Quantum Moduli Spaces of Flat Connections

ANTON YU. ALEKSEEV\textsuperscript{1} and VOLKER SCHOMERUS\textsuperscript{2}

\textsuperscript{1} Institute of Theoretical Physics, Uppsala University, Box 803 S-75108, Uppsala, Sweden; e-mail: alekseev@teorfys.uu.se
\textsuperscript{2} II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, 22761 Hamburg, Germany; e-mail: vschomer@x4u2.desy.de

Abstract

Using the formalism of discrete quantum group gauge theory, one can construct the quantum algebras of observables for the Hamiltonian Chern-Simons model. The resulting moduli algebras provide quantizations of the algebra of functions on the moduli spaces of flat connections on a punctured 2-dimensional surface. In this note we describe some features of these moduli algebras with special emphasis on the natural action of mapping class groups. This leads, in particular, to a closed formula for representations of the mapping class groups on conformal blocks.

1 Introduction

Moduli spaces of flat connections on a Riemann surface $\Sigma$ enter physics as phase spaces of the so-called Chern-Simons gauge theory on a three dimensional manifold $\Sigma \times \mathbb{R}$. They possess a famous Poisson structure that has been studied extensively in the mathematical literature (see e.g. [1]). The information about a particular flat connection on $\Sigma$ can be encoded in its holonomies along nontrivial closed curves. Hence, one may regard points on the moduli spaces as classes of homomorphisms from the fundamental group $\pi_1(\Sigma)$ into the gauge group $G$. Such a description based on $\pi_1(\Sigma)$ furnishes a natural action of the mapping class
group of $\Sigma$ on moduli spaces. The latter is known to respect the canonical Poisson structure.

Different methods have been employed to quantize the Hamiltonian Chern-Simons theory (see e.g. [2, 3]). The combinatorial quantization developed in [3] resulted in a noncommutative algebra $\mathcal{A}_{CS}$ of ‘functions on the quantum moduli space of flat connections’. It is one feature of this approach that it follows closely the classical construction of moduli spaces with certain quantum holonomies replacing their classical counterparts (see below). The deformed universal enveloping algebras $\mathcal{G} = U_q(g)$ play the role of gauge symmetries in the quantized theory. As it is expected, the action of the mapping class group survives quantization and gives rise to automorphisms of the algebras $\mathcal{A}_{CS}$ of observables in the quantum theory.

In this note we reconstruct quantum moduli spaces of flat connections from the described properties, namely that (1) they are built up from quantum holonomies and (2) they admit an action of the mapping class group. The interplay of classical and quantum symmetries (i.e. the action of mapping class groups and deformed universal enveloping algebras) naturally leads to the algebras $\mathcal{A}_{CS}$ which were found in [3, 4] by quantizing the Poisson structure on moduli spaces of flat connections. There the analysis was essentially based on a lattice formulation of the Hamiltonian Chern-Simons theory due to V.V. Fock and A.A. Rosly [1]. Even without making this connection to the classical Chern-Simons theory explicit, our constructions below have an immediate mathematical application: they provide a closed and very elegant formula for the Reshetikhin-Turaev representations of mapping class groups.

2 Products on the Dual of $U_q(g)$

The space $\mathcal{G}'$ of linear forms on the quantum universal enveloping algebras $\mathcal{G} = U_q(g)$ can be equipped with a number of different multiplications. In this section we describe three such product structures and discuss some of their properties thereby discovering natural quantum analogues of holonomies along open and closed curves.

2.1 Quantum group algebras

It is well known that the Hopf algebra structure of $\mathcal{G}$ admits to define a natural product $a \circ b$ of two linear forms $a, b : \mathcal{G} \to \mathbb{C}$. By definition, we evaluate $a \circ b$ on elements $\xi \in \mathcal{G}$ with the help of $(a \circ b)(\xi) = (a \otimes b)(\Delta(\xi))$. The
associative algebra \( \mathcal{F} \) which emerges from this simple construction is known as *quantum group algebra*. Representations \( \tau \) of \( \mathcal{G} \) may be reinterpreted as matrices of linear forms on \( \mathcal{G} \) and hence – with the natural product on \( \mathcal{G}' \) – as matrices with elements in the non-commutative quantum group algebra \( \mathcal{F} \). Let us consider one particular representation \( \tau \) and denote the associated \( \mathcal{F} \)-valued matrix by \( U \). From the definition of \( \circ \) in terms of \( \Delta \) and the famous intertwining property \( R \Delta(\cdot) = \Delta'(\cdot)R \) one deduces the following quadratic relations:

\[
R \, U \, U = U \, U \, R .
\] (2.1)

Here \( U = U \otimes 1, \overline{U} = 1 \otimes U \) and multiplication of matrix elements with \( \circ \) is understood. \( R \) denotes the universal \( R \)-matrix of \( \mathcal{G} \) evaluated in the representation \( \tau \). In the standard example of \( \mathcal{G} = U_q(sl_2) \) we may fix \( \tau \) to be the two dimensional fundamental representation. Then the quadratic relations (2.1) become equations between products of algebra valued \( 4 \times 4 \) matrices and \( R \) is the famous \( 4 \times 4 \)-matrix-solution of the Yang-Baxter equation. When, in addition to these quadratic relations, the value of the quantum determinant is fixed to \( 1 \), we obtain a complete description of \( \mathcal{F} \). We disregard such determinant relations in the following to simplify our discussion. For a precise treatment it is preferable to assemble the matrices \( U \) into *universal elements*. The latter obey certain functoriality relations which imply quadratic relations and constrain the quantum determinant at the same time. This and other topics, such as the existence of a \( \ast \)-operation, are explained in [3, 4].

To investigate the quantum symmetries of \( \mathcal{F} \) we introduce another copy \( T \) of our matrix \( U \) such that

\[
R \, T \, T = T \, T \, R
\] (2.2)

and matrix elements of \( T \) commute with matrix elements of \( U \). Together, matrix elements of \( U, T \) generate the algebra \( \mathcal{F} \otimes \mathcal{F} \) and we can use \( T \) in defining a homomorphism \( \Phi : \mathcal{F} \rightarrow \mathcal{F} \otimes \mathcal{F} \),

\[
\Phi : \ U \mapsto U \, T .
\]

On the right hand side, \( U \, T \) is an ordinary product of algebra valued matrices. Even though it is quite elementary, let us demonstrate once that \( \Phi \) indeed extends to a homomorphism. Similar computations will then be left to the reader in what follows.

\[
\Phi( R \, U \, U ) = R \, U \, T \, U \, T = R \, U \, U \, T \, T = U \, U \, T \, T \, R
\]

\[
= U \, T \, U \, T \, R = \Phi( U \, U \, R) .
\]

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We call the above symmetry $\Phi = \Phi_R$ a right regular transformation because objects $T$ are multiplied from the right.

### 2.2 Link- and loop-algebras

One would certainly expect to find a similar left regular transformation $\Phi_L$ which maps $U$ to $T_L^{-1}U$. A short computation reveals that such a map gives rise to a homomorphism, if we replace $R$ by $R' = PRP$ in the exchange relations \(2.2\) for $T = T_L$. Let us follow a slightly different route here: we will leave the relations for $T$ untouched and manipulate the exchange relations for $U$ instead so that the resulting algebra has the desired two commuting quantum symmetries (QS) under left/right-regular transformations with $T$. To be more concrete, we introduce the link-algebra $\mathcal{J}$ that is generated by elements of an algebra valued matrix $G$ such that

\[
R' \begin{pmatrix} 1 & 2 \\ G & G \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ G & G \end{pmatrix} R \quad (2.3)
\]

$\mathcal{J}$ has QS: $\Phi_R : G \mapsto G T_R$, $\Phi_L : G \mapsto T_L^{-1}G$.

Here $T_R, T_L$ obey the exchange relations \(2.2\) and commute with each other and with matrix elements of $G$. The difference between eqs. \(2.1\) and \(2.3\) is that one of the $R$-matrices has been replaced by $R'$. We wish to emphasize that the precise meaning of eq. \(2.3\) is the same as in eq. \(2.1\). Namely, the think of $\mathcal{J}$ as being generated by elements of $G'$ subject to a new multiplication $(a \odot b)(\xi) := (a \otimes b)(R_\Delta(\xi))$. One may check that this prescription defines an associative product on $G'$ such that matrix elements of the representations obey the exchange relations \(2.3\).

The algebra $\mathcal{J}$ that is generated by the components of $G$ was constructed so that it admits two commuting quantum symmetries which we called left- and right-regular transformations. Clearly, these symmetries are remnants of the usual left- and right regular actions of a group on itself. Classical groups, however, possess another important symmetry: the adjoint action, i.e. the action of the group on itself by conjugation. It is easy to see that this symmetry is lost when we pass from the group to the quantum object $G$. In other words, the map $G \mapsto T^{-1}GT$ does not extend to a homomorphism of algebras, if $T$ and $G$ obey the relations we have specified above. But from our experience with $G$ we know already, how a manipulation of the fundamental exchange relations may influence the quantum symmetry of the corresponding noncommutative algebra. It is not too hard to
come up with new relations for an algebra valued matrix \( M \) which are consistent with the adjoint transformation that we were looking for,

\[
R' \ M \ R \ \mapsto \ \ M \ R' \ M \ R \quad (2.4)
\]

\( \mathcal{L} \) has QS: \( \Phi_{\text{Ad}} : M \mapsto T^{-1}MT \).

Matrix elements of \( M \) generate the loop-algebra \( \mathcal{L} \) and we call \( \Phi_{\text{Ad}} \) the \textit{adjoint transformation}. Again, there exits a product on the dual of \( \mathcal{G} \) which implies that representation matrices obey eq. (2.4) and, strictly speaking, it is \( \mathcal{G}' \) endowed with this product that we denote by \( \mathcal{L} \). The relations which we have just constructed are not new. In fact, they appeared in a work of Reshetikhin and Semenov-Tian-Shansky on the deformations of universal enveloping algebras. For finite dimensional semisimple modular Hopf algebras \( \mathcal{G} \), the loop-algebra \( \mathcal{L} \) is even isomorphic to \( \mathcal{G} \).[4]

Elements in the loop-algebra which transform trivially under the symmetry \( \Phi_{\text{Ad}} \) form a subalgebra \( \mathcal{L}^\mathcal{G} \subset \mathcal{L} \).[2] The quantum trace \( \text{tr}_q \) is especially designed to produce such an element \( c = \text{tr}_q(M) \in \mathcal{L}^\mathcal{G} \) from the algebra valued matrix \( M \). We will see more examples of this construction later when we consider algebras composed from several copies of \( \mathcal{L} \). In our particular case here, the element \( c = \text{tr}_q(M) \) is central, i.e. it commutes with all the matrix elements of \( M \). Further computations reveal that \( c \) generates the \textit{fusion algebra} (or \textit{Verlinde algebra} [7]) which is defined by the multiplicities in the Clebsch Gordan decomposition of \( \mathcal{G} = U_q(\mathfrak{g}) \).

The behaviour of \( G \) and \( M \) under quantum symmetry transformations may be attributed to different geometric situations. In some sense, \( G \) behaves very much like a holonomy along an open curve (link) on which gauge transformations at the two endpoints act independently from the left and from the right. The quantum symmetry of \( M \) should be compared with the transformation law of holonomies along a closed curve (loop). On such holonomies, gauge transformations act by conjugation. Our pictorial presentations display these differences and encode the symmetry properties of \( G \) and \( M \).

\[\text{In more mathematical terms, } \Phi_{\text{Ad}} \text{ is called a co-action and elements in } \mathcal{L}^\mathcal{G} \text{ are said to be co-invariant with respect to } \Phi_{\text{Ad}}.\]
3 Braided Tensor Products and Braid Groups

In the previous section we met three quite different algebraic structures. From now on, we concentrate on the loop-algebra and use it as a basic building block to construct more complicated algebras. New classical symmetries will emerge in addition to the quantum symmetries that we have discussed already.

3.1 Multiloop-algebras $\mathcal{L}_N$

Whereas the tensor product is one of the most fundamental constructions in algebra, it is well known to break quantum symmetries and hence becomes useless in our present context. If we want to combine several copies of $\mathcal{L}$ into one algebra while preserving the quantum symmetry, we are forced to use braided tensor products (see e.g. [9]). The $N$-fold braided tensor product $\mathcal{L}_N$ of the loop-algebra $\mathcal{L}$ is generated by matrix elements of $M_\nu$, $\nu = 1, \ldots, N$ satisfying eq. (2.4) and the exchange relations

$$R^{-1} M_\nu R M_\mu = M_\mu R^{-1} M_\nu R \quad \nu < \mu$$

$\mathcal{L}_N$ has QS: 

$$\Phi_{Ad}: M_\nu \mapsto T^{-1} M_\nu T .$$

An ordinary tensor product of loop-algebras would mean to replace all $R$’s by unit-matrices in the exchange relations of $M_\nu$ with $M_\mu$. The reader is invited to check that such commutation relations are inconsistent with the quantum symmetry. On the other hand, $\Phi_{Ad}$ preserves the exchange relations with nontrivial $R$-matrices. We call $\mathcal{L}_N$ a multiloop-algebra and write $\mathcal{L}^G_N$ for the algebra of elements in $\mathcal{L}_N$ which transform trivially under $\Phi_{Ad}$. Note that the picture we have drawn to illustrate the $N$-fold braided tensor product is in agreement with our previous rules. In particular, it has one vertex keeping track of the quantum symmetry and $N$ loops which symbolize the $N$ loop-algebras that generate $\mathcal{L}_N$.

It turns out that the multiloop-algebras $\mathcal{L}_N$ have interesting classical symmetries in addition to the quantum symmetry $\Phi_{Ad}$. In fact, elements of the braid group $B_N$ on $N$ strands act as automorphisms on $\mathcal{L}_N$. Let $\sigma_\rho$ denote the generator which corresponds to an exchange of the $\rho^{th}$ with the $(\rho + 1)^{st}$ strand. Their action on the multiloop-algebras $\mathcal{L}_N$ may be defined by

$$\sigma_\rho(M_\nu) = M_\nu, \quad \text{for all } \nu \neq \rho, \rho + 1 \quad (3.5)$$

$$\sigma_\rho(M_\rho) = M_{\rho+1}, \quad \sigma_\rho(M_{\rho+1}) \sim M_{\rho+1}^{-1} M_\rho M_{\rho+1} . \quad (3.6)$$
This is consistent with all exchange relations of $\mathcal{L}_N$ and respects the defining relations in $B_N$. The ‘$\sim$’ in the second line means ‘up to a scalar factor’. On the level of quadratic relations, such factors are certainly irrelevant and their discussion is beyond the scope of this text.

It is possible to give a more geometric interpretation of these formulas. To this end, let us consider an $N$-punctured disc $D_N$. Its fundamental group $\pi_1(D_N)$ is freely generated by $N$ elements $l_\nu$. We can represent these generators by $N$ loops which surround the punctures as in Fig. 1. It was known to Artin already that the braid group $B_N$ acts on the fundamental group $\pi_1(D_N)$ and that this action is given by

$$\sigma_\rho(l_\nu) = l_\nu \quad \text{for all } \nu \neq \rho, \rho + 1$$
$$\sigma_\rho(l_\rho) = l_{\rho+1}, \quad \sigma_\rho(l_{\rho+1}) = l_{\rho+1}^{-1} l_\rho l_{\rho+1}.$$

Automorphisms of the fundamental group and of the multiloop-algebra are denoted by the same letters. In any case, their action is formally identical. We may push this even further, if we assign algebra valued matrices $M(p)$ to arbitrary elements $p \in \pi_1(D_N)$ by the obvious recursive definition

$$M(p_1 p_2) \sim M(p_1) M(p_2) \quad \text{for all } p_i \in \pi_1(D_N).$$

Notice that the objects $M(p)$ are products of the fundamental matrices $M_\nu$ (up to scalar factors) and that they obey the same exchange relations (2.4) and transformation law under $\Phi_{\text{Ad}}$ as the elements $M_\nu$. Using this notation it is possible to lift the action of the braid group on the fundamental group to an action on the multiloop-algebra with the help of the formula

$$\sigma(M(p)) = M(\sigma(p)).$$

Let us close this subsection by comparing Fig. 1 with the picture we have drawn for $\mathcal{L}_N$: while our pictorial presentations were originally designed as a bookkeeping for the quantum symmetries of an algebra, they have turned out to encode all the classical symmetries at the same time.
3.2 Representations of the pure braid group $PB_N$

The braid group $B_N$ contains an important subgroup $PB_N$ known as the pure braid group. By definition, $PB_N$ is the kernel of the canonical homomorphism from the braid group into the symmetric group. One can show that $PB_N$ is generated by the elements $\eta_{\nu\mu}$ which are depicted in Fig. 2. In the context of multiloop-algebras, the pure braid group is singled out because its elements $\eta$ give rise to inner automorphisms of $L_N$. This means that for given $\eta \in PB_N$ there exists an element $v_\eta \in L_N$ such that

$$\eta(M) = v_\eta M v_\eta^{-1} \text{ holds for all } M \in L_n.$$ 

Moreover, when $\eta$ is one of the standard generators $\eta_{\nu\mu} \in PB_N$, it is possible to give an explicit formula for the corresponding element $v_\eta = v_{\nu\mu}$.

$$v_{\nu\mu} = V[tr_q(M l_{\nu\mu})] \sim V[tr_q(M_{\nu\mu})] \in L_N^G.$$ 

Here $V[.]$ is a universal function which does not depend on $\nu, \mu$. It is intimately related with the so-called ribbon element of $U_q(g)$ (see [4] for details).

**Proposition 1** [4] (Representations of $PB_N$) There exists an involution on $L_N^G$ such that the elements $v_\eta$ become unitary and the map $\eta \mapsto v_\eta \in L_N$ provides a unitary (projective) representation of the pure braid group $PB_N$ by elements in the multiloop-algebra.

![Figure 2: Elements $\eta_{\nu\mu}$ generate the pure braid group $PB_N$. They wrap the $\nu^{th}$ strand once around the $\mu^{th}$.](image)

![Figure 3: Dehn twists along the curves $l_{\nu\mu}$ generate the mapping class group of $D_N$. They correspond to the elements $\eta_{\nu\mu}$ of the pure braid group.](image)

This simple result admits for more interesting generalizations that we discuss in the next section. As a preparation we note that the pure braid group may be
reinterpreted as the mapping class group of the $N$-punctured disc. The mapping class group is the group of diffeomorphisms of a surface (in this case of $D_N$) modulo its identity component. It is generated by so-called Dehn twists which correspond to cutting the surface along a given circle, rotating it by $2\pi$ and gluing back into the surface. When we perform this operation on a circle $l_\nu l_\mu$ in $D_N$, we obtain an element associated with the generator $\eta_{\nu\mu}$ of the pure braid group (cf. Fig. 2 and Fig. 3). Hence, we may rephrase the main result this subsection by saying that our homomorphism $\eta \mapsto \nu_\eta$ represents the mapping class group of the $N$-punctured disc.

4 Quantum Moduli Spaces and Mapping Class Groups

We are finally in a position to generalize the theory of the preceding section to surfaces of higher genus. This leads us to the construction of quantum moduli spaces and to projective unitary representations of mapping class groups.

4.1 Theory on a torus

It is instructive to study the torus $T$ first. It has two nontrivial cycles, the $a$- and the $b$-cycle, which generate the fundamental group $\pi_1(T)$. For simplicity, let us remove a disc $D$ from the torus $T$ and look at the group $\pi_1(T \setminus D)$ which is freely generated by $a, b$. The mapping class group of $T \setminus D$ is generated by two Dehn twists $\alpha, \beta$ along the $a, b$-cycle. They act on the fundamental group according to

$$
\alpha(a) = a, \quad \alpha(b) = ba,
\beta(a) = b^{-1}a, \quad \beta(b) = b.
$$

This action obviously respects the standard commutator relation $[a, b] := ba^{-1}b^{-1}a = e$ and hence descend to the fundamental group $\pi_1(T)$ of the torus.

Next we have to understand what replaces the multiloop-algebra in the genus 1 example we are dealing with now. The basic idea is to assign two copies of the loop-algebra $\mathcal{L}$ to the two generators $a, b$ of the fundamental group. They will
be denoted by $A = M(a)$ and $B = M(b)$. We want to combine them so that the resulting algebra $\mathcal{H}$ preserves the quantum symmetry of the loop-algebras and admits an action of the mapping class group. The second requirement implies that

$$\alpha(A) = A, \quad \alpha(B) \sim BA,$$

$$\beta(A) \sim B^{-1}A, \quad \beta(B) = B$$

(4.1)

(4.2)

extend to automorphisms of $\mathcal{H}$. Observe that the formulas for the action of $\alpha, \beta$ on $\mathcal{H}$ are obtained by lifting the action of the mapping class group on the fundamental group.

With these to basic requirements one is lead to an algebra $\mathcal{H}$ which is generated by matrix elements of $A, B$ subject to the defining relations of $L$ and the exchange relation

$$R^{-1}\hat{A}R^2 = \hat{B}R'\hat{A}R$$

(4.3)

$\mathcal{H}$ has QS:

$$\Phi_{Ad}: A \mapsto T^{-1}AT, \quad \Phi_{Ad}: B \mapsto T^{-1}BT.$$ 

We call $\mathcal{H}$ the handle-algebra. A more detailed investigation reveals that elements of the mapping class group act as inner automorphisms on the handle-algebra. The action of $\alpha, \beta$ is implemented by conjugation with the elements

$$v_\alpha = V[tr_q(A)] \quad \text{and} \quad v_\beta = V[tr_q(B)].$$

Here $V$ is the same universal function we encountered in Subsection 3.2. It can be shown that the map $\alpha \mapsto v_\alpha, \beta \mapsto v_\beta$ gives rise to a projective unitary representation of the mapping class group of $T \setminus D$. One may consistently impose the constraint $BA^{-1}B^{-1}A \sim 1$ to obtain a theory for the torus $T$.

### 4.2 Theory for arbitrary genus

It should be clear by now, how to proceed in the general case of an arbitrary Riemann surface $\Sigma_{g,N}$ of genus $g$ and with $N$ punctures. The fundamental group of $\Sigma_{g,N} \setminus D$ is freely generated by $2g + N$ cycles $l_\nu, a_i, b_i$. Within $\pi_1(\Sigma_{g,N})$, these generators obey one relation

$$r = [a_g, b_g] \cdots [a_1, b_1]l_N \cdots l_1 = e.$$ 

As in the two examples we have studied, the fundamental group $\pi_1(\Sigma_{g,N} \setminus D)$ carries an action of the mapping class group of $\Sigma_{g,N} \setminus D$. Explicit formulas can be worked out.
Let us define an algebra $L_{g,N}$ to be the braided tensor product of $N$ copies of the loop-algebra $L$ and $g$ copies of the handle-algebra $H$ so that the generating matrices $M_{\nu}, A_i$ and $B_i$ ($i = 1, \ldots, g$) obey the following additional exchange relations

\begin{align}
R^{-1} M_{\nu} R A_i &= A_i R^{-1} M_{\nu} R \quad \text{for all } i, \nu \tag{4.4}
\end{align}

\begin{align}
R^{-1} A_i R A_j &= A_j R^{-1} A_i R \quad i < j \tag{4.5}
\end{align}

and similarly when $A_i$ and/or $A_j$ are replaced by $B_i, B_j$. By construction, $L_{g,N}$ contains $2g + N$ copies of $L$ which we think of as being assigned to the $2g + N$ fundamental cycles of the surface. This allows to lift the action of the mapping class group from the fundamental group to the algebra $L_{g,N}$. All the automorphisms of $L_{g,N}$ which arise in this way are inner and we end up with a projective (unitary) representation of the mapping class group by elements in the algebra $L^{G}_{g,N}$ of elements transforming trivially under $\Phi_{Ad}$.

**Proposition 2** [4] (Representations of the mapping class group) Let $p$ denote an arbitrary cycle on $\Sigma_{g,N} \setminus D$ and $\gamma_p$ the associated Dehn twist in the mapping class group. Then there exists an involution on $L^{G}_{g,N}$ such that

\[ \gamma_p \mapsto \nu(\gamma_p) := V[tr_q(M(p))] \in L^{G}_{g,N} \subset L_{g,N} \]

extends to a projective unitary representation of the mapping class group of $\Sigma_{g,N} \setminus D$. Here $M(p)$ is defined recursively as in Subsection 3.2. The map $\gamma_p \mapsto \nu(\gamma_p)$ descends to the mapping class group of $\Sigma_{g,N}$, if $L^{G}_{g,N}$ is factored by the relation $M(r) \sim 1$.

Representations of the algebras $L_{g,N}$ and $L^{G}_{g,N}$ have been constructed in [4]. They give rise to projective unitary actions of mapping class groups on finite dimensional Hilbert spaces. The latter are equivalent to the representations obtained by Reshetikhin and Turaev [3].

We are finally able to explain how the quantum moduli spaces $A_{CS}$ arise within the described framework. The idea is to descend from the algebra $L_{g,N}$ generated by the quantum connections in two steps. First one restricts to the subspace of elements on which the quantum gauge symmetry $\Phi_{Ad}$ acts trivially. This simply gives our algebra $L^{G}_{g,N}$. Then, in a second step, the flatness condition $M(r) \sim 1$ is imposed, i.e.

\[ A_{CS} := L^{G}_{g,N} / \langle M(r) \sim 1 \rangle. \]
It was shown in [4] that this algebra $\mathcal{A}_{CS}$ possesses irreducible representations on the spaces of conformal blocks of the corresponding WZNW-model. This is in perfect agreement with the results of [2].

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