BPS spectrum, indices and wall crossing in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories

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Abstract: BPS states in $\mathcal{N} = 4$ supersymmetric SU($N$) gauge theories in four dimensions can be represented as planar string networks with ends lying on D3-branes. We introduce several protected indices which capture information on the spectrum and various quantum numbers of these states, give their wall crossing formula and describe how using the wall crossing formula we can compute all the indices at all points in the moduli space.

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1 Introduction

BPS states on the Coulomb branch of $\mathcal{N} = 4$ supersymmetric Yang-Mills theories in four space-time dimensions have provided us with important tools for understanding various non-perturbative aspects of the theory [1–6]. A convenient way to represent these states in SU($N$) gauge theories is to regard the gauge theory as the world-volume theory of $N$ D3-branes. The BPS states are then described as planar networks of open strings ending on D3-branes [5, 6]. The half BPS states correspond to single open strings stretched between a pair of D3-branes whereas quarter BPS states correspond to more general planar string network constructed by joining many three string vertices [7–12]. Of these the spectrum of half BPS states is by now completely understood and can be obtained simply by a duality transformation of the spectrum of the massive gauge bosons. However despite the simple representation of the quarter BPS states as string network on D3-branes, and several extensive studies of the spectrum of these states from the study of bound states of multiple monopoles [13–17] and other techniques [18, 19] the full spectrum of quarter BPS states is still unknown. The goal of this paper will be to provide a complete answer to this problem.

We shall address the problem in two steps. The first step will be to introduce appropriate protected indices which get contribution from BPS states but not from a non-BPS state. These indices contain information on the BPS spectrum which are stable under quantum corrections. In the second step we shall describe the wall crossing formula [20–70] for these indices.¹ Combining the wall crossing formula with the observation that for

¹Application of wall crossing formula to special configurations in $\mathcal{N} = 4$ supersymmetric gauge theories can be found in [19].
collinear configuration of D3-branes there are no BPS string network except the half BPS states, we can derive the formula for the index at a generic point in the moduli space by identifying the walls of marginal stability which need to be crossed as we deform the moduli from the collinear configuration to the configuration of interest. This is a purely kinematic problem and can in fact be solved diagrammatically by following the deformation of the string network configuration as we deform the locations of the D3-branes.

We now summarize our main results and the organization of the paper. After reviewing the string network representation of BPS states in section 2 we introduce in section 3 three different indices for the BPS states in $\mathcal{N} = 4$ supersymmetric gauge theories. The first one is the standard sixth helicity trace index $B_6$ [71, 72] — given in eq. (3.3) — which can be defined everywhere in the moduli space. This receives contribution from only three string junctions supported on three D3-branes, and the corresponding walls of marginal stability have simple structure, dividing the moduli space into two regions — the one where the state exists and the one where the state does not exist. Using the wall crossing formula for this index derived earlier in the context of $\mathcal{N} = 4$ supersymmetric string theories [73, 74] we derive in eq. (4.5) a simple formula for $B_6$ in the region where the state exists.

The other two indices $B_2(z)$ and $B_1(y, z)$, which we introduce in eqs. (3.8) and (3.9), do not exist everywhere in the moduli space but can be defined when only two of the six adjoint Higgs fields of the gauge theory take vacuum expectation values. Equivalently this corresponds to a configuration of D3-branes where all the D3-branes lie in a plane. On such subspaces of the moduli space the gauge theory has an unbroken $\text{SO}(4) \equiv \text{SU}(2)_L \times \text{SU}(2)_R$ R-symmetry, and the indices $B_2(z)$ and $B_1(y, z)$ are twisted indices which keep track of the R-charges and the angular momentum of the system. These receive contribution from all planar string network with ends lying on the D3-branes and have more complicated structure of the walls of marginal stability compared to the three pronged string which contributes to the sixth helicity trace. For half BPS states these indices are easy to compute and are given in eq. (3.13). We give wall crossing formulæ for these indices in section 4.2 following the physical derivation of the Kontsevich-Soibelman (KS) formula [35–37] given in [56]. As in the case of KS formula, the wall crossing formula is the statement that the quantities given in eqs. (4.11), (4.14) remain unchanged as we cross a wall of marginal stability. Using these wall crossing formulæ and the fact that for collinear D3-brane configurations (i.e. with only one adjoint Higgs field getting vacuum expectation value) the only BPS states are the half BPS states whose spectrum and the indices are known explicitly, we can compute the indices everywhere in the moduli space. We do not have a closed form expression for these indices since the result depends not only on the charges but also on the region of the moduli space we are in. However in section 5 we illustrate the procedure for computing these indices with the help of three examples.

2 Review of string network

In this section we shall review some aspects of the string network on D3-branes which represent BPS states in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories. We begin by reviewing the basic rules for constructing supersymmetric string network [5–12]:
Figure 1. A string network representation of half BPS states in $\mathcal{N} = 4$ supersymmetric SU(5) theory. 1,2,3,4,5 denote the positions of the five D3-branes on which the open strings can end. Although we have displayed all the D3-branes in a plane, each of these D3-branes can actually move along six directions transverse to the D3-brane, representing the vacuum expectation values of the six higgs fields in the adjoint representation of the gauge group.

1. The links of the network are made of $(p,q)$ strings, where we use the convention that a $(1,0)$ string represents a D-string and a $(0,1)$ string represents a fundamental string. Each such string must end either on an external D3-brane or on an internal 3-string vertex.

2. The $(p,q)$ values associated with a given external or internal string need not be relatively prime. If for example $(p,q) = s(m,n)$ with $m,n$ relatively prime, then $(p,q)$ represents $s$ copies of the $(m,n)$ string.

3. At any junction the sum of the $(p_i,q_i)$ charges carried by the outgoing strings must vanish.

4. The network must be planar.

5. If $\tau$ denotes the complex coupling constant of the theory, with its real part given by $\theta/2\pi$ and its imaginary part given by $4\pi/g^2_\text{YM}$, then the $(p,q)$ string must lie along the direction $e^{i\alpha}(p\tau + q)$ in the two dimensional plane of the string network.\(^2\) Here $\alpha$ is an arbitrary constant, but it must take the same value for all the strings in a given network. The charge conservation at each junction then also guarantees that the net force on the junction due to the tensions of different strings cancel.

Some examples of string network on multiple D3-branes have been shown in figures 1 and 2. Figure 1, containing a single $(p,q)$ string stretched between two D3-branes, represents a half-BPS state, whereas more general planar string networks of the kind shown in figure 2 describe quarter BPS states.

Although we have drawn the network in figure 2 with the topology of a tree there can also be networks with internal faces [6]. For each such face the network possesses a bosonic zero mode that corresponds to changing the size of the face by parallel transport.

\(^2\)We could also consider another class of supersymmetric network for which the $(p,q)$ string lies along $e^{i\alpha}(p\tau + q)$. These two classes will be called respectively class A and class B quarter BPS states in section 3. They differ from each other in the way their unbroken supersymmetries transform under R-symmetry.
Figure 2. A string network representation of quarter BPS states in $\mathcal{N} = 4$ supersymmetric SU(5) theory. Unless otherwise labelled the label $(p,q)$ on an external string will denote a $(p,q)$ string entering the D3-brane. The electric and the magnetic charges carried by such a configuration are given by $Q = (q_1, q_2, q_3, q_4, q_5)$ and $P = (p_1, p_2, p_3, p_4, p_5)$ respectively. Since $\sum_i q_i = 0 = \sum_i p_i$ we see that the configuration is neutral under the overall U(1) factor of the U(5) gauge theory living on the five D3-branes.

Figure 3. Changing the size of an internal face in a string network. The solid lines represent the initial configuration and the dashed lines the deformed configuration.

The general rule for such deformations is that a $(p_i, q_i)$ string along the perimeter is moved by $\epsilon e^{i\alpha}(p_i - q_i\bar{\tau})/(p_i^2 + q_i^2)$ for some small real number $\epsilon$. For deformations associated with a given face $\epsilon$ is constant for all the edges, but for different faces $\epsilon$ can be chosen differently. Such deformations can hit boundaries when the face hits an external D3-brane (see figure 4(a)) or shrinks to zero size (see figure 4(b)).
Figure 4. Boundaries of the deformations of the internal face shown in figure 3.

Figure 5. Dual grid diagrams corresponding to the string network of (a) figure 3 and (b) figure 4(b). The circles represent the vertices of the grid dual to the faces of the original diagram.

A convenient way of representing a string network is the dual grid diagram [6, 9] in which the faces are represented as vertices, vertices are represented as faces, and the links are represented as links. The precise rule for drawing the grid diagram is as follows. We take any face of the original diagram and declare it as the origin of the dual diagram. Then the other vertices in the dual grid diagram are chosen such that if two adjacent faces in the original network are separated by a \((p, q)\) string then the corresponding vertices in the dual diagram are separated by a vector \((q, -p)\). Since at each vertex of the original diagram we have charge conservation, this guarantees that in the dual diagram the links forming the boundary of the face close. Figure 5(a) shows the grid diagram dual to the string network of figure 3. Note that the internal points in a grid diagram represent internal faces in the original diagram. The grid diagram remains invariant under the deformation described in figure 3, but when the internal face shrinks to zero size, as in figure 4(b), in the dual grid diagram the lines ending at the corresponding vertex gets removed. For example
Figure 6. Marginal stability walls for the string networks displayed in figure 4.

Figure 5(b) shows the grid diagram corresponding to the string network shown in figure 4(b). Conversely, existence of an integral lattice point in the interior of the grid implies that the network admits a deformation where an internal face grows and the corresponding dual grid diagram would correspond to connecting the internal lattice point to its neighbors by links.

For a given network characterized by the charges carried by the external strings one can move around in the moduli space of the theory by moving the positions of the D3-branes. As long as the D3-branes all lie in a plane the network remains planar and one can preserve the BPS nature of the network. During such movements of the moduli one can hit walls of marginal stability along which the original network becomes marginally unstable against decay into two or more smaller networks carrying the same total mass and charge. Typically this happens as one or more of the external strings shrink to zero size. We have shown in figure 6 two such examples, both involving the networks shown in figure 4. In the first example the original network displayed in figure 4(a) becomes unstable against decay into a single string (labelled by A) stretched from 1 to 2 and the rest of the network. In the second example the network becomes unstable against decay into a single string stretched between 1 and 3 and the rest of the network ending on the D3-branes 1, 2 and 4.

For identifying all the walls of marginal stability correctly it is always best to work with the fully deformed diagram where all possible internal faces have finite size. In the dual grid diagram this will require that each internal point is connected to its neighbors. The networks where some of the internal faces shrink to zero size can be regarded as special points in the deformation space of this more general network parametrized by the bosonic zero modes. Conversely, given any network with internal faces, we can associate with it a tree graph where all internal faces have been shrunk to zero size. Thus the necessary condition for a network to exist in some chamber of the moduli space is that the its associated tree must exist in the same chamber.

Finally we note that when all the D3-branes are along a line, the only string networks which exist are single \((p,q)\) strings stretched between a pair of D3-branes. Any more complicated configuration can be ruled out as follows. First of all we note that the string network must lie along the same line along which the D3-branes lie, since otherwise the
vertex in the network farthest from this line will be pulled towards this line by all the strings ending at the vertex and such a system cannot be in equilibrium. Thus all the strings in the network must be collinear to the line along which the D3-branes lie. Since the relative orientation of different strings are fixed by the charges they carry, the strings can be collinear iff they carry parallel \((p,q)\) charges. In particular the strings entering or leaving different D3-branes must also carry parallel \((p,q)\) charges. This means that the total electric and magnetic charges carried by the network are parallel. Such a configuration can be rotated to purely electrically charged configuration using S-duality and the spectrum is that of half-BPS W-bosons of the theory, represented by single \((0,1)\) strings stretched between pairs of D3-branes. After reversing the S-duality transformations they correspond to single \((p,q)\) strings stretched between a pair of D3-branes with \((p,q)\) relatively prime.

### 3 Three indices

Suppose we have a BPS state that breaks \(2n\) supersymmetries. Then there will be \(2n\) fermion zero modes (goldstinos) on the world-line of the state. To see the effect of these zero modes consider a pair of fermion zero modes \(\psi_0, \psi_0^\dagger\) satisfying

\[
\{\psi_0, \psi_0^\dagger\} = 1.
\]  

(3.1)

Let us denote by \(J_3\) the third component of the angular momentum and suppose that we have chosen the basis of zero modes such that \(\psi_0\) has \(J_3 = -1/2\) and \(\psi_0^\dagger\) has \(J_3 = 1/2\). If \(|0\rangle\) is the state annihilated by \(\psi_0\) then \(|0\rangle\) and \(\psi_0^\dagger|0\rangle\) will carry \(J_3\) eigenvalues \(-1/4\) and \(1/4\) respectively. Thus we have

\[
\text{Tr} e^{2i\pi J_3} = 0, \quad \text{Tr} e^{2i\pi J_3} (2J_3) = i
\]  

(3.2)

Thus the usual Witten index \(\text{Tr} (-1)^F = \text{Tr} e^{2i\pi J_3}\) will receive vanishing contribution from this sector reflecting the fact that the quantization of the fermion zero modes produces equal number of bosonic and fermionic states. To remedy this situation, we define a new index called the helicity trace index \([71, 72]\):

\[
B_n = \frac{(-i)^n}{n!} \text{Tr} \left\{ e^{2i\pi J_3} (2J_3)^n \right\}.
\]  

(3.3)

The trace is taken over states carrying a fixed set of charges. To see how this solves the problem let us denote by \(J_3^{(1)}, \ldots, J_3^{(n)}\) the contribution to \(J_3\) from the \(n\) pairs of fermion zero modes and by \(J_3^{\text{rest}}\) the contribution to \(J_3\) from the rest of the degrees of freedom. Then we have

\[
B_n = \frac{(-i)^n}{n!} \text{Tr}_{\text{rest}} \text{Tr}_{\text{zero}} \left\{ e^{2i\pi \left( J_3^{(1)} + \cdots J_3^{(n)} + J_3^{\text{rest}} \right)} \left( 2J_3^{(1)} + \cdots 2J_3^{(n)} + 2J_3^{\text{rest}} \right)^n \right\}.
\]  

(3.4)

For every pair of fermion zero modes, \(\text{Tr} \left\{ e^{2i\pi J_3^{(i)}} \right\}\) vanishes but \(\text{Tr} \left\{ e^{2i\pi J_3^{(i)}} (2J_3^{(i)}) \right\}\) gives a non-vanishing result \(i\). Thus the only non-vanishing contribution to (3.4) comes from the term \(n! 2J_3^{(1)} \times 2J_3^{(2)} \times \cdots 2J_3^{(n)}\) in the binomial expansion of \((2J_3^{(1)} + \cdots 2J_3^{(n)} + 2J_3^{\text{rest}})^n\).
For this term the trace over the fermion zero modes gives a contribution of $i^n n!$. Cancelling this against the explicit factor of $(-i)^n / n!$ included in the definition of $B_n$ we are left with

$$B_n = \text{Tr}_{\text{rest}} \left\{ e^{2i\pi J^3_{\text{rest}}} \right\}.$$  

(3.5)

This is in general non-vanishing. On the other hand, any state that breaks more than $2n$ supersymmetries will have more then $n$ pairs of fermion zero modes and will give vanishing contribution to this trace. In particular, non-BPS states will not contribute. This shows that the index cannot change under a continuous change in the moduli and hence is protected from quantum corrections. It can however change discontinuously across the walls of marginal stability which will be discussed later.

We can generalize this construction as follows. Suppose that the theory has a global symmetry $g$ under which $2m$ of the broken supersymmetries and certain number of unbroken supersymmetries of the BPS state are invariant. Then it follows from the argument given above that the index

$$B_{m}^{g} = \frac{(-i)^m}{m!} \text{Tr} \left\{ e^{2i\pi J^3_{g}} g (2J^3)^m \right\},$$

(3.6)

is protected, and is in general non-zero. Such an index contains information about the $g$ quantum numbers of the BPS states.

Let us now apply these general considerations to $\mathcal{N} = 4$ supersymmetric Yang-Mills theories in four dimensions. This theory has 16 supersymmetries. Thus half BPS states break 8 supersymmetries and the relevant index is $B_4$. As we have already mentioned, the result for $B_4$ is known completely. In particular for the configuration displayed in figure 1 we have

$$B_4(p, q) = \begin{cases} 1 & \text{for } \gcd(q, p) = 1 \\ 0 & \text{otherwise} \end{cases}.$$  

(3.7)

Quarter BPS states break 12 supersymmetries and the relevant index for counting these states is $B_6$. However the only configurations which contribute to $B_6$ are those containing three external strings ending on three D3-branes, e.g. the one shown in figure 7. The reason for this is that only planar string networks describe BPS configurations. Since we can always draw a plane through three points, any configuration that has ends on at most three D3-branes can always be made planar. In contrast a more general configuration, like the one shown in figure 2, is necessarily non-planar if the D3-brane coordinates do not lie in a plane. Such a configuration is non-BPS and hence does not contribute to $B_6$. Since we have argued that $B_6$ is invariant under continuous deformation of the moduli, it follows that even when all the D3-branes lie in a plane, planar networks ending on four or more D3-branes must have vanishing $B_6$. Physically this has its origin in the fact that such planar networks have some additional fermion zero modes besides the ones associated with the 12 broken supersymmetries [6], and the trace over these fermion zero modes makes the index vanish.

Are there other protected indices which can capture information about the BPS states associated with the planar string network? From the discussion above it should be clear
that any index that can be defined at a generic point in the moduli space must vanish for planar networks with four or more external strings since in that case we can compute that index by going to a non-planar configuration of D3-branes where the state is manifestly non-supersymmetric and hence gives vanishing contribution to the index. Thus we need to look for indices which are defined only for planar configuration of the D3-branes.\footnote{Twisted indices which are defined on a subspace of the full moduli space played an important role in testing the correspondence between black holes and microstates at the non-perturbative level [75, 76]. At the level of supersymmetric quantum mechanics describing the dynamics of multiple monopoles in $\mathcal{N} = 4$ supersymmetric Yang-Mills theories, such indices have been introduced by Stern and Yi [16].} To this end we note that when all the D3-branes lie in a plane, the theory has an additional unbroken $\mathrm{SO}(4) \simeq \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry corresponding to rotation in the four directions transverse to the plane of the D3-brane. In the language of the supersymmetric Yang-Mills theory this $\mathrm{SO}(4)$ symmetry is a subgroup of the $\mathrm{SO}(6)$ $R$-symmetry group that remains unbroken when only two of the six adjoint Higgs fields acquire vacuum expectation values. The quarter BPS states of the theory can be divided into two classes, which we shall call class A and class B states, according to the transformations properties of the unbroken supersymmetries under the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ transformation. We have shown in table 1 the transformation laws of the unbroken and broken supersymmetries under the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_{\text{rotation}}$ group for different types of BPS states, with $\mathrm{SU}(2)_{\text{rotation}}$ denoting the usual rotation group in the three space dimensions. In particular we note that for class A quarter BPS states and also half BPS states there are four $\mathrm{SU}(2)_L$ invariant

| BPS state          | unbroken supersymmetries | broken supersymmetries |
|--------------------|---------------------------|------------------------|
| Half BPS           | $(1,2,2) + (2,1,2)$       | $(1,2,2) + (2,1,2)$   |
| Class A quarter BPS| $(1,2,2)$                 | $(1,2,2) + 2(2,1,2)$ |
| Class B quarter BPS| $(2,1,2)$                 | $2(1,2,2) + (2,1,2)$ |

*Table 1. SU(2)$_L \times$ SU(2)$_R \times SU(2)_{\text{rotation}}$ transformation laws of various supersymmetries.*

Figure 7. A string network representation of quarter BPS states in $\mathcal{N} = 4$ supersymmetric SU(5) theory with three external strings ending on three D3-branes. Such configurations contribute to $B_6$. 
unbroken supersymmetries, and four SU(2)$_L$ invariant broken super symmetries. We shall denote by $I_{3L}$ and $I_{3R}$ the third components of the generators of SU(2)$_L$ and SU(2)$_R$ respectively. It now follows from (3.6) that the index

$$B_2(z) = -\frac{1}{2!} \text{Tr} \left\{ e^{2i\pi J_3 z^2 I_{3L}} (2J_3)^2 \right\}, \quad z \in \mathbb{C}, \quad |z| = 1,$$

(3.8)
is protected and will receive contribution from half BPS states and the class A quarter BPS states. We can also introduce another index by replacing $I_{3L}$ by $I_{3R}$ in (3.8) which will receive contribution from half BPS and class B quarter BPS states but for definiteness we shall concentrate on the index given in (3.8). Since this index is defined only for planar configuration of D3-branes, it can receive contribution from general string network ending on arbitrary number of D3-branes.

Table 1 also shows that for half BPS and class A quarter BPS states 2 of the SU(2)$_L$ invariant unbroken generators and 2 of the SU(2)$_L$ invariant broken generators are invariant under $(I_{3R} + J_3)$. This allows us to to define yet another protected index

$$B_1(y, z) = -\frac{1}{y - y^{-1}} \text{Tr} \left\{ e^{2i\pi J_3 z^2 I_{3L}} y^{2I_{3R} + 2J_3} (2J_3) \right\}, \quad y, z \in \mathbb{C}, \quad |z| = 1, \quad |y| = 1.$$  

(3.9)
The index (3.9) is analogous to the protected spin character defined in [52] (with the replacement $y \to -y$).

We have adjusted the normalizations of $B_2(z)$ and $B_1(y, z)$ such that after factoring out the contribution from the SU(2)$_L$ invariant fermion zero modes in the trace we are left with $\text{Tr}_{\text{rest}} \left\{ e^{2i\pi J_3 z^2 I_{3L}} \right\}$ and $\text{Tr}_{\text{rest}}' \left\{ e^{2i\pi J_3 z^2 I_{3L}} y^{2I_{3R} + 2J_3} \right\}$ respectively as in (3.5). The prime on $\text{Tr}$ denotes that it includes traces over the SU(2)$_L$ non-invariant fermion zero modes. Thus we have the relation

$$B_2(z) = \lim_{y \to 1} B_1(y, z).$$  

(3.10)

We can also find a relation between $B_2(z)$ and $B_6$ by factoring out the contribution from SU(2)$_L$ non-invariant fermion zero modes from $B_2(z)$. From table 1 we see that for class A quarter BPS states these transform in two $(2,1,2)$ representation of SU(2)$_L \times$ SU(2)$_R \times$ SU(2)$_{\text{rotation}}$. Their contribution to $B_2(z)$ corresponds to a factor of $(z + z^{-1} - 2)^2$. Thus $B_2(z)$ can be expressed as $(z + z^{-1} - 2)^2 \text{Tr}_{\text{rest}} \left\{ e^{2i\pi J_3 z^2 I_{3L}} \right\}$. Comparing this with (3.5) we get

$$B_6 = \lim_{z \to 1} (z + z^{-1} - 2)^2 B_2(z).$$  

(3.11)

For a half BPS state represented by a $(p, q)$ string stretched between two D3-branes with $(p, q)$ relatively prime, the contribution to $B_2(z)$ and $B_1(y, z)$ can be evaluated by knowing the $(I_{3L}, I_{3R}, J_3)$ quantum numbers carried by the 8 fermion zero modes associated with broken supersymmetries. The $(I_{3L}, I_{3R}, J_3)$ assignments are as follows:

$$\begin{align*}
(0, 1/2, 1/2), & \quad (0, 1, 2, -1/2), \quad (0, -1/2, 1/2), \quad (0, -1/2, -1/2), \\
(1/2, 0, 1/2), & \quad (1/2, 0, -1/2), \quad (1, 2, 0, 1/2), \quad (1, -2, 0, -1/2).
\end{align*}$$

(3.12)
Upon quantization of these zero modes we get the following contribution to the indices $B_2(z)$ and $B_1(y, z)$:

\[
B_2(z) = (z + z^{-1} - 2), \\
B_1(y, z) = (z + z^{-1} - y - y^{-1}).
\]  

Equation (3.13)

In the next section we shall describe the wall crossing formula for the jump in these indices across walls of marginal stability. Since we have argued at the end of section 2 that for collinear configuration of D3-branes the only surviving BPS configurations are the half BPS states, the computation of the various indices for such configurations is straightforward. The values of the indices elsewhere in the moduli space can then be determined using the wall crossing formulæ for the various indices.

4 Wall crossing formulæ

In this section we shall describe the wall crossing formulæ for the three indices introduced in section 3.

4.1 Sixth helicity trace index

As argued in section 3, the sixth helicity trace $B_6$ receives contribution only from configurations with three external strings. The walls of marginal stability across which $B_6$ jumps are those on which the state becomes unstable against decay into a pair of half BPS states. In terms of string network such decays correspond to one of the external strings shrinking to zero size, as shown in figure 8. If $(Q_1, P_1)$ and $(Q_2, P_2)$ denote the (electric, magnetic) charges carried by the two half BPS states into which the state decays, then the jump in the $B_6$ value as we cross the wall from the side in which it does not exist to the one in

\[
\begin{align*}
\text{Figure 8. A marginal stability wall of the string network shown in figure 7. The decay products are a } (p_2, q_2) \text{ string stretched from 1 to 2, and a } (p_3, q_3) \text{ string stretched from 1 to 3.}
\end{align*}
\]
which it exists is given by [73, 74]:

\[
\Delta B_6(Q, P) = (-1)^{Q_1 P_2 - Q_2 P_1 + 1} |Q_1 P_2 - Q_2 P_1| \sum_{L_1 \mid (Q_1, P_1)} B_4(Q_1/L_1, P_1/L_1) \\
\times \sum_{L_2 \mid (Q_2, P_2)} B_4(Q_2/L_2, P_2/L_2),
\]

(4.1)

where \(L_i \mid (Q_i, P_i)\) means that \(L_i\) must be a common factor of all components of \(Q_i\) and \(P_i\).

For the decay displayed in figure 8 we have

\[
P_1 = (-p_2, p_2, 0), \quad Q_1 = (-q_2, q_2, 0), \quad P_2 = (-p_3, 0, p_3), \quad Q_2 = (-q_3, 0, q_3).
\]

(4.2)

Thus we have

\[
Q_1 P_2 - Q_2 P_1 = (q_2 p_3 - q_3 p_2).
\]

(4.3)

Let \(s_2 = \gcd(p_2, q_2)\) and \(s_3 = \gcd(p_3, q_3)\). Now \((Q_1/L_1, P_1/L_1)\) represents a \((q_2/L_2, p_2/L_1)\) string stretched between the D3-branes 1 and 2. It follows from (3.7) that the index \(B_4\) for such a state is non-zero iff \(\gcd(q_2/L_1, p_2/L_1) = 1\), i.e. iff \(L_1 = s_2\). Similarly \(B_4(Q_2/L_2, P_2/L_2)\) is non-vanishing iff \(L_2 = s_3\). Furthermore we have \(B_4(Q_1/s_2, P_1/s_2) = 1\) and \(B_4(Q_2/s_3, P_2/s_3) = 1\). Substituting these and (4.3) into (4.1) we get

\[
\Delta B_6(Q, P) = (-1)^q_2 p_3 - q_3 p_2 + 1 |q_2 p_3 - q_3 p_2|.
\]

(4.4)

When the D3 brane 1 crosses the wall of marginal stability the configuration displayed in figure 8 ceases to exist and hence \(B_6\) vanishes. Thus (4.4) represents the value of \(B_6\) on the side of the wall where the configuration exists, and we can write

\[
B_6(Q, P) = (-1)^q_2 p_3 - q_3 p_2 + 1 |q_2 p_3 - q_3 p_2|.
\]

(4.5)

We note that this formula is symmetric under the exchange of the three external strings as a consequence of the ‘conservation law’

\[
p_1 + p_2 + p_3 = 0, \quad q_1 + q_2 + q_3 = 0,
\]

(4.6)

and hence we shall arrive at the same formula if we apply the wall crossing across the other two walls of marginal stability where either the \((p_2, q_2)\) string or the \((p_3, q_3)\) string shrinks to zero size.

We should also add that we can consider string network with three external strings and internal faces. Such configurations are planar and in principle could contribute to \(B_6\). However the marginal stability walls on which such a string network breaks apart into a pair of half BPS states are always of the type shown in figure 8 where the internal face has shrunk to zero size. Along other marginal stability walls where the original network contains internal faces, at least one of the decay products will be quarter BPS (see e.g. figure 14) and such decays do not contribute to jumps in \(B_6 \) [77–79]. Thus we can ignore them for computation of \(B_6\), although, as we shall see later, they will contribute to jumps in \(B_2(z)\) and \(B_1(y, z)\).
Figure 9. A special class of string network configurations.

For special configurations of the type shown in figure 9 the expression for $B_6$ can be derived from the results of [16, 17] based on the study of supersymmetric quantum mechanics of monopole system. The result is $(-1)^{s+1}|s|$ in agreement with (4.5). For the same configuration the formula was also derived in [19] using primitive wall crossing formula. However for deriving the result for most general set of charges we need to use the general wall crossing formula for decays into non-primitive charge vectors as given in (4.1).

4.2 The twisted and motivic helicity trace indices

As discussed before, the index $B_6$ vanishes for planar string network with four or more external legs but the indices $B_2(z)$ and $B_1(y, z)$ defined in (3.8) and (3.9) do not vanish in general. However for collinear configuration of D3-branes these indices do vanish except for half BPS states. The latter indices have been computed in (3.13). Thus if we can write down the general wall crossing formula for these indices, then we can compute them at any point in the moduli space by starting with the known values of the indices for collinear configurations and then successively applying the wall crossing formula across each wall of marginal stability.

Since the supersymmetry subalgebra that commutes with $SU(2)_L$ is the $\mathcal{N} = 2$ supersymmetry algebra, one expects that the wall crossing formulæ for $B_2(z)$ and $B_1(y, z)$ will be similar to the KS wall crossing formula [35–37]. Indeed by now there are many physical ‘derivations’ of the KS wall crossing formula [42, 44, 52, 53, 56, 60, 62, 63, 70] and we can use any of them to derive the wall crossing formula for the indices $B_2(z)$ and $B_1(y, z)$. We have derived these formulæ using the arguments given in [56, 62] and the result of [70] proving the equivalence of the wall crossing formulæ of [56] and the KS wall crossing formula. Since the logic is identical to those in [56, 70], we shall not give the details of the argument but only quote the final results.

We begin by introducing some notations. Let $\alpha = (Q, P)$ denote the charge vector and given two such vectors we define

$$\langle \alpha, \alpha' \rangle = Q \cdot P' - P \cdot Q'. \quad (4.7)$$
We shall denote by $Z_\gamma$ the central charge of a charge vector $\gamma$ under the SU(2)$_L$ invariant $\mathcal{N} = 2$ subalgebra. For a set of D3-branes at positions $z_1, z_2, \cdots$ in the complex plane, the central charge of a planar network with $(p_i, q_i)$ string entering the $i$-th D3-brane is given by

$$Z_\gamma = \frac{1}{\sqrt{\tau_2}} \sum_i \bar{z}_i (p_i \bar{\tau} + q_i),$$  \hspace{1cm} (4.8)

up to a constant of proportionality.\textsuperscript{4} Near any wall of marginal stability we can find a pair of vectors $\gamma_1$ and $\gamma_2$ satisfying the following properties:

1. Along the wall of marginal stability of interest the central charges $Z_{\gamma_1}$ and $Z_{\gamma_2}$ get aligned.
2. Any charge vector lying in the plane of $\gamma_1$ and $\gamma_2$ can be expressed as $m\gamma_1 + n\gamma_2$ with integer $m, n$.
3. Near the wall of marginal stability BPS states of charge $m\gamma_1 + n\gamma_2$ exist only for $m, n \geq 0$ or $m, n \leq 0$ \cite{53}.

First we shall give the wall crossing formula for $B_2(z)$ across such a wall. We denote by $B_2(\alpha; z)$ the $B_2(z)$ index for charge vector $\alpha$, and introduce the rational index

$$\bar{B}_2(\alpha; z) = \sum_{m|\alpha} m^{-2} B_2(\alpha/m; z^m).$$  \hspace{1cm} (4.9)

We also introduce an infinite dimensional algebra with generators $e_\alpha$ satisfying the commutations relations:

$$[e_\alpha, e_{\alpha'}] = (-1)^{\langle \alpha, \alpha' \rangle} \langle \alpha, \alpha' \rangle e_{\alpha + \alpha'}. \hspace{1cm} (4.10)$$

The KS wall crossing formula is the statement that

$$P \left( \prod_{M \geq 0, N \geq 0} \exp \left[ \bar{B}_2(M\gamma_1 + N\gamma_2; z) e_{M\gamma_1 + N\gamma_2} \right] \right)$$  \hspace{1cm} (4.11)

remains unchanged across a wall of marginal stability. $P$ denotes a phase ordered product of the exponentials such that the phase of $Z_{M\gamma_1 + N\gamma_2}$ decreases monotonically as we move from the left most element to the right-most element of the product. As we cross a wall of marginal stability, the phases of $Z_{\gamma_1}$ and $Z_{\gamma_2}$ switch order and as a result the order in the product in (4.11) is reversed. The wall crossing formula tells us that the indices $B_2(\alpha; z)$ on one side of the wall of marginal stability if we know their values on the other side.

\textsuperscript{4}Using the charge conservation at each vertex we can express (4.8) as $\tau_2^{-1/2} \sum_{\text{links}} \Delta z_m (p_m \bar{\tau} + q_m)$ where $\Delta z_m$ is the complex number describing the length and orientation of the $m$-th link and $(p_m, q_m)$ is the charge of the string along the $m$-th link. Using the fact that $\Delta z_m \propto \epsilon^{i\alpha_m} (p_m \bar{\tau} + q_m)$ we can express this as $\tau_2^{-1/2} e^{i\alpha_m} \sum_{\text{links}} |\Delta z_m| |p_m\tau + q_m|$. Since $|p_m\tau + q_m|/\sqrt{\tau_2}$ is the tension of the $(p_m, q_m)$ string, we see that $|Z_\gamma|$ is proportional to the total mass of the network as expected.
The wall crossing formula for \( B_1(\alpha; y, z) \) is a generalization of the motivic wall crossing formula of KS. For this we define\(^5\)

\[
\tilde{B}_1(\alpha; y, z) = \sum_{m|\alpha} m^{-1} \frac{y}{y^m-y^{-m}} B_1(\alpha/m; y^m, z^m),
\]

(4.12)

and introduce the infinite dimensional algebra generated by \( \tilde{e}_\alpha \) satisfying the commutation relations:

\[
[\tilde{e}_\alpha, \tilde{e}_{\alpha'}] = \frac{(-y)^{\langle \alpha, \alpha' \rangle} - (-y)^{-\langle \alpha, \alpha' \rangle}}{y-y^{-1}} \tilde{e}_{\alpha+\alpha'}.
\]

(4.13)

The wall crossing formula for \( B_1(\alpha; y, z) \) then tells us that the product

\[
P\left( \prod_{M \geq 0, N \geq 0} \exp \left[ B_1(M\gamma_1 + N\gamma_2; y, z)\tilde{e}_{M\gamma_1+N\gamma_2} \right] \right)
\]

(4.14)

remains unchanged across the wall of marginal stability. Using (4.14) we can determine the indices \( B_1(\alpha; y, z) \) on one side of the wall of marginal stability if we know their values on the other side. Note that as \( y \to 1 \) the wall crossing formula for \( B_1(\alpha; y, z) \) tends to that for \( B_2(\alpha; z) \).

Eqs. (4.11) and (4.14) give implicit relations which determine the index on one side in terms of the index on the other side. Explicit formulæ for the indices on one side in terms of their values on the other side can be found in [56]. The equivalence of these explicit formulæ and (4.11), (4.14) has been proved in [70].

Special cases of these general wall crossing formulæ are the primitive and the semiprimitive wall crossing formulæ [31]. Let us for definiteness denote by \( B_2^+ \) and \( B_1^+ \) the indices on the side of the wall in which

\[
\langle \gamma_1, \gamma_2 \rangle \text{Im}(Z_{\gamma_1}Z_{\gamma_2}) < 0,
\]

(4.15)

and by \( B_2^- \) and \( B_1^- \) the indices on the other side.\(^6\) Then the primitive wall crossing formula tells us that

\[
B_2^- (\gamma_1 + \gamma_2; y, z) - B_2^+ (\gamma_1 + \gamma_2; y, z) = (-1)^{\langle \gamma_1, \gamma_2 \rangle+1} |\langle \gamma_1, \gamma_2 \rangle| B_2^+ (\gamma_1; z) B_2^+(\gamma_2; z),
\]

\[
B_1^- (\gamma_1 + \gamma_2; y, z) - B_1^+ (\gamma_1 + \gamma_2; y, z) = \frac{(-y)^{\langle \gamma_1, \gamma_2 \rangle} - (-y)^{-\langle \gamma_1, \gamma_2 \rangle}}{y-y^{-1}} B_1^+(\gamma_1; y, z) B_1^+(\gamma_2; y, z).
\]

(4.16)

The semiprimitive wall crossing formulæ for the index \( B_2(\gamma_1 + N\gamma_2; z) \) tells us that

\[
\tilde{B}_2^- (\gamma_1 + N\gamma_2; z) - \tilde{B}_2^+ (\gamma_1 + N\gamma_2; z) = \sum_{\ell=0}^{N-1} \tilde{B}_2^+(\gamma_1 + \ell\gamma_2; z) \Omega_{\text{halo}}(\gamma_1, \gamma_2, N - \ell; z),
\]

(4.17)

---

\(^5\)In the analysis of [56] the \((y-y^{-1})/(y^m-y^{-m})\) factor in (4.12) arose from the fact that for motion in a magnetic field the orbital angular momentum grows with the magnetic field. In contrast the SU(2)_L quantum number to which \( z \) couples is not affected by the magnetic field and hence there is no such factor involving \( z \).

\(^6\)Physically the – side corresponds to the side in which there are multi-centered loosely bound configurations with individual centers carrying charges of the form \( n\gamma_1 + n\gamma_2 \). On the + side there are no such bound states. Hence the jump in the index can be identified as the contribution from these loosely bound states.
where \[31\]
\[
\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; z) q^N = \exp \left[ -\sum_{s=1}^{\infty} s q^s (-1)^s \langle \gamma_1, \gamma_2 \rangle | B_2^+(s \gamma_2; z) \right].
\] (4.18)

Since \(\gamma_1 + \ell \gamma_2\) is primitive for all \(\ell\), we can replace \(B_2^+(\gamma_1 + N \gamma_2; z)\) and \(B_2^+(\gamma_1 + \ell \gamma_2; z)\) in (4.17) by \(B_2^+(\gamma_1 + N \gamma_2; z)\) and \(B_2^+(\gamma_1 + \ell \gamma_2; z)\) respectively. On the other hand using (4.9) and (4.18) we get
\[
\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; z) q^N = \exp \left[ -\sum_{s=1}^{\infty} s q^s (-1)^s \langle \gamma_1, \gamma_2 \rangle \sum_{m|s} m^{-2} B_2^+(s \gamma_2/m; z^m) \right]
= \exp \left[ -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} m^{-1} k q^{mk} (-1)^{mk} \langle \gamma_1, \gamma_2 \rangle | B_2^+(k \gamma_2; z^m) \right].
\] (4.19)

If
\[
B_2^+(k \gamma_2; z) = \sum_p B_{2,p}^+(k \gamma_2) z^p,
\] (4.20)

then (4.19) may be expressed as
\[
\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; z) q^N = \prod_p \prod_{k=1}^{\infty} \left( 1 - q^k z^p (-1)^{k \gamma_2} \right)^{k \langle \gamma_1, \gamma_2 \rangle} B_{2,p}^+(k \gamma_2).
\] (4.21)

This gives an expression for the change in \(B_2(\gamma_1 + N \gamma_2; z)\) across the wall of marginal stability.

Finally the semiprimitive wall crossing formula for the index \(B_1(\gamma_1 + N \gamma_2; y, z)\) tells us that \[46\]
\[
B_1^-(\gamma_1 + N \gamma_2; y, z) - B_1^+(\gamma_1 + N \gamma_2; y, z) = \sum_{\ell=0}^{N-1} B_1^+(\gamma_1 + \ell \gamma_2; y, z) \Omega_{\text{halo}}(\gamma_1, \gamma_2, N-\ell; y, z),
\] (4.22)

where
\[
\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; y, z) q^N = \exp \left[ \sum_{s=1}^{\infty} q^s \frac{(-y)^{-s \gamma_2} - (-y)^{s \gamma_2}}{y^{1-s}} B_1^+(s \gamma_2; y, z) \right].
\] (4.23)

Again using (4.12) we can express (4.23) as
\[
\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; y, z) q^N = \exp \left[ \sum_{m=1}^{\infty} \sum_{k=1}^{\infty} m^{-1} q^{mk} \frac{(-y)^{-mk \gamma_2} - (-y)^{mk \gamma_2}}{y^m - y^{-m}} B_1^+(k \gamma_2; y^m, z^m) \right].
\] (4.24)

If
\[
B_1^+(k \gamma_2; y, z) = \sum_{n,p} B_{1,n,p}^+(k \gamma_2) y^n z^p,
\] (4.25)
Figure 10. The string network configuration of example 1. Here \( s_i \) and \( r_i \) are positive integers.

Figure 11. The dual grid diagram corresponding to the network of figure 10.

then (4.24) may be expressed as

\[
\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; y, z) q^N = \prod_{p,n} \prod_{k=1}^{\infty} \prod_{r=1}^{\infty} \left( 1 - (-1)^{k(\gamma_1, \gamma_2)} q^{k} z^{p} y^{n+2r-1-k(\gamma_1, \gamma_2)} \right) B_{1,n,p}^{+}(k\gamma_2). \tag{4.26}
\]

In the next section we shall see some examples of how using these wall crossing formulæ we can calculate the indices \( B_2(z) \) and \( B_1(y, z) \) for planar string network.

5 Three examples

In this section we shall apply the wall crossing formulæ to compute the indices \( B_2(z) \) and \( B_1(y, z) \) of three different string network configurations.

5.1 Example 1

We begin with the planar string network shown in figure 10. This system has been analyzed extensively in [16, 17] as bound states of distinct monopoles, and we shall compare our results with the known results later. The corresponding grid diagram is shown in figure 11. The important point to note is that the two horizontal lines of the grid diagram are separated by unit distance along the vertical direction and as a result there are no integral lattice points in the interior of the diagram. Thus we cannot deform the original network...
by growing an internal face, and figure 10 represents the most general network with these external strings.

To compute the index of the state let us approach the wall of marginal stability where the \((0, s_1)\) string shrinks to zero size. Since the configuration ceases to exist on the other side of the wall the index vanishes there and hence the jump in the index across this wall gives the index of the configuration. This jump on the other hand can be computed using the primitive wall crossing formula (4.16), and the difference in the index of the initial configuration and that of the final configuration is given by \((-1)^{s_1+1} s_1\) (for \(B_2(z)\)) or \((-1)^{s_1+1} (y^{s_1} - y^{-s_1})/(y - y^{-1})\) (for \(B_1(y, z)\)) times the product of the index of a half BPS state and the index of a quarter BPS network in which the \((-1, j)\) and \((0, s_1)\) strings are removed. The index of the half BPS state can be computed from (3.13). On the other hand to compute the index of the quarter BPS state we repeat the analysis, this time approaching the marginal stability wall along which the \((0, -r_1)\) string shrinks to zero size. By repeating this process we can arrive at the following final expressions for the indices:

\[
B_2(z) = (-1)^{\sum s_i + \sum r_j + N - 2} (z + z^{-1} - 2)^{N-1} \prod_i s_i \prod_j r_j, \\
B_1(y, z) = (-1)^{\sum s_i + \sum r_j + N - 2} \{z + z^{-1} - y - y^{-1}\}^{N-1} \prod_i y^{s_i} - y^{-s_i} \prod_j y^{r_j} - y^{-r_j},
\]

(5.1)

where \(N\) denotes the total number of external strings.

The system described by the string network shown in figure 10 in fact represents a system of \((N - 1)\) distinct monopoles and the supersymmetric quantum mechanics associated with this system has been thoroughly analyzed in [16, 17]. In particular Stern and Yi [16] computed an index in this supersymmetric quantum mechanics which led to a net protected degeneracy of \(16 \times \prod_i 4s_i \prod_j 4r_j\). The value of \(B_2(z = -1)\) computed from (5.1) is \(4 \times (-1)^{\sum s_i + \sum r_i + 1} \prod_i (4 s_i) \prod_j (4 r_j)\). Multiplying the magnitude of this by 4 — the degeneracy due to the \(\text{SU}(2)_L\) invariant fermion zero modes which was factored out from the definition of \(B_2(z)\) — we get the same result \(16 \times \prod_i 4s_i \prod_j 4r_j\). In fact this result was already rederived in [19] by making repeated use of wall crossing formula in the manner we have described above.

There is a more detailed result on Stern-Yi dyon chain in the context of \(\mathcal{N} = 2\) supersymmetric theories due to Denef [28]. To compare our result with that of [28], we need to first extract the result for the Stern-Yi dyon chain in \(\mathcal{N} = 2\) supersymmetric theories from our results. The dynamics of distinct monopoles in \(\mathcal{N} = 2\) supersymmetric theories can be obtained from those in the \(\mathcal{N} = 4\) supersymmetric theories by projecting out the \(\text{SU}(2)_L\) non-invariant fermion zero modes from each constituent monopole. Since each constituent monopole is half-BPS, we see from table 1 that the \(\text{SU}(2)_L\) non-invariant fermion zero modes transform in the \((2, 1, 2)\) representation of \(\text{SU}(2)_L \times \text{SU}(2)_R \times \text{SU}(2)_{\text{rotation}}\) and hence gives a factor of \((z + z^{-1} - y - y^{-1})\) to \(B_1\). Since there are \((N - 1)\) distinct constituents these zero modes give a net factor of \((z + z^{-1} - y - y^{-1})^{N-1}\). Factoring out this contribution from the expression for \(B_1\) given in (5.1) we can get the result for \(B_1(y)\)
for the Stern-Yi dyon chain in the $\mathcal{N} = 2$ supersymmetric theory:

$$B_1(y)|_{\mathcal{N}=2} = (-1)^{\sum s_i + \sum r_j + N - 2} \prod_i y^s_i - y^{-s_i} \prod_j y^r_j - y^{-r_j}.$$  (5.2)

This agrees with the result of [28].

Before leaving this example we note that instead of shrinking the $(0, s_1)$ string in the first step we could have also shrunk the $(-1, j)$ string. The wall crossing we shall now encounter is semi-primitive involving decay into $\gamma_1$ and $s_1 \gamma_2$, with $\gamma_1$ corresponding to the quarter BPS state represented by the part of the network without the $(-1, j)$ and $(0, s_1)$ string, and $\gamma_2$ corresponding to the half BPS state represented by the $(0, 1)$ string. Thus $B_1^+(k \gamma_2, z)$ and $B_1^+(k \gamma_2, y, z)$ both vanish for $k > 1$ and is given by (3.13) for $k = 1$. Furthermore one can check that here $\langle \gamma_1, \gamma_2 \rangle = 1$. Thus using (4.21), (4.26) we get

$$\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; z) q^N = (1 + qz)(1 + qz^{-1})(1 + q)^{-2},$$

$$\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; y, z) q^N = (1 + qz)(1 + qz^{-1})(1 + qy)^{-1}(1 + qy^{-1})^{-1}.  \quad (5.3)$$

From this we get

$$\Omega_{\text{halo}}(\gamma_1, \gamma_2, N; z) = (-1)^{N+1} N (z + z^{-1} - 2),$$

$$\Omega_{\text{halo}}(\gamma_1, \gamma_2, N; y, z) = (-1)^{N+1} \{z + z^{-1} - y - y^{-1}\} \frac{y^N - y^{-N}}{y - y^{-1}}.  \quad (5.4)$$

We can now use the semi-primitive wall crossing formulae (4.17) and (4.22) with $N = s_1$, together with the fact that only $\ell = 0$ terms on the right hand of these formulae contribute. To see the latter we have shown in figure 12 the string network corresponding to the charge vector $\gamma_1 + \ell \gamma_2$ for $\ell > 0$. This has a marginal stability wall corresponding to shrinking of the $(-1, j + s_1 - \ell)$ string and in the moduli space this wall coincides with the corresponding wall of $\gamma_1 + s_1 \gamma_2$ on which the $(-1, j)$ string shown in figure 10 shrinks to zero size. Thus
Figure 13. (a) The string network configuration of example 2 and (b) its deformation.

$B^+_{2}(\gamma_1 + \ell\gamma_2; z)$ and $B^+_{1}(\gamma_1 + \ell\gamma_2; y, z)$, which correspond to the index measured on the other side of this wall, vanish for $\ell > 0$. This shows that the jump in the index across the wall is given by the product of (5.4) with $N$ replaced by $s_1$ and the index of $\gamma_1$ — a quarter BPS state in which the $(-1, j)$ and $(0, s_1)$ strings are removed from figure 10. This gives us back the same result we have found before. This provides a consistency check of our approach and the wall crossing formula.

5.2 Example 2

We shall now consider the string network shown in figure 13. The tree configuration is shown in figure 13(a), but the analysis of the grid diagram shows that the network can be deformed to include an internal face as shown in figure 13(b). We shall see that including the contribution from this deformed configuration is essential for the consistency of the wall crossing formulæ.

First consider the limit in which the length of the $(2,0)$ string in figure 13(a) shrinks to zero size. In this limit the internal face in figure 13(b) also shrinks to zero size and we reach the marginal stability wall on which the system becomes unstable against decay into a pair of half BPS states, one containing a $(1,2)$ string stretched between D3-branes 2 and 3 and a $(-3, -2)$ string stretched between D3-branes 1 and 3. The jump in the index, which also gives the index since the configuration ceases to exist on the other side of the wall, is given by the primitive wall crossing formula. The result is

\[
B_2(z) = -4(z + z^{-1} - 2)^2,
\]

\[
B_1(y, z) = -\{z + z^{-1} - y - y^{-1}\}^2 \frac{y^4 - y^{-4}}{y - y^{-1}}.
\]

Next consider the limit in which the $(-3, -2)$ string in figure 13(a) shrinks to zero size. Again in this case neither of the configurations shown in figure 13 will survive on the other side of this wall and hence the index vanishes. Thus the jump in the index gives the
index. However in this case the decay is semi-primitive, involving the (1,2) string stretched between 1 and 2 and the (2,0) string stretched between 1 and 3. Thus we can use the semi-primitive wall crossing formulæ with $\gamma_1$ representing the (1,2) string stretched between 1 and 2 and $\gamma_2$ representing the (1,0) string stretched between 1 and 3, with $\langle \gamma_1, \gamma_2 \rangle = 2$. As in the case of example 1 $B_2^2(k\gamma_2, z)$ and $B_1^1(k\gamma_2, y, z)$ both vanish for $k > 1$ and is given by (3.13) for $k = 1$. Thus using (4.21), (4.26) we get

$$\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; z) q^N = (1 - qz)^2(1 - qz^{-1})^2(1-q)^{-4},$$

$$\sum_{N=0}^{\infty} \Omega_{\text{halo}}(\gamma_1, \gamma_2, N; y, z) q^N = (1 - qzy)(1 - qzy^{-1})(1 - qz^{-1}y)(1 - qz^{-1}y^{-1})$$

$$(1 - q)^{-2}(1 - qy^2)^{-1}(1 - qy)^{-2}. \quad (5.6)$$

The relevant quantities we need for the decay into $\gamma_1$ and $2\gamma_2$ are $\Omega_{\text{halo}}(\gamma_1, \gamma_2, N = 2; z)$ and $\Omega_{\text{halo}}(\gamma_1, \gamma_2, N = 2; y, z)$. These can be read out from (5.6):

$$\Omega_{\text{halo}}(\gamma_1, \gamma_2, N = 2; z) = z^2 + \frac{1}{z^2} - 8z - \frac{8}{z} + 14,$$

$$\Omega_{\text{halo}}(\gamma_1, \gamma_2, N = 2; y, z) = y^4 + \frac{1}{y^4} - y^3z - \frac{y^3}{z} - \frac{z}{y^3z} + \frac{1}{y^3} + 3y^2 + \frac{3}{y^2}$$

$$- 3yz - \frac{3y}{z} - \frac{3z}{y} + \frac{3}{yz} + z^2 + \frac{1}{z^2} + 6. \quad (5.7)$$

Using (4.17) and (4.22) with $N = 2, \ell = 0$ we get the indices $B_2(z)$ and $B_1(y, z)$, which we shall denote by $B_2^2(z)$ and $B_1^1(y, z)$ to distinguish them from (5.5). The results are

$$B_2^2(z) = (z + z^{-1} - 2)\left(z^2 + \frac{1}{z^2} - 8z - \frac{8}{z} + 14\right),$$

$$B_1^1(y, z) = (z + z^{-1} - y - y^{-1})\left(y^4 + \frac{1}{y^4} - y^3z - \frac{y^3}{z} - \frac{z}{y^3z} + \frac{1}{y^3} + 3y^2 + \frac{3}{y^2} - 3yz - \frac{3y}{z} - \frac{3z}{y} + \frac{3}{yz} + z^2 + \frac{1}{z^2} + 6\right). \quad (5.8)$$

These are different from (5.5). In fact we have

$$B_2^2(z) - B_2(z) = (z + z^{-1} - 2)^3,$$

$$B_1^1(y, z) - B_1(y, z) = (z + z^{-1} - y - y^{-1})^3. \quad (5.9)$$

The fact that $B_2(z), B_1(y, z)$ are different from $B_2^2(z), B_1^1(y, z)$ is not an immediate contradiction since they represent indices computed in different regions in the moduli space — the former in a region where the (2,0) string is short and the latter in a region where the (−3, −2) string is short. However for consistency we need to show that these two regions are separated by a new wall of marginal stability and that the jump in the index across this wall accounts for the differences shown in (5.9). This new wall can be identified by considering the network shown in figure 13(b). When the (−3, −2) string is short then the maximal size of the internal face is set by the configuration where the face touches the
Figure 14. A wall of marginal stability of the string network of figure 13(b).

D3-brane 1. On the other hand when the (2,0) string is short then the maximal size of the internal face is set by the configuration where the face touches the D3-brane 3. The boundary between these two regions of the moduli space corresponds to an arrangement of the D3-branes 1 and 3 such that when the internal face touches the D3-brane 1 it also touches the D3-brane 3.\footnote{During this deformation of the moduli we can keep the D3-brane 2 far away so that the internal face never touches it.} This situation has been shown in figure 14. From this diagram it is clear that this represents a wall of marginal stability along which the original network is unstable against decay into a (−2, −1) string stretched between D3-branes 1 and 3 and the rest of the network containing the (1, 1), (0, −1) and (1, 2) strings. The jump across this wall can be computed using the primitive wall crossing formula and involves the product of the index of a half BPS state represented by the (−2, −1) string and a quarter BPS state containing the (1, 1), (0, −1) and (1, 2) strings. The former is known from (3.13) while the latter can be found by applying the wall crossing formula again across the wall on which the (0, −1) string shrinks to zero size. The result is the following expression for the jump in the index across the marginal stability wall shown in figure 14:

\[
\Delta B_2(z) = (z + z^{-1} - 2)^3, \\
\Delta B_1(y, z) = (z + z^{-1} - y - y^{-1})^3. 
\] (5.10)

This accounts for the difference (5.9). By carefully calculating the phase of $Z_\gamma$ one can verify that (5.10) actually represents the jump in the index that we encounter as we cross from the side in which the (2,0) string is short towards the side on which the (−3, −2) string is short. This is precisely what is needed to explain the difference between $B'_2$, $B'_1$ and $B_2$, $B_1$ given in (5.9).

5.3 Example 3

The final example we shall consider is the string network shown in figure 15 and its possible deformations. We shall compute the index in the chamber in which the (−5, −15) string is short, i.e. near the wall where it can break apart into (2, 0) = 2(1, 0) and (3, 15) = 3(1, 5) string. For brevity we shall only compute the index $B_1(y, z)$ since $B_2(z)$ can be obtained by...
taking the $y \to 1$ limit of $B_1(y, z)$. Since the decay across the wall on which the $(-5, -15)$ string shrinks to zero size is neither primitive nor semi-primitive, we need the full power of the KS wall crossing formula. Labelling by $\gamma_1$ the charge carried by the $(1, 0)$ string and by $\gamma_2$ the charge carried by the $(1, 5)$ string, we see that

$$\gamma \equiv \langle \gamma_1, \gamma_2 \rangle = -5.$$  \hfill (5.11)

We also define

$$\kappa(x) = (-1)^x \frac{y^x - y^{-x}}{y - y^{-1}}.$$  \hfill (5.12)

Now the index we want to compute is $B_1^-(2\gamma_1 + 3\gamma_2; y, z)$. Using logic similar to the one used in the earlier examples we see that on the other side (+ side) of the wall of marginal stability the only non-zero indices of relevance are those of the half BPS states carrying charges $\gamma_1$ or $\gamma_2$:

$$B_1^+(\gamma_1; y, z) = z + z^{-1} - y - y^{-1},$$
$$B_1^+(m\gamma_1 + n\gamma_2) = 0 \text{ otherwise}.  \hfill (5.13)$$

Eq. (4.12) now gives

$$\tilde{B}_1^+(m\gamma_1; y, z) = \tilde{B}_1^+(m\gamma_2; y, z) = (z^m + z^{-m} - y^m - y^{-m}) \frac{1}{m} \frac{y - y^{-1}}{y^m - y^{-m}},$$
$$\tilde{B}_1^+(m\gamma_1 + n\gamma_2) = 0 \text{ otherwise}.  \hfill (5.14)$$

Using eq. (A.4) of [56] we now get

$$B_1^-(2\gamma_1 + 3\gamma_2; y, z) = \kappa(6\gamma) \tilde{B}_1^+(2\gamma_1; y, z) \tilde{B}_1^+(3\gamma_2; y, z) + \frac{1}{2} \kappa(3\gamma)^2 \tilde{B}_1^+(\gamma_1; y, z) \tilde{B}_1^+(3\gamma_2; y, z)$$
$$+ \kappa(2\gamma) \kappa(4\gamma) \tilde{B}_1^+(2\gamma_1; y, z) \tilde{B}_1^+(2\gamma_2; y, z) \tilde{B}_1^+(\gamma_2; y, z)$$
$$+ \frac{1}{2} \kappa(\gamma) \kappa(2\gamma) \{ \kappa(\gamma) + \kappa(3\gamma) \} \tilde{B}_1^+(\gamma_1; y, z) \tilde{B}_1^+(2\gamma_2; y, z) \tilde{B}_1^+(\gamma_2; y, z)$$
$$+ \frac{1}{6} \kappa(2\gamma)^3 \tilde{B}_1^+(2\gamma_1; y, z) \tilde{B}_1^+(\gamma_2; y, z)^3$$
$$+ \frac{1}{12} \kappa(\gamma)^3 \{ 3\kappa(\gamma) + \kappa(3\gamma) \} \tilde{B}_1^+(\gamma_1; y, z) \tilde{B}_1^+(\gamma_2; y, z)^3.  \hfill (5.15)$$

Eqs. (5.11)–(5.15) gives us the complete expression for $B_1^-(2\gamma_1 + 3\gamma_2; y, z)$.
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