An explicit formula of the normalized Mumford form

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Abstract: We give an explicit formula of the normalized Mumford form which expresses the second tautological line bundle by the Hodge line bundle defined on the moduli space of algebraic curves of any genus. This formula is represented by an infinite product which is a higher genus version of the Ramanujan delta function under the trivialization by normalized abelian differentials and Eichler integrals of their products. By this formula, we have a universal expression of the normalized Mumford form as a computable power series with integral coefficients by the moduli parameters of algebraic curves.

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1. Introduction

For integers $g > 1$ and $k > 0$, denote by $\mathcal{M}_g$ the moduli space of (algebraic) curves, and denote by $\lambda_{g,k}$ the $k$-th tautological line bundle on $\mathcal{M}_g$ whose each fiber is given by the determinant of the space of $k$-differentials (i.e., regular $k$-forms) on the corresponding curve. Then Mumford [15] that there exists an isomorphism

$$\mu_{g,k} : \lambda_{g,k} \sim \lambda_{g,1}^\otimes(6k^2-6k+1)$$

with certain boundary condition. Since $\mathcal{M}_g$ can be constructed as the moduli stack over $\mathbb{Z}$ (cf. [4]), $\mu_{g,k}$ is uniquely determined up to a sign, and then it is called the normalized Mumford form.

In physics and mathematics, explicit formulas of $\mu_{g,k}$ are studied by Belavin-Knizhnik [2], Verlinde-Verlinde [17], Beilinson-Manin [1], Fay [5], Matone-Volpato [12] and others, especially in the case $k = 2$ since $|\mu_{g,2}|^2$ is the Polyakov string measure (cf. [2]). The formulas given in [1, 5, 12, 17] by theta functions use points on corresponding curves and do not express the normalized form. Then it is required to give a precise formula of the normalized Mumford form without using points on curves. When $g = 1$, $\mu_{1,2}$ is essentially the Ramanujan delta function

$$e^{2\pi\sqrt{-1}z} \prod_{n=1}^{\infty} \left(1 - e^{2\pi\sqrt{-1}nz}\right)^{24} (\text{Im}(z) > 0),$$

and when $g = 2$ or $3$, $\mu_{g,2}$ becomes an integral Teichmüller modular form of degree $g$ and is expressed in [9] by the product of even theta constants.

The aim of this paper is to give a precise formula of the normalized Mumford form $\mu_{g,2}$ for any $g$ without using points on curves. Our formula is expressed by an
infinite product which is a higher genus version of the Ramanujan delta function via the trivialization by normalized abelian (i.e., 1-)differentials and Eichler integrals of their products. By this formula, one can obtain a universal expression of \( \mu_{g,2} \) as a computable power series with integral coefficients by local coordinates on \( M_g \).

This formula is obtained by combining the arithmetic Schottky uniformization theory [8, 9] with the formulas of Zograf [19, 20] and of McIntyre-Takhtajan [13] on determinants of Laplacians on Riemann surfaces. A key point of the proof is to show the fact that the exterior products of normalized \( k \)-differentials give a local generator of \( \lambda_{g,k} \) in the case \( k = 2 \) by expanding Eichler integrals of certain products of normalized abelian differentials. This fact is extended in [10] for general \( k \), and hence there exists a similar formula of any \( \mu_{g,k} \) if one can take a basis of the space of \( k \)-differentials which consists of products of normalized abelian differentials (such products exist by a theorem of Max Noether).

2. Normalized differential

2.1. Schottky uniformization. A Schottky group \( \Gamma \) of rank \( g \) is a free group \( \langle \gamma_1, \ldots, \gamma_g \rangle \) with generators \( \gamma_i \in PSL_2(\mathbb{C}) \) which map Jordan curves \( C_i \subset \mathbb{P}^1_{\mathbb{C}} = \mathbb{C} \cup \{ \infty \} \) to other Jordan curves \( C_{-i} \subset \mathbb{P}^1_{\mathbb{C}} \) with orientation reversed, where the interiors of \( C_{\pm 1}, \ldots, C_{\pm g} \) are mutually disjoint. Each element \( \gamma \in \Gamma - \{ 1 \} \) is conjugated in \( PSL_2(\mathbb{C}) \) to \( z \mapsto q_\gamma z \) for some \( q_\gamma \in \mathbb{C}^\times \) with \(|q_\gamma| < 1 \) which is called the multiplier of \( \gamma \). Therefore, one has

\[
\frac{\gamma(z) - a_\gamma}{\gamma(z) - b_\gamma} = q_\gamma \frac{z - a_\gamma}{z - b_\gamma}
\]

for some element \( a_\gamma, b_\gamma \) of \( \mathbb{P}^1_{\mathbb{C}} \) called the attractive, repulsive fixed points of \( \gamma \) respectively. Then the discontinuity set \( \Omega_\Gamma \subset \mathbb{P}^1_{\mathbb{C}} \) under the action of \( \Gamma \) has a fundamental domain \( D_\Gamma \) which is given by the complement of the union of the interiors of \( C_{\pm 1}, \ldots, C_{\pm g} \). The quotient space \( \Omega_\Gamma / \Gamma \) is a (compact) Riemann surface of genus \( g \) which we denote by \( R_\Gamma \). Furthermore, by a result of Koebe, every Riemann surface of genus \( g \) can be represented in this manner. A Schottky group \( \Gamma \) is marked if its free generators \( \gamma_1, \ldots, \gamma_g \) are fixed, and a marked Schottky group \( (\Gamma; \gamma_1, \ldots, \gamma_g) \) is normalized if \( a_{\gamma_1} = 0, b_{\gamma_1} = \infty \) and \( a_{\gamma_2} = 1 \). By definition, the Schottky space \( \mathcal{S}_g \) of degree \( g \) is the space of marked Schottky groups of rank \( g \) modulo conjugation in \( PSL_2(\mathbb{C}) \) which becomes the space of normalized Schottky groups of rank \( g \) if \( g > 1 \). Then \( \mathcal{S}_g \) is a covering space of the moduli space of Riemann surfaces of genus \( g \).

Let \( R \) be a Riemann surface of genus \( g > 1 \), and \( \{ \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \} \) be a set of standard generators of \( \pi_1(R, x_0) \) for some \( x_0 \in R \) satisfying

\[
(\alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1}) \cdots (\alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}) = 1.
\]

Then one can take a marked Schottky group \( (\Gamma; \gamma_1, \ldots, \gamma_g) \) such that \( R = R_\Gamma \) and that each \( C_k \) is homotopic to \( \alpha_k \). Therefore, there is uniquely a basis \( \varphi_1, \ldots, \varphi_g \) of holomorphic 1-forms such that \( \int_{\alpha_j} \varphi_i \) is equal to the Kronecker delta \( \delta_{ij} \), and then
the period matrix $\tau = \left( \int_{\beta_j} \varphi_i \right)$ becomes a symmetric matrix whose imaginary part is positive definite. For each Schottky group $\Gamma$ of rank $g$, $\{ \varphi_1, ..., \varphi_g \}$ is a basis of the space $H^0(R_\Gamma, \Omega^k_{R_\Gamma})$. Therefore, the Hodge line bundle $\lambda_1$ consisting of $\bigwedge^H H^0(R_\Gamma, \Omega^k_{R_\Gamma})$ becomes a holomorphic line bundle on $\mathfrak{S}_g$ with a holomorphic canonical section $\varphi_1 \wedge \cdots \wedge \varphi_g$. From the viewpoint of arithmetic geometry, we call $2\pi \sqrt{-1} \varphi (1 \leq i \leq g)$ the normalized abelian differentials.

2.2. Eichler integral and normalized differential. Assume that $g > 1$, and take an integer $k > 1$. Let $(\Gamma; \gamma_1, ..., \gamma_g)$ be a marked normalized Schottky group, and $\mathbb{C}[z]_{2k-2}$ be the $\mathbb{C}$-vector space of polynomials $f = f(z)$ of $z$ with degree $\leq 2k - 2$ on which $\Gamma$ acts as

$$\gamma(f)(z) = f(\gamma(z)) \cdot \gamma'(z)^{1-k} \quad (\gamma \in \Gamma, \ f \in \mathbb{C}[z]_{2k-2}).$$

Take $\xi_{1,k-1}$, $\xi_{2,1}, ..., \xi_{2,k-2}$, $\xi_{i,0}, ..., \xi_{i,2k-2}$ $(3 \leq i \leq g)$ as elements of the Eichler cohomology group $H^1(\Gamma, \mathbb{C}[z]_{2k-2})$ of $\Gamma$ which are uniquely determined by the condition:

$$\xi_{i,j}(\gamma) = \begin{cases} \delta_{2l}(z^i l - 1)^j & (i = 2), \\ \delta_{2l}z^j & (i \neq 2) \end{cases}$$

for $1 \leq l \leq g$. Then it is shown in [13, Section 4] that the Eichler integral

$$\Psi_{g,k}(\psi, \xi) := \frac{1}{2\pi \sqrt{-1}} \sum_{i=1}^{g} \oint_{C_i} \psi \cdot \xi(\gamma_i)dz$$

for $\psi(dz)^k \in H^0(R_\Gamma, \Omega^k_{R_\Gamma})$, $\xi \in H^1(\Gamma, \mathbb{C}[z]_{2k-2})$ is a non-degenerate pairing on

$$H^0(R_\Gamma, \Omega^k_{R_\Gamma}) \times H^1(\Gamma, \mathbb{C}[z]_{2k-2}).$$

Denote by

$\{ \psi_{1,k-1}, \psi_{2,1}, ..., \psi_{2,k-2}, \psi_{i,0}, ..., \psi_{i,2k-2} \ (3 \leq i \leq g) \}$

the basis of $H^0(R_\Gamma, \Omega^k_{R_\Gamma})$ dual to $\{ \xi_{l,m} \}$, namely $\Psi_{g,k}(\psi_{i,j}, \xi_{l,m}) = \delta_{il} \cdot \delta_{jm}$, and call $\psi_{i,j}$ normalized $k$-differentials.

Remark. Since $-\pi \cdot \Psi_{g,k}$ is the pairing given in [13, 4.1],

$$\left\{ -\frac{\psi_{1,k-1}}{\pi}, -\frac{\psi_{2,1}}{\pi}, ..., -\frac{\psi_{2,k-2}}{\pi}, -\frac{\psi_{i,0}}{\pi}, ..., -\frac{\psi_{i,2k-2}}{\pi} \ (3 \leq i \leq g) \right\}$$

is the natural basis for $k$-differentials defined in [13].

In what follows, put

$$\left\{ \psi_{1}^{(k)}, ..., \psi_{(2k-1)(g-1)}^{(k)} \right\} = \left\{ \psi_{1,k-1}, \psi_{2,1}, ..., \psi_{2,k-2}, \psi_{i,0}, ..., \psi_{i,2k-2} \ (3 \leq i \leq g) \right\},$$
and $\psi^{(k)} = \psi_1^{(k)} \wedge \cdots \wedge \psi_{(2k-1)(g-1)}^{(k)}$.

3. Explicit Mumford form

3.1. Normalized Mumford form. For an integer $g > 1$, let $\overline{M}_g$ denote the moduli stack over $\mathbb{Z}$ of stable curves of genus $g$, and $\mathcal{M}_g$ denotes the open substack of $\overline{M}_g$ classifying proper smooth curves of genus $g$ (cf. [4]). Denote by $\overline{M}_g^{an}$ and $\mathcal{M}_g^{an}$ the complex orbifolds associated with $\overline{M}_g$ and $\mathcal{M}_g$ respectively. By definition, there exists the universal curve $\pi : \mathcal{C}_g \to \mathcal{M}_g$. Then from the relative dualizing sheaf $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ and the complement $\partial \mathcal{M}_g = \overline{M}_g - \mathcal{M}_g$ of $\mathcal{M}_g$, one can obtain the following line bundles over $\mathcal{M}_g$:

$$
\lambda_{g,k} := \det R\pi_* \left( \omega_{\mathcal{C}_g/\mathcal{M}_g} \right),
$$

$$
\delta_g := \mathcal{O}_{\overline{M}_g} (\partial \mathcal{M}_g).
$$

Furthermore, let $\kappa_g$ be the line bundle over $\overline{M}_g$ defined as the following Deligne pairing:

$$
\kappa_g = \left< \omega_{\mathcal{C}_g/\overline{M}_g}, \omega_{\mathcal{C}_g/\mathcal{M}_g} \right>.
$$

Then it is known (cf. [3, 6, 7, 18]) that these line bundles have canonical hermitian metric over $\mathcal{M}_g^{an}$ which is the Quillen metric [16] for $\lambda_{g,k}$. Furthermore, put

$$
d_k = 6k^2 - 6k + 1, \quad a(g) = (2g - 2) \left( -12\zeta_\mathbb{Q}'(-1) + \frac{1}{2} \right),
$$

where $\zeta_\mathbb{Q}'(-1)$ denotes the derivative of the Riemann zeta function $\zeta_\mathbb{Q}$ at $-1$. Then it is shown in [3, 6, 7, 18] that there exists a unique (up to a sign) isomorphism

$$
\rho_{g,k} : \lambda_{g,k}^{\otimes 12} \xrightarrow{\sim} \kappa_g^{\otimes d_k} \otimes \delta_g \cdot e^{a(g)}
$$

between the line bundles over $\overline{M}_g$ which is an isometry between the line bundles over $\mathcal{M}_g^{an}$ for these hermitian structure.

**Theorem 3.1.** There exists a unique (up to a sign) isomorphism

$$
\mu_{g,k} : \lambda_{g,k}^{\otimes d_k} \xrightarrow{\sim} \lambda_{g,1}^{\otimes d_k} \otimes \delta_g^{\otimes (k-k^2)/2} \cdot \exp \left( (k - k^2)a(g)/2 \right)
$$

between the line bundles over $\overline{M}_g$ which is also an isometry between the line bundles over $\mathcal{M}_g^{an}$ for these hermitian structure. We call $\mu_{g,k}$ the Mumford isomorphism or the normalized Mumford form.

**Proof.** Mumford [15] shows the existence of $\mu_{g,k}$, and since $\rho_{g,k} = \rho_{g,1}^{\otimes d_k} \circ \mu_{g,k}^{\otimes 12}$ is an isometry. □

3.2. Arithmetic Schottky uniformization. The theory of arithmetic Schottky uniformization given in [9] constructs a higher genus version of the Tate curve, and
its 1-forms and periods. We review the theory for the special case concerned with universal deformations of irreducible degenerate curves.

Denote by $\Delta$ the graph with one vertex and $g$ loops. Let $x_{\pm 1}, \ldots, x_{\pm g}, y_1, \ldots, y_g$ be variables, and put

$$A_g = \mathbb{Z} \left[ \frac{1}{x_{m} - x_{n}} \right] \ (l, m, n \in \{ \pm 1, \ldots, \pm g \}, \ m \neq n),$$

$$A_\Delta = A_g[[y_1, \ldots, y_g]],$$

$$B_\Delta = A_\Delta [1/y_i \ (1 \leq i \leq g)].$$

Then it is shown in [9, Section 3] that there exists a stable curve $C_\Delta$ of genus $g$ over $A_\Delta$ which satisfies the followings:

- $C_\Delta$ is a universal deformation of the universal degenerate curve with dual graph $\Delta$ which is obtained from $\mathbb{P}^1_{A_g}$ by identifying $x_i$ and $x_{-i} \ (1 \leq i \leq g)$. The ideal of $A_\Delta$ generated by $y_1, \ldots, y_g$ corresponds to the boundary $\partial M_g = \overline{M}_g - M_g$ of $\overline{M}_g$ via the morphism $\text{Spec}(A_\Delta) \to \overline{M}_g$ associated with $C_\Delta$.

- $C_\Delta$ is smooth over $B_\Delta$, and is Mumford uniformized (cf. [14]) by the subgroup $\Gamma_\Delta$ of $\text{PGL}_2(B_\Delta)$ with $g$ generators

$$\phi_i = \left( \begin{array}{cc} x_i & x_{-i} \\ 1 & 1 \end{array} \right) \left( \begin{array}{cc} y_i & 0 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cc} x_i & x_{-i} \\ 1 & 1 \end{array} \right)^{-1} \mod (B_\Delta^\times) \ (1 \leq i \leq g).$$

Furthermore, $C_\Delta$ has the following universality: for a complete integrally closed noetherian local ring $R$ with quotient field $K$ and a Mumford curve $C$ over $K$ such that $\Delta$ is the dual graph of its degenerate reduction, there is a ring homomorphism $A_\Delta \to R$ which gives rise to $C_\Delta \otimes_{A_\Delta} K \cong C$.

- Let $\Gamma = \langle \gamma_1, \ldots, \gamma_g \rangle$ be a Schottky group of rank $g$, where each $\gamma_i$ has the attractive (resp. repulsive) fixed points $a_i$ (resp. $a_{-i}$), and it has the multiplier $q_i$. Then substituting $a_{\pm i}$ to $x_{\pm i}$ and $q_i$ to $y_i \ (1 \leq i \leq g)$, $C_\Delta$ becomes the Riemann surface $R_\Gamma$ uniformized by $\Gamma$ if $|q_i|$ are sufficiently small.

Actually, $C_\Delta$ is constructed in [9] as the quotient of a certain subspace of $\mathbb{P}^1_{B_\Delta}$ by the action of $\Gamma$ using the theory of formal schemes. Furthermore, as is shown in [11] and [8, Section 3], there exists a basis of sections

$$\omega_i = \sum_{\phi \in \Gamma_\Delta / \langle \phi_i \rangle} \left( \frac{1}{z - \phi(x_i)} - \frac{1}{z - \phi(x_{-i})} \right) dz \ (1 \leq i \leq g)$$

of the dualizing sheaf $\omega_{C_\Delta/A_\Delta}$ on $C_\Delta$ with coefficients in

$$A_g \left[ \prod_{k=1}^{g} \frac{1}{(z-x_k)(z-x_{-k})} \right] [[y_1, \ldots, y_g]].$$
These ω_i are regarded as the universal expression of normalized abelian differentials since ω_i|_{x=±h,y_h=q_h} = 2π√−1φ_i on R_Γ. Put

ω = ω_1 ∧ · · · ∧ ω_g.

In the case when we consider the Schottky space S_g of degree g > 1 as the moduli space of normalized Schottky groups, we assume that the above φ_1, ..., φ_g are normalized by putting x_1 = 0, x_−1 = ∞, namely,

φ_1 = \begin{pmatrix} 1 & 0 \\ 0 & y_1 \end{pmatrix} \text{ mod } (B^\wedge_\Delta),

and by putting x_2 = 1. Then as is shown in [9, 1.1], the corresponding generalized Tate curve C_\Delta is defined over ˜A_\Delta = ˜A_g[[y_1, ..., y_g]], where ˜A_g is obtained from A_g by deleting x_−1 and putting x_1 = 0, x_2 = 1. Therefore, one has the associated morphism Spec(˜A_\Delta) → M_g.

3.3. Explicit formula. For a Schottky group Γ such that the Hausdorff dimension δ_Γ of limit set of Γ satisfies δ_Γ < 1, the Zograf infinite product F_1(Γ) is defined as the following absolutely convergent infinite product:

\prod_{\{γ\}} \prod_{m=0}^{∞} \left( 1 - q_γ^{1+m} \right),

where \{γ\} runs over primitive conjugacy classes in Γ − \{1\}, and q_γ denotes the multiplier of γ. For an integer k > 1 and a marked normalized Schottky group (Γ; γ_1, ..., γ_g), the McIntyre-Takhtajan infinite product F_k(Γ; γ_1, ..., γ_g) is defined as the absolutely convergent infinite product

(1 - q_{γ_1})^2 \cdots (1 - q_{γ_1}^{k-1})^2 \left( 1 - q_{γ_2}^{k-1} \right) \prod_{\{γ\}} \prod_{m=0}^{∞} \left( 1 - q_γ^{k+m} \right).

Proposition 3.2. Let the notation be as above. Then the infinite products

\prod_{\{γ\}} \prod_{m=0}^{∞} \left( 1 - q_γ^{1+m} \right), \quad (1 - q_{γ_1})^2 \cdots (1 - q_{γ_1}^{k-1})^2 \left( 1 - q_{γ_2}^{k-1} \right) \prod_{\{γ\}} \prod_{m=0}^{∞} \left( 1 - q_γ^{k+m} \right)

have universal expression as invertible elements of ˜A_\Delta which we denote by F_1, F_k respectively.

Proof. Let (Γ; γ_1, ..., γ_g) be a normalized Schottky group, and for i = 1, ..., g, put γ_−i = γ_i^{-1}. Then by Proposition 1.3 of [9] and its proof, if γ ∈ Γ − \{1\} has the reduced expression γ_σ(1) · · · γ_σ(l) (σ(i) ∈ \{±1, ..., ±g\}) such that σ(1) ≠ −σ(l), then its multiplier q_γ has universal expression as an element of ˜A_\Delta divisible by y_{σ(1)} · · · y_{σ(l)}. Therefore, the assertion holds. □
Proposition 3.3. There exists a nonzero constant \( c(g,k) \) such that
\[
c(g,k) \cdot \mu_{g,k}(\psi^{(k)}) = \frac{F_{d_k}}{F_k} \omega^\otimes d_k.
\]

Proof. By Theorem 3.1, \( \mu_{g,k} \) gives rise to an isometry
\[
\lambda_{g,k} \sim \lambda_{g,k} \cdot \exp \left( (k-k^2)a(g)/2 \right)
\]
between the metrized tautological line bundles with Quillen metric on \( M_g \). As is shown by Zograf [19, 20], \( F_1 \) can be extended to a holomorphic function on the Schottky space \( \mathfrak{S}_g \) which we denote by the same symbol. Let \( S_L \) denote the classical Liouville action given in Zograf-Takhtajan [21]. Then by the formula of Zograf [19, 20],
\[
\| \omega \|_Q \cdot |F_1| = c_g \cdot \exp \left( \frac{S_L}{24\pi} \right),
\]
and by the formula of McIntyre-Takhtajan [13],
\[
\| \psi^{(k)} \|_Q \cdot |F_k| = c_{g,k} \cdot \exp \left( \frac{S_L}{24\pi} \right)^{d_k},
\]
where \( \| \cdot \|_Q \) denotes the Quillen metric, \( c_g \) (resp. \( c_{g,k} \)) means constants depending only on \( g \) (resp. \( g,k \)). Therefore, there exists a holomorphic function \( c(g,k) \) on \( \mathfrak{S}_g \) satisfying the above formula such that \( |c(g,k)| \) is a constant function. Since \( \mathfrak{S}_g \) is a connected complex manifold, \( c(g,k) \) is also a constant function. □

We consider the case \( k = 2 \), and give an explicit formula of \( \mu_{g,2} \). Put
\[
\left\{ \omega_1^{(2)}, \ldots, \omega_3^{(2)(g-1)} \right\} = \left\{ \omega_1^2 (1 \leq l \leq g), \omega_1 \omega_1 (2 \leq l \leq g), \omega_2 \omega_1 (3 \leq l \leq g) \right\},
\]
and \( \omega_2 = \omega_1 \wedge \cdots \wedge \omega_3^{(2)(g-1)} \). Let \( \zeta_{i,j} \) (\( 1 \leq i \leq g, 0 \leq j \leq 2 \)) be the map from the set \( \{ \phi_1, \ldots, \phi_g \} \) of generators of \( \Gamma_\Delta \) into \( \tilde{A}_g[z] \) defined as
\[
\zeta_{i,j}(\phi_l) = \delta_{il}(z-x_i)^j \ (1 \leq l \leq g).
\]
Since the coefficients of \( \omega_1^{(2)}, \ldots, \omega_3^{(2)(g-1)} \) belong to \( \tilde{A}_\Delta \), the residue theorem implies that there exists an element \( \Psi_{g,2} \left( \omega_m^{(2)}, \zeta_{i,j} \right) \) of \( \tilde{A}_\Delta \) such that
\[
\Psi_{g,2} \left( \omega_m^{(2)}, \zeta_{i,j} \right) \bigg|_{x_{\pm h}=a_{\pm h}, y_h=q_h} = \Psi_{g,2} \left( \omega_m^{(2)} \big|\big| x_{\pm h}=a_{\pm h}, y_h=q_h, \zeta_{i,j} \big|\big| x_{h}=a_{h} \right).
\]
Therefore, one can define \( \det(\Lambda_{g,2}) \in \tilde{A}_\Delta \) as the determinant of a \( 3(g-1) \times 3(g-1) \)-matrix \( \Lambda_{g,2} \) consisting of the values of \( \Psi_{g,2} \) on
\[
\left\{ \omega_1^{(2)}, \ldots, \omega_3^{(2)(g-1)} \right\} \times \{ \zeta_{1,1}, \zeta_{1,2}, \zeta_{1,0}, \zeta_{2,1}, \zeta_{i,2} (3 \leq i \leq g) \}.
\]
Note that \( \det(\Lambda_{g,2}) \) gives (up to a sign) the determinant of the matrix consisting of the values of \( \Psi_{g,2} \) on

\[
\{ \omega_1^{(2)}, \ldots, \omega_{g(g-1)}^{(2)} \} \times \{ \xi_{1,1}, \xi_{2,1}, \xi_{2,2,0}, \xi_{i,1}, \xi_{i,2} \ (3 \leq i \leq g) \},
\]

and hence it gives the universal expression of the values of \( \det(\Psi_{g,2}) \) on the products of normalized abelian differentials.

**Theorem 3.4.** Let the notation be as above. Then

\[
\mu_{g,2}(\omega^{(2)}) = \pm \det(\Lambda_{g,2}) \frac{F_{13}^{1}}{F_2} \omega^{\otimes 13}.
\]

**Proof.** Since \( \{ \omega_i^{(2)} \} \) is a basis of the space of regular 2-forms on genus 2 or non-hyperelliptic curves, \( \det(\Lambda_{g,2}) \) is non-zero and \( \omega^{(2)}/\det(\Lambda_{g,2}) \) gives the exterior product \( \psi^{(2)} \) of the normalized 2-differentials on \( \bar{\mathcal{S}}_g \). Then by Theorem 3.1 and Propositions 3.2, 3.3, \( \mu_{g,2}(\omega^{(2)}) \), \( \det(\Lambda_{g,2})F_{13}^{1}F_{2}^{-1}\omega^{\otimes 13} \) are represented by elements of \( \tilde{A}_{\Delta} \), and are equal to each other up to a non-zero constant. Hence this constant is a rational number. Furthermore, Proposition 3.2 implies that \( F_{13}^{1}F_{2}^{-1}\omega^{\otimes 13} \) is primitive, namely is not congruent to 0 modulo any prime number. In the following, we will prove that \( \det(\Lambda_{g,2}) \) is primitive in \( \tilde{A}_{\Delta} \) by calculating its leading term. Then \( \omega_1^{(2)}, \ldots, \omega_{g(g-1)}^{(2)} \) are linearly independent over the quotient field of \( \tilde{A}_{\Delta} \otimes K \) for a field \( K \) of any characteristic, and hence \( \mu_{g,2}(\omega^{(2)}) \) is primitive. Therefore, the assertion follows from that \( \det(\Lambda_{g,2}) \) is primitive in \( \tilde{A}_{\Delta} \).

If \( \psi(dz)^2 \) is a 2-differential on a Schottky uniformized Riemann surface \( R_I \), then

\[
\Psi_{g,2}(\psi, \zeta_{i,j}) \]

is the sum of the residues of \( \psi(z - x_i)^j dz \) in the interior of \( C_l \). By [8, Proposition 3.2],

\[
\omega_i = \left( \frac{1}{z-x_i} - \frac{1}{z-x_i} + \sum_{\phi \in \Gamma_\Delta/\langle \phi_i \rangle - \{1\}} \frac{\phi(x_i) - \phi(x_{-i})}{(z-\phi(x_i))(z-\phi(x_{-i}))} \right) dz,
\]

and \( \phi(x_i) - \phi(x_{-i}) \in \prod_{j=1}^{l} y_{\sigma(j)} \cdot \tilde{A}_{\Delta} \) if \( \phi \) has the reduced expression \( \phi_{\sigma(1)} \cdots \phi_{\sigma(l)} \), where \( \sigma(j) \in \{ \pm 1, \ldots, \pm g \}, \phi_{-\sigma(j)} = \phi_{\sigma(j)}^{-1} \) and \( \sigma(l) \neq \pm i \). Denote by \( I_{\Delta} \) the ideal of \( \tilde{A}_{\Delta} \) generated by \( y_1, \ldots, y_g \). Then we have

\[
\Psi_{g,2}(\omega_i^2, \zeta_{i,1}) = \text{Res}_{x_i} \left( \frac{z-x_i}{(z-x_i)^2} \right) = 1 \mod (I_{\Delta}),
\]

and for \( l \neq i \),

\[
\Psi_{g,2}(\omega_i\omega_l, \zeta_{i,0}) = \text{Res}_{x_i} \left( \frac{1}{z-x_i} - \frac{1}{z-x_l} \right) \left( \frac{1}{z-x_l} - \frac{1}{z-x_l} \right)
\]

\[
= \frac{x_l - x_{-l}}{(x_l - x_i)(x_i - x_{-l})} \mod (I_{\Delta}).
\]
If \( l \neq i \), then
\[
\phi_l(x_l)^{\pm 1} \equiv x_{\pm i} + \frac{(x_l - x_{\pm i})(x_{\pm i} - x_i)}{(x_l - x_{\pm i})} y_i \mod (I_3^2),
\]
and hence
\[
\Psi_{g,2} (\omega l \omega_l, \zeta_{i,2}) = \frac{\phi_l(x_l)}{z - \phi_l(x_l)} \left( \frac{1}{z - x_i} - \frac{1}{z - x_{-i}} \right) \left( \frac{1}{z - \phi_l(x_l)} - \frac{1}{z - \phi_l(x_{-l})} \right) (z - x_l)^2
+ \frac{\phi_l(x_{-l})}{z - \phi_l(x_{-l})} \left( \frac{1}{z - x_i} - \frac{1}{z - x_{-i}} \right) \left( \frac{1}{z - \phi_l(x_l)} - \frac{1}{z - \phi_l(x_{-l})} \right) (z - x_{-l})^2
+ \cdots
= \frac{\phi_l(x_l)}{z - \phi_l(x_l)} \left( \frac{1}{z - x_i} - \frac{1}{z - x_{-i}} \right) \left( \frac{1}{z - \phi_l(x_l)} - \frac{1}{z - \phi_l(x_{-l})} \right) (z - x_l)^2
+ \cdots
= \phi_l(x_l) - \phi_l(x_{-l}) + \cdots
= \frac{(x_l - x_{-l})(x_i - x_{-i})^2}{(x_l - x_{-i})(x_{-l} - x_{-i})} y_i + \cdots.
\]
Furthermore, for \( l, m \neq i \),
\[
\Psi_{g,2} (\omega l \omega_m, \zeta_{i,2}) = \frac{\phi_l(x_l)}{z - \phi_l(x_l)} \left( \frac{1}{z - x_m} - \frac{1}{z - x_{-m}} \right) (z - x_l)^2
- \frac{\phi_l(x_{-l})}{z - \phi_l(x_{-l})} \left( \frac{1}{z - x_m} - \frac{1}{z - x_{-m}} \right) (z - x_{-l})^2
+ \cdots
= \frac{1}{(x_l - x_m) - (x_l - x_{-m})} \left\{ (\phi_l(x_l) - x_i) - (\phi_l(x_{-l}) - x_i) \right\}^2
+ \cdots
= \frac{(x_l - x_{-l})(x_m - x_{-m})}{(x_l - x_m)(x_{-l} - x_{-m})} \left\{ (x_l - x_i)^2 - (x_{-l} - x_i)^2 \right\} y_i^2
+ \cdots
= \frac{(x_l - x_{-l})(x_l - x_{-l})}{(x_l - x_{-i})(x_{-l} - x_{-i})} \left\{ (x_l - x_i)^2 - (x_{-l} - x_i)^2 \right\} y_i^2 + \cdots.
\]
Therefore, we have
\[
\begin{align*}
\Psi_{g,2}(\omega_1, \zeta_{i,0}) &\equiv \frac{\delta_{il}(x_1 - x_{-1})}{(x_i - x_1)(x_i - x_{-1})} \mod (I_\Delta) \quad (2 \leq l \leq g), \\
\Psi_{g,2}(\omega_2, \zeta_{i,0}) &\equiv \frac{\delta_{il}(x_2 - x_{-2})}{(x_i - x_2)(x_i - x_{-2})} \mod (I_\Delta) \quad (3 \leq l \leq g)
\end{align*}
\]
for \(3 \leq i \leq g\),
\[
\begin{align*}
\Psi_{g,2}(\omega_1^2, \zeta_{i,1}) &\equiv \delta_{il} \mod (I_\Delta) \quad (1 \leq l \leq g), \\
\Psi_{g,2}(\omega_1\omega_2, \zeta_{i,1}) &\equiv 0 \mod (I_\Delta) \quad (2 \leq l \leq g), \\
\Psi_{g,2}(\omega_2\omega_2, \zeta_{i,1}) &\equiv 0 \mod (I_\Delta) \quad (3 \leq l \leq g)
\end{align*}
\]
for \(1 \leq i \leq g\), and
\[
\begin{align*}
\Psi_{g,2}(\omega_1\omega_1, \zeta_{i,2}) &\equiv \frac{\delta_{il}(x_1 - x_{-1})(x_i - x_{-1})^2}{(x_i - x_1)(x_i - x_{-1})} y_i \mod (I_\Delta^2) \quad (2 \leq l \leq g), \\
\Psi_{g,2}(\omega_2\omega_1, \zeta_{i,2}) &\equiv \frac{\delta_{il}(x_2 - x_{-2})(x_i - x_{-1})^2}{(x_i - x_2)(x_i - x_{-2})} y_i \mod (I_\Delta^2) \quad (i \neq 2, \ 3 \leq l \leq g)
\end{align*}
\]
for \(2 \leq i \leq g\). Hence the leading term of \(\det(\Lambda_{g,2}) \in \tilde{A}_\Delta\) is \(\pm \prod_{i=2}^{g} \tau_i\), where
\[
\tau_2 \equiv \Psi_{g,2}(\omega_1\omega_1, \zeta_{i,2}) \mod (I_\Delta) = \frac{(x_1 - x_{-1})(x_2 - x_{-2})^2}{(x_1 - x_2)(x_{-1} - x_{-2})} y_2,
\]
and for \(3 \leq i \leq g\),
\[
\tau_i \equiv \Psi_{g,2}(\omega_i\omega_2, \zeta_{i,0}) \Psi_{g,2}(\omega_i\omega_1, \zeta_{i,2}) - \Psi_{g,2}(\omega_i\omega_1, \zeta_{i,0}) \Psi_{g,2}(\omega_i\omega_2, \zeta_{i,2}) \mod (I_\Delta^2)
\]
\[
= \frac{1}{(x_i - x_1)(x_i - x_{-1})} - \frac{(x_i - x_2)(x_i - x_{-2})}{(x_i - x_2)(x_i - x_{-2})(x_i - x_{-1})(x_i - x_{-1})}
\times \frac{(x_1 - x_{-1})(x_2 - x_{-2})(x_i - x_{-1})^2}{(x_i - x_2)(x_i - x_{-2})} y_i.
\]
Therefore, this product is primitive in \(\tilde{A}_\Delta\), and hence the assertion holds. \(\square\)

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