Towards complex (rational) powers of free fields, generalized $\beta\gamma$ systems and non-polynomial quantum field theory

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Abstract

The $\beta\gamma$ system is generalized by complex (rational) powers of the fields, which leads to a corresponding extension of the Fock space. Two different approaches to compute the Green functions of the physical operators are proposed. First the complex (rational) powers are defined via an integral representation, that allows to compute the conformal blocks, Green functions and structure constants of OPA. Next an approach based on a system of recursion equations for the Green functions is developed. A number of solutions of the system is found. A lot of possible applications is briefly discussed.

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1 Introduction

In fact the problem addressed in this work is an old one. Quantum Mechanics deals with Hamiltonians
\[ H = \Delta + U, \]
(1.1)
where \( \Delta \) is a kinetic term and \( U \) is a potential. It should be noted that the choice of \( U \) is rather unrestricted. On the other hand in Quantum Field Theory \( U \) is, as a rule, a polynomial of fields and their derivatives. The problem is to extend a variety of Quantum Field Theory potentials to Quantum Mechanical ones. It is clear that the extension will lead to new theories which may give us a hope for realistic models of Particle Physics, Statistical Mechanics etc.

In recent years, the most promising avenue to this extension has been to explore string field theory (see e.g.\cite{1}). The Lagrangian of a string field \( \Phi \) is written in a form
\[ L = \langle \Phi | \hat{Q} | \Phi \rangle + \sum_{n=3}^{+\infty} \alpha_n \langle \Phi^n \rangle, \]
(1.2)
The first term is kinetic, the second represents a potential as an infinite sum of string vertex functions. \( \langle \Phi^n \rangle \) corresponds to a string vertex function, such that the \( n \) strings meet together.

There are a number of other examples. Some of them are provided by the 2d Conformal Field Theory(CFT).

For illustration, let me consider the \( SL(2) \) Wess-Zumino-Witten (WZW) model. The well-known Wakimoto free field description of the model (at fixed point) is built in terms of one free boson \( \varphi \) coupled to a background charge and a first order bosonic \( (\omega, \omega^\dagger) \) system of weight \((0,1)\) \cite{2}, whose actions on the plane are
\[ S_{\omega,\omega^\dagger} \sim \int d^2 z \omega^\dagger \bar{\partial} \omega, \quad S_{\varphi} \sim \int d^2 z \partial \varphi \bar{\partial} \varphi, \]
(1.3)
where \( \partial = \partial/\partial z, \bar{\partial} = \partial/\partial \bar{z} \). The currents are represented as
\[ J^\dagger(z) = \omega^\dagger(z), \]
\[ J^0(z) = -i(\omega \omega^\dagger(z) + \frac{1}{2\alpha_0} \partial \varphi(z)), \]
\[ J^-(z) = \omega^2 \omega^\dagger(z) + ik \partial \omega(z) + \frac{1}{\alpha_0} \omega \partial \varphi(z), \]
(1.4)
Here \( k \) is the level, \( 2\alpha_0^2 = 1/(k + 2) \).

The primary (physical) fields of the model are given by
\[ \Phi^j(z) = e^{-2j\alpha_0 \varphi(z)}. \]
(1.5)
for the highest weight vector and
\[ \Phi^j_{-j}(z) = \omega^{2j} e^{-2ij\omega \varphi(z)} \] (1.5a)
for the lowest weight vector.

\( j \) is the weight of the representation. In the above I omitted \( \bar{z} \)-dependence.

From a physical point of view it is interesting to explore the weights \( j \) which correspond to reducible (with singular vectors) representations of the \( \hat{sl}_2 \) algebra. An irreducible representation is obtained by setting the singular vectors to zero. This leads to differential equations for correlation functions [3]. Kac and Kazhdan [4] found that there are singular vectors if \( j \) takes the values \( j_{m,n} \) defined by
\[ j_{m,n} = -\frac{m}{2} j_+ + \frac{n}{2} j_- \], \quad (1.6)
\[ j_{m,n} = -\frac{m}{2} j_+ + \frac{n}{2} j_- \], \quad (1.6a)
where \( j_+ = 1, j_- = -k - 2, \{ m, n \} \in \mathbb{N}, k \in \mathbb{C} \).

From this set it is worth to distinguish the so-called admissible representations [5], which correspond to the rational level \( k \). In the case \( j_- = -p/q \) it is possible to recover the minimal models (series with \( c < 1 \)) via the Drinfeld-Sokolov reduction. On the other hand \( j_- = p/q \) leads to the Liouville series with \( c > 25 \). The second point is an existence of modular invariants for such representations.

Now it is evident from (1.5a) and (1.6),(1.6a) that there are the physical fields which are non-polynomial in the free fields! One has the free fields in complex(rational, for the admissible reps.) powers.

Due to a connection of the \( \hat{sl}_2 \) algebra with \( N = 2 \), parafermionic and topological models (see e.g.[6] and refs. therein) it is expected that there are physical fields in such theories which are non-polynomial in free fields too.

For instance, a topological primary field in the Witten free field realization [6] of the topological conformal algebra can be represented by
\[ \Psi_j = \varphi^{2j} \], \quad (1.7)
with \( j \) is exactly given by (1.6). Note that the topological charge is given by \( q = -2j/(k + 2) \).

In this work I report a more handable problem, namely an extended \((\omega, \omega^\dagger)\) system. The extension is done by the complex(rational) powers of the free fields. This model is of interest for several reasons. First of all, as it is known one can use the model to build more complicated theories like \( \hat{sl}_2(\hat{sl}_n), N = 2 \), topological etc [6]. Second, it is a particular case of the so-called \( \beta\gamma \) systems [7] which play a crucial role in string theory. Of course, all results presented may be generalized to an arbitrary \( \beta\gamma \) system.
The last reason is a relative simplicity which allows one to focus on an effect of the complex(rational) powers of the fields only.

The structure of the paper is as follows:

In section 2, the brief review of the $$(\omega, \omega^\dagger)$$ system is given. In particular, the differential equations for correlation functions are represented. These standard equations follow from the $SL(2)$ invariance of vacua.

Section 3 provides a definition of the complex(rational) powers of the free fields via an integral representation like the Mellin transform. By using this definition a general conformal block of fields is computed. The Green functions are defined through conformal blocks in the spirit of the 2d Conformal Field Theory. As a result structure constants are calculated. Generalized Fock spaces are proposed.

In section 4 alternative approach to the problem is developed. It is based on other definitions for the fields which allow to obtain recursion equations for the Green functions. A number of solutions is found and their correspondence with the results of sec.3 is established.

The last, section 5, contains some conclusions and speculations.

2 General properties of $$(\omega, \omega^\dagger)$$ system

As a preparation for a discussion of complex(rational) powers of the free fields in the later sections, let me briefly recall the main properties of $$(\omega, \omega^\dagger)$$ system [7,8].

Consider the action
\[ S \sim \int d^2 z \omega^\dagger \bar{\partial} \omega + (c.c) \]  
(2.1)

where $\omega$ and $\omega^\dagger$ denote a pair of conjugate bosonic fields of dimension 0 and 1 respectively. $\langle \omega(z)\omega^\dagger(z') \rangle = \frac{i}{z - z'}$. (2.2)

In terms of mode expansions one has
\[ \omega(z) = i \sum_{n=\infty}^{+\infty} \frac{\omega_n}{z^n}, \quad \omega^\dagger(z) = \sum_{n=-\infty}^{+\infty} \frac{\omega^\dagger_n}{z^{n+1}} \]  
(2.3)

Canonical quantization gives the following commutation relations
\[ [\omega_n, \omega^\dagger_{-n}] = 1 \]  
(2.4)

\[ ^1 \text{This normalization is chosen to build the free field representation of } \hat{sl}_2[9]. \]
The stress tensor and central charge are

\[ T(z) = i \omega^\dagger \partial \omega(z) \quad , \]
\[ c = 2 \quad . \]

In terms of mode expansion \( T(z) \) is given by

\[ T(z) = \sum_{n=-\infty}^{+\infty} \frac{L_n}{z^{n+2}} \quad , \quad L_n = \sum_{m=-\infty}^{+\infty} m : \omega^\dagger_{n-m} \omega_m : \quad . \]

Define a vacuum \( |0\rangle \) as

\[ \omega_{n+1}|0\rangle = \omega^\dagger_n|0\rangle = 0 \quad , \quad n \in \mathbb{Z}_+ \quad . \]

It is easy to see that \( |0\rangle \) is \( SL(2) \) invariant. Indeed, by using (2.7) and (2.8) one can check that

\[ L_1|0\rangle = L_0|0\rangle = L_{-1}|0\rangle = 0 \quad , \quad |0\rangle = |sl_2\rangle \quad . \]

Now let me introduce a conjugate vacuum \( \langle 0 | \) as

\[ \langle 0 | \omega_{-n} = \langle 0 | \omega^\dagger_{n-1} = 0 \quad , \quad \langle 0 | 0 \rangle = 1 \quad , \quad n \in \mathbb{Z}_+ \quad . \]

It is found that \( \langle 0 | \) is not \( SL(2) \) invariant. Namely,

\[ \langle 0 | L_0 = \langle 0 | L_{-1} = 0 \quad , \quad \langle 0 | L_1 = \langle 0 | \omega_1 \omega^\dagger_0 \neq 0 \quad . \]

It is known that the vacuum \( \langle 0 | \) is expressed through the \( SL(2) \) invariant one as

\[ \langle 0 | = \langle sl_2 | \delta(\omega_0) \quad , \]

where \( \delta(\omega_0) \) is the picture changing operator\(^2\).

The correlation functions (conformal blocks) are defined as

\[ \Upsilon_{ab}(z, u) = \langle 0 | \prod_{n=1}^{N} \omega_a^n(z_n) \prod_{m=1}^{M} \omega^\dagger_{b_m}(u_m) |0\rangle \quad , \]

where \( a = (a_1, ... a_N), \ b = (b_1, ... b_M), \ z = (z_1, ... z_N), \ u = (u_1, ... u_M), \ \{ a_n, b_m \} \in \mathbb{Z}_+ \).

The balance of charges (zero modes) is

\[ \sum_{n=1}^{N} a_n = \sum_{m=1}^{M} b_m \quad . \]

It should be stressed that the dimension of the conformal block space (2.13) is given by \( D = 1 \).

\(^2\)I consider the case when \( \langle 0 | = \lim_{z_0 \rightarrow \infty} \langle sl_2 | \delta(\omega(z_0)) \).
From the $SL(2)$ invariance of vacua one can derive the following equations

\[
\left( \sum_{n=1}^{N} \frac{\partial}{\partial z_n} + \sum_{m=1}^{M} \frac{\partial}{\partial u_m} \right) \Upsilon_{ab}(z, u) = 0, \quad (2.15)
\]

\[
\left( \sum_{n=1}^{N} z_n \frac{\partial}{\partial z_n} + \sum_{m=1}^{M} u_m \frac{\partial}{\partial u_m} + b_m \right) \Upsilon_{ab}(z, u) = 0, \quad (2.16)
\]

\[
\left( \sum_{n=1}^{N} z_n^2 \frac{\partial}{\partial z_n} + \sum_{m=1}^{M} u_m^2 \frac{\partial}{\partial u_m} + 2b_m u_m \right) \Upsilon_{ab}(z, u) + i \sum_{n=1}^{N} \sum_{m=1}^{M} \Upsilon_{a_n b_m - 1}(z, u) = 0, \quad (2.17)
\]

where $\Upsilon_{a_n b_m - 1}(z, u) = \langle 0 | \omega_{a_1}^{n_1}(z_1) ... \omega_{a_n - 1}^{n_n}(z_n) ... \omega_{b_M}^{M_1}(u_M) ... \omega_{b_m}^{M_m}(u_m) | 0 \rangle$.

Note that the last term in (2.17) is due to the picture changing operator.

The Fock space $F$ is obtained by acting on the vacuum $|0\rangle$ with the mode $\omega_0$ and all the negative frequency modes of the fields $\omega, \omega^\dagger$. The basis of $F$ is given by the states

\[
\omega_0^{A_0} \omega_1^{A_1} ... \omega_{n-1}^{A_{n-1}} B_1^{B_1} ... B_M^{B_M} | 0 \rangle, \quad \{A_n, B_m\} \in Z_+. \quad (2.18)
\]

To compute the physical Green functions one has to combine the conformal blocks (2.13) with their conjugate ones as

\[
G_{ab}(z, u, \bar{z}, \bar{u}) = \Upsilon_{ab}(z, u) \overline{\Upsilon_{ab}(z, u)} \quad . \quad (2.19)
\]

It is clear that so obtained Green functions are local i.e. they don’t depend on a mutual position of operators in the Euclidian region $\bar{z} = z^*$. $*$ means complex conjugation.

### 3 Generalized $\beta\gamma$ system. Complex(rational) powers of the fields

The purpose of this section is to develop all the machinery of sec.2 in the case of the complex(rational) powers of the fields. It is curious and essential that it leads to nontrivial results.

Define the complex power of the fields as

\[
\omega^a(z) = \oint_C dt \, t^{-1-a} e^{t \omega(z)}, \quad \omega^b(z) = \oint_C dt \, t^{-1-b} e^{t \omega^\dagger(z)}, \quad \{a, b\} \not\in Z_- . \quad (3.1)
\]

The integration contours will be shown later. Also I suppressed normalization factors. Note that the definition is, in fact, the Mellin transform [10]. It was used in [11] in order to define complex power of $SL(2)$ generators (differential operators).

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3 The $(\omega, \omega^\dagger)$ system has infinitely many non-equivalent Fock spaces. I choose one of them by defining the vacuum as in (2.8). It is done in anticipation of an application to the $\hat{sl}_2$ algebra.
By using the two-point function (2.2) one can easily check that the so-defined $\omega^a(z)$, $\omega^b(z)$ are the primary fields with respect to the stress tensor (2.6) with the conformal dimensions 0 and $b$, respectively.

Now let me say a few more words about this definition. It is clear that $\omega^a(z)$, $\omega^b(z)$ are rather complicated objects and, in fact, depend on a pair variables $(z, C)$. In some sense it is like a construction introduced in [12] where the admissible representation of $\hat{\mathfrak{sl}}_2$ is attached to a pair (point on a curve, Borel subalgebra of the underlying finite dimensional Lie algebra which the representation is induced from). I will denote them as $\omega^a_C(z)$, $\omega^b_C(z)$ below.

To clarify this formal definition, consider the correlator (conformal block, see (2.13))

$$\Upsilon_{ab}(z, u, C) = \langle 0 | \prod_{n=1}^N \omega^a_{C_n}(z_n) \prod_{m=1}^M \omega^b_{C_m}(u_m) | 0 \rangle ,$$

where $a = (a_1, \ldots a_N)$, $b = (b_1, \ldots b_M)$, $z = (z_1, \ldots z_N)$, $u = (u_1, \ldots u_M)$, $\{a_n, b_m\} \not\in \mathbb{Z}$; $C = (C_1, \ldots C_N, C_1^\dagger, \ldots C_M^\dagger)$. $\langle 0 |, | 0 \rangle$ are vacua defined in sec.2.

To get a non-zero result for the conformal block one has to take into account the constraint

$$\sum_{n=1}^N a_n = \sum_{m=1}^M b_m .$$

(3.3)

It will be shown that only in this case the conformal block is non-zero. The constraint (3.3) is a generalization of the balance of charges in correlator $\omega, \omega^\dagger$ fields (see (2.14)).

From (3.1) one finds that

$$\Upsilon_{ab}(z, u, C) = \langle 0 | \prod_{n=1}^N \prod_{m=1}^M dx_n \int_{C_n} dy_n x_n^{1-a_n} y_n^{1-b_m} e^{x_n \omega(z_n) y_n \omega^\dagger(u_m)} | 0 \rangle .$$

(3.4)

By using the two-point function of the $\omega, \omega^\dagger$ fields I arrive at

$$\Upsilon_{ab}(z, u, C) = \prod_{n=1}^N \prod_{m=1}^M \int_{C_n} dx_n \int_{C_m} dy_m x_n^{1-a_n} y_m^{1-b_m} e^{i x_n y_m / (z_n - u_m)} \frac{e^{x_n y_m / (z_n - u_m)}} .$$

(3.5)

Now one can use the definition (3.1) in order to integrate over $y_m$:

$$\Upsilon_{ab}(z, u, C) = \prod_{n=1}^N \prod_{m=1}^M \int_{C_n} dx_n x_n^{1-a_n} \left( \sum_{k=1}^N x_k / (z_k - u_m) \right)^{b_m} .$$

(3.6)

By changing the variables $x_1 = x$, $x_2 = x t_1$, $\ldots x_N = x t_{N-1}$ I obtain

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4 I assume $N \leq M$. Otherwise one has to integrate over $x_n$ to get the minimal number of integrals.
\[ \Upsilon_{ab}(z, u, C) = \oint_{C_0} dx x^{D} \prod_{n=1}^{M} \prod_{m=1}^{N-1} dt_n t_n^{1-a_n+1} \left( \frac{1}{z_1 - u_m} + \sum_{k=1}^{N-1} \frac{t_k}{z_{k+1} - u_m} \right)^{b_m} \]  

(3.7)

where \( D = \sum_{m=1}^{M} b_m - \sum_{n=1}^{N} a_n - 1 \).

Choosing \( C_0 \) around 0 one gets\(^5\)

\[ \Upsilon_{ab}(z, u, C) = \prod_{n=1}^{M} \prod_{m=1}^{N-1} \oint_{C_n} dt_n t_n^{1-a_n+1} \left( \frac{1}{z_1 - u_m} + \sum_{k=1}^{N-1} \frac{t_k}{z_{k+1} - u_m} \right)^{b_m}. \]  

(3.8)

It should be noted that the constrain (3.3) is crucial in order to get a non-zero result. Because only in this case the integral over \( x \) does not give rise to zero.

Now let me turn to some simple examples.

**\( N = 1 \).** In this case the expression (3.8) reduces to

\[ \Upsilon_{ab}(z, u) = \prod_{m=1}^{M} (z_1 - u_m)^{-b_m}. \]  

(3.9)

The dependence on \( C \) is trivial. The dimension of a conformal block space is given by \( D = 1 \).

**\( N = 2 \).** The relevant correlator is

\[ \Upsilon_{ab}(z, u, C) = \oint_C dt \left( t^{1-a_2} \prod_{m=1}^{M} \left( \frac{1}{z_1 - u_m} + \frac{t}{z_2 - u_m} \right)^{b_m} \right). \]  

(3.10)

One can rewrite it as

\[ \Upsilon_{ab}(z, u, C) = (\frac{z_1 - u_1}{z_2 - u_1})^{a_2} \prod_{m=1}^{M} (z_1 - u_m)^{-b_m} I_{ab}(\eta_2, \ldots, \eta_M, C), \]  

(3.11)

where

\[ I_{ab}(\eta_2, \ldots, \eta_M, C) = \oint_C dv v^{1-a_2} (1 - v)^{b_1} \prod_{m=2}^{M} (1 - \eta_m v)^{b_m}. \]  

(3.12)

with the harmonic ratios \( \eta_m = \frac{u_1 - z_1}{u_1 - u_m} \frac{u_m - z_1}{u_m - z_2}, \ldots \frac{u_m - z_1}{u_m - u_M} \).

The integral (3.12) looks as a conformal block of the minimal models with one screening operator [13]. One can choose the integration contour in a similar way. It is easy to see that the dimension of a conformal block space is given by \( D = M \).

Let me consider the case \( M = 2 \) in more detail. For convenience set \( u_1 = \infty, z_1 = 1, u_2 = z, z_2 = 0 \). Then

5Assume \( \sum_{m=1}^{M} b_m \geq \sum_{n=1}^{N} a_n \) and take the limit \( r \to 0 \), here \( r \) is a radius of an integration contour around 0. After this an analytic continuation is used.
\[ \langle \omega_{C_1}^{b_1}(\infty)\omega_{C_2}^{a_1}(1)\omega_{C_2}^{b_2}(z)\omega_{C_1}^{a_2}(0) \rangle \sim (1-z)^{-b_2}\oint_C dv v^{-1-a_2}(1-v)^{b_1}(1-z^{-1})^{-1}v^{b_2}. \quad (3.13) \]

To explore the behavior of the integral under \( z \to 0 \), take the basic integration contours as shown in fig.1 (see [13])

![Fig.1: Contours used in the definition of the basic conformal blocks in the case of \( N = 2, M = 2 \).](image)

A simple algebra leads to

\[ \Upsilon_{ab}(z, C_1) \sim z^{-a_2}(1 - z)^{a_2 - b_2}F(-b_1, -a_2; 1 - a_2 + b_2; z/z - 1) \quad , \quad (3.14) \]

\[ \Upsilon_{ab}(z, C_2) \sim z^{-b_2}F(-b_2, -a_1; 1 + a_2 - b_2; z/z - 1) \quad , \quad (3.14a) \]

where \( F \) is the hypergeometric function.

Under \( z \to 0 \) one has

\[ \Upsilon_{ab}(z, C_1) \sim z^{-a_2}(1 + O(z)) \quad , \quad (3.15) \]

\[ \Upsilon_{ab}(z, C_2) \sim z^{-b_2}(1 + O(z)) \quad . \quad (3.15a) \]

Because an arbitrary contour \( C \) is a linear combination of \( C_1 \) and \( C_2 \), \( \Upsilon_{a,b}(z, C) \) is given by

\[ \Upsilon_{a,b}(z, C) \sim z^{-a_2}(1 + O(z)) + z^{-b_2}(1 + O(z)) \quad . \quad (3.16) \]

In above I omitted a relative factor.

It is evident that such asymptotic behavior corresponds to the following OP Algebra

\[ [\omega^1(z)\bar{\omega}^1(z)]^{b_2}[\omega(0)\bar{\omega}(0)]^{a_2} = \]

\[ = C_1(a_2, b_2)[\omega^1(0)\bar{\omega}^1(0)]^{b_2-a_2}/|z|^{2a_2} + \ldots + C_2(a_2, b_2)[\omega(0)\bar{\omega}(0)]^{a_2-b_2}/|z|^{2b_2} + \ldots , \quad (3.17) \]

where dots mean terms differ from the first by the integer powers of \( z, \bar{z} \). \( C_1 \) and \( C_2 \) are the structure constants.

In the case of integer powers one term disappears, yielding the usual OPA. For example
\[
\omega_{\dagger b}^b(z)\omega^a(0) = \omega_{\dagger b}^b(z) \int_C dt \, t^{-1-a} e^{t \omega(0)} \sim \\
\sim \frac{1}{z^b} \int_C dt \, t^{-1-a+b} e^{t \omega(0)} + \ldots \sim C_1(a, b) \omega^{a-b}(0) / z^b + \ldots \quad , \quad b \in \mathbb{N} . \quad (3.18)
\]

In above I omitted the \( \bar{z} \)-dependence for the sake of simplicity.

Note that it is possible to choose the another basis for the conformal blocks (3.13).
In that case there is a diagonal monodromy under \( z \to 1 \).

\( N = 2, M = 3 \). Choosing the contours as shown in fig.2

\[
\text{Fig.2: Contours used in the definition of the basic conformal blocks in the case of } \quad N = 2, \quad M = 3.
\]

one has

\[
\Upsilon_{ab}(z, z', C) = \langle \omega_{C_1}^{t b_1} (\infty) \omega_{C_2}^{a_1} (1) \omega_{C_3}^{t b_3} (z) \omega_{C_2}^{a_3} (z') \omega_{C_3}^{a_3} (0) \rangle \quad (3.19)
\]

with

\[
\Upsilon_{ab}(z, z', C_1) \sim z'^{-a_2} (1 - z)^{-b_2} (1 - z')^{-a_2 - b_3} F_1(-a_2, -b_1, -b_2, 1 - a_2 + b_3; \frac{z'}{z'-1}, \frac{z - 1}{z'}), \quad (3.20a)
\]

\[
\Upsilon_{ab}(z, z', C_2) \sim z^{-b_2} z'^{-b_3} F_1(-a_1, -b_3, -b_2, 1 - a_1 + b_1; \frac{z'}{z'-1}, \frac{z}{z'}), \quad (3.20b)
\]

\[
\Upsilon_{ab}(z, z', C_3) \sim (1 - z)^{-b_2} (1 - z')^{-b_3} F_1(-a_2, -b_3, -b_2, 1 - a_2 + b_1; \frac{z'}{z'-1}, \frac{z - 1}{z'}), \quad (3.20c)
\]

where \( F_1 \) is the hypergeometric function in \( M - 1 \) variables. I got its integral representation as a single integral of Euler’s type.

Note that in the above I have got the Picard’s integral representation for \( F_1 \) (single integral of Euler type) [14]. I fixed \( u_1 = \infty, z_1 = 1, u_2 = z, u_3 = z', z_2 = 0 \).

For an arbitrary \( M \) a basic conformal block is given by

\[
I_{ab}(\eta_2, \ldots \eta_M, C) \sim F_1(\alpha, \beta_2, \ldots \beta_M, \gamma; f_2, \ldots f_M) \quad , \quad (3.21)
\]

where \( \alpha, \beta_i, \gamma \) are functions in \( a_i, b_i \) and \( f_i \) are functions in \( \eta_i \). \( F_1 \) is the hypergeometric function in \( M - 1 \) variables. I got its integral representation as a single integral of Euler’s type.

The conformal blocks (3.8) for \( N \geq 3 \) are more involved. In particular, one can try to use an integral representation to turn them into more standard form like the one introduced in [13].
In order to compute the physical Green functions one has to combine the conformal blocks \((3.8)\) with the ones from \(\tilde{\omega}, \tilde{\omega}^\dagger\) fields (see \((2.19)\)). This is now less trivial. Define the Green functions as a monodromy invariant forms

\[
G_{ab}(z, u, \bar{z}, \bar{u}) = \sum_{C, \bar{C}} h_{C\bar{C}} \Upsilon_{ab}(z, u, C) \overline{\Upsilon_{ab}(\bar{z}, \bar{u}, C)}. \tag{3.22}
\]

Here \(h_{C\bar{C}}\) is a metric on the space of conformal blocks. It is clear that the so defined Green functions are local. Note that in the case of integer powers the above definition becomes a simple product of analytic and antianalytic factors (see \((2.19)\)).

Now let me sketch a computation of the structure constants of the OPA \((3.17)\).

The Green function to be calculated is

\[
G_{ab}(z, \bar{z}) = \langle \left[ \omega^\dagger(\infty)\omega^\dagger(\infty) \right]^{a+\varepsilon} \left[ \omega(1)\omega(1) \right]^{b+\varepsilon} \left[ \omega^\dagger(z)\omega^\dagger(z) \right]^{b} \left[ \omega(0)\omega(0) \right]^{a} \rangle, \tag{3.23}
\]

or

\[
G_{ab}(z, \bar{z}) = \sum_{C, \bar{C}} h_{C\bar{C}} \Upsilon_{a, b}(z, C) \overline{\Upsilon_{a, b}(\bar{z}, C)}. \tag{3.24}
\]

The relevant conformal block is

\[
\Upsilon_{a, b}(z, C) = \oint_{C} dv v^{-1-a}(1-v)^{a+\varepsilon}(1/(1-z)+v/z)^{b} . \tag{3.24}
\]

Denote \(I_{k}(z) = \Upsilon_{a, b}(z, C_{k}), I^{k}(z) = \overline{\Upsilon_{a, b}(z, C^{k})}\), where \(k = 1, 2\) and fix the phases of the integrands so that \(I_{k}(z), I^{k}(z)\) are real. The integration contours are shown in fig.3.

![Fig.3: Two basic types of contours used for the definition of the conformal blocks.](image)

The basis \(I_{k}\) corresponds to the diagonal monodromy of the conformal blocks under \(z \rightarrow e^{2i\pi}z\). The other has the diagonal monodromy at \(z = 1\).

There is the following relation between them

\[
I_{1} = \frac{\sin \pi(a + \varepsilon)}{\sin \pi \varepsilon} I^{1} - \frac{\sin \pi(b + \varepsilon)}{\sin \pi \varepsilon} I^{2} \tag{3.25},
\]

\[
I_{2} = \frac{\sin \pi a}{\sin \pi \varepsilon} I^{1} - \frac{\sin \pi b}{\sin \pi \varepsilon} I^{2} . \tag{3.25a}
\]

\[^{6}\text{I shift parameters in order to get rid of the degeneration.}\]
The monodromy invariant Green function is
\[ G_{a,b}(z, \bar{z}) \sim \left\{ I_1 I_1 - \frac{\sin \pi (a + \varepsilon) \sin \pi (b + \varepsilon)}{\sin \pi a \sin \pi b} I_2 \bar{I}_2 \right\} \tag{3.26} \]

In terms of the normalized conformal blocks [3] it is rewritten as
\[ G_{a,b}(z, \bar{z}) \sim \left\{ B^2(-a, 1 + b) F_1 \bar{F}_1 - N B^2(-b - \varepsilon, 1 + a + \varepsilon) F_2 \bar{F}_2 \right\} \tag{3.27} \]

where \( B \) means the B-function, \( N = \frac{\sin \pi (a+\varepsilon) \sin \pi (b+\varepsilon)}{\sin \pi a \sin \pi b} \).

Finally normalizing the two-point function as
\[ \langle [\omega(z)\omega(z')]^a[\omega^\dagger(z')\omega^\dagger(z')]^a \rangle = 1/|z - z'|^{2a} \tag{3.28} \]

and taking the limit \( \varepsilon \to 0 \), one gets
\[ C_1(a,b) = C(a,b) \quad , \quad C_2(a,b) = C(b,a) \quad , \quad C(a,b) = \frac{\Gamma(1 + b)}{\Gamma(1 + a)\Gamma(1 - a + b)} \quad . \tag{3.29} \]

Before ending this section, I wish to make several remarks.

First, in (3.1) complex powers are defined for \( \{a, b\} \not\in \mathbb{Z} \). It is not hard to see from (3.29) that fields in negative integer powers don’t appear in operator expansions.

In the case of \( a(b) \in \mathbb{Z}_+ \) there is a degeneracy i.e. it is possible to build the Green function via \( I_1(I_2) \) only. This leads to one term on the right hand side of (3.17) (also see (3.18)).

One can find the structure constants using analytic continuation from integer \( a, b \) to general ones. However in this case only one term appears on the r.h.s. and the full OPA is disguised.

Finally, the Fock space \( \mathcal{F} \) is extended to \( \mathcal{F}^{(0)} \) by the complex(rational) powers of \( \omega, \omega^\dagger \). For this space the basis is given by the states
\[ \omega_0^{A_0} \omega_{-1}^{A_1} \ldots \omega_{-1}^{B_1} \omega_{-2}^{B_2} \ldots |0\rangle \quad , \tag{3.30} \]

where \( \{A_0, B_1\} \in \mathbb{C}\backslash\mathbb{Z}_- \), the other powers are the same as in (2.18). Next one can define \( \mathcal{F}^{(1)} \) with \( \{A_0, A_1, B_1, B_2\} \in \mathbb{C}\backslash\mathbb{Z}_- \). It is implemented by the complex(rational) powers of \( \omega, \omega^\dagger, \partial\omega, \partial\omega^\dagger \) and so on.

### 4 Construction of conformal blocks (Green functions) via recursion equations

In the previous section, I have obtained the conformal blocks (Green functions) of the \( \omega, \omega^\dagger \) fields by the integral representation. I now wish to develop an alternative approach to this problem, namely via a system of recursion equations for the Green functions.
Consider the Green function
\[
G_{ab}(z, u, \bar{z}, \bar{u}) = \left\langle \prod_{n=1}^{N} \left[ \omega(z_n) \right]^{a_n} \prod_{m=1}^{M} \left[ \omega^\dagger(u_m) \right]^{b_m} \right\rangle .
\] (4.1)

For the sake of simplicity I will suppress the \(\bar{z}, \bar{u}\)-dependence for the time being.

Define the complex(rational) powers of the fields as
\[
\left[ \omega(z) \right]^a = \omega(z)\left[ \omega(z) \right]^{a-1},
\] (4.2)
\[
\left[ \omega^\dagger(z) \right]^b = \omega^\dagger(z)\left[ \omega^\dagger(z) \right]^{b-1}.
\]

Next define the derivatives of these fields
\[
\frac{\partial}{\partial z}[\omega(z)]^a = a\partial \omega(z)[\omega(z)]^{a-1},
\] (4.3)
\[
\frac{\partial}{\partial z}[\omega^\dagger(z)]^b = b\partial \omega^\dagger(z)[\omega^\dagger(z)]^{b-1}.
\]

Finally assume the following OP expansions
\[
\omega(z)[\omega^\dagger(0)]^b = ib[\omega^\dagger(0)]^{b-1}/z + O(1),
\] (4.4)
\[
\omega^\dagger(z)[\omega(0)]^a = -ia[\omega(0)]^{a-1}/z + O(1).
\]

All definitions are, of course, generalizations of the usual ones for the arbitrary powers.

Now from (4.2) and (4.4) one can easily find
\[
G_{ab}(z, u, \bar{z}, \bar{u}) = \sum_{n=1}^{N} \frac{ia_n}{z_n - u_m} G_{a_n-1, b_m-1}(z, u, \bar{z}, \bar{u}),
\] (4.5)
\[
G_{ab}(z, u, \bar{z}, \bar{u}) = \sum_{m=1}^{M} \frac{ib_m}{z_n - u_m} G_{a_n-1, b_m-1}(z, u, \bar{z}, \bar{u}),
\] (4.5a)

where \(G_{a_n-1, b_m-1}(z, u, \bar{z}, \bar{u}) = G_{a_1, a_2, \ldots, a_N, b_1, \ldots, b_M}(z, u, \bar{z}, \bar{u})\).

It is clear that recursion equations in \(\bar{z}, \bar{u}\) is obtained by a similar way.

On the other hand (4.3) and (4.4) lead to
\[
\frac{\partial}{\partial u_m} G_{ab}(z, u, \bar{z}, \bar{u}) = \sum_{n=1}^{N} \frac{ia_n b_m}{(z_n - u_m)^2} G_{a_n-1, b_m-1}(z, u, \bar{z}, \bar{u}),
\] (4.6)
\[
\frac{\partial}{\partial z_n} G_{ab}(z, \bar{z}, \bar{u}) = -\sum_{m=1}^{M} \frac{i a_n b_m}{(z_n - u_m)^2} G_{a_n-1,b_m-1}(z, u, \bar{z}, \bar{u}) ,
\] (4.6a)

and the same equations in \( \bar{z}, \bar{u} \).

Before solving the equations, let me point out some important consequences.

After a simple algebra with (4.5),(4.5a) the reader can derive

\[
(N \sum_{n=1}^{N} a_n - M \sum_{m=1}^{M} b_m) G_{ab}(z, u, \bar{z}, \bar{u}) = 0 .
\] (4.7)

Using the above equation, one immediately deduces the constraint (2.13).

Algebraic manipulations with (4.6),(4.6a) give

\[
\left( \sum_{n=1}^{N} \frac{\partial}{\partial z_n} + \sum_{m=1}^{M} \frac{\partial}{\partial u_m} \right) G_{ab}(z, u, \bar{z}, \bar{u}) = 0 .
\] (4.8)

By looking at this equation it is evident that it coincides with (2.15), which reflects the \( SL(2) \) invariance of vacua with respect to \( L_{-1} \).

In addition to these equations, one may also derive the last two equations following from the \( SL(2) \) invariance (see (2.16),(2.17)). In order to do this, one needs (4.5), (4.5a) as well as (4.6),(4.6a). After some algebra it is possible to arrive at

\[
\left( \sum_{n=1}^{N} z_n \frac{\partial}{\partial z_n} + \sum_{m=1}^{M} u_m \frac{\partial}{\partial u_m} + b_m \right) G_{ab}(z, u, \bar{z}, \bar{u}) = 0 ,
\] (4.9)

\[
\left( \sum_{n=1}^{N} z_n^2 \frac{\partial}{\partial z_n} + \sum_{m=1}^{M} u_m^2 \frac{\partial}{\partial u_m} + 2 b_m u_m \right) G_{ab}(z, u, \bar{z}, \bar{u}) + i \sum_{n=1}^{N} \sum_{m=1}^{M} G_{a_n-1,b_m-1}(z, u, \bar{z}, \bar{u}) = 0 .
\] (4.10)

It should be stressed that in (4.7)-(4.10) \( \{a, b\} \in \mathbb{C} \setminus \mathbb{Z}_- \).

A solution of the equations (4.5),(4.5a),(4.6),(4.6a) and corresponding equations in \( \bar{z}, \bar{u} \) is given by

\[
G_{ab}(z, u, \bar{z}, \bar{u}) = \sum_{ij} \eta_{ij} \overline{\chi^i_{ab}(z, u)} \overline{\chi^j_{ab}(z, u)} ,
\] (4.11)

where \( \chi^i_{ab}(z, u) \) is a solution of the system (4.5)-(4.6a) and \( \overline{\chi^j_{ab}(z, u)} \) is one of the system in \( \bar{z}, \bar{u} \). \( h_{ij} \) is a metric on the space of solutions.

Now let me solve the system for the simple cases.

\( N = 1 \). Then (4.5)-(4.6a) are written in a form

\[
\chi_{ab}(z, u) = \frac{ia_1}{z_1 - u_m} \chi_{a_1-1,b_m-1}(z, u) ,
\] (4.12)

\[
\overline{\chi_{ab}(z, u)} = \sum_{m=1}^{M} \frac{ib_m}{\bar{z}_1 - \bar{u}_m} \chi_{a_1-1,b_m-1}(z, u) .
\] (4.12a)
and
\[
\frac{\partial}{\partial z_1} \Upsilon_{ab}(z, u) = -\sum_{m=1}^{M} \frac{ia_1 b_m}{(z_1 - u_m)^2} \Upsilon_{a_1-1,b_m-1}(z, u) ,
\]
(4.13)
\[
\frac{\partial}{\partial u_m} \Upsilon_{ab}(z, u) = i a_1 b_m \frac{b_m}{(z_1 - u_m)^2} \Upsilon_{a_1-1,b_m-1}(z, u) ,
\]
(4.13a)
where \( \Upsilon_{a_1-1,b_m-1}(z, u) = \Upsilon_{a_1-1,b_1,...,b_m-1}(z, u) \). It is now a simple exercise to derive differential equations for the conformal blocks \( \Upsilon_{ab}(z, C) \)
\[
\frac{\partial}{\partial u_m} \Upsilon_{ab}(z, u) = \frac{b_m}{z_1 - u_m} \Upsilon_{ab}(z, u) ,
\]
(4.14)
\[
\frac{\partial}{\partial z_1} \Upsilon_{ab}(z, u) = -\sum_{m=1}^{M} \frac{b_m}{z_1 - u_m} \Upsilon_{ab}(z, u) .
\]
(4.14a)
The solution of eqs.(4.14),(4.14a) is given by
\[
\Upsilon_{ab}(z, u) \sim \prod_{m=1}^{M} (z_1 - u_m)^{-b_m} .
\]
(4.15)
It is the same as the one (3.9) derived via the integral representation.

\( N = 2, M = 2 \). After a substitution
\[
\Upsilon_{ab}(z, u) = (z_1 - u_1)^{b_2-a_1}(u_1 - z_2)^{-a_2}(z_1 - u_2)^{-b_2} f_{ab}(z, u)
\]
the equations (4.5)-(4.6a) are written in a form
\[
\begin{align*}
\frac{d}{du_1} f_{ab}(z, u) &= \frac{z_{12}}{(z_1 - u_1)(z_2 - u_1)} \{(b_1 - a_2)f_{ab}(z, u) - ia_1 b_1 \Upsilon_{a_1-1,b_1-1}(z, u)\} , \\
\frac{d}{du_2} f_{ab}(z, u) &= \frac{z_{12}}{(z_1 - u_2)(z_2 - u_2)} \{(b_2 - a_1)f_{ab}(z, u) + ia_1 b_1 \Upsilon_{a_1-1,b_1-1}(z, u)\} ,
\end{align*}
\]
(4.17a)
Now it is possible to find equations to cover the results found in sec. 3 via the integral representation. It is not hard to see that the result coincides with the one obtained in sec. 3.

Next combining (4.19a) with (4.19d) one gets

\[ \eta \frac{d}{d \eta} f_{ab} = (a_2 - b_1)f_{a,b} + i a_1 b_1 f_{a_1-1,b_1-1} \]  \hspace{1cm} (4.19a)

\[ \eta \frac{d}{d \eta} f_{ab} = a_2 f_{a,b} + i a_2 b_1 f_{a_2-1,b_1-1} \]  \hspace{1cm} (4.19b)

\[ \eta \frac{d}{d \eta} f_{ab} = b_2 f_{a,b} - i a_1 b_2 f_{a_1-1,b_2-1} \]  \hspace{1cm} (4.19c)

\[ \eta \frac{d}{d \eta} f_{ab} = -i a_2 b_2 f_{a_2-1,b_2-1} \]  \hspace{1cm} (4.19d)

A change of variables \((z_1, z_2, u_1, u_2) \rightarrow (z_1, z_2, \eta, u_2)\), where \( \eta = \frac{u_1 - u_2}{u_1 - z_1} \frac{u_2 - z_1}{u_1 - z_1} \), leads to

\[ \frac{\partial}{\partial z_1} f_{ab}(z, u) = \frac{\partial}{\partial z_2} f_{ab}(z, u) = \frac{\partial}{\partial u_2} f_{ab}(z, u) = 0 \]  \hspace{1cm} (4.18)

From this it follows that \( f_{ab}(z, u) \) depends on \( \eta \) only, i.e. \( f_{ab}(z, u) = f_{ab}(\eta) \equiv f_{ab} \). Now it is possible to find equations

\[ \eta \frac{d}{d \eta} f_{ab} = (a_2 - b_1)f_{a,b} + i a_1 b_1 f_{a_1-1,b_1-1} \]  \hspace{1cm} (4.19a)

\[ \eta \frac{d}{d \eta} f_{ab} = a_2 f_{a,b} + i a_2 b_1 f_{a_2-1,b_1-1} \]  \hspace{1cm} (4.19b)

\[ \eta \frac{d}{d \eta} f_{ab} = b_2 f_{a,b} - i a_1 b_2 f_{a_1-1,b_2-1} \]  \hspace{1cm} (4.19c)

\[ \eta \frac{d}{d \eta} f_{ab} = -i a_2 b_2 f_{a_2-1,b_2-1} \]  \hspace{1cm} (4.19d)

Next combining (4.19a) with (4.19d) one gets

\[ \eta \frac{d^2}{d \eta^2} f_{ab} + (1 - a_2 + b_1) \frac{d}{d \eta} f_{ab} - a_1 a_2 b_1 b_2 f_{a-1,b-1} = 0 \]  \hspace{1cm} (4.20)

The same procedure with (4.19b),(4.19c) gives

\[ \eta^2 \frac{d^2}{d \eta^2} f_{ab} + (1 - a_2 - b_2) \eta \frac{d}{d \eta} f_{ab} + a_2 b_2 f_{ab} - a_1 a_2 b_1 b_2 f_{a-1,b-1} = 0 \]  \hspace{1cm} (4.20a)

Finally I obtain the Riemann equation

\[ \eta (1 - \eta) \frac{d^2}{d \eta^2} f_{ab} + \{(1 - a_2 + b_1) - (1 - a_2 - b_2) \eta\} \frac{d}{d \eta} f_{ab} - a_2 b_2 f_{ab} = 0 \]  \hspace{1cm} (4.21)

It is well known that the solution of this equation is given by [14]

\[ f_{ab} = \oint_C dy y^{-1-a_2} (1 - y)^{b_1} (1 - \eta y)^{b_2} \]  \hspace{1cm} (4.22)

It is not hard to see that the result coincides with the one obtained in sec. 3. Unfortunately computations for other cases are tedious. However it is expected that they cover the results found in sec. 3 via the integral representation.
5 Conclusions and remarks

First let me say a few words about the results.

At present the problem of extension a variety of Quantum Field Theory potentials to Quantum Mechanical ones is an open problem. Some examples of such extension are provided by the 2d Conformal Field Theory. Because the free field representation plays a significant role in computations of the conformal blocks, of the Green functions and of Operator Algebras of the 2d Conformal Field Theory it seems reasonable at first to extend the free field theories. Next one may build more nontrivial theories via the free field representation by the standard methods of CFT. The \((\omega, \omega^\dagger)\) system is interesting for some reasons. First of all, it has a lot of applications in the 2d CFT, namely in \(SL(N)\) WZW, \(N = 2\), parafermionic and topological models [6]. Second this is a particular case of the \(\beta\gamma\) systems which play crucial role in String Theory. All the results for \((\omega, \omega^\dagger)\) can be generalized to an arbitrary case. Finally it is a simple system that allows one to focus purely on the effect of non-integer powers.

In this work I have generalized the \((\omega, \omega^\dagger)\) system by the complex(rational) powers of the fields that leads to the corresponding extension on the Fock space. Next two different approaches for a computation of the Green functions are proposed. First the complex(rational) powers of the fields are defined via the integral representation. It allows to compute the Green functions of the physical operators in the spirit of the 2d CFT, namely through the conformal blocks. The structure constants of OPA are calculated. Next the alternative approach is developed. It is based on the system of recursion equations for the Green functions. A number of solutions is found. They coincide with the results obtained via the integral representation.

Let me conclude by mentioning some open problems.

(i) The main problem is, of course, to extend to more real theories, i.e. models of the 4d Quantum Field Theory.

As to the 2d Field Theory one can, for example, introduce the following action

\[ S \sim \int d^2z \omega^\dagger \bar{\partial} \omega + (c.c) + g \int d^2z [\omega \bar{\omega}]^a [\omega^\dagger \bar{\omega}^\dagger]^b \quad , \quad (5.1) \]

with a coupling constant \(g\) and some noninteger parameters \(a,b\). It is clear that the results obtained in sec.3,4 allow to consider the perturbation theory for such action.

In the context of topological models [6], one can define the action

\[ S \sim \int d^2z \omega^\dagger \partial \omega + w^\dagger \bar{\partial} w + (c.c) + g \int d^2z \bar{w} \bar{\omega}^\dagger [\omega \bar{\omega}]^a \quad , \quad (5.2) \]

where \((w, w^\dagger)\) is a first order fermionic system of weight \((0,1)\). Now \(a\) is a noninteger parameter that leads to the nonpolynomial potential

(ii) The second problem is impressive too. It concerns a lot of applications of the techniques developed in the 2d CFT, for instance \(sl_2\), topological models. Some steps in this direction are available [15].
(iii) As it was shown in sec.3 there is no factorization for the operators $[\omega(z)\overline{\omega(z)}]^a$, $[\omega^\dagger(z)\overline{\omega^\dagger(z)}]^b$ in a general case. However it is clear that they decompose as

$$[\omega(z)\overline{\omega(z)}]^a = \sum_{C,\overline{C}} \omega_C^a(z)\omega_{\overline{C}}^\dagger(z) \quad .$$  \hspace{1cm} (5.3)$$

It resembles a formula by Moore-Reshetikhin [16]. This time $C, \overline{C}$ play a role of quantum group labels. At this point it would be interesting to understand an underlying quantum group structure.

(iv) One puzzling aspect of the approach developed in sec.3 is that from the definition (3.1) one has more contours than it is necessary to define the conformal blocks. This means that only a part of them makes sense. To remedy this situation, one can bosonize the $(\omega, \omega^\dagger)$ system [7]. In terms of new variables (scalar fields with background charges) the $\omega, \omega^\dagger$ fields are written as

$$\omega(z) = e^{a(\phi(z)+i\varphi(z))} \quad ,$$

$$\omega^\dagger(z) = \partial \varphi e^{-a(\phi(z)-i\varphi(z))} \quad .$$ \hspace{1cm} (5.4a)

It is easy to check that the operator $\omega^a(z)$ corresponds to

$$\omega^a(z) = e^{a(\phi(z)+i\varphi(z))} \quad .$$ \hspace{1cm} (5.5)

Due to this the conformal blocks don’t depend on $C$. However this is not the case for $\omega^\dagger^b(z)$.

It seems interesting to use bosonization in order to obtain integral identities or something else in the spirit of the usual bosonization. It will be a generalization of the last.

(v) Although I have considered the complex(rational) powers of the free fields it is possible to develop such technique for more complicated objects. For example, one can apply the approach of sec.4 to the Kac-Moody algebras.

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\footnote{Indeed, as it was shown in sec.3 one can integrate over $C(C^\dagger)$ contours, as a result the conformal blocks are defined in terms $C(C^\dagger)$ only.}
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