Scalar perturbations in a class of extended symmetric teleparallel gravity models

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We have investigated the accelerating behaviour of the universe in $f(Q, T)$ gravity at the backdrop of an isotropic and homogeneous space-time. We have initially derive the dynamical parameters in the general form of $f(Q, T) = aQ^m + bT$ [Xu et al., Eur. Phys. J. C, 79, 708 (2019)] and then split it into two cases (i) one with $m = 1$ and the (ii) other with $b = 0$. In the first case, it reduces to the linear form of the functional $f(Q, T)$ and second case leads to the higher power of the nonmetricity $Q$. In an assumed form of the hyperbolic scale factor, the models are constructed and its evolutionary behaviours are studied. The geometrical parameters as well the equation of state parameter are obtained and found to be in the preferred range of the cosmological observations. Marginal variation has been noticed in the behaviour of $\omega$ and $\omega_{\text{eff}}$ at present time. The violation of strong energy conditions in both the cases are shown. The scalar perturbation of the models has been derived and stability has been shown.

Keywords: Symmetric teleparallel gravity, Dynamical parameters, Energy conditions, Scalar perturbation.

I. INTRODUCTION

Recently another attempt has been taken to address the late time cosmic acceleration issue by proposing a new gravitational theory, the $f(Q, T)$ theory of gravity. Xu et al. [1] have proposed this modified theory of gravity by extending the symmetric teleparallel gravity. In $f(Q, T)$ gravity, $Q$ be the non-metricity and $T$ be the trace of the energy momentum tensor. We shall give here a brief discussion on the development of this modified theory. Weitzenböck has mathematically developed the Weitzenböck space [2]. The Weitzenböck manifold characterized by the metric tensor, curvature tensor and the torsion tensor of the manifold, and on different regions of Weitzenböck manifold, the torsion tensor has different values. As the Riemannian curvature tensor of the Weitzenböck space vanishes, the geometry follows the property of distant parallelism, known as teleparallelism. In this approach the set of tetrad vectors $e_i^\mu$ replace the metric tensor of the space-time $g^\mu\nu$. The gravitational field now uses the torsion instead of the curvature and the tetrad generates the torsion. Because of this replacement of curvature by torsion in the gravitational field, this is known as the teleparallel equivalent of general relativity(TEGR) or the $f(T)$ gravity [3–5]. The advantage of this gravity is that its field equation deals with second order, which is easy to solve as compared to other gravitational theories. Then Haghani et al. [6] have formulated an extension of teleparallel gravity in a four dimensional curved space-time, known as the Weyl-Cartan-Weitzenböck gravity. In this gravity theory, the Weitzenböck condition of the vanishing of sum of the curvature and torsion scalar has been imposed. It has been believed that this arrangement may lead to the possibility of geometrical description of dark energy, which may address late time cosmic acceleration phenomena. It can be inferred that GR can be represented geometrically at least with curvature representation and teleparallel representation. The torsion and curvature vanishes respectively in the curvature and teleparallel representation whereas the non-metricity vanishes in both the approaches. Hence another approach is on the description of non-vanishing of the basic geometrical variable, the non-metricity. This approach is known as the symmetric teleparallel gravity proposed by Nester and Yo [7]. Most recently it has been developed as the $f(Q)$ gravity by Jimenez et al. [8]. Latorre et al. [9] have investigated that the non-metricity produces observable effects in the quantum fields. Conroy and Koivisto [10] have given a note on the spectrum of symmetric teleparallel gravity. Soudi et al. [11] have studied the gravitational waves in $f(Q)$ gravity and obtained the same speed and polarization as in GR. Lu et al. [12] in $f(Q)$ gravity, have indicated that the role of dark energy

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The field equation of \( f(Q) \) extended symmetric teleparallel gravity can be obtained as [1], conditions are given in Sec. III. In Sec. IV, we have presented two cases by considering (i) \( m \) presented. Finally the results and conclusions are given in Sec. VI.

Xu et al. [1] extended \( f(Q) \) gravity by introducing the non-minimal coupling between the non-metricity and the trace of energy momentum tensor \( T \). The Lagrangian density of the gravitational field would be described with respect to \( Q \) and \( T \) in the form \( L = f(Q, T) \). The motivation behind this action is to study the cosmological implications such as, to describe the decelerating and accelerating evolutionary phase of the universe. Some \( f(Q, T) \) gravity cosmological models are available in the literature. Xu et al. [17] have given the Weyl type \( f(Q, T) \) gravity and its cosmological implications. Zia et al. [18] have presented transit cosmological model aligning with the observational value of the deceleration parameter in \( f(Q, T) \) gravity. Pati et al. [19] have obtained quintessence model in the context of hybrid scale factor. Agrawal et al. [20] have shown the non-singular matter bouncing scenario with the violation of null and strong energy conditions. Najera and Fajardo [21] have tested five \( f(Q) \) cosmological implications. Zia et al. [18] have presented transit cosmological model aligning with the observational

The paper is organised as follows: in Sec. II, the \( f(Q, T) \) gravity has been discussed and the field equations are derived in FLRW space-time. For the general case of \( f(Q, T) = aQ + \beta T \), the dynamical parameters and energy conditions are given in Sec. III. In Sec. IV, we have presented two cases by considering (i) \( m = 1 \), which reduces to linear case of \( f(Q, T) \) and (ii) \( \beta = 0 \) that reduces to the \( f(Q) \) gravity. The cosmological models for both the cases are constructed with the hyperbolic Hubble parameter. In Sec. V the scalar perturbation of the models has been presented. Finally the results and conclusions are given in Sec. VI.

II. OVERVIEW OF THE \( f(Q, T) \) GRAVITY AND DERIVATION OF THE DYNAMICAL PARAMETERS

Every gravitational theory begins with the action and so here the action for the \( f(Q, T) \) gravity, which is the extended symmetric teleparallel gravity can be obtained as [1],

\[
S = \int \left[ \frac{1}{16\pi} f(Q, T) d^4x \sqrt{-g} + L_m d^4x \sqrt{-g} \right]
\]

where \( Q \) and \( T \) in the functional \( f(Q, T) \) respectively represents the non-metricity and the trace of the energy momentum tensor \( T_{\mu\nu} \). The non-metricity can be expressed as \( Q = -g^{\mu\nu} (L^k_{\mu\nu} T^l_k - L^k_{lk} T^l_{\mu\nu}) \), where \( L^l_{\mu\nu} Q^k_l = -\frac{1}{2} g^{kl} (\nabla_\gamma g_{l\lambda} + \nabla_l g_{\lambda\gamma} - \nabla_\lambda g_{l\gamma}) \). \( L_m \) denotes the matter Lagrangian and \( g = \det(g_{\mu\nu}) \) be the determinant of the metric tensor \( g_{\mu\nu} \).

The field equation of \( f(Q, T) \) gravity can be obtained by varying the action as [1],

\[
-2 \sqrt{-g} \nabla_k (F \sqrt{-g} p^k_{\mu\nu}) - \frac{1}{2} f g_{\mu\nu} - F(p_{\mu\kappa} Q_{\nu\kappa} - 2 Q_{\mu\nu} p_{\kappa\nu}) = 8\pi T_{\mu\nu} (1 - \kappa) - 8\pi\kappa \Theta_{\mu\nu}.
\]

For brevity, we represent \( f \equiv f(Q, T) \) and \( F = \frac{\delta f}{\delta Q} \) \( 8\pi\kappa = \frac{\delta f}{\delta T} \). Further the super potential of the model, the energy momentum tensor and the trace of the non-metricity can be obtained respectively as,

\[
p^k_{\mu\nu} = -\frac{1}{2} L^k_{\mu\nu} + \frac{1}{4} (Q^k - Q^k) g_{\mu\nu} - \frac{1}{4} \delta^k_\nu Q_{\mu},
\]

\[
T_{\mu\nu} = -2 \sqrt{-g} \frac{\delta (\sqrt{-g} L_m)}{\delta g^{\mu\nu}}; \quad \Theta_{\mu\nu} = g^{kl} \delta T_{kl}^{\mu\nu},
\]

\[
Q_k = Q_k^\mu \mu, \quad \dot{Q}_k = Q_k^\mu k_\mu.
\]
We wish to study the cosmological model of the Universe in $f(Q,T)$ theory of gravity at the backdrop of an isotropic and homogeneous FLRW space-time considered in the form,

$$ds^2 = -N^2(t)dt^2 + \mathcal{R}^2(t)(dx^2 + dy^2 + dz^2),$$

(4)

where the lapse function $N(t)$ and scale factor $\mathcal{R}(t)$ are the function of cosmic time. Also, the dilation rate can be defined as, $\tilde{T} = \frac{N(t)}{N'(t)}$. In an FLRW space-time, the lapse function can be $N(t) = 1$ and subsequently the dilation rate, $\tilde{T} = 0$ and the non-metricity reduces to $Q = 6H^2$, where $H = \frac{\mathcal{R}'}{\mathcal{R}}$ is the Hubble parameter. We consider the energy momentum tensor that of a perfect fluid distribution $T^μ_ν = \text{diag}(-ρ, ρ, ρ, ρ)$. Also, $\Theta^μ_ν = \text{diag}(2ρ + p, -p, -p, -p)$. So, the field equations of $f(Q,T)$ gravity (2) in FLRW space-time can be obtained as,

$$-16πp = f - 12\frac{\chi^2}{F} - 4\dot{\chi},$$

(5)

$$16πρ = f - 12\frac{\chi^2}{F} - 4\dot{\chi}\chi_1,$$

(6)

where $\chi = FH$ and $κ_1 = \frac{κ}{1 + κ}$. Also, we have $\dot{\chi} = FH + FH$. From Eqs. (5) and (6), the evolution equation for $\chi$ can be obtained as,

$$\ddot{\chi} = 4π (ρ + p) (1 + κ).$$

(7)

For a constant value of $F$, the above evolution equation reduces to the evolution equation for the Hubble parameter as,

$$\dot{H} = \frac{4π}{F} (1 + κ) (ρ + p).$$

(8)

Assuming a barotropic relationship $p = ωρ$, the above relation reduces to,

$$\rho = \frac{F}{4π (1 + κ) (1 + ω)} θ.$$

(9)

The equation of state (EoS) parameter $ω = \frac{p}{ρ}$ for the $f(Q,T)$ gravity theory may be obtained from Eqs. (5) and (6) as,

$$ω = -1 + \frac{4\dot{χ}}{(1 + κ)(f - 12FH^2) - 4χκ}.$$ 

(10)

The EoS parameter will decide the possibility of the accelerating Universe at least at the late times of the cosmic evolution.

The equivalent Friedman equations for the present gravity theory may be written as,

$$2\dot{H} + 3H^2 = \frac{1}{F} \left[ \frac{f}{4} - 2FH + 4π [(1 + κ)ρ + (2 + κ)p] \right] = -8πρ_{eff},$$

(11)

$$3H^2 = \frac{1}{F} \left[ \frac{f}{4} - 4π [(1 + κ)ρ + κp] \right] = 8πρ_{eff},$$

(12)

Obviously, the effective energy density $ρ_{eff}$ and the effective pressure $p_{eff}$ satisfy the conservation equation

$$\dot{ρ}_{eff} + 3H (ρ_{eff} + p_{eff}) = 0.$$ 

(13)

Consequently, we may define an effective EoS parameter as

$$ω_{eff} = -1 - \frac{8FH}{f - 16π [(1 + κ)ρ + κp]}.$$ 

(14)

In order to investigate viable cosmological scenario in the framework of the above discussed $f(Q,T)$ gravity theory, it is required to consider certain assumed form of the functional $f(Q,T)$. In the seminal work, Xu et al. [1] have considered three different forms for $f(Q,T)$ such as (i) $f(Q,T) = aQ + bT$ (ii) $f(Q,T) = aQ^m + bT$ and (iii) $f(Q,T) = -\left(aQ + bT^2\right)$. Here $a, b$ and $m$ are constants.
III. DYNAMICAL PARAMETERS AND ENERGY CONDITIONS

In the present work, we will consider the functional \( f(Q, T) = aQ^m + bT \) to model the Universe. Also, we wish to obtain the corresponding models for the choice \( m = 1 \) and \( b = 0 \). It should be remarked here that, for \( b = 0 \), the model reduces to \( f(Q) \) gravity. For this choice of the functional, we have \( F = amQ^{m-1}, \dot{F} = 2(m-1)F \frac{H}{H}, \chi = amQ^{(m-1)}H, \dot{\chi} = FH(2m - 1) \) and \( b = 8\pi\kappa \). We may now have the dynamical parameters as,

\[
p = \frac{(2m - 1) aQ^m + 2\dot{\chi}[2 + \kappa - 3\kappa_1]}{4\pi[(2 + \kappa)(2 + 3\kappa) - 3\kappa^2]},
\]

\[
\rho = \frac{(1 - 2m)aQ^m + 2\dot{\chi}[3\kappa - (2 + 3\kappa)\kappa_1]}{4\pi[(2 + \kappa)(2 + 3\kappa) - 3\kappa^2]},
\]

\[
\omega = -1 + \frac{4\dot{\chi}[(1 + 2\kappa)(1 - \kappa_1)]}{(1 - 2m)aQ^m + 2\dot{\chi}[3\kappa - (2 + 3\kappa)\kappa_1]}.
\]

We may express the above equations, Eqns. (15)-(17) in the term of the Hubble parameter and the deceleration parameter \( q = -1 - \frac{\ddot{H}}{H^2} \) as

\[
p = -\frac{2m3^{(m-1)}(1 - 2m)aH^{2m}[3 - m(1 + q)(2 + \kappa - 3\kappa_1)]}{4\pi[(2 + \kappa)(2 + 3\kappa) - 3\kappa^2]},
\]

\[
\rho = \frac{2m3^{(m-1)}(1 - 2m)aH^{2m}[3 + m(1 + q)(3\kappa - (2 + 3\kappa)\kappa_1)]}{4\pi[(2 + \kappa)(2 + 3\kappa) - 3\kappa^2]},
\]

\[
\omega = -1 + \frac{2m(1 + q)[(1 + 2\kappa)(1 - \kappa_1)]}{3 + m(1 + q)(3\kappa - (2 + 3\kappa)\kappa_1)}.
\]

The effective EoS parameter (14) becomes

\[
\omega_{\text{eff}} = -1 + \frac{2^{(m+2)}3^{m-1}amH^{2m}(1 + q)}{6^n aH^{2m} - 16\pi\kappa[(1 + \kappa) + k\omega]}.
\]

Since the study of energy energy conditions is an important aspect of the gravitational theory, we have given below the energy conditions of the proposed problem as,

\[
\rho + p = \frac{2m3^{m-1}(1 - 2m)aH^{2m}}{8\pi}[m(1 + q)(1 - \kappa_1)], \quad \text{[NEC/WEC]}
\]

\[
\rho + 3p = \frac{2m3^{m-1}(1 - 2m)aH^{2m}}{8\pi(1 + 2\kappa)}[-3 + m(1 + q)(3 + 3\kappa - \kappa_1 - 3\kappa\kappa_1)], \quad \text{[SEC]}
\]

\[
\rho - p = \frac{2m3^{m-1}(1 - 2m)aH^{2m}}{8\pi(1 + 2\kappa)}[3 + m(1 + q)(-1 + \kappa - \kappa_1 - \kappa\kappa_1)], \quad \text{[DEC]},
\]

where NEC, WEC, SEC and DEC respectively represents null, weak, strong and dominant energy condition. All the dynamical parameters and the energy conditions are expressed in terms of Hubble and deceleration parameter. So, we consider in the subsequent section, a specific form of the Hubble parameter.

IV. MODELS WITH HYPERBOLIC SCALE FACTOR

To analyse the evolutionary behaviour of the dynamical parameters, the Hubble parameter involved in eqns. (18)-(20) required to be expressed in terms of cosmic time or redshift, \( 1 + z = \frac{1}{a(t)} \). In the literature several scale factors such as, de Sitter expansion [24], power law expansion [25], hybrid scale factor [26], bouncing scale factor [20] and many more are introduced from time to time to address the astrophysical and cosmological issues of the Universe.
In a similar approach, here we shall consider the Hubble parameter in such a manner that its corresponding scale factor would have a quadratic term in its exponent. The hyperbolic scale factor can be expressed in the form,

\[ R(t) \propto \left[ \sinh \left( \frac{3}{2} \sqrt{\frac{\Lambda}{3}} t \right) \right]^{\frac{2}{3}}, \tag{25} \]

where \( \Lambda \) is constant. The Hubble parameter and deceleration parameter of the scale factor can be respectively obtained as,

\[ H = \frac{\dot{R}}{R} = \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3\Lambda}}{2} t \right)}{\sqrt{3}} \quad \text{and} \quad q = -\frac{\ddot{R}}{\dot{R}} = \frac{\frac{3}{\cosh \left( \sqrt{3\Lambda} t \right)} - 1}{\frac{3}{\cosh \left( \sqrt{3\Lambda} t \right)} + 1} \]

Also, the slope is obtained as,

\[ \frac{\Lambda}{1 - \cosh \left( \sqrt{3\Lambda} t \right)} \]

There is always a solution for \( \Lambda < 0 \), that is the cosmos will ultimately collapse. Depending on the relative sizes of the terms, it is possible to find a solution for \( \Lambda > 0 \), but the universe typically expands indefinitely until its density is high enough to collapse it before the cosmological constant term takes over. We have given below the graphical behaviour of the Hubble and deceleration parameter in FIG. 1. The Hubble parameter decreases over time and at late time vanishes. The present value of \( H \) has been noted as 71.63. The deceleration parameter also decrease over time and found to be \(-0.70\) at present time and at late times it approaches to \(-1\).

![Graph of Hubble parameter and deceleration parameter](image)

FIG. 1. Hubble parameter (left panel) and deceleration parameter (right panel) in redshift. The parameter scheme, \( \Lambda = e^{3\pi} \).

A. Case-I: \( m = 1 \)

It is to be noted that, substituting \( m = 1 \) in the above set of equations, one can retrieve the equation of the dynamical parameter for the case \( f(Q, T) = aQ + bT \). In this case, we have a constant value for the partial derivative of \( f \) with respect to the nonmetricity as \( F = a \). Consequently, the evolution equation becomes

\[ \dot{H} = \frac{4\pi \lambda (1 + \omega)}{a} \rho, \tag{26} \]

where \( \lambda = 1 + \frac{b}{4\pi} \). On integration of Eq.(26), we get

\[ R(t) = R_0 \exp \left[ \frac{4\pi \lambda}{a} \int \bar{\xi}(t) dt \right], \tag{27} \]

where \( \bar{\xi}(t) = \int (1 + \omega) \rho \, dt \).

For this case with \( m = 1 \), the dynamical parameters of the model may be expressed as,
\[
p = \frac{a}{8\pi(1+2\kappa)} \left[ \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda t}{2} \right)}{\sqrt{3}} \right]^2 \left( 3 - \frac{3}{1 + \cosh \left( \sqrt{3} \Lambda t \right)} (2 + \kappa - \kappa \kappa_1) \right)
\]
\[
\rho = \frac{-a}{8\pi(1+2\kappa)} \left[ \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda t}{2} \right)}{\sqrt{3}} \right]^2 \left( 3 + \frac{3}{1 + \cosh \left( \sqrt{3} \Lambda t \right)} (3\kappa - (2 + 3\kappa) \kappa_1) \right)
\]
\[
\omega = -1 + \frac{2 \left( \frac{3}{1 + \cosh \left( \sqrt{3} \Lambda t \right)} \right) (1 + 2\kappa)(1 - \kappa_1)}{3 + \left( \frac{3}{1 + \cosh \left( \sqrt{3} \Lambda t \right)} \right) (3\kappa - (2 + 3\kappa) \kappa_1)}
\]

The graphical behaviour of energy density and EoS parameter has been presented in FIG. 2. We have considered here three representative values of the model parameter \( b = 0.01, 0.51, 1.01 \) to assess its impact on the evolution of the curve of energy density and EoS parameter. The other model parameter \( a \) kept considered to be a fixed value. The energy density reduces from high positive value to lower one and remain entirely in the positive domain. Lower is the value of the model parameter \( b \), the evolution starts from higher \( \rho \) value and maintain the same behaviour throughout the evolution. Whereas the EoS parameter reduces from higher to lower value entirely in the negative domain. Though a slight variation noticed for the different value of \( b \) at early epoch, however at late time all merged together and remains at \(-1\). This behavior is in agreement with the concordant \( \Lambda \)CDM model. For the representative values of the parameter \( b \), at present time \((z = 0)\), \( \omega \in [-0.803, -0.791] \).

FIG. 2. Energy density (left panel) and EoS parameter (right panel) in redshift. The parameter scheme, \( \Lambda = e^{3\pi}, a = -4.4 \).

The effective EoS parameter for \( m = 1 \) becomes

\[
\omega_{\text{eff}} = -1 + \frac{4(1 + q)}{3 - 16\pi \rho' \left[ (1 + k) + k\omega \right]} = -1 + \frac{4 \left( \frac{3}{\cosh \left( \sqrt{3} \Lambda t \right) + 1} \right)}{3 - 16\pi \rho' \left[ (1 + k) + k\omega \right]}
\]

(28)

where \( \rho' = \frac{\rho'}{\pi H^2} \). Also, the energy conditions can be obtained as,
\[
\rho + p = \frac{-a}{4\pi} \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right)}{\sqrt{3}} \right)^2 \left[ \frac{3}{1 + \cosh(\sqrt{3}\Lambda t)} (1 - \kappa) \right]
\]

\[
\rho + 3p = \frac{-a}{8\pi(1 + 2\kappa)} \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right)}{\sqrt{3}} \right)^2 \left[ -3 + \left( \frac{3}{1 + \cosh(\sqrt{3}\Lambda t)} \right) (3 + 3\kappa - \kappa_1 - 3\kappa\kappa_1) \right]
\]

\[
\rho - p = \frac{-a}{8\pi(1 + 2\kappa)} \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right)}{\sqrt{3}} \right)^2 \left[ 3 - 2 \left( \frac{3}{1 + \cosh(\sqrt{3}\Lambda t)} \right) (-1 + \kappa - \kappa_1 - \kappa\kappa_1) \right]
\]

FIG. 3. Effective EoS parameter (left panel) and Energy conditions (right panel) in redshift. The parameter scheme, \( \Lambda = e^{3\pi} \), \( a = -4.4 \).

FIG. 3 represents the effective EoS parameter (left panel) and energy conditions (right panel). The behaviour of effective EoS parameter remains similar to that of EoS parameter except the present value. In this case, the present value is -0.799. All the energy conditions are reducing from higher to lower value. The violation of SEC and positive behaviour of DEC has been notice, whereas the NEC remains positive and at late times merged with the null line. It confirms the validation of the model.

**B. Case-II: \( b = 0 \)**

With a substitution of \( b = 0 \) into the functional \( f(Q, T) \) and that in the action, we obtain the \( f(Q) \) gravity with the functional behaving as \( f(Q) = aQ^m \). For this specific case, the dynamical parameters of the model reduces to

\[
p = \frac{1}{16\pi} \left[ 2^m 3(m-1)(2m-1)a \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right)}{\sqrt{3}} \right)^{2m} \left( 3 - 2m \left( \frac{3}{1 + \cosh(\sqrt{3}\Lambda t)} \right) \right) \right]
\]

\[
\rho = \frac{1}{16\pi} \left[ 6^m (1 - 2m)a \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3} \Lambda}{2} t \right)}{\sqrt{3}} \right)^{2m} \right]
\]
\[ \omega = -1 + \frac{2}{3} m \left( \frac{3}{1 + \cosh \left( \sqrt{3} \Lambda t \right)} \right) \]  

FIG. 4. Energy density (left panel) and EoS parameter (right panel) in redshift. The parameter scheme, \( \Lambda = e^{3\pi}, a = -4.4 \).

The evolutionary behaviour of energy density and EoS parameter has been given in FIG. 4 for the representative value of the exponent \( m = 0.6, 0.8, 1 \). For \( m = 1 \), \( f(Q) = Q \). The energy density reduces gradually for \( m = 1 \) whereas for \( m = 0.6 \) it remains flat throughout and for \( m = 0.8 \), there is a slight reduction at the start of the evolution else remains flat throughout. Whereas the EoS parameter reduces from early to late times and merge together at some finite future and approaches to \( -1 \) at late time. This shows the \( \Lambda \)CDM behaviour of the Universe. Also higher the value of the exponent \( b \), the evolution starts from higher value of EoS parameter. At present, the EoS value has been recorded in the range \([-0.878, -0.802]\). Now, the effective EoS parameter becomes

\[ \omega_{\text{eff}} = -1 + \frac{2^{m+1}3^{m-1}a m H^2}{a 2^m 3^m H^2 - 16\pi \rho} = -1 + \frac{2^{m+1}3^{m-1}a m H^2}{a 2^m 3^m H^2 - 16\pi \rho} \left( \frac{\sqrt{3} \coth \left( \frac{\sqrt{3} \Lambda t}{2} \right)}{\sqrt{3}} \right) \left( \frac{3}{1 + \cosh \left( \sqrt{3} \Lambda t \right)} \right) \]

\[ 3^m 2^{m-1}a \left( \frac{\sqrt{3} \coth \left( \frac{\sqrt{3} \Lambda t}{2} \right)}{\sqrt{3}} \right) ^{2m} - 8\pi \rho \]  

FIG. 5. Effective EoS parameter (left panel) and Energy conditions(right panel)in redshift. The parameter scheme, \( R_0 = 0.5, \Lambda = e^{3\pi}, a = -4.4 \).
and the energy conditions for Case-II becomes,

\[
\begin{align*}
\rho + p &= \frac{1}{16\pi} \left[ 2^{m+1}3^{m-1}a(1-2m) \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3}\Lambda t}{2} \right)}{\sqrt{3}} \right)^{2m} \left( \frac{3}{1 + \cosh \left( \frac{\sqrt{3}\Lambda t}{2} \right)} \right) \right], \\
\rho + 3p &= \frac{1}{16\pi} \left[ 2^{m+1}3^{m}a(1-2m) \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3}\Lambda t}{2} \right)}{\sqrt{3}} \right)^{2m} \left( -1 + m \left( \frac{3}{1 + \cosh \left( \frac{\sqrt{3}\Lambda t}{2} \right)} \right) \right) \right], \\
\rho - p &= \frac{1}{16\pi} \left[ 2^{m+1}3^{m-1}a(1-2m) \left( \frac{\sqrt{\Lambda} \coth \left( \frac{\sqrt{3}\Lambda t}{2} \right)}{\sqrt{3}} \right)^{2m} \left( 3 - m \left( \frac{3}{1 + \cosh \left( \frac{\sqrt{3}\Lambda t}{2} \right)} \right) \right) \right].
\end{align*}
\] (33)

The behaviour of effective EoS shows the similar behavior that of EoS, in fact the present value of \(\omega_{\text{eff}} = -0.802\) same as that of EoS for \(m = 1\). It indicates that the dark energy phase dominates the evolution and the matter part becomes suppressed. For different values of \(m\), no change has been noticed and all are lying on the same curve. The energy conditions for Case-II also shows the similar behaviour that of Case-I. The DEC satisfies, NEC initially satisfies and over the time merged with the null line and at infinite future shows violation. The SEC violates entirely and this behaviour has been inevitable in modified theories of gravity.

V. SCALAR PERTURBATIONS

The cosmological perturbations plays a vital role in converting a complicated mathematical problem into a simpler one. There are different forms of perturbations such as homogeneous, inhomogeneous, isotropic and anisotropic can be studied. The homogeneous and inhomogeneous perturbations again classified into three categories such as: scalar, vector and tensor perturbations. In this section, we will study the linear homogeneous and isotropic perturbations for a class of cosmological solution of \(f(Q, T)\) gravity described by the action (1). The first order perturbation can be expressed as [27, 28],

\[
H(t) \rightarrow H_0(t)(1 + \delta(t)), \quad \rho(t) \rightarrow \rho_0(t)(1 + \delta_m(t))
\]  

(34)

Here, \(H(t)\) and \(\rho(t)\) represents the zero order quantities, and therefore satisfy Eqns. (5),(6) and energy conservation equation. These quantities here represented as \(H_0\) and \(\rho_0\). Also, \(\delta\) and \(\delta_m\) respectively represent the isotropic deviation of the Hubble parameter and matter over-density. To study the linear perturbations, we expand the function \(f(Q, T)\) in powers of \(Q_0\) and \(T_0\) evaluated at the solution \(H = H_0(t)\) as,

\[
f(Q, T) = f(Q_0, T_0) + f_1^0(Q - Q_0) + f_2^0(T - T_0) + \mathcal{O}^2
\]  

(35)

Though we shall examine the linear terms of the induced perturbations are, however the \(\mathcal{O}^2\) term includes all terms proportional to the square or higher powers of \(Q\) and \(T\). Now by introducing expression (34) in the FLRW background eqns.(6) and using the expansion (35), the equation for the perturbation \(\delta(t)\) in the linear approximation becomes,

\[
c_1 + c_2\delta(t) + c_3\delta(t) = \epsilon_m\delta_m(t),
\]  

(36)

Where,
\[ c_1 = f^0 + f^0_0(T - T_0) - 12H_0^2f^0_0 - 12H_0^2f^0_0f^0_0(T - T_0) - 4H_0f^0_0 - 4H_0f^0_0f^0_0(T - T_0) - 96H_0^2f^0_0 - 48H_0^2Hf^0_0f^0_0(T - T_0) - 8H_0f^0_0f^0_0 - 4H_0f^0_0f^0_0(T - T_0) \]
\[ c_2 = 12H_0^2f^0_0 - 144H_0^2f^0_0 - 24H_0^2f^0_0f^0_0(T - T_0) - 48H_0^2f^0_0 - 4H_0f^0_0 - 4H_0f^0_0f^0_0(T - T_0) - 576H_0^4f^0_0 \]
\[ c_3 = -96H_0^2f^0_0 - 48H_0^2f^0_0f^0_0(T - T_0) - 4H_0f^0_0 - 4H_0f^0_0f^0_0(T - T_0) \]
\[ c_m = 16\pi\rho_0 \]

The coefficients contain the trace of energy momentum tensor, which on simplification yield the linear homogeneous and isotropic perturbation equation as,

\[ c_1\dot{m}(t) + c_2\dot{(t)} + c_3\dot{t}(t) = c_m\delta_m(t), \tag{37} \]

In this paper, we have considered two forms of \( f(Q, T) \) with \( m = 1 \) and \( b = 0 \). SO, we shall simplify further the perturbation equation based on the form of \( f(Q, T) \).

**Case I:** For \( f(Q, T) = aQ + bT \), eqn. (37), reduces to,

\[ c_2\dot{t}(t) + c_3\dot{t}(t) = c_m\delta_m(t), \tag{38} \]

where, \( c_2 = -12aH_0^2 - 4aH_0, \quad c_3 = -4aH_0, \quad c_m = (16\pi + b)\rho_0 \). The conservation equation of matter yields the relation,

\[ \delta_m(t) + 3H_0(t)\delta(t) = 0, \tag{39} \]

From eqns. (38) and (39) and substituting the coefficients we obtain,

\[ \dot{\delta}(t) + \left(3H_0 - \frac{H_0}{H_0}\right)\dot{\delta}(t) - \frac{3(16\pi + b)\left(bH_0 - 3H_0^2(b + 8\pi)\right)}{4}\delta(t) = 0 \tag{40} \]

On Solving, we obtain the geometrical perturbation as,

\[ \delta(t) = Ae^{\alpha_1}t + Be^{\alpha_2}t \]

where \( A \) and \( B \) are constant and for brevity we consider this as unity. The exponents,

\[ \alpha_1 = \frac{-(3H_0 - \frac{H_0}{H_0}) + \sqrt{\left(3H_0 - \frac{H_0}{H_0}\right)^2 + 3(16\pi + b)\left(bH_0 - 3H_0^2(b + 8\pi)\right)}}{2} \]
\[ \alpha_2 = \frac{-(3H_0 - \frac{H_0}{H_0}) - \sqrt{\left(3H_0 - \frac{H_0}{H_0}\right)^2 + 3(16\pi + b)\left(bH_0 - 3H_0^2(b + 8\pi)\right)}}{2} \]

Subsequently, we can find the matter perturbation as,

\[ \delta_m(t) = \frac{(c_2 + c_1c_3)}{c_m}Ae^{\alpha_1}t + \frac{(c_2 + c_2c_3)}{c_m}Be^{\alpha_2}t \]

The behaviour of the linear perturbation depend on the model parameter \( a \) and \( b \) and the constant \( \Lambda \). With the best fit values of these parameter, the graphical behaviour of both \( \delta \) and \( \delta_m \) are shown in FIG. 6. We wish to mention here that the due to the hyperbolic scale factor, the oscillatory behaviour of the deviations has been noticed, therefore to keep it in the positive region, we consider the modulus of the deviations everywhere in the paper. In both the geometric and
On solving, Eqn. (42) provides the geometrical deviation and subsequently the matter deviation respectively as,

\[ \delta(t) = D e^{\delta_1 t} + E e^{\delta_2 t} \]  

and

\[ \delta_m(t) = \left( \frac{c_2 + c_3 \delta_1}{c_m} \right) D e^{\delta_1 t} + \frac{c_2 + c_3 \delta_2}{c_m} E e^{\delta_2 t}, \]
where,

\[
\beta_1 = \frac{-A_1 + \sqrt{A_1^2 - 4A_2}}{2}, \quad \beta_2 = \frac{-A_1 - \sqrt{A_1^2 - 4A_2}}{2}.
\]  

For simplicity we consider the coefficients, \( D = E = 1 \).

In this case both the geometrical and matter deviation depends on the model parameter \( a \) and \( m \) with the constant of the scale factor. Using the same set of values of these parameter as in the model, the graphical behaviour of the deviations \( \delta \) and \( \delta_m \) are shown in FIG.7. In this case also the deviations are decaying over time and confirms the stability behaviour of the model.

VI. RESULTS AND DISCUSSIONS

The dynamical parameters of \( f(Q,T) \) gravity have been presented in the most general form for the function, \( f(Q,T) = aQ^m + \beta T \). Two cases pertaining to \( m = 1 \) and \( b = 0 \) are investigated for the late time behaviour of the Universe. The basic geometrical parameters such as the Hubble parameter and deceleration parameter are found to be in the preferred range of the cosmological observations and so also the EoS parameter. The geometrical parameters are scale factor dependent and hence depend on the value of \( \Lambda \) only. Therefore since it is independent of model parameters, the present value of \( H \) and \( q \) does not change with the varying values of the model parameters. In both the cases the Universe shows the \( \Lambda \)CDM behaviour at late times. Though there is some difference in the present value of EoS and effective EoS parameter, but not very significant effect on the evolution. The key results and values of the parameters of both the cases are given in Table I and Table II below.

| Model | Varying \( b \) | \( H(z = 0) \) | \( q(z = 0) \) | \( \omega(z = 0) \) | \( \omega_{\text{eff}}(z = 0) \) |
|-------|----------------|--------------|-------------|----------------|----------------|
| Case-I | \( b = 0.01 \) | 71.808 | -0.701 | -0.803 | -0.866 |
|        | \( b = 0.51 \) | 71.808 | -0.701 | -0.797 | -0.863 |
|        | \( b = 1.51 \) | 71.808 | -0.701 | -0.791 | -0.860 |
| Case-II | Varying \( m \) |
| \( m = 0.8 \) | 71.808 | -0.701 | -0.838 | -0.799 |
| \( m = 1.0 \) | 71.808 | -0.701 | -0.799 | -0.799 |
| \( m = 1.2 \) | 71.808 | -0.701 | -0.761 | -0.799 |

TABLE I. Present value of Hubble parameter \( H \), deceleration parameter \( q \), EoS parameter \( \omega \) and \( \omega_{\text{eff}} \).
| Model  | Energy Conditions | Early Time ($z >> 0$) | Present Time ($z = 0$) | Late Time ($z << 0$) |
|--------|-------------------|----------------------|------------------------|----------------------|
| Case-I | SEC               | Violated             | Violated               | Violated             |
|        | NEC               | Satisfied            | Satisfied              | Satisfied            |
|        | DEC               | Satisfied            | Satisfied              | Satisfied            |
| Case-II| SEC               | Violated             | Violated               | Violated             |
|        | NEC               | Satisfied            | Satisfied              | Satisfied            |
|        | DEC               | Satisfied            | Satisfied              | Satisfied            |

TABLE II. Behaviour of energy conditions.

The scalar perturbation expression has been derived for the general form of $f(Q, T)$ along with the explicit expression of its coefficients. For both the cases: (i) $m = 1$ and (ii) $b = 0$, the form and its coefficients are simplified. The geometric and matter deviations for both the cases are derived. The models parameters and the scale factor parameter are controlling the behaviour of the deviations, hence we have used the value of the parameters used in the models. In both the case, the deviations are decaying out over time; in Case-I, the decay in geometric deviation appears to be earlier than that of matter deviation whereas in Case-II, the decay of both the deviations are happening around the same time.

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