Hybrid Trigonometric Varieties

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Abstract

In this paper we introduce the notion of hybrid trigonometric parametrization as a tuple of real rational expressions involving circular and hyperbolic trigonometric functions as well as monomials, with the restriction that variables in each block of functions are different. We analyze the main properties of the varieties defined by these parametrizations and we prove that they are exactly the class of real unirational varieties. In addition, we provide algorithms to implicitize and to convert a hybrid trigonometric parametrization into a unirational one, and viceversa.

keywords: Trigonometric parametrization, hyperbolic parametrization, implicitization algorithm, unirational algebraic variety.

1 Introduction

In many problems, the parametric representation of geometric objects turns to be a fundamental tool. Clear examples of this claim may be found in some geometric constructions in computer aided design, like plotting, computing intersections, etc. (see [9]), or in computing integrals or solving differential equations (see e.g. [5], [6]).
Probably the most common used parametrizations are the rational parametrizations (see [9] and [25]), but other types of parametrizations can also be applied, as radical parametrizations (see [23], [24]) or trigonometric parametrizations (see [8], [11], [17], [22]). Alternatively, one may work locally with parametrizations using rational or trigonometric splines (see [7], [9], [11], [21]).

In this paper, we focus on the trigonometric-like type of parametrizations. The class of varieties studied in this paper is extended from the trigonometric curves considered in [8], i.e. curves parametrized in terms of truncated Fourier series. This extension is made in three respects. On one hand, we analyze varieties associated to the so-called hybrid trigonometric parametrizations in which not only circular trigonometric functions may appear, but also hyperbolic trigonometric and monomials; the idea of considering hyperbolic functions already appears in [20]. On the other hand, not only polynomials are accounted, but also rational parametrizations of the previous form. Finally, the study is done for general real algebraic varieties and it is not restricted to the case of curves or surfaces.

So, we may be leading with a parametrization of the form (see Definition 1 for further details)

\[
\left( \frac{\sin(t_1)}{\cos(t_2) + \cosh(t_3)}, t_4 + \sinh(t_3), t_4 \cos(t_1), \frac{\cosh(t_3)}{t_4}, \sin(t_2) \right).
\]

We call this type of parametrizations hybrid trigonometric in the sense that the combine rationally elements from three different sets, namely

\[
\{\sin(t_i), \cos(t_i)\}_{1 \leq i \leq m_1}, \{\sinh(t_i), \cosh(t_i)\}_{m_1+1 \leq i \leq m_2}, \{t_i\}_{m_2+1 \leq i \leq m_3}.
\]

Considering the parametrization as a real-valued function, and taking the Zariski closure of its image, we introduce the notion of hybrid trigonometric variety (see Definition 6). We prove that hybrid trigonometric varieties are irreducible. Furthermore, we prove that they are precisely the (real) unirational varieties; we recall that real unirational means that it can be parametrized by means of real rational functions, but the corresponding function associated to the parametrization might not be injective. In addition, we provide algorithms to implicitize the hybrid trigonometric parametrization and to convert unirational parametrizations into hybrid trigonometric parametrization, and vice versa; for any prescribed triple \((m_1, m_2, m_3)\) such that \(m_1 + m_2 + m_3\) is the dimension of the variety (see above the meaning of \(m_i\)).

At first glance one may notice no advance on this approach due to the absence of an enlargement of the class of unirational varieties when considering hybrid trigonometric parametrizations. However, a deepen study reveals that the appropriate point of view, considering either rational or trigonometric parametrizations, may lead to a more accurate solution of a problem under consideration; and hence being provided with conversion and implicitation algorithms enhances the applicability of the unirational varieties.

We devote a section to illustrate by examples some potential applications of hybrid
trigonometric varieties. We comment some of them in this introduction. Trigono-
metric curves emerge in numerous areas, as stated in [8]: classical curves, differential
equations, Fourier analysis, etc. In addition, the important role of the varieties under
study here are also shown up in recent studies. In the work [16] (see also [4], [12], and
[13]) the extended generalized Riccati Equation Mapping Method is applied for the
(1+1)-Dimensional Modified KdV Equation (see Subsection 5.4). The authors arrive
at different families of solutions, classified into soliton and soliton-like solutions (writ-
ten in terms of rational functions of hyperbolic ones), and periodic solutions (written
as rational functions of trigonometric ones), under different cases of the parameters in-
volved. Applying the ideas in [5], and using the results in this paper, one may approach
the problem transforming the trigonometric parametric parametrization induced by the
solution into a rational one.

Another source of applications is the use of trigonometric functions to describe geo-
metric constructions like offsets, conchoids, cissoids, epicycloids, hypocycloids, etc. In
this case, one usually introduces polar parametrizations. A polar representation of a
surface is of the form

$$f(u, v) = \rho(u, v)k(u, v),$$

where $\|k(u, v)\| = 1$ is a parametrization of the unit sphere, and $\rho(u, v)$ is a posi-
tive radius function; for references on this topic, we refer to [18, 19]. This work is
concerned with the case in which both $\rho$ and $k$ are expressed as rational functions of
$\{\cos(u), \sin(u), \cos(v), \sin(v)\}$. Indeed, Subsection 5.2 deals with the study of epicy-
cloid and hypocycloid surfaces in which $k$ is chosen as the spherical coordinates in
$\mathbb{R}^3$.

Trigonometric curves and surfaces provide a wide catalog of shapes to be used in the
application of the Hough transform to image processing. However, in order to apply the
method one needs to deal with the implicit representation of the curves and surfaces
in the catalog (see [2], [3]). So, implicitization processes, as detailed in this paper, are
required.

Other applications of this family of parametrizations are the interpolation of certain
functions via quotients of trigonometric polynomials, as described in [10], the plotting
of curves and surfaces via trigonometric parametrizations (see Subsection 5.1) or the
computation of intersection of geometric objects given in trigonometric form; in this
last case, it is useful to transform the given parametrization into a rational one (see
Subsection 5.3).

The paper is structured as follows. Section 2 is devoted to introduce the notions of
hybrid trigonometric parametrization and variety and the first properties are studied.
In Section 3 we analyze the fundamental properties of the hybrid trigonometric varieties
and we see that they are characterized as the real unirational varieties. In Section 4 we
outline the algorithms derived from the proofs in the previous section, and in Section
5 we illustrate by examples the potential applicability of our results. The paper ends
with a summary of conclusions.
Throughout this paper, we will be working with both the usual Euclidean topology and the Zariski topology. In case of ambiguity we will specify which topology is used.

2 Hybrid Trigonometric Parametrizations and Varieties

We denote by $\text{dom}(f)$ the domain of a function $f$, and by $\text{Jac}(f)$ its Jacobian. Let $m, n \in \mathbb{N}$ such that $0 < m < n$; $n$ will represent the dimension of the affine space where we work and $m$ the dimension of the variety. We denote $\vec{t} = (t_1, \ldots, t_m)$. In addition, in the sequel we consider non-negative integers $m_1, m_2, m_3$ such that $m_1 + m_2 + m_3 = m$. We will use the notation $\vec{m} = (m_1, m_2, m_3)$.

In the next, we introduce the notion of $\vec{m}$–hybrid trigonometric parametrization that essentially is an $n$-tuple depending on $m$ parameters such that $m_1$ of them appear as the variable of a sine or a cosine, $m_2$ appear as the variable of a hyperbolic sine or cosine and $m_3$ of them appear rationally. More precisely, we have the following the definition.

**Definition 1** We say that an $n$-tuple 
$$ T(\vec{t}) = (T_1(\vec{t}), \ldots, T_n(\vec{t})),$$

is an $\vec{m}$–hybrid trigonometric parametrization if there exists a decomposition 
$$ \{1, \ldots, m\} = J_1 \cup J_2 \cup J_3$$

with $\#(J_i) = m_i$, $i = 1, 2, 3$ and there exist $\alpha_{ij}, \alpha_{ij}^* \in \mathbb{Q} \setminus \{0\}$ and $\omega_{ij}, \omega_{ij}^* \in \mathbb{R}$ such that

$$ T_i(\vec{t}) \in \mathbb{R}(A_1, A_2, A_3)$$

where

$$\begin{cases} 
A_1 &= \{\cos(\alpha_{i1}t_i + \omega_{i1}), \sin(\alpha_{i1}^*t_i + \omega_{i1}^*)\}_{i \in J_1}, \\
A_2 &= \{\cosh(\alpha_{i2}t_i + \omega_{i2}), \sinh(\alpha_{i2}^*t_i + \omega_{i2}^*)\}_{i \in J_2}, \\
A_3 &= \{t_j\}_{j \in J_3},
\end{cases}$$

and all parameters in $\vec{t}$ do appear in $T(\vec{t})$. Associated to $T$ we will consider the real function $T : \text{dom}(T) \subset \mathbb{R}^m \rightarrow \mathbb{R}^n; \vec{t} \mapsto T(\vec{t})$.

**Remark 1** We will refer to $(m, 0, 0)$–parametrizations as circular trigonometric, and to $(0, m, 0)$–parametrizations as hyperbolic trigonometric. Note that $(0, 0, m)$–parametrizations are precisely rational parametrizations. In Theorem 12, we prove that three notions, namely hybrid trigonometric, circular trigonometric and hyperbolic trigonometric, are related.

**Example 2** The clearest examples of trigonometric parametrizations are the circle $(r \cos(t), r \sin(t))$ and the hyperbola $(r \cosh(t), r \sinh(t))$, with $r \in \mathbb{R}$, that are $(1, 0, 0)$ and $(0, 1, 0)$ parametrizations, respectively. Similarly the spherical coordinates of an
sphere generates a \((2, 0, 0)\)-parametrization. All the previous examples are polynomial expressions of the trigonometric functions. However, in our case, we allow denominators. For instance, \((1/\cos(t), \sin(t))\) is a circular trigonometric parametrization of the rational quartic \(x^2y^2 - x^2 + 1 = 0\). However, observe that \((t, \sin(t))\) is not a trigonometric parametrization (in the sense of our definition) since \(t \not\in \mathbb{R}(\cos(t), \sin(t))\); note that, any rational expression of \(\{\cos(t), \sin(t)\}\) is periodic. A similar case happens with the helix \((\cos(t), \sin(t), t)\). Finally, \((t_2 \cos(t_1), t_2 \sin(t_1), t_2)\) is a \((1, 0, 1)\)-parametrization of the cone \(x^2 + y^2 = z^2\).

**Remark 2** Note that in Definition 1 we have asked \(\alpha_{ij}\) and \(\alpha_{ij}^*\) to be rational numbers. The reason to exclude irrational numbers is that, in general, the Zariski closure of the image of the real function defined by the parametrization could be the whole affine space. For instance, if we take \(T(t) = (\sin(t), \sin(\alpha t))\), with \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\), we prove that the Zariski closure of \(T(\text{dom}(T))\) is \(\mathbb{R}^2\) and hence it does not define an algebraic curve as one may expect. More precisely, let \(\beta \in \mathbb{R}\), and let \(\Omega_\beta := \{(\sin(\beta + 2\pi n), \sin(\alpha\beta + 2\pi n)) \mid n \in \mathbb{N}\} \subset \{T(t) \mid t \in \mathbb{R}\} \cap \{(\sin(\beta), \lambda) \mid \lambda \in \mathbb{R}\}\). Now, we prove that \(\Omega_\beta\) has infinitely many elements. Indeed, \(\sin(\alpha\beta + 2\pi n)\) turns out to be the imaginary part of \(e^{i(\alpha\beta + 2\pi n)}\). Two of these exponentials coincide for different \(n_1, n_2 \in \mathbb{N}\) if and only if
\[
\alpha\beta + 2\alpha n_1\pi = \alpha\beta + 2\alpha n_2\pi + 2\pi n,
\]
for some \(n \in \mathbb{Z}\). Then, it holds that \(\alpha(n_1 - n_2) = n\), which yields that \(n_1 = n_2\) and \(n = 0\) under the assumption that \(\alpha \in \mathbb{R} \setminus \mathbb{Q}\). This implies that \(\Omega_\beta\) must be an infinite set. Therefore, by Bézout's theorem, the Zariski closure of \(T(\text{dom}(T))\) contains all lines \(x = \sin(\beta)\) with \(\beta \in \mathbb{R}\). Thus, \(\overline{T(\text{dom}(T))} = \mathbb{R}^2\).

**Lemma 3** For every \(\overline{m}\)-hybrid trigonometric parametrization there exists a linear transformation \(L : \mathbb{R}^m \rightarrow \mathbb{R}^m\) such that \(T(L(\overline{t}))\) is a tuple which entries belong to \(\mathbb{R}(\{\cos(t_i), \sin(t_i)\}_{i \in J_1}, \{\sinh(t_i), \cosh(t_i)\}_{i \in J_2}, \{t_i\}_{i \in J_3})\), where \(J_1, J_2, J_3\) are as in Definition 1. Furthermore, for every \(i \in J_1\) and for every \(j \in J_2\) the functions \(\cos(t_i), \sin(t_i), \cosh(t_j), \sinh(t_j)\) appear in \(T(L(\overline{t}))\).

**Proof:** Let \(T(\overline{t})\) be expressed as in Definition 1. First we observe that, using the formulas of the sine, cosine (both circular and hyperbolic) of the addition of angles, \(T(\overline{t})\) can be expressed as a tuple with entries in
\[
\mathbb{R}(\{\cos(\alpha_{i,1} t_i), \sin(\alpha_{i,1}^* t_i)\}_{i \in J_1}, \{\sinh(\alpha_{i,2} t_i), \cosh(\alpha_{i,2}^* t_i)\}_{i \in J_2}, \{t_i\}_{i \in J_3}).
\]
Let us assume that \(T(\overline{t})\) is already expressed as mentioned above. Let \(\ell_1\) be the lcm of all denominators of \(\{\alpha_{i,1}, \alpha_{i,1}^*\}_{i \in J_1}\), and let \(\ell_2\) be the lcm of all denominators of \(\{\alpha_{i,2}, \alpha_{i,2}^*\}_{i \in J_2}\). Now, we consider the linear map defined as
\[
L_1(\overline{t}) = \begin{cases} 
\ell_1 t_i & \text{if } i \in J_1 \\
\ell_2 t_i & \text{if } i \in J_2 \\
t_i & \text{if } i \in J_3
\end{cases}
\]
Then, $\mathcal{T}(\mathcal{L}_1(\vec{t}))$ is a tuple with entries in
\[
\mathbb{R}(\{\cos(n_1t_i), \sin(n_1^*t_i)\}_{i \in J_1}, \{\sinh(n_2t_i), \cosh(n_2^*t_i)\}_{i \in J_2}, \{t_i\}_{i \in J_3})
\]
with $n_1, n_1^*, n_2, n_2^* \in \mathbb{N}$. Finally, based on the expression of $\sin(kt)$, $\cos(kt)$, $\sinh(kt)$ and $\cosh(kt)$, with $k \in \mathbb{N}$, as polynomials expressions of $\sin(t)$, $\cos(t)$, $\sinh(t)$ and $\cosh(t)$, via Chebyshev polynomials (see 22.3.15 and the derivative of this formula; and 4.5.31 and 4.5.32 in [1]) one gets that the entries of $\mathcal{T}(\mathcal{L}_1(\vec{t}))$ belong to
\[
\mathbb{R}(\{\cos(t_i), \sin(t_i)\}_{i \in J_1}, \{\sin(t_i), \cosh(t_i)\}_{i \in J_2}, \{t_i\}_{i \in J_3}).
\]
For the last part of the proof, let $K_1 \subset J_1$ consist in those $i \in J_1$ for which either $\sin(t_i)$ or $\cos(t_i)$ does not appear in the $\mathcal{T}(\mathcal{L}_1(\vec{t}))$; similarly, we introduce $K_2 \subset J_2$. Then, we introduce the linear map
\[
\mathcal{L}_2(\vec{t}) = \begin{cases} 
2t_i & \text{if } i \in K_1 \\
2t_i & \text{if } i \in K_2 \\
t_i & \text{otherwise}
\end{cases} \tag{2}
\]
Using the expressions of $\cos(2t_i)$ and $\sin(2t_i)$ in terms of $\cos(t_i), \sin(t_i)$, and the corresponding expressions of $\cosh(2t_i)$ and $\sinh(2t_i)$ in terms of $\cosh(t_i), \sinh(t_i)$ we have that $\mathcal{T}(\mathcal{L}_2(\mathcal{L}_1(\vec{t})))$ satisfies the required property.

**Remark 3** Let $\mathcal{T}(\vec{t})$ be a hybrid trigonometric parametrization as in Definition 1 and $\mathcal{T}^*(\vec{t}) = \mathcal{T}(\mathcal{L}(\vec{t}))$ be the reparametrization in Lemma 3. Then, taking into account that $\mathcal{L}$ is a bijection we get that $\mathcal{T}(\text{dom}(\mathcal{T})) = \mathcal{T}^*(\text{dom}(\mathcal{T}^*))$.

**Example 4** In this example, we illustrate the statement in Lemma 3, see also Algorithm 4. Let $n = 4, m = 3$, $\overline{m} = (1, 1, 1)$, and $a_1, a_2 \in \mathbb{R}$. Let $\mathcal{T}$ be defined by
\[
\mathcal{T}(\vec{t}) = \begin{pmatrix}
\frac{\cos(a_1 + \frac{1}{3}t_1) + t_3}{\sinh(\frac{1}{3}t_2) + t_3^2}, & \frac{\cos(\frac{1}{3}t_1) + t_3^2}{\sin(\frac{1}{3}t_2) + t_3^2}, & \frac{\cos(\frac{1}{3}t_1) + t_3}{\sinh(\frac{1}{3}t_2) + t_3}
\end{pmatrix},
\]
that reads as follows
\[
\mathcal{T}(\vec{t}) = \begin{pmatrix}
A_1 \cos(\frac{1}{3}t_1) - A_2 \sin(\frac{1}{3}t_1) + t_3, & A_1 \sin(\frac{1}{3}t_1) + A_2 \cos(\frac{1}{3}t_1) + t_3
\end{pmatrix},
\]
for $A_1 = \cos(a_1), A_2 = \sin(a_1), A_3 = \cosh(a_2), A_4 = \sinh(a_2)$. The linear changes of parameters in (4) and (2) are $\mathcal{L}_1(\vec{t}) = (3t_1, 2t_2, t_3)$ and $\mathcal{L}_2(\vec{t}) = (t_1, t_2, t_3)$. So we get
\[
\mathcal{T}(\mathcal{L}_2(\mathcal{L}_1(\vec{t}))) = \begin{pmatrix}
\frac{A_1 \cos(t_1) - A_2 \sin(t_1) + t_3}{\sinh(t_2) + t_3^2}, & \frac{\cos(t_1) + t_3^2}{\sinh(t_2) A_3 + \cosh(t_2) A_4 + t_3},
\end{pmatrix}.
\]
Definition 5 We say that an \(\overline{m}\)-hybrid trigonometric parametrization \(\mathcal{T}(\overline{t})\) is pure if it satisfies the properties in Lemma \(\tilde{3}\) i.e. it is a tuple which entries belong to \(\mathbb{R}(\{\cos(t_i), \sin(t_i)\}_{i \in J_1}, \{\sinh(t_i), \cosh(t_i)\}_{i \in J_2}, \{t_i\}_{i \in J_3})\), where \(J_1, J_2, J_3\) are as in Definition \(\tilde{4}\) and for every \(i \in J_1\) and for every \(j \in J_2\) the functions \(\cos(t_i), \sin(t_i), \cosh(t_j), \sinh(t_j)\) appear in \(\mathcal{T}(\overline{t})\).

General assumptions: In the sequel we will always assume that all parametrizations are pure. If this would not be the case, reparametrizing with \(\mathcal{L}_2 \circ \mathcal{L}_1\) (see \(\tilde{1}\) and \(\tilde{2}\)) the parametrization is transformed in pure form. In addition, for simplicity in the explanation, we assume w.l.o.g. that \(J_1, J_2, J_3\) in Definition \(\tilde{4}\) are taken as \(J_1 = \{1, \ldots, m_1\}, J_2 = \{m_1 + 1, \ldots, m_1 + m_2\}, J_3 = \{m_1 + m_2 + 1, \ldots, m\}\), understanding that if \(m_i = 0\) the corresponding \(J_i\) is empty and the sets are swift to the left.

Definition 6 Let \(\mathcal{V} \subset \mathbb{R}^n\) be a real algebraic variety of dimension \(m\). We say that \(\mathcal{V}\) is a hybrid trigonometric variety if there exists a hybrid trigonometric parametrization \(\mathcal{T}(\overline{t})\) such that \(\mathcal{V}\) is the Zariski closure of the image of \(\mathcal{T}\). In this case, we say that \(\mathcal{T}(\overline{t})\) is a hybrid trigonometric parametrization of \(\mathcal{V}\).

Remark 4 In Definition \(\tilde{6}\), if \(\mathcal{T}(\overline{t})\) is given as in Definition \(\tilde{2}\) because of Remark \(\tilde{3}\) the pure parametrization \(\mathcal{T}^*(\overline{t})\), provided by Lemma \(\tilde{3}\) also satisfies the conditions imposed in Definition \(\tilde{6}\).

3 Main Properties

In this section, we use the notation as well as the hypotheses introduced in Section \(\tilde{2}\). In addition, in the sequel, we will consider the hybrid \(m\)-dimensional torus

\[
(\text{HT})_{\overline{m}} := S^1 \times \cdots \times S^1 \times \mathcal{H}^1 \times \cdots \times \mathcal{H}^1 \times \mathbb{R}^{m_3} \subset \mathbb{R}^{2m_1 + 2m_2 + m_3},
\]

where \(S^1\) is the unit circle centered at the origin, and \(\mathcal{H}^1\) stands for the hyperbola \(\{(x, y) : x^2 - y^2 = 1\}\). Observe that the implicit equations of \((\text{HT})_{\overline{m}}\) are

\[
\{x_1^2 + x_2^2 = 1, \ldots, x_{2m_1-1}^2 + x_{2m_1}^2 = 1, x_{2m_1+1}^2 - x_{2m_1+2}^2 = 1, \ldots, x_{2m_1+2m_2-1}^2 - x_{2m_1+2m_2}^2 = 1\}.
\]

Furthermore, let

\[
\xi(t) = \left(\frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1}\right), \quad \nu(t) = \left(\frac{t^2 + 1}{2t}, \frac{t^2 - 1}{2t}\right)
\]

be a proper parametrization of \(S^1\) and \(\mathcal{H}^1\), respectively. Then

\[
\mathcal{M} : \mathbb{R}^{m_1} \times (\mathbb{R} \setminus \{0\})^{m_2} \times \mathbb{R}^{m_3} \longrightarrow (\text{HT})_{\overline{m}}
\]

\[
\overline{t} \mapsto (\xi(t_1), \ldots, \xi(t_{m_1}), \nu(t_{m_1+1}), \ldots, \nu(t_{m_1+m_2}), t_{m_1+m_2+1}, \ldots, t_m).
\]
is a proper parametrization of \((HT)_m\) which inverse is

\[
\mathcal{M}^{-1} : (HT)_m \setminus F_{m_1} \subset \mathbb{R}^{2m_1+2m_2+m_3} \to \mathbb{R}^m
\]

\[
\begin{pmatrix}
\frac{x_1}{1-x_2}, \ldots, \frac{x_{2m_1-1}}{1-x_{2m_1}}, \frac{1}{x_{2m_1+1} - x_{2m_1+2}}, \ldots, \frac{1}{x_{2m_1+2m_2} - x_{2m_1+2m_2}}
\end{pmatrix},
\]

and \(F_{m_1}\) stands for the closed set

\[
F_{m_1} := \{(x_1, \ldots, x_{2m_1+2m_2+m_3}) \in (HT)_m \mid x_{2i} = 1, \text{ for } i = 1, \ldots, m_1\}.
\]

**Proposition 7** Every hybrid trigonometric variety is irreducible.

*Proof:* Let \(V\) be hybrid of dimension \(m\), and let \(\mathcal{T}(\vec{t})\), with \(\vec{t} = (t_1, \ldots, t_m)\), be an \(m\)–hybrid trigonometric parametrization of \(V\), which by our assumption is taken pure. Let \(\mathcal{F}(\vec{y})\), where \(\vec{y} = (y_{1,1}, y_{1,2}, \ldots, y_{m_1,1}, y_{m_1,2}, \ldots, y_{m_1+m_2,1}, y_{m_1+m_2,2}, y_{m_1+m_2+1,1}, \ldots, y_m)\), be the tuple obtained from \(\mathcal{T}(\vec{t})\) by replacing \(\cos(t_i)\) (resp. \(\sin(t_i)\)) by \(y_{i_1}\) (resp. \(y_{i_2}\)), with \(1 \leq i \leq m_1\) and \(\cosh(t_j)\) (resp. \(\sinh(t_j)\)) by \(y_{j_1}\) (resp. \(y_{j_2}\)), with \(m_1 + 1 \leq j \leq m_1 + m_2\), and \(t_k\) by \(y_k\), with \(m_1 + m_2 + 1 \leq k \leq m\). We introduce the map

\[
\Psi : \text{dom}(\mathcal{T}) \to (HT)_m \subset \mathbb{R}^{2m_1+2m_2+m_3}
\]

\[
\vec{t} \mapsto (\cos(t_1), \sin(t_1), \ldots, \cos(t_{m_1}), \sin(t_{m_1}), \cosh(t_{m_1+1}), \sinh(t_{m_1+1}), \ldots, \cosh(t_{m_1+m_2}), \sinh(t_{m_1+m_2}), \ldots, t_m).
\]

Let \(H\) be the lcm of all denominators in the tuple of rational function \(\mathcal{F}(\vec{y})\), and \(\Omega = (HT)_m \setminus \{\vec{y} \mid H(\vec{y}) = 0\}\); observe that \(\Omega \neq \emptyset\) since \(\emptyset \neq \text{ dom}(\mathcal{T})\) because \(V\) is its Zariski closure. So \(\mathcal{F}\) induces the rational map

\[
\mathcal{F} : \Omega \subset (HT)_m \to \mathbb{R}^n
\]

\[
\vec{y} \mapsto \mathcal{F}(\vec{y}).
\]

Moreover, let \(\mathcal{Z}\) be the Zariski closure \(\mathcal{Z} = \overline{\mathcal{F}(\Omega)}\). Since \((HT)_m\) is irreducible, we have that \(\mathcal{Z}\) is irreducible. Furthermore, \(\dim(\mathcal{Z}) \leq \dim((HT)_m) = m\) (see [14], page 73, for a reference).
Since \( \text{dom}(\mathcal{T}) = \Psi^{-1}(\text{dom}(\mathcal{F})) = \Psi^{-1}(\Omega) \) (see Diagram 13), then \( \mathcal{T}(\vec{t}) = \mathcal{F}(\Psi(\vec{t})) \) for \( \vec{t} \in \text{dom}(\mathcal{T}) \). Thus, \( \mathcal{V} = \overline{\mathcal{T}(\text{dom}(\mathcal{T}))} \subset \mathcal{F}(\Omega) = \mathcal{Z} \). Therefore, since \( m = \dim(\mathcal{V}) \leq \dim(\mathcal{Z}) \leq m \), then \( \dim(\mathcal{Z}) = m \). Thus, since \( Z \) is irreducible and \( \mathcal{V} \subset \mathcal{Z} \), by [26] (see Theorem 1 in p. 68), we get that \( \mathcal{Z} = \mathcal{V} \), and hence \( \mathcal{V} \) is irreducible. \( \square \)

**Proposition 8** Every hybrid trigonometric variety is unirational over \( \mathbb{R} \).

**Proof:** Let \( \mathcal{T}(\vec{t}) \), with \( \vec{t} = (t_1, \ldots, t_m) \), be an \( m \)-hybrid trigonometric parametrization of a hybrid trigonometric variety \( \mathcal{V} \), where \( \dim(\mathcal{V}) = m \). Let \( \mathcal{F}(\vec{y}), \Psi \) and \( \mathcal{Z} \) as in the proof of Proposition 7, and (HT) \( m \) as in (5). Let \( \mathcal{M} \) be as in (7).

\[
\mathcal{V} = \mathcal{Z} \subset \mathbb{R}^n \quad \text{(HT)} \quad \mathbb{R}^{m_1} \times (\mathbb{R} \setminus \{0\})^{m_2} \times \mathbb{R}^{m_3}
\]

Then, \( \mathcal{G} = \mathcal{F}(\mathcal{M}(\vec{t})) \) (see Diagram 14) is a real unirational parametrization with image in \( \mathcal{Z} \). Since \( \mathcal{M} \) is dominant in (HT) \( m \) and \( \mathcal{F} \) is dominant in \( \mathcal{Z} \), then \( \mathcal{G} \) is a real unirational parametrization of \( \mathcal{Z} = \mathcal{V} \). \( \square \)

**Lemma 9** Let \( p(\vec{z}) \in \mathbb{R}[\vec{z}] \), with \( \vec{z} = (z_1, \ldots, z_m) \), be a non-zero polynomial and let \( L(\vec{x}) \), with \( \vec{x} = (x_1, \ldots, x_{2m_1+2m_2+m_3}) \), be as in (9). Then

1. \( p(L(\vec{x})) \) is not identically zero.

2. Let \( M(\vec{x}) \) be the numerator of \( p(L(\vec{x})) \). It holds that (HT) \( m \) \( \not\subset \{ \vec{a} \in \mathbb{R}^{2m_1+2m_2+m_3} \mid M(\vec{a}) = 0 \} \).

**Proof:** Let

\[
\vec{x}^* = (x_1, 0, x_3, 0, \ldots, x_{2m_1-1}, 0, \frac{1}{x_{2m_1+1}}, 0, \frac{1}{x_{2m_1+3}}, 0, \ldots, \frac{1}{x_{2m_1+2m_2-1}}, 0, \frac{1}{x_{2m_1+2m_2+1}}, \ldots, x_{2m_1+2m_2+m}).
\]

Then, one has that \( p(\vec{x}) = p(L(\vec{x}^*)) \). It holds that, under the assumption that \( p(L(x_1, \ldots, x_{2m})) = 0 \) then \( p(x_1, x_3, \ldots, x_{2m-1}) = 0 \). Hence, the first part of the statement follows.
Let us prove the second statement in the result. First we observe $M = p(L)N$ where $N$ is a polynomial of the form

$$(1 - x_2)^{\ell_1} \cdots (1 - x_{2m_1})^{\ell_{m_1}}(x_{2m_1+1} - x_{2m_1+2})^{\ell_{m_1+1}} \cdots (x_{2m_1+2m_2-1} - x_{2m_1+2m_2})^{\ell_{m_1+m_2}},$$

for some $\ell_i \in \mathbb{N} \cup \{0\}$, $i = 1, \ldots, m_1 + m_2$. Substituting by $M(\vec{t})$, see [7], we get

$$M(\mathcal{M}(\vec{t})) = p(L(\mathcal{M}(\vec{t})))N(\mathcal{M}(\vec{t})) = p(\vec{t})N(\mathcal{M}(\vec{t})).$$

Since $p$ is not zero, and $N(\mathcal{M}(\vec{t}))$ is not zero either, then $M(\mathcal{M}(\vec{t})) \neq 0$ and hence then result follows. \qed

In the following lemma, let $(\text{HT})_\overline{m}$, $\Psi$ be as in [5, 11].

**Lemma 10** Let $\Theta$ be such that $\Psi(\Theta)$ is Zariski-dense in $(\text{HT})_\overline{m}$, and let $\Omega$ be a Zariski non–empty open subset of $(\text{HT})_\overline{m}$. Then, $\Psi^{-1}(\Omega) \cap \Theta \neq \emptyset$.

**Proof:** Let $\Omega = (\text{HT})_\overline{m} \setminus \Sigma$, with $\Sigma$ close. We first prove that $\Omega \cap \Psi(\Theta) \neq \emptyset$. Indeed, let us assume that $\Omega \cap \Psi(\Theta) = \emptyset$. Then, $\Psi(\Theta) \subset \Sigma$. Taking the Zariski closures we get $(\text{HT})_\overline{m} = \Sigma$, which implies that $\Omega = \emptyset$, that is a contradiction. Now, let $\overline{x} \in \Omega \cap \Psi(\Theta)$, then there exists $\overline{t} \in \Theta$ such that $\Psi(\overline{t}) = \overline{x}$. So, $\overline{t} \in \Psi^{-1}(\Omega) \cap \Theta$. \qed

**Proposition 11** Every unirational variety over $\mathbb{R}$ is hybrid trigonometric.

**Proof:** Let $V \subset \mathbb{R}^n$ be a unirational variety over $\mathbb{R}$ with $\dim(V) = m$. Fix a triple of non-negative integers $\overline{m} = (m_1, m_2, m_3)$ such that $m_1 + m_2 + m_3 = m$. Let $(\text{HT})_{\overline{m}} \subset \mathbb{R}^{2m_1+2m_2+m_3}$, $\mathcal{M}$, $\mathcal{M}^{-1}$ and $\Psi$ as in [5, 7, 8, 11]. Let

$$\mathcal{P}(\overline{t}) = \left(\frac{p_1(\overline{t})}{q_1(\overline{t})}, \ldots, \frac{p_n(\overline{t})}{q_n(\overline{t})}\right), \text{ with } \overline{t} = (t_1, \ldots, t_m),$$

be a rational real parametrization of $V$. We consider the map $\mathcal{Q} = \mathcal{P} \circ \mathcal{M}^{-1} \circ \Psi$ (see Diagram [16].

\begin{align*}
\begin{array}{cccc}
\mathcal{P} & : & V \subset \mathbb{R}^n & \rightarrow \mathbb{R}^{2m_1+2m_2+m_3} \\
\mathcal{M}^{-1} & : & (\text{HT})_{\overline{m}} \subset \mathbb{R}^{2m_1+2m_2+m_3} & \rightarrow \mathbb{R}^{2m_1+2m_2+m_3} \\
\Psi & : & \mathbb{R}^{m_1} \times (\mathbb{R} \setminus \{0\})^{m_2} \times \mathbb{R}^{m_3} & \rightarrow \mathbb{R}^m
\end{array}
\end{align*}
Let us prove that $Q(\bar{\theta})$ is an $m$-hybrid trigonometric parametrization (see Definition [1]). For this purpose, we have to check that the formal substitutions in $Q$ are well defined. We first observe that by Lemma [9] (1), the substitution $P \circ M^{-1}(\bar{x}) = P(L(\bar{x}))$ is well defined. Now, let $M(\bar{x})$ be the numerator of $q_i(L(\bar{x}))$ for some $i$, and let us assume that $M(\Psi(\bar{\theta}))$ is identically zero. This implies that $\Psi(\mathbb{R}^m)$ is included in the variety $W$ defined by $M$. Therefore, taking Zariski closures, and using that $\Psi$ is dominant in $(HT)_m^\infty$, we get that $(HT)_m^\infty = \overline{\Psi(\mathbb{R}^m)} \subset W$, which contradicts Lemma [9] (2).

Now, we check that $Q$ satisfies the condition in Definition [6], namely that the Zariski closure of $Q(\text{dom}(Q))$ is $V$. For this purpose, we prove that there exists an Euclidean open set $\emptyset \neq \Theta \subset \text{dom}(Q)$, such that the Zariski closure of $Q(\Theta)$ is $V$; from where one concludes the result since

$$V = \overline{Q(\Theta)} \subset \overline{Q(\text{dom}(Q))} \subset V = V.$$

(1) We prove that $P \circ M^{-1}$ is defined on a non-empty Euclidean open subset (in the induced topology) $\Omega_1$ of $(HT)_m^\infty$. We know that $P(\mathcal{M}^{-1}(\bar{x}))$ is well-defined. Let $H(\bar{x})$ be the lcm of the denominators of $P(\bar{\theta})$ and $g(\bar{x})$ the numerator of $H(L(\bar{x}))$. We consider the close subsets of the $(HT)_m^\infty$, $F_{m_1}$ (see [10]) and $\Sigma := \{ \bar{x} \in (HT)_m^\infty | g(\bar{x}) = 0 \}$. Clearly, the open subset $(HT)_m^\infty \setminus F_{m_1}$ is not empty, and by Lemma [9] (2) it holds that $(HT)_m^\infty \setminus \Sigma \neq \emptyset$. Moreover, since $(HT)_m^\infty$ is irreducible, then $\Omega_1 := (HT)_m^\infty \setminus (F_{m_1} \cup \Sigma)$ is a nonempty Euclidean open subset of the torus. Finally, let us see that $\Omega_1$ is included in the domain of $P \circ M^{-1}$. Indeed, if $\bar{x} \in \Omega_1$, then $\bar{x} \notin F_{m_1}$ and hence $\mathcal{M}^{-1}(\bar{x}) = L(\bar{x})$ is well-defined. Moreover, since $\bar{x} \notin \Sigma$ then $g(\bar{x}) \neq 0$, and thus $H(L(\bar{x})) \neq 0$. So, $P(\mathcal{M}^{-1}(\bar{x}))$ is well-defined.

(2) We prove that $Q$ is defined on a non-empty Euclidean open subset $\Theta$ of $\mathbb{R}^m$. Since $\Psi$ is continuous and $\Omega_1$ (see above) is open, then $\Theta := \Psi^{-1}(\Omega_1)$ is Euclidean open in $\mathbb{R}^m$. Furthermore, using that $\Psi(\mathbb{R}^m)$ is Zariski dense in $(HT)_m^\infty$, and that $\Omega_1$ is a non-empty Zariski subset of $(HT)_m^\infty$, by Lemma [10] we get that $\Theta \neq \emptyset$.

We prove that $Q(\Theta)$ is Zariski dense in $V$. Let $f := P \circ \mathcal{M}^{-1}$. Since $\Theta$ is Euclidean open in $\mathbb{R}^n$, there exists open intervals $A_i, B_i, C_i$ such that $\Theta^* := \prod_{i=1}^{m_1} A_i \times \prod_{i=m_1+1}^{m_1+m_2} B_i \times \prod_{i=m_1+m_2+1}^{m_1+m_2+m_3} C_i \subset \Theta$. Therefore, $\Psi(\Theta^*)$ is the product of non-empty arcs in the unit circle, in the hyperbola, and segments in $\mathbb{R}$. Thus, the Zariski closure of $\Psi(\Theta^*)$ is $(HT)_m^\infty$. So, $\Psi(\Theta)$ is Zariski dense in $(HT)_m^\infty$.

Since $f$ is continuous, we have that (see e.g. Theorem 7.2. pag. 44 in [27]) $f(\overline{\Psi(\Theta)}) \subset f(\overline{\Psi(\Theta)})$, where all closures are w.r.t. the corresponding Zariski topologies. Now, since $\Psi(\Theta)$ is dense and $f$ is a dominant map, we finally get that

$$V = f(\overline{\Psi(\Theta)}) \subset f(\overline{\Psi(\Theta)}) = Q(\Theta) \subset V.$$
Therefore, $V = \overline{Q(\Theta)}$. □

**Remark 5** Observe that in the previous proposition, the tuple $\overline{m}$ is freely chosen.

**Remark 6** We observe that if in the proof of Proposition 11 we consider a map

$$\Psi^* = (\psi_1, \ldots, \psi_m) : \mathbb{R}^m \to \mathbb{R}^m,$$

with $\psi_i \in \mathbb{R}(\{\cos(t_i), \sin(t_i)\}_{i \in J_1}, \{\sinh(t_i), \cosh(t_i)\}_{i \in J_2}, \{t_i\}_{i \in J_3})$, such that $P(\Psi^*(\overline{t}))$ is well-defined, and $\Psi^*(\text{dom}(\Psi^*))$ is Zariski dense in $\mathbb{R}^m$, then the trigonometric parametrization $Q(\overline{t})$ can be taken as $P(\Psi^*(\overline{t}))$.

We finish this section with the main theorem.

**Theorem 12** Let $V$ an irreducible variety. The following statements are equivalent

1. $V$ is unirational over $\mathbb{R}$.
2. $V$ is hybrid trigonometric.
3. $V$ is circular trigonometric.
4. $V$ is hyperbolic trigonometric.

Proof: (1) implies (2), (3), (4) follows from Proposition 11 taking $\overline{m} = (m_1, m_2, m_3)$, $\overline{m} = (m, 0, 0)$ and $\overline{m} = (0, m, 0)$, respectively. (2), (3), (4) follows from Proposition 8. □

4 **Parametrization and Implicitization Algorithms**

The proofs in the previous sections are constructive, and hence provide algorithms to deal with hybrid trigonometric varieties. In this section, we derive these algorithms that, essentially, show how to change from hybrid trigonometric parametrizations to rational parametrizations and how to implicitize.

We start outlining the procedure to transform a trigonometric parametrization in pure form (see Definition 5).
Algorithm 1 [ConvertPure] Convert a hybrid trigonometric parametrization in pure form (see Def. 5).

Require: An $m$–hybrid trigonometric parametrization $T(\vec{t})$ as in Def. 1.
Ensure: A pure $m$–hybrid trigonometric reparametrization of $T(\vec{t})$.

1: Apply to $T(\vec{t})$ the formulas of the sine, cosine (both circular and hyperbolic) of the addition of angles.
2: Compute $T(L_1(\vec{t}))$, where $L_1(\vec{t})$ is as in (1).
3: Apply to $T(\vec{t})$ the expression of $\sin(kt)$, $\cos(kt)$, $\sinh(kt)$ and $\cosh(kt)$, with $k \in \mathbb{N}$, as polynomials expressions of $\sin(t), \cos(t), \sinh(t)$ and $\cosh(t)$, via Chebyshev polynomials.
4: Compute $L_2(\vec{t})$ (see (2)) and return $T(L_2(L_1(\vec{t})))$.

The next two algorithms focus on the conversion from trigonometric to rational and vice-versa.

Algorithm 2 [FromTrigToRat] Obtains a rational parametrization from a hybrid trigonometric parametrization.

Require: A hybrid trigonometric parametrization $T(\vec{t})$ of an algebraic variety $V$.
Ensure: A rational parametrization $G(\vec{t})$ of $V$.

1: If $T(\vec{t})$ is not pure, apply Algorithm 1 end if.
2: Determine the rational parametrization $G(\vec{t}) = F(M(\vec{t}))$ (see (12) for the definition of $F$ and (7) for the definition of $M$).
3: return $G(\vec{t})$.

Example 13 We consider the $(1, 1, 0)$-hybrid trigonometric parametrization

$$T(t_1, t_2) = \left( \frac{\cos(t_1)^2 \sin(t_1)}{\sinh(t_2)}, \frac{\sin(t_1)}{2 \sinh(t_2) \cosh(t_2)}, \sin(t_1)^3 \right)$$

of $V$. We apply Algorithm 2. In the first step, we observe that $T(\vec{t})$ is not pure, since $\cosh(t_2)$ does not appear in the tuple. So, we replace $T(t_1, t_2)$ by $T(t_1, 2t_2)$. The new parametrization is

$$T(t_1, t_2) = \left( \frac{\cos(t_1)^2 \sin(t_1)}{2 \sinh(t_2) \cosh(t_2)}, \frac{\sin(t_1)}{2 \sinh(t_2) \cosh(t_2)}, \sin(t_1)^3 \right).$$

So, we have that

$$F(y_{11}, y_{12}, y_{21}, y_{22}) = \left( \frac{y_{11}^2 y_{12}}{2 y_{21} y_{22}}, \frac{y_{12}^3}{y_{11}^2} \right).$$

Finally, we get the rational parametrization of $V$ is

$$G(t_1, t_2) = \left( \frac{t_1^2 (t_1^2 - 1)}{(t_1^2 + 1)^3}, 2 \frac{(t_1^2 - 1) t_2^2}{(t_1^2 + 1) (t_1^2 - 1)^3}, \frac{(t_1^2 - 1)^3}{(t_1^2 + 1)^3} \right).$$

In Example 15, we see that $V$ is the surface $x_1^3 + 3x_2^2x_3 + 3x_1x_3^2 + x_3^3 - x_3 = 0$. 

13
Algorithm 3 \textbf{[FromRatToTrig]} Obtains a hybrid trigonometric parametrization from a rational parametrization.

\textbf{Require:} A rational parametrization $P(t)$ of an $m$–dimensional unirational variety $V$ as well as a non-negative integer triple $\overline{m} = (m_1, m_2, m_3)$ such that $m_1 + m_2 + m_3 = m$.

\textbf{Ensure:} A $\overline{m}$–hybrid trigonometric parametrization $Q(\overline{t})$ of $V$.

1: Compute $Q(\overline{t}) = P(M^{-1}(\Psi(\overline{t})))$ (see Diagram (16)): for $M^{-1}$ and $\Psi$ see (8) and (11), respectively.
2: \textbf{return} $Q(\overline{t})$.

\textbf{Example 14} We apply Algorithm 3 to the unit circle parametrized by

$$P(t) = \left( \frac{2t}{t^2 + 1}, \frac{t^2 - 1}{t^2 + 1} \right)$$

taking $\overline{m} = (1, 0, 0)$. In this case,

$$M^{-1} = \frac{x_1}{1 - x_2}, \quad \text{and} \quad \Psi(t) = (\cos(t), \sin(t))$$

and the algorithm returns the expected parametrization $Q(t) = (\cos(t), \sin(t))$. Alternatively, we may consider the same parametrization $P(t)$ and $\overline{m} = (0, 1, 0)$. In this case, we

$$M^{-1} = \frac{1}{x_1 - x_2}, \quad \text{and} \quad \Psi(t) = (\cosh(t), \sinh(t))$$

and the algorithm returns the hyperbolic trigonometric parametrization

$$Q(t) = \left( \frac{1}{\cosh(t)}, \frac{\sinh(t)}{\cosh(t)} \right).$$

Now we deal with the problem of implicitizing a hybrid trigonometric parametrization. Since we already have an algorithm to compute a rational parametrization of the variety, namely Algorithm 2, one can simply apply the existing implicitization techniques to that rational parametrization. Alternatively, one may use the implicit equations of the circles and hyperbolas involved in the input parametrization. More precisely, one has the following algorithm.
Algorithm 4 [FromTrigParamToImpl] Obtains the implicit equations from a hybrid trigonometric parametrization.

Require: A hybrid trigonometric parametrization $\mathcal{T}(\bar{t})$ of the hybrid trigonometric variety $\mathcal{V}$.

Ensure: A set of polynomials defining $\mathcal{V}$.

Option 1
1: Apply Algorithm 2 to get a rational parametrization $\mathcal{G}(\bar{t})$ of $\mathcal{V}$.
2: Implicitize $\mathcal{G}(\bar{t})$ and return the output.

Option 2
3: If $\mathcal{T}(\bar{t})$ is not pure, apply Algorithm 1 end if.
4: Determine $\mathcal{F}$ (see (12)); say $\mathcal{F} = (f_1(\bar{y}), \ldots, f_n(\bar{y}), g_n(\bar{y}))$.
5: Eliminate $\{W, \bar{y}\}$ from $\{g_i(\bar{y}) x_i - f_i(\bar{y})\}_{i=1,\ldots,n} \cup \{W \lcm(g_1, \ldots, g_n) - 1\} \cup \mathcal{H}(\bar{y})$, where $\mathcal{H}$ is the set of generators of $(HT)_m$ (see [6]).
6: return the result of the previous step.

Example 15 Let $\mathcal{T}(\bar{t})$ be the parametrization in Example 13 and $\mathcal{G}$ be the rational parametrization generated by Algorithm 2 (see Example 13). Implicitizing $\mathcal{G}$, that is using Option 1 in Algorithm 4, one gets that $x_1^3 + 3x_1^2x_3 + 3x_1x_2^2 + x_3^3 - x_3$ is the implicit equation of $\mathcal{V}$. Alternatively, one may use Option 2 in Algorithm 4. Proceeding as in Example 13, we get

$$\mathcal{F}(y_{11}, y_{12}, y_{21}, y_{22}) = \left(\frac{y_{12}^2}{y_{11}y_{12}}, \frac{y_{12}}{2y_{21}y_{22}}, y_{12}^3\right).$$

Moreover, the implicit equations of $\mathcal{H}$ are $\mathcal{H} = \{y_{11}^2 + y_{12}^2 - 1, y_{21}^2 - y_{22}^2 - 1\}$. Let $\mathcal{J}$ be the ideal generated by $\{x_1 - y_{11}y_{12}, x_2y_{12} - 2y_{21}y_{22}, x_3 - y_{12}^3, y_{11}^2 + y_{12}^2 - 1, y_{21}^2 - y_{22}^2 - 1, W y_{21}y_{22} - 1\}$. Using a suitable Gröbner basis we get that $\mathcal{J} \cap \mathbb{C}[\bar{t}]$ is generated by $\{x_1^3 + 3x_1^2x_3 + 3x_1x_2^2 + x_3^3 - x_3\}$.

Example 16 We consider the trigonometric variety $\mathcal{V}$ in $\mathbb{R}^4$ with associated $\bar{m} = (2, 0, 0)$-parametrization

$$\mathcal{T}(t_1, t_2) = \left(\frac{1}{\cos(t_1)}, \frac{\cos(t_2)}{\sin(t_1)}, \frac{1}{\sin(t_1)}, \frac{\cos(t_2)}{\sin(t_2)}\right).$$

Applying Algorithm 4 we obtain the rational parametrization

$$\mathcal{G}(\bar{t}) = \left(\frac{t_1^2 + 1}{2t_1}, \frac{2t_2 (t_1^2 + 1)}{(t_2^2 + 1)(t_1^2 - 1)}, \frac{t_1^2 + 1}{t_1^2 - 1}, \frac{2t_2}{t_2^2 - 1}\right).$$

Applying Algorithm 4, we get that $\mathcal{V}$ is the surface of $\mathbb{R}^4$ defined by

$$\{x_1^2x_3 - x_1^2 - x_3^2 = 0, x_2^2x_4 - x_3^2x_4 + x_2^2 = 0\}.$$
5 Motivating Examples of Applicability

In this section, by means of some examples we illustrate some potential applications that motivate the use of the theory developed in the previous sections.

5.1 Plotting

It is well known that plotting geometric objects using a parametric representation is more suitable. In the next example we show that using an \((m,0,0)\)-trigonometric parametrization can have advantages over a unirational parametrization; the key idea is that in the first case the behavior of the parametrization is controlled when the parameters take values in a bounded set.

Example 17 We consider the \((1,0,0)\)-trigonometric parametrization

\[
T(t) = \left(\frac{(1 + \cos(5t))\sin(t)}{1 - \cos(t)}, (1 + \cos(5t))(\cos(t))\right).
\]

Applying Algorithm 4 we obtain the rational parametrization

\[
G(t) = \left(\frac{(t + 1)^3(t^4 + 4t^3 - 14t^2 + 4t + 1)^2}{(t - 1)(t^2 + 1)^3}, 2 \frac{t(t + 1)^2(t^4 + 4t^3 - 14t^2 + 4t + 1)^2}{(t^2 + 1)^6}\right).
\]

Applying Algorithm 4 we get that \(V\) is the curve of \(\mathbb{R}^2\) defined by

\[
x_1^{12}x_2 - 2x_1^{12} + 42x_1^{10}x_2^2 - 344x_1^8x_2^4 + 32x_1^6x_2^6 + 23x_1^{10}y - 623x_1^8x_2^3 + 3304x_1^6x_2^5 + 11824x_1^4x_2^7
\]
\[
+ 3712x_1^2x_2^9 + 256x_2^{11} + 2x_1^{10} - 80x_1^8x_2^2 + 1120x_1^6x_2^4 - 6400x_1^4x_2^6 + 12800x_1^2x_2^8 = 0.
\]

In Fig. 7 we plot the real curve \(V\). Both plots have been generated using Maple. The plot on the left was obtained using the trigonometric parametrization \(T(t)\) with \(t \in [-2\pi, 2\pi]\). The plot on the right used the rational parametrization \(G(t)\) with \(t \in [-10, 10]\). One may observe that Maple, using the rational parametrization introduces wrongly the line \(y = 2\) which corresponds to the asymptote of the curve when \(t\) tends to \(-1\).

5.2 Epicycloid and hypocycloid surfaces

In this subsection, we show how the classical epicycloid and hypocycloid constructions for circles can be generalized to the case of spheres, generating naturally examples of trigonometric parametrizations. An epicycloid is a plane curve drawn by a fixed point in a circle rolling without slipping around a second fixed circle. This is a very classical curve which has been widely studied (e.g. see [15]).
A natural generalization of such construction to a higher number of variables describes
the trail of a fixed point in a sphere within the affine space of dimension 3, rolling around
a second fixed sphere (see Fig. 2). This phenomenon can be generally described in
terms of a parametrization of a surface. Assume the fixed sphere is centered at the
origin, with radius $R > 0$, and the moving one has radius $0 < r \leq R$. In the case of
the rolling sphere being of larger radius, an analogous construction can be made.

After the application of an affinity on the space, one can describe the generalized
epicycloid in terms of the following parametrization:

$$\mathcal{T} : [0, \pi] \times [0, 2\pi) \rightarrow \mathbb{R}^3$$

$$(t_1, t_2) \mapsto \left( (R + r) \sin(t_1) \cos(t_2) - r \sin \left( (1 + \frac{R}{r})t_1 \right) \cos(t_2), \\
(R + r) \sin(t_1) \sin(t_2) - r \sin \left( (1 + \frac{R}{r})t_1 \right) \sin(t_2), \\
(R + r) \cos(t_1) - r \cos \left( (1 + \frac{R}{r})t_2 \right) \right).$$ (17)

In the sequel, let us assume that $R/r$ is a rational number. In this situation, $\mathcal{T}$ is a
$(2, 0, 0)$–trigonometric parametrization (see Definition[1]) and the epicycloid the surface
that it generates. Moreover, one can rewrite it in pure form (see Lemma 3) as well as Algorithm 1.

Let us illustrate the construction with a particular example.

**Example 18** Consider the case of $R = 5$ and $r = 1$ (see Fig. 3). Then, $\mathcal{T}$ can be written in the form

$$
\mathcal{T}(t) = (\mathcal{T}_1(t_1, t_2), \mathcal{T}_2(t_1, t_2), \mathcal{T}_3(t_1, t_2)),
$$

where

$$
\mathcal{T}_1(t) = 6 \sin(t_1) \cos(t_2) - 32 \cos(t_2) \sin(t_1) \cos(t_1)^5 + 32 \cos(t_2) \sin(t_1) \cos(t_1)^3 - 6 \cos(t_2) \sin(t_1) \cos(t_1),
$$
$$
\mathcal{T}_2(t) = 6 \sin(t_1) \sin(t_2) - 32 \sin(t_2) \sin(t_1) \cos(t_1)^5 + 32 \sin(t_2) \sin(t_1) \cos(t_1)^3 - 6 \sin(t_2) \sin(t_1) \cos(t_1),
$$
$$
\mathcal{T}_3(t) = 6 \cos(t_1) - 32 \cos(t_1)^6 + 48 \cos(t_1)^4 - 18 \cos(t_1)^2 + 1.
$$

(18)

**Applying Algorithm 4** we get that the implicit equation of the epicycloid is given by the following polynomial

$$
-52521875 - 42x_1^{10} + 90x_1^4x_2^4 - 252x_2^2x_1^3x_3 - 252x_2^2x_1^2x_3^2 - 252x_2^4x_1^2x_3^2 - 840x_2^6x_1^2x_3^2 + 60x_2^8x_1^4 + 30x_2^6x_2^2x_1^2 - 2436x_2^6x_1x_3^3 + 60x_2^8x_3^2x_1^2 + x_2^{12} - 812x_3^6 - 4872x_2^6x_3^2x_1^2 - 2436x_3^4x_2^4 - 210x_3^8x_2^4 - 1286250x_1^2 - 64050x_2^2x_3^2 - 1286250x_3^2 - 42x_1^{10} + x_1^{12} + 60x_3^4x_1^2x_3 + 20x_3^6x_2^6 - 64050x_2^3 + 20x_2^6x_1^6 - 84x_2^6x_3^2 + 466560x_3^4x_1^2 - 1260x_1^2x_1^2x_3^2 - 1260x_2^4x_1^2x_3^2 - 1260x_2^4x_1x_3^3 - 210x_2^8x_1^4 + 6x_2^8x_3^2 + 15x_3^4x_1^2 + x_2^{12} + 933120x_3x_1^2x_2^2 - 84x_2^6x_1^2 - 21x_1^8 - 933120x_2x_3^2 x_1^2 + 15x_2^8x_1^4 + 466560x_3x_2^2 x_1^2 - 210x_2^2x_1^8 - 2436x_1^4x_1^2 + 60x_3^4x_1^2x_3 + 84x_2^6x_1^4 - 420x_1^6x_3 - 420x_1^6x_3^2 - 1286250x_2^3 - 126x_1^4x_2^2 - 933120x_1^2x_3^3 + 933120x_3x_1^2x_3^2 - 840x_3^2x_2^6x_3^2 + 60x_3^4x_2^4 + 30x_3^6x_2^2x_3^2 - 30x_3^6x_2^2x_2^2 + 210x_3^8x_2^4 - 420x_2^8x_1^2 - 84x_2^6x_3^2 + 15x_3^4x_2^2 + 20x_3^6x_1^6 - 15x_3^4x_2^2 - 2436x_2^2x_1^4 + 6x_1^10x_2^2 - 420x_1^4x_3^2 - 84x_1^6x_3^2 - 3205x_4^2 - 126x_1^4x_3^2 + 15x_3^4x_2^4 - 2436x_2^2x_3^2 - 21x_1^8 - 812x_2^6 - 420x_2^4x_1^6 + 15x_3^4x_2^2 - 21x_1^8 - 840x_3^2x_2^2x_3^2 - 210x_2^2x_3^2 - 3205x_1^2 - 2436x_2^2x_3^2 + 6x_2^6x_1^10 - 812x_2^6 - 42x_1^{10} + 6x_2^6x_1^6 + 6x_2^6x_1^6 - 126x_2^4x_1^4 - 64050x_3^2x_1^2 - 84x_3^2x_1^2.
$$
Applying Algorithm 2 we get the following rational parametrization of the epicycloid

\[ G(t) = \left( 4 \left( \frac{t_1^2 - 1}{t_1^2 + 1} \right)^{\frac{1}{5}}, \frac{2(t_2^2 - 1)}{(t_2^2 + 1)(t_1^2 + 1)^6}, \frac{g_1(t)}{(t_1^2 + 1)^6} \right) , \]

where \( g_1(t) = 3t_1^{10} - 6t_1^9 + 15t_1^8 + 104t_1^7 + 30t_1^6 - 292t_1^5 + 30t_1^4 + 104t_1^3 + 15t_1^2 - 6t_1 + 3 \)
and \( g_2(t) = t_1^{12} + 12t_1^{11} - 66t_1^{10} + 60t_1^9 + 495t_1^8 + 120t_1^7 - 924t_1^6 + 120t_1^5 + 495t_1^4 + 60t_1^3 - 66t_1^2 + 12t_1 + 1 \).

We can adapt the previous reasoning to the case of hypocycloids. The construction of a classical hypocycloid is analogous to that of a cycloid. Here, the moving disc is rolling inside the fixed one (see Fig 4). We consider the generalization in which a sphere of radius \( r > 0 \) is rolling inside a fixed one of radius \( r < R \).

We assume again that \( R/r \) is a natural number. After the application of an affinity on the space, one can describe the generalized hypocycloid in terms of the following \((2,0,0)\)-trigonometric parametrization:

\[ \mathcal{T} : \ [0, \pi] \times [0,2\pi] \rightarrow \mathbb{R}^3 \]
\[ (t_1, t_2) \mapsto ((R - r) \sin(t_1) \cos(t_2) - r \sin \left( \frac{R}{r} t_1 \right) \cos(t_2) , (R - r) \sin(t_1) \sin(t_2) - r \sin \left( \frac{R}{r} t_1 \right) \sin(t_2) , (R - r) \cos(t_1) - r \cos \left( \frac{R}{r} t_2 \right)) . \]

**Example 19** Consider the case of \( R = 7 \) and \( r = 1 \) (see Fig 4). Then, \( \mathcal{T} \) can be written in the form

\[ \mathcal{T}(t_1,t_2) = (\mathcal{T}_1(t_1,t_2), \mathcal{T}_2(t_1,t_2), \mathcal{T}_3(t_1,t_2)) , \]
where

\[ \mathcal{T}_1(\bar{t}) = 5 \sin(t_1) \cos(t_2) + 64 \cos(t_2) \sin(t_1) \cos(t_1)^6 - 80 \cos(t_2) \sin(t_1) \cos(t_1)^4 \\
+ 24 \cos(t_2) \sin(t_1) \cos(t_1)^2, \]

\[ \mathcal{T}_2(\bar{t}) = 5 \sin(t_1) \sin(t_2) + 64 \sin(t_2) \sin(t_1) \cos(t_1)^6 - 80 \sin(t_2) \sin(t_1) \cos(t_1)^4 \\
+ 24 \sin(t_2) \sin(t_1) \cos(t_1)^2, \]

\[ \mathcal{T}_3(\bar{t}) = 13 \cos(t_1) - 64 \cos(t_1)^7 + 112 \cos(t_1)^5 - 56 \cos(t_1)^3. \]

Applying Algorithm 4, we get the following polynomial that defines the hypocycloid.

\[-2251875390625 + 4245x_3^2x_1^3 + 35x_2^6x_1^8 + x_2^{14} + 584965x_2^5 + x_2^{14} + 174x_2^2x_1^2 + 105x_2^2x_1^3 + 140x_2^2x_1^6 + 16980x_2^3x_1^4 + 105x_2^2x_1^2 + 5803x_2^4x_1^2 + 210x_2^4x_1^2x_1 + 16980x_2^3x_1^2 + 25470x_2^4x_1^2 + 42x_2^2x_1^4 + 25470x_2^3x_1^2x_1^7 + 7x_2^2x_1^2 + 8490x_2^3x_1^5 + 8490x_2^3x_1^7 - 46728132x_2^2x_1^7x_2^3 + 78683196x_2^2x_1^6 + 4490850x_2^2x_1^2 - 34 - 46728132x_2^2x_1^2x_2^3 + 42x_2^2x_1^6 + 25470x_2^4x_1^2 + 140x_2^6x_1^3 + 16980x_2^3x_1^4 + 105x_2^2x_1^3 + 210x_2^4x_1^2x_1^7 + 20x_2^4x_1^2x_1^4 + 105x_2^4x_1^4 + 140x_2^6x_1^4 + 1740x_2^3x_1^4x_1^4 + 2610x_2^3x_1^2x_1^7 + 1740x_2^3x_1^4x_1^2 + 1740x_2^3x_1^4x_1^2 + 1740x_2^3x_1^4x_1^2 - 15576044x_2^6x_1^2 + 2339860x_2^2x_1^3 + 2339860x_2^2x_1^3 + 245x_2^8x_1^2 + 8490x_2^6x_1^4 + 748475x_2^6 + 29x_1^2 - 15576044x_2^6x_1^3 + 39341598x_2^4x_1^4 + 4245x_2^8x_1^2 + 3509790x_2^4x_1^4 + 849x_2^10 + 352x_2^8x_1^2 + 2245425x_2^6x_1^4 + 435x_2^6x_1^4 + 174x_2^3x_1^4 + 7x_2^4x_1^4 + 8490x_2^6x_1^4 + 4890x_2^4x_1^4 + 849x_2^4x_1^4 - 15576044x_2^6x_1^4 + 45018750x_2^2x_1^3 + 45018750x_2^2x_1^3 + 4245x_2^8x_1^2 + 7x_2^3x_1^4 + 35x_2^4x_1^4 + 21x_2^4x_1^3 + 35x_2^4x_1^3 + 7x_2^3x_1^4 + 7x_2^3x_1^4 + 35x_2^4x_1^3 + 435x_2^8x_1^4 + 174x_2^3x_1^4 + 174x_2^3x_1^4 + 435x_2^4x_1^4 + 580x_2^6x_1^4 + 2245425x_2^4x_1^2 + 435x_2^4x_1^4 + 174x_2^3x_1^4 + 174x_2^3x_1^4 + 435x_2^4x_1^4 + 2245425x_2^4x_1^2 + 174x_2^3x_1^4 + 435x_2^4x_1^4 + 35x_2^3x_1^4 + 45018750x_2^3x_1^3 + 35x_2^3x_1^4 + 21x_2^2x_1^4 - 15576044x_2^6x_1^2 + 21x_1^10 + 21x_1^4x_1^10 + 245x_2^8x_1^2 + 174x_2^3x_1^2 + 2245425x_2^3x_1^2 + 245x_2^8x_1^2 + 174x_2^3x_1^2 + 2245425x_2^3x_1^2 + 748475x_2^6 + 748475x_2^6 + 7x_2^3x_1^2 + 39341598x_2^4x_1^3 + 4245x_2^8x_1^2 + 849x_2^10 + 22509375x_2^4 + 22509375x_2^4 + 21x_2^10 + 79045421875x_2^3 + 584965x_2^3 + 2245425x_2^3x_1^2 + 21x_1^3x_1^10 + 435x_2^8x_1^3 + 580x_2^6x_1^3 + 29x_2^10 + 79045421875x_2^3 + 79045421875x_2^3 + 29x_2^{12}.\]
5.3 Computing Intersections

Let us say that we want to compute the intersection of two algebraic surfaces. Usually one takes, if possible, a parametrization of one of the surfaces, and substitute it in the implicit equation of the other. This provides an equation that encodes the parameter values to be substituted in the parametrization to achieve the intersection set. In the following example we see that if we are given a trigonometric parametrization (for instance when dealing with an epicycloid) the task is more difficult than using a rational parametrization.

Example 20 In this example, we consider the epicycloid of Example 18, let us call it \( V_1 \), and the sphere \( V_2 \) of equation \( x_1^2 + x_2^2 + x_3^2 = 36 \). We want to compute \( V_1 \cap V_2 \).

The construction of the generalized epicycloid with such sphere suggests a nonempty intersection, as it can be observed in Figure 6.

Although the parametrization of \( V_2 \) is simple, the implicit equation of \( V_1 \) is huge. So, we try to use a parametrization of \( V_1 \) and the implicit equation of the sphere. If we use the trigonometric parametrization \( T(t) \) (see (18)), we get the equation

\[
-192 \cos(t_1)^5 + 240 \cos(t_1)^3 - 60 \cos(t_1) + 1 = 0.
\]

Maple provides only the solution \( t_1 = 0.297473248 \) that generates the circle parametrized as

\[(0.7814521186 \cos(t_2), 0.7814521186 \sin(t_2), 5.94889339)\].

Taking into account Fig. 6, this solution looks incomplete. However, if we use the rational parametrization \( G(t) \) given in (19) we get the equation

\[
t_1^{10} - 120t_1^9 + 5t_1^8 + 1440t_1^7 + 10t_1^6 - 3024t_1^5 + 10t_1^4 + 1440t_1^3 + 5t_1^2 - 120t_1 + 1 = 0,
\]

which roots are all real and can be approximated as

\[
\{-2.992499717, -1.400811244, -0.7138720538, -0.3341687869, 0.008343202240, 0.3157206162, 0.739367548, 1.352507292, 3.167357305, 119.8580558\}.
\]
Substituting these two roots in $G(t)$ we get the following five circles of intersection

$$
\left(\frac{t_2}{t_2^2+1}, 0.781452121\frac{t_2^2}{t_2^2+1}, 5.948893399\right),
$$

$$
\left(\frac{t_2}{t_2^2+1}, 2.864463754\frac{t_2^2}{t_2^2+1}, -5.272081897\right),
$$

$$
\left(\frac{t_2}{t_2^2+1}, 4.128879847\frac{t_2^2}{t_2^2+1}, -4.353429820\right),
$$

$$
\left(\frac{t_2}{t_2^2+1}, 5.416251839\frac{t_2^2}{t_2^2+1}, 2.581514281\right),
$$

$$
\left(\frac{t_2}{t_2^2+1}, 5.899215810\frac{t_2^2}{t_2^2+1}, 1.095104026\right).
$$

In order to check that this last result is correct, we compute a Gröbner basis of the ideal of $V_1 \cap V_2$ w.r.t. a lexicographic order to get

$$
\{1492992x_3^5 - 67184640x_3^3 + 604661760x_3 - 576284939, x_1^2 + x_2^2 + x_3^2 - 36\}.
$$

The roots of the univariate polynomial in the basis are

$$
\{-5.272081883, -4.353429821, 1.095104026, 2.581514286, 5.948893392\}
$$

that are the level planes where the circles lie on. On the other hand substituting these 5 roots in the second polynomial we get the circles. In Fig. 7 we plot the five intersection circles.
5.4 Solving Differential Equations

In the work [16] the extended generalized Riccati Equation Mapping Method is applied to the (1+1)-Dimensional Modified KdV Equation

\[ u_t - u^2 u_x + \delta u_{xxx} = 0, \]  

(21)

for some fixed \( \delta > 0 \). More precisely, solutions of the generalized Riccati equation, namely

\[ G' = r + pG + qG^2, \]  

(22)

are used in order to provide solutions of (21). The authors arrive at different families of solutions, classified into soliton and soliton-like solutions (written in terms of rational functions of hyperbolic ones), and periodic solutions (written as rational functions of trigonometric ones), under different cases of the parameters involved.

In the following, we see how taking the particular solutions provided in [16] of (22) and using the ideas of this paper, we can generate all families of solutions in [16]. More precisely, using the notation in [16], we take the solution of (22)

\[ G_1(\eta) = -\frac{1}{2q} \left( p + \delta \tanh \left( \frac{\delta}{2} \eta \right) \right) \]

where \( \delta = \sqrt{p^2 - 4qr} \). Therefore,

\[ \mathcal{P}(\eta) = (G_1(\eta), G_1'(\eta)) \]

is a parametrization of the algebraic variety \( \mathcal{V} \) associated with (22), namely the conic \( y = r + px + qx^2 \) (see [5]). Since we do not whether \( \delta \) is a rational number, \( \mathcal{P}(\eta) \) may not satisfy the conditions in Definition 1. However the reparametrization

\[ \mathcal{T}(\eta) = \mathcal{P} \left( \frac{2}{\delta} \eta \right) \]

does, and it is a (0, 1, 0)–trigonometric parametrization of \( \mathcal{V} \). Algorithm 2 provides a rational parametrization of the variety, given by

\[ \left( -\frac{1}{2q} \left( p + \frac{\delta (\eta^2 - 1)}{\eta^2 + 1} \right), -\frac{\delta^2 \eta^2}{q (\eta^2 + 1)^2} \right), \]

that can be properly reparametrized as

\[ \mathcal{G}(\eta) := (g_1(\eta), g_2(\eta)) = \left( -\frac{\delta \eta + \eta p - \delta + p}{2q (\eta + 1)}, -\frac{\delta^2 \eta}{q (\eta + 1)^2} \right). \]

The previous parametrization is no longer a solution of (22), so we search for a function \( t \mapsto \phi(t) \) such that \( \mathcal{G}(\phi(t)) \) provides a solution of (22). We ask the derivative of \( g_1(\phi(t)) \)
with respect to \( t \) to coincide with \( g_2(\phi(t)) \) to obtain a differential condition on \( \phi \). Namely,

\[-\delta\phi(t) + \phi(t)' = 0,
\]

with general solution

\[ \phi(t) = Ce^{\delta t}. \]

So, we get the general solution

\[ S(\eta, C) = -\frac{\delta C e^{\delta \eta} + C e^{\delta \eta}p - \delta + p}{2q(C e^{\delta \eta} + 1)} \]

of the generalized Ricatti equation. Now, from \( S(\eta, C) \) one may obtain the families of solutions in [16]. For instance, using the notation in [16], \( S(\eta, 1) = G_1(\eta), S(\eta, -1) = G_2(\eta), S(\eta, \pm i) = G_3(\eta), S(\eta, \mp 1) = G_4(\eta), \) etc.

We observe that if one proceeds analogously replacing the rational parametrization by the hyperbolic parametrization \( T(\eta) = (h_1(\eta), h_2(\eta)) \), the procedure does not succeed. More precisely, we consider an unknown function \( \psi(t) \) and search for all such functions which satisfy \( h_2(\psi(t)) = \frac{d}{dt}(h_1(\psi(t))) \). We only get \( \psi(t) = t + C \), for \( C \) being an arbitrary constant.

### 6 Conclusions

We have introduced a new type of parametrizations, namely those involving rationally circular and hyperbolic trigonometric functions and monomials being each of these three block depending of different sets of parameters. We have seen that the algebraic varieties defined by these new objects are precisely the real unirational varieties. In addition, we provide algorithms to deal with the computation of the generators of the variety, and to convert from trigonometric to unirational and vice versa. We have also illustrated by means of examples that having the option of parametrizing in these two different ways is an advantage for dealing with some applications; for some a rational parametrization is better, for others a trigonometric parametrization is more suitable.

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