Two-dimensional random walk in a bounded domain

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Abstract – In a recent letter Ciftci and Cakmak (EPL, 87 (2009) 60003) showed that the two-dimensional random walk in a bounded domain, where walkers which cross the boundary return to a base curve near the origin with deterministic rules, can produce regular patterns. Our numerical calculations suggest that the cumulative probability distribution function of the returning walkers along the base curve is a Devil’s staircase, which can be explained from the mapping of these walks to a non-linear stochastic map. The non-trivial probability distribution function (PDF) is a universal feature of CCRW characterized by the fractal dimension $d=1.75(0)$ of the curve which bounds this distribution.

Diffusion is a basic physical process [1] that moves substances randomly from the high concentration regimes to the low ones. A simple mathematical model of this random Brownian motion [2,3] is well described by a random walk [4] where the walker takes unit steps successively in an arbitrary direction. Several variations of the random walk with different boundary conditions [5] have been studied in $d$-dimensions. Such walks can also be defined on a lattice where, starting from the origin, the walker moves randomly to one of its neighbour. The individual random walks are known to become scale invariant after a large number of steps and their radial distances from the origin follow a normal distribution. Being simple, the models of random walk and some of their variations, have found applications in several branches of science. Reaction-diffusion systems, percolation, network dynamics and stock fluctuations [6] are few to name.

Different boundary conditions [5] are known to have strikingly unusual effects on the longtime behavior of the random walks. Recently Ciftci and Cakmak [7] studied the two-dimensional random walk (2DRW) in a bounded region, where the walker deterministically returns to a pre-defined base curve near the origin after crossing the domain boundary. For example, in one particular case, the walker chooses a new coordinate $(x \rightarrow x/\sqrt{x^2+y^2}, y \rightarrow y/\sqrt{x^2+y^2})$ after crossing the boundary $(|x|<6$ or $|y|<2)$ and starts a fresh walk. Interestingly these walks, hereafter named as Ciftci and Cakmak random walk (CCRW) produce regular patterns.

In this letter we show that the patterns are simple repetition of the base curves with centers placed at all the points $(i,j)$ within the domain where the sum of these integers $(i+j)$ is even. The cumulative distribution of returning walkers along the base curve is found to be a Devil’s staircase (DS) [8], which could be explained from the mapping of CCRW to a stochastic non-linear map. These DS structures, which are generic features of CCRW, can be characterized by the fractal dimension $d$ of the curve which bounds the corresponding PDF. Numerical calculations suggest that $d=1.75(0)$ is universal.

In CCRW, the walker initiates a discrete-time walk on a two-dimensional square lattice starting from the origin, by taking a random unit step $\pm 1$ both in the $x$- and $y$-direction. Thus in each time step its coordinate changes from $(x,y)$ to $(x+\sigma, y+\sigma)$, where $\sigma_{x,y} = \pm 1$ chosen randomly. The walk is confined in a bounded domain, usually taken as a rectangle $|x|<b_x$ and $|y|<b_y$. If the walker crosses the boundary, e.g. when $|x|>b_x$ or $|y|>b_y$, it returns immediately to a new coordinate $(f(\theta), g(\theta))$, where $\theta = \tan^{-1}(y/x)$. The new co-ordinates define a curve,

$$x = f(\theta); \quad y = g(\theta),$$

parametrized by $\theta$, which will be referred to as the base curve (BC). Case-I of [7] corresponds to $f(\theta) = \cos \theta$ and $g(\theta) = \sin \theta$ which generate regular patterns in a bounded domain, compared to a simple two-dimensional random

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Fig. 1: (Colour on-line) Patterns: (a) uncorrelated random walk starting from the base curve (2), (b) corresponding CCRW after $10^5$ steps.

walk which visits only a set of integer points \{\vec{r}\} = \{(x = i, y = j)\} where \(i + j\) is even.

The walker in CCRW eventually returns to the base curve and initiates a fresh walk from there. Starting from any point \(\vec{r}\) on the base curve each walker further visits only the points \{\vec{r} + \vec{v}\}. Hence, the patterns which are produced by CCRW are the repetition of the base curve shifted by the vectors \{\vec{v}\}. Such patterns can simply be generated by a set of uncorrelated random walks (URW), when a large number of walkers, initially distributed arbitrarily on a specific base curve, start performing the usual two-dimensional random walk. In URW, however, one must use an absorbing boundary condition that the walk terminates when the walker crosses the boundary.

To show that URW can generate any desired pattern, we take an example,

Boundary: \(|x| \leq 3\) and \(|y| \leq 2\),

Base curve: \(f(\theta) = a \sin(\theta) \cos^2(\theta)\),

\(g(\theta) = a \sin^2(\theta) \cos(\theta)\). \hspace{1cm} (2)

Let the distribution of walkers \(P(\theta)\) along the base curve at an angle \(\theta\) be uniform (i.e., \(\theta\) is a random number uniform in \((0, 2\pi)\)). The patterns generated by this URW with \(a = 1\) is shown in fig. 1. In this pattern the repeating base curves are non-overlapping as their lobes are bounded within a circle of radius \(1/\sqrt{2}\). One can, in fact, generate more complicated and overlapping patterns either by choosing \(a > \sqrt{2}\) or by taking different BCs.

Patterns similar to fig. 1(a) can also be produced by CCRW. The walker in this case initiates the walk from origin and immediately after crossing the boundary it returns to the base curve (2). The return is deterministic as \(\theta = \tan^{-1}(y/x)\) in eq. (2) depends on the final co-ordinates \((x, y)\) of the walker. Corresponding pattern is shown in fig. 1(b). Unlike fig. 1(a), here, the patterns appear discontinuous, which indicates that certain regions along the BC are never visited. This is also confirmed from the numerical results of the distribution of returning walkers \(P(\theta)\) along the BC in the range \(\theta \in (0, \pi)\). Figure 2 shows the distribution \(P(\theta)\), where some of the forbidden regions \((P(\theta) = 0)\) are marked with an arrows. A quantitative measure of the forbidden regions is

\[ \phi = \int_0^{2\pi} \Theta(\epsilon - P(\theta)) \, d\theta, \]  

where \(\epsilon \simeq 0\) is a pre-determined positive number and \(\Theta(x)\) is the Heaviside step function. In fact \(\phi_s\) saturates to \(\phi_s = 0.232(3)\) as the number of time steps \(N\) increases (inset of fig. 2(a)). A non-zero \(\phi_s\) is not a typical characteristic of CCRW. When the BC is a circle (Case I of [7]), we find that the walker returns to all values of \(\theta\) with finite probability (as shown in fig. 2(b)) and thus \(\phi_s \rightarrow 0\) (inset of fig. 2(b)).

A unique feature of the CCRW that emerges from the distribution of returning walkers shown in fig. 2 is that this probability measure may not be represented in any functional form. To get a functional form, namely the PDF, one needs to count the fraction of walkers \(P(\theta)\Delta\theta\) that comes back to a bin of size \(\Delta\theta\) about \(\theta\) and then take the limit \[ P(\theta) = \lim_{\Delta\theta \to 0} P(\theta)\Delta\theta. \] The PDF is well defined only when above limit exists. In fig. 3(a) we have plotted \(P(\theta)\Delta\theta\) against \(\Delta\theta\) for CCRW.
with BC (2) and for two different values of $\theta = 0.7, 0.4$. It is evident that the limit $\Delta \theta \to 0$ does not exist. Figure 3(b) there shows the same for Case I of [7] and for $\theta = 0.7, 0.3$.

It is natural to ask if the cumulative distribution defined by $P(\theta \geq \phi) = \int_0^\phi P(\theta) \, d\theta$ is a well-defined function. Figure 4 shows that $P(\theta \geq \phi)$ against $\phi$, for the base curve $f(\theta) = \cos(\theta)$, $g(\theta) = \sin(\theta)$ is a Devil’s staircase [8], not differentiable at infinitely many points. Thus it is not surprising that $P(\theta|\Delta \theta)$ does not have a well-defined limit for $\Delta \theta \to 0$.

Contrary to the naive expectation that any deterministic dynamics, when added to an existing stochastic system, does not alter the stochastic behavior, here we observe that the deterministic return of a two-dimensional random walker to a pre-defined base curve partially destroy the randomness resulting in a non-trivial probability distribution. To understand it better, let us take the following CCRW:

Boundary: \( |x| < 1/2 \) and \( |y| < 1/2 \),

Base curve: \( x = \cos(\theta)/2 \) and \( y = \sin(\theta)/2 \),

where $\theta$ is angle made with $x$-axis by the returning random walker. In this case, the walker crosses the boundary in every attempted walk and returns to the base curve. Effectively the walker traces different points on the base curve using a stochastic map \( x_{t+1} = (x_t + \sigma^x_t) / r_t; y_{t+1} = (y_t + \sigma^y_t) / r_t \) where \( \sigma_t^x, \sigma_t^y \) is chosen randomly (by the two-dimensional walker during its walk) and \( r_t = \sqrt{(x_t + \sigma_t^x)^2 + (y_t + \sigma_t^y)^2} \). One can, in fact write an equivalent one-dimensional map in terms of \( z = y/x = \tan(\theta) \) as

\[
z_{t+1} = h(z_t, \sigma^x_t, \sigma^y_t),
\]

with $h(z, \sigma^x, \sigma^y) = \frac{z + 2\sigma^y\sqrt{1 + z^2}}{1 + 2\sigma^x\sqrt{1 + z^2}}$.

On each iteration $z$ takes a new value by choosing one of the four non-linear functions $h(z, \pm 1, \pm 1)$ randomly as shown in fig. 5. These functions $h(z, \sigma^x, \sigma^y)$ have attractive fixed points at $z^* = \sigma^x \sigma^y$. Note, that if $\sigma_z$ were not random (say $\sigma_z = 1$) then $z$ would evolve using the map $z_{t+1} = h(z_t, 1, \sigma_t^y)$ and ultimately remains confined in the region $|z| < 1$. The corresponding distribution of $\theta = \tan^{-1}(z)$ (shown as a cumulative distribution in fig. 6) compares well with fig. 4, which indicates that the CCRW (5) has all the characteristic features of the generic two-dimensional bounded random walk.

We argue that this non-trivial behavior, which occurs even for the CCRW (5), is the effect of the non-linearity that exists in the stochastic map (6). In the following we show that a linear stochastic map having identical fixed points would not show any such features. In particular, the steady-state probability of $z$ has a well-defined PDF. Let us take the following stochastic linear map:

\[
z_{t+1} = \frac{\sigma^x_t}{2} z_t + \left(1 - \frac{\sigma^x_t}{2}\right) \sigma^y_t,
\]

which has the same fixed points as (6) and calculate the steady-state distribution $g(z)$. The direct iteration of this
map yields

\[ z_{t+1} = \sum_{n=0}^{t} \sigma_n^y \left( 1 - \frac{\sigma_n^x}{2} \right) - \frac{1}{\beta} \prod_{k=n+1}^{t} \sigma_k^x. \]

Since the product of \( \sigma^x \)s produces a random sign \( \pm \), we have

\[ z_{t+1} = \sum_{n=0}^{t} \tau_n^y \left( 1 - \frac{\tau_n^x}{2} \right), \quad (8) \]

where the term containing the initial value \( z_0 \) (being exponentially small for large \( t \)) is dropped, and we have taken \( \tau_n^x = \sigma_n^x \). The right-hand side of the above equation resembles the Hamiltonian of a spin system on a lattice of size \( L = t \), where the spins \( \tau_n^x, \tau_n^y \) of two different kinds of particles \( x \) and \( y \) at the site \( n \) interact with an inhomogeneous magnetic field \( B_n = 2^{-n} \). Thus the expression for the energy of the system is

\[ E = \sum_{i=0}^{L} B_i \tau_i^y \left( 1 - \frac{\tau_i^x}{2} \right). \quad (9) \]

The energy distribution of this model \( P(E; \beta) \) in equilibrium is related to the stationary distribution \( g(z) \) of \( (8) \);

\[ g(z) = \lim_{\beta \to 0} P(E = z; \beta). \quad (10) \]

Since, for classical systems with partition function \( Z_L(\beta) \),

\[ P(E, \beta) = \mathcal{L}^{-1} \left[ \frac{Z_L(s + \beta)}{Z_L(\beta)} \right]; E, \]

where \( \mathcal{L}^{-1} [f(s); x] = \int_{x}^{\infty} e^{-s} f(s) ds \) is the inverse Laplace transform, we have

\[ g(z) = \mathcal{L}^{-1} \left[ \frac{Z_L(s)}{Z_L(0)} ; z \right]. \quad (11) \]

The partition function of the model \( (9) \)

\[ Z_L(\beta) = \frac{\sinh \beta \sinh 2 \beta}{\sinh(2 - \beta) \sinh(2 - 2 \beta)} \]

is used in eq. (11) to obtain

\[ g(z) = \begin{cases} 
\frac{1}{4}, & |z| < 1, \\
\frac{3 - \epsilon(z) z}{8}, & 1 < |z| < 3, \\
0, & |z| > 3, 
\end{cases} \quad (12) \]

in thermodynamic limit \( L \to \infty \). Here \( \epsilon(z) \) is the signum function. Correspondingly, the distribution of \( \theta = \tan^{-1} z \) is \( P(\theta) = g(\tan \theta) \sec^2 \theta \), and its cumulative distribution is

\[ P(\theta > \phi) = \begin{cases} 
\frac{2 + \tan(\phi)}{4}, & |\tan \phi| \leq 1, \\
\frac{8 + 6 \tan \phi + \epsilon(\tan \phi) \sec^2 \phi}{16}, & 1 < |\tan \phi| < 3, \\
0, & |\tan \phi| \geq 3.
\end{cases} \]

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Fig. 7: (Colour on-line) Plot of \( \nu \) vs. \( \Delta \phi \) in log scale for the PDF of the CCRW with BC (2), a unit circle, and a unit square (see footnote\(^1\)) are shown with symbols \( ×, o \) and \( □ \), respectively. A line with slope \(-7/4\) is drawn to guide the eye.

In fig. 6 we have plotted the cumulative distribution \( P(\theta > \phi) \) for both the non-linear map \( (6) \) and the stochastic linear map \( (8) \) which has identical fixed points. It is quite evident from the figure that \( P(\theta > \phi) \) for the non-linear map, which corresponds to the CCRW \( (5) \), has a structure of Devil’s staircase, whereas the same for the linear map is a continuous and differentiable function. Thus, it is suggestive that the non-trivial distribution of returning walkers on the base curve is an artifact of non-linearity.

It is evident from fig. 2 that the curve which bounds the distribution of the returning walkers \( P(\theta) \) is a fractal. The fractal dimension of this curve, named hereafter as the bounding curve, can be used to characterize the non-trivial distribution \( P(\theta) \) of CCRW. From the naive box-counting methods one expects that the bounding curve, is covered by \( \nu \sim \Delta \phi^{-d} \) segments of size \( \Delta \phi \) with \( d \neq 1 \). For the CCRW with BC \((2)\) a plot of \( \nu \) vs. \( \Delta \phi \) in log-scale (fig. 7) shows that \( d = 1.75(8) \). The same calculation for a few other BCS\(^1\) results in \( d \approx 1.75 \), which made us to conjecture that possibly the fractal dimension of PDF bounding curve of CCRW is universal (\( d = 7/4 \)).

In conclusion, we explain that patterns of Ciftci\-Cakmak random walk [7] (CCRW) are simple repetition of the base curve. A unique feature of CCRWs is that the distribution of returning walkers along the base curve cannot be represented by a functional form; the corresponding cumulative distributions are Devil’s staircases. A correspondence of CCRW with stochastic Wiley non-linear maps reveals that this unusual distribution is an artifact of non-linearity. A quantitative characterization of the distribution could be the fractal dimension \( d \) of its bounding curve. Our numerical calculations of CCRWs with different base curves show that \( d \approx 1.75 \).
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