Fourier transform acting on the functions
defined in the infinite LCA groups.

A Grand Lebesgue Spaces approach.

Ostrovsky E., Sirota L.

Israel, Bar - Ilan University, department of Mathematic and Statistics, 59200,
E - mails: eugostrovsky@list.ru, sirota3@bezeqint.net

Abstract

We derive in this article the exact norm in the Grand Lebesgue Spaces (GLS)
estimates for Fourier transform acting on the functions defined in the infinite local
compact Abelian (LCA) group, compact or discrete.

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and dual Haar measures, distance, Fourier transform, Lebesgue - Riesz and Grand
Lebesgue Spaces (GLS), fundamental functions, compactness, discreetness, reaar-
nagement invariant space, exponential Orlicz spaces, natural function.

1 1. Definitions. Notations. Previous results. Statement of problem.

Let \( X \) be infinite local compact Abelian (LCA) group, compact or discrete; \( T \)
be unit circle on the plane,

\[
Y = \{ \gamma : X \to T \} \tag{1.0}
\]

be its dual group, \( \alpha(\cdot) \) be the Haar’s measure on the group \( X \); \( \beta(\cdot) \) be the
Haar’s measure on the dual group \( Y \); \( F[f](\cdot) = \hat{f}(\cdot) \) be the Fourier transform of
the measurable function \( f : X \to C : \)

\[
F[f](\gamma) = \hat{f}(\gamma) \overset{def}{=} \int_X f(x) \gamma(-x) \alpha(dx). \tag{1.1}
\]

The reciprocal norming of the Haar’s measures \( \alpha, \beta \) will be presumed such that

\[
f(x) = \int_Y \hat{f}(\gamma) \gamma(x) \beta(d\gamma). \tag{1.2}
\]

Put also
A := α(X),  B := β(Y).

Recall that the classical Lebesgue - Riesz \( L_p = L_p(X), \ p \geq 1 \) norm for the (measurable) function \( f : X \to C \) is defined as ordinary by the formula

\[
|f|_{L_p(X)} = |f|_p := \left[ \int_X |f(x)|^p \alpha(dx) \right]^{1/p}.
\]

The correspondent \( L_q = L_q(Y), \ q \geq 1 \) norm for the Fourier transform of the function \( f(\cdot) \) has by definition a form

\[
|\hat{f}|_{L_q(Y)} = |\hat{f}|_q := \left[ \int_Y |\hat{f}(\gamma)|^q \beta(d\gamma) \right]^{1/q}.
\]

Denote for each number \( p, \ p > 1 \) by \( p' \) its conjugate number:

\[
p' = \frac{p}{p-1}, \ \Leftrightarrow \frac{1}{p} + \frac{1}{p'} = 1.
\]

Analogously \( q' = q/(q-1), \ q > 1. \)

Denote following the authors of the very interest article [7] by \( K(p,q) \) the norm of the Fourier transform in the \( L_q(Y) \to L_p(X) \) sense:

\[
K(p,q) := \sup_{|f|_p = 1} |\hat{f}|_q = \sup_{0 \neq f \in L_p(X)} \frac{|\hat{f}|_q(Y)}{|f|_p(X)}
\]

so that

\[
|\hat{f}|_q(Y) \leq K(p,q) |f|_p(X).
\]

Let us introduce the following domain

\[
Q := \{(p,q) \in (0,\infty)^2 : 1/p + 1/q \leq 1, \ q \geq 2\}.
\]

Mokshay Madiman and Peng Xu in the article [7] calculated the exact value of the variable \( K(p,q) \). Namely, they proved that if \( (p,q) \in Q \), then

\[
K(p,q) = A^{1-1/p'-1/q}
\]

and \( K(p,q) = \infty \) otherwise.

The ”conjugate” proposition has the following form. Define the following domain and variable:

\[
\hat{Q} := \{(p,q) : p \in (0,2), \ 1/p + 1/q \geq 1, \ p \leq 2\};
\]

\[
\hat{K}(q,p) := B^{1/p+1/q-1}, \ (q,p) \in \hat{Q}; \ \hat{K}(q,p) = \infty, \ (q,p) \notin \hat{Q}; \quad (1.6)
\]

then
\[ \hat{K}(q,p) = \sup_{0 \neq f \in L^p(X)} \frac{|f|_p(X)}{|\hat{f}|_q(Y)}, \quad (1.7) \]

so that

\[ |f|_p(X) \leq B^{1/p+1/q-1} |\hat{f}|_q(Y), \quad (p,q) \in \hat{Q}, \quad (1.8) \]

and the “constant” \( B^{1/p+1/q-1} \) is the best possible.

Our target in this article is to extend the estimates (1.5) - (1.8) into the case when instead the classical Lebesgue - Riesz spaces \( L_p, L_q \) stands the so - called Grand Lebesgue Spaces (GLS).

2 2. Grand Lebesgue Spaces (GLS). Fundamental functions.

Let \( Z = (Z, M, \mu) \) be measurable space with non - trivial sigma - finite measure \( \mu \). Let also \( \psi = \psi(p), \ p \in [1, b], \ b = \text{const} \in (1, \infty] \) (or \( p \in [1, \hat{b}] \) ) be certain bounded from below: \( \inf \psi(p) > 0 \) continuous inside the semi - open interval \( p \in [1, b) \) numerical function. We can and will suppose \( b = \sup\{p, \psi(p) < \infty\} \), so that \( \text{supp } \psi = [1, b) \) or \( \text{supp } \psi = [1, b] \). The set of all such a functions will be denoted by \( \Psi(b) = \{\psi(\cdot)\} ; \Psi := \Psi(\infty) \).

By definition, the (Banach) Grand Lebesgue Space (GLS) space \( G\psi = G\psi(b) \) consists on all the complex numerical valued measurable functions \( \zeta \) defined on our measurable space and having a finite norm

\[ ||\zeta|| = ||\zeta||_{G\psi} \overset{def}{=} \sup_{p \in [1,b]} \left\{ \frac{|\zeta|_p}{\psi(p)} \right\}, \quad (2.0) \]

here

\[ |\zeta|_p = |\zeta|_{L_p(Z)}. \]

These spaces are Banach functional space, are complete, and rearrangement invariant in the classical sense, see [1], chapters 1, 2; and were investigated in particular in many works, see e.g. [3], [4], [5], [6], [8], [15], [16], [17]. We refer here some used in the sequel facts about these spaces and supplement more.

Suppose temporarily that the measure \( \mu \) is probabilistic: \( \mu(A) = P(A) ; \mu(Z) = P(Z) = 1 \).

It is known that if \( \zeta \neq 0 \),

\[ P(|\zeta| > y) \leq \exp\left( -v_\psi^*(\ln(y/||\zeta||) \right), \quad y \geq e \cdot ||\zeta||, \quad (2.2) \]

Conversely, the last inequality may be reversed in the following version: if
\[ P(|\zeta| > y) \leq \exp \left( -v_\psi^*(\ln(y/K)) \right), \quad y \geq e \cdot K, \quad K = \text{const} \in (0, \infty), \]
and if the function \( v_\psi(p), \; 1 \leq p < \infty \) is positive, continuous, convex and such that
\[ \lim_{p \to \infty} \ln \psi(p) = \infty, \]
then \( \zeta \in G\psi \) and besides \( ||\zeta|| \leq C(\psi) \cdot K \):
\[ ||\zeta||_{G\psi} \leq C_1 ||\zeta||_{L(M)} \leq C_2 ||\zeta||_{G\psi}, \; 0 < C_1 < C_2 < \infty. \]  

(2.3)

Furthermore, let now \( \eta = \eta(z), \; z \in W \) be arbitrary family of random variables defined on any set \( W \) such that
\[ \exists b \in (1, \infty) \; \forall p \in [1, b) \Rightarrow \psi_W(p) := \sup_{z \in W} |\eta(z)|_p < \infty. \]  

(2.4)

The function \( p \to \psi_W(p) \) is named as a \textit{natural} function for the family of random variables \( W \). Obviously,
\[ \sup_{z \in W} ||\eta(z)||_{G\Psi_W} = 1. \]

\textbf{Definition 2.1.} The \textit{fundamental function} for GLS \( G\psi_b \), \( \phi[G\psi](\delta), \; \delta \in (0, \infty) \) is defined by a formula
\[ \phi[G\psi](\delta) := \sup_{p \in [1,b)} \left\{ \delta^{1/p} \right\} \psi(p). \]  

(2.5)

This notion play a very important role in the Functional Analysis, theory of Fourier series, Operator Theory, Theory of Random Processes etc., see the classical monograph [1]. For the GLS this function was investigated in the preprint [17].

\textbf{Definition 2.2.} The \textit{low truncated fundamental function} for the GLS \( G\psi_b \), namely, \( \phi_s[G\psi](\delta), \; \delta \in (0, \infty), 0 < s < b \) is defined by a formula
\[ \phi_s[G\psi](\delta) := \sup_{p \in [s,b)} \left\{ \delta^{1/p} \right\} \psi(p), \; 1 \leq s < b. \]  

(2.5a)

\textbf{Definition 2.3.} The \textit{upper truncated fundamental function} for the GLS \( G\psi_b \), indeed: \( \phi^s[G\psi](\delta), \; \delta \in (0, \infty), 0 < s < b \) is defined by a formula
\[ \phi^s[G\psi](\delta) := \sup_{p \in [s,b)} \left\{ \delta^{1/p} \right\} \psi(p), \; 1 \leq s < b. \]  

(2.5b)
3 3. Main results.

A. Compact case.
Suppose for beginning that the LCA group $X$ is compact. Then $A = \alpha(X) < \infty$ and the dual group is discrete with correspondent counting Haar’s measure $\beta$.

Assume in this sub-section that the measurable function $f : X \to C$ belongs to some $G_{\psi_b}$ space:

$$|f|_p \leq \psi(p) \cdot ||f||_{G_{\psi_b}}, \ b = \text{const} \in (1, \infty]; \ p \in [1, b). \quad (3.0)$$

We can and will suppose without loss of generality $||f||_{G_{\psi_b}} = 1$. Of course, the function $\psi = \psi(p)$ may be picked as a natural function for the function $f : \psi(p) = |f|_p(X)$, if it is non-trivial.

Let $(p, q)$ be the numbers from the set $Q$; this implies $q \geq 2$, $q \geq p'$; $\iff q \geq \max(2, p')$.

(3.1)

It follows from the result of Mokshay Madiman and Peng Xu [7], (1.5)

$$|\hat{f}|_q \leq A^{1-1/p - 1/q} \cdot |f|_p \leq A^{1-1/p - 1/q} \psi(p), \quad (3.2a)$$

or equally

$$A^{1/q - 1}|\hat{f}|_q \leq A^{-1/p} \psi(p). \quad (3.2b)$$

We deduce from (3.2a) taking maximum over $p$ and denoting $t(q) = \min(q', 2)$:

$$A^{1/q - 1}|\hat{f}|_q \leq \inf_{p \leq t(q)} [A^{-1/p} \psi(p)]. \quad (3.3)$$

The right-hand side of (3.3) may be rewritten as follows:

$$\inf_{p \leq t(q)} [A^{-1/p} \psi(p)] = \sup_{p \leq t(q)} \left\{\frac{A^{1/p}}{\psi(p)}\right\}^{-1} = \left\{\phi[G_{\psi}]_{t(q)}(A)\right\}^{-1},$$

therefore

$$A^{-1}|\hat{f}|_q \leq \frac{A^{1/q}}{\phi[G_{\psi}]_{t(q)}(A)}. \quad (3.4)$$

Suppose in addition that the function $\phi[G_{\psi}]_{t(q)}(A)$ allows a following representation (factorization)

$$\phi[G_{\psi}]_{t(q)}(A) = \frac{\theta_A(q)}{\nu_A(q)}, \ q \geq 2, \quad (3.5)$$

where both the functions $\theta_A(\cdot), \nu_A(\cdot)$ are from the set $G[2, d]$; $d = \text{const} \in (2, b)$.

Then the inequality (3.4) may be rewritten as follows.
\[
\frac{\|\hat{f}\|_q}{\nu_A(q)} \leq A \cdot \frac{A^{1/q}}{\theta_A(q)},
\]  
and we obtain after taking the maximum over both the sides of the inequality (3.6)  

\[
\|\hat{f}\|_{G\nu_A} \leq A \cdot \phi[G\theta_A](A).
\]  

Thus, we obtained really the following result.

**Theorem 2.1.** We conclude under formulated in this section conditions, restrictions and notations, in particular, under the condition (3.5),  

\[
\|\hat{f}\|_{G\nu_A} \leq A \cdot \phi[G\theta_A](A) \cdot \|f\|_{G\psi}.
\]  

**B. Discrete case.**

Suppose now that the LCA group \( X \) is discrete. Then \( B = \beta(Y) < \infty \) and the dual group \( Y \) is compact with correspondent Haar’s measure \( \alpha \).

Our considerations in this subsection are alike the foregoing ones in the investigation of theorem 2.1.

Assume here that the measurable function \( \hat{f} : Y \to C \) belongs to some \( G\psi_b \) space:  

\[
|\hat{f}|_q \leq \psi(q) \cdot \|f\|_{G\psi_b}, \quad b = \text{const} \in (2, \infty]; \quad q \in [2, b), .
\]  

We can and will suppose without loss of generality \( \|\hat{f}\|_{G\psi_b} = 1 \). Of course, the function \( \psi = \psi(q) \) may be choosed as a natural function for the function \( f : \psi(q) = |\hat{f}|_q(Y) \), if the last function is non-trivial.

Let \( (p, q) \) be the numbers from the set \( \hat{Q} \); this implies

\[
1 \leq p \leq 2, \quad p \leq q'; \quad 1 \leq p \leq \min(2, q').
\]  

It follows again from the second result of Mokshay Madiman and Peng Xu [7], theorem 4.1; (1.7)

\[
|f|_p \leq B^{-1+1/p+1/q} \cdot |\hat{f}|_q \leq B^{-1+1/p+1/q} \psi(q),
\]  

or equally

\[
B^{1-1/p}|f|_p \leq B^{1/q} \psi(q).
\]  

We deduce from (3.11a) taking maximum over \( p \) and denoting \( s(p) = \max(p', 2) \):

\[
B^{1-1/p}|f|_p \leq \inf_{q \geq s(p)} \left[ B^{1/q} \psi(q) \right].
\]  

The right-hand side of (3.12) may be rewritten as follows:

\[
\inf_{q \geq s(p)} \left[ B^{1/q} \psi(q) \right] = \sup_{q \geq s(p)} \left\{ \frac{B^{-1/q}}{\psi(q)} \right\}^{-1} =
\]
\[
\left\{ \phi[G\psi]^{s(p)}(B^{-1}) \right\}^{-1},
\]
therefore
\[
B^{-1}|f|_p \leq \frac{B^{1/p}}{\phi[G\psi]^{s(p)}(B^{-1})}.
\] (3.13)

Suppose in addition that the function \( \phi[G\psi]^{s(p)}(B) \) allows a following representation (factorization)
\[
\phi[G\psi]^{s(p)}(B^{-1}) = \frac{\tau_B(q)}{\kappa_B(p)}, \ p \geq q',
\] (3.14)
where both the functions \( \tau_B(\cdot) \), \( \kappa_B(\cdot) \) are from the set \( G[1,b); \ b = \text{const} \in (1,2] \).

Then the inequality (3.13) may be rewritten as follows.
\[
\frac{|f|_p}{\kappa_B(p)} \leq B^{-1} \cdot \frac{B^{-1/p}}{\tau_B(p)},
\] (3.15)
and we obtain after taking the maximum over both the sides of the inequality (3.15)
\[
\|f\|_{G\kappa_B} \leq B^{-1} \cdot \phi[G\tau_B](B^{-1}).
\] (3.16)

Thus, we obtained really the following result.

**Theorem 2.2.** We conclude under formulated in this section conditions, restrictions and notations, in particular, under the condition (3.14),
\[
\|f\|_{G\kappa_B} \leq B^{-1} \cdot \phi[G\tau_B](B^{-1}) \cdot \|f\|_{G\psi}.
\] (3.17)

**Remark 3.1.** It is easily to verify that the source estimated of Mokshay Madiman and Peng Xu \[7\] are the particular cases of obtained here (3.8) and (3.17), as long as the classical Lebesgue - Riesz spaces \( L_p \) are the particular cases of Grand Lebesgue Spaces.

This implies as a consequence that both the constants in the right - hands sides of inequalities (3.8) and (3.17) are in general case non - improvable.

4 4. Concluding remarks.

It is interest by our opinion to generalize obtained in \[?\] and in this report estimates to the non - commutative infinite local compact groups, as well as in the other rearrangement invariant spaces builded over \( X \) or \( Y \).

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