqs. (14) and (15) reduce to

$$\sigma = -\frac{(d-2)}{4\pi a} \sqrt{f},\tag{17}$$

$$\mathcal{P} = \frac{a f' + 2(d-3) f}{8\pi a \sqrt{f}}.\tag{18}$$

A general equation of state of the form $\mathcal{P} = \mathcal{P}(\sigma)$ turns the terms in $f(a)$, such as $M$, $Q$, $a$, and $\Lambda$ into related terms, that is, there is a relation between these terms, making them dependent upon each other.

### III. WORMHOLE PROPERTIES: ENERGY CONDITIONS AND TEST PARTICLE MOTION

#### A. Wormhole and domain walls

These traversable wormhole solutions are a natural extension of the corresponding black hole solutions upon the addition of exotic matter. For the plane and hyperbolic wormhole solutions, infinity carries the same topology as the the topology of the $t = \text{constant}$, $r = \text{constant}$ $(d-2)$-dimensional surfaces of the background spacetime. Thus, the construction of these wormholes does not alter the topology of the background spacetime (i.e., spacetime is not multiply-connected). Therefore, these solutions can instead be considered higher-dimensional domain walls or branes connecting two universes, and as such, in general, do not allow time travel.
B. Energy conditions

1. The throat shell

Now we turn to the issue of the energy conditions on the shell. In our case the weak energy condition is given by $\sigma + \mathcal{P} \geq 0$ and $\sigma \geq 0$. Since, in the reference frame we are working, the surface energy density is negative, see Eq. (17), this means that the weak energy condition is violated as usual for wormholes. The null energy condition requires only that $\sigma + \mathcal{P} \geq 0$. Using (17)-(18) this implies the following inequality

$$a^{d-3}(2a^{d-3}k - (d-1)M) + 2(d-2)Q^2 \geq 0.$$  \tag{19}

We can further require that the strong energy condition holds, i.e., $\sigma + (d-2)\mathcal{P} \geq 0$, which using (17)-(18) yields

$$(d-4)k - \frac{(d-3)}{3} \Lambda a^2 - \frac{(d-5)M}{2a^{d-3}} - \frac{Q^2}{a^{2(d-3)}} \geq 0.$$  \tag{20}

2. The two exterior regions to the shell

Now we turn to the issue of the energy conditions outside the shell, i.e., for $r > a$. Here only the electromagnetic field and $\Lambda$ contribute to the energy-momentum tensor of the Einstein equations, see (10) for the case of spherical symmetry ($k = 1$). The energy-momentum tensor of the electromagnetic field is $T_{\alpha\beta}^{em} = \frac{1}{2} \left( -F^\alpha_\gamma F^\gamma_\beta - \frac{1}{4} g_{\alpha\beta} F^\gamma_\delta F^\delta_\gamma \right)$. The only nonzero component of the electromagnetic field is $E_r = \frac{d\rho}{dr}$. Calculating the $T_{00}^{em}$ component of the energy-momentum tensor yields $T_{00}^{em} = \frac{1}{8\pi} f(r) \frac{Q^2}{r^{2(d-3)}}$. We can write the $T_{11}^{em}$ component as $T_{11}^{em} = -\frac{1}{8\pi} f(r)^{-1} \frac{Q^2}{r^{2(d-3)}}$. We have also to include the vacuum energy represented by the cosmological constant. The corresponding vacuum energy-momentum tensor can be written as $T_{\alpha\beta}^{vac} = -\frac{\Lambda}{8\pi} g_{\alpha\beta}$. Thus, $T_{00}^{vac} = \frac{\Lambda}{8\pi} f(r)$, $T_{11}^{vac} = -\frac{\Lambda}{8\pi} f^{-1}(r)$, $T_{22}^{vac} = -\frac{\Lambda}{8\pi} r^2$, and so on. Using $u_\alpha = (-\sqrt{f}, 0, \ldots, 0)$ for a timelike, future directed vector field, and $T_{00} = \rho f(r)$, $T_{11} = p_r f(r)^{-1}$, $T_{22} = \rho_\theta r^2$, $T_{33}^{k=0} = p_\theta r^2$, $T_{33}^{k=1} = p_\theta r^2 \sin^2 \theta^1$, we can establish that the weak energy condition, $T_{\mu\nu} u^\mu u^\nu \geq 0$, is satisfied if

$$Q^2 \geq -\Lambda r^{2(d-2)},$$  \tag{21}

implying that if $\Lambda < 0$, then the weak energy condition is satisfied only if $|Q^2| \geq \sqrt{|\Lambda|} r^{d-2}$, and if $\Lambda < 0$, then the weak energy condition is always satisfied, because $Q^2 > 0$ and $\Lambda > 0$ make it always true. There is no dependence on the topological factor $k$. So, outside the shell, where the contributions to the energy-momentum tensor come from the electromagnetic field and the cosmological constant, the weak energy condition is not satisfied except in a finite region, if $|Q|$ is large enough, in the case of anti-de Sitter spacetimes. As to the null energy condition, it always holds. This can be gleaned from the fact that $T_{\mu\nu} k^\mu k^\nu = 0$, where the $k^\mu$ are null vectors, defined as $k^\mu = \left( \frac{1}{\sqrt{f}}, \sqrt{f}, 0, \ldots, 0 \right)$, where $f$ is again the metric function, and the $T_{\mu\nu}$ is the same as above, the sum of the electromagnetic and vacuum energy-momentum tensors.

C. Attraction and repulsion of the wormhole on test particles

In order to complete the general considerations on the properties of the static wormholes of the family of solutions under study, we address the issue of the attractive or repulsive character of the traversable wormhole on test particles. First, let us recall the 4-velocity in Eq. (12) now for a test particle. The expression contains the term $\sqrt{f}$ inside the square root. For the static wormhole case under consideration it holds that $\dot{a}^2 = 0$. So the 4-velocity is now written as

$$u_\pm^\alpha = \left( \frac{1}{\sqrt{f(r)}}, 0, \ldots, 0 \right).$$  \tag{22}

The 4-acceleration is $a^\alpha = u^\alpha_\beta u^\beta$. Now, given the expression for $u^\alpha$ in Eq. (22) we have

$$a^\alpha = u^\alpha_{\alpha,0} u^0 = u^\alpha_{\alpha,0} \frac{1}{\sqrt{f(r)}}.$$  \tag{23}
We can show that \( u^{a\cdot 0} = u^{\alpha\cdot 0} + \Gamma^\alpha_{a\theta} u^\alpha = \Gamma^\alpha_{00} u^0 \). The only nonzero component of this is \( \alpha = 1 \), such that \( u^1 = \Gamma^1_{00} u^0 \).

From this we know that \( a^\alpha = (0, a^r, 0, \ldots, 0) \), where \( a^r = \frac{1}{2} f'(r) \). Expanding, we get

\[
a^r = - \frac{\Lambda}{3} r^2 + \frac{(d-3)M}{2 r^{d-2}} - \frac{(d-3)Q^2}{r^{2d-5}}.
\]  

(24)

The geodesic equation, \( \frac{d^2 r}{d\tau^2} + \Gamma^r_{00} \dot{x}^0 = 0 \), allows us to write

\[
\frac{d^2 r}{d\tau^2} = - \Gamma^r_{00} \left( \frac{d^2 t}{d\tau^2} \right) = -a^r.
\]

(25)

As in [15], an observer must maintain a proper acceleration with the radial component given in Eq. (24) in order to remain at rest. Again, the wormhole is attractive if \( a^r > 0 \) and repulsive if \( a^r < 0 \), which depends on the balancing of the parameters in Eq. (24).

IV. STABILITY ANALYSIS: LINEAR STABILITY

A. General considerations: the stability equations and criteria

The equations of motion (14) and (16) can be put in the more useful form

\[
\ddot{a}^\alpha - \left\{ - \frac{\Lambda}{3} a^\alpha - \frac{M}{a^{d-3}} + \frac{Q^2}{a^{2(d-3)}} - \frac{16 \pi^2 a^2}{(d-2)^2} \sigma^2 (a) \right\} = -k,
\]

(26)

and

\[
\dot{\sigma} = -(d-2) \frac{\dot{a}}{a} (\sigma + \mathcal{P}),
\]

(27)

respectively. One can choose a particular equation of state, for instance, an equation of state taken from the generic cold equation of state \( \mathcal{P} = \mathcal{P}(\sigma) \). Then we can integrate Eq. (27) to yield the general solution \( \ln a = \frac{1}{\pi} \int \frac{d\sigma}{\sigma + \mathcal{P}(\sigma)} \), which, if we wish, can be formally inverted to provide the function \( \sigma = \sigma (a) \), i.e., the wormhole surface density as a function of its own radius. Substituting this equation for \( \ln a \) into Eq. (26) determines the the motion of the throat, and so its stability.

We follow other path provided by Poisson and Visser [12]. In order to test stability we consider a linear perturbation around those static solutions found in Sec. II B. Let us take \( a_0 \) as the radius of the static solution. The respective static values of the surface energy density and the surface pressure are given by Eqs. (17)-(18). To know whether the equilibrium solution is stable or not, one must analyze the shell’s equation of motion near the equilibrium solution. Following [12] we put quite generally

\[
\mathcal{P} = \mathcal{P}(\sigma),
\]

(28)

i.e., we impose a generic, as opposed to a particular, cold equation of state. Now, Eq. (26) can be written in the more useful form

\[
\ddot{a}^\alpha = -V(a),
\]

(29)

with \( V(a) \) given by

\[
V(a) = k - \frac{\Lambda}{3} a^\alpha - \frac{M}{a^{d-3}} + \frac{Q^2}{a^{2(d-3)}} - \frac{16 \pi^2 a^2}{(d-2)^2} \sigma(a)^2,
\]

(30)

or more compactly as,

\[
V(a) = f(a) - \frac{16 \pi^2 a^2}{(d-2)^2} \sigma(a)^2.
\]

(31)

In the study of the stability of a static solution of radius \( a_0 \), of course we are considering \( f(a_0) \) in the interval \( f(a_0) > 0 \) so that no event horizon is present. Our initial condition before the cutting and gluing operation is that \( a > r_g \), hence
$a_0 > r_g$. A linearization is going to be done in order to determine whether and in what conditions the static solutions with a radius $a_0$ are stable under a linear perturbation around $a_0$. The values of the density and pressure are given in Eqs. (17) and (18) for the case of a static solution for the throat. For the linearization, we make a Taylor expansion of the function $V(a)$ around the static radius $a_0$,

$$V(a) = V(a_0) + V'(a_0)(a - a_0) + \frac{1}{2}V''(a_0)(a - a_0)^2 + O[(a - a_0)^3],$$

with a prime corresponding to a derivative with respect to $a$. By definition, if we are dealing with a static configuration then $V(a_0) = 0$, because of Eq. (29). One can also show that $\ddot{a} = -\frac{1}{2}V'(a)$. With this relation, as a static configuration also demands that $\ddot{a} = 0$, we also have $V'(a_0) = 0$, where $V'(a) = V/\dot{a}$. From Eq. (28) one can define a quantity $\eta$ as,

$$\eta(\sigma) \equiv \frac{dP}{d\sigma},$$

or, when preferable, $\eta(\sigma) = \frac{\sigma}{d\sigma}$. With this definition the second derivative $V''(a)$ can be written as,

$$V''(a) = -\frac{2\Lambda}{3} - \frac{(d-2)(d-3)M}{a^{d-1}} + \frac{2(d-3)(2d-5)Q^2}{a^{2d-2}} - \frac{32\pi^2}{(d-2)^2} \left\{ [(d-3)\sigma + (d-2)P^2] + (d-2)\sigma(\sigma + P)(d-3 + (d-2)\eta) \right\}.$$

From Eqs. (29) and (32), the equation of motion of the wormhole throat is given by

$$\ddot{a} = -\frac{1}{2}V''(a_0)(a - a_0)^2 + O[(a - a_0)^3].$$

In order to insure stability we have to guarantee that the second derivative evaluated at the radius of the static configuration $a_0$ is positive, i.e., $V''(a_0) > 0$. This is going to present us with conditions on $\eta(\sigma)$. To write the conditions in a manageable form we define $f_0 \equiv f(a_0)$ and $\eta_0 \equiv \eta(\sigma_0)$, for the quantities considered at $a_0$. A prime on $f$ indicates derivation with respect to $a$. The conditions can be put thus

$$\eta_0 < \frac{a_0^2 (f_0')^2 - 2 a_0^3 f_0'' f_0}{2(d-2) f_0} \left( -2k + \frac{(d-1)M}{a_0^{d-3}} - \frac{2(d-2)Q^2}{a_0^{2d-3}} \right) - \frac{d-3}{d-2} \quad \text{if} \quad -2k + \frac{(d-1)M}{a_0^{d-3}} - \frac{2(d-2)Q^2}{a_0^{2d-3}} < 0,$$

$$\eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching} \quad \text{if} \quad -2k + \frac{(d-1)M}{a_0^{d-3}} - \frac{2(d-2)Q^2}{a_0^{2d-3}} = 0,$$

$$\eta_0 > \frac{a_0^2 (f_0')^2 - 2 a_0^3 f_0'' f_0}{2(d-2) f_0} \left( -2k + \frac{(d-1)M}{a_0^{d-3}} - \frac{2(d-2)Q^2}{a_0^{2d-3}} \right) - \frac{d-3}{d-2} \quad \text{if} \quad -2k + \frac{(d-1)M}{a_0^{d-3}} - \frac{2(d-2)Q^2}{a_0^{2d-3}} > 0.$$

The quantity $f_0$ is given by Eq. (3) evaluated at $a_0$. The quantities $f_0'$ and $f_0''$ are given by

$$f_0' = \frac{2\Lambda a_0}{3} + \frac{(d-3)M}{a_0^{d-2}} - \frac{2(d-3)Q^2}{a_0^{2d-5}}$$

and

$$f_0'' = -\frac{2\Lambda}{3} - \frac{(d-3)(d-2)M}{a_0^{d-1}} + \frac{2(d-3)(2d-5)Q^2}{a_0^{2d-4}},$$

evaluated at $a_0$ respectively. Eqs. (36) - (38), together with Eqs. (39) - (41), generalize what has been found in previous papers.

It is interesting to see to what does this lead in the case that the cold equation of state $P = P(\sigma)$ in Eq. (28), assumes a particular form, $P = \omega \sigma$, with $\omega < 0$. This is a dark energy equation of state. The expression for the second derivative $V''(a_0)$ is now

$$V''(a_0) = f_0'' - \frac{32\pi^2}{(d-2)^2} \left\{ [(d-3) + (d-2)\omega] - (d-2)(1 + \omega) \right\} \left\{ (d-3) + (d-2)\omega \right\},$$

so that the condition $V''(a_0) > 0$ implies

$$f_0'' > \frac{32\pi^2}{(d-2)^2} \left\{ [(d-3) + (d-2)\omega] - (d-2)(1 + \omega) \right\} \left\{ (d-3) + (d-2)\omega \right\}.$$
B. Particular cases studied in the literature and a new example

It is now interesting to reduce our general results to some well known cases already studied for particular choices of the parameters \(d, k, \Lambda, \) and \(Q\). We also give a new example.

1. Poisson-Visser, \(d = 4, k = 1, \Lambda = 0, \) and \(Q = 0\)

Poisson and Visser \[12\] were the first to study the linear stability of wormholes. Perturbations were done around some four-dimensional static wormhole solution with spherical symmetry with no cosmological constant and no charge. Putting \(d = 4, k = 1, \Lambda = 0, Q = 0\), calling \(m\) the ADM mass, so that for this case our mass parameter \(M\) is \(M \rightarrow 2m\), our formulas \[36\]-\[38\] yield

\[
\eta_0 < -\frac{1 - 3m/a_0 + 3(m/a_0)^2}{2(1 - 2m/a_0)(1 - 3m/a_0)} \quad \text{if} \quad -1 + \frac{3m}{a_0} < 0, \quad (41)
\]

\[
\eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching} \quad \text{if} \quad -1 + \frac{3m}{a_0} = 0, \quad (42)
\]

\[
\eta_0 > -\frac{1 - 3m/a_0 + 3(m/a_0)^2}{2(1 - 2m/a_0)(1 - 3m/a_0)} \quad \text{if} \quad -1 + \frac{3m}{a_0} > 0. \quad (43)
\]

This is precisely what Poisson and Visser \[12\] obtained.

2. Lobo-Crawford, \(d = 4, k = 1, \Lambda \neq 0, \) and \(Q = 0\)

Lobo and Crawford \[13\] extended the Poisson and Visser study by considering \(\Lambda \neq 0\) in the analysis of linear stability of wormholes. Then putting again \(M \rightarrow 2m, \) and \(d = 4, k = 1, \Lambda \) generic, and \(Q = 0\) in our formulas \[36\]-\[38\], one finds

\[
\eta_0 < -\frac{1 - 3m/a_0 + 3m^2/a_0^2 - \Lambda m a_0}{2 \left(1 - 2m/a_0 - \frac{1}{3}\Lambda a_0^2\right) \left(1 - 3m/a_0\right)} \quad \text{if} \quad -1 + \frac{3m}{a_0} < 0, \quad (44)
\]

\[
\eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching} \quad \text{if} \quad -1 + \frac{3m}{a_0} = 0, \quad (45)
\]

\[
\eta_0 > -\frac{1 - 3m/a_0 + 3m^2/a_0^2 - \Lambda m a_0}{2 \left(1 - 2m/a_0 - \frac{1}{3}\Lambda a_0^2\right) \left(1 - 3m/a_0\right)} \quad \text{if} \quad -1 + \frac{3m}{a_0} > 0. \quad (46)
\]

This is what Lobo and Crawford \[13\] obtained. Putting further \(\Lambda = 0\) in Eqs. \[44\]-\[46\] one obtains Eqs. \[41\]-\[43\].

3. Eiroa-Romero, \(d = 4, k = 1, \Lambda = 0, \) and \(Q \neq 0\)

Eiroa and Romero \[14\] discussed for the first time wormhole stability for systems with electric charge \(Q \neq 0\). Putting \(M = 2m, \) with \(m\) being the ADM mass of the \(d = 4\) solution, \(Q = q, \) with \(q\) being the ADM charge of the \(d = 4\) solution, \(k = 1, \Lambda = 0, \) we recover the Reissner-Nordström wormhole system and stability given in \[14\]. The relevant condition, i.e., the condition on the right hand side of Eqs. \[36\]-\[38\], yields that either \(a_0\) is larger or smaller than \(3m + \sqrt{3m^2 - 2q^2}\), which is the same result as of \[14\]. The no gravitational radius condition, \(a_0 > r_g\), implies \(a_0 > m + \sqrt{m - q^2}\). In this case, our formulas \[36\]-\[38\] give for the case of \(\frac{q}{2m} < 1,\)

\[
\eta_0 < -\frac{\left(1 - \frac{m}{a_0}\right)^3 + m \frac{a_0}{a_0^2} \left(m^2 - q^2\right)}{2 \left(1 - 2m/a_0 + q^2/a_0^2\right) \left(1 - 3m/a_0 + 2q^2/a_0^2\right)} \quad \text{if} \quad -1 + \frac{3m}{a_0} = \frac{2q^2}{a_0^2} < 0, \quad (47)
\]

\[
\eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching} \quad \text{if} \quad -1 + \frac{3m}{a_0} = \frac{2q^2}{a_0^2} = 0. \quad (48)
\]
\[ \eta_0 > -\frac{(1 - \frac{m}{a_0})^3 + \frac{m}{a_0^2} (m^2 - q^2)}{2 \left(1 - \frac{2m}{a_0} + \frac{q^2}{a_0^2} \right)} \left(1 - \frac{3m}{a_0} + \frac{2q^2}{a_0^2} \right) \text{ if } -1 + \frac{3m}{a_0} - \frac{2q^2}{a_0^2} > 0. \]  

(49)

This is what Eiroa and Romero \[14\] obtained. Putting further \( q = 0 \) in Eqs. (47)-(49) one obtains Eqs. (41)-(43). One deduces from Eqs. (17)-(19) that electric charge tends to destabilize the system. We have presented the case \( \frac{m}{2m} < 1 \) for way of comparison, for the other cases and a detailed analysis see \[14\].

4. Lemos-Lobo, \( d = 4, k = 0, \Lambda \neq 0, \text{ and } Q = 0 \)

Lemos and Lobo \[15\] studied the stability of four-dimensional planar wormholes, i.e., wormholes with \( k = 0 \). Putting \( d = 4, k = 0, \Lambda \neq 0, \text{ and } Q = 0 \), and noting that for the planar case \( M \) can be consider the ADM mass \( m \) \[21\], \( M = m \), our formulas (36)-(38) give

\[ \eta_0 < -\frac{m}{2a_0} - \frac{1}{2} \text{ if } \frac{3m}{a_0} < 0, \]  

(50)

\[ \eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching } \]  

(51)

\[ \eta_0 > -\frac{m}{2a_0} - \frac{1}{2} \text{ if } \frac{3m}{a_0} > 0. \]  

(52)

This is what Lemos and Lobo obtained \[15\]. In fact, Lemos and Lobo worked out the case with negative cosmological constant and have defined \( \alpha \) such that \( \alpha^2 = -\frac{\Lambda}{8} \). One does not need to make such a restriction and can consider wormholes valid for \( \Lambda < 0 \) as well as \( \Lambda \geq 0 \) as we show here. The mass \( m \) here and the mass in \[15\] are different, \( m_\text{here} = m_\text{LL}/\alpha \). The case \( m = 0 \) is the case of no wormhole, where spacetime is just de Sitter, flat, or anti-de Sitter, depending on \( \Lambda \).

5. Rahaman-Kalam-Chakraborty, \( d = d, k = 1, \Lambda = 0, \text{ and } Q \neq 0 \)

Another particular case seen in the literature was studied by Rahaman-Kalam-Chakraborty \[16\], where \( d \) is left general, the spherical \( k = 1 \) geometry is chosen, the cosmological constant is set to zero \( \Lambda = 0 \), and there is a nonzero electric charge \( Q \neq 0 \). Their main expression for \( \eta_0 \) is taken from our general expression in Eqs. (36) and (37), with the appropriate replacements. Putting \( d = d, k = 1, \Lambda = 0, \text{ and } Q \neq 0 \), our formulas (36)-(38) give

\[ \eta_0 < -\frac{a_0^2 (f_0)^2 - 2 a_0^2 f_0' f_0}{2(d-2) f_0 \left[-2 + \frac{(d-1)M}{a_0^2} - \frac{2(d-2)Q^2}{a_0^2} \right]} - \frac{d-3}{d-2} \text{ if } -2 + \frac{(d-1)M}{a_0^2} - \frac{2(d-2)Q^2}{a_0^2} < 0, \]  

(53)

\[ \eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching } \text{ if } -2 + \frac{(d-1)M}{a_0^2} - \frac{2(d-2)Q^2}{a_0^2} = 0, \]  

(54)

\[ \eta_0 > -\frac{a_0^2 (f_0)^2 - 2 a_0^2 f_0' f_0}{2(d-2) f_0 \left[-2 + \frac{(d-1)M}{a_0^2} - \frac{2(d-2)Q^2}{a_0^2} \right]} - \frac{d-3}{d-2} \text{ if } -2 + \frac{(d-1)M}{a_0^2} - \frac{2(d-2)Q^2}{a_0^2} > 0. \]  

(55)

Eqs. (53)-(55) complete the stability conditions given in \[16\], in \[16\] only Eq. (53) is given. In the inequalities (53)-(55), one can explicitly write the expressions for \( f_0, f_0', \text{ and } f_0'' \) in terms of the ADM mass \( m \), the ADM charge \( q \), and the other quantities. In this case, since the geometry is spherical, \( k = 1 \), the relation between the mass parameter \( M \) and the ADM mass \( m \) is given by

\[ M = \frac{16 \pi \Gamma \left(\frac{d-1}{2}\right)}{(d-2) 2\pi \frac{1}{a_0^2}} m, \]  

(56)
where \( \Gamma(z) \) is the Gamma Function, and between the charge parameter \( Q \) and the ADM charge \( q \) is given by

\[
Q^2 = \frac{2}{(d-2)(d-3)} q^2.
\]

(57)

Thus, the inequality (63), can be written as

\[
\eta_0 < -d - 3 \frac{1}{d - 2} \left( 1 - \frac{16 \pi \Gamma \left( \frac{d+1}{2} \right)}{(d-2) 2^{d-2} \pi^{d/2}} m \right) - \frac{16 \pi \Gamma \left( \frac{d+1}{2} \right)}{(d-2) 2^{d-2} \pi^{d/2}} m \left( \frac{1}{d-2} a^2_{d-3} q^2 \right) + \frac{16 \pi \Gamma \left( \frac{d+1}{2} \right)}{(d-2) 2^{d-2} \pi^{d/2}} m \left( \frac{1}{d-2} a^2_{d-3} q^2 \right) - \frac{4(d-4)}{(d-2) 2^{d-2} \pi^{d/2}} m^2 (d-3) \left( \frac{1}{d-2} a^2_{d-3} q^2 \right)
\]

(58)

\[
\eta_0 < -d - 3 \frac{1}{d - 2} \left( 1 - \frac{16 \pi \Gamma \left( \frac{d+1}{2} \right)}{(d-2) 2^{d-2} \pi^{d/2}} m \right) \frac{-2(d-2)}{(d-2)(d-3) a^2_{d-3}} q^2 < 0.
\]

To rewrite the conditions (64) and (65) one has only to make the appropriate changes in the inequality sign.

6. A new example in several dimensions \( d = 4, 5, \infty, k = 1, 0, -1, \ Lambda \neq 0, \text{ and } Q = 0 \)

An illustration of the above general stability conditions, Eqs. (30)-(38) can be displayed if we put \( Q = 0 \). Then Eqs. (30)-(38) turn into

\[
\eta_0 < \frac{a_0^2 \left( f_0' \right)^2 - 2 a_0^2 \left( f_0'' \right) f_0}{2(d-2) f_0 \left( -2k + \frac{(d-1)M}{a_0^{d-3}} \right)} - \frac{d - 3}{d - 2} \text{ if } -2k + \frac{(d-1)M}{a_0^{d-3}} < 0,
\]

(59)

\[
\eta_0 = -\infty, \text{ or } \eta_0 = +\infty, \text{ depending on the branching}
\]

(60)

\[
\eta_0 > \frac{a_0^2 \left( f_0' \right)^2 - 2 a_0^2 \left( f_0'' \right) f_0}{2(d-2) f_0 \left( -2k + \frac{(d-1)M}{a_0^{d-3}} \right)} - \frac{d - 3}{d - 2} \text{ if } -2k + \frac{(d-1)M}{a_0^{d-3}} > 0.
\]

(61)

In Fig. 11 \( \eta_0 \) is plotted as a function of \( a_0 \) and the regions of stability are displayed in some chosn particular cases, namely, \( d = 4, 5, \infty, k = 1, 0, -1, \ Lambda \neq 0, \text{ and } Q = 0 \). For all plots we put further \( M = 10 \) and \( Lambda = -\frac{1}{2} \), in natural units, i.e., \( G = 1 \) and \( c = 1 \). The items (a), (b), and (c) refer to the three possible geometries of the AdS spacetime, namely, \( k = 1 \) spherical, \( k = 0 \) planar, \( k = -1 \) hyperbolic, respectively. For each item (a), (b), and (c)) we display three plots, namely, for \( d = 4, 5, \text{ and } d = \infty \). In the plots, the physically relevant region is always to the right of the left vertical asymptote, which marks the gravitational radius (or the horizon) of the solution. Given \( d, k, M, \text{ and } Lambda, \) Eqs. (59)-(61) tell us the regions in a plot \( \eta_0 \times a_0 \) where the stability conditions are satisfied. Depending in which slot of the rhs of (59)-(61) one is, the inequality on the left hand side gives the region above or under the curves of Fig. 11.

For \( k = 1 \), Fig. 11(a), there are two intervals worth of mentioning. The first interval is between the left asymptote and the right asymptote. The region of stability is then above the curve shown. The second interval is to the right of the right asymptote. The stability region is given below the curve shown. In this interval \( \eta_0 < 0 \) for which some justification can be given, see, e.g., 12. At the point where (60) holds one gets for stability that \( \eta_0 = +\infty \) or \( \eta_0 = -\infty \) depending on the branch one is.

For \( k = 0 \), Fig. 11(b), there is only one interval worth of mentioning. This interval is to the right of the left asymptote. The region of stability is then above the curve shown and \( \eta_0 > 0 \) in this region.

For \( k = -1 \), Fig. 11(c), there is also only one interval worth of mentioning. This interval is to the right of the left asymptote. The region of stability is then above the curve shown and \( \eta_0 > 0 \) in this region. The parameter \( \eta_0 \) can be negative in this region.

Now in each geometry (a), (b), or (c), for finite \( d \) the curves do not change qualitatively, as can be seen displayed in the figure. However if we take the limit \( d = \infty \) we find that interesting things happen. Firstly, the gap between the asymptotes and the curves is reduced, and is the shorter as the dimension \( d \) increases. This shows that in this limit, the region of stability is going to be the area limited by the respective asymptotes, both vertical and horizontal, as
FIG. 1: These are plots for $\eta_0$ as function of $a_0$, as given on the right hand side (rhs) of the inequalities (36)-(38). The inequality refers to the area above or under the curves given by the rhs of (36)-(38). For every graphic we have $M = 10$, $Q = 0$, and $\Lambda = -\frac{1}{5}$, in natural units, $G = 1$ and $c = 1$. In every case, from (a) to (c), the left side is for $d = 4$, the center is for $d = 5$ and the right is for the limit of large $d$, illustrating the limit $d \to \infty$. The (a) graphics are for spherical geometry, $k = 1$, the (b) are for planar geometry, $k = 0$, and the (c) are for hyperbolic geometry, $k = -1$. The physically relevant region is always to the right of the left vertical asymptote, which marks the gravitational radius (or the horizon) of the vacuum black hole solution. For (a) the region of stability is to the right of the left asymptote, above the curve. To the right of the right asymptote, the stability region is below the curve. For (b) and (c), there is only the left asymptote, and the stability regions are always above the curve. This is true for our choice of parameters, which renders the expression (38) the relevant one for this particular case, because of eqcondition expression being positive. Note that the separation of the two asymptotes in the spherical case, $k = 1$, tends to zero as $d \to \infty$. This does not apply to planar, $k = 0$, and hyperbolic, $k = -1$. What is notorious is the closing of the curves to their respective asymptotes in all cases, as the dimension increases. Also note that the higher the dimension, the clearer the horizontal asymptotes, as $a_0$ tends to infinity. For spherical, $k = 1$, and hyperbolic geometries, $k = -1$, the horizontal asymptote is at $\eta_0 = -1$, despite the fact that for spherical geometry the asymptote is approached from below, whereas for hyperbolic geometry it is approached from above; for planar geometry, $k = 0$, the horizontal asymptote is at $\eta_0 = 0$, approached from above.

the actual limiting curves given by the expressions (59)-(61) are identified with the asymptotes. Secondly, only for the spherical case, with our choice of parameters, the two vertical asymptotes, limiting the two different branches of stability, the left one given by (61), and the right one given by (59), in the limit of very large dimension, $d = \infty$, will merge, eliminating the interval between the asymptotes, thus ending the branch given by (61). Finally, we note that in the limit of large $a_0$, each geometry has a horizontal asymptote at a certain value of $\eta_0$. Now, when $d$ is increased, this horizontal asymptote approaches a certain value. In the limit $d = \infty$, this value is $\eta_0 = -1$ for spherical and hyperbolic geometries, and $\eta_0 = 0$ for planar geometry (see Fig. 1 right column).
V. CONCLUSIONS

We have used the cut and paste procedure in order to build a class of $d$-dimensional wormholes, with a $(d-1)$-dimensional timelike throat, by gluing together the spacetimes of geometric-topological $d$-dimensional charged vacuum solutions, with a negative cosmological constant, cut somewhere above the respective gravitational radii. After obtaining the static solutions, through the use of the Darmois-Israel formalism, we analyzed the energy conditions, and performed a linearized stability analysis, where the purpose was to establish the response of the solutions to a linear perturbation around a static configuration. We obtained general results. Previous results are obtainable from the present’s work general results for the appropriate choices of $d$, $k$, $\Lambda$ and $Q$.

Acknowledgments

GASD thanks Centro Multidisciplinar de Astrofísica - CENTRA for hospitality. GASD is supported by FCT fellowship SFRH/BPD/63022/2009. This work was partially supported by FCT - Portugal through projects CERN/FP/109276/2009 and PTDC/FIS/098962/2008.

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