Emergence of Unstable Modes for Shock Waves in Ideal MHD

Heinrich Freistühler*, Felix Kleber*, and Johannes Schropp*

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Abstract

This note studies classical magnetohydrodynamic shock waves in an inviscid fluidic plasma that is assumed to be a perfect conductor of heat as well as of electricity. For this mathematically prototypical material, it identifies a critical manifold in parameter space, across which slow classical MHD shock waves undergo emergence of a complex conjugate pair of unstable transverse modes. In the reflectionally symmetric case of parallel shocks, this emergence happens at the spectral value \( \hat{\lambda} \equiv \lambda/|\omega| = 0 \), and the critical manifold possesses a simple explicit algebraic representation. Results of refined numerical treatment show that for only almost parallel shocks the unstable mode pair emerges from two spectral values \( \hat{\lambda} = \pm i\gamma, \gamma > 0 \).

1 The equations of ideal isothermal MHD

We consider ideal MHD in twodimensional space,

\[
\begin{align*}
0 &= \rho_t + \text{div}(\rho V) \\
0 &= (\rho V)_t + \text{div}(\rho V \otimes V + (p + \frac{1}{2} |B|^2)I - B \otimes B) \quad (1) \\
0 &= B_t + \text{div}(B \otimes V - V \otimes B).
\end{align*}
\]

The dependent variables \( \rho > 0, p > 0, V \in \mathbb{R}^2 \) denote the fluid’s density, pressure, and velocity. In addition to (1), the magnetic field \( B \in \mathbb{R}^2 \) satisfies

\[
\text{div } B = 0. \quad (2)
\]

\*Department of Mathematics, University of Konstanz, 78457 Konstanz, Germany
The fluid is assumed to be polytropic, \( p = R \rho T \), and have a constant temperature \( T \), so that \( p = c^2 \rho \) with constant sound speed \( c \). By scaling, we assume without loss of generality that \( p = \rho \), i.e., the speed of sound is 1.

We abbreviate (1) as

\[
U_t + F(U)_x + G(U)_y = 0
\]

with

\[
U = \begin{pmatrix} \rho \\ \rho v_1 \\ b_1 \\ \rho v_2 \\ b_2 \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho v_1 \\ \rho v_1 v_1 + p + \frac{1}{2} (b_2^2 - b_1^2) \\ \rho v_1 v_2 - b_1 b_2 \\ 0 \\ b_2 v_1 - v_2 b_1 \end{pmatrix}, \quad G(U) = \begin{pmatrix} \rho v_2 \\ \rho v_2 v_1 - b_2 b_1 \\ \rho v_2 v_2 + p + \frac{1}{2} (b_1^2 - b_2^2) \\ b_1 v_2 - v_1 b_2 \\ 0 \end{pmatrix}.
\]

Using (2), we also write it as a symmetric hyperbolic system\[\]
\[
D(\tilde{U}) \tilde{U}_t + \tilde{A}(\tilde{U}) \tilde{U}_x + \tilde{B}(\tilde{U}) \tilde{U}_y = 0
\]

with \( \tilde{U} = (\rho, v_1, v_2, b_1, b_2)^T \), \( D(\tilde{U}) = \text{diag}(1/\rho, \rho, \rho, 1, 1) \), and

\[
\tilde{A}(\tilde{U}) = \begin{pmatrix}
\frac{v_1}{\rho} & 1 & 0 & 0 & 0 \\
1 & \rho v_1 & 0 & 0 & b_2 \\
0 & 0 & \rho v_1 & 0 & -b_1 \\
0 & 0 & 0 & v_1 & 0 \\
0 & b_2 & -b_1 & 0 & v_1
\end{pmatrix}, \quad \tilde{B}(\tilde{U}) = \begin{pmatrix}
v_2/\rho & 0 & 1 & 0 & 0 \\
0 & \rho v_2 & 0 & -b_2 & 0 \\
1 & 0 & \rho v_2 & b_1 & 0 \\
0 & -b_2 & b_1 & v_2 & 0 \\
0 & 0 & 0 & 0 & v_2
\end{pmatrix}.
\]

Applying the chain rule, we rewrite (5) as

\[
U_t + A(U)U_x + B(U)U_y = 0
\]

where

\[
A = TD^{-1} \tilde{A}T^{-1}, \quad B = TD^{-1} \tilde{B}T^{-1}
\]

with

\[
T = \frac{\partial U}{\partial \tilde{U}} = \begin{pmatrix}
1 & 0 & 0 \\
V & \rho I_2 & 0 \\
0 & 0 & I_2
\end{pmatrix}.
\]

Note that, as we have used (2) on the way from (1) to (5), the matrices \( A \) and \( B \) in (6) are not the Jacobians of the fluxes \( F \) and \( G \).
2 Slow and fast, parallel and non-parallel shock waves

Ideal MHD shock waves, in their prototypical form, have the structure

\[ U(t, x, y) = \begin{cases} U^- = (\rho^-, \rho^- V^-, H^-), & (x, y) \cdot N < s_t, \\ U^+ = (\rho^+, \rho^+ V^+, H^+), & (x, y) \cdot N > s_t, \end{cases} \]

where \( N = (N_1, N_2) \in S^1 \) is the direction of propagation and \( s \) the speed of the shock wave. Function \( U \) being a weak solution of (1) is equivalent to the Rankine-Hugoniot conditions

\[-s(U^+ - U^-) + N_1(F(U^+) - F(U^-)) + N_2(G(U^+) - G(U^-)) = 0.\]

Due to rotational and Galilean invariance it is without loss of generality that we henceforth assume that

\[ N = (1, 0) \quad \text{and} \quad s = 0; \]

i. e. we exclusively consider shock waves of the form

\[ U(t, x, y) = \begin{cases} U^- = (\rho^-, \rho^- V^-, H^-), & x < 0, \\ U^+ = (\rho^+, \rho^+ V^+, H^+), & x > 0, \end{cases} \]

and the Rankine-Hugoniot conditions read

\[ F(U^-) = F(U^+). \]

Note now first that for waves (9), as for any solutions of (1) whose spatial dependence is only via \( x \), the divergence-free condition (2) reduces to

\[ b_1 = a, \quad a \] any constant. (11)

We assume (11) and simply write \( b, v, w \) instead of \( b_2, v_1, v_2 \).

In this paper, we are interested in Lax shocks. Following [8, 13, 4], two states

\[ U^- = (\rho^-, \rho^- v^-, \rho^- w^-, a, b^-) \quad \text{and} \quad U^+ = (\rho^+, \rho^+ v^+, \rho^+ w^+, a, b^+) \]

that satisfy the Rankine-Hugoniot conditions (10) constitute a slow Lax shock iff \( 0 < \rho^+(v^+)^2 < \rho^-(v^-)^2 < a^2 \)

and a fast Lax shock iff \( a^2 < \rho(v^+)^2 < \rho^-(v^-)^2. \)

Two states (12) do satisfy the Rankine-Hugoniot conditions (10) if and only if the two quadruples \((\rho^-, v^-, w^-, b^-)\) and \((\rho^+, v^+, w^+, b^+)\) have coinciding images under the mapping

\[
\begin{pmatrix} \rho \\ v \\ w \\ b \end{pmatrix} \mapsto \begin{pmatrix} \rho v \\ \rho v^2 + \rho + \frac{1}{2} b^2 \\ \rho v w - ab \\ bv - aw \end{pmatrix}
\]

(15)
that \( F \) induces by omitting its forth, trivial component, in other words if both quadruples satisfy the four equations

\[
\begin{align*}
\rho v &= m \\
\rho v^2 + \rho + \frac{1}{2} b^2 &= j \\
v b - a w &= c \\
m w - a b &= d
\end{align*}
\]

(16) – (19)

for the same values of the four parameters \( m, j, c, d \in \mathbb{R} \). As simple arguments\(^2\) show, we lose no generality in assuming that

\[
m > 0, \quad d = 0, \quad \text{and} \quad \rho v^2 \neq a^2.
\]

(20)

Using (19) in (18) and inserting the result and (16) in (17) then yields

\[
g^{amc}(v) \equiv m \frac{1 + v^2}{v} + \frac{1}{2} \left( \frac{mc}{mv - a^2} \right)^2 = j, \quad \text{to be solved for } v \in (0, a^2/m) \cup (a^2/m, \infty). \quad (21)
\]

As for every solution \( v \) of (21), relations (16), (18), (19) provide unique associated values for \( v, w \) and \( b \), understanding (21) will give a complete picture. One distinguishes two cases.

c = 0: parallel shocks. In this case, (21) has two solutions

\[
v^\pm = \frac{1}{2} \left( \frac{j}{m} \mp \sqrt{\left( \frac{j}{m} \right)^2 - 4} \right) \quad \text{if } \frac{j}{m} > 2, \quad \text{with } 0 < v^+ < v^-.
\]

The corresponding states (12) constitute a slow parallel shock iff \( mv^- < a^2 \)

(22)

and a fast parallel shock iff \( a^2 < mv^+ \).

(23)

The fact that the value of \( a \) has no influence on the \( \rho, v, w, b \) components of parallel shocks is easily understood by noticing that they have \( b = w = 0 \) and thus are purely gas dynamical.

c \neq 0: non-parallel shocks. In this case, \( g^{amc} \) tends to \( \infty \) not only for \( v \searrow 0 \) und \( v \nearrow \infty \), but also for \( v \to a^2/m \). Thus for every

\[
\frac{j}{m} > j_{\text{min}}^s(a, m, c) = \min_{(0, a^2/m)} g^{amc},
\]

(21) has two solutions

\[
v^+_s(a, m, c, j) < v^-_s(a, m, c, j) < a^2/m
\]

that constitute a slow shock. Similarly, for every

\[
\frac{j}{m} > j_{\text{min}}^f(a, m, c) = \min_{(a^2/m, \infty)} g^{amc},
\]

there are two solutions

\[
a^2/m < v^+_f(a, m, c, j) < v^-_f(a, m, c, j)
\]

that define a fast shock.

\(^2\)reversing, shifting, scaling, and the observation that cases with \( m = 0 \) or \( \rho v^2 = a^2 \) give no Lax shocks
3 Lopatinski determinant and critical manifold

According to Majda’s theory\textsuperscript{[10, 11]} on the persistence of shock fronts, the local-in-time stability of the planar discontinuous wave (9) is determined by the behaviour of the Lopatinski determinant

\[ \Delta : S_+ = (\mathbb{C}_+ \times \mathbb{R}) \setminus \{(0, 0)\} \rightarrow \mathbb{C}, \quad \Delta(\lambda, \omega) := \det(R^-(\lambda, \omega), J(\lambda, \omega), R^+(\lambda, \omega)), \quad (24) \]

where \( \mathbb{C}_+ = \{ \lambda \in \mathbb{C} : \Re\lambda > 0 \} \). While uniform stability corresponds to the non-vanishing of \( \Delta \) on all of \( S_+ \), shocks with \( \emptyset \neq \Delta^{-1}(0) \subset i\mathbb{R} \times \mathbb{R} \) or \( \emptyset \neq \Delta^{-1}(0) \cap (\mathbb{C}_+ \times \mathbb{R}) \) (25) are neutrally stable or strongly unstable, respectively. The ingredients of the Lopatinski determinant are

\[ J(\lambda, \omega) := \lambda(U^+ - U^-) + i\omega(G(U^+) - G(U^-)), \text{“jump vector”}, \]
\[ R^-(\lambda, \omega), \text{ base of the stable space of } L^- := (\lambda I + i\omega B^-)(A^-)^{-1}, \]
\[ R^+(\lambda, \omega), \text{ base of the unstable space of } L^+ := (\lambda I + i\omega B^+)(A^+)^{-1}, \]

where \( A^\pm, B^\pm \) denote \( A(U^\pm), B(U^\pm) \). The theory of hyperbolic initial-boundary value problems \textsuperscript{[7, 10]} implies that \( R^\pm \) are well-defined bundles of constant dimension. To be precise, it is on \( \mathbb{C}_+ \times \mathbb{R} = S_+ \setminus (i\mathbb{R} \times \mathbb{R}) \) that the Lopatinski matrices \( L^\pm \) have constantly trivial neutral spaces and thus “consistent splitting”, i.e., stable and unstable spaces of constant dimensions, so that in particular

\[ d^- = \dim \text{span } R^-(\lambda, \omega) \quad \text{and} \quad d^+ = \dim \text{span } R^+(\lambda, \omega) \]

are constant; for points \( (\lambda, \omega) \in S_+ \) with purely imaginary values of \( \lambda \), the \( R^\pm(\lambda, \omega) \) are defined as limits from the interior of \( S_+ \) \textsuperscript{[7]}. From the one-dimensional ‘Lax counting’ of characteristic speeds \textsuperscript{[8, 11]}, we know that

\[ d^- = 1 \quad \text{and} \quad d^+ = 3 \quad \text{for slow MHD shocks}, \quad (26) \]

while

\[ d^- = 0 \quad \text{and} \quad d^+ = 4 \quad \text{for fast MHD shocks}. \quad (27) \]

The Lopatinski determinant \( \Delta \) being degree-one homogeneous in \( (\lambda, \omega) \), we from now on fix the transverse wave number to

\[ \omega = \pm 1. \]

To avoid abundant notation, we also fix from now, again without loss of generality,

\[ \rho^- = 1 \quad (28) \]

and use the two parameters \( \rho^+, c \) instead of the three parameters \( j, m, c \). For parallel shocks, our choice \textsuperscript{[28]} implies

\[ v^- = \sqrt{\rho^+}, \quad v^+ = 1/\sqrt{\rho^+}. \]

\textsuperscript{3} This is what our passing, in Sec. 1, through the symmetric hyperbolic formulation \textsuperscript{[5]} is needed for.
In this paper we concentrate on slow shocks. The following is a key observation.

**Theorem 1.** For slow parallel MHD shocks in (1), (2) with (3) and $\rho^{-} = 1$,

$$\Delta(0, \pm 1) = 0 \text{ if } \rho^{+} = \frac{a^{2} + 2}{a^{2} + 1}. \quad (29)$$

**Proof.** Interesting manipulations show that one can take

$$R^{-} = \begin{pmatrix} \frac{1}{\sqrt{\rho^{+}}} \\ -i \sqrt{\left(a^{2} - \rho^{+}\right)\left(\frac{a^{2}}{\rho^{+}} - \frac{1}{1-\rho^{+}}\right)} \\ a \\ 0 \end{pmatrix}, \quad R^{+} = \begin{pmatrix} \sqrt{\rho^{+}} & a(\rho^{+} - 1) & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 1 \end{pmatrix}.$$  

Together with $J = (0, 0, i, 0, 0)$, this yields

$$\Delta(0, 1) = 2i[\rho^{+}(a^{2} + 1) - (a^{2} + 2)].$$

Figure 1: Slow parallel shock with $\rho^{-} = 1$, cf. Theorem 1. The black boundary is the Lax condition (22): $\rho^{-}(v^{-})^{2} < a^{2}$.

4 **Symmetry breaking**

The situation of parallel shocks is degenerate as it possesses a reflectional symmetry in the transverse ($y$-)direction. For the Lopatinski determinant this symmetry means that

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4Cf. Trakhinin’s paper [12] (and also [2, 5]) for other results.
$\Delta(\lambda, -\omega)$ vanishes exactly if $\Delta(\lambda, \omega)$ does. Perturbing the parameter $c$ away from 0 breaks this symmetry, and the zero of $\Delta$ that we found, for $c = 0$ at $\lambda = 0$, splits.

For all values of $\rho^+, a, c$ that permit a (then unique) slow MHD shock wave, we write $\Delta^{\rho^+, a, c}$ for the corresponding Lopatinski determinant. Starting from Theorem 1, we found the following.

**Numerical Observation 1.** There are an $\epsilon > 0$ and two functions,

$$R(a, c), \text{ even in } c \text{ and with } R(a, 0) = \frac{a^2 + 2}{a^2 + 1}, \text{ and } \gamma(a, c), \text{ odd in } c,$$

both defined on $\Omega_\epsilon = \{(a, c) : a \geq a_{\text{min}}(c), -\epsilon < c < \epsilon \}$, such that

$$\Delta^{R(a, c), a, c}_\epsilon (\pm i \gamma(a, c), \pm 1) = 0 \text{ for all } c \in (-\epsilon, \epsilon).$$

Figure 2: Curves $a \mapsto (R(a, c), \gamma(a, c))$ for some values of $c$ between $-0.01$ and $0.01$. The red curve corresponds to $c = 0$ and thus to the red curve in Fig. 1.

A detailed description of the numerics is postponed to a later publication.
5 Emergence of unstable modes

Do unstable modes emerge in families of shock waves that correspond to parameter values which cross the critical manifold? The following is what we conclude from numerical computations.

**Numerical Observation 2.** There are a $\delta > 0$ and a smooth function $\alpha + i \beta : \Omega \times [0, \delta) \to \mathbb{C}$ with

$$\alpha(a_0, c, 0) = 0 \quad \text{and} \quad \beta(a_0, c, 0) = \gamma(a_0, c)$$

and

$$\alpha(a_0, c, \xi) > 0 \quad \text{for} \quad \xi > 0$$

such that

$$\Delta^{R(a_0, c), a_0 + \xi, c}(\alpha(a_0, c, \xi) \pm i \beta(a_0, c, \xi), \pm 1) = 0 \quad \text{for all} \quad \xi \in [0, \delta).$$

This means that for $\xi > 0$,

$$\lambda = \alpha \pm i \beta$$

is an unstable eigenvalue for $\omega = \pm 1$.

![Figure 3: Curves $\xi \mapsto (a_0 + \xi, R(a_0, 0), \alpha(a_0, 0, \xi))$ for various values of $a_0$.](image)

A detailed description of the numerics is again postponed to a later publication.
Remark. Both from a physics perspective and as the Evans function for non-ideal shock waves is intimately related to the Lopatinski determinant for their non-ideal counterparts [13], one expects the galloping instability described in this paper to occur also in the presence of viscosity and and electrical resistivity.

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