CRITERION FOR SURJECTIVITY OF LOCALIZATION IN GALOIS COHOMOLOGY OF A REDUCTIVE GROUP OVER A NUMBER FIELD

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With an appendix by Zev Rosengarten

Abstract. Let $G$ be a connected reductive group over a number field $F$, and let $S$ be a set (finite or infinite) of places of $F$. We give a necessary and sufficient condition for the surjectivity of the localization map from $H^1(F, G)$ to the “direct sum” of the sets $H^1(F_v, G)$ where $v$ runs over $S$. In the appendices, we give a new construction of the abelian Galois cohomology of a reductive group over a field of arbitrary characteristic.

Résumé. Soit $G$ un groupe réductif connexe sur un corps de nombres $F$, et soit $S$ un ensemble (fini ou infini) de places de $F$. On donne une condition nécessaire et suffisante pour la surjectivité de l’application de localisation de $H^1(F, G)$ vers la “somme directe” des ensembles $H^1(F_v, G)$, où $v$ parcourt $S$. Dans les appendices on donne une nouvelle construction de la cohomologie galoisienne abélienne d’un groupe réductif sur un corps de caractère quelconque.

1. Introduction

1.1. Let $G$ be a (connected) reductive group over a number field $F$ (we follow the convention of SGA3, where reductive groups are assumed to be connected). Let $\overline{F}$ be a fixed algebraic closure of $F$. We denote by $\mathcal{V}(F)$ the set of places of $F$. For $v \in \mathcal{V}(F)$, we denote by $F_v$ the completion of $F$ at $v$. We refer to Serre’s book [20] for the definition of the first Galois cohomology set $H^1(F, G)$.

In general, $H^1(F, G)$ is just a pointed set and has no natural groups structure. Let $H^1_{ab}(F, G)$ denote the abelian Galois cohomology group of $G$ introduced in [4, Section 2]; see also Labesse [14, Section 1.3]. This is an abelian group depending functorially on $G$ and $F$. There is a canonical abelianization map

$$\text{ab}: H^1(F, G) \to H^1_{ab}(F, G).$$

We give a new, better construction of $H^1_{ab}(F, G)$ in Appendix A.

Let $S \subseteq \mathcal{V}(F)$ be a subset (finite or infinite). We consider the localization map

$$H^1_{ab}(F, G) \to \prod_{v \in S} H^1_{ab}(F_v, G).$$

In fact this map takes values in the subgroup $\bigoplus_{v \in S} H^1_{ab}(F_v, G) \subseteq \prod_{v \in S} H^1_{ab}(F_v, G)$; see [4, Corollary 4.6]. Thus we obtain a localization map

$$\text{loc}^S_{ab}: H^1_{ab}(F, G) \to \bigoplus_{v \in S} H^1_{ab}(F_v, G).$$

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Similarly, consider the localization map
\[ H^1(F, G) \to \prod_{v \in S} H^1(F_v, G). \]

In fact it takes values in the subset \( \bigoplus_{v \in S} H^1(F_v, G) \) consisting of the families \((\xi_v)_{v \in S}\) with \( \xi_v \in H^1(F_v, G) \) and such that \( \xi_v = 1 \) for all \( v \) except maybe finitely many of them. This well-known fact follows, for instance, from the corresponding assertion for (1.1.1) together with [4, Theorem 5.11 and Corollary 5.4.1]. Thus we obtain a localization map
\[
(1.1.3) \quad \text{loc}_S : H^1(F, G) \to \bigoplus_{v \in S} H^1(F_v, G).
\]

We wish to find conditions under which the localization maps (1.1.2) and (1.1.3) are surjective.

1.2. We denote by \( M = \pi_1(G) \) the algebraic fundamental group of \( G \) (also known as the Borovoi fundamental group of \( G \)) introduced in [4, Section 1], and also introduced by Merkurjev [16, Section 10.1] and Colliot-Thélène [8, Proposition-Definition 6.1]. See Subsection 2.3 for our definition of \( \pi_1(G) \). This is a finitely generated abelian group, on which the absolute Galois group \( \text{Gal}(\overline{F}/F) \) naturally acts. Let \( E/F \) be a finite Galois extension in \( \overline{F} \) such that \( \text{Gal}(\overline{F}/E) \) acts on \( M \) trivially and that \( E \) has no real places. Then the Galois group \( \Gamma := \text{Gal}(E/F) \) naturally acts on \( M \) and on the set of places \( \mathcal{V}(E) \) of the field \( E \).

1.3. We denote by \( \mathfrak{U}^1_{S}(F, G) \) the cokernel of the homomorphism (1.1.2), that is,
\[
\mathfrak{U}^1_{S}(F, G) = \text{coker} \left( \text{loc}_{S}^{\text{ab}} : H^{1}_{\text{ab}}(F, G) \to \bigoplus_{v \in S} H^{1}_{\text{ab}}(F_v, G) \right).
\]

After explaining our notation in Section 2, we compute in Section 3 the finite abelian group \( \mathfrak{U}^1_{S}(F, G) \) in terms of the action of \( \Gamma \) on \( M \) and on \( \mathcal{V}(E) = \mathcal{V}_f(E) \cup \mathcal{V}_C(E) \); see Corollary 3.8. See Subsection 2.4 for the notations \( \mathcal{V}_f \) and \( \mathcal{V}_C \).

Concerning the map \( \text{loc}_S \) of (1.1.3), in Section 3 we compute the image of this map; see Main Theorem 3.7. Using this result, we give a criterion (necessary and sufficient condition) for the map \( \text{loc}_S \) to be surjective; see Corollary 3.9. This is also a criterion for the vanishing of \( \mathfrak{U}^1_{S}(F, G) \). Again, our criterion is given in terms of the action of \( \Gamma \) on \( M \) and on \( \mathcal{V}(E) = \mathcal{V}_f(E) \cup \mathcal{V}_C(E) \).

Using this criterion, we give a simple proof of the result of Borel and Harder [2, Theorem 1.7] (see also Prasad and Rapinchuk [19, Proposition 1]) on the surjectivity of the map \( \text{loc}_S \) when \( G \) is semisimple and there exists a finite place \( v_0 \) of \( F \) outside \( S \); see Proposition 3.14 below.

Let \( \Gamma \) be a finite group. In Section 4 we construct an exact sequence arising from a short exact sequence of \( \Gamma \)-modules. In Section 5 using this exact sequence and Main Theorem 3.7 we generalize a result of Prasad and Rapinchuk giving a sufficient condition for the surjectivity of the localization map \( \text{loc}_S \) when \( G \) is reductive, in terms of the radical (largest central torus) of \( G \); see Theorem 5.1. As a particular case, we obtain the following corollary.

**Corollary 1.4** (of Theorem 5.1). Let \( G \) be a reductive group over a number field \( F \), and let \( C \) denote the radical of \( G \) (the identity component of the center of \( G \)). Let \( S \subset \mathcal{V}(F) \) be a set of places of \( F \). Assume that the \( F \)-torus \( C \) splits over a finite Galois extension of \( F \) of prime degree \( p \) and that there exists a finite place \( v_0 \) in the complement \( S^C := \mathcal{V}(F) \setminus S \) of \( S \) such that \( C \) does not split over \( F_{v_0} \). Then the localization map \( \text{loc}_S \) of (1.1.3) is surjective.

For \( p = 2 \) this assertion was earlier proved by Prasad and Rapinchuk [19, Proposition 2(b)].

1.5. Let \( G \) be a reductive group over a field \( F \) of characteristic 0. In [4], the author defined the abelian group \( H^{1}_{\text{ab}}(F, G) \) as a set in a canonical way as the Galois hypercohomology of a certain
crossed module. However, the definition of the structure of abelian group on \( H^1_{\text{ab}}(F,G) \) in [4] was complicated. In Appendix A we define \( H^1_{\text{ab}}(F,G) \) (in arbitrary characteristic) following the letter of Breen to the author [7] and the article by Noohi [17] (written at the author’s request), as the Galois hypercohomology \( H^1(F,G_{\text{ab}}) \) of a certain stable crossed module, that is, a crossed module endowed with a symmetric braiding. The structure of abelian group comes from the symmetric braiding. Note that our specific crossed module and specific symmetric braiding were constructed by Deligne [11].

In Appendix B Zev Rosengarten shows that certain equivalences of crossed modules of algebraic groups over a field \( F \) of arbitrary characteristic induce equivalences on \( F_s \)-points where \( F_s \) is a separable closure of \( F \). This permits us to use in Appendix A the Galois hypercohomology of these crossed modules rather than fppf hypercohomology.

2. Notation

2.1. Let \( A \) be an abelian group. We denote by \( A_{\text{Tors}} \) the torsion subgroup of \( A \). We set \( A_{t.f.} = A/A_{\text{Tors}} \), which is a torsion-free group.

2.2. Let \( \Gamma \) be a finite group, and let \( B \) be a \( \Gamma \)-module. We denote by \( B_{\Gamma} \) the group of coinvariants of \( \Gamma \) in \( B \), that is,

\[
B_{\Gamma} = B / \left\{ \sum_{\gamma \in \Gamma} (\gamma^{-1}b_{\gamma} - b_{\gamma}) \mid b_{\gamma} \in B \right\}.
\]

We write \( B_{\Gamma,\text{Tors}} := (B_{\Gamma})_{\text{Tors}} \) (which is the torsion subgroup of \( B_{\Gamma} \)), \( B_{\Gamma,\text{t.f.}} = B_{\Gamma}/B_{\Gamma,\text{Tors}} \) (which is a torsion-free group).

2.3. Let \( G \) be a reductive group over a field \( F \). Let \([G,G]\) denote the commutator subgroup of \( G \), which is semisimple. Let \( G^{\text{sc}} \) denote the universal cover of \([G,G]\), which is simply connected; see [3, Proposition (2.24)(ii)] or [10, Corollary A.4.11]. Following Deligne [11, Section 0.2], we consider the composite homomorphism

\[
\rho: G^{\text{sc}} \rightarrow [G,G] \hookrightarrow G,
\]

which in general is neither injective nor surjective.

For a maximal torus \( T \subseteq G \), we write \( T^{\text{sc}} = \rho^{-1}(T) \subseteq G^{\text{sc}} \) and consider the natural homomorphism

\[
\rho: T^{\text{sc}} \rightarrow T.
\]

We consider the algebraic fundamental group \( M = \pi_1(G) \) of \( G \) defined by

\[
\pi_1(G) = X_*(T)/\rho_*(X_*(T^{\text{sc}}))
\]

where \( X_* \) denotes the cocharacter group. The Galois group \( \text{Gal}(F_s/F) \) naturally acts on \( M \), and the \( \text{Gal}(F_s/F) \)-module \( M \) is well defined (does not depend on the choice of \( T \) up to a transitive system of isomorphisms); see [3, Lemma 1.2].

2.4. From now on (except for the appendices), \( F \) is a number field. We denote by \( \mathcal{V}(F) \), \( \mathcal{V}_f(F) \), \( \mathcal{V}_{\infty}(F) \), \( \mathcal{V}_R(F) \), and \( \mathcal{V}_C(F) \) the sets of all places of \( F \), of finite places, of infinite places, of real places, and of complex places, respectively.

Let \( E/F \) be a finite Galois extension of number fields with Galois group \( \Gamma = \text{Gal}(E/F) \); then \( \Gamma \) acts on \( \mathcal{V}(F) \). If \( w \in \mathcal{V}(F) \), we write \( \Gamma_w \) for the stabilizer of \( w \) in \( \Gamma \); then \( \Gamma_w \cong \text{Gal}(E_w/F_v) \) where \( v \in \mathcal{V}(F) \) is the restriction of \( w \) to \( F \).
3. Main theorem

In this section we state and prove Main Theorem 3.7 computing the images of the localization maps \(1.1.2\) and \(1.1.3\). We deduce Corollary 3.8 computing the group \(\Psi^1_F(G, F)\), and Corollary 3.9 giving a necessary and sufficient condition for the surjectivity of the localization map \((1.1.3)\).

3.1. Let \(G\) be a reductive group over a number field \(F\), and let \(v \in \mathcal{V}_f(F)\) be a finite place of \(F\). In [4] we computed \(H^1_{ab}(F_v, G)\). Write \(M = \pi(G)\). Let \(E/F\) be a finite Galois extension in \(\mathcal{F}\) such that \(\text{Gal}(\mathcal{F}/E)\) acts on \(M\) trivially and that \(E\) has no real places. Write \(\Gamma = \text{Gal}(E/F)\).

**Theorem 3.2** ([4] Proposition 4.1(i) and Corollary 5.4.1). With the notation and assumptions of Subsection 3.7, for any finite place \(v\) of \(F\) there is a canonical isomorphism of abelian groups

\[
\alpha_v^{ab}: H^1_{ab}(F_v, G) \xrightarrow{\sim} M_{\Gamma_v, \text{Tors}}
\]

where \(w\) is a place of \(E\) over \(v\), and a canonical bijection

\[
ab_v: H^1(F_v, G) \rightarrow H^1_{ab}(F_v, G).
\]

3.3. Let \(v\) be a finite place of \(F\). We have a surjective (even bijective) map

\[
\alpha_v: H^1(F_v, G) \xrightarrow{ab} H^1_{ab}(F_v, G) \xrightarrow{\alpha_v^{ab}} M_{\Gamma_v, \text{Tors}}.
\]

We consider two composite maps with the same image

\[
\begin{align*}
\lambda_v^{ab}: H^1_{ab}(F_v, G) \xrightarrow{\alpha_v^{ab}} M_{\Gamma_v, \text{Tors}} & \xrightarrow{\omega_v} M_{\Gamma, \text{Tors}}, \\
\lambda_v: H^1(F_v, G) \xrightarrow{\alpha_v} M_{\Gamma_v, \text{Tors}} & \xrightarrow{\omega_v} M_{\Gamma, \text{Tors}},
\end{align*}
\]

where \(\omega_v: M_{\Gamma_v, \text{Tors}} \rightarrow M_{\Gamma, \text{Tors}}\) is the homomorphism induced by the inclusion \(\Gamma_v \hookrightarrow \Gamma\). Since the maps \(\alpha_v^{ab}\) and \(\alpha_v\) are surjective (even bijective), and \(\omega_v\) is a homomorphism, we see that the set \(\text{im } \lambda_v^{ab} = \text{im } \lambda_v\) is a subgroup of \(M_{\Gamma, \text{Tors}}\), namely, \(\text{im } \lambda_v^{ab} = \text{im } \lambda_v = \text{im } \omega_v\).

Let \(v \in \mathcal{V}_f(F)\) be a complex place. We have zero maps

\[
\lambda_v^{ab}: H^1_{ab}(F_v, G) = \{1\} \rightarrow \{0\} \subseteq M_{\Gamma_v, \text{Tors}}, \quad \lambda_v: H^1(F_v, G) = \{1\} \rightarrow \{0\} \subseteq M_{\Gamma, \text{Tors}}.
\]

Clearly, the set \(\text{im } \lambda_v^{ab} = \text{im } \lambda_v\) is a subgroup of \(M_{\Gamma, \text{Tors}}\), namely, the subgroup \(\{0\}\).

3.4. Let \(v \in \mathcal{V}_R(F)\) be a real place; then \(\Gamma_v\) is a group of order 2, \(\Gamma_v = \{1, \gamma\}\) where \(\gamma = \gamma_w\) induces the nontrivial automorphism of \(E_w\) over \(F_v\). We consider the Tate cohomology group

\[
\tilde{H}^{-1}(\Gamma_w, M) = \{m \in M \mid \gamma m = -m\} / \{m' - \gamma m' \mid m' \in M\}.
\]

We see immediately that the abelian group \(\tilde{H}^{-1}(\Gamma_w, M)\) naturally embeds into \(M_{\Gamma_w}\). If \(m \in M\) is a \((-1)\)-cocycle, that is, \(\gamma m = -m\), then \(2m = m + m = -\gamma m\), whence \(2 \cdot \tilde{H}^{-1}(\Gamma_w, M) = 0\). We conclude that \(\tilde{H}^{-1}(\Gamma_w, M)\) naturally embeds into \(M_{\Gamma_w, \text{Tors}}\).

There is a canonical surjective map of Kottwitz [13, Theorem 1.2] (see also [4] Theorem 5.4.1)

\[
\text{ab}_v: H^1(F_v, G) \rightarrow H^1_{ab}(F_v, G),
\]

a canonical isomorphism of [3] Proposition 8.21

\[
H^1_{ab}(F_v, G) \xrightarrow{\sim} \tilde{H}^{-1}(\Gamma_w, M),
\]

and a canonical embedding

\[
\tilde{H}^{-1}(\Gamma_w, M) \hookrightarrow M_{\Gamma_w, \text{Tors}}.
\]
Thus we obtain composite maps
\[ \alpha_v^{ab} : H^1_{ab}(F_v, G) \xrightarrow{\sim} \hat{H}^{-1}(\Gamma_w, M) \xrightarrow{\alpha_v} M_{\Gamma_w, \text{Tors}}, \]
\[ \alpha_v : H^1(F_v, G) \rightarrow H^1_{ab}(F_v, G) \xrightarrow{\sim} \hat{H}^{-1}(\Gamma_w, M) \xrightarrow{\alpha_v} M_{\Gamma_w, \text{Tors}}, \]
with the same image \( \text{im} \alpha_v^{ab} = \text{im} \alpha_v \), which is a subgroup of \( M_{\Gamma_w, \text{Tors}} \). Consider the composite maps with the same image
\[ \lambda_v^{ab} : H^1_{ab}(F_v, G) \xrightarrow{\alpha_v^{ab}} M_{\Gamma_w, \text{Tors}} \xrightarrow{\omega_v} M_{\Gamma, \text{Tors}}, \]
\[ \lambda_v : H^1(F_v, G) \xrightarrow{\alpha_v} M_{\Gamma_w, \text{Tors}} \xrightarrow{\omega_v} M_{\Gamma, \text{Tors}}. \]
Since the set \( \text{im} \alpha_v^{ab} = \text{im} \alpha_v \) is a subgroup of \( M_{\Gamma_w, \text{Tors}} \), and \( \omega_v \) is a homomorphism, we conclude that the set \( \text{im} \lambda_v^{ab} = \text{im} \lambda_v \) is a subgroup of \( M_{\Gamma, \text{Tors}} \), namely, \( \text{im} \lambda_v^{ab} = \text{im} \lambda_v = \omega_v(\hat{H}^{-1}(\Gamma_w, M)) \).

**Lemma 3.5.** Let \( S \subseteq \mathcal{V}(F) \) be any subset, finite or infinite. Consider the summation maps
\[ \Sigma_S^{ab} : \bigoplus_{v \in S} H^1_{ab}(F_v, G) \xrightarrow{\sim} M_{\Gamma, \text{Tors}}, \quad \xi_S^{ab} = (\xi_v^{ab})_{v \in S} \mapsto \sum_{v \in S} \lambda_v^{ab}(\xi_v), \]
\[ \Sigma_S : \bigoplus_{v \in S} H^1(F_v, G) \xrightarrow{\sim} M_{\Gamma, \text{Tors}}, \quad \xi_S = (\xi_v)_{v \in S} \mapsto \sum_{v \in S} \lambda_v(\xi_v). \]
Then the sets \( \text{im} \Sigma_S^{ab} \) and \( \text{im} \Sigma_S \) are subgroups of \( M_{\Gamma, \text{Tors}} \), and they are equal.

**Proof.** Indeed, we have
\[ \text{im} \Sigma_S^{ab} = \text{im} \Sigma_S = \langle \text{im} \lambda_v \rangle_{v \in S} \quad \text{where} \quad \text{im} \lambda_v = \begin{cases} \text{im} \omega_v & \text{if } v \in \mathcal{V}_f(F), \\ \omega_v(\hat{H}^{-1}(\Gamma_w, M)) & \text{if } v \in \mathcal{V}_R(F), \\ 0 & \text{if } v \in \mathcal{V}_C(F). \end{cases} \]
Here we write \( \langle \text{im} \lambda_v \rangle_{v \in S} \) for the subgroup of \( M_{\Gamma, \text{Tors}} \) generated by the subgroups \( \text{im} \lambda_v \) for \( v \in S \).

**Theorem 3.6.** The following sequences are exact:
\[ H^1_{ab}(F, G) \xrightarrow{\text{loc}_S^{ab}} \bigoplus_{v \in \mathcal{V}} H^1_{ab}(F_v, G) \xrightarrow{\Sigma_S^{ab}} M_{\Gamma, \text{Tors}}, \]
\[ H^1(F, G) \xrightarrow{\text{loc}_S} \bigoplus_{v \in \mathcal{V}} H^1(F_v, G) \xrightarrow{\Sigma_S} M_{\Gamma, \text{Tors}}, \]
where for brevity we write \( \mathcal{V} \) for \( \mathcal{V}(F) \).

Here (3.6.1) is an exact sequence of abelian groups, and (3.6.2) is an exact sequence of pointed sets.

**Proof.** In view of [4, Proposition 4.8], exact sequence (3.6.1) is actually a part of the exact sequence [4 (4.3.1)]. For (3.6.2), see [13, Proposition 2.6] or [4, Theorem 5.15].

**Main Theorem 3.7.** Let \( G \) be a reductive group over a number field \( F \). Let \( S \subseteq \mathcal{V} := \mathcal{V}(F) \) be a subset. Write \( S^c = \mathcal{V} \setminus S \), the complement of \( S \) in \( \mathcal{V} \). Then:
\[ \text{im} \text{loc}_S^{ab} = \left\{ \xi_S^{ab} \in \bigoplus_{v \in S} H^1_{ab}(F_v, G) \mid \Sigma_S^{ab}(\xi_S) \in \text{im} \Sigma_S^{ab} \cap \text{im} \Sigma_S \right\}, \]
\[ \text{im} \text{loc}_S = \left\{ \xi_S \in \bigoplus_{v \in S} H^1(F_v, G) \mid \Sigma_S(\xi_S) \in \text{im} \Sigma_S \cap \text{im} \Sigma_S^c \right\}. \]
Proof. By Lemma 3.5 the sets \( \text{im} \Sigma_{\mathcal{S}}^{ab} \) and \( \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}} \) are (equal) subgroups of \( M_{T_{\text{tors}}} \), and therefore it suffices to prove (3.7.1) with \((- \text{im} \Sigma_{\mathcal{S}}^{ab})\) instead of \( \text{im} \Sigma_{\mathcal{S}}^{ab} \), and to prove (3.7.2) with \((- \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}})\) instead of \( \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}} \). Now the corresponding assertions follow easily from the exactness of (3.6.1) and (3.6.2), respectively.

For the reader’s convenience, we provide an easy proof of (3.7.2) with \((- \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}})\) instead of \( \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}} \). Let

\[
\xi_S = (\xi_v)_{v \in S} \in \text{im} \text{loc}_S \subseteq \bigoplus_{v \in S} H^1(F_v, G),
\]

that is, \( \xi_S = \text{loc}_S(\xi) \) for some \( \xi \in H^1(F, G) \). Write \( \eta_{\mathcal{S}}^{\mathcal{C}} = (\eta_v)_{v \in \mathcal{S}^\mathcal{C}} = \text{loc}_{\mathcal{S}}(\xi) \). Since the sequence (3.6.2) is exact, we have \( (\Sigma_{\mathcal{V}} \circ \text{loc}_{\mathcal{V}})(\xi) = 0 \), whence

\[
\Sigma_S(\xi_S) + \Sigma_{\mathcal{S}}^{\mathcal{C}}(\eta_{\mathcal{S}}^{\mathcal{C}}) = 0 \quad \text{and} \quad \Sigma_S(\xi_S) = -\Sigma_{\mathcal{S}}^{\mathcal{C}}(\eta_{\mathcal{S}}^{\mathcal{C}}).
\]

We conclude that \( \Sigma_S(\xi_S) \in \text{im} \Sigma_S \cap (- \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}}) \), as required.

Conversely, let an element \( \xi_S = (\xi_v)_{v \in S} \in \bigoplus_{v \in S} H^1(F_v, G) \) be such that

\[
\Sigma_S(\xi_S) \in \text{im} \Sigma_S \cap (- \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}}).
\]

Write \( a = \Sigma_S(\xi_S) \). Then \(-a \in \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}} \), that is,

\[
-a = \Sigma_{\mathcal{S}}^{\mathcal{C}}(\eta_{\mathcal{S}}^{\mathcal{C}}) \quad \text{for some} \quad \eta_{\mathcal{S}}^{\mathcal{C}} = (\eta_v)_{v \in \mathcal{S}^\mathcal{C}} \in \bigoplus_{v \in \mathcal{S}^\mathcal{C}} H^1(F_v, G).
\]

Define

\[
\zeta_{\mathcal{V}} = (\zeta_v)_{v \in \mathcal{V}} \in \bigoplus_{v \in \mathcal{V}} H^1(F_v, G), \quad \zeta_v = \begin{cases} 
\xi_v & \text{if } v \in S, \\
\eta_v & \text{if } v \in S^{\mathcal{C}}.
\end{cases}
\]

Then

\[
\Sigma_{\mathcal{V}}(\zeta_{\mathcal{V}}) = a + (-a) = 0.
\]

Since the sequence (3.6.2) is exact, we have \( \zeta_{\mathcal{V}} = \text{loc}_{\mathcal{V}}(\zeta) \) for some \( \zeta \in H^1(F, G) \). Then \( \text{loc}_S(\zeta) = \xi_S \), whence \( \xi_S \in \text{im} \text{loc}_S \), as required. \( \square \)

**Corollary 3.8.** The homomorphism

\[
\chi_S^{ab} : \bigoplus_{v \in \mathcal{V}} H^1_{ab}(F_v, G) \xrightarrow{\Sigma_S^{ab}} \text{im} \Sigma_S^{ab} \longrightarrow \text{im} \Sigma_S^{ab} / (\text{im} \Sigma_S^{ab} \cap \text{im} \Sigma_S^{ab})
\]

induces a canonical isomorphism

\[
\Psi_S^1(F, G) \xrightarrow{\sim} \text{im} \Sigma_S^{ab} / (\text{im} \Sigma_S^{ab} \cap \text{im} \Sigma_S^{ab}).
\]

**Proof.** The homomorphism \( \chi_S^{ab} \) is clearly surjective, and by Theorem 3.7 its kernel is the image \( \text{im} \text{loc}_S^{ab} \) of the localization homomorphism \( \text{loc}_S^{ab} \) of (1.1.2). The corollary follows. \( \square \)

**Corollary 3.9.** The localization map \( \text{loc}_S \) of (1.1.3) is surjective if and only if

\[
(3.9.1) \quad \text{im} \Sigma_S \subseteq \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}}.
\]

**Proof.** Consider the map

\[
\chi_S : \bigoplus_{v \in S} H^1(F_v, G) \xrightarrow{\Sigma_S} \text{im} \Sigma_S \longrightarrow \text{im} \Sigma_S / (\text{im} \Sigma_S \cap \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}}).
\]

By Lemma 3.5 the sets \( \text{im} \Sigma_S = \text{im} \Sigma_S^{ab} \) and \( \text{im} \Sigma_S \cap \text{im} \Sigma_{\mathcal{S}}^{\mathcal{C}} = \text{im} \Sigma_S^{ab} \cap \text{im} \Sigma_{\mathcal{S}}^{ab} \) are abelian groups. The morphism of pointed sets \( \chi_S \) is clearly surjective, and by Theorem 3.7 its kernel is \( \text{im} \text{loc}_S \). We see that the following assertions are equivalent:
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(a) the map \( \text{loc}_S \) is surjective, that is, \( \text{im} \text{loc}_S = \bigoplus_{v \in S} H^1(F_v, G) \);
(b) \( \ker \chi_S = \bigoplus_{v \in S} H^1(F_v, G) \);
(c) \( \#(\text{im} \chi_S) = 1 \);
(d) \( \text{im} \Sigma_S \cap \text{im} \Sigma_{S^c} = \text{im} \Sigma_S \);
(e) \( \text{im} \Sigma_S \subseteq \text{im} \Sigma_{S^c} \).

This completes the proof. \( \square \)

Remark 3.10. Since by Lemma 3.5 we have \( \text{im} \Sigma_{ab} = \text{im} \Sigma_S \) and \( \text{im} \Sigma_{ab}^c = \text{im} \Sigma_{S^c} \), we see from (d) in the proof above and from Corollary 3.8 that the localization map \( \text{loc}_S \) of \( (1.1.3) \) is surjective if and only if \( \Psi^1_S(F, G) = \{1\} \).

Corollary 3.11. Let \( v_0 \in \mathcal{V}(F) \), \( S = \mathcal{V}(F) \setminus \{v_0\} \). Then the localization map \( \text{loc}_S \) of \( (1.1.3) \) is surjective if and only if

\[
\text{im} \lambda_v \subseteq \text{im} \lambda_{v_0} \quad \text{for all } v \in \mathcal{V}(F).
\]

Proof. Indeed, in our case condition (3.11.1) is equivalent to (3.9.1), and we conclude by Corollary 3.9. \( \square \)

Corollary 3.12. For a subset \( S \subset \mathcal{V}(F) \), let \( v_0 \in S^c \), and assume that

\[
\text{im} \lambda_v \subseteq \text{im} \lambda_{v_0} \quad \text{for all } v \in S.
\]

Then the localization map \( \text{loc}_S \) of \( (1.1.3) \) is surjective.

Proof. Indeed, (3.12.1) implies (3.9.1), and we conclude by Corollary 3.9. \( \square \)

Corollary 3.13. Let \( v_0 \in S^c \), and assume that the map \( \lambda_{v_0} : H^1(F_{v_0}, G) \to M_{\Gamma, \text{Tors}} \) is surjective. Then the localization map \( \text{loc}_S \) of \( (1.1.3) \) is surjective.

Proof. Indeed, then

\[
\text{im} \Sigma_S \subseteq M_{\Gamma, \text{Tors}} = \text{im} \lambda_{v_0} \subseteq \text{im} \Sigma_{S^c},
\]

and we conclude by Corollary 3.9. \( \square \)

Proposition 3.14 (Borel and Harder [2, Theorem 1.7]). Let \( G \) be a semisimple group over a number field \( F \), and let \( S \subset \mathcal{V}(F) \) be a subset such that the complement \( S^c \) of \( S \) contains a finite place \( v_0 \in \mathcal{V}_f(F) \). Then the localization map \( \text{loc}_S \) of \( (1.1.3) \) is surjective.

Proof. Since \( G \) is semisimple, the \( \Gamma \)-module \( M \) is finite, and so are the groups \( M_{\Gamma} \) and \( M_{\Gamma, w} \) where \( w \) is a place of \( E \) over \( v_0 \). It follows that

\[
M_{\Gamma, \text{Tors}} = M_{\Gamma} \quad \text{and} \quad M_{\Gamma, \text{Tors}} = M_{\Gamma}.
\]

The natural homomorphism \( M_{\Gamma, \text{Tors}} \to M_{\Gamma} \) is clearly surjective. Therefore, the homomorphism

\[
\omega_{v_0} : M_{\Gamma, \text{Tors}} = M_{\Gamma, w} \to M_{\Gamma} = M_{\Gamma, \text{Tors}}
\]

is surjective. Since \( v_0 \) is finite, we have \( \text{im} \lambda_{v_0} = \text{im} \omega_{v_0} \), whence the map \( \lambda_{v_0} \) is surjective. We conclude by Corollary 3.13. \( \square \)
4. Exact sequence

In this section we construct an exact sequence that we shall use in Section 5.

**Theorem 4.1.** A finite group $\Gamma$ and a short exact sequence of $\Gamma$-modules

\[
(4.1.1) \quad 0 \to B_1 \xrightarrow{i} B_2 \xrightarrow{j} B_3 \to 0
\]

give rises to an exact sequence

\[
(4.1.2) \quad (B_1)_{\Gamma,\text{tors}} \xrightarrow{i_*} (B_2)_{\Gamma,\text{tors}} \xrightarrow{j_*} (B_3)_{\Gamma,\text{tors}} \xrightarrow{\delta} Q/\mathbb{Z} \otimes \mathbb{Z} (B_1)_\Gamma \xrightarrow{i} Q/\mathbb{Z} \otimes \mathbb{Z} (B_2)_\Gamma \xrightarrow{j} Q/\mathbb{Z} \otimes \mathbb{Z} (B_3)_\Gamma \to 0
\]

depending functorially on $\Gamma$ and on the sequence (4.1.1).

**4.2.** We specify the homomorphism $\delta$. Let $x_3 \in B_3$ be such that the image $(x_3)_\Gamma$ of $x_3$ in $(B_3)_\Gamma$ is contained in $(B_3)_{\Gamma,\text{tors}}$. This means that there exist $n \in \mathbb{Z}_{>0}$ and $y_{3,\gamma} \in B_3$ such that

\[
nx_3 = \sum_{\gamma \in \Gamma} (y_{3,\gamma} - y_{3,\gamma}).
\]

We lift $x_3$ to some $x_2 \in B_2$, we lift each $y_{3,\gamma}$ to some $y_{2,\gamma} \in B_2$, and we consider the element

\[
z_2 = nx_2 - \sum_{\gamma \in \Gamma} (y_{2,\gamma} - y_{2,\gamma}).
\]

Then $j(z_2) = 0 \in B_3$, whence $z_2 = i(z_1)$ for some $z_1 \in B_1$. We consider the image $(z_1)_{\Gamma,\text{t.f.}}$ of $z_1 \in B_1$ in $(B_1)_{\Gamma,\text{t.f.}}$, and we put

\[
delta((x_3)_\Gamma) = \frac{1}{n} \otimes (z_1)_{\Gamma,\text{t.f.}} \in Q/\mathbb{Z} \otimes \mathbb{Z} (B_1)_{\Gamma,\text{t.f.}} = Q/\mathbb{Z} \otimes \mathbb{Z} (B_1)_\Gamma
\]

where we write $\frac{1}{n}$ for the image in $Q/\mathbb{Z}$ of $\frac{1}{n} \in Q$.

Below we give the proof of Theorem 4.1 suggested by Vladimir Hinich (private communication). For another proof, due to Alexander Petrov, see [18].

**4.3.** Proof Theorem 4.1 due to Vladimir Hinich. The functor from the category $\Gamma$-modules to the category of abelian groups

\[
B \rightsquigarrow Q/\mathbb{Z} \otimes \mathbb{Z} B_\Gamma
\]

is the same as

\[
B \rightsquigarrow Q/\mathbb{Z} \otimes \Lambda B
\]

where $\Lambda = \mathbb{Z}[\Gamma]$ is the group ring of $\Gamma$. From the short exact sequence of $\Gamma$-modules (4.1.1), we obtain a long exact sequence

\[
\cdots \to \text{Tor}^1_\Lambda(Q/\mathbb{Z}, B_1) \xrightarrow{i_*} \text{Tor}^1_\Lambda(Q/\mathbb{Z}, B_2) \xrightarrow{j_*} \text{Tor}^1_\Lambda(Q/\mathbb{Z}, B_3) \xrightarrow{\delta} Q/\mathbb{Z} \otimes \Lambda B_1 \xrightarrow{i_*} Q/\mathbb{Z} \otimes \Lambda B_2 \xrightarrow{j} Q/\mathbb{Z} \otimes \Lambda B_3 \to 0
\]

depending functorially on $\Gamma$ and on (4.1.1); see Weibel [21]. Now Theorem 4.1 follows from the next proposition. \hfill \Box

**Proposition 4.4.** For a finite group $\Gamma$ and a $\Gamma$-module $B$, there is a canonical and functorial isomorphism

\[
\text{Tor}^1_\Lambda(Q/\mathbb{Z}, B) \xrightarrow{\sim} B_{\Gamma,\text{tors}}
\]

where $\Lambda = \mathbb{Z}[\Gamma]$. 

Proof. Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

regarded as a short exact sequence of $\Gamma$-modules with trivial action of $\Gamma$. Tensoring with $B$, we obtain a long exact sequence

$$(4.4.1) \quad \cdots \to \text{Tor}_1^\Lambda(\mathbb{Q}, B) \to \text{Tor}_1^\Lambda(\mathbb{Q}/\mathbb{Z}, B) \to \mathbb{Z} \otimes_\Lambda B \to \mathbb{Q} \otimes_\Lambda B \to \mathbb{Q}/\mathbb{Z} \otimes_\Lambda B \to 0.$$

We have canonical isomorphisms

$$\mathbb{Z} \otimes_\Lambda B = B_\Gamma \quad \text{and} \quad \ker [\mathbb{Z} \otimes_\Lambda B \to \mathbb{Q} \otimes_\Lambda B] = B_{\Gamma, \text{Tors}}.$$

By Lemma 4.5 below, we have $\text{Tor}_1^\Lambda(\mathbb{Q}, B) = 0$, and the proposition follows from (4.4.1).

□

Lemma 4.5. For a finite group $\Gamma$ and any $\Gamma$-module $B$, we have

$$\text{Tor}_1^\Lambda(\mathbb{Q}, B) = 0$$

where $\Lambda = \mathbb{Z}[\Gamma]$.

Proof. Let

$$P_* : \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

be a $\Lambda$-free resolution of the trivial $\Gamma$-module $\mathbb{Z}$, for example, the standard complex; see Atiyah and Wall [1, Section 2]. Tensoring with $\mathbb{Q}$ over $\mathbb{Z}$, we obtain a flat resolution of $\mathbb{Q}$

$$\cdots \to \mathbb{Q} \otimes_\mathbb{Z} P_2 \to \mathbb{Q} \otimes_\mathbb{Z} P_1 \to \mathbb{Q} \otimes_\mathbb{Z} P_0 \to \mathbb{Q} \to 0.$$

Tensoring with $B$ over $\Lambda = \mathbb{Z}[\Gamma]$, we obtain the complex $(\mathbb{Q} \otimes_\mathbb{Z} P_*) \otimes_\Lambda B$:

$$(4.5.1) \quad \cdots \to (\mathbb{Q} \otimes_\mathbb{Z} P_2) \otimes_\Lambda B \to (\mathbb{Q} \otimes_\mathbb{Z} P_1) \otimes_\Lambda B \to (\mathbb{Q} \otimes_\mathbb{Z} P_0) \otimes_\Lambda B \to \mathbb{Q} \otimes_\Lambda B \to 0.$$

By definition, $\text{Tor}_1^\Lambda(\mathbb{Q}, B)$ is the first homology group of this complex.

However, we can obtain the complex (4.5.1) from $P_*$ by tensoring first with $B$ over $\Lambda$, and after that with $\mathbb{Q}$ over $\mathbb{Z}$:

$$\mathbb{Q} \otimes_\mathbb{Z} (P_* \otimes_\Lambda B) \cong (\mathbb{Q} \otimes_\mathbb{Z} P_*) \otimes_\Lambda B.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, we obtain canonical isomorphisms

$$\text{Tor}_1^\Lambda(\mathbb{Q}, B) \cong \mathbb{Q} \otimes_\mathbb{Z} \text{Tor}_1^\Lambda(\mathbb{Z}, B) = \mathbb{Q} \otimes_\mathbb{Z} H_1(\Gamma, B).$$

Now, since the group $\Gamma$ is finite, the abelian group $H_1(\Gamma, B)$ is killed by multiplication by $\#\Gamma$; see, for instance, Atiyah and Wall [1, Section 6, Corollary 1 of Proposition 8]. It follows that $\mathbb{Q} \otimes_\mathbb{Z} H_1(\Gamma, B) = 0$. Thus $\text{Tor}_1^\Lambda(\mathbb{Q}, B) = 0$, which completes the proofs of Lemma 4.5, Proposition 4.4 and Theorem 4.1.

□

Alternatively, one can check directly that the map $\delta$ constructed in Subsection 4.2 is well-defined (does not depend on the choices made) and that the sequence (4.1.2) is exact.

5. Surjectivity for a reductive group with nice radical

In this section we prove the following theorem that gives a sufficient condition for the surjectivity of the localization map (1.1.3) for a reductive $F$-group $G$ in terms of the radical (largest central torus) of $G$.
**Theorem 5.1.** Let $G$ be a reductive group over a number field $F$, and let $C$ denote the radical of $G$. Write $\overline{G} = G/C$, which is a semisimple group, and consider the short exact sequence of fundamental groups [4] Lemma 1.5]

$$0 \to M_C \to M \to \overline{M} \to 0$$

where

$$M_C = \pi_1(C) = X_*(C), \quad M = \pi_1(G), \quad \overline{M} = \pi_1(\overline{G}).$$

We define $\Gamma = \text{Gal}(E/F)$ for $M$ as in Subsection 3.7. Let $S \subset \mathcal{V}(F)$ be a subset, and assume that $S^\mathbf{E}$ contains a finite place $v_0$ such that

$$\text{im}[\Gamma_w \to \text{Aut } M_C] = \text{im}[\Gamma \to \text{Aut } M_C]$$

where $w$ is a place of $E$ over $v_0$. Then the localization map $\text{loc}_S$ of (1.1.3) is surjective.

**Proof.** It follows from (5.1.1) that $(M_C)_{\Gamma_w} = (M_C)_{\Gamma}$, whence

$$(M_C)_{\Gamma_w, \text{Tors}} = (M_C)_{\Gamma, \text{Tors}} \quad \text{and} \quad Q/Z \otimes_{\mathbb{Z}} (M_C)_{\Gamma_w} = Q/Z \otimes_{\mathbb{Z}} (M_C)_{\Gamma}.$$

Using Theorem 4.1, we construct an exact commutative diagram

$$
\begin{array}{ccc}
(M_C)_{\Gamma_w, \text{Tors}} & \longrightarrow & M_{\Gamma_w, \text{Tors}} \\
\downarrow & & \downarrow \omega \\
(M_C)_{\Gamma, \text{Tors}} & \longrightarrow & M_{\Gamma, \text{Tors}}
\end{array}
\longrightarrow
\begin{array}{ccc}
\overline{M}_{\Gamma_w, \text{Tors}} & \longrightarrow & Q/Z \otimes_{\mathbb{Z}} (M_C)_{\Gamma_w} \\
\downarrow & & \downarrow \overline{\omega} \\
\overline{M}_{\Gamma, \text{Tors}} & \longrightarrow & Q/Z \otimes_{\mathbb{Z}} (M_C)_{\Gamma}
\end{array}
$$

Since $\overline{G}$ is semisimple, its algebraic fundamental group $\overline{M}$ is finite, and therefore the homomorphism $\overline{\omega}$ in the diagram above is surjective; see the proof of Proposition 3.14. By a four lemma, the homomorphism

$$\omega = \omega_{v_0} : M_{\Gamma_w, \text{Tors}} \to M_{\Gamma, \text{Tors}}$$

is surjective as well. Since $v_0$ is finite, the map

$$\alpha_{v_0} : H^1(F_{v_0}, G) \to H^1_{\text{ab}}(F_{v_0}, G) \to M_{\Gamma_w, \text{Tors}}$$

is bijective, and therefore the map

$$\lambda_{v_0} : H^1(F_{v_0}, G) \to H^1_{\text{ab}}(F_{v_0}, G) \to M_{\Gamma_w, \text{Tors}} \xrightarrow{\omega} M_{\Gamma, \text{Tors}}$$

is surjective. We conclude by Corollary 3.13. \hfill $\square$

**Corollary 5.2** (Prasad and Rapinchuk [19, Proposition 2(a)]). Let $G$ be a reductive group over a number field $F$, and let $C$ denote the radical of $G$. Assume that the $F$-torus $C$ is split and that $S^\mathbf{E}$ contains a finite place $v_0$. Then the localization map $\text{loc}_S$ of (1.1.3) is surjective.

**Proof.** We define $E$, $\Gamma$, and $\Gamma_w$ for $M = \pi_1(G)$ as in Subsection 5.1. Then $\text{im}[\Gamma \to \text{Aut } M_C] = \{1\}$, and hence (5.1.1) holds. We conclude by Theorem 5.1. \hfill $\square$

**Proof of Corollary 4.4**. We define $E$, $\Gamma$, and $\Gamma_w$ for $M = \pi_1(G)$ as in Subsection 3.1. We have

$$\text{im } [\Gamma_w \to \text{Aut } M_C] \subseteq \text{im } [\Gamma \to \text{Aut } M_C], \quad \# \text{im } [\Gamma_w \to \text{Aut } M_C] \mid p, \quad \text{im } [\Gamma_w \to \text{Aut } M_C] \neq \{1\}.$$ 

It follows that (5.1.1) holds. We conclude by Theorem 5.1. \hfill $\square$
Appendix A. Abelianization

A.1. Let \( G \) be a reductive group over a field \( F \) of arbitrary characteristic. We consider the homomorphism \( \rho: G^{sc} \to G \) of Subsection 2.3.

The group \( G \) acts by conjugation on itself on the left, and by functoriality \( G \) acts on \( G^{sc} \). We obtain an action

\[
\theta: G \times G^{sc} \to G^{sc}, \quad (g, s) \mapsto gs.
\]

On \( F \)-points, if \( s \in G^{sc}(F) \), \( g_1 \in G(F) \), \( g_1 = \rho(s_1) \cdot z_1 \) with \( s_1 \in G^{sc}(F) \), \( z_1 \in Z_G(F) \), then

\[
\theta(g_1, s) = g_1s = s_1gs_1^{-1}.
\]

Since the groups \( G \) and \( G^{sc} \) are smooth, this formula uniquely determines \( \theta \). The action \( \theta \) has the following properties:

\[
\rho(s)s' = ss's^{-1},
\]

\[
\rho(g_1s') = g_1\rho(s')g_1^{-1}
\]

for \( g_1 \in G(F) \), \( s, s' \in G^{sc}(F) \). In other words, \((G^{sc}, G, \rho, \theta)\) is a (left) crossed module of algebraic groups; see for instance [4, Definition 3.2.1]. We write it as \((G^{sc}, G, \rho, \theta)\), and we regard it as a complex in degrees \(-1, 0\). On \( F_s \)-points we obtain a \( \text{Gal}(F_s/F) \)-equivariant crossed module \((G^{sc}(F_s), \rho, \theta)\) where \( F_s \) is the separable closure of \( F \) in \( F \).

A.2. Deligne [11, Section 2.0.2] noticed that the commutator map

\[
[\cdot, \cdot]: G \times G \to G, \quad g_1, g_2 \mapsto [g_1, g_2] := g_1g_2g_1^{-1}g_2^{-1}
\]

lifts to a certain map (morphism of \( F \)-varieties)

\[
\{\cdot, \cdot\}: G \times G \to G^{sc}, \quad g_1, g_2 \mapsto \{g_1, g_2\}
\]

as follows. The commutator map

\[
G^{sc} \times G^{sc} \to G^{sc}, \quad s_1, s_2 \mapsto [s_1, s_2] := s_1s_2s_1^{-1}s_2^{-1}
\]

clearly factors via a morphism of \( F \)-varieties

\[
(G^{sc})^{\text{ad}} \times (G^{sc})^{\text{ad}} \to G^{\text{ad}}
\]

where \((G^{sc})^{\text{ad}} = G^{sc}/Z_{G^{sc}} \) and \( Z_{G^{sc}} \) denotes the center of \( G^{sc} \). Identifying \((G^{sc})^{\text{ad}} \) with \( G^{\text{ad}} := G/Z_G \), we obtain the desired morphism of \( F \)-varieties

\[
\{\cdot, \cdot\}: G \times G \to G^{\text{ad}} \times G^{\text{ad}} \to G^{sc}.
\]

On \( F \)-points, if \( g_1, g_2 \in G(F) \), \( g_1 = \rho(s_1)z_1 \), \( g_2 = \rho(s_2)z_2 \) where \( s_1, s_2 \in G^{sc}(F) \), \( z_1, z_2 \in Z_G(F) \), then

\[
\{g_1, g_2\} = [s_1, s_2] = s_1s_2s_1^{-1}s_2^{-1}.
\]

Since \( G \) and \( G^{sc} \) are smooth, this formula uniquely determines \( \{\cdot, \cdot\} \). The constructed map \( \{\cdot, \cdot\} \) satisfies the following equalities of Conduché [9, (3.11))]:

\[
\rho(\{g_1, g_2\}) = [g_1, g_2];
\]

\[
\{\rho(s_1), \rho(s_2)\} = [s_1, s_2];
\]

\[
\{g_1, g_2\} = \{g_2, g_1\}^{-1};
\]

\[
\{g_1g_2, g_3\} = \{g_1g_2g_1^{-1}, g_1g_3g_1^{-1}\}\{g_1, g_3\}.
\]

In other words, the map \( \{\cdot, \cdot\} \) is a symmetric braiding of the crossed module \((G^{sc}, G, \rho, \theta)\). We denote by \( G_{ab} \) the corresponding stable (=symmetrically braided) crossed module:

\[
G_{ab} = (G^{sc} \xrightarrow{\rho} G, \theta, \{\cdot, \cdot\}).
\]
Let \( \varphi: G \to H \) be a homomorphism of reductive \( F \)-groups. It induces a homomorphism \( \varphi^\text{sc}: G^\text{sc} \to H^\text{sc} \). It is easy to see that
\[
\varphi(g) \varphi^\text{sc}(s) = \varphi^\text{sc}(gs) \quad \text{for all } g \in G(F), s \in G^\text{sc}(F).
\]
Thus we obtain a morphism of crossed modules
\[
(G^\text{sc} \to G, \theta_G) \to (H^\text{sc} \to H, \theta_H)
\]
with obvious notations. Moreover, we have
\[
\{\varphi(g_1), \varphi(g_2)\}_H = \varphi^\text{sc}(\{g_1, g_2\}_G) \quad \text{for all } g_1, g_2 \in G(F)
\]
with obvious notations; see [5] for a proof. Thus we obtain a morphism of stable crossed modules
\[
(A.2.1) \quad (G^\text{sc} \to G, \theta_G, \{\cdot\}_G) \to (H^\text{sc} \to H, \theta_H, \{\cdot\}_H).
\]

A.3. In this appendix, we denote by \( H^1 \) and \( \Pi^1 \) the first Galois cohomology and hypercohomology. One can define the first Galois (hyper)cohomology of the \( \text{Gal}(F_s/F) \)-equivariant crossed module
\[
(A.3.1) \quad \Pi^1(F, G^\text{sc} \overset{\rho}{\to} G, \theta) := \Pi^1(\text{Gal}(F_s/F), G^\text{sc}(F_s) \overset{\rho}{\to} G(F_s), \theta);
\]
see [4] Section 3 or Noohi [17, Section 4]. A priori it is just a pointed set. However, using the symmetric braiding \( \{\cdot, \cdot\} \), one can define a structure of abelian group on the pointed set \((A.3.1)\); see Noohi [17, Corollaries 4.2 and 4.5]. We denote the obtained abelian group by
\[
H^1_{\text{ab}}(F, G) = \Pi^1(F, G_{\text{ab}}) := \Pi^1(\text{Gal}(F_s/F), G^\text{sc}(F_s) \overset{\rho}{\to} G(F_s), \theta, \{\cdot\}).
\]
A homomorphism of reductive \( F \)-groups \( \varphi: G \to H \) induces a morphism of stable crossed modules \((A.2.1)\), which in turn induces a homomorphism of abelian groups
\[
\varphi_{\text{ab}}: H^1_{\text{ab}}(F, G) \to H^1_{\text{ab}}(F, H).
\]
Thus \( G \hookrightarrow H^1_{\text{ab}}(F, G) \) is a functor from the category of reductive \( F \)-group to the category of abelian groups.

A.4. The morphism of crossed modules (but not of stable crossed modules)
\[
i_G: (1 \to G) \hookrightarrow (G^\text{sc} \to G)
\]
induces a morphism of pointed sets
\[
(i_G)_*: \Pi^1(F, 1 \to G) \to \Pi^1(F, G^\text{sc} \overset{\rho}{\to} G).
\]
The abelianization map is the composite morphism of pointed sets
\[
\text{ab}: H^1(F, G) = \Pi^1(F, 1 \to G) \overset{(i_G)_*}{\to} \Pi^1(F, G^\text{sc} \overset{\rho}{\to} G, \theta) = \Pi^1(F, G_{\text{ab}}) =: H^1_{\text{ab}}(F, G).
\]
Here \( \Pi^1(F, G^\text{sc} \overset{\rho}{\to} G, \theta) \) and \( \Pi^1(F, G_{\text{ab}}) \) are the same sets, but \( \Pi^1(F, G_{\text{ab}}) \) is endowed with the structure of abelian group coming from the symmetric braiding \( \{\cdot, \cdot\} \).

A.5. For a maximal torus \( T \subseteq G \), we consider the homomorphism
\[
\rho: T^\text{sc} \to T
\]
of Subsection 2.3, which we regard as a stable crossed module with the trivial action \( \theta_T \) of \( T \) on \( T^\text{sc} \) and the trivial symmetric braiding \( \{\cdot\}_T: T \times T \to T^\text{sc} \). We may and shall identify the first Galois hypercohomology of this stable crossed module with the usual first Galois hypercohomology of the complex \( T^\text{sc} \overset{\rho}{\to} T \) in degrees \(-1, 0\):
\[
\Pi^1(F, T^\text{sc} \overset{\rho}{\to} T, \theta_T, \{\cdot\}_T) = \Pi^1(F, T^\text{sc} \overset{\rho}{\to} T).
\]
The morphism of stable crossed modules
\[
(A.5.1) \quad j_T: (T^\text{sc} \overset{\rho}{\to} T, \theta_T, \{\cdot\}_T) \hookrightarrow (G^\text{sc} \overset{\rho}{\to} G, \theta, \{\cdot\})
\]
is an equivalence (quasi-isomorphism), that is, it induces isomorphisms of $F$-group schemes

$$\ker[T^{sc} \to T] \xrightarrow{\sim} \ker[G^{sc} \to G] \quad \text{and} \quad \coker[T^{sc} \to T] \xrightarrow{\sim} \coker[G^{sc} \to G].$$

Following an idea sketched by Labesse and Lemaire [15], we observe that (A.5.1) induces isomorphisms on groups of $F_s$-points

$$\ker\left[T^{sc}(F_s) \to T(F_s)\right] \xrightarrow{\sim} \ker\left[G^{sc}(F_s) \to G(F_s)\right]$$

and

$$\coker\left[T^{sc}(F_s) \to T(F_s)\right] \xrightarrow{\sim} \coker\left[G^{sc}(F_s) \to G(F_s)\right]$$

(in arbitrary characteristic); see Theorem B.1 in Appendix B below. It follows that the induced map on Galois hypercohomology

$$(j_T)_*: \mathcal{H}^1(F, T^{sc} \to T) \longrightarrow \mathcal{H}^1(F, G^{sc} \xrightarrow{\rho} G, \theta, \{,\}) =: H^1_{ab}(F, G)$$

is an isomorphism of abelian groups; see Noohi [17, Proposition 5.6]. This shows that the abelian group structure on the pointed set $\mathcal{H}^1(F, G^{sc} \xrightarrow{\rho} G, \theta) =: \mathcal{H}^1_{ab}(F, G)$ defined using the bijection $(j_T)_*$ (as in [4, Section 3.8]) coincides with the abelian group structure defined by the symmetric braiding $\{,\}$.

**Remark A.6.** González-Avilés [12] defined the abelian fppf cohomology group $H^1_{fppf, ab}(X, G)$ and the abelianization map

$$\text{ab}: H^1_{fppf}(X, G) \rightarrow H^1_{fppf, ab}(X, G)$$

for a reductive group scheme $G$ over an arbitrary base scheme $X$, which includes the case of a reductive group over a field $F$ of arbitrary characteristic. However, his definition uses the center $Z_G$ of $G$, and hence it is functorial only with respect to the normal homomorphisms $G_1 \rightarrow G_2$ (homomorphisms with normal image, hence sending $Z_{G_1}$ to $Z_{G_2}$), whereas our definition above (over a field only) is functorial with respect to all homomorphisms.

**Appendix B. Equivalence on $F_s$-points in arbitrary characteristic**

Zev Rosengarten

In this appendix we prove the following theorem:

**Theorem B.1.** Let $F$ be a field of arbitrary characteristic and let $F_s$ be a fixed separable closure of $F$. Let

$$\rho: G^{sc} \rightarrow [G, G] \hookrightarrow G$$

be as in Subsection 2.3. Let $T \subseteq G$ be a maximal torus. We write $T^{sc} = \rho^{-1}(T)$. Then the morphism of crossed modules

$$(T^{sc}(F_s) \to T(F_s)) \longrightarrow (G^{sc}(F_s) \to G(F_s))$$

is an equivalence (quasi-isomorphism).

**Proof.** We must show that the maps

$$i_{\ker} : \ker[T^{sc}(F_s) \to T(F_s)] \longrightarrow \ker[G^{sc}(F_s) \to G(F_s)]$$

and

$$i_{\coker} : \coker[T^{sc}(F_s) \to T(F_s)] \longrightarrow \coker[G^{sc}(F_s) \to G(F_s)]$$

are isomorphisms.

For (B.1.1), the injectivity is obvious. Moreover, any element of $\ker[G^{sc}(F_s) \to G(F_s)]$ lies in the preimage $T^{sc}$ of $T$, hence it is an element of $T^{sc}(F_s)$ and of $\ker[T^{sc}(F_s) \to T(F_s)]$, which gives the surjectivity of $i_{\ker}$.
We prove the injectivity of (B.1.2). Let \([t] \in \ker [T^{sc}(F_s) \to T(F_s)], t \in T(F_s),\) and \([t] \in \ker i_{\text{cok}};\) then \(t = \rho(s)\) for some \(s \in G^{sc}(F_s).\) Since \(T^{sc} = \rho^{-1}(T),\) we see that \(s \in T^{sc}(F_s),\) whence \([t] = 1,\) as required.

We prove the surjectivity of (B.1.2). Let \(C \subseteq G\) denote the radical (largest central torus) of \(G.\) Then the map 
\[
\psi: C \times G^{sc} \to G, \quad (c, s) \mapsto c \cdot \rho(s) \quad \text{for} \quad c \in C, \ s \in G^{sc}
\]
is surjective with central kernel \(Z \cong \rho^{-1}(C \cap [G, G])\) (which might be non-smooth). We have an exact commutative diagram of \(F\)-group schemes
\[
\begin{array}{cccccc}
1 & \longrightarrow & Z & \longrightarrow & C \times T^{sc} & \psi_T & \longrightarrow & T & \longrightarrow & 1 \\
\| & & \| & & \| & & \| & & \| & & \\
1 & \longrightarrow & Z & \longrightarrow & C \times G^{sc} & \psi & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]
in which the maps on \(F_s\)-points
\[
\psi_T: C(F_s) \times T^{sc}(F_s) \to T(F_s) \quad \text{and} \quad \psi: C(F_s) \times G^{sc}(F_s) \to G(F_s)
\]
might not be surjective. This diagram gives rise to an exact commutative diagram of fppf cohomology groups
\[
\begin{array}{cccccccc}
C(F_s) \times T^{sc}(F_s) & \stackrel{\psi_T}{\longrightarrow} & T(F_s) & \longrightarrow & H^1_{\text{fppf}}(F_s, Z) & \longrightarrow & H^1_{\text{fppf}}(F_s, C \times T^{sc}) = 1 \\
\| & & \| & & \| & & \| & & \\
C(F_s) \times G^{sc}(F_s) & \stackrel{\psi}{\longrightarrow} & G(F_s) & \longrightarrow & H^1_{\text{fppf}}(F_s, Z) & \longrightarrow & H^1_{\text{fppf}}(F_s, C \times G^{sc}) = 1
\end{array}
\]
in which the rightmost term in both rows is trivial because \(F_s\) is separably closed and the \(F\)-groups \(C \times T^{sc},\ C \times G^{sc}\) are smooth. The latter diagram shows that 
\[
G(F_s) = T(F_s) \cdot \psi(C(F_s) \times G^{sc}(F_s)) = T(F_s) \cdot C(F_s) \cdot \rho(G^{sc}(F_s)) = T(F_s) \cdot \rho(G^{sc}(F_s)),
\]
whence the surjectivity of (B.1.2). \(\square\)

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