Research Article

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Reflected BSDEs with two completely separated barriers and regulated trajectories in general filtration.

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Abstract: In this paper, we study doubly reflected Backward Stochastic Differential Equations defined on probability spaces equipped with filtration satisfying only the usual assumptions of right continuity and completeness in the case where the barriers $L$ and $U$ are not necessarily right continuous. We suppose that the barriers $L$ and $U$ and their left limits are completely separated and we show existence and uniqueness of the solution.

Keywords: Reflected RBSDE, Doubly RBSDs, General filtration, Regulated trajectories, Local-global solution.

Classification: 60K35, 82B43, 60H05, 60H15

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1 Introduction.

In this paper, we study the problem of existence and uniqueness of the solution of backward stochastic differential equations (BSDE) with two reflecting optional barriers (or obstacles) $L$ and $U$. Our main in this work is to deal with equations on probability space with general filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying only usual conditions of right continuity and completeness. Also, we assume that the lower barrier $L$ and the upper barrier $U$ are completely separated in the sense that $(L_t < U_t)$ and $(L_t^- < U_t^-)$ for all $t \in [0,T]$ and which are two regulated process, i.e. processes whose trajectories have left and right finite limit. Consequently, the solution of these equations need not be càdlàg but are called regulated processes.

Precisely, a solution for the BSDE with two reflecting barriers associated with a generator $f(t,y)$, a terminal value $\xi$, a lower barrier $L$ and an upper barrier $U$ (RBSDE($\xi, f, L, U$) for short), is a quadruple of processes $(Y, M, K, A)$ which mainly satisfies:

\[
\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_s)ds + (K_T - K_t) - (A_T - A_t) - \int_t^T dM_s, t \in [0,T], \\
L_t \leq Y_t \leq U_t, t \in [0,T], \text{and} \\
\int_0^T (U_s - Y_s^-)dA_s^+ + \sum_{s<T} (U_s - Y_s^-)\Delta^+ A_s = 0, \text{and} \\
\int_0^T (Y_s^- - L_s^-)dK_s^+ + \sum_{s<T} (Y_s^- - L_s^-)\Delta^+ K_s = 0,
\end{cases}
\]

where $Y$ has regulated trajectories, $K, A$ are increasing processes such that $K_0 = A_0 = 0$, $M$ is a local martingale with $M_0 = 0$, $K^+$ (resp. $A^+$) the càdlàg part of $K$ (resp. $A$) and $\Delta^+ K$ (resp. $\Delta^+ A$) the right jump of $K$ (resp. of $A$). The reason we chose the minimality conditions

\[
\int_0^T (Y_s^- - L_s^-)dK_s^+ + \sum_{s<T} (Y_s^- - L_s^-)\Delta^+ K_s = 0 \quad \text{and} \quad \int_0^T (U_s^-- Y_s^-)dA_s^+ + \sum_{s<T} (U_s^-- Y_s^-)\Delta^+ A_s = 0,
\]

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is to use the penalization method for regulated BSDE with regulated trajectories proposed by Klimsiak et al. in [21]. Note that if \( L \) and \( U \) are càdlàg, then this condition reduces to the classical condition: [21] (1.3).

Generally speaking, in BSDE theory, during several years, there have been a lot of works which study the problem of existence and uniqueness of BSDE with two reflecting barriers under these three conditions:

(a) one of the obstacles is a semimartingale.

(b) the Mokobodski condition: between \( U \) and \( L \) one can find a process \( X \) such that \( X \) is a difference of nonnegative càdlàg supermartingales.

(c) the barriers are completely separated: \( L_t < U_t \) and \( L_{t-} < U_{t-} \) for all \( t \in [0, T] \) a.s.

Under the assumption \( b \), the problem is studied in [20], [19], [5], [4] ... in the case of continuous or right-continuous obstacles and/or a larger filtration than the Brownian, but the issue with this condition is that it is quite difficult to check in practice. Then, it has been removed by Hamadène and Hassani in [12], when they showed that if the assumption \( c \) hold, the two barriers reflected BSDE has a unique solution. Under the same assumption there are also a lot of works which dealt with the problem of existence and uniqueness, for instance, the papers [14], [17], [18], [23] ... In all of the above-mentioned works (and others) on double reflected BSDEs, the barriers are assumed to be at least right-continuous.

The only paper dealing with BSDEs with two reflecting barriers that are not càdlàg, in our knowledge, is the paper by Grigorova et al. in [10]. The authors proved the existence and uniqueness of the solution of double reflected BSDE with two irregular barriers satisfying the generalized Mokobodzki’s condition. First they showed the existence and uniqueness in the case where the driver does not depend on solution, then they proved a priori estimates for the doubly reflected BSDE by using Gal’chuk-Lenglart’s formula and from these they derived the existence and uniqueness of the solution with general Lipschitz driver by using the Banach fixed point theorem.

BSDE with two reflecting barriers have been introduced by Cvitanic and Karatzas in [3] in the case of continuous barriers and a Brownian filtration. The solutions of such equations are constrained to stay between two adapted barriers \( L \) and \( U \) with \( L \leq U \) and \( L_T = U_T \). In the case of the continuous/càdlàg barriers, reflected doubly BSDE have been studied by several authors in [14], [17], [18], [23], [13], [15], [20], [11], [5], [3], [4], [6], [16], [10] (for regulated barriers case).

This paper is organized as follows:

In the second and third section, we give some preliminary and some result related to BSDE with one barrier (definition, existence). In section four, we recall the doubly reflected BSDE definition and we prove a comparison and uniqueness result. In the fifth section, we deal with the notion of local solution of doubly reflected BSDE, which is a solution of that equation but between two comparable stopping times. Some local solution properties are also given. Section six is reserved to our main result of this paper.

2 Preliminaries.

Let us consider a filtered probability space \((\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F} = \{\mathcal{F}_t, t \geq 0\})\). The filtration is assumed to be complete, right continuous and quasi-left continuous.

Let \( T > 0 \) be a fixed positive real number. We recall that a function \( y : [0, T] \to \mathbb{R}^d \) is called regulated if for every \( t \in [0, T] \) the limit \( y_{t+} = \lim_{u \downarrow t} y_u \) exists, and for every \( t \in [0, T] \) the limit \( y_{t-} = \lim_{u \uparrow t} y_u \) exists. For any regulated function \( y \) on \([0, T]\), we denote by \( \Delta^+ y_t = y_{t+} - y_t \) the size of the right jump of \( y \) at \( t \), and by \( \Delta^- y_t = y_t - y_{t-} \) the size of the left jump of \( y \) at \( t \). In this paper, we consider an \( \mathcal{F}\)-adapted process \( X \) with regulated trajectories of the form \( X_t = X^*_t + \sum_{s \leq t} \Delta^+ X_s \), \( t \in [0, T] \), where \( X^* \) is an \( \mathcal{F}\)-adapted semimartingale with càdlàg trajectories and \( \sum_{s \leq t} |\Delta^+ X_s| < \infty, \mathbb{P} - \text{a.s.} \) We denote:

- \( \mathcal{T}_{0,T} \) is the set of all stopping times \( \tau \) such that \( \mathbb{P}(t \leq \tau \leq T) = 1 \). More generally, for a given stopping time \( \nu \) in \( \mathcal{T}_{0,T} \), we denote by \( \mathcal{T}_{\nu,T} \) the set of all stopping times \( \tau \) such that \( \mathbb{P}(\nu \leq \tau \leq T) = 1 \).
- \( L^2(\mathcal{F}_T) \) is the set of random variables which are \( \mathcal{F}_T \)-measurable and square-integrable.
Now to define the solution of our reflected backward stochastic differential equation, let us introduce the following spaces:

- $S^2$ is the set of all $\mathcal{F}$-progressively measurable process with regulated trajectories $\phi$ such that:
  \[
  E\left[\sup_{0 \leq t \leq T} |\phi_t|^2\right] < \infty.
  \]
- $M^2$ is the subspace of $\mathcal{M}_{\text{loc}}$ of all martingales such that: $E([M]^T) < +\infty$.

The random variable $\xi$ is $\mathcal{F}_T$-measurable with values in $\mathbb{R}^d$ ($d \geq 1$) and $f : \Omega \times [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a random function measurable with respect to $\text{Prog} \times \mathcal{B}(\mathbb{R}^d)$ where $\text{Prog}$ denotes the $\sigma$-field of progressive subsets of $\Omega \times [0, T]$. A sequence $\{\tau_k\} \subset \mathcal{T}_{0,T}$ is called stationary if $\forall \omega \in \Omega, \exists n \in \mathbb{N} : \forall k \geq n, \tau_k(\omega) = T$.

We will need the following assumptions:

1. There is $\mu \in \mathbb{R}$ such that $|f(t, y) - f(t, y')| \leq \mu |y - y'|$ for all $t \in [0, T]$, $y, y' \in \mathbb{R}$.
2. $\xi, f_0^T \, \, |f(r, 0)|dr \in L^2$
3. $[0, T] \ni t \mapsto f(t, y) \in L^1(0, T)$ for every $y \in \mathbb{R}$.

3 Reflected BSDE with one barrier.

In that follows, we assume that $\xi$ is an $\mathcal{F}$-measurable random variable, $L$ and $U$ are $\mathcal{F}$-adapted optional processes in $S^2$ and $L_t \leq U_t$, for all $t \leq T$ and $L_T \leq \xi \leq U_T$. We assume that the lower $L$ obstacle is right-upper-continuous (r.u.s.c.) and the upper $U$ obstacle is left-upper-continuous (r.l.s.c.).

**Definition 1.** We say that a triple $(Y, M, K)$ of $\mathcal{F}$-progressively measurable processes is a solution of the reflected BSDE with driver $f$, terminal value $\xi$ and lower barrier $L$ ($\text{RBSDE}(\xi, f, L)$ for short) if

1. $Y, K \in S^2$, $M \in \mathcal{M}_{\text{loc}}$ with $M_0 = 0$.
2. $Y_t \geq L_t$ for all $t \in [0, T]$ a.s., and $\int_0^T (Y_{s-} - L_{s-})dK_s + \sum_{s \leq T} (Y_s - L_s)\Delta^+ K_s = 0$
3. $\int_0^T |f(s, Y_s)|ds < \infty$ a.s.
4. $Y_t = \xi + \int_t^T f(s, Y_s)ds + K_T - K_t - \int_t^T dM_s$, for all $t \in [0, T]$, a.s.

**Remark 1.** We note that if $L$ and $K$ are càdlàg, then (2) in Definition 1 reduces to $\int_0^T (Y_{s-} - L_{s-})dK_s = 0$.

**Definition 2.** We say that a triple $(Y, M, A)$ of $\mathcal{F}$-progressively measurable processes is a solution of the reflected BSDE with driver $f$, terminal value $\xi$ and upper barrier $U$ ($\text{RBSDE}(\xi, f, U)$ for short) if

1. $Y, A \in S^2$, $M \in \mathcal{M}_{\text{loc}}$ with $M_0 = 0$.
2. $Y_t \leq U_t$ for all $t \in [0, T]$ a.s., and $\int_0^T (U_{s-} - Y_{s-})dA_s + \sum_{s \leq T} (U_s - Y_s)\Delta^+ A_s = 0$
3. $\int_0^T |f(s, Y_s)|ds < \infty$ a.s.
4. $Y_t = \xi + \int_t^T f(s, Y_s)ds - (A_T - A_t) - \int_t^T dM_s$ for all $t \in [0, T]$ a.s.

**Remark 2.** If $(Y, M, K) \in S^2 \times \mathcal{M}_{\text{loc}} \times S^2$ satisfies definition 1 then the process $Y$ has left and right limits. Moreover, the process given by $(Y_t + \int_t^T f(s, Y_s)ds)_{t \in [0, T]}$ is a strong martingale (H.1 Definition A.1).

In the theorem below we recall some results on reflecting BSDEs with one barrier. They will play important role in the proof of our main result. In the penalization method for reflected BSDEs proposed by Klimešik et al in [21], they defined arrays of stopping times $\{\{\sigma_{n,i}\}\}$ exhausting right-side jumps of $L$ inductively as follow: $\sigma_{1,0} = 0$ and

$$
\sigma_{1,i} = \inf\{t > \sigma_{1,i-1} : \Delta^+ L_s < -1\} \wedge T, \quad i = 1, \ldots, k_1
$$
for some $k_1 \in \mathbb{N}$. Next for $n \in \mathbb{N}$ and given array $\{\{\sigma_{n,i}\}\}$, $\sigma_{n+1,0} = 0$ and

\[
\sigma_{n+1,i} = \inf\{t > \sigma_{n+1,i-1} : \Delta^+ L_s < -\frac{1}{n+1}\} \wedge T
\]

for $i = 1, ..., j_n+1$ where $j_n+1$ is chosen so that $P(\sigma_{n+1,j_n+1} < T) \to 0$ as $n \to \infty$ and

\[
\sigma_{n+1,i} = \sigma_{n+1,j_n+1} \vee \sigma_{n,i-j_n+1}, \quad i = j_n+1, ..., k_n+1, \quad k_n+1 = j_n+1+k_n.
\]

**Theorem 1.** Assume that (H1) – (H4) are satisfied. Then

(i) There exists a unique solution $(Y, M, K)$ of RBSDE$(\xi, f, L)$. Moreover if $(Y^n, M^n)$, $n \in \mathbb{N}$ are solution of BSDEs of the form

\[
Y^n_t = \xi + \int_t^T f(s, Y^n_s)ds - \int_t^T dM^n_s + n \int_t^T (Y^n_s - L_s)^- ds + \sum_{t \leq \sigma_{n,i} < T} (Y^n_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-
\]

then $Y^n_t \not\geq Y_t$, $t \in [0, T]$ $P$-a.s.

(ii) There exists a unique solution $(\overline{Y}, \overline{M}, \overline{A})$ of $\overline{RBSDE}(\xi, f, U)$. Moreover if $(\overline{Y}^n, \overline{M}^n)$, $n \in \mathbb{N}$ are solution of BSDEs of the form

\[
\overline{Y}^n_t = \xi + \int_t^T f(s, \overline{Y}^n_s)ds - \int_t^T d\overline{M}^n_s - n \int_t^T (U_s - \overline{Y}^n_s)^- ds - \sum_{t \leq \sigma_{n,i} < T} (U_{\sigma_{n,i}} - \overline{Y}^n_{\sigma_{n,i}})^-
\]

then $\overline{Y}^n_t \not\geq \overline{Y}_t$, $t \in [0, T]$ $P$-a.s.

**Proof.** The first part in (i) is proved in [1] (see also [21] $(p > 1)$ and [11] $(p = 2)$ in the case of Brownian filtration and [2] for the case of a filtration that supports a Brownian motion and an independent Poisson random measure).

The second part in (i) is proved for the case of a Brownian filtration in [21, Theorem 4.1]. To show the results in a general filtration we use the Itô formula for the regulated process (see [2, Theorem 2.5] or [21, Appendix]) to get this inequality:

\[
\int_\sigma^\tau d[M^n - M^{n^1}]_s \leq |Y_\tau - Y^n_\tau|^2 + 2 \int_\sigma^\tau (Y^n_s - Y^n_s^-)(f(s, Y^n_s) - f(s, Y^n_s^-))ds + 2 \int_\sigma^\tau (Y^n_s - Y^n_s^-)d(K^n_s - K^n_s^*)
\]

\[
- 2 \int_\sigma^\tau (Y^n_s - Y^n_s^-)(M^n_s - M^n_s^-) - 2 \sum_{\sigma \leq s < \tau} (Y^n_s - Y^n_s^-)\Delta^+ (Y^n_s - Y^n_s^-)
\]

with $(Y^n, M^n)$ defined in [11], $\sigma, \tau \in T_{0,T}$, $\sigma \leq \tau$, and $K^n_t = n \int_0^t (Y^n_s - L_s)^- ds + \sum_{0 \leq \sigma_{n,i} \leq t} (Y^n_{\sigma_{n,i}} - L_{\sigma_{n,i}})^-$. By the fact that $\Delta^+ (Y^n_s - Y^n_s^-) = -\Delta^+ (K^n_s - K^n_s^*)$, we have

\[
\int_\sigma^\tau d[M^n - M^{n^1}]_s \leq |Y_\tau - Y^n_\tau|^2 + 2 \int_\sigma^\tau (Y^n_s - Y^n_s^-)(f(s, Y^n_s) - f(s, Y^n_s^-))ds + 2 \int_\sigma^\tau (Y^n_s - Y^n_s^-)d(K^n_s - K^n_s^*)
\]

\[
- 2 \int_\sigma^\tau (Y^n_s - Y^n_s^-)(M^n_s - M^n_s^-) + 2 \sum_{\sigma \leq s < \tau} (Y^n_s - Y^n_s^-)\Delta^+ (K^n_s - K^n_s^*)
\]

Then $E \int_\sigma^\tau d[M^n - M^{n^1}]_s \leq E|Y_\tau - Y^n_\tau|^2 + 2E \int_\sigma^\tau |Y^n_s - Y^n_s^-|(f(s, Y^n_s) - f(s, Y^n_s^-))ds$. And with the Theorem 4.1 assumptions in [21] we get the existence of a stationary sequence $\{\tau_k\}$ of stopping times such that: $E \int_0^{\tau_k} d[M^n - M^{n^1}]_s = E \int_0^{\tau_k} d[M - M^{n^1}]_s \to 0$.

Therefore to prove that $Y^n \not\geq Y$, $t \in [0, T]$ it suffices to repeat step by step the proof of [21, Theorem 4.1]. The assertion (ii) follows directly from (i). Indeed, let $(Y, M, A)$ be the solution for the $RBSDE(\xi, f, U)$. Set $f'(s, y) = -f(s, -y)$. Then the process $(-Y, -M, A)$ is the solution of the reflected BSDE $\overline{RBSDE}(-\xi, f', -U)$. 

\[\square\]
4 BSDEs with two reflecting barriers.

In this section, \( \xi, f, L \) and \( U \) are as in above. We also suppose that \( L_t \leq U_t \) for \( t \in [0,T] \) \( \mathbb{P} \)-a.s.

**Definition 3.** We say that a quadruplet \((Y, M, K, A)\) of \( \mathbb{F} \)-progressively measurable processes is a solution of the reflected BSDE with driver \( f \), terminal value \( \xi \), lower barrier \( L \) and upper barrier \( U \), if \( (RBSDE(\xi, f, L, U)) \) for short.

\begin{align*}
\text{(LU1)} & \quad Y, K, A \in \mathcal{S}^2, M \in \mathcal{M}_{loc} \text{ with } M_0 = 0. \\
\text{(LU2)} & \quad L_t \leq Y_t \leq U_t, \quad t \in [0,T] \quad \mathbb{P}\text{-a.s.} \\
\text{(LU3)} & \quad \int_0^T (U_s - Y_s) dA_s^+ + \sum_{s \leq T} (U_s - Y_s) \Delta^+ A_s = \int_0^T (Y_s - L_s) dK_s^+ + \sum_{s \leq T} (Y_s - L_s) \Delta^+ K_s = 0, \\
\text{a.s.} \\
\text{(LU4)} & \quad Y_t = \xi + \int_t^T f(s,Y_s) ds + (K_T - k_t) - (A_T - A_t) - \int_t^T dM_s, \quad t \in [0,T] \quad \mathbb{P}\text{-a.s.}
\end{align*}

**Remark 3.** We note that, due to equation \((LU4)\), we have \( \Delta^+ Y_t = - \Delta^+ (K_t - A_t) \).

We are now going to focus on the uniqueness of the solution of the doubly reflected BSDE associated with \((f, \xi, L, U)\). However, the first step is to provide a comparison result between the components \( Y \) of two solutions (in Definition 3). Actually, we have:

**Proposition 1.** Let \((f, \xi, L, U)\) and \((f', \xi', L', U')\) be two sets of data satisfying (H1)-(H3). Let \((Y, M, K, A)\) and \((Y', M', K', A')\) be two solutions of the doubly reflected BSDE associated with \((f, \xi, L, U)\) and \((f', \xi', L', U')\) respectively. Assume that \( \xi \leq \xi' \), \( L \leq L' \), \( U \leq U' \) and \( f \leq f' \). Then \( \mathbb{P} \)-a.s. \( Y_t \leq Y'_t \).

**Proof.** Let \((\tau_k)_{k \geq 0}\) be a non-decreasing sequence of stationary type and converges to \( T \) such that:

\[ \tau_k = \inf \{ t \geq 0, |M_t| + |M'_t| \geq k \} \land T. \]

we have \( \mathbb{P} - a.s., |M|_T + |M'|_T < \infty \). Now, by Itô-Tanaka’s formula for the regulated process (see [22, Section 3, page 539]) with \((Y - Y')^+\) on \([t \land \tau_k, \tau_k]\) we get:

\[ (Y_{t \land \tau_k} - Y'_{t \land \tau_k})^+ \leq (Y_{\tau_k} - Y'_{\tau_k})^+ + \int_{\tau_k}^{T} 1_{Y_{s} > Y'_{s}} (f(s, Y_s) - f'(s, Y'_s)) ds \]

\[ - \int_{\tau_k}^{T} 1_{Y_{s} > Y'_{s}} d(M_s - M'_s) + \int_{\tau_k}^{T} 1_{Y_{s} > Y'_{s}} d(K_s - K'_s - A_s + A'_s). \]

From definition of solution we have \( \int_{\tau_k}^{T} 1_{Y_{s} > Y'_{s}} d(K_s - K'_s - A_s + A'_s) \leq 0 \), and by using the Lipschitz condition of \( f \), we have

\[ (Y_{t \land \tau_k} - Y'_{t \land \tau_k})^+ \leq (Y_{\tau_k} - Y'_{\tau_k})^+ + \mu \int_{\tau_k}^{T} (Y_s - Y'_s)^+ ds - \int_{\tau_k}^{T} 1_{Y_{s} > Y'_{s}} d(M_s - M'_s). \]

where \( \mu \) the Lipschitz constant of \( f \). Therefore taking expectation, the limit as \( k \to \infty \), we have \( E[(Y_{\tau_k} - Y'_{\tau_k})^+] \to E[(Y_T - Y'_T)^+] = 0 \) since \( L \leq Y \leq U \) and \( L \leq Y' \leq U' \). And by using Grönwall’s Lemma we get \( E[(Y_t - Y'_t)^+] = 0 \) for any \( t \leq T \), a.s. \( Y_t \leq Y'_t \), which is the desired result.

**Proposition 2.** The RBSDE\((\xi, f, L, U)\) has at most one solution, i.e., if \((Y, M, K, A)\) and \((Y', M', K', A')\) are two solutions of \( \text{RBSDE}(\xi, f, L, U) \), then \( \mathbb{P} - a.s., Y = Y', M = M' \) and \( K = K' = A = A' \).

**Proof.** Let \((Y, M, K, A)\) and \((Y', M', K', A')\) be two solutions of \( \text{RBSDE}(f, \xi, L, U) \). Then from the comparison result (Proposition 1), we have \( Y_t = Y'_t, \ t \leq T, \mathbb{P} \)-a.s. and then \( M = M' \) and by \((LU4)\) in Definition 3, we get \( K = K' = A = A' \).

**Remark 4.** We have also \( K = K' = A = A' \) since \( L_t < U_t, \forall t < T \). (see, [7, Proposition 2.1]).
5 Local solutions of BSDEs with two optional reflecting barriers

We are going to construct a solution for the doubly reflected BSDE associated with \((f, \xi, L, U)\) step by step under \((H1)\) – \((H3)\). For this we need to construct a process \(Y\) which satisfies locally the RBSDE\((f, \xi, L, U)\), that is to say, for any stopping time \(\tau\), we can find another appropriate stopping time \(\lambda_\tau\) such that between \(\tau\) and \(\lambda_\tau\), \(Y\) satisfies the doubly reflected BSDE. This local solution will be constructed as a limit of a penalization scheme, which leads to study BSDEs with one reflecting barrier. Thus our first task is to provide the results we need later on BSDEs with one reflecting barrier. We first introduce the notion of a local solution of the RBSDE\((f, \xi, L, U)\).

**Definition 4.** Let \(\tau\) and \(\sigma\) be two stopping times such that \(\tau \leq \sigma\) \(\mathbb{P}\)-a.s.. We say that \((Y_t, M_t, K_t, A_t)_{t \leq T}\) is a local solution on \([\tau, \sigma]\) for the doubly reflected BSDE associated with two barriers \(L\) and \(U\), the terminal condition \(\xi\) and the generator \(f\) if: \(\mathbb{P}\)-a.s., \(Y, K, A \in \mathcal{S}^2, M \in \mathcal{M}_{loc}, M_0 = 0\) and

\[
\begin{align*}
Y_t &= \xi, \forall t \in [\tau, \sigma] \\
Y_t &= Y_t + \int_{T}^{\tau} f(s, Y_s)ds + (K_{\sigma} - K_t) - (A_{\sigma} - A_t) - \int_{T}^{\tau} dM_s, t \in [\tau, \sigma], \mathbb{P}\text{-a.s.}, \\
L_t &\leq Y_t \leq U_t, \forall t \in [\tau, \sigma], \\
\int_{T}^{\tau} (U_s - Y_s) dA_s^+ + \sum_{\tau \leq s < \sigma} (U_s - Y_s) \Delta^+ A_s &= \int_{T}^{\tau} (Y_s - L_s) dK_s^+ + \sum_{\tau \leq s < \sigma} (Y_s - L_s) \Delta^+ K_s = 0, \text{a.s.}
\end{align*}
\]

In this section, we are going to show the existence of an appropriate local solution which later will allow us to construct a global solution for the RBSDE\((f, \xi, L, U)\) with regulated processes in a general filtration. But we assume only that \(L\) is right upper-semicontinuous (r.u.s.c) and \(U\) is right lower-semicontinuous (r.l.s.c).

The idea of the proof is the same as in the paper of Hamadène and Hassani \cite{Ref1}, in which the authors proved the results for the double RBSDE with continuous processes and Brownian filtration.

5.1 The increasing penalization scheme

Let us introduce the following increasing penalization scheme. For any \(n \geq 0\), let \((Y^n_t, M^n_t, A^n_t)\) be the triple of \(\mathcal{F}_t\)-adapted processes with values in \(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}\), unique solution of the RBSDE\((f(s, y) + n(y - L_s)^-, \xi, U)\) such that: \(Y^n, A^n \in \mathcal{S}^2, M^n \in \mathcal{M}_{loc}\) with \(M^n_0 = 0\) and

\[
\begin{align*}
Y^n_t &= \xi + \int_{t}^{T} f(s, Y^n_s) + n(Y^n_s - L_s)^-)ds + \sum_{t \leq \sigma_n, \xi}(Y^n_{\sigma_n}^- - L_{\sigma_n})^- - A^n_t + A^n_t - \int_{t}^{T} dM^n_s, \text{ a.s.}, \\
\forall t \in [0, T], \quad Y^n_t &\leq U_t \quad \text{and} \quad \int_{0}^{T} (U_s - Y^n_s) dA^n_s^+ + \sum_{s \leq T} (U_s - Y^n_s) \Delta^+ A^n_s = 0, \text{ a.s.}
\end{align*}
\]

We set \(f_n(t, y) = f(t, y) + n(y - L_t)^- + \sum_{\sigma_n \leq t}(y_{\sigma_n}^- - L_{\sigma_n})^-\).

By Theorem 1 there exist a unique solution \((Y^n_t, M^n_t, A^n_t)\) of RBSDE\((f_n(s, y), \xi, U)\). We have \(f_n(s, y) \leq f_{n+1}(s, y)\) which implies from the comparison result that for any \(n \geq 0\), we have \(Y^n \leq Y^{n+1} \leq U\). And by consequence there exist \(Y = (Y_t)_{t \leq T}\) such that \((Y^n_t)_{t \leq T}\) converges increasingly to \((Y_t)_{t \leq T}\) and for any \(t \leq T, Y_t \leq U_t\). Besides for a stopping time \(\tau\) let us set: \(\delta_\tau^n = \inf\{s \geq \tau, Y^n_s = U_s\} \wedge T\).

Since \(Y^n \leq Y^{n+1}\) then the sequence \((\delta^n_\tau)_{n \geq 0}\) is decreasing and converges to \(\delta_\tau\). Let us now focus on some properties of \(Y\) and especially show that \(Y \geq L\).

**Proposition 3.** The following properties are fulfilled \(\mathbb{P}\)-a.s.:

(i) \(Y_{\delta_\tau} 1_{[\delta_\tau, < T]} = U_{\delta_\tau} 1_{[\delta_\tau, < T]}\).

(ii) \(\forall t \leq T, Y_t \geq L_t\).
Proof. We begin with the proof of (i). For $n \geq 0$ and $t \leq T$ the process $A^n$ does not increase before $Y^n$ reaches the barrier $U$, then for any $t \in [\tau, \delta^\tau_Y]$, we have, $A^n \geq 0$ and then

$$Y^n_t = Y^n_{\delta^\tau_Y} + \int_0^t (f(s, Y^n_s) + n(Y^n_s - L_s))\,ds + \sum_{\tau \leq \sigma, n, i < \delta^\tau_Y} (Y^n_{\sigma, n, i} - L_{\sigma, n, i}) - \int_0^t dM^n_s.$$  \hfill (5)

For $n \geq 0$, writing (5) between $\delta_\tau$ and $\delta^\tau_Y$ ($\delta^\tau_Y \setminus \delta_\tau$) yields:

$$Y^n_{\delta^\tau_Y} = Y^n_{\delta_\tau} + \int_{\delta_\tau}^t (f(s, Y^n_s) + n(Y^n_s - L_s))\,ds + \sum_{\delta_\tau \leq \sigma, n, i < \delta^\tau_Y} (Y^n_{\sigma, n, i} - L_{\sigma, n, i}) - \int_{\delta_\tau}^t dM^n_s.$$ \hfill (6)

and then $Y_{\delta_\tau} \geq U_{\delta^\tau_Y} 1_{[\delta^\tau_Y < T]} + \xi 1_{[\delta^\tau_Y = T]} + \int_{\delta_\tau}^{\delta^\tau_Y} f(s, Y^n_s)\,ds - \int_{\delta_\tau}^{\delta^\tau_Y} dM^n_s$, which implies that

$$1_{[\delta_\tau < T]} Y_{\delta_\tau} \geq 1_{[\delta_\tau < T]} (U_{\delta^\tau_Y} 1_{[\delta^\tau_Y < T]} + \xi 1_{[\delta^\tau_Y = T]} + \int_{\delta_\tau}^{\delta^\tau_Y} f(s, Y^n_s)\,ds - \int_{\delta_\tau}^{\delta^\tau_Y} 1_{[\delta_\tau < T]} dM^n_s.$$ \hfill (7)

By (H1) we have: $|f(s, Y^n_s)| \leq |f(s, 0)| + \mu|Y^n_s|$ where $\mu$ is a constant. We have also $Y^n \leq U$ which implies that $E[\int_{\delta_\tau}^{\delta^\tau_Y} |f(s, Y^n_s)|\,ds]$ converges to 0. Consequently, $\lim_{n \to \infty} E[\int_{\delta_\tau}^{\delta^\tau_Y} |f(s, Y^n_s)|\,ds] = 0$.

Using now inequality (7) and taking expectation in both hand-sides then the limit as $n$ goes to infinity to obtain: $E[1_{\delta_\tau < T} Y_{\delta_\tau}] \geq E[1_{\delta_\tau < T} U_{\delta^\tau_Y}] \geq E[1_{\delta_\tau < T} U_{\delta^\tau_Y}]$, since $U$ is optional r.l.s.c. process. By $Y \leq U$, we have the desired result.

We now prove (ii). For any $n \geq 0$ and any stopping time $\tau \leq T$, the following property holds true:

$$Y^n_\tau = E\left[\int_{\tau}^{\delta^\tau_Y} (f(s, Y^n_s) + n(Y^n_s - L_s))\,ds + \sum_{\tau \leq \sigma, n, i < \delta^\tau_Y} (Y^n_{\sigma, n, i} - L_{\sigma, n, i}) - \int_0^\tau dM^n_s \bigg| \mathcal{F}_\tau\right],$$

since the process $A^n$ does not increase before $Y^n$ reaches the barrier $U$ by definition of $\delta^\tau_Y$. From last equality we have

$$E\left[\int_{\tau}^{\delta^\tau_Y} (Y^n_s - L_s)^-\,ds + \frac{1}{n} \sum_{\tau \leq \sigma, n, i < \delta^\tau_Y} (Y^n_{\sigma, n, i} - L_{\sigma, n, i})^- \right] \leq E\left[\int_{\tau}^{\delta^\tau_Y} f(s, Y^n_s)\,ds + |Y^n_\tau| + |U_{\delta^\tau_{\delta_\tau}} 1_{[\delta^\tau_Y < T]} + \xi 1_{[\delta^\tau_Y = T]}\right].$$ \hfill (8)

by (H3) we have $E[\int_0^T |f(s, Y^n_s)|\,ds] < \infty$ when $n$ goes to infinity, and by Fatou’s lemma we deduce from (8) that: $E\left[\liminf_{n \to \infty} \int_{\tau}^{\delta^\tau_Y} (Y_s - L_s)^-\,ds\right] \leq \liminf_{n \to \infty} E\left[\int_{\tau}^{\delta^\tau_Y} (Y_s - L_s)^-\,ds\right] = 0$, then

$$\int_{\tau}^{\delta^\tau_Y} (Y_s - L_s)^-\,ds = 0 \text{ a.s.}$$ \hfill (9)

Since $Y^n \leq Y^{n+1}$, note that if $L$ is a càdlàg process the limit $Y$ of $\{Y^n\}$ is càdlàg (Theorem 3.1 and Lemma 2.2) on $[\tau, \delta_\tau]$ and $\delta^\tau_{\delta_\tau}$) on $[\tau, \delta_\tau]$. But in our case $Y$ need not to be càdlàg. Henceforth from (9) we obtain that $Y_\tau \geq L_\tau$ on the set $[\tau, \delta_\tau]$. If $\tau = \delta_\tau < T$ we have $Y_\tau = U_\tau \geq L_\tau$ and if $\tau = \delta_\tau = T$ we have $Y_\tau = \xi \geq L_\tau$. By consequence for all $\tau$, $Y_\tau \geq L_\tau$. As the barriers $L$ and $U$ are optional, using the optional section theorem [11, Proposition A.4] we have $P$-a.s., $Y \geq U$. The proof is complete. \hfill \Box \hfill \Box

Next, we show the existence of the local solution of the reflected BSDE$(f, \xi, L, U)$ on $[\tau, \delta_\tau]$.
Proposition 4. There exists two measurable processes \((\bar{K}_t^n)_{t \leq T}\) and \((M_t^n)_{t \leq T}\) such that \((Y_t, M_t^n, \bar{K}_t^n, 0)_{t \leq T}\) is a local solution of RBSDE in Definition 3 on \([\tau, \delta_t]\). That is: \(\bar{K}_t^n \in \mathcal{S}^2, \bar{M}_t^n \in \mathcal{M}_{loc}, \bar{M}_0^n = 0\) and

\[
\begin{align*}
Y_t = Y_{\delta_t} + \int_t^{\delta_t} & f(s, Y_s)ds + \bar{K}_s^n - \bar{K}_{\delta_t} - \int_t^{\delta_t} d\bar{M}_s^n, a.s., \quad \text{and} \quad Y_T = \xi, \\
\forall t \in [\tau, \delta_t], L_t \leq Y_t \leq U_t & \quad \text{and} \quad \int_t^{\delta_t} (Y_s - L_s) - d\bar{K}_s^n + \sum_{\tau \leq s < \delta_t} (Y_s - L_s) \Delta^+ \bar{K}_s^n = 0, a.s. 
\end{align*}
\]

(10)

Proof. For any \(n \geq 0\) and \(t \in [\tau, \delta_t]\) and since the process \(A^n\) moves only when \(Y^n\) reaches the barrier \(U\) (then \(A^n = A^n_{\delta_t}\)), we have

\[
Y^n_t = Y^n_{\delta_t} + \int_t^{\delta_t} f(s, Y^n_s)ds + \int_t^{\delta_t} (Y^n_s - L_s)^- ds + \sum_{t \leq s < \delta_t} (Y^n_{\sigma_a,i} - L_{\sigma_a,i})^- - \int_t^{\delta_t} dM^n_s.
\]

On the other hand, for \(n \geq 0\), let \((\bar{Y}_n, \bar{M}_n)_{t \leq \delta_t}\) be the unique solution of the BSDE associated with the coefficient \(f(t, y) = n(y - L_t)^- + \sum_{s \leq \delta_t, (y_{\sigma_a,i} - L_{\sigma_a,i})^-}\), the terminal value \(Y^n_{\delta_t}\) and a bounded terminal time \(\delta_t\), that is,

\[
\begin{align*}
E(\sup_{s \leq \delta_t} | \bar{Y}_s |^2 + |\bar{M}_s|_{\delta_t}) < & \infty, \\
\bar{Y}_t = Y^n_{\delta_t} + \int_t^{\delta_t} f(s, \bar{Y}_s)ds + \int_t^{\delta_t} n(\bar{Y}_s - L_s)^- ds + \sum_{t \leq s < \delta_t} (\bar{Y}_{\sigma_a,i} - L_{\sigma_a,i})^- - \int_t^{\delta_t} d\bar{M}_s^n.
\end{align*}
\]

(11)

The proof of existence and uniqueness is obtained by the same arguments such that in [12, 3.2, Proposition 4] since \(\delta_t\) is bounded. We have \((Y^n_{\delta_t})_{n \geq 0} \nrightarrow Y_{\delta_t} \leq U_{\delta_t}\), hence from the Lepesgue dominated convergence theorem we get \(E(|Y^n_{\delta_t} - Y_{\delta_t}|) = 0\) as \(n \to \infty\). Therefore the sequence of processes \((\bar{Y}_n, \bar{M}_n, \int_0^T n(\bar{Y}_s - L_s)^- ds + \sum_{t \leq s < \delta_t (\bar{Y}_{\sigma_a,i} - L_{\sigma_a,i})^-)_{t \leq \delta_t} n \geq 0\) converges in \(S^2_{\delta_t} \times M^2_{\delta_t} \times S^2_{\delta_t} \) (\(S^2_{\delta_t}\) and \(M^2_{\delta_t}\) are the same as \(S^2\) and \(M^2\) except for that \(T\) is replaced by the stopping time \(\delta_t\)) to \((\hat{Y}_t, \hat{M}_t, \hat{K}_t)_{t \leq \delta_t}\), such that

\[
\begin{align*}
E(\sup_{s \leq \delta_t} | \hat{Y}_s |^2 + |\hat{M}_s|_{\delta_t}) < & \infty, K_s \in S^2_{\delta_t} \quad \text{and} \quad \hat{K}_0 = 0, \\
\hat{Y}_t = Y_{\delta_t} + \int_t^{\delta_t} f(s, \hat{Y}_s)ds + (\hat{K}_s - \hat{K}_t) - \int_t^{\delta_t} d\hat{M}_s, \forall t \leq \delta_t, \\
\hat{Y}_t \geq L_t \quad \text{and} \quad \int_t^{\delta_t} (Y_s - L_s)^- d\hat{K}_s + \sum_{t \leq s < \delta_t} (Y_s - L_s) \Delta^+ \hat{K}_s = 0.
\end{align*}
\]

Now by (11), (11) and uniqueness of the solution on \([\tau, \delta_t]\) implies that for any \(t \in [\tau, \delta_t]\), \(Y^n_t = \bar{Y}_t^n\) and \(M^n_t = \bar{M}_t^n\). Therefore \(Y_t = \bar{Y}_t^n\) for any \(t \in [\tau, \delta_t]\),

\[
\begin{align*}
E(\sup_{s \leq \delta_t} | Y_s |^2 + |M_s|_{\delta_t}) < & \infty, \hat{K}_s \in S^2_{\delta_t} \quad \text{and} \quad \hat{K}_0 = 0, \\
Y_t = Y_{\delta_t} + \int_t^{\delta_t} f(s, \hat{Y}_s)ds + (\hat{K}_s - \hat{K}_t) - \int_t^{\delta_t} d\hat{M}_s, \forall t \leq \delta_t, \\
\forall t \leq \delta_t L_t \leq Y_t \leq U_t \quad \text{and} \quad \int_t^{\delta_t} (Y_s - L_s)^- d\hat{K}_s + \sum_{t \leq s < \delta_t} (Y_s - L_s) \Delta^+ \hat{K}_s = 0.
\end{align*}
\]

For any \(t \leq T\), let us set \(\bar{K}_T = (\hat{K}_T, \delta_t - \hat{K}_T)_{1_{[\tau \geq \delta_t]}}\) and \(\bar{M}_T = \hat{M}_T \cdot 1_{[\tau \leq \delta_t]}\) (see Remark 4), we deduce that \((Y_t, \hat{M}_t, \hat{K}_t, 0)_{t \leq T}\) is a local solution of RBSDE in Definition 3 on \([\tau, \delta_t]\).

\[\Box\quad \Box\]

5.2 The decreasing penalization scheme

We now consider the following decreasing penalization scheme for any \(n \geq 0\):

\[
\begin{align*}
\hat{Y}_t &= \xi + \int_t^T f(s, \hat{Y}_s)ds - n(U_s - \hat{Y}_s)^- ds - \sum_{t \leq s < T} (U_{\sigma_a,i} - \hat{Y}_{\sigma_a,i})^- + \hat{K}_s - \hat{K}_t - \int_t^T d\hat{M}_s, a.s., \\
\forall t \in [0, T], \hat{Y}_t \geq L_t, \quad \text{and} \quad \int_0^T (\hat{Y}_{s,-} - L_{s,-})d\hat{K}_s^n + \sum_{s < T} (\hat{Y}_{s,-} - L_{s,-}) \Delta^+ \hat{K}_s^n = 0, a.s.
\end{align*}
\]

(12)

First we note that the existence of the triple \((\hat{Y}_n, \hat{M}_n, \hat{K}_n)\) is due to [21, Theorem 4.1] and the following remark.
Remark 5. A triple \((Y, M, K)\) is a solution for the BSDE with a lower reflecting barrier associated with \((f, \xi, L)\) iff \((-Y, -M, K)\) is a solution of the BSDE with an upper reflecting barrier associated with \((-f(t, \omega, -y), -\xi, -L)\).

For any stopping time \(\tau \leq T\) and any \(n \geq 0\), let us set \(\theta^n = \inf\{s \geq \tau, \bar{Y}^n_s = \bar{L}_s\} \land T\). By Proposition 1, we have \(\bar{Y}^n \geq \bar{Y}^{n+1} \geq L\) then the sequence \((\bar{Y}^n)_{n \geq 0}\) converges to \(\bar{Y}\) and \((\theta^n)_{n \geq 0}\) is decreasing and converges to another stopping time \(\theta_\tau = \lim_{n \to \infty} \theta^n\). Using the same arguments in Propositions 3 and 4 and by Remark 5 we get:

**Proposition 5.** The following properties hold true \(\mathbb{P}\)-a.s.:

(i) \(\bar{Y}_\theta, 1[\theta_\tau < T] = \bar{L}_\theta, 1[\theta_\tau < T]\).

(ii) \(\forall t \leq T, \bar{Y}_t \leq \bar{U}_t\).

(iii) There exists two measurable processes \((\bar{A}^n_t)_{t \leq T}\) and \((\bar{M}^n_t)_{t \leq T}\) such that \((\bar{Y}_t, \bar{M}^n_t, 0, \bar{A}^n_t)_{t \leq T}\) is a local solution of RBSDE in Definition 3 on \([\tau, \theta_\tau]\). That is: \(\bar{A}^n \in \mathcal{S}^2, \bar{M}^n \in \mathcal{M}_{loc}, \bar{M}^n_0 = 0\) and

\[
\begin{aligned}
\forall t \in [\tau, \theta_\tau], & L_t \leq \bar{Y}_t \leq \bar{U}_t \text{ and } \int_\tau^{\theta_\tau} (U_s - \bar{Y}_s)ds + \sum_{\tau < s \leq \theta_\tau} (U_s - \bar{Y}_s)\Delta^+ \bar{A}^n_s = 0, \text{ a.s.}.
\end{aligned}
\]

(13)

### 5.3 Existence of the local solution

Recall that \(Y\) (resp. \(\bar{Y}\)) is the limit of the increasing (resp. decreasing) approximating scheme. We are going to show that the processes \(Y\) and \(\bar{Y}\) are undistinguishable.

**Proposition 6.** \(\mathbb{P}\)-a.s., for any \(t \leq T\), \(Y_t \equiv \bar{Y}_t\).

**Proof.** We prove the equality in two steps, first we show that \(Y \leq \bar{Y}\), and second we show the other inequality. For that, let \(J^0(Y^n - \bar{Y}^m)\) denote the local time of \(Y^n - \bar{Y}^m\) at 0. For any \(t \leq T\) and any \(n, m \geq 0\), by the Itô-Tanaka formula for regulated processes (see [22, Section 3, page 539]) applied to \((Y^n - \bar{Y}^m)^+\) we have

\[
(Y^n_t - \bar{Y}^m_t)^+ \leq (Y^n_T - \bar{Y}^m_T)^+ + \int_t^T 1_{\{Y^n_s > \bar{Y}^m_s\}} (f(s, Y^n_s) - f(s, \bar{Y}^m_s))ds + \int_t^T 1_{\{Y^n_s > \bar{Y}^m_s\}} d(K^n_s - A^n_s - K^m_s + \bar{A}^m_s) - \int_t^T 1_{\{Y^n_s > \bar{Y}^m_s\}} d(M^n_s - \bar{M}^m_s)
\]

As in the proof of the comparison result (see Proposition 1) we show that \(Y^n_t \leq \bar{Y}^m_t\) and we get \(Y_t \leq \bar{Y}_t\), for any \(t \leq T\).

Now we prove that \(Y_t \geq \bar{Y}_t, \forall t \leq T\). Let \(\tau\) be a stopping time and let \(\delta_\tau\) and \(\theta_\tau\) be the stopping times introduced in Propositions 3 and 5 respectively. We have:

\[
Y_{\delta_\tau \wedge \theta_\tau} = Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] + Y_{\delta_\tau} 1[\delta_\tau > \theta_\tau] + Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau = T] \geq L_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] + U_{\theta_\tau} 1[\delta_\tau > \theta_\tau] + Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau = T] \\
Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] \geq L_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] + \bar{Y}_{\theta_\tau} 1[\delta_\tau > \theta_\tau] + \bar{Y}_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau = T] = \bar{Y}_{\delta_\tau \wedge \theta_\tau}
\]

since \(Y \geq L, \bar{Y}_{\delta_\tau} 1[\delta_\tau > \theta_\tau] = U_{\theta_\tau} 1[\delta_\tau > \theta_\tau]\) (Proposition 5) and

\[
Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] = Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] \cap [\delta_\tau < T] + Y_{\delta_\tau} 1[\delta_\tau > \theta_\tau] = T] \cap [\delta_\tau = T] \\
Y_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] \geq L_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T] \cap [\delta_\tau < T] + \bar{Y}_{\delta_\tau} 1[\delta_\tau < T] \cap [\delta_\tau = T] = \bar{Y}_{\delta_\tau} 1[\delta_\tau \leq \theta_\tau < T]
\]

Since \(Y\) and \(\bar{Y}\) satisfy the BSDEs \((10)\) and \((13)\) respectively between \(\tau\) and \(\delta_\tau \wedge \theta_\tau\), then using comparison result of solution (Proposition 1) of BSDEs with \((1[\tau < \delta_\tau \wedge \theta_\tau]) \bar{Y}_t\) and \((1[\tau < \delta_\tau \wedge \theta_\tau]) Y_t\) we have:

\[
Y_t \leq \bar{Y}_t, \forall t \leq T.
\]
we get $1_{[\tau < \delta_s \land \delta_t]}Y_\tau \geq 1_{[\tau < \delta_s \land \delta_t]}\tilde{Y}_\tau$. On the other hand from (15), we have $1_{[\tau = \delta_s \land \delta_r]}Y_\tau \geq 1_{[\tau = \delta_s \land \delta_r]}\tilde{Y}_\tau$ which implies that $Y_\tau \geq \tilde{Y}_\tau$. As $\tau$ is an arbitrary stopping time and $Y$ and $\tilde{Y}$ are optional processes then $\mathbb{P}$-a.s., $Y \geq \tilde{Y}$ by [3, Proposition 2.4]. We conclude that $Y = \tilde{Y}$ $\mathbb{P}$-a.s.

As a consequence of the result obtained in Propositions 4, 5, 6 and 7, we have:

**Theorem 2.** There exists a unique measurable process $(Y_t)_{t \leq T}$ such that:

i) $\forall t \leq T$, $L_t \leq Y_t \leq U_t$ and $Y_T = \xi$;

ii) for any stopping time $\tau$, there exist another stopping time $\lambda_\tau \geq \tau$ $\mathbb{P}$-a.s., and a triple of measurable processes $(M_\tau^t, K_\tau^t, A_\tau^t)_{t \leq T}$ such that on $[\tau, \lambda_\tau]$ the process $(Y_t, M_t^\tau, K_t^\tau, A_t^\tau)_{t \leq T}$ is a local solution for the reflected BSDE associated $(f, \xi, L_t, U_t)$, (3) in Definition 4.

**Remark 6.** If we set $\nu_\tau = \inf\{t \geq \tau, Y_t = U_t\} \wedge T$, $\sigma_\tau = \inf\{t \geq \tau, Y_t = L_t\} \wedge T$, when $\nu_\tau \vee \sigma_\tau \leq \lambda_\tau$, that is $Y$ reaches $L$ and $U$ between the times $\tau$ and $\lambda_\tau$ when $\lambda_\tau < T$.

**Proof.** By Propositions 4, 5, 6 and 7 we have the first point (i). Let $\tau$ be a fixed stopping time and let $(Y, \lambda_\tau, M^\tau, K^\tau, A^\tau)$ and $(Y', \lambda'_\tau, M'^{\tau'}, K'^{\tau'}, A'^{\tau'})$ two solutions for the reflected BSDE associated $(f, \xi, L_t, U_t)$. Then by the same argument of [16, Theorem 3.2.], we can prove that $Y = Y'$.

Let $(Y_t, M_t^\tau, K_t^\tau, 0)_{t \leq T}$ (resp. $(Y_t, M_t^\tau, 0, A_t^\tau)_{t \leq T}$) be a local solution of reflected BSDE in Definition 4 on $[\tau, \delta_s]$ (resp. on $[\delta_s, \tau]$) which exists according to Proposition 4 (resp. Proposition 5), where $\lambda_\tau$ is a stopping time such that $\tau \leq \lambda_\tau \leq T$. Now for $t \leq T$, let $M_t = M_t^\tau_{[\tau \leq \delta_s]} + M_t^\tau_{[\delta_s \leq \lambda_\tau]}$ (see Remark 7), $K_t = K_t^\tau_{[\tau \leq \delta_s]}$ and $A_t = A_t^\tau_{[\tau \leq \delta_s]}1_{[\tau \leq \delta_s]}$. For any $t \in [\tau, \lambda_\tau]$ we have,

\[
\begin{align*}
Y_t &= Y_{\lambda_\tau} + \int_{\tau}^{\lambda_\tau} f(s, Y_s)ds + K_{\lambda_\tau}^t - K_{\tau}^t - (A_{\lambda_\tau}^t - A_{\tau}^t) - \int_{\tau}^{\lambda_\tau} dM_s, \quad \mathbb{P} \text{-a.s.} \\
&\quad \forall t \in [\tau, \lambda_\tau], L_t \leq Y_t \leq U_t \text{ a.s.}, \quad \text{and} \\
&\int_{\tau}^{\lambda_\tau} (Y_s - L_s) dK_s^t + \int_{\tau}^{\lambda_\tau} (Y_s - U_s) dA_s^t = 0 \quad \text{a.s., and} \\
&\int_{\tau}^{\lambda_\tau} (U_s - Y_s) dA_s^t + \int_{\tau}^{\lambda_\tau} (Y_s - L_s) dK_s^t = 0 \quad \text{a.s.}
\end{align*}
\]

(16)

Indeed, if $t \in [\delta_s, \tau]$, we have $K_{\lambda_\tau}^t - K_{\tau}^t = 0$ and (10) is satisfied from (13). And if $t \in [\tau, \delta_s]$, then from (3) we have, $Y_t = Y_{\delta_s} + \int_{\tau}^{\delta_s} f(s, Y_s)ds + K_{\delta_s}^t - K_{\tau}^t - \int_{\tau}^{\delta_s} dM_s$. As $Y_s = Y_{\lambda_\tau} + \int_{\delta_s}^{\lambda_\tau} f(s, Y_s)ds - (A_{\lambda_\tau}^t - A_{\tau}^t) - \int_{\delta_s}^{\lambda_\tau} dM_s$, then (10) is also satisfied since $K_{\lambda_\tau}^t - K_{\tau}^t = 0$.

Now for any $t \in [\tau, \lambda_\tau]$, $L_t \leq Y_t \leq U_t$ a.s. and $\int_{\tau}^{\lambda_\tau} (Y_s - L_s) dK_s^t + \int_{\tau}^{\lambda_\tau} (Y_s - U_s) dA_s^t + \int_{\tau}^{\lambda_\tau} (U_s - Y_s) dA_s^t + \int_{\tau}^{\lambda_\tau} (Y_s - L_s) dK_s^t + \int_{\tau}^{\lambda_\tau} (Y_s - U_s) dA_s^t + \int_{\tau}^{\lambda_\tau} (U_s - Y_s) dA_s^t = 0$.

Finally $Y_T = \xi$ and the process $(Y_t, M_t^\tau, K_t^\tau, A_t^\tau)$ is a local solution for (3) on $[\tau, \lambda_\tau]$.

**Remark 7.** If $M_t$ is a local martingale w.r.t $\mathcal{F}_t$ and if $\tau$ and $\delta_s$ are two $\mathcal{F}_t$-stopping times such that $\tau \leq \delta_s$, then $M_t 1_{[\tau \leq \delta_s]}$ is a $\mathcal{F}_t$-martingale. Indeed:

\[
M_t 1_{[\tau \leq \delta_s]}(t) = \int_{0}^{t} 1_{[\tau \leq \delta_s]}(s)dM_s + \int_{\tau}^{t} M_s d(1_{[\tau \leq \delta_s]}(s)) = \int_{0}^{t} 1_{[\tau \leq \delta_s]}(s)dM_s - M_{\delta_s \wedge t} + M_{\tau \wedge t}
\]

(17)

The construction of $Y$ does not depend on $\tau$ but the ones of $M$, $K$ and $A$ do.

6 Existence of a global solution for reflected BSDE with two completely separated barriers.

We are now ready to give the main result of this paper. Let us assume that the barriers $L$ and $U$ and their left limits are completely separated, i.e., they satisfy the following assumption:

[H]: $\mathbb{P}$-a.s., $\forall t \in [0, T]$, $L_t < U_t$ and $L_{t-} < U_{t-}$.
Theorem 3. Under Assumption $[H]$, there exists a unique process $(Y_t, M_t, K_t, A_t)_{t \leq T}$ solution of the reflected BSDE associated with $(f, \xi, L, U)$, i.e., $Y_t, K_t, A_t \in \mathcal{S}^2$, $M_t \in \mathcal{M}_{loc}$ with $M_0 = 0$ and

$$
\left\{
\begin{array}{ll}
Y_t = \xi \pm \int_{t}^{T} f(s, Y_s) ds + K_T - K_t - (A_T - A_t) - \int_{t}^{T} dM_s, & \forall t \leq T \\
L_t \leq Y_t \leq U_t, & \text{a.s. } \forall t \leq T, \text{ and}
\end{array}
\right.
$$

$$
\int_{0}^{T} (U_s - Y_s) dA_s^* + \sum_{0 \leq s < T} (U_s - Y_s) \Delta^+ A_s = \int_{0}^{T} (Y_s - L_s) dK^*_s + \sum_{0 \leq s < T} (Y_s - L_s) \Delta^+ K_s = 0, \text{ a.s.}
$$

(18)

Proof. Let $(Y_t)_{t \leq T}$ be the process defined in Theorem 2 then $L \leq Y \leq U$ and $Y_T = \xi$. Now let $(\tau_n)_{n \geq 0}$ a sequence of stopping times such that $\tau_0 = 0$ and $\tau_{n+1} = \inf\{t \geq \tau_n, Y_t = U_t\} \wedge T$ and $\tau_{n+2} = \inf\{t \geq \tau_{n+1}, Y_t = L_t\} \wedge T$. Henceforth, for any $n \geq 0$ there exists a triple $(M^n_t, K^n_t, A^n_t)_{t \leq T}$ of processes such that the process $(Y_t, M^n_t, K^n_t, A^n_t)_{t \leq T}$ is a local solution for the reflected BSDE associated with $(f, \xi, L, U)$ on the set $[\tau_n, \tau_{n+1}]$ (by Theorem 2).

By the same argument in [12, 3.7. Theorem.] (see also [14, Theorem 4.1], [18, Theorem 4.1] or [17, Theorem 4.1]) we show that $\mathbb{P}(\tau_n < T, \forall n \geq 0) = 0$, $\mathbb{P}$-a.s. since $\mathbb{P}$-a.s., $L_{t-} < U_{t-}$. Which means that for $\omega \in \Omega$ there exists $n_0(\omega) \geq 0$ such that $\tau_{n_0}(\omega) = T$. Next let us introduce the following processes $M$, $K$, $A$: $\mathbb{P}$-a.s., for any $t \leq T$, one sets

$$
K_t = K_{\tau_n} + (K^n_t - K^n_{\tau_n}) \quad \text{if } t \in [\tau_n, \tau_{n+1}] \quad (K_0 = 0)
$$

$$
A_t = A_{\tau_n} + (A^n_t - A^n_{\tau_n}) \quad \text{if } t \in [\tau_n, \tau_{n+1}] \quad (A_0 = 0)
$$

$$
M_t = M_{t1}[0, \tau_n] + \sum_{n \geq 1} M^n_{t1}[\tau_n, \tau_{n+1}].
$$

Since the sequence $(\tau_n)_{n \geq 0}$ is $\mathbb{P}$-a.s. of stationary type and for any $n \geq 0$, $\mathbb{E}(|M|_{\tau_n}) < \infty$ then $\mathbb{E}(|M|_T) < \infty$, $\mathbb{P}$-a.s.

Next let us show that $(Y, M, K, A)$ is the solution of the reflected BSDE$(\xi, f, L, U)$. For any $n \geq 0$ we have: $\mathbb{P}$-a.s. for all $t \in [\tau_n, \tau_{n+1}]$,

$$
Y_t = Y_{\tau_{n+1}} + \int_{t}^{\tau_{n+1}} f(s, Y_s) ds + K_{\tau_{n+1}} - K_t - (A_{\tau_{n+1}} - A_t) - \int_{t}^{\tau_{n+1}} dM_s. \quad (19)
$$

For any $n \geq 0$ we have: $\mathbb{P}$-a.s.

$$
Y_{\tau_{n+1}} = Y_{\tau_{n+1}} + \int_{\tau_{n+1}}^{T} f(s, Y_s) ds + K_{\tau_{n+1}} - K_{\tau_n} - (A_{\tau_{n+1}} - A_{\tau_n}) - \int_{\tau_{n+1}}^{T} dM_s.
$$

Now for any $n \geq 0$ and $m \geq n$ we have: $Y_{\tau_n} = Y_{\tau_n} + \int_{\tau_n}^{\tau_m} f(s, Y_s) ds + K_{\tau_{m+1}} - K_{\tau_n} - (A_{\tau_{m+1}} - A_{\tau_n}) - \int_{\tau_n}^{\tau_m} dM_s$. By the fact that $(\tau_n)_{n \geq 0}$ is of stationary type and taking $m$ large enough we obtain: $\forall n \geq 0$, $\mathbb{P}$-a.s.,

$$
Y_{\tau_n} = \xi + \int_{\tau_n}^{T} f(s, Y_s) ds + K_T - K_{\tau_n} - (A_T - A_{\tau_n}) - \int_{\tau_n}^{T} dM_s. \quad (20)
$$

Now let $t \in [0, T]$ then there exists $n_0$ such that $t \in [\tau_{n_0}, \tau_{n_0+1}]$. Using $(19)$ then $(20)$ we obtain: $Y_t = \xi + \int_{t}^{T} f(s, Y_s) ds + K_T - K_t - (A_T - A_t) - \int_{t}^{T} dM_s$, which means that $(Y, M, K, A)$ verify equation $(LUA4)$ of Definition 4. Finally the processes satisfy $K_T$ and $Y$ satisfy

$$
\int_{0}^{T} (U_s - Y_s) ds = \sum_{0 \leq s < T} (Y_s - L_s) \Delta^+ K_s + \sum_{0 \leq s < T} (Y_s - L_s) \Delta^+ A_s = \sum_{0 \leq s < T} (Y_s - L_s) \Delta^+ K^*_s + \sum_{0 \leq s < T} (Y_s - L_s) \Delta^+ K^*_s = 0.
$$

(18) In the same way we have

$$
\int_{0}^{T} (U_s - Y_s) dA^*_s + \sum_{0 \leq s < T} (U_s - Y_s) \Delta^+ A_s = 0.
$$

Then the process $(Y, M, K, A)$ is a solution for the reflected BSDE$(\xi, f, L, U)$. Uniqueness is a direct consequence of the comparison theorem (Proposition 1). \bbox

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