Asymptotics of spinfoam amplitude on simplicial manifold: Lorentzian theory

Muxin Han and Mingyi Zhang

Centre de Physique Théorique\(^1\), CNRS-Luminy, Case 907, F-13288 Marseille, France

E-mail: Muxin.Han@cpt.univ-mrs.fr and Mingyi.Zhang@cpt.univ-mrs.fr

Received 17 December 2012, in final form 1 July 2013
Published 22 July 2013
Online at stacks.iop.org/CQG/30/165012

Abstract

This paper studies the large-\(j\) asymptotics of the Lorentzian Engle–Pereira–Rovelli–Livine (EPRL) spinfoam amplitude on a 4D simplicial complex with an arbitrary number of simplices. The asymptotics of the spinfoam amplitude is determined by the critical configurations. Here we show that, given a critical configuration in general, there exists a partition of the simplicial complex into three types of regions \(R_{\text{Nondeg}}, R_{\text{Deg-A}}\) and \(R_{\text{Deg-B}}\), where the three regions are simplicial sub-complexes with boundaries. The critical configuration implies different types of geometries in different types of regions, i.e. (1) the critical configuration restricted to \(R_{\text{Nondeg}}\) implies a nondegenerate discrete Lorentzian geometry, (2) the critical configuration restricted to \(R_{\text{Deg-A}}\) is degenerate of type-A in our definition of degeneracy, but it implies a nondegenerate discrete Euclidean geometry in \(R_{\text{Deg-A}}\), (3) the critical configuration restricted to \(R_{\text{Deg-B}}\) is degenerate of type-B, and it implies a vector geometry in \(R_{\text{Deg-B}}\). With the critical configuration, we further make a subdivision of the regions \(R_{\text{Nondeg}}\) and \(R_{\text{Deg-A}}\) into sub-complexes (with boundaries) according to their Lorentzian/Euclidean oriented 4-volume \(V_4(v)\) of the 4-simplices, such that \(\text{sgn}(V_4(v))\) is a constant sign on each sub-complex. Then in each sub-complex \(R_{\text{Nondeg}}\) or \(R_{\text{Deg-A}}\), the spinfoam amplitude at the critical configuration gives the Regge action in a Lorentzian signature or an Euclidean signature respectively. The Regge action reproduced here contains a sign prefactor \(\text{sgn}(V_4(v))\) related to the oriented 4-volume of the 4-simplices. Therefore the Regge action reproduced here can be viewed as a discretized Palatini action with an on-shell connection. Finally, the asymptotic formula of the spinfoam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated with different types of geometries.

PACS number: 04.60.Pp

(Some figures may appear in colour only in the online journal)

\(^1\)Unité mixte de recherche (UMR 6207) du CNRS et des Universités de Provence (Aix-Marseille I), de la Méditerranée (Aix-Marseille II) et du Sud (Toulon-Var); laboratoire affilié à la FRUMAM (FR 2291).
1. Introduction

Loop quantum gravity (LQG) is an attempt to make a background independent, non-perturbative quantization of four-dimensional General Relativity (GR)—for reviews, see [1–3]. It is inspired by the classical formulation of GR as a dynamical theory of connections. Starting from this formulation, the kinematics of LQG is well studied and results in a successful kinematical framework (see the corresponding chapters in the books [1]), which is also unique in a certain sense. However, the framework of the dynamics in LQG is still largely open so far. There are two main approaches to the dynamics of LQG; they are (1) the operator formalism of LQG, which follows the spirit of Dirac quantization or reduced phase-space quantization of a constrained dynamical system and performs a canonical quantization of GR [4]; (2) the covariant formulation of LQG, which is currently understood in terms of the spinfoam models [3, 5–8]. The relation between these two approaches is well understood in the case of 3D quantum gravity [9], while in 4D the situation is much more complicated and there are some recent attempts [10] for relating these two approaches.}

2 Rigorously speaking, it is so far unclear whether the spinfoam formulation truly gives the dynamics for LQG. For example, the natural boundary space of spinfoam models is projective spin networks, but not SU(2) spin networks.
This paper is concerning the spinfoam approach of LQG. The current spinfoam models for quantum gravity are mostly inspired by the four-dimensional Plebanski formulation of GR [11] (or Plebanski–Holst formulation by including the Barbero–Immirzi parameter $\gamma$), which is a background field (BF) theory constrained by the condition that the $B$ field should be ‘simple’, i.e. there is a tetrad field $e^I$ such that $B = \ast(e \wedge e)$. Currently, one of the successful spinfoam models is the EPRL model defined in [6], whose implementation of simplicity constraint is understood in the sense of [12]. The EPRL vertex amplitude is shown to reproduce the classical discrete GR in the large-$j$ asymptotics [13]. Recently, The fermion coupling is included in the framework of an EPRL spinfoam model [14], and a $q$-deformed EPRL spinfoam model is defined and gives discrete GR with a cosmological constant in the large-$j$ asymptotics [15, 16].

The semiclassical behavior of a spinfoam model is currently understood in terms of the \textit{large-$j$ asymptotics} of the spinfoam amplitude, i.e. if we consider a spinfoam model as a state sum

$$A(K) = \sum_{j_f} \mu(j_f) A_{j_f}(K), \quad (1.1)$$

where $\mu(j_f)$ is a measure, we investigate the asymptotic behavior of the (partial) amplitude $A_{j_f}$ as all the spins $j_f$ are taken to be large uniformly. The area spectrum in LQG is given approximately by $A_f = \gamma j_f \ell_p^2$, so the semiclassical limit of spinfoam models is argued to be achieved by taking $\ell_p^2 \to 0$ while keeping the area $A_f$ fixed, which results in $j_f \to \infty$ uniformly as $\gamma$ is a fixed Barbero–Immirzi parameter. There is another argument relating the large-$j$ asymptotics of the spinfoam amplitude to the semiclassical limit, by imposing the semiclassical boundary state to the vertex amplitude [17]. Mathematically, the asymptotic problem is posed by making a uniform scaling for the spins $j_f \mapsto \lambda j_f$ and studying the asymptotic behavior of the amplitude $A_{\lambda j_f}(K)$ as $\lambda \to \infty$.

There were various investigations for the large-$j$ asymptotics of the spinfoam models. The asymptotics of the Barrett–Crane vertex amplitude (10$j$-symbol) was studied in [18], which showed that the degenerate configurations in the Barrett–Crane model were non-Doscillatory, but dominant. The large-$j$ asymptotics of the Freidel–Krasnov model was studied in [19], concerning the nondegenerate Riemannian geometry, in the case of a simplicial manifold without a boundary. The large-$j$ asymptotics of the EPRL model was initially investigated in [13] for both Euclidean and Lorentzian cases, where the analysis concerned a single 4-simplex amplitude (EPRL vertex amplitude). It was shown that the asymptotics of the vertex amplitude is mainly a Cosine of the Regge action in a 4-simplex if the boundary data admits a nondegenerate 4-simplex geometry, and the asymptotics is non-oscillatory if the boundary data does not admit a nondegenerate 4-simplex geometry. There were also recent works to find the Regge gravity from the Euclidean/Lorentzian spinfoam amplitude on a simplicial complex via a certain ‘double scaling limit’ [20].

This work analyzes the large-$j$ asymptotic analysis of the Lorentzian EPRL spinfoam amplitude to the general situation of a 4D simplicial manifold with or without a boundary, with an arbitrary number of simplices. The analysis for the Euclidean EPRL model is presented in [21]. The asymptotics of the spinfoam amplitude is determined by the critical configurations of the ‘spinfoam action’, and is given by a sum of the amplitudes evaluated at the critical configurations. Therefore, the large-$j$ asymptotics is clarified once we find all the critical configurations and clarify their geometrical implications. Here for the Lorentzian EPRL spinfoam amplitude, a critical configuration in general is given by the data $(j_f, g_{ve}, \xi_{ef}, z_{vf})$ that solves the critical point equations, where $j_f$ is an SU(2) spin assigned to each triangle, $g_{ve}$ is an $\text{SL}(2, \mathbb{C})$ group variable, and $\xi_{ef}$ and $z_{vf}$ are two types of spinors. Here in this work
we show that given a general critical configuration, there exists a partition of the simplicial complex $K$ into three types of regions $R_{\text{Nondeg}}, R_{\text{Deg-A}}$ and $R_{\text{Deg-B}}$, where the three regions are simplicial sub-complexes with boundaries, and they may be disconnected regions. The critical configuration implies different types of geometries in different types of regions.

- The critical configuration restricted to $R_{\text{Nondeg}}$ is nondegenerate in our definition of degeneracy. It implies a nondegenerate discrete Lorentzian geometry on the simplicial sub-complex $R_{\text{Nondeg}}$.
- The critical configuration restricted to $R_{\text{Deg-A}}$ is degenerate of type-A in our definition of degeneracy. However, it implies a nondegenerate discrete Euclidean geometry on the simplicial sub-complex $R_{\text{Deg-A}}$.
- The critical configuration restricted to $R_{\text{Deg-B}}$ is degenerate of type-B in our definition of degeneracy. It implies a vector geometry on the simplicial sub-complex $R_{\text{Deg-B}}$.

With the critical configuration, we further make a subdivision of the regions $R_{\text{Nondeg}}$ and $R_{\text{Deg-A}}$ into sub-complexes (with boundary) $K_{\text{i}}(R_{\text{Nondeg}})$, $\ldots$, $K_{n}(R_{\text{Nondeg}})$ according to their Lorentzian/Euclidean-oriented 4-volume $V_{4}(v)$ of the 4-simplices, such that $\text{sgn}(V_{4}(v))$ is a constant sign on each $K_{i}(R_{\text{Nondeg}})$. Then in each sub-complex $K_{i}(R_{\text{Nondeg}})$ or $K_{i}(R_{\text{Deg-A}})$, the spinfoam amplitude at the critical configuration gives an exponential of the Regge action in a Lorentzian signature or an Euclidean signature respectively. However we emphasize that the Regge action reproduced here contains a sign factor $\text{sgn}(V_{4}(v))$ related to the oriented 4-volume of the 4-simplices, i.e.

$$S_{i} = \text{sgn}(V_{4}) \sum_{\text{Internal}} A_{f} \Theta_{f} + \text{sgn}(V_{4}) \sum_{\text{Boundary}} A_{f} \Theta_{f}^{B}, \quad (1.2)$$

where $A_{f}$ is the area of the triangle $f$, and $\Theta_{f}$ and $\Theta_{f}^{B}$ are the deficit angle and dihedral angle respectively. Recall that the Regge action without $\text{sgn}(V_{4})$ is a discretization of the Einstein–Hilbert action of GR. Therefore, the Regge action reproduced here is actually a discretized Palatini action with the on-shell connection (compatible with the tetrad).

The asymptotic formula of the spinfoam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated with different types of geometries.

Additionally, we also show in section 5 that given a spinfoam amplitude $A_{j_{f}}(K)$ with the spin configuration $j_{f}$, any pair of the nondegenerate critical configurations associated with $j_{f}$ is related with each other by a local parity transformation. The parity transformation is the one studied in [13] in the case of a single 4-simplex. A similar result holds for any pair of the degenerate configuration of type-A associated with $j_{f}$, since it implies a nondegenerate Euclidean geometry.

2. Lorentzian spinfoam amplitude

Given a simplicial complex $K$ (with or without boundary), the Lorentzian spinfoam amplitude on $K$ can be expressed in the coherent state representation\(^3\)\(^4\):

$$A(K) = \sum_{j_{f}} \prod_{f} \mu(j_{f}) \prod_{(e,f)} \int_{\text{SL}(2,C)} dg_{ee} \prod_{(e,f)} \int_{S^{2}} d\hat{n}_{ef} \prod_{v \in f} (j_{f}, \xi_{ef}) Y^{\dagger} g_{ve} g_{ve} Y |j_{f}, \xi_{ef}) \quad (2.1)$$

\(^3\) The spinfoam amplitude written here requires an orientation on each face. However, it turns out that it does not depend on the face orientation in the bulk, although it depends on the boundary face orientation.

\(^4\) The boundary state of the spinfoam amplitude is a spin-network state with all intertwiners being coherent intertwiners.
Here $\mu(j_f)$ is the face amplitude of the spinfoam given by $\mu(j_f) = (2j_f + 1)$. \(|j_f, \xi_{ef}\rangle\) is an SU(2) coherent state in the spin-\(j\) representation. The coherent state is labeled by the spin-\(j\) and a normalized two-component spinor \(|\xi_{ef}\rangle = g(\xi_{ef})|\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle (g(\xi_{ef}) \in \text{SU}(2))\), while \(\hat{\nu}_{ef} := g(\xi_{ef}) \otimes \hat{z}\) is a unit 3-vector. \(Y\) is an embedding map from the spin-\(j\) irrep \(\mathcal{H}_j\) of SU(2) to the unitary irrep \(\mathcal{H}^{(j)}\) of SL(2, \(\mathbb{C}\)) with \((k, p) = (j, \gamma j)\). The embedding \(Y\) identifies \(\mathcal{H}_j\) with the lowest level in the decomposition \(\mathcal{H}^{(j)} = \bigoplus_{m=0}^{2j} \mathcal{H}_m\). Therefore, we define an SL(2, \(\mathbb{C}\)) coherent state by the embedding

\[
|\langle j_f, \gamma j_f; j_f, \xi_{ef}\rangle| := Y|j_f, \xi_{ef}\rangle = \Pi^{(j_f, \gamma j_f)}(g(\xi_{ef}))|\langle j_f, \gamma j_f; j_f, j_f\rangle|.
\]

In order to write the \(\mathcal{H}^{(j_f, \gamma j_f)}\) inner product in equation (2.1) explicitly, we express the SL(2, \(\mathbb{C}\)) coherent state in terms of the canonical basis [23]. The Hilbert space \(\mathcal{H}^{(k, p)}\) can be represented as a space of homogeneous functions of two complex variables \((z^0, z^1)\) with degree \((-1 + ip + k; -1 + ip - k)\), i.e.

\[
f(\lambda z^\alpha) = \lambda^{-1+ip+k} \bar{\lambda}^{-1-ip-k} f(z^\alpha).
\]

Given a normalized two-component spinor \(z^\alpha (\alpha = 0, 1)\) with \((z, z) := \delta_{\alpha \bar{\alpha}} z^\alpha \bar{z}^\alpha = 1\), we construct the SU(2) matrix

\[
g(z) = \begin{pmatrix}
  z^0 \\
  z^1
\end{pmatrix} \equiv (z, Jz),
\]

where \(J(z^0, z^1) := (-z^1, \bar{z}^0)^T\). When restricted to the normalized spinors, the canonical basis \(f_{\alpha}(z)^{(k, p)} = |(k, p); j, m\rangle\) in the SL(2, \(\mathbb{C}\)) unitary irrep \(\mathcal{H}^{(k, p)}\) is given by the following:

\[
f_{\alpha}(z)^{(k, p)} = \frac{\text{dim}(j)}{\pi} D_{\alpha k}(g(z)),
\]

where \(D_{\alpha k}(g(z))\) is the SU(2) representation matrix. The canonical basis \(f_{\alpha}(z)^{(k, p)}\) evaluated on the non-normalized spinor \(z^\alpha\) is then given by the homogeneity

\[
f_{\alpha}(z)^{(k, p)} = \frac{\text{dim}(j)}{\pi} (z, z)^{ip-1-j} D_{\alpha k}(g(z)),
\]

while here \(D_{\alpha k}(g(z))\) is an analytic continuation of the SU(2) representation matrix. Thus, we can write down explicitly the highest weight state in the \(j\)-representation and in the case of \((k, p) = (j, \gamma j)\),

\[
f_j^j(z)^{(j_f, \gamma j_f)} = \frac{\text{dim}(j)}{\pi} (z, z)^{ip-1-j} (z^0)^2j.
\]

Therefore, the coherent state is given explicitly by

\[
|\langle j, \gamma j; j, \xi\rangle| = f_j^j(z)^{(j_f, \gamma j_f)} = f_j^j(g(\xi)^\gamma z)^{(j_f, \gamma j_f)} = \frac{\text{dim}(j)}{\pi} (z, z)^{ip-1-j} (\bar{z}, \xi)^{2j}.
\]

As a result, we can write down explicitly the inner product in equation (2.1) in terms of an \(L^2\) inner product on \(\mathbb{C}^2\) between the coherent states \(f_j^j(z)^{(j_f, \gamma j_f)}\):

\[
\langle j_f, \xi_{ef}|Y^\dagger g_{\alpha \bar{\alpha}} g_{\gamma \bar{\gamma}} Y|j_f, \xi_{ef}\rangle = \langle (j_f, \gamma j_f); j_f, \xi_{ef}|g_{\alpha \bar{\alpha}} g_{\gamma \bar{\gamma}} (j_f, \gamma j_f); j_f, \xi_{ef}\rangle = \int_{\mathbb{C}^2} \Omega_{\xi_{ef}} \langle f_j^j(z)^{(j_f, \gamma j_f)}(g_{\alpha \bar{\alpha}} z_{\alpha \bar{\alpha}})^\gamma \cdot f_j^j(z)^{(j_f, \gamma j_f)}(g_{\gamma \bar{\gamma}} z_{\gamma \bar{\gamma}})^{\gamma}_{\xi_{ef}} \rangle,
\]

where \(\Omega_{\xi} = \frac{1}{2} (z_0 d\bar{z}_1 - z_1 d\bar{z}_0) \wedge (\bar{z}_0 d\bar{z}_1 - \bar{z}_1 d\bar{z}_0)\) is a homogeneous measure on \(\mathbb{C}^2\).

We insert the result equation (2.9) back into equation (2.1) and define a new spinor variable \(Z_{\alpha \bar{\alpha}}\) and a measure on \(\mathbb{C}^{p^2}\) (a scaling invariant measure):

\[
Z_{\alpha \bar{\alpha}} := g_{\alpha \bar{\alpha}} z_{\alpha \bar{\alpha}} \\
\Omega_{\alpha \bar{\alpha}} := \langle Z_{\alpha \bar{\alpha}}, Z_{\alpha \bar{\alpha}}\rangle (Z_{\alpha \bar{\alpha}}, Z_{\alpha \bar{\alpha}}).
\]

\[\text{Class. Quantum Grav. 30 (2013) 165012}\]

M Han and M Zhang
Then the spinfoam amplitude $A(K)$ can be written as
\begin{equation}
A(K) = \sum_{j_f} \prod_{f} \mu(j_f) \prod_{(v,c)} \int_{\text{SL}(2,\mathbb{C})} \int_{\mathbb{CP}^1} \left( \frac{\dim(j_f)}{\pi} \Omega_{te} \right) e^{S}, \tag{2.11}
\end{equation}
where we have a ‘spinfoam action’ $S = \sum_f S_f$ and
\begin{equation}
S_f = \sum_{v \in f} (j_f \ln \frac{(|\xi_{te}|^2)(|\xi_{te}|^2)}{(|\xi_{te}|^2)(|\xi_{te}|^2)} + iy_j f \ln \frac{(|\xi_{te}|^2)(|\xi_{te}|^2)}{(|\xi_{te}|^2)(|\xi_{te}|^2)}). \tag{2.12}
\end{equation}

Note that the above derivation follows [13], where the above spinfoam action was given firstly. Here we reproduce the derivation in detail in order to make a self-contained presentation of the work. For the same reason, we reproduce the derivation of some critical equations given in [13].

In this paper, we consider the large-$j$ regime of the spinfoam amplitude $A(K)$. Concretely, we define the partial amplitude
\begin{equation}
A_{j_f}(K) := \prod_{(v,c)} \int_{\text{SL}(2,\mathbb{C})} \int_{\mathbb{CP}^1} \left( \frac{\dim(j_f)}{\pi} \Omega_{te} \right) e^{S}, \tag{2.13}
\end{equation}
and consider the regime in the sum $\sum_{j_f}$ where all the spins $j_f$ are large. In this regime, the spinfoam amplitude is a sum over the asymptotics of partial amplitude $A_{j_f}(K)$ with large spins $j_f$. In the following, we study the large-$j$ asymptotics of the partial amplitudes $A_{j_f}(K)$ by making the uniform scaling $j_f \mapsto \lambda j_f$ and taking the limit $\lambda \to \infty$. Each face action $S_f \mapsto \lambda S_f$ scales linearly with $\lambda$, so we can use the generalized stationary phase approximation [24] to study the asymptotical behavior of $A_{j_f}(K)$ in the large-$j$ regime.

Before coming to the asymptotic analysis, we note that in all the following discussions, the geometric tetrahedron with the oriented area $j_f \partial_{te}$, $f \subset t_c$ is always assumed to be nondegenerate.

### 2.1. Derivation of critical point equations

We use the generalized stationary phase method to study the large-$j$ asymptotics of the above spinfoam amplitude. The spinfoam amplitude has been reduced to the following type of integral:
\begin{equation}
f(\lambda) = \int_{D} \prod_{x} a(x) e^{S(x)}, \tag{2.14}
\end{equation}
where $D$ is a closed manifold, $S(x)$ and $a(x)$ are smooth, complex-valued functions, and $\text{Re} \ S \leq 0$ (this will be shown in the following for the spinfoam amplitude). For a large parameter $\lambda$, the dominant contributions for the above integral come from the critical points $x_c$, which are the stationary points of $S(x)$ and satisfy $\text{Re} \ S(x_c) = 0$. The asymptotic behavior of the above integral for large $\lambda$ is given by
\begin{equation}
f(\lambda) = \sum_{x_c} a(x_c) \left( \frac{2\pi}{\lambda} \right)^{\dim x_c} \frac{e^{\ln H(x_c)}}{\sqrt{\det \left( H'(x_c) \right)}} e^{S(x_c)} \left[ 1 + o\left( \frac{1}{\lambda} \right) \right], \tag{2.15}
\end{equation}
for isolated critical points, where $r(x_c)$ is the rank of the Hessian matrix $H_{ij}(x_c) = \partial_i \partial_j S(x_c)$ at a critical point and $H'(x_c)$ is the invertible restriction on $\ker H(x_c)^+$. When the critical points are not isolated, the above $\sum_{x_c}$ is replaced by an integral over a submanifold of critical points. If the $S(x)$ does not have any critical point, $f(\lambda)$ decreases faster than any power of $\lambda^{-1}$.  

From the above asymptotic formula, we see that the asymptotics of the spinfoam amplitude is clarified by finding all the critical points of the action and evaluating the integrand at each critical point.

In order to find the critical points of the spinfoam action, first of all, we show that the spinfoam action \( S \) satisfies \( \text{Re} \ S \leq 0 \). For each \( S_{ef} \), by using the Cauchy–Schwarz inequality
\[
\text{Re} \ S_{ef} = j_f \ln \frac{(\xi_{ef}^*(Z_{ref} f, Z_{ref} f))}{(Z_{ref} f, Z_{ref} f)} \leq j_f \ln \frac{(\xi_{ef}^*(Z_{ref} f, Z_{ref} f)\xi_{ef})(Z_{ref} f, Z_{ref} f)}{(Z_{ref} f, Z_{ref} f)} \leq 0.
\]
(2.16)

Therefore,
\[
\text{Re} \ S = \sum_{f,v} \text{Re} \ S_{ef} \leq 0.
\]
(2.17)

From \( \text{Re} \ S = 0 \), we obtain the following equations:
\[
\xi_{ef} = e^{i\phi_{e\nu}} Z_{ref} \quad \text{and} \quad \xi_{ef} = e^{i\phi_{\nu\nu}} Z_{vef},
\]
(2.18)

where \( \|Z_{vef}\| = \|Z_{vef} f, Z_{vef} f\|^{1/2} \). If we define \( \phi_{e\nu} = \phi_{\nu\nu} \), the above equation results in the fact that
\[
(g^{i}_{e\nu})^{-1} \xi_{ef} = \left(\frac{\|Z_{vef}\|}{\|Z_{vef} f, Z_{vef} f\|}\right)^{-1} e^{i\phi_{\nu\nu}} Z_{ref} e_{ef}.
\]
(2.19)

Here we use the property of anti-linear map \( J \)
\[
Jg^{-1} = (g^{i}_{e\nu})^{-1}
\]
(2.20)
in equation (2.19), we find
\[
g_{ve}(J\xi_{ef}) = \left(\frac{\|Z_{vef}\|}{\|Z_{vef} f, Z_{vef} f\|}\right)^{1/2} e^{-i\phi_{\nu\nu}} g_{ve}(J\xi_{ef}).
\]
(2.21)

Now we compute the derivative of the action \( S \) on the variables \( z_{ef}, \xi_{ef}, g_{ve} \) to find the stationary point of \( S \). We first consider the derivative with respect to the \( \mathbb{C}^3 \) variable \( z_{ef} \). Given a spinor \( \epsilon' = (\epsilon_0, \epsilon_1) \), \( \epsilon' \) and \((J\epsilon)' = (\epsilon_{\overline{1}}, \epsilon_{\overline{0}}) \) are the bases of the space \( \mathbb{C}^3 \) of two-component spinors. The following variation can be written in general by
\[
\delta \epsilon' = \epsilon(J\epsilon)' + \omega \epsilon',
\]
(2.22)

where \( \epsilon, \omega \) are the complex numbers. Since \( z \in \mathbb{C}^3 \), we can choose a partial gauge fixing that \((z, \bar{z}) = 1 \), which gives \((\delta z, z) = -(z, \delta z) \). Thus, we obtain \( \omega = i\eta \) with a real number \( \eta \). Moreover, if we choose the variation with \( \epsilon = 0 \), it leads to \( \delta \epsilon' = i\eta \epsilon' \), which gives \( \eta = 0 \) for \( z \in \mathbb{C}^3 \). Using the variation \( \delta \epsilon' = \epsilon_{ef}(J\xi_{ef})' \) and \( \delta \epsilon'' = \bar{\epsilon}_{ef}(J\xi_{ef})'' \), we obtain that
\[
0 = \delta_{z_{ef}} S_{ef}
\]
(2.23)
Using equation (2.18), we obtain the following equation:
\[
\langle Jz_{ef}, g_{ve} \xi_{ef} \rangle = \left\| Z_{vef} \right\| e^{i\theta_{ve}} \langle Jz_{ef}, g_{ve} \xi_{ef} \rangle.
\]
(2.24)

Also, from equation (2.18), because of \( \langle \xi_{ef}, \xi_{ef} \rangle = \langle z_{ef}, z_{ef} \rangle = 1 \),
\[
\langle z_{ef}, g_{ve} \xi_{ef} \rangle = \frac{\left\| Z_{vef} \right\|}{\left\| Z_{vef} \right\|} e^{i\theta_{ve}} \langle z_{ef}, g_{ve} \xi_{ef} \rangle.
\]
(2.25)

Therefore, since \( z^\alpha \) and \( (Jz)^\mu \) are the bases of the space \( \mathbb{C}^2 \) of two-component spinors,
\[
g_{ve} \xi_{ef} = \frac{\left\| Z_{vef} \right\|}{\left\| Z_{vef} \right\|} e^{i\theta_{ve}} g_{ve} \xi_{ef}.
\]
(2.26)

The above critical equations were firstly given in [13]. We reproduce the derivation in detail for the completeness. However, the variation with respect to \( \xi_{ef} \) was not discussed previously in the literature, in particular [13], where a single 4-simplex is considered. Since the spinor \( \xi_{ef} \) is normalized, we should use \( \delta \xi^\alpha = \omega_{ef} (J\xi_{ef})^\alpha + i\eta_{ef} \xi^\alpha_\xi \) for the complex infinitesimal parameters \( \omega \in \mathbb{C} \) and \( \eta \in \mathbb{R} \). The variation of the action vanishes automatically
\[
\delta_{\xi_{ef}} S = j_f \left( 2 \frac{\delta_{\xi_{ef}} (\xi_{ef}, Z_{vef})}{(\xi_{ef}, Z_{vef})} + 2 \frac{\delta_{\xi_{ef}} (Z_{vef}, \xi_{ef})}{(Z_{vef}, \xi_{ef})} \right)
= j_f \left( 2 \frac{iJ^{I\mu} Z_{vef}, Z_{ef}}{(\xi_{ef}, Z_{vef})} + 2 \frac{iJ^{I\mu} Z_{vef}, Z_{ef}}{(Z_{vef}, \xi_{ef})} \right)
= 0
\]
(2.27)

by using equation (2.18) and the identity \( (J\xi_{ef}, \xi_{ef}) = 0 \).

Finally, we consider the stationary point for the group variables \( g_{ve} \). We parameterize the group with the parameter \( \theta_{IJ} \) around a saddle point \( g_{ve} \), i.e. \( \theta_{IJ} = g_{ve} e^{-i\phi \mu J^{I\mu}} \), where \( J^{I\mu} \) is the generator of the Lie algebra \( sl_2(\mathbb{C}) \). Then, we have
\[
0 = \frac{\delta S_{vef}}{\delta \theta_{IJ}} \big|_{\delta \theta = 0} = \sum_{f \text{ incoming } e} j_f \left[ \frac{2 (\xi_{ef}, iJ^{I\mu} Z_{vef})}{(\xi_{ef}, Z_{vef})} - \frac{i (J^{I\mu} Z_{vef}, Z_{ef}) + (Z_{vef}, iJ^{I\mu} Z_{ef})}{(Z_{vef}, Z_{ef})} \right]
+ i\gamma j_f \left[ \frac{i (J^{I\mu} Z_{vef}, \xi_{ef})}{(\xi_{ef}, Z_{vef})} - \frac{i (J^{I\mu} Z_{vef}, \xi_{ef})}{(Z_{vef}, \xi_{ef})} \right]
+ \sum_{f \text{ outgoing } e} j_f \left[ \frac{2 (iJ^{I\mu} Z_{vef}, Z_{ef})}{(\xi_{ef}, Z_{vef})} - \frac{i (J^{I\mu} Z_{vef}, Z_{ef}) + (Z_{vef}, iJ^{I\mu} Z_{ef})}{(Z_{vef}, Z_{ef})} \right]
+ i\gamma j_f \left[ \frac{i (J^{I\mu} Z_{vef}, Z_{ef}) + (Z_{vef}, iJ^{I\mu} Z_{ef})}{(Z_{vef}, Z_{ef})} \right].
\]
(2.28)

We again apply equation (2.18) and find
\[5\]
\[5\]
\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]

\[5\]
Given a spinor \( \xi_{ef} \) given in \( [13] \).

Individual 4-simplices, which is essentially the same as many copies of the critical equations in the case of an arbitrary simplicial complex, the spinfoam critical equations factorize into

\[
0 = \sum_{f \text{ incoming}} \text{e}_{\text{f}} (\xi_{ef}, iJ^{H \xi_{ef}}) + \text{f}_{\text{o}} (\xi_{ef}, iJ^{H \xi_{ef}}) + \text{g}_{\text{e}} (\xi_{ef}, iJ^{H \xi_{ef}}) - (\xi_{ef}, iJ^{H \xi_{ef}})
\]

for the triangle \( f \) shared by the tetrahedra \( t_e \) and \( t_f \) in the 4-simplex \( \sigma_{e} \), and the dual edge \( e = (v, v') \). As usual we can rewrite the Lorentz Lie algebra generator \( J^H \) in terms of the rotation part \( J^R \) and boost part \( K \), where \( J_i = \frac{1}{2} \epsilon_{ijk} J^H \) and \( K = -iJ^0 \). In the spin-1/2 representation, the rotation generators \( J^H = \frac{1}{2} \hat{x} \) and the boost generators \( K = \frac{1}{2} \hat{y} \). Recall that

\[
\langle \xi | \sigma | \xi \rangle = \hat{n}_e \quad \text{with} \quad \hat{n}_e = (\xi^0 \xi^1 + \xi^1 \xi^2) \hat{x} = i(\xi^0 \xi^1 - \xi^1 \xi^2) \hat{y} = (\xi^0 \xi^2 - \xi^1 \xi^3) \hat{z},
\]

we have

\[
\langle \xi_{ef}, J^R \xi_{ef} \rangle = - (\xi_{ef}, J^R \xi_{ef}) = \frac{1}{2} \hat{n}_{ef}
\]

(2.33)

\[
\langle \xi_{ef}, K^R \xi_{ef} \rangle = (\xi_{ef}, K^R \xi_{ef}) = \frac{1}{2} \hat{n}_{ef}
\]

(2.34)

Using the above relations, equation (2.31) results in the closure condition \([13]\)

\[
\sum_{f \subset t_e} \text{e}_{\text{f}} (v) jf \hat{n}_{ef} = 0.
\]

(2.35)

Thus, we finish the derivation of all the critical point equations.

2.2. Analysis of critical point equations

We summarize the critical point equations for a spinfoam configuration \((f, g_{e}, \xi_{ef}, z_{ef})\)

\[
g_{e} (J \xi_{ef}) = \frac{|Z_{e}|}{|Z_{ef}|} e^{-i\phi_{e} \sigma_{e}} g_{e} (J \xi_{ef})
\]

(2.36)

\[
g_{e} \xi_{ef} = \frac{|Z_{e}|}{|Z_{ef}|} e^{i\phi_{e} \sigma_{e}} g_{e} \xi_{ef}
\]

(2.37)

\[
0 = \sum_{f \subset t_e} \text{e}_{\text{f}} (v) jf \hat{n}_{ef},
\]

(2.38)

where equation (2.38) stands for the closure condition for each tetrahedron. We find that in the case of an arbitrary simplicial complex, the spinfoam critical equations factorize into individual 4-simplices, which is essentially the same as many copies of the critical equations given in \([13]\).

In the following, we show that equations (2.36) and (2.37) give the parallel transportation condition of the bivectors. Given a spinor \( \xi^a \), it naturally constructs a null vector \( \bar{\xi}^a = i(\xi)^{\dagger} a^a \), where \( a^a = (1, \hat{x}) \). It is straightforward to check that

\[
\bar{\xi} \bar{\xi} = \frac{1}{2} (1 + \hat{x} \cdot \hat{n}_e) \quad \text{with} \quad \hat{n}_e = (\xi^0 \xi^1 + \xi^1 \xi^2) \hat{x} = i(\xi^0 \xi^1 - \xi^1 \xi^2) \hat{y} = (\xi^0 \xi^2 - \xi^1 \xi^3) \hat{z},
\]

(2.39)
where $\hat{n}_\xi$ is a unit 3-vector since $\xi$ is a normalized spinor. Thus, we obtain that
\[
\iota(\xi) = \frac{1}{\xi}(1, \hat{n}_\xi).
\] (2.40)
Similarly for the spinor $J\xi$, we define the null vector $J\xi^a\xi^a = \iota(J\xi)^a a_f^{aa}$ and obtain that
\[
\iota(J\xi) = \frac{1}{\xi}(1, -\hat{n}_\xi).
\] (2.41)
We can write equations (2.36) and (2.37) in their spin-1 representation:
\[
\hat{g}_{ae} \iota(J\xi) = \frac{\|Z_{aef}\|^2}{\|Z_{aef}\|^2} \hat{g}_{ae'} \iota(J\xi e_f) \quad \text{and} \quad \hat{g}_{ae} \iota(\xi e_f) = \frac{\|Z_{aef}\|^2}{\|Z_{aef}\|^2} \hat{g}_{ae'} \iota(\xi e_f).
\] (2.42)
It is obvious that if we construct a bivector\(^6\)
\[
X_{ef}^{IJ} = -4\gamma j_f (\iota(\xi e_f) \wedge \iota(J\xi e_f))^{IJ},
\] (2.43)
$X_{ef}$ satisfies the parallel transportation condition within a 4-simplex:
\[
(\hat{g}_{ae})^I_K (\hat{g}_{ae'})^J_K X_{ef}^{KL} = (\hat{g}_{ae})^I_K (\hat{g}_{ae'})^J_K X_{ef}^{KL}. \tag{2.44}
\]
We define the bivector $X_{ef}^{IJ}$ located at each vertex $v$ of the dual face $f$ by the parallel transportation
\[
X_{ef}^{IJ}(v) := (\hat{g}_{ae})^I_K (\hat{g}_{ae'})^J_K X_{ef}^{KL}(v'),
\]
which is independent of the choice of $e$ by the above parallel transportation condition. Then we have the parallel transportation relation of $X_{ef}^{IJ}(v)$:
\[
X_{ef}^{IJ}(v) = (\hat{g}_{ae})^I_K (\hat{g}_{ae'})^J_K X_{ef}^{KL}(v') \tag{2.46}
\]
because the spinor $\xi e_f$ belonging to the tetrahedron $t_e$ is shared as the boundary data by two neighboring 4-simplex.

On the other hand, we can write the bivector $X_{ef}^{IJ}$ as a matrix:
\[
X_{ef}^{IJ} = 2\gamma j_f \begin{pmatrix}
0 & \hat{n}_1^{ef} & \hat{n}_2^{ef} & \hat{n}_3^{ef} \\
-\hat{n}_1^{ef} & 0 & 0 & 0 \\
-\hat{n}_2^{ef} & 0 & 0 & 0 \\
-\hat{n}_3^{ef} & 0 & 0 & 0
\end{pmatrix}
\]
\[
|X_{ef}^{IJ}| = \sqrt{\frac{1}{2} X_{ef}^{IJ} X_{ef}^{IJ}} = 2\gamma j_f. \tag{2.47}
\]
However, the matrix $(X_{ef})^I_J = X_{ef}^{IJ} \eta_{IJ}$ reads
\[
X_{ef} \equiv (X_{ef})^I_J = 2\gamma j_f \begin{pmatrix}
0 & \hat{n}_1^{ef} & \hat{n}_2^{ef} & \hat{n}_3^{ef} \\
\hat{n}_1^{ef} & 0 & 0 & 0 \\
\hat{n}_2^{ef} & 0 & 0 & 0 \\
\hat{n}_3^{ef} & 0 & 0 & 0
\end{pmatrix} = 2\gamma j_f \hat{n}_e \cdot \vec{K}. \tag{2.48}
\]
where $\vec{K}$ denotes the boost generator of the Lorentz Lie algebra $\mathfrak{sl}_2 \mathbb{C}$ in the spin-1 representation. The rotation generator in $\mathfrak{sl}_2 \mathbb{C}$ is denoted by $\vec{J}$. The generators in $\mathfrak{sl}_2 \mathbb{C}$ satisfy the commutation relations $[\vec{J}, \vec{J}] = -\epsilon^{ijk} J^k$, $[\vec{J}, \vec{K}'] = -\epsilon^{ijk} K^k$, $[\vec{K}', \vec{K}'] = \epsilon^{ijk} J^j$. The relation $X_{ef} = 2\gamma j_f \hat{n}_e \cdot \vec{K}$ gives a representation of the bivector in terms of the $\mathfrak{sl}_2 \mathbb{C}$ lie algebra generators. Moreover, it is not difficult to verify that in the spin-1/2 representation $\vec{J} = \frac{1}{2} \vec{\sigma}$ and $\vec{K} = \frac{1}{2} \vec{\sigma}^2$. Thus, in the spin-1/2 representation
\[
X_{ef} = \gamma j_f \vec{\sigma} \cdot \hat{n}_e. \tag{2.49}
\]
\(^6\) The pre-factor is a convention for simplifying the notation in the following discussion.
For this $\mathfrak{sl}_2 \mathbb{C}$ Lie algebra representation of the bivector $X_{ef}$, the parallel transportation is represented by the adjoint action of the Lie group on its Lie algebra. Therefore, we have

\[ g_{ve}X_{ef}g_{ev} = g_{ve}X_{ef}g_{ev}, \quad X_f(v) := g_{ve}X_{ef}(v)g_{ve}, \quad X_f(v) := g_{ve}X_f(v')g_{ve}, \]

(2.50)

where $g_{ve} = g_{ev}^{-1} = g_{ve}^{-1}$. We note that the above equations are valid for all the representations of $\text{SL}(2, \mathbb{C})$.

There is the duality map acting on $\mathfrak{sl}_2 \mathbb{C}$ by $\ast \hat{J} = -\hat{K}, \hat{K} = \ast \hat{J}$. For self-dual/anti-self-dual bivector $\hat{T}_k := \frac{1}{2}(\hat{J} \pm i\hat{K})$, One can verify that $\ast \hat{T}_k = \pm i\hat{T}_k$. In the spin-1 representation, the duality map is represented by $\ast X^{ij} = \frac{1}{2}\epsilon^{ijk}X_{k\ell}$. In the spin-1/2 representation, the duality map is represented by $\ast X = iX$ since $\hat{J} = \frac{i}{2}\sigma$ and $\hat{K} = \frac{1}{2}\sigma$ in the spin-1/2 representation. From equation (2.48), we see that

\[ X_{ef} = -\ast (2\gamma_{jf}\hat{n}_{ef} \cdot \hat{J}). \]

(2.51)

From its bivector representation, one can see that

\[ \eta_{ij}\mu^i X_{ef}^{jk} = 0, \quad \mu^i = (1, 0, 0, 0). \]

(2.52)

It motivates us to define a unit vector at each vertex $v$ for each tetrahedron $t_v$ by

\[ N_l^i(v) := (\hat{g}_{ve})^j_i \mu^j. \]

(2.53)

Then for all triangles $f$ in the tetrahedron $t_v$, $N_l^i(v)$ is orthogonal to all the bivectors $\ast X_f(v)$ with $f$ belonging to $t_v$.

\[ \eta_{ij}N_l^i(v) \ast X_f^{jk}(v) = 0. \]

(2.54)

In addition, from the closure constraint equation (2.38), we obtain for each tetrahedron $t_v$,

\[ \sum_{f \subset t_v} \epsilon_{ef}(v)X_f(v) = 0. \]

(2.55)

We summarize the above analysis of the critical point equations (2.36)–(2.38) into the following proposition.

**Proposition 2.1.** Given that the data $(j_f, g_{ve}, \xi_{ef}, \gamma_{ef})$ is a spinfoam configuration that solves the critical point equations (2.36)–(2.38), we construct the bivector variables (in the $\mathfrak{sl}_2 \mathbb{C}$ Lie algebra representation) for the spinfoam amplitude $X_{ef} = -\ast (2\gamma_{jf}\hat{n}_{ef} \cdot \hat{J})$ and $X_f(v) := g_{ve}X_{ef}g_{ev}$, where $|X_f(v)| = \sqrt{\frac{1}{2}\text{tr}(X_f(v)X_f(v))} = 2\gamma_{jf}$. The critical point equations imply the following equations for the bivector variables:

\[ X_{ef}(v) = X_{ef}(v) \equiv X_f(v), \quad X_f(v) := g_{ve}X_f(v')g_{ve}, \]

\[ \eta_{ij}N_l^i(v) \ast X_f^{jk}(v) = 0, \quad \sum_{f \subset t_v} \epsilon_{ef}(v)X_f(v) = 0, \]

(2.56)

where $t_v$ and $t_{ef}$ are two different tetrahedra of a 4-simplex dual to $v$, $f$ is a triangle shared by the two tetrahedra $t_v$ and $t_{ef}$, and $N_l^i(v) = (\hat{g}_{ve})^j_i \mu^j$ with $\mu^i = (1, 0, 0, 0)$ is a unit vector associated with the tetrahedron $t_v$.

### 3. Geometric interpretation of nondegenerate critical configuration

#### 3.1. Classical geometry from spinfoam critical configuration

Now we come back to the discussion of the critical point of spinfoam amplitude. The purpose of this section is to make a relation between the solutions of the critical point equations (2.36)–(2.38) and the Lorentzian discrete geometry. The detailed discussion of the nondegenerate
discrete Lorentzian geometry in terms of the discrete cotetrad and connection is presented in appendix A, see also [19, 25].

Given a spin foam configuration \((j_f, g_{ef}, \xi_{ef}; z_f)\) that solves the critical point equations, let us recall proposition 2.1 and consider a triangle \(f\) shared by two tetrahedra \(t_e\) and \(t_f\) of a 4-simplex \(\sigma_v\). In equation (2.56), there are the simplicity conditions \(N^v_{ef}(v) = 0\) and \(N^v_{ef}(v) + X^v_{ef}(v) = 0\) from the viewpoint of the two tetrahedra \(t_e\) and \(t_f\). The two simplicity conditions imply that there exist two 4-vectors \(M^v_{ef}(v)\) and \(M^v_{ef}(v)\), such that \(X_{ef}(v) = N_{ef}(v) \land M_{ef}(v)\) and \(X_{ef}(v) = N_{ef}(v) \land M_{ef}(v)\). However, we have in equation (2.56) the gluing condition \(X_{ef}(v) = X_{ef}(v) = X_f(v)\), which implies that \(N_{ef}(v)\) belongs to the plane spanned by \(N_{ef}(v), M_{ef}(v)\), i.e. \(N_{ef}(v) = a_{ef}M_{ef}(v) + b_{ef}N_{ef}(v)\). If we assume the following nondegeneracy condition\(^7\):

\[
\prod_{e_1, e_2, e_3, e_4 = 1}^5 \det(N_{e_1}(v), N_{e_2}(v), N_{e_3}(v), N_{e_4}(v)) \neq 0, \quad (3.1)
\]

then \(N_{ef}(v)\) and \(N_{ef}(v)\) cannot be parallel with each other, for all pairs of \(e, e'\), which excludes the case of vanishing \(a_{ef}\) in the above. Denoting \(a_{ef}' = a_{ef}^{-1}\), we obtain that \(M_{ef}(v) = a_{ef}N_{ef}(v) - a_{ef}b_{ef}N_{ef}(v)\). Therefore,

\[
X_f(v) = a_{ef}'(v)[N_{ef}(v) \land N_{ef}(v)] \quad (3.2)
\]

for all \(f\) shared by \(t_e\) and \(t_f\). Note that within a simplex \(\sigma_v\) there is a one-to-one correspondence between a pair of tetrahedra \(t_e\) and \(t_f\) and a triangle \(f\) shared by them. Thus, we can write the bivector \(X_f(v) = X_{ef}(v) = a_{ef}(v)[N_{ef}(v) \land N_{ef}(v)]\).

We label the five tetrahedra of \(\sigma_v\) by \(t_i, i = 1, \ldots, 5\). Then equation (3.2) reads

\[
X_{ef}(v) = a_{ij}(v)[N_{ef}(v) \land N_{ef}(v)]. \quad (3.3)
\]

Then the closure condition \(\sum_{j=1}^{4} e_{ef}(v)X_{ef}(v) = 0\)\(^8\) gives that \(\forall i = 1, \ldots, 5,\)

\[
0 = \sum_{j=1}^{4} e_{ef}(v)a_{ij}(v)[N_{ef}(v) \land N_{ef}(v)] = N_{ef}(v) \land \sum_{j=1}^{4} e_{ef}(v)a_{ij}(v)N_{ef}(v), \quad (3.4)
\]

which implies that for a choice of the diagonal element \(\beta_i(v)\),

\[
\sum_{j=1}^{5} \beta_j(v)N_{ef}(v) = 0, \quad (3.5)
\]

where we denote \(\beta_j(v) := e_{ef}(v)a_{ij}(v).\) Here, \(\beta_i(v)\) must be chosen as nonzero, because if \(\beta_i(v) = 0\), equation (3.5) would reduce to \(\sum_{j \neq i} \beta_j(v)N_{ef}(v) = 0\), which gives all the coefficients \(\beta_j(v) = 0\) by linearly independence of any four \(N_{ef}(v)\) (from the nondegeneracy equation (3.1)).

**Lemma 3.1.** **Assuming the nondegeneracy condition, equations (3.1) and (3.5) imply a factorization of \(\beta_j(v)\):**

\[
\beta_j(v) = \hat{\varepsilon}(v)\beta_i(v)\beta_j(v), \quad (3.6)
\]

where \(\varepsilon(v) = \pm 1.\) Thus, we have the following expression of the bivector \(e_{ef}(v)X_{ef}(v)\):

\[
e_{ef}(v)X_{ef}(v) = \hat{\varepsilon}(v)\beta_i(v)N_{ef}(v) \land (\beta_j(v)N_{ef}(v)). \quad (3.7)
\]

\(^7\) Note that the nondegenerate here is purely a condition for the group variables \(g_{ef}\), since \(N_{ef}(v) = g_{ef}(1, 0, 0, 0)^t\).

\(^8\) Here \(e_{ef}(v) = -e_{ef}(v)\) and \(X_{ef}(v) = X_{ef}(v)\).
Equation (3.5) then takes the form
\[
\sum_{j=1}^{5} \beta_j(v) N_{ej}(v) = 0.
\]
(3.8)

\[\beta_i(v)\] satisfies the following relation with \(j_f\) and \(g_{se}\):
\[
4v^2 j_f^2 = \beta_e(v)^2 \beta_{ef}(v)^2 [1 - \cosh^2 \theta_{se}(v)], \quad \cosh \theta_{se}(v) = N_{se}(v) \cdot N_{fe}(v).
\]
(3.9)

Such a result was firstly shown in [19]. The proof is also given in appendix B.

Now we construct the frame vectors \(U_{er}(v)\) for a classical discrete geometry at each vertex \(v^0\):
\[
U_{er}(v) := \pm \frac{\beta_e(v) N_{er}(v)}{\sqrt{|V_4(v)|}}
\]
with \(V_4(v) := \det(\beta_2(v) N_{e2}(v), \beta_3(v) N_{e3}(v), \beta_4(v) N_{e4}(v), \beta_5(v) N_{e5}(v))\),
(3.10)

where \(U_{er}(v)\) are time-like 4-vectors by equation (2.53), and any four of the five frame vectors \(U_{er}(v)\) span a four-dimensional vector space by the assumption of nondegeneracy. Moreover, the frame vectors satisfy the closure conditions
\[
\sum_{j=1}^{5} U_{er}(v) = 0
\]
(3.11)

and
\[
\frac{1}{V_4(v)} = \det(U_{e2}(v), U_{e3}(v), U_{e4}(v), U_{e5}(v)),
\]
(3.12)

and
\[
\varepsilon_{efj}(v) X_{efj}^{(0)}(v) = \varepsilon(v) |V_4(v)|[U_{er}(v) \wedge U_{er}(v)]_I = \varepsilon(v) V_4(v)[U_{er}(v) \wedge U_{er}(v)]_I,
\]
(3.13)

where \(\varepsilon(v) = \varepsilon(v) |V_4(v)|\). We emphasize that these frame vectors \(U_{er}(v)\) are constructed from the spinfoam configuration \((j_f, g_{se}, \xi_{ef}, \zeta_{ef})\) that solves the critical point equations. Note that the oriented 4-volume \(V_4(v)\) in general can be either positive or negative for different 4-simplices. However, for a nondegenerate critical configuration \((j_f, g_{se}, \xi_{ef}, \zeta_{ef})\), we can always make a subdivision of the triangulation, such that \(\text{sgn}(V_4(v))\) is a constant within each sub-triangulation.

Fixing an edge \(e_1\) at the vertex \(v\), we construct the inverse of the nondegenerate matrix \((U_{e2}(v), U_{e3}(v), U_{e4}(v), U_{e5}(v))\)' denoted by \(E^I_{e1}(v)\), such that
\[
U^I_{e1}(v) E^I_{e1}(v) = \delta^I_j, \quad i, j = 2, 3, 4, 5.
\]
(3.14)

Explicitly, for example,
\[
E^I_{e1}(v) = V_4(v) e^{ijkl} U^I_{e1}(v) U^I_{e2}(v) U^I_{e3}(v) U^I_{e5}(v).
\]
(3.15)

Note that \(E^I_{e1}(v)\) is determined only up to a sign by equation (3.10). However, if we fix \(e_2\) instead of \(e_1\) and find the inverse of \((U_{e1}(v), U_{e3}(v), U_{e4}(v), U_{e5}(v))\)' denoted by \(E^I_{e2}(v)\), then
\[
U^I_{e1}(v) E^I_{e2}(v) = \delta^I_j, \quad i, j = 1, 3, 4, 5
\]
(3.16)

and
\[
E^I_{e1}(v) = -V_4(v) e^{ijkl} U^I_{e1}(v) U^I_{e2}(v) U^I_{e3}(v) U^I_{e5}(v),
\]
(3.17)

9 We denote the dual vector \(N_f\) by \(N^e\) and the vector \(N_f\) by \(N_e\), and the same convention holds for \(U_e\) and \(U^e\).
where the minus sign comes from $V_{4}(v)$, because from the closure condition
\[ \sum_{j=1}^{5} U_{ej}(v) = 0, \]
\[ \det(U^{e_{2}}(v), U^{e_{3}}(v), U^{e_{4}}(v), U^{e_{5}}(v)) = - \det(U^{e_{1}}(v), U^{e_{4}}(v), U^{e_{5}}(v), U^{e_{3}}(v)). \]  \tag{3.18} \]

Therefore, we find
\[ E^{l}_{e_{ej}}(v) = - E^{l}_{e_{ej}}(v). \]  \tag{3.19} \]

Then we can fix $e_{3}$, $e_{4}$ and $e_{5}$, and do the same manipulation as above, to obtain $E_{e_{ij}}(v)$, $i, j = 1, \ldots, 5$, such that
\[ U^{e_{ij}}(v) E^{l}_{e_{ej}}(v) = \delta_{ij} - \delta_{jk} \]
and
\[ E^{l}_{e_{ij}}(v) = - E^{l}_{e_{ij}}(v) \]  \tag{3.20} \]
from which we can see that all $E^{l}_{e_{ij}}(v)$ are spatial vectors. One can also verify immediately that
\[ U^{e_{ij}}(v) \left( E^{l}_{e_{ij}}(v) + E^{l}_{e_{ij}}(v) + E^{l}_{e_{ij}}(v) \right) = 0 \quad \forall i = 1, \ldots, 5. \]  \tag{3.21} \]

By the nondegeneracy of $U^{e_{ij}}(v)$, one has
\[ E^{l}_{e_{ij}}(v) + E^{l}_{e_{ij}}(v) + E^{l}_{e_{ij}}(v) = 0. \]  \tag{3.22} \]

Comparing equations (3.20) and (3.22) with equation (A.8), we see that the collection of $E_{e_{ij}}(v)$ at $v$ is a co-frame at the vertex $v$. The bivector $X^{e}_{e_{ij}}(v)$ can also be expressed by $E^{l}_{e_{ij}}(v)$:
\[ \varepsilon_{e_{ij}}^{e_{kl}}(v) X^{e_{kl}}_{e_{ij}}(v) = \varepsilon(v) \ast [E_{e_{ij}}(v) \wedge E_{e_{ij}}(v)]^{e_{kl}} \]  \tag{3.23} \]

which will also be denoted by $\varepsilon_{e_{ij}}(v) X_{e_{ij}}^{e_{kl}}(v) = \varepsilon(v) \ast [E_{e_{ij}}(v) \wedge E_{e_{ij}}(v)]^{e_{kl}}$.

The above work are done essentially with in a single 4-simplex $\sigma_{e}$. Now we consider two neighboring 4-simplices $\sigma_{v}$ and $\sigma_{v'}$ while their respective centers $v$ and $v'$ are connected by the dual edge $e$. We have the following result (see appendix C for a detailed proof).

**Lemma 3.2.** We define a sign $\varepsilon_{e}(v) = \pm 1$ by $\varepsilon_{e}(v) \frac{U^{e}(v)}{U^{e}(v)} = N_{e}(v)$. Given an edge $e = (v, v')$, $X_{e}(v) = g_{vv} X_{e}(v') g_{v'}$ and $\varepsilon_{e}(v) \frac{U^{e}(v)}{U^{e}(v')} = g_{vv} \varepsilon_{e}(v') \frac{U^{e}(v)}{U^{e}(v')} \implies$ that

1. $\varepsilon(v) = \varepsilon(v') = \varepsilon$ is a global sign on the triangulation.
2. For all edge $e$ of a triangulation $t_{c}$ shared by the 4-simplices $\sigma_{e}$ and $\sigma_{v}$, we have the parallel transportation relation for $E_{e}(v)$ and $E_{e}(v')$ up to a sign, i.e.
\[ \mu_{e} E_{e}(v) = g_{vv} E_{e}(v'), \]  \tag{3.24} \]
where $\mu_{e} = - \varepsilon_{e}(v) \varepsilon_{e}(v') \text{sgn}(V_{4}(v) V_{4}(v'))$. If $\Omega_{vv'} \in \text{SO}(4)$ is the unique discrete spin connection determined by $E_{e}(v)$, $E_{e}(v')$, then
\[ g_{vv'} = \mu_{e} \Omega_{vv'}. \]  \tag{3.25} \]

Since $E_{e}(v)$ and $U_{e}(v)$ are determined by the critical data up to $\pm$, the sign $\varepsilon_{e}(v)$ is determined up to $\pm$, and $\mu_{e}$ is determined up to $\pm \varepsilon_{e}(v)$. The signs unambiguously determined by the critical data are $\varepsilon_{e}(v) \varepsilon_{e}(v')$ and $\prod_{e \in \partial f} \mu_{e}$.
3.2. Boundary data for spinfoam critical configuration

Given a spinfoam configuration \((j_f, g_{ve}, \xi_{ef}, z_{ef})\) that solves critical point equations. The boundary data of the spinfoam amplitude is given by the boundary spins and the normalized spinors \((j_f, \xi_{ef})\) for the boundary triangles \(f\). Equation (2.43) naturally associates a bivector \(X_{ef}\) with each pair \((e, f)\) for each \((e, f)\). From equation (2.47),

\[
X_{ef}^{ij} = 2\gamma_{jf}[\hat{n}_{ef} \wedge u].
\]

(3.26)

The spatial 3-vectors \(j_f \hat{n}_{ef}\) satisfy the critical point equation (2.38)

\[
\sum_f \varepsilon_{ef} j_f \hat{n}_{ef} = 0,
\]

(3.27)

where \(v\) is the vertex connecting to the edge \(e\). We define \(V_3(e)\), such that

\[
\det(\varepsilon_{ef}, j_f \hat{n}_{ef}, \varepsilon_{ef}, j_f \hat{n}_{ef}, \varepsilon_{ef}, j_f \hat{n}_{ef}, \varepsilon_{ef}, j_f \hat{n}_{ef}) = \text{sgn}(V_3(e)) |V_3(e)|^2.
\]

(3.28)

We rescale each vector \(\varepsilon_{ef} j_f \hat{n}_{ef}\) by

\[
n_{ef} := \frac{\varepsilon_{ef} j_f \hat{n}_{ef}}{|V_3(e)|} \quad \text{then} \quad \sum_f n_{ef} = 0 \quad \text{and} \quad \det(n_{ef}, n_{ef}, n_{ef}) = \frac{1}{V_3(e)}.
\]

(3.29)

We assume the nondegeneracy of the boundary data, i.e. any three of the 4-vectors \(n_{ef}\) span the three-dimensional spatial subspace; in other words, the following product of determinants is nonvanishing:

\[
\prod_{f_1, f_2, f_3} \det(n_{ef_1}, n_{ef_2}, n_{ef_3}) \neq 0.
\]

(3.30)

The nondegeneracy of the tetrahedron equation (3.30) is implied by the nondegeneracy condition in the bulk equation (3.1). The reason is as follows. By the parallel transportation relations \(X_f(v) = g_{vn}X_f g_{vn}\) and \(X_f = 2\gamma_{jf} \hat{n}_{ef} \wedge u\), the bivector \(X_f(v)\) is then given by \(X_f(v) = V_{ef}(v) \wedge \gamma_{vf}(v)\), where \(V_{ef}(v) = g_{vn}u\) and \(V_{ef}(v) := 2\gamma_{jf} g_{vn} \hat{n}_{ef}\) is orthogonal to \(N_e(v)\). For \(f\), the triangle shared by \(t_e\) and \(t_{ef}\) \((i = 1, \ldots, 4)\), we know that \(X_f(v) = \alpha_{ef}(v) N_e(v) \wedge N_e(v)\). Therefore, the vector \(V_{ef}(v)\) is a linear combination of \(N_e(v)\) and \(N_e(v)\). The nondegeneracy condition (equation (3.1)) in four dimensions implies that the four unit vectors, say \(N_e\) and any three out of four vectors \(N_{ef}\), are linearly independent and span a four-dimensional vector space. Thus any three out of the four vectors \(V_{ef}(v)\) must be linearly independent and span a three-dimensional subspace orthogonal to \(N_e(v)\). Then equation (3.30) is a result from parallel transporting \(V_{ef}(v)\) back to the center of \(t_e\).

We now denote \(n_{ef} = n_{p_\ell}(e)\), where the triangle \(f\) is determined by \((p_2, p_3, p_4)\). Now we construct the spatial 3-vectors \(E_{p_\ell p_r}(e)\), such that the matrix \((E_{p_\ell p_r}(e), E_{p_\ell p_r}(e), E_{p_\ell p_r}(e))\) is the inverse of \((n_{p_\ell}(e), n_{p_\ell}(e), n_{p_\ell}(e))\). Therefore, we have

\[
n_{p_\ell}(e) \cdot E_{p_\ell p_r}(e) = \delta_{ij} - \delta_{ik}.
\]

(3.31)

The 3-vectors \(E_{p_\ell p_r}(e)\) are associated with the edges \(\ell = (p_i, p_j)\) of the tetrahedron \(t_e\), so it can be denoted by \(E_\ell(e)\). Note that \(E_\ell(e)\) is determined up to an overall rescaling \(\alpha \in \mathbb{R}\). In the following, we are going to show that the vectors \(E_\ell(e)\) are co-frame vectors on the boundary.

**Lemma 3.3.**

1. The set of vectors \(E_\ell(e)\) gives a co-frame on the boundary, i.e. the vectors \(E_\ell(e)\) satisfy the definition in section A.2.

2. The bivector can be expressed by \(X_{ef} = \varepsilon \ast [E_\ell(e) \wedge E_\ell(e)]\), where \(\varepsilon\) is the same global sign mentioned in lemma 3.2.
(3) The global sign $\varepsilon$ on the entire triangulation can be fixed by the boundary data $\varepsilon \equiv \text{sgn}(V_3(e))$ as the uniform boundary orientation. Thus prior to the construction, one has to choose a consistent orientation of the boundary triangulation, such that $\text{sgn}(V_3(e)) = \text{sgn}(V_3(e'))$ for each pair of tetrahedra $t_e$ and $t_e'$.

The proof of the above lemma is given in appendix D.

By the following relations (we choose the orientation of the 4-simplex $\sigma_v = [p_0, p_1, p_2, p_3, p_4]$):

$$V_3 = \varepsilon_{IJK} E_{21}^I E_{31}^J E_{41}^K, \quad V'_3 = \varepsilon_{IJK} E_{21}^I E_{31}^J E_{41}^K (g_{0v} U^0_l) = -\frac{1}{V_4} \varepsilon_{IJKL} E_{21}^I E_{31}^J E_{41}^K,$$

we obtain that

$$V'_3 = -V_4 U^0_l (g_{0v} u)^l = -V_4 U^0_l N_0^l, \quad \text{where} \quad u = (1, 0, 0, 0)^l.$$

Then for an edge $e$ connecting to the boundary

$$\mu_e = -\varepsilon \text{sgn}(V_4(v)) \text{sgn}(U^0_l(v) N_0^l(v))$$

which implies that if we choose $\varepsilon = \text{sgn}(V_3(e)) = +1$ globally on the boundary, and if $V_3(v) > 0$, $\mu_e = +1$ when $U^0_l(v)$ is future-pointing and $\mu_e = -1$ when $U^0_l(v)$ is past-pointing, while $N_0(v) = g_{0v} u$ is always future-pointing.

**Lemma 3.4.** Given $f$ either an internal face or a boundary face, the product $\prod_{e \subset \partial f} \mu_e$ does not change when $U_e(v)$ flips the sign for any 4-simplex $\sigma_v$; recall that the five normals $U_e(v)$ at $\sigma_v$ are defined up to an overall sign. Therefore, the product $\prod_{e \subset \partial f} \mu_e$ is determined by the spinfoam critical configuration.

**Proof.** For an internal edge $e = (v, v')$, we have

$$\mu_e = -\varepsilon \text{sgn}(V_4(v) V_4(v')) = \text{sgn}(U^0_l(v) (g_{0v} U^0_l(v'))) \text{sgn}(V_4(v) V_4(v')), \tag{3.35}$$

where we recall that $\varepsilon = \text{sgn}(U_e(v)) = g_{0v} U_e(v'/v)/|U_e(v')|$. Combining with equation (3.34), it is easy to see that if we flip simultaneously the sign of all the five $U_e(v)$ at any $\sigma_v$ ($v \in \partial f)$, the product $\prod_{e \subset \partial f} \mu_e$ does not change, for $f$ either an internal face or a boundary face. \qed

We recall figure A1, where the triangle $f_1$ is shared by two boundary tetrahedra $t_{e_0}$ and $t_{e_1}$. Because of equation (D.12), we parallel transport three co-frame vectors $E_l(e_0)$ corresponding to the three edges of the triangle $f_1$,

$$\left( \prod_{e} \mu_e \right) E_l(e_1) = G_f(e_1, e_0) E_l(e_0), \quad \forall \ell \subset f_1, \tag{3.36}$$

where $G_f(e_1, e_0) := \prod g_{e_0}$ is a product of the edge holonomy $g_e$ over all the internal edges $e$ of the dual face $f_1$. Therefore, the triangle formed by the three $E_l(e_0)$ ($\ell \subset f_1$) matches in shape with the triangle formed by $E_l(e_1)$ ($\ell \subset f_1$), since both $E_l(e_0)$ and $E_l(e_1)$ are orthogonal to the unit time-like vector $u = (1, 0, 0, 0)$. There exists an $O(3)$ matrix $\hat{g}_f$, such that

$$\hat{g}_f E_l(e_0) = \hat{E}_l(e_1) \quad \text{and} \quad \hat{g} \hat{n}_{e_0 f_1} = \hat{n}_{e_1 f_1}. \tag{3.37}$$

These relations give the restrictions of the boundary data for the spinfoam amplitude. We call the boundary condition given by equation (3.37) the (nondegenerate) Regge boundary condition. The above analysis shows that the spinfoam boundary data must satisfy the Regge boundary condition in order to have nondegenerate solutions of the critical point equations (2.36)–(2.38).
3.3. Summary

Now we summarize the results in this section as a theorem\(^{10}\).

**Theorem 3.5** (Construction of classical geometry from spinfoam critical configuration).

- Given that the data \((j_f, g_{v'v}, \xi_{ef}, z_{ef})\) is a nondegenerate spinfoam configuration that solves the critical point equations, there exists a discrete classical Lorentzian geometry on \(\mathcal{M}\), represented by a set of spatial co-frame vectors \(E_{v}(v)\) in the bulk, and \(E_{\ell}(e)\) on the boundary, such that the bivectors \(X_{jf}(v)\) and \(X_{ef}\) are written by\(^{11}\)

\[
X_{j1}^{(v)} = \varepsilon \ast [E_{\ell_{1}}(v) \wedge E_{\ell_{2}}(v)]^{\dagger}, \quad X_{j2}^{(v)} = \varepsilon \ast [E_{\ell_{1}}(e) \wedge E_{\ell_{2}}(e)]^{\dagger},
\]

where \(\ell_{1}\) and \(\ell_{2}\) are the edges of the triangle \(f\). The above equation is a relation between the spinfoam data \(X_{jf}(v), X_{ef}\) and a classical geometric data \(E_{\ell}(v)\). Such a relation is determined up to a global sign \(\varepsilon\) on the whole triangulation. Moreover, the above co-frame is unique up to inversion \(E_{\ell} \mapsto -E_{\ell}\) at each \(v\) or \(e\).

- The norm of the bivector \(X_{j}(v)\) is \(|E_{\ell_{1}}(v) \wedge E_{\ell_{2}}(v)| = 2\gamma j_f\). Thus, \(\gamma j_f\) is understood as the area of the triangle \(f\)\(^{12}\).

- If the triangulation has a boundary, and a consistent orientation of the boundary triangulation is chosen, the global sign \(\varepsilon\) is specified by the orientation of the boundary, i.e. \(\varepsilon = \text{sgn}(V_3(e))\).

- Given a dual edge \(e\), for all tetrahedron edge \(\ell\) of the tetrahedron \(t_e\) dual to \(e = (v, v')\), the associated co-frame vectors \(E_{\ell}(v)\) and \(E_{\ell}(v')\) at neighboring vertices \(v\) and \(v'\) are related by parallel transportation up to a sign \(\mu_{e}\), i.e.

\[
\mu_{e}E_{\ell}(v) = g_{v'v}E_{\ell}(v') \quad \forall \ \ell \in t_e. \tag{3.39}
\]

If the dual edge \(e\) connects the boundary, we have similarly

\[
\mu_{e}E_{\ell}(v) = g_{v'\ell}E_{\ell}(e) \quad \forall \ \ell \subseteq t_e. \tag{3.40}
\]

We define the \(\text{SO}(1,3)\) matrices \(\Omega_{vv'}, \Omega_{ve}\) by

\[
\Omega_{vv'} = \mu_{e}g_{v'v} \quad \Omega_{ve} = \mu_{e}g_{v'e}. \tag{3.41}
\]

The simplicial complex \(K\) can be subdivided into sub-complexes \(K_1, \ldots, K_n\) such that

1. (each \(K_i\) is a simplicial complex with a boundary,
2. within each sub-complex \(K_i\), \(\text{sgn}(V_3(e))\) is a constant.

Then within each sub-complex \(K_i\), the \(\text{SO}(1,3)\) matrices \(\Omega_{vv'}, \Omega_{ve}\) are the discrete spin connection compatible with the co-frame \(E_{v}(v)\) and \(E_{\ell}(v')\).

- Given the boundary triangles \(f\) and boundary tetrahedra \(t_e\), in order to have nondegenerate solutions of the critical point equations (2.36)–(2.38), the spinfoam boundary data \((j_f, \xi_{ef})\) must satisfy the (nondegenerate) Regge boundary condition. (1) For each boundary tetrahedron \(t_e\) and its triangles \(f\), \((j_f, \xi_{ef})\) determines four triangle normals \(\hat{n}_{ef}\) that span a three-dimensional spatial subspace. (2) Given the tetrahedra \(t_e\) and \(t_{e'}\) sharing the triangle \(f\), the triangle normals \(\hat{n}_{e_{1}f}\) and \(\hat{n}_{e_{1}f}\) are related by an \(O(3)\) matrix \(g_{1}\) (1 is the link dual to \(f\) on the boundary)

\[
\hat{g}_{1}\hat{n}_{e_{1}f} = \hat{n}_{e_{1}f}. \tag{3.42}
\]

\(^{10}\) The theorem here generalizes the result in [13, 22] to a simplicial complex with arbitrarily many simplices, and generalizes the result in [19] to the Lorentzian case and the case of the simplicial complex with a boundary. Moreover, the issues of orientations and the relating sign issues are emphasized and clarified in our result.

\(^{11}\) The sign factor \(s_{ef}(v)\) for the triangle orientation does not show up here because of a compatible choice of edge orientations of \(\ell_{1}\) and \(\ell_{2}\).

\(^{12}\) \(|E_{1} \wedge E_{2}|^2 = |E_{1}E_{2}^{\dagger} - E_{1}^{\dagger}E_{2}|^{2} = |E_{1}E_{2}^{\dagger} - E_{1}^{\dagger}E_{2}|^{2}(1 - \cos^2\theta) = (2A_{f})^2\) where \(E_{1}E_{2} = |E_{1}| |E_{2}| \cos \theta\).

\(|E_{1} \wedge E_{2}|\) corresponds to the area of a parallelogram (two times the area of the triangle) determined by \(E_{1}\) and \(E_{2}\).
(3) The boundary triangulation is consistently oriented, such that the orientation sgn($V_4(v)$) (recall equation (3.28)) is a constant on the boundary. If the Regge boundary condition is satisfied, there are nondegenerate solutions of the critical point equations, and the solutions imply the shape-matching of the triangle $f$ shared by the tetrahedra $t_e$ and $t_z$. If the Regge boundary condition is not satisfied, there is no nondegenerate critical configuration.

4. Spinfoam amplitude at nondegenerate critical configuration

Given a nondegenerate critical configuration $(j_f, g_{ee'}, \xi_{ef}, z_{ef})$, the previous discussions show us that we can construct a discrete classical geometry from the critical configuration. Moreover, we can make a subdivision of the triangulation into sub-triangulations $K_1, \ldots, K_n$, such that (1) each $K_i$ is a simplicial complex with a boundary, (2) within each sub-complex $K_i$, sgn($V_4(v)$) is a constant. To study the spinfoam (partial) amplitude $A_j(K)$ at a nondegenerate critical configuration, we only need to study the amplitude $A_j(K_i)$ on the sub-triangulation $K_i$ where sgn($V_4(v)$) is a constant. Then the behavior of $A_j(K)$ can be expressed as a product

$$A_j(K) \bigg|_{\text{critical}} = \prod_i A_j(K_i) \bigg|_{\text{critical}}.$$ (4.1)

Therefore, in the following analysis of this section we always assume that the triangulation has a boundary and sgn($V_4$) is a constant on the triangulation.

4.1. Internal faces

We have shown previously that the action $S$ of the spinfoam amplitude can be written as a sum

$$S = \sum_j S_j.$$ (4.2)

Each internal ‘face action’ $S_j$ evaluated at the critical point defined by equations (2.36)–(2.38) takes the form

$$S_j = 2i\gamma j_f \sum_{v \in \partial f} \ln \frac{||Z_{ee'}||}{||Z_{ee'}||} - 2i j_f \sum_{v \in \partial f} \Phi_{ee'} = -2i j_f \left( \gamma \sum_{v \in \partial f} \theta_{ee'} + \sum_{v \in \partial f} \Phi_{ee'} \right).$$ (4.3)

where we have denoted

$$\frac{||Z_{ee'}||}{||Z_{ee'}||} := e^{\theta_{ee'}}.$$ (4.4)

Recall equations (2.36) and (2.37), and consider the following successive actions on $\xi_{ef}$ of $g_{ee'} g_{ee''}$ around the entire boundary of the face $f$:

$$\prod_{v \in \partial f} g_{ee'} g_{ee''} \xi_{ef} = e^{-\sum_{e' \neq e''} \theta_{ee'} - \sum_{e''} \theta_{ee''}} \xi_{ef}$$

$$\prod_{v \in \partial f} g_{ee'} g_{ee''} \xi_{ef} = e^{-\sum_{e' \neq e''} \theta_{ee'} + \sum_{e''} \theta_{ee''}} \xi_{ef}.$$ (4.5)

Thus, $\xi_{ef}$ is an eigenvector of the loop holonomy $\prod_{v \in \partial f} g_{ee'} g_{ee''}$. We obtain the following expression of the loop holonomy $G_f(e)$:

$$G_f(e) = \exp \left[ \sum_{v \in \partial f} (\theta_{ee'} + i \Phi_{ee'}) \vec{a} \cdot \vec{n}_{ef} \right].$$ (4.6)

which is an exponential map from the Lie algebra variable$^{13}$.

$^{13}$ Note that not all the elements in $\text{SL}(2, \mathbb{C})$ can be written in an exponential form, because of the noncompactness.
Consider the following identity: for any complex number $\alpha$ and unit vector $\hat{n}$,

$$\text{tr}\left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}) e^{\omega \sigma \cdot \hat{n}} \right] = e^{\alpha},$$

(4.6)

which can be proved by the identities of Pauli matrices: $(\sigma \cdot \hat{n})^{2k} = 1_{2 \times 2}$ and $(\sigma \cdot \hat{n})^{2k+1} = \sigma \cdot \hat{n}$. Using this identity, we have

$$\ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f (e) \right] = \sum_{v \in \partial f} \theta_{ev} + i \sum_{v \in \partial f} \phi_{ev},$$

(4.7)

where we use the fact that $\sigma$ are the Hermitian matrices. Insert these into the expression of the face action $S_f$:

$$S_f = -ij_f \gamma \left[ \ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f (e) \right] + \ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f^\dagger (e) \right] \right]$$

$$- j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f (e) \right] - \ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f^\dagger (e) \right]$$

$$= -(i \gamma + 1) j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f (e) \right] - (i \gamma - 1) j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \sigma \cdot \hat{n}_f) G_f^\dagger (e) \right].$$

(4.8)

We define the following variables by making a parallel transport to a vertex $v$:

$$\hat{X}_f (v) := g_{ev} \sigma \cdot \hat{n}_f g_{ee}, \quad \hat{X}_f^\dagger (v) := g_{ev}^\dagger \sigma \cdot \hat{n}_f g_{ee}^\dagger,$$

$$G_f (v) := g_{ev} G_f (e) g_{ee}, \quad G_f^\dagger (v) := g_{ev}^\dagger G_f (e) g_{ee}^\dagger,$$

(4.9)

where one can see that $\hat{X}_f (v)$ is related to the bivector in proposition 2.1 by $\hat{X}_f (v) = \hat{X}_f (v)/\gamma j_f$. In terms of these new variables at the vertex $v$, the face action is written as

$$S_f = -(i \gamma + 1) j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \hat{X}_f (v)) G_f (e) \right] - (i \gamma - 1) j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \hat{X}_f^\dagger (v)) G_f^\dagger (e) \right].$$

(4.10)

According to theorem 3.5, at the critical point, the bivector $\hat{X}_f (v)$ is written as

$$\hat{X}_f (v) = 2e^i \star \left[ \star G_f (v) \wedge E_c (v) \right],$$

(4.11)

and the spinfoam edge holonomy $g_{ev}$ equals the spin connection $\Omega_{ev}$ up to a sign $\mu_e = e^{\pi \alpha_e}$, i.e.

$$g_{ev} = e^{\pi \alpha_e} \Omega_{ev}.$$

(4.12)

The spinfoam loop holonomy (in its spin-1 representation) at the critical point satisfies

$$G_f (v) E_c (v) = e^{\pi \sum_{e \subset f} \alpha_e} \star E_c (v) = \cos \left( \pi \sum_{e \subset f} \alpha_e \right) E_c (v).$$

(4.13)

Then the spin-1 representation of the loop holonomy $G_f (v)$ can be expressed as

$$G_f (v) = \exp \left[ \frac{\star E_c (v) \wedge E_c (v)}{\star \star E_c (v) \wedge E_c (v)} \theta_f + \frac{E_c (v) \wedge E_c (v)}{\star \star E_c (v) \wedge E_c (v)} \pi \sum_{e \subset f} \alpha_e \right]$$

$$= \exp \left( \frac{1}{2} \theta_f \hat{X}_f (v) + \frac{1}{2} \pi \sum_{e \subset f} \alpha_e \hat{X}_f (v) \right),$$

(4.14)
where $\vartheta_f$ is an arbitrary number. Since the duality map $\ast = i$ in the spin-1/2 representation,

$$G_f(v) = \exp\left(i \frac{1}{2} \vartheta_f \hat{X}_f(v) + i \frac{\pi}{2} \sum_{e \in f} n_e \hat{X}_f(v)\right)$$  \hspace{1cm} (4.15)

in the spin-1/2 representation, where $G_f(v) \in \text{SL}(2, \mathbb{C})$.

We now determine the physical meaning of the parameter $\vartheta_f$. $\text{sgn}(V_4(v))$ is a constant on the triangulation for the oriented 4-volumes of the 4-simplices. By the relation between the spinfoam variable $g_{uv}$ and the spin connection: $g_{uv} = \mu_e \Omega_{e_v}$, for the spin connection, we have

$$\Omega_f(v) = e^{i \pi \sum_{e} n_e} G_f(v)$$

$$= e^{i \pi \sum_{e} n_e} \exp\left(\frac{\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)}{\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)} \vartheta_f + \frac{E_{\ell_1}(v) \wedge E_{\ell_2}(v)}{|E_{\ell_1}(v) \wedge E_{\ell_2}(v)|} \pi \sum_e n_e\right) \in \text{SO}(1, 3).$$  \hspace{1cm} (4.16)

We consider a discretization of classical Einstein–Hilbert action $\int R \sqrt{-g} d^2 x$: For each dual face $f$

$$\text{tr} \left[ \int_{\Delta_f} \text{sgn} \det (e_{\mu}^f) \ast [e \wedge e] \int_f R \right] \simeq \text{sgn}(V_4) \frac{1}{2} \text{tr} \left[ \ast (E_{\ell_1}(v) \wedge E_{\ell_2}(v)) \ln \Omega_f^{\text{boost}}(v) \right]$$

$$= \text{sgn}(V_4) A_f \vartheta_f$$  \hspace{1cm} (4.17)

This formula should be understood by ignoring the higher order correction in the continuum limit. Here we use $\Delta_f$ to denote the triangle dual to $f$. $e_{\mu}^f$ is a co-tetrad in the continuum. $R$ is the local curvature from the $\mathfrak{sl}_2 \mathbb{C}$-valued local spin connection compatible with $e_{\mu}^f$. Only the pure boost part $\Omega_f^{\text{boost}}(v) = \exp(\frac{\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)}{\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)} \vartheta_f)$ of the spin connection $\Omega_f(v)$ contributes the curvature $R$ in the discrete context. When $e^{i \pi \sum_{e} n_e} = -1$, the factor $e^{i \pi \sum_{e} n_e} \exp(\frac{\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)}{\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)} \pi \sum_e n_e)$ flips the overall sign of the reference frame at $v$ and rotates $\pi$ on the 2-plane spanned by $E_{\ell_1}(v)$ and $E_{\ell_2}(v)$. It serves for the case that the time orientation of the reference frame is flipped by $\Omega_f(v)$, while the triangle spanned by $E_{\ell_1}(v)$ and $E_{\ell_2}(v)$ is kept unchanged. Such an operation does not change the quantity$^{14}$

$$\text{tr} \left[ \int_{\Delta_f} \text{sgn} \det (e_{\mu}^f) \ast [e \wedge e] \int_f R \right].$$  \hspace{1cm} (4.18)

$A_f = \frac{1}{2} |\ast E_{\ell_1}(v) \wedge E_{\ell_2}(v)|$ is the area of the triangle dual to $f$. Comparing equation (4.17) with the Regge action of discrete GR, we identify that $\text{sgn}(V_4) \vartheta_f$ is the deficit angle $\Theta_f$ of $f$ responsible to the curvature $R$ from the spin connection (see also [25]).

$$\Theta_f = \text{sgn}(V_4) \vartheta_f,$$  \hspace{1cm} (4.19)

where we keep in mind that $\text{sgn}(V_4)$ is a constant sign on the (sub-)triangulation.

Inserting the expression of $G_f(v)$ into equation (4.10), we obtain for an internal face $f$,

$$S_f = - \frac{(i \gamma + 1)}{2} \int_f \left[ \varepsilon \text{sgn}(V_4) \Theta_f + i \pi \sum_{e \in f} n_e \right] - \frac{(i \gamma - 1)}{2} \int_f \left[ \varepsilon \text{sgn}(V_4) \Theta_f - i \pi \sum_{e \in f} n_e \right]$$

$$= - i \varepsilon \text{sgn}(V_4) \gamma \Theta_f - i \pi \sum_{e \in f} n_e.$$  \hspace{1cm} (4.20)

$^{14} \Omega_f(v) \in \text{SO}^+(1, 3)$ comes from an oriented but time-unoriented orthonormal frame bundle, where the co-tetrad $e_{\mu}^f$ can flip the sign. However, the local spin connection $\Gamma^e_{\alpha \mu} = e_{\alpha}^e \text{sgn} e_{\mu}^f$ does not change as $e_{\mu}^f \mapsto -e_{\mu}^f$ and coincides with the spin connection on the oriented and time-oriented orthonormal frame bundle. The same holds also for the curvature $R$ from the spin connection.
Finally, we sum over all the internal faces and construct the total internal action
\[
S_{\text{internal}} = -i \varepsilon \sgn(V) \sum_{f} \gamma_j f_0 - i \tau \sum_{e \in \partial f} j_0 n_e, \tag{4.21}
\]
where \( \gamma_j f_0 \) is understood as the area of the triangle \( f \) and \( \sum_f \gamma_j f_0 \Theta_f \) is the Regge action for
discrete GR.

4.2. Boundary faces

Let us consider a face \( f \) dual to a boundary triangle (see figure A1). The corresponding face action \( S_f \) reads
\[
S_f = 2i \gamma j_f \sum_v \ln \frac{|Z_{\text{eff}}|}{|Z_{\text{ref}}|} - 2i j_f \sum_v \phi_{\text{ev}} = -2i j_f \left( \gamma \sum_v \theta_{\text{ev}} - \sum_v \phi_{\text{ev}} \right), \tag{4.22}
\]
where the sum is over all the internal vertices \( v \) around the face \( f \), and we have also used the notation \( \ln \frac{|Z_{\text{eff}}|}{|Z_{\text{ref}}|} := e^{\theta_{\text{ev}}}. \)

On the boundary of the face \( f \), there are at least two edges connecting to the nodes on the boundary of the triangulation. We suppose that there is an edge \( e_0 \) of the face \( f \) connecting a boundary node associated with a boundary spinor \( \xi_{v_0}. \) Recall equations (2.36) and (2.37), and consider the following successive action on \( \xi_{v_0} \) of \( g_{v_0} \) along the boundary of the face \( f \), until reaching another edge \( e_1 \) connecting to another boundary node. We denote by \( p_{v_0} \) the path from \( e_0 \) to \( e_1, \)
\[
g_{e_1} g_{e_0} \cdots g_{e_v} g_{e_0} J \xi_{v_0} = J \xi_{v_1} \exp \left[ - \sum_{v \in p_{v_0}} \theta_{\text{ev}} - i \sum_{v \in p_{v_0}} \phi_{\text{ev}} \right] \tag{4.23}
\]
We denote the holonomy along the path \( p_{v_0} \) by
\[
G_f(e_1, e_0) := g_{e_1} g_{e_0} \cdots g_{e_v} g_{e_0} \tag{4.24}
\]
and construct an SU(2) matrix from the normalized spinor \( \xi \) by
\[
g(\xi) = (\xi, J\xi) \in \text{SU}(2). \tag{4.25}
\]
If we denote by
\[
\alpha = \sum_{v \in p_{v_0}} \theta_{\text{ev}} + i \sum_{v \in p_{v_0}} \phi_{\text{ev}}, \tag{4.26}
\]
equation (4.23) can be expressed as a matrix equation
\[
G_f(e_1, e_0) = g(\xi_{v_0}) \left( e^{\alpha} 0 \right). \tag{4.27}
\]
Therefore \( G_f(e_1, e_0) \) can be solved immediately
\[
G_f(e_1, e_0) = g(\xi_{v_1}) \exp \left( \sum_{v} (\theta_{\text{ev}} + i \phi_{\text{ev}}) \hat{\sigma} \cdot \hat{z} \right) g(\xi_{v_0})^{-1}. \tag{4.28}
\]
We again employ the identity (4.6) to obtain
\[
\ln \left[ \frac{1}{2} \left( 1 + \hat{\sigma} \cdot \hat{z} \right) g(\xi_{v_1})^{-1} G_f(e_1, e_0) g(\xi_{v_0}) \right] = \sum_{v} (\theta_{\text{ev}} + i \phi_{\text{ev}}) \tag{4.29}
\]
\[
\ln \left[ \frac{1}{2} \left( 1 + \hat{\sigma} \cdot \hat{z} \right) g(\xi_{v_0})^{-1} G_f(e_1, e_0) g(\xi_{v_1}) \right] = \sum_{v} (\theta_{\text{ev}} - i \phi_{\text{ev}}). \]
Insert these relations into the face action $S_f$:

$$S_f = -(iy + 1) j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \hat{\sigma} \cdot \hat{\tau}) g(\xi_\ell f) \right] G_f(e_1, e_0) g(\xi_\ell f) \]$$

$$- (iy - 1) j_f \ln \text{tr} \left[ \frac{1}{2} (1 + \hat{\sigma} \cdot \hat{\tau}) g(\xi_\ell f) \right] G_f^\dagger(e_1, e_0) g(\xi_\ell f). \] \quad (4.30)$$

Recall that at the critical configuration, $G_f(e_1, e_0)$ coincides with the spin connection $\Omega_f(e_1, e_0)$ up to a sign. Given the co-frame vectors $E_\ell(e_0)$ and $E_\ell(e_1)$ with $\ell$, the edges of the triangle $f$:

$$\left( \prod_c \mu_c \right) E_\ell(e_1) = G_f(e_1, e_0) E_\ell(e_0) \quad \forall \ell \subset f \quad (4.31)$$

$$G_f(e_1, e_0) = \left( \prod_c \mu_c \right) \Omega_f(e_1, e_0),$$

where the product $\prod_c$ is over all the edges along the path $p_{e_1 e_0}$.

Here we are going to give an explicit expression for $G_f(e_1, e_0)$ from equation (4.31). We first define three new vectors $\tilde{E}_\ell(e_i)$ for the three $\ell$s of the triangle $f$:

$$\tilde{E}_\ell(e_i) = \tilde{g}(\xi_\ell f)^{-1} E_\ell(e_i), \quad i = 0, 1, \quad (4.32)$$

where $\tilde{g}(\xi_\ell f)$ is the spin-1 representation of $g(\xi_\ell f) \in \text{SU}(2)$. Thus,

$$\tilde{g}(\xi_\ell f)^{-1} G_f(e_1, e_0) \tilde{g}(\xi_\ell f) \tilde{E}_\ell(e_0) = \left( \prod_c \mu_c \right) \tilde{E}_\ell(e_1). \quad (4.33)$$

The co-frame vectors $E_\ell(e)$ of a triangle $f$ are orthogonal to $\hat{n}_f$, which is given by $\hat{n}_f = \hat{g}(\xi_\ell f) \hat{\xi}$. Thus, the triangles formed by $\tilde{E}_\ell(e_i)$ ($i = 0, 1$) are both on the 2-plane (the $xy$-plane) orthogonal to $u = (1, 0, 0, 0)$ and $\hat{\xi} = (0, 0, 1, 1)$; then they are related by a rotation $e^{i j h}$ on the $xy$-plane:

$$\tilde{E}_\ell(e_1) = e^{i j h} \tilde{E}_\ell(e_0) \quad \forall \ell \subset f. \quad (4.34)$$

Therefore, $\tilde{g}(\xi_\ell f)^{-1} G_f(e_1, e_0) \tilde{g}(\xi_\ell f)$ is the above rotation plus a pure boost along the $z$-direction and a rotation taking care of the sign factor $\prod_c \mu_c$, both of which leave the vector on the $xy$-plane invariant. Hence,

$$G_f(e_1, e_0) = \tilde{g}(\xi_\ell f) e^{\theta^B} e^{\pi \sum n_j \tilde{g}(\xi_\ell f)^{-1}}, \quad (4.35)$$

where $\theta^B$ is an arbitrary number. The rotation $e^{i j h}$ corresponds to a gauge transformation in the context of the twisted geometry [26]. Here we can always absorb $e^{i j h}$ into one of $\tilde{g}(\xi_\ell f)$, which leads to a redefinition of the boundary data $\xi_\ell f$. Such a redefinition does not change the triangle normal $\hat{n}_f$, and thus does not change the bivector $X_f$. Then all the above analysis about constructing discrete geometry is unaffected. The boundary data after this redefinition is the Regge boundary data employed in [13]. With this setting, we obtain

$$G_f(e_1, e_0) = \tilde{g}(\xi_\ell f) e^{\theta^B} e^{\pi \sum n_j \tilde{g}(\xi_\ell f)^{-1}} \quad (4.36)$$

for an explicit expression of $G_f(e_1, e_0)$ and

$$\tilde{E}_\ell(e_0) = \tilde{E}_\ell(e_1) = \tilde{E}_\ell \quad (4.37)$$

for the edges of triangle $\ell$. The three vectors $\tilde{E}_\ell$ determine the triangle geometry of $f$ in the frame at $\tilde{f}$. From equation (4.31), we obtain the spin connection compatible with the co-frame

$$\Omega_f(e_1, e_0) = e^{i \pi \sum n_j \tilde{g}(\xi_\ell f)} e^{\theta^B} e^{\pi \sum n_j \tilde{g}(\xi_\ell f)^{-1}}. \quad (4.38)$$
When $e^{in} \Sigma_n = 1$, the spin connection $\Omega_f(e_1, e_0) \in \text{SO}^+(1, 3)$, and when $e^{in} \Sigma_n = -1$, $\Omega_f(e_1, e_0) \in \text{SO}^-(1, 3)$.

We now determine the physical meaning of the parameter $\vartheta_f^B$ in the expression of $G_f(e_1, e_0)$. It is related to the dihedral angle $\Theta_f^B$ of the two boundary tetrahedra $t_{e_0}$ and $t_{e_1}$ at the triangle $f$ shared by them. The two tetrahedra $t_{e_0}$ and $t_{e_1}$ belong to different 4-simplices $\sigma_{e_0}$ and $\sigma_{e_1}$, while the curvature from the spin connection between $\sigma_{e_0}$ and $\sigma_{e_1}$ is given by the pure boost part of $\Omega_f(v_1, v_0)$ along the internal edges of the face $f$. This curvature is responsible to the dihedral angle between $t_{e_0}$ and $t_{e_1}$. The dihedral boost between the normals of $t_{e_0}$ and $t_{e_1}$ at the triangle $f$ is given by the pure boost part of

$$\hat{g}(\xi_{e,f})^{-1} \Omega_f(e_1, e_0) \hat{g}(\xi_{e_0,f}) = e^{i\pi \sum_n n_e} e^{\vartheta_f^B K_1} e^{i\pi \sum_n n_f}.$$  \hspace{1cm} (4.39)

The above transformation leaves the triangle geometry $\tilde{E}_e$ invariant in both cases of $e^{in} \Sigma_n = \pm 1$. We consider the unit normal of the tetrahedron $t_{e_0}$ (viewed in its own frame) $u' = (1, 0, 0, 0)^i$, parallel transported by $G_f(e_1, e_0)$ (from the frame of $t_{e_0}$ to the frame of $t_{e_1}$)

$$G_f(e_1, e_0)^{jf} u' = e^{\vartheta_f^B \sigma_{eiston}} u = (\cosh \vartheta_f^B, 0, 0, \sinh \vartheta_f^B)^j.$$  \hspace{1cm} (4.40)

Contracting this equation with the unit normal $u' = (1, 0, 0, 0)^i$ viewed in the frame of $t_{e_1}$, we obtain that for the dihedral angle $\Theta_f^B$,

$$\cosh \Theta_f^B = -u_1 G_f(e_1, e_0)^{jf} u' = \cosh \vartheta_f^B,$$  \hspace{1cm} (4.41)

which implies that $\Theta_f^B = \pm \vartheta_f^B$. By a generalization of the analysis in [13], we can conclude that (see appendix E for a proof)

**Lemma 4.1.** The dihedral angle $\Theta_f^B$ at the triangle $f$ relates to the parameter $\vartheta_f^B$ by

$$\Theta_f^B = \varepsilon \text{ sgn}(V_4) \vartheta_f^B.$$  \hspace{1cm} (4.42)

Equation (4.36) is now related to the dihedral angle $\Theta_f^B$:}

$$\hat{g}(\xi_{e,f})^{-1} G_f(e_1, e_0) \hat{g}(\xi_{e_0,f}) = e^{i\varepsilon \text{ sgn}(V_4) \vartheta_f^B} e^{i\pi \sum_n n_f}.$$  \hspace{1cm} (4.43)

Recall that in the spin-1/2 representation $\hat{J} = \frac{i}{2} \sigma$ and $\hat{K} = \frac{1}{2} \sigma$; thus in the spin-1/2 representation,

$$g(\xi_{e,f})^{-1} g_f(e_1, e_0) g(\xi_{e_0,f}) = e^i \varepsilon \text{ sgn}(V_4) \vartheta_f^B \sigma_{eiston} e^{i\pi \sum_n n_f}.$$  \hspace{1cm} (4.44)

Insert this relation back into equation (4.30),

$$S_f = -\frac{(i\gamma + 1)}{2} j_f \begin{bmatrix} \varepsilon \text{ sgn}(V_4) \Theta_f^B + i\pi \sum_{e \in P_{e_1 e_0}} n_e \\ - \frac{(i\gamma - 1)}{2} j_f \varepsilon \text{ sgn}(V_4) \Theta_f^B - i\pi \sum_{e \in P_{e_1 e_0}} n_e \end{bmatrix}$$

$$= -i\varepsilon \text{ sgn}(V_4) \gamma j_f \Theta_f^B - i j_f \pi \sum_{e \in P_{e_1 e_0}} n_e.$$  \hspace{1cm} (4.45)

Then the total boundary action $S_{\text{boundary}} = \sum_{\text{boundary}} S_f$:

$$S_{\text{boundary}} = -i \varepsilon \text{ sgn}(V_4) \sum_{\text{boundary}} \gamma j_f \Theta_f^B - i j_f \pi \sum_{e \in P_{e_1 e_0}} n_e.$$  \hspace{1cm} (4.46)
4.3. Spinfoam amplitude at nondegenerate critical configuration

In this subsection we summarize our result and give spinfoam amplitude at a general nondegenerate critical configuration. First of all, we say a spin configuration $j_f$ is Regge-like, if with $j_f$ on each face the critical point equations (2.36)–(2.38) have a nondegenerate solution $(j_f, g_{ve}, \xi_{ef}, z_{ef})$. For a non-Regge-like spin configuration $j_f$, the critical point equations have no nondegenerate solutions.

Given a Regge-like spin configuration $j_f$ and a solution $(j_f, g_{ve}, \xi_{ef}, z_{ef})$ of the critical point equations, as in section 3, a nondegenerate solution $(j_f, g_{ve}, \xi_{ef}, z_{ef})$ of the spinfoam critical point equations specifies uniquely a set of variables $(g_{\ell, f}, n_e, \varepsilon)$, which include a discrete metric and two types of sign factors.

The previous analysis shows that, given a general critical configuration $(j_f, g_{ve}, \xi_{ef}, z_{ef})$, we can divide the triangulation $\mathcal{K}$ into sub-triangulations $\mathcal{K}_1, \ldots, \mathcal{K}_n$, where each of the sub-triangulations is a triangulation with a boundary, with a constant $\text{sgn}(V_4(v))$. On each of the sub-triangulation $\mathcal{K}_i$, the spinfoam action $S$ evaluated at $(j_f, g_{ve}, \xi_{ef}, z_{ef})_{\mathcal{K}_i}$ is a function of the variables $(g_{\ell, f}, n_e, \varepsilon)$ and behaves mainly as a Regge action:

$$S(g_{\ell, f}, n_e, \varepsilon)|_{\mathcal{K}_i} = S_{\text{internal}}(g_{\ell, f}, n_e, \varepsilon) + S_{\text{boundary}}(g_{\ell, f}, n_e, \varepsilon)$$

where $\text{sgn}$ is the overall sign factor. For each tetrahedron $t$, the sum of face spins $\sum_{f \subset t} j_f$ is an integer. If the spins $j_f$ are integers, then $\sum_{f \subset t} j_f$ is an even integer, such that $e^{i \pi \sum_{f \subset t} j_f} = 1$, so the second term in the above formula does not contribute the exponential $e^{i \pi \sum_{f \subset t} j_f}$. Therefore, in general, at a nondegenerate spinfoam configuration $(j_f, g_{ve}, \xi_{ef}, z_{ef})$ that solves the critical point equations,

$$e^{i S}|_{\mathcal{K}_i} = \pm \exp \left[ -i \varepsilon \text{sgn}(V_4) \sum_{f \subset t} \gamma j_f \Theta_f - i \varepsilon \text{sgn}(V_4) \sum_{f \subset t} \gamma j_f \Theta_f^B \right]. \quad (4.48)$$

There exist two ways to make the overall sign factor disappear: (1) only consider integer spins $j_f$ or (2) modify the embedding from $SU(2)$ unitary irreps to $SL(2, \mathbb{C})$ unitary irreps by $j_f \mapsto \langle p_f, k_f \rangle := (2j_f, 2j_f)$; then the spinfoam action $S$ is replaced by $2S$. In these two cases, the exponential $e^{i S}$ at the critical configuration is independent of the variable $n_e$.

On the triangulation $\mathcal{K} = \bigcup_{i=1}^n \mathcal{K}_i$, $e^{i S}$ is given by a product over all the sub-triangulations:

$$e^{i S} = \prod_{i=1}^n e^{i S}|_{\mathcal{K}_i} = \prod_{i=1}^n \exp \left[ -i \varepsilon \text{sgn}(V_4) \sum_{f \subset t} \gamma j_f \Theta_f - i \varepsilon \text{sgn}(V_4) \sum_{f \subset t} \gamma j_f \Theta_f^B - i \varepsilon \sum_{f \subset t} n_e \sum_{f \subset t} j_f \right]. \quad (4.49)$$
Suppose that the oriented 4-volumes are different between two sub-triangulations \( K_i \) and \( K_j \) sharing a boundary; then the spinfoam amplitude at this critical configuration exhibits a transition between two different spacetime regions with different spacetime orientation. The spacetime orientation is not continuous on the boundary between \( K_i \) and \( K_j \).

We recall the difference between the Einstein–Hilbert action and Palatini action

\[
\mathcal{L}_{EH} = R \xi = \text{sgn det} \left( e_i^a \right) * [e \wedge e]_{AB} \wedge R^{AB} = \text{sgn det} \left( e_i^a \right) \mathcal{L}_P, \tag{4.50}
\]

where \( \mathcal{L}_{EH} \) and \( \mathcal{L}_P \) denote the Lagrangian densities of the Einstein–Hilbert action and Palatini action respectively, and \( \xi \) is a chosen volume form compatible with the metric \( g_{\mu \nu} = \eta_{\mu \nu} e_i^a e_i^b \). Since the Regge action is a discretization of the Einstein–Hilbert action, we may consider the resulting action

\[
-i \varepsilon \sum_{i=1}^{n} \left[ \text{sgn}(V_c) \sum_{\text{internal } f} \gamma j_f \Theta_f + \text{sgn}(V_c) \sum_{\text{boundary } f} \gamma j_f \Theta_f^R \right]_{K_i}, \tag{4.51}
\]

as a discretized Palatini action with the on-shell connection, where the on-shell connection means that the discrete connection is the spin connection compatible with the co-frame.

According to the properties of the Regge geometry, given a collection of Regge-like areas \( \gamma j_f \), the discrete metric \( g_{\ell_1 \ell_2}(v) \) is uniquely determined at each vertex \( v \). Furthermore, since the areas \( \gamma j_f \) are Regge-like, there exists a discrete metric \( g_{\ell_1 \ell_2} \) in the entire bulk of the triangulation, such that the neighboring 4-simplices are consistently glued together, as we constructed previously. This discrete metric \( g_{\ell_1 \ell_2} \) is obviously unique by the uniqueness of \( g_{\ell_1 \ell_2}(v) \) at each vertex. Therefore, given the partial amplitude \( A_{j_f}(K) \) in equation (2.13) with a specified Regge-like \( j_f \), all the critical configurations \( (j_f, g_{\ell_1 \ell_2}, \xi_\ell_\ell, z_{\ell_\ell}) \) of \( A_{j_f}(K) \) correspond to the same discrete metric \( g_{\ell_1 \ell_2} \), provided a Regge boundary data. The critical configurations from the same Regge-like \( j_f \) are classified in the following section.

As a result, given a Regge-like spin configuration \( j_f \) and a Regge boundary data, the partial amplitude \( A_{j_f}(K) \) has the following asymptotics:

\[
A_{j_f}(K)_{\text{Nondeg}} \sim \sum_{x_c} a(x_c) \left( \frac{2\pi}{\lambda} \right)^{\text{dim}(j_f) - N(v,f)} \frac{e^{\text{ln} H'(x_c)}}{\sqrt{|\text{det} H'(x_c)|}} \left[ 1 + \mathcal{O} \left( \frac{1}{\lambda} \right) \right]
\]

\[
\times \exp -i\varepsilon \sum_{\ell}(x_c) \left[ \delta \text{sgn}(V_c) \sum_{\text{internal } f} \gamma j_f \Theta_f \right]
\]

\[
+ \varepsilon \text{sgn}(V_c) \sum_{\text{boundary } f} \gamma j_f \Theta_f^\beta + \pi \sum_{e} \mathcal{N}_e \sum_{j_f \in \mathcal{K}_e(x_c)} \gamma j_f \Theta_f^\beta, \tag{4.52}
\]

where \( x_c \equiv (j_f, g_{\ell_1 \ell_2}, \xi_\ell_\ell, z_{\ell_\ell}) \) labels the nondegenerate critical configurations, \( r(x_c) \) is the rank of the Hessian matrix at \( x_c \) and \( N(v,f) \) is the number of the pair \( (v,f) \) with \( v \in \partial f \) (recall equation (2.13), there is a factor of \( \text{dim}(j_f) \) for each pair of \( (v,f) \)). \( a(x_c) \) is the evaluation of the integration measures at \( x_c \), which does not scale with \( \lambda \). Here \( \Theta_f \) and \( \Theta_f^\beta \) only depend on the metric \( g_{\ell_1 \ell_2} \), which is uniquely determined by the Regge-like spin configuration \( j_f \) and the Regge boundary data. Note that different critical configurations \( x_c \) may have different subdivisions of the triangulation into sub-triangulations \( \mathcal{K}_1(x_c), \ldots, \mathcal{K}_{\text{dim}(j_f)}(x_c) \).

Finally, the additional term \( \pi \sum_{e} \mathcal{N}_e \sum_{j_f \in \mathcal{K}_e(x_c)} \gamma j_f \) appearing or not relates to the properties that if the spinfoam critical data is ‘time-oriented’ or not, which is analyzed in [31].
5. Parity inversion

We consider a tetrahedron $t$, associated with spins $j_f^L, \ldots, j_f^R$; we know that the set of four spinors $\xi_{ef}^L, \ldots, \xi_{ef}^R$, modulo diagonal SU(2) gauge transformation, is equivalent to the shape of the tetrahedron, if the closure condition is satisfied [27]. Given a nondegenerate critical configuration $(j_f^L, g_{ve}, \xi_{ef}, z_{ef})$, as we discussed previously, the Regge-like spin configuration $j_f$ determines a discrete metric $g_{ef}$, which determines the shape of all the tetrahedra in the triangulation. At the critical configuration, the closure condition of tetrahedron is always satisfied, so the spinors $\xi_{ef}^L, \ldots, \xi_{ef}^R$ for each tetrahedron are determined by the Regge-like spins $j_f$, up to a diagonal SU(2) action on the spinors $\xi_{ef}^L, \ldots, \xi_{ef}^R$, which is a gauge transformation of the spinfoam action\(^{15}\). Therefore, the gauge equivalence class of the critical configurations $(j_f^L, g_{ve}, \xi_{ef}, z_{ef})$ with the same Regge-like spins $j_f$ must have the same set of spinors $\xi_{ef}^L, \ldots, \xi_{ef}^R$. Thus, with a given Regge-like spin configuration $j_f$, the degrees of freedom of the nondegenerate critical configurations are the variables $g_{ve}$ and $z_{ef}$. The degrees of freedom of $g_{ve}$ and $z_{ef}$ are factorized into the 4-simplices. Given the Regge-like spins $j_f$ and spinors $\xi_{ef}^L, \ldots, \xi_{ef}^R$, within each 4-simplex, the solutions of $g_{ve}$ and $z_{ef}$ from the critical point equations are completely classified in [13], which are the two solutions related by a parity transformation.

Given a nondegenerate critical configuration $(j_f^L, g_{ve}, \xi_{ef}, z_{ef})$, it generates many other nondegenerate critical configurations $(j_f^L, \tilde{g}_{ve}, \tilde{\xi}_{ef}, \tilde{z}_{ef})$, which are the solutions of the critical point equations (2.36)–(2.38). In at least one simplex or some 4-simplices $\tilde{\sigma}_v$,

$$\tilde{g}_{ve} = J g_{ve} J^{-1} = (g_{ve}^*)^{-1} \quad \text{and} \quad \frac{||\tilde{Z}_{vef}||}{||Z_{vef}||} = \frac{||\tilde{Z}_{vef}||}{||Z_{vef}||}, \quad (5.1)$$

while in the other 4-simplices $\tilde{g}_{ve} = g_{ve}$ and $\tilde{z}_{ef} = z_{ef}$. In [13], such solution-generating maps $g_{ve} \mapsto \tilde{g}_{ve}$ and $z_{ef} \mapsto \tilde{z}_{ef}$ are called a parities, because $N_e(v) = g_{ve} \gg (1, 0, 0, 0)^t$ and $\tilde{N}_e(v) = \tilde{g}_{ve} \gg (1, 0, 0, 0)^t$ are different by a parity inversion. The parity inversion between $N_e(v)$ and $\tilde{N}_e(v)$ can be shown by using the Hermitian matrix representation of the vectors $V = V^0 I + V^1 \sigma$; thus,

$$\tilde{N}_e(v) = \tilde{g}_{ve} g_{ve}^t = J g_{ve} g_{ve}^t J^{-1} = J N_e(v) J^{-1} = J N_e(v) J^{-1} = N_{ve}^0(v) I - N_{ve}^1(v) \sigma_j, \quad (5.2)$$

since $J \sigma J^{-1} = -\sigma$. We denote the parity inversion in $\mathbb{R}^4$ by $P = \text{diag}(1, -1, -1, -1)$; then we have $N_{ve} (v) = P N_{ve} (v)$ in the simplices $\tilde{\sigma}_v$, where $g_{ve} \neq \tilde{g}_{ve}$.

Within a single 4-simplex, there are in total two parity-related solutions of $(g_{ve}, z_{ef})$ in the nondegenerate case [13]. Therefore, in a general simplicial complex with $N$ simplices, given a Regge-like spin configuration $j_f$, there are in total $2^N$ nondegenerate critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{ef})$ that solve the critical point equations. Any two critical configurations are related by the parity transformation in one 4-simplex or many 4-simplices.

We define the bivectors $\hat{X}_f (v) = \tilde{g}_{ve} \otimes g_{ve} \triangleright X_f$ within the 4-simplices $\tilde{\sigma}_v$, where

$$X_{ef}^{I J} = 2y j_f [\tilde{\hat{n}}_{ef} \wedge u], \quad u = (1, 0, 0, 0)^t. \quad (5.3)$$

Considering the Hermitian matrix representation of $\tilde{\hat{n}}_{ef}$, the action $\tilde{g}_{ve} \triangleright \tilde{\hat{n}}_{ef}$ is given by (note that $J^2 = -1$)

$$\tilde{g}_{ve} (\tilde{\hat{n}}_{ef} \cdot \tilde{\sigma})^I_{ve} = J g_{ve} J^{-1} (\hat{n}_{ef} \cdot \sigma) J g_{ve} J^{-1} = -J g_{ve} (\hat{n}_{ef} \cdot \sigma) J g_{ve} J^{-1} = -P g_{ve} (\hat{n}_{ef} \cdot \sigma) g_{ve} J^{-1}, \quad (5.4)$$

while we have shown $\tilde{g}_{ve} \triangleright u = P (g_{ve} \triangleright u)$; thus, we obtain that

$$\tilde{X}_f (v) = - (P \otimes P) X_f (v). \quad (5.5)$$

\(^{15}\) The SU(2) transformations $\xi_{ef} \mapsto h_{ef} \xi_{ef}$ and $g_{ve} \mapsto g_{ve} h_{ve}^{-1}$ ($h_{v} \in \text{SU}(2)$) are the gauge transformations of the spinfoam action $S$. 
Recall the construction in section 3 and equation (3.2):

$$X_f(v) = \alpha_\sigma(v) N_\sigma(v) \wedge N_\epsilon(v).$$

(5.6)

Following the same argument towards equation (3.2), we obtain that for the bivectors and normals constructed from $\tilde{g}_\sigma$:

$$\tilde{X}_f(v) = \tilde{\alpha}_\sigma(v) \tilde{N}_\sigma(v) \wedge \tilde{N}_\epsilon(v) \Rightarrow -(P \otimes P) X_f(v) = \tilde{\alpha}_\sigma(v) P N_\epsilon(v) \wedge P N_\epsilon(v).$$

(5.7)

Then we have the relation

$$\tilde{\alpha}_\sigma(v) = -\alpha_\sigma(v) \quad \text{and} \quad \tilde{\beta}_\sigma(v) = -\beta_\sigma(v),$$

(5.8)

where $\beta_\sigma(v) = \alpha_\sigma(v) e_\sigma(v)$. Following the same procedure as in section 3, we denote $\tilde{\beta}_{\sigma,\epsilon}$ by $\tilde{\beta}_{\sigma,\epsilon}$ and construct the closure condition for the 4-simplex $\tilde{\sigma}_v$:

$$\sum_{j=1}^{5} \tilde{\beta}_{\sigma,\epsilon}(v) \tilde{N}_\epsilon(v) = 0$$

(5.9)

by choosing the nonvanishing diagonal elements $\tilde{\beta}_{ij}$. Since we have the closure condition $\sum_{j=1}^{5} \tilde{\beta}_{ij} \tilde{N}_\epsilon(v) = 0$, the parity inversion $\tilde{N}_\epsilon(v) = P N_\epsilon(v)$ and $\tilde{\beta}_{ij}(v) = -\tilde{\beta}_{ij}(v)$ for $i \neq j$, we obtain that the diagonal elements $\tilde{\beta}_{ij}(v) = -\beta_{ij}(v)$. Furthermore, we can show that $\tilde{\beta}_{ij}$ can be factorized into the same way as in section 3

$$\tilde{\beta}_{\sigma,\epsilon}(v) = \text{sgn}(\tilde{\beta}_{\sigma,\epsilon}(v)) \tilde{\beta}_{\epsilon}(v) \tilde{\beta}_j(v) = \tilde{\beta}_{\sigma,\epsilon}(v) \sqrt{\text{sgn}(\tilde{\beta}_{\sigma,\epsilon}(v))},$$

(5.10)

which results in the fact that

$$\text{sgn}(\tilde{\beta}_{\sigma,\epsilon}(v)) = -\text{sgn}(\beta_{\sigma,\epsilon}(v)) \quad \text{and} \quad \tilde{\beta}_j(v) = -\beta_j(v).$$

(5.11)

We construct the 4-volume for $\tilde{V}_f(v) N_\epsilon(v)$:

$$\tilde{V}_4(v) := \det(\tilde{\beta}_2(v) \tilde{N}^{\sigma}(v), \tilde{\beta}_3(v) \tilde{N}^{\epsilon}(v), \tilde{\beta}_4(v) \tilde{N}^{\sigma}(v), \tilde{\beta}_5(v) \tilde{N}^{\epsilon}(v)) = -V_4(v)$$

(5.12)

by the parity inversion. Since in section 3 we define the sign factor $\varepsilon(v) = \text{sgn}(\beta_{\sigma,\epsilon}(v)) \text{sgn}(V_4(v))$, we have for the parity inversion

$$\tilde{\varepsilon}(v) = \text{sgn}(\tilde{\beta}_{\sigma,\epsilon}(v)) \text{sgn}(\tilde{V}_4(v)) = \varepsilon(v).$$

(5.13)

Note that one should not confuse the $\tilde{\varepsilon}$ here with the $\tilde{\varepsilon}$ appeared in section 3. This result shows that the parity configuration $(j_f, \tilde{g}_\sigma, \tilde{\epsilon}_f, \tilde{z}_f)$ results in an identical global sign factor $\varepsilon$ for the bivector (recall the proof of theorem 3.5).

The fact that the parity flips the sign of the oriented 4-volume, $\tilde{V}_4(v) = -V_4(v)$, has some interesting consequences. First of all, we mentioned that given a set of Regge-like spins, different nondegenerate critical configurations $x_\sigma = (j_f, g_{\epsilon,\sigma}, \xi_{\epsilon,\sigma}, z_{\sigma})$ may lead to different subdivisions of the triangulation $K$ into sub-triangulation $K_1(x_\sigma), \ldots, K_n(x_\sigma)$, where on each sub-triangulation, $\text{sgn}(V_4(v))$ is a constant. Now we understand that the difference of the subdivisions comes from a local parity transformation, which flips the sign of the oriented 4-volume. On the other hand, given a nondegenerate critical configuration $x_\sigma = (j_f, g_{\epsilon,\sigma}, \xi_{\epsilon,\sigma}, z_{\sigma})$, there exists another nondegenerate critical configuration $\tilde{x}_\sigma = (j_f, \tilde{g}_{\epsilon,\sigma}, \tilde{\xi}_{\epsilon,\sigma}, \tilde{z}_{\sigma})$, naturally associated with $x_\sigma$, obtained by a global parity (parity transformation in all simplices) on the triangulation. The global parity flips the sign of the oriented volume $V_4(v)$ everywhere, thus flipping the sign of the spinfoam action at the nondegenerate critical configuration (the deficit angle, dihedral angle and $\sum_{e \in \partial f} \alpha_e$ are unchanged under the global parity, which is shown in the following), i.e.\cite{16}

$$S(\tilde{x}_\sigma) = -S(x_\sigma),$$

(5.14)

if $\tilde{x}_\sigma$ and $x_\sigma$ are related by a global parity transformation.

\cite{16} The sign in front of the term $i\pi \sigma \sum_{e \in \partial f} I_j$ is unimportant.
Since the frame vectors \( U_e(v) = \pm \tilde{g}_e(v) \frac{N_e(v)}{\sqrt{\det g}} \) are defined up to a sign, the frame \( \tilde{U}_e(v) \) constructed from the parity configuration relates \( U_e(v) \) only by a parity inversion
\[
\tilde{U}_e(v) = P U_e(v). \tag{5.15}
\]
The same relation holds for the co-frame \( \tilde{E}_e(v) \),
\[
\tilde{E}_e(v) = P E_e(v), \tag{5.16}
\]
from the relation
\[
\tilde{U}_e^j (v) \tilde{E}_{j\ell\epsilon} (v) = \delta_\ell^j - \delta_\ell^j. \tag{5.17}
\]
We then obtain the same relation relating the bivector and co-frame/frame as in theorem 3.5:
\[
\tilde{X}_f(v) = \varepsilon \tilde{V}_f[U_e(v) \wedge \tilde{U}_e(v)] \quad \text{and} \quad \tilde{X}_f(v) = \varepsilon * [\tilde{E}_e(v) \wedge \tilde{E}_{e\epsilon}(v)], \tag{5.18}
\]
which is consistent because of the relations \( \tilde{X}_f(v) = -(P \otimes P) X_f(v), \tilde{U}_e(v) = P U_e(v), \tilde{E}_e(v) = P E_e(v), \tilde{V}_f(v) = -V_f(v) \) and \( \delta_{ijkl} P_i^\mu P_j^\nu P_k^\rho P_l^\sigma = -\varepsilon_{MNPH} \). Here we emphasize that the sign factor \( \varepsilon \) for the parity configuration \( (j_f, \tilde{g}_{ef}, \tilde{\xi}_{ef}, \tilde{z}_{ef}) \) is the same as the original configuration \( (j_f, g_{ef}, \xi_{ef}, z_{ef}) \), and thus is consistent with the fact that \( \varepsilon \) is a global sign factor on the entire triangulation, i.e. the local/global parity inversion of the critical configuration does not change the global sign \( \varepsilon \).

The local/global parity inversion \( \tilde{E}_e(v) = P E_e(v) \) does not change the discrete metric \( g_{e\ell\epsilon}(v) = \eta_{ef} E_{e\ell}(v) E_{e\ell}(v) \), so the parity configuration \( (j_f, \tilde{g}_{ef}, \tilde{\xi}_{ef}, \tilde{z}_{ef}) \) leads to the same discrete metric as \( (j_f, g_{ef}, \xi_{ef}, z_{ef}) \), but gives an O(1,3) gauge transformation (parity inversion) for the co-frame \( E_e(v) \). The SO(1,3) matrix \( \Omega_{ef} \in SO(1, 3) \) is uniquely compatible with the co-frame \( E_e(v) \) and is a discrete spin connection when \( \text{sgn}(V_4(v)) = \text{sgn}(V_4(v')) \), as was shown in appendix A. Given a nondegenerate critical configuration with a subdivision of the triangulation into sub-triangulations, in each of which \( \text{sgn}(V_4(v)) \) is a constant, we consider a global parity transformation which does not change the subdivision but flip the signs of \( \text{sgn}(V_4(v)) \) in all sub-triangulations. Given a spin connection \( \Omega_{uv} \) with both \( \sigma_u, \sigma_v \) being in the same sub-triangulation, i.e. \( \text{sgn}(V_4(v)) = \text{sgn}(V_4(v')) \), the spin connection \( \Omega_{uv} \in SO(1, 3) \) after a parity transformation in both \( \sigma_u, \sigma_v \) is given by
\[
\Omega_{uv'} = P \Omega_{uv} P, \tag{5.19}
\]

since \( \tilde{\Omega}_{uv'} \) is uniquely determined by
\[
\tilde{\Omega}_{uv'} \tilde{E}_e(v') = \tilde{E}_e(v), \quad e \subset t_e, \quad e = (v, v'). \tag{5.20}
\]

On the other hand, we can check from
\[
\tilde{g} = J g J^{-1}, \quad \tilde{g} (-\tilde{\sigma}) \tilde{g} = P \gg \gg \tilde{g} \tag{5.21}
\]
that given a 4-vector \( V' \),
\[
\tilde{g} P (V' \sigma_1) \tilde{g} = P (\gg (V' \sigma_1) \gg), \quad \text{i.e.} \quad \tilde{g} P V = P g V \quad \text{in spin-1 representation}. \tag{5.22}
\]
Let \( V = E_e(v) \), using \( g_{euv} = \mu_e \Omega_{euv} \),
\[
\tilde{g}_{euv} \tilde{E}_e(v') = \tilde{g}_{euv} P E_e(v') = P g_{euv} E_e(v') = \mu_e P E_e(v) \equiv \mu_e \tilde{E}_e(v). \tag{5.23}
\]
Therefore, we obtain from \( \tilde{g}_{euv} = \tilde{\mu}_e \tilde{\Omega}_{euv} \) that the sign \( \mu_e \) is invariant under the parity transformation:
\[
\mu_e = \tilde{\mu}_e, \tag{5.24}
\]
where \( e \) is a, internal edge. In the case when \( t_e \) is a boundary tetrahedron, the parity transformation changes the co-frame \( E_e(v) \mapsto \tilde{E}_e(v) = PE_e(v) \) at the vertex \( v \), while leaves
the boundary co-frame $E_i(e)$ invariant. Therefore, the spin connection $\tilde{\Omega}_{ve} \in SO(1,3)$ is uniquely determined by

$$\tilde{\Omega}_{ve} E_i(e) = \tilde{E}_i(v), \quad \ell \subset t_e.$$ (5.25)

Before the parity transformation, $\Omega_{ve} E_i(e) = E_i(v)$ determines uniquely the spin connection $\Omega_{ve}$. Then the relation between $\tilde{\Omega}_{ve}$ and $\Omega_{ve}$ is given by

$$\tilde{\Omega}_{ve} = \mathbf{P} \Omega_{ve} \mathbf{T} \quad \text{where} \quad \mathbf{T} = \text{diag}(-1, 1, 1, 1)$$ (5.26)

by the fact that the co-frame vectors $E_i(e)$ are orthogonal to $(1, 0, 0, 0)^\ell$ and both $\tilde{\Omega}_{ve}$ and $\Omega_{ve}$ belong to $SO(1,3)$. Here the matrix $\mathbf{T}$ is a time-reversal in the Minkowski space, which leaves $E_i(e)$ invariant. Given a spatial vector $V^f$ orthogonal to $(1, 0, 0, 0)^f$,

$$\tilde{g}(V^f \sigma_i)g^f_\sigma = -P g(V^f \sigma_i)g^f_\sigma, \quad \text{i.e.} \quad \tilde{g} = -Pg \quad \text{in spin-1 representation.}$$ (5.27)

Let $V = E_i(e)$, using $g_{ve} = \mu_e \Omega_{ve}$ in the spin-1 representation:

$$\tilde{g}_{ve} E_i(e) = -P g_{ve} E_i(e) = -\mu_e P E_i(v) = -\mu_e \tilde{E}_i(v).$$ (5.28)

Therefore, we obtain from $\tilde{g}_{ve} E_i(e) = \mu_e \tilde{E}_i(v)$ that

$$\mu_e = -\tilde{\mu}_e$$ (5.29)

for an edge connecting to the boundary. A boundary triangle is shared by exactly two boundary tetrahedra; in the dual language, a boundary face has exactly two edges connecting to the boundary. Thus, the product $\prod_{e \in \partial f} \mu_e$ is invariant under the parity transformation, i.e.

$$\prod_{e \in \partial f} \mu_e = \prod_{e \in \partial f} \tilde{\mu}_e$$ (5.30)

for either a boundary face or an internal face. If we write $\mu_e = e^{i\pi n_e}$ and $\tilde{\mu}_e = e^{i\pi \tilde{n}_e}$, then we have

$$\sum_{e \in \partial f} n_e = \sum_{e \in \partial f} \tilde{n}_e.$$ (5.31)

We consider $\tilde{\Omega}_f(v)$ a loop holonomy of the spin connection along the boundary of an internal face $f$, based at the vertex $v$, which is constructed from a global parity configuration $(f_j, \tilde{g}_{ve}, \tilde{e}_{jv}, \tilde{e}_jv)$, with $\tilde{g}_{ve} \neq g_{ve}$ at all the vertices. It is different from the original $\Omega_f(v)$ by

$$\tilde{\Omega}_f(v) = \mathbf{P} \Omega_f(v) \mathbf{T}.$$ (5.32)

From equation (4.16), $\Omega_f(v)$ can be expressed in terms of the co-frame vectors $E_i(v)$ and $E_i(e)$ for the edges $\ell_1$ and $\ell_2$ respectively of the triangle $f$:

$$\Omega_f(v) = e^{i\pi \sum_{\ell} n_{\ell}} \exp\left( \frac{\# E_{\ell_1}(v) \wedge E_{\ell_2}(v)}{|E_{\ell_1}(v) \wedge E_{\ell_2}(v)|} \text{sgn}(V_4) \Theta_f + \frac{E_{\ell_1}(v) \wedge E_{\ell_2}(v)}{|E_{\ell_1}(v) \wedge E_{\ell_2}(v)|} \pi \sum_{e} n_e \right)$$

$$\tilde{\Omega}_f(v) = e^{i\pi \sum_{\ell} \tilde{n}_{\ell}} \exp\left( \frac{\# \tilde{E}_{\ell_1}(v) \wedge \tilde{E}_{\ell_2}(v)}{|\tilde{E}_{\ell_1}(v) \wedge \tilde{E}_{\ell_2}(v)|} \text{sgn}(\tilde{V}_4) \tilde{\Theta}_f + \frac{\tilde{E}_{\ell_1}(v) \wedge \tilde{E}_{\ell_2}(v)}{|\tilde{E}_{\ell_1}(v) \wedge \tilde{E}_{\ell_2}(v)|} \pi \sum_{e} \tilde{n}_e \right).$$ (5.33)

From the previous results $\text{sgn}(\tilde{V}_4) = -\text{sgn}(V_4)$, $\sum_e n_e = \sum_e \tilde{n}_e$ and the relation $\mathbf{P} \otimes \mathbf{P} (\# E_1 \wedge E_2) = -\# \mathbf{P} E_1 \wedge \mathbf{P} E_2$, we obtain that

$$\Theta_f = \tilde{\Theta}_f$$ (5.34)

which is consistent with the fact that the deficit angle $\Theta_f$ is determined by the metric $g_{\ell_1 \ell_2}$ which is invariant under the parity transformation.

For the holonomy $\Omega_f(e_1, e_0)$ for a boundary face $f$, under a global parity

$$\tilde{\Omega}_f(e_1, e_0) = \mathbf{T} \Omega_f(e_1, e_0) \mathbf{T}.$$ (5.35)
Recall equation (4.38), we have for both $\tilde{\Omega}_f(e_1, e_0)$ and $\Omega_f(e_1, e_0)$:

\[
\hat{g}(\xi_{ef})^{-1}\tilde{\Omega}_f(e_1, e_0)\hat{g}(\xi_{ef}) = e^{\pi \sum n_e \epsilon_e \text{sgn}(v_i) \tilde{g}_f^e} \cdot e^{\pi \sum n_e \epsilon_e \text{sgn}(v_i) \tilde{g}_f^e} \cdot e^{\pi \sum n_e \epsilon_e \text{sgn}(v_i) \tilde{g}_f^e}.
\]

(5.36)

Since $T$ commutes with $\hat{g}(\xi_{ef}) \in \text{SU}(2)$ and $TK_3T = -K_3$, $TJ_f = J_3$, we obtain that

\[
\Theta_f^B = \tilde{\Theta}_f^B
\]

and consistent with the fact that the dihedral angle $\Theta_f^B$ is determined by the metric $g_{t_1t_2}$, which is invariant under the parity transformation.

Before we reach the following section, we emphasize that given a Regge-like spin configuration $f_j$, there exist only two nondegenerate critical configurations $(j_f, g_{ef}, \bar{\xi}_{ef}, \bar{z}_{ef})$, such that the oriented 4-volume has a constant sign on the triangulation, i.e. $\text{sgn}(V_4(v))$ is a constant for all $v$. The existence can be shown in the following way: given a nondegenerate critical configuration $(j_f, g_{ef}, \bar{\xi}_{ef}, \bar{z}_{ef})$, it determines a subdivision of the triangulation into sub-triangulations $K_1, \ldots, K_n$, where on each $K_i$, $\text{sgn}(V_4(v))$ is a constant, but $\text{sgn}(V_4(v))$ is not a constant for neighboring $K_i$ and $K_j$. However, we can always make a parity transformation for all the simplices within some sub-triangulations, to flip the sign of the oriented 4-volume, such that $\text{sgn}(V_4(v))$ is a constant on the entire triangulation. Any two nondegenerate solutions $(j_f, g_{ef}, \tilde{\xi}_{ef}, \tilde{z}_{ef})$ are related by a (local) parity transformation, which flips the sign of $V_4(v)$ at least locally. There exist two nondegenerate critical configurations $(j_f, g_{ef}, \bar{\xi}_{ef}, \bar{z}_{ef})$, such that the oriented 4-volume has a constant sign on the entire triangulation, while the two configurations are related by a global parity transformation. If there was another nondegenerate critical configuration such that the oriented 4-volume has a constant sign on the entire triangulation, it must relate the existed two configurations by a local parity transformation, which flips $\text{sgn}(V_4(v))$ only locally, and thus breaks the constancy of $\text{sgn}(V_4(v))$.

6. Asymptotics of degenerate amplitudes

6.1. Degenerate critical configurations

The previous discussions of the critical configuration and asymptotic formula are under the nondegenerate assumption:

\[
\prod_{e_1, e_2, e_3, e_4} \det(N_e(v), N_{e_2}(v), N_{e_3}(v), N_{e_4}(v)) \neq 0,
\]

(6.1)

where $N_e(v) = g_{ve}(1, 0, 0, 0)'$, i.e. any four of the five normal vectors $N_e(v)$ form a linearly independent set and span the four-dimensional Minkowski space.

Now we consider a degenerate critical configuration $(j_f, g_{ef}, \bar{\xi}_{ef}, \bar{z}_{ef})$ that solves the critical point equations (2.36)–(2.38), but violates the above nondegenerate assumption at all vertices on a triangulation (with boundary). If we assume the nondegeneracy of the tetrahedra, i.e. given a tetrahedron $t_e$, the 4-vectors $\vec{n}_{ef}$ obtained from the spinors $\bar{\xi}_{ef}$ span a three-dimensional subspace, then lemma 3 in the first reference of [13] shows that within each 4-simplex, all five normals $N_e(v)$ from the degenerate critical configuration $(j_f, g_{ef}, \bar{\xi}_{ef}, \bar{z}_{ef})$ are parallel and more precisely $N_e(v) = u = (1, 0, 0, 0)'$. By definition $N_e(v) = g_{ve}(1, 0, 0, 0)'$, we find that all the group variables $g_{ve} \in \text{SU}(2)$ for a degenerate critical configuration $(j_f, g_{ef}, \bar{\xi}_{ef}, \bar{z}_{ef})$. For the bivectors $\ast X_f(v)$, they are all orthogonal to the same unit vector $u = (1, 0, 0, 0)$.

17 Recall that we have fixed $g_{ve} = 1$ to make the vertex amplitude finite.
From \( \ast X_f(v) \cdot u = 0 \), we can write the bivector \( X_f(v) = V_f(v) \wedge u \) for a vector \( V_f(v) \) orthogonal to \( u \). The vector \( V_f(v) \) can be determined by the parallel transportation \( X_f(v) = g_{ve}X_{ef}g_{ev} \) and \( X_{ef} = 2\gamma_j j \hat{n}_{ef} \wedge u \); thus,

\[
V_f(v) = 2\gamma_j j \hat{n}_{ef} \quad V_f(v) = 2\gamma_j j g_{ve} \hat{n}_{ef}.
\]

The above relation does not depend on the choice of \( e \) (recall proposition 2.1). From the closure condition equation (2.38), we have

\[
\sum_{f \in C_v} \varepsilon_{ef}(v)V_f(v) = 0.
\]

Therefore a degenerate critical configuration \( (j_f, g_{ve}, \xi_{ef}) \) assigns uniquely a spatial vector \( V_f(v) \perp u \) at the vertex \( v \) for each triangle \( f \), satisfying the closure condition (6.3). The collection of the vectors \( V_f(v) \) is referred to as a vector geometry in [13]18.

Since \( g_{ve} \in \text{SU}(2) \) in the degenerate critical configuration \( (j_f, g_{ve}, \xi_{ef}) \), we have immediately \( ||Z_{ef}||^2 = 1 \). Then for each face action \( S_f \) (internal face or boundary face),

\[
S_f = 2i\gamma_j j \sum_v \ln \frac{||Z_{ef}||}{||Z_{ef}||} - 2i j_f \sum_v \phi_{ve'} = -2i j_f \sum_v \phi_{ve'}.
\]

In the same way as we did for the nondegenerate amplitude, we make use of equations (2.36) and (2.37), which now take the following forms:

\[
g_{ve}(j \xi_{ef}) = e^{-i \phi_{ve'}} g_{ve'}(j \xi_{ef})
g_{ve} \xi_{ef} = e^{i \phi_{ve'}} g_{ve} \xi_{ef}.
\]

First of all, for an internal face \( f \), we again consider the successive actions on \( \xi_{ef} \) of \( g_{ve}g_{ve} \) around the entire boundary of the face \( f \),

\[
\prod_{v \in \partial f} g_{ve}g_{ve} j \xi_{ef} = e^{-i \sum_v \phi_{ve'}} j \xi_{ef}
\]

\[
\prod_{v \in \partial f} g_{ve}g_{ve} \xi_{ef} = e^{i \sum_v \phi_{ve'}} \xi_{ef},
\]

where \( g_{ve} \in \text{SU}(2) \). In the same way as we did for the nondegenerate case, the above equations imply that for the loop holonomy \( G_f(e) = \prod_{v \in \partial f} g_{ve}g_{ve} \),

\[
G_f(e) = \exp \left[ i \sum_{v \in \partial f} \phi_{ve'} \sigma \cdot \hat{n}_{ef} \right].
\]

For a boundary face \( f \), again in the same way as we did for the nondegenerate case, we obtain

\[
G_f(e_1, e_0) = g(\xi_{ef}) e^{i \sum_{v \in \partial f} \phi_{ve'}} g(\xi_{ef})^{-1}.
\]

We then need to determine the physical interpretation of the angle \( \sum_{v \in \partial f} \phi_{ve'} \) in different cases.

Recalling the degenerate critical equations (6.5) together with the closure condition (2.38), we find that they are essentially the same as the critical equations in [21] for a Euclidean

18 The geometrical interpretation of degenerate critical configurations is also understood as the geometrical interpretation of critical configuration in the SU(2) BF theory [29].
spinfoam amplitude:

\[ g_{ve}^\pm(J_{\xi_{ef}}) = e^{-i\phi_{\xi_{ef}}} g_{ve}\pm(J_{\xi_{ef}}) \]

\[ g_{ve}^\pm \xi_{ef} = e^{i\phi_{\xi_{ef}}} g_{ve}\pm \xi_{ef} \]

\[ 0 = \sum_{j \in \sigma_v} \xi_{ef}(v) \hat{n}_{ef} \]

where the equations for self-dual or anti-self-dual sector are essentially the same, and both of them are the same as the above degenerate critical equation for the Lorentzian amplitude.

Therefore, given a degenerate critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{ef}) \) for the Lorentzian amplitude, there exists a critical configuration \( (j_f, \tilde{g}_{ve}, \tilde{\xi}_{ef}) \) for the Euclidean amplitude in [21], such that \( g_{ve} = \tilde{g}_{ve} \). In the following, we classify the degenerate Lorentzian critical configurations into two types (type A and type B) and discuss the uniqueness of the corresponding Euclidean critical configurations.

**Type-A configuration.** A degenerate Lorentzian critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{ef}) \) corresponds to a Euclidean critical configuration \( (j_f, \tilde{g}_{ve}, \tilde{\xi}_{ef}) \), which is nondegenerate at each 4-simplex \( \sigma_v \) of the triangulation, i.e. any four of the five normals \( N_v = (g_{ve}, g_{ve}) \triangleright (1, 0, 0, 0)' \) span a four-dimensional vector space. Since the Euclidean spins \( j_f \) and spinors \( \xi_{ef} \) are uniquely specified by the Lorentzian configuration \( (j_f, g_{ve}, \xi_{ef}, z_{ef}) \), we only need to consider how many solutions \( (\tilde{g}_{ve}, \tilde{\xi}_{ef}) \) in equation (6.9) if the variables \( j_f \) and \( \xi_{ef} \) are fixed. It is shown in [13] that for a 4-simplex \( \sigma_v \), there are only two solutions in the nondegenerate case\(^{19}\):

\[ (g_{ve}^+, g_{ve}^-) = (g_{ve}^+, g_{ve}^-) \quad \text{and} \quad (g_{ve}^+, g_{ve}^-) = (g_{ve}^+, g_{ve}^-). \]

Then the correspondence \( g_{ve} = \tilde{g}_{ve} \) uniquely fixes a solution \( (\tilde{g}_{ve}, \tilde{\xi}_{ef}) \) for the Euclidean critical configuration \( (j_f, \tilde{g}_{ve}, \tilde{\xi}_{ef}) \).

**Type-B configuration.** The degenerate Lorentzian critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{ef}) \) could always correspond to a degenerate Euclidean critical configuration \( (j_f, g_{ve}, \xi_{ef}) \) with \( g_{ve} = \tilde{g}_{ve} \) by \( (g_{ve}^+, g_{ve}^-) = (g_{ve}, g_{ve}) \), even the data \( j_f \) and \( \xi_{ef} \) can have two nondegenerate solutions as above. Then in this case, we always make the above nondegenerate choice as the canonical choice.

**Type-B configuration.** The data \( j_f \) and \( \xi_{ef} \) in a degenerate Lorentzian critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{ef}) \) lead to only one Euclidean solutions \( (g_{ve}, g_{ve}) \in \text{SO}(4) \) for equation (6.9) in each 4-simplex \( \sigma_v \). Then the Euclidean configuration \( (j_f, g_{ve}, \xi_{ef}) \) is degenerate in \( \sigma_v \) in the sense of [13]. Then obviously the correspondence is unique by \( g_{ve} \mapsto (g_{ve}, g_{ve}) \).

6.2. **Type-A degenerate critical configuration: Euclidean geometry**

First of all, we consider a type A degenerate Lorentzian critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{ef}) \) on the triangulation (with a boundary). The corresponding Euclidean critical configuration \( (j_f, g_{ve}^\pm, \xi_{ef}) \) is nondegenerate everywhere. We can construct a nondegenerate discrete Euclidean geometry on the triangulation such that (see [21], see also [19])

\(^{19}\) The notion of nondegeneracy here is different from the notion in [13]. In lemma 4 of the first reference of [13], there are four solutions in a 4-simplex \( (g_{ve}, g_{ve}), (g_{ve}, g_{ve}), (g_{ve}, g_{ve}), (g_{ve}, g_{ve}) \) for the nondegenerate case (in the sense of [13]). However, the two solutions \( (g_{ve}, g_{ve}), (g_{ve}, g_{ve}) \) are degenerate in our notion of degeneracy.
A Euclidean co-tetrad $E_\ell(v), E_\ell(e)$ of the triangulation (bulk and boundary) can be constructed from $(j_f, g^{+}_v, \xi_{ef})$, unique up to a sign flipping $E_\ell \rightarrow -E_\ell$, such that the spin $j_f$ satisfies

$$2 \gamma j_f = |E_{\ell_1}(v) \wedge E_{\ell_2}(v)|.$$  

(6.11)

From the co-tetrad, we can construct a unique discrete metric with the Euclidean signature on the whole triangulation (bulk and boundary)

$$E g^{\ell_1\ell_2}_v(v) = \delta_{II} E^I_{\ell_1}(v)E^I_{\ell_2}(v)$$

(6.12)

$$E g^{\ell_1\ell_2}_e(e) = \delta_{II} E^I_{\ell_1}(e)E^I_{\ell_2}(e).$$

So $\gamma j_f$ is the triangle area from the discrete metric $E g^{\ell_1\ell_2}$.

For the bivectors in the bulk,

$$j_f(g^+_{ve}, g^-_{ve})(\hat{n}_{ef}, \hat{\ell}_{ef}) = \varepsilon \ast E_{\ell_1}(v) \wedge E_{\ell_2}(v).$$

(6.13)

For the bivector on the boundary

$$j_f(\hat{n}_{ef}, \hat{\ell}_{ef}) = \varepsilon \ast E_{\ell_1}(e) \wedge E_{\ell_2}(e),$$

(6.14)

where $\varepsilon$ is a global sign on the entire triangulation. If the triangulation has a boundary, the sign factor $\varepsilon$ is specified by the orientation of the boundary triangulation, i.e. $\varepsilon = \text{sgn}(V_3)$ for the boundary tetrahedra.

The SO(4) group variable $(g^+_v, g^-_v)$ equals the Euclidean spin connection $E \Omega_e$ compatible with $E_\ell(v)$, up to a sign $\mu_v = e^\pi n_v$ ($n_v = 0, 1$), i.e.

$$(g^+_v, g^-_v) = \mu_v E \Omega_e$$

(6.15)

in the spin-1 representation. Here, $E \Omega_e \in $ SO(4) is compatible with the co-frame $E_\ell(v), E_\ell(e)$

$$(E \Omega_{ve}^I)j_E^I(v') = E_{\ell_1}(v) \quad \text{and} \quad (E \Omega_{ve}^I)j_E^I(e) = E_{\ell_1}(v)$$

(6.16)

where $\text{sgn}(V_4(v)) = \text{sgn}(V_4(v'))$, $E \Omega_e$ is the unique discrete spin connection compatible with the co-frame. In addition, we note that each $\mu_v$ is not invariant under the sign flipping $E_\ell \rightarrow -E_\ell$, but the product $\prod_{v \in \bar{f}} \mu_v$ is invariant for any (internal or boundary) face $f$ (see lemma 3.4).

Therefore, in this way, a type-A degenerate Lorentzian critical configuration determines uniquely a triple of (Euclidean) variables $(E g^{\ell_1\ell_2}, n_v, \varepsilon)$ corresponding to a Euclidean geometry and two types of sign factors.

Given a nondegenerate Euclidean critical configuration $(j_f, g^{+}_v, \xi_{ef})$, in the same way as the nondegenerate Lorentzian critical configuration, it determines a subdivision of the triangulation into sub-triangulations (with boundaries) $K_1, \ldots, K_n$; on each of the sub-triangulations, the sign of the oriented 4-volume $\text{sgn}(V_4(v))$ is a constant.

Now we discuss the spin foam amplitude at a type-A degenerate configuration, while we restrict our attention to a sub-triangulation $K_0$ where $\text{sgn}(V_4(v))$ is a constant. For an internal face $f$, it is shown in [21] that the loop holonomy along the boundary of $f$ is given by

$$(G_j^f(e), G^-_j(e)) = (e^{\frac{1}{2} [\varepsilon \text{sgn}(V_4)E \Theta_f + \pi \sum_{n} n_v]} \hat{\alpha}_j, e^{-\frac{1}{2} [\varepsilon \text{sgn}(V_4)E \Theta_f - \pi \sum_{n} n_v]} \hat{\alpha}_j),$$

(6.17)

where $E \Theta_f$ is the deficit angle from the Euclidean spin connection compatible with the metric $E g^{\ell_1\ell_2}$. By the above identification, $g_{ve} = g^+_v$ between the degenerate Lorentzian critical configuration $(j_f, g_v^+, \xi_{ef}, z_{ef})$ and a nondegenerate Euclidean critical configuration $(j_f, g^+_v, \xi_{ef})$. We obtain that for the degenerate Lorentzian critical configuration, the loop holonomy $G_j^f(e) = G_{-j}^f(e)$. Comparing with equation (6.7),

$$\sum_{v \in \partial f} \phi_{ve} = \frac{1}{2} \left[ \varepsilon \text{sgn}(V_4)E \Theta_f + \pi \sum_{v \in \partial f} n_v \right].$$

(6.18)
Therefore, the angle $\sum_{v \in \partial f} \phi_{ev'}$ has the physical meaning as a deficit angle in a corresponding Euclidean geometry. Then the face action (as a function of $(E_{g_{f12}}, n_e, \epsilon)$) reads

$$S_f(E_{g_{f12}}, n_e, \epsilon) = -i \epsilon \text{ sgn}(V_4) \sum_{e \in \partial f} n_e$$

for an internal face $f$.

For a boundary face $f$, along the path holonomy along its internal boundary $p_{e_i e_0}$ given by

$$\left(G_f^j(e_i, e_0), G_f^j(e_1, e_0)\right) = \left(g(\xi_{e_i e_0}) e^{i \epsilon \text{ sgn}(V_4) \Theta_f + \pi \sum_{e \in \partial f} n_e}, g(\xi_{e_1 e_0})^{-1}\right),$$

where $E \Theta_f^B$ is the dihedral angle (determined by the metric $E_{g_{f12}}$) between two boundary tetrahedra $t_{e_0}$ and $t_{e_1}$ at the triangle $f$ shared by them. The degenerate Lorentzian critical configuration $G_f(e_i, e_0)$ is identified with $G_f^j(e_1, e_0)$ here. Comparing to equation (6.8), we obtain that

$$\sum_{v \in \partial p_{e_i e_0}} \phi_{ev'} = \frac{1}{2} \left[ \epsilon \text{ sgn}(V_4) E \Theta_f^B + \pi \sum_{e} n_e \right].$$

Therefore, the face action $S_f$ for a boundary face $f$ is given by

$$S_f(E_{g_{f12}}, n_e, \epsilon) = -i \epsilon \text{ sgn}(V_4) \sum_{e \in \partial f} n_e.$$

As a result, at a type-A degenerate critical configuration (restricted to a sub-triangulation $K_0$), the Lorentzian spinfoam action $S_f$ is a function of the variables $(E_{g_{f12}}, n_e, \epsilon)$ and behaves mainly as a Euclidean Regge action:

$$S(E_{g_{f12}}, n_e, \epsilon)|_{K_0} = \sum_{f \text{ internal}} S_f(E_{g_{f12}}, n_e, \epsilon) + \sum_{f \text{ boundary}} S_f(E_{g_{f12}}, n_e, \epsilon)$$

$$= \left[ -i \epsilon \text{ sgn}(V_4) \sum_{e \in \partial f} n_e \sum_{f \in \partial C_i} j_f \right],$$

where we note that the areas $\gamma_{j_f}$, deficit angles $E \Theta_f$ and dihedral angles $E \Theta_f^B$ are uniquely determined by the discrete metric $g_{f12}$. Moreover, for each tetrahedron $t$, the sum of face spin $\sum_{f \in t} j_f$ is an integer. For half-integer spins, $e^{-i \pi \sum_{e \in \partial f} n_e}$ gives an overall sign factor. Therefore, in general, at a type-A degenerate critical configuration $(j_f, g_{f12}, \xi_{ef}, \gamma_{j_f})$ for the Lorentzian amplitude,

$$e^{S}|_{K_0} = \pm \exp \left[ -i \epsilon \text{ sgn}(V_4) \sum_{e \in \partial f} n_e \sum_{f \in \partial C_i} j_f \right].$$

Again there exist two ways to make the overall sign factor disappear: (1) only consider integer spins $j_f$ or (2) modify the embedding from SU(2) unitary irreps to SL(2, C) unitary irreps by $j_f \mapsto (p_f, k_f) := (2\gamma_{j_f}, 2j_f)$; then the spinfoam action $S$ is replaced by $2S$. In these two cases, the exponential $e^{S}$ at the critical configuration is independent of the variable $n_e$.

According to the properties of the Euclidean Regge geometry, given a collection of (Euclidean) Regge-like areas $\gamma_f$, the discrete Euclidean metric $E_{g_{f12}}(v)$ is uniquely determined at each vertex $v$. Furthermore, since the areas $\gamma_f$ are Regge-like, there exists a discrete Euclidean metric $E_{g_{f12}}$ in the entire bulk of the triangulation, such that the
neighboring 4-simplices are consistently glued together, as we constructed in [21]. This discrete metric $\delta g_{f,t_2}$ is obviously unique by the uniqueness of $g_{t_1,t_2}(v)$. Therefore, given the partial amplitude $A_{\delta g_{f,t_2}}(K)$ in equation (2.13) with a specified Euclidean Regge-like $j_f$, all the degenerate critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{ef})$ of type-A corresponds to the same discrete Euclidean metric $\delta g_{f,t_2}$, provided a Regge boundary data. Any two type-A critical configurations $(j_f, g_{ve}, \xi_{ef}, z_{ef}) = (j_f, g_{ve}', \xi_{ef}')$ with the same $j_f$ are related by local or global parity transformation in the Euclidean theory, see [21], similar to the Lorentzian nondegenerate case.

As a result, given an Euclidean Regge-like spin configuration $j_f$ and a Regge boundary data, the degenerate critical configurations of type-A give the following asymptotics:

$$A_{j_f}(K)|_{\text{Dir-A}} \sim \sum_{x_c} a(x_c) \left( \frac{2\pi}{\lambda} \right)^{\frac{|\delta x_c|}{2} - N(v,f)} \exp \frac{i \text{Ind} \mathcal{H}(x_c)}{\sqrt{|\det \mathcal{H}(x_c)|}} \left[ 1 + o \left( \frac{1}{\lambda} \right) \right]$$

$$\times \prod_{i=1}^{n(x_c)} \exp -i\lambda \left[ \varepsilon \text{sgn}(V_i) \sum_{\text{internal } f} j^E_f \Theta_f + \varepsilon \text{sgn}(V_i) \sum_{\text{boundary } f} j^E_f \Theta^B_f \right]$$

$$+ \pi \sum_{\varepsilon} n_e \sum_{j \in \mathcal{C}_e} \delta_{j_f} j_{\mathcal{C}_e}$$

where $x_c = (j_f, g_{ve, \xi_{ef}}, z_{ef}) = (j_f, g_{ve}', \xi_{ef}')$ labels the degenerate critical configurations of type-A, $r(x_c)$ is the rank of the Hessian matrix at $x_c$ and $N(v,f)$ is the number of the pair $(v,f)$ with $v \in \partial f$ (recall equation (2.13), there is a factor of $\text{dim}(j_f)$ for each pair of $(v,f)$). $a(x_c)$ is the evaluation of the integration measures at $x_c$, which does not scale with $\lambda$. Here, $E \Theta_f$ and $E \Theta^B_f$ only depend on the Euclidean metric $\delta g_{f,t_2}$, which is uniquely determined by the Euclidean Regge-like spin configuration $j_f$ and the Regge boundary data.

### 6.3. Type-B degenerate critical configuration: vector geometry

Given a type-B degenerate Lorentzian critical configuration $(j_f, g_{ve, \xi_{ef}}, z_{ef})$, the data $\xi_{ef}$ lead to only one Euclidean solution $(g_{ve}, g_{ve}) \in \text{SU(2)} \times \text{SU(2)}$ for equation (6.9) in each 4-simplex $\sigma_v$. Then the Euclidean configuration $(j_f, g_{ve}', \xi_{ef})$ is degenerate in $\sigma_v$ in the sense of [13]. Therefore, there is no nondegenerate geometric interpretation of a type-B degenerate Lorentzian critical configuration $(j_f, g_{ve, \xi_{ef}}, z_{ef})$. It can only be interpreted as a vector geometry in terms of $V_f(v), V_f(e)$ on the triangulation (bulk and boundary), where all the vectors $V_f(v), V_f(e)$ are orthogonal to the unit time-like vector $u = (1, 0, 0, 0)$ and $|V_f(v)| = |V_f(e)| = 2\gamma j_f$. The vectors $V_f(v)$ and $V_f(e)$ are uniquely determined by $j_f$ and $\xi_{ef}$ by $V_f(v) = 2\gamma j_f \tilde{\xi}_{ef}$ and $V_f(e) = 2\gamma j_f g_{ve} \tilde{\xi}_{ef}$, since the group variable $g_{ve}$ is uniquely determined by $\xi_{ef}$. We have the parallel transportation using the spin-1 representation of $g_{ve}$:

$$g_{ve} \triangleright V_f(v') = V_f(v)$$

for all triangles $f$ in the tetrahedron $t_e$ (shared by $v, v'$ if not a boundary tetrahedron). Then the unique group variables $g_{ve}, g_{ve} \in \text{SU(2)}$ are said to be compatible with the vector geometry $V_f(v), V_f(e)$. Therefore, a type-B degenerate Lorentzian critical configuration $(j_f, g_{ve}, \xi_{ef})$ uniquely determines a vector geometry $V_f(v), V_f(e)$. Conversely, given a vector geometry $V_f(v), V_f(e)$, it uniquely determines the SU(2) group variables $g_{ve}$ up to a sign $e^{i\pi \alpha_v}$, due to the two-to-one correspondence between SU(2) and SO(3).

Since we have shown from the critical point equations that

$$G_f(e) = e^{i \sum (\xi_{ef} + \Phi_{ve} d_{\Phi_{ve}})}$$

$$G_f(e_0) = g(\xi_{ef}) e^{i \sum (\xi_{ef} + \Phi_{ve} d_{\Phi_{ve}})} g(\xi_{ef})^{-1}$$

(6.27)
the above SU(2) angle $\sum_{v\in\delta f} \phi_{ve}$ is determined uniquely by the group variables $g_{ve}$ (which is uniquely compatible with the vector geometry $V_f(\nu)$, $V_f(e)$ up to a sign $e^{\pm m}$)

$$\sum_{v\in\delta f} \phi_{ve} = \frac{1}{2} \Phi_f + \pi \sum_{e\in\delta f} n_e \quad \text{and} \quad \sum_{v\in\delta_{1\nu}} \phi_{ve} = \frac{1}{2} \Phi_f + \pi \sum_{e\in\delta_{1\nu}} n_e$$

(6.28)

respectively for an internal face and a boundary face, where the SO(3) angle $\Phi_f$ is uniquely determined by the vector geometry $V_f$ only (the factor 1/2 shows the relation between an SU(2) angle and an SO(3) angle). Therefore, for the face action (internal face and boundary face),

$$S_f(V_f, n_e) = i j_f \Phi_f - 2i\pi \sum_{e\in\delta f} n_e j_f \quad \text{and} \quad S_f(V_f, n_e) = i j_f \Phi^\nu_f - 2i\pi \sum_{e\in\delta_{1\nu}} n_e j_f.$$

(6.29)

As a result, at a type-B degenerate critical configuration, the Lorentzian spinfoam action $S$ is a function of the variables $(V_f, n_e)$:

$$S(V_f, n_e) = -i \sum_{f\in\text{internal}} j_f \Phi_f - i \sum_{f\in\text{boundary}} j_f \Phi_f - 2i\pi \sum_{e\in\delta f} n_e \sum_{f\in\delta_e} j_f.$$

(6.30)

Moreover, for each tetrahedron $t$, the sum of face spins $\sum_{f\in\delta t} j_f$ is an integer. Therefore, in general at a type-B degenerate critical configuration $(j_f, g_{ve}, \xi_{ef}, \varsigma_{ef})$ for the Lorentzian amplitude, $e^{iS}$ is a function of the vector geometry $V_f$ only:

$$e^{iS} = \exp \lambda \left[ -i \sum_{f\in\text{internal}} j_f \Phi_f - i \sum_{f\in\text{boundary}} j_f \Phi_f \right].$$

(6.31)

where the area $\gamma j_f = \frac{1}{2} |V_f|$ and the angle $\Phi_f$ is uniquely determined by the vector geometry $V_f$.

As a result, given a spin configuration $j_f$ and a boundary data that admit a vector geometry on the triangulation, the degenerate critical configurations of type-B give the following asymptotics:

$$A_{j_f}(\mathcal{K})|_{\text{Deg-B}} \sim \sum_{x_e} a(x_e) \left( \frac{2\pi}{\lambda} \right)^{\frac{n_{\text{int}} - N(V_f)}{2}} \frac{e^{i\text{det}\mathcal{H}(x_e)}}{\sqrt{\text{det}\mathcal{H}(x_e)}} \left[ 1 + o \left( \frac{1}{\lambda} \right) \right]$$

$$\times \exp \lambda \left[ -i \sum_{f\in\text{internal}} j_f \Phi_f - i \sum_{f\in\text{boundary}} j_f \Phi_f \right].$$

(6.32)

where $x_e = (j_f, g_{ve}, \xi_{ef}, \varsigma_{ef})$ labels the degenerate critical configurations of type-B. Note that if we make a suitable gauge fixing for the boundary data, we can always set $\Phi^\nu_f = 0$ [13].

7. Transition between Lorentzian, Euclidean and vector geometry

All the previous analyses assume that on the entire triangulation, the critical configuration $(j_f, g_{ve}, \xi_{ef}, \varsigma_{ef})$ is one of the three types: nondegenerate, degenerate of type-A and degenerate of type-B. However, they are not the most general case. In principle, one should admit the critical configuration that mixes the three types on the triangulation. Given a most general critical configuration $(j_f, g_{ve}, \xi_{ef}, \varsigma_{ef})$ that mixes the three types, one can always make a partition of the triangulation into three regions (maybe disconnected regions) $\mathcal{R}_{\text{Nondeg}}, \mathcal{R}_{\text{Deg-A}}, \mathcal{R}_{\text{Deg-B}}$. Each of the three regions $\mathcal{R}_*$, $* = \text{Nondeg}, \text{Deg-A}, \text{Deg-B}$ is a triangulation with a boundary, on which the critical configuration $(j_f, g_{ve}, \xi_{ef}, \varsigma_{ef})|_{\mathcal{R}_*}$ is of single type $* = \text{Nondeg}, \text{Deg-A}, \text{Deg-B}$. See figure 1 for an illustration.
Therefore, for a generic spin configuration \( j_f \), the asymptotics of the partial amplitude \( A_{j_f}(\mathcal{K}) \) is given by

\[
A_{j_f}(\mathcal{K}) \sim \sum_{\kappa} a(x_c) \left( \frac{2\pi}{\lambda} \right)^{\frac{\mu-1}{2}} e^{i\pi H'(x_c)} \frac{1}{\sqrt{|\det_H'(x_c)|}} \\
\times \left[ 1 + o \left( \frac{1}{\lambda} \right) \right] A_{j_f}(\mathcal{R}_{\text{Nondeg}}) A_{j_f}(\mathcal{R}_{\text{Deg-A}}) A_{j_f}(\mathcal{R}_{\text{Deg-B}}),
\]

where \( x_c \) labels the general critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{vf}) \) admitted by the spin configuration \( j_f \) and boundary data, and \( (j_f, g_{ve}, \xi_{ef}, z_{vf}) \) determines the regions \( \mathcal{R}_* \), \( * = \text{Nondeg}, \text{Deg-A}, \text{Deg-B} \), such that \( (j_f, g_{ve}, \xi_{ef}, z_{vf}) \mathcal{R}_* \) is of single type. The amplitudes \( A_{j_f}(\mathcal{R}_{\text{Nondeg}}) \), \( A_{j_f}(\mathcal{R}_{\text{Deg-A}}) \) and \( A_{j_f}(\mathcal{R}_{\text{Deg-B}}) \) are given respectively by

\[
A_{j_f}(\mathcal{R}_{\text{Nondeg}}) = \prod_{j=1}^{n(\kappa)} \exp -i\lambda \left[ \varepsilon \text{ sgn}(V_4) \sum_{\text{internal } f} \gamma_j \Theta_j + \varepsilon \sum_{\text{boundary } f} j_j \frac{E_j}{\Phi_j} \right] \]

and

\[
A_{j_f}(\mathcal{R}_{\text{Deg-A}}) = \prod_{j=1}^{n'(\kappa)} \exp -i\lambda \left[ \varepsilon \text{ sgn}(V_4) \sum_{\text{internal } f} j_j \frac{E_j}{\Phi_j} + \varepsilon \sum_{\text{boundary } f} j_j \frac{E_j}{\Phi_j} \right] \]

and

\[
A_{j_f}(\mathcal{R}_{\text{Deg-B}}) = \exp -i\lambda \left[ \sum_{\text{internal } f} j_j \Phi_j + \sum_{\text{boundary } f} j_j \Phi_j \right] .
\]

As we discussed previously, given a general critical configuration \( (j_f, g_{ve}, \xi_{ef}, z_{vf}) \), the regions \( \mathcal{R}_{\text{Nondeg}} \) and \( \mathcal{R}_{\text{Deg-A}} \) should be respectively divided into sub-triangulations \( \mathcal{K}_1, \ldots, \mathcal{K}_{n(\kappa)} \) and \( \mathcal{K}_{1}', \ldots, \mathcal{K}_{n'(\kappa)}' \), such that in each \( \mathcal{K}_i \) or \( \mathcal{K}_i' \), \( \text{sgn}(V_4) \) is a constant.
Interestingly, from equation (7.1) we find a transition between a nondegenerate Lorentzian geometry and a nondegenerate Euclidean geometry through the boundary shared by $R_{\text{Nondeg}}$ and $R_{\text{Deg-A}}$. In $R_{\text{Nondeg}}$, the asymptotics gives a Regge action in the Lorentzian signature (plus an additional term):

$$S_{\text{Nondeg}} = -i \varepsilon \text{sgn}(V_4) \sum_{\text{internal } f} A_f \Theta_f - i \varepsilon \text{sgn}(V_4) \sum_{\text{boundary } f} A_f \Theta_f^\beta - \frac{i\pi}{\gamma} \sum_{\text{internal } f} n_f \sum_{f \in C} A_f,$$

(7.3)

where we set the physical area $A_f = \gamma j_f$ (in Planck unit). In $R_{\text{Deg-A}}$, the asymptotics gives a Euclidean Regge action divided by the Barbero–Immirzi parameter (plus an additional term)

$$S_{\text{Deg-A}} = -i \varepsilon \text{sgn}(V_4) \sum_{\text{internal } f} A_f^E \Theta_f - i \varepsilon \text{sgn}(V_4) \sum_{\text{boundary } f} A_f^E \Theta_f^\beta - \frac{i\pi}{\gamma} \sum_{\text{internal } f} n_f \sum_{f \in C} A_f.$$

(7.4)

In the case of a single simplex, this asymptotics has been presented in [13]. One might expect the transition between the Lorentzian and Euclidean geometries is a quantum tunneling effect. But surprisingly in the large-$j$ regime, $e^{S_{\text{Deg-A}}}$ does not damp exponentially but oscillatory. Similarly, there is also a transition between a nondegenerate Lorentzian/Euclidean geometry and a vector geometry through the boundary of $R_{\text{Deg-B}}$, and in the region $R_{\text{Deg-B}}$, the asymptotics gives

$$S_{\text{Deg-B}} = -i \varepsilon \text{sgn}(V_4) \sum_{\text{internal } f} A_f \Phi_f - i \varepsilon \text{sgn}(V_4) \sum_{\text{boundary } f} A_f \Phi_f^\beta.$$

(7.5)

Thus, $e^{S_{\text{Deg-A}}}$ is also oscillatory and gives nontrivial transition in the large-$j$ regime. The behaviors mean that the transitions are not the quantum tunneling effect. It means that there are semiclassical configurations from the spinfoam model, which do not correspond to classical (discrete) GR. It presents a mixture of the gravity-like configurations and SU(2) BF-like configurations.

However, there are some specialties for the phases $e^{S_{\text{Deg-A}}}$ and $e^{S_{\text{Deg-B}}}$. These phases oscillate much more violently than the Regge action part in $e^{S_{\text{Nondeg}}}$ when the Barbero–Immirzi parameter $\gamma$ is small, unless $\xi \Theta_f, \xi \Theta_f^\beta, \xi \Phi_f, \xi \Phi_f^\beta$ are all vanishing. We expect that when we take into account the sum over spins $j_f$, the violently oscillating phases $e^{S_{\text{Deg-A}}}$ and $e^{S_{\text{Deg-B}}}$ may only have relatively small contribution to the total amplitude $A(K) = \sum_j A_j(K)$, as is suggested by the Riemann–Lebesgue lemma. But surely the nontrivial transition between different types of geometries is an interesting phenomenon exhibiting the semiclassical analysis of the Lorentzian spinfoam amplitude; thus it requires further investigation and clarification.

8. Conclusion and discussion

This work studies the large-$j$ asymptotics of the Lorentzian EPRL spinfoam amplitude on a 4D simplicial complex with an arbitrary number of simplices. The asymptotics of the spinfoam amplitude is determined by the critical configurations of the spinfoam action. Here we show that, given a critical configuration $(j_f, g_{ij}, \xi_{ij}, z_{ij})$ in general, there exists a partition of the simplicial complex $K$ into three types of regions $R_{\text{Nondeg}}, R_{\text{Deg-A}}$ and $R_{\text{Deg-B}}$, where the three regions are simplicial sub-complexes with boundaries. The critical configuration implies

20 The term $\sum n_f \sum_{f \in C} A_f$ in both $S_{\text{Nondeg}}$ and $S_{\text{Deg-A}}$ may need special treatment by imposing the boundary semiclassical state carefully.

21 The Riemann–Lebesgue lemma states that for all complex $L^1$-function $f(x)$ on $\mathbb{R}$,

$$\int_{-\infty}^{\infty} f(x) e^{x\alpha} dx = 0 \quad \text{as} \quad \alpha \to \pm \infty.$$  

(7.6)
different types of geometries in different types of regions, i.e. (1) the critical configuration restricted to $R_{\text{Nondeg}}$ implies a nondegenerate discrete Lorentzian geometry in $R_{\text{Nondeg}}$. (2) the critical configuration restricted to $R_{\text{Deg-A}}$ is degenerate of type-A in our definition of degeneracy, but implies a nondegenerate discrete Euclidean geometry in $R_{\text{Deg-A}}$. (3) the critical configuration restricted into $R_{\text{Deg-B}}$ is degenerate of type-B and implies a vector geometry in $R_{\text{Deg-B}}$.

With the critical configuration $(f_j, g_{ve}, \xi_{ef}, z_{vf})$, we further make a subdivision of the regions $R_{\text{Nondeg}}$ and $R_{\text{Deg-A}}$ into sub-complexes (with boundaries) $K_1(R_\ast), \ldots, K_n(R_\ast)$ ($\ast = \text{Nondeg, Deg-A}$) according to their Lorentzian/Euclidean-oriented 4-volume $V_4(v)$ of the 4-simplices, such that $\text{sgn}(V_4(v))$ is a constant sign on each $K_\ast(R_\ast)$. Then in each sub-complex $K_\ast(R_{\text{Nondeg}})$ or $K_\ast(R_{\text{Deg-A}})$, the spinfoam amplitude at the critical configuration gives an exponential of the Regge action in the Lorentzian or Euclidean signature respectively. However, we should note that the Regge action reproduced here contains a sign prefactor $\text{sgn}(V_4(v))$ related to the oriented 4-volume of the 4-simplices. Therefore, the Regge action reproduced here is actually a discretized Palatini action with the on-shell connection.

In principle, for a generic critical configuration, the sub-complex may consist of only a single 4-simplex. In some sense, the gluing of the simplices is not perfect as far as the orientation and degeneracy/nondegeneracy are concerned. The resulting asymptotics from this paper is not completely consistent with discrete GR, but contains additional contributions from non-GR-like configurations. All the classical discrete geometry can be reproduced by the spinfoam critical data, but there are also many other spinfoam critical data which are not GR-like (not the Regge geometry). As it is pointed out in the last section, there are many critical data mixing Lorentzian geometries with different orientations and Euclidean/vector geometries.

Finally, the asymptotic formula of the spinfoam amplitude is given by a sum of the amplitudes evaluated at all possible critical configurations, which are the products of the amplitudes associated with different types of geometries.

This work gives explicitly the critical configurations of the spinfoam amplitude and their geometrical interpretations. However we did not answer the question such as whether or not the nondegenerate critical configurations are dominating the large-$j$ asymptotic behavior, although we expect that the Lorentzian nondegenerate configurations are dominating when the Barbero–Immirzi parameter $\gamma$ is small. Answer to this question in general requires a detailed investigation about the rank of the Hessian matrix in general circumstances. In appendix F, we compute the Hessian matrix of the spinfoam action. However, we leave the detailed study about its rank to the future research.

In this work, we show that given a Regge-like spin configuration $j_f$ on the simplicial complex, the critical configurations $(f_j, g_{ve}, \xi_{ef}, z_{vf})$ with the Regge-like $j_f$ are nondegenerate, and there is a unique critical configuration $(f_j, g'_{ve}, \xi'_{ef}, z'_{vf})$ with the oriented 4-volume $V_4(v) > 0$ (or $V_4(v) < 0$) everywhere. We can regard the critical configuration $(f_j, g'_{ve}, \xi'_{ef}, z'_{vf})$ with $V_4(v) > 0$ as a classical background geometry, and define the perturbation theory with the background field method. Thus, with the background field method, the $n$-point functions in spinfoam formulation should be investigated as a generalization of [28] to the context with the arbitrary simplicial complex, which is a research undergoing.

Acknowledgments

The authors would like to thank E Bianchi, L Freidel, T Krajewski, S Speziale and C Rovelli for discussions and communications. MZ is supported by the CSC scholarship no. 2010601003.
Appendix A. Nondegenerate geometry on a simplicial complex

A.1. Discrete bulk geometry

In order to relate the spinfoam configurations solving the critical point equations to the discrete Regge geometry, here we introduce the classical geometric variables for the discrete Lorentzian geometry on a 4-manifold. In the presentation of this appendix, we closely follow the presentation of that in [19, 25], but make the issue of various orientations more explicit.

Given a simplicial complex \( K \) triangulating, the 4-manifold \( M \) with the Lorentzian metric \( g_{\mu\nu} \), we associate each 4-simplex \( \sigma_v \) (dual to the vertex \( v \)) a reference frame. In this reference frame, the vertices \( \{p_1(v), \ldots, p_5(v)\} \) of the 4-simplex \( \sigma_v \) have the coordinates

\[
p_i(v) = \{x_i^e(v)\}_{i=1,\ldots,5}.
\]

Consider another 4-simplex \( \sigma_v' \) neighboring \( \sigma_v \), an edge \( e \) connecting \( v \) and \( v' \), and a tetrahedron \( t_e \) shared by \( \sigma_v, \sigma_v' \) with vertices \( \{p_2(v), \ldots, p_5(v)\} = \{p_2(v'), \ldots, p_5(v')\} \). Then it is possible to associate the edge \( e = (v, v') \) uniquely with an element of the Poincaré group \( \{(\Omega_e)_1^I, (\Omega_e)_2^I\} \), such that for the vertices \( p_2, \ldots, p_5 \) of \( t_e \),

\[
(\Omega_e)_1^I x_I^{e(v')} + (\Omega_e)_2^I x_I^{e(v)} = x_I^e(v), \quad i = 2, \ldots, 5.
\]

Here the matrix \( (\Omega_e)_1^I \) describes the change of the reference frames in \( \sigma_v \) and \( \sigma_v' \), while \( (\Omega_e)_2^I \) describes the transportation of the frame origins from \( \sigma_v \) to \( \sigma_v' \). We assume that the triangulation is orientable, and we choose the reference frames in \( \sigma_v, \sigma_v' \) in such a way that \( \Omega_e \in \text{SO}(1, 3) \).

We focus on a 4-simplex \( \sigma_v \) whose center is the vertex \( v \). For each oriented edge \( \ell = \{p_0(v), p_1(v)\} \) in the 4-simplex, we associate an edge vector \( E^I_\ell(v) = x^I_1(v) - x^I_0(v) \).

Thus, under the change of the reference frame from \( \sigma_v \) to \( \sigma_v' \),

\[
(\Omega_v)_1^I E^I_\ell(v') = E^I_\ell(v) \quad \forall \ell \subset t_e.
\]

In this paper, we assume that all the edge vectors \( E^I_\ell(v) \) are spatial in the sense of the flat metric \( \eta_{IJ} = \text{diag}(-1, 1, 1, 1) \). It is straightforward to check from the definition that the edge vectors \( E^I_\ell(v) \) satisfy

- if we reverse the orientation of \( \ell \), then \( E^I_{\ell'}(v) = -E^I_\ell(v) \), (A.4)
- for all triangle \( f \) in the simplex \( \sigma_v \) with edges \( \ell_1, \ell_2, \ell_3 \), the vectors \( E^I_\ell(v) \) close, i.e.

\[
E^I_{\ell_1}(v) + E^I_{\ell_2}(v) + E^I_{\ell_3}(v) = 0.
\]

The set of \( E^I_\ell(v) \) at \( v \) satisfying equations (A.4) and (A.5) is called a co-frame at the vertex \( v \).

- Moreover, given a tetrahedron \( t \) shared by two 4-simplices \( \sigma_v, \sigma_v' \), for all pair of edges \( \ell_1, \ell_2 \) of the tetrahedron, we further require that

\[
\eta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v) = \eta_{IJ} E^I_{\ell_1}(v') E^J_{\ell_2}(v').
\]

**Definition A.1.** The collection of the vectors \( E^I_\ell(v) \) satisfying equations (A.4)–(A.6) at all the vertices is called a co-frame on the simplicial complex \( K \). The discrete (spatial) metric on the each tetrahedron \( t \) induced from \( g_{\mu\nu} \) is given by

\[
g_{\ell_1\ell_2}(v) = \eta_{IJ} E^I_{\ell_1}(v) E^J_{\ell_2}(v),
\]

which is actually independent of \( v \) because of equation (A.6).
We assume that the co-frame $E^I_i(v)$ is nondegenerate, i.e. for each 4-simplex $\sigma_v$, the set of $E^I_i(v)$ with $I \in \partial \sigma_v$ spans a four-dimensional vector space.

An edge $\ell$ can be denoted by its end points, say $p_1, p_2$, i.e. $\ell = [p_1, p_2]$. There are five vertices $p_i, i = 1, \ldots, 5$ for a 4-simplex $\sigma_v$. Then each $p_i$ corresponds one-to-one to a tetrahedron $t_{\ell}$ of the 4-simplex $\sigma_v$. Therefore, we can denote the edge $\ell = [p_1, p_2]$ also by $\ell = (e_1, e_2)$, once a 4-simplex $\sigma_v$ is specified. Thus, we can also write the co-frame $E^I_i(v)$ at the vertex $v$ by $E^I_{e_\ell}(v)$. In this notation, for example, equations (A.4) and (A.5) become

$$E^I_{e_\ell}(v) = -E^I_{e_\ell'}(v), \quad E^I_{e_{\ell'\ell}}(v) + E^I_{e_{\ell\ell'}}(v) + E^I_{e_{e\ell}}(v) = 0. \quad (A.8)$$

In the following, we use both of the notations, according to the convenience by the context.

**Lemma A.1.** Given a co-frame $E^I_i(v)$ on the triangulation, an $SO(1,3)$ matrix $(\Omega_x)_{ij}$ associated with each edge $e = (v, v')$ is uniquely determined, such that for all the edge of the tetrahedron $t_e$ shared by $\sigma_v$ and $\sigma_{v'}$:

$$(\Omega_x)^I_{J'} E^I_i(v') = E^I_i(v) \quad \forall I \subset t_e. \quad (A.9)$$

We can associate a reference frame in each 4-simplex, such that the $SO(1,3)$ matrix $(\Omega_x)_{ij}$ changes the frame from $\sigma_v$ to $\sigma_{v'}$.

**Proof.** Given a tetrahedron $t_e$ shared by two 4-simplices $\sigma_v, \sigma_{v'}$, we consider the relation between the co-frame vectors $E^I_i(v)$ at the vertex $v$ and $E^I_i(v')$ at $v'$, for all six edges $\ell$ of the tetrahedron $t_e$. The spatial vectors $E^I_i(v)$ $\ell \subset t_e$ spans a three-dimensional subspace, and the same holds for $E^I_i(v')$. We choose the time-like unit normal vectors $\hat{U}(v)$ and $\hat{U}(v')$ orthogonal to $E^I_i(v)$ and $E^I_i(v')$ respectively, and require that

$$\text{sgn det}(E^I_i(v), E^I_i(v), E^I_i(v), \hat{U}(v)) = \text{sgn det}(E^I_i(v'), E^I_i(v'), E^I_i(v'), \hat{U}(v')), \quad (A.10)$$

where $E^I_i(v), E^I_i(v)$ and $E^I_i(v)$ form a basis in the three-dimensional subspace spanned by $E^I_i(v) \ell \subset t_e$. From equations (A.10), (A.6) and $E^I_i(v) \cdot \hat{U}(v) = E^I_i(v') \cdot \hat{U}(v') = 0$, $i = 1, 2, 3$, an $SO(1,3)$ matrix $\Omega_x$ is determined by

$$(\Omega_x)^I_{J'} E^I_i(v') = E^I_i(v) \quad (\Omega_x)^I_{J'} \hat{U}^I(v') = \hat{U}^I(v). \quad (A.11)$$

Suppose that there are two $SO(1,3)$ matrices $\Omega_e, \Omega_e'$ satisfying

$$(\Omega_e)^I_{J'} E^I_i(v') = E^I_i(v) \quad (\Omega_e')^I_{J'} E^I_i(v') = E^I_i(v); \quad (A.12)$$

we then have $\Omega_e = \Omega_e'$.

We choose a numbering $[p_1, \ldots, p_5]$ of the vertices of $\sigma_v, \sigma_{v'}$, such that $[p_2(v), \ldots, p_5(v)] = [p_2(v'), \ldots, p_5(v')]$ are the vertices of the tetrahedron $t_e$. Two reference frames in the 4-simplices $\sigma_v, \sigma_{v'}$ are specified by the coordinates $\{E^I_{e_\ell}(v), E^I_{e_{\ell'\ell}}(v), E^I_{e_\ell}(v), E^I_{e_\ell'}(v)\}$ and $\{E^I_{e_{e\ell}}(v'), E^I_{e_{e\ell'}}(v'), E^I_{e_{e\ell}}(v'), E^I_{e_{e\ell'}}(v')\}$ by defining $x^I_j(v) := E^I_{e_\ell}(v)$ and similarly for $x^I_j(v')$. Since

$$E^I_{e_{e\ell}} = E^I_{e_e} - E^I_{e_{\ell'}}, \quad E^I_{e_{e\ell'}} = E^I_{e_e} - E^I_{e_{\ell}}, \quad E^I_{e_{e\ell}} = E^I_{e_e} - E^I_{e_{\ell'}}, \quad E^I_{e_{e\ell'}} = E^I_{e_e} - E^I_{e_{\ell}} \quad (A.13)$$

and there exists a unique $(\Omega_x)^I_{J'} \in SO(1,3)$ such that $E^I_{e_{e\ell}}(v) = (\Omega_x)^I_{J'} E^I_{e_{e\ell'}}(v')$, $i, j = 2, \ldots, 5$, we can relate the coordinates $\{E^I_{e_{e\ell}}(v), E^I_{e_{e\ell'}}(v), E^I_{e_{e\ell}}(v), E^I_{e_{e\ell'}}(v)\}$ and $\{E^I_{e_{e\ell}}(v'), E^I_{e_{e\ell'}}(v'), E^I_{e_{e\ell}}(v'), E^I_{e_{e\ell'}}(v')\}$ in two different 4-simplices by

$$E^I_{e_{e\ell}}(v) = (\Omega_x)^I_{J'} E^I_{e_{e\ell'}}(v') + (\Omega_x)^I_{J'} E^I_{e_{e\ell}}(v') - E^I_{e_{e\ell}}(v)$$

$$E^I_{e_{e\ell'}}(v) = (\Omega_x)^I_{J'} E^I_{e_{e\ell'}}(v') + (\Omega_x)^I_{J'} E^I_{e_{e\ell}}(v') - E^I_{e_{e\ell'}}(v) \quad (A.14)$$

$$E^I_{e_{e\ell}}(v) = (\Omega_x)^I_{J'} E^I_{e_{e\ell'}}(v') + (\Omega_x)^I_{J'} E^I_{e_{e\ell}}(v') - E^I_{e_{e\ell}}(v)$$

$$E^I_{e_{e\ell'}}(v) = (\Omega_x)^I_{J'} E^I_{e_{e\ell'}}(v') + (\Omega_x)^I_{J'} E^I_{e_{e\ell}}(v') - E^I_{e_{e\ell'}}(v).$$
The coordinates of \( p_1, \ldots, p_5 \) are given by \( x_j(v) := E_{j\nu}(v) \) with respect to the reference frame in \( \sigma \); thus, the Poincaré transformation relating two reference frames is given by an SO(1,3) matrix and a translation \( \{(\Omega_\nu)_j^I, (\Omega_\nu)_l^I\} \), where the translation vector \((\Omega_\nu)_j^I\) is given by
\[
(\Omega_\nu)_j^I := (\Omega_\nu)_j^I E_{\nu r s}(v') - E_{\nu r s}(v). \tag{A.15}
\]

The orientation of a 4-simplex \( \sigma \) is represented by an ordering of its five vertices, i.e. a tuple \([p_1, \ldots, p_5]\). Two orientations are opposite to each other if the two orderings are related by an odd permutation, e.g. \([p_1, p_2, \ldots, p_5] = -[p_2, p_1, \ldots, p_5]\). We say that two neighboring 4-simplices \( \sigma, \sigma' \) are consistently oriented, if the orientation of their shared tetrahedron \( t \) induced from \( \sigma \) is the opposite orientation induced from \( \sigma' \). For example, \( \sigma = [p_1, p_2, \ldots, p_5] \) and \( \sigma' = -[p_1', p_2', \ldots, p_5'] \) are consistently oriented since the opposite orientations \( t = \pm[p_2, \ldots, p_5] \) are induced respectively from \( \sigma \) and \( \sigma' \). The simplicial complex \( K \) is said to be orientable if it is possible to orient consistently all pair of neighboring 4-simplices. Such a choice of consistent 4-simplex orientations is called a global orientation.

We assume that we define a global orientation of the triangulation \( K \). Then for each 4-simplex \( \sigma = [p_1, p_2, \ldots, p_5] \), we define an oriented volume (assumed to be nonvanishing as the nondegeneracy)
\[
V_4(v) := \det(E_{\nu r s}(v), E_{\nu r e}(v), E_{\nu e r}(v), E_{\nu e s}(v)). \tag{A.16}
\]
In general, the oriented 4-volume \( V_4(v) \) can be positive or negative for different 4-simplices.

**Definition A.2.** Given two neighboring 4-simplices \( \sigma \) and \( \sigma' \), if their oriented volumes are both positive and negative, i.e. \( \text{sgn}(V_4(v)) = \text{sgn}(V_4(v')) \), then the SO(1,3) matrix \((\Omega_\nu)_j^I\) for \( \sigma \) and \( \sigma' \) is the discrete spin connection compatible with \( E_{\nu}(v)^I \).

For each vertex \( v \) and a dual edge \( e \) connecting \( v \), we define a time-like vector \( U_e(v) \) at the vertex \( v \) by (no summing over \( j \) and choosing any \( j \neq k \), the definition is independent of the choice of \( j \) by equations (A.4) and (A.5). Einstein summation of \( I, J, K, L \)
\[
U^\nu_{j e}(v) := \frac{1}{3V_4(v)} \sum_{l,m,n} e^{jlnm} e_{IJKL} E^I_{\nu r s}(v) e^K_{\nu r e}(v) e^L_{\nu e s}(v). \tag{A.17}
\]
In total there are five vectors \( U_e(v) \) at each vertex \( v \). Using equations (A.4) and (A.5), one can show that
\[
U^\nu_{j e}(v) E^I_{\nu r s}(v) = \delta_{jk} - \delta_{jl}, \tag{A.18}
\]
which shows that \( U^\nu_{j e}(v) \) is orthogonal to \( E^I_{\nu r s}(v) \) as \( j \neq k, l \). Thus, we call the collection of \( U_e(v) \) a discrete frame, since \( E_{\nu r s}(v) \) is called a discrete co-frame. Moreover, from this equation we see that \( U^\nu_{j e}(v) \) is a vector at \( v \) normal to the tetrahedron \( t_e \). If we sum over all five frame vectors \( U_e(v) \) at \( v \) in equation (A.18),
\[
\sum_{j=1}^{5} U^\nu_{j e}(v) E^I_{\nu r s}(v) = \sum_{j=1}^{5} \delta_{jk} - \sum_{j=1}^{5} \delta_{jl} = 0 \quad \forall e_k, e_l, \tag{A.19}
\]
which shows the closure of \( U_e(v) \) at each vertex \( v \), i.e.
\[
\sum_{e=1}^{5} U_e(v) = 0 \tag{A.20}
\]
by the nondegeneracy of \( E_{\nu r s}(v) \). Equation (A.20) shows that the 5-vectors \( U_e(v) \) are all out-pointing or all in-pointing normal vectors to the tetrahedra. Also following
from equation (A.18) (fix \( l = 1 \) and let \( j = 2, 3, 4, 5 \)), we have that the \( 4 \times 4 \) matrix \((U^{\ell}(v), U^o(v), U^s(v), U^{fs}(v))^T\) is the inverse of the matrix \((E_{\ell\ell}(v), E_{o\ell}(v), E_{s\ell}(v), E_{fs\ell}(v)).\) Therefore,
\[
\frac{1}{V_4(v)} = \det(U^{\ell}(v), U^o(v), U^s(v), U^{fs}(v)). \tag{A.21}
\]
It implies \((i, j, k, l = 2, 3, 4, 5)\)
\[
V_4(v)\varepsilon^{\ell j k l}U^\ell_j(v)U^o_k(v)U^s_l(v) = \varepsilon^{ijkl}
\]
\[
V_4(v)\varepsilon_{ijkl}U^\ell_j(v)U^o_k(v)U^s_l(v) = \varepsilon_{ijkl}, \tag{A.22}
\]
where the above \(\varepsilon^{ijkl} = \varepsilon_{ijkl} = \varepsilon^{ijkl}\) are all Levi-Civita symbols. Then using the fact that the matrix \(U^\ell_j(v)\) is the inverse of \(E^{\ell\ell}_j(v),\) we can verify that
\[
E^{\ell\ell}_j(v) = \frac{V_4(v)}{3!} \sum_{l,m,n} \varepsilon_{ijklm} \varepsilon^{ijkl}U^\ell_j(v)U^o_k(v)U^s_l(v)
\]
\[
V_4(v)U^o_j(v)U^{fs}_l(v) = \frac{1}{2} \sum_{m,n} \varepsilon_{ijklm} \varepsilon^{ijkl} E^K_{m\ell}(v)E^L_{n\ell}(v), \tag{A.23}
\]
where the last equation is a relation for the area bivector \(E_\ell(v) \wedge E^\ell(v)\) of each triangle \(f.\) For example, given a triangle \(f\) shared by \(t_{e_1}\) and \(t_{e_2}\) in a 4-simplex \(\sigma_v,\) one has
\[
* [E_{\ell\ell}(v) \wedge E_{\ell\ell}(v)] = V_4(v) [U^o(v) \wedge U^{fs}(v)], \tag{A.24}
\]
where \(*[E_1 \wedge E_2] = \varepsilon_{ijkl} E^K_{i\ell} E^L_{j\ell}.\)

### A.2. Discrete boundary geometry

All the above discussions consider the discrete geometry in the bulk of the triangulation, where all the co-frame vectors \(E_\ell(v)\) and frame vectors \(U_o(v)\) are located at internal vertices \(v.\) Now we consider a triangulation with a boundary, where the boundary is a simplicial complex \(\partial K\) built by tetrahedra triangulating a boundary 3-manifold. On the boundary \(\partial K,\) each triangle is shared by precisely two boundary tetrahedra. This triangle is dual to a unique boundary link \(l,\) connecting the centers of the two boundary tetrahedra sharing the triangle. We denote this triangle \(f_l.\) On the other hand, from the viewpoint of the whole triangulation \(K,\) there is a unique face dual to the triangle \(f_l,\) where two edges \(e_0, e_1\) of this dual face are dual to the two boundary tetrahedra \(t_{e_0}\) and \(t_{e_1}\) sharing \(f_l.\) This dual face intersects the boundary uniquely by the link \(^{22}\) Thus, we denote this dual face also by \(f_l\) because of the one-to-one correspondence of the duality for \(K.\) See figure A1 for an example of a face dual to a boundary triangle.

The end-points \(s(l), t(l)\) of the boundary link \(l\) are centers of the tetrahedra \(t_{e_0}\) and \(t_{e_1}\) respectively. For each edge \(\ell\) of the tetrahedron \(t_e, (i = 0, 1,\) we associate a spatial vector \(E_\ell(e_i)\) at the center of \(t_e,\) satisfying the following requirement.

- Given the time-like unit vector \(u^\ell = (1, 0, 0, 0),\) all the vectors \(E_\ell(e_i) (i = 0, 1)\) are orthogonal to \(u^\ell,\) i.e.,
\[
u^\ell E_\ell(e_i) = 0 \quad \forall \ell \in t_{e_i}. \tag{A.25}
\]

\(^{22}\) If the dual face intersects the boundary by more than one link, then it means that the triangle \(f_l\) is shared by more than two tetrahedra, which is impossible for a three-dimensional triangulation.

\(^{23}\) It is compatible the time-gauge condition in the construction of the spinfoam model. The time-gauge makes the spinfoam amplitude not manifestly Lorentzian invariant. However, it is shown in [30] that the amplitude is Lorentzian covariant after the time-gauge fixing.
• If we reverse the orientation of $\ell$, then

$$E_{-\ell}(e_i) = -E_{\ell}(e_i) \quad \forall \, \ell \in t_e.$$  \hfill (A.26)

• For all triangle $f$ of the boundary tetrahedron $t_e$ with edges $\ell_1, \ell_2, \ell_3$, the vectors $E_{\ell_i}(e_i)$ close, i.e.

$$E_{\ell_1}(e_i) + E_{\ell_2}(e_i) + E_{\ell_3}(e_i) = 0.$$  \hfill (A.27)

• There is an internal vertex $v_i$ as one of the end-points of the dual edge $e_i$ ($i = 0, 1$), i.e. the boundary tetrahedron $t_e$ belongs to the boundary of the 4-simplex $\sigma_v$. Then we require that

$$\eta_{IJ}E_{\ell_1}^I(e_i)E_{\ell_2}^J(e_i) = \eta_{IJ}E_{\ell_1}^I(v_i)E_{\ell_2}^J(v_i) \quad \forall \, \ell_1, \ell_2 \in t_e.$$  \hfill (A.28)

The set of $E_{\ell_i}^I(e_i) (i = 0, 1)$ at the center of $t_e$ satisfying the above requirements is called a boundary (three-dimensional) co-frame at the center of $t_e$ (at the node $s(l)$). The discrete metric

$$g_{l_1l_2}(e_i) := \eta_{IJ}E_{l_1}^I(e_i)E_{l_2}^J(e_i)$$  \hfill (A.29)

is the induced metric on the boundary $\partial\mathcal{K}$.

Consider a boundary tetrahedron $t_e$ belonging to a 4-simplex $\sigma_v$; then the edge $e_i$ dual to $t_e$ connects to a boundary node (the center of $t_e$). We choose three linearly independent co-frame vectors $E_{l_1}(e_i), E_{l_2}(e_i), E_{l_3}(e_i)$ at the center of $t_e$, associated with three edges $\ell_1, \ell_2, \ell_3$, and also choose three linearly independent co-frame vectors $E_{l_1}(v_i), E_{l_2}(v_i), E_{l_3}(v_i)$ at the vertex $v_i$ associated with the same set of edges. Given a unit vector $\hat{U}(v_i)$ orthogonal to $E_{l_1}(v_i), E_{l_2}(v_i), E_{l_3}(v_i)$, such that

$$\text{sgn det}(E_{l_1}(v_i), E_{l_2}(v_i), E_{l_3}(v_i), \hat{U}(v_i)) = \text{sgn det}(E_{l_1}(e_i), E_{l_2}(e_i), E_{l_3}(e_i), u)$$  \hfill (A.30)

by the requirement (A.28), there exist a unique SO(1,3) matrix $\Omega_{e_i}$, such that

$$(\Omega_{e_i})^J_{\ell_i} E_{\ell_i}^I(e_i) = E_{\ell_i}^I(v_i) \quad (\Omega_{e_i})^J_{\ell_i} u^I = \hat{U}^I(v_i).$$  \hfill (A.31)

Thus, $\Omega_{e_i}$ is identify as the spin connection compatible with $E_{l_i}(v_i), E_{l_i}(e_i)$.

Consider a dual face bounded by a boundary link $l$ (see, e.g., figure A1), by using the defining requirement of the co-frames in the bulk and on the boundary, i.e. equations (A.6) and (A.28), we have

$$\eta_{IJ}E_{l_j}^I(e_0)E_{l_k}^J(e_0) = \eta_{IJ}E_{l_j}^I(e_1)E_{l_k}^J(e_1).$$  \hfill (A.32)
where $\ell_j$, $\ell_l$ are two of the three edges of the triangle $f_l$ dual to the face. Therefore, we obtain the shape-matching condition between the triangle geometries of $f_l$ viewed in the frame of $t_{e_0}$ and $t_{e_i}$. More precisely, there exists an SO(3) matrix $\tilde{g}_l$, such that for all the three $\ell$s forming the boundary of the triangle $f_l$:

$$
(\tilde{g}_l)^T L^T(\ell_0) = L^T(\ell_1)
$$

by the fact that both $L(\ell_0)$ and $L(\ell_1)$ are orthogonal to $u' = (1, 0, 0, 0)$.

Now we consider a single boundary tetrahedron $t_e$ dual to an edge $e$ connecting to the boundary. Since all the boundary co-frame vectors $E_i(e)$ at the center of $t_e$ are orthogonal to the time-like unit vector $u' = (1, 0, 0, 0)$, we now only consider the three-dimensional spatial subspace orthogonal to $u' = (1, 0, 0, 0)$. We further assume that the boundary tetrahedral geometry is nondegenerate, i.e. the (oriented) 3-volume of the tetrahedron

$$
V_3(e) = \det(E_{i_1}(e), E_{i_2}(e), E_{i_3}(e))
$$

is nonvanishing, where $\ell_1, \ell_2, \ell_3$ are the three edges of $t_e$ connecting to a vertex $p$ of $t_e$. Since there are four vertices of $t_e$ and an edge $\ell$ is determined by its end-points $p_1, p_j$, we denote $E_l(e)$ by $E_{p_i p_j}(e)$. Fixing a vertex $p_1$ and constructing the nondegenerate $3 \times 3$ matrix

$$
(E_{p_1 p_2}(e), E_{p_1 p_3}(e), E_{p_1 p_4}(e))
$$

we construct is inverse

$$
(n_{p_2}(e), n_{p_3}(e), n_{p_4}(e))^T
$$

with $n_{p_i}(e) \cdot E_{p_i p_j}(e) = \delta_{ij}$. Repeat the same construction for all the other three vertices $p_2, p_3, p_4$; then we obtain four 3-vector $n_{p_i}(e)$, such that

$$
n_{p_i}(e) \cdot E_{p_j p_k}(e) = \delta_{ij} - \delta_{ik}.
$$

From this relation, one can verify that (i) the 3-vector $n_{p_j}(e)$ is orthogonal to the triangle $(p_j, p_k, p_l)$ spanned by $E_{p_j p_k}(e), E_{p_j p_l}(e), E_{p_k p_l}(e)$ with $i \neq j, k, l$. Therefore, we denote $n_{p_i}(e)$ by $n_{e_f}$, where $f$ is the triangle determined by the three vertices other than $p$; (ii) the four $n_{e_f}$ satisfy the closure condition

$$
\sum_{j=1}^{4} n_{e_f} = 0.
$$

We call the set of $n_{e_f}$ a three-dimensional frame at the center of $t_e$. Explicitly, the vector $n_{e_f}$ is given by

$$
n_{e_f} = V_3(e)^{-1} E_{i_1}(e) \times E_{i_2}(e) \quad \text{or} \quad n_{p_i}(e) = V_3(e)^{-1} E_{p_i p_j}(e) \times E_{p_k p_l}(e)
$$

The norm $|n_{e_f}| = 2A_f / |V_3(e)|$ is proportional to the area of the triangle $A_f = \frac{1}{2} \left| E_{i_1}(e) \times E_{i_2}(e) \right|$.

### Appendix B. Proof of lemma 3.1

We consider

$$
0 = \beta_{mn}(v) \sum_{j=1}^{5} \beta_{jj}(v) N_{e_j}(v) - \beta_{lm}(v) \sum_{j=1}^{5} \beta_{kj}(v) N_{e_j}(v)
$$

$$
= \sum_{j \neq m} [\beta_{mn}(v) \beta_{jj}(v) - \beta_{lm}(v) \beta_{kj}(v)] N_{e_j}(v).
$$

\[\text{(B.1)}\]
Since we assume the nondegeneracy condition (3.1), any four of the five \( N_e(v) \) are linearly independent. Thus,
\[
\beta_{km}(v)\beta_{lj}(v) = \beta_{km}(v)\beta_{lj}(v). 
\]
Let us pick one \( j_0 \) for each 4-simplex, and asking \( l = j = j_0 \) we obtain
\[
\beta_{km}(v) = \frac{\beta_{km}(v)\beta_{lj}(v)}{\beta_{lj}(v)}. 
\]
Therefore, we have the factorization of \( \beta_j(v) \),
\[
\beta_j(v) = \text{sgn}(\beta_{lj}(v))\beta_{lj}(v), 
\]
where \( \beta_j(v) = \beta_{lj}(v)/\sqrt{|\beta_{lj}(v)|} \). We denote \( \text{sgn}(\beta_{lj}(v)) = \tilde{\epsilon}(v) \) which is a constant within a 4-simplex \( \sigma_v \). Thus, we have the following expression of the bivector \( \epsilon_{\epsilon'\epsilon'j}(v)X_{\epsilon'\epsilon'j}(v) \):
\[
\epsilon_{\epsilon'\epsilon'j}(v)X_{\epsilon'\epsilon'j}(v) = \tilde{\epsilon}(v)(\beta_j(v)N_e(v)) \wedge (\beta_j(v)N_e(v)). 
\]
Equation (3.5) takes the form
\[
\sum_{j=1}^{5} \beta_j(v)N_e(v) = 0. 
\]

Appendix C. Proof of lemma 3.2

Since we only consider two simplices, we introduce a short-hand notation:
\[
\begin{align*}
U_0 &:= U_e(v) \\
U_0' &:= g_{v'v}U_e(v') \\
U_i &:= U_e(v) \\
U_i' &:= g_{v'v}U_e(v') \\
E_{ij} &:= E_{\epsilon'\epsilon'j}(v) \\
E_{ij}' &:= g_{v'v}E_{\epsilon'\epsilon'j}(v'), 
\end{align*}
\]
where \( i, j = 1, \ldots, 4 \) labels the edges connecting to \( v \) or \( v' \) other than \( e \), \( E_{ij} \) and \( E_{ij}' \) are orthogonal to \( U_e \) and \( U_0' \) respectively from equation (3.20). Here \( g_{v'v} = g_{e'v}g_{v'v} \) comes from the spinfoam configuration \( (j_f, g_{e'v}, \tilde{\epsilon}_f, \epsilon_{e'f}) \) that solves the critical point equations. From the closure condition of \( U_e(v) \), we have
\[
\begin{align*}
\tilde{U}_0 &= -\sum_{i} U_i \\
\tilde{U}_0' &= -\sum_{i} U_i'. 
\end{align*}
\]
By definition \( N_e(v) = g_{v'v}u \) and \( N_e(v') = g_{v'v}u \), where \( u = (1, 0, 0, 0) \); thus \( N_e(v) = g_{v'v}N_e(v') \) with \( e = (v, v') \). Thus from the definition of \( U_e(v) \) in equation (3.10), we find
\[
\frac{U_0}{|U_0|} = \tilde{\epsilon} \frac{U_0}{|U_0|}. 
\]
where \( \tilde{\epsilon} = \pm \). On the other hand, from the parallel transportation relation \( X_f(v) = g_{v'v}X_f(v')g_{v'v} \) and \( \epsilon_{e'f}(v) = -\epsilon_{e'f}(v') \) for \( e = (v, v') \), we have
\[
\epsilon_0 X_{ij}^0 = \epsilon V(U^0 \wedge U^i)_{ij} = -\epsilon' V'(U^0 \wedge U^i)_{ij},
\]
where \( X_{ij}^0 \) is the bivector corresponding to the dual face \( f \) determined by \( e, e', \), the sign factor \( \epsilon_0 = \epsilon_{e'f}(v) \), the sign factors \( \epsilon \) and \( \epsilon' \) are short-hand notations of \( \epsilon(v) \) and \( \epsilon(v') \) respectively, and
\[
\frac{1}{V} = \det(U^1, U^2, U^3, U^4) = \frac{1}{V'} = \det(U'^1, U'^2, U'^3, U'^4). 
\]
Here the minus sign for $1/V'$ is because the compatible orientations of $\sigma_0$ and $\sigma_0'$ are $[p_0, p_1, p_2, p_3, p_4]$ and $-[p_0, p_1, p_2, p_3, p_4]$, respectively. Thus, we should set $\epsilon_01234(v) = -\epsilon_01234(v') = 1$. Equations (C.3) and (C.4) tell us that $U_0^1$ is proportional to $U_0^{i_0}$ and $U_0^0$ is a linear combination of $U_i^1$ and $U_i^{i_0}$. Explicitly,

$$U_i^1 = -\epsilon_{ijk} \frac{[U_0]_V}{[U_0'_0]_V} U_j^1 + a_i U_{i_0}^0,$$

(C.6)

where $a_i$ are the coefficients, such that $\sum_i U_i^1 = -U_0'$, Using this expression of $U^0$, we have

$$-\frac{1}{V'} = \det(U^1, U^2, U^3, U^4) = \det(U^0, U^1, U^2, U^3)$$

$$= \tilde{\epsilon}_0 \left( \frac{[U_0']_V}{[U_0]_V} \right)^3 \det(U^0, U^1, U^2, U^3) = -\epsilon_{ijk} \left( \frac{[U_0]_V}{[U_0'_0]_V} \right)^2 \frac{1}{V'}$$

(C.7)

which results in $\epsilon = \epsilon'$. Therefore, $\epsilon(v) = \epsilon(v') = \epsilon$ is a global sign on the entire triangulation.

Now for the bivectors $X_0(v)$ and $X_0(v')$ ($X^I(v) = X^I(v)$ and $e_{ij}(v) = -e_{ji}(v)$):

$$\epsilon_0(v) X_0^I(v) = \frac{1}{2} \sum_{m,n} \epsilon^{kmn} \epsilon e_{ijkl} E_{ik}^J E_{mn}^L (v)$$

$$\epsilon_0(v') X_0^I(v') = \frac{1}{2} \sum_{m,n} \epsilon^{kmn} \epsilon e_{ijkl} E_{ik}^J E_{mn}^L (v').$$

(C.8)

Since $\epsilon_0(v) = -\epsilon_0(v')$, $\epsilon e_{ijkl} (v) = -\epsilon e_{ijkl} (v')$, we can set $\epsilon_0(v) \epsilon^{kmn} = \epsilon_0(v') \epsilon^{kmn}$. Therefore,

$$\epsilon_0 X_0^I(v) = \frac{1}{2} \sum_{m,n} \epsilon^{kmn} \epsilon e_{ijkl} E_{ik}^J E_{mn}^L (v)$$

$$\epsilon_0 X_0^I(v') = \frac{1}{2} \sum_{m,n} \epsilon^{kmn} \epsilon e_{ijkl} E_{ik}^J E_{mn}^L (v').$$

(C.9)

Given a triangle $f$, we can choose $E_{i_1}(v), E_{i_2}(v)$ (e.g. $\ell_1 = (p_m, p_k)$ and $\ell_2 = (p_n, p_k)$ with $\epsilon_{ij} = 1$ and $\epsilon e_{ijkl}$ = 1), such that

$$X^I_{ij}(v) = \epsilon *[E_{i_1}(v) \wedge E_{i_2}(v)]^I$$

and

$$X^I_{ij}(v') = \epsilon *[E_{i_1}(v') \wedge E_{i_2}(v')]^I$$

(C.10)

On the other hand, equation (C.7) also implies that $[U_0]_V = \pm [U_0'_0]_V$. Thus, we define a sign factor $\mu := -\tilde{\epsilon} [U_0]_V / [U_0'_0]_V = \pm 1$, such that from equation (C.6)

$$U_i^0 = \mu U_i^1 + a_i U_{i_0}^0 \quad \mu = -\tilde{\epsilon} \mathrm{sgn}(VV').$$

(C.11)

Therefore, we obtain the relation between $E_{ij}$ and $E_{ij}'$ (using $\epsilon_{jklm}(v') = -\epsilon_{jklm}(v)$),

$$E_{ij}' = \epsilon_{ijkl} e^{JJKL} U_{ij}^{J_1} U_{J_1}^{J_2} U_{J_2}^{J_3} U_{J_3}^{J_4} = \mu^2 \epsilon_{ijkl} e^{JJKL} U_{ik}^I U_{j}^{J_1} U_{J_1}^{J_2} U_{J_2}^{J_3} U_{J_3}^{J_4}$$

(C.12)

which means that for all tetrahedron edge $\ell$ of the tetrahedron $t_v$ dual to $e = (v, v')$, the co-frame vectors $E_{\ell}(v)$ and $E_{\ell}(v')$ at neighboring vertices $v$ and $v'$ are related by parallel transportation up to a sign $\mu_{e\ell}$, i.e.

$$\mu_{e\ell} E_{\ell}(v) = g_{e\ell} E_{\ell}(v') \quad \forall \ e \subset t_v.$$  

(C.13)

This relation shows that the vectors $E_{\ell}(v)$ (constructed from spinfoam critical point configuration) satisfy the metricity condition (A.6). Therefore, the collection of co-frame
vectors $E_i(v)$ at different vertices consistently forms a discrete co-frame of the whole triangulation. At the critical configuration, we define an SO(1,3) matrix $\Omega_{vv'}$ relating $g_{vv'}$ (in the spin-1 representation) by the sign $\mu_v$, i.e.

$$g_{vv'} = \mu_v \Omega_{vv'}.$$  

(C.14)

By lemma A.1 and definition A.2, the SO(1,3) matrix $\Omega_{vv'}$ is a discrete spin connection compatible with the co-frame if $\text{sgn}(V_4(v)) = \text{sgn}(V_4(v'))$.

If $\text{sgn}(V_4(v)) = \text{sgn}(V_4(v'))$, $\mu_v = -\bar{\epsilon} \text{sgn}(V_4(v)V_4(v')) = -\bar{\epsilon}$. Thus, from equation (C.3),

$$\frac{U'_0}{|U'_0|} = -\mu_v \frac{U_0}{|U_0|};$$  

(C.15)

the tetrahedron normal $U_e(v)/|U_e(v)|$ is always opposite to $\Omega_e U_e(v')/|U_e(v')|$ when $\text{sgn}(V_4(v)) = \text{sgn}(V_4(v'))$.

Since in the spin-1 representation $g_{vv} \in \text{SO}^+(1,3)$ and $\Omega \in \text{SO}(1,3)$, $\mu_v = -1$ corresponds the case that $\Omega_{vv} \in \text{SO}^-(1,3)$. It means that in the case of $\mu_v = -1$ if we choose the unit vectors $\hat{U}(v), \hat{U}(v')$ orthogonal to $E_i(v), E_i(v')$ ($\ell \subset t_v$), such that

$$\text{sgn det}(E_i(v), E_{i'}(v), E_{i''}(v'), \hat{U}(v')) = \text{sgn det}(E_i(v'), E_{i'}(v'), E_{i''}(v'), \hat{U}(v')),$$

then one of $\hat{U}(v), \hat{U}(v')$ is future-pointing and the other is past-pointing.

Appendix D. Proof of lemma 3.3

First of all, equations (A.25), (A.26) and (A.28) can be verified immediately from equation (3.31). Since $(E_{p,p_1}(e), E_{p,p_2}(e), E_{p,p_3}(e))$ is the inverse of $(n_{p_1}(v), n_{p_2}(v), n_{p_3}(v))'$, we have

$$\text{det}(E_{p,p_1}(e), E_{p,p_2}(e), E_{p,p_3}(e)) = V_3(e)$$  

(D.1)

and

$$\epsilon_{ef} j f \hat{n}_{p_1}(e) = |V_3(e)| n_{p_1}(v) = \epsilon(e) V_3(e) n_{p_1}(v) = \epsilon(e) \frac{1}{2} \sum_{k,l} \epsilon_{ijkl} E_{p,k}(e) \times E_{p,l}(e);$$  

(D.2)

where we have defined a sign factor $\epsilon(e) = \text{sgn}(V_3(e))$. Equivalently, for the bivector $X_{ef}$, there exist $E_{i_1}(e), E_{i_2}(e)$, such that

$$X_{ef}^{ij} = 2 \gamma_{j} [\hat{n}_{i_1} \wedge u']^j = \epsilon(e) [E_{i_1}(e) \wedge E_{i_2}(e)]^{ij}.$$  

(D.3)

Consider an internal vertex $v$ which is connected by the edge $e$; we introduce the shorthand notation

$$E_{ij} := E_{p,p_1}(e) \quad E'_{ij} := g_{vv'} E_{p,p_1}(v) \quad \epsilon(e) := \epsilon' \quad \hat{n}_j := \hat{n}_{p_1}(e).$$  

(D.4)

Since $X_f(v) = g_{vv'} X_{ef} g_{evv}$, for each triangle determined by $(p_1, p_2, p_3)$,

$$\epsilon' \frac{1}{2} \sum_{k,l} \epsilon_{ijkl} * E_{kl} \wedge E_{li} = \epsilon' \frac{1}{2} \sum_{k,l} \epsilon_{ijkl} * E'_{kl} \wedge E'_{li} = 2 \epsilon_{ef} j f [\hat{n}_{j} \wedge u].$$  

(D.5)

We also have $N_e(v) = g_{vv} u$. So $E'_{ij}$ is orthogonal to $u' = (1, 0, 0, 0)$, since $E_i(v)$ ($\ell \subset t_v$) is orthogonal to $N_e(v)$. Thus,

$$\epsilon' V n_j = 2 \epsilon_{ef} j f \hat{n}_j = \epsilon' \frac{1}{2} \sum_{k,l} \epsilon_{ijkl} E'_{kl} \times E'_{li},$$  

(D.6)
which implies that the $3 \times 3$ matrix given by $E'_{ik}$ (with $i$ fixed) is the inverse of the matrix given by $n_j$, $j \neq i$, up to an overall constant, i.e.

$$n_i \cdot E'_{ik} = \varepsilon \varepsilon V_3^{\prime} \left( \delta_{ij} - \delta_{ik} \right); \quad \text{(D.7)}$$

we have used the short-hand notation

$$V_3 = V_3(e) = \det(E_{31}(e), E_{31}(e), E_{41}(e)) \quad V_3' = V_3'(e) = \det(E'_{31}(e), E'_{31}(e), E'_{41}(e)). \quad \text{(D.8)}$$

Comparing equations (D.7) and (3.31), we determine that $E'_{jk}$ is proportional to $E''_{jk}$:

$$E'_{jk} = \varepsilon \varepsilon V_3 V_3' E_{jk}, \quad \text{(D.9)}$$

since the matrix given by $n_1$ has a unique inverse. Inserting this relation back into equation (D.5), we obtain that

$$\varepsilon \left( V_3' \right)^2 = \varepsilon', \quad \text{(D.10)}$$

which tell us that

$$\varepsilon' = \varepsilon \quad \text{and} \quad \left| \frac{V_3'}{V_3} \right| = 1. \quad \text{(D.11)}$$

As a result, we find the relations

$$X_{ij}^{\ell} = \varepsilon \left[ E_{\ell}(e) \wedge E'_{\ell}(e) \right] \quad \text{and} \quad \mu_{\ell} E_{\ell}(e) = g_{e\ell} E_{\ell}(v) \quad \forall \ell \subset t_\varepsilon, \quad \text{(D.12)}$$

where $\varepsilon = \pm 1$ is the global sign factor of the whole triangulation, and $\mu_{\ell} = \text{sgn}(V_3) \text{sgn}(V_3') = \pm 1$. From the second relation above, we obtain the metricity condition (A.28). Therefore, we confirm that $E_{\ell}(e)$ is a boundary co-frame constructed from the spinfoam critical configuration. The group element $g_{e\ell}$ equals the spin connection $\Omega_{e\ell}$ up to a sign, i.e.

$$g_{e\ell} = \mu_{\ell} \Omega_{e\ell}. \quad \text{(D.13)}$$

Since $\varepsilon$ is a global sign of the entire triangulation and $\varepsilon = \text{sgn}(V_3(e))$ on the boundary, prior to the construction, one has to choose a consistent orientation of the boundary triangulation, such that $\text{sgn}(V_3(e)) = \text{sgn}(V_3'(e))$ for each pair of tetrahedra $t_\varepsilon$ and $t_{e'}$.

### Appendix E. Proof of lemma 4.1

In the tetrahedra $t_{e_0}$ and $t_{e_1}$, both pairs of the vectors $E_{\ell}(e_0), E_{\ell}(e_0)$ and $E_{\ell}(e_1), E_{\ell}(e_1)$ are orthogonal to $u = (1, 0, 0, 0)^T$. Thus, at the vertex $v$, both $E_{\ell}(v)$ and $E'_{\ell}(v)$ are orthogonal to

$$F_{e_0}(v) = G_{\ell}(v, e_0) \triangleright u \quad F_{e_1}(v) = G_{\ell}(v, e_1) \triangleright u. \quad \text{(E.1)}$$

Thus, both $F_{e_0}(v)$ and $F_{e_1}(v)$ are future-pointing, since $G_{\ell}(v, e) \in \text{SL}(2, \mathbb{C})$. Equation (4.41) implies that

$$|\eta_{ij} F_{e_0}^I(v) F_{e_1}^I(v) | = \cosh \Theta^0_{ij}. \quad \text{(E.2)}$$

We define a dihedral boost from the dihedral angle $\Theta^0_{ij}$ by

$$D(e_1, e_0) = \exp \left[ \Theta^0_{ij} \frac{F_{e_0}(v) \wedge F_{e_1}(v)}{|F_{e_0}(v) \wedge F_{e_1}(v)|} \right] = \exp \left[ \Theta^0_{ij} \frac{U_{e}(v) \wedge U_{e'}(v)}{|U_{e}(v) \wedge U_{e'}(v)|} \right] \quad \text{(E.3)}$$
where we have chosen the sign of the dihedral angle, such that [13]

$$\begin{align*}
\text{If } & F_n(v) \wedge F_e(v) = \frac{U_e(v) \wedge U_e(v)}{|U_e(v) \wedge U_e(v)|} = \Theta_f^B : |\Theta_f^B| = -\Theta_f^B \\
\text{If } & F_n(v) \wedge F_e(v) = \frac{U_e(v) \wedge U_e(v)}{|U_e(v) \wedge U_e(v)|} = -\frac{U_e(v) \wedge U_e(v)}{|U_e(v) \wedge U_e(v)|} : |\Theta_f^B| = \Theta_f^B.
\end{align*}$$

(E.4)

with \( V_4(v)U_e(v) \wedge U_e(v) = G_f(v, e_0) \nabla G_f(v, e_0) \cap E_1(e_0) \cap E_2(e_0). \)

On the other hand, the boost generator \( K_3 \) can be related to the bivector \( X_{ef}^{IJ} = 2\gamma f (\hat{\nu}_{ef} \wedge u)^IJ \):

$$K_3 = -\hat{z} \wedge u = -g(\xi_{ef})^{-1} \nabla g(\xi_{ef})^{-1}(\hat{\nu}_{ef} \wedge u) = -g(\xi_{ef})^{-1} \nabla g(\xi_{ef})^{-1} \frac{1}{2\gamma f} X_{ef}.$$  

(E.5)

At the critical configuration, the bivector \( X_{ef} \) is given by equation (3.38), which results in that

$$K_3 = -\epsilon g(\xi_{ef})^{-1} \nabla g(\xi_{ef})^{-1} \epsilon E_1(e) \wedge E_2(e) \left| E_1(e) \wedge E_2(e) \right| = \epsilon \frac{\epsilon E_1(e) \wedge E_2(e)}{|E_1(e) \wedge E_2(e)|}.$$  

(E.6)

where \( \frac{\epsilon E_1(e) \wedge E_2(e)}{|E_1(e) \wedge E_2(e)|} \) is the (unit) bivector corresponding to the triangle \( f \). Therefore, for the bivector at the vertex \( v \),

$$\frac{U_e(v) \wedge U_e(v)}{|U_e(v) \wedge U_e(v)|} = \sgn(V_4)G_f(v, e_0) \nabla G_f(v, e_0) \left| \frac{U_e(v) \wedge U_e(v)}{|U_e(v) \wedge U_e(v)|} \right| = -\sgn(V_4) \epsilon G_f(v, e_0)g(\xi_{ef})^{-1}G_f(v, e_0)^{-1}.$$  

(E.7)

Then we obtain the following expression of \( D(e_1, e_0) \):

$$D(e_1, e_0) = G_f(v, e_0)g(\xi_{ef})e^{-\epsilon \sgn(V_4)\epsilon^f K_f}g(\xi_{ef})^{-1}G_f(v, e_0)^{-1}.$$  

(E.8)

One can check that \( D(e_1, e_0) \) gives a dihedral boost from \( F_n(v) \) to \( F_e(v) \), i.e.

$$D(e_1, e_0)F_n(v) = F_e(v).$$  

(E.9)

If we represent the vector \( F_e(v) \) by the \( 2 \times 2 \) matrix \( F_e = F_e^T \sigma_f \), and we have \( F_e(v) = G_f(v, e)G_f(v, e)^T \), then equation (E.9) can be expressed as

$$D(e_1, e_0)G_f(v, e_0)G_f(v, e_0)^T D(e_1, e_0)^T = G_f(v, e_1)G_f(v, e_1)^T.$$  

(E.10)

By using equation (E.8), we obtain that \( (J_1^\dagger = -J_3) \)

$$G_f(v, e_0)g(\xi_{ef})e^{-2\epsilon \sgn(V_4)\epsilon^f K_f}g(\xi_{ef})^{-1}G_f(v, e_0)^{T} = G_f(v, e_1)G_f(v, e_1)^{T}.$$  

(E.11)

From expression (4.36) of \( G_f(e_1, e_0) = G_f(v, e_1)^{-1}G_f(v, e_0) \) in terms of \( \Theta_f^B \), we obtain

$$G_f(e_0, e_1)G_f(e_0, e_1)^{T} = \hat{g}(\xi_{ef})e^{-2\epsilon \Theta_f^B K_f}\hat{g}(\xi_{ef})^{-1}.$$  

(E.12)

Combining equations (E.11) and (E.12), we obtain

$$e^{-2\epsilon \sgn(V_4)\Theta_f^B K_f} = e^{-2\epsilon \Theta_f^B K_f},$$  

(E.13)

which results in

$$\Theta_f^B = \epsilon \sgn(V_4)\Theta_f^B.$$  

(E.14)
Appendix F. Hessian matrix

In this section, we compute the Hessian matrix of the spin foam action

$$S_f = \sum_{v \in f} S_{vf} = \sum_{v \in f} \left( j_f \ln \left( \frac{\langle \xi_{vf}, Z_{vf} \rangle^2}{\langle Z_{vf}, Z_{vf} \rangle} \right) + i \gamma j_f \ln \left( \frac{\langle Z_{vf}, Z_{vf} \rangle}{\langle Z_{vf}, Z_{vf} \rangle} \right) \right).$$  \hspace{1cm} (F.1)

First of all, we compute the double variation $\delta_{\xi_e \xi_f} \delta_{\xi_e \xi_f} S$, by using $\delta \xi_{ef} = \omega_{ef}(\langle \xi_{ef}, Z_{ef} \rangle) + i \eta_{ef} \langle \xi_{ef}, Z_{ef} \rangle$ for the complex infinitesimal parameter $\omega_{ef} \in \mathbb{C}$ and $\eta_{ef} \in \mathbb{R}$. We see immediately from the variation in equation (2.27) that $\delta_{\xi_e \xi_f} \delta_{\xi_e \xi_f} S = 0$ if $e \neq e'$. Then for the double variation $\delta_{\xi_e \xi_f}^2 S$ for the same $\xi_{ef}$,

$$\delta_{\xi_e \xi_f}^2 S = j_f \delta_{\xi_e \xi_f} \left( 2 \frac{\delta_{\xi_e}(\xi_{ef}, Z_{vf})}{\langle \xi_{ef}, Z_{vf} \rangle} + 2 \frac{\delta_{\xi_e}(Z_{vf}, \xi_{ef})}{\langle Z_{vf}, \xi_{ef} \rangle} \right)$$

$$= 2 j_f \left( - \frac{\delta_{\xi_e}(\xi_{ef}, Z_{vf})}{\langle \xi_{ef}, Z_{vf} \rangle^2} \delta_{\xi_e}(Z_{vf}, \xi_{ef}) + \frac{\delta_{\xi_e}(\xi_{ef}, Z_{vf})}{\langle \xi_{ef}, Z_{vf} \rangle^2} \right)$$

$$- \frac{\delta_{\xi_e}(Z_{vf}, \xi_{ef})}{\langle \xi_{ef}, \xi_{ef} \rangle^2} \delta_{\xi_e}(\xi_{ef}, Z_{vf}) + \frac{\delta_{\xi_e}(Z_{vf}, \xi_{ef})}{\langle \xi_{ef}, \xi_{ef} \rangle^2} \right),$$  \hspace{1cm} (F.2)

where we use the relation for the double variation $\delta(X^{-1} \delta X) = -X^{-2}(\delta X)^2 + X^{-1} \delta^2 X$. We compute the above double variation term by term, by using the following relations:

$$\delta_{\xi_e}(\xi_{ef}, Z_{vf}) = \bar{\omega}(\langle \xi_{ef}, Z_{vf} \rangle) - i \eta \langle \xi_{ef}, Z_{vf} \rangle$$

$$\delta_{\xi_e}(\langle \xi_{ef}, Z_{vf} \rangle) = -\omega(\xi_{ef}, Z_{vf}) + i \eta \langle \xi_{ef}, Z_{vf} \rangle$$

$$\delta_{\xi_e}(\xi_{ef}, Z_{vf}) = \bar{\omega} \delta_{\xi_e}(\xi_{ef}) - i \eta \delta_{\xi_e}(\xi_{ef})$$

$$= -\bar{\omega} \omega(\xi_{ef}, Z_{vf}) - \eta^2 \langle \xi_{ef}, Z_{vf} \rangle$$

$$\delta_{\xi_e}(\langle \xi_{ef}, Z_{vf} \rangle) = \omega(\langle \xi_{ef}, Z_{vf} \rangle) + i \eta \langle \xi_{ef}, Z_{vf} \rangle$$

$$\delta_{\xi_e}(\xi_{ef}, \xi_{ef}) = \bar{\omega} \delta_{\xi_e}(\xi_{ef}) - i \eta \delta_{\xi_e}(\xi_{ef})$$

$$= -\omega \omega(\xi_{ef}, \xi_{ef}) - \eta^2 \langle \xi_{ef}, \xi_{ef} \rangle.$$

Then each term in the above $\delta_{\xi_e \xi_f}^2 S$ can be computed

$$- \frac{\delta_{\xi_e}(\xi_{ef}, Z_{vf})}{\langle \xi_{ef}, Z_{vf} \rangle^2} \delta_{\xi_e}(\xi_{ef}, Z_{vf}) = - \frac{(\bar{\omega}^2(\langle \xi_{ef}, Z_{vf} \rangle)^2 - 2i \eta \bar{\omega}(\langle \xi_{ef}, Z_{vf} \rangle)(\langle \xi_{ef}, Z_{vf} \rangle) - \eta^2 (\langle \xi_{ef}, Z_{vf} \rangle)^2)}{\langle \xi_{ef}, Z_{vf} \rangle^2}$$

$$= - \frac{(\bar{\omega}^2(\langle \xi_{ef}, Z_{vf} \rangle)^2 - 2i \eta \bar{\omega}(\langle \xi_{ef}, Z_{vf} \rangle)}{\langle \xi_{ef}, Z_{vf} \rangle^2} + \frac{\eta^2}{\langle \xi_{ef}, Z_{vf} \rangle^2}$$

$$\delta_{\xi_e}(\xi_{ef}, \xi_{ef}) = - \frac{\omega \omega(\xi_{ef}, \xi_{ef}) - \eta^2 \langle \xi_{ef}, \xi_{ef} \rangle}{\langle \xi_{ef}, \xi_{ef} \rangle^2} = - \bar{\omega} \omega - \eta^2$$

$$- \frac{\delta_{\xi_e}(Z_{vf}, \xi_{ef})}{\langle \xi_{ef}, \xi_{ef} \rangle^2} \delta_{\xi_e}(Z_{vf}, \xi_{ef})$$

$$= - \frac{(\omega^2 \langle Z_{vf}, J_{\xi_{ef}} \rangle)^2 + 2i \eta \omega \langle Z_{vf}, J_{\xi_{ef}} \rangle (\langle Z_{vf}, \xi_{ef} \rangle) - \eta^2 \langle Z_{vf}, \xi_{ef} \rangle^2)}{\langle Z_{vf}, \xi_{ef} \rangle^2}$$

$$= - \frac{(\omega^2 \langle Z_{vf}, J_{\xi_{ef}} \rangle)^2}{\langle Z_{vf}, \xi_{ef} \rangle^2} - \frac{2i \eta \omega \langle Z_{vf}, J_{\xi_{ef}} \rangle}{\langle Z_{vf}, \xi_{ef} \rangle^2} + \frac{\eta^2}{\langle Z_{vf}, \xi_{ef} \rangle^2}$$

51
\[
\frac{\delta^2 S_{\xi e}}{\langle \xi e, \xi e \rangle} = -\omega \tilde{\omega} - \eta^2. \tag{F.3}
\]

Therefore, \(\delta^2 S_{\xi e}\) is obtained explicitly
\[
\delta^2 S_{\xi e} = 2j_f \left( -\frac{\tilde{\omega}^2 (J\xi e, Z_{\text{vef}})^2}{\langle \xi e, \xi e \rangle} + \frac{2\eta \tilde{\omega} (J\xi e, Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} - \frac{\omega^2 (Z_{\text{vef}}, J\xi e)^2}{\langle \xi e, \xi e \rangle} - \frac{2\eta \omega (Z_{\text{vef}}, J\xi e)}{\langle \xi e, \xi e \rangle} - 2\omega \tilde{\omega} \right).
\]

Because of equation (2.19), at the critical configuration \(Z_{\text{vef}} \sim \xi e\). Therefore, by using the relation \((J\xi e, \xi e) = 0\), we obtain the result
\[
\delta^2 S_{\xi e} = -4j_f \omega \tilde{\omega},
\]
which means that the \(H_{\xi e\xi e}\) components of the Hessian matrix are the only nonvanishing components in the Hessian submatrix with respect to the spinorial variables \(\xi e\), and
\[
H_{\xi e\xi e} = H_{\xi e\xi e} = -4j_f \tilde{\omega} \omega.
\]

Secondly, we compute the double variation with respect to both the spinorial variable \(\xi e\) and the group variable \(g_{\text{ve}}\), where \(\delta g_{\text{ve}} := \partial \theta J J^\dagger \bar{g}_{\text{ve}} = g_{\text{ve}} \partial \theta J J^\dagger \bar{g}_{\text{ve}}\) at a critical configuration \(g_{\text{ve}}\),
\[
\delta g_{\text{ve}} \delta g_{\text{ve}} = \frac{j_f \delta g_{\text{ve}}}{2j_f} \begin{pmatrix} \frac{\delta \xi e, (\xi e, Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} + \frac{2\delta \xi e, (Z_{\text{vef}}, \xi e)}{\langle \xi e, \xi e \rangle} + \frac{2\delta \xi e, (\xi e, Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} \end{pmatrix}
\]
\[
= 2j_f \begin{pmatrix} -\frac{\delta \xi e, (\xi e, Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} - \frac{\delta \xi e, (\xi e, Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} + \frac{\delta \xi e, (\xi e, Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} \end{pmatrix}
\]
\[
= 2j_f \begin{pmatrix} \frac{\delta \xi e, (\xi e, J^\dagger Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} - \frac{\delta \xi e, (\xi e, J^\dagger Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} + \frac{\delta \xi e, (\xi e, J^\dagger Z_{\text{vef}})}{\langle \xi e, \xi e \rangle} \end{pmatrix}
\]
\[
\tag{F.6}
\]

We use the following relations:
\[
\delta g_{\text{ve}} (\xi e, Z_{\text{vef}}) = \langle \xi e, J^\dagger Z_{\text{vef}} \rangle
\]
\[
\delta g_{\text{ve}} (\xi e, \xi e, Z_{\text{vef}}) = \omega (J \xi e, J^\dagger Z_{\text{vef}}) - \bar{\eta} (\xi e, J^\dagger Z_{\text{vef}}).
\]

Thus using equation (2.19), we have
\[
\delta g_{\text{ve}} \delta g_{\text{ve}} = 2j_f \left[ \frac{\langle \xi e, J^\dagger Z_{\text{vef}} \rangle (\bar{\omega} (J \xi e, Z_{\text{vef}}) - \bar{\eta} (\xi e, J^\dagger Z_{\text{vef}}))}{\langle \xi e, \xi e \rangle} \right]
\]
\[
= 2j_f \left[ \frac{\langle \xi e, J^\dagger Z_{\text{vef}} \rangle ^2}{\langle \xi e, \xi e \rangle} \right]
\]
\[
= 2j_f \frac{\bar{\omega} (J \xi e, J^\dagger \xi e)}{\langle \xi e, \xi e \rangle}.
\]

where explicitly for \((J \xi e, \sigma^i \xi e), \xi = (\xi_0, \xi_1)^\prime\) we have
\[
\langle J \xi e, \sigma^1 \xi e \rangle = -2 \xi_0 \xi_1
\]
\[
\langle J \xi e, \sigma^2 \xi e \rangle = 0
\]
\[
\langle J \xi e, \sigma^3 \xi e \rangle = - (\xi_0^2 - \xi_1^2).
\]

However, there are also nonvanishing components \(\delta_{\xi e} \delta_{\xi e} S\) with \(e = (v, v')\). Similarly, we obtain
\[
\delta_{\xi e} \delta_{\xi e} S = 2j_f \omega (J^\dagger \xi e, J \xi e).
\]
\[
\tag{F.9}
\]
Thus, the nonvanishing components of the Hessian matrix are
\[
H_{v_1\nu_1} = H_{\bar{v}_1\nu_1} = 2j_f (\langle J_{\xi,v_1}, J_{\xi,v_1} \rangle) \quad \text{and} \quad H_{v_1\bar{v}_1} = H_{\bar{v}_1\nu_1} = 2j_f (\langle J_{\xi,v_1}, J_{\xi,\bar{v}_1} \rangle). \tag{F.10}
\]

Next we compute the double variation \( \delta v_1 \delta \bar{v}_1 S \). Here, \( Z_{\xi,v} \) is a \( \mathbb{CP}^1 \) variable, so \( \delta Z_{\xi,v} = \epsilon v_1 J_{\xi,v} \):

\[
\delta v_1 \delta \bar{v}_1 S = j_f \delta v_1 \left( 2 \frac{\delta \bar{v}_1 \langle \xi_{\xi,v}, Z_{\xi,v} \rangle}{\langle \xi_{\xi,v}, Z_{\xi,v} \rangle} + 2 \frac{\delta \bar{v}_1 \langle Z_{\xi,v}, \xi_{\xi,v} \rangle}{\langle Z_{\xi,v}, \xi_{\xi,v} \rangle} \right) = 2j_f \left( \delta v_1 \langle \xi_{\xi,v}, Z_{\xi,v} \rangle + \delta \bar{v}_1 \langle Z_{\xi,v}, \xi_{\xi,v} \rangle \right). \tag{F.11}
\]

We use the following relations:
\[
\delta v_1 \langle \xi_{\xi,v}, Z_{\xi,v} \rangle = \epsilon \langle \xi_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle \tag{F.12}
\]
\[
\delta v_1 \langle Z_{\xi,v}, \xi_{\xi,v} \rangle = \bar{\epsilon} \delta v_1 \langle \xi_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle = \bar{\epsilon} \delta v_1 \langle J_{\xi,v}, g_{\xi,v} \xi_{\xi,v} \rangle - \bar{\epsilon} e^{2i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle. \tag{F.13}
\]

Using equation (2.19), we have at a critical configuration
\[
\frac{\delta v_1 \delta \bar{v}_1 S}{\langle \xi_{\xi,v}, Z_{\xi,v} \rangle} = 2j_f \left( - \frac{e \langle \xi_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle (\bar{\epsilon} \langle J_{\xi,v}, g_{\xi,v} \xi_{\xi,v} \rangle - \bar{\epsilon} e^{2i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle)}{\langle \xi_{\xi,v}, Z_{\xi,v} \rangle} \right) + \frac{\bar{\epsilon} e \langle J_{\xi,v}, g_{\xi,v} \xi_{\xi,v} \rangle - \bar{\epsilon} e^{2i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle}{\langle \xi_{\xi,v}, Z_{\xi,v} \rangle} \right) = 2j_f \bar{\epsilon} e^{2i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle. \tag{F.14}
\]

Similarly, we also have, for \( e = (v, v') \), that
\[
\delta v_1 \delta \bar{v}_1 S = 2j_f e^{2i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle \tag{F.15}
\]

Then the nonvanishing components of the Hessian matrix are
\[
H_{v_1\nu_1} = H_{\bar{v}_1\nu_1} = 2j_f e^{2i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle \quad \text{and} \quad H_{v_1\bar{v}_1} = H_{\bar{v}_1\nu_1} = 2j_f e^{-2i\phi_v}. \tag{F.16}
\]

Note that in the degenerate case \( g_{\xi,v} \in SU(2) \),

\[
H_{v_1\nu_1}|_{\text{deg}} = H_{\bar{v}_1\nu_1}|_{\text{deg}} = 2j_f e^{2i\phi_v} \quad \text{and} \quad H_{v_1\bar{v}_1}|_{\text{deg}} = H_{\bar{v}_1\nu_1}|_{\text{deg}} = 2j_f e^{-2i\phi_v}.
\]

For the double variations \( \delta v_1 \delta \bar{v}_1 S \), it is obvious that the nonvanishing components are \( \delta v_1^2 S \):

\[
\delta v_1^2 S_{ov} = j_f \left( 2 \frac{\delta v_1 \langle \xi_{\xi,v}, Z_{\xi,v} \rangle}{\langle \xi_{\xi,v}, Z_{\xi,v} \rangle} + 2 \frac{\delta v_1 \langle Z_{\xi,v}, \xi_{\xi,v} \rangle}{\langle Z_{\xi,v}, \xi_{\xi,v} \rangle} \right) + \bar{\epsilon} j_f \left( \frac{\delta v_1 \langle Z_{\xi,v}, \xi_{\xi,v} \rangle}{\langle Z_{\xi,v}, \xi_{\xi,v} \rangle} - \frac{\delta v_1 \langle \xi_{\xi,v}, Z_{\xi,v} \rangle}{\langle \xi_{\xi,v}, Z_{\xi,v} \rangle} \right) \tag{F.19}
\]

In the following, we compute \( \delta v_1^2 S_{ov} \) term by term:

\[
2 \delta v_1 \langle \xi_{\xi,v}, Z_{\xi,v} \rangle = -2\bar{\epsilon} e^{4i\phi_v} \langle g_{\xi,v} \xi_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle - 2\bar{\epsilon} e^{4i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} \xi_{\xi,v} \rangle
\]

\[
2 \delta v_1 \langle Z_{\xi,v}, \xi_{\xi,v} \rangle = -2\bar{\epsilon} e^{4i\phi_v} \langle g_{\xi,v} \xi_{\xi,v}, g_{\xi,v} J_{\xi,v} \rangle - 2\bar{\epsilon} e^{4i\phi_v} \langle g_{\xi,v} J_{\xi,v}, g_{\xi,v} \xi_{\xi,v} \rangle
\]
Thus, we have explicitly
\[
\begin{align*}
-\delta_{\bar{f}f} \frac{\delta_{\bar{f}f} (Z_{v\bar{f}}, Z_{v\bar{f}})}{(Z_{v\bar{f}}, Z_{v\bar{f}})} &= (\varepsilon e^{2\delta_{\bar{f}f}} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}} + \bar{\varepsilon} e^{-2\delta_{\bar{f}f}} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
&\times 2\bar{\varepsilon} - 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}) \nonumber \\
-\delta_{\bar{f}f} \frac{\delta_{\bar{f}f} (Z_{v\bar{f}}, Z_{v\bar{f}})}{(Z_{v\bar{f}}, Z_{v\bar{f}})} &= (\varepsilon e^{2\delta_{\bar{f}f}} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}} + \bar{\varepsilon} e^{-2\delta_{\bar{f}f}} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
&\times 2\bar{\varepsilon} - 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}) \nonumber.
\end{align*}
\]

We obtain explicitly the expression of \( \delta_{\bar{f}f} S_{v\bar{f}} \):
\[
\delta_{\bar{f}f} S_{v\bar{f}} = j f (2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) + 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
- 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) + 2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
+ i\varepsilon j f (2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) - 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \nonumber \\
+ 2i\bar{\varepsilon} j f (2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) - 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}).
\]

Therefore, we obtain the nonvanishing components of the Hessian matrix
\[
\begin{align*}
\left( H_{v\bar{f},\bar{f}v}, H_{v\bar{f},\bar{f}v} \right) &= H_{v\bar{f},\bar{f}v} = 2 j f (2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
&\quad + (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
&\quad - (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \\
&\quad + 2i\bar{\varepsilon} j f (2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) - 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) \nonumber \\
&\quad + 2i\bar{\varepsilon} j f (2\varepsilon (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) - 2\bar{\varepsilon} (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}) (g_{v\bar{f}}\xi_{\bar{f}f}, g_{v\bar{f}}\xi_{\bar{f}f}}). \tag{F.19}
\end{align*}
\]

Note that in the degenerate case \( g_{v\bar{f}} \in SU(2) \),
\[
H_{v\bar{f},v\bar{f},v\bar{f},v\bar{f}} \mid \text{deg} = H_{v\bar{f},v\bar{f},v\bar{f},v\bar{f}} \mid \text{deg} = -4 j f. \tag{F.20}
\]

For the double variation \( \delta_{\bar{g}_w} \delta_{\bar{g}_w} S_{v\bar{f}} \), we have
\[
\delta_{\bar{g}_w} \delta_{\bar{g}_w} S_{v\bar{f}} = j f \left( 2 \delta_{\bar{g}_w} \frac{\delta_{\bar{g}_w} (\xi_{\bar{f}f}, Z_{\bar{v}f})}{(\xi_{\bar{f}f}, Z_{\bar{v}f})} - \delta_{\bar{g}_w} \frac{\delta_{\bar{g}_w} (Z_{v\bar{f}}, Z_{v\bar{f}})}{(Z_{v\bar{f}}, Z_{v\bar{f}})} \right) = i\varepsilon j f \delta_{\bar{g}_w} \frac{\delta_{\bar{g}_w} (Z_{v\bar{f}}, Z_{v\bar{f}})}{(Z_{v\bar{f}}, Z_{v\bar{f}})}, \quad \tag{F.21}
\]
while in the degenerate case

\[
H_{\theta, \epsilon_v}|_{\text{deg}} = H_{\epsilon_v}^\theta|_{\text{deg}} = 2J_f e^{2ib_v} \langle \xi_{ef}, J^\dagger \xi_{ef} \rangle \\
- (1 + iy) J_f [e^{2ib_v} \langle J^\dagger \xi_{ef}, J\xi_{ef} \rangle + e^{2ib_v} \langle \xi_{ef}, J^\dagger J\xi_{ef} \rangle]
\]

\[
H_{\theta, \epsilon_v}|_{\text{deg}} = H_{\epsilon_v}^\theta|_{\text{deg}} = - (1 + iy) J_f [e^{-2ib_v} \langle J^\dagger J\xi_{ef}, \xi_{ef} \rangle + e^{-2ib_v} \langle J\xi_{ef}, J^\dagger \xi_{ef} \rangle].
\] (F.23)

However, there are nonvanishing components of the Hessian matrix from \(\delta_{S_{\epsilon_v}}\delta_{\epsilon_v} S_{\epsilon_v}\):

\[
\delta_{S_{\epsilon_v}}\delta_{\epsilon_v} S_{\epsilon_v} = J_f \left( 2\delta_{\epsilon_v} \delta_{\epsilon_v} (Z_{ve\epsilon}, \xi_{ef}) - \delta_{\epsilon_v} (Z_{ve\epsilon}, Z_{ve\epsilon}) \right) + iy J_f \delta_{\epsilon_v} \delta_{\epsilon_v} (Z_{ve\epsilon}, Z_{ve\epsilon})
\]

\[
= 2J_f [\delta_{\epsilon_v} (-\delta_{\epsilon_v} (g_{ve\epsilon} J\xi_{ef}, g_{ve\epsilon} \xi_{ef}) (J^\dagger \xi_{ef}, \xi_{ef}) + \delta_{\epsilon_v} (J^\dagger g_{ve\epsilon} J\xi_{ef}, g_{ve\epsilon} \xi_{ef}))

- (1 - iy) J_f [- (e^{2ib_v} \langle g_{ve\epsilon} \xi_{ef}, g_{ve\epsilon} \xi_{ef} \rangle + \bar{e} e^{-2ib_v} \langle g_{ve\epsilon} J\xi_{ef}, g_{ve\epsilon} \xi_{ef} \rangle)

+ \bar{e} e^{-2ib_v} \langle J^\dagger g_{ve\epsilon} J\xi_{ef}, g_{ve\epsilon} \xi_{ef} \rangle + \bar{e} e^{-2ib_v} \langle g_{ve\epsilon} g_{ve\epsilon} \xi_{ef}, J^\dagger \xi_{ef} \rangle]

+ (\langle J^\dagger \xi_{ef}, \xi_{ef} \rangle + \langle \xi_{ef}, J^\dagger \xi_{ef} \rangle)]

\] (F.24)

Therefore, we obtain the components of the Hessian matrix:

\[
H_{\epsilon_v}^\theta = H_{\epsilon_v}^{\theta}|_{\text{deg}} = - (1 - iy) J_f [e^{2ib_v} \langle J^\dagger \xi_{ef}, \xi_{ef} \rangle + e^{2ib_v} \langle \xi_{ef}, J^\dagger J\xi_{ef} \rangle]

+ e^{2ib_v} \langle J^\dagger \xi_{ef}, g_{ve\epsilon} \xi_{ef} \rangle \langle g_{ve\epsilon} J\xi_{ef}, g_{ve\epsilon} \xi_{ef} \rangle + e^{2ib_v} \langle \xi_{ef}, J^\dagger \xi_{ef} \rangle]

\]

while in the degenerate case,

\[
H_{\epsilon_v}^\theta = H_{\epsilon_v}^{\theta}|_{\text{deg}} = - (1 - iy) J_f [e^{-2ib_v} \langle J^\dagger \xi_{ef}, J\xi_{ef} \rangle + e^{-2ib_v} \langle J\xi_{ef}, J^\dagger \xi_{ef} \rangle]

\]

Finally, the nonvanishing Hessian components \(H_{\theta, \theta_v}\) are computed in [13]

\[
H_{\theta, \theta_v} = \frac{1}{2} \sum_f J_f (-\delta^{ij} + \hat{n}_{ij} \hat{n}_{ef} + i\epsilon^{ijk} \hat{r}_{ef})
\]

\[
H_{\theta, \theta_v} = -\frac{1}{2} \sum_f J_f (-\delta^{ij} + \hat{n}_{ij} \hat{n}_{ef} + i\epsilon^{ijk} \hat{r}_{ef})
\]

\[
H_{\theta, \theta_v} = -\frac{1}{2} \sum_f J_f (-\delta^{ij} + \hat{n}_{ij} \hat{n}_{ef} + i\epsilon^{ijk} \hat{r}_{ef})
\]

\[
H_{\theta, \theta_v} = 2 \left( 1 + \frac{i}{2} \right) \sum_f J_f (-\delta^{ij} + \hat{n}_{ij} \hat{n}_{ef} + i\epsilon^{ijk} \hat{r}_{ef})
\]

where \(r\) and \(b\) label respectively the rotation and boost parts of the generators.

References

[1] Thiemann T 2007 Modern Canonical Quantum General Relativity (Cambridge: Cambridge University Press)
Rovelli C 2004 Quantum Gravity (Cambridge: Cambridge University Press)

[2] Ashtekar A and Lewandowski J 2004 Background independent quantum gravity: a status report Class. Quantum Grav. 21 R53
Han M, Huang W and Ma Y 2007 Fundamental structure of loop quantum gravity Int. J. Mod. Phys. D 16 1397–474 (arXiv:gr-qc/0509064)

[3] Rovelli C 2011 Zakopane lectures on loop gravity arXiv:1102.3660
Rovelli C 2010 Simple model for quantum general relativity from loop quantum gravity arXiv:1010.1939
Rovelli C 2011 A new look at loop quantum gravity Class. Quantum Grav. 28 114005 (arXiv:1004.1780)
Perez A 2003 Spin-foam models for quantum gravity Class. Quantum Grav. 20 R43–R104

[4] Thiemann T 1998 Quantum spin dynamics (QSD) Class. Quantum Grav. 15 839–73 (arXiv:gr-qc/9606089)
Thiemann T 2006 Quantum spin dynamics: VIII. The master constraint Class. Quantum Grav. 23 2249–66 (arXiv:gr-qc/0510011)
Han M and Ma Y 2006 Master constraint operator in loop quantum gravity Phys. Lett. B 635 225–31 (arXiv:gr-qc/0510014)
Giesel K and Thiemann T 2007 Algebraic quantum gravity (AQG) I,II,III,IV Class. Quantum Grav. 24 2465–588
Giesel K and Thiemann T 2010 Class. Quantum Grav. 27 175009

[5] Barrett J and Crane L 2000 Relativistic spin-networks and quantum gravity J. Math. Phys. 39 3296
Barrett J and Crane L 2000 A Lorentzian signature model for quantum general relativity Class. Quantum Grav. 17 3101–18

[6] Engle J, Pereira R and Rovelli C 2007 The loop-quantum-gravity vertex-amplitude Phys. Rev. Lett. 99 161301
Engle J, Livine E, Pereira R and Rovelli C 2008 LQG vertex with finite Immirzi parameter Nucl. Phys. B 799 136

[7] Freidel L and Krasnov K 2008 New spin foam model for 4d gravity Class. Quantum Grav. 25 125018
Livine E and Speziale S 2007 A new spin foam vertex for quantum gravity Phys. Rev. D 76 084028
Livine E and Speziale S 2008 Consistently solving the simplicity constraints for spin foam quantum gravity Europhys. Lett. 81 50004

[8] Han M and Thiemann T 2010 Commuting simplicity and closure constraints for 4D spin foam models arXiv:1010.5444

[9] Noui K and Perez A 2006 Three dimensional loop quantum gravity: physical scalar product and spin foam models Class. Quantum Grav. 22 1739–62

[10] Han M and Thiemann T 2010 On the relation between operator constraint, master constraint, reduced phase space, and path integral quantisation Class. Quantum Grav. 27 225019 (arXiv:0911.3428)
Han M and Thiemann T 2010 On the relation between rigging inner product and master constraint direct integral decomposition J. Math. Phys. 51 092501 (arXiv:0911.3431)
Han M 2010 Path integral for the master constraint of loop quantum gravity Class. Quantum Grav. 27 215009 (arXiv:0911.3432)
Engle J, Han M and Thiemann T 2010 Canonical path-integral measure for Holst and Plebanski gravity: I. Reduced phase space derivations Class. Quantum Grav. 27 245014 (arXiv:0911.3433)
Han M 2010 Canonical path-integral measure for Holst and Plebanski gravity: II. Gauge invariance and physical inner product Class. Quantum Grav. 27 245015 (arXiv:0911.3436)

[11] Plebanski J 1977 On the separation of Einsteinian substructures J. Math. Phys. 18 2511–20
Reisenberger M P 1998 Classical Euclidean general relativity from ‘left-handed area = righthanded area’ arXiv:gr-qc/9804061
De Pietri R and Freidel L 1999 SO(4) Plebanski action and relativistic spin foam model Class. Quantum Grav. 16 2187–96

[12] Ding Y, Han M and Rovelli C 2011 Generalized spinfoams Phys. Rev. D 83 124020 (arXiv:1011.2149 [gr-qc])
Ding Y and Rovelli C 2010 The volume operator in covariant quantum gravity Class. Quantum Grav. 27 165003 (arXiv:0911.0543 [gr-qc])
Ding Y and Rovelli C 2010 Physical boundary Hilbert space and volume operator in the Lorentzian new spin foam theory Class. Quantum Grav. 27 205003 (arXiv:1006.1294 [gr-qc])

[13] Barrett J W, Dowdall R J, Fairbairn W J, Gomes H and Hellmann F 2009 Asymptotic analysis of the EPRL four-simplex amplitude J. Math. Phys. 50 112504
Barrett J W, Dowdall R J, Fairbairn W J, Hellmann F and Pereira R 2009 Lorentzian spin foam amplitudes: graphical calculus and asymptotics arXiv:0907.2440

[14] Han M and Rovelli C 2011 Spin foam fermions: PCT symmetry, Dirac determinant, and correlation functions arXiv:1101.3264
Bianchi E, Han M, Magliaro E, Perini C, Rovelli C and Wieland W 2010 Spin foam fermions arXiv:1012.4719
Han M 2011 4-dimensional spin foam model with quantum Lorentz group J. Math. Phys. 52 072501 (arXiv:1012.4216)
Fairbairn W J and Meusburger C 2010 Quantum deformation of two four-dimensional spin foam models arXiv:1012.4784

[15] Han M 2011 Cosmological constant in LQG vertex amplitude Phys. Rev. D 84 064010 (arXiv:1105.2212)
Ding Y and Han M 2011 On the asymptotics of quantum group spin foam model arXiv:1103.1597
[17] Bianchi E, Magliaro E and Perini C 2010 Spinfoams in the holomorphic representation Phys. Rev. D 82 124031
[18] Freidel L and Louapre D 2003 Asymptotics of 6j and 10j symbols Class. Quantum Grav. 20 1267
Barrett J W and Steele C M 2003 Asymptotics of relativistic spin networks Class. Quantum Grav. 20 1341
[19] Conrady F and Freidel L 2008 On the semiclassical limit of 4d spin foam models Phys. Rev. D 78 104023
[20] Magliaro E and Perini C 2011 Emergence of gravity from spinfoams Europhys. Lett. 95 30007
Magliaro E and Perini C 2011 Regge gravity from spinfoams arXiv:1105.0216
[21] Han M and Zhang M 2012 Asymptotics of spinfoam amplitude on simplicial manifold: Euclidean theory Class. Quantum Grav. 29 165004 (arXiv:1109.0500)
[22] Hellmann F 2011 State sums and geometry PhD Thesis arXiv:1102.1688
Rühl W 1970 Lorentz Group and Harmonic Analysis (New York: Benjamin)
[23] Hörmander L 1990 The Analysis of Linear Partial Differential Operators I. Distribution Theory and Fourier Analysis 2nd edn (Berlin: Springer)
[24] Caselle M, D’Adda A and Magna L 1989 Regge calculus as a local theory of the Poincare group Phys. Lett. B 232 457
Gionti G S J 1998 Discrete approaches towards the definition of a quantum theory of gravity arXiv:gr-qc/9812080
Sorkin R 1975 Time-evolution problem in Regge calculus Phys. Rev. D 12 385–396
[25] Freidel L and Speziale S 2010 Twisted geometries: a geometric parametrisation of SU(2) phase space Phys. Rev. D 82 084040
Conrady F and Freidel L 2009 Quantum geometry from phase space reduction J. Math. Phys. 50 123510
Bianchi E, Dona’ P and Speziale S 2011 Polyhedra in loop quantum gravity Phys. Rev. D 83 044035
Bianchi E, Magliaro E and Perini C 2009 LQG propagator from the new spin foams Nucl. Phys. B 822 245–69
Rovelli C and Zhang M 2011 Euclidean three-point function in loop and perturbative gravity arXiv:1105.0566
Barrett J W, Fairbairn W J and Hellmann F 2009 Quantum gravity asymptotics from the SU(2) 15j symbol arXiv:0912.4907
Engle J 2011 The Plebanski sectors of the EPRL vertex arXiv:1107.0709
[26] Rovelli C and Speziale S 2011 Lorentz covariance of loop quantum gravity Phys. Rev. D 83 104029
[27] Han M and Krajewski T 2013 Path integral representation of Lorentzian spinfoam model, asymptotics, and simplicial geometries arXiv:1304.5626