TEST VECTORS FOR RANKIN–SELBERG $L$-FUNCTIONS

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Abstract. We study the local zeta integrals attached to a pair of generic representations $(\pi, \tau)$ of $\text{GL}_n \times \text{GL}_m$, $n > m$, over a $p$-adic field. Through a process of unipotent averaging we produce a pair of corresponding Whittaker functions whose zeta integral is non-zero, and we express this integral in terms of the Langlands parameters of $\pi$ and $\tau$. In many cases, these Whittaker functions also serve as a test vector for the associated Rankin–Selberg (local) $L$-function.

1. Introduction

Let $F$ be a non-archimedean local field with ring of integers $\mathfrak{o}$ and residue field of cardinality $q$. For $m < n$, let $\pi$ and $\tau$ be irreducible admissible representations of $\text{GL}_n(F)$ and $\text{GL}_m(F)$, respectively. We fix an additive character $\psi$ of $F$ with conductor $\mathfrak{o}$ and assume that $\pi$ and $\tau$ are generic relative to $\psi$.

Recall that the local zeta integral $\Psi(s; W, W')$ is defined by

$$\Psi(s; W, W') = \int_{U_m(F) \backslash \text{GL}_m(F)} W\left(h I_{n-m}\right) W'(h) \| \det h \|^{-\frac{n-m}{2}} dh,$$

where $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\tau, \psi^{-1})$ are Whittaker functions in the corresponding Whittaker spaces, and $U_m$ is the group of unipotent matrices. It converges for $\Re(s) > 1$, and the collection of such zeta integrals spans a fractional ideal $\mathbb{C}[q^s, q^{-s}]L(s, \pi \boxtimes \tau)$ of the ring $\mathbb{C}[q^s, q^{-s}]$. We may choose the generator to satisfy $1/L(s, \pi \boxtimes \tau) \in \mathbb{C}[q^{-s}]$ and $\lim_{s \to \infty} L(s, \pi \boxtimes \tau) = 1$, and this gives the local Rankin–Selberg factor attached to the pair $(\pi, \tau)$ in §2.7.

In particular, if we define a map

$$\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\tau, \psi^{-1}) \longrightarrow \mathbb{C}(q^{-s})$$

via

$$W \otimes W' \mapsto \Psi(s; W, W'),$$

then there is an element in $\mathcal{W}(\pi, \psi) \otimes \mathcal{W}(\tau, \psi^{-1})$ that maps to $L(s, \pi \boxtimes \tau)$. However, a priori this element need not be a pure tensor. In this paper, we produce a pure tensor $W \otimes W'$ for which the associated zeta integral is explicitly computable and non-zero. The precise result that we prove is the following.

Theorem 1.1. Let $\{\alpha_i\}_{i=1}^n$ and $\{\gamma_j\}_{j=1}^m$ denote the Langlands parameters of $\pi$ and $\tau$, respectively, and let $L(s, \pi \times \tau)$ be the naive Rankin–Selberg $L$-factor defined by

$$L(s, \pi \times \tau) = \prod_{i=1}^n \prod_{j=1}^m (1 - \alpha_i \gamma_j q^{-s})^{-1}.$$

Then there is a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$, described explicitly in §2, such that

$$\Psi(s; W, W') = L(s, \pi \times \tau).$$

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When \( \Psi(s; W, W') = L(s, \pi \boxtimes \tau) \), the pair \((W, W')\) is called a test vector for \((\pi, \tau)\). Hence the theorem produces a test vector whenever \( L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau) \) — for instance, if either \( \pi \) or \( \tau \) is unramified or if \( L(s, \pi \boxtimes \tau) = 1 \). In general, one has \( L(s, \pi \times \tau) = P(q^{-s}) L(s, \pi \boxtimes \tau) \) for a non-zero polynomial \( P \in \mathbb{C}[X] \) (see Lemma 2.1).

The overview of our method is as follows. Let \( \xi^0 \) (resp. \( \varphi^0 \)) denote the “essential vector” in the space of \( \pi \) (resp. \( \tau \)), and let \( W_{\xi^0} \in \mathcal{W}(\pi, \psi) \) (resp. \( W_{\varphi^0} \in \mathcal{W}(\tau, \psi^{-1}) \)) be the associated essential Whittaker functions, as described in detail in [2]. When \( \tau \) is unramified, it follows from [10, Corollary 3.3] that

(1.2) \[
L(s, \pi \boxtimes \tau) = \int_{U_m(F) \backslash GL_m(F)} W_{\xi^0} \left( h \begin{pmatrix} I_{n-m} & \ast \\ \ast & 1 \end{pmatrix} \right) W_{\varphi^0}(h) \det h^{s - \frac{m}{2}} \, dh
\]

for a suitable normalization of the measure on \( U_m(F) \backslash GL_m(F) \). When \( m = n - 1 \), the above equality is part of the characterization of the essential vector in [4, 6]; the fact that it holds for any \( m < n \) is the result of a concrete realization of essential functions in [10]. On the other hand, if \( \tau \) is ramified then the local integral in (1.2) vanishes. Through a process of unipotent averaging (see (3.1) below), we modify \( W_{\xi^0} \) to obtain a Whittaker function \( W \in \mathcal{W}(\pi, \psi) \) such that the resulting zeta integral \( \Psi(s; W, W_{\varphi^0}) \) equals \( cL(s, \pi \times \tau) \) for a non-zero number \( c \in \mathbb{C} \), depending on the conductor of \( \tau \), its central character \( \omega_{\tau} \), and \( \psi \). Setting \( W' = c^{-1} W_{\varphi^0} \), we obtain the required pair \((W, W')\).

We mention some related results in the literature. First, if \( \pi \) and \( \tau \) are discrete series representations, then the existence of a test vector \((W, W')\) was shown in [9], but the Whittaker function \( W' \) there is taken to be in a larger space, namely the Whittaker space associated to the standard module of \( \tau \). Second, the so-called local Birch lemma, arising in the context of \( p \)-adic interpolation of special values of twisted Rankin–Selberg (global) \( L \)-functions, is also related. It concerns evaluation of a local integral in the special case that \( \pi \) is unramified and \( \tau \) is the twist of an unramified representation by a character with non-trivial conductor; see [8, Proposition 3.1] and [7, Theorem 2.1]. The approach in [8] is similar to ours in that it also uses a process of unipotent averaging in order to modify the Whittaker function on the larger general linear group. We can of course apply Theorem (1.1) to their setup: Since \( L(s, \pi \times \tau) = 1 \) in this case, the pair \((W, W_{\varphi^0})\) described above has the property that \( \Psi(s; W, W_{\varphi^0}) \) is an explicit constant (independent of \( s \)).

Finally, note that one can obtain a global version of Theorem (1.1) by combining the test vectors at all (finite) places. In work in progress, we study the analogous question over an archimedean local field.

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2. PRELIMINARIES

Let \( \mathfrak{p} \) be the unique maximal ideal in \( \mathfrak{a} \). We fix a generator \( \varpi \) of \( \mathfrak{p} \) with absolute value \( \|\varpi\| = q^{-1} \). Let \( \nu : F^\times \to \mathbb{Z} \) denote the valuation map, and extend it to fractional ideals in the usual way. For any \( n > 1 \), let \( B_n = T_n U_n \) be the Borel subgroup of \( GL_n \) consisting of upper triangular matrices; let \( P'_n \supset B_n \) be the standard parabolic subgroup of type \((n - 1, 1)\) with Levi decomposition \( P'_n = M_n N_n \). Then \( M_n = GL_{n-1} \times GL_1 \) and

\[
N_n = \left\{ \begin{pmatrix} I_{n-1} & * \\ & 1 \end{pmatrix} \right\}.
\]
Also, we write $Z_n$ to denote the center consisting of scalar matrices and $A_n \subset T_n$ to denote the subtorus consisting of diagonal matrices with lower-right entry 1.

If $R$ is any $F$-algebra and $H$ is any algebraic $F$-group, we write $H(R)$ to denote the corresponding group of $R$-points. Let $P_n(R) \subset P'_n(R)$ denote the mirabolic subgroup consisting of matrices whose last row is of the form $(0, \ldots, 0, 1)$, i.e.,

$$P_n(R) = \left\{ \begin{pmatrix} h & y \\ 1 \end{pmatrix} : h \in GL_{n-1}(R), y \in R^{n-1} \right\} \cong GL_{n-1}(R) \ltimes N_n(R).$$

The character

$$u \mapsto \psi \left( \sum_{i=1}^{n-1} u_{i,i+1} \right)$$

for $u \in U_n(F)$ defines a \textit{generic character} of $U_n(F)$, and by abuse of notation we continue to denote this character by $\psi$. Further, for any algebraic subgroup $V \subseteq U_n$, $\psi$ defines a character of $V(F)$ via restriction. In particular, we may consider the character $\psi|_{N_n(F)}$; its stabilizer in $M_n(F)$ is then $P_{n-1}(F)$, where we regard $P_{n-1}$ as a subgroup of $M_n$ via $h \mapsto \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}$.

An irreducible representation $(\pi, V_n)$ of $GL_n(F)$ is said to be \textit{generic} if

$$\text{Hom}_{GL_n(F)}(V_{\pi}, \text{Ind}_{U_n(F)}^{GL_n(F)} \psi) \neq 0.$$

By Frobenius reciprocity, this means that there is a non-zero linear form $\lambda : V_{\pi} \to \mathbb{C}$ satisfying $\lambda(\pi(u)v) = \psi(u)\lambda(v)$ for $v \in V_{\pi}, u \in U_n(F)$. It is known (see [3]) that for a generic $\pi$ the space of such linear functionals, or equivalently the space $\text{Hom}_{GL_n(F)}(V_{\pi}, \text{Ind}_{U_n(F)}^{GL_n(F)} \psi)$, is of dimension 1. Let $W(\pi, \psi)$ denote the Whittaker model of $\pi$, viz. the space of functions $W_v$ on $GL_n(F)$ defined by $W_v(g) = \lambda(\pi(g)v)$ for $v \in V_{\pi}$. Then $W(\pi, \psi)$ is independent of the choice of $\lambda$, and for $u \in U_n(F), g \in GL_n(F)$,

$$W_v(ug) = \psi(u)W(g),$$

$$W_v(g) = W_{\pi(g)v}(I_n).$$

We will consider certain compact open subgroups of $GL_n(F)$; namely, for any integer $f \geq 0$, set

$$K_1(p^f) = \left\{ g \in GL_n(\mathfrak{o}) : g \equiv \begin{pmatrix} * & \cdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \pmod{p^f} \right\},$$

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so that $K_1(p^f)$ is a normal subgroup of $K_0(p^f)$, with quotient $K_0(p^f)/K_1(p^f) \cong (\mathfrak{o}/p^f)^{\times}$.

Next we introduce our choice of measures. For $n \geq 1$ we normalize the Haar measure on $GL_n(F)$ and $GL_n(\mathfrak{o})$ so that vol($GL_n(\mathfrak{o})$) = 1, and we fix the Haar measure on $U_n(F)$ for which vol($U_n(F) \cap GL_n(\mathfrak{o})$) = 1. From these, we obtain a right-invariant measure on $U_n(F) \setminus GL_n(F)$. We may make this explicit using the Iwasawa decomposition. For instance, let $dx$ be the Haar measure on $F$ such that $\mathfrak{o}$ has unit volume, and let $d^\times x$ be the multiplicative measure on $F^\times$ such that vol$(\mathfrak{o}^\times) = 1$, i.e., $d^\times x = \frac{q}{q-1} \frac{dx}{|x||}$, and let $da$ be the corresponding measures on the center $Z_n(F) \cong F^\times$ and the subtorus $A_n(F) \cong (F^\times)^{n-1}$, respectively. We fix the isomorphism $(F^\times)^{n-1} \cong A_n(F)$ via $(a_1, \ldots, a_{n-1}) \mapsto$
a = t(a_1, \ldots, a_{n-1}), \text{ where}

\begin{equation}
(2.1) \quad t(a_1, \ldots, a_{n-1}) = \begin{pmatrix}
a_1a_2 \ldots a_{n-1} \\
\vdots \\
a_1 \\
1
\end{pmatrix}.
\end{equation}

Then \( da = d^x a_1 d^x a_2 \cdots d^x a_{n-1} \). If \( f \in C^\infty_c(\text{GL}_n(F)) \) is \( U_n(F) \)-invariant on the left, we then have the integration formula

\begin{equation}
(2.2) \quad \int_{U_n(F)\backslash GL_n(F)} f(g) \, dg = \int_{Z_n(F) \times A_n(F) \times GL_n(o)} f(za) \delta_{B_n}(a)^{-1} \, dz \, da \, dk,
\end{equation}

where \( \delta_{B_n} \) is the modulus character, defined so that

\begin{equation}
(2.3) \quad \delta_{B_n}(a) = \prod_{i=1}^{n-1} \|a_i\|^{(n-1)}.
\end{equation}

Next we review the notion of conductor and the theory of the essential vector associated to an irreducible, admissible, generic representation \( \pi \). According to [4] (see also [6]), there is a unique positive integer \( m(\pi) \) such that the space of \( K_1(p^{m(\pi)}) \)-fixed vectors is 1-dimensional. Further, as alluded to in the introduction, by loc. cit. there is a unique positive integer \( q \in \mathbb{N} \) such that the space of \( \text{GL}_n(o) \times GL_n(o) \)-invariant on the left, we

\begin{equation}
(2.4) \quad W_{\psi^0}(t(a_1, \ldots, a_{n-1})) \neq 0 \implies a_1, \ldots, a_{n-1} \in o.
\end{equation}

If \( \pi \) is unramified, let \( W_{\psi}^{0,\psi} \in \mathcal{W}(\pi, \psi) \) denote the normalized spherical function [6, p. 2]. If \( m(\pi) = 0 \) then by uniqueness of essential functions, one has the equality \( W_{\psi^0} = W_{\pi,\psi}^{0,\psi} \). The integral ideal \( \mathfrak{p}^{m(\pi)} \) is called the conductor of \( \pi \). In passing, we mention that the integer \( m(\pi) \) can also be characterized as the degree of the monomial in the local \( \epsilon \)-factor \( \epsilon(s, \pi, \psi) \) [4], i.e. so that

\begin{equation}
\epsilon(s, \pi, \psi) = \epsilon(\pi, \psi)q^{m(\pi)(\frac{1}{2} - s)}
\end{equation}

for some \( \epsilon(\pi, \psi) \in \mathbb{C}^\times \).

A crucial property of the conductor is that \( K_0(p^{m(\pi)}) \) acts on the space of \( K_1(p^{m(\pi)}) \)-fixed vectors via the central character \( \omega_\pi \) (cf. [2] Section 8]). Precisely, for \( g = (g_{i,j}) \in K_0(p^{m(\pi)}) \), define

\begin{equation}
\chi_\pi(g) = \begin{cases}
1 & \text{if } m(\pi) = 0, \\
\omega_\pi(g_{1,1}) & \text{if } m(\pi) > 0.
\end{cases}
\end{equation}

It is shown in loc. cit. that \( \chi_\pi \) is a character of \( K_0(p^{m(\pi)}) \) trivial on \( K_1(p^{m(\pi)}) \), and

\begin{equation}
\pi(g)\xi^0 = \chi_\pi(g)\xi^0 \quad \text{for all } g \in K_0(p^{m(\pi)}).
\end{equation}

We end this section by recalling the definition of conductor of a multiplicative character \( \chi \) of \( F^\times \). If \( \chi \) is trivial on \( o^\times \) then the conductor of \( \chi \) is \( o \); otherwise, the conductor is \( p^n \), where \( n \geq 1 \) is the least integer such that \( \chi \) is trivial on \( 1 + p^n \).
2.1. Rankin–Selberg L-functions. In this subsection alone we drop the assumption that \( m < n \) and allow \((m, n)\) to be an arbitrary pair of positive integers. For \( \pi \) and \( \tau \) irreducible, admissible, generic representations of \( \text{GL}_n(F) \) and \( \text{GL}_m(F) \), respectively, let \( L(s, \pi \boxtimes \tau) \) be as defined in \([3]\). When \( m < n \), \( L(s, \pi \boxtimes \tau) \) is defined as in the introduction. For \( m > n \), one defines \( L(s, \pi \boxtimes \tau) = L(s, \tau \boxtimes \pi) \). For \( m = n \), the defining local integrals are different and involve a Schwartz function on \( F^n \); see loc. cit.

Next we elaborate on the definition of the naive Rankin–Selberg L-factor, \( L(s, \pi \times \tau) \), introduced in Theorem \([5, §9.5]\). By definition, the L-function \( L(s, \pi) \) is of the form \( P_{\pi}(q^{-s})^{-1} \), where \( P_{\pi} \in \mathbb{C}[X] \) has degree at most \( n \) and satisfies \( P_{\pi}(0) = 1 \). We may then find \( n \) complex numbers \( \{\alpha_i\}_{i=1}^n \) (allowing some of them to be zero) satisfying

\[
L(s, \pi) = \prod_{i=1}^n (1 - \alpha_i q^{-s})^{-1}.
\]

We call the set \( \{\alpha_i\} \) the Langlands parameters of \( \pi \); if \( \pi \) is spherical, they agree with the usual Satake parameters. Let \( \{\gamma_{ij}\}_{j=1}^m \) be the Langlands parameters of \( \tau \), and set

\[
L(s, \pi \times \tau) = \prod_{i=1}^n \prod_{j=1}^m (1 - \alpha_i \gamma_{ij} q^{-s})^{-1}.
\]

Of course, \( L(s, \pi \times \tau) = L(s, \pi \boxtimes \tau) \) if both \( \pi \) and \( \tau \) are spherical.

In the following lemma we describe the connection between \( L(s, \pi \times \tau) \) and \( L(s, \pi \boxtimes \tau) \). To that end, we first recall the classification of irreducible admissible representations of \( \text{GL}_n(F) \). Let \( \mathcal{A}_n \) denote the set of equivalence classes of such representations, and put \( \mathcal{A} = \bigcup \mathcal{A}_n \). The essentially square-integrable representations of \( \text{GL}_n(F) \) have been classified by Bernstein and Zelevinsky, and they are as follows. If \( \sigma \) is an essentially square-integrable representation of \( \text{GL}_n(F) \), then there is a divisor \( a \mid n \) and a supercuspidal representation \( \eta \) of \( \text{GL}_a(F) \) such that if \( b = \frac{n}{a} \) and \( Q \) is the standard (upper) parabolic subgroup of \( \text{GL}_b(F) \) of type \((a, \ldots, a)\), then \( \sigma \) can be realized as the unique quotient of the (normalized) induced representation

\[
\text{Ind}_{Q}^{\text{GL}_b(F)}(\eta, \eta \| \cdot, \ldots, \eta \| \cdot b^{-1}).
\]

The integer \( a \) and the class of \( \eta \) are uniquely determined by \( \sigma \). In short, \( \sigma \) is parametrized by \( b \) and \( \eta \), and we denote this by \( \sigma = \sigma_b(\eta) \); further, \( \sigma \) is square-integrable (also called “discrete series”) if and only if the representation \( \eta \| \cdot b^{-1} \) of \( \text{GL}_b(F) \) is unitary.

Now, let \( P \) be an upper parabolic subgroup of \( \text{GL}_b(F) \) of type \((n_1, \ldots, n_r)\). For each \( i = 1, \ldots, r \), let \( \tau_i^0 \) be a discrete series representation of \( \text{GL}_{n_i}(F) \). Let \((s_1, \ldots, s_r)\) be a sequence of real numbers satisfying \( s_1 \geq \cdots \geq s_r \), and put \( \tau_i = \tau_i^0 \boxtimes \| \cdot \|^{s_i} \) (an essentially square-integrable representation). Then the induced representation

\[
\xi = \text{Ind}_{P}^{\text{GL}_b(F)}(\tau_1 \boxtimes \cdots \boxtimes \tau_r)
\]

is said to be a representation of \( \text{GL}_b(F) \) of Langlands type. If \( \tau \in \mathcal{A}_n \), then it is well known that it is uniquely representable as the quotient of an induced representation of Langlands type. We write \( \tau = \tau_1 \boxplus \cdots \boxplus \tau_r \) to denote this realization of \( \tau \). Thus one obtains a sum operation on the set \( \mathcal{A} \) \([5, §9.5]\). It follows easily from the definition that \( L(s, \pi \times \tau) \) is bi-additive, i.e.

\[
\begin{align*}
L(s, \pi \times (\tau \boxplus \tau')) &= L(s, \pi \times \tau)L(s, \pi \times \tau') \\
L(s, (\pi \boxplus \pi') \times \tau) &= L(s, \pi \times \tau)L(s, \pi' \times \tau)
\end{align*}
\]
for all $\pi, \pi', \tau, \tau' \in \mathcal{A}$. The local factor $L(s, \pi \boxtimes \tau)$ is also bi-additive in the above sense, by [5] §9.5, Theorem]

**Lemma 2.1.** Let $m$ and $n$ be positive integers, and consider $\pi \in \mathcal{A}_n$, $\tau \in \mathcal{A}_m$. Then

$$L(s, \pi \times \tau) = P(q^{-s})L(s, \pi \boxtimes \tau)$$

for a polynomial $P \in \mathbb{C}[X]$ (depending on $\pi$ and $\tau$) satisfying $P(0) = 1$.

**Proof.** Since $\pi$ and $\tau$ are sums of essentially square-integrable representations and $L(s, \pi \times \tau)$ and $L(s, \pi \boxtimes \tau)$ are both additive with respect to $\boxtimes$, it suffices to prove the lemma for a pair $(\pi, \tau)$ of essentially square-integrable representations. In particular, assume $\pi = \sigma_b(\eta)$ as above.

We proceed by induction on $m$. If $m = 1$ then $\tau = \chi$ is a quasi-character of $F^\times$ and $L(s, \pi \boxtimes \tau) = L(s, \pi \otimes \chi)$, where $\pi \otimes \chi$ is the representation of $\text{GL}_n(F)$ defined by $g \mapsto \pi(g)\chi(\det g)$. If $\chi$ is unramified, then

$$L(s, \pi \otimes \chi) = L(s, \pi \times \chi),$$

and consequently $P = 1$. On the other hand, if $\chi$ is ramified then $L(s, \pi \times \chi) = 1$, and the assertion follows since $L(s, \pi \otimes \chi)^{-1}$ is a polynomial in $q^{-s}$.

We now assume $m > 1$ and $\tau$ is an essentially square-integrable representation of $\text{GL}_m(F)$, say $\tau = \sigma_{b'}(\eta')$, where $\eta' \in \mathcal{A}_{b'}$ is supercuspidal and $a'b' = m$. Then the standard $L$-factor $L(s, \tau)$ is given by $L(s, \tau) = L(s + b' - 1, \eta')$ [5]. Therefore, $L(s, \tau) = 1$ unless $a' = 1$ and $\eta' = \chi$ is an unramified quasi-character of $F^\times$. On the other hand, if $L(s, \tau) = 1$, then $L(s, \pi \times \tau) = 1$ and the assertion of the lemma follows. Hence we may assume $\tau = \sigma_m(\chi)$ for an unramified quasi-character $\chi$ of $F^\times$, in which case

$$L(s, \pi \times \tau) = L(s, \pi \otimes \chi)^\| \cdot \|^{m-1}) = L(s, \sigma_b(\eta) \otimes \chi)^\| \cdot \|^{m-1}) = L(s + m - 1 + b - 1, \eta \otimes \chi).$$

On the other hand, it follows from [5] §8.2, Theorem] that

$$L(s, \pi \boxtimes \tau) = \begin{cases} \prod_{j=0}^{m-1} L(s + j + b - 1, \eta \otimes \chi) & \text{if } m \leq n, \\ \prod_{i=0}^{n-1} L(s + m - 1 + i, \eta \otimes \chi) & \text{if } m > n. \end{cases}$$

From (2.6) and (2.7), one sees that the ratio $\frac{L(s, \pi \times \tau)}{L(s, \pi \boxtimes \tau)}$ is a polynomial in $q^{-s}$, thus proving the lemma.

**Corollary 2.2.** If $L(s, \pi \boxtimes \tau) = 1$ then either $L(s, \pi) = 1$ or $L(s, \tau) = 1$.

**Proof.** If $L(s, \pi \boxtimes \tau) = 1$ then Lemma 2.1 implies that $L(s, \pi \times \tau)$ is a polynomial in $q^{-s}$, and hence must be 1. This in turn implies the conclusion. \qed

### 3. The main calculation

Recall that $\xi^0$ and $\varphi^0$ are the essential vectors of $\pi$ and $\tau$, respectively. Here we construct a pair $(W, W') \in \mathcal{W}(\pi, \psi) \times \mathcal{W}(\tau, \psi^{-1})$ as in Theorem 1.1. Let $\mathfrak{n}$, $\mathfrak{q}$ and $\mathfrak{c}$ denote the conductors of $\pi$, $\tau$ and $\omega_\tau$, respectively. If $\tau$ is an unramified representation of $\text{GL}_m(F)$, then by (1.2) we have

$$\Psi(s; W_{\xi^0}, W_{\varphi^0}) = L(s, \pi \boxtimes \tau) = L(s, \pi \times \tau).$$

Thus, in this case we can take $(W, W') = (W_{\xi^0}, W_{\varphi^0})$.

Let us assume from now on that $\tau$ is ramified, meaning $v(\mathfrak{q}) > 0$. Since $\mathfrak{c} \supseteq \mathfrak{q}$, we have $v(c) \leq v(q)$. Consider $\beta = (\beta_1, \ldots, \beta_m) \in F^m$, with $\beta_i \in q^{-1}$ for $i = 1, \ldots, m$, and
let \( u(\beta) \) denote the \( n \times n \) matrix with 1s on the diagonal and \( \beta' \) embedded above the diagonal in the \((m + 1)\)st column. Let \( \xi_0^{\beta} \) denote the vector \( \xi_0^\beta = \pi(u(\beta))\xi_0 \), and define

\[
\overline{\xi} = \frac{1}{|\mathfrak{o} : q|^{m-1}} \sum_{(\beta_1, ..., \beta_{m-1}) \in (q^{-1}\mathfrak{o})^{m-1}} \xi_0^{\beta_1, ..., \beta_{m-1}, w^{-v(i)}}.
\]

(When \( m = 1 \) we understand there to be one summand, so that \( \overline{\xi} = \xi_0^{w^{-v(i)}}. \) We will now calculate \( \Psi(s; W_\overline{\xi}, W_{\varphi_0}) \), which by linearity equals

\[
\frac{1}{|\mathfrak{o} : q|^{m-1}} \sum_{(\beta_1, ..., \beta_{m-1}) \in (q^{-1}\mathfrak{o})^{m-1}} \Psi(s; W_{\varphi_0}^{\beta_1, ..., \beta_{m-1}, w^{-v(i)}}, W_{\varphi_0}).
\]

Put \( K = \text{GL}_m(\mathfrak{o}). \) By (2.2), for fixed \( \beta = (\beta_1, ..., \beta_m) \), we have

\[
\Psi(s; W_{\overline{\xi}}^{\beta}, W_{\varphi_0}) = \int_{Z_m(F) \times A_m(F) \times K} \omega_r(z) W_{\overline{\xi}}^\beta \left( \begin{pmatrix} z & \varphi \cdot a & I_{n-m} \end{pmatrix} \right) W_{\varphi_0}^a (ak)\cdot \delta_B(a)^{-1} \|z\|^{s-n-m} d z d a d k
\]

\[
= \int_{F^\times \times A_m(F) \times K} \omega_r(z) \left( \sum_{j=1}^{m} \psi(zk_j \beta_j) \right) W_{\overline{\xi}}^\beta \left( \begin{pmatrix} z & \varphi \cdot a & I_{n-m} \end{pmatrix} \right) W_{\varphi_0}^a (ak)\cdot \delta_B(a)^{-1} \|z\|^m \det a^{\frac{s-n-m}{2}} d^m z d a d k,
\]

where \( (k_1, ..., k_m) \) is the bottom row of the matrix \( k \). Here we have used the fact that the function \( h \mapsto W_{\overline{\xi}}^\beta (h I_{n-m}, h \in \text{GL}(F), \) is right \( K \)-invariant. Now, performing the average over \( \beta_j \in q^{-1}\mathfrak{o} \) for each \( j < m \), we see that \( \Psi(s; W_{\overline{\xi}}, W_{\varphi_0}) \) equals

\[
\int_{\{F^\times \times A_m(F) \times K \mid zk_j \in q \forall j < m\}, \{zk_j \in q \forall j < m\}} \omega_r(z) \psi(zk_m w^{-v(i)}) W_{\overline{\xi}}^\beta \left( \begin{pmatrix} z & \varphi \cdot a & I_{n-m} \end{pmatrix} \right) W_{\varphi_0}^a (ak)\cdot \delta_B(a)^{-1} \|z\|^m \det a^{\frac{s-n-m}{2}} d^m z d a d k.
\]

By (2.4) we have \( W_{\overline{\xi}}^\beta (t(a_1, ..., a_{n-1}) = 0 \) unless \( a_1, ..., a_{n-1} \in \mathfrak{o} \). In view of (2.1), it follows that the integrand vanishes unless \( z \) is integral.

Note that

\[
z k_j \in q \forall j < m \iff k \in K_0(z^{-1}q \cap \mathfrak{o}).
\]

For \( r \in \mathbb{Z}_{\geq 0} \), put

\[
\Psi_r = \int_{A_m(F) \times K_0(q^{-r} \cap \mathfrak{o})} \mathcal{G}(\omega_r, \psi, w^{r-v(i)} k_m) W_{\overline{\xi}}^\beta \left( \begin{pmatrix} w^r a & \varphi \cdot a & I_{n-m} \end{pmatrix} \right) W_{\varphi_0}^a (ak)\cdot \delta_B(a)^{-1} \|a\|^s \frac{n-m}{2} \|d a d k.
\]

where \( \mathcal{G}(\omega, \psi, y) \) denotes the Gauss sum

\[
\mathcal{G}(\omega, \psi, y) = \int_{\mathfrak{o}^\times} \omega_r(z) \psi(yz) d^m z.
\]

Then we have

\[
\Psi(s; W_{\overline{\xi}}, W_{\varphi_0}) = \sum_{r \geq 0} \omega_r(w^r) q^{rm(s-n-m)} \Psi_r.
\]
Suppose that $\omega_r$ is ramified, so that $v(c) > 0$. Then $G(\omega_r, \psi, \varpi^{-v(c)}k_m)$ vanishes unless $v(\varpi^{-v(c)}k_m) = -v(c)$, which implies $v(k_m) = -r$. Since $k_m$ is integral, it follows that $r = 0$ is the only contributing term to $\Psi(s; W_\xi, W_{\varphi^0})$, so that

$$
\Psi(s; W_\xi, W_{\varphi^0}) = \int_{A_m(F) \times K_0(q)} G(\omega_r, \psi, \varpi^{-v(c)}k_m) W_{\xi^0} \left( a I_{n-m} \right) W_{\varphi^0}(ak) \cdot \delta_B(a)^{-1/2} \| \det a \|^{s - \frac{n-m}{2}} \, da \, dk.
$$

Moreover, since $k_m \in \mathfrak{o}^\times$, we have $G(\omega_r, \psi, \varpi^{-v(c)}k_m) = \omega_r(k_m)^{-1}G(\omega_r, \psi, \varpi^{-v(c)})$, and thus

$$
(3.2) \quad \Psi(s; W_\xi, W_{\varphi^0}) = c \int_{A_m(F)} W_{\xi^0} \left( a I_{n-m} \right) W_{\varphi^0}(a) \delta_B(a)^{-1/2} \| \det a \|^{s - \frac{n-m}{2}} \, da,
$$

where

$$
c = \frac{G(\omega_r, \psi, \varpi^{-v(c)})}{[\text{GL}_m(\mathfrak{o}) : K_0(q)]} \neq 0.
$$

Suppose now that $\omega_r$ is unramified, so that $c = 0$ and $m > 1$. Then $G(\omega_r, \psi, \varpi^{-v(c)}k_m) = 1$ for all $r$. If $r > 0$ then $K_0(qp^{-r} \cap \mathfrak{o}) \supseteq K_0(q)$; since the conductor of $\tau$ is $q$, it follows from [4, Theorem 5.1] that

$$
\int_{K_0(qp^{-r} \cap \mathfrak{o})} W_{\varphi^0}(ak) \, dk = 0,
$$

which in turn implies that $\Psi_r = 0$. Hence only the $r = 0$ term contributes, and again we arrive at (3.2).

It remains only to identify the integral over $A_m(F)$.

**Lemma 3.1.** When $\tau$ is ramified, we have

$$
\int_{A_m(F)} W_{\xi^0} \left( a I_{n-m} \right) W_{\varphi^0}(a) \delta_B(a)^{-1/2} \| \det a \|^{s - \frac{n-m}{2}} \, da = L(s, \pi \times \tau).
$$

**Proof.** Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $(\gamma_1, \ldots, \gamma_m)$ denote the Langlands parameters of $\pi$ and $\tau$, respectively. Since $\tau$ is ramified, we may take $\gamma_m = 0$. We set $\gamma_{m+1} = \ldots = \gamma_n = 0$ and write $\gamma = (\gamma_1, \ldots, \gamma_m)$.

Writing $a = t(a_1, \ldots, a_{m-1})$ as in (2.1), by (2.1) we see that $W_{\varphi^0}(a)$ vanishes unless each $a_i$ is integral. Setting

$$
\lambda_i = \sum_{1 \leq j \leq m-i} v(a_j) \quad \text{for} \quad i = 1, \ldots, n,
$$

the integral in question may be written as

$$
\sum_{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n} W_{\xi^0}(d_n(\lambda)) W_{\varphi^0}(d_m(\lambda)) \delta_B(d_m(\lambda))^{-1} \| \det d_m(\lambda) \|^{s - \frac{n-m}{2}},
$$

where

$$
d_n(\lambda) = \text{diag}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_n}) \quad \text{and} \quad d_m(\lambda) = \text{diag}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_m}).
$$

On the other hand, by [11, Theorem 4.1], we have

$$
W_{\xi^0}(d_n(\lambda)) = \delta_B(d_n(\lambda))^{\frac{1}{2}} s_\lambda(\alpha) \quad \text{and} \quad W_{\varphi^0}(d_m(\lambda)) = \delta_B(d_m(\lambda))^{\frac{1}{2}} s_\lambda(\gamma),
$$
where \( s_\lambda \) denotes the Schur polynomial

\[
s_\lambda(X_1, \ldots, X_n) = \det (X_j^{\lambda_i+n-i})_{1 \leq i, j \leq n} / \prod_{1 \leq i < j \leq n} (X_i - X_j)
\]

(see [1, Theorem 38.1]). Thus the integral becomes

\[
\sum_{\lambda=(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n \atop \lambda_1 \geq \cdots \geq \lambda_m} s_\lambda(\alpha)s_\lambda(\gamma) \left( \frac{\delta_B_x(d_n(\lambda))}{\delta_B_x(d_m(\lambda))} \right)^{1/2} \| \det d_m(\lambda) \|^s \gamma_m^{n-m}
\]

where the last line follows from (2.3) and the fact that \( \gamma_m = \ldots = \gamma_n = 0 \).

By the Cauchy identity [1, Theorem 43.1], for sufficiently small \( x \in \mathbb{C} \), we have

\[
\prod_{i=1}^n \prod_{j=1}^n (1 - x\alpha_i \gamma_j)^{-1} = \sum_{\lambda=(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n \atop \lambda_1 \geq \cdots \geq \lambda_n} s_\lambda(x\alpha)s_\lambda(\gamma) = \sum_{\lambda=(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n \atop \lambda_1 \geq \cdots \geq \lambda_n} s_\lambda(\alpha)s_\lambda(\gamma)x^{\lambda_1 + \cdots + \lambda_n}.
\]

Therefore,

\[
\sum_{\lambda=(\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}_+^n \atop \lambda_1 \geq \cdots \geq \lambda_n} s_\lambda(\alpha)s_\lambda(\gamma) \| \det d_m(\lambda) \|^s = \prod_{i=1}^n \prod_{j=1}^n (1 - \alpha_i \gamma_j q^{-s})^{-1} = L(s, \pi \times \tau).
\]

Finally, we take \( W = W_0^\tau \) and \( W' = c^{-1} W_0^{\sigma} \) to conclude the proof of Theorem 1.1.

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