ANNULAR REPRESENTATION THEORY FOR RIGID C*-TENSOR CATEGORIES

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ABSTRACT. We introduce the notion of annular categories over rigid C*-tensor categories, generalizing both the tube algebra and the affine annular category of a planar algebra. We study the representation theory of annular categories, which provides classes of admissible representations on annular centralizer algebras, and we identify the resulting universal C*-algebras for generic Temperley-Lieb-Jones categories, which have interesting topological characterizations. The weight 0 centralizer algebra of an annular category over C is canonically isomorphic to the fusion algebra of C, and we show that the annular representations of the fusion algebra correspond precisely to the class of admissible representations introduced by Popa and Vaes, allowing the approximation and rigidity properties introduced by them be interpreted in the annular context. Inspired by the work of Brown and Guentner, we present definitions of approximation and rigidity properties for other weights, and show that if the category has the approximation property (Haagerup property or amenability), then it has the corresponding property for all weights. We include a result of Stefaan Vaes, which shows that the monoidal category of annular representations is contravariantly equivalent to the unitary Drinfeld center of the inductive category of C studied by Neshveyev and Yamashita.

1. INTRODUCTION

The affine annular category of a planar algebra AP was introduced by Jones and Reznikoff in [17], and its structure was further studied in [5]. The unitary representation theory of the annular Temperley-Lieb planar algebras has played an important role in the construction and classification of small index subfactors [16], [18]. Annular categories also play a role in the realm of 2 + 1 TQFT's [32]. The braided monoidal category of Hilbert representations of the affine annular category of a planar algebra, denoted Rep(AP), was introduced in [5], where it was shown that in the case of finite depth planar algebras, Rep(AP) ≃ ⊗ Z(Proj(P)), where Z(Proj(P)) denotes the unitary Drinfeld center of the underlying projection category. Ocneanu’s tube algebra has also been shown to be closely related to the Drinfeld center of a fusion categories, with equivalence classes of irreducible representations of the tube algebra being in 1−1 correspondence with simple objects in the center in the case |Irr(C)| < ∞, [12], [22]. Thus the tube algebra of a category can be viewed as a “substitute” for the Drinfeld double in the absence of any quantum group symmetries. Analyzing the tube algebra has also been useful for computing the S and T matrices of Z(C), and hence the corresponding TQFT, [13].

In this paper, we introduce the notion of an annular category over C with weight set W ⊆ Obj(C). An annular category generalizes both the tube algebra, where the weight set is isomorphism classes of simple objects, and the affine annular category of a planar algebra, where the weight set is naturally determined by string labels in the planar algebra. Motivated by the representation theory developed for planar algebras in [5], we study the representation theory of annular categories with arbitrary weight sets, and show all annular categories over C have equivalent representation theories. The endomorphism algebras of the annular categories are called centralizer algebras, and annular representations produce an admissible class of...
representations of these centralizer algebras. In analogy with groups, we may construct universal $C^*$-closures of these algebras with respect to admissible representations. We identify the resulting universal $C^*$-algebras for pointed categories, and the generic Temperley-Lieb-Jones categories based on the analysis of affine annular representations of Jones and Reznikoff \[17\], \[31\], which is in turn based inspired by the work of Graham and Lehrer \[9\].

The $C^*$-algebras that appear as centralizer algebras in the Temperley-Lieb-Jones category are unital, abelian $C^*$-algebras hence isomorphic to the continuous functions on compact Hausdorff spaces. The spaces appear to be rather interesting. Let $\delta \geq 2$. We define the following topological spaces:

For $k$ even, $k > 0$, $X_k := \cdots \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes$. For $k$ odd, define $X_k := \cdots \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes$.

For $k$ even, $k > 0$ define $Y_k := \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes$. For $k$ odd, define $Y_k := \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes \bigotimes$.

We define $X_0 = Y_0 := [-\delta, \delta]$.

We let $T_{k,k}$ be the centralizer in the tube algebra of $TLJ(\delta)$ corresponding to the $k^{th}$ Jones-Wenzl idempotent. The main result of this paper is the following:

**Theorem 1.1.** If $\delta > 2$, then $C^*(T_{k,k}) \cong X_k$. If $\delta = 2$, $C^*(T_{k,k}) \cong Y_k$.

This result yields a topological characterization of the centralizer algebras of the tube algebras and hopefully will yield a deeper topological understanding of the category $Rep(ATL)$. This result highlights a key point that was uncovered by Jones and Reznikoff in \[17\]: in terms of annular representation theory, the category $TLJ(2)$ is non-generic. Here we see that the space arising is topologically distinct from the case $\delta > 2$, and thus we see that the universal $C^*(T_{k,k})$ distinguish these two cases. In general, it seems that the “non-smooth” points arise precisely from the existence of an actual unitary half-braiding. In the $\delta = 2$ case, the many “non-smooth” points are the result of the standard braidings on $TLJ(\delta)$ being unitary (whereas they are not for $\delta > 2$). In future work, we hope to characterize unitary half-braidings by discontinuities of functors on the spectra of the algebras $C^*(T_{X,X})$. We see that the results here may open the door for methods of non-commutative geometry, particularly $K$-theory, to be applied in the study of annular representation theory.

Recently in a remarkable paper \[30\], Popa and Vaes defined approximation and rigidity properties for rigid $C^*$-tensor categories in terms of a class of admissible representations of the fusion algebra, and showed that in the case when the category is equivalent to the category of $M - M$ bimodules in the standard invariant of a finite index inclusion of $II_1$-factors $N \subseteq M$, these definitions agree with the previously existing definitions for subfactors given by Popa, \[27\], \[28\]. Incidentally, the centralizer algebra of the identity object in an annular category is canonically isomorphic to the fusion algebra. We show that the class of admissible representations of the fusion algebra which extend to representations of the entire annular category is precisely the class identified by Popa and Vaes. This allows approximation and rigidity properties defined therein to be interpreted in the annular context, and provides further motivation for the study of $Rep(AW)$. Inspired by the work of Brown and Guentner in \[4\], we define notions of amenability, the Haagerup property, and
Property (T) for higher centralizer algebras. We then show that the approximation properties (amenability or Haagerup) of Popa and Vaes imply the approximation property for all centralizer algebras.

In recent work [24], Neshveyev and Yamashita introduce the category ind-$C$. From an object in the Drinfeld center of this category, they construct a representation of the fusion algebra of $C$. They show that this class of representations coincides precisely with the class identified by Popa and Vaes. Due to a result of Stefaan Vaes (which he has generously allowed us to include in this paper, Proposition 4.5) $Rep(AW)$ is contravariantly equivalent to the unitary Drinfeld center of the inductive category, $Z(\text{ind-}C)$. Thus representations of the fusion algebra which are restrictions of representations of the tube algebra can be realized as representations constructed from an inductive object and a unitary half-braiding, described by Neshveyev and Yamashita. Thus the equivalence between the annular weight 0 representations and the admissible representations of Popa and Vaes in fact follows from the theorem of Neshveyev and Yamashita [24]. We include our own proof of the equivalence in section 6 for the convenience of the reader, because it is relatively short.

The structure of the paper is as follows: In section 2 we briefly review rigid $C^*$-tensor categories and the graphical calculus. In section 3, we define annular categories over $C$, and discuss the tube algebra. In section 4 we describe the basic annular representation theory and include the result of Stefaan Vaes establishing a categorical contravariant equivalence between $Rep(TC)$ and $Z(\text{ind-}C)$. We also describe universal norms for centralizer algebras. In section 5 we present our analysis of the examples. Section 6 discusses approximation and rigidity properties and the relationship to the work of Popa and Vaes, as well as our definitions for higher weights.

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2. Preliminaries

2.1. Rigid $C^*$-Tensor Categories. In this paper we will be concerned with semi-simple, $C^*$-categories with strict tensor functor, simple unit and duals. We also assume that $C$ has countably many isomorphism classes of simple objects. Often in the literature, this is the definition of a rigid $C^*$-tensor category. We elaborate on the meaning of each of these terms.

A $C^*$-category is a $C$-linear category $C$, with each hom space $(X,Y)$ a Banach space, and a conjugate linear, involutive, contravariant functor $*:C\to C$ which fixes objects and satisfies for every hom $f$, $||f^*f|| = ||ff^*|| = ||f||^2$. We say the category is semi-simple if the category has direct sums, sub-objects, and each $(X,Y)$ is finite dimensional.

A strict tensor structure is a bi-linear functor $\otimes:C\times C\to C$, which is associative and has a distinguished unit $id\in Obj(C)$ such that $X\otimes id = X = id\otimes X$. We note that we don’t lose anything by assuming strictness since every $C^*$-monoidal category is unitarily equivalent to a strict one. The category is rigid if for each
$X \in \text{Obj}(\mathcal{C})$, there exists $\overline{X} \in \text{Obj}(\mathcal{C})$ and morphism $R \in (id, \overline{X} \otimes X)$ and $\overline{R} \in (id, X \otimes \overline{X})$ satisfying the so-called conjugate equations:

$$(1_X \otimes \overline{R}^\ast) \circ (R \otimes 1_{\overline{X}}) = 1_{\overline{X}} \text{ and } (1_X \otimes R^\ast) \circ (\overline{R} \otimes 1_X) = 1_X$$

We say two objects are $X, Y$ are isomorphic if there exists $f \in (X,Y)$ such that $f^* \circ f = 1_X$ and $f \circ f^* = 1_Y$. We call an object $X$ simple if $(X, X) \cong \mathbb{C}$. We note that for any simple objects $X$ and $Y$, $(X, Y)$ is either isomorphic to $\mathbb{C}$ or 0. Two simple objects are isomorphic iff $(X, Y) \cong \mathbb{C}$. Isomorphism defines an equivalence relation on the collection of all objects and we denote the equivalence class of an object by $[X]$, and the set of isomorphism classes of simple objects $\text{Irr}(\mathcal{C})$.

The semi-simplicity axiom implies that for any object $X$, $(X, X)$ is a finite dimensional $C^*$-algebra over $\mathbb{C}$, hence a multi-matrix algebra. It is easy to see that each summand of the matrix algebra corresponds to an equivalence class of simple objects, and the dimension of the matrix algebra corresponding to a simple object $Y$ is the square of the multiplicity with which $Y$ occurs in $X$. In general for a simple object $Y$ and any object $X$, we denote by $N^Y_X$ the natural number describing the multiplicity with which $[Y]$ appears in the simple object decomposition of $X$. If $X$ is equivalent to a subobject of $Y$, we write $X \prec Y$. We often write $X \otimes Y$ simply as $XY$ for objects $X$ and $Y$.

For two simple objects $X$ and $Y$, we have that $[X \otimes Y] \cong \bigoplus Z N_{XY}^Z[Z]$. These means that the the tensor product of $X$ and $Y$ decomposes as a direct sum of simple objects of which $N_{XY}^Z$ are equivalent to the simple object $Z$. The $N_{XY}^Z$ specify the fusion rules of the tensor category and are a critical piece of data.

For a more detailed discussion and analysis of the axioms of a rigid $C^*$-tensor category, see the paper of Longo and Roberts [19], where these categories were first defined and studied with this axiomatization.

In a rigid $C^*$-tensor category, we can define the statistical dimension of an object $d(X) = \inf_{(R, \overline{R})} ||R|| ||\overline{R}||$, where the infimum is taken over all solutions to the conjugate equations for an object $X$. The function $d(\cdot) : \text{Obj}(\mathcal{C}) \to \mathbb{R}_+$ depends on objects only up to unitary isomorphism. It is multiplicative and additive and satisfies $d(X) = d(\overline{X})$ for any dual of $X$. We called solutions to the conjugate equations standard if $||R|| = ||\overline{R}|| = d(X)^{\frac{1}{2}}$, and such solutions are essentially unique. For the rest of this paper, all solutions to the conjugate equations will be assumed standard. For standard solutions of the conjugate equations, we have a well defined trace $Tr_X$ on endomorphism spaces $(X, X)$ given by

$$Tr_X(f) = R^*(1_{\overline{X}} \otimes f)R = \overline{R}^*(f \otimes 1_{\overline{X}})\overline{R} \in (id, id) \cong \mathbb{C}$$

This trace does not depend on the choice of dual for $X$ or on the choice of standard solutions. See [19] for details.

2.2. **Graphical Calculus and Spherical Structures.** We recall the graphical calculus, a diagrammatic description of the hom spaces which elucidates the axioms of a rigid $C^*$-tensor category, and which we will use heavily in this paper. A hom $f \in (X, Y)$ will be represented by a box or disk in the plane with a string labelled by $X$ emanating from the bottom of the box, and a string labeled by $Y$ coming out the top. The the unit morphism $1_X$ is presented as a single string labeled with $X$ (with no box). For example,
We emphasize that diagrams are read from bottom to top. The hom $1_{id}$ is represented by an empty diagram, (no string). Composition of morphisms is given by vertical stacking of diagrams:

$$f = \begin{array}{c}
\text{Y} \\
\text{f} \\
\text{X}
\end{array} \in (X,Y)$$

Tensor product of objects is given by horizontal juxtaposition of strings, and tensor product of morphisms is given by horizontal juxtaposition of diagrams:

$$\begin{array}{cc}
\text{Z} & \text{g} \\
\text{g} & \text{f} \\
\text{X} & \text{Y}
\end{array} \otimes \begin{array}{c}
\text{z} \\
\text{g} \\
\text{X}
\end{array} \in (X,Z)$$

$$\begin{array}{cc}
\text{z} & \text{g} \\
\text{g} & \text{f} \\
\text{X} & \text{Y}
\end{array} \otimes \begin{array}{c}
\text{Z} \\
\text{W} \\
\text{X}
\end{array}$$

We represent standard solutions to the conjugate equations, $R \in (id, XX)$, $\overline{R} \in (id, X X)$ in pictures by the “cups” $\overline{R} = \begin{array}{c}
\text{X} \\
\text{X}
\end{array}$ and $R = \begin{array}{c}
\text{X} \\
\text{X}
\end{array}$. Their ∗’s will be represented by “caps” (the same diagram as cup reflected across a horizontal line). The duality equations are then conveniently expressed in pictures by:

$$\begin{array}{c}
\text{X} \\
\text{X}
\end{array} = \begin{array}{c}
\text{X} \\
\text{X}
\end{array}$$
In general there is no good reason to choose a specific dual for each object, but for our construction of the tube algebra, we need to choose one anyways. When choosing duals, it is helpful to make a good choice. This is formalized by the notion of a spherical structure. It is a theorem of Yamagami (see [33]) that every rigid-$\mathcal{C}^*$-tensor category can be given a spherical structure (called frobenius duality pairing by Yamagami), which is unique up to unitary equivalence.

First, we define a *pivotal structure*, which is a specific choice of dual for each object $X$, and we require that this choice satisfies a compatibility condition with the tensor product given by

$$X \otimes Y = X \otimes Y \otimes X \otimes Y \otimes X \otimes Y \otimes X \otimes Y$$

We also require that

$$f_X f_Y = f_{X \otimes Y}$$

The pivotal structure is said to be *spherical*, if for our choice of dual and duality maps, we have for all objects $X$ and all homs $f \in (X, X)$,

$$Tr_X(f) := \begin{array}{c}
\begin{array}{c}
f \\
X
\end{array}
\end{array} = \begin{array}{c}
\begin{array}{c}
f \\
X
\end{array}
\end{array} \in (id, id) = \mathbb{C}$$

One can show that if the duality maps are compatible with $*$ in this spherical structure, then they can be chosen to be standard solutions to the conjugate equations, and thus this trace necessarily agrees with the canonical trace described above for any spherical structure that is compatible with the $*$-structure. For an in depth discussion of these properties for $*$ categories see [33] and for pivotal and spherical structures see [21]. We note that while we do not need to choose a spherical structure to define annular categories, it is certainly more convenient to have one at hand.
For all statements and definitions in this paper that are not given in pictures, we encourage the reader to draw a picture using the graphical calculus, as it helps immensely to parse complicated equations involving tensor products.

3. Annular Categories

Let $\mathcal{C}$ be rigid $C^*$-tensor category, equipped with a spherical structure whose duality maps are standard solutions to the conjugate equations. Then each object $\alpha$ has a chosen dual $\overline{\alpha}$, and there is an unambiguously defined map from $(X,Y) \to (\overline{Y},\overline{X})$, denoted by $f \to \overline{f}$. Let $\mathcal{W} \subseteq \text{Obj}(\mathcal{C})$ such that for all $X \in \text{Irr}(\mathcal{C})$, there exists $k \in \mathcal{W}$ such that $X \prec k$. We define a bilinear operation $(k,\overline{m}\beta) \otimes (l,\overline{n}\alpha) \to (l,\overline{(\overline{\beta}\alpha)m(\alpha\beta)})$ which we call multiplication. For $f \in (k,\overline{m}\beta)$ and $g \in (l,\overline{n}\alpha)$ multiplication is given by

$$fg = (1_\overline{\pi} \otimes f \otimes 1_\alpha) \circ g$$

Let $\tilde{A}\mathcal{W}_{m,n} := \bigoplus_{\alpha \in \text{Obj}(\mathcal{C})}(m,\overline{n}\alpha)$, and extend multiplication linearly. If $f \in (m,\overline{n}\alpha)$ we denote its inclusion in $\tilde{A}\mathcal{W}_{m,n}$ by $\tilde{f}$. We have a conjugate linear involution $*: \tilde{A}\mathcal{W}_{m,n} \to \tilde{A}\mathcal{W}_{n,m}$ defined by

$$(\tilde{f})^* := (1_n \otimes f^* \otimes 1_\overline{\alpha}) \circ \overline{R}_{\alpha} \otimes 1_n \otimes R_{\alpha} \in (n,\overline{am}\overline{\pi})$$

We often abuse notation and omit the $\tilde{\cdot}$ over $f$. We use the $\ast$ to represent both the conjugate linear involution given here and the $\ast$ from the original category. Hopefully it will be clear from context which $\ast$ we are using. The collection $\tilde{A}\mathcal{W} := \{\tilde{A}\mathcal{W}_{m,n}\}_{m,n \in \mathcal{W}}$ forms a $\ast$-algebroid with respect to multiplication and the $\ast$-structure defined above.

We proceed to define a $\ast$-algebroid ideal of $\tilde{A}\mathcal{W}$. Let $U_{m,n} := \text{span}\{(g \otimes 1_n \otimes 1_\alpha - 1_\overline{\pi} \otimes 1_n \otimes \overline{g}) \circ f \in \tilde{A}\mathcal{W} : f \in (m,\overline{m}\alpha), \ g \in (\overline{\beta},\overline{\alpha}), \ \alpha,\beta \in \text{Obj}(\mathcal{C})\} \subseteq \mathcal{A}\mathcal{W}_{m,n}$. It is straightforward to check that this is a $\ast$-algebroid ideal with respect to the multiplication defined above. This implies the collection

$$\mathcal{A}\mathcal{W}_{m,n} := \tilde{A}\mathcal{W}_{m,n}/U_{m,n}$$

forms a $\ast$-algebroid with respect to multiplication. Note that $U_{m,n}$ is not graded, i.e. it is spanned by differences of homogeneous components, and thus the quotient does not preserve the grading. We denote the quotient map $\Psi^o : (m,\overline{n}\alpha) \to \mathcal{A}\mathcal{W}_{m,n}$, and use $\Psi$ for the extension to $\tilde{A}\mathcal{W}$.

DEFINITION 3.1. Let $\mathcal{C}$ be rigid $C^*$-tensor category. Let $\mathcal{W} \subseteq \text{Obj}(\mathcal{C})$ such that

1. $id \in \mathcal{W}$

2. and for all $X \in \text{Irr}(\mathcal{C})$, there exists $k \in \mathcal{W}$ such that $X \prec k$.

We define the annular category over $\mathcal{C}$ with weight set $\mathcal{W}$ as the $\ast$-category $\mathcal{A}\mathcal{W}$, with $\text{Obj}(\mathcal{A}\mathcal{W}) = \mathcal{W}$ and $\text{Hom}(k,m) = \mathcal{A}\mathcal{W}_{k,m}$ for $k,m \in \mathcal{W}$. Composition is given by annular multiplication. We call $\mathcal{A}\mathcal{W}_{k,k}$ the weight $k$ centralizer algebra.

Condition (1) ensures there is a “weight 0” space $(\mathcal{A}\mathcal{W}_{id,id})$ so that we can more easily see the approximation properties discussed in section 6. Condition (2) on $\mathcal{W}$ is a fullness condition, which is necessary and sufficient to guarantee that the representation category $\text{Rep}(\mathcal{A}\mathcal{W})$ described in the next section contains the entirety of the information available. In short, neither of these conditions are essential to the definition of an annular category, but it seems that there is not great loss of generality when including them since there is no previously defined example of an annular category that does not satisfy these two conditions.

We introduce a graphical calculus for annular categories. For $f \in (m,\overline{n}\alpha)$, we represent $\tilde{f} \in \mathcal{A}\mathcal{W}_{m,n}$ by
We see that composition described above is given in the graphical calculus by stacking diagrams. We can represent our $\ast$ in this graphical calculus by

$$
\left( \begin{array}{c}
\begin{array}{c}
\alpha \\
\end{array}
\begin{array}{c}
\beta \\
\end{array}
\begin{array}{c}
\gamma \\
\end{array}
\begin{array}{c}
f \\
\end{array}
\begin{array}{c}
\delta \\
\end{array}
\begin{array}{c}
m \\
\end{array}
\end{array}
\right)
\ast
\left( \begin{array}{c}
\begin{array}{c}
\alpha \\
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\right)
= 
\left( \begin{array}{c}
\begin{array}{c}
\alpha \\
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\begin{array}{c}
\gamma \\
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\begin{array}{c}
f \\
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\delta \\
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\begin{array}{c}
m \\
\end{array}
\end{array}
\right)

In the graphical calculus, we may describe $U_{m,n}$ as

$$U_{m,n} = \text{span} \left\{ \begin{array}{c}
\begin{array}{c}
\alpha \\
\end{array}
\begin{array}{c}
\beta \\
\end{array}
\begin{array}{c}
\gamma \\
\end{array}
\begin{array}{c}
f \\
\end{array}
\begin{array}{c}
\delta \\
\end{array}
\begin{array}{c}
m \\
\end{array}
\end{array}
\right| f \in (m, \beta n \alpha), \ g \in (\beta, \alpha), \ \alpha, \beta \in \text{Obj}(\mathcal{C}) \right\}
$$

We see then that the quotient lets us move diagrams from the left side around to the right. This map can be visualized by attaching the left and right strings around the bottom of an annulus. Then taking the quotient by $U_{m,n}$ intuitively forces the pictures

$$
\begin{array}{c}
\begin{array}{c}
\alpha \\
\end{array}
\begin{array}{c}
\beta \\
\end{array}
\begin{array}{c}
\gamma \\
\end{array}
\begin{array}{c}
f \\
\end{array}
\begin{array}{c}
\delta \\
\end{array}
\begin{array}{c}
m \\
\end{array}
\end{array}
= 
\begin{array}{c}
\begin{array}{c}
\alpha \\
\end{array}
\begin{array}{c}
\beta \\
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\left( \begin{array}{c}
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f \\
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\right)
\left( \begin{array}{c}
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\beta \\
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\begin{array}{c}
\gamma \\
\end{array}
\begin{array}{c}
f \\
\end{array}
\begin{array}{c}
\delta \\
\end{array}
\begin{array}{c}
m \\
\end{array}
\end{array}
\right)

This is the origin of the term “annular”. In the planar algebra setting, these annular pictures can be formalized and it is shown in [5] that the result is isomorphic to our definition.

**Conventions:**
(1) We use lowercase Latin letters \((k, m, l, \text{etc.})\) for objects in \(W\).
(2) Capital Latin letters \(X, Y, Z, \text{etc.}\) will be used to denote simple objects (usually in a chosen set of representatives).
(3) Lowercase Greek letters \((\alpha, \beta, \text{etc.})\) will be typically be used to represent arbitrary objects in \(C\) (we do not strictly adhere to this convention, but it will be clear from context what we mean).
(4) Diagrams with box labeled \(f\), and strings labeled \(m, n\) on the bottom and top respectively, and \(\alpha\) on the sides will be taken to mean the image \(\tilde{f} \in \tilde{A}W_{m, n}\) or \(\Psi(f) \in AW_{m, n}\). Which one we mean should be clear from context.
(5) For vertical strings in a diagram, a label to the left will mean the object and a label to the right will be its dual (given by the pivotal structure).
(6) For horizontal strings, a label above the string will signify the object (as introduced in the picture above), and below will be its dual.
(7) All diagrams are to be read bottom to top.

There are several nice choices of \(W\). For example, we could choose \(W = \text{Obj}(C)\), and this choice is obviously canonical. If \(C = \text{Proj}(P)\) for some spherical \(C^*\)-planar algebra \(P\) with string label set \(L\) (see [2] or [15] for definitions), we set \(W := \{\text{words on } L\}\). Then \(AW\) is called the affine annular category of the planar algebra, and has been studied in the subfactor context [17], [5]. In fact this is the motivation for our work. The case where \(W\) is a set of representatives of isomorphism classes of simple objects is called the tube algebra has been studied in various contexts, first appearing in the work of Ocneanu. We will see that in fact this definition is equivalent to the standard one, see [12], [22]. We will discuss this in detail, but first we continue our general analysis of annular categories, and obtain a nice description of the dual space that facilitates the definition of tensor product on the representations category.

For \(m, n \in W\), we define the space of commutativity constraints:

\[
CC_{m, n} := \left\{ \prod_{\alpha \in \text{Obj}(C)} c_{\alpha} : c_{\alpha} \in (am, na), \text{and } \forall g \in (\beta, \alpha), \right\}
\]

Another way to define commutativity constraints is a family of morphism \(c_{\alpha} \in (am, na)\) natural in the argument \(\alpha \in \text{Obj}(C)\).

We define a duality pairing between \((an, ma)\) and \((m, an\overline{a}) \subseteq AW_{m, n}\) by the standard trace duality pairing, given for \(f \in (m, an\overline{a})\) and \(g \in (an, ma)\), by

\[
(f, g) = Tr_n \left( (R_{\alpha}^* \otimes 1_n \otimes \overline{R}_{\alpha})(1_{\overline{\pi}} \otimes f \otimes 1_{\alpha})(1_{\overline{\pi}} \otimes g)(R_{\alpha} \otimes 1_n) \right)
\]

This is given in pictures by
Proposition 3.2. \( CC_{n,m} \) is isomorphic to the (algebraic) dual space \( (AW_{m,n})^\# \), given by the pairing

\[
c(f) = \sum_{i} (u_i^* \otimes 1_n \otimes (u_i^*)^*) \circ f
\]

Proof. First, by definition, \((\tilde{A}W_{m,n})^\# \cong \prod_{\alpha \in \text{Obj}(C)} (an, ma)\) where each \((an, ma)\) acts on the summand \((m, an\alpha) \subseteq \tilde{A}W_{m,n}\) by the pairing described above. Viewing \(c \in CC_{n,m}\) as an element in \((\tilde{A}W_{m,n})^\#\), we see that our pairing described in the proposition agrees on each component with the standard pairing. Now by definition \(c(U_{m,n}) = 0\), and thus \(c\) descends to a linear functional on \(AW_{m,n}\). To see that every linear functional is of this form, we note that if \(\phi \in (AW_{m,n})^\#\), then \(\phi\) can be viewed as a functional on \((\tilde{A}W_{m,n})\) such that \(\phi(U_{m,n}) = 0\). Thus using our standard pairing along with the quotient map \(\Psi\) we see that \(\phi\) can be realized as \(c := \prod_{\alpha \in \text{Obj}(C)} c_\alpha\), where each \(c_\alpha \in (an, ma)\). To see that \(c\) is a commutativity constraint, for any \(g \in (\beta, \alpha)\),

Since this is true for all \(f\), by non-degeneracy of the categorical trace, this implies \(c \in CC_{n,m}\). 

Here we follow [5] to analyze the morphism space, with one minor difference: here we use the categorical (unnormalized) trace, while in [5] they use the normalized trace from the planar algebra.

We define a map \(E : AW_{m,n} \to (m, n)\) as follows: For each object \(\alpha\), let \(\{u^i\}_\alpha\) be an orthonormal basis (with respect to the trace pairing) of \((id, \alpha)\). We then define \(E_\alpha : (m, \overline{\alpha}na) \to (m, n)\) by

\[
E_\alpha(f) = \sum_i \left( u^i_\alpha \otimes 1_n \otimes (u^i_\alpha)^* \right) \circ f
\]

Extending by linearity, we have a map \(E : \tilde{A}W_{m,n} \to (m, n)\). It is clear that this map does not depend on choice of basis. Simple linear algebra shows that \(E(U_{m,n}) = 0\), so this map descends to a map on \(AW_{m,n}\).

Definition 3.3. We define a tracial functional \(\omega : AW_{m,m} \to \mathbb{C}\) by \(Tr_m \circ E\).
That this is tracial follows from sphericity of the categorical trace. This defines an a priori positive semi-definite inner product on \( AW_{m,n} \) given by \( \langle f,g \rangle = \omega(g^*f) \). We will show this form is positive definite.

First we consider a convenient decomposition of \( AW_{m,n} \) which will make these facts clear.

Let \( \Lambda \) be a choice of simple objects, one for each equivalence class of simple objects.

**Lemma 3.4.** Every \( f \in AW_{m,n} \) can be written as \( f := \sum f_X \) where \( f_X \in \Psi^X(m, \underline{X}nX) \), \( X \in \Lambda \).

**Proof.** If \( f \in (m, \underline{\alpha}n\alpha) \), for each \( X \sim \alpha \), we let \( \{ b_{i,X}^\alpha \}_{1 \leq i \leq N_\alpha} \) be a basis of \((\alpha, X)\) such that \( b_{j,X}^\alpha (b_{i,X}^\alpha)^* = \delta_{i,j}1_X \). Then we see that

\[
1\alpha = \sum_{X \sim \alpha} \sum_{1 \leq i \leq N_\alpha} (b_{i,X}^\alpha)^* b_{i,X}^\alpha
\]

Thus \( f = \sum_{X \sim \alpha} \sum_{1 \leq i \leq N_\alpha} (1\alpha \otimes 1_n \otimes (b_{i,X}^\alpha)^* b_{i,X}^\alpha) \circ f \)

Therefore

\[
\Psi(f) = \sum_{X \sim \alpha} \sum_{1 \leq i \leq N_\alpha} \Psi((b_{i,X}^\alpha)^* \otimes 1_n \otimes b_{i,X}^\alpha \circ f)
\]

But each of the terms in the sum is of the desired form. Extend by linearity to obtain the desired result \( \square \).

Now that we have a nice form for elements in \( AW_{m,n} \), we have the following

**Proposition 3.5.** The sesquilinear form \( \langle \cdot, \cdot \rangle : AW_{m,n} \times AW_{m,n} \to \mathbb{C} \) given by \( \langle f,g \rangle = \omega(g^*f) \) is positive definite.

**Proof.** Let \( f = \sum f_X \in AW_{m,n} \) be as above. Then we have \( f^* = \sum_{X,Y} f_X^* f_Y \) where here \(*\) is taken in \( AW \). We notice that \( f_X^* f_X \in \Psi(m, \underline{X}Y nY X) \). Since \( X,Y \) are simple we see that \((id, YX) = 0\) unless \( Y \equiv X \) (recall that \( X,Y \in \Lambda \) are chosen representatives, hence we actually have this hom space if 0 if \( Y \neq X \)). If \( Y = X \), we have a choice of orthonormal basis given a normalized solutions to the conjugate equations, namely \( \frac{1}{d(X)^2} R \), where \( R \) is standard and is dictated by the pivotal structure. By definition of \(*\) on \( AW \), we see that if \( f_X = \Psi(g_X) \) with \( g_X \in (m, \underline{X}nX) \), then unraveling the definition of \(*\), we see that \( E(f_X^* f_X) = \frac{1}{\pi^{1/2}} g_X^* g_X \), where the \(*\) on \( g_X \) and composition is taken in \( C \). Thus

\[
\omega(f^* f) = Tr_m(E(f^* f)) = \sum_X \frac{1}{d(X)^2} Tr_m(g_X^* g_X)
\]

Since \( Tr_m \) is positive definite in \( C \), we see that this is positive, and 0 if and only if each \( g_X \) is 0. But then

\[
f = \sum \Psi(g_X) = 0 \square.
\]

The above proof implies several things. First, we see that \( \Psi : \bigoplus_{X \in \Lambda} (m, \underline{X}nX) \to AW_{m,n} \) is injective. We have already seen that it is surjective. If we define \( AW_{m,n}^X := \Psi(m, \underline{X}nX) \) we see that \( AW_{m,n} \cong (m, \underline{X}nX) \), as finite dimensional Hilbert spaces. This leads to the following corollary:

**Corollary 3.6.** \( AW_{m,n} \cong \bigoplus_{X \in \Lambda} AW_{m,n}^X \), where each \( AW_{m,n}^X \cong (m, \underline{X}nX) \) as finite dimensional vector spaces. Furthermore, this direct sum is orthogonal with respect to \( \omega \).
Definition 3.7. Let $\Lambda$ be a choice of representatives for the set $\text{Irr}(C)$. Then we define the tube algebra of $C$ as the $\ast$-category $TC := AA$. We see that $TC$ does not depend on the choice of representatives $\Lambda$ up to unitary equivalence.

We single out this particular choice of $W$ due to its special importance. For any annular category over $C$, its representations theory, which we define in the next section, will be equivalent to that of the tube algebra. The tube algebra was first introduced by Ocneanu in the finite depth case, where instead of a category it is viewed a single algebra. It was further studied in the type III subfactor context by Izumi [12] and in the categorical setting by Müger [22]. To see that our definition is really the same as the standard definition of the tube algebra, up to normalization.

For each admissible triplet of simple objects $U,Z,W$, we pick an orthogonal basis $B_{ZW}^{U} := \{t_{j}^{U} : 1 \leq j \leq N_{ZW}^{U}\} \subseteq \text{Hom}(U,ZW)$, with standard normalization so that $\sum_{U \prec Z,W} \sum_{1 \leq j \leq N_{ZW}^{U}} t_{j}^{U} (t_{j}^{U})^{*} = 1_{ZW}$, and $(t_{j}^{U})^{*} t_{j}^{U} = \delta_{j,1_{U}}$.

For $f \in T_{V,X}^{W}$ and $g \in T_{V,Y}$, we see that the annular multiplication is given by

$$fg = \sum_{U \prec Z,W} \sum_{1 \leq j \leq N_{ZW}^{U}} U \sum_{t_{j}^{U}} f \sum_{Z} \sum_{X} g \sum_{Y}.$$ 

This is precisely the standard definition of the tube algebra, up to normalization.

We fix a set $\Lambda$ of representatives and identify it as a set with $W$. Then resolving the identities $1_{k}$ and $1_{m}$ as above $1_{k} = \sum_{X \prec k} \sum_{1 \leq i \leq N_{k}^{X}} (b_{i,k}^{X})^{*} b_{i,k}^{X}$, and similarly for $1_{m}$, it is straightforward to check that

$$AW_{m,k} \cong \bigoplus_{X \prec k, Y \prec m} (X,k) \otimes T_{Y,X} \otimes (m,Y).$$

Since we require that each for $X \in \Lambda$ there is some $m \in W$ such that $X \prec m$, we see that the category $AW$ contains the tube category $TC$ as a “subcategory”. In fact $AW$ is simply a sequence of finite dimensional “amplifications” of the tube algebra.

For each $X \in \Lambda$, let $\hat{X} \in \Lambda$ denote the object chosen in the equivalence class $[X]$ (recall $X$ is dictated by the pivotal structure). Let $t_{X}^{\hat{X}} \in (\hat{X}, X)$ such that $t_{X}^{\hat{X}} t_{X} = 1_{\hat{X}}$ and $t_{X} t_{X}^{\hat{X}} = 1_{\hat{X}}$. We define $r : T_{X,Y} \rightarrow T_{\hat{Y},\hat{X}}$ as follows. If $f \in (X, \pi Y \alpha)$, recall $f \in (\pi Y \alpha, X)$, and define

$$r(f) = (1_{\alpha} \otimes (t_{X}^{\hat{X}} \circ \overline{f}) \otimes 1_{\pi}) \circ (\overline{R_{\alpha}} \otimes t_{X}^{\hat{X}} \otimes R_{\alpha}) \in (\hat{Y}, \alpha \hat{X} \alpha)$$

where $\overline{R_{\alpha}}, R_{\alpha}$ are standard solutions to the conjugate equations for $\alpha$ dictated by the pivotal structure we have chosen. Then $r$ extends to a linear map on the pre-annular category $AA$, and it is straightforward to check that this preserves the algebroid ideal $U$ (draw pictures!), and thus $r$ is well-defined on the quotient. It is easy to see that this map is an anti-homomorphism, $r(fg) = r(g)r(f)$ for $f \in T_{Y,Z}, g \in T_{X,Y}$. 

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We call \( r : TC \to TC \) the global duality map. If we view our category as sitting on a vertical cylinder (as in the annular picture above) this map corresponds to a 180 degree rotation of the cylinder.

4. Representations

**Definition 4.1.** A Hilbert representation of \( AW \) is a representation of the \( * \)-category \( AW \) in the category of Hilbert spaces and bounded operators. This means

1. A sequence of Hilbert spaces \( (H_k)_{k \in W} \).
2. A sequence of linear maps \( \pi_{k,l} : AW_{k,l} \to B(H_k, H_l) \) which are \( * \)-homomorphisms with respect to annular multiplication.

We will often denote a Hilbert representation simply by \((\pi, H)\) or even just \(H\), and use \( \pi \) to denote all \( \pi_{k,l} \).

**Definition 4.2.** A bounded intertwiner between two Hilbert representations \((\pi, H)\) and \((\gamma, V)\) is a collection of maps \(\{(f_k)_{k \in W} : f_k \in B(H_k, V_k), \sup_k ||f_k|| < \infty, \text{ and for all } x \in AW_{k,m}, \gamma(x)f_k = f_m\pi(x)\}\)

**Definition 4.3.** \( \text{Rep}(AW) \) is the \( C^* \)-category whose objects are Hilbert representations of \( AW \) and whose morphisms are bounded intertwiners.

We will define a tensor functor on \( \text{Rep}(AW) \). If \((\pi, H) \in \text{Rep}(AW)\), then we can define for \( \eta \in H_k \) and \( \xi \in H_m \) a linear functional \( \langle \pi(\cdot)\eta, \xi \rangle : AW_{k,m} \to \mathbb{C} \). We denote this functional \( \phi_{\eta,\xi} \in (AW_{k,m})^* \cong CC_{k,m} \).

Now, if \( H, K \in \text{Rep}(AW) \), we define for \( n \in W \) the linear space

\[
(H \otimes K)_n := \bigoplus_{k,m,l} AW_{l,n} \otimes \text{Hom}(km,l) \otimes (H_k \otimes K_m)
\]

A closure of a suitable quotient of this space will be the weight \( n \) space of the tensor product of the two modules. First we need an inner product. We define this inner product on simple tensors \( f_1 \otimes x_1 \otimes (\eta_1 \otimes \xi_1) \in AW_{l_1,n} \otimes (k_1 m_1, l_1) \otimes H_{k_1} \otimes K_{m_1} \) and \( f_2 \otimes x_2 \otimes (\eta_2 \otimes \xi_2) \in AW_{l_2,n} \otimes (k_2 m_2, l_2) \otimes H_{k_2} \otimes K_{m_2} \) by

\[
\langle f_1 \otimes x_1 \otimes (\eta_1 \otimes \xi_1), f_2 \otimes x_2 \otimes (\eta_2 \otimes \xi_2) \rangle = \ldots
\]

In the above pictures, \( \alpha \) and \( \beta \) are arbitrary objects representing the \( f \)s. That this is well defined follows from the definition of commutativity constraints. Following the proof in [5] one can show that this is positive semidefinite, and the natural action of the annular morphisms are bounded and leave the kernel of the form \( \langle \cdot, \cdot \rangle \) invariant. Taking the completion of \((H \otimes K)_n\) for each \( n \in W \) yields Hilbert spaces \( H_n \) and the natural action of the annular category by action on the first component in the tensor product extends to a full Hilbert
AW representation, denotes $H \boxtimes K$. One can show that indeed this is a tensor functor, and that the tensor product of bounded intertwiners is again bounded intertwiners. The tensor functor is not strict. The unit is given by the trivial representation, as described in the next section. There is also a unitary braiding on this category. For more details, see [5].

**Theorem 4.4.** For any annular category $AW$ of $C$, the $C^*$-category $\text{Rep}(AW)$ is equivalent to the $C^*$-category $\text{Rep}(TC)$.

**Proof.** First we will establish a functor $F : \text{Rep}(AW) \to \text{Rep}(TC)$. Let $(\pi, \{H_k\}) \in \text{Rep}(AW)$. For each $X \in \text{Irr}(C)$, we fix a projection $P_X \in (m, m) \subseteq AW_{m,m}$ equivalent to $X$ for some $m \in W$ such that $X \prec m$. When we write $(k, X)$ we mean the subset of $(k, m)$ such that $Xf = f$. We define $H_X := (\pi((k, X))H_k)_{k \in W} \subset H_m$. We use the natural isomorphism described in the previous subsection to identify $T_{X,Y} \cong P_Y AW_{m,n}P_X$. Then for $f \in T_{X,Y}$, $\pi(f) \in B(H_Y, H_X)$, and the induced $*$ structure on $TC$ is the right one, since it is correct for $AW$.

We have define a map $F : \text{Rep}(AW) \to \text{Rep}(TC)$ on objects. If $f = (f_k)_{k \in W}$ is an intertwiner between representations $(\pi, H)$ and $(\gamma, V)$ then we define $F(f) = (F(f)_X)_{X \in \text{Irr}(C)}$ as follows. If we have chosen $m$ as above so that $X$ appears in $m \in W$, every $\eta \in H_X$ is of the form $\pi(y)\mu$ for $y \in (k, X) \subseteq (k, m)$ and $\mu \in H_k$. Then $F(f)_X(\eta) = f_m(\eta) \in \gamma(y)f_k(V_k) \subset V_X$. Thus in fact $F(f)_X = f_m|\pi(k, X)H_k$. It is easy to see that in fact this is still an intertwiner between the tube algebra actions, and that $\|F(f)_X\| \leq ||f_m||$. Thus $\sup_{X \in \text{Irr}(C)} ||F(f)_X|| \leq \sup_{m \in W} ||f_m|| < \infty$.

To show our functor is an equivalence, we first claim that $H_k \cong \bigoplus_{X \prec k} (X, k) \otimes H_X$.

To see this isomorphism, we define the map by the map $Q : \bigoplus_{X \prec k} (X, k) \otimes H_X \to H_k$ defined by $Q(y \otimes \eta) = \pi(y)\eta$. Note that this direct sum is orthogonal, and that $Q$ preserves this orthogonality. For each $X \prec k$, let $\{x_\alpha\}$ be an orthogonal basis for $(X, k)$, normalized so that $x_\beta^*x_\alpha = \delta_{\alpha,\beta}1_X$. Since $(X, k)$ is finite dimensional, an arbitrary vector in $(X, k) \otimes H_X$ can be written as $\sum_\alpha x_\alpha \otimes \eta_\alpha$ for $\eta_\alpha \in H_X$. We note that if $\sum_\alpha \pi(x_\alpha)\eta_\alpha = 0$, we see that for a given $\beta$, $\pi(x_\beta^*)\sum_\alpha \pi(x_\alpha)\eta_\alpha = \pi(1_X)\eta_\beta = \eta_\beta = 0$. This is true for all $\beta$, and thus we have that $Q$ is injective. It is obviously surjective.

To show our functor is faithful, we need $\text{Hom}(H, V) \cong \text{Hom}(F(H), F(V))$. In particular, we need to show that $F$ is surjective on $\text{Hom}$ spaces, i.e. for a tube intertwiner $f = (f_X)_{X \in \text{Irr}(C)} \in \text{Hom}(F(H), F(V))$, there is some annular intertwiner $g = (g_k)_{k \in W} \in \text{Hom}(H, V)$ with with $F(g) = f$. To define $g$, using our isomorphism above, we define

$$g_k(\sum_{X \prec k} \sum_\alpha x_\alpha \otimes \eta_\alpha) := \sum_{X \prec k} \sum_\alpha x_\alpha \otimes f_X(\eta_\alpha)$$

It is clear that this is an $H-V$ intertwiner and that $F(g) = f$. From our construction, we see that $g$ is unique among intertwiners between the annular modules such that $F(g) = f$, and thus $F$ provides an isomorphism between $\text{Hom}(H, V)$ and $\text{Hom}(F(H), F(V))$. Thus our functor is fully faithful.

Given a $(\pi, H) \in \text{Rep}(TC)$, we can easily define a representation of $AW$. Define $\tilde{H}_k := \bigoplus_{X \prec k} (X, k) \otimes H_X$. Using our tube algebra decomposition $AW_{m,k} \cong \bigoplus_{X \prec k, Y \prec m} (X, k) \otimes T_{Y,X} \otimes (m, Y)$ we can define the obvious
actions of the annular category. This gives us a representation \((\hat{\pi}, \hat{H}) \in \text{Rep}(\text{AW})\), and it is clear from construction that \(F(\hat{\pi}, \hat{H}) \cong (\pi, H)\).

We remark that in fact the functor \(F\) can define a monoidal equivalence between \(\text{Rep}(\text{AW})\) and \(\text{Rep}(\text{TC})\), with the obvious choice of morphisms. We do not include a proof here since it is beyond the scope of this paper.

4.1. \(\text{Rep}(\text{TC})\) and \(\text{Z}(\text{ind-}C)\). We now give another realization of the category \(\text{Rep}(\text{TC})\) as the oppositite category of the unitary Drinfeld center on inductive objects, introduced by Neshveyev and Yamashita [24]. This equivalence was pointed out to us by Stefaan Vaes, and we include this section to elaborate on his proof.

We give the definitions from [24] for \(\text{ind-}C\).

An inductive object of \(C\) will be an inductive system \(\{u_{ji} : X_i \rightarrow X_j\}_{i < j}\) where \(X_i, X_j \in \text{Obj}(C)\) and \(u_{ji}\) are isometries. A morphism between two inductive objects \(\{u_{ji} : X_i \rightarrow X_j\}_{i < j}\) and \(\{v_{lk} : Y_k \rightarrow Y_l\}_{k < l}\) is a collection \(T\) of morphisms \(T_{ki} : X_i \rightarrow Y_k\) in \(C\) such that

\[ u^*_{lk}T_{ki} = T_{ki} \text{ if } k < l, \quad T_{kj}u_{ji} = T_{ki} \text{ if } i < j, \quad \text{and } \|T\| = \sup_{k,i} \|T_{ki}\| < \infty \]

If \(S, T\) are morphisms between inductive objects, we define their composition by

\[ (ST)_{ni} = \lim_{k} S_{nk}T_{ki} \]

In [24], they show that this composition is well defined and associative. Furthermore, they show that every inductive object is isomorphic to one of the form \(\bigoplus_{i \in I} X_i\), for some index set \(I\). We can refine this by saying any object is determined, up to isomorphism, by collection of cardinalities associated to each simple object (objects in the actual category \(C\) will have finite cardinalities, and only finitely many simply objects will have non-zero cardinalities). We call the category with inductive objects and morphisms \(\text{ind-}C\). Neshveyev and Yamashita show that this category is a \(C^*\) tensor category, with direct sums, subobjects, and simple unit. This category not rigid.

A unitary half-braiding in \(\text{ind-}C\) is an object \(Z \in \text{Obj}(\text{ind-}C)\) together with a collection of unitaries for \(X \in \text{Obj}(C)\), \(c_X : X \otimes Z \rightarrow Z \otimes X\), natural in \(X\), satisfying \(c_{X \otimes Y} = (c_X \otimes i_Y)((i_X \otimes c Y)\). Although we only require \(c_X\) for \(X \in \text{Obj}(C)\), this uniquely defines a family for \(X \in \text{Obj}(\text{ind-}C)\).

Following the definition for Drinfeld center in the typical case, we have the braided tensor category \(\text{Z}(\text{ind-}C)\), where objects are given by inductive objects with a unitary half-braiding, and homs are homs between the objects which intertwine the half-braiding.

\(\text{ind-}C\) can also be realized as a category of Hilbert space valued presheafs on \(C\). More explicitly, we define a new category \(D\) as follows: \(\text{Obj}(D)\) will be covariant additive functors \(F : C \rightarrow \text{Hilb}\). To be clear, this means that to each object \(\alpha\), we associate a Hilbert space \(F(\alpha)\), and \(F(\alpha, \beta) \subset B(F(\alpha), F(\beta))\). Additivity implies that \(F(\alpha + \beta) \cong F(\alpha) \oplus F(\beta)\). This implies among other things that \(F\) is determined by its values on simple objects. Thus an object of \(D\) will be a collection of Hilbert spaces for each simple object, and unitaries between equivalent objects. If \(F, G\) are two such functors, \(\text{Hom}(F, G)\) will be natural transformations \(\{a_\alpha \in B(F(\alpha), G(\alpha)) : \alpha \in \text{Obj}(C)\}\) such that \(\sup_{\alpha \in \text{Obj}(C)} \|a_\alpha\| < \infty\).

We have a natural inclusion of \(C^{op}\) (the same category with arrows reversed) into \(D\) by sending \(\alpha\) to the functor \(\text{Hom}_C(\alpha, \bullet)\), which sends elements of \(\text{Obj}(C)\) to finite dimensional morphism spaces with inner
product coming from the unnormalized canonical categorical trace. It is straightforward to check that this is an equivalence of $\mathcal{C}^{\text{op}}$ onto a full subcategory of $\mathcal{D}$.

We choose a set of representatives of simple objects $\Lambda$. We define a tensor functor with respect to this collection of simple objects $\otimes : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ on objects by

$$(F \otimes G)(\alpha) = \oplus_{i,j} F(X_i) \otimes (X_i X_j, \alpha) \otimes G(X_j)$$

where $(X_i X_j, \alpha)$ is the finite dimensional morphism space of $\mathcal{C}$ with inner product induced by the unnormalized canonical categorical trace. It is straightforward to check that this object depends on $\Lambda$ up to isomorphism in $\mathcal{D}$. The definition of tensor product of natural transformations is the obvious ones. That this tensor functor is associative follows from the associativity of the tensor functor in $\mathcal{C}$.

We have a contravariant functor from $\text{Ind-}\mathcal{C}$ to $\mathcal{D}$ as follows. For each object $\alpha \in \text{Obj}(\mathcal{C})$ and $Z \in \text{Obj}(\text{ind-}\mathcal{C})$, we define a Hilbert space $\hat{Z}(\alpha) = \text{Hom}_{\text{ind-}\mathcal{C}}(Z, \alpha)$, with inner product given by the categorical trace $(f, g) = \text{Tr}(fg^*)$. Then we can view $\hat{Z}(\cdot)$ as an additive, contravariant functor from $\mathcal{C}$ to $\text{Hilb}$, where a morphism $g \in \text{Hom}_{\text{ind-}\mathcal{C}}(Z, W)$ defines a natural transformation in $\text{Hom}_{\text{op}}(\hat{W}, \hat{Z})$ by $\hat{g}_\alpha(x) = x \circ g \in \hat{Z}(\alpha)$ for $x \in \hat{W}(\alpha)$. It is easy to see that $\hat{g} \in \text{B}(\hat{W}(\alpha), \hat{Z}(\alpha))$ for all $\alpha \in \text{Obj}(\mathcal{C})$. Furthermore, it is easy to see that the adjoint operator $\hat{g}^*(x) = x \circ (g^*)$ for all $x \in \hat{Z}(\alpha)$ and for all $\alpha \in \text{Obj}(\mathcal{C})$. It is straightforward to check that functor induces a monoidal (contravariant) equivalence (via [24 Proposition 1.4]).

We now present the following result, which is due to Stefaan Vaes. We thank Stefaan for allowing us to include his argument in our paper (the argument presented here is modified to fit with our previously established language).

**Proposition 4.5. [Vaes]** The categories $\text{Rep}(\mathcal{T}\mathcal{C})$ and $\text{Z}(\text{ind-}\mathcal{C})$ are contravariantly equivalent.

**Proof.** Suppose we have an inductive object $Z$ with unitary half-braiding $c = \{c_\alpha : \alpha \in \text{Obj}(\mathcal{C})\}$. Let $\Lambda$ be a collection of simple objects representing $\text{Irr}(\mathcal{C})$. For each simple object $X_i \in \Lambda$, define the Hilbert space $K_i := \text{Hom}_{\text{ind-}\mathcal{C}}(Z, X_i)$, with inner product given by $\langle f, g \rangle = \text{Tr}_{X_i}(fg^*)$. For each $\xi \in K_i, \eta \in K_j$, we define for every $\alpha \in \text{Obj}(\mathcal{C})$ the morphism

$$c_\alpha(\xi, \eta) = (\xi \otimes 1_\alpha) c_\alpha(1_\alpha \otimes \eta^*) \in (\alpha X_j, X_i \alpha)$$

Since $c_\alpha$ is a half-braiding, the family

$$c(\xi, \eta) := \frac{1}{\sqrt{d(X_i)d(X_j)}} \prod_{\alpha \in \text{Obj}(\mathcal{C})} c_\alpha(\xi, \eta)$$

is an $i, j$ commutativity constraint, $c(\xi, \eta) \in CC_{X_i, X_j}$. We then define a representation of $\pi : \mathcal{T}\mathcal{C} \rightarrow \mathcal{B}(K)$ by

$$\langle \pi(f)\xi, \eta \rangle = c(\xi, \eta)(f)$$

for $f \in T_{X_i, X_j}$, where here we use the duality paring on commutativity constraints described in the previous section (Proposition 3.2).

For $f \in T_{X_i, X_j}$, $g \in T_{X_j, X_k}$, the formula $\pi(g \circ f) = \pi(g) \circ \pi(f)$ follows from the fact that the half-braiding $\{c_\alpha : \alpha \in \text{Obj}(\mathcal{C})\}$ is a natural tensor transformation, whereas $\pi(f^*) = \pi(f)^*$ follows from its unitarity. For boundedness of the operator $\pi(f)$, $f \in T_{X_i, X_j}^a$, note that $K_i \times K_j \ni (\xi, \eta) \mapsto \text{Tr}((1_\alpha X_j \otimes R_\alpha) \circ ((f \circ \xi) \otimes 1_\alpha)) c_\alpha(1_\alpha \otimes \eta^*) \in \mathbb{C}$ is a bounded sesquilinear form.
This gives us a map on objects \( F : \text{Z(ind-C)} \to \text{Rep(TC)} \). To see that this map induces a contravariant functor, let \((Z, c^Z)\) and \((W, c^W)\) be inductive objects with unitary, and \{K_i^Z\} and \{K_i^W\} be the Hilbert spaces with induced representations of the tube algebra described above. Let \( g \in \text{Hom}_{Z(\text{ind-C})}(\{Z, c^Z\}, \{W, c^W\}) \). Then we have the natural transformation \( \hat{g} \), which induces maps \( \hat{g}_i : K_i^W \to K_i^Z \). We claim that \( \hat{g} \) is a tube algebra intertwiner. To see this, let \( \xi \in K_i^W \), \( \eta \in K_j^Z \). We claim that \( c^Z_\alpha(\hat{g}_\alpha(\xi), \eta) = c^W_\alpha(\xi, \hat{g}_\alpha^*(\eta)) \). To see this, we have

\[
c^Z_\alpha(\hat{g}_\alpha(\xi), \eta) = (\xi \otimes g) \otimes 1_\alpha \otimes \eta^* = (\xi \otimes 1_\alpha) \otimes (g \otimes \eta^*) = (\xi \otimes 1_\alpha) c^W_\alpha(1 \otimes (g \otimes \eta^*))
\]

Thus \( c^Z(\hat{g}_\alpha(\xi), \eta) = c^W(\xi, \hat{g}_\alpha^*(\eta)) \). Now, for \( x \in T_{X_1, X_2} \), we have

\[
\langle \hat{g}_\alpha(x) \xi, \eta \rangle = \langle \pi(x) \xi, \hat{g}_\alpha^*(\eta) \rangle = c^Z(\xi, \hat{g}_\alpha^*(\eta))(x) = c^W(\hat{g}_\alpha(\xi), \eta)(x) = \langle \pi(x) \hat{g}_\alpha(\xi), \eta \rangle
\]

Hence, we have established a contravariant functor \( F : \text{Z(ind-C)} \to \text{Rep(TC)} \), which sends morphisms \( F(g) = \hat{g} \). It is straightforward to check this functor is fully faithful.

For essential surjectivity of \( F \), we construct an element of \((\hat{Z}, c) \in Z(D)\) from \( K \in \text{Rep(TC)} \), whose \( F \)-image is \( K \) modulo the (contravariant) equivalence between \text{ind-C} and \( D \). We set \( \hat{Z} \) as the covariant functor \( \otimes \text{[K} \otimes (X_j, \bullet)\] : C \to \text{Hilb} \). Now, we will define the half-braiding \( c : \bullet \otimes \hat{Z} \to \hat{Z} \otimes \bullet \). For each \( G \in \text{Obj}(D) \), \( \alpha \in \text{Obj}(C) \), we need to have an unitary \( c_{G, \alpha} \in B((\hat{Z} \otimes \hat{Z})(\alpha), (\hat{Z} \otimes \hat{Z})(\alpha)^\ast) \). From the definition of \( \otimes \) on \( D \), one can easily deduce that \( (G \otimes \hat{Z})(\alpha) = \bigoplus_{j \in \Lambda} \left(G(X_j) \otimes \left[\bigoplus_{i < \alpha} K_i \otimes (X_j, \alpha)\right]\right) \), whereas \( (\hat{Z} \otimes G)(\alpha) = \bigoplus_{j \in \Lambda} \left[\left[\bigoplus_{i < \alpha} K_i \otimes (X_j, \alpha)\right] \otimes G(X_j)\right] \). For \( \alpha \in \text{Obj}(C) \), \( j \in \Lambda \), we first define the operator

\[
c_{\alpha, j} = (c^{i_1, i_2}_{\alpha, j}) : \bigoplus_{i_1 < \alpha} [(X_j, X_{i_1}, \alpha) \otimes K_{i_1}] \to \bigoplus_{i_2 < \alpha} [K_{i_2} \otimes (X_{i_2}X_j, \alpha)]
\]

by

\[
c^{i_1, i_2}_{\alpha, j}(a_1 \otimes \xi_1) = \sqrt{d(i_1)d(i_2)} \sum_{a_2 \in \text{onb}(X_{i_2}X_j, \alpha)} \Psi^j \left([1_{X_j} \otimes (a_2^\ast \circ a_1)] \circ [R_{X_j} \otimes 1_{X_{i_2}}]\right)(\xi_1) \otimes a_2
\]

for \( \xi_1 \in K_{i_1}, a_1 \in (X_jX_{i_1}, \alpha) \) where the choice of orthonormal basis of \((X_{i_2}X_j, \alpha)\) does not matter, and \( \Psi^j : (X_{i_1}, \bar{J}X_{i_2}) \to T_{X_{i_1}, X_{i_2}} \) is the canonical map. In order to establish that \( c_{\alpha, j} \) is an isometry, we compute

\[
\langle c_{\alpha, j}(a_1 \otimes \xi_1), c_{\alpha, j}(a_3 \otimes \xi_3) \rangle = \sqrt{d(i_1)d(i_3)} \sum_{i_2 < \alpha} d(i_2) \sum_{a_2 \in \text{onb}(X_{i_2}X_j, \alpha)} \langle \Psi^j \left([1_{X_j} \otimes (a_2^\ast \circ a_1)] \circ [R_{X_j} \otimes 1_{X_{i_2}}]\right)(\xi_1) \otimes a_2, \Psi^j \left([1_{X_j} \otimes (a_2^\ast \circ a_3)] \circ [R_{X_j} \otimes 1_{X_{i_2}}]\right)(\xi_3) \otimes a_2 \rangle
\]
for $\xi_1 \in K_{i_1}, \xi_3 \in K_{i_3}, a_1 \in (X_j X_{i_1}, \alpha), a_3 \in (X_j X_{i_3}, \alpha)$. The second equality follows from the relation

$$\sum_{i_2 < \alpha} \sum_{a_2 \in \onb(X_{i_2} X_j, \alpha)} d(i_2) = 1_{\alpha X_j}$$

Pulling the cap from the right side of the commutativity constraint $c_{jj}(\xi_1, \xi_3)$ to its left, we get

$$\langle c_{\alpha,j}(a_1 \otimes \xi_1), c_{\alpha,j}(a_3 \otimes \xi_3) \rangle = \delta_{i_1,i_3} \langle a_1, a_3 \rangle \langle \xi_1, \xi_3 \rangle$$

For proving that $c_{\alpha,j}$ is a co-isometry, one needs to first deduce

$$(c_{\alpha,j}^{i_1,i_2})^* (\xi_2 \otimes a_2) = \sqrt{d(i_1)d(i_3)} \sum_{a_1 \in \onb(X_{i_1} X_j, \alpha)} a_1 \otimes \Psi_j \left( ([a_1^* \circ a_2] \otimes 1_{X_j}) \circ [1_{X_{i_2}} \otimes R_{X_j}] \right) (\xi_2)$$

for $\xi_2 \in K_{i_2}, a_2 \in (X_{i_2} X_j, \alpha)$, which follows since our representation is a $*$-representation. Then follow the same steps as above. Thus, $c_{\alpha,j}$ is a unitary. Now, extend $c_{\alpha,j}$ to the unitary $c_G,\alpha$ by acting trivially on the elements of $G(X_j)$'s. It immediately follows from the definition that $c_G = (c_{G,\alpha})_{\alpha \in \text{Obj}(\mathbb{G})} : G \otimes \hat{Z} \rightarrow \hat{Z} \otimes G$ and $c = (c_G)_{G \in \text{Obj}(\mathbb{D})} : \bullet \otimes \hat{Z} \rightarrow \hat{Z} \otimes \bullet$ are natural. The proof $(c_{G_1} \otimes 1_{G_2}) \circ (1_{G_1} \otimes c_{G_2}) = c_{G_1 \otimes G_2}$ is a bit long but completely routine $\Box$.

We notice that we can simply pull back the tensor functor naturally defined on $Z(\text{ind} - \mathcal{C})$ to induce a tensor structure on $\text{Rep}(T\mathcal{C})$, instead of the more complicated one described above, giving a somewhat simpler tensor functor. These two tensor functors are essentially the same, a point which will be addressed in future work.

### 4.2. Affine states and Universal norms.

**Definition 4.6.** A linear functional $\phi \in (AW_{k,k})^\#$ is called an **affine state** if:

1. $\phi(1_k) = 1$
2. $\phi(a^* \circ a) \geq 0$ for all $a \in AW_{k,m}, m \in \mathbb{W}$

We denote that set of affine states on $AW_{k,k}$ by $\Phi_{AW_k}$. 
If we have \( V \in \text{Rep}(AW) \), then for \( \eta \in V_k \) with \( ||\eta|| = 1 \), \( \phi_\eta(\cdot) = \langle \pi_V(\cdot)\eta, \eta \rangle \in \Phi AW_k \). We will see that all affine states are of this form, by performing a GNS type construction.

**Definition 4.7.** An affine module \( V \in \text{Rep}(AW) \) is called \( k \)-cyclic for \( k \in \mathcal{W} \) if there exists \( \omega \in V_k \) such that \( V_m = \pi_V(AW_{k,m})\omega \).

For any \( V \in \text{Rep}(AW) \) with \( 0 \neq \eta \in V_k \), we can define a cyclic module by taking the closure of the range of our annular operators applied to this vector.

**Lemma 4.8.** For each \( b \in AW_{m,n} \), there is a positive constant \( M_b \) such that for all \( k \in \mathcal{W} \), \( \phi \in \Phi AW_k \), and every \( a \in AW_{k,m} \), \( \phi(a^*ba) \leq M_b \phi(a^*a) \).

**Proof.** We pick \( x \in (k,\pi ms) \), \( y \in (m,\tilde{m} t) \) such that \( a = \Psi(x), b = \Psi(y) \) for some \( s, t \in \text{Obj}(\mathcal{C}) \). Then we have

\[
\phi(a^* \circ b^* \circ b \circ a) = (\phi \circ \Psi)
\]

Define

\[
T :=
\]

We consider \( T(\cdot) \) as a map from \((tm\bar{t},tm\bar{t})\) to \( AW_{k,k} \) by insertion in the unlabeled internal disk, and applying \( \Psi \). Then \( \phi \circ T(\cdot) \) defines a linear functional on the finite dimensional \( C^* \)-algebra \((tm\bar{t},tm\bar{t})\).

We claim that this linear functional is positive. Let \( X \in \text{Irr}(\mathcal{C}) \) such that \( X \prec l \) and let \( l \in \mathcal{W} \) such that \( X \prec l \) and \( t_X \in (X,l) \) such that \( t_X^* t_X = 1_X \) and \( t_X t_X^* \) is a minimal projection in \((l,l)\). Then let \( \{b_{X,i}\}_{1 \leq i \leq N_{X,tm\bar{t}}} \subseteq (tm\bar{t},X) \) be a standard basis. Then define \( g_{i,X} := t_X b_{X,i} \in (tm\bar{t},l) \). We have

\[
1_{tm\bar{t}} = \sum_{X \prec tm\bar{t}} \sum_{1 \leq i \leq N_{X,tm\bar{t}}} g_{i,X}^* g_{i,X}.
\]

If \( w \in (tm\bar{t},tm\bar{t}) \) is positive, then set \( w_{X,i} := g_{i,X}(w^{1/2}) \in (tm\bar{t},l) \). Then we have

\[
w = \sum_{X \prec tm\bar{t}} \sum_{1 \leq i \leq N_{X,tm\bar{t}}} w_{X,i}^* w_{X,i}.
\]

Now we consider \( \hat{w}_{X,i} := (1_{\bar{t}} \otimes w_{X,i} \otimes 1_{\bar{t}}) \circ (R_t \otimes 1_m \otimes \overline{R_t}) \in (m,\tilde{m}l) \).

Then we have that \( \Psi^t(\hat{w}_{X,i}) \Psi(x) \in AW_{k,l} \), and since \( \phi \) is an affine state,

\[
\phi \circ T(w_{X,i}^* w_{X,i}) = \phi \left( [\Psi^t(\hat{w}_{X,i}) \Psi(x)]^* [\Psi^t(\hat{w}_{X,i}) \Psi(x)] \right) \geq 0
\]
Thus
\[ \phi \circ T(w) = \sum_{X \in tmT} \sum_{1 \leq i \leq N^x_{tmT}} \phi \circ T(w^*_X, w_X, i) \geq 0 \]

Therefore \( \phi \circ T(.) \) defines a positive linear functional on \( (tmT, tmT) \). Set

\[ \eta_y := \begin{pmatrix} \vdots & \vdots \\ t \end{pmatrix} \in (tmT, tmT) \]

Then \( \eta_y \) is positive, thus \( \langle b \circ a, b \circ a \rangle = \phi(\Psi(T(\eta))) \leq \|\eta_y\|\phi(1_{tmT}) = d(t)\|\eta_y\|(a, a) \). Take \( M_b = \inf\{d(t)\|\eta_y\| : y \in (m, tmT), \Psi(y) = b\} \).

**Corollary 4.9.** For each affine state \( \phi \in \Phi\mathcal{AW}_k \), there exists a unique (up to unitary equivalence) \( k \)-cyclic module \( V_\phi \in \text{Rep}(\mathcal{AW}) \) such that \( \phi(x) = \langle \phi_{V_\phi}(x) \omega, \omega \rangle \).

**Proof.** Let \( \phi \in \Phi\mathcal{AW}_k \). Set \( U_m := \mathcal{AW}_{k,m} \). We define a sesquilinear form \( U_m \times U_m \mapsto \langle a, b \rangle \phi = \phi(b^*a) \). Clearly this is a positive semi-definite sesquilinear form. We have a natural representation of \( \mathcal{AW}_{m,n} \) as linear operators from \( U_m \) to \( U_n \). The above lemma shows us that the kernel of \( \langle \ldots \rangle_{\phi} \) is preserved under the action on \( \mathcal{AW}_{m,n} \), and \( \mathcal{AW}_{m,n} \) is bounded with respect to \( \langle \ldots \rangle_{\phi} \). We denote the Hilbert space completion of \( U_m \) by \( V^\phi_m \). The above lemma shows that the actions extend to bounded operators between the Hilbert spaces, and we have a Hilbert representation \( (\pi_{\phi}, V^\phi) \in \text{Rep}(\mathcal{AW}) \). Uniqueness is clear.

**Corollary 4.10.** \( \Phi\mathcal{AW}_k = \{ \langle \eta, \eta \rangle : \eta \in V_k \text{ for some } V \in \text{Rep}(\mathcal{AW}), \|\eta\| = 1 \} \).

**Definition 4.11.** For \( x \in \mathcal{AW}_{m,k} \), we define the universal norm \( \|x\|_u = \sup_{V \in \text{Rep}(\mathcal{AW})} \|x\|_V \).

**Proposition 4.12.** For \( x \in \mathcal{AW}_{m,k} \) with \( x \neq 0 \), we have
\[ 0 < \|x\|_u = \sup_{\phi \in \Phi\mathcal{AP}_m} \phi(x^*x)^{1 \over 2} = \sup_{\phi \in \Phi\mathcal{AP}_k} \phi(xx^*)^{1 \over 2} < \infty \]

**Proof.** The first inequality follows from the fact that the trace \( \omega_m \) is positive definite, and an affine state. The first equality follows from the above corollary. The equalities in the middle are easy, since every affine state is realized in a \( \mathcal{AW} \) module by a vector state and \( \|x^*x\|_V = \|xx^*\|_V \). The last inequality follows from Lemma 4.3 \( \square \)

We now remark on a corollary of the above proof. If \( n = W \) happens to be a simple object (as in the tube algebra), then \( \eta_y \) in the above proof happened to be a scalar multiple of a projection. Replacing \( n \) with the simple object \( W \) in the picture, we set \( \lambda_y := \begin{pmatrix} \vdots & \vdots \\ \end{pmatrix} \), and we have that \( \frac{d(W)}{\lambda_y} \) is a projection. Therefore \( \|\eta_y\| = \frac{\lambda_y}{d(W)} \). We record this in a corollary:
Corollary 4.13. Let \( f \in T^Z_{X,Y} \). Then setting \( \lambda_f := \frac{d(Z)}{d(Y)} \), we have \( \|f\|^2_u \leq \lambda_f \).

Proposition 4.14.

1. \( \| \|_u \) is a \( C^* \)-algebra norm on the centralizer algebras \( AW_{k,k} \). We denote the completion \( C^*(AW_{k,k}) \).

2. The completion of \( AW_{k,m} \) with respect to \( \| \|_u \), which we denote \( B(AW_{k,m}) \), is a Hilbert \( C^* \) bimodule over \( C^*(AW_{m,m}) - C^*(AW_{k,k}) \).

For the first item, we simply take a direct sum of cyclic representations over all affine states, \( V_u = \bigoplus_{\phi \in \Phi_{AW_k}} V_\phi \). Then it is clear that \( \| \|_u \) is a \( C^* \)-norm on \( AW_{k,k} \) since it agrees with the operator norm in this representations. For the second item, for \( x, y \in C^*(AW_{k,m}) \) we define the left \( C^*(AW_{m,m}) \) valued inner product by \( \langle x, y \rangle_L = xy^* \) and the right \( C^*(AP_{k,k}) \) valued inner product \( \langle x, y \rangle_R = x^*y \) it is straightforward to check that these maps satisfy the axioms for a Hilbert \( C^* \) bimodule.

We use this fact to provide a framework for understanding the universal tube algebras.

Definition 4.15. We define the universal annular category, \( C^*(AW) \) is the category with objects \( k \in \mathcal{W} \) and \( Hom(k, m) = B(AW_{k,m}), k \neq m \) and \( End(k) = C^*(AW_{k,k}) \).

We can also view both each category as a single \( C^* \)-algebra, by setting the product to be the composition of morphisms, and 0 if the objects labels do not match up. Equivalently this can be described as the linking algebra for the system of algebras and bimodules. It was from this perspective that the tube algebra was first studied, hence the name tube algebra. We prefer to think of it in the categorical sense, which seems more natural from the point of view of representation theory. We end this section with a result that essentially says the representation theory of the entire tube algebra is contained in the centralizer algebras \( AW_{k,k} \).

Proposition 4.16. Let \( (\pi, V) \) be a Hilbert space \( \ast \)-representation of \( AW_{k,k} \). Then \( (\pi, V) \) extends to a Hilbert \( P \)-module if and only if \( \| \|_V \leq \| \|_u \).

Proof. We perform the obvious construction generalizing the GNS construction, namely, we set \( V_k = V \), and \( \hat{V}_m := AW_{k,m} \otimes V_k \). We define a sesquilinear form on \( \hat{V}_m \) by

\[
\langle \sum_\alpha x_\alpha \otimes \eta_\alpha, \sum_\beta y_\beta \otimes \mu_\beta \rangle = \sum_{\alpha, \beta} \langle \pi(y_\beta^* x_\alpha) \eta_\alpha, \mu_\beta \rangle
\]

Let \( x \in AW_{k,m} \) for \( m \in \mathcal{W} \). If we set \( \eta = \bigoplus_{1 \leq \alpha \leq n} \eta_\alpha \in \mathcal{H} \otimes \mathbb{C}^n \), we have

\[
\langle \sum_\alpha x_\alpha \otimes \eta_\alpha, \sum_\alpha y_\alpha \otimes \eta_\alpha \rangle = \langle \pi((y_\alpha^* x_\alpha^* y_\beta)) \rangle_{1 \leq \alpha, \beta \leq n}
\]

Here we view \( \pi(y_\alpha^* x_\alpha^* y_\beta) \in AW_{k,k} \otimes M_n(\mathbb{C}) \). But since \( \| \|_V \leq \| \|_u \), \( \pi \) extends to a representation of the universal \( C^* \) algebra \( C^*(AW_{k,k}) \), and thus to the tensor product with \( n \times n \) matrices. Since \( 0 \leq (y_\alpha^* x_\alpha^* y_\beta)_{\alpha, \beta} \leq \|x\|^2 (y_\alpha^* y_\beta)_{\alpha, \beta} \) in \( C^*(AW_{k,k}) \otimes M_n(\mathbb{C}) \) and homomorphisms are ucp maps, then \( 0 \leq (\pi(y_\alpha^* y_\beta))_{\alpha, \beta} \leq \|x\|^2 (\pi(y_\alpha^* x_\alpha^* y_\beta))_{\alpha, \beta} \). Setting \( x = 1_k \), we have that our inner product is positive semi-definite, and in general we see that the action of \( AW_{k,m} \) on these spaces is bounded and preserves the kernel. This everything extends to a Hilbert space completion, yielding an element of \( Rep(AW) \).
Definition 4.17. A $(\pi,V)$ be a Hilbert space $*$-representation of $AW_{k,k}$ with $\|\cdot\|_V \leq \|\cdot\|_U$ is called admissible. The Hilbert representation constructed in the above proposition from an admissible representation is called the canonical extension, and is unique up to unitary equivalence.

We now note a corollary of the equivalence $\text{Rep}(T\mathcal{C})$ and $\text{Rep}(AW)$.

Corollary 4.18. For a simple object $X \in \mathcal{C}$, let $P_X$ be a minimal projection equivalent to $X$ in $(k,k)$. Then $C^*(T_{X,X}) \cong C^*(P_X AW_{k,k} P_X)$.

Examples of Affine States:
(1) The normalized trace functional $\omega$ on $AW_{k,k}$. Performing the GNS construction we get what we call the $k$-left regular representation.
(2) We have, as we shall see later, if $0 \in \Lambda$ denotes the identity object, then $AW_{0,0} \cong T_{0,0} \cong C[Irr(\mathcal{C})]$, where by $C[Irr(\mathcal{C})]$ we mean the fusion algebra, with multiplication a $C$ linear extension of the fusion rules, and $(X)^* = \overline{X}$, extend by conjugate linearity. Then the trivial representation, denoted $1_{\mathcal{C}}$ will be defined by the affine state sending $1_{\mathcal{C}}(X) = d(X)$ for $X \in Irr(\mathcal{C})$. Performing the GNS construction from the planar algebra perspective, we see that $V^1_{kX} \cong (0,k)$ with the action given by the natural action of $AW_{m,k}$ as operators from $(0,m) \rightarrow (0,k)$. See, for example [10], for more details from the planar algebraic perspective. This implies in the tube algebra, $V^1_X = 0$ for $X \neq 0$.

5. Examples

We will now analyze the tube algebra and, in particular, the centralizer algebras for two classes of categories: pointed categories (with trivial cocycle), which are group-like and quite simple, and the Temperly-Lieb-Jones categories, which are more interesting. In the pointed category case, we see the centralizer algebras are exactly the groups algebras of centralizer subgroups of elements, hence the name centralizer algebras.

5.1. Pointed Categories.

Definition 5.1. A pointed category is a rigid-$C^*$ tensor category $\mathcal{C}$ such that $d(X) = 1$ for all $X \in Irr(\mathcal{C})$.

There several major consequences of this definition:
Let $X, Y \in Irr(\mathcal{C})$. Then
(1) $XX \cong id$
(2) $XY \in Irr(\mathcal{C})$
(3) The set $Irr(\mathcal{C})$ forms a group under multiplication, which we will denote $G$. Inverse is given by duality.

Thus pointed categories are strongly group-like in the sense that there fusion rules are group multiplication. Two simple objects are equivalent if they represent the same element in $G$. If $X \in Irr(\mathcal{C})$ we abuse notation and view it as an element of $G$. The tube algebra of this example is known, and is one of the earliest examples of a tube algebra. The reason is that the tube algebra is essentially the Drinfeld double of the Hopf algebra $\mathbb{C}[G]$, which was one of the motivating examples in the definition of the Drinfeld center. This example is typically presented in the case of finite groups, while here we consider discrete groups in general. Here we consider the case when our pointed category is trivial, i.e. the 6-j symbols are the identity. It is well known that we may twist these 6-j symbols by a cocycle on the group resulting in another pointed category with
the same group fusion rules. Any pointed category is determined up to equivalence by the group $G$ and the cocycle in its 6-j symbols.

Recalling the definition of the tube algebra, we have $T^X_{X,Y} := Hom(X, ZYZ)$ which is 1 dimensional if $X = Z^{-1}YZ$ as group elements, and 0 otherwise. Thus in the tube category language, there is a non-zero hom between $X,Y$ iff $X$ is conjugate to $Y$. If we set $\Lambda$ := set of conjugacy classes of $G$, then we have a first decomposition $TC = \bigoplus_{\lambda \in \Lambda} T_{\lambda}$, where $T_{\lambda} := \bigoplus_{X,Y \in \Lambda} T_{X,Y}$.

It suffices to determine the structure of $T_{\lambda}$ for each conjugacy class $\lambda$. For $X \in G$, $T_{X,X} := \bigoplus_{Y \in ZG(X)} T^Y_{X,X}$, where $ZG(X)$ is the centralizer of $X$ in $G$. Since each $T^Y_{X,X} \cong \mathbb{C}$, we have a natural vector space isomorphism $\alpha : T_{X,X} \cong \mathbb{C}[ZG(X)]$. Furthermore, it is easy to check that this is an algebra isomorphism. More specifically $Y \in ZG(X)$, we can choose $f^Y_X \in T^Y_{X,X} \cong \mathbb{C}$ such that $(f^Y_X)^* f^Y_X = 1_X$. Then we have from the tube algebra multiplication $f^Y_X f^2 = f^2 f^Y_X \in T_{X,X}$. Now, for each $X,Y \in \lambda$, $ZG(X) \cong ZG(Y)$. In fact these are conjugate by any group element that conjugates $Y$ to $X$. The number of possible conjugators from $X$ to $Y$ is $|Z_G(C)|$. It is now easy to see that $T_{\lambda} = \mathbb{C}[ZG(X)] \otimes B_{0,0}(L^2(\lambda))$, where $B_{0,0}(L^2(\lambda))$ is the algebra of finite rank operators on a Hilbert space $L^2(\lambda)$. The diagonal copies of $Z_G(X)$ are the $T_{X,X}$, and the matrix unit copies are given by $T_{X,Y}$.

We have the following claim: Let $X \in Irr(\mathcal{C}) \cong G$, and let $Z_G(X)$ be the centralizer subgroup of $X$ in $G$. Then if $(\pi,H)$ is a unitary representation of $ZG(X)$, then $(\pi,H)$ extends to a representation of $T_{\lambda}$, where $\lambda$ is the conjugacy class of $X$. To see that this is true we simply note that $T_{\lambda} \cong ZG(X) \otimes B_{0,0}(L^2(\lambda))$. Thus we define $H_{\lambda} := H \otimes L^2(\lambda)$, with the obvious action. It is clear that this is a * representation by bounded operators of $T_{\lambda}$. Therefore

$$C^*(T_{X,X}) \cong C^*_u(Z_G(X))$$

5.2. TLJ Categories. The Temperley-Lieb-Jones categories are related to $Rep(SU_q(2))$ and provide a fundamental class of infinite depth rigid $C^*$-tensor categories. They are also an example of a category that has, simultaneously, a nice planar algebra description and a nice categorical description. Fix a positive real number with $\delta \geq 2$. Then there is a unique $q \in \mathbb{R}$ such that $q + q^{-1} = \delta$. We can then define for $n \in \mathbb{N}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ if $q \neq 1$, and $[n]_1 = n$.

The rigid $C^*$-tensor category $TLJ(\delta)$ consists of:

- (1) Self dual simple objects indexed by natural numbers, with 0 indexing the identity.
- (2) $d(k) = |k + 1|_q$
- (3) $k \otimes m \cong (k + m) \oplus (k + m - 2) \oplus \cdots \oplus |k - m|$

For the rest of this section, we use $[n]$ to denote $[n]_q$, assuming $q$ is given by context. The above properties are merely a summary of some relevant categorical data. These categories have much more structure than this, for example there are complicated 6-j symbols, and these categories naturally have a braiding (non-unitary unless $q = 1$). These categories also can be realized as the projection categories of particularly nice planar algebras.

Define the unoriented, unshaded planar algebra $TL(\delta)$ as follows:

- (1) $P_0 \cong \mathbb{C}$
- (2) $P_{2n+1} = 0$
- (3) $P_{2n} := \text{Linear span of disks with } 2n \text{ boundary points with strings connecting boundary points}$
- (4) strings do not cross
- (5) All boundary points are connected to some other boundary point with a string
- (6) Closed circles multiply the diagram by a factor of $\delta$
We note that in our generic case \( \delta \geq 2 \), this is a spherical \( C^* \)-planar algebra (see \cite{2,10} for definitions of spherical \( C^* \)-planar algebras). We have \( \text{dim}(\mathcal{P}_n) = \frac{1}{n+1} \binom{2n}{n} \). We remark that this is perhaps the most important example of a planar algebra since it appears in some form as a sub-algebra of an arbitrary planar algebra. It is usually presented as a shaded planar algebra in the subfactor planar algebra context, and there exists many detailed expositions of this planar algebra. See \cite{15,16}. We can realize the category described above as \( \text{TLJ}(\delta) = \text{Proj}(\text{TL}(\delta)) \) (see \cite{2} for definition of the projection category of a planar algebra). The object \( k \) in \( \text{TLJ}(\delta) \) corresponds to the \( k^{th} \) Jones-Wenzl idempotent in the planar algebra \( \text{TL}(\delta) \), denoted \( f_k \). These projections satisfy the property that apply a cap or cup to the top or bottom of \( f_k \) results in 0. \( f_k \) is a minimal projection in \( \text{TL}_{k,k} \) and can be defined by an inductive formula, see \cite{20} or \cite{15} for more details.

The affine annular representations of this planar algebra have been well studied \cite{17,16}, and we will make use of know results to analyze the universal algebra structure on the centralizer algebras of the tube algebra of this category. The beginning of this section can be deduced in its entirety from \cite{17}, \cite{16}, and \cite{31}. We include these results here for the purpose of self-containment, and due to the slight differences in our setting.

A planar algebra \( \mathcal{P} \) naturally provides an annular category \( \mathcal{AP} \). For details on this see \cite{17,15,5}. In this case, the annular category \( \mathcal{ATL} \) is easy to describe. The weights will simply be natural numbers, and they will signify the number of strings on the boundaries of disks. The object in \( \text{Proj}(\text{TLJ}(\delta)) \) corresponding to \( k \in \mathbb{N} \) is \( 1_k \in \text{TL}_{k,k} \). Then \( \text{ATL}_{k,m} \) will consist of all \( TL \) diagrams in an annulus with \( k \) boundary points on the internal circle and \( m \) on the external circle. This means there are \( \frac{k+m}{2} \) non-intersecting strings in the annulus, and each string touches precisely one boundary point (on either the inner or outer disk). We consider these diagrams only up to affine annular isotopy. That the set of affine annular pictures described here (isotopy classes of non-intersecting string diagrams) is really a basis for the annular category of the planar algebra follows from the analysis of \cite{5} and the fact that \( \text{TL}(\delta) \) for \( \delta \geq 2 \) has no local skein relations. Composition is the obvious one, and homologically trivial circles in the annulus multiply the diagram by a factor of \( \delta \). For more details on this annular category in particular see \cite{17}.

We consider here a subcategory of \( \text{Rep(\mathcal{ATL})} \) consisting of all locally finite representations. By this we mean the set of Hilbert representations of \( \mathcal{ATL} (\pi, V_k) \) such that each \( V_k \) is a finite dimensional Hilbert space. While not closed under tensor products, it is closed under finite direct sums. In the literature, Hilbert representations of a planar algebra \( \mathcal{P} \) are called Hilbert \( \mathcal{P} \) modules, and so we use these terms interchangeably in the planar algebra setting.

DEFINITION 5.2. A lowest weight \( k \) Hilbert \( \mathcal{ATL} \)-module is a representation \( (\pi, V_m) \) such that \( V_m = 0 \) for all \( m < k \).

Irreducible representations have their obvious meanings. Following the proof in \cite{10}, one can show that every locally finite Hilbert \( TL \)-module is isomorphic to the direct some of irreducible lowest weight \( k \) modules. It then becomes our task to classify and construct these.

To do so we start by noting that \( \text{ATL}_{0,0} \) is isomorphic to the fusion algebra \( \mathbb{C}[\text{Irr(\text{TLJ}(\delta))}] \), which is abelian. Thus an irreducible lowest weight 0 module will be a 1 dimensional representation of the fusion rules. Let \( v_0 \) be a non-zero vector in the one dimensional space normalized so that \( \langle v_0, v_0 \rangle = 1 \). We notice the identity object \( (f_0) \) must go to the identity and we may identify \( \pi(f_k) \) with some number (its eigenvalue on \( v_0 \)). But from the fusion rules, all these numbers are determined by \( \pi(f_1) \). Since \( f_1 \) in \( \text{ATL}_{0,0} \) is self dual and this must be a \(*\)-representation, we see that \( \pi(f_1) \) (hence \( \pi(f_k) \) for all \( k \)) must be a real number. Furthermore, by the bounds on the universal norm for the weight 0 case (Corollary 4.13), we must have...
\[ |\pi(f_1)| \leq \delta. \] Let \( t := \pi(f_1) \in [-\delta, \delta]. \) Then this parameter determines \( \pi \) completely. We still must see which of these extend to Hilbert TL modules, but we will see that all of them will.

Now, consider \( k > 0. \) Let \( ATL_{k,k}^\leq \) be the ideal in \( ATL_{k,k} \) spanned by diagrams with less than \( k \) through strings. We see that in a lowest weight \( k \) representation, this ideal must act by 0. An irreducible lowest weight \( k \) representation will then necessarily be an irreducible representation of the algebra \( ATL_{k,k}/ATL_{k,k}^\leq. \)

Let in \( \rho_k \in ATL_{k,k} \) we define the element “rotation by one” as \( \begin{array}{c}
\text{f} \\
\text{m} \\
\text{g}_\alpha
\end{array} \). We see that this element is invertible in \( ATL_{k,k} \), and we denote its inverse \( \rho_k^{-1} \) the “left rotation by one”. The powers of \( \rho_k \) form a subgroup of the algebra \( ATL_{k,k} \) isomorphic to \( \mathbb{Z} \). We see then that \( ATL_{k,k}/ATL_{k,k}^\leq \cong \mathbb{C}[\mathbb{Z}] \), which again is abelian, hence an irreducible lowest weight \( k \) \(*\)-representation will be an irreducible unitary representation of \( \mathbb{Z} \), hence determined by some \( \omega \in S^1. \)

We have now found all candidates for irreducible lowest weight \( m \) representations of \( ATL \) for all \( m \). The question that remains is which of these representations of the fusion algebra and \( \mathbb{Z} \) extend to a representation of the entire annular category, i.e. have a canonical extension. Since all the spaces are finite dimensional (as we shall see), the annular actions are bounded, hence it suffices to demonstrate that the inner products of the canonical extension are positive semi-definite. We follow the method prescribed in section 4, namely if we have a representation of \( ATL_{m,m}/ATL_{m,m}^\leq \) (or \( ATL_{0,0} \)) representation determined by the parameter \( \alpha \in S^1 \) (or in \([-\delta, \delta]\)) on the one dimensional vector space \( V_m^\alpha \), define \( \hat{V}_n^\alpha := ATL_{m,n} \otimes_{ATL_{m,m}} V_m^\alpha \). If we let \( s_\alpha \in V_m^\alpha \) be normalized, we can represent simple tensors in the vector space \( \hat{V}_n^\alpha \) by

\[ f \otimes s_\alpha := \begin{array}{c}
\text{f} \\
\text{m} \\
\text{s}_\alpha
\end{array}. \]

Connecting the bottom \( m \) strings to the rotation eigenvector signifies that we are taking a relative tensor product over \( ATL_{m,m} \). Now, we can easily see that \( dim(f_k ATL_{m,k} \otimes_{ATL_{m,m}} V_m^\alpha) \) is at most one. This is because only non-rectangular caps can appear, and there are only finitely many. In particular, \( f_k \hat{V}_k^\alpha \) is spanned by the vector \( g_{m,k}^\alpha := \begin{array}{c}
\text{f}_k \\
\text{m} \\
\text{s}_\alpha
\end{array} \). We note that \( g_{m,k}^\alpha = 0 \) for \( k < m \).

To understand \( \hat{V}_n^\alpha \), for each Jones-Wenzl idempotent \( f_{n-2j} \prec n \), let \( (f_{n-2j}, n) \) denote the planar algebra elements \( x \in P_{n-2j,n} \) such that \( xf_{n-2j} = x \), for \( 0 \leq j \leq \lfloor \frac{n}{2} \rfloor \) as in section 4. It is clear that \( \hat{V}_n^\alpha \cong \bigoplus_{0 \leq j \leq \lfloor \frac{n}{2} \rfloor} (f_{n-2j}, n) \otimes g_{m,n-2j}^\alpha \).

With this nice decomposition, we want to see how the canonical inner product behaves. First, we will do some diagrammatics that will allow us to clearly see the canonical inner product is positive semi-definite.
We closely follow the work of [17]. Let \( \alpha \) be the parameter of a lowest weight \( m \) representation. We define the numbers \( B_{m,l}^k(\alpha) \) by the following:

\[
\begin{array}{cccc}
\fbox{f_k} & 1 & \fbox{m} & \fbox{l} \\
& k \alpha & k-2 \\
\fbox{m} & k \alpha & k-2 \\
\end{array}
= B_{m,l}^k(\alpha)
\begin{array}{cccc}
\fbox{f_1} & 1 & \fbox{m} & \fbox{l} \\
& k \alpha & k-2 \\
\fbox{m} & k \alpha & k-2 \\
\end{array}.
\]

Note that \( B_{m,k}^k = 1 \) for all \( k \geq m \).

**Lemma 5.3.** \( |k|^2 - \left[ \frac{k-m}{2} \right]^2 - \left[ \frac{m+k}{2} \right]^2 = (q^k + q^{-k}) \left[ \frac{k-m}{2} \right] \left[ \frac{m+k}{2} \right] \)

**Proof.** Direct computation.

Recall that \( TL_{k,k} \) as a vector space is the linear span of all isotopy classes of rectangular diagrams and strings, with \( k \) boundary points on the top and bottom of the rectangle, and all strings are attached to exactly two of these boundary points. Strings are not allowed to intersect. Since \( f_k \in TL_{k,k} \) can be written as a linear combination of diagrams in \( TL_{k,k} \). We will need the coefficients for several of these diagrams. First the coefficient of the identity diagram \( 1_k \in TL_{k,k} \) is one. Next, we have that the coefficient of the diagram

\[
\begin{array}{c}
\fbox{n-1} \\
\fbox{k-n-1} \\
\end{array}
\]

in \( f_k \) is \( (-1)^{n-k} \frac{[n]}{[k]} \). For a proof of these formulas and other coefficient formulas for the Jones-Wenzl idempotents, see the paper of Scott Morrison [20]. We note that the \( f_k \) are invariant under vertical and horizontal reflection. In particular, a horizontal or vertical reflection of the above diagram will have the same coefficient in \( f_k \) as this diagram. Using these results, we have the following proposition:

**Proposition 5.4.**

1. For \( m > 0 \) and \( \omega \in S^1 \), \( k \) even, \( B_{m,j}^k(\omega) = \frac{[k-m][m+k]}{[k][k-1]} (q^k + q^{-k} - \omega^2 - \omega^{-2}) B_{m,j}^{k-2}(\omega) \)

2. \( B_{0,j}^k(t) = \frac{1}{[k][k-1]} ([k]^2 - t^2[k]^2) B_{0,j}^{k-2} \)

3. For \( m > 0 \), \( \omega \in S^1 \) and \( k \) odd, we have \( B_{m,j}^k(\omega) = \frac{[k-m][m+k]}{[k][k-1]} (q^k + q^{-k} - (i\omega)^2 - (i\omega)^{-2}) B_{m,j}^{k-2}(\omega) \)

First assume \( m > 0 \). Then we have

\[
\begin{array}{cccc}
\fbox{f_k} & 1 & \fbox{m} & \fbox{l} \\
& k \alpha & k-2 \\
\fbox{m} & k \alpha & k-2 \\
\end{array}
= \begin{array}{cccc}
\fbox{f_k} & 1 & \fbox{m} & \fbox{l} \\
& k \alpha & k-2 \\
\fbox{m} & k \alpha & k-2 \\
\end{array}
\]

We see that there are precisely 3 diagrams that can be inserted into the bottom \( f_k \). A diagram with no caps or cups, i.e. the identity \( 1_k \) is the first. Then there can only be one cap in the top, which must be on the top right. Such a diagram must have exactly one cap on the bottom, and it can be either at position \( \frac{k-m}{2} \) and or \( \frac{k+m}{2} \). The coefficient of such a diagram (see [20]) is \( -1 \frac{[k-m][k+m]}{[k]} \) for the former and \( -1 \frac{[k-m][k+m]}{[k]} \) for the latter. We see then pick up a value of \( \omega^{-1} \) for the first diagram, and an \( \omega \) for the second diagram. Then the above is equal to
We want an expression just involving the diagram to the right in the sum, so we consider the left diagram in the sum and apply the same sort of argument. We notice the diagram on the left is in fact equal to

\[ (-1)^{k-m} \left( \left\lfloor \frac{k-m}{2} \right\rfloor \omega^{-1} + \left\lceil \frac{k+m}{2} \right\rceil \omega \right) \]

Now here we see that the identity gives 0. There is only one possible place for a cup, and that is on the top left. A careful consideration of caps and through strings shows a cap at the bottom right gives 0. There there are precisely 2 places for a bottom cap that give non-zero contributions, namely with positions at \( \frac{k-m}{2} - 1 \) and \( \frac{k+m}{2} - 1 \). The coefficients in \( f_{k-1} \) of these in diagrams are \((-1)^{\frac{k-m}{2}} + \left\lfloor \frac{k+m}{2} \right\rceil \omega^{-1} \) and \((-1)^{\frac{k+m}{2}} + \left\lceil \frac{k-m}{2} \right\rceil \omega \) respectively. Again the first coefficient picks up an \( \omega^{-1} \) and the second picks up an \( \omega \).

Thus this diagram is equal to

\[ (-1)^{\frac{k+m}{2}} \left( \left\lfloor \frac{k+m}{2} \right\rfloor \omega^{-1} + (-1)^k \left\lceil \frac{k-m}{2} \right\rceil \omega \right) \]

Putting everything together we end up with

\[ \frac{1}{k|k-1|} \left( [k]^2 - \left\lceil \frac{k-m}{2} \right\rceil^2 - \left\lfloor \frac{k+m}{2} \right\rfloor^2 - (-1)^k (\omega^2 + \omega^{-2}) \left\lceil \frac{k-m}{2} \right\rceil \left\lceil \frac{k+m}{2} \right\rceil \right) \]

By quantum number identity, \( [k]^2 - \left\lceil \frac{k-m}{2} \right\rceil^2 - \left\lfloor \frac{k+m}{2} \right\rfloor^2 = (q^k + q^{-k}) \left\lceil \frac{k-m}{2} \right\rceil \left\lceil \frac{k+m}{2} \right\rceil \), so the above coefficient is

\[ \frac{\left\lfloor \frac{k-m}{2} \right\rceil \left\lceil \frac{k+m}{2} \right\rceil}{k|k-1|} (q^k + q^{-k} - (-1)^k (\omega^2 + \omega^{-2})) \]

we see that we immediately see the formulas for \( k \) even and \( k \) odd (for \( k \) odd the \( i \) comes from the \((-1)^k = -1\), which we then bring inside the \( (\omega^2) \) as an \( i \)).

Now for \( m = 0 \), we must have that \( k, l \) are even. We perform the same analysis. The main difference here is that there is only one diagram with caps that survives, i.e. we have
Evaluating the bottom $f_k$ in Temperley-Lieb diagram, we see the identity in $f_k$ yields \([k]\) \([k-1]\). The cap at the bottom yields a factor of $t$ since it produces a homologically non-trivial circle. Thus this leads to

$$(-1)^{k} t^{\frac{k}{2}} \binom{\frac{k}{2}}{k}. $$

As in the case $m > 0$, the identity yields 0, and thus there is precisely one diagram which gives a non-zero contribution, with a cap in the upper left hand corner, and a cap on the bottom at position $\frac{k}{2} - 1$. The coefficient of this diagram in $f_{k-1}$ is \((-1)^{k-1} \binom{\frac{k}{2}}{k-1}\). Again a factor of $t$ pops out. Combining all the terms, we end up with

$$\frac{1}{|k|} \left( \left\lfloor \frac{k}{2} \right\rfloor^2 - t^2 \left\lceil \frac{k}{2} \right\rceil^2 \right).$$

This gives us the desired formula $\square$.

We are ready to analyze the inner product space described above. Let $\alpha$ be the parameter of a lowest weight $m$ representation. $V^\alpha_n \cong \bigoplus_{0 \leq j \leq \frac{n}{2}} (f_{n-2j}, n) \otimes g^\alpha_{m,n-2j}$. We see that this decomposition is orthogonal with respect to the sesquilinear form defined by our lowest weight $m$ representation. If $x \otimes g^\alpha_{m,n-2j}, y \otimes g^\alpha_{m,n-2j} \in (f_{n-2j}, n) \otimes g^\alpha_{m,n-2j}$, we see that $\langle x \otimes g^\alpha_{m,n-2j}, y \otimes g^\alpha_{m,n-2j} \rangle = \langle x, y \rangle g^\alpha_{m,n-2j} g^\alpha_{m,n-2j} / \alpha = \langle x, y \rangle B^\alpha_{m,m^2}$, where $\langle x, y \rangle$ denotes the positive definite inner product in the planar algebra. An inspection of the formulas shows that $B^\alpha_{m,m^2} \geq 0$. Thus our inner product is positive semidefinite, hence, taking the quotient, we obtain a sequence of finite dimensional Hilbert spaces $\{V^\alpha_k\}$ with $V^\alpha_k = 0$ for $k < m$ (and 0 if the parity of $k$ is distinct from the parity of $m$). We notice also that this inner product is uniquely determined.
by $\alpha$, thus for a given lowest weight $k$ and parameter $\alpha$, there is a unique Hilbert $TL$ module constructed as above.

In some cases, however, even with $k \geq m$, it may be that $g_{m,k}^\alpha = 0$ in the quotient with respect to the positive semi-definite inner product. This happens precisely when $B_{m,m}(\alpha) = 0$. Inspecting the coefficients as in [17], we can determine when this happens. For the weight 0 case, we see that this happens precisely when $k > 0$ and $t = \pm \delta$. For $\delta > 2$, all other coefficients are positive. When $\delta = 2$ and hence $q = 1$, the weight 0 story is the same, but for higher weights we see that we run into a problem in two places: For $m > k > 0$ and $t = \pm \delta$, we can determine when this happens. For the weight 0 case, we see that this happens precisely when $\omega$ is even.

For $k > m$, it may be that $B_{m,m}(\pm i) = 0$ for all $k > m$. For $m$ even, $\omega = \pm 1$, $B_{m,m}(\pm 1) = 0$ for all $k > m$. For $m$ odd, we see that the problem occurs at $\omega = \pm i$. In this case, we see that $B_{m,m}(\pm i) = 0$ for all $k > m$. This will be relevant when we analyze the tube algebra representations of $TLJ(\delta)$, so we record the results in the following proposition.

**Proposition 5.5.** [17, 31]: For any $m$, we classify irreducible lowest weight $m$ representations as follows: Let for a lowest weight $m$ representation with parameter $\alpha$, let $g_{m,k}^\alpha$ be the vector described above. Recall that $g_{m,k}^\alpha = 0$ if $k < m$.

1. For $t \in [-\delta, \delta]$, there exists a unique irreducible lowest weight 0 Hilbert $TL$-module $V_t^0 := \{V_k^t : k \text{ is even}\}$. For $t \in (-\delta, \delta)$, $g_{0,k}^t \neq 0$ for all even $k$. $g_{0,k}^t = 0$ for all $k > 0$.
2. For $m > 0$, $\omega \in S^1$, there exists a unique irreducible lowest weight $m$ Hilbert $TL$ module $V_{m,m}^\omega := \{V_{m,m}^\omega : m - k \text{ is even}\}$. For $\delta > 2$, $g_{m,k}^\omega \neq 0$ for all $k \geq m$ with $k - m$ even. For $\delta = 2$, $k$ even, $g_{m,m}^\omega = 1$ and $g_{m,k}^\omega = 0$ for all $k > m$. If $m$ is odd, then $g_{m,m}^\omega = 1$ and $g_{m,k}^\omega = 0$ for all $k > m$.
3. Define the space $X_\infty^+ := [-\delta, \delta] \cup S_1 \cup S_2 \cup \ldots$, with infinitely many copies of $S_1$, and $X_\infty := S_1 \cup S_2 \cup \ldots$. Then irreducible representations in $\text{Rep}(ATL)$ are parameterized (as a set) by $X_\infty^+ \cup X_\infty$.

We notice that (3) agrees with the parameterization of irreducible representations of the quantum Lorentz group found in [29], which is to be expected from the quantum group realization of $TLJ(\delta)$. We thank Makoto Yamashita for pointing this out to us.

Now we are ready to analyze the centralizer algebras $T_{k,k}$ of $TLJ(\delta)$. To study the tube algebra we construct a nice basis for $T_{k,k}$ which will allow us to exploit the planar algebra description of this category. Recalling from our work from section 2 on the tube algebra and affine annular planar algebras that the tube centralizer $T_{k,k} \cong ATL_k f_k$, i.e. $T_{k,k}$ is the cut down of the affine annular Temperley-Lieb $ATL_k$ space by the rectangular $k^{th}$ Jones-Wenzl projection $f_k$. Thus we can construct a basis of $T_{k,k}$ which consist of diagrams as follows:

For $k$ even, $x_{0,j}^k := \begin{array}{c}
\begin{array}{l}
\text{f}_k \\
\text{m}
\end{array}
\end{array}$, then, in general for $n \in \mathbb{Z}$ and $0 \leq m \leq k$ with $k - m$ even, define

$x_{m,n}^k := \begin{array}{c}
\begin{array}{l}
\text{f}_k \\
\text{m}
\end{array}
\end{array}$

These pictures are meant to represent annular tangles, with the strings on the left connecting to strings on the right around the bottom of an annulus. In the center of $x_{m,n}^k$ is the $n^{th}$ power
of the rotation $\rho_m$. We define the rank of the diagram as $\text{Rank}(x^k_{m,n}) = m$. We see that the rank of a diagram in $T_{k,k}$ must be the same parity as $k$. The rank corresponds to the number of strings starting from the bottom $f_k$ and going all the way to the top $f_k$.

**Proposition 5.6.** Define the set $B := \{x^k_{m,n} : m \in \mathbb{N}, 0 \leq m \leq k, k - m = 0 \mod 2, n \in \mathbb{Z} \text{ or } n \in \mathbb{N} \text{ for } m = 0\}$. Then $B$ is a basis for $T_{k,k}$.

**Proof.** Since $T_{k,k} \cong f_k ATL_{k,k} f_k$, we see that the only diagrams that survive are in $B$, hence $B$ is a spanning set. To see that these are linearly independent, we note that the diagrams listed above without the $f_k$ (i.e. replacing each $f_k$ by $1_k \in ATL_{k,k}$ are linearly independent in $ATL_{k,k}$ (since they correspond to distinct isotopy classes of diagrams), and have no rectangular caps on their boundaries. If denote our set $B := \{x_{n,m}\}$, we have a bijective correspondence between $B$ and $ATL_{k,k}$ diagrams with no rectangular caps on the top and bottom boundaries, given by replacing the JWs in $x_{n,m}$ with the $1_k \in TL$. We also note that by definition, the diagrams in $ATL_{k,k}$ with no rectangular caps on their boundaries must be linearly independent from the set of diagrams with some rectangular caps on their boundaries. Suppose there exists some $\{b_i\}_{1 \leq i \leq n} \subseteq B$ and $\lambda_i \in \mathbb{C}$ such that $\sum \lambda_i b_i = 0$. Let $\hat{b}_i \in ATL_{k,k}$ be the diagram obtained by replacing the top and bottom Jones-Wenzl idempotents in $b_i$ with the identity. Then evaluating the Jones-Wenzl idempotents at the top and the bottom of the diagrams in terms of $TL$ diagrams, we see that the only terms in both the top and bottom JWs that give no rectangular caps on the boundary are the identity diagrams $1_k \in TL$, and these have coefficient $1$ in $f_k$. Since these diagrams are independent from the diagrams with caps, we notice that our equation implies $\sum \lambda_i \hat{b}_i = 0$. But our correspondence is bijective, and these are independent in $ATL_{k,k}$, hence there is no solution. □

**Proposition 5.7.** For every $k$, $T_{k,k}$ is abelian.

**Proof.** Recall that since $f_k = \overline{f_k}$, by the symmetry of our basis diagrams it is easy to see that the global duality duality map defined at the end of section 3, $r : T_{k,k} \to T_{k,k}$ given by a global 180 rotation, (i.e. rotate the entire cylinder 180 degrees) is in fact the identity map on $B$, hence on all of $T_{k,k}$. Since $r$ is in general an anti-automorphism, we have that for any $x, y \in T_{k,k}$, $xy = r(xy) = r(y)r(x) = yx$. Thus $T_{k,k}$ is abelian. □

This means the universal $C^*$ algebras $C^*(T_{k,k})$ will be unital, abelian $C^*$ algebras. Thus they will be isomorphic to the algebra of continuous functions of some compact Hausdorff space. We describe these spaces below.

1. Define $X_0 := [-\delta, \delta]$.
2. For $k$ even, $k > 0$, we define

$$X_k := \begin{array}{c}
\begin{array}{c}
\text{...}
\end{array}
\end{array}$$
(3) For $k$ odd, define

$$X_k := \begin{array}{c}
\begin{array}{c}
\circ \quad \cdots \quad \circ
\end{array}
\end{array}\frac{k+1}{2}$$

We will demonstrate the following:

**Theorem 5.8.** If $C \cong TLJ(\delta)$, $\delta > 2$ then $C^*(T_{k,k}) \cong C(X_k)$.

For $\delta = 2$, the situation is different. As discovered in [17], the annular representation theory of $ATL(2)$ is non generic. In particular, there are some “missing” one dimensional representations. This will force us to identify points, resulting in some interesting topological spaces.

(1) Define $Y_0 := [-2, 2]$.

(2) For $k$ even, $k > 0$ define

$$Y_k := \begin{array}{c}
\begin{array}{c}
\bigcirc \quad \cdots
\end{array}
\end{array}\frac{k}{2}$$

(3) For $k$ odd, define

$$Y_k := \begin{array}{c}
\begin{array}{c}
\bigcirc \quad \cdots
\end{array}
\end{array}\frac{k+1}{2}$$

**Theorem 5.9.** If $C \cong TLJ(2)$, then $C^*(T_{k,k}) \cong C(Y_k)$.

We note that in the case $k = 0$, this essentially recovers the result of Popa and Vaes. The only difference is that they use the even part of the $TLJ(\delta)$ category while we take the category as a whole, thus they have the “square" of this interval $[0, \delta^2]$, see [30].

To understand the one dimensional representations of $T_{k,k}$ (which we often call characters) we note that (almost) every lowest weight $m$ representation with parameter $\alpha$ and $k - m$ even gives a one dimensional representation of $T_{k,k}$. We simply take the vector $g^\alpha_{k,m}$. Then this will be an eigenvector of $T_{k,k}$ viewed as a subalgebra of $ATL_{k,k}$. Thus if we understand the action of $T_{k,k}$ on the vector $g^\alpha_{m,k}$ we will understand the characters. There is a snag, however. From the above proposition, some of these $g^\alpha_{m,k}$ are 0 in the semi-simple quotient, hence do not produce characters on $T_{k,k}$. Furthermore, it is not a priori clear that
every admissible representation of $T_{k,k}$ comes from $ATL$ in the manner described here. For example, it seems feasible that a one dimensional representation of $T_{k,k}$ may have its canonical extension infinite dimensional in other weight spaces. We will show that this is not the case.

**Lemma 5.10.** For $\delta > 2$, one dimensional representations of $T_{k,k}$ are parameterized as a set by

1. If $k$ is even, $k > 0$, the space $X_k := (-\delta, \delta) \sqcup S^1 \sqcup \cdots \sqcup S^1$ if with $\frac{k}{2}$ copies of $S^1$
2. If $k$ is odd, the space $X_k := S^1 \sqcup \cdots \sqcup S^1$ with $\frac{k+1}{2}$ copies of $S^1$
3. If $k = 0$, the space $X_0 := [-\delta, \delta]$.

**Lemma 5.11.** For $\delta = 2$, one dimensional representations are parameterized by:

1. If $k > 0$ is even, the space $Y_k := (-2, 2) \sqcup (S^1 - \{1, -1\}) \sqcup (S^1 - \{1, -1\}) \sqcup \cdots \sqcup S^1$ with $\frac{k+1}{2} - 1$ copies of $S^1 - \{1, -1\}$ and one copy of $S^1$.
2. If $k$ is odd, $Y_k := (S^1 - \{-1, 1\}) \sqcup \cdots \sqcup S^1$ with $\frac{k+1}{2} - 1$ copies of $S^1 - \{-1, 1\}$ and one copy of $S^1$.
3. $Y_0 := [-\delta, \delta]$.

**Proof.** This set produces characters by evaluating the action of $T_{k,k}$ on the vectors $g_{m,k}^\omega$ for $k \geq 0$ and $k - m$ even. Now, the reason that $\pm \delta$ is missing in the interval $(-\delta, \delta)$ from all but $k = 0$ is that the trivial representation of $T_{0,0}$ does not extend to higher weight spaces by Proposition 5.5 (1), i.e. $g_{0,k}^{+\delta} = 0$ in the semi-simple quotient of the canonical extension. In the case $\delta = 2$, we have from Proposition 5.5(2) that the characters corresponding to the lowest weight $k$ representations are “missing”, i.e. the corresponding $g_{m,k}^\omega$ are 0 for the parameters $\omega = \pm 1$ for even $m > 0$ and $\omega = \pm i$ for odd $m > 0$. Thus the set listed describes all possible characters coming from $ATL$, by [17]. Applying Corollary 4.18 we see that this yields all possible characters. In particular, suppose we have a character $\alpha$ on $T_{k,k}$. Let $m$ be the smallest $m$ such that the canonical extension to $T_{m,m}$ is non-zero. It is straightforward to check that since $\alpha$ is irreducible, the canonical extension to $T_{m,m}$ is one dimensional. Then when extended to an $ATL$ representation, this extends to an irreducible lowest weight $m$ representation, and we apply the classification of these described in the begging of this section $\Box$.

We can identify the circles as weight $m$ characters, i.e. each distinct circle corresponds to characters of distinct weights. The interior of the interval $(-\delta, \delta)$ corresponds to weight 0 characters. We know now that all characters must be given by $X_k$, but we do not yet know that distinct points in $X_k$ yield truly distinct characters on $T_{k,k}$. They yield distinct representations for $ATL$, but independent characters might become the same when restricted to the tube algebra. In fact, we will show that they are distinct, but first we see how to evaluate characters on a special subset of our basis, namely elements in $T_{k,k}$ of the form $x_{m,1}^k$.

Let $t \in (-\delta, \delta)$, and $k$ even. Then we see that $t(x_{0,j}^k) = t^j B_{0,0}^k(t)$. This is non-zero for $t \in (-\delta, \delta)$. For $m > 0$, and $\omega$ the eigenvalue for a lowest weight $m$ representation, we see that for $m \geq m$, $\omega(x_{n,1}^m) g_{m,k}^\omega = x_{n,1}^{\omega} g_{m,k}^\omega$ where here, we identify $\rho_n$ in $T_{n,n}$ and the $\omega$ as a character on $T_{n,n}$. Thus to compute the value of $\omega(x_{n,1}^m)$, we simply need to determine the value of $\omega(\rho_n)$. In pictures, we have to compute the scalar

that pops out when we substitute $TL$ diagrams in the bottom $f_n$ of the picture

If $n > m$, we see that there are precisely two diagrams which give non-zero contributions, a cup in the upper right hand
corner, and a bottom cap at positions $\frac{n-m}{2}$ and $\frac{n+m}{2}$. The coefficients of these diagrams in $JW_n$ are given by $(-1)\frac{n+m}{2} [\frac{n-m}{2}]/[n]$ and $(-1)\frac{n+m}{2} [\frac{n+m}{2}]/[n]$ respectively. The first diagram gives an eigenvalue of $\omega^{-1}$ and the second gives eigenvalue of $\omega$, and thus we get $\omega(\rho_n) = (-1)\frac{n+m}{2} (\omega^{-1} (1) [\frac{n-m}{2}] + \omega [\frac{n+m}{2}]) / [n]$. If $n = m$, we simply get $\omega$. We apply the same procedure for $m = 0$, which is even easier since there is only one TL diagram to evaluate.

We also notice that applying this same procedure to arbitrary basis diagrams, we see that an element of $T_{k,k}$ evaluated at a character $\alpha$ will depend on $\alpha$ only as polynomial either in $\alpha$ and $\alpha^{-1}$ if $\alpha \in S^1$ or just in $\alpha$ if $|\alpha| \in (-\delta, \delta)$. We record these results in the following lemma, which expresses our knowledge of how to evaluate characters:

**Lemma 5.12.** Let $k > 0$.

1. For $k$ even, $t \in (-\delta, \delta)$, $k$ even, we have $t(x^k_{0,0}) = t B^k_{0,0}(t)$.
2. For $k$ even, $t \in (-\delta, \delta)$, $t(x^k_{n,0}) = B^k_{n,0}(t)$, $t(x^k_{n,1}) = (-1)^t \frac{n+m}{2} / [n] B^k_{0,n}(t)$.
3. For $\omega \in S^1$ of lowest weight $m > 0$, for $k, n \geq m$, $\omega(x^k_{n,1}) = (-1)^t \frac{n+m}{2} / [n] \omega^{-1} [\frac{n-m}{2}] \omega B^k_{m,n}(\omega)$, where here $0 = 0$.
4. $\omega \in S^1$, then $\omega(x^k_{1,m}) \in \mathbb{C}[\omega, \omega^{-1}]$, and if $t \in (-\delta, \delta)$, then $t(x^k_{1,m}) \in \mathbb{C}[t]$.

**Lemma 5.13.** For $\delta \geq 2$, and $X_k$ as above, $T_{k,k}$ separates the points of $X_{k,k}$.

First consider the $k = 0$ case. Then $t(x^k_{0,1}) = t$ separates all points in $\{-\delta, \delta\}$. For each pair of distinct characters $\alpha_1, \alpha_2 \in X_k$, we must show that there exists $f \in T_{k,k}$ such that $\alpha_1(f) \neq \alpha_2(f)$.

If $\alpha_1$ and $\alpha_2$ correspond to different weights (i.e. live in different connected components of $X_k$), assume without loss of generality, that the weight of $\alpha_1$ is strictly less than the weight of $\alpha_2$. Then suppose the weight of $\alpha_1$ is $m$. Then we pick the diagram $x^k_{m,0}$. Then from the above proposition, we have that $\alpha_1(x^k_{m,0}) = B^k_{m,m}(\alpha_1) \neq 0$, while $\alpha_2(x^k_{m,0}) = 0$ since $x^k_{m,0}$ has rank $m$. Thus we can separate characters of different weights.

We now consider the case when $\delta > 2$, and $k$ even.

Suppose $\alpha_1, \alpha_2 \in (-\delta, \delta)$. Then we have $B^k_{0,0}(\alpha_1)$ and $B^k_{0,0}(\alpha_2) \neq 0$, and thus if $\alpha_1(x^k_{0,0}) = B^k_{0,0}(\alpha_1) \neq B^k_{0,0}(\alpha_2)$ we are done. If $B^k_{0,0}(\alpha_1) = B^k_{0,0}(\alpha_2) \neq 0$, then $\alpha_1(x^k_{0,1}) = \alpha_1 B^k_{0,0}(\alpha_1) \neq \alpha_2 B^k_{0,0}(\alpha_2) = \alpha_2 B^k_{0,0}(\alpha_2)$. Thus we can separate the weight 0 characters with $T_{k,k}$.

Now suppose the weight of $\alpha_1, \alpha_2 \in X_k$ are of the same weight $m > 0$ but $\alpha_1 \neq \alpha_2$. Then $\alpha_1(x^k_{m,0}) = B^k_{m,m}(\alpha_1)$, and $\alpha_2(x^k_{m,0}) = B^k_{m,m}(\alpha_2)$. If $B^k_{m,m}(\alpha_1) \neq B^k_{m,m}(\alpha_2)$ we are done. Suppose they are not equal. They are not 0 by Proposition 5.5. Then $\alpha_1(x^k_{m,1}) = \alpha_1 B^k_{m,m}(\alpha_1)$ while $\alpha_2(x^k_{m,1}) = \alpha_2 B^k_{m,m}(\alpha_2)$. Since $\alpha_1 \neq \alpha_2$ we are finished.

The other cases are the same. For $\delta = 2$, we simply remove the points in the domain where $B^k_{m,m} = 0$, and the above proof applies □.

Now, we know that $C^*(T_{k,k})$ will be a unital ($f_k$ is the unit) abelian $C^*$-algebra thus it must be isomorphic to the continuous functions on some compact Hausdorff space. Since the characters evaluated on $T_{k,k}$ are simply polynomials in the parameters of $X_k$ (Lemma 5.12(4)), away from the “missing” points ($\pm \delta$, and when $\delta = 2$, the points corresponding to $\pm 1$ on the even circles and $\pm 2$ on the odd circles), the topology on the set of characters precisely agrees with the natural topology on the spaces. Let us now consider the case when $\delta > 2$. The only “missing” points are $t = \pm \delta$. In other words, since our character space is compact and the topology on $X_k$ as characters agrees with the natural topology on $(-\delta, \delta)$, if we have a sequence of
characters $t_n \subseteq (-\delta, \delta)$, such that $t_n \to \pm \delta$, this sequence must be converging to some other character in $X_k$. Thus to identify the topology on $X_k$ as the space of characters, we must identify which character such a sequence $t_n$ converges to. It must live in $X_k$ since $X_k$ contains all characters.

**Lemma 5.14.** Let $\delta > 2$, and let $k = 2m$ be even. Let $\omega_1$ be the point $-1 \in S^1 \subseteq X_k$ corresponding to the weight 2 copy of $S^1$ and similarly, $\omega_1$ the point in the same circle corresponding to 1. Then for any $f \in T_{k,k}$, if $\{t_n\} \subseteq (-\delta, \delta)$ is a sequence such that $t_n \to \delta$, $t_n(f) \to \omega_1(f)$. If $t_n \to -\delta$, then $t_n(f) \to \omega_1(f)$.

**Proof.** First from the list of coefficients above $B_{0,0}^k \to 0$ as $t_j \to \pm \delta$, and thus $t_n(x_{0,j}^k) = \nu B_{0,0}^k \to 0$ $t_j \to \pm \delta$, $t_j(x_1) \to 0$ for all $l$. Thus the limit of $t_j$ must be some higher weight character. We see that

$$B_{0,2}^k(t_j) = \prod_{1 \leq i \leq \frac{k}{2}} \frac{[2i]^2 - t_j^2[i]^2}{[2i - 1][2i]} \to \prod_{1 \leq i \leq \frac{k}{2}} \frac{[2i]^2 - [2i]^2[i]^2}{[2i - 1][2i]}$$

On the other hand using our formula for the $B$’s and lemma 5.3,

$$B_{2,2}^k(\pm 1) = \prod_{i=2}^n \frac{[2i]^2 - [i - 1]^2 - [i + 1]^2 - 2[i + 1][i - 1]}{[2i][2i - 1]}$$

Using the fact that $[2i][i] = [i + 1] + [i - 1]$, and comparing each term in the product with the same denominator, we see that the term in the limit of the $t_j$ is $[2i] - [2i]^2[i]^2 = [2i] - ([i + 1] + [i - 1])^2 = [2i] - [i + 1]^2 - [i - 1]^2 - 2[i + 1][i - 1]$, which is precisely the term in $B_{2,2}^k(\pm 1)$. Therefore we see that $\lim t_n$ must be a lowest weight 2 character, and it must be $\omega_\pm$. The problem is, we do not know which it is. To determine this, we notice that $\alpha(x_{2,1}^k) = \alpha B_{0,2}^k$.

Therefore, as $t_n \to \delta$, $t_n(x_{2,1}^k) \to -B_{0,2}^k(-1) = \omega_1(x_{2,1}^k)$. Since $x_{2,1}^k$ separates points, we see that $\lim_{t \to -\delta} t = \omega_1. □$

**Lemma 5.15.** Let $k > 0$, $\delta = 2$.

1. Suppose $k$ is even. Let $\omega_{\pm 1}$ be the points on the highest weight $k$ circle. If $\omega_n$ is a subset of the lowest weight $m$ circle for some $m \leq k$ then $\omega_n \to \pm 1$, then $\omega_n(f) \to \omega_{\pm(1)^{k-m}}(f)$.

2. Let $k$ be odd and $\omega_{\pm 1}$ be the points on the weight $k$ circle corresponding to $\pm 1$. If $\omega_n$ is a subset of a weight $m$ circle for some $m \leq k$ such that $\omega_n \to \pm 1$, then $\omega_n(f) \to \omega_{\mp(1)^{m-2}}(f)$.

**Proof.** If $\omega_n \to \pm 1$ by examining coefficients, we see that $B_{m,n}^k(\omega_n) \to 0$. Since this coefficient occurs in the evaluation of $\omega_n(x)$ for all diagrams of rank $< k$, we see that $\omega_n$ must be converging to a lowest weight $k$ character. To determine which one, we note that $x_{k,1}^k = \rho_m$ compute $\omega_n(x_{k,1}^k) = (-1)^{k-m} \frac{1}{k} \left((-1)^k \frac{k-m}{2} \omega_n^{-1} + \frac{k+m}{2} \omega_n\right)$.

If $k$ is even, then as $\omega_n \to \pm 1$, $\omega_n(x_{k,1}^k) \to \pm(-1)^{k-m}$. Since $x_{k,1}^k$ separates lowest weight $k$ representations, we are done.

If $k$ is odd, then as $\omega_n \to \pm 1$, $\omega_n(x_{k,1}^k) \to \mp(-1)^{k-m}$. Since $x_{k,1}^k$ separates lowest weight $k$ representations, we are done.

**Proof of Theorem 5.9** Now, we have found the appropriate topology on the set $X_k$. For $k$ odd, we are done since there are no “missing” points. For $k$ even, we must identify the points $\pm \delta$ on the interval $[-\delta, \delta]$ with the points $\mp 1$ respectively, on the weight 2 circle. Thus we have that $C^*(T_{k,k})$ is an abelian $C^*$ algebra whose spectrum is the compact Hausdorff space $X_k$. This concludes the case when $\delta > 2$.

Now assume $\delta = 2$. For $k$ even, by the above lemma, the weight $m$ circle for $m > 0$ will be glued on to the weight $k$ circle at the points $\pm 1$, and it alternates which endpoint goes to which endpoint on the circle
as \( \frac{k-m}{2} \) changes parity. We know by the above lemma that the interval is glued at the points \( \pm 1 \) on the weight 2 circle which in turn is glued to the points \( \pm 1 \) on the weight \( k \) circle, resulting in the space pictured as \( Y_k \). For \( k \) odd, we glue the points \( \pm 1 \) to the highest weight circle in an alternating fashion as described in the above lemma. Topologically, we obtain the space \( Y_k \) pictured □.

**Remark.** It would be interesting to understand the tensor structure on \( \text{Rep}(ATL) \) in terms of this topological characterization. We also have not characterized \( T_{k,m} \) for \( m \neq n \). We notice that its completion is a Hilbert \( C^* \) bimodule over \( C(X_m) - C(X_k) \), and thus \( T_{k,m} \) should have some characterization as vector fields on \( X_k \times X_m \), which may be interesting. Understanding this structure will help determine the tensor structure on the category \( \text{Rep}(ATL) \).

### 6. Approximation and Rigidity Properties

#### 6.1. Weight 0 Representations and Approximation/Rigidity Properties.

In a recent remarkable paper [30], Popa and Vaes defined representation theory for rigid \( C^* \)-tensor categories. In particular, they introduce \( C^* \)-multipliers for \( C \), thus identifying a class of admissible states on the fusion algebra. This provides a framework for a representation theory for \( C \). In this context, they define approximation and rigidity properties, which in the case that \( C \) is pointed, are equivalent to the corresponding property for the group. They show that if \( C \) is equivalent to the category of \( M - M \) bimodules in the standard invariant of a finite index inclusion \( N \subseteq M \) of \( II_1 \) factors, then the definitions of approximation and rigidity properties given via \( C^* \)-multipliers are equivalent to the definitions defined via the symmetric envelopping algebra for the subfactor \( N \subseteq M \).

In a recent paper of Neshveyev and Yamashita [24], given an object of \( Z(\text{ind} - C) \), they construct a representation of the fusion algebra. They then show that the class of representations of the fusion algebra that arises in this way is exactly the class identified by Popa and Vaes. By the proposition due to Stefaan Vaes in section 3, the representations of the fusion algebra coming from \( Z(\text{ind} - C) \) are precisely the representations of the fusion algebra coming from \( \text{Rep}(AW) \). Thus the results of this subsection follow from [24]. However, we prefer to give a proof of this fact in our language since it is relatively short, and perhaps more convenient for the reader.

Here, we let \( \Lambda \) be a choice of representatives of equivalence classes of simple objects in \( \text{Irr}(C) \). We identify \( \Lambda \) and \( \text{Irr}(C) \) as sets, and use them interchangeably.

We recall that the fusion algebra is isomorphic to \( \mathbb{C}[\Lambda] \), with multiplication given by the fusion rules extended linearly, with \( * \) structure given by the duality map. If we let \( 0 \in \Lambda \) denote the strict tensor identity \( \text{id} \), it is easy to see, thanks to our tube composition in section 3, that \( AW_{0,0} \cong \mathbb{C}[\Lambda] \) as a vector space. Multiplication in the annular category clearly gives the fusion rules, and thus \( AW_{0,0} \cong \mathbb{C}[\Lambda] \) as a \( * \) algebra. If \( \phi \) is a function on \( \text{Irr}(C) \), trivially, it acts on \( f = \sum_X f_X \in AW_{0,0} \) (where \( f_X \in AW_{0,0}^X \)) in pictures by

\[
\phi(f) = \sum_X \frac{\phi(X)}{d(X)} X_f X
\]
This is because $f_X$ is really a scalar times the single string labeled $X$. Now since any annular category has $AW_{0,0} \cong \mathbb{C}[\Lambda]$, we can naturally identify $(T_{0,0})^\#$ with $(AP_{0,0})^\#$, both as functions $\phi : \text{Irr}(\mathcal{C}) \to \mathbb{C}$. We have the following lemma:

**Lemma 6.1.** If $\phi : \text{Irr}(\mathcal{C}) \to \mathbb{C}$, then $\phi \in \Phi AW_0$ if and only if $\phi \in \Phi T_0$.

**Proof.** Since we can imbed $TC$ as a subcategory of $AW$, it is clear that $\phi \in \Phi AW_0 \Rightarrow \phi \in \Phi T_0$. For the converse, suppose $\phi \in \Phi T_0$. Let $f \in AW_{0,m}$, $m \in \mathcal{W}$. Then $f = \sum_X f_X$, where $f_X \in AW_{0,m}$ and $X \in \Lambda$, by the tube algebra decomposition of $AW_{0,m}$. For each $Y \in \Lambda$ with $Y \prec m$, let $B_Y^Y$ be the orthogonal basis of $(Y, m)$ in $\mathcal{C}$. Then for each $Y \prec m$, and $\mu \in B_Y^Y$, we set $f_{\mu, Y} := \sum_X \phi_X(f_{\mu, Y}) \in T_{0,Y}$. Then we have, resolving the identity $1_m$ into central projections, that

$$\phi(f^*f) = \sum_{Y < m, \mu \in B_Y^Y} \phi(f_{\mu, Y}^*f_{\mu, Y})$$

Since $\phi \in \Phi T_0$ each term in the above sum is positive, hence the entire sum is positive, thus $\phi \in \Phi AW_0$.

**Lemma 6.2.** If $\phi : \text{Irr}(\mathcal{C}) \to \mathbb{C}$. Define $\phi^{\text{op}} : \text{Irr}(\mathcal{C}) \to \mathbb{C}$ by $\phi^{\text{op}}(X) = \phi(\overline{X})$. Then $\phi$ is an affine state if and only if $\phi^{\text{op}}$ is an affine state.

**Proof.** We only need to check the positivity condition. Suppose $\phi \in AW_{0,0}$. Recall the global duality map defined at the end of section 3. Then if $f \in AW_{0,k}$, $\phi^{\text{op}}(f^*f) = \phi(r(f^*)r(f)) = \phi(r(f)r(f^*))$. Since $r$ is an anti-isomorphism on the tube category, the result follows.

Now we recall several definitions from [30]. Let $\mathcal{C}$ be a rigid $C^*$ tensor category and let $\text{Irr}(\mathcal{C})$ be the set of simple objects.

**Definition 6.3.** A *multiplier* on a rigid $C^*$ tensor category is a family of linear maps $\Theta_{\alpha, \beta} : \text{End}(\alpha \otimes \beta) \to \text{End}(\alpha \otimes \beta)$ for all $\alpha, \beta \in \text{Obj}(\mathcal{C})$ such that

1. Each $\Theta_{\alpha, \beta}$ is $\text{End}(\alpha) \otimes \text{End}(\beta)$-bimodular
2. $\Theta_{\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2}(1 \otimes X \otimes 1) = 1 \otimes \Theta_{\alpha_2, \beta_1}(X) \otimes 1$ for all $\alpha_i, \beta_i \in \mathcal{C}, X \in \text{End}(\alpha_2, \beta_2)$
3. A multiplier is a $cp$ – multiplier if each $\Theta_{\alpha, \beta}$ is completely positive.

In [30], Proposition 3.6, it is shown that multipliers are in one-one correspondence with functions $\phi : \text{Irr}(\mathcal{C}) \to \mathbb{C}$ as follows. For such a $\phi$, we define a multiplier $\Theta^\phi_{\alpha, \beta}$ as follows:

For an object $\alpha \in \mathcal{C}$, define the central projection in $\text{End}(\alpha \otimes \pi)$,
where \( \{ t_{Y}^{\alpha,i} \}_{1 \leq i \leq N_{Y}} \subseteq \text{Hom}(Y, \alpha \otimes \overline{\alpha}) \) is an orthogonal basis normalized so that \( (t_{Y}^{\alpha,i})^{\ast} t_{Y}^{\alpha,i} = 1_{Y} \) for each \( i \). Then for \( x \in \text{End}(\alpha, \beta) \),

\[
\Theta_{\alpha,\beta}^{\phi}(x) = \sum_{Y \prec \overline{\alpha}} \phi(Y) s_{Y}^{\alpha} s_{Y}^{\ast}_{\alpha} x = \sum_{Y \prec \overline{\beta}} \phi(Y) s_{Y}^{\alpha} s_{Y}^{\ast}_{\alpha} x
\]

Furthermore, they show every multiplier is of this form. We abuse notation notation, and say a function \( \phi : \text{Irr}(C) \to \mathbb{C} \) is a cp-multiplier (cpm) if \( \Theta^{\phi} \) is a cp-multiplier. It is shown in PV that if \( \phi : \text{Irr}(C) \to \mathbb{C} \) is a cp-multiplier, that \( d(.)\phi(.) : \mathbb{C}[\text{Irr}(C)] \to \mathbb{C} \) is a state on the fusion algebra.

**Definition 6.4.**

1. A function \( \phi : \text{Irr}(C) \to \mathbb{C} \) is called an **admissible state** if \( \frac{\phi(.)}{d(.)} \) is a cp-multiplier.
2. A * representation \( \pi \) of \( AW_{0,0} \cong \mathbb{C}[\text{Irr}(C)] \) is called **admissible** if every vector state in the representation is admissible.
3. Define \( ||| \cdot |||_{u} := \sup_{\pi \text{ admissible}} ||| \cdot |||_{\pi} \) on \( AW_{0,0} \cong \mathbb{C}[\text{Irr}(C)] \). \( C^\ast(\text{Irr}(C)) \) is the completion of \( AW_{0,0} \cong \mathbb{C}[\text{Irr}(C)] \) with respect to this universal norm. It is shown in [30] that this is finite and a \( C^\ast \)-algebra norm.

We will show that admissible states are exactly the same thing as affine states on \( AW_{0,0} \).

**Theorem 6.5.** \( \phi \) is an affine state if and only if \( \phi \) is admissible.

First, a lemma due to Popa and Vaes:

**Lemma 6.6 ([30] Lemma 3.7).** Let \( C \) be a rigid \( C^\ast \)-tensor category and \( \Theta \) a multiplier on \( C \). Then the following are equivalent:

1. For all \( \alpha, \beta \in C \), the map \( \Theta_{\alpha,\beta} : \text{End}(\alpha \otimes \beta) \to \text{End}(\alpha \otimes \beta) \) is completely positive.
2. For all \( \alpha, \beta \in C \), the map \( \Theta_{\alpha,\beta} : \text{End}(\alpha \otimes \beta) \to \text{End}(\alpha \otimes \beta) \) is positive.
3. For all \( \alpha \in C \) we have that \( \Theta_{\alpha,\overline{\alpha}}(s_{\alpha}s_{\alpha}^\ast) \) is positive. (Here \( s_{\alpha}s_{\alpha}^\ast \) is the unnormalized \( \alpha \)-Jones projection described below).

The \( \alpha \) Jones projection is give in picture by

\[
\begin{array}{c}
\alpha \\
\alpha
\end{array}
\]

**Proof of Theorem.** First let \( \phi : \text{Irr}(C) \to \mathbb{C} \) be an arbitrary function. We define the multiplier \( \Theta^{\psi} \) associated to the function \( \psi(.) := \frac{\phi(.)}{d(.)} \) as above. Take any vector \( v \in \text{End}(\alpha \otimes \overline{\alpha}) \). Then we have

\[
\langle \Theta^{\psi}_{\alpha,\overline{\alpha}}(s_{\alpha}s_{\alpha}^\ast)v, v \rangle = Tr(\Theta^{\psi}_{\alpha,\overline{\alpha}}(s_{\alpha}s_{\alpha}^\ast)vv^\ast)
\]
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$$= \sum_{Y \prec \alpha} \frac{\phi(Y)}{d(Y)}$$

Now, since affine states are the same for all annular categories, we set $W = \text{Obj}(C)$.

$$\phi^{\text{op}} \circ \Psi^{\alpha}$$

where in the last equality, we view $\phi \in (AW_{0,0})^\#$. Now, we see that $\phi \in \Phi AW_0$ if and only if $\phi^{\text{op}} \in \Phi AW_0$, and so then the above expression is non-negative for all $v, \alpha$, hence $\Theta^\psi$ is a cp-multiplier by. Conversely, if $\Theta^\psi$ is a cp-multiplier for all then we need to show that $\phi$ is an affine state, but it suffices to show $\phi^{\text{op}} \in \Phi AW_0$.

But by lemma 6.1 it suffices to check this for the tube algebra. Let $f \in T_{0,Y}$, $Y \in \Lambda$. Then $f = \sum f_X$ where $X \in \Lambda$ and $f_X \in T^X_{0,Y}$. Note that if we set $\alpha := \oplus X$ where $X$ appears as components of $f$, let $i_X \in (X, \alpha)$ be the canonical inclusions. Then we have, by definition that $f_X \neq 0$ implies $Y \prec \alpha X$. Let $t_Y \in (Y, \alpha X)$ such that $t^*_Y t_Y = 1_Y$ and $t_Y t^*_Y \in (\alpha X, \alpha X)$ is a minimal projection. Then set

$$v^* := \sum_{\alpha} f_X \in (\alpha X, \alpha X)$$

Then since $\Theta^\psi$ is a cp-multiplier,

$$\phi^{\text{op}}(f^* f) = (\Theta^\psi_{\alpha, \pi} v, v) \geq 0$$

Thus $\phi$ is an affine state.
Corollary 6.7. (π, H) be a * representation of the fusion algebra AW_{0,0}. Then the following are equivalent:

1. (π, H) is admissible.
2. (π, H) is the weight 0 part of a Hilbert AW representation.
3. (π, H) is the weight 0 part of a Hilbert TC representation.

We consider the affine state 1_C corresponding to the trivial representation, given by 1_C(X) = d(X) for each X ∈ Irr(C). We note that if φ ∈ ΦAW_0, then φ is a state on C*(AW_{0,0}). Furthermore, for each simple object X ∈ Irr(C), ||X||_u = d(X), since we have ||X|| ≤ d(X) by corollary 4.13, and this value is realized in the trivial representation. Furthermore, C*(AW_{0,0}) contains the one dimensional subspace \( \hat{X} \cong \mathbb{C}[X] \) for X ∈ Irr(C). Hence for an affine state φ ∈ ΦAW_0, when viewed as a state on C*(AW_{0,0}), ||φ|_X|| = \( |\frac{φ(X)}{d(X)}| \). Hence the numbers \( |\frac{φ(X)}{d(X)}| \) are “local norms” of the state φ. Now, we recall the definitions of approximation and rigidity properties given by Popa and Vaes, but present them translated into our annular language.

Definition 6.8. [30] A rigid C* tensor category (with Irr(C) countable) is said

1. to be amenable if there exists a sequence of finitely supported affine state φ_n that converges to 1_C pointwise on Irr(C).
2. to have Property T if for every sequence of affine states φ_n which converges pointwise to 1_C, the sequence of functions \( \frac{φ_n(\cdot)}{n} \) converges uniformly to 1 on Irr(C).
3. to have the Haagerup property if there exists a sequence of affine states φ_n each of which vanish at ∞ (for every ϵ, there exists a finite subset K ⊆ Irr(C) such that \( |\frac{φ(X)}{d(X)}| < \epsilon \) for all X ∈ K′), which converge to 1_C pointwise.

The statements above can thus be interpreted as statements about the convergence of states in the “local norms” of affine states.

It is shown in [30] that these definitions are equivalent to the usual ones given in terms of symmetric enveloping algebras in the case where C is the even part of some subfactor standard invariant. Popa and Vaes also give several very interesting examples of categories with each of these approximation properties. We recall the following results of Popa and Vaes, which can be seen in our setting:

Proposition 6.9. [30]

1. Let C be a pointed category with Irr(C) ≅ G for a discrete group G. Then C has the above approximation property if and only if G has the corresponding property for groups as a discrete group.
2. TLJ(2) is amenable.
3. TLJ(δ) has the Haagerup property for δ ≥ 2.

The first two items in the above proposition are due to Popa and have been known for some time. We direct the reader to the paper of Popa and Vaes [30] for more details on the third item. They use results about the corresponding quantum groups SU_q(2) obtained in [6]. We note that while Popa and Vaes consider only the even part of TLJ(δ), their proofs work more generally. One could deduce these results from the analysis of our examples above, by choosing weight 0 characters t ∈ (−δ, δ) such that t → δ. That these affine states are c_0 is straightforward. We have not emphasized this because the arguments are exactly the same as in [30]. We remark that in [1], the authors give a proof that TLJ(δ) has the Haagerup property using annular representations to directly construct a bi-module over a symmetric enveloping inclusion of a subfactor whose standard invariant is TLJ(δ).
6.2. Higher Weights and Approximation Properties. If we try to define approximation and rigidity properties for the higher weights (i.e. \( k \in \mathcal{W} \) with \( k \neq 0 \)) in the vein of Popa and Vaes for higher weight centralizer algebras \( AW_{k,k} \), we do not have the notion of a weight \( k \) trivial representation. However, we do have the notions of affine states and universal norms. In [4], Brown and Guentner have characterized the three approximation/rigidity properties described in the previous section in terms of \( C^* \) algebra completions.

We will recast their work in the language of affine annular categories, and briefly show that definitions defined here for weight 0 (the fusion algebra) agree with the definitions of Popa and Vaes from the previous section. This will allow us to define approximation and rigidity properties for higher weight spaces in their terms. We would like to thank Ben Hayes for pointing us in this direction.

First we define a “pointwise product” of affine states and show that they are again affine states. Our definition requires some knowledge of the functor \( \text{Rep}(AW) \) as in [5], however the final result will be easy to understand. Let \( \phi \in \Phi_{AW_k} \) and \( \psi \in \Phi_{AW_0} \). We define the tensor product of the two states as follows. Let \( \xi_{\phi} \) be a vector in a Hilbert representations \( H_{\phi} \) realizing \( \phi \), i.e. \( \phi(x) = \langle \pi(x) \xi_{\phi}, \xi_{\phi} \rangle \) and similarly for \( \psi \). Then we define \( \phi \boxtimes \psi \in \Phi_{AW_k} \) as the vector state in the tensor product representation corresponding to the vector \( \xi_{\phi} \otimes 1_k \otimes \xi_{\psi} \) in the weight \( k \) space of \( H_{\phi} \otimes H_{\psi} \). This vector state is an affine state by [5].

**Lemma 6.10.** Let \( x = \sum_W x_w \in AP_{k,k} \), where each \( x_w \in AP_{W,k,k} \). Then \( \phi \boxtimes \psi(x) = \sum_W \frac{\psi(W)}{d(W)} \phi(x_w) \).

**Proof.** To compute the value of this state, let \( \prod_n c_1^{\phi} \) and \( \prod_n c_1^{\psi} \) be the commutativity constraints corresponding to \( \phi \) and \( \psi \) respectively. Then we see that for \( x \in AP_{k,k} \), \( W \prec m \), we have

\[
\phi \boxtimes \psi(x) = \sum_{Y \prec W} \frac{\psi(W)}{d(W)} \phi(x_w)
\]

Where \( P_{WW}^Y \in \text{End}(WW) \) is the central projection of \( WW \) onto the summand of subobjects isomorphic to \( Y \). Reading the diagram sideways, we see that we have a hom from identity to identity factoring through the \( P_{WW}^Y \), but since each \( Y \) is simple, all these terms are 0 except when \( Y = \text{id} \). But \( Z_{WW}^0 \) is the \( W \) Jones projection, and thus the above is equal to

\[
\frac{1}{d(W)} \psi(W) \phi(x)
\]
Extending by linearity, we see that for \( x = \sum_w x_w \in AW_{k,k} \), where each \( x_w \in AW_{k,k}^W \),
\[
\phi \otimes \psi(x) = \sum_w \frac{\psi(W)}{d(W)} \phi(x_w). 
\]

Although the fact that this is an affine state in general requires \([5]\), the reader unfamiliar with this work can simply use the above lemma for the rest of this section to understand this state. Now, we are ready to define universal norms with respect to certain ideals, following \([4]\).

Consider the annular centralizer algebra of weight \( k \), \( AW_{k,k} \cong \bigoplus_{X \in A} AW_{k,k}^X \). We equip each finite dimensional vector space \( AW_{k,k}^X \) with the restriction of the universal norm \( \| \cdot \|_u \).

**Definition 6.11.** Let \( D \trianglelefteq \ell^\infty(\Lambda) \) be an algebraic ideal. If \( \phi \in (\text{C}^*(AW_{k,k}))^\# \) (the space of continuous linear functionals on \( \text{C}^*(AW_{k,k}) \)), then we say \( \phi \) is \( D \)-class if the function \( \hat{\phi} : \text{Irr}(C) \to \mathbb{R}_+ \) defined by \( \hat{\phi}(X) := \|\phi|_{AW_{k,k}^X}\| \) is in \( D \).

We notice that \( \hat{\phi} \in \ell^\infty(\text{Irr}(C)) \) since \( \|\phi|_{AW_{k,k}^X}\| \leq \|\phi\| \), so the definition above makes sense.

**Definition 6.12.** An admissible representation \( (\pi, V) \) of \( AW_{k,k} \) is \( D \)-class if there is a dense subspace \( \{\eta_\alpha\} \subseteq V \) such that all the matrix coefficients \( \langle \pi(\cdot), \eta_\alpha, \eta_\beta \rangle \) are \( D \)-class.

Since we only require dense subspaces, this class is closed under direct sums.

**Definition 6.13.** We define a \( \text{C}^* \) semi-norm on \( AW_{k,k} \) by \( ||f||_D := \sup\{|f|_V : V \text{ is class } D\} \).

We remark that this semi-norm is finite since it is bounded by the universal norm, and a \( \text{C}^* \) semi-norm since arbitrary direct sums are class \( D \) hence we may take such a direct sum to realize this semi-norm. We call this representation \( \pi_D \). Let \( \text{Ker}_D \trianglelefteq AW_{k,k} \) denote the kernel of this representation. Then we define the \( \text{C}^* \)-algebra \( C_D^*(AW_{k,k}) := \overline{AW_{k,k}/\text{Ker}_D}||\cdot||_D \). We have a natural homomorphism \( \gamma_D : \text{C}^*(AW_{k,k}) \to C_D^*(AW_{k,k}) \), and we notice that if \( D = \ell^\infty(\text{Irr}(C)) \), then the natural homomorphism is an isomorphism \( C_D^*(AP_{k,k}) \cong C_{\ell^\infty}(AP_{k,k}) \).

**Definition 6.14.** We say an ideal \( D \trianglelefteq \ell^\infty(\text{Irr}(C)) \) is \( k \)-translation invariant if for all \( \phi \in (AP_{k,k})^\# \) such that \( \phi \) is \( D \)-class, then for any \( x, y \in AP_{k,k}, \phi(x(\cdot)y) \) is \( D \)-class.

For \( k = 0 \), this is equivalent to \( D \) being invariant under left and right actions of the fusion algebras. We present the following lemma, following \([4]\), to underline the role of translation invariance for the rest of the section:

**Lemma 6.15.** If \( D \) is \( k \)-translation invariant, \( \phi \in \Phi AW_k \), and \( \phi \) is \( D \)-class, then the GNS representation of \( \text{C}^*(AW_{k,k}) \) with respect to \( \phi \) is \( D \)-class.

**Proof.** Let \( \omega \) be a cyclic vector for \( \phi \). Then for \( f, g, h \in AW_{k,k} \) we have \( \langle \pi_\phi(f)g\omega, h\omega \rangle_\phi = \phi(g^*fh) \). By translation invariance, this implies \( \langle \pi_\phi(\cdot)g\omega, h\omega \rangle \) is \( D \)-class. Since \( \{g\omega : g \in AW_{k,k}\} \) is dense in the GNS representation, this representation class \( D \).

**Corollary 6.16.** If \( D \) is translation invariant, \( f \in AW_{k,k} \), then \( \|f\|_D^2 = \sup_{\phi \in \Phi AW_k \cap D\text{-class}} \phi(f^*f) \).

**Lemma 6.17.** If \( \phi \in \Phi AW_0 \), then \( \phi \) is \( D \)-class if and only if \( |\phi(\cdot)|_D \in D \).
Proof. We note that $\phi$ is $D$-class if and only if the function from $Irr(C) \to \mathbb{C}$ given by $\hat{\phi}(X) = ||\phi|_{AW_0}||$ is $D$-class. But $AW_0 \cong CX$, and for $\lambda \in \mathbb{C}$, $\phi(\lambda X) = \lambda \phi(X)$. But $||X||_n = d(X)$, and thus $\hat{\phi}(X) = |\frac{\phi(X)}{d(X)}|$, \Box.

The following lemma is a direct adaptaion of \cite[Theorem 3.2]{H}. We use an almost identical proof with the exception that we are now using affine states and $\mathbb{E}$ defined above, instead of positive definite functions and pointwise product.

**Lemma 6.18.** If $D \subseteq \ell^\infty(Irr(C))$ is a 0-translation invariant ideal, then the canonical homomorphism $\gamma_D : C^*(AP_0) \to C^*_D(AW_0)$ is an isomorphism if and only if there exists a sequence $\{\phi_n\} \subseteq \Phi AW_0 \cap D – \text{class}$ such that $\frac{\phi_n(\cdot)}{d(\cdot)} \to 1$ pointwise.

**Proof.** First suppose the canonical map $C^*(AW_0) \to C^*_D(AW_0)$ is an isomorphism. Then there exists a faithful $D$-class representation $\pi$ of $C^*(AW_0)$. Taking infinite direct sums if necessary, we can assume that $\pi(C^*(AW_0))$ contains no compact operators. Then by Glimm’s Lemma (see, for example \cite[Lemma 1.4.11]{H}), for any state $\phi$ of $C^*(AW_0)$ there exists a sequence of vector states $\omega_i \to \phi$. By definition of $D$-class, there is a dense subspace of vectors whose vector states are $D$-class. We can thus approximate the vector states with $D$-class vector states. Setting $\phi = 1_C$, the trivial representation vector state described in the previous section, we have one direction of our lemma.

Now suppose that there exists a sequence of functions $\phi_n \in \Phi AW_0$ such that $|\frac{\phi_n(\cdot)}{d(\cdot)}| \in D$, and $\frac{\phi_n(\cdot)}{d(\cdot)} \to 1$ pointwise. By the above corollary, we simply need to show that the collection of $D$-class affine states is weak- * dense in $\Phi AW_0$. Let $\psi \in \Phi AW_0$ be arbitrary. Then $\psi \boxtimes \phi_n : Irr(C) \to \mathbb{C}$ is $D$-class since $|\psi \boxtimes \phi_n(\cdot)| = |\psi(\cdot) \frac{\phi_n(\cdot)}{d(\cdot)}|$. Since $|\frac{\phi_n(\cdot)}{d(\cdot)}| \in D$, $|\psi(\cdot) \frac{\phi_n(\cdot)}{d(\cdot)}| \in D$ since $D$ is an ideal. Now $\psi \boxtimes \phi_n(X) = \psi(X) \frac{\phi_n(X)}{d(X)} \to \psi(X)$ by hypothesis, and thus $\psi \boxtimes \phi_n \to \psi$ in the weak * topology on $\Phi AW_0$.

We let $c_c \subseteq \ell^\infty(Irr(C))$ be the algebraic ideal of finitely supported functions, and $c_0$ the ideal of functions vanishing at $\infty$.

**Lemma 6.19.** $c_c$ and $c_0$ are $k$-translation invariant ideals for all $k \in \mathbb{W}$.

**Proof.** First we remark that to check translation invariance, it suffices to check that $\phi(\hat{j} \cdot)$ and $\phi(\hat{f} \cdot)$ are in $D$ independently. Furthermore by linearity it suffices to check for $f \in AW^X_0$, $Y \in Irr(C)$. Now, we claim that for a fixed $X$, $\# \{Y \in Irr(C) : N^Y_{XZ} \neq 0\} \leq d(X)^2$ for all $Z$. To see this, note that if $Y \prec XZ$ by Frobenius reciprocity, $\overline{Z} \prec \overline{Y}X$, and thus $d(Z) = d(\overline{Y}d(X) = d(X)d(Y)$, hence $1 \leq \frac{d(X)d(Y)}{d(Z)}$. But we have $\#\{Y \in Irr(C) : N^Y_{XZ} \neq 0\} \leq \sum_{Y < XZ} \frac{d(X)d(Y)}{d(Z)} N^Y_{XZ} = d(X)^2$.

Now, to show $c_0$ is $k$-translation invariant, we want to show that if $\hat{\phi} \in c_0$, then the function $\phi(\hat{j} \cdot)(X) := ||\phi(\cdot)||_{AW^X_0} \in \ell^\infty(Irr(C))$ is in fact in $c_0$, where $f \in AW^X_0$. For $c > 0$, there exists a finite subset $K \subset Irr(C)$ such that $||\phi||_{AW^W_0} \leq c d(X)^2$ for all $W \notin K$. Now, for $Y \in K$, define $K'_Y = \{Z \in Irr(C) : Y < XZ\}$. This set is clearly finite by Frobenius reciprocity. Thus $K' = \bigcup_{Y \in K} K'_Y$ is finite. For $Z \notin K'$, since $fAP^Z_0 \subseteq \oplus_{W < XZ} AW^W_0$, we have $||\phi(\cdot)||_{AW^X_0} \leq ||f|| \sum_{W < XZ} ||\phi(\cdot)||_{AW^W_0} \leq \#\{W < XZ\} \frac{c}{d(X)^2} < c$.

The proof for right invariance works exactly the same. Putting $c = 0$ and carrying out the same argument gives the $c_c$ case. \Box.
We are now ready for the generalization of Popa and Vaes’s approximation properties to higher weights $k \in \mathcal{W}$

**Definition 6.20.** Let $\mathcal{AW}$ be an annular category. Let $k \in \mathcal{W}$. Then $\mathcal{AW}$:

1. is $k$-amenable if $C^*(\mathcal{AW}_{k,k}) = C^*_{\mathcal{C}}(\mathcal{AW}_{k,k})$.
2. has the $k$-Haagerup property if $C^*(\mathcal{AW}_{k,k}) = C^*_{\mathcal{A}}(\mathcal{AW}_{k,k})$

**Corollary 6.21.** The definitions of the approximation properties (amenability and Haagerup) above for $k = 0$ are equivalent to the definitions of Popa and Vaes.

This follows easily from lemma 6.18 □.

**Theorem 6.22.** Let $\mathcal{AW}$ be an annular category over $\mathcal{C}$.

1. If $\mathcal{C}$ is amenable, then it is $k$-amenable for all $k \in \mathcal{W}$.
2. If $\mathcal{C}$ has the Haagerup property, it has the $k$-Haagerup property for all $k \in \mathcal{W}$.

**Proof.** Let $D$ be either $c_0(Irr(\mathcal{C}))$ or $c_0(Irr(\mathcal{C}))$. By the above corollary, if $C^*_D(\mathcal{AP}_{0,0}) = C^*_{\mathcal{AW}_{0,0}}$, then there exists $\phi_n \in \Phi \mathcal{AW}_0$ with $|\phi_n(x, y)| \in D$ such that $\phi_n(x, y)$ converges to 1 for each $X \in Irr(\mathcal{C})$. Now, if $\psi \in \Phi \mathcal{AW}_0$, we see that the function defined on $Irr(\mathcal{C})$ by $h_n(X) := ||\psi \boxtimes \phi_n|_{\mathcal{AW}_{k,k}^0}|| = ||\phi_n(X)| \mathcal{AW}_{k,k}^0|| \in D$. By the definition $\psi \boxtimes \phi_n$ is a $D$-class affine state, and for every $f \in \mathcal{AW}_{k,k}^0$ with $f = \sum X f_X$ where $f_X \in \mathcal{AW}_{k,k}^0$, $\psi \boxtimes \phi_n(f) = \sum X \psi(f_X) \phi_n(X) \rightarrow \psi(f)$. Thus by the set of $D$-class states is weak * dense in the set of all states of $C^*(\mathcal{AW}_{k,k})$, and thus $C^*_D(\mathcal{AW}_{k,k}) = C^*(\mathcal{AW}_{k,k})$ □.

Therefore weight 0 approximation properties imply the approximation properties for all higher weights, thus supporting the notion that the definition for weight 0 is certainly the “right” place to define these properties. Using the pointed category example, it is easy to find examples with higher approximation properties, but not weight 0, i.e. find a group which is not amenable, or does not have the Haagerup property, but some centralizer subgroup does.

**Remark:** We can define property $T$ for higher weights as follows: $\mathcal{AW}$ has $k$-property ($T$) if the only translation invariant ideal $D \subseteq \ell^\infty$ with $C^*(\mathcal{AW}_{k,k}) = C^*_D(\mathcal{AW}_{k,k})$ is the ideal $D = \ell^\infty(Irr(\mathcal{C}))$. The difficulty here is that it is not totally obvious that this is equivalent in the weight 0 case to the work of Popa and Vaes. We give a sketch of the proof following [4], Proposition 3.6. Suppose $C^*_D(\mathcal{AP}_{0,0}) = C^*(\mathcal{AP}_{0,0})$. Then by the above lemma, there exists a sequence of functions $h_n(\cdot) := \phi_n\frac{\cdot}{X} \in D$ converging to 1 pointwise. Thus if $\mathcal{C}$ has property $T$, this sequence of functions converges uniformly on $Irr(\mathcal{C})$. Thus for some $n$, $h_n$ is bounded away from 0, hence is invertible. Thus $1 \in D$, so $D = \ell^\infty(\mathcal{C})$. Now, suppose $\mathcal{C}$ does not have property $T$. Define the ideal $D := \{f \in \ell^\infty(Irr(\mathcal{C})) \inf_{s \in F} |f(s)| = 0, F \subset Irr(\mathcal{C}) \}$. It is easy to check that this is proper, and translation invariant. But one can modify the definitions/proofs for the group case to show that not Property $T$ implies there exists an unbounded conditionally negative function $\Psi$ on $Irr(\mathcal{C})$, and then show that $h_n := e^{-\frac{s}{x}} \in D$, and $d(\cdot)h_n(\cdot) \in \Phi \mathcal{AP}_0$ and $d(\cdot)h_n(\cdot) \rightarrow 1\mathcal{C}$. This implies $C^*(\mathcal{AP}_{0,0}) = C^*_D(\mathcal{AP}_{0,0})$ by the above lemma. It will be interesting to analyze the higher centralizer algebras of the categories $Rep(SU_q(n))$ for $n$ an odd natural number greater than or equal to 3, where in [30] they showed that these categories have property $T$.

**Corollary 6.23.**

1. If $\mathcal{C}$ is a pointed category with $Irr(\mathcal{C}) \cong G$, then we consider the tube algebra, and hence the set of weights $\mathcal{W}$ is indexed by elements of $G$. $\mathcal{C}$ has $X$-property (amenability, Haagerup, $T$) if and only if $Z_G(\mathcal{X})$ has the corresponding property for discrete groups.
(2) For all weight sets $W$ and all $k \in W$, $TLJ(2)$ is $k$-amenable
(3) For all weight sets $W$ and all $k \in W$, $TLJ(\delta)$ has the $k$-Haagerup property.

Proof. The first item follows from the fact that $C^*(T_X, X) \cong C^*_u(Z_G(X))$. The second two follow from 6.22 and 6.9 □.

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