Rigidity of noncompact complete Bach-flat manifolds

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Abstract

Let \((M, g)\) be a noncompact complete Bach-flat manifold with positive Yamabe constant. We prove that \((M, g)\) is flat if \((M, g)\) has zero scalar curvature and sufficiently small \(L^2\) bound of curvature tensor. When \((M, g)\) has nonconstant scalar curvature, we prove that \((M, g)\) is conformal to the flat space if \((M, g)\) has sufficiently small \(L^2\) bound of curvature tensor and \(L^4/3\) bound of scalar curvature.

1 Introduction

Let \((M, g)\) be a noncompact complete Riemannian 4-manifold with scalar curvature \(R\), Weyl curvature \(W\), Ricci curvature \(R_{ij}\) and curvature tensor \(Riem\). A metric is Bach-flat if it is a critical metric of the functional

\[
g \rightarrow \int_M |W|^2 \, dV_g. \tag{1}\]

Bach-flat condition is equivalent to the vanishing of Bach tensor \(B_{ij}\), which is defined by

\[
B_{ij} \equiv \nabla^k \nabla^l W_{kij} + \frac{1}{2} R^{kl} W_{kij}. \tag{2}\]

(see [1]). Important examples of Bach manifolds are Einstein manifolds, self-dual (anti-self-dual) manifolds, conformally flat manifolds and Kähler surfaces with zero scalar curvature (see [2]).
Einstein metrics and Bach-flat metrics share many important properties. When the curvature of a given Einstein metric \((M, g)\) is sufficiently close to that of the constant curvature space, in \(L_2^2\) sense, it is known that \((M, g)\) is isometric to a quotient of the constant curvature space \([3, 4, 5, 6, 7]\). In this paper, we study this rigidity phenomena for noncompact complete Bach-flat manifolds. First, we study rigidity of metrics with positive Yamabe constant, zero scalar curvature and small \(L_2\) bound of curvature. Next we look for rigidity of nonconstant scalar curvature spaces with a conformal change of the given metric.

There are known rigidity of Bach-flat metrics. For a compact Bach-flat manifold \((M, g)\) with positive Yamabe constant, Chang, Ji and Yang \([8]\) proved that there is only finite diffeomorphism class with an \(L_2\) bound of Weyl tensor, and \((M, g)\) is conformal to the standard sphere if \(L_2\) norm of Weyl tensor is small enough. For a noncompact complete Bach-flat manifold \((M, g)\) with positive Yamabe constant and zero scalar curvature, Tian and Viaclovsky \([9]\) proved that \((M, g)\) is almost locally Euclidean of order 0 with \(L_2\) bounds of curvature, bounded first Betti number and the uniform volume growth for any geodesic ball.

For rigidity of Bach-flat manifolds, we use an elliptic estimation on the Laplacian of curvature tensor. For this, we introduce the Yamabe constant on \((M, g)\). Let \((M, g)\) be a Riemannian manifold of dimension \(n \geq 3\) with scalar curvature \(R\). The Yamabe constant \(Q(M, g)\) is defined by

\[
Q(M, g) \equiv \inf_{0 \neq u \in C^\infty_0 (M)} \frac{4(n-1)}{n-2} \int_M |\nabla u|^2 + R_g u^2 \, dV_g 
\left( \int_M |u|^{2n/(n-2)} \, dV_g \right)^{(n-2)/n}.
\]

\(Q(M, g)\) is conformally invariant and any locally conformally flat manifold and manifolds with zero scalar curvature satisfy \(Q(M, g) > 0\) (see \([10]\)).

## 2 Bach-flat metric with constant scalar curvature

In this section, we study noncompact complete Bach-flat manifolds with nonnegative constant scalar curvature. First, we consider Bach-flat manifolds whose \(L_2\) curvature norm is small. By an elliptic estimation for the Laplacian of curvature tensor, we have:
Theorem 1 Let $(M, g)$ be a noncompact complete Bach-flat Riemannian 4-manifold with zero scalar curvature and $Q(M, g) > 0$. Then there exists a small number $c_0$ such that if $\int_M |\text{Riem}|^2 dV_g \leq c_0$, then $(M, g)$ is flat, i.e., $\text{Riem} = 0$, where $\text{Riem}$ is curvature tensor.

Proof. We need to prove that $|\text{Riem}| = 0$. The Laplacian of curvature tensor is

$$\Delta R_{ijkl} = 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) + \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} - \nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik} + g^{pq}(R_{pijkl} R_{qij} + R_{ipjk} R_{qij}).$$

(3)

where $B_{ijkl} = g^{ps} R_{pinj} R_{rksl}$ (see [11]). Multiplying $R_{ijkl}$ on (3),

$$R_{ijkl} \Delta R_{ijkl} = 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) R_{ijkl} + \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk} + \nabla_j \nabla_k R_{il} - \nabla_j \nabla_l R_{ik} + g^{pq}(R_{pijkl} R_{qij} + R_{ipjk} R_{qij}) R_{ijkl}.$$ (4)

To simplify notations, we will work in an orthonormal frame. By the Bianchi identity,

$$\nabla^i R_{ijkl} = \nabla_k R_{jl} - \nabla_l R_{jk}.$$ (5)

For a smooth compact supported function $\phi$ and small $\epsilon > 0$, we integrate the second term in (3),

$$\int_M \phi^2 (\nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk}) R_{ijkl} dV_g$$

$$= - \int_M \nabla_i \phi^2 (\nabla_k R_{jl} - \nabla_l R_{jk}) R_{ijkl} + \phi^2 (\nabla_k R_{jl} - \nabla_l R_{jk}) \nabla_i R_{ijkl} dV_g$$

$$= - \int_M \nabla_i \phi^2 \nabla_l R_{ijkl} R_{ijkl} + \phi^2 |\nabla_i R_{ijkl}|^2 dV_g$$ (6)

$$\geq - \int_M \frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + (1 + \epsilon) \phi^2 |\nabla_i R_{ijkl}|^2 dV_g.$$ (7)

Using the same method,

$$\int_M \phi^2 (-\nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik}) R_{ijkl} dV_g$$

$$\geq - \int_M \frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + (1 + \epsilon) \phi^2 |\nabla_j R_{ijkl}|^2 dV_g.$$ (8)
The first and fourth terms in (4) are contractions of cubic terms of curvature tensor which can be bounded by $c|Riem|^3$ for a constant $c$. In this paper, we use $c$ and $c'$ to denote some positive constant, which can be varied. By the Kato inequality,

$$|\nabla Riem|^2 \geq |\nabla |Riem||^2$$

and

$$-\int_M \phi^2 |Riem| \Delta |Riem| \, dV_g \quad (9)$$

$$= -\int_M \phi^2 \left( |\nabla Riem|^2 - |\nabla |Riem||^2 + R_{ijkl} \Delta R_{ijkl} \right) \, dV_g \quad (10)$$

$$\leq \int_M 2 \left( \frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + (1 + \epsilon) \phi^2 |\nabla R_{ijkl}|^2 \right) + c|Riem|^3 \phi^2 \, dV_g \quad (11)$$

For a general Riemannian $n$-manifold, the following hold:

$$\langle \delta W \rangle_{ijkl} = \nabla^i W_{ijkl} = \frac{(n-3)}{(n-2)} \left( \nabla_k R_{jl} - \nabla_l R_{jk} - \frac{1}{6} \nabla_k R g_{jl} + \frac{1}{6} \nabla_l R g_{jk} \right) \quad (12)$$

and

$$|\nabla^i W_{ijkl}|^2 = \left( \frac{n-3}{n-2} \right)^2 \left( |\nabla^i R_{ijkl}|^2 - \frac{1}{6} |\nabla R|^2 \right). \quad (13)$$

Let $E_{ij}$ be the traceless Ricci tensor, i.e $E_{ij} = R_{ij} - \frac{1}{4} R g_{ij}$. Multiplying $\phi E_{ij}$ on Bach-flat equation (2) (cf. [12, (3. 24)]),

$$0 = -\int_M \phi^2 E_{ij} \left( \nabla_k \nabla_i W_{iklj} - \frac{1}{2} W_{iklj} E_{kl} \right) \, dV_g \quad (14)$$

$$= \int_M \phi^2 \left( |\delta W|^2 - \frac{1}{2} W_{iklj} E_{kl} E_{ij} \right) - 2 \phi \nabla_k \phi \nabla_l W_{iklj} E_{ij} \, dV_g$$

$$= \int_M \phi^2 \left( |\delta W|^2 - \frac{1}{2} W_{iklj} E_{kl} E_{ij} \right) - 2 \phi \nabla_k \phi \nabla_l W_{iklj} E_{ij} \, dV_g$$

$$\geq \int_M (1 - \epsilon_2) \phi^2 |\delta W|^2 - \frac{1}{2} \phi^2 W_{iklj} E_{kl} E_{ij} - \frac{1}{\epsilon_2} |\nabla \phi|^2 |E_{ij}|^2 dV_g. \quad (15)$$

where $W_{iklj} g_{kl} = 0$ is used in (14). Therefore,

$$\int_M \phi^2 |\delta W|^2 \leq (1 - \epsilon_2)^{-1} \int_M \frac{1}{2} \phi^2 W_{iklj} E_{kl} E_{ij} + \frac{1}{\epsilon_2} |\nabla \phi|^2 |E_{ij}|^2 dV_g \quad (17)$$
\begin{align*}
\int_M \phi^2 |\text{Riem}| \Delta |\text{Riem}| \; dV_g \\
\leq \int_M 2 \left[ \frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + \frac{4(1 + \epsilon)}{1 - \epsilon^2} \phi^2 W_{ikjl} E_{kl} E_{ij} + \frac{8(1 + \epsilon)}{(1 - \epsilon^2)\epsilon^2} |\nabla \phi|^2 |E_{ij}|^2 \\
+ \frac{1}{3}(1 + \epsilon)\phi^2 |\nabla R|^2 \right] + c|\text{Riem}|^3 \phi^2 \; dV_g. \tag{18}
\end{align*}

Note that second term in (18) is also cubic term of curvature tensor. Now we can bound all cubic terms in the above equation by $c|\text{Riem}|^3$, and the first and third terms by $c|\nabla \phi|^2 |\text{Riem}|^2$ for a suitable constant $c$. Next we use the fact that scalar curvature is zero. For simplicity of notations, we let $u = |\text{Riem}|$. Using the Yamabe constant $\Lambda_0 \equiv Q(M, g)$,

\begin{align*}
\Lambda_0 \left( \int_M (\phi u)^4 \; dV_g \right)^{1/2} \\
\leq \int_M |u \nabla \phi + \phi \nabla u|^2 \; dV_g + \frac{1}{6} Ru^2 \phi^2 \; dV_g \\
\leq \int_M u^2 |\nabla \phi|^2 + |\nabla u|^2 \phi^2 + 2u \phi \nabla \phi \cdot \nabla u + \frac{1}{6} Ru^2 \phi^2 \; dV_g \\
\leq \int_M (c + 1)|\nabla \phi|^2 u^2 + cu^3 \phi^2 \; dV_g \\
\leq \int_M (c + 1)|\nabla \phi|^2 u^2 \; dV_g \\
+ c \left( \int_M (\phi u)^4 \; dV_g \right)^{1/2} \left( \int_M u^2 \; dV_g \right)^{1/2}. \tag{20}
\end{align*}

Since $\int_M |\text{Riem}|^2 \; dV_g$ is sufficiently small, there exists a constant $c'$ such that

\begin{equation}
\begin{aligned}
c'(\int_M (\phi u)^4 \; dV_g)^{1/2} \leq \int_M |\nabla \phi|^2 u^2 \; dV_g. \tag{21}
\end{aligned}
\end{equation}

Now we choose $\phi$ as

\begin{equation}
\phi = \begin{cases}
1 & \text{on } B_t \\
0 & \text{on } M - B_{2t} \\
\frac{1}{t} |\nabla \phi| & \text{on } B_{2t} - B_t
\end{cases} \tag{22}
\end{equation}
with \(0 \leq \phi \leq 1\) and \(B_t = \{x \in M \mid d(x, x_0) \leq t\}\) for some fixed \(x_0 \in M\). From (21)

\[
c' \left( \int_M u^4 \phi^4 \, dV_g \right)^{1/2} \leq \frac{4}{t^2} \int_{B_t(B(2t))} u^2 \, dV_g \leq \frac{4 \sqrt{3}}{t^2} c_0.
\]  

By taking \(t \to \infty\), we have \(u = 0\). Therefore \((M, g)\) is flat.

Next we consider complete Bach-flat metric with positive constant scalar curvature. Using an elliptic estimation for traceless Ricci tensor, we prove

**Theorem 2** Let \((M, g)\) be a noncompact complete Riemannian 4-manifold with nonnegative constant scalar curvature \(R\), Weyl curvature \(W\) and traceless Ricci curvature \(E_{ij}\). Assume that \((M, g)\) is Bach-flat and \(Q(M, g) > 0\). Then there exists a small number \(c_0\) such that if \(\int_M |W|^2 + |E_{ij}|^2 \, dV_g \leq c_0\), then \((M, g)\) is an Einstein manifold.

There is no noncompact complete Einstein manifold of positive scalar curvature. By Theorem 2 we have an obstruction for the existence of a noncompact complete Bach-flat manifold.

**Theorem 3** Let \((M, g)\) be a noncompact complete Riemannian 4-manifold with positive constant scalar curvature \(R\), Weyl curvature \(W\) and traceless Ricci curvature \(E_{ij}\). Assume that \((M, g)\) is Bach-flat and \(Q(M, g) > 0\). Then there exists a positive number \(c_1\) such that \(\int_M |W|^2 + |E_{ij}|^2 \, dV_g \geq c_1\).

**Proof** of Theorem 2.

Let \(E_{ij}\) be the traceless Ricci tensor and \(|E| = |E_{ij}|\). Using Bianchi identity, Bach tensor can be expressed in the following way (cf. [13] (1.18)),

\[
B_{ij} = -\frac{1}{2} \Delta E_{ij} + \frac{1}{6} \nabla_i \nabla_j R - \frac{1}{24} \Delta R g_{ij} - E^{kl} W_{ikjl} + E_i^k E_{jk} - \frac{1}{4} |E|^2 g_{ij} + \frac{1}{6} R E_{ij}.
\]  

By the Kato inequality, \(|\nabla E|^2 \geq |\nabla |E||^2\) and \(\text{tr}E^3 \leq \frac{1}{\sqrt{3}} |E|^3\), there exists a positive constant \(c\) satisfying the following equation for a Bach-flat metric
\[ |E| \triangle |E| = |\nabla E|^2 - |\nabla |E||^2 - 2E^{ki}W_{kj}E^{ij} + 2\text{tr}E^3 + \frac{1}{3}R|E|^2 \] 
(25)

\[ \geq -2E^{ki}W_{kj}E^{ij} + 2\text{tr}E^3 + \frac{1}{3}R|E|^2 \] 
(26)

\[ \geq -c|W||E|^2 - \frac{2}{\sqrt{3}}|E|^3 + \frac{1}{3}R|E|^2. \] 
(27)

Let \( u = |E| \). Multiplying a smooth compact supported function \( \phi \) to (27) and integrating on \( M \),

\[ \int_M \phi^2|\nabla u|^2 + 2\phi u\nabla \phi \cdot \nabla u \, dV_g \leq \int_M c|W|u^2\phi^2 + \frac{2}{\sqrt{3}}u^3\phi^2 - \frac{1}{3}Ru^2\phi^2 \, dV_g. \]

Using the Yamabe constant,

\[ \Lambda_0 \left( \int_M (\phi u)^4 \, dv_g \right)^{1/2} \]
\[ \leq \int_M |u\nabla \phi + \phi \nabla u|^2 \, dV_g + \frac{1}{6}Ru^2\phi^2 \]
\[ \leq \int_M u^2|\nabla \phi|^2 + c|W|u^2\phi^2 + \frac{2}{\sqrt{3}}u^3\phi^2 - \frac{1}{6}Ru^2\phi^2 \, dV_g. \] 
(28)

Note that the second term of (28) is bounded by

\[ c \left( \int_M |W|^2 \, dV_g \right)^{1/2} \left( \int_M u^4 \phi^4 \, dV_g \right)^{1/2} \]

and the third term is bounded by

\[ \frac{2}{\sqrt{3}} \left( \int_M u^4 \phi^4 \, dV_g \right)^{1/2} \left( \int_M u^2 \, dV_g \right)^{1/2}. \]

Assume that

\[ c \left( \int_M |W|^2 \, dV_g \right)^{1/2} + \frac{2}{\sqrt{3}} \left( \int_M u^2 \, dV_g \right)^{1/2} \leq \Lambda_0. \] 
(29)

Then, three terms in the right hand side of (28) can be absorbed in the left hand side. Therefore, there exists a constant \( c' > 0 \) such that

\[ c' \left( \int_M u^4 \phi^4 \, dV_g \right)^{1/2} \leq \int_M u^2|\nabla \phi|^2 \, dV_g \] 
(30)
Now we choose $\phi$ as

$$\psi = \begin{cases} 1 & \text{on } B_t \\ 0 & \text{on } M - B_{2t} \\ |\nabla \psi| \leq \frac{t}{2} & \text{on } B_{2t} - B_t \end{cases}$$

with $0 \leq \psi \leq 1$. From (29) and (30)

$$c' \left( \int_M u^4 \phi^4 \, dV_g \right)^{1/2} \leq \frac{4}{t^2} \int_{B(2t) - B(t)} u^2 \, dV_g$$

$$\leq \frac{4}{t^2} \frac{\sqrt{3}}{2} \Lambda_0.$$  

By taking $t \to \infty$, we have $u = 0$. Therefore $(M, g)$ is Einstein.

3 Bach-flat metric with nonconstant scalar curvature

In this section, we study noncompact complete Bach-flat metric with nonconstant scalar curvature. We apply a result of the Yamabe problem on noncompact manifold to study rigidity. For a given manifold $(M, g)$, we find a conformal metric $\overline{g} = u^{4/(n-2)} g$ whose scalar curvature is zero. This is equivalent to find a solution for the following partial differential equation

$$-\Delta_g u + \frac{1}{6} R_g u = 0. \quad (33)$$

The following existence of a conformal metric with zero scalar curvature was proved by Kim [14].

**Theorem 4** Let $(M, g)$ be a noncompact complete Riemannian manifold of dimension $n \geq 3$ with scalar curvature $R$. Assume that $Q(M, g) > 0$ and $\int_M |R|^{2n/(n+2)} + |R|^{n/2} \, dV_g < \infty$. Then, there exists a conformal metric $\overline{g} = u^{4/(n-2)} g$ whose scalar curvature is zero. Moreover, $u$ satisfies the following:

$$\int_M |\nabla (u - 1)|^2 + |u - 1|^{2n/(n-2)} \, dV_g < \infty \quad (34)$$
\[
\int_M \left| \nabla (u - 1) \right|^2 + |u - 1|^{2n/(n-2)} \, dV_g \rightarrow 0 \quad \text{as} \quad \int_M |R|^{2n/(n+2)} + |R|^{n/2} \, dV_g \rightarrow 0.
\] 

(35)

By Theorem 4 and an elliptic estimation for solutions of (33), new metric \((M, \overline{g})\) in Theorem 4 is also complete (cf. [15, ch. 8]). By a standard elliptic estimation, \(C^{2,\alpha}\) norm of \(u - 1\) is bounded by \(L_{n/2}\) and \(L_{2n/(n+2)}\) norm of \(R\). Therefore, in dimension 4, if \((M, g)\) has sufficiently small \(L_2\) bound of \(|Riem|\) and \(L_{4/3}\) bound of \(R\), there is a conformal metric \(\overline{g}\) with zero scalar curvature and small \(L_2\) norm of \(|Riem\overline{g}|\) with respect to metric \(\overline{g}\). Applying Theorem 4 to new metric \((M, \overline{g})\), we have:

**Theorem 5** Let \((M, g)\) be a noncompact complete Bach-flat Riemannian 4-manifold with scalar curvature \(R\) and \(Q(M, g) > 0\). Then there exists a small number \(c_0\) such that if \(\int_M |Riem|^2 + |R|^{4/3} \, dV_g \leq c_0\), then \((M, g)\) is conformal to a flat space.

**Remark 1** The constant \(c_0\) in Theorem 4, 5 and \(c_1\) in Theorem 3 depend on \(Q(M, g)\).

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