Rationality, aggregate monotonicity and consistency in cooperative games: some (im)possibility results

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Received: 26 October 2015 / Accepted: 5 May 2016 / Published online: 20 May 2016
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Abstract On the domain of cooperative games with transferable utility, we investigate if there are single-valued solutions that reconcile individual rationality, core selection, consistency and monotonicity (with respect to the worth of the grand coalition). This paper states some impossibility results for the combination of core selection with either complement consistency (Moulin, J Econ Theory 36:120–148, 1985) or projected consistency (Funaki, Dual axiomatizations of solutions of cooperative games. Mimeo, Tokyo, 1998), and core selection, max consistency (Davis and Maschler, Naval Res Logist Q 12:223–259, 1965) and monotonicity. By contrast, possibility results are manifest when combining individual rationality, projected consistency and monotonicity.

1 Introduction

A cooperative game with transferable utility (a game) is a concise description of a situation in which a society can profit from agreeing to cooperate. It is specified by a finite set of players and a real-valued function defined on the set of coalitions of players. A (single-valued) solution is a function that assigns a feasible playoff vector for each game. Efficiency (or Pareto-optimality) states that all the gains from cooperation are distributed among the players, where one of the objectives of the axiomatic approach is to determine whether efficiency can be combined with other suitable properties. In this

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In the literature, several notions of monotonicity have been introduced to characterize solutions on different frameworks (e.g., Kalai and Smorodinsky 1975; Kalai 1977; Kalai and Samet 1985 or Thomson 1987). On the domain of cooperative games with transferable utility, coalitional monotonicity (Young 1985) says that if the worth of a given coalition increases, whereas the worth of all other coalitions remains the same, then the payoff of every member in that coalition should weakly increase. Aggregate monotonicity (Megiddo 1974) is a particular case in which only the worth of the grand coalition changes. The core (Gillies 1953) of a game is the set of feasible outcomes that cannot be improved upon by any coalition of players. A solution satisfies core selection (or coalitional rationality) if it selects a core element for any game with a non-empty core. Young (1985) and Housman and Clark (1998) show that core selection and coalitional monotonicity are incompatible.1 Arin and Feltkamp (2012) and Arin (2013) consider new monotonicity properties to overcome this incompatibility. However, core selection and aggregate monotonicity are compatible when considering, for instance, the per-capita prenucleolus (Grotte 1970). Individual rationality imposes that every player obtains at least his individual worth whenever this is possible. Although the per-capita prenucleolus is not individually rational, Calleja et al. (2012) show that core selection, individual rationality and aggregate monotonicity can be reconciled.2

Consistency is a relevant principle in game theory. A solution is consistent if whenever we reduce the game to a subset of agents and the excluded agents are paid according to a solution payoff, the projection of this payoff to the remaining agents still belongs to the solution of the reduced game. Depending on how reduced games are defined, different notions of consistency are obtained. Here, we consider three kinds of consistency property widely used in the axiomatic formulation of solutions: max consistency (Davis and Maschler 1965), complement consistency (Moulin 1985) and projected consistency (Funaki 1998).3 In line with Calleja et al. (2012), we investigate if there are single-valued solutions that satisfy individual rationality, core selection, aggregate monotonicity and consistency.

When combining individual rationality or core selection with different notions of consistency a number of drawbacks emerge. Hwang (2013) shows that individual rationality is incompatible with both complement and max consistency. On the other hand, although the core itself obeys the above three consistency properties, for single-valued solutions core selection is compatible with neither complement consistency nor projected consistency. Hence, individual rationality and core selection together are incompatible with any of these consistency properties. By contrast, the prenucleolus

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1 Young (1985) formulates his impossibility result for games with five or more players. Housman and Clark (1998) extend this result to games with four players.
2 In fact, Calleja et al. (2012) solve an open question noted by Young et al. (1982) showing that core selection, individual rationality, aggregate monotonicity, symmetry and dummy player property are compatible for single-valued solutions.
3 See Thomson (2011) for an essay on the consistency principle.
(Schmeidler 1969) satisfies core selection and max consistency, and the center of imputations (Driessen and Funaki 1991) is individually rational and projected consistent.

Most solutions that are aggregate monotonic, such as the Shapley value (Shapley 1953), the per-capita prenucleolus, the consensus value (Ju et al. 2007), the equal division solution and the center of imputations, satisfy equal surplus division, which implies that any variation in the worth of the grand coalition is shared equally among all players. Capturing the idea that players may distribute variations in their wealth monotonically, but not necessarily equally, we introduce regular aggregate monotonicity. The center of imputations, which has recently been axiomatized by van den Brink (2007), Casajus and Huettner (2014a) and Yokote and Funaki (2015) invoking other monotonicity properties, turns out to be the unique single-valued solution satisfying individual rationality and equal surplus division (or, alternatively, individual rationality, regular aggregate monotonicity and symmetry\(^4\)). Dropping the monotonicity principle, further characterization can be found in Béal et al. (2015) and Chun and Park (2012).

Finally, we explore the possibility of reconciling individual rationality, core selection and consistency with some aggregate monotonicity properties. We provide a characterization of the family of single-valued solutions satisfying individual rationality and projected consistency together with (regular) aggregate monotonicity. Despite these positive results, accommodating core selection and max consistency presents some difficulties. We show that core selection, regular aggregate monotonicity and max consistency are incompatible on the domain of all games and also on the domain of balanced games. However, if we restrict ourselves to convex games we observe that these properties can be reconciled, but compatibility is lost if we consider equal surplus division.

The remainder of the paper is organized as follows. In Sect. 2, we introduce some preliminaries on games. In Sect. 3, we formalize the aforementioned properties. In Sect. 4, we analyze the (in)compatibilities between individual rationality, core selection and either consistency or monotonicity. In Sect. 5, we discuss how well individual rationality, core selection, consistency and monotonicity combine. The “Appendix” contains the proofs that have been omitted in Sect. 5. Some final remarks and open questions conclude the paper.

2 Preliminaries

The set of natural numbers \(\mathbb{N}\) denotes the universe of potential players. A coalition is a non-empty finite subset of \(\mathbb{N}\) and let \(\mathcal{N}\) denote the set of all coalitions of \(\mathbb{N}\). Given \(S, T \in \mathcal{N}\), we use \(S \subseteq T\) to indicate strict inclusion, that is, \(S \subseteq T\) and \(S \neq T\). By \(|S|\) we denote the cardinality of the coalition \(S \in \mathcal{N}\). A transferable utility coalitional game (a game) is a pair \((\mathcal{N}, \nu)\) where \(\mathcal{N} \in \mathcal{N}\) is the set of players and \(\nu : 2^\mathcal{N} \rightarrow \mathbb{R}\) is the characteristic function that assigns to each coalition \(S \subseteq \mathcal{N}\) a real number \(\nu(S)\), representing what \(S\) can achieve by agreeing to cooperate, with the convention that \(\nu(\emptyset) = 0\). By \(\Gamma\) we denote the class of all games. Given \(\mathcal{N} \in \mathcal{N}\) and \(\emptyset \neq S \subseteq \mathcal{N}\),

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\(^4\) Symmetry says that if two players contribute equal amounts to all coalitions, their payoff should be equal.
the *unanimity game* \((N, u_S)\) associated to \(S\) is defined as \(u_S(R) := 1\) if \(S \subseteq R\) and \(u_S(R) := 0\) otherwise. Given a game \((N, v)\) and \(\emptyset \neq N' \subset N\), the subgame \((N', v|_{N'})\) is defined as \(v|_{N'}(S) := v(S)\) for all \(S \subseteq N'\).

Given \(N \in \mathcal{N}\), let \(\mathbb{R}^N\) stand for the space of real-valued vectors indexed by \(N\), \(x = (x_i)_{i \in N}\), and for all \(S \subseteq N, x(S) = \sum_{i \in S} x_i\), with the convention \(x(\emptyset) = 0\). Given \(\emptyset \neq S \subseteq N, e_S \in \mathbb{R}^N\) is defined as \(e_S := 1\) if \(i \in S\) and \(e_S := 0\) otherwise. For each \(x \in \mathbb{R}^N\) and \(\emptyset \neq T \subseteq N, x|_T\) denotes the restriction of \(x\) to \(T\): \(x|_T = (x_i)_{i \in T} \in \mathbb{R}^T\).

Given two vectors \(x, y \in \mathbb{R}^N\), \(x \geq y\) if \(x_i \geq y_i\), for all \(i \in N\).

The set of *feasible payoff vectors* of \((N, v)\) is defined by \(X^*(N, v) := \{x \in \mathbb{R}^N \mid x(N) \leq v(N)\}\), the *preimputation set* by \(X(N, v) := \{x \in \mathbb{R}^N \mid x(N) = v(N)\}\) and the set of *imputations* by \(I(N, v) := \{x \in X(N, v) \mid x_i \geq v(i)\}\) for all \(i \in N\). A game with a non-empty imputation set is called *essential*. We denote by \(\Gamma_E\) the class of essential games. The *core* of \((N, v)\) is the set of those imputations where each coalition obtains at least its worth, that is \(C(N, v) := \{x \in X(N, v) \mid x(S) \geq v(S)\}\) for all \(S \subseteq N\). A game \((N, v)\) is *balanced* if it has a non-empty core. By \(\Gamma_B\) we denote the class of balanced games. A game \((N, v)\) is *convex* if \(v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\) for all \(S, T \subseteq N\). We denote by \(\Gamma_C\) the class of convex games.

A *solution* on a class of games \(\Gamma' \subseteq \Gamma\) is a correspondence \(\sigma\) that associates with each game \((N, v) \in \Gamma'\) a subset of feasible payoff vectors \(\sigma(N, v)\) of \(X^*(N, v)\). A solution \(\sigma\) on \(\Gamma' \subseteq \Gamma\) satisfies *efficiency* if for all \((N, v) \in \Gamma'\), \(\sigma(N, v) \subseteq X(N, v)\). A solution \(\sigma\) on \(\Gamma' \subseteq \Gamma\) is said to be *single-valued* if \(|\sigma(N, v)| = 1\) for all \((N, v) \in \Gamma'\).

For our purposes, we introduce some well-known single-valued solutions defined on \(\Gamma\). Let \(N \in \mathcal{N}\) and \((N, v) \in \Gamma\). The *Shapley value*, \(Sh_i\), is defined by

\[
Sh_i(N, v) := \sum_{S \subseteq N \setminus \{i\}} \frac{|S|! (|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S)) \quad \text{for all } i \in N.
\]

Let \(N \in \mathcal{N}\) and \((N, v) \in \Gamma\). With any preimputation \(x \in X(N, v)\) we associate the vector of all excesses \(e(S, x) = v(S) - x(S), \emptyset \neq S \subset N\), the components of which are non-increasingly ordered. The *prenucleolus*, \(\nu_\ast\), is the preimputation that minimizes with respect to the lexicographic order\(^5\) the vector of excesses over the set of preimputations. With any preimputation \(x \in X(N, v)\) we associate the vector of all per-capita excesses \(\bar{e}(S, x) = \frac{v(S) - x(S)}{|S|}, \emptyset \neq S \subset N\), the components of which are non-increasingly ordered. The *per-capita prenucleolus*, \(\bar{\nu}_\ast\), is the preimputation that minimizes with respect to the lexicographic order the vector of per-capita excesses over the set of preimputations. The *equal division solution*, \(ED\), is defined by \(ED_i(N, v) = \frac{v(N)}{|N|}\) for all \(i \in N\). The *center of imputations* solution, \(CI\), is defined by\(^6\)

\[
CI_i(N, v) := v(i) + \left(\frac{1}{|N|}\right) \left(v(N) - \sum_{i \in N} v(i)\right) \quad \text{for all } i \in N.
\]

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\(^5\) Given two vector \(x, y \in \mathbb{R}^N\), we say that \(x \leq y\) if either \(x = y\), or \(x_1 < y_1\) or there exists \(k \in \{2, \ldots, |N|\}\) such that \(x_i = y_i\) for all \(1 \leq i \leq k - 1\) and \(x_k < y_k\).

\(^6\) For simplicity of notation, and if no confusion arises, we write \(v(i), v(ij), \ldots\) instead of \(v(\{i\}), v(\{i, j\}), \ldots\)
The weighted center of imputations solution, $CI^w$, relative to a list of positive weights $w = (w_i)_{i \in \mathbb{N}} \in \mathbb{R}^+_{++}$ is defined by

$$CI^w_i(N, v) := v(i) + \frac{w_i}{\sum_{j \in N} w_j} \left(v(N) - \sum_{i \in N} v(i)\right)$$

for all $i \in N$.

Note that when $w_i = 1$ for all $i \in \mathbb{N}$, then $CI^w(N, v) = CI(N, v)$.

Next, we define a single-valued solution similar to that provided by Calleja et al. (2012). Let $N \in \mathcal{N}$ and $(N, v) \in \Gamma$. By $(N, v_r)$ we denote the balanced game with smallest worth for the grand coalition such that $v_r(S) = v(S)$ for all $S \subset N$. The single-valued solution $\varphi$ is defined by

$$\varphi_i(N, v) := \begin{cases} 
    v(i) + \frac{v(N) - \sum_{i \in N} v(i)}{v_r(N) - \sum_{i \in N} v(i)} (v_{s_i}(N, v_r) - v(i)) & \text{if } v_r(N) \neq \sum_{i \in N} v(i), \\
    CI_i(N, v) & \text{otherwise},
\end{cases}$$

(1)

for all $i \in N$.

Given a game $(N, v)$, $\varphi(N, v)$ is the unique efficient allocation on the straight line joining $v_{s}(N, v_r)$ and $(v(1), \ldots, v(n))$, except when these two allocations coincide, which happens when $v_r(N) = \sum_{i \in N} v(i)$. In this last case, $\varphi(N, v) = CI(N, v)$.

3 Properties

Our main goal is to determine whether or not it is possible, for single-valued solutions, to make a property of rationality compatible with a property of consistency and a property of monotonicity.

By rationality, we refer to properties that allude to the idea that players will accept a payoff if they cannot achieve a higher utility by refusing to cooperate.

A single-valued solution $\sigma$ on $\Gamma' \subseteq \Gamma$ satisfies

- Individual rationality (IR): if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$ with $I(N, v) \neq \emptyset$, it holds $\sigma_i(N, v) \geq v(i)$ for all $i \in N$.
- Core selection (CS): if for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$ with $C(N, v) \neq \emptyset$, it holds $\sigma(N, v) \in C(N, v)$.

Individual rationality means that no single player can improve the payoff proposed by the solution without cooperation, while core selection extends this impossibility to any coalition. Clearly, individual rationality does not imply core selection. Moreover, core selection does not imply individual rationality since a single-valued solution may select a non-individually rational allocation in an essential game with an empty core.

Consistency is a sort of internal stability requirement that relates the solution of a game to the solution of the game that appears when some agents leave. The variety of ways in which the remaining agents evaluate the possible coalitions give rise to different notions of reduced game. We focus on three types of reduced game widely used. The terminology is taken from Thomson (2003).
Let $N \in \mathcal{N}$, $(N, v) \in \Gamma$, $x \in \mathbb{R}^N$ and $\emptyset \neq N' \subset N$.

1. The max reduced game (Davis and Maschler 1965) relative to $N'$ at $x$ is the game

$\left( N', r_{M,x}^N(v) \right)$ defined by

$$r_{M,x}^N(v)(S) := \begin{cases} 0 & \text{if } S = \emptyset, \\ \max_{Q \subseteq N \setminus N'} \{ v(S \cup Q) - x(Q) \} & \text{if } \emptyset \neq S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases}$$

2. The complement reduced game (Moulin 1985) relative to $N'$ at $x$ is the game

$\left( N', r_{C,x}^N(v) \right)$ defined by

$$r_{C,x}^N(v)(S) := \begin{cases} 0 & \text{if } S = \emptyset, \\ v(S \cup N \setminus N') - x(N \setminus N') & \text{if } \emptyset \neq S \subseteq N'. \end{cases}$$

3. The projected reduced game (Funaki 1998) relative to $N'$ at $x$ is the game

$\left( N', r_{P,x}^N(v) \right)$ defined by

$$r_{P,x}^N(v)(S) := \begin{cases} v(S) & \text{if } S \subset N', \\ v(N) - x(N \setminus N') & \text{if } S = N'. \end{cases}$$

In the max-reduced game (relative to $N'$ at $x$), the worth of a coalition $S \subset N'$ is determined under the assumption that $S$ can choose the best partners in $N \setminus N'$, provided they are paid according to $x$. The complement and the projected reduced games represent the two extreme cases. In the complement reduced game each coalition $S \subset N'$ is required to join all the members of $N \setminus N'$ while, by contrast, in the projected reduced game when players in $N \setminus N'$ leave the game, for a proper subcoalition $S \subset N'$ cooperation is no longer possible with them.

Different notions of consistency rely on the above definitions of reduced game.

A single-valued solution $\sigma$ on $\Gamma' \subseteq \Gamma$ satisfies

- Max consistency (M-CON): if for all $N \in \mathcal{N}'$, $(N, v) \in \Gamma'$, all $\emptyset \neq N' \subset N$, and $x = \sigma(N, v)$, then $\left( N', r_{M,x}^N(v) \right) \in \Gamma'$ and $x_{|N'} = \sigma \left( N', r_{M,x}^N(v) \right)$.

- Complement consistency (C-CON): if for all $N \in \mathcal{N}'$, $(N, v) \in \Gamma'$, all $\emptyset \neq N' \subset N$, and $x = \sigma(N, v)$, then $\left( N', r_{C,x}^N(v) \right) \in \Gamma'$ and $x_{|N'} = \sigma \left( N', r_{C,x}^N(v) \right)$.

- Projected consistency (P-CON): if for all $N \in \mathcal{N}'$, $(N, v) \in \Gamma'$, all $\emptyset \neq N' \subset N$, and $x = \sigma(N, v)$, then $\left( N', r_{P,x}^N(v) \right) \in \Gamma'$ and $x_{|N'} = \sigma \left( N', r_{P,x}^N(v) \right)$.

Roughly speaking, a solution is consistent if it assigns the same payoff to players in both the original game and the reduced game.

Finally, we consider monotonicity properties with respect to the worth of the grand coalition.

A single-valued solution $\sigma$ on $\Gamma' \subseteq \Gamma$ satisfies
– Aggregate monotonicity (AM): if for all \( N \in \mathcal{N} \) and all \((N, v), (N, v') \in \Gamma'\) with 
\[ v(S) = v'(S) \] for all \( S \subseteq N \) and \( v(N) < v'(N) \), it holds \( \sigma(N, v) \leq \sigma(N, v') \).

Aggregate monotonicity states that nobody is worse off whenever the worth of the grand coalition increases, while the worth of every other coalition remains unchanged.

In examining the compatibility of rationality, consistency and monotonicity for single-valued solutions, the idea of regular aggregate monotonicity will be of help. Assume that whenever a set of players \( N \in \mathcal{N} \) agree on how to distribute monotonically an amount \( t \in \mathbb{R} \), representing the difference of the worth of the grand coalitions between two games, they will respect this agreement by following the same rule, regardless of the games they eventually play. A monotone path is just a complete list of these monotonic agreements.

**Definition 1** A monotone path is a function \( f : \mathcal{N} \times \mathbb{R} \to \bigcup_{N \in \mathcal{N}} \mathbb{R}^N \) satisfying the following conditions: for all \( N \in \mathcal{N} \) and all \( t \in \mathbb{R} \),

(i) \( f(N, 0) = (0, \ldots, 0) \in \mathbb{R}^N \),
(ii) \( f(N, t) \in \mathbb{R}^N \) and \( \sum_{i \in N} f_i(N, t) = t \),
(iii) if \( t' \in \mathbb{R} \) is such that \( t' > t \), then \( f_i(N, t') \geq f_i(N, t) \) for all \( i \in N \).

Note that a monotone path assigns non-negative (non-positive) vectors to positive (negative) real numbers.

Let \( \mathcal{F}_{mon} \) denote the class of functions satisfying the above conditions. Examples of functions in \( \mathcal{F}_{mon} \) that will be used in the rest of the paper are:

1. For all \( N \in \mathcal{N} \), all \( t \in \mathbb{R} \) and all \( i \in N \), define \( \tilde{f}i(N, t) = \frac{t}{|N|} \).

   \( \tilde{f} \) distributes \( t \) equally among players in \( N \).

2. Let \( w \in \mathbb{R}_{++}^N \) be a list of positive weights. For all \( N \in \mathcal{N} \), all \( t \in \mathbb{R} \) and all \( i \in N \), define \( f^w_i(N, t) = \frac{w_i t}{\sum_{j \in N} w_j} \).

   \( f^w \) distributes \( t \) among players in \( N \) proportionally according to their weights \( w \).

3. Let \( \pi \) be a permutation on \( N \). For all \( N \in \mathcal{N} \) and all \( t \in \mathbb{R} \), define \( f^\pi(N, t) = t \cdot e_{\pi(1)} \).

   where \( j \in N \) is such that \( \pi(j) \geq \pi(i) \) for all \( i \in N \).

\( f^\pi \) assigns all of amount \( t \) to the last player in \( N \) according to \( \pi \).

By using the notion of a monotone path, we introduce a new monotonicity property.

A single-valued solution \( \sigma \) on \( \Gamma' \subseteq \Gamma \) satisfies

– Regular aggregate monotonicity (RAM): if there exists a monotone path \( f \in \mathcal{F}_{mon} \) such that, for all \( N \in \mathcal{N} \) and all \((N, v), (N, v') \in \Gamma'\) with \( v(S) = v'(S) \) for all \( S \subseteq N \), it holds \( \sigma(N, v') - \sigma(N, v) = f(N, v'(N) - v(N)) \).

Regular aggregate monotonicity requires that any set of players \( N \in \mathcal{N} \) reaches an agreement (which can be different for different sets) on how to distribute monotonically any change in the worth of the grand coalition.

When a single-valued solution satisfies regular aggregate monotonicity according to \( f = \tilde{f} \), we say that it satisfies equal surplus division (ESD).\(^7\) Remarkably, most

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\(^7\) Properties related with equal surplus division are weak fairness (van den Brink et al. 2016) and strong aggregate monotonicity (Arin 2013). These two properties, together with efficiency, imply equal surplus division.
single-valued solutions satisfying aggregate monotonicity, such as $Sh$, $\bar{v}_s$, $ED$ and $CI$, also satisfy equal surplus division. Clearly, regular aggregate monotonicity implies aggregate monotonicity and it is implied by equal surplus division.

4 Rationality and monotonicity or consistency

In this section, we examine how well two of these three groups of properties combine. It turns out that combining monotonicity with consistency properties is always possible, since $ED$ satisfies equal surplus division (and, therefore, regular aggregate monotonicity and aggregate monotonicity), max consistency, complement consistency and projected consistency.

If we focus on how well properties of rationality combine with properties of monotonicity, note that $core$ selection and equal surplus division are compatible since $\bar{v}_s$ satisfies both properties, and individual rationality and equal surplus division by means of $CI$. In fact, these last two properties characterize $CI$.

Theorem 1 The center of imputations solution is the unique single-valued solution on $\Gamma$ that satisfies individual rationality and equal surplus division.

The proof of Theorem 1 is not difficult and can be obtained following similar arguments as those in the proof of Lemma 2 (see Sect. 5) by replacing regular aggregate monotonicity by equal surplus division. Clearly, individual rationality and equal surplus division are independent on $\Gamma$. Indeed, $CI$ can also be characterized by using regular aggregate monotonicity together with individual rationality and symmetry.

A single-valued solution $\sigma$ on $\Gamma' \subseteq \Gamma$ satisfies

- Symmetry\(^8\) (SYM): if for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma'$ and all $i, j \in N$, if $v(S U i) = v(S U j)$ for all $S \subseteq N \setminus \{i, j\}$, then $\sigma_i(N, v) = \sigma_j(N, v)$.

Symmetry demands that players who are interchangeable for any coalition not containing them obtain the same payoff. Not surprisingly, regular aggregate monotonicity and symmetry imply equal surplus division.

Lemma 1 On the domain of all games $\Gamma$, regular aggregate monotonicity together with symmetry imply equal surplus division.

Proof Let $\sigma$ be a single-valued solution on $\Gamma$ satisfying regular aggregate monotonicity and symmetry. Let $N \in \mathcal{N}, t \in \mathbb{R}$ and $(N, v)$ be a symmetric game.\(^9\) Now, consider the game $(N, v')$ defined by $v' = v + t \cdot u_N$. By regular aggregate monotonicity, there exists $f \in \mathcal{F}_{mon}$ such that $\sigma(N, v') = \sigma(N, v) + f(N, t)$. By symmetry, and taking into account that $\sum_{i \in N} f_i(N, t) = t$, it follows $f(N, t) = \left(\frac{t}{|N|}, \ldots, \frac{t}{|N|}\right)$, which means that $\sigma$ satisfies equal surplus division. □

Since $CI$ satisfies symmetry, Theorem 1 and Lemma 1 lead to the next characterization.

\(^8\) This property is also referred to as equal treatment of equals.

\(^9\) A game $(N, v)$ is symmetric if for all $S, T \subseteq N$ with $|S| = |T|$, $v(S) = v(T)$. 

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Table 1: Compatibilities of rationality and monotonicity on $\Gamma$

|          | CS | IR  | CS + IR |
|----------|----|-----|---------|
| ESD      | Yes ($\tilde{v}_k$) | Yes (Th. 1) | No (Th. 3) |
| RAM      | Yes ($\tilde{v}_k$) | Yes (Th. 1 or Th. 2) | No (Th. 3) |
| AM       | Yes ($\tilde{v}_k$) | Yes (Th. 1 or Th. 2) | Yes ($\varphi$) |

**Theorem 2** The center of imputations solution is the unique single-valued solution on $\Gamma$ that satisfies individual rationality, regular aggregate monotonicity and symmetry.

The properties in Theorem 2 are independent. 10

Although core selection, individual rationality and aggregate monotonicity are compatible by considering $\varphi$ (see expression (1)), the next result shows that core selection and individual rationality are incompatible with regular aggregate monotonicity.

**Theorem 3** There is no single-valued solution on $\Gamma$ that satisfies core selection, individual rationality and regular aggregate monotonicity.

**Proof** Suppose, on the contrary, that there exists a single-valued solution $\sigma$ satisfying core selection, individual rationality and regular aggregate monotonicity on $\Gamma$.

Let $(N, v^1)$ be a game with set of players $N = \{1, 2, 3\}$ and characteristic function: $v^1(12) = v^1(13) = v^1(123) = 1$ and $v^1(S) = 0$ otherwise. Clearly, $C(N, v^1) = \{(1, 0, 0)\}$ and, by core selection, $\sigma(N, v^1) = (1, 0, 0)$. Now, define the game $(N, (v^1)^I)$ as $(v^1)^I = v^1 + (-1) \cdot u_N$. Since $I(N, (v^1)^I) = \{(0, 0, 0)\}$, by individual rationality (and feasibility), $\sigma(N, (v^1)^I) = (0, 0, 0)$. By regular aggregate monotonicity, there exists $f \in F_{mon}$ such that $\sigma(N, (v^1)^I) - \sigma(N, v^1) = f(N, -1)$, from which it follows $f(N, -1) = (-1, 0, 0)$.

Let us now consider the game $(N, v^2)$ with set of players $N = \{1, 2, 3\}$ and characteristic function $v^2(12) = v^2(23) = v^2(123) = 1$ and $v^2(S) = 0$ otherwise. Let $(N, (v^2)^I)$ be defined as $(v^2)^I = v^2 + (-1) \cdot u_N$. Following a similar line of argument to that used before we have, by core selection, $\sigma(N, v^2) = (0, 1, 0)$, by individual rationality (and feasibility), $\sigma(N, (v^2)^I) = (0, 0, 0)$ and by regular aggregate monotonicity, $f(N, -1) = (0, -1, 0)$, in contradiction with $f(N, -1) = (-1, 0, 0)$.

Table 1 summarizes the (in)compatibilities of properties of rationality combined with properties of monotonicity for single-valued solutions on $\Gamma$.

Finally, we study whether individual rationality and core selection are compatible with any of the notions of consistency considered in Sect. 3. Although formally it would be admissible to consider two consistency properties simultaneously, from a behavioral point of view it makes no sense for players to agree in two different ways to reduce the game.

10 ED satisfies regular aggregate monotonicity and symmetry but not individual rationality, $\varphi$ (see expression (1)) satisfies individual rationality and symmetry but not regular aggregate monotonicity and, for a suitable list of positive weights $w$, $C1^w$ satisfies individual rationality and regular aggregate monotonicity but not symmetry.
On the domain of all games \( \Gamma \), \( v_\pi \) satisfies core selection and max consistency (Potters 1991). Moreover, the core has been characterized using max consistency (Peleg 1986), complement consistency (Tadenuma 1992) and projected consistency (Llerena and Rafels 2007). But, surprisingly, for single-valued solutions core selection together with either complement consistency or projected consistency are incompatible.

**Theorem 4** There is no single-valued solution on \( \Gamma \) that satisfies core selection and complement consistency.

**Proof** Suppose, on the contrary, that there exists a single-valued solution \( \sigma \) satisfying core selection and complement consistency on \( \Gamma \). Let \( (N, v^1) \) and \( (N, v^2) \) be two balanced games with set of players \( N = \{1, 2, 3\} \) and characteristic functions as follow: \( v^1(1) = v^2(2) = 0, v^1(2) = v^2(1) = 1 \) and \( v^1(S) = v^2(S) \) for any other \( S \subseteq N \), being \( v^1(3) = 1, v^1(12) = 0, v^1(13) = v^1(23) = 1 \) and \( v^1(N) = 2 \). Notice that \( C(N, v^1) = \{(0, 1, 1)\} \) and \( C(N, v^2) = \{(1, 0, 1)\} \). By core selection, \( \sigma(N, v^1) = (0, 1, 1) \) and \( \sigma(N, v^2) = (1, 0, 1) \). Let \( N' = \{1, 2\} \). By complement consistency,

\[
\sigma \left( N', r_{C,(0,1,1)}^{N'}(v^1) \right) = (0, 1) \in C \left( N', r_{C,(0,1,1)}^{N'}(v^1) \right)
\]

and

\[
\sigma \left( N', r_{C,(1,0,1)}^{N'}(v^2) \right) = (1, 0) \in C \left( N', r_{C,(1,0,1)}^{N'}(v^2) \right).
\]

But \( \left( N', r_{C,(0,1,1)}^{N'}(v^1) \right) = \left( N', r_{C,(1,0,1)}^{N'}(v^2) \right) \), which leads to a contradiction. \( \square \)

**Theorem 5** There is no single-valued solution on \( \Gamma \) that satisfies core selection and projected consistency.

**Proof** Suppose, on the contrary, that there exists a single-valued solution \( \sigma \) satisfying core selection and projected consistency on \( \Gamma \). Let \( (N, v^1) \) and \( (N, v^2) \) be two balanced games with set of players \( N = \{1, 2, 3\} \) and characteristic functions as follow: \( v^1(i) = 0 \) for all \( i \in N \), \( v^1(12) = v^1(13) = v^1(23) = 1 \) and \( v^2(23) = 0 \); \( v^2(S) = v^1(S) \) for \( S \neq \{13\} \) and \( S \neq \{23\} \). Notice that \( C(N, v^1) = \{(1, 0, 0)\} \) and \( C(N, v^2) = \{(0, 1, 0)\} \). By core selection, \( \sigma(N, v^1) = (1, 0, 0) \) and \( \sigma(N, v^2) = (0, 1, 0) \). Let \( N' = \{1, 2\} \). By projected consistency,

\[
\sigma \left( N', r_{P,(1,0,0)}^{N'}(v^1) \right) = (1, 0) \in C \left( N', r_{P,(1,0,0)}^{N'}(v^1) \right)
\]

and

\[
\sigma \left( N', r_{P,(0,1,0)}^{N'}(v^2) \right) = (0, 1) \in C \left( N', r_{P,(0,1,0)}^{N'}(v^2) \right).
\]

But \( \left( N', r_{P,(1,0,0)}^{N'}(v^1) \right) = \left( N', r_{P,(0,1,0)}^{N'}(v^2) \right) \), which leads to a contradiction. \( \square \)
Table 2  Compatibilities of rationality and consistency on $\Gamma$

|        | CS       | IR                  |
|--------|----------|---------------------|
| M-CON  | Yes ($v_*$) | No (Hwang 2013)    |
| C-CON  | No (Th. 4) | No (Hwang 2013)    |
| P-CON  | No (Th. 5) | Yes (Th. 9)        |

If we replace core selection by individual rationality, Hwang (2013) shows that there is no single-valued solution satisfying simultaneously individual rationality and max consistency, or individual rationality and complement consistency on $\Gamma$.$^{11}$ Thus, combining Theorem 5 and Hwang’s results we obtain the following corollary.

**Corollary 1** There is no single-valued solution on $\Gamma$ that satisfies core selection, individual rationality and either max consistency or complement consistency or projected consistency.

In order to fulfill all possible combinations we point out that individual rationality and projected consistency are compatible by considering CI. This result emerges as a consequence of Theorem 9 in Sect. 5.

Table 2 summarizes the (in)compatibilities of properties of rationality combined with properties of consistency for single-valued solutions on $\Gamma$.

**Remark 1** The results in this section, summarized in Tables 1 and 2, are stated on the domain of all games $\Gamma$. However, Theorems 1, 2, 3 and Corollary 1 hold on the domain of essential games $\Gamma_E$, and Theorems 4 and 5 hold on the domain of balanced games $\Gamma_B$.

5 **Rationality, monotonicity and consistency**

The main concern of this section is to determine whether it is possible to combine, for a single-valued solution, a property of rationality with a property of monotonicity and a property of consistency.

Table 2 above shows that core selection can only be combined with max consistency, and individual rationality with projected consistency. We begin by focusing on the compatibility of core selection, max consistency and either equal surplus division or (regular) aggregate monotonicity.

5.1 (Im)possibility results

First, it is worth noting that core selection, max consistency and either equal surplus division or (regular) aggregate monotonicity are compatible in combinations of two properties. For instance, $v_*$ satisfies core selection and max consistency, $\bar{v}_*$

---

$^{11}$ Hwang (2013) uses conditional individual rationality, which states that for all $N \in \mathcal{N}$ and all $(N, v) \in \Gamma'$ with $I(N, v) \neq \emptyset$, it holds $\sigma(N, v) \in I(N, v)$. However, his impossibility results (Theorem 5 and Theorem 6) also hold replacing conditional individual rationality by individual rationality (and feasibility).
satisfies core selection and equal surplus division (and, therefore, regular aggregate monotonicity and aggregate monotonicity) and ED satisfies equal surplus division and max consistency. Unfortunately, we encounter problems when trying to combine all these properties simultaneously.

**Theorem 6** There is no single-valued solution on $\Gamma$ that satisfies core selection, regular aggregate monotonicity and max consistency.

**Proof** Suppose, on the contrary, that there exists a single-valued solution $\sigma$ satisfying core selection, regular aggregate monotonicity and max consistency on $\Gamma$.

Let $\{(N, v^k)\}_{k \in N}$ be a family of balanced games defined as follows: for each $k \in N = \{1, 2, 3\}$, $v^k(i) = 0$ for all $i \in N$, $v^k(ik) = 1$ for all $i \in N \setminus \{k\}$, $v^k(ij) = 0$ for all $i, j \in N \setminus \{k\}$, and $v^k(N) = 3$. For each $k \in N$, let $(N, v^k_\ast)$ be given by

$$v^k_\ast = v^k - 2 \cdot u_N.$$  

By regular aggregate monotonicity, there exists $f \in \mathcal{F}_{mon}$ such that $\sigma(N, v^1) - \sigma(N, v^1_\ast) = f(N, v^1(N) - v^1_\ast(N))$. Since $C(N, v^1_\ast) = \{(1, 0, 0)\}$, by core selection, $\sigma(N, v^1_\ast) = (1, 0, 0)$ and thus $\sigma(N, v^1) = (1, 0, 0) + f(N, 2)$. Following similar arguments, and taking into account that $C(N, v^2_\ast) = \{(0, 1, 0)\}$ and $C(N, v^3_\ast) = \{(0, 0, 1)\}$, we can write

$$\sigma(N, v^2) = (0, 1, 0) + f(N, 2) \quad \text{and} \quad \sigma(N, v^3) = (0, 0, 1) + f(N, 2). \quad (2)$$

We claim that $f_i(N, 2) = 0$ for all $i \in N$, in contradiction with $\sum_{i \in N} f_i(N, 2) = 2$.

First, we see that $f_1(N, 2) = 0$. Since $f \in \mathcal{F}_{mon}$, we have $f_1(N, 2) \geq 0$. Let us denote $x^k = \sigma(N, v^k)$, $k \in \{1, 2, 3\}$, and take $N' = \{2, 3\}$. Two cases can be distinguished:

- **Case 1**: $1 - f_1(N, 2) \leq 0$

  In this situation, for all $S \subseteq N'$, it holds $r^{N'}_{M,x^2}(v^2(S)) = r^{N'}_{M,x^3}(v^3(S))$. Thus,

  $$\sigma\left(N', r^{N'}_{M,x^2}(v^2)\right) = \sigma\left(N', r^{N'}_{M,x^3}(v^3)\right). \quad (3)$$

  By max consistency, and considering expression (2), we obtain

  $$\sigma_2\left(N', r^{N'}_{M,x^2}(v^2)\right) = 1 + f_2(N, 2) \quad \text{and} \quad \sigma_2\left(N', r^{N'}_{M,x^3}(v^3)\right) = f_2(N, 2),$$

  in contradiction with (3).

- **Case 2**: $1 - f_1(N, 2) > 0$

  In this situation, it holds

  $$r^{N'}_{M,x^2}(v^2)(2) = r^{N'}_{M,x^3}(v^3)(3) = 1 - f_1(N, 2),$$

  $$r^{N'}_{M,x^2}(v^2)(3) = r^{N'}_{M,x^3}(v^3)(2) = 0 \quad \text{and} \quad r^{N'}_{M,x^2}(v^2)(N') = r^{N'}_{M,x^3}(v^3)(N') = 3 - f_1(N, 2).$$
For each $k \in N' = \{2, 3\}$, consider the game $\left(N', \left( r^{N'}_{M,x^k} (v^k) \right)_* \right)$ with characteristic function $\left( r^{N'}_{M,x^k} (v^k) \right)_* = r^{N'}_{M,x^k} (v^k) - 2 \cdot u_{N'}$. Thus,

$$C \left(N', \left( r^{N'}_{M,x^2} (v^2) \right)_* \right) = \{(1 - f_1(N, 2), 0)\} \text{ and}$$

$$C \left(N', \left( r^{N'}_{M,x^3} (v^3) \right)_* \right) = \{(0, 1 - f_1(N, 2))\}.$$

By core selection, we have

$$\sigma \left(N', \left( r^{N'}_{M,x^2} (v^2) \right)_* \right) = (1 - f_1(N, 2), 0) \text{ and}$$

$$\sigma \left(N', \left( r^{N'}_{M,x^3} (v^3) \right)_* \right) = (0, 1 - f_1(N, 2)).$$

By core selection and regular aggregate monotonicity, we can write:

$$\sigma \left(N', r^{N'}_{M,x^2} (v^2) \right) = (1 - f_1(N, 2), 0) + f(N', 2) \text{ and}$$

$$\sigma \left(N', r^{N'}_{M,x^3} (v^3) \right) = (0, 1 - f_1(N, 2)) + f(N', 2).$$

Now, applying max consistency and taking into account expression (2), we obtain $f_3(N, 2) = f_3(N', 2)$ and $f_2(N, 2) = f_2(N', 2)$. But due to $f \in F_{mon}$, we have $2 = \sum_{i \in N'} f_i(N', 2) = f_2(N, 2) + f_3(N, 2)$, which implies $f_1(N, 2) = 0$.

Following symmetric arguments for $N' = \{1, 3\}$ and $N' = \{1, 2\}$ we obtain $f_2(N, 2) = 0$ and $f_3(N, 2) = 0$, respectively, which proves the claim. But then, $\sum_{i \in N} f_i(N, 2) = 0$, in contradiction with $f \in F_{mon}$. □

Remark 2 It is not difficult to check that Theorem 6 remains valid on the domain of balanced games. Thus, core selection and max consistency can only be combined with aggregate monotonicity. On the domain of balanced games (and also on the domain of all games) with at most three players, core selection, aggregate monotonicity and max consistency are compatible by means of $\nu_\pi$ (Housman and Clark 1998). However, it is still an open question if these three properties are compatible on the domain of balanced games with at least four players.

Despite these negative results, we find some possibilities if we restrict our attention to convex games. Core selection, aggregate monotonicity and max consistency are compatible on $\Gamma_C$ by considering, for instance, the marginal contribution solution. Let $\pi$ be a permutation on $\mathbb{N}$, the marginal contribution solution relative to $\pi$, denoted by $mc^\pi$, is defined as follows: for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in N$

$$mc^\pi_i := v \left(\{ j \in N \mid \pi(j) \leq \pi(i) \}\right) - v \left(\{ j \in N \mid \pi(j) < \pi(i) \}\right).$$
It is not difficult to check that \( mc^\pi \) satisfies regular aggregate monotonicity (and thus aggregate monotonicity) on \( \Gamma_C \) according to \( f^\pi \) as defined in Sect. 3. Moreover, it is well-known that \( mc^\pi \) satisfies core selection and max consistency on \( \Gamma_C \) (see, for instance, Hokari and van Gellekom 2002). This leads to the following result.

**Theorem 7** Core selection, regular aggregate monotonicity and max consistency are compatible on \( \Gamma_C \).

By contrast, if we consider equal surplus division we find an impossibility.

**Theorem 8** There is no single-valued solution on \( \Gamma_C \) that satisfies core selection, equal surplus division and max consistency.

**Proof** Suppose, on the contrary, that there exists a single-valued solution \( \sigma \) satisfying core selection, equal surplus division and max consistency on \( \Gamma_C \).

Let \( (N, v) \) be the convex game with set of players \( N = \{1, 2, 3\} \) and characteristic function as follows: \( \forall i \in N \, v(i) = v(i23) = 1 \) and \( v(S) = 0 \) otherwise. Note that for all \( z \in C(N, v), z_3 = 0 \). Let us denote \( x = \sigma(N, v) \). By core selection, \( x_3 = 0 \). Take \( N' = \{1, 2\} \). Then, \( (N', r_{M,x}^{N'}(v)) \in \Gamma_C \) with \( r_{M,x}^{N'}(v)(1) = r_{M,x}^{N'}(v)(2) = 0 \) and \( r_{M,x}^{N'}(v)(N') = 1 \). Now, define the convex game \( (N', r_{M,x}^{N'}(v)) \) as \( r_{M,x}^{N'}(v) = r_{M,x}^{N'}(v) - u_{N'} \). By core selection, \( \sigma(N', r_{M,x}^{N'}(v)) = (0, 0) \). Thus, by equal surplus division \( \sigma(N', r_{M,x}^{N'}(v)) = \left( \frac{1}{2}, \frac{1}{2} \right) \) and, by max consistency, \( \sigma(N, v) = \left( \frac{1}{2}, \frac{1}{2}, 0 \right) \).

Now, let \( (N, v') \) be the convex game given by \( v' = v + 3 \cdot u_N \). By equal surplus division, \( \sigma(N, v') = \left( \frac{3}{2}, \frac{3}{2}, 1 \right) \in C(N, v') \). Take \( N' = \{1, 3\} \). Then, \( (N', r_{M,y}^{N'}(v')) \in \Gamma_C \) with \( r_{M,y}^{N'}(v')(1) = r_{M,y}^{N'}(v')(3) = 0 \) and \( r_{M,y}^{N'}(v')(N') = \frac{5}{2} \). Define the convex game \( (N', r_{M,y}^{N'}(v')) \) as \( r_{M,y}^{N'}(v') = r_{M,y}^{N'}(v') - \frac{5}{2} \cdot u_{N'} \). By core selection, \( \sigma(N', r_{M,y}^{N'}(v')) = (0, 0) \) and, by equal surplus division, \( \sigma(N', r_{M,y}^{N'}(v')) = \left( \frac{5}{4}, \frac{5}{4} \right) \neq \left( \frac{1}{2}, 1 \right) \), which contradicts max consistency. \( \Box \)

**5.2 Possibility results**

Compatibilities show up when combining individual rationality, projected consistency and different monotonicity properties. In this subsection, and for a clearer presentation, all proofs (and non-redundancies of the properties) are consigned to the “Appendix”.

We first characterize the family of single-valued solutions satisfying individual rationality, regular aggregate monotonicity and projected consistency. This family happens to be a generalization of \( CI \).
Definition 2 Let $f \in \mathcal{F}_{\text{mon}}$. The $f$–center of imputations, $CI^f$, is defined as follows: for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in N$, 

$$CI^f_i(N, v) := v(i) + f_i \left( N, v(N) - \sum_{i \in N} v(i) \right).$$

Given a monotone path $f$, $CI^f$ can be interpreted as a two-stage rule: after assigning to every player what they can achieve for themselves alone, it distributes monotonically (but not necessarily equally) what is left of the gains of cooperation. Note that if $f = \bar{f}$ or $f = f^w$ we recover $CI$ or $CI^w$, respectively.

Now, we introduce the notions of additive and consistent monotone paths.

Definition 3 A monotone path $f \in \mathcal{F}_{\text{mon}}$ is said to be additive if for all $N \in \mathcal{N}$ and all $t, t' \in \mathbb{R}$, it holds $f(N, t + t') = f(N, t) + f(N, t')$.

The next lemma, of help to prove Theorem 9, extends the characterization in Theorem 1 when we replace equal surplus division by regular aggregate monotonicity.

Lemma 2 A single-valued solution $\sigma$ satisfies individual rationality and regular aggregate monotonicity on $\Gamma$ if and only if there exists an additive monotone path $f \in \mathcal{F}_{\text{mon}}$ such that $\sigma = CI^f$.

The consistency of a monotone path guarantees that the agreements reached by a set of players are invariant with respect to a reduction in population.

Definition 4 A monotone path $f \in \mathcal{F}_{\text{mon}}$ is said to be consistent if for all $N \in \mathcal{N}$, all $\emptyset \neq N' \subset N$ and all $t \in \mathbb{R}$, it holds

$$f_j \left( N', \sum_{i \in N'} f_i(N, t) \right) = f_j(N, t)$$

for all $j \in N'$.

The idea to relate monotone paths to consistency properties was already used by Hokari (2002) to provide non-symmetric and non-homogeneous generalizations of the egalitarian solution of Dutta and Ray (1989) on the domain of convex games.

Now, we are ready to state our first characterization result.

Theorem 9 A single-valued solution $\sigma$ satisfies individual rationality, regular aggregate monotonicity and projected consistency on $\Gamma$ if and only if there exists an additive and consistent monotone path $f \in \mathcal{F}_{\text{mon}}$ such that $\sigma = CI^f$.

Instances of single-valued solutions satisfying individual rationality, regular aggregate monotonicity and projected consistency are $CI$ and $CI^w$, since $\bar{f}$ and $f^w$ are consistent and additive monotone paths. Consequently, as $CI$ is the unique single-valued solution that obeys individual rationality and equal surplus division (Theorem 1), these two properties together imply projected consistency.

On the other hand, it is not difficult to check that for a given consistent, but not additive monotone path $f$, $CI^f$ satisfies individual rationality, aggregate monotonicity...
(but not regular aggregate monotonicity) and projected consistency. However, there are other single-valued solutions which do not belong to the family of \( f \) — center of imputation solutions that also satisfy these three properties. Let us consider an example.

Example 1 Define the single-valued solution \( \rho \) as follows. For all \( N \in \mathcal{N} \), all \( (N, v) \in \Gamma \) and all \( i \in N \)

\[
\rho_i(N, v) := \begin{cases} 
  v(i) + \left( v(N) - \sum_{i \in N} v(i) \right) & \text{if } i = i^*, \\
  v(i) & \text{otherwise,}
\end{cases}
\]

where \( i^* \in N \) is such that \( v(i^*) \geq v(i) \), for all \( i \in N \), and \( i^* \leq j \), for all \( j \in N \) with \( v(i^*) = v(j) \).

It is easy to check that \( \rho \) satisfies individual rationality, aggregate monotonicity and projected consistency. However, there is no \( f \in \mathcal{F}_{\text{mon}} \) such that \( \rho = CI_f \). Indeed, consider the games \((N, v^1), (N, v^2)\) with player set \( N = \{1, 2\} \) and characteristic functions: \( v^1(1) = v^2(2) = 1, v^1(2) = v^2(1) = 0 \) and \( v^1(N) = v^2(N) = 2 \). Now define the associated games \((N, (v^1)^I) \) and \((N, (v^2)^I) \) as follows: \( (v^1)^I = v^1 - u_N \) and \( (v^2)^I = v^2 - u_N \). Clearly, (a) \( \rho_1(N, v^1) = 2, \rho_1(N, (v^1)^I) = 1 \) and (b) \( \rho_1(N, v^2) = \rho_1(N, (v^2)^I) = 0 \). Suppose, on the contrary, that there exists \( f \in \mathcal{F}_{\text{mon}} \) such that \( \rho = CI_f \). Then, from (a), \( f_1(N, 1) = 1 \) and, from (b), \( f_1(N, 1) = 0 \), resulting in a contradiction.

According to \( \rho \), players 1 and 2 reach different agreements on how to distribute \( t = 1 \) if they play game \( v^1 \) or \( v^2 \).

In view of Example 1, to characterize the class of single-valued solutions satisfying individual rationality, aggregate monotonicity and projected consistency, it is crucial that players have some freedom to reach different agreements to share \( t \), depending on the specificities of the game played.

Given \((N, v) \in \Gamma \), we define the associated game \((N, v^I)\) as follows: \( v^I(S) := v(S) \) for all \( S \subseteq N \), and \( v^I(N) := \sum_{i \in N} v(i) \). Notice that \((N, v)\) can be written as \( v = v^I + (v(N) - v^I(N)) \cdot u_N \). Let us denote \( \Gamma^I := \{(N, v) \in \Gamma \mid v(N) = \sum_{i \in N} v(i)\} \).

Definition 5 A monotone \( \Gamma^I - \text{selection} \) is a function \( F : \Gamma^I \to \mathcal{F}_{\text{mon}} \) that associates with each game \((N, v) \in \Gamma^I \) a monotone path \( F(N, v) \in \mathcal{F}_{\text{mon}} \).

For simplicity, and if there is no confusion, we denote \( F(N, v) \) by \( f^v \). Making use of the notion of a monotone \( \Gamma^I \)-selection, we introduce the \( F \)-center of imputation solutions.

Definition 6 Let \( F \) be a monotone \( \Gamma^I \)-selection. The \( F \)-center of imputations, \( CI^F \), is defined as follows: for all \( N \in \mathcal{N} \), all \( (N, v) \in \Gamma \) and all \( i \in N \),

\[
CI^F_i(N, v) := v(i) + f^v(N, v(N) - \sum_{i \in N} v(i)).
\]
Given a monotone $\Gamma_f$-selection $F$ and $(N, v) \in \Gamma_f$, $CIF(N, v)$ assigns to every player his individual worth and distributes the amount $v(N) - \sum_{i \in N} v(i)$ according to a monotone path $f^v$ which will depend on the associated game $(N, v^I)$. On the contrary, all $CIF$ allocate this amount by means of a fix monotone path $f$, that does not depend on the game $(N, v^I)$. We go back to Example 1 to illustrate this difference.

**Example 1 (continuation)** The single-valued solution $\rho$ is a particular $CIF$ according to the following monotone $\Gamma_f$-selection $F$. For all $(N, w) \in \Gamma_f$, let $F(N, w) = f^{\pi_w}$ where $f^{\pi_w}$ is the monotone path as defined in Sect. 3, being $\pi_w$ a permutation on $N$ such that $\pi_w(k) < \pi_w(i^*)$ for all $k \in N \setminus \{i^*\}$. For an arbitrary game $(N, v) \in \Gamma_f$, $\rho(N, v)$ allocates $v(N) - \sum_{i \in N} v(i)$ according to $f^{\pi_v}$ which depends on $(N, v^I)$. In particular, for the games $(N, v^1), (N, v^2)$ defined in Example 1 we have $f^{\pi_v}(N, 1) = (1, 0)$ and $f^{\pi_v}(N, 1) = (0, 1)$.

The next lemma extends the characterizations in Theorem 1 and Lemma 2 when we consider aggregate monotonicity instead of either equal surplus division or regular aggregate monotonicity, and it will be useful to prove Theorem 10.

**Lemma 3** A single-valued solution $\sigma$ satisfies efficiency, individual rationality and aggregate monotonicity on $\Gamma$ if and only if there exists a monotone $\Gamma_f$-selection $F$ such that $\sigma = CIF$.

Next, we generalize the notion of consistent monotone paths.

**Definition 7** A monotone $\Gamma_f$-selection $F$ is said to be consistent if for all $N \in N$, all $(N, v) \in \Gamma_f$, all $\emptyset \neq N' \subset N$ and all $t \in \mathbb{R}$, it holds

$$f_j^{(v|_{N'})}(N', \sum_{i \in N'} f_i^{v}(N, t)) = f_j^{v}(N, t) \quad \text{for all } j \in N',$$

being $(N', v|_{N'})$ the subgame of $(N, v)$ associated to $N'$.

The interpretation of a consistent monotone $\Gamma_f$-selection $F$ is as follows: assume a set of players $N$ agree on how to distribute $t$ according to the monotone path $f^v$ selected by $F$ to the game $(N, v) \in \Gamma_f$. Now, suppose that some players leave with their part of $t$. Consistency says that the remaining players $\emptyset \neq N' \subset N$ reach the same agreement to distribute what is left of $t$ according to the monotone path $f^{(v|_{N'})}$ selected by $F$ to the game $(N', (v|_{N'})^I) \in \Gamma_f$.

Now, we have all the tools to characterize the class of single-valued solutions satisfying individual rationality, aggregate monotonicity and projected consistency. Previously, we need to connect individual rationality, projected consistency and efficiency.

**Lemma 4** On the domain of all games $\Gamma$, individual rationality together with projected consistency imply efficiency.

**Theorem 10** A single-valued solution $\sigma$ satisfies individual rationality, aggregate monotonicity and projected consistency on $\Gamma$ if and only if there exists a consistent monotone $\Gamma_f$-selection $F$ such that $\sigma = CIF$. 

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Table 3  Compatibilities of rationality, consistency and monotonicity on $\Gamma$

|                | ESD | RAM | AM    |
|----------------|-----|-----|-------|
| IR + P-CON     | Yes (Th. 9) | Yes (Th. 9) | Yes (Th. 10) |
| CS + M-CON     | No (Th. 6)  | No (Th. 6)  | Open   |

An instance of a single-valued solution satisfying individual rationality, aggregate monotonicity and projected consistency is the solution $\rho$ defined in Example 1.

Table 3 summarizes the (in)compatibilities of properties of rationality combined with properties of consistency and properties of monotonicity for single-valued solutions on $\Gamma$.

6 Final remarks

For single-valued solutions, we study the compatibility between rationality properties (core selection and individual rationality), properties of monotonicity with respect to the worth of the grand coalition (equal surplus division, regular aggregate monotonicity and aggregate monotonicity) and consistency properties (max consistency, complement consistency and projected consistency). We characterize the set of single-valued solutions satisfying either individual rationality, regular aggregate monotonicity and projected consistency or individual rationality, aggregate monotonicity and projected consistency. We also show that core selection, regular aggregate monotonicity and max consistency are compatible on the domain of convex games but incompatible on the domain of all games, and also on the domain of balanced games. However, whether core selection, aggregate monotonicity and max consistency are compatible for games with at least four players is still an open question.

We have focused on three consistency properties often used in the axiomatic approach of solutions, showing that none of them are compatible with core selection and individual rationality together. A prominent consistency property is self consistency, invoked in Hart and Mas-Colell (1989) in order to characterize the Shapley value. Thus, in future works it might be interesting to study how well self consistency combines with rationality and monotonicity properties. Finally, it could be also worthwhile to analyze these (in)compatibilities considering other monotonicity properties, like strong monotonicity (Young 1985) or weak monotonicity (see van den Brink et al. 2013; Casajus and Huettner 2014b), that do not only apply to games that exclusively differ from the worth of the grand coalition.

Acknowledgements  We want to sincerely thank two anonymous referees for their useful comments and suggestions. All errors are our own. We also acknowledge the support from research grant ECO2014-52340-P (Ministerio de Economía y Competitividad), 2014SGR40 and 2014SGR631 (Generalitat de Catalunya).

Appendix

We begin proving lemmas. To give general arguments, Theorem 10 is proved before Theorem 9. Finally, we show the independence of the properties used in the characterization results.
Proof (Lemma 2) Let $\sigma$ be a single-valued solution satisfying individual rationality and regular aggregate monotonicity on $\Gamma$. Let $N \in \mathcal{N}$ and $(N, v) \in \Gamma$. Define $(N, v')$ as $v'(S) = v(S)$ for all $S \subseteq N$, and $v'(N) = \sum_{i \in N} v(i)$. By individual rationality (and feasibility), $\sigma_i(N, v') = v(i)$ for all $i \in N$. By regular aggregate monotonicity, there exists a monotone path $f \in \mathcal{F}_{\text{mon}}$ such that, for all $i \in N$,

$$\sigma_i(N, v) = v(i) + f_i \left( N, v(N) - \sum_{i \in N} v(i) \right).$$

Let us see that $f$ is additive. Let $N \in \mathcal{N}$ and $t, t' \in \mathbb{R}$. Consider three games $(N, v), (N, v')$ and $(N, v'')$ defined as follows: for all $S \subseteq N$, $v(S) = v'(S) = v''(S)$, $v(N) - v'(N) = t$ and $v'(N) - v''(N) = t'$. Then,

$$f(N, t + t') = f(N, (v'(N) + t) - (v'(N) - t')) = f(N, v(N) - v''(N)) = \sigma(N, v) - \sigma(N, v'') = \sigma(N, v) - \sigma(N, v') + \sigma(N, v') - \sigma(N, v'') = f(N, v(N) - v'(N)) + f(N, v'(N) - v''(N)) = f(N, t) + f(N, t'),$$

where third and fifth equalities follow from $\sigma$ satisfying regular aggregate monotonicity. Hence, $\sigma = CI f$ where $f \in \mathcal{F}_{\text{mon}}$ is additive.

To show the reverse implication, let $\sigma$ be a single-valued solution on $\Gamma$ such that $\sigma = CI f$, for some additive monotone path $f \in \mathcal{F}_{\text{mon}}$. Let $N \in \mathcal{N}$ and $(N, v)$ be an essential game. Then, $v(N) - \sum_{i \in N} v(i) \geq 0$ and thus, from the monotonicity of $f$, $f_i(N, v(N) - \sum_{i \in N} v(i)) \geq 0$, for all $i \in N$. Hence, $\sigma_i(N, v) = v(i) + f_i(N, v(N) - \sum_{i \in N} v(i)) \geq v(i)$ for all $i \in N$, which proves individual rationality.

To see regular aggregate monotonicity, let $N \in \mathcal{N}$ and consider two games $(N, v), (N, v')$ such that $v(S) = v'(S)$, for all $S \subseteq N$. Taking into account the additivity of $f$ we have, for all $i \in N$,

$$\sigma_i(N, v) - \sigma_i(N, v') = f_i \left( N, v(N) - \sum_{i \in N} v(i) \right) - f_i \left( N, v'(N) - \sum_{i \in N} v(i) \right) = f_i \left( N, v(N) - \sum_{i \in N} v(i) \right) + f_i \left( N, v'(N) - \sum_{i \in N} v(i) \right).$$

This, together with $f \in \mathcal{F}_{\text{mon}}$, proves regular aggregate monotonicity. \qed

Proof (Lemma 3) Let $\sigma$ be a single-valued solution satisfying efficiency, individual rationality and aggregate monotonicity on $\Gamma$. \qed
Define a monotone $\Gamma_I$ – selection $F$ as follows. Take $f \in \mathcal{F}_{\text{mon}}$. For all $N' \in \mathcal{N}$ and all $(N', v) \in \Gamma_I$, define

$$f^v(N, t) := \begin{cases} \sigma(N, v + t \cdot u_N) - \sigma(N, v) & \text{if } N = N', \\ f(N, t) & \text{if } N \neq N', \end{cases}$$

for all $N \in \mathcal{N}$ and all $t \in \mathbb{R}$.

Let us show that $f^v \in \mathcal{F}_{\text{mon}}$. Clearly, $f^v(N, 0) = (0, \ldots, 0) \in \mathbb{R}^\mathcal{N}$ and, by the efficiency of $\sigma$ and the definition of $f \in \mathcal{F}_{\text{mon}}$, \(\sum_{i \in \mathcal{N}} f^v_i(N, t) = t\), for all $N \in \mathcal{N}$ and $t \in \mathbb{R}$. Moreover, if $t' \in \mathbb{R}$ is such that $t' > t$ and $N = N'$, then

$$f^v(N, t') = \sigma(N, v + t' \cdot u_N) - \sigma(N, v) \geq \sigma(N, v + t \cdot u_N) - \sigma(N, v) = f^v(N, t)$$

where the inequality follows from the aggregate monotonicity of $\sigma$. If $N \neq N'$, we have $f(N, t') \geq f(N, t)$ since $f \in \mathcal{F}_{\text{mon}}$. Consequently, $f^v \in \mathcal{F}_{\text{mon}}$ for all $(N', v) \in \Gamma_I$, and thus $F$ is a monotone $\Gamma_I$ – selection.

Let $N \in \mathcal{N}$ and $(N, v) \in \Gamma$. Since $(N, v)$ can be expressed as $v = v^I + (v(N) - v^I(N)) \cdot u_N$, it is easy to see that

$$\sigma(N, v) - \sigma(N, v^I) = f^v^I\left(N, v(N) - \sum_{i \in \mathcal{N}} v(i)\right).$$

being $f^v^I$ as defined in (5).

By individual rationality (and feasibility), $\sigma_i\left(N, v^I\right) = v(i)$ for all $i \in \mathcal{N}$, and thus

$$\sigma_i\left(N, v\right) = v(i) + f^v_i\left(N, v(N) - \sum_{i \in \mathcal{N}} v(i)\right).$$

Hence, $\sigma = CI^F$ being $F$ a monotone $\Gamma_I$-selection.

To show the reverse implication, let $\sigma$ be a single-valued solution on $\Gamma$ such that $\sigma = CI^F$, for some monotone $\Gamma_I$ – selection $F$. Hence, for all $N \in \mathcal{N}$, all $(N, v) \in \Gamma$ and all $i \in \mathcal{N}$ it holds

$$\sigma_i(N, v) = v(i) + f^v_i\left(N, v(N) - \sum_{i \in \mathcal{N}} v(i)\right).$$

From (6) it is not difficult to check individual rationality by using symmetric arguments as in the proof of Lemma 2. Efficiency and aggregate monotonicity comes from $f^v^I \in \mathcal{F}_{\text{mon}}$.

**Proof (Lemma) 4** Let $\sigma$ be a single-valued satisfying individual rationality and projected consistency on $\Gamma$. For one person games, efficiency follows directly from
individual rationality (and feasibility). Let \( N \in \mathcal{N} \) with \( |N| \geq 2 \), \((N, v) \in \Gamma \) and \( i \in N \). Then, efficiency for one person games implies \( \sigma_i([i], r_{P,x}^{[i]}(v)) = r_{P,x}^{[i]}(v)i = v(N) - \sum_{j \in N \setminus [i]} \sigma_j(N, v) \), where \( x = \sigma(N, v) \). By projected consistency, \( \sigma_i(N, v) = \sigma_i([i], r_{P,x}^{[i]}(v)) \) and thus \( \sigma_i(N, v) = v(N) - \sum_{j \in N \setminus [i]} \sigma_j(N, v) \), which proves efficiency.

**Proof (Theorem 10)** Let \( \sigma \) be a single-valued solution satisfying individual rationality, aggregate monotonicity and projected consistency on \( \Gamma \). By Lemma 4, \( \sigma \) satisfies efficiency, and by Lemma 3 we conclude that \( \sigma = CI^F \) for some monotone \( \Gamma_l \)-selection \( F \). To show that \( F \) is consistent, let \( N \in \mathcal{N}, (N, v) \in \Gamma_l, \emptyset \neq N' \subset N \), and \( t \in \mathbb{R} \). Define the game \((N, v')\) as \( v' = v + t \cdot u_N \). Notice that \((v')^I = v \). Let us denote \( x = \sigma(N, v') \). Then, for all \( j \in N' \) we have

\[
\sigma_j(N, v') = v'(j) + f_j(v') \left( N, v'(N) - \sum_{i \in N} v'(i) \right) = v'(j) + f_j(v)(N, t). \tag{7}
\]

From (7), and taking into account the efficiency of \( \sigma \), the definition of projected reduced game and the fact that \( \left(N', \left(r_{N',x}^{N'}(v')\right)^I\right) = \left(N', (v|_{N'})^I\right) \), we obtain

\[
\sigma_j \left(N', r_{N',x}^{N'}(v')\right) = r_{N',x}^{N'}(v')(j) + f_j \left(r_{N',x}^{N'}(v')\right)^I \left(N', r_{N',x}^{N'}(v')(N') - \sum_{i \in N'} r_{N,x}^{N'}(v')(i) \right) = v'(j) + f_j \left(v|_{N'}\right)^I \left(N', \sum_{i \in N'} \sigma_i(N, v') - \sum_{i \in N'} v'(i) \right) = v'(j) + f_j \left(v|_{N'}\right)^I \left(N', \sum_{i \in N'} f_i^v(N, t) \right). \tag{8}
\]

By projected consistency, (7) and (8) must coincide and thus

\[
f_j^v(N, t) = f_j \left(v|_{N'}\right)^I \left(N', \sum_{i \in N'} f_i^v(N, t) \right),
\]

which proves that \( F \) is consistent.

Hence, \( \sigma = CI^F \) being \( F \) a consistent monotone \( \Gamma_l \)-selection.

To show the reverse implication, let \( \sigma \) be a single-valued solution on \( \Gamma \) such that \( \sigma = CI^F \), for some consistent monotone \( \Gamma_l \)-selection \( F \). From Lemma 3, \( \sigma \) satisfies individual rationality and aggregate monotonicity. To check projected consistency, let \((N, v)\) be a game and \( \emptyset \neq N' \subset N \). Let us denote \( x = \sigma(N, v) \). For all \( j \in N' \) we have

\(
\sum_{i \in N'} f_i^v(N, t) = f_j \left(v|_{N'}\right)^I \left(N', \sum_{i \in N'} f_i^v(N, t) \right). \tag{8}
\)
Theorem 9: The proof of this theorem can be obtained following the same lines as in Theorem 10’s proof by using Lemma 2 instead of Lemma 3 and the notion of a consistent monotone path \( f \) (Definition 4) instead of the notion of a consistent monotone \( \Gamma_I \)-selection \( F \) (Definition 7).

**Independence of the properties**

To see that the properties in both Lemma 2 and Theorem 9 are independent, we introduce the following monotone paths:

1. Let \( \pi \) be a permutation on \( \mathbb{N} \). For all \( N \in \mathcal{N} \) and all \( t \in \mathbb{R} \), define \( f^\pi(N, t) = t \cdot e_{\{j\}} \), being \( j \in N \) such that \( \pi(j) \leq \pi(i) \) for all \( i \in N \) if \( |N| \) is even, and \( \pi(j) \geq \pi(i) \) for all \( i \in N \) if \( |N| \) is odd.
   
   If the cardinality of \( N \) is even, \( f^\pi \) assigns all of amount \( t \) to the first player in \( N \) according to \( \pi \); otherwise, \( f^\pi \) assigns \( t \) to the last player in \( N \) according to \( \pi \).

2. Let \( \pi \) be a permutation on \( \mathbb{N} \). For all \( N \in \mathcal{N} \) and all \( t \in \mathbb{R} \), define

   \[
   \hat{f}^\pi(N, t) := \begin{cases} 
   \left\lfloor \frac{t}{|N|} \right\rfloor \cdot e_N + \sum_{i \in S^*} e_i & \text{if } t \geq 0, \\
   -\hat{f}^\pi(N, -t) & \text{if } t < 0,
   \end{cases}
   \]

   where \( S^* \subset N \) is formed by the first \( \lfloor t \mod |N| \rfloor \) players according to \( \pi \) (if any) and \( k \in N \setminus S^* \) with \( \pi(k) \leq \pi(j) \), for all \( j \in N \setminus S^* \).

The interpretation of \( \hat{f}^\pi \) when \( t \geq 0 \) is as follows: if \( 0 \leq t \leq 1 \), then \( \hat{f}^\pi \) assigns amount \( t \) to the first player in \( N \) according to \( \pi \). If \( 1 < t \leq 2 \), the first player receives a unit of \( t \) and the second player \( t - 1 \), etc. If \( t > |N| \), then after distributing

---

12 For all \( x, y \in \mathbb{R} \), \( \lfloor x \rfloor := \max\{k \in \mathbb{Z} \mid k \leq x\} \), and \( x \mod y := x - y \cdot \left\lfloor \frac{x}{y} \right\rfloor \).
one unit of \( t \) to every player, one additional unit is again assigned to the first player, and so on until the amount \( t \) is exhausted.

It is not difficult to check that \( f^\sharp \) is an additive but not consistent monotone path. Thus, \( CI f^\sharp \) satisfies individual rationality and regular aggregate monotonicity but not projected consistency. In contrast, \( \widehat{f}^\pi \) is consistent but not additive, which means that \( CI \widehat{f}^\pi \) satisfies individual rationality and projected consistency but not regular aggregate monotonicity. Finally, \( ED \) satisfies regular aggregate monotonicity and projected consistency but not individual rationality.

To see that the properties in both Lemma 3 and Theorem 10 are independent, we introduce two single-valued solutions:

1. Let \( \sigma^\ast \) be the single-valued solution on \( \Gamma \) defined as follows: for all \( N \in \mathcal{N} \), all \((N, v)\) and all \( i \in N \), \( \sigma^\ast_i (N, v) := CI_i (N, v) \) if \((N, v) \in \Gamma_E \), and \( \sigma^\ast_i (N, v) := CI_i (N, v) - 1 \) otherwise.

2. For all \( N \in \mathcal{N} \) and all \( t \in \mathbb{R} \), let \( SN(t) = \{ i \in N | i \geq |t| \} \), where \( |t| = t \) if \( t \geq 0 \) and \( |t| = -t \) otherwise. If \( SN(t) \neq \emptyset \), choose \( i^* \in SN(t) \) to be such that \( i^* \leq j \), for all \( j \in SN(t) \). If \( SN(t) = \emptyset \), choose \( i^* \in N \) to be such that \( i^* \geq j \), for all \( j \in N \). Let \( g : \mathcal{N} \times \mathbb{R} \rightarrow \bigcup_{N \in \mathcal{N}} \mathbb{R}^N \) be a function defined as \( g(N, t) = t \cdot e_{\{i^*\}} \). Then, define \( \sigma^g \) to be the single-valued solution on \( \Gamma \) defined as follows: for all \( N \in \mathcal{N} \), all \((N, v) \in \Gamma \) and all \( i \in N \),

\[
\sigma^g_i (N, v) := v(i) + g_i \left( N, v(N) - \sum_{i \in N} v(i) \right)
\]

The single-valued solution \( \sigma^\ast \) satisfies individual rationality and aggregate monotonicity but neither efficiency nor projected consistency. The function \( g \) satisfies conditions (i) and (ii) in the definition of a monotone path (Definition 1), but not condition (iii). Moreover, \( g \) satisfies expression (4) in the definition of consistent monotone path (Definition 4). Then, \( \sigma^g \) satisfies individual rationality and projected consistency (and thus efficiency) but not aggregate monotonicity. Finally, \( ED \) satisfies efficiency, aggregate monotonicity and projected consistency but not individual rationality.

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