Invariance of the Goresky–Hingston algebra on reduced Hochschild homology

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Abstract
We prove that two quasi-isomorphic simply connected differential graded associative Frobenius algebras have isomorphic Goresky–Hingston algebras on their reduced Hochschild homology. Our proof is based on relating the Goresky–Hingston algebra on reduced Hochschild homology to the singular Hochschild cohomology algebra. For any simply connected oriented closed manifold $M$ of dimension $k$, the Goresky–Hingston algebra on reduced Hochschild homology induces an algebra structure of degree $k-1$ on $\overline{H}^*(LM; \mathbb{Q})$, the reduced rational cohomology of the free loop space of $M$. As a consequence of our algebraic result, we deduce that the isomorphism class of the induced algebra structure on $\overline{H}^*(LM; \mathbb{Q})$ is an invariant of the homotopy type of $M$.

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1 | INTRODUCTION

1.1 | String topology and motivation

String topology is concerned with the algebraic structure of the free loop space $LM$ of a manifold $M$. The field began with the construction of a graded associative and commutative product on the homology of $LM$ defined by combining the intersection product on the underlying manifold $M$ with the concatenation product on pairs of loops in $M$ with a common base point [4]. Later on, a
graded coassociative and cocommutative coproduct on the homology of \( LM \) relative to constant loops \( M \subset LM \) was described in [20] and [8] by considering self intersections in a single family of loops and cutting at these intersection points to obtain two new families. These two operations are part of a rich family of compatible operations, which may be constructed at the chain level by intersecting, cutting, and reconnecting families of loops according to combinatorial patterns associated to moduli spaces of surfaces (for example, [7, 20]).

There are different ways of making choices to construct string topology operations rigorously. Some use geometric and topological methods to describe intersections [7–9, 20], others homotopy theoretic techniques [6], and others start with an algebraic chain or cochain model for a manifold, such as the commutative dg algebra of differential forms, and then make choices algebraically in order to construct operations using the relationship between Hochschild homology and the free loop space [2, 12, 17, 21, 23]. This paper is concerned with an algebra structure defined through this last approach on a reduced version of the Hochschild homology of a differential graded Frobenius algebra over a field of characteristic zero. Our main result is a purely algebraic statement regarding the quasi-isomorphism invariance of this algebra structure under a simply connected hypothesis on the underlying dg Frobenius algebra. The algebraic product studied in this paper resembles a geometric construction known as the Goresky–Hingston coproduct [8, 9, 20], which has been computed to be non-trivial in specific cases. The question of whether this operation is a homotopy invariant has been proven to be particularly subtle [10, 16].

1.2 Main theorem

Fix a field \( \mathbb{K} \) of characteristic zero and write \( \otimes = \otimes \mathbb{K} \). Let \( A \) be a unital dg \( \mathbb{K} \)-algebra equipped with a non-degenerate symmetric pairing \( \langle -,- \rangle : A \otimes A \to \mathbb{K} \) of degree \( k > 0 \), which is compatible with the product and the differential of \( A \). Such an object is called a dg Frobenius algebra of degree \( k \) (see Definition 2.1). In particular, any dg Frobenius algebra of degree \( k \) comes equipped with special degree \( k \) element \( \sum_i e_i \otimes f_i \in A \otimes A \) (called the Casimir element) satisfying \( \sum_i x e_i \otimes f_i = \sum_i (-1)\varepsilon_i || x || e_i \otimes f_i = \sum_i (-1)^{\varepsilon_i \varepsilon_j} e_i \otimes x f_j \) for any \( x \in A \). If we think of \( A \) as a cochain model for a closed manifold \( M \) of dimension \( k \) and \( \langle -,- \rangle : A \otimes A \to \mathbb{K} \) as a pairing inducing the Poincaré duality pairing, then \( \sum_i e_i \otimes f_i \) plays the role of a representative for the Thom class of the diagonal embedding \( M \hookrightarrow M \times M \).

Suppose \( A \) is connected (that is, \( A^0 \cong \mathbb{K} \)) and consider the double complex of Hochschild chains \( C_{n,*}(A, A) \) where \( C_{-p,n}(A, A) = ((sA)^{\otimes p} \otimes A)^n, \) \( sA \subset A \) denotes the positive degree elements of \( A \), and for any graded vector space \( V = \bigoplus_{j \in \mathbb{Z}} V^j \), \( s^i V \) denotes the \( i \)-th shifted graded vector space given by \( (s^i V)^j = V^{i+j} \) (we write \( s = s^1 \)). Denote by \( C_n(A, A) = \bigoplus_p C_{-p,n}(A, A) \) and \( C_*(A, A) = \bigoplus_n C_n(A, A) \). In this paper we study a product

\[ \star : C_*(A, A) \otimes C_*(A, A) \to C_*(A, A) \]

of degree \( k - 1 \) defined by the formula (see Definition 2.8)

\[
(a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) \star (b_1 \otimes \cdots \otimes b_q \otimes b_{q+1}) = \sum_i (-1)^{\varepsilon_i} b_1 \otimes \cdots \otimes b_{q+1} e_i \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{q+1} f_i.
\]
The above product appears as a secondary operation in the family of algebraic constructions on the Hochschild chain complex of a Frobenius algebra constructed in [11, 21, 23]. In the context when the underlying dg Frobenius is commutative, the product $\star$ has been studied in [2] and [14]. In Section 2, we observe that the product $\star$ defines a (non-unital) dg algebra structure on the (shifted) reduced Hochschild complex $s^{1-k}\mathcal{C}_e(A, A)$ of any connected (not necessarily commutative) dg Frobenius algebra of degree $k$. The reduced Hochschild complex is the subcomplex of $C_e(A, A)$ defined as the complement of $C_{0,0}(A, A) = A^0 \cong \mathbb{k}$. We call the induced algebra structure on homology $(s^{1-k}\mathcal{HH}_e(A, A), \star)$ the Goresky–Hingston algebra on the reduced Hochschild homology of a dg Frobenius algebra $A$. The reason for this name is the analogy and similarities between the properties of the algebraic product $\star$ and the geometrically defined operation of [8] of the same degree. Recently, this analogy has been made mathematically precise: It has been announced in [17] that if $A$ is a Poincaré duality model for the rational polynomial differential forms on a simply connected oriented closed manifold $M$ (as constructed in [15]), then the product $\star$ at the level of a relative version of Hochschild homology of $A$ corresponds to a topologically defined version of the Goresky–Hingston product on $H^*(\mathcal{L}M, M; \mathbb{Q})$, the rational cohomology of the free loop space $\mathcal{L}M$ relative to constant loops.

In this paper we give an algebraic proof of the invariance of the isomorphism class of the algebra $(s^{1-k}\mathcal{HH}_e(A, A), \star)$ under quasi-isomorphisms of simply connected dg Frobenius algebras in the following sense. Let $\text{DGA}_k^1$ be the category of unital dg $\mathbb{k}$-algebras $A$ which are simply connected and non-negatively graded, that is, $A^{<0} = 0$, $A^0 \cong \mathbb{k}$ and $A^1 = 0$.

**Theorem 1.1.** Let $(A, \langle -, -, \rangle_A)$ and $(B, \langle -, -, \rangle_B)$ be two dg Frobenius algebras of degree $k$ such that $A, B \in \text{DGA}_k^1$. Suppose that there is a zig-zag of quasi-isomorphisms of dg algebras

$$A \rightleftharpoons \cdots \rightleftharpoons B.$$

Then there is an isomorphism of Goresky–Hingston algebras

$$(s^{1-k}\mathcal{HH}_e(A, A), \star) \cong (s^{1-k}\mathcal{HH}_e(B, B), \star).$$

In the above statement there is no required compatibility between the quasi-isomorphisms in the zig-zag and the pairings $\langle -, -, \rangle_A$ and $\langle -, -, \rangle_B$. However, the isomorphism we construct between the Goresky–Hingston algebras of $A$ and $B$ uses both pairings $\langle -, -, \rangle_A$ and $\langle -, -, \rangle_B$ as well as the quasi-isomorphisms in the zig-zag.

### 1.3 Outline of the proof

One of the main purposes of this paper is to highlight the techniques used in the proof of the above theorem which rely on the invariance properties of the *singular Hochschild cochain complex*, as introduced in the second author’s thesis and in [24] and extended in [19] to the dg setting. Given any dg algebra $A$ (no Frobenius structure required) the singular Hochschild cochain complex $C^*_{\text{sg}}(A, A)$ is a dg algebra, with a cup product $\cup$ extending the classical cup product on Hochschild cochains (see Section 3.2). We denote the cohomology of $C^*_{\text{sg}}(A, A)$ by $\mathcal{HH}^*_{\text{sg}}(A, A)$ and call it the *singular Hochschild cohomology* of $A$ (also called *Tate–Hochschild cohomology*). Under mild finiteness conditions on $A$, $\mathcal{HH}^*_{\text{sg}}(A, A)$ is the graded algebra of morphisms from $A$ to itself.
In the singularity category

\[ \mathcal{D}_{\text{sg}}(A \otimes A^{\text{op}}) = \mathcal{D}^b(A \otimes A^{\text{op}})/\text{Perf}(A \otimes A^{\text{op}}), \]

that is, the Verdier quotient of the bounded derived category of finitely generated dg \( A\)-\( A\)-bimodules by the full subcategory of perfect \( A\)-\( A\)-bimodules \([19, 24]\). The singularity category was introduced in \([3]\) and used in \([18]\) to study singularities of algebraic varieties.

The proof of our main theorem may be outlined in three steps.

**Step 1:** The isomorphism class of the graded algebra \( \text{HH}^*_\text{sg}(A, A) \) is invariant under quasi-isomorphisms (see Proposition 3.11). Namely, if \( A \) and \( B \) are quasi-isomorphic dg algebras, then there is an isomorphism of graded algebras \( \text{HH}^*_\text{sg}(A, A) \cong \text{HH}^*_\text{sg}(B, B) \).

**Step 2:** When the dg algebra \( A \) is equipped with a dg Frobenius structure of degree \( k \), there is a smaller complex \( D^*(A, A) \), called the Tate–Hochschild complex of \( A \), that also computes \( \text{HH}^*_\text{sg}(A, A) \) (see Definition 4.1). The complex \( D^*(A, A) \) is defined as the mapping cone of a chain map (see Equation (17))

\[ \gamma : s^{-k}C_*(A, A) \to C^*(A, A) \]

from the (shifted) Hochschild chain complex of \( A \) to the Hochschild cochain complex of \( A \). The map \( \gamma \) is defined using the Frobenius structure of \( A \). The Tate–Hochschild complex \( D^*(A, A) \) carries a natural product \( \star \) extending the Goresky–Hingston product on \( C_*(A, A) \), also denoted by \( \star \) above, and the classical Hochschild cup product on \( C^*(A, A) \). In fact, the Goresky–Hingston algebra \( (s^{1-k}\text{HH}_*(A, A), \star) \) is a subalgebra of the cohomology algebra \( (\text{HH}^*(D^*(A, A)), \star) \). We temporarily denote \( (\text{HH}^*(D^*(A, A)), \star) \) by \( (\text{TH}^*(A, A), \star) \). The isomorphism of graded algebras

\[ (\text{HH}^*_\text{sg}(A, A), \cup) \cong (\text{TH}^*(A, A), \star), \]

which follows from Theorem 4.2 and Proposition 4.4, implies that \( (\text{TH}^*(A, A), \star) \) is also invariant under quasi-isomorphisms.

**Step 3:** To conclude the desired invariance for \( (s^{1-k}\text{HH}_*(A, A), \star) \) from the invariance property of \( (\text{TH}^*(A, A), \star) \), we must show the following: If \( A \) and \( B \) are simply connected dg Frobenius algebras that are quasi-isomorphic as dg algebras, then the isomorphism

\[ (\text{TH}^*(A, A), \star) \cong (\text{TH}^*(B, B), \star) \]

restricts to an isomorphism between subalgebras

\[ (s^{1-k}\text{HH}_*(A, A), \star) \cong (s^{1-k}\text{HH}_*(B, B), \star). \]

To show this we consider two cases: when the Euler characteristic of the underlying Frobenius algebras is non-zero or zero. In the first case, when the Euler characteristic of \( A \) (and hence of \( B \)) is non-zero, the result follows immediately from our description of \( \text{TH}^*(A, A) \) in terms of \( \text{HH}^*(A, A) \) and \( \text{HH}_*(A, A) \) (see Proposition 5.1). Namely, in this case \( s^{1-k}\text{HH}_*(A, A) \) (resp. \( s^{1-k}\text{HH}_*(B, B) \)) coincides with the truncation \( \text{TH}^{*\geq k}(A, A) \) (resp. \( \text{TH}^{*\geq k}(B, B) \)) and note that \( \text{TH}^{*\geq k}(A, A) \cong \text{TH}^{*\geq k}(B, B) \) as graded algebras. The second case, when the Euler characteristic of \( A \) (and hence of \( B \)) is zero, is more subtle since by Proposition 5.1 we have \( s^{1-k}\text{HH}_*(A, A) \oplus \text{HH}^k(A, A) \cong \text{TH}^{*\geq k}(A, A) \). This leads us to take a closer look at the functoriality properties of the
quasi-isomorphism

\[ \iota : D^*(A, A) \to C^*_{sg}(A, A) \]

constructed in Theorem 4.2. In Proposition 4.7, we factor this quasi-isomorphism as the composition of two quasi-isomorphisms

\[ D^*(A, A) \to E^*(A, A) \to C^*_{sg}(A, A), \]

where \( E^*(A, A) \) is a new complex constructed using the inverse dualizing complex of \( A \). The complex \( E^*(A, A) \) has a better functorial behavior (see Proposition 4.8 and Corollary 4.9) that allows us to conclude the desired result, namely, the algebra isomorphism \( TH^{* \geq k}(A, A) \cong THH^{* \geq k}(B, B) \) restricts to \( s^{1-k}HH_*(A, A) \cong s^{1-k}HH_*(B, B) \).

In this paper we use \( E^*(A, A) \) merely to treat the Euler characteristic zero case in the proof of our main result. The main advantage of \( E^*(A, A) \) is that it shares similarities with both the Tate–Hochschild complex and the singular Hochschild complex, that is, \( E^*(A, A) \) is constructed as a mapping cone, like \( D^*(A, A) \), but without using a Frobenius structure, like \( C^*_{sg}(A, A) \). In fact, it is plausible that one may develop the whole theory of singular Hochschild cohomology at the chain level entirely in terms of the complex \( E^*(A, A) \) without alluding to \( C^*_{sg}(A, A) \). This will be explored elsewhere.

1.4 Application to string topology

Let \( A \) be a commutative \( \text{dg (cdg) algebra whose cohomology } H^*(A) \text{ is a simply connected graded Frobenius algebra of degree } k. \) By the main result of [15], there exists a cdg Frobenius algebra \( (A, \langle - , - \rangle_A) \) such that \( A^0 \cong k, A^1 = 0 \), and \( A \) is quasi-isomorphic to \( A \) through a zig-zag of cdg algebras such that the induced isomorphism on cohomology preserves the graded Frobenius algebra structure. Following [15], we call \( A \) a Poincaré duality \( (\text{cdg}) \text{ model for } A. \) For any simply connected oriented closed manifold \( M \) of dimension \( k \), the Goresky–Hingston algebra on reduced Hochschild homology induces an algebra structure on \( s^{1-k}HH_*(LM; \mathbb{Q}) \), the shifted reduced rational cohomology of the free loop space on \( M \), by choosing a Poincré duality cdg model \( A \) for the cdg \( \mathbb{Q} \)-algebra \( \mathcal{A}(M) \) of rational polynomial differential forms on \( M \) and using the isomorphisms of graded vector spaces

\[ \overline{H}^*(LM; \mathbb{Q}) \cong \overline{HH}_*(\mathcal{A}(M), \mathcal{A}(M)) \cong \overline{HH}_*(A, A). \]

The first isomorphism above is induced by the Chen’s classical iterated integrals construction or by a well-known result of J.D.S. Jones. The second isomorphism follows from the quasi-isomorphism invariance of Hochschild homology. As an immediate consequence of our main theorem we have the following result.

**Corollary 1.2.**

(1) Let \( M \) be a simply connected oriented closed manifold of dimension \( k \) and \( A \) a Poincaré duality model for the cdg algebra of rational differential forms \( \mathcal{A}(M) \). The isomorphism class of the algebra structure on \( s^{1-k}H^*(LM; \mathbb{Q}) \) induced by the product \( \star \) through the isomorphism \( \overline{H}^*(LM; \mathbb{Q}) \cong \overline{HH}_*(A, A) \) is independent of the choice of Poincaré duality model \( A \) for \( \mathcal{A}(M) \).
If $M$ and $M'$ are homotopy equivalent simply connected oriented closed manifolds of dimension $k$, then the algebra structures on $s^{1-k} \overline{H}^\ast (LM; \mathbb{Q})$ and $s^{1-k} \overline{H}^\ast (LM'; \mathbb{Q})$ are isomorphic.

1.5 Related results in the literature

The geometric constructions for the Goresky–Hingston operation describe an operation defined at the level at the homology (or chains) of the free loop space relative to constant loops, as described in [8, 9, 17, 20]. The identification between the algebraic product on the reduced Hochschild homology of a Poincaré duality cdga model for a simply connected manifold $M$ and a geometric construction of the product on $H^\ast (LM, M)$ using configuration spaces of two points has been announced in [17]. Hingston and Wahl have announced in [10] a version of homotopy invariance for the geometric Goresky–Hingston coproduct on $H_\ast (LM, M)$ (with arbitrary coefficients) for certain type of homotopy equivalences satisfying suitable restrictions. In the non-simply connected case, a counterexample for the homotopy invariance of the geometric Goresky–Hingston coproduct has been announced in [16]. Naef shows that the coproduct can distinguish homotopy equivalent Lens spaces $L(1, 7)$ and $L(2, 7)$.

Proposition 5.1, together with [5, Theorem 1.10], yields that the singular Hochschild cohomology of the dg algebra of cochains (with real coefficients) on a simply connected oriented closed manifold $M$ is isomorphic to the Rabinowitz–Floer homology of the unit cotangent bundle of $M$. We expect that the chain-level algebraic structure of the Tate–Hochschild complex may provide a better understanding of the structure of the geometric operations on Rabinowitz–Floer theory.

For the relationship between the singular Hochschild cochain complex of an algebra and the Hochschild cochain complex of the dg singularity category, we refer to the recent article [13].

1.6 Conventions

Throughout this paper we will work over a fixed field $\mathbb{k}$ of characteristic zero. The sign conventions are obtained from the Koszul sign rule: When $a$ moves past $b$, a sign change of $(-1)^{|a||b|}$ is required. We refer to the appendix of [2] for more on sign conventions. By dg algebras we mean differential graded associative algebras over the field $\mathbb{k}$.

2 GORESKY–HINGSTON ALGEBRA ON THE REDUCED HOCHSCHILD HOMOLOGY OF A CONNECTED DG FROBENIUS ALGEBRA

In this section we start by recalling some classical definitions and constructions. Then we describe the Goresky–Hingston algebra on the reduced Hochschild homology of a dg Frobenius algebra.

2.1 Differential graded associative Frobenius algebras

Let $\mathbb{k}$ be a field. Recall that a dg vector space $(V, d_V)$ is a graded $\mathbb{k}$-vector space $V = \bigoplus_{j \in \mathbb{Z}} V^j$ together with a graded $\mathbb{k}$-linear map $d_V : V \to V$ of degree one (that is, $d_V(V^j) \subset V^{j+1}$) such that $d_V \circ d_V = 0$. For any element $a \in V^j$ we write $|a| = j$. 
The dual of a dg vector space \((V, d_V)\) is defined as the dg vector space \((V^\vee, d_{V^\vee})\) with \((V^\vee)^{-j} = \text{Hom}_K(V^j, K)\) and \(d_{V^\vee}(\alpha)(x) = -(-1)^{|\alpha||x|} \alpha(d_V(x))\) for any homogeneous elements \(\alpha \in V^\vee\) and \(x \in V\). Let \((U, d_U)\) and \((V, d_V)\) be two dg vector spaces. There is a natural inclusion of dg vector spaces

\[
\sigma_{U, V} : U^\vee \otimes V^\vee \hookrightarrow (V \otimes U)^\vee, \quad \alpha \otimes \beta \mapsto (v \otimes u \mapsto \beta(v) \alpha(u)).
\]

If either \(U\) or \(V\) is finite dimensional as a \(K\)-vector space then the above inclusion becomes an isomorphism.

A dg algebra \(A = (A, d, \mu)\) over \(K\) is a dg \(K\)-vector space \((A, d)\) equipped with an associative product \(\mu : A \otimes A \rightarrow A\) of degree zero which is a cochain map, that is, \(\mu\) satisfies the identity

\[
\mu \circ (d \otimes \text{id} + \text{id} \otimes d) = d \circ \mu.
\]

A dg algebra is said to be unital if there is a map \(u : K \rightarrow A\) such that

\[
\mu \circ (u \otimes \text{id}) = \text{id} = \mu \circ (\text{id} \otimes u).
\]

Denote by \(1_A\) or just by 1 the image of \(1_K\) under \(u\).

**Definition 2.1.** Let \(k\) be a positive integer. A dg Frobenius algebra of degree \(k\) is a non-negatively graded dg algebra \((A, d, \mu)\) equipped with a pairing \(\langle -,- \rangle : A \otimes A \rightarrow K\) such that

(i) \(\langle -,- \rangle\) is of degree \(-k\), namely, it is possibly non-zero only on \(A^i \otimes A^{k-i}\) for any \(i = 0, \ldots, k\);

(ii) \(\langle -,- \rangle\) is non-degenerate, namely, the induced map

\[
\rho : A \rightarrow A^\vee, \quad a \mapsto (b \mapsto \langle a, b \rangle)
\]

is an isomorphism of degree \(-k\);

(iii) \(\langle \mu(a \otimes b), c \rangle = \langle a, \mu(b \otimes c) \rangle\) for any \(a, b, c \in A\);

(iv) \(\langle a, b \rangle = (-1)^{|a||b|} \langle b, a \rangle\) for any \(a, b \in A\);

(v) \(\langle d(a), b \rangle = -(-1)^{|a|} \langle a, d(b) \rangle\) for any \(a, b \in A\).

The non-degeneracy of the pairing implies that \(A\) is finite dimensional as a \(K\)-vector space and \(A^i = 0\) for \(i > k\). It follows that the inclusion in (1) induces an isomorphism \(\sigma_{A, A} : A^\vee \otimes A^\vee \xrightarrow{\cong} (A \otimes A)^\vee\). We will write \(\mu(a \otimes b) = ab\).

Conditions (iii)–(v) imply that \(\rho : A \rightarrow A^\vee\) is a map of dg \(A\)-\(A\)-bimodules of degree \(-k\). Recall the \(A\)-\(A\)-bimodule structure on \(A^\vee\) given by

\[
(a \otimes b) \cdot \beta(c) = (-1)^{|\beta||a|+|b|+|a||b|} \beta(bca), \quad \text{for any } \beta \in A^\vee \text{ and } a, b, c \in A.
\]

Denote by \(\Delta : A \rightarrow A \otimes A\) the coproduct of degree \(k\) defined by the composition

\[
A \xrightarrow{\rho} A^\vee \xrightarrow{\mu^\vee} (A \otimes A)^\vee \xrightarrow{\sigma_{A, A}^{-1}} A^\vee \otimes A^\vee \xrightarrow{(\rho \otimes \rho)^{-1}} A \otimes A.
\]

Denote \(\Delta(1) := \sum_i e_i \otimes f_i \in A \otimes A\). In the algebraic literature \(\sum_i e_i \otimes f_i\) is called the Casimir element of the Frobenius algebra \(A\). If we think of \(A\) as a cochain model for a manifold \(M\), then we may think of the Casimir element as a representative for the Thom class of the diagonal embedding \(M \hookrightarrow M \times M\).

The following identities regarding the Carimir element will be useful in our algebraic manipulations.

**Lemma 2.2.** Let \(a \in A\). We have the following identities:
\(\begin{align*}
&\text{(1)} \, (-1)^{|a|k} \sum_i (f_i, a)e_i = a = (-1)^{k-|a|} \sum_i (e_i, a)f_i,
&\text{(2)} \, \sum_i e_i \otimes f_i = \sum_i (-1)^{|e_i||f_i|+k} f_i \otimes e_i,
&\text{(3)} \, \sum_i de_i \otimes f_i = -\sum_i (-1)^{|e_i|} e_i \otimes df_i,
&\text{(4)} \, \sum_i ae_i \otimes f_i = (-1)^{|a|k} \sum_i e_i \otimes f_i a,
&\text{(5)} \, \sum_i (-1)^{|e_i||a|} e_i a \otimes f_i = \sum_i (-1)^{|e_i||a|} e_i \otimes af_i.
\end{align*}\)

**Proof.** By definition, we have \(\sum \sigma_{A,A} \circ (\rho \otimes \rho) \circ \Delta = \mu \circ \rho\). It follows that for any \(a, b \in A\)

\[\sum_i (1-)^{|e_i|k} (f_i, a)\langle e_i, b \rangle = \langle a, b \rangle.\]

By the non-degeneracy of \(\langle -,- \rangle\) we obtain

\[(-1)^{|a|k} \sum_i (f_i, a)e_i = a = (-1)^{k-|a|} \sum_i (e_i, a)f_i,\]

where we implicitly use the fact that \(\langle f_i, a \rangle\) is zero if \(|f_i| + |a| \neq k\).

The second assertion follows by noting

\[\sum_i e_i \otimes f_i = \sum_i (-1)^{|f_i||e_i|} f_i \otimes (f_j, f_i)e_j = \sum_i (-1)^{|f_i||f_j|+|f_i||f_j|} (f_i, f_j)e_i \otimes e_j = \sum_i (-1)^{|e_i||f_j|+k} f_j \otimes e_j,\]

where the first and third equalities follow from the first assertion. Here, we also use the fact that \(\langle f_i, f_j \rangle = 0\) if \(|f_i| \neq |e_j|\). The similar argument yields the remaining assertions. \(\square\)

**Remark 2.3.** The following equations are useful in keeping track of the signs. Using the Koszul sign rule, Lemma 2.2 yields the following identities in \(A \otimes V \otimes A\) for any dg vector space \(V\):

\[\sum_i (-1)^{|x||e_i|} ae_i \otimes x \otimes f_i = \sum_i (-1)^{|x||e_i|+|a||x|+k} e_i \otimes x \otimes f_i a,\]

\[\sum_i (-1)^{|e_i||x|} e_i a \otimes x \otimes f_i = \sum_i (-1)^{|e_i||x|+|a||x|} e_i \otimes x \otimes af_i,\]

where \(x \in V\).

It follows from Lemma 2.2 that \(\Delta : A \rightarrow A \otimes A\) is a map of dg \(A\)-\(A\)-bimodules of degree \(k\). Lemma 2.2 also implies that \(\Delta\) defines a dg coassociative coalgebra structure on \(A\) (of degree \(k\)) with counit \(\varepsilon : A \rightarrow \mathbb{K}\), \(\varepsilon(a) = \langle a, 1 \rangle\), where \(1 \in A^0\) denotes the unit of \(A\). Namely, we have \((\Delta \otimes id) \circ \Delta = (-1)^k (id \otimes \Delta) \circ \Delta\) and \((id \otimes \varepsilon) \circ \Delta = id = (-1)^k (\varepsilon \otimes id) \circ \Delta\).

**Proposition 2.4.** Any dg Frobenius algebra of degree \(k\) satisfies the ‘\(k\)-Calabi-Yau’ property, that is, there is a quasi-isomorphism of dg \(A\)-\(A\)-bimodules between \(s^{-k} A\) and the inverse dualizing complex \(A^! := \text{RHom}_A(A, A^e)\) introduced in [22].
**Proof.** Since $A$ is dg Frobenius of degree $k$, it follows that the enveloping algebra $A^e = A \otimes A^{op}$ is dg Frobenius of degree $2k$ with pairing $\langle a_1 \otimes b_1, a_2 \otimes b_2 \rangle := (-1)^{|a_1||b_1|} \langle a_1, a_2 \rangle \langle b_1, b_2 \rangle$. In particular, it induces a natural isomorphism $s^{2k}A^e \cong (A^e)^\vee$ of right dg $A^e$-modules. Thus, we have the following natural quasi-isomorphisms of dg $A$-$A$-bimodules (that is, dg left $A^e$-modules):

$$s^{-k}A \cong \text{RHom}_{\text{mod}-A^e}(s^{2k}A^e, s^kA) \cong \text{RHom}_{\text{mod}-A^e}((A^e)^\vee, A^\vee) \cong \text{RHom}_{A^e}(A, A^e),$$

(2)

where the first (quasi-)isomorphism follows since $s^{2k}A^e$ is dg projective and we have a natural isomorphism $s^{-k}A \cong \text{Hom}_{\text{mod}-A^e}(s^{2k}A^e, s^kA)$ of left dg $A^e$-modules, the second one follows since $s^{2k}A^e \cong (A^e)^\vee$ and $s^kA \cong A^\vee$ as right dg $A^e$-modules, and the last one follows since the functor

$$(-)^\vee : A^e\text{-mod} \rightarrow \text{mod}-A^e, \quad X \mapsto X^\vee$$

is an equivalence between left and right finite-dimensional dg $A^e$-modules (since $(X^\vee)^\vee \cong X$).

On the other hand, by resolving $A$ as an $A$-$A$-bimodule using the bar resolution, $A!$ can be modeled as the Hochschild cochain complex $C^*(A, A^e)$. Recall that the $A$-$A$-bimodule structure on $C^0(A, A^e)$ is induced by the ‘inner action’ of $A^e$ on $A^e$:

$$(a \otimes b) \cdot (m \otimes n) = (-1)^{|a||b|+|m||b||m|} mb \otimes an.$$ (3)

Then the quasi-isomorphism (2) can be lifted to the following map of dg $A$-$A$-bimodules:

$$\varphi_A : s^{-k}A \rightarrow C^*(A, A^e),$$

(4)

which sends $s^{-k}a$ to $\sum_i (-1)^{|a|+|a-k||e_i|} e_i a \otimes f_i \in C^0(A, A^e) = A^e$. Here, we stress that the $A$-$A$-bimodule structure on $C^0(A, A^e) = A^e$ is given by the inner action (3). □

### 2.2 Hochschild chains and cochains

Let $(A = \mathbb{k}.1 \oplus \overline{A}, d, \mu)$ be an augmented dg algebra where $\overline{A}$ is the kernel of the augmentation map. Recall that there is an isomorphism $\overline{A} \cong A/\mathbb{k}.1$.

Recall that for any dg vector space $(V, d)$ we denote by $(s^iV, s^i d)$ the $i$-th shifted space given by $(s^iV)^j = V^{i+j}$ and $s^i d(v) = (-1)^i d(s^i v)$ for any $v \in V$. For simplicity, we write $\overline{a}$ for the element $sa \in s\overline{A}$ where $a \in \overline{A}$.

**Definition 2.5.** Denote by $C_{-m,n}(A, A) = ((s\overline{A})^\otimes m \otimes A)^n$, that is, elements in $(s\overline{A})^\otimes m \otimes A$ of total degree $n$. Let $C_n(A, A) = \bigoplus_{m \in \mathbb{Z}} C_{-m,n}(A, A)$ and $C^*_n(A, A) = \bigoplus_{n \in \mathbb{Z}} C_n(A, A)$. The **Hochschild chain complex of $A$** is the complex $(C_n(A, A), \partial = \partial_v + \partial_h)$ where $\partial_v$ is the *internal* differential given by

$$\partial_v(\overline{a}_1 \otimes \cdots \otimes \overline{a}_m \otimes a_{m+1}) = - \sum_{i=1}^m (-1)^{\delta_{i-1}} \overline{a}_1 \otimes \cdots \otimes \overline{a}_{i-1} \otimes d(a_i) \otimes \overline{a}_{i+1} \otimes \cdots \otimes a_{m+1}$$

$$+ (-1)^{\delta_m} \overline{a}_1 \otimes \cdots \otimes \overline{a}_m \otimes d(a_{m+1})$$

where

$$\delta_i := \begin{cases} 0 & \text{if } i \text{ odd} \\ 1 & \text{if } i \text{ even} \end{cases}$$
and \( \partial_h \) is the external differential given by

\[
\partial_h(a_1 \otimes \cdots \otimes a_m \otimes a_{m+1}) = \sum_{i=1}^{m-1} (-1)^{\varepsilon_i} a_1 \otimes \cdots \otimes \overline{a}_{i-1} \otimes a_i a_{i+1} \otimes \overline{a}_{i+2} \otimes \cdots \otimes a_{m+1} \\
- (-1)^{\varepsilon_{m-1}} a_1 \otimes \cdots \otimes \overline{a}_{m-1} \otimes a_m a_{m+1} \\
+ (-1)^{|a_2| + \cdots + |a_m+1| - m+1} |a_1| a_2 \otimes \cdots \otimes \overline{a}_m \otimes a_{m+1} a_1.
\]

Here we denote \( \varepsilon_i = |a_1| + \cdots + |a_i| - i \) and \( \varepsilon_0 = 0 \).

Remark 2.6. Note that an element \( a_1 \otimes \cdots \otimes a_m \otimes a_{m+1} \in (sA)^{\otimes m} \otimes A \) belongs to \( C_n(A, A) \) if and only if \( |a_1| + |a_2| + \cdots + |a_m+1| - m = n \). The differential \( \partial \) on \( C_*(A, A) \) is of degree \(+1\).

We use lower index notation in \( C_*(A, A) \) to distinguish from the Hochschild cochain complex defined below.

Definition 2.7. Denote by \( C^{m,n}(A,A) = \text{Hom}_\mathbb{K}^n((sA)^{\otimes m}, A) \), that is, \( \mathbb{K} \)-linear maps \( (sA)^{\otimes m} \to A \) of degree \( n \in \mathbb{Z} \). Let \( C^n(A,A) = \prod_{m \in \mathbb{Z} \geq 0} C^{m,n}(A,A) \) and \( C^*(A,A) = \bigoplus_{n \in \mathbb{Z}} C^n(A,A) \). The Hochschild cochain complex of \( A \) is the complex \( (C^*(A,A), \delta = \delta_v + \delta_h) \) where \( \delta_v \) is the internal differential given by

\[
\delta_v(f)(a_1 \otimes \cdots \otimes a_m) = d(f(a_1 \otimes \cdots \otimes a_m)) + \sum_{i=1}^{m} (-1)^{|f|+\varepsilon_{i-1}} f(a_1 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_m),
\]

and \( \delta_h \) is the external differential given by

\[
\delta_h(f)(a_1 \otimes \cdots \otimes a_{m+1}) = -(-1)^{|a_1|-1}|f| a_1 f(a_2 \otimes \cdots \otimes a_{m+1}) \\
- \sum_{i=1}^{m} (-1)^{|f|+\varepsilon_i} f(a_1 \otimes \cdots \otimes \overline{a}_{i-1} \otimes a_i a_{i+1} \otimes \overline{a}_i \otimes \cdots \otimes a_{m+1}) \\
+ (-1)^{|f|+\varepsilon_m} f(a_1 \otimes \cdots \otimes \overline{a}_m) a_{m+1},
\]

where \( \varepsilon_i = |a_1| + \cdots + |a_i| - i \) and \( \varepsilon_0 = 0 \).

Using similar formulas as the ones above, we may define for any dg \( A \)-\( A \)-bimodule \( M \) the Hochschild chain and cochain complexes \( C_*(A,M) \) and \( C^*(A,M) \) by setting, respectively,

\[
C_n(A,M) = \bigoplus_{m \geq 0} ((sA)^{\otimes m} \otimes M)^n \quad \text{and} \quad C^n(A,M) = \prod_{m \geq 0} \text{Hom}_\mathbb{K}^n((sA)^{\otimes m}, M).
\]

We denote \( \text{HH}_i(A,M) = H^i(C^*(A,M)) \) and \( \text{HH}_i(A,M) = H^{-i}(C_*(A,M)) \) for any \( i \in \mathbb{Z} \).

2.3 Goresky–Hingston algebra on reduced Hochschild homology

Let \( A \) be a connected dg Frobenius algebra of degree \( k > 0 \). In particular \( A^0 \cong \mathbb{K} \cong A^k \). We follow the notation of Section 2.1.
Definition 2.8. Define a product $\star : C_*(A, A) \otimes C_*(A, A) \to C_*(A, A)$ of degree $k - 1$ as follows:

For any $\alpha = a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}$ and $\beta = b_1 \otimes \cdots \otimes b_q \otimes b_{q+1}$ let

$$
\alpha \star \beta = \sum_{i} (-1)^{\eta_i} b_1 \otimes \cdots \otimes b_{q+1} e_i \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i,
$$

where $\eta_i = |\alpha||f_i| + |b_{q+1}| + (|\alpha| + k - 1)(|\beta| + k - 1)$. The product $\star$ induces a degree zero product on the $(1 - k)$-shifted graded vector space $s^{1-k}C_*(A, A)$.

Note that $\star$ does not satisfy the Leibniz rule with respect to the Hochschild chains differential $\delta$. In fact, if $p > 0$ and $q > 0$, we have (cf. Remark 4.5)

$$
\delta(\alpha \star \beta) - \delta(\alpha) \star \beta - (-1)^{|\alpha|+k-1} \alpha \star \delta(\beta) = 0,
$$

but if $p = 0$, so that $\alpha = a_1 \in C_{0,0}(A, A) = A$, then

$$
\delta(\alpha \star \beta) - \delta(\alpha) \star \beta - (-1)^{|\alpha|+k-1} \alpha \star \delta(\beta) = \sum_{i} (-1)^{|\beta|+1-|b_{q+1}|} b_1 \otimes \cdots \otimes b_q \otimes b_{q+1} e_i a_1 f_i.
$$

A completely analogous computation yields that if $q = 0$ there is a similar obstruction for $\star$ to satisfy the Leibniz rule. However, note that by degree reasons $e_i a_1 f_i$ is only non-zero if $a_1 \in A^0 \cong \mathbb{K}$, and, in such a case, $e_i a_1 f_i \in A^k \cong \mathbb{K}$. Hence, $\star$ induces a non-unital dg algebra structure on the reduced Hochschild chain complex, defined as follows.

Definition 2.9. The reduced Hochschild chain complex, denoted by $\overline{C}_*(A, A)$, of a connected non-negatively graded dg algebra $A$ is the subcomplex of $C_*(A, A)$ defined as the complement of $C_{0,0}(A, A) = A^0 \cong \mathbb{K} \subset C_*(A, A)$. More precisely, $\overline{C}_*(A, A)$ is the total complex of the sub-double complex $\overline{C}_{i,j}(A, A) \subset C_{i,j}(A, A)$ given by $\overline{C}_{0,0}(A, A) = 0$ and $\overline{C}_{i,j}(A, A) = C_{i,j}(A, A)$ for all pairs of integers $(i, j) \neq (0, 0)$.

Remark 2.10. Note that if $A$ is a simply connected dg algebra (that is, $A^0 \cong \mathbb{K}$ and $A^1 = 0$), then

$$
\overline{HH}_i(A, A) = \begin{cases} 
HH_i(A, A) & \text{if } i > 0 \\
0 & \text{otherwise},
\end{cases}
$$

since in this case we have $\overline{C}_{\leq 0}(A, A) = 0$ and $\overline{C}_{> 0}(A, A) = C_{>0}(A, A)$. In other words, we have $\overline{HH}_*(A, A) \cong HH_0(A, A) \oplus \overline{HH}_*(A, A) \cong \mathbb{K} \oplus \overline{HH}_*(A, A)$.

The following proposition is now easy to check.

Proposition 2.11. Let $A$ be a connected dg Frobenius algebra of degree $k$. Then the triple $(s^{1-k}\overline{C}_*(A, A), \delta, \star)$ is a (non-unital) dg algebra.

Definition 2.12. We call $(s^{1-k}\overline{HH}_*(A, A), \star)$ the Goresky–Hingston algebra on the reduced Hochschild homology of the connected dg Frobenius algebra $A$. 
Remark 2.13. The product $\star$ has been studied in the context of commutative dg Frobenius algebras in [2, 14]. In the commutative case, the complement of $C_{0,*}(A, A) = A \subset C_*(A, A)$ is a subcomplex of $C_*(A, A)$. The fact that $\star$ can be extended to the associative possibly non-commutative case has also been observed in [12].

Remark 2.14. If the Euler characteristic $\chi(A) := \mu \circ \Delta(1) = \sum_i e_i f_i = 0$, then the Leibniz rule (5) for $\star$ and the differential $\partial$ of the Hochschild chain complex always hold, namely, the failure of $\star$ in being a chain map vanishes in this case. Hence, when $\chi(A) = 0$ the Goersky–Hingston product is well defined on the whole complex $HH_*(A, A)$.

3 | SINGULAR HOCHSCHILD COHOMOLOGY ALGEBRA

We recall the definition of the singular Hochschild cochain complex and its cup product, as well as its invariance properties, following [24] and [19], that will be used in the proof of our main theorem. Throughout this section, we fix $A$ to be a unital dg algebra (no Frobenius structure is assumed).

3.1 | Singular Hochschild cochain complex

For any unital dg algebra $A$ and non-negative integer $p$ let $\text{Bar}_{-p}(A) := A \otimes (sA)^{\otimes p} \otimes A$. Note that $\text{Bar}_{-p}(A)$ is a dg $A$-$A$-bimodule with differential

$$d_{-p}(a_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) = d(a_0) \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}$$

$$- \sum_{i=1}^{p} (-1)^{|a_0|+\varepsilon_i} a_0 \otimes \cdots \otimes d(a_i) \otimes \cdots \otimes a_{p+1}$$

$$+ (-1)^{|a_0|+\varepsilon_p} a_0 \otimes \cdots \otimes a_p \otimes d(a_{p+1}),$$

where $\varepsilon_i = |a_1| + \cdots + |a_i| - i$. For each positive integer $p$ denote by

$$b_{-p} : \text{Bar}_{-p}(A) \rightarrow \text{Bar}_{-p+1}(A)$$

the map of degree one

$$b_{-p}(a_0 \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}) = (-1)^{|a_0|} a_0 a_1 \otimes a_2 \otimes \cdots \otimes a_p \otimes a_{p+1}$$

$$+ \sum_{i=1}^{p-1} (-1)^{|a_0|+\varepsilon_i} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \otimes a_{p+1}$$

$$- (-1)^{|a_0|+\varepsilon_{p-1}} a_0 \otimes \cdots \otimes a_{p-1} \otimes a_p a_{p+1},$$

where $\varepsilon_i = |a_1| + \cdots + |a_i| - i$. 
Set $\text{Bar}_s(A) := \bigoplus_{p=0}^{\infty} \text{Bar}_p(A)$. Note that $\text{Bar}_s(A)$, equipped with the differential $b_s + d_s$ defined as above, is the normalized bar resolution of $A$ (cf., for example, [1, Section 1]). In particular,

\[ b_{-p+1} \circ b_p = 0, \quad d_{-p} \circ d_p = 0, \quad b_{-p} \circ d_p + d_{-p+1} \circ b_p = 0, \quad \text{for any } p \geq 0. \]

It follows from the above identity $b_{-p} \circ d_p + d_{-p+1} \circ b_p = 0$ that $b_{-p}$ is a morphism of dg $A$-$A$-bimodules of degree one. Define $\Omega_{nc}^{p-1}(A) := \text{Coker}(b_{-p})$ for $p \geq 1$. Since each $b_{-p}$ is a map of dg $A$-$A$-bimodules, $\Omega_{nc}^{p-1}(A)$ inherits a dg $A$-$A$-bimodule structure. Let $\pi : A \twoheadrightarrow sA$ be the natural projection of degree $-1$. Then we have the following description of $\Omega_{nc}^{p}(A)$.

**Lemma 3.1.** For each $p \in \mathbb{Z}_{\geq 0}$, there is a natural isomorphism of dg $A$-$A$-bimodules

\[ \alpha : \Omega_{nc}^{p}(A) \xrightarrow{\cong} (sA)^{\otimes p} \otimes A, \]

where the left $A$-module structure in $(sA)^{\otimes p} \otimes A$ is given by

\[ a \triangleright (\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1}) := (\pi \otimes \text{id}^{\otimes p})(b_{-p}(a \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1})), \]

the right $A$-module structure is given by multiplication on the right $A$ factor of $(sA)^{\otimes p} \otimes A$, and the differential on $(sA)^{\otimes p} \otimes A$ is the tensor differential, that is, given by

\[ d(\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1}) = (-1)^{\varepsilon_p} \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes d(a_{p+1}) \]

\[ - \sum_{i=1}^{p} (-1)^{\varepsilon_{i-1}} \overline{a}_1 \otimes \cdots \otimes d(\overline{a}_i) \otimes \cdots \otimes a_{p+1}, \]

where $\varepsilon_i = |a_1| + \cdots + |a_{i-1}| + i - 1$.

**Proof.** Since $b_{-p} \circ b_{-p-1} = 0$, the map $b_{-p}$ factors through $\Omega_{nc}^{p}(A)$. Namely,

\[
\begin{array}{c}
\Omega_{nc}^{p}(A) \\
\xrightarrow{\tilde{b}_{-p}} \\
\xrightarrow{\hat{b}_{-p}} \\
\text{Bar}_{-p-1} \xrightarrow{b_{-p-1}} \text{Bar}_{-p}(A) \xrightarrow{b_{-p}} \text{Bar}_{-p+1}(A),
\end{array}
\]

where $\tilde{b}_{-p}$ and $\hat{b}_{-p}$ are the natural maps induced by the cokernel.

Define $\alpha : \Omega_{nc}^{p}(A) \to (sA)^{\otimes p} \otimes A$ as the composition

\[
\Omega_{nc}^{p}(A) \xrightarrow{\tilde{b}_{-p}} A \otimes (sA)^{\otimes p-1} \otimes A \xrightarrow{\pi \otimes \text{id}^{\otimes p-1} \otimes \text{id}_A} sA \otimes (sA)^{\otimes p-1} \otimes A = (sA)^{\otimes p} \otimes A.
\]
It is clear that $\alpha$ is a morphism of dg $A$-$A$-bimodules. The inverse of $\alpha$ is given by the composition $\beta : (sA) \otimes^p A \rightarrow A \otimes (sA) \otimes^p A \rightarrow \Omega^p_{nc}(A)$, where the first map is

$$
\bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes a_{p+1} \mapsto 1 \otimes \bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes a_{p+1}.
$$

Note that we have $\alpha \circ \beta = \text{id}$ and $\beta \circ \alpha = \text{id}$.

**Remark 3.2.** We have an exact sequence of dg $A$-$A$-bimodules (of degree one)

$$
0 \rightarrow s^{-1} \Omega^1_{nc}(A) \rightarrow \text{Bar}^{-1}(A) \rightarrow \text{Bar}^{-1}(A) \rightarrow 0
$$

for any $p > 0$. In particular, we have a short exact sequence of dg $A$-$A$-bimodules

$$
0 \rightarrow s^{-1} \Omega^p_{nc}(A) \rightarrow A \otimes A \rightarrow 0.
$$

For all $p, q \geq 0$ there is a natural isomorphism of dg $A$-$A$-bimodules

$$
\chi_{p,q} : \Omega^p_{nc}(A) \otimes_A \Omega^q_{nc}(A) \rightarrow \Omega^{p+q}_{nc}(A),
$$

which sends $(\bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes a_{p+1}) \otimes (\bar{b}_1 \otimes \cdots \otimes \bar{b}_p \otimes b_{q+1})$ to

$$
\bar{a}_1 \otimes \cdots \otimes \bar{a}_p \otimes (a_{p+1} \triangleright (\bar{b}_1 \otimes \cdots \otimes \bar{b}_p \otimes b_{q+1})).
$$

In particular, we get $\Omega^p_{nc}(A) \cong \Omega^1_{nc}(A) \otimes_A \cdots \otimes_A \Omega^1_{nc}(A)$ for $p > 0$. For this reason, we call $\Omega^p_{nc}(A)$ the noncommutative differential $p$-forms of $A$.

We identify $\Omega^p_{nc}(A)$ with $(sA) \otimes^p A$ via the isomorphism $\alpha$ in Lemma 3.1 from now on. Consider the Hochschild cochain complex $C^*(A, \Omega^p_{nc}(A))$ with coefficients in the dg $A$-$A$-bimodule $\Omega^p_{nc}(A)$. Namely, we have

$$
C^{m,n}(A, \Omega^p_{nc}(A)) = \text{Hom}_A((sA)^{m}, (sA)^{p} \otimes A)^n,
$$

$$
C^n(A, \Omega^p_{nc}(A)) = \prod_{m \in \mathbb{Z} \geq 0} C^{m,n}(A, \Omega^p_{nc}(A)).
$$
Define a morphism of (total) degree zero

$$\theta_{m,p} : C^{m,*}(A, \Omega^p_{nc}(A)) \to C^{m+1,*}(A, \Omega^{p+1}_{nc}(A)),$$

which sends $f \in \text{Hom}_K((sA)^{\otimes m}, (sA)^{\otimes p} \otimes A)$ to $\theta_{m,p}(f) \in \text{Hom}_K((sA)^{\otimes m+1}, (sA)^{\otimes p+1} \otimes A)$ given by the following formula:

$$\theta_{m,p}(f)(a_1 \otimes \cdots \otimes a_{m+1}) = (-1)^{|a_1|-1 |f|} a_1 \otimes f(a_2 \otimes \cdots \otimes a_{m+1}).$$

Write $\theta_p = \prod_{m \in \mathbb{Z}_{\geq 0}} \theta_{m,p}$. We have that $\theta_p$ is compatible with the differentials. Namely, the following diagram commutes:

$$
\begin{array}{ccc}
C^*(A, \Omega^p_{nc}(A)) & \xrightarrow{\theta_p} & C^*(A, \Omega^{p+1}_{nc}(A)) \\
\downarrow{\delta} & & \downarrow{\delta} \\
C^{*+1}(A, \Omega^p_{nc}(A)) & \xrightarrow{\theta_p} & C^{*+1}(A, \Omega^{p+1}_{nc}(A)).
\end{array}
$$

The commutativity of the above diagram can also be verified immediately from formula (9) below. Hence, the maps $\theta_p$ for $p \geq 0$ form an inductive system of cochain complexes.

**Definition 3.3.** Define

$$C^*_s(A, A) := \lim_{\theta_p} C^*(A, \Omega^0_{nc}(A)),$$

as the colimit of the inductive system of cochain complexes

$$C^*(A, A) \xrightarrow{\theta_0} C^*(A, \Omega^1_{nc}(A)) \xrightarrow{\theta_1} \cdots \xrightarrow{\theta_{p-1}} C^*(A, \Omega^p_{nc}(A)) \xrightarrow{\theta_p} C^*(A, \Omega^{p+1}_{nc}(A)) \xrightarrow{\theta_{p+1}} \cdots.$$

Since the $\theta_p$ are compatible with the Hochschild differentials, there is an induced differential

$$\delta_s : C^*_s(A, A) \to C^*_s(A, A).$$

We call $(C^*_s(A, A), \delta_s)$ the *singular Hochschild cochain complex* of $A$. We denote its cohomology by $HH^s(A, A)$ and call it *singular Hochschild cohomology* of $A$ (also called Tate–Hochschild cohomology of $A$).

### 3.2 Cup product on $C^*_s(A, A)$

Recall that $C^{m,*}(A, \Omega^p_{nc}(A)) = \text{Hom}_K((sA)^{\otimes m}, (sA)^{\otimes p} \otimes A)$. 

Definition 3.4. Let $f \in C^{m,*}(A, \Omega^p_{nc}(A))$ and $g \in C^{n,*}(A, \Omega^q_{nc}(A))$. Define the cup product $f \cup g \in C^{m+n,*}(A, \Omega^{p+q}_{nc}(A))$ by

$$f \cup g := (id \otimes f \otimes \mu) \circ (id \otimes g \otimes \mu) \circ (id \otimes f \otimes \mu),$$

where we have identified $\Omega^p_{nc}(A)$ with $(sA)^p \otimes A$ as in Lemma 3.1 and denoted by $\mu : A \otimes A \to A$ the multiplication of $A$. More precisely, $f \cup g$ is given by the following composition:

$$(sA)^{m+n} \xrightarrow{id \otimes f \otimes \mu} (sA)^{m+n} \otimes A \xrightarrow{id \otimes g \otimes f \otimes \mu} (sA)^{m+n} \otimes A \otimes A \xrightarrow{id \otimes \mu} (sA)^{m+n} \otimes A.$$

When applying the above composition to an element the Koszul sign rule is used. For instance,

$$g(a_{m+1} \otimes \cdots \otimes a_{m+n}) = \sum g(b_1, i) \otimes g(b_2, i) \otimes \cdots \otimes g(b_q, i) \otimes b_{q+1, i}.$$ 

If $m \geq q$ we have

$$f \cup g(a_1 \otimes \cdots \otimes a_{m+n}) = \sum (-1)^{\varepsilon} a_1 \otimes \cdots \otimes a_q \otimes f(a_{q+1} \otimes \cdots \otimes a_m \otimes b_1, i \otimes \cdots \otimes b_{q+1, i}).$$

(8)

Here the isomorphism $\Omega^p_{nc}(A) \cong \text{Hom}_\kappa(\kappa, \Omega^p_{nc}(A))$ is given by $\alpha \mapsto (1 \mapsto \alpha)$. The above identity will be used in the proof of Proposition 4.4.

Consider the de Rham cocycle $d_{dR} : a \mapsto a \otimes 1$ in $C^{1,*}(A, \Omega^1_{nc}(A))$. Note that $\delta(d_{dR}) = 0$. So $d_{dR}$ is indeed a cocycle. Each structure map $\theta_p : C^{*(A, \Omega^p_{nc}(A))} \to C^{*(A, \Omega^{p+1}_{nc}(A))}$ is given by the cup product with $d_{dR}$, namely,

$$\theta_p(f) = f \cup d_{dR}, \quad \text{for any } f \in C^{*(A, \Omega^p_{nc}(A))}.$$ 

(9)

Since $d_{dR} \cup f = f \cup d_{dR}$, it follows that $\cup$ is compatible with the structure maps $\theta_p$, so it induces a well-defined product

$$\cup : C^{*}_{sg}(A, A) \otimes C^{*}_{sg}(A, A) \to C^{*}_{sg}(A, A).$$

The following is straightforward to verify. We refer to [24, section 4.1] for details.

Proposition 3.5. The cup product defines a unital dg algebra structure $(C^{*}_{sg}(A, A), \delta_{sg}, \cup)$. This induces a graded commutative algebra structure $(\text{HH}^*_{sg}(A, A), \cup)$. 

In the rest of this subsection, we will show that the cup product $\cup$ on $\text{HH}^*_{sg}(A, A)$ may be obtained from the classical Hochschild cup product construction. Consider the short exact
sequence of dg $A$-$A$-bimodules (cf. (6))

$$0 \to s^{-1} \Omega_{\text{nc}}^{p+1}(A) \hookrightarrow \text{Bar}_{-p}(A) \to \Omega_{\text{nc}}^{p}(A) \to 0.$$ 

Apply the derived functor $\text{HH}^*(A,-)$ to the above to obtain a long exact sequence

$$\cdots \to \text{HH}^*(A, \text{Bar}_{-p}(A)) \to \text{HH}^*(A, \Omega_{\text{nc}}^{p}(A)) \xrightarrow{\partial_{s,p}} \text{HH}^{*+1}(A, s^{-1} \Omega_{\text{nc}}^{p+1}(A)) \to \cdots,$$

where $\partial_{s,p}$ denotes the connecting morphism. Note that there is a natural isomorphism

$$\text{HH}^{*+1}(A, s^{-1} \Omega_{\text{nc}}^{p+1}(A)) \cong \text{HH}^*(A, \Omega_{\text{nc}}^{p+1}(A)).$$

Consider the following inductive system of graded vector spaces:

$$\text{HH}^*(A, A) \xrightarrow{\partial_{s,0}} \cdots \xrightarrow{\partial_{s,-1}} \text{HH}^*(A, \Omega_{\text{nc}}^{p}(A)) \xrightarrow{\partial_{s,p}} \text{HH}^*(A, \Omega_{\text{nc}}^{p+1}(A)) \xrightarrow{\partial_{s,p+1}} \cdots$$

and denote its colimit by $\varinjlim_{p} \text{HH}^*(A, \Omega_{\text{nc}}^{p}(A))$.

For any $p, q \geq 0$ and $m, n \in \mathbb{Z}$, consider the product defined as the composition

$$\mathcal{U}': \text{HH}^m(A, \Omega_{\text{nc}}^{p}(A)) \otimes \text{HH}^n(A, \Omega_{\text{nc}}^{q}(A)) \to \text{HH}^{m+n}(A, \Omega_{\text{nc}}^{p}(A) \otimes_A \Omega_{\text{nc}}^{q}(A)) \cong \text{HH}^{m+n}(A, \Omega_{\text{nc}}^{p+q}(A)),$$

where the first map is given by the classical Hochschild cup product construction and the second is induced by the isomorphism $\kappa_{p,q}: \Omega_{\text{nc}}^{p}(A) \otimes_A \Omega_{\text{nc}}^{q}(A) \xrightarrow{\cong} \Omega_{\text{nc}}^{p+q}(A)$ in Remark 3.2. Note that $\mathcal{U}'$ can be lifted to the cochain complex level

$$\text{HH}^m(A, \Omega_{\text{nc}}^{p}(A)) \otimes \text{HH}^n(A, \Omega_{\text{nc}}^{q}(A)) \to \text{HH}^{m+n}(A, \Omega_{\text{nc}}^{p+q}(A))$$

by

$$f \otimes g(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{\epsilon} \kappa_{p,q}(f(a_1 \otimes \cdots \otimes a_m) \otimes_A g(a_{m+1} \otimes \cdots \otimes a_{m+n})),$$

where $\epsilon = (|a_1| + \cdots + |a_m| - m)|g|$. In general, $\mathcal{U}'$ is not compatible with the structure maps $\partial_{s,p}$ at the cochain complex level. However, by the functoriality of the classical Hochschild cup product, $\mathcal{U}'$ is compatible with the connecting morphism $\partial_{m,p}$ at the cohomology level. This induces a well-defined product (still denoted by $\mathcal{U}'$) on the colimit $\varinjlim_{p} \text{HH}^m(A, \Omega_{\text{nc}}^{p}(A))$.

**Proposition 3.6.** There is a natural isomorphism of graded algebras

$$(\text{HH}^*_\text{sg}(A, A), \mathcal{U}) \cong (\varinjlim_{p} \text{HH}^*(A, \Omega_{\text{nc}}^{p}(A)), \mathcal{U}')$$.
Proof. First we claim that $H^*(\partial_p) = \partial_p$ for any $p \in \mathbb{Z}_{\geq 0}$. Indeed, since the bar resolution $\text{Bar}_*(A)$ is a projective resolution of $A$ as a dg $A$-$A$-bimodule, we have

$$\text{HH}^m(A, \Omega^p_{nc}(A)) \cong \text{Hom}_{\mathcal{D}(A \otimes A^{op})}(A, s^m \Omega^p_{nc}(A)) \cong \text{Hom}_{\mathcal{K}(A \otimes A^{op})}(\text{Bar}^*(A), s^m \Omega^p_{nc}(A)),$$

where $\mathcal{D}(A \otimes A^{op})$ is the derived category of dg $A$-$A$-bimodules and $\mathcal{K}(A \otimes A^{op})$ is the homotopy category of dg $A$-$A$-bimodules. For any dg $A$-$A$-bimodule morphism $f : \text{Bar}^{-n}(A) \to s^m \Omega^p_{nc}(A)$ of degree zero, we have a commutative diagram

$$
\begin{array}{ccc}
\text{Bar}^{-n}(A) & \xrightarrow{b-1} & \text{Bar}^{-n}(A) \\
\downarrow{\theta_{n+1,p+1}(f)} & & \downarrow{f} \\
\text{s}^{m-1}(\bar{s}A)^{p+1} \otimes A & \rightarrow & \text{s}^m \text{Bar}^p(A) \\
\end{array}
$$

where the maps in the bottom row are from the two middle maps in the short exact sequence (6), the map $\text{id} \otimes f : \text{Bar}^{-n}(A) \to s^m \text{Bar}^{-p}(A)$ is defined by

$$(\text{id} \otimes f)(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = (-1)^{|a_0|} s^m(a_0 \otimes s^{-m} f(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1})),$$

and $\theta_{n+1,p+1}(f)$ is given by (7). This shows that $H^*(\partial_p)$ is precisely the connecting morphism

$$\partial_p : H^*(A, \Omega^p_{nc}(A)) \rightarrow H^*(A, \Omega^{p+1}_{nc}(A))$$

of the long exact sequence (10).

Since filtered colimits commute with homology, there is an isomorphism of vector spaces

$$\text{HH}^*_{sg}(A, A) \cong \lim_{\theta_p} \text{HH}^*(A, \Omega^p_{nc}(A)) \cong \lim_{\theta_p} \text{HH}^*(A, \Omega^p_{nc}(A)).$$

The fact that this is an isomorphism of algebras follows from the observation that the two products $\cup, \cup' : C^*(A, \Omega^p_{nc}(A)) \otimes C^*(A, \Omega^q_{nc}(A)) \rightarrow C^*(A, \Omega^{p+q}_{nc}(A))$ (see (11) and Definition 3.4) agree up to chain homotopy. More precisely, let $f \in C^{m,*}(A, \Omega^p_{nc}(A))$ and $g \in C^{n,*}(A, \Omega^q_{nc}(A))$. The chain homotopy for $f \cup g = f \cup' g$ is given by

$$g \cdot_{<0} f := \sum_{i=1}^q g \cdot_{-i} f,$$

where $g \cdot_f f$ is given in Definition 3.7 below. That is, we have

$$f \cup g - f \cup' g = \delta(g \cdot_{<0} f) - \delta(g) \cdot_{<0} f - (-1)^{|g|} g \cdot_{<0} \delta(f).$$

When $q = 0$, we note that $f \cup g = f \cup' g$ and $g \cdot_{<0} f = 0$. For the general case, identity (12) may be verified by a direct computation. □
Definition 3.7. Let \( f \in C^{m,*}(A, \Omega^p_{nc}(A)) \) and \( g \in C^{n,*}(A, \Omega^q_{nc}(A)) \). Define a bracket \( \{f, g\} \in C^{m+n-1,*}(A, \Omega^{p+q}_{nc}(A)) \) as

\[
\{f, g\} = f \cdot g - (-1)^{|f||g|+1} g \cdot f,
\]

where

\[
f \cdot g = \sum_{i=1}^{m} f \cdot_i g - \sum_{i=1}^{p} f \cdot_{-i} g
\]

and \( f \cdot_i g \) is defined by

\[
f \cdot_i g := \begin{cases} 
(id^{\otimes q} \otimes f) \circ (id^{i-1} \otimes ((id^{\otimes q} \otimes \pi) \circ g) \otimes id^{\otimes m-i}) & \text{for } 1 \leq i \leq m, \\
(id^{\otimes p+i} \otimes ((id^{\otimes q} \otimes \pi) \circ g) \otimes id^{\otimes -i}) \circ (id^{\otimes n-1} \otimes f) & \text{for } -p \leq i \leq -1,
\end{cases}
\]

where \( \pi : A \rightarrow sA \) is the natural projection map of degree \(-1\), and we identify \( \Omega^p_{nc}(A) \) with \((sA)^{\otimes p} \otimes A\) as in Lemma 3.1. In particular, when \( p = q = 0 \) we recover the classical Gerstenhaber bracket on \( C^*(A, A) \). It follows from a direct calculation that the bracket is compatible with the colimit construction, thus the bracket is well defined on \( C^*_sg(A, A) \).

We will not use the following result in this paper but we recall it for general interest.

Theorem 3.8 [24, Corollary 5.3]. The cup product \( \cup \) and the bracket \( \{\cdot, \cdot\} \) defined above give the singular Hochschild cochain complex \( C^*_sg(A, A) \) the structure of a unital dg algebra and a dg Lie algebra of degree \(-1\), respectively. Moreover, \( \cup \) and \( \{\cdot, \cdot\} \) induce a Gerstenhaber algebra structure on \( HH^*_sg(A, A) \).

3.3 Invariance of singular Hochschild cohomology under quasi-isomorphisms

Let \( A \) and \( B \) be two unital dg algebras. Let \( \varphi : A \rightarrow B \) be a morphism of dg algebras. Any dg \( B\)-\( B \)-bimodule can be viewed as a dg \( A\)-\( A \)-bimodule via \( \varphi \). Then \( \varphi \) induces a map of dg \( A\)-\( A \)-bimodules \( \Omega^p_{nc}(\varphi) : \Omega^p_{nc}(A) \rightarrow \Omega^p_{nc}(B) \) for \( p \in \mathbb{Z}_{\geq 0} \) given by

\[
\Omega^p_{nc}(\varphi)(\overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}) := \overline{\varphi(a_1)} \otimes \cdots \otimes \overline{\varphi(a_p)} \otimes \varphi(a_{p+1}),
\]

where we use Lemma 3.1 to identify \( \Omega^p_{nc}(A) \) with \((sA)^{\otimes p} \otimes A\) and \( \Omega^p_{nc}(B) \) with \((sB)^{\otimes p} \otimes B\). Moreover, if \( \varphi \) is a quasi-isomorphism, then so is \( \Omega^p_{nc}(\varphi) \).

Now let us construct the singular Hochschild cochain complex \( C^*_sg(A, B) \) with coefficients in \( B \). Consider the Hochschild cochain complex \( C^*(A, \Omega^p_{nc}(B)) \) with coefficients in the dg \( A\)-\( A \)-bimodule \( \Omega^p_{nc}(B) \). Similar to (7), we define a morphism of cochain complexes

\[
\Theta^{A,B}_p : C^*(A, \Omega^p_{nc}(B)) \rightarrow C^*(A, \Omega^{p+1}_{nc}(B)),
\]
which sends \( f \in \text{Hom}_\text{K}(\bar{sA}^\otimes m, \bar{sB}^\otimes p \otimes B) \) to \( \theta_p^{A,B}(f) \in \text{Hom}_\text{K}(\bar{sA}^\otimes m+1, \bar{sB}^\otimes p+1 \otimes B) \) by
\[
\theta_p^{A,B}(f)(\bar{a}_1 \otimes \cdots \otimes \bar{a}_{m+1}) = (-1)^{(\lfloor |a_1|-1 \rfloor + |f|)} \varphi(a_1) \otimes f(\bar{a}_2 \otimes \cdots \otimes \bar{a}_{m+1}).
\]

The maps \( \theta_p^{A,B} \) form an inductive system of cochain complexes and we may define the singular Hochschild cochain complex of \( A \) with coefficients in \( B \) as
\[
C^*_\text{sg}(A,B) := \lim_\longrightarrow \text{C}^*(\text{C}_\otimes(\Lambda, \Omega^p_{\text{nc}}(B))
\]
with the induced differential. We denote its cohomology by \( \text{HH}^*_\text{sg}(A,B) \).

Observe that, for each \( p \in \mathbb{Z}_{\geq 0} \), there is a zig-zag of morphisms of cochain complexes
\[
\begin{align*}
C^*(A, \Omega^p_{\text{nc}}(A)) &\xrightarrow{C^*(A, \Omega^p_{\text{nc}}(f))} C^*(A, \Omega^p_{\text{nc}}(B)) &\xrightarrow{C^*(f, \Omega^p_{\text{nc}}(B))} C^*(B, \Omega^p_{\text{nc}}(B)).
\end{align*}
\]

These zig-zags are compatible with the inductive systems, thus we obtain an induced zig-zag of morphisms between singular Hochschild cochain complexes
\[
\begin{align*}
C^*_\text{sg}(A,A) &\xrightarrow{C^*_\text{sg}(A,\varphi)} C^*_\text{sg}(A,B) &\xleftarrow{C^*_\text{sg}(\varphi,B)} C^*_\text{sg}(B,B).
\end{align*}
\]

**Lemma 3.9.** Let \( \varphi : A \to B \) be a quasi-isomorphism of dg algebras. Then \( C^*_\text{sg}(A, \varphi) \) and \( C^*_\text{sg}(\varphi, B) \) are both quasi-isomorphisms, namely, the zig-zag above is one of quasi-isomorphisms.

**Proof.** Note that all three complexes in the zig-zag (13) have complete decreasing filtrations with the associated quotients
\[
\begin{align*}
\text{Hom}_\text{K}(\bar{sA}^\otimes m, \Omega^p_{\text{nc}}(A)) &\longrightarrow \text{Hom}_\text{K}(\bar{sA}^\otimes m, \Omega^p_{\text{nc}}(B)) &\longrightarrow \text{Hom}_\text{K}(\bar{sB}^\otimes m, \Omega^p_{\text{nc}}(B)).
\end{align*}
\]

These associated quotients are quasi-isomorphic, thus by the usual spectral sequence argument we may conclude that the morphisms in the zig-zag in (13) are quasi-isomorphisms for each \( p \in \mathbb{Z}_{\geq 0} \). It follows that \( C^*_\text{sg}(A, \varphi) \) and \( C^*_\text{sg}(\varphi, B) \) are quasi-isomorphisms. \( \Box \)

**Remark 3.10.** As in Proposition 3.6, we have a natural isomorphism of graded vector spaces
\[
\text{HH}^*_\text{sg}(A,B) \cong \lim_\longrightarrow \text{HH}^*(A, \Omega^p_{\text{nc}}(B)),
\]
where the colimit on the right-hand side is taken along the connecting morphisms
\[
\delta_p^{A,B} : \text{HH}^*(A, \Omega^p_{\text{nc}}(B)) \to \text{HH}^{*+1}(A, s^{-1} \Omega^p_{\text{nc}}(B)) = \text{HH}^*(A, \Omega^{p+1}_{\text{nc}}(B)).
\]
In general, there is no natural cup product on $C_{sg}^*(A, B)$ as in Definition 3.4. However, we have a well-defined cup product $\cup'$ at the cohomology level induced by the composition

$$HH^m(A, \Omega^p_{nc}(B)) \otimes HH^p(A, \Omega^q_{nc}(B)) \to HH^{m+p}(A, \Omega^p_{nc}(B) \otimes_A \Omega^q_{nc}(B)) \to HH^{m+n}(A, \Omega^{p+q}_{nc}(B)),$$

where the first map is given by the classical Hochschild cup product construction using the dg $A$-$A$-bimodule structure on $\Omega^i_{nc}(B)$ for $i = p, q$ via $\phi : A \to B$, and the second isomorphism is induced by the following natural composition:

$$\Omega^p_{nc}(B) \otimes_A \Omega^q_{nc}(B) \to \Omega^p_{nc}(B) \otimes_B \Omega^q_{nc}(B) \xrightarrow{\kappa_{p,q}} \Omega^{p+q}_{nc}(B),$$

where $\kappa_{p,q}$ is defined in Remark 3.2. This yields a product

$$\cup' : HH^m(A, \Omega^p_{nc}(B)) \otimes HH^p(A, \Omega^q_{nc}(B)) \to HH^{m+n}(A, \Omega^{p+q}_{nc}(B)). \quad (16)$$

By the functoriality of the classical Hochschild cup product, we have

$$\hat{\phi}^A_B(f) \cup' g = f \cup' \hat{\phi}^A_B(g) = \hat{\phi}^{A,B}(f \cup' g)$$

for any $f \in HH^m(A, \Omega^p_{nc}(B))$ and $g \in HH^p(A, \Omega^q_{nc}(B))$, so the cup product $\cup'$ in (16) induces a well-defined product on the colimit. Thus we obtain a product (via the isomorphism (15))

$$\cup' : HH^*_{sg}(A, B) \otimes HH^*_{sg}(A, B) \to HH^*_{sg}(A, B).$$

**Proposition 3.11.** The zig-zag of quasi-isomorphisms

$C^*_sg(A, A) \xrightarrow{C^*_sg(A, \phi)} C^*_sg(A, B) \xleftarrow{C^*_sg(B, \phi)} C^*_sg(B, B)$

induces isomorphisms of graded algebras

$$(HH^*_{sg}(A, A), \cup) \xrightarrow{HH^*_{sg}(A, \phi)} (HH^*_{sg}(A, B), \cup') \xrightarrow{HH^*_{sg}(B, \phi)^{-1}} (HH^*_{sg}(B, B), \cup).$$

**Proof.** First for any $m, n \in \mathbb{Z}$ and $p, q \in \mathbb{Z}_{\geq 0}$ we have the following commutative diagram

$$\begin{array}{c}
HH^m(A, \Omega^p_{nc}(A)) \otimes HH^p(A, \Omega^q_{nc}(A)) \xrightarrow{\cup'} HH^{m+n}(A, \Omega^{p+q}_{nc}(A)) \\
HH^m(A, \Omega^p_{nc}(A)) \otimes HH^p(A, \Omega^q_{nc}(A)) \xrightarrow{\cup'} HH^{m+n}(A, \Omega^{p+q}_{nc}(A))
\end{array}$$

The above commutative diagram is compatible with the structure maps $\hat{\phi}$. It follows from Proposition 3.6 that $\cup = \cup'$ on $HH^*_{sg}(A, A)$ and $HH^*_{sg}(B, B)$, respectively. \qed
4 | A QUASI-ISOMORPHISM RELATING THE GORESKY–HINGSTON PRODUCT AND THE SINGULAR HOCHSCHILD CUP PRODUCT

In Sections 4.1 and 4.2, we assume that $A$ is a dg Frobenius algebra of degree $k > 0$ and we follow the notation of Section 2.

4.1 | A homotopy retract between the Tate–Hochschild complex and the singular Hochschild cochain complex

Definition 4.1. Let $A$ be a dg Frobenius algebra of degree $k > 0$. The Tate–Hochschild complex of $A$, denoted by $(D^* (A, A), \delta)$, is the totalization complex of the double complex $D^{*, *}(A, A)$ obtained by connecting the Hochschild chains and cochains as

$$D^{*, *}(A, A) = \cdots \to s^{1-k} C_{-1, *} (A, A) \to s^{1-k} C_{0, *} (A, A) \overset{\gamma}{\to} C^{0, *}(A, A) \to C^{1, *}(A, A) \to \cdots,$$

where the differential $\gamma$ is given by

$$\gamma(a) = \sum_i (-1)^{|f_i|} |a| e_i a f_i, \text{ for any } a \in A,$$

where we recall that $\sum_i e_i \otimes f_i = \Delta(1)$ and $C^{0, *}(A, A) \cong A \cong C_{0, *}(A, A)$. By totalization we mean the direct sum totalization in the Hochschild chains direction and the direct product totalization in the Hochschild cochains direction. More precisely, for each $n \in \mathbb{Z}$ we have

$$D^n(A, A) = \prod_{p \in \mathbb{Z}_{\geq 0}} \text{Hom}_{\mathbb{K}}((sA)^{\otimes p}, A)^n \oplus \bigoplus_{p \in \mathbb{Z}_{\geq 0}} ((sA)^{\otimes p} \otimes A)^{n-k+1}$$

$$= C^n(A, A) \oplus C_{n-k+1}(A, A).$$

We denote by $\delta : D^*(A, A) \to D^{*+1}(A, A)$ the differential of the totalization.

In other words, the Tate–Hochschild complex $(D^* (A, A), \delta)$ is the mapping cone of the morphism

$$\gamma : s^{-k} C_*(A, A) \to C^*(A, A), \quad (17)$$

where $\gamma(\alpha) = 0$ if $\alpha \in C_{-m,*}(A, A)$ for $m \neq 0$ and $\gamma(a) = \sum_i (-1)^{|f_i|} |a| e_i a f_i$ if $a \in A = C_{0,*}(A, A)$.

Theorem 4.2. There exists a homotopy retract of cochain complexes.

$$D^*(A, A) \overset{\iota}{\leftarrow} \Pi \overset{\delta_{sg}}{\rightarrow} C^*_s(A, A) \bowtie H. \quad (18)$$

Namely, $\iota$ and $\Pi$ are morphisms of cochain complexes such that

$$\Pi \circ \iota = \text{id} \quad \text{and} \quad \text{id} - \iota \Pi = \delta_{sg} \circ H + H \circ \delta_{sg}.$$
Combining Theorem 4.2 and the mapping cone construction (17), we obtain a long exact sequence

\[ \cdots \to \text{HH}_{-k}(A, A) \to \text{HH}^i(A, A) \to \text{HH}^i_{\text{sg}}(A, A) \to \text{HH}_{-k+1}(A, A) \to \cdots, \]

where we identify $\text{HH}^i_{\text{sg}}(A, A)$ with $H^i(D^n(A, A))$ via the natural isomorphism $H^i(\iota)$.

For the purposes of this paper, we only use the fact that the map $\iota$ is a quasi-isomorphism from the above theorem. We define $\iota$ in Definition 4.3 below. However, for the sake of clarity and organization, we refer to the Appendix at the end of the paper for the definition of $\Pi$ and $H$ and for the proof of Theorem 4.2.

**Definition 4.3.** Define

\[ \iota : D^r(A, A) \hookrightarrow C^\ast_{\text{sg}}(A, A) \]

as follows.

1. If $f \in C^n(A, A)$, define $\iota(f) = f \in C^n(A, A) \subset C^\ast_{\text{sg}}(A, A)$.
2. If $\alpha = a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \in C^\ast(A, A)$ define

\[ \iota(\alpha) \in \text{Hom}_{\mathbb{K}}(\mathbb{K}, (sA)\otimes^{p+1} \otimes A) \subset C^\ast_{\text{sg}}(A, A) \]

by

\[ \iota(\alpha)(1_\mathbb{K}) = \sum_i (-1)^{\|f_i\|} e_i \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i. \]

The map $\iota$ is clearly an injection. We refer to the Appendix for the proof that $\iota$ is compatible with the differentials.

### 4.2 Relating the Goresky–Hingston product and the singular Hochschild cup product

We now show that $\iota$ intertwines the products $\star$ and $\cup$.

**Proposition 4.4.** Let $\alpha = a_1 \otimes \cdots \otimes a_p \otimes a_{p+1}$ and $\beta = b_1 \otimes \cdots \otimes b_q \otimes b_{q+1}$ be two elements in $C^\ast(A, A)$. Then we have

\[ \iota(\alpha) \cup \iota(\beta) = \iota(\alpha \star \beta). \]

**Proof.** By Definition 2.8 we have

\[ \iota(\alpha \star \beta)(1_\mathbb{K}) = \sum_i (-1)^{\|\beta\|} \iota\left( b_1 \otimes \cdots \otimes b_{q+1} e_i \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i \right)(1_\mathbb{K}) \]

\[ = \sum_{i,j} (-1)^{\|\beta\| + \gamma} e_j \otimes b_1 \otimes \cdots \otimes b_q \otimes b_{q+1} e_i \otimes a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i f_j, \]
where $\eta_i = |\alpha| |f_i| + |b_{q+1}| + (|\alpha| + k - 1)(|\beta| + k - 1)$ and $\eta'_j = |f_j|(|\alpha| + |\beta| + k - 1)$. Using the identities in Remark 2.3, the above term equals

$$
\sum_{i,j} (-1)^{\eta_i + \eta'_j + |b_{q+1}|(k+|\alpha|)} e_j \otimes b_1 \otimes \cdots \otimes b_q \otimes e_i \otimes a_1 \otimes \cdots \otimes a_{p+1} f_i b_{q+1} f_j,$$

which is precisely $(\iota(\alpha) \cup \iota(\beta))(1_{A_q})$ (cf. (8)).

**Remark 4.5.** Note that Proposition 4.4 yields (5) since $\iota$ is an injection of complexes.

### 4.3 Another mapping cone model for singular Hochschild cohomology

We now define a new cochain complex $\mathcal{E}^n(A, A)$ associated to any dg algebra $A$ without requiring any Frobenius structure. This new complex $\mathcal{E}^n(A, A)$ satisfies the following properties:

1. $\mathcal{E}^n(A, A)$ arises as the mapping cone of a natural map $A \otimes_{A^e} A^! \to \text{RHom}_{A^e}(A, A)$ associated to any dg algebra $A$.
2. When $A$ is equipped with a dg Frobenius structure we factor the quasi-isomorphism $\iota : D^s(A, A) \to C_{sg}^*(A, A)$ (Definition 4.3) as a composition of quasi-isomorphisms

$$ D^s(A, A) \xrightarrow{\bar{\iota}} \mathcal{E}^n(A, A) \xrightarrow{\bar{\iota}} C_{sg}^*(A, A). $$

3. $\mathcal{E}^n(A, A)$ enjoys better functoriality properties than $D^s(A, A)$ and we will use these in the proof of our main theorem.

Recall that the inverse dualizing complex $A^! = \text{RHom}_{A^e}(A, A^e)$ can be modeled as the Hochschild cochain complex $C^*(A, A^e)$ equipped with the $A^e$-$A^e$-bimodule structure given by the inner $A^e$-$A^e$-bimodule action of $A \otimes A$. From now on we will denote $A^! = C^*(A, A^e)$. By the functoriality of $C^*(A, -)$, there is a natural morphism of complexes

$$ C^*(A, \mu) : A^! \to C^*(A, A) \quad (20) $$

induced by the multiplication $\mu$ of $A$.

Consider the Hochschild chain complex $C_*^n(A, A^!)$ of $A$ with coefficients in $A^!$. The above morphism (20) induces a morphism of complexes

$$ \bar{\gamma} : C_*^n(A, A^!) \to C^*(A, A) \quad (21) $$

given by

$$ \bar{\gamma}(\alpha) = \begin{cases} 
C^*(A, \mu)(\alpha) & \text{if } \alpha \in C_{0,*}^n(A, A^!) = A^! \\
0 & \text{otherwise.}
\end{cases} $$
Definition 4.6. Given any dg algebra $A$, define the cochain complex $\mathcal{E}^*(A, A)$ as the mapping cone of the morphism $\overline{\gamma}$.

Similar to (19), since $\mathcal{E}^*(A, A)$ is defined as a mapping cone, we have a long exact sequence

$$\cdots \to \text{HH}^i(A, A^1) \to \text{HH}^i(A, A) \to H^i(\mathcal{E}^*(A, A)) \to \text{HH}^{i+1}(A, A^1) \to \text{HH}^{i+1}(A, A) \to \cdots. \quad (22)$$

The complex $\mathcal{E}^*(A, A)$ may be viewed as a subcomplex of $C^*_{sg}(A, A)$ as follows. Define a morphism of complexes

$$\overline{\tau} : \mathcal{E}^*(A, A) \hookrightarrow C^*_{sg}(A, A), \quad (23)$$

which sends $\varphi \in C^*(A, A)$ to $\varphi \in C^*(A, A) \subset C^*_{sg}(A, A)$ and sends $a_1 \otimes \cdots \otimes a_m \otimes \varphi_n \in C_{-m,n}(A, A^1)$, where $\varphi_n \in C_{n,*}(A, A \otimes A)$, to an element in $C_{n,*}(A, \Omega_{nc}^{m+1}(A)) \subset C^*_{sg}(A, A)$ given by

$$\overline{b_1} \otimes \cdots \otimes \overline{b_n} \mapsto \sum_j (-1)^{\varepsilon_j} \overline{u_j} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m} \otimes v_j \in \Omega_{nc}^{m+1}(A),$$

where we write $\varphi_n(\overline{b_1} \otimes \cdots \otimes \overline{b_n}) = \sum_j u_j \otimes v_j \in A \otimes A$ and $\varepsilon_j = |v_j| - (|a_1| + \cdots + |a_m| - m)$. Similar to the map $\tau$ in Definition 4.3, $\overline{\tau}$ is an injective map of complexes.

Let $A$ be a dg Frobenius algebra of degree $k$. Recall from (4) that we have a natural quasi-isomorphism $\varphi_A : s^{-k}A \simeq A^1$ of dg $A$-$A$-bimodules. There is an induced commutative square

$$\begin{array}{ccc}
C^*(A, s^{-k}A) & \overset{\gamma}{\longrightarrow} & C^*(A, A) \\
\downarrow \varphi_A & & \downarrow = \\
C^*(A, A^1) & \overset{\overline{\gamma}}{\longrightarrow} & C^*(A, A),
\end{array}$$

where $C^*(A, \varphi_A)$ is a quasi-isomorphism and $\gamma$ is given in (17). This yields a quasi-isomorphism between the mapping cones

$$\tau : D^*(A, A) \hookrightarrow \mathcal{E}^*(A, A).$$

Proposition 4.7. Let $A$ be a dg Frobenius algebra of degree $k$. Then the quasi-isomorphism $\tau : D^*(A, A) \to C^*_{sg}(A, A)$ in Definition 4.3 is the composition of the morphisms

$$D^*(A, A) \overset{\tau}{\longrightarrow} \mathcal{E}^*(A, A) \overset{\overline{\tau}}{\longrightarrow} C^*_{sg}(A, A).$$

As a result, $\overline{\tau}$ is a quasi-isomorphism and there is a commutative diagram between the long exact sequences (19) and (22).
The first part is clear. The cochain map $\tilde{t}$ is a quasi-isomorphism since both $t$ and $\tau$ are quasi-isomorphisms. The commutativity of the diagram follows since $\tau$ is a map between mapping cones. Note that in the diagram we identify $\text{HH}^i_{\text{sg}}(A, A)$ with $H^i(D^\ast(A, A))$ via $H^i(t)$. □

We now discuss the functoriality properties of $\mathcal{E}^\ast(A, A)$ that will be used in the proof of our main theorem in Section 5.

Let $\varphi : A \to B$ be a morphism of dg algebras. Then $B^e$ may be viewed as an $A$-$A$-bimodule. We have a morphism of complexes induced by the multiplication $\mu_B$ of $B$:

$$C^\ast(A, \mu_B) : C^\ast(A, B^e) \to C^\ast(A, B).$$

We denote by $\mathcal{E}^\ast(B, C^\ast(A, B^e))$ the mapping cone of the following natural map (cf. (21)):

$$C^\ast(B, C^\ast(A, B^e)) \to C^\ast(A, B),$$

which sends $C_{0,\ast}(B, C^\ast(A, B^e)) = C^\ast(A, B^e)$ to $C^\ast(A, B)$ via the map $C^\ast(A, \mu_B)$ and sends $C_{i,\ast}(B, C^\ast(A, B^e))$ to zero for $i \neq 0$. Similar to (23), we have a morphism of complexes

$$\tilde{t}_{A,B} : \mathcal{E}^\ast(B, C^\ast(A, B^e)) \to C^\ast_{\text{sg}}(A, B).$$

By functoriality, $\varphi$ induces two morphisms

$$\mathcal{E}^\ast(A, A) \to \mathcal{E}^\ast(B, C^\ast(A, B^e)), \quad \mathcal{E}^\ast(B, B) \to \mathcal{E}^\ast(B, C^\ast(A, B^e)). \tag{24}$$

**Proposition 4.8.** Any morphism $\varphi : A \to B$ of dg algebras induces a commutative diagram

$$\begin{array}{ccc}
\mathcal{E}^\ast(A, A) & \longrightarrow & \mathcal{E}^\ast(B, C^\ast(A, B^e)) \\
\downarrow & & \downarrow \\
C^\ast_{\text{sg}}(A, A) & \longrightarrow & C^\ast_{\text{sg}}(A, B)
\end{array} \tag{25}$$

$$\begin{array}{ccc}
\mathcal{E}^\ast(B, B) & \longleftarrow & \mathcal{E}^\ast(B, C^\ast(A, B^e)) \\
\downarrow & & \downarrow \\
C^\ast_{\text{sg}}(B, B) & \longleftarrow & C^\ast_{\text{sg}}(B, A)
\end{array}$$

where the horizontal maps on the top are given in (24). Furthermore, if $\varphi$ a quasi-isomorphism, then all the horizontal morphisms in (25) are quasi-isomorphisms.

Finally, we relate the long exact sequences associated to quasi-isomorphic dg Frobenius algebras.
Corollary 4.9. Let \((A, \langle -,- \rangle_A)\) and \((B, \langle -,- \rangle_B)\) be two dg Frobenius algebras of degree \(k\). Suppose that there is a zig-zag of quasi-isomorphisms of dg algebras

\[
A \cong \cdots \cong B.
\]

Then there is a commutative diagram between long exact sequences

\[
\cdots \rightarrow \text{HH}_{i-k}(A, A) \rightarrow \text{HH}^i(A, A) \rightarrow \text{HH}^i_{\text{sg}}(A, A) \rightarrow \text{HH}_{i-k+1}(A, A) \rightarrow \cdots
\]

where the vertical isomorphisms in the middle row are induced by the zig-zag of maps in (24), and the vertical isomorphisms in the top and bottom rows are given in Proposition 4.7.

Moreover, the isomorphism between \(\text{HH}^i_{\text{sg}}(A, A)\) and \(\text{HH}^i_{\text{sg}}(B, B)\) in the diagram coincides with the isomorphism induced by the zig-zag (14).

Proof. The commutativity of the squares in the top and bottom rows follows from Proposition 4.7. This commutativity of the squares in the middle row directly follows from the fact that \(\mathcal{E}^*\) is defined as mapping cones. The last statement follows from Proposition 4.8. 

\[\square\]

5 | PROOF OF THE MAIN THEOREM

We will now prove that, for simply connected dg Frobenius algebras, the isomorphism class of the Goresky–Hingston algebra is invariant under quasi-isomorphisms. To conclude this, we use the invariance of the singular Hochschild cohomology algebra with respect to quasi-isomorphisms together with the quasi-isomorphism relating the Tate–Hochschild complex and the singular Hochschild cochain complex described in the previous section.

5.1 | Tate–Hochschild cohomology in the simply connected case

For simply connected dg Frobenius algebras, Tate–Hochschild cohomology may be described in terms of Hochschild homology and cohomology as follows.

Proposition 5.1. Let \(A\) be a simply connected dg Frobenius algebra of degree \(k\). Then
(1) if \( \chi(A) := \mu \circ \Delta(1) \neq 0 \), we have vector space isomorphisms

\[
\HH^i_{sg}(A, A) \cong H^i(D^s(A, A)) \cong \begin{cases} 
\HH^i(A, A) & \text{if } i \leq k - 1, \\
\HH_{i-k+1}(A, A) & \text{if } i \geq k;
\end{cases}
\]

(2) if \( \chi(A) := \mu \circ \Delta(1) = 0 \), we have vector space isomorphisms

\[
\HH^i_{sg}(A, A) \cong H^i(D^s(A, A)) \cong \begin{cases} 
\HH^i(A, A) & \text{if } i < k - 1, \\
\HH^{k-1}(A, A) \oplus \HH_0(A, A) & \text{if } i = k - 1, \\
\HH_1(A, A) \oplus \HH^k(A, A) & \text{if } i = k, \\
\HH_{i-k+1}(A, A) & \text{if } i > k.
\end{cases}
\]

**Proof.** The isomorphism \( \HH^i_{sg}(A, A) \cong H^i(D^s(A, A)) \) is induced by the homotopy retract (18).

Let us use the long exact sequence (19) to calculate \( H^i(D^s(A, A)) \) in terms of Hochschild homology and cohomology. Since \( A \) is simply connected, we have \( C^i(A, A) = 0 \) if \( i > k \) and \( C_i(A, A) = 0 \) if \( i < 0 \) by degree reasons. It follows that

\[
\HH^i(A, A) = \begin{cases} 
0 & \text{if } i > k \\
A^k \cong \mathbb{K} & \text{if } i = k
\end{cases}
\]

and \( \HH^i_{sg}(A, A) \) is a graded subspace of \( \HH^i(A, A) \).

Then by the long exact sequence (19) we obtain

\[
\HH^i_{sg}(A, A) \cong \begin{cases} 
\HH^i(A, A) & \text{if } i < k - 1 \\
\HH_{i-k+1}(A, A) & \text{if } i > k
\end{cases}
\]

and an exact sequence involving \( \HH^{k-1}_{sg}(A, A) \) and \( \HH^k_{sg}(A, A) \)

\[0 \to \HH^{k-1}(A, A) \to \HH^{k-1}_{sg}(A, A) \to A^0 \to A^k \to \HH^k_{sg}(A, A) \to \HH_1(A, A) \to 0. \quad (26)\]

If \( \chi(A) \neq 0 \) then the differential \( \gamma \) in (26) is an isomorphism. Hence, we have

\[
\HH^{k-1}_{sg}(A, A) \cong \HH^{k-1}(A, A) \quad \text{and} \quad \HH^k_{sg}(A, A) \cong \HH_1(A, A).
\]

If \( \chi(A) = 0 \) then the differential \( \gamma \) is zero, so we have

\[
\HH^{k-1}_{sg}(A, A) \cong \HH^{k-1}(A, A) \oplus \HH_0(A, A) \quad \text{and} \quad \HH^k_{sg}(A, A) \cong \HH_1(A, A) \oplus \HH^k(A, A).
\]

**Remark 5.2.** It follows from Remark 2.10 and Proposition 5.1 that the (shifted) reduced Hochschild homology \( s^{1-k}\HH^i_{sg}(A, A) \) is a graded subspace of \( \HH^*_sg(A, A) \). Theorem 5.4 below shows that it is actually a graded subalgebra of \( \HH^*_sg(A, A) \).

**Remark 5.3.** Proposition 5.1 should be compared with the computation of Rabinowitz–Floer homology in [5, Theorem 1.10]. In fact, when \( M \) is a simply connected oriented closed manifold of dimension \( k \), Proposition 5.1 implies that the singular Hochschild cohomology of the dg algebra
of cochains on $M$ with real coefficients is isomorphic to the Rabinowitz–Floer homology of the unit cotangent bundle of $M$. The long exact sequence (19) should be compared with the long exact sequence in [5, Theorem 1.2].

Recall that $\text{DGA}_k^{1}$ is the category of unital dg $\kappa$-algebras $A$ which are simply connected and non-negatively graded (that is, $A^{-\infty} = 0$, $A^{0} \cong \kappa$ and $A^{1} = 0$). Our main result, Theorem 1.1 in the introduction, can now be proved by considering two cases: when the Euler characteristic is zero and when it is non-zero.

**Theorem 5.4.** Let $(A, \langle -, -, \rangle_A)$ and $(B, \langle -, -, \rangle_B)$ be two dg Frobenius algebras of degree $k$ such that $A, B \in \text{DGA}_k^{1}$. Suppose that there is a zig-zag of quasi-isomorphisms of dg algebras

$$A \rightsquigarrow \cdots \rightsquigarrow B.$$ 

(1) If $\chi(A) \neq 0$, then $\chi(B) \neq 0$ and the composition of isomorphisms

$$s^{1-k} \mathbb{H}_*(A, A) \cong \mathbb{H}_*(B, B)$$

preserves the Goresky–Hingston algebra structures.

(2) If $\chi(A) = 0$, then $\chi(B) = 0$ and the composition of isomorphisms

$$s^{1-k} \mathbb{H}_*(A, A) \oplus \mathbb{H}_k(A, A) \cong \mathbb{H}_*(B, B)$$

restricts to an algebra isomorphism

$$s^{1-k} \mathbb{H}_*(A, A) \cong s^{1-k} \mathbb{H}_*(B, B).$$

**Proof.** First observe that $\chi(A) \neq 0$ if and only if $\chi(B) \neq 0$. This follows, for instance, from the quasi-isomorphism invariance of (singular) Hochschild cohomology and Proposition 5.1, which tells us that if $\chi(A) \neq 0$ then $\mathbb{H}_*^{k-1}(A, A) \cong \mathbb{H}_*^{k-1}(A, A)$ and if $\chi(A) = 0$ then $\mathbb{H}_*^{k-1}(A, A) \cong \mathbb{H}_*^{k-1}(A, A) \oplus \kappa$.

In both cases (1) and (2), the middle isomorphism is obtained from the invariance of the singular Hochschild cohomology algebra along quasi-isomorphisms, as proven in Proposition 3.11. The outer isomorphisms of graded algebras are obtained from Propositions 5.1 and 4.4 and Remark 2.10. It follows that the compositions in both cases (1) and (2) are isomorphisms of graded algebras.

It remains to show that the composition in the case (2) restricts to an isomorphism

$$s^{1-k} \mathbb{H}_*(A, A) \cong s^{1-k} \mathbb{H}_*(B, B).$$

This follows from the functoriality of the long exact sequences with respect to quasi-isomorphisms in Corollary 4.9. □
Remark 5.5. The argument in the above proof actually shows that, if $\chi(A) = 0 = \chi(B)$, the isomorphism $\text{HH}^*_\text{sg}(A, A) \cong \text{HH}^*_\text{sg}(B, B)$ restricts to an isomorphism of (ordinary) Hochschild homology $\text{s}^{1-k} \text{HH}_*(A, A) \cong \text{s}^{1-k} \text{HH}_*(B, B)$ which preserves the Goresky–Hingston algebra structures, which exist without restricting to reduced Hochschild homology in this particular case as mentioned in Remark 2.14.

Proof of Theorem 1.1. Theorem 1.1 in the introduction follows directly from Theorem 5.4.

Proof of Corollary 1.2. Part (1) of the corollary follows from Theorem 1.1 since any two Poincaré duality models for $\mathcal{A}(M)$ are connected by a zig-zag of quasi-isomorphisms of simply connected cdg algebras. Part (2) follows since if $A$ and $A'$ are Poincaré duality models for $\mathcal{A}(M)$ and $\mathcal{A}(M')$ and $M$ and $M'$ are homotopy equivalent oriented closed manifolds of dimension $k$, then $A$ and $A'$ are connected by a zig-zag of quasi-isomorphisms of simply connected cdg algebras.

APPENDIX A: PROOF OF THEOREM 4.2

In this appendix we will prove Theorem 4.2.

A.1 | Basic identities regarding the action

The following identities will be useful when keeping track of the signs in the computations in this appendix.

Lemma A.1. Let $\partial_h$ be the external differential of $C_*(A, A)$ as in Definition 2.5. Let $\triangleright$ be the action described in Lemma 3.1. Define $\epsilon(a) = \langle a, 1 \rangle$ for any $a \in A$. For any element $\alpha = a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} \in (sA)^{\otimes p} \otimes A$, we have the following identities:

1. $\sum_i (-1)^{|f_i|} (\epsilon \otimes \text{id}^{\otimes p})(e_i \triangleright (a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i)) = \partial_h(\alpha)$,
2. $\sum_i (-1)^{|f_i|} (1-k)(1-k) e_i \otimes (\epsilon \otimes \text{id}^{\otimes p})(f_i \triangleright \alpha) = -\alpha$,
3. $\sum_i (-1)^{|e_i|} (1-k) e_i \triangleright (\epsilon \otimes \text{id}^{\otimes p})(f_i \triangleright \alpha) = 0$.

Proof. For the first identity, we have

$$
\sum_i (-1)^{|f_i|} (\epsilon \otimes \text{id}^{\otimes p})(e_i \triangleright (a_1 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i))
= \sum_i (-1)^{|f_i|} (\epsilon(a_1) a_2 \otimes \cdots \otimes a_p \otimes a_{p+1} f_i - (-1)^{p-1} \epsilon(a_1) a_1 \otimes \cdots \otimes a_{p+1} \otimes a_p a_{p+1} f_i)
+ \sum_{j=1}^{p-1} (-1)^j \epsilon(a_j) a_1 \otimes \cdots \otimes a_{j-1} \otimes a_j a_{j+1} \otimes \cdots \otimes a_p \otimes a_{p+1} f_i)
= \partial_h(a_1 \otimes \cdots \otimes a_{p+1}),
$$

where $\epsilon_j = |a_1| + \cdots + |a_j| - j$ and the second equality follows from Lemma 2.2 (1).

Let us verify the second identity. We have

$$
\sum_i (-1)^{|f_i|} (1-k)(1-k) e_i \otimes (\epsilon \otimes \text{id}^{\otimes p})(f_i \triangleright (a_1 \otimes \cdots \otimes a_p \otimes a_{p+1})).
$$
\[
\sum_i \left( (-1)^{[f_i]\cdots[1]-[1-k]+[f_i]\cdots[1]}\varepsilon(f_i,a_1)\overline{e_i} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1} \right)
\]

\[+
\sum_{j=1}^{p-1} (-1)^{[f_i]\cdots[1]-[1-k]+[f_i]\cdots[1]}\varepsilon(f_i)\overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{j-1}} \otimes \overline{a_j} a_{j+1} \otimes \cdots \otimes a_{p+1} \]

\[- (-1)^{[f_i]\cdots[1]-[1-k]+[f_i]\cdots[1]}\varepsilon(f_i)\overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{p-1}} \otimes a_p a_{p+1} \]

\[= - \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}, \]

where in the second equality we use the fact that \( \sum_i \varepsilon(f_i)\overline{e_i} = 1 = 0 \) in \( sA \). The third identity can be verified in a similar way. \( \square \)

### A.2 The injection \( \iota \) is a cochain map

Recall the injection \( \iota : D^*(A,A) \hookrightarrow C^*_{sg}(A,A) \) is defined in Definition 4.3. We now check \( \iota \) is compatible with differentials.

**Lemma A.2.** The map \( \iota \) is a cochain map.

**Proof.** Let us first check that \( \iota \) is compatible with the external differentials. It suffices to prove that the following diagram commutes for \( p > 1 \) :

\[
s^{1-k}C^{-(p-1),*}_{C}(A,A) \xrightarrow{\iota} C^0,*(A,\Omega^p_{nc}(A)) \]

\[
\downarrow (-1)^{1-k}\delta_h
\]

\[
s^{1-k}C^{-(p-2),*}_{C}(A,A) \xrightarrow{\iota} C^0,*(A,\Omega^{p-1}_{nc}(A)) \xrightarrow{\delta^n} C^{1,*}(A,\Omega^p_{nc}(A)).
\]

Let \( \alpha = \overline{a_1} \otimes \cdots \otimes \overline{a_{p-1}} \otimes a_p \). We have

\[
(-1)^{1-k}\delta_h \circ \iota \circ \delta_h \overline{(a_1 \otimes \cdots \otimes a_{p-1} \otimes a_p)(a_0)}
\]

\[
= \sum_i \sum_{j=1}^{p-2} (-1)^{1-k+[f_i][[\alpha]+1]+(|a_0|-1)(|\alpha|+k)+\varepsilon_j} \overline{a_0} \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_j} a_{j+1} \otimes \cdots \otimes \overline{a_{p-1}} \otimes a_p f_i
\]

\[- \sum_i (-1)^{1-k+[f_i][[\alpha]+1]+(|a_0|-1)(|\alpha|+k)+\varepsilon_{p-2}} \overline{a_0} \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{p-2}} \otimes a_{p-1} a_p f_i
\]

\[+ \sum_i (-1)^{1-k+[f_i][[\alpha]+1]+(|a_0|-1)(|\alpha|+k)+\varepsilon_{p-2}} \overline{a_0} \otimes \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{p-1}} \otimes a_p a_1 f_i \]

and

\[
\delta^h \circ \iota \circ \delta_h(a_0) = - \sum_i (-1)^{[f_i][\alpha]+(|a_0|-1)(|\alpha|+k-1)} a_0 \triangleright (\overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes a_p f_i)
\]

\[+ \sum_i (-1)^{[f_i][\alpha]+|\alpha|+k-1} \overline{e_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{p-1}} \otimes a_p f_i a_0, \]
where \( \varepsilon_j = |a_1| + \cdots + |a_j| - j \). By Lemma 3.1, we may cancel the common terms and obtain

\[
(−1)^{1−k}\partial t o_\partial_h − \delta^h o_t)(a_1 \otimes \cdots \otimes a_{p−1} \otimes a_p)(a_0)
= \sum_i \left( (−1)^{1−k}[f_i]|x|+(|a_0|−1)(|x|+k−1)+|a_0|\right) \varepsilon_i \otimes a_1 \otimes \cdots \otimes a_{p−1} \otimes a_p f_i
+ (−1)[f_i]|x|+(|a_0|−1)(|x|+k−1)+|a_0|+|x|−1 \varepsilon_0 \otimes a_1 \otimes \cdots \otimes a_{p−1} \otimes a_p f_i
− (−1)[f_i]|x|+(|a_0|−1)(|x|+k−1) \varepsilon_i \otimes a_1 \otimes \cdots \otimes a_{p−1} \otimes a_p f_i.
\]

It follows from Remark 2.3 that the first term cancels with the third one and the second term cancels with the fourth one. We obtain \( (−1)^{1−k}\partial t o_\partial_h − \delta^h o_t = 0 \). Similarly, we may check that \( t \) is compatible with the internal differential.

### A.3 | The surjection \( \Pi \)

We will now construct \( \Pi : C^*_{sg}(A, A) \to D^*(A, A) \). If \( m, p \in \mathbb{Z}_{>0} \) define

\[
\pi_{m, p} : C^{m, *}(A, \Omega^p_{nc}(A)) \to C^{m−1, *}(A, \Omega^{p−1}_{nc}(A))
\]
on any \( f \in C^{m, *}(A, (sA) \otimes_p A) = \text{Hom}_\mathbb{K}((sA)^\otimes m, (sA)^\otimes p \otimes A) \) by letting

\[
\pi_{m, p}(f)(a_1 \otimes \cdots \otimes a_{m−1}) = \sum_i (−1)^{(i−1)(|f_i|+k)} \varepsilon_i \otimes \text{id}^\otimes p(f_i \otimes a_1 \otimes \cdots \otimes a_{m−1}),
\]

where \( \varepsilon : sA \to \mathbb{K} \) is the degree \( 1−k \) map given by \( \varepsilon(a) = \langle a, 1 \rangle \) and \( \text{\textbullet} \) is the left action of \( A \) on \( \Omega^{p−1}_{nc}(A) \) defined in Lemma 3.1. For convenience, set \( \pi_{m, 0} = \text{id} : C^{m, *}(A, A) \to C^{m, *}(A, A) \) for \( m \geq 0 \).

Define

\[
\pi_{0, p} : C^{0, *}(A, \Omega^p_{nc}(A)) \to C^{−(p−1), *}(A, A), \quad \text{for } p > 0
\]
as follows. If \( f \in C^{0, *}(A, \Omega^p_{nc}(A)) = \text{Hom}_\mathbb{K}(\mathbb{K}, \Omega^p_{nc}(A)) \) and \( f(1) = \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1} \in \Omega^p_{nc}(A) = (sA)^\otimes p \otimes A \) let

\[
\pi_{0, p}(f) = (−1)^k \varepsilon(a_1) \overline{a_2} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}.
\]

**Lemma A.3.** \( \pi_{*, *} \) is compatible with the differentials.

**Proof.** First we check \( \pi_{>0, *} \) is compatible with the external differential \( \delta^h \). Let \( f \in C^{m+1, *}(A, \Omega^p_{nc}(A)) \), \( m \geq 0 \). Then for any \( \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}} \in (sA)^\otimes m+1 \) we have
\[ \pi_{*,*} \circ \delta^h(f)(\overline{a_1} \otimes \cdots \otimes a_{m+1}) \]
\[ = - \sum_i (-1)^{(|f| - 1)(k+1)} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f_i \triangleright f(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})) \]
\[ - \sum_i (-1)^{|f_i|} f_i^{+|(f_i| - 1)^k} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})) \]
\[ - \sum_{j=1}^m \sum_i (-1)^{|f_i|} f_i^{+|(f_i| - 1)^k + \varepsilon_i e_i} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})) \]
\[ + \sum_i (-1)^{|f_i|} f_i^{+|(f_i| - 1)^k + \varepsilon_m e_i} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m})a_{m+1}), \]

where \( \varepsilon_j = |a_1| + \cdots + |a_j| - j \). On the other hand, we have
\[ \delta^h \circ \pi_{*,*}(f)(\overline{a_1} \otimes \cdots \otimes a_{m+1}) \]
\[ = - \sum_i (-1)^{|a_1| + |f_i| + (|f| - 1)^k} a_1 \triangleright (e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}}))) \]
\[ - \sum \sum_i (-1)^{|f_i|} f_i^{+|(f_i| - 1)^k + \varepsilon_i e_i} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})) \]
\[ + \sum (-1)^{|f_i|} f_i^{+|(f_i| - 1)^k + \varepsilon_m e_i} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m})a_{m+1}). \]

Thus, we may cancel the common terms to obtain:
\[ (\pi_{*,*} \circ \delta^h - \delta^h \circ \pi_{*,*})(f)(\overline{a_1} \otimes \cdots \otimes a_{m+1}) \quad \text{(A1)} \]
\[ = - \sum_i (-1)^{|f_i| - 1}(k+1) e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f_i \triangleright f(\overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})) \]
\[ - \sum_i (-1)^{|f_i|} f_i^{+|(f_i| - 1)^k} e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})) \]
\[ + \sum_i (-1)^{|a_1| + |f_i|} f_i^{+|(f_i| - 1)^k} a_1 \triangleright (e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}}))). \]

It follows from Lemma A.1 (3) that the first sum on the right-hand side of the equality in (A1) vanishes. Since \( \cdot \) defines a left action of \( A \) on \( \Omega_{\text{nc}}(A) \), we have
\[ a_1 \triangleright (e_i \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}}))) = (a_1 e_i) \triangleright (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{f_i} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_{m+1}})), \]

thus the second sum on the right-hand side in (A1) cancels with the third sum since \( \sum a_1 e_i \otimes f_i = \sum (-1)^{|a_1|} e_i \otimes f_i a_1 \). Therefore, \( (\pi_{*,*} \circ \delta^h - \delta^h \circ \pi_{*,*})(f) = 0 \). Similarly, we may check that \( \pi_{>0,*} \) is compatible with the internal differential \( \delta^v \), which we leave to the reader.
We now check that $\pi_{0,*}$ are compatible with the external differentials $\delta^h$ and $\delta_h$. Namely, that the following diagram commutes for any $p \in \mathbb{Z}_{>0}$:

$$
\begin{array}{ccc}
C^0,*(A, \Omega^p_{nc}(A)) & \xrightarrow{\pi_{0,p}} & s^{1-k}C_{-(p-1),*}(A, A) \\
\downarrow{\delta^h} & & \downarrow{(-1)^{1-k}\delta_h} \\
C^1,*(A, \Omega^p_{nc}(A)) & \xrightarrow{\pi_{1,p}} & C^0,*(A, \Omega^{p-1}_{nc}(A)) & \xrightarrow{\pi_{0,p-1}} & s^{1-k}C_{-(p-2),*}(A, A).
\end{array}
$$

The commutativity of diagram (A2) follows since

$$
\pi_{0,p-1} \circ \pi_{1,p} \circ \delta^h(\bar{a_1} \otimes \cdots \otimes \bar{a_p} \otimes a_{p+1})
= - \sum_i (-1)^{(|f_i|-1)(k-1)}\pi_{0,p-1}(\eta_i \cdot (\varepsilon \otimes \text{id}^{\otimes p})(f_i \cdot (\bar{a_1} \otimes \cdots \otimes \bar{a_p} \otimes a_{p+1})))
+ \sum_i (-1)^{(|f_i|-1)(|\beta|+k+1)+|\alpha|}\varepsilon(a_1)\pi_{0,p-1}(\eta_i \cdot (\bar{a_2} \otimes \cdots \otimes \bar{a_p} \otimes a_{p+1}f_i))
= \sum_i (-1)^{(|f_i|-1)(|\beta|+k+1)+|\alpha|}\varepsilon(a_1)\pi_{0,p-1}(\eta_i \cdot (\bar{a_2} \otimes \cdots \otimes \bar{a_p} \otimes a_{p+1}f_i))
= (-1)^{1-k}\delta_h \circ \pi_{0,p}(\bar{a_1} \otimes \cdots \otimes \bar{a_{p+1}}),
$$

where the second identity follows from Lemma A.1 (3) and the third identity from Lemma A.1 (1). Similarly, we may check that $\pi_{0,*}$ are compatible with the internal differentials. □

**Lemma A.4.** We have the following identities:

$$
\pi_{m+1,p+1} \circ \theta_{m,p} = \text{id} \quad \text{for } m, p \in \mathbb{Z}_{\geq 0},
$$

$$
\pi_{0,p+1} \circ \alpha = \text{id} \quad \text{for } p \in \mathbb{Z}_{\geq 0}.
$$

**Proof.** Recall that $\theta_{m,p}$ is defined in (7). We have

$$
\pi_{m+1,p+1} \circ \theta_{m,p}(f)(\bar{a_1} \otimes \cdots \otimes \bar{a_m})
= \sum_i (-1)^{(|f_i|-1)(|f_i|+k)}\eta_i \cdot (\varepsilon \otimes \text{id}^{\otimes p+1})(\theta_{m,p}(f)(\bar{a_1} \otimes \cdots \otimes \bar{a_m}))
= \sum_i (-1)^{(|f_i|-1)k}\eta_i \cdot (\varepsilon(\bar{a_1}f_i \cdot (\bar{a_2} \otimes \cdots \otimes \bar{a_{m-1}}))
= f(\bar{a_1} \otimes \cdots \otimes \bar{a_{m-1}}),
$$

where the last identity follows from the fact that $\sum_i \eta_i \varepsilon(\bar{a_1}f_i) = \sum_i \eta_i (f_i, 1) = 1$; see Lemma 2.2.
Similarly, let \( \alpha = \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1} \in C_{-p,*}^-(A, A) \), then we have
\[
\pi_{0,p+1} \circ (\overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1}) = \sum_i (-1)^{|f_i|} |\alpha| \pi_{0,p+1}(e_i \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1} f_i) \\
= \sum_i (-1)^{|f_i| |\alpha|+k} (e_i \overline{a}_1 \otimes \cdots \overline{a}_p \otimes a_{p+1} f_i) \\
= \overline{a}_1 \otimes \cdots \otimes \overline{a}_p \otimes a_{p+1},
\]
where the last identity follows from Lemma 2.2.

**Definition A.5.** Define \( \Pi : C^*_{sg}(A, A) \to D^*(A, A) \) as follows. For an element \( \bar{f} \in C^*_\text{sg}(A, A) \) represented by \( f \in C^m,\sigma(A, \Omega^\text{nc}_p(A)) \), let
\[
\Pi(\bar{f}) := \begin{cases}
\pi_{m-p,0} \circ \cdots \circ \pi_{m,p}(f) & \text{if } m - p \geq 0, \\
\pi_{0,p-m} \circ \pi_{1,m-p+1} \circ \cdots \circ \pi_{m,p}(f) & \text{if } m - p < 0.
\end{cases}
\]

Lemma A.4 implies that this is indeed well defined, namely, \( \Pi \) does not depend on the representative of \( \bar{f} \). Moreover, it follows from Lemmas A.3 and A.4 that \( \Pi \) is a morphism of cochain complexes such that \( \Pi \alpha = \text{id} \).

**A.4 | The chain homotopy \( H \) and Theorem 4.2**

We shall now define the chain homotopy \( H : C^*_{sg}(A, A) \to C^{*-1}_{sg}(A, A) \). Suppose \( m, p \in \mathbb{Z}_{>0} \). Given any \( f \in C^m,\sigma(A, (sA) \otimes^p A) \) define
\[
h_{m,p}(f) \in C^{m-1,\sigma}(A, (sA) \otimes^p A)
\]
by
\[
h_{m,p}(f) (\overline{a}_1 \otimes \cdots \otimes \overline{a}_{m-1}) = \sum_i (-1)^{|f_i|(|f|+k)} e_i \otimes (e \otimes \text{id} \otimes^p) (f (f_i \otimes \overline{a}_1 \otimes \cdots \otimes \overline{a}_{m-1})).
\]
We also define \( h_{m,p} := 0 \) if either \( m \leq 0 \) or \( p \leq 0 \). Note that \( h_{m,p} \) is of degree \(-1\). It follows from the identity \( \sum_i e_i \epsilon(f_i) = 1 \), that \( h_{m,p} \circ \epsilon = 0 \).

**Lemma A.6.** For any \( p > 0 \) we have the following identities:
\[
\delta \circ h_{m,p} + h_{m+1,p} \circ \delta = \begin{cases}
\text{id} - \theta_{0,p} & \text{if } m = 0, \\
\text{id} - \theta_{m-1,p-1} \circ \pi_{m,p} & \text{if } m > 0.
\end{cases} \tag{A3}
\]

**Proof.** By a similar computation in the proof of Lemma A.3, we can show that \( h_{\text{sg}} \) commutes with the internal differentials (that is, \( \delta^u \circ h_{\text{sg}} + h_{\text{sg}} \circ \delta^u = 0 \)). Thus, it is sufficient to prove that we have the following homotopy diagram:
For any \( x = \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1} \in C^{0,*}(A, \Omega^p_{nc}(A)) \), we have

\[
(id - \tau \circ \pi_{0,p})(x) = x - \sum_i (-1)^{|f_i| - 1} e_i \otimes (\epsilon \otimes \text{id}^\otimes p)(f_i(x))
\]

and

\[
h_{1,p} \circ \delta^h(x) = \sum_i (-1)^{|f_i| - 1} \epsilon(f_i) \otimes (\epsilon \otimes \text{id}^\otimes p)^\otimes (e_i \circ (\epsilon \otimes \text{id}^\otimes p)(x f_i))
\]

where, in the third identity, we use Lemma A.1 (2) and \( \epsilon(\overline{a_1}) = 0 \) if \( |a_1| \neq k \).

Similarly, for \( m > 0 \) we have

\[
(id - \theta_{m-1,p-1} \circ \pi_{m,p})(f) = f - \sum_i (-1)^{|f_i| - 1} \epsilon(f_i) \otimes (\epsilon \otimes \text{id}^\otimes p)(f_i(x f_i \otimes \epsilon(\overline{a_1}) \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1} f_i))
\]
and

\[ h_{m+1,p} \circ \delta^h(f)(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \]

\[ = - \sum_i (-1)^{|f_i|+1} \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f_i \mapsto f(\overline{a_1} \otimes \cdots \otimes \overline{a_m})) \]

\[ - \sum_i (-1)^{|f_i|+1} \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f_i \mapsto f(\overline{a_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_m})) \]

\[ - \sum_{j=1}^{m-1} (-1)^{|f_i|+1} \varepsilon_j \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f_i \mapsto f(\overline{a_j} \otimes \overline{a_{j+1}} \otimes \cdots \otimes \overline{a_m})) \]

\[ + \sum_i (-1)^{|f_i|+1} \varepsilon_j \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f_i \mapsto f(\overline{a_j} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_m})), \]

where \( \varepsilon_j = |a_1| + \cdots + |a_j| - j \). After canceling the common terms in the above two identities, we obtain

\[ (\delta^h \circ h_{m,p} + h_{m+1,p} \circ \delta^h)(f)(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) \]

\[ = - \sum_i (-1)^{|a_1|-1} \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{a_1} \otimes \overline{a_2} \otimes \cdots \otimes \overline{a_m})) \]

\[ - \sum_i (-1)^{|a_1|-1} \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{a_1} \otimes \cdots \otimes \overline{a_m})) \]

\[ - \sum_i (-1)^{|a_1|-1} \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{a_1} \otimes \cdots \otimes \overline{a_m})) \]

\[ = f(\overline{a_1} \otimes \cdots \otimes \overline{a_m}) - \sum_i (-1)^{|f|(|a_1|-1)+|f|-1} \overline{e_i} \otimes (\varepsilon \otimes \text{id}^\otimes p)(f(\overline{a_1} \otimes \cdots \otimes \overline{a_m})), \]

where in the second equality we use Lemma A.1 (2).

For \( m, p \in \mathbb{Z}_{>0} \), define

\[ H_{m,p} : \mathcal{C}^{m,*}(A, \Omega_{\text{nc}}^p(A)) \to \mathcal{C}^{m-1,*}(A, \Omega_{\text{nc}}^p(A)) \]

as the composition

\[ H_{m,p} := \min\{p-1,m-1\} \sum_{i=0}^{\min\{p-1,m-1\}} \theta_{m-2,p-1} \circ \cdots \circ \theta_{m-i-1,p-i} \circ h_{m-i,p-i} \circ \pi_{m-i+1,p-i+1} \circ \cdots \circ \pi_{m,p}, \]

where the term for \( i = 0 \) is \( h_{m,p} \). For convenience, we set \( H_{m,p} = 0 \) if either \( m = 0 \) or \( p = 0 \).
Lemma A.7. For $m, p \in \mathbb{Z}_{>0}$ the following diagram commutes:

\[
\begin{array}{ccc}
C^{m-1,*}(A, \Omega^{b-1}_{nc}(A)) & \xrightarrow{e_{m-1,p-1}} & C^{m,*}(A, \Omega^{b}_{nc}(A)) \\
\downarrow H_{m-1,p-1} & & \downarrow H_{m,p} \\
C^{m-2,*}(A, \Omega^{b-1}_{nc}(A)) & \xrightarrow{e_{m-2,p-1}} & C^{m-1,*}(A, \Omega^{b}_{nc}(A)).
\end{array}
\]

Proof. We have that

\[
H_{m,p} \circ \theta_{m-1,p-1} = \sum_{i=0}^{\min\{p-1,m-1\}} \theta_{m-2,p-1} \circ \cdots \circ \theta_{m-i+1,p-i} \circ h_{m-i,p-i} \circ \pi_{m-i+1,p-i+1} \circ \cdots \circ \pi_{m,p} \circ \theta_{m-1,p-1}
\]

\[
= \theta_{m-2,p-1} \circ H_{m-1,p-1},
\]

where the second identity follows from Lemma A.4 and the fact that $h_{m,p} \circ \theta_{m-1,p-1} = 0$. \qed

The above lemma allows us to make the following definition.

Definition A.8. Define $H : C^*_{sg}(A,A) \rightarrow C^{*,-1}_{sg}(A,A)$ to be the map induced by the maps $H_{m,p}$ after passing to the colimit. That is, for any $\tilde{f} \in C^*_{sg}(A,A)$ which is represented by $f \in C^{m,*}(A,\Omega^{b}_{nc}(A))$ for some $m,p$, we define

\[
H(\tilde{f}) = H_{m,p}(f) \in C^{*,1}_{sg}(A,A).
\]

Proof of Theorem 4.2. It follows directly from Lemmas A.6 and A.7 that the map $H$ is a chain homotopy between $id$ and $\iota \circ \Pi$. Namely,

\[
id - \iota \circ \Pi = \delta_{sg} \circ H + H \circ \delta_{sg}.
\]

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