ON THE TRACE OF UNIMODAL LÉVY PROCESSES ON LIPSCHITZ DOMAINS

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ABSTRACT. We show that the second term in the asymptotic expansion as $t \to 0$ of the trace of the Dirichlet heat kernel on Lipschitz domains for unimodal Lévy processes, satisfying some weak scaling conditions, is given by the surface area of the boundary of the domain. This brings the asymptotics for the trace of unimodal Lévy processes in domains of Euclidean space on par with those of symmetric stable processes as far as boundary smoothness is concerned.

1. Introduction

The following two-term estimate for the trace of the heat kernel corresponding to the symmetric $\alpha$-stable processes, $\alpha \in (0, 2]$, on an $\mathbb{R}$-smooth domain $D \subset \mathbb{R}^d$ was given by Bañuelos and Kulczycki [1]:

$$
|Z_D(t) - C_1 |D| + C_2 t^{1/\alpha} |\partial D| | \leq C_3 t^{2/\alpha} |D| \mathcal{H}^{d-1}(\partial D) + o(t^{1/\alpha}).
$$

(1.1)

Bañuelos et al. [2] expanded this idea, in analogy with a result for Brownian motion in Brown [9], to bounded Lipschitz domains:

$$
Z_D(t) = C_1 |D| - C_2 t^{1/\alpha} \mathcal{H}^{d-1}(\partial D) + o(t^{1/\alpha}).
$$

(1.2)

In another direction, this first bound (1.1) was generalized by Bogdán and Siudeja [8] to unimodal Lévy processes satisfying certain weak lower and upper scaling conditions on $\mathbb{R}$-smooth domains:

$$
|Z_D(t) - p_t(0)|D| + C_3(t) |\partial D| | \leq c(\varepsilon) T(t)^{1-d},
$$

(1.3)

where $c(\varepsilon) \to 0$ and $\varepsilon \to 0$.

In this paper we combine the results of Bañuelos et al. [2] and Bogdán and Siudeja [8] to obtain generalizations of both (1.2) and (1.3). This generalization says that for a unimodal Lévy processes on a bounded Lipschitz domain we have

$$
|Z_D(t) - p_t(0)|D| + C_3(t) |\partial D| | \leq c(\varepsilon) T(t)^{1-d},
$$

(1.4)

where $c(\varepsilon) \to 0$ and $\varepsilon \to 0$.

2. Preliminaries

We call a measure isotropic if it is absolutely continuous on $\mathbb{R}^d \setminus \{0\}$ with respect to Lebesgue measure and is invariant under linear isometries of $\mathbb{R}^d$. We call a measure isotropic unimodal, or unimodal in short, if its density function is also radially non-increasing. A Lévy process is called isotropic unimodal if all its density functions are isotropic unimodal, see [6, 21]. Unimodal Lévy processes are characterized by Lévy-Khintchine (characteristic) exponents of the form

$$
\psi(\xi) = \sigma^2 |\xi|^2 + \int_{\mathbb{R}^d} (1 - \cos(\xi, x)) \nu(dx),
$$

(2.1)

where $\nu(dx) = \nu(x)dx = \nu(|x|)dx$ is a unimodal Lévy measure and $\sigma \geq 0$. Since $\psi(\xi)$ is a radial function, we often let $\xi(r) = \psi(\xi)$ where $\xi \in \mathbb{R}^d$ and $r = |\xi| \geq 0$.

In what follows, we assume that we have a unimodal Lévy measure and we consider the pure-jump, $\sigma = 0$, Lévy process $X = (X_t)_{t \geq 0}$ on $\mathbb{R}^d$ determined by the Lévy-Khintchine formula:

$$
\mathbb{E} e^{i \xi \cdot X_t} = \int_{\mathbb{R}^d} e^{i \xi \cdot x} p_t(dx) = e^{-t \psi(\xi)}.
$$

(2.2)
Here \( p_t(dx) \) is the distribution of \( X_t \). It turns out that \( p_t(dx) \) is also unimodal; therefore we can call the process \( X \) (isotropic) unimodal. We wish for \( p_t(dx) \) to have bounded and smooth density functions, \( p_t(x) \) for \( t > 0 \). This is achieved as a consequence of the Hartman-Wintner condition, see Lemma 1.1 of [5].

\[
\lim_{|\xi| \to \infty} \frac{\psi(\xi)}{\ln(\xi)} = \infty. \tag{2.3}
\]

The Hartman-Wintner condition itself will be a consequence of our assumption that \( \psi(\xi) \) satisfies some weak lower scaling condition, yet to be defined. We always assume that the Lévy-Khintchine exponent, \( \psi(\xi) \), is unbounded, that is, \( \nu(\mathbb{R}^d) = \infty \). Clearly \( \psi(0) = 0 \) and \( \psi(u) > 0 \) for \( u > 0 \).

2.1. Renewal function of the ladder-height processes.

Let \( X^1_t \) be the first coordinate process of \( X_t \). We define the running maximum of \( X_t \) by

\[
M_t = \sup_{0 \leq s \leq t} X^1_s. \tag{2.4}
\]

We define \( L^0(t) \) to be the local time of \( M_t - X^1_t \) at 0. That is, \( L^0(t) \) is the amount of time, up to time \( t \), that \( M_t - X^1_t \) spends at 0:

\[
L^0(t) = \int_0^t \delta (M_s - X^1_s) \, ds, \tag{2.5}
\]

where \( \delta(\cdot) \) is the Dirac delta function. Consider the right-continuous inverse of \( L^0(t), \) \( (L^0)^{-1}(s) \), this is called the ascending ladder time process for \( X^1_t \). Composing \( X^1_t \) with \( (L^0)^{-1}(s) \) gives us the ascending ladder-height process:

\[
H_s = X^1_{(L^0)^{-1}(s)} = M_{(L^0)^{-1}(s)}. \tag{2.6}
\]

The accumulated potential of our ascending ladder-height process is then defined by

\[
V(x) = \mathbb{E} \int_0^\infty 1_{[0,x]}(H_s) \, ds = \int_0^\infty \mathbb{P}(H_s \leq x) \, ds. \tag{2.7}
\]

The function \( V(x) \) is continuous and strictly increasing from \([0, \infty)\) onto \([0, \infty)\). In particular, \( \lim_{r \to \infty} V(r) = \infty \) and \( V(x) \) is sub-additive:

\[
V(x + y) \leq V(x) + V(y), \quad \text{for all } x, y \in \mathbb{R}. \tag{2.8}
\]

For example, if \( \psi(\xi) = |\xi|^\alpha \) with \( \alpha \in (0, 2) \), then \( V(x) = x^{\alpha/2} \). See Example 3.7 in [19]. For more details on the ascending ladder-height process and accumulated potential see [6] and [18].

Remark 2.1. The relationship between \( V(x) \) and \( \psi(x) \) is given in Lemma 1.2 of [5] by

\[
V^2(r) \simeq \frac{1}{\psi(1/r)}, \quad r > 0.
\]

The notation \( \simeq \) means that there is some constant \( C \in (0, \infty) \) such that for all \( r > 0 \) we have

\[
C^{-1}V^2(r) \leq \frac{1}{\psi(1/r)} \leq CV^2(r).
\]

It also worth noting that throughout this paper we use many different constants. The value of these constants is not usually of importance and the same specific constant is rarely required more than once. Hence the letter \( C \) is often used generically to refer to a constant, but it almost never refers to the same constant more than once.
2.2. Scaling.

We are interested in the (relative) power-type behavior of \( \psi(r) \) at infinity.

**Definition 2.2.** We say that \( \psi(r) \) satisfies the weak lower scaling condition at infinity, \( WLSC(\alpha, \underline{\theta}, \underline{C}) \), if there are numbers \( \alpha > 0, \underline{\theta} > 0, \) and \( \underline{C} \in (0, 1] \) such that

\[
\psi(\lambda r) \geq \underline{C}\lambda^\alpha \psi(r),
\]

for \( \lambda \geq 1, r > \underline{\theta} \). In general, we write \( \psi \in WLSC(\alpha, \underline{\theta}, \underline{C}) \).

Or, in short, we write \( \psi \in WLSC(\alpha, \underline{\theta}, \underline{C}) \), \( \psi \in WLSC(\alpha, \underline{\theta}) \), or \( \psi \in WLSC(\alpha) \) depending on how specific we want to be. Further, we say that \( \psi(r) \) satisfies the global weak lower scaling condition at infinity (global WLSC) if \( \psi \in WLSC(\alpha, 0) \). If \( \underline{\theta} > 0 \), then we can emphasize this by calling the scaling “local at infinity”.

**Definition 2.3.** We say that \( \psi(r) \) satisfies the weak upper scaling condition at infinity, \( WUSC(\overline{\pi}, \overline{\theta}, \overline{C}) \), if there are numbers \( \overline{\pi} < 2, \overline{\theta} > 0, \) and \( \overline{C} \in [1, \infty) \) such that

\[
\psi(\lambda r) \leq \overline{C}\lambda^{\overline{\pi}} \psi(r),
\]

for \( \lambda \geq 1, r > \overline{\theta} \). In general, we write \( \psi \in WUSC(\overline{\pi}, \overline{\theta}, \overline{C}) \).

Or, in short, we write \( \psi \in WUSC(\overline{\pi}, \overline{\theta}, \overline{C}) \), \( \psi \in WUSC(\overline{\pi}, \overline{\theta}) \), or \( \psi \in WUSC(\overline{\pi}) \) depending on how specific we want to be. Further, we say that \( \psi(r) \) satisfies the global weak upper scaling condition at infinity (global WUSC) if \( \psi \in WUSC(\overline{\pi}, 0) \). If \( \overline{\theta} > 0 \), then we can emphasize this by calling the scaling “local at infinity”.

**Remark 2.4.** As pointed out in Remark 1.4 of [3], by inflating (or deflating) \( \underline{C} \) and \( \overline{C} \) we can deflate (or inflate) \( \underline{\theta} \) and \( \overline{\theta} \) so that \( \underline{\theta} = \overline{\theta} = \overline{C} > 0 \) in both WLSC and WUSC.

These scalings are natural conditions on \( \psi(r) \) in the unimodal setting and there are many examples of Lévy-Khintchine exponents which satisfy WLSC or WUSC. For example, as is shown in [3], for any unimodal Lévy process we have

\[
\psi \in WLSC(0, 0, 1/\pi^2) \cap WUSC(2, 0, \pi^2).
\]

Another example is \( \psi(\xi) = |\xi|^\alpha \), the Lévy-Khintchine exponent of the isotropic \( \alpha \)-stable Lévy process in \( \mathbb{R}^d \) with \( \alpha \in (0, 2) \). This satisfies WLSC(\( \alpha, 0, 1 \)) and WUSC(\( \alpha, 0, 1 \)). Alternatively, a non-stable example is \( \psi(\xi) = |\xi|^\alpha_1 + |\xi|^\alpha_2 \), for which we have \( \psi(\xi) \in WLSC(\alpha_1, 0, 1) \cap WUSC(\alpha_2, 0, 1) \), where \( 0 < \alpha_1 < \alpha_2 < 2 \). Finally, if \( \psi(r) \) is \( \alpha \)-regular varying at infinity and \( 0 < \alpha < 2 \), then \( \psi \in WLSC(\alpha) \cap WUSC(\alpha) \), for any \( 0 < \alpha < \alpha < \pi \leq 2 \). See [3] for more details on WLSC and WUSC.

**Remark 2.5.** By definition, if \( \psi \in WLSC(\alpha, \underline{\theta}) \), then there exists some constant \( \underline{C} \) such that

\[
\frac{V(\lambda \xi)}{V(\xi)} \leq \underline{C}\lambda^{-\alpha/2},
\]

for \( \lambda \geq 1 \) and \( r > \underline{\theta} \). That is,

\[
\frac{V(\varepsilon s)}{V(s)} \leq \underline{C}\varepsilon^{-\alpha/2},
\]

for \( 0 < \varepsilon \leq 1 \) and \( s < 1/\underline{\theta} \). Similarly, if \( \psi \in WUSC(\overline{\pi}, \overline{\theta}) \) then there exists some constant \( \overline{C} \) such that

\[
\frac{V(s)}{V(\varepsilon s)} \leq \overline{C}\varepsilon^{-\overline{\pi}/2},
\]

for \( 0 < \varepsilon \leq 1 \) and \( s < 1/\overline{\theta} \).

**Lemma 2.6** (Potter-like Bound). If \( \psi \in WLSC(\alpha, \underline{\theta}) \cap WUSC(\overline{\pi}, \overline{\theta}) \), \( 0 < x < 1/\overline{\theta} \), and \( 0 < y < 1/\underline{\theta} \), then there exists some constant \( C \) such that

\[
\frac{V(x)}{V(y)} \leq C \left( \left( \frac{x}{y} \right)^{\alpha/2} \lor \left( \frac{x}{y} \right)^{\overline{\pi}/2} \right),
\]

(2.13)
Proof. Using (2.11) and (2.12) we have
\[
V(x) / V(y) = \begin{cases} 
\frac{V(ty)}{V(y)}, & \text{if } t = \frac{x}{y} \leq 1, \\
\frac{V(y)}{V(x)}, & \text{if } t^{-1} = \frac{y}{x} \leq 1.
\end{cases}
\]
\[
\leq \begin{cases} 
C \sqrt{\frac{\alpha}{2}}, & \text{if } t = \frac{x}{y} \leq 1 \text{ and } y < 1 / \theta, \\
C \sqrt{\frac{\alpha}{2}}, & \text{if } t^{-1} = \frac{y}{x} \leq 1 \text{ and } x < 1 / \theta.
\end{cases}
\]
\[
\leq C \left( \frac{x}{y} \right)^{\frac{\alpha}{2}} \lor \left( \frac{y}{x} \right)^{\frac{\alpha}{2}}, \quad \text{for } x < 1 / \theta, \ y < \theta.
\]
\[
\square
\]

Note 2.7. We heavily use the inverse function of \( V(x) \) on \([0, \infty)\) in this paper. Thus we choose the notation
\[
T(t) := V^{-1} \left( \sqrt{t} \right). \tag{2.14}
\]
This is equivalent to \( V^2(T(t)) = t \). For example, \( T(t) = t^{1/\alpha} \) for the isotropic \( \alpha \)-stable Lévy process. The scaling properties of \( T(t) \) at zero reflect those of \( \psi(\xi) \) at infinity. See [8] for further discussion of \( T(t) \).

Throughout the rest of this paper we will make the following assumptions:
- Our Lévy measure \( \nu \) is unimodal and infinite on \( \mathbb{R}^d \) with \( d \geq 2 \)
- Our Lévy-Khintchine exponent satisfies
  \[
  0 \neq \psi \in WLSC(\alpha, \theta) \cap WUSC(\overline{\alpha}, \theta),
  \]
  for some constants \( 0 < \alpha \leq \overline{\alpha} < 2 \) and \( 0 \leq \theta \leq \inf_{x \in D} (1 / \delta_D(x)) \).

Note 2.8. These assumptions guarantee that the Hartman-Wintner condition, mentioned above in (2.3), is satisfied. It is also worth noting that many partial results below require less assumptions, but for simplicity of the presentation we ignore such extensions.

2.3. Heat Kernel.

Let \( p_t(x - y) = p(t, x, y) \) denote the (smooth) transition density function associated to the distribution of our Lévy process, \( X_t \), starting at the point \( x \).

Definition 2.9. The first exit time of \( X \) from \( D \) is defined by
\[
\tau_D = \inf \{ t > 0 : X_t \notin D \}. \tag{2.15}
\]

Definition 2.10. For \( t > 0 \) and \( x, y \in \mathbb{R}^d \) the heat remainder of \( X_t \) is defined to be
\[
r_D(t, x, y) = \mathbb{E}^x [\tau_D < t, \ p_{t-\tau_D} (X(\tau_D) - y)]. \tag{2.16}
\]

Definition 2.11. The Dirichlet heat kernel of \( X_t \) is the transition density of the process killed upon exiting \( D \) and is given by the Hunt formula:
\[
p_D(t, x, y) = p_t(y - x) - r_D(t, x, y). \tag{2.17}
\]

Definition 2.12. The trace of the heat kernel \( p_D(t, x, x) \) is given by
\[
Z_D(t) = \int_{\mathbb{R}^d} p_D(t, x, x) dx. \tag{2.18}
\]

Eventually we will refer to the Green function of \( X \) on \( D \) using the following notation:

Definition 2.13. Let \( M \geq 0 \). The truncated Green function of the process \( X \) on \( D \) is defined by
\[
G_D^M(x, y) = \int_0^M p_D(t, x, y) dt. \tag{2.19}
\]
We will also refer to the Poisson kernel using the following notation:

**Definition 2.14.** Let \( M \geq 0 \). The **truncated Poisson kernel** of the process \( X \) on \( D \) is defined by

\[
K^M_D(x, z) = \int_D G^M_D(x, y) \nu(y - z) \, dy. \tag{2.20}
\]

3. **Main Theorem**

Our main theorem coincides exactly with what would be predicted based on previous work in [8] and [2].

**Theorem 3.1.** Let \( D \subset \mathbb{R}^d, d \geq 2, \) be a bounded Lipschitz domain. Let \( |D| \) denote the \( d \)-dimensional Lebesgue measure of \( D \), and let \( \mathcal{H}^{d-1}(\partial D) \) denote the \((d - 1)\)-dimensional Hausdorff measure of \( \partial D \). Given any unimodal Lévy process and any \( \varepsilon > 0 \), there exists a \( t_0 > 0 \) such that for any \( 0 < t < t_0 \) the trace of the heat kernel satisfies

\[
|Z_D(t) - p_t(0)|_D + C_H(t) \mathcal{H}^{d-1}(\partial D) \leq c(\varepsilon) T(t)^{1 - d}, \tag{3.1}
\]

where \( c(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and

\[
C_H(t) = T(t)^{1 - d} \int_0^\infty r_H(t, (q, 0, ..., 0), (q, 0, ..., 0)) \, dq. \tag{3.2}
\]

Here \( \mathbb{H} = \{(x_1, ..., x_d) \in \mathbb{R}^d : x_1 > 0\} = \mathbb{R}^d_+ \) is the upper half-space of \( \mathbb{R}^d \).

3.1. **Domain.**

Let \( D \) be a bounded Lipschitz domain. In order to prove our theorem we treat our Lipschitz domain \( D \), as it was treated in [2] and [9]; by dividing it into good and bad sets.

**Definition 3.2.** Let \( \varepsilon, r > 0 \). We say that \( G \subset \partial D \) is \((\varepsilon, r)\)-**good** if for each point \( q \in G \) the unit inner normal, \( v(q) \), exists and

\[
B(q, r) \cap \partial D \subset \{ x : (x - q) \cdot v(q) < \varepsilon|x - q| \}. \tag{3.4}
\]

Here \( \varphi_\varepsilon \in [0, \pi/2] \) denotes the angle, measured from \( v(q) \), such that \( \cos(\varphi_\varepsilon) = \varepsilon \).

**Definition 3.3.** If \( G \) is an \((\varepsilon, r)\)-good set, then a **good** subset, \( G \), of \( D \) is a set of points of the form

\[
G := \bigcup_{q \in G} \Gamma_r(q, \varepsilon), \tag{3.4}
\]

where \( \Gamma_r(q, \varepsilon) \) is a cone given by

\[
\Gamma_r(q, \varepsilon) := \left\{ x : (x - q) \cdot v(q) > \sqrt{1 - \varepsilon^2}|x - q| \right\} \cap B(q, r). \tag{3.5}
\]

Let us define \( \delta_D(x) := \text{dist}(x, \partial D), x \in \mathbb{R}^d \). In [2], the results Lemma 2.7 and Lemma 2.8 are combined to give the following result:
Lemma 3.4. Let $0 < \varepsilon < 1/4$ and $r > 0$. There exists a measurable $(\varepsilon, r)$-good set $G \subset \partial D$ and $s_0(\partial D, G)$ such that for all $s < s_0$
\[ |\{x \in D : \delta_D(x) < s\} \setminus G| \leq s\varepsilon \left(4 + H^{d-1}(\partial D)\right). \] (3.6)

3.2. Inner and Outer Cone.

Let $G \subset \partial D$ be an $(\varepsilon, r)$-good set and let $\mathcal{G}$ be good subset of $D$. If $x \in \mathcal{G}$, then, by definition, there exists a point $q(x) \in \partial D$ such that $x \in \Gamma_r(q(x), \varepsilon)$.

We define the Inner and Outer cones of $B(q(x), r)$ as follows

\[ I_r(q(x)) := \{ y : (y - q(x)) \cdot v(q(x)) > \varepsilon|y - q(x)| \} \cap B(q(x), r), \] (3.7)

\[ U_r(q(x)) := \{ y : (y - q(x)) \cdot v(q(x)) < -\varepsilon|y - q(x)| \} \cap B(q(x), r). \] (3.8)

Note, for $x \in \mathcal{G}$, we have

\[ \Gamma_r(q(x), \varepsilon) \subset I_r(q(x)) \subset D \subset U_r(q(x)). \]

It is shown in [2] that for any $x \in \mathcal{G}$ there exists a half-space $H^*(x)$ such that:

\[ x \in H^*(x), \quad \delta_{H^*(x)}(x) = \delta_D(x) \quad I_r(q(x)) \subseteq H^*(x) \subseteq U_r(q(x)). \] (3.9)

4. Proof of the Main Theorem

The transition densities of isotropic processes killed upon exiting a domain $D$ are given by the Hunt formula

\[ p_D(t, x, y) = p_t(y - x) - r_D(t, x, y). \] (4.1)

It follows that

\[ -\int_D r_D(t, x, x)dx = \int_D p_D(t, x, x)dx - \int_D p_t(0)dx = Z_D(t) - p_t(0)|D|. \] (4.2)
Hence in order to prove Theorem 3.1 it is sufficient to show that for an arbitrary $\varepsilon > 0$ there exists a $t_0 > 0$ such that for any $0 < t < t_0$ we have

$$\left| \int_D r_D(t, x, x) \, dx - C_{\text{H}}(t) \mathcal{H}^{d-1}(\partial D) \right| \leq c(\varepsilon) T(t)^{1-d}, \quad (4.3)$$

where $c(\varepsilon) \to 0$ as $\varepsilon \to 0$.

We need to estimate

$$\int_D r_D(t, x, x) \, dx.$$  

Fix $0 < \varepsilon < 1/4$. Let us define $G \subset \partial D$ to be the $(\varepsilon, \tau)$-good set as described above in Lemma 3.4. Let $G$ be the corresponding good subset of $D$. Then we divide $D$ into the following domains

$$\begin{align*}
D_1 &= \{ x \in D \setminus G : \delta_D(x) < s \}, \\
D_2 &= \{ x \in D \cap G : \delta_D(x) < s \}, \\
D_3 &= \{ x \in D : \delta_D(x) \geq s \},
\end{align*}$$

where $s$ must be smaller than the $s_0$ given in Lemma 3.4. For small enough $t$ we can let $s = T(t)/\sqrt{\varepsilon}$.

### 4.1. The domain $D_1$:

The following estimate for $r_D(t, x, y)$ comes from Lemma 2.4 of [8].

**Lemma 4.1.** Suppose $\psi \in WLSC(\alpha, \theta)$ and $T(t) < 1/\theta$. Then

$$r_D(t, x, y) \leq C \left\{ T(t)^{-d} \wedge \frac{t}{\delta_D^2(x) V^2(\delta_D(x))} \right\}. \quad (4.4)$$

By assumption $\psi \in WLSC(\alpha, \theta)$ and so, for us, this lemma implies that if $T(t) < 1/\theta$, then

$$\int_{D_1} r_D(t, x, x) \, dx \leq C \int_{D_1} T(t)^{-d} \, dx \quad (4.5)$$

$$= CT(t)^{-d} |D_1|. \quad (4.6)$$

But, by Lemma 3.4 we know that the measure of the set of bad points near the boundary is small. Hence if $T(t) < 1/\theta$, then

$$\int_{D_1} r_D(t, x, x) \, dx \leq C(\partial D) \varepsilon T(t)^{-d} \quad (4.7)$$

where $C$ is a constant depending on $d$, $\alpha$, and $\partial D$.

### 4.2. The domain $D_3$:

By assumption $\psi \in WLSC(\alpha, \theta)$, and so, again by Lemma 4.1 if $T(t) < 1/\theta$, then

$$\int_{D_3} r_D(t, x, x) \, dx \leq CT(t)^{-d} \int_{D_3} \left\{ 1 \wedge \frac{T(t)^d}{\delta_D^2(x) V^2(\delta_D(x))} \right\} \, dx. \quad (4.8)$$

Next, our Potter-like bound in Lemma 2.6 tells us that if $T(t) < 1/\theta$, then

$$\int_{D_3} r_D(t, x, x) \, dx \leq CT(t)^{-d} \int_{D_3} \left\{ 1 \wedge \frac{T(t)^d}{\delta_D^2(x)} \left( \frac{T(t)\alpha}{\delta_D^2(x)} \sqrt[\alpha]{T(t)} \right) \right\} \, dx. \quad (4.9)$$
By definition of $D_3$, for any $x \in D_3$ we have $\delta_D(x) \geq s = T(t)/\varepsilon$. Or equivalently $1 \leq \frac{\delta_D(x)}{T(t)} \varepsilon$. Hence

$$
\int_{D_3} r_D(t,x,x) \, dx \leq C(t)^{-d} \int_{D_3} \left\{ 1 \wedge \varepsilon T(t)^{d-1} \left( \frac{T(t)^{\alpha}}{\delta_D(x)} \right) \sqrt{\frac{T(t)}{\delta_D(x)}} \right\} \, dx
$$

(4.10)

$$
\leq C(t)^{-d} \int_D \left\{ 1 \wedge \varepsilon T(t)^{d+\alpha-1} \frac{T(t)^{\alpha}}{\delta_D(x)} + 1 \wedge \varepsilon T(t)^{d+\alpha-1} \frac{T(t)^{\alpha}}{\delta_D(x)} \right\} \, dx
$$

(4.11)

$$
= C(t)^{-d} \frac{1}{T(t)} \int_D \left\{ 1 \wedge \varepsilon \left( \frac{\delta_D(x)}{T(t)} \right)^{-d+\alpha+1} + 1 \wedge \varepsilon \left( \frac{\delta_D(x)}{T(t)} \right)^{-d+\alpha+1} \right\} \, dx. \tag{4.12}
$$

We are now in a position to apply the following important proposition from [2]:

**Proposition 4.2.** Let $D \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Suppose that $f : (0, \infty) \to \mathbb{R}$ is continuous and satisfies $f(s) \leq c (1 \wedge s^{-\beta})$, $s > 0$, for some $\beta > 1$, and suppose that for any $0 < R_1 < R_2 < \infty$, $f(s)$ is Lipschitz on $[R_1, R_2]$. Then

$$
\lim_{\eta \to 0^+} \frac{1}{\eta} \int_D f \left( \frac{\delta_D(x)}{\eta} \right) \, dx = H^{d-1}(\partial D) \int_0^\infty f(s) \, ds. \tag{4.13}
$$

Letting $\eta = T(t)$ and $f(s) = 1 \wedge \sqrt{\varepsilon s}^{-d+\alpha+1}$ and $f(s) = 1 \wedge \sqrt{\varepsilon s}^{-d+\alpha+1}$, respectively, we can apply Proposition 4.2 to both of the integrals in (4.12). Thus for small values of $t$ we get

$$
\int_{D_3} r_D(t,x,x) \, dx \leq C(t)^{1-d} H^{d-1}(\partial D) \int_0^\infty \left\{ (1 \wedge \varepsilon r^{-d+\alpha+1}) + (1 \wedge \varepsilon r^{-d+\alpha+1}) \right\} \, dr. \tag{4.14}
$$

Using substitution this becomes

$$
\int_{D_3} r_D(t,x,x) \, dx \leq C(t)^{1-d} H^{d-1}(\partial D) \int_0^\infty \left\{ (1 \wedge r^{-d+\alpha+1}) + (1 \wedge r^{-d+\alpha+1}) \right\} \, dr
$$

$$
\leq C(t)^{1-d} \left( \frac{1}{\varepsilon (\alpha + \beta - 1)} + \frac{1}{\varepsilon (\alpha + \beta - 1)} \right). \tag{4.15}
$$

This covers domains $D_1$ and $D_3$.

### 4.3. The domain $D_2$:

It remains to show that $r_D(t,x,x)$ is comparable to $r_{H^*}(t,x,x)$ for $x \in D_2$.

Suppose $x \in D_2 \subset \mathcal{G}$. Let $q(x)$ be as above. Then $x \in \Gamma_r(q(x), \varepsilon)$. For the purposes of brevity we will use the following notation $\mathcal{I} := I_r(q(x))$ and $\mathcal{U}^c := U^c_r(q(x))$.

Notice that

$$
H^*(x) \subseteq \mathcal{U}^c \quad \text{and} \quad \mathcal{I} \subseteq D.
$$
Using (4.16) and Proposition 4.3 we get
\[ T \leq r_Z(t, x, x) \leq \frac{c_1}{T(t)^d} \sqrt{\varepsilon} \left( \frac{1 + T(t)^{d-1}}{\delta(x)} \right), \]
(4.17)
We have the following important proposition:

Proposition 4.3. Let \( v(q) \in \mathbb{R}^d \) be a unit vector. Assume that \( 0 < \varepsilon < 1/4 \) and \( r > 0 \). If \( x \in \Gamma_2(v(q), \varepsilon) \) and \( s = T(t)/\sqrt{\varepsilon} < r/4 \), then
\[ 0 \leq r_Z(t, x, x) - r_H^\dagger(t, x, x) \leq \frac{c_1}{T(t)^d} \sqrt{\varepsilon} \left( \frac{1 + T(t)^{d-1}}{\delta(x)} \right). \]
(4.18)
We postpone the proof of this proposition until Section 5.

Using (4.16) and Proposition 4.3 we get
\[ \int_{D_2} |r_D(t, x, x) - r_H^\dagger(t, x, x)| \, dx \leq \int_{D_2} (r_Z(t, x, x) - r_H^\dagger(t, x, x)) \, dx \]
(4.19)
Notice that since \( x \in \Gamma_2(v(q), \varepsilon) \), \( \partial D \cap B(q, r) \subset B(q, r) \cap D \), and \( \varepsilon < 1/4 \), we have
\[ \delta(x) \geq |x - q| \sin(2\varphi - \pi/2) = |x - q| \cos(2\varphi) \]
\[ = (1 - 2\varepsilon^2) |x - q| \geq (1 - 2\varepsilon^2) \delta_D(x) > \frac{7}{8} \delta_D(x). \]
(4.20)
Hence
\[ \int_{D_2} |r_D(t, x, x) - r_H^\dagger(t, x, x)| \, dx \leq \frac{C(\varepsilon)}{T(t)^d} \int_{D_2} \left( \frac{1 + T(t)^{d-1}}{\delta(x)} \right) \, dx. \]
(4.21)
We can use our Potter-like bounds from Lemma 2.6 again: if \( T(t) < 1/\theta \), then
\[ \int_{D_2} |r_D(t, x, x) - r_H^\dagger(t, x, x)| \, dx \leq \frac{C(\varepsilon)}{T(t)^d} \int_{D_2} \left( \frac{1 + T(t)^{d-1}}{\delta(x)} \right) \, dx. \]
(4.22)
Letting \( \eta = T(t) \) as above, we can apply Proposition 4.2 to get, for small enough \( t \), that
\[ \int_{D_2} |r_D(t, x, x) - r_H^\dagger(t, x, x)| \, dx \leq \frac{C(\varepsilon)}{T(t)^{d-1}} \eta^{d-1}(\partial D) \int_0^\infty \left( (1 + \eta^{-d-\alpha+1}) + (1 + \eta^{-d-\alpha+1}) \right) \, d\eta \]
(4.23)
Finally, it remains to show that
\[ \left| \int_{D_2} r_H^\dagger(t, x, x) \, dx - \int_0^\infty r_H(t, q, 0, 0, ..., 0) \, dq \right| \leq c(\varepsilon) T(t). \]
(4.24)
To do this we apply Proposition 4.2 to \( \int_{D_2} r_H^\dagger(t, x, x) \, dx \). Note that, by construction, we have
\[ r_H^\dagger(t, x, x) = r_H^\dagger(t, (\delta_H^\dagger(x), 0, ..., 0), (\delta_H^\dagger(x), 0, ..., 0)) \]
(4.25)
\[ = r_H^\dagger(t, (\delta_D(x), 0, ..., 0), (\delta_D(x), 0, ..., 0)) \]
(4.26)
\[ = r_H(t, (\delta_D(x), 0, ..., 0), (\delta_D(x), 0, ..., 0)) \]
(4.27)
\[ = r_H(t, \delta_D(x)). \]
(4.28)
Thus we can change from \( D_2 \) to \( D \) by remarking that
\[ \int_{D_2} r_H^\dagger(t, x, x) \, dx = \int_D r_H(t, \delta_D(x)) \, dx - \int_{D_1 \cup D_3} r_H(t, \delta_D(x)) \, dx \]
(4.29)
and that by the same arguments as \eqref{4.7} and \eqref{4.15} we also have that
\[
\int_{D(t) \cup D_\delta} r_{\infty}(t, \delta_D(x)) \, dx \leq c(\varepsilon) T(t)^{1-d},
\]  
(4.31)
where \(c(\varepsilon) \to 0\), as \(\varepsilon \to 0\). Lemma \ref{4.1} tells us
\[
r_{\infty}(t, \delta_D(x)) \leq CT(t)^{-d} \left( 1 \wedge \frac{T(t)^d}{\delta_D^d} \frac{\mathcal{V}^2(T(t))}{\delta_D} \right).
\]  
(4.32)
Applying our Potter-like bounds from Lemma \ref{2.6} gives us
\[
r_{\infty}(t, \delta_D(x)) \leq \frac{C}{T(t)^d} \left\{ 1 \wedge \left( \frac{T(t)^d}{\delta_D} \right)^{d+\alpha} + 1 \wedge \left( \frac{T(t)^d}{\delta_D} \right)^{d+\beta} \right\}.
\]  
(4.33)
We wish to show that \(r_{\infty}(t, \delta_D(x))\) satisfies the assumptions of Proposition \ref{4.2}. Hence we must show that \(r_{\infty}(t, \delta_D(x))\) is Lipschitz. Firstly, the following bound is provided by \cite{10}:

**Lemma 4.4.** Let \(\psi \in WLSC(\alpha, \theta)\). Then for \(T(t) < 1/\theta\) we have
\[
|\nabla_x p_t(x)| \leq \frac{c}{T(t)} \min \left\{ p_t(0), \frac{t}{|x|^d \mathcal{V}^2(|x|)} \right\}.
\]  
(4.34)
Next

**Lemma 4.5.** Let \(D \subset \mathbb{R}^d\) be an open nonempty set. Fix \(\varepsilon > 0\). For any \(y \in D\) and \(w, z \in D\) with \(\delta_D(w) > \varepsilon\), \(\delta_D(z) > \varepsilon\), there exists \(c(\varepsilon, t)\) such that
\[
|r_D(t, w, y) - r_D(t, z, y)| \leq c(\varepsilon, t) |w - z|.
\]  
(4.35)
**Proof.** The mean value theorem and Lemma \ref{4.4} tells us that there exists some \(0 \leq l \leq 1\) such that
\[
|p_t(w) - p_t(z)| \leq |\nabla_x p_t(lw + (1 - l)w) - \nabla_x p_t(0) - (1 - l)w| |w - z|
\]  
(4.36)
\[
\leq \frac{c}{T(t)} \min \left\{ p_t(0), \frac{|lw + (1 - l)z|}{|lw + (1 - l)z|} \right\} |w - z|,
\]  
(4.37)
\[
\leq \frac{c}{T(t)} \min \left\{ p_t(0), \frac{t}{|w| \wedge |z|} \right\} |w - z|.
\]  
(4.38)
By definition of the heat remainder, \cite{2.10}, we have
\[
r_D(t, x, y) = \mathbb{E}^y [\tau_D < t \mid p_t - \tau_D (X(\tau_D) - x)].
\]
Thus
\[
|r_D(t, w, y) - r_D(t, z, y)| \leq \mathbb{E}^y [\tau_D < t \mid p_t - \tau_D (X(\tau_D) - w) - p_t - \tau_D (X(\tau_D) - z)]
\]  
(4.39)
\[
\leq c \mathbb{E}^y \left[ \tau_D < t; \frac{1}{T(t - \tau_D)} \min \left\{ p_t - \tau_D (0), \frac{|X(\tau_D) - w| \wedge |X(\tau_D) - z|}{(|X(\tau_D) - w| \wedge |X(\tau_D) - z|)^d \mathcal{V}^2(|X(\tau_D) - w| \wedge |X(\tau_D) - z|)} \right\} |w - z| \right]
\]  
(4.40)
\[
\leq c \mathbb{E}^y \left[ \tau_D < t; \frac{1}{T(t - \tau_D)} \min \left\{ p_t - \tau_D (0), \frac{|X(\tau_D) - w| \wedge |X(\tau_D) - z|}{(|X(\tau_D) - w| \wedge |X(\tau_D) - z|)^d \mathcal{V}^2(|X(\tau_D) - w| \wedge |X(\tau_D) - z|)} \right\} |w - z| \right]
\]  
(4.41)
\[
\leq \frac{c}{(|\delta_D(w) - \delta_D(z)|)^d \mathcal{V}^2(|\delta_D(w) - \delta_D(z)|)} \mathbb{E}^y \left[ \tau_D < t; \frac{1}{T(t - \tau_D)} \min \left\{ p_t - \tau_D (0), \frac{|w - z|}{(|\delta_D(w) - \delta_D(z)|)^d \mathcal{V}^2(|\delta_D(w) - \delta_D(z)|)} \right\} \right]
\]  
(4.42)
\[
\leq c(\varepsilon, t) |w - z|.
\]  
(4.43)
where, in the last inequality, we have used our assumption that both \(\delta_D(w)\) and \(\delta_D(z)\) are larger than \(\varepsilon\). \qed

Finally we can now show that \(r_{\infty}(t, \delta_D(x))\) is Lipschitz:
Lemma 4.6. Let $D \subset \mathbb{R}^d$ be an open nonempty set. Fix $\varepsilon > 0$. For any $y \in D$ and $w, z \in D$ with $\delta_D(w) > \varepsilon$, $\delta_D(z) > \varepsilon$, there exists $c(\varepsilon, t)$ such that

$$|r_D(t, w, w) - r_D(t, z, z)| \leq c(\varepsilon, t) |w - z|.$$  \hfill (4.44)

Proof. By Lemma 4.5 and the symmetry of the heat remainder, that is $r_D(t, w, z) = r_D(t, z, w)$, we get

$$|r_D(t, w, w) - r_D(t, z, z)| \leq |r_D(t, w, w) - r_D(t, z, w)| + |r_D(t, w, z) - r_D(t, z, z)|$$  \hfill (4.45)

$$\leq c(\varepsilon, t) |w - z|.$$  \hfill (4.46)

Lemma 4.6 tells us that $r_{\mathbb{H}}(t, \delta_D(x))$ is Lipschitz. Thus $r_{\mathbb{H}}(t, \delta_D(x))$ satisfies the assumptions of Proposition 4.2. Hence, for small $t$, we have

$$\left| \int_D r_{\mathbb{H}}(t, \delta_D(x)) \, dx - C_{\mathbb{H}}(t) \mathcal{H}^{d-1}(\partial D) \right| \leq \varepsilon T(t)^{1-d}.$$  \hfill (4.47)

This completes the proof of Theorem 3.1.

5. Proof of Proposition 4.3

Proof of Proposition 4.3. We wish to show that

$$0 \leq r_{I}(t, x, x) - r_{U^c}(t, x, x) \leq \left( \varepsilon^{1-\alpha/2} + \varepsilon^{1-\alpha/2} \right) \sqrt{\varepsilon} \left( 1 \wedge \frac{T(t)^{d-1}}{\delta_D^{-1}(x) V^2(\delta_D(x))} \right).$$  \hfill (5.1)

In order to show this inequality we combine different aspects of similar proofs given in Proposition 3.2 of [8] and Proposition 3.1 of [2].

Firstly, by definition, we have

$$r_{I}(t, x, x) - r_{U^c}(t, x, x) = p_{U^c}(t, x, x) - p_{I}(t, x, x)$$  \hfill (5.2)

$$= E^x [r_{I}(t, X(t)) - X(t) \in U^c \setminus I; p_{U^c}(t - \tau_{I}, X(\tau_{I}), x)].$$  \hfill (5.3)

The space-time Ikeda-Watanabe formula from Corollary 2.8 in [14] then tells us that

$$r_{I}(t, x, x) - r_{U^c}(t, x, x) = \int_0^t \int_{U^c \setminus I} p_{I}(t, l, y) \int_{U^c \setminus I} \nu(y - z) p_{U^c}(t - l, x, z) \, dz \, dl \, dy.$$  \hfill (5.4)

Without loss of generality we may assume that $q = 0$ and $v(0) = (1, 0, \ldots, 0)$. Let

$$I = \{ y : y \cdot v(0) > \varepsilon |y| \},$$  \hfill (5.5)

$$U = \{ y : y \cdot v(0) < -\varepsilon |y| \},$$  \hfill (5.6)

$$\Gamma(0, \varepsilon) = \left\{ y : y \cdot v(0) > \sqrt{1 - \varepsilon^2} |y| \right\}.$$  \hfill (5.7)
Notice that
\[ U^c \setminus I = B^c(0, r) \cup (U^c \setminus I) \quad \text{and} \quad I \subset I. \]

Hence (5.4) can be broken up as
\[
\begin{align*}
 r_x(t, x) - r_{U^c}(t, x, x) & \leq \int_I \int_0^t p_x(l, x, y) \int_{(U^c \setminus I) \cap B(0, r)} \nu(y - z) p_{U^c}(t - l, x, z) \, dz \, dl \, dy \\
 & \quad + \int_I \int_0^t p_x(l, x, y) \int_{B^c(0, r)} \nu(y - z) p_{U^c}(t - l, x, z) \, dz \, dl \, dy \\
 & = A_t(x) + B_t(x). \quad (5.8)
\end{align*}
\]

\( A_t(x) \): Lemma 1.5 in [5] gives a bound for the heat kernel under certain scaling conditions:

**Lemma 5.1.** Suppose \( \psi \in WLSC(\omega, \theta) \) and \( T(t) < 1/\theta \). Then there exists a constant \( C \) such that
\[
 p_t(x - z) \leq C \left( T^{-d}(t) \wedge \frac{t}{|x - z|^d V^2(|x - z|)} \right). \quad (5.10)
\]

Notice that if \( x \in \Gamma(0, \varepsilon) \) and \( z \in U^c \setminus I = \{ y : -\varepsilon |y| < y \cdot v(0) < \varepsilon |y| \} \), then
\[
|x - z| \geq |x| \sin \left( 2\varphi_x - \frac{\pi}{2} \right) = |x| \left( 1 - 2\cos^2(\varphi_x) \right) = |x|(1 - 2\varepsilon^2). \quad (5.11)
\]

Lemma 5.1 and the monotonicity of \( V(r) \) thus imply that
\[
 p_{t-l}(x - z) \leq C \left( 1 - 2\varepsilon^2 \right)^{-d} \frac{1}{|x|^d V^2(|x|)} \frac{t}{|x|^d V^2(|x|)} \leq C \frac{t}{|x|^d V^2(|x|)}. \quad (5.12)
\]

By assumption \( \psi \in WUSC(\omega, \theta) \) and \( \varepsilon < 1/4 \), hence:
\[
 p_{t-l}(x - z) \leq C \left( 1 - 2\varepsilon^2 \right)^{-d} \frac{1}{|x|^d V^2(|x|)} \frac{t}{|x|^d V^2(|x|)} \leq C \frac{t}{|x|^d V^2(|x|)}. \quad (5.13)
\]

We can now apply this bound directly to \( A_t(x) \):
\[
 A_t(x) \leq \int_I \int_0^t p_x(l, x, y) \int_{(U^c \setminus I) \cap B(0, r)} \nu(y - z) p_{U^c}(t - l, x, z) \, dz \, dl \, dy \\
\leq \int_I \int_0^t \frac{C}{|x|^d V^2(|x|)} \int_{(U^c \setminus I) \cap B(0, r)} p_x(l, x, y) \int_{(U^c \setminus I) \cap B(0, r)} \nu(y - z) \, dz \, dl \, dy \\
\leq \int_I \int_0^t \frac{C}{|x|^d V^2(|x|)} \int_{(U^c \setminus I) \cap B(0, r)} p_x(l, x, y) \int_{(U^c \setminus I) \cap B(0, r)} \nu(y - z) \, dz \, dl \, dy \\
\leq \int_I \int_0^t \frac{C}{|x|^d V^2(|x|)} \int_{(U^c \setminus I) \cap B(0, r)} K_{2}^{2(1/\theta)}(x, z) \, dz, \quad (5.15)
\]

where in the last two equations we have used definitions of the truncated Green function and the truncated Poisson kernel, (2.19) and (2.20) respectively. We can then apply the bound for truncated Poisson kernels on convex sets that is given in Lemma 2.9 of [8]:

\[
 A_t(x) \leq \int_{(U^c \setminus I) \cap B(0, r)} \frac{c_y}{|x - z|^d V(\delta_{I^c}(z))} \, dz. \quad (5.19)
\]
Our Potter-like bounds in Lemma 2.30 tell us that
\[
\int_{(U \setminus I) \cap B(0,r)} \frac{1}{|x-z|^d} \frac{V(\delta_T(x))}{V(\delta_T(z))} \, dz \leq \int_{(U \setminus I) \cap B(0,r)} \frac{1}{|x-z|^d} \left\{ \left( \frac{\delta_T(x)}{\delta_T(z)} \right)^{\alpha/2} \vee \left( \frac{\delta_T(x)}{\delta_T(z)} \right)^{\alpha/2} \right\} \, dz \tag{5.20}
\]
\[
\leq \delta_T(x) \int_{(U \setminus I) \cap B(0,r)} \frac{dz}{\delta_T(z)|x-z|^d} + \delta_T(x) \int_{(U \setminus I) \cap B(0,r)} \frac{dz}{\delta_T(z)|x-z|^d}.
\]

In Lemma 3.2 of [2] it is shown that:

**Lemma 5.2.** For any \( \varepsilon \in (0,1/4) \), \( w \in \Gamma(0, \varepsilon) \), \( M \in (0, \infty) \) we have
\[
\int_{(U \setminus I) \cap B(0,M)} \frac{dz}{\delta_T^\varepsilon(z)|z-w|^\gamma} \leq \begin{cases} c_\gamma \varepsilon^{1-\alpha/2} M^{-\alpha/2-\gamma} & \text{for } \gamma > d - \alpha/2, \\ c_\gamma \varepsilon^{1-\alpha/2} M^{d-\alpha/2-\gamma} & \text{for } 0 < \gamma < d - \alpha/2. \end{cases} \tag{5.21}
\]

Notice that for \( z \in (U \setminus I) \cap B(0,r) \) we must have \( \delta_{T_x}(z) = \delta_T(z) \). Thus for \( \gamma = d \) we get:
\[
\int_{(U \setminus I) \cap B(0,r)} \frac{1}{|x-z|^d} \frac{V(\delta_T(x))}{V(\delta_T(z))} \, dz \leq C \left\{ \delta_T(x) \varepsilon^{1-\alpha/2} |x|^{-\alpha/2} + \delta_T(x) \varepsilon^{1-\alpha/2} |x|^{-\alpha/2} \right\} \tag{5.22}
\]
\[
\leq C \left\{ \delta_T(x) \varepsilon^{1-\alpha/2} \delta_T(x) \varepsilon^{1-\alpha/2} \delta_T(x) \varepsilon^{1-\alpha/2} \delta_T(x) \right\} \tag{5.23}
\]
\[
\leq C \left\{ \varepsilon^{1-\alpha/2} + \varepsilon^{1-\alpha/2} \right\}. \tag{5.24}
\]

This gives us one bound for \( A_t(x) \):
\[
A_t(x) \leq \boxed{C \left( \varepsilon^{1-\alpha/2} + \varepsilon^{1-\alpha/2} \right) \frac{1}{|x|^d} \frac{V^2(T(t))}{V^2(|x|)}} \tag{5.25}
\]

Let us now consider \( A_t(x) \) from another perspective. We divide \( A_t(x) \) into the following subregions:
\[
A_t(x) = \int_{\mathbb{T}} \int_{0}^{t/2} P_T(l,x,y) \int_{(U \setminus I) \cap B(0,r)} \nu(y-z) \rho_T(t-l,x,z) \, dz \, dl \, dy \tag{5.26}
\]
\[
+ \int_{\mathbb{T}} \int_{t/2}^{t} P_T(l,x,y) \int_{(U \setminus I) \cap B(0,r) \cap \{|x-z| \leq T\}} \nu(y-z) \rho_T(t-l,x,z) \, dz \, dl \, dy \tag{5.27}
\]
\[
+ \int_{\mathbb{T}} \int_{t/2}^{t} P_T(l,x,y) \int_{(U \setminus I) \cap B(0,r) \cap \{|x-z| > T\}} \nu(y-z) \rho_T(t-l,x,z) \, dz \, dl \, dy \tag{5.28}
\]
\[
= \mathbf{I} + \mathbf{II} + \mathbf{III}. \tag{5.29}
\]

### 5.1. Short jump time: I.

For \( l \in [0,t/2] \) we can use the bound for the heat kernel given in (5.10) of Lemma 5.1:
\[
\rho_T(t-l,x,z) \leq p(t-l,x,z) \leq CT(t-l)^{-d}. \tag{5.30}
\]

Monotonicity of \( T(t) \) then implies
\[
\rho_T(t-l,x,z) \leq CT(t/2)^{-d}. \tag{5.31}
\]

The scaling of \( \psi(\xi) \) at infinity implies the scaling of \( T(t) \) at 0, as is shown in Lemma 2.1 of [8]. Hence
\[
\rho_T(t-l,x,z) \leq C (1/2)^{-d/2} T(t)^{-d} = CT(t)^{-d}. \tag{5.32}
\]

Thus
\[
\mathbf{I} \leq CT(t)^{-d} \int_{\mathbb{T}} \int_{0}^{t/2} P_T(l,x,y) \int_{(U \setminus I) \cap B(0,r)} \nu(y-z) \, dz \, dl \, dy \tag{5.33}
\]
\[
\leq CT(t)^{-d} \int_{(U \setminus I) \cap B(0,r)} K_T^{v(1/2)}(x,z) \, dz. \tag{5.34}
\]
Lemma 5.3. It now follows from our calculations between (5.18) and (5.24) above that
\[ I \leq C \left( \varepsilon^{1-\omega/2} + \varepsilon^{1-\pi/2} \right) T(t)^{-d}. \]  

(5.35)

5.2. Long exit time and short jumps: II.

The following bound for the heat kernel is given in Lemma 2.6 of [8]:

**Lemma 5.3.** Assume \( D \) is convex. There exists a constant \( c_0 \) such that if \( T(t) < 1/\theta \vee |x - y| \), then
\[ p_D(t, x, y) \leq c_0 \left( \frac{V(\delta_D(x))}{V(T(t))} \wedge 1 \right) \left( \frac{V(\delta_D(y))}{V(T(t))} \wedge 1 \right) \left( \frac{t}{|x - y|^d V^2(|x - y|)} \wedge T(t)^{-d} \right). \]  

(5.36)

Let \( S := (U^c \setminus J) \cap B(0, r) \cap \{|x - z| \leq T\} \). For \( l \in [t/2, t) \) we can use the bounds from Lemma 5.1 and Lemma 5.3 to get
\[ II = \int_{t/2}^{t} p_T(l, x, y) \int_S \nu(y - z) \eta(t - l, x, z, y, z) dz \, dl \, dy 
\leq CT(t)^{-d} \int_{t/2}^{t} \int_S \frac{V(\delta_T(y))}{V(T(t))} \frac{1}{|y - z|^d V^2(|y - z|)} \int_{t/2}^{t} \eta(t - l, x, z, y, z) dz \, dl \, dy 
\leq CT(t)^{-d} \int_{t/2}^{t} \int_S \frac{V(\delta_T(y))}{V(T(t))} \frac{1}{|y - z|^d V^2(|y - z|)} G_{t/2}^{I_d}(x, z) dy dz. \]

(5.37)

By construction \( \delta_T(y), \delta_T(z) \leq |y - z| \) and so
\[ II \leq CT(t)^{-d} \frac{V(|x|)}{V(T(t))} \int_S \frac{1}{|y - z|^d V^2(|y - z|)} \frac{V(\delta_T(z))}{|x - z|^d} dy \]  
\[ \leq CT(t)^{-d} \frac{V(|x|)}{V(T(t))} \int_S \frac{\delta_T^{d/2}(z)}{|y - z|^d + \omega^2 V(\delta_T(z))} \frac{V(\delta_T(z))}{|x - z|^d} dy dz \]  
\[ \leq CT(t)^{-d} \frac{V(|x|)}{V(T(t))} \int_S \frac{\delta_T^{d/2}(z)}{|x - z|^d V(\delta_T(z))} \int_{I_r} \frac{1}{|y - z|^d + \omega^2 dy dz} \]  
\[ \leq CT(t)^{-d} \frac{V(|x|)}{V(T(t))} \int_S \frac{\delta_T^{d/2}(z)}{|x - z|^d V(\delta_T(z))} \int_{B(z, \delta_T(z))} \frac{1}{|y - z|^d + \omega^2 dy dz}. \]

(5.42)

We have seen in (5.11) that \( |x - z| > (1 - 2\varepsilon^2)|x| \). Thus for these short jumps we have \((1 - 2\varepsilon^2)|x| < T(t)\) and hence \( V(|x|) < c V(T(t)) \), for some constant \( c \). Therefore
\[ II \leq CT(t)^{-d} \frac{\delta_T^{d/2}(z)}{|x - z|^d V(\delta_T(z))} \int_{I_r} \frac{1}{|y - z|^d + \omega^2 dy dz} \]  
\[ \leq CT(t)^{-d} \frac{\delta_T^{d/2}(z)}{|x - z|^d V(\delta_T(z))} \int_{B(z, \delta_T(z))} \frac{1}{|y - z|^d + \omega^2 dy dz}. \]

(5.43)

(5.44)

Changing to polar coordinates:
\[ II \leq CT(t)^{-d} \int_{S} \frac{\delta_T^{d/2}(z)}{|x - z|^d V(\delta_T(z))} \int_{\delta_T(z)}^{\infty} \frac{1}{r^d + \omega^2} r^{d-1} dr dz \]  
\[ = CT(t)^{-d} \int_{S} \frac{\delta_T^{d/2}(z)}{|x - z|^d V(\delta_T(z))} \frac{1}{\delta_T^{d/2}(z)} \]  
\[ = CT(t)^{-d} \int_{S} \frac{1}{|x - z|^d V(\delta_T(z))} dz. \]

(5.45)

(5.46)

(5.47)
Lemma 5.4. For any $\varepsilon \in (0, 1/4)$, $x \in \Gamma(0, \varepsilon)$, $r \in (0, \infty)$ we have

$$
\int_{(U^c \setminus I) \cap B(0, r)} \frac{1}{|x-z|^d} \frac{\delta_{\varepsilon}^{\alpha/2}(z)}{\delta_{L}^{\alpha/2}(z)} \, dz \leq c \varepsilon^{1-\alpha/2}.
$$

Proof. Let us use polar coordinates $(\rho, \varphi_1, ..., \varphi_d)$, with center $q = 0$ and principal axis $v(0) = (1, 0, ..., 0)$. We prove this lemma for the case $d \geq 3$, the case with $d = 2$ is essentially the same but with different restrictions on the angle. As above, we let $\varphi \in [0, \pi/2]$ be the angle such that $\cos(\varphi) = \varepsilon$. Then

$$
U^c \setminus I = \{(\rho, \varphi_1, ..., \varphi_d) : \varphi_1 \in (\varphi, \pi - \varphi)\}, \quad \delta_{\varepsilon}(z) = \rho \sin(\varphi - \varphi_1), \quad \text{and} \quad \delta_{\varepsilon}(z) = \rho \sin(\varphi + \varphi_1)
$$

for $z \in U^c \setminus I$.

Let $V_1 = (U^c \setminus I) \cap B(0, |x|)$ and $V_2 = (U^c \setminus I) \cap B(0, |x|) \cap B(0, r)$. Recall, $(1 - 2\varepsilon^2)|x|, (1 - 2\varepsilon^2)|z| \leq |x-z|$ and notice that for $z \in V_1$ we have $|x-z| \leq 2|x|$, thus $|x-z| \simeq |x|$ for $z \in V_1$. Similarly, if $z \in V_2$, then $|x-z| \simeq |z|$. Thus

$$
\int_{V_1} \frac{1}{|x-z|^d} \frac{\delta_{\varepsilon}^{\alpha/2}(z)}{\delta_{L}^{\alpha/2}(z)} \, dz \leq c \frac{1}{|x|^d} \int_{V_1} \frac{\delta_{\varepsilon}^{\alpha/2}(z)}{\delta_{L}^{\alpha/2}(z)} \, dz
$$

(5.49)

$$
\leq c \int_{0}^{2\pi} \frac{\rho^{\alpha/2} \sin(\varphi + \varphi_1) \rho^{d-1} \sin^{d-2}(\varphi_1)}{\rho^{\alpha/2} \sin^{\alpha/2}(\varphi_1 - \varphi)} \, d\varphi_1 d\rho
$$

(5.50)

$$
\leq c \int_{0}^{\pi} \frac{\rho^{d-1} \rho \int_{0}^{\pi - \varphi} \frac{1}{\sin^{\alpha/2}(\varphi_1 - \varphi)} \, d\varphi_1}{\rho^{\alpha/2} \sin^{\alpha/2}(\varphi_1 - \varphi)} \, d\varphi
$$

(5.51)

$$
\leq c \int_{\pi - \varphi}^{\pi} \frac{1}{\varphi^{\alpha/2}} \, d\varphi
$$

(5.52)

$$
\leq c \varepsilon^{1-\alpha/2}.
$$

(5.53)

The last inequality follows from the fact that for $\varepsilon \in (0, 1/4)$, we have $\sin(\pi - 2\varphi) \simeq 2\sin(\pi/2 - \varphi)$, so $\pi - 2\varphi \leq \varepsilon \pi$. On the remaining domain we have

$$
\int_{V_2} \frac{1}{|x-z|^d} \frac{\delta_{\varepsilon}^{\alpha/2}(z)}{\delta_{L}^{\alpha/2}(z)} \, dz \leq \int_{V_2} \frac{\delta_{\varepsilon}^{\alpha/2}(z)}{\delta_{L}^{\alpha/2}(z)} \, dz
$$

(5.54)

$$
\leq \int_{|x|}^{r} \frac{\rho^{d-1} \rho \int_{0}^{\pi - \varphi} \frac{1}{\varphi^{\alpha/2}} \, d\varphi_1 \, d\rho}{\rho^{d+\alpha/2} \sin^{\alpha/2}(\varphi_1 - \varphi)}
$$

(5.55)

$$
\leq \int_{|x|}^{r} \frac{\rho^{d-1} \int_{0}^{\varphi} \frac{1}{\varphi^{\alpha/2}} \, d\varphi}{\rho^{d+\alpha/2} \sin^{\alpha/2}(\varphi_1 - \varphi)}
$$

(5.56)

$$
\leq c \varepsilon^{1-\alpha/2}.
$$

(5.57)

It now follows from (5.47) and Lemma 5.4 that

$$
\text{II} \leq C \left( \varepsilon^{1-\alpha/2} + \varepsilon^{1-\pi/2} \right) T(t)^{-d}.
$$

(5.58)

5.3. Long exit time and large jumps: III.

We now suppose that $|x-z| > T$. Let $Q := (U^c \setminus I) \cap B(0, r) \cap \{|x-z| > T\}$. Again using the bound from (5.39) of Lemma 5.3 we get

$$
\text{III} \leq C \int_{Q} \int_{Q} p_t(l, x, y) \frac{1}{V(T(t) - l)} \left( 1 \wedge \frac{V(T(t) - l)}{|x-z|^d} \right) \, dz \, dy
$$

(5.59)
We can again use the Poisson kernel bound from Lemma 2.9 in [5]:

\[
\begin{align*}
\text{III} & \leq CT(t)^{-d} \int_Q \frac{V(|x|)}{V(\delta_2(z))} \frac{1}{|x-z|^d} \frac{V(\delta_{t^2}(z))}{V(T(t))} \left(1 \wedge \frac{T(t)^d V^2(T(t))}{|x-z|^d V^2(|x-z|)} \right) \, dz \\
& \leq CV(T(t)) \int_Q \frac{V(|x|)}{|x-z|^{2d}} \frac{V(\delta_t(z))}{V(\delta_2(z))} \, dz \\
& \leq CV(T(t)) \frac{1}{T(t)^d} \frac{V(\delta_t(z))}{V(\delta_2(z))} \int_Q \frac{1}{|x-z|^d} \, dz \\
& \leq C \frac{V(T(t))}{(T(t))^d} V(T(t)) \int_Q \frac{1}{|x-z|^d} \, dz \\
& \leq CT(t)^{-d} \left(1 \wedge \frac{T(t)^d V^2(T(t))}{|x-z|^d V^2(|x-z|)} \right) \, dz.
\end{align*}
\]

We can use Lemma 5.4 again to get

\[
\text{III} \leq C \left(\epsilon^{1-\alpha/2} + \epsilon^{1-\pi/2} \right) T(t)^{-d}
\]

Therefore

\[
A_t(x) \leq C \left(\epsilon^{1-\alpha/2} + \epsilon^{1-\pi/2} \right) \left(\frac{T(t)^{d-\alpha}}{|x|^d} \wedge \frac{1}{|x|^d} \frac{V^2(T(t))}{V^2(|x|)} \right)
\]

**B_t(x):** It remains to find a bound for

\[
B_t(x) \leq \int_Q \int_Q p_t(l, x, y) \int_{B^c(0,r)} \nu(y-z) p(t-l, x, z) \, dz \, dl \, dy.
\]

By assumption \(x \in \Gamma_2(v(q), \epsilon)\), \(s < r/4\), and \(z \in B^c(0, r)\). Thus \(|x-z| > r/2 > 2s\). Combining this with the bound for the heat kernel in Lemma 5.1 we get:

\[
p(t-l, x, z) \leq C \left(\frac{T(t-l)^{-d}}{|x-z|^d} \wedge \frac{1}{|x-z|^d} \frac{t-l}{V^2(|x-z|)} \right)
\]

\[
\leq C \left(\frac{T(t-l)^{-d}}{|x-z|^d} \wedge \frac{1}{s^d} \frac{t-l}{V^2(s)} \right).
\]

Thus

\[
B_t(x) \leq C \left(\frac{T(t-l)^{-d}}{|x-z|^d} \wedge \frac{1}{s^d} \frac{V^2(T(t))}{V^2(s)} \right) \int_Q \int_Q p_t(l, x, y) \int_{B^c(0,r)} \nu(y-z) \, dz \, dl \, dy
\]

\[
\leq C \left(\frac{T(t-l)^{-d}}{s^d} \wedge \frac{1}{s^d} \frac{V^2(T(t))}{V^2(s)} \right) \int_{B^c(0,r)} \nu(z) \, dz
\]

\[
\leq C \frac{1}{s^d} \frac{V^2(T(t))}{V^2(s)}.
\]

We chose \(s = T(t)/\sqrt{\epsilon}\). Thus

\[
B_t(x) \leq C \frac{(\sqrt{\epsilon})^d}{T(t)^d} \frac{V^2(T(t))}{V^2 \left(\frac{T(t)}{\sqrt{\epsilon}}\right)}.
\]

Since \(x \in \Gamma_2(v(q), \epsilon)\) it also tells us that \(|x| < 2s = 2T(t)/\sqrt{\epsilon}\). Hence

\[
B_t(x) \leq C \frac{(\sqrt{\epsilon})^d}{T(t)^d} \frac{V^2(T(t))}{|x|^d} \leq C \frac{(\sqrt{\epsilon})^d}{T(t)^d} \frac{1}{|x|^d} \frac{V^2(T(t))}{V^2(|x|)}.
\]

Letting \(\beta = d\) and \(\beta = 1\) we get

\[
B_t(x) \leq C \frac{(\sqrt{\epsilon})^d}{T(t)^d} \frac{1}{|x|^d} \frac{V^2(T(t))}{V^2(|x|)} \leq C \sqrt{T} T(t)^{-d} \left(1 \wedge \frac{T(t)^d V^2(T(t))}{|x|^{d-1} V^2(|x|)} \right).
\]
Therefore, combining our bounds for $A_t(x)$ and $B_t(x)$, we get

$$r_I(t,x,x) - r_{U_c}(t,x,x) \leq C \left( \varepsilon^{1-\alpha/2} + \varepsilon^{1-\alpha/2} \right) T(t)^{-d} \left( 1 \wedge \frac{T(t)^{d-1} V^2(T(t))}{|x|^{d-1} V^2(|x|)} \right).$$

(5.72)

□

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