Cylindric-Polyadic algebras have the super amalgamation property

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May 10, 2014

Abstract

We show that the cylindric- polyadic algebras, introduced by Ferenczi, have the superamalgamation property

1 Introduction

So called relativization started as a technique for generalizing representations of cylindric algebras, while also, in some cases, ‘defusing’ undesirable properties, like undecidability or lack of definability (like Beth definability). These ideas have counterparts in logic, and they have been influential in several ways. Relativization in cylindric-like algebras lends itself to a modal perspective where transitions are viewed as objects in their own right, in addition to states, while algebraic terms now correspond to modal formulas defining the essential properties of transitions. The modal perspective also suggests that first order predicate logic reflects only part of the expressive resources of abstract state models.

Indeed, why insist on standard models? This is a voluntary commitment to only one mathematical implementation, whose undesirable complexities can pollute the laws of logics needed to describe the core phenomena. Set theoretic cartesian squares modelling as the intended vehicle may not be an orthogonal concern, it can be detrimental, repeating hereditary sins of old paradigms.

Indeed in [6] square units got all the attention and relativization was treated as a side issue. Extending original classes of models for logics to manipulate their properties is common. This is no mere tactical opportunism, general models just do the right thing.

The famous move from standard models to generalized models is Henkin’s turning round second order logic into an axiomatizable two sorted first order logic. Such moves are most attractive when they get an independent motivation.
The idea is that we want to find a semantics that gives just the bare bones of action, while additional effects of square set theoretic modelling are separated out as negotiable decisions of formulation that threatens completeness, decidability, and interpolation.

And indeed by using relativized representations Ferenzci, proved that if we weaken commutativity of cylindrifiers and allow relativized representations, then we get a finitely axiomatizable variety of representable quasi-polyadic equality algebras (analogous to the Andréka-Resek Thompson CA version); even more this can be done without the merry go round identities. This is in sharp view with complexity results proved recently by the author for quasi poyadic equality algebras, of non finite axiomatizability over their diagonal free reduct and transposition free reducts. Ferenzci’s results can be seen as establishing a hitherto fruitful contact between neat embedding theorems and relativized representations, with enriching repercussions for both notions.

One can find well motivated appropriate notions of semantics by first locating them while giving up classical semantical prejudices. It is hard to give a precise mathematical underpinning to such intuitions. What really counts at the end of the day is a completeness theorem stating a natural fit between chosen intuitive concrete-enough, but perhaps not excessively concrete, semantics and well behaved axiomatizations. The move of altering semantics has radical phiosophical repercussions, taking us away from the conventional Tarskian semantics captured by Fregean-Godel-like axiomatization; the latter completeness proof is effective but highly undecidable; and this property is inherited by finite variable fragments of first order logic as long as we insist on Tarskian semantics.

Now we use two techniques to get positive results, concerning the supermalgamation property for what Ferenzci calls cylindric-polyadic algebras. The term is justified by the fact that their signature ahs only finite cylindrifiers (like cylindric algebras) but they also have all substitutions like polyadic algebras. The first is yet again a Henkin construction (carefully implemented because we have changed the semantics, so that desired Henkin ultrafilters, on which our relativized models, will be based are more involved), the other is inspired by the well-developed duality theory in modal logic between Kripke frames and complex algebras. This last technique was first implemented by Németi in the context of relativized cylindric set algebras which are complex algebras of weak atom structures.

We need some preliminaries. The part to follow, which has to do only with representation, is due to Ferenczi, which we include with his permission. We thank him for sending us the tex file of his manuscript, which made the writing much easier. Using one modified version of a Henkin construction, which Ferenczi calls perfect, proves a very elegant completeness theorem for infinitary extensions of first order logic with equality. In fact, such a result can...
be regarded as one possible solution to the finitizability problem for first order logic with equality!

To get the stronger result of interpolation (indeed the former result can be easily distilled from the proof of the latter), we need two perfect ultrafilters that agree on the common subalgebra, or the common language of the two formulas to be interpolated.

The Daigneault–Monk–Keisler, neat embedding theorem, says that if \( \mathfrak{A} \in \mathcal{PA}_\alpha \), \( \mathfrak{B} \in \mathcal{SM}_\alpha \mathcal{B} \) for some \( \mathfrak{B} \in \mathcal{PA}_{\alpha + \varepsilon} \), where \( \alpha \) is a fixed infinite ordinal and \( \varepsilon > 1 \), then \( \mathfrak{A} \) is representable. (see [2] and [6] Thm. 5.4.17). The neat embedding part does work when we add diagonal elements, every algebra neatly embeds into arbitrary higher dimensions; however, this does not enforce representability like the diagonal free case. Therefore, it is quite an achievement to obtain a representation theorem for polyadic-like algebras with diagonal elements, even if the representation is relativized.

On the face of it, it seems that the process of relativization in this context, is not a free choice, it is necessary, to get over the hurdle of incompatibility of the presence of infinitary substitutions and diagonal elements together. This phenomena manifests itself prominently, like for example in the case of Sain’s algebras prohibiting a solution to the finitizability problem for (algebraisable extensions of) first order logic with equality.

This unfavourable contact is explained with care in [8], showing that such a precarious combination blows up ultraproducts, taking us out of the representant class, if we stick to Tarskian semantics.

To cut a long story short, Tarskian square semantics breaks down here, yet again, confirming our earlier stipulation, that square semantics was not the best choice in the world, and we also need to weaken the Rosser condition of commutativity of cylindrifiers, so we are actually varying both syntax and semantics, in the ultimate aim of obtaining a perfect match represented in a strong completeness theorem.

This is indeed, analogous to celebrated the breath-taking Andréka-Resek-Thompson result, which opened many windows establishing a fruitful dichotomy in algebraic logic, between set algebras that are so resilient to nice axiomtiizations, to Stone like neat representation theorems for algebraisations of a whole landscape of multi dimensional modal logics of quantifiers, with the ill-behaved multi dimensional cylindric modal logic, just appearing as the top of an ice-berg, with a lot of hidden treasures below the surface.

This form of the neat embedding theorem for \( \mathcal{PA} \) is due to Daigneault and Monk. Keisler published the proof theoretical variant of the theorem in the same issue. Here we are going to refer to the proof of Theorem 4.3 in [2] and its variant for polyadic equality algebras ([6] II. 5.4.17.)

Several abstract classes \( \mathcal{CPEA}_\alpha, \mathcal{CPES}_\alpha, m_\mathcal{CPEA}_\alpha \) \((m < \alpha)\), \( \mathcal{LM}_\alpha \) are defined in [3] p. 156, which are polyadic equality algebras, without full fledged com-
mutativity of cylindrifiers. Te class $L_m$ are those algebras in $mCPE\alpha$ such that $|\Delta b| < \alpha$ for all $b$.

**Definition 1.1.**

1. Assume $\alpha$ is an infinite ordinal and $m < \alpha$ is infinite.

Given a set $U$ and a fixes sequence $p \in ^\alpha U$, the set

$$\alpha^mU(p) = \{ x \in ^\alpha U : x \text{ and } p \text{ are different at most in } m \text{ many spaces} \}$$

2. A transformaton $\tau$ on $\alpha$ is said to be an $m$ transformatin if $\tau i = i$ excepty for $m$-many $i\alpha$. The set of al such transformations is denoted by $mT_\alpha$.

Concrete classes of relativized set algebras $mGwp_\alpha$ and $Gp_\alpha$ are aso defined on the next page. The units of such algebras are unions of squares or weak spaces, but disjointness of the bases is omitted, and the top elements satisfy certain closure conditions depending on the substitutions used. So here we are encountered with a different geometry too. Such new axioms or principles are geometric conditions on a par with standard geometric axioms about, points, lines, squares and rectangles in multi dimensional geometric spaces.

With so many defined classes, and so many representation theorems proved by Ferenczi, we make the choice of sticking to one of them. The other cases can be approached in exactly the same manner undergoing the obvious modifications, with the aid of Ferenczi’s work.

The axiom refered to as $(CP9)^*$ is also defined in [3], which roughly states that cylindrifiers do not commute even with substitutions, when it is consistent that they can as in the case of square representations of $\text{PEA}$.

### 2 First proof

Ferenczi shows that the Diagneault Monk theorem holds if the class $\text{PA}_\alpha$ is replaced by $mCPE_\alpha$ and $\text{PEA}_{\alpha+\varepsilon}$ is replaced by such a class $mCPE_{\alpha+\varepsilon}$ such that the $mCPE_{\alpha+\varepsilon}$ axioms hold, except for the axiom $(CP9)^*$ which merely holds for every $i, j \in \alpha, \sigma \in mT_\alpha$ and, in additional, the following two instances of $(CP9)^*$ are satisfied:

$$c_i x = c_{m^{|i/m|}} x \text{ if } i \in \alpha, m \notin \alpha, x \in A \quad (1)$$

$$c_{m^\tau z} = c_{m^\tau} z \text{ if } \tau m = m, m \notin \alpha, \tau \in mT_\beta, z \in B \quad (2)$$

By $r$-representability, we mean relativized representability.

The following theorem implies that a more sophisticated kind of neat embeddability of an algebra in $mCPE_\alpha \cap L_m$ is equivalent to $r$-representability.
Theorem 2.1. (Ferenzci) Assume that $A \in m \text{CPE}_{\alpha} \cap L_{m} \alpha$, where $m$ is infinite, $m < \alpha$. Then $A \in SNr_{\alpha}B$ for some $B \in m \text{CPE}_{\alpha+\varepsilon}$, where $\varepsilon$ is infinite, if and only if $A \in I_{m}Gwp_{\alpha}$.

Let us consider the hard direction, namely, that $A \in SNr_{\alpha}B$ implies $A \in I_{m}Gwp_{\alpha}$.

Fix an algebra $B$. Let us denote by $adm$ the class of $m$-transformations $\tau \in \alpha \beta$, i.e., $\tau \in mT_{\alpha} \cap \alpha \beta$, where $\alpha + \varepsilon$ is denoted by $\beta$. We introduce a concept needed in the proof: A Boolean ultrafilter $F$ in $B$ is a perfect ultrafilter if for any element of the form $s\tau_{c}jx$ included in $F$, where $j \in \alpha$, $x \in A$ and $\tau \in adm$, there exists an $m$, $m \notin \alpha$, $\tau_{m} = m$ such that $s\tau_{[j/m]}x \in F$.

As is known, neat embeddability into a class with $\varepsilon$ extra dimensions, where $\varepsilon$ is infinite, implies neat embeddability into the class with any infinitely many extra dimensions ([5] I. 2.6.35). In otherwords, stretching extra dimensions to $\omega$ is enough to do any infinite stretching using arbitrary more dimensions. Therefore, from now on, we may assume that $\varepsilon > \max(\alpha, |A|)$, where $\varepsilon + \alpha$ is a regular cardinal.

Lemma 2.2. Let $a$ be an arbitrary, but fixed non-zero element of $A$ and $\varepsilon > \max(\alpha, |A|)$, where $\varepsilon + \alpha$ is regular, and assume that $A \in SNr_{\alpha}B$ for some $B \in m \text{CPE}_{\alpha+\varepsilon}$. Then, there exists a proper Boolean filter $D$ in $B$, such that $a \in D$ and any arbitrary ultrafilter containing $D$ is a perfect ultrafilter in $B$.

Proof. Henkin's completeness proof is appropriately adapted to the algebras in question. We use freely the axiomatization $(CP_{0}) - (E_{3})$ in [3] p.156. Let

$$X = \{s_{\tau_{c}j}x : \tau \in adm, j \in \alpha, x \in A\}.$$ 

Let $\beta$ denote the ordinal $\alpha + \varepsilon$, $|X| \leq m\beta \cdot \alpha \cdot |A|$. $\beta > \alpha$, $\beta$ is regular, therefore, $m\beta = \beta$. $\beta > \max(\alpha, |A|)$ and $m\beta = \beta$ imply that $|X| \leq \beta$. Let $\rho : \beta \to X$ be a fixed enumeration of $X$.

Let $F_{0}$ be the Boolean (BA) filter of $B$ generated by $a$. We define recursively an increasing sequence $(F_{i} : i < \beta)$ of proper BA filters in $B$.

Assume that $\rho_{1} = s_{\tau'}_{c}j'x'$, where $\tau' \in adm$, $j' \in \alpha$ and $x' \in A$. Let $F_{1}$ be the filter generated by the set $G_{1}$ in $B$, where

$$G_{1} = F_{0} \cup \{s_{\tau'_{c}j'x'} : \tau'_{c}j'x' \to s_{\tau'_{c}j'x' / m'}x'\}$$

and $m' \in \beta$ is such that $m' \notin \text{Rg} \tau' \cup \alpha$. We will show that $F_{1}$ is a proper filter in $B$. 

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Let $n$ be a fixed ordinal ($n < \beta$). Assume that $F_i$ ($0 \leq i < n - 1$) has been defined by the fixed generator system $G_i$ ($G_0 \subset G_1 \subset \ldots \subset G_{n-1}$). Let $ho_n = s_\tau c_j x$, where $\tau \in \text{adm}$, $j \in \alpha$ and $x \in A$.

If $n$ is a successor ordinal, let $F_n$ be the filter in $B$ generated by the set

$$G_n = G_{n-1} \cup \{s_\tau c_j x \rightarrow s_\tau s[j/ m_n] x\}$$

(3)

where $m_n \in \beta$ is such an ordinal that $m_n \notin \alpha \cup \bigcup_{g \in G_{n-1}} \dim g$ and $\tau m_n = m_n$.

Such an ordinal exists because $|\dim g| \leq m$, $|G_{n-1}| < n$, hence $\bigcup_{g \in G_{n-1}} \dim g < m \cdot n < \beta$, furthermore $\alpha < \beta$, too. Let us denote by $g_n$ the generator element $s_\tau c_j x \rightarrow s_\tau s[j/ m] x$.

If $n$ is a limit ordinal, then let $F_n$ be $\bigcup_{i<n} F_i$. Obviously, $F_j \subseteq F_k$ if $j < k$.

It remains to show that the filter $F_n$ generated by the set in (3) is a proper filter. The only case worthwhile considering is the case when $n$ is a successor ordinal. Indirectly, assume that $F_{n-1}$ is proper and assume, seeking a contradiction, that $F_n$ is not.

In what follows, let us denote $m_n$ by $m$, for short. Suppose on the contrary that $-(s_\tau c_j x \rightarrow s_\tau s[j/ m] x)$ belongs to $F_n$. The property of generating filters in BA’s implies that there are finitely many generators in $F_{n-1}$ such that

$$a \cdot (s_\tau c_j x_1 \rightarrow s_\tau s[j_1/ m_1] x_1) \cdot \ldots \cdot (s_\tau c_j x_k \rightarrow s_\tau s[j_k/ m_k] x_k) \leq$$

$$\leq -(s_\tau c_j x \rightarrow s_\tau s[j/ m] x)$$

(4)

where $x_1, x_2, \ldots, x_k, x$ are in $A$. Let us apply $c_m^\partial$ to both sides of this inequality ($c_m^\partial$ denotes the operator $-c_{m-}$).

If $x$ is any factor of the left-hand side, then the conditions $m \notin \dim g$, $g \in G_{n-1}$, $x \in F_{n-1}$ and (2) imply that

$$c_m(s_\tau c_j x_i \rightarrow s_\tau s[j/ m_1] x) = s_\tau c_j x_i \rightarrow s_\tau s[j/ m_1] x.$$  

(5)

But (5) is true for $c_m^\partial$ instead of $c_m$, using that $c_m(-c_m x) = -c_m x$, $x \in B$. Thus, applying $c_m^\partial$ to the left-hand side of (4) does not change it and it must be different from 0 because $F_{n-1}$ is a proper filter. Here we have used that

$$c_m^\partial(u + v) = c_m^\partial u + c_m^\partial v,$$

which is a consequence of (CP3).

Applying $c_m^\partial$ to the right-hand side of (4), we can show that we obtain zero. We have

$$c_m^\partial(-(s_\tau c_j x \rightarrow s_\tau s[j/ m] x)) = -c_m[-s_\tau c_j x + s_\tau s[j/ m] x]$$

$$= -(c_m(-s_\tau c_j x) + c_m s_\tau s[j/ m] x)$$

(6)
because \(c_m(u + v) = c_m u + c_m v\).

On the one hand, as regards \(c_m(-s_r c_j x)\) in (6), by \(m \notin \alpha, \tau m = m\) and (2),

\[
c_m(-s_r c_j x) = c_m(-c_m s_r c_j x) = -c_m s_r c_j x.
\]

Here \(c_m s_r c_j x = s_r c_m s[j / m] c_m x\) because

\[
c_j x = c_m s[j / m] x
\]

by (2) if \(x \in A\). Therefore,

\[
c_m(-s_r c_j x) = -s_r c_m s[j / m] c_m x
\]

On the other hand, as regards \(c_m s_r s[j / m] c_m x\), i.e., \(c_m s_r s[j / m] c_m x\) in (6),

\[
c_m s_r s[j / m] c_m x = s_r c_m s[j / m] c_m x.
\]

by (1), where \(m \notin \alpha, \tau m = m\).

From (9) and (10) we get that (6) is zero. It is a contradiction because the left-hand side of (6) is different from zero. Therefore we have shown that, in fact, \(F_n\) is a proper filter.

Now, we have a sequence \(G_0 = F_0 \subset F_1 \subset F_2 \subset \ldots \subset F_n \subset \ldots\) of proper filters. Now let

\[
\mathcal{D} = \bigcup_n \{F_n : n < \beta\}.
\]

\(\mathcal{D}\) is a proper filter, too. \(\mathcal{D}\) contains all the elements of the form \(s_r c_j x, x \in A\). It is easily seen that \(\mathcal{D}\) is the desired filter.

But we are not quite finished. We need an extra technical trick to make things tic.

*The required perfect ultrafilter \(F\) in \(\mathcal{B}\), extending the filter \(\mathcal{D}\) will be defined as follows:* 

Take the *minimal completion* \(\mathcal{B}'\) of \(\mathcal{B}\) (see [6], I., 2.7.2). exists because operations are additive (here diagonals play an essential role; for in fact our variety is conjugated). Take the filter \(F''\) in \(\mathcal{B}'\), generated by (the generators of) \(\mathcal{D}\) – such a filter \(F''\) obviously exists. Next consider any fixed ultrafilter \((F')^+\) in \(\mathcal{B}'\), which extends \(F''\). The restriction \(F\) of \((F')^+\) to \(\mathcal{B}\) is an ultrafilter in \(\mathcal{B}\), which we take for the required extension of the filter \(\mathcal{D}\) in \(\mathcal{B}\). \(\square\)
Now using the ideas above, let us prove interpolation. Let $\mathfrak{A}$ be the free algebra, $X_1, X_2 \subseteq \mathfrak{A}$, $a \in \mathfrak{S}gX_1$ and $b \in \mathfrak{S}gX_2$, such that that $a \leq b$. Assume that an interpolant does not exist. Neatly embed the free algebra into the full neat reduct of a dilation having enough spare dimensions, the dimension of the dilation should also be a regular cardinal as in the above proof. This gives enough space to maneuver, in the process, eliminating cylindrifiers. This part is easy, because we have so many substitutions.

The second part consists of the construction the two perfect ultrafilters, the first contains $a$, the second contains $-c$, constructed in such a way that they agree on $\mathfrak{S}g^{\alpha}(X_1 \cap X_2)$, then one constructs yet a third perfect ultrafilter in the last algebra, and obtains (by Ferenczi’s result) a relativized representation $h$ of $\mathfrak{A}$ using its freeness, on the set of generators $X_1 \cup X_2$ such that $h(a, -c) \neq 0$, but this is a contradiction.

Let us implement the above sketch, but we consider the most basic cylindric polyadic algebras, namely the class $m\text{-CPEA}$. The rest of the cases for non-commutative cylindric polyadic algebras, are entirely analogous.

Ferenczi proves that if we have a neat embedding into enough infinite spare dimensions, we have representability. The next lemma shows that we can always neatly embed our algebras in enough spare dimension. The idea is that our algebras form what Diagneault and Monk call transformation systems [2]. Strictly speaking this applies to their reduct obtained by discarding all operations except all substitutions.

**Lemma 2.3.** Let $\mathfrak{A}$ be a cylindric polyadic algebra of dimension $\alpha$. Then for every $\beta > \alpha$ there exists a cylindric polyadic algebra of dimension $\beta$ such that $\mathfrak{A} \subseteq \mathfrak{N}r_{\alpha}\mathfrak{B}$, and furthermore, for all $X \subseteq \mathfrak{A}$ we have

$$\mathfrak{S}g^{\alpha}X = \mathfrak{S}g^{\mathfrak{N}r_{\alpha}\mathfrak{A}}X = \mathfrak{N}r_{\alpha}\mathfrak{S}g^{\mathfrak{B}}X.$$  

In particular, $\mathfrak{A} = \mathfrak{N}r_{\alpha}\mathfrak{B}$. $\mathfrak{S}g^{\mathfrak{B}}\mathfrak{A}$ is called the minimal dilation of $\mathfrak{A}$.

**Proof.** The proof depends essentially on the abundance of substitutions; we have all of them, which makes stretching dimensions possible. We provide a proof for cylindric polyadic algebras; the rest of the cases are like the corresponding proof in [2] for Boolean polyadic algebras.

We extensively use the techniques in [2], but we have to watch out, for we only have finite cylindrifications. Let $(\mathfrak{A}, \alpha, S)$ be a transformation system. That is to say, $\mathfrak{A}$ is a Boolean algebra and $S : \circ \alpha \to \text{End}(\mathfrak{A})$ is a homomorphism. For any set $X$, let $F(\alpha, \mathfrak{A})$ be the set of all functions from $\circ \alpha$ to $\mathfrak{A}$ endowed with Boolean operations defined pointwise and for $\tau \in \circ \alpha$ and $f \in F(\circ \alpha, \mathfrak{A})$, $s_\tau f(x) = f(x \circ \tau)$. This turns $F(\circ \alpha, \mathfrak{A})$ to a transformation system as well. The map $H : \mathfrak{A} \to F(\circ \alpha, \mathfrak{A})$ defined by $H(p)(x) = s_\tau p$ is easily checked to be an isomorphism. Assume that $\beta \supseteq \alpha$. 

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Then $K : F(\alpha, \mathfrak{A}) \rightarrow F(\beta, \mathfrak{A})$ defined by $K(f)x = f(x \upharpoonright \alpha)$ is an isomorphism. These facts are straightforward to establish, cf. theorem 3.1, 3.2 in [2]. $F(\beta, \mathfrak{A})$ is called a minimal dilation of $F(\alpha, \mathfrak{A})$. Elements of the big algebra, or the cylindifier free dilation, are of form $s_{\alpha}p, p \in F(\beta, \mathfrak{A})$ where $\sigma$ is one to one on $\alpha$, cf. [2] theorem 4.3-4.4.

We say that $J \subseteq I$ supports an element $p \in A$, if whenever $\sigma_1$ and $\sigma_2$ are transformations that agree on $J$, then $s_{\sigma_1}p = s_{\sigma_2}p$. $\mathfrak{N}_f A$, consisting of the elements that $J$ supports, is just the neat $J$ reduct of $\mathfrak{A}$; with the operations defined the obvious way as indicated above. If $\mathfrak{A}$ is an $\mathfrak{B}$ valued $I$ transformation system with domain $X$, then the $J$ compression of $\mathfrak{A}$ is isomorphic to a $\mathfrak{B}$ valued $J$ transformation system via $H : \mathfrak{N}_J \mathfrak{A} \rightarrow F(\mathfrak{A})$ by setting for $f \in \mathfrak{N}_J \mathfrak{A}$ and $x \in J$, $H(f)x = f(y)$ where $y \in X$ and $y \upharpoonright J = x$, cf. [2] theorem 3.10.

Now let $\alpha \subseteq \beta$. If $|\alpha| = |\beta|$ then the the required algebra is defined as follows. Let $\mu$ be a bijection from $\beta$ onto $\alpha$. For $\tau \in \beta \beta$, let $s_{\tau} = s_{\mu \tau \mu^{-1}}$ and for each $i \in \beta$, let $c_i = c_{\mu(i)}$. Then this defined $\mathfrak{B} \in GPHA_\beta$ in which $\mathfrak{A}$ neatly embeds via $s_{\mu|\alpha}$, cf. [2] p.168. Now assume that $|\alpha| < |\beta|$. Let $\mathfrak{A}$ be a given polyadic algebra of dimension $\alpha$; discard its cylindrifications and then take its minimal dilation $\mathfrak{B}$, which exists by the above. We need to define cylindrifications on the big algebra, so that they agree with their values in $\mathfrak{A}$ and to have $\mathfrak{A} \cong \mathfrak{N}_\alpha \mathfrak{B}$. We let (*):

$$c_{\rho}s_{\rho}p = s_{\rho^{-1}}c_{\rho(\rho\sigma|\alpha)}s_{\rho\sigma|\alpha}p.$$ 

Here $\rho$ is any permutation such that $\rho \circ \sigma(\alpha) \subseteq \sigma(\alpha)$. Then we claim that the definition is sound, that is, it is independent of $\rho, \sigma, p$. Towards this end, let $q = s_{\sigma}p = s_{\sigma\rho^{-1}p}$ and $(\rho_1 \circ \sigma_1)(\alpha) \subseteq \alpha$.

We need to show that (***)

$$s_{\rho^{-1}}c_{\rho|\sigma(\alpha)}s_{\rho\sigma|\alpha}p = s_{\rho_1^{-1}}c_{\rho_1|\sigma_1|\alpha}p.$$ 

Let $\mu$ be a permutation of $\beta$ such that $\mu(\sigma(\alpha) \cup \sigma_1(\alpha)) \subseteq \alpha$. Now applying $s_{\mu}$ to the left hand side of (***) we get that

$$s_{\mu}s_{\rho^{-1}}c_{\rho|\sigma(\alpha)}s_{\rho\sigma|\alpha}p = s_{\rho\mu^{-1}}c_{\rho(\rho\sigma|\alpha)}s_{\rho\sigma|\alpha}p.$$ 

The latter is equal to $c_{\mu|\sigma(\alpha)}s_{\rho}q$. Now since $\mu(\sigma(\alpha) \cap \sigma_1(\alpha)) \subseteq \alpha$, we have $s_{\mu}|p = s_{\mu|\sigma(\alpha)}p = s_{\mu|\sigma_1(\alpha)}p \in A$. It thus follows that

$$s_{\rho^{-1}}c_{\rho|\sigma(\alpha)}s_{\rho\sigma|\alpha}p = c_{\rho\mu|\sigma(\alpha)}s_{\rho\mu|\sigma(\alpha)}s_{\rho}q.$$ 

By exactly the same method, it can be shown that

$$s_{\rho_1^{-1}}c_{\rho_1|\sigma_1(\alpha)}s_{\rho_1\sigma|\alpha}p = c_{\rho_1|\sigma(\alpha) \cap \sigma_1(\alpha)}s_{\rho}q.$$
By this we have proved (**).

Furthermore, it defines the required algebra \( \mathfrak{B} \). Let us check this. Since our definition is slightly different than that in [2], by restricting cylindrifications to be only finite, we need to check the polyadic axioms which is tedious but routine. The idea is that every axiom can be pulled back to its corresponding axiom holding in the small algebra \( \mathfrak{A} \). We check only the axiom

\[
c_k(q_1 \land c_k q_2) = c_k q_1 \land c_k q_2.
\]

We follow closely [2] p. 166. Assume that \( q_1 = s^\alpha_p \rho_1 \) and \( q_2 = s^\alpha_p \rho_2 \). Let \( \rho \) be a permutation of \( I \) such that \( \rho(\sigma_1 I \cup \sigma_2 I) \subseteq I \) and let

\[
p = s^\alpha_p [q_1 \land c_k q_2].
\]

Then

\[
p = s^\alpha_p q_1 \land s^\alpha_p c_k q_2 = s^\alpha_p s^\alpha_{\sigma_1} p_1 \land s^\alpha_p c_k s^\alpha_{\sigma_2} p_2.
\]

Now we calculate \( c_k s^\alpha_{\sigma_2} p_2 \). We have by (*)

\[
c_k s^\alpha_{\sigma_2} p_2 = s^\alpha_{\sigma_2}^{-1} c_\rho(\{k\} \cap \sigma_2 I) s^\alpha_{(\rho \sigma_2 | I)} p_2.
\]

Hence

\[
p = s^\alpha_p s^\alpha_{\sigma_1} p_1 \land s^\alpha_p s^\alpha_{\sigma_1}^{-1} c_\rho(\{k\} \cap \sigma_2 I) s^\alpha_{(\rho \sigma_2 | I)} p_2.
\]

Now

\[
c_k s^\alpha_{\rho^{-1}} p = c_k s^\alpha_{\rho^{-1}} s^\alpha_p (q_1 \land c_k q_2) = c_k (q_1 \land c_k q_2).
\]

We next calculate \( c_k s^\alpha_{\rho^{-1}} p \). Let \( \mu \) be a permutation of \( I \) such that \( \mu \rho^{-1} I \subseteq I \). Let \( j = \mu(\{k\} \cap \rho^{-1} I) \). Then applying (*), we have:

\[
c_k s^\alpha_{\rho^{-1}} p = s^\alpha_{\mu^{-1}} c_j s^\alpha_{(\mu \rho^{-1} | \rho^{-1} I)} p_1,
\]

\[
= s^\alpha_{\mu^{-1}} c_j s^\alpha_{(\mu \rho^{-1} | \rho^{-1} I)} s^\alpha_{\rho \sigma_1 | I} p_1 \land c_\rho(\{k\} \cap \sigma_2 I) s^\alpha_{(\rho \sigma_2 | I)} p_2,
\]

\[
= s^\alpha_{\mu^{-1}} c_j [s^\alpha_{\mu \sigma_1 | I} p_1 \land r].
\]

where

\[
r = s^\alpha_{\mu \rho^{-1}} c_j s^\alpha_{\rho \sigma_2 | I} p_2.
\]

Now \( c_k r = r \). Hence, applying the axiom in the small algebra, we get:

\[
s^\alpha_{\mu^{-1}} c_j [s^\alpha_{\mu \sigma_1 | I} p_1] \land c_k q_2 = s^\alpha_{\mu^{-1}} c_j [s^\alpha_{\mu \sigma_1 | I} p_1 \land r].
\]

But

\[
c_\mu(k)\cap \rho^{-1} I s^\alpha_{(\mu \sigma_1 | I)} p_1 = c_\mu(\{k\} \cap \sigma_1 I) s^\alpha_{(\mu \sigma_1 | I)} p_1.
\]
So

\[ s^{\alpha}_{\mu-1}c_k[s^{\alpha}_{\mu\sigma}1P_1] = c_kq_1, \]

and we are done.

For the second part, let \( \mathfrak{A} \subseteq \mathfrak{M}_a\mathfrak{B} \) and \( A \) generates \( \mathfrak{B} \) then \( \mathfrak{B} \) consists of all elements \( s^\beta x \) such that \( x \in A \) and \( \sigma \) is a transformation \( \mathfrak{B} \) such that \( \sigma \upharpoonright \alpha \) is one to one \([2]\) theorem 3.3 and 4.3. Now suppose \( x \in \mathfrak{M}_a\mathfrak{S}g^X \) and \( \Delta x \subseteq \alpha \). There exists \( y \in \mathfrak{S}g^X \) and a transformation \( \sigma \) of \( \beta \) such that \( \sigma \upharpoonright \alpha \) is one to one and \( x = s^\beta y \). Let \( \tau \) be a transformation of \( \beta \) such that \( \tau \upharpoonright \alpha = Id \) and \((\tau \circ \sigma)\alpha \subseteq \alpha \). Then \( x = s^\beta x = s^\beta s_\sigma y = s^\beta s_\sigma s_\rho y = s^\beta s_\sigma s_\rho s_\alpha y \).

From now on cylindric polyadic algebras are denoted by \( \mathfrak{mCPEA} \).

**Theorem 2.4.** Let \( \beta \) be a cardinal, and \( \mathfrak{A} = \mathfrak{F}_{\beta m}\mathfrak{CPEA}_\alpha \) be the free algebra on \( \beta \) generators. Let \( X_1, X_2 \subseteq \beta \), \( a \in \mathfrak{S}g^X \) and \( c \in \mathfrak{S}g^X \) be such that \( a \leq c \). Then there exists \( b \in \mathfrak{S}g^X \) such that \( a \leq b \leq c \).

**Proof.** Let \( a \in \mathfrak{S}g^X \) and \( c \in \mathfrak{S}g^X \) be such that \( a \leq c \). We want to find an interpolant in \( \mathfrak{S}g^X \). Assume that \( \mu \) is a regular cardinal \( > max(|\alpha|, |A|) \). Let \( \mathfrak{B} \subseteq \mathfrak{mCPEA}_\mu \), such that \( \mathfrak{A} = \mathfrak{N}_a\mathfrak{B} \), and \( A \) generates \( \mathfrak{B} \). Let \( H_{\mu} = \{ \rho \in \mu^\mu : |\rho(\alpha) \cap (\mu \sim \alpha)| < \omega \} \). Let \( S \) be the semigroup generated by \( H_{\mu} \). Let \( \mathfrak{B}' \) be an ordinary dilation of \( \mathfrak{A} \) where all transformations in \( \mu^\mu \) are used. (This can be easily defined like in the case of ordinary polyadic algebras). Then \( \mathfrak{A} = \mathfrak{N}_a\mathfrak{B}' \). We take a suitable reduct of \( \mathfrak{B}' \). Let \( \mathfrak{B} \) be the subalgebra of \( \mathfrak{B}' \) generated from \( A \) by all operations except for substitutions indexed by transformations not in \( S \). Then, of course \( A \subseteq \mathfrak{B} \); in fact, \( \mathfrak{A} = \mathfrak{N}_a\mathfrak{B} \), since for each \( \tau \in \mu^\alpha \), \( \tau \cup Id \subseteq S \). Then one can show inductively that for \( b \in B \), if \( |\Delta b \sim \alpha| < \omega \), and \( \rho \in S \), then \( |\rho(\Delta b) \sim \alpha| < \omega \). Next one defines filters in the dilations \( \mathfrak{S}g^X \) and in \( \mathfrak{S}g^X \) like in Ferenczi \([3]\), but they have to be compatible on the common subalgebras. This needs some work. Assume that no interpolant exists in \( \mathfrak{S}g^X \). Then no interpolant exists in \( \mathfrak{S}g^X \), for if one does, then it can be pulled back, using the first part, by cylindrifiers only on finitely many indices, to an interpolant in \( \mathfrak{S}g^X \), which we assume does not exists. We eventually arrive at a contradiction. Arrange \( adm \times \mu \times \mathfrak{S}g^X \) and \( adm \times \mu \times \mathfrak{S}g^X \) into \( \omega \)-termed sequences:

\[ \langle (\tau_i, k_i, x_i) : i \in \mu \rangle \]

is as desired. Thus we can define by recursion (or step-by-step) \( \mu \)-termed sequences of witnesses:

\[ \langle u_i : i \in \mu \rangle \]

such that for all \( i \in \mu \) we have:

\[ u_i \in \mu \setminus \{\Delta a \cup \Delta c\} \cup \{\Delta x_j \cup \Delta y_j \cup Do\tau_j \cup Rg\tau_j \cup Do\sigma_j \cup Rg\sigma_j\} \cup \{u_j : j < i\} \cup \{v_j : j < i\} \]
and
\[ v_i \in \mu \setminus (\Delta a \cup \Delta c) \cup \cup_{j \leq i} (\Delta x_j \cup \Delta y_j \cup Do\sigma_j \cup Rg\tau_j \cup Do\sigma_j \cup Rg\sigma_j)) \cup \{u_j : j \leq i\} \cup \{v_j : j < i\}. \]

For a cylindric algebra \( D \) we write \( BlD \) to denote its boolean reduct. For \( i,j, i \neq j, s_i^j x = c_i(d_{ij} \cdot x) \) and \( s_i^j x = x \). \( s_i^j \) is a unary operation that abstracts the operation of substituting the variable \( v_i \) for the variable \( v_j \) such that the substitution is free. For a boolean algebra \( C \) and \( Y \subseteq C \), we write \( f.r.Y \) to denote the boolean filter generated by \( Y \) in \( C \). Now let
\[
Y_1 = \{a\} \cup \{-s_{r_i}c_{k_i}\cdot x_i + s_{r_i}s_{i\cdot x_i} : i \in \mu\},
\]
\[
Y_2 = \{-c\} \cup \{-s_{r_i}c_{l_i}\cdot y_i + s_{r_i}s_{l_i\cdot y_i} : i \in \mu\},
\]
\[
H_1 = f.lBlg^B(X_1)Y_1, \quad H_2 = f.lBlg^B(X_2)Y_2,
\]
and
\[
H = f.lBlg^B(X_1 \cap X_2)[(H_1 \cap g^B(X_1 \cap X_2)) \cup (H_2 \cap g^B(X_1 \cap X_2))].
\]

We claim that \( H \) is a proper filter of \( g^B(X_1 \cap X_2) \). To prove this it is sufficient to consider any pair of finite, strictly increasing sequences of natural numbers
\[
\eta(0) < \eta(1) \cdots < \eta(n - 1) < \mu \quad \text{and} \quad \xi(0) < \xi(1) < \cdots < \xi(m - 1) < \mu,
\]
and to prove that the following condition holds:

1. For any \( b_0, b_1 \in g^B(X_1 \cap X_2) \) such that
\[
a \prod_{i < n} (-s_{r_{\eta(i)}}c_{k_{\eta(i)}}\cdot x_{\eta(i)} + s_{r_{\eta(i)}}s_{u_{\eta(i)}\cdot x_{\eta(i)}}) \leq b_0
\]
and
\[
(-c) \prod_{i < m} (-s_{r_{\xi(i)}}c_{l_{\xi(i)}}\cdot y_{\xi(i)} + s_{r_{\xi(i)}}s_{l_{\xi(i)}\cdot y_{\xi(i)}}) \leq b_1
\]
we have
\[ b_0, b_1 \neq 0. \]

We prove this by induction on \( n + m \). If \( n + m = 0 \), then (1) simply expresses the fact that no interpolant of \( a \) and \( c \) exists in \( g^B(X_1 \cap X_2) \). In more detail: if \( n + m = 0 \), then \( a_0 \leq b_0 \) and \(-c \leq b_1 \). So if \( b_0,b_1 = 0 \), we get \( a \leq b_0 \leq -b_1 \leq c \). Now assume that \( n + m > 0 \) and for the time being suppose that \( \eta(n - 1) > \xi(m - 1) \). Apply \( c_{u_{\eta(n-1)}} \) to both sides of the first inclusion of (1). By \( u_{\eta(n-1)} \notin \Delta a \), i.e. \( c_{u_{\eta(n-1)}}a = a \), and by recalling that \( c_i(x,y) = c_i\cdot x\cdot c_i\cdot y \), we get (2)
\[
a \cdot c_{u_{\eta(n-1)}} \prod_{i < n} (-s_{r_{\eta(i)}}c_{k_{\eta(i)}}\cdot x_{\eta(i)} + s_{r_{\eta(i)}}s_{u_{\eta(i)}\cdot x_{\eta(i)}}) \leq c_{u_{\eta(n-1)}}b_0.
\]
Let \( c^\partial_i(x) = -c_i(-x) \). \( c^\partial_i \) is the algebraic counterpart of the universal quantifier \( \forall x_i \). Now apply \( c^\partial_{u\eta(n-1)} \) to the second inclusion of (1). By noting that \( c^\partial_i \), the dual of \( c_i \), distributes over the boolean meet and by \( u\eta(n-1) \notin \Delta c = \Delta(-c) \) we get (3)

\[
(-c) \prod_{j<m} c^\partial_{u\eta(n-1)} (-s_{\sigma_{\xi(i)}} c_{l_{\xi(i)}} \eta_{\xi(i)} + s_{\sigma_{\xi(i)}} l_{\xi(i)} \eta_{\xi(i)}) \leq c^\partial_{u\eta(n-1)} b_1.
\]

Before going on, we formulate (and prove) a claim that will enable us to eliminate the quantifier \( c_{u\eta(n-1)} \) (and its dual) from (2) (and (3)) above.

For the sake of brevity set for each \( i < n \) and each \( j < m \) :

\[
z_i = -c_{k_{\eta(i)}} x_{\eta(i)} + k_{\eta(i)} y_{\xi(i)}
\]

and

\[
t_i = -c_{l_{\xi(i)}} y_{\xi(i)} + l_{\xi(i)} y_{\xi(i)}.
\]

Then (i) and (ii) below hold:

(i) \( c_{u\eta(n-1)} z_i = z_i \) for \( i < n - 1 \) and \( c_{u\eta(n-1)} z_{n-1} = 1 \). (ii) \( c^\partial_{u\eta(n-1)} t_j = t_j \) for all \( j < m \).

Proof of \( c_{u\eta(n-1)} z_i = z_i \) for \( i < n - 1 \).

Let \( i < n - 1 \). Then by the choice of witnesses we have

\( u\eta(n-1) \neq u\eta(i) \).

Also it is easy to see that for all \( i, j \in \alpha \) we have

\[ \Delta c_j x \subseteq \Delta x \] and that \( \Delta s_j x \subseteq \Delta x \setminus \{i\} \cup \{j\} \).

In particular,

\[ u\eta(n-1) \notin \Delta c_{k_{\eta(i)}} x_{\eta(i)} \] and \( u\eta(n-1) \notin \Delta(s_{u\eta(i)}) x_{\eta(i)} \).

It thus follows that

\[ c_{u\eta(n-1)} (-c_{k_{\eta(i)}} x_{\eta(i)}) = -c_{k_{\eta(i)}} x_{\eta(i)} \] and \( c_{u\eta(n-1)} (s_{u\eta(i)}) x_{\eta(i)} \).

Finally, by \( c_{u\eta(n-1)} \) distributing over the boolean join, we get

\[ c_{u\eta(n-1)} z_i = z_i \] for \( i < n - 1 \).

Proof of \( c_{u\eta(n-1)} z_{n-1} = 1 \).
Computing we get, by $u_\eta(n-1) \notin \Delta x_{\xi(n-1)}$ and by [5, 1.5.8(i), 1.5.8(ii)] the following:

$$c_{u_\eta(n-1)} \left( -c_{k_\eta(n-1)} x_\eta(n-1) + s_{u_\eta(n-1)}^k x_\eta(n-1) \right)$$

$$= c_{u_\eta(n-1)} - c_{k_\eta(n-1)} x_\eta(n-1) + c_{u_\eta(n-1)} s_{u_\eta(n-1)}^k x_\eta(n-1)$$

$$= -c_{k_\eta(n-1)} x_\eta(n-1) + c_{u_\eta(n-1)} s_{u_\eta(n-1)}^k x_\eta(n-1)$$

$$= -c_{k_\eta(n-1)} x_\eta(n-1) + c_{u_\eta(n-1)} s_{k_\eta(n-1)} c_{u_\eta(n-1)} x_\eta(n-1)$$

$$= -c_{k_\eta(n-1)} x_\eta(n-1) + c_{u_\eta(n-1)} c_{u_\eta(n-1)} x_\eta(n-1)$$

$$= -c_{k_\eta(n-1)} x_\eta(n-1) + c_{u_\eta(n-1)} x_\eta(n-1) = 1.$$

With this the proof of (i) in our claim is complete. Now we prove (ii). Let $j < m$. Then we have

$$c_{u_\eta(n-1)}^\partial \left( -c_{l_\xi(j)} y_\xi(j) \right) = -c_{l_\xi(j)} y_\xi(j)$$

and

$$c_{u_\eta(n-1)}^\partial \left( s_{l_\xi(j)} y_\xi(j) \right) = s_{l_\xi(j)} y_\xi(j).$$

Indeed, computing we get

$$c_{u_\eta(n-1)}^\partial \left( -c_{l_\xi(j)} y_\xi(j) \right) = -c_{u_\eta(n-1)} - c_{l_\xi(j)} y_\xi(j) = -c_{u_\eta(n-1)} c_{l_\xi(j)} y_\xi(j) = -c_{l_\xi(j)} y_\xi(j).$$

Similarly, we have

$$c_{u_\eta(n-1)}^\partial \left( s_{l_\xi(j)} y_\xi(j) \right) = -c_{u_\eta(n-1)} \left( s_{l_\xi(j)} y_\xi(j) \right)$$

$$= -c_{u_\eta(n-1)} \left( s_{l_\xi(j)} y_\xi(j) \right) = -s_{l_\xi(j)} y_\xi(j) = s_{l_\xi(j)} y_\xi(j).$$

By $c_i^\partial (c_i^\partial x + y) = c_i^\partial x + c_i^\partial y$ we get from the above that

$$c_{u_\eta(n-1)}^\partial \left( t_j \right) = c_{u_\eta(n-1)}^\partial \left( c_{l_\xi(j)} y_\xi(j) + s_{l_\xi(j)} y_\xi(j) \right)$$

$$= c_{l_\xi(j)}^\partial c_{l_\xi(j)} y_\xi(j) + c_{u_\eta(n-1)}^\partial s_{l_\xi(j)} y_\xi(j)$$

$$= c_{l_\xi(j)} y_\xi(j) + s_{l_\xi(j)} y_\xi(j) = t_j.$$

By the above proven claim we have

$$c_{u_\eta(n-1)} \prod_{i<n} z_i = c_{u_\eta(n-1)} \left[ \prod_{i<n-1} z_i \cdot z_n \right].$$
\[ c_i(c_i x.y) = c_i x . c_i y. \] Combined with (2) we obtain
\[ a. \prod_{i<n} \left( -c_{k(i)} x_{\eta(i)} + s_{k(i)} x_{\eta(i)} \right) \leq c_{u_{\eta(n-1)}} b_0. \]

On the other hand, from our claim and (3), it follows that
\[ (-c). \prod_{j<m} \left( -c_{l(j)} y_{\xi(j)} + s_{l(j)} y_{\xi(j)} \right) \leq c_{u_{\eta(n-1)}} b_1. \]

Now making use of the induction hypothesis we get
\[ c_{u_{\eta(n-1)}} b_0, c_{u_{\eta(n-1)}} b_1 \neq 0; \]
and hence that
\[ b_0, c_{u_{\eta(n-1)}} b_1 \neq 0. \]

From
\[ b_0. c_{u_{\eta(n-1)}} b_1 \leq b_0. b_1 \]
we reach the desired conclusion, i.e. that
\[ b_0. b_1 \neq 0. \]

The other case, when \( \eta(n-1) \leq \xi(m-1) \) can be treated analogously and is therefore left to the reader. We have proved that \( H \) is a proper filter.

Proving that \( H \) is a proper filter of \( Sg^B(X_1 \cap X_2) \), let \( H^* \) be a (proper boolean) ultrafilter of \( Sg^B(X_1 \cap X_2) \) containing \( H \). We obtain ultrafilters \( F_1 \) and \( F_2 \) of \( Sg^B(X_1) \) and \( Sg^B(X_2) \), respectively, such that
\[ H^* \subseteq F_1, \quad H^* \subseteq F_2, \]
and (**)
\[ F_1 \cap Sg^B(X_1 \cap X_2) = H^* = F_2 \cap Sg^B(X_1 \cap X_2). \]

Now for all \( x \in Sg^B(X_1 \cap X_2) \) we have
\[ x \in F_1 \text{ if and only if } x \in F_2. \]

Also from how we defined our ultrafilters, \( F_i \) for \( i \in \{1, 2\} \) satisfy the following condition:

(*) For all \( k < \mu \), for all \( x \in Sg^B X_i \) if \( c_k x \in F_i \) then \( s^l x \) is in \( F_i \) for some \( l \notin \Delta x \). We obtain ultrafilters \( F_1 \) and \( F_2 \) of \( Sg^B X_1 \) and \( Sg^B X_2 \), respectively, such that
\[ H^* \subseteq F_1, \quad H^* \subseteq F_2. \]
and (**) 
\[ F_1 \cap \mathcal{S}g^B(\cap X_1 X_2) = H^* = F_2 \cap \mathcal{S}g^B(\cap X_1 X_2). \]

Now for all \( x \in \mathcal{S}g^B(\cap X_1 X_2) \) we have 
\[ x \in F_1 \text{ if and only if } x \in F_2. \]

Also from how we defined our ultrafilters, \( F_i \) for \( i \in \{1, 2\} \) are perfect.

Then define the homomorphisms, one on each subalgebra, like in [9] p. 128-129, using the perfect ultrafilters, then freeness will enable pase these homomorphisms, to a single one defined to the set of free generators, which we can assume to be, without any loss, to be \( X_1 \cap X_2 \) and it will satisfy \( h(a. c) \neq 0 \) which is a contradiction.

\[ \Box \]

### 3 Second proof

A technique used in constructing cylindric algebras with certain desirable properties is to construct atom structure with certain first order corresponding properties (like Monk’s algebras, Maddux’s algebras, and the Hirsch-Hodkinson numerous constructions. Sayed Ahmed, too, has implemented such constructions in a few publications of his).

Algebraic logicians and Modal logicians, frequently talk about the same thing with different names which has caused a communication problem in the past.

This was partially surpassed by the pioneering work of Venema on cylindric modal logic, later enhanced by the Van Bentham-Andréka-Németi-Goldblatt (and their students, to name a few, Marx, Mikulas, and Kurusz) collaboration.

We know how to build atom structures, or indeed frames, from atomic algebras, and conversely subalgebras of complex algebras from frames. We are happy when we are able to preserve crucial logical properties.

But something seems to be missing. Modal logicians rarely study frames in isolation, rather they are interested in constructing new frames from old ones using bounded morphisms, generated subframes, disjoint unions, zig-zag products (the latter is a less familiar notion).

An algebraic logician adopts an analogous perspective but on different (algebraic) level, via such constructions as homomorphisms, subalgebras and direct products. So it appears that modal logicians work in a universe that is distant from that of the algebraic logicians. In this connection, it is absolutely natural to ask whether these universes are perhaps systematically related. And the answer is: they are, very much so, and duality theory is devoted to studying such links. For example a representation theorem for algebras is the dual of representing abstract state frames by what Venema calls assignment frames.
Here we give an application of this duality, worked out by Marx, to show that cylindric-polyadic algebras have the $SUPAP$. This proof also works for all varieties of relativized cylindric polyadic algebras studied by Ferenzci and reported in [3].

The result follows from the simple observation that such varieties can be axiomatized with positive, hence Sahlqvist equations, and therefore they are canonical; and also we do not have a Rosser condition on cylindifiers; cylindifiers do not commute, this allows that the first order correspondants of such equations are *clausifiable*, see [7] for the definition of this.

This proof is inspired by the modal perspective of cylindric-like algebras, applied to cylindric-polyadic algebras that suggests a whole landscape below standard predicate logic, with a minimal modal logic at the base ascending to standard semantics via frame constraints. In particular, this landscape contains nice sublogics of the full predicate logic, sharing its desirable meta properties and at the same time avoiding its negative accidents due to its Tarskian 'square frames' modelling. In such nice numerous sublogics quantifiers do not commute, and the merry go round identities hold. Such mutant logics are currently a very rich area of research.

The technique used here can be traced back to Németi, when he proved that relativized cylindric set algebras have $SUPAP$; using (classical) duality between atom structures and cylindric algebras. Marx 'modalized' the proof, and slightly strengthenened Németi results, using instead the well-established duality between modal frames and complex algebras.

While Németi talks about subalgebras of finite direct products of frames, Marx talks about finite zig-zag products of frames, and this is a non-trivial very useful generalization for proving strong amalgamation for a lot of relativized set algebras whose units satisfy certain closure properties.

On the other hand, such strong amalgamation results cannot be obtained from Németi’s technique which seems to work only for very relativized set algebras, namely, the class $\mathbf{Cr}_\alpha$, for any $\alpha$. This class is referred to as the class of cylindric relativized set algebras in [6]. As a matter of fact, this latter class is just the first step along a radical path, obtained by deconstructing, so to speak, the semantics of first order logic, designing lighter modal versions of this system by locating implicit choice points in this step up. This gives a whole landscape of decidable version with different computational constraints over universes of states (assignments) related by variable updates. An off shoot of such a arelativization technique is also regaining definability (interpolation) and for tha matter an easier match between syntax and semantics.

We consider the non-commute cylindric polyadic algebras introduced by Ferenzci; we consider, this time, the case $\mathbf{CPEA}$. There is no deep motivation for such a choice, except that varying the studies algebras, suggests tht our proofs work for all.a A frame is a first order structure $\mathfrak{F} = (F, T_i, S_\tau)_{i \in \alpha, \tau \in \alpha}$
where \( T \) is an arbitrary set and and both \( T_i \) and \( S_\tau \) are binary relations on \( T \) for all \( i \in \alpha \); and \( \tau \in ^{\alpha} \alpha \).

Given a frame \( \mathfrak{F} \), its complex algebra will be denoted by \( \mathfrak{F}^+ \); \( \mathfrak{F}^+ \) is the algebra \((\wp(\mathfrak{F}), c_i, s_\tau)_{i\in \alpha, \tau \in ^\alpha \alpha} \) where for \( X \subseteq V \), \( c_i(X) = \{ s \in V : \exists t \in X, (t, s) \in T_i \} \), and similarly for \( s_\tau \).

For \( K \subseteq \text{CPEA}_\alpha \), we let \( \text{Str}K = \{ \mathfrak{F} : \mathfrak{F}^+ \in K \} \).

For a variety \( V \), it is always the case that \( \text{Str}V \subseteq \text{At}V \) and equality holds if the variety is atom-canonical. If \( V \) is canonical, then \( \text{Str}V \) generates \( V \) in the strong sense, that is \( V = \mathbb{C}m\text{Str}V \). For Sahlqvist varieties, that are completely additive, as is our case, \( \text{Str}V \) is elementary.

**Definition 3.1.**

Given a family \((\mathfrak{F}_i)_{i \in I}\) of frames, a zigzag product of these frames is a substructure of \( \prod_{i \in I} \mathfrak{F}_i \) such that the projection maps restricted to \( S \) are onto.

**Definition 3.2.**

Let \( \mathfrak{F}, \mathfrak{G}, \mathfrak{H} \) be frames, and \( f : \mathfrak{G} \to \mathfrak{F} \) and \( h : \mathfrak{F} \to \mathfrak{H} \). Then
\[
\text{INSEP} = \{(x, y) \in \mathfrak{G} \times \mathfrak{H} : f(x) = h(y)\}.
\]

**Lemma 3.3.** The frame \( \text{INSEP} \upharpoonright G \times H \) is a zigzag product of \( G \) and \( H \), such that \( \pi \circ \pi_0 = h \circ \pi_1 \), where \( \pi_0 \) and \( \pi_1 \) are the projection maps.

**Proof.** [7] 5.2.4

For an algebra \( \mathfrak{A}, \mathfrak{A}^+ \) denotes its ultrafilter atom structure. For \( h : \mathfrak{A} \to \mathfrak{B}, \) \( h^+ \) denotes the function from \( \mathfrak{B}^+ \to \mathfrak{A}^+ \) defined by \( h^+(u) = h^{-1}[u] \) where the latter is \( \{ x \in a : h(x) \in u \} \).

**Theorem 3.4.** ([7] lemma 5.2.6) Assume that \( K \) is a canonical variety and \( \text{Str}K \) is closed under finite zigzag products. Then \( K \) has the superamalgamation property.

**Sketch of proof.** Let \( \mathfrak{A}, \mathfrak{B}, \mathfrak{C} \in K \) and \( f : \mathfrak{A} \to \mathfrak{B} \) and \( h : \mathfrak{A} \to \mathfrak{C} \) be given monomorphisms. Then \( f^+ : \mathfrak{B}^+ \to \mathfrak{A}^+ \) and \( h^+ : \mathfrak{C}^+ \to \mathfrak{A}^+ \). We have \( \text{INSEP} = \{(x, y) : f^+(x) = h^+(y)\} \) is a zigzag connection. Let \( \mathfrak{F} \) be the zigzag product of \( \text{INSEP} \upharpoonright \mathfrak{A}^+ \times \mathfrak{B}^+ \). Then \( \mathfrak{F}^+ \) is a superamalgam.

**Theorem 3.5.** The variety \( \text{CPEA}_\alpha \) has \( \text{SUPAP} \).

**Proof.** \( \text{CPEA}_\alpha \) can be easily defined by positive equations then it is canonical. The first order correspondents of the positive equations translated to the class of frames will be Horn formulas, hence clausifiable [7] theorem 5.3.5, and so \( \text{Str}K \) is closed under finite zigzag products. Marx’s theorem finishes the proof. □
If one views relativized models as the natural semantics for predicate logic rather than some tinkering devise which is the approach adopted in [5], then many well-established taboos of the field must be challenged.

In standard textbooks one learns that predicate logical validity is one unique notion specified once and for all by the usual Tarskian (square) semantics and canonized by Gödel’s completeness theorem. Moreover, it is essentially complex, being undecidable by Church’s theorem.

On the present view, however standard predicate logic has arisen historically by making several ad-hoc semantic decisions that could have gone differently. Its not all about ‘one completeness theorem’ but rather about several completeness theorems obtained by varying both the semantic and syntactical parameters. This can be implemented from a classical relativized representability theory, like that adopted in the monograph [5], though such algebras were treated in op.cit offf main stream, and they were only brought back to the front of the scence by the work of Resek, Thompson, Andr´eka and last but not least Ferenczi, or from a modal perspective, that has enriched the subject considerably.

But on the other hand, careful scrutiny of the situation reveals that things are not so clear cut, and the borderlines are hazy. Within the polyadic cylindric dichotomty there is the square relativisation dichotomy, and also vice versa.

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